Applied Probability: Digital Communication

G V V Sharma*

Contents

***Abstract*—The manual frames the problems of receiver design and performance analysis in digital communication as applications of probability theory.**

1 Multivariate Gaussian: Coherent BFSK

1.1 The multivariate Gaussian distribution is defined as

$$p_{\mathbf{x}}(x_1, \dots, x_k) = \frac{1}{\sqrt{(2\pi)^k |\Sigma|}} \exp\left\{-\frac{1}{2}(\mathbf{x}-\mu)^T \Sigma^{-1} (\mathbf{x}-\mu)\right\} \tag{1}$$

where μ is the mean vector, $\Sigma = E\left[(\mathbf{x}-\mu)(\mathbf{x}-\mu)^T\right]$ is the covariance matrix and $|\Sigma|$ is the determinant of Σ.

1.2 Show that

$$p(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)} \times \left\{\frac{(x-\mu_x)^2}{\sigma_x^2} + \frac{\left(y-\mu_y\right)^2}{\sigma_y^2}\right\} - \frac{2\rho(x-\mu_x)\left(y-\mu_y\right)}{\sigma_x\sigma_y}\right] \tag{2}$$

where

$$\mu = \begin{pmatrix}\mu_x \\ \mu_y\end{pmatrix}, \Sigma = \begin{pmatrix}\sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2\end{pmatrix} \tag{3}$$

1.3 If

$$\mathbf{y}|0 = \begin{pmatrix}\sqrt{A}+n_1 \\ n_2\end{pmatrix}, \tag{4}$$

and

$$\mathbf{y}|1 = \begin{pmatrix}n_1 \\ \sqrt{A}+n_2\end{pmatrix}, \tag{5}$$

use the MAP criterion to reach a decision.

1.4 Derive and plot the probability of error. Verify through simulation.

2 QPSK

2.1 Let

$$\mathbf{r} = \mathbf{s} + \mathbf{n} \tag{6}$$

where $\mathbf{s} \in \{s_0, s_1, s_2, s_3\}$ and

$$\mathbf{s}_0 = \begin{pmatrix}A \\ 0\end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix}0 \\ A\end{pmatrix}, \mathbf{s}_2 = \begin{pmatrix}-A \\ 0\end{pmatrix}, \mathbf{s}_3 = \begin{pmatrix}0 \\ -A\end{pmatrix}, \tag{7}$$

$$E[\mathbf{n}] = \mathbf{0}, E\left[\mathbf{n}\mathbf{n}^T\right] = \sigma^2\mathbf{I} \tag{8}$$

2.2 Show that the MAP decision for detecting $\mathbf{s}_0$ results in

$$|r|_2 < r_1 \tag{9}$$

2.3 Express $\Pr(\hat{\mathbf{s}} = \mathbf{s}_0 | \mathbf{s} = \mathbf{s}_0)$ in terms of r_1, r_2. Let $X = n_2 - n_1, Y = -n_2 - n_1$, where $\mathbf{n} = (n_1, n_2)$. Their correlation coefficient is defined as

$$\rho = \frac{E\left[(X-\mu_x)\left(Y-\mu_y\right)\right]}{\sigma_x\sigma_y} \tag{10}$$

X and Y are said to be uncorrelated if $\rho = 0$

2.4 Show that if X and Y are uncorrelated and verify this numerically.

*The author is with the Department of Electrical Engineering, Indian Institute of Technology, Hyderabad 502205 India e-mail: gadepall@iith.ac.in.

2.5 Show that X and Y are independent, i.e. $p_{XY}(x,y) = p_X(x)p_Y(y)$.
2.6 Show that $X, Y \sim \mathcal{N}\left(0, 2\sigma^2\right)$.
2.7 Show that

$$\Pr\left(\hat{\mathbf{s}} = \mathbf{s}_0 | \mathbf{s} = \mathbf{s}_0\right) = \Pr\left(X < A, Y < A\right). \tag{11}$$

2.8 Find $\Pr\left(X < A, Y < A\right)$.
2.9 Verify the above through simulation.

3 Noncoherent BFSK

3.1 Show that

$$I_0(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{x\cos\theta}\,d\theta \tag{12}$$

$$I_0(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{x\cos(\theta-\phi)}\,d\theta \tag{13}$$

$$\frac{1}{2\pi}\int_0^{2\pi} e^{m_1\cos\theta + m_2\sin\theta}\,d\theta = I_0\left(\sqrt{m_1^2 + m_2^2}\right) \tag{14}$$

where the modified Bessel function of order n (integer) is defined as

$$I_n(x) = \frac{1}{\pi}\int_0^{\pi} e^{x\cos\theta}\cos n\theta\,d\theta \tag{15}$$

