Digital Modulation Techniques

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Contents

***Abstract*—The manual frames the problems of receiver design and performance analysis in digital communication as applications of probability theory.**

1 BPSK

Problem 1. The *signal constellation diagram* for BPSK is given by Fig. 1. The symbols s_0 and s_1 are equiprobable. $\sqrt{E_b}$ is the energy transmitted per bit. Assuming a zero mean additive white gaussian noise (AWGN) with variance $\frac{N_0}{2}$, obtain the symbols that are received.

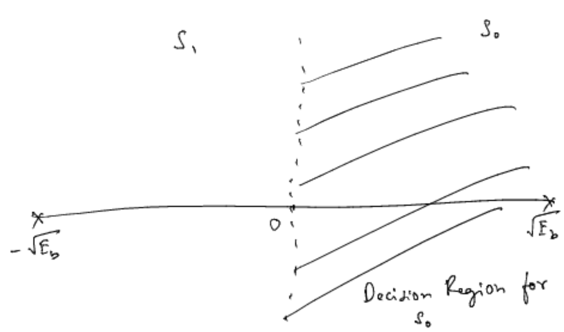

Fig. 1

Solution: The possible received symbols are

$$y|s_0 = \sqrt{E_b} + n \tag{1}$$
$$y|s_1 = -\sqrt{E_b} + n \tag{2}$$

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where the AWGN $n \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$.

Problem 2. From Fig. 1 obtain a decision rule for BPSK

Solution: The decision rule is

$$y \underset{s_1}{\overset{s_0}{\gtrless}} 0 \tag{3}$$

Problem 3. Repeat the previous exercise using the MAP criterion.

Problem 4. Using the decision rule in Problem 2, obtain an expression for the probability of error for BPSK.

Solution: Since the symbols are equiprobable, it is sufficient if the error is calculated assuming that a 0 was sent. This results in

$$P_e = \Pr\left(y < 0|s_0\right) = \Pr\left(\sqrt{E_b} + n < 0\right) \tag{4}$$
$$= \Pr\left(-n > \sqrt{E_b}\right) = \Pr\left(n > \sqrt{E_b}\right) \tag{5}$$

since n has a symmetric pdf. Let $w \sim \mathcal{N}(0, 1)$. Then $n = \sqrt{\frac{N_0}{2}}w$. Substituting this in (5),

$$P_e = \Pr\left(\sqrt{\frac{N_0}{2}}w > \sqrt{E_b}\right) = \Pr\left(w > \sqrt{\frac{2E_b}{N_0}}\right) \tag{6}$$
$$= Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \tag{7}$$

where $Q(x) \triangleq \Pr(w > x), x \geq 0$.

Problem 5. The PDF of $w \sim \mathcal{N}(0, 1)$ is given by

$$p_w(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty \tag{8}$$

and the complementary error function is defined as

$$\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt. \tag{9}$$

Show that

$$Q(x) = \frac{1}{2} \text{erfc}\left(\frac{x}{\sqrt{2}}\right) \tag{10}$$

Problem 6. Verify the bit error rate (BER) plots for

BPSK through simulation and analysis for 0 to 10 dB.

Solution: The following code

```
import numpy as np
import mpmath as mp
import matplotlib.pyplot as plt

def qfunc(x):
        return 0.5*mp.erfc(x/mp.
            sqrt(2))

#Number of SNR samples
snrlen = 10
#SNR values in dB
snrdb = np.linspace(0,9,10)
#Number of samples
simlen = int(1e5)
#Simulated BER declaration
err = []
#Analytical BER declaration
ber = []

#for SNR 0 to 10 dB
for i in range(0,snrlen):
        #Generating AWGN, 0 mean
            unit variance
        noise = np.random.normal
            (0,1,simlen)
        #from dB to actual SNR
        snr = 10**(0.1*snrdb[i])
        #Received symbol in
            baseband
        rx = mp.sqrt(snr) + noise
        #storing the index for the
             received symbol
        #in error
        err_ind = np.nonzero(rx <
            0)
        #calculating the total
            number of errors
        err_n = np.size(err_ind)
        #calcuating the simulated
            BER
        err.append(err_n/simlen)
        #calculating the
            analytical BER
        ber.append(qfunc(mp.sqrt(
            snr)))
plt.semilogy(snrdb.T,ber,label='
    Analysis')
plt.semilogy(snrdb.T,err,'o',label
    ='Sim')
plt.xlabel('SNR$\\left(\\frac{E_b
    }{N_0}\\right)$')
plt.ylabel('$P_e$')
plt.legend()
plt.grid()
plt.savefig('../figs/bpsk_ber.eps'
    )
plt.show()
```

