

C. Shruti, P. N. V. S. S. K. HAVISH, S. S. Ashish and G V V Sharma\*

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**Abstract**—The manual frames the problems of receiver design and performance analysis in digital communication as applications of probability theory.

Download all codes in this manual from

svn co <https://github.com/gadepall/comm/trunk/modulation/manual/codes>

## 1 BPSK

1.1. The *signal constellation diagram* for BPSK is given by Fig. 1.1. The symbols  $s_0$  and  $s_1$  are equiprobable.  $\sqrt{E_b}$  is the energy transmitted per bit. Assuming a zero mean additive white gaussian noise (AWGN) with variance  $\frac{N_0}{2}$ , obtain the symbols that are received. **Solution:** The possible received symbols are

$$y|s_0 = \sqrt{E_b} + n \quad (1.1.1)$$

$$y|s_1 = -\sqrt{E_b} + n \quad (1.1.2)$$

where the AWGN  $n \sim \mathcal{N}(0, \frac{N_0}{2})$ .

1.2. From Fig. 1.1 obtain a decision rule for BPSK  
**Solution:** The decision rule is

$$y \underset{s_1}{\overset{s_0}{\geq}} 0 \quad (1.2.1)$$

\*The author is with the Department of Electrical Engineering, Indian Institute of Technology, Hyderabad 502285 India e-mail: gadepall@iith.ac.in. All content in this manual is released under GNU GPL. Free and open source.

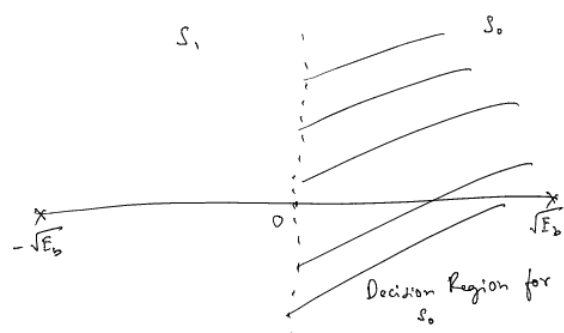


Fig. 1.1

1.3. Repeat the previous exercise using the MAP criterion.

1.4. Using the decision rule in Problem 1.2, obtain an expression for the probability of error for BPSK.

**Solution:** Since the symbols are equiprobable, it is sufficient if the error is calculated assuming that a 0 was sent. This results in

$$P_e = \Pr(y < 0|s_0) = \Pr(\sqrt{E_b} + n < 0) \quad (1.4.1)$$

$$= \Pr(-n > \sqrt{E_b}) = \Pr(n > \sqrt{E_b}) \quad (1.4.2)$$

since  $n$  has a symmetric pdf. Let  $w \sim \mathcal{N}(0, 1)$ .

Then  $n = \sqrt{\frac{N_0}{2}}w$ . Substituting this in (1.4.2),

$$P_e = \Pr\left(\sqrt{\frac{N_0}{2}}w > \sqrt{E_b}\right) = \Pr\left(w > \sqrt{\frac{2E_b}{N_0}}\right) \quad (1.4.3)$$

$$= Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \quad (1.4.4)$$

where

$$Q(x) \triangleq \Pr(w > x), x \geq 0. \quad (1.4.5)$$

$$= \frac{1}{\pi} \int_0^{\frac{\pi}{2}} e^{-\frac{x^2}{2 \sin^2 \theta}} d\theta \quad (1.4.6)$$

