Sequences

J. Balasubramaniam† and G V V Sharma*

***Abstract*—This manual covers the properties of sequences through examples. Python scripts are provided for understanding the properties of sequences.**

1 Limit

Problem 1.1. Sketch the following sequence.

$$x_n = \frac{2n}{n + 4\sqrt{n}}, \quad n \geq 0 \tag{1.1}$$

Solution:

```
import numpy as np
import matplotlib.pyplot as plt

n = np.linspace(0,1e9,100)
T_n = (2.0*n)/(n+4.0*np.sqrt(n))
plt.plot(n,T_n)
plt.grid()
plt.xlabel('$n$')
plt.ylabel('$x_n$')
#Comment the following line
#plt.savefig('../figs/seq_converge.eps')
plt.show()
```

Definition 1.1. The sequence x_n converges to a limit L if for $\epsilon > 0$, there exists an integer $K(\epsilon)$ such that for all $n > K(\epsilon)$,

$$|x_n - L| < \epsilon \tag{1.2}$$

Proposition 1.1. Archimedian Property: For any real number x, there exists an integer $n > x$.

Problem 1.2. Guess the value of L for x_n in (1.1) as $n \to \infty$.

† The author is with the Department of Mathematics, IIT Hyderabad. *The author is with the Department of Electrical Engineering, IIT, Hyderabad 502285 India e-mail: {jbala,gadepall}@iith.ac.in.

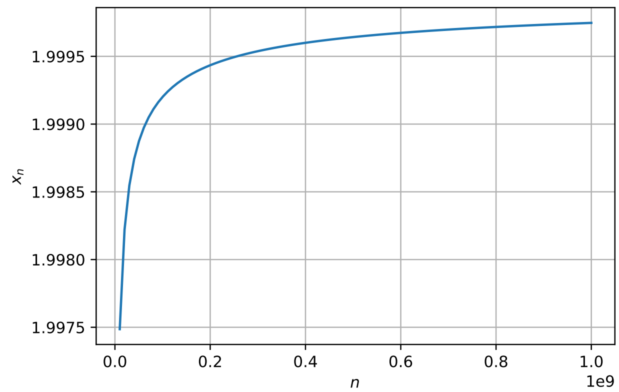

Fig. 1.1

Solution: From Fig. 1.1, $L = 2$.

Problem 1.3. Show that x_n in (1.1) converges to $L = 2$ using Definition 1.1.

Solution:

$$|x_n - 2| = \left| \frac{2n}{n + 4\sqrt{n}} - 2 \right| \tag{1.3}$$

$$= \frac{8\sqrt{n}}{n + 4\sqrt{n}} \tag{1.4}$$

$$< \frac{8\sqrt{n}}{n} = \frac{8}{\sqrt{n}} \tag{1.5}$$

Using Proposition 1.1, choose $K(\epsilon) > \frac{64}{\epsilon^2}$ to be an integer.

$$n > K(\epsilon) \Rightarrow n > \frac{64}{\epsilon^2} \Rightarrow \frac{8}{\sqrt{n}} < \epsilon \tag{1.6}$$

Thus, there exists $K(\epsilon)$ such that $|x_n - 2| < \epsilon$.

Problem 1.4. Let $x_n = \frac{1}{\ln(n+1)}$.

1) Find the value to which x_n converges.
2) Find $K(\epsilon)$ when $\epsilon = \frac{1}{2}$ and $\epsilon = \frac{1}{10}$.

2 Monotonicity and Boundedness

Definition 2.1. A sequence x_n is said to be monotonically *increasing* if

$$x_{n+1} > x_n \tag{2.1}$$

x_n is monotonically *decreasing* if $x_{n+1} < x_n$.

Definition 2.2. The sequence x_n is said to be bounded if for all $n > N$

$$|x_n| < M \tag{2.2}$$

for some positive real number M.

Problem 2.1. Consider the sequence defined by

$$x_n = \begin{cases} 1 & n = 1 \\ \frac{x_{n-1}+1}{3} & n > 1 \end{cases} \tag{2.3}$$

Is the sequence

1) Monotonic?
2) Bounded?

Solution: The following code plots Fig. 2.1. It is obvious from the figure that x_n is both monotonically decreasing as well as bounded.

```
import numpy as np
import matplotlib.pyplot as plt

x = []
temp = 1
for i in range(100):
     x.append(temp)
     temp = (temp+1.0)/3.0

plt.plot(range(100),x)
plt.grid()
plt.xlabel('$n$')
plt.ylabel('$x_n$')
#Comment the following line
#plt.savefig('../figs/seq_monotone
    .eps')
plt.show()
```

Problem 2.2. Prove that x_n is monotonically decreasing.

