Digital Modulation Techniques

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Contents

***Abstract*—The manual frames the problems of receiver design and performance analysis in digital communication as applications of probability theory.**

Download all codes in this manual from

```
svn co https://github.com/gadepall/comm/trunk/
    modulation/manual/codes
```

1 BPSK

1.1. The *signal constellation diagram* for BPSK is given by Fig. 1.1. The symbols s_0 and s_1 are equiprobable. $\sqrt{E_b}$ is the energy transmitted per bit. Assuming a zero mean additive white gaussian noise (AWGN) with variance $\frac{N_0}{2}$, obtain the symbols that are received. **Solution:** The possible received symbols are

$$y|s_0 = \sqrt{E_b} + n \tag{1.1.1}$$
$$y|s_1 = -\sqrt{E_b} + n \tag{1.1.2}$$

where the AWGN $n \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$.

1.2. From Fig. 1.1 obtain a decision rule for BPSK **Solution:** The decision rule is

$$y \underset{s_1}{\overset{s_0}{\gtrless}} 0 \tag{1.2.1}$$

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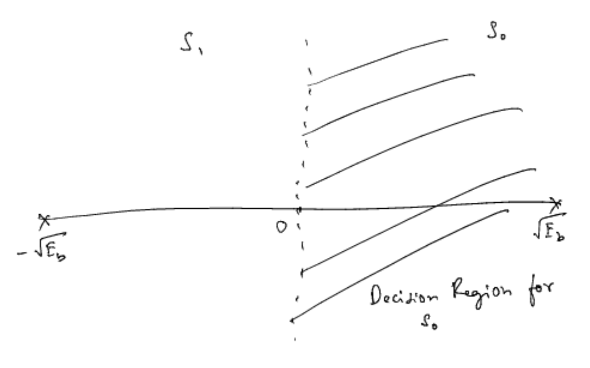

Fig. 1.1

1.3. Repeat the previous exercise using the MAP criterion.

1.4. Using the decision rule in Problem 1.2, obtain an expression for the probability of error for BPSK.
Solution: Since the symbols are equiprobable, it is sufficient if the error is calculated assuming that a 0 was sent. This results in

$$P_e = \Pr\left(y < 0 | s_0\right) = \Pr\left(\sqrt{E_b} + n < 0\right) \tag{1.4.1}$$
$$= \Pr\left(-n > \sqrt{E_b}\right) = \Pr\left(n > \sqrt{E_b}\right) \tag{1.4.2}$$

since n has a symmetric pdf. Let $w \sim \mathcal{N}(0,1)$. Then $n = \sqrt{\frac{N_0}{2}}w$. Substituting this in (1.4.2),

$$P_e = \Pr\left(\sqrt{\frac{N_0}{2}}w > \sqrt{E_b}\right) = \Pr\left(w > \sqrt{\frac{2E_b}{N_0}}\right) \tag{1.4.3}$$

$$= Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \tag{1.4.4}$$

where

$$Q(x) \triangleq \Pr\left(w > x\right), x \geq 0. \tag{1.4.5}$$
$$= \frac{1}{\pi}\int_0^{\frac{\pi}{2}} e^{-\frac{x^2}{2\sin^2\theta}}\, d\theta \tag{1.4.6}$$

The PDF of $w \sim \mathcal{N}(0,1)$ is given by

$$p_w(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty \tag{1.4.7}$$

and the complementary error function is defined as

$$\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt. \tag{1.4.8}$$

1.5. Show that

$$Q(x) = \frac{1}{2}\,\text{erfc}\left(\frac{x}{\sqrt{2}}\right) \tag{1.5.1}$$

Solution: From (1.4.5)

$$Q(x) = \Pr(w > x),\, x \geq 0 \tag{1.5.2}$$

$$= \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{t^2}{2}}\, dt. \tag{1.5.3}$$

$$= \frac{1}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{2}}}^\infty e^{-y^2}\, dy. \quad \left(t = \sqrt{2}y\right) \tag{1.5.4}$$

