



# Solutions: Linear Algebra by Hoffman and Kunze



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**Abstract—This book provides solutions to the Linear Algebra book by Hoffman and Kunze.**

## 1 LINEAR EQUATIONS

### 1.1 Fields and Linear Equations

1.1.1. Verify that the set of complex numbers numbers described in the form of  $c$  where  $x$  and  $y$  are rational is a sub-field of  $\mathbb{C}$ .

**Solution:** Lets consider the set  $S = \{x + y\sqrt{2}, x, y \in \mathbb{Q}\}$ ,  $S \subset \mathbb{C}$  We must verify that  $S$  meets the following two conditions:

$$0, 1 \in S \quad (1.1.1.1)$$

$$a, b \in S, a + b, -a, ab, a^{-1} \in S \quad (1.1.1.2)$$

Throughout let

$$a = x + y\sqrt{2}, b = w + z\sqrt{2} \quad (1.1.1.3)$$

If

a)

$$x = 0, y = 0 \in \mathbb{Q}, a = 0 + \sqrt{2}.0 = 0, 0 \in S \quad (1.1.1.4)$$

b)

$$x = 1, y = 0, a = 1 + \sqrt{2}.0 = 1, 1 \in S \quad (1.1.1.5)$$

c)

$$a + b = x + y\sqrt{2} + w + z\sqrt{2} = b + a \quad (1.1.1.6)$$

d)

$$-a = -x - y\sqrt{2}, x, y \in \mathbb{Q} \text{ so } -x, -y \in \mathbb{Q}, a \in S \quad (1.1.1.7)$$

e)

$$ab = (x + y\sqrt{2})(w + z\sqrt{2}) = ba, ab \in S \quad (1.1.1.8)$$

f)

$$a^{-1}a = (x + y\sqrt{2})^{-1}(x + y\sqrt{2}) = 1, a^{-1} \in S \quad (1.1.1.9)$$

Hence (1.1.1.1), (1.1.1.2) is verified. Therefore by considering the (1.1.1.1) and (1.1.1.2) we can say set complex numbers of given form  $x + y\sqrt{2}$  is subfield of  $\mathbb{C}$ .

1.1.2. Let  $\mathbb{F}$  be the field of complex numbers. Are the following two systems of linear equations equivalent? If so, express each equation in each

system as a linear combination of the equations in the other system.

$$x_1 - x_2 = 0$$

$$2x_1 + x_2 = 0$$

and

$$3x_1 + x_2 = 0$$

$$x_1 + x_2 = 0$$

**Solution:** The given system of linear equations can be written as,

$$\mathbf{A}\mathbf{x} = 0 \quad (1.1.2.1)$$

$$\Rightarrow \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \mathbf{x} = 0 \quad (1.1.2.2)$$

$$\mathbf{B}\mathbf{x} = 0 \quad (1.1.2.3)$$

$$\Rightarrow \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \mathbf{x} = 0 \quad (1.1.2.4)$$

Now we can obtain  $\mathbf{B}$  from matrix  $\mathbf{A}$  by performing elementary row operations given as,

$$\mathbf{B} = \mathbf{C}\mathbf{A} \quad (1.1.2.5)$$

$$\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} = \mathbf{C} \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \quad (1.1.2.6)$$

where  $\mathbf{C}$  is product of elementary matrices given as,

$$\begin{aligned} \mathbf{C} &= (\mathbf{E}_7\mathbf{E}_6\mathbf{E}_5\mathbf{E}_4\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1) \\ &= \begin{pmatrix} 1 & 0 \\ \frac{1}{3} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{3} & \frac{4}{3} \\ \frac{-1}{3} & \frac{2}{3} \end{pmatrix} \quad (1.1.2.7) \end{aligned}$$

Now, performing elementary operations on the right side of  $\mathbf{A}$  we obtain matrix  $\mathbf{B}$  given as,

$$\mathbf{B} = \mathbf{A}\mathbf{P} \quad (1.1.2.8)$$

$$\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \mathbf{P} \quad (1.1.2.9)$$

where,  $\mathbf{P}$  is product of elementary matrices

given by,

$$\begin{aligned}\mathbf{P} &= (\mathbf{E}_1 \mathbf{E}_2 \mathbf{E}_3 \mathbf{E}_4 \mathbf{E}_5) \\ &= \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{4}{3} & \frac{2}{3} \\ \frac{-5}{3} & \frac{-1}{3} \end{pmatrix} \quad (1.1.2.10)\end{aligned}$$

Similarly,  $\mathbf{A}$  can be obtained from matrix  $\mathbf{B}$  from (1.1.2.5) as,

$$\mathbf{A} = \mathbf{C}^{-1} \mathbf{B} \quad (1.1.2.11)$$

$$\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} = \mathbf{C}^{-1} \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \quad (1.1.2.12)$$

Matrix  $\mathbf{C}$  is product of elementary matrices and hence invertible and is given as,

$$\begin{aligned}\mathbf{C}^{-1} &= (\mathbf{E}_1^{-1} \mathbf{E}_2^{-1} \mathbf{E}_3^{-1} \mathbf{E}_4^{-1} \mathbf{E}_5^{-1} \mathbf{E}_6^{-1} \mathbf{E}_7^{-1}) \\ &= \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{pmatrix} \\ &\quad \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{3}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{3} & 0 \\ -\frac{1}{3} & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -2 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad (1.1.2.13)\end{aligned}$$

Matrix  $\mathbf{A}$  can also be obtained from (1.1.2.8) given as,

$$\mathbf{A} = \mathbf{B} \mathbf{P}^{-1} \quad (1.1.2.14)$$

$$\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \mathbf{P}^{-1} \quad (1.1.2.15)$$

where,

$$\begin{aligned}\mathbf{P}^{-1} &= (\mathbf{E}_5^{-1} \mathbf{E}_4^{-1} \mathbf{E}_3^{-1} \mathbf{E}_2^{-1} \mathbf{E}_1^{-1}) \\ &= \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{-1}{2} & -1 \\ \frac{5}{2} & 2 \end{pmatrix} \quad (1.1.2.16)\end{aligned}$$

Thus (1.1.2.4) can be obtained from (1.1.2.2) by multiplying it with matrix  $\mathbf{C}$ , and by inverse row operations (1.1.2.2) can be obtained back from (1.1.2.4) since  $\mathbf{C}$  is product of elementary matrices and hence invertible.

Thus the two given homogeneous systems are row equivalent.

Now writing equations in matrix-vector form

as,

$$3x_1 + x_2 = \begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} \quad (1.1.2.17)$$

$$\Rightarrow \begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} = \frac{1}{3} \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} + \frac{4}{3} \begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} \quad (1.1.2.18)$$

$$x_1 + x_2 = \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} \quad (1.1.2.19)$$

$$\Rightarrow \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = \frac{-1}{3} \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} + \frac{2}{3} \begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} \quad (1.1.2.20)$$

(1.1.2.18), (1.1.2.20) is same as multiplying  $\mathbf{C}$  with  $\mathbf{A}$  as it takes the linear combination of each rows of matrix  $\mathbf{A}$  i.e, (1.1.2.6)

$$x_1 - x_2 = \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} \quad (1.1.2.21)$$

$$\Rightarrow \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = (1) \begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} + (-2) \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} \quad (1.1.2.22)$$

$$2x_1 + x_2 = \begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} \quad (1.1.2.23)$$

$$\Rightarrow \begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = \frac{1}{2} \begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} + \frac{1}{2} \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} \quad (1.1.2.24)$$

(1.1.2.22), (1.1.2.24) is same as multiplying  $\mathbf{C}^{-1}$  with  $\mathbf{B}$  as it takes the linear combination of each rows of matrix  $\mathbf{B}$  i.e, (1.1.2.12)

Thus each equation in each system can be expressed as a linear combination of the equations in the other system when they are equivalent.

1.1.3. Are the following two systems of linear equations equivalent?

$$\begin{aligned}-x_1 + x_2 + 4x_3 &= 0 \\ x_1 + 3x_2 + 8x_3 &= 0 \\ \frac{1}{2}x_1 + x_2 + \frac{5}{2}x_3 &= 0\end{aligned} \quad (1.1.3.1)$$

**Solution:**

$$\begin{aligned}x_1 - x_3 &= 0 \\ x_2 + 3x_3 &= 0\end{aligned} \quad (1.1.3.2)$$

System of linear equations in (1.1.3.1) can be

expressed in matrix form as,

$$\mathbf{Ax} = 0 \quad (1.1.3.3)$$

$$\begin{pmatrix} -1 & 1 & 4 \\ 1 & 3 & 8 \\ \frac{1}{2} & 1 & \frac{5}{2} \end{pmatrix} \mathbf{x} = 0 \quad (1.1.3.4)$$

System of linear equations in (1.1.3.2) can be expressed in matrix form as,

$$\mathbf{Bx} = 0 \quad (1.1.3.5)$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \end{pmatrix} \mathbf{x} = 0 \quad (1.1.3.6)$$

Two system of linear equations are equivalent if one system can be expressed as a linear combination of other system.

Matrix  $\mathbf{B}$  can be obtained from matrix  $\mathbf{A}$  as,

$$\mathbf{B} = \mathbf{CA} \quad (1.1.3.7)$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \end{pmatrix} = \mathbf{C} \begin{pmatrix} -1 & 1 & 4 \\ 1 & 3 & 8 \\ \frac{1}{2} & 1 & \frac{5}{2} \end{pmatrix} \quad (1.1.3.8)$$

$$\mathbf{C} = \begin{pmatrix} -1 & 1 & -2 \\ \frac{1}{2} & -\frac{1}{2} & 2 \end{pmatrix} \quad (1.1.3.9)$$

Now, writing equations in matrix-vector form,

$$x_1 - x_3 = \begin{pmatrix} 1 & 0 & -1 \end{pmatrix} \mathbf{x}$$

$$\begin{aligned} \Rightarrow (1 \ 0 \ -1) \mathbf{x} &= -1(-1 \ 1 \ 4) \mathbf{x} \\ &+ 1(1 \ 3 \ 8) \mathbf{x} - 2\left(\frac{1}{2} \ 1 \ \frac{5}{2}\right) \mathbf{x} \end{aligned} \quad (1.1.3.10)$$

$$x_2 + 3x_3 = \begin{pmatrix} 0 & 1 & 3 \end{pmatrix} \mathbf{x}$$

$$\begin{aligned} \Rightarrow (0 \ 1 \ 3) \mathbf{x} &= \frac{1}{2}(-1 \ 1 \ 4) \mathbf{x} \\ &- \frac{1}{2}(1 \ 3 \ 8) \mathbf{x} + 2\left(\frac{1}{2} \ 1 \ \frac{5}{2}\right) \mathbf{x} \end{aligned} \quad (1.1.3.11)$$

Equations (1.1.3.10) and (1.1.3.11) is same as multiplying  $\mathbf{C}$  with  $\mathbf{A}$  which is the linear combination of rows of matrix  $\mathbf{A}$ .

Thus each equation in second system can be expressed as linear combination of the equations in first system.

Therefore, the two system of linear equations are equivalent.

1.1.4. Let  $\mathbb{F}$  be the field of complex numbers. Are the following two systems of linear equations

equivalent? If so, express each equation in each system as a linear combination of equations in other system. First system of equations:

$$2x_1 + (-1 + i)x_2 + x_4 = 0 \quad (1.1.4.1)$$

$$3x_2 - 2ix_3 + 5x_4 = 0 \quad (1.1.4.2)$$

The second system of equations:

$$\left(1 + \frac{i}{2}\right)x_1 + 8x_2 - ix_3 - x_4 = 0 \quad (1.1.4.3)$$

$$\frac{2}{3}x_1 - \frac{1}{2}x_2 + x_3 + 7x_4 = 0 \quad (1.1.4.4)$$

**Solution:** Let  $\mathbf{R}_1$  and  $\mathbf{R}_2$  be the reduced row echelon forms of the augmented matrices of the following systems of homogeneous equations respectively.

$$\mathbf{AX} = \mathbf{0} \quad (1.1.4.5)$$

$$\mathbf{BX} = \mathbf{0} \quad (1.1.4.6)$$

Where  $\mathbf{A}$  and  $\mathbf{B}$  as follows

$$\mathbf{A} = \begin{pmatrix} 2 & -1 + i & 0 & 1 \\ 0 & 3 & -2i & 5 \end{pmatrix} \quad (1.1.4.7)$$

$$\mathbf{B} = \begin{pmatrix} 1 + \frac{i}{2} & 8 & -i & -1 \\ \frac{2}{3} & -\frac{1}{2} & 1 & 7 \end{pmatrix} \quad (1.1.4.8)$$

On performing elementary row operations on (1.1.4.7),

$$\mathbf{R}_1 = \mathbf{CA} \quad (1.1.4.9)$$

where  $\mathbf{C}$  is the product of all elementary matrices. Reducing the first system of linear equations, we get,

$$\mathbf{C} = \begin{pmatrix} 1 & \frac{1-i}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} \quad (1.1.4.10)$$

$$\mathbf{R}_1 = \begin{pmatrix} 1 & 0 & \frac{-1-i}{3} & \frac{4}{3} - \frac{5i}{6} \\ 0 & 1 & \frac{-2i}{3} & \frac{5}{3} \end{pmatrix} \quad (1.1.4.11)$$

On performing elementary row operations on (1.1.4.8),

$$\mathbf{R}_2 = \mathbf{DA} \quad (1.1.4.12)$$

where  $\mathbf{D}$  is the product of all elementary matrices. Reducing the second system of linear equations, we get,

$$\mathbf{D} = \begin{pmatrix} \frac{4}{5}(1 - \frac{i}{2}) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{-2}{3} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{-6(143+43i)}{4909} \end{pmatrix} \begin{pmatrix} 1 & \frac{16(-2+i)}{5} \\ 0 & 1 \end{pmatrix} \quad (1.1.4.13)$$

$$\mathbf{R}_2 = \begin{pmatrix} 1 & 0 & \frac{6702}{4909} - \frac{708i}{4909} & \frac{46620}{4909} - \frac{1998i}{4909} \\ 0 & 1 & \frac{-2(441+472i)}{4909} & \frac{-2(3283+1332i)}{4909} \end{pmatrix} \quad (1.1.4.14)$$

From the equations (1.1.4.11) and (1.1.4.14), we can say that

$$\mathbf{R}_1 \neq \mathbf{R}_2 \quad (1.1.4.15)$$

Hence the given systems of linear equations are not equivalent.

1.1.5. Let  $\mathbb{F}$  be a set which contains exactly two elements, 0 and 1. Define an addition and multiplication by tables. Verify that the set  $\mathbb{F}$ ,

$$\begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array}$$

$$\begin{array}{c|cc} \cdot & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array}$$

together with these two operations, is a field.

**Solution:**

To prove that  $(\mathbb{F}, +, \cdot)$  is a field we need to satisfy the following,

a)  $+$  and  $\cdot$  should be closed

- For any  $a$  and  $b$  in  $\mathbb{F}$ ,  $a+b \in \mathbb{F}$  and  $a \cdot b \in \mathbb{F}$ . For example  $0+0=0$  and  $0 \cdot 0=0$ .

b)  $+$  and  $\cdot$  should be commutative

- For any  $a$  and  $b$  in  $\mathbb{F}$ ,  $a+b = b+a$  and  $a \cdot b = b \cdot a$ . For example  $0+1=1+0$  and  $0 \cdot 1=1 \cdot 0$ .

c)  $+$  and  $\cdot$  should be associative

- For any  $a$  and  $b$  in  $\mathbb{F}$ ,  $a+(b+c) = (a+b)+c$  and  $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ . For example  $0+(1+0)=(0+1)+0$  and  $0 \cdot (1 \cdot 0)=(0 \cdot 1) \cdot 0$ .

d)  $+$  and  $\cdot$  operations should have an identity element

- If we perform  $a + 0$  then for any value of  $a$  from  $\mathbb{F}$  the result will be  $a$  itself. Hence  $0$  is an identity element of  $+$  operation. If we perform  $a \cdot 1$  then for any value of  $a$  from  $\mathbb{F}$  the result will be  $a$  itself. Hence  $1$  is an identity element of  $\cdot$  operation.

e)  $\forall a \in \mathbb{F}$  there exists an additive inverse

- For additive inverse to exist,  $\forall a$  in  $\mathbb{F}$   $a+(-a)=0$ . For example.  $1-1=0$  and  $0-0=0$ .

f)  $\forall a \in \mathbb{F}$  such that  $a$  is non zero there exists a multiplicative inverse

- For multiplicative inverse to exist,  $\forall a$  such that  $a$  is non zero in  $\mathbb{F}$ ,  $a \cdot a^{-1}=1$ . For example  $1 \cdot 1^{-1} = 1$ .

g)  $+$  and  $\cdot$  should hold distributive property

- For any  $a, b$  and  $c$  in  $\mathbb{F}$  the property  $a \cdot (b+c) = a \cdot b + a \cdot c$  should always hold true. For example  $0 \cdot (1+1) = 0 \cdot 1 + 0 \cdot 1$ .

Since the above properties are satisfied we can say that  $(\mathbb{F}, +, \cdot)$  is a field.

1.1.6. Prove that if two homogenous systems of linear equations in two unknowns have the same solutions, then they are equivalent.

**Solution:** Let the two systems of homogenous equations be

$$\mathbf{Ax} = \mathbf{0} \quad (1.1.6.1)$$

$$\mathbf{By} = \mathbf{0} \quad (1.1.6.2)$$

We can write

$$\mathbf{CAx} = \mathbf{0} \quad (1.1.6.3)$$

$$\mathbf{DBy} = \mathbf{0} \quad (1.1.6.4)$$

where  $\mathbf{C}$  and  $\mathbf{D}$  are product of elementary matrices that reduce  $\mathbf{A}$  and  $\mathbf{B}$  into their reduced row echelon forms  $\mathbf{R}_1$  and  $\mathbf{R}_2$

(1.1.6.3) and (1.1.6.4) imply

$$\mathbf{R}_1 \mathbf{x} = \mathbf{0} \quad (1.1.6.5)$$

$$\mathbf{R}_2 \mathbf{y} = \mathbf{0} \quad (1.1.6.6)$$

Given that they have same solution, we can write

$$\mathbf{R}_1 \mathbf{x} = \mathbf{0} \quad (1.1.6.7)$$

$$\mathbf{R}_2 \mathbf{x} = \mathbf{0} \quad (1.1.6.8)$$

$$\Rightarrow (\mathbf{R}_1 - \mathbf{R}_2) \mathbf{x} = \mathbf{0} \quad (1.1.6.9)$$

Note that for a solution to exist,  $\mathbf{R}_1$  and  $\mathbf{R}_2$  can be either of matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (1.1.6.10)$$

Case 1 Let us assume that the solution is unique. The unique solution is

$$\mathbf{x} = \mathbf{0} \quad (1.1.6.11)$$

Since they have the same solution, both  $\mathbf{R}_1, \mathbf{R}_2$  must have their rank as 2.

So,

$$\mathbf{R}_1 = \mathbf{R}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (1.1.6.12)$$

Case 2 Let us assume that (1.1.6.3),(1.1.6.4) have infinitely many solutions

So,

$$\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{B}) = 1 \quad (1.1.6.13)$$

equation (1.1.6.9) for solutions other than zero solution implies

$$\mathbf{R}_1 = \mathbf{R}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (1.1.6.14)$$

So, in both the cases, we have

$$\mathbf{R}_1 = \mathbf{R}_2 \quad (1.1.6.15)$$

$$\implies \mathbf{CA} = \mathbf{DB} \quad (1.1.6.16)$$

Since  $\mathbf{C}, \mathbf{D}$  are product of elementary matrices, they are invertible.

$$\implies \mathbf{A} = \mathbf{C}^{-1}\mathbf{DB} \quad (1.1.6.17)$$

$$\mathbf{B} = \mathbf{D}^{-1}\mathbf{CA} \quad (1.1.6.18)$$

$$\text{Let } \mathbf{C}^{-1}\mathbf{D} = \mathbf{E} \quad (1.1.6.19)$$

where  $\mathbf{E}$  is also a product of elementary matrices

(1.1.6.17) and (1.1.6.18) hence become

$$\mathbf{A} = \mathbf{EB} \quad (1.1.6.20)$$

$$\mathbf{B} = \mathbf{E}^{-1}\mathbf{A} \quad (1.1.6.21)$$

Hence the two systems of equations are equivalent.

1.1.7. Prove that each subfield of the field of complex number contains every rational number

**Solution:**

**Complex Numbers:** A complex number is a number that can be expressed in the form  $a + bi$ , where  $a$  and  $b$  are real numbers, and  $i$  represents the imaginary unit, satisfying the equation  $i^2 = -1$ . The set of complex numbers is denoted by  $\mathbb{C}$

$$\mathbb{C} = \{(a, b) : a, b \in \mathbb{R}\} \quad (1.1.7.1)$$

**Rational Numbers:** A number in the form  $\frac{p}{q}$ , where both  $p$  and  $q$  (non-zero) are integers, is called a rational number. The set of rational numbers is denoted by  $\mathbb{Q}$ . Let  $\mathbb{Q}$  be the set of

rational numbers.

$$\mathbb{Q} = \left\{ \frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{Z}_{\neq 0} \right\} \quad (1.1.7.2)$$

Let  $\mathbb{C}$  be the field of complex numbers and given  $\mathbb{F}$  be the subfield of field of complex numbers  $\mathbb{C}$ . Since  $\mathbb{F}$  is the subfield, we could say that

$$0 \in \mathbb{F} \quad (1.1.7.3)$$

$$1 \in \mathbb{F} \quad (1.1.7.4)$$

Closed under addition: Here  $\mathbb{F}$  is closed under addition since it is subfield

$$1 + 1 = 2 \in \mathbb{F} \quad (1.1.7.5)$$

$$1 + 1 + 1 = 3 \in \mathbb{F} \quad (1.1.7.6)$$

$$\vdots$$

$$1 + 1 + \dots + 1 (p \text{ times}) = p \in \mathbb{F} \quad (1.1.7.7)$$

$$1 + 1 + \dots + 1 (q \text{ times}) = q \in \mathbb{F} \quad (1.1.7.8)$$

By using the above property we could say that zero and other positive integers belong to  $\mathbb{F}$ . Since  $p$  and  $q$  are integers we say,

$$p \in \mathbb{Z} \quad (1.1.7.9)$$

$$q \in \mathbb{Z} \quad (1.1.7.10)$$

**Additive Inverse:** Let  $x$  be the positive integer belong  $\mathbb{F}$  and by additive inverse we could say,

$$\forall x \in \mathbb{F} \quad (1.1.7.11)$$

$$(-x) \in \mathbb{F} \quad (1.1.7.12)$$

Therefore field  $\mathbb{F}$  contains every integers. Let  $n$  be a integer then,

$$n \in \mathbb{Z} \implies n \in \mathbb{F} \quad (1.1.7.13)$$

$$\mathbb{Z} \subseteq \mathbb{F} \quad (1.1.7.14)$$

Where  $\mathbb{Z}$  is subset of  $\mathbb{F}$  **Multiplicative Inverse:** Every element except zero in the subfield  $\mathbb{F}$  has an multiplicative inverse. From equation (1.1.7.8), since  $q \in \mathbb{F}$  we could say,

$$\frac{1}{q} \in \mathbb{F} \quad \text{and } q \neq 0 \quad (1.1.7.15)$$

Closed under multiplication: Also,  $\mathbb{F}$  is closed under multiplication and thus, from equation

(1.1.7.7) and (1.1.7.15) we get ,

$$p \cdot \frac{1}{q} \in \mathbb{F} \quad (1.1.7.16)$$

$$\Rightarrow \frac{p}{q} \in \mathbb{F} \quad (1.1.7.17)$$

where ,  $p \in \mathbb{Z}$  and  $q \in \mathbb{Z}_{\neq 0}$  (from equation (1.1.7.3) and (1.1.7.15)) Conclusion From (1.1.7.2) and (1.1.7.17) we could say ,

$$\mathbb{Q} \subseteq \mathbb{F} \quad (1.1.7.18)$$

From equation (1.1.7.18) we could say that each subfield of the field of complex number contains every rational number

Hence Proved

1.1.8. Prove that, each field of the characteristic zero contains a copy of the rational number field.

**Solution:** The characteristic of a field is defined to be the smallest number of times one must use the field's multiplicative identity (1) in a sum to get the additive identity (0), then the field is said to have characteristic zero.

Let  $\mathbb{Q}$  be the rational number field. Hence,

$$0 \in \mathbb{Q} \quad [\text{Additive Identity}] \quad (1.1.8.1)$$

$$1 \in \mathbb{Q} \quad [\text{Multiplicative Identity}] \quad (1.1.8.2)$$

As addition is defined on  $\mathbb{Q}$  hence we have,

$$1 \neq 0 \quad (1.1.8.3)$$

$$1 + 1 = 2 \neq 0 \quad (1.1.8.4)$$

And so on,

$$1 + 1 + \cdots + 1 = n \neq 0 \quad (1.1.8.5)$$

From the definition of characteristic of a field and from (1.1.8.3), (1.1.8.4) and so on up to (1.1.8.5), the rational number field,  $\mathbb{Q}$  has characteristic 0.

can be expressed in matrix form as,

$$\mathbf{A}\mathbf{x} = 0 \quad (1.2.1.2)$$

$$\begin{pmatrix} 1-i & -i \\ 2 & 1-i \end{pmatrix} \mathbf{x} = 0 \quad (1.2.1.3)$$

By row reduction ,

$$\begin{pmatrix} 1-i & -i \\ 2 & 1-i \end{pmatrix} \xrightarrow[R_1 \leftarrow R_1/2]{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & \frac{1-i}{2} \\ 1-i & -i \end{pmatrix} \quad (1.2.1.4)$$

$$\xrightarrow{R_2 \leftarrow R_2 - (1-i)R_1} \begin{pmatrix} 1 & \frac{1-i}{2} \\ 0 & 0 \end{pmatrix} \quad (1.2.1.5)$$

$$\begin{pmatrix} 1 & \frac{1-i}{2} \end{pmatrix} \mathbf{x} = 0 \quad (1.2.1.6)$$

$$\begin{pmatrix} 1 & \frac{1-i}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \quad (1.2.1.7)$$

$$x_1 = -\frac{1-i}{2} x_2 \quad (1.2.1.8)$$

$$\mathbf{x} = \begin{pmatrix} -\frac{1-i}{2} x_2 \\ x_2 \end{pmatrix} \quad (1.2.1.9)$$

$$\Rightarrow \mathbf{x} = x_2 \begin{pmatrix} -\frac{1-i}{2} \\ 1 \end{pmatrix} \quad (1.2.1.10)$$

1.2.2. If

$$A = \begin{pmatrix} 3 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 0 \end{pmatrix} \quad (1.2.2.1)$$

Find all solutions of  $AX = 0$  by row reducing  $A$ .

**Solution:** For the given equation  $AX = 0$  can be defined as follows:

$$\begin{pmatrix} 3 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (1.2.2.2)$$

Now, we can apply Row Reduction Methodol-

## 1.2 Matrices and Elementary Row Operations

1.2.1. Find all solutions to the system of equations

$$\begin{aligned} (1-i)x_1 - ix_2 &= 0 \\ 2x_1 + (1-i)x_2 &= 0 \end{aligned} \quad (1.2.1.1)$$

**Solution:** System of Linear Equations (1.2.1.1)

ogy of matrix  $A$  :

1.2.3.

$$\begin{pmatrix} 3 & -1 & 2 & 0 \\ 2 & 1 & 1 & 0 \\ 1 & -3 & 0 & 0 \end{pmatrix} \xleftrightarrow{R_1=R_1+R_2} \begin{pmatrix} 5 & 0 & 3 & 0 \\ 2 & 1 & 1 & 0 \\ 1 & -3 & 0 & 0 \end{pmatrix} \quad (1.2.2.3)$$

$$\xleftrightarrow{R_2=R_2-2R_3} \begin{pmatrix} 5 & 0 & 3 & 0 \\ 0 & 7 & 1 & 0 \\ 1 & -3 & 0 & 0 \end{pmatrix} \quad (1.2.2.4)$$

$$\xleftrightarrow{R_3=R_3-\frac{1}{5}R_1} \begin{pmatrix} 5 & 0 & 3 & 0 \\ 0 & 7 & 1 & 0 \\ 0 & -3 & -\frac{3}{5} & 0 \end{pmatrix} \quad (1.2.2.5)$$

$$\xleftrightarrow{R_1=\frac{1}{5}R_1} \begin{pmatrix} 1 & 0 & \frac{3}{5} & 0 \\ 0 & 7 & 1 & 0 \\ 0 & -3 & -\frac{3}{5} & 0 \end{pmatrix} \quad (1.2.2.6)$$

$$\xleftrightarrow{R_2=\frac{1}{7}R_2} \begin{pmatrix} 1 & 0 & \frac{3}{5} & 0 \\ 0 & 1 & \frac{1}{7} & 0 \\ 0 & -3 & -\frac{3}{5} & 0 \end{pmatrix} \quad (1.2.2.7)$$

$$\xleftrightarrow{R_3=R_3+3R_2} \begin{pmatrix} 1 & 0 & \frac{3}{5} & 0 \\ 0 & 1 & \frac{1}{7} & 0 \\ 0 & 0 & -\frac{6}{35} & 0 \end{pmatrix} \quad (1.2.2.8)$$

$$\xleftrightarrow{R_3=-\frac{35}{6}R_3} \begin{pmatrix} 1 & 0 & \frac{3}{5} & 0 \\ 0 & 1 & \frac{1}{7} & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad (1.2.2.9)$$

$$\xleftrightarrow{R_2=R_2-\frac{1}{7}R_3} \begin{pmatrix} 1 & 0 & \frac{3}{5} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad (1.2.2.10)$$

$$\xleftrightarrow{R_1=R_1-\frac{3}{5}R_3} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad (1.2.2.11)$$

So, as we can see the only solution we got after row reducing of matrix  $A$  is zero vector. Thus, the solution is :

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (1.2.2.12)$$

$$A = \begin{pmatrix} 6 & -4 & 0 \\ 4 & -2 & 0 \\ -1 & 0 & 3 \end{pmatrix} \quad (1.2.3.1)$$

Find all solutions of  $AX = 2X$  and all solutions of  $AX = 3X$ . The symbol  $cX$  denotes the matrix each entry of which is  $c$  times corresponding entry.

**Solution:**

$$A = \begin{pmatrix} 6 & -4 & 0 \\ 4 & -2 & 0 \\ -1 & 0 & 3 \end{pmatrix} \quad (1.2.3.2)$$

To calculate solution of  $AX = 2X$  and all solutions of  $AX = 3X$  we calculate eigen values of  $A$ :

$$(A - \lambda I)X = 0 \quad (1.2.3.3)$$

Substituting values in (1.2.3.3),

$$\begin{pmatrix} 6-\lambda & -4 & 0 \\ 4 & -2-\lambda & 0 \\ -1 & 0 & 3-\lambda \end{pmatrix} X = 0 \quad (1.2.3.4)$$

Simplifying:

$$\begin{pmatrix} 6-\lambda & -4 & 0 \\ 4 & -2-\lambda & 0 \\ -1 & 0 & 3-\lambda \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 - R_2} \begin{pmatrix} 2-\lambda & -2+\lambda & 0 \\ 4 & -2-\lambda & 0 \\ -1 & 0 & 3-\lambda \end{pmatrix} \quad (1.2.3.5)$$

Taking  $(3-\lambda)$  and  $(2-\lambda)$  common from  $C_3$  and  $R_1$

$$(3-\lambda)(2-\lambda) \begin{pmatrix} 1 & -1 & 0 \\ 4 & -2-\lambda & 0 \\ -1 & 0 & 1 \end{pmatrix} \quad (1.2.3.6)$$

$$\begin{pmatrix} 1 & -1 & 0 \\ 4 & -2-\lambda & 0 \\ -1 & 0 & 1 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - 4R_1} \begin{pmatrix} 1 & -1 & 0 \\ 0 & -\lambda+2 & 0 \\ -1 & 0 & 1 \end{pmatrix} \quad (1.2.3.7)$$

Taking  $(2-\lambda)$  common from  $R_2$ :

$$(2-\lambda)^2(3-\lambda) \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \quad (1.2.3.8)$$



Eigen values are:

$$\lambda_1 = 2 \quad (1.2.3.9)$$

$$\lambda_2 = 3 \quad (1.2.3.10)$$

solution to  $\mathbf{AX} = 2\mathbf{X}$  is eigen vector corresponding to  $\lambda = 2$

$$(\mathbf{A} - 2\mathbf{I})\mathbf{X} = 0 \quad (1.2.3.11)$$

Substituting values:

$$\begin{pmatrix} 4 & -4 & 0 & 0 \\ 4 & -4 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{R_1}{4}} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 4 & -4 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - 4R_1}$$

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 + R_1} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{pmatrix} \xleftrightarrow{R_3 \leftrightarrow R_2}$$

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow -R_2} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 + R_2}$$

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (1.2.3.12)$$

So,  $x_3$  is a free variable: Let  $x_3 = c$ .

$$x_2 - x_3 = 0 \implies x_2 = x_3 = c \quad (1.2.3.13)$$

$$x_1 - x_3 = 0 \implies x_1 = x_3 = c \quad (1.2.3.14)$$

So, the solution to  $\mathbf{AX} = 2\mathbf{X}$  is

$$\mathbf{X} = c \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad (1.2.3.15)$$

solution of  $\mathbf{AX} = 3\mathbf{X}$  is eigen vector corresponding to  $\lambda = 3$

$$(\mathbf{A} - 3\mathbf{I})\mathbf{X} = 0 \quad (1.2.3.16)$$

substituting we have:

$$\begin{pmatrix} 3 & -4 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{R_1}{3}} \begin{pmatrix} 1 & -\frac{4}{3} & 0 & 0 \\ 4 & -5 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - 4R_1}$$

$$\begin{pmatrix} 1 & -\frac{4}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 + R_1} \begin{pmatrix} 1 & -\frac{4}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & -\frac{4}{3} & 0 & 0 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow \frac{R_2}{\frac{1}{3}}}$$

$$\begin{pmatrix} 1 & -\frac{4}{3} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\frac{4}{3} & 0 & 0 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 - \frac{4}{3}R_2} \begin{pmatrix} 1 & -\frac{4}{3} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 + \frac{4}{3}R_2}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (1.2.3.17)$$

So  $x_3$  is a free variable:

$$x_1 = 0 \quad (1.2.3.18)$$

$$x_2 = 0 \quad (1.2.3.19)$$

$$x_3 = c \quad (1.2.3.20)$$

So, the solution to  $\mathbf{AX} = 3\mathbf{X}$  is,

$$\mathbf{X} = c \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad (1.2.3.21)$$

1.2.4. Find a row-reduced matrix which is row equivalent to,

$$\mathbf{A} = \begin{pmatrix} i & -(1+i) & 0 \\ 1 & -2 & 1 \\ 1 & 2i & -1 \end{pmatrix} \quad (1.2.4.1)$$

**Solution: Step 1:** Performing scaling operation to matrix  $\mathbf{A}$  as  $R_1 \leftarrow \frac{1}{i}R_1$  by scaling matrix  $D_1$  given as,

$$\mathbf{D}_1 = \begin{pmatrix} \frac{1}{i} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (1.2.4.2)$$

$$\mathbf{D}_1\mathbf{A} = \begin{pmatrix} \frac{1}{i} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} i & -(1+i) & 0 \\ 1 & -2 & 1 \\ 1 & 2i & -1 \end{pmatrix} \quad (1.2.4.3)$$

$$\implies \mathbf{D}_1\mathbf{A} = \begin{pmatrix} 1 & -1+i & 0 \\ 1 & -2 & 1 \\ 1 & 2i & -1 \end{pmatrix} \quad (1.2.4.4)$$

**Step 2:** Performing  $R_2 \leftarrow R_2 - R_1$  and  $R_3 \leftarrow R_3 - R_1$  given by elementary matrix  $\mathbf{E}_{31}\mathbf{E}_{21}$  on

equation (1.2.4.4),

$$\mathbf{E}_{31}\mathbf{E}_{21} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \quad (1.2.4.5)$$

$$\mathbf{E}_{31}\mathbf{E}_{21}\mathbf{D}_1\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1+i & 0 \\ 1 & -2 & 1 \\ 1 & 2i & -1 \end{pmatrix} \quad (1.2.4.6)$$

$$\Rightarrow \mathbf{A}_1 = \mathbf{E}_{31}\mathbf{E}_{21}\mathbf{D}_1\mathbf{A} = \begin{pmatrix} 1 & -1+i & 0 \\ 0 & -1-i & 1 \\ 0 & 1+i & -1 \end{pmatrix} \quad (1.2.4.7)$$

**Step 3:** Performing  $R_2 \leftarrow \frac{-1}{1+i}R_2$  given by  $\mathbf{D}_2$  on equation (1.2.4.7),

$$\mathbf{D}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2}(-1+i) & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (1.2.4.8)$$

$$\mathbf{D}_2\mathbf{A}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2}(-1+i) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1+i & 0 \\ 0 & -1-i & 1 \\ 0 & 1+i & -1 \end{pmatrix} \quad (1.2.4.9)$$

$$\Rightarrow \mathbf{A}_2 = \mathbf{D}_2\mathbf{A}_1 = \begin{pmatrix} 1 & -1+i & 0 \\ 0 & 1 & \frac{1}{2}(-1+i) \\ 0 & 1+i & -1 \end{pmatrix} \quad (1.2.4.10)$$

**Step 4:** Performing  $R_3 \leftarrow R_3 - (1+i)R_2$  given by  $\mathbf{E}_{32}$  on equation (1.2.4.10),

$$\mathbf{E}_{32} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -(1+i) & 1 \end{pmatrix} \quad (1.2.4.11)$$

$$\mathbf{E}_{32}\mathbf{A}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1-i & 1 \end{pmatrix} \begin{pmatrix} 1 & -1+i & 0 \\ 0 & 1 & \frac{-1+i}{2} \\ 0 & 1+i & -1 \end{pmatrix} \quad (1.2.4.12)$$

$$\Rightarrow \mathbf{A}_3 = \mathbf{E}_{32}\mathbf{A}_2 = \begin{pmatrix} 1 & -1+i & 0 \\ 0 & 1 & \frac{-1+i}{2} \\ 0 & 0 & 1 \end{pmatrix} \quad (1.2.4.13)$$

**Step 5:** Performing  $R_1 \leftarrow R_1 - (-1+i)R_2$  given

by  $\mathbf{E}_{12}$  on equation (1.2.4.13),

$$\mathbf{E}_{12} = \begin{pmatrix} 1 & 1-i & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (1.2.4.14)$$

$$\mathbf{E}_{12}\mathbf{A}_3 = \begin{pmatrix} 1 & 1-i & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1+i & 0 \\ 0 & 1 & \frac{-1+i}{2} \\ 0 & 0 & 1 \end{pmatrix} \quad (1.2.4.15)$$

$$\Rightarrow \mathbf{A}_4 = \mathbf{E}_{12}\mathbf{A}_3 = \begin{pmatrix} 1 & 0 & i \\ 0 & 1 & \frac{-1+i}{2} \\ 0 & 0 & 1 \end{pmatrix} \quad (1.2.4.16)$$

**Step 6:** Performing  $R_1 \leftarrow R_1 - iR_3$  and  $R_2 \leftarrow R_2 - \frac{-1+i}{2}R_3$  given by  $\mathbf{E}_{13}\mathbf{E}_{23}$  on equation (1.2.4.16),

$$\mathbf{E}_{13}\mathbf{E}_{23} = \begin{pmatrix} 1 & 0 & -i \\ 0 & 1 & -\left(\frac{-1+i}{2}\right) \\ 0 & 0 & 1 \end{pmatrix} \quad (1.2.4.17)$$

$$\mathbf{E}_{13}\mathbf{E}_{23}\mathbf{A}_4 = \begin{pmatrix} 1 & 0 & -i \\ 0 & 1 & -\left(\frac{-1+i}{2}\right) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & i \\ 0 & 1 & \frac{-1+i}{2} \\ 0 & 0 & 1 \end{pmatrix} \quad (1.2.4.18)$$

$$\Rightarrow \mathbf{A}_5 = \mathbf{E}_{13}\mathbf{E}_{23}\mathbf{A}_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (1.2.4.19)$$

$\therefore$  Row-reduced matrix of  $\mathbf{A}$  given by equation (1.2.4.1) is,

$$\mathbf{A} = \begin{pmatrix} i & -1-i & 0 \\ 1 & -2 & 1 \\ 1 & 2i & -1 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I} \quad (1.2.4.20)$$

1.2.5. Prove that the following two matrices are not row equivalent

$$\begin{pmatrix} 2 & 0 & 0 \\ a & -1 & 0 \\ b & c & 3 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 2 \\ -2 & 0 & -1 \\ 1 & 3 & 5 \end{pmatrix} \quad (1.2.5.1)$$

**Solution:** Call the first matrix  $\mathbf{A}$  and the second matrix  $\mathbf{B}$ .

$$\mathbf{A}^T = \begin{pmatrix} 2 & a & b \\ 0 & -1 & c \\ 0 & 0 & 3 \end{pmatrix} \quad (1.2.5.2)$$

$\mathbf{A}^T$  is a upper triangular matrix with non-zero diagonal. Hence it has full rank = 3.

$$\mathbf{B}^T = \begin{pmatrix} 1 & -2 & 1 \\ 1 & 0 & 3 \\ 2 & -1 & 5 \end{pmatrix} \xrightarrow[R_3 \rightarrow R_3 - 2R_1]{R_2 \rightarrow R_2 - R_1} \begin{pmatrix} 1 & -2 & 1 \\ 0 & 2 & 2 \\ 0 & 3 & 3 \end{pmatrix} \quad (1.2.5.3)$$

$$\xrightarrow[R_2 \leftarrow R_2/2]{R_3 \leftarrow R_3/3} \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad (1.2.5.4)$$

$$\xrightarrow{R_3 \leftarrow R_3 - R_2} \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad (1.2.5.5)$$

$\mathbf{B}^T$  is a upper triangular matrix with zero diagonal. Hence it doesn't have full rank. Therefore both matrices have different rank, so it cannot be row equivalent.

1.2.6. Let

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (1.2.6.1)$$

be a  $2 \times 2$  matrix with complex entries. Suppose  $\mathbf{A}$  is row-reduced and also that  $a+b+c+d=0$ . Prove that there are exactly three such matrices.

**Solution:** A matrix is in row echelon form if it follows the following conditions

1. All nonzero rows are above any rows of all zeros.
  2. Each leading entry (i.e. left most nonzero entry) of a row is in a column to the right of the leading entry of the row above it.
  3. All entries in a column below a leading entry are zero
- Row Reduced Echelon Form A matrix is in row reduced echelon form if it follows the following conditions
1. The matrix should be row echelon form
  2. The leading entry in each nonzero row is 1.
  3. Each leading 1 is the only nonzero entry in its column. Proof

Given ,

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (1.2.6.2)$$

**Condition 1 :** Matrix  $\mathbf{A}$  should be in row-reduced echelon form

**Condition 2 :**  $a+b+c+d=0$  where  $a, b, c$  and  $d$  are the elements of the matrix  $\mathbf{A}$

Reducing the matrix  $\mathbf{A}$  from equation (1.2.6.2)

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \xrightarrow{R_1 = \frac{1}{a}R_1} \begin{pmatrix} 1 & \frac{b}{a} \\ c & d \end{pmatrix} \quad (1.2.6.3)$$

$$\xrightarrow{R_2 = R_2 - cR_1} \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & \frac{ad-bc}{a} \end{pmatrix} \quad (1.2.6.4)$$

$$\xrightarrow{R_2 = \frac{a}{ad-bc}R_2} \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix} \quad (1.2.6.5)$$

$$\xrightarrow{R_1 = R_1 - \frac{b}{a}R_2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (1.2.6.6)$$

**Case 1:** Matrix  $\mathbf{A}$  of Rank 2

From the equation (1.2.6.4), for the matrix to be in row reduced echelon form,

$$b = 0$$

$$a \neq 0$$

$$d = 1$$

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (1.2.6.7)$$

For the condition 2 to get satisfied,

$$a + 0 + c + 1 = 0 \quad (1.2.6.8)$$

$$\implies a = -(c + 1) \quad (1.2.6.9)$$

$$\implies c \neq -1 \quad (1.2.6.10)$$

Both the condition gets satisfied and so exactly one matrix  $\mathbf{A}$  can be formed of Rank 2 with given conditions

**Case 2:** Matrix  $\mathbf{A}$  of Rank 1

From the equation (1.2.6.4), for the matrix to be in row reduced echelon form,

$$a \neq 0$$

$$d = 0$$

$$c = 0$$

For the condition 2 to get satisfied,

$$a + b + 0 + 0 = 0 \quad (1.2.6.11)$$

$$\implies b = -a \quad (1.2.6.12)$$

$$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \quad (1.2.6.13)$$

Both the condition gets satisfied and so exactly one matrix  $\mathbf{A}$  can be formed of Rank 1 with given conditions

**Case 3:** Matrix  $\mathbf{A}$  of Rank 0

From equation (1.2.6.2), for the matrix to be in row reduced echelon form,

$$\begin{aligned} a &= 0 \\ b &= 0 \\ c &= 0 \\ d &= 0 \\ \mathbf{A} &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned} \quad (1.2.6.14)$$

Both the condition gets satisfied and so exactly one matrix  $\mathbf{A}$  can be formed of Rank 0 with given conditions

Therefore matrix  $\mathbf{A}$  shown in equation (1.2.6.7), (1.2.6.13) and (1.2.6.14) are the exactly three such matrices that can be formed with given conditions.

1.2.7. Prove that the interchange of two rows of a matrix can be accomplished by a finite sequence of elementary row operations of the other two types.

**Solution:** Let  $\mathbf{A}$  be a  $3 \times 3$  matrix with having row vectors  $\mathbf{a}_1, \mathbf{a}_2$  and  $\mathbf{a}_3$ .

$$\mathbf{A} = \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} \quad (1.2.7.1)$$

Let's exchange row  $\mathbf{a}_1$  and  $\mathbf{a}_2$ . Let's call this elementary operation  $\mathbf{E}_1$ .

$$\mathbf{E}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (1.2.7.2)$$

$$(1.2.7.3)$$

Now performing operation  $\mathbf{E}_1$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{a}_2 \\ \mathbf{a}_1 \\ \mathbf{a}_3 \end{pmatrix} \quad (1.2.7.4)$$

Now, to prove that same matrix can be obtained by elementary operations let's call them  $\mathbf{E}_2$  and  $\mathbf{E}_3$ . Now performing operation  $\mathbf{E}_2$  by adding

row 2 to row 1.

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{a}_1 + \mathbf{a}_2 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} \quad (1.2.7.5)$$

Using elementary operation  $\mathbf{E}_2$  we will subtract row 1 from row 2.

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}_1 + \mathbf{a}_2 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{a}_1 + \mathbf{a}_2 \\ -\mathbf{a}_1 \\ \mathbf{a}_3 \end{pmatrix} \quad (1.2.7.6)$$

Using elementary operation  $\mathbf{E}_2$  we will add row 2 to row 1.

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}_1 + \mathbf{a}_2 \\ -\mathbf{a}_1 \\ \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{a}_2 \\ -\mathbf{a}_1 \\ \mathbf{a}_3 \end{pmatrix} \quad (1.2.7.7)$$

Using elementary operation  $\mathbf{E}_3$  we will multiply row 2 by -1.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}_2 \\ -\mathbf{a}_1 \\ \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{a}_2 \\ \mathbf{a}_1 \\ \mathbf{a}_3 \end{pmatrix} \quad (1.2.7.8)$$

Hence, we can say that,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} \quad (1.2.7.9)$$

Let us assume a matrix  $\mathbf{A}$

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \quad (1.2.7.10)$$

Let's exchange row  $\mathbf{a}_1$  and  $\mathbf{a}_2$  by applying operation  $\mathbf{E}_1$ .

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & 3 \\ 1 & 1 & 0 \end{pmatrix} \quad (1.2.7.11)$$

Now, to prove that same matrix can be obtained by other two elementary operations. We will first perform elementary operation  $\mathbf{E}_2$  by adding row 2 to row 1.

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \quad (1.2.7.12)$$

Using elementary operation  $\mathbf{E}_2$  we will subtract row 1 from row 2.

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 3 \\ -1 & -2 & -3 \\ 1 & 1 & 0 \end{pmatrix} \quad (1.2.7.13)$$

Using elementary operation  $\mathbf{E}_2$  we will add row 2 to row 1.

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 3 \\ -1 & -2 & -3 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & -2 & -3 \\ 1 & 1 & 0 \end{pmatrix} \quad (1.2.7.14)$$

Using elementary operation  $\mathbf{E}_3$  we will multiply row 2 by -1.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & -2 & -3 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & 3 \\ 1 & 1 & 0 \end{pmatrix} \quad (1.2.7.15)$$

Hence, we can say that,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \times \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \quad (1.2.7.16)$$

1.2.8. Consider the system of equations  $\mathbf{A}\mathbf{X} = 0$  where

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a  $2 \times 2$  matrix over the field  $F$ . Prove the following -

- If every entry of  $\mathbf{A}$  is 0, then every pair  $x_1$  and  $x_2$  is a solution of  $\mathbf{A}\mathbf{X} = 0$ .
- If  $ad - bc \neq 0$ , then the system  $\mathbf{A}\mathbf{X} = 0$  has only the trivial solution  $x_1 = x_2 = 0$
- If  $ad - bc = 0$  and some entry of  $\mathbf{A}$  is different from 0, then there is a solution  $x_1^0$  and  $x_2^0$  such that  $x_1$  and  $x_2$  is a solution if and only if there is a scalar  $y$  such that  $x_1 = yx_1^0$  and  $x_2 = yx_2^0$

**Solution:** Solution 1 If every entry of  $\mathbf{A}$  is 0

then the equation  $\mathbf{A}\mathbf{X} = 0$  becomes,

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \quad (1.2.8.1)$$

$$\Rightarrow 0.x_1 + 0.x_2 = 0 \quad \forall x_1, x_2 \in F \quad (1.2.8.2)$$

Hence proved, every pair  $x_1$  and  $x_2$  is a solution for the equation  $\mathbf{A}\mathbf{X} = 0$ . **Solution 2 Case 1:** Let  $a = 0$ . Since  $ad - bc \neq 0$ . As  $bc \neq 0$  therefore  $b \neq 0$  and  $c \neq 0$ . Hence, we can perform row reduction on the augmented matrix of equation  $\mathbf{A}\mathbf{X}=0$  as follows,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & b & 0 \\ c & d & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{c} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c & d & 0 \\ 0 & b & 0 \end{pmatrix} \quad (1.2.8.3)$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{b} \end{pmatrix} \begin{pmatrix} 1 & \frac{d}{c} & 0 \\ 0 & b & 0 \end{pmatrix} \quad (1.2.8.4)$$

$$= \begin{pmatrix} 1 & -\frac{d}{c} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{d}{c} & 0 \\ 0 & b & 0 \end{pmatrix} \quad (1.2.8.5)$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (1.2.8.6)$$

**Case 2:** Let  $a, b, c, d \neq 0$ . Considering the following case,

$$\mathbf{A}\mathbf{X} = \mathbf{u} \quad (1.2.8.7)$$

$$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad (1.2.8.8)$$

Row Reducing the augmented matrix of (1.2.8.8) we get,

$$\begin{pmatrix} \frac{1}{a} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b & u_1 \\ c & d & u_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -c & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{b}{a} & \frac{u_1}{a} \\ c & d & u_2 \end{pmatrix} \quad (1.2.8.9)$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & \frac{a}{ad-bc} \end{pmatrix} \begin{pmatrix} 1 & \frac{b}{a} & \frac{u_1}{a} \\ 0 & \frac{ad-bc}{a} & \frac{au_2-cu_1}{a} \end{pmatrix} \quad (1.2.8.10)$$

$$= \begin{pmatrix} 1 & -\frac{b}{a} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{b}{a} & \frac{u_1}{a} \\ 0 & \frac{ad-bc}{a} & \frac{au_2-cu_1}{a} \end{pmatrix} \quad (1.2.8.11)$$

$$= \begin{pmatrix} 1 & 0 & \frac{du_1-bu_2}{ad-bc} \\ 0 & 1 & \frac{au_2-cu_1}{ad-bc} \end{pmatrix} \quad (1.2.8.12)$$

From (1.2.8.12) we get,

$$x_1 = \frac{du_1 - bu_2}{ad - bc} \quad (1.2.8.13)$$

$$x_2 = \frac{au_2 - cu_1}{ad - bc} \quad (1.2.8.14)$$

Since  $u_1 = 0$  and  $u_2 = 0$  then from (1.2.8.13) and (1.2.8.14),

$$x_1 = 0 \quad (1.2.8.15)$$

$$x_2 = 0 \quad (1.2.8.16)$$

Hence we get,

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (1.2.8.17)$$

In (1.2.8.6) and (1.2.8.17), we can see that  $\mathbf{AX} = 0$  has only one trivial solution i.e  $x_1 = x_2 = 0$  in all cases. Hence proved, the equation  $\mathbf{AX}=0$  has only one trivial solution  $x_1 = x_2 = 0$  **Solution 3 Case 1:** Let,  $a \neq 0$  for  $\mathbf{A}$ . Given  $ad - bc = 0$ , we can perform row reduction on augmented matrix of equation  $\mathbf{AX} = 0$  as follows,

$$\begin{pmatrix} \frac{1}{a} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b & 0 \\ c & d & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -c & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{b}{a} & 0 \\ c & d & 0 \end{pmatrix} \quad (1.2.8.18)$$

$$= \begin{pmatrix} 1 & \frac{b}{a} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad [\because ad - bc = 0] \quad (1.2.8.19)$$

Hence from (1.2.8.19),  $\mathbf{AX} = 0$  if and only if

$$x_1 = -\frac{b}{a}x_2 \quad [a \neq 0] \quad (1.2.8.20)$$

Letting  $x_1^0 = -\frac{b}{a}$  and  $x_2^0 = 1$  we get for  $y = 1$ ,

$$x_1 = yx_1^0 \quad (1.2.8.21)$$

$$x_2 = yx_2^0 \quad (1.2.8.22)$$

which is a solution of the equation  $\mathbf{AX} = 0$ .

**Case 2:** Let,  $b \neq 0$  for  $\mathbf{A}$ . Given  $ad - bc = 0$ , at first we multiply by elementary matrix to change the columns and then we can perform row reduction on augmented matrix of equation  $\mathbf{AX} = 0$  as follows,

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} b & a & 0 \\ d & c & 0 \end{pmatrix} \quad (1.2.8.23)$$

Hence using the result obtained from (1.2.8.19)

we can conclude for (1.2.8.23),  $\mathbf{AX} = 0$  if and only if

$$x_2 = -\frac{a}{b}x_1 \quad [b \neq 0] \quad (1.2.8.24)$$

Letting  $x_2^0 = -\frac{a}{b}$  and  $x_1^0 = 1$  we get for  $y = 1$ ,

$$x_1 = yx_1^0 \quad (1.2.8.25)$$

$$x_2 = yx_2^0 \quad (1.2.8.26)$$

which is a solution of the equation  $\mathbf{AX} = 0$ .

**Case 3:** Let,  $c \neq 0$  for  $\mathbf{A}$ . Given  $ad - bc = 0$ , we can perform row reduction on augmented matrix of equation  $\mathbf{AX} = 0$  as follows,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b & 0 \\ c & d & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{c} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c & d & 0 \\ a & b & 0 \end{pmatrix} \quad (1.2.8.27)$$

$$= \begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{d}{c} & 0 \\ a & b & 0 \end{pmatrix} \quad (1.2.8.28)$$

$$= \begin{pmatrix} 1 & \frac{d}{c} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad [\because ad - bc = 0] \quad (1.2.8.29)$$

Hence from (1.2.8.29),  $\mathbf{AX} = 0$  if and only if

$$x_1 = -\frac{d}{c}x_2 \quad [a \neq 0] \quad (1.2.8.30)$$

Letting  $x_1^0 = -\frac{d}{c}$  and  $x_2^0 = 1$  we get for  $y = 1$ ,

$$x_1 = yx_1^0 \quad (1.2.8.31)$$

$$x_2 = yx_2^0 \quad (1.2.8.32)$$

which is a solution of the equation  $\mathbf{AX} = 0$ .

**Case 4:** Let,  $d \neq 0$  for  $\mathbf{A}$ . Given  $ad - bc = 0$ , at first we multiply by elementary matrix to change the columns and then we can perform row reduction on augmented matrix of equation  $\mathbf{AX} = 0$  as follows,

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b & a & 0 \\ d & c & 0 \end{pmatrix} \quad (1.2.8.33)$$

$$= \begin{pmatrix} d & c & 0 \\ b & a & 0 \end{pmatrix} \quad (1.2.8.34)$$

Hence using the result from (1.2.8.29) we can conclude for (1.2.8.34),  $\mathbf{AX} = 0$  if and only if

$$x_2 = -\frac{c}{d}x_1 \quad [a \neq 0] \quad (1.2.8.35)$$

Letting  $x_2^0 = -\frac{c}{d}$  and  $x_1^0 = 1$  we get for  $y = 1$ ,

$$x_1 = yx_1^0 \quad (1.2.8.36)$$

$$x_2 = yx_2^0 \quad (1.2.8.37)$$

which is a solution of the equation  $\mathbf{AX} = 0$ .

are:

$$\begin{pmatrix} \frac{1}{3} & 2 & -6 \\ -4 & 0 & 5 \\ -3 & 6 & -13 \\ -\frac{7}{3} & 2 & -\frac{8}{3} \end{pmatrix} \xleftrightarrow[R_1 \leftarrow R_1 \times 3]{R_4 \leftarrow R_4 \times 3} \begin{pmatrix} 1 & 6 & -18 \\ -4 & 0 & 5 \\ -3 & 6 & -13 \\ -7 & 6 & -8 \end{pmatrix} \quad (1.3.1.6)$$

$$\xleftrightarrow{R_3 \leftarrow R_2 + R_3} \begin{pmatrix} 1 & 6 & -18 \\ -4 & 0 & 5 \\ -7 & 6 & -8 \\ -7 & 6 & -8 \end{pmatrix} \xleftrightarrow{R_4 \leftarrow R_4 - R_3} \quad (1.3.1.7)$$

$$\begin{pmatrix} 1 & 6 & -18 \\ -4 & 0 & 5 \\ -7 & 6 & -8 \\ 0 & 0 & 0 \end{pmatrix} \xleftrightarrow[R_3 \leftarrow R_3 + 7R_1]{R_2 \leftarrow R_2 + 4R_1} \begin{pmatrix} 1 & 6 & -18 \\ 0 & 24 & -67 \\ 0 & 48 & -138 \\ 0 & 0 & 0 \end{pmatrix} \quad (1.3.1.8)$$

$$\xleftrightarrow{R_3 \leftarrow R_3 / 2} \begin{pmatrix} 1 & 6 & -18 \\ 0 & 24 & -67 \\ 0 & 24 & -69 \\ 0 & 0 & 0 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 - R_2} \quad (1.3.1.9)$$

$$\begin{pmatrix} 1 & 6 & -18 \\ 0 & 24 & -67 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow \frac{R_3}{(-2)}} \begin{pmatrix} 1 & 6 & -18 \\ 0 & 24 & -67 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad (1.3.1.10)$$

$$\xleftrightarrow{R_2 \leftarrow \frac{R_2}{4}} \begin{pmatrix} 1 & 6 & -18 \\ 0 & 6 & -\frac{67}{4} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 - R_2} \quad (1.3.1.11)$$

$$\begin{pmatrix} 1 & 0 & -\frac{5}{4} \\ 0 & 6 & -\frac{67}{4} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow \frac{R_2}{6}} \begin{pmatrix} 1 & 0 & -\frac{5}{4} \\ 0 & 1 & -\frac{67}{24} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad (1.3.1.12)$$

$$\xleftrightarrow[R_1 \leftarrow R_1 + \frac{5R_3}{4}]{R_2 \leftarrow R_2 + \frac{67R_3}{24}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad (1.3.1.13)$$

Now,

$$\mathbf{AX} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \quad (1.3.1.14)$$

### 1.3 Row Reduced Echelon Matrices

1.3.1. Find all solutions to the following system of equations by row-reducing the co-efficient matrix:

$$\frac{1}{3}x_1 + 2x_2 - 6x_3 = 0 \quad (1.3.1.1)$$

$$-4x_1 + 5x_3 = 0 \quad (1.3.1.2)$$

$$-3x_1 + 6x_2 - 13x_3 = 0 \quad (1.3.1.3)$$

$$-\frac{7}{3}x_1 + 2x_2 - \frac{8}{3}x_3 = 0 \quad (1.3.1.4)$$

**Solution:** The coefficient matrix is:

$$A = \begin{pmatrix} \frac{1}{3} & 2 & -6 \\ -4 & 0 & 5 \\ -3 & 6 & -13 \\ -\frac{7}{3} & 2 & -\frac{8}{3} \end{pmatrix} \quad (1.3.1.5)$$

The number of rows of this coefficient matrix is  $m = 4$  and the number of columns is  $n = 3$ . So in this case,  $n < m$ . Now the row operations

So,

$$\mathbf{I}_3 \mathbf{x} = 0 \quad (1.3.1.15)$$

$$\Rightarrow \mathbf{x} = 0 \quad (1.3.1.16)$$

1.3.2. Find a row-reduced matrix which is row equivalent to A. What are the solutions of  $\mathbf{Ax} = \mathbf{0}$  ?

$$\mathbf{A} = \begin{pmatrix} 1 & -i \\ 2 & 2 \\ i & 1+i \end{pmatrix} \quad (1.3.2.1)$$

**Solution:** Let  $\mathbf{R}$  be a row-reduced echelon matrix which is row equivalent to  $\mathbf{A}$ . Then the systems

$$\mathbf{Ax} = \mathbf{0}, \mathbf{Rx} = \mathbf{0} \quad (1.3.2.2)$$

have the same solutions. On performing elementary row operations on (1.3.2.1),

$$\mathbf{R} = \mathbf{BA} \quad (1.3.2.3)$$

where  $\mathbf{B}$  is the product of all elementary matrices. Reducing the given matrix, we get

$$\begin{aligned} \mathbf{B} &= (\mathbf{E}_5 \mathbf{E}_4 \mathbf{E}_3 \mathbf{E}_2 \mathbf{E}_1) \\ &= \begin{pmatrix} 1 & i & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -i & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4}(1-i) & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &\quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -i & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2}(1-i) & \frac{1}{4}(1+i) & 0 \\ \frac{1}{2}(-1+i) & \frac{1}{4}(1-i) & 0 \\ \frac{1}{2}(1-i) & \frac{1}{4}(-1-i) & 1 \end{pmatrix} \quad (1.3.2.4) \end{aligned}$$

$$\mathbf{R} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (1.3.2.5)$$

$\therefore$  Row-reduced matrix of  $\mathbf{A}$  is,

$$\mathbf{A} = \begin{pmatrix} 1 & -i \\ 2 & 2 \\ i & 1+i \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (1.3.2.6)$$

From (1.3.2.2) and (1.3.2.6),

$$\mathbf{Ax} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \quad (1.3.2.7)$$

The solution of  $\mathbf{Ax} = \mathbf{0}$  is,

$$\mathbf{I}_2 \mathbf{x} = 0 \quad (1.3.2.8)$$

$$\Rightarrow \mathbf{x} = 0 \quad (1.3.2.9)$$

As  $\mathbf{I}_2$  is invertible.

1.3.3. Describe explicitly all 2x2 row-reduced echelon matrices.

**Solution:**

2x2 matrices which are row-reduced echelon matrix can be represented as a linear combination of three matrices:-

$$c_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (1.3.3.1)$$

1.3.4. Consider the system of the equations

$$x_1 - x_2 + 2x_3 = 1 \quad (1.3.4.1)$$

$$x_1 - 0x_2 + 2x_3 = 1 \quad (1.3.4.2)$$

$$x_1 - 3x_2 + 4x_3 = 2 \quad (1.3.4.3)$$

Does this system have a solution ? If so describe explicitly all solutions.

**Solution:** Let  $\mathbf{V}$  is the set of all  $(x_1, x_2, x_3) \in \mathbb{R}^3$  which satisfy the (1.3.4.1), (1.3.4.2) and (1.3.4.3)

From equation (1.3.4.1) to (1.3.4.3) we can write,

$$\begin{pmatrix} 1 & -1 & 2 \\ 1 & 0 & 2 \\ 1 & -3 & 4 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \quad (1.3.4.4)$$

$$\Rightarrow \mathbf{Ax} = \mathbf{b} \quad (1.3.4.5)$$

Where,

$$(1.3.4.6)$$

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 2 \\ 1 & 0 & 2 \\ 1 & -3 & 4 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad (1.3.4.7)$$



Solving the matrix  $\mathbf{A}$  for rank we get,

$$\begin{pmatrix} 1 & -1 & 2 \\ 2 & 0 & 2 \\ 1 & -3 & 4 \end{pmatrix} \xleftrightarrow{R_2=R_1-2R_1} \begin{pmatrix} 1 & -1 & 2 \\ 0 & 2 & -2 \\ 1 & -3 & 4 \end{pmatrix} \quad (1.3.4.8)$$

$$\xleftrightarrow{R_3=R_3-R_1} \begin{pmatrix} 1 & -1 & 2 \\ 0 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix} \quad (1.3.4.9)$$

$$\xleftrightarrow{R_3=R_3+R_2} \begin{pmatrix} 1 & -1 & 2 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{pmatrix} \quad (1.3.4.10)$$

Hence, rank  $(\mathbf{A}) = 2$ . Now solving the augmented matrix of (1.3.4.5) we get,

$$\begin{pmatrix} 1 & -1 & 2 & 1 \\ 2 & 0 & 2 & 1 \\ 1 & -3 & 4 & 2 \end{pmatrix} \xleftrightarrow{R_2=R_1-2R_1} \begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 2 & -2 & -1 \\ 1 & -3 & 4 & 2 \end{pmatrix} \quad (1.3.4.11)$$

$$\xleftrightarrow{R_3=R_3-R_1} \begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 2 & -2 & -1 \\ 0 & -2 & 2 & 1 \end{pmatrix} \quad (1.3.4.12)$$

$$\xleftrightarrow{R_3=R_3+R_2} \begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 2 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (1.3.4.13) \quad 1.3.5.$$

We have rank  $(\mathbf{A}) = \text{rank}(\mathbf{A} : \mathbf{b}) = 2 < n$ , where  $n = 3$ . Hence we have infinite no of solutions for given system of equations.

Using Gauss - Jordan elimination method to getting the solution,

$$\begin{pmatrix} 1 & -1 & 2 & 1 \\ 2 & 0 & 2 & 1 \\ 1 & -3 & 4 & 2 \end{pmatrix} \xleftrightarrow{R_2=R_1-2R_1} \begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 2 & -2 & -1 \\ 1 & -3 & 4 & 2 \end{pmatrix} \quad (1.3.4.14)$$

$$\xleftrightarrow{R_3=R_3-R_1} \begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 2 & -2 & -1 \\ 0 & -2 & 2 & 1 \end{pmatrix} \quad (1.3.4.15)$$

$$\xleftrightarrow{R_2=\frac{R_2}{2}} \begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & -\frac{1}{2} \\ 0 & -2 & 2 & 1 \end{pmatrix} \quad (1.3.4.16)$$

$$\xleftrightarrow{R_3=R_3+2R_2} \begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (1.3.4.17)$$

$$\xleftrightarrow{R_1=R_1+R_2} \begin{pmatrix} 1 & 0 & 1 & \frac{1}{2} \\ 0 & 1 & -1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (1.3.4.18)$$

$$\Rightarrow x_1 + x_3 = \frac{1}{2}, x_2 - x_3 = -\frac{1}{2} \quad (1.3.4.19)$$

$$\Rightarrow x_2 = -\frac{1}{2} + x_3, x_1 = \frac{1}{2} - x_3 \quad (1.3.4.20)$$

From equation (1.3.4.19) and (1.3.4.20)

$$\mathbf{x} = \begin{pmatrix} \frac{1}{2} - x_3 \\ -\frac{1}{2} + x_3 \\ x_3 \end{pmatrix} \quad (1.3.4.21)$$

which can be written as,

$$\mathbf{x} = x_3 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{pmatrix} \quad (1.3.4.22)$$

from 1.3.4.22 we can say that for any value  $x_3$ ,  $\mathbf{V}$  will not be gives zero vector. Hence the given solution space will not span of the vector space  $\mathbf{V}$

1.3.5. Give an example of a system of two linear equations in two unknowns which has no solution.

**Solution:** Let us assume two equations as given below  $\begin{pmatrix} 5 & 2 \end{pmatrix} \mathbf{x} = 7$  and  $\begin{pmatrix} 10 & 4 \end{pmatrix} \mathbf{x} = -3$

Let the coefficient matrix be given as

$$\mathbf{A} = \begin{pmatrix} 5 & 2 \\ 10 & 4 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 7 \\ -3 \end{pmatrix} \quad (1.3.5.1)$$

the augmented matrix be given as matrix be given as

$$\mathbf{A}|\mathbf{B} = \begin{pmatrix} 5 & 2 & 7 \\ 10 & 4 & -3 \end{pmatrix} \quad (1.3.5.2)$$

Applying row reduction

$$\begin{pmatrix} 5 & 2 & 7 \\ 10 & 4 & -3 \end{pmatrix} \xrightarrow{R_2=R_2-2R_1} \begin{pmatrix} 5 & 2 & 7 \\ 0 & 0 & -17 \end{pmatrix} \quad (1.3.5.3)$$

$$\xrightarrow{R_1=\frac{R_1}{5}} \begin{pmatrix} 1 & \frac{2}{5} & \frac{7}{5} \\ 0 & 0 & -17 \end{pmatrix} \quad (1.3.5.4)$$

$$\xrightarrow{R_2=\frac{R_2}{-17}} \begin{pmatrix} 1 & \frac{2}{5} & \frac{7}{5} \\ 0 & 0 & 1 \end{pmatrix} \quad (1.3.5.5)$$

$$\xrightarrow{R_1=R_1-\frac{7}{5}R_2} \begin{pmatrix} 1 & \frac{2}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (1.3.5.6)$$

$$(1.3.5.7)$$

Clearly, On comparing the ranks of matrix  $\mathbf{A}$  and  $\mathbf{A}|\mathbf{B}$ , we find that rank of matrix  $\mathbf{A}|\mathbf{B} \neq \mathbf{A}$ . Hence the system of linear equation have no solutions. Consider the system  $\mathbf{Ax} = \mathbf{b}$ , with coefficient matrix  $\mathbf{A}$  and augmented matrix  $\mathbf{A}|\mathbf{B}$ .

As above, the sizes of  $\mathbf{b}$ ,  $\mathbf{A}$ , and  $\mathbf{A}|\mathbf{B}$  are  $m \times 1$ ,  $m \times n$ , and  $m \times (n + 1)$ , respectively; in addition, the number of unknowns is  $n$ .

$\mathbf{Ax}$  is inconsistent (i.e., no solution exists) if 1.3.7. Find all solutions of and only if  $\text{rank } \mathbf{A} < \text{rank } \mathbf{A}|\mathbf{B}$ .

1.3.6. Find all solutions of

$$x_1 - 2x_2 + x_3 + 2x_4 = 1$$

$$x_1 + x_2 - x_3 + x_4 + x_5 = 2$$

$$x_1 + 7x_2 - 5x_3 - x_4 = 3$$

**Solution:** The given equations can be written as,

$$\mathbf{Ax} = \mathbf{B} \quad (1.3.6.1)$$

$$\begin{pmatrix} 1 & -2 & 1 & 2 \\ 1 & 1 & -1 & 1 \\ 1 & 7 & -5 & -1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad (1.3.6.2)$$

Now, we form the augmented matrix and per-

form Row reduction,

$$\begin{pmatrix} 1 & -2 & 1 & 2 & | & 1 \\ 1 & 1 & -1 & 1 & | & 2 \\ 1 & 7 & -5 & -1 & | & 3 \end{pmatrix} \quad (1.3.6.3)$$

$$\xrightarrow{R_2=R_2-R_1, R_3=R_3-R_1} \begin{pmatrix} 1 & -2 & 1 & 2 & | & 1 \\ 0 & 3 & -2 & -1 & | & 1 \\ 0 & 9 & -6 & -3 & | & 2 \end{pmatrix} \quad (1.3.6.4)$$

$$\xrightarrow{R_2=\frac{1}{3}R_2} \begin{pmatrix} 1 & -2 & 1 & 2 & | & 1 \\ 0 & 1 & -\frac{2}{3} & -\frac{1}{3} & | & \frac{1}{3} \\ 0 & 9 & -6 & -3 & | & 2 \end{pmatrix} \quad (1.3.6.5)$$

$$\xrightarrow{R_3=R_3-9R_2} \begin{pmatrix} 1 & -2 & 1 & 2 & | & 1 \\ 0 & 1 & -\frac{2}{3} & -\frac{1}{3} & | & \frac{1}{3} \\ 0 & 0 & 0 & 0 & | & -1 \end{pmatrix} \quad (1.3.6.6)$$

Rank of  $\mathbf{A}$  is less than rank of the augmented matrix. Hence, the given system has no solution.

1.3.7. Find all solutions of

$$2x_1 - 3x_2 - 7x_3 + 5x_4 + 2x_5 = -2 \quad (1.3.7.1)$$

$$x_1 - 2x_2 - 4x_3 + 3x_4 + x_5 = -2 \quad (1.3.7.2)$$

$$2x_1 - 4x_3 + 2x_4 + x_5 = 3 \quad (1.3.7.3)$$

$$x_1 - 5x_2 - 7x_3 + 6x_4 + 2x_5 = -7 \quad (1.3.7.4)$$

**Solution:** The given equations can be written as,

$$\begin{pmatrix} 2 & -3 & -7 & 5 & 2 \\ 1 & -2 & -4 & 3 & 1 \\ 2 & 0 & -4 & 2 & 1 \\ 1 & -5 & -7 & 6 & 2 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -2 \\ -2 \\ 3 \\ 7 \end{pmatrix} \quad (1.3.7.5)$$

Now, we form the augmented matrix and per-

form Row reduction,

$$\left(\begin{array}{ccccc|c} 2 & -3 & -7 & 5 & 2 & -2 \\ 1 & -2 & -4 & 3 & 1 & -2 \\ 2 & 0 & -4 & 2 & 1 & 3 \\ 1 & -5 & -7 & 6 & 2 & 7 \end{array}\right) \quad (1.3.7.6)$$

$$\xleftrightarrow{R_3=R_3-R_1} \left(\begin{array}{ccccc|c} 2 & -3 & -7 & 5 & 2 & -2 \\ 1 & -2 & -4 & 3 & 1 & -2 \\ 0 & 3 & 3 & -3 & -1 & 5 \\ 1 & -5 & -7 & 6 & 2 & 7 \end{array}\right) \quad (1.3.7.7)$$

$$\xleftrightarrow{R_1=\frac{1}{2}R_1} \left(\begin{array}{ccccc|c} 1 & -\frac{3}{2} & -\frac{7}{2} & \frac{5}{2} & 1 & -1 \\ 1 & -2 & -4 & 3 & 1 & -2 \\ 0 & 3 & 3 & -3 & -1 & 5 \\ 1 & -5 & -7 & 6 & 2 & 7 \end{array}\right) \quad (1.3.7.8)$$

$$\xleftrightarrow{R_2=R_2-R_1, R_4=R_4-R_1} \left(\begin{array}{ccccc|c} 1 & -\frac{3}{2} & -\frac{7}{2} & \frac{5}{2} & 1 & -1 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & -1 \\ 0 & 3 & 3 & -3 & -1 & 5 \\ 0 & -\frac{7}{2} & -\frac{7}{2} & \frac{7}{2} & 1 & -6 \end{array}\right) \quad (1.3.7.9)$$

$$\xleftrightarrow{R_1=R_1-3R_2} \left(\begin{array}{ccccc|c} 1 & 0 & -2 & 1 & 1 & 2 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & -1 \\ 0 & 3 & 3 & -3 & -1 & 5 \\ 0 & -\frac{7}{2} & -\frac{7}{2} & \frac{7}{2} & 1 & -6 \end{array}\right) \quad (1.3.7.10)$$

$$\xleftrightarrow{R_3=R_3+6R_2, R_4=R_4-7R_2} \left(\begin{array}{ccccc|c} 1 & 0 & -2 & 1 & 1 & 2 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right) \quad (1.3.7.11)$$

$$\xleftrightarrow{R_2=-2R_2} \left(\begin{array}{ccccc|c} 1 & 0 & -2 & 1 & 1 & 2 \\ 0 & 1 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right) \quad (1.3.7.12)$$

$$\xleftrightarrow{R_1=R_1+R_3, R_4=R_4+R_3, R_3=-R_3} \left(\begin{array}{ccccc|c} 1 & 0 & -2 & 1 & 0 & 1 \\ 0 & 1 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad (1.3.7.13)$$

So,

$$x_1 - 2x_3 + x_4 = 1 \quad (1.3.7.14)$$

$$x_2 + x_3 - x_4 = 2 \quad (1.3.7.15)$$

$$x_5 = 1 \quad (1.3.7.16)$$

Solving the equations we get,

$$x_1 = 1 + 2x_3 - x_4 \quad (1.3.7.17)$$

$$x_2 = 2 - x_3 + x_4 \quad (1.3.7.18)$$

$$x_5 = 1 \quad (1.3.7.19)$$

which can be written as,

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \quad (1.3.7.20)$$

$$\Rightarrow \mathbf{x} = \begin{pmatrix} 1 + 2x_3 - x_4 \\ 2 - x_3 + x_4 \\ x_3 \\ x_4 \\ 1 \end{pmatrix} \quad (1.3.7.21)$$

We can express (1.3.7.21) as a sum of linear combination of vectors,

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} x_3 + \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} x_4 \quad (1.3.7.22)$$

where  $x_3, x_4 \in \mathbb{R}$ .

Note that the above solution space is not closed on vector addition and scalar multiplication. As  $x_5 = 1$ , the zero vector is not included in the solution space. Hence,  $\mathbf{x}$  is not a vector space. Since,  $\mathbf{x}$  is not a vector space, it cannot be expressed in the form of linear combination of basis vectors.

1.3.8. Let

$$\mathbf{A} = \begin{pmatrix} 3 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 0 \end{pmatrix} \quad (1.3.8.1)$$

For which triples  $(y_1, y_2, y_3)$  does the system  $\mathbf{AX} = \mathbf{Y}$  have a solution ?

**Solution:**

Given ,

$$\mathbf{AX} = \mathbf{Y} \quad (1.3.8.2)$$

$$\begin{pmatrix} 3 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 0 \end{pmatrix} \mathbf{X} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \quad (1.3.8.3)$$

Now we try to find the matrix  $\mathbf{B}$  such that  $\mathbf{BA}$  gives the row echelon form of matrix  $\mathbf{A}$ .

Here,  $\mathbf{B}$  is given by ,

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ -\frac{7}{5} & \frac{8}{5} & 1 \end{pmatrix} \quad (1.3.8.4)$$

$$\Rightarrow \mathbf{BA} = \begin{pmatrix} 3 & -1 & 2 \\ 0 & \frac{5}{3} & -\frac{1}{3} \\ 0 & 0 & -\frac{6}{5} \end{pmatrix} \quad (1.3.8.5)$$

Therefore, from (1.3.8.5) rank of matrix  $\mathbf{A}$  is 3 and it is a full rank matrix.

Hence the columns of  $\mathbf{A}$  are linearly independent.

Therefore, the triples  $(y_1, y_2, y_3)$  are linear combination of columns of matrix  $\mathbf{A}$ .

$$\Rightarrow \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = a \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + b \begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix} + c \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \quad (1.3.8.6)$$

where a,b,c can be any real value.

1.3.9. Let

$$\mathbf{A} = \begin{pmatrix} 3 & -6 & 2 & -1 \\ -2 & 4 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 1 & -2 & 1 & 0 \end{pmatrix} \quad (1.3.9.1)$$

For which  $(y_1, y_2, y_3, y_4)$  does the system of equations  $\mathbf{AX} = \mathbf{Y}$  have a solution ? **Solution:** Given ,

$$\mathbf{AX} = \mathbf{Y} \quad (1.3.9.2)$$

$$\begin{pmatrix} 3 & -6 & 2 & -1 \\ -2 & 4 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 1 & -2 & 1 & 0 \end{pmatrix} \mathbf{X} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} \quad (1.3.9.3)$$

Now we try to find the matrix  $\mathbf{B}$  such that  $\mathbf{BA}$  gives the row echelon form of matrix  $\mathbf{A}$  Here,  $\mathbf{B}$  is given by ,

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{2}{3} & 1 & 0 & 0 \\ -\frac{2}{7} & -\frac{3}{7} & 1 & 0 \\ 0 & \frac{1}{2} & -\frac{3}{2} & 1 \end{pmatrix} \quad (1.3.9.4)$$

$$\mathbf{BA} = \begin{pmatrix} 3 & -6 & 2 & -1 \\ 0 & 0 & \frac{7}{3} & \frac{7}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (1.3.9.5)$$

Therefore, rank of matrix  $\mathbf{A}$  is 2 Now  $\mathbf{B}$  is

expressed in terms of two block matrices

$$\mathbf{B} = \begin{pmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{pmatrix} \quad (1.3.9.6)$$

$$\mathbf{B}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{2}{3} & 1 & 0 & 0 \end{pmatrix} \quad (1.3.9.7)$$

$$\mathbf{B}_2 = \begin{pmatrix} -\frac{2}{7} & -\frac{3}{7} & 1 & 0 \\ 0 & \frac{1}{2} & -\frac{3}{2} & 1 \end{pmatrix} \quad (1.3.9.8)$$

Multiplying matrix  $\mathbf{B}$  to both sides on the equation (1.3.9.2), we get ,

$$\begin{pmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{pmatrix} \mathbf{AX} = \begin{pmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{pmatrix} \mathbf{Y} \quad (1.3.9.9)$$

We know that , matrix  $\mathbf{A}$  is of rank 2 The augmented matrix of (1.3.9.9) is given by

$$\left( \begin{array}{cc|cc} \mathbf{B}_1\mathbf{A} & & \mathbf{B}_1\mathbf{Y} \\ \mathbf{B}_2\mathbf{A} & & \mathbf{B}_2\mathbf{Y} \end{array} \right) \quad (1.3.9.10)$$

$$\mathbf{B}_1\mathbf{A} = \begin{pmatrix} 3 & -6 & 2 & -1 \\ 0 & 0 & \frac{7}{3} & \frac{7}{3} \end{pmatrix} \quad (1.3.9.11)$$

$$\mathbf{B}_2\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (1.3.9.12)$$

Since  $\mathbf{B}_2\mathbf{A}$  is zero matrix and for the given system  $\mathbf{AX} = \mathbf{Y}$  to have a solution,

$$\mathbf{B}_2\mathbf{Y} = 0 \quad (1.3.9.13)$$

$$\begin{pmatrix} -\frac{2}{7} & -\frac{3}{7} & 1 & 0 \\ 0 & \frac{1}{2} & -\frac{3}{2} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = 0 \quad (1.3.9.14)$$

The augmented matrix of (1.3.9.14) is given by,

$$\left( \begin{array}{cccc|c} -\frac{2}{7} & -\frac{3}{7} & 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{3}{2} & 1 & 0 \end{array} \right) \quad (1.3.9.15)$$

By row reduction technique,

$$\xleftrightarrow{R_1 = -\frac{7}{2}R_1} \left( \begin{array}{cccc|c} 1 & \frac{3}{2} & -\frac{7}{2} & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{3}{2} & 1 & 0 \end{array} \right) \quad (1.3.9.16)$$

$$\xleftrightarrow{R_2 = 2R_2} \left( \begin{array}{cccc|c} 1 & \frac{3}{2} & -\frac{7}{2} & 0 & 0 \\ 0 & 1 & -3 & 2 & 0 \end{array} \right) \quad (1.3.9.17)$$

$$\xleftrightarrow{R_1 = R_1 - \frac{3}{2}R_2} \left( \begin{array}{cccc|c} 1 & 0 & 1 & -3 & 0 \\ 0 & 1 & -3 & 2 & 0 \end{array} \right) \quad (1.3.9.18)$$

Equation (1.3.9.14) can be modified as ,

$$\begin{pmatrix} 1 & 0 & 1 & -3 \\ 0 & 1 & -3 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = 0 \quad (1.3.9.19)$$

Here  $y_3$  and  $y_4$  are free variables

If  $y_3 = a$  and  $y_4 = b$ , then the solution to the system of equation  $\mathbf{AX} = \mathbf{Y}$  is given by,

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = a \begin{pmatrix} -1 \\ 3 \\ 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 3 \\ -2 \\ 0 \\ 1 \end{pmatrix} \quad (1.3.9.20)$$

One of the solution when  $a = 1$  and  $b = 2$  is given by ,

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 1 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 3 \\ -2 \\ 0 \\ 1 \end{pmatrix} \quad (1.3.9.21)$$

1.3.10. Suppose  $\mathbf{R}$  and  $\mathbf{R}'$  are  $2 \times 3$  row-reduced echelon matrices and that the system  $\mathbf{RX}=0$  and  $\mathbf{R}'\mathbf{X}=0$  have exactly the same solutions. Prove that  $\mathbf{R} = \mathbf{R}'$ .

**Solution:**

Since  $\mathbf{R}$  and  $\mathbf{R}'$  are  $2 \times 3$  row-reduced echelon matrices they can be of following three types:-

- a) Suppose matrix  $\mathbf{R}$  has one non-zero row then  $\mathbf{RX}=0$  will have two free variables. Since  $\mathbf{R}'\mathbf{X}=0$  will have the exact same solution as  $\mathbf{RX} = 0$ ,  $\mathbf{R}'\mathbf{X}=0$  will also have two free variables. Thus  $\mathbf{R}'$  have one non zero row. Now let's consider a matrix  $\mathbf{A}$  with the first row as the non-zero row  $\mathbf{R}$  and second row as the second row of  $\mathbf{R}'$ .

$$\mathbf{R} = \begin{pmatrix} 1 & a & b \\ 0 & 0 & 0 \end{pmatrix} \quad (1.3.10.1)$$

$$\mathbf{R}' = \begin{pmatrix} 1 & c & d \\ 0 & 0 & 0 \end{pmatrix} \quad (1.3.10.2)$$

$$(1.3.10.3)$$

Let  $\mathbf{X}$  satisfy

$$\mathbf{RX} = 0 \quad (1.3.10.4)$$

$$\begin{pmatrix} 1 & \mathbf{a}^T \end{pmatrix} \begin{pmatrix} x \\ \mathbf{y} \end{pmatrix} = 0 \quad (1.3.10.5)$$

$$x + \mathbf{a}^T \mathbf{y} = 0 \quad (1.3.10.6)$$

where

$$\mathbf{a} = \begin{pmatrix} a \\ b \end{pmatrix} \quad (1.3.10.7)$$

$$\mathbf{R}'\mathbf{X} = 0 \quad (1.3.10.8)$$

$$\begin{pmatrix} 1 & \mathbf{b}^T \end{pmatrix} \begin{pmatrix} x \\ \mathbf{y} \end{pmatrix} = 0 \quad (1.3.10.9)$$

$$x + \mathbf{b}^T \mathbf{y} = 0 \quad (1.3.10.10)$$

where

$$\mathbf{b} = \begin{pmatrix} c \\ d \end{pmatrix} \quad (1.3.10.11)$$

Subtracting (1.3.10.10) from (1.3.10.6),

$$x + \mathbf{a}^T \mathbf{y} - x - \mathbf{b}^T \mathbf{y} = 0 \quad (1.3.10.12)$$

$$(\mathbf{a}^T - \mathbf{b}^T) \mathbf{y} = 0 \quad (1.3.10.13)$$

Since  $\mathbf{y}$  is a  $2 \times 1$  vector,

$$\Rightarrow y_1 \mathbf{a} - y_2 \mathbf{b} = 0 \quad (1.3.10.14)$$

Which can be written as,

$$\mathbf{a} = k\mathbf{b} \quad (1.3.10.15)$$

where,  $k = \frac{y_2}{y_1}$  assuming  $y_1 \neq 0$ . Now, Substituting (1.3.10.15) in (1.3.10.6)

$$x + k\mathbf{b}^T \mathbf{y} = 0 \quad (1.3.10.16)$$

Comparing (1.3.10.16) with (1.3.10.10)

$$x + \mathbf{b}^T \mathbf{y} = 0 \quad (1.3.10.17)$$

$$x + k\mathbf{b}^T \mathbf{y} = 0 \quad (1.3.10.18)$$

Hence  $k=1$  which means  $y_1=y_2$  and from this we can say that  $\mathbf{a}=\mathbf{b}$ . If in the above case we take  $y_1=0$  then

$$y_1 \mathbf{a} - y_2 \mathbf{b} = 0 \quad (1.3.10.19)$$

$$y_2 \mathbf{b} = 0 \quad (1.3.10.20)$$

Hence for the (1.3.10.20) to be always true  $\mathbf{b}$  should be zero. Now from (1.3.10.15) we will see that  $\mathbf{a}$  will also be 0. Hence,  $\mathbf{R}=\mathbf{R}'$

- b) Let  $\mathbf{R}$  and  $\mathbf{R}'$  have all rows as non zero.

$$\mathbf{R} = \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & c \end{pmatrix} \quad (1.3.10.21)$$

$$\mathbf{R}' = \begin{pmatrix} 1 & 0 & e \\ 0 & 1 & f \end{pmatrix} \quad (1.3.10.22)$$

Let  $\mathbf{X}$  satisfy

$$\mathbf{R}\mathbf{X} = 0 \quad (1.3.10.23)$$

$$\mathbf{X}^T \mathbf{R}^T = 0 \quad (1.3.10.24)$$

Here,

$$\mathbf{R} = \begin{pmatrix} \mathbf{I} & \mathbf{a} \end{pmatrix} \quad (1.3.10.25) \quad 1.4 \text{ Matrix Multiplication}$$

$$\mathbf{a} = \begin{pmatrix} b \\ c \end{pmatrix} \quad (1.3.10.26) \quad 1.4.1. \text{ Let}$$

$$\mathbf{R}^T = \begin{pmatrix} \mathbf{I} \\ \mathbf{a}^T \end{pmatrix} \quad (1.3.10.27)$$

Let,

$$\mathbf{X}^T = \begin{pmatrix} \mathbf{y}^T & z \end{pmatrix} \quad (1.3.10.28)$$

where  $z$  is a scalar constant. Now, substituting (1.3.10.28) and (1.3.10.25) in (1.3.10.24)

$$\begin{pmatrix} \mathbf{y}^T & z \end{pmatrix} \begin{pmatrix} \mathbf{I} \\ \mathbf{a}^T \end{pmatrix} = 0 \quad (1.3.10.29)$$

$$\mathbf{y}^T + z\mathbf{a}^T = 0 \quad (1.3.10.30)$$

Now for,

$$\mathbf{R}'\mathbf{X} = 0 \quad (1.3.10.31)$$

$$\mathbf{X}^T \mathbf{R}'^T = 0 \quad (1.3.10.32)$$

Here,

$$\mathbf{R}' = \begin{pmatrix} \mathbf{I} & \mathbf{b} \end{pmatrix} \quad (1.3.10.33)$$

$$\mathbf{b} = \begin{pmatrix} e \\ f \end{pmatrix} \quad (1.3.10.34)$$

Let,

$$\mathbf{X}^T = \begin{pmatrix} \mathbf{y}^T & z \end{pmatrix} \quad (1.3.10.35)$$

where  $z$  is a scalar constant. Now, substituting (1.3.10.35) and (1.3.10.33) in (1.3.10.32)

$$\begin{pmatrix} \mathbf{y}^T & z \end{pmatrix} \begin{pmatrix} \mathbf{I} \\ \mathbf{b}^T \end{pmatrix} = 0 \quad (1.3.10.36)$$

$$\mathbf{y}^T + z\mathbf{b}^T = 0 \quad (1.3.10.37)$$

Subtracting (1.3.10.37) from (1.3.10.30)

$$\mathbf{y}^T + z\mathbf{a}^T - \mathbf{y}^T - z\mathbf{b}^T = 0 \quad (1.3.10.38)$$

$$(\mathbf{a}^T - \mathbf{b}^T)z = 0 \quad (1.3.10.39)$$

$$\mathbf{a}^T = \mathbf{b}^T \quad (1.3.10.40)$$

c) Suppose matrix  $\mathbf{R}$  have all the rows as zero

then  $\mathbf{R}\mathbf{X}=0$  will be satisfied for all values of  $\mathbf{X}$ . We know that  $\mathbf{R}'\mathbf{X}=0$  will have the exact same solution as  $\mathbf{R}\mathbf{X}=0$  then we can say that for all values of  $\mathbf{X}=0$  equation  $\mathbf{R}'\mathbf{X}=0$  will be satisfied. Hence,  $\mathbf{R}'=\mathbf{R}=0$ .

Compute  $\mathbf{ABC}$  and  $\mathbf{CAB}$ .

**Solution:** Given,

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 2 & 1 \end{pmatrix} \quad (1.4.1.2)$$

$$\mathbf{B} = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} \quad (1.4.1.3)$$

$$\mathbf{C} = \begin{pmatrix} 1 & -1 \end{pmatrix} \quad (1.4.1.4)$$

Take,  $\mathbf{ABC} = (\mathbf{AB}) \mathbf{C}$

$$\mathbf{AB} = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} \quad (1.4.1.5)$$

$$\mathbf{AB} = \begin{pmatrix} 6-1-1 \\ 3+2-1 \end{pmatrix} \quad (1.4.1.6)$$

$$\mathbf{AB} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \quad (1.4.1.7)$$

Now,

$$\mathbf{ABC} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \begin{pmatrix} 1 & -1 \end{pmatrix} \quad (1.4.1.8)$$

$$\mathbf{ABC} = \begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix} \quad (1.4.1.9)$$

similarly,  $\mathbf{CAB} = \mathbf{C}(\mathbf{AB})$

$$\mathbf{CAB} = \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 4 \\ 4 \end{pmatrix} \quad (1.4.1.10)$$

$$\Rightarrow \mathbf{CAB} = 0 \quad (1.4.1.11)$$

therefore,

$$\mathbf{ABC} = \begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix} \quad (1.4.1.12)$$

$$\mathbf{CAB} = 0 \quad (1.4.1.13)$$

1.4.2. Let

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 2 & -2 \\ 1 & 3 \\ 4 & 4 \end{pmatrix} \quad (1.4.2.1)$$

Verify directly that  $A(AB) = A^2B$  **Solution:**

$$A^2 = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{pmatrix} \quad (1.4.2.2)$$

$$A^2 = \begin{pmatrix} 2 & -1 & 1 \\ 5 & -2 & 3 \\ 6 & -3 & 4 \end{pmatrix} \quad (1.4.2.3)$$

and

$$AB = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & -2 \\ 1 & 3 \\ 4 & 4 \end{pmatrix} \quad (1.4.2.4)$$

$$AB = \begin{pmatrix} 5 & -1 \\ 8 & 0 \\ 10 & -2 \end{pmatrix} \quad (1.4.2.5)$$

Now RHS is

$$A^2B = \begin{pmatrix} 2 & -1 & 1 \\ 5 & -2 & 3 \\ 6 & -3 & 4 \end{pmatrix} \begin{pmatrix} 2 & -2 \\ 1 & 3 \\ 4 & 4 \end{pmatrix} \quad (1.4.2.6)$$

$$A^2B = \begin{pmatrix} 7 & -3 \\ 20 & -4 \\ 25 & -5 \end{pmatrix} \quad (1.4.2.7)$$

Now LHS is

$$A(AB) = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 5 & -1 \\ 8 & 0 \\ 10 & -2 \end{pmatrix} \quad (1.4.2.8)$$

$$A(AB) = \begin{pmatrix} 7 & -3 \\ 20 & -4 \\ 25 & -5 \end{pmatrix} \quad (1.4.2.9)$$

Hence verified.

1.4.3. Find two different  $2 \times 2$  matrices  $\mathbf{A}$  such that  $\mathbf{A}^2 = 0$  but  $\mathbf{A} \neq 0$

**Solution:** The matrix  $\mathbf{A}$  can be given by,

$$\mathbf{A} = \begin{pmatrix} \mathbf{m} & \mathbf{n} \end{pmatrix} \quad (1.4.3.1)$$

$$\mathbf{m} = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, \mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \quad (1.4.3.2)$$

Now,

$$\mathbf{A}^2 = \mathbf{A}\mathbf{A} = \mathbf{0} \quad (1.4.3.3)$$

$$\Rightarrow \mathbf{A}^2 = \begin{pmatrix} \mathbf{A}\mathbf{m} & \mathbf{A}\mathbf{n} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \end{pmatrix} \quad (1.4.3.4)$$

From (1.4.3.4), we say that the null space of  $\mathbf{A}$  contains columns of matrix  $\mathbf{A}$ . Also atleast one of the columns must be non-zero since given  $\mathbf{A} \neq 0$ . Thus, the null space of  $\mathbf{A}$  contains non zero vectors,  $\text{rank}(\mathbf{A}) < 2$ . Hence,  $\mathbf{A}$  is a singular matrix. This implies that the columns of  $\mathbf{A}$  are linearly dependent.

$$\mathbf{A}\mathbf{x} = 0 \quad (1.4.3.5)$$

$$\begin{pmatrix} \mathbf{m} & \mathbf{n} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \quad (1.4.3.6)$$

$$x_1\mathbf{m} + x_2\mathbf{n} = 0 \quad (1.4.3.7)$$

$$\mathbf{n} = \frac{-x_1}{x_2}\mathbf{m} \quad (1.4.3.8)$$

$$\Rightarrow \mathbf{n} = k\mathbf{m} \quad (1.4.3.9)$$

where  $\mathbf{m} \neq 0$  as  $\mathbf{A} \neq 0$

Now from (1.4.3.4),

$$\mathbf{A}\mathbf{m} = 0 \quad (1.4.3.10)$$

$$m_1\mathbf{m} + m_2\mathbf{n} = 0 \quad (1.4.3.11)$$

$$(m_1 + km_2)\mathbf{m} = 0 \quad (1.4.3.12)$$

Thus we get,  $m_1 = -km_2$

$$\mathbf{A} = \begin{pmatrix} -km_2 & -k^2m_2 \\ m_2 & km_2 \end{pmatrix}; m_2 \neq 0 \quad (1.4.3.13)$$

(1.4.3.9) can be written as,

$$\Rightarrow \mathbf{m} = \frac{1}{k}\mathbf{n} \quad (1.4.3.14)$$

$$\Rightarrow \mathbf{m} = c\mathbf{n} \quad (1.4.3.15)$$

where  $\mathbf{n} \neq 0$  as  $\mathbf{A} \neq 0$

From (1.4.3.4),

$$\mathbf{A}\mathbf{n} = 0 \quad (1.4.3.16)$$

$$n_1\mathbf{m} + n_2\mathbf{n} = 0 \quad (1.4.3.17)$$

$$(cn_1 + n_2)\mathbf{n} = 0 \quad (1.4.3.18)$$

Thus we get,  $n_2 = -cn_1$

$$\mathbf{A} = \begin{pmatrix} cn_1 & n_1 \\ -c^2n_1 & -cn_1 \end{pmatrix}; n_1 \neq 0 \quad (1.4.3.19)$$

From (1.4.3.13), (1.4.3.19) two different  $2 \times 2$

matrices  $\mathbf{A}$  can be given as,

$$\mathbf{A} = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \quad (1.4.3.20)$$

$$\mathbf{A} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \quad (1.4.3.21)$$

1.4.4. For the matrix  $\mathbf{A} = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{pmatrix}$ , find elementary matrices  $\mathbf{E}_1, \mathbf{E}_2, \dots, \mathbf{E}_k$  such that

$$\mathbf{E}_k \dots \mathbf{E}_2 \mathbf{E}_1 \mathbf{A} = \mathbf{I} \quad (1.4.4.1)$$

**Solution:** Given,

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{pmatrix} \quad (1.4.4.2)$$

Take,

$$\mathbf{E}_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (1.4.4.3)$$

$$\mathbf{E}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix} \quad (1.4.4.4)$$

$$\mathbf{E}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (1.4.4.5)$$

$$\mathbf{E}_4 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (1.4.4.6)$$

$$\mathbf{E}_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix} \quad (1.4.4.7)$$

$$\mathbf{E}_6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{2}{7} \end{pmatrix} \quad (1.4.4.8)$$

$$\mathbf{E}_7 = \begin{pmatrix} 1 & 0 & \frac{-1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (1.4.4.9)$$

$$\mathbf{E}_8 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix} \quad (1.4.4.10)$$

Now, we calculate

$$\mathbf{E}_8 \mathbf{E}_7 \mathbf{E}_6 \mathbf{E}_5 \mathbf{E}_4 \mathbf{E}_3 \mathbf{E}_2 \mathbf{E}_1 = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 0 & 3 & -2 \end{pmatrix} \quad (1.4.4.11)$$

Hence,

$$(\mathbf{E}_8 \mathbf{E}_7 \mathbf{E}_6 \mathbf{E}_5 \mathbf{E}_4 \mathbf{E}_3 \mathbf{E}_2 \mathbf{E}_1) \mathbf{A} = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 0 & 3 & -2 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (1.4.4.12)$$

1.4.5. Let  $A = \begin{pmatrix} 1 & -1 \\ 2 & 2 \\ 1 & 0 \end{pmatrix}$  and  $B = \begin{pmatrix} 3 & 1 \\ -4 & 4 \end{pmatrix}$  Is there any matrix  $C$  such that  $CA = B$ ?

**Solution:** The matrix  $B$  is obtained by multiplying the matrix  $A$  with matrix  $C$ .  $B$  is a  $2 \times 2$  matrix and  $A$  is a  $3 \times 2$  matrix. so matrix  $C$  must be a  $2 \times 3$  matrix. Let the matrix  $C$  is:

$$C = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix} \quad (1.4.5.1)$$

$$\Rightarrow C^T = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \end{pmatrix} \quad (1.4.5.2)$$

So, after multiplying with  $A$  matrix we get,

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 2 & 2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a_1 + 2b_1 + c_1 & -a_1 + 2b_1 \\ a_2 + 2b_2 + c_2 & -a_2 + 2b_2 \end{pmatrix} \quad (1.4.5.3)$$

Matrix  $A$  is a rectangular matrix. Now, Considering  $CA = B$  and by transposing both side,

$$(CA)^T = B^T \quad (1.4.5.4)$$

$$\Rightarrow A^T C^T = B^T \quad (1.4.5.5)$$

$$\Rightarrow \begin{pmatrix} 1 & 2 & 1 \\ -1 & 2 & 0 \end{pmatrix} \begin{pmatrix} c_1 & c_2 \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 1 & 4 \end{pmatrix} \quad (1.4.5.6)$$



We can represent it like this:

$$\begin{pmatrix} 1 & 2 & 1 \\ -1 & 2 & 0 \end{pmatrix} \mathbf{c}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad (1.4.5.7)$$

$$(1.4.5.8)$$

Now the augmented matrix is:

$$\begin{pmatrix} 1 & 2 & 1 & 3 \\ -1 & 2 & 0 & 1 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_1 + R_2} \begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 4 & 1 & 4 \end{pmatrix}$$

$$\xrightarrow{R_2 \leftarrow R_2/2} \begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 2 & \frac{1}{2} & 2 \end{pmatrix} \xrightarrow{R_1 \leftarrow R_1 - R_2} \begin{pmatrix} 1 & 0 & \frac{1}{2} & 1 \\ 0 & 2 & \frac{1}{2} & 2 \end{pmatrix}$$

$$\xrightarrow{R_2 \leftarrow R_2/2} \begin{pmatrix} 1 & 0 & \frac{1}{2} & 1 \\ 0 & 1 & \frac{1}{4} & 1 \end{pmatrix} \quad (1.4.5.9)$$

Similarly,

$$\begin{pmatrix} 1 & 2 & 1 \\ -1 & 2 & 0 \end{pmatrix} \mathbf{c}_2 = \begin{pmatrix} -4 \\ 4 \end{pmatrix} \quad (1.4.5.10)$$

$$(1.4.5.11)$$

Now the augmented matrix is:

$$\begin{pmatrix} 1 & 2 & 1 & -4 \\ -1 & 2 & 0 & 4 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_1 + R_2} \begin{pmatrix} 1 & 2 & 1 & -4 \\ 0 & 4 & 1 & 0 \end{pmatrix}$$

$$\xrightarrow{R_2 \leftarrow R_2/2} \begin{pmatrix} 1 & 2 & 1 & -4 \\ 0 & 2 & \frac{1}{2} & 0 \end{pmatrix} \xrightarrow{R_1 \leftarrow R_1 - R_2} \begin{pmatrix} 1 & 0 & \frac{1}{2} & -4 \\ 0 & 2 & \frac{1}{2} & 0 \end{pmatrix}$$

$$\xrightarrow{R_2 \leftarrow R_2/2} \begin{pmatrix} 1 & 0 & \frac{1}{2} & -4 \\ 0 & 1 & \frac{1}{4} & 0 \end{pmatrix} \quad (1.4.5.12)$$

From equations 1.4.5.9 and 1.4.5.12, it can be observed that solutions exist and there is a matrix  $C$  such that  $CA = B$ . Now,

$$\mathbf{c}_1 = \begin{pmatrix} 1 - \frac{c_1}{2} \\ 1 - \frac{c_1}{4} \\ c_1 \end{pmatrix} \quad (1.4.5.13)$$

$$\Rightarrow \mathbf{c}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_1 \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{4} \\ 1 \end{pmatrix} \quad (1.4.5.14)$$

$$\mathbf{c}_2 = \begin{pmatrix} -4 - \frac{c_2}{2} \\ -\frac{c_2}{4} \\ c_2 \end{pmatrix} \quad (1.4.5.15)$$

$$\Rightarrow \mathbf{c}_2 = \begin{pmatrix} -4 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{4} \\ 1 \end{pmatrix} \quad (1.4.5.16)$$

Now,

$$C^T = \begin{pmatrix} 1 & -4 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} + c_1 \begin{pmatrix} -\frac{1}{2} & 0 \\ -\frac{1}{4} & 0 \\ 1 & 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 & -\frac{1}{2} \\ 0 & -\frac{1}{4} \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow C = \begin{pmatrix} 1 & 1 & 0 \\ -4 & 0 & 0 \end{pmatrix} + c_1 \begin{pmatrix} -\frac{1}{2} & -\frac{1}{4} & 1 \\ 0 & 0 & 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{4} & 1 \end{pmatrix} \quad (1.4.5.17)$$

Now,

$$CA = \begin{pmatrix} 1 & 1 & 0 \\ -4 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 2 & 2 \\ 1 & 0 \end{pmatrix}$$

$$+ c_1 \begin{pmatrix} -\frac{1}{2} & -\frac{1}{4} & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 2 & 2 \\ 1 & 0 \end{pmatrix}$$

$$+ c_2 \begin{pmatrix} 0 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{4} & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 2 & 2 \\ 1 & 0 \end{pmatrix}$$

$$\Rightarrow CA = \begin{pmatrix} 3 & 1 \\ -4 & 4 \end{pmatrix} + c_1 \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow CA = B \quad (1.4.5.18)$$

Hence, it is proved that there exist a matrix  $C$  such that  $CA = B$ .

