Convex Optimisation (22/1/19) :

* **Perspective Map :-**

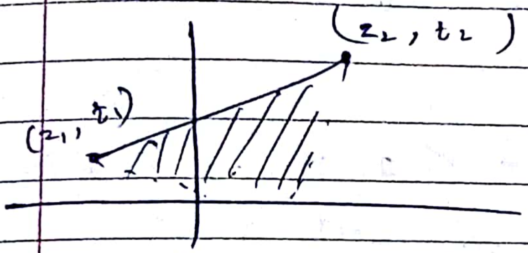
$P(z,t) = \frac{z}{t}$ $\quad R_{++} = \{x \in R, x > 0\}$

$R_{+} = \{x \in R, x \geq 0\}$

dom $P = R^n \times R_{++}$

ran $P = R^n$

- If P is a perspective map & C, $C \subseteq$ dom P, then $P(C)$ is convex.
- If C is convex in R^n; $P^{-1}(C)$ is convex in $R^n \times R_{++}$.
- Proof :-



$(\theta z_1 + (1-\theta) z_2 , \theta t_1 + (1-\theta) t_2)$

$$\frac{\theta z_1 + (1-\theta) z_2}{\theta t_1 + (1-\theta) t_2}$$

$$= \underbrace{\left(\frac{\theta t_1}{\theta t_1 + (1-\theta) t_2}\right)}_{\mu} \frac{z_1}{t_1} + \underbrace{\left(\frac{(1-\theta) t_2}{\theta t_1 + (1-\theta) t_2}\right)}_{1-\mu} \frac{z_2}{t_2}$$

Q.) Exercise : P.T. If C is convex then $P^{-1}(C)$ is convex

- 1) $S = \{(x,y,z) : x^2+y^2 \leq z^2 , z > 0\} \Rightarrow$ 2 norm-cone in R^3
- 2) $S = \{(x,z) : x^2+1 \leq z^2 ; z > 0\} \Rightarrow$ Hyperbolic sets

Is (2) convex ? ~~Exercise~~

3) $S = \{(x,y) : xy \geq 1 , x \geq 0 , y \geq 0\}$

Exercise : start from def^n & show that set is convex

4) $S_n^+ = \{(x,y,z) ; xy \geq z^2 , x \geq 0 ; y \geq 0\}$

$P((x,y,z)) = \left(\frac{x}{z}, \frac{y}{z}\right)$

$$S_n^+ = \left\{ \left(\frac{x}{z}\right) \cdot \left(\frac{y}{z}\right) \geq 1 \right\}$$

S is perspective map of S_n^+ $\Rightarrow$ Claim

- $x \in \mathbb{R}^{n+1}$

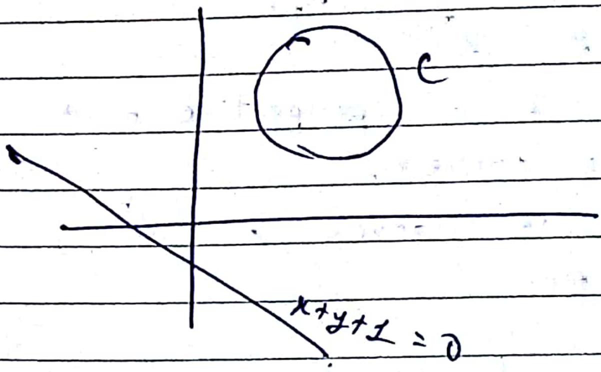
$$Ax+b = \begin{pmatrix} a_1^T x + b_1 \\ a_2^T x + b_2 \\ \vdots \\ a_{n+1}^T x + b_{n+1} \end{pmatrix} \Rightarrow \text{Affine mapping}$$

perspective map provided $a_{n+1}^T x + b_{n+1} \geq 0$

$$\begin{pmatrix} \dfrac{a_1^T x + b_1}{a_{n+1}^T x + b_{n+1}} \\ \dfrac{a_2^T x + b_2}{a_{n+1}^T x + b_{n+1}} \\ \vdots \\ \vdots \end{pmatrix}$$

[Linear fractional

suppose C is convex, & a



$$f(x,y) = \frac{x}{x+y+1}$$

- $\Omega = \langle 1, 2, \ldots 6 \rangle \times \langle 1, 2, \ldots 6 \rangle$

$S_1 = \{$ all possible pmf on $\Omega \}$

S_1 : convex set, simplex, polyhedron

$S_2 = \{$ all possible conditional pmfs on $\Omega \}$

prob $(X_i = i \mid X_j = j)$

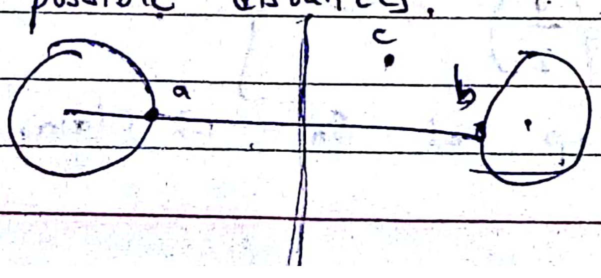
$$P(X_1 = i \mid X_2 = j) = \frac{P(X_1 = i, X_2 = j)}{\sum_{i=1}^{6} P(X_1 = \ldots)}$$

* Separating hyperplane theorem :-

- If A & B are disjoint convex sets then there exists $\lambda \neq 0$ & μ such that $\lambda^T x \leq \mu$ for every $x \in A$ & $\lambda^T x \geq \mu$ for every $x \in B$.

$D(A,B) = \inf \{ \|a-b\| ; a \in A, b \in B \}$

$a \in A$; $b \in B$ such that $\|a-b\|$ is smallest among all possible distances.

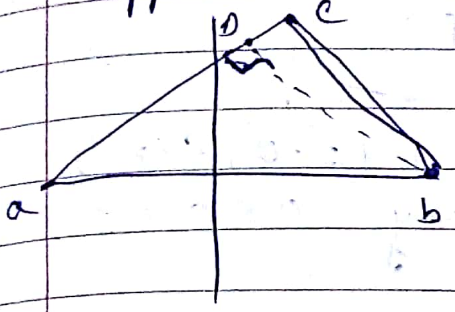


$\|x-a\|^2 = \|x-b\|^2$

Scanned by CamScanner

$(x-a)^T(x-a)$

Suppose $c \in A$ $\quad \|c-b\| < \|c-a\|$



c is in A

D is closer now than a from b.

$\Rightarrow$ Contradiction. ($Db < ab$).

* Converse :-

- A & B are convex sets that have a separating ~~(atleast one of them has boundary~~ removed) hyperplane, then A & B are disjoint.

* Supporting hyperplane theorem :-

A is convex, at any point x_0 on the boundary, we can draw a hyperplane passing through x_0 such ~~that~~ A is contained in one halfspace.

* Theorem of alternates :-

$-Ax > 0$ | $S_1 = \{-Ax\}$ $\quad S_2 = R^n_{++}$

* Gordon's lemma :- ; Farkas Lemma :-

① $\lambda^T y \leq \mu$, $y \in S_1$ $\quad$ ② $\lambda^T x \geq \mu$, $x \in S_2$

$\Rightarrow$ From ①, $\mu = 0$ and $\lambda^T y = 0$ for all $y \in S_1$

$\Rightarrow$ From ②, $\lambda \geq 0$

* Gordon's :- Exactly one of those is feasible

$-Ax > 0$	$\lambda^T A = 0$
	$\lambda \geq 0$