yields Fig. 2

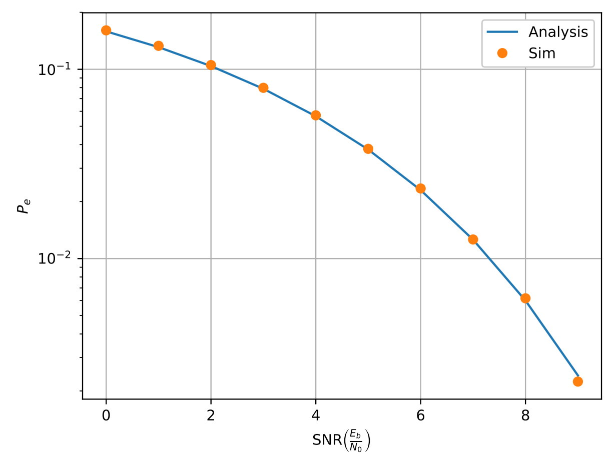

Fig. 2

Problem 7. Show that

$$Q(x) = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} e^{-\frac{x^2}{2\sin^2\theta}}\, d\theta \tag{11}$$

2 Coherent BFSK

Problem 8. The signal constellation for binary frequency shift keying (BFSK) is given in Fig. 3. Obtain the equations for the received symbols.

Solution: The received symbols are given by

$$\mathbf{y}|s_0 = \begin{pmatrix}\sqrt{E_b}\\ 0\end{pmatrix} + \begin{pmatrix}n_1\\ n_2\end{pmatrix}, \tag{12}$$

and

$$\mathbf{y}|s_1 = \begin{pmatrix}0\\ \sqrt{E_b}\end{pmatrix} + \begin{pmatrix}n_1\\ n_2\end{pmatrix}, \tag{13}$$

where $n_1, n_2 \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$. and $\mathbf{y} = \begin{pmatrix}y_1\\ y_2\end{pmatrix}$.

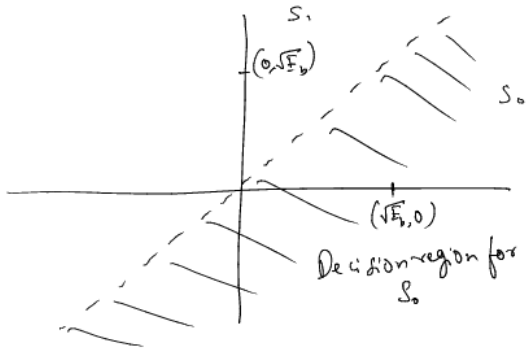

Fig. 3

Problem 9. Obtain a decision rule for BFSK from Fig. 3.

Solution: The decision rule is

$$y_1 \underset{s_1}{\overset{s_0}{\gtrless}} y_2 \tag{14}$$

Problem 10. Repeat the previous exercise using the MAP criterion.

Problem 11. Derive and plot the probability of error. Verify through simulation.

Solution: Given we transmitted s_0, the probability of error is given by

$$P_e = \Pr\left(y_1 < y_2 | s_0\right) = \Pr\left(\sqrt{E_b} + n_1 < n_2\right) \tag{15}$$

$$= \Pr\left(n_2 - n_1 > \sqrt{E_b}\right) = \Pr\left(X > \sqrt{E_b}\right) \tag{16}$$

where $X \sim \mathcal{N}(0, N_0)$
Let $w \sim \mathcal{N}(0, 1)$. Then $X = \sqrt{N_0}w$. Substituting this in (**??**),

$$P_e = \Pr\left(\sqrt{N_0}w > \sqrt{E_b}\right) = \Pr\left(w > \sqrt{\frac{E_b}{N_0}}\right) \tag{17}$$

