BCH Codes

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Contents

***Abstract*—This manual provides an introduction to BCH codes.**

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1 Generator Polynomial

1.1 For a BCH code, the minimal polynomials are given by

$$g_1(x) = 1 + x + x^3 + x^5 + x^{14} \tag{1.1}$$
$$g_2(x) = 1 + x^6 + x^8 + x^{11} + x^{14} \tag{1.2}$$
$$g_3(x) = 1 + x + x^2 + x^6 + x^9 + x^{10} + x^{14} \tag{1.3}$$
$$g_4(x) = 1 + x^4 + x^7 + x^8 + x^{1}0 + x^{12} + x^{14} \tag{1.4}$$
$$g_5(x) = 1 + x^2 + x^4 + x^6 + x^8 + x^9 + x^{11} + x^{13} + x^{14} \tag{1.5}$$
$$g_6(x) = 1 + x^3 + x^7 + x^8 + x^9 + x^{13} + x^{14} \tag{1.6}$$
$$g_7(x) = 1 + x^2 + x^5 + x^6 + x^7 + x^{10} + x^{11} + x^{13} + x^{14} \tag{1.7}$$
$$g_8(x) = 1 + x^5 + x^8 + x^9 + x^{10} + x^{11} + x^{14} \tag{1.8}$$
$$g_9(x) = 1 + x + x^2 + x^3 + x^9 + x^{10} + x^{14} \tag{1.9}$$
$$g_{10}(x) = 1 + x^3 + x^6 + x^9 + x^{11} + x^{12} + x^{14} \tag{1.10}$$
$$g_{11}(x) = 1 + x^4 + x^{11} + x^{12} + x^{14} \tag{1.11}$$
$$g_{12}(x) = 1 + x + x^2 + x^3 + x^5 + x^6 + x^7 + x^8 + x^{10} + x^{13} + x^{14} \tag{1.12}$$

Obtain the minimal polynomial matrix.
Solution:

```
https://raw.githubusercontent.com/gadepall/
    EE6317/master/BCH/codes/
    min_poly_mat.py
```

1.2 Obtain the generator polynomial vector.
Solution: The generator polynomial is obtained as

$$g(x) = \prod_{i=1}^{m} g_i(x) \tag{1.13}$$

where $m = 12$. The following code computes $\mathbf{g}$. What is the length of $\mathbf{g}$?

```
https://raw.githubusercontent.com/gadepall/
    EE6317/master/BCH/codes/gen_poly.py
```

1.3 What is the maximum number of errors that can be corrected by $\mathbf{g}$?
Solution: $m = 12$.

2 Encoding

2.1 Let $\mathbf{m}$ be a $k \times 1$ message vector and

$$m(x) = m_{k-1}x^{k-1} + m_{k-2}x^{k-2} + \cdots + m_1 x + m_0 \tag{2.1}$$

be the corresponding Message polynomial.

2.2 Let

$$m(x)x^{n-k} = q(x)g(x) + d(x) \tag{2.2}$$

and

$$c(x) = m(x)x^{n-k} + d(x) \tag{2.3}$$

2.3 If $k = 3072$ find the length of $\mathbf{c}$.

2.4 Write a program to compute the corresponding coefficient vector $\mathbf{c}$. This is the output of the BCH encoder.
Solution:

```
https://raw.githubusercontent.com/gadepall/
    EE6317/master/BCH/codes/encoder.py
```

3 Berlekamp's Decoding Algorithm

3.1 Find the number of minimal polynomials for $g(x)$.
Solution: This is given by $2m = 24$.

3.2 Find t such that

$$n - k \leq mt \tag{3.1}$$

Solution: Since $m = 12, n = 3240, k = 3072$,

$$t \geq \frac{n-k}{m} = 14 \tag{3.2}$$

3.3 Generate the elements table for $t = 14$.
Solution:

```
wget https://raw.githubusercontent.com/
    gadepall/EE6317/master/BCH/codes/
    alpha_tables.py
```

3.4 Find the size of the elements table $\mathbf{A}$ for $t = 14$.
Solution: The size is given by

$$(2^t - 1) \times t = 16383 \times 14 \tag{3.3}$$

3.5 Let $\alpha_i, 2 \leq i \leq 2m+1$ be the ith row of $\mathbf{A}$ and $\alpha_i(x)$ be the corresponding polynomial. Let $\mathbf{r}$ be the received codeword (noisy). $r(x)$ is then defined to be the received polynomial. Find the correspondig syndromes.
Solution:

$$S_i(x) = r(\alpha_i(x)) \tag{3.4}$$

3.6 Write a program to compute the matrix $\mathbf{S}$ given $\mathbf{r}$.
Solution:

```
wget https://raw.githubusercontent.com/
    gadepall/EE6317/master/BCH/codes/
    syndromes_calc.py
```

3.7 Find the number of errors in $\mathbf{r}$.
Solution: The number of nonzero $S_i(x)$ is equal to the number of errors in $\mathbf{r}$.

