Digital Modulation Techniques

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Contents

***Abstract*—The manual frames the problems of receiver design and performance analysis in digital communication as applications of probability theory.**

Download all codes in this manual from

```
svn co https://github.com/gadepall/comm/trunk/
    modulation/manual/codes
```

1 BPSK

1.1. The *signal constellation diagram* for BPSK is given by Fig. 1.1. The symbols s_0 and s_1 are equiprobable. $\sqrt{E_b}$ is the energy transmitted per bit. Assuming a zero mean additive white gaussian noise (AWGN) with variance $\frac{N_0}{2}$, obtain the symbols that are received. **Solution:** The possible received symbols are

$$y|s_0 = \sqrt{E_b} + n \tag{1.1.1}$$

$$y|s_1 = -\sqrt{E_b} + n \tag{1.1.2}$$

where the AWGN $n \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$.

1.2. From Fig. 1.1 obtain a decision rule for BPSK **Solution:** The decision rule is

$$y \underset{s_1}{\overset{s_0}{\gtrless}} 0 \tag{1.2.1}$$

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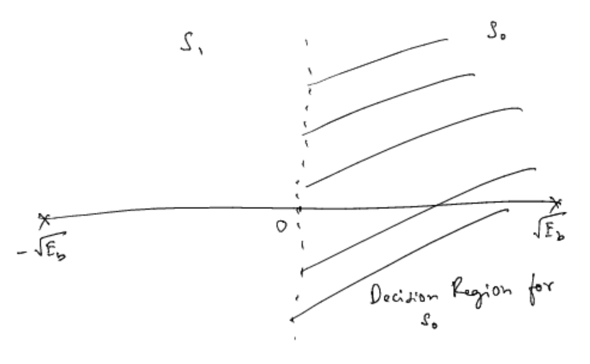

Fig. 1.1

1.3. Repeat the previous exercise using the MAP criterion.

1.4. Using the decision rule in Problem 1.2, obtain an expression for the probability of error for BPSK.
Solution: Since the symbols are equiprobable, it is sufficient if the error is calculated assuming that a 0 was sent. This results in

$$P_e = \Pr\left(y < 0 | s_0\right) = \Pr\left(\sqrt{E_b} + n < 0\right) \tag{1.4.1}$$

$$= \Pr\left(-n > \sqrt{E_b}\right) = \Pr\left(n > \sqrt{E_b}\right) \tag{1.4.2}$$

since n has a symmetric pdf. Let $w \sim \mathcal{N}(0,1)$. Then $n = \sqrt{\frac{N_0}{2}}w$. Substituting this in (1.4.2),

$$P_e = \Pr\left(\sqrt{\frac{N_0}{2}}w > \sqrt{E_b}\right) = \Pr\left(w > \sqrt{\frac{2E_b}{N_0}}\right) \tag{1.4.3}$$

$$= Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \tag{1.4.4}$$

where

$$Q(x) \triangleq \Pr\left(w > x\right), x \geq 0. \tag{1.4.5}$$

$$= \frac{1}{\pi}\int_0^{\frac{\pi}{2}} e^{-\frac{x^2}{2\sin^2\theta}}\, d\theta \tag{1.4.6}$$

The PDF of $w \sim \mathcal{N}(0,1)$ is given by

$$p_w(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty \tag{1.4.7}$$

and the complementary error function is defined as

$$\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt. \tag{1.4.8}$$

1.5. Show that

$$Q(x) = \frac{1}{2}\text{erfc}\left(\frac{x}{\sqrt{2}}\right) \tag{1.5.1}$$

Solution: From (1.4.5)

$$\begin{aligned} Q(x) &= \Pr(w > x), x \geq 0 && (1.5.2)\\ &= \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{t^2}{2}}\, dt. && (1.5.3)\\ &= \frac{1}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{2}}}^\infty e^{-y^2}\, dy. \quad \left(t = \sqrt{2}y\right) && (1.5.4) \end{aligned}$$

resulting in

1.6. Verify the bit error rate (BER) plots for BPSK through simulation and analysis for 0 to 10 dB.
Solution: The following code

```
codes/bpsk_ber.py
```

