

Linear Forms

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Abstract—This manual provides solved problems in linear algebra from CBSE Class 10 and 12 board exam papers.

1 VECTORS

1.1. Find the value of k , if the points

$$\mathbf{A} = \begin{pmatrix} k \\ 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 6 \\ -2 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -3 \\ 4 \end{pmatrix} \quad (1.1.1)$$

are collinear.

Solution:

Let

$$\mathbf{M} = (\mathbf{A} - \mathbf{B} \quad \mathbf{B} - \mathbf{C})^T = \begin{pmatrix} k - 6 & 3 - (-2) \\ 6 - (-3) & -2 - 4 \end{pmatrix} \quad (1.1.2)$$

$$= \begin{pmatrix} k - 6 & 5 \\ 9 & -6 \end{pmatrix} \quad (1.1.3)$$

Upon row reduction,

$$\begin{pmatrix} k - 6 & 5 \\ 9 & -6 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 9 & -6 \\ k - 6 & 5 \end{pmatrix} \quad (1.1.4)$$

$$\leftrightarrow \begin{pmatrix} 9 & -6 \\ 0 & 9 + 6k \end{pmatrix} \quad (1.1.5)$$

$$\xrightarrow{R_2 \rightarrow \frac{R_2}{6}} \begin{pmatrix} 9 & -6 \\ 0 & \frac{3}{2} + k \end{pmatrix} \quad (1.1.6)$$

$$\xrightarrow{R_1 \rightarrow \frac{R_1}{3}} \begin{pmatrix} 1 & \frac{-2}{3} \\ 0 & \frac{3}{2} + k \end{pmatrix} \quad (1.1.7)$$

\therefore the points are collinear, $\text{rank}(\mathbf{M}) = 1$. Hence,

$$\frac{3}{2} + k = 0 \quad (1.1.8)$$

$$\Rightarrow k = \frac{-3}{2} \quad (1.1.9)$$

This is verified in Fig. 1.1.1.

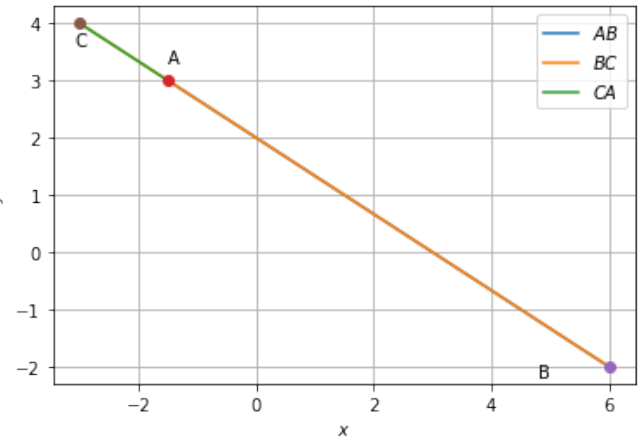


Fig. 1.1.1: Graphical solution

1.2. Find the value of p for which the points

$$\mathbf{A} = \begin{pmatrix} -5 \\ 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ p \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 4 \\ -2 \end{pmatrix} \quad (1.2.1)$$

are collinear

Solution:

\therefore the points are collinear, we create a matrix

$$\mathbf{M} = (\mathbf{B} - \mathbf{A} \quad \mathbf{C} - \mathbf{A})^T \quad (1.2.2)$$

$$= \begin{pmatrix} 1 + 5 & p - 1 \\ 4 + 5 & -2 - 1 \end{pmatrix} \quad (1.2.3)$$

$$= \begin{pmatrix} 6 & p - 1 \\ 9 & -3 \end{pmatrix} \quad (1.2.4)$$

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Now we row reduce the matrix \mathbf{M} ,

$$\begin{pmatrix} 6 & p-1 \\ 9 & -3 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 9 & -3 \\ 6 & p-1 \end{pmatrix} \quad (1.2.5)$$

$$\xrightarrow{R_1 \rightarrow \frac{R_1}{3}} \begin{pmatrix} 3 & -1 \\ 6 & p-1 \end{pmatrix} \quad (1.2.6)$$

$$\xrightarrow{R_2 \rightarrow R_2 - 2R_1} \begin{pmatrix} 3 & -1 \\ 0 & p+1 \end{pmatrix} \quad (1.2.7)$$

$$\xrightarrow{R_1 \rightarrow \frac{R_1}{3}} \begin{pmatrix} 1 & -\frac{1}{3} \\ 0 & p+1 \end{pmatrix} \quad (1.2.8)$$

Since $\text{rank}(\mathbf{M}) = 1$, we have

$$p+1 = 0 \quad (1.2.9)$$

$$\Rightarrow p = -1 \quad (1.2.10)$$

Fig. 1.2.1 verifies that the points are indeed collinear for $p = -1$.