$$= Q\left(\sqrt{\frac{E_b}{N_0}}\right) \tag{18}$$

where $Q(x) \triangleq \Pr(w > x), x \geq 0$. The following code

```
from __future__ import division
import numpy as np
import mpmath as mp
import matplotlib.pyplot as plt


#the ber expression is sqrt(E_b)+
   n1-n2<0
def qfunc(x):
        return 0.5*mp.erfc(x/mp.
           sqrt(2))

#Number of SNR samples
snrlen = 10
#SNR values in dB
snrdb = np.linspace(0,9,10)
#Number of samples
simlen = int(1e5)
#Simulated BER declaration
err = []
#Analytical BER declaration
ber = []
noise1 = np.random.normal(0,1,
   simlen)
noise2=np.random.normal(0,1,simlen
   )
#for SNR 0 to 10 dB
for i in range(0,snrlen):
        #Generating AWGN, 0 mean
           unit variance

        #from dB to actual SNR
        snr = 10**(0.1*snrdb[i])
        #Received symbol in
           baseband
        y1 = mp.sqrt(2*snr) +
           noise1
        y2=noise2
        #storing the index for the
            received symbol
        #in error
        err_ind = np.nonzero(y1 <
           y2)
        #calculating the total
           number of errors
        err_n = np.size(err_ind)
        #calcuating the simulated
           BER
        err.append(err_n/simlen)
        #calculating the
           analytical BER
        ber.append(qfunc(mp.sqrt(
           snr)))

plt.semilogy(snrdb.T,ber,label='
   Analysis')
plt.semilogy(snrdb.T,err,'o',label
```

```
    ='Sim')
plt.xlabel('SNR$\\left(\\frac{E_b
    }{N_0}\\right)$')
plt.ylabel('$P_e$')
plt.legend()
plt.grid()
plt.savefig('../figs/bfsk_ber.eps'
    )
plt.show()
```

yields Fig. 4

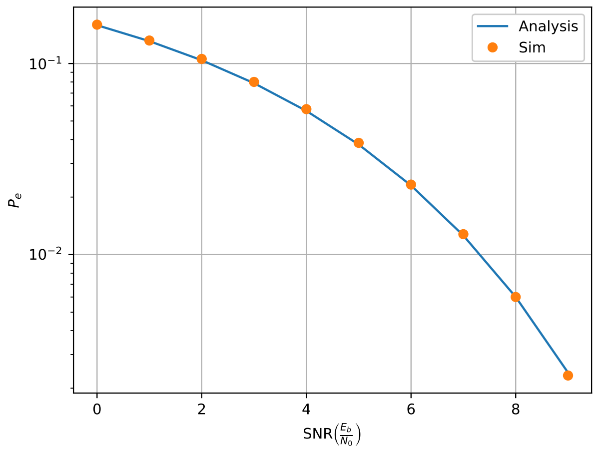

Fig. 4

3 QPSK

Problem 12. Let

$$\mathbf{r} = \mathbf{s} + \mathbf{n} \tag{19}$$

where $\mathbf{s} \in \{s_0, s_1, s_2, s_3\}$ and

$$\mathbf{s}_0 = \begin{pmatrix} \sqrt{E_b} \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ \sqrt{E_b} \end{pmatrix}, \tag{20}$$

$$\mathbf{s}_2 = \begin{pmatrix} -\sqrt{E_b} \\ 0 \end{pmatrix}, \mathbf{s}_3 = \begin{pmatrix} 0 \\ -\sqrt{E_b} \end{pmatrix}, \tag{21}$$

$$E[\mathbf{n}] = \mathbf{0}, E\left[\mathbf{n}\mathbf{n}^T\right] = \sigma^2 \mathbf{I} \tag{22}$$