yields Fig. 1.6

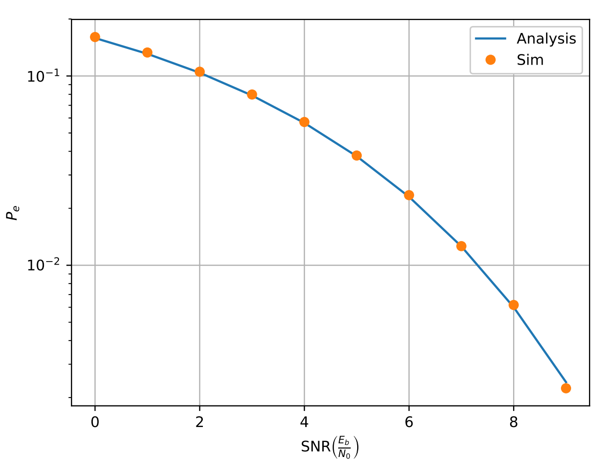

Fig. 1.6

2 Coherent BFSK

2.1. The signal constellation for binary frequency shift keying (BFSK) is given in Fig. 2.1. Obtain

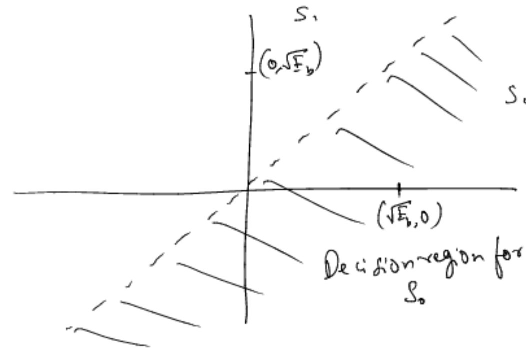

Fig. 2.1

the equations for the received symbols.
Solution: The received symbols are given by

$$\mathbf{y}|s_0 = \begin{pmatrix}\sqrt{E_b}\\0\end{pmatrix} + \begin{pmatrix}n_1\\n_2\end{pmatrix}, \tag{2.1.1}$$

and

$$\mathbf{y}|s_1 = \begin{pmatrix}0\\\sqrt{E_b}\end{pmatrix} + \begin{pmatrix}n_1\\n_2\end{pmatrix}, \tag{2.1.2}$$

where $n_1, n_2 \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$. and $\mathbf{y} = \begin{pmatrix}y_1\\y_2\end{pmatrix}$.

2.2. Obtain a decision rule for BFSK from Fig. 2.1.
Solution: The decision rule is

$$y_1 \underset{s_1}{\overset{s_0}{\gtrless}} y_2 \tag{2.2.1}$$

Definition 1. *The joint PDF of X, Y is given by*

$$p(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)} \times \left\{\frac{(x-\mu_x)^2}{\sigma_x^2} + \frac{\left(y-\mu_y\right)^2}{\sigma_y^2} - \frac{2\rho(x-\mu_x)\left(y-\mu_y\right)}{\sigma_x\sigma_y}\right\}\right] \tag{2.2.2}$$

where

$$\begin{aligned} \mu_x &= E[X], && (2.2.3)\\ \sigma_x^2 &= var(X), && (2.2.4)\\ \rho &= \frac{E\left[(X-\mu_x)\left(Y-\mu_y\right)\right]}{\sigma_x\sigma_y}. && (2.2.5) \end{aligned}$$

For equiprobably symbols, the MAP criterion is defined as

$$p(\mathbf{y}|s_0) \underset{s_1}{\overset{s_0}{\gtrless}} p(\mathbf{y}|s_1) \tag{2.2.6}$$

2.3. Use (2.2.2) in (2.2.6) to obtain (2.2.1).
Solution: According to the MAP criterion, assuming equiprobably symbols,

$$p\left(\mathbf{y}|s_0\right) \underset{s_1}{\overset{s_0}{\gtrless}} p\left(\mathbf{y}|s_1\right) \tag{2.3.1}$$

