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**Abstract**—The manual frames the problems of receiver design and performance analysis in digital communication as applications of probability theory.

Download all codes in this manual from

svn co <https://github.com/gadepall/comm/trunk/modulation/codes>

## 1 BPSK

**Problem 1.** The *signal constellation diagram* for BPSK is given by Fig. 1. The symbols  $s_0$  and  $s_1$  are equiprobable.  $\sqrt{E_b}$  is the energy transmitted per bit. Assuming a zero mean additive white gaussian noise (AWGN) with variance  $\frac{N_0}{2}$ , obtain the symbols that are received.

**Solution:** The possible received symbols are

$$y|s_0 = \sqrt{E_b} + n \quad (1)$$

$$y|s_1 = -\sqrt{E_b} + n \quad (2)$$

where the AWGN  $n \sim \mathcal{N}(0, \frac{N_0}{2})$ .

**Problem 2.** From Fig. 1 obtain a decision rule for BPSK

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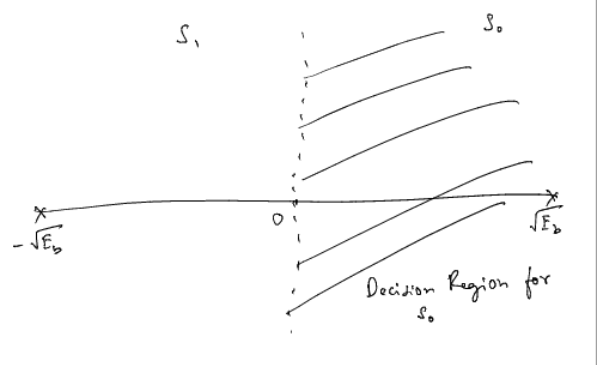


Fig. 1

**Solution:** The decision rule is

$$y \underset{s_1}{\overset{s_0}{\geq}} 0 \quad (3)$$

**Problem 3.** Repeat the previous exercise using the MAP criterion.

**Problem 4.** Using the decision rule in Problem 2, obtain an expression for the probability of error for BPSK.

**Solution:** Since the symbols are equiprobable, it is sufficient if the error is calculated assuming that a 0 was sent. This results in

$$P_e = \Pr(y < 0 | s_0) = \Pr(\sqrt{E_b} + n < 0) \quad (4)$$

$$= \Pr(-n > \sqrt{E_b}) = \Pr(n > \sqrt{E_b}) \quad (5)$$

since  $n$  has a symmetric pdf. Let  $w \sim \mathcal{N}(0, 1)$ . Then  $n = \sqrt{\frac{N_0}{2}}w$ . Substituting this in (5),

$$P_e = \Pr\left(\sqrt{\frac{N_0}{2}}w > \sqrt{E_b}\right) = \Pr\left(w > \sqrt{\frac{2E_b}{N_0}}\right) \quad (6)$$

$$= Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \quad (7)$$

where  $Q(x) \triangleq \Pr(w > x)$ ,  $x \geq 0$ .

**Problem 5.** The PDF of  $w \sim \mathcal{N}(0, 1)$  is given by

$$p_w(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty \quad (8)$$

and the complementary error function is defined as

$$\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt. \quad (9)$$

Show that

$$Q(x) = \frac{1}{2} \text{erfc}\left(\frac{x}{\sqrt{2}}\right) \quad (10)$$

**Problem 6.** Verify the bit error rate (BER) plots for BPSK through simulation and analysis for 0 to 10 dB.

**Solution:** The following code

```
codes / bpsk_ber . py
```

yields Fig. 2

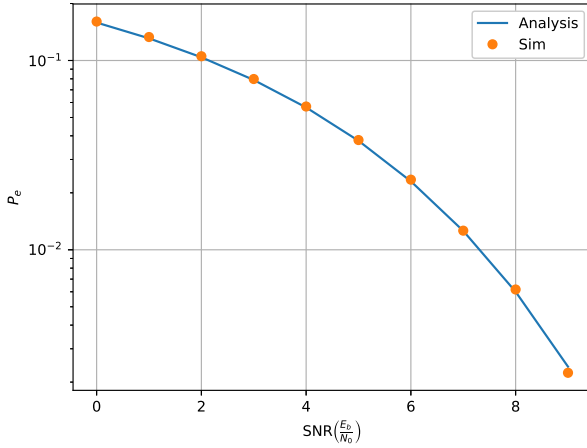


Fig. 2

**Problem 7.** Show that

$$Q(x) = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} e^{-\frac{x^2}{2\sin^2\theta}} d\theta \quad (11)$$

## 2 COHERENT BFSK

**Problem 8.** The signal constellation for binary frequency shift keying (BFSK) is given in Fig. 3. Obtain the equations for the received symbols.

