Digital Modulation Techniques

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Contents

***Abstract*—The manual frames the problems of receiver design and performance analysis in digital communication as applications of probability theory.**

Download all codes in this manual from

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svn co https://github.com/gadepall/comm/trunk/
    modulation/manual/codes
```

1 BPSK

1.1. The *signal constellation diagram* for BPSK is given by Fig. 1.1. The symbols s_0 and s_1 are equiprobable. $\sqrt{E_b}$ is the energy transmitted per bit. Assuming a zero mean additive white gaussian noise (AWGN) with variance $\frac{N_0}{2}$, obtain the symbols that are received.

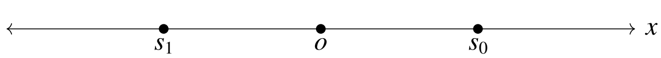

Fig. 1.1

Solution: The possible received symbols are

$$y|s_0 = \sqrt{E_b} + n \tag{1.1.1}$$

$$y|s_1 = -\sqrt{E_b} + n \tag{1.1.2}$$

where the AWGN $n \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$.

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1.2. From Fig. 1.1 obtain a decision rule for BPSK
Solution: The decision rule is

$$y \underset{s_1}{\overset{s_0}{\gtrless}} 0 \tag{1.2.1}$$

1.3. Repeat the previous exercise using the MAP criterion.
Solution: According to MAP detection rule

$$\hat{s} = \max_{s\in\{s_0,s_1\}} p\left(s|y\right) \tag{1.3.1}$$

$$\implies p\left(s_0|y\right) \underset{s_1}{\overset{s_0}{\gtrless}} p\left(s_1|y\right) \tag{1.3.2}$$

Using Bayes rule,

$$p\left(s_0|y\right) = \frac{p\left(y|s_0\right)p\left(s_0\right)}{p\left(y\right)} \tag{1.3.3}$$

$$p\left(s_1|y\right) = \frac{p\left(y|s_1\right)p\left(s_1\right)}{p\left(y\right)} \tag{1.3.4}$$

Since symbols are equi probable, $p\left(s_0\right) = p\left(s_1\right)$. Hence the decision becomes

$$\frac{p\left(y|s_0\right)p\left(s_0\right)}{p\left(y\right)} \underset{s_1}{\overset{s_0}{\gtrless}} \frac{p\left(y|s_1\right)p\left(s_1\right)}{p\left(y\right)} \tag{1.3.5}$$

$$\implies p\left(y|s_0\right) \underset{s_1}{\overset{s_0}{\gtrless}} p\left(y|s_1\right) \tag{1.3.6}$$

The above condition is known as the maximum-likelihood (ML) criterion. (1.3.6) can be expressed as

$$\frac{1}{\sqrt{2\pi}}\exp-\frac{(y-\sqrt{E_b})^2}{\frac{N_oN_0}{2}} \underset{s_1}{\overset{s_0}{\gtrless}} \frac{1}{\sqrt{2\pi}}\exp-\frac{(y+\sqrt{E_b})^2}{\frac{N_oN_0}{2}} \tag{1.3.7}$$

$$\implies (y+\sqrt{E_b})^2 \underset{s_1}{\overset{s_0}{\gtrless}} (y-\sqrt{E_b})^2 \tag{1.3.8}$$

$$\implies y \underset{s_1}{\overset{s_0}{\gtrless}} 0 \tag{1.3.9}$$

Fig. 1.3 shows the decision regions D_1 and D_2 for s_0 and s_1 respectively.