The PDF of  $w \sim \mathcal{N}(0, 1)$  is given by

$$p_w(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty \quad (1.4.7)$$

and the complementary error function is defined as

$$\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt. \quad (1.4.8)$$

1.5. Show that

$$Q(x) = \frac{1}{2} \text{erfc}\left(\frac{x}{\sqrt{2}}\right) \quad (1.5.1)$$

**Solution:** From (1.4.5)

$$Q(x) = \Pr(w > x), x \geq 0 \quad (1.5.2)$$

$$= \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{t^2}{2}} dt. \quad (1.5.3)$$

$$= \frac{1}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{2}}}^\infty e^{-y^2} dy. \quad (t = \sqrt{2}y) \quad (1.5.4)$$

resulting in

1.6. Verify the bit error rate (BER) plots for BPSK through simulation and analysis for 0 to 10 dB.

**Solution:** The following code

```
codes/bpsk_ber.py
```

yields Fig. 1.6

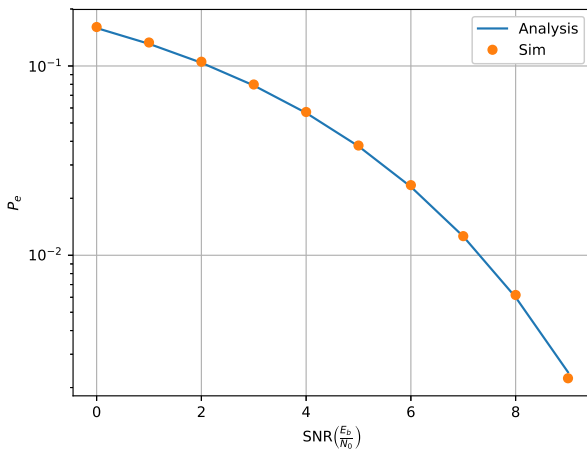


Fig. 1.6

## 2 COHERENT BFSK

2.1. The signal constellation for binary frequency shift keying (BFSK) is given in Fig. 2.1. Obtain

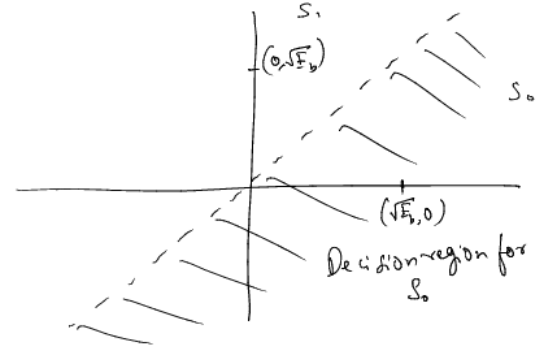


Fig. 2.1

the equations for the received symbols.

**Solution:** The received symbols are given by

$$\mathbf{y}|s_0 = \begin{pmatrix} \sqrt{E_b} \\ 0 \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, \quad (2.1.1)$$

and

$$\mathbf{y}|s_1 = \begin{pmatrix} 0 \\ \sqrt{E_b} \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, \quad (2.1.2)$$

where  $n_1, n_2 \sim \mathcal{N}(0, \frac{N_0}{2})$ . and  $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ .

2.2. Obtain a decision rule for BFSK from Fig. 2.1.

**Solution:** The decision rule is

$$y_1 \underset{s_1}{\overset{s_0}{\geq}} y_2 \quad (2.2.1)$$

**Definition 1.** The joint PDF of  $X, Y$  is given by

$$p(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}\right] \times \left\{ \frac{(x-\mu_x)^2}{\sigma_x^2} + \frac{(y-\mu_y)^2}{\sigma_y^2} - \frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y} \right\} \quad (2.2.2)$$

where

$$\mu_x = E[X], \quad (2.2.3)$$

$$\sigma_x^2 = \text{var}(X), \quad (2.2.4)$$

$$\rho = \frac{E[(X-\mu_x)(Y-\mu_y)]}{\sigma_x\sigma_y}. \quad (2.2.5)$$

For equiprobably symbols, the MAP criterion is defined as

$$p(\mathbf{y}|s_0) \underset{s_1}{\overset{s_0}{\geq}} p(\mathbf{y}|s_1) \quad (2.2.6)$$

2.3. Use (2.2.2) in (2.2.6) to obtain (2.2.1).

**Solution:** According to the MAP criterion, assuming equiprobably symbols,

$$p(\mathbf{y}|s_0) \underset{s_1}{\overset{s_0}{\geq}} p(\mathbf{y}|s_1) \quad (2.3.1)$$

