EE5603:Conceention Inequalities

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1 Convergence

1.1 Definitions

- Today:
- review basic inqulities
- law of large numbers (proof)
- demo on LLN and CLT
- hoeffohing's inqulity:

  Review: Markor inequality: for a non-negative RVX and for any$\varepsilon > 0$

$$\Pr(x \geqslant \varepsilon) \leqslant \frac{EX}{\varepsilon} \tag{1.1}$$

$$\tag{1.2}$$

Chebyshow inequality:for any RVX with bounded variance,

$$\Pr\left(|x - E[X]| \geqslant \varepsilon\right) \leqslant \frac{Var(x)}{\varepsilon^2} \tag{1.3}$$

$$\tag{1.4}$$

Cheruff bouned :for any RVX with bounded variance,and for any $t > 0$,

$$\Pr(e^{tx} \geqslant e^{t\varepsilon}) \leqslant \frac{Ee^{tx}}{e^{t\varepsilon}} \tag{1.5}$$

$$\tag{1.6}$$

2 Law of large numbers (LLN)

$$\Pr(lim_{n\longrightarrow\infty}|\frac{1}{n}s_n - m| \geqslant \varepsilon) = 0 \tag{2.1}$$

$$\tag{2.2}$$

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$s_n = \Sigma_{1=1}^n X_i$ are iid RVS with mean μ and bounded variance σ^2

applying chebyshev inequality,

$$\Pr\left(|\frac{s_n}{n} - \mu| \leqslant \varepsilon\right) \leqslant \frac{Var(\frac{s_n}{n})}{\varepsilon^2} \tag{2.3}$$

$$\tag{2.4}$$

$Var(s_n) = \Sigma_{1=1}^n Var(x_i) = \frac{\sigma^2}{n}$

$$\therefore \Pr(|\frac{s_n}{n} - \mu| \geqslant \varepsilon) \leqslant \frac{\sigma^2}{n\varepsilon^2} \tag{2.5}$$

$$\tag{2.6}$$

$\Rightarrow lim_n \to \infty \Pr(|\frac{s_n}{n} - \mu| \geqslant \varepsilon) \leqslant lim_n \to \infty \frac{\sigma^2}{n\varepsilon^2}$ =0

3 Hoeffding's inequalitity:

$$mativation: s_n = \Sigma_{1=1}^n X_i[x_i areiid] \tag{3.1}$$

$$\tag{3.2}$$

applying chebysher inequality to $\frac{s_n}{n}$ gives us.

$$\Pr(|\frac{s_n}{n} - \mu| \geqslant \varepsilon) \leqslant \frac{\sigma^2}{n\varepsilon^2} \tag{3.3}$$

$$\tag{3.4}$$

where $Var(X_i) = \sigma^2 EX_i = \mu$.

- **how about the tail problitity** $\Pr(|\frac{s_n}{n} - \mu \geqslant \varepsilon)$?

we know $Pr(\frac{s_n}{n} - \mu \geqslant \varepsilon) = \Pr(e^{\lambda(\frac{s_n}{n})} - \mu \geqslant e^{\lambda\varepsilon}$ apply

$$\mu.I. \leqslant \frac{E[e^{\lambda}(\frac{s_n}{n} - \mu)]}{e^{\lambda}\varepsilon} \tag{3.5}$$

$$\tag{3.6}$$

$$= e^{-\lambda\varepsilon} E[e^{\lambda}[\frac{1}{n}\Sigma_{1=1}^{n} X_i - E[Xi] \tag{3.7}$$

$$\tag{3.8}$$

$$= e^{-}\lambda\varepsilon\Pi_{1=1}^{n} E[e^{\lambda}\frac{X_i - E[xi]}{n}] \tag{3.9}$$

$$\tag{3.10}$$

Question: How canwe come with a toght bound for the
moment guerating function of (X-E[X]).

4 HOEFFDING'S LEMMA:

if x is a radom vaniable with E[X]=0 and a$\leqslant X \leqslant b$, then for a any s>0

$E[e^{sx}] \leqslant e^{\frac{s^2(b-a)^2}{8}}$