MUSIC

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Contents

***Abstract*—This manual explains the MUSIC algorithm through examples.**

1 Source signal generation

The system parameters are listed in Table 0.

1.1 See Table 0

Parameter	Symbol
Number of transmitters	M
Number of receivers	N
Number of Samples	p
Sigal frequency	f_c
Sampling frequency	f_s
Speed of light	c
Transmitted signal	$s(t)$
Received Signal	$x(t)$
AWGN	$n(t)$
Direction of arrival	θ
Signal Amplitude	A

TABLE 1.1

1.2 Explain the optimization problem with a figure.
Solution: Fig. 1.1 explains (1.3) where z_0 is the set of points comprising of the intersection of the smallest circle Γ : with the largest circle Ω : $r_2 \geq \sqrt{5}$ with radii r_1 and $r_2 \geq \sqrt{5}$ respectively.

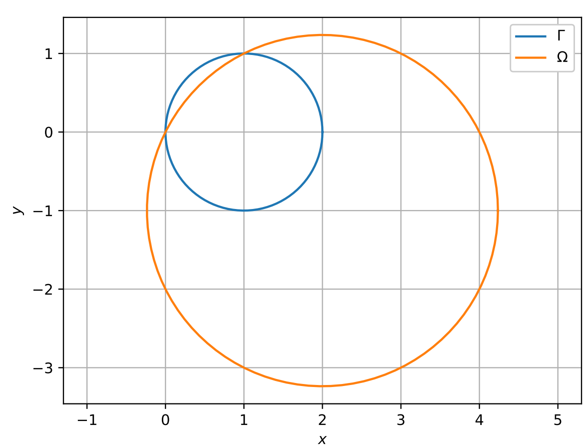

Fig. 1.1

1.3 Obtain the Lagrangian.

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Solution: The Lagrangian is

$$L(\mathbf{x}, \lambda) = \|\mathbf{x} - \mathbf{c}_1\|^2 - \lambda \left\{\|\mathbf{x} - \mathbf{c}_2\|^2 - r_2^2\right\} \quad (1.1)$$

1.4 Use the KKT conditions to obtain the minima.
Solution: From the KKT conditions,

$$\frac{\partial L(\mathbf{x}, \lambda)}{\partial \mathbf{x}} = 0 \quad (1.2)$$

$$\implies \mathbf{x} - \mathbf{c}_1 - \lambda(\mathbf{x} - \mathbf{c}_2) = 0 \quad (1.3)$$

$$\implies \mathbf{x} = \frac{\mathbf{c}_1 - \lambda\mathbf{c}_2}{1 - \lambda} \quad (1.4)$$

and

$$\frac{\partial L(\mathbf{x}, \lambda)}{\partial \lambda} = 0 \quad (1.5)$$

$$\implies \|\mathbf{x} - \mathbf{c}_2\|^2 - r_2^2 = 0 \quad (1.6)$$

Substituting from (1.9) in (1.11),

$$\left\|\frac{\mathbf{c}_1 - \lambda\mathbf{c}_2}{1 - \lambda} - \mathbf{c}_2\right\|^2 - r_2^2 = 0 \quad (1.7)$$

$$\implies \lambda = 1 \pm \frac{\|\mathbf{c}_1 - \mathbf{c}_2\|}{r_2} \quad (1.8)$$

$$= 1 \pm \sqrt{\frac{2}{5}} \quad (1.9)$$

Fig. 1.2 plots Γ for

$$\lambda = 1 - \sqrt{\frac{2}{5}} \quad (1.10)$$

1.5 If the maximum value is obtained at z_0, find the principal argument of

$$\frac{4 - z_0 - \bar{z}_0}{z_0 - \bar{z}_0 + 2\jmath} \quad (1.11)$$

Solution: From (1.9),

$$\mathbf{x}_0 = \frac{\mathbf{c}_1 - \lambda\mathbf{c}_2}{1 - \lambda} \quad (1.12)$$

$$\implies z_0 = \frac{1}{1 - \lambda}(1 - 2\lambda + \jmath\lambda) \quad (1.13)$$

$$\text{or, } \arg\frac{4 - z_0 - \bar{z}_0}{z_0 - \bar{z}_0 + 2\jmath} = \frac{2 - \Re\{z_0\}}{\jmath(\Im\{z_0\} + 1)} \quad (1.14)$$

$$= \frac{2(1 - \lambda) - (1 - 2\lambda)}{\jmath} \quad (1.15)$$

$$= -\jmath \quad (1.16)$$

Thus, the principal argument is $-\frac{\pi}{2}$.

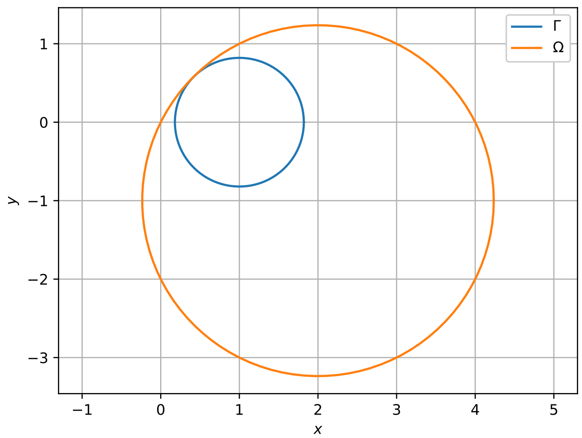

Fig. 1.2

1.6 Show that the set

$$D = \{\mathbf{x} : \|\mathbf{x} - \mathbf{C}_2\| \geq r_2\}, r_2 > 0 \quad (1.17)$$