**Solution:** The received symbols are given by

$$\mathbf{y}|_{s_0} = \begin{pmatrix} \sqrt{E_b} \\ 0 \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, \quad (12)$$

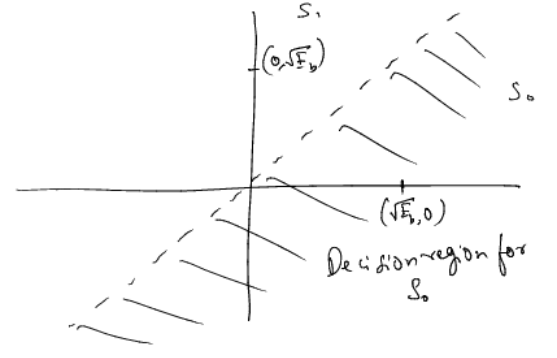


Fig. 3

and

$$\mathbf{y}|_{s_1} = \begin{pmatrix} 0 \\ \sqrt{E_b} \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, \quad (13)$$

where  $n_1, n_2 \sim \mathcal{N}(0, \frac{N_0}{2})$ . and  $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ .

**Problem 9.** Obtain a decision rule for BFSK from Fig. 3.

**Solution:** The decision rule is

$$y_1 \underset{s_1}{\overset{s_0}{\geq}} y_2 \quad (14)$$

**Definition 2.1.** The joint PDF of  $X, Y$  is given by

$$p(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}\right] \times \left\{ \frac{(x-\mu_x)^2}{\sigma_x^2} + \frac{(y-\mu_y)^2}{\sigma_y^2} - \frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y} \right\} \quad (15)$$

where

$$\mu_x = E[X], \sigma_x^2 = \text{var}(X), \rho = \frac{E[(X-\mu_x)(Y-\mu_y)]}{\sigma_x\sigma_y}. \quad (16)$$

**Problem 10.** For equiprobably symbols, the MAP criterion is defined as

$$p(\mathbf{y}|s_0) \underset{s_1}{\overset{s_0}{\geq}} p(\mathbf{y}|s_1) \quad (17)$$

Use (15) in (17) to obtain (14).

**Solution:** According to the MAP criterion, assum-

ing equiprobably symbols,

$$p(\mathbf{y}|s_0) \stackrel{s_0}{\geq} p(\mathbf{y}|s_1) \quad (18)$$

**Problem 11.** Derive and plot the probability of error. Verify through simulation.

**Solution:** Given that  $s_0$  was transmitted, the received symbols are

$$\mathbf{y}|s_0 = \begin{pmatrix} \sqrt{E_b} \\ 0 \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, \quad (19)$$

From (14), the probability of error is given by

$$P_e = \Pr(y_1 < y_2|s_0) = \Pr(\sqrt{E_b} + n_1 < n_2) \quad (20)$$

$$= \Pr(n_2 - n_1 > \sqrt{E_b}) \quad (21)$$

Note that  $n_2 - n_1 \sim \mathcal{N}(0, N_0)$ . Thus,

$$P_e = \Pr(\sqrt{N_0}w > \sqrt{E_b}) = \Pr\left(w > \sqrt{\frac{E_b}{N_0}}\right) \quad (22)$$

$$\Rightarrow P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right) \quad (23)$$

where  $w \sim \mathcal{N}(0, 1)$ . The following code plots the BER curves in Fig. 4

```
codes / fsk_ber.py
```

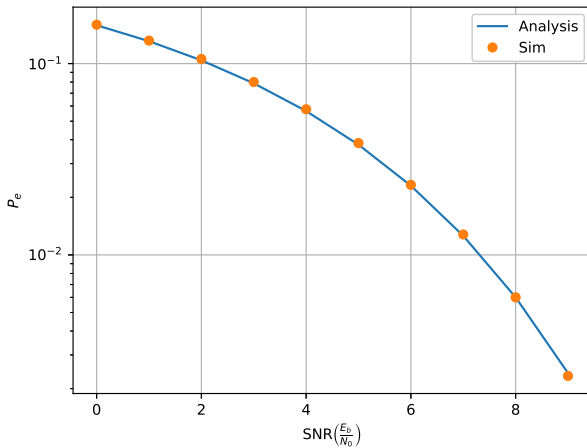


Fig. 4

### 3 QPSK

Let

$$\mathbf{y} = \mathbf{s} + \mathbf{n} \quad (24)$$

where  $\mathbf{s} \in \{\mathbf{s}_0, \mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}$  and

$$\mathbf{s}_0 = \begin{pmatrix} \sqrt{E_s} \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ \sqrt{E_s} \end{pmatrix}, \quad (25)$$

$$\mathbf{s}_2 = \begin{pmatrix} -\sqrt{E_s} \\ 0 \end{pmatrix}, \mathbf{s}_3 = \begin{pmatrix} 0 \\ -\sqrt{E_s} \end{pmatrix}, \quad (26)$$

$$E[\mathbf{n}] = \mathbf{0}, E[\mathbf{n}\mathbf{n}^T] = \sigma^2 \mathbf{I} \quad (27)$$