Let  $\mathbf{A}$  be an  $m \times n$  matrix and  $\mathbf{B}$  be an  $n \times k$  matrix. Show that the columns of  $\mathbf{C} = \mathbf{AB}$  are linear combinations of columns of  $\mathbf{A}$ . If  $\alpha_1, \alpha_2, \dots, \alpha_n$  are the columns of  $\mathbf{A}$  and  $\gamma_1, \gamma_2, \dots, \gamma_k$  are the columns of  $\mathbf{C}$  then,

$$\gamma_j = \sum_{r=1}^n B_{rj} \alpha_r \quad (1.4.6.1)$$

**Solution:**

$$\mathbf{C} = \mathbf{AB} \quad (1.4.6.2)$$

$$\mathbf{C} = (\gamma_1 \ \gamma_2 \ \dots \ \gamma_k) \quad (1.4.6.3)$$

$$\mathbf{A} = (\alpha_1 \ \alpha_2 \ \dots \ \alpha_n) \quad (1.4.6.4)$$

$$\mathbf{B} = (\beta_1 \ \beta_2 \ \dots \ \beta_k) \quad (1.4.6.5)$$

$$= \begin{pmatrix} B_{11} & B_{12} & \dots & B_{1k} \\ B_{21} & B_{22} & \dots & B_{2k} \\ \vdots & \vdots & \dots & \vdots \\ B_{n1} & B_{n2} & \dots & B_{nk} \end{pmatrix} \quad (1.4.6.6)$$

By matrix multiplication, we can write

$$(\gamma_1 \ \gamma_2 \ \dots \ \gamma_k) = (A\beta_1 \ A\beta_2 \ \dots \ A\beta_k) \quad (1.4.6.7)$$

Consider  $\gamma_1$

$$\gamma_1 = A\beta_1 \quad (1.4.6.8)$$

$$= (\alpha_1 \ \alpha_2 \ \dots \ \alpha_n) \begin{pmatrix} B_{11} \\ B_{21} \\ \vdots \\ B_{n1} \end{pmatrix} \quad (1.4.6.9)$$

$$= B_{11}\alpha_1 + B_{21}\alpha_2 + \dots + B_{n1}\alpha_n \quad (1.4.6.10) \quad 1.4.8. \text{ Let,}$$

Similarly, considering  $j^{th}$  column of  $C$

$$\gamma_j = (\alpha_1 \ \alpha_2 \ \dots \ \alpha_n) \begin{pmatrix} B_{1j} \\ B_{2j} \\ \vdots \\ B_{nj} \end{pmatrix} \quad (1.4.6.11)$$

$$= B_{1j}\alpha_1 + B_{2j}\alpha_2 + \dots + B_{nj}\alpha_n \quad (1.4.6.12)$$

$$\Rightarrow \gamma_j = \sum_{r=1}^n B_{rj}\alpha_r \quad (1.4.6.13)$$

which proves that columns of  $C$  are linear combinations of columns of  $A$

1.4.7. Let  $A$  and  $B$  be  $n \times n$  matrices such that  $AB = I$ . Prove that  $BA = I$ . **Solution:** Let  $BX = 0$  be a system of linear equation with  $n$  unknowns and  $n$  equations as  $B$  is  $n \times n$  matrix. Hence,

$$BX = 0 \quad (1.4.7.1)$$

$$\Rightarrow A(BX) = 0 \quad (1.4.7.2)$$

$$\Rightarrow (AB)X = 0 \quad (1.4.7.3)$$

$$\Rightarrow IX = 0 \quad [\because AB = I] \quad (1.4.7.4)$$

$$\Rightarrow X = 0 \quad (1.4.7.5)$$

From (1.4.7.5) since  $X = 0$  is the only solution of (1.4.7.1), hence  $\text{rank}(B) = n$ . Which implies all columns of  $B$  are linearly independent. Hence  $B$  is invertible. Therefore, every left inverse of  $B$  is also a right inverse of  $B$ . Hence there exists a  $n \times n$  matrix  $C$  such that,

$$BC = CB = I \quad (1.4.7.6)$$

Again given that  $AB = I$ . Hence,

$$AB = I \quad (1.4.7.7)$$

$$\Rightarrow ABC = C \quad (1.4.7.8)$$

$$\Rightarrow A(BC) = C \quad (1.4.7.9)$$

$$\Rightarrow A = C \quad [\because BC = I] \quad (1.4.7.10)$$

Hence using (1.4.7.10) and (1.4.7.6) we can write,

$$BA = I \quad (1.4.7.11)$$

Hence Proved.

$$C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \quad (1.4.8.1)$$

be a  $2 \times 2$  matrix. We inquire when it is possible to find  $2 \times 2$  matrices  $A$  and  $B$  such that  $C = AB - BA$ . Prove that such matrices can be found if and only if  $C_{11} + C_{22} = 0$ . **Solution:** We have to find,

$$\text{tr}(C) = C_{11} + C_{22} = \text{tr}(AB - BA) \quad (1.4.8.2)$$

$$\Rightarrow \text{tr}(C) = \text{tr}(AB) - \text{tr}(BA) \quad (1.4.8.3)$$

We know that,

$$\text{tr}(AB) = \sum_{i=1}^2 (AB)_{ii} \quad (1.4.8.4)$$

$$\Rightarrow \sum_{i=1}^2 \sum_{j=1}^2 a_{ij}b_{ji} \quad (1.4.8.5)$$

$$\Rightarrow \sum_{j=1}^2 \sum_{i=1}^2 b_{ji}a_{ij} \quad (1.4.8.6)$$

$$\Rightarrow \text{tr}(AB) = \sum_{j=1}^2 BA_{jj} \quad (1.4.8.7)$$

$$\Rightarrow \text{tr}(AB) = \text{tr}(BA) \quad (1.4.8.8)$$

Substituting equation (1.4.8.8) to (1.4.8.3) we get

$$\Rightarrow \text{tr}(C) = \text{tr}(AB) - \text{tr}(BA) = 0 \quad (1.4.8.9)$$

## 1.5 Invertible Matrices

1.5.1. Let

$$A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ -1 & 0 & 3 & 5 \\ 1 & -2 & 1 & 1 \end{pmatrix} \quad (1.5.1.1)$$

Find a row-reduced echelon matrix  $\mathbf{R}$  which is row-equivalent to  $\mathbf{A}$  and an invertible  $3 \times 3$  matrix  $\mathbf{P}$  such that  $\mathbf{R} = \mathbf{P} \mathbf{A}$ . **Solution:** Given

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 & 0 \\ -1 & 0 & 3 & 5 \\ 1 & -2 & 1 & 1 \end{pmatrix} \quad (1.5.1.2)$$

Row reduce  $\mathbf{A}$  by applying the elementary row operations and equivalently at each operations find the elementary matrix  $\mathbf{E}$

$$\mathbf{A}|\mathbf{I} = \left( \begin{array}{cccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 3 & 5 & 0 & 1 & 0 \\ 1 & -2 & 1 & 1 & 0 & 0 & 1 \end{array} \right) \quad (1.5.1.3)$$

$$\xleftrightarrow{R_2=R_2+R_1} \left( \begin{array}{cccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 4 & 5 & 1 & 1 & 0 \\ 1 & -2 & 1 & 1 & 0 & 0 & 1 \end{array} \right) \quad (1.5.1.4)$$

$$\xleftrightarrow{R_3=R_3-R_1} \left( \begin{array}{cccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 4 & 5 & 1 & 1 & 0 \\ 0 & -4 & 0 & 1 & -1 & 0 & 1 \end{array} \right) \quad (1.5.1.5)$$

$$\xleftrightarrow{R_1=R_1-R_2} \left( \begin{array}{cccc|ccc} 1 & 0 & -3 & -5 & 0 & -1 & 0 \\ 0 & 2 & 4 & 5 & 1 & 1 & 0 \\ 0 & -4 & 0 & 1 & -1 & 0 & 1 \end{array} \right) \quad (1.5.1.6)$$

$$\xleftrightarrow{R_3=R_3+2R_2} \left( \begin{array}{cccc|ccc} 1 & 0 & -3 & -5 & 0 & -1 & 0 \\ 0 & 2 & 4 & 5 & 1 & 1 & 0 \\ 0 & 0 & 8 & 11 & 1 & 2 & 1 \end{array} \right) \quad (1.5.1.7)$$

$$\xleftrightarrow{R_2=\frac{R_2}{2}} \left( \begin{array}{cccc|ccc} 1 & 0 & -3 & -5 & 0 & -1 & 0 \\ 0 & 1 & 2 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 8 & 11 & 1 & 2 & 1 \end{array} \right) \quad (1.5.1.8)$$

$$\xleftrightarrow{R_3=\frac{R_3}{8}} \left( \begin{array}{cccc|ccc} 1 & 0 & -3 & -5 & 0 & -1 & 0 \\ 0 & 1 & 2 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{11}{8} & \frac{1}{8} & \frac{1}{4} & \frac{1}{8} \end{array} \right) \quad (1.5.1.9)$$

$$\xleftrightarrow{R_1=R_1+3R_3} \left( \begin{array}{cccc|ccc} 1 & 0 & 0 & -\frac{7}{8} & \frac{3}{8} & -\frac{1}{4} & \frac{3}{8} \\ 0 & 1 & 2 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{11}{8} & \frac{1}{8} & \frac{1}{4} & \frac{1}{8} \end{array} \right) \quad (1.5.1.10)$$

$$\xleftrightarrow{R_2=R_2-2R_3} \left( \begin{array}{cccc|ccc} 1 & 0 & 0 & -\frac{7}{8} & \frac{3}{8} & -\frac{1}{4} & \frac{3}{8} \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{1}{4} & 0 & -\frac{1}{4} \\ 0 & 0 & 1 & \frac{11}{8} & \frac{1}{8} & \frac{1}{4} & \frac{1}{8} \end{array} \right) \quad (1.5.1.11)$$

Hence, row reduced echelon matrix that is row equivalent to  $\mathbf{A}$  is

$$\mathbf{R} = \left( \begin{array}{cccc} 1 & 0 & 0 & -\frac{7}{8} \\ 0 & 1 & 0 & -\frac{1}{4} \\ 0 & 0 & 1 & \frac{11}{8} \end{array} \right) \quad (1.5.1.12)$$

where  $\mathbf{E}$  is the elementary matrices that transform  $\mathbf{A}$  to  $\mathbf{R}$  Thus:-

$$\mathbf{E} \mathbf{A} = \mathbf{R} \quad (1.5.1.13)$$

Since elementary matrices is invertible

$$\mathbf{P} = \mathbf{E} \quad (1.5.1.14)$$

is invertible.

From (1.5.1.11)

$$\mathbf{P} = \left( \begin{array}{ccc} \frac{3}{8} & -\frac{1}{4} & \frac{3}{8} \\ \frac{1}{4} & 0 & -\frac{1}{4} \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{8} \end{array} \right) \quad (1.5.1.15)$$

$$\mathbf{R} = \left( \begin{array}{cccc} 1 & 0 & 0 & -\frac{7}{8} \\ 0 & 1 & 0 & -\frac{1}{4} \\ 0 & 0 & 1 & \frac{11}{8} \end{array} \right) \quad (1.5.1.16)$$

such that  $\mathbf{R} = \mathbf{P} \mathbf{A}$ .

1.5.2. Let  $\mathbf{A} = \begin{pmatrix} 2 & 0 & i \\ 1 & -3 & -i \\ i & 1 & 1 \end{pmatrix}$ , find a row-reduced echelon matrix  $\mathbf{R}$  which is row-equivalent to  $\mathbf{A}$  and an invertible  $3 \times 3$  matrix  $\mathbf{P}$  such that  $\mathbf{R} = \mathbf{P} \mathbf{A}$ . **Solution:** Given,

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & i \\ 1 & -3 & -i \\ i & 1 & 1 \end{pmatrix} \quad (1.5.2.1)$$

Row reduce  $\mathbf{A}$  by applying the elementary row operations and equivalently at each operations find the elementary matrix  $\mathbf{E}$

$$[\mathbf{A} \mid \mathbf{I}] = \left( \begin{array}{ccc|ccc} 2 & 0 & i & 1 & 0 & 0 \\ 1 & -3 & -i & 0 & 1 & 0 \\ i & 1 & 1 & 0 & 0 & 1 \end{array} \right) \quad (1.5.2.2)$$

$$\xleftrightarrow{R_1 \leftrightarrow R_2} \left( \begin{array}{ccc|ccc} 1 & -3 & -i & 0 & 1 & 0 \\ 2 & 0 & i & 1 & 0 & 0 \\ i & 1 & 1 & 0 & 0 & 1 \end{array} \right) \quad (1.5.2.3)$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - 2R_1} \left( \begin{array}{ccc|ccc} 1 & -3 & -i & 0 & 1 & 0 \\ 0 & 6 & 3i & 1 & -2 & 0 \\ i & 1 & 1 & 0 & 0 & 1 \end{array} \right) \quad (1.5.2.4)$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - iR_1} \left( \begin{array}{ccc|ccc} 1 & -3 & -i & 0 & 1 & 0 \\ 0 & 6 & 3i & 1 & -2 & 0 \\ 0 & 1 + 3i & 0 & 0 & -i & 1 \end{array} \right) \quad (1.5.2.5)$$

$$\xleftrightarrow{R_2 \leftarrow \frac{R_2}{6}} \left( \begin{array}{ccc|ccc} 1 & -3 & -i & 0 & 1 & 0 \\ 0 & 1 & \frac{i}{2} & \frac{1}{6} & -\frac{1}{3} & 0 \\ 0 & 1 + 3i & 0 & 0 & -i & 1 \end{array} \right) \quad (1.5.2.6)$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + 3R_2} \left( \begin{array}{ccc|ccc} 1 & 0 & \frac{i}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{i}{2} & \frac{1}{6} & -\frac{1}{3} & 0 \\ 0 & 1 + 3i & 0 & 0 & -i & 1 \end{array} \right) \quad (1.5.2.7)$$

$$\xleftrightarrow{R_3 \leftarrow R_3 / (3-i)/2} \left( \begin{array}{ccc|ccc} 1 & 0 & \frac{i}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{i}{2} & \frac{1}{6} & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & -\frac{(i)}{3} & \frac{3+i}{15} & \frac{3+i}{5} \end{array} \right) \quad (1.5.2.8)$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - \frac{i}{2}R_3} \left( \begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & \frac{1-3i}{30} & \frac{1-3i}{10} \\ 0 & 1 & \frac{i}{2} & \frac{1}{6} & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & -\frac{(i)}{3} & \frac{3+i}{15} & \frac{3+i}{5} \end{array} \right) \quad (1.5.2.9)$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - \frac{i}{2}R_3} \left( \begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & \frac{1-3i}{30} & \frac{1-3i}{10} \\ 0 & 1 & 0 & 0 & -\frac{3+i}{15} & \frac{1-3i}{5} \\ 0 & 0 & 1 & -\frac{i}{3} & \frac{3+i}{15} & \frac{3+i}{5} \end{array} \right) \quad (1.5.2.10)$$

$$= [\mathbf{I} \ \mathbf{E}]$$

Hence, the row reduced matrix that is row equivalent to  $\mathbf{A}$  is

$$\mathbf{R} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I} \quad (1.5.2.11)$$

Using Gauss-Jordan Elimination, if there exists an elementary matrix  $\mathbf{E}$  such that  $\mathbf{E}[\mathbf{A} \ \mathbf{I}] = [\mathbf{I} \ \mathbf{E}]$  then  $\mathbf{E}$  is the inverse of  $\mathbf{A}$  i.e

$$\mathbf{E} = \mathbf{A}^{-1}$$

$$\mathbf{E} = \mathbf{A}^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{1-3i}{30} & \frac{1-3i}{10} \\ 0 & -\frac{3+i}{15} & \frac{1-3i}{5} \\ -\frac{i}{3} & \frac{3+i}{15} & \frac{3+i}{5} \end{pmatrix} \quad (1.5.2.12)$$

Since,

$$\mathbf{R} = \mathbf{PA} \implies \mathbf{P} = \mathbf{A}^{-1}\mathbf{R} \quad (1.5.2.13)$$

$$\mathbf{P} = \begin{pmatrix} \frac{1}{3} & \frac{1-3i}{30} & \frac{1-3i}{10} \\ 0 & -\frac{3+i}{15} & \frac{1-3i}{5} \\ -\frac{i}{3} & \frac{3+i}{15} & \frac{3+i}{5} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (1.5.2.14)$$

Thus,

$$\mathbf{P} = \begin{pmatrix} \frac{1}{3} & \frac{1-3i}{30} & \frac{1-3i}{10} \\ 0 & -\frac{3+i}{15} & \frac{1-3i}{5} \\ -\frac{i}{3} & \frac{3+i}{15} & \frac{3+i}{5} \end{pmatrix}$$

$$\mathbf{R} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

1.5.3. For each of the two matrices use elementary row operations to discover whether it is invertible, and to find the inverse in case it is invertible.

$$\mathbf{A} = \begin{pmatrix} 2 & 5 & -1 \\ 4 & -1 & 2 \\ 6 & 4 & 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 2 & 4 \\ 0 & 1 & -2 \end{pmatrix}$$

**Solution:** Given

$$\mathbf{A} = \begin{pmatrix} 2 & 5 & -1 \\ 4 & -1 & 2 \\ 6 & 4 & 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 2 & 4 \\ 0 & 1 & -2 \end{pmatrix} \quad (1.5.3.1)$$

By applying row reductions on  $\mathbf{A}$

$$\begin{pmatrix} 2 & 5 & -1 \\ 4 & -1 & 2 \\ 6 & 4 & 1 \end{pmatrix} \xleftrightarrow{R_2 = R_2 - 2R_1} \mathbf{A} = \begin{pmatrix} 2 & 5 & -1 \\ 0 & -11 & 4 \\ 6 & 4 & 1 \end{pmatrix} \quad (1.5.3.2)$$

$$\xleftrightarrow{R_3 = R_3 - 3R_1} \begin{pmatrix} 2 & 5 & -1 \\ 0 & -11 & 4 \\ 0 & -11 & 4 \end{pmatrix} \quad (1.5.3.3)$$

$$\xleftrightarrow{R_1 = \frac{R_1}{2}} \begin{pmatrix} 1 & \frac{5}{2} & -\frac{1}{2} \\ 0 & -11 & 4 \\ 0 & -11 & 4 \end{pmatrix} \quad (1.5.3.4)$$

$$\xleftrightarrow{R_3=R_3-R_2} \begin{pmatrix} 1 & \frac{5}{2} & \frac{-1}{2} \\ 0 & -11 & 4 \\ 0 & 0 & 0 \end{pmatrix} \quad (1.5.3.5)$$

$$\xleftrightarrow{R_2=\frac{-R_2}{11}} \begin{pmatrix} 1 & \frac{5}{2} & \frac{-1}{2} \\ 0 & 1 & \frac{-4}{11} \\ 0 & 0 & 0 \end{pmatrix} \quad (1.5.3.6)$$

$$\xleftrightarrow{R_1=R_1-\frac{5}{2}R_2} \begin{pmatrix} 1 & 0 & \frac{9}{22} \\ 0 & 1 & \frac{-4}{11} \\ 0 & 0 & 0 \end{pmatrix} \quad (1.5.3.7) \quad 1.5.4. \text{ Let}$$

For a matrix to be invertible, it has to be a matrix of full rank. However the matrix  $\mathbf{A}$  is not of full rank ( $\text{Rank}(\mathbf{A}) < 3$ ). Therefore  $\mathbf{A}$  is not invertible.

Let us now consider augmented matrix  $\mathbf{B}|\mathbf{I}$ . By applying row reductions on  $\mathbf{B}|\mathbf{I}$

$$\begin{pmatrix} 1 & -1 & 2 & | & 1 & 0 & 0 \\ 3 & 2 & 4 & | & 0 & 1 & 0 \\ 0 & 1 & -2 & | & 0 & 0 & 1 \end{pmatrix} \xleftrightarrow{R_2=R_2-3R_1} \begin{pmatrix} 1 & -1 & 2 & | & 1 & 0 & 0 \\ 0 & 5 & -2 & | & -3 & 1 & 0 \\ 0 & 1 & -2 & | & 0 & 0 & 1 \end{pmatrix} \quad (1.5.3.8)$$

$$\xleftrightarrow{R_2=\frac{R_2}{5}} \begin{pmatrix} 1 & -1 & 2 & | & 1 & 0 & 0 \\ 0 & 1 & \frac{-2}{5} & | & \frac{-3}{5} & \frac{1}{5} & 0 \\ 0 & 1 & -2 & | & 0 & 0 & 1 \end{pmatrix} \quad (1.5.3.9)$$

$$\xleftrightarrow{R_1=R_1+R_2} \begin{pmatrix} 1 & 0 & \frac{8}{5} & | & \frac{2}{5} & \frac{1}{5} & 0 \\ 0 & 1 & \frac{-2}{5} & | & \frac{-3}{5} & \frac{1}{5} & 0 \\ 0 & 1 & -2 & | & 0 & 0 & 1 \end{pmatrix} \quad (1.5.3.10)$$

$$\xleftrightarrow{R_3=R_3-R_2} \begin{pmatrix} 1 & 0 & \frac{8}{5} & | & \frac{2}{5} & \frac{1}{5} & 0 \\ 0 & 1 & \frac{-2}{5} & | & \frac{-3}{5} & \frac{1}{5} & 0 \\ 0 & 0 & \frac{-8}{5} & | & \frac{3}{5} & \frac{-1}{5} & 1 \end{pmatrix} \quad (1.5.3.11)$$

$$\xleftrightarrow{R_1=R_1+R_3} \begin{pmatrix} 1 & 0 & 0 & | & 1 & 0 & 1 \\ 0 & 1 & \frac{-2}{5} & | & \frac{-3}{5} & \frac{1}{5} & 0 \\ 0 & 0 & \frac{-8}{5} & | & \frac{3}{5} & \frac{-1}{5} & 1 \end{pmatrix} \quad (1.5.3.12)$$

$$\xleftrightarrow{R_3=\frac{-5}{8}R_3} \begin{pmatrix} 1 & 0 & 0 & | & 1 & 0 & 1 \\ 0 & 1 & \frac{-2}{5} & | & \frac{-3}{5} & \frac{1}{5} & 0 \\ 0 & 0 & 1 & | & \frac{3}{8} & \frac{1}{8} & \frac{-5}{8} \end{pmatrix} \quad (1.5.3.13)$$

$$\xleftrightarrow{R_3=R_2+\frac{2}{5}R_3} \begin{pmatrix} 1 & 0 & 0 & | & 1 & 0 & 1 \\ 0 & 1 & 0 & | & \frac{-3}{4} & \frac{1}{4} & \frac{-1}{4} \\ 0 & 0 & 1 & | & \frac{3}{8} & \frac{1}{8} & \frac{-5}{8} \end{pmatrix} \quad (1.5.3.14)$$

For a matrix to be invertible, it has to be a matrix of full rank. Here, the matrix  $\mathbf{B}$  is of full

rank ( $\text{Rank}(\mathbf{B}) = 3$ ). Therefore  $\mathbf{B}$  is invertible and the inverse matrix  $\mathbf{B}^{-1}$  can be written from (1.5.3.14):

$$\mathbf{B}^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ \frac{-3}{4} & \frac{1}{4} & \frac{-1}{4} \\ \frac{3}{8} & \frac{1}{8} & \frac{-5}{8} \end{pmatrix} \quad (1.5.3.15)$$

$$\mathbf{A} = \begin{pmatrix} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 1 & 5 \end{pmatrix} \quad (1.5.4.1)$$

For which  $\mathbf{X}$  does there exist a scalar  $c$  such that  $\mathbf{A}\mathbf{X} = c\mathbf{X}$

**Solution:** Given

$$\mathbf{A} = \begin{pmatrix} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 1 & 5 \end{pmatrix} \quad (1.5.4.2)$$

The given matrix has single eigenvalue as it is the lower triangular matrix and has equal diagonal elements. Hence  $c_1 = c_2 = c_3 = 5$ . To find the corresponding eigenvector, consider the following

$$(\mathbf{A} - c\mathbf{I})\mathbf{X} = 0 \quad (1.5.4.3)$$

$$\Rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (1.5.4.4)$$

Solving the homogeneous system of linear equations by performing rref, we get

$$\begin{pmatrix} 30 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xleftrightarrow[R_3 \leftrightarrow R_2]{R_2 \leftrightarrow R_1} \begin{pmatrix} 31 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (1.5.4.5)$$

Hence we get,

$$x_1 = 0, x_2 = 0, x_3 = t \quad (1.5.4.6)$$

where,  $x_3$  is arbitrary. Therefore,

$$\mathbf{X} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} t \quad (1.5.4.7)$$

Hence, the given matrix has single eigenvector and is not diagonalizable.

1.5.5. Discover whether

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{pmatrix} \quad (1.5.5.1)$$

is invertible, and find  $\mathbf{A}^{-1}$  if it exists.

**Solution:** The matrix  $\mathbf{A}$  is in row reduced echelon form with four pivot elements. Therefore the rank( $\mathbf{A}$ ) is 4. Hence the rows of matrix  $\mathbf{A}$  constitute of 4 linearly independent vectors. Thus it can be concluded that matrix  $\mathbf{A}$  is invertible. Using Gauss-Jordan Elimination, if there exists an elementary matrix  $\mathbf{E}$  such that  $\mathbf{E}[\mathbf{A} \ \mathbf{I}] = [\mathbf{I} \ \mathbf{E}]$  then  $\mathbf{E}$  is the inverse of  $\mathbf{A}$  i.e  $\mathbf{E} = \mathbf{A}^{-1}$ .

$$[\mathbf{A} \ \mathbf{I}] = \left( \begin{array}{cccc|cccc} 1 & 2 & 3 & 4 & 1 & 0 & 0 & 0 \\ 0 & 2 & 3 & 4 & 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 1 \end{array} \right) \quad (1.5.5.2)$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - R_2} \left( \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 2 & 3 & 4 & 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 1 \end{array} \right) \quad (1.5.5.3)$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - R_3} \left( \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & 4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 1 \end{array} \right) \quad (1.5.5.4)$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - R_4} \left( \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 1 \end{array} \right) \quad (1.5.5.5)$$

$$\xleftrightarrow{\begin{array}{l} R_4 \leftarrow \frac{R_4}{4} \\ R_2 \leftarrow \frac{R_2}{2} \quad R_3 \leftarrow \frac{R_3}{3} \end{array}} \left( \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{4} \end{array} \right) = [\mathbf{I} \ \mathbf{E}] \quad (1.5.5.6)$$

Therefore, for the given problem,

$$\mathbf{A}^{-1} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix} \quad (1.5.5.7)$$

Generalization of above result to a matrix of any arbitrary size: Let

$$\mathbf{A} = \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_N \\ 0 & a_2 & a_3 & \dots & a_N \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & \dots & \dots & a_N \end{pmatrix} \quad (1.5.5.8)$$

Then

$$\mathbf{E}_1 \mathbf{A} = \begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & \dots & \dots & 1 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_N \\ 0 & a_2 & a_3 & \dots & a_N \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & \dots & \dots & a_N \end{pmatrix} \quad (1.5.5.9)$$

$$= \begin{pmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & a_3 & \dots & a_N \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & \dots & \dots & a_N \end{pmatrix} \quad (1.5.5.10)$$

$$\mathbf{E}_2 \mathbf{E}_1 \mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & -1 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & \dots & \dots & 1 \end{pmatrix} \begin{pmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & a_3 & \dots & a_N \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & \dots & \dots & a_N \end{pmatrix} \quad (1.5.5.11)$$

$$= \begin{pmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & \dots & \dots & a_N \end{pmatrix} \quad (1.5.5.12)$$

Proceeding in similar manner, we get

$$\mathbf{E}_N \mathbf{E}_{N-1} \dots \mathbf{E}_2 \mathbf{E}_1 \mathbf{A} = \mathbf{U} = \begin{pmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & 0 & \dots & 0 \\ 0 & 0 & a_3 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & \dots & \dots & a_N \end{pmatrix} \quad (1.5.5.13)$$

$$= \text{diag}(a_1 \ a_2 \ \dots \ a_N) \quad (1.5.5.14)$$

$$\Rightarrow \mathbf{A} = \mathbf{L} \mathbf{U} \quad (1.5.5.15)$$

where  $\mathbf{L} = \mathbf{E}_1^{-1} \mathbf{E}_2^{-1} \dots \mathbf{E}_N^{-1}$

$$\Rightarrow \mathbf{A}^{-1} = \mathbf{U}^{-1} \mathbf{L}^{-1} \quad (1.5.5.16)$$

$$\Rightarrow \mathbf{A}^{-1} = \begin{pmatrix} \frac{1}{a_1} & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{a_2} & 0 & \dots & 0 \\ 0 & 0 & \frac{1}{a_3} & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & \dots & \dots & \frac{1}{a_N} \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & -1 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & \dots & \dots & 1 \end{pmatrix} \quad (1.5.5.17)$$

Therefore

$$\mathbf{A}^{-1} = \begin{pmatrix} \frac{1}{a_1} & -\frac{1}{a_1} & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{a_2} & -\frac{1}{a_2} & 0 & \dots & 0 \\ 0 & 0 & \frac{1}{a_3} & -\frac{1}{a_3} & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & \frac{1}{a_N} \end{pmatrix} \quad (1.5.5.18)$$

From (1.5.5.18) for the above problem

$$\mathbf{A}^{-1} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix} \quad (1.5.5.19)$$

1.5.6. Suppose  $\mathbf{A}$  is a  $2 \times 1$  matrix and  $\mathbf{B}$  is  $1 \times 2$  matrix. Prove that  $\mathbf{C} = \mathbf{AB}$  is non invertible.

**Solution:** Let's take  $\mathbf{A}$  and  $\mathbf{B}$  to be non zero vectors. Now, we know that for  $\mathbf{C}$  to be non invertible  $\mathbf{Cx} = 0$  should have a non trivial solution. So,

$$\mathbf{Cx} = 0 \quad (1.5.6.1)$$

$$\Rightarrow \mathbf{ABx} = 0 \quad (1.5.6.2)$$

Here, we know that  $\mathbf{B}$  is  $1 \times 2$  matrix and  $\mathbf{x}$  is  $2 \times 1$  matrix then  $\mathbf{Bx}$  will result to a scalar constant  $k$ .

$$\Rightarrow \mathbf{Ak} = 0 \quad (1.5.6.3)$$

For (1.5.6.3) to be true  $k$  should be zero. We also know that  $\mathbf{B}$  is  $1 \times 2$  matrix i.e. rows are less than column hence,

$$\mathbf{Bx} = 0 \quad (1.5.6.4)$$

will have a non trivial solution. Hence, using (1.5.6.3) and (1.5.6.4) we can say,

$$\mathbf{ABx} = 0 \quad (1.5.6.5)$$

will have a non trivial solution so,  $\mathbf{C}$  is non

invertible.

1.5.7. Let  $\mathbf{A}$  be an  $n \times n$  (square) matrix, Prove the following two statements:

a) If  $\mathbf{A}$  is invertible and  $\mathbf{AB} = 0$  for some  $n \times n$  matrix  $\mathbf{B}$ , then  $\mathbf{B} = 0$ .

b) If  $\mathbf{A}$  is not invertible, then there exists an  $n \times n$  matrix  $\mathbf{B}$  such that  $\mathbf{AB} = 0$  but  $\mathbf{B} \neq 0$ .

**Solution:**

a) Given  $\mathbf{A}$  is an invertible matrix and  $\mathbf{AB} = 0$  then,

$$\mathbf{AB} = 0 \quad (1.5.7.1)$$

$$\Rightarrow \mathbf{A}^{-1}(\mathbf{AB}) = 0 \quad (1.5.7.2)$$

$$\Rightarrow (\mathbf{A}^{-1}\mathbf{A})\mathbf{B} = 0 \quad (1.5.7.3)$$

$$\Rightarrow \mathbf{IB} = 0 \quad [\because \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}] \quad (1.5.7.4)$$

$$\Rightarrow \mathbf{B} = 0 \quad (1.5.7.5)$$

b) If  $\mathbf{A}$  is not invertible, then there exists an  $n \times n$  matrix  $\mathbf{B}$  such that  $\mathbf{AB} = 0$  but  $\mathbf{B} \neq 0$ . Since  $\mathbf{A}$  is not invertible,  $\mathbf{AX} = 0$  must have a non-trivial solution. Let the non-trivial solution be,

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \quad (1.5.7.6)$$

Let  $\mathbf{B}$  which is an  $n \times n$  matrix have all its columns as  $\mathbf{y}$ .

$$\mathbf{B} = (\mathbf{y} \quad \mathbf{y} \quad \dots \quad \mathbf{y}) \quad (1.5.7.7)$$

From equation (1.5.7.7), we can say that  $\mathbf{B} \neq 0$  but  $\mathbf{AB} = 0$

1.5.8. Let

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (1.5.8.1)$$

Prove, using elementary row operations that  $\mathbf{A}$  is invertible if and only if  $(ad - bc) \neq 0$

**Solution:**

The goal is to effect the transformation  $(\mathbf{A}|\mathbf{I}) \rightarrow (\mathbf{I}|\mathbf{A}^{-1})$ . Augmenting  $\mathbf{A}$  with the  $2 \times 2$  identity matrix, we get:

$$\begin{pmatrix} a & b & 1 & 0 \\ c & d & 0 & 1 \end{pmatrix} \quad (1.5.8.2)$$

Now, if  $a = 0$ , switch the rows. If  $c$  is also

0, then the process of reducing  $\mathbf{A}$  to  $\mathbf{I}$  cannot even begin. So, one necessary condition for  $\mathbf{A}$  to be invertible is that the entries  $a$  and  $c$  are not both 0.

a) Assume that  $a \neq 0$ , Then :

$$\begin{aligned} \begin{pmatrix} a & b & 1 & 0 \\ c & d & 0 & 1 \end{pmatrix} &\xleftrightarrow{R_1=R_1/a} \begin{pmatrix} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ c & d & 0 & 1 \end{pmatrix} \\ &\xleftrightarrow{R_2=R_2-cR_1} \begin{pmatrix} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & \frac{ad-bc}{a} & \frac{-c}{a} & 1 \end{pmatrix} \end{aligned} \quad (1.5.8.3)$$

Next, assuming that  $ad - bc \neq 0$ , we get:

$$\begin{aligned} &\xleftrightarrow{R_1=R_1-\frac{b}{ad-bc}R_2} \begin{pmatrix} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & \frac{ad-bc}{a} & \frac{-c}{a} & 1 \end{pmatrix} \\ &\xleftrightarrow{R_2=R_2\frac{a}{ad-bc}} \begin{pmatrix} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix} \end{aligned}$$

Therefore, if  $ad - bc \neq 0$ , then the matrix is invertible and its inverse is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad (1.5.8.4)$$

b) In (1.5.8.3), we have assumed that  $a \neq 0$ . Now consider  $a = 0$ , then, as we have seen before, it is mandatory that  $c \neq 0$ :

$$\begin{pmatrix} 0 & b & 1 & 0 \\ c & d & 0 & 1 \end{pmatrix} \xleftrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} c & d & 0 & 1 \\ 0 & b & 1 & 0 \end{pmatrix} \quad (1.5.8.5)$$

$$\begin{aligned} &\xleftrightarrow{R_1=R_1/c} \begin{pmatrix} 1 & \frac{d}{c} & 0 & \frac{1}{c} \\ 0 & b & 1 & 0 \end{pmatrix} \\ &\xleftrightarrow{R_1=R_1-R_2 \times \frac{d}{bc}} \begin{pmatrix} 1 & 0 & -\frac{d}{bc} & \frac{1}{c} \\ 0 & b & 1 & 0 \end{pmatrix} \\ &\xleftrightarrow{R_2=R_2/b} \begin{pmatrix} 1 & 0 & -\frac{d}{bc} & \frac{1}{c} \\ 0 & 1 & \frac{1}{b} & 0 \end{pmatrix} \end{aligned}$$

Therefore, When we consider  $a = 0$  the matrix is invertible if  $bc \neq 0$ , which is included in the condition  $ad - bc \neq 0$ .

c) Similarly, consider  $c = 0$ , then, as we have

seen before, it is mandatory that  $a \neq 0$ :

$$\begin{aligned} \begin{pmatrix} a & b & 1 & 0 \\ 0 & d & 0 & 1 \end{pmatrix} &\xleftrightarrow{R_1=R_1/a} \begin{pmatrix} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & d & 0 & 1 \end{pmatrix} \\ &\xleftrightarrow{R_1=R_1-R_2 \times \frac{b}{ad}} \begin{pmatrix} 1 & 0 & \frac{1}{a} & -\frac{b}{ad} \\ 0 & d & 0 & 1 \end{pmatrix} \\ &\xleftrightarrow{R_2=R_2/d} \begin{pmatrix} 1 & 0 & \frac{1}{a} & -\frac{b}{ad} \\ 0 & 1 & 0 & \frac{1}{d} \end{pmatrix} \end{aligned} \quad (1.5.8.6)$$

Therefore, When we consider  $c = 0$ , the matrix is invertible if  $ad \neq 0$ , which is included in the condition  $ad - bc \neq 0$ .

Hence, it is proved from above three cases that the given matrix is invertible iff  $ad - bc \neq 0$ .

1.5.9. An  $n \times n$  matrix  $\mathbf{A}$  is called upper-triangular if  $\mathbf{A}_{ij} = 0$  for  $i > j$ , that is, if every entry below the main diagonal is 0. Prove that an upper-triangular (square) matrix is invertible if and only if every entry on its main diagonal is different from 0. **Solution:** An  $n \times n$  matrix  $\mathbf{A}$  is called upper-triangular if  $\mathbf{A}_{ij} = 0$  for  $i > j$ , that is, if every entry below the main diagonal is 0. Prove that an upper-triangular (square) matrix is invertible if and only if every entry on its main diagonal is different from 0. Considering  $\mathbf{A}$ , an upper triangular matrix. Using the property that determinant of upper triangular matrix is the product of diagonal elements,

$$|\mathbf{A}| = \prod_{i=1}^n a_{i,i} \quad (1.5.9.1)$$

If  $\mathbf{A}$  be invertible then  $|\mathbf{A}| \neq 0$ . Hence from (1.5.9.1) we get,

$$\prod_{i=1}^n a_{i,i} \neq 0 \quad (1.5.9.2)$$

if any diagonal element is 0 then (1.5.9.2) won't be right hence no diagonal elements should be 0. Hence Proved.

1.5.10. Let  $A$  be a  $m \times n$  matrix. Show that by a finite number of elementary row and/or column operations one can pass from  $A$  to a matrix  $R$  which is both row-reduced echelon and column-reduced echelon, i.e.,  $R_{ij} = 0$  if  $i \neq j$ ,  $R_{ii} = 1$ ,  $1 \leq i \leq r$ ,  $R_{ii} = 0$ , if  $i > r$ . Show that  $R = PAQ$ , where  $P$  is an invertible  $m \times m$  matrix and  $Q$  is an invertible  $n \times n$  matrix.



**Solution:**

Lemma Every elementary matrix is invertible and the inverse is again an elementary matrix. If an elementary matrix  $E$  is obtained from  $I$  by using a certain row or column operation  $q$ , then  $E^{-1}$  is obtained from  $I$  by the "inverse" operation  $q^{-1}$ .

Solution Given  $\mathbf{A}$  is a  $m \times n$  matrix. Converting  $\mathbf{A}$  into row reduced echelon form by performing a series of elementary row operations  $\mathbf{P}$ . Let  $\mathbf{R}'$  be the row reduced echelon matrix. Also, by using the lemma we can tell that  $\mathbf{P}$  is invertible and order  $m \times m$ .

$$\mathbf{R}' = \mathbf{PA} \quad (1.5.10.1)$$

where,

$$\mathbf{R}' = \begin{pmatrix} \mathbf{I} & \mathbf{F} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$$

$\mathbf{I}$  is an identity matrix,  $\mathbf{F}$  is Free variables matrix and  $\mathbf{0}$  represents a block of zeroes

$\mathbf{R}'$  is in row-reduced echelon form. To perform column operations, elementary matrices should be multiplied on the right side in order to convert the  $\mathbf{R}'$  into column-reduced echelon form

$$\mathbf{R} = \mathbf{R}'\mathbf{Q} \quad (1.5.10.2)$$

But performing column operations on a matrix is equivalent to performing row operations on the transposed matrix.

$$\begin{aligned} \mathbf{R}^T &= (\mathbf{R}'\mathbf{Q})^T \\ \Rightarrow \mathbf{R}^T &= \mathbf{Q}^T \mathbf{R}'^T \end{aligned} \quad (1.5.10.3)$$

Hence, by using lemma it can be observed that  $\mathbf{Q}^T$  is invertible and of the order  $n \times n$ . Converting  $\mathbf{R}^T$  to row-reduced echelon is equivalent to converting  $\mathbf{R}$  to column-reduced echelon.

$$\mathbf{R} = \mathbf{PAQ} \quad (1.5.10.4)$$

where,

$$\mathbf{R} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \quad (1.5.10.5)$$

$\mathbf{I}$  is an identity matrix and  $\mathbf{0}$  represents a block

of zeroes.  $\mathbf{Q}$  is a upper triangular matrix.  $\mathbf{R}$  in (1.5.10.4) is in both row and column reduced echelon form. Hence proved.

Example Let,

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 5 & 7 \\ 1 & 2 & 3 & 4 \end{pmatrix} \quad (1.5.10.6)$$

To convert (1.5.10.6) into row reduced echelon form,  $\mathbf{A}$  has to be multiplied by  $\mathbf{P}$

$$\mathbf{P} = \begin{pmatrix} -5 & 3 & 0 \\ 2 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \quad (1.5.10.7)$$

$$\mathbf{R}' = \mathbf{PA} = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (1.5.10.8)$$

$\mathbf{R}'$  is in row reduced echelon form. To convert (1.5.10.8) into column-reduced echelon form, elementary operations have to be performed on  $\mathbf{R}'^T$ . By multiplying all the elementary matrices,

$$\mathbf{Q}^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & 1 & 0 & 0 \\ -1 & 0 & -1 & 1 \end{pmatrix} \quad (1.5.10.9)$$

$$\Rightarrow \mathbf{Q} = \begin{pmatrix} 1 & 0 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (1.5.10.10)$$

So  $\mathbf{PAQ}$  is in both row-reduced and column-reduced echelon form.

$$\mathbf{R} = \mathbf{PAQ} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (1.5.10.11)$$

The inverses of  $\mathbf{P}$  and  $\mathbf{Q}$  are,

$$\mathbf{P}^{-1} = \begin{pmatrix} 1 & 3 & 0 \\ 2 & 5 & 0 \\ 1 & 3 & 1 \end{pmatrix}; \quad \mathbf{Q}^{-1} = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (1.5.10.12)$$

1.5.11. Prove that the given matrix is invertible and

$\mathbf{A}^{-1}$  has integer values.

$$\mathbf{A} = \begin{pmatrix} 1 & \frac{1}{2} & \cdot & \cdot & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \cdot & \cdot & \frac{1}{n+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{1}{n} & \frac{1}{n+1} & \cdot & \cdot & \frac{1}{2n-1} \end{pmatrix} \quad (1.5.11.1)$$

**Solution: Proof that  $\mathbf{A}$  is Invertible.**

The elements of the given matrix is of the form:

$$A_{ij} = \frac{1}{i+j-1} = \int_0^1 t^{i+j-2} dt = \int_0^1 t^{i-1} t^{j-1} dt \quad (1.5.11.2)$$

We will prove that the matrix  $\mathbf{A}$  is positive definite:

$\mathbf{A}$  is Positive definite, if  $\mathbf{XAX}^T > 0$

$$\text{Let } \mathbf{X} = (x_i)_{1 \leq i \leq n} \in \mathbb{R}^N \quad (1.5.11.3)$$

$$\mathbf{XAX}^T = \sum_{1 \leq i, j \leq n} \frac{x_i x_j}{i+j-1} \quad (1.5.11.4)$$

From (1.5.11.2),

$$\begin{aligned} \mathbf{XAX}^T &= \sum_{1 \leq i, j \leq n} x_i x_j \int_0^1 t^{i+j-2} dt \quad (1.5.11.5) \\ &= \int_0^1 \left( \sum_{i=1}^n x_i t^{i-1} \right) \left( \sum_{j=1}^n x_j t^{j-1} \right) dt \quad (1.5.11.6) \end{aligned}$$

$$\Rightarrow \mathbf{XAX}^T = \int_0^1 \left( \sum_{i=1}^n x_i t^{i-1} \right)^2 dt > 0 \quad (1.5.11.7)$$

Thus, Matrix  $\mathbf{A}$  is Positive definite.

Now, let's say  $\lambda$  is an eigen value of  $\mathbf{A}$ . Then, for the corresponding eigen vector  $\mathbf{X} = (x_1, x_2, \dots, x_n)$ , we can write:

$$\mathbf{XAX}^T = \mathbf{X}\lambda\mathbf{X}^T \quad [\because \mathbf{AX}^T = \lambda\mathbf{X}^T] \quad (1.5.11.8)$$

$$= \|\mathbf{X}\|^2 \lambda \quad (1.5.11.9)$$

$$\Rightarrow \lambda = \frac{\mathbf{XAX}^T}{\|\mathbf{X}\|^2} > 0 \quad (1.5.11.10)$$

So, all of the eigenvalues belonging to  $\mathbf{A}$  must be positive. The product of the eigenvalues of a matrix equals the determinant.

$$\therefore \det(\mathbf{A}) > 0 \quad (1.5.11.11)$$

Thus, the given matrix  $\mathbf{A}$  is non-singular and

its inverse exist (Invertible).

**Proof that  $\mathbf{A}^{-1}$  has integer values.**

Let us consider a set of shifted legendre polynomials as follow:

$$P_i(x) = \begin{cases} P_1(x) = P_{11} \\ P_2(x) = P_{21} + P_{22}x \\ P_3(x) = P_{31} + P_{32}x + P_{33}x^2 \\ \vdots \\ P_n(x) = P_{n1} + P_{n2}x + P_{n3}x^2 + \dots + P_{nn}x^{n-1} \end{cases} \quad (1.5.11.12)$$

Where, the coefficients  $P_{ij}$  are given as:

$$P_{ij} = (-1)^{i+j-1} \binom{j-1}{i-1} \binom{i+j-2}{i-1} \quad (1.5.11.13)$$

The shifted legendre polynomials are analogous to legendre polynomials, but defined on the interval  $[0, 1]$  (whereas the interval is  $[-1, 1]$  for legendre polynomial).

A set of shifted legendre polynomials obey the following orthogonal relationship:

$$\int_0^1 P_i(x)P_j(x)dx = 0 \text{ for } i \neq j \quad (1.5.11.14)$$

$$\int_0^1 P_i(x)P_j(x)dx = \frac{1}{2i+1} \text{ for } i = j \quad (1.5.11.15)$$

Forming a matrix  $\mathbf{P}$  whose elements are the coefficients of polynomials in (1.5.11.12)

$$\mathbf{P} = \begin{pmatrix} P_{11} & 0 & \cdot & \cdot & 0 \\ P_{21} & P_{22} & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ P_{n1} & P_{n2} & \cdot & \cdot & P_{nn} \end{pmatrix} \quad (1.5.11.16)$$

Forming a matrix  $\mathbf{PAP}^T$ , the elements of the matrix  $\mathbf{PAP}^T$  can be written as:

$$\mathbf{PAP}_{ij}^T = \sum_{s=1}^N \sum_{r=1}^N P_{ir}P_{js}A_{rs} \quad (1.5.11.17)$$

From (1.5.11.2) with suitable change in variable notations, we can write:

$$A_{rs} = \int_0^1 x^{r-1}x^{s-1}dx \quad (1.5.11.18)$$

From (1.5.11.17) and (1.5.11.18),

$$\mathbf{PAP}_{ij}^T = \int_0^1 \sum_{s=1}^N \sum_{r=1}^N P_{ir}P_{js}x^{r-1}x^{s-1}dx \quad (1.5.11.19)$$

$$= \int_0^1 \sum_{s=1}^N P_{ir}x^{r-1} \sum_{r=1}^N P_{js}x^{s-1}dx \quad (1.5.11.20)$$

$$= \int_0^1 P_i(x)P_j(x)dx \quad (1.5.11.21)$$

From (1.5.11.15)

$$\mathbf{PAP}_{ij}^T = \begin{cases} 0 & i \neq j \\ \frac{1}{2i+1} & i = j \end{cases} \quad (1.5.11.22)$$

Thus, Matrix  $\mathbf{PAP}^T$  is diagonal matrix:

$$\mathbf{PAP}^T = \mathbf{D} = \begin{pmatrix} \frac{1}{3} & 0 & \cdot & \cdot & 0 \\ 0 & \frac{1}{5} & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \frac{1}{2n+1} \end{pmatrix} \quad (1.5.11.23)$$

From (1.5.11.23), the inverse of matrix  $\mathbf{A}$  can be written as:

$$\mathbf{A} = \mathbf{P}^{-1}\mathbf{D}(\mathbf{P}^T)^{-1} \quad (1.5.11.24)$$

$$\Rightarrow \mathbf{A}^{-1} = \mathbf{P}^T\mathbf{D}^{-1}\mathbf{P} \quad (1.5.11.25)$$

From (1.5.11.13) and (1.5.11.23), It can be clearly observed that the elements of matrix  $\mathbf{P}$ ,  $\mathbf{P}^T$  and  $\mathbf{D}^{-1}$  are all integers given as:

$$\mathbf{P}_{ij} = (-1)^{i+j-1} \binom{j-1}{i-1} \binom{i+j-2}{i-1} \quad (1.5.11.26)$$

$$\mathbf{D}_{ij}^{-1} = \frac{1}{\mathbf{D}_{ij}} = \begin{cases} 0 & i \neq j \\ 2i+1 & i = j \end{cases} \quad (1.5.11.27)$$

Since, matrix  $\mathbf{P}$ ,  $\mathbf{P}^T$  and  $\mathbf{D}^{-1}$  are integer matrices, therefore  $\mathbf{A}^{-1}$  is also an integer matrix.

Hence proved.

## Solution 2

[https://github.com/Arko98/EE5609/blob/master/Assignment\\_14](https://github.com/Arko98/EE5609/blob/master/Assignment_14)

Let  $\mathbf{A}_3$  be  $3 \times 3$  matrix i.e

$$\mathbf{A}_3 = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{pmatrix} \quad (1.5.11.28)$$

Now we find the inverse of the matrix  $\mathbf{A}_3$  as

follows,

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & 0 & 0 & 1 \end{pmatrix} \quad (1.5.11.29)$$

$$\begin{matrix} \xleftarrow{R_2=R_2-\frac{1}{2}R_1} \\ \xleftarrow{R_3=R_3-\frac{1}{3}R_1} \end{matrix} \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & \frac{1}{12} & \frac{1}{12} & -\frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{12} & \frac{1}{45} & -\frac{1}{3} & 0 & 1 \end{pmatrix} \quad (1.5.11.30)$$

$$\xleftarrow{R_3=R_3-R_2} \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & \frac{1}{12} & \frac{1}{12} & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & \frac{1}{180} & -\frac{1}{6} & -1 & 1 \end{pmatrix} \quad (1.5.11.31)$$

$$\begin{matrix} \xleftarrow{R_2=12R_2} \\ \xleftarrow{R_3=180R_3} \end{matrix} \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & 1 & 1 & -6 & 12 & 0 \\ 0 & 0 & 1 & 30 & -180 & 180 \end{pmatrix} \quad (1.5.11.32)$$

$$\begin{matrix} \xleftarrow{R_2=R_2-R_3} \\ \xleftarrow{R_1=R_1-R_3} \end{matrix} \begin{pmatrix} 1 & \frac{1}{2} & 0 & -9 & 60 & -60 \\ 0 & 1 & 0 & -36 & 192 & -180 \\ 0 & 0 & 1 & 30 & -180 & 180 \end{pmatrix} \quad (1.5.11.33)$$

$$\xleftarrow{R_1=R_1-\frac{1}{2}R_2} \begin{pmatrix} 1 & 0 & 0 & 9 & -36 & 30 \\ 0 & 1 & 0 & -36 & 192 & -180 \\ 0 & 0 & 1 & 30 & -180 & 180 \end{pmatrix} \quad (1.5.11.34)$$

Hence we see that  $\mathbf{A}_3$  is invertible and the inverse contains integer entries and  $\mathbf{A}_3^{-1}$  is given by,

$$\mathbf{A}_3^{-1} = \begin{pmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{pmatrix} \quad (1.5.11.35)$$

Let,  $\mathbf{A}_4$  be  $4 \times 4$  matrix as follows,

$$\mathbf{A}_4 = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{pmatrix} \quad (1.5.11.36)$$

Now, expressing  $\mathbf{A}_4$  using  $\mathbf{A}_3$  we get,

$$\mathbf{A}_4 = \begin{pmatrix} \mathbf{A}_3 & \mathbf{u} \\ \mathbf{u}^T & d \end{pmatrix} \quad (1.5.11.37)$$

where,

$$\mathbf{u} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{5} \\ \frac{1}{6} \\ \frac{1}{7} \end{pmatrix} \quad (1.5.11.38)$$

$$d = \frac{1}{7} \quad (1.5.11.39)$$

Now assuming  $\mathbf{A}_4$  has an inverse, then from (1.5.11.37), the inverse of  $\mathbf{A}_4$  can be written using block matrix inversion,

#### Block matrix inversion

If a matrix is partitioned into four blocks, it can be inverted blockwise as follows:

$$\text{If } \mathbf{M} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} \text{ then,} \quad \mathbf{M}^{-1} = \begin{pmatrix} \mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{B}(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1}\mathbf{C}\mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{B}(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1} \\ -(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1}\mathbf{C}\mathbf{A}^{-1} & (\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1} \end{pmatrix} \quad (1.5.11.40)$$

$$\therefore \mathbf{A}_4^{-1} = \begin{pmatrix} \mathbf{A}_3^{-1} + \mathbf{A}_3^{-1}\mathbf{u}x_4^{-1}\mathbf{u}^T\mathbf{A}_3^{-1} & -\mathbf{A}_3^{-1}\mathbf{u}x_4^{-1} \\ -x_4^{-1}\mathbf{u}^T\mathbf{A}_3^{-1} & x_4^{-1} \end{pmatrix} \quad (1.5.11.41)$$

$$= x_4^{-1} \begin{pmatrix} \mathbf{A}_3^{-1}x_4 + \mathbf{A}_3^{-1}\mathbf{u}\mathbf{u}^T\mathbf{A}_3^{-1} & -\mathbf{A}_3^{-1}\mathbf{u} \\ \mathbf{u}^T\mathbf{A}_3^{-1} & 1 \end{pmatrix} \quad (1.5.11.42)$$

$$\text{where, } x_4 = d - \mathbf{u}^T\mathbf{A}_3^{-1}\mathbf{u} \quad (1.5.11.43)$$

Now, the assumption of  $\mathbf{A}_4$  being invertible will hold if and only if  $\mathbf{A}_3$  is invertible, which has been proved in (1.5.11.35) and  $x_4$  from (1.5.11.43) is invertible or  $x_4$  is a nonzero scalar. We now prove that  $x_4$  is invertible as follows,

$$x_4 = \frac{1}{7} - \begin{pmatrix} \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{pmatrix} \begin{pmatrix} \frac{1}{4} \\ \frac{1}{5} \\ \frac{1}{6} \end{pmatrix} \quad (1.5.11.44)$$

$$\Rightarrow x_4 = \frac{1}{2800} \quad (1.5.11.45)$$

Hence,  $x_4$  is a scalar, hence  $x_4^{-1}$  exists and is given by,

$$x_4^{-1} = 2800 \quad (1.5.11.46)$$

Hence,  $\mathbf{A}_4$  is invertible. Now putting the values

of  $\mathbf{A}_3^{-1}$ ,  $x_4^{-1}$  and  $\mathbf{u}$  we get,

$$\mathbf{A}_3^{-1} + \mathbf{A}_3^{-1}\mathbf{u}x_4^{-1}\mathbf{u}^T\mathbf{A}_3^{-1} = \begin{pmatrix} 16 & -120 & 240 \\ -120 & 1200 & -2700 \\ 240 & -2700 & 6480 \end{pmatrix} \quad (1.5.11.47)$$

$$-\mathbf{A}_3^{-1}\mathbf{u}x_4^{-1} = \begin{pmatrix} -140 \\ 1680 \\ -4200 \end{pmatrix} \quad (1.5.11.48)$$

$$x_4^{-1}\mathbf{u}^T\mathbf{A}_3^{-1} = \begin{pmatrix} -140 & 1680 & -4200 \end{pmatrix} \quad (1.5.11.49)$$

$$x_4^{-1} = 2800 \quad (1.5.11.50)$$

Putting values from (1.5.11.47), (1.5.11.48), (1.5.11.49) and (1.5.11.50) into (1.5.11.41) we get,

$$\mathbf{A}_4^{-1} = \begin{pmatrix} 16 & -120 & 240 & -140 \\ -120 & 1200 & -2700 & 1680 \\ 240 & -2700 & 6480 & -4200 \\ -140 & 1680 & -4200 & 2800 \end{pmatrix} \quad (1.5.11.51)$$

Hence, from (1.5.11.51) we proved that,  $\mathbf{A}_4$  is invertible and has integer entries.

By successively repeating this method, we can prove that  $\mathbf{A}_5$ ,  $\mathbf{A}_6$ ,  $\mathbf{A}_7$ , ..., and so on, are invertible and have integer values. Thus, we can say,  $\mathbf{A}_{n-1}$  will be invertible with integer entries. Then we can represent  $\mathbf{A}_n$  as follows,

$$\mathbf{A}_n = \begin{pmatrix} \mathbf{A}_{n-1} & \mathbf{u} \\ \mathbf{u}^T & d \end{pmatrix} \quad (1.5.11.52)$$

where,

$$\mathbf{u} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{5} \\ \vdots \\ \frac{1}{2n-2} \end{pmatrix} \quad (1.5.11.53)$$

$$d = \frac{1}{2n-1} \quad (1.5.11.54)$$

Now assuming  $\mathbf{A}_n$  has an inverse, then from (1.5.11.52), the inverse of  $\mathbf{A}_n$  can be written

using block matrix inversion as follows,

$$\mathbf{A}_n^{-1} = \begin{pmatrix} \mathbf{A}_{n-1}^{-1} + \mathbf{A}_{n-1}^{-1}\mathbf{u}x_n^{-1}\mathbf{u}^T\mathbf{A}_{n-1}^{-1} & -\mathbf{A}_{n-1}^{-1}\mathbf{u}x_n^{-1} \\ -x_n^{-1}\mathbf{u}^T\mathbf{A}_{n-1}^{-1} & x_n^{-1} \end{pmatrix} \quad (1.5.11.55)$$

$$= x_n^{-1} \begin{pmatrix} x_n\mathbf{A}_{n-1}^{-1} + \mathbf{A}_{n-1}^{-1}\mathbf{u}^T\mathbf{A}_{n-1}^{-1} & -\mathbf{A}_{n-1}^{-1}\mathbf{u} \\ -\mathbf{u}^T\mathbf{A}_{n-1}^{-1} & 1 \end{pmatrix} \quad (1.5.11.56)$$

where,

$$x_n = d - \mathbf{u}^T\mathbf{A}_{n-1}^{-1}\mathbf{u} \quad (1.5.11.57)$$

Now, the assumption of  $\mathbf{A}_n$  being invertible will hold if and only if  $\mathbf{A}_{n-1}$  is invertible, which is intuitively proved and  $x$  from (1.5.11.57) is invertible or  $x_n$  is a nonzero scalar. We now prove that  $x_n$  is invertible as follows,

$$x_n = \frac{1}{2n-1} - \left( \frac{1}{4} \quad \frac{1}{5} \quad \cdots \quad \frac{1}{2n-2} \right) \mathbf{A}_{n-1}^{-1} \begin{pmatrix} \frac{1}{4} \\ \frac{1}{5} \\ \vdots \\ \frac{1}{2n-2} \end{pmatrix} \quad (1.5.11.58)$$

In equation (1.5.11.58)  $\mathbf{u}$  contains no negative or zero entries, again  $\mathbf{A}_{n-1}^{-1}$  has non zero integer entries, hence  $\mathbf{u}^T\mathbf{A}_{n-1}^{-1}\mathbf{u}$  is a non zero scalar. Moreover  $d$  is not equal to  $\mathbf{u}^T\mathbf{A}_{n-1}^{-1}\mathbf{u}$  hence in (1.5.11.58)  $x$  is non-zero scalar and invertible and hence it has an inverse. Hence  $\mathbf{A}_n$  is invertible, proved.

#### Proof for $\mathbf{A}_{n+1}$ :

Expressing  $\mathbf{A}_{n+1}$  using  $\mathbf{A}_n$  we get:

$$\mathbf{A}_{n+1} = \begin{pmatrix} \mathbf{A}_n & \mathbf{u} \\ \mathbf{u}^T & d \end{pmatrix} \quad (1.5.11.59)$$

where,

$$\mathbf{u} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{5} \\ \vdots \\ \frac{1}{2n-1} \end{pmatrix} \quad (1.5.11.60)$$

$$d = \frac{1}{2n} \quad (1.5.11.61)$$

Now assuming  $\mathbf{A}_{n+1}$  has an inverse, then from (1.5.11.59), the inverse of  $\mathbf{A}_{n+1}$  can be written

using block matrix inversion as follows,

$$\mathbf{A}_{n+1}^{-1} = \begin{pmatrix} \mathbf{A}_n^{-1} + \mathbf{A}_n^{-1} \mathbf{u} x_{n+1}^{-1} \mathbf{u}^T \mathbf{A}_n^{-1} & -\mathbf{A}_n^{-1} \mathbf{u} x_{n+1}^{-1} \\ -x_{n+1}^{-1} \mathbf{u}^T \mathbf{A}_n^{-1} & x_{n+1}^{-1} \end{pmatrix} \quad (1.5.11.62)$$

$$= x_{n+1}^{-1} \begin{pmatrix} x_{n+1} \mathbf{A}_n^{-1} + \mathbf{A}_n^{-1} \mathbf{u}^T \mathbf{A}_n^{-1} & -\mathbf{A}_n^{-1} \mathbf{u} \\ -\mathbf{u}^T \mathbf{A}_n^{-1} & 1 \end{pmatrix} \quad (1.5.11.63)$$

where,

$$x_{n+1} = d - \mathbf{u}^T \mathbf{A}_n^{-1} \mathbf{u} \quad (1.5.11.64)$$

Now, the assumption of  $\mathbf{A}_{n+1}$  being invertible will hold if and only if  $\mathbf{A}_n$  is invertible, which is proved and  $x$  from (1.5.11.64) is invertible or  $x_{n+1}$  is a nonzero scalar. We now prove that  $x_{n+1}$  is invertible as follows,

$$x_{n+1} = \frac{1}{2n} - \begin{pmatrix} \frac{1}{4} & \frac{1}{5} & \cdots & \frac{1}{2n-1} \end{pmatrix} \mathbf{A}_n^{-1} \begin{pmatrix} \frac{1}{4} \\ \frac{1}{5} \\ \vdots \\ \frac{1}{2n-1} \end{pmatrix} \quad (1.5.11.65)$$

In equation (1.5.11.65)  $\mathbf{u}$  contains no negative or zero entries, again  $\mathbf{A}_n^{-1}$  has non zero integer entries, hence  $\mathbf{u}^T \mathbf{A}_n^{-1} \mathbf{u}$  is a non zero scalar. Moreover  $d$  is not equal to  $\mathbf{u}^T \mathbf{A}_n^{-1} \mathbf{u}$  hence in (1.5.11.65)  $x$  is non-zero scalar and invertible and hence it has an inverse. Hence  $\mathbf{A}_{n+1}$  is also invertible.

**Problem statement:** If  $\mathbf{A}_{n-1}^{-1}$  is invertible and has integer values, Then  $\mathbf{A}_n^{-1}$  also has integer values.

**Proof:**

The matrix  $\mathbf{A}_n^{-1}$  can be expressed as:

$$\mathbf{A}_n = \begin{pmatrix} \mathbf{A}_{n-1} & \mathbf{u} \\ \mathbf{u}^T & d \end{pmatrix} \quad (1.5.11.66)$$

where,

$$\mathbf{u} = \begin{pmatrix} \frac{1}{n+1} \\ \frac{1}{n+1} \\ \vdots \\ \frac{1}{2n-2} \end{pmatrix} \quad (1.5.11.67)$$

$$d = \frac{1}{2n-1} \quad (1.5.11.68)$$

The inverse of  $\mathbf{A}_n$  can be written using block matrix inversion,  
**Block matrix inversion**

If a matrix is partitioned into four blocks, it can be inverted blockwise as follows:

$$\text{If } \mathbf{M} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} \text{ then,} \quad \mathbf{M}^{-1} = \begin{pmatrix} \mathbf{A}^{-1} + \mathbf{A}^{-1} \mathbf{B} (\mathbf{D} - \mathbf{C} \mathbf{A}^{-1} \mathbf{B})^{-1} \mathbf{C} \mathbf{A}^{-1} & -\mathbf{A}^{-1} \mathbf{B} (\mathbf{D} - \mathbf{C} \mathbf{A}^{-1} \mathbf{B})^{-1} \\ -(\mathbf{D} - \mathbf{C} \mathbf{A}^{-1} \mathbf{B})^{-1} \mathbf{C} \mathbf{A}^{-1} & (\mathbf{D} - \mathbf{C} \mathbf{A}^{-1} \mathbf{B})^{-1} \end{pmatrix} \quad (1.5.11.69)$$

$$\mathbf{A}_n^{-1} = \begin{pmatrix} \mathbf{A}_{n-1}^{-1} + \mathbf{A}_{n-1}^{-1} \mathbf{u} x_n^{-1} \mathbf{u}^T \mathbf{A}_{n-1}^{-1} & -\mathbf{A}_{n-1}^{-1} \mathbf{u} x_n^{-1} \\ -x_n^{-1} \mathbf{u}^T \mathbf{A}_{n-1}^{-1} & x_n^{-1} \end{pmatrix} \quad (1.5.11.70)$$

where,

$$x_n = d - \mathbf{u}^T \mathbf{A}_{n-1}^{-1} \mathbf{u} \quad (1.5.11.71)$$

For  $\mathbf{A}_n^{-1}$  to have integer values, each of the four blocks in (1.5.11.70) should have integer values.

Let us first prove that  $x_n^{-1}$  have integer values as follow:

$$x_n = d - \mathbf{u}^T \mathbf{A}_{n-1}^{-1} \mathbf{u} \quad (1.5.11.72)$$

$$= \frac{1}{2n-1} - \sum_i^{n-1} \sum_j^{n-1} u_i u_j (A_{n-1}^{-1})_{ij} \quad (1.5.11.73)$$

$$= \frac{1}{2n-1} - \sum_i^{n-1} \sum_j^{n-1} \frac{1}{n+i-1} \frac{1}{n+j-1} (A_{n-1}^{-1})_{ij} \quad (1.5.11.74)$$

Thus,  $x_n$  is a scalar of the form  $\frac{p}{q}$ . Also, by

calculation,  $x_2 = \frac{1}{12}$ ,  $x_3 = \frac{1}{180}$  and  $x_3 = \frac{1}{2800}$ . Thus, by induction, we can say  $p = 1$  for any value of  $n$ . Thus,  $x_n^{-1}$  can be written as:

$$x_n^{-1} = \frac{1}{x_n} = \frac{q}{p} \quad (1.5.11.75)$$

Since,  $p = 1$  for any value of  $n$ , Hence  $x_n^{-1} = q$  is always integer.

Now, let us consider the block  $-\mathbf{A}_{n-1}^{-1}\mathbf{u}x_n^{-1}$ , we can write:

$$(-\mathbf{A}_{n-1}^{-1}\mathbf{u}x_n^{-1})_{i,1} = -q \sum_j \frac{1}{n+j-1} (A_{n-1}^{-1})_{i,j} \quad (1.5.11.76)$$

Here, the considered block is a  $(n-1) \times 1$  matrix, with each element of rational form  $\frac{p_1}{q_1}$ . For  $n = 2, 3, 4$  the value of  $q_1$  comes as 1. Thus by induction, the value of  $q_1$  is always 1 for any value of  $n$ . Hence, this matrix always has integer values.

Now, let us consider the block  $-x_n^{-1}\mathbf{u}^T\mathbf{A}_{n-1}^{-1}$ , we can write:

$$(-x_n^{-1}\mathbf{u}^T\mathbf{A}_{n-1}^{-1})_{1,j} = -q \sum_i \frac{1}{n+i-1} (A_{n-1}^{-1})_{i,j} \quad (1.5.11.77)$$

Here, the considered block is a  $1 \times (n-1)$  matrix, with each element of rational form  $\frac{p_2}{q_2}$ . For  $n = 2, 3, 4$  the value of  $q_2$  comes as 1. Thus by induction, the value of  $q_2$  is always 1 for any value of  $n$ . Hence, this matrix always has integer values.

Considering the block  $\mathbf{A}_{n-1}^{-1} + \mathbf{A}_{n-1}^{-1}\mathbf{u}x_n^{-1}\mathbf{u}^T\mathbf{A}_{n-1}^{-1}$ , let denote it as  $V_{n-1}^{-1}$ :

$$V_{n-1}^{-1} = \mathbf{A}_{n-1}^{-1} + \mathbf{A}_{n-1}^{-1}\mathbf{u}x_n^{-1}\mathbf{u}^T\mathbf{A}_{n-1}^{-1} \quad (1.5.11.78)$$

$$= \mathbf{A}_{n-1}^{-1} + \mathbf{A}_{n-1}^{-1}\mathbf{u} \left( d - \mathbf{u}^T\mathbf{A}_{n-1}^{-1}\mathbf{u} \right)^{-1} \mathbf{u}^T\mathbf{A}_{n-1}^{-1} \quad (1.5.11.79)$$

The Woodbury matrix identity is given by:

$$(A + UCV)^{-1} = A^{-1} - A^{-1}U \left( C^{-1} + VA^{-1}U \right)^{-1} VA^{-1}U^T \quad (1.5.11.80) \quad 2.1.1.$$

Comparing (1.5.11.79) and (1.5.11.80), we can

write:

$$V_{n-1}^{-1} = \left( A_{n-1} - (2n-1)\mathbf{u}\mathbf{u}^T \right)^{-1} \quad (1.5.11.81)$$

$$\therefore V_{n-1} = A_{n-1} - (2n-1)\mathbf{u}\mathbf{u}^T \quad (1.5.11.82)$$

Here,  $V_{n-1}$  can be expanded as:

$$V_{n-1} = A_{n-1} - (2n-1)\mathbf{u}\mathbf{u}^T \quad (1.5.11.83)$$

$$= \frac{1}{i+j-1} - \frac{(2n-1)}{(n+i-1)(n+j-1)} \quad (1.5.11.84)$$

$$= \frac{n^2 - ni - nj - ij}{(i+j-1)(n+i-1)(n+j-1)} \quad (1.5.11.85)$$

$$= \frac{(n-i)(n-j)}{(i+j-1)(n+i-1)(n+j-1)} \quad (1.5.11.86)$$

$$\therefore (V_{n-1})_{ij} = \left( (A_{n-1})_{ij} \frac{(n-i)(n-j)}{(n+i-1)(n+j-1)} \right) \quad (1.5.11.87)$$

Using (1.5.11.87), The inverse  $V_{n-1}^{-1}$  can be written as:

$$(V_{n-1}^{-1})_{ij} = \left( (A_{n-1}^{-1})_{ij} \frac{(n+i-1)(n+j-1)}{(n-i)(n-j)} \right) \quad (1.5.11.88)$$

Here, the considered block is a  $(n-1) \times (n-1)$  matrix, with each element of rational form  $\frac{p_3}{q_3}$ . For  $n = 2, 3, 4$  the value of  $q_3$  comes as 1. Thus by induction, the value of  $q_3$  is always 1 for any value of  $n$ . Hence, this matrix always has integer values.

Since, all the four blocks has integer values, the inverse  $A_n^{-1}$  has integer values. **Observations:**

- The given matrix is a  $n \times n$  Hilbert matrix. Which is always invertible with its inverse having integer values.
- The Hilbert matrix is symmetric and positive definite.

## 2 VECTOR SPACES

### Vector Spaces

If  $\mathbf{F}$  is a field, verify that vector space of all ordered  $n$ -tuples  $\mathbf{F}^n$  is a vector space over the field  $\mathbf{F}$ .

**Solution:** Let  $\mathbf{F}^n$  be a set of all ordered  $n$ -tuples

over  $\mathbf{F}$  i.e

$$\mathbf{F}^n = \left\{ \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} : a_1, a_2, \dots, a_n \in \mathbf{F} \right\} \quad (2.1.1.1)$$

For  $\mathbf{F}^n$  to be a vector space over  $\mathbf{F}$  it must satisfy the closure property of vector addition and scalar multiplication.

### Vector Addition in $\mathbf{F}^n$ :

Let  $\alpha = (a_i)$  and  $\beta = (b_i) \forall i = 1, 2, \dots, n \in \mathbf{F}^n$  then

$$\alpha + \beta = (a_i) + (b_i) \quad (2.1.1.2)$$

$$= (a_i + b_i) \quad (2.1.1.3)$$

Since

$$a_i + b_i \in \mathbf{F} \forall i = 1, 2, \dots, n \quad (2.1.1.4)$$

$$\implies \alpha + \beta \in \mathbf{F}^n \quad (2.1.1.5)$$

### Scalar multiplication in $\mathbf{F}^n$ over $\mathbf{F}$ :

Let  $\alpha = (a_i) \forall i = 1, 2, \dots, n \in \mathbf{F}^n$  and  $a \in \mathbf{F}$  then

$$a\alpha = (aa_i) \quad (2.1.1.6)$$

Since

$$aa_i \in \mathbf{F} \forall i = 1, 2, \dots, n \quad (2.1.1.7)$$

$$\implies a\alpha \in \mathbf{F}^n \quad (2.1.1.8)$$

### Associativity of addition in $\mathbf{F}^n$ :

Let  $\alpha = (a_i)$ ,  $\beta = (b_i)$ ,  $\gamma = (g_i) \forall i = 1, 2, \dots, n \in \mathbf{F}^n$  then

$$\alpha + (\beta + \gamma) = (a_i) + (b_i + g_i) \quad (2.1.1.9)$$

$$= (a_i + b_i + g_i) \quad (2.1.1.10)$$

$$= (a_i + b_i) + (g_i) \quad (2.1.1.11)$$

$$= (\alpha + \beta) + \gamma \quad (2.1.1.12)$$

### Existence of additive identity in $\mathbf{F}^n$ :

We have  $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbf{F}^n$  and  $\alpha = (a_i) \forall i =$

$1, 2, \dots, n \in \mathbf{F}^n$  then

$$(a_i) + (0) = (a_i + 0) \quad (2.1.1.13)$$

$$= (a_i) \quad (2.1.1.14)$$

Therefore  $\mathbf{0}$  is the additive identity in  $\mathbf{F}^n$ .

### Existence of additive inverse of each element of $\mathbf{F}^n$ :

If  $\alpha = (a_i) \forall i = 1, 2, \dots, n \in \mathbf{F}^n$  then  $(-a_i) \in \mathbf{F}^n$ . Also we have

$$(-a_i) + (a_i) = \mathbf{0} \quad (2.1.1.15)$$

Therefore  $-\alpha = (-a_i)$  is the additive inverse of  $\alpha$ . Thus  $\mathbf{F}^n$  is an abelian group with respect to addition.

Futher we observe that

a) If  $a \in \mathbf{F}$  and  $\alpha = (a_i)$ ,  $\beta = (b_i) \forall i = 1, 2, \dots, n \in \mathbf{F}^n$  then

$$a(\alpha + \beta) = a(a_i + b_i) \quad (2.1.1.16)$$

$$= (a[a_i + b_i]) \quad (2.1.1.17)$$

$$= (aa_i + ab_i) \quad (2.1.1.18)$$

$$(aa_i) + (ab_i) \quad (2.1.1.19)$$

$$= a(a_i) + a(b_i) \quad (2.1.1.20)$$

$$= a\alpha + a\beta \quad (2.1.1.21)$$

b) If  $a, b \in \mathbf{F}$  and  $\alpha = (a_i) \forall i = 1, 2, \dots, n \in \mathbf{F}^n$  then

$$(a + b)\alpha = ([a + b]a_i) \quad (2.1.1.22)$$

$$= (aa_i + ba_i) \quad (2.1.1.23)$$

$$= (aa_i) + (ba_i) \quad (2.1.1.24)$$

$$= a(a_i) + b(a_i) \quad (2.1.1.25)$$

$$= a\alpha + b\alpha \quad (2.1.1.26)$$

c) If  $a, b \in \mathbf{F}$  and  $\alpha = (a_i) \forall i = 1, 2, \dots, n \in \mathbf{F}^n$



then

$$(ab)\alpha = ([ab]a_i) \quad (2.1.1.27)$$

$$= (a[ba_i]) \quad (2.1.1.28)$$

$$= a(ba_i) \quad (2.1.1.29)$$

$$= a(b\alpha) \quad (2.1.1.30)$$

d) If 1 is the unity element of  $\mathbf{F}$  and  $\alpha = (a_i) \forall i = 1, 2, \dots, n \in \mathbf{F}^n$  then

$$1\alpha = (1a_i) \quad (2.1.1.31)$$

$$= (a_i) \quad (2.1.1.32)$$

$$= \alpha \quad (2.1.1.33)$$

Hence  $\mathbf{F}^n$  is a vector space over  $\mathbf{F}$ .

2.1.2. If  $\mathbf{V}$  is a vector space over field  $\mathbf{F}$ , verify that:

$$(\alpha_1 + \alpha_2) + (\alpha_3 + \alpha_4) = [\alpha_2 + (\alpha_3 + \alpha_1)] + \alpha_4 \quad (2.1.2.1)$$

**Solution:** Using property of commutativity of (+) in  $\mathbf{V}$

$$(\alpha_1 + \alpha_2) + (\alpha_3 + \alpha_4) = (\alpha_2 + \alpha_1) + (\alpha_3 + \alpha_4) \quad (2.1.2.2) \quad 2.1.4.$$

Using property of associativity of (+) in  $\mathbf{V}$

$$(\alpha_2 + \alpha_1) + (\alpha_3 + \alpha_4) = \alpha_2 + [\alpha_1 + (\alpha_3 + \alpha_4)] \quad (2.1.2.3)$$

Using property of commutativity of (+) in  $\mathbf{V}$

$$\alpha_2 + [\alpha_1 + (\alpha_3 + \alpha_4)] = \alpha_2 + (\alpha_3 + \alpha_1) + \alpha_4 \quad (2.1.2.4)$$

Using property of associativity of (+) in  $\mathbf{V}$

$$\alpha_2 + (\alpha_3 + \alpha_1) + \alpha_4 = [\alpha_2 + (\alpha_3 + \alpha_1)] + \alpha_4 \quad (2.1.2.5)$$

2.1.3. If  $\mathbb{C}$  is the field of complex numbers, which

vectors in  $\mathbb{C}^3$  are linear combinations of  $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ ,

$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$  and  $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ ?

**Solution:** Expressing the given vectors as the columns of a matrix,

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \quad (2.1.3.1)$$

The row reduced echelon form of the matrix on performing elementary row operations can be given as,

$$\mathbf{R} = \mathbf{CA} \quad (2.1.3.2)$$

where  $\mathbf{C}$  is the product of elementary matrices,

$$\mathbf{C} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} \quad (2.1.3.3)$$

Thus we get,

$$\mathbf{R} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (2.1.3.4)$$

From (2.1.3.4),  $\text{rank}(\mathbf{A}) = 3$ . Thus  $\mathbf{A}$  is a full rank matrix. Hence the columns of  $\mathbf{A}$  are linearly independent i.e., the given vectors are linearly independent and forms the basis for  $\mathbb{C}^3$ .

Hence any vector  $\mathbf{Y} \in \mathbb{C}^3$  can be written as the linear combinations of  $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ ,  $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$  and  $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ .

2.1.4. Let  $\mathbf{V}$  be the set of all pairs  $(x, y)$  of real numbers and let  $\mathbf{F}$  be the field of real numbers. Define

$$(x, y) + (x_1, y_1) = (x + x_1, y + y_1) \quad (2.1.4.1)$$

$$c(x, y) = (cx, y) \quad (2.1.4.2)$$

Is  $\mathbf{V}$  with these operations, a vector space over the field of real numbers?

**Solution:**  $\mathbf{V} = \{(x, y) \mid x, y \in \mathbb{R}\}$ , consider  $u = (x_1, y_1) \in \mathbf{V}$ ,  $a, b, c \in \mathbb{R}$ . Axioms with respect to addition and scalar multiplication.

a)

$$(a + b)u = (a + b)(x_1, y_1) \quad (2.1.4.3)$$

$$= ((a + b)x_1, (a + b)y_1) \neq au + bu \quad (2.1.4.4)$$

Since  $\mathbf{V}$  with the given operations the equation (2.1.4.4) contradicts the axioms of scalar multiplication. Hence it is not vector space over real number with these operations.

2.1.5. On  $\mathbb{R}^n$  define two operations

$$\alpha \oplus \beta = \alpha - \beta \quad (2.1.5.1)$$

$$c \cdot \alpha = -c\alpha \quad (2.1.5.2)$$

The operations on the right are usual ones.

Which of the axioms for a vector space are satisfied by  $(\mathbb{R}^n, \oplus, \cdot)$ ?

**Solution:** Let  $(\alpha, \beta, \gamma) \in \mathbb{R}^n$  and  $c, c_1, c_2$  are scalars taken from the field  $\mathbb{R}$  where the vector space is defined on. Table 2.1.5 lists the axioms satisfied and not satisfied for  $(\mathbb{R}^n, \oplus, \cdot)$ .

- 2.1.6. Let  $\mathbf{V}$  be the set of pairs  $(x, y)$  of real numbers and let  $\mathbf{F}$  be the field of real numbers. Define

$$(x, y) + (x_1, y_1) = (x + x_1, 0) \quad (2.1.6.1)$$

$$c(x, y) = (cx, 0) \quad (2.1.6.2)$$

Is  $\mathbf{V}$ , with these operations, a vector space?

**Solution:**  $\mathbf{V}$  is a vector space if it satisfies all properties of the vector space. Let us consider the property of Existence of additive identity. According to Existence of additive identity, there is a unique vector  $\mathbf{0}$  in  $\mathbf{V}$  called the zero vector, such that  $\alpha + \mathbf{0} = \alpha$  for all  $\alpha$  in  $\mathbf{V}$ .

Let  $u = (x_1, y_1) \in \mathbf{V}$

$$\begin{aligned} u + \mathbf{0} &= (x_1, y_1) + (0, 0) \\ &= (x_1 + 0, 0) \\ &= (x_1, 0) \\ &\neq u \end{aligned} \quad (2.1.6.3)$$

From (2.1.6.3), there does not exist an additive identity for  $\mathbf{V}$ .

Hence  $\mathbf{V}$  is not a vector space.

- 2.1.7. Let  $\mathbb{V}$  be the set of all complex-valued functions  $f$  on the real line such that

$$f(-t) = \overline{f(t)} \quad (2.1.7.1)$$

The bar denotes complex conjugation. Show that  $\mathbb{V}$ , with the operations

$$(f + g)(t) = f(t) + g(t) \quad (2.1.7.2)$$

$$(cf)(t) = cf(t) \quad (2.1.7.3)$$

is a vector space over the field of real numbers. Give an example of a function in  $\mathbb{V}$  which is not real valued.

**Solution:** To prove that  $\mathbb{V}$  with the given operations is a vector space over the field of real numbers, we have to start by proving that additivity and homogeneity both hold true. So, we have to prove that  $(cf+g)(t)$  is equal to

$cf(t)+g(t)$ .

$$(cf + g)(t) \quad (2.1.7.4)$$

$$= (cf)(t) + g(t) \quad (2.1.7.5)$$

$$= cf(t) + g(t) \quad (2.1.7.6)$$

Now, we know that  $f(-t) = \overline{f(t)}$  and so  $(cf+g)(t)$  should also satisfy the property,

$$(cf + g)(-t) \quad (2.1.7.7)$$

$$= cf(-t) + g(-t) \quad (2.1.7.8)$$

$$= \overline{cf(t)} + \overline{g(t)} \quad (2.1.7.9)$$

$$= \overline{cf(t) + g(t)} \quad (2.1.7.10)$$

$$= \overline{(cf + g)(t)} \quad (2.1.7.11)$$

**Example** Let's take  $f(x)=a+ix$

$$f(1) = a + i \quad (2.1.7.12)$$

Hence,  $f(x)$  is not real valued. Now,

$$f(x) = a + ix \quad (2.1.7.13)$$

$$f(-x) = a - ix \quad (2.1.7.14)$$

$$f(-x) = \overline{f(x)} \quad (2.1.7.15)$$

Since  $a$  and  $x \in \mathbb{R}$ , so  $f \in \mathbb{V}$

## 2.2 Subspaces

- 2.2.1. Which of the following set of vectors

$$\alpha = (a_1, a_2, \dots, a_n)$$

in  $\mathbf{R}^n$  are subspace of  $\mathbf{R}^n$  ( $n \geq 3$ )?

a) All  $\alpha$  such that  $a_1 \geq 0$

b) All  $\alpha$  such that  $a_1 + 3a_2 = a_3$

c) All  $\alpha$  such that  $a_2 = a_1^2$

d) All  $\alpha$  such that  $a_1 a_2 = 0$

e) All  $\alpha$  such that  $a_2$  is rational

**Solution:** Table 2.2.1 lists the summary of which set of vectors in  $\mathbf{R}^n$  are subspace of  $\mathbf{R}^n$  ( $n \geq 3$ ).

- 2.2.2. Let  $\mathbf{V}$  be the (real) vector space of all functions  $f$  from  $\mathbf{R}$  into  $\mathbf{R}$ .

a) Is  $f(x^2) = f(x)^2$ . **Solution:** For each of the function to be a subspace, it must be

| UNSATISFIED  | SATISFIED  |
|--|--|
| <b>Associativity of addition</b><br>$\alpha \oplus (\beta \oplus \gamma) = \alpha - \beta + \gamma$<br>$(\alpha \oplus \beta) \oplus \gamma = \alpha - \beta - \gamma$<br>$\alpha \oplus (\beta \oplus \gamma) \neq (\alpha \oplus \beta) \oplus \gamma$                     | <b>Additive identity</b><br>$\alpha \oplus \beta = \alpha - \beta = \alpha$<br>Additive identity is $\beta$<br>unique $\beta = (0, 0, \dots, 0)$ |
| <b>Commutativity of addition</b><br>$\alpha \oplus \beta = \alpha - \beta$<br>$\beta \oplus \alpha = \beta - \alpha$<br>$\alpha \oplus \beta \neq \beta \oplus \alpha$   | <b>Additive inverse</b><br>$\alpha \oplus \alpha = \alpha - \alpha = 0$<br>Additive inverse is $\alpha$  |
| <b>Scalar multiplication with field multiplication</b><br>$(c_1 c_2) \cdot \alpha = (-c_1 c_2) \alpha$<br>$c_1 \cdot (c_2 \cdot \alpha) = c_1 c_2 \alpha$<br>$(c_1 c_2) \cdot \alpha \neq c_1 \cdot (c_2 \cdot \alpha)$  |  |
| <b>Identity element of scalar multiplication</b><br>$1 \cdot \alpha = -\alpha = \alpha$ for $\alpha = (0, 0, \dots, 0)$<br>$1 \cdot \alpha = -\alpha \neq \alpha \quad \forall \quad \alpha \neq (0, 0, \dots, 0)$   |  |
| <b>Distributivity of scalar multiplication w.r.t vector addition</b><br>$c \cdot (\alpha \oplus \beta) = -c(\alpha - \beta)$<br>$c \cdot \alpha \oplus c \cdot \beta = -c\alpha - (-c\beta)$<br>$c \cdot (\alpha \oplus \beta) \neq c \cdot \alpha \oplus c \cdot \beta$     |  |
| <b>Distributivity of scalar multiplication w.r.t field addition</b><br>$(c_1 + c_2) \cdot \alpha = -(c_1 + c_2) \alpha$<br>$c_1 \cdot \alpha \oplus c_2 \cdot \beta = -c_1 \alpha - (-c_2 \beta)$<br>$(c_1 + c_2) \cdot \alpha \neq c_1 \cdot \alpha \oplus c_2 \cdot \beta$ |  |

TABLE 2.1.5: Axioms of vector space  $(\mathbb{R}^n, \oplus, \cdot)$ 

| $\alpha = (a_1, a_2, \dots, a_n)$  |   |
|--|---|
| Vector space   | Subspace summary  |
| $\alpha = (a_1, a_2, a_3, a_4, \dots, a_n); \quad a_1 \geq 0$              | Not a subspace. Scalar multiplication is not satisfied.<br>$-1(\alpha) \neq \alpha$ |
| $\alpha = (a_1, a_2, a_3, a_4, \dots, a_n); \quad a_1 + 3a_2 = a_3$        | It is a subspace  |
| $\alpha = (a_1, a_2, a_3, a_4, \dots, a_n); \quad a_2 = a_1^2$             | Not a subspace. Addition is not satisfied.<br>$(a_1 + b_1)^2 \neq a_1^2 + b_1^2$    |
| $\alpha = (a_1, a_2, a_3, a_4, \dots, a_n); \quad a_1 a_2 = 0$             | Not a subspace. Addition is not satisfied.<br>$a_1 b_1 \neq 0$                      |
| $\alpha = (a_1, a_2, a_3, a_4, \dots, a_n); \quad a_2 \text{ is rational}$ | Not a subspace. Scalar multiplication is not satisfied.<br>$a_2 \neq \sqrt{2} a_1$  |

TABLE 2.2.1: Summary

closed with respect to addition and scalar multiplication in  $\mathbf{V}$  defined as, for  $f, g \in \mathbf{W}$

Then,

$$(f + g)(x^2) = f(x^2) + g(x^2) \quad (2.2.2.1)$$

$$= f(x)^2 + g(x)^2 \quad (2.2.2.2)$$

$$\neq (f + g)(x)^2 \quad (2.2.2.3)$$

Since  $\mathbf{W}$  is not closed with respect to addi-

tion. So It is not a subspace of  $\mathbf{V}$ .

- b) Is  $f(0) = f(1)$  a subspace of  $\mathbf{V}$  ?

**Solution:** A non-empty subset  $\mathbf{W}$  of  $\mathbf{V}$  is a subspace of  $\mathbf{V}$  if and only if for each pair of vectors  $\alpha, \beta$  in  $\mathbf{W}$  and each scalar  $c$  in  $\mathbf{R}$  the vector  $c\alpha + \beta$  is again in  $\mathbf{W}$ . For each of the function to be a subspace, it must be closed with respect to addition and scalar multiplication in  $\mathbf{V}$  defined as, for  $f, g \in \mathbf{W}$  and  $c \in \mathbf{R}$

Then,

$$h = cf + g \quad (2.2.2.4)$$

$$h(0) = cf(0) + g(0) \quad (2.2.2.5)$$

$$= cf(1) + g(1) \quad (2.2.2.6)$$

$$= h(1) \quad (2.2.2.7)$$

Thus,  $h(0) = h(1)$ . Therefore,  $\mathbf{W}$  is a subset of  $\mathbf{V}$  and also a vector space. Therefore  $\mathbf{W}$  is a subspace of  $\mathbf{V}$ .

Hence,  $f(0) = f(1)$  is a subspace of  $\mathbf{V}$ .

- c) Let  $\mathbf{V}$  be the (real) vector space of all functions  $f$  from  $\mathbf{R}$  into  $\mathbf{R}$ . Is  $f(3) = 1 + f(-5)$  a subspace of  $\mathbf{V}$

**Solution:** For each of the function to be a subspace, it must be closed with respect to addition and scalar multiplication in  $\mathbf{V}$  defined as, for  $f, g \in \mathbf{W}$

Then,

$$(f + g)(3) = f(3) + g(3) \quad (2.2.2.8)$$

$$= 1 + f(-5) + 1 + g(-5) \quad (2.2.2.9)$$

$$= 2 + f(-5) + g(-5) \quad (2.2.2.10)$$

$$= 2 + (f + g)(-5) \quad (2.2.2.11)$$

$$\neq 1 + (f + g)(-5) \quad (2.2.2.12)$$

Since  $\mathbf{W}$  is not closed with respect to addition  $\therefore$  It is not a subspace of  $\mathbf{V}$ .

- d) Let  $\mathbf{V}$  be the (real) vector space of all functions  $f$  from  $\mathbf{R}$  into  $\mathbf{R}$ . Is  $f(-1) = 0$  a subspace of  $\mathbf{V}$

**Solution:** A non-empty subset  $\mathbf{W}$  of  $\mathbf{V}$  is a subspace of  $\mathbf{V}$  if and only if for each pair of vectors  $\alpha, \beta$  in  $\mathbf{W}$  and each scalar  $c$  in  $\mathbf{R}$  the vector  $c\alpha + \beta$  is again in  $\mathbf{W}$ .

For each of the function to be a subspace, it must be closed with respect to addition

and scalar multiplication in  $\mathbf{V}$  defined as, for  $f, g \in \mathbf{W}$  and  $c \in \mathbf{R}$

Then,

$$(cf + g)(-1) = cf(-1) + g(-1) \quad (2.2.2.13)$$

$$= c(0) + 0 \quad (2.2.2.14)$$

$$= 0 \quad (2.2.2.15)$$

Thus,  $(cf + g)(-1) = 0$ . Therefore,  $\mathbf{W}$  is a subset of  $\mathbf{V}$  and also a vector space. Therefore  $\mathbf{W}$  is a subspace of  $\mathbf{V}$ .

Hence,  $f(-1) = 0$  is a subspace of  $\mathbf{V}$ .

- e) Whether the set containing all functions which are continuous is subspace of  $\mathbf{V}$

**Solution:**

If  $\mathbf{V}$  is a vector Space over field  $\mathbf{F}$ . A subspace of  $\mathbf{V}$  is a subset  $\mathbf{W}$  of  $\mathbf{V}$  which is itself a vector space over  $\mathbf{F}$  with the operations of vector addition and scalar multiplication on  $\mathbf{V}$

Let  $f$  and  $g$  be any continuous functions from  $\mathbf{R} \rightarrow \mathbf{R}$

and let  $c$  be any scalar  $\in \mathbf{R}$

From real analysis we know that sum and product of continuous functions is continuous. So  $cf + g$  is also a continuous function. Proof:  $f$  and  $g$  are continuous at  $a$ , condition for continuity will be satisfied

$$\lim_{x \rightarrow a} f(x) = k_1 \quad (2.2.2.16)$$

$$\lim_{x \rightarrow a} g(x) = k_2 \quad (2.2.2.17)$$

Applying limits to  $cf + g$

$$\lim_{x \rightarrow a} cf(x) + g(x) = \lim_{x \rightarrow a} g(x) + \lim_{x \rightarrow a} f(x) = k_1 + k_2 \quad (2.2.2.18)$$

So  $cf + g$  is also continuous at  $a$ . We know from theorem that any non-empty subset  $\mathbf{W}$  of  $\mathbf{V}$  is a subspace of  $\mathbf{V}$  if and only if for each pair of vectors  $\alpha, \beta$  in  $\mathbf{W}$  and each scalar  $c$  in  $\mathbf{F}$  the vector  $c\alpha + \beta$  is again in  $\mathbf{W}$ . So given vector space is a subspace.

2.2.3. Is the vector  $\begin{pmatrix} 3 \\ -1 \\ 0 \\ -1 \end{pmatrix}$  in the subspace of  $\mathbf{R}^4$

spanned by the vectors  $\begin{pmatrix} 2 \\ -1 \\ 3 \\ 2 \end{pmatrix}$ ,  $\begin{pmatrix} -1 \\ 1 \\ 1 \\ -3 \end{pmatrix}$  and  $\begin{pmatrix} 1 \\ 1 \\ 9 \\ -5 \end{pmatrix}$

?

**Solution:** Expressing the given three vectors as columns of a matrix,

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 1 & 1 \\ 3 & 1 & 9 \\ 2 & -3 & -5 \end{pmatrix} \quad (2.2.3.1)$$

and

$$\mathbf{b} = \begin{pmatrix} 3 \\ -1 \\ 0 \\ -1 \end{pmatrix} \quad (2.2.3.2)$$

For the vector  $\mathbf{b}$  to be in the subspace of  $\mathbf{R}^4$  spanned by the three vectors.

$$\mathbf{Ax} = \mathbf{b} \quad (2.2.3.3)$$

must have a solution.

$$\begin{pmatrix} 2 & -1 & 1 \\ -1 & 1 & 1 \\ 3 & 1 & 9 \\ 2 & -3 & -5 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 3 \\ -1 \\ 0 \\ -1 \end{pmatrix} \quad (2.2.3.4)$$

Forming the augmented matrix and row reducing it by elementary row operations,

$$\begin{pmatrix} 2 & -1 & 1 & 3 \\ -1 & 1 & 1 & -1 \\ 3 & 1 & 9 & 0 \\ 2 & -3 & -5 & -1 \end{pmatrix} \xrightarrow[R_4 \leftarrow R_4 - R_1]{R_2 \leftarrow -2R_2 + R_1, R_3 \leftarrow R_3 - \frac{3}{2}R_1} \quad (2.2.3.5)$$

$$\begin{pmatrix} 2 & -1 & 1 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & \frac{5}{2} & \frac{15}{2} & -\frac{9}{2} \\ 0 & -2 & -6 & -4 \end{pmatrix} \xrightarrow[R_4 \leftarrow R_4 + 2R_2]{R_3 \leftarrow 2R_3 - 5R_2} \begin{pmatrix} 2 & -1 & 1 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & -14 \\ 0 & 0 & 0 & -2 \end{pmatrix} \quad (2.2.3.6)$$

From (2.2.3.6), it is clear that the system does

not have a solution. Hence the vector  $\begin{pmatrix} 3 \\ -1 \\ 0 \\ -1 \end{pmatrix}$  does

not lie in the subspace of  $\mathbf{R}^4$  spanned by the given three vectors.

2.2.4. Let  $\mathbf{W}$  be the set of all  $(x_1, x_2, x_3, x_4, x_5)$  in  $\mathbf{R}^5$

which satisfy

$$2x_1 - x_2 + \frac{4}{3}x_3 - x_4 = 0 \quad (2.2.4.1)$$

$$x_1 + \frac{2}{3}x_3 - x_5 = 0 \quad (2.2.4.2)$$

$$9x_1 - 3x_2 + 6x_3 - 3x_4 - 3x_5 = 0 \quad (2.2.4.3)$$

Find a finite set of vectors which spans  $\mathbf{W}$ .

**Solution:** The given equations are,

$$2x_1 - x_2 + \frac{4}{3}x_3 - x_4 = 0 \quad (2.2.4.4)$$

$$x_1 + \frac{2}{3}x_3 - x_5 = 0 \quad (2.2.4.5)$$

$$9x_1 - 3x_2 + 6x_3 - 3x_4 - 3x_5 = 0 \quad (2.2.4.6)$$

which can be written as,

$$\begin{pmatrix} 2 & -1 & \frac{4}{3} & -1 & 0 \\ 1 & 0 & \frac{2}{3} & 0 & -1 \\ 9 & -3 & 6 & -3 & -3 \end{pmatrix} \mathbf{x} = 0 \quad (2.2.4.7)$$

Now, the augmented matrix,

$$\begin{pmatrix} 2 & -1 & \frac{4}{3} & -1 & 0 & | & 0 \\ 1 & 0 & \frac{2}{3} & 0 & -1 & | & 0 \\ 9 & -3 & 6 & -3 & -3 & | & 0 \end{pmatrix} \quad (2.2.4.8)$$

$$\xleftrightarrow{R_3 = R_3 - 3R_1 - 3R_2} \begin{pmatrix} 2 & -1 & \frac{4}{3} & -1 & 0 & | & 0 \\ 1 & 0 & \frac{2}{3} & 0 & -1 & | & 0 \\ 0 & 0 & 0 & 0 & 0 & | & 0 \end{pmatrix} \quad (2.2.4.9)$$

$$\xleftrightarrow{R_2 = R_2 - \frac{1}{2}R_1} \begin{pmatrix} 2 & -1 & \frac{4}{3} & -1 & 0 & | & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & -1 & | & 0 \\ 0 & 0 & 0 & 0 & 0 & | & 0 \end{pmatrix} \quad (2.2.4.10)$$

$$\xleftrightarrow{R_2 = 2R_2} \begin{pmatrix} 2 & -1 & \frac{4}{3} & -1 & 0 & | & 0 \\ 0 & 1 & 0 & 1 & -2 & | & 0 \\ 0 & 0 & 0 & 0 & 0 & | & 0 \end{pmatrix} \quad (2.2.4.11)$$

$$\xleftrightarrow{R_1 = R_1 + R_2} \begin{pmatrix} 2 & 0 & \frac{4}{3} & 0 & -2 & | & 0 \\ 0 & 1 & 0 & 1 & -2 & | & 0 \\ 0 & 0 & 0 & 0 & 0 & | & 0 \end{pmatrix} \quad (2.2.4.12)$$

So,

$$2x_1 + \frac{4}{3}x_3 - 2x_5 = 0 \quad (2.2.4.13)$$

$$x_2 + x_4 - 2x_5 = 0 \quad (2.2.4.14)$$

Solving the equations we get,

$$x_1 = -\frac{2}{3}x_3 + x_5 \quad (2.2.4.15)$$

$$x_2 = -x_4 + 2x_5 \quad (2.2.4.16)$$

which can be written as,

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \quad (2.2.4.17)$$

$$= \begin{pmatrix} -\frac{2}{3}x_3 + x_5 \\ -x_4 + 2x_5 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \quad (2.2.4.18)$$

$$= x_3 \begin{pmatrix} -\frac{2}{3} \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad (2.2.4.19)$$

where  $x_3, x_4$  and  $x_5 \in \mathbb{R}$ . Hence, the vectors

$$\begin{pmatrix} -\frac{2}{3} \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 1 \end{pmatrix} \text{ will span } \mathbf{W}$$

2.2.5. Let  $\mathbf{F}$  be a field and let  $n$  be a positive integer ( $n \geq 2$ ). Let  $\mathbf{V}$  be the vector space of all  $n \times n$  matrices over  $\mathbf{F}$ . Which of the following set of matrices  $\mathbf{A}$  in  $\mathbf{V}$  are subspaces of  $\mathbf{V}$ ?

- a) all invertible  $\mathbf{A}$ ;
- b) all non-invertible  $\mathbf{A}$ ;
- c) all  $\mathbf{A}$  such that  $\mathbf{AB} = \mathbf{BA}$ , where  $\mathbf{B}$  is some fixed matrix in  $\mathbf{V}$ ;
- d) all  $\mathbf{A}$  such that  $\mathbf{A}^2 = \mathbf{A}$ .

**Solution:**

- a) Let the matrices  $\mathbf{A}$  and  $\mathbf{B} \in \mathbf{V}$ , be set of invertible matrix. For them to be a subspace they need to be closed under addition. Let,

$$\mathbf{A} = \mathbf{I} \quad (2.2.5.1)$$

$$\mathbf{B} = -\mathbf{I} \quad (2.2.5.2)$$

It could be easily proven that both matrices

$\mathbf{A}$  and  $\mathbf{B}$  are invertible as,

$$\text{rank}(\mathbf{I}_{n \times n}) = \text{rank} \left( \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}_{n \times n} \right) \quad (2.2.5.3)$$

$$\implies \text{rank}(-\mathbf{I}_{n \times n}) = \text{rank}(\mathbf{I}_{n \times n}) = n \quad (2.2.5.4)$$

or it is a full rank matrix as there are  $n$  pivots.

$$\therefore \mathbf{A} + \mathbf{B} = \mathbf{0}. \quad (2.2.5.5)$$

But the zero matrix  $\mathbf{0}$  is non-invertible as,

$$\text{rank}(\mathbf{0}_{n \times n}) = \text{rank} \left( \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}_{n \times n} \right) \quad (2.2.5.6)$$

$$\implies \text{rank}(\mathbf{0}_{n \times n}) = 0 \quad (2.2.5.7)$$

$\therefore$  the set of invertible matrices are not closed under addition. Hence not a subspace of  $\mathbf{V}$ .

- b) Let the matrices  $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n \in \mathbf{V}$ , be set of non-invertible matrix. For them to be a subspace they need to be closed under addition. Let,

$$\mathbf{A}_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}_{n \times n} \quad (2.2.5.8)$$

$$\mathbf{A}_2 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}_{n \times n} \quad (2.2.5.9)$$

$$\mathbf{A}_n = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}_{n \times n} \quad (2.2.5.10)$$

$$(2.2.5.11)$$

It could be proven that matrices  $\mathbf{A}_1$ ,

$A_2, \dots, A_n$  are non-invertible as,

$$\text{rank}(A_1) = \text{rank} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \quad (2.2.5.12)$$

$$\Rightarrow \text{rank}(A_1) = 1 \quad (2.2.5.13)$$

or there is only one pivot hence rank is 1.

$$\Rightarrow A_1 + A_2 + A_3 + \cdots + A_n = I_{n \times n} \quad (2.2.5.14)$$

Now the identity matrix  $I$  is invertible as shown in equation (2.2.5.4).  $\therefore$  **the set of non-invertible matrices are not closed under addition. Hence not a subspace of  $V$ .**

- c) **Theorem 1:** A non-empty subset  $W$  of  $V$  is a subspace of  $V$  if and only if for each pair of vectors  $\alpha, \beta$  in  $W$  and each scalar  $c \in F$ , the vector  $c\alpha + \beta \in W$ .

Let the matrices  $A_1$  and  $A_2$  satisfy,

$$A_1 B = B A_1 \quad (2.2.5.15)$$

$$A_2 B = B A_2 \quad (2.2.5.16)$$

Let,  $c \in F$  be any constant.

$$\therefore (cA_1 + A_2) B = cA_1 B + A_2 B \quad (2.2.5.17)$$

Substituting from equations (2.2.5.15) and (2.2.5.16) to (2.2.5.17),

$$\Rightarrow (cA_1 + A_2) B = cBA_1 + BA_2 \quad (2.2.5.18)$$

$$\Rightarrow BcA_1 + BA_2 \quad (2.2.5.19)$$

$$\Rightarrow B(cA_1 + A_2) \quad (2.2.5.20)$$

**Thus,  $(cA_1 + A_2)$  satisfy the criteria and from Theorem-1 it can be seen that the set is a subspace of  $V$ .**

- d) Let  $A$  and  $B \in V$  be set of matrices such that,

$$A^2 = A \quad (2.2.5.21)$$

$$B^2 = B \quad (2.2.5.22)$$

Now for them to be closed under addition,

$$(A + B)^2 = A + B \quad (2.2.5.23)$$

Which is not always same. Example let,

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad (2.2.5.24)$$

$$B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (2.2.5.25)$$

Clearly,

$$A^2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = A \quad (2.2.5.26)$$

$$B^2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = B \quad (2.2.5.27)$$

Now,

$$A + B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad (2.2.5.28)$$

$$\Rightarrow (A + B)^2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad (2.2.5.29)$$

Hence, clearly from equations (2.2.5.28) and (2.2.5.29),

$$(A + B)^2 \neq A + B \quad (2.2.5.30)$$

**$\therefore$  the set of all  $A$  such that  $A^2 = A$  is not closed under addition. Hence, not a subspace of  $V$ .**

- 2.2.6. a. Prove that only subspace of  $\mathbb{R}^1$  are  $\mathbb{R}^1$  and the zero subspace

- b. Prove that a subspace of  $\mathbb{R}^2$  is  $\mathbb{R}^2$ , or the zero subspace, or consists of all scalar multiples of some fixed vector in  $\mathbb{R}^2$ . (The last type of subspace is, intuitively, a straight line through the origin.)

- c. Can you describe the subspaces of  $\mathbb{R}^3$  ?

**Solution:**

- a. Let  $W \neq 0$  be subspace of  $\mathbb{R}^1$ . Then  $W$  is a nonempty subset of  $\mathbb{R}^1$  and there exist  $w \in W$  such that  $w \neq 0$  which gives us that there exist  $w^{-1}$ .

Let  $x \in \mathbb{R}^1$ . Since  $W$  is in  $\mathbb{R}^1$  we have that it is closed under scalar

multiplication which gives us that  $(xw^{-1})w = x(w^{-1}w) = x.1 = x \in W$

Hence  $\mathbb{R}^1 \subset W$  and therefore  $W = \mathbb{R}^1$

Thus the only subspace of  $\mathbb{R}^1$  distinct of 0 is  $\mathbb{R}^1$  and therefore only subspaces of  $\mathbb{R}^1$  are 0 and  $\mathbb{R}^1$ .

- b. Clearly, 0 and  $\mathbb{R}^2$  itself are subspaces of  $\mathbb{R}^2$ . If  $u \neq 0$  and  $u \in \mathbb{R}^2$  then  $\text{span}\{\mathbf{u}\} = c\mathbf{u} : c \in \mathbb{R} = \text{set of all scalar multiples of } \mathbf{u}$  is a subspace of  $\mathbb{R}^2$ .

To show that these are the only subspaces of  $\mathbb{R}^2$ , assume that  $W \subset \mathbb{R}^2$  is any subspace of  $\mathbb{R}^2$ . Since  $W \subset \mathbb{R}^2$  is a subspace of  $\mathbb{R}^2$ , we have that  $\mathbf{0} \in W$ . If  $W \neq \mathbf{0}$  then there is a vector  $\mathbf{u} \neq \mathbf{0}$  and  $\mathbf{u} \in W$ , and hence  $W$  contains  $c\mathbf{u}$  for every  $c \in \mathbb{R}$ . If  $W \neq \text{span}\{\mathbf{u}\}$ , then there is a vector  $\mathbf{v} \in W$  so that  $\mathbf{v} \neq k\mathbf{u}$  for any  $k \in \mathbb{R}$ .

Then  $\mathbf{z} = c\mathbf{u} + d\mathbf{v} \in \text{span}\{\mathbf{u}, \mathbf{v}\}$  for any  $c, d \in \mathbb{R}$ . Since  $W$  is a subspace  $c\mathbf{u}$  and  $d\mathbf{v} \in W$  for any  $c, d \in \mathbb{R}$ , and hence so does  $\mathbf{z} = c\mathbf{u} + d\mathbf{v}$ . Thus  $\mathbf{z} \in \text{span}\{\mathbf{u}, \mathbf{v}\} \implies \mathbf{z} \in W$ , and so  $\text{span}\{\mathbf{u}, \mathbf{v}\} \subset W \subset \mathbb{R}^2$ .

Let  $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$  be any vector in  $\mathbb{R}^2$ , and let  $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$  and let  $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ . We show that there are real numbers  $c$  and  $d$  so that  $c\mathbf{u} + d\mathbf{v} = \mathbf{x}$

$$\begin{pmatrix} cu_1 \\ cu_2 \end{pmatrix} + \begin{pmatrix} dv_1 \\ dv_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad (2.2.6.1)$$

$$\begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad (2.2.6.2)$$

Since  $\mathbf{v} \neq k\mathbf{u}$  for any  $k \in \mathbb{R}$  and since  $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$  assume that  $u_1 \neq 0$ , and since

$k\mathbf{u} \neq \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$  assume that  $v_2 \neq 0$ .

Then

$$A = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (2.2.6.3)$$

Hence  $A$  is row equivalent to  $I_2$  and so  $A$  is invertible and so (2.2.6.2) has unique solution for  $c$  and  $d$ . Thus for any  $\mathbf{x} \in \mathbb{R}^2$  we can find real numbers  $c$  and  $d$  such that  $\mathbf{x} = c\mathbf{u} + d\mathbf{v}$ . Hence  $\mathbf{x} \in \mathbb{R}^2 \implies \mathbf{x} \in \text{span}\{\mathbf{u}, \mathbf{v}\}$ . Thus  $\mathbb{R}^2 \subset \text{span}\{\mathbf{u}, \mathbf{v}\} \subset W \subset \mathbb{R}^2$ .

Hence  $\text{span}\{\mathbf{u}, \mathbf{v}\} = W = \mathbb{R}^2$ , and so the only subspace of  $\mathbb{R}^2$  are  $\mathbf{0}$ ,  $\mathbb{R}^2$ , and  $L = c\mathbf{u} : \mathbf{u} \neq \mathbf{0}, c \in \mathbb{R}$ .

- c. The following are the subspaces of  $\mathbb{R}^3$  :

1. Origin is a trivial subspace of  $\mathbb{R}^3$ .
2.  $\mathbb{R}^3$  itself is a trivial subspace of  $\mathbb{R}^3$ .
3. Every line through origin is subspace of  $\mathbb{R}^3$ .
4. Every plane in  $\mathbb{R}^3$  passing through origin is a subspace  $\mathbb{R}^3$ .

*Proof :* Let  $W$  be a plane passing through origin. We need  $\mathbf{0} \in W$ , but we have that since we're only considering planes that contain origin. Next, we need  $W$  is closed under vector addition. If  $\mathbf{w}_1$  and  $\mathbf{w}_2$  both belong to  $W$ , then so does  $\mathbf{w}_1 + \mathbf{w}_2$  because it's found by constructing a parallelogram, and the whole parallelogram lies in the plane  $W$ . Finally, we need  $W$  is closed under scalar products, but it is since scalar multiples lie in a straight line through the origin, and that line lies in  $W$ . Thus, each plane  $W$  passing through the origin is a subspace of  $\mathbb{R}^3$ .

5. The intersection of any of the above subspaces will also be a subspace of  $\mathbb{R}^3$ . Because intersection of subspaces of a vector space is also a subspace of vector space.

*Proof :* Let  $W$  be a collection of subspaces of  $V$ , and let  $W = \cap W_i$  be their intersection. Since each  $W_i$  is a subspace, each of it contains the zero vector. Thus the zero vector is in the



intersection  $W$ , and  $W$  is non-empty. Let  $\alpha$  and  $\beta$  be vectors in  $W$  and let  $c$  be a scalar. By definition of  $W$ , both  $\alpha$  and  $\beta$  belong to each  $W_i$ , and because each  $W_i$  is a subspace, the vector  $(c\alpha + \beta)$  is again in  $W$ . Hence by definition of subspace,  $W$  is a subspace of  $V$ .

These 5 are only subspaces of  $\mathbb{R}^3$  possible. Because dimension of vector space  $\mathbb{R}^3$  is 3. Any subspace of  $\mathbb{R}^3$  should have dimension less than or equal to its dimension. Hence possible dimensions of subspaces are 0,1,2,3. Only subspace with 0 dimension is origin. Subspaces of dimension 1 with zero vector are lines passing through origin. Subspaces of dimension 2 with zero vector are plane passing through origin. Subspace of dimension 3 are all of  $\mathbb{R}^3$  itself.