2.4. Derive and plot the probability of error. Verify through simulation.
Solution: Given that s_0 was transmitted, the received symbols are

$$\mathbf{y}|s_0 = \begin{pmatrix}\sqrt{E_b}\\ 0\end{pmatrix} + \begin{pmatrix}n_1\\ n_2\end{pmatrix}, \tag{2.4.1}$$

From (2.2.1), the probability of error is given by

$$P_e = \Pr\left(y_1 < y_2|s_0\right) = \Pr\left(\sqrt{E_b} + n_1 < n_2\right) \tag{2.4.2}$$

$$= \Pr\left(n_2 - n_1 > \sqrt{E_b}\right) \tag{2.4.3}$$

Note that $n_2 - n_1 \sim \mathcal{N}\left(0, N_0\right)$. Thus,

$$P_e = \Pr\left(\sqrt{N_0}w > \sqrt{E_b}\right) \tag{2.4.4}$$

$$= \Pr\left(w > \sqrt{\frac{E_b}{N_0}}\right) \tag{2.4.5}$$

$$\implies P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right) \tag{2.4.6}$$

where $w \sim \mathcal{N}\left(0, 1\right)$. The following code plots the BER curves in Fig. 2.4

```
codes/fsk_ber.py
```

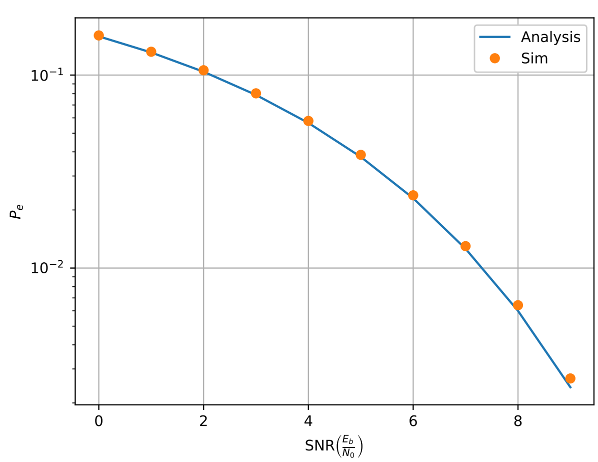

Fig. 2.4

3 QPSK

1. Let

$$\mathbf{y} = \mathbf{s} + \mathbf{n} \tag{3.1.1}$$

where $\mathbf{s} \in \{\mathbf{s}_0, \mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}$ and

$$\mathbf{s}_0 = \begin{pmatrix}\sqrt{E_s}\\ 0\end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix}0\\ \sqrt{E_s}\end{pmatrix}, \tag{3.1.2}$$

$$\mathbf{s}_2 = \begin{pmatrix}-\sqrt{E_s}\\ 0\end{pmatrix}, \mathbf{s}_3 = \begin{pmatrix}0\\ -\sqrt{E_s}\end{pmatrix}, \tag{3.1.3}$$

$$E\left[\mathbf{n}\right] = \mathbf{0}, E\left[\mathbf{n}\mathbf{n}^T\right] = \frac{N_0}{2}\mathbf{I} \tag{3.1.4}$$

2. Using (2.2.2), show that the MAP decision for detecting $\mathbf{s}_0$ results in

$$|y_2| < y_1 \tag{3.2.1}$$

3. Express $\Pr\left(\hat{\mathbf{s}} = \mathbf{s}_0|\mathbf{s} = \mathbf{s}_0\right)$ in terms of y_1, y_2.
Solution: From (3.2.1) and (3.1.2) ,

$$\Pr\left(\hat{\mathbf{s}} = \mathbf{s}_0|\mathbf{s} = \mathbf{s}_0\right)$$
$$= \Pr\left(|y_2| < y_1|y_1 = \sqrt{E}_s, y_2 = 0\right) \tag{3.3.1}$$

4. Let

$$X = n_2 - n_1, \tag{3.4.1}$$
$$Y = -n_2 - n_1, \tag{3.4.2}$$

where $\mathbf{n} = \begin{pmatrix}n_1\\ n_2\end{pmatrix}$.
Show that $X, Y \sim \mathcal{N}\left(0, N_0\right)$.

