Digital Modulation Techniques

P. N. V. S. S. K. HAVISH, S. S. Ashish and G V V Sharma

Contents

Abstract—**The manual frames the problems of receiver design and performance analysis in digital communication as applications of probability theory.**

Download all codes in this manual from

```
svn co https://github.com/gadepall/comm/trunk/
    modulation/codes
```

1 BPSK

Problem 1. The *signal constellation diagram* for BPSK is given by Fig. 1. The symbols s_0 and s_1 are equiprobable. $\sqrt{E_b}$ is the energy transmitted per bit. Assuming a zero mean additive white gaussian noise (AWGN) with variance $\frac{N_0}{2}$, obtain the symbols that are received.

Solution: The possible received symbols are

$$y|s_0 = \sqrt{E_b} + n \tag{1}$$

$$y|s_1 = -\sqrt{E_b} + n \tag{2}$$

where the AWGN $n \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$.

Problem 2. From Fig. 1 obtain a decision rule for BPSK

The authors are with the Department of Electrical Engineering, IIT, Hyderabad 502285 India e-mail: {ee16btech11023,ee16btech11043,jbala,gadepall}@iith.ac.in.

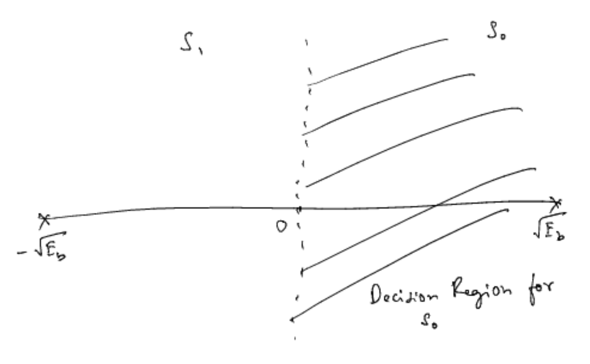

Fig. 1

Solution: The decision rule is

$$y \underset{s_1}{\overset{s_0}{\gtrless}} 0 \tag{3}$$

Problem 3. Repeat the previous exercise using the MAP criterion.

Problem 4. Using the decision rule in Problem 2, obtain an expression for the probability of error for BPSK.

Solution: Since the symbols are equiprobable, it is sufficient if the error is calculated assuming that a 0 was sent. This results in

$$P_e = \Pr\left(y < 0|s_0\right) = \Pr\left(\sqrt{E_b} + n < 0\right) \tag{4}$$

$$= \Pr\left(-n > \sqrt{E_b}\right) = \Pr\left(n > \sqrt{E_b}\right) \tag{5}$$

since n has a symmetric pdf. Let $w \sim \mathcal{N}(0, 1)$. Then $n = \sqrt{\frac{N_0}{2}}w$. Substituting this in (5),

$$P_e = \Pr\left(\sqrt{\frac{N_0}{2}}w > \sqrt{E_b}\right) = \Pr\left(w > \sqrt{\frac{2E_b}{N_0}}\right) \tag{6}$$

$$= Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \tag{7}$$

where $Q(x) \triangleq \Pr(w > x), x \geq 0$.

Problem 5. The PDF of $w \sim \mathcal{N}(0,1)$ is given by

$$p_w(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty \tag{8}$$

and the complementary error function is defined as

$$\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt. \tag{9}$$

Show that

$$Q(x) = \frac{1}{2} \text{erfc}\left(\frac{x}{\sqrt{2}}\right) \tag{10}$$

Problem 6. Verify the bit error rate (BER) plots for BPSK through simulation and analysis for 0 to 10 dB.

Solution: The following code

codes/bpsk_ber.py

yields Fig. 2

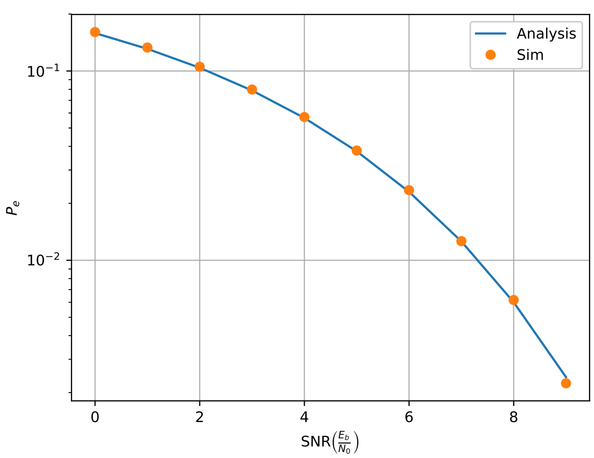

Fig. 2

Problem 7. Show that

$$Q(x) = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} e^{-\frac{x^2}{2\sin^2\theta}}\, d\theta \tag{11}$$

2 Coherent BFSK

Problem 8. The signal constellation for binary frequency shift keying (BFSK) is given in Fig. 3. Obtain the equations for the received symbols.