is nonconvex.
Solution: Let $\mathbf{x}_1 \in D$ and

$$\mathbf{x}_2 = 2\mathbf{C}_2 - \mathbf{x}_1 \quad (1.18)$$

Then

$$\|\mathbf{x}_2 - \mathbf{C}_2\| = \|\mathbf{C}_2 - \mathbf{x}_1\| \geq r_2 \quad (1.19)$$

$$\implies \mathbf{x}_2 \in D. \quad (1.20)$$

Suppose

$$\mathbf{x} = \theta\mathbf{x}_1 + (1 - \theta)\mathbf{x}_2 \quad (1.21)$$

For $\theta = \frac{1}{2}$,

$$\mathbf{x} = \mathbf{C}_2 \quad (1.22)$$

$$\implies \|\mathbf{x} - \mathbf{C}_2\| = 0, \quad (1.23)$$

$$\text{or, } \mathbf{x} \notin D \quad (1.24)$$

Thus, by definition, D is not a convex set.

2 Matrices: Cayley-Hamilton Theorem

2.1 Let

$$\mathbf{M} = \begin{pmatrix} \sin^4\theta & -1 - \sin^2\theta \\ 1 + \cos^2\theta & \cos^4\theta \end{pmatrix} = \alpha\mathbf{I} + \beta\mathbf{M}^{-1} \quad (2.1)$$

where α, β are real functions of θ and $\mathbf{I}$ is the identity matrix. Find the characteristic equation

of $\mathbf{M}$.
Solution: (2.1) can be expressed as

$$\mathbf{M}^2 - \alpha\mathbf{M} - \beta\mathbf{I} = 0 \tag{2.2}$$

which yields the characteristic equation of $\mathbf{M}$ as

$$\lambda^2 - \alpha\lambda - \beta = 0 \tag{2.3}$$

2.2 Find α and β.
Solution: Since the sum of the eigenvalues is equal to the trace and the determinant is the product of eigenvalues,

$$\alpha = \sin^4\theta + \cos^4\theta \tag{2.4}$$

$$\beta = -\sin^4\theta\cos^4\theta + \left(1+\sin^2\theta\right)\left(1+\cos^2\theta\right) \tag{2.5}$$

2.3 If

$$\alpha^* = \min_\theta \alpha(\theta) \tag{2.6}$$

$$\beta^* = \min_\theta \beta(\theta), \tag{2.7}$$

find $\alpha^* + \beta^*$.
Solution:

$$\because \alpha = \sin^4\theta + \cos^4\theta = 1 - \frac{\sin^2 2\theta}{2}, \tag{2.8}$$

$$\alpha^* = \frac{1}{2}, \tag{2.9}$$

Similarly,

$$-\beta = \sin^4\theta\cos^4\theta + \left(1+\sin^2\theta\right)\left(1+\cos^2\theta\right) \tag{2.10}$$

$$= 2 + \frac{\sin^2 2\theta}{4} + \frac{\sin^4 2\theta}{16} \tag{2.11}$$

$$= \left(\frac{\sin^2 2\theta}{4} + \frac{1}{2}\right)^2 + \frac{7}{4} \tag{2.12}$$

Thus,

$$\beta^* = -\frac{37}{16} \tag{2.13}$$

$$\implies \alpha^* + \beta^* = -\frac{29}{16} \tag{2.14}$$

3 Vector Algebra

3.1 The line

$$\Gamma : \mathbf{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ m \end{pmatrix} \tag{3.1}$$

intersects the circle

$$\Omega : \left\| \mathbf{x} - \begin{pmatrix} 3 \\ -2 \end{pmatrix} \right\| = 5 \tag{3.2}$$

at points $\mathbf{P}$ and $\mathbf{Q}$ respectively. The mid point of PQ is $\mathbf{R}$ such that

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{R} = -\frac{3}{5} \tag{3.3}$$

Find m.
Solution: Let

$$\mathbf{c} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \mathbf{O} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \text{ and } \mathbf{m} = \begin{pmatrix} 1 \\ m \end{pmatrix} \tag{3.4}$$

The intersection of (3.1) and (3.2) is

$$\|\mathbf{c} + \lambda\mathbf{m} - \mathbf{O}\|^2 = 25 \tag{3.5}$$

$$\implies \lambda^2 \|\mathbf{m}\|^2 + 2\lambda\mathbf{m}^T(\mathbf{c} - \mathbf{O}) + \|\mathbf{c} - \mathbf{O}\|^2 - 25 = 0 \tag{3.6}$$

Since $\mathbf{P}, \mathbf{Q}$ lie on Γ,

$$\mathbf{P} = \mathbf{c} + \lambda_1\mathbf{m} \tag{3.7}$$

$$\mathbf{Q} = \mathbf{c} + \lambda_2\mathbf{m} \tag{3.8}$$

$$\implies \frac{\mathbf{P}+\mathbf{Q}}{2} = \mathbf{c} + \frac{\lambda_1 + \lambda_2}{2}\mathbf{m} \tag{3.9}$$

$$\implies \begin{pmatrix} 1 & 0 \end{pmatrix} \frac{\mathbf{P}+\mathbf{Q}}{2} = \begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{c} + \frac{\lambda_1 + \lambda_2}{2}\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{m} \tag{3.10}$$

$$= \begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{c} - \frac{\mathbf{m}^T(\mathbf{c} - \mathbf{O})}{\|\mathbf{m}\|^2} \tag{3.11}$$

using the sum of roots in (3.6). From (3.3) and (3.4),

$$-\begin{pmatrix} 1 & m \end{pmatrix}\begin{pmatrix} -3 \\ 3 \end{pmatrix} = -\frac{3}{5}\left(1 + m^2\right) \tag{3.12}$$

$$\implies m^2 - 5m + 6 = 0 \tag{3.13}$$

$$\implies m = 2 \text{ or } 3 \tag{3.14}$$

From (3.6),

$$\lambda = \frac{-\mathbf{m}^T(\mathbf{c} - \mathbf{O})}{\|\mathbf{m}\|^2} \pm \frac{\sqrt{\left(\mathbf{m}^T(\mathbf{c} - \mathbf{O})\right)^2 - \|\mathbf{c} - \mathbf{O}\|^2 + 25}}{\|\mathbf{m}\|^2} \tag{3.15}$$