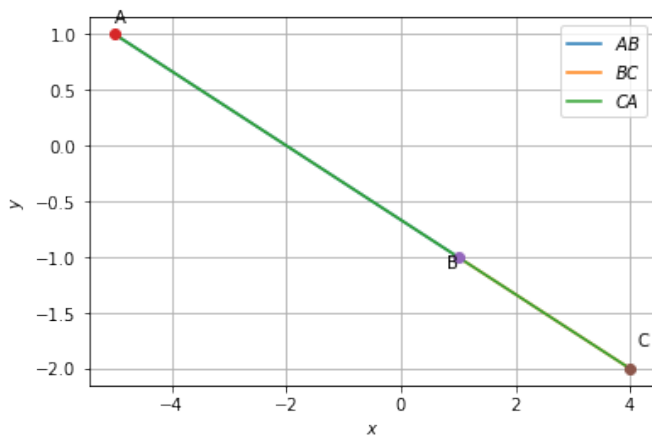


Fig. 1.2.1: Collinear

1.3. Show that the points

$$\mathbf{A} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 3 \\ 8 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} -1 \\ 6 \end{pmatrix} \quad (1.3.1)$$

are the vertices of a square.

Solution:

Lemma 1.3.1. *The diagonals of a square bisect each other at the right angles*

\therefore

$$\frac{\mathbf{A} + \mathbf{C}}{2} = \frac{1}{2} \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ 8 \end{pmatrix} \right\} \quad (1.3.2)$$

$$= \begin{pmatrix} 2 \\ 5 \end{pmatrix} = \frac{\mathbf{B} + \mathbf{D}}{2}, \quad (1.3.3)$$

and

$$(\mathbf{A} - \mathbf{C})^\top (\mathbf{B} - \mathbf{D}) = \begin{pmatrix} -2 & 6 \end{pmatrix} \begin{pmatrix} 6 \\ -2 \end{pmatrix} = 0, \quad (1.3.4)$$

$$\Rightarrow (\mathbf{A} - \mathbf{B})^\top (\mathbf{B} - \mathbf{C}) = 0 \quad (1.3.5)$$

from Lemma 1.3.1, the given points form a square. This is verified in Fig. 1.3.1.

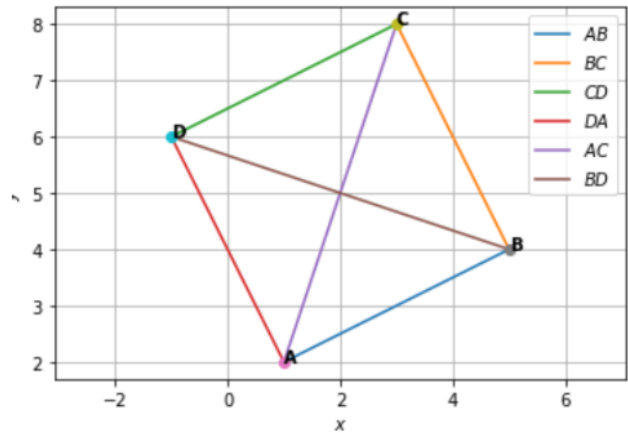


Fig. 1.3.1: Square ABCD

1.4. Represent the following pair of equation graphically and write the coordinates of points where the line is intersect y axis.

$$x + 3y - 6 = 0 \quad (1.4.1)$$

$$2x - 3y - 12 = 0 \quad (1.4.2)$$

Solution: The given lines can be represented in vector form as

$$\begin{pmatrix} 1 & 3 \end{pmatrix} \mathbf{x} = 6 \quad (1.4.3)$$

$$\begin{pmatrix} 2 & -3 \end{pmatrix} \mathbf{x} = 12 \quad (1.4.4)$$

and the equation of y axis is

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 0 \quad (1.4.5)$$

a) Let line (1.4.3) and line (1.4.4) meet at point \mathbf{P} . Then,

$$\begin{pmatrix} 1 & 3 \\ 2 & -3 \end{pmatrix} \mathbf{P} = \begin{pmatrix} 6 \\ 12 \end{pmatrix} \quad (1.4.6)$$

$$\Rightarrow \mathbf{P} = \begin{pmatrix} 1 & 3 \\ 2 & -3 \end{pmatrix}^{-1} \begin{pmatrix} 6 \\ 12 \end{pmatrix} \quad (1.4.7)$$

