Solutions: Linear Algebra by Hoffman and Kunze

G V V Sharma*

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***Abstract*—This book provides solutions to the Linear Algebra book by Hoffman and Kunze.**

1 Linear Equations

1.1 Fields and Linear Equations

1.1.1. Verify that the set of complex numbers numbers described in the form of c where x and y are rational is a sub-field of C.

Solution: Lets consider the set S = $\{x + y\sqrt{2}, x, y \in Q\}$, $S \subset C$ We must verify that S meets the following two conditions:

$$0, 1 \in S \tag{1.1.1.1}$$

$$a, b \in S, a + b, -a, ab, a^{-1} \in S \tag{1.1.1.2}$$

Throughout let

$$a = x + y\sqrt{2}, b = w + z\sqrt{2} \tag{1.1.1.3}$$

If

a)

$$x = 0, y = 0 \in Q, a = 0 + \sqrt{2}.0 = 0, 0 \in S \tag{1.1.1.4}$$

b)

$$x = 1, y = 0, a = 1 + \sqrt{2}.0 = 1, 1 \in S \tag{1.1.1.5}$$

c)

$$a + b = x + y\sqrt{2} + w + z\sqrt{2} = b + a \tag{1.1.1.6}$$

d)

$$-a = -x - y\sqrt{2}, x, y \in Qso - x, -y \in Q, a \in S \tag{1.1.1.7}$$

e)

$$ab = (x + y\sqrt{2})(w + z\sqrt{2}) = ba, ab \in S \tag{1.1.1.8}$$

f)

$$a^{-1}a = (x + y\sqrt{2})^{-1}(x + y\sqrt{2}) = 1, a^{-1} \in S \tag{1.1.1.9}$$

Hence (1.1.1.1) ,(1.1.1.2) is verfied . Therefore by considering the (1.1.1.1) and (1.1.1.2) we can say set complex numbers of given form $x + y\sqrt{2}$ is subfield of C .

1.1.2. Let $\mathbb{F}$ be a set which contains exactly two elements,0 and 1.Define an addition and mul-

*The author is with the Department of Electrical Engineering, Indian Institute of Technology, Hyderabad 502285 India e-mail: gadepall@iith.ac.in. All content in this manual is released under GNU GPL. Free and open source.

tiplication by tables. Verify that the set $\mathbb{F}$, together with these two operations, is a field.

+	0	1
0	0	1
1	1	0

·	0	1
0	0	0
1	0	1

Solution:
To prove that ($\mathbb{F}$,+,·) is a field we need to satisfy the following,

a) + and · should be closed
   - For any a and b in $\mathbb{F}$, a+b $\in \mathbb{F}$ and a · b $\in \mathbb{F}$.For example 0+0=0 and 0·0=0.

b) + and · should be commutative
   - For any a and b in $\mathbb{F}$, a+b = b+a and a · b = b· a. For example 0+1=1+0 and 0· 1=1· 0.

c) + and · should be associative
   - For any a and b in $\mathbb{F}$, a+(b+c) = (a+b)+c and a· (b · c) = (a · b) · c. For example 0+(1+0)=(0+1)+0 and 0·(1·0)=(0·1)·0.

d) + and · operations should have an identity element
   - If we perform a + 0 then for any value of a from $\mathbb{F}$ the result will be a itself. Hence 0 is an identity element of + operation.If we perform a · 1 then for any value of a from $\mathbb{F}$ the result will be a itself. Hence 1 is an identity element of · operation.

e) ∀ a $\in \mathbb{F}$ there exists an additive inverse
   - For additive inverse to exist, ∀ a in $\mathbb{F}$ a+(-a)=0. For example. 1-1=0 and 0-0=0.

f) ∀ a $\in \mathbb{F}$ such that a is non zero there exists a multiplicative inverse
   - For multiplicative inverse to exist, ∀ a such that a is non zero in $\mathbb{F}$, $a \cdot a^{-1}$=1.For example $1 \cdot 1^{-1} = 1$.

g) + and · should hold distributive property
   - For any a,b and c in $\mathbb{F}$ the property a·(b+c)=a·b+a·c should always hold true.For example 0·(1+1)=0·1+0·1.

Since the above properties are satisfied we can say that ($\mathbb{F}$,+,·) is a field.