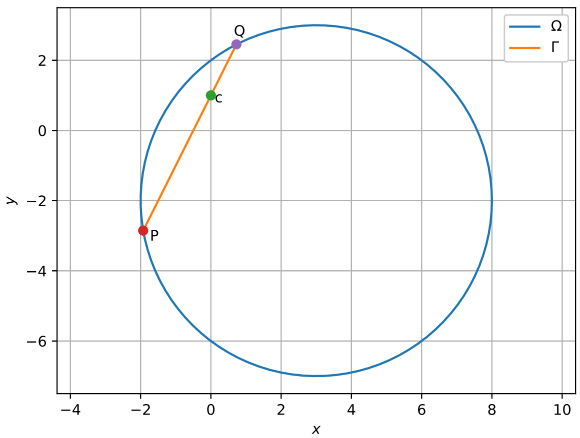

Fig. 3.1

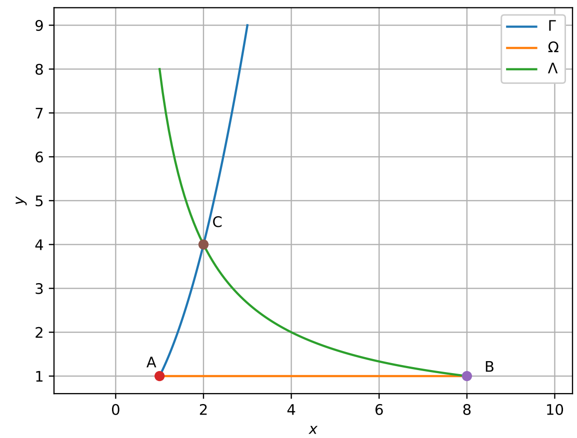

Fig. 4.1

Fig. 3.1 summarizes the solution for $m = 2$.

4 Calculus: Integration

4.1 Sketch the region

$$\begin{pmatrix} x \\ y \end{pmatrix} : xy \leq 8, 1 \leq y \leq x^2 \tag{4.1}$$

4.2 Find the area of the region.
Solution: The intersection of $y = 1, y = x^2$ is

$$\mathbf{A} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{4.2}$$

The intersection of $y = 1, xy = 8$ is

$$\mathbf{B} = \begin{pmatrix} 8 \\ 1 \end{pmatrix} \tag{4.3}$$

The intersection of $y = x^2, xy = 8$ is

$$\mathbf{C} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} \tag{4.4}$$

The desired region is enclosed by the vertices $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ Thus, the area is obtained as

$$\int_1^2 x^2 \, dx + \int_2^8 \frac{8}{x} \, dx = \left[\frac{x^3}{3} \right]_1^2 + 8 \left[\ln x \right]_2^8 - 7 \tag{4.5}$$

$$= 16 \ln 2 - \frac{14}{3} \tag{4.6}$$

5 Signal Processing: Z Transform

5.1 Let

$$a(n) = \frac{\alpha^n - \beta^n}{\alpha - \beta} u(n) \tag{5.1}$$

$$b(n) = a(n-1) + a(n+1) - \delta(n) \tag{5.2}$$

where α, β are the roots of the equation

$$z^2 - z - 1 = 0 \tag{5.3}$$

and

$$u(n) = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases} \tag{5.4}$$

$$\delta(n) = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases} \tag{5.5}$$

5.2 Verify your results through a C program.

5.3 Show that the Z transform of $u(n)$

$$U(z) \triangleq \sum_{n=-\infty}^{\infty} u(n) z^{-n} \tag{5.6}$$

$$= \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{5.7}$$

5.4 Show that

$$A(z) = \frac{z^{-1}}{1 - z^{-1} - z^{-2}} \tag{5.8}$$

5.5 Let

$$y(n) = a(n) * u(n) \triangleq \sum_{k=-\infty}^{\infty} a(k) u(n-k) \tag{5.9}$$

Show that

$$y(n) = \sum_{k=0}^{n} a(k) \tag{5.10}$$

5.6 Show that

$$\begin{aligned} Y(z) &= A(z)U(z) && (5.11) \\ &= \frac{z^{-1}}{(1 - z^{-1} - z^{-2})(1 - z^{-1})} && (5.12) \end{aligned}$$

5.7 Show that

$$\begin{aligned} w(n) &= [a(n+2) - 1]\, u(n-1) && (5.13) \\ &= a(n+2) - u(n+1) + 2\delta(n) && (5.14) \end{aligned}$$

5.8 Is $W(z) = Y(z)$?

5.9 Verify if

$$\sum_{n=1}^{\infty} \frac{a(n)}{10^n} = \frac{10}{89} \tag{5.15}$$

5.10 Verify if

$$\sum_{n=1}^{\infty} \frac{b(n)}{10^n} = \frac{8}{89} \tag{5.16}$$

6 Matrices: Adjugate

Let

$$\mathbf{M} = \begin{pmatrix} 0 & 1 & a \\ 1 & 2 & 3 \\ 3 & b & 1 \end{pmatrix}, \quad \operatorname{adj}(\mathbf{M}) = \begin{pmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{pmatrix} \tag{6.1}$$

6.1 Show that $a + b = 3$

Solution:

$$\because \mathbf{M} \operatorname{adj}(\mathbf{M}) = \det(\mathbf{M})\, \mathbf{I}, \tag{6.2}$$

$$\begin{pmatrix} 0 & 1 & a \end{pmatrix} \begin{pmatrix} 1 \\ -6 \\ 3 \end{pmatrix} = 0 \tag{6.3}$$

$$\begin{pmatrix} 3 & b & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 8 \\ -5 \end{pmatrix} = 0 \tag{6.4}$$

resulting in

$$a = 2, b = 1 \tag{6.5}$$

Hence, $a + b = 3$.

