Solutions: Linear Algebra by Hoffman and Kunze

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***Abstract*—This book provides solutions to the Linear Algebra book by Hoffman and Kunze.**

1 Fields and Linear Equations

1.1. Let $\mathbb{F}$ be a set which contains exactly two elements,0 and 1.Define an addition and multiplication by tables. Verify that the set $\mathbb{F}$,

+	0	1
0	0	1
1	1	0

$\cdot$	0	1
0	0	0
1	0	1

together with these two operations, is a field.
Solution:

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To prove that ($\mathbb{F}$,+,$\cdot$) is a field we need to satisfy the following,

a) + and $\cdot$ should be closed
   - For any a and b in $\mathbb{F}$, a+b $\in \mathbb{F}$ and a $\cdot$ b $\in \mathbb{F}$.For example 0+0=0 and 0$\cdot$0=0.

b) + and $\cdot$ should be commutative
   - For any a and b in $\mathbb{F}$, a+b = b+a and a $\cdot$ b = b$\cdot$ a. For example 0+1=1+0 and 0$\cdot$ 1=1$\cdot$ 0.

c) + and $\cdot$ should be associative
   - For any a and b in $\mathbb{F}$, a+(b+c) = (a+b)+c and a$\cdot$ (b $\cdot$ c) = (a $\cdot$ b) $\cdot$ c. For example 0+(1+0)=(0+1)+0 and 0$\cdot$(1$\cdot$0)=(0$\cdot$1)$\cdot$0.

d) + and $\cdot$ operations should have an identity element
   - If we perform a + 0 then for any value of a from $\mathbb{F}$ the result will be a itself. Hence 0 is an identity element of + operation.If we perform a $\cdot$ 1 then for any value of a from $\mathbb{F}$ the result will be a itself. Hence 1 is an identity element of $\cdot$ operation.

e) $\forall$ a $\in \mathbb{F}$ there exists an additive inverse
   - For additive inverse to exist, $\forall$ a in $\mathbb{F}$ a+(-a)=0. For example. 1-1=0 and 0-0=0.

f) $\forall$ a $\in \mathbb{F}$ such that a is non zero there exists a multiplicative inverse
   - For multiplicative inverse to exist, $\forall$ a such that a is non zero in $\mathbb{F}$, $a \cdot a^{-1}$=1.For example $1 \cdot 1^{-1} = 1$.

g) + and $\cdot$ should hold distributive property
- For any a,b and c in $\mathbb{F}$ the property a$\cdot$(b+c)=a$\cdot$b+a$\cdot$c should always hold true.For example 0$\cdot$(1+1)=0$\cdot$1+0$\cdot$1.

Since the above properties are satisfied we can say that ($\mathbb{F}$,+,$\cdot$) is a field.

2 Matrices and Elementary Row Operations

2.1. Prove that the interchange of two rows of a matrix can be accomplished by a finite sequence of elementary row operations of the other two types.

Solution: Let $\mathbf{A}$ be a 3×3 matrix with having row vectors $\mathbf{a}_1$,$\mathbf{a}_2$ and $\mathbf{a}_3$.

$$\mathbf{A} = \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} \tag{2.1.1}$$

Let's exchange row $\mathbf{a}_1$ and $\mathbf{a}_2$. Let's call this elementary operation $\mathbf{E_1}$.

$$\mathbf{E}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.1.2}$$

$$\tag{2.1.3}$$

Now performing operation $\mathbf{E}_1$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{a}_2 \\ \mathbf{a}_1 \\ \mathbf{a}_3 \end{pmatrix} \tag{2.1.4}$$

Now, to prove that same matrix can be obtained by elementary operations let's call them $\mathbf{E_2}$ and $\mathbf{E_3}$.Now performing operation $\mathbf{E}_2$ by adding row 2 to row 1.

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{a}_1 + \mathbf{a}_2 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} \tag{2.1.5}$$

Using elementary operation $\mathbf{E_2}$ we will subtract row 1 from row 2.

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}_1 + \mathbf{a}_2 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{a}_1 + \mathbf{a}_2 \\ -\mathbf{a}_1 \\ \mathbf{a}_3 \end{pmatrix} \tag{2.1.6}$$

Using elementary operation $\mathbf{E_2}$ we will add row 2 to row 1.

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}_1 + \mathbf{a}_2 \\ -\mathbf{a}_1 \\ \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{a}_2 \\ -\mathbf{a}_1 \\ \mathbf{a}_3 \end{pmatrix} \tag{2.1.7}$$

Using elementary operation $\mathbf{E_3}$ we will multiply row 2 by -1.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}_2 \\ -\mathbf{a}_1 \\ \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{a}_2 \\ \mathbf{a}_1 \\ \mathbf{a}_3 \end{pmatrix} \tag{2.1.8}$$

Hence, we can say that,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} \tag{2.1.9}$$

Let us assume a matrix $\mathbf{A}$

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \tag{2.1.10}$$

Let's exchange row $\mathbf{a}_1$ and $\mathbf{a}_2$ by applying operation $\mathbf{E_1}$.

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & 3 \\ 1 & 1 & 0 \end{pmatrix} \tag{2.1.11}$$

Now, to prove that same matrix can be obtained by other two elementary operations.We will first perform elementary operation $\mathbf{E_2}$ by adding row 2 to row 1.

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \tag{2.1.12}$$

Using elementary operation $\mathbf{E_2}$ we will subtract row 1 from row 2.

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 3 \\ -1 & -2 & -3 \\ 1 & 1 & 0 \end{pmatrix} \tag{2.1.13}$$

Using elementary operation $\mathbf{E_2}$ we will add row 2 to row 1.

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 3 \\ -1 & -2 & -3 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & -2 & -3 \\ 1 & 1 & 0 \end{pmatrix} \tag{2.1.14}$$

Using elementary operation $\mathbf{E_3}$ we will multiply row 2 by -1.

$$\begin{pmatrix}1 & 0 & 0\\0 & -1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1 & 0\\-1 & -2 & -3\\1 & 1 & 0\end{pmatrix}=\begin{pmatrix}0 & 1 & 0\\1 & 2 & 3\\1 & 1 & 0\end{pmatrix} \tag{2.1.15}$$

Hence,we can say that,

$$\begin{pmatrix}1 & 0 & 0\\0 & -1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 1 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\-1 & 1 & 0\\0 & 0 & 1\end{pmatrix}$$
$$\times\begin{pmatrix}1 & 1 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\0 & 1 & 0\\1 & 1 & 0\end{pmatrix}=\begin{pmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\0 & 1 & 0\\1 & 1 & 0\end{pmatrix} \tag{2.1.16}$$