1.1.3. Prove that each subfield of the field of complex number contains every rational number

Solution:
Complex Numbers: A complex number is a number that can be expressed in the form $a + bi$, where a and b are real numbers, and i represents the imaginary unit, satisfying the equation $i^2 = -1$.The set of complex numbers is denoted by $\mathbb{C}$

$$\mathbb{C} = \{(a, b) : a, b \in \mathbb{R}\} \tag{1.1.3.1}$$

Rational Numbers: A number in the form $\frac{p}{q}$, where both p and q(non-zero) are integers, is called a rational number.The set of rational numbers is dentoed by $\mathbb{Q}$ Let $\mathbb{Q}$ be the set of rational numbers.

$$\mathbb{Q} = \left\{ \frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{Z}_{\neq 0} \right\} \tag{1.1.3.2}$$

Let $\mathbb{C}$ be the field of complex numbers and given $\mathbb{F}$ be the subfield of field of complex numbers $\mathbb{C}$ Since $\mathbb{F}$ is the subfield , we could say that

$$0 \in \mathbb{F} \tag{1.1.3.3}$$
$$1 \in \mathbb{F} \tag{1.1.3.4}$$

Closed under addition: Here $\mathbb{F}$ is closed under addition since it is subfield

$$1 + 1 = 2 \in \mathbb{F} \tag{1.1.3.5}$$
$$1 + 1 + 1 = 3 \in \mathbb{F} \tag{1.1.3.6}$$
$$\vdots$$
$$1 + 1 + \cdots + 1(\text{p times}) = p \in \mathbb{F} \tag{1.1.3.7}$$
$$1 + 1 + \cdots + 1(\text{q times}) = q \in \mathbb{F} \tag{1.1.3.8}$$

By using the above property we could say that zero and other positive integers belongs to $\mathbb{F}$.Since p and q are integers we say,

$$p \in \mathbb{Z} \tag{1.1.3.9}$$
$$q \in \mathbb{Z} \tag{1.1.3.10}$$

Additive Inverse: Let x be the positive integer belong $\mathbb{F}$ and by additive inverse we could say,

$$\forall x \in \mathbb{F} \tag{1.1.3.11}$$
$$(-x) \in \mathbb{F} \tag{1.1.3.12}$$

Therefore field $\mathbb{F}$ contains every integers. Let n be a integer then,

$$n \in \mathbb{Z} \implies n \in \mathbb{F} \tag{1.1.3.13}$$
$$\mathbb{Z} \subseteq \mathbb{F} \tag{1.1.3.14}$$

Where $\mathbb{Z}$ is subset of $\mathbb{F}$ Multiplicative Inverse: Every element except zero in the subfield $\mathbb{F}$ has an multiplicative inverse. From equation (1.1.3.8), since q $\in \mathbb{F}$ we could say ,

$$\frac{1}{q} \in \mathbb{F} \quad \text{and } q \neq 0 \tag{1.1.3.15}$$

Closed under multiplication: Also, $\mathbb{F}$ is closed under multiplication and thus, from equation (1.1.3.7) and (1.1.3.15) we get ,

$$p \cdot \frac{1}{q} \in \mathbb{F} \tag{1.1.3.16}$$
$$\implies \frac{p}{q} \in \mathbb{F} \tag{1.1.3.17}$$

where , $p \in \mathbb{Z}$ and $q \in \mathbb{Z}_{\neq 0}$ (from equation (1.1.3.3) and (1.1.3.15)) Conclusion From (1.1.3.2) and (1.1.3.17) we could say ,

$$\mathbb{Q} \subseteq \mathbb{F} \tag{1.1.3.18}$$

From equation (1.1.3.18) we could say that each subfield of the field of complex number contains every rational number

Hence Proved

1.1.4. Prove that, each field of the characteristic zero contains a copy of the rational number field.
Solution: The characteristic of a field is defined to be the smallest number of times one must use the field's multiplicative identity (1) in a sum to get the additive identity. If this sum never reaches the additive identity (0), then the field is said to have characteristic zero.
Let $\mathbb{Q}$ be the rational number field. Hence,

$$0 \in \mathbb{Q} \qquad \text{[Additive Identity]} \tag{1.1.4.1}$$
$$1 \in \mathbb{Q} \qquad \text{[Multiplicative Identity]} \tag{1.1.4.2}$$

As addition is defined on $\mathbb{Q}$ hence we have,

$$1 \neq 0 \tag{1.1.4.3}$$
$$1 + 1 = 2 \neq 0 \tag{1.1.4.4}$$

And so on,

$$1 + 1 + \cdots + 1 = n \neq 0 \tag{1.1.4.5}$$

From the definition of characteristic of a field and from (1.1.4.3), (1.1.4.4) and so on upto (1.1.4.5), the rational number field, $\mathbb{Q}$ has characteristic 0.