6.2 Verify if

$$(\operatorname{adj}(\mathbf{M}))^{-1} + \operatorname{adj}\left(\mathbf{M}^{-1}\right) = -\mathbf{M} \tag{6.6}$$

Solution: From (6.2)

$$(\operatorname{adj}(\mathbf{M}))^{-1} = \frac{\mathbf{M}}{\det(\mathbf{M})} \tag{6.7}$$

and

$$\begin{aligned} \left(\operatorname{adj}\left(\mathbf{M}^{-1}\right)\right) &= \frac{\mathbf{M}^{-1}}{\det(\mathbf{M}^{-1})} && (6.8) \\ &= \mathbf{M}^{-1} \det(\mathbf{M}) && (6.9) \end{aligned}$$

Thus,

$$\begin{aligned} &\left(\operatorname{adj}\left(\mathbf{M}^{-1}\right)\right) + \operatorname{adj}\left(\mathbf{M}^{-1}\right) \\ &\quad = \mathbf{M}^{-1} \det(\mathbf{M}) + \frac{\mathbf{M}}{\det(\mathbf{M})} \\ &\quad = \operatorname{adj}(\mathbf{M}) + \frac{\mathbf{M}}{\det(\mathbf{M})} && (6.10) \end{aligned}$$

From (6.2)

$$\begin{pmatrix} 0 & 1 & a \end{pmatrix} \begin{pmatrix} -1 \\ 8 \\ -5 \end{pmatrix} = \det(\mathbf{M}) \tag{6.11}$$

$$\implies \det(\mathbf{M}) = 8 - 5a = -2 \tag{6.12}$$

If

$$\begin{aligned} \left(\operatorname{adj}\left(\mathbf{M}^{-1}\right)\right) + \operatorname{adj}\left(\mathbf{M}^{-1}\right) &= -\mathbf{M}, \\ \operatorname{adj}(\mathbf{M}) - \frac{\mathbf{M}}{2} &= -\mathbf{M} \\ \implies \mathbf{M} &= -\operatorname{adj}(\mathbf{M}) \end{aligned}$$

which is incorrect.

6.3 Verify if

$$\det\left(\operatorname{adj}\left(\mathbf{M}^2\right)\right) = 81 \tag{6.13}$$

Solution:

$$\begin{aligned} \operatorname{adj}\left(\mathbf{M}^2\right) &= \mathbf{M}^{-2} \det(\mathbf{M})^2 && (6.14) \\ &= 4\mathbf{M}^{-2} && (6.15) \\ \implies \det\left(\operatorname{adj}\left(\mathbf{M}^2\right)\right) &= 4^3 \det(\mathbf{M})^{-2} && (6.16) \\ &= 16 \neq 81 && (6.17) \end{aligned}$$

6.4 If

$$\mathbf{M} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \tag{6.18}$$

show that

$$\alpha - \beta + \gamma = 3 \tag{6.19}$$

Solution:

$$\mathbf{M}\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix}, \tag{6.20}$$

$$\implies \text{adj}(\mathbf{M})\,\mathbf{M}\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix} = \text{adj}(\mathbf{M})\begin{pmatrix}1\\2\\3\end{pmatrix}, \tag{6.21}$$

which can be expressed as

$$\det(\mathbf{M})\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix} = \text{adj}(\mathbf{M})\begin{pmatrix}1\\2\\3\end{pmatrix}, \tag{6.22}$$

$$\text{or, } \begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix} = -\frac{1}{2}\,\text{adj}(\mathbf{M})\begin{pmatrix}1\\2\\3\end{pmatrix}, \tag{6.23}$$

Thus,

$$\alpha - \beta + \gamma = \begin{pmatrix}1 & -1 & 1\end{pmatrix}\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix} \tag{6.24}$$

$$= -\frac{1}{2}\begin{pmatrix}1 & -1 & 1\end{pmatrix}\,\text{adj}(\mathbf{M})\begin{pmatrix}1\\2\\3\end{pmatrix} \tag{6.25}$$

$$= \begin{pmatrix}7 & -5 & 2\end{pmatrix}\begin{pmatrix}1\\2\\3\end{pmatrix} = 3 \tag{6.26}$$

7 Probability

Table 4 lists the number of red (R) and green (G) balls in bags B_1, B_2 and B_3. Also listed are the probabilities of each bag.

