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CONTENTS

1	Multivariate Gaussian	1
2	QPSK	1
3	Noncoherent BFSK	1
4	M-PSK	2
5	Craig's Formula and MGF	3

Abstract—The manual frames the problems of receiver design and performance analysis in digital communication as applications of probability theory.

1 MULTIVARIATE GAUSSIAN

The multivariate Gaussian distribution is defined as

$$p_{\mathbf{x}}(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^k |\Sigma|}} \exp \left\{ -\frac{1}{2} (\mathbf{x} - \boldsymbol{\mu})^T \Sigma^{-1} (\mathbf{x} - \boldsymbol{\mu}) \right\} \quad (1)$$

where $\boldsymbol{\mu}$ is the mean vector, $\Sigma = E[(\mathbf{x} - \boldsymbol{\mu})(\mathbf{x} - \boldsymbol{\mu})^T]$ is the covariance matrix and $|\Sigma|$ is the determinant of Σ .

2 QPSK

2.1 Let

$$\mathbf{r} = \mathbf{s} + \mathbf{n} \quad (2)$$

where $\mathbf{s} \in \{s_0, s_1, s_2, s_3\}$ and

$$\mathbf{s}_0 = \begin{pmatrix} A \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ A \end{pmatrix}, \mathbf{s}_2 = \begin{pmatrix} -A \\ 0 \end{pmatrix}, \mathbf{s}_3 = \begin{pmatrix} 0 \\ -A \end{pmatrix}, \quad (3)$$

$$E[\mathbf{n}] = \mathbf{0}, E[\mathbf{nn}^T] = \sigma^2 \mathbf{I} \quad (4)$$

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a) Show that the MAP decision for detecting \mathbf{s}_0 results in

$$|r|_2 < r_1 \quad (5)$$

b) Express $\Pr(\hat{\mathbf{s}} = \mathbf{s}_0 | \mathbf{s} = \mathbf{s}_0)$ in terms of r_1, r_2 . Let $X = n_2 - n_1, Y = -n_2 - n_1$, where $\mathbf{n} = (n_1, n_2)$. Their correlation coefficient is defined as

$$\rho = \frac{E[(X - \mu_x)(Y - \mu_y)]}{\sigma_x \sigma_y} \quad (6)$$

X and Y are said to be uncorrelated if $\rho = 0$

c) Show that if X and Y are uncorrelated Verify this numerically.

d) Show that X and Y are independent, i.e. $p_{XY}(x, y) = p_X(x)p_Y(y)$.

e) Show that $X, Y \sim \mathcal{N}(0, 2\sigma^2)$.

f) Show that $\Pr(\hat{\mathbf{s}} = \mathbf{s}_0 | \mathbf{s} = \mathbf{s}_0) = \Pr(X < A, Y < A)$.

g) Find $\Pr(X < A, Y < A)$.

h) Verify the above through simulation.

3 NONCOHERENT BFSK

3.1 Show that

$$I_0(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{x \cos \theta} d\theta \quad (7)$$

$$I_0(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{x \cos(\theta - \phi)} d\theta \quad (8)$$

$$\frac{1}{2\pi} \int_0^{2\pi} e^{m_1 \cos \theta + m_2 \sin \theta} d\theta = I_0(\sqrt{m_1^2 + m_2^2}) \quad (9)$$

where the modified Bessel function of order n (integer) is defined as

$$I_n(x) = \frac{1}{\pi} \int_0^\pi e^{x \cos \theta} \cos n\theta d\theta \quad (10)$$

3.2 Let

$$\mathbf{r}|0 = \sqrt{E_b} \begin{pmatrix} \cos \phi_0 \\ \sin \phi_0 \\ 0 \\ 0 \end{pmatrix} + \mathbf{n}_0, \mathbf{r}|1 = \sqrt{E_b} \begin{pmatrix} 0 \\ 0 \\ \cos \phi_1 \\ \sin \phi_1 \end{pmatrix} + \mathbf{n}_1 \quad (11)$$

where $\mathbf{n}_0, \mathbf{n}_1 \sim \mathcal{N}\left(\mathbf{0}, \frac{N_0}{2}\mathbf{I}\right)$.

- Taking $\mathbf{r} = (r_1, r_2, r_3, r_4)^T$, find the pdf $p(\mathbf{r}|0, \phi_0)$ in terms of $r_1, r_2, r_3, r_4, \phi, E_b$ and N_0 . Assume that all noise variables are independent.
- If ϕ_0 is uniformly distributed between 0 and 2π , find $p(\mathbf{r}|0)$. Note that this expression will no longer contain ϕ_0 .
- Show that the ML detection criterion for this scheme is

$$I_0\left(k\sqrt{r_1^2 + r_2^2}\right) \stackrel{0}{\geq} I_0\left(k\sqrt{r_3^2 + r_4^2}\right) \quad (12)$$

where k is a constant.

