Linear Forms

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Contents

***Abstract*—This manual provides solved problems in linear algebra from CBSE Class 10 and 12 board exam papers.**

1 Vectors

1.1. Find the coordinates of the point which divides the line joining the points $\mathbf{A} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 \\ 7 \end{pmatrix}$ in the ratio $3:4$
Solution:
The desired point is given by

$$\mathbf{P} = \left(\frac{k\mathbf{B}+\mathbf{A}}{k+1}\right) \tag{1.1.1}$$

$$= \frac{\frac{3}{4}\begin{pmatrix} 2 \\ 7 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \end{pmatrix}}{\frac{3}{4}+1} \tag{1.1.2}$$

$$= \frac{1}{7}\begin{pmatrix} 10 \\ 33 \end{pmatrix} \tag{1.1.3}$$

See Fig. 1.1.1.

1.2. Find the value of k, if the points

$$\mathbf{A} = \begin{pmatrix} k \\ 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 6 \\ -2 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -3 \\ 4 \end{pmatrix} \tag{1.2.1}$$

are collinear.
Solution:
Let

$$\mathbf{M} = \begin{pmatrix} \mathbf{A}-\mathbf{B} & \mathbf{B}-\mathbf{C} \end{pmatrix}^\top = \begin{pmatrix} k-6 & 3-(-2) \\ 6-(-3) & -2-4 \end{pmatrix} \tag{1.2.2}$$

$$= \begin{pmatrix} k-6 & 5 \\ 9 & -6 \end{pmatrix} \tag{1.2.3}$$

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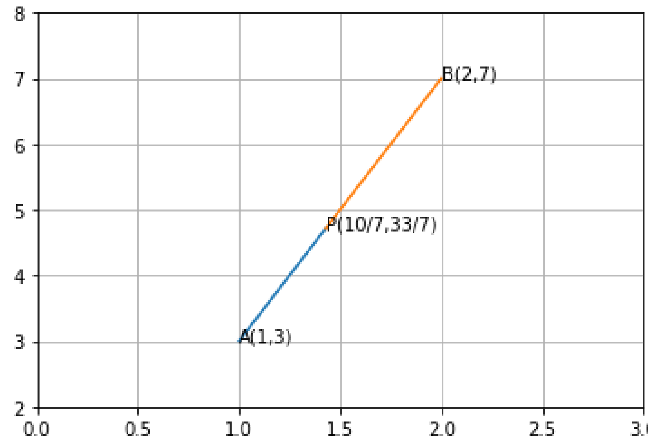

Fig. 1.1.1: Two lines representing given equations meet at point $\begin{pmatrix} 2 & -1 \end{pmatrix}$

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Upon row reduction,

$$\begin{pmatrix} k-6 & 5 \\ 9 & -6 \end{pmatrix} \xleftrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 9 & -6 \\ k-6 & 5 \end{pmatrix} \tag{1.2.4}$$

$$\leftrightarrow \begin{pmatrix} 9 & -6 \\ 0 & 9+6k \end{pmatrix} \tag{1.2.5}$$

$$\xleftrightarrow{R_2 \to \frac{R_2}{6}} \begin{pmatrix} 9 & -6 \\ 0 & \frac{3}{2}+k \end{pmatrix} \tag{1.2.6}$$

$$\xleftrightarrow{R_1 \to \frac{R_1}{3}} \begin{pmatrix} 1 & \frac{-2}{3} \\ 0 & \frac{3}{2}+k \end{pmatrix} \tag{1.2.7}$$

$\because$ the points are collinear, $\text{rank}(\mathbf{M}) = 1$. Hence,

$$\frac{3}{2}+k = 0 \tag{1.2.8}$$

$$\implies k = \frac{-3}{2} \tag{1.2.9}$$

This is verified in Fig. 1.2.1.

