Digital Modulation Techniques

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Contents

***Abstract*—The manual frames the problems of receiver design and performance analysis in digital communication as applications of probability theory.**

1 BPSK

Problem 1. The *signal constellation diagram* for BPSK is given by Fig. 1. The symbols s_0 and s_1 are equiprobable. $\sqrt{E_b}$ is the energy transmitted per bit. Assuming a zero mean additive white gaussian noise (AWGN) with variance $\frac{N_0}{2}$, obtain the symbols that are received.

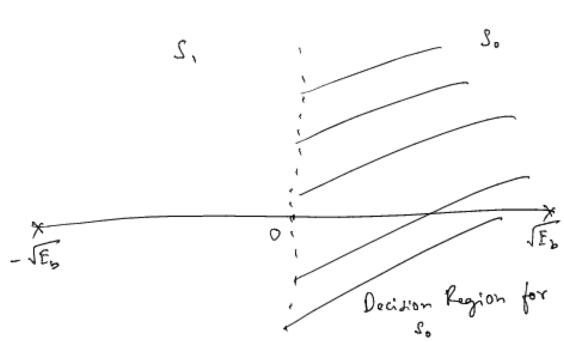

Fig. 1

Solution: The possible received symbols are

$$y|s_0 = \sqrt{E_b} + n \tag{1}$$
$$y|s_1 = -\sqrt{E_b} + n \tag{2}$$

where the AWGN $n \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$.

Problem 2. From Fig. 1 obtain a decision rule for BPSK

Solution: The decision rule is

$$y \underset{s_1}{\overset{s_0}{\gtrless}} 0 \tag{3}$$

Problem 3. Repeat the previous exercise using the MAP criterion.

Problem 4. Using the decision rule in Problem 2, obtain an expression for the probability of error for BPSK.

Solution: Since the symbols are equiprobable, it is sufficient if the error is calculated assuming that a 0 was sent. This results in

$$P_e = \Pr\left(y < 0|s_0\right) = \Pr\left(\sqrt{E_b} + n < 0\right) \tag{4}$$
$$= \Pr\left(-n > \sqrt{E_b}\right) = \Pr\left(n > \sqrt{E_b}\right) \tag{5}$$

since n has a symmetric pdf. Let $w \sim \mathcal{N}(0,1)$. Then $n = \sqrt{\frac{N_0}{2}}w$. Substituting this in (5),

$$P_e = \Pr\left(\sqrt{\frac{N_0}{2}}w > \sqrt{E_b}\right) = \Pr\left(w > \sqrt{\frac{2E_b}{N_0}}\right) \tag{6}$$
$$= Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \tag{7}$$

where $Q(x) \triangleq \Pr(w > x), x \geq 0$.

Problem 5. The PDF of $w \sim \mathcal{N}(0,1)$ is given by

$$p_w(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty \tag{8}$$

and the complementary error function is defined as

$$\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt. \tag{9}$$

Show that

$$Q(x) = \frac{1}{2} \text{erfc}\left(\frac{x}{\sqrt{2}}\right) \tag{10}$$

Problem 6. Verify the bit error rate (BER) plots for

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BPSK through simulation and analysis for 0 to 10 dB.

Solution: The following code

```
import numpy as np
import mpmath as mp
import matplotlib.pyplot as plt

def qfunc(x):
        return 0.5*mp.erfc(x/mp.
           sqrt(2))

#Number of SNR samples
snrlen = 10
#SNR values in dB
snrdb = np.linspace(0,9,10)
#Number of samples
simlen = int(1e5)
#Simulated BER declaration
err = []
#Analytical BER declaration
ber = []

#for SNR 0 to 10 dB
for i in range(0,snrlen):
        #Generating AWGN, 0 mean
           unit variance
        noise = np.random.normal
           (0,1,simlen)
        #from dB to actual SNR
        snr = 10**(0.1*snrdb[i])
        #Received symbol in
           baseband
        rx = mp.sqrt(snr) + noise
        #storing the index for the
            received symbol
        #in error
        err_ind = np.nonzero(rx <
           0)
        #calculating the total
           number of errors
        err_n = np.size(err_ind)
        #calcuating the simulated
           BER
        err.append(err_n/simlen)
        #calculating the
           analytical BER
        ber.append(qfunc(mp.sqrt(
           snr)))
plt.semilogy(snrdb.T,ber,label='
   Analysis')
plt.semilogy(snrdb.T,err,'o',label
   ='Sim')
plt.xlabel('SNR$\\left(\\frac{E_b
   }{N_0}\\right)$')
plt.ylabel('$P_e$')
plt.legend()
plt.grid()
#plt.savefig('../figs/bpsk_ber.eps
   ')
plt.show()
```

yields Fig. 2

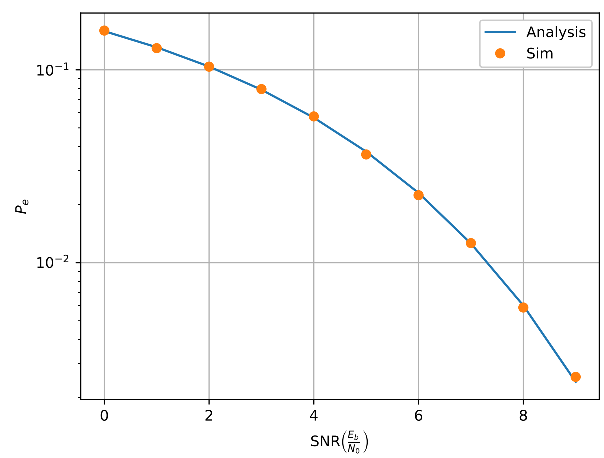

Fig. 2

Problem 7. Show that

$$Q(x) = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} e^{-\frac{x^2}{2\sin^2\theta}}\, d\theta \tag{11}$$

2 Coherent BFSK

Problem 8. The signal constellation for binary frequency shift keying (BFSK) is given in Fig. 3. Obtain the equations for the received symbols.