2.2.7. Let  $W_1$  and  $W_2$  be subspaces of a vector space  $V$  such that the set-theoretic union of  $W_1$  and  $W_2$  is also a subspace. Prove that one of the spaces  $W_i$  is contained in the other. **Solution:** Given  $W_1 \cup W_2$  is a subspace, we need to prove that

$$W_1 \subseteq W_2 \quad \text{or} \quad W_2 \subseteq W_1 \quad (2.2.7.1)$$

Let us assume that

$$W_1 \not\subseteq W_2 \quad (2.2.7.2)$$

We need to show that

$$W_2 \subseteq W_1 \quad (2.2.7.3)$$

i.e., the generators of  $W_2$  are in  $W_1$ . Consider a vector,  $w_1 \in W_1 \setminus W_2$  and a vector  $w_2 \in W_2$ . Since  $W_1 \cup W_2$  is a subspace,

$$w_1 + w_2 \in W_1 \cup W_2 \quad (2.2.7.4)$$

$$\Rightarrow w_1 + w_2 \in W_1 \quad \text{or} \quad (2.2.7.5)$$

$$w_1 + w_2 \in W_2 \quad (2.2.7.6)$$

But,  $w_1 + w_2 \notin W_2$  because for some vector  $-w_2 \in W_2$ ,

$$(w_1 + w_2) - w_2 = w_1 \notin W_2 \quad (2.2.7.7)$$

Hence it must be that,  $w_1 + w_2 \in W_1$  because for some vector  $-w_1 \in W_1$ ,

$$(w_1 + w_2) - w_1 = w_2 \in W_1 \quad (2.2.7.8)$$

Thus, we have shown that every vector  $w_2$  in  $W_2$  is also in  $W_1$ . Hence,  $W_2 \subseteq W_1$

2.2.8. Let  $V$  be the vector space of all functions from  $\mathbf{R}$  into  $\mathbf{R}$ ; let  $V_e$  be the subset of even functions,  $f(-x) = f(x)$ ; let  $V_o$  be the subset of odd functions,  $f(-x) = -f(x)$ .

a) Prove that  $V_e$  and  $V_o$  are subspaces of  $V$

b) Prove that  $V_e + V_o = V$

c) Prove that  $V_e \cap V_o = \{0\}$

**Solution:**

a) Prove that  $V_e$  and  $V_o$  are subspaces of  $V$ .

A non-empty subset  $W$  of  $V$  is a subspace of  $V$  if and only if for each pair of vectors  $\alpha, \beta$  in  $W$  and each scalar  $c$  in  $\mathbf{F}$  the vector  $c\alpha + \beta$  is again in  $W$ .

Let  $u, v \in V_e$  and  $c \in \mathbf{R}$  and let  $h = cu + v$ . Then,

$$\begin{aligned} h(-x) &= cu(-x) + v(-x) \\ &= cu(x) + v(x) \\ &= h(x) \end{aligned} \quad (2.2.8.1)$$

From (2.2.8.1)

$$\Rightarrow h(-x) = h(x) \quad (2.2.8.2)$$

$$\Rightarrow h \in V_e \quad (2.2.8.3)$$

Let  $u, v \in V_o$  and  $c \in \mathbf{R}$  and let  $h = cu + v$ . Then,

$$\begin{aligned} h(-x) &= cu(-x) + v(-x) \\ &= -cu(x) - v(x) \\ &= -h(x) \end{aligned} \quad (2.2.8.4)$$

From (2.2.8.4)

$$\Rightarrow h(-x) = -h(x) \quad (2.2.8.5)$$

$$\Rightarrow h \in V_o \quad (2.2.8.6)$$

From (2.2.8.3) and (2.2.8.6),  $V_e$  and  $V_o$  are subspaces of  $V$ .

a) Prove that  $V_e + V_o = V$ .

Let  $u \in V$

$$u_e(x) = \frac{u(x) + u(-x)}{2} \quad (2.2.1.7)$$

$$u_o(x) = \frac{u(x) - u(-x)}{2} \quad (2.2.1.8)$$

Equation (2.2.1.7) and (2.2.1.8),  $u_e$  is

even and  $\mathbf{u}_o$  is odd. Adding both the equations,

$$\mathbf{u} = \mathbf{u}_e + \mathbf{u}_o \quad (2.2.1.9)$$

a) Prove that  $\mathbf{V}_e \cap \mathbf{V}_o = \{0\}$ .

Let  $\mathbf{u} \in \mathbf{V}_e \cap \mathbf{V}_o$

$$\mathbf{u} \in \mathbf{V}_e \implies \mathbf{u}(-x) = \mathbf{u}(x) \quad (2.2.2.10)$$

$$\mathbf{u} \in \mathbf{V}_o \implies \mathbf{u}(-x) = -\mathbf{u}(x) \quad (2.2.2.11)$$

Equating (2.2.2.10) and (2.2.2.11),

$$\mathbf{u}(x) = -\mathbf{u}(x) \quad (2.2.2.12)$$

$$\implies 2\mathbf{u}(x) = 0 \quad (2.2.2.13)$$

$$\implies \mathbf{u} = 0 \quad (2.2.2.14)$$

Equations (2.2.8.3), (2.2.8.6), (2.2.1.9), (2.2.2.14) proves 1, 2 and 3.

2.2.3. Let  $\mathbf{W}_1$  and  $\mathbf{W}_2$  be subspaces of a vector space  $\mathbf{V}$  such that

$$\mathbf{W}_1 + \mathbf{W}_2 = \mathbf{V} \quad (2.2.3.1)$$

$$\text{and } \mathbf{W}_1 \cap \mathbf{W}_2 = \mathbf{0} \quad (2.2.3.2)$$

Prove that for each vector  $\alpha$  in  $\mathbf{V}$  there are unique vectors  $\alpha_1$  in  $\mathbf{W}_1$  and  $\alpha_2$  in  $\mathbf{W}_2$  such that

$$\alpha = \alpha_1 + \alpha_2 \quad (2.2.3.3)$$

**Solution:** Suppose, vectors  $\alpha_1$  and  $\alpha_2$  are not unique.

Consider

$$\alpha'_1 \in \mathbf{W}_1, \quad (2.2.3.4)$$

$$\alpha'_2 \in \mathbf{W}_2 \quad (2.2.3.5)$$

$$\text{such that } \alpha = \alpha'_1 + \alpha'_2 \quad (2.2.3.6)$$

(2.2.3.3) and (2.2.3.6) indicate

$$\alpha_1 + \alpha_2 = \alpha'_1 + \alpha'_2 \quad (2.2.3.7)$$

$$\implies \alpha_1 - \alpha'_1 = \alpha'_2 - \alpha_2 \quad (2.2.3.8)$$

For  $\alpha_1$  and  $\alpha'_1$  lying in subspace  $\mathbf{W}_1$ , defined on field  $\mathbb{F}$ , the following holds

$$\alpha_1 + c\alpha'_1 \in \mathbf{W}_1, c \in \mathbb{F} \quad (2.2.3.9)$$

$$c = -1 \implies \alpha_1 - \alpha'_1 \in \mathbf{W}_1 \quad (2.2.3.10)$$

$$\text{Similarly, } \alpha'_2 - \alpha_2 \in \mathbf{W}_2 \quad (2.2.3.11)$$

$$(2.2.3.8) \implies \alpha_1 - \alpha'_1 \in \mathbf{W}_2 \quad (2.2.3.12)$$

(2.2.3.2), (2.2.3.10), (2.2.3.12) indicate

$$\alpha_1 - \alpha'_1 = \alpha'_2 - \alpha_2 = \mathbf{0} \quad (2.2.3.13)$$

$$\implies \alpha_1 = \alpha'_1 \quad (2.2.3.14)$$

$$\alpha_2 = \alpha'_2 \quad (2.2.3.15)$$

So, there exists a unique  $\alpha_1 \in \mathbf{W}_1$  and  $\alpha_2 \in \mathbf{W}_2$  such that

$$\alpha = \alpha_1 + \alpha_2 \quad (2.2.3.16)$$

where  $\alpha \in \mathbf{V}$

2.2.4. Let  $\mathbf{V}$  be the set of all 2 X 2 matrices  $\mathbf{A}$  with complex entries which satisfy  $\mathbf{A}_{11} + \mathbf{A}_{22} = 0$  and Show that  $\mathbf{V}$  is a vector space over the field of real numbers, with the usual operations of matrix addition and multiplication of a matrix by a scalar.

**Solution:** A non-empty subset  $\mathbf{W}$  of  $\mathbf{V}$  is a subspace of  $\mathbf{V}$  if and only if for each pair of vectors  $\mathbf{a}, \mathbf{b}$  in  $\mathbf{W}$  and each scalar  $c$  in  $\mathbb{F}$  the vector  $c\mathbf{a} + \mathbf{b}$  is again in  $\mathbf{W}$

Let  $\mathbf{M}$  be the vector space of all 2 x 2 matrices over  $\mathbb{C}$  i.e set of complex numbers and let  $\mathbb{R}$  be set of real numbers.

Consider  $\mathbf{A}, \mathbf{B} \in \mathbf{V}$  and  $c \in \mathbb{C}$

$$\text{Let } \mathbf{A} = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \quad (2.2.4.1)$$

$$\text{and } \mathbf{B} = \begin{pmatrix} x' & y' \\ z' & w' \end{pmatrix} \quad (2.2.4.2)$$

$$x + w = x' + w' = 0 \quad (2.2.4.3)$$

$$c\mathbf{A} + \mathbf{B} = \begin{pmatrix} cx + x' & cy + y' \\ cz + z' & cw + w' \end{pmatrix} \quad (2.2.4.4)$$

$$\therefore (cx + x') + (cw + w') = c(x + w) + (x' + w') = 0 \quad (2.2.4.5)$$

Since  $\mathbf{V}$  is a subset of  $\mathbf{M}$ .

Therefore using above Theorem,

$$c\mathbf{A} + \mathbf{B} \in \mathbf{V} \quad (2.2.4.6)$$

Hence  $\mathbf{V}$  is a subspace of  $\mathbf{M}$  as a vector space over  $\mathbb{C}$ . Hence  $\mathbf{V}$  is a vector space over  $\mathbb{R}$ .

### 2.3 Bases and Dimension

2.3.1. Prove that if two vectors are linearly dependent, one of them is a scalar multiple of the

other.

**Solution:** consider the row reduced matrix

$$\begin{pmatrix} 1 & 1 & 2 & 4 \\ 2 & -1 & -5 & 2 \\ 1 & -1 & -4 & 0 \\ 2 & 1 & 1 & 6 \end{pmatrix} \xrightarrow[R_3 \rightarrow R_3 - R_1]{R_2 \rightarrow R_2 - 2R_1} \begin{pmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -9 & -6 \\ 0 & -2 & -6 & -4 \\ 2 & 1 & 1 & 6 \end{pmatrix} \quad (2.3.1.1)$$

$$\xrightarrow[R_2 \leftarrow R_4]{R_4 \leftarrow R_4 - 2R_1} \begin{pmatrix} 1 & 1 & 2 & 4 \\ 0 & -1 & -3 & -2 \\ 0 & -2 & -6 & -4 \\ 0 & -3 & -9 & -6 \end{pmatrix} \quad (2.3.1.2)$$

$$\xrightarrow[R_2 \leftarrow -R_2]{R_4 \leftarrow R_2} \begin{pmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & -3 & -9 & -6 \\ 0 & -2 & -6 & -4 \end{pmatrix} \quad (2.3.1.3)$$

$$\xrightarrow[R_4 \leftarrow R_4 + 2R_2]{R_3 \leftarrow R_3 + 3R_2} \begin{pmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (2.3.1.4)$$

Therefore the rank = no. of pivot columns = 2 (less than no. of columns). Thus the four vectors are not linearly independent.

2.3.2. Are the vectors

$$\alpha_1 = (1, 1, 2, 4), \alpha_2 = (2, -1, -5, 2) \quad (2.3.2.1)$$

$$\alpha_3 = (1, -1, -4, 0), \alpha_4 = (2, 1, 1, 6) \quad (2.3.2.2)$$

linearly independent in  $R^4$

**Solution:** consider the row reduced matrix

$$\begin{pmatrix} 1 & 1 & 2 & 4 \\ 2 & -1 & -5 & 2 \\ 1 & -1 & -4 & 0 \\ 2 & 1 & 1 & 6 \end{pmatrix} \xrightarrow[R_3 \rightarrow R_3 - R_1]{R_2 \rightarrow R_2 - 2R_1} \begin{pmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -9 & -6 \\ 0 & -2 & -6 & -4 \\ 2 & 1 & 1 & 6 \end{pmatrix} \quad (2.3.2.3)$$

$$\xrightarrow[R_2 \leftarrow R_4]{R_4 \leftarrow R_4 - 2R_1} \begin{pmatrix} 1 & 1 & 2 & 4 \\ 0 & -1 & -3 & -2 \\ 0 & -2 & -6 & -4 \\ 0 & -3 & -9 & -6 \end{pmatrix} \quad (2.3.2.4)$$

$$\xrightarrow[R_2 \leftarrow -R_2]{R_4 \leftarrow R_2} \begin{pmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & -3 & -9 & -6 \\ 0 & -2 & -6 & -4 \end{pmatrix} \quad (2.3.2.5)$$

$$\xrightarrow[R_4 \leftarrow R_4 + 2R_2]{R_3 \leftarrow R_3 + 3R_2} \begin{pmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (2.3.2.6)$$

Therefore the rank = no. of pivot columns = 2 (less than no. of columns). Thus the four vectors are not linearly independent.

2.3.3. Find a basis for the subspace of  $\mathbb{R}^4$  spanned by the four vectors

$$\alpha_1 = (1 \ 1 \ 2 \ 4) \quad (2.3.3.1)$$

$$\alpha_2 = (2 \ -1 \ -5 \ 2) \quad (2.3.3.2)$$

$$\alpha_3 = (1 \ -1 \ -4 \ 0) \quad (2.3.3.3)$$

$$\alpha_4 = (2 \ 1 \ 1 \ 6) \quad (2.3.3.4)$$

**Solution:** The basis of the given four vectors is equivalent to finding the basis of column-space  $C(\mathbf{A})$  of a matrix  $\mathbf{A}$  defined as follows,

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 & 2 \\ 1 & -1 & -1 & 1 \\ 2 & -5 & -4 & 1 \\ 4 & 2 & 0 & 6 \end{pmatrix} \quad (2.3.3.5)$$

Now we calculate the row echelon form of  $\mathbf{A}$

as follows,

$$\begin{pmatrix} 1 & 2 & 1 & 2 \\ 1 & -1 & -1 & 1 \\ 2 & -5 & -4 & 1 \\ 4 & 2 & 0 & 6 \end{pmatrix} \xleftrightarrow[R_2=R_2-R_1]{R_2=R_2-R_1} \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & -3 & -2 & -1 \\ 0 & -9 & -6 & -3 \\ 4 & 2 & 0 & 6 \end{pmatrix} \quad (2.3.3.6)$$

$$\xleftrightarrow{R_4=R_4-R_1} \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & -3 & -2 & -1 \\ 0 & -9 & -6 & -3 \\ 0 & -6 & -4 & -2 \end{pmatrix} \quad (2.3.3.7)$$

$$\xleftrightarrow{R_2=-\frac{1}{3}R_2} \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & -\frac{2}{3} & \frac{1}{3} \\ 0 & -9 & -6 & -3 \\ 0 & -6 & -4 & -2 \end{pmatrix} \quad (2.3.3.8)$$

$$\xleftrightarrow{R_3=R_3-9R_2} \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & -\frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 \\ 0 & -6 & -4 & -2 \end{pmatrix} \quad (2.3.3.9)$$

$$\xleftrightarrow{R_4=R_4+6R_2} \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & -\frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (2.3.3.10)$$

From (2.3.3.10) we can see that the first column and second column of  $\mathbf{A}$  contains pivot values. Hence the column 1 and column 2 are the basis of the subspace of  $\mathbb{R}^4$  spanned by the given vectors  $\alpha_1, \alpha_2, \alpha_3, \alpha_4$

Hence the required basis vectors are,

$$\mathbf{a}_1 = \begin{pmatrix} 1 & 1 & 2 & 4 \end{pmatrix} \quad (2.3.3.11)$$

$$\mathbf{a}_2 = \begin{pmatrix} 2 & -1 & -5 & 2 \end{pmatrix} \quad (2.3.3.12)$$

2.3.4. Show that the vectors

$$\alpha_1 = \begin{pmatrix} 1 & 0 & -1 \end{pmatrix} \quad \alpha_2 = \begin{pmatrix} 1 & 2 & 1 \end{pmatrix} \quad (2.3.4.1)$$

$$\alpha_3 = \begin{pmatrix} 0 & -3 & 2 \end{pmatrix} \quad (2.3.4.2)$$

form a basis for  $\mathbb{R}^3$ . Express each of the standard basis vectors as linear combinations of  $(\alpha_1 \ \alpha_2 \ \alpha_3)$

**Solution:**

**Theorem 2.1.** Let  $\mathbf{V}$  be an  $n$ -dimensional vector space over the field  $\mathbf{F}$ , and let  $\beta$  and  $\beta'$  be two ordered basis of  $\mathbf{V}$ . Then, there is a

unique, necessarily invertible,  $n \times n$  matrix  $\mathbf{P}$  with entries in  $\mathbf{F}$  such that

$$\begin{aligned} a) \quad & \begin{bmatrix} \alpha \end{bmatrix}_{\beta} = \mathbf{P} \begin{bmatrix} \alpha \end{bmatrix}_{\beta'} \\ b) \quad & \begin{bmatrix} \alpha \end{bmatrix}_{\beta'} = \mathbf{P}^{-1} \begin{bmatrix} \alpha \end{bmatrix}_{\beta} \end{aligned}$$

for every vector  $\alpha$  in  $\mathbf{V}$ . The columns of  $\mathbf{P}$  are given by

$$\mathbf{P}_j = \begin{bmatrix} \alpha_j \end{bmatrix}_{\beta} \quad j = 1, 2, \dots, n \quad (2.3.4.3)$$

In order to show that the set of vectors  $\alpha_1, \alpha_2, \alpha_3$  are basis for  $\mathbb{R}^3$ . We first show that  $\alpha_1, \alpha_2, \alpha_3$  are linearly independent in  $\mathbb{R}^3$  and also they span  $\mathbb{R}^3$ . Consider,

$$\mathbf{A} = (\alpha_1^T \ \alpha_2^T \ \alpha_3^T) \quad (2.3.4.4)$$

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & -3 \\ -1 & 1 & 2 \end{pmatrix} \quad (2.3.4.5)$$

Now, by row reduction

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & -3 \\ -1 & 1 & 2 \end{pmatrix} \xleftrightarrow{R_3=R_3+R_1} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & -3 \\ 0 & 2 & 2 \end{pmatrix} \quad (2.3.4.6)$$

$$\xleftrightarrow{R_3=R_3-R_2} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & -3 \\ 0 & 0 & 5 \end{pmatrix} \quad (2.3.4.7)$$

$$\xleftrightarrow{R_2=\frac{R_2}{2}} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -\frac{3}{2} \\ 0 & 0 & 5 \end{pmatrix} \quad (2.3.4.8)$$

$$\xleftrightarrow{R_1=R_1-R_2} \begin{pmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 1 & -\frac{3}{2} \\ 0 & 0 & 5 \end{pmatrix} \quad (2.3.4.9)$$

$$\xleftrightarrow{R_3=\frac{R_3}{5}} \begin{pmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 1 & -\frac{3}{2} \\ 0 & 0 & 1 \end{pmatrix} \quad (2.3.4.10)$$

$$\xleftrightarrow{R_1=R_1-\frac{3}{2}R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{3}{2} \\ 0 & 0 & 1 \end{pmatrix} \quad (2.3.4.11)$$

$$\xleftrightarrow{R_2=R_2+\frac{3}{2}R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (2.3.4.12)$$

(2.3.4.12) is the row reduced echelon form of  $\mathbf{A}$  and since it is identity matrix of order 3, we say that vectors  $\alpha_1, \alpha_2, \alpha_3$  are linearly independent and their column space is  $\mathbb{R}^3$  which means vectors  $\alpha_1, \alpha_2, \alpha_3$  span  $\mathbb{R}^3$ . Hence, vectors  $\alpha_1, \alpha_2, \alpha_3$  form a basis for

$\mathbb{R}^3$ .

Now, use theorem (2.1), and calculate the inverse of (2.3.4.5) then the columns of  $\mathbf{A}^{-1}$  will give the coefficients to write the standard basis vectors in terms of  $\alpha_i$ 's. We try to find the inverse of  $\mathbf{A}$  by row-reducing the augmented matrix  $\mathbf{A}|\mathbf{I}$

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & -3 \\ -1 & 1 & 2 \end{pmatrix} \quad (2.3.4.13)$$

Now, by row reducing  $\mathbf{A}|\mathbf{I}$  as follows

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & -3 & 0 & 1 & 0 \\ -1 & 1 & 2 & 0 & 0 & 1 \end{pmatrix} \quad (2.3.4.14)$$

$$\xleftrightarrow{R_3=R_3+R_1} \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & -3 & 0 & 1 & 0 \\ 0 & 2 & 2 & 1 & 0 & 1 \end{pmatrix} \quad (2.3.4.15)$$

$$\xleftrightarrow{R_3=R_3-R_2} \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & -3 & 0 & 1 & 0 \\ 0 & 0 & 5 & 1 & -1 & 1 \end{pmatrix} \quad (2.3.4.16)$$

$$\xleftrightarrow{R_2=\frac{R_2}{2}} \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -\frac{3}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 5 & 1 & -1 & 1 \end{pmatrix} \quad (2.3.4.17)$$

$$\xleftrightarrow{R_1=R_1-R_2} \begin{pmatrix} 1 & 0 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{3}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 5 & 1 & -1 & 1 \end{pmatrix} \quad (2.3.4.18)$$

$$\xleftrightarrow{R_3=\frac{R_3}{5}} \begin{pmatrix} 1 & 0 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{3}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{1}{5} & \frac{1}{5} \end{pmatrix} \quad (2.3.4.19)$$

$$\xleftrightarrow{R_1=R_1-\frac{3R_3}{2}} \begin{pmatrix} 1 & 0 & 0 & \frac{7}{10} & -\frac{1}{5} & -\frac{3}{10} \\ 0 & 1 & -\frac{3}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{1}{5} & \frac{1}{5} \end{pmatrix} \quad (2.3.4.20)$$

$$\xleftrightarrow{R_2=R_2+\frac{3R_3}{2}} \begin{pmatrix} 1 & 0 & 0 & \frac{7}{10} & -\frac{1}{5} & -\frac{3}{10} \\ 0 & 1 & 0 & \frac{3}{10} & \frac{1}{5} & \frac{3}{10} \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{1}{5} & \frac{1}{5} \end{pmatrix} \quad (2.3.4.21)$$

Thus, by (2.3.4.21), we have

$$\mathbf{A}^{-1} = \begin{pmatrix} \frac{7}{10} & -\frac{1}{5} & -\frac{3}{10} \\ \frac{3}{10} & \frac{1}{5} & \frac{3}{10} \\ \frac{1}{5} & -\frac{1}{5} & \frac{1}{5} \end{pmatrix} \quad (2.3.4.22)$$

Now, let  $\mathbf{e}_1 = (1 \ 0 \ 0)$ ,  $\mathbf{e}_2 = (0 \ 1 \ 0)$ , and  $\mathbf{e}_3 = (0 \ 0 \ 1)$  be the standard basis for  $\mathbb{R}^3$ . Hence, each of the standard basis vectors as

linear combinations of  $\alpha_1, \alpha_2, \alpha_3$  is as under

$$\mathbf{e}_1 = \frac{7}{10}\alpha_1 + \frac{3}{10}\alpha_2 + \frac{1}{5}\alpha_3 \quad (2.3.4.23)$$

$$\mathbf{e}_2 = -\frac{1}{5}\alpha_1 + \frac{1}{5}\alpha_2 - \frac{1}{5}\alpha_3 \quad (2.3.4.24)$$

$$\mathbf{e}_3 = \frac{-3}{10}\alpha_1 + \frac{3}{10}\alpha_2 + \frac{1}{5}\alpha_3 \quad (2.3.4.25)$$

2.3.5. Find three vectors in  $\mathbb{R}^3$  which are linearly dependent and are such that any two of them are linearly independent

**Solution:** Let

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad (2.3.5.1)$$

$$\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad (2.3.5.2)$$

$$\mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad (2.3.5.3)$$

Then,

$$\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (2.3.5.4)$$

Thus, they are linearly dependent. As we know that  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are linearly independent as they form the standard basis vectors.

Now, Suppose

$$\mathbf{a}\mathbf{v}_1 + \mathbf{b}\mathbf{v}_2 = \mathbf{0} \quad (2.3.5.5)$$

Then

$$\begin{pmatrix} a+b \\ b \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (2.3.5.6)$$

Hence, the second co-ordinate implies  $\mathbf{b} = 0$  and the first coordinate in turn implies  $\mathbf{a} = 0$ .

Thus  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are linearly independent. Similarly,  $\mathbf{v}_2$  and  $\mathbf{v}_3$  are linearly independent

2.3.6. Let  $\mathbf{V}$  be the vector space of all  $2 \times 2$  matrices over the field  $\mathbf{F}$ . Prove that  $\mathbf{V}$  has dimension 4 by exhibiting a basis for  $\mathbf{V}$  which has four elements.

**Solution:**

Let

$$v_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad v_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (2.3.6.1)$$

$$v_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad v_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (2.3.6.2)$$

Suppose  $av_{11} + bv_{12} + cv_{21} + dv_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ . Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad (2.3.6.3)$$

The only values of  $a, b, c, d$  which makes the equation (2.3.6.3) satisfied is, when  $a = b = c = d = 0$ . Thus  $v_1, v_2, v_3, v_4$  are linearly independent.

Now, let  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  be any  $2 \times 2$  matrix. Then  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = av_{11} + bv_{12} + cv_{21} + dv_{22}$ . Thus  $v_{11}, v_{12}, v_{21}, v_{22}$  span the space of  $2 \times 2$  matrix.

Thus  $v_{11}, v_{12}, v_{21}, v_{22}$  are both linearly independent and they span the span of all  $2 \times 2$  matrices. So,  $v_{11}, v_{12}, v_{21}, v_{22}$  constitute a basis for the space of all  $2 \times 2$  matrices.

We know that, the dimension of a vector space  $\mathbf{V}$ , denoted by  $\dim(\mathbf{V})$ , is the number of basis for  $\mathbf{V}$ . Therefore,  $\dim(\mathbf{V}) = 4$ .

2.3.7. Let  $\mathbf{V}$  be the vector space of all  $2 \times 2$  matrices over the field  $\mathbb{F}$ . Let  $W_1$  be the set of matrices of the form

$$\begin{pmatrix} x & -x \\ y & z \end{pmatrix} \quad (2.3.7.1)$$

and let  $W_2$  be the set of matrices of the form

$$\begin{pmatrix} a & b \\ -a & c \end{pmatrix} \quad (2.3.7.2)$$

- Prove that  $W_1$  and  $W_2$  are subspaces of  $\mathbf{V}$ .
- Find the dimension of  $W_1, W_2, W_1 + W_2$  and  $W_1 \cap W_2$ .

**Solution:** A non-empty subset  $W$  of  $\mathbf{V}$  is a subspace of  $\mathbf{V}$  if and only if for each pair of vectors  $\alpha, \beta$  in  $W$  and each scalar  $c \in \mathbb{F}$ , the vector  $c\alpha + \beta \in W$ .

a) Let  $A_1, A_2 \in W_1$  where,

$$A_1 = \begin{pmatrix} x_1 & -x_1 \\ y_1 & z_1 \end{pmatrix}, A_2 = \begin{pmatrix} x_2 & -x_2 \\ y_2 & z_2 \end{pmatrix} \quad (2.3.7.3)$$

Let  $c \in \mathbb{F}$  then,

$$cA_1 + A_2 = \begin{pmatrix} cx_1 + x_2 & -cx_1 - x_2 \\ cy_1 + y_2 & cz_1 + z_2 \end{pmatrix} = \begin{pmatrix} u & -u \\ v & w \end{pmatrix} \quad (2.3.7.4)$$

Thus  $cA_1 + A_2 \in W_1$ . **Hence  $W_1$  is a subspace.** Similarly, let  $A_1, A_2 \in W_2$  where,

$$A_1 = \begin{pmatrix} a_1 & b_1 \\ -a_1 & c_1 \end{pmatrix}, A_2 = \begin{pmatrix} a_2 & b_2 \\ -a_2 & c_2 \end{pmatrix} \quad (2.3.7.5)$$

Let  $c \in \mathbb{F}$  then,

$$cA_1 + A_2 = \begin{pmatrix} ca_1 + a_2 & cb_1 + b_2 \\ -ca_1 - a_2 & cc_1 + c_2 \end{pmatrix} = \begin{pmatrix} u & v \\ -u & w \end{pmatrix} \quad (2.3.7.6)$$

Thus  $cA_1 + A_2 \in W_2$ . **Hence  $W_2$  is a subspace.**

b) The subspace  $W_1$  can be given as,

$$\begin{pmatrix} x & -x \\ y & z \end{pmatrix} = x \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} + y \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + z \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (2.3.7.7)$$

$$= xA_1 + yA_2 + zA_2 \quad (2.3.7.8)$$

Now,

$$x \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} + y \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + z \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad (2.3.7.9)$$

$$\implies x = y = z = 0 \quad (2.3.7.10)$$

$A_1, A_2, A_3$  are linearly independent and spans  $W_1$ . Thus  $\{A_1, A_2, A_3\}$  forms basis for  $W_1$ .

$\therefore$  **dimension of  $W_1$  is 3.**

The subspace  $W_2$  can be given as,

$$\begin{pmatrix} a & b \\ -a & c \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (2.3.7.11)$$

$$= aA_1 + bA_2 + cA_2 \quad (2.3.7.12)$$

Now,

$$a \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad (2.3.7.13)$$

$$\implies a = b = c = 0 \quad (2.3.7.14)$$

$A_1, A_2, A_3$  are linearly independent and spans  $W_2$ . Thus  $\{A_1, A_2, A_3\}$  forms basis for  $W_2$ .

$\therefore$  **dimension of  $W_2$  is 3.**

Subspace  $W_1 + W_2$  is given by,

$$\begin{pmatrix} x+a & -x+b \\ y-a & z+c \end{pmatrix} \quad (2.3.7.15)$$

For  $x+a \neq -x+b \neq y-a \neq z+c$ ,

$$\begin{pmatrix} x+a & -x+b \\ y-a & z+c \end{pmatrix} = \begin{pmatrix} j & k \\ l & m \end{pmatrix} \quad (2.3.7.16)$$

$$= j \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + k \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + l \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + m \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (2.3.7.17)$$

$$= jA_1 + kA_2 + lA_3 + mA_4 \quad (2.3.7.18)$$

Now,

$$jA_1 + kA_2 + lA_3 + mA_4 = 0 \quad (2.3.7.19)$$

$$\implies j = k = l = m = 0 \quad (2.3.7.20)$$

$A_1, A_2, A_3, A_4$  are linearly independent and spans  $W_1 + W_2$ . Thus  $\{A_1, A_2, A_3, A_4\}$  forms a basis.

$\therefore$  **dimension of  $W_1 + W_2$  is 4.**

The subspace  $W_1 \cap W_2$  is given as,

$$\begin{pmatrix} x & -x \\ -x & y \end{pmatrix} = x \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix} + y \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (2.3.7.21)$$

$$= xA_1 + yA_2 \quad (2.3.7.22)$$

Now,

$$x \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix} + y \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad (2.3.7.23)$$

$$\implies x = y = 0 \quad (2.3.7.24)$$

$A_1, A_2$  are linearly independent and spans  $W_1 \cap W_2$ . Thus,  $\{A_1, A_2\}$  forms a basis.

$\therefore$  **dimension of  $W_1 \cap W_2$  is 2.**

2.3.8. Let  $\mathbf{V}$  be the space of  $2 \times 2$  matrices over  $\mathbf{F}$ .

Find a basis  $\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4\}$  for  $\mathbf{V}$  such that  $\mathbf{A}_j^2 = \mathbf{A}_j$  for each  $j$

**Solution:** Every  $2 \times 2$  matrix may be written as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (2.3.8.1)$$

This shows that

$$\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4\} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \quad (2.3.8.2)$$

can be the basis for the space  $\mathbf{V}$  of all  $2 \times 2$  matrices. However  $\mathbf{A}_2$  and  $\mathbf{A}_3$  doesn't satisfy the property of  $\mathbf{A}^2 = \mathbf{A}$ . Consider  $b = 0$  and  $c = 0$ , then the matrix

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \quad (2.3.8.3)$$

can't be a basis as it is the linear combination of  $\mathbf{A}_1$  and  $\mathbf{A}_4$ . Hence either  $b$  or  $c$  or both must be non zero. Hence,

$$\mathbf{A}_2 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \quad (2.3.8.4)$$

$$\mathbf{A}_3 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \quad (2.3.8.5)$$

Here,  $\mathbf{A}_2^2 = \mathbf{A}_2$  and  $\mathbf{A}_3^2 = \mathbf{A}_3$ . Therefore the basis can be

$$\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4\} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \quad (2.3.8.6)$$

$\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4\}$  forms the basis, iff they are linearly independent and the linear combination of them span the space  $\mathbf{V}$ . To show that they are linearly independent, we show that the equation has a trivial solution.

$$a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad (2.3.8.7)$$

$$\implies a + b = 0 \quad (2.3.8.8)$$

$$b = 0 \quad (2.3.8.9)$$

$$c = 0 \quad (2.3.8.10)$$

$$c + d = 0 \quad (2.3.8.11)$$

The corresponding matrix form is  $\mathbf{Ax} = 0$

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (2.3.8.12)$$

Row reducing the augmented matrix,

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \xleftrightarrow[R_4 \leftarrow R_4 - R_3]{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad (2.3.8.13)$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - R_2} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad (2.3.8.14)$$

Therefore,  $a = b = c = d = 0$ . Hence the matrices are linearly independent. To show that the linear combination of  $\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4\}$  span the space  $\mathbf{V}$ , consider an arbitrary matrix,

$$\begin{pmatrix} w & x \\ y & z \end{pmatrix} \quad (2.3.8.15)$$

Compute  $a, b, c, d$  such that

$$\begin{pmatrix} w & x \\ y & z \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (2.3.8.16)$$

$$= \begin{pmatrix} a+b & c \\ b & c+d \end{pmatrix} \quad (2.3.8.17)$$

Equating the entries, this produces system of linear equations,

$$a + b = w, y = b, x = c, z = c + d \quad (2.3.8.18) \quad 2.3.10.$$

$$\Rightarrow a = w - y \quad (2.3.8.19)$$

$$b = y \quad (2.3.8.20)$$

$$c = x \quad (2.3.8.21)$$

$$d = z - x \quad (2.3.8.22)$$

In particular, there exists atleast one solution regardless of the values of  $w, x, y, z$ . For example, consider the following matrix,

$$\begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ -2 & 7 \end{pmatrix} \quad (2.3.8.23)$$

Here,  $a = 5, b = -2, c = 4, d = 3$ . Using (2.3.8.16), we get

$$5 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - 2 \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + 4 \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} + 3 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ -2 & 7 \end{pmatrix} \quad (2.3.8.24)$$

Hence  $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$  forms

the basis for the given space  $\mathbf{V}$ .

2.3.9. Let  $\mathbf{V}$  be a vector space over a subfield  $\mathbf{F}$  of complex numbers. Suppose  $\alpha, \beta$  and  $\gamma$  are linearly independent vectors in  $\mathbf{V}$ . Prove that  $(\alpha+\beta), (\beta+\gamma)$  and  $(\gamma+\alpha)$  are linearly independent.

**Solution:** Let  $\alpha, \beta$  and  $\gamma$  be three  $n \times 1$  dimensional vectors. We need to prove that,

$$(\alpha + \beta \quad \beta + \gamma \quad \gamma + \alpha) \mathbf{x} = 0 \quad (2.3.9.1)$$

will only have a trivial solution. The above equation can be written as

$$(\alpha \quad \beta \quad \gamma) \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \mathbf{x} = 0 \quad (2.3.9.2)$$

$$\mathbf{x}^T \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha^T \\ \beta^T \\ \gamma^T \end{pmatrix} = 0 \quad (2.3.9.3)$$

Since,  $\alpha, \beta$  and  $\gamma$  are independent.

$$\mathbf{x}^T \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} = 0 \quad (2.3.9.4)$$

In the above equation we can see that the  $3 \times 3$  matrix has linearly independent rows and hence will have a trivial solution. So,  $\mathbf{x}$  is a zero vector. Hence,  $(\alpha+\beta), (\beta+\gamma)$  and  $(\gamma+\alpha)$  are linearly independent.

2.3.10. Prove that the space of all  $\mathbf{m} \times \mathbf{n}$  matrices over the field  $\mathbf{F}$  has dimension  $mn$ , by exhibiting a basis for this space.

**Solution:** Let  $\mathbf{M}$  be the space of all  $\mathbf{m} \times \mathbf{n}$  matrices. Let,  $\mathbf{M}_{ij} \in \mathbf{M}$  be,

$$\mathbf{M}_{ij} = \begin{cases} 0 & m \neq i, n \neq j \\ 1 & m = i, n = j \end{cases} \quad (2.3.10.1)$$

For example,

$$\mathbf{M}_{12} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{m \times n} \quad (2.3.10.2)$$

$$(2.3.10.3)$$



Let  $\mathbf{A} \in \mathbf{M}$  given as,

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}_{m \times n} \quad (2.3.10.4)$$

Now clearly,

$$\mathbf{a}_{11} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \quad (2.3.10.5)$$

$$\Rightarrow \mathbf{a}_{11} = \mathbf{A}\mathbf{M}_{11} \quad (2.3.10.6)$$

$$\therefore \mathbf{A} = \sum_{i=1}^m \sum_{j=1}^n a_{ij} \mathbf{M}_{ij} \quad (2.3.10.7)$$

$\Rightarrow \mathbf{M}_{ij}$  span  $\mathbf{M}$ . Also from the above equation  $\mathbf{A} = 0$  if and only if all elements are zero, that is,

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \quad (2.3.10.8)$$

$$\Rightarrow a_{ij} = 0 \quad (2.3.10.9)$$

Hence,  $\mathbf{M}_{ij}$  are linearly independent as well. Hence,  $\mathbf{M}_{ij}$  constitutes a basis for  $\mathbf{M}$ . and number of elements in basis are  $mn$ . Hence dimension of space of all  $m \times n$  matrices  $\mathbf{M}$  is  $mn$ .

2.3.11. Let  $V$  be a vector space over the field  $F = \{0, 1\}$ . Suppose  $\alpha, \beta$  and  $\gamma$  are linearly independent vectors in  $V$ . Comment on  $(\alpha + \beta)$ ,  $(\beta + \gamma)$  and  $(\gamma + \alpha)$

**Solution:** The addition of elements in the field  $F$  is defined as,

$$\begin{aligned} 0 + 0 &= 0 \\ 1 + 1 &= 0 \end{aligned} \quad (2.3.11.1)$$

A set are vectors  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  are linearly independent if

$$a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3 = 0 \quad (2.3.11.2)$$

has only one trivial solution

$$a = b = c = 0 \quad (2.3.11.3)$$

Now,

$$a(\alpha + \beta) + b(\beta + \gamma) + c(\gamma + \alpha) = 0 \quad (2.3.11.4)$$

$$\Rightarrow (a + c)\alpha + (a + b)\beta + (b + c)\gamma = 0 \quad (2.3.11.5)$$

Writing (2.3.11.5) in matrix form,

$$\begin{pmatrix} \alpha & \beta & \gamma \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \mathbf{x} = 0 \quad (2.3.11.6)$$

where,

$$\mathbf{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\mathbf{x}^T \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = 0 \quad (2.3.11.7)$$

Since  $\alpha, \beta$  and  $\gamma$  are linearly independent vectors,

$$\mathbf{x}^T \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} = (0 \ 0 \ 0) \quad (2.3.11.8)$$

Transposing on both sides,

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \mathbf{x} = 0 \quad (2.3.11.9)$$

By using the properties from (2.3.11.1) and reducing (2.3.11.9) to row echelon form,

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_1 + R_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{R_3 \leftarrow R_2 + R_3} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad (2.3.11.10)$$

Expressing (2.3.11.10) as a linear combination

of vectors,

$$\begin{aligned} a \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ \Rightarrow \begin{pmatrix} a+c \\ b+c \\ 0 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ \Rightarrow a+c=0; \quad b+c=0 &\quad (2.3.11.11) \end{aligned}$$

The solutions to (2.3.11.11) are,

$$a=b=c=0; \quad a=b=c=1 \quad (2.3.11.12)$$

Since there is no trivial solution,  $(\alpha+\beta)$ ,  $(\beta+\gamma)$  and  $(\gamma+\alpha)$  are linearly dependent

2.3.12. Let  $\mathbf{V}$  be the set of real numbers. Regard  $\mathbf{V}$  as a vector space over the field of rational numbers, with usual operations. Prove that this vector space is not finite-dimensional.

**Solution:** Given  $\mathbf{V}$  is a vector space over field  $\mathbb{Q}$  (rational numbers)

It is finite dimensional with dimensionality  $n$  if every vector  $\mathbf{v}$  in  $\mathbf{V}$  can be written as

$$\mathbf{v} = \sum_{i=0}^{n-1} c_i \alpha_i \quad (2.3.12.1)$$

$$\text{where } c_i \in \mathbb{Q} \quad (2.3.12.2)$$

$$\text{and } \mathbf{B} = \{\alpha_0, \alpha_1, \dots, \alpha_{n-1}\} \quad (2.3.12.3)$$

is the basis with linearly independent  $\alpha_i$  that is, basis is the largest set with linearly independent vectors from  $\mathbf{V}$

Consider the set of vectors  $\{1, x\}$ , where  $x$  is irrational.

Assume there exists non zero  $\beta_0, \beta_1 \in \mathbb{Q}$  such that

$$\beta_0 + \beta_1 x = 0 \quad (2.3.12.4)$$

$$\Rightarrow x = -\frac{\beta_0}{\beta_1} \quad (2.3.12.5)$$

But  $x$  is irrational and  $-\frac{\beta_0}{\beta_1}$  is rational so (2.3.12.5) can't be possible so  $\beta_0, \beta_1 = 0$

Hence  $\{1, x\}$  are independent.

Similarly for the set  $\{1, x, x^2\}$

for  $\beta_0, \beta_1, \beta_2 \in \mathbb{Q}$

$$\beta_0 + \beta_1 x + \beta_2 x^2 = 0 \quad (2.3.12.6)$$

$\beta_1 x + \beta_2 x^2$  is irrational and  $\beta_0$  is rational.

Therefore

$$\beta_0 = 0 \quad (2.3.12.7)$$

$$\text{and } \beta_1 x + \beta_2 x^2 = 0, (x \neq 0) \quad (2.3.12.8)$$

$$\Rightarrow \beta_1 + \beta_2 x = 0 \quad (2.3.12.9)$$

$$\Rightarrow \beta_1, \beta_2 = 0 \quad (2.3.12.10)$$

$$\therefore \beta_0 + \beta_1 x + \beta_2 x^2 = 0 \quad (2.3.12.11)$$

$$\Leftrightarrow \beta_0, \beta_1, \beta_2 = 0 \quad (2.3.12.12)$$

Hence  $\{1, x, x^2\}$  are independent

By induction, let us say the set  $\{1, x, x^2, \dots, x^n\}$  is independent

$$\text{for } \beta_0, \beta_1, \beta_2, \dots, \beta_n \in \mathbb{Q} \quad (2.3.12.13)$$

$$\beta_0 + \beta_1 x + \beta_2 x^2 + \dots + \beta_n x^n = 0 \quad (2.3.12.14)$$

$$\Leftrightarrow \beta_0, \beta_1, \beta_2, \dots, \beta_n = 0 \quad (2.3.12.15)$$

To prove this for the set  $\mathbf{A} = \{1, x, x^2, \dots, x^{n+1}\}$

$$\text{for } \beta_0, \beta_1, \beta_2, \dots, \beta_n, \beta_{n+1} \in \mathbb{Q} \quad (2.3.12.16)$$

$$\beta_0 + \beta_1 x + \dots + \beta_n x^n + \beta_{n+1} x^{n+1} = 0 \quad (2.3.12.17)$$

Comparing to (2.3.12.7) and (2.3.12.8)

$$\beta_0 = 0 \quad (2.3.12.18)$$

$$\beta_1 + \beta_2 x + \dots + \beta_{n+1} x^n = 0 \quad (2.3.12.19)$$

Comparing with (2.3.12.14), we have

$$\beta_1, \beta_2, \dots, \beta_{n+1} = 0$$

$$\therefore \beta_0 + \beta_1 x + \dots + \beta_n x^n + \beta_{n+1} x^{n+1} = 0 \quad (2.3.12.20)$$

$$\Leftrightarrow \beta_0, \beta_1, \beta_2, \dots, \beta_n, \beta_{n+1} = 0 \quad (2.3.12.21)$$

Hence  $\mathbf{A}$  has linearly independent vectors

Let the set  $\mathbf{B} = \{1, x, x^2, \dots, x^m\}$  be the largest linearly independent set in  $\mathbf{V}$  and hence can form the basis leading to dimensionality  $m+1$

But from induction, we have proved that  $\{1, x, x^2, \dots, x^m, x^{m+1}\}$  is also independent which is a contradiction to dimensionality being  $m+1$

Hence we deduce that the vector space  $\mathbf{V}$  is not finite dimensional over the field  $\mathbb{Q}$

## 2.4 Coordinates

2.4.1. Show that the vectors

$$\alpha_1 = \begin{pmatrix} 1 & 1 & 0 & 0 \end{pmatrix} \quad \alpha_2 = \begin{pmatrix} 0 & 0 & 1 & 1 \end{pmatrix} \quad (2.4.1.1)$$

$$\alpha_3 = \begin{pmatrix} 1 & 0 & 0 & 4 \end{pmatrix} \quad \alpha_4 = \begin{pmatrix} 0 & 0 & 0 & 2 \end{pmatrix} \quad (2.4.1.2)$$

form a basis for  $\mathfrak{R}^4$ . Find the coordinates of each of the standard basis vectors in the ordered basis  $(\alpha_1 \ \alpha_2 \ \alpha_3 \ \alpha_4)$

**Solution:**

**Theorem 2.2.** Let  $\mathbf{V}$  be an  $n$ -dimensional vector space over the field  $\mathbf{F}$ , and let  $\beta$  and  $\beta'$  be two ordered basis of  $\mathbf{V}$ . Then, there is a unique, necessarily invertible,  $n \times n$  matrix  $\mathbf{P}$  with entries in  $\mathbf{F}$  such that

$$a) \begin{bmatrix} \alpha \end{bmatrix}_{\beta} = \mathbf{P} \begin{bmatrix} \alpha \end{bmatrix}_{\beta'}$$

$$b) \begin{bmatrix} \alpha \end{bmatrix}_{\beta'} = \mathbf{P}^{-1} \begin{bmatrix} \alpha \end{bmatrix}_{\beta}$$

for every vector  $\alpha$  in  $\mathbf{V}$ . The columns of  $\mathbf{P}$  are given by

$$\mathbf{P}_j = \begin{bmatrix} \alpha_j \end{bmatrix}_{\beta} \quad j = 1, 2, \dots, n \quad (2.4.1.3)$$

First, we need to show that the set of vectors  $\alpha_1, \alpha_2, \alpha_3$  and  $\alpha_4$  are basis for  $\mathfrak{R}^4$ . For, this we first show that  $\alpha_1, \alpha_2, \alpha_3$  and  $\alpha_4$  are linearly independent in  $\mathfrak{R}^4$  and also they span  $\mathfrak{R}^4$ . Consider,

$$\mathbf{A} = (\alpha_1^T \ \alpha_2^T \ \alpha_3^T \ \alpha_4^T) \quad (2.4.1.4)$$

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 2 \end{pmatrix} \quad (2.4.1.5)$$

Now,

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 2 \end{pmatrix} \xleftrightarrow{r_2=r_2-r_1} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 2 \end{pmatrix} \quad (2.4.1.6)$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 2 \end{pmatrix} \xleftrightarrow{r_2 \leftrightarrow r_3} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 4 & 2 \end{pmatrix} \quad (2.4.1.7)$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 4 & 2 \end{pmatrix} \xleftrightarrow{r_4=r_4-r_2} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 4 & 2 \end{pmatrix} \quad (2.4.1.8)$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 4 & 2 \end{pmatrix} \xleftrightarrow{r_3=-r_3} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & 2 \end{pmatrix} \quad (2.4.1.9)$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & 2 \end{pmatrix} \xleftrightarrow{r_4=r_4-4r_3} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \quad (2.4.1.10)$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \xleftrightarrow{r_1=r_1-r_3} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \quad (2.4.1.11)$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \xleftrightarrow{r_4=\frac{r_4}{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (2.4.1.12)$$

(2.4.1.12) is the row reduced echelon form of  $\mathbf{A}$  and since it is identity matrix of order 4, we say that vectors  $\alpha_1, \alpha_2, \alpha_3$  and  $\alpha_4$  are linearly independent and their column space is  $\mathfrak{R}^4$  which means vectors  $\alpha_1, \alpha_2, \alpha_3$  and  $\alpha_4$  span  $\mathfrak{R}^4$ . Hence, vectors  $\alpha_1, \alpha_2, \alpha_3$  and  $\alpha_4$  form a basis for  $\mathfrak{R}^4$ .

Now, we use theorem (2.2), and if we calculate

the inverse of

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 2 \end{pmatrix} \quad (2.4.1.13)$$

then the columns of  $\mathbf{A}^{-1}$  will give the coefficients to write the standard basis vectors in terms of  $\alpha'_i$ 's. We try to find the inverse of  $\mathbf{A}$  by row-reducing the augmented matrix.

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 4 & 2 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (2.4.1.14)$$

Now, we solve for  $\mathbf{A}^{-1}$  as follows

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 4 & 2 & 0 & 0 & 0 & 1 \end{pmatrix} \xleftrightarrow{r_2=r_2-r_1} \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 4 & 2 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (2.4.1.15)$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 4 & 2 & 0 & 0 & 0 & 1 \end{pmatrix} \xleftrightarrow{r_2 \leftrightarrow r_3} \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 4 & 2 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (2.4.1.16)$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 4 & 2 & 0 & 0 & 0 & 1 \end{pmatrix} \xleftrightarrow{r_4=r_4-r_2} \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 & 0 & -1 & 1 \end{pmatrix} \quad (2.4.1.17)$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 & 0 & -1 & 1 \end{pmatrix} \xleftrightarrow{r_3=-r_3} \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 & 0 & -1 & 1 \end{pmatrix} \quad (2.4.1.18)$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 & 0 & -1 & 1 \end{pmatrix} \xleftrightarrow{r_4=r_4-4r_3} \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & -4 & 4 & -1 & 1 \end{pmatrix} \quad (2.4.1.19)$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & -4 & 4 & -1 & 1 \end{pmatrix} \xleftrightarrow{r_1=r_1-r_3} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & -4 & 4 & -1 & 1 \end{pmatrix} \quad (2.4.1.20)$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & -4 & 4 & -1 & 1 \end{pmatrix} \xleftrightarrow{r_4=\frac{r_4}{2}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 2 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad (2.4.1.21)$$

Thus, by (2.4.1.21), we have

$$\mathbf{A}^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ -2 & 2 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad (2.4.1.22)$$

Now, let  $\mathbf{e}_1 = (1 \ 0 \ 0 \ 0)$ ,  $\mathbf{e}_2 = (0 \ 1 \ 0 \ 0)$ ,  $\mathbf{e}_3 = (0 \ 0 \ 1 \ 0)$  and  $\mathbf{e}_4 = (0 \ 0 \ 0 \ 1)$  be

the standard basis for  $\mathfrak{R}^4$ . Hence,

$$\mathbf{e}_1 = \alpha_3 - 2\alpha_4 \quad (2.4.1.23)$$

$$\mathbf{e}_2 = \alpha_1 - \alpha_3 + 2\alpha_4 \quad (2.4.1.24)$$

$$\mathbf{e}_3 = \alpha_2 - \frac{1}{2}\alpha_4 \quad (2.4.1.25)$$

$$\mathbf{e}_4 = \frac{1}{2}\alpha_4 \quad (2.4.1.26)$$

2.4.2. Find the coordinate matrix of the vector  $\begin{pmatrix} 1 & 0 & 1 \end{pmatrix}$  in the basis of  $C^3$  consisting of the vectors  $\begin{pmatrix} 2i & 1 & 0 \end{pmatrix}$ ,  $\begin{pmatrix} 2 & -1 & 1 \end{pmatrix}$ ,  $\begin{pmatrix} 0 & 1+i & 1-i \end{pmatrix}$  in that order.

**Solution:**

$$\begin{pmatrix} 1 & 0 & 1 \end{pmatrix} = \alpha_1 \begin{pmatrix} 2i & 1 & 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 2 & -1 & 1 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 & 1+i & 1-i \end{pmatrix} \quad (2.4.2.1)$$

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2i & 2 & 0 \\ 1 & -1 & 1+i \\ 0 & 1 & 1-i \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \quad (2.4.2.2)$$

Now we find  $\alpha_i$  by row reducing augmented matrix.

$$\begin{pmatrix} 2i & 2 & 0 & 1 \\ 1 & -1 & 1+i & 0 \\ 0 & 1 & 1-i & 1 \end{pmatrix} \xrightarrow[R_2 \rightarrow R_2 - 2iR_3]{R_1 \rightarrow R_2} \begin{pmatrix} 1 & -1 & 1+i & 0 \\ 0 & 2+2i & 2-2i & 1 \\ 0 & 1 & 1-i & 1 \end{pmatrix} \quad (2.4.2.3)$$

$$\xleftrightarrow[R_3 \leftarrow R_3 - R_2]{R_2 \leftarrow R_2/2} \begin{pmatrix} 1 & -1 & 1+i & 0 \\ 0 & 1+i & 1-i & \frac{1}{2} \\ 0 & -i & 0 & \frac{1}{2} \end{pmatrix} \quad (2.4.2.4)$$

Therefore the coordinate matrix of the vector is

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} \frac{-1-i}{2} \\ \frac{i}{2} \\ \frac{3+i}{4} \end{pmatrix} \quad (2.4.2.5)$$

2.4.3. Let  $\mathbf{B} = (\alpha_1 \ \alpha_2 \ \alpha_3)$  be the ordered basis for  $R^3$  consisting of

$$\alpha_1 = \begin{pmatrix} 1 & 0 & -1 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}.$$

What are the coordinates of vector  $\begin{pmatrix} a & b & c \end{pmatrix}$  in the ordered basis  $\mathbf{B}$ ?

**Solution:** Given

$$\mathbf{B} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix} \quad (2.4.3.1)$$

be the ordered basis for  $R^3$ , then the coordinates of vector,

$$\alpha = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad (2.4.3.2)$$

in the ordered basis  $R^3$  is the vector,

$$[\alpha]_{\mathbf{B}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad (2.4.3.3)$$

hence

$$x_1\alpha_1 + x_2\alpha_2 + x_3\alpha_3 = \alpha \quad (2.4.3.4)$$

substituting (2.4.3.1) and (2.4.3.2) in (2.4.3.4)

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad (2.4.3.5)$$

augmented matrix form

$$\begin{pmatrix} 1 & 1 & 1 & a \\ 0 & 1 & 0 & b \\ -1 & 1 & 0 & c \end{pmatrix} \quad (2.4.3.6)$$

converting above matrix into row reduced echelon form

$$\begin{pmatrix} 1 & 1 & 1 & a \\ 0 & 1 & 0 & b \\ -1 & 1 & 0 & c \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 + R_1} \begin{pmatrix} 1 & 1 & 1 & a \\ 0 & 1 & 0 & b \\ 0 & 2 & 1 & c+a \end{pmatrix} \quad (2.4.3.7)$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - 2R_2} \begin{pmatrix} 1 & 1 & 1 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & a-2b+c \end{pmatrix} \quad (2.4.3.8)$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - R_2} \begin{pmatrix} 1 & 0 & 1 & a-b \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & a-2b+c \end{pmatrix} \quad (2.4.3.9)$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - R_3} \begin{pmatrix} 1 & 0 & 0 & b-c \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & a-2b+c \end{pmatrix} \quad (2.4.3.10)$$

∴ The coordinates of  $\alpha$  w.r.t  $\mathbf{B}$  is

$$[\alpha]_{\mathbf{B}} = \begin{pmatrix} b-c \\ b \\ a-2b+c \end{pmatrix} \quad (2.4.3.11)$$

2.4.4. Let  $\mathbf{W}$  be the subspace of  $\mathbf{C}^3$  spanned by  $\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}$  and  $\alpha_2 = \begin{pmatrix} 1 \\ i \\ 1+i \end{pmatrix}$ .

a) Show that  $\alpha_1$  and  $\alpha_2$  form a basis for  $\mathbf{W}$ .

b) Show that the vectors  $\beta_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$  and

$\beta_2 = \begin{pmatrix} 1 \\ i \\ 1+i \end{pmatrix}$  are in  $\mathbf{W}$  and form another basis for  $\mathbf{W}$ .

c) What are the coordinates of  $\alpha_1$  and  $\alpha_2$  in the ordered basis  $\{\beta_1, \beta_2\}$  for  $\mathbf{W}$ .

**Solution:**

a) It is given that  $\alpha_1$  and  $\alpha_2$  span  $\mathbf{W}$ . For  $\alpha_1$  and  $\alpha_2$  to be the basis for  $\mathbf{W}$  they must be linearly independent. Let

$$S_1 = \{\alpha_1, \alpha_2\} = \left\{ \begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}, \begin{pmatrix} 1+i \\ 1 \\ -1 \end{pmatrix} \right\} \quad (2.4.4.1)$$

Using row reduction on matrix  $\mathbf{A} = (\alpha_1 \ \alpha_2)$

$$\begin{pmatrix} 1 & 1+i \\ 0 & 1 \\ i & -1 \end{pmatrix} \xrightarrow{R_3 \leftarrow R_3 - iR_1} \begin{pmatrix} 1 & 1+i \\ 0 & 1 \\ 0 & -i \end{pmatrix} \quad (2.4.4.2)$$

$$\xrightarrow{R_3 \leftarrow R_3 + iR_2} \begin{pmatrix} 1 & 1+i \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (2.4.4.3)$$

$$\xrightarrow{R_1 \leftarrow R_1 - (i+1)R_2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (2.4.4.4)$$

Since  $\mathbf{A}$  is a full-rank matrix the column vectors are linearly independent. Therefore  $S_1 = \{\alpha_1, \alpha_2\}$  is a basis set for  $\mathbf{W}$ .

b)

$$\beta_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad (2.4.4.5)$$

$$\beta_2 = \begin{pmatrix} 1 \\ i \\ 1+i \end{pmatrix} \quad (2.4.4.6)$$

Since column vectors of  $\mathbf{A} = (\alpha_1 \ \alpha_2)$  are basis for  $\mathbf{W}$  and if  $\beta_1$  and  $\beta_2 \in \mathbf{W}$  there exist a unique solution  $\mathbf{x}$  such that

$$(\alpha_1 \ \alpha_2)\mathbf{x} = (\beta_1 \ \beta_2) \quad (2.4.4.7)$$

Using row reduction on augmented matrix

$$\left( \begin{array}{cc|cc} 1 & 1+i & 1 & 1 \\ 0 & 1 & 1 & i \\ i & -1 & 0 & 1+i \end{array} \right) \quad (2.4.4.8)$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - iR_1} \left( \begin{array}{cc|cc} 1 & 1+i & 1 & 1 \\ 0 & 1 & 1 & i \\ 0 & -i & -i & 1 \end{array} \right) \quad (2.4.4.9)$$

$$\xleftrightarrow{R_3 \leftarrow R_3 + iR_2} \left( \begin{array}{cc|cc} 1 & 1+i & 1 & 1 \\ 0 & 1 & 1 & i \\ 0 & 0 & 0 & 0 \end{array} \right) \quad (2.4.4.10)$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - (i+1)R_2} \left( \begin{array}{cc|cc} 1 & 0 & -i & 2-i \\ 0 & 1 & 1 & i \\ 0 & 0 & 0 & 0 \end{array} \right) \quad (2.4.4.11)$$

$$\Rightarrow \mathbf{x} = \begin{pmatrix} -i & 2-i \\ 1 & i \end{pmatrix} \quad (2.4.4.12)$$

Therefore  $\beta_1$  and  $\beta_2 \in \mathbf{W}$ .

Consider

$$S_2 = \{\beta_1, \beta_2\} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ i \\ 1+i \end{pmatrix} \right\} \quad (2.4.4.13)$$

and also let

$$\mathbf{B} = \begin{pmatrix} 1 & 1 \\ 1 & i \\ 0 & 1+i \end{pmatrix} \quad (2.4.4.14)$$

Using row reduction on matrix  $\mathbf{B}$

$$\begin{pmatrix} 1 & 1 \\ 1 & i \\ 0 & 1+i \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 1 \\ 0 & i-1 \\ 0 & 1+i \end{pmatrix} \quad (2.4.4.15)$$

$$\xleftrightarrow{R_2 \leftarrow \frac{R_2}{i-1}} \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1+i \end{pmatrix} \quad (2.4.4.16)$$

$$\xleftrightarrow{\begin{matrix} R_1 \leftarrow R_1 - R_2 \\ R_3 \leftarrow R_3 - (i-1)R_2 \end{matrix}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (2.4.4.17)$$

Since  $\mathbf{B}$  is a full rank matrix the column vectors are linearly independent.

Let  $\alpha$  be any vector in the subspace  $\mathbf{W}$ , then it can be expressed as  $\text{span}\{\alpha_1, \alpha_2\}$  i.e

$$\alpha = (\alpha_1 \quad \alpha_2) \mathbf{x}_1 = \mathbf{A} \mathbf{x}_1 \quad (2.4.4.18)$$

$S_2 = \{\beta_1, \beta_2\}$  spans  $\mathbf{W}$  if any vector  $\alpha \in \mathbf{W}$  can be expressed as

$$\alpha = (\beta_1, \beta_2) \mathbf{x}_2 = \mathbf{B} \mathbf{x}_2 \quad (2.4.4.19)$$

From (2.4.4.18) and (2.4.4.19) we conclude

$$\mathbf{B} \mathbf{x}_2 = \mathbf{A} \mathbf{x}_1 \quad (2.4.4.20)$$

$$\Rightarrow \mathbf{x}_2 = \mathbf{B}^{-1} \mathbf{A} \mathbf{x}_1 \quad (2.4.4.21)$$

Therefore from (2.4.4.21)  $\mathbf{x}_2$  exists if  $\mathbf{B}$  is invertible. From (2.4.4.17) we conclude  $\mathbf{x}_2$  exists and hence any vector  $\alpha \in \mathbf{W}$  can be expressed as  $\text{span}\{\beta_1, \beta_2\}$ . Therefore  $\{\beta_1, \beta_2\}$  is basis for  $\mathbf{W}$ .

- c) Since  $\alpha_1, \alpha_2 \in \mathbf{W}$  and  $\{\beta_1, \beta_2\}$  are ordered basis for  $\mathbf{W}$  there must exist unique value of  $\mathbf{x}$  such that

$$(\beta_1 \quad \beta_2) \mathbf{x} = (\alpha_1 \quad \alpha_2) \quad (2.4.4.22)$$

Using row reduction on (2.4.4.22) we get,

$$\begin{pmatrix} 1 & 1 & | & 1 & 1+i \\ 1 & i & | & 0 & 1 \\ 0 & 1+i & | & i & -1 \end{pmatrix} \quad (2.4.4.23)$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 1 & | & 1 & 1+i \\ 0 & i-1 & | & -1 & -i \\ 0 & 1+i & | & i & -1 \end{pmatrix} \quad (2.4.4.24)$$

$$\xleftrightarrow{R_2 \leftarrow \frac{R_2}{i-1}} \begin{pmatrix} 1 & 1 & | & 1 & 1+i \\ 0 & 1 & | & \frac{1+i}{2} & \frac{-1+i}{2} \\ 0 & 1+i & | & i & -1 \end{pmatrix} \quad (2.4.4.25)$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - (i+1)R_2} \begin{pmatrix} 1 & 1 & | & 1 & 1+i \\ 0 & 1 & | & \frac{1+i}{2} & \frac{-1+i}{2} \\ 0 & 0 & | & 0 & 0 \end{pmatrix} \quad (2.4.4.26)$$

$$\xleftrightarrow{R_1 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 0 & | & \frac{1-i}{2} & \frac{3+i}{2} \\ 0 & 1 & | & \frac{1+i}{2} & \frac{-1+i}{2} \\ 0 & 0 & | & 0 & 0 \end{pmatrix} \quad (2.4.4.27)$$

$$\Rightarrow \mathbf{x} = \frac{1}{2} \begin{pmatrix} 1-i & 3+i \\ 1+i & -1+i \end{pmatrix} \quad (2.4.4.28)$$

Thus the column vectors of (2.4.4.28) are corresponding coordinates of  $\alpha_1$  and  $\alpha_2$  in ordered basis  $\{\beta_1, \beta_2\}$ .

- 2.4.5. let  $\alpha = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$  and  $\beta = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$  be vectors in  $\mathbb{R}^2$  such that

$$x_1 y_1 + x_2 y_2 = 0; \quad x_1^2 + x_2^2 = y_1^2 + y_2^2 = 1.$$

Proove that  $\mathcal{B} = \{\alpha, \beta\}$  is a basis of  $\mathbb{R}^2$ . Find the coordinates of the vector  $(a, b)$  in the ordered basis  $\mathcal{B} = \{\alpha, \beta\}$ . (The conditions on  $\alpha$  and  $\beta$  say, geometrically, that  $\alpha$  and  $\beta$  are perpendicular and each has length 1).

**Solution:** we need to show that  $\alpha$  and  $\beta$  are linearly independent in order to prove that  $\mathcal{B} = \{\alpha, \beta\}$  is a basis of  $\mathbb{R}^2$ .

Given in the question are:

$$\alpha^T \beta = 0 \quad (2.4.5.1)$$

$$\|\alpha\|^2 = \|\beta\|^2 = 1 \quad (2.4.5.2)$$

Let,

$$\mathbf{A} = (\alpha \ \beta) = \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} \quad (2.4.5.3)$$

then,

$$\mathbf{A}^T \mathbf{A} = \begin{pmatrix} \|\alpha\|^2 & \alpha^T \beta \\ \alpha^T \beta & \|\beta\|^2 \end{pmatrix} \quad (2.4.5.4)$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (2.4.5.5)$$

$$\therefore \mathbf{A}^T \mathbf{A} = \mathbf{I} \quad (2.4.5.6)$$

Inverse of  $\mathbf{A}$  exist.  $\mathbf{A}^T$  is the inverse of  $\mathbf{A}$ . Thus, the columns of  $\mathbf{A}$  are linearly independent i.e,  $\alpha$  and  $\beta$  are linearly independent. Hence,  $\beta = \{\alpha, \beta\}$  is a basis of  $\mathbb{R}^2$ .

To, find the coordinates of the vector  $(a, b)$  in the ordered basis  $\beta = \{\alpha, \beta\}$ .

$$(\alpha \ \beta) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \quad (2.4.5.7)$$

$$\mathbf{A}^T \mathbf{A} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \mathbf{A}^T \begin{pmatrix} a \\ b \end{pmatrix} \quad (2.4.5.8)$$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \mathbf{A}^T \begin{pmatrix} a \\ b \end{pmatrix} \quad (2.4.5.9)$$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \quad (2.4.5.10)$$

2.4.6. Let  $\mathbf{V}$  be the vector space over the complex numbers of all functions from  $\mathbf{R}$  into  $\mathbf{C}$ , i.e. the space of all complex-valued functions on the real line. Let  $f_1(x) = 1$ ,  $f_2(x) = e^{ix}$ ,  $f_3(x) = e^{-ix}$ .

(a) Prove that  $f_1$ ,  $f_2$ , and  $f_3$  are linearly dependent.

(b) Let  $g_1(x) = 1$ ,  $g_2(x) = \cos(x)$ ,  $g_3(x) = \sin(x)$ . Find an invertible  $3 \times 3$  matrix  $P$  such that  $g_i = \sum_{j=1}^3 P_{ij} f_j$

**Solution:**

(a) To check for independence, lets represent the function in a polynomial format as

$$\alpha f_1 + \beta f_2 + \gamma f_3 = 0 \quad (2.4.6.1)$$

$$\alpha + \beta e^{ix} + \gamma e^{-ix} = 0 \quad (2.4.6.2)$$

Multiply the whole equation with  $e^{ix}$

$$\beta(e^{ix})^2 + \alpha e^{ix} + \gamma = 0 \quad (2.4.6.3)$$

Let  $y = e^{ix}$

$$\beta y^2 + \alpha y + \gamma = 0 \quad (2.4.6.4)$$

The above quadratic polynomial in  $y$  can be zero for atmost two values of  $y$ . But,

$$y = e^{ix} \quad x \in \mathbf{R} \quad (2.4.6.5)$$

So (2.4.6.4) cannot be zero for all  $y = e^{ix}$ . Which implies there is a contradiction.

Then, the only case when (2.4.6.4) gets satisfied is

$$\alpha = \beta = \gamma = 0 \quad (2.4.6.6)$$

Therefore,  $f_1, f_2, f_3$  are linearly independent.

(b) We need to find the coordinates of vectors  $g_i$  where  $i = 1, 2, 3$  in ordered basis

$$B = (f_1 \ f_2 \ f_3) \quad (2.4.6.7)$$

It is given that  $g_1 = 1$ , which can be written as

$$g_1 = f_1 \quad (2.4.6.8)$$

$$\Rightarrow g_1 = (f_1 \ f_2 \ f_3) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad (2.4.6.9)$$

We can use the following identities:-

$$\cos(x) = \frac{1}{2}e^{ix} + \frac{1}{2}e^{-ix} \quad (2.4.6.10)$$

$$\sin(x) = \frac{1}{2i}e^{ix} - \frac{1}{2i}e^{-ix} \quad (2.4.6.11)$$

Comparing equations (2.4.6.10) and (2.4.6.11) with  $f_2, f_3$ , we can write  $g_2$  and  $g_3$  as

$$g_2 = \frac{1}{2}f_2 + \frac{1}{2}f_3 \quad (2.4.6.12)$$

$$\Rightarrow g_2 = (f_1 \ f_2 \ f_3) \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \quad (2.4.6.13)$$



$$g_3 = \frac{1}{2i}f_2 - \frac{1}{2i}f_3 \quad (2.4.6.14)$$

$$\Rightarrow g_3 = \begin{pmatrix} f_1 & f_2 & f_3 \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{2i} \\ -\frac{1}{2i} \end{pmatrix} \quad (2.4.6.15)$$

Therefore, the required matrix  $P$  is

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2i} \\ 0 & \frac{1}{2} & -\frac{1}{2i} \end{pmatrix} \quad (2.4.6.16)$$

2.4.7. Let  $V$  be the real vector space of all polynomial functions from  $\mathbb{R}$  to  $\mathbb{R}$  of degree 2 or less, i.e, the space of all functions  $f$  of the form,

$$f(x) = c_0 + c_1x + c_2x^2$$

Let  $t$  be a fixed real number and define

$$g_1(x) = 1, g_2(x) = x + t, g_3(x) = (x + t)^2$$

Prove that  $\beta = \{g_1, g_2, g_3\}$  is a basis for  $V$ . If

$$f(x) = c_0 + c_1x + c_2x^2$$

what are the coordinates of  $f$  in the ordered basis  $\beta$

**Solution:** We start by taking,

$$\mathbf{f} = \begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix} \quad (2.4.7.1)$$

Let's start by proving that  $\mathbf{g}$  is linearly independent.

$$\mathbf{g} = \mathbf{B}\mathbf{f} \quad (2.4.7.2)$$

where,

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ t^2 & 2t & 1 \end{pmatrix} \quad (2.4.7.3)$$

Now,

$$\mathbf{v}^T \mathbf{g} = 0 \quad (2.4.7.4)$$

$$\Rightarrow \mathbf{v}^T \mathbf{B}\mathbf{f} = 0 \quad (2.4.7.5)$$

Since  $\mathbf{f}$  is linearly independent,

$$\mathbf{v}^T \mathbf{B} = 0 \quad (2.4.7.6)$$

$$\mathbf{B}^T \mathbf{v} = 0 \quad (2.4.7.7)$$

Since  $\mathbf{B}^T$  is an upper triangular matrix with

non zero values in principal diagonal, it is invertible matrix and hence  $\mathbf{v}$  will be zero vector. Now, Finding the inverse of  $\mathbf{B}^T$

$$\left( \begin{array}{ccc|ccc} 1 & t & t^2 & 1 & 0 & 0 \\ 0 & 1 & 2t & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right) \quad (2.4.7.8)$$

$$\xleftrightarrow{R_1 = R_1 - tR_2} \left( \begin{array}{ccc|ccc} 1 & 0 & -t^2 & 1 & -t & 0 \\ 0 & 1 & 2t & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right) \quad (2.4.7.9)$$

$$\xleftrightarrow{R_1 = R_1 + t^2 R_3} \left( \begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -t & t^2 \\ 0 & 1 & 2t & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right) \quad (2.4.7.10)$$

$$\xleftrightarrow{R_2 = R_2 - 2tR_3} \left( \begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -t & t^2 \\ 0 & 1 & 0 & 0 & 1 & -2t \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right) \quad (2.4.7.11)$$

So,

$$(\mathbf{B}^T)^{-1} = \begin{pmatrix} 1 & -t & t^2 \\ 0 & 1 & -2t \\ 0 & 0 & 1 \end{pmatrix} \quad (2.4.7.12)$$

Now, to find the coordinates,

$$f(x) = \mathbf{w}^T \mathbf{g} \quad (2.4.7.13)$$

So,

$$\mathbf{c}^T \mathbf{f} = \mathbf{w}^T \mathbf{g} \quad (2.4.7.14)$$

$$\mathbf{c}^T \mathbf{f} = \mathbf{w}^T \mathbf{B}\mathbf{f} \quad (2.4.7.15)$$

$$(\mathbf{c}^T - \mathbf{w}^T \mathbf{B})\mathbf{f} = 0 \quad (2.4.7.16)$$

Since,  $\mathbf{f}$  is linearly independent,

$$\mathbf{c}^T - \mathbf{w}^T \mathbf{B} = 0 \quad (2.4.7.17)$$

$$\mathbf{c}^T = \mathbf{w}^T \mathbf{B} \quad (2.4.7.18)$$

$$\mathbf{c}^T \mathbf{B}^{-1} = \mathbf{w}^T \quad (2.4.7.19)$$

$$(\mathbf{B}^{-1})^T \mathbf{c} = \mathbf{w} \quad (2.4.7.20)$$

Using (2.4.7.12) in (2.4.7.20)

$$\mathbf{w} = \begin{pmatrix} 1 & -t & t^2 \\ 0 & 1 & -2t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} \quad (2.4.7.21)$$

## 2.5 Summary of Row Equivalence

2.5.1. Consider the vectors in  $\mathbb{R}^4$  defined by:

$$\alpha_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 2 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 3 \\ 4 \\ -2 \\ 5 \end{pmatrix} \text{ and } \alpha_3 = \begin{pmatrix} 1 \\ 4 \\ 0 \\ 9 \end{pmatrix}.$$

Find a system of homogeneous linear equations for which the space of solutions is exactly the subspace of  $\mathbb{R}^4$  spanned by the given three vectors.

**Solution:** A system of linear equations is homogeneous if all of the constant terms are zero. It can be represented as,

$$\mathbf{A}\mathbf{X} = 0 \quad (2.5.1.1)$$

Let  $\mathbf{R}$  be a echelon matrix which is reduced to  $\mathbf{A}$ . Then the systems  $\mathbf{A}\mathbf{X} = 0$  and  $\mathbf{R}\mathbf{X} = 0$  have the same solutions. Here,

$$\mathbf{A} = \begin{pmatrix} -1 & 0 & 1 & 2 \\ 3 & 4 & -2 & 5 \\ 1 & 4 & 0 & 9 \end{pmatrix} \quad (2.5.1.2)$$

By row reducing on  $\mathbf{A}$ , we get:

$$\mathbf{A} = \begin{pmatrix} -1 & 0 & 1 & 2 \\ 3 & 4 & -2 & 5 \\ 1 & 4 & 0 & 9 \end{pmatrix} \xleftrightarrow{R_3=R_3-2R_1-R_2} \begin{pmatrix} -1 & 0 & 1 & 2 \\ 3 & 4 & -2 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (2.5.1.3)$$

$$\xleftrightarrow{R_2=R_2+3R_1} \begin{pmatrix} -1 & 0 & 1 & 2 \\ 0 & 4 & 1 & 11 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (2.5.1.4)$$

$$\xleftrightarrow{R_1=-R_1} \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 4 & 1 & 11 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (2.5.1.5)$$

The basis vector is non zero vector which are given from 2.5.1.5,

$$\rho_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ -2 \end{pmatrix}, \rho_2 = \begin{pmatrix} 0 \\ 4 \\ 1 \\ 11 \end{pmatrix} \quad (2.5.1.6)$$

$\rho_1, \rho_2$  forms the basis of the solution space. The subspace spanned by  $b_1$  and  $b_2$  is given

as:

$$(\rho_1 \ \rho_2) \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \mathbf{X} \quad (2.5.1.7)$$

Using 2.5.1.7, we can write the augmented matrix as:

$$\begin{pmatrix} 1 & 0 & x_1 \\ 0 & 4 & x_2 \\ -1 & 1 & x_3 \\ -2 & 11 & x_4 \end{pmatrix} \xleftrightarrow{R_3=4R_3+4R_1-R_2} \quad (2.5.1.8)$$

$$\begin{pmatrix} 1 & 0 & x_1 \\ 0 & 4 & x_2 \\ 0 & 0 & 4x_1 - x_2 + 4x_3 \\ -2 & 11 & x_4 \end{pmatrix} \xleftrightarrow{R_4=4R_4+8R_1-11R_2} \quad (2.5.1.9)$$

$$\begin{pmatrix} 1 & 0 & x_1 \\ 0 & 4 & x_2 \\ 0 & 0 & 4x_1 - x_2 + 4x_3 \\ 0 & 0 & 8x_1 - 11x_2 + 4x_4 \end{pmatrix} \quad (2.5.1.10)$$

Using 2.5.1.10, The required homogeneous equation is given as:

$$\begin{pmatrix} 4 & -1 & 4 & 0 \\ 8 & -11 & 0 & 4 \end{pmatrix} \mathbf{X} = 0 \quad (2.5.1.11)$$

2. Let  $s < n$  and  $\mathbf{A}$  an  $s \times n$  matrix with entries in the field  $\mathbb{F}$ . Use Theorem 4 to show that there is a non-zero  $\mathbf{x}$  in  $\mathbb{F}^{n \times 1}$  such that  $\mathbf{A}\mathbf{x} = \mathbf{0}$ .

**Solution: Theorem 4:** Let  $\mathbb{V}$  be a vector space which is spanned by a finite set of vectors  $\beta_1, \beta_2, \dots, \beta_m$ . Then any independent set of vectors in  $\mathbb{V}$  is finite and contains no more than  $m$  elements. Let  $\mathbb{V}$  be a vector space spanned by  $a_1, a_2, \dots, a_n$ , where  $a_i, i=1, 2, \dots, n$  are columns of matrix  $\mathbf{A}_{s \times n}$ .

$$\mathbf{A} = (a_1 \ a_2 \ \dots \ a_n) \quad (2.5.2.1)$$

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{s1} & a_{s2} & \dots & a_{sn} \end{pmatrix} \quad (2.5.2.2)$$

Let us take  $a_i, i=1, 2, \dots, n$  as standard  $s \times 1$  bases.

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & \dots & 0 & a_{1,s+1} & \dots & a_{1n} \\ 0 & 1 & \dots & 0 & a_{2,s+1} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 & a_{s,s+1} & \dots & a_{sn} \end{pmatrix} \quad (2.5.2.3)$$

From (2.5.2.3), it is clear that

$$\dim(\text{col}(\mathbf{A})) \leq s \quad (2.5.2.4)$$

$$\implies \text{rank}(\mathbf{A}) \leq s \quad (2.5.2.5)$$

Now, from rank-nullity theorem,

$$\text{rank}(\mathbf{A}) + \text{nullity}(\mathbf{A}) = n \quad (2.5.2.6)$$

$$\text{nullity}(\mathbf{A}) = n - \text{rank}(\mathbf{A}) \quad (2.5.2.7)$$

$$\implies \text{nullity}(\mathbf{A}) > 0 \quad (2.5.2.8)$$

From equation (2.5.2.8) it is clear that there will be a non zero  $\mathbf{x}$  such that  $\mathbf{A}\mathbf{x} = \mathbf{0}$

2.5.3. Let

$$\alpha_1 = \begin{pmatrix} 1 & 1 & -2 & 1 \end{pmatrix}^T \quad (2.5.3.1)$$

$$\alpha_2 = \begin{pmatrix} 3 & 0 & 4 & -1 \end{pmatrix}^T \quad (2.5.3.2)$$

$$\alpha_3 = \begin{pmatrix} -1 & 2 & 5 & 2 \end{pmatrix}^T \quad (2.5.3.3)$$

Let

$$\alpha = \begin{pmatrix} 4 & -5 & 9 & -7 \end{pmatrix}^T \quad (2.5.3.4)$$

$$\beta = \begin{pmatrix} 3 & 1 & -4 & 4 \end{pmatrix}^T \quad (2.5.3.5)$$

$$\gamma = \begin{pmatrix} -1 & 1 & 0 & 1 \end{pmatrix}^T \quad (2.5.3.6)$$

- Which of the vectors  $\alpha, \beta, \gamma$  are in the subspace of  $\mathbb{R}^4$  spanned by  $\alpha_i$ ?
- Which of the vectors  $\alpha, \beta, \gamma$  are in the subspace of  $\mathbb{C}^4$  spanned by  $\alpha_i$ ?
- Does this suggest a theorem?

**Solution:**

- The linear combination of  $\alpha_i$  for  $i = 1, 2, 3$  spans subspace S. We can write,

$$c_1 \begin{pmatrix} 1 \\ 1 \\ -2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 0 \\ 4 \\ -1 \end{pmatrix} + c_3 \begin{pmatrix} -1 \\ 2 \\ 5 \\ 2 \end{pmatrix} = \text{span}(\mathbf{S}) \quad (2.5.3.7)$$

where  $c_1, c_2, c_3$  are scalars. Vectors in matrix

form is given by

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & -1 \\ 1 & 0 & 2 \\ -2 & 4 & 5 \\ 1 & -1 & 2 \end{pmatrix} \quad (2.5.3.8)$$

We can observe that the columns of matrix  $\mathbf{A}$  formed by vectors  $\alpha_i$  are independent as the rank of matrix is 3. Hence  $\alpha_i$  forms basis for subspace S.

- Checking for  $\alpha$ :** To check if a solution exists for  $\mathbf{A}\mathbf{x} = \alpha$ . The corresponding augmented matrix can be written as,

$$(\mathbf{A} \ \alpha) = \begin{pmatrix} 1 & 3 & -1 & 4 \\ 1 & 0 & 2 & -5 \\ -2 & 4 & 5 & 9 \\ 1 & -1 & 2 & -7 \end{pmatrix} \quad (2.5.3.9)$$

On performing row-reduction on (2.5.3.9),

$$(\mathbf{A} \ \alpha) = \begin{pmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (2.5.3.10)$$

As  $\text{Rank}((\mathbf{A} \ \alpha)) = \text{Rank}(\mathbf{A}) = 3$ , the vector  $\alpha$  can be represented as linear combination of  $\alpha_i$ . From equation (2.5.3.10), we can write

$$-3 \begin{pmatrix} 1 \\ 1 \\ -2 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} 3 \\ 0 \\ 4 \\ -1 \end{pmatrix} - 1 \begin{pmatrix} -1 \\ 2 \\ 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ -5 \\ 9 \\ -7 \end{pmatrix} \quad (2.5.3.11)$$

Hence  $\alpha$  is in the subspace S.

- Checking for  $\beta$ :** To check if a solution exists for  $\mathbf{A}\mathbf{x} = \beta$ . The corresponding augmented matrix can be written as,

$$(\mathbf{A} \ \beta) = \begin{pmatrix} 1 & 3 & -1 & 3 \\ 1 & 0 & 2 & 1 \\ -2 & 4 & 5 & -4 \\ 1 & -1 & 2 & 4 \end{pmatrix} \quad (2.5.3.12)$$

On performing row-reduction on

(2.5.3.12),

$$(\mathbf{A} \ \beta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (2.5.3.13)$$

As  $\text{Rank}((\mathbf{A} \ \beta))=4$  and  $\text{Rank}(\mathbf{A})=3$ , Solution doesn't exist for  $AX = \beta$  and hence  $\beta$  is not in the subspace S.

- iii) **Checking for  $\gamma$ :** To check if a solution exists for  $\mathbf{A}\mathbf{X} = \gamma$ . The corresponding augmented matrix can be written as,

$$(\mathbf{A} \ \gamma) = \begin{pmatrix} 1 & 3 & -1 & -1 \\ 1 & 0 & 2 & 1 \\ -2 & 4 & 5 & 0 \\ 1 & -1 & 2 & 1 \end{pmatrix} \quad (2.5.3.14)$$

On performing row-reduction on (2.5.3.14),

$$(\mathbf{A} \ \gamma) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (2.5.3.15)$$

As  $\text{Rank}((\mathbf{A} \ \gamma))=4$  and  $\text{Rank}(\mathbf{A})=3$ , Solution doesn't exist for  $AX = \gamma$  and hence  $\gamma$  is not in the subspace S.

- b) In part 1, we haven't considered the field to be either  $\mathbb{R}$  or  $\mathbb{C}$ . The above equations solved holds for field  $\mathbb{C}$  and that implies, they hold for field  $\mathbb{R}$  also. Hence  $\alpha$  is in the subspace and  $\beta$  and  $\gamma$  are not in the subspace.
- c) **Theorem suggested:** Let  $\mathbb{F}_1$  and  $\mathbb{F}_2$  are two fields where  $\mathbb{F}_2$  is subfield of  $\mathbb{F}_1$ . Let  $\alpha_i$ ,  $i=1,2,3,\dots,n$  forms basis for subspace of  $\mathbb{F}_2^n$  and a vector  $\alpha \in \mathbb{F}_2^n$ . Then  $\alpha$  is in the subspace of  $\mathbb{F}_2^n$  spanned by  $\alpha_i$ ,  $i=1,2,3,\dots,n$  if only if  $\alpha$  is in the subspace of  $\mathbb{F}_1^n$  spanned by  $\alpha_i$ ,  $i=1,2,3,\dots,n$ .

2.5.4. In  $C^3$ , let  $\alpha_1 = (1, 0, -i)$ ,  $\alpha_2 = (1 + i, 1 - i, 1)$ ,  $\alpha_3 = (i, i, i)$ . Prove that these vectors form a basis for  $C^3$ . What are the coordinates of the vector  $(a, b, c)$  in the basis?

**Solution:** Now,

$$C_1\alpha_1 + C_2\alpha_2 + C_3\alpha_3 = \mathbf{0} \quad (2.5.4.1)$$

$$\Rightarrow C_1 \begin{pmatrix} 1 \\ 0 \\ -i \end{pmatrix} + C_2 \begin{pmatrix} 1+i \\ 1-i \\ 1 \end{pmatrix} + C_3 \begin{pmatrix} i \\ i \\ i \end{pmatrix} = \mathbf{0} \quad (2.5.4.2)$$

So,

$$\begin{pmatrix} 1 & 1+i & i \\ 0 & 1-i & i \\ -i & 1 & i \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (2.5.4.3)$$

Considering the co-efficient matrix A:

$$\begin{aligned} \begin{pmatrix} 1 & 1+i & i \\ 0 & 1-i & i \\ -i & 1 & i \end{pmatrix} &\xleftrightarrow{R_3 \leftarrow R_3 + iR_1} \begin{pmatrix} 1 & 1+i & i \\ 0 & 1-i & i \\ 0 & i & i-1 \end{pmatrix} \\ &\xleftrightarrow{R_3 \leftarrow R_3/i} \begin{pmatrix} 1 & 1+i & i \\ 0 & 1-i & i \\ 0 & 1 & 1+i \end{pmatrix} \xleftrightarrow{R_3 \leftarrow (1-i)R_3} \\ &\begin{pmatrix} 1 & 1+i & i \\ 0 & 1-i & i \\ 0 & 1-i & 2 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 - R_2} \begin{pmatrix} 1 & 1+i & i \\ 0 & 1-i & i \\ 0 & 0 & 2-i \end{pmatrix} \\ &\xleftrightarrow{R_2 \leftarrow \frac{1+i}{1-i}R_2} \begin{pmatrix} 1 & 1+i & i \\ 0 & 1+i & -1 \\ 0 & 0 & 2-i \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 - R_2} \\ &\begin{pmatrix} 1 & 0 & i+1 \\ 0 & 1+i & -1 \\ 0 & 0 & 2-i \end{pmatrix} \quad (2.5.4.4) \end{aligned}$$

Now let

$$R = \begin{pmatrix} 1 & 0 & i+1 \\ 0 & 1+i & -1 \\ 0 & 0 & 2-i \end{pmatrix} \quad (2.5.4.5)$$

Where  $R$  is the row reduced form of matrix A. So  $\alpha_1, \alpha_2$  and  $\alpha_3$  are linearly independent which implies that these 3 vectors form a basis of vector space  $C^3$ .

Now, consider a vector  $\beta = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$  and let the

coordinates are  $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$  such that

$$Ax = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad (2.5.4.6)$$

$$\Rightarrow x = A^{-1} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad (2.5.4.7)$$

Let us consider a matrix  $(A|I)$  where  $I$  is a 3x3 identity matrix. Now, applying the Gauss-

Jordan theorem we can get  $A^{-1}$

$$\begin{aligned}
 & \begin{pmatrix} 1 & 1+i & i & 1 & 0 & 0 \\ 0 & 1-i & i & 0 & 1 & 0 \\ -i & 1 & i & 0 & 0 & 1 \end{pmatrix} \\
 & \xleftrightarrow{R_3 \leftarrow R_3 + iR_1} \\
 & \begin{pmatrix} 1 & 1+i & i & 1 & 0 & 0 \\ 0 & 1-i & i & 0 & 1 & 0 \\ 0 & i & i-1 & i & 0 & 1 \end{pmatrix} \\
 & \xleftrightarrow{R_3 \leftarrow R_3 / i} \\
 & \begin{pmatrix} 1 & 1+i & i & 1 & 0 & 0 \\ 0 & 1-i & i & 0 & 1 & 0 \\ 0 & 1 & 1+i & 1 & 0 & -i \end{pmatrix} \\
 & \xleftrightarrow{R_3 \leftarrow (1-i)R_3} \\
 & \begin{pmatrix} 1 & 1+i & i & 1 & 0 & 0 \\ 0 & 1-i & i & 0 & 1 & 0 \\ 0 & 1-i & 2 & 1-i & 0 & -i-1 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 - R_2} \\
 & \begin{pmatrix} 1 & 1+i & i & 1 & 0 & 0 \\ 0 & 1-i & i & 0 & 1 & 0 \\ 0 & 0 & 2-i & 1-i & -1 & -i-1 \end{pmatrix} \\
 & \xleftrightarrow{R_2 \leftarrow \frac{1+i}{1-i} R_2} \\
 & \begin{pmatrix} 1 & 1+i & i & 1 & 0 & 0 \\ 0 & 1+i & -1 & 0 & i & 0 \\ 0 & 0 & 2-i & 1-i & -1 & -i-1 \end{pmatrix} \\
 & \xleftrightarrow{R_1 \leftarrow R_1 - R_2} \\
 & \begin{pmatrix} 1 & 0 & i+1 & 1 & -i & 0 \\ 0 & 1+i & -1 & 0 & i & 0 \\ 0 & 0 & 2-i & 1-i & -1 & -i-1 \end{pmatrix} \\
 & \xleftrightarrow{R_3 \leftarrow \frac{R_3}{2-i}} \\
 & \xleftrightarrow{R_1 \leftarrow R_1 + R_3} \\
 & \begin{pmatrix} 1 & 0 & 3 & 2-i & -i-1 & -i-1 \\ 0 & 1+i & -1 & 0 & i & 0 \\ 0 & 0 & 1 & \frac{3-i}{5} & -\frac{2+i}{5} & -\frac{3i+1}{5} \end{pmatrix} \\
 & \xleftrightarrow{R_2 \leftarrow R_2 + R_3} \\
 & \begin{pmatrix} 1 & 0 & 3 & 2-i & -i-1 & -i-1 \\ 0 & 1+i & 0 & \frac{3-i}{5} & \frac{-2+4i}{5} & -\frac{3i+1}{5} \\ 0 & 0 & 1 & \frac{3-i}{5} & -\frac{2+i}{5} & -\frac{3i+1}{5} \end{pmatrix} \\
 & \xleftrightarrow{R_2 \leftarrow \frac{R_2}{1+i}} \\
 & \begin{pmatrix} 1 & 0 & 3 & 2-i & -i-1 & -i-1 \\ 0 & 1 & 0 & \frac{3-i}{5(1+i)} & \frac{-2+4i}{5(1+i)} & -\frac{3i+1}{5(1+i)} \\ 0 & 0 & 1 & \frac{3-i}{5} & -\frac{2+i}{5} & -\frac{3i+1}{5} \end{pmatrix} \\
 & \xleftrightarrow{R_1 \leftarrow R_1 - 3R_3} \\
 & \begin{pmatrix} 1 & 0 & 0 & \frac{3-i}{5} & \frac{-2+4i}{5} & -\frac{3i+1}{5} \\ 0 & 1 & 0 & \frac{3-i}{5(1+i)} & \frac{-2+4i}{5(1+i)} & -\frac{3i+1}{5(1+i)} \\ 0 & 0 & 1 & \frac{3-i}{5} & -\frac{2+i}{5} & -\frac{3i+1}{5} \end{pmatrix}
 \end{aligned} \quad (2.5.4.8)$$

$$\begin{pmatrix} 1 & 0 & 0 & \frac{1-4i}{5} & \frac{1-2i}{5} & \frac{-2+4i}{5} \\ 0 & 1 & 0 & \frac{1-2i}{5} & \frac{1+3i}{5} & \frac{-2+i}{5} \\ 0 & 0 & 1 & \frac{3-i}{5} & \frac{-2+i}{5} & \frac{-3i+1}{5} \end{pmatrix} \quad (2.5.4.9)$$

$$\Rightarrow (I|A^{-1}) = \begin{pmatrix} 1 & 0 & 0 & \frac{1-4i}{5} & \frac{1-2i}{5} & \frac{-2+4i}{5} \\ 0 & 1 & 0 & \frac{1-2i}{5} & \frac{1+3i}{5} & \frac{-2+i}{5} \\ 0 & 0 & 1 & \frac{3-i}{5} & \frac{-2+i}{5} & \frac{-3i+1}{5} \end{pmatrix} \quad (2.5.4.10)$$

$$\Rightarrow A^{-1} = \begin{pmatrix} \frac{1-4i}{5} & \frac{1-2i}{5} & \frac{-2+4i}{5} \\ \frac{1-2i}{5} & \frac{1+3i}{5} & \frac{-2+i}{5} \\ \frac{3-i}{5} & \frac{-2+i}{5} & \frac{-3i+1}{5} \end{pmatrix} \quad (2.5.4.11)$$

So,

$$x = A^{-1} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad (2.5.4.12)$$

$$\Rightarrow x = \begin{pmatrix} \frac{1-4i}{5} & \frac{1-2i}{5} & \frac{-2+4i}{5} \\ \frac{1-2i}{5} & \frac{1+3i}{5} & \frac{-2+i}{5} \\ \frac{3-i}{5} & \frac{-2+i}{5} & \frac{-3i+1}{5} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad (2.5.4.13)$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1-4i}{5} & \frac{1-2i}{5} & \frac{-2+4i}{5} \\ \frac{1-2i}{5} & \frac{1+3i}{5} & \frac{-2+i}{5} \\ \frac{3-i}{5} & \frac{-2+i}{5} & \frac{-3i+1}{5} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad (2.5.4.14)$$

2.5.5. Give an explicit description of the type  $b_j = \sum_{i=1}^r b_{ki} R_{ij}$  for the vectors

$$\beta = (b_1, b_2, b_3, b_4, b_5)$$

in  $R^5$  which are linear combinations of the 2.5.6. vectors

$$\alpha_1 = (1, 0, 2, 1, -1), \quad (2.5.5.1)$$

$$\alpha_2 = (-1, 2, -4, 2, 0), \quad (2.5.5.2)$$

$$\alpha_3 = (2, -1, 5, 2, 1), \quad (2.5.5.3)$$

$$\alpha_4 = (2, 1, 3, 5, 2) \quad (2.5.5.4)$$

**Solution:** Above matrix represented as:  $Ax = \beta$

$$\begin{pmatrix} 1 & -1 & 2 & 2 \\ 0 & 2 & -1 & 1 \\ 2 & -4 & 5 & 3 \\ 1 & 2 & 2 & 5 \\ -1 & 0 & 1 & 2 \end{pmatrix} x = \beta \quad (2.5.5.5)$$

$$\begin{pmatrix} 1 & -1 & 2 & 2 \\ 0 & 2 & -1 & 1 \\ 2 & -4 & 5 & 3 \\ 1 & 2 & 2 & 5 \\ -1 & 0 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{pmatrix} \quad (2.5.5.6)$$

$AX^T = Y^T$ , where  $A$ ,  $X$  and  $Y$  matrices have shown below:

$$A = \begin{pmatrix} 1 & -1 & 2 & 2 \\ 0 & 2 & -1 & 1 \\ 2 & -4 & 5 & 3 \\ 1 & 2 & 2 & 5 \\ -1 & 0 & 1 & 2 \end{pmatrix} \quad (2.5.5.7)$$

$$X = \begin{pmatrix} \frac{67}{100} & -\frac{167}{100} & -2 & \frac{133}{100} \\ \frac{1}{2} & -\frac{3}{2} & -\frac{5}{2} & \frac{3}{2} \\ -\frac{4}{25} & \frac{117}{100} & \frac{3}{2} & -\frac{83}{100} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad (2.5.5.8)$$

$$Y = \begin{pmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (2.5.5.9)$$

Now since the  $Y^T$  matrix is full rank thus we can say that columns of matrix  $A$  are linearly independent and  $b$  is described by (2.5.5.6)

Let  $\mathbb{V}$  be a vector space which is spanned by the rows of matrix

$$A = \begin{pmatrix} 3 & 21 & 0 & 9 & 0 \\ 1 & 7 & -1 & -2 & -1 \\ 2 & 14 & 0 & 6 & 1 \\ 6 & 42 & -1 & 13 & 0 \end{pmatrix} \quad (2.5.6.1)$$

a. Find a basis for  $\mathbb{V}$

b. Tell which vectors  $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$  are elements of  $\mathbb{V}$

c. If  $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$  is in  $\mathbb{V}$ , what are its coordinates in

the basis chosen in part(a)?

**Solution:** Row reducing (2.5.6.1)

$$\begin{aligned}
 & \begin{pmatrix} 3 & 21 & 0 & 9 & 0 \\ 1 & 7 & -1 & -2 & -1 \\ 2 & 14 & 0 & 6 & 1 \\ 6 & 42 & -1 & 13 & 0 \end{pmatrix} \\
 & \xleftrightarrow{R_1 \leftarrow \frac{R_1}{3}} \begin{pmatrix} 1 & 7 & 0 & 3 & 0 \\ 1 & 7 & -1 & -2 & -1 \\ 2 & 14 & 0 & 6 & 1 \\ 6 & 42 & -1 & 13 & 0 \end{pmatrix} \\
 & \xleftrightarrow{\begin{matrix} R_3 \leftarrow R_3 - 2R_1 \\ R_2 \leftarrow R_2 - R_1 \end{matrix}} \begin{pmatrix} 1 & 7 & 0 & 3 & 0 \\ 0 & 0 & -1 & -5 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -5 & 0 \end{pmatrix} \\
 & \xleftrightarrow{R_4 \leftarrow R_4 - R_2} \begin{pmatrix} 1 & 7 & 0 & 3 & 0 \\ 0 & 0 & -1 & -5 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \\
 & \xleftrightarrow{R_2 \leftarrow -R_2} \begin{pmatrix} 1 & 7 & 0 & 3 & 0 \\ 0 & 0 & 1 & 5 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \\
 & \xleftrightarrow{\begin{matrix} R_4 \leftarrow R_4 - R_3 \\ R_2 \leftarrow R_2 - R_3 \end{matrix}} \begin{pmatrix} 1 & 7 & 0 & 3 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (2.5.6.2)
 \end{aligned}$$

a. For the basis of  $\mathbb{V}$ , we can take the non zero rows of (2.5.6.2)

$$\rho_1 = \begin{pmatrix} 1 \\ 7 \\ 0 \\ 3 \\ 0 \end{pmatrix} \rho_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 5 \\ 0 \end{pmatrix} \rho_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad (2.5.6.3)$$

b. Vectors which are elements of  $\mathbb{V}$  are of the form:

$$\begin{aligned}
 & c_1\rho_1 + c_2\rho_2 + c_3\rho_3 \\
 & = \begin{pmatrix} c_1 \\ 7c_1 \\ c_2 \\ 3c_1 + 5c_2 \\ c_3 \end{pmatrix} \quad (2.5.6.4)
 \end{aligned}$$

where  $c_1, c_2, c_3$  are scalars.

c. Expressing (2.5.6.4) in matrix form, if  $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$  is

in  $\mathbb{V}$ , it must be of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 7 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \quad (2.5.6.5)$$

The augmented matrix form

$$\begin{pmatrix} 1 & 0 & 0 & x_1 \\ 7 & 0 & 0 & x_2 \\ 0 & 1 & 0 & x_3 \\ 3 & 5 & 0 & x_4 \\ 0 & 0 & 1 & x_5 \end{pmatrix} \quad (2.5.6.6)$$

Converting the above matrix into row reduced echelon form

$$\begin{aligned}
 & \begin{pmatrix} 1 & 0 & 0 & x_1 \\ 7 & 0 & 0 & x_2 \\ 0 & 1 & 0 & x_3 \\ 3 & 5 & 0 & x_4 \\ 0 & 0 & 1 & x_5 \end{pmatrix} \xleftrightarrow{\begin{matrix} R_4 \leftarrow R_4 - 3R_1 \\ R_2 \leftarrow R_2 - 7R_1 \end{matrix}} \begin{pmatrix} 1 & 0 & 0 & x_1 \\ 0 & 0 & 0 & x_2 - 7x_1 \\ 0 & 1 & 0 & x_3 \\ 0 & 5 & 0 & x_4 - 3x_1 \\ 0 & 0 & 1 & x_5 \end{pmatrix} \\
 & \xleftrightarrow{R_2 \leftarrow R_3} \begin{pmatrix} 1 & 0 & 0 & x_1 \\ 0 & 1 & 0 & x_3 \\ 0 & 0 & 0 & x_2 - 7x_1 \\ 0 & 5 & 0 & x_4 - 3x_1 \\ 0 & 0 & 1 & x_5 \end{pmatrix} \\
 & \xleftrightarrow{R_4 \leftarrow R_4 - 5R_2} \begin{pmatrix} 1 & 0 & 0 & x_1 \\ 0 & 1 & 0 & x_3 \\ 0 & 0 & 0 & x_2 - 7x_1 \\ 0 & 0 & 0 & x_4 - 3x_1 - 5x_3 \\ 0 & 0 & 1 & x_5 \end{pmatrix} \\
 & \xleftrightarrow{R_5 \leftarrow R_3} \begin{pmatrix} 1 & 0 & 0 & x_1 \\ 0 & 1 & 0 & x_3 \\ 0 & 0 & 1 & x_5 \\ 0 & 0 & 0 & x_4 - 3x_1 - 5x_3 \\ 0 & 0 & 0 & x_2 - 7x_1 \end{pmatrix} \quad (2.5.6.7)
 \end{aligned}$$

From (2.5.6.7), the coordinates of  $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$  in the

basis is

$$\begin{pmatrix} x_1 \\ x_3 \\ x_5 \end{pmatrix} \quad (2.5.6.8)$$

2.5.7. Let  $\mathbf{A}$  be an  $m \times n$  matrix over the field  $\mathbf{F}$ , and consider the system of equations  $\mathbf{AX} = \mathbf{Y}$ . Prove that this system of equations has a solutions if and only if the row rank of  $\mathbf{A}$  is equal to the row rank of augmented matrix of the system. **Solution:** Consider  $\mathbf{A}$  as,

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}_{m \times n} \quad (2.5.7.1)$$

Now to solve the equation  $\mathbf{AX} = \mathbf{Y}$  for  $\mathbf{X}$ , we first write its augmented matrix  $\mathbf{A}_Y$  and then convert it into row reduced echelon form given as  $\mathbf{R}_Y$ . Here  $\mathbf{R}$  is the row reduced form of matrix  $\mathbf{A}$ . Also we know that, for  $\mathbf{AX} = \mathbf{Y}$  to have a solution,  $\mathbf{Y}$  should be in column space of  $\mathbf{A}$ .

$$\mathbf{A}_Y = (\mathbf{A} \mid \mathbf{Y}) \quad (2.5.7.2)$$

Assume that the last  $k$  rows of  $\mathbf{R}$  are zero rows. This implies that we have  $k$  number of linear dependent rows in matrix  $\mathbf{A}$ . Hence the row rank of matrix  $\mathbf{A}$  is  $r = m - k$ . As there are  $m - k$  number of non-zero vectors in the row of  $\mathbf{R}$ . Now,

$$\mathbf{R}_Y = \mathbf{E}\mathbf{A}_Y \quad (2.5.7.3)$$

$$\Rightarrow \mathbf{C} \left( \begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & y_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & y_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & y_m \end{array} \right) \quad (2.5.7.4)$$

Here,  $\mathbf{E}$  is product of matrix responsible for performing operation on matrix such as scaling, Row exchange etc, given as,

$$\mathbf{E} = \mathbf{E}_i \mathbf{E}_{i-1} \cdots \mathbf{E}_1 \quad (2.5.7.5)$$

Hence row reduced echelon form,  $\mathbf{R}_Y$  is given as,

$$\mathbf{R}_Y = \mathbf{E}\mathbf{A}_Y = (\mathbf{E}\mathbf{A} \mid \mathbf{E}\mathbf{Y}) \quad (2.5.7.6)$$

Now RREF can be represented in Block matrix

as,

$$\Rightarrow \mathbf{R}_Y = \left( \begin{array}{cc|c} \mathbf{I} & \mathbf{F} & \mathbf{Y} \\ 0 & 0 & \end{array} \right) \quad (2.5.7.7)$$

Also from equation (2.5.7.7) it can be observed that for  $\mathbf{AX} = \mathbf{Y}$  to have a solution,

$$y'_{m-k} = y'_{m-k-1} = \cdots = y'_{m-1} = y'_m = 0 \quad (2.5.7.8)$$

Hence, the rank of  $\mathbf{R}_Y$  is, also  $r$ . This implies rank of augmented matrix  $\mathbf{A}_Y$  is also  $r$ .

*Example:* Let  $\mathbf{A}$  be a  $3 \times 4$  matrix given as,

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 5 & 7 \\ 2 & 9 & 3 & 6 \\ 1 & 7 & 2 & 9 \end{pmatrix} \quad (2.5.7.9)$$

$$\mathbf{Y} = \begin{pmatrix} 4 \\ 8 \\ 14 \end{pmatrix} \quad (2.5.7.10)$$

For the time being it could be concluded that the following set of linear equation have a solution as  $\mathbf{Y} \in \text{Column-space}(\mathbf{A})$ . Hence to solve  $\mathbf{AX} = \mathbf{Y}$  first the augmented matrix  $\mathbf{A}_Y$  is given as,

$$\mathbf{A}_Y = \left( \begin{array}{cccc|c} 1 & 2 & 5 & 7 & 4 \\ 2 & 9 & 3 & 6 & 8 \\ 1 & 7 & 2 & 9 & 14 \end{array} \right) \quad (2.5.7.11)$$

Now reducing in RREF we get,

$$\mathbf{R}_Y = \mathbf{E}\mathbf{A}_Y \quad (2.5.7.12)$$

Here  $\mathbf{E}$  is elementary matrix given as,

$$\mathbf{E} = \begin{pmatrix} -3/20 & 31/20 & -39/20 \\ -1/20 & -3/20 & 7/20 \\ 1/4 & -1/4 & 1/4 \end{pmatrix} \quad (2.5.7.13)$$

$$\Rightarrow \mathbf{R}_Y = \mathbf{E}\mathbf{A}_Y = (\mathbf{E}\mathbf{A} \mid \mathbf{E}\mathbf{Y}) \quad (2.5.7.14)$$

$$\Rightarrow \mathbf{R}_Y = \left( \begin{array}{cc|c} \mathbf{I} & \mathbf{F} & \mathbf{Y} \end{array} \right) \quad (2.5.7.15)$$



$$\mathbf{I}_{3 \times 3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (2.5.7.16)$$

$$\mathbf{F}_{3 \times 1} = \begin{pmatrix} -93/10 \\ 19/10 \\ 5/2 \end{pmatrix} \quad (2.5.7.17)$$

$$\mathbf{Y}_{3 \times 1} = \begin{pmatrix} -31/2 \\ 7/2 \\ 5/2 \end{pmatrix} \quad (2.5.7.18)$$

Hence, rank = 3 as there are 0 non zero rows in row reduced echelon form. Also it could be observed that both  $\mathbf{R}$  and  $\mathbf{R}_Y$  have non zero rows hence we have a solution and same rank.

### 3 LINEAR TRANSFORMATIONS

#### 3.1 Linear Transformations

3.1.1. Find whether given functions  $\mathbf{T}$  from  $\mathbb{R}^2$  into  $\mathbb{R}^2$  are linear transformations or not

a)

$$T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 + x_1 \\ x_2 \end{pmatrix} \quad (3.1.1.1)$$

**Solution:** Counter example can be given as follows:-

$$x_1 = x_2 = 0 \quad (3.1.1.2)$$

Substituting (3.1.1.2) in (3.1.1.1) we get,

$$T \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.1.1.3)$$

(3.1.1.3) is clearly false because linear transformation on  $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$  will always be equal to  $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

b)

$$\mathbf{T} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix} \quad (3.1.1.4)$$

Does function  $\mathbf{T}$  from  $\mathbb{R}^2$  into  $\mathbb{R}^2$  is Linear Transformation.

**Solution:** Let,

$$\mathbf{x}, \mathbf{y} \in \mathbb{R}^2 \quad (3.1.1.5)$$

Using transformation on  $\mathbf{T}$ ,

$$\mathbf{T}(\mathbf{x}) = \mathbf{Ax} \quad (3.1.1.6)$$

From (3.1.1.4) we get,

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (3.1.1.7)$$

With  $c$  being a scalar,

$$\mathbf{T}(c\mathbf{x} + \mathbf{y}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (c\mathbf{x} + \mathbf{y}) \quad (3.1.1.8)$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} c\mathbf{x} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{y} \quad (3.1.1.9)$$

$$= \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{y} \quad (3.1.1.10)$$

$$= c \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{y} \quad (3.1.1.11)$$

From (3.1.1.11) we can say,

$$\mathbf{T}(c\mathbf{x} + \mathbf{y}) = c\mathbf{T}(\mathbf{x}) + \mathbf{T}(\mathbf{y}) \quad (3.1.1.12)$$

Hence from (3.1.1.12) we can say  $\mathbf{T}$  is a Linear Transformation from  $\mathbb{R}^2$  to  $\mathbb{R}^2$

c)

$$\mathbf{T}(x_1, x_2) = (x_1^2, x_2) \quad (3.1.1.13)$$

**Solution:** If  $\mathbf{T}$  were a linear transformation then we would have

$$\mathbf{T} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.1.1.14)$$

$$\Rightarrow \mathbf{T} \left( -1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) = -1 \cdot \mathbf{T} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.1.1.15)$$

$$\Rightarrow -1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad (3.1.1.16)$$

which is a contradiction, since

$$\mathbf{T} \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.1.1.17)$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \neq \begin{pmatrix} -1 \\ 0 \end{pmatrix}. \quad (3.1.1.18)$$

Hence non-linear transformation.

d) Is the following function  $\mathbf{T}$  from  $\mathbb{R}^2$  into  $\mathbb{R}^2$  is linear transformation?

$$\mathbf{T} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \sin(x_1) \\ x_2 \end{pmatrix}$$

**Solution:** Let,

$$\mathbf{x} = \begin{pmatrix} \pi \\ 0 \end{pmatrix}; \quad \mathbf{y} = \begin{pmatrix} \frac{\pi}{2} \\ 0 \end{pmatrix}$$

$$\mathbf{T}(\mathbf{x} + \mathbf{y}) = \mathbf{T}\left(\begin{pmatrix} 3\pi \\ 2 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad (3.1.1.19)$$

$$\mathbf{T}(\mathbf{x}) + \mathbf{T}(\mathbf{y}) = \mathbf{T}\left(\begin{pmatrix} \pi \\ 0 \end{pmatrix}\right) + \mathbf{T}\left(\begin{pmatrix} \pi \\ 2 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.1.1.20)$$

From (3.1.1.19) and (3.1.1.20) additive transformation property is not satisfied. Hence not a linear transformation.