1.2 Matrices and Elementary Row Operations

1.2.1. Find a row-reduced matrix which is row equivalent to,

$$\mathbf{A} = \begin{pmatrix} i & -(1+i) & 0 \\ 1 & -2 & 1 \\ 1 & 2i & -1 \end{pmatrix} \tag{1.2.1.1}$$

Solution: Step 1: Performing scaling operation to matrix **A** as $R_1 \leftarrow \frac{1}{i} R_1$ by scaling matrix D_1 given as,

$$\mathbf{D_1} = \begin{pmatrix} \frac{1}{i} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.2.1.2}$$

$$\mathbf{D_1 A} = \begin{pmatrix} \frac{1}{i} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} i & -(1+i) & 0 \\ 1 & -2 & 1 \\ 1 & 2i & -1 \end{pmatrix} \tag{1.2.1.3}$$

$$\implies \mathbf{D_1 A} = \begin{pmatrix} 1 & -1+i & 0 \\ 1 & -2 & 1 \\ 1 & 2i & -1 \end{pmatrix} \tag{1.2.1.4}$$

Step 2: Performing $R_2 \leftarrow R_2 - R_1$ and $R_3 \leftarrow R_3 - R_1$ given by elementary matrix $\mathbf{E_{31} E_{21}}$ on equation (1.2.1.4),

$$\mathbf{E_{31} E_{21}} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \tag{1.2.1.5}$$

$$\mathbf{E_{31} E_{21} D_1 A} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1+i & 0 \\ 1 & -2 & 1 \\ 1 & 2i & -1 \end{pmatrix} \tag{1.2.1.6}$$

$$\implies \mathbf{A_1} = \mathbf{E_{31} E_{21} D_1 A} = \begin{pmatrix} 1 & -1+i & 0 \\ 0 & -1-i & 1 \\ 0 & 1+i & -1 \end{pmatrix} \tag{1.2.1.7}$$

Step 3: Performing $R_2 \leftarrow \frac{-1}{1+i}R_2$ given by $\mathbf{D_2}$ on equation (1.2.1.7),

$$\mathbf{D_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2}(-1+i) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.2.1.8}$$

$$\mathbf{D_2A_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2}(-1+i) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1+i & 0 \\ 0 & -1-i & 1 \\ 0 & 1+i & -1 \end{pmatrix} \tag{1.2.1.9}$$

$$\implies \mathbf{A_2} = \mathbf{D_2A_1} = \begin{pmatrix} 1 & -1+i & 0 \\ 0 & 1 & \frac{1}{2}(-1+i) \\ 0 & 1+i & -1 \end{pmatrix} \tag{1.2.1.10}$$

Step 4: Performing $R_3 \leftarrow R_3 - (1+i)R_2$ given by $\mathbf{E_{32}}$ on equation (1.2.1.10),

$$\mathbf{E_{32}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -(1+i) & 1 \end{pmatrix} \tag{1.2.1.11}$$

$$\mathbf{E_{32}A_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1-i & 1 \end{pmatrix} \begin{pmatrix} 1 & -1+i & 0 \\ 0 & 1 & \frac{-1+i}{2} \\ 0 & 1+i & -1 \end{pmatrix} \tag{1.2.1.12}$$

$$\implies \mathbf{A_3} = \mathbf{E_{32}A_2} = \begin{pmatrix} 1 & -1+i & 0 \\ 0 & 1 & \frac{-1+i}{2} \\ 0 & 0 & 1 \end{pmatrix} \tag{1.2.1.13}$$

Step 5: Performing $R_1 \leftarrow R_1 - (-1+i)R_2$ given by $\mathbf{E_{12}}$ on equation (1.2.1.13),

$$\mathbf{E_{12}} = \begin{pmatrix} 1 & 1-i & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.2.1.14}$$

$$\mathbf{E_{12}A_3} = \begin{pmatrix} 1 & 1-i & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1+i & 0 \\ 0 & 1 & \frac{-1+i}{2} \\ 0 & 0 & 1 \end{pmatrix} \tag{1.2.1.15}$$

$$\implies \mathbf{A_4} = \mathbf{E_{12}A_3} = \begin{pmatrix} 1 & 0 & i \\ 0 & 1 & \frac{-1+i}{2} \\ 0 & 0 & 1 \end{pmatrix} \tag{1.2.1.16}$$

