

Digital Modulation Techniques

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Abstract—The manual frames the problems of receiver design and performance analysis in digital communication as applications of probability theory.

Download all codes in this manual from

svn co <https://github.com/gadepall/comm/trunk/modulation/manual/codes>

1 BPSK

1.1. The *signal constellation diagram* for BPSK is given by Fig. 1.1. The symbols s_0 and s_1 are equiprobable. $\sqrt{E_b}$ is the energy transmitted per bit. Assuming a zero mean additive white gaussian noise (AWGN) with variance $\frac{N_0}{2}$, obtain the symbols that are received.

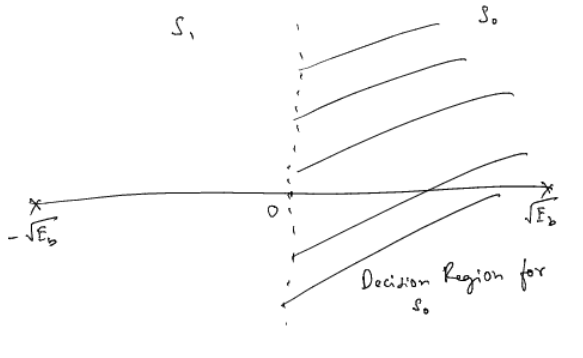


Fig. 1.1

Solution: The possible received symbols are

$$y|s_0 = \sqrt{E_b} + n \quad (1.1.1)$$

$$y|s_1 = -\sqrt{E_b} + n \quad (1.1.2)$$

where the AWGN $n \sim \mathcal{N}(0, \frac{N_0}{2})$.

1.2. From Fig. 1.1 obtain a decision rule for BPSK.

Solution: The decision rule is

$$y \underset{s_1}{\overset{s_0}{\geq}} 0 \quad (1.2.1)$$

1.3. Find the PDFs of $y|s_0$ and $y|s_1$

Solution: $\because y|s_0$ is Gaussian with

$$E[y|s_0] = \sqrt{E_b} \quad (1.3.1)$$

$$\text{var}(y|s_0) = \frac{N_0}{2}, \quad (1.3.2)$$

$$\Rightarrow p(y|s_0) = \frac{1}{\sqrt{2\pi}} \exp \left\{ -\frac{(y - \sqrt{E_b})^2}{\frac{N_0}{2}} \right\} \quad (1.3.3)$$

Similarly,

$$p(y|s_1) = \frac{1}{\sqrt{2\pi}} \exp \left\{ -\frac{(y + \sqrt{E_b})^2}{\frac{N_0}{2}} \right\} \quad (1.3.4)$$

1.4. Obtain (1.2.1) using the MAP criterion.

Solution: According to the MAP rule,

$$\hat{s} = \max_{s \in \{s_0, s_1\}} p(s|y) \quad (1.4.1)$$

$$\Rightarrow p(s_0|y) \underset{s_1}{\overset{s_0}{\geq}} p(s_1|y) \quad (1.4.2)$$

Using Bayes' rule,

$$p(s_0|y) = \frac{p(y|s_0)p(s_0)}{p(y)} \quad (1.4.3)$$

$$p(s_1|y) = \frac{p(y|s_1)p(s_1)}{p(y)} \quad (1.4.4)$$

which, upon substituting in (1.4.2), yields

$$\frac{p(y|s_0)p(s_0)}{p(y)} \underset{s_1}{\overset{s_0}{\geq}} \frac{p(y|s_1)p(s_1)}{p(y)} \quad (1.4.5)$$

$$p(y|s_0) \underset{s_1}{\overset{s_0}{\geq}} p(y|s_1) \quad (1.4.6)$$

\because the symbols are equiprobable. Substituting

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from and in

$$\frac{1}{\sqrt{2\pi}} \exp \left\{ -\frac{(y - \sqrt{E_b})^2}{\frac{N_0}{2}} \right\} \underset{s_1}{\overset{s_0}{\geq}} \frac{1}{\sqrt{2\pi}} \exp \left\{ -\frac{(y + \sqrt{E_b})^2}{\frac{N_0}{2}} \right\} \quad (1.4.7)$$

$$\Rightarrow (y + \sqrt{E_b})^2 \underset{s_1}{\overset{s_0}{\geq}} (y - \sqrt{E_b})^2 \quad (1.4.8)$$

$$\text{or, } y \underset{s_1}{\overset{s_0}{\geq}} 0 \quad (1.4.9)$$

after some algebra.