1) Show that the MAP decision for detecting $\mathbf{s}_0$ results in
$$|r|_2 < r_1 \tag{23}$$
2) Express $\Pr(\hat{\mathbf{s}} = \mathbf{s}_0 | \mathbf{s} = \mathbf{s}_0)$ in terms of r_1, r_2. Let $X = n_2 - n_1, Y = -n_2 - n_1$, where $\mathbf{n} = (n_1, n_2)$. Their correlation coefficient is defined as
$$\rho = \frac{E\left[(X - \mu_x)\left(Y - \mu_y\right)\right]}{\sigma_x \sigma_y} \tag{24}$$
X and Y are said to be uncorrelated if $\rho = 0$
3) Show that if X and Y are uncorrelated Verify this numerically.
4) Show that X and Y are independent, i.e. $p_{XY}(x, y) = p_X(x) p_Y(y)$.
5) Show that $X, Y \sim \mathcal{N}\left(0, 2\sigma^2\right)$.
6) Show that $\Pr(\hat{\mathbf{s}} = \mathbf{s}_0 | \mathbf{s} = \mathbf{s}_0) = \Pr(X < A, Y < A)$.
Solution: Given we transmitted s_0, the probability of decoding it as s_0 is given by
$$\Pr(\hat{\mathbf{s}} = \mathbf{s}_0 | \mathbf{s} = \mathbf{s}_0) = \Pr(-n_2 < A + n_1, A + n_1 > n_2) \tag{25}$$
$$\implies \Pr(\hat{\mathbf{s}} = \mathbf{s}_0 | \mathbf{s} = \mathbf{s}_0) = \Pr(X < A, Y < A) \tag{26}$$
Where, $X = n_2 - n_1, Y = -n_2 - n_1$. Also $X, Y \sim \mathcal{N}\left(0, 2\sigma^2\right)$ and are independent.
7) Find $\Pr(X < A, Y < A)$.
Solution:
$$\Pr(X < A, Y < A) = \Pr(X < A) \Pr(Y < A) \tag{27}$$
$$\implies \Pr(X < A, Y < A) = \left(1 - Q\left(\frac{A}{\sqrt{2}\sigma}\right)\right)^2 \tag{28}$$
$$\implies \Pr(e | s_0) = 1 - \Pr(X < A, Y < A) \tag{29}$$
8) Verify the above through simulation. **Solution:** The first result can be verified with the help of following plot.

```
from __future__ import division
import numpy as np
import mpmath as mp
import matplotlib.pyplot as plt




def qfunc(x):
        return 0.5*mp.erfc(x/np
            .sqrt(2))

#Number of SNR samples
snrlen = 10
#SNR values in dB
snrdb = np.linspace(0,9,10)
#Number of samples
```

```
simlen = int(1e5)
#Simulated BER declaration
err = []
#Analytical BER declaration
ber = []
temp=0
noise1 = np.random.normal(0,1,
    simlen)
noise2=np.random.normal(0,1,
    simlen)

#for SNR 0 to 10 dB
for i in range(0,snrlen):
        snr = 10**(0.1*snrdb[i
            ])         #Received
            symbol in baseband
        rx = mp.sqrt(2*snr) +
            noise1
        ry = noise2
        temp=0
        for j in range (0,len(
            rx)):
              if ((rx[j]>ry[j])
                and (rx[j]>-ry[j
                ])):
                  temp=temp+1

        #calculating the total
            number of errors
        #err_n = np.size(
            err_ind)
        #calcuating the
            simulated BER
        err.append(temp/simlen)
        #calculating the
            analytical BER
        ber.append((1-qfunc(mp.
            sqrt(snr)))**2)

plt.semilogy(snrdb.T,ber,label=
    'Analysis')
plt.semilogy(snrdb.T,err,'o',
    label='Sim')
plt.xlabel('SNR$\\left(\\frac{
   E_b}{N_0}\\right)$')
#plt.ylabel('$Pr(\hat{s}=s_0|s=
    s_0)$')
plt.legend()
plt.grid()
plt.savefig('../figs/qpsk.eps')
plt.show()
```