Step 6: Performing $R_1 \leftarrow R_1 - iR_3$ and $R_2 \leftarrow R_2 - \frac{-1+i}{2}R_3$ given by $\mathbf{E_{13}E_{23}}$ on equation (1.2.1.16),

$$\mathbf{E_{13}E_{23}} = \begin{pmatrix} 1 & 0 & -i \\ 0 & 1 & -\left(\frac{-1+i}{2}\right) \\ 0 & 0 & 1 \end{pmatrix} \tag{1.2.1.17}$$

$$\mathbf{E_{13}E_{23}A_4} = \begin{pmatrix} 1 & 0 & -i \\ 0 & 1 & -\left(\frac{-1+i}{2}\right) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & i \\ 0 & 1 & \frac{-1+i}{2} \\ 0 & 0 & 1 \end{pmatrix} \tag{1.2.1.18}$$

$$\implies \mathbf{A_5} = \mathbf{E_{13}E_{23}A_4} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.2.1.19}$$

$\therefore$ Row-reduced matrix of $\mathbf{A}$ given by equation (1.2.1.1) is,

$$\mathbf{A} = \begin{pmatrix} i & -1-i & 0 \\ 1 & -2 & 1 \\ 1 & 2i & -1 \end{pmatrix} \xleftrightarrow{RREF} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I} \tag{1.2.1.20}$$

1.2.2. Let

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{1.2.2.1}$$

be a 2×2 matrix with complex entries. Suppose A is row-reduced and also that $a+b+c+d = 0$. Prove that there are exactly three such matrices.

Solution: A matrix is in row echelon form if it follows the following conditions
1. All nonzero rows are above any rows of all zeros.
2. Each leading entry (i.e. left most nonzero entry) of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zero Row Reduced Echelon Form A matrix is in row reduced echelon form if it follows the following conditions
1. The matrix should be row echelon form
2. The leading entry in each nonzero row is 1.
3. Each leading 1 is the only nonzero entry in its column. Proof

Given ,

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{1.2.2.2}$$

Condition 1 : Matrix $\mathbf{A}$ should be in row-reduced echelon form

Condition 2 : $a + b + c + d = 0$ where a,b,c and d are the elements of the matrix **A**
Reducing the matrix **A** from equation (1.2.2.2)

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \xleftrightarrow{R_1=\frac{1}{a}R_1} \begin{pmatrix} 1 & \frac{b}{a} \\ c & d \end{pmatrix} \tag{1.2.2.3}$$

$$\xleftrightarrow{R_2=R_2-cR_1} \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & \frac{ad-bc}{a} \end{pmatrix} \tag{1.2.2.4}$$

$$\xleftrightarrow{R_2=\frac{a}{ad-bc}R_2} \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix} \tag{1.2.2.5}$$

$$\xleftrightarrow{R_1=R_1-\frac{b}{a}R_2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.2.2.6}$$

Case 1: Matrix **A** of Rank 2
From the equation (1.2.2.4),for the matrix to be in row reduced echelon form,

$$b = 0$$
$$a \neq 0$$
$$d = 1$$
$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.2.2.7}$$

For the condition 2 to get satisfied,

$$a + 0 + c + 1 = 0 \tag{1.2.2.8}$$
$$\implies a = -(c+1) \tag{1.2.2.9}$$
$$\implies c \neq -1 \tag{1.2.2.10}$$

Both the condition gets satisfied and so exactly one matrix **A** can be formed of Rank 2 with given conditions
Case 2: Matrix **A** of Rank 1
From the equation (1.2.2.4),for the matrix to be in row reduced echelon form,

$$a \neq 0$$
$$d = 0$$
$$c = 0$$

For the condition 2 to get satisfied,

$$a + b + 0 + 0 = 0 \tag{1.2.2.11}$$
$$\implies b = -a \tag{1.2.2.12}$$

$$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \tag{1.2.2.13}$$

Both the condition gets satisfied and so exactly one matrix **A** can be formed of Rank 1 with given conditions

Case 3: Matrix **A** of Rank 0
From equation (1.2.2.2),for the matrix to be in row reduced echelon form,

$$a = 0$$
$$b = 0$$
$$c = 0$$
$$d = 0$$
$$\mathbf{A} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{1.2.2.14}$$

Both the condition gets satisfied and so exactly one matrix **A** can be formed of Rank 0 with given conditions
Therefore matrix **A** shown in equation (1.2.2.7),(1.2.2.13) and (1.2.2.14) are the exactly three such matrices that can be formed with given conditions.