1.3. Find the value of p for which the points

$$\mathbf{A} = \begin{pmatrix} -5 \\ 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ p \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 4 \\ -2 \end{pmatrix} \tag{1.3.1}$$

are collinear
Solution:

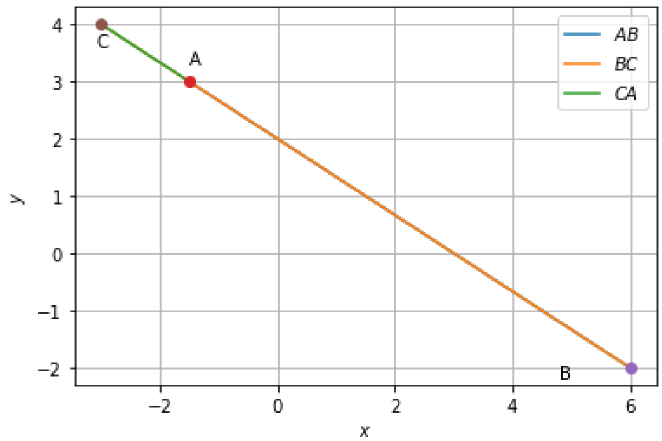

Fig. 1.2.1: Graphical solution

∵ the points are collinear, we create a matrix

$$\mathbf{M} = \begin{pmatrix} \mathbf{B}-\mathbf{A} & \mathbf{C}-\mathbf{A} \end{pmatrix}^\top \tag{1.3.2}$$

$$= \begin{pmatrix} 1+5 & p-1 \\ 4+5 & -2-1 \end{pmatrix} \tag{1.3.3}$$

$$= \begin{pmatrix} 6 & p-1 \\ 9 & -3 \end{pmatrix} \tag{1.3.4}$$

Now we row reduce the matrix $\mathbf{M}$,

$$\begin{pmatrix} 6 & p-1 \\ 9 & -3 \end{pmatrix} \xleftrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 9 & -3 \\ 6 & p-1 \end{pmatrix} \tag{1.3.5}$$

$$\xleftrightarrow{R_1 \rightarrow \frac{R_1}{3}} \begin{pmatrix} 3 & -1 \\ 6 & p-1 \end{pmatrix} \tag{1.3.6}$$

$$\xleftrightarrow{R_2 \rightarrow R_2 - 2R_1} \begin{pmatrix} 3 & -1 \\ 0 & p+1 \end{pmatrix} \tag{1.3.7}$$

$$\xleftrightarrow{R_1 \rightarrow \frac{R_1}{3}} \begin{pmatrix} 1 & \frac{-1}{3} \\ 0 & p+1 \end{pmatrix} \tag{1.3.8}$$

Since $rank(\mathbf{M}) = 1$,we have

$$p+1 = 0 \tag{1.3.9}$$

$$\implies p = -1 \tag{1.3.10}$$

Fig. 1.3.1 verifies that the points are indeed collinear for $p = -1$.

1.4. Show that the points

$$\mathbf{A} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 3 \\ 8 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} -1 \\ 6 \end{pmatrix} \tag{1.4.1}$$

are the vertices of a square.
Solution:

Lemma 1.4.1. *The diagonals of a square bisect each other at the right angles*

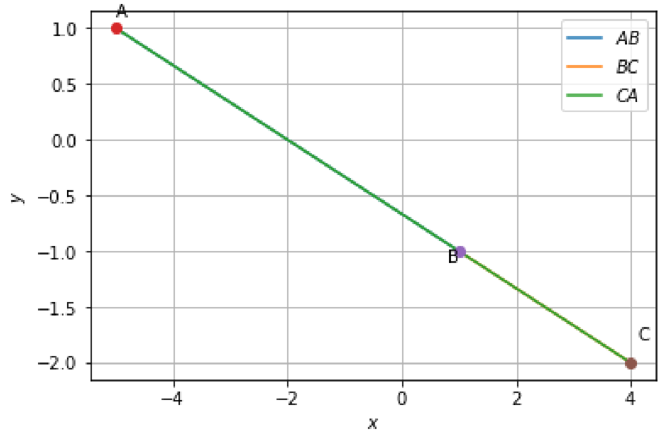

Fig. 1.3.1: Collinear

∵

$$\frac{\mathbf{A}+\mathbf{C}}{2} = \frac{1}{2}\left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ 8 \end{pmatrix}\right\} \tag{1.4.2}$$