$$= \begin{pmatrix} 6 \\ 0 \end{pmatrix} \quad (1.4.8)$$

- b) Let line (1.4.3) and line (1.4.5) meet at point **Q**. Then,

$$\begin{pmatrix} 1 & 3 \\ 1 & 0 \end{pmatrix} \mathbf{Q} = \begin{pmatrix} 6 \\ 0 \end{pmatrix} \quad (1.4.9)$$

$$\Rightarrow \mathbf{Q} = \begin{pmatrix} 1 & 3 \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 6 \\ 0 \end{pmatrix} \quad (1.4.10)$$

$$= \begin{pmatrix} 0 \\ 2 \end{pmatrix} \quad (1.4.11)$$

- c) Let line (1.4.4) and line (1.4.5) meet at point **R**. Then,

$$\begin{pmatrix} 2 & -3 \\ 1 & 0 \end{pmatrix} \mathbf{R} = \begin{pmatrix} 12 \\ 0 \end{pmatrix} \quad (1.4.12)$$

$$\Rightarrow \mathbf{R} = \begin{pmatrix} 2 & -3 \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 12 \\ 0 \end{pmatrix} \quad (1.4.13)$$

$$= \begin{pmatrix} 0 \\ -4 \end{pmatrix} \quad (1.4.14)$$

So $\triangle PQR$ is formed by intersection of (1.4.3), (1.4.4) and (1.4.5). See Fig. 1.4.1.

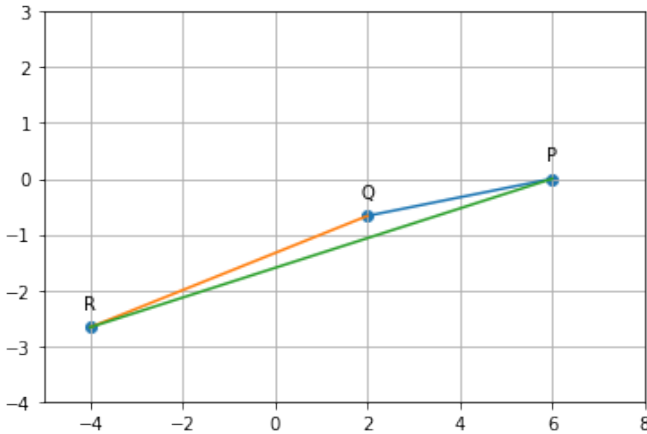


Fig. 1.4.1: Graphical solution

2 LINEAR FORMS

2.1. Draw the graphs of the following equations:

$$(3 \quad -4) \mathbf{x} = -6 \quad (2.1.1)$$

$$(3 \quad 1) \mathbf{x} = 9 \quad (2.1.2)$$

Also determine the co-ordinates of the vertices of the triangle formed by these lines and the x-axis.

Solution:

- a) The intersection of the lines is given by

$$\begin{pmatrix} 3 & -4 \\ 3 & 9 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -6 \\ 9 \end{pmatrix} \quad (2.1.3)$$

for which, the augmented matrix is

$$\begin{pmatrix} 3 & -4 & -6 \\ 3 & 1 & 9 \end{pmatrix} \quad (2.1.4)$$

which can be reduced as

$$\begin{pmatrix} 3 & -4 & -6 \\ 3 & 1 & 9 \end{pmatrix} \xrightarrow[R_1 \leftarrow R_2]{R_2 \leftarrow R_1} \begin{pmatrix} 3 & 1 & 9 \\ 3 & -4 & -6 \end{pmatrix} \quad (2.1.5)$$

$$\xrightarrow{R_1 \leftarrow \frac{R_1}{3}} \begin{pmatrix} 1 & \frac{1}{3} & 3 \\ 3 & -4 & -6 \end{pmatrix} \quad (2.1.6)$$

$$\xrightarrow{R_2 \leftarrow R_2 - 3R_1} \begin{pmatrix} 1 & \frac{1}{3} & 3 \\ 0 & -5 & -15 \end{pmatrix} \quad (2.1.7)$$

$$\xrightarrow{R_2 \leftarrow -\frac{1}{5}R_2} \begin{pmatrix} 1 & \frac{1}{3} & 3 \\ 0 & 1 & 3 \end{pmatrix} \quad (2.1.8)$$

$$\xrightarrow{R_1 \leftarrow R_1 - \frac{1}{3}R_2} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix} \quad (2.1.9)$$

$$\therefore \mathbf{P} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad (2.1.10)$$

is the point of intersection of the lines and the vertex of the triangle formed by the two lines with x-axis as base.

