EE5603:Concentration Inequalities

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1 Markov Inequality

1.1 Let $X \geq 0$ be a positive random integer. Show that

$$E[X] = \sum_{m=0}^{\infty} \Pr\left(X \geqslant m\right) \tag{1.1}$$

Solution: By definition,

$$E[X] = \sum_{m=0}^{\infty} m \Pr\left(X = m\right) \tag{1.2}$$
$$= \Pr\left(X = 1\right) + 2\Pr\left(X = 2\right) + 3\Pr\left(X = 3\right)$$
$$+ \dots \tag{1.3}$$
$$= \{\Pr\left(X = 1\right) + \Pr\left(X = 2\right) + \Pr\left(X = 3\right)$$
$$+ \dots\} \tag{1.4}$$
$$+ \{\Pr\left(X = 2\right) + \Pr\left(X = 3\right) + \dots\} \tag{1.5}$$
$$+ \{\Pr\left(X = 3\right) + \dots\} + \dots \tag{1.6}$$
$$= \Pr\left(X \geqslant 1\right) + 2\Pr\left(X \geqslant 2\right) + 3\Pr\left(X \geqslant 3\right)$$
$$+ \dots \tag{1.7}$$

resulting in (1.2).

1.2 For a continuious r.v $X \geq 0$, show that

$$E\left[X\right] = \int_0^{\infty} \Pr(x \geqslant t)dt \tag{1.8}$$

1.3 For r.v $X \geq 0$ and $\varepsilon > 0$, show that

$$\Pr\left(X \geqslant \varepsilon\right) \leqslant \frac{E\left[X\right]}{\varepsilon} \tag{1.9}$$

Solution: $\because X \geqslant 0$,

$$E\left[X\right] = \int_0^{\infty} x p_X(x)\, dx \tag{1.10}$$
$$= \int_0^{\varepsilon} x p_X(x)\, dx + \int_{\varepsilon}^{\infty} x p_X(x)\, dx \tag{1.11}$$
$$\geqslant \int_{\varepsilon}^{\infty} x p_X(x)\, dx \tag{1.12}$$

which can be expressed as

$$E\left[X\right] \geqslant \int_{\varepsilon}^{\infty} \varepsilon p_X(x)\, dx \tag{1.13}$$
$$= \varepsilon \int_{\varepsilon}^{\infty} p_X(x)\, dx = \varepsilon \Pr\left(X \geqslant \varepsilon\right) \tag{1.14}$$

resulting in (1.9).

1.4 *Chernoff Bound :* For any r.v X with bounded variance, and for any $t > 0$, show using (1.9) that

$$\Pr(e^{tX} \geqslant e^{t\varepsilon}) \leqslant \frac{E\left(e^{tX}\right)}{e^{t\varepsilon}} \tag{1.15}$$

1.5 Show that

$$\Pr(X \geqslant \varepsilon) = \Pr(e^{tX} \geqslant e^{t\varepsilon}) \tag{1.16}$$

Solution: This is true for any monotonic function.

2 Chebyschev inequality

2.1 Let

$$Y = \left(X - E[X]\right)^2 \tag{2.1}$$

and $\varepsilon > 0$. Show using (1.9) that

$$\Pr\left(Y \geqslant \varepsilon^2\right) \leqslant \frac{E\left(Y\right)}{\varepsilon^2} \tag{2.2}$$

2.2 Show that

$$\Pr\left(Y \geqslant \varepsilon^2\right) = \Pr\left(\sqrt{Y} \geqslant \varepsilon\right) + \Pr\left(\sqrt{Y} \leqslant -\varepsilon\right) \tag{2.3}$$

2.3 Show that

$$\Pr\left(\sqrt{Y} \leqslant -\varepsilon\right) = 0, \tag{2.4}$$

2.4 Show that

$$\Pr\left(\sqrt{Y} \geqslant \varepsilon\right) \leqslant \frac{E\left(Y\right)}{\varepsilon^2} \tag{2.5}$$

2.5 Show that

$$\Pr\left(\left|X - E[X]\right| \geqslant \varepsilon\right) \leqslant \frac{\mathrm{Var}(X)}{\varepsilon^2} \tag{2.6}$$

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3 Law of large numbers (LLN)

3.1 Let

$$S_n = \frac{1}{n}\sum_{i=1}^{n} X_i \tag{3.1}$$

where X_i are i.i.d r.v. with mean μ and bounded variance σ^2. Show that

$$E(S_n) = \mu \tag{3.2}$$

$$\text{Var}(S_n) = \frac{\sigma^2}{n} \tag{3.3}$$

3.2 Using Chebyschev inequality in (2.6), show that

$$\Pr(|S_n - \mu| \geqslant \varepsilon) \leqslant \frac{\sigma^2}{n\varepsilon^2} \tag{3.4}$$

3.3 Show that

$$\lim_{n\to\infty} \Pr(|S_n - \mu| \geqslant \varepsilon) = 0 \tag{3.5}$$

4 Hoeffding's Lemma

4.1 A convex function g is defined as

$$g(ax + b(1-x)) \leqslant xg(a) + (1-x)g(b), \quad 0 \leqslant x \leqslant 1 \tag{4.1}$$

4.2 Show that

$$g(x) = e^{sx}, s > 0 \tag{4.2}$$

is convex.

4.3 Find the equation of the line joining the points (a, e^{as}) and $\left(b, e^{bs}\right)$.

4.4 (*Jensen's Inequality*) Show that

$$g(E[X]) \leqslant E[g(X)] \tag{4.3}$$

4.5 Show that $\Pr(|S_n - \mu| \geqslant \varepsilon)$

$$mativation : s_n = \Sigma_{1=1}^{n} X_i[x_i areiid] \tag{4.4}$$

$$\tag{4.5}$$

applying chebysher inequality to $\frac{s_n}{n}$ gives us.

$$\Pr(|\frac{s_n}{n} - \mu| \geqslant \varepsilon) \leqslant \frac{\sigma^2}{n\varepsilon^2} \tag{4.6}$$

$$\tag{4.7}$$

where $Var(X_i) = \sigma^2 EX_i = \mu$.
how about the tail problitity $\Pr(|\frac{s_n}{n} - \mu \geqslant \varepsilon)$?

we know $Pr(\frac{s_n}{n} - \mu \geqslant \varepsilon) = \Pr(e^{\lambda(\frac{s_n}{n})} - \mu \geqslant e^{\lambda\varepsilon}$ apply

$$\mu.I. \leqslant \frac{E[e^{\lambda}(\frac{s_n}{n} - \mu)]}{e^{\lambda}\varepsilon} \tag{4.8}$$

$$\tag{4.9}$$

$$= e^{-\lambda\varepsilon}E[e^{\lambda}[\frac{1}{n}\Sigma_{1=1}^{n}X_i - E[Xi] \tag{4.10}$$

$$\tag{4.11}$$

$$= e^{-}\lambda\varepsilon\Pi_{1=1}^{n}E[e^{\lambda}\frac{X_i - E[xi]}{n}] \tag{4.12}$$

$$\tag{4.13}$$

Question: How canwe come with a toght bound for the
moment guerating function of (X-E[X]).

5 hoeffding's lemma:

if x is a radom vaniable with E[X]=0 and a$\leqslant X \leqslant b$, then for a any s>0
$E[e^{sx}] \leqslant e^{\frac{s^2(b-a)^2}{8}}$