$$= \begin{pmatrix} 2 \\ 5 \end{pmatrix} = \frac{\mathbf{B}+\mathbf{D}}{2}, \tag{1.4.3}$$

and

$$(\mathbf{A}-\mathbf{C})^\top(\mathbf{B}-\mathbf{D}) = \begin{pmatrix} -2 & 6 \end{pmatrix}\begin{pmatrix} 6 \\ -2 \end{pmatrix} = 0, \tag{1.4.4}$$

$$\implies (\mathbf{A}-\mathbf{B})^\top(\mathbf{B}-\mathbf{C}) = 0 \tag{1.4.5}$$

from Lemma 1.4.1, the given points form a square. This is verified in Fig. 1.4.1.

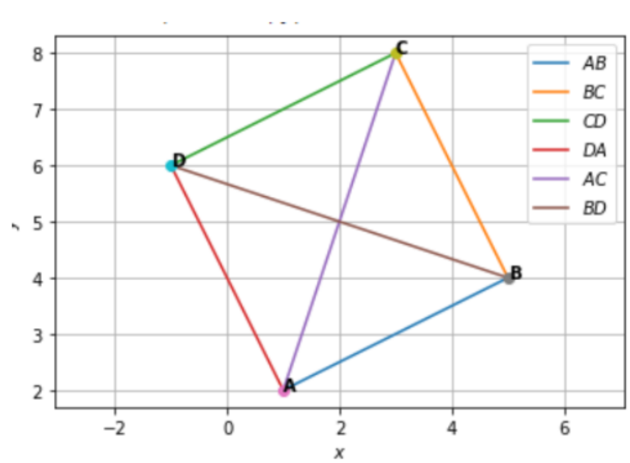

Fig. 1.4.1: Square ABCD

1.5. Represent the following pair of equation graphically and write the coordinates of points where the line is intersect y axis.

$$x + 3y - 6 = 0 \tag{1.5.1}$$
$$2x - 3y - 12 = 0 \tag{1.5.2}$$

Solution: The given lines can be represented in vector form as

$$\begin{pmatrix} 1 & 3 \end{pmatrix} \mathbf{x} = 6 \tag{1.5.3}$$
$$\begin{pmatrix} 2 & -3 \end{pmatrix} \mathbf{x} = 12 \tag{1.5.4}$$

and the equation of y axis is

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 0 \tag{1.5.5}$$

a) Let line (1.5.3) and line (1.5.4) meet at point $\mathbf{P}$. Then,

$$\begin{pmatrix} 1 & 3 \\ 2 & -3 \end{pmatrix} \mathbf{P} = \begin{pmatrix} 6 \\ 12 \end{pmatrix} \tag{1.5.6}$$
$$\implies \mathbf{P} = \begin{pmatrix} 1 & 3 \\ 2 & -3 \end{pmatrix}^{-1} \begin{pmatrix} 6 \\ 12 \end{pmatrix} \tag{1.5.7}$$
$$= \begin{pmatrix} 6 \\ 0 \end{pmatrix} \tag{1.5.8}$$

b) Let line (1.5.3) and line (1.5.5) meet at point $\mathbf{Q}$. Then,

$$\begin{pmatrix} 1 & 3 \\ 1 & 0 \end{pmatrix} \mathbf{Q} = \begin{pmatrix} 6 \\ 0 \end{pmatrix} \tag{1.5.9}$$
$$\implies \mathbf{Q} = \begin{pmatrix} 1 & 3 \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 6 \\ 0 \end{pmatrix} \tag{1.5.10}$$
$$= \begin{pmatrix} 0 \\ 2 \end{pmatrix} \tag{1.5.11}$$

c) Let line (1.5.4) and line (1.5.5) meet at point $\mathbf{R}$. Then,

$$\begin{pmatrix} 2 & -3 \\ 1 & 0 \end{pmatrix} \mathbf{R} = \begin{pmatrix} 12 \\ 0 \end{pmatrix} \tag{1.5.12}$$
$$\implies \mathbf{R} = \begin{pmatrix} 2 & -3 \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 12 \\ 0 \end{pmatrix} \tag{1.5.13}$$
$$= \begin{pmatrix} 0 \\ -4 \end{pmatrix} \tag{1.5.14}$$

So $\triangle PQR$ is formed by intersection of (1.5.3), (1.5.4) and (1.5.5). See Fig. 1.5.1.