- b) The equation of the x axis is

$$(0 \quad 1) \mathbf{x} = 0 \quad (2.1.11)$$

Thus, the intersection of (2.1.1) with the x axis is given by the set

$$(3 \quad -4) \mathbf{x} = -6 \quad (2.1.12)$$

$$(0 \quad 1) \mathbf{x} = 0 \quad (2.1.13)$$

The augmented matrix for above is

$$\begin{pmatrix} 3 & -4 & -6 \\ 0 & 1 & 0 \end{pmatrix} \quad (2.1.14)$$

which can be reduced as

$$\begin{pmatrix} 3 & -4 & -6 \\ 0 & 1 & 0 \end{pmatrix} \xrightarrow{R_1 \leftarrow -\frac{1}{3}R_1} \begin{pmatrix} 1 & \frac{1}{3} & 2 \\ 0 & 1 & 0 \end{pmatrix} \quad (2.1.15)$$

$$\xrightarrow{R_1 \leftarrow R_1 + \frac{1}{3}R_2} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix} \quad (2.1.16)$$

$$(2.1.17)$$

$$\therefore \mathbf{Q} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \quad (2.1.18)$$

is the point of intersection of the line (2.1.1) with the x axis.

- c) Similarly, the intersection of (2.1.2) with the x axis is given by the set

$$(3 \ 1)\mathbf{x} = 9 \quad (2.1.19)$$

$$(0 \ 1)\mathbf{x} = 0 \quad (2.1.20)$$

with augmented matrix

$$\begin{pmatrix} 3 & 1 & 9 \\ 0 & 1 & 0 \end{pmatrix} \quad (2.1.21)$$

Which can be reduced as

$$\begin{pmatrix} 3 & 1 & 9 \\ 0 & 1 & 0 \end{pmatrix} \xrightarrow{R_1 \leftarrow \frac{1}{3}R_1} \begin{pmatrix} 1 & \frac{1}{3} & 3 \\ 0 & 1 & 0 \end{pmatrix} \quad (2.1.22)$$

$$\xrightarrow{R_1 \leftarrow R_1 - \frac{1}{3}R_2} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \end{pmatrix} \quad (2.1.23)$$

$$(2.1.24)$$

resulting in

$$\mathbf{R} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \quad (2.1.25)$$

as the point of intersection of the line (2.1.2) with the x axis.

These points are then plotted in Fig. 2.1.1 for verification.

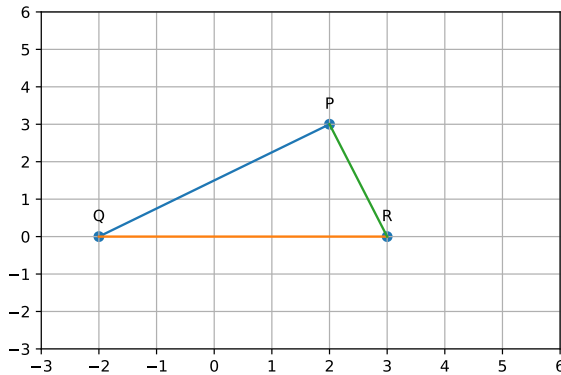


Fig. 2.1.1: Two lines representing given equations meet at point $(2 \ 3)$

the two digits exceeds the given number by 18. Find the number.

Solution: Let the tens digit of the required number be a_1 and the units digit be a_0 . Then

$$a_1 + a_0 = 12 \quad (2.2.1)$$

Which can be expressed as,

$$(1 \ 1)\mathbf{x} = 12 \quad (2.2.2)$$

where

$$\mathbf{x} = \begin{pmatrix} a_1 \\ a_0 \end{pmatrix} \quad (2.2.3)$$

Thus, the desired number is

$$(10a_1 + a_0) = (10 \ 1)\mathbf{x} \quad (2.2.4)$$

and the Number obtained by reversing the digits is

$$(1 \ 10)\mathbf{x} \quad (2.2.5)$$

From the given information,

$$\Rightarrow (10 \ 1)\mathbf{x} - (1 \ 10)\mathbf{x} = 18 \quad (2.2.6)$$

$$\Rightarrow (9 \ -9)\mathbf{x} = 18 \quad (2.2.7)$$

$$\Rightarrow (1 \ -1)\mathbf{x} = 2 \quad (2.2.8)$$

(2.2.2) and (2.2.8) can be expressed as the matrix equation

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 12 \\ 2 \end{pmatrix} \quad (2.2.9)$$