5. The correlation coefficient of X, Y is defined as

$$\rho = \frac{E\left[\left(X - \mu_x\right)\left(Y - \mu_y\right)\right]}{\sigma_x\sigma_y} \tag{3.5.1}$$

X and Y are said to be uncorrelated if $\rho = 0$ Show that X and Y are uncorrelated. Verify this numerically.
Solution: From (3.1.4),

$$\mu_x = E\left[X\right] = 0 \tag{3.5.2}$$
$$\mu_y = E\left[Y\right] = 0 \tag{3.5.3}$$
$$\implies \rho = E\left[XY\right] = E\left[\left(n_2 - n_1\right)\left(-n_2 - n_1\right)\right]$$
$$= 0 \tag{3.5.4}$$

upon substituting from (3.4.1) and (3.4.2).

6. Show that X and Y are independent, i.e.

$$p_{XY}(x, y) = p_X(x)p_Y(y). \tag{3.6.1}$$

Solution: Use (3.5.4) in (2.2.2) to get (3.6.1).

Uncorrelated Gaussians are independent.

7. Show that

$$\Pr\left(\hat{\mathbf{s}} = \mathbf{s}_0 | \mathbf{s} = \mathbf{s}_0\right) = \Pr\left(X < \sqrt{E_s}\right) \Pr\left(Y < \sqrt{E_s}\right). \tag{3.7.1}$$

Solution: From (3.3.1) and (3.1.1)

$$\Pr\left(\hat{\mathbf{s}} = \mathbf{s}_0 | \mathbf{s} = \mathbf{s}_0\right) = \Pr\left(|n_2| < \sqrt{E_s} + n_1\right) \tag{3.7.2}$$

which can be expressed as

$$\Pr\left(n_2 < \sqrt{E_s} + n_1, -n_2 > \sqrt{E_s} + n_1\right) \tag{3.7.3}$$
$$= \Pr\left(X < \sqrt{E_s}, Y < \sqrt{E_s}\right) \tag{3.7.4}$$
$$= \Pr\left(X < \sqrt{E_s}\right) \Pr\left(Y < \sqrt{E_s}\right) \tag{3.7.5}$$

after some algebra, using the fact that X, Y are independent.

8. Show that

$$\Pr\left(\hat{\mathbf{s}} = \mathbf{s}_0 | \mathbf{s} = \mathbf{s}_0\right) = \left(1 - Q\left(\sqrt{\frac{E_s}{N_0}}\right)\right)^2 \tag{3.8.1}$$

Solution: From ,

$$\Pr\left(X > \sqrt{E_s}\right) = \Pr\left(Y > \sqrt{E_s}\right) = Q\left(\sqrt{\frac{E_s}{N_0}}\right) \tag{3.8.2}$$

yielding (3.8.1).

9. Verify the above through simulation.
Solution: This is shown in Fig. 3.9 through the following code.

```
codes/qpsk.py
```

10. Modify the above script to obtain the probability of symbol error.

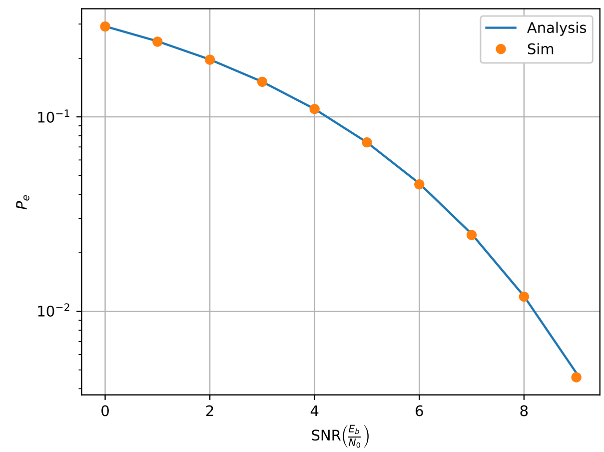

Fig. 3.9

4 M-PSK

1. Consider a system where $\mathbf{s}_i = \begin{pmatrix} \cos\left(\frac{2\pi i}{M}\right) \\ \sin\left(\frac{2\pi i}{M}\right) \end{pmatrix}, i = 0, 1, \dots M-1$. Let