1.2.3. Prove that the interchange of two rows of a matrix can be accomplished by a finite sequence of elementary row operations of the other two types.
Solution: Let **A** be a 3 × 3 matrix with having row vectors $\mathbf{a}_1$,$\mathbf{a}_2$ and $\mathbf{a}_3$.

$$\mathbf{A} = \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} \tag{1.2.3.1}$$

Let's exchange row $\mathbf{a}_1$ and $\mathbf{a}_2$. Let's call this elementary operation $\mathbf{E_1}$.

$$\mathbf{E}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.2.3.2}$$

$$\tag{1.2.3.3}$$

Now performing operation $\mathbf{E}_1$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{a}_2 \\ \mathbf{a}_1 \\ \mathbf{a}_3 \end{pmatrix} \tag{1.2.3.4}$$

Now, to prove that same matrix can be obtained by elementary operations let's call them $\mathbf{E_2}$ and $\mathbf{E_3}$.Now performing operation $\mathbf{E}_2$ by adding row 2 to row 1.

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{a}_1 + \mathbf{a}_2 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} \tag{1.2.3.5}$$

Using elementary operation $\mathbf{E_2}$ we will subtract row 1 from row 2.

$$\begin{pmatrix}1&0&0\\-1&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}\mathbf{a}_1+\mathbf{a}_2\\\mathbf{a}_2\\\mathbf{a}_3\end{pmatrix}=\begin{pmatrix}\mathbf{a}_1+\mathbf{a}_2\\-\mathbf{a}_1\\\mathbf{a}_3\end{pmatrix} \quad (1.2.3.6)$$

Using elementary operation $\mathbf{E_2}$ we will add row 2 to row 1.

$$\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}\mathbf{a}_1+\mathbf{a}_2\\-\mathbf{a}_1\\\mathbf{a}_3\end{pmatrix}=\begin{pmatrix}\mathbf{a}_2\\-\mathbf{a}_1\\\mathbf{a}_3\end{pmatrix} \quad (1.2.3.7)$$

Using elementary operation $\mathbf{E_3}$ we will multiply row 2 by -1.

$$\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}\begin{pmatrix}\mathbf{a}_2\\-\mathbf{a}_1\\\mathbf{a}_3\end{pmatrix}=\begin{pmatrix}\mathbf{a}_2\\\mathbf{a}_1\\\mathbf{a}_3\end{pmatrix} \quad (1.2.3.8)$$

Hence, we can say that,

$$\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\-1&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}\mathbf{a}_1\\\mathbf{a}_2\\\mathbf{a}_3\end{pmatrix}= \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}\mathbf{a}_1\\\mathbf{a}_2\\\mathbf{a}_3\end{pmatrix} \quad (1.2.3.9)$$

Let us assume a matrix $\mathbf{A}$

$$\mathbf{A}=\begin{pmatrix}1&2&3\\0&1&0\\1&1&0\end{pmatrix} \quad (1.2.3.10)$$

Let's exchange row $\mathbf{a}_1$ and $\mathbf{a}_2$ by applying operation $\mathbf{E_1}$.

$$\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&2&3\\0&1&0\\1&1&0\end{pmatrix}=\begin{pmatrix}0&1&0\\1&2&3\\1&1&0\end{pmatrix} \quad (1.2.3.11)$$

Now, to prove that same matrix can be obtained by other two elementary operations.We will first perform elementary operation $\mathbf{E_2}$ by adding row 2 to row 1.

$$\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&2&3\\0&1&0\\1&1&0\end{pmatrix}=\begin{pmatrix}1&3&3\\0&1&0\\1&1&0\end{pmatrix} \quad (1.2.3.12)$$

Using elementary operation $\mathbf{E_2}$ we will subtract row 1 from row 2.

$$\begin{pmatrix}1&0&0\\-1&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&3&3\\0&1&0\\1&1&0\end{pmatrix}=\begin{pmatrix}1&3&3\\-1&-2&-3\\1&1&0\end{pmatrix} \quad (1.2.3.13)$$

Using elementary operation $\mathbf{E_2}$ we will add row 2 to row 1.

$$\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&3&3\\-1&-2&-3\\1&1&0\end{pmatrix}=\begin{pmatrix}0&1&0\\-1&-2&-3\\1&1&0\end{pmatrix} \quad (1.2.3.14)$$

Using elementary operation $\mathbf{E_3}$ we will multiply row 2 by -1.

$$\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}\begin{pmatrix}0&1&0\\-1&-2&-3\\1&1&0\end{pmatrix}=\begin{pmatrix}0&1&0\\1&2&3\\1&1&0\end{pmatrix} \quad (1.2.3.15)$$

Hence,we can say that,

$$\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\-1&1&0\\0&0&1\end{pmatrix} \times\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&2&3\\0&1&0\\1&1&0\end{pmatrix}=\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&2&3\\0&1&0\\1&1&0\end{pmatrix} \quad (1.2.3.16)$$