Solution: The received symbols are given by

$$\mathbf{y}|s_0 = \begin{pmatrix}\sqrt{E_b}\\0\end{pmatrix} + \begin{pmatrix}n_1\\n_2\end{pmatrix}, \tag{12}$$

and

$$\mathbf{y}|s_1 = \begin{pmatrix}0\\\sqrt{E_b}\end{pmatrix} + \begin{pmatrix}n_1\\n_2\end{pmatrix}, \tag{13}$$

where $n_1, n_2 \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$. and $\mathbf{y} = \begin{pmatrix}y_1\\y_2\end{pmatrix}$.

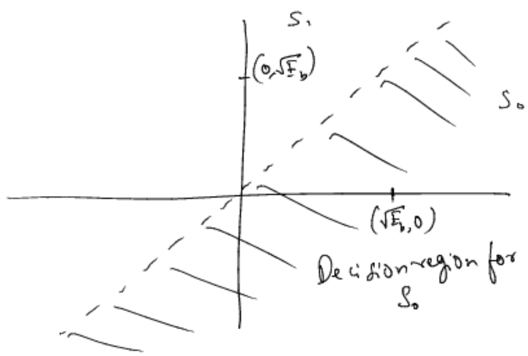

Fig. 3

Problem 9. Obtain a decision rule for BFSK from Fig. 3.

Solution: The decision rule is

$$y_1 \underset{s_1}{\overset{s_0}{\gtrless}} y_2 \tag{14}$$

Problem 10. Repeat the previous exercise using the MAP criterion.

Problem 11. Derive and plot the probability of error. Verify through simulation.

3 QPSK

Problem 12. Let

$$\mathbf{r} = \mathbf{s} + \mathbf{n} \tag{15}$$

where $\mathbf{s} \in \{s_0, s_1, s_2, s_3\}$ and

$$\mathbf{s}_0 = \begin{pmatrix} \sqrt{E_b} \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ \sqrt{E_b} \end{pmatrix}, \tag{16}$$

$$\mathbf{s}_2 = \begin{pmatrix} -\sqrt{E_b} \\ 0 \end{pmatrix}, \mathbf{s}_3 = \begin{pmatrix} 0 \\ -\sqrt{E_b} \end{pmatrix}, \tag{17}$$

$$E[\mathbf{n}] = \mathbf{0}, E\left[\mathbf{n}\mathbf{n}^T\right] = \sigma^2\mathbf{I} \tag{18}$$

1) Show that the MAP decision for detecting $\mathbf{s}_0$ results in
$$|r|_2 < r_1 \tag{19}$$
2) Express $\Pr(\hat{\mathbf{s}} = \mathbf{s}_0 | \mathbf{s} = \mathbf{s}_0)$ in terms of r_1, r_2. Let $X = n_2 - n_1, Y = -n_2 - n_1$, where $\mathbf{n} = (n_1, n_2)$. Their correlation coefficient is defined as
$$\rho = \frac{E\left[(X - \mu_x)\left(Y - \mu_y\right)\right]}{\sigma_x \sigma_y} \tag{20}$$
X and Y are said to be uncorrelated if $\rho = 0$
3) Show that if X and Y are uncorrelated Verify this numerically.
4) Show that X and Y are independent, i.e. $p_{XY}(x, y) = p_X(x) p_Y(y)$.
5) Show that $X, Y \sim \mathcal{N}\left(0, 2\sigma^2\right)$.
6) Show that $\Pr(\hat{\mathbf{s}} = \mathbf{s}_0 | \mathbf{s} = \mathbf{s}_0) = \Pr(X < A, Y < A)$.
7) Find $\Pr(X < A, Y < A)$.
8) Verify the above through simulation.

4 M-PSK

Problem 13. Consider a system where $\mathbf{s}_i = \begin{pmatrix} \cos\left(\frac{2\pi i}{M}\right) \\ \cos\left(\frac{2\pi i}{M}\right) \end{pmatrix}, i = 0, 1, \dots M-1$. Let

$$\mathbf{r}|s_0 = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} \sqrt{E_s} + n_1 \\ n_2 \end{pmatrix} \tag{21}$$

where $n_1, n_2 \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$.

1) Substituting
$$r_1 = R\cos\theta \tag{22}$$
$$r_2 = R\sin\theta \tag{23}$$
show that the joint pdf of R, θ is
$$p(R, \theta) = \frac{R}{\pi N_0} \exp\left(-\frac{R^2 - 2R\sqrt{E_s}\cos\theta + E_s}{N_0}\right) \tag{24}$$
2) Show that
$$\lim_{\alpha \to \infty} \int_0^\infty (V - \alpha) e^{-(V-\alpha)^2} \, dV = 0 \tag{25}$$
$$\lim_{\alpha \to \infty} \int_0^\infty e^{-(V-\alpha)^2} \, dV = \sqrt{\pi} \tag{26}$$
3) Using the above, evaluate
$$\int_0^\infty V \exp\left\{-\left(V^2 - 2V\sqrt{\gamma}\cos\theta + \gamma\right)\right\} dV \tag{27}$$
for large values of γ.
4) Find a compact expression for
$$I = 1 - \sqrt{\frac{\gamma}{\pi}} \int_{-\frac{\pi}{M}}^{\frac{\pi}{M}} e^{-\gamma \sin^2\theta} \cos\theta \, d\theta \tag{28}$$
5) Find $P_{e|\mathbf{s}_0}$.