2.4. Derive and plot the probability of error. Verify through simulation.

**Solution:** Given that  $s_0$  was transmitted, the received symbols are

$$\mathbf{y}|s_0 = \begin{pmatrix} \sqrt{E_b} \\ 0 \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, \quad (2.4.1)$$

From (2.2.1), the probability of error is given by

$$P_e = \Pr(y_1 < y_2|s_0) = \Pr(\sqrt{E_b} + n_1 < n_2) \quad (2.4.2)$$

$$= \Pr(n_2 - n_1 > \sqrt{E_b}) \quad (2.4.3)$$

Note that  $n_2 - n_1 \sim \mathcal{N}(0, N_0)$ . Thus,

$$P_e = \Pr(\sqrt{N_0}w > \sqrt{E_b}) \quad (2.4.4)$$

$$= \Pr\left(w > \sqrt{\frac{E_b}{N_0}}\right) \quad (2.4.5)$$

$$\Rightarrow P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right) \quad (2.4.6)$$

where  $w \sim \mathcal{N}(0, 1)$ . The following code plots the BER curves in Fig. 2.4

```
codes/fsk_ber.py
```

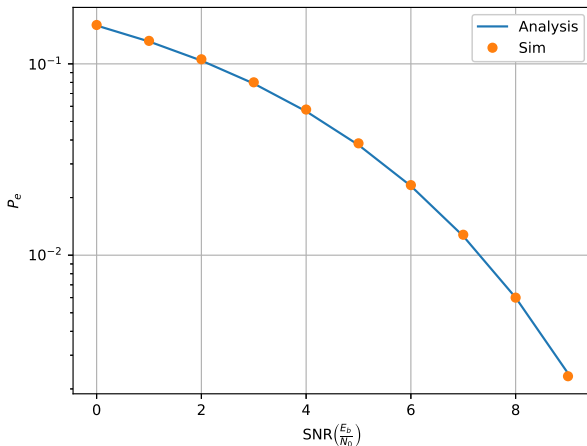


Fig. 2.4

### 3 QPSK

1. Let

$$\mathbf{y} = \mathbf{s} + \mathbf{n} \quad (3.1.1)$$

where  $\mathbf{s} \in \{\mathbf{s}_0, \mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}$  and

$$\mathbf{s}_0 = \begin{pmatrix} \sqrt{E_s} \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ \sqrt{E_s} \end{pmatrix}, \quad (3.1.2)$$

$$\mathbf{s}_2 = \begin{pmatrix} -\sqrt{E_s} \\ 0 \end{pmatrix}, \mathbf{s}_3 = \begin{pmatrix} 0 \\ -\sqrt{E_s} \end{pmatrix}, \quad (3.1.3)$$

$$E[\mathbf{n}] = \mathbf{0}, E[\mathbf{n}\mathbf{n}^T] = \sigma^2 \mathbf{I} \quad (3.1.4)$$

2. Show that the MAP decision for detecting  $s_0$  results in

$$|y_2| < y_1 \quad (3.2.1)$$

3. Express  $\Pr(\hat{\mathbf{s}} = \mathbf{s}_0 | \mathbf{s} = \mathbf{s}_0)$  in terms of  $r_1, r_2$ . Let  $X = n_2 - n_1, Y = -n_2 - n_1$ , where  $\mathbf{n} = (n_1, n_2)$ . Their correlation coefficient is defined as

$$\rho = \frac{E[(X - \mu_x)(Y - \mu_y)]}{\sigma_x \sigma_y} \quad (3.3.1)$$

$X$  and  $Y$  are said to be uncorrelated if  $\rho = 0$

4. Show that if  $X$  and  $Y$  are uncorrelated Verify this numerically.

5. Show that  $X$  and  $Y$  are independent, i.e.  $p_{XY}(x, y) = p_X(x)p_Y(y)$ .

6. Show that  $X, Y \sim \mathcal{N}(0, N_0)$ .

7. Show that

$$\Pr(\hat{\mathbf{s}} = \mathbf{s}_0 | \mathbf{s} = \mathbf{s}_0) = \Pr(X < \sqrt{E_s}, Y < \sqrt{E_s}). \quad (3.7.1)$$