$$\mathbf{y}|s_0 = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \sqrt{E_s} + n_1 \\ n_2 \end{pmatrix} \tag{4.1.1}$$

where $n_1, n_2 \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$.
Substituting

$$y_1 = R\cos\theta \tag{4.1.2}$$
$$y_2 = R\sin\theta \tag{4.1.3}$$

show that the joint pdf of R, θ is

$$p(R, \theta) = \frac{R}{\pi N_0} \exp\left(-\frac{R^2 - 2R\sqrt{E_s}\cos\theta + E_s}{N_0}\right) \tag{4.1.4}$$

2. Show that

$$\lim_{\alpha \to \infty} \int_0^\infty (V - \alpha) e^{-(V-\alpha)^2} \, dV = 0 \tag{4.2.1}$$
$$\lim_{\alpha \to \infty} \int_0^\infty e^{-(V-\alpha)^2} \, dV = \sqrt{\pi} \tag{4.2.2}$$

3. Using the above, show that

$$\int_0^\infty V \exp\left\{-\left(V^2 - 2V\sqrt{\gamma}\cos\theta + \gamma\right)\right\} dV = e^{-\gamma \sin^2\theta} \sqrt{\gamma\pi}\cos\theta \tag{4.3.1}$$

for large values of γ.
Solution: The integrand in (4.3.1) can be expressed as

$$Ve^{-\left(V^2 - 2V\sqrt{\gamma}\cos\theta + \gamma\right)} = \left\{\left(V - \sqrt{\gamma}\cos\theta\right) + \left(\sqrt{\gamma}\cos\theta\right)\right\} \times e^{-\left(V - \sqrt{\gamma}\cos\theta\right)^2} e^{-\sqrt{\gamma}\sin^2\theta}$$

$$\implies \int_0^\infty Ve^{-\left(V^2 - 2V\sqrt{\gamma}\cos\theta + \gamma\right)} dV = e^{-\sqrt{\gamma}\sin^2\theta}$$
$$\times \left\{\int_0^\infty \left(V - \sqrt{\gamma}\cos\theta\right) e^{-\left(V - \sqrt{\gamma}\cos\theta\right)^2} d\theta + \int_0^\infty \left(\sqrt{\gamma}\cos\theta\right) e^{-\left(V - \sqrt{\gamma}\cos\theta\right)^2} d\theta\right\} \tag{4.3.2}$$

yielding (4.3.1) from (4.2.1) and (4.2.2)

4. Find a compact expression for

$$I = 1 - \sqrt{\frac{\gamma}{\pi}} \int_{-\frac{\pi}{M}}^{\frac{\pi}{M}} e^{-\gamma \sin^2 \theta} \cos \theta \, d\theta \tag{4.4.1}$$

Solution: The above integral can be expressed as

$$\begin{aligned} I &= 1 - 2\sqrt{\frac{\gamma}{\pi}} \int_{0}^{\frac{\pi}{M}} e^{-\gamma \sin^2 \theta} \cos \theta \, d\theta && (4.4.2) \\ &= 1 - 2\left\{ Q(0) - Q\left(\sqrt{2\gamma} \sin \frac{\pi}{M} \right) \right\} && (4.4.3) \\ &= 2Q\left(\sqrt{2\gamma} \sin \frac{\pi}{M} \right) && (4.4.4) \end{aligned}$$

$\because Q(0) = \frac{1}{2}$.

5. Find the decision region for the symbol $\mathbf{s}_0$.
6. Show that

$$P_{e|\mathbf{s}_0} = 2Q\left(\sqrt{2\left(\frac{E_s}{N_o}\right)} \sin \frac{\pi}{M} \right) \tag{4.6.1}$$

7. Verify the SER through simulation.