EE5603:Concentration Inequalities

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1 Markov Inequality

1.1 Let $X \geq 0$ be a positive random integer. Show that

$$E[X] = \sum_{m=0}^{\infty} \Pr\left(X \geqslant m\right) \tag{1.1}$$

Solution: By definition,

$$E[X] = \sum_{m=0}^{\infty} m \Pr\left(X = m\right) \tag{1.2}$$
$$= \Pr\left(X = 1\right) + 2\Pr\left(X = 2\right) + 3\Pr\left(X = 3\right)$$
$$+ \dots \tag{1.3}$$
$$= \{\Pr\left(X = 1\right) + \Pr\left(X = 2\right) + \Pr\left(X = 3\right)$$
$$+ \dots\} \tag{1.4}$$
$$+ \{\Pr\left(X = 2\right) + \Pr\left(X = 3\right) + \dots\} \tag{1.5}$$
$$+ \{\Pr\left(X = 3\right) + \dots\} + \dots \tag{1.6}$$
$$= \Pr\left(X \geqslant 1\right) + 2\Pr\left(X \geqslant 2\right) + 3\Pr\left(X \geqslant 3\right)$$
$$+ \dots \tag{1.7}$$

resulting in (1.2).

1.2 For a continuious r.v $X \geq 0$, show that

$$E\left[X\right] = \int_0^{\infty} \Pr(x \geqslant t)dt \tag{1.8}$$

1.3 For r.v $X \geq 0$ and $\varepsilon > 0$, show that

$$\Pr\left(X \geqslant \varepsilon\right) \leqslant \frac{E\left[X\right]}{\varepsilon} \tag{1.9}$$

Solution: $\because X \geqslant 0$,

$$E\left[X\right] = \int_0^{\infty} x p_X(x)\, dx \tag{1.10}$$
$$= \int_0^{\varepsilon} x p_X(x)\, dx + \int_{\varepsilon}^{\infty} x p_X(x)\, dx \tag{1.11}$$
$$\geqslant \int_{\varepsilon}^{\infty} x p_X(x)\, dx \tag{1.12}$$

which can be expressed as

$$E\left[X\right] \geqslant \int_{\varepsilon}^{\infty} \varepsilon p_X(x)\, dx \tag{1.13}$$
$$= \varepsilon \int_{\varepsilon}^{\infty} p_X(x)\, dx = \varepsilon \Pr\left(X \geqslant \varepsilon\right) \tag{1.14}$$

resulting in (1.9).

2 Chebyschev inequality

2.1 For any $\varepsilon > 0$, show that

$$\Pr\left(|X - E[X]| \geqslant \varepsilon\right) \leqslant \frac{\mathrm{Var}(X)}{\varepsilon^2} \tag{2.1}$$

Solution: Let

$$Y = \left(X - E[X]\right)^2 \tag{2.2}$$

From (1.9),

$$\Pr\left(Y \geqslant \varepsilon^2\right) \leqslant \frac{E\left(Y\right)}{\varepsilon^2} \tag{2.3}$$
$$\implies \Pr\left(\sqrt{Y} \geqslant \varepsilon\right) + \Pr\left(\sqrt{Y} \leqslant -\varepsilon\right) \leqslant \frac{E\left(Y\right)}{\varepsilon^2} \tag{2.4}$$

$$\because \sqrt{Y} = |X - E[X]|, \tag{2.5}$$
$$\Pr\left(\sqrt{Y} \leqslant -\varepsilon\right) = 0, \tag{2.6}$$

substituting in (2.4) results in (2.1).

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