resulting in

1.6. Verify the bit error rate (BER) plots for BPSK through simulation and analysis for 0 to 10 dB.
Solution: The following code

```
codes/bpsk_ber.py
```

yields Fig. 1.6

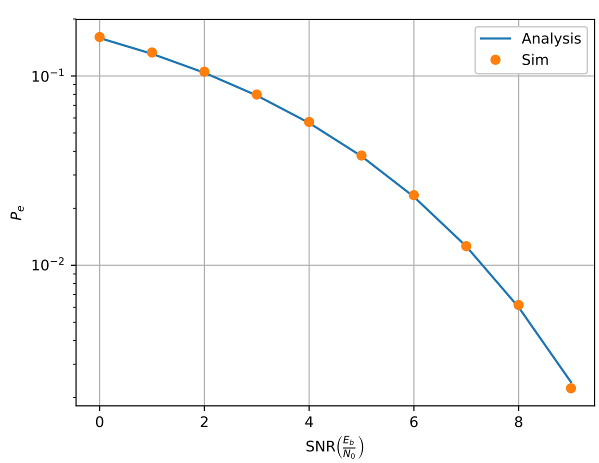

Fig. 1.6

2 Coherent BFSK

2.1. The signal constellation for binary frequency shift keying (BFSK) is given in Fig. 2.1. Obtain

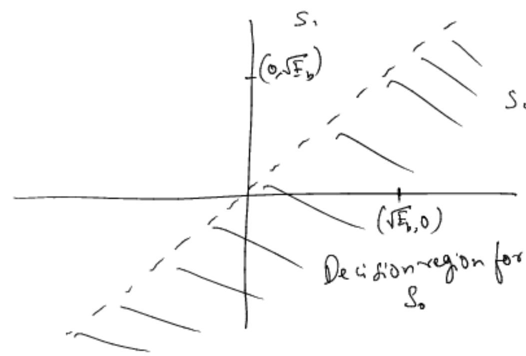

Fig. 2.1

the equations for the received symbols.
Solution: The received symbols are given by

$$\mathbf{y}|s_0 = \begin{pmatrix} \sqrt{E_b} \\ 0 \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, \tag{2.1.1}$$

and

$$\mathbf{y}|s_1 = \begin{pmatrix} 0 \\ \sqrt{E_b} \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, \tag{2.1.2}$$

where $n_1, n_2 \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$. and $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$.

2.2. Obtain a decision rule for BFSK from Fig. 2.1.
Solution: The decision rule is

$$y_1 \underset{s_1}{\overset{s_0}{\gtrless}} y_2 \tag{2.2.1}$$

Definition 1. *The joint PDF of X, Y is given by*

$$p(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)} \times \left\{\frac{(x-\mu_x)^2}{\sigma_x^2} + \frac{\left(y-\mu_y\right)^2}{\sigma_y^2} - \frac{2\rho(x-\mu_x)\left(y-\mu_y\right)}{\sigma_x\sigma_y}\right\}\right] \tag{2.2.2}$$

where

$$\mu_x = E[X], \tag{2.2.3}$$

$$\sigma_x^2 = var(X), \tag{2.2.4}$$

$$\rho = \frac{E\left[(X-\mu_x)\left(Y-\mu_y\right)\right]}{\sigma_x\sigma_y}. \tag{2.2.5}$$

For equiprobably symbols, the MAP criterion is defined as

$$p(\mathbf{y}|s_0) \underset{s_1}{\overset{s_0}{\gtrless}} p(\mathbf{y}|s_1) \tag{2.2.6}$$

2.3. Use (2.2.2) in (2.2.6) to obtain (2.2.1).
Solution: According to the MAP criterion, assuming equiprobably symbols,

$$p(\mathbf{y}|s_0) \underset{s_1}{\overset{s_0}{\gtrless}} p(\mathbf{y}|s_1) \tag{2.3.1}$$