The augmented matrix for the above equation is row reduced as follows

$$\begin{pmatrix} 1 & 1 & 12 \\ -1 & 1 & 2 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 + R_1} \begin{pmatrix} 1 & 1 & 12 \\ 0 & 2 & 14 \end{pmatrix} \quad (2.2.10)$$

$$\xrightarrow{R_2 \leftarrow \frac{1}{2}R_2} \begin{pmatrix} 1 & 1 & 12 \\ 0 & 1 & 7 \end{pmatrix} \quad (2.2.11)$$

$$\xrightarrow{R_1 \leftarrow R_1 - R_2} \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 7 \end{pmatrix} \quad (2.2.12)$$

yielding

$$a_1 = 5, a_0 = 7 \Rightarrow 10a_1 + a_0 = 57 \quad (2.2.13)$$

Fig. 2.2.1 verifies the solution .

2.2. The sum of the digits of a two-digit number is 12. The number obtained by interchanging

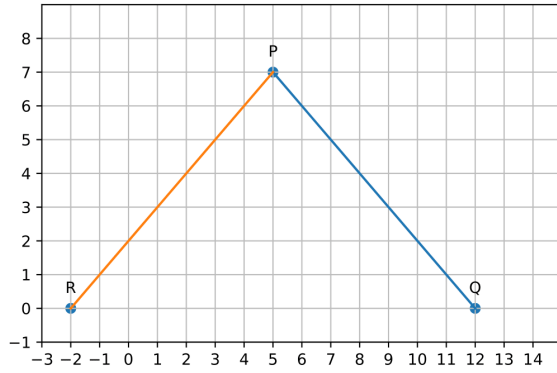


Fig. 2.2.1: Graphical solution

2.3. Solve for x and y :

$$\begin{pmatrix} 47 & 31 \end{pmatrix} \mathbf{x} = 63 \quad (2.3.1)$$

$$\begin{pmatrix} 31 & 47 \end{pmatrix} \mathbf{x} = 15 \quad (2.3.2)$$

Solution:

Both equations are written together in matrix form as

$$\begin{pmatrix} 47 & 31 \\ 31 & 47 \end{pmatrix} \mathbf{Z} = \begin{pmatrix} 63 \\ 15 \end{pmatrix} \quad (2.3.3)$$

Augmented matrix for above is

$$\begin{pmatrix} 47 & 31 & 63 \\ 31 & 47 & 15 \end{pmatrix} \quad (2.3.4)$$

This can be reduced as follows:

$$\begin{pmatrix} 47 & 31 & 63 \\ 31 & 47 & 15 \end{pmatrix} \xrightarrow{R_1 \leftarrow \frac{R_1}{47}} \begin{pmatrix} 1 & \frac{31}{47} & \frac{63}{47} \\ 31 & 47 & 15 \end{pmatrix} \quad (2.3.5)$$

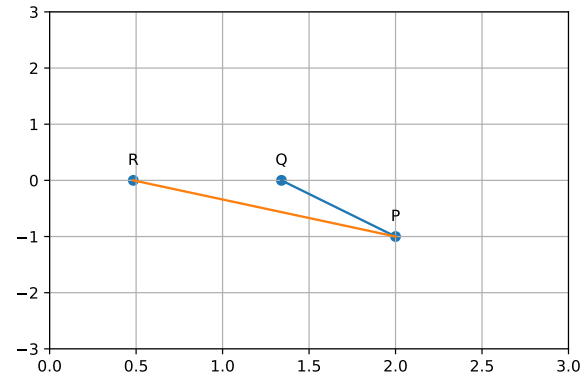
$$\xrightarrow{R_2 \leftarrow R_2 - 31R_1} \begin{pmatrix} 1 & \frac{31}{47} & \frac{63}{47} \\ 0 & \frac{1248}{47} & -\frac{1248}{47} \end{pmatrix} \quad (2.3.6)$$

$$\xrightarrow{R_2 \leftarrow \frac{47}{1248} R_2} \begin{pmatrix} 1 & \frac{31}{47} & \frac{63}{47} \\ 0 & 1 & -1 \end{pmatrix} \quad (2.3.7)$$

$$\xrightarrow{R_1 \leftarrow R_1 - \frac{31}{47} R_2} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{pmatrix} \quad (2.3.8)$$

$$\therefore \mathbf{P} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad (2.3.9)$$

is the desired solution as can be seen from Fig. 2.3.1.

Fig. 2.3.1: Two lines representing given equations meet at point $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$