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Abstract—The manual frames the problems of receiver design and performance analysis in digital communication as applications of probability theory.

Download all codes in this manual from

svn co <https://github.com/gadepall/comm/trunk/modulation/manual/codes>

1 BPSK

1.1. The *signal constellation diagram* for BPSK is given by Fig. 1.1. The symbols s_0 and s_1 are equiprobable. $\sqrt{E_b}$ is the energy transmitted per bit. Assuming a zero mean additive white gaussian noise (AWGN) with variance $\frac{N_0}{2}$, obtain the symbols that are received. **Solution:** The possible received symbols are

$$y|s_0 = \sqrt{E_b} + n \quad (1.1.1)$$

$$y|s_1 = -\sqrt{E_b} + n \quad (1.1.2)$$

where the AWGN $n \sim \mathcal{N}(0, \frac{N_0}{2})$.

1.2. From Fig. 1.1 obtain a decision rule for BPSK
Solution: The decision rule is

$$y \underset{s_1}{\overset{s_0}{\geq}} 0 \quad (1.2.1)$$

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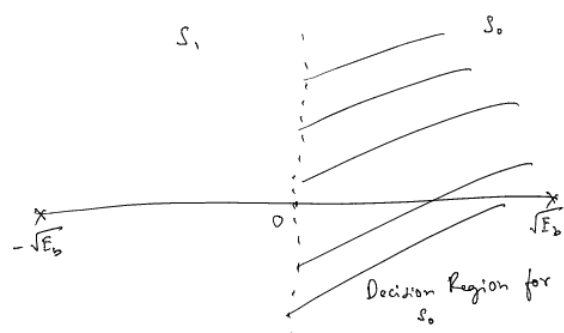


Fig. 1.1

1.3. Repeat the previous exercise using the MAP criterion.

1.4. Using the decision rule in Problem 1.2, obtain an expression for the probability of error for BPSK.

Solution: Since the symbols are equiprobable, it is sufficient if the error is calculated assuming that a 0 was sent. This results in

$$P_e = \Pr(y < 0|s_0) = \Pr(\sqrt{E_b} + n < 0) \quad (1.4.1)$$

$$= \Pr(-n > \sqrt{E_b}) = \Pr(n > \sqrt{E_b}) \quad (1.4.2)$$

since n has a symmetric pdf. Let $w \sim \mathcal{N}(0, 1)$.

Then $n = \sqrt{\frac{N_0}{2}}w$. Substituting this in (1.4.2),

$$P_e = \Pr\left(\sqrt{\frac{N_0}{2}}w > \sqrt{E_b}\right) = \Pr\left(w > \sqrt{\frac{2E_b}{N_0}}\right) \quad (1.4.3)$$

$$= Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \quad (1.4.4)$$

where

$$Q(x) \triangleq \Pr(w > x), x \geq 0. \quad (1.4.5)$$

$$= \frac{1}{\pi} \int_0^{\frac{\pi}{2}} e^{-\frac{x^2}{2 \sin^2 \theta}} d\theta \quad (1.4.6)$$

The PDF of $w \sim \mathcal{N}(0, 1)$ is given by

$$p_w(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty \quad (1.4.7)$$

and the complementary error function is defined as

$$\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt. \quad (1.4.8)$$

1.5. Show that

$$Q(x) = \frac{1}{2} \text{erfc}\left(\frac{x}{\sqrt{2}}\right) \quad (1.5.1)$$

Solution: From (1.4.5)

$$Q(x) = \Pr(w > x), x \geq 0 \quad (1.5.2)$$

$$= \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{t^2}{2}} dt. \quad (1.5.3)$$

$$= \frac{1}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{2}}}^\infty e^{-y^2} dy. \quad (t = \sqrt{2}y) \quad (1.5.4)$$