Bag	**R**	**G**	**Probability**
B_1	5	5	$\Pr(B_1) = \frac{3}{10}$
B_2	3	5	$\Pr(B_2) = \frac{3}{10}$
B_3	5	3	$\Pr(B_3) = \frac{4}{10}$

TABLE 4

7.1 Show that

$$\Pr(G|B_3) = \frac{3}{8} \tag{7.1}$$

7.2 Show that

$$\Pr(G) = \frac{39}{80} \tag{7.2}$$

Solution:

$$\because \Pr(G|B_1) = \frac{1}{2}, \Pr(G|B_2) = \frac{5}{8}, \Pr(G|B_3) = \frac{3}{8},$$

$$\Pr(G) = \sum_{i=1}^{3} \Pr(G|B_i)\Pr(B_i) \tag{7.3}$$

$$= \frac{1}{2}\times\frac{3}{10} + \frac{5}{8}\times\frac{3}{10} + \frac{3}{8}\times\frac{4}{10} \tag{7.4}$$

$$= \frac{39}{80} \tag{7.5}$$

7.3 Is

$$\Pr(B_3|G) = \frac{5}{13}\ ? \tag{7.6}$$

Solution:

$$\Pr(B_3|G) = \frac{\Pr(G|B_3)\Pr(B_3)}{\Pr(G)} \tag{7.7}$$

$$= \frac{\frac{3}{8}\times\frac{4}{10}}{\frac{39}{80}} = \frac{4}{13} \neq \frac{5}{13} \tag{7.8}$$

7.4 Is

$$\Pr(B_3 \cap G) = \frac{3}{10}\ ? \tag{7.9}$$

Solution:

$$\Pr(B_3 \cap G) = \Pr(G|B_3)\Pr(B_3) \tag{7.10}$$

$$\tag{7.11}$$

$$= \frac{3}{8}\times\frac{4}{10} = \frac{3}{20} \neq \frac{3}{10} \tag{7.12}$$

8 Trigonometry

8.1 In $\triangle PQR$, which is not right angled, let

$$PQ = r, QR = p, RP = q \tag{8.1}$$

The median RS and the altitude PE intersect at $\mathbf{O}$. $p = \sqrt{3}, q = 1$ and the radius of the circumcircle of $\triangle PQR = k = 1$.

8.2 Find RS

Solution: Using the sine formula,

$$\frac{p}{\sin P} = \frac{q}{\sin Q} = 2k \tag{8.2}$$

$$\implies \sin P = \frac{\sqrt{3}}{2}, \sin Q = \frac{1}{2} \tag{8.3}$$

If $\angle R \neq \frac{\pi}{2}$, the only possible solution is

$$\angle P = \frac{2\pi}{3}, \angle Q = \frac{\pi}{6}, \angle R = \frac{\pi}{6} \tag{8.4}$$

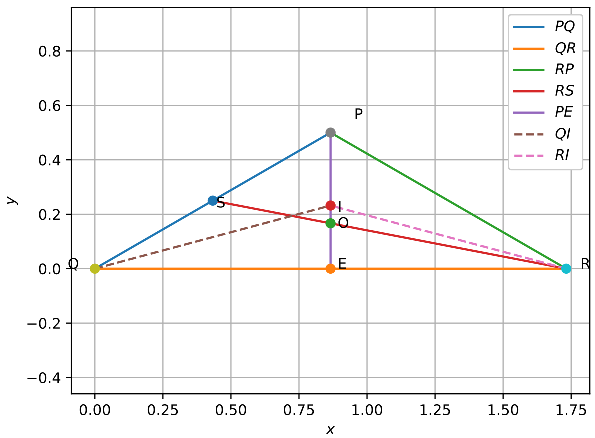

Fig. 8.1

$\because \angle Q = \angle R, q = r = 1$. The given information is shown in Fig. 8.1 Using the cosine formula,

$$RS = \sqrt{q^2 + \left(\frac{r}{2}\right)^2 - qr\cos P} \tag{8.5}$$

$$= \sqrt{1 + \frac{1}{4} + \frac{1}{2}} = \sqrt{\frac{7}{2}} \tag{8.6}$$

8.3 Find OE.
Solution: Using Baudhayana's theorem,

$$OE = \sqrt{OR^2 - ER^2} \tag{8.7}$$

$$= \sqrt{\left(\frac{2RS}{3}\right)^2 - \left(\frac{p}{2}\right)^2} \tag{8.8}$$

$$= \sqrt{\frac{7}{9} - \frac{3}{4}} = \frac{1}{6} \tag{8.9}$$

8.4 Find the area of $\triangle SOE$
Solution: $\because$ PE and RS are medians,

$$\text{ar}(\triangle SOE) = \frac{1}{4}\text{ar}(\triangle POR), \tag{8.10}$$

$$\text{ar}(\triangle POR) = \frac{2}{3}\text{ar}(\triangle PER), \tag{8.11}$$

$$\text{ar}(\triangle PER) = \frac{1}{2}\text{ar}(\triangle PQR), \tag{8.12}$$

$$\implies \text{ar}(\triangle SOE) = \frac{1}{12}\text{ar}(\triangle PQR) = \frac{\sqrt{3}}{24} \tag{8.13}$$

8.5 Find the radius of the incircle of $\triangle PQR$.
Solution: I is the incentre in Fig. 8.1. The radius of the incircle is

$$\frac{p}{2\cos\frac{Q}{2}} = \frac{p}{\sqrt{2(1+\cos Q)}} \tag{8.14}$$

$$= \sqrt{\frac{3}{1+\sqrt{3}}} \tag{8.15}$$

8.6 Repeat all the above exercises using vector algebra and plot Fig. 8.1.

9 Coordinate Geometry

Let the ellipse $E_1, n = 1, 2, \dots$ have the equation

$$\mathbf{x}^T\mathbf{D}\mathbf{x} = 1, \tag{9.1}$$

where

$$\mathbf{D} = \begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{b^2} \end{pmatrix} \tag{9.2}$$