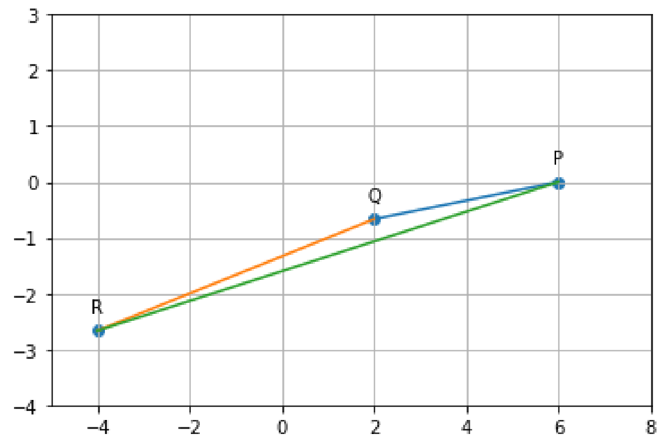

Fig. 1.5.1: Graphical solution

2 Linear Forms

2.1. Draw the graphs of the following equations:

$$\begin{pmatrix} 3 & -4 \end{pmatrix} \mathbf{x} = -6 \tag{2.1.1}$$
$$\begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} = 9 \tag{2.1.2}$$

Also determine the co-ordinates of the vertices of the triangle formed by these lines and the x-axis.

Solution:

a) The intersection of the lines is given by

$$\begin{pmatrix} 3 & -4 \\ 3 & 9 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -6 \\ 9 \end{pmatrix} \tag{2.1.3}$$

for which, the augmented matrix is

$$\begin{pmatrix} 3 & -4 & -6 \\ 3 & 1 & 9 \end{pmatrix} \tag{2.1.4}$$

which can be reduced as

$$\begin{pmatrix} 3 & -4 & -6 \\ 3 & 1 & 9 \end{pmatrix} \xleftrightarrow[R_1 \leftarrow R_2]{R_2 \leftarrow R_1} \begin{pmatrix} 3 & 1 & 9 \\ 3 & -4 & -6 \end{pmatrix} \tag{2.1.5}$$
$$\xleftrightarrow{R_1 \leftarrow \frac{R_1}{3}} \begin{pmatrix} 1 & \frac{1}{3} & 3 \\ 3 & -4 & -6 \end{pmatrix} \tag{2.1.6}$$
$$\xleftrightarrow{R_2 \leftarrow R_2 - 3R_1} \begin{pmatrix} 1 & \frac{1}{3} & 3 \\ 0 & -5 & -15 \end{pmatrix} \tag{2.1.7}$$
$$\xleftrightarrow{R_2 \leftarrow \frac{1}{5} R_2} \begin{pmatrix} 1 & \frac{1}{3} & 3 \\ 0 & 1 & 3 \end{pmatrix} \tag{2.1.8}$$
$$\xleftrightarrow{R_1 \leftarrow R_1 - \frac{1}{3} R_2} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix} \tag{2.1.9}$$

$$\therefore \mathbf{P} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \tag{2.1.10}$$

is the point of intersection of the lines and the vertex of the triangle formed by the two lines with x-axis as base.

b) The equation of the x axis is

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{2.1.11}$$

Thus, the intersection of (2.1.1) with the x axis is given by the set

$$\begin{pmatrix} 3 & -4 \end{pmatrix} \mathbf{x} = -6 \tag{2.1.12}$$

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{2.1.13}$$

The augmented matrix for above is

$$\begin{pmatrix} 3 & -4 & -6 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.1.14}$$

which can be reduced as

$$\begin{pmatrix} 3 & -4 & -6 \\ 0 & 1 & 0 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{1}{3}R_1} \begin{pmatrix} 1 & \frac{1}{3} & 3 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.1.15}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + \frac{4}{3}R_2} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.1.16}$$

$$\tag{2.1.17}$$

$$\therefore \mathbf{Q} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \tag{2.1.18}$$

is the point of intersection of the line (2.1.1) with the x axis.