3.2 Let

$$\mathbf{r}|0 = \sqrt{E_b}\begin{pmatrix}\cos\phi_0\\ \sin\phi_0\\ 0\\ 0\end{pmatrix} + \mathbf{n}_0, \mathbf{r}|1 = \sqrt{E_b}\begin{pmatrix}0\\ 0\\ \cos\phi_1\\ \sin\phi_1\end{pmatrix} + \mathbf{n}_1 \tag{16}$$

where $\mathbf{n}_0, \mathbf{n}_1 \sim \mathcal{N}\left(\mathbf{0}, \frac{N_0}{2}\mathbf{I}\right)$.
3.3 Taking $\mathbf{r} = (r_1, r_2, r_3, r_4)^T$,, find the pdf $p\left(\mathbf{r}|0, \phi_0\right)$ in terms of $r_1, r_2, r_3, r_4, \phi, E_b$ and N_0. Assume that all noise variables are independent.
3.4 If ϕ_0 is uniformly distributed between 0 and 2π, find $p\left(\mathbf{r}|0\right)$. Note that this expression will no longer contain ϕ_0.
3.5 Show that the ML detection criterion for this scheme is

$$I_0\left(k\sqrt{r_1^2 + r_2^2}\right) \underset{1}{\overset{0}{\gtrless}} I_0\left(k\sqrt{r_3^2 + r_4^2}\right) \tag{17}$$

where k is a constant.
3.6 The above criterion reduces to something simpler. Can you guess what it is? Justify your answer.
3.7 Show that

$$P_{e|0} = \Pr\left(r_1^2 + r_2^2 < r_3^2 + r_4^2 | 0\right) \tag{18}$$

3.8 Show that the pdf of $Y = r_3^2 + r_4^2$ id

$$p_Y(y) = \frac{1}{N_0}e^{-\frac{y}{N_0}}, y > 0 \tag{19}$$

3.9 Find

$$g\left(r_1, r_2\right) = \Pr\left(r_1^2 + r_2^2 < X | 0, r_1, r_2\right). \tag{20}$$

3.10 Show that $E\left[e^{-\frac{X^2}{2\sigma^2}}\right] = \frac{1}{\sqrt{2}}e^{-\frac{\mu^2}{2\sigma^2}}$ for $X \sim \mathcal{N}\left(\mu, \sigma^2\right)$.
3.11 Now show that

$$E\left[g\left(r_1, r_2\right)\right] = \frac{1}{2}e^{-\frac{E_b}{2N_0}}. \tag{21}$$

4 Fun with Probability

4.1 Let $U, V \sim \mathcal{N}\left(0, \frac{k}{2}\right)$ be i.i.d. Assuming that

$$U = \sqrt{R}\cos\Theta \tag{22}$$

$$V = \sqrt{R}\sin\Theta \tag{23}$$

4.2 Compute the jacobian for U, V with respect to X and Θ defined by

$$J = \det\begin{pmatrix}\frac{\partial U}{\partial R} & \frac{\partial U}{\partial \Theta}\\ \frac{\partial V}{\partial R} & \frac{\partial V}{\partial \Theta}\end{pmatrix} \tag{24}$$

4.3 The joint pdf for R, Θ is given by,

$$p_{R,\Theta}\left(r, \theta\right) = p_{U,V}\left(u, v\right) J|_{u=\sqrt{r}\cos\theta, v=\sqrt{r}\sin\theta} \tag{25}$$

Show that

$$p_R(r) = \begin{cases}\frac{1}{k}e^{-\frac{r}{k}} & r > 0,\\ 0 & r < 0,\end{cases} \tag{26}$$

assuming that Θ is uniformly distributed between 0 to 2π.
4.4 Show that the pdf of $Y = R_1 - R_2$, where R_1 and R_2 are i.i.d. and have the same distribution as R is

$$p_Y(y) = \frac{1}{2k}e^{-\frac{|y|}{k}} \tag{27}$$

4.5 Find the pdf of

$$Z = p + \sqrt{p}\left[U\cos\phi + V\sin\phi\right] \tag{28}$$

where ϕ is a constant.