Solution: The received symbols are given by

$$\mathbf{y}|s_0 = \begin{pmatrix} \sqrt{E_b} \\ 0 \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, \tag{12}$$

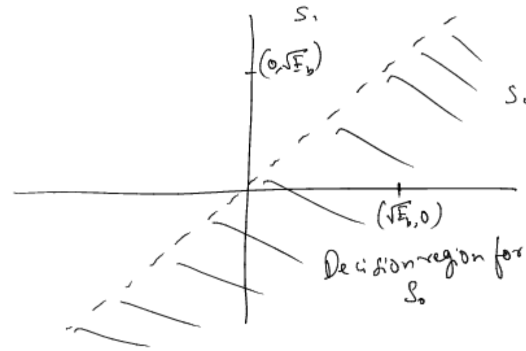

Fig. 3

and

$$\mathbf{y}|s_1 = \begin{pmatrix} 0 \\ \sqrt{E_b} \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, \tag{13}$$

where $n_1, n_2 \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$. and $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$.

Problem 9. Obtain a decision rule for BFSK from Fig. 3.

Solution: The decision rule is

$$y_1 \underset{s_1}{\overset{s_0}{\gtrless}} y_2 \tag{14}$$

Definition 2.1. The joint PDF of X, Y is given by

$$p(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)} \times \left\{\frac{(x-\mu_x)^2}{\sigma_x^2} + \frac{\left(y-\mu_y\right)^2}{\sigma_y^2} - \frac{2\rho(x-\mu_x)\left(y-\mu_y\right)}{\sigma_x\sigma_y}\right\}\right] \tag{15}$$

where

$$\mu_x = E[X], \sigma_x^2 = \text{var}(X), \rho = \frac{E\left[(X-\mu_x)\left(Y-\mu_y\right)\right]}{\sigma_x\sigma_y}. \tag{16}$$

Problem 10. For equiprobably symbols, the MAP criterion is defined as

$$p(\mathbf{y}|s_0) \underset{s_1}{\overset{s_0}{\gtrless}} p(\mathbf{y}|s_1) \tag{17}$$

Use (15) in (17) to obtain (14).

Solution: According to the MAP criterion, assuming equiprobably symbols,

$$p(\mathbf{y}|s_0) \underset{s_1}{\overset{s_0}{\gtrless}} p(\mathbf{y}|s_1) \tag{18}$$

Problem 11. Derive and plot the probability of error. Verify through simulation.

Solution: Given that s_0 was transmitted, the received symbols are

$$\mathbf{y}|s_0 = \begin{pmatrix} \sqrt{E_b} \\ 0 \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, \tag{19}$$

From (14), the probability of error is given by

$$P_e = \Pr(y_1 < y_2|s_0) = \Pr\left(\sqrt{E_b} + n_1 < n_2\right) \tag{20}$$

$$= \Pr\left(n_2 - n_1 > \sqrt{E_b}\right) \tag{21}$$

Note that $n_2 - n_1 \sim \mathcal{N}(0, N_0)$. Thus,

$$P_e = \Pr\left(\sqrt{N_0}w > \sqrt{E_b}\right) = \Pr\left(w > \sqrt{\frac{E_b}{N_0}}\right) \tag{22}$$

$$\implies P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right) \tag{23}$$