- e) Verify whether  $T(x_1, x_2) = (x_1 - x_2, 0)$  is a linear transformation or not. **Solution:** Let  $\mathbf{V}$  and  $\mathbf{W}$  be the vector spaces. The function  $T : \mathbf{V} \rightarrow \mathbf{W}$  is called a linear transformation of  $\mathbf{V}$  into  $\mathbf{W}$  if for all  $\mathbf{u}$  and  $\mathbf{v}$  in  $\mathbf{V}$  and for any scalar  $k$  in field  $\mathbf{F}$ ,

$$T(k\mathbf{u} + \mathbf{v}) = kT(\mathbf{u}) + T(\mathbf{v}) \quad (3.1.1.21)$$

Given,

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad (3.1.1.22)$$

Consider,

$$\begin{aligned} T(k\mathbf{x} + \mathbf{y}) &= \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} (k\mathbf{x} + \mathbf{y}) \quad (3.1.1.23) \\ &= \begin{pmatrix} k & -k \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \mathbf{y} \end{aligned} \quad (3.1.1.24)$$

$$= k \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \mathbf{y} \quad (3.1.1.25)$$

$$T(\mathbf{x} + \mathbf{y}) = kT(\mathbf{x}) + T(\mathbf{y}) \quad (3.1.1.26)$$

Therefore, the given function  $T$  is a linear transformation.

- 3.1.2. Find the range, rank, null space, and nullity for the zero transformation and the identity transformation on a finite-dimensional space  $\mathbf{V}$ .

**Solution:**

Suppose vector space  $\mathbf{V}$  has  $\dim(\mathbf{V}) = n$ . Table 3.1.2 provides the properties of range, rank, null space and nullity of zero and identity transformation on a vector space  $\mathbf{V}$

- 3.1.3. a) Let  $\mathbf{F}$  be a field and let  $\mathbf{V}$  be the space of polynomial functions  $f$  from  $\mathbf{F}$  into  $\mathbf{F}$ , given by

$$f(x) = c_0 + c_1x + \cdots + c_nx^n$$

Let  $\mathbf{D}$  be a linear differentiation transforma-

tion defined as

$$(\mathbf{D}f)(x) = \frac{df(x)}{dx}$$

Then find the range and null space of  $\mathbf{D}$ .

- b) Let  $\mathbf{R}$  be the field of real numbers and let  $\mathbf{V}$  be the space of all functions from  $\mathbf{R}$  into  $\mathbf{R}$  which are continuous. Let  $\mathbf{T}$  be linear transformation defined by

$$(\mathbf{T}f)(x) = \int_0^x f(t) dt$$

Find the range and null space of  $\mathbf{T}$ .

**Solution:** Let the vector space of  $n$ -dimension be defined as

$$\mathbf{V} = \left\{ f : \mathbf{F} \rightarrow \mathbf{F} : f(x) = \sum_{k=0}^n c_k x^k, c_k \in \mathbf{F} \right\} \quad (3.1.3.1)$$

The corresponding standard basis for  $\mathbf{V}$  is

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x \\ \vdots \\ 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ x^{n-1} \end{pmatrix} \right\} \quad (3.1.3.2)$$

- a) Let  $f$  and  $g \in \mathbf{V}$  and let  $\alpha$  and  $\beta \in \mathbf{F}$  then

$$\mathbf{D}(\alpha f + \beta g) = \frac{d(\alpha f(x) + \beta g(x))}{dx} \quad (3.1.3.3)$$

$$= \alpha \frac{df(x)}{dx} + \beta \frac{dg(x)}{dx} \quad (3.1.3.4)$$

$$= \alpha(\mathbf{D}f) + \beta(\mathbf{D}g) \quad (3.1.3.5)$$

Therefore  $\mathbf{D}$  is a linear transformation.

The  $\mathbf{D}$  transformation maps the  $k^{th}$  basis vector as follows

$$\mathbf{D} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ x^k \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ kx^{k-1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad (3.1.3.6)$$

| Properties     | Zero Transformation            | Identity Transformation        |
|----------------|--------------------------------|--------------------------------|
| Transformation | $T_0(\mathbf{v}) = \mathbf{0}$ | $T_I(\mathbf{v}) = \mathbf{v}$ |
| Range          | Zero subspace $\{0\}$          | $\mathbf{V}$                   |
| Rank           | $\dim(\mathbf{0}) = 0$         | $\dim(\mathbf{V}) = n$         |
| Null space     | $\mathbf{V}$                   | Zero subspace $\{0\}$          |
| Nullity        | $\dim(\mathbf{V}) = n$         | $\dim(\mathbf{0}) = 0$         |

TABLE 3.1.2: Properties of Zero and Identity transformation

Since the transformed vector

$$\begin{pmatrix} 0 \\ \vdots \\ kx^{k-1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbf{V} \quad (3.1.3.7)$$

the range of  $\mathbf{D}$  is the vector space  $\mathbf{V}$ . Thus the transformation is defined as  $\mathbf{D} : \mathbf{V} \rightarrow \mathbf{V}$ . Therefore the  $\mathbf{D}$  Transformation on the basis vector set is

$$\mathbf{D} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \quad (3.1.3.8)$$

$$= \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & n-2 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \quad (3.1.3.9)$$

Thus the  $\mathbf{D}$  transformation coefficient matrix is

$$D = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & n-2 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \quad (3.1.3.10)$$

Since  $D$  contains a zero row hence  $|D| = 0$ . Therefore  $\mathbf{D}$  transformation matrix is singular. The nullspace for differentiation trans-

formation is defined as

$$\mathbf{N} = \{f \in \mathbf{V} : \mathbf{D}f = 0\} \quad (3.1.3.11)$$

Let the coefficient matrix of  $f \in \mathbf{V}$  be

$$\mathbf{f} = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{pmatrix} \quad (3.1.3.12)$$

then

$$\mathbf{D}f = 0 \quad (3.1.3.13)$$

$$\Rightarrow \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & n-2 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{pmatrix} = \mathbf{0} \quad (3.1.3.14)$$

Since  $D$  is in row reduced echolon form and  $\text{rank}(D) = n - 1$  the solution of (3.1.3.14) is

$$\mathbf{f} = \begin{pmatrix} k \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad (3.1.3.15)$$

where  $k \in \mathbf{R}$ . Therefore the nullspace for  $\mathbf{D} : \mathbf{V} \rightarrow \mathbf{V}$  is

$$\mathbf{N} = \left\{ \begin{pmatrix} k \\ 0 \\ \vdots \\ 0 \end{pmatrix} : k \in \mathbf{R} \right\} \quad (3.1.3.16)$$

b) Let  $f$  and  $g \in \mathbf{V}$  and let  $\alpha$  and  $\beta \in \mathbf{F}$  then

$$\mathbf{T}(\alpha f + \beta g) = \int_0^x (\alpha f(t) + \beta g(t)) dt \quad (3.1.3.17)$$

$$= \alpha \int_0^x f(t) dt + \beta \int_0^x g(t) dt \quad (3.1.3.18)$$

$$= \alpha(\mathbf{T}f) + \beta(\mathbf{T}g) \quad (3.1.3.19)$$

Therefore  $\mathbf{T}$  is a linear transformation.

The  $\mathbf{T}$  transformation maps the  $k^{\text{th}}$  basis vector as follows

$$\mathbf{T} \begin{pmatrix} 0 \\ \vdots \\ x^k \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{x^{k+1}}{k+1} \\ \vdots \\ 0 \end{pmatrix} \quad (3.1.3.20)$$

Since the transformed vector

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{x^{k+1}}{k+1} \\ \vdots \\ 0 \end{pmatrix} \in \mathbf{V} \quad (3.1.3.21)$$

the range of  $\mathbf{T}$  is the vector space  $\mathbf{V}$ . Thus the transformation is defined as  $\mathbf{T} : \mathbf{V} \rightarrow \mathbf{V}$ . Therefore the  $\mathbf{T}$  Transformation on the basis vector set is

$$\mathbf{T} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \quad (3.1.3.22)$$

$$= \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & \frac{1}{n-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{n} \end{pmatrix} \quad (3.1.3.23)$$

Thus the  $\mathbf{T}$  transformation coefficient matrix

is

$$T = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & \frac{1}{n-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{n} \end{pmatrix} \quad (3.1.3.24)$$

Since  $T$  contains a zero row hence  $|T| = 0$ . Therefore  $\mathbf{T}$  transformation matrix is singular. The nullspace for integration transformation is defined as

$$\mathbf{N} = \{f \in \mathbf{V} : \mathbf{T}f = 0\} \quad (3.1.3.25)$$

Let the coefficient matrix of  $f \in \mathbf{V}$  be

$$\mathbf{f} = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{pmatrix} \quad (3.1.3.26)$$

then

$$\mathbf{T}f = 0 \quad (3.1.3.27)$$

$$\Rightarrow \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & \frac{1}{n-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{n} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{pmatrix} = \mathbf{0} \quad (3.1.3.28)$$

Since  $T$  is in row reduced echolon form and  $\text{rank}(T) = n$  the solution of (3.1.3.28) is

$$\mathbf{f} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad (3.1.3.29)$$

where  $k \in \mathbf{R}$ . Therefore the nullspace for  $\mathbf{T} : \mathbf{V} \rightarrow \mathbf{V}$  is

$$\mathbf{N} = \left\{ \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} : k \in \mathbf{R} \right\} \quad (3.1.3.30)$$

3.1.4. Is there a linear transformation  $\mathbf{T}$  from  $\mathbb{R}^3$  into

$\mathbb{R}^2$  such that,

$$\mathbf{T} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.1.4.1)$$

$$\mathbf{T} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (3.1.4.2)$$

**Solution:** *Linear Transformation* A linear transformation is a function  $\mathbf{T} : \mathbb{R}^n \rightarrow \mathbb{R}^m$  which satisfies:

1.  $\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ ,

$$\mathbf{T}(\mathbf{x} + \mathbf{y}) = \mathbf{T}(\mathbf{x}) + \mathbf{T}(\mathbf{y}) \quad (3.1.4.3)$$

2.  $\forall \mathbf{x} \in \mathbb{R}^n$  and  $c \in \mathbb{R}$ ,

$$\mathbf{T}(c\mathbf{x}) = c\mathbf{T}(\mathbf{x}) \quad (3.1.4.4)$$

*Matrix of the Linear Transformation* Let,  $\mathbf{T} : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is a linear transformation and  $\mathbf{x} \in \mathbb{R}^n$  is given by ,

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad (3.1.4.5)$$

$$= x_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \cdots + x_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \quad (3.1.4.6)$$

Let  $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$  be the standard basis of  $\mathbb{R}^n$  and the equation (3.1.4.6) can be rewritten as,

$$\mathbf{x} = \sum_{i=1}^n x_i \mathbf{e}_i \quad (3.1.4.7)$$

$$\mathbf{T}(\mathbf{x}) = \sum_{i=1}^n x_i \mathbf{T}(\mathbf{e}_i) \quad (3.1.4.8)$$

$$= (\mathbf{T}(\mathbf{e}_1) \quad \mathbf{T}(\mathbf{e}_2) \quad \dots \quad \mathbf{T}(\mathbf{e}_n)) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad (3.1.4.9)$$

$$\mathbf{T}(\mathbf{x}) = \mathbf{A}\mathbf{x} \quad (3.1.4.10)$$

Where,

$$\mathbf{A} = (\mathbf{T}(\mathbf{e}_1) \quad \mathbf{T}(\mathbf{e}_2) \quad \dots \quad \mathbf{T}(\mathbf{e}_n)) \quad (3.1.4.11)$$

If  $\mathbf{T}$  is any linear transformation which maps  $\mathbb{R}^n \rightarrow \mathbb{R}^m$  there is always an  $m \times n$  matrix  $\mathbf{A}$  with the property that

$$\mathbf{T}(\mathbf{x}) = \mathbf{A}\mathbf{x}$$

where ,  $\mathbf{x} \in \mathbb{R}^n$  *Solution* Let,

$$\mathbf{v} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad (3.1.4.12)$$

$$\mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad (3.1.4.13)$$

Given,

$$\mathbf{T}(\mathbf{v}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.1.4.14)$$

$$\mathbf{T}(\mathbf{u}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (3.1.4.15)$$

Let the standard basis vectors is denoted as,

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad (3.1.4.16)$$

$$\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad (3.1.4.17)$$

$$\mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad (3.1.4.18)$$

Let,  $\mathbf{T} : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be a linear transformation. Then the function  $\mathbf{T}$  is just matrix-vector multiplication  $\mathbf{T}(\mathbf{x}) = \mathbf{A}\mathbf{x}$  for some matrix  $\mathbf{A}$  as shown in equation (3.1.4.10)

Matrix  $\mathbf{A}$  of order  $2 \times 3$  is given by,

$$\mathbf{A} = (\mathbf{T}(\mathbf{e}_1) \quad \mathbf{T}(\mathbf{e}_2) \quad \mathbf{T}(\mathbf{e}_3)) \quad (3.1.4.19)$$

Consider the vector  $\mathbf{b} \in \mathbb{R}^3$  which is the linear combinations of the vectors  $\mathbf{v}$  and  $\mathbf{u}$ .

For  $x_1, x_2 \in \mathbb{R}$ ,

$$\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad (3.1.4.20)$$

$$\mathbf{T} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = x_1 \mathbf{T} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + x_2 \mathbf{T} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad (3.1.4.21)$$

To find  $x_1, x_2$ , we solve the linear system,  $\mathbf{M}\mathbf{x} = \mathbf{b}$  where  $\mathbf{M}$  is the  $3 \times 2$  matrix obtained by stacking the given vectors  $\mathbf{v}$  and  $\mathbf{u}$  as columns

$$\mathbf{M} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 1 \end{pmatrix} \quad (3.1.4.22)$$

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \quad (3.1.4.23)$$

The augmented matrix of the equation (3.1.4.23) is given by ,

$$\left( \begin{array}{cc|c} 1 & 1 & b_1 \\ -1 & 1 & b_2 \\ 1 & 1 & b_3 \end{array} \right) \quad (3.1.4.24)$$

By row reducing the above equation (3.1.4.24),

$$\left( \begin{array}{cc|c} 1 & 1 & b_1 \\ -1 & 1 & b_2 \\ 1 & 1 & b_3 \end{array} \right) \xleftrightarrow{R_2=R_2+R_1} \left( \begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & 2 & b_2+b_1 \\ 1 & 1 & b_3 \end{array} \right) \quad (3.1.4.25)$$

$$\left( \begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & 2 & b_2+b_1 \\ 1 & 1 & b_3 \end{array} \right) \xleftrightarrow{R_3=R_3-R_1} \left( \begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & 2 & b_2+b_1 \\ 0 & 0 & b_3-b_1 \end{array} \right) \quad (3.1.4.26)$$

$$\left( \begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & 2 & b_2+b_1 \\ 0 & 0 & b_3-b_1 \end{array} \right) \xleftrightarrow{R_2=\frac{R_2}{2}} \left( \begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & 1 & \frac{b_2+b_1}{2} \\ 0 & 0 & b_3-b_1 \end{array} \right) \quad (3.1.4.27)$$

$$\left( \begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & 1 & \frac{b_2+b_1}{2} \\ 0 & 0 & b_3-b_1 \end{array} \right) \xleftrightarrow{R_1=R_1-R_2} \left( \begin{array}{cc|c} 1 & 0 & \frac{b_1-b_2}{2} \\ 0 & 1 & \frac{b_2+b_1}{2} \\ 0 & 0 & b_3-b_1 \end{array} \right) \quad (3.1.4.28)$$

Now equation (3.1.4.23) can be written as,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{b_1-b_2}{2} \\ \frac{b_2+b_1}{2} \\ b_3-b_1 \end{pmatrix} \quad (3.1.4.29) \quad 3.1.5.$$

Solving the above equation we get ,

$$x_1 = \frac{b_1-b_2}{2} \quad (3.1.4.30)$$

$$x_2 = \frac{b_1+b_2}{2} \quad (3.1.4.31)$$

Substituting the above equations (3.1.4.30),(3.1.4.31) in equation (3.1.4.21), we

get,

$$\mathbf{T} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \left( \frac{b_1-b_2}{2} \right) \mathbf{T} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \left( \frac{b_1+b_2}{2} \right) \mathbf{T} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad (3.1.4.32)$$

Substituting the equations (3.1.4.1) and (3.1.4.2) in equation (3.1.4.32) we get,

$$\mathbf{T} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \left( \frac{b_1-b_2}{2} \right) \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \left( \frac{b_1+b_2}{2} \right) \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad (3.1.4.33)$$

$$\mathbf{T} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} \frac{b_1-b_2}{2} \\ \frac{b_1+b_2}{2} \\ \frac{b_1+b_2}{2} \end{pmatrix} \quad (3.1.4.34)$$

Using the above equation (3.1.4.34) we compute,

$$\mathbf{T}(\mathbf{e}_1) = \mathbf{T} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \quad (3.1.4.35)$$

$$\mathbf{T}(\mathbf{e}_2) = \mathbf{T} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \quad (3.1.4.36)$$

$$\mathbf{T}(\mathbf{e}_3) = \mathbf{T} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad (3.1.4.37)$$

Substituting the equations (3.1.4.35),(3.1.4.36) and (3.1.4.37) in equation (3.1.4.19) we get,

$$\mathbf{A} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \quad (3.1.4.38)$$

$$\Rightarrow \mathbf{A} = \frac{1}{2} \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \quad (3.1.4.39)$$

Therefore from the above matrix  $\mathbf{A}$  we can say that there is a linear transformation  $\mathbf{T}$  from  $\mathbb{R}^3$  into  $\mathbb{R}^2$  which satisfies the given conditions (3.1.4.1) and (3.1.4.2).

3.1.5. Describe explicitly the linear transformation  $\mathbf{T}$  from  $\mathbb{F}^2$  into  $\mathbb{F}^2$  such that  $T(\mathbf{e}_1) = (a, b)$ ,  $T(\mathbf{e}_2) = (c, d)$ . **Solution:** We are given a linear transformation,

$$T : \mathbb{F}^2 \rightarrow \mathbb{F}^2 \quad (3.1.5.1)$$

The transformation for  $\mathbf{e}_1$  and  $\mathbf{e}_2$  can be written

as,

$$T(\epsilon_1) = \begin{pmatrix} a \\ b \end{pmatrix} \quad (3.1.5.2)$$

$$T(\epsilon_2) = \begin{pmatrix} c \\ d \end{pmatrix} \quad (3.1.5.3)$$

Now, let's assume  $\epsilon_1$  and  $\epsilon_2$  as linearly independent. So the linear transformation  $T$  for any vector  $\mathbf{v}$  in two dimensional space will be,

$$T(\mathbf{v}) = \begin{pmatrix} T(\epsilon_1) & T(\epsilon_2) \end{pmatrix} \mathbf{v} \quad (3.1.5.4)$$

$$= \begin{pmatrix} a & c \\ b & d \end{pmatrix} \mathbf{v} \quad (3.1.5.5)$$

Now, there can be two cases here, transformation of linearly independent vector can be independent or it can be dependent. Considering the first case and (3.1.5.5) we can say that,

$$\text{Range}(T) = \text{columnspace of } \begin{pmatrix} a & c \\ b & d \end{pmatrix} \quad (3.1.5.6)$$

Now, considering the case when linear transformation will be linearly dependent,

$$\text{Range}(T) = \text{columnspace of } \begin{pmatrix} a \\ b \end{pmatrix} \quad (3.1.5.7)$$

Now, considering that vectors  $\epsilon_1$  and  $\epsilon_2$  itself are linearly dependent. Let  $\mathbf{v} = \epsilon_1 + \epsilon_2$

$$T(\mathbf{v}) = T(\epsilon_1) + T(\epsilon_2) \quad (3.1.5.8)$$

$$= T(\epsilon_1) + T(k\epsilon_1) \quad (3.1.5.9)$$

$$= (k+1)T(\epsilon_1) \quad (3.1.5.10)$$

$$= (k+1) \begin{pmatrix} a \\ b \end{pmatrix} \quad (3.1.5.11)$$

We can see from above equation that when  $\epsilon_1$  and  $\epsilon_2$  as linearly dependent then the transformation  $T$  will be along the line only.

| Vectors Independent   | Vectors Dependent  |
|---|--|
| $T(\mathbf{v}) = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \mathbf{v}$ | $T(\mathbf{v}) = (k+1) \begin{pmatrix} a \\ b \end{pmatrix}$ |
| Output:<br>On the plane   | Output:<br>On the line                                       |

TABLE 3.1.5

by

$$\mathbb{T}(x_1, x_2, x_3) = \quad (3.1.6.1)$$

$$(x_1 - x_2 + 2x_3, 2x_1 + x_2, -x_1 - 2x_2 + 2x_3) \quad (3.1.6.2)$$

- Verify that  $\mathbb{T}$  is a linear transformation.
- If  $(a, b, c)$  is a vector in  $\mathbb{F}^3$ , what are the conditions on  $a, b, c$  that the vector be in the range of  $\mathbb{T}$ ? What is the rank of  $\mathbb{T}$ ?
- What are the conditions on  $a, b, c$  that  $(a, b, c)$  be in the null space of  $\mathbb{T}$ ? What is the nullity of  $\mathbb{T}$ ?

**Solution:** Representing the transformation in matrix form

$$\mathbb{T}(x_1, x_2, x_3) = \mathbf{T}\mathbf{x} \quad (3.1.6.3)$$

$$\mathbf{T} = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ -1 & -2 & 2 \end{pmatrix} \quad (3.1.6.4)$$

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad (3.1.6.5)$$

Part (a) Consider the matrices  $\mathbf{x}, \mathbf{y} \in \mathbb{F}^3$  and the scalar  $c \in \mathbb{F}$

By the associativity of matrix multiplications, we can write

$$\mathbf{T}(c\mathbf{x} + \mathbf{y}) = \mathbf{T}(c\mathbf{x}) + \mathbf{T}\mathbf{y} \quad (3.1.6.6)$$

$$= c\mathbf{T}\mathbf{x} + \mathbf{T}\mathbf{y} \quad (3.1.6.7)$$

So,  $\mathbf{T}$  is a linear transformation. Part (b)

$$\text{range}(\mathbf{T}) = \{ \mathbf{y} : \mathbf{T}\mathbf{x} = \mathbf{y} \text{ where } \mathbf{x}, \mathbf{y} \in \mathbb{F}^3 \} \quad (3.1.6.8)$$

3.1.6. Let  $\mathbb{F}$  be a subfield of the complex numbers and let  $\mathbb{T}$  be the function from  $\mathbb{F}^3$  into  $\mathbb{F}^3$  defined

$$\mathbf{y} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad (3.1.6.9)$$

$$\mathbf{T}\mathbf{x} = \mathbf{y} \quad (3.1.6.10)$$

$$\Rightarrow \mathbf{B}\mathbf{T}\mathbf{x} = \mathbf{B}\mathbf{y} \quad (3.1.6.11)$$

$$\Rightarrow \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{-2}{3} & \frac{1}{3} & 0 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ -1 & -2 & 2 \end{pmatrix} \mathbf{x} = \quad (3.1.6.12)$$

$$\begin{pmatrix} \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{-2}{3} & \frac{1}{3} & 0 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad (3.1.6.13)$$

$$\begin{pmatrix} 1 & 0 & \frac{2}{3} \\ 0 & 1 & \frac{-4}{3} \\ 0 & 0 & 0 \end{pmatrix} \mathbf{x} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{-2}{3} & \frac{1}{3} & 0 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad (3.1.6.14)$$

So,  $\text{rank}(\mathbf{T})=2$  and comparing the third row element in LHS and RHS of (3.1.6.14)

$$-a + b + c = 0 \quad (3.1.6.15)$$

All vectors  $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{F}^3$  that satisfy (3.1.6.15) lie in the range of  $\mathbf{T}$  Part (c)

$$\text{nullspace}(\mathbf{T}) = \left\{ \mathbf{x} : \mathbf{T}\mathbf{x} = \mathbf{0} \text{ where } \mathbf{x} \in \mathbb{F}^3 \right\} \quad (3.1.6.16)$$

$$\mathbf{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad (3.1.6.17)$$

$$\mathbf{T}\mathbf{x} = \mathbf{0} \quad (3.1.6.18)$$

$$\mathbf{B}\mathbf{T}\mathbf{x} = \mathbf{0} \quad (3.1.6.19)$$

where  $\mathbf{B}\mathbf{T}$  is in reduced row echelon form

$$\begin{pmatrix} \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{-2}{3} & \frac{1}{3} & 0 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ -1 & -2 & 2 \end{pmatrix} \mathbf{x} = \mathbf{0} \quad (3.1.6.20)$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & \frac{2}{3} \\ 0 & 1 & \frac{-4}{3} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (3.1.6.21)$$

$$\Rightarrow a + \frac{2}{3}c = 0 \quad (3.1.6.22)$$

$$b - \frac{4}{3}c = 0 \quad (3.1.6.23)$$

The number of free variables in the reduced row echelon form of  $\mathbf{T}$  is 1 hence nullity( $\mathbf{T}$ )=1

So, the null space of  $\mathbf{T}$  is set of all vectors

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{F}^3 \text{ that satisfy (3.1.6.22) and (3.1.6.23)}$$

**Note**

$$\text{rank}(\mathbf{T}) + \text{nullity}(\mathbf{T}) = 2 + 1 = \dim(\mathbb{F}^3)$$

3.1.7. Describe explicitly a linear transformation from  $R^3$  into  $R^3$  which has as its range the subspace spanned by  $\begin{pmatrix} 1 & 0 & -1 \end{pmatrix}$  and  $\begin{pmatrix} 1 & 2 & 2 \end{pmatrix}$ .

**Solution:** Transformation  $T$  from  $R^3$  to  $R^3$  range gives the column space. Hence,

$$T(\mathbf{x}) = \mathbf{A}\mathbf{x} \quad (3.1.7.1)$$

$$T(\mathbf{x}) = \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ -1 & 2 \end{pmatrix} \mathbf{x} \quad (3.1.7.2)$$

3.1.8. Let  $\mathbf{V}$  be the vector space of all  $n \times n$  matrices over the field  $\mathbb{F}$ , and let  $\mathbf{B}$  be a fixed  $n \times n$  matrix. If a transformation  $T$  defined as follows,

$$T(\mathbf{A}) = \mathbf{A}\mathbf{B} - \mathbf{B}\mathbf{A}$$

Prove that  $T$  is a linear transformation from  $\mathbf{V}$  into  $\mathbf{V}$  **Solution:** Let,

$$\mathbf{A}_1 \in \mathbf{V} \quad (3.1.8.1)$$

$$\mathbf{A}_2 \in \mathbf{V} \quad (3.1.8.2)$$

If  $c$  be any scalar of the field  $\mathbb{F}$  we get,

$$c\mathbf{A}_1 + \mathbf{A}_2 \in \mathbf{V} \quad (3.1.8.3)$$

Applying transformation  $T$  on  $(c\mathbf{A}_1 + \mathbf{A}_2)$  we get,

$$T(c\mathbf{A}_1 + \mathbf{A}_2) = (c\mathbf{A}_1 + \mathbf{A}_2)\mathbf{B} - \mathbf{B}(c\mathbf{A}_1 + \mathbf{A}_2) \quad (3.1.8.4)$$

$$= c\mathbf{A}_1\mathbf{B} + \mathbf{A}_2\mathbf{B} - c\mathbf{B}\mathbf{A}_1 - \mathbf{B}\mathbf{A}_2 \quad (3.1.8.5)$$

$$= c(\mathbf{A}_1\mathbf{B} - \mathbf{B}\mathbf{A}_1) + (\mathbf{A}_2\mathbf{B} - \mathbf{B}\mathbf{A}_2) \quad (3.1.8.6)$$

$$= cT(\mathbf{A}_1) + T(\mathbf{A}_2) \quad (3.1.8.7)$$

From (3.1.8.7) we conclude that  $T$  is a linear transformation from vector space  $\mathbf{V}$  to  $\mathbf{V}$ .

3.1.9. Let  $\mathbf{V}$  be the set of all complex numbers regarded as a vector space over the field of real numbers (usual operations). Find a function from  $\mathbf{V}$  into  $\mathbf{V}$  which is a linear transformation on the above vector space, but which is not a linear transformation on  $\mathbb{C}$  i.e., which is not complex linear.



**Solution:** Let

$$T : V \rightarrow V \quad (3.1.9.1)$$

be a function such that,

$$T(x + iy) = \operatorname{Re}(x + iy) = x \quad (3.1.9.2)$$

$$\implies T : x + iy \rightarrow x \quad (3.1.9.3)$$

where  $x, y \in \mathbb{R}$ .

Let,  $\alpha = a + ib, \beta = c + id$ .

$$\implies T(k\alpha + \beta) = T(ka + ikb + c + id) \quad (3.1.9.4)$$

$$= T(ka + c + i(kb + d)) \quad (3.1.9.5)$$

$$= ka + c \quad (3.1.9.6)$$

$$= kT(\alpha) + T(\beta) \quad (3.1.9.7)$$

Now, let  $z \in V$  such that,

$$z = i \quad (3.1.9.8)$$

$$\implies T(z) = T(i) = 0 \quad (3.1.9.9)$$

We can also write,

$$T(i) = T(i(1)) = iT(1) = i \neq 0 \quad (3.1.9.10)$$

Thus from (3.1.9.7),  $T$  is real linear transformation and from (3.1.9.10),  $T$  is not complex linear.

3.1.10. Let  $\mathbf{V}$  be the space of  $n \times 1$  matrices over  $F$  and let  $\mathbf{W}$  be the space of  $m \times 1$  matrices over  $F$ . Let  $\mathbf{A}$  be a fixed  $m \times n$  matrix over  $F$  and let  $T$  be the linear transformation from  $\mathbf{V}$  into  $\mathbf{W}$  defined by  $T(\mathbf{X}) = \mathbf{AX}$ . Prove that  $T$  is the zero transformation if and only if  $\mathbf{A}$  is the zero matrix. **Solution:** If  $\mathbf{A}_{m \times n}$  is a zero transformation and  $\mathbf{X}_{n \times 1}$  is a vector, then

$$\mathbf{AX} = \mathbf{0}_{m \times 1} \quad (3.1.10.1)$$

Let,

$$\mathbf{A} = (\mathbf{A}_1 \quad \dots \quad \mathbf{A}_j \quad \dots \quad \mathbf{A}_n)_{1 \times n} \text{ and} \quad (3.1.10.2)$$

$$\mathbf{X}_j = \begin{pmatrix} x_1 \\ \vdots \\ x_j \\ \vdots \\ x_n \end{pmatrix}, \text{ where } x_i = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \quad (3.1.10.3)$$

If  $\mathbf{A}_{m \times n}$  is zero transformation, then for any vector  $\mathbf{X}_{n \times 1}$ ,  $\mathbf{AX} = \mathbf{0}$ .

Consider ,

$$\mathbf{AX}_j = \mathbf{0}_{m \times 1} \quad (3.1.10.4)$$

$$(\mathbf{A}_1 \dots \mathbf{A}_j \dots \mathbf{A}_n) \begin{pmatrix} x_1 \\ \vdots \\ x_j \\ \vdots \\ x_n \end{pmatrix} = \mathbf{0}_{m \times 1} \quad (3.1.10.5)$$

From (3.1.10.3) and (3.1.10.5)

$$\mathbf{A}_j = \mathbf{0}_{m \times 1} \text{ for } j = 1, 2, \dots, n \quad (3.1.10.6)$$

Substitute (3.1.10.6) in (3.1.10.2)

$$\mathbf{A} = (\mathbf{0}_{m \times 1} \quad \mathbf{0}_{m \times 1} \quad \dots \quad \mathbf{0}_{m \times 1})_{1 \times n} \quad (3.1.10.7)$$

$$\therefore \mathbf{A} = \mathbf{0}_{m \times n} \quad (3.1.10.8)$$

Hence  $\mathbf{A}$  is zero matrix.

Let us assume  $\mathbf{A}_{m \times n}$  is a zero matrix

$$\mathbf{A} = \mathbf{0}_{m \times n} \quad (3.1.10.9)$$

Then,

$$T(\mathbf{X}) = \mathbf{AX} \quad (3.1.10.10)$$

$$= \mathbf{0.X} \quad (3.1.10.11)$$

$$= \mathbf{0}_{m \times 1}, \forall \mathbf{X} \in F \quad (3.1.10.12)$$

Hence  $T(\mathbf{X}) = \mathbf{AX}$  is the zero transformation.

From (3.1.10.8) and (3.1.10.12) it is proved that  $T$  is the zero transformation if and only if  $\mathbf{A}$  is the zero matrix.

3.1.11. Let  $\mathbf{V}$  be an  $n$ -dimensional vector space over

the field  $\mathbf{F}$  and let  $\mathbf{T}$  be a linear transformation from  $\mathbf{V}$  into  $\mathbf{V}$  such that the range and null space of  $\mathbf{T}$  are identical. Prove that  $n$  is even. (Can you give an example of such a linear transformation  $\mathbf{T}$ )?

**Solution:** Let  $\mathbf{V}$  and  $\mathbf{W}$  be vector spaces over the field  $\mathbf{F}$  and let  $\mathbf{T}$  be a linear transformation from  $\mathbf{V}$  into  $\mathbf{W}$ . Then,

$$\text{rank}(\mathbf{T}) + \text{nullity}(\mathbf{T}) = \dim \mathbf{V} \quad (3.1.11.1)$$

It is given that range and null space of  $\mathbf{T}$  are same, let us assume it to be  $m$ . Substituting in equation (3.1.11.1)

$$m + m = n \quad (3.1.11.2)$$

$$\Rightarrow n = 2m \quad (3.1.11.3)$$

From equation (3.1.11.3), we can say that  $n$  is even.

Example: Let us consider a vector space  $\mathbf{V}$ , such that  $\mathbf{V} \in \mathbb{R}^2$  and let us consider a linear transformation  $\mathbf{T} : \mathbf{V} \rightarrow \mathbf{V}$  defined by  $\mathbf{T} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ 0 \end{pmatrix}$  and is given by matrix  $\mathbf{M}$

$$\mathbf{T} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ 0 \end{pmatrix} \quad (3.1.11.4)$$

$$\mathbf{M} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (3.1.11.5)$$

Let us consider basis of  $\mathbb{R}^2$   $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$  and apply linear transformation on it.

$$\mathbf{T} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (3.1.11.6)$$

$$\mathbf{T} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.1.11.7)$$

From (3.1.11.5),

The range of matrix can be found from row reduced echelon form. But as matrix  $\mathbf{M}$  is in RREF form,

the basis for range is given by  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ .

The null space of matrix is,

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (3.1.11.8)$$

$$\Rightarrow x_1 = t \quad x_2 = 0 \quad (3.1.11.9)$$

$$\Rightarrow \mathbf{X} = \begin{pmatrix} t \\ 0 \end{pmatrix} = t \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.1.11.10)$$

The basis for null space is  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ .

$$\text{rank}(\mathbf{T}) = 1 \quad \text{nullity}(\mathbf{T}) = 1 \quad (3.1.11.11)$$

$$\dim(\mathbf{V}) = 2 \quad (3.1.11.12)$$

Thus the range and null space are equal, and  $n$  is even.

3.1.12. Let  $\mathbb{V}$  be a vector space and  $\mathbf{T}$  a linear transformation from  $\mathbb{V}$  into  $\mathbb{V}$ . Prove that the following two statements about  $\mathbf{T}$  are equivalent.

- (a) The intersection of the range of  $\mathbf{T}$  and null space of  $\mathbf{T}$  is the zero subspace of  $\mathbb{V}$ .
- (b) If  $\mathbf{T}(\mathbf{T}\alpha) = 0$ , then  $\mathbf{T}\alpha = 0$ .

**Solution:**

|          |   |
|----------|---|
| Given    | $\mathbf{T} : \mathbb{V} \rightarrow \mathbb{V}$  |
| To prove | a) $\text{range}(\mathbf{T}) \cap \text{nullspace}(\mathbf{T}) = \{0\}$<br>b) If $\mathbf{T}(\mathbf{T}\alpha) = 0$ , then $\mathbf{T}\alpha = 0$ . |

### 3.2 The Algebra of Linear Transformations

3.2.1. Let  $\mathbf{T}$  and  $\mathbf{U}$  be the linear operators on  $\mathbb{R}^2$  defined by  $\mathbf{T}(x_1, x_2) = (x_2, x_1)$  and  $\mathbf{U}(x_1, x_2) = (x_1, 0)$ .

- a) Let  $\mathbf{T}$  and  $\mathbf{U}$  be the linear operators on  $\mathbb{R}^2$  defined by

$$\mathbf{T}(x_1, x_2) = (x_2, x_1) \quad (3.2.1.1)$$

and

$$\mathbf{U}(x_1, x_2) = (x_1, 0) \quad (3.2.1.2)$$

How would you describe  $\mathbf{T}$  and  $\mathbf{U}$  geometrically ?

**Solution:** Geometrically, in the  $x$ - $y$  plane,  $\mathbf{T}$  is the reflection about the diagonal  $x = y$

|                 |  |  |
|-----------------|--|--|
| <p>Proof(a)</p> | <p>Let <math>\mathbf{x} \in \mathbb{V}</math><br/>and<br/><math>\mathbf{x} \in \text{range}(\mathbf{T}) \cap \text{nullspace}(\mathbf{T})</math><br/>then,<br/><math>\mathbf{x} \in \text{range}(\mathbf{T})</math><br/><math>\mathbf{x} \in \text{nullspace}(\mathbf{T})</math></p> <p>Consider <math>\mathbf{y} \in \mathbb{V}</math> whose<br/>linear transformation into <math>\mathbb{V}</math> is <math>\mathbf{x}</math>.<br/><math>\Rightarrow \mathbf{T}(\mathbf{y}) = \mathbf{x}</math></p> <p>since <math>\mathbf{x} \in \text{null space}(\mathbf{T})</math><br/>and the sub space is linearly independent<br/><math>\mathbf{T}(\mathbf{x}) = 0</math><br/>from above equations<br/><math>\mathbf{T}(\mathbf{T}(\mathbf{y})) = 0</math></p> <p>from the definition of linear<br/>transformation of independent vector space<br/><math>\mathbf{T}(\mathbf{y}) = 0</math><br/><math>\Rightarrow \mathbf{x} = 0</math><br/><math>\Rightarrow \{0\} \subseteq \text{range}(\mathbf{T}) \cap \text{nullspace}(\mathbf{T})</math><br/><math>\therefore \text{range}(\mathbf{T}) \cap \text{nullspace}(\mathbf{T}) = \{0\}</math><br/>Hence Proved.</p> | <p>Eg:</p> <p>Let <math>\alpha \in \mathbb{V}</math> and</p> $\alpha = \begin{pmatrix} 1 & 7 & -1 & -1 \\ -1 & 1 & 2 & 1 \\ 4 & -2 & 0 & -4 \\ 2 & 3 & 4 & -2 \end{pmatrix}$ <p>linear transformation of <math>\alpha</math> into <math>\mathbb{V}</math><br/><math>\mathbf{T}(\alpha) = c\alpha</math></p> <p>then row reduced echelon form of <math>\mathbf{T}</math> is</p> $\text{rref}(\mathbf{T}) = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ <p><math>\Rightarrow \text{rank}(\mathbf{T}) = 3,</math><br/><math>\text{nullity}(\mathbf{T}) = 1</math></p> <p><math>\Rightarrow \text{range}(\mathbf{T}) = \begin{pmatrix} 1 &amp; 7 &amp; -1 \\ -1 &amp; 1 &amp; 2 \\ 4 &amp; -2 &amp; 0 \\ 2 &amp; 3 &amp; 4 \end{pmatrix},</math></p> <p><math>\text{nullspace}(\mathbf{T}) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}</math></p> <p><math>\therefore \text{range}(\mathbf{T}) \cap \text{nullspace}(\mathbf{T}) = \{0\}</math></p> <p>Hence proved that the intersection<br/>of the range of <math>\mathbf{T}</math> and<br/>null space of <math>\mathbf{T}</math> is the zero<br/>subspace of <math>\mathbb{V}</math>.</p> |
| <p>Proof(b)</p> | <p>If <math>\mathbf{T}(\mathbf{T}\alpha) = 0</math><br/>then, from the definition of linear<br/>transformation, <math>\mathbf{T}\alpha</math> will<br/>be in the null space of linear<br/>transformation <math>\mathbf{T}</math> and is linearly<br/>independent<br/><math>\therefore \mathbf{T}\alpha = 0</math></p>  |  |

and  $\mathbf{U}$  is a projection onto the x-axis.

#### i) Reflection

Let Consider Matrix  $\mathbf{A}$  as

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (3.2.1.3)$$

The matrix  $\mathbf{A}$  is representation of the  
linear transformation  $\mathbf{T}$  across the line  
 $y=x$  with respect to the standard basis.

Let suppose

$$\mathbf{x}_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \mathbf{x}_2 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \quad (3.2.1.4)$$

After applying linear operator  $\mathbf{T}$  on it,

$$\mathbf{T}(x_1, x_2) = \mathbf{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad (3.2.1.5)$$

$$\Rightarrow \mathbf{A} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad (3.2.1.6)$$

Similarly

$$\mathbf{A} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \quad (3.2.1.7)$$

Hence after applying Operator  $\mathbf{T}$  on  $\mathbf{x}_1$  and  $\mathbf{x}_2$

$$\mathbf{x}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \mathbf{x}_2 = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \quad (3.2.1.8)$$

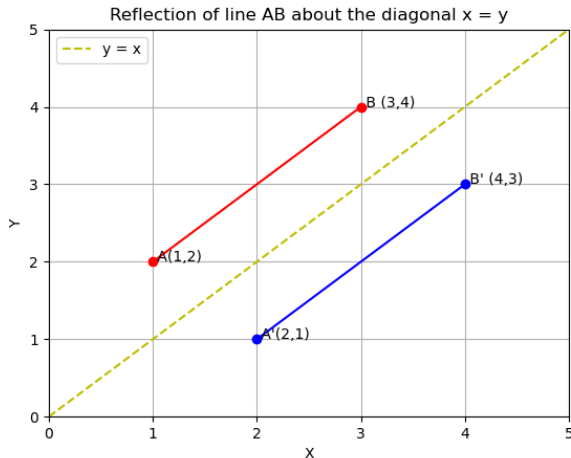


Fig. 3.2.1.1: Reflection of line AB about the  $x = y$

ii) Projection

For projection let Consider Matrix  $\mathbf{B}$  as

$$\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (3.2.1.9)$$

The matrix  $\mathbf{B}$  is representation of the linear transformation  $\mathbf{U}$  that is projection on x-axis.

Let suppose

$$\mathbf{x}_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \mathbf{x}_2 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \quad (3.2.1.10)$$

After applying linear operator  $\mathbf{U}$  on  $\mathbf{x}_1$  and 3.2.2. Let  $\mathbf{T}$  be the unique linear operator on  $C^3$  for

$x_2$ ,

$$\mathbf{T}(x_1, x_2) = \mathbf{U} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad (3.2.1.11)$$

$$\Rightarrow \mathbf{B} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.2.1.12)$$

Similarly

$$\mathbf{A} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \quad (3.2.1.13)$$

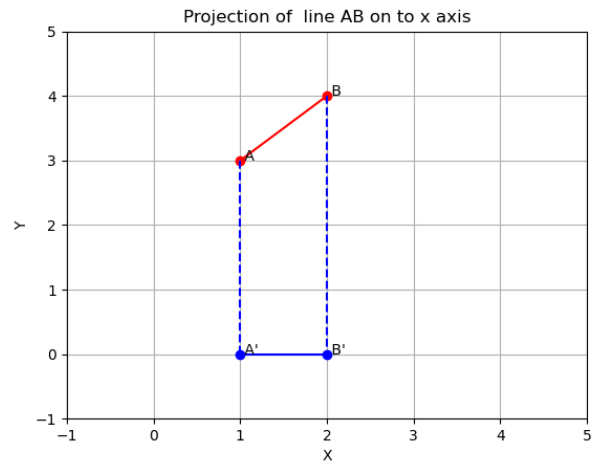


Fig. 3.2.1.2: Projection of AB onto x-axis

Hence after applying Operator  $\mathbf{U}$  on  $\mathbf{x}_1$  and  $\mathbf{x}_2$

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{x}_2 = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \quad (3.2.1.14)$$

b) Give rules like the ones defining  $\mathbf{T}$  and  $\mathbf{U}$  for each of the transformations  $\mathbf{U} + \mathbf{T}$ ,  $\mathbf{UT}$ ,  $\mathbf{TU}$ ,  $\mathbf{T}^2$ ,  $\mathbf{U}^2$ .  $\mathbb{R}^2$  into  $\mathbb{R}^2$  is linear transformation?

**Solution:** Let  $\mathbf{T}$  and  $\mathbf{U}$  defined by matrices  $\mathbf{A}$  and  $\mathbf{B}$  such that ,

$$\mathbf{T}(\mathbf{x}) = \mathbf{A}\mathbf{x}; \quad \mathbf{U}(\mathbf{x}) = \mathbf{B}\mathbf{x} \quad (3.2.1.15)$$

Where,

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (3.2.1.16)$$

Table 3.2.1.1 lists the summary of each Transformations.

| Transformations           | Matrix   | Vector   |
|---------------------------|--|--|
| $\mathbf{U} + \mathbf{T}$ | $(\mathbf{B} + \mathbf{A}) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ | $\begin{pmatrix} x_1 + x_2 \\ x_1 \end{pmatrix}$ |
| $\mathbf{UT}$             | $\mathbf{BA} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$               | $\begin{pmatrix} x_2 \\ 0 \end{pmatrix}$         |
| $\mathbf{TU}$             | $\mathbf{AB} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$               | $\begin{pmatrix} 0 \\ x_1 \end{pmatrix}$         |
| $\mathbf{T}^2$            | $\mathbf{A}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$              | $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$       |
| $\mathbf{U}^2$            | $\mathbf{B}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$              | $\begin{pmatrix} x_1 \\ 0 \end{pmatrix}$         |

TABLE 3.2.1.1: Summary

which

$$T(\epsilon_1) = \begin{pmatrix} 1 & 0 & i \end{pmatrix}, T(\epsilon_2) = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix},$$

$$T(\epsilon_3) = \begin{pmatrix} i & 1 & 0 \end{pmatrix} \quad (3.2.2.1)$$

Is  $\mathbf{T}$  invertible ?

**Solution:** Let  $\epsilon_i$  is basis for  $C^3$  such that  $T(\epsilon_i)$  is basis for  $C^3$   $\mathbf{T}$  is said to be singular if

$$T(\epsilon) = 0 \implies \epsilon \neq 0 \quad (3.2.2.2)$$

now,

$$\begin{pmatrix} 1 & 0 & i \\ 0 & 1 & 1 \\ i & 1 & 0 \end{pmatrix} \epsilon = 0 \quad (3.2.2.3)$$

consider the row reduced matrix

$$\begin{pmatrix} 1 & 0 & i \\ 0 & 1 & 1 \\ i & 1 & 0 \end{pmatrix} \xrightarrow[R_3 \rightarrow R_3 - R_2]{R_3 \rightarrow R_3 - iR_1} \begin{pmatrix} 1 & 0 & i \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad (3.2.2.4)$$

$$\epsilon = c \begin{pmatrix} -i \\ -1 \\ 1 \end{pmatrix} \quad (3.2.2.5)$$

Hence it holds the condition of singularity therefore  $\mathbf{T}$  is not invertible .

3.2.3. For the linear operator  $\mathbf{T}$

$$\mathbf{T} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3x_1 \\ x_1 - x_2 \\ 2x_1 + x_2 + x_3 \end{pmatrix} \quad (3.2.3.1)$$

3.2.4. Let  $\mathbf{T}$  be a linear operator on  $\mathbf{R}^3$  defined by

$$\mathbf{T} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3x_1 \\ x_1 - x_2 \\ 2x_1 + x_2 + x_3 \end{pmatrix}$$

Is  $\mathbf{T}$  invertible? If so, find a rule for  $\mathbf{T}^{-1}$  like the one which defines  $\mathbf{T}$ .

**Solution:** The transformed vector can be re-written by expanding the columns as follows

$$\begin{pmatrix} 3x_1 \\ x_1 - x_2 \\ 2x_1 + x_2 + x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} x_1 + \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} x_2 + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} x_3 \quad (3.2.4.1)$$

$$= \begin{pmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad (3.2.4.2)$$

$$\implies \mathbf{T} = \begin{pmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & 1 & 1 \end{pmatrix} \quad (3.2.4.3)$$

Using Gauss-Jordan Elimination to find the inverse of  $\mathbf{T}$ , if it exists

$$\left( \begin{array}{ccc|ccc} 3 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{array} \right) \quad (3.2.4.4)$$

$$\xleftrightarrow{R_1 \leftarrow \frac{R_1}{3}} \left( \begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{array} \right) \quad (3.2.4.5)$$

$$\xleftrightarrow[R_3 \leftarrow R_3 - 2R_1]{R_2 \leftarrow R_2 - R_1} \left( \begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & -1 & 0 & -\frac{1}{3} & 1 & 0 \\ 0 & 1 & 1 & -\frac{2}{3} & 0 & 1 \end{array} \right) \quad (3.2.4.6)$$

$$\xleftrightarrow{R_2 \leftarrow -R_2} \left( \begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{3} & -1 & 0 \\ 0 & 1 & 1 & -\frac{2}{3} & 0 & 1 \end{array} \right) \quad (3.2.4.7)$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - R_2} \left( \begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{3} & -1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array} \right) \quad (3.2.4.8)$$

Since  $\text{rank}(\mathbf{T}) = 3$ ,  $\mathbf{T}$  is invertible and the

inverse is

$$\mathbf{T}^{-1} = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & -1 & 0 \\ -1 & 1 & 1 \end{pmatrix} \quad (3.2.4.9)$$

Now consider any vector  $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbf{R}^3$ , then

$$\mathbf{T}^{-1} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & -1 & 0 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad (3.2.4.10)$$

$$= \begin{pmatrix} \frac{x_1}{3} \\ \frac{x_1}{3} - x_2 \\ -x_1 + x_2 + x_3 \end{pmatrix} \quad (3.2.4.11)$$

Therefore the transformation  $\mathbf{T}^{-1}$  is defined on  $\mathbf{R}^3$  as

$$\mathbf{T}^{-1} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{x_1}{3} \\ \frac{x_1}{3} - x_2 \\ -x_1 + x_2 + x_3 \end{pmatrix} \quad (3.2.4.12)$$

Prove that

$$(\mathbf{T}^2 - I)(\mathbf{T} - 3I) = 0 \quad (3.2.4.13)$$

**Solution:** Expressing (3.2.3.1) in matrix form

$$\mathbf{T} = \begin{pmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & 1 & 1 \end{pmatrix} \quad (3.2.4.14)$$

The characteristic equation of  $\mathbf{T}$  is given as follows,

$$|\mathbf{T} - \lambda \mathbf{I}| = \begin{vmatrix} 3 - \lambda & 0 & 0 \\ 1 & -1 - \lambda & 0 \\ 2 & 1 & 1 - \lambda \end{vmatrix} = 0 \quad (3.2.4.15)$$

$$\begin{aligned} \implies (3 - \lambda)(-1 - \lambda)(1 - \lambda) &= 0 \\ \implies (\lambda - 3)(1 + \lambda)(1 - \lambda) &= 0 \\ \implies (\lambda - 3)(1 - \lambda^2) &= 0 \\ \implies (\lambda^2 - 1)(\lambda - 3) &= 0 \end{aligned} \quad (3.2.4.16)$$

By the Cayley-Hamilton theorem, We can write (3.2.4.16) as

$$(\mathbf{T}^2 - I)(\mathbf{T} - 3I) = 0 \quad (3.2.4.17)$$

3.2.5. Let  $\mathbb{C}$  be the complex vector space of  $2 \times 2$

matrices with complex entries. Let

$$\mathbf{B} = \begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix} \quad (3.2.5.1)$$

and let  $\mathbf{T}$  be the linear operator on  $\mathbb{C}^{2 \times 2}$  defined by  $\mathbf{T}(\mathbf{A}) = \mathbf{BA}$ . What is the rank of  $\mathbf{T}$ ? Can you describe  $\mathbf{T}^2$ ?

**Solution:** An ordered basis for  $\mathbb{C}^{2 \times 2}$  is given by

$$\mathbf{A}_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \mathbf{A}_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (3.2.5.2)$$

$$\mathbf{A}_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \mathbf{A}_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (3.2.5.3)$$

Now, we compute

$$\mathbf{T}(\mathbf{A}_{11}) = \mathbf{BA}_{11} \quad (3.2.5.4)$$

$$= \begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (3.2.5.5)$$

$$= \begin{pmatrix} 1 & 0 \\ -4 & 0 \end{pmatrix} \quad (3.2.5.6)$$

from (3.2.5.6) we have

$$\mathbf{T}(\mathbf{A}_{11}) = \mathbf{A}_{11} - 4\mathbf{A}_{21} \quad (3.2.5.7)$$

$$\mathbf{T}(\mathbf{A}_{12}) = \mathbf{BA}_{12} \quad (3.2.5.8)$$

$$= \begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (3.2.5.9)$$

$$= \begin{pmatrix} 0 & 1 \\ 0 & -4 \end{pmatrix} \quad (3.2.5.10)$$

from (3.2.5.10), we have

$$\mathbf{T}(\mathbf{A}_{12}) = \mathbf{A}_{12} - 4\mathbf{A}_{22} \quad (3.2.5.11)$$

$$\mathbf{T}(\mathbf{A}_{21}) = \mathbf{BA}_{21} \quad (3.2.5.12)$$

$$= \begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad (3.2.5.13)$$

$$= \begin{pmatrix} -1 & 0 \\ 4 & 0 \end{pmatrix} \quad (3.2.5.14)$$

from (3.2.5.14), we have

$$\mathbf{T}(\mathbf{A}_{21}) = -\mathbf{A}_{11} + 4\mathbf{A}_{21} \quad (3.2.5.15)$$

$$\mathbf{T}(\mathbf{A}_{22}) = \mathbf{B}\mathbf{A}_{22} \quad (3.2.5.16)$$

$$= \begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (3.2.5.17)$$

$$= \begin{pmatrix} 0 & -1 \\ 0 & 4 \end{pmatrix} \quad (3.2.5.18)$$

from (3.2.5.18), we have

$$\mathbf{T}(\mathbf{A}_{22}) = -\mathbf{A}_{12} + 4\mathbf{A}_{22} \quad (3.2.5.19)$$

Now, by (3.2.5.7), (3.2.5.11), (3.2.5.15) and (3.2.5.19) we write matrix of the linear transformation as follows

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -4 & 0 & 4 & 0 \\ 0 & -4 & 0 & 4 \end{pmatrix} \quad (3.2.5.20)$$

Also, we know that the rank of a linear transformation is same as the rank of the matrix of the linear transformation. Thus, we find the rank of matrix  $\mathbf{P}$ .

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -4 & 0 & 4 & 0 \\ 0 & -4 & 0 & 4 \end{pmatrix} \xleftrightarrow{r_3=r_3+4r_1} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 4 \end{pmatrix} \quad (3.2.5.21)$$

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 4 \end{pmatrix} \xleftrightarrow{r_4=r_4+4r_1} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (3.2.5.22)$$

from (3.2.5.22), we found out that  $\text{rank}(\mathbf{T}) = 2$ . Now, we compute

$$\mathbf{T}^2(\mathbf{A}) = \mathbf{T}(\mathbf{T}(\mathbf{A})) \quad (3.2.5.23)$$

$$= \mathbf{T}(\mathbf{B}\mathbf{A}) \quad (3.2.5.24)$$

$$= \mathbf{B}^2\mathbf{A} \quad (3.2.5.25)$$

where

$$\mathbf{B}^2 = \begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix} \quad (3.2.5.26)$$

$$= \begin{pmatrix} 5 & -5 \\ -20 & 20 \end{pmatrix} \quad (3.2.5.27)$$

3.2.6. Let  $T$  be a linear transformation from  $\mathbb{R}^3$  into  $\mathbb{R}^2$ , and let  $U$  be a linear transformation from  $\mathbb{R}^2$  into  $\mathbb{R}^3$ . Prove that the transformation  $UT$  is not invertible. Generalize the theorem.

**Solution:** Let  $\mathbf{v}, \mathbf{x} \in \mathbb{R}^3$  and  $\mathbf{w} \in \mathbb{R}^2$ . Table 3.2.6.1 shows that maximum rank the transformation matrix  $\mathbf{C}$  can have is 2.

$$\text{Rank}(\mathbf{C}) = 2 \quad (3.2.6.1)$$

$$\dim(\mathbf{C}) = 3 \quad (3.2.6.2)$$

$$\implies \text{Rank}(\mathbf{C}) < \dim(\mathbf{C}) \quad (3.2.6.3)$$

Therefore from the equation (3.2.6.3), we can say transformation  $UT$  is not invertible. Generalizing the proof, for  $n > m$  and considering vectors  $\mathbf{v}, \mathbf{x} \in \mathbb{R}^n$  and  $\mathbf{w} \in \mathbb{R}^m$ . From the Table 3.2.6.2,

$$\text{Rank}(\mathbf{C}) = m \quad (3.2.6.4)$$

$$\dim(\mathbf{C}) = n \quad (3.2.6.5)$$

$$\implies \text{Rank}(\mathbf{C}) < \dim(\mathbf{C}) \quad (3.2.6.6)$$

From equation (3.2.6.6) we can say that the transformation  $UT$  is not invertible. Let the vectors  $\mathbf{v} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \in \mathbb{R}^3$  and  $\mathbf{w} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} \in \mathbb{R}^2$ .

a) Calculating transformation matrix  $\mathbf{A}$ ,

$$T(\mathbf{v}) = \mathbf{A}\mathbf{v} \quad (3.2.6.7)$$

$$\begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \quad (3.2.6.8)$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \text{rref}(\mathbf{A}) \quad (3.2.6.9)$$

$$\implies \text{Rank}(\mathbf{A}) = 2 \quad (3.2.6.10)$$

b) Calculating transformation matrix  $\mathbf{B}$ ,

$$U(\mathbf{w}) = \mathbf{B}\mathbf{w} \quad (3.2.6.11)$$

$$\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & 2 \\ 1 & -1 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix} \quad (3.2.6.12)$$

$$\begin{pmatrix} \frac{3}{4} & 2 \\ 1 & -1 \\ 0 & \frac{1}{2} \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \text{rref}(\mathbf{B}) \quad (3.2.6.13)$$

$$\implies \text{Rank}(\mathbf{B}) = 2 \quad (3.2.6.14)$$

c) Now for the transformation  $UT$ , calculating

| Transformation                               | Matrix Representation  | Dimension                 | Max Rank of transformation matrix  |
|--|--|---------------------------|--|
| $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  | $T(\mathbf{v}) = \mathbf{A}\mathbf{v}$   | $\mathbf{A} : 2 \times 3$ | $\text{Rank}(\mathbf{A}) = 2$  |
| $U : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  | $U(\mathbf{w}) = \mathbf{B}\mathbf{w}$   | $\mathbf{B} : 3 \times 2$ | $\text{Rank}(\mathbf{B}) = 2$  |
| $UT : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ | $UT(\mathbf{x}) = \mathbf{C}\mathbf{x}$<br>$U(T(\mathbf{x})) = \mathbf{B}(\mathbf{A}\mathbf{x})$<br>$\mathbf{C} = \mathbf{AB}$ | $\mathbf{C} : 3 \times 3$ | $\text{Rank}(\mathbf{C}) \leq \min(\text{Rank}(\mathbf{B}), \text{Rank}(\mathbf{A}))$<br>$\text{Rank}(\mathbf{C}) = 2$ |

TABLE 3.2.6.1: Proof for non-invertibility of the transformation  $UT$  where  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  and  $U : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ 

| Transformation                               | Matrix Representation  | Dimension                 | Max Rank of transformation matrix  |
|--|--|---------------------------|--|
| $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  | $T(\mathbf{v}) = \mathbf{A}\mathbf{v}$   | $\mathbf{A} : m \times n$ | $\text{Rank}(\mathbf{A}) = m$  |
| $U : \mathbb{R}^m \rightarrow \mathbb{R}^n$  | $U(\mathbf{w}) = \mathbf{B}\mathbf{w}$   | $\mathbf{B} : n \times m$ | $\text{Rank}(\mathbf{B}) = m$  |
| $UT : \mathbb{R}^n \rightarrow \mathbb{R}^n$ | $UT(\mathbf{x}) = \mathbf{C}\mathbf{x}$<br>$U(T(\mathbf{x})) = \mathbf{B}(\mathbf{A}\mathbf{x})$<br>$\mathbf{C} = \mathbf{AB}$ | $\mathbf{C} : n \times n$ | $\text{Rank}(\mathbf{C}) \leq \min(\text{Rank}(\mathbf{B}), \text{Rank}(\mathbf{A}))$<br>$\text{Rank}(\mathbf{C}) = m$ |

TABLE 3.2.6.2: Generalization of the proof

the transformation matrix  $\mathbf{C}$ ,

$$UT : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \quad (3.2.6.15)$$

$$\implies UT(\mathbf{x}) = \mathbf{C}\mathbf{x} \quad (3.2.6.16)$$

$$U(T(\mathbf{x})) = \mathbf{B}(\mathbf{A}\mathbf{x}) \quad (3.2.6.17)$$

$$\implies \mathbf{C} = \mathbf{BA} \quad (3.2.6.18)$$

$$\mathbf{C} = \begin{pmatrix} \frac{3}{4} & 2 \\ 1 & -1 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & 0 & \frac{3}{4} \\ 0 & 0 & 2 \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix} \quad (3.2.6.19)$$

$$\begin{pmatrix} \frac{3}{4} & 0 & \frac{3}{4} \\ 0 & 0 & 2 \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \text{rref}(\mathbf{C}) \quad (3.2.6.20)$$

$$\implies \text{Rank}(\mathbf{C}) = 2 \quad (3.2.6.21)$$

$$\dim(\mathbf{C}) = 3 \quad (3.2.6.22)$$

As  $\text{Rank}(\mathbf{C}) < \dim(\mathbf{C})$ , transformation  $UT$  is not invertible.

3.2.7. Find two linear operators  $\mathbf{T}$  and  $\mathbf{U}$  on  $\mathbf{R}^2$  such that  $\mathbf{TU} = 0$  but  $\mathbf{UT} \neq 0$

**Solution:** Let,

$$\mathbf{x}, \mathbf{y} \in \mathbf{R}^2 \quad (3.2.7.1)$$

Let  $\mathbf{T}$  and  $\mathbf{U}$  be given by the matrices

$$\mathbf{T}(\mathbf{x}) = \mathbf{A}\mathbf{x}; \quad \mathbf{U}(\mathbf{x}) = \mathbf{B}\mathbf{x} \quad (3.2.7.2)$$

where,

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \quad \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (3.2.7.3)$$

$$\mathbf{T}(a\mathbf{x} + \mathbf{y}) = a\mathbf{T}\mathbf{x} + \mathbf{T}\mathbf{y} \quad (3.2.7.4)$$

$$\mathbf{U}(a\mathbf{x} + \mathbf{y}) = a\mathbf{U}\mathbf{x} + \mathbf{U}\mathbf{y} \quad (3.2.7.5)$$

From (3.2.7.4) and (3.2.7.5), we can tell that  $\mathbf{T}$  and  $\mathbf{U}$  are linear operators. Now,

$$\mathbf{TU} = \mathbf{AB} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbf{0} \quad (3.2.7.6)$$

$$\mathbf{UT} = \mathbf{BA} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq \mathbf{0} \quad (3.2.7.7)$$

From (3.2.7.6) and (3.2.7.7) it can be observed that  $\mathbf{TU} = 0$  but  $\mathbf{UT} \neq 0$

Let  $\mathbf{V}$  be a vector space over the field  $\mathbf{F}$  and  $\mathbf{T}$  is a linear operator on  $\mathbf{V}$ . If  $\mathbf{T}^2 = 0$ , what can you say about the relation of the range of  $\mathbf{T}$  to the null space of  $\mathbf{T}$ ? Give an example of linear operator  $\mathbf{T}$  on  $\mathbf{R}^2$  such that  $\mathbf{T}^2 = 0$  but  $\mathbf{T} \neq 0$ .



**Solution:** Given,

$$\mathbf{T} : \mathbf{V} \rightarrow \mathbf{V} \quad (3.2.8.1)$$

Now,  $\mathbf{T}^2$  is also a linear operator as,

$$\mathbf{T}^2(c\alpha) = \mathbf{T}(\mathbf{T}(c\alpha)) = \mathbf{T}(c\mathbf{T}(\alpha)) \quad (3.2.8.2)$$

$$= c\mathbf{T}(\mathbf{T}(\alpha)) = c\mathbf{T}^2(\alpha) \quad (3.2.8.3)$$

Let some vector  $\mathbf{y} \in \text{Range}(\mathbf{T})$  then there exists  $\mathbf{x} \in \mathbf{V}$  such that,

$$\mathbf{T}(\mathbf{x}) = \mathbf{y} \quad (3.2.8.4)$$

Now given that,

$$\mathbf{T}^2(\mathbf{x}) = \mathbf{0} \quad (3.2.8.5)$$

$$\implies \mathbf{T}(\mathbf{T}(\mathbf{x})) = \mathbf{0} \quad (3.2.8.6)$$

$$\mathbf{T}(\mathbf{y}) = \mathbf{0} \quad (3.2.8.7)$$

$\therefore \mathbf{y}$  lies in the Null space of  $\mathbf{T}$ . Hence  $\mathbf{T}$  is singular. Thus, the range of  $\mathbf{T}$  must be contained in Null space of  $\mathbf{T}$  i.e.,  $\text{Range}(\mathbf{T}) \subseteq \text{NullSpace}(\mathbf{T})$

*Example:*

$$\mathbf{T} : \mathbf{R}^2 \rightarrow \mathbf{R}^2 \quad (3.2.8.8)$$

Consider,

$$\mathbf{T}(\mathbf{x}) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mathbf{x} \quad (3.2.8.9)$$

$$\implies \mathbf{T} \neq \mathbf{0} \quad (3.2.8.10)$$

Now,

$$\mathbf{T}^2 : \mathbf{R}^2 \rightarrow \mathbf{R}^2 \quad (3.2.8.11)$$

$$\mathbf{T}^2(\mathbf{x}) = \mathbf{T}(\mathbf{T}(\mathbf{x})) \quad (3.2.8.12)$$

$$= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mathbf{x} = \mathbf{0} \quad (3.2.8.13)$$

$$\implies \mathbf{T}^2(\mathbf{x}) = \mathbf{0} \quad (3.2.8.14)$$

Thus  $\mathbf{T}^2$  is a zero transformation,  $\mathbf{T}^2 = \mathbf{0}$ .

Now, Kernel of  $\mathbf{T}$  is given by,

$$\mathbf{T}(\mathbf{x}) = \mathbf{0} \quad (3.2.8.15)$$

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (3.2.8.16)$$

$$\implies x = 0 \quad (3.2.8.17)$$

Thus,

$$\text{Ker}(\mathbf{T}) = y \begin{pmatrix} 0 \\ 1 \end{pmatrix}; y \in \mathbf{R} \quad (3.2.8.18)$$

Now,

$$\text{Range}(\mathbf{T}) = \text{ColumnSpace} \left\{ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\} \quad (3.2.8.19)$$

$$= k \begin{pmatrix} 0 \\ 1 \end{pmatrix}; k \in \mathbf{R} \quad (3.2.8.20)$$

Thus for the example,  $\text{Range}(\mathbf{T}) = \text{Kernel}(\mathbf{T})$  and from (3.2.8.10), (3.2.8.14) it is clear that  $\mathbf{T}^2 = \mathbf{0}$  but  $\mathbf{T} \neq \mathbf{0}$ .

3.2.9. Let  $\mathbf{T}$  be a linear operator on the finite-dimensional space  $\mathbf{V}$ . Suppose there is a linear operator  $\mathbf{U}$  on  $\mathbf{V}$  such that  $\mathbf{T}\mathbf{U} = \mathbf{I}$ . Prove that  $\mathbf{T}$  is invertible and  $\mathbf{U} = \mathbf{T}^{-1}$ . Give an example which shows that this is false when  $\mathbf{V}$  is not finite-dimensional.

**Solution:** Theorem

**Theorem 3.1.** Let  $f$  be a function from  $X$  into  $Y$ . We say that  $f$  is invertible if there is a function  $g$  from  $Y$  to  $X$  such that

- $g \circ f$  is the identity function on  $X$  i.e.  $g \circ f = \mathbf{I}$ . Here,  $g$  will be onto and  $f$  will be one-one.
- $f \circ g$  is the identity function on  $Y$  i.e.  $f \circ g = \mathbf{I}$ . Here,  $f$  will be onto and  $g$  will be one-one.

**Theorem 3.2.** Let  $V$  and  $W$  be finite dimensional vector spaces such that  $\dim V = \dim W$ . If  $T$  is a linear transformation from  $V$  into  $W$ , then the following are equivalent:

- $T$  is non-singular
- $T$  is onto

If any of the above two condition is satisfied then  $T$  is invertible.

- We are given  $\mathbf{V}$  which is a finite dimensional vector space, with the following linear operators defined as:-

$$\mathbf{T} : \mathbf{V} \rightarrow \mathbf{V} \quad (3.2.9.1)$$

$$\mathbf{U} : \mathbf{V} \rightarrow \mathbf{V} \quad (3.2.9.2)$$

The linear operators also satisfies the condition

$$\mathbf{T}\mathbf{U} = \mathbf{I} \quad (3.2.9.3)$$

Where  $\mathbf{I}$  is an Identity transformation. This identity transformation can be written as

$$\mathbf{I} : \mathbf{V} \rightarrow \mathbf{V} \quad (3.2.9.4)$$

$$\Rightarrow \mathbf{TU} : \mathbf{V} \rightarrow \mathbf{V} \quad (3.2.9.5)$$

$$\Rightarrow \mathbf{T}[\mathbf{U}(\mathbf{V})] = \mathbf{V} \quad (3.2.9.6)$$

From theorem (3.1) we can say that  $\mathbf{U}$  must be one-one and  $\mathbf{V}$  must be onto.

From theorem (3.2) we can say that  $\mathbf{T}$  is invertible.

Now we know that

$$\mathbf{TT}^{-1} = \mathbf{I} \quad (3.2.9.7)$$

Comparing (3.2.9.3) and (3.2.9.7) we get

$$\mathbf{TT}^{-1} = \mathbf{I} = \mathbf{TU} \quad (3.2.9.8)$$

Multiply both sides with  $\mathbf{T}^{-1}$

$$\mathbf{T}^{-1}(\mathbf{TT}^{-1}) = \mathbf{T}^{-1}(\mathbf{TU}) \quad (3.2.9.9)$$

$$\mathbf{T}^{-1}\mathbf{I} = (\mathbf{T}^{-1}\mathbf{T})\mathbf{U} \quad (3.2.9.10)$$

$$\mathbf{T}^{-1} = \mathbf{IU} \quad (3.2.9.11)$$

$$\therefore \mathbf{T}^{-1} = \mathbf{U} \quad (3.2.9.12)$$

- b) Let  $\mathbf{D}$  be a differential operator  $\mathbf{D} : \mathbf{V} \rightarrow \mathbf{V}$ , where  $\mathbf{V}$  is a space of polynomial functions in one variable over  $\mathbf{R}$ .

$$\begin{aligned} \mathbf{D}(c_0 + c_1x + \dots + c_nx^n) &= c_1 + c_2x + \dots \\ &\quad + c_nx^{n-1} \end{aligned} \quad (3.2.9.13)$$

And  $\mathbf{U} : \mathbf{V} \rightarrow \mathbf{V}$  be linear operator such that

$$\begin{aligned} \mathbf{U}(c_0 + c_1x + \dots + c_nx^n) &= c_0x + \frac{c_1x^2}{2} + \dots \\ &\quad + \frac{c_nx^{n+1}}{n+1} \end{aligned} \quad (3.2.9.14)$$

Therefore, the linear operator  $\mathbf{UD} : \mathbf{V} \rightarrow \mathbf{V}$

will be  $\mathbf{UD}(c_0 + c_1x + \dots + c_nx^n)$

$$\begin{aligned} &= \mathbf{U}[\mathbf{D}(c_0 + c_1x + \dots + c_nx^n)] \\ &= \mathbf{U}\left[c_1 + c_2x + \dots + c_nx^{n-1}\right] \\ &= c_1x + \frac{c_2x^2}{2} + \dots + \frac{c_nx^n}{n} \\ &= c_1x + c_2x^2 + \dots + c_nx^n \\ &\neq \mathbf{I} \end{aligned} \quad (3.2.9.15)$$

Now, the linear operator  $\mathbf{DU} : \mathbf{V} \rightarrow \mathbf{V}$  will be  $\mathbf{DU}(c_0 + c_1x + \dots + c_nx^n)$

$$\begin{aligned} &= \mathbf{D}[\mathbf{U}(c_0 + c_1x + \dots + c_nx^n)] \\ &= \mathbf{D}\left[c_0x + \frac{c_1x^2}{2} + \dots + \frac{c_nx^{n+1}}{n+1}\right] \\ &= c_0 + \frac{2c_1x}{2} + \dots + \frac{(n+1)c_nx^n}{n+1} \\ &= c_0 + c_1x + c_2x^2 + \dots + c_nx^n \\ &= \mathbf{I} \end{aligned} \quad (3.2.9.16)$$

From (3.2.9.15) and (3.2.9.16) we see that  $\mathbf{DU} = \mathbf{I}$ , but  $\mathbf{UD} \neq \mathbf{I}$ .

3.2.10. Let  $\mathbf{A}$  be an  $m \times n$  matrix with entries in  $F$  and let  $T$  be the linear transformation from  $F^{n \times 1}$  into  $F^{m \times 1}$  defined by  $T(\mathbf{X}) = \mathbf{AX}$ . Show that

- if  $m < n$  it may happen that  $T$  is onto without being non-singular
- if  $m > n$  we may have  $T$  non-singular but not onto.

**Solution:** Proof

- a)  $m < n$

$$\text{Let, } T : \mathbb{R}^3 \rightarrow \mathbb{R}^2 \quad (3.2.10.1)$$

$$T(\mathbf{X}) = \mathbf{AX} = \mathbf{b} \quad (3.2.10.2)$$

$$\text{Let, } \mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad (3.2.10.3)$$

$$\text{Consider, } \mathbf{X} = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} \quad (3.2.10.4)$$

$$\Rightarrow \mathbf{AX} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} \quad (3.2.10.5)$$

$$= \begin{pmatrix} 6 \\ 5 \end{pmatrix} \quad (3.2.10.6)$$

|              |   |
|--------------|---|
| singular     | <p>A linear transformation <math>T : \mathbb{R}^n \rightarrow \mathbb{R}^n</math> is said to be singular if <math>\exists</math> some non-zero <math>\mathbf{X} \in \mathbb{R}^n</math> s.t <math>\mathbf{AX} = \mathbf{0}</math> i.e <math>\text{Nullity}(A) \neq 0</math>.</p> <p>From rank-nullity theorem we can say <math>\text{rank}(A) &lt; n</math></p>   |
| non-singular | <p>A linear transformation <math>T : \mathbb{R}^n \rightarrow \mathbb{R}^m</math> is said to be non-singular if <math>\mathbf{AX} = \mathbf{0}</math> implies <math>\mathbf{X} = \mathbf{0}</math> i.e <math>\text{Nullity}(A) = 0</math></p>   |
| onto         | <p>A linear transformation <math>T : \mathbb{R}^n \rightarrow \mathbb{R}^m</math>, <math>m \leq n</math> is said to be onto if for every <math>\mathbf{b} \in \mathbb{R}^m</math>, <math>\mathbf{AX} = \mathbf{b}</math> has atleast one solution <math>\mathbf{X} \in \mathbb{R}^n</math>.</p> <p>i.e <math>\dim(\text{Col}(\mathbf{A})) = m</math> or <math>\text{Rank}(\mathbf{A}) = m</math></p> <p>If <math>m &gt; n</math>, then <math>\mathbf{AX} = \mathbf{b}</math> has no solution because rank-nullity theorem is not satisfied.</p> |

TABLE 3.2.10.1

Hence T is onto.

$$\text{Consider, } \mathbf{X} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad (3.2.10.7)$$

$$\Rightarrow \mathbf{AX} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad (3.2.10.8)$$

$$= \mathbf{0} \quad (3.2.10.9)$$

Since  $\exists \mathbf{X} \neq \mathbf{0}$  such that  $\mathbf{AX} = \mathbf{0}$ , T is singular.

$\therefore$  T is both onto and singular.

| Let $\mathbf{A}$ be an $m \times n$ matrix with entries in $F$ and<br>let $T$ be the linear transformation from $F^{n \times 1}$ into $F^{m \times 1}$<br>defined by $T(\mathbf{X}) = \mathbf{AX}$ . If, |   |  |
|--|---|--|
|  | $m < n$   | $m > n$  |
| singular   | Since $\text{rank}(\mathbf{A}) < n$ , by definition T is singular | Consider an non-singular $T$ such that $\text{rank}(\mathbf{A}) > n$ |
| onto   | Since $m < n$ , by definition $T$ can be onto                     | Since $m > n$ , by definition $T$ is not onto.                       |

TABLE 3.2.10.2

|  |   |
|--|---|
| Given  | $\mathbf{T} : \mathbf{V} \rightarrow \mathbf{V}$ be a linear operator.<br>$\text{Rank}(T^2) = \text{Rank}(T)$   |
| Rank of $T$ and $T^2$  | Let $(e_1, e_2, \dots, e_n)$ be a basis for $T$ , then $\text{Rank}(T)$ is linearly independent vectors in the set $(Te_1, Te_2, \dots, Te_n)$<br>Let, $\text{Rank}(T) = r = \text{Rank}(T^2)$  |
| Rank Nullity Theorem   | If $\text{Rank}(T) = r$ then $(Te_1, Te_2, \dots, Te_r)$ is the basis of range $T$ .<br>Similarly for $T^2$ , $(T^2e_1, T^2e_2, \dots, T^2e_r)$ is the basis of range $T^2$   |
| $\mathbf{v} \in \text{range}(T)$<br>$\mathbf{v} \in \text{nullspace}(T)$ | $\mathbf{v} = c_1Te_1 + c_2Te_2 + \dots + c_rTe_r$<br>$T(\mathbf{v}) = 0$<br>$T(c_1Te_1 + c_2Te_2 + \dots + c_rTe_r) = 0$<br>$c_1T^2e_1 + c_2T^2e_2 + \dots + c_rT^2e_r = 0$<br>But, $(T^2e_1, T^2e_2, \dots, T^2e_r)$ is the basis of range $T^2$<br>So, $c_1 = c_2 = \dots = c_r = 0$<br>Substituting these in $\mathbf{v}$ we get $\mathbf{v} = 0$ |
| Conclusion   | Hence from above it can be seen that when $\mathbf{v}$ belongs to both $\text{range}(T)$ and $\text{nullspace}(T)$ then $\mathbf{v}$ is a zero vector.<br>Hence, $\text{range}(T)$ and $\text{nullspace}(T)$ are disjoint.  |

TABLE 3.2.11.1

b)  $m > n$ 

$$\text{Let, } T : \mathbb{R}^3 \rightarrow \mathbb{R}^2 \quad (3.2.10.10)$$

$$T(\mathbf{X}) = \mathbf{A}\mathbf{X} = \mathbf{b} \quad (3.2.10.11)$$

$$\text{Let, } \mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (3.2.10.12)$$

$$\text{Consider, } \mathbf{X} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad (3.2.10.13)$$

$$\Rightarrow \mathbf{A}\mathbf{X} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad (3.2.10.14)$$

$$= \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \quad (3.2.10.15)$$

$$(3.2.10.16)$$

$\therefore T$  is not onto, and is also non-singular.

3.2.11. Let  $V$  be a finite-dimensional vector space and let  $T$  be a linear operator on  $V$ . Suppose that  $\text{rank}(T^2) = \text{rank}(T)$ . Prove that the range and null space of  $T$  are disjoint, i.e., have only the

zero vector in common. **Solution:** See Table 3.2.11.1.

3.2.12. Let  $p, m, n$  be positive integers and  $\mathbb{F}$  a field. Let  $\mathbf{V}$  be the space of  $m \times n$  matrices over  $\mathbb{F}$  and  $\mathbf{W}$  the space of  $p \times n$  matrices over  $\mathbb{F}$ . Let  $\mathbf{B}$  be a fixed  $p \times m$  matrix and let  $\mathbb{T}$  be the linear transformation from  $\mathbf{V}$  into  $\mathbf{W}$  defined by  $\mathbb{T}(\mathbf{A}) = \mathbf{B}\mathbf{A}$ . Prove that  $\mathbb{T}$  is invertible if and only if  $p = m$  and  $\mathbf{B}$  is an invertible  $m \times m$  matrix. **Solution:**

| Parameter  | Description                                      |
|--|--|
| $p, m, n$  | Positive integers                                |
| $\mathbb{F}$   | Field  |
| $\mathbf{V}$   | Space of $m \times n$ matrices over $\mathbb{F}$ |
| $\mathbf{W}$   | Space of $p \times n$ matrices over $\mathbb{F}$ |
| $\mathbf{B}$   | Fixed $p \times m$ matrix                        |
| Linear transformation $\mathbb{T} : \mathbf{V} \rightarrow \mathbf{W}$ | $\mathbb{T}(\mathbf{A}) = \mathbf{B}\mathbf{A}$  |

TABLE 3.2.12.1: Input Parameters

$$\mathbb{T}(\mathbf{A}) = \mathbf{BA} \quad (3.2.12.1)$$

So,  $\mathbf{B}$  is the transformation matrix.

$\mathbf{B}$  is invertible if

a)  $\mathbb{T}$  is one to one mapping, that is

$$\mathbf{BA} = \mathbf{BA}' \quad (3.2.12.2)$$

$$\implies \mathbf{A} = \mathbf{A}' \quad (3.2.12.3)$$

b)  $\mathbb{T}$  must be onto, that is  $\text{range}(\mathbf{B}) = \mathbf{W}$

Case 1: Let us assume that  $\mathbb{T}$  is invertible with inverse transformation  $\mathbb{T}_1$  from  $\mathbf{W}$  to  $\mathbf{V}$  that satisfies

$$\mathbb{T}(\mathbf{A}) = \mathbf{BA} \in \mathbf{W} \quad (3.2.12.4)$$

$$\implies \mathbb{T}_1(\mathbf{BA}) = \mathbf{A} \in \mathbf{V} \quad (3.2.12.5)$$

$$\dim(\mathbf{V}) = mn, \dim(\mathbf{W}) = pn \quad (3.2.12.6)$$

Since  $\mathbb{T}$  is one-one mapping, the zero vector in  $\mathbf{V}$ ,  $\mathbf{0}_{m \times n}$  is uniquely mapped to

$$\mathbb{T}(\mathbf{0}_{m \times n}) = \mathbf{B0}_{m \times n} = \mathbf{0}_{p \times n} \quad (3.2.12.7)$$

$$\text{So, } \mathbf{BA} = \mathbf{0} \iff \mathbf{A} = \mathbf{0} \quad (3.2.12.8)$$

Let  $\{\mathbf{V}_1, \mathbf{V}_2, \dots, \mathbf{V}_{mn}\}$  be the basis for  $\mathbf{V}$

$$c_1 \mathbf{V}_1 + c_2 \mathbf{V}_2 + \dots + c_{mn} \mathbf{V}_{mn} = \mathbf{0} \quad (3.2.12.9)$$

$$\iff c_1, c_2, \dots, c_{mn} \in \mathbb{F} = 0 \quad (3.2.12.10)$$

Any matrix  $\mathbf{A} \in \mathbf{V}$  can be written as

$$\mathbf{A} = \sum_{i=1}^{mn} \alpha_i \mathbf{V}_i \quad (3.2.12.11)$$

Since  $\mathbb{T}$  is onto, any matrix  $\mathbf{C} \in \mathbf{W}$  can be expressed as

$$\mathbf{C} = \mathbf{B} \left( \sum_{i=1}^{mn} \alpha_i \mathbf{V}_i \right) \quad (3.2.12.12)$$

$$= \sum_{i=1}^{mn} \alpha_i (\mathbf{BV}_i) \quad (3.2.12.13)$$

So, the set  $\mathbf{S} = \{\mathbf{BV}_1, \mathbf{BV}_2, \dots, \mathbf{BV}_{mn}\}$  forms basis of  $\mathbf{W}$  if all matrices in it are linearly

independent.

$$c_1(\mathbf{BV}_1) + c_2(\mathbf{BV}_2) + \dots + c_{mn}(\mathbf{BV}_{mn}) = \mathbf{0} \quad (3.2.12.14)$$

$$\mathbf{B}(c_1 \mathbf{V}_1 + c_2 \mathbf{V}_2 + \dots + c_{mn} \mathbf{V}_{mn}) = \mathbf{0} \quad (3.2.12.15)$$

$$(3.2.12.8) \implies c_1 \mathbf{V}_1 + \dots + c_{mn} \mathbf{V}_{mn} = \mathbf{0} \quad (3.2.12.16)$$

$$\iff c_1, c_2, \dots, c_{mn} = 0 \text{ (from (3.2.12.10))} \quad (3.2.12.17)$$

So, the set  $\mathbf{S}$  with cardinality  $mn$  is basis for  $\mathbf{W}$

$$(3.2.12.6) \implies pn = mn \quad (3.2.12.18)$$

$$p = m \quad (3.2.12.19)$$

(3.2.12.8), (3.2.12.19) prove that  $\mathbf{B}$  is invertible  $m \times m$  matrix. Case 2: Consider  $p = m$  and  $\mathbf{B}$  is an invertible  $m \times m$  matrix.

Verifying if  $\mathbb{T}$  is onto,

Let the set of matrices  $\{\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_{mn}\}$  be the basis for  $\mathbf{V}$

Any matrix  $\mathbf{A} \in \mathbf{V}$  can be written as

$$\mathbf{A} = \sum_{i=1}^{mn} \alpha_i \mathbf{A}_i \quad (3.2.12.20)$$

where  $\alpha_i \in \mathbb{F}$

The set  $\mathbf{M} = \{\mathbf{BA}_1, \mathbf{BA}_2, \dots, \mathbf{BA}_{mn}\}$  lie in  $\mathbf{W}$

$$c_1(\mathbf{BA}_1) + c_2(\mathbf{BA}_2) + \dots + c_{mn}(\mathbf{BA}_{mn}) = \mathbf{0} \quad (3.2.12.21)$$

$$\implies \mathbf{B}(c_1 \mathbf{A}_1 + c_2 \mathbf{A}_2 + \dots + c_{mn} \mathbf{A}_{mn}) = \mathbf{0} \quad (3.2.12.22)$$

Since  $\mathbf{B}$  is non-singular,

$$(c_1 \mathbf{A}_1 + c_2 \mathbf{A}_2 + \dots + c_{mn} \mathbf{A}_{mn}) = \mathbf{0} \quad (3.2.12.23)$$

$$\iff c_1, c_2, \dots, c_{mn} = 0 \quad (3.2.12.24)$$

because  $\{\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_{mn}\}$  are linearly independent

So,  $\mathbf{M}$  forms basis for  $\mathbf{W}$

Any vector  $\mathbf{C} \in \mathbf{W}$  can be written as

$$\mathbf{C} = \sum_{i=1}^{mn} \beta_i \mathbf{BA}_i \text{ where } \beta_i \in \mathbb{F} \quad (3.2.12.25)$$

$$= \mathbf{B} \left( \sum_{i=1}^{mn} \beta_i \mathbf{A}_i \right) \quad (3.2.12.26)$$

$$= \mathbf{BA} \text{ (from (3.2.12.20))} \quad (3.2.12.27)$$

So,  $\text{range}(\mathbf{B}) = \mathbf{W}$

Consider the matrix  $\mathbf{A}, \mathbf{A}' \in \mathbf{V}$  such that

$$\mathbf{BA} = \mathbf{BA}' \quad (3.2.12.28)$$

$$\mathbf{B}^{-1}(\mathbf{BA}) = \mathbf{B}^{-1}(\mathbf{BA}') \quad (3.2.12.29)$$

$$(\mathbf{B}^{-1}\mathbf{B})\mathbf{A} = (\mathbf{B}^{-1}\mathbf{B})\mathbf{A}' \quad (3.2.12.30)$$

$$\Rightarrow \mathbf{A} = \mathbf{A}' \quad (3.2.12.31)$$

So,  $\mathbb{T}$  is invertible. Conclusion: From case 1, case 2  $\mathbb{T}$  is invertible if and only if  $p = m$  and  $\mathbf{B}$  is an invertible  $m \times m$  matrix. Example: Let  $p = m = 3, n = 4$  Let  $\mathbb{T} : \mathbf{V} \rightarrow \mathbf{W}$  adds row 2 to row 3 for a matrix  $\mathbf{A} \in \mathbf{V}$

The elementary matrix that performs this is

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad (3.2.12.32)$$

$$\text{Let } \mathbf{A} = \begin{pmatrix} 1 & 2 & 2 & 5 \\ 1 & 3 & 6 & 7 \\ 4 & 9 & 2 & 6 \end{pmatrix} \quad (3.2.12.33)$$

$$\mathbb{T}(\mathbf{A}) = \mathbf{BA} \quad (3.2.12.34)$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 2 & 5 \\ 1 & 3 & 6 & 7 \\ 4 & 9 & 2 & 6 \end{pmatrix} \quad (3.2.12.35)$$

$$= \begin{pmatrix} 1 & 2 & 2 & 5 \\ 1 & 3 & 6 & 7 \\ 5 & 12 & 8 & 13 \end{pmatrix} \quad (3.2.12.36)$$

$$= \mathbf{C} \in \mathbf{W} \quad (3.2.12.37)$$

Let transformation  $\mathbb{T}_1 : \mathbf{W} \rightarrow \mathbf{V}$  subtracts row 2 from row 3 for a matrix  $\mathbf{C} \in \mathbf{W}$  and is

performed by elementary matrix

$$\mathbf{U} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \quad (3.2.12.38)$$

$$\text{Let } \mathbf{C} = \begin{pmatrix} 1 & 2 & 2 & 5 \\ 1 & 3 & 6 & 7 \\ 5 & 12 & 8 & 13 \end{pmatrix} \quad (3.2.12.39)$$

$$\mathbb{T}_1(\mathbf{C}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 2 & 5 \\ 1 & 3 & 6 & 7 \\ 5 & 12 & 8 & 13 \end{pmatrix} \quad (3.2.12.40)$$

$$= \begin{pmatrix} 1 & 2 & 2 & 5 \\ 1 & 3 & 6 & 7 \\ 4 & 9 & 2 & 6 \end{pmatrix} \quad (3.2.12.41)$$

$$= \mathbf{A} \quad (3.2.12.42)$$

$$\Rightarrow \mathbb{T}_1(\mathbf{C}) = \mathbf{A} \quad (3.2.12.43)$$

$$\mathbb{T}_1(\mathbb{T}(\mathbf{A})) = \mathbf{A} \quad (3.2.12.44)$$

$$\text{and } \mathbb{T}(\mathbf{A}) = \mathbf{C} \quad (3.2.12.45)$$

$$\Rightarrow \mathbb{T}(\mathbb{T}_1(\mathbf{C})) = \mathbf{C} \quad (3.2.12.46)$$

So,  $\mathbb{T}_1$  is the inverse transformation of  $\mathbb{T}$  and

$$\mathbb{T}_1 = \mathbb{T}^{-1} \quad (3.2.12.47)$$

$$\mathbf{UB} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad (3.2.12.48)$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (3.2.12.49)$$

$$\mathbf{BU} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \quad (3.2.12.50)$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (3.2.12.51)$$

$$\Rightarrow \mathbf{B}^{-1} = \mathbf{U} \quad (3.2.12.52)$$

So,  $\mathbb{T}$  is invertible and  $\mathbf{B}$  is an invertible  $3 \times 3$  matrix.

### 3.3 Isomorphism

3.3.1. Let  $\mathbf{V}$  be the set of complex numbers and let  $\mathbf{F}$  be the field of real numbers. With the usual operations,  $\mathbf{V}$  is a vector space over  $\mathbf{F}$ . Describe explicitly an isomorphism of this space onto  $\mathbb{R}^2$ .

**Solution:** Let,

$$\mathbf{T} : \mathbf{V} \rightarrow \mathbb{R}^2 \quad (3.3.1.1)$$

$$\mathbf{T}(x + iy) = \begin{pmatrix} x \\ y \end{pmatrix} \quad (3.3.1.2)$$

$$x, y \in \mathbb{R} \quad i \in \mathbb{C} \quad (3.3.1.3)$$

Consider two vectors,

$$\mathbf{u}, \mathbf{v} \in \mathbf{V} \quad (3.3.1.4)$$

Using (3.3.1.2) if  $\mathbf{T}$  is Linear Transformation.

$$\mathbf{T}(\mathbf{u} + c\mathbf{v}) = \mathbf{T}(\mathbf{u}) + \mathbf{T}(c\mathbf{v}) = \mathbf{T}(\mathbf{u}) + c\mathbf{T}(\mathbf{v}) \quad (3.3.1.5)$$

Hence this is a Linear transformation. Now, checking if  $\mathbf{T}$  is one-one. let,

$$\mathbf{u} = 0 = 0 + j0 \quad (3.3.1.6)$$

$$(3.3.1.7)$$

from (3.3.1.2),

$$\mathbf{T}(\mathbf{u}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (3.3.1.8)$$

From (3.3.1.8) we see  $\mathbf{T}$  is one-one. Now, checking if  $\mathbf{T}$  is onto. let,

$$\mathbf{T}(\mathbf{u}) = \begin{pmatrix} a \\ b \end{pmatrix} \quad (3.3.1.9)$$

where,  $a, b$  are scalars. Using (3.3.1.2), we see there exists a solution for (3.3.1.8) for all  $a, b$ . Hence  $\mathbf{T}$  is onto. Therefore  $\mathbf{T}$  is isomorphic over  $\mathbf{V}$  and  $\mathbb{R}^2$

3.3.2. Let  $\mathbf{V}$  be a vector space over the field of complex numbers, and suppose there is an isomorphism  $\mathbf{T}$  of  $\mathbf{V}$  onto  $C^3$ . Let  $\alpha_1, \alpha_2,$

$\alpha_3, \alpha_4$  be vectors in  $\mathbf{V}$  such that

$$\begin{aligned} T(\alpha_1) &= \begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}, T(\alpha_2) = \begin{pmatrix} -2 \\ 1+i \\ 0 \end{pmatrix}, \\ T(\alpha_3) &= \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, T(\alpha_4) = \begin{pmatrix} \sqrt{2} \\ i \\ 3 \end{pmatrix} \end{aligned} \quad (3.3.2.1)$$

a) Is  $\alpha_1$  in the subspace spanned by  $\alpha_2$  and  $\alpha_3$ ?

**Solution:**  $\mathbf{T} : \mathbf{V} \rightarrow \mathbf{W}$  is an isomorphism if

(1)  $\mathbf{T}$  is one one.

(2)  $\mathbf{T}$  is onto.

$$\begin{pmatrix} 1 & -2 & -1 & \sqrt{2} \\ 0 & 1+i & 1 & i \\ i & 0 & 1 & 3 \end{pmatrix} \xrightarrow{\text{ref}} \begin{pmatrix} 1 & 0 & -i & -3i \\ 0 & 2 & 1-i & i+1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (3.3.2.2)$$

$\mathbf{T}$  is one one over  $C^3$  if

$$T(\alpha) = 0 \implies \alpha = 0 \quad (3.3.2.3)$$

now,

$$\begin{pmatrix} 1 & -2 & \sqrt{2} \\ 0 & 1+i & i \\ i & 0 & 3 \end{pmatrix} \alpha = 0 \quad (3.3.2.4)$$

consider the row reduced matrix

$$\begin{pmatrix} 1 & -2 & \sqrt{2} \\ 0 & 1+i & i \\ i & 0 & 3 \end{pmatrix} \xleftrightarrow[R_3 \rightarrow iR_3]{R_3 \rightarrow R_3 - iR_1} \begin{pmatrix} 1 & -2 & \sqrt{2} \\ 0 & 1+i & i \\ 0 & -2 & \sqrt{2} + 3i \end{pmatrix} \quad (3.3.2.5)$$

$$\xleftrightarrow[R_3 \leftarrow R_3 + R_2]{R_2 \leftarrow (1-i)R_2} \begin{pmatrix} 1 & -2 & \sqrt{2} \\ 0 & 2 & i+1 \\ 0 & 0 & \sqrt{2} + 4i + 1 \end{pmatrix} \quad (3.3.2.6)$$

$$\alpha = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (3.3.2.7)$$

Therefore it holds the condition of one one and the rank = no. of pivot columns = 3 (equal to no. of columns). Thus the vectors are linearly independent hence it is onto. Since  $\mathbf{T}$  is an isomorphism onto  $C^3$ .

$$T(\alpha_1) = c_1 T(\alpha_2) + c_2 T(\alpha_3) \quad (3.3.2.8)$$

$c_1$  and  $c_2$  are scalar.

$$\begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix} = c_1 \begin{pmatrix} -2 \\ 1+i \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \quad (3.3.2.9)$$

$$\begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix} = \begin{pmatrix} -2 & -1 \\ 1+i & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \quad (3.3.2.10)$$

Now we find  $c_i$  by row reducing augmented matrix.

$$\begin{pmatrix} -2 & -1 & 1 \\ 1+i & 1 & 0 \\ 0 & 1 & i \end{pmatrix} \xrightarrow[R_2 \rightarrow R_3]{R_1 \rightarrow -R_1/2} \begin{pmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & i \\ 1+i & 1 & 0 \end{pmatrix} \quad (3.3.2.11)$$

$$\xrightarrow[R_3 \leftarrow R_3 - (1+i)R_1]{R_1 \leftarrow R_1 - R_2/2} \begin{pmatrix} 1 & 0 & \frac{-1-i}{2} \\ 0 & 1 & i \\ 0 & \frac{1-i}{2} & \frac{1+i}{2} \end{pmatrix} \quad (3.3.2.12)$$

$$\xrightarrow{R_3 \leftarrow R_3 - (1-i)/2 R_2} \begin{pmatrix} 1 & 0 & \frac{-1-i}{2} \\ 0 & 1 & i \\ 0 & 0 & 0 \end{pmatrix} \quad (3.3.2.13)$$

Therefore the coordinate matrix of the vector is

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \frac{-1-i}{2} \\ i \end{pmatrix} \quad (3.3.2.14)$$

substituting the  $c_i$  in (3.3.2.8)

$$T(\alpha_1) = -\frac{1+i}{2}T(\alpha_2) + iT(\alpha_3) \quad (3.3.2.15)$$

Hence  $\alpha_1$  belongs to the subspace spanned by  $\alpha_2$  and  $\alpha_3$ .

- b) Let  $V$  be a vector space over the field of complex numbers, and suppose there is an isomorphism  $T$  of  $V$  onto  $C^3$ . Let  $\alpha_1, \alpha_2, \alpha_3, \alpha_4$  be vectors in  $V$  such that

$$T(\alpha_1) = \begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}, T(\alpha_2) = \begin{pmatrix} -2 \\ 1+i \\ 0 \end{pmatrix}, \quad (3.3.2.16)$$

$$T(\alpha_3) = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, T(\alpha_4) = \begin{pmatrix} \sqrt{2} \\ i \\ 3 \end{pmatrix}$$

Let  $W_1$  be the subspace spanned by  $\alpha_1$  and  $\alpha_2$ , and let  $W_2$  be the subspace spanned by  $\alpha_3$  and  $\alpha_4$ . What is the intersection of  $W_1$

and  $W_2$ .

**Solution:**  $T : V \rightarrow W$  is an isomorphism if

(1)  $T$  is one one.

(2)  $T$  is onto. If  $W_1$  and  $W_2$  are finite-dimensional subspaces of a vector space  $V$ , then  $W_1 + W_2$  is finite-dimensional and

$$\dim(W_1) + \dim(W_2) = \dim(W_1 \cap W_2) + \dim(W_1 + W_2) \quad (3.3.2.17)$$

We have,

$$T = \begin{pmatrix} 1 & -2 & -1 & \sqrt{2} \\ 0 & 1+i & 1 & i \\ i & 0 & 1 & 3 \end{pmatrix} \quad (3.3.2.18)$$

$$\begin{pmatrix} 1 & -2 & -1 & \sqrt{2} \\ 0 & 1+i & 1 & i \\ i & 0 & 1 & 3 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & 0 & -i & 0 \\ 0 & 1 & \frac{1-i}{2} & 0 \\ 0 & 0 & \frac{1-i}{2} & 1 \end{pmatrix} \quad (3.3.2.19)$$

$T$  is one one over  $C^3$  if

$$T(\alpha) = 0 \implies \alpha = 0 \quad (3.3.2.20)$$

now,

$$\begin{pmatrix} 1 & -2 & \sqrt{2} \\ 0 & 1+i & i \\ i & 0 & 3 \end{pmatrix} \alpha = 0 \quad (3.3.2.21)$$

consider the row reduced matrix

$$\begin{pmatrix} 1 & -2 & \sqrt{2} \\ 0 & 1+i & i \\ i & 0 & 3 \end{pmatrix} \xrightarrow[R_3 \rightarrow iR_3]{R_3 \rightarrow R_3 - iR_1} \begin{pmatrix} 1 & -2 & \sqrt{2} \\ 0 & 1+i & i \\ 0 & -2 & \sqrt{2} + 3i \end{pmatrix} \quad (3.3.2.22)$$

$$\xrightarrow[R_3 \leftarrow R_3 + R_2]{R_2 \leftarrow (1-i)R_2} \begin{pmatrix} 1 & -2 & \sqrt{2} \\ 0 & 2 & i+1 \\ 0 & 0 & \sqrt{2} + 4i + 1 \end{pmatrix} \quad (3.3.2.23)$$

$$\alpha = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (3.3.2.24)$$

Therefore it holds the condition of one one and the rank = no. of pivot columns = 3 (equal to no. of columns). Thus the vectors are linearly independent hence it is onto.



Hence,  $T$  is an isomorphism onto  $C^3$ .  
Also, from (3.3.2.19), we observe,

$$T(\alpha_3) = -iT(\alpha_1) + \frac{i-1}{2}T(\alpha_2) \quad (3.3.2.25)$$

Hence  $T(\alpha_3)$  belongs to the subspace spanned by  $T(\alpha_1)$  and  $T(\alpha_2)$ .

Therefore,  $\alpha_3$  is in subspace spanned by  $\alpha_1$  and  $\alpha_2$ . Therefore,

$$\alpha_3 \in W_1 \quad (3.3.2.26)$$

$$\Rightarrow \alpha_3 \in W_1 \cap W_2 \quad (3.3.2.27)$$

Since  $T(\alpha_1)$  and  $T(\alpha_2)$  are linearly independent, and  $T(\alpha_3)$  and  $T(\alpha_4)$  are linearly independent, we have,

$$\dim(W_1) = \dim(W_2) = 2 \quad (3.3.2.28)$$

From (3.3.2.17),

$$\dim(W_1) + \dim(W_2) = \dim(W_1 \cap W_2) + \dim(W_1 + W_2) \quad (3.3.2.29)$$

$$\dim(W_1 + W_2) = 3 \quad (3.3.2.30)$$

$$\Rightarrow \dim(W_1 \cap W_2) = 2 + 2 - 3 \quad (3.3.2.31)$$

$$\Rightarrow \dim(W_1 \cap W_2) = 1 \quad (3.3.2.32)$$

Therefore,

$$W_1 \cap W_2 = c\alpha_3 \quad (3.3.2.33)$$

- c) Find a basis for the subspace of  $V$  spanned by the 4 vectors  $\alpha_i$ .

**Solution:**  $V$  is a vector space and  $V$  is isomorphic to  $C^3$  via isomorphism  $T$  which implies that  $C^3$  is also isomorphic to  $V$  via isomorphism  $T^{-1}$ .

As  $V$  is isomorphic to  $C^3$ , so

$$\dim(V) = \dim(C^3) = 3 \quad (3.3.2.34)$$

Now,

$$\begin{aligned} & \begin{pmatrix} 1 & 0 & i \\ -2 & 1+i & 0 \\ -1 & 1 & 1 \\ \sqrt{2} & i & 3 \end{pmatrix} \xleftrightarrow[R_4 \leftarrow \sqrt{2}R_4]{R_3 \leftarrow R_3 + R_1} \begin{pmatrix} 1 & 0 & i \\ -2 & 1+i & 0 \\ 0 & 1 & 1+i \\ 2 & i\sqrt{2} & 3\sqrt{2} \end{pmatrix} \\ & \xleftrightarrow{R_4 \leftarrow R_4 + R_2} \begin{pmatrix} 1 & 0 & i \\ -2 & 1+i & 0 \\ 0 & 1 & 1+i \\ 0 & 1+i(1+\sqrt{2}) & 3\sqrt{2} \end{pmatrix} \\ & \xleftrightarrow{R_2 \leftarrow R_2 + 2R_1} \begin{pmatrix} 1 & 0 & i \\ 0 & 1+i & 2i \\ 0 & 1 & 1+i \\ 0 & 1+i(1+\sqrt{2}) & 3\sqrt{2} \end{pmatrix} \\ & \xleftrightarrow{R_3 \leftarrow (1+i)R_3} \begin{pmatrix} 1 & 0 & i \\ 0 & 1+i & 2i \\ 0 & 1+i & 2i \\ 0 & 1+i(1+\sqrt{2}) & 3\sqrt{2} \end{pmatrix} \\ & \xleftrightarrow{R_3 \leftarrow R_3 - R_2} \begin{pmatrix} 1 & 0 & i \\ 0 & 1+i & 2i \\ 0 & 0 & 0 \\ 0 & 1+i(1+\sqrt{2}) & 3\sqrt{2} \end{pmatrix} \quad (3.3.2.35) \end{aligned}$$

From here we can get that  $T\alpha_3$  is dependent vector while  $T\alpha_1, T\alpha_2$  and  $T\alpha_4$  are independent vector. These  $T\alpha_1, T\alpha_2$  and  $T\alpha_4$  also span the vector space  $C^3$ , so these 3 vectors are the basis of  $C^3$ .

As  $\dim(V) = 3$ , so it must have 3 basis and as  $V$  and  $C^3$  are isomorphic so  $\alpha_1, \alpha_2$  and  $\alpha_4$  are the basis of  $V$ .

3.3.3. Let  $\mathbb{W}$  be the set of all  $2 \times 2$  complex Hermitian matrices, that is the set of  $2 \times 2$  complex matrices  $\mathbf{A}$  such that  $\mathbf{A}_{ij} = \overline{\mathbf{A}_{ji}}$  (the bar denoting complex conjugation).  $\mathbb{W}$  is a vector space over the field of real numbers, under the usual operations. Verify that

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \rightarrow \begin{pmatrix} t+x & y+iz \\ y-iz & t-x \end{pmatrix} \quad (3.3.3.1)$$

is an isomorphism of  $\mathbb{R}^4$  onto  $\mathbb{W}$ .

**Solution:**

- a) **Check for linearity:** The transformation  $T$

is given by

$$T: \mathbb{R}^4 \rightarrow \mathbb{W} \quad (3.3.3.2)$$

$$T \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} t+x & y+iz \\ y-iz & t-x \end{pmatrix} \quad (3.3.3.3)$$

Let  $\mathbf{x} = \begin{pmatrix} x \\ y \\ t \\ z \end{pmatrix}$ . Expressing R.H.S of equation (3.3.3.3) using Kronecker Product,

$$T(\mathbf{x}) = \begin{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix} \mathbf{x} & \begin{pmatrix} 0 & 1 & i & 0 \end{pmatrix} \mathbf{x} \\ \begin{pmatrix} 0 & 1 & -i & 0 \end{pmatrix} \mathbf{x} & \begin{pmatrix} -1 & 0 & 0 & 1 \end{pmatrix} \mathbf{x} \end{pmatrix} \quad (3.3.3.4)$$

$$= \begin{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix} \mathbf{x} & \begin{pmatrix} 0 & 1 & i & 0 \end{pmatrix} \mathbf{x} \\ \begin{pmatrix} 0 & 1 & -i & 0 \end{pmatrix} \mathbf{x} & \begin{pmatrix} -1 & 0 & 0 & 1 \end{pmatrix} \mathbf{x} \end{pmatrix} \quad (3.3.3.5)$$

$$= \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & i & 0 \\ 0 & 1 & -i & 0 & -1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \\ 0 \\ x \\ 0 \\ y \\ 0 \\ z \\ 0 \\ t \end{pmatrix} \quad (3.3.3.6)$$

$$\Rightarrow T(\mathbf{x}) = (\mathbf{A} \ \mathbf{B}) \begin{pmatrix} \mathbf{x} & \mathbf{0}_{4 \times 1} \\ \mathbf{0}_{4 \times 1} & \mathbf{x} \end{pmatrix} \quad (3.3.3.7)$$

Where  $\mathbf{A}$  and  $\mathbf{B}$  are block matrices.

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -i & 0 \end{pmatrix} \quad (3.3.3.8)$$

$$\mathbf{B} = \begin{pmatrix} 0 & 1 & i & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \quad (3.3.3.9)$$

The Kronecker Product of  $\mathbf{I}_2$  and  $\mathbf{x}$  gives the block matrix in equation (3.3.3.7).

$$\mathbf{I}_{2 \times 2} \otimes \mathbf{x}_{4 \times 1} = \begin{pmatrix} \mathbf{x} & \mathbf{0} \\ \mathbf{0} & \mathbf{x} \end{pmatrix}_{8 \times 2} \quad (3.3.3.10)$$

Hence we can write equation (3.3.3.7) as,

$$T(\mathbf{x}) = (\mathbf{A} \ \mathbf{B})(\mathbf{I} \otimes \mathbf{x}) \quad (3.3.3.11)$$

Let  $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^4$  and  $\alpha, \beta \in \mathbb{R}$ .

$$T(\alpha \mathbf{x}_1 + \beta \mathbf{x}_2) = (\mathbf{A} \ \mathbf{B})(\mathbf{I} \otimes (\alpha \mathbf{x}_1 + \beta \mathbf{x}_2)) \quad (3.3.3.12)$$

$$= \alpha (\mathbf{A} \ \mathbf{B})(\mathbf{I} \otimes \mathbf{x}_1) + \beta (\mathbf{A} \ \mathbf{B})(\mathbf{I} \otimes \mathbf{x}_2) \quad (3.3.3.13)$$

$$= \alpha T\mathbf{x}_1 + \beta T\mathbf{x}_2 \quad (3.3.3.14)$$

Therefore from equation (3.3.3.14), we can say T is linear transformation.

b) **Check for one-one property:** For transformation T to be one-one, we can prove if  $T(\mathbf{x}) = \mathbf{0}$ , that implies  $\mathbf{x} = \mathbf{0}$ . From the equation (3.3.3.11),

$$T(\mathbf{x}) = \mathbf{0} \quad (3.3.3.15)$$

$$(\mathbf{A} \ \mathbf{B})(\mathbf{I} \otimes \mathbf{x}) = \mathbf{0} \quad (3.3.3.16)$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & i & 0 \\ 0 & 1 & -i & 0 & -1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \\ 0 \\ x \\ 0 \\ y \\ 0 \\ z \\ 0 \\ t \end{pmatrix} = \mathbf{0}_{2 \times 2} \quad (3.3.3.17)$$

From equation (3.3.3.3),

$$\begin{pmatrix} t+x & y+iz \\ y-iz & t-x \end{pmatrix} = \mathbf{0}_{2 \times 2} \quad (3.3.3.18)$$

$$\Rightarrow x = 0, y = 0, z = 0, t = 0 \quad (3.3.3.19)$$

$$\Rightarrow \mathbf{x} = \mathbf{0} \quad (3.3.3.20)$$

Hence from (3.3.3.15) and (3.3.3.20), T is one-one and that implies  $T: \mathbb{R}^4 \rightarrow \mathbb{W}$  is isomorphism.