9.1 Let the largest rectangle inside E_1 with sides parallel to the axebe be R_1. Show that the coordinates of the R_1 have the form

$$\begin{pmatrix} \pm p_1 \\ \pm p_2 \end{pmatrix} \tag{9.3}$$

Solution: Let R_1 be the rectangle $PQRS$, where $PQ \parallel RS \parallel x-axis$, $QR \parallel PS \parallel y-axis$. Their corresponding equations are

$$PQ : \mathbf{x} = \mathbf{P} + \lambda_1\mathbf{m}_1 \tag{9.4}$$

$$PS : \mathbf{x} = \mathbf{P} + \lambda_2\mathbf{m}_2 \tag{9.5}$$

$$QR : \mathbf{x} = \mathbf{Q} + \lambda_3\mathbf{m}_2 \tag{9.6}$$

where

$$\mathbf{m}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{m}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{9.7}$$

The intersection of PQ with E_1 is

$$[\mathbf{P} + \lambda_1\mathbf{m}_1]^T \mathbf{D}[\mathbf{P} + \lambda_1\mathbf{m}_1] = 1$$

$$\implies \lambda_1^2\|\mathbf{m}_1\|^2 + 2\lambda_1\mathbf{m}_1^T\mathbf{P} + \mathbf{P}^T\mathbf{D}\mathbf{P} - 1 = 0$$

$$\because \mathbf{P} \in E_1, \|\mathbf{m}_1\|^2 = 1, \mathbf{P}^T\mathbf{D}\mathbf{P} - 1 = 0$$

$$\implies \lambda_1 = 0, -2\mathbf{m}_1^T\mathbf{P} \tag{9.8}$$

Thus,

$$\mathbf{Q} = \mathbf{P} - 2\mathbf{m}_1^T\mathbf{P}\mathbf{m}_1$$

$$= \begin{pmatrix} -p_1 \\ p_2 \end{pmatrix} \tag{9.9}$$

Similarly,

$$\mathbf{S} = \mathbf{P} - 2\mathbf{m}_2^T\mathbf{P}\mathbf{m}_2 \tag{9.10}$$
$$= \begin{pmatrix} p_1 \\ -p_2 \end{pmatrix}$$

and

$$\mathbf{R} = \mathbf{Q} - 2\mathbf{m}_2^T\mathbf{Q}\mathbf{m}_2 \tag{9.11}$$
$$= \begin{pmatrix} -p_1 \\ -p_2 \end{pmatrix}$$

9.2 Find an expression for the square of the area of R_1.
Solution:

$$\because \frac{p_1^2}{a^2} + \frac{p_2^2}{b^2} = 1,$$
$$p_2 = b\sqrt{1 - \frac{p_1^2}{a^2}}. \tag{9.12}$$

Hence the desired expression is

$$F = (PQ \times QR)^2 = 16p_1^2p_2^2 = 16p_1^2b^2\left(1 - \frac{p_1^2}{a^2}\right). \tag{9.13}$$

9.3 Find p_1 that maximises F.
Solution: (9.13) can be expressed as

$$F = a^2b^2\left(16a^2p_1^2 - 16p_1^4\right)$$
$$= a^2b^2\left\{a^4 - \left(a^2 - 4p_1^2\right)^2\right\} \tag{9.14}$$

Thus, F is maximum when

$$\left(a^2 - 4p_1^2\right)^2 = 0$$
$$\implies p_1 = \pm\frac{a}{2} \tag{9.15}$$

9.4 Verify the above result graphically.
9.5 Find p_2.
Solution: From (9.12)

$$p_2 = \pm\frac{\sqrt{3}}{2}b \tag{9.16}$$

9.6 Find E_2, the largest ellipse within R_1. **Solution:** From (9.15) and (9.16), the and semi-major/minor axes of E_2 are

$$E_2 : \left(\frac{a}{2}, \frac{\sqrt{3}}{2}b\right) \tag{9.17}$$

9.7 find E_n and R_n,
Solution: From (9.15) and (9.16), the vertices of R_n and semi-major/minor axes of E_n are

$$R_n : \left\{\pm\frac{a}{2^n}, \pm\left(\frac{\sqrt{3}}{2}\right)^n b\right\}$$
$$E_n : \left\{\frac{a}{2^{n-1}}, \left(\frac{\sqrt{3}}{2}\right)^{n-1} b\right\} \tag{9.18}$$

In the following questions, $a = 3, b = 2$. Use a computer program.
9.8 Is the eccentricity $e_18 = e_19$?
9.9 Verify if

$$\sum_{n=1}^{N} (\text{Area of } R_n) < 24, \tag{9.19}$$

for each positive integer N.
9.10 Is the length of the latus rectum of $E_9 = \frac{1}{6}$?
9.11 Is the distance of a focus from the centre in $E_9 = \frac{\sqrt{5}}{32}$?

10 Calculus: Differentiation

Let

$$f(x) = \begin{cases} x^5 + 5x^4 + 10x^3 + 10x^2 + 3x + 1 & x < 0 \\ x^2 - x + 1 & 0 \le x < 1 \\ \frac{2}{3}x^3 - 4x^2 + 7x - \frac{8}{3} & 1 \le x < 3 \\ (x-2)\ln(x-2) - x + \frac{10}{3} & x \ge 3 \end{cases} \tag{10.1}$$

10.1 Is f increasing in $(-\infty, 0)$?
Solution:

$$f'(x) = 5x^4 + 20x^3 + 30x^2 + 20x + 3 \quad x < 0$$
$$\implies f'(-1) = 5 - 20 + 30 - 20 + 3 = -2 < 0 \tag{10.2}$$

Hence $f'(x)$ is non-increasing.
10.2 Does f' have a local maximum at $x = 1$?
Solution:

$$f'(x) = \begin{cases} 2x - 1 > 0, & \frac{1}{2} < x < 1, \\ 2(x-2)^2 - 1 < 0 & 1 \le x < 3 \end{cases} \tag{10.3}$$

Hence, f is increasing in $\left(\frac{1}{2}, 1\right)$ and decreasing between $(1, 3) \implies f$ has a local maximum at $x = 1$
10.3 Show that f' is differentiable at $x = 1$.
Solution: Since

$$f'(1-) = f'(1) = 1, \tag{10.4}$$

f is differentiable at $x = 1$.
10.4 Is f onto?