8. Show that

$$\Pr(X < \sqrt{E_s}, Y < \sqrt{E_s}) = \left(1 - Q\left(\sqrt{\frac{E_s}{N_0}}\right)\right)^2 \quad (3.8.1)$$

9. Verify the above through simulation.

**Solution:** This is shown in Fig. 3.9 through the following code.

```
codes/qpsk.py
```

10. Modify the above script to obtain the probability of symbol error.

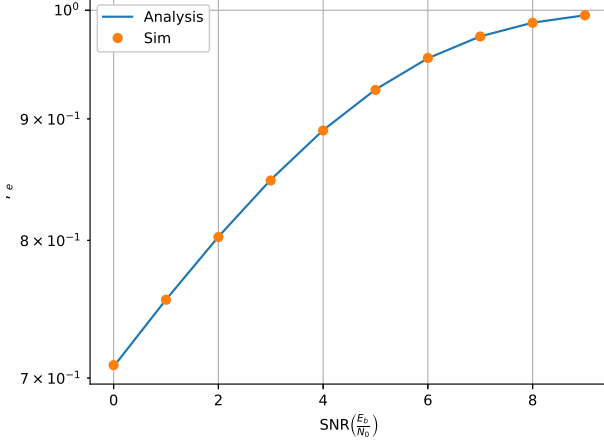


Fig. 3.9

#### 4 $M$ -PSK

1. Consider a system where  $\mathbf{s}_i = \begin{pmatrix} \cos\left(\frac{2\pi i}{M}\right) \\ \cos\left(\frac{2\pi i}{M}\right) \end{pmatrix}$ ,  $i = 0, 1, \dots, M-1$ . Let

$$\mathbf{y}|s_0 = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \sqrt{E_s} + n_1 \\ n_2 \end{pmatrix} \quad (4.1.1)$$

where  $n_1, n_2 \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$ .

Substituting

$$y_1 = R \cos \theta \quad (4.1.2)$$

$$y_2 = R \sin \theta \quad (4.1.3)$$

show that the joint pdf of  $R, \theta$  is

$$p(R, \theta) = \frac{R}{\pi N_0} \exp\left(-\frac{R^2 - 2R\sqrt{E_s}\cos\theta + E_s}{N_0}\right) \quad (4.1.4)$$

2. Show that

$$\lim_{\alpha \rightarrow \infty} \int_0^\infty (V - \alpha) e^{-(V-\alpha)^2} dV = 0 \quad (4.2.1)$$

$$\lim_{\alpha \rightarrow \infty} \int_0^\infty e^{-(V-\alpha)^2} dV = \sqrt{\pi} \quad (4.2.2)$$

3. Using the above, show that

$$\begin{aligned} \int_0^\infty V \exp\left\{-\left(V^2 - 2V\sqrt{\gamma}\cos\theta + \gamma\right)\right\} dV \\ = e^{-\gamma\sin^2\theta} \sqrt{\gamma\pi} \cos\theta \end{aligned} \quad (4.3.1)$$

for large values of  $\gamma$ .

4. Find a compact expression for

$$I = 1 - \sqrt{\frac{\gamma}{\pi}} \int_{-\frac{\pi}{M}}^{\frac{\pi}{M}} e^{-\gamma\sin^2\theta} \cos\theta d\theta \quad (4.4.1)$$

5. Show that

$$P_{e|s_0} = 2Q\left(\sqrt{2\left(\frac{E_s}{N_0}\right)\sin\frac{\pi}{M}}\right) \quad (4.5.1)$$

6. Verify the SER through simulation.