- 1.5. Using the decision rule in Problem 1.2, obtain an expression for the probability of error for BPSK.

Solution: Since the symbols are equiprobable, it is sufficient if the error is calculated assuming that a 0 was sent. This results in

$$P_e = \Pr(y < 0 | s_0) = \Pr(\sqrt{E_b} + n < 0) \quad (1.5.1)$$

$$= \Pr(-n > \sqrt{E_b}) = \Pr(n > \sqrt{E_b}) \quad (1.5.2)$$

since n has a symmetric pdf. Let $w \sim \mathcal{N}(0, 1)$. Then $n = \sqrt{\frac{N_0}{2}}w$. Substituting this in (??),

$$P_e = \Pr\left(\sqrt{\frac{N_0}{2}}w > \sqrt{E_b}\right) = \Pr\left(w > \sqrt{\frac{2E_b}{N_0}}\right) \quad (1.5.3)$$

$$= Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \quad (1.5.4)$$

where

$$Q(x) \triangleq \Pr(w > x), x \geq 0. \quad (1.5.5)$$

$$= \frac{1}{\pi} \int_0^{\frac{\pi}{2}} e^{-\frac{x^2}{2\sin^2\theta}} d\theta \quad (1.5.6)$$

The PDF of $w \sim \mathcal{N}(0, 1)$ is given by

$$p_w(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty \quad (1.5.7)$$

and the complementary error function is defined as

$$\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt. \quad (1.5.8)$$

1.6. Show that

$$Q(x) = \frac{1}{2} \text{erfc}\left(\frac{x}{\sqrt{2}}\right) \quad (1.6.1)$$

Solution: From (??)

$$Q(x) = \Pr(w > x), x \geq 0 \quad (1.6.2)$$

$$= \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{t^2}{2}} dt. \quad (1.6.3)$$

$$= \frac{1}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{2}}}^\infty e^{-y^2} dy. \quad (t = \sqrt{2}y) \quad (1.6.4)$$

resulting in

- 1.7. Verify the bit error rate (BER) plots for BPSK through simulation and analysis for 0 to 10 dB.

Solution: The following code

```
codes/bpsk_ber.py
```

yields Fig. ??

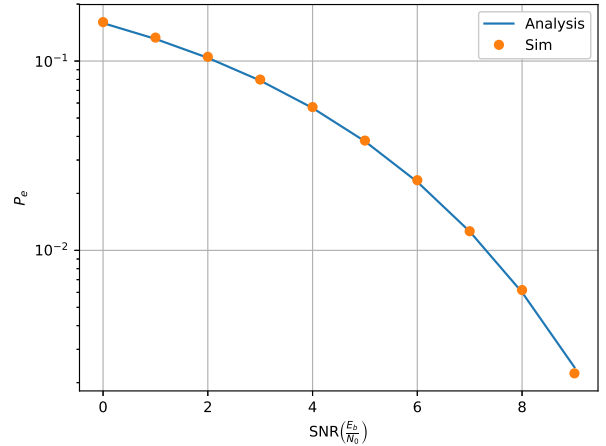


Fig. 1.7

2 COHERENT BFSK

- 2.1. The signal constellation for binary frequency shift keying (BFSK) is given in Fig. ?. Obtain the equations for the received symbols.

Solution: The received symbols are given by

$$\mathbf{y}|_{s_0} = \begin{pmatrix} \sqrt{E_b} \\ 0 \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, \quad (2.1.1)$$

and

$$\mathbf{y}|_{s_1} = \begin{pmatrix} 0 \\ \sqrt{E_b} \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, \quad (2.1.2)$$

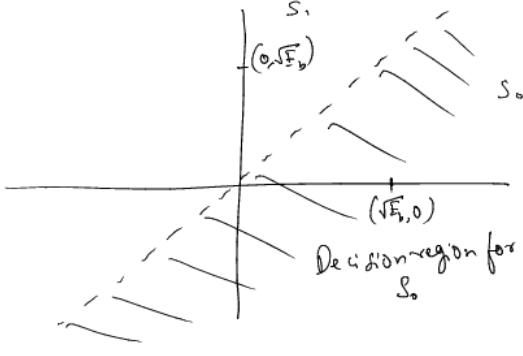


Fig. 2.1

where $n_1, n_2 \sim \mathcal{N}(0, \frac{N_0}{2})$. and $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$.