3.8 Write a program to compute the number of errors in $\mathbf{r}$.

3.9 Define the syndrome polynomial $S(x)$

3.10 Intialization : $k = 0, \Lambda^{(0)}(x) = 1, T^{(0)} = 1$

3.11 Let $\Delta^{(2k)}$ be the coefficient of x^{2k+1} in $\Lambda^{(2k)}[1 + S(x)]$

3.12 Compute

$$\Delta^{(2k+2)}(x) = \Lambda^{(2k)}(x) + \Delta^{(2k)}[x.T^{(2k)}(x)] \tag{3.5}$$

3.13 Compute

$$T^{(2k+2)}(x) = \begin{cases} x^2 T^{(2k)}(x) & \text{if } \Delta^{(2k)} = 0 \text{ or } \deg\left[\Lambda^{(2k)}(x)\right] > k \\ \frac{x\Lambda^{(2k)}(x)}{\Delta^{(2k)}} & \text{if } \Delta^{(2k)} \neq 0 \text{ or } \deg\left[\Lambda^{(2k)}(x)\right] \leq k \end{cases} \tag{3.6}$$

3.14 Set $k = k + 1$.If $k < t$ then go to step3.

3.15 Return the Error Locator polynomial $\Lambda(x) = \Lambda^{(2k)}(x)$

4 The Chien's Search Algorithm

4.1 Take α^j as test root . $0 \leq j \leq n-1$.

4.2 if Λ_i test every root and if its equals to zero. Then that is root.

4.3 Flip the bit values at root positions.

4.4 Let the Received polynomial be $r(x)$ i.e which contains both transmitted codeword polynomial $c(x)$ and the error polynomial $e(x)$

$$e(x) = e_0 + e_1 x^1 + ... + e_{n-1}x^{n-1} \tag{4.1}$$

Where e_i represents the value of the error at the location. For binary BCH codes e_i is either

0 or 1.

$$r(x) = c(x)+e(x) = r_{n-1}x^{n-1}+r_{n-2}x^{n-2}+\cdots+r_1x+r_0 \tag{4.2}$$

Definie, Syndrome

$$S_i = r(\alpha^i) = c(\alpha^i) + e(\alpha^i) = e(\alpha^i) \tag{4.3}$$

Where α^i is a root of the codeword.
Suppose that v errors occurred, and $0 \leq v \leq t$.
Let the error occurs at $i_1, i_2, \ldots, i_v$.
The Decoding Process, for a t-error correcting code will follows the basic steps,

4.5 Compute the Syndrome $S = (S_1, S_2, \ldots, S_{2t})$ from the received polynomial $r(x)$

4.6 Determine the error-location polynomial $\sigma(x)$ from the syndrome components $S_1, S_2, \ldots, S_{2t}$ using the Berlekamp's Algorithm.

4.7 Using the Chain searching algorithm, determine the error-locations by finding the roots of $\sigma(x)$, the flip the posisions in $r(x)$. Which is the estimated message vector polynomial $\hat{c}(x)$.

4.8 where $\mathbf{w}$ is $p \times 1$ and $\mathbf{X}$ is $N \times p$. Show that

$$E\left(\hat{\mathbf{w}}\right) = \mathbf{w} \tag{4.4}$$

4.9 If the covariance matrix of $\mathbf{y}$ is

$$\mathbf{C_y} = \sigma^2\mathbf{I} \tag{4.5}$$

show that

$$\mathbf{C_w} = \sigma^2\left(\mathbf{X}^T\mathbf{X}\right)^{-1} \tag{4.6}$$

4.10 Let

$$\hat{\mathbf{y}} = \mathbf{X}\hat{\mathbf{w}} \tag{4.7}$$

$$\hat{\sigma}^2 = \frac{1}{N-p}\|\mathbf{y} - \hat{\mathbf{y}}\|^2 \tag{4.8}$$

$$\mathbf{y} - \hat{\mathbf{y}} \sim \mathcal{N}\left(\mathbf{0}, \sigma^2\mathbf{I}\right) \tag{4.9}$$

Show that

$$(N-p)\,\hat{\sigma}^2 \sim \sigma^2\chi^2_{N-p} \tag{4.10}$$

4.11 Let

$$\hat{z} = \frac{w_j}{\hat{\sigma}\sqrt{v_j}} \tag{4.11}$$

where v_j is the diagonal element of $\left(\mathbf{X}^T\mathbf{X}\right)^{-1}$. If $w_j = 0$, show that z_j has a t_{N-p} distribution.

4.12 Plot $\Pr\left(|Z| > z\right)$ for t_{30}, t_{100} and the standard normal distribution.

5 Applications

5.1 Explain how (**??**) can be used to obtain the Nearest Neighbour approximation.

5.2 Repeat the exercise for the least squares method.