c) Similarly, the intersection of (2.1.2) with the x axis is given by the set

$$\begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} = 9 \tag{2.1.19}$$

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{2.1.20}$$

with augmented matrix

$$\begin{pmatrix} 3 & 1 & 9 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.1.21}$$

Twhich can be reduced as

$$\begin{pmatrix} 3 & 1 & 9 \\ 0 & 1 & 0 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{1}{3}R_1} \begin{pmatrix} 1 & \frac{1}{3} & 3 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.1.22}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - \frac{1}{3}R_2} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.1.23}$$

$$\tag{2.1.24}$$

resulting in

$$\mathbf{R} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \tag{2.1.25}$$

as the point of intersection of the line (2.1.2) with the x axis.

These points are then plotted in Fig. 2.1.1 for verification.

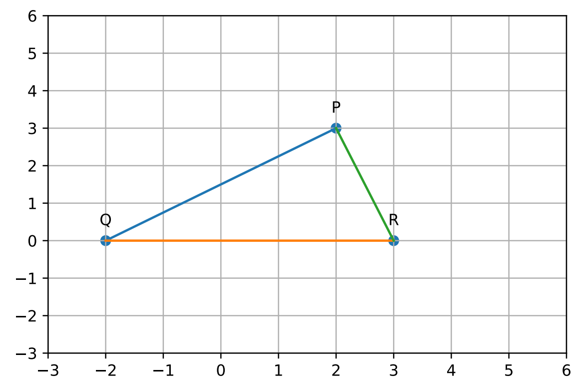

Fig. 2.1.1: Two lines representing given equations meet at point $\begin{pmatrix} 2 & 3 \end{pmatrix}$

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2.2. The sum of the digits of a two-digit number is 12. The number obtained by interchanging the two digits exceeds the given number by 18. Find the number.

Solution: Let the tens digit of the required number be a_1 and the units digit be a_0. Then

$$a_1 + a_0 = 12 \tag{2.2.1}$$

Which can be expressed as,

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 12 \tag{2.2.2}$$

where

$$\mathbf{x} = \begin{pmatrix} a_1 \\ a_0 \end{pmatrix} \tag{2.2.3}$$

Thus, the desired number is

$$(10a_1 + a_0) = \begin{pmatrix} 10 & 1 \end{pmatrix} \mathbf{x} \tag{2.2.4}$$

and the Number obtained by reversing the digits is

$$\begin{pmatrix} 1 & 10 \end{pmatrix} \mathbf{x} \tag{2.2.5}$$

From the given information,

$$\implies \begin{pmatrix}10 & 1\end{pmatrix}\mathbf{x} - \begin{pmatrix}1 & 10\end{pmatrix}\mathbf{x} = 18 \tag{2.2.6}$$

$$\implies \begin{pmatrix}9 & -9\end{pmatrix}\mathbf{x} = 18 \tag{2.2.7}$$

$$\implies \begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = 2 \tag{2.2.8}$$

(2.2.2)and (2.2.8) can be expressed as the matrix equation

$$\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}12\\ 2\end{pmatrix} \tag{2.2.9}$$

The augmented matrix for the above equation is row reduced as follows

$$\begin{pmatrix}1 & 1 & 12\\ -1 & 1 & 2\end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 + R_1} \begin{pmatrix}1 & 1 & 12\\ 0 & 2 & 14\end{pmatrix} \tag{2.2.10}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{1}{2}R_2} \begin{pmatrix}1 & 1 & 12\\ 0 & 1 & 7\end{pmatrix} \tag{2.2.11}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - R_2} \begin{pmatrix}1 & 0 & 5\\ 0 & 1 & 7\end{pmatrix} \tag{2.2.12}$$

yielding

$$a_1 = 5, a_0 = 7 \implies 10a_1 + a_0 \quad = 57 \tag{2.2.13}$$

Fig. 2.2.1 verifies the solution .