2.2. Obtain a decision rule for BFSK from Fig. ??.

Solution: The decision rule is

$$y_1 \stackrel{s_0}{\geq} y_2 \quad (2.2.1)$$

Definition 1. The joint PDF of X, Y is given by

$$p(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)} \times \left\{ \frac{(x-\mu_x)^2}{\sigma_x^2} + \frac{(y-\mu_y)^2}{\sigma_y^2} - \frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y} \right\}\right] \quad (2.2.2)$$

where

$$\mu_x = E[X], \quad (2.2.3)$$

$$\sigma_x^2 = \text{var}(X), \quad (2.2.4)$$

$$\rho = \frac{E[(X-\mu_x)(Y-\mu_y)]}{\sigma_x\sigma_y}. \quad (2.2.5)$$

For equiprobably symbols, the MAP criterion is defined as

$$p(\mathbf{y}|s_0) \stackrel{s_0}{\geq} p(\mathbf{y}|s_1) \quad (2.2.6)$$

2.3. Use (??) in (??) to obtain (??).

Solution: According to the MAP criterion, assuming equiprobably symbols,

$$p(\mathbf{y}|s_0) \stackrel{s_0}{\geq} p(\mathbf{y}|s_1) \quad (2.3.1)$$

2.4. Derive and plot the probability of error. Verify through simulation.

Solution: Given that s_0 was transmitted, the

received symbols are

$$\mathbf{y}|s_0 = \begin{pmatrix} \sqrt{E_b} \\ 0 \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, \quad (2.4.1)$$

From (??), the probability of error is given by

$$P_e = \Pr(y_1 < y_2|s_0) = \Pr(\sqrt{E_b} + n_1 < n_2) \quad (2.4.2)$$

$$= \Pr(n_2 - n_1 > \sqrt{E_b}) \quad (2.4.3)$$

Note that $n_2 - n_1 \sim \mathcal{N}(0, N_0)$. Thus,

$$P_e = \Pr(\sqrt{N_0}w > \sqrt{E_b}) \quad (2.4.4)$$

$$= \Pr\left(w > \sqrt{\frac{E_b}{N_0}}\right) \quad (2.4.5)$$

$$\Rightarrow P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right) \quad (2.4.6)$$

where $w \sim \mathcal{N}(0, 1)$. The following code plots the BER curves in Fig. ??

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codes/fsk_ber.py
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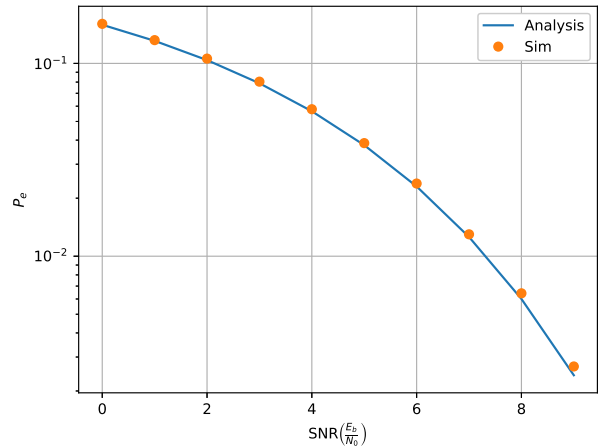


Fig. 2.4

3 QPSK

1. Let

$$\mathbf{y} = \mathbf{s} + \mathbf{n} \quad (3.1.1)$$

where $\mathbf{s} \in \{\mathbf{s}_0, \mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}$ and

$$\mathbf{s}_0 = \begin{pmatrix} \sqrt{E_s} \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ \sqrt{E_s} \end{pmatrix}, \quad (3.1.2)$$

$$\mathbf{s}_2 = \begin{pmatrix} -\sqrt{E_s} \\ 0 \end{pmatrix}, \mathbf{s}_3 = \begin{pmatrix} 0 \\ -\sqrt{E_s} \end{pmatrix}, \quad (3.1.3)$$

$$E[\mathbf{n}] = \mathbf{0}, E[\mathbf{n}\mathbf{n}^T] = \frac{N_0}{2}\mathbf{I} \quad (3.1.4)$$