yields Fig. 5

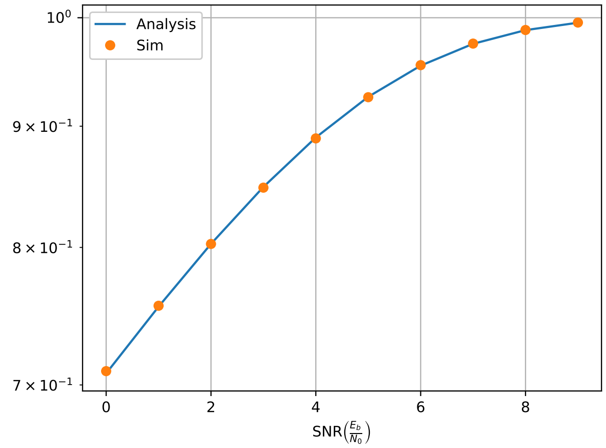

Fig. 5

The second result can be verified with the help of following plot.

```
from __future__ import division
import numpy as np
import mpmath as mp
import matplotlib.pyplot as plt




def qfunc(x):
        return 0.5*mp.erfc(x/np
            .sqrt(2))

#Number of SNR samples
snrlen = 10
#SNR values in dB
snrdb = np.linspace(0,9,10)
#Number of samples
simlen = int(1e5)
#Simulated BER declaration
err = []
#Analytical BER declaration
ber = []
temp=0
noise1 = np.random.normal(0,1,
    simlen)
noise2=np.random.normal(0,1,
    simlen)
```

```
#for SNR 0 to 10 dB
for i in range(0,snrlen):
        snr = 10**(0.1*snrdb[i
           ])        #Received
           symbol in baseband
        rx = mp.sqrt(2*snr) +
           noise1
        ry = noise2
        temp=0
        for j in range (0,len(
           rx)):
             if ((rx[j]>ry[j])
                and (rx[j]>-ry[j
                ])):
                  temp=temp+1

        #calculating the total
           number of errors
        #err_n = np.size(
           err_ind)
        #calcuating the
           simulated BER
        err.append(1-temp/
           simlen)
        #calculating the
           analytical BER
        ber.append(1-(1-qfunc(
           mp.sqrt(snr)))**2)

plt.semilogy(snrdb.T,ber,label=
    'Analysis')
plt.semilogy(snrdb.T,err,'o',
    label='Sim')
plt.xlabel('SNR$\\left(\\frac{
   E_b}{N_0}\\right)$')
plt.ylabel('$P_e$')
plt.legend()
plt.grid()
plt.savefig('qpsk_err.eps')
plt.show()
```

yields Fig. 6

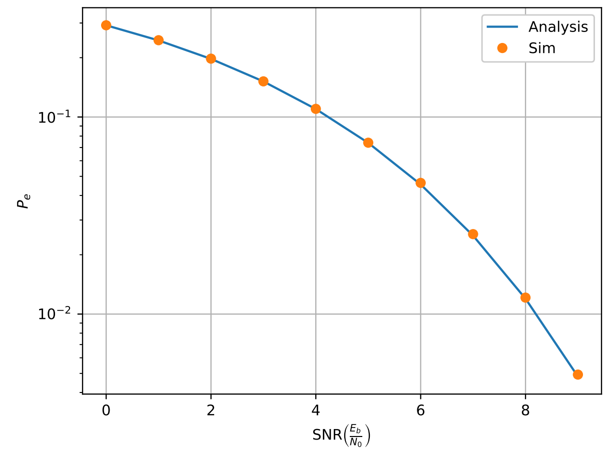

Fig. 6

4 M-PSK

Problem 13. Consider a system where $\mathbf{s}_i = \begin{pmatrix} \cos\left(\frac{2\pi i}{M}\right) \\ \cos\left(\frac{2\pi i}{M}\right) \end{pmatrix}, i = 0, 1, \dots M-1$. Let

$$\mathbf{r}|s_0 = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} \sqrt{E_s} + n_1 \\ n_2 \end{pmatrix} \tag{30}$$

where $n_1, n_2 \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$.

1) Substituting

$$r_1 = R\cos\theta \tag{31}$$

$$r_2 = R\sin\theta \tag{32}$$

show that the joint pdf of R, θ is

$$p(R,\theta) = \frac{R}{\pi N_0}\exp\left(-\frac{R^2 - 2R\sqrt{E_s}\cos\theta + E_s}{N_0}\right) \tag{33}$$

Solution:

$$p(r_1, r_2) = p(r_1)\,p(r_2) = \frac{1}{N_0\pi}\exp\left(-\frac{(x-\sqrt{E_s})^2 + y^2}{N_0}\right) \tag{34}$$

We know that $p(u,v) = |J|p(x,y)$ where J is the jacobian matrix. In this case, $|J| = R$. This implies that

$$p(R,\theta) = \frac{R}{\pi N_0}\exp\left(-\frac{R^2 - 2R\sqrt{E_s}\cos\theta + E_s}{N_0}\right) \tag{35}$$

Hence proved.

2) Show that

$$\lim_{\alpha\to\infty}\int_0^\infty (V-\alpha)\,e^{-(V-\alpha)^2}\,dV = 0 \tag{36}$$

$$\lim_{\alpha\to\infty}\int_0^\infty e^{-(V-\alpha)^2}\,dV = \sqrt{\pi} \tag{37}$$

Solution: Say, $V - \alpha = t$. $\implies$

$$\lim_{\alpha\to\infty}\int_0^\infty (V-\alpha)\,e^{-(V-\alpha)^2}\,dV = \lim_{\alpha\to\infty}\int_{-\alpha}^\infty te^{-(t)^2}\,dt \tag{38}$$

Applying the limit, we end up with the following expression

$$\int_{-\infty}^{\infty} te^{-(t)^2}\,dt = 0 \tag{39}$$

because the integrand is an odd function. For the next part of the problem, using similar substitution, we end up with

$$\int_{-\infty}^{\infty} e^{-t^2}\,dt \tag{40}$$

which equals to $\sqrt{\pi}$ by the relation

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\frac{-t^2}{2}} = 1 \tag{41}$$

Hence proved.