Proof. From (2.3), it can be shown that

$$x_n = \frac{1}{2}\left(1 + \frac{1}{3^{n-1}}\right) \tag{2.4}$$

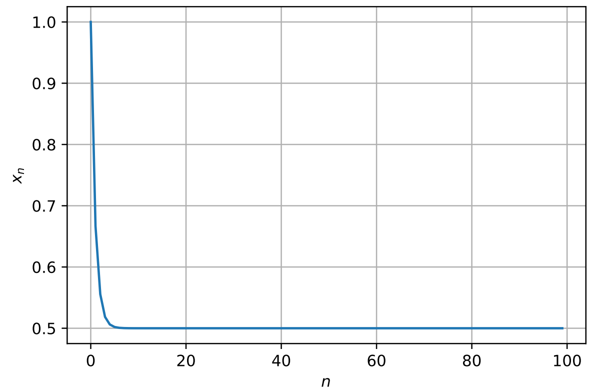

Fig. 2.1

Thus,

$$x_{n-1} - x_n = \frac{2}{3^{n-1}} > 0 \implies x_{n-1} > x_n \tag{2.5}$$

which is the condition for x_n to be monotonically decreasing.

Problem 2.3. Prove that x_n is monotonically decreasing using induction.

Proof. Since $x_1 = 1, x_2 = \frac{2}{3} < x_1$. Let $x_k < x_{k-1}$. Then

$$x_{k+1} - x_k = \frac{x_k + 1}{3} - \frac{x_{k-1} + 1}{3} \tag{2.6}$$

$$= \frac{x_k - x_{k-1}}{3} < 0 \tag{2.7}$$

Thus, $x_k < x_{k-1} \implies x_{k+1} < x_k$. This shows that x_n is decreasing.

Problem 2.4. Show that x_n is bounded.

Proof. From (2.4), it is obvious that

$$|x_n| \le 1 \tag{2.8}$$

Thus, x_n is bounded. □

Problem 2.5. Find the limit of x_n.

Solution: From Fig. 2.1, it is clear that the limit is $\frac{1}{2}$.

Problem 2.6. Show that the limit of x_n is $\frac{1}{2}$.

Problem 2.7. Show that the sequence defined by

$$x_n = \begin{cases} 2 & n = 1 \\ \sqrt{2x_{n-1} + 1} & n > 1 \end{cases} \tag{2.9}$$

is monotone as well as bounded. Find its limit.

Proposition 2.1. Any monotone sequence that is bounded is convergent.

Problem 2.8. Graphically show that $x_n = \sqrt{n+1} - 1$ is divergent.

Solution: The following code results in Fig. 2.8. It is obvious that the series does not converge.

```
import numpy as np
import matplotlib.pyplot as plt

n = np.linspace(0,1e3,100)
x_n = np.sqrt(n+1) - 1
plt.plot(n,x_n)
plt.grid()
plt.xlabel('$n$')
plt.ylabel('$x_n$')
#Comment the following line
#plt.savefig('../figs/seq_diverge.
    eps')
plt.show()
```

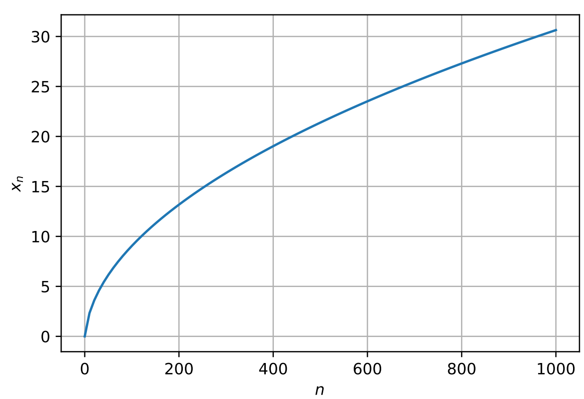

Fig. 2.8

Problem 2.9. Show that x_n in Problem 2.8 is increasing.

Proof. Since $x_n > x_{n-1}$ for all n, the sequence is increasing. □

Problem 2.10. Show that x_n in Problem 2.8 is unbounded.

Proof. For every M, it is possible to find an integer $n > M^2 - 1 \implies \sqrt{n+1} > M$. Thus, x_n is unbounded. □

Proposition 2.2. A monotone unbounded sequence is divergent.

3 Cauchy Sequence

Definition 3.1. The sequence x_n is Cauchy if for every $\epsilon > 0$, there exists an integer N such that

$$|x_m - x_n| < \epsilon \quad \text{whenever } n, m \geq N \tag{3.1}$$

Problem 3.1. Show that

$$x_n = \frac{1}{n^2} \tag{3.2}$$

is a Cauchy sequence.

Proof. Let $m > n > N$.

$$|x_m - x_n| = \left|\frac{1}{m^2} - \frac{1}{n^2}\right| = \left|\frac{(m-n)(m+n)}{m^2 n^2}\right| \tag{3.3}$$

The numerator in (3.3)

$$|(m-n)(m+n)| < 2m^2 \tag{3.4}$$

Thus,

$$|x_m - x_n| < \frac{2}{n^2} < \frac{2}{N^2} < \epsilon \tag{3.5}$$

Thus it is possible to find an N given ϵ for x_n such that x_n is Cauchy.

Proposition 3.1. Every Cauchy sequence is convergent and vice versa.

Problem 3.2. Show that

$$x_n = 1 + \frac{1}{2!} + \frac{1}{3!} \cdots + \frac{1}{n!} \tag{3.6}$$

is a Cauchy sequence.

Problem 3.3. Is $x_n = \sqrt{n}$ a Cauchy sequence?