1.4. Using the decision rule in Problem 1.2, obtain

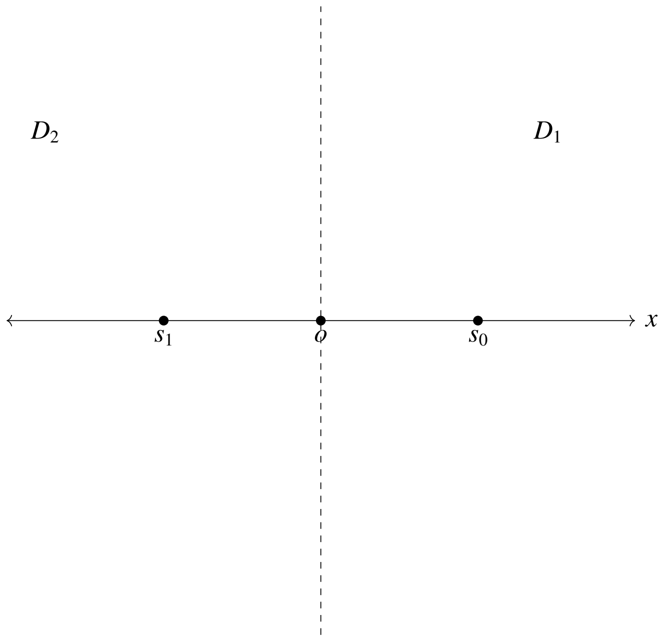

Fig. 1.3: Decision region for BPSK

an expression for the probability of error for BPSK.
Solution: Since the symbols are equiprobable, it is sufficient if the error is calculated assuming that a 0 was sent. This results in

$$P_e = \Pr\left(y < 0|s_0\right) = \Pr\left(\sqrt{E_b} + n < 0\right) \tag{1.4.1}$$
$$= \Pr\left(-n > \sqrt{E_b}\right) = \Pr\left(n > \sqrt{E_b}\right) \tag{1.4.2}$$

since n has a symmetric pdf. Let $w \sim \mathcal{N}(0,1)$. Then $n = \sqrt{\frac{N_0}{2}}w$. Substituting this in (1.4.2),

$$P_e = \Pr\left(\sqrt{\frac{N_0}{2}}w > \sqrt{E_b}\right) = \Pr\left(w > \sqrt{\frac{2E_b}{N_0}}\right) \tag{1.4.3}$$
$$= Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \tag{1.4.4}$$

where

$$Q(x) \triangleq \Pr\left(w > x\right), x \geq 0. \tag{1.4.5}$$
$$= \frac{1}{\pi}\int_0^{\frac{\pi}{2}} e^{-\frac{x^2}{2\sin^2\theta}}\,d\theta \tag{1.4.6}$$

The PDF of $w \sim \mathcal{N}(0,1)$ is given by

$$p_w(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty \tag{1.4.7}$$

and the complementary error function is defined as

$$\text{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^{\infty} e^{-t^2}\,dt. \tag{1.4.8}$$

1.5. Show that

$$Q(x) = \frac{1}{2}\text{erfc}\left(\frac{x}{\sqrt{2}}\right) \tag{1.5.1}$$

Solution: From (1.4.5)

$$Q(x) = \Pr\left(w > x\right), x \geq 0 \tag{1.5.2}$$
$$= \frac{1}{\sqrt{2\pi}}\int_x^{\infty} e^{-\frac{t^2}{2}}\,dt. \tag{1.5.3}$$
$$= \frac{1}{\sqrt{\pi}}\int_{\frac{x}{\sqrt{2}}}^{\infty} e^{-y^2}\,dy. \quad \left(t = \sqrt{2}y\right) \tag{1.5.4}$$

resulting in (1.5.1)

1.6. Verify the bit error rate (BER) plots for BPSK through simulation and analysis for 0 to 10 dB.
Solution: The following code

```
codes/bpsk_ber.py
```

yields Fig. 1.6

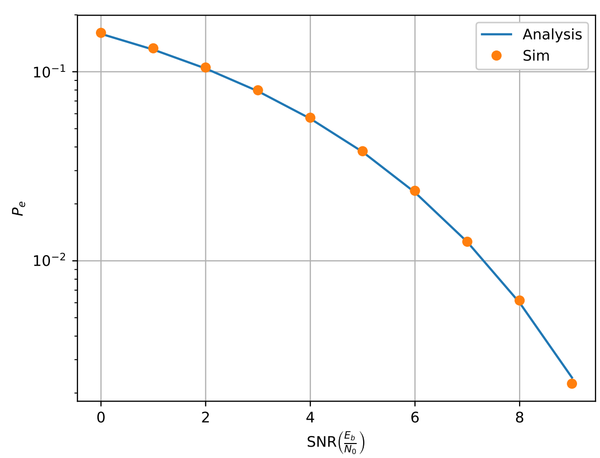

Fig. 1.6

2 Coherent BFSK

2.1. The signal constellation for binary frequency shift keying (BFSK) is given in Fig. 2.1. Obtain the equations for the received symbols.