**Problem 12.** Show that the MAP decision for detecting  $\mathbf{s}_0$  results in

$$|y_2| < y_1 \quad (28)$$

**Problem 13.** Express  $\Pr(\hat{\mathbf{s}} = \mathbf{s}_0|\mathbf{s} = \mathbf{s}_0)$  in terms of  $r_1, r_2$ . Let  $X = n_2 - n_1, Y = -n_2 - n_1$ , where  $\mathbf{n} = (n_1, n_2)$ . Their correlation coefficient is defined as

$$\rho = \frac{E[(X - \mu_x)(Y - \mu_y)]}{\sigma_x \sigma_y} \quad (29)$$

$X$  and  $Y$  are said to be uncorrelated if  $\rho = 0$

**Problem 14.** Show that if  $X$  and  $Y$  are uncorrelated Verify this numerically.

**Problem 15.** Show that  $X$  and  $Y$  are independent, i.e.  $p_{XY}(x, y) = p_X(x)p_Y(y)$ .

**Problem 16.** Show that  $X, Y \sim \mathcal{N}(0, N_0)$ .

**Problem 17.** Show that

$$\Pr(\hat{\mathbf{s}} = \mathbf{s}_0|\mathbf{s} = \mathbf{s}_0) = \Pr(X < \sqrt{E_s}, Y < \sqrt{E_s}). \quad (30)$$

**Problem 18.** Show that

$$\Pr(X < \sqrt{E_s}, Y < \sqrt{E_s}) = \left(1 - Q\left(\sqrt{\frac{E_s}{N_0}}\right)\right)^2 \quad (31)$$

**Problem 19.** Verify the above through simulation.

**Solution:** This is shown in Fig. 5 through the following code.

```
codes / qpsk.py
```

**Problem 20.** Modify the above script to obtain the probability of symbol error.

### 4 M-PSK

Consider a system where  $\mathbf{s}_i = \begin{pmatrix} \cos\left(\frac{2\pi i}{M}\right) \\ \sin\left(\frac{2\pi i}{M}\right) \end{pmatrix}, i = 0, 1, \dots, M-1$ . Let

$$\mathbf{y}|s_0 = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \sqrt{E_s} + n_1 \\ n_2 \end{pmatrix} \quad (32)$$

**Problem 26.** Verify the SER through simulation.

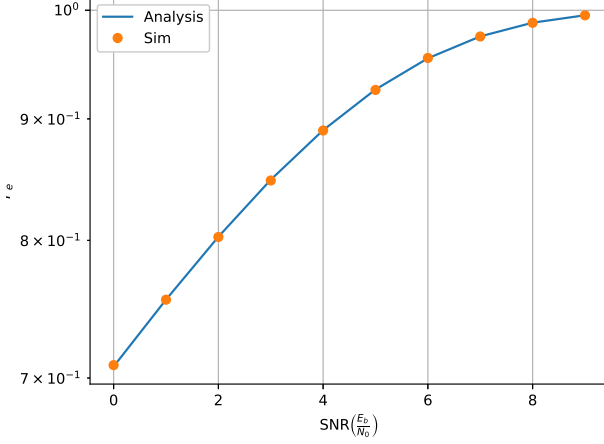


Fig. 5

where  $n_1, n_2 \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$ .

**Problem 21.** Substituting

$$y_1 = R \cos \theta \quad (33)$$

$$y_2 = R \sin \theta \quad (34)$$

show that the joint pdf of  $R, \theta$  is

$$p(R, \theta) = \frac{R}{\pi N_0} \exp\left(-\frac{R^2 - 2R\sqrt{E_s}\cos\theta + E_s}{N_0}\right) \quad (35)$$

**Problem 22.** Show that

$$\lim_{\alpha \rightarrow \infty} \int_0^\infty (V - \alpha) e^{-(V-\alpha)^2} dV = 0 \quad (36)$$

$$\lim_{\alpha \rightarrow \infty} \int_0^\infty e^{-(V-\alpha)^2} dV = \sqrt{\pi} \quad (37)$$

**Problem 23.** Using the above, show that

$$\begin{aligned} \int_0^\infty V \exp\left\{-\left(V^2 - 2V\sqrt{\gamma}\cos\theta + \gamma\right)\right\} dV \\ = e^{-\gamma\sin^2\theta} \sqrt{\gamma\pi} \cos\theta \end{aligned} \quad (38)$$

for large values of  $\gamma$ .

**Problem 24.** Find a compact expression for

$$I = 1 - \sqrt{\frac{\gamma}{\pi}} \int_{-\frac{\pi}{M}}^{\frac{\pi}{M}} e^{-\gamma\sin^2\theta} \cos\theta d\theta \quad (39)$$

**Problem 25.** Show that

$$P_{e|s_0} = 2Q\left(\sqrt{2\left(\frac{E_s}{N_o}\right)} \sin\frac{\pi}{M}\right) \quad (40)$$