10.5 Sketch $f(x)$ in Python to verify your answeres.

11 Calculus: Differential Equations

Γ is a curve in the first qudrant and

$$\mathbf{R} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{11.1}$$

lies on it. The tangent to Γ at $\mathbf{P}$ intersects the $y-$axis at $\mathbf{Y}_P$. The line segment $PY_P = 1$.

11.1 Find the differential equation of Γ.
Solution: Let

$$\mathbf{P} = \begin{pmatrix} x \\ y \end{pmatrix}, \mathbf{Y}_P = \begin{pmatrix} 0 \\ c \end{pmatrix}. \tag{11.2}$$

Then using the equation of a line,

$$\mathbf{Y}_P = \mathbf{P} + \lambda \mathbf{m}, \tag{11.3}$$

where

$$\mathbf{m} = \begin{pmatrix} 1 \\ y' \end{pmatrix}. \tag{11.4}$$

Thus,

$$\begin{pmatrix} 0 \\ c \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ y' \end{pmatrix} \tag{11.5}$$

$$\implies \lambda = -x. \tag{11.6}$$

$$\because PY_P = \|\mathbf{P} - \mathbf{Y}_P\| = |\lambda| \|\mathbf{m}\| = 1, \tag{11.7}$$

$$x^2 \left(1 + (y')^2\right) = 1 \tag{11.8}$$

$$\implies xy' \pm \sqrt{1-x^2} = 0 \tag{11.9}$$

11.2 Find the equation of Γ.
Solution: From (11.9),

$$dy = \pm \frac{\sqrt{1-x^2}}{x} dx \tag{11.10}$$

$$\implies \int dy = \pm \int \frac{\sqrt{1-x^2}}{x} dx \tag{11.11}$$

Letting

$$z = \sqrt{1-x^2}, dz = -\frac{x}{\sqrt{1-x^2}} dx$$

$$\begin{aligned} \implies \int \frac{\sqrt{1-x^2}}{x} dx &= -\int \frac{z^2}{1-z^2} dz \\ &= \int dz - \int \frac{1}{1-z^2} dz \\ &= z + \frac{1}{2} \ln \frac{1-z}{1+z} + C \end{aligned} \tag{11.12}$$

Thus,

$$y = \pm \left(\sqrt{1-x^2} + \frac{1}{2} \ln \frac{1-\sqrt{1-x^2}}{1+\sqrt{1-x^2}} \right) \tag{11.13}$$

since $C = 0$ after substituting $x = 0, y = 1$.

11.3 Verify your result through a python sketch.

12 Linear Algebra: Orthogonality

12.1 Let

$$L_1 : \quad \mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} \tag{12.1}$$

$$L_2 : \quad \mathbf{x} = \lambda_1 \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \tag{12.2}$$

Given that $L_3 \perp L_1, L_3 \perp L_2$, find L_3.
Solution: Let

$$L_3 : \quad \mathbf{x} = \mathbf{c} + \lambda \mathbf{m}_3 \tag{12.3}$$

Then

$$\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \end{pmatrix} \mathbf{m}_3 = \mathbf{0} \tag{12.4}$$

Row reducing the coefficient matrix,

$$\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & -2 & -2 \\ 0 & 1 & 2 \end{pmatrix} \tag{12.5}$$

$$\leftrightarrow \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{pmatrix} \implies \mathbf{m}_3 = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \tag{12.6}$$

Also, $L_1 \perp L_2$, but $L_1 \cup L_2 = \phi$. The given information can be summarized as

$$L_1 : \quad \mathbf{x} = \mathbf{c}_1 + \lambda_1 \mathbf{m}_1 \tag{12.7}$$

$$L_2 : \quad \mathbf{x} = \lambda_2 \mathbf{m}_2 \tag{12.8}$$

$$L_3 : \quad \mathbf{x} = \mathbf{c}_3 + \lambda \mathbf{m}_3 \tag{12.9}$$

where

$$\mathbf{c}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{m}_1 = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}, \mathbf{m}_2 = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \tag{12.10}$$

The objective is to find $\mathbf{c}_3$. Since $L_1 \cup L_3 \neq \phi, L_2 \cup L_3 \neq \phi$, from (12.7)-(12.9),

$$\mathbf{c}_1 + \lambda_1 \mathbf{m}_1 = \mathbf{c}_3 + \lambda_3 \mathbf{m}_3 \tag{12.11}$$

$$\lambda_2 \mathbf{m}_2 = \mathbf{c}_3 + \lambda_4 \mathbf{m}_3 \tag{12.12}$$

Using the fact that $L_1 \perp L_2 \perp L_3$, (12.11)-(12.12) can be expressed as

$$\mathbf{m}_1^T\mathbf{c}_1 + \lambda_1 \|\mathbf{m}\|_1^2 = \mathbf{m}_1^T\mathbf{c}_3 \tag{12.13}$$
$$\mathbf{m}_2^T\mathbf{c}_1 = \mathbf{m}_2^T\mathbf{c}_3 \tag{12.14}$$
$$\mathbf{m}_3^T\mathbf{c}_1 = \mathbf{m}_3^T\mathbf{c}_3 + \lambda_3 \|\mathbf{m}_3\|^2 \tag{12.15}$$
$$0 = \mathbf{m}_1^T\mathbf{c}_3 \tag{12.16}$$
$$\lambda_2 \|\mathbf{m}_2\|^2 = \mathbf{m}_2^T\mathbf{c}_3 \tag{12.17}$$
$$0 = \mathbf{m}_3^T\mathbf{c}_3 + \lambda_4 \|\mathbf{m}_3\|^2 \tag{12.18}$$