3.3.4. Show that  $\mathbf{F}^{m \times n}$  is isomorphic to  $\mathbf{F}^{mn}$ .

**Solution:** See Tables 3.3.4.1, 3.3.4.2 and 3.3.4.3.

$\mathbb{R}^{2 \times 2}$  is isomorphic to  $\mathbb{R}^4$  ie,  $\mathbb{R}^{2 \times 2} \cong \mathbb{R}^4$ .

3.3.5. Let  $\mathbf{V}$  be the set of complex numbers regarded as a vector space over the field of real numbers. We define a function T from  $\mathbf{V}$  into the space of  $2 \times 2$  real matrices, as follows. If  $z = x + iy$

|                          |   |
|--------------------------|---|
| Invertible Linear Map    | <p>A linear map <math>\mathbf{T} \in \mathbf{L}(\mathbf{V}, \mathbf{W})</math> is called invertible if there exists a linear map <math>\mathbf{S} \in \mathbf{L}(\mathbf{W}, \mathbf{V})</math> such that <math>\mathbf{ST}</math> equals the identity map on <math>\mathbf{V}</math> and <math>\mathbf{TS}</math> equals the identity map on <math>\mathbf{W}</math>.</p> <p>A linear map <math>\mathbf{S} \in \mathbf{L}(\mathbf{W}, \mathbf{V})</math> satisfying <math>\mathbf{ST} = \mathbf{I}_\mathbf{V}</math> and <math>\mathbf{TS} = \mathbf{I}_\mathbf{W}</math> is called an inverse of <math>\mathbf{T}</math>.</p> |
| Isomorphic Vector Spaces | Two vector spaces $\mathbf{V}$ and $\mathbf{W}$ are called isomorphic if there is an isomorphism from one vector space onto the other one. An isomorphism is an invertible linear map.  |
| Rank Nullity Theorem     | Let $\mathbf{V}$ and $\mathbf{W}$ be finite dimensional vector spaces. Let $\mathbf{T}: \mathbf{V} \rightarrow \mathbf{W}$ be a linear transformation<br>$\text{Rank}(\mathbf{T}) + \text{Nullity}(\mathbf{T}) = \dim \mathbf{V}$   |

TABLE 3.3.4.1: Definition

|          |  |
|----------|--|
| Result 1 | The space of all $m \times n$ matrices over the field $\mathbf{F}$ has dimension $mn$ .  |
| Result 2 | Let $\mathbf{V}$ and $\mathbf{W}$ be finite-dimensional vector spaces over the field $\mathbf{F}$ such that $\dim \mathbf{V} = \dim \mathbf{W}$ . If $\mathbf{T}$ is a linear transformation from $\mathbf{V}$ into $\mathbf{W}$ , then the following are equivalent:<br>(a). $\mathbf{T}$ is invertible.<br>(b). $\mathbf{T}$ is non-singular.<br>(c). $\mathbf{T}$ is onto, that is, range of $\mathbf{T}$ is $\mathbf{W}$ . |

TABLE 3.3.4.2: Results Used

with  $x$  and  $y$  real numbers, then

$$\mathbf{T}(z) = \begin{pmatrix} x + 7y & 5y \\ -10y & x - 7y \end{pmatrix}$$

- a) Verify that  $\mathbf{T}$  is a one-one (real) linear transformation of  $\mathbf{V}$  into the space of  $2 \times 2$  real matrices.

**Solution:** The kronecker product also called as matrix direct product is defined as

$$\mathbf{A} \otimes \mathbf{B} = \begin{pmatrix} a_{11}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ \vdots & \ddots & \vdots \\ a_{m1}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{pmatrix} \quad (3.3.5.1)$$

Also,

$$\mathbf{A} \otimes (\mathbf{B} + \mathbf{C}) = \mathbf{A} \otimes \mathbf{B} + \mathbf{A} \otimes \mathbf{C} \quad (3.3.5.2)$$

$$\mathbf{A} \otimes (k\mathbf{B}) = k(\mathbf{A} \otimes \mathbf{B}) \quad (3.3.5.3)$$

Given,

$$\mathbf{T}: \mathbf{C} \rightarrow \mathbf{R}^{2 \times 2}$$

$$\mathbf{T}(x + iy) = \begin{pmatrix} x + 7y & 5y \\ -10y & x - 7y \end{pmatrix} \quad (3.3.5.4)$$

Let,

$$z = x + iy; \quad w = a + ib; \quad z, w \in \mathbf{C}$$

Also the RHS of (3.3.5.4) can be expressed as,

$$\begin{aligned} \mathbf{T}(z) &= \left( \begin{pmatrix} 1 & 7 \\ 0 & -10 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right) \quad \left( \begin{pmatrix} 0 & 5 \\ 1 & -7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right) \\ &= \begin{pmatrix} 1 & 7 & 0 & 5 \\ 0 & -10 & 1 & -7 \end{pmatrix} \begin{pmatrix} x & 0 \\ y & 0 \\ 0 & x \\ 0 & y \end{pmatrix} \\ &= (\mathbf{A} \quad \mathbf{B}) \begin{pmatrix} \mathbf{x} & 0 \\ 0 & \mathbf{x} \end{pmatrix} \quad (3.3.5.5) \end{aligned}$$

where  $\mathbf{A}$  and  $\mathbf{B}$  are block matrices and,

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$

The diagonal block matrix can be expressed as the kronecker product of  $\mathbf{I}$  and  $\mathbf{x}$

$$\mathbf{I} \otimes \mathbf{x} = \begin{pmatrix} \mathbf{x} & 0 \\ 0 & \mathbf{x} \end{pmatrix} \quad (3.3.5.6)$$

|                               |  |
|-------------------------------|--|
| Defining Sets                 | We define set $S$ and set $T$ as<br>$S = \{(a, b) : a, b \in \mathbb{N}, 1 \leq a \leq m, 1 \leq b \leq n\}, \quad T = \{1, 2, \dots, mn\}$  |
| Defining Bijection            | We now define a bijection $\sigma : S \rightarrow T$ as<br>$(a, b) \rightarrow (a-1)n + b$   |
| Defining Function $G$         | We now define a function $G$ from $F^{m \times n}$ to $F^{mn}$ as follows. Let $\mathbf{A} \in F^{m \times n}$ .<br>Then map $\mathbf{A}$ to the $mn$ tuple that has $\mathbf{A}_{ij}$ in the $\sigma(i, j)$ position. In other words,<br>$\mathbf{A} \rightarrow (\mathbf{A}_{11}, \mathbf{A}_{12}, \dots, \mathbf{A}_{1n}, \dots, \mathbf{A}_{m1}, \mathbf{A}_{m2}, \dots, \mathbf{A}_{mn})$   |
| Proving $G$ to be Linear      | Since, addition in $F^{m \times n}$ and in $F^{mn}$ is performed component-wise, $G(\mathbf{A} + \mathbf{B}) = G(\mathbf{A}) + G(\mathbf{B})$ and scalar multiplication in $F^{m \times n}$ and in $F^{mn}$ is also defined as $G(c\mathbf{A}) = cG(\mathbf{A})$ .   |
| Proving $G$ to be One-One     | $G(\mathbf{A}) = G(\mathbf{B})$<br>$\implies (\mathbf{A}_{11}, \mathbf{A}_{12}, \dots, \mathbf{A}_{1n}, \dots, \mathbf{A}_{m1}, \mathbf{A}_{m2}, \dots, \mathbf{A}_{mn}) = (\mathbf{B}_{11}, \mathbf{B}_{12}, \dots, \mathbf{B}_{1n}, \dots, \mathbf{B}_{m1}, \mathbf{B}_{m2}, \dots, \mathbf{B}_{mn})$<br>$\implies \mathbf{A}_{ij} = \mathbf{B}_{ij} \quad \forall 1 \leq i \leq m, 1 \leq j \leq n$<br>$\implies \mathbf{A} = \mathbf{B}$ |
| Proving $G$ to be Onto        | Since $G$ is one to one, so $\text{Null}(G) = 0$ . Thus, by Rank-Nullity Theorem $\dim(\text{Range}(G)) = mn$ , proving $G$ to be a surjective (onto) map as by Result 1 dimension of $F^{m \times n} = mn$  |
| $F^{m \times n} \cong F^{mn}$ | Since $G$ has an inverse and is an isomorphism of $\mathbf{T}$ . Thus, by Result 2<br>$F^{m \times n} \cong F^{mn}$  |

TABLE 3.3.4.3: Proof

Where  $\mathbf{I}$  is an identity matrix. (3.3.5.5) can be rewritten as,

$$\mathbf{T}(\mathbf{z}) = \begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} (\mathbf{I} \otimes \mathbf{x}) \quad (3.3.5.7)$$

Consider,

$$\mathbf{T}(\alpha \mathbf{z} + \mathbf{w}) = \begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} (\mathbf{I} \otimes (\alpha \mathbf{z} + \mathbf{w}))$$

Using properties (3.3.5.2), (3.3.5.3), the above equation can be expressed as,

$$\begin{aligned} \mathbf{T}(\alpha \mathbf{z} + \mathbf{w}) &= \begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} (\mathbf{I} \otimes (\alpha \mathbf{z})) + \begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} (\mathbf{I} \otimes \mathbf{w}) \\ &= \alpha \begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} (\mathbf{I} \otimes \mathbf{z}) + \begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} (\mathbf{I} \otimes \mathbf{w}) \\ &= \alpha \mathbf{T}(\mathbf{z}) + \mathbf{T}(\mathbf{w}) \end{aligned} \quad (3.3.5.8)$$

From (3.3.5.8), it can be proved that  $\mathbf{T}$  is a linear operator.

b) Verify that

$$\mathbf{T}(z_1 z_2) = \mathbf{T}(z_1) \mathbf{T}(z_2) \quad (3.3.5.9)$$

**Solution:** The product of two Kronecker products yields another Kronecker product:

$$(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = (\mathbf{AC}) \otimes (\mathbf{BD}) \quad (3.3.5.10)$$

Given,

$$\mathbf{T}(\mathbf{z}) = \begin{pmatrix} x + 7y & 5y \\ -10y & x - 7y \end{pmatrix} \quad (3.3.5.11)$$

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \quad (3.3.5.12)$$

$$\mathbf{T}(\mathbf{x}) = \begin{pmatrix} (1 \ 7)\mathbf{x} & (0 \ 5)\mathbf{x} \\ (0 \ -10)\mathbf{x} & (1 \ -7)\mathbf{x} \end{pmatrix} \quad (3.3.5.13)$$

$$= \left( \begin{pmatrix} 1 & 7 \\ 0 & -10 \end{pmatrix} \mathbf{x} \quad \begin{pmatrix} 0 & 5 \\ 1 & -7 \end{pmatrix} \mathbf{x} \right) \quad (3.3.5.14)$$

$$\text{Let } \mathbf{A} = \begin{pmatrix} 1 & 7 \\ 0 & -10 \end{pmatrix} \quad (3.3.5.15)$$

$$\mathbf{B} = \begin{pmatrix} 0 & 5 \\ 1 & -7 \end{pmatrix} \quad (3.3.5.16)$$

$$\Rightarrow \mathbf{T}(\mathbf{x}) = (\mathbf{A}\mathbf{x} \quad \mathbf{B}\mathbf{x}) \quad (3.3.5.17)$$

$$\mathbf{T}(\mathbf{x}) = (\mathbf{A} \quad \mathbf{B}) \begin{pmatrix} \mathbf{x} & \mathbf{0} \\ \mathbf{0} & \mathbf{x} \end{pmatrix} \quad (3.3.5.18)$$

The diagonal block matrix can be expressed as the kronecker product of  $\mathbf{I}$  and  $\mathbf{x}$

$$\mathbf{I} \otimes \mathbf{x} = \begin{pmatrix} \mathbf{x} & \mathbf{0} \\ \mathbf{0} & \mathbf{x} \end{pmatrix} \quad (3.3.5.19)$$

We can write (3.3.5.11) as

$$\mathbf{T}(\mathbf{x}) = (\mathbf{A} \quad \mathbf{B})(\mathbf{I} \otimes \mathbf{x}) \quad (3.3.5.20)$$

Starting with RHS of (3.3.5.10)

$$\mathbf{T}(\mathbf{z}_1)\mathbf{T}(\mathbf{z}_2) = (\mathbf{A} \quad \mathbf{B})(\mathbf{I} \otimes \mathbf{z}_1)(\mathbf{A} \quad \mathbf{B})(\mathbf{I} \otimes \mathbf{z}_2) \quad (3.3.5.21)$$

If

$$(\mathbf{I} \otimes \mathbf{z}_1)(\mathbf{A} \quad \mathbf{B})(\mathbf{I} \otimes \mathbf{z}_2) = (\mathbf{I} \otimes \mathbf{z}_1 \mathbf{z}_2) \quad (3.3.5.22)$$

then, we can write (3.3.5.21) as

$$\mathbf{T}(\mathbf{z}_1)\mathbf{T}(\mathbf{z}_2) = (\mathbf{A} \quad \mathbf{B})(\mathbf{I} \otimes \mathbf{z}_1 \mathbf{z}_2) = \mathbf{T}(\mathbf{z}_1 \mathbf{z}_2) \quad (3.3.5.23)$$

c) How would you describe the range of  $\mathbb{T}$ ?

**Solution:**

$$\mathbb{T} : \mathbf{V} \rightarrow \mathbb{R}^{2 \times 2} \quad (3.3.5.24)$$

where  $\mathbb{R}^{2 \times 2}$ , is the space of all  $2 \times 2$  real matrices

$$\mathbb{T}(\mathbf{z}) = \begin{pmatrix} x + 7y & 5y \\ -10y & x - 7y \end{pmatrix} \quad (3.3.5.25)$$

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \quad (3.3.5.26)$$

$$\mathbb{T}(\mathbf{x}) = \left( \begin{pmatrix} 1 & 7 \\ 0 & -10 \end{pmatrix} \mathbf{x} \quad \begin{pmatrix} 0 & 5 \\ 1 & -7 \end{pmatrix} \mathbf{x} \right) \quad (3.3.5.27)$$

$$= \left( \begin{pmatrix} 1 & 7 \\ 0 & -10 \end{pmatrix} \mathbf{x} \quad \begin{pmatrix} 0 & 5 \\ 1 & -7 \end{pmatrix} \mathbf{x} \right) \quad (3.3.5.28)$$

$$\text{Let } \mathbf{A} = \begin{pmatrix} 1 & 7 \\ 0 & -10 \end{pmatrix} \quad (3.3.5.29)$$

$$\mathbf{B} = \begin{pmatrix} 0 & 5 \\ 1 & -7 \end{pmatrix} \quad (3.3.5.30)$$

$$\Rightarrow \mathbb{T}(\mathbf{x}) = (\mathbf{A}\mathbf{x} \quad \mathbf{B}\mathbf{x}) \quad (3.3.5.31)$$

$$\mathbb{T}(\mathbf{x}) = (\mathbf{A} \quad \mathbf{B}) \begin{pmatrix} \mathbf{x} & \mathbf{0}_{2 \times 1} \\ \mathbf{0}_{2 \times 1} & \mathbf{x} \end{pmatrix} \quad (3.3.5.32)$$

The kronecker product of  $\mathbf{I}$ ,  $\mathbf{x}$  gives the block matrix

$$\mathbf{I}_{2 \times 2} \otimes \mathbf{x}_{2 \times 1} = \begin{pmatrix} \mathbf{x} & \mathbf{0} \\ \mathbf{0} & \mathbf{x} \end{pmatrix}_{4 \times 2} \quad (3.3.5.33)$$

$$(3.3.5.32) \Rightarrow \mathbb{T}(\mathbf{x}) = (\mathbf{A} \quad \mathbf{B})\mathbf{I} \otimes \mathbf{x} \quad (3.3.5.34)$$

$$= (\mathbf{A} \quad \mathbf{B})\mathbf{I} \otimes \left[ \begin{pmatrix} x \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ y \end{pmatrix} \right] \quad (3.3.5.35)$$

$$= x(\mathbf{A} \quad \mathbf{B})\mathbf{I} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y(\mathbf{A} \quad \mathbf{B})\mathbf{I} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (3.3.5.36)$$

$$\mathbf{I} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (3.3.5.37)$$

$$\mathbf{I} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (3.3.5.38)$$

Kronecker product in the first term of (3.3.5.36) picks out 1st columns of  $\mathbf{A}, \mathbf{B}$  and in the second term picks out 2nd columns of

$\mathbf{A}, \mathbf{B}$  so basis for  $\text{range}(\mathbb{T})$  is

$$\left\{ (\mathbf{A} \ \mathbf{B}) \mathbf{I} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}, (\mathbf{A} \ \mathbf{B}) \mathbf{I} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} = \quad (3.3.5.39)$$

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 7 & 5 \\ -10 & -7 \end{pmatrix} \right\} \quad 3.3.8. \quad (3.3.5.40)$$

$$\text{range}(\mathbb{T}) = \quad (3.3.5.41)$$

$$\text{span of } \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 7 & 5 \\ -10 & -7 \end{pmatrix} \right\} \quad (3.3.5.42)$$

3.3.6. Let  $\mathbf{V}$  and  $\mathbf{W}$  be finite-dimensional vector spaces over the field  $\mathbf{F}$ . Prove that  $\mathbf{V}$  and  $\mathbf{W}$  are isomorphic if and only if  $\dim \mathbf{V} = \dim \mathbf{W}$

**Solution:** Refer Table 3.3.6.1.

Therefore from equation (3.3.6.4) and (3.3.6.7) in Table 3.3.6.1,  $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \dots, T(\mathbf{v}_n)\}$  are linearly independent and span  $\mathbf{W}$ .

$$\implies \dim \mathbf{W} = n = \dim \mathbf{V}$$

$\mathbf{V}$  and  $\mathbf{W}$  are isomorphic if and only if  $\dim \mathbf{V} = \dim \mathbf{W}$

3.3.7. Let  $V$  and  $W$  be finite-dimensional vector spaces over the field  $F$  and let  $U$  be an isomorphism of  $V$  and  $W$ . Prove that  $T \rightarrow UTU^{-1}$  is an isomorphism of  $L(V, V)$  onto  $L(W, W)$ .

**Solution:**

Example Let

$$U = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad (3.3.7.1)$$

here  $U$  is an isomorphism from  $\mathbb{R}^{2 \times 2}$  to  $\mathbb{R}^{2 \times 2}$  since inverse of  $U$  exists and

$$U^{-1} = \begin{pmatrix} -2 & -\frac{3}{2} \\ -1 & -\frac{1}{2} \end{pmatrix} \quad (3.3.7.2)$$

Consider

$$T = \begin{pmatrix} -1 & 2 \\ 3 & 1 \end{pmatrix} \in \mathbb{R}^{2 \times 2} \quad (3.3.7.3)$$

Now

$$UTU^{-1} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} -2 & -\frac{3}{2} \\ -1 & -\frac{1}{2} \end{pmatrix} \quad (3.3.7.4)$$

$$= \begin{pmatrix} -16 & -7 \\ -33 & -14 \end{pmatrix} \in \mathbb{R}^{2 \times 2} \quad (3.3.7.5)$$

Also inverse exists for  $T$

$$S = T^{-1} = \begin{pmatrix} -\frac{1}{2} & \frac{2}{7} \\ \frac{3}{7} & \frac{1}{7} \end{pmatrix} \quad (3.3.7.6)$$

Since  $T$  inverse exists  $\mathcal{T}(T) = UTU^{-1}$  is an isomorphism from  $\mathbb{R}^{2 \times 2}$  onto  $\mathbb{R}^{2 \times 2}$ .

Let  $T$  be a linear operator on the finite-dimensional space  $\mathbb{V}$ . Suppose there is a linear operator  $U$  on  $\mathbb{V}$  such that  $TU = I$ . Prove that  $T$  is invertible and  $U = T^{-1}$ . Give an example which shows that this is false when  $\mathbb{V}$  is not finite-dimensional.

**Solution:** Let  $T : \mathbb{V} \rightarrow \mathbb{V}$  be a linear operator, where  $\mathbb{V}$  is a finite dimensional vectors space and  $U : \mathbb{V} \rightarrow \mathbb{V}$  is also a linear operator such that,

$$TU = I \quad (3.3.8.1)$$

Where,  $I$  is an identity transformation. Now we know that linear transformations are functions. Hence,

$$TU = I \text{ is a function} \quad (3.3.8.2)$$

$$\implies I : \mathbb{V} \rightarrow \mathbb{V} \quad (3.3.8.3)$$

Such that  $T(V) = V$ . Defining  $TU : \mathbb{V} \rightarrow \mathbb{V}$  to be a linear operator, we have,

$$T[U(V_i)] = V_i \quad [V_i \in \mathbb{V}] \quad (3.3.8.4)$$

Now we show in the below Table that  $T$  is one-one and onto as follows,

$$TT^{-1} = I \quad (3.3.8.5)$$

Hence we get from Table 3.3.8.1 that,  $T$  is invertible. Hence we get the following,

Where  $T^{-1}$  is an inverse function of linear operator  $T$ . Hence,

$$TT^{-1} = I = TU \quad (3.3.8.6)$$

$$\implies T^{-1}(TT^{-1}) = T^{-1}(TU) \quad (3.3.8.7)$$

$$\implies T^{-1}(I) = IU \quad (3.3.8.8)$$

$$\implies T^{-1} = U \quad (3.3.8.9)$$

Hence from (3.3.8.9) it is proven that  $T$  is invertible and  $T^{-1} = U$

**Example:** Let  $D$  be the differential operator  $D : \mathbb{V} \rightarrow \mathbb{V}$  where  $\mathbb{V}$  is a space of polynomial

| $T : \mathbb{V} \rightarrow \mathbb{W} \quad \mathbb{V} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\} \quad \mathbb{W} = \{T(\mathbf{v}_1), T(\mathbf{v}_2), \dots, T(\mathbf{v}_n)\}$ |   |
|--|---|
| Property Used  | Derivation  |
| T is one-one   | <p>Linear combination of vectors in <math>\mathbb{W}</math></p> $\sum_{k=1}^n \alpha_k T(\mathbf{v}_k) = \mathbf{0} \quad (3.3.6.1)$ $\sum_{k=1}^n T(\alpha_k \mathbf{v}_k) = \mathbf{0} \quad (3.3.6.2)$ $\Rightarrow \sum_{k=1}^n \alpha_k \mathbf{v}_k = \mathbf{0} \quad (3.3.6.3)$ $\Rightarrow \alpha_1 = \alpha_2 = \dots = \alpha_n = 0 \quad (3.3.6.4)$ <p>From equation (3.3.6.1) and (3.3.6.4), the set of vectors <math>\{T(\mathbf{v}_1), T(\mathbf{v}_2), \dots, T(\mathbf{v}_n)\}</math> are linearly independent</p>  |
| T is onto  | <p>For any <math>\mathbf{y} \in \mathbb{W}</math> there exists an <math>\mathbf{x} \in \mathbb{V}</math> such that <math>T(\mathbf{x}) = \mathbf{y}</math>.</p> $\mathbf{x} = \sum_{k=1}^n \alpha_k \mathbf{v}_k \quad (3.3.6.5)$ $T(\mathbf{x}) = T\left(\sum_{k=1}^n \alpha_k \mathbf{v}_k\right) = \mathbf{y} \quad (3.3.6.6)$ $\sum_{k=1}^n \alpha_k T(\mathbf{v}_k) = \mathbf{y} \quad (3.3.6.7)$ <p>From equation (3.3.6.7), any vector in <math>\mathbb{W}</math> can be represented as linear combination of <math>\{T(\mathbf{v}_1), T(\mathbf{v}_2), \dots, T(\mathbf{v}_n)\}</math>. That is it spans <math>\mathbb{W}</math>.</p> |

TABLE 3.3.6.1: Derivation

functions in one variable  $x$  over  $\mathbb{R}$  as follows,

$$D(c_0 + c_1x + \dots + c_nx^n) = c_1 + c_2'x + \dots + c_n'x^{n-1} \quad (3.3.8.10)$$

We first prove that the vector space  $\mathbb{V}$  is infinite dimensional.

Suppose to the contrary that  $\mathbb{V}$  is finite dimensional vector space and is given by the span of  $k$  polynomials in  $\mathbb{V}$  as follows,

$$\text{span}(\mathbb{V}) = \{p_1, p_2, \dots, p_k\} \quad (3.3.8.11)$$

Also let  $m$  be the maximum of the degree of these  $k$  polynomials in (3.3.8.11). Now let an element of the vector space  $\mathbb{V}$  be,

$$cx^{m+1} \in \mathbb{V} \quad (3.3.8.12)$$

As maximum degree of the basis of  $\mathbb{V}$  is  $m$  hence  $cx^{m+1}$  cannot be represented by any

linear combination of the basis of  $\mathbb{V}$ . If  $\mathbb{F}$  is field corresponding to  $\mathbb{V}$  then we have,

$$cx^{m+1} \neq \sum_{i=1}^k \alpha_i p_i \quad [\alpha_i \in \mathbb{F} \forall i] \quad (3.3.8.13)$$

Hence,  $cx^{m+1}$  is not in the span of  $p_1, p_2, \dots, p_k$ . Hence,  $\mathbb{V}$  is infinite dimensional vector space.

Next we prove that  $D$  is not one-one operator. Let, two different elements from the vector space  $\mathbb{V}$  be as follows,

$$c_1 + x^m \in \mathbb{V} \quad (3.3.8.14)$$

$$c_2 + x^m \in \mathbb{V} \quad (3.3.8.15)$$

From definition (3.3.8.10) of operator  $D$  we

|          |   |
|----------|---|
| Given    | $\mathcal{T}(T) : T \rightarrow UTU^{-1}$<br>$U$ is isomorphism of $V$ onto $W$<br>that means $U$ is <i>one – one</i><br>$\mathcal{T} : L(V, V) \rightarrow L(W, W)$  |
| To prove | $\mathcal{T}$ is isomorphism of $L(V, V)$<br>onto $L(W, W)$<br><br>It is same as proving<br>$\mathcal{T}$ is invertible, because<br><br><i>isomorphism</i> $\implies$ <i>one – one</i><br>$\implies$ <i>invertible</i> by definition  |
| Proof    | Consider inverse transformation<br>$\mathcal{S} : L(W, W) \rightarrow L(V, V)$<br>$\mathcal{S} : S \rightarrow U^{-1}SU$<br><br>where $U^{-1}SU$ is a composition<br>of 3 linear transformations<br>$V \xrightarrow{U} W \xrightarrow{S} W \xrightarrow{U^{-1}} V$<br><br>Now consider $\mathcal{S}(UTU^{-1})$ ,<br>$\mathcal{S}(UTU^{-1}) = U^{-1}(UTU^{-1})U = T$<br><br>Similarly consider $\mathcal{T}(U^{-1}SU)$ ,<br>$\mathcal{T}(U^{-1}SU) = U(U^{-1}SU)U^{-1} = S$<br><br>$\implies TS = I$ and $ST = I$<br><br>we can say $\mathcal{T}$ is invertible<br>since we have found an inverse $\mathcal{S}$<br><br>Hence $\mathcal{T}$ is one-one implies<br>$\mathcal{T}$ isomorphism of $V$ onto $W$ |

TABLE 3.3.7.1: Proof

have,

$$D(c_1 + x^m) = mx^{m-1} \quad (3.3.8.16)$$

$$D(c_2 + x^m) = mx^{m-1} \quad (3.3.8.17)$$

From (3.3.8.16) and (3.3.8.17),

$$c_1 + x^m \neq c_2 + x^m \quad (3.3.8.18)$$

$$D(c_1 + x^m) = D(c_2 + x^m) \quad (3.3.8.19)$$

Hence from (3.3.8.19) we see that  $D$  is not One-One operator.

And,  $U : \mathbb{V} \rightarrow \mathbb{V}$  is another linear operator such that,

$$U(c_0 + c_1x + \cdots + c_nx^n) = c_0x + c_1\frac{x^2}{2} + \cdots + c_n\frac{x^{n+1}}{n+1} \quad (3.3.8.20)$$

Now,  $DU : \mathbb{V} \rightarrow \mathbb{V}$  is a linear operator such that,

$$DU(c_0 + c_1x + \cdots + c_nx^n) \quad (3.3.8.21)$$

$$= D[U(c_0x + c_1\frac{x^2}{2} + \cdots + c_n\frac{x^{n+1}}{n+1})] \quad (3.3.8.22)$$

$$= D[c_0x + c_1\frac{x^2}{2} + \cdots + c_n\frac{x^{n+1}}{n+1}] \quad (3.3.8.23)$$

$$= c_0 + c_1\frac{2x}{2} + \cdots + c_n\frac{(n+1)x^n}{n+1} \quad (3.3.8.24)$$

$$= c_0 + c_1x + \cdots + c_nx^n \quad (3.3.8.25)$$

Hence, from (3.3.8.25),

$$DU = I \quad (3.3.8.26)$$

Again  $UD : \mathbb{V} \rightarrow \mathbb{V}$  is a linear operator such that,

$$UD(c_0 + c_1x + \cdots + c_nx^n) \quad (3.3.8.27)$$

$$= U[D(c_0x + c_1\frac{x^2}{2} + \cdots + c_n\frac{x^{n+1}}{n+1})] \quad (3.3.8.28)$$

$$= U[c_1 + c'_2x + \cdots + c'_nx^{n-1}] \quad (3.3.8.29)$$

$$= c_1x + c_2\frac{x^2}{2} + \cdots + c_n\frac{x^n}{n} \quad (3.3.8.30)$$

Hence, from (3.3.8.30),

$$UD \neq I \quad (3.3.8.31)$$

Hence, from (3.3.8.26) and (3.3.8.31),  $D$  is not invertible.

### 3.4 Representation of Transformations by Matrices

3.4.1. Let  $T$  be the linear operator on  $\mathbb{C}^2$  defined by  $T(x_1, x_2) = (x_1, 0)$ . Let  $\beta$  be the standard ordered basis for  $\mathbb{C}^2$  and  $\beta' = \{\alpha_1, \alpha_2\}$  be the ordered basis defined by  $\alpha_1 = (1, i)$ ,  $\alpha_2 = (-i, 2)$ .



|                       |  |
|-----------------------|--|
| linear transformation | <p>Let <math>V</math> and <math>W</math> be vector spaces over field <math>F</math>.</p> <p>A <b>linear transformation</b> <math>V</math> into <math>W</math> is a function <math>T</math> from <math>V</math> into <math>W</math> such that</p> $T(c\alpha + \beta) = c(T\alpha) + T\beta$ <p>for all <math>\alpha</math> and <math>\beta</math> in <math>V</math> and all scalars in <math>c</math> in <math>F</math>.</p>                               |
| isomorphism           | <p>If <math>V</math> and <math>W</math> are vector spaces over the field <math>F</math>, any <i>one – one</i> linear transformation <math>T : V \rightarrow W</math> is called <b>isomorphism of <math>V</math> onto <math>W</math></b></p>  |
| one-one               | <p>A linear transformation <math>T : \mathbb{R}^n \rightarrow \mathbb{R}^m</math> is said to be <b>one-one</b> if for every <math>\mathbf{b} \in \mathbb{R}^m</math>, <math>\mathbf{A}\mathbf{X} = \mathbf{b}</math> has atmost one solution in <math>\mathbb{R}^n</math>.</p> <p>Equivalently, if <math>T(\mathbf{u}) = T(\mathbf{v})</math>, then <math>u = v</math>.</p> <p>By definition, all <i>invertible</i> transformations are <b>one-one</b></p> |
| invertible            | <p>A linear transformation <math>T : V \rightarrow W</math> is <b>invertible</b> if there exists another linear transformation <math>U : W \rightarrow V</math> such that <math>UT</math> is the <i>identity</i> transformation on <math>V</math> and <math>TU</math> is the identity transformation on <math>W</math>.</p> <p><math>T</math> is <b>invertible</b> if and only if <math>T</math> is <i>one – one</i> and <i>onto</i></p>                   |

TABLE 3.3.7.2: Definitions

| Proof  | Conclusion              |
|--|-------------------------|
| Let $\mathbf{V}_1, \mathbf{V}_2 \in \mathbb{V}$ then,<br>If $\mathbf{V}_1 \neq \mathbf{V}_2$ then,<br>$T[U(\mathbf{V}_1)] \neq T[U(\mathbf{V}_2)]$ | $T$ is one-one function |
| $T$ is linear operator on<br>finite dimensional<br>vector space  | $T$ is onto function    |

TABLE 3.3.8.1: Proof of Invertibility of transformation

- a) What is the matrix of  $\mathbf{T}$  relative to the pair  $\beta, \beta'$ ?

**Solution:**

$$\beta = \{\epsilon_1, \epsilon_2\} \implies \epsilon_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \epsilon_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (3.4.1.1)$$

Hence,  $\beta$  as matrix

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (3.4.1.2)$$

$$\beta' = \{\alpha_1, \alpha_2\} \implies \alpha_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}, \alpha_2 = \begin{pmatrix} -i \\ 2 \end{pmatrix} \quad (3.4.1.3)$$

Hence,  $\beta'$  as matrix

$$\beta' = \begin{pmatrix} 1 & -i \\ i & 2 \end{pmatrix} \quad (3.4.1.4)$$

Geometrically,  $\mathbf{T}$  is a projection onto the x-axis.

For projection, let Consider Matrix  $\mathbf{A}$  as

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (3.4.1.5)$$

The matrix  $\mathbf{A}$  is representation of the linear transformation  $\mathbf{T}$  that is projection on x-axis. After applying linear operator  $\mathbf{T}$  on it,

$$\mathbf{T}(\beta) = \mathbf{A}\beta = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (3.4.1.6)$$

$$\begin{pmatrix} 1 & -i & 1 & 0 \\ i & 2 & 0 & 0 \end{pmatrix} \xrightarrow{R_2=R_2-iR_1} \begin{pmatrix} 1 & -i & 1 & 0 \\ 0 & 1 & -i & 0 \end{pmatrix} \quad (3.4.1.7)$$

$$\xrightarrow{R_1=R_1+iR_2} \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -i & 0 \end{pmatrix} \quad (3.4.1.8)$$

Therefore matrix of relative to the pair  $\beta$ ,  $\beta'$

$$\mathbf{T}(\beta) = \begin{pmatrix} 2 & 0 \\ -i & 0 \end{pmatrix} \beta' \quad (3.4.1.9)$$

b) What is the matrix of  $\mathbf{T}$  relative to the pair  $\beta$ ,  $\beta'$ ?

**Solution:** Transformation  $\mathbf{T}$  from  $C^2$  to  $C^2$ .  
Let

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (3.4.1.10)$$

$$\beta' = \begin{pmatrix} 1 & -i \\ i & 2 \end{pmatrix} \quad (3.4.1.11)$$

$\mathbf{T}$  is defined by

$$\mathbf{T}(\mathbf{x}) = \mathbf{A}\mathbf{x} \quad (3.4.1.12)$$

$$\mathbf{T}(\beta) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \beta \quad (3.4.1.13)$$

$$\mathbf{T}(\beta) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (3.4.1.14)$$

To find relative matrix we will use row reduce augmented matrix.

$$\begin{pmatrix} 1 & -i & 1 & 0 \\ i & 2 & 0 & 0 \end{pmatrix} \quad (3.4.1.15)$$

$$\begin{pmatrix} 1 & -i & 1 & 0 \\ i & 2 & 0 & 0 \end{pmatrix} \xrightarrow[R_1 \rightarrow R_1 + iR_2]{R_2 \rightarrow R_2 - iR_1} \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -i & 0 \end{pmatrix} \quad (3.4.1.16)$$

Hence the matrix of  $\mathbf{T}$  in the order basis of  $\beta'$

$$\mathbf{B} = \begin{pmatrix} 2 & 0 \\ -i & 0 \end{pmatrix} \quad (3.4.1.17)$$

Therefore matrix of relative to the pair  $\beta$ ,

$\beta'$

$$\mathbf{T}(\beta) = \mathbf{A}\beta = \mathbf{B}\beta' = \begin{pmatrix} 2 & 0 \\ -i & 0 \end{pmatrix} \beta' \quad (3.4.1.18)$$

c) What is the matrix of  $\mathbf{T}$  in the ordered basis  $\beta'$

**Solution:**

Let

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (3.4.1.19)$$

We have

$$\alpha_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}, \alpha_2 = \begin{pmatrix} -i \\ 2 \end{pmatrix} \quad (3.4.1.20)$$

So,

$$\beta' = \begin{pmatrix} 1 & -i \\ i & 2 \end{pmatrix} \quad (3.4.1.21)$$

Geometrically,  $\mathbf{T}$  is a projection onto the x-axis.

For projection, let Consider Matrix  $\mathbf{A}$  as

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (3.4.1.22)$$

The matrix  $\mathbf{A}$  is representation of the linear transformation  $\mathbf{T}$  that is projection on x-axis. After applying linear operator  $\mathbf{T}$  on it,

$$\mathbf{T}(\beta') = \mathbf{A}\beta' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -i \\ i & 2 \end{pmatrix} = \begin{pmatrix} 1 & -i \\ 0 & 0 \end{pmatrix} \quad (3.4.1.23)$$

Now, for finding the matrix of  $\mathbf{T}$  in the ordered basis  $\beta'$ , we combine the 3.4.1.20 and 3.4.1.23 and use concept of row-reduction of the augmented matrix:

$$\begin{pmatrix} 1 & -i & 1 & -i \\ i & 2 & 0 & 0 \end{pmatrix} \xrightarrow{R_2=R_2-iR_1} \begin{pmatrix} 1 & -i & 1 & -i \\ 0 & 1 & -i & -1 \end{pmatrix} \quad (3.4.1.24)$$

$$\xrightarrow{R_1=R_1+iR_2} \begin{pmatrix} 1 & 0 & 2 & -2i \\ 0 & 1 & -i & -1 \end{pmatrix} \quad (3.4.1.25)$$

Hence, the matrix of  $\mathbf{T}$  in the ordered basis

$\beta'$  is

$$\mathbf{B} = \begin{pmatrix} 2 & 2i \\ -i & -1 \end{pmatrix} \quad (3.4.1.26)$$

And this can also be represented using  $\beta$  and  $\beta'$ , which shows the relation between  $\mathbf{T}$ ,  $\beta$  and  $\beta'$ .

$$\mathbf{T}(\beta) = \mathbf{A}\beta = \mathbf{B}\beta' \quad (3.4.1.27)$$

- d) What is the matrix of  $\mathbf{T}$  in the ordered basis  $\{\alpha_2, \alpha_1\}$ ?

**Solution:** Transformation  $\mathbf{T}$  from  $\mathbb{C}^2$  to  $\mathbb{C}^2$ .  
Let

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (3.4.1.28)$$

$$\beta = (\mathbf{e}_1 \quad \mathbf{e}_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (3.4.1.29)$$

$$\beta' = (\alpha_2 \quad \alpha_1) = \begin{pmatrix} -i & 1 \\ 2 & i \end{pmatrix} \quad (3.4.1.30)$$

$\mathbf{T}$  in the ordered basis  $\beta$  is:

$$[\mathbf{T}]_\beta = \begin{pmatrix} -i & 1 \\ 2 & i \end{pmatrix} \quad (3.4.1.31)$$

$\mathbf{T}$  is defined by

$$\mathbf{T}(\mathbf{x}) = \mathbf{A}\mathbf{x} \quad (3.4.1.32)$$

$$\mathbf{T}(\mathbf{x}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} \quad (3.4.1.33)$$

$$\mathbf{T}(\alpha_2) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -i \\ 2 \end{pmatrix} = \begin{pmatrix} -i \\ 0 \end{pmatrix} \quad (3.4.1.34)$$

$$\mathbf{T}(\alpha_1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.4.1.35)$$

$$[\alpha]_\beta = \begin{pmatrix} -i & 1 \\ 0 & 0 \end{pmatrix} \quad (3.4.1.36)$$

The matrix of  $\mathbf{T}$  in the ordered basis  $\{\alpha_2, \alpha_1\}$  is given as:

$$[\mathbf{T}_\alpha]_\beta = [\mathbf{T}]_\beta [\alpha]_\beta = \begin{pmatrix} -i & 1 \\ 2 & i \end{pmatrix} \begin{pmatrix} -i & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & -i \\ -2i & 2 \end{pmatrix} \quad (3.4.1.37)$$

3.4.2. Let  $\mathbf{T}$  be the linear transformation from  $\mathbb{R}^3$  into  $\mathbb{R}^2$  defined by

$$\mathbf{T}(x_1, x_2, x_3) = (x_1 + x_2, 2x_3 - x_1) \quad (3.4.2.1)$$

If  $\beta = (\alpha_1, \alpha_2, \alpha_3)$  and  $\beta' = (\beta_1, \beta_2)$  where

$$\alpha_1 = (1, 0, -1), \alpha_2 = (1, 1, 1), \alpha_3 = (1, 0, 0)$$

$$\beta_1 = (0, 1), \beta_2 = (1, 0)$$

- a) What is the matrix of  $\mathbf{T}$  relative to the pair  $\beta, \beta'$

**Solution:** Let

$$\beta = (\alpha_1, \alpha_2, \alpha_3) \quad (3.4.2.2)$$

$$\Rightarrow \beta = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix} \quad (3.4.2.3)$$

and

$$\beta' = \{\beta_1, \beta_2\} \quad (3.4.2.4)$$

$$\Rightarrow \beta' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (3.4.2.5)$$

$\mathbf{T}$  is defined by

$$\mathbf{T}(\mathbf{x}) = \mathbf{A}\mathbf{x} \quad (3.4.2.6)$$

using equation (3.4.2.1)

$$\mathbf{T} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ 2x_3 - x_1 \end{pmatrix} \quad (3.4.2.7)$$

R.H.S of the equation can be written as a product of  $2 \times 3$  and  $3 \times 1$  matrices,

$$= \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad (3.4.2.8)$$

$$\Rightarrow \mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \end{pmatrix} \quad (3.4.2.9)$$

Now,

$$\mathbf{T}(\beta) = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \end{pmatrix} \beta \quad (3.4.2.10)$$

$$\mathbf{T}(\beta) = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ -3 & 1 & -1 \end{pmatrix} \quad (3.4.2.11)$$

To find relative matrix we will use row

reduce augmented matrix.

$$\left( \begin{array}{ccc|cc} 1 & 2 & 1 & 0 & 1 \\ -3 & 1 & -1 & 1 & 0 \end{array} \right) \quad (3.4.2.12)$$

$$\left( \begin{array}{ccc|cc} 1 & 2 & 1 & 0 & 1 \\ -3 & 1 & -1 & 1 & 0 \end{array} \right) \xrightarrow{R_1 \leftrightarrow R_2} \left( \begin{array}{ccc|cc} -3 & 1 & -1 & 1 & 0 \\ 1 & 2 & 1 & 0 & 1 \end{array} \right) \Rightarrow \lambda^2 - (a+d)\lambda + (ad-bc) = 0 \quad (3.4.2.13)$$

Hence the matrix of  $\mathbf{T}$  in the order basis of  $\beta'$

$$\mathbf{B} = \begin{pmatrix} -3 & 1 & -1 \\ 1 & 2 & 1 \end{pmatrix} \quad (3.4.2.14)$$

Therefore matrix of relative to the pair  $\beta, \beta'$

$$T(\beta) = \mathbf{A}\beta = \mathbf{B}\beta' = \begin{pmatrix} -3 & 1 & -1 \\ 1 & 2 & 1 \end{pmatrix} \beta' \quad (3.4.2.15)$$

3.4.3. Let  $T$  be the linear transformation from  $\mathbb{R}^3$  into  $\mathbb{R}^2$  defined by,

$$T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ 2x_3 - x_1 \end{pmatrix} \quad (3.4.3.1)$$

a) If  $\beta$  is the standard ordered basis for  $\mathbb{R}^3$  and  $\beta'$  is the standard ordered basis for  $\mathbb{R}^2$ , what is the matrix of  $T$  relative to the pair  $\beta, \beta'$

3.4.4. Let  $T$  be a linear operator on  $\mathbf{F}^n$ , let  $\mathbf{A}$  be the matrix of  $T$  in the standard ordered basis for  $\mathbf{F}^n$ , and let  $W$  be the subspace of  $\mathbf{F}^n$  spanned by the column vectors of  $\mathbf{A}$ . What does  $W$  have to do with  $T$ ?

**Solution:** See Table 3.4.4.1.

3.4.5. Let  $V$  be a two-dimensional vector space over the field  $F$  and let  $B$  be an ordered basis for  $V$ . If  $T$  is a linear operator on  $V$  and

$$[T]_B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (3.4.5.1)$$

Prove that

$$T^2 - (a+d)T + (ad-bc)I = 0 \quad (3.4.5.2)$$

**Solution:** Here  $T$  is a linear operator on  $V$  and  $B$  is an ordered basis of  $V$ . Let us consider  $[T]_B = A$ , so  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ .

Now, the characteristic equation of  $A$  is:

$$|A - \lambda I| = 0 \quad (3.4.5.3)$$

$$\Rightarrow \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0 \quad (3.4.5.4)$$

$$\Rightarrow \lambda^2 - (a+d)\lambda + (ad-bc) = 0 \quad (3.4.5.5)$$

According to the Cayley-Hamilton's Theorem, every square matrix satisfies its own characteristic equation. Here  $A$  is a  $2 \times 2$  square matrix, so it should also satisfy its characteristic equation. Now,

$$\lambda^2 - (a+d)\lambda + (ad-bc) = 0 \quad (3.4.5.6)$$

$$\Rightarrow A^2 - (a+d)A + (ad-bc)I = 0 \quad (3.4.5.7)$$

We can also write the equation 3.4.5.7 as:

$$[T]_B^2 - (a+d)[T]_B + (ad-bc)I = 0 \quad (3.4.5.8)$$

$$\text{or, } T^2 - (a+d)T + (ad-bc)I = 0 \quad (3.4.5.9)$$

3.4.6. Let  $T$  be a linear operator on  $\mathbb{R}^3$ , the matrix of which in the standard ordered basis is,

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 3 & 4 \end{pmatrix} \quad (3.4.6.1)$$

Find a basis for the range of  $T$  and a basis for the null-space of  $T$ .

**Solution:** The basis of the range of linear transformation  $T$  is the basis of the column-space of  $\mathbf{A}$  or basis of  $C(\mathbf{A})$ . Hence the basis of the range of the linear transformation  $T$  is derived by reducing  $\mathbf{A}$  into Reduced-Row Echelon form as follows,

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 3 & 4 \end{pmatrix} \xrightarrow[R_1=R_1-2R_2]{R_3=R_3+R_1} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 5 & 5 \end{pmatrix} \quad (3.4.6.2)$$

$$\xrightarrow{R_3=R_3-5R_2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad (3.4.6.3)$$

From (3.4.6.3) the basis of the range of linear operator  $T$  are as follows,

$$\mathbf{a}_1 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \quad (3.4.6.4)$$

$$\mathbf{a}_2 = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \quad (3.4.6.5)$$

Again, the basis for null-space of linear oper-

| Let $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be an ordered basis of $\mathbf{F}^n$ |   |
|---|---|
| Given   | Explanation   |
| T is a linear operator $\mathbf{F}^n$   | <p>As T is linear,</p> $T(\mathbf{x}) = \mathbf{A}\mathbf{x} \quad (3.4.4.1)$ $\mathbf{A} = (T\epsilon_1 \ T\epsilon_2 \ \dots \ T\epsilon_n) \quad (3.4.4.2)$ <p>From equation (3.4.4.2), columns of <math>\mathbf{A}</math> are the images of the standard basis elements of <math>\mathbf{F}^n</math>.</p> |
| Range of T  | $\text{range}(T) = \{T\epsilon_1, T\epsilon_2, \dots, T\epsilon_n\} \quad (3.4.4.3)$ <p>From equation (3.4.4.2) and (3.4.4.3), columns of <math>\mathbf{A}</math> generate the range of T.</p>  |
| W spanned by column vectors $\mathbf{A}$  | <p>Since any generating set contains a basis for the generated space, we can say that the columns of <math>\mathbf{A}</math> contains a basis of the range of T. As W is spanned by column vectors of <math>\mathbf{A}</math>, we can say that W contains a basis for the range of T.</p>                     |

TABLE 3.4.4.1: Expanation

ator  $T$  or  $N(\mathbf{A})$  is a solution of the equation  $\mathbf{A}\mathbf{x} = 0$ . From (3.4.6.3) we have,

$$\mathbf{A}\mathbf{x} = 0 \quad (3.4.6.6)$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \quad (3.4.6.7)$$

Setting the value of the free variable  $x_3 = 1$  we get the solution,

$$\mathbf{x} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad (3.4.6.8)$$

Hence, the basis of the null-space of the linear operator  $T$  is given by,

$$\mathbf{b} = (1 \ -1 \ 1) \quad (3.4.6.9)$$

3.4.7. Let T be the linear operator on  $\mathbb{R}^2$  defined by

$$T(x_1, x_2) = (-x_2, x_1) \quad (3.4.7.1)$$

a) What is the matrix of  $\mathbf{T}$  in the standard ordered basis of  $\mathbb{R}^2$ ?

**Solution:**

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad (3.4.7.2)$$

$$\mathbf{T}(\mathbf{x}) = \mathbf{T}\mathbf{x} \quad (3.4.7.3)$$

$$\Rightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} \quad (3.4.7.4)$$

The matrix of  $\mathbf{T}$  in the standard ordered basis

from (3.4.7.1) is

$$\mathbf{T} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (3.4.7.5)$$

b) Let  $T$  be the linear operator on  $\mathbf{R}^2$  defined by

$$T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} \quad (3.4.7.6)$$

What is the matrix of  $T$  in the ordered basis  $\mathbf{B} = \{\alpha_1, \alpha_2\}$ , where  $\alpha_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$  and  $\alpha_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ ?

**Solution:**

Let  $\mathbf{B}'$  be the standard ordered basis for  $\mathbf{R}^2$ . Then,

$$T(\alpha'_1) = T \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0\alpha_1 + 1\alpha_2 \quad (3.4.7.7)$$

$$T(\alpha'_2) = T \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} = -1\alpha_1 + 0\alpha_2 \quad (3.4.7.8)$$

Hence, the matrix of  $T$  in the standard ordered basis  $\mathbf{B}'$  is

$$T_{\mathbf{B}'} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (3.4.7.9)$$

Given,

$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (3.4.7.10)$$

$$\alpha_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (3.4.7.11)$$

Then,

$$\alpha'_1 = 1\alpha_1 + 2\alpha_2 \quad (3.4.7.12)$$

$$\alpha'_2 = 1\alpha_1 - 1\alpha_2 \quad (3.4.7.13)$$

So the  $\mathbf{P}$  matrix is

$$\mathbf{P} = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \quad (3.4.7.14)$$

And

$$\mathbf{P}^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{pmatrix} \quad (3.4.7.15)$$

Hence

$$[T]_{\mathbf{B}} = \mathbf{P}^{-1}[T]_{\mathbf{B}'}\mathbf{P} \quad (3.4.7.16)$$

$$= \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \quad (3.4.7.17)$$

$$= \begin{pmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \quad (3.4.7.18)$$

$$[T]_{\mathbf{B}} = \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{pmatrix} \quad (3.4.7.19)$$

Hence,  $[T]_{\mathbf{B}}$  is the required matrix for the given ordered basis  $\mathbf{B}$ .

- c) Prove that for every real number  $c$ , the operator  $(T - cI)$  is invertible.

**Solution:** From the equation (3.4.7.1), the matrix of  $T$  in standard order basis is,

$$\mathbf{T} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (3.4.7.20)$$

To find the invertibility of the operator  $(\mathbf{T} - c\mathbf{I})$  for every real number  $c$ , let us start with

$$(\mathbf{T} - c\mathbf{I})(\mathbf{T} + c\mathbf{I}) \quad (3.4.7.21)$$

$$= \mathbf{T}^2 - c^2\mathbf{I} \quad (3.4.7.22)$$

Consider  $\mathbf{T}^2$

$$\mathbf{T}^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow \mathbf{T}^2 = -\mathbf{I} \quad (3.4.7.23)$$

Substituting equation (3.4.7.23) in (3.4.7.22),

$$(\mathbf{T} - c\mathbf{I})(\mathbf{T} + c\mathbf{I}) = -(1 + c^2)\mathbf{I} \quad (3.4.7.24)$$

As  $c$  is a real number,  $c^2 \geq 0$  and hence factor  $-(1 + c^2)$  is always non-zero. Therefore, from the equation (3.4.7.24),

$$(\mathbf{T} - c\mathbf{I})^{-1} = \frac{-1}{1 + c^2} (\mathbf{T} + c\mathbf{I}) \quad (3.4.7.25)$$

Hence the operator  $(T - cI)$  is invertible and its inverse is given by the equation (3.4.7.25)

- d) Prove that if  $\beta$  is any ordered basis for  $\mathbb{R}^2$  and  $[\mathbf{T}]_{\beta} = \mathbf{A}$  then  $\mathbf{A}_{12}\mathbf{A}_{21} \neq 0$ .

**Solution:** If the matrix  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  is invertible, then  $ad - bc \neq 0$ . It implies that both  $a$  and  $b$  cannot become 0 at the same time.

$$\text{Let } \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad (3.4.7.26)$$

$$T(\mathbf{x}) = \mathbf{A}\mathbf{x} \quad (3.4.7.27)$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad (3.4.7.28)$$

$$\Rightarrow \mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (3.4.7.29)$$

Let  $\{\alpha_1, \alpha_2\}$  be any basis and  $\alpha_1 = (a, b)$  and  $\alpha_2 = (c, d)$ . Then

$$T(\beta) = \mathbf{A}\beta \quad (3.4.7.30)$$

$$T(\beta) = \begin{pmatrix} -b & -d \\ a & c \end{pmatrix} \quad (3.4.7.31)$$

Now, for finding the matrix of  $\mathbf{T}$  in the ordered basis  $\beta$ , We use the concept of row reducing the augmented matrix.

$$\begin{pmatrix} a & c & -b & -d \\ b & d & a & c \end{pmatrix} \quad (3.4.7.32)$$

i) Assuming  $a \neq 0$ ,

$$\begin{aligned}
 & \begin{pmatrix} a & c & -b & -d \\ b & d & a & c \end{pmatrix} \\
 & \xleftrightarrow{R_1=R_1/a} \begin{pmatrix} 1 & \frac{c}{a} & -\frac{b}{a} & -\frac{d}{a} \\ b & d & a & c \end{pmatrix} \\
 & \xleftrightarrow{R_2=R_2-R_1 \times b} \begin{pmatrix} 1 & \frac{c}{a} & -\frac{b}{a} & -\frac{d}{a} \\ 0 & \frac{ad-bc}{a} & \frac{a^2+b^2}{a} & \frac{ac+bd}{a} \end{pmatrix} \\
 & \xleftrightarrow{R_2=R_2 \frac{a}{ad-bc}} \begin{pmatrix} 1 & \frac{c}{a} & -\frac{b}{a} & -\frac{d}{a} \\ 0 & 1 & \frac{a^2+b^2}{ad-bc} & \frac{ac+bd}{ad-bc} \end{pmatrix} \\
 & \xleftrightarrow{R_1=R_1-R_2 \times \frac{c}{a}} \begin{pmatrix} 1 & 0 & -\frac{ac+bd}{ad-bc} & -\frac{c^2+d^2}{ad-bc} \\ 0 & 1 & \frac{a^2+b^2}{ad-bc} & \frac{ac+bd}{ad-bc} \end{pmatrix}
 \end{aligned}$$

ii) Assuming  $b \neq 0$ , We arrive at the same result as follows

$$\begin{aligned}
 & \begin{pmatrix} a & c & -b & -d \\ b & d & a & c \end{pmatrix} \\
 & \xleftrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} b & d & a & c \\ a & c & -b & -d \end{pmatrix} \\
 & \xleftrightarrow{R_1=R_1/b} \begin{pmatrix} 1 & \frac{d}{b} & \frac{a}{b} & \frac{c}{b} \\ a & c & -b & -d \end{pmatrix} \\
 & \xleftrightarrow{R_2=R_2-R_1 \times a} \begin{pmatrix} 1 & \frac{d}{b} & \frac{a}{b} & \frac{c}{b} \\ 0 & \frac{bc-ad}{b} & -\frac{a^2+b^2}{b} & -\frac{ac+bd}{b} \end{pmatrix} \\
 & \xleftrightarrow{R_2=R_2 \frac{b}{bc-ad}} \begin{pmatrix} 1 & \frac{d}{b} & \frac{a}{b} & \frac{c}{b} \\ 0 & 1 & \frac{a^2+b^2}{ad-bc} & \frac{ac+bd}{ad-bc} \end{pmatrix} \\
 & \xleftrightarrow{R_1=R_1-R_2 \times \frac{d}{b}} \begin{pmatrix} 1 & 0 & -\frac{ac+bd}{ad-bc} & -\frac{c^2+d^2}{ad-bc} \\ 0 & 1 & \frac{a^2+b^2}{ad-bc} & \frac{ac+bd}{ad-bc} \end{pmatrix}
 \end{aligned}$$

Thus, from above two cases:

$$[\mathbf{T}]_{\beta} = \mathbf{A} = \begin{pmatrix} -\frac{ac+bd}{ad-bc} & -\frac{c^2+d^2}{ad-bc} \\ \frac{a^2+b^2}{ad-bc} & \frac{ac+bd}{ad-bc} \end{pmatrix}. \quad (3.4.7.33)$$

Now  $ad - bc \neq 0$  implies that atleast one of  $a$  or  $b$  is non-zero and atleast one of  $c$  or  $d$  is non-zero, which follows that:

$$a^2 + b^2 > 0 \text{ and } c^2 + d^2 > 0 \quad (3.4.7.34)$$

$$\text{Thus, } (a^2 + b^2)(c^2 + d^2) \neq 0. \quad (3.4.7.35)$$

Hence, it is proved that  $\mathbf{A}_{12}\mathbf{A}_{21} \neq 0$ .

3.4.8. Let  $\mathbb{T}$  be the linear operator on  $\mathbb{R}^3$  defined by

$$\mathbb{T}(x_1, x_2, x_3) = \quad (3.4.8.1)$$

$$(3x_1 + x_3, -2x_1 + x_2, -x_1 + 2x_2 + 4x_3) \quad (3.4.8.2)$$

What is the matrix of  $\mathbb{T}$  in the standard ordered

basis of  $\mathbb{R}^3$ ?

**Solution:**

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad (3.4.8.3)$$

$$\mathbb{T}(\mathbf{x}) = \mathbf{T}\mathbf{x} \quad (3.4.8.4)$$

The matrix of  $\mathbb{T}$  in the standard ordered basis from (3.4.8.2) is

$$\mathbf{T} = \begin{pmatrix} 3 & 0 & 1 \\ -2 & 1 & 0 \\ -1 & 2 & 4 \end{pmatrix} \quad (3.4.8.5)$$

3.4.9. The linear operator  $\mathbf{T}$  on  $\mathbb{R}^2$  defined by

$$\mathbf{T} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \quad (3.4.9.1)$$

is represented by the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (3.4.9.2)$$

3.4.10. Let  $\theta$  be a real number. Prove that the following two matrices are similar over the field of complex numbers:

$$\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

**Solution:** Two matrices  $\mathbf{A}$  and  $\mathbf{B}$  are said to be similar iff there exists a invertible matrix  $\mathbf{P}$  such that:

$$\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} \quad (3.4.10.1)$$

Let,

$$\mathbf{A} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}; \quad \mathbf{B} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \quad (3.4.10.2)$$

Finding the characteristic polynomial of  $\mathbf{A}$ ,

$$\begin{aligned}
 |A - \lambda I| &= \begin{vmatrix} \cos \theta - \lambda & -\sin \theta \\ \sin \theta & \cos \theta - \lambda \end{vmatrix} \\
 &= (\cos \theta - \lambda)^2 + \sin^2 \theta \\
 &= 1 + \lambda^2 + 2\lambda \cos \theta \quad (3.4.10.3)
 \end{aligned}$$

The eigenvalues can be calculated by equating the characteristic polynomial to zero. The

eigenvalues are,

$$\lambda_1 = \cos \theta + i \sin \theta; \lambda_2 = \cos \theta - i \sin \theta \quad (3.4.10.4)$$

The eigenvectors corresponding to (3.4.10.4) are,

$$\alpha_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}; \alpha_2 = \begin{pmatrix} -i \\ 1 \end{pmatrix} \quad (3.4.10.5)$$

$$\mathbf{P} = (\alpha_1 \quad \alpha_2) = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \quad (3.4.10.6)$$

Now,

$$\begin{aligned} \mathbf{P}^{-1}\mathbf{A}\mathbf{P} &= \frac{1}{2i} \begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos \theta + i \sin \theta & 0 \\ 0 & \cos \theta - i \sin \theta \end{pmatrix} \\ &= \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} = \mathbf{B} \end{aligned} \quad (3.4.10.7)$$

Hence, from (3.4.10.7),  $\mathbf{A}$  and  $\mathbf{B}$  are similar matrices.

3.4.11. Let  $\mathbb{V}$  be finite dimensional vector space over the field  $\mathbb{F}$ , and let  $\mathbf{S}$  and  $\mathbf{T}$  be linear operators on  $\mathbb{V}$ . When do there exist ordered bases  $\mathcal{B}$  and  $\mathcal{B}'$  for  $\mathbb{V}$  such that  $[\mathbf{S}]_{\mathcal{B}} = [\mathbf{T}]'_{\mathcal{B}}$ ? Prove that such bases exist if and only if there is an invertible linear operator  $\mathbf{U}$  on  $\mathbb{V}$  such that  $\mathbf{T} = \mathbf{U}\mathbf{S}\mathbf{U}^{-1}$

**Solution:** See Table 3.4.11.1.

3.4.12. Prove that if  $\mathbf{S}$  is a linear operator on  $\mathbf{R}^2$  such that  $\mathbf{S}^2 = \mathbf{S}$ , then  $\mathbf{S} = \mathbf{0}$ , or  $\mathbf{S} = \mathbf{I}$ , or there is an ordered basis  $\mathbf{B}$  for  $\mathbf{R}^2$  such that  $[\mathbf{S}]_{\mathbf{B}} = \mathbf{A}$ .

**Solution:** If a linear operator  $\mathbf{S}$  is defined on  $\mathbf{R}^2$  such that  $\mathbf{S}^2 = \mathbf{S}$ , then

$$\mathbf{S}^2 - \mathbf{S} = \mathbf{0} \quad (3.4.12.1)$$

$$\mathbf{S}(\mathbf{S} - \mathbf{I}) = \mathbf{0} \quad (3.4.12.2)$$

$$\implies \mathbf{S} = \mathbf{0}, \mathbf{S} = \mathbf{I} \quad (3.4.12.3)$$

The transformation of a vector  $\mathbf{x} \in \mathbf{R}^2$  can be represented as

$$\mathbf{S}\mathbf{x} = \mathbf{y} \quad (3.4.12.4)$$

$$\implies \mathbf{S}(\mathbf{S}\mathbf{x}) = \mathbf{S}\mathbf{y} \quad (3.4.12.5)$$

$$\implies \mathbf{S}^2\mathbf{x} = \mathbf{S}\mathbf{y} \quad (3.4.12.6)$$

$$\implies \mathbf{S}\mathbf{x} = \mathbf{S}\mathbf{y} \quad (3.4.12.7)$$

$$\implies \mathbf{x} = \mathbf{y} \quad (3.4.12.8)$$

Therefore the transformation of a vector  $\mathbf{x} \in \mathbf{R}^2$

can be given as

$$\mathbf{S}\mathbf{x} = \mathbf{x} \quad \forall \mathbf{x} \in \mathbf{R}^2 \quad (3.4.12.9)$$

Consider the ordered basis set

$$\mathbf{B} = \{\epsilon_1, \epsilon_2\} \in \mathbf{R}^2 \quad (3.4.12.10)$$

and if

$$[\mathbf{S}]_{\mathbf{B}} = \mathbf{A} \quad (3.4.12.11)$$

$$\implies [\mathbf{S}]_{\mathbf{B}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (3.4.12.12)$$

Thus we can re-write the column vectors of  $[\mathbf{S}]_{\mathbf{B}}$  using (3.4.12.9) as

$$\mathbf{S} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (3.4.12.13)$$

$$\mathbf{S} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (3.4.12.14)$$

Therefore, any vector  $\mathbf{x}$  in column space of  $[\mathbf{S}]_{\mathbf{B}}$  can be uniquely expressed by  $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$ , hence it forms the basis for column space of  $[\mathbf{S}]_{\mathbf{B}}$ . Therefore one of the basis vector of  $\mathbf{B}$  is  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ . The other basis vector can be any vector which is linearly independent to  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ . One of the ordered basis set can be

$$\mathbf{B} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \quad (3.4.12.15)$$

3.4.13. Let  $\mathbf{W}$  be the space of all  $n \times 1$  column matrices over a field  $\mathbf{F}$ . If  $\mathbf{A}$  is an  $n \times n$  matrix over  $\mathbf{F}$ , then  $\mathbf{A}$  defines a linear operator  $\mathbf{L}_{\mathbf{A}}$  on  $\mathbf{W}$  through left multiplication:  $\mathbf{L}_{\mathbf{A}}(\mathbf{X}) = \mathbf{A}\mathbf{X}$ . Prove that every linear operator on  $\mathbf{W}$  is left multiplication by some  $n \times n$  matrix, i.e., is  $\mathbf{L}_{\mathbf{A}}$  for some  $\mathbf{A}$ .

Now suppose  $\mathbf{V}$  is an  $n$ -dimensional vector space over the field  $\mathbf{F}$ , and let  $\beta$  be an ordered basis for  $\mathbf{V}$ . For each  $\alpha$  in  $\mathbf{V}$ , define  $\mathbf{U}_{\alpha} = [\alpha]_{\beta}$ . Prove that  $\mathbf{U}$  is an isomorphism of  $\mathbf{V}$  onto  $\mathbf{W}$ . If  $\mathbf{T}$  is a linear operator on  $\mathbf{V}$ , then  $\mathbf{U}\mathbf{T}\mathbf{U}^{-1}$  is a linear operator on  $\mathbf{W}$ . Accordingly,  $\mathbf{U}\mathbf{T}\mathbf{U}^{-1}$  is left multiplication by some  $n \times n$  matrix  $\mathbf{A}$ . What is  $\mathbf{A}$ ?

**Solution:** See Tables 3.4.13.1 and 3.4.13.2.

3.4.14. Let  $\mathbf{V}$  be an  $n$ -dimensional vector space over the field  $\mathbf{F}$ , and let  $\mathcal{B} = \{\alpha_1, \dots, \alpha_n\}$  be an



|  |   |
|--|---|
| Assume $[S]_{\mathcal{B}} = [T]_{\mathcal{B}'}$  | Assume $\mathbf{T} = \mathbf{U}\mathbf{S}\mathbf{U}^{-1}$   |
| <b>Given</b><br><br>$\mathbb{V}$ is a finite dimensional vector space over field $\mathbb{F}$<br>$\mathbf{S}$ and $\mathbf{T}$ are linear operators on $\mathbb{V}$<br>$\mathcal{B}$ and $\mathcal{B}'$ are ordered bases for $\mathbb{V}$<br>$[S]_{\mathcal{B}} = [T]_{\mathcal{B}}'$   | <b>Given</b><br><br>$\mathbb{V}$ is a finite dimensional vector space over field $\mathbb{F}$<br>$\mathbf{S}$ and $\mathbf{T}$ are linear operators on $\mathbb{V}$<br>$\mathcal{B}$ and $\mathcal{B}'$ are ordered bases for $\mathbb{V}$<br>There is an invertible linear operator $\mathbf{U}$ on $\mathbb{V}$<br>such that $\mathbf{T} = \mathbf{U}\mathbf{S}\mathbf{U}^{-1}$   |
| <b>To prove</b><br><br>There is an invertible linear operator $\mathbf{U}$ on $\mathbb{V}$<br>such that $\mathbf{T} = \mathbf{U}\mathbf{S}\mathbf{U}^{-1}$   | <b>To prove</b><br><br>$[S]_{\mathcal{B}} = [T]_{\mathcal{B}}'$   |
| <b>Assumptions</b><br><br>Let $\mathbf{U}$ be the operator which carries $\mathcal{B}$ to $\mathcal{B}'$<br>$\mathcal{B} = \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$<br>$\mathcal{B}' = \{\mathbf{x}'_1, \mathbf{x}'_2, \dots, \mathbf{x}'_n\}$   | <b>Assumptions</b><br><br>Let $\mathbf{U}$ be the operator which carries $\mathcal{B}$ to $\mathcal{B}'$<br>$\mathcal{B} = \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$<br>$\mathcal{B}' = \{\mathbf{x}'_1, \mathbf{x}'_2, \dots, \mathbf{x}'_n\}$  |
| <b>Proof</b><br><br>For $\mathbf{v} \in \mathbb{V}$ , expressed as a linear combination<br>of the vectors of $\mathcal{B}$<br>$\mathbf{v} = a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \dots + a_n\mathbf{x}_n$<br><br>$\mathbf{w} = \mathbf{U}(\mathbf{v}) = a_1\mathbf{x}'_1 + a_2\mathbf{x}'_2 + \dots + a_n\mathbf{x}'_n$<br><br>$\because [S]_{\mathcal{B}} = [T]_{\mathcal{B}}'$<br>$\mathbf{S}(\mathbf{v}) = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \dots + c_n\mathbf{x}_n$ and<br>$\mathbf{T}(\mathbf{w}) = c_1\mathbf{x}'_1 + c_2\mathbf{x}'_2 + \dots + c_n\mathbf{x}'_n$<br><br>$\mathbf{U}^{-1}\mathbf{T}\mathbf{U}(\mathbf{v})$<br>$= \mathbf{U}^{-1}\mathbf{T}(\mathbf{w})$<br>$= \mathbf{U}^{-1}(c_1\mathbf{x}'_1 + c_2\mathbf{x}'_2 + \dots + c_n\mathbf{x}'_n)$<br>$= c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \dots + c_n\mathbf{x}_n$<br>$= \mathbf{S}(\mathbf{v})$<br><br>$\implies \mathbf{U}^{-1}\mathbf{T}\mathbf{U} = \mathbf{S}$<br>$\implies \mathbf{T} = \mathbf{U}\mathbf{S}\mathbf{U}^{-1}$ | <b>Proof</b><br><br>For $\mathbf{v} \in \mathbb{V}$ , expressed as a linear combination<br>of the vectors of $\mathcal{B}'$<br>$\mathbf{v} = a_1\mathbf{x}'_1 + a_2\mathbf{x}'_2 + \dots + a_n\mathbf{x}'_n$<br><br>$\mathbf{U}\mathbf{S}\mathbf{U}^{-1}(\mathbf{v}) = \mathbf{T}(\mathbf{v})$<br>$= \mathbf{T}(a_1\mathbf{x}'_1 + a_2\mathbf{x}'_2 + \dots + a_n\mathbf{x}'_n)$<br>$= c_1\mathbf{x}'_1 + c_2\mathbf{x}'_2 + \dots + c_n\mathbf{x}'_n$<br><br>But we know that<br>$\mathbf{U}\mathbf{S}\mathbf{U}^{-1}(\mathbf{v}) = \mathbf{U}\mathbf{S}\mathbf{U}^{-1}(a_1\mathbf{x}'_1 + a_2\mathbf{x}'_2 + \dots + a_n\mathbf{x}'_n)$<br>$= \mathbf{U}\mathbf{S}(a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \dots + a_n\mathbf{x}_n)$<br><br>So, $\mathbf{S}$ in basis $\mathcal{B}$ has the same entries as $\mathbf{T}$ in basis $\mathcal{B}'$<br>$\therefore [S]_{\mathcal{B}} = [T]_{\mathcal{B}}'$ |

TABLE 3.4.11.1

ordered basis for  $\mathbb{V}$  then there is a unique linear

operator  $\mathbf{T}$  on  $\mathbb{V}$  such that

$$T\alpha_j = \alpha_{j+1}, j = 1, \dots, n-1 \quad (3.4.14.1)$$

$$T\alpha_n = 0. \quad (3.4.14.2)$$

|  |   |
|--|---|
| Defining<br>Linear<br>Map $T$  | <p>Let <math>\mathbf{T} : \mathbf{W} \rightarrow \mathbf{W}</math> be a linear operator and <math>(e_1, e_2, \dots, e_n)</math> be a basis for <math>\mathbf{W}</math>.<br/>Now,</p> $\begin{aligned}\mathbf{T}e_1 &= \alpha_{11}e_1 + \alpha_{12}e_2 + \dots + \alpha_{1n}e_n \\ \mathbf{T}e_2 &= \alpha_{21}e_1 + \alpha_{22}e_2 + \dots + \alpha_{2n}e_n \\ &\vdots \\ \mathbf{T}e_n &= \alpha_{n1}e_1 + \alpha_{n2}e_2 + \dots + \alpha_{nn}e_n\end{aligned}$ |
| Matrix of<br>Linear Map $T$  | <p>Let <math>\mathbf{A}</math> be matrix of linear transformation <math>\mathbf{T}</math>. Then <math>\mathbf{A}</math> is</p> $\mathbf{A} = \begin{pmatrix} \alpha_{11} & \alpha_{21} & \dots & \alpha_{n1} \\ \alpha_{12} & \alpha_{22} & \dots & \alpha_{n2} \\ \vdots & \vdots & \dots & \vdots \\ \alpha_{1n} & \alpha_{2n} & \dots & \alpha_{nn} \end{pmatrix}$   |
| Proving every linear<br>operator on $\mathbf{W}$ is left<br>multiplication by<br>some $n \times n$ matrix, i.e, is<br>$\mathbf{L}_A$ for some $\mathbf{A}$ | $\begin{aligned}\mathbf{A}e_1 &= (\alpha_{11}, \alpha_{12}, \dots, \alpha_{1n}) \\ &= \alpha_{11}e_1 + \alpha_{12}e_2 + \dots + \alpha_{1n}e_n \\ &= \mathbf{T}e_1\end{aligned}$ <p>Hence,</p> $\mathbf{T}e_i = \mathbf{A}e_i$ <p>Since <math>\mathbf{T}</math> and <math>\mathbf{A}</math> are linear. Then</p> $\begin{aligned}\mathbf{T}\mathbf{x} &= \mathbf{A}\mathbf{x} \\ \mathbf{T}\mathbf{x} &= \mathbf{L}_A\mathbf{x}\end{aligned}$                     |

TABLE 3.4.13.1

- a) What is the matrix  $\mathbf{A}$  of  $\mathbf{T}$  in the ordered basis  $\mathcal{B}$ ?

**Solution:** Given that,

$$T : V \rightarrow V \quad (3.4.14.3)$$

$$[T(\alpha)]_{\mathcal{B}} = A[\alpha]_{\mathcal{B}} \quad (3.4.14.4)$$

$$T\alpha_j = \alpha_{j+1} \quad (3.4.14.5)$$

$$T\alpha_n = 0 \quad (3.4.14.6)$$

where  $j = 1, \dots, n-1$ . The matrix  $\mathbf{A}$  of  $\mathbf{T}$  in the ordered basis  $\mathcal{B}$  is given by,

$$\Rightarrow A = ([T\alpha_1]_{\mathcal{B}} \quad \dots \quad [T\alpha_n]_{\mathcal{B}}) \quad (3.4.14.7)$$

For  $j = 1, \dots, n-1$  we have,

$$T\alpha_j = \alpha_{j+1} \quad (3.4.14.8)$$

we can write,

$$T\alpha_j = 0\alpha_1 + \dots + 0\alpha_j + 1\alpha_{j+1} + \dots + 0\alpha_n \quad (3.4.14.9)$$

$$\Rightarrow [T\alpha_j]_{\mathcal{B}} = (0, \dots, 0, 1, 0, \dots, 0)^T \quad (3.4.14.10)$$

where 1 is in  $(j+1)$ th position. Now,

$$T\alpha_n = 0 \quad (3.4.14.11)$$

$$\Rightarrow [T\alpha_n]_{\mathcal{B}} = 0 \quad (3.4.14.12)$$

Thus from (3.4.14.7), (3.4.14.10) and (3.4.14.12) we get matrix  $\mathbf{A}$  of  $\mathbf{T}$  in the ordered basis  $\mathcal{B}$  as,

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix} \quad (3.4.14.13)$$

|   |   |
|---|---|
| Proving $\mathbf{U}$ as Linear and defining a linear map $\mathbf{T}$ | $\begin{aligned}\mathbf{U}(c\alpha_1 + \alpha_2) &= [c\alpha_1 + \alpha_2]_\beta \\ &= c[\alpha_1]_\beta + [\alpha_2]_\beta \\ &= c\mathbf{U}(\alpha_1) + \mathbf{U}(\alpha_2)\end{aligned}$ <p>Suppose <math>\beta = \{\alpha_1, \alpha_2, \dots, \alpha_n\}</math> be the ordered basis for <math>\mathbf{V}</math>.<br/>Let <math>\mathbf{T}</math> be the function from <math>\mathbf{W}</math> to <math>\mathbf{V}</math> as follows:</p> $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \rightarrow a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n$  |
| Proving $\mathbf{U}$ to be an isomorphism                             | For isomorphism, we must show that $\mathbf{TU}$ is identity map on $\mathbf{V}$ and $\mathbf{UT}$ is an identity map on $\mathbf{W}$ .   |
| $\mathbf{TU} = \mathbf{I}_V$  | $\begin{aligned}\mathbf{TU}(\mathbf{x}) &= \mathbf{TU}(a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n) \\ &= \mathbf{TU}(a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n) \\ &= a_1\mathbf{T}\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + a_2\mathbf{T}\begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \dots + a_n\mathbf{T}\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \\ &= a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n\end{aligned}$ <p>Hence, <math>\mathbf{TU}</math> is identity map on <math>\mathbf{V}</math>.</p>  |
| $\mathbf{UT} = \mathbf{I}_W$  | $\begin{aligned}\mathbf{UT}(\mathbf{x}) &= \mathbf{UT}(a_1e_1 + a_2e_2 + \dots + a_ne_n) \\ &= \mathbf{UT}((a_1e_1 + a_2e_2 + \dots + a_ne_n)) \\ &= \mathbf{U}(a_1\mathbf{T}\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + a_2\mathbf{T}\begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \dots + a_n\mathbf{T}\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}) \\ &= a_1\mathbf{U}(\alpha_1) + a_2\mathbf{U}(\alpha_2) + \dots + a_n\mathbf{U}(\alpha_n) \\ &= a_1e_1 + a_2e_2 + \dots + a_ne_n\end{aligned}$ <p>Hence, <math>\mathbf{UT}</math> is identity map on <math>\mathbf{W}</math>.</p>   |
| Matrix of $\mathbf{UTU}^{-1}$   | <p>Now, we define the matrix of <math>\mathbf{UTU}^{-1}</math>. Since <math>\mathbf{U}\alpha_i</math> is the standard <math>n \times 1</math> matrix with all zeros except in the <math>i</math>th place which equals one. Let <math>\beta'</math> be the standard basis for <math>\mathbf{W}</math>. Then the matrix of <math>\mathbf{U}</math> with respect to <math>\beta</math> and <math>\beta'</math> is the identity matrix. Likewise the matrix of <math>\mathbf{U}^{-1}</math> with respect to <math>\beta'</math> and <math>\beta</math> is the identity matrix. Thus,</p> $[\mathbf{UTU}^{-1}]_\beta = \mathbf{I}[\mathbf{T}]_\beta \mathbf{I}^{-1} = [\mathbf{T}]_\beta$ <p>Thus, the matrix <math>\mathbf{A}</math> is simply <math>[\mathbf{T}]_\beta</math>, the matrix of <math>\mathbf{T}</math> with respect to <math>\beta</math>.</p> |

TABLE 3.4.13.2

3.4.15. Let  $V$  and  $W$  be finite-dimensional vector spaces over the field  $F$  and let  $T$  be a linear transformation from  $V$  into  $W$ . If

$$\mathcal{B} = \{\alpha_1, \dots, \alpha_n\} \text{ and } \mathcal{B}' = \{\beta_1, \dots, \beta_m\} \quad (3.4.15.1)$$

are ordered bases for  $V$  and  $W$ , respectively, define the linear transformation  $E^{p,q}$  as in the proof of Theorem 5:  $E^{p,q}(\alpha_i) = \delta_{iq}\beta_p$ . Then the  $E^{p,q}$ ,  $1 \leq p \leq m$ ,  $1 \leq q \leq n$ , form a basis for  $L(V, W)$  and so

$$T = \sum_{p=1}^m \sum_{q=1}^n A_{pq} E^{p,q} \quad (3.4.15.2)$$

for certain scalars  $A_{pq}$  (the coordinates of  $T$  in this basis for  $L(V, W)$ ). Show that the matrix  $A$  with entries  $A(p, q) = A_{pq}$  is precisely the matrix of  $T$  relative to the pair  $\mathcal{B}, \mathcal{B}'$ .

**Solution:** Given,

$$T = \sum_{p=1}^m \sum_{q=1}^n A_{pq} E^{p,q} \quad (3.4.15.3)$$

where

$$E^{p,q}(\alpha_i) = \begin{cases} \beta_p & p = i \\ 0 & \text{otherwise} \end{cases} \quad (3.4.15.4)$$

$$= \delta_{iq}\beta_p \quad (3.4.15.5)$$

where  $\mathcal{B} = \{\alpha_1, \dots, \alpha_n\}$  is basis of  $V$  and  $\mathcal{B}' = \{\beta_1, \dots, \beta_m\}$  is basis of  $W$ .

### 3.5 Linear Functionals

Consider a vector  $\mathbf{x} = \{x_1, x_2, \dots, x_n\} \in V$ ,

$$\mathbf{x} = \sum_{q=1}^n x_q \alpha_q \quad (3.4.15.6)$$

$$\therefore E^{p,q}(\mathbf{x}) = \sum_{q=1}^n x_q E^{p,q}(\alpha_q) \quad (3.4.15.7)$$

$$= x_q \delta_{iq} \beta_p \quad (3.4.15.8)$$

Consider  $T(\mathbf{x})$ , from (3.4.15.3)

$$T(\mathbf{x}) = \sum_{p=1}^m \sum_{q=1}^n A_{pq} E^{p,q}(\mathbf{x}) \quad (3.4.15.9)$$

Substitute (3.4.15.8) in (3.4.15.9)

$$T(\mathbf{x}) = \sum_{p=1}^m \sum_{q=1}^n A_{pq} x_p \delta_{iq} \beta_q \quad (3.4.15.10)$$

From (3.4.15.5),  $\delta_{iq}\beta_q$  is the transformation of basis of  $V$  to  $W$ . Hence  $T : V \rightarrow W$  is

$$T = \begin{pmatrix} \sum_{p=1}^n A_{p1}x_p \\ \sum_{p=1}^n A_{p2}x_p \\ \vdots \\ \sum_{p=1}^n A_{pm}x_p \end{pmatrix} \quad (3.4.15.11)$$

$$T = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & \dots & \vdots \\ A_{m1} & A_{m2} & \dots & A_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad (3.4.15.12)$$

$\therefore$  We showed that the matrix  $A$  with entries  $A(p, q) = A_{pq}$  is precisely the matrix of  $T$  relative to the pair  $\mathcal{B}, \mathcal{B}'$ .

3.5.1. In  $R^3$ , let  $\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ ,  $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$  and  $\alpha_3 = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$ .

a) if  $f$  is a linear functional on  $R^3$  such that  $f(\alpha_1) = 1$ ,  $f(\alpha_2) = -1$ ,  $f(\alpha_3) = 3$ ,

And if  $\alpha = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ , find  $f(\alpha)$ .

**Solution:** Given,  $\alpha = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ . Let,

$$\mathbf{A} = (\alpha_1 \quad \alpha_2 \quad \alpha_3) \quad (3.5.1.1)$$

$$\mathbf{A}\mathbf{X} = \alpha \quad (3.5.1.2)$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & -2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad (3.5.1.3)$$

$\mathbf{X} = A^{-1}\alpha$  will give solution of the equation.

$$\begin{pmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 1 & -2 & 0 & 0 & 0 & 1 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 - R_1} \quad (3.5.1.4)$$

$$\begin{pmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & -2 & 1 & -1 & 0 & 1 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 + 2R_2} \quad (3.5.1.5)$$

$$\begin{pmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 2 & 1 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 / (-1)} \quad (3.5.1.6)$$

$$\begin{pmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & -1 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 + R_3} \quad (3.5.1.7)$$

$$\begin{pmatrix} 1 & 0 & 0 & 2 & -2 & -1 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & -1 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 + R_3} \quad (3.5.1.8)$$

$$\begin{pmatrix} 1 & 0 & 0 & 2 & -2 & -1 \\ 0 & 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 & -2 & -1 \end{pmatrix} \quad (3.5.1.9)$$

Thus,

$$A^{-1} = \begin{pmatrix} 2 & -2 & -1 \\ 1 & -1 & -1 \\ 1 & -2 & -1 \end{pmatrix} \quad (3.5.1.10)$$

$$\mathbf{X} = A^{-1}\alpha = \begin{pmatrix} 2 & -2 & -1 \\ 1 & -1 & -1 \\ 1 & -2 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad (3.5.1.11)$$

Given,  $f$  is a linear functional on  $R^3$ ,

$$\alpha = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 \quad (3.5.1.12)$$

$$\Rightarrow f(\alpha) = \mathbf{X}^T \begin{pmatrix} f(\alpha_1) \\ f(\alpha_2) \\ f(\alpha_3) \end{pmatrix} \quad (3.5.1.13)$$

Given,  $f(\alpha_1) = 1$ ,  $f(\alpha_2) = -1$  and  $f(\alpha_3) = 3$ .

$$f(\alpha) = \mathbf{X}^T \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \quad (3.5.1.14)$$

$$\Rightarrow f(\alpha) = \begin{pmatrix} a \\ b \\ c \end{pmatrix}^T \begin{pmatrix} 2 & 1 & 1 \\ -2 & -1 & -2 \\ -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \quad (3.5.1.15)$$

$$f(\alpha) = \begin{pmatrix} a \\ b \\ c \end{pmatrix}^T \begin{pmatrix} 4 \\ -7 \\ -3 \end{pmatrix} \quad (3.5.1.16)$$

Hence,

$$f(\alpha) = 4a - 7b - 3c \quad (3.5.1.17)$$

- b) Describe explicitly a linear functional  $f$  on  $R^3$  such that  $f(\alpha_1) = f(\alpha_2) = 0$  but  $f(\alpha_3) \neq 0$ .

**Solution:** Let us consider  $\alpha = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$  such that

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = \alpha \quad (3.5.1.18)$$

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad (3.5.1.19)$$

The coefficient matrix is :

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & -2 & 0 \end{pmatrix} \quad (3.5.1.20)$$

So,

$$A\mathbf{x} = \alpha \quad (3.5.1.21)$$

$$\Rightarrow \mathbf{x} = A^{-1}\alpha \quad (3.5.1.22)$$

$$(3.5.1.23)$$

Now to get  $A^{-1}$ , we will consider Gauss-Jordan theorem. So, we will take  $(A|I)$ ,

where  $I$  is a  $3 \times 3$  identity matrix.

$$\begin{aligned}
 & \begin{pmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 1 & -2 & 0 & 0 & 0 & 1 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 - R_1} \\
 & \begin{pmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & -2 & 1 & -1 & 0 & 1 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 + 2R_2} \\
 & \begin{pmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 2 & 1 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 / (-1)} \\
 & \begin{pmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & -1 \end{pmatrix} \xleftrightarrow{\begin{matrix} R_2 \leftarrow R_2 + R_3 \\ R_1 \leftarrow R_1 + R_3 \end{matrix}} \\
 & \begin{pmatrix} 1 & 0 & 0 & 2 & -2 & -1 \\ 0 & 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 & -2 & -1 \end{pmatrix} \quad (3.5.1.24)
 \end{aligned}$$

Now, we can say that

$$A^{-1} = \begin{pmatrix} 2 & -2 & -1 \\ 1 & -1 & -1 \\ 1 & -2 & -1 \end{pmatrix} \quad (3.5.1.25)$$

As

$$\mathbf{x} = A^{-1}\alpha \quad (3.5.1.26)$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = A^{-1} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad (3.5.1.27)$$

$$\Rightarrow \mathbf{x} = \begin{pmatrix} 2 & -2 & -1 \\ 1 & -1 & -1 \\ 1 & -2 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad (3.5.1.28)$$

Now, as  $f$  is a linear functional on  $R^3$ ,

$$\alpha = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_2 x_2 \quad (3.5.1.29)$$

$$\Rightarrow f(\alpha) = f(\alpha_1 x_1 + \alpha_2 x_2 + \alpha_2 x_2) \quad (3.5.1.30)$$

$$\Rightarrow f(\alpha) = x_1 f(\alpha_1) + x_2 f(\alpha_2) + x_3 f(\alpha_3) \quad (3.5.1.31)$$

$$\Rightarrow f(\alpha) = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} f(\alpha_1) \\ f(\alpha_2) \\ f(\alpha_3) \end{pmatrix} \quad (3.5.1.32)$$

$$\Rightarrow f(\alpha) = \mathbf{x}^T \begin{pmatrix} f(\alpha_1) \\ f(\alpha_2) \\ f(\alpha_3) \end{pmatrix} \quad (3.5.1.33)$$

As mentioned in the problem statement,  $f(\alpha_1) = f(\alpha_2) = 0$  and  $f(\alpha_3) \neq 0$ .

Now,

$$f(\alpha) = \mathbf{x}^T \begin{pmatrix} f(\alpha_1) \\ f(\alpha_2) \\ f(\alpha_3) \end{pmatrix} \quad (3.5.1.34)$$

$$\Rightarrow f(\alpha) = \mathbf{x}^T \begin{pmatrix} 0 \\ 0 \\ f(\alpha_3) \end{pmatrix} \quad (3.5.1.35)$$

$$\Rightarrow f(\alpha) = \begin{pmatrix} a \\ b \\ c \end{pmatrix}^T \begin{pmatrix} 2 & 1 & 1 \\ -2 & -1 & -2 \\ -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ f(\alpha_3) \end{pmatrix} \quad (3.5.1.36)$$

$$\Rightarrow f(\alpha) = \begin{pmatrix} a \\ b \\ c \end{pmatrix}^T \begin{pmatrix} f(\alpha_3) \\ -2f(\alpha_3) \\ -f(\alpha_3) \end{pmatrix} \quad (3.5.1.37)$$

$$\Rightarrow f(a, b, c) = f(\alpha_3) \begin{pmatrix} a \\ b \\ c \end{pmatrix}^T \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} \quad (3.5.1.38)$$

So, the function can be defined as:

$$f(a, b, c) = f(\alpha_3) \begin{pmatrix} a \\ b \\ c \end{pmatrix}^T \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} \quad (3.5.1.39)$$

$$\text{or, } f(\alpha) = f(\alpha_3)(a - 2b - c) \quad (3.5.1.40)$$

3.5.2. Let  $\mathbf{B} = \{\alpha_1, \alpha_2, \alpha_3\}$  be the basis for  $\mathbf{C}^3$  defined by

$$\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} \quad (3.5.2.1)$$

Find the dual basis of  $\mathbf{B}$ .

**Solution:** Let  $\{f_1, f_2, f_3\}$  be the dual basis of  $\mathbf{B}$  such that,

$$f_i(\alpha_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \quad (3.5.2.2)$$

and

$$f_i(\alpha_1, \alpha_2, \alpha_3) = \sum_{j=1}^3 \delta_{ij} \alpha_j \quad (3.5.2.3)$$

Given,

$$\mathbf{B} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ -1 & 1 & 0 \end{pmatrix} \quad (3.5.2.4)$$

Then,

$$\mathbf{B}^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 1 \\ -\frac{1}{2} & 1 & -\frac{1}{2} \end{pmatrix} \quad (3.5.2.5)$$

So the elements of dual basis are rows of matrix  $\mathbf{B}^{-1}$ . Therefore we get,

$$(\delta_{11}, \delta_{12}, \delta_{13}) = (1, -1, 0) \quad (3.5.2.6)$$

$$(\delta_{21}, \delta_{22}, \delta_{23}) = (1, -1, 1) \quad (3.5.2.7)$$

$$(\delta_{31}, \delta_{32}, \delta_{33}) = \left(-\frac{1}{2}, 1, -\frac{1}{2}\right) \quad (3.5.2.8)$$

Using (3.5.2.3), we get

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 1 \\ -\frac{1}{2} & 1 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \quad (3.5.2.9)$$

$$= \begin{pmatrix} \alpha_1 - \alpha_2 \\ \alpha_1 - \alpha_2 + \alpha_3 \\ -\frac{1}{2}\alpha_1 + \alpha_2 - \frac{1}{2}\alpha_3 \end{pmatrix} \quad (3.5.2.10)$$

Hence,  $\{f_1, f_2, f_3\}$  is the required dual basis for  $\mathbf{B}$ .

3.5.3. If  $\mathbf{A}$  and  $\mathbf{B}$  are  $n \times n$  matrices over the field  $\mathbf{F}$ , show that  $\text{trace}(\mathbf{AB}) = \text{trace}(\mathbf{BA})$ . Now show that similar matrices have the same trace.

**Solution:**

$$\text{trace}(\mathbf{AB}) = \sum_{i=1}^n (\mathbf{AB})_{ii} \quad (3.5.3.1)$$

$$= \sum_{i=1}^n \sum_{k=1}^n \mathbf{A}_{ik} \mathbf{B}_{ki} \quad (3.5.3.2)$$

$$= \sum_{i=1}^n \sum_{k=1}^n \mathbf{B}_{ki} \mathbf{A}_{ik} \quad (3.5.3.3)$$

$$= \sum_{i=1}^n (\mathbf{BA})_{ii} \quad (3.5.3.4)$$

$$= \text{trace}(\mathbf{BA}) \quad (3.5.3.5)$$

Hence proved  $\text{trace}(\mathbf{AB}) = \text{trace}(\mathbf{BA})$ . Let  $\mathbf{A}$  and  $\mathbf{B}$  be similar matrices then  $\exists \mathbf{S}$  such that,

$$\mathbf{A} = \mathbf{S}^{-1} \mathbf{B} \mathbf{S} \quad (3.5.3.6)$$

Taking trace of  $\mathbf{A}$  we get,

$$\text{trace}(\mathbf{A}) = \text{trace}(\mathbf{S}^{-1} \mathbf{B} \mathbf{S}) = \text{trace}(\mathbf{S}^{-1} (\mathbf{B} \mathbf{S})) \quad (3.5.3.7)$$

Using (3.5.3.5) in (3.5.3.7)

$$\text{trace}(\mathbf{A}) = \text{trace}(\mathbf{S}^{-1} \mathbf{S} \mathbf{B}) \quad (3.5.3.8)$$

$$= \text{trace}((\mathbf{S}^{-1} \mathbf{S}) \mathbf{B}) \quad (3.5.3.9)$$

$$= \text{trace}(\mathbf{IB}) \quad (3.5.3.10)$$

$$= \text{trace}(\mathbf{B}) \quad (3.5.3.11)$$

Hence Proved

3.5.4. Let  $\mathbf{V}$  be the vector space of all polynomial functions  $p$  from  $\mathbf{R}$  into  $\mathbf{R}$  which have degree 2 or less:

$$p(x) = c_0 + c_1 x + c_2 x^2$$

Define three linear functionals on  $\mathbf{V}$  by

$$f_1(p) = \int_0^1 p(x) dx; \quad f_2(p) = \int_0^2 p(x) dx;$$

$$f_3(p) = \int_0^{-1} p(x) dx$$

Show that  $\{f_1, f_2, f_3\}$  is a basis for  $\mathbf{V}^*$  by exhibiting the basis for  $\mathbf{V}$  of which it is the dual.

**Solution:** Given the basis  $\mathbf{F}$  and corresponding dual basis  $\mathbf{G}$ , the defining property of the dual

basis states that:

$$\begin{aligned} \mathbf{G}^T \mathbf{F} &= \mathbf{I} \\ \Rightarrow \mathbf{G} &= (\mathbf{F}^{-1})^T \end{aligned} \quad (3.5.4.1)$$

Let the indexed vector sets,

$$\mathbf{V} = \{f_1, f_2, f_3\}; \quad \mathbf{V}^* = \{\alpha_1, \alpha_2, \alpha_3\}$$

1. Let,

$$\mathbf{p}' = \mathbf{c}^T \mathbf{x} \quad (3.5.4.2)$$

where,

$$\mathbf{c} = \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix}; \quad \mathbf{x} = \begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix}$$

2. Representing the functionals as vector,

$$\mathbf{f} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \quad (3.5.4.3)$$

3. Representing the integrations as vector,

$$\mathbf{I} = \begin{pmatrix} \int_0^1 dx \\ \int_0^2 dx \\ \int_0^{-1} dx \end{pmatrix} \quad (3.5.4.4)$$

4. So,

$$\mathbf{f} = \mathbf{I} \mathbf{c}^T \mathbf{x} = \mathbf{I} \mathbf{p}' \quad (3.5.4.5)$$

(3.5.4.5) can be written in matrix format as,

$$\mathbf{f} = \mathbf{P} \mathbf{c} \quad (3.5.4.6)$$

where,

$$\mathbf{P} = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 2 & 2 & \frac{8}{3} \\ -1 & \frac{1}{2} & \frac{-1}{3} \end{pmatrix} \quad (3.5.4.7)$$

5.  $\mathbf{P}$  is one-one if it has an inverse. Calculating the determinant of  $\mathbf{P}$ ,

$$\Rightarrow |\mathbf{P}| = -2 \quad (3.5.4.8)$$

From, (3.5.4.8),  $\mathbf{P}$  is one-one. Also,

$$\mathbf{V} = \mathbf{f}^T = (f_1 \quad f_2 \quad f_3) \quad (3.5.4.9)$$

From (3.5.4.6), (3.5.4.8) and (3.5.4.9), the rows of  $\mathbf{P}$  are isomorphic to  $\mathbf{V}$ . So, finding the dual

basis by performing matrix operations on  $\mathbf{P}^T$

$$\begin{pmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ \frac{1}{2} & 2 & \frac{1}{2} & 0 & 1 & 0 \\ \frac{1}{3} & \frac{8}{3} & \frac{-1}{3} & 0 & 0 & 1 \end{pmatrix} \begin{array}{l} R_3 \leftarrow R_3 - \frac{R_1}{3} \\ R_2 \leftarrow R_2 - \frac{R_1}{2} \end{array}$$

$$\begin{pmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{-1}{2} & 1 & 0 \\ 0 & 2 & 0 & \frac{-1}{3} & 0 & 1 \end{pmatrix} \begin{array}{l} R_3 \leftarrow R_3 - \frac{R_2}{2} \end{array}$$

$$\begin{pmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{-1}{2} & 1 & 0 \\ 0 & 0 & 1 & \frac{2}{3} & 1 & \frac{-1}{2} \end{pmatrix} \begin{array}{l} R_1 \leftarrow R_1 + R_3 \\ R_2 \leftarrow R_2 - R_3 \end{array}$$

$$\begin{pmatrix} 1 & 2 & 0 & \frac{2}{3} & 1 & \frac{-1}{2} \\ 0 & 1 & 0 & \frac{-3}{6} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{-1}{3} & 1 & \frac{-1}{2} \end{pmatrix} \begin{array}{l} R_1 \leftarrow R_1 - 2R_2 \end{array}$$

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & \frac{-3}{2} \\ 0 & 1 & 0 & \frac{-1}{6} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{-1}{3} & 1 & \frac{-1}{2} \end{pmatrix} \quad (3.5.4.10)$$

From (3.5.4.10), the dual of  $\mathbf{V} \Rightarrow \mathbf{V}^*$  can be written in matrix form as,

$$\mathbf{V}^* = \mathbf{A} \mathbf{x} \quad (3.5.4.11)$$

where,

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & \frac{-3}{2} \\ \frac{-1}{6} & 0 & \frac{1}{2} \\ \frac{-1}{3} & 1 & \frac{-1}{2} \end{pmatrix}; \quad \mathbf{x} = \begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix}$$

3.5.5. If  $\mathbf{A}$  and  $\mathbf{B}$  are  $n \times n$  matrices, show that

$$\mathbf{AB} - \mathbf{BA} = \mathbf{I} \quad (3.5.5.1)$$

is impossible.

**Solution:**

$$\text{tr}(\mathbf{AB}) = \text{tr}(\mathbf{BA}) \quad (3.5.5.2)$$

$$\text{tr}(\mathbf{I}) = \sum_{j=1}^n I_{jj} \quad (3.5.5.3)$$

$$\Rightarrow \sum_{j=1}^n 1 = n \quad (3.5.5.4)$$

Taking tr on both sides (3.5.5.1)

$$\text{tr}(\mathbf{AB} - \mathbf{BA}) = \text{tr}(\mathbf{I}) \quad (3.5.5.5)$$

$$\Rightarrow \text{tr}(\mathbf{AB}) - \text{tr}(\mathbf{BA}) = \text{tr}(\mathbf{I}) \quad (3.5.5.6)$$



From (3.5.5.2) and (3.5.5.4)

$$tr(\mathbf{AB}) - tr(\mathbf{BA}) = 0 \quad (3.5.5.7)$$

$$tr(\mathbf{I}) = n \quad (3.5.5.8)$$

$$= 0 \neq n \quad (3.5.5.9)$$

Thus:-

$$tr(\mathbf{AB} - \mathbf{BA}) \neq tr(\mathbf{I}) \quad (3.5.5.10)$$

hence no solution possible.

3.5.6. Let  $m$  and  $n$  be positive integers and field  $\mathbf{F}$ . Let  $f_1, \dots, f_m$  be linear functions in  $F^n$ . For  $\alpha$  in  $F^n$  define

$$T\alpha = (f_1(\alpha), \dots, f_m(\alpha)) \quad (3.5.6.1)$$

show that  $T$  is a linear transformation from  $F^n$  into  $F^m$ . Then show that every linear transformation from  $F^n$  into  $F^m$  is of the above form, for some  $f_1, \dots, f_m$ .

**Solution:** Let  $\mathbf{b}, \alpha \in F^n$  and  $a$  is a scalar

$$\begin{aligned} T(a\alpha + \mathbf{b}) &= (f_1(a\alpha + \mathbf{b}), \dots, f_m(a\alpha + \mathbf{b})) \\ &= (af_1(\alpha) + f_1(\mathbf{b}), \dots, af_m(\alpha) + f_m(\mathbf{b})) \\ &= a(f_1(\alpha), \dots, f_m(\alpha)) + (f_1(\mathbf{b}), \dots, f_m(\mathbf{b})) \end{aligned} \quad (3.5.6.2)$$

The equation (3.5.6.2) can be written as

$$T(a\alpha + \mathbf{b}) = aT(\alpha) + T(\mathbf{b}) \quad (3.5.6.3)$$

So,  $T$  is a linear transformation.

Let the matrix  $A$  of order  $m \times n$  represent any linear transformation  $\mathbf{X} \mapsto A\mathbf{X}$  from  $F^n$  into  $F^m$ . For  $i=1, \dots, m$ , let

$$f_i(x_1, \dots, x_n) = \sum_{j=1}^n A_{ij}x_j \quad (3.5.6.4)$$

The transformation into  $F^m$ ,  $A\mathbf{X}$  can be written as

$$(f_1(\mathbf{X}), \dots, f_m(\mathbf{X})) \quad (3.5.6.5)$$

This is of the form (3.5.6.1)

3.5.7. Let  $\alpha_1 = (1, 0, -1, 2)$  and  $\alpha_2 = (2, 3, 1, 1)$  and let  $\mathbf{W}$  be the subspace of  $\mathbb{R}^4$  spanned by  $\alpha_1$  and  $\alpha_2$ . Which linear functionals  $\mathbf{f}$  :

$$f(x_1, x_2, x_3, x_4) = c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 \quad (3.5.7.1)$$

are in the annihilator of  $\mathbf{W}$ ?

**Solution:** (3.5.7.1), can be expressed as

|                            |  |
|----------------------------|--|
| Linear Functional          | If $\mathbf{V}$ is a vector space over the field $\mathbf{F}$ , field $\mathbf{F}$ is also called a linear functional                |
| Dual Space of $\mathbf{V}$ | If $\mathbf{V}$ is a vector space, the collection of It is the space $\mathbf{L}(\mathbf{V}, \mathbf{F})$ which we denote            |
| Annihilator of $S$         | $\mathbf{V}$ is a vector space over the field $\mathbf{F}$ and $S^\circ$ of linear functionals $\mathbf{f}$ on $\mathbf{V}$ such tha |

$$\mathbf{f}(\mathbf{x}) = \mathbf{x}^T \mathbf{c} \quad (3.5.7.2)$$

$$\text{where } \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \text{ and } \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} \quad (3.5.7.3)$$

Given two vectors

$$\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 2 \end{pmatrix} \text{ and } \alpha_2 = \begin{pmatrix} 2 \\ 3 \\ 1 \\ 1 \end{pmatrix} \quad (3.5.7.4)$$

Since,  $\alpha_1$  is not a scalar multiple of  $\alpha_2$ . Thus,  $\alpha_1$  and  $\alpha_2$  are linearly independent. Also, given  $\alpha_1$  and  $\alpha_2$  span  $\mathbf{W}$ , thus  $\{\alpha_1, \alpha_2\}$  form basis for  $\mathbf{W}$ . Hence,  $\mathbf{W}$  has dimension 2.

Now, a functional  $\mathbf{f}$  is in the annihilator of  $\mathbf{W}$  if and only if  $\mathbf{f}(\alpha_1) = \mathbf{f}(\alpha_2) = 0$ . We find such  $\mathbf{f}$  by solving the system

$$\mathbf{f}(\alpha_1) = \mathbf{f}(\alpha_2) = 0 \quad (3.5.7.5)$$

From, (3.5.7.2) and (3.5.7.5), we have

$$\begin{pmatrix} 1 & 0 & -1 & 2 \\ 2 & 3 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = 0 \quad (3.5.7.6)$$

Converting (3.5.7.6) into row reduced echelon form

$$\begin{pmatrix} 1 & 0 & -1 & 2 \\ 2 & 3 & 1 & 1 \end{pmatrix} \xrightarrow{r_2=r_2-2r_1} \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 3 & 3 & -3 \end{pmatrix} \quad (3.5.7.7)$$

$$\begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 3 & 3 & -3 \end{pmatrix} \xrightarrow{r_2=r_2/3} \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \end{pmatrix} \quad (3.5.7.8)$$

from (3.5.7.8), we have

$$\begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = 0 \quad (3.5.7.9)$$

The general element of  $\mathbf{W}^0$  is therefore

$$\mathbf{f}(\mathbf{x}) = \mathbf{x}^T \mathbf{a} \quad (3.5.7.10)$$

where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \quad \text{and} \quad \mathbf{a} = \begin{pmatrix} c_3 - 2c_4 \\ -c_3 + c_4 \\ c_3 \\ c_4 \end{pmatrix} \quad (3.5.7.11)$$

for arbitrary constants  $c_3$  and  $c_4$ .

Also,  $\mathbf{W}^0$  has dimension 2.

3.5.8. Let  $\mathbf{W}$  be the subspace of  $\mathbb{R}^5$  which is spanned by the vectors

$$\begin{aligned} \alpha_1 &= \epsilon_1 + 2\epsilon_2 + \epsilon_3, \\ \alpha_2 &= \epsilon_2 + 3\epsilon_3 + 3\epsilon_4 + \epsilon_5, \\ \alpha_3 &= \epsilon_1 + 4\epsilon_2 + 6\epsilon_3 + 4\epsilon_4 + \epsilon_5 \end{aligned} \quad (3.5.8.1)$$

Find a basis for  $\mathbf{W}^0$

**Solution:** If  $\mathbf{V}$  is a vector space over the field  $\mathbb{F}$  and  $\mathbf{W}$  is a subset of  $\mathbf{V}$ , the annihilator of  $\mathbf{W}$  is the set  $\mathbf{W}^0$  of linear functionals  $\mathbf{f}$  on  $\mathbf{V}$  such that  $\mathbf{f}(\alpha) = 0$  for every  $\alpha$  in  $\mathbf{W}$ .