resulting in

1.6. Verify the bit error rate (BER) plots for BPSK through simulation and analysis for 0 to 10 dB.

Solution: The following code

```
codes/bpsk_ber.py
```

yields Fig. 1.6

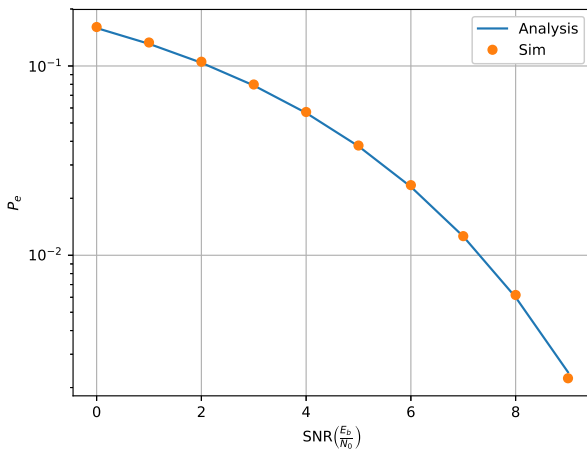


Fig. 1.6

2 COHERENT BFSK

2.1. The signal constellation for binary frequency shift keying (BFSK) is given in Fig. 2.1. Obtain

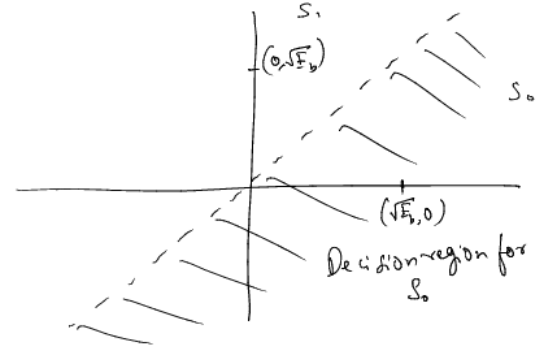


Fig. 2.1

the equations for the received symbols.

Solution: The received symbols are given by

$$\mathbf{y}|s_0 = \begin{pmatrix} \sqrt{E_b} \\ 0 \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, \quad (2.1.1)$$

and

$$\mathbf{y}|s_1 = \begin{pmatrix} 0 \\ \sqrt{E_b} \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, \quad (2.1.2)$$

where $n_1, n_2 \sim \mathcal{N}(0, \frac{N_0}{2})$. and $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$.

2.2. Obtain a decision rule for BFSK from Fig. 2.1.

Solution: The decision rule is

$$y_1 \underset{s_1}{\overset{s_0}{\geq}} y_2 \quad (2.2.1)$$

Definition 1. The joint PDF of X, Y is given by

$$p(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}\right] \\ \times \left\{ \frac{(x-\mu_x)^2}{\sigma_x^2} + \frac{(y-\mu_y)^2}{\sigma_y^2} - \frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y} \right\} \quad (2.2.2)$$

where

$$\mu_x = E[X], \quad (2.2.3)$$

$$\sigma_x^2 = \text{var}(X), \quad (2.2.4)$$

$$\rho = \frac{E[(X-\mu_x)(Y-\mu_y)]}{\sigma_x\sigma_y}. \quad (2.2.5)$$

For equiprobably symbols, the MAP criterion is defined as

$$p(\mathbf{y}|s_0) \underset{s_1}{\overset{s_0}{\geq}} p(\mathbf{y}|s_1) \quad (2.2.6)$$

2.3. Use (2.2.2) in (2.2.6) to obtain (2.2.1).

Solution: According to the MAP criterion, assuming equiprobably symbols,

$$p(\mathbf{y}|s_0) \underset{s_1}{\overset{s_0}{\gtrless}} p(\mathbf{y}|s_1) \quad (2.3.1)$$

2.4. Derive and plot the probability of error. Verify through simulation.

Solution: Given that s_0 was transmitted, the received symbols are

$$\mathbf{y}|s_0 = \begin{pmatrix} \sqrt{E_b} \\ 0 \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, \quad (2.4.1)$$

From (2.2.1), the probability of error is given by

$$P_e = \Pr(y_1 < y_2|s_0) = \Pr(\sqrt{E_b} + n_1 < n_2) \quad (2.4.2)$$

$$= \Pr(n_2 - n_1 > \sqrt{E_b}) \quad (2.4.3)$$

Note that $n_2 - n_1 \sim \mathcal{N}(0, N_0)$. Thus,

$$P_e = \Pr(\sqrt{N_0}w > \sqrt{E_b}) \quad (2.4.4)$$

$$= \Pr\left(w > \sqrt{\frac{E_b}{N_0}}\right) \quad (2.4.5)$$

$$\Rightarrow P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right) \quad (2.4.6)$$

where $w \sim \mathcal{N}(0, 1)$. The following code plots the BER curves in Fig. 2.4

```
codes/fsk_ber.py
```