Solution: The received symbols are given by

$$\mathbf{y}|s_0 = \begin{pmatrix}\sqrt{E_b}\\0\end{pmatrix} + \begin{pmatrix}n_1\\n_2\end{pmatrix}, \tag{2.1.1}$$

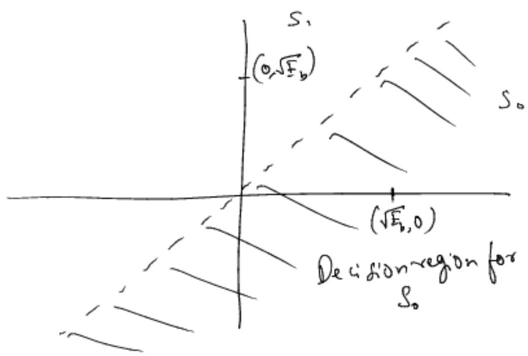

Fig. 2.1

and

$$\mathbf{y}|s_1 = \begin{pmatrix} 0 \\ \sqrt{E_b} \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, \tag{2.1.2}$$

where $n_1, n_2 \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$. and $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$.

2.2. Obtain a decision rule for BFSK from Fig. 2.1.
Solution: The decision rule is

$$y_1 \underset{s_1}{\overset{s_0}{\gtrless}} y_2 \tag{2.2.1}$$

2.3. The multivariate Gaussian distribution is defined as

$$p_{\mathbf{x}}(x_1, \dots, x_k) = \frac{1}{\sqrt{(2\pi)^k |\boldsymbol{\Sigma}|}} \exp\left\{-\frac{1}{2} (\mathbf{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu})\right\} \tag{2.3.1}$$

where $\boldsymbol{\mu}$ is the mean vector, $\boldsymbol{\Sigma} = E\left[(\mathbf{x} - \boldsymbol{\mu})(\mathbf{x} - \boldsymbol{\mu})^T\right]$ is the covariance matrix and $|\boldsymbol{\Sigma}|$ is the determinant of $\boldsymbol{\Sigma}$. Show that the PDF of the *bivariate* Gaussian is

$$p(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)} \times \left\{\frac{(x-\mu_x)^2}{\sigma_x^2} + \frac{\left(y-\mu_y\right)^2}{\sigma_y^2} - \frac{2\rho(x-\mu_x)\left(y-\mu_y\right)}{\sigma_x\sigma_y}\right\}\right] \tag{2.3.2}$$

where

$$\boldsymbol{\mu} = \begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix}, \boldsymbol{\Sigma} = \begin{pmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{pmatrix} \tag{2.3.3}$$

For equiprobably symbols, the MAP criterion is defined as

$$p(s_0|\mathbf{y}) \underset{s_1}{\overset{s_0}{\gtrless}} p(s_1|\mathbf{y}) \tag{2.3.4}$$

2.4. Use (2.3.2) in (2.3.4) to obtain (2.2.1).
Solution: According to the MAP criterion, assuming equiprobably symbols,

$$p(s_0|\mathbf{y}) \underset{s_1}{\overset{s_0}{\gtrless}} p(s_1|\mathbf{y}) \tag{2.4.1}$$