*Properties of Annihilator:* If  $f$  is a linear functional on  $\mathbb{R}^n$ :

$$f(x_1, x_2, \dots, x_n) = \sum_{j=1}^n c_j x_j \quad (3.5.8.2)$$

Then  $f$  is in  $\mathbf{W}^0$  if and only if

$$\forall \alpha \in \mathbf{W}: f(\alpha) = 0 \iff f = 0 \quad (3.5.8.3)$$

(3.5.8.2) can be expressed as:

$$\mathbf{f}(\mathbf{x}) = \mathbf{x}^T \mathbf{c} \quad (3.5.8.4)$$

$$\text{where } \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \quad \text{and} \quad \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{pmatrix} \quad (3.5.8.5)$$

Given three vectors

$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 3 \\ 3 \\ 1 \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} 1 \\ 4 \\ 6 \\ 4 \\ 1 \end{pmatrix} \quad (3.5.8.6)$$

Let matrix  $A$  with column vectors  $\alpha_1, \alpha_2, \alpha_3$ :

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 4 \\ 1 & 3 & 6 \\ 0 & 3 & 4 \\ 0 & 1 & 1 \end{pmatrix} \quad (3.5.8.7)$$

Given that  $\mathbf{f}$  is a linear functional on  $\mathbb{R}^5$ , then  $\mathbf{f}$  is in  $\mathbf{W}^0$  if and only if,

$$f(\alpha_i) = 0, i = 1, 2, 3 \quad (3.5.8.8)$$

$$\implies \mathbf{A}^T \mathbf{c} = \mathbf{0} \quad (3.5.8.9)$$

Converting the (3.5.8.9) into system of equations, we have,

$$\begin{pmatrix} 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & 3 & 1 \\ 1 & 4 & 6 & 4 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{pmatrix} = 0 \quad (3.5.8.10)$$

Converting (3.5.8.10) into row reduced echelon form,

$$\begin{pmatrix} 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & 3 & 1 \\ 1 & 4 & 6 & 4 & 1 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & 0 & 0 & 4 & 3 \\ 0 & 1 & 0 & -3 & -2 \\ 0 & 0 & 1 & 2 & 1 \end{pmatrix} \quad (3.5.8.11)$$

From (3.5.8.11), we have,

$$c_1 = -(4c_4 + 3c_5) \quad (3.5.8.12)$$

$$c_2 = (3c_4 + 2c_5) \quad (3.5.8.13)$$

$$c_3 = -(2c_4 + c_5) \quad (3.5.8.14)$$

Therefore, general element of  $\mathbf{W}^0$  is therefore,

$$\begin{aligned} f(x_1, \dots, x_5) &= -(4c_4 + 3c_5)x_1 + (3c_4 + 2c_5)x_2 \\ &\quad -(2c_4 + c_5)x_3 + c_4x_4 + c_5x_5 \end{aligned} \quad (3.5.8.15)$$

Therefore, dimension of  $\mathbf{W}^0$  is 2 and a basis  $\{f_1, f_2\}$  can be obtained by putting  $c_4 = 0, c_5 =$

1 and  $c_4 = 1, c_5 = 0$  in (3.5.8.15)

$$f_1(x_1, \dots, x_5) = -3x_1 + 2x_2 - x_3 + x_5 \quad (3.5.8.16)$$

$$f_2(x_1, \dots, x_5) = -4x_1 + 3x_2 - 2x_3 + x_4 \quad (3.5.8.17)$$

3.5.9. Let  $\mathbb{V}$  be the vector space of all  $2 \times 2$  matrices over the field of real numbers, and let

$$\mathbf{B} = \begin{pmatrix} 2 & -2 \\ -1 & 1 \end{pmatrix} \quad (3.5.9.1)$$

Let  $\mathbb{W}$  be the subspace of  $\mathbb{V}$  consisting of all  $\mathbf{A}$  such that  $\mathbf{AB} = 0$ . Let  $f$  be a linear functional on  $\mathbb{V}$  which is in the annihilator of  $\mathbb{W}$ . Suppose that  $f(\mathbf{I}) = 0$  and  $f(\mathbf{C}) = 3$ , where  $\mathbf{I}$  is the  $2 \times 2$  identity matrix and

$$\mathbf{C} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (3.5.9.2)$$

Find  $f(\mathbf{B})$

**Solution:** The general Linear functional  $f$  on  $\mathbb{V}$  is of the form,

$$f(\mathbf{A}) = aA_{11} + bA_{12} + cA_{21} + dA_{22} \quad (3.5.9.3)$$

for  $a, b, c, d \in \mathbf{R}$  Let  $\mathbf{A} \in \mathbf{W}$  be,

$$\mathbf{A} = \begin{pmatrix} p & q \\ q & s \end{pmatrix} \quad (3.5.9.4)$$

$$\therefore \mathbf{AB} = 0 \quad (3.5.9.5)$$

$$\Rightarrow \begin{pmatrix} p & q \\ q & s \end{pmatrix} \begin{pmatrix} 2 & -2 \\ -1 & 1 \end{pmatrix} = 0 \quad (3.5.9.6)$$

$$\Rightarrow \begin{pmatrix} 2p - q & -2p + q \\ 2q - s & -2q + s \end{pmatrix} = 0 \quad (3.5.9.7)$$

$\therefore q=2p$  and  $s=2q$ . Hence  $\mathbf{W}$  consists of all matrices of the form

$$\begin{pmatrix} p & 2p \\ q & 2q \end{pmatrix} \quad (3.5.9.8)$$

Now  $\mathbf{V}$  is an annihilator of  $\mathbf{W}$ . Hence,  $f \in \mathbf{W}^0$

$$\Rightarrow f \begin{pmatrix} p & 2p \\ q & 2q \end{pmatrix} = 0 \forall p, q \in \mathbf{R} \quad (3.5.9.9)$$

from equation (3.5.9.3)

$$\Rightarrow ap + 2bp + cq + 2dq = 0 \forall p, q \in \mathbf{R} \quad (3.5.9.10)$$

$$\Rightarrow (a + 2b)p + (c + 2d)q = 0, \forall p, q \in \mathbf{R} \quad (3.5.9.11)$$

Hence,  $b = -\frac{1}{2}a$  and  $d = -\frac{1}{2}c$ . Hence general  $f \in \mathbf{W}^0$  is of the form,

$$f(\mathbf{A}) = aA_{11} - \frac{1}{2}aA_{12} + cA_{21} - \frac{1}{2}cA_{22} \quad (3.5.9.12)$$

Now,  $f(\mathbf{C}) = 3 \Rightarrow d = 3 \Rightarrow c = -6$ . Also given that,  $f(\mathbf{I}) = 0 \Rightarrow a - \frac{1}{2}c = 0 \Rightarrow a = -3$ . Substituting the above parameters in equation (3.5.9.12) we get,

$$\therefore f(\mathbf{A}) = -3A_{11} + \frac{3}{2}A_{12} - 6A_{21} + 3A_{22} \quad (3.5.9.13)$$

$$\text{Now, } f(\mathbf{B}) = f \begin{pmatrix} 2 & -2 \\ -1 & 1 \end{pmatrix} \quad (3.5.9.14)$$

$$\Rightarrow f(\mathbf{B}) = -3(2) + \frac{3}{2}(-2) - 6(-1) + 3(1) = 0 \quad (3.5.9.15)$$

3.5.10. Let  $F$  be a subfield of the complex numbers. We define  $n$  linear functionals on  $F^n (n \geq 2)$  by

$$f_k(x_1, \dots, x_n) = \sum_{j=1}^n (k - j)x_j, 1 \leq k \leq n. \quad (3.5.10.1)$$

What is the dimension of the subspace annihilated by  $f_1, f_2, \dots, f_n$ ?

**Solution:** See Tables 3.5.10.1 and 3.5.10.2.

|         |   |
|---------|---|
| Given   | $F$ be a subfield of the complex numbers<br>Definition of $n$ linear functionals on $F^n (n \geq 2)$ by   |
| To find | The dimension of the subspace annihilated by $f_1, f_2, \dots, f_n$   |
| $f_k$   | $f_k(x_1, \dots, x_n) = \sum_{j=1}^n (k - j)x_j$<br>$f_k(x_1, \dots, x_n) = k \sum_{j=1}^n x_j$<br>All $f_k$ are linear combinations of the $x_j$ |

TABLE 3.5.10.1

3.5.11. Let  $W_1$  and  $W_2$  be subspaces of a finite-dimensional vector space  $\mathbb{V}$ . Prove that

|        |   |
|--------|---|
| Vector | <p> <math>f_k(x_1, \dots, x_n) = \sum_{j=1}^n (k_j) x_j</math> </p> <p> <math>\mathbf{f} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}</math> </p> <p> <math>\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}</math> </p> <p> <math>f_1(\mathbf{x}) = 0.x_1 - x_2 - 2x_3 - \dots - (n-1)x_n</math><br/> <math>f_2(\mathbf{x}) = x_1 + 0.x_2 - 1.x_3 - \dots - (n-2)x_n</math><br/> <math>\vdots</math><br/> <math>f_n(\mathbf{x}) = (n-1)x_1 + (n-2).x_2 + \dots + (n-2)x_{n-1} + 0.x_n</math> </p> <p> <math>A_{n \times n} = \begin{pmatrix} 0 &amp; -1 &amp; -2 &amp; \cdot &amp; \cdot &amp; -(n-1) \\ 1 &amp; 0 &amp; -1 &amp; \cdot &amp; \cdot &amp; -(n-2) \\ \cdot &amp; \cdot &amp; \cdot &amp; \cdot &amp; \cdot &amp; \cdot \\ (n-1) &amp; (n-2) &amp; \cdot &amp; \cdot &amp; (n-2) &amp; 0 \end{pmatrix}</math> </p> <p> a) <math>(W_1 + W_2)^0 = W_1^0 \cap W_2^0</math><br/> b) <math>(W_1 \cap W_2)^0 = W_1^0 + W_2^0</math><br/> <b>Solution:</b> See Table 3.5.11.1 </p> |
| Matrix | <p> <math>AX = 0</math><br/> where the <math>i^{th}</math> row is defined by<br/> <math>A_i = (i-1, i-2, \dots, i-n)</math><br/> <math>1 \leq i \leq n</math> </p> <p> For the <math>n = 4</math>, matrix <math>A</math> is: <math display="block">\begin{pmatrix} 0 &amp; -1 &amp; -2 &amp; -3 \\ 1 &amp; 0 &amp; -1 &amp; -2 \\ 2 &amp; 1 &amp; 0 &amp; -1 \\ 3 &amp; 2 &amp; 1 &amp; 0 \end{pmatrix}</math> </p> <p> For <math>i \geq 3</math>, perform the following elementary operations of <math>n</math> linear functionals as defined below<br/> (a) <math>A_i \longrightarrow (1-i)A_2 + A_i</math><br/> <math>A_i = (0, i-2, 2(i-2), 3(i-2), \dots, (n-1)(i-2))</math><br/> (b) <math>A_i \longrightarrow \frac{1}{i-2}A_i</math><br/> <math>A_i = -A_1</math><br/> (c) <math>A_i \longrightarrow A_i + A_1</math><br/> <math>A_i = 0</math> </p> <p> Since, <math>A_1</math> and <math>A_2</math> are linearly independent<br/> Thus, the dimension of the subspace annihilated <math>= n-2</math> </p>             |

TABLE 3.5.10.2

|   |  |
|---|--|
| Given   | $W_1$ and $W_2$ are subspaces of a finite dimensional vector space $\mathbb{V}$  |
| 1. To prove   | $(W_1 + W_2)^0 = W_1^0 \cap W_2^0$   |
| Proof<br>of<br>$(W_1 + W_2)^0$<br>$\subseteq$<br>$(W_1^0 \cap W_2^0)$ | <p>Let <math>f \in (W_1 + W_2)^0</math></p> <p><math>\forall \mathbf{v} \in (W_1 + W_2)</math><br/> <math>f(\mathbf{v}) = 0</math><br/> <math>\implies \forall \mathbf{w}_1 \in W_1, \mathbf{w}_2 \in W_2</math><br/> <math>\mathbf{w}_1 + \mathbf{w}_2 \in (W_1 + W_2)</math><br/> <math>\therefore f(\mathbf{w}_1 + \mathbf{w}_2) = 0</math></p> <p><math>\implies</math> When <math>\mathbf{w}_2 = 0</math>, then <math>\forall \mathbf{w}_1 \in W_1</math><br/> <math>f(\mathbf{w}_1) = 0</math><br/> <math>\therefore f \in W_1^0</math></p> <p>And when <math>\mathbf{w}_1 = 0, \forall \mathbf{w}_2 \in W_2</math><br/> <math>f(\mathbf{w}_2) = 0</math><br/> <math>\therefore f \in W_2^0</math></p> <p><math>\therefore f \in W_1^0, f \in W_2^0</math><br/> <math>\implies f \in (W_1^0 \cap W_2^0)</math></p> <p><math>\therefore (W_1 + W_2)^0 \subseteq (W_1^0 \cap W_2^0)</math></p> |
| Proof<br>of<br>$(W_1^0 \cap W_2^0)$<br>$\subseteq$<br>$(W_1 + W_2)^0$ | <p>Let <math>f \in (W_1^0 \cap W_2^0)</math></p> <p><math>\implies f \in W_1^0, f \in W_2^0</math><br/> <math>\implies \forall \mathbf{w}_1 \in W_1, \mathbf{w}_2 \in W_2</math><br/> <math>f(\mathbf{w}_1) = 0</math> and <math>f(\mathbf{w}_2) = 0</math></p> <p><math>\forall \mathbf{v} \in (W_1 + W_2),</math></p>  |
|   | <p><math>\mathbf{v} = \mathbf{w}_1 + \mathbf{w}_2</math><br/> <math>\implies f(\mathbf{v}) = f(\mathbf{w}_1 + \mathbf{w}_2)</math><br/> <math>\implies f(\mathbf{v}) = f(\mathbf{w}_1) + f(\mathbf{w}_2)</math><br/> <math>\implies f(\mathbf{v}) = 0</math><br/> <math>\implies f \in (W_1 + W_2)^0</math></p> <p><math>\therefore (W_1^0 \cap W_2^0) \subseteq (W_1 + W_2)^0</math></p>  |
|   | Hence, $(W_1 + W_2)^0 = (W_1^0 \cap W_2^0)$  |
| 2. To prove   | $(W_1 \cap W_2)^0 = W_1^0 + W_2^0$   |

|   |   |
|---|---|
| <p>Proof<br/>of<br/><math>(W_1^0 + W_2^0)</math><br/><math>\subseteq</math><br/><math>(W_1 \cap W_2)^0</math></p> | <p>Let <math>f \in (W_1^0 + W_2^0)</math><br/>for some <math>f_1 \in W_1^0, f_2 \in W_2^0</math>,<br/><math>f = f_1 + f_2</math></p> <p>Now, for <math>\mathbf{v} \in (W_1 \cap W_2)</math><br/><math>f(\mathbf{v}) = (f_1 + f_2)(\mathbf{v})</math><br/><math>\implies f(\mathbf{v}) = f_1(\mathbf{v}) + f_2(\mathbf{v})</math></p> <p><math>\because \mathbf{v} \in (W_1 \cap W_2)</math><br/><math>\implies \mathbf{v} \in W_1</math>, and <math>\mathbf{v} \in W_2</math><br/>So, <math>f_1(\mathbf{v}) = 0</math>, and <math>f_2(\mathbf{v}) = 0</math></p> <p><math>\implies f(\mathbf{v}) = 0 + 0 = 0</math><br/><math>\implies f \in (W_1 \cap W_2)^0</math></p> <p><math>\therefore (W_1^0 + W_2^0) \subseteq (W_1 \cap W_2)^0</math></p>  |
| <p>Proof<br/>of<br/><math>(W_1 \cap W_2)^0</math><br/><math>\subseteq</math><br/><math>(W_1^0 + W_2^0)</math></p> | <p>Let <math>f \in (W_1 \cap W_2)^0</math></p> <p>Assuming<br/>Basis of <math>W_1</math> as <math>\{\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_l\}</math><br/>Basis of <math>W_2</math> as <math>\{\alpha_1, \dots, \alpha_k, \gamma_1, \dots, \gamma_m\}</math></p> <p><math>\therefore</math> Basis of <math>(W_1 \cap W_2)</math> is <math>\{\alpha_1, \dots, \alpha_k\}</math><br/>and Basis of <math>W_1 + W_2</math> is <math>\{\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_l, \gamma_1, \dots, \gamma_m\}</math></p> <p>Now, for <math>\mathbf{v} \in (W_1 + W_2)</math><br/><math>\mathbf{v} = \sum_{i=1}^k x_i \alpha_i + \sum_{i=1}^l y_i \beta_i + \sum_{i=1}^m z_i \gamma_i</math><br/><math>f(\mathbf{v}) = \sum_{i=1}^k a_i x_i + \sum_{i=1}^l b_i y_i + \sum_{i=1}^m c_i z_i</math></p> <p><math>\forall \mathbf{v} \in (W_1 \cap W_2)</math><br/><math>\mathbf{v} = \sum_{i=1}^k x_i \alpha_i</math></p> |
|   | <p><math>f(\mathbf{v}) = \sum_{i=1}^k a_i x_i</math><br/>But since <math>f \in (W_1 \cap W_2)^0</math>, thus <math>f(\mathbf{v}) = 0</math><br/><math>\therefore a_1 = a_2 = \dots = a_k = 0</math></p> <p>So, we can now write<br/><math>f(\mathbf{v}) = \sum_{i=1}^l b_i y_i + \sum_{i=1}^m c_i z_i</math></p> <p>Now, <math>\forall \mathbf{v} \in W_1</math>,<br/><math>\mathbf{v} = \sum_{i=1}^k x_i \alpha_i + \sum_{i=1}^l y_i \beta_i</math><br/><math>\implies f_1(\mathbf{v}) = \sum_{i=1}^k a_i x_i + \sum_{i=1}^l b_i y_i</math><br/>Comparing this to the original equation, we can say <math>c_i = 0</math></p> <p>And <math>\forall \mathbf{v} \in W_2</math>,<br/><math>\mathbf{v} = \sum_{i=1}^k x_i \alpha_i + \sum_{i=1}^m z_i \gamma_i</math></p>   |

|              |   |
|--------------|---|
|              | $\Rightarrow f_2(\mathbf{v}) = \sum_{i=1}^k a_i x_i + \sum_{i=1}^m c_i z_i$ <p>Comparing this to the original equation, we can say <math>b_i = 0</math></p> $f = f_1 + f_2$ $\because a_i = b_i = c_i = 0,$ $f_1(\mathbf{v}) = 0$ $\Rightarrow f_1 \in W_1^0$ <p>Also, <math>f_2(\mathbf{v}) = 0</math></p> $\Rightarrow f_2 \in W_2^0$ <p>So, <math>f_1 + f_2 \in (W_1^0 + W_2^0)</math></p> $\Rightarrow f \in (W_1^0 + W_2^0)$ $\therefore (W_1 \cap W_2)^0 \subseteq (W_1^0 + W_2^0)$ |
|              | Hence, $(W_1 \cap W_2)^0 = (W_1^0 + W_2^0)$   |
| Verification | <p>1. Since the annihilator <math>(W_1 + W_2)^0</math> is a complement of <math>(W_1 + W_2)</math>, Using Demorgan's Laws of the complement of union of two sets is the intersection of their complements, it is verified</p> $(W_1 + W_2)^0 = W_1^0 \cap W_2^0$ <p>2. And using De Morgan's laws of the complement of intersection of two sets is the union of their complements, it is verified</p> $(W_1 \cap W_2)^0 = W_1^0 + W_2^0$  |

TABLE 3.5.11.1: Proving properties of vectorspaces and subspaces

3.5.12. Let  $\mathbf{F}$  be a subfield of the field of complex numbers and let  $\mathbf{V}$  be any vector space over  $\mathbf{F}$ . Suppose that  $f$  and  $g$  are linear functionals on  $\mathbf{V}$  such that the function  $h$  defined by  $h(\alpha) = f(\alpha)g(\alpha)$  is also a linear functional on  $\mathbf{V}$ . Prove that either  $f = 0$  or  $g = 0$ .

**Solution:** Refer Table 3.5.12.1.

3.5.13. Let  $\mathbb{F}$  be a field of characteristic zero and let  $\mathbf{V}$  be a finite dimensional vector space over  $\mathbb{F}$ . If  $\alpha_1, \alpha_2, \dots, \alpha_m$  are finitely many vectors in  $\mathbf{V}$ , each different from the zero vector, prove that there is a linear functional  $f$  on  $\mathbf{V}$  such that

$$f(\alpha_i) \neq 0, i = 1, 2, \dots, m \quad (3.5.13.1)$$

**Solution:** We will make use of the following theorem

Let  $\mathbf{V}$  be a finite dimensional vector space over the field  $\mathbb{F}$  and let  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  be a basis for  $\mathbf{V}$ . Then there exists a unique dual basis  $\{f_1, f_2, \dots, f_n\}$  for  $\mathbf{V}^*$  such that

$$f_i(\mathbf{v}_j) = \delta_{ij} \quad (3.5.13.2)$$

and  $\mathbf{V}^*$  is the space of all linear functionals on  $\mathbf{V}$

3.5.14. Similar matrices have the same trace. Thus we can define the trace of a linear operator on a finite-dimensional space to the trace of any matrix which represents the operator in a ordered basis. This is well-defined since all such representing matrices for one operator are similar.

Now let  $\mathbf{V}$  be the space of all  $2 \times 2$  matrices over the field  $\mathbf{F}$  and let  $\mathbf{P}$  be a fixed  $2 \times 2$  matrix. Let  $\mathbf{T}$  be the linear operator on  $\mathbf{V}$  defined by  $\mathbf{T}(\mathbf{A}) = \mathbf{P}\mathbf{A}$ . Prove that  $\text{tr}(\mathbf{T}) = 2\text{tr}(\mathbf{P})$ .

**Solution:** Given  $\mathbf{V}$  is the space of all  $2 \times 2$  matrices over field  $\mathbf{F}$ .  $\mathbf{P}$  is a  $2 \times 2$  matrix,

$$\mathbf{P} = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \quad (3.5.14.1)$$

Let  $\mathcal{B} = \{e_{11}, e_{12}, e_{21}, e_{22}\}$  be the ordered basis of  $\mathbf{V}$  where,

$$e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (3.5.14.2)$$

$$e_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (3.5.14.3)$$

Given,  $\mathbf{T}(\mathbf{A}) = \mathbf{P}\mathbf{A}$ ,

$$\mathbf{T}(e_{11}) = \mathbf{P}e_{11} = \begin{pmatrix} p_{11} & 0 \\ p_{21} & 0 \end{pmatrix} \quad (3.5.14.4)$$

$$= p_{11}e_{11} + p_{21}e_{21} \quad (3.5.14.5)$$

$$\mathbf{T}(e_{12}) = \mathbf{P}e_{12} = \begin{pmatrix} 0 & p_{11} \\ 0 & p_{21} \end{pmatrix} \quad (3.5.14.6)$$

$$= p_{11}e_{12} + p_{21}e_{22} \quad (3.5.14.7)$$

$$\mathbf{T}(e_{21}) = \mathbf{P}e_{21} = \begin{pmatrix} p_{12} & 0 \\ p_{22} & 0 \end{pmatrix} \quad (3.5.14.8)$$

$$= p_{12}e_{11} + p_{22}e_{21} \quad (3.5.14.9)$$

$$\mathbf{T}(e_{22}) = \mathbf{P}e_{22} = \begin{pmatrix} 0 & p_{12} \\ 0 & p_{22} \end{pmatrix} \quad (3.5.14.10)$$

$$= p_{12}e_{12} + p_{22}e_{22} \quad (3.5.14.11)$$

The matrix representation of linear functional  $\mathbf{T}$  in the ordered basis  $\mathcal{B}$  is given as,

$$\mathbf{T} = \begin{pmatrix} [\mathbf{T}(e_{11})]_{\mathcal{B}} & [\mathbf{T}(e_{12})]_{\mathcal{B}} & [\mathbf{T}(e_{21})]_{\mathcal{B}} & [\mathbf{T}(e_{22})]_{\mathcal{B}} \end{pmatrix} \quad (3.5.14.12)$$

$$\mathbf{T} = \begin{pmatrix} p_{11} & 0 & p_{12} & 0 \\ 0 & p_{11} & 0 & p_{12} \\ p_{21} & 0 & p_{22} & 0 \\ 0 & p_{21} & 0 & p_{22} \end{pmatrix} \quad (3.5.14.13)$$

$$= \begin{pmatrix} p_{11} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & p_{12} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ p_{21} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & p_{22} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} \quad (3.5.14.14)$$

$$= \begin{pmatrix} p_{11}I & p_{12}I \\ p_{21}I & p_{22}I \end{pmatrix} = \mathbf{P} \otimes I \quad (3.5.14.15)$$

Now by using the property of kronecker product we get,

$$\text{tr}(\mathbf{T}) = \text{tr}(\mathbf{P} \otimes I) = \text{tr}(\mathbf{P})\text{tr}(I) \quad (3.5.14.16)$$

$$= 2\text{tr}(\mathbf{P}) \quad (3.5.14.17)$$

$$\therefore \text{tr}(\mathbf{T}) = 2\text{tr}(\mathbf{P}) \quad (3.5.14.18)$$

3.5.15. Show that the trace functional on  $n \times n$  matrices is unique in the following sense. If  $\mathbf{W}$  is the space of  $n \times n$  matrices over the field  $\mathbf{F}$  and if  $f$  is a linear functional on  $\mathbf{W}$  such that  $f(\mathbf{AB}) = f(\mathbf{BA})$  for each  $\mathbf{A}$  and  $\mathbf{B}$  in  $\mathbf{W}$ , then  $f$  is a scalar multiple of the trace function. If, in addition,  $f(I) = n$ , then  $f$  is the trace function.

**Solution:** See Tables 3.5.15.1 and 3.5.15.2

3.5.16. Let  $\mathbf{W}$  be the space of  $n \times n$  matrices over the field  $\mathbf{F}$ , and let  $\mathbf{W}_0$  be the subspace spanned by



| Given   | Derivation   |
|---|--|
| $f, g, h$ are linear functionals of $\mathbf{V}$      | <p>By contradiction, let us assume <math>f \neq 0</math> and <math>g \neq 0</math>. For all <math>\mathbf{v} \in \mathbf{V}</math></p> $h(\mathbf{v}) = f(\mathbf{v})g(\mathbf{v}) \quad (3.5.12.1)$ $h(2\mathbf{v}) = f(2\mathbf{v})g(2\mathbf{v}) \quad (3.5.12.2)$ $= 2f(\mathbf{v})2g(\mathbf{v}) \quad (3.5.12.3)$ $= 4f(\mathbf{v})g(\mathbf{v}) \quad (3.5.12.4)$ <p>Similarly,</p> $h(2\mathbf{v}) = 2h(\mathbf{v}) \quad (3.5.12.5)$ $= 2f(\mathbf{v})g(\mathbf{v}) \quad (3.5.12.6)$ <p>From equation (3.5.12.4) and (3.5.12.6),</p> $\implies 4f(\mathbf{v})g(\mathbf{v}) = 2f(\mathbf{v})g(\mathbf{v}) \quad (3.5.12.7)$ $\implies f(\mathbf{v}).g(\mathbf{v}) = 0 \quad (3.5.12.8)$   |
| Choosing Basis  | <p>Let <math>\mathbf{B}</math> be a basis for <math>\mathbf{V}</math>. Let,</p> $\mathbf{B}_1 = \{\mathbf{b} \in \mathbf{B} \mid f(\mathbf{b}) = 0\}, \quad (3.5.12.9)$ $\mathbf{B}_2 = \{\mathbf{b} \in \mathbf{B} \mid g(\mathbf{b}) = 0\} \quad (3.5.12.10)$ <p>Since,</p> $f(\mathbf{b}).g(\mathbf{b}) = 0 \quad \forall \mathbf{b} \in \mathbf{B} \quad (3.5.12.11)$ $\implies f(\mathbf{b}) = 0 \text{ or } g(\mathbf{b}) = 0 \quad (3.5.12.12)$ $\implies \mathbf{b} \in \mathbf{B}_1 \text{ or } \mathbf{b} \in \mathbf{B}_2 \quad (3.5.12.13)$  |
| Choosing $\mathbf{b}_1$ and $\mathbf{b}_2$ from basis | <p>Let us choose <math>\mathbf{b}_1 \in \mathbf{B}_1 - \mathbf{B}_2</math> and <math>\mathbf{b}_2 \in \mathbf{B}_2 - \mathbf{B}_1</math></p> $\implies f(\mathbf{b}_2) \neq 0 \text{ and } g(\mathbf{b}_1) \neq 0$ $f(\mathbf{b}_1 + \mathbf{b}_2).g(\mathbf{b}_1 + \mathbf{b}_2) = (f(\mathbf{b}_1) + f(\mathbf{b}_2)).(g(\mathbf{b}_1) + g(\mathbf{b}_2)) \quad (3.5.12.14)$ $= f(\mathbf{b}_1).g(\mathbf{b}_1) + f(\mathbf{b}_1).g(\mathbf{b}_2) + f(\mathbf{b}_2).g(\mathbf{b}_1) + f(\mathbf{b}_2).g(\mathbf{b}_2) \quad (3.5.12.15)$ $= 0 + 0 + f(\mathbf{b}_2).g(\mathbf{b}_1) + 0 \quad (3.5.12.16)$ $= f(\mathbf{b}_2).g(\mathbf{b}_1) \neq 0 \quad (3.5.12.17)$ <p>Equation (3.5.12.17) is contradiction to the fact that <math>f(\mathbf{v}).g(\mathbf{v}) = 0</math>.</p> $\implies \boxed{f = 0 \text{ or } g = 0}$ |

TABLE 3.5.12.1: Expanation

the matrices  $C$  of the form  $C = AB - BA$ . Prove that  $\mathbf{W}_0$  is exactly the subspace of matrices which have trace zero.

**Solution:** Let there be two subspaces defined

| PARAMETERS                              | DESCRIPTION                                       |
|---|---|
| $\mathbb{F}$                            | Field   |
| $\mathbf{V}$                            | Finite dimensional vector space over $\mathbb{F}$ |
| $\alpha_1, \alpha_2, \dots, \alpha_m$   | non zero vectors in $\mathbf{V}$                  |
| $f : \mathbf{V} \rightarrow \mathbb{F}$ | Linear functional on $\mathbf{V}$                 |

TABLE 3.5.13.1: Input Parameters

as

$$\mathbf{W}_0 = \{A \in \mathbf{W} : \text{trace}(A) = 0\} \quad (3.5.16.1)$$

and

$$\mathbf{W}_1 = \{C \in \mathbf{W} : C = AB - BA\} \quad (3.5.16.2)$$

Consider  $C \in \mathbf{W}$  such that  $C = AB - BA$  where  $A, B \in \mathbf{R}^{N \times N}$  and since

$$\text{trace}(AB) = \text{trace}(BA) \quad (3.5.16.3)$$

$$\implies \text{trace}(AB) - \text{trace}(BA) = 0 \quad (3.5.16.4)$$

$$\implies \text{trace}(C) = 0 \quad (3.5.16.5)$$

Thus any linear combination of matrices  $C = AB - BA$  represents the subspace of all  $N \times N$  matrices with trace equal to 0. Hence, the subspace  $\mathbf{W}_0$  and  $\mathbf{W}_1$  are equivalent.

### 3.6 The Double Dual

3.6.1. Let  $n$  be a positive integer and  $\mathbf{F}$  a field. let  $\mathbf{W}$  be the set of all vectors  $(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n) \in \mathbf{F}^n$  such that  $\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 + \dots + \mathbf{x}_n = 0$ .

a) Prove that  $\mathbf{W}^0$  consists of all linear functional  $f$  of the form

$$f(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n) = c \sum_{j=1}^n \mathbf{x}_j \quad (3.6.1.1)$$

**Solution:** *Definition 1:* If the vector space  $\mathbf{V}$  is finite dimensional (say dimension  $=n$ ), the dimension of null-space  $\mathbf{N}_f$  by rank nullity theorem is given by,

$$|\mathbf{N}_f| = |\mathbf{V}| - 1 = n - 1 \quad (3.6.1.2)$$

*Definition 2:* If  $\mathbf{V}$  is a vector space over the field  $\mathbf{F}$  and  $\mathbf{S}$  is a subset of  $\mathbf{V}$ , the annihilator of  $\mathbf{S}$  is the set  $\mathbf{S}^0$  of linear functional  $f$  on  $\mathbf{V}$  such that  $f(\alpha) = 0$  for every  $\alpha$  in  $\mathbf{S}$ . Solution

Let  $h$  be the functional,

$$h(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n) = \mathbf{x}_1 + \mathbf{x}_2 + \dots + \mathbf{x}_n \quad (3.6.1.3)$$

$$\text{Let, } \mathbf{X} = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_n \end{pmatrix} \quad (3.6.1.4)$$

$$\implies h(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n) = 0 \quad (3.6.1.5)$$

Then  $\mathbf{W}$  is in null-space of  $h$ . Hence by definition-1, the dimension of  $\mathbf{W}$  is,

$$|\mathbf{W}| = n - 1 \quad (3.6.1.6)$$

Now let,

$$a_j = \epsilon_1 - \epsilon_{i+1}, \text{ for } i = (1, \dots, n-1) \quad (3.6.1.7)$$

Hence clearly  $\{a_1, a_2, \dots, a_{n-1}\}$  are linearly independent. Hence from (3.6.1.6) and above statement we can conclude that  $\{a_1, a_2, \dots, a_{n-1}\}$  are all in  $\mathbf{W}$  so they must form basis for  $\mathbf{W}$ . Now, it is given that  $f$  is linear functional hence,

$$f(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n) = \sum_{j=1}^n c_j \mathbf{x}_j \quad (3.6.1.8)$$

$$\implies f(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n) = C^T \mathbf{X} \quad (3.6.1.9)$$

Where,

$$C = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \quad (3.6.1.10)$$

Now  $f \in \mathbf{W}^0$  from definition-2 is given as,

$$f(a_1) = f(a_2) = \dots = f(a_n) = 0 \text{ for every } a_j \in \mathbf{W} \quad (3.6.1.11)$$

$$\Rightarrow c_1 - c_i = 0 \forall i = 2 \cdots n \quad (3.6.1.12)$$

$$\Rightarrow c_i = c_1 = c \forall i \quad (3.6.1.13)$$

$$\Rightarrow C = c \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \quad (3.6.1.14)$$

Hence,

$$f(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n) = C^T \mathbf{X} \quad (3.6.1.15)$$

$$\Rightarrow f(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n) = c \sum_{j=1}^n \mathbf{x}_j \quad (3.6.1.16)$$

**Hence, proved**

- b) Show that the dual space  $\mathbf{W}^*$  of  $\mathbf{W}$  can be ‘naturally’ identified with the linear functionals

$$f(x_1, \dots, x_n) = c_1 x_1 + \dots + c_n x_n$$

on  $\mathbf{F}^n$  which satisfy  $c_1 + \dots + c_n = 0$

**Solution:** See Table 3.6.1.1  
Pictorial Representation

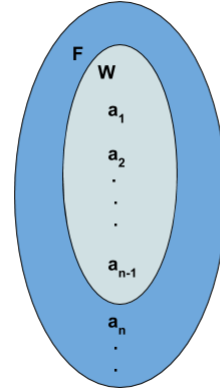


Fig. 3.6.1.1:  $\mathbf{W}$  of dimension  $n-1$ , is the null space of  $\mathbf{F}$ , where  $(a_1, \dots, a_{n-1})$  are basis vectors for  $\mathbf{W}$

| PARAMETERS  | MATRIX REPRESENTATION  |
|---|--|
| Basis for $\mathbf{V}$  | $\mathbf{B} = (\mathbf{v}_1 \ \mathbf{v}_2 \ \dots \ \mathbf{v}_n)$  |
| Basis for $\mathbf{V}^*$  | $\mathbf{B}^* = (\mathbf{f}_1 \ \mathbf{f}_2 \ \dots \ \mathbf{f}_n)$  |
| $f_i(\mathbf{v}_j) = \delta_{ij}$   | $(\mathbf{f}_i)^T \mathbf{v}_j = \delta_{ij}$  |
| $\mathbf{B}, \mathbf{B}^*$ are dual basis   | $(\mathbf{B}^*)^T \mathbf{B} = \mathbf{I}$   |
| $\alpha_i = \sum_{k=1}^n a_k \mathbf{v}_k, i \in [1, m]$                                      | $\alpha_i = \mathbf{B} \mathbf{a}$ where $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}, a_l \in \mathbb{F}, l \in [1, n]$  |
| $\alpha_i \neq \mathbf{0}$  | $\mathbf{a} \neq \mathbf{0}$   |
| Any linear functional $f$ over $\mathbf{V}$   | $\mathbf{f} = \mathbf{B}^* \mathbf{c}$ , where $\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}, c_l \in \mathbb{F}, l \in [1, n]$                                  |
| $f(\alpha_i)$   | $\begin{aligned} & \mathbf{f}^T \alpha_i \\ &= (\mathbf{B}^* \mathbf{c})^T \alpha_i \\ &= \mathbf{c}^T (\mathbf{B}^*)^T \mathbf{B} \mathbf{a} \\ &= \mathbf{c}^T \mathbf{a} \end{aligned}$ |
| $f$ with $c_l \neq 0 \forall l \in [1, n]$ lies in $\mathbf{V}^*, \mathbf{a} \neq \mathbf{0}$ | $f(\alpha_i) = \mathbf{c}^T \mathbf{a} \neq 0, i = 1, 2, \dots, m$   |

TABLE 3.5.13.2: Proof

|              |  |
|--------------|--|
| Given        | <p><math>W</math> is the space of <math>n \times n</math> matrices over the field <math>F</math></p> <p><math>f</math> is a linear functional on <math>W</math>, that <math>f(AB) = f(BA)</math></p>   |
| To prove     | <p><math>f</math> is scalar multiple of trace</p> <p><math>f(A) = c \times \text{trace}(A)</math></p> <p>and</p> <p>If <math>f(I) = n</math>, then <math>f</math> is the trace</p> <p><math>f(A) = \text{trace}(A)</math></p>                      |
| Basis of $W$ | <p>Let <math>\{E^{pq}, 1 \leq p, q \leq n\}</math> be the basis of <math>W</math> where <math>E^{pq}</math> is a <math>n \times n</math> matrix</p> $E^{pq} = \begin{cases} 1 & pq^{th}\text{-position} \\ 0 & \text{everywhere else} \end{cases}$ |

TABLE 3.5.15.1

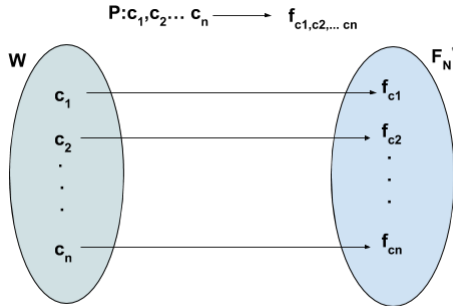


Fig. 3.6.1.2: Mapping from  $W \xrightarrow{P} F_N^*$ , where  $P(c_1, \dots, c_n) = f_{c_1, \dots, c_n}$ ,  $f_{c_1, \dots, c_n}(x_1, \dots, x_n) = c_1 x_1 + \dots + c_n x_n$

3.6.2. Use theorem 20 to prove the following. If  $W$  is a subspace of a finite dimensional vector space  $V$  and if  $\{g_1, g_2, \dots, g_r\}$  is any basis for  $W^0$  then

$$W = \bigcap_{i=1}^r N_{g_i} \quad (3.6.2.1) \quad 3.6.3.$$

**Solution:**

*Theorem 20:* Let  $g, f_1, f_2, \dots, f_r$  be linear functionals on vector space  $V$  with respective null spaces  $N, N_1, \dots, N_r$ .

|                         |  |
|-------------------------|--|
| Find $f(E^{pq})$        | <p>If <math>p = q</math>, we know <math>I = \sum_{p,q=1}^n E^{pp}</math> then</p> $f(I) = \sum_{p,q=1}^n f(E^{pp}) \implies f(I) = n f(E^{pp})$ $\implies f(E^{pp}) = \frac{f(I)}{n}$ <p>If <math>p \neq q</math>, then</p> $f(E^{pq}) = f(E^{p1} E^{1q}) = f(E^{1q} E^{p1}) = 0$ $\text{trace}(E^{pq}) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$ $\therefore f(E^{pq}) = \frac{f(I)}{n} \text{trace}(E^{pq})$  |
| Proof<br>(find $f(A)$ ) | <p>Let <math>A \in W</math>, then <math>A = \sum_{p,q=1}^n c^{pq} E^{pq}</math></p> <p>Also, <math>\text{trace}(A) = \sum_{p,q=1}^n c^{pq} \text{trace}(E^{pq})</math></p> <p>Since <math>f</math> is linear, <math>f(A) = \sum_{p,q=1}^n c^{pq} f(E^{pq})</math></p> $f(A) = \sum_{p,q=1}^n c^{pq} \left( \frac{f(I)}{n} \text{trace}(E^{pq}) \right)$ $f(A) = \frac{f(I)}{n} \text{trace}(A)$ <p>Hence <math>f</math> is a scalar multiple of trace.</p> <p>If <math>f(I) = n</math>, then <math>f(A) = \text{trace}(A)</math></p> <p>Hence <math>f</math> is the trace function</p> |

TABLE 3.5.15.2

Then  $g$  is a linear combination of  $f_1, \dots, f_r$  if and only if  $N$  contains the intersection  $N_1 \cap N_2 \cap \dots \cap N_r$ .

See Tables 3.6.2.1, 3.6.2.2 for the proof.

Let  $S$  be a set,  $F$  a field, and  $V(S; F)$  the space of all functions from  $S$  into  $F$ :

$$(f + g)(x) = f(x) + g(x)$$

$$(cf)(x) = cf(x)$$

Let  $W$  be any  $n$ -dimensional subspace

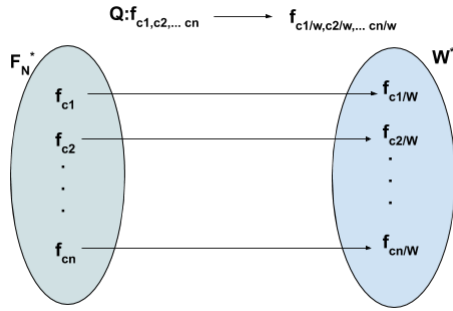


Fig. 3.6.1.3: Mapping from  $\mathbf{F}_N^* \xrightarrow{Q} \mathbf{W}^*$ , where  $f_{c_1, \dots, c_n}(x_1, \dots, x_n) = c_1 x_1 + \dots + c_n x_n$  is the linear functional with  $c_1 + \dots + c_n = 0$

of  $V(S, F)$ . Show that there exist points  $x_1, x_2, \dots, x_n$  in  $S$  and functions  $f_1, \dots, f_n$  in  $W$  such that  $f_i(x_j) = \delta_{ij}$

**Solution:**

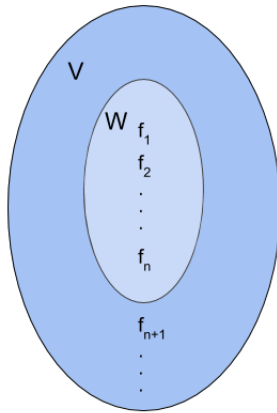


Fig. 3.6.3.1: Vector space of all function  $V$  and it's  $n$ -dimensional subspace  $W$

*Example:*

Consider points  $\{x_1, x_2\} \in S$ , let

$$x_1 = 1 \quad (3.6.3.1)$$

$$x_2 = 0 \quad (3.6.3.2)$$

Also consider functions  $\{f_1, f_2\} \in W$  where

$$f_1(x) = x \quad (3.6.3.3)$$

$$f_2(x) = 1 - x \quad (3.6.3.4)$$

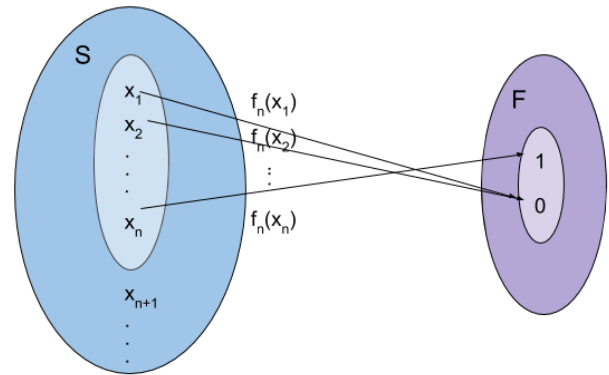
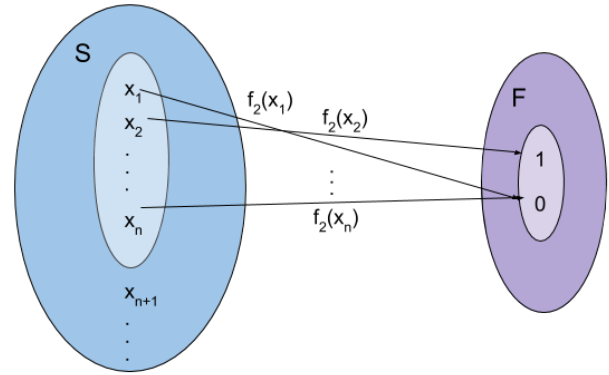
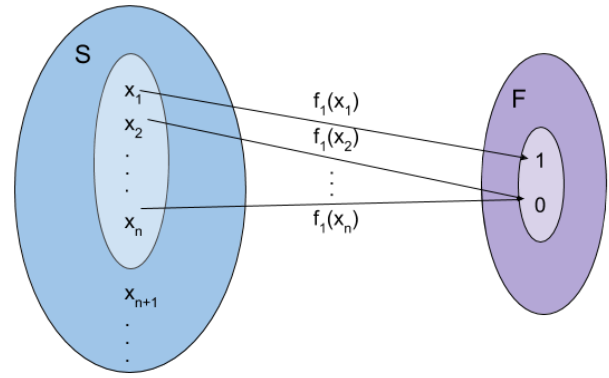


Fig. 3.6.3.2: Functions  $f_1, f_2, \dots, f_n$  where  $f : S \rightarrow F$  and  $f_i(x_j) = \delta_{ij}$  for  $i, j = 1, \dots, n$

Now, we have

$$f_1(x_1) = f_1(1) = 1 \quad (3.6.3.5)$$

$$f_1(x_2) = f_1(0) = 0 \quad (3.6.3.6)$$

Also,

$$f_2(x_1) = f_2(1) = 0 \quad (3.6.3.7)$$

$$f_2(x_2) = f_2(0) = 1 \quad (3.6.3.8)$$

Hence  $f_i(x_j) = \delta_{ij}$ .

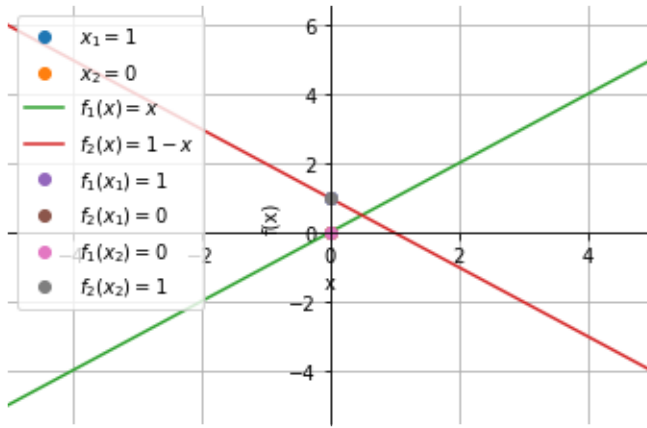


Fig. 3.6.3.3: Functions  $f_1, f_2$  and points  $x_1, x_2$

### 3.7 The Transpose of a Linear Transformation

3.7.1. Let  $\mathbb{F}$  be a field and let  $f$  be the linear functional on  $\mathbb{F}^2$  defined by,

$$f(x_1, x_2) = ax_1 + bx_2 \quad (3.7.1.1)$$

a) For the linear operator  $T(x_1, x_2) = (x_1, 0)$

Let,  $g = T^t f$  and find  $g(x_1, x_2)$

**Solution:** The linear functional  $f$  on  $\mathbb{F}^2$  is defined by,

$$f(x_1, x_2) = \mathbf{a}^T \mathbf{x} \quad \forall (x_1, x_2) \in \mathbb{F}^2 \quad (3.7.1.2)$$

where,

$$\mathbf{a} = \begin{pmatrix} a \\ b \end{pmatrix} \quad (3.7.1.3)$$

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad (3.7.1.4)$$

We use the following theorem,

Let  $\mathbb{V}$  and  $\mathbb{W}$  be vector spaces, over the field  $F$ . For each linear transformation  $T : \mathbb{V} \rightarrow \mathbb{W}$ , there is a unique linear transformation  $T^t : \mathbb{W}^* \rightarrow \mathbb{V}^*$  such that,

$$(T^t g)(\alpha) = g(T\alpha) \quad (3.7.1.5)$$

The given linear operator  $T$  defined as,

$$T(x_1, x_2) = \mathbf{A}\mathbf{x} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \quad \forall (x_1, x_2) \in \mathbb{F}^2 \quad (3.7.1.6)$$

Where,

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (3.7.1.7)$$

$$(3.7.1.8)$$

Consider the following mapping,

$$g = T^t f \quad (3.7.1.9)$$

Then,  $\forall (x_1, x_2) \in \mathbb{F}^2$  we have,

$$g(x_1, x_2) = T^t f(x_1, x_2) \quad [\text{From (3.7.1.9)}] \quad (3.7.1.10)$$

$$= f(T(x_1, x_2)) \quad [\text{From (3.7.1.5)}] \quad (3.7.1.11)$$

$$= \mathbf{a}^T \mathbf{A}\mathbf{x} \quad (3.7.1.12)$$

$$= \mathbf{a}^T \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \quad [\text{From (3.7.1.6)}] \quad (3.7.1.13)$$

$$= ax_1 \quad [\text{From (3.7.1.2)}] \quad (3.7.1.14)$$

Hence, (3.7.1.14) is the required answer.

b) For given linear operator  $\mathbf{T}$ , such that

$$\mathbf{T}(x_1, x_2) = (-x_2, x_1) \quad (3.7.1.15)$$

Let

$$g = \mathbf{T}^t f \quad (3.7.1.16)$$

Then find  $g(x_1, x_2)$

**Solution:**

The linear operator  $\mathbf{T}$  can be represented as a matrix  $\mathbf{A}$  as follows

$$\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (3.7.1.17)$$

$$(3.7.1.18)$$

Let suppose,

$$\mathbf{X}_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \mathbf{X}_2 = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} \quad (3.7.1.19)$$

And,

$$\mathbf{U} = \begin{pmatrix} a & b \end{pmatrix} \quad (3.7.1.20)$$

$$\mathbf{T}(x_1, x_2) = \mathbf{A}\mathbf{X}_1 \quad (3.7.1.21)$$

$$\Rightarrow \mathbf{A}\mathbf{X}_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{X}_1 = \mathbf{X}_2 \quad (3.7.1.22)$$

And (3.7.1.15) can be written as

$$f(x_1, x_2) = \mathbf{U}\mathbf{X}_1 \quad (3.7.1.23)$$

Now, we have given,

$$g = \mathbf{T}'f \quad (3.7.1.24)$$

$$\Rightarrow g(x_1, x_2) = \mathbf{T}'f(x_1, x_2) \quad (3.7.1.25)$$

We know that, if  $\mathbf{V}$  and  $\mathbf{W}$  be vector spaces over the field  $\mathbb{F}$ . For each linear transformation  $\mathbf{T}$  from  $\mathbf{V}$  into  $\mathbf{W}$ , there is a unique linear transformation  $\mathbf{T}'$  from  $\mathbf{W}^*$  into  $\mathbf{V}^*$  such that,

$$(\mathbf{T}'g)(\alpha) = g(\mathbf{T}\alpha) \quad (3.7.1.26)$$

Where for every  $g$  in  $\mathbf{W}^*$  and  $\alpha$  in  $\mathbf{V}$ .

Now using (3.7.1.26) in (3.7.1.25) we can write,

$$\begin{aligned} \mathbf{T}'f(x_1, x_2) &= f(\mathbf{T}(x_1, x_2)) \\ &\quad (3.7.1.27) \end{aligned}$$

$$\Rightarrow f(\mathbf{T}(x_1, x_2)) = \mathbf{U}\mathbf{A}\mathbf{X}_1 \quad (3.7.1.28)$$

$$f(\mathbf{T}(x_1, x_2)) = \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{U}\mathbf{X}_2 \quad (3.7.1.29)$$

$$\Rightarrow f(\mathbf{T}(x_1, x_2)) = -ax_2 + bx_1 \quad (3.7.1.30)$$

Hence,

$$\begin{aligned} f(\mathbf{T}(x_1, x_2)) &= -ax_2 + bx_1 = g(x_1, x_2) \\ &= \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} = \mathbf{U}\mathbf{X}_2 \quad (3.7.1.31) \end{aligned}$$

c) For the linear operator  $T(x_1, x_2) = (x_1 - x_2, x_1 + x_2)$

Let,  $g = T'y$  and find  $g(x_1, x_2)$

**Solution:** Given,  $\alpha = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ . Let,

$$\mathbf{A} = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \end{pmatrix} \quad (3.7.1.32)$$

$$\mathbf{A}\mathbf{X} = \alpha \quad (3.7.1.33)$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & -2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad (3.7.1.34)$$

$\mathbf{X} = A^{-1}\alpha$  will give solution of the equation.

$$\begin{pmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 1 & -2 & 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_3 \leftarrow R_3 - R_1} \quad (3.7.1.35)$$

$$\begin{pmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & -2 & 1 & -1 & 0 & 1 \end{pmatrix} \xrightarrow{R_3 \leftarrow R_3 + 2R_2} \quad (3.7.1.36)$$

$$\begin{pmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 2 & 1 \end{pmatrix} \xrightarrow{R_3 \leftarrow R_3 / (-1)} \quad (3.7.1.37)$$

$$\begin{pmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & -1 \end{pmatrix} \xrightarrow{R_1 \leftarrow R_1 + R_3} \quad (3.7.1.38)$$

$$\begin{pmatrix} 1 & 0 & 0 & 2 & -2 & -1 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & -1 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 + R_3} \quad (3.7.1.39)$$

$$\begin{pmatrix} 1 & 0 & 0 & 2 & -2 & -1 \\ 0 & 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 & -2 & -1 \end{pmatrix} \quad (3.7.1.40)$$

Thus,

$$A^{-1} = \begin{pmatrix} 2 & -2 & -1 \\ 1 & -1 & -1 \\ 1 & -2 & -1 \end{pmatrix} \quad (3.7.1.41)$$

$$\mathbf{X} = A^{-1}\alpha = \begin{pmatrix} 2 & -2 & -1 \\ 1 & -1 & -1 \\ 1 & -2 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad (3.7.1.42)$$

Given,  $f$  is a linear functional on  $R^3$ ,

$$\alpha = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 \quad (3.7.1.43)$$

$$\Rightarrow f(\alpha) = \mathbf{X}^T \begin{pmatrix} f(\alpha_1) \\ f(\alpha_2) \\ f(\alpha_3) \end{pmatrix} \quad (3.7.1.44)$$

Given,  $f(\alpha_1) = 0$ ,  $f(\alpha_2) = 0$  and  $f(\alpha_3) \neq 0$ .

$$f(\alpha) = \mathbf{X}^T \begin{pmatrix} 0 \\ 0 \\ f(\alpha_3) \end{pmatrix} \quad (3.7.1.45)$$

$$\Rightarrow f(\alpha) = \begin{pmatrix} a \\ b \\ c \end{pmatrix}^T \begin{pmatrix} 2 & 1 & 1 \\ -2 & -1 & -2 \\ -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ f(\alpha_3) \end{pmatrix} \quad (3.7.1.46)$$

$$f(\alpha) = \begin{pmatrix} a \\ b \\ c \end{pmatrix}^T \begin{pmatrix} f(\alpha_3) \\ -2f(\alpha_3) \\ -f(\alpha_3) \end{pmatrix} \quad (3.7.1.47)$$

Hence,

$$f(\alpha) = f(\alpha_3)(a - 2b - c) \quad (3.7.1.48)$$

Given  $\alpha = \begin{pmatrix} 2 & 3 & -1 \end{pmatrix}$  Substituting value of  $\alpha$  in equation (3.7.1.48) we have,

$$f(\alpha) = -3f(\alpha_3) \neq 0 \quad (3.7.1.49)$$

Hence proved,  $f(\alpha) \neq 0$

3.7.2. Let  $V$  be the vector space of all polynomial function over the field of real numbers. Let  $a$  and  $b$  be fixed real numbers and let  $f$  be the linear functional on  $V$  defined by

$$f(p) = \int_a^b p(x) dx \quad (3.7.2.1)$$

If  $D$  is the differentiation operator on  $V$ , what is  $D^t f$ ?

**Solution:** Let

$$p(x) = c_0 + c_1 x + c_2 x^2 + \dots + c_n x^n \quad (3.7.2.2)$$

$$D^t f(p) = f(D(p)) \quad (3.7.2.3)$$

$$D^t f(p) = f(c_1 + 2c_2 x + 3c_3 x^2 + \dots + nc_n x^{n-1}) \quad (3.7.2.4)$$

$$D^t f(p) = \int_a^b (c_1 + 2c_2 x + 3c_3 x^2 + \dots + nc_n x^{n-1}) dx \quad (3.7.2.5)$$

$$D^t f(p) = c_1 x + c_2 x^2 + c_3 x^3 + \dots + c_n x^n \Big|_a^b \quad (3.7.2.6)$$

$$D^t f(p) = p(b) - p(a) \quad (3.7.2.7)$$

3.7.3. Let,  $V$  be the space of all  $n \times n$  matrices over a field  $F$  and let  $B$  be a fixed  $n \times n$  matrix. If  $T$  is the linear operator on  $V$  defined by  $T(A) = AB - BA$ , and if  $f$  is the trace function, what is  $T^t f$ ?

**Solution:**

$$T^t f(A) = f(T(A)) \quad (3.7.3.1)$$

$$= f(AB - BA) \quad (3.7.3.2)$$

$$= \text{trace}(AB - BA) \quad (3.7.3.3)$$

using  $\text{trace}(AB) = \text{trace}(BA)$  in (3.7.3.3),

$$T^t f(A) = \text{trace}(AB) - \text{trace}(BA) = 0 \quad (3.7.3.4)$$

Hence  $T^t f = 0$

3.7.4. Let  $A$  be an  $m \times n$  matrix with real entries. Prove that  $A = 0$  is and only if  $\text{tr}(A^T A) = 0$ .

**Solution:** The proof is given in the table (3.7.4.2) and the properties used for the proof are listed in the table (3.7.4.1)

3.7.5. Let  $V$  be the vector space of all polynomial functions over the field of real numbers. Let  $a$  and  $b$  be fixed real numbers and let  $f$  be the linear functional on  $V$  defined by

$$f(p) = \int_a^b p(x) dx \quad (3.7.5.1)$$

If  $D$  is the differentiator operator on  $V$ , what is  $D^t f$ ?

**Solution:** See Tables 3.7.5.1 and 3.7.5.2

3.7.6. Let  $V$  be a finite-dimensional vector space over the field  $F$ . Show that  $T \rightarrow T^t$  is an isomorphism of  $L(V, V)$  onto  $L(V^*, V^*)$ .

**Solution:** See Tables 3.7.6.1 and 3.7.6.1



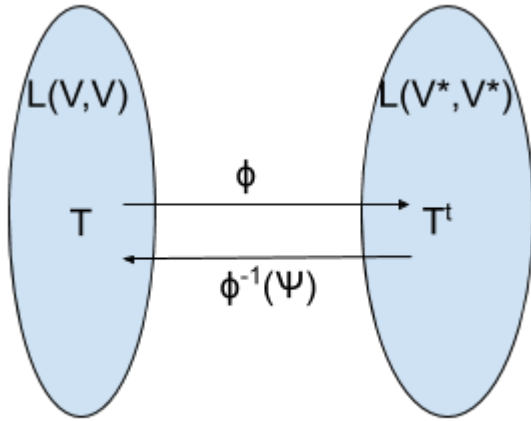


Fig. 3.7.6.1:  $\phi : L(V, V) \rightarrow L(V^*, V^*)$

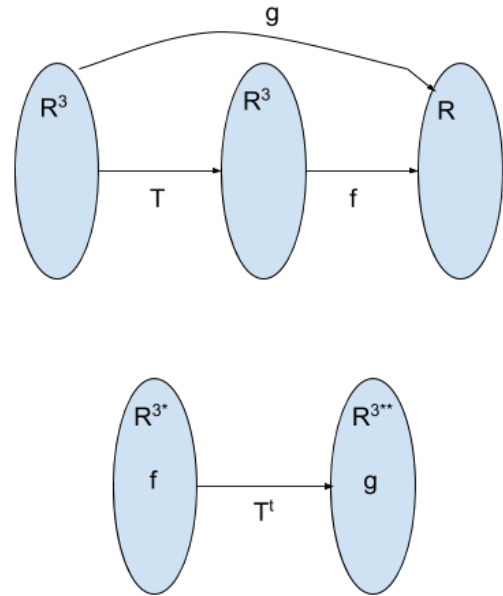


Fig. 3.7.6.2:  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  and  $T' : \mathbb{R}^{3*} \rightarrow \mathbb{R}^{3**}$

*Example* Consider,  $T \in L(\mathbb{R}^3, \mathbb{R}^3)$

$$T = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and} \quad (3.7.6.1)$$

$$T' = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad (3.7.6.2)$$

If  $T \in L(\mathbb{R}^3, \mathbb{R}^3)$  we can show that  $T' \in L(\mathbb{R}^{3*}, \mathbb{R}^{3*})$  as follows.

Consider linear functional  $f : \mathbb{R}^{3*} \rightarrow F$

$$f(x, y, z) = 2x + 3y + 4z \quad (3.7.6.3)$$

Let  $g : \mathbb{R}^3 \rightarrow F$ . By definition of transpose,

$$g = T' f \quad (3.7.6.4)$$

$$= \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \\ 6 \end{pmatrix} \quad (3.7.6.5)$$

$$g(x, y, z) = 7x + 5y + 6z \quad (3.7.6.6)$$

Consider vector in  $v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \in \mathbb{R}^3$ ,

$$Tv = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \\ 3 \end{pmatrix} \quad (3.7.6.7)$$

$$f(Tv) = f(7, 3, 3) \quad (3.7.6.8)$$

$$= 2 * 7 + 3 * 3 + 4 * 3 = 35 \quad (3.7.6.9)$$

$$g(1, 2, 3) = 7 * 1 + 5 * 2 + 6 * 3 = 35 \quad (3.7.6.10)$$

Hence verified  $g = T' f$ , and  $T'$  is transpose of  $T$ .

Now we will show that  $\phi$  is invertible.

Given,

$$\phi T = T' \quad (3.7.6.11)$$

$$\phi = T' T^{-1} \quad (3.7.6.12)$$

$$= \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \quad (3.7.6.13)$$

$$= \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & -1 \\ -1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad (3.7.6.14)$$

$$\phi = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (3.7.6.15)$$

$$\phi^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (3.7.6.16)$$

Since  $\phi^{-1}$  exists,  $\phi$  is isomorphism of  $L(V, V)$  onto  $L(V^*, V^*)$

3.7.7. Let  $V$  be the vector space of  $n \times n$  matrices over the field  $F$ . If  $\mathbf{B}$  is a fixed  $n \times n$  matrix, define a function  $f_B$  on  $V$  by  $f_B(\mathbf{A}) = \text{tr}(\mathbf{B}^t \mathbf{A})$ .

a) Show that,  $f_B$  is a linear functional on  $V$ .

**Solution:**

See Table 3.7.7.1

Hence, it can be said that  $f_B$  is a linear functional on  $V$ .

b) Let  $\mathbf{V}$  be the vector space of  $n \times n$  matrices over the field  $\mathbb{F}$ . If  $\mathbf{B}$  is a fixed  $n \times n$  matrix, define a function  $f_B$  on  $\mathbf{V}$  by  $f_B(\mathbf{A}) = \text{Tr}(\mathbf{B}^T \mathbf{A})$ . Show that every linear functional on  $\mathbf{V}$  is of the form  $f_B$  for some  $\mathbf{B}$ .

**Solution:** For  $\mathbf{A} = (a_{ij}) \in \mathbf{V}$ , a linear functional  $f_B : \mathbf{V} \rightarrow \mathbb{F}$  is defined as:

$$f_B(\mathbf{A}) = \mathbf{A}^T \mathbf{c} \quad (3.7.7.1)$$

(3.7.7.1) can be written as,

$$f_B(\mathbf{A}) = \sum_{i=1}^n \sum_{j=1}^n c_{ij} a_{ij} \quad (3.7.7.2)$$

Now,

$$\text{Tr}(\mathbf{B}^t \mathbf{A}) = \sum_{i=1}^n (\mathbf{B}^t \mathbf{A})_{ii} \quad (3.7.7.3)$$

$$= \sum_{j=1}^n \sum_{i=1}^n (b_{ji})^T a_{ij} \quad (3.7.7.4)$$

$$= \sum_{j=1}^n \sum_{i=1}^n b_{ij} a_{ij} \quad (3.7.7.5)$$

Let  $c_{ij} = b_{ij}$ , from (3.7.7.2) and (3.7.7.5),

$$\text{Tr}(\mathbf{B}^T \mathbf{A}) = f_B \quad (3.7.7.6)$$

Hence, Proved.

## 4 POLYNOMIALS

### 4.1 The Algebra of Polynomials

4.1.1. Let  $F$  be a sub field of the complex numbers, and let  $\mathbf{A}$  be the following  $2 \times 2$  matrix over  $F$ .

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix} \quad (4.1.1.1)$$

Compute  $f(\mathbf{A})$  for the polynomial

a)

$$f(x) = x^2 - x + 2 \quad (4.1.1.2)$$

**Solution:** consider

$$|\mathbf{A} - \lambda \mathbf{I}| = 0 \quad (4.1.1.3)$$

$$\begin{vmatrix} 2-\lambda & 1 \\ -1 & 3-\lambda \end{vmatrix} = 0 \quad (4.1.1.4)$$

The characteristics equation of the matrix

$$\lambda^2 - 5\lambda + 7 = 0 \quad (4.1.1.5)$$

The characteristics equation will satisfy its own matrix

$$\mathbf{A}^2 - 5\mathbf{A} + 7\mathbf{I} = 0 \quad (4.1.1.6)$$

$$\mathbf{A}^2 = 5\mathbf{A} - 7\mathbf{I} \quad (4.1.1.7)$$

The given polynomial

$$f(x) = x^2 - x + 2 \quad (4.1.1.8)$$

$$f(\mathbf{A}) = \mathbf{A}^2 - \mathbf{A} + 2\mathbf{I} \quad (4.1.1.9)$$

substituting (4.1.1.7) in (4.1.1.9) we get

$$f(\mathbf{A}) = 5\mathbf{A} - 7\mathbf{I} - \mathbf{A} + 2\mathbf{I} \quad (4.1.1.10)$$

$$f(\mathbf{A}) = 4\mathbf{A} - 5\mathbf{I} \quad (4.1.1.11)$$

$$f(\mathbf{A}) = 4 \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix} - 5 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (4.1.1.12)$$

$$f(\mathbf{A}) = \begin{pmatrix} 3 & 4 \\ -4 & 7 \end{pmatrix} \quad (4.1.1.13)$$

b)

$$f = x^3 - 1 \quad (4.1.1.14)$$

**Solution:** We first find the eigen values of the  $\mathbf{A}$ . We get the characteristic equation of  $\mathbf{A}$  as follows,

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0 \quad (4.1.1.15)$$

$$\Rightarrow \lambda^2 - 5\lambda + 7 = 0 \quad (4.1.1.16)$$

Where  $\mathbf{I}$  is  $2 \times 2$  Identity matrix. Now using Cayley Hamilton Theorem we get from (4.1.1.16) the following,

$$\mathbf{A}^2 - 5\mathbf{A} + 7\mathbf{I} = 0 \quad (4.1.1.17)$$

Hence the polynomial  $f(\mathbf{A})$  can be written using the characteristic function of  $\mathbf{A}$  as follows,

$$f(\mathbf{A}) = \mathbf{A}^3 - \mathbf{I} \quad (4.1.1.18)$$

$$= (\mathbf{A} - \mathbf{I})(\mathbf{A}^2 + \mathbf{A} + \mathbf{I}) \quad (4.1.1.19)$$

$$= (\mathbf{A} - \mathbf{I})(\mathbf{A}^2 - 5\mathbf{A} + 7\mathbf{I} + 6\mathbf{A} - 6\mathbf{I}) \quad (4.1.1.20)$$

$$= 6(\mathbf{A} - \mathbf{I})^2 \quad [\text{From (4.1.1.17)}] \quad (4.1.1.21)$$

$$= 6 \left[ \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right]^2 \quad (4.1.1.22)$$

$$= \begin{pmatrix} 0 & 18 \\ -18 & 18 \end{pmatrix} \quad (4.1.1.23)$$

Here, (4.1.1.23) is the required answer.

c)  $f(x) = x^2 - 5x + 7$

**Solution:** Characteristic equation of  $\mathbf{A}$  can be written as:

$$|\mathbf{A} - \lambda\mathbf{I}| = 0 \quad (4.1.1.24)$$

$$\begin{pmatrix} 2 - \lambda & 1 \\ -1 & 3 - \lambda \end{pmatrix} = 0 \quad (4.1.1.25)$$

The characteristics equation of the matrix will be,

$$\lambda^2 - 5\lambda + 7 = 0 \quad (4.1.1.26)$$

The characteristics equation will satisfy its own matrix

$$\mathbf{A}^2 - 5\mathbf{A} + 7\mathbf{I} = \mathbf{0} \quad (4.1.1.27)$$

The given polynomial

$$f(x) = x^2 - 5x + 7 \quad (4.1.1.28)$$

$$f(\mathbf{A}) = \mathbf{A}^2 - 5\mathbf{A} + 7\mathbf{I} \quad (4.1.1.29)$$

substituting (4.1.1.27) in (4.1.1.29) we get

$$f(\mathbf{A}) = \mathbf{0} \quad (4.1.1.30)$$

4.1.2. Let  $T$  be the linear operator on  $\mathbb{R}^3$  defined by

$$T(x_1, x_2, x_3) = (x_1, x_3, -2x_2 - x_3) \quad (4.1.2.1)$$

Let  $f$  be the polynomial over  $\mathbb{R}$  defined by  $f = -x^3 + 2$

**Solution:** The given transformation can be written as,

$$T(\mathbf{x}) = \mathbf{A}\mathbf{x} \quad (4.1.2.2)$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -1 \end{pmatrix} \mathbf{x} \quad (4.1.2.3)$$

Hence,

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -1 \end{pmatrix} \quad (4.1.2.4)$$

Now the characteristic equation of  $\mathbf{A}$  is given by,

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0 \quad (4.1.2.5)$$

$$= \begin{vmatrix} 1 - \lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ 0 & -2 & -1 - \lambda \end{vmatrix} \quad (4.1.2.6)$$

$$\Rightarrow (1 - \lambda)(\lambda^2 + \lambda + 2) = 0 \quad (4.1.2.7)$$

Now, simplifying the above equation,

$$(1 - \lambda)(\lambda^2 + \lambda + 2) = 0 \quad (4.1.2.8)$$

$$\lambda^2 + \lambda + 2 - \lambda^3 - \lambda^2 - 2\lambda = 0 \quad (4.1.2.9)$$

$$\lambda^3 = 2 - \lambda \quad (4.1.2.10)$$

Now using Cayley Hamilton Theorem we get,

$$\mathbf{A}^3 = 2\mathbf{I} - \mathbf{A} \quad (4.1.2.11)$$

Hence the polynomial  $f(\mathbf{A})$  can be written using the characteristic function of  $\mathbf{A}$  as follows,

$$f(\mathbf{A}) = -\mathbf{A}^3 + 2\mathbf{I} \quad (4.1.2.12)$$

$$= 2\mathbf{I} - \mathbf{A} + 2\mathbf{I} \quad (4.1.2.13)$$

$$= \mathbf{A} \quad (4.1.2.14)$$

Hence,

$$f(T)(\mathbf{x}) = \mathbf{A}\mathbf{x} \quad (4.1.2.15)$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -1 \end{pmatrix} \mathbf{x} \quad (4.1.2.16)$$

4.1.3. Let  $\mathbf{A}$  be an  $n \times n$  diagonal matrix over the field  $\mathbf{F}$ , i.e., a matrix satisfying  $\mathbf{A}_{ij} = 0$  for  $i \neq j$ . Let  $f$  be the polynomial over  $\mathbf{F}$  defined by  $f = (x - A_{11}) \dots (x - A_{nn})$ . What is the matrix  $f(\mathbf{A})$ ?

**Solution:** The given transformation can be written as,

$$T(\mathbf{x}) = \mathbf{A}\mathbf{x} \quad (4.1.3.1)$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -1 \end{pmatrix} \mathbf{x} \quad (4.1.3.2)$$

Hence,

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -1 \end{pmatrix} \quad (4.1.3.3)$$

Now the characteristic equation of  $\mathbf{A}$  is given by,

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0 \quad (4.1.3.4)$$

$$= \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ 0 & -2 & -1-\lambda \end{vmatrix} \quad (4.1.3.5)$$

$$\implies (1 - \lambda)(\lambda^2 + \lambda + 2) = 0 \quad (4.1.3.6)$$

Now, simplifying the above equation,

$$(1 - \lambda)(\lambda^2 + \lambda + 2) = 0 \quad (4.1.3.7)$$

$$\lambda^2 + \lambda + 2 - \lambda^3 - \lambda^2 - 2\lambda = 0 \quad (4.1.3.8)$$

$$\lambda^3 = 2 - \lambda \quad (4.1.3.9)$$

Now using Cayley Hamilton Theorem we get,

$$\mathbf{A}^3 = 2\mathbf{I} - \mathbf{A} \quad (4.1.3.10)$$

Hence the polynomial  $f(\mathbf{A})$  can be written using the characteristic function of  $\mathbf{A}$  as follows,

$$f(\mathbf{A}) = -\mathbf{A}^3 + 2\mathbf{I} \quad (4.1.3.11)$$

$$= 2\mathbf{I} - \mathbf{A} + 2\mathbf{I} \quad (4.1.3.12)$$

$$= \mathbf{A} \quad (4.1.3.13)$$

Hence,

$$f(T)(\mathbf{x}) = \mathbf{A}\mathbf{x} \quad (4.1.3.14)$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -1 \end{pmatrix} \mathbf{x} \quad (4.1.3.15)$$

4.1.4. If  $f$  and  $g$  are independent polynomials over a field  $\mathbf{F}$  and  $h$  is a non-zero polynomial over  $\mathbf{F}$ , show that  $fh$  and  $gh$  are independent.

**Solution:** Polynomials  $(f_1, f_2, \dots, f_m) \in k[x_1, \dots, x_n]$  are called algebraically independent over a field  $\mathbf{F}$ , if there is no nonzero  $m$ -variate polynomial  $A \in k[y_1, \dots, y_m]$  such that  $A(f_1, \dots, f_m) = 0$

**Example:** The smallest degree independent polynomials are:  $1+x$  and  $1-x$ .

$$a(1+x) + b(1-x) = 0 \quad (4.1.4.1)$$

Simplifying,

$$a + b = 0 \quad (4.1.4.2)$$

$$a - b = 0 \quad (4.1.4.3)$$

solving, we get  $a=0$  and  $b=0$ . So, polynomials are linearly independent. Given  $f$  and  $g$  are independent polynomials over a field  $\mathbf{F}$ . Consider scalars  $a$  and  $b \in \mathbf{F}$  Hence,

$$af + bg = 0 \quad (4.1.4.4)$$

Since  $f$  and  $g$  are independent Hence  $f$  and  $g \neq 0$

$$\implies a, b = 0. \quad (4.1.4.5)$$

Given  $h$  a non zero polynomial over  $\mathbf{F}$ . Substituting in equation (4.1.4.4) we have,

$$a(fh) + b(gh) = 0 \quad (4.1.4.6)$$

$$(af)h + (bg)h = 0 \quad (4.1.4.7)$$

$$(af + bg)h = 0 \quad (4.1.4.8)$$

$$af + bg = 0 \quad (4.1.4.9)$$

$f$  and  $g$  are independent polynomial. Also from equation (4.1.4.5)  $a=0$  and  $b=0$ .

Hence proved  $fh$  and  $gh$  are independent.

4.1.5. Let  $\mathbf{S}$  be a set of non-zero polynomials over a field  $F$ . If no two elements of  $\mathbf{S}$  have the same degree, show that  $\mathbf{S}$  is an independent set in  $\mathbf{F}[x]$ .

**Solution:** See Tables 4.1.5.1, 4.1.5.2 and 4.1.5.3

Hence, it is proved that  $\mathbf{S}$  is an independent set in  $\mathbf{F}[x]$ .

4.1.6. If  $a$  and  $b$  are element of a field  $\mathbb{F}$  and  $a \neq 0$ , show that the polynomials  $1, ax + b, (ax + b)^2, (ax + b)^3, \dots$  form a basis of  $\mathbb{F}[x]$ .

**Solution:**

Let consider we have a set  $S$  such that,

$$S = \{1, ax + b, (ax + b)^2, (ax + b)^3, \dots\} \quad (4.1.6.1)$$

And let  $\langle S \rangle$  be the subspace, that is spanned by  $S$ .

Since

$$1 \in S \quad (4.1.6.2)$$

and

$$ax + b \in S, \quad (4.1.6.3)$$

$$\Rightarrow b.1 + \frac{1}{a}(b + ax) \in \langle S \rangle \quad (4.1.6.4)$$

and hence, it follows

$$\Rightarrow x \in \langle S \rangle \quad (4.1.6.5)$$

Now to prove

$$x^2 \in \langle S \rangle \quad (4.1.6.6)$$

let consider another element form  $S$  which is

$$(ax + b)^2 \quad (4.1.6.7)$$

Subtracting  $1.a^2 + 2.a.b.x$  from  $(ax + b)^2$

$$\Rightarrow (ax + b)^2 - a^2 - 2.a.b.x = a^2.x^2 \quad (4.1.6.8)$$

$$\Rightarrow a^2.x^2 \in \langle S \rangle \quad (4.1.6.9)$$

$$\Rightarrow \frac{1}{a^2}.a^2.x^2 \in S. \quad (4.1.6.10)$$

$$\Rightarrow x^2 \in \langle S \rangle. \quad (4.1.6.11)$$

Now, Thus Hence using this concept with higher degree we can prove that,

$$x^n \in \langle S \rangle, \forall n \quad (4.1.6.12)$$

Consider,

$$S' = \{1, x, x^2, x^3, \dots\} \quad (4.1.6.13)$$

Hence we can say that, (4.1.6.13) span the space of all polynomials which form with the help of

$$(ax + b)^n \quad (4.1.6.14)$$

Hence we conclude that  $S$  spans the space of all polynomials. We can summarize our procedure step by step using Table 4.1.6.1.

|          |   |
|----------|---|
| Given    | $x_1 + \dots + x_n = 0$ $(x_1, \dots, x_n) \in \mathbf{W}$ $F \text{ is a field}$ $\mathbf{W}^* \text{ is dual space of } \mathbf{W}$   |
| To prove | $\mathbf{W} \rightarrow \mathbf{W}^* \text{ is a natural isomorphism}$ $f(x_1, \dots, x_n) = c_1 x_1 + \dots c_n x_n$ $\text{which satisfy } c_1 + \dots + c_n = 0$   |
| Proof    | <p>Let <math>\alpha_i = \epsilon_1 - \epsilon_{i+1}</math><br/> <math>i \in \{1, \dots, n-1\}</math></p> $\sum_{i=1}^{n-1} c_i \alpha_i = 0$ $\Rightarrow \left( \sum_{i=1}^{n-1} c_i \right) \epsilon_1 - \sum_{i=1}^{n-1} c_i \epsilon_{i+1} = 0$ <p><math>(\alpha_1, \dots, \alpha_{n-1})</math> are linearly independent and form a basis for <math>\mathbf{W}</math></p> $\mathbf{W} \xrightarrow{P} (\mathbf{F}^n)^* \xrightarrow{Q} \mathbf{W}^*$ <p>The function <math>P</math> is defined as<br/> <math>P(c_1, \dots, c_n) = f_{c_1, \dots, c_n}</math>; where,<br/> <math>f_{c_1, \dots, c_n}(x_1, \dots, x_n) = c_1 x_1 + \dots c_n x_n</math></p> <p>Let <math>Q \circ P(c_1, \dots, c_n) = 0</math>;<br/> <math>(c_1, \dots, c_n) \in \mathbf{W}</math><br/> <math>Q(f_{c_1, \dots, c_n}) = 0 \Rightarrow f_{c_1, \dots, c_n} _{\mathbf{W}} = 0</math><br/> <math>\Rightarrow f_{c_1, \dots, c_n}(x_1, \dots, x_n) = 0</math></p> $f_{c_1, \dots, c_n}(\alpha_i) = 0; \quad i = 1, \dots, n-1$ $\Rightarrow c_1 = c_i; \quad i = 2, \dots, n$ $\Rightarrow \sum_{i=2}^n c_i = (n-1)c_1$ <p>since <math>(c_1, \dots, c_n) \in \mathbf{W}</math><br/> <math>\sum_{i=1}^n c_i = 0</math><br/> <math>\Rightarrow c_1 = 0</math><br/> <math>\Rightarrow c_i = 0; \quad i = 1, \dots, n</math></p> <p>Hence, <math>f_{c_1, \dots, c_n}</math> is a zero function.<br/> Thus the mapping <math>\mathbf{W} \rightarrow \mathbf{W}^*</math><br/> is a natural isomorphism</p> |

TABLE 3.6.1.1

| PARAMETERS                 | DESCRIPTION   |
|----------------------------|---|
| $\mathbf{V}$               | Finite dimensional vector space with $\dim(\mathbf{V}) = n$ |
| $\mathbf{W}$               | Subspace of $\mathbf{V}$                                    |
| $\mathbf{W}^0$             | Annihilator of $\mathbf{W}$                                 |
| $g \in \mathbf{W}^0$       | $g(\mathbf{w}) = 0 \forall \mathbf{w} \in \mathbf{W}$       |
| $\{g_1, g_2, \dots, g_r\}$ | basis for $\mathbf{W}^0$                                    |
| $\mathbf{N}_{g_i}$         | Null space of $g_i$   |

TABLE 3.6.2.1: Input Parameters

| STATEMENT  | DESCRIPTION  |
|--|--|
| $g \in \mathbf{W}^0$   | $g = \sum_{i=1}^r c_i g_i \quad (3.6.2.2)$   |
| If $\mathbf{N}$ is nullspace of $g$                                      | From the theorem,<br>$\bigcap_{i=1}^r \mathbf{N}_{g_i} \subseteq \mathbf{N} \quad (3.6.2.3)$   |
| $\forall \mathbf{w} \in \mathbf{W}$                                      | $g(\mathbf{w}) = 0 \quad (3.6.2.4)$  |
| $(3.6.2.3), (3.6.2.4) \implies$  | $\mathbf{W} \subseteq \bigcap_{i=1}^r \mathbf{N}_{g_i} \quad (3.6.2.5)$  |
| Starting with method of contradiction                                    | $\mathbf{W} \neq \bigcap_{i=1}^r \mathbf{N}_{g_i} \quad (3.6.2.6)$   |
| From (3.6.2.5),(3.6.2.6) there exists<br>a vector $\mathbf{e}$ such that | $\mathbf{e} \in \bigcap_{i=1}^r \mathbf{N}_{g_i}, \mathbf{e} \notin \mathbf{W} \quad (3.6.2.7)$  |
|  | So $\forall g \in \mathbf{W}^0, g(\mathbf{e}) = 0 \quad (3.6.2.8)$   |
| Since $g$ is a linear functional on $\mathbf{V}$                         | $(3.6.2.8) \implies \mathbf{e} \in \mathbf{V} \quad (3.6.2.9)$   |
| $f$  | A linear functional on $\mathbf{V}$ defined as<br>$f(\alpha) \begin{cases} = 0 & \text{for } \alpha \in \mathbf{W} \\ \neq 0 & \text{for } \alpha \in \mathbf{V}, \alpha \notin \mathbf{W} \end{cases} \quad (3.6.2.10)$ |
|  | $\therefore f \in \mathbf{W}^0 \text{ and } f(\mathbf{e}) \neq 0 \quad (3.6.2.11)$   |
| $(3.6.2.8), (3.6.2.11)$ contradict each other                            | $\mathbf{W} = \bigcap_{i=1}^r \mathbf{N}_{g_i} \quad (3.6.2.12)$   |

TABLE 3.6.2.2: Proof

|          |  |
|----------|--|
| Given    | <p> <math>S</math> is a set<br/> <math>F</math> is a field<br/> <math>V(S, F)</math> is a linear functional<br/> such that<br/> <math>W</math> be <math>n</math>-dim subspace of <math>V(S, F)</math>. </p> <p>Also, <math>\delta_{ij} = \begin{cases} 1 &amp; i = j \\ 0 &amp; i \neq j \end{cases}</math></p>  |
| To prove | <p> <math>f_i(x_j) = \delta_{ij}</math> </p> <p> where <math>x_1, x_2, \dots, x_n \in S</math><br/> and <math>f_1, f_2, \dots, f_n \in W</math> </p>   |
| Proof    | <p>Let <math>\phi_x : W \rightarrow F</math></p> <p>Suppose <math>\phi_x(f) = 0 \quad \forall x \in S \quad \&amp; \quad f \in W</math><br/> <math>\implies f(x) = 0</math></p> <p>If <math>\forall x, \phi_x(f) \neq 0</math> for some <math>f \in W</math><br/> If <math>n &gt; 0 \exists x \in S</math> such that<br/> <math>\phi_x(f) \neq 0</math> for some <math>f \in W</math><br/> <math>\implies f_1(x_1) \neq 0</math></p> <p>By scaling we can have<br/> <math>f_1(x_1) = 1</math></p> <p>Hence <math>f_i(x_j) = \delta_{ij}</math></p> |



| Properties Used               |   |
|-------------------------------|---|
| SVD of matrix $\mathbf{A}$    | $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$    |
| $\mathbf{U}$ is unitary       | $\mathbf{U}^T\mathbf{U} = \mathbf{I}$                   |
| $\mathbf{V}$ is unitary       | $\mathbf{V}^T\mathbf{V} = \mathbf{I}$                   |
| $\mathbf{\Sigma}$ is diagonal | $\mathbf{\Sigma}^T\mathbf{\Sigma} = \mathbf{\Sigma}^2$  |
| Cyclic property               | $tr(\mathbf{ABC}) = tr(\mathbf{CAB})$                   |
| $Rank(\mathbf{A})$            | $Rank(\mathbf{A}) = \# \text{non-zero singular values}$ |

TABLE 3.7.4.1: Properties

| Statement   | Proof  |
|---|--|
| $tr(\mathbf{A}^T\mathbf{A}) = 0 \implies \mathbf{A} = \mathbf{0}$ | $tr(\mathbf{A}^T\mathbf{A}) = tr(\mathbf{V}\mathbf{\Sigma}^T\mathbf{U}^T\mathbf{U}\mathbf{\Sigma}\mathbf{V}^T) \quad (3.7.4.1)$ $= tr(\mathbf{V}\mathbf{\Sigma}^T\mathbf{\Sigma}\mathbf{V}^T) \quad (3.7.4.2)$ $= tr(\mathbf{V}\mathbf{\Sigma}^2\mathbf{V}^T) \quad (3.7.4.3)$ $= tr(\mathbf{V}^T\mathbf{V}\mathbf{\Sigma}^2) \quad (3.7.4.4)$ $= tr(\mathbf{\Sigma}^2) \quad (3.7.4.5)$ $= \sum_{i=1}^r \sigma_i^2 \quad (3.7.4.6)$ $tr(\mathbf{A}^T\mathbf{A}) = 0 \quad (3.7.4.7)$ $\implies \sum_{i=1}^r \sigma_i^2 = 0 \quad \forall i = 1, 2, \dots, r \quad (3.7.4.8)$ $\implies \sigma_i = 0 \quad \forall i = 1, 2, \dots, r \quad (3.7.4.9)$ $\therefore Rank(\mathbf{A}) = \# \text{non-zero singular values} = 0 \quad (3.7.4.10)$ $\implies \mathbf{A} = \mathbf{0} \quad (3.7.4.11)$ |
| $\mathbf{A} = \mathbf{0} \implies tr(\mathbf{A}^T\mathbf{A}) = 0$ | $\mathbf{A} = \mathbf{0} \quad (3.7.4.12)$ $\implies \mathbf{A}^T\mathbf{A} = \mathbf{0} \quad (3.7.4.13)$ $\implies tr(\mathbf{A}^T\mathbf{A}) = 0 \quad (3.7.4.14)$  |

TABLE 3.7.4.2: Proofs

| PARAMETERS   | DESCRIPTION                                       |
|--------------|---|
| $\mathbb{R}$ | Field of real numbers                             |
| $\mathbf{V}$ | Vector space of all polynomials over $\mathbb{R}$ |
| $a, b$       | Fixed real numbers                                |
| $f$          | Linear functional on $\mathbf{V}$                 |
| $D$          | Differentiator operator on $\mathbf{V}$           |
| $D^t$        | Transpose of $D$                                  |

TABLE 3.7.5.1: Input Parameters

| STATEMENTS   | DERIVATIONS  |
|--|--|
| $D^t$ is transpose of $D$                                  | $(D^t f)(p) = f[D(p)] \quad (3.7.5.2)$   |
| A polynomial of degree $n$                                 | $p = \mathbf{c}^T \mathbf{x} \quad (3.7.5.3)$  |
| $p(x) = c_0 + c_1 x + c_2 x^2 + \dots + c_n x^n$           | $\mathbf{c} = \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}, \mathbf{x} = \begin{pmatrix} 1 \\ x \\ x^2 \\ \vdots \\ x^n \end{pmatrix} \quad (3.7.5.4)$   |
| $f(p) = \int_a^b p(x) dx$                                  | $f(p) = \mathbf{c}^T \mathbf{F} \quad (3.7.5.5)$   |
|  | $\mathbf{F} = \begin{pmatrix} b-a \\ \frac{b^2-a^2}{2} \\ \frac{b^3-a^3}{3} \\ \vdots \\ \frac{b^{n+1}-a^{n+1}}{n+1} \end{pmatrix} \quad (3.7.5.6)$  |
| $D(p) = c_1 + 2c_2 x + 3c_3 x^2 + \dots + nc_n x^{n-1}$    | $D(p) = \mathbf{c}^T \mathbf{D} \mathbf{x} \quad (3.7.5.7)$  |
|  | $\mathbf{D} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 2 & 0 & \dots & 0 & 0 \\ 0 & 0 & 3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & n & 0 \end{pmatrix} \quad (3.7.5.8)$ |
| $f[D(p)] = c_1(b-a) + c_2(b^2-a^2) + \dots + c_n(b^n-a^n)$ | Let $D(p) = p' \quad (3.7.5.9)$  |
|  | $\implies p' = \mathbf{c}'^T \mathbf{x} \quad (3.7.5.10)$  |
|  | where $\mathbf{c}'^T = \begin{pmatrix} c_1 \\ 2c_2 \\ 3c_3 \\ \vdots \\ nc_n \\ 0 \end{pmatrix} \quad (3.7.5.11)$  |
|  | $f[D(p)] = \mathbf{c}'^T \mathbf{F} \quad (3.7.5.12)$  |
| $(D^t f)(p) = p(b) - p(a)$                                 | $(D^t f)(p) = c_1(b-a) + c_2(b^2-a^2) + \dots + c_n(b^n-a^n) + c_0 - c_0 \quad (3.7.5.13)$   |
|  | $\dots + c_n(b^n-a^n) + c_0 - c_0 \quad (3.7.5.14)$  |
|  | $= (c_0 + c_1 b + c_2 b^2 + \dots + c_n b^n) - (c_0 + c_1 a + c_2 a^2 + \dots + c_n a^n) \quad (3.7.5.15)$   |
|  | $= p(b) - p(a) \quad (3.7.5.16)$   |
|  | $= p(b) - p(a) \quad (3.7.5.17)$   |

TABLE 3.7.5.2: Proof

|                     |  |
|---------------------|--|
| <i>iso-morphism</i> | A linear transformation $T : V \rightarrow W$ is <i>isomorphism of V onto W</i> if $T$ is <i>one – one</i>   |
| <i>one-one</i>      | A linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is <i>one-one</i> if $\forall \mathbf{b} \in \mathbb{R}^m, \mathbf{A}\mathbf{X} = \mathbf{b}$ has at most one solution in $\mathbb{R}^n$<br><br>By definition, all <i>invertible</i> linear transformations are <i>one-one</i> |
| <i>invertible</i>   | A linear transformation $T : V \rightarrow W$ is <i>invertible</i> if there exists another linear transformation $U : W \rightarrow V$ such that<br>$UT$ is identity transformation on $V$ and<br>$TU$ is identity transformation on $W$   |

TABLE 3.7.6.1: Definitions

|          |  |
|----------|--|
| Given    | $\phi(T) = T'$ where<br>$\phi : L(V, V) \rightarrow L(V^*, V^*)$<br>$T \in L(V, V)$ and $T' \in L(V^*, V^*)$   |
| To prove | $\phi$ is an <i>isomorphism of</i><br>$L(V, V)$ <i>onto</i> $L(V^*, V^*)$<br><i>i.e</i> $\phi$ is <i>one – one</i> $\implies \phi$ is <i>invertible</i>  |
| Proof    | Consider a linear transformation<br>$\psi : L(V^*, V^*) \rightarrow L((V^*)^*, (V^*)^*)$<br><br>We know, $L((V^*)^*, (V^*)^*) = L(V, V)$<br><br>Hence $\psi : L(V^*, V^*) \rightarrow L(V, V)$<br><br>We have $\phi \circ \psi = I$ and $\psi \circ \phi = I$<br><br>$\therefore \phi$ is invertible with $\psi$ its inverse.<br><br>Hence $\phi$ is <i>isomorphic</i> . |

TABLE 3.7.6.2: Solution

|                               |  |
|-------------------------------|--|
| Given                         | <p><math>V</math> is a vector space of <math>n \times n</math> matrices over the field <math>F</math></p> <p><math>\mathbf{B}</math> is a fixed <math>n \times n</math> matrix</p> <p>A function <math>f_B</math> on <math>V</math> is defined such that</p> $f_B(\mathbf{A}) = \text{tr}(\mathbf{B}^t \mathbf{A})$  |
| To prove                      | $f_B$ is a linear functional on $V$  |
| Properties of Trace of Matrix | <p>1. <math>\text{tr}(\mathbf{M} + \mathbf{N}) = \text{tr}(\mathbf{M}) + \text{tr}(\mathbf{N})</math></p> <p>2. <math>\text{tr}(c\mathbf{M}) = c(\text{tr}(\mathbf{M}))</math></p> <p>Where <math>\mathbf{M}</math> and <math>\mathbf{N}</math> are two square matrices</p>  |
| Proof                         | <p>As <math>f_B(\mathbf{A}) = \text{tr}(\mathbf{B}^t \mathbf{A})</math></p> $\Rightarrow f_B(c\mathbf{A}_1 + \mathbf{A}_2) = \text{tr}(\mathbf{B}^t(c\mathbf{A}_1 + \mathbf{A}_2))$ $\Rightarrow f_B(c\mathbf{A}_1 + \mathbf{A}_2) = \text{tr}(c\mathbf{B}^t \mathbf{A}_1) + \text{tr}(\mathbf{B}^t \mathbf{A}_2)$ $\Rightarrow f_B(c\mathbf{A}_1 + \mathbf{A}_2) = c(\text{tr}(\mathbf{B}^t \mathbf{A}_1)) + \text{tr}(\mathbf{B}^t \mathbf{A}_2)$ $\Rightarrow f_B(c\mathbf{A}_1 + \mathbf{A}_2) = cf_B(\mathbf{A}_1) + f_B(\mathbf{A}_2)$ |

TABLE 3.7.7.1

|                     |  |
|---------------------|--|
| Given               | <p><math>\mathbf{S}</math> be a set of non-zero polynomial over a field <math>F</math></p> <p>No two elements of <math>\mathbf{S}</math> have the same degree</p>  |
| To prove            | $\mathbf{S}$ is an independent set in $\mathbf{F}[\mathbf{x}]$   |
| Linear Independency | <p>Let <math>f_1, f_2, \dots, f_n</math> are the polynomials and they will be linearly independent if</p> $a_1 f_1 + a_2 f_2 + \dots + a_n f_n = \theta$ <p>for <math>a_1 = a_2 = \dots = a_n = 0</math> where <math>a_1, a_2, \dots, a_n</math> are scalars from field <math>F</math></p> |

TABLE 4.1.5.1

|       |  |
|-------|--|
| Proof | <p>Let the degrees of <math>f_1, f_2, \dots, f_n</math> are <math>d_1, d_2, \dots, d_n</math> respectively such that the degree of <math>f_i = d_i \neq d_j</math> for <math>j=1, 2, \dots, n</math> and <math>i \neq j</math> and <math>d_1 &lt; d_2 &lt; \dots &lt; d_n</math> so <math>d_n</math> is the largest degree</p>   |
|       | <p>Now, let <math>a_1 f_1 + a_2 f_2 + \dots + a_n f_n = \theta</math></p> <p>where <math>f_1 = \sum_{i=0}^{d_1} k_{1i} x^i</math></p> <p><math>f_2 = \sum_{i=0}^{d_2} k_{2i} x^i</math><br/> <b>or,</b> <math>f_2 = \sum_{i=0}^{d_1} k_{2i} x^i + \sum_{i=d_1+1}^{d_2} k_{2i} x^i</math></p> <p>Similarly, <math>f_{n-1} = \sum_{i=0}^{d_{n-1}} k_{(n-1)i} x^i</math><br/> <math>\Rightarrow f_{n-1} = \sum_{i=0}^{d_{n-2}} k_{(n-1)i} x^i + \sum_{i=d_{n-2}+1}^{d_{n-1}} k_{(n-1)i} x^i</math></p> <p>and <math>f_n = \sum_{i=0}^{d_{n-1}} k_{ni} x^i + \sum_{i=d_{n-1}+1}^{d_n} k_{ni} x^i</math></p>  |
|       | <p>Now,</p> $a_1 f_1 + a_2 f_2 + \dots + a_n f_n$ $= a_1 \sum_{i=0}^{d_1} k_{1i} x^i + a_2 \left( \sum_{i=0}^{d_1} k_{2i} x^i + \sum_{i=d_1+1}^{d_2} k_{2i} x^i \right)$ $+ \dots + a_n \left( \sum_{i=0}^{d_{n-1}} k_{ni} x^i + \sum_{i=d_{n-1}+1}^{d_n} k_{ni} x^i + \dots \right)$ $= \sum_{i=0}^{d_1} (a_1 k_{1i} + a_2 k_{2i} + \dots + a_n k_{ni}) x^i$ $+ \sum_{i=d_1+1}^{d_2} (a_2 k_{2i} + \dots + a_n k_{ni}) x^i$ $+ \dots + \sum_{i=d_{n-1}+1}^{d_n} a_n k_{ni} x^i$ <p>Now, as <math>a_1 f_1 + a_2 f_2 + \dots + a_n f_n = \theta</math> for <math>d_{n-1} + 1 \leq i \leq d_n</math>, <math>k_{ni} \neq 0</math> so <math>a_n</math> must be 0</p> |
|       | <p>Now, discarding <math>a_n</math> associated term, we get</p> $\sum_{i=0}^{d_1} (a_1 k_{1i} + a_2 k_{2i} + \dots + a_n k_{ni}) x^i$ $+ \sum_{i=d_1+1}^{d_2} (a_2 k_{2i} + \dots + a_n k_{ni}) x^i$ $+ \dots + \sum_{i=d_{n-2}+1}^{d_{n-1}} a_{n-1} k_{(n-1)i} x^i = 0$ <p>so, for <math>d_{n-2} + 1 \leq i \leq d_{n-1}</math>, <math>k_{(n-1)i} \neq 0</math><br/> <math>\Rightarrow a_{n-1} = 0</math></p>   |

TABLE 4.1.5.2

|       |   |
|-------|---|
| Proof | <p>Similarly, for <math>d_1 + 1 \leq i \leq d_2</math></p> $\sum_{i=0}^{d_1} (a_1 k_{1i} + a_2 k_{2i}) x^i + \sum_{i=d_1+1}^{d_2} a_2 k_{2i} x^i$ $\Rightarrow a_2 = 0 \text{ as } k_{2i} \neq 0$ $\Rightarrow \sum_{i=0}^{d_1} a_1 k_{1i} x^i = 0$ $\Rightarrow a_1 = 0 \text{ as } k_{1i} \neq 0$ |
|       | <p>In this way, it can be proved that <math>a_1 = a_2 = \dots = a_{n-1} = a_n = 0</math> for <math>a_1 f_1 + a_2 f_2 + \dots + a_n f_n = \theta</math><br/> <math>\Rightarrow f_1, f_2, \dots, f_n</math> are linearly independent</p>  |

TABLE 4.1.5.3

| Sr. No | Description   | Mathematical representation   |
|--------|---|---|
| 1.     | Consider a set $S$  | $S = \{1, ax + b, \dots\}$  |
| 2.     | Provide a proof that subset $S$ span the subspace $\langle S \rangle$   | Since $1 \in S$ and $ax + b \in S$ ,<br>$\implies b.1 + \frac{1}{a}(b + ax) \in \langle S \rangle$<br>$\implies x \in \langle S \rangle$ Given element are $\in S$  |
| 3.     | Repeat step 2 for the higher degree of polynomial also lie in the subspace and the also lie in the subset $S$ . | Since $(ax + b)^2 \in S$<br>$\implies (ax + b)^2 - a^2 - 2.a.b.x = a^2.x^2$<br>$\implies a^2.x^2 \in \langle S \rangle$<br>$\implies \frac{1}{a^2}.a^2.x^2 \in S$ .<br>$\implies x^2 \in \langle S \rangle$ Given element are $\in S$ |
| 4.     | After providing proof for all element $\in S$ find the basis .  | $S' = \{1, x, x^2, x^3, \dots\}$  |
| 5.     | Show the element $\in S'$ are able to form all element $S$ over $\mathbb{F}$ .                                  | Hence $S$ form basis of $\mathbb{F}$  |

TABLE 4.1.6.1: Step of solution for given problem

| PARAMETER  | DESCRIPTION  |
|--|--|
| $\mathbb{F}$   | Field of complex numbers   |
| $\mathbb{F}^\infty$                                  | Vector space defined on the field $\mathbb{F}$                         |
| $\mathbb{F}[x]$                                      | Subspace of $\mathbb{F}^\infty$ spanned by $\{1, x, x^2, x^3, \dots\}$ |
| $T: \mathbb{F}[x] \rightarrow \mathbb{F}[x]$         | Transformation T   |
| $f = \sum_{i=0}^n c_i x^i$                           | Polynomial $f \in \mathbb{F}[x]$                                       |
| $f' = \sum_{i=0}^n c'_i x^i$                         | Polynomial $f' \in \mathbb{F}[x]$                                      |
| $c_i, c'_i \forall i = 0, 2, \dots, n$               | Scalars in $\mathbb{F}$ and coefficients of polynomials $f$ and $f'$   |
| $T(f) = g = \sum_{i=0}^n \frac{c_i}{i+1} x^{i+1}$    | Transformed polynomial $g \in \mathbb{F}[x]$                           |
| $T(f') = g' = \sum_{i=0}^n \frac{c'_i}{i+1} x^{i+1}$ | Transformed polynomial $g' \in \mathbb{F}[x]$                          |
| $\mathbf{M}_T$                                       | Transformation matrix for T  |
| $N(T)$   | Null Space of T  |

TABLE 4.1.7.1: Parameters

4.1.7. Let  $\mathbb{F}$  be a subfield of complex numbers and let T be the transformation on  $\mathbb{F}(x)$  defined by

$$T\left(\sum_{i=0}^n c_i x^i\right) = \sum_{i=0}^n \frac{c_i}{i+1} x^{i+1} \quad (4.1.7.1)$$

- a) Show that T is a non-singular linear operator on  $\mathbb{F}[x]$ . Also show that T is not invertible.

**Solution:** The transformation T does integral of a polynomial. Table refeq:solutions/4/1/9/a/table:2 provides proof that the transformation T is a linear operator and non-singular. Table refeq:solutions/4/1/9/a/table:3 provides proof that T is not invertible, however there exists a left inverse. The parameters used in the proof are listed in the table refeq:solutions/4/1/9/a/table:1.

- b) Let  $\mathbf{F}$  be a sub-field of the complex numbers and let  $\mathbf{D}$  be the transformation on  $\mathbf{F}[x]$  defined by

$$\mathbf{D}\left(\sum_{i=0}^n c_i x^i\right) = \sum_{i=0}^n i c_i x^{i-1} \quad (4.1.7.14)$$

$$\mathbf{D}(\mathbf{x}^T \mathbf{c}) = \mathbf{y}^T \mathbf{b} \quad (4.1.7.15)$$

where

$$\mathbf{x} = \begin{pmatrix} 1 \\ x \\ \vdots \\ x^n \end{pmatrix} \quad \text{and} \quad \mathbf{c} = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{pmatrix} \quad (4.1.7.16)$$

$$\mathbf{y} = \begin{pmatrix} 1 \\ x \\ \vdots \\ x^{n-1} \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} c_1 \\ 2c_2 \\ \vdots \\ nc_n \end{pmatrix} \quad (4.1.7.17)$$

Show that  $\mathbf{D}$  is a linear operator on  $\mathbf{F}[x]$  and find its null space.

**Solution:** See Tables 4.1.7.4 and 4.1.7.5

## 4.2 Lagrange Interpolation

- 4.2.1. Use the Lagrange interpolation formula to find a polynomial f with real coefficients such that f has degree  $\leq 3$  and  $f(-1)=-6$ ,  $f(0)=2$ ,  $f(1)=-2$ ,  $f(2)=6$ .

**Solution:** Let the required polynomial be

$$f(x) = a + bx + cx^2 + dx^3 \quad (4.2.1.1)$$

Testing points are

$$t_0 = -1, t_1 = 0, t_2 = 1, t_3 = 2 \quad (4.2.1.2)$$



| Statement                  | Derivation   |
|----------------------------|--|
| $f = \sum_{i=0}^n c_i x^i$ | $f = (c_0 \ c_1 \ c_2 \ \cdots \ c_n)_{(n+1) \times 1}^T$  |
| $T[f] = \mathbf{M}_T f$    | $T[f] = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{2} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{n+1} \end{pmatrix}_{(n+2) \times (n+1)} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}_{(n+1) \times 1} = \begin{pmatrix} 0 \\ c_0 \\ \frac{c_1}{2} \\ \frac{c_2}{3} \\ \vdots \\ \frac{c_n}{n+1} \end{pmatrix}_{(n+2) \times 1} = g \in \mathbb{F}[x]$  |
| T is a linear operator     | $T[\alpha f + f'] \quad (4.1.7.2)$ $= \mathbf{M}_T (\alpha f + f') \quad (4.1.7.3)$ $= \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{2} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{n+1} \end{pmatrix} \begin{pmatrix} \alpha c_0 + c'_0 \\ \alpha c_1 + c'_1 \\ \alpha c_2 + c'_2 \\ \vdots \\ \alpha c_n + c'_n \end{pmatrix} \quad (4.1.7.4)$ $= \begin{pmatrix} 0 \\ \alpha c_0 + c'_0 \\ \frac{\alpha c_1 + c'_1}{2} \\ \frac{\alpha c_2 + c'_2}{3} \\ \vdots \\ \frac{\alpha c_n + c'_n}{n+1} \end{pmatrix} = \alpha \begin{pmatrix} 0 \\ c_0 \\ \frac{c_1}{2} \\ \frac{c_2}{3} \\ \vdots \\ \frac{c_n}{n+1} \end{pmatrix} + \begin{pmatrix} 0 \\ c'_0 \\ \frac{c'_1}{2} \\ \frac{c'_2}{3} \\ \vdots \\ \frac{c'_n}{n+1} \end{pmatrix} \quad (4.1.7.5)$ $= \alpha T[f] + T[f'] \quad (4.1.7.6)$ $= \alpha g + g' \quad (4.1.7.7)$ $\therefore T[\alpha f + f'] = \alpha T[f] + T[f'] \quad (4.1.7.8)$ |
| T is non-singular          | $T[f] = 0 \quad (4.1.7.9)$ $\implies \mathbf{M}_T f = \mathbf{0} \implies f = 0 \because \mathbf{M}_T \neq \mathbf{0} \quad (4.1.7.10)$ $\implies N(T) = \{0\} \quad (4.1.7.11)$   |

TABLE 4.1.7.2: Proof for Non-Singular and linear transformation T

Substituting the values of testing points in (4.2.1.1) we form matrix equation

$$\begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} -6 \\ 2 \\ -2 \\ 6 \end{pmatrix} \quad (4.2.1.3)$$

By row reducing augmented matrix we get

$$\begin{pmatrix} 1 & -1 & 1 & -1 & -6 \\ 1 & 0 & 0 & 0 & 2 \\ 1 & 1 & 1 & 1 & -2 \\ 1 & 2 & 4 & 8 & 6 \end{pmatrix} \quad (4.2.1.4)$$

| Statement  | Derivation  |
|--|---|
| T is not invertible                              | As $\mathbf{M}_T$ is a non-square matrix with dimensions $(n+2) \times (n+1)$ , the transformation T is not invertible  |
| $\mathbf{M}_D$ is left inverse of $\mathbf{M}_T$ | $\mathbf{M}_D \mathbf{M}_T = \mathbf{I}_{n+1} \quad (4.1.7.12)$ $\Rightarrow \mathbf{M}_D = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & n+1 \end{pmatrix} \quad (4.1.7.13)$ |

TABLE 4.1.7.3: Non-Invertibility of transformation T

|                                |  |
|--------------------------------|--|
| Linear Transformation          | <p>A linear transformation from <math>\mathbf{V}</math> into <math>\mathbf{W}</math> is a function <math>\mathbf{T}</math> from <math>\mathbf{V}</math> into <math>\mathbf{W}</math> such that</p> $\mathbf{T}(c\alpha + \beta) = c\mathbf{T}(\alpha) + \mathbf{T}(\beta)$ <p><math>\forall \alpha</math> and <math>\beta</math> in <math>\mathbf{V}</math> and <math>\forall</math> scalars <math>c</math> in <math>\mathbf{F}</math>.</p>  |
| $\mathbf{F}[x]$                | <p>Let <math>\mathbf{F}[x]</math> be the subspace of <math>\mathbf{F}^\infty</math> spanned by the vectors <math>1, x, x^2, \dots</math>. An element of <math>\mathbf{F}[x]</math> is called a polynomial over <math>\mathbf{F}</math>.</p>  |
| Differentiation Transformation | <p>Let <math>\mathbf{F}</math> be a field and let <math>\mathbf{V}</math> be the space of polynomial functions <math>f</math> from <math>\mathbf{F}</math> into <math>\mathbf{F}</math>, given by</p> $f(x) = c_0 + c_1x + \dots + c_kx^k$ <p>Then,</p> $\mathbf{D}f(x) = c_1 + 2c_2x + \dots + kc_kx^{k-1}$ <p>is called Differentiation Transformation. The Differentiation Transformation is a Linear map because</p> $\begin{aligned} \mathbf{D}(cf + g)(x) &= (c.c_1 + c'_1) + 2(c.c_2 + c'_2)x + \dots + k(c.c_k + c'_k)x^{k-1} \\ &= c.c_1 + 2c.c_2x + \dots + kc.c_kx^{k-1} + c'_1 + 2c'_2x + \dots + kc'_kx^{k-1} \\ &= c\mathbf{D}f(x) + \mathbf{D}g(x) \end{aligned}$ |

TABLE 4.1.7.4

$$\begin{matrix} \xleftrightarrow{R_2 \leftarrow R_2 - R_1} \\ \xleftrightarrow{R_3 \leftarrow R_3 - R_1} \end{matrix} \begin{pmatrix} 1 & -1 & 1 & -1 & -6 \\ 0 & 1 & -1 & 1 & 8 \\ 0 & 2 & 0 & 2 & 4 \\ 1 & 2 & 4 & 8 & 6 \end{pmatrix} \quad (4.2.1.5)$$

$$\begin{matrix} \xleftrightarrow{R_4 \leftarrow R_4/3} \\ \xleftrightarrow{R_3 \leftarrow R_3 - R_2} \end{matrix} \begin{pmatrix} 1 & -1 & 1 & -1 & -6 \\ 0 & 1 & -1 & 1 & 8 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 1 & 1 & 3 & 4 \end{pmatrix} \quad (4.2.1.7)$$

$$\begin{matrix} \xleftrightarrow{R_4 \leftarrow R_4 - R_1} \\ \xleftrightarrow{R_3 \leftarrow R_3/2} \end{matrix} \begin{pmatrix} 1 & -1 & 1 & -1 & -6 \\ 0 & 1 & -1 & 1 & 8 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 3 & 3 & 9 & 12 \end{pmatrix} \quad (4.2.1.6)$$

$$\begin{matrix} \xleftrightarrow{R_1 \leftarrow R_1 + R_2} \\ \xleftrightarrow{R_4 \leftarrow R_4 - R_2} \end{matrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & -1 & 1 & 8 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 2 & 2 & -4 \end{pmatrix} \quad (4.2.1.8)$$

|                            |  |
|----------------------------|--|
| Proving <b>D</b> is Linear | <p>From (4.1.7.14), clearly <b>D</b> is a function from <math>\mathbf{F}[\mathbf{x}]</math> to <math>\mathbf{F}[\mathbf{x}]</math>. We must show that <b>D</b> is linear. Clearly <b>D</b> is a Differentiation Transformation, and hence is linear. In other words</p> <p>Let <math>\mathbf{m} = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{pmatrix}</math>, <math>\mathbf{n} = \begin{pmatrix} c'_0 \\ c'_1 \\ \vdots \\ c'_n \end{pmatrix}</math> and <math>\alpha</math> be a scalar. Then</p> $\mathbf{D}(\mathbf{x}^T(\alpha\mathbf{m} + \mathbf{n})) = \mathbf{y}^T\mathbf{p}, \text{ where } \mathbf{p} = \begin{pmatrix} \alpha c_1 + c'_1 \\ 2(\alpha c_2 + c'_2) \\ \vdots \\ n(\alpha c_n + c'_n) \end{pmatrix} \text{ and } \alpha\mathbf{m} + \mathbf{n} = \begin{pmatrix} \alpha c_0 + c'_0 \\ \alpha c_1 + c'_1 \\ \vdots \\ \alpha c_n + c'_n \end{pmatrix}$ $\mathbf{D}(\mathbf{x}^T(\alpha\mathbf{m} + \mathbf{n})) = \mathbf{y}^T(\mathbf{m}' + \mathbf{n}'), \text{ where } \mathbf{m}' = \alpha \begin{pmatrix} c_1 \\ 2c_2 \\ \vdots \\ nc_n \end{pmatrix} \text{ and } \mathbf{n}' = \begin{pmatrix} c'_1 \\ 2c'_2 \\ \vdots \\ nc'_n \end{pmatrix}$ <p>Thus,</p> $\mathbf{D}(\mathbf{x}^T(\alpha\mathbf{m} + \mathbf{n})) = \mathbf{y}^T\mathbf{m}' + \mathbf{y}^T\mathbf{n}'$ <p>Now,</p> $\mathbf{D}(\mathbf{x}^T(\alpha\mathbf{m} + \mathbf{n})) = \alpha\mathbf{D}(\mathbf{x}^T\mathbf{m}) + \mathbf{D}(\mathbf{x}^T\mathbf{n})$ <p>Hence, <b>D</b> is a linear transformation.</p> |
| Null Space of <b>D</b>     | <p>Let <math>\mathbf{N}(\mathbf{D})</math> denotes the nullspace of <b>D</b>. Then</p> $\mathbf{N}(\mathbf{D}) = \{f \in \mathbf{F}[\mathbf{x}] : \mathbf{D}f(x) = 0\}$ <p>A polynomial is zero if and only if its every coefficient is zero. Thus, it must be such that each <math>c_1 = c_2 = \dots = 0</math>. Since, <b>D</b> is a Differentiation Transformation and we know that derivative of a constant polynomial is zero. Thus, the nullspace of <b>D</b> contains the constant polynomials. Hence,</p> $\mathbf{N}(\mathbf{D}) = \{f \in \mathbf{F}[\mathbf{x}] : f(x) = c\}$   |

TABLE 4.1.7.5

$$\begin{array}{c} \xleftrightarrow{R_1 \leftarrow R_1 + R_2} \\ \xleftrightarrow{R_4 \leftarrow R_4 - R_2} \end{array} \begin{pmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & -1 & 1 & 8 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 2 & 2 & -4 \end{pmatrix} \quad (4.2.1.9)$$

$$\begin{array}{c} \xleftrightarrow{R_4 \leftarrow R_4 - R_3} \\ \xleftrightarrow{R_2 \leftarrow R_2 - R_4} \end{array} \begin{pmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & 4 \end{pmatrix} \quad (4.2.1.11)$$

Therefore ,

$$\begin{array}{c} \xleftrightarrow{R_4 \leftarrow R_4/2} \\ \xleftrightarrow{R_2 \leftarrow R_2 + R_3} \end{array} \begin{pmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 1 & -2 \end{pmatrix} \quad (4.2.1.10)$$

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ -6 \\ 4 \end{pmatrix} \quad (4.2.1.12)$$

Hence required polynomial is

$$f(x) = 2 - 2x - 6x^2 + 4x^3 \quad (4.2.1.13)$$

4.2.2. Let  $T$  be the linear operator on  $R^3$  defined by  $T(x_1, x_2, x_3) = (x_1, x_3, -2x_2 - x_3)$

Let  $f$  be the polynomial over  $R$  defined by  $f = -x^3 + 2$ . Find  $f(T)$ .