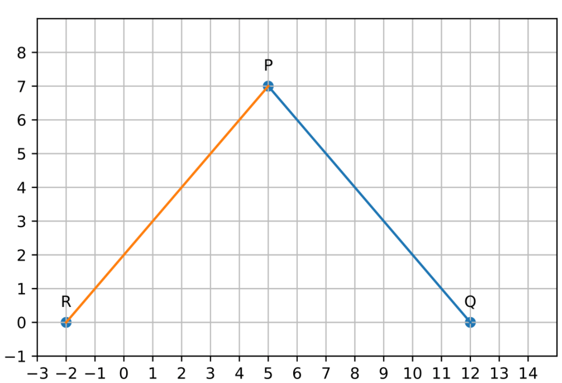

Fig. 2.2.1: Graphical solution

2.3. Solve for x and y:

$$\begin{pmatrix}47 & 31\end{pmatrix}\mathbf{x} = 63 \tag{2.3.1}$$

$$\begin{pmatrix}31 & 47\end{pmatrix}\mathbf{x} = 15 \tag{2.3.2}$$

Solution:

2.4. Find a relation between x and y if the points $\mathbf{A} = \begin{pmatrix}x\\ y\end{pmatrix}, \mathbf{B} = \begin{pmatrix}1\\ 2\end{pmatrix} and \mathbf{C} = \begin{pmatrix}7\\ 0\end{pmatrix}$ are collinear.

Solution:
Let

$$\mathbf{x} = \begin{pmatrix}x\\ y\end{pmatrix}, \mathbf{B} = \begin{pmatrix}1\\ 2\end{pmatrix} and \mathbf{C} = \begin{pmatrix}7\\ 0\end{pmatrix} \tag{2.4.1}$$

Then,

$$\mathbf{m} = \mathbf{B} - \mathbf{C} = \begin{pmatrix}-6\\ 2\end{pmatrix}, \tag{2.4.2}$$

and

$$\mathbf{n}^T\mathbf{m} = 0 \tag{2.4.3}$$

$$\implies \mathbf{n}^T\begin{pmatrix}-6\\ 2\end{pmatrix} = 0 \implies \mathbf{n}^T = \begin{pmatrix}2 & 6\end{pmatrix} \tag{2.4.4}$$

Equation of line is given by

$$\mathbf{n}^T(\mathbf{x} - \mathbf{B}) = 0 \tag{2.4.5}$$

$$\implies \mathbf{n}^T\left(\mathbf{x} - \begin{pmatrix}1\\ 2\end{pmatrix}\right) = 0 \tag{2.4.6}$$

$$\begin{pmatrix}2 & 6\end{pmatrix}\left(\mathbf{x} - \begin{pmatrix}1\\ 2\end{pmatrix}\right) = 0 \tag{2.4.7}$$

$$\begin{pmatrix}2 & 6\end{pmatrix}\mathbf{x} - \begin{pmatrix}2 & 6\end{pmatrix}\begin{pmatrix}1\\ 2\end{pmatrix} = 0 \tag{2.4.8}$$

$$\begin{pmatrix}2 & 6\end{pmatrix}\mathbf{x} = 14 \tag{2.4.9}$$

is the equation of the desired line. See Fig. 2.4.1

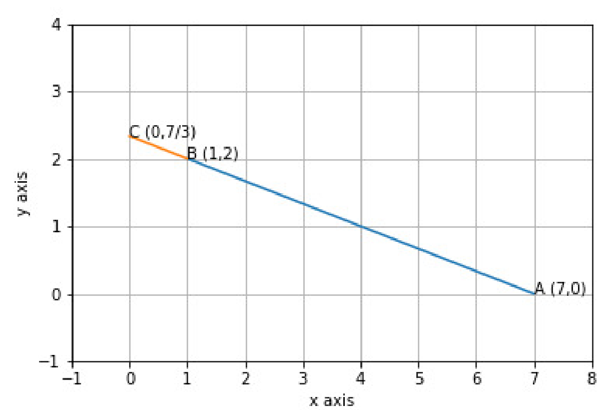

Fig. 2.4.1: Three points $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are collinear

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