MA 1010 - Calculus

Problem Sheet 3: Continuity of Single Variable Functions

1. Show that the following functions are continuous by using $\varepsilon - \delta$ definition.
(i) x^n (ii) $\sin x$ (iii) e^{2x} (iv) $\ln x$
(v) $\sqrt{x}$ (vi) $\sin^{-1}(x)$ (vii) $x^2 + x$ (viii) $\dfrac{x+1}{x-2}$, $x \neq 2$
(ix) $\sinh 4x$ (x) $\ln(\sin x)$ (xi) $\sin(2x) + x^2$ (xii) $(x+2)^3$

2. Investigate the continuity of each of the following functions at the indicated point x_0.
(i) $f(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}; x_0 = 0$. (ii) $f(x) = x - |x|; x_0 = 0$.
(iii) $f(x) = \begin{cases} \dfrac{x^3 - 8}{x^2 - 4}, & x \neq 2, \\ 3, & x = 2 \end{cases}; x_0 = 2$. (iv) $f(x) = \begin{cases} \sin \pi x, & 0 < x < 1 \\ \ln x, & 1 < x < 2 \end{cases}; x_0 = 1.$

3. Give examples of functions with the following properties:
   (a) A function f which is continuous at *only* a finite number of points.
   (b) A function f which is discontinuous at *only* a finite number of points.
   (c) An f which is nowhere continuous on the real line.
   (d) An f which is continuous at every rational number.
   (e) An f which is discontinuous at every rational and continuous at every irrational on $(0, \infty)$.
   (f) A $c \in \mathbb{R}$ and two functions f, g that are discontinuous at c but such that $f + g$ and fg are continuous at c.

4. Prove that any polynomial of finite degree over $\mathbb{R}$ is continuous. Hence show that rational functions are continuous over their admissible domain.

5. Let f, g be two continuous functions on (a, b) such that $f(x) = g(x)$ for every rational $x \in (a, b)$. Prove that $f(x) = g(x)$ for all $x \in (a, b)$.

6. Determine the domain of continuity of the following functions.
(i) $\sqrt{1 - x^2}$ (ii) $\sin(e^{-x^2})$ (iii) $\ln(1 + \sin x)$
(iv) $\dfrac{1 + \cos x}{3 + \sin x}$ (v) $\dfrac{1}{\sqrt{10 + x}}$ (vi) $\sin \dfrac{1}{(x-1)^2}$
(vii) $\sin\left(\dfrac{1}{\cos x}\right)$ (viii) $\sqrt{(x-3)(6-x)}$ (ix) $\begin{cases} x^2 \sin\left(\dfrac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$
(x) $\sqrt{x + \sqrt{x}}$ (xi) $\cos(\sqrt{1 + x^2})$ (xii) $\dfrac{\sqrt{1 + |\sin x|}}{x}$

7. (a) Prove that $f(x) = \begin{cases} x \sin(\frac{1}{x}), & x \neq 0 \\ 5, & x = 0 \end{cases}$, is not continuous at $x = 0$.
(b) Can you redefine $f(0)$ so that f is continuous at $x = 0$.

8. Let $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ be continuous on $\mathbb{R}$.

   (a) If $f(r) = 0$ for every rational number r, then prove that $f(x) = 0$ for all $x \in \mathbb{R}$.

   (b) If $f(r) = 0$ for every *irrational* number r, then is it true that $f(x) = 0$ on whole of $\mathbb{R}$?

9. Show that every polynomial of odd degree with real coefficients has atleast one real root.

10. Prove that $\displaystyle\lim_{x\to 0} \frac{x^2 \sin\left(\frac{1}{x}\right)}{\sin x} = 0.$

11. Let $I = [a, b]$ and $f : I \to \mathbb{R}$ be a continuous function.

   (a) If $f(x) > 0$ for all $x \in I$, show that there exists an $\alpha > 0$ such that $f(x) \geqslant \alpha$ for all $x \in I$.

   (b) If for each $x \in I$ there exists a $y \in I$ such that $|f(y)| \leqslant \frac{1}{2} f(x)$, then show that there exists a $c \in I$ such that $f(c) = 0$.

   (c) Show that f is bounded on I, i.e., there exists an $M > 0$ such that $|f(x)| < M$ for all $x \in I$. What if the interval $I = (-\infty, b]$ or $I = [a, \infty)$, will f still be bounded ?

   (d) Show that f attains both maximum and minimum values over I, i.e., there exists $p, q \in I$ such that $f(p) \leqslant f(x) \leqslant f(q)$ for all $x \in I$. What happens if either $a = -\infty$ or $b = \infty$?

