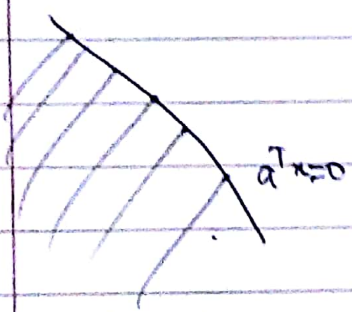
Convex optimisation

* Linear, Affine & Convex combinations :-

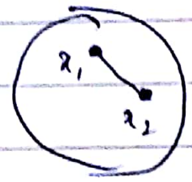
1) If an affine set includes the origin, then it is a subspace. Suppose $x_1, x_2, x_3 \ldots, x_k \in S$
$\theta_1 x_1 + \theta_2 x_2 + \ldots + \theta_k x_k + (1-\theta_1-\theta_2 \ldots \theta_k) 0$
Affine sets are translated subspaces.
Affine sets are high-dimensional dialogue of line.

- Hyperplanes :-
$$S = \{x : a^T x = b\}, x \in R$$

Half-space are convex but not affine.



- Balls :-
$$S = \{x . \|x\| \le 1\}$$



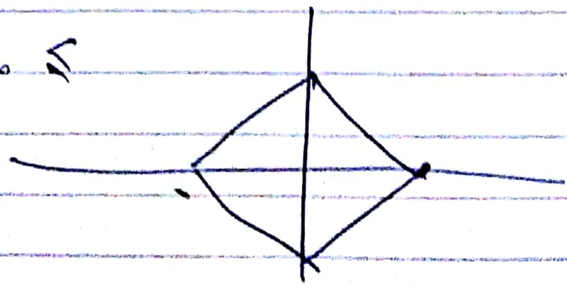
S is convex.

$\|\theta x_1 + (1-\theta) x_2\| \le \theta \|x_1\| + (1-\theta)\|x_2\|$
≤ 1

* Euclidean Norm :-
$\|x_p\| = \sqrt[p]{x_1^p + x_2^p + \ldots}$
Ball is 2 norm ball

* 1 norm ball :- $S = \langle x . |x_1| + |x_2| \le 1 \rangle$
$\|x\|_\infty = \max_{i=1}^{n} \langle |x_i| \rangle$

$S = \{x : \|x\|_\infty \le$



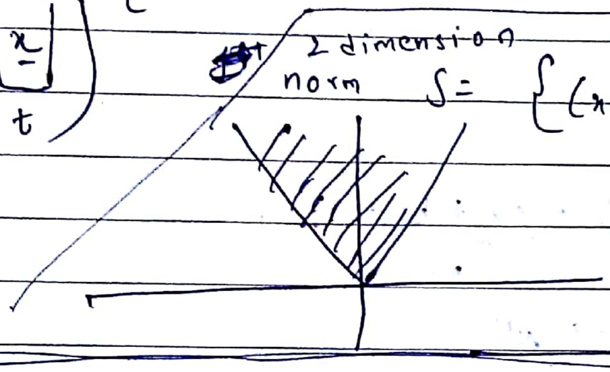
DATE | | PAGE

* Cones :- (ray must lie inside cone)
- A set S is a cone if $x \in S \Rightarrow \theta x \in S$, $\theta \geq 0$

* Convex Cone θ :-
- S is convex cone if for every $x_1, x_2, \ldots x_k \in S$
$\theta_1 x_1 + \theta_2 x_2 + \ldots + \theta_k x_k \in S$ where $\theta_i \geq 0$.
Conic combination of $x_1, x_2, \ldots x_k$ is $\theta_1 x_1 + \theta_2 x_2 + \ldots \theta_k x_k$; $\theta_i \geq 0$
- Example :- Norm cones
$S = \{ (x,t) : \|x\| \leq t, t > 0 \}$

$\begin{pmatrix} \boxed{x} \\ t \end{pmatrix}$

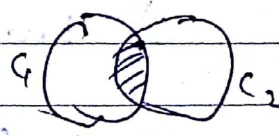
2 dimension norm $S = \{ (x,t), |x| \leq t, t > 0 \}$



Q.) Prove that norm cones are convex cones.
$\theta_1 x_1 + \theta_2 x_2 + \ldots \theta_k x_k$
$0 \leq \theta_1 \leq 1$; $\Sigma \theta_i = 1$
Generalisation
$$\int_{x \in C} p(x)\, x \, dx \in C$$

$p(x) \geq 0$ and $\int_{x \in C} p(x)\, dx = 1$

* Operations that preserve convexity :-



Intersection :- If C_1 & C_2 are convex ; then $C_1 \cap C_2$ is convex

Q.) Exercise : Complete the proof.

Scanned by CamScanner

Polyhedra :-

$$S = \{ x : A x \le b \}$$

If C_i is a family of convex sets, then $\bigcap C_i$ is convex. S is an intersection of m half spaces.

$$S = \{ x : |x_1 \cos t + x_2 \cos 2t + \dots + x_n \cos nt| \le 1 \text{ for all } 0 \le t \le 1 \}$$

S is convex.

Affine functions :-

$$f(x) = \underset{m \times n}{A} x + \underset{n \times 1}{b} \leftarrow m \times 1$$

$$f : R^n \to R^m.$$

If $x_1, x_2, \dots x_k \in \text{dom } f$
then $f(\theta_1 x_1 + \theta_2 x_2 + \dots \theta_k x_k) = \sum_{i=1}^{k} \theta_i f(x_i)$

If C is convex ; $f(C)$ is convex.
Proof :- Consider $f(x_1)$, $f(x_2)$ $x_1, x_2 \in C$.
$(0 < \theta \le 1)$

$$\theta f(x_1) + (1-\theta) f(x_2) = f(\theta x_1 + (1-\theta) x_2) \in f(C)$$

$f(C)$ is the image of C under f.
$f^{-1}(C)$ is the preimage of C under f.
Exercise : Complete the proof :-

$C \subseteq R^3$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

* Positive semidefinite matrix

$A \in S^n$ is psd if $x^T A x \geq 0$ for all x

Set of all positive semidefinite matrices is denoted S_+^n

* Theorem S_+^n is a convex cone.

Scanned by CamScanner