2. Using (??), show that the MAP decision for detecting \mathbf{s}_0 results in

$$|y_2| < y_1 \quad (3.2.1)$$

3. Express $\Pr(\hat{\mathbf{s}} = \mathbf{s}_0 | \mathbf{s} = \mathbf{s}_0)$ in terms of y_1, y_2 .

Solution: From (??) and (??) ,

$$\begin{aligned} \Pr(\hat{\mathbf{s}} = \mathbf{s}_0 | \mathbf{s} = \mathbf{s}_0) \\ = \Pr(|y_2| < y_1 | y_1 = \sqrt{E_s}, y_2 = 0) \end{aligned} \quad (3.3.1)$$

4. Let

$$X = n_2 - n_1, \quad (3.4.1)$$

$$Y = -n_2 - n_1, \quad (3.4.2)$$

where $\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$.

Show that $X, Y \sim \mathcal{N}(0, N_0)$.

5. The correlation coefficient of X, Y is defined as

$$\rho = \frac{E[(X - \mu_x)(Y - \mu_y)]}{\sigma_x \sigma_y} \quad (3.5.1)$$

X and Y are said to be uncorrelated if $\rho = 0$. Show that X and Y are uncorrelated. Verify this numerically.

Solution: From (??),

$$\mu_x = E[X] = 0 \quad (3.5.2)$$

$$\mu_y = E[Y] = 0 \quad (3.5.3)$$

$$\begin{aligned} \Rightarrow \rho &= E[XY] = E[(n_2 - n_1)(-n_2 - n_1)] \\ &= 0 \end{aligned} \quad (3.5.4)$$

upon substituting from (??) and (??).

6. Show that X and Y are independent, i.e.

$$p_{XY}(x, y) = p_X(x)p_Y(y). \quad (3.6.1)$$

Solution: Use (??) in (??) to get (??). Uncorrelated Gaussians are independent.

7. Show that

$$\Pr(\hat{\mathbf{s}} = \mathbf{s}_0 | \mathbf{s} = \mathbf{s}_0) = \Pr(X < \sqrt{E_s}) \Pr(Y < \sqrt{E_s}). \quad (3.7.1)$$

Solution: From (??) and (??)

$$\begin{aligned} \Pr(\hat{\mathbf{s}} = \mathbf{s}_0 | \mathbf{s} = \mathbf{s}_0) \\ = \Pr(|n_2| < \sqrt{E_s} + n_1) \end{aligned} \quad (3.7.2)$$

which can be expressed as

$$\Pr(n_2 < \sqrt{E_s} + n_1, -n_2 > \sqrt{E_s} + n_1) \quad (3.7.3)$$

$$= \Pr(X < \sqrt{E_s}, Y < \sqrt{E_s}) \quad (3.7.4)$$

$$= \Pr(X < \sqrt{E_s}) \Pr(Y < \sqrt{E_s}) \quad (3.7.5)$$

after some algebra, using the fact that X, Y are independent.

8. Show that

$$\Pr(\hat{\mathbf{s}} = \mathbf{s}_0 | \mathbf{s} = \mathbf{s}_0) = \left(1 - Q\left(\sqrt{\frac{E_s}{N_0}}\right)\right)^2 \quad (3.8.1)$$

Solution: From ,

$$\Pr(X > \sqrt{E_s}) = \Pr(Y > \sqrt{E_s}) = Q\left(\sqrt{\frac{E_s}{N_0}}\right) \quad (3.8.2)$$

yielding (??).

9. Verify the above through simulation.

Solution: This is shown in Fig. ?? through the following code.

codes/qpsk.py

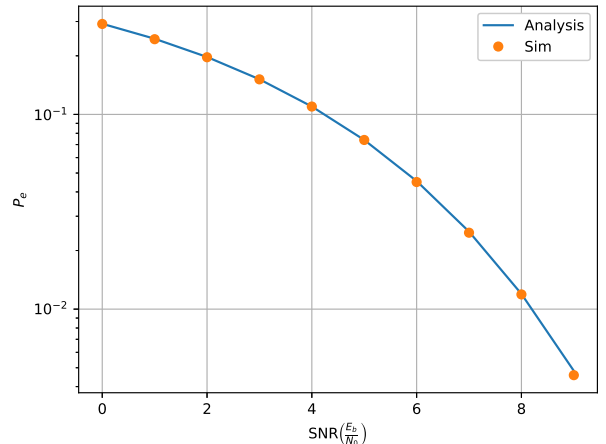


Fig. 3.9

10. Modify the above script to obtain the probability of symbol error.

4 M-PSK

1. Consider a system where $\mathbf{s}_i = \begin{pmatrix} \cos\left(\frac{2\pi i}{M}\right) \\ \sin\left(\frac{2\pi i}{M}\right) \end{pmatrix}$, $i = 0, 1, \dots, M-1$. Let