**Solution:** The transformation is given as:

$$T(\mathbf{x}) = \begin{pmatrix} x_1 \\ x_3 \\ -2x_2 - x_3 \end{pmatrix} \quad (4.2.2.1)$$

$$T(\mathbf{x}) = \mathbf{TX} \quad (4.2.2.2)$$

$$T(\mathbf{x}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -1 \end{pmatrix} \mathbf{X} \quad (4.2.2.3)$$

Characteristic equation of  $\mathbf{T}$  can be written as:

$$|\mathbf{T} - \lambda \mathbf{I}| = 0 \quad (4.2.2.4)$$

$$\begin{vmatrix} 1 - \lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ 0 & -2 & -1 - \lambda \end{vmatrix} = 0 \quad (4.2.2.5)$$

The characteristics equation of the matrix will be,

$$-\lambda^3 - \lambda + 2 = 0 \quad (4.2.2.6)$$

$$-\lambda^3 + 2 = \lambda \quad (4.2.2.7)$$

Given, Polynomial

$$f(x) = -x^3 + 2 \quad (4.2.2.8)$$

$$f(\lambda) = -\lambda^3 + 2 \quad (4.2.2.9)$$

Using 4.2.2.7,

$$f(\lambda) = \lambda \quad (4.2.2.10)$$

Substituting  $\mathbf{T}$  for  $\lambda$  in equation 4.2.2.10, We get

$$f(\mathbf{T}) = \mathbf{T} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -1 \end{pmatrix} \quad (4.2.2.11)$$

4.2.3. Let  $\mathbb{F}$  be the field of real numbers,

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (4.2.3.1)$$

$$p = (x - 2)(x - 3)(x - 1) \quad (4.2.3.2)$$

a) Let  $\mathbb{F}$  be the field of real numbers,

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (4.2.3.3)$$

$$p = (x - 2)(x - 3)(x - 1) \quad (4.2.3.4)$$

**Solution:**

- i) Show that  $p(\mathbf{A}) = 0$
- ii) Let  $P_1, P_2, P_3$  be the Lagrange polynomials for  $t_1 = 2, t_2 = 3, t_3 = 1$ . Compute  $E_i = P_i(\mathbf{A})$ ,  $i=1,2,3$
- i) Since  $\mathbf{A}$  is a diagonal matrix, It's characteristic polynomial is,

$$\det(\mathbf{A} - x\mathbf{I}) = 0 \quad (4.2.3.5)$$

$$f(x) = (x - 2)^2(x - 3)(x - 1) = 0 \quad (4.2.3.6)$$

From, (4.2.3.6) and using Cayley Hamilton Theorem,

$$(\mathbf{A} - 2)^2(\mathbf{A} - 3)(\mathbf{A} - 1) = 0 \quad (4.2.3.7)$$

We can also see that  $(x-2)(x-3)(x-1)$  is a minimal polynomial for  $\mathbf{A}$ , Hence  $p(\mathbf{A})=0$ .

ii) Using Lagrange Interpolation,

$$P_1(x) = \frac{(x - 3)(x - 1)}{(2 - 3)(2 - 1)} \quad (4.2.3.8)$$

$$= -(x - 3)(x - 1) \quad (4.2.3.9)$$

$$P_2(x) = \frac{(x - 2)(x - 1)}{(3 - 2)(3 - 1)} \quad (4.2.3.10)$$

$$= \frac{(x - 2)(x - 1)}{2} \quad (4.2.3.11)$$

$$P_3(x) = \frac{(x - 2)(x - 3)}{(1 - 2)(1 - 3)} \quad (4.2.3.12)$$

$$= \frac{(x - 2)(x - 3)}{2} \quad (4.2.3.13)$$

Now, Substituting the value of  $\mathbf{A}$ ,

$$P_1(\mathbf{A}) = - \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (4.2.3.14)$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (4.2.3.15)$$

$$P_2(\mathbf{A}) = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (4.2.3.16)$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (4.2.3.17)$$

$$P_3(\mathbf{A}) = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} \quad (4.2.3.18)$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (4.2.3.19)$$

- b) Let  $P_1, P_2, P_3$  be the Lagrange polynomials for  $t_1 = 2, t_2 = 3, t_3 = 1$ . Compute  $E_i = P_i(\mathbf{A}) \quad i = 1, 2, 3$ .

**Solution:** Lagrange polynomials are given

by,

$$P_i = \prod_{i \neq j} \frac{x - t_j}{t_i - t_j} \quad (4.2.3.20)$$

$$P_1 = \frac{(x-3)(x-1)}{(2-3)(2-1)} = -(x-3)(x-1) \quad (4.2.3.21)$$

$$P_2 = \frac{(x-2)(x-1)}{(3-2)(3-1)} = \frac{1}{2}(x-2)(x-1) \quad (4.2.3.22)$$

$$P_3 = \frac{(x-2)(x-3)}{(1-2)(1-3)} = \frac{1}{2}(x-2)(x-3) \quad (4.2.3.23)$$

Now calculating  $E_i$ ,

$$E_1 = P_1(\mathbf{A}) = -(\mathbf{A} - 3\mathbf{I})(\mathbf{A} - \mathbf{I}) \quad (4.2.3.24)$$

$$= - \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (4.2.3.25)$$

$$\Rightarrow E_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (4.2.3.26)$$

$$E_2 = P_2(\mathbf{A}) = \frac{1}{2}(\mathbf{A} - 2\mathbf{I})(\mathbf{A} - \mathbf{I}) \quad (4.2.3.27)$$

$$= \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (4.2.3.28)$$

$$\Rightarrow E_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (4.2.3.29)$$

$$E_3 = P_3(\mathbf{A}) = \frac{1}{2}(\mathbf{A} - 2\mathbf{I})(\mathbf{A} - 3\mathbf{I}) \quad (4.2.3.30)$$

$$= \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} \quad (4.2.3.31)$$

$$\Rightarrow E_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (4.2.3.32)$$

4.2.4. Let  $n$  be a positive integer and  $\mathbb{F}$  be a field. Suppose  $\mathbf{A}$  is an  $n \times n$  matrix over field  $\mathbb{F}$  and  $\mathbf{P}$  is an invertible  $n \times n$  matrix over field  $\mathbb{F}$ . If  $f$  is any polynomial over  $\mathbb{F}$ , prove that,

$$f(\mathbf{P}^{-1}\mathbf{A}\mathbf{P}) = \mathbf{P}^{-1}f(\mathbf{A})\mathbf{P} \quad (4.2.4.1)$$

**Solution:** First we observe the following,

$$(\mathbf{P}^{-1}\mathbf{A}\mathbf{P})^2 = (\mathbf{P}^{-1}\mathbf{A}\mathbf{P})(\mathbf{P}^{-1}\mathbf{A}\mathbf{P}) \quad (4.2.4.2)$$

$$= \mathbf{P}^{-1}\mathbf{A}^2\mathbf{P} \quad (4.2.4.3)$$

Let the (4.2.4.3) be true for a positive integer  $m$  i.e.,

$$(\mathbf{P}^{-1}\mathbf{A}\mathbf{P})^m = \mathbf{P}^{-1}\mathbf{A}^m\mathbf{P} \quad (4.2.4.4)$$

Now for the integer  $m+1$  we get from (4.2.4.4),

$$(\mathbf{P}^{-1}\mathbf{A}\mathbf{P})^{m+1} = (\mathbf{P}^{-1}\mathbf{A}^m\mathbf{P})(\mathbf{P}^{-1}\mathbf{A}\mathbf{P}) \quad (4.2.4.5)$$

$$= \mathbf{P}^{-1}\mathbf{A}^{m+1}\mathbf{P} \quad (4.2.4.6)$$

From (4.2.4.4) and (4.2.4.6) we get for any positive integer  $n$ ,

$$(\mathbf{P}^{-1}\mathbf{A}\mathbf{P})^n = \mathbf{P}^{-1}\mathbf{A}^n\mathbf{P} \quad (4.2.4.7)$$

Again we have,

$$\mathbf{P}^{-1}\mathbf{P} = \mathbf{I} \quad (4.2.4.8)$$

The general form of polynomial  $f(\mathbf{A})$  is defined as,

$$f(\mathbf{A}) = a_0 + a_1\mathbf{A} + a_2\mathbf{A}^2 + \cdots + a_n\mathbf{A}^n \quad (4.2.4.9)$$

Now we have,

$$f(\mathbf{P}^{-1}\mathbf{A}\mathbf{P}) = a_0 + a_1(\mathbf{P}^{-1}\mathbf{A}\mathbf{P}) + \cdots + a_n(\mathbf{P}^{-1}\mathbf{A}\mathbf{P})^n \quad (4.2.4.10)$$

$$= (\mathbf{P}^{-1}a_0\mathbf{P}) + (\mathbf{P}^{-1}a_1\mathbf{A}\mathbf{P}) + \cdots + (\mathbf{P}^{-1}a_n\mathbf{A}^n\mathbf{P}) \quad (4.2.4.11)$$

$$= (\mathbf{P}^{-1}a_0\mathbf{P}) + (\mathbf{P}^{-1}a_1\mathbf{A}\mathbf{P}) + \cdots + (\mathbf{P}^{-1}a_n\mathbf{A}^n\mathbf{P}) \quad (4.2.4.12)$$

$$= \mathbf{P}^{-1}(a_0 + a_1\mathbf{A} + a_2\mathbf{A}^2 + \cdots + a_n\mathbf{A}^n)\mathbf{P} \quad (4.2.4.13)$$

$$= \mathbf{P}^{-1}f(\mathbf{A})\mathbf{P} \quad [\text{From (4.2.4.9)}] \quad (4.2.4.14)$$

Hence proved (4.2.4.14)

4.2.5. Let  $\mathbf{F}$  be a field. We have considered certain special linear functionals on  $\mathbf{F}[x]$  obtained via 'evaluation at  $t$ ':  $L(f) = f(t)$ . Such functionals are not only linear but also have the property that  $L(fg) = L(f)L(g)$ . Prove that if  $L$  is any linear functional on  $\mathbf{F}[x]$  such that  $L(fg) = L(f)L(g)$  for all  $f$  and  $g$ , then either  $L = 0$  or there is a  $t$  in  $\mathbf{F}$  such that  $L(f) = f(t)$  for all  $f$ .

**Solution:** Let  $L$  be a non zero linear transformation.

$$f(x) = a_0 + a_1(x) + \cdots + a_n(x^n) \quad (4.2.5.1)$$

$$L(f) = L(a_0 + a_1(x) + \cdots + a_n(x^n)). \quad (4.2.5.2)$$

Given,  $L$  is any linear functional on  $\mathbf{F}[x]$ .

Hence,

$$L(f) = L(f.1) = L(f)L(1) \quad (4.2.5.3)$$

$$\Rightarrow L(1) \neq 0.$$

Similarly,

$$L(1) = L(1)L(1) \quad (4.2.5.4)$$

$$\Rightarrow L(1) = 1 \quad (4.2.5.5)$$

Similarly,  $\forall a \in \mathbf{F}$ .

$$L(a) = L(a.1) \Rightarrow aL(1) = a \quad (4.2.5.6)$$

Simplifying (4.2.5.2) we have,

$$L(f) = L(a_0) + L(a_1)L(x) + \cdots + L(a_n)L(x^n) \quad (4.2.5.7)$$

From (4.2.5.6) we have,  $L(x) = t$ .

$$L(f) = a_0 + a_1L(x) + \cdots + a_nL(x)^n \quad (4.2.5.8)$$

$$\Rightarrow L(f) = a_0 + a_1(t) + \cdots + a_n(t)^n \quad (4.2.5.9)$$

Hence proved,  $L(f) = f(t)$ .

4.2.6. If  $F$  is a field and  $h$  is a polynomial over  $F$  of degree  $\geq 1$ , show that the mapping  $f \rightarrow f(h)$  is a one-one linear transformation of  $F[x]$  into  $F[x]$ . Show that this transformation is an isomorphism of  $F[x]$  onto  $F[x]$  if and only if  $\deg h = 1$ .

**Solution:** Here,  $F[x]$  is a set of polynomials over field  $F$ , written as:

$$F[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \mid a_i \in F \right\} \quad (4.2.6.1)$$

Let,

$$G(f) = f(h) \quad (4.2.6.2)$$

Thus,  $G(f)$  is clearly a function from  $F[x]$  into  $F[x]$ .

Now, we need to show that the function  $G$  is one-one linear transformation. Let us first show that  $G$  is a linear transformation:

Let,  $f, g \in F[x]$  and  $\alpha \in F$

$$\begin{aligned} G(\alpha f + g) &= (\alpha f + g)(h) \\ &= (\alpha f)(h) + g(h) \\ &= \alpha f(h) + g(h) \\ &= \alpha G(f) + G(g) \end{aligned} \quad (4.2.6.3)$$

From (4.2.6.3),  $G$  is a linear transformation.

For  $G$  to be one-one linear transformation, it should map a set of linearly independent polynomials in  $F(x)$  to another set of linearly independent polynomials in  $F(x)$ . let us consider the following basis set for  $F(x)$ :

$$S = \{f_0, f_1, f_2, f_3, f_4, \dots\} \quad (4.2.6.4)$$

Where,

$$f_i = x^i \quad (4.2.6.5)$$

Since, the set  $S$  forms the basis for  $F(x)$ , the set  $S$  is a set of linearly independent polynomials. Let us apply the transformation  $G$  to set  $S$ , then we obtain another set  $S'$  as:

$$S' = \{f_0(h), f_1(h), f_2(h), f_3(h), f_4(h), \dots\} \quad (4.2.6.6)$$

Where,

$$f_i = x^i \quad (4.2.6.7)$$

Here, The degree of each polynomial in set  $S'$  is distinct and given by  $i \cdot \deg(h)$ . Thus, set  $S'$  is also a set of linearly independent polynomials.

**Conclusion:**  $G$  will maps any arbitrary set  $S_a$  of linearly independent polynomials in  $F(x)$  to another set  $S'_a$  of linearly independent polynomials in  $F(x)$ . (Since any arbitrary set  $S_a$  can be written in terms of basis set  $S$ ). Hence,  $G$  is one-one linear transformation.

Now, Let us prove that  $G$  is an isomorphism of  $F(x)$  onto  $F(x)$  if and only if  $\deg(h) = 1$ .

Let  $\deg(h) = 1$ , then  $h$  can be written as:

$$h = a + bx, \quad \text{Where, } b \neq 0 \quad (4.2.6.8)$$

Let us define  $h'$  such that:

$$h' = \frac{1}{b}x - \frac{a}{b} \quad (4.2.6.9)$$

Let  $G'$  be the linear transformation from  $F(x)$  to  $F(x)$  given by:

$$G'(f) = f\left(\frac{1}{b}x - \frac{a}{b}\right) \quad (4.2.6.10)$$

It can be shown that  $G'$  is inverse of  $G$  as follow:

$$G(G'(f)) = G\left(f\left(\frac{1}{b}x - \frac{a}{b}\right)\right) \quad (4.2.6.11)$$

$$= f\left(a\left(\frac{1}{a}x - \frac{b}{a}\right) + b\right) \quad (4.2.6.12)$$

$$= f(x) \quad (4.2.6.13)$$

Similarly,

$$G'(G(f)) = G'(f(ax + b)) \quad (4.2.6.14)$$

$$= f\left(\frac{1}{a}(ax + b) - \frac{b}{a}\right) \quad (4.2.6.15)$$

$$= f(x) \quad (4.2.6.16)$$

Thus,  $G'$  is inverse of  $G$ . Therefore,  $G$  is isomorphism and we can say:

$$\boxed{\deg(h) = 1 \implies G \text{ is isomorphism.}} \quad (4.2.6.17)$$

Let  $\deg(h) > 1$ , then

$$\deg f(h) = \deg f \cdot \deg h \quad (4.2.6.18)$$

$$\implies \deg f(h) \geq 1 \quad (4.2.6.19)$$

$$\implies G(f) = f(h) \neq x \quad (4.2.6.20)$$

This means the image of  $G$  does not contain polynomials of degree one. Hence  $G$  is not onto and therefore  $G$  can not be an isomorphism. Thus we can write:

$$\deg(h) > 1 \implies G \text{ is not isomorphism.} \quad (4.2.6.21)$$

From (4.2.6.17) and (4.2.6.21), We can conclude:

$$G \text{ is isomorphism.} \iff \deg(h) = 1$$

(4.2.6.22)

### 4.3 Polynomial Ideals

4.3.1. Let  $\mathbb{Q}$  be the field of rational numbers. Determine if the following subset of  $\mathbb{Q}[x]$  is ideal or not.

a) The subset is defined by all  $f$  with degree  $\geq 5$

**Solution:** See Table 4.3.1.1

|             |  |
|-------------|--|
| Example     | The defined subset of $\mathbb{Q}[x]$ be $\mathbb{U}$ ,<br>$f(x) = c_1x^5 + c_2x^4 \in \mathbb{U}$<br>$g(x) = -c_1x^5 + c_3x^4 \in \mathbb{U}$   |
| Proof       | If $\mathbb{U}$ is an ideal then,<br>$\mathbb{U}$ must be a subset.<br>$\mathbb{U}$ must be closed under addition.<br>$f \in \mathbb{U}$<br>$g \in \mathbb{U}$<br>$\implies f + g \in \mathbb{U}$<br>But here,<br>$f + g = c_1x^5 + c_2x^4 - c_1x^5 + c_3x^4$<br>$\implies f + g = (c_2 + c_3)x^4 \notin \mathbb{U}$ |
| Observation | $f \in \mathbb{U}$<br>$g \in \mathbb{U}$<br>But, $f + g \notin \mathbb{U}$   |
| Conclusion  | $\mathbb{U}$ is not closed under addition<br>$\implies \mathbb{U}$ is not a subset of $\mathbb{Q}[x]$<br>$\implies \mathbb{U}$ is not an ideal of $\mathbb{Q}[x]$  |

TABLE 4.3.1.1

4.3.2. a) Find the g.c.d of each of the following pairs of polynomials.

$$2x^5 - x^3 - 3x^2 - 6x + 4, x^4 + x^3 - x^2 - 2x - 2 \quad (4.3.2.1)$$

**Solution:** Refer Table 4.3.2.1.

b) Let  $\mathbb{Q}$  be the field of rational numbers. Determine which of the following subsets of  $\mathbb{Q}[x]$  are ideals. When the set is an ideal, find its monic generator.

All  $f$  such that  $f(2) = f(4) = 0$ .

**Solution:** *Definition:* Let  $F$  be a field. An ideal in  $F[x]$  is a subspace  $M$  of  $F[x]$  such that  $fg$  belongs to  $M$  whenever  $f$  is in  $F[x]$  and  $g$  is in  $M$ .

See Table 4.3.2.2

4.3.3. Let  $A$  be an  $n \times n$  matrix over a field  $\mathbf{F}$ . Show that the set of all polynomials  $f$  in  $\mathbf{F}[x]$  such that  $f(A) = 0$  is an ideal.

**Solution:** Given a square matrix of order  $n$  and  $f(A) = 0$ . Let  $\mathbf{I}$  be an ideal of  $C[z_1, \dots, z_n]$ .

$$\mathbf{I} = \{f \in \mathbf{F}(x) | f(A) = 0\} \quad (4.3.3.1)$$

Now, Consider the polynomials,

$$f = f_0 + f_1z_n + \dots + f_dz_n^d \quad (4.3.3.2)$$

$$g = g_0 + g_1z_n + \dots + g_ez_n^e \quad (4.3.3.3)$$

$$f, g \in \mathbf{I} \ \& \ k \in \mathbf{F}.$$

$$(kf + g)(A) = kf(A) + g(A) \quad (4.3.3.4)$$

$$\implies c.0 + 0 = 0 \quad (4.3.3.5)$$

From equation (4.3.3.5)  $\mathbf{I}$  is a subspace of  $\mathbf{F}(x)$

Now,  $f \in \mathbf{I}$  and  $g \in F(x)$  then,

$$(gf)A = g(A)f(A) \quad (4.3.3.6)$$

$$\implies g(A).0 = 0 \quad (4.3.3.7)$$

Hence, proved  $\mathbf{I}$  is ideal

4.3.4. Let  $\mathbf{F}$  be a subfield of complex numbers, and let

$$\mathbf{A} = \begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix} \quad (4.3.4.1)$$

Find the monic generator of the ideal of all polynomials  $f$  in  $F[z]$  such that  $f(\mathbf{A}) = 0$ .

**Solution:** let, ideal  $I = \{f(x) \in F[x] \text{ such that}$



| Let the field be rational numbers  |  |
|------------------------------------|--|
| Steps                              | Explanation  |
| Say $f(x)$ and $g(x)$              | $f(x) = 2x^5 - x^3 - 3x^2 - 6x + 4 \quad (4.3.2.2)$ $g(x) = x^4 + x^3 - x^2 - 2x - 2 \quad (4.3.2.3)$  |
| Expanding $f(x)$ in term of $g(x)$ | $2x^5 - x^3 - 3x^2 - 6x + 4 = (x^4 + x^3 - x^2 - 2x - 2)(2x - 2) + (3x^3 - x^2 - 6x) \quad (4.3.2.4)$  |
| Expanding degree four polynomial   | $x^4 + x^3 - x^2 - 2x - 2 = (3x^3 - x^2 - 6x)\left(\frac{1}{3}x + \frac{4}{9}\right) + \left(\frac{13}{9}x^2 + \frac{2}{3}x - 2\right) \quad (4.3.2.5)$  |
| Expanding degree three polynomial  | $3x^3 - x^2 - 6x = \left(\frac{13}{9}x^2 + \frac{2}{3}x - 2\right)\left(\frac{27}{13}x - \frac{279}{169}\right) + \left(-\frac{126}{169}x - \frac{558}{169}\right) \quad (4.3.2.6)$  |
| Expanding degree two polynomial    | $\frac{13}{9}x^2 + \frac{2}{3}x - 2 = \left(-\frac{126}{169}x - \frac{558}{169}\right)\left(-\frac{2197}{1134}x + \frac{61009}{7938}\right) + \left(\frac{10309}{441}\right) \quad (4.3.2.7)$ <p>Since it contains scalar polynomial hence the g.c.d of <math>f(x)</math>, <math>g(x)</math> is 1.</p> |

TABLE 4.3.2.1: Solution

$f(\mathbf{A}) = 0$ . where,

$$f(x) = \sum_{i=0}^n a_i x^i \quad a_n = 1 \quad (4.3.4.2)$$

Computing  $f(\mathbf{A})$  for  $\deg(f) \leq 1$  we get,

$$\mathbf{A}^0 = \mathbf{I} \quad (4.3.4.3)$$

$$\mathbf{A}^1 = \begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix} \quad (4.3.4.4)$$

$$f(\mathbf{A}) = a_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + a_1 \begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix} \quad (4.3.4.5)$$

As  $\mathbf{I}$  and  $\mathbf{A}$  are linearly independent so from (4.3.4.5) we see  $f(\mathbf{A}) = 0$  only if  $a_1 = 0$  and  $a_0 = 0$  but this can't happen from (4.3.4.2). Hence for  $\deg(f) \leq 1$ ,  $f(\mathbf{A}) \neq 0$ . Hence for any  $f(x) \in I$  such that  $\deg(f) = 2$  then  $f$  is Monic generator or minimal polynomial. We can

write,

$$f(x) = x^2 + a_1 x^1 + a_0 \quad (4.3.4.6)$$

Minimal polynomial or monic generator can be found using Characteristic equation,

$$|\mathbf{A} - \lambda \mathbf{I}| = 0 \quad (4.3.4.7)$$

$$\begin{vmatrix} 1 - \lambda & -2 \\ 0 & 3 - \lambda \end{vmatrix} = 0 \quad (4.3.4.8)$$

$$(1 - \lambda)(3 - \lambda) = 0 \quad (4.3.4.9)$$

$$\Rightarrow \lambda^2 - 4\lambda + 3 = 0 \quad (4.3.4.10)$$

comparing (4.3.4.6) and (4.3.4.10) we get,

$$f(x) = x^2 - 4x + 3 = 0 \quad (4.3.4.11)$$

using Cayley-Hamilton equation,

$$f(\mathbf{A}) = \mathbf{A}^2 - 4\mathbf{A} + 3 = 0 \quad (4.3.4.12)$$

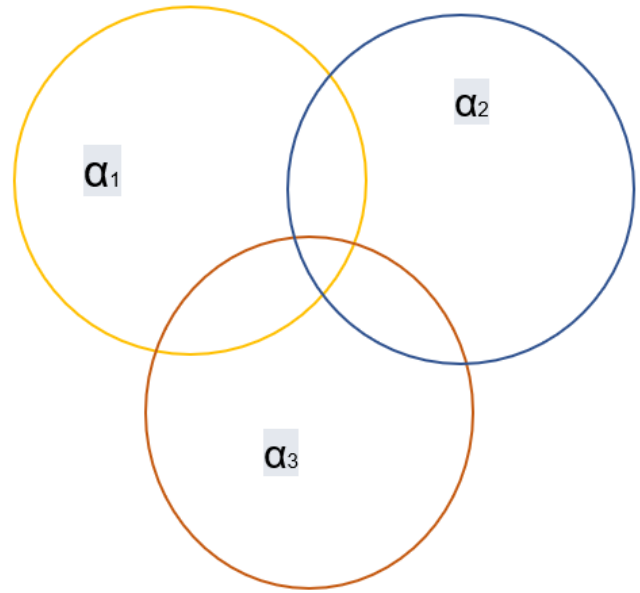
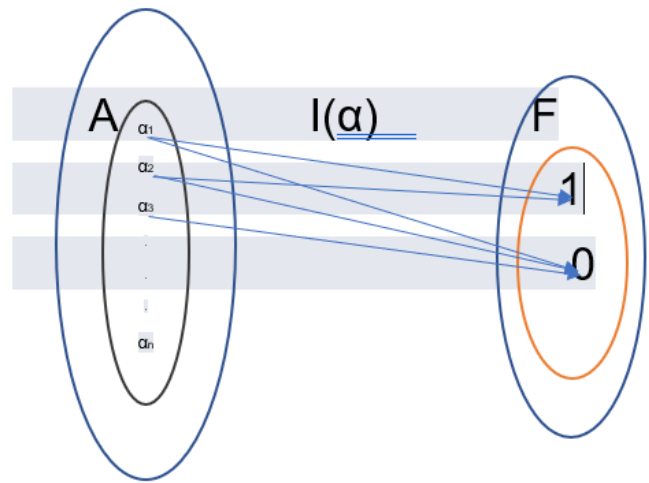
Hence,  $f = x^2 - 4x + 3$  is the monic generator.

|          |   |
|----------|---|
| Given    | $Q$ be the field of rational numbers.<br>Subset: All $f$ such that $f(2) = f(4) = 0$  |
| To prove | Given subset is an Ideal.<br>If Ideal then find its monotic generator.  |
| Proof    | <p>Let, <math>M = \{f \in F[x]   f(2) = f(4) = 0\}</math>.<br/> <math>g(2) = g(4) = 0</math> and <math>f(2) = f(4) = 0</math></p> <p><math>(df + g)(2) = df(2) + g(2) = 0</math><br/> <math>(df + g)(4) = df(4) + g(4) = 0</math></p> <p><math>\Rightarrow M</math> is a subspace of <math>F[x]</math></p> <p>Let suppose <math>f \in M</math> and <math>g \in F[x]</math>.<br/> <math>(fg)(2) = f(2)g(2) = 0</math><br/> <math>(fg)(4) = f(4)g(4) = 0</math><br/> <math>\therefore fg \in M</math></p> <p>Hence, <math>M</math> is an ideal.</p> <p>Representation of polynomial<br/> <math>f(x) = (x - c)q(x) + r</math><br/> When <math>f(c) = 0</math> then <math>r = 0</math>.</p> <p><math>\Rightarrow f(x) = (x - 2)(x - 4)q(x)</math><br/> <math>\therefore f(x) = g(x)q(x)</math></p> <p>Hence, <math>f(x)</math> is generated by <math>g(x)</math>.</p> |

TABLE 4.3.2.2

4.3.5. Let  $F$  be a field, show that the intersection of any numbers of ideals in  $F[x]$  is an ideal.

**Solution:** See Figs. 4.3.5.1, 4.3.5.2 and Table 4.3.5.1

Fig. 4.3.5.1: Index set containing  $\alpha$ Fig. 4.3.5.2: ideal containing  $I_\alpha$ 

## 5 DETERMINANTS

### 6 ELEMENTARY CANONICAL FORMS

#### 6.1 Characteristic Values

6.1.1. In each of the following cases, let  $T$  be the linear operator on  $R^2$  which is represented by matrix  $A$  in the standard ordered basis for  $R^2$ , and let  $U$  be the linear operator on  $C^2$  represented by  $A$  in the standard ordered basis. Find the characteristic polynomial for  $T$  and that for  $U$ , find the characteristic value of each operator, and for each characteristic value  $c$

find a basis for the corresponding space of characteristic vectors.

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (6.1.1.1)$$

$$A_2 = \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix} \quad (6.1.1.2)$$

$$A_3 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad (6.1.1.3)$$

**Solution:** See Table 6.1.1.1 and 6.1.1.2.

Let  $T$  be the linear operator on  $R^3$  which is represented in the standard ordered basis by

|          |  |
|----------|--|
| Given    | $F$ is a field   |
| To prove | $I = \bigcap_{\alpha \in A} I_\alpha$ is an ideal  |
| Proof    | <p>Let <math>A</math> be an index set and</p> <p><math>I_\alpha</math> be an in <math>F[x]</math> for each <math>\alpha \in A</math></p> <p>Obviously <math>I</math> is the subspace</p> <p>since <math>I_\alpha</math> is a subspace of <math>F[x]</math></p> <p>and arbitrary intersection of subspace</p> <p>is also a subspace.</p> <p>Let <math>g(x) \in F[x]</math> and <math>f(x) \in I</math></p> <p>Since <math>f(x) \in I</math></p> <p>and <math>I_\alpha</math> is an ideal follows that</p> <p><math>f(x) g(x) \in I_\alpha \forall \alpha \in A</math></p> <p>Thus <math>f(x) g(x) \in I</math>.</p> |

TABLE 4.3.5.1

the matrix

$$\begin{pmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{pmatrix} \quad (6.1.2.1)$$

Prove that  $T$  is diagonalizable by exhibiting a basis for  $\mathbf{R}^3$ , each vector of which is a characteristic vector of  $T$ .

**Solution:** Let  $T$  be a linear operator on a finite-dimensional space  $V$ . Let  $c_1, \dots, c_k$  be the distinct characteristic values of  $T$  and let  $W_i$  be the null space of  $T - c_i I$ . The following are equivalent.

- (i)  $T$  is diagonalizable.
- (ii) The characteristic polynomial for  $T$  is,

$$f = (x - c_1)^{d_1} \cdots (x - c_k)^{d_k} \quad (6.1.2.2)$$

- and  $\dim W_i = d_i, i = 1, \dots, k$
- (iii)  $\dim W_1 + \dots + \dim W_k = \dim V$

Now let,

$$A = \begin{pmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{pmatrix} \quad (6.1.2.3)$$

Solving  $|\lambda I - A| = 0$

$$|\lambda I - A| = \begin{vmatrix} \lambda + 9 & -4 & -4 \\ 8 & \lambda - 3 & -4 \\ 16 & -8 & \lambda - 7 \end{vmatrix} \quad (6.1.2.4)$$

$$\xrightarrow{C_2 \leftarrow C_2 - C_3} \begin{vmatrix} \lambda + 9 & 0 & -4 \\ 8 & \lambda + 1 & -4 \\ 16 & -\lambda - 1 & \lambda - 7 \end{vmatrix} \quad (6.1.2.5)$$

$$\therefore |\lambda I - A| = (\lambda + 1) \begin{vmatrix} \lambda + 9 & 0 & -4 \\ 8 & 1 & -4 \\ 16 & -1 & \lambda - 7 \end{vmatrix} \quad (6.1.2.6)$$

$$\xrightarrow{R_3 \leftarrow R_3 + R_2} (\lambda + 1) \begin{vmatrix} \lambda + 9 & 0 & -4 \\ 8 & 1 & -4 \\ 24 & 0 & \lambda - 11 \end{vmatrix} \quad (6.1.2.7)$$

$$\Rightarrow (\lambda + 1) \begin{vmatrix} \lambda + 9 & -4 \\ 24 & \lambda - 11 \end{vmatrix} \quad (6.1.2.8)$$

$$\Rightarrow |\lambda I - A| = (\lambda + 1)^2 (\lambda - 3) = 0 \quad (6.1.2.9)$$

$$\Rightarrow \lambda_1 = -1, \lambda_2 = -1, \lambda_3 = 3 \quad (6.1.2.10)$$

Now at  $\lambda_1$  and  $\lambda_3$ ,

$$A + I = \begin{pmatrix} -8 & 4 & 4 \\ -8 & 4 & 4 \\ -16 & 8 & 8 \end{pmatrix} \quad (6.1.2.11)$$

$$A - 3I = \begin{pmatrix} -12 & 4 & 4 \\ -8 & 0 & 4 \\ -16 & 8 & 4 \end{pmatrix} \quad (6.1.2.12)$$

Now we know that  $A - 3I$  is singular and  $\text{rank}(A - 3I) \geq 2$ . Therefore,  $\text{rank}(A - 3I) = 2$ . Hence from the theorem-1 (iii) it is evident that  $\text{rank}(A + I) = 1$ . Let  $X_1$  and  $X_3$  be the spaces of characteristic vectors associated with the characteristic values 1 and 3 respectively. We know from rank nullity theorem that  $\dim X_1 = 2$  and  $\dim X_3 = 1$ . Hence by Theorem-2 (i)  $T$  is diagonalizable.

|         |   |
|---------|---|
| Given   | <p><math>T</math> be the linear operator on <math>R^2</math> which is represented by matrix <math>A</math> in the standard ordered basis for <math>R^2</math></p> <p><math>U</math> be the linear operator on <math>C^2</math> which is represented by matrix <math>A</math> in the standard ordered basis</p> <p>In all cases, denoting <math>B_c</math> the basis for the subspace corresponding to characteristic value <math>c</math></p> |
| To find | <p>Characteristic value of each operator</p> <p>For each characteristic value <math>c</math> find a basis for the corresponding space of characteristic vectors</p>   |

TABLE 6.1.1.1

The null-space of  $\mathbf{T} + \mathbf{I}$  is spanned by the vectors,

$$\alpha_1 = \begin{pmatrix} 1 & 0 & 2 \end{pmatrix} \quad (6.1.2.13)$$

$$\alpha_2 = \begin{pmatrix} 1 & 2 & 0 \end{pmatrix} \quad (6.1.2.14)$$

As both  $\alpha_1$  and  $\alpha_2$  are independent, hence they form basis for  $X_1$ . The null-space of  $\mathbf{T} - 3\mathbf{I}$  is spanned by the vector,

$$\alpha_3 = \begin{pmatrix} 1 & 1 & 2 \end{pmatrix} \quad (6.1.2.15)$$

Here  $\alpha_3$  is a characteristic vector and a basis for  $X_3$ . Also the matrix  $P$  which enables us to change coordinates from the basis  $\beta$  to the standard basis is the matrix which has transposes of  $\alpha_1, \alpha_2$  and  $\alpha_3$  as it's column vectors:

$$\mathbf{P} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 2 & 0 & 2 \end{pmatrix} \quad (6.1.2.16)$$

6.1.3. Let

$$\mathbf{A} = \begin{pmatrix} 6 & -3 & -2 \\ 4 & -1 & -2 \\ 10 & -5 & -3 \end{pmatrix} \quad (6.1.3.1)$$

Is  $A$  similar over the field  $R$  to a diagonal matrix ?

Is  $A$  similar over the field  $C$  to a diagonal matrix?

**Solution:** See Tables 6.1.3.1 and 6.1.3.2

| Matrix   | Characteristic Polynomial   | Basis  |
|--|---|--|
| $\mathbf{A}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$  | $\det(xI - A) = \begin{vmatrix} (x-1) & 0 \\ 0 & x \end{vmatrix}$ <p>Characteristic Polynomial = <math>x(x-1)</math><br/> To find characteristic values of the operator<br/> <math>\det(xI - A) = 0</math>, which gives</p> $c_1 = 0 \text{ and } c_2 = 1$ <p>Both <math>c_1</math> and <math>c_2</math> are the characteristic values<br/> Assume <math>B_1</math> and <math>B_2</math> are Basis for <math>c_1</math> and <math>c_2</math></p>  | <p>Basis for characteristic value <math>c_1 = 0</math><br/> will be obtained by solving<br/> homogeneous equation, <math>(A - c_1 I)x = 0</math><br/> After solving, basis for characteristic value<br/> <math>c_1</math> is <math>B_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}</math><br/> Similarly, we can find out the Basis for<br/> <math>c_2 = 1</math> which is <math>B_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}</math></p>  |
| $\mathbf{A}_2 = \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix}$ | $\det(xI - A) = \begin{vmatrix} (x-2) & -3 \\ 1 & (x-1) \end{vmatrix}$ <p>Characteristic Polynomial = <math>x^2 - 3x + 5</math><br/> To find characteristic values of the operator<br/> <math>\det(xI - A) = 0</math>, which gives</p> $c_1 = \frac{3+i\sqrt{11}}{2} \text{ and } c_2 = \frac{3-i\sqrt{11}}{2}$ <p>Both <math>c_1</math> and <math>c_2</math> are the characteristic values<br/> Assume <math>B_1</math> and <math>B_2</math> are Basis for <math>c_1</math> and <math>c_2</math></p> | <p>Basis for characteristic value <math>c_1 = \frac{3+i\sqrt{11}}{2}</math><br/> will be obtained by solving<br/> homogeneous equation, <math>(A - c_1 I)x = 0</math><br/> After solving, basis for characteristic value<br/> <math>c_1</math> is <math>B_1 = \begin{pmatrix} \frac{1+i\sqrt{11}}{2} \\ -1 \end{pmatrix}</math><br/> Similarly, we can find out the Basis for<br/> <math>c_2</math> which is <math>B_2 = \begin{pmatrix} \frac{1-i\sqrt{11}}{2} \\ -1 \end{pmatrix}</math></p> |
| $\mathbf{A}_3 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$  | $\det(xI - A) = \begin{vmatrix} (x-1) & -1 \\ -1 & (x-1) \end{vmatrix}$ <p>Characteristic Polynomial = <math>x(x-2)</math><br/> To find characteristic values of the operator<br/> <math>\det(xI - A) = 0</math>, which gives</p> $c_1 = 0 \text{ and } c_2 = 2$ <p>Both <math>c_1</math> and <math>c_2</math> are the characteristic values<br/> Assume <math>B_1</math> and <math>B_2</math> are Basis for <math>c_1</math> and <math>c_2</math></p>  | <p>Basis for characteristic value <math>c_1 = 0</math><br/> will be obtained by solving<br/> homogeneous equation, <math>(A - c_1 I)x = 0</math><br/> After solving, basis for characteristic value<br/> <math>c_1</math> is <math>B_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}</math><br/> Similarly, we can find out the Basis for<br/> <math>c_2 = 2</math> which is <math>B_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}</math></p>   |

TABLE 6.1.1.2: Finding of Characteristic Polynomial, Characteristic value and corresponding Basis

|                             |   |
|-----------------------------|---|
| Theorem 2                   | <p>Let <math>T</math> be the linear operator on a finite dimensional space <math>V</math> and <math>c_1, \dots, c_k</math> be a distinct characteristic values of <math>T</math> and let <math>W_i</math> be the null space of <math>(T - c_i I)</math> then</p> <p>1) <math>T</math> is diagonalizable</p> <p>2) characteristic polynomial for <math>T</math> is <math>f = (x - c_1)^{d_1} \dots (x - c_k)^{d_k}</math> and</p> <p>3) <math>\dim W_i = d_i, i = 1, \dots, k</math></p> |
| Concept for diagonalization | <p>A linear operator <math>T</math> on a <math>n</math>-dimensional space <math>V</math> is diagonalizable, if and only if <math>T</math> has <math>n</math> distinct characteristic vectors or null spaces corresponding to the characteristic values</p>  |

TABLE 6.1.3.1: Illustration of theorem.

|  |  |
|--|--|
| Given  | <p>Let the given matrix be</p> $\mathbf{A} = \begin{pmatrix} 6 & -3 & -2 \\ 4 & -1 & -2 \\ 10 & -5 & -3 \end{pmatrix}$   |
| Finding Characteristics polynomial                                   | <p>Characteristics polynomial of the matrix <math>A</math> is <math>\det(xI - A)</math></p> $\det(xI - A) = \begin{vmatrix} (x-6) & 3 & 2 \\ -4 & (x+1) & 2 \\ -10 & 5 & x+3 \end{vmatrix}$ <p>Characteristic Polynomial = <math>(x-2)(x^2+1) = (x-2)(x-i)(x+i)</math></p>         |
| Checking whether $A$ similar over the field $R$ to a diagonal matrix | <p>As the characteristics polynomial is not product of linear factors over <math>R</math>. Therefore from Theorem 2, <math>A</math> is not diagonalizable over <math>R</math></p>  |
| Checking whether $A$ similar over the field $C$ to a diagonal matrix | <p>The Characteristic Polynomial can be written as a product of linear factors over <math>C</math> i.e</p> $\det(xI - A) = (x-2)(x-i)(x+i)$ <p>To find characteristic values of the operator <math>\det(xI - A) = 0</math> which gives <math>c_1 = 2, c_2 = i, c_3 = -i</math></p> |

|            |  |
|------------|--|
|            | <p>Thus over <math>C</math> matrix <math>A</math> has three distinct characteristic values.</p> <p>There will be atleast one characteristics vector i.e., one dimension with each characteristics value .</p> <p>From Theorem 2;<br/> <math>\sum_i W_i = n = 3</math> , which is equal to <math>\dim</math> of <math>A</math>.</p> <p>Thus , <math>A</math> is diagonalizable over <math>C</math>.</p> |
| Conclusion | <p>1) <math>A</math> is not diagonalizable over <math>R</math>.</p> <p>2) <math>A</math> is diagonalizable over <math>C</math>.</p>  |

TABLE 6.1.3.2: Finding of characterisctics values and diagonalization over different field

6.1.4. Let  $\mathbf{T}$  be the linear operator on  $\mathbf{R}^4$  which is represented in the standard basis by the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \end{pmatrix} \quad (6.1.4.1)$$

Under what conditions on  $a$ ,  $b$  and  $c$  in  $\mathbf{T}$  is diagonalizable?

**Solution:**

**Theorem 6.1.** A linear operator  $\mathbf{T}$  on a  $n$ -dimensional space  $\mathbf{V}$  is diagonalizable, if and only if  $\mathbf{T}$  has an  $n$  distinct characteristic vectors (or) null spaces corresponding to the characteristic values.

**Theorem 6.2.** Let  $\mathbf{T}$  be a linear operator on a finite-dimensional space  $\mathbf{V}$ . Let  $c_1, c_2, \dots, c_k$  be the distinct characteristic values of  $\mathbf{T}$  and let  $\mathbf{W}_i$  be the null space of  $(\mathbf{T} - c_i\mathbf{I})$ . The following are equivalent:

- a)  $\mathbf{T}$  is diagonalizable
- b)  $\dim \mathbf{W}_1 + \dots + \dim \mathbf{W}_k = \dim \mathbf{V}$

Let the given matrix be,

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \end{pmatrix} \quad (6.1.4.2)$$

As per theorem 6.1, we need to find the characteristic polynomial for the matrix  $\mathbf{A}$ . Characteristic equation is given by  $\det(x\mathbf{I} - \mathbf{A})$ .

$$\det(x\mathbf{I} - \mathbf{A}) = \begin{vmatrix} x-0 & 0 & 0 & 0 \\ -a & x-0 & 0 & 0 \\ 0 & -b & x-0 & 0 \\ 0 & 0 & -c & x-0 \end{vmatrix} \quad (6.1.4.3)$$

$$\det(x\mathbf{I} - \mathbf{A}) = x^4 \quad (6.1.4.4)$$

The characteristic equation will be,

$$\det(x\mathbf{I} - \mathbf{A}) = 0 \quad (6.1.4.5)$$

$$x^4 = 0 \quad (6.1.4.6)$$

From (6.1.4.6) we get the characteristic value as  $c_1 = 0$  with a multiplicity of 4.

The basis for the characteristic value  $c_1 = 0$  can be obtained by solving the equation

$$(\mathbf{A} - c_1\mathbf{I})\mathbf{x} = \mathbf{0} \quad (6.1.4.7)$$

i.e.

$$(\mathbf{A} - (0)\mathbf{I})\mathbf{x} = \mathbf{0} \quad (6.1.4.8)$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \mathbf{0} \quad (6.1.4.9)$$

Solving the above equation we get

$$ax = 0, by = 0, cz = 0 \quad (6.1.4.10)$$

We know that the null space of  $(\mathbf{A} - (0)\mathbf{I})$ , is spanned by the vector  $\mathbf{x}$ , where the basis for the space  $\mathbf{W}_1$  need to satisfy the condition of (6.1.4.10). If we assume that

$$a \neq 0, b \neq 0, c \neq 0 \quad (6.1.4.11)$$

This will correspond that the elements in the basis of the vector  $\mathbf{x}$  will be

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ t \end{pmatrix} \quad (6.1.4.12)$$

Which implies that the  $\dim \mathbf{W}_1 = 1$ . From theorem 6.2, for  $\mathbf{T}$  to be diagonalizable, the null space  $\mathbf{W}_1$  of  $\mathbf{A}$  must have the  $\dim \mathbf{W}_1 = 4$ , since  $\dim \mathbf{R}^4 = 4$ . So, there is a contradiction with (6.1.4.11).

$\therefore \mathbf{A}$  is diagonalizable only if

$$a = b = c = 0 \quad (6.1.4.13)$$

i.e.  $\mathbf{A}$  is a zero matrix.



6.1.5. Let  $\mathbf{T}$  be the linear operator on a  $n$ -dimensional vector space  $\mathbf{V}$  and suppose that  $\mathbf{T}$  has  $n$  distinct characteristic values. Prove that  $\mathbf{T}$  is diagonalizable.

**Solution:** See Table 6.1.5.2

|                |   |
|----------------|---|
| Diagonalizable | A linear operator $\mathbf{T}$ on a finite-dimensional vector space $\mathbf{V}$ is diagonalizable if and only if there exists an basis of $\mathbf{V}$ , consisting of eigen vectors of $\mathbf{T}$   |
| Theorem        | <p>If <math>\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k</math> are eigenvectors of a linear operator <math>\mathbf{T}</math> with distinct eigen values <math>\lambda_1, \lambda_2, \dots, \lambda_k</math>, then <math>\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k</math> are linearly independent.</p> <p>Let <math>S_k = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}</math>. Let <math>P(k) : S_k</math> is linearly independent. <math>S_1</math> is linearly independent. So, <math>P(1)</math> holds. Assume <math>P(k)</math> holds for <math>1 \leq k \leq n</math>. Therefore, <math>S_k</math> is linearly independent.</p> $\text{Let } \sum_{i=1}^{k+1} a_i \mathbf{v}_i = 0 \quad (6.1.5.1)$ <p>Applying <math>\mathbf{T}</math> on both sides, we get</p> $\Rightarrow \mathbf{T}(\sum_{i=1}^{k+1} a_i \mathbf{v}_i) = 0 \quad (6.1.5.2)$ $\Rightarrow \sum_{i=1}^{k+1} a_i \mathbf{T}(\mathbf{v}_i) = 0 \quad (6.1.5.3)$ $\Rightarrow \sum_{i=1}^{k+1} a_i \lambda_i \mathbf{v}_i = 0 \quad (6.1.5.4)$ $\Rightarrow \sum_{i=1}^k a_i \lambda_i \mathbf{v}_i + a_{k+1} \lambda_{k+1} \mathbf{v}_{k+1} = 0 \quad (6.1.5.5)$ <p>Multiplying (6.1.5.1) by <math>\lambda_{k+1}</math>, we get</p> $\lambda_{k+1} (\sum_{i=1}^{k+1} a_i \mathbf{v}_i) = 0 \quad (6.1.5.6)$ $\Rightarrow \sum_{i=1}^{k+1} a_i \lambda_{k+1} \mathbf{v}_i = 0 \quad (6.1.5.7)$ $\Rightarrow \sum_{i=1}^k a_i \lambda_{k+1} \mathbf{v}_i + a_{k+1} \lambda_{k+1} \mathbf{v}_{k+1} = 0 \quad (6.1.5.8)$ <p>Subtracting (6.1.5.5) and (6.1.5.8), we get</p> $\sum_{i=1}^k a_i (\lambda_i - \lambda_{k+1}) \mathbf{v}_i = 0 \quad (6.1.5.9)$ <p>As <math>\lambda_i</math> are distinct <math>\forall i \leq k, a_i = 0</math> (6.1.5.10)</p> <p>Substituting this in (6.1.5.1)</p> $\sum_{i=1}^{k+1} a_i \mathbf{v}_i = 0 \quad (6.1.5.11)$ $\Rightarrow a_{k+1} \mathbf{v}_{k+1} = 0 \quad (6.1.5.12)$ <p>As <math>\mathbf{v}_{k+1} \neq 0 \Rightarrow a_{k+1} = 0</math></p> |

Since  $\forall i \leq k+1, a_i = 0, S_{k+1}$  is linearly independent  
 By principle of mathematic induction,  $S_n$  is linearly independent.

TABLE 6.1.5.1: Definitions and theorem used

|                            |  |
|----------------------------|--|
| Given                      | <b>T</b> has an n distinct characteristic values and $\dim(\mathbf{V}) = n$  |
| <b>T</b> is diagonalizable | Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be distinct eigen values of <b>T</b> and let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ be the eigen vectors of <b>T</b> . From above results we can state that $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is linearly independent. And also given that $\dim(\mathbf{V}) = n$ . So, this set forms a basis of <b>V</b> . $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for <b>V</b> consisting of eigen vectors of <b>T</b> . So, <b>T</b> is diagonalizable. |

TABLE 6.1.5.2: Solution

6.1.6. Let  $\mathbf{A}$  and  $\mathbf{B}$  be  $n \times n$  matrices over the field  $F$ . Prove that if  $(\mathbf{I} - \mathbf{AB})$  is invertible

a)  $(\mathbf{I} - \mathbf{BA})$  is invertible and

b)  $(\mathbf{I} - \mathbf{BA})^{-1} = \mathbf{I} + \mathbf{B}(\mathbf{I} - \mathbf{AB})^{-1} \mathbf{A}$

**Solution:** See Table 6.1.6.1

6.1.7. Using the result of the previous exercise, prove that, if  $A$  and  $B$  are  $n \times n$  matrices over the field  $F$ , then  $AB$  and  $BA$  have precisely the same characteristic values. **Solution:** See 6.1.7.1

6.1.8. Suppose that  $\mathbf{A}$  is a  $2 \times 2$  matrix with real entries which is symmetric ( $\mathbf{A}^t = \mathbf{A}$ ). Prove that  $\mathbf{A}$  is similar over  $\mathbb{R}$  to a diagonal matrix.

**Solution:** See Table 6.1.8.1

|             |  |
|-------------|--|
| Invertible  | A matrix <b>M</b> is invertible if it is non-singular i.e. the null space of <b>M</b> contains only zero vector.If <b>x</b> is a vector such that $\mathbf{Mx} = 0 \implies \mathbf{x} = 0$  |
| Proof for 1 | <p>Consider a vecor <b>y</b> such that <math>(\mathbf{I} - \mathbf{BA})\mathbf{y} = 0</math></p> $(\mathbf{I} - \mathbf{BA})\mathbf{y} = 0 \implies \mathbf{y} = \mathbf{BAy}$ $\mathbf{Ay} = \mathbf{ABAy} \implies (\mathbf{I} - \mathbf{AB})\mathbf{Ay} = 0$ <p>since the matrix <math>(\mathbf{I} - \mathbf{AB})</math> is invertible,<math>\mathbf{Ay} = 0</math></p> $\mathbf{y} = \mathbf{B}(\mathbf{Ay}) \implies \mathbf{y} = 0$ <p>Hence the matrix <math>(\mathbf{I} - \mathbf{BA})</math> is invertible.</p> |
| Observation | <p>Let <math>\mathbf{C} = (\mathbf{I} - \mathbf{AB})^{-1}</math>,then</p> $(\mathbf{I} - \mathbf{BA})(\mathbf{BCA}) = \mathbf{BCA} - \mathbf{BABCA} = \mathbf{B}(\mathbf{I} - \mathbf{AB})\mathbf{CA} = \mathbf{BC}^{-1}\mathbf{CA} = \mathbf{BA}$ $(\mathbf{I} - \mathbf{BA})(\mathbf{BCA}) = \mathbf{BA}$  |
| Proof for 2 | <p>Let us consider the product <math>(\mathbf{I} - \mathbf{BA})(\mathbf{BCA} + \mathbf{I})</math></p> $(\mathbf{I} - \mathbf{BA})(\mathbf{BCA} + \mathbf{I}) = (\mathbf{I} - \mathbf{BA})(\mathbf{BCA}) + (\mathbf{I} - \mathbf{BA})$ $= \mathbf{BA} + \mathbf{I} - \mathbf{BA} = \mathbf{I}$ $(\mathbf{I} - \mathbf{BA})^{-1} = \mathbf{I} + \mathbf{B}(\mathbf{I} - \mathbf{AB})^{-1}\mathbf{A}$ <p>Hence proved.</p>  |

TABLE 6.1.6.1: Proof

|             |  |
|-------------|--|
| Given       | <p><math>A</math> and <math>B</math> are <math>n \times n</math> matrices over the field <math>F</math>.</p> <p>In Exercise 8, If <math>(I - AB)</math> is invertible<br/>then <math>(I - BA)</math> is invertible.</p>  |
| To prove    | $AB$ and $BA$ have precisely the same characteristic values.   |
| Observation | <p>We have to show that if <math>c</math> is a characteristic value for <math>AB</math><br/>then <math>c</math> is a characteristic value for <math>BA</math>.<br/>Conversely, This is equivalent to the statement<br/>if <math>c</math> is not a characteristic value for <math>AB</math><br/>then it is not a characteristic value for <math>BA</math>.</p>  |
| Proof       | <p>Suppose that <math>c</math> is not a characteristic value for <math>BA</math>,<br/>this means that <math> cI - AB  \neq 0</math>.</p> $\implies c^n  I - \frac{1}{c}AB  \neq 0$ <p><math>\therefore (I - \frac{1}{c}AB)</math> is invertible <math>\implies (I - \frac{1}{c}BA)</math> is invertible.<br/><math>\implies  I - \frac{1}{c}AB  \neq 0</math></p> $\implies c^n  I - \frac{1}{c}AB  = c^n  I - \frac{1}{c}AB  \neq 0$ <p>Hence, If <math>c</math> is not a characteristic value for <math>AB</math><br/>then it is not a characteristic value for <math>BA</math>.</p> <p>Hence, <math>AB</math> and <math>BA</math> have precisely the same characteristic value.</p> |

TABLE 6.1.7.1

|          |   |
|----------|---|
| Given    | $\mathbf{A}$ is a $2 \times 2$ matrix with real entries and $\mathbf{A}$ is symmetric ( $\mathbf{A}^t = \mathbf{A}$ )   |
| To Prove | $\mathbf{A}$ is similar to diagonal matrix over $\mathbb{R}$  |
| Theory   | $\mathbf{A}$ is similar to diagonal matrix $\mathbf{\Lambda}$ if $\exists$ an invertible matrix $\mathbf{P}$ such that:<br>$\mathbf{A} = \mathbf{P}\mathbf{\Lambda}\mathbf{P}^{-1}$   |
| Proof    | <p>Let <math>\mathbf{A} = \begin{pmatrix} a &amp; c \\ c &amp; b \end{pmatrix}, a, b, c \in \mathbb{R}</math></p> <p>Characteristic polynomial:<br/> <math>p(t) =  \mathbf{A} - \lambda\mathbf{I} </math><br/> <math>p(t) = \begin{vmatrix} a-t &amp; c \\ c &amp; b-t \end{vmatrix}</math></p> <p><math>\implies p(t) = t^2 - (a+b)t + ab - c^2 = 0</math><br/> Roths of <math>p(t)</math> are eigenvalues of <math>\mathbf{A}</math></p> <p>Discriminant of <math>p(t)</math> is given by<br/> <math>(a+b)^2 - 4(ab - c^2) = a^2 + b^2 - 2ab + c^2</math><br/> <math>= (a-b)^2 + 4c^2 &gt; 0</math></p> <p>We observe that the above equation has positive discriminant, hence <math>\lambda</math> has real values</p>   |
|          | <p>Eigen vectors are obtained by:<br/> <math>(\mathbf{A} - \lambda\mathbf{I})\mathbf{X} = \mathbf{0}</math></p> <p>Let <math>\mathbf{v}_1</math> and <math>\mathbf{v}_2</math> be the eigen vectors corresponding to eigen values <math>\lambda_1</math> and <math>\lambda_2</math><br/> <math>\implies \mathbf{A}\mathbf{v}_1 = \lambda_1\mathbf{v}_1</math> and<br/> <math>\mathbf{A}\mathbf{v}_2 = \lambda_2\mathbf{v}_2</math></p> <p>Let linear combination of the two eigen vectors be,<br/> <math>c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}</math></p> <p>Multiplying both sides by <math>\lambda_1</math>, we have,<br/> <math>\implies c_1\lambda_1\mathbf{v}_1 + c_2\lambda_1\mathbf{v}_2 = \mathbf{0} \quad \dots(1)</math></p> <p>Consider,<br/> <math>\mathbf{A}\mathbf{0} = \mathbf{0}</math><br/> <math>\implies \mathbf{A}(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = \mathbf{0}</math><br/> <math>\implies c_1(\mathbf{A}\mathbf{v}_1) + c_2(\mathbf{A}\mathbf{v}_2) = \mathbf{0}</math><br/> <math>\implies c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 = \mathbf{0} \quad \dots(2)</math></p> |

Subtracting equation (1) and (2), we have,

$$c_2(\lambda_1 - \lambda_2)\mathbf{v}_2 = \mathbf{0}$$

Since,  $\lambda_1$  and  $\lambda_2$  are real and distinct,

$$c_2 = 0$$

Similarly,

$$c_1 = 0$$

Therefore, eigen vectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are linearly independent.

$$\text{Let } \mathbf{P} = (\mathbf{v}_1 \ \mathbf{v}_2)$$

$$\Rightarrow \mathbf{AP} = (\lambda_1\mathbf{v}_1 \ \lambda_2\mathbf{v}_2)$$

$$\Rightarrow \mathbf{AP} = \mathbf{P}\mathbf{\Lambda},$$

$$\text{where } \mathbf{\Lambda} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$\Rightarrow \mathbf{A} = \mathbf{P}\mathbf{\Lambda}\mathbf{P}^{-1}$$

Therefore,  $\mathbf{A}$  is similar to diagonal matrix  $\mathbf{\Lambda}$

Hence, Proved.

TABLE 6.1.8.1: Proving that eigen vectors are linearly independent for real eigen values and symmetric matrix is similar to diagonal matrix



6.1.9. Let  $\mathbf{N}$  be a  $2 \times 2$  complex matrix such that  $\mathbf{N}^2 = 0$ . Prove that either  $\mathbf{N} = 0$  or  $\mathbf{N}$  is similar over  $\mathbb{C}$  to

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad (6.1.9.1)$$

**Solution:** See Table 6.1.9.1

| Statement  | Solution   |
|--|--|
| <p>Proof that <math>\mathbf{N} = 0</math></p>  | <p>Let <math>\mathbf{N} = \begin{pmatrix} a &amp; b \\ c &amp; d \end{pmatrix}</math> (6.1.9.2)</p> <p>Since <math>\mathbf{N}^2 = 0</math> (6.1.9.3)</p> <p>If <math>\begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix}</math> are linearly independent then <math>\mathbf{N}</math> is diagonalizable to <math>\begin{pmatrix} 0 &amp; 0 \\ 0 &amp; 0 \end{pmatrix}</math>.</p> <p>If <math>\mathbf{P}\mathbf{N}\mathbf{P}^{-1} = 0</math> (6.1.9.4)</p> <p>then <math>\mathbf{N} = \mathbf{P}^{-1}\mathbf{0}\mathbf{P} = 0</math> (6.1.9.5)</p> <p>So in this case <math>\mathbf{N}</math> itself is the zero matrix.</p> <p>This contradicts the assumption that <math>\begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix}</math> are linearly independent.</p> <p><math>\therefore</math> we can assume that <math>\begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix}</math> are linearly dependent if both are equal to the zero vector</p> <p>then <math>\mathbf{N} = 0</math>. (6.1.9.6)</p> |
| <p>Assuming <math>\begin{pmatrix} b \\ d \end{pmatrix}</math> as the zero vector</p> | <p>Therefore we can assume at least one vector is non-zero.</p> <p>Therefore <math>\mathbf{N} = \begin{pmatrix} a &amp; 0 \\ c &amp; 0 \end{pmatrix}</math></p> <p>So <math>\mathbf{N}^2 = 0</math> (6.1.9.7)</p> <p><math>\implies a^2 = 0</math> (6.1.9.8)</p> <p><math>\therefore a = 0</math> (6.1.9.9)</p> <p>Thus <math>\mathbf{N} = \begin{pmatrix} a &amp; 0 \\ c &amp; 0 \end{pmatrix}</math> (6.1.9.10)</p> <p>In this case <math>\mathbf{N}</math> is similar to <math>\mathbf{N} = \begin{pmatrix} 0 &amp; 0 \\ 1 &amp; 0 \end{pmatrix}</math> via the matrix <math>\mathbf{P} = \begin{pmatrix} c &amp; 0 \\ 0 &amp; 1 \end{pmatrix}</math></p>   |
| <p>Assuming <math>\begin{pmatrix} a \\ c \end{pmatrix}</math> as the zero vector</p> | <p>Therefore <math>\mathbf{N} = \begin{pmatrix} 0 &amp; b \\ 0 &amp; d \end{pmatrix}</math></p> <p>Then <math>\mathbf{N}^2 = 0</math> (6.1.9.11)</p> <p><math>\implies d^2 = 0</math> (6.1.9.12)</p> <p><math>\therefore d = 0</math> (6.1.9.13)</p> <p>Thus <math>\mathbf{N} = \begin{pmatrix} 0 &amp; b \\ 0 &amp; 0 \end{pmatrix}</math> (6.1.9.14)</p>   |

|  |  |
|--|--|
|  | <p>In this case <math>\mathbf{N}</math> is similar to <math>\mathbf{N} = \begin{pmatrix} 0 &amp; 0 \\ b &amp; 0 \end{pmatrix}</math> via the matrix <math>\mathbf{P} = \begin{pmatrix} 0 &amp; 1 \\ 1 &amp; 0 \end{pmatrix}</math>, which is similar to <math>\begin{pmatrix} 0 &amp; 0 \\ 1 &amp; 0 \end{pmatrix}</math> as above.</p>  |
| Hence  | <p>we can assume neither <math>\begin{pmatrix} a \\ c \end{pmatrix}</math> or <math>\begin{pmatrix} b \\ d \end{pmatrix}</math> is the zero vector.</p>  |
| Consequences of linear independence  | <p>Since they are linearly dependent we can assume,</p> $\begin{pmatrix} b \\ d \end{pmatrix} = x \begin{pmatrix} a \\ c \end{pmatrix} \quad (6.1.9.15)$ $\therefore \mathbf{N} = \begin{pmatrix} a & ax \\ c & cx \end{pmatrix} \quad (6.1.9.16)$ $\therefore \mathbf{N}^2 = 0 \quad (6.1.9.17)$ $\implies a(a + cx) = 0 \quad (6.1.9.18)$ $c(a + cx) = 0 \quad (6.1.9.19)$ $ax(a + cx) = 0 \quad (6.1.9.20)$ $cx(a + cx) = 0 \quad (6.1.9.21)$   |
| Proof that $\mathbf{N}$ is similar over $\mathbb{C}$ to $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ | <p>We know that at least one of <math>a</math> or <math>c</math> is not zero.<br/>         If <math>a = 0</math> then <math>c \neq 0</math>, it must be that <math>x = 0</math>.<br/>         So in this case <math>\mathbf{N} = \begin{pmatrix} 0 &amp; 0 \\ c &amp; 0 \end{pmatrix}</math> which is similar to <math>\begin{pmatrix} 0 &amp; 0 \\ 1 &amp; 0 \end{pmatrix}</math> as before.</p> <p>If <math>a \neq 0 \quad (6.1.9.22)</math><br/>         then <math>x \neq 0 \quad (6.1.9.23)</math><br/>         else <math>a(a + cx) = 0 \quad (6.1.9.24)</math><br/> <math>\implies a = 0 \quad (6.1.9.25)</math><br/>         Thus <math>a + cx = 0 \quad (6.1.9.26)</math><br/>         Hence <math>\mathbf{N} = \begin{pmatrix} a &amp; ax \\ \frac{-a}{x} &amp; -a \end{pmatrix} \quad (6.1.9.27)</math></p> <p>This is similar to <math>\begin{pmatrix} a &amp; a \\ -a &amp; -a \end{pmatrix}</math> via <math>\mathbf{P} = \begin{pmatrix} \sqrt{x} &amp; 0 \\ 0 &amp; \frac{1}{\sqrt{x}} \end{pmatrix}</math>.<br/>         And <math>\begin{pmatrix} a &amp; a \\ -a &amp; -a \end{pmatrix}</math> is similar to <math>\begin{pmatrix} 0 &amp; 0 \\ -a &amp; 0 \end{pmatrix}</math> via <math>\mathbf{P} = \begin{pmatrix} -1 &amp; -1 \\ 1 &amp; 0 \end{pmatrix}</math><br/>         And this finally is similar to <math>\begin{pmatrix} 0 &amp; 0 \\ 1 &amp; 0 \end{pmatrix}</math> as before.</p> |

|            |   |
|------------|---|
| Conclusion | Thus either $\mathbf{N} = 0$ or $\mathbf{N}$ is similar over $\mathbb{C}$ to $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ . |
|------------|---|

TABLE 6.1.9.1: Solution summary

6.1.10. Let  $A$  be the an  $n \times n$  diagonal matrix with characteristic polynomial

$$(x - c_1)^{d_1}(x - c_2)^{d_2} \dots (x - c_k)^{d_k} \quad (6.1.10.1)$$

Where  $c_1, c_2, \dots, c_k$  are distinct. Let  $\mathbf{V}$  the space of  $n \times n$  matrices  $B$  such that

$$AB = BA \quad (6.1.10.2)$$

Prove that the dimension of  $\mathbf{V}$  is,

$$d_1^2 + d_2^2 \dots + d_k^2 \quad (6.1.10.3)$$

**Solution:** Let consider we have a matrix  $A$  which is a diagonal matrix, which is given as

$$A = \begin{pmatrix} c_1 I & 0 & 0 & \dots & 0 & 0 \\ 0 & c_2 I & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & . & . \\ . & . & . & \dots & . & . \\ 0 & 0 & . & \dots & . & c_k I \end{pmatrix} \quad (6.1.10.4)$$

Consider  $B$  as :

$$B = \begin{pmatrix} B_{11} & B_{12} & . & \dots & . & B_{1k} \\ B_{21} & B_{22} & . & \dots & . & B_{2k} \\ . & . & . & \dots & . & . \\ . & . & . & \dots & . & . \\ B_{k1} & B_{k2} & . & \dots & . & B_{kk} \end{pmatrix} \quad (6.1.10.5)$$

Where  $B_{ij}$  has dimension  $d_i \times d_j$ . Since we have given ,

$$AB = BA \quad (6.1.10.6)$$

$$\Rightarrow \begin{pmatrix} c_1 B_{11} & c_1 B_{12} & . & \dots & . & c_1 B_{1k} \\ c_2 B_{21} & c_2 B_{22} & . & \dots & . & c_2 B_{2k} \\ . & . & . & \dots & . & . \\ . & . & . & \dots & . & . \\ c_k B_{k1} & c_k B_{k2} & . & \dots & . & c_k B_{kk} \end{pmatrix} =$$

$$\begin{pmatrix} c_1 B_{11} & B_{12} & . & \dots & . & c_1 B_{1k} \\ c_2 B_{21} & B_{22} & . & \dots & . & c_2 B_{2k} \\ . & . & . & \dots & . & . \\ . & . & . & \dots & . & . \\ c_k B_{k1} & c_k B_{k2} & . & \dots & . & c_k B_{kk} \end{pmatrix} \quad (6.1.10.7)$$

Hence, from above equation (6.1.10.7) we can conclude,

$$c_i \neq c_j, \forall i \neq j \quad (6.1.10.8)$$

$$\Rightarrow B_{ij} = 0, \forall i \neq j \quad (6.1.10.9)$$

We can have  $B_{11}, B_{22} \dots$  any arbitrary matrices. From (6.1.10.7) we can have

$$D(B_{ij}) = d_i^2 \quad (6.1.10.10)$$

Where  $D$  represents dimension of matrix. Therefore the dimension of the space of all such  $B_{ij}$ 's matrices is given as :

$$d_1^2 + d_2^2 \dots + d_k^2 \quad (6.1.10.11)$$

Let suppose we have matrix  $A$  as :

$$A = \begin{pmatrix} c_1 I & \mathbf{0} \\ \mathbf{0} & c_2 I \end{pmatrix} \quad (6.1.10.12)$$

Where,

$$c_1 I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, c_2 I = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \quad (6.1.10.13)$$

$$\Rightarrow A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \quad (6.1.10.14)$$

and  $B$  as :

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \quad (6.1.10.15)$$

Where,

$$B_{11} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, B_{12} = \begin{pmatrix} b_{13} & b_{14} \\ b_{23} & b_{24} \end{pmatrix} \quad (6.1.10.16)$$

$$(6.1.10.17)$$

$$B_{21} = \begin{pmatrix} b_{31} & b_{32} \\ b_{41} & b_{42} \end{pmatrix}, B_{22} = \begin{pmatrix} b_{33} & b_{34} \\ b_{43} & b_{44} \end{pmatrix} \quad (6.1.10.18)$$

$$(6.1.10.19)$$

$$\Rightarrow B = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{pmatrix} \quad (6.1.10.20)$$

Consider,

$$C = AB \quad (6.1.10.21)$$

$$\Rightarrow C = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ 2b_{31} & 2b_{32} & 2b_{33} & 2b_{34} \\ 2b_{41} & 2b_{42} & 2b_{43} & 2b_{44} \end{pmatrix} \quad (6.1.10.22)$$

Let another matrix D as :

$$D = BA \quad (6.1.10.23)$$

$$B = \begin{pmatrix} b_{11} & b_{12} & 2b_{13} & 2b_{14} \\ b_{21} & b_{22} & 2b_{23} & 2b_{24} \\ b_{31} & b_{32} & 2b_{33} & 2b_{34} \\ b_{41} & b_{42} & 2b_{43} & 2b_{44} \end{pmatrix} \quad (6.1.10.24)$$

We have given as,

$$BA = AB \quad (6.1.10.25)$$

$$\Rightarrow C = D \quad (6.1.10.26)$$

it is possible only when,

$$b_{13} = b_{14} = b_{23} = b_{24} = 0 \quad (6.1.10.27)$$

And,

$$b_{31} = b_{32} = b_{41} = b_{42} = 0 \quad (6.1.10.28)$$

$$B_{12} = \begin{pmatrix} b_{13} & b_{14} \\ b_{23} & b_{24} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad (6.1.10.29)$$

And,

$$B_{21} = \begin{pmatrix} b_{31} & b_{32} \\ b_{41} & b_{42} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad (6.1.10.30)$$

Hence, therefore matrix B becomes,

$$B = \begin{pmatrix} b_{11} & b_{12} & 0 & 0 \\ b_{21} & b_{22} & 0 & 0 \\ 0 & 0 & 2b_{33} & 2b_{34} \\ 0 & 0 & 2b_{43} & 2b_{44} \end{pmatrix} \quad (6.1.10.31)$$

$$\Rightarrow B = \begin{pmatrix} c_1 B_{11} & \mathbf{0} \\ \mathbf{0} & c_2 B_{22} \end{pmatrix} \quad (6.1.10.32)$$

$$\Rightarrow B_{ij} = 0, \forall i \neq j \quad (6.1.10.33)$$

Now the basis of the  $n \times n$  matrices for vector

space of all  $n \times n$  matrix B are,

$$\beta_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \beta_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (6.1.10.34)$$

$$\beta_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \beta_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (6.1.10.35)$$

$$\beta_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \beta_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (6.1.10.36)$$

$$\beta_7 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \beta_8 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (6.1.10.37)$$

Thus, Dimension of  $\mathbf{V}$  (vector space of all  $n \times n$  matrices  $B$ ) = 8,

Also

$$d_1^2 + d_2^2 = 2^2 + 2^2 = 8. \quad (6.1.10.38)$$

Therefore, Dimension of  $\mathbf{V}$  (vector space of all  $n \times n$  matrix  $B$  is :

$$d_1^2 + d_2^2 \quad (6.1.10.39)$$

6.1.11. Let  $\mathbf{V}$  be the space of  $n \times n$  matrices over  $F$ . Let  $\mathbf{A}$  be a fixed  $n \times n$  matrix over  $F$ . Let  $\mathbf{T}$  be the linear operator 'left multiplication by  $\mathbf{A}$ ' on  $\mathbf{V}$ . Is it true that  $\mathbf{A}$  and  $\mathbf{T}$  have the same characteristic values?

**Solution:** See Table 6.1.11.1

|                   |  |
|-------------------|--|
| <b>Given</b>      | <p><math>V</math> is the space of <math>n \times n</math> matrices over <math>F</math></p> <p><math>A</math> is a fixed <math>n \times n</math> matrix over <math>F</math></p> <p><math>T</math> be the linear operator on <math>V</math> such that<br/> <math>T(B) = AB</math></p>  |
| <b>To prove</b>   | $A$ and $T$ have the same characteristic values  |
| <b>Theorem</b>    | <p>Let <math>\lambda</math> be a characteristic value of <math>T</math> and <math>\lambda \in F</math> and <math>v</math> is the corresponding characteristic vector which is a <math>n \times n</math> matrix, then if <math>T</math> is a linear operator on a finite dimensional space <math>V</math>, it must be <math>\det(T - \lambda I) = 0</math> and for <math>(T - \lambda I)v = 0</math>, <math>v \neq 0</math></p> |
| <b>Proof</b>      | <p>As per the problem statement, <math>Tv = \lambda v</math><br/> <b>or</b>, <math>Av = \lambda v</math>, as <math>Tv = Av</math><br/> <b>or</b>, <math>(A - \lambda I)v = 0</math><br/> From here 2 cases can be arrived.</p>   |
| Case 1:           | $\det((A - \lambda I)) = 0$ , where $v \neq 0$<br>$\Rightarrow \lambda$ is characteristic value of $A$ .   |
| Case 2:           | $\det((A - \lambda I)) \neq 0$<br>$\Rightarrow (A - \lambda I)$ is invertible and $v = 0$<br>so, for $(T - \lambda I)v = 0$ and<br>$(T - \lambda I) \neq 0$<br>$\Rightarrow v$ is not a characteristic vector of $T$<br>which is a contradiction. So, case 2 is not possible.  |
| <b>Conclusion</b> | <p>So, from the above 2 cases and from the theorem, it can be concluded that <math>A</math> and <math>T</math> have the same characteristic values</p>   |

TABLE 6.1.11.1: Solution summary

## 6.2 Annihilating Polynomials

6.2.1. Let  $a, b$  and  $c$  be the elements of a field  $\mathbf{F}$ , and let  $\mathbf{A}$  be the following  $3 \times 3$  matrix over  $\mathbf{F}$

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & c \\ 1 & 0 & b \\ 0 & 1 & a \end{pmatrix} \quad (6.2.1.1)$$

Prove that the characteristic polynomial for  $\mathbf{A}$  is  $x^3 - ax^2 - bx - c$  and that this is also minimal polynomial for  $\mathbf{A}$ . **Solution:** Minimal polynomial of  $\mathbf{A}$  is a polynomial which satisfies,

- 1)  $P(\mathbf{A}) = 0$
- 2)  $P(x)$  is monic.
- 3) If there is some other annihilating polynomial  $q(x)$  such that,  $q(\mathbf{A}) = 0$ , then  $q$  does not divide  $p$ .

The characteristic polynomial is calculated by solving  $|\mathbf{A} - \lambda \mathbf{I}| = 0$

$$\Rightarrow |\mathbf{A} - \lambda \mathbf{I}| = \begin{vmatrix} -\lambda & 0 & c \\ 1 & -\lambda & b \\ 0 & 1 & a - \lambda \end{vmatrix} \quad (6.2.1.2)$$

$$\xleftrightarrow{R_2 \leftarrow R_2 + \lambda R_3} \begin{vmatrix} -\lambda & 0 & c \\ 1 & 0 & b + a\lambda - \lambda^2 \\ 0 & 1 & a - \lambda \end{vmatrix} \quad (6.2.1.3)$$

$$\Rightarrow |\mathbf{A} - \lambda \mathbf{I}| = 1 \begin{vmatrix} -\lambda & c \\ 1 & b + a\lambda - \lambda^2 \end{vmatrix} \quad (6.2.1.4)$$

$$\Rightarrow |\mathbf{A} - \lambda \mathbf{I}| = (-\lambda)(b + a\lambda - \lambda^2) - c \quad (6.2.1.5)$$

Hence the characteristic polynomial of  $\mathbf{A}$  is,

$$\lambda^3 - a\lambda^2 - b\lambda - c \quad (6.2.1.6)$$

Now for any  $r, s \in \mathbf{F}$  and considering the annihilating polynomial  $f$  with degree 2.

$$f(\mathbf{A}) = \mathbf{A}^2 + r\mathbf{A} + s \quad (6.2.1.7)$$

$$\Rightarrow \begin{pmatrix} 0 & c & ac \\ 0 & b & c + ba \\ 1 & a & b + a^2 \end{pmatrix} + \begin{pmatrix} 0 & 0 & rc \\ r & 0 & rb \\ 0 & r & ra \end{pmatrix} + \begin{pmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{pmatrix} \quad (6.2.1.8)$$

$$\therefore f(\mathbf{A}) = \mathbf{A}^2 + r\mathbf{A} + s = \begin{pmatrix} s & c & ac + rc \\ r & b + s & c + ba + br \\ 1 & a + r & b + a^2 + ra + s \end{pmatrix} \neq 0 \quad (6.2.1.9)$$

Element positioned at row-3 and column-1 is non-zero, hence for any  $r, s \in \mathbf{F} \Rightarrow f(\mathbf{A}) \neq 0 \forall f \in \mathbf{F}$ . Hence minimal polynomial cannot have degree 2. Hence degree of minimal polynomial is 3. Also  $x^3 - ax^2 - bx - c$  divides  $f$ . Hence from definition-1 we can conclude that,

$$p(x) = x^3 - ax^2 - bx - c \quad (6.2.1.10)$$

is a minimal polynomial.

*Example:* Let  $a = 0, b = 0, c = 0 \in \mathbf{F}$ . Hence,

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (6.2.1.11)$$

Now finding characteristic polynomial by substituting the values of  $a, b$ , and  $c$  in equation (6.2.1.6) we get,

$$\lambda^3 = 0 \quad (6.2.1.12)$$

Now let  $r=0, s=0 \in \mathbf{F}$ , Hence  $f(\mathbf{A})$  is given by using the equation (6.2.1.9),

$$\Rightarrow f(\mathbf{A}) = \mathbf{A}^2 + 0\mathbf{A} + 0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \neq 0 \quad (6.2.1.13)$$

Hence,  $f(\mathbf{A}) \neq 0$ , Hence degree of minimal polynomial is 3 and is equal to,

$$p(x) = x^3 \quad (6.2.1.14)$$

Verification by calculating  $p(\mathbf{A})$ ,

$$p(\mathbf{A}) = \mathbf{A}^3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (6.2.1.15)$$

$$\Rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (6.2.1.16)$$

$$\Rightarrow f(\mathbf{A}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0 \quad (6.2.1.17)$$

Hence, from definition-1 it can be concluded that  $p(x) = x^3$  is a minimal polynomial. For



|                                  |   |
|----------------------------------|---|
| Characteristic Polynomial        | For an $n \times n$ matrix $\mathbf{A}$ , characteristic polynomial is defined by,<br>$p(x) =  x\mathbf{I} - \mathbf{A} $       |
| Cayley-Hamilton Theorem          | If $p(x)$ is the characteristic polynomial of an $n \times n$ matrix $\mathbf{A}$ , then,<br>$p(\mathbf{A}) = \mathbf{0}$       |
| Minimal Polynomial               | Minimal polynomial $m(x)$ is the smallest factor of characteristic polynomial $p(x)$ such that,<br>$m(\mathbf{A}) = \mathbf{0}$ |
| Algebraic Multiplicity ( $A_M$ ) | No. of times an eigen value appears in a characteristic equation.   |

TABLE 6.2.2.1: Definitions

the  $\mathbf{A}$  given by equation (6.2.1.1), characteristic and minimal polynomial is given by,

$$x^3 - ax^2 - bx - c \quad (6.2.1.18)$$

6.2.2. Let  $\mathbf{A}$  be the  $4 \times 4$  real matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -2 & -2 & 2 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix} \quad (6.2.2.1)$$

Show that the characteristic polynomial for  $\mathbf{A}$  is  $x^2(x-1)^2$  and that it is also the minimal polynomial

**Solution:** See Tables 6.2.2.1 and 6.2.2.2

|                           |   |
|---------------------------|---|
| Characteristic polynomial | $  \begin{aligned}  p(x) &=  x\mathbf{I} - \mathbf{A}  \\  &= \begin{vmatrix} x-1 & -1 \\ 1 & x+1 \end{vmatrix} \begin{vmatrix} x-2 & -1 \\ 1 & x \end{vmatrix} \\  &= ((x-1)(x+1)+1)((x-2)x+1) \\  &= x^2(x^2-2x+1) \\  &= x^2(x-1)^2  \end{aligned}  $  |
| $A_M$                     | <p>For <math>\lambda = 0</math>, <math>A_M = 2</math><br/> For <math>\lambda = 1</math>, <math>A_M = 2</math></p>   |
| Minimal Polynomial        | $p(x) = x^a(x-1)^b, \quad a \leq 2, b \leq 2$   |
| $a = 1, b = 1$            | $m(x) = x(x-1)$ $\Rightarrow m(\mathbf{A}) = \mathbf{A}(\mathbf{A} - \mathbf{I})$ $  = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -2 & -2 & 2 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 \\ -2 & -2 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix}  $ $  = \begin{pmatrix} -1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & -2 & -2 \end{pmatrix} \neq \mathbf{0}  $ $\Rightarrow x(x-1) \text{ is not a minimal polynomial}$    |
| $a = 2, b = 1$            | $m(x) = x^2(x-1)$ $\Rightarrow m(\mathbf{A}) = \mathbf{A}^2(\mathbf{A} - \mathbf{I})$ $  = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -3 & -3 & 3 & 2 \\ 2 & 2 & -2 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 \\ -2 & -2 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix}  $ $  = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix} \neq \mathbf{0}  $ $\Rightarrow x^2(x-1) \text{ is not a minimal polynomial}$ |
| $a = 1, b = 2$            |   |

|                |   |
|----------------|---|
|                | $m(x) = x(x-1)^2$ $\Rightarrow m(\mathbf{A}) = \mathbf{A}(\mathbf{A} - \mathbf{I})^2$ $= \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -2 & -2 & 2 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -2 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $= \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq \mathbf{0}$ $\Rightarrow x^2(x-1) \text{ is not a minimal polynomial}$ |
| $a = 2, b = 2$ | $m(x) = x^2(x-1)^2$ $\Rightarrow m(\mathbf{A}) = \mathbf{A}^2(\mathbf{A} - \mathbf{I})^2$ $= p(\mathbf{A})$ $= \mathbf{0} \text{ (Cayley-Hamilton Theorem)}$ $\Rightarrow x^2(x-1)^2 \text{ is a minimal polynomial}$   |
| Conclusion     | <p>For the given matrix <math>\mathbf{A}</math>,<br/> <math>x^2(x-1)^2</math> is the characteristic polynomial as well as minimal polynomial.</p>   |

TABLE 6.2.2.2: Checking for minimal polynomial

|                           |   |
|---------------------------|---|
| Characteristic Polynomial | For an $n \times n$ matrix $\mathbf{A}$ , characteristic polynomial is defined by,<br>$p(x) =  x\mathbf{I} - \mathbf{A} $   |
| Cayley-Hamilton Theorem   | If $p(x)$ is the characteristic polynomial of an $n \times n$ matrix $\mathbf{A}$ , then,<br>$p(\mathbf{A}) = \mathbf{0}$   |
| Minimal Polynomial        | Minimal polynomial $m(x)$ is the smallest factor of characteristic polynomial $p(x)$ such that,<br>$m(\mathbf{A}) = \mathbf{0}$<br>Every root of characteristic polynomial should be the root of minimal polynomial |

TABLE 6.2.3.1: Definitions

6.2.3. Find a  $3 \times 3$  matrix for which the minimal polynomial is  $x^2$ .

**Solution:** See Tables 6.2.3.1 and 6.2.3.2

| Statement  | Solution   |
|--|--|
| Assuming matrix $\mathbf{A}$ as follows:   | Let us Consider $3 \times 3$ upper triangular matrix,<br>$\mathbf{A} = \begin{pmatrix} e & a & b \\ 0 & f & c \\ 0 & 0 & d \end{pmatrix}$  |
| Characteristic polynomial of $\mathbf{A}$  | $ x\mathbf{I} - \mathbf{A}  = \begin{vmatrix} x-e & -a & -b \\ 0 & x-f & -c \\ 0 & 0 & x-d \end{vmatrix}$ $= (x-e)(x-f)(x-d)$  |
| Given  | <p>The minimum polynomial is</p> $p(x) = x^2$ <p>Therefore <math>p(x)</math> must divide characteristic polynomial.<br/>This will be satisfied only if the values <math>e, f, d</math> are zeros.</p>  |
| Characteristic polynomial<br>when $e=0, f=0$ and $d=0$   | $ x\mathbf{I} - \mathbf{A}  = x^3$   |
| <p>Since <math>p(x) = x^2</math><br/>Hence <math>p(\mathbf{A}) = \mathbf{A}^2 = \mathbf{0}_{3 \times 3}</math></p> | <p>Therefore calculating <math>p(\mathbf{A})</math> as follows:</p> $\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \mathbf{0}_{3 \times 3}$ $\begin{pmatrix} 0 & 0 & ac \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathbf{0}_{3 \times 3}$ |

|  |   |
|--|---|
| <p>With entries <math>a=0, e=0, f=0, d=0</math><br/>The matrix <math>\mathbf{A}</math> will be:</p> <p>For <math>b=1, c=1</math></p> | <p>For <math>\mathbf{A}^2</math> to be a zero matrix, either <math>a=0</math> or <math>c=0</math></p> $\mathbf{A} = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}$ $\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ |
| <p>Conclusion</p>  | <p>Thus the matrix,</p> $\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ <p>has the minimal polynomial as <math>x^2</math>.</p>   |

TABLE 6.2.3.2: Solution summary

6.2.4. Let  $n$  be a positive integer, and let  $V$  be the space of polynomials over  $\mathbb{R}$  which have degree at most  $n$  (throw in the 0-polynomial). Let  $\mathbf{D}$  be the differentiation operator on  $V$ . What is the minimal polynomial for  $\mathbf{D}$ ?

**Solution:** See Tables 6.2.4.1 and 6.2.4.2

|                           |  |
|---------------------------|--|
| Characteristic Polynomial | For an $n \times n$ matrix $\mathbf{A}$ , characteristic polynomial is defined by,<br>$p(x) =  x\mathbf{I} - \mathbf{A} $  |
| Minimal Polynomial        | Minimal polynomial $m(x)$ is the smallest factor of characteristic polynomial $p(x)$ such that,<br>$m(\mathbf{A}) = 0$<br>Every root of characteristic polynomial should be the root of minimal polynomial |

TABLE 6.2.4.1: Definitions and theorem used

|                       |  |
|-----------------------|--|
| Given                 | $V$ is the space of polynomials over $\mathbb{R}$ which have degree at most $n$ .  |
| Matrix Representation | <p>The basis for the space <math>V</math> is</p> $\mathcal{B} = \{1, x, x^2, \dots, x^n\} \quad (6.2.4.1)$ <p>Given that <math>\mathbf{D}</math> is the differentiation operator. So,</p> $\mathbf{D}(1) = 0 \quad (6.2.4.2)$ $\mathbf{D}(x) = 1 \quad (6.2.4.3)$ $\vdots$ $\mathbf{D}(x^n) = nx^{n-1} \quad (6.2.4.4)$ <p>The vectors of differentiation operator with respect to basis <math>\mathcal{B}</math></p> $[\mathbf{D}(1)]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{(n+1) \times 1}, [\mathbf{D}(x)]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{(n+1) \times 1} \dots [\mathbf{D}(x^n)]_{\mathcal{B}} = \begin{pmatrix} 0 \\ \vdots \\ n \\ 0 \end{pmatrix}_{(n+1) \times 1} \quad (6.2.4.5)$ <p>The matrix representation can be written as:</p> $\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & n \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \quad (6.2.4.6)$ |



|                           |   |
|---------------------------|---|
| Characteristic polynomial | $p(x) =  x\mathbf{I} - \mathbf{A}  = \begin{vmatrix} x & -1 & 0 & \dots & 0 \\ 0 & x & -2 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & -n \\ 0 & 0 & 0 & \dots & x \end{vmatrix} \quad (6.2.4.7)$ <p>It is equal to the product of diagonal entries.</p> $p(x) = x^{n+1} \quad (6.2.4.8)$  |
| Minimal Polynomial        | <p>The minimal polynomial of <math>\mathbf{A}</math> can be any of <math>x, x^2, \dots, x^{n+1}</math> such that,</p> $m(\mathbf{A}) = 0 \quad (6.2.4.9)$   |
| Explanation               | <p>Let <math>P(n)</math>: Minimum polynomial of <math>\mathbf{D}=x^{n+1}</math> i.e <math>\mathbf{A}^{n+1} = 0</math><br/>For <math>n=1</math></p> $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (6.2.4.10)$ $\mathbf{A}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad (6.2.4.11)$ <p>So, <math>P(1)</math> is true.<br/>Assume <math>P(k)</math> holds for <math>1 \leq k \leq n</math>.</p> $\mathbf{A}_k = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & k \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}_{(k+1 \times k+1)} \Rightarrow \mathbf{A}_k^{k+1} = \mathbf{0} \quad (6.2.4.12)$ <p>We need to show that <math>P(k+1)</math> is true.</p> $\mathbf{A}_{k+1} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & k & 0 \\ 0 & 0 & 0 & \dots & 0 & k+1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}_{(k+2 \times k+2)} \quad (6.2.4.13)$ <p>Expressing in terms of block matrices</p> |

|                                  |   |
|----------------------------------|---|
|                                  | $\mathbf{A}_{k+1} = \begin{pmatrix} \mathbf{A}_k & \mathbf{x} \\ \mathbf{0}_{1 \times k+1} & 0 \end{pmatrix}, \mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ k+1 \end{pmatrix}_{k+1 \times 1} \quad (6.2.4.14)$  |
| Finding $\mathbf{A}_{k+1}^{k+2}$ | $\mathbf{A}_{k+1}^2 = \begin{pmatrix} \mathbf{A}_k & \mathbf{x} \\ \mathbf{0} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{A}_k & \mathbf{x} \\ \mathbf{0} & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{A}_k^2 & \mathbf{A}_k \mathbf{x} \\ 0 & 0 \end{pmatrix} \quad (6.2.4.15)$ $\mathbf{A}_{k+1}^3 = \begin{pmatrix} \mathbf{A}_k^2 & \mathbf{A}_k \mathbf{x} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{A}_k & \mathbf{x} \\ \mathbf{0} & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{A}_k^3 & \mathbf{A}_k^2 \mathbf{x} \\ 0 & 0 \end{pmatrix} \quad (6.2.4.16)$ $\mathbf{A}_{k+1}^{k+2} = \begin{pmatrix} \mathbf{A}_k^{k+2} & \mathbf{A}_k^{k+1} \mathbf{x} \\ 0 & 0 \end{pmatrix} \quad (6.2.4.17)$ <p>From (6.2.4.12), We know that <math>\mathbf{A}_k^{k+1} = \mathbf{0}</math></p> $\implies \mathbf{A}_{k+1}^{k+2} = \mathbf{0} \quad (6.2.4.18)$ <p>So, <math>P(k+1)</math> is true.</p> |
| Conclusion                       | <p>From above, by using the principle of induction we can say that the minimal polynomial is</p> $x^{n+1} \quad (6.2.4.19)$   |

TABLE 6.2.4.2: Finding minimal polynomial

|                           |   |
|---------------------------|---|
| Characteristic Polynomial | For an $n \times n$ matrix $\mathbf{A}$ , characteristic polynomial is defined by,<br>$p(x) =  x\mathbf{I} - \mathbf{A} $   |
| Cayley-Hamilton Theorem   | If $p(x)$ is the characteristic polynomial of an $n \times n$ matrix $\mathbf{A}$ , then,<br>$p(\mathbf{A}) = \mathbf{0}$   |
| Minimal Polynomial        | Minimal polynomial $m(x)$ is the smallest factor of characteristic polynomial $p(x)$ such that,<br>$m(\mathbf{A}) = \mathbf{0}$<br>Every root of characteristic polynomial should be the root of minimal polynomial |

TABLE 6.2.6.1: Definitions

6.2.5. Let  $\mathbf{A}$  and  $\mathbf{B}$  be  $n \times n$  matrices over the field

$F$ . Prove that if  $(\mathbf{I} - \mathbf{AB})$  is invertible

a)  $(\mathbf{I} - \mathbf{BA})$  is invertible and

b)  $(\mathbf{I} - \mathbf{BA})^{-1} = \mathbf{I} + \mathbf{B}(\mathbf{I} - \mathbf{AB})^{-1}\mathbf{A}$

6.2.6. Let  $P$  be the operator on  $R^2$  which projects each vector onto the x-axis, parallel to the y-axis:  $p(x, y) = (x, 0)$ . Show that  $P$  is linear.

What is the minimal polynomial for  $P$ ?

**Solution:** See Tables 6.2.6.1 and 6.2.6.2.

|                           |  |
|---------------------------|--|
| Proof of $P$ is linear    | <p>Consider two vectors <math>(x_1, y_1)</math> and <math>(x_2, y_2)</math>, then</p> $P((x_1, y_1) + (x_2, y_2)) = P(x_1 + x_2, y_1 + y_2)$ $P((x_1, y_1) + (x_2, y_2)) = P(x_1 + x_2, 0)$ $P((x_1, y_1) + (x_2, y_2)) = P((x_1, 0), (x_2, 0))$ <p>Now, consider some scalar <math>k</math>, then</p> $P(k(x_1, y_1)) = P((kx_1, ky_1))$ $P(k(x_1, y_1)) = P((kx_1, 0))$ $P(k(x_1, y_1)) = kP(x_1, 0)$ <p>Thus, using above observations we can conclude that <math>P</math> is linear.</p> |
| Matrix form of Projection | <p>For the projection <math>P(x, y) = (x, 0)</math>, the matrix of linear transform is,</p> $P(x, y) = \mathbf{A} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = (x, 0)$ <p>So, <math>\mathbf{A} = \begin{pmatrix} 1 &amp; 0 \\ 0 &amp; 0 \end{pmatrix}</math></p>   |
| Characteristic polynomial | $p(x) =  x\mathbf{I} - \mathbf{A} $ $= \begin{vmatrix} x-1 & 0 \\ 0 & x-0 \end{vmatrix}$ $= x(x-1)$  |
| Minimal Polynomial        | $p(x) = x^a(x-1)^b, \quad a \leq 1, b \leq 1$  |
| $a = 1, b = 1$            | $m(x) = x(x-1)$ $\Rightarrow m(\mathbf{A}) = \mathbf{A}(\mathbf{A} - \mathbf{I}) = 0$ $\Rightarrow x(x-1) \text{ is a minimal polynomial}$   |
| Conclusion                | <p>For the given matrix <math>\mathbf{A}</math>,<br/> <math>x(x-1)</math> is the characteristic polynomial as well as minimal polynomial.</p>  |

TABLE 6.2.6.2: Illustration of Proof and finding of minimal polynomial

6.2.7. Let  $\mathbf{A}$  be an  $n \times n$  matrix with characteristics polynomial

$$f = (x - c_1)^{d_1} \dots (x - c_k)^{d_k} \quad (6.2.7.1)$$

Show that

$$c_1 d_1 + \dots + c_k d_k = \text{tr}(\mathbf{A}) \quad (6.2.7.2)$$

**Solution:** See Table 6.2.7.1.

|          |  |
|----------|--|
| Given    | <p>Let <math>\mathbf{A}</math> be an <math>n \times n</math></p> $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$ <p>and Characteristics polynomial</p> $f = (x - c_1)^{d_1} \cdots (x - c_k)^{d_k}$  |
| To prove | $c_1 d_1 + \cdots + c_k d_k = \text{tr}(\mathbf{A})$   |
| proof    | <p>Characteristics polynomial;<br/> <math>f = (x - c_1)^{d_1} \cdots (x - c_k)^{d_k}</math><br/> here,<br/> <math>c_1, \dots, c_k</math> are the distinct eigen values.<br/> and <math>d_1, \dots, d_k</math> denotes the repetition of eigen values</p> <p>Therefore,</p> <div style="border: 1px solid black; padding: 5px; display: inline-block;"> <math display="block">c_1 d_1 + c_2 d_2 + \dots + c_k d_k = \sum_i \lambda_i</math> </div> =Sum of all eigen values. <p>As we know ,<br/> Trace of a matrix is the sum of its eigen values.</p> <p><math>\Rightarrow \text{tr}(\mathbf{A}) = \sum_i \lambda_i</math><br/> therefore,</p> <div style="border: 1px solid black; padding: 5px; display: inline-block;"> <math display="block">\Rightarrow c_1 d_1 + c_2 d_2 + \dots + c_k d_k = \sum_i \lambda_i = \text{tr}(\mathbf{A})</math> </div> <p>Hence, Proved.</p> |

TABLE 6.2.7.1: Solution Summary

6.2.8. Let  $\mathbf{V}$  be the vector space of  $n \times n$  matrices over field  $\mathbf{F}$ . Let  $\mathbf{A}$  be a fixed  $n \times n$  matrix. Let  $\mathbf{T}$  be the linear operator on  $\mathbf{V}$  defined by

$$\mathbf{T}(\mathbf{B}) = \mathbf{AB} \quad (6.2.8.1)$$

Show that the minimal polynomial for  $\mathbf{T}$  is the minimal polynomial for  $\mathbf{A}$ .

**Solution:** See Tables 6.2.8.1, 6.2.8.2 and 6.2.8.3.

|                                       |   |
|---------------------------------------|---|
| Given                                 | $\mathbf{A}$ is a fixed matrix from the vector space $\mathbf{V}$ of $n \times n$ matrices. A linear operator on the finite dimensional vector space $\mathbf{V}$ , $\mathbf{T}$ is defined as $\mathbf{T}(\mathbf{B}) = \mathbf{A}\mathbf{B}$ .  |
| Minimal polynomial                    | The minimal polynomial of a linear operator $\mathbf{T}$ is a monic polynomial which annihilates $\mathbf{T}$ .   |
| Matrix representation of $\mathbf{T}$ | <p>If we stack up the columns of the matrix <math>\mathbf{B}</math>, the linear operator <math>\mathbf{T}</math> can be represented in the equivalent form as</p> <p>If <math>\mathbf{B} = \begin{pmatrix} b_1 &amp; b_2 &amp; \cdot &amp; \cdot &amp; b_n \end{pmatrix}</math>, then the linear transformation of <math>\mathbf{B}</math> will be</p> <p><math>\mathbf{T}(\mathbf{B}) = \begin{pmatrix} \mathbf{A}b_1 &amp; \mathbf{A}b_2 &amp; \cdot &amp; \cdot &amp; \mathbf{A}b_n \end{pmatrix}</math></p> <p><math display="block">\mathbf{M}_{\mathbf{T}}(\mathbf{B}) = \begin{pmatrix} \mathbf{T}(b_1) \\ \mathbf{T}(b_2) \\ \cdot \\ \cdot \\ \mathbf{T}(b_n) \end{pmatrix} = \begin{pmatrix} \mathbf{A} &amp; &amp; &amp; &amp; \\ &amp; \mathbf{A} &amp; &amp; &amp; \\ &amp; &amp; \mathbf{O} &amp; &amp; \\ &amp; &amp; &amp; \cdot &amp; \\ &amp; \mathbf{O} &amp; &amp; &amp; \cdot \\ &amp; &amp; &amp; &amp; \mathbf{A} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ b_n \end{pmatrix}</math></p> |
|                                       | $\mathbf{M}_{\mathbf{T}} = \begin{pmatrix} \mathbf{A} & & & & \\ & \mathbf{A} & & & \\ & & \mathbf{O} & & \\ & & & \cdot & \\ & \mathbf{O} & & & \cdot \\ & & & & \mathbf{A} \end{pmatrix}$   |

TABLE 6.2.8.1: Construction

|   |   |
|---|---|
| Properties of minimal polynomial                | The roots of the characteristic polynomial, eigen values and the minimal polynomial are same, except for multiplicities. The roots of the minimal polynomial of $\mathbf{A}$ are the roots of $\det(\mathbf{A} - \lambda\mathbf{I})$  |
| The roots of minimal polynomial of $\mathbf{T}$ | <p>The roots of the minimal polynomial of <math>\mathbf{T}</math> are the roots of <math>\det(\mathbf{T} - \lambda\mathbf{I})</math></p> <p><math display="block">\det(\mathbf{T} - \lambda\mathbf{I}) = \begin{vmatrix} (\mathbf{A} - \lambda\mathbf{I}) &amp; &amp; &amp; &amp; \\ &amp; (\mathbf{A} - \lambda\mathbf{I}) &amp; &amp; &amp; \mathbf{O} \\ &amp; &amp; \mathbf{O} &amp; &amp; \cdot \\ &amp; &amp; &amp; \cdot &amp; \\ &amp; &amp; &amp; &amp; (\mathbf{A} - \lambda\mathbf{I}) \end{vmatrix}</math></p> <p><math display="block">= (\det(\mathbf{A} - \lambda\mathbf{I}))^n</math></p> <p>Therefore we can see that the eigen values of <math>\mathbf{A}</math> are also the eigen values of the linear operator <math>\mathbf{T}</math></p> |
| Minimal polynomial of $\mathbf{T}$              | <p>The minimal polynomial of <math>\mathbf{A}</math> divides the characteristic polynomial of <math>\mathbf{A}</math> and <math>\mathbf{T}</math>. Let the minimal polynomial of <math>\mathbf{A}</math> is of degree <math>p \leq n</math></p> <p><math>f(x) = a_0 + a_1x + a_2x^2 \dots a_px^p</math> such that <math>f(\mathbf{A}) = \mathbf{O}</math></p> <p><math>f(\mathbf{T}) = a_0\mathbf{I} + a_1\mathbf{T} + a_2\mathbf{T}^2 + \dots + a_p\mathbf{T}^p</math></p>   |
|   | <p><math display="block">f(\mathbf{T}) = \begin{pmatrix} f(\mathbf{A}) &amp; &amp; &amp; &amp; \\ &amp; f(\mathbf{A}) &amp; &amp; &amp; \mathbf{O} \\ &amp; &amp; \mathbf{O} &amp; &amp; \cdot \\ &amp; &amp; &amp; \cdot &amp; \\ &amp; &amp; &amp; &amp; f(\mathbf{A}) \end{pmatrix} = \mathbf{O}_{n^2 \times n^2}</math></p> <p>Therefore the minimal polynomial for <math>\mathbf{T}</math> is the minimal polynomial for <math>\mathbf{A}</math>.</p>  |

TABLE 6.2.8.2: Proof



|  |   |
|--|---|
| Assuming matrix $\mathbf{A}$ as follows: | Let us Consider $2 \times 2$ matrix,<br>$\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 0 & 2 \end{pmatrix}$   |
| Minimal polynomial of $\mathbf{A}$       | The eigen values of $\mathbf{A}$ are 1, 2.<br>So, the minimal polynomial is $f(x) = (x - 1)(x - 2)$   |
| Matrix of linear operator                | So, the matrix of the linear operator $\mathbf{T}$ with respect to the basis<br>$\mathbf{e}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{e}_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{e}_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$<br>$\mathbf{T}(\mathbf{e}_1) = \begin{pmatrix} 1 & 4 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 1.\mathbf{e}_1 + 0.\mathbf{e}_2 + 0.\mathbf{e}_3 + 0.\mathbf{e}_4$<br>$\mathbf{T}(\mathbf{e}_2) = \begin{pmatrix} 1 & 4 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 2 & 0 \end{pmatrix} = 4.\mathbf{e}_1 + 2.\mathbf{e}_2 + 0.\mathbf{e}_3 + 0.\mathbf{e}_4$<br>$\mathbf{T}(\mathbf{e}_3) = \begin{pmatrix} 1 & 4 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0.\mathbf{e}_1 + 0.\mathbf{e}_2 + 1.\mathbf{e}_3 + 0.\mathbf{e}_4$ |
|  | $\mathbf{T}(\mathbf{e}_4) = \begin{pmatrix} 1 & 4 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 4 \\ 0 & 2 \end{pmatrix} = 0.\mathbf{e}_1 + 0.\mathbf{e}_2 + 4.\mathbf{e}_3 + 2.\mathbf{e}_4$<br>So the matrix of the linear operator will be<br>$\mathbf{M}_T = \begin{pmatrix} 1 & 4 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} \mathbf{A} & \mathbf{O} \\ \mathbf{O} & \mathbf{A} \end{pmatrix}$   |
| Characteristic equation of $\mathbf{T}$  | The characteristic equation of $\mathbf{T}$ is $(x - 1)^2(x - 2)^2$<br>So the eigen values are 1, 1, 2, 2   |
| Minimal polynomial of $\mathbf{T}$       | $f(\mathbf{M}_T) = (\mathbf{T} - \mathbf{I})(\mathbf{T} - 2\mathbf{I}) = \begin{pmatrix} \mathbf{A} - \mathbf{I} & \mathbf{O} \\ \mathbf{O} & \mathbf{A} - \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{A} - 2\mathbf{I} & \mathbf{O} \\ \mathbf{O} & \mathbf{A} - 2\mathbf{I} \end{pmatrix}$<br>$f(\mathbf{M}_T) = \begin{pmatrix} (\mathbf{A} - \mathbf{I})(\mathbf{A} - 2\mathbf{I}) & \mathbf{O} \\ \mathbf{O} & (\mathbf{A} - \mathbf{I})(\mathbf{A} - 2\mathbf{I}) \end{pmatrix} = \begin{pmatrix} f(\mathbf{A}) & \mathbf{O} \\ \mathbf{O} & f(\mathbf{A}) \end{pmatrix} = \mathbf{O}$<br>We know that eigen values of $\mathbf{T}$ should be roots of minimal polynomial of $\mathbf{T}$ , thus minimal polynomial should be of the form $(x - 1)^p(x - 2)^q$ where $p, q \in \mathbb{N}, 1 \leq p, q \leq 2$<br>Therefore the minimal polynomial $f(\mathbf{A})$ of $\mathbf{A}$ annihilates $\mathbf{T}$ , thus we can conclude that $f(x)$ is the minimal polynomial of linear operator $\mathbf{T}$   |

TABLE 6.2.8.3: Example

6.2.9. Let  $\mathbf{A}$  and  $\mathbf{B}$  be  $n \times n$  matrices over the field  $\mathbf{F}$ . If the matrices  $\mathbf{AB}$  and  $\mathbf{BA}$  have the same characteristic values. Do they have the same characteristic polynomial? Do they have the same minimal polynomial?