3) Using the above, evaluate

$$\int_0^{\infty} V\exp\left\{-\left(V^2 - 2V\sqrt{\gamma}\cos\theta + \gamma\right)\right\}\,dV \tag{42}$$

for large values of γ.

Solution: On some careful adjustments, the integral becomes

$$e^{-\gamma\sin^2\theta}\int_0^{\infty}(V - \sqrt{\gamma}\cos\theta)e^{-\left(V-\sqrt{\gamma}\cos\theta\right)^2}\,dV \tag{43}$$

$$+e^{-\gamma\sin^2\theta}\int_0^{\infty}\left(\sqrt{\gamma}\cos\theta\right)e^{-\left(V-\sqrt{\gamma}\cos\theta\right)^2}\,dV \tag{44}$$

This turns out to be(on proper integration),

$$\frac{e^{-\gamma}}{2} + e^{-\gamma\sin^2\theta}\sqrt{\gamma}\cos\theta\int_0^{\infty} e^{-\left(V-\sqrt{\gamma}\cos\theta\right)^2}dV \tag{45}$$

For larger values of γ, this tend to the following equation:

$$e^{-\gamma\sin^2\theta}\sqrt{\gamma\pi}\cos\theta \tag{46}$$

4) Find a compact expression for

$$I = 1 - \sqrt{\frac{\gamma}{\pi}}\int_{-\frac{\pi}{M}}^{\frac{\pi}{M}} e^{-\gamma\sin^2\theta}\cos\theta\,d\theta \tag{47}$$

Solution: Substituting $\sqrt{\gamma}\sin\theta = t$

$$\implies I = 1 - \frac{1}{\sqrt{\pi}}\int_{-\sqrt{\gamma}\sin\frac{\pi}{M}}^{\sqrt{\gamma}\sin\frac{\pi}{M}} e^{-t^2}dt \tag{48}$$

This can be converted into standard normal distribution using the substitution $t = \frac{t}{\sqrt{2}}$. So, the expression turns out to be

$$I = 1 - \frac{1}{\sqrt{2\pi}}\int_{-\sqrt{2\gamma}\sin\frac{\pi}{M}}^{\sqrt{2\gamma}\sin\frac{\pi}{M}} e^{\frac{-t^2}{2}}dt \tag{49}$$

$$\implies I = 2Q(a) \tag{50}$$

where $a = \sqrt{2\gamma}\sin\frac{\pi}{M}$.

5) Find $P_{e|\mathbf{s}_0}$.

Solution:

$$p(\theta) = \int_0^{\infty} p(R,\theta)\,dR \tag{51}$$

$$= \int_0^{\infty}\frac{R}{\pi N_0}\exp\left(-\frac{R^2 - 2R\sqrt{E_s}\cos\theta + E_s}{N_0}\right)dR \tag{52}$$

On substituting

$$V = \frac{R}{\sqrt{N_o}} \tag{53}$$

$$\gamma = \frac{E_s}{N_o} \tag{54}$$

$$= \frac{1}{\pi}\int_0^{\infty} V\exp\left\{-\left(V^2 - 2V\sqrt{\gamma}\cos\theta + \gamma\right)\right\}\,dV \tag{55}$$

From eq(45) we can conclude that

$$p(\theta) = e^{-\gamma\sin^2\theta}\sqrt{\frac{\gamma}{\pi}}\cos\theta \tag{56}$$

We know that $P_{\mathbf{s}_0|\mathbf{s}_0}$ is.

$$P_{\mathbf{s}_0|\mathbf{s}_0} = \int_{-\frac{\pi}{M}}^{\frac{\pi}{M}} e^{-\gamma\sin^2\theta}\sqrt{\frac{\gamma}{\pi}}\cos\theta\,d\theta \tag{57}$$

Now,

$$P_{e|\mathbf{s}_0} = 1 - P_{\mathbf{s}_0|\mathbf{s}_0} \tag{58}$$

$$= 1 - \int_{-\frac{\pi}{M}}^{\frac{\pi}{M}} e^{-\gamma\sin^2\theta}\sqrt{\frac{\gamma}{\pi}}\cos\theta\,d\theta \tag{59}$$

From eq(49) and $\gamma = \frac{E_s}{N_o}$

$$P_{e|\mathbf{s}_0} = 2Q\left(\sqrt{2\left(\frac{E_s}{N_o}\right)}\sin\frac{\pi}{M}\right) \tag{60}$$