   (e) Let $I = [0, 1]$ and $f(0) = f(1)$. Prove that there exists a point $c \in [0, \frac{1}{2}]$ such that $f(c) = f\left(c + \frac{1}{2}\right)$.

   (f) Let $I = [0, 1]$ and let $0 \leqslant f(x) \leqslant 1$ for all $x \in I$. Show that there exists a point $c \in I$ such that $f(c) = c$, i.e., f has a fixed point.

   (g) Show that if $f(a) \leqslant a$ and $f(b) \geqslant b$ then f has a fixed point in I, i.e., there exists a point $c \in I$ such that $f(c) = c$.

12. Let $K > 0$ and let $f : \mathbb{R} \to \mathbb{R}$ satisfy the condition $|f(x) - f(y)| \leqslant K|x - y|$ for all $x, y \in \mathbb{R}$. Show that f is continuous at every point $c \in \mathbb{R}$.

13. Define $g : \mathbb{R} \to \mathbb{R}$ as follows: $g(x) = \begin{cases} 2x, & x \in \mathbb{Q} \\ x + 3, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$. Find all the points at which g is continuous.

14. Let $A \subseteq B \subseteq \mathbb{R}$, $f : B \to \mathbb{R}$ and g be the restriction of f to A, i.e., $g(x) = f(x)$ for $x \in A$.

   - If f is continuous at some $c \in A$, then show that g is also continuous at c.
   - Show by example that g is continuous at some $c \in A$ does not necessarily imply that f is continuous at that c.

15. Let $A, B \subseteq \mathbb{R}$, $f : A \to B$ and $g : B \to \mathbb{R}$. If f is continuous on A and g is continuous on B, show that $g \circ f$ is continuous on $f(A)$.

16. While the intermediate value theorem assures you of a root of a continuous function which assumes both negative and positive values on a bounded interval, how do you actually find the roots ? Read up on some such methods, viz., Bisection Method, Newton-Raphson method, etc. Investigate the additional requirements on f for such methods to be applicable.

17. Let $f : [a, b] \to \mathbb{R}$ be integrable on $[a, x]$ for every $x \in [a, b]$ and let the indefinite integral $A(x)$ be defined as
$$A(x) = \int_a^x f(t)dt\ .$$
Show that A is continuous on whole of $[a, b]$. (At each end point we have one-sided continuity.)

MA 1010 - Calculus

Problem Sheet 3: Differentiation

18. Find the derivatives of the following functions from the definition.

(i) $\dfrac{3+x}{3-x}, x \neq 3$ (ii) $\sqrt{2x-1}$ (iii) $\ln(1+\sin x)$

(iv) $\dfrac{1+\cos x}{3+\sin x}$ (v) $\dfrac{1}{\sqrt{10+x}}$ (vi) $\sin \dfrac{1}{(x-1)^2}$

(vii) $\sin\left(\dfrac{1}{\cos x}\right)$ (viii) $\sqrt{(x-3)(6-x)}, 3 \leqslant x \leqslant 6$ (ix) $x^2 \sin(\dfrac{1}{x}), x \neq 0; f(0) = 0$

(x) $\sqrt{x+\sqrt{x}}$ (xi) $\cos(\sqrt{1+x^2})$ (xii) $\dfrac{\sqrt{1+|\sin x|}}{x}$

19. Let $f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$

(a). Is f differentiable at $x = 0$?

(b). Is f' is continuous at $x = 0$?

20. Use L'Hopital's rule to evaluate the following limits.

(i) $\displaystyle\lim_{x \longrightarrow 0} \frac{e^{2x}-1}{x}$ (ii) $\displaystyle\lim_{x \longrightarrow 0} \frac{1+\cos \pi x}{x^2-2x+1}$ (iii) $\displaystyle\lim_{x \longrightarrow \infty} \frac{3x^2-x+5}{5x^2+6x-3}$

(iv) $\displaystyle\lim_{x \longrightarrow \infty} x^2 e^{-x}$ (v) $\displaystyle\lim_{x \longrightarrow 0} (\cos x)^{\frac{1}{x^2}}$ (vi) $\displaystyle\lim_{x \longrightarrow 1} \frac{(2x-x^4)^{1/2}-x^{1/3}}{1-x^{3/4}}$

(vii) $\displaystyle\lim_{x \longrightarrow \infty} \frac{5x^2-3x}{7x^2+1}$ (viii) $\displaystyle\lim_{x \longrightarrow \infty} (x-\sqrt{x+x^2})$ (ix) $\displaystyle\lim_{x \longrightarrow \infty} \frac{\sqrt{x+2}}{\sqrt{x+1}}$