2.5. Derive and plot the probability of error. Verify through simulation.
Solution: Given that s_0 was transmitted, the received symbols are

$$\mathbf{y}|s_0 = \begin{pmatrix} \sqrt{E_b} \\ 0 \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, \tag{2.5.1}$$

From (2.2.1), the probability of error is given by

$$P_e = \Pr(y_1 < y_2|s_0) = \Pr\left(\sqrt{E_b} + n_1 < n_2\right) \tag{2.5.2}$$

$$= \Pr\left(n_2 - n_1 > \sqrt{E_b}\right) \tag{2.5.3}$$

Note that $n_2 - n_1 \sim \mathcal{N}(0, N_0)$. Thus,

$$P_e = \Pr\left(\sqrt{N_0}w > \sqrt{E_b}\right) \tag{2.5.4}$$

$$= \Pr\left(w > \sqrt{\frac{E_b}{N_0}}\right) \tag{2.5.5}$$

$$\implies P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right) \tag{2.5.6}$$

where $w \sim \mathcal{N}(0, 1)$. The following code plots the BER curves in Fig. 2.5

```
codes/fsk_ber.py
```

3 QPSK

1. See Fig.3.1 for the constellation diagram. The transmitted symbol set is given by

$$\mathbf{s}_m = \begin{pmatrix} \cos\frac{2m\pi}{4} \\ \sin\frac{2m\pi}{4} \end{pmatrix}, \quad m \in \{0, 1, \dots, 3\}. \tag{3.1.1}$$

The numerical values and encoding scheme for $\mathbf{s}_m$ are listed in Table 3.1

2. Let

$$\mathbf{y} = \sqrt{E_s}\mathbf{s} + \mathbf{n} \tag{3.2.1}$$

where $\mathbf{s} \in \{\mathbf{s}_0, \mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}$

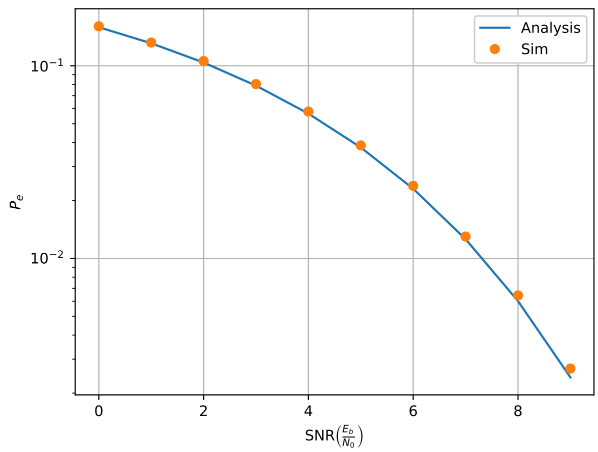

Fig. 2.5

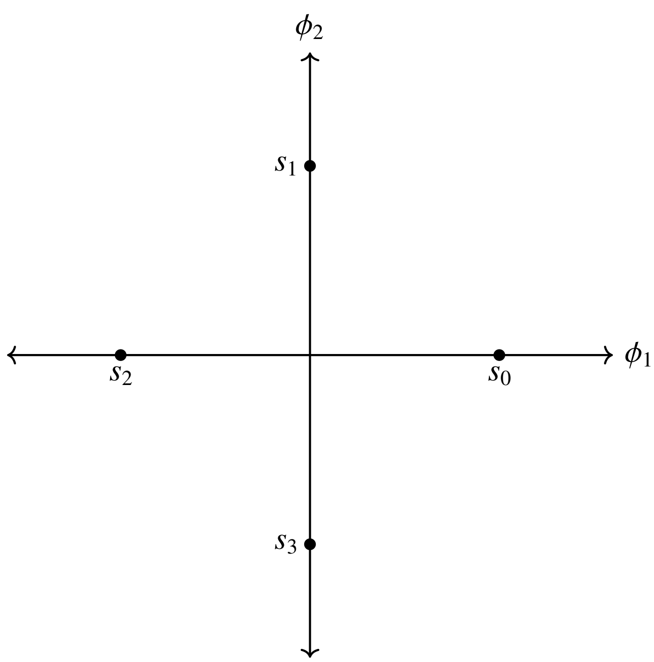

Fig. 3.1: constellation diagram

