Differentiation

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***Abstract*—This manual discusses problems related to differentiation through examples. Python scripts are provided to supplement the theory.**

Definition 1. Suppose f is a real function and c is a point in its domain.The derivative of f at c is defined by

$$\lim_{h\to 0}\frac{f(c+h)-f(c)}{h} \tag{0.1}$$

provided this limit exists. In general

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \tag{0.2}$$

Problem 1. Find the derivative of the function $f(x) = \sqrt{2x-1}$ from the definition

Proof. Using Definition 1,

$$f'(x)=\lim_{h\to 0}\frac{\sqrt{2(x+h)-1}-\sqrt{2x-1}}{h} \tag{1.1}$$

$$=\lim_{h\to 0}\frac{2(x+h)-1-2x+1}{h(\sqrt{2(x+h)-1}+\sqrt{2x-1})} \tag{1.2}$$

$$=\lim_{h\to 0}\frac{2h}{h(\sqrt{2(x+h)-1}+\sqrt{2x-1})} \tag{1.3}$$

$$f'(x)=\frac{1}{\sqrt{2x-1}} \tag{1.4}$$

The following code plots the derivative of $f(x)$. Fig. 1 verifies that the derivative is indeed $\frac{1}{\sqrt{2x-1}}$.

```
from __future__ import division
import numpy as np
import matplotlib.pyplot as plt
k=1e-5
x = np.linspace(0.6,10,50)
y=[]
for i in x:
        l=np.sqrt(2*(i+k)-1)-np.
           sqrt(2*i-1)
        l=l/k
        y.append(l)
fx=1/np.sqrt(2*x-1)
plt.plot(x,y,'o')
plt.plot(x,fx)
plt.grid()
plt.xlabel('$x$')
plt.ylabel('$f(x)$')
# # # #Comment the following line
plt.savefig('../figs/1.eps')
plt.show()
```

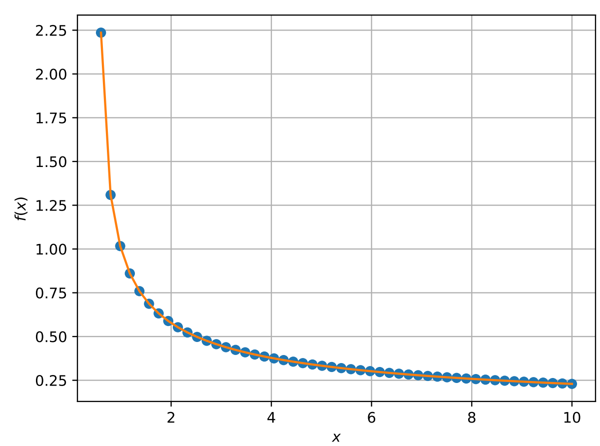

Fig. 1

□

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Problem 2. Let

$$f(x)=\begin{cases} x^2\sin\left(\frac{1}{x}\right) & x\neq 0\\ 0 & x=0\end{cases} \tag{2.1}$$

- Is f differentiable at x=0
- Is f' continuous at x=0

Proof. From Definition 1,

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{h^2 \sin\left(\frac{1}{h}\right) - 0}{h} = 0 \tag{2.2}$$

From eq(2.2) we can conclude that $f(x)$ is differentiable at x=0.
Now,

$$f'(x) = \begin{cases} 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases} \tag{2.3}$$

which is not continuous as the $\cos\left(\frac{1}{x}\right)$ oscillates between -1 and 1 as $x \to 0$ whereas $f(x) = 0$ at $x = 0$.
Thus $f'(x)$ is not continuous at x=0.
Hence $f(x)$ is not continuously differentiable. The following code plots $f(x)$ as $x \to 0$. Fig. 2 verifying that $f'(x)$ is indeed discontinuous at $x = 0$.

```
from __future__ import division
import numpy as np
import matplotlib.pyplot as plt

x = np.linspace(1e-5,1,100)
fx = 2*x*np.sin(1/x)-np.cos(1/x)

plt.plot(x,fx)
plt.grid()
plt.xlabel('$x$')
plt.ylabel('$f(x)$')
plt.ylim([0, 2])
# # # #Comment the following line
plt.savefig('../figs/2.eps')
plt.show()
```