- The above criterion reduces to something simpler. Can you guess what it is? Justify your answer.
- Show that

$$P_{e|0} = \Pr\left(r_1^2 + r_2^2 < r_3^2 + r_4^2 | 0\right) \quad (13)$$

- Show that the pdf of $Y = r_3^2 + r_4^2$ id

$$p_Y(y) = \frac{1}{N_0} e^{-\frac{y}{N_0}}, y > 0 \quad (14)$$

- Find

$$g(r_1, r_2) = \Pr\left(r_1^2 + r_2^2 < Y | 0, r_1, r_2\right). \quad (15)$$

- Show that $E\left[e^{-\frac{X^2}{2\sigma^2}}\right] = \frac{1}{\sqrt{2}} e^{-\frac{\mu^2}{4\sigma^2}}$ for $X \sim \mathcal{N}(\mu, \sigma^2)$.
- Now show that

$$E[g(r_1, r_2)] = \frac{1}{2} e^{-\frac{E_b}{2N_0}}. \quad (16)$$

3.3 Let $U, V \sim \mathcal{N}\left(0, \frac{k}{2}\right)$ be i.i.d. Assuming that

$$U = \sqrt{R} \cos \Theta \quad (17)$$

$$V = \sqrt{R} \sin \Theta \quad (18)$$

- Compute the jacobian for U, V with respect

to X and Θ defined by

$$J = \det \begin{pmatrix} \frac{\partial U}{\partial R} & \frac{\partial U}{\partial \Theta} \\ \frac{\partial V}{\partial R} & \frac{\partial V}{\partial \Theta} \end{pmatrix} \quad (19)$$

- The joint pdf for R, Θ is given by,

$$p_{R,\Theta}(r, \theta) = p_{U,V}(u, v) J|_{u=\sqrt{r}\cos\theta, v=\sqrt{r}\sin\theta} \quad (20)$$

Show that

$$p_R(r) = \begin{cases} \frac{1}{k} e^{-\frac{r}{k}} & r > 0, \\ 0 & r < 0, \end{cases} \quad (21)$$

assuming that Θ is uniformly distributed between 0 to 2π .

- Show that the pdf of $Y = R_1 - R_2$, where R_1 and R_2 are i.i.d. and have the same distribution as R is

$$p_Y(y) = \frac{1}{2k} e^{-\frac{|y|}{k}} \quad (22)$$

- Find the pdf of

$$Z = p + \sqrt{p}[U \cos \phi + V \sin \phi] \quad (23)$$

where ϕ is a constant.

- Find $\Pr(Y > Z)$.
- If $U \sim \mathcal{N}\left(m_1, \frac{k}{2}\right), V \sim \mathcal{N}\left(m_2, \frac{k}{2}\right)$, where m_1, m_2, k are constants, show that the pdf of

$$R = \sqrt{U^2 + V^2} \quad (24)$$

is

$$p_R(r) = \frac{e^{-\frac{r+m}{k}}}{k} I_0\left(\frac{2\sqrt{mr}}{k}\right), \quad m = \sqrt{m_1^2 + m_2^2} \quad (25)$$

- Show that

$$I_0(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{4^n (n!)^2} \quad (26)$$

- If

$$p_Z(z) = \begin{cases} \frac{1}{k} e^{-\frac{z}{k}} & z \geq 0 \\ 0 & z < 0 \end{cases} \quad (27)$$

find $\Pr(R < Z)$.

4 M-PSK

4.1 Consider a system where $\mathbf{s}_i = \begin{pmatrix} \cos\left(\frac{2\pi i}{M}\right) \\ \cos\left(\frac{2\pi i}{M}\right) \end{pmatrix}$, $i = 0, 1, \dots, M-1$. Let

$$\mathbf{r}|_{s_0} = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} \sqrt{E_s} + n_1 \\ n_2 \end{pmatrix} \quad (28)$$

where $n_1, n_2 \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$.

a) Substituting

$$r_1 = R \cos \theta \quad (29)$$

$$r_2 = R \sin \theta \quad (30)$$

show that the joint pdf of R, θ is

$$p(R, \theta) = \frac{R}{\pi N_0} \exp\left(-\frac{R^2 - 2R\sqrt{E_s} \cos \theta + E_s}{N_0}\right) \quad (31)$$

b) Show that

$$\lim_{\alpha \rightarrow \infty} \int_0^\infty (V - \alpha) e^{-(V-\alpha)^2} dV = 0 \quad (32)$$

$$\lim_{\alpha \rightarrow \infty} \int_0^\infty e^{-(V-\alpha)^2} dV = \sqrt{\pi} \quad (33)$$

c) Using the above, evaluate

$$\int_0^\infty V \exp\left\{-\left(V^2 - 2V\sqrt{\gamma} \cos \theta + \gamma\right)\right\} dV \quad (34)$$

for large values of γ .

d) Find a compact expression for

$$I = 1 - \sqrt{\frac{\gamma}{\pi}} \int_{-\frac{\pi}{M}}^{\frac{\pi}{M}} e^{-\gamma \sin^2 \theta} \cos \theta d\theta \quad (35)$$

e) Find $P_{e|s_0}$.

5 CRAIG'S FORMULA AND MGF

5.1 The Moment Generating Function (MGF) of X is defined as

$$M_X(s) = E[e^{sX}] \quad (36)$$

where X is a random variable and $E[\cdot]$ is the expectation.

a) Let $Y \sim \mathcal{N}(0, 1)$. Define

$$Q(x) = \Pr(Y > x), x > 0 \quad (37)$$

Show that

$$Q(x) = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} e^{-\frac{x^2}{2 \sin^2 \theta}} d\theta \quad (38)$$

b) Let $h \sim \mathcal{CN}\left(0, \frac{\Omega}{2}\right)$, $n \sim \mathcal{CN}\left(0, \frac{N_0}{2}\right)$. Find the distribution of $|h|^2$.

c) Let

$$P_e = \Pr(\Re\{h^* y\} < 0), \text{ where } y = (\sqrt{E_s} h + n), \quad (39)$$

Show that

$$P_e = \int_0^\infty Q(\sqrt{2x}) p_A(x) dx \quad (40)$$

where $A = \frac{E_s |h|^2}{N_0}$.

d) Show that

$$P_e = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} M_A\left(-\frac{1}{\sin^2 \theta}\right) d\theta \quad (41)$$

e) compute $M_A(s)$.

f) Find P_e .

g) If $\gamma = \frac{\Omega E_s}{N_0}$, show that $P_e < \frac{1}{2\gamma}$.