3. Obtain the decision rule for QPSK by inspection.
   Solution: The decision rule is given by Fig.3.3
4. Using (2.3.2), show that the MAP decision for detecting $\mathbf{s}_0$ results in

$$|y_2| < y_1 \tag{3.4.1}$$

Solution: The MAP criterion reduces to

$$\hat{s} = \min_{\mathbf{s} \in \mathbf{s}_i} \|\mathbf{y} - \mathbf{s}\|, i \in \{0, \dots, 3\} \tag{3.4.2}$$

From eq.3.4.2,$\mathbf{s}_0$ is chosen if

Symbol	Grey Code	Co-ordinates
s_0	00	$\begin{pmatrix}1\\0\end{pmatrix}$
s_1	01	$\begin{pmatrix}0\\1\end{pmatrix}$
s_2	11	$\begin{pmatrix}-1\\0\end{pmatrix}$
s_3	10	$\begin{pmatrix}0\\-1\end{pmatrix}$

TABLE 3.1

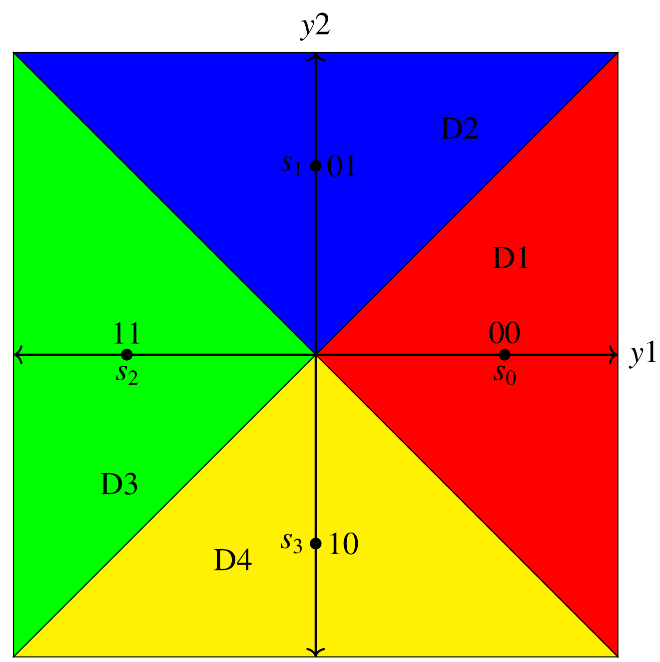

Fig. 3.3: decision regions

$$\|\mathbf{y} - \mathbf{s}_0\|^2 < \|\mathbf{y} - \mathbf{s}_i\|^2 \tag{3.4.3}$$

$$\implies (\mathbf{s}_0 - \mathbf{s}_i)^T \mathbf{y} > 0 \quad i \in \{1, 2, 3\} \tag{3.4.4}$$

or,

$$\begin{pmatrix}1 & -1\\1 & 0\\1 & 1\end{pmatrix}\mathbf{y} \geq 0 \tag{3.4.5}$$

The above condition can be simplified to obtain the region

$$D_1 : |y_2| < y_1 \tag{3.4.6}$$

Table 3.4 summarizes the decisions for all symbols.

Symbol	Decision region	Decision Rule
s_0	$D1$	$y1 > y2, y1 > -y2$
s_1	$D2$	$y1 < y2, y1 > -y2$
s_2	$D3$	$y1 < y2, y1 < -y2$
s_3	$D3$	$y1 > y2, y1 < -y2$

TABLE 3.4

5. Express $\Pr(\hat{\mathbf{s}} = \mathbf{s}_0 | \mathbf{s} = \mathbf{s}_0)$ in terms of y_1, y_2.
Solution: From (3.4.1) and (3.4.6) ,

$$\Pr(\hat{\mathbf{s}} = \mathbf{s}_0 | \mathbf{s} = \mathbf{s}_0) \\ = \Pr\left(|y_2| < y_1 | y_1 = \sqrt{E}_s, y_2 = 0\right) \quad (3.5.1)$$

6. Let

$$X = n_2 - n_1, \quad (3.6.1)$$
$$Y = -n_2 - n_1, \quad (3.6.2)$$

where $\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$.