□

Proposition 1. (L'Hospital's Rule)
If f and g are two differentiable functions, and

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{0}{0} \tag{2.4}$$

$$OR \tag{2.5}$$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\infty}{\infty} \tag{2.6}$$

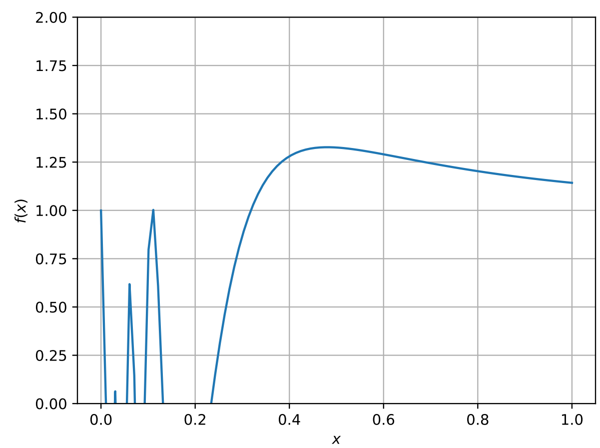

Fig. 2

in these cases we have

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f^n(x)}{g^n(x)} \tag{2.7}$$

as long as the indeterminate form holds and the derivatives exist.

Problem 3. Use L'Hospital's Rule to evaluate $\lim_{x\to 0} \frac{e^{2x}-1}{x}$.

Solution:

$$\lim_{x\to 0} \frac{e^{2x} - 1}{x} = \frac{0}{0} \tag{3.1}$$

By using Proposition 1

$$\lim_{x\to 0} \frac{e^{2x} - 1}{x} = \lim_{x\to 0} \frac{\frac{d(e^{2x}-1)}{dx}}{\frac{x}{dx}} = \lim_{x\to 0} \frac{2e^{2x}}{1} = 2 \tag{3.2}$$

Problem 4. Use L'Hospital's Rule to evaluate $\lim_{x\to 0} (cos\,(x))^{\frac{1}{x^2}}$.

Solution:

$$f(x) = (cos\,(x))^{\frac{1}{x^2}} \implies \lim_{x\to 0} f(x) = 1^{\infty} \tag{4.1}$$

$$g(x) = \log\,(f(x)) = \frac{\log\,(cosx)}{x^2} \tag{4.2}$$

By using Proposition 1

$$\lim_{x\to 0} g(x) = \lim_{x\to 0} \frac{\log\,(cosx)}{x^2} = \lim_{x\to 0} \frac{-tanx}{2x} \tag{4.3}$$

$$= \lim_{x\to 0} \frac{-\,(secx)^2}{2} \implies \lim_{x\to 0} f(x) = e^{-0.5} \tag{4.4}$$

$\therefore$ required limit is $e^{-0.5}$.

Problem 5. Use L'Hospital's Rule to evaluate $\lim_{x\to\infty}(x - \sqrt{x + x^2})$.

Solution: On rationalising $f(x)$

$$\lim_{x\to\infty}(x - \sqrt{x+x^2}) = \lim_{x\to\infty}\frac{(x - \sqrt{x+x^2})(x + \sqrt{x+x^2})}{(x + \sqrt{x+x^2})} \tag{5.1}$$

$$= \lim_{x\to\infty}\frac{-x}{(x + \sqrt{x+x^2})} \tag{5.2}$$

Now,

$lim_{x\to\infty}\frac{-x}{(x+\sqrt{x+x^2})}$ is of the form $\frac{\infty}{\infty}$ By using Proposition 1

$$\lim_{x\to\infty}\frac{-x}{(x + \sqrt{x+x^2})} = \lim_{x\to\infty}\frac{\frac{d(-x)}{dx}}{\frac{d(x+\sqrt{x+x^2})}{dx}} \tag{5.3}$$

$$= \lim_{x\to\infty}\frac{-1}{1 + \frac{1+2x}{2\sqrt{x+x^2}}} = 0 \tag{5.4}$$