4.6 Find $\Pr(Y > Z)$.

4.7 If $U \sim \mathcal{N}\left(m_1, \frac{k}{2}\right), V \sim \mathcal{N}\left(m_2, \frac{k}{2}\right)$, where m_1, m_2, k are constants, show that the pdf of

$$R = \sqrt{U^2 + V^2} \tag{29}$$

is

$$p_R(r) = \frac{e^{-\frac{r+m}{k}}}{k} I_0\left(\frac{2\sqrt{mr}}{k}\right), \quad m = \sqrt{m_1^2 + m_2^2} \tag{30}$$

4.8 Show that

$$I_0(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{4^n (n!)^2} \tag{31}$$

4.9 If

$$p_Z(z) = \begin{cases} \frac{1}{k} e^{-\frac{z}{k}} & z \geq 0 \\ 0 & z < 0 \end{cases} \tag{32}$$

find $\Pr(R < Z)$.

5 M-PSK

5.1 Consider a system where $\mathbf{s}_i = \begin{pmatrix} \cos\left(\frac{2\pi i}{M}\right) \\ \cos\left(\frac{2\pi i}{M}\right) \end{pmatrix}, i = 0, 1, \dots M-1$. Let

$$\mathbf{r}|s_0 = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} \sqrt{E_s} + n_1 \\ n_2 \end{pmatrix} \tag{33}$$

where $n_1, n_2 \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$. Find the decision rule.

5.2 Substituting

$$r_1 = R\cos\theta \tag{34}$$

$$r_2 = R\sin\theta \tag{35}$$

show that the joint pdf of R, θ is

$$p(R, \theta) = \frac{R}{\pi N_0} \exp\left(-\frac{R^2 - 2R\sqrt{E_s}\cos\theta + E_s}{N_0}\right) \tag{36}$$

5.3 Show that

$$\lim_{\alpha \to \infty} \int_0^{\infty} (V - \alpha) e^{-(V-\alpha)^2} \, dV = 0 \tag{37}$$

$$\lim_{\alpha \to \infty} \int_0^{\infty} e^{-(V-\alpha)^2} \, dV = \sqrt{\pi} \tag{38}$$

5.4 Using the above, evaluate

$$\int_0^{\infty} V \exp\left\{-\left(V^2 - 2V\sqrt{\gamma}\cos\theta + \gamma\right)\right\} dV \tag{39}$$

for large values of γ.

5.5 Find a compact expression for

$$I = 1 - \sqrt{\frac{\gamma}{\pi}} \int_{-\frac{\pi}{M}}^{\frac{\pi}{M}} e^{-\gamma \sin^2\theta} \cos\theta \, d\theta \tag{40}$$

5.6 Find $P_{e|\mathbf{s}_0}$.

6 Craig's Formula and MGF

6.1 The Moment Generating Function (MGF) of X is defined as

$$M_X(s) = E\left[e^{sX}\right] \tag{41}$$

where X is a random variable and $E[\cdot]$ is the expectation.

6.2 Let $Y \sim \mathcal{N}(0, 1)$. Define

$$Q(x) = \Pr(Y > x), x > 0 \tag{42}$$

Show that

$$Q(x) = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} e^{-\frac{x^2}{2\sin^2\theta}} \, d\theta \tag{43}$$

6.3 Let $h \sim \mathcal{CN}\left(0, \frac{\Omega}{2}\right), n \sim \mathcal{CN}\left(0, \frac{N_0}{2}\right)$. Find the distribution of $|h|^2$.

6.4 Let

$$P_e = \Pr\left(\Re\left\{h^* y\right\} < 0\right), \text{ where } y = \left(\sqrt{E_s} h + n\right), \tag{44}$$

Show that

$$P_e = \int_0^{\infty} Q\left(\sqrt{2x}\right) p_A(x) \, dx \tag{45}$$

where $A = \frac{E_s |h|^2}{N_0}$.

6.5 Show that

$$P_e = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} M_A\left(-\frac{1}{\sin^2\theta}\right) d\theta \tag{46}$$

6.6 compute $M_A(s)$.

6.7 Find P_e.

6.8 If $\gamma = \frac{\Omega E_s}{N_0}$, show that $P_e < \frac{1}{2\gamma}$.