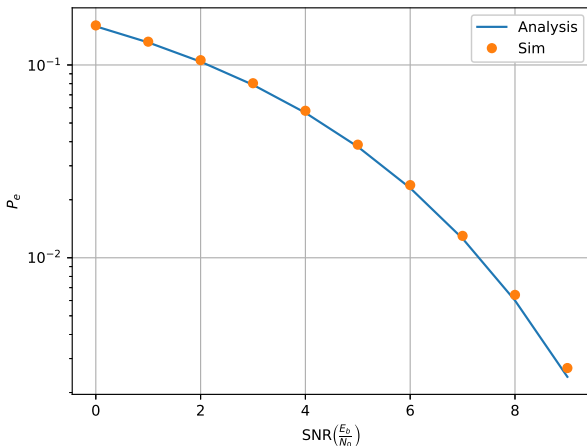


Fig. 2.4

3 QPSK

1. Let

$$\mathbf{y} = \mathbf{s} + \mathbf{n} \quad (3.1.1)$$

where $\mathbf{s} \in \{\mathbf{s}_0, \mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}$ and

$$\mathbf{s}_0 = \begin{pmatrix} \sqrt{E_s} \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ \sqrt{E_s} \end{pmatrix}, \quad (3.1.2)$$

$$\mathbf{s}_2 = \begin{pmatrix} -\sqrt{E_s} \\ 0 \end{pmatrix}, \mathbf{s}_3 = \begin{pmatrix} 0 \\ -\sqrt{E_s} \end{pmatrix}, \quad (3.1.3)$$

$$E[\mathbf{n}] = \mathbf{0}, E[\mathbf{n}\mathbf{n}^T] = \frac{N_0}{2}\mathbf{I} \quad (3.1.4)$$

2. Using (2.2.2), show that the MAP decision for detecting \mathbf{s}_0 results in

$$|y_2| < y_1 \quad (3.2.1)$$

3. Express $\Pr(\hat{\mathbf{s}} = \mathbf{s}_0|\mathbf{s} = \mathbf{s}_0)$ in terms of y_1, y_2 .

Solution: From (3.2.1) and (3.1.2),

$$\begin{aligned} \Pr(\hat{\mathbf{s}} = \mathbf{s}_0|\mathbf{s} = \mathbf{s}_0) \\ = \Pr(|y_2| < y_1 | y_1 = \sqrt{E_s}, y_2 = 0) \end{aligned} \quad (3.3.1)$$

4. Let

$$X = n_2 - n_1, \quad (3.4.1)$$

$$Y = -n_2 - n_1, \quad (3.4.2)$$

where $\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$.

Show that $X, Y \sim \mathcal{N}(0, N_0)$.

5. The correlation coefficient of X, Y is defined as

$$\rho = \frac{E[(X - \mu_x)(Y - \mu_y)]}{\sigma_x \sigma_y} \quad (3.5.1)$$

X and Y are said to be uncorrelated if $\rho = 0$

Show that X and Y are uncorrelated. Verify this numerically.

Solution: From (3.1.4),

$$\mu_x = E[X] = 0 \quad (3.5.2)$$

$$\mu_y = E[Y] = 0 \quad (3.5.3)$$

$$\begin{aligned} \Rightarrow \rho &= E[XY] = E[(n_2 - n_1)(-n_2 - n_1)] \\ &= 0 \end{aligned} \quad (3.5.4)$$

upon substituting from (3.4.1) and (3.4.2).

6. Show that X and Y are independent, i.e.

$$p_{XY}(x, y) = p_X(x)p_Y(y). \quad (3.6.1)$$

Solution: Use (3.5.4) in (2.2.2) to get (3.6.1).

Uncorrelated Gaussians are independent.