Show that $X, Y \sim \mathcal{N}(0, N_0)$.

7. The correlation coefficient of X, Y is defined as

$$\rho = \frac{E\left[(X - \mu_x)\left(Y - \mu_y\right)\right]}{\sigma_x \sigma_y} \quad (3.7.1)$$

X and Y are said to be uncorrelated if $\rho = 0$ Show that X and Y are uncorrelated. Verify this numerically.
Solution: From (3.6.1) and (3.6.2)

$$\mu_x = E[X] = 0 \quad (3.7.2)$$
$$\mu_y = E[Y] = 0 \quad (3.7.3)$$
$$\implies \rho = E[XY] = E[(n_2 - n_1)(-n_2 - n_1)] \\ = 0 \quad (3.7.4)$$

upon substituting from (3.6.1) and (3.6.2).

8. Show that X and Y are independent, i.e.

$$p_{XY}(x, y) = p_X(x) p_Y(y). \quad (3.8.1)$$

Solution: Use (3.7.4) in (2.3.2) to get (3.8.1). Uncorrelated Gaussians are independent.

9. Show that

$$\Pr(\hat{\mathbf{s}} = \mathbf{s}_0 | \mathbf{s} = \mathbf{s}_0) = \Pr\left(X < \sqrt{E_s}\right) \Pr\left(Y < \sqrt{E_s}\right). \quad (3.9.1)$$

Solution: From (3.5.1) and (3.2.1)

$$\Pr(\hat{\mathbf{s}} = \mathbf{s}_0 | \mathbf{s} = \mathbf{s}_0) \\ = \Pr\left(|n_2| < \sqrt{E}_s + n_1\right) \quad (3.9.2)$$

which can be expressed as

$$\Pr\left(n_2 < \sqrt{E}_s + n_1, -n_2 > \sqrt{E}_s + n_1\right) \quad (3.9.3)$$
$$= \Pr\left(X < \sqrt{E}_s, Y < \sqrt{E}_s\right) \quad (3.9.4)$$
$$= \Pr\left(X < \sqrt{E}_s\right) \Pr\left(Y < \sqrt{E}_s\right) \quad (3.9.5)$$

after some algebra, using the fact that X, Y are independent.

10. Show that

$$\Pr(\hat{\mathbf{s}} = \mathbf{s}_0 | \mathbf{s} = \mathbf{s}_0) = \left(1 - Q\left(\sqrt{\frac{E_s}{N_0}}\right)\right)^2 \quad (3.10.1)$$

Solution: From (3.6),

$$\Pr\left(X > \sqrt{E}_s\right) = \Pr\left(Y > \sqrt{E}_s\right) = Q\left(\sqrt{\frac{E_s}{N_0}}\right) \quad (3.10.2)$$

yielding (3.10.1).

11. Verify the above through simulation.
Solution: This is shown in Fig. 3.11 through the following code.

```
codes/qpsk.py
```

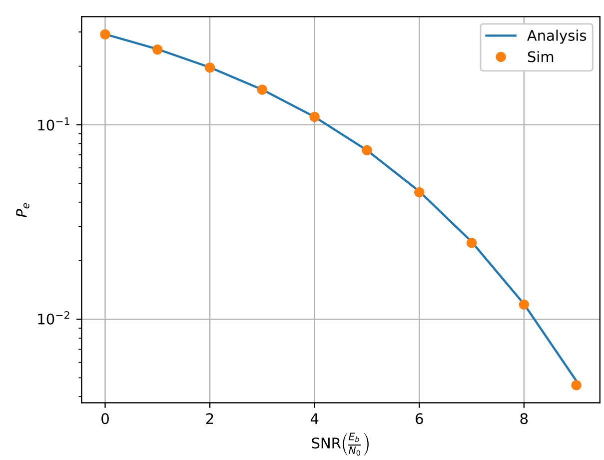

Fig. 3.11

12. Modify the above script to obtain the probability of symbol error.

4 M-PSK

1. Consider a system where $\mathbf{s}_i = \begin{pmatrix} \cos\left(\frac{2\pi i}{M}\right) \\ \sin\left(\frac{2\pi i}{M}\right) \end{pmatrix}, i = 0, 1, \dots M-1$. Let

$$\mathbf{y}|s_0 = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \sqrt{E_s} + n_1 \\ n_2 \end{pmatrix} \tag{4.1.1}$$

where $n_1, n_2 \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$.
