EE5603:Conceention Inequalities

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1 Convergence

1.1 Definitions

Review: •Hoeffding's lemma: For a random variable x with $E[Xi] = 0$ and a⩽x⩽b,

$$E[e^{sx}] \leqslant e^{s2}\frac{(b-a)^2}{8} \tag{1.1}$$

Recall

$$L(h) = -hp + log[(1-p) + p.e^h] where \tag{1.2}$$

$$h = s.(b-a), p = \frac{-a}{(b-a)} \tag{1.3}$$

$$L^{"}(h) \leqslant \frac{1}{4} foranyh. \tag{1.4}$$

•Hoeffding's inequality: if $s_n\Sigma_{1=1}^n$ xi where xi's are independent RVs with $ai \leqslant xi \leqslant bi$
then

$$\Pr s_n - ES_n \leqslant t \leqslant exp[\frac{-2t^2}{\Sigma_{1=1}^n (bi - ai.)^2}], \tag{1.5}$$

$$\Pr s_n - Es_n \leqslant -t \leqslant exp[\frac{-2t^2}{\Sigma_{1=1}^n (bi - ai)^2}] \tag{1.6}$$

Taylor's theorm: if $f(x)$ is acontinus function in the bound intirral [a,b] and has $f^1(x)$ and $f"(x)$define in this interal
then

$$f(h) = f(0).h^0 + f^1(0).\frac{h^1}{1!} + f"(v)\frac{h^2}{2!} \tag{1.7}$$

sub- ganssion RV: A real valud RV x is s and to be σ^2-sub.lynmian if there exist a σ such that

$$E[e^{\lambda x} \leqslant exp(\frac{\lambda^2\sigma^2}{2}) forany \tag{1.8}$$

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• **Observation about the Hoeffding's inequality:** The bound does not inuber the varianuc of the R.V .

• **Question:**can we find afighter bound when the RV has low variance.? Yes, the Bennrtt's inequality.

2 Benntt's

let,

$$X_1, \ldots, X_n \tag{2.1}$$

be independent RVS with finite variance and $xi \leqslant b$ for $b > 0$almost surely for $i \leqslant n$.
let

$$\vartheta = \Sigma_{1=1}^n E[xi^2].if s = \Sigma_{1=1}^n (xi - Exi), then \tag{2.2}$$

$$log[Ee^{\lambda} s] \leqslant n.log[1 + \frac{\vartheta}{nb^2}\phi(\lambda b)] \leqslant \tag{2.3}$$

where

$$\phi(u) = e^u - u - 1 u\epsilon R \tag{2.4}$$

Proof:• let no assume b=1.

• show that $u^1.\phi(u)$ is a non decreasing function.

$$g(u) = \frac{\phi(u)}{n^2} \Rightarrow g^1(u) = \frac{e^u(u-2) + (u+2)}{u^3} \tag{2.5}$$

$$(\lambda xi)^2\phi(\lambda xi) \leqslant \lambda^2\phi(\lambda) \tag{2.6}$$

$$\Rightarrow \phi(\lambda xi) \leqslant xi^2\phi(\lambda) \tag{2.7}$$

$$i.e.[e^{\lambda xi} - \lambda xi - 1 \leqslant xi^2(e^{\lambda} - \lambda - 1)] \tag{2.8}$$

$$E[e^{\lambda xi - \lambda xi - 1}] \leqslant \phi(\lambda).E[xi^2] \tag{2.9}$$

$$E[e^{\lambda xi}] \leqslant (E[\lambda XI] + 1 + E[xi^2].\phi(\lambda)) \tag{2.10}$$

$$\Sigma_{1=1}^{n} logE[e^{\lambda xi]} \leqslant \Sigma_{1=1}^{n} log(E[\lambda xi]) + 1 + E[XI^2].\phi(X) \tag{2.11}$$

$$log[\Pi_{1=1}^{n} E[e^{\lambda .xi}] \leqslant\longrightarrow 1 \tag{2.12}$$

$$log[\Pi_{1=1}^{n} E[e^{\lambda xi - Exi}] + \Sigma_{1=1}^{n} \lambda .E[xi] \leqslant \Sigma_{1=1}^{n} log(E[\lambda xi]1 + E[xi^2]\phi(\lambda)] \tag{2.13}$$

$\psi(\lambda)$ where $\psi_s(\lambda)$ =log $[E[e^{\lambda s}]]$

$$\psi_s(\lambda) \leqslant \Sigma_{1=1}^{n} log[E(\lambda xi) + 1 + E[xi^2]\phi(\lambda)] \tag{2.14}$$

$$-\Sigma_{1=1}^{n} .Exi. \tag{2.15}$$

$$\leqslant n\Sigma_{1=1}^{n} \frac{log}{n}[] - \Sigma_{1=1}^{n} \lambda E[xi] \tag{2.16}$$