2.4. Derive and plot the probability of error. Verify through simulation.
Solution: Given that s_0 was transmitted, the received symbols are

$$\mathbf{y}|s_0 = \begin{pmatrix} \sqrt{E_b} \\ 0 \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, \tag{2.4.1}$$

From (2.2.1), the probability of error is given by

$$P_e = \Pr(y_1 < y_2|s_0) = \Pr\left(\sqrt{E_b} + n_1 < n_2\right) \tag{2.4.2}$$

$$= \Pr\left(n_2 - n_1 > \sqrt{E_b}\right) \tag{2.4.3}$$

Note that $n_2 - n_1 \sim \mathcal{N}(0, N_0)$. Thus,

$$P_e = \Pr\left(\sqrt{N_0}w > \sqrt{E_b}\right) \tag{2.4.4}$$

$$= \Pr\left(w > \sqrt{\frac{E_b}{N_0}}\right) \tag{2.4.5}$$

$$\implies P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right) \tag{2.4.6}$$

where $w \sim \mathcal{N}(0, 1)$. The following code plots the BER curves in Fig. 2.4

```
codes/fsk_ber.py
```

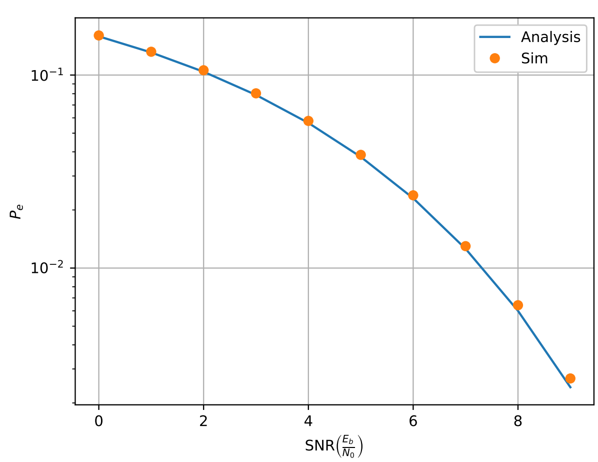

Fig. 2.4

3 QPSK

1. Let

$$\mathbf{y} = \mathbf{s} + \mathbf{n} \tag{3.1.1}$$

where $\mathbf{s} \in \{\mathbf{s}_0, \mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}$ and

$$\mathbf{s}_0 = \begin{pmatrix} \sqrt{E_s} \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ \sqrt{E_s} \end{pmatrix}, \tag{3.1.2}$$

$$\mathbf{s}_2 = \begin{pmatrix} -\sqrt{E_s} \\ 0 \end{pmatrix}, \mathbf{s}_3 = \begin{pmatrix} 0 \\ -\sqrt{E_s} \end{pmatrix}, \tag{3.1.3}$$

$$E[\mathbf{n}] = \mathbf{0}, E\left[\mathbf{n}\mathbf{n}^T\right] = \sigma^2\mathbf{I} \tag{3.1.4}$$

2. Using (2.2.2), show that the MAP decision for detecting $\mathbf{s}_0$ results in

$$|y_2| < y_1 \tag{3.2.1}$$

3. Express $\Pr(\hat{\mathbf{s}} = \mathbf{s}_0|\mathbf{s} = \mathbf{s}_0)$ in terms of y_1, y_2.
Solution: From (3.2.1),

$$\Pr(\hat{\mathbf{s}} = \mathbf{s}_0|\mathbf{s} = \mathbf{s}_0)$$
$$= \Pr\left(|y_2| < y_1|y_1 = \sqrt{E_s}, y_2 = 0\right) \tag{3.3.1}$$

4. Let $X = n_2 - n_1, Y = -n_2 - n_1$, where $\mathbf{n} = (n_1, n_2)$. Their correlation coefficient is defined as

$$\rho = \frac{E\left[(X - \mu_x)\left(Y - \mu_y\right)\right]}{\sigma_x\sigma_y} \tag{3.4.1}$$