Substituting

$$y_1 = R\cos\theta \tag{4.1.2}$$
$$y_2 = R\sin\theta \tag{4.1.3}$$

show that the joint pdf of R, θ is

$$p(R, \theta) = \frac{R}{\pi N_0} \exp\left(-\frac{R^2 - 2R\sqrt{E_s}\cos\theta + E_s}{N_0}\right) \tag{4.1.4}$$

2. Show that

$$\lim_{\alpha\to\infty} \int_0^\infty (V - \alpha) e^{-(V-\alpha)^2} \, dV = 0 \tag{4.2.1}$$
$$\lim_{\alpha\to\infty} \int_0^\infty e^{-(V-\alpha)^2} \, dV = \sqrt{\pi} \tag{4.2.2}$$

3. Using the above, show that

$$\int_0^\infty V \exp\left\{-\left(V^2 - 2V\sqrt{\gamma}\cos\theta + \gamma\right)\right\} dV = e^{-\gamma\sin^2\theta}\sqrt{\gamma\pi}\cos\theta \tag{4.3.1}$$

for large values of γ.
Solution: The integrand in (4.3.1) can be expressed as

$$\begin{aligned} &Ve^{-\left(V^2 - 2V\sqrt{\gamma}\cos\theta + \gamma\right)} \\ &= \{(V - \sqrt{\gamma}\cos\theta) + (\sqrt{\gamma}\cos\theta)\} \\ &\quad \times e^{-\left(V - \sqrt{\gamma}\cos\theta\right)^2} e^{-\sqrt{\gamma}\sin^2\theta} \\ &\implies \int_0^\infty Ve^{-\left(V^2 - 2V\sqrt{\gamma}\cos\theta + \gamma\right)} dV \\ &= e^{-\sqrt{\gamma}\sin^2\theta} \\ &\times \left\{\int_0^\infty (V - \sqrt{\gamma}\cos\theta) e^{-\left(V - \sqrt{\gamma}\cos\theta\right)^2} d\theta \right. \\ &+ \left. \int_0^\infty (\sqrt{\gamma}\cos\theta) e^{-\left(V - \sqrt{\gamma}\cos\theta\right)^2} d\theta \right\} \end{aligned} \tag{4.3.2}$$

yielding (4.3.1) from (4.2.1) and (4.2.2)

4. Find a compact expression for

$$I = 1 - \sqrt{\frac{\gamma}{\pi}} \int_{-\frac{\pi}{M}}^{\frac{\pi}{M}} e^{-\gamma\sin^2\theta} \cos\theta \, d\theta \tag{4.4.1}$$

Solution: The above integral can be expressed as

$$I = 1 - 2\sqrt{\frac{\gamma}{\pi}} \int_0^{\frac{\pi}{M}} e^{-\gamma\sin^2\theta} \cos\theta \, d\theta \tag{4.4.2}$$
$$= 1 - 2\left\{Q(0) - Q\left(\sqrt{2\gamma}\sin\frac{\pi}{M}\right)\right\} \tag{4.4.3}$$
$$= 2Q\left(\sqrt{2\gamma}\sin\frac{\pi}{M}\right) \tag{4.4.4}$$

$\because Q(0) = \frac{1}{2}$.

5. Find the decision region for the symbol $\mathbf{s}_0$.
6. Show that

$$P_{e|\mathbf{s}_0} = 2Q\left(\sqrt{2\left(\frac{E_s}{N_o N_0}\right)}\sin\frac{\pi}{M}\right) \tag{4.6.1}$$

7. Verify the SER through simulation.