7. Show that

$$\Pr(\hat{s} = s_0 | s = s_0) = \Pr(X < \sqrt{E_s}) \Pr(Y < \sqrt{E_s}). \quad (3.7.1)$$

Solution: From (3.3.1) and (3.1.1)

$$\begin{aligned} \Pr(\hat{s} = s_0 | s = s_0) \\ = \Pr(|n_2| < \sqrt{E_s} + n_1) \end{aligned} \quad (3.7.2)$$

which can be expressed as

$$\Pr(n_2 < \sqrt{E_s} + n_1, -n_2 > \sqrt{E_s} + n_1) \quad (3.7.3)$$

$$= \Pr(X < \sqrt{E_s}, Y < \sqrt{E_s}) \quad (3.7.4)$$

$$= \Pr(X < \sqrt{E_s}) \Pr(Y < \sqrt{E_s}) \quad (3.7.5)$$

after some algebra, using the fact that X, Y are independent.

8. Show that

$$\Pr(\hat{s} = s_0 | s = s_0) = \left(1 - Q\left(\sqrt{\frac{E_s}{N_0}}\right)\right)^2 \quad (3.8.1)$$

Solution: From ,

$$\Pr(X > \sqrt{E_s}) = \Pr(Y > \sqrt{E_s}) = Q\left(\sqrt{\frac{E_s}{N_0}}\right) \quad (3.8.2)$$

yielding (3.8.1).

9. Verify the above through simulation.

Solution: This is shown in Fig. 3.9 through the following code.

codes/qpsk.py

10. Modify the above script to obtain the probability of symbol error.

4 M-PSK

1. Consider a system where $\mathbf{s}_i = \begin{pmatrix} \cos\left(\frac{2\pi i}{M}\right) \\ \cos\left(\frac{2\pi i}{M}\right) \end{pmatrix}$, $i = 0, 1, \dots, M-1$. Let

$$\mathbf{y}|s_0 = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \sqrt{E_s} + n_1 \\ n_2 \end{pmatrix} \quad (4.1.1)$$

where $n_1, n_2 \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$.

Substituting

$$y_1 = R \cos \theta \quad (4.1.2)$$

$$y_2 = R \sin \theta \quad (4.1.3)$$

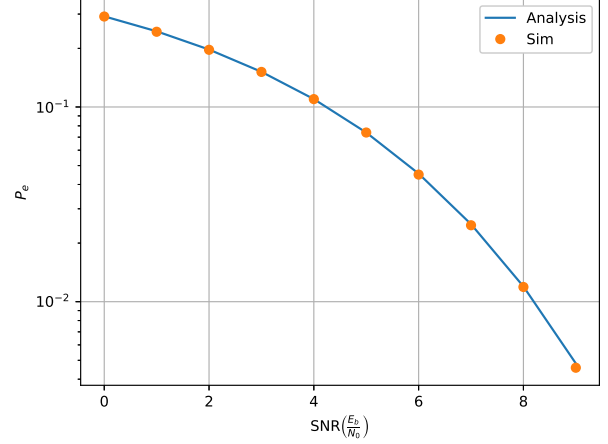


Fig. 3.9

show that the joint pdf of R, θ is

$$p(R, \theta) = \frac{R}{\pi N_0} \exp\left(-\frac{R^2 - 2R\sqrt{E_s}\cos\theta + E_s}{N_0}\right) \quad (4.1.4)$$

2. Show that

$$\lim_{\alpha \rightarrow \infty} \int_0^\infty (V - \alpha) e^{-(V-\alpha)^2} dV = 0 \quad (4.2.1)$$

$$\lim_{\alpha \rightarrow \infty} \int_0^\infty e^{-(V-\alpha)^2} dV = \sqrt{\pi} \quad (4.2.2)$$

3. Using the above, show that

$$\begin{aligned} \int_0^\infty V \exp\left\{-\left(V^2 - 2V\sqrt{\gamma}\cos\theta + \gamma\right)\right\} dV \\ = e^{-\gamma\sin^2\theta} \sqrt{\gamma\pi} \cos\theta \end{aligned} \quad (4.3.1)$$

for large values of γ .

4. Find a compact expression for

$$I = 1 - \sqrt{\frac{\gamma}{\pi}} \int_{-\frac{\pi}{M}}^{\frac{\pi}{M}} e^{-\gamma\sin^2\theta} \cos\theta d\theta \quad (4.4.1)$$

5. Show that

$$P_{e|s_0} = 2Q\left(\sqrt{2\left(\frac{E_s}{N_0}\right)} \sin\frac{\pi}{M}\right) \quad (4.5.1)$$

6. Verify the SER through simulation.