X and Y are said to be uncorrelated if $\rho = 0$
5. Show that X and Y are uncorrelated. Verify this numerically.
6. Show that X and Y are independent, i.e. $p_{XY}(x, y) = p_X(x)p_Y(y)$.
7. Show that $X, Y \sim \mathcal{N}(0, N_0)$.
8. Show that

$$\Pr(\hat{\mathbf{s}} = \mathbf{s}_0|\mathbf{s} = \mathbf{s}_0) = \Pr\left(X < \sqrt{E_s}, Y < \sqrt{E_s}\right). \tag{3.8.1}$$

9. Show that

$$\Pr\left(X < \sqrt{E_s}, Y < \sqrt{E_s}\right) = \left(1 - Q\left(\sqrt{\frac{E_s}{N_0}}\right)\right)^2 \tag{3.9.1}$$

10. Verify the above through simulation.
Solution: This is shown in Fig. 3.10 through the following code.

```
codes/qpsk.py
```

11. Modify the above script to obtain the probability of symbol error.

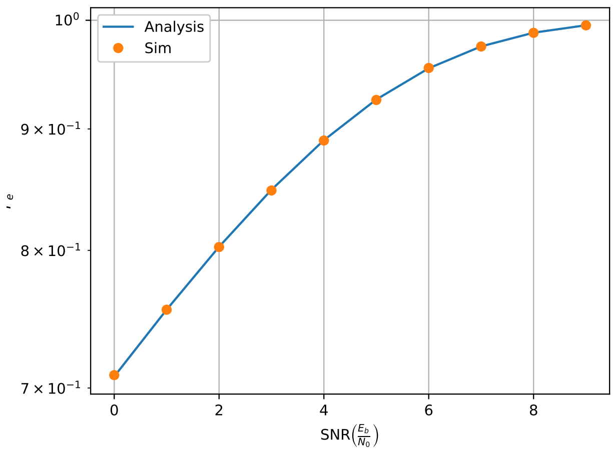

Fig. 3.10

4 M-PSK

1. Consider a system where $\mathbf{s}_i = \begin{pmatrix} \cos\left(\frac{2\pi i}{M}\right) \\ \cos\left(\frac{2\pi i}{M}\right) \end{pmatrix}, i = 0, 1, \dots M-1$. Let
$$\mathbf{y}|s_0 = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \sqrt{E_s} + n_1 \\ n_2 \end{pmatrix} \tag{4.1.1}$$
where $n_1, n_2 \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$.
Substituting
$$y_1 = R\cos\theta \tag{4.1.2}$$
$$y_2 = R\sin\theta \tag{4.1.3}$$
show that the joint pdf of R, θ is
$$p(R,\theta) = \frac{R}{\pi N_0}\exp\left(-\frac{R^2 - 2R\sqrt{E_s}\cos\theta + E_s}{N_0}\right) \tag{4.1.4}$$
2. Show that
$$\lim_{\alpha\to\infty}\int_0^\infty (V-\alpha)e^{-(V-\alpha)^2}\,dV = 0 \tag{4.2.1}$$
$$\lim_{\alpha\to\infty}\int_0^\infty e^{-(V-\alpha)^2}\,dV = \sqrt{\pi} \tag{4.2.2}$$
3. Using the above, show that
$$\int_0^\infty V\exp\left\{-\left(V^2 - 2V\sqrt{\gamma}\cos\theta + \gamma\right)\right\}dV = e^{-\gamma\sin^2\theta}\sqrt{\gamma\pi}\cos\theta \tag{4.3.1}$$
for large values of γ.
4. Find a compact expression for
$$I = 1 - \sqrt{\frac{\gamma}{\pi}}\int_{-\frac{\pi}{M}}^{\frac{\pi}{M}} e^{-\gamma\sin^2\theta}\cos\theta\,d\theta \tag{4.4.1}$$
5. Show that
$$P_{e|\mathbf{s}_0} = 2Q\left(\sqrt{2\left(\frac{E_s}{N_o}\right)}\sin\frac{\pi}{M}\right) \tag{4.5.1}$$
6. Verify the SER through simulation.