where $w \sim \mathcal{N}(0, 1)$. The following code plots the BER curves in Fig. 4

```
codes/fsk_ber.py
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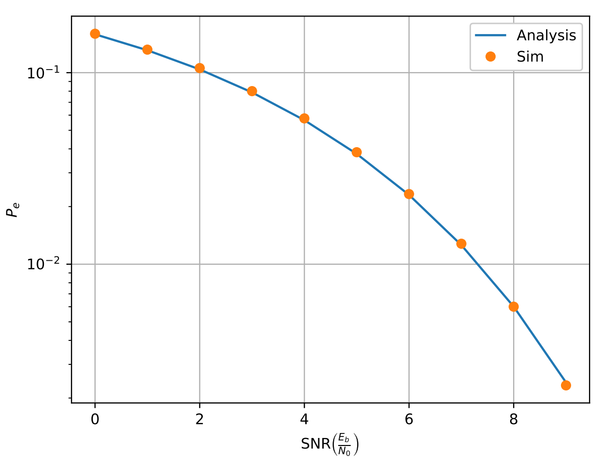

Fig. 4

3 QPSK

Let

$$\mathbf{y} = \mathbf{s} + \mathbf{n} \tag{24}$$

where $\mathbf{s} \in \{\mathbf{s}_0, \mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}$ and

$$\mathbf{s}_0 = \begin{pmatrix} \sqrt{E_s} \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ \sqrt{E_s} \end{pmatrix}, \tag{25}$$

$$\mathbf{s}_2 = \begin{pmatrix} -\sqrt{E_s} \\ 0 \end{pmatrix}, \mathbf{s}_3 = \begin{pmatrix} 0 \\ -\sqrt{E_s} \end{pmatrix}, \tag{26}$$

$$E[\mathbf{n}] = \mathbf{0}, E\left[\mathbf{n}\mathbf{n}^T\right] = \sigma^2\mathbf{I} \tag{27}$$

Problem 12. Show that the MAP decision for detecting $\mathbf{s}_0$ results in

$$|y_2| < y_1 \tag{28}$$

Problem 13. Express $\Pr(\hat{\mathbf{s}} = \mathbf{s}_0|\mathbf{s} = \mathbf{s}_0)$ in terms of r_1, r_2. Let $X = n_2 - n_1, Y = -n_2 - n_1$, where $\mathbf{n} = (n_1, n_2)$. Their correlation coefficient is defined as

$$\rho = \frac{E\left[(X - \mu_x)\left(Y - \mu_y\right)\right]}{\sigma_x\sigma_y} \tag{29}$$

X and Y are said to be uncorrelated if $\rho = 0$

Problem 14. Show that if X and Y are uncorrelated Verify this numerically.

Problem 15. Show that X and Y are independent, i.e. $p_{XY}(x, y) = p_X(x)p_Y(y)$.

Problem 16. Show that $X, Y \sim \mathcal{N}(0, N_0)$.

Problem 17. Show that

$$\Pr(\hat{\mathbf{s}} = \mathbf{s}_0|\mathbf{s} = \mathbf{s}_0) = \Pr\left(X < \sqrt{E_s}, Y < \sqrt{E_s}\right). \tag{30}$$

Problem 18. Show that

$$\Pr\left(X < \sqrt{E_s}, Y < \sqrt{E_s}\right) = \left(1 - Q\left(\sqrt{\frac{E_s}{N_0}}\right)\right)^2 \tag{31}$$

Problem 19. Verify the above through simulation.

Solution: This is shown in Fig. 5 through the following code.

```
codes/qpsk.py
```

Problem 20. Modify the above script to obtain the probability of symbol error.

4 M-PSK

Consider a system where $\mathbf{s}_i = \begin{pmatrix} \cos\left(\frac{2\pi i}{M}\right) \\ \cos\left(\frac{2\pi i}{M}\right) \end{pmatrix}, i = 0, 1, \dots M - 1$. Let

$$\mathbf{y}|s_0 = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \sqrt{E_s} + n_1 \\ n_2 \end{pmatrix} \tag{32}$$

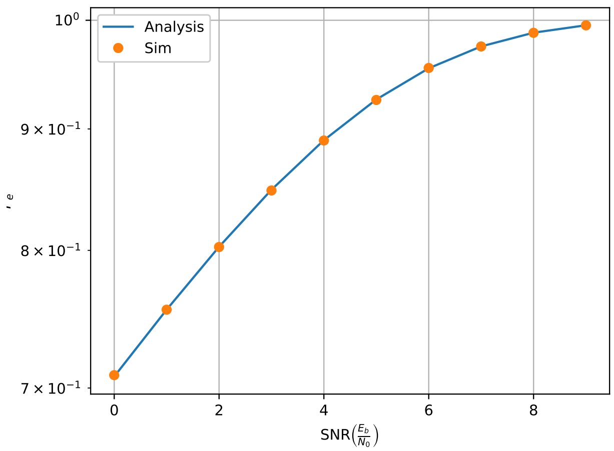

Fig. 5

where $n_1, n_2 \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$.

Problem 21. Substituting

$$y_1 = R\cos\theta \tag{33}$$
$$y_2 = R\sin\theta \tag{34}$$

show that the joint pdf of R, θ is

$$p(R,\theta) = \frac{R}{\pi N_0}\exp\left(-\frac{R^2 - 2R\sqrt{E_s}\cos\theta + E_s}{N_0}\right) \tag{35}$$

Problem 22. Show that

$$\lim_{\alpha\to\infty}\int_0^\infty (V-\alpha)\,e^{-(V-\alpha)^2}\,dV = 0 \tag{36}$$
$$\lim_{\alpha\to\infty}\int_0^\infty e^{-(V-\alpha)^2}\,dV = \sqrt{\pi} \tag{37}$$

Problem 23. Using the above, show that

$$\int_0^\infty V\exp\left\{-\left(V^2 - 2V\sqrt{\gamma}\cos\theta + \gamma\right)\right\}dV$$
$$= e^{-\gamma\sin^2\theta}\sqrt{\gamma\pi}\cos\theta \tag{38}$$

for large values of γ.

Problem 24. Find a compact expression for

$$I = 1 - \sqrt{\frac{\gamma}{\pi}}\int_{-\frac{\pi}{M}}^{\frac{\pi}{M}} e^{-\gamma\sin^2\theta}\cos\theta\,d\theta \tag{39}$$

Problem 25. Show that

$$P_{e|\mathbf{s}_0} = 2Q\left(\sqrt{2\left(\frac{E_s}{N_o}\right)}\sin\frac{\pi}{M}\right) \tag{40}$$

Problem 26. Verify the SER through simulation.