EE5603:Conceention Inequalities

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1 Markov Inequality

1.1 Let $X \geq 0$ be a positive random integer. Show that

$$E[X] = \sum_{m=0}^{\infty} \Pr(X \geqslant m) \tag{1.1}$$

Solution: By definition,

$$E[X] = \sum_{m=0}^{\infty} m \Pr(X = m) \tag{1.2}$$

$$= \Pr(X = 1) + 2\Pr(X = 2) + 3\Pr(X = 3) + \dots \tag{1.3}$$

$$= \{\Pr(X = 1) + \Pr(X = 2) + \Pr(X = 3) + \dots\} \tag{1.4}$$

$$+ \{\Pr(X = 2) + \Pr(X = 3) + \dots\} \tag{1.5}$$

$$+ \{\Pr(X = 3) + \dots\} + \dots \tag{1.6}$$

$$= \Pr(X \geqslant 1) + 2\Pr(X \geqslant 2) + 3\Pr(X \geqslant 3) + \dots \tag{1.7}$$

resulting in (1.2).

1.2 For r.v $X \geq 0$ and $\varepsilon > 0$

$$\Pr(X \geqslant \varepsilon) \leqslant \frac{E[X]}{\varepsilon} \tag{1.8}$$

$$Ex = \int_0^\infty \Pr(x \geqslant t) dt \tag{1.9}$$

$$= \int_0^\infty \int_t^\infty f_x(x) d_x dt \tag{1.10}$$

$$= \int_0^\infty t.f_x(t) dt \tag{1.11}$$

2 discretc case:

$x = nn\epsilon IN, \Pr(n)$

$$E[x] = \Sigma_{n=1}^{\infty} n.\Pr(n) = 1.\Pr(1) + 2p(2).... \tag{2.1}$$

n=1 $\Updownarrow \rightarrow p(x \geqslant 1)$

$$= 1.\Pr(1) + 1.\Pr(2).... \tag{2.2}$$

$$\tag{2.3}$$

$$= \Sigma_{m=0}^{\infty} \Pr(x \geqslant m) \tag{2.4}$$

$$\tag{2.5}$$

E[x]=$\int_{-\infty}^{\infty} x f_x(\lambda) dx$=$\int_0^\infty x.f_x(\lambda) dx (\because x \geqslant 0)$

$$= \int_0^\varepsilon x.f_x(\lambda).dx. + \int_\varepsilon^\infty x.f_x(\lambda) dx \tag{2.6}$$

$$\geqslant \int_\varepsilon^\infty x.f_x(\lambda) dx \tag{2.7}$$

$$\geqslant \int_\varepsilon^\infty \varepsilon.f_x(\lambda) dx \tag{2.8}$$

(since x is non negetive)

$$= \varepsilon \int_\varepsilon^\infty f_x(\lambda) dx \tag{2.9}$$

$$= \varepsilon.\Pr(x \geqslant \varepsilon) \tag{2.10}$$

$$i.e E[X] \geqslant \varepsilon.\Pr(x \geqslant \varepsilon) \tag{2.11}$$

$$Or \Pr(x \geqslant \varepsilon) \leqslant \frac{EX}{\varepsilon}; \varepsilon > 0 \tag{2.12}$$

3 Chebysher ineqnality:

For any random variable x,and for any $\varepsilon > 0$,

$$\Pr[1x - E[x]| \geqslant \varepsilon] \leqslant \frac{Var(x)}{\varepsilon^2} \tag{3.1}$$

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if $\phi(\lambda)$ is stricthy monitrmic and incaring,

$$\Pr(x \geqslant \varepsilon) = \Pr(\phi(x) \geqslant \phi(\varepsilon)). \tag{3.2}$$

using this result and lelting

$$Y = |x - E[x]| \tag{3.3}$$

$\Pr(y \geqslant \varepsilon) = \Pr(\phi(Y) \geqslant \phi(c))$ Where $\phi(\lambda) = x^2$,

$$= \Pr[(x - E[x])^2 \geqslant \varepsilon^2] \tag{3.4}$$

Cheorrly $\phi(y)$ is non-negative and the μI is Valid.

$$\Rightarrow \Pr(y \geqslant \varepsilon) \leqslant \frac{E(x - Ex)}{\varepsilon^2} \tag{3.5}$$