$$\mathbf{y}|s_0 = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \sqrt{E_s} + n_1 \\ n_2 \end{pmatrix} \quad (4.1.1)$$

where $n_1, n_2 \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$.

Substituting

$$y_1 = R \cos \theta \quad (4.1.2)$$

$$y_2 = R \sin \theta \quad (4.1.3)$$

show that the joint pdf of R, θ is

$$p(R, \theta) = \frac{R}{\pi N_0} \exp\left(-\frac{R^2 - 2R\sqrt{E_s}\cos\theta + E_s}{N_0}\right) \quad (4.1.4)$$

2. Show that

$$\lim_{\alpha \rightarrow \infty} \int_0^\infty (V - \alpha) e^{-(V-\alpha)^2} dV = 0 \quad (4.2.1)$$

$$\lim_{\alpha \rightarrow \infty} \int_0^\infty e^{-(V-\alpha)^2} dV = \sqrt{\pi} \quad (4.2.2)$$

3. Using the above, show that

$$\begin{aligned} \int_0^\infty V \exp\left\{-\left(V^2 - 2V\sqrt{\gamma}\cos\theta + \gamma\right)\right\} dV \\ = e^{-\gamma\sin^2\theta} \sqrt{\gamma\pi} \cos\theta \end{aligned} \quad (4.3.1)$$

for large values of γ .

Solution: The integrand in (??) can be expressed as

$$\begin{aligned} & V e^{-(V^2 - 2V\sqrt{\gamma}\cos\theta + \gamma)} \\ &= \{(V - \sqrt{\gamma}\cos\theta) + (\sqrt{\gamma}\cos\theta)\} \\ &\quad \times e^{-(V - \sqrt{\gamma}\cos\theta)^2} e^{-\sqrt{\gamma}\sin^2\theta} \\ &\Rightarrow \int_0^\infty V e^{-(V^2 - 2V\sqrt{\gamma}\cos\theta + \gamma)} dV \\ &\quad = e^{-\sqrt{\gamma}\sin^2\theta} \\ &\quad \times \left\{ \int_0^\infty (V - \sqrt{\gamma}\cos\theta) e^{-(V - \sqrt{\gamma}\cos\theta)^2} d\theta \right. \\ &\quad \left. + \int_0^\infty (\sqrt{\gamma}\cos\theta) e^{-(V - \sqrt{\gamma}\cos\theta)^2} d\theta \right\} \end{aligned} \quad (4.3.2)$$

yielding (??) from (??) and (??)

4. Find a compact expression for

$$I = 1 - \sqrt{\frac{\gamma}{\pi}} \int_{-\frac{\pi}{M}}^{\frac{\pi}{M}} e^{-\gamma\sin^2\theta} \cos\theta d\theta \quad (4.4.1)$$

Solution: The above integral can be expressed as

$$I = 1 - 2 \sqrt{\frac{\gamma}{\pi}} \int_0^{\frac{\pi}{M}} e^{-\gamma\sin^2\theta} \cos\theta d\theta \quad (4.4.2)$$

$$= 1 - 2 \left\{ Q(0) - Q\left(\sqrt{2\gamma}\sin\frac{\pi}{M}\right) \right\} \quad (4.4.3)$$

$$= 2Q\left(\sqrt{2\gamma}\sin\frac{\pi}{M}\right) \quad (4.4.4)$$

$\because Q(0) = \frac{1}{2}$.

5. Find the decision region for the symbol \mathbf{s}_0 .

6. Show that

$$P_{e|s_0} = 2Q\left(\sqrt{2\left(\frac{E_s}{N_0}\right)\sin\frac{\pi}{M}}\right) \quad (4.6.1)$$

7. Verify the SER through simulation.