Simplifying the above,

$$\lambda_1 = -\frac{\mathbf{m}_1^T\mathbf{c}_1}{\|\mathbf{m}\|_1^2} = \frac{1}{9} \tag{12.19}$$
$$\lambda_2 = \frac{\mathbf{m}_2^T\mathbf{c}_1}{\|\mathbf{m}\|_2^2} = \frac{2}{9} \tag{12.20}$$

Substituting in (12.11) and (12.12),

$$L_3: \quad \mathbf{x} = \frac{2}{9}\begin{pmatrix}4\\1\\1\end{pmatrix} + \lambda_3\begin{pmatrix}2\\2\\-1\end{pmatrix} \text{ or} \tag{12.21}$$
$$L_3: \quad \mathbf{x} = \frac{2}{9}\begin{pmatrix}2\\-1\\2\end{pmatrix} + \lambda_3\begin{pmatrix}2\\2\\-1\end{pmatrix} \tag{12.22}$$

The key concept in this question is that orthogonality of L_1 and L_2 doesnot mean that they intersect. They are skew lines.

13 Linear Algebra: Eigenvalue and Eigenvector

13.1 Obtain the 3×3 matrices $\{\mathbf{P}_k\}_{k=1}^6$ from the vectors

$$\mathbf{v}_1 = \begin{pmatrix}0 & 0 & 1\end{pmatrix} \tag{13.1}$$
$$\mathbf{v}_2 = \begin{pmatrix}0 & 1 & 0\end{pmatrix} \tag{13.2}$$
$$\mathbf{v}_3 = \begin{pmatrix}1 & 0 & 0\end{pmatrix} \tag{13.3}$$

and let

$$\mathbf{X} = \sum_{k=1}^{6} \mathbf{P}_k \begin{pmatrix}2 & 1 & 3\\1 & 0 & 2\\3 & 2 & 1\end{pmatrix} \mathbf{P}_k^T \tag{13.4}$$

Verify if

a) $\lambda = 30$ is an eigenvalue of $\mathbf{X}$ and $\mathbf{x} = \begin{pmatrix}1 & 1 & 1\end{pmatrix}^T$ the corresponding eigenvector.

b) $\mathbf{X}$ is symmetric.

c) $\operatorname{tr}(\mathbf{X}) = 18$.

d) $\mathbf{X} - 30\mathbf{I}$ is invertible.

13.2 Let

$$\mathbf{P} = \begin{pmatrix}1 & 1 & 1\\0 & 2 & 2\\0 & 0 & 3\end{pmatrix}, \mathbf{Q} = \begin{pmatrix}2 & x & x\\0 & 4 & 0\\x & x & 6\end{pmatrix} \tag{13.5}$$

Verify if

a) $PQ = QP$ for some x.

b) $\det R = \det \begin{pmatrix}2 & x & x\\0 & 4 & 0\\x & x & 5\end{pmatrix}$ for all x.

c) for $x = 0$, if $\mathbf{R}\begin{pmatrix}1\\a\\b\end{pmatrix} = 6\begin{pmatrix}1\\a\\b\end{pmatrix}$, then $a + b = 5$. *Use property of eigenvector.*

d) For $x = 1$, there exists a vector $\mathbf{y}$ for whcih $\mathbf{R}\mathbf{y} = \mathbf{0}$. This implies that the null space of $\mathbf{R}$ is nonempty. Also, $\mathbf{R}$ is noninvertible, $\det(R) = 0$ and has a 0 eigenvalue.

13.3

13.4

13.5

13.6

13.7

13.8 Let

$$L_1: \quad \mathbf{r} = \lambda_1\begin{pmatrix}1\\0\\0\end{pmatrix} \tag{13.6}$$
$$L_2: \quad \mathbf{r} = \begin{pmatrix}0\\0\\1\end{pmatrix} + \lambda_2\begin{pmatrix}0\\1\\0\end{pmatrix} \tag{13.7}$$
$$L_3: \quad \mathbf{r} = \begin{pmatrix}1\\1\\0\end{pmatrix} + \lambda_3\begin{pmatrix}0\\0\\1\end{pmatrix} \tag{13.8}$$

Let $\mathbf{P} \in L_1, \mathbf{Q} \in L_2, \mathbf{R} \in L_3$. Verify if $\mathbf{Q}$ can be

a) $\begin{pmatrix}0\\-\frac{1}{2}\\1\end{pmatrix}$

b) $\begin{pmatrix}0\\0\\1\end{pmatrix}$

c) $\begin{pmatrix}0\\\frac{1}{2}\\1\end{pmatrix}$

d) $\begin{pmatrix}0\\1\\1\end{pmatrix}$

given that $\mathbf{P}, \mathbf{Q}, \mathbf{R}$ are collinear.

Solution: If $\mathbf{P}, \mathbf{Q}, \mathbf{R}$ are collinear,

$$\frac{PQ}{QR} = k, \tag{13.9}$$

$$(k+1)\,\mathbf{Q} = k\mathbf{P} + \mathbf{R}, \tag{13.10}$$

From (13.6), (13.7) and (13.8),

$$k\lambda_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$
$$= (k+1) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + (k+1)\,\lambda_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \tag{13.11}$$

which can be expressed as

$$\begin{pmatrix} k & 0 & 0 \\ 0 & -(k+1) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ k+1 \end{pmatrix} \tag{13.12}$$

Thus,

$$\mathbf{Q} = \begin{pmatrix} 0 \\ \frac{1}{k+1} \\ 1 \end{pmatrix} \tag{13.13}$$