**Solution:**

**Theorem 6.3.** *If  $\mathbf{T}$  is a linear operator on a finite-dimensional space  $\mathbf{V}$  and  $c$  is a characteristic value of  $\mathbf{T}$ , then the operator  $(c\mathbf{I} - \mathbf{T})$  is singular, i.e.*

$$\det(c\mathbf{I} - \mathbf{T}) = 0$$

**Theorem 6.4.** *If  $\mathbf{A}$  and  $\mathbf{B}$  are  $n \times n$  matrix, then  $(\mathbf{I} - \mathbf{AB})$  is invertible if and only if  $(\mathbf{I} - \mathbf{BA})$  is invertible.*

To prove that  $\mathbf{AB}$  and  $\mathbf{BA}$  have the same characteristic values in  $\mathbf{F}$ , we can use theorem 6.3 and show

$$\det(c\mathbf{I} - \mathbf{AB}) = \det(c\mathbf{I} - \mathbf{BA}) = 0 \quad (6.2.9.1)$$

That is, if  $c$  is a characteristic value for  $\mathbf{AB}$  then  $c$  is a characteristic value for  $\mathbf{BA}$  as well. Which can be also said as, if  $c$  is not the characteristic value of  $\mathbf{BA}$  then it won't be the characteristic value for  $\mathbf{AB}$ .

Let's say  $c$  is not a characteristic value of the matrix  $\mathbf{BA}$ , which would imply that

$$\det(c\mathbf{I} - \mathbf{BA}) \neq 0 \quad (6.2.9.2)$$

There are two cases for this:

a)  $c = 0$

In this case  $\det(-(BA)) \neq 0$ .

$$\begin{aligned} \det(-(BA)) &= (-1)^n \det(\mathbf{B}) \det(\mathbf{A}) \\ &= (-1)^n \det(\mathbf{A}) \det(\mathbf{B}) \\ &= \det(-\mathbf{AB}) \\ &= \det(c\mathbf{I} - \mathbf{AB}) \end{aligned}$$

From (6.2.9.2) we can write this as

$$\det(c\mathbf{I} - \mathbf{AB}) \neq 0 \quad (6.2.9.3)$$

b)  $c \neq 0$

In this case

$$\begin{aligned} c\mathbf{I} - \mathbf{BA} &= c \left( \mathbf{I} - \frac{1}{c}\mathbf{BA} \right) \\ \Rightarrow \det \left( c \left( \mathbf{I} - \frac{1}{c}\mathbf{BA} \right) \right) &= c^n \det \left( \mathbf{I} - \frac{1}{c}\mathbf{BA} \right) \end{aligned}$$

From (6.2.9.2), we get

$$c^n \det \left( \mathbf{I} - \frac{1}{c}\mathbf{BA} \right) \neq 0$$

Therefore,  $(\mathbf{I} - \frac{1}{c}\mathbf{BA})$  is invertible. From theorem 6.4 we can say  $(\mathbf{I} - \frac{1}{c}\mathbf{AB})$  is invertible.

$$\begin{aligned} \Rightarrow \det \left( \mathbf{I} - \frac{1}{c}\mathbf{AB} \right) &\neq 0 \\ \Rightarrow c^n \det \left( \mathbf{I} - \frac{1}{c}\mathbf{AB} \right) &\neq 0 \\ \Rightarrow \det(c\mathbf{I} - \mathbf{AB}) &\neq 0 \end{aligned} \quad (6.2.9.4)$$

After proving both the cases of  $c = 0$  and  $c \neq 0$  we say that the characteristic value  $c$  must be the same.

$\therefore \mathbf{AB}$  and  $\mathbf{BA}$  have the same characteristic values.

After proving both the above statements, we can say that the two polynomials of degree  $n$  with exactly the same roots, will be equal in nature. So, the characteristic polynomials are equal.

Even though the characteristic polynomials are same, they may not necessarily be the same minimal polynomial.

Lets take the below examples

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (6.2.9.5)$$

$$\mathbf{B} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (6.2.9.6)$$

We get the matrices  $\mathbf{AB}$  and  $\mathbf{BA}$  as

|                           |   |
|---------------------------|---|
| Characteristic Polynomial | For an $n \times n$ matrix $\mathbf{A}$ , characteristic polynomial is defined by,<br>$p(x) =  x\mathbf{I} - \mathbf{A} $                                 |
| Theory                    | $\mathbf{A}$ is similar to triangular matrix $\mathbf{J}$ if<br>$\exists$ an invertible matrix $\mathbf{P}$ such that<br>$\mathbf{A} = \mathbf{PJP}^{-1}$ |

TABLE 6.3.1.1: Definitions

$$\mathbf{AB} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (6.2.9.7)$$

$$\mathbf{BA} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad (6.2.9.8)$$

Then  $\mathbf{AB} = \mathbf{A}$ , whereas  $\mathbf{BA}$  is the zero matrix. Since  $\mathbf{A}^2 = 0$  and  $\mathbf{A} \neq 0$ , the minimal polynomial of  $\mathbf{AB}$  is  $x^2$ , whereas the minimal polynomial of  $\mathbf{BA}$  is  $x$ .  
 $\therefore$ ,  $\mathbf{AB}$  and  $\mathbf{BA}$  have the same characteristic polynomial, but the minimal polynomials are not the same.

### 6.3 Invariant Subspaces

6.3.1. Let

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 2 & -2 & 2 \\ 2 & -3 & 2 \end{pmatrix} \quad (6.3.1.1)$$

Is  $\mathbf{A}$  similar over the field of real numbers to a triangular matrix? If so, find such a triangular matrix.

**Solution:** See Tables 6.3.1.1 and 6.3.1.2

|   |  |
|---|--|
| Characteristic polynomial   | $  \begin{aligned}  p(x) &=  x\mathbf{I} - \mathbf{A}  \\  &= \begin{vmatrix} x & -1 & 0 \\ -2 & x+2 & -2 \\ -2 & 3 & x-2 \end{vmatrix} \\  &= x((x+2)(x-2) + 6) + (-2)(x-2) + 4 + 0 \\  &= x(x^2 + 2) - 2x \\  &= x^3 \\  \implies \lambda &= 0  \end{aligned}  $   |
| $\dim(\text{Ker}(\mathbf{A} - \lambda\mathbf{I}))$                    | $  \begin{aligned}  (\mathbf{A} - 0\mathbf{I})\mathbf{X} &= \mathbf{0} \\  \implies \begin{pmatrix} 0 & 1 & 0 \\ 2 & -2 & 2 \\ 2 & -3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\  \implies x &= -z, y = 0 \\  \text{So,} \\  \mathbf{v}_1 &= \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}  \end{aligned}  $ |
| Find $\mathbf{v}_2$ such that $\mathbf{A}\mathbf{v}_2 = \mathbf{v}_1$ | $  \begin{aligned}  \begin{pmatrix} 0 & 1 & 0 \\ 2 & -2 & 2 \\ 2 & -3 & 2 \end{pmatrix} \mathbf{v}_2 &= \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \\  \implies \mathbf{v}_2 &= \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}  \end{aligned}  $  |
| Find $\mathbf{v}_3$ such that $\mathbf{A}\mathbf{v}_3 = \mathbf{v}_2$ | $  \begin{aligned}  \begin{pmatrix} 0 & 1 & 0 \\ 2 & -2 & 2 \\ 2 & -3 & 2 \end{pmatrix} \mathbf{v}_3 &= \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} \\  \implies \mathbf{v}_3 &= \begin{pmatrix} -\frac{3}{2} \\ -1 \\ 0 \end{pmatrix}  \end{aligned}  $   |
|   | <p>Let <math>\mathbf{P} = (\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3)</math></p> <p>Then,</p> $\mathbf{AP} = (\mathbf{Av}_1 \ \mathbf{Av}_2 \ \mathbf{Av}_3)$   |

|            |  |
|------------|--|
|            | $\begin{aligned} \Rightarrow \mathbf{AP} &= \begin{pmatrix} 0 & \mathbf{v}_1 & \mathbf{v}_2 \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \mathbf{PJ} \\ \Rightarrow \mathbf{AP} &= \mathbf{PJ} \\ \Rightarrow \mathbf{A} &= \mathbf{PJP}^{-1} \end{aligned}$ |
| Conclusion | <p>Therefore, <math>\mathbf{A}</math> is similar to triangular matrix and the triangular matrix is,</p> $\mathbf{J} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$   |

TABLE 6.3.1.2: Checking triangularizability of  $\mathbf{A}$

|                           |  |
|---------------------------|--|
| Characteristic Polynomial | <p>For an <math>n \times n</math> matrix <math>\mathbf{A}</math>, characteristic polynomial is defined by,</p> $p(x) =  x\mathbf{I} - \mathbf{A} $   |
| Cayley-Hamilton Theorem   | <p>If <math>p(x)</math> is the characteristic polynomial of an <math>n \times n</math> matrix <math>\mathbf{A}</math>, then,</p> $p(\mathbf{A}) = \mathbf{0}$  |
| Minimal Polynomial        | <p>Minimal polynomial <math>m(x)</math> is the smallest factor of characteristic polynomial <math>p(x)</math> such that,</p> $m(\mathbf{A}) = \mathbf{0}$ <p>Every root of characteristic polynomial should be the root of minimal polynomial</p>  |
| Theorem                   | <p>Let <math>\mathbf{V}</math> be a finite-dimensional vector space over the field <math>\mathbf{F}</math> and let <math>\mathbf{T}</math> be a linear operator on <math>\mathbf{V}</math>. Then <math>\mathbf{T}</math> is diagonalizable if and only if the minimal polynomial for <math>\mathbf{T}</math> has the form <math>p = (x - c_1) \dots (x - c_k)</math>, where <math>c_1, c_2, \dots, c_k</math> are distinct elements of <math>F</math>.</p> |

TABLE 6.3.2.1: Definitions

6.3.2. Every matrix  $A$  such that  $A^2 = A$  is similar to a diagonal matrix.

**Solution:** See Tables 6.3.2.1 and 6.3.2.2

|                    |   |
|--------------------|---|
| Proof              | $A^2 = A$ (Given)<br>$\implies A$ satisfies the polynomial $x^2 - x = x(x - 1)$   |
| Minimal Polynomial | $p(x) = x^a(x - 1)^b$ , $a \leq 1, b \leq 1$<br><br>Minimal Polynomial $m_A(x)$ , of $A$ divides $x^2 - x$ , that is $m_A(x) = x$ or $m_A(x) = x - 1$ or $m_A(x) = x(x - 1)$<br>If $m_A(x) = x$ , then $A = 0$ .<br>If $m_A(x) = x - 1$ , then $A = I$ .<br>If $m_A(x) = x(x - 1)$<br>In all above three cases, the minimal polynomial factors into distinct linears. So, it follows $A$ is diagonalisable. |
| Conclusion         | In all three cases, the minimal polynomial splits into distinct linear factors, so it follows that $A$ is diagonalisable according to the Theorem mentioned in definitions section.   |

TABLE 6.3.2.2: Illustration of Proof

6.3.3. True or false? If the triangular matrix  $\mathbf{A}$  is similar to a diagonal matrix, then  $\mathbf{A}$  is already diagonal.

**Solution:** See Tables 6.3.3.1 and 6.3.3.2



|                           |  |
|---------------------------|--|
| Characteristic Polynomial | For an $n \times n$ matrix $\mathbf{A}$ , characteristic polynomial is defined by,<br>$p(x) =  x\mathbf{I} - \mathbf{A} $  |
| Minimal Polynomial        | Minimal polynomial $m(x)$ is the smallest factor of characteristic polynomial $p(x)$ such that,<br>$m(\mathbf{A}) = 0$   |
| Theorem                   | Let $\mathbf{V}$ be a finite-dimensional vector space over the field $\mathbf{F}$ and let $\mathbf{T}$ be a linear operator on $\mathbf{V}$ . Then $\mathbf{T}$ is diagonalizable if and only if the minimal polynomial for $\mathbf{T}$ has the form<br>$p = (x - c_1) \dots (x - c_k) \quad (6.3.3.1)$ where $c_1, c_2, \dots, c_k$ are distinct elements of $F$ . |
| Diagonalizable            | $\mathbf{A}$ is called diagonalizable if it is similar to diagonal matrix $\mathbf{B}$ i.e., if $\exists$ an invertible matrix $\mathbf{P}$ such that<br>$\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} \quad (6.3.3.2)$  |

TABLE 6.3.3.1: Definitions and theorem used

|                           |   |
|---------------------------|---|
| Given                     | The triangular matrix $\mathbf{A}$ is similar to a diagonal matrix.   |
| Example                   | Let<br>$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \quad (6.3.3.3)$ We can see that $\mathbf{A}$ is triangular but not diagonal. |
| Characteristic polynomial | $ x\mathbf{I} - \mathbf{A}  = \begin{vmatrix} x-1 & 2 \\ 0 & x-3 \end{vmatrix} \quad (6.3.3.4)$ $= (x-1)(x-3) \quad (6.3.3.5)$                  |
| Minimal polynomial        | As the eigen values are distinct, minimal polynomial  |

|                                    |  |
|------------------------------------|--|
|                                    | $m(x) = (x - 1)(x - 3) \quad (6.3.3.6)$  |
| Diagonalizable                     | From theorem (3.7.3.3), We can say that <b>A</b> diagonalizable i.e., it is similar to a diagonal matrix.  |
| Finding matrix similar to <b>A</b> | <p>The eigen values are</p> $\lambda_1 = 1, \lambda_2 = 3 \quad (6.3.3.7)$ <p>The eigen vectors are</p> $(\mathbf{A} - \lambda_i \mathbf{I}) \mathbf{x}_i = 0 \quad (6.3.3.8)$ $\lambda_1 = 1 \Rightarrow \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix} \mathbf{x}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (6.3.3.9)$ $\lambda_2 = 3 \Rightarrow \begin{pmatrix} -2 & 2 \\ 0 & 0 \end{pmatrix} \mathbf{x}_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \mathbf{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (6.3.3.10)$ <p>The invertible matrix</p> $\mathbf{P} = (\mathbf{x}_1 \ \mathbf{x}_2) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad (6.3.3.11)$ <p>The diagonal matrix similar to <b>A</b></p> $\mathbf{B} = \mathbf{P}^{-1} \mathbf{A} \mathbf{P} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad (6.3.3.12)$ $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \quad (6.3.3.13)$ |
| Conclusion                         | From above, we can say that <b>A</b> need not be diagonal to satisfy given conditions. So, given statement is false.   |

TABLE 6.3.3.2: Finding minimal polynomial and similar matrix

|              |   |
|--------------|---|
| Theorem      | According to theorem , if a $2 \times 2$ matrix has two characteristics values then the $\mathbf{P}$ that diagonalize $\mathbf{A}$ will necessarily also diagonalize any $\mathbf{B}$ that commutes with $\mathbf{A}$ .   |
| Common Basis | <p>Let there exist a <math>\mathbf{P}</math> in basis <math>\beta = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}</math> of <math>\mathbb{V}</math> consisting of eigen vector which are common to both <math>\mathbf{A}</math> and <math>\mathbf{B}</math> such that</p> $\mathbf{A}\mathbf{b}_i = \lambda_i\mathbf{b}_i$ $\mathbf{B}\mathbf{b}_i = \mu_i\mathbf{b}_i$ $\Lambda_A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ $\Lambda_B = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}$ $\Lambda_A = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ $\Lambda_B = \mathbf{P}^{-1}\mathbf{B}\mathbf{P}$ |

TABLE 6.4.1.1: Theorems

#### 6.4 Simultaneous Triangulation; Simultaneous Diagonalization

6.4.1. Find an invertible matrix  $\mathbf{P}$  such that  $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$  and  $\mathbf{P}^{-1}\mathbf{B}\mathbf{P}$  are both diagonal where  $\mathbf{A}$  and  $\mathbf{B}$  are real matrices.

a)

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 3 & -8 \\ 0 & -1 \end{pmatrix} \quad (6.4.1.1)$$

**Solution:** See Tables 6.4.1.1 and 6.4.1.2

| Operations                       | Matrix A   | Matrix B   |
|----------------------------------|--|--|
| Characteristic Polynomial        | $p(x) =  x\mathbf{I} - \mathbf{A} $ $= \begin{vmatrix} x-1 & -2 \\ 0 & x-2 \end{vmatrix}$ $= (x-1)(x-2)$   | $p(x) =  x\mathbf{I} - \mathbf{B} $ $= \begin{vmatrix} x-3 & 8 \\ 0 & x+1 \end{vmatrix}$ $= (x-3)(x+1)$  |
| Characteristic values            | $p(x) = 0$ $(x-1)(x-2) = 0$ $\lambda_1 = 1, \lambda_2 = 2$   | $p(x) = 0$ $(x-3)(x+1) = 0$ $\mu_1 = 3, \mu_2 = -1$  |
| Basis for Characteristics Values | <p>For <math>\lambda_1 = 1</math></p> $(\mathbf{A} - \lambda_1 \mathbf{I})\mathbf{b}_1 = 0$ $\left( \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \mathbf{b}_1 = 0$ $\left( \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix} \right) \mathbf{b}_1 = 0$ $\mathbf{b}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ <p>For <math>\lambda_2 = 2</math></p> $(\mathbf{A} - \lambda_2 \mathbf{I})\mathbf{b}_2 = 0$ $\left( \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} - 2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \mathbf{b}_2 = 0$ $\left( \begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix} \right) \mathbf{b}_2 = 0$ $\mathbf{b}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\mathbf{P} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ | <p>For <math>\mu_1 = 3</math></p> $(\mathbf{B} - \mu_1 \mathbf{I})\mathbf{b}_1 = 0$ $\left( \begin{pmatrix} 3 & -8 \\ 0 & -1 \end{pmatrix} - 3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \mathbf{b}_1 = 0$ $\left( \begin{pmatrix} 0 & -8 \\ 0 & -4 \end{pmatrix} \right) \mathbf{b}_1 = 0$ $\mathbf{b}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ <p>For <math>\mu_2 = -1</math></p> $(\mathbf{B} - \mu_2 \mathbf{I})\mathbf{b}_2 = 0$ $\left( \begin{pmatrix} 3 & -8 \\ 0 & -1 \end{pmatrix} - (-1) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \mathbf{b}_2 = 0$ $\left( \begin{pmatrix} 4 & -8 \\ 0 & 0 \end{pmatrix} \right) \mathbf{b}_2 = 0$ $\mathbf{b}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\mathbf{P} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ |
| Verification                     | $\Lambda_A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ $\Lambda_A = \mathbf{P}^{-1} \mathbf{A} \mathbf{P}$ $\Rightarrow \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$   | $\Lambda_B = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$ $\Lambda_B = \mathbf{P}^{-1} \mathbf{B} \mathbf{P}$ $\Rightarrow \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & -8 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$  |

|                                       |  |   |
|---------------------------------------|--|---|
|                                       | $= \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \Lambda_A$ | $= \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} = \Lambda_B$ |
| Answer.The Invertible Matrix <b>P</b> | $\mathbf{P} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$  | $\mathbf{P} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$   |

TABLE 6.4.1.2: Solution Table

|                           |  |
|---------------------------|--|
| Characteristic Polynomial | <p>For an <math>n \times n</math> matrix <math>\mathbf{A}</math>, characteristic polynomial is defined by,</p> $p(x) =  x\mathbf{I} - \mathbf{A} $   |
| Theorem                   | <p>According to theorem 8, if a <math>2 \times 2</math> matrix has two characteristics values then the <math>\mathbf{P}</math> that diagonalize <math>\mathbf{A}</math> will necessarily also diagonalize any <math>\mathbf{B}</math> that commutes with <math>\mathbf{A}</math>.</p>  |
| Basis                     | <p>Let there exist a <math>\mathbf{P}</math> in basis <math>\beta = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}</math> of <math>\mathbb{V}</math> consisting of eigen vector which are common to both <math>\mathbf{A}</math> and <math>\mathbf{B}</math> such that</p> $\mathbf{A}\mathbf{b}_i = \lambda_i\mathbf{b}_i \qquad \mathbf{B}\mathbf{b}_i = \mu_i\mathbf{b}_i$ $\Lambda_A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \qquad \Lambda_B = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}$ $\Lambda_A = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} \qquad \Lambda_B = \mathbf{P}^{-1}\mathbf{B}\mathbf{P}$ |

TABLE 6.4.1.3: Definitions and theorem used

b)

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad (6.4.1.2)$$

$$\mathbf{B} = \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix} \qquad (6.4.1.3)$$

**Solution:** See Tables 6.4.1.3 and 6.4.1.4

| Operations                       | Matrix A   | Matrix B   |
|----------------------------------|--|--|
| Characteristic Polynomial        | $p(x) =  x\mathbf{I} - \mathbf{A} $ $= \begin{vmatrix} x-1 & -1 \\ -1 & x-1 \end{vmatrix}$ $= (x-1)(x-1) - 1$  | $p(x) =  x\mathbf{I} - \mathbf{B} $ $= \begin{vmatrix} x-1 & -a \\ -a & x-1 \end{vmatrix}$ $= (x-1)(x-1) - a^2$  |
| Characteristic values            | $p(x) = 0$ $x(x-1) = 0$ $\lambda_1 = 0, \lambda_2 = 2$   | $p(x) = 0$ $(x-1)^2 - a^2 = 0$ $\mu_1 = (1-a), \mu_2 = (1+a)$  |
| Basis for Characteristics Values | $(\mathbf{A} - \lambda_i \mathbf{I})\mathbf{X} = 0$ $\Rightarrow \text{For } \lambda_1 = 0$ $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$ $\text{So, } \mathbf{b}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $\Rightarrow \text{For } \lambda_2 = 2$ $\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$ $\text{So, } \mathbf{b}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ | $(\mathbf{B} - \mu_i \mathbf{I})\mathbf{X} = 0$ $\Rightarrow \text{For } \mu_1 = (1-a)$ $\begin{pmatrix} a & a \\ a & a \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$ $\text{So, } \mathbf{b}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $\Rightarrow \text{For } \mu_2 = (1+a)$ $\begin{pmatrix} -a & a \\ a & -a \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$ $\text{So, } \mathbf{b}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ |
| Invertible matrix                | <p>Let <math>\mathbf{P} = (\mathbf{b}_1 \ \mathbf{b}_2)</math></p> <p>Then,</p> $\mathbf{P} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$   | <p>Let <math>\mathbf{P} = (\mathbf{b}_1 \ \mathbf{b}_2)</math></p> <p>Then,</p> $\Rightarrow \mathbf{P} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$   |
| Verification                     | $\Lambda_A = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$   | $\Lambda_B = \begin{pmatrix} 1-a & 0 \\ 0 & 1+a \end{pmatrix}$   |

|            |   |   |
|------------|---|---|
|            | $\Lambda_A = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$<br>$\Lambda_A = \begin{pmatrix} \frac{-1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$<br>$= \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} = \Lambda_A$ | $\Lambda_B = \mathbf{P}^{-1}\mathbf{B}\mathbf{P}$<br>$\Lambda_B = \begin{pmatrix} \frac{-1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$<br>$= \begin{pmatrix} 1-a & 0 \\ 0 & 1+a \end{pmatrix} = \Lambda_B$ |
| Conclusion | $\mathbf{P} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$  | $\mathbf{P} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$  |

TABLE 6.4.1.4: Finding an Invertible matrix



6.4.2. Let  $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$  be  $n \times n$  complex matrices which commute. Let  $\mathbf{E}$  be the  $2n \times 2n$  matrix

$$\mathbf{E} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} \quad (6.4.2.1)$$

prove that  $\det(\mathbf{E}) = \det(\mathbf{AD} - \mathbf{BC})$

**Solution:** Given matrices  $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$  commute.

Let  $\mathbf{P}$  be an invertible matrix that can simultaneously diagonalize matrices  $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$  as below

$$\mathbf{A} = \mathbf{P}\mathbf{\Lambda}_a\mathbf{P}^{-1} \quad (6.4.2.2)$$

$$\mathbf{B} = \mathbf{P}\mathbf{\Lambda}_b\mathbf{P}^{-1} \quad (6.4.2.3)$$

$$\mathbf{C} = \mathbf{P}\mathbf{\Lambda}_c\mathbf{P}^{-1} \quad (6.4.2.4)$$

$$\mathbf{D} = \mathbf{P}\mathbf{\Lambda}_d\mathbf{P}^{-1} \quad (6.4.2.5)$$

where  $\mathbf{\Lambda}_a, \mathbf{\Lambda}_b, \mathbf{\Lambda}_c, \mathbf{\Lambda}_d$  are diagonal matrices whose diagonal values are eigenvalues of matrices  $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$  respectively and matrix  $\mathbf{P}$  is formed by  $n$ -linearly independent eigen vectors.

Now (6.4.2.1) can be written as

$$\mathbf{E} = \begin{pmatrix} \mathbf{P}\mathbf{\Lambda}_a\mathbf{P}^{-1} & \mathbf{P}\mathbf{\Lambda}_b\mathbf{P}^{-1} \\ \mathbf{P}\mathbf{\Lambda}_c\mathbf{P}^{-1} & \mathbf{P}\mathbf{\Lambda}_d\mathbf{P}^{-1} \end{pmatrix} \quad (6.4.2.6)$$

Using block matrix multiplication, we get

$$\Rightarrow \mathbf{E} = \begin{pmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{P} \end{pmatrix} \begin{pmatrix} \mathbf{\Lambda}_a & \mathbf{\Lambda}_b \\ \mathbf{\Lambda}_c & \mathbf{\Lambda}_d \end{pmatrix} \begin{pmatrix} \mathbf{P}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{P}^{-1} \end{pmatrix} \quad (6.4.2.7)$$

$$\Rightarrow \mathbf{E} = \mathbf{MDM}^{-1} \quad (6.4.2.8)$$

where

$$\mathbf{M} = \begin{pmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{P} \end{pmatrix} \quad (6.4.2.9)$$

$$\mathbf{D} = \begin{pmatrix} \mathbf{\Lambda}_a & \mathbf{\Lambda}_b \\ \mathbf{\Lambda}_c & \mathbf{\Lambda}_d \end{pmatrix} \quad (6.4.2.10)$$

Now we will calculate  $\det(\mathbf{E})$ ,

$$|\mathbf{E}| = |\mathbf{MDM}^{-1}| \quad (6.4.2.11)$$

$$\Rightarrow |\mathbf{E}| = |\mathbf{M}| |\mathbf{D}| |\mathbf{M}^{-1}| \quad (6.4.2.12)$$

$$\Rightarrow |\mathbf{E}| = |\mathbf{M}| |\mathbf{D}| |\mathbf{M}|^{-1} \quad (6.4.2.13)$$

$$\Rightarrow |\mathbf{E}| = |\mathbf{D}| \quad (6.4.2.14)$$

$$\Rightarrow |\mathbf{E}| = \begin{vmatrix} \mathbf{\Lambda}_a & \mathbf{\Lambda}_b \\ \mathbf{\Lambda}_c & \mathbf{\Lambda}_d \end{vmatrix} \quad (6.4.2.15)$$

$$= \begin{vmatrix} \lambda_{1a} & 0 & \dots & 0 & \lambda_{1b} & 0 & \dots & 0 \\ 0 & \lambda_{2a} & \dots & 0 & 0 & \lambda_{2b} & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \lambda_{na} & 0 & 0 & \dots & \lambda_{nb} \\ \lambda_{1c} & 0 & \dots & 0 & \lambda_{1d} & 0 & \dots & 0 \\ 0 & \lambda_{2c} & \dots & 0 & 0 & \lambda_{2d} & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \lambda_{nc} & 0 & 0 & \dots & \lambda_{nd} \end{vmatrix} \quad (6.4.2.16)$$

Using row reduction,

$$\begin{array}{c} R_{n+1} = R_{n+1} - \frac{\lambda_{1c}}{\lambda_{1a}} R_1 \\ \hline \end{array} \begin{vmatrix} \lambda_{1a} & 0 & \dots & 0 & \lambda_{1b} & 0 & \dots & 0 \\ 0 & \lambda_{2a} & \dots & 0 & 0 & \lambda_{2b} & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \lambda_{na} & 0 & 0 & \dots & \lambda_{nb} \\ 0 & 0 & \dots & 0 & \lambda_{1d} - \frac{\lambda_{1c}\lambda_{1b}}{\lambda_{1a}} & 0 & \dots & 0 \\ 0 & \lambda_{2c} & \dots & 0 & 0 & \lambda_{2d} & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \lambda_{nc} & 0 & 0 & \dots & \lambda_{nd} \end{vmatrix} \quad (6.4.2.17)$$

similarly doing elementary row operations for rows  $R_{n+2}$  to  $R_{2n}$ , we get

$$|\mathbf{E}| = \begin{vmatrix} \lambda_{1a} & \dots & 0 & \lambda_{1b} & \dots & 0 \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ 0 & \dots & \lambda_{na} & 0 & \dots & \lambda_{nb} \\ 0 & \dots & 0 & \lambda_{1d} - \frac{\lambda_{1c}\lambda_{1b}}{\lambda_{1a}} & \dots & 0 \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ 0 & \dots & 0 & 0 & \dots & \lambda_{nd} - \frac{\lambda_{nc}\lambda_{nb}}{\lambda_{na}} \end{vmatrix} \quad (6.4.2.18)$$

Since it is upper triangular matrix, then  $|\mathbf{E}|$  will

be multiplication of diagonal elements.

$$\begin{aligned} \Rightarrow |\mathbf{E}| &= \lambda_{1a}\lambda_{2a}\dots\lambda_{na} \\ &\times \left(\lambda_{1d} - \frac{\lambda_{1c}\lambda_{1b}}{\lambda_{1a}}\right) \dots \left(\lambda_{nd} - \frac{\lambda_{nc}\lambda_{nb}}{\lambda_{na}}\right) \end{aligned} \quad (6.4.2.19)$$

$$\begin{aligned} \Rightarrow |\mathbf{E}| &= (\lambda_{1a}\lambda_{1d} - \lambda_{1c}\lambda_{1b}) \\ &\times (\lambda_{2a}\lambda_{2d} - \lambda_{2c}\lambda_{2b}) \dots (\lambda_{na}\lambda_{nd} - \lambda_{nc}\lambda_{nb}) \end{aligned} \quad (6.4.2.20)$$

Now we will calculate  $\det(\mathbf{AD} - \mathbf{BC})$  by substituting (6.4.2.2), (6.4.2.3), (6.4.2.4), (6.4.2.5)

$$|\mathbf{AD} - \mathbf{BC}| = |\mathbf{P}\mathbf{\Lambda}_a\mathbf{P}^{-1}\mathbf{P}\mathbf{\Lambda}_d\mathbf{P}^{-1} - \mathbf{P}\mathbf{\Lambda}_b\mathbf{P}^{-1}\mathbf{P}\mathbf{\Lambda}_c\mathbf{P}^{-1}| \quad (6.4.2.21)$$

$$= |\mathbf{P}\mathbf{\Lambda}_a\mathbf{\Lambda}_d\mathbf{P}^{-1} - \mathbf{P}\mathbf{\Lambda}_b\mathbf{\Lambda}_c\mathbf{P}^{-1}| \quad (6.4.2.22)$$

$$= |\mathbf{P}(\mathbf{\Lambda}_a\mathbf{\Lambda}_d - \mathbf{\Lambda}_b\mathbf{\Lambda}_c)\mathbf{P}^{-1}| \quad (6.4.2.23)$$

$$= |\mathbf{P}| |\mathbf{\Lambda}_a\mathbf{\Lambda}_d - \mathbf{\Lambda}_b\mathbf{\Lambda}_c| |\mathbf{P}^{-1}| \quad (6.4.2.24)$$

$$= |\mathbf{P}| |\mathbf{P}|^{-1} |\mathbf{\Lambda}_a\mathbf{\Lambda}_d - \mathbf{\Lambda}_b\mathbf{\Lambda}_c| \quad (6.4.2.25)$$

$$|\mathbf{AD} - \mathbf{BC}| = |\mathbf{\Lambda}_a\mathbf{\Lambda}_d - \mathbf{\Lambda}_b\mathbf{\Lambda}_c| \quad (6.4.2.26)$$

Since  $\mathbf{\Lambda}_a, \mathbf{\Lambda}_b, \mathbf{\Lambda}_c, \mathbf{\Lambda}_d$  are diagonal matrices, we get

$$\mathbf{\Lambda}_a\mathbf{\Lambda}_d = \begin{pmatrix} \lambda_{1a}\lambda_{1d} & 0 & \dots & 0 \\ 0 & \lambda_{2a}\lambda_{2d} & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \lambda_{na}\lambda_{nd} \end{pmatrix} \quad (6.4.2.27)$$

$$\mathbf{\Lambda}_b\mathbf{\Lambda}_c = \begin{pmatrix} \lambda_{1b}\lambda_{1c} & 0 & \dots & 0 \\ 0 & \lambda_{2b}\lambda_{2c} & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \lambda_{nb}\lambda_{nc} \end{pmatrix} \quad (6.4.2.28)$$

$$(6.4.2.29)$$

Substitute (6.4.2.27) and (6.4.2.28) in

(6.4.2.26), we get

$$|\mathbf{AD} - \mathbf{BC}| = \begin{vmatrix} \lambda_{1a}\lambda_{1d} - \lambda_{1b}\lambda_{1c} & \dots & 0 \\ 0 & \dots & 0 \\ \vdots & \dots & \vdots \\ 0 & \dots & \lambda_{na}\lambda_{nd} - \lambda_{nb}\lambda_{nc} \end{vmatrix} \quad (6.4.2.30)$$

$$\begin{aligned} \Rightarrow |\mathbf{AD} - \mathbf{BC}| &= (\lambda_{1a}\lambda_{1d} - \lambda_{1b}\lambda_{1c}) \\ &\times (\lambda_{2a}\lambda_{2d} - \lambda_{2b}\lambda_{2c}) \dots (\lambda_{na}\lambda_{nd} - \lambda_{nb}\lambda_{nc}) \end{aligned} \quad (6.4.2.31)$$

Comparing (6.4.2.20) and (6.4.2.31) we can say that

$$|\mathbf{E}| = |\mathbf{AD} - \mathbf{BC}| \quad (6.4.2.32)$$

Hence proved.

*Example:* Let us consider below matrices  $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$  be  $n \times n$  complex matrices which commute.

$$\mathbf{A} = \begin{pmatrix} 1-i & -2 \\ 3 & -1-i \end{pmatrix} \quad (6.4.2.33)$$

$$\mathbf{B} = \begin{pmatrix} 3-3i & -6 \\ 9 & -3-3i \end{pmatrix} \quad (6.4.2.34)$$

$$\mathbf{C} = \begin{pmatrix} 7-7i & -14 \\ 21 & -7-7i \end{pmatrix} \quad (6.4.2.35)$$

$$\mathbf{D} = \begin{pmatrix} -2+2i & 4 \\ -6 & 2+2i \end{pmatrix} \quad (6.4.2.36)$$

Lets find eigenvalues of matrix  $\mathbf{A}$

$$|\mathbf{A} - \lambda\mathbf{I}| = 0 \quad (6.4.2.37)$$

$$\Rightarrow \begin{vmatrix} 1-i-\lambda & -2 \\ 3 & -1-i-\lambda \end{vmatrix} = 0 \quad (6.4.2.38)$$

$$\Rightarrow \lambda^2 + 2\lambda i + 4 = 0 \quad (6.4.2.39)$$

$$\Rightarrow \lambda_{1a} = -(1 + \sqrt{5})i \quad \lambda_{2a} = (-1 + \sqrt{5})i \quad (6.4.2.40)$$

(6.4.2.40) are eigenvalues of matrix  $\mathbf{A}$ .

The eigenvectors of matrix  $\mathbf{A}$  are

$$\mathbf{v}_1 = \begin{pmatrix} \frac{-1+\sqrt{5}i}{3} \\ 1 \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} \frac{-1-\sqrt{5}i}{3} \\ 1 \end{pmatrix} \quad (6.4.2.41)$$

Since matrices  $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$  commute, from (6.4.2.33), (6.4.2.34), (6.4.2.35), (6.4.2.36) we

can say that

$$\mathbf{B} = 3\mathbf{A} \quad (6.4.2.42)$$

$$\mathbf{C} = 7\mathbf{A} \quad (6.4.2.43)$$

$$\mathbf{D} = -2\mathbf{A} \quad (6.4.2.44)$$

Then the eigenvalues of matrices  $\mathbf{B}, \mathbf{C}, \mathbf{D}$  are

$$\lambda_{1b} = -3(1 + \sqrt{5})i \quad \lambda_{2b} = 3(-1 + \sqrt{5})i \quad (6.4.2.45)$$

$$\lambda_{1c} = -7(1 + \sqrt{5})i \quad \lambda_{2c} = 7(-1 + \sqrt{5})i \quad (6.4.2.46)$$

$$\lambda_{1d} = 2(1 + \sqrt{5} + 1)i \quad \lambda_{2d} = -2(-1 + \sqrt{5})i \quad (6.4.2.47)$$

But the eigenvectors of matrices  $\mathbf{B}, \mathbf{C}, \mathbf{D}$  are same as of matrix  $\mathbf{A}$ .

The eigenvalue decomposition of matrices  $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$  is done as in (6.4.2.2), (6.4.2.3), (6.4.2.4), (6.4.2.5). Here eigenvector matrix  $\mathbf{P}$  and  $\Lambda_a, \Lambda_b, \Lambda_c, \Lambda_d$  are

$$\mathbf{P} = \begin{pmatrix} \frac{-1+\sqrt{5}i}{3} & \frac{-1-\sqrt{5}i}{3} \\ 1 & 1 \end{pmatrix} \quad (6.4.2.48)$$

$$\Lambda_a = \begin{pmatrix} mi & 0 \\ 0 & ni \end{pmatrix} \quad (6.4.2.49)$$

$$\Lambda_b = \begin{pmatrix} 3mi & 0 \\ 0 & 3ni \end{pmatrix} \quad (6.4.2.50)$$

$$\Lambda_c = \begin{pmatrix} 7mi & 0 \\ 0 & 7ni \end{pmatrix} \quad (6.4.2.51)$$

$$\Lambda_d = \begin{pmatrix} -2mi & 0 \\ 0 & -2ni \end{pmatrix} \quad (6.4.2.52)$$

where

$$m = -(1 + \sqrt{5}) \quad n = (-1 + \sqrt{5}) \quad (6.4.2.53)$$

From (6.4.2.8), we got

$$\mathbf{E} = \mathbf{MDM}^{-1} \quad (6.4.2.54)$$

where

$$\mathbf{M} = \begin{pmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{P} \end{pmatrix} \quad (6.4.2.55)$$

$$= \begin{pmatrix} \frac{-1+\sqrt{5}i}{3} & \frac{-1-\sqrt{5}i}{3} & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & \frac{-1+\sqrt{5}i}{3} & \frac{-1-\sqrt{5}i}{3} \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad (6.4.2.56)$$

$$\mathbf{D} = \begin{pmatrix} \Lambda_a & \Lambda_b \\ \Lambda_c & \Lambda_d \end{pmatrix} = \begin{pmatrix} mi & 0 & 3mi & 0 \\ 0 & ni & 0 & 3ni \\ 7mi & 0 & -2mi & 0 \\ 0 & 7ni & 0 & -2ni \end{pmatrix} \quad (6.4.2.57)$$

Now we will calculate  $\det(\mathbf{E})$ , from (6.4.2.14) we got

$$|\mathbf{E}| = |\mathbf{D}| \quad (6.4.2.58)$$

$$= \begin{vmatrix} mi & 0 & 3mi & 0 \\ 0 & ni & 0 & 3ni \\ 7mi & 0 & -2mi & 0 \\ 0 & 7ni & 0 & -2ni \end{vmatrix} \quad (6.4.2.59)$$

Using row reduction technique,

$$\begin{vmatrix} mi & 0 & 3mi & 0 \\ 0 & ni & 0 & 3ni \\ 7mi & 0 & -2mi & 0 \\ 0 & 7ni & 0 & -2ni \end{vmatrix} \quad (6.4.2.60)$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - 7R_1} \begin{vmatrix} mi & 0 & 3mi & 0 \\ 0 & ni & 0 & 3ni \\ 0 & 0 & -23mi & 0 \\ 0 & 7ni & 0 & -2ni \end{vmatrix} \quad (6.4.2.61)$$

$$\xleftrightarrow{R_4 \leftarrow R_4 - 7R_2} \begin{vmatrix} mi & 0 & 3mi & 0 \\ 0 & ni & 0 & 3ni \\ 0 & 0 & -23mi & 0 \\ 0 & 0 & 0 & -23ni \end{vmatrix} \quad (6.4.2.62)$$

Now

$$|\mathbf{E}| = \begin{vmatrix} mi & 0 & 3mi & 0 \\ 0 & ni & 0 & 3ni \\ 0 & 0 & -23mi & 0 \\ 0 & 0 & 0 & -23ni \end{vmatrix} \quad (6.4.2.63)$$

$$\Rightarrow |\mathbf{E}| = 529m^2n^2 \quad (6.4.2.64)$$

Substitute (6.4.2.53) in (6.4.2.64), we get

$$|\mathbf{E}| = 529(1 + \sqrt{5})^2(-1 + \sqrt{5})^2 = 8464 \quad (6.4.2.65)$$

Now we will calculate  $\det(\mathbf{AD} - \mathbf{BC})$ , from (6.4.2.26) we got

$$|\mathbf{AD} - \mathbf{BC}| = |\Lambda_a \Lambda_d - \Lambda_b \Lambda_c| \quad (6.4.2.66)$$

Using (6.4.2.49), (6.4.2.50), (6.4.2.51),

(6.4.2.52) we get

$$\Lambda_a \Lambda_d = \begin{pmatrix} mi & 0 \\ 0 & ni \end{pmatrix} \begin{pmatrix} -2mi & 0 \\ 0 & -2ni \end{pmatrix} \quad (6.4.2.67)$$

$$\Rightarrow \Lambda_a \Lambda_d = \begin{pmatrix} 2m^2 & 0 \\ 0 & 2n^2 \end{pmatrix} \quad (6.4.2.68)$$

$$\Lambda_b \Lambda_c = \begin{pmatrix} 3mi & 0 \\ 0 & 3ni \end{pmatrix} \begin{pmatrix} 7mi & 0 \\ 0 & 7ni \end{pmatrix} \quad (6.4.2.69)$$

$$\Rightarrow \Lambda_b \Lambda_c = \begin{pmatrix} -21m^2 & 0 \\ 0 & -21n^2 \end{pmatrix} \quad (6.4.2.70)$$

Substitute (6.4.2.68) and (6.4.2.70) in (6.4.2.66), we get

$$|\mathbf{AD} - \mathbf{BC}| = \left| \begin{pmatrix} 2m^2 & 0 \\ 0 & 2n^2 \end{pmatrix} - \begin{pmatrix} -21m^2 & 0 \\ 0 & -21n^2 \end{pmatrix} \right| \quad (6.4.2.71)$$

$$= \begin{vmatrix} 23m^2 & 0 \\ 0 & 23n^2 \end{vmatrix} \quad (6.4.2.72)$$

$$= 529m^2n^2 \quad (6.4.2.73)$$

Substitute (6.4.2.53) in (6.4.2.66), we get

$$|\mathbf{AD} - \mathbf{BC}| = 529(1 + \sqrt{5})^2(-1 + \sqrt{5})^2 = 8464 \quad (6.4.2.74)$$

Comparing (6.4.2.65) and (6.4.2.74), we get

$$|\mathbf{E}| = |\mathbf{AD} - \mathbf{BC}| \quad (6.4.2.75)$$

## 6.5 Direct Sum Decomposition

6.5.1. Let  $\mathbb{V}$  be a finite-dimensional vector space and let  $\mathbb{W}_1$  be any subspace of  $\mathbb{V}$ . Prove that there is a subspace  $\mathbb{W}_2$  of  $\mathbb{V}$  such that  $\mathbb{V} = \mathbb{W}_1 \oplus \mathbb{W}_2$

**Solution:** See Table 6.5.1.1

|   |  |
|---|--|
| Assumption and Claim                                | <p>Let <math>\beta = \{\mathbf{u}_1, \dots, \mathbf{u}_n\}</math> be a basis for <math>\mathbb{W}_1</math>. Since <math>\mathbb{W}_1</math> is the subspace of <math>\mathbb{V}</math>.</p> <p>therefore let us take <math>\alpha = \{\mathbf{u}_1, \dots, \mathbf{u}_n, \mathbf{u}_{n+1}, \dots, \mathbf{u}_m\}</math> the basis of <math>\mathbb{V}</math></p> <p>So <math>\mathbb{W}_2 = \text{span}(\{\mathbf{u}_{n+1}, \dots, \mathbf{u}_m\})</math></p> <p>Claim that <math>\mathbb{V} = \mathbb{W}_1 \oplus \mathbb{W}_2</math>.</p>  |
| Proof of $\mathbb{V} = \mathbb{W}_1 + \mathbb{W}_2$ | <p>if <math>\mathbf{v} \in \mathbb{V}</math>, then</p> <p><math>\mathbf{v} = \sum_{i=1}^m a_i \mathbf{u}_i = \sum_{i=1}^n a_i \mathbf{u}_i + \sum_{i=n+1}^m a_i \mathbf{u}_i \in \mathbb{W}_1 + \mathbb{W}_2</math> for scalar <math>a_i</math>, <math>i = 1, \dots, m</math></p> <p>This implies that <math>\mathbb{V} \subseteq \mathbb{W}_1 + \mathbb{W}_2</math> But by the definition of <math>\mathbb{W}_1 + \mathbb{W}_2</math></p> <p>we know that <math>\mathbb{W}_1 + \mathbb{W}_2 \subseteq \mathbb{V}</math>.</p> <p>Hence <math>\mathbb{V} = \mathbb{W}_1 + \mathbb{W}_2</math></p>                                 |
| Proof of $\mathbb{W}_1 \cap \mathbb{W}_2 = \{0\}$   | <p>Let <math>\mathbf{u} \in \mathbb{W}_1 \cap \mathbb{W}_2</math></p> <p>Then <math>\mathbf{u} = \sum_{i=1}^n b_i \mathbf{u}_i = \sum_{i=n+1}^m c_i \mathbf{u}_i</math> for some scalar <math>b_1, \dots, b_n, c_{n+1}, \dots, c_m</math></p> <p><math>\implies \sum_{i=1}^n b_i \mathbf{u}_i + \sum_{i=n+1}^m (-c_i) \mathbf{u}_i = 0</math></p> <p>But <math>\alpha</math> is linearly independent, since <math>\alpha</math> is a basis.</p> <p>Hence <math>b_1 = \dots = b_n = c_{n+1} \dots = c_m = 0</math>. This implies <math>\mathbf{u} = 0</math>.</p> <p>Thus <math>\mathbb{W}_1 \cap \mathbb{W}_2 = \{0\}</math></p> |
| Combining Both the proof                            | <p><math>\mathbb{V} = \mathbb{W}_1 + \mathbb{W}_2</math></p> <p><math>\mathbb{W}_1 \cap \mathbb{W}_2 = \{0\}</math></p>  |
|   | <p>From the above two conditions we can say that <math>\mathbb{V}</math> is the direct sum of subspaces <math>\mathbb{W}_1</math> and <math>\mathbb{W}_2</math>. Hence it is represented as</p> <p><math>\mathbb{V} = \mathbb{W}_1 \oplus \mathbb{W}_2</math></p> <p>Hence Proved.</p>   |

TABLE 6.5.1.1: Solution Table

6.5.2. Find a projection  $\mathbf{E}$  which projects  $\mathbb{R}^2$  onto the subspace spanned by  $(1, -1)$  along the subspace spanned by  $(1, 2)$ .

**Solution:** See Table 6.5.2.1

|                                   |   |
|-----------------------------------|---|
| Given                             | <p>Let <math>\mathbf{x} \in \mathbb{R}^2</math></p> $\mathbf{x} = a \begin{pmatrix} 1 \\ -1 \end{pmatrix} + b \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (6.5.2.1)$ <p>where <math>\begin{pmatrix} a \\ b \end{pmatrix}</math> is representation of <math>\mathbf{x}</math> in new basis.</p>   |
| To find                           | $\mathbf{E}(\mathbf{x}) = a \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (6.5.2.2)$  |
| Finding a Projection $\mathbf{E}$ | <p>As, <math>\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}</math> are linearly independent.</p> <p>Therefore, <math>\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\}</math> is a basis of <math>\mathbb{R}^2</math></p> <p>As <math>\mathbf{x} \in \mathbb{R}^2</math></p> $\Rightarrow \mathbf{x} = a \begin{pmatrix} 1 \\ -1 \end{pmatrix} + b \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (6.5.2.3)$   |
|                                   | $\Rightarrow \mathbf{x} = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \quad (6.5.2.4)$ $\Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}^{-1} \mathbf{x} \quad (6.5.2.5)$ $\Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & \frac{-1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix} \mathbf{x} \quad (6.5.2.6)$ <p>Therefore,</p> $a = \left( \frac{2}{3} \quad \frac{-1}{3} \right) \mathbf{x} \quad (6.5.2.7)$ <p>Projection of <math>\mathbf{x}</math> on subspace spanned by <math>\begin{pmatrix} 1 \\ -1 \end{pmatrix}</math>;</p> |

|              |   |
|--------------|---|
|              | $\mathbf{E}(\mathbf{x}) = a \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (6.5.2.8)$ <p>Substituting (6.5.2.7) in (6.5.2.8)</p> $\mathbf{E}(\mathbf{x}) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} \frac{2}{3} & \frac{-1}{3} \end{pmatrix} \mathbf{x} \quad (6.5.2.9)$ $\mathbf{E}(\mathbf{x}) = \begin{pmatrix} \frac{2}{3} & \frac{-1}{3} \\ \frac{-2}{3} & \frac{1}{3} \end{pmatrix} \mathbf{x} \quad (6.5.2.10)$ $\Rightarrow \mathbf{E} = \begin{pmatrix} \frac{2}{3} & \frac{-1}{3} \\ \frac{-2}{3} & \frac{1}{3} \end{pmatrix} \quad (6.5.2.11)$ |
| Verification | <p>If <math>n \times n</math> matrix <math>\mathbf{E}</math> is projection matrix, then<br/> <math>\mathbf{E}^2 = \mathbf{E}</math></p> $\mathbf{E}^2 = \begin{pmatrix} \frac{2}{3} & \frac{-1}{3} \\ \frac{-2}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} \frac{2}{3} & \frac{-1}{3} \\ \frac{-2}{3} & \frac{1}{3} \end{pmatrix} \Rightarrow \mathbf{E} = \mathbf{E}^2 = \begin{pmatrix} \frac{2}{3} & \frac{-1}{3} \\ \frac{-2}{3} & \frac{1}{3} \end{pmatrix} \quad (6.5.2.12)$ <p>Hence, Verified.</p>  |

TABLE 6.5.2.1: Finding Projection Matrix



6.5.3. If  $E_1$  and  $E_2$  are the projections of independent subspaces, then  $E_1 + E_2$  is a projection. True or False?

**Solution:** See Table 6.5.3.1

|            |  |
|------------|--|
| Definition | If $V$ is a vector space, a projection of $V$ is a linear operator $E$ on $V$ such that $E^2 = E$  |
| Proof      | <p>Assume projection matrices in <math>R^{2 \times 2}</math></p> <p>Let <math>E_1 = \begin{pmatrix} 1 &amp; 1 \\ 0 &amp; 0 \end{pmatrix}</math></p> <p>Let <math>E_2 = \begin{pmatrix} 0 &amp; 0 \\ 0 &amp; 1 \end{pmatrix}</math></p> <p>Now, <math>E_1</math> is a projection of <math>R^2</math> onto its first coordinate<br/> And <math>E_2</math> is a projection of <math>R^2</math> onto its second coordinate</p> <p><math>E = E_1 + E_2 = \begin{pmatrix} 1 &amp; 1 \\ 0 &amp; 1 \end{pmatrix}</math></p> <p><math>E^2 = \begin{pmatrix} 1 &amp; 2 \\ 0 &amp; 1 \end{pmatrix}</math></p> <p><math>E \neq E^2</math></p> <p>Here, <math>E_1 + E_2</math> is not equal to its square and hence its not a projection.</p> |
| Conclusion | With the help of above proof we can say that the given statement is false.   |

TABLE 6.5.3.1: Illustration of Proof

6.5.4. If  $\mathbf{E}$  is the projection and  $f$  is a polynomial, then  $f(\mathbf{E}) = a\mathbf{I} + \mathbf{bE}$ . What are  $a$  and  $\mathbf{b}$  in terms of the coefficient vector of  $f$ ?

**Solution:** Given ,

$\mathbf{E}$  is the projection

$f$  is a polynomial

$$f(\mathbf{E}) = a\mathbf{I} + \mathbf{bE} \quad (6.5.4.1)$$

Let ,

$$f(x) = c_0 + c_1x + c_2x^2 + \cdots + c_nx^n \quad (6.5.4.2)$$

Then,  $f(\mathbf{E})$  can be written as,

$$f(\mathbf{E}) = c_0\mathbf{I} + c_1\mathbf{E} + c_2\mathbf{E}^2 + \cdots + c_n\mathbf{E}^n \quad (6.5.4.3)$$

Since  $\mathbf{E}$  is the projection,

$$\mathbf{E}^2 = \mathbf{E} \quad (6.5.4.4)$$

$$\mathbf{E}^k = \mathbf{E} \quad \text{for any } k > 1 \quad (6.5.4.5)$$

Using equations (6.5.4.4) and (6.5.4.5), equation (6.5.4.3) can be modified as ,

$$f(\mathbf{E}) = c_0\mathbf{I} + c_1\mathbf{E} + c_2\mathbf{E} + \cdots + c_n\mathbf{E} \quad (6.5.4.6)$$

$$f(\mathbf{E}) = c_0\mathbf{I} + \mathbf{C}^T\mathbf{E} \quad (6.5.4.7)$$

where,  $\mathbf{C}$  is the column vector and it is given as follows.

$$\mathbf{C} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \quad (6.5.4.8)$$

Comparing the equations (6.5.4.1) and (6.5.4.7) we get,

$$a = c_0 \quad (6.5.4.9)$$

$$\mathbf{b} = \mathbf{C}^T = (c_1 \ c_2 \ \dots \ c_n) \quad (6.5.4.10)$$

Here,  $a$  is the constant term of  $f$  and  $\mathbf{b}$  is the coefficient vector of projection  $\mathbf{E}$  which contains the all co-efficients of polynomial other than constant term  $c_0$

6.5.5. True or False? If a diagonalizable operator has only the characteristic values 0 and 1, it is a projection.

**Solution:** See Tables 6.5.5.1 and 6.5.5.2

|                          |  |
|--------------------------|--|
| Diagonalizable Operator  | <p>For a linear operator <math>\mathbf{T}: \mathbf{V} \longrightarrow \mathbf{V}</math>, <math>\mathbf{T}</math> is a diagonalizable operator if <math>\exists</math> some basis for <math>\mathbf{V}</math> such that the matrix representing <math>\mathbf{T}</math> is a diagonal matrix i.e.</p> $\mathbf{T}(\mathbf{X}) = \mathbf{A}\mathbf{X},$ $\implies \mathbf{A} \text{ is a diagonalizable matrix}$ |
| Properties of Projection | <p>If <math>n \times n</math> matrix <math>\mathbf{A}</math> is projection matrix, then</p> $\mathbf{A}^2 = \mathbf{A}$  |

TABLE 6.5.5.1: Definitions

|                   |  |
|-------------------|--|
| Diagonalizability | <p>Let <math>\mathbf{A}</math> be <math>n \times n</math> matrix.<br/> Given that <math>\mathbf{A}</math> is diagonalizable, it can be expressed as,</p> $\mathbf{A} = \mathbf{P}\mathbf{\Lambda}\mathbf{P}^{-1}$ $\Rightarrow \mathbf{A}\mathbf{P} = \mathbf{P}\mathbf{\Lambda} \quad \dots(1)$ <p>where, <math>\mathbf{\Lambda} = \begin{pmatrix} \lambda_1 &amp; 0 &amp; \dots &amp; 0 \\ 0 &amp; \lambda_2 &amp; \dots &amp; 0 \\ \vdots &amp; \vdots &amp; \dots &amp; \vdots \\ 0 &amp; 0 &amp; \dots &amp; \lambda_n \end{pmatrix}</math></p>   |
| Eigen values      | <p>Given that <math>\mathbf{A}</math> has eigen values 0 and 1<br/> <math>\Rightarrow \mathbf{\Lambda}</math> has diagonal entries of 0s and 1s only</p> $\Rightarrow \lambda_i = 0 \text{ or } 1, \quad i = 0, 1, \dots, n$ $\Rightarrow \lambda_i^2 = 0 \text{ or } 1$ $\Rightarrow \lambda_i^2 = \lambda_i$ $\Rightarrow \mathbf{\Lambda}^2 = \mathbf{\Lambda}\mathbf{\Lambda} = \begin{pmatrix} \lambda_1^2 & 0 & \dots & 0 \\ 0 & \lambda_2^2 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \lambda_n^2 \end{pmatrix} = \mathbf{\Lambda} \quad \dots(2)$                             |
| Projection        | $\mathbf{A} = \mathbf{P}\mathbf{\Lambda}\mathbf{P}^{-1}$ $\Rightarrow \mathbf{A}\mathbf{A} = \mathbf{A}\mathbf{P}\mathbf{\Lambda}\mathbf{P}^{-1}$ $\Rightarrow \mathbf{A}^2 = (\mathbf{A}\mathbf{P})\mathbf{\Lambda}\mathbf{P}^{-1}$ <p>From (1),</p> $\Rightarrow \mathbf{A}^2 = \mathbf{P}\mathbf{\Lambda}\mathbf{\Lambda}\mathbf{P}^{-1}$ $= \mathbf{P}\mathbf{\Lambda}^2\mathbf{P}^{-1}$ <p>From (2),</p> $\Rightarrow \mathbf{A}^2 = \mathbf{P}\mathbf{\Lambda}\mathbf{P}^{-1}$ $= \mathbf{A}$ <p>Therefore,</p> $\mathbf{A}^2 = \mathbf{A}$ <p>Hence, <math>\mathbf{A}</math> is a projection matrix</p> |
| Example           | <p>Consider a <math>2 \times 2</math> matrix</p> $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ <p>Characteristic polynomial,</p>   |

|            |   |
|------------|---|
|            | $  \begin{aligned}  p(x) &=  x\mathbf{I} - \mathbf{A}  \\  &= \begin{vmatrix} x & -1 \\ 0 & x-1 \end{vmatrix} \\  &= x(x-1)  \end{aligned}  $ $\implies \lambda_1 = 0, \lambda_2 = 1$ <p>Also, <math>\mathbf{A}</math> is diagonalizable,<br/> <math>\mathbf{A} = \mathbf{PJP}^{-1}</math><br/> where<br/> <math display="block">\mathbf{P} = \begin{pmatrix} 1 &amp; 1 \\ 0 &amp; 1 \end{pmatrix}, \quad \mathbf{J} = \begin{pmatrix} 0 &amp; 0 \\ 0 &amp; 1 \end{pmatrix}</math> </p> <p>Now we check if <math>\mathbf{A}</math> is projection matrix.</p> $  \begin{aligned}  \mathbf{A}^2 &= \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \\  &= \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \\  &= \mathbf{A}  \end{aligned}  $ <p>Therefore,<br/> <math display="block">\mathbf{A}^2 = \mathbf{A}</math> </p> <p>Hence, if <math>\mathbf{A}</math> is diagonalizable and has eigen values 0 and 1, then <math>\mathbf{A}</math> is a projection matrix.</p> |
| Conclusion | Given statement is True   |

TABLE 6.5.5.2: Checking for projection matrix

6.5.6. Prove that if  $\mathbf{E}$  is the projection on  $\mathbf{R}$  along  $\mathbf{N}$ , then  $(\mathbf{I} - \mathbf{E})$  is the projection on  $\mathbf{N}$  along  $\mathbf{R}$

**Solution:**

**Theorem 6.5.** If  $\mathbf{V}$  is a vector space, a projection of  $\mathbf{V}$  is a linear operator  $\mathbf{E}$  on  $\mathbf{V}$  such that  $\mathbf{E}^2 = \mathbf{E}$ . Let  $\mathbf{R}$  be the range and let  $\mathbf{N}$  be the nullspace of  $\mathbf{E}$ . Then the vector space  $\mathbf{V}$  can be written as  $\mathbf{V} = \mathbf{R} \oplus \mathbf{N}$ . This operator is called as projection on  $\mathbf{R}$  along  $\mathbf{N}$ .

It is given that  $\mathbf{E}$  is the projection. From theorem 6.5, the linear operator  $\mathbf{E}$  will satisfy  $\mathbf{E}^2 = \mathbf{E}$ . Let's check whether  $\mathbf{I} - \mathbf{E}$  is also a projection.

$$\begin{aligned} (\mathbf{I} - \mathbf{E})^2 &= \mathbf{I}^2 + \mathbf{E}^2 - 2\mathbf{I}\mathbf{E} \\ &= \mathbf{I} + \mathbf{E} - 2\mathbf{E} \\ &= (\mathbf{I} - \mathbf{E}) \end{aligned} \quad (6.5.6.1)$$

From (6.5.6.1), we can say that  $(\mathbf{I} - \mathbf{E})$  is also a projector. But  $(\mathbf{I} - \mathbf{E})$  is called as the "Complementary Projector", i.e.

$$\text{range}(\mathbf{I} - \mathbf{E}) = \text{null}(\mathbf{E}) \quad (6.5.6.2)$$

$$\text{null}(\mathbf{I} - \mathbf{E}) = \text{range}(\mathbf{E}) \quad (6.5.6.3)$$

Lets take a vector  $\mathbf{v}$  such that  $\mathbf{E}\mathbf{v} = 0$ , where  $\mathbf{v}$  is in the null space of  $\mathbf{E}$ . Then,

$$\begin{aligned} (\mathbf{I} - \mathbf{E})\mathbf{v} &= \mathbf{v} - \mathbf{v}\mathbf{E} \\ &= \mathbf{v} \end{aligned} \quad (6.5.6.4)$$

In other words, any  $\mathbf{v}$  in the nullspace of  $\mathbf{E}$  is also in the range of  $(\mathbf{I} - \mathbf{E})$ . We know that any  $\mathbf{x} \in \text{range}(\mathbf{I} - \mathbf{E})$  is characterized by

$$\begin{aligned} \mathbf{x} &= (\mathbf{I} - \mathbf{E})\mathbf{v} \quad , \text{ for some } \mathbf{v} \\ &= \mathbf{v} - \mathbf{E}\mathbf{v} \\ &= -(\mathbf{E}\mathbf{v} - \mathbf{v}) \end{aligned} \quad (6.5.6.5)$$

Now we need to check if  $\mathbf{x}$  is in the nullspace of  $\mathbf{E}$ . i.e.  $\mathbf{E}\mathbf{x} = 0$

$$\begin{aligned} \mathbf{E}(-(\mathbf{E}\mathbf{v} - \mathbf{v})) &= -(\mathbf{E}^2\mathbf{v} - \mathbf{E}\mathbf{v}) \\ &= -(\mathbf{E}\mathbf{v} - \mathbf{E}\mathbf{v}) \quad (\because \mathbf{E} \text{ is a projection}) \\ &= 0 \end{aligned} \quad (6.5.6.6)$$

Thus, if  $\mathbf{x} \in \text{range}(\mathbf{I} - \mathbf{E})$ , then  $\mathbf{x} \in \text{null}(\mathbf{E})$ .

Therefore, we can say that  $\text{null}(\mathbf{E}) = \text{range}(\mathbf{I} - \mathbf{E})$ .

We can use the same argument as above for proving (6.5.6.3), by taking  $\mathbf{E} = \mathbf{I} - (\mathbf{I} - \mathbf{E})$ .

$\therefore$  we can say that  $(\mathbf{I} - \mathbf{E})$  is the projection on  $\mathbf{N}$  along  $\mathbf{R}$ .

As an example, lets take the below matrix.

$$\mathbf{A} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{pmatrix} \quad (6.5.6.7)$$

We can check that the matrix in (6.5.6.7) satisfies the condition  $\mathbf{A}^2 = \mathbf{A}$ . Thus,  $\mathbf{A}$  is a projection matrix.

$$\begin{aligned} (\mathbf{I} - \mathbf{A}) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{pmatrix} \\ (\mathbf{I} - \mathbf{A}) &= \begin{pmatrix} \frac{1}{3} & \frac{-1}{3} & \frac{-1}{3} \\ \frac{-1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{-1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \end{aligned} \quad (6.5.6.8)$$

We can check that the matrix in (6.5.6.8) satisfies the condition  $(\mathbf{I} - \mathbf{A})^2 = (\mathbf{I} - \mathbf{A})$ . Thus,  $(\mathbf{I} - \mathbf{A})$  is a projection matrix.

Null space of  $\mathbf{A}$  is given by

$$\text{null}(\mathbf{A}) = b \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \quad ('b' \text{ is any real number}) \quad (6.5.6.9)$$

Range of  $(\mathbf{I} - \mathbf{A})$  is given by

$$\begin{aligned} \text{range}(\mathbf{I} - \mathbf{A}) &= a \begin{pmatrix} \frac{1}{3} \\ \frac{-1}{3} \\ \frac{-1}{3} \end{pmatrix} \quad ('a' \text{ is any real number}) \\ &= a \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \\ &= a \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \end{aligned} \quad (6.5.6.10)$$

$\therefore$  from (6.5.6.9) and (6.5.6.10), we can say that  $(\mathbf{I} - \mathbf{A})$  is a projection matrix on  $\mathbf{N}$  along  $\mathbf{R}$ .

6.5.7. Let  $E_1, \dots, E_k$  be linear operators on the space  $\mathbf{V}$  such that  $E_1 + \dots + E_k = \mathbf{I}$ .

a) Prove that if  $E_i E_j = 0$  for  $i \neq j$ , then  $E_i^2 = E_i$  for each  $i$ .

b) In the case  $k = 2$ , prove the converse of (a). That is, if  $E_1 + E_2 = \mathbf{I}$  and  $E_1^2 = E_1$ ,  $E_2^2 = E_2$ , then  $E_1 E_2 = 0$

**Solution:**

a) From the given,

$$E_1 + \dots + E_k = \mathbf{I} \quad \mathbf{I} + \mathbf{E} \quad (6.5.8.1)$$

$$(6.5.7.1)$$

$$E_i = \mathbf{I} - E_1 - \dots - E_{i-1} - E_{i+1} - \dots - E_k \quad (6.5.7.2)$$

$$E_i^2 = E_i(\mathbf{I} - E_1 - \dots - E_{i-1} - E_{i+1} - \dots - E_k) \quad (6.5.7.3)$$

$$E_i^2 = E_i - E_i E_1 - \dots - E_i E_{i-1} - E_i E_{i+1} - \dots - E_i E_k \quad (6.5.7.4)$$

$$\Rightarrow E_i^2 = E_i - \sum_{i \neq j} E_i E_j \quad (6.5.7.5)$$

substituting  $E_i E_j = 0$  for  $i \neq j$  in the above equation we get,

$$E_i^2 = E_i - 0 \quad (6.5.7.6)$$

$$\Rightarrow E_i^2 = E_i \quad (6.5.7.7)$$

Hence proved if  $E_i E_j = 0$  for  $i \neq j$ , then  $E_i^2 = E_i$

b) Using,

$$E_1 + E_2 = \mathbf{I} \quad (6.5.7.8)$$

Multiplying both sides by  $E_1$ ,

$$E_1(E_1 + E_2) = E_1 \quad (6.5.7.9)$$

$$\Rightarrow E_1^2 + E_1 E_2 = E_1 \quad (6.5.7.10)$$

Substituting  $E_1^2 = E_1$  in (6.5.7.8) we get,

$$\Rightarrow E_1 + E_1 E_2 = E_1 \quad (6.5.7.11)$$

$$\Rightarrow E_1 E_2 = 0 \quad (6.5.7.12)$$

Similarly multiplying on both sides of (6.5.7.8) and substituting  $E_2^2 = E_2$  we get,

$$\Rightarrow E_1 E_2 + E_2^2 = E_2 \quad (6.5.7.13)$$

$$\Rightarrow E_1 E_2 + E_2 = E_2 \quad (6.5.7.14)$$

$$\Rightarrow E_1 E_2 = 0 \quad (6.5.7.15)$$

Hence proved from (6.5.7.12) and (6.5.7.15) that if  $E_1 + E_2 = \mathbf{I}$  and  $E_1^2 = E_1$ ,  $E_2^2 = E_2$ , then  $E_1 E_2 = 0$

Prove that  $(\mathbf{I} + \mathbf{E})$  is invertible. Find  $(\mathbf{I} + \mathbf{E})^{-1}$ .  
**Solution:** we have  $\mathbf{E}$  and it is idempotent. And we know that the eigen value of idempotent matrix is either 0 or 1. When we add the identity matrix in this:

Then eigen value will be either 1 or 2. Hence  $(\mathbf{I} + \mathbf{E})$  is invertible. Since  $\mathbf{E}$  is an idempotent matrix, that is :

$$\mathbf{E}^2 = \mathbf{E} \quad (6.5.8.2)$$

Let,

$$\mathbf{A} = \mathbf{I} + \mathbf{E} \quad (6.5.8.3)$$

$$\Rightarrow \mathbf{E} = \mathbf{A} - \mathbf{I} \quad (6.5.8.4)$$

$$\Rightarrow \mathbf{E}^2 = (\mathbf{A} - \mathbf{I})(\mathbf{A} - \mathbf{I}) = \mathbf{A}^2 - 2\mathbf{A} + \mathbf{I}^2 \quad (6.5.8.5)$$

From 6.5.8.2,

$$\Rightarrow \mathbf{E} = \mathbf{A}^2 - 2\mathbf{A} + \mathbf{I} \quad (6.5.8.6)$$

Using (6.5.8.4) we have,

$$\Rightarrow \mathbf{A} - \mathbf{I} = \mathbf{A}^2 - 2\mathbf{A} + \mathbf{I} = \mathbf{A}^2 - 3\mathbf{A} + 2\mathbf{I} = 0 \quad (6.5.8.7)$$

$$\Rightarrow \mathbf{I} = \frac{3\mathbf{A} - \mathbf{A}^2}{2} \quad (6.5.8.8)$$

multiplying  $\mathbf{A}^{-1}$  both side,

$$\mathbf{A}^{-1} = \frac{3\mathbf{I} - \mathbf{A}}{2} = \frac{3\mathbf{I} - (\mathbf{I} + \mathbf{E})}{2} \quad (6.5.8.9)$$

$$\Rightarrow \mathbf{A}^{-1} = \frac{2\mathbf{I} - \mathbf{E}}{2} \quad (6.5.8.10)$$

Using (6.5.8.3), we have,

$$(\mathbf{I} + \mathbf{E})^{-1} = \mathbf{I} - \frac{1}{2}\mathbf{E} \quad (6.5.8.11)$$

Let consider a matrix  $\mathbf{E}$  as :

$$\mathbf{E} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (6.5.8.12)$$

$$\Rightarrow \mathbf{E}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (6.5.8.13)$$

$$\Rightarrow \mathbf{E}^2 = \mathbf{E} \quad (6.5.8.14)$$

6.5.8. Let  $\mathbf{V}$  be a real vector space and  $\mathbf{E}$  an idempotent linear operator on  $\mathbf{V}$ , i.e., a projection.



Now,

$$\mathbf{I} + \mathbf{E} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (6.5.8.15)$$

$$\Rightarrow \mathbf{I} + \mathbf{E} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \quad (6.5.8.16)$$

$$(6.5.8.17)$$

Now let find the eigen value of matrix  $(\mathbf{I} + \mathbf{E})$  :

$$\Rightarrow \begin{pmatrix} 2 - \lambda & 0 \\ 0 & 1 - \lambda \end{pmatrix} = 0 \quad (6.5.8.18)$$

$$\Rightarrow (2 - \lambda)(1 - \lambda) = 0 \quad (6.5.8.19)$$

$$\Rightarrow \lambda_1 = 2, \lambda_2 = 1 \quad (6.5.8.20)$$

The eigen values of the matrix  $(\mathbf{I} + \mathbf{E})$  from (6.5.8.20) are 2 and 1. Since none of the eigen value is zero, hence matrix is invertible.

Inverse of the matrix from (6.5.8.11) is :

$$(\mathbf{I} + \mathbf{E})^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} \quad (6.5.8.21)$$

## 6.6 Invariant Direct Sums

6.6.1. Let  $E$  be a projection of  $V$  and let  $T$  be a linear operator on  $V$ . Prove that the range of  $E$  is invariant under  $T$  if and only if  $ETE = TE$ . Prove that both the range and null space of  $E$  are invariant under  $T$  if and only if  $ET = TE$ .

**Solution:** See Table 6.6.1.1

|                     |  |
|---------------------|--|
| Proof of $ETE = TE$ | <p>Any projection <math>E</math> is represented by a matrix that is a part of an identity matrix<br/> Assume the Basis can be defined as follows:<br/> <math>B = \{\alpha_1, \dots, \alpha_r, \dots, \alpha_n\}</math><br/> such that <math>E_{ii} = 1</math> for <math>i \leq r</math> and 0 elsewhere<br/> <math>E_B = \begin{pmatrix} I &amp; 0 \\ 0 &amp; 0 \end{pmatrix}</math><br/> where <math>I</math> is the <math>r \times r</math> matrix<br/> Let <math>\alpha = (a_1, \dots, a_n)</math><br/> <math>\implies T(E\alpha) = T(a_1, \dots, a_r, \dots, 0) = \beta</math><br/> If we assume <math>T</math> to be invariant over the range <math>W</math> of <math>E</math>, then <math>\beta \in W</math><br/> <math>\beta = (\beta_1, \dots, \beta_r, \dots, 0)</math>, <math>E\beta = \beta</math><br/> Therefore, <math>ETE = TE</math></p>    |
| Proof of $ET = TE$  | <p>Consider the same assumption for basis <math>B</math> and projection <math>E</math> as defined above.<br/> <math>B = \{\alpha_1, \dots, \alpha_r, \dots, \alpha_n\}</math><br/> <math>E_B = \begin{pmatrix} I &amp; 0 \\ 0 &amp; 0 \end{pmatrix}</math><br/> where <math>I</math> is the <math>r \times r</math> matrix<br/> Let <math>TE \neq ET</math>, then there exists some vector <math>V</math> such that<br/> <math>T(a_1, \dots, a_r, \dots, 0) \neq (Ta_1, \dots, Ta_r, \dots, 0)</math><br/> But for this case, <math>T</math> is not an invariant of <math>W</math>.<br/> Assuming that <math>T</math> is an invariant of <math>W</math>, <math>T(a_1, \dots, a_r, \dots, 0) \in W</math> for all <math>\alpha \in W</math>.<br/> Therefore, <math>T(a_1, \dots, a_r, \dots, 0) = (Ta_1, \dots, Ta_r, \dots, 0) \implies ET = TE</math></p> |
| Conclusion          | <p>Hence, it is proved that the range of <math>E</math> is invariant under <math>T</math> if and only if <math>ETE = TE</math>.<br/> And both the range and null space of <math>E</math> are invariant under <math>T</math> if and only if <math>TE = ET</math>.</p>   |

TABLE 6.6.1.1: Illustration of Proof

6.6.2. Let  $T$  be the linear operator on  $R^2$ , the matrix of which in the standard ordered basis is

$$\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

Let  $W_1$  be the subspace of  $R^2$  spanned by the vector  $e_1 = (1, 0)$

- a) Prove that  $W_1$  is invariant under  $T$ .
- b) Prove that there is no subspace  $W_2$  which is invariant under  $T$  and is complementary to  $W_1$ :

$$R^2 = W_1 \oplus W_2$$

(Compare with exercise 1 of section 6.5.)

**Solution:** See Table 6.6.2.1

| Statement   | Solution   |
|-------------|--|
| Given       | <p><math>T</math> be the linear operator on <math>R^2</math> the matrix of which in the standard ordered basis is <math>\begin{pmatrix} 2 &amp; 1 \\ 0 &amp; 2 \end{pmatrix}</math></p> <p><math>W_1</math> be the subspace of <math>R^2</math> spanned by the vector <math>\epsilon_1 = (1, 0)</math>.</p>  |
| To Prove    | <p>a) <math>W_1</math> is invariant under <math>T</math>.</p> <p>b) There is no subspace <math>W_2</math> which is invariant under <math>T</math> and is complementary to <math>W_1</math>: <math>R^2 = W_1 \oplus W_2</math></p> <p>c) Compare with exercise 1 of section 6.5.</p>  |
| Proof (a)   | $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \quad (6.6.2.1)$ $ A - \lambda I  = 0 \quad (6.6.2.2)$ $\Rightarrow \begin{pmatrix} 2 - \lambda & 1 \\ 0 & 2 - \lambda \end{pmatrix} \quad (6.6.2.3)$ $= (2 - \lambda)^2 = 0 \quad (6.6.2.4)$ $\therefore \lambda = 2 \quad (6.6.2.5)$ <p>for <math>\lambda = 2</math>, the corresponding vector is</p> $(\mathbf{A} - \lambda I)X = 0 \quad (6.6.2.6)$ $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} X = 0 \quad (6.6.2.7)$ $\therefore X = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (6.6.2.8)$ <p>Hence, <math>W_1</math> be the subspace of <math>R^2</math> spanned by the vector <math>\epsilon_1 = (1, 0)</math> is invariant under <math>T</math>.</p> |
| Proof (b)   | <p>Corresponding to <math>\lambda = 2</math>,</p> <p>Among, two eigen vectors only one is independent and other one is dependent.</p> <p>Thus, <math>P^{-1}</math> does not exist and <math>A</math> can not be diagonalized.</p> <p>Hence, there is no subspace <math>W_2</math> which is invariant under <math>T</math> and is complementary to <math>W_1</math>:</p> <p><math>R^2 = W_1 \oplus W_2</math></p>   |
| Observation | <p>In exercise 1 of section 6.5, for <math>2 \times 2</math> matrix there is 2 distinct characteristic value, corresponding to which there is a eigen vector.</p> <p>Hence, <math>P^{-1}</math> exists.</p> <p><math>\therefore</math> the given matrix is diagonalizable.</p>   |

TABLE 6.6.2.1: Solution

6.6.3. Let  $\mathbf{T}$  be the diagonalizable linear operator on  $\mathbb{R}^3$  which we discussed in example 3 of section 6.2. Use the Lagrange polynomials to write the representing matrix  $\mathbf{A}$  in the form

$$\mathbf{A} = \mathbf{E}_1 + 2\mathbf{E}_2, \quad \mathbf{E}_1 + \mathbf{E}_2 = \mathbf{I}, \mathbf{E}_1\mathbf{E}_2 = \mathbf{0} \quad (6.6.3.1)$$

**Solution:** See Tables 6.6.3.2 and 6.6.3.2

|  |  |
|--|--|
| Diagonalizable Operator                              | <p>For a linear operator <math>\mathbf{T}: \mathbf{V} \longrightarrow \mathbf{V}</math>, <math>\mathbf{T}</math> is a diagonalizable operator if <math>\exists</math> some basis for <math>\mathbf{V}</math> such that the matrix representing <math>\mathbf{T}</math> is a diagonal matrix i.e.</p> $\mathbf{T}(\mathbf{X}) = \mathbf{A}\mathbf{X},$ $\implies \mathbf{A} \text{ is a diagonalizable matrix}$   |
| Characteristic Polynomial                            | <p>For an <math>n \times n</math> matrix <math>\mathbf{A}</math>, characteristic polynomial is defined by,</p> $p(x) =  x\mathbf{I} - \mathbf{A} $   |
| Minimal Polynomial                                   | <p>Minimal polynomial <math>m(x)</math> is the smallest factor of characteristic polynomial <math>p(x)</math> such that,</p> $m(\mathbf{A}) = 0$ <p>Every root of characteristic polynomial should be the root of minimal polynomial</p>   |
| Lagrange Polynomials                                 | <p>For a set of scalars <math>c_0, c_1, \dots, c_n \in \mathbb{F}</math>, Lagrange Polynomial is defined as:</p> $p_j = \prod_{i \neq j} \frac{(x - c_i)}{(c_j - c_i)}$  |
| Theorem  | <p>If <math>\mathbf{T}</math> is a diagonalizable linear operator on a finite dimensional space <math>\mathbf{V}</math>, and if <math>c_1, c_2, \dots, c_k</math> are distinct characteristic values of <math>\mathbf{T}</math>, then there exist linear operators <math>\mathbf{E}_1, \mathbf{E}_2, \dots, \mathbf{E}_k</math> such that:</p> <ol style="list-style-type: none"> <li>(1) <math>\mathbf{T} = c_1\mathbf{E}_1 + \dots + c_k\mathbf{E}_k</math></li> <li>(2) <math>\mathbf{E}_1 + \dots + \mathbf{E}_k = \mathbf{I}</math></li> <li>(3) <math>\mathbf{E}_i\mathbf{E}_j = \mathbf{0}, \quad i \neq j</math></li> <li>(4) <math>\mathbf{E}_i = \mathbf{E}_i^2, \quad (\mathbf{E}_i \text{ is a projection})</math></li> <li>(5) <math>\alpha = \mathbf{E}_i\alpha, \forall \alpha \in \mathbf{V}</math></li> </ol> |
| Relation between Lagrange Polynomials and Projection | <p>We have:</p> $\mathbf{T} = c_1\mathbf{E}_1 + \dots + c_k\mathbf{E}_k$ <p>If <math>g</math> is any polynomial over field <math>\mathbb{F}</math>,</p> $g(\mathbf{T}) = g(c_1)\mathbf{E}_1 + \dots + g(c_k)\mathbf{E}_k \quad \dots (1)$ <p>Now,</p> $p_j = \prod_{i \neq j} \frac{(x - c_i)}{(c_j - c_i)}$   |

|  |  |
|--|--|
|  | $\Rightarrow p_j(c_i) = \delta_{ij} \text{ (Kronecker Delta)} \quad \dots (2)$ <p>From (1) and (2),</p> $\Rightarrow p_j(\mathbf{T}) = \sum_{i=1}^k \delta_{ij} \mathbf{E}_i = \mathbf{E}_j$ $\Rightarrow \boxed{p_j(\mathbf{T}) = \mathbf{E}_j}$ <p><math>\Rightarrow</math> Projections <math>\mathbf{E}_j</math> are polynomials in <math>\mathbf{T}</math></p> |
|--|--|

TABLE 6.6.3.1: Definitions and results used

|                           |   |
|---------------------------|---|
| Given                     | <p>Matrix of <math>\mathbf{T}</math> in the standard basis of <math>\mathbb{R}^3</math> :</p> $\mathbf{A} = \begin{pmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{pmatrix}$   |
| Characteristic polynomial | $p(x) =  \mathbf{xI} - \mathbf{A} $ $= \begin{vmatrix} x & -1 & 0 \\ -2 & x+2 & -2 \\ -2 & 3 & x-2 \end{vmatrix}$ $= x^3 - 5x^2 + 8x - 4$ $= (x-1)(x-2)^2$ <p><math>\Rightarrow \lambda = 1, 2</math></p>   |
| Minimal Polynomial        | $p(x) = (x-1)(x-2)^b, \quad b \leq 2$ $(\mathbf{A} - \mathbf{I})(\mathbf{A} - 2\mathbf{I}) = \begin{pmatrix} 4 & -6 & -6 \\ -1 & 3 & 2 \\ 3 & -6 & -5 \end{pmatrix} \begin{pmatrix} 3 & -6 & -6 \\ -1 & 2 & 2 \\ 3 & -6 & -6 \end{pmatrix} = \mathbf{0}$ <p>Therefore, <math>(x-1)(x-2)</math> is the minimal polynomial.</p> |
| Lagrange Polynomial       | $p_j = \prod_{i \neq j} \frac{(x - c_i)}{(c_j - c_i)}$ <p>For characteristic values <math>c_1 = 1, \quad c_2 = 2</math>,</p> $\Rightarrow p_1 = \frac{(x-1)}{2-1}, \quad p_2 = \frac{(x-2)}{1-2}$ <p><math>\Rightarrow p_1 = (x-1)</math>, and<br/> <math>p_2 = (2-x)</math></p>  |

|                 |  |
|-----------------|--|
| Projection Maps | <p>We know that,</p> $\mathbf{E}_j = p_j(\mathbf{T})$ $\Rightarrow \mathbf{E}_1 = \mathbf{A} - \mathbf{I} \text{ and } \mathbf{E}_2 = 2\mathbf{I} - \mathbf{A}$ $\Rightarrow \mathbf{E}_1 = \begin{pmatrix} 4 & -6 & -6 \\ -1 & 3 & 2 \\ 3 & -6 & -5 \end{pmatrix}, \text{ and}$ $\mathbf{E}_2 = \begin{pmatrix} -3 & 6 & 6 \\ 1 & -2 & -2 \\ -3 & 6 & 6 \end{pmatrix}$  |
| Verification    | <p>We have,</p> $\mathbf{E}_1 = \mathbf{A} - \mathbf{I}$ $\Rightarrow \mathbf{A} - \mathbf{E}_1 = \mathbf{I} \quad \dots (1)$ $\mathbf{E}_2 = 2\mathbf{I} - \mathbf{A}$ <p>From (1),</p> $\Rightarrow \mathbf{E}_2 = 2(\mathbf{A} - \mathbf{E}_1) - \mathbf{A}$ $\Rightarrow \boxed{\mathbf{A} = 2\mathbf{E}_1 + \mathbf{E}_2}$ <p>Also,</p> $\mathbf{E}_1 = \begin{pmatrix} 4 & -6 & -6 \\ -1 & 3 & 2 \\ 3 & -6 & -5 \end{pmatrix}, \quad \mathbf{E}_2 = \begin{pmatrix} -3 & 6 & 6 \\ 1 & -2 & -2 \\ -3 & 6 & 6 \end{pmatrix}$ $\Rightarrow \mathbf{E}_1 + \mathbf{E}_2 = \begin{pmatrix} 4 & -6 & -6 \\ -1 & 3 & 2 \\ 3 & -6 & -5 \end{pmatrix} + \begin{pmatrix} -3 & 6 & 6 \\ 1 & -2 & -2 \\ -3 & 6 & 6 \end{pmatrix}$ $= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\Rightarrow \boxed{\mathbf{E}_1 + \mathbf{E}_2 = \mathbf{I}}$ $\mathbf{E}_1\mathbf{E}_2 = \begin{pmatrix} 4 & -6 & -6 \\ -1 & 3 & 2 \\ 3 & -6 & -5 \end{pmatrix} \begin{pmatrix} -3 & 6 & 6 \\ 1 & -2 & -2 \\ -3 & 6 & 6 \end{pmatrix}$ $= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\Rightarrow \boxed{\mathbf{E}_1\mathbf{E}_2 = \mathbf{0}}$ |

TABLE 6.6.3.2: Using Lagrange Polynomials to represent  $\mathbf{A}$



|                           |  |
|---------------------------|--|
| Characteristic Polynomial | For an $n \times n$ matrix $\mathbf{A}$ , characteristic polynomial is defined by,<br>$p(x) =  x\mathbf{I} - \mathbf{A} $  |
| Minimal Polynomial        | Minimal polynomial $m(x)$ is the smallest factor of characteristic polynomial $p(x)$ such that,<br>$m(\mathbf{A}) = 0$<br>Every root of characteristic polynomial should be the root of minimal polynomial and the minimal polynomial divides the characteristic polynomial. |
| Basis Theorem             | Let $V$ be a subspace of dimension $m$ . Then:<br>Any $m$ linearly independent vectors in $V$ forms a basis for $V$ .<br>Any $m$ vectors that span $V$ forms a basis for $V$ .   |

TABLE 6.7.1.1: Definitions

### 6.7 The Primary Decomposition Theorem

6.7.1. Let  $T$  be a linear operator on  $R^3$  which is represented in the standard ordered basis by the matrix

$$\begin{pmatrix} 6 & -3 & -2 \\ 4 & -1 & -2 \\ 10 & -5 & -3 \end{pmatrix} \quad (6.7.1.1)$$

Express the minimal polynomial  $p$  for  $T$  in the form  $p = p_1 p_2$ , where  $p_1$  and  $p_2$  are monic and irreducible over the field of real numbers. Let  $W_i$  be the null space of  $p_i(T)$ . Find the basis  $B_i$  for the spaces  $W_1$  and  $W_2$ . If  $T_i$  is the operator induced on  $W_i$  by  $T$ , find the matrix of  $T_i$  in the basis  $B_i$  above. **Solution:** See Tables 6.7.1.1 6.7.1.2 and 6.7.1.3

|                              |  |
|------------------------------|--|
| Express Minimal Polynomial   | $A = \begin{pmatrix} 6 & -3 & -2 \\ 4 & -1 & -2 \\ 10 & -5 & -3 \end{pmatrix}$ <p>Characteristic Polynomial = <math> xI - A  = \begin{vmatrix} x-6 &amp; 3 &amp; 2 \\ -4 &amp; x+1 &amp; 2 \\ -10 &amp; 5 &amp; x+3 \end{vmatrix}</math></p> <p>By solving above determinant, we find out that</p> $x^3 - 2x^2 + x - 2 = (x - 2)(x^2 + 1)$ <p>Since, <math>T - 2I \neq 0</math> and the minimal polynomial divides the characteristic polynomial, thus minimal polynomial <math>p</math> for <math>T</math> is <math>p = m(x)</math></p> $p = (x - 2)(x^2 + 1)$ <p>Put <math>p_1 = (x - 2)</math> and <math>p_2 = (x^2 + 1)</math></p> <p>Thus, <math>p = p_1 p_2</math></p>   |
| Bases $B_1$ and matrix $T_1$ | <p>Let <math>W_1 = \{\alpha \in R^3 / p_1(T)\alpha = 0, (T - 2I)\alpha = 0\}</math></p> <p>Therefore, <math>A - 2I = \begin{pmatrix} 4 &amp; -3 &amp; -2 \\ 4 &amp; -3 &amp; -2 \\ 10 &amp; -5 &amp; -5 \end{pmatrix} \rightarrow \begin{pmatrix} -4 &amp; 3 &amp; 2 \\ -2 &amp; 1 &amp; 1 \\ 0 &amp; 0 &amp; 0 \end{pmatrix} \rightarrow \begin{pmatrix} -2 &amp; 1 &amp; 1 \\ 0 &amp; 1 &amp; 0 \\ 0 &amp; 0 &amp; 0 \end{pmatrix}</math></p> <p>Rank of <math>A - 2I</math> is 2</p> <p>Nullity of <math>A - 2I</math> = no of columns - Rank = <math>3 - 2 = 1</math></p> <p>That means the dimension of <math>W_1</math> is 1</p> <p>Thus we can let, <math>\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \in W_1</math> (Basis theorem mentioned in Definitions)</p> <p>Therefore, <math>B_1 = \{\alpha_1\}</math> is the basis for <math>W_1</math></p> <p>Let <math>T_1</math> be the matrix induced by <math>T</math> on <math>W_1</math></p> $T_1 \alpha_1 = T \alpha_1 = \begin{pmatrix} 6 & -3 & -2 \\ 4 & -1 & -2 \\ 10 & -5 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = 2\alpha_1$ <p><math>[T_1]_{B_1} = [2]</math></p> |
| Bases $B_2$ and matrix $T_2$ | <p>Let <math>W_2 = \{\alpha \in R^3 / p_2(T)\alpha = 0, (T^2 + I)\alpha = 0\}</math></p> <p>Therefore, <math>A^2 + I = \begin{pmatrix} 6 &amp; -3 &amp; -2 \\ 4 &amp; -1 &amp; -2 \\ 10 &amp; -5 &amp; -3 \end{pmatrix} \begin{pmatrix} 6 &amp; -3 &amp; -2 \\ 4 &amp; -1 &amp; -2 \\ 10 &amp; -5 &amp; -3 \end{pmatrix} + \begin{pmatrix} 1 &amp; 0 &amp; 0 \\ 0 &amp; 1 &amp; 0 \\ 0 &amp; 0 &amp; 1 \end{pmatrix} = \begin{pmatrix} 5 &amp; -5 &amp; 0 \\ 0 &amp; 0 &amp; 0 \\ 10 &amp; -10 &amp; 0 \end{pmatrix}</math></p> <p>Rank of <math>A^2 + I</math> is 1</p> <p>Nullity of <math>A^2 + I</math> = no of columns - Rank = <math>3 - 1 = 2</math></p> <p>That means the dimension of <math>W_2</math> is 2</p> <p>Thus we can let, <math>\alpha_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in W_2</math> (Basis theorem in Definitions)</p> <p>Therefore, <math>B_2 = \{\alpha_2, \alpha_3\}</math> is the basis for <math>W_2</math></p> <p>Let <math>T_2</math> be the matrix induced by <math>T</math> on <math>W_2</math></p>  |

|  |
|--|
| $T_2\alpha_2 = T\alpha_2 = \begin{pmatrix} 6 & -3 & -2 \\ 4 & -1 & -2 \\ 10 & -5 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 5 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 5 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 3\alpha_2 + 5\alpha_3$ $T_2\alpha_3 = T\alpha_3 = \begin{pmatrix} 6 & -3 & -2 \\ 4 & -1 & -2 \\ 10 & -5 & -3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ -3 \end{pmatrix} = -2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + -3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = -2\alpha_2 - 3\alpha_3$ $(\alpha_2 \ \alpha_3)[T_2] = (\alpha_2 \ \alpha_3) \begin{pmatrix} 3 & -2 \\ 5 & -3 \end{pmatrix}$ $\Rightarrow [T_2]_{B_2} = \begin{pmatrix} 3 & -2 \\ 5 & -3 \end{pmatrix}$ |
|--|

TABLE 6.7.1.2: Finding of Basis and corresponding matrix

|                            |   |
|----------------------------|---|
| Express Minimal Polynomial | $A = \begin{pmatrix} 6 & -3 & -2 \\ 4 & -1 & -2 \\ 10 & -5 & -3 \end{pmatrix}$ <p>We get, <math>p_1 = (x - 2)</math> and <math>p_2 = (x^2 + 1)</math><br/> Thus, <math>p = p_1 p_2</math></p> |
| $W_i$                      | $W_1 = \{\alpha \in R^3/p_1(T)\alpha = 0, (T - 2I)\alpha = 0\}$<br>$W_2 = \{\alpha \in R^3/p_2(T)\alpha = 0, (T^2 + I)\alpha = 0\}$   |
| $B_1$                      | $B_1 = \{\alpha_1\}$ is the basis for $W_1$<br>where, $\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \in W_1$  |
| $B_2$                      | $B_2 = \{\alpha_2, \alpha_3\}$ is the basis for $W_2$<br>where, $\alpha_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in W_2$          |
| $T_1$                      | $[T_1]_{B_1} = (2)$   |
| $T_2$                      | $[T_2]_{B_2} = \begin{pmatrix} 3 & -2 \\ 5 & -3 \end{pmatrix}$  |

TABLE 6.7.1.3: Conclusion of above Results

6.7.2. Let  $\mathbf{V}$  be the space of  $n \times n$  matrices over a field  $\mathbb{F}$ , and let  $A$  be a fixed  $n \times n$  matrix over field  $\mathbb{F}$ . Define a linear operator  $\mathbf{T}$  on  $\mathbf{V}$  by the equation

$$\mathbf{T}(\mathbf{B}) = \mathbf{AB} - \mathbf{BA} \quad (6.7.2.1)$$

Prove that if  $\mathbf{A}$  is a nilpotent matrix, then  $\mathbf{T}$  is a nilpotent operator. **Solution:** Since  $\mathbf{A}$  is a nilpotent matrix, hence for a positive value  $K$ :

$$\mathbf{A}^K = \mathbf{0} \quad (6.7.2.2)$$

Now we have

$$\mathbf{T}(\mathbf{B}) = \mathbf{AB} - \mathbf{BA} \quad (6.7.2.3)$$

$$\implies \mathbf{T}^2(\mathbf{B}) = \mathbf{T}(\mathbf{T}(\mathbf{B})) = \mathbf{T}(\mathbf{AB} - \mathbf{BA}) \quad (6.7.2.4)$$

$$\begin{aligned} \mathbf{T}(\mathbf{AB} - \mathbf{BA}) &= \mathbf{A}^2\mathbf{B} - \mathbf{ABA} - \mathbf{ABA} + \mathbf{BA}^2 \\ &= \mathbf{A}^2 - 2\mathbf{ABA} + \mathbf{BA}^2 \end{aligned} \quad (6.7.2.5)$$

$$\begin{aligned} \implies \mathbf{T}^3(\mathbf{B}) &= \mathbf{T}(\mathbf{T}^2(\mathbf{B})) \\ &= \mathbf{A}^3 - 3\mathbf{A}^2\mathbf{BA} + 3\mathbf{ABA}^2 - \mathbf{BA}^3 \end{aligned} \quad (6.7.2.6)$$

Hence, from (6.7.2.6) we can say that, as we are increasing the power of operator  $\mathbf{T}$ , the power of  $\mathbf{A}$  is also increasing in every terms. Hence there exist a value  $P$  such that :

$$\mathbf{T}^P(\mathbf{B}) = \mathbf{0} \quad (6.7.2.7) \quad 6.7.3.$$

Hence, if  $\mathbf{A}$  is a nilpotent matrix then operator  $\mathbf{T}$  is also a nilpotent operator.

Let consider  $K = 2$  for which  $\mathbf{A}^K = \mathbf{0}$  that is :

$$\mathbf{A}^2 = \mathbf{0} \quad (6.7.2.8)$$

$$\implies \mathbf{A}^3 = \mathbf{0} \quad (6.7.2.9)$$

Now using (6.7.2.8) and (6.7.2.9) in (6.7.2.6), we get:

$$\mathbf{T}^3(\mathbf{B}) = \mathbf{0} \quad (6.7.2.10)$$

Hence  $P = 3$ . Let consider a matrices  $\mathbf{A}$  and,

$\mathbf{B}$  as :

$$\mathbf{A} = \begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad (6.7.2.11)$$

$$\mathbf{A} \cdot \mathbf{B} = \begin{pmatrix} -4 & -4 \\ -4 & -4 \end{pmatrix} \neq \mathbf{0} \quad (6.7.2.12)$$

Now,

$$\implies \mathbf{A}^2 = \begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad (6.7.2.13)$$

$$\implies \mathbf{A}^3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad (6.7.2.14)$$

Hence  $K = 2$ .

And we have also,

$$\mathbf{ABA} = \begin{pmatrix} -16 & 16 \\ -16 & 16 \end{pmatrix} \neq \mathbf{0} \quad (6.7.2.15)$$

Hence from (6.7.2.4) we conclude,  $P \neq 2$ . Now putting the value of  $\mathbf{A}^3$  from (6.7.2.14) and value of  $\mathbf{A}^2$  from (6.7.2.13) in (6.7.2.6) we get,

$$\mathbf{T}^3(\mathbf{B}) = \mathbf{0} \quad (6.7.2.16)$$

Hence,  $P = 3$  for this example.

6.7.3. Give an example of two  $4 \times 4$  nilpotent matrices which have the same minimal polynomial (they necessarily have the same characteristic polynomial) but which are not similar. **Solution:** See Tables 6.7.3.1 and 6.7.3.2

|                           |   |
|---------------------------|---|
| Characteristic Polynomial | <p>For an <math>n \times n</math> matrix <math>\mathbf{A}</math>, characteristic polynomial is defined by,</p> $p(x) =  x\mathbf{I} - \mathbf{A} $  |
| Minimal Polynomial        | <p>Minimal polynomial <math>m(x)</math> is the smallest factor of characteristic polynomial <math>p(x)</math> such that,</p> $m(\mathbf{A}) = \mathbf{0}$   |
| Cayley-Hamilton Theorem   | <p>If <math>p(x)</math> is the characteristic polynomial of an <math>n \times n</math> matrix <math>\mathbf{A}</math>, then,</p> $p(\mathbf{A}) = \mathbf{0}$   |
| Similar matrices          | <p>Two matrices <math>\mathbf{A}</math> and <math>\mathbf{B}</math> are said to be similar if</p> <ul style="list-style-type: none"> <li>a) <math>\det(\mathbf{A}) = \det(\mathbf{B})</math></li> <li>b) <math>\text{tr}(\mathbf{A}) = \text{tr}(\mathbf{B})</math></li> <li>c) <math>\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{B})</math></li> </ul> |
| Nilpotent Matrix          | <p>A square matrix <math>\mathbf{A}</math> is called a Nilpotent matrix if there exist a positive integer '<math>m</math>' such that <math>\mathbf{A}^m = \mathbf{0}</math> and '<math>m</math>' is called Index of nilpotent matrix <math>\mathbf{A}</math>.</p> <p>The determinant and trace of a nilpotent matrix are always zero.</p>                 |

TABLE 6.7.3.1: Definitions

|                           |   |
|---------------------------|---|
| Given                     | <p>Let <b>A</b> and <b>B</b> be two nilpotent matrix.</p> $\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $\mathbf{B} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ |
| Characteristic polynomial | <p>For Matrix <b>A</b>;</p> $p_A(x) =  x\mathbf{I} - \mathbf{A} $   |

$$p_A(x) = \begin{vmatrix} x & -1 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & x & -1 \\ 0 & 0 & 0 & x \end{vmatrix} = x^4$$

For Matrix **B**;

$$p_B(x) = |x\mathbf{I} - \mathbf{B}|$$

$$p_B(x) = \begin{vmatrix} x & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & x & -1 \\ 0 & 0 & 0 & x \end{vmatrix} = x^4$$

Therefore, Characteristics polynomial are same for both matrix **A** and **B**.

Minimal Polynomial

For Matrix **A**;

$$p_A(x) = x^4$$

Let, minimal polynomial of **A** is  $m_A(x)$

$m_A(x)$  always divide  $p_A(x)$

$$m_A(x) = \{x, x^2, x^3, x^4\}$$

Minimal polynomial always annihilates its matrix.

$$m_A(\mathbf{A}) = \mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq \mathbf{0}$$

$$m_A(\mathbf{A}) = \mathbf{A}^2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \mathbf{0}$$

$$\Rightarrow m_A(x) = x^2$$

$\Rightarrow x^2$  is a minimal polynomial of Matrix **A**

For Matrix **B**;

$$m_B(\mathbf{B}) = \mathbf{B} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq \mathbf{0}$$

|  |   |
|--|---|
|  | $m_B(\mathbf{B}) = \mathbf{B}^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \mathbf{0}$ $\Rightarrow m_B(x) = x^2$ <p><math>\Rightarrow x^2</math> is a minimal polynomial of Matrix <math>\mathbf{B}</math></p> <p>Therefore, minimal polynomial for both Matrix <math>\mathbf{A}</math> and <math>\mathbf{B}</math> are same.</p> |
| <p>Checking whether Matrix <math>\mathbf{A}</math> and <math>\mathbf{B}</math> are similar .</p> | $\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ <p>rank of matrix = no of linearly independent row or column vectors in the matrix</p> $\text{rank}(\mathbf{A}) = 2$ $\text{tr}(\mathbf{A}) = 0$ $\det(\mathbf{A}) = 0$   |
|  | $\mathbf{B} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $\text{rank}(\mathbf{B}) = 1$ $\text{tr}(\mathbf{B}) = 0$ $\det(\mathbf{B}) = 0$ $\Rightarrow \text{rank}(\mathbf{A}) \neq \text{rank}(\mathbf{B})$ <p>Therefore, Matrix <math>\mathbf{A}</math> and <math>\mathbf{B}</math> are not similar.</p>   |
| Conclusion   | <p>a) Characteristics polynomial for both Matrix <math>\mathbf{A}</math> and <math>\mathbf{B}</math> are same.</p> <p>b) Minimal polynomial for both matrix are same i.e <math>m_A(x) = m_B(x) = x^2</math>.</p> <p>c) Matrix <math>\mathbf{A}</math> and <math>\mathbf{B}</math> are not similar .</p>   |

TABLE 6.7.3.2: Solution Summary



|   |   |
|---|---|
| Invariant Subspaces                           | Suppose $\mathbf{T} \in \mathbf{L}(\mathbf{V})$ . A subspace $\mathbf{U}$ of $\mathbf{V}$ is called invariant under $\mathbf{T}$ if $\mathbf{u} \in \mathbf{U}$ implies $\mathbf{T}(\mathbf{u}) \in \mathbf{U}$ . Suppose $\mathbf{T} \in \mathbf{L}(\mathbf{V})$ , then null $\mathbf{T}$ and range $\mathbf{T}$ are invariant subspaces of $\mathbf{T}$ . |
| Complementary $\mathbf{T}$ invariant subspace | Suppose we have a vector space $\mathbf{V}$ , if $\mathbf{V}$ is written as direct sum of its subspaces $\mathbf{W}$ and $\mathbf{W}'$ , i.e $\mathbf{V} = \mathbf{W} \oplus \mathbf{W}'$ and each of $\mathbf{W}$ and $\mathbf{W}'$ is invariant under $\mathbf{T}$ , then we say $\mathbf{W}$ has a complementary $\mathbf{T}$ invariant subspace.        |

TABLE 7.1.1.1: Definition and Result used

## 7 THE RATIONAL AND JORDAN FORMS

## 7.1 Cyclic Decompositions and the Rational Form

7.1.1. Let,  $\mathbf{T}$  be the linear operator on  $\mathbb{R}^3$  which is represented in the standard ordered basis by the matrix

$$\begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \quad (7.1.1.1)$$

Let  $\mathbf{W}$  be the null space of  $\mathbf{T} - 2\mathbf{I}$ . Prove that  $\mathbf{W}$  has no complementary  $\mathbf{T}$ -invariant subspace.

**Solution:** See Tables 7.1.1.1 and 7.1.1.2

## 7.2 The Jordan Form

7.2.1. If  $\mathbf{A}$  is a complex  $5 \times 5$  matrix with characteristic polynomial  $f = (x - 2)^3(x + 7)^2$  and minimal polynomial  $p = (x - 2)^2(x + 7)$ , what is the Jordan form for  $\mathbf{A}$ ?

**Solution:** *Theory:* Table 7.2.1.1 gives the overview of properties of a Jordan block based on characteristic polynomial, minimal polynomial, algebraic multiplicity and geometric multiplicity.

From the properties stated in table 7.2.1.1, the Jordan blocks for eigenvalues of  $\mathbf{A}$  can be written as,

$$\mathbf{J}_1 = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}; \quad \mathbf{J}_2 = \begin{pmatrix} -7 & 0 \\ 0 & -7 \end{pmatrix} \quad (7.2.1.1)$$

Where  $\mathbf{J}_1$  and  $\mathbf{J}_1$  are the Jordan blocks corresponding to  $\lambda_1 = 2$  and  $\lambda_1 = -7$  respectively.

The Jordan form for  $\mathbf{A}$  can be written as,

$$\mathbf{J} = \begin{pmatrix} \mathbf{J}_1 & 0 \\ 0 & \mathbf{J}_2 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -7 & 0 \\ 0 & 0 & 0 & 0 & -7 \end{pmatrix} \quad (7.2.1.2)$$

*Inference* An  $n \times n$  matrix with  $\lambda$  as diagonal elements, ones on the super diagonal and zeroes in all other entries is nilpotent with minimal polynomial  $(A - \lambda I)^n$  *Example* Let,

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad (7.2.1.3)$$

(7.2.1.3) is nilpotent for minimal polynomial  $A^3$

7.2.2. How many possible Jordan forms are there for a  $6 \times 6$  complex matrix with characteristic polynomial  $(x + 2)^4(x - 1)^2$ ?

**Solution:** See Tables 7.2.2.1, 7.2.2.2 and 7.2.2.3

7.2.3. The differentiation operator on the space of polynomials of degree less than or equal to 3 is represented in the natural ordered basis by the matrix,

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (7.2.3.1)$$

What is the Jordan form of this matrix? ( $\mathbb{F}$  a subfield of the complex numbers.)

**Solution:** First, we find the characteristic poly-

|   |   |
|---|---|
| Nullspace of $\mathbf{T} - 2\mathbf{I}$ | <p>We Know that Nullspace of a linear operator <math>\mathbf{T}</math> is the nullspace of its matrix representation of <math>\mathbf{T}</math> w.r.t standard basis. Thus, <math>\text{Nullspace}(\mathbf{W}) = \text{Nullspace}(\mathbf{T} - 2\mathbf{I})</math>.</p> <p>Now, <math>\text{Nullspace}(\mathbf{T} - 2\mathbf{I}) = \text{Nullspace} \begin{pmatrix} 0 &amp; 0 &amp; 0 \\ 1 &amp; 0 &amp; 0 \\ 0 &amp; 0 &amp; 1 \end{pmatrix}</math></p> <p>Hence, <math>\text{Nullspace}(\mathbf{T} - 2\mathbf{I}) = \left\{ \begin{pmatrix} 0 \\ k \\ 0 \end{pmatrix} : k \in \mathbb{R} \right\}</math></p> <p style="text-align: center;"><math>= \left\{ k \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} : k \in \mathbb{R} \right\}</math></p>  |
| Proof                                   | <p>Let <math>\beta = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}</math>. Then</p> $(\mathbf{T} - 2\mathbf{I})\beta = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \gamma \in \mathbf{W}$ <p>Now,</p> $(\mathbf{T} - 2\mathbf{I})\gamma = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in \mathbf{W}.$ <p>Now, we assume that <math>\mathbf{W}</math> has a complementary <math>\mathbf{T}</math>-invariant subspace <math>\mathbf{S}</math>. Then <math>\beta</math> can be written as <math>\beta = s + w</math>, <math>s \in \mathbf{W}</math>, <math>w \in \mathbf{W}'</math>.</p> <p>Finally, we see that</p> $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = (\mathbf{T} - 2\mathbf{I})\beta = (\mathbf{T} - 2\mathbf{I})(s + w) = (\mathbf{T} - 2\mathbf{I})w \in \mathbf{W}' \text{ as } \mathbf{W}' \text{ is invariant under } \mathbf{T} \text{ and } s \in \text{Nullspace } \mathbf{W}.$ <p>Thus, we conclude that <math>\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \in \mathbf{W} \cap \mathbf{W}'</math>, which is a contradiction. Since, <math>\mathbf{V} = \mathbf{W} \oplus \mathbf{W}'</math>,</p> <p>thus <math>\mathbf{W} \cap \mathbf{W}' = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}</math>.</p> <p>Therefore, <math>\mathbf{W}</math> has no complementary <math>\mathbf{T}</math>-invariant subspace.</p> |

TABLE 7.1.1.2: Solution

nomial of  $\mathbf{A}$ ,

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0 \quad (7.2.3.2)$$

$$\Rightarrow \begin{vmatrix} -\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 2 & 0 \\ 0 & 0 & -\lambda & 3 \\ 0 & 0 & 0 & -\lambda \end{vmatrix} = 0 \quad (7.2.3.3)$$

$$\Rightarrow \lambda^4 = 0 \quad (7.2.3.4)$$

(7.2.3.4) is the required characteristic equation and  $\lambda_1 = 0$  is the only eigen value of  $\mathbf{A}$ . Hence the characteristic polynomial of  $\mathbf{A}$  is,

$$f(\lambda) = \lambda^4 \quad (7.2.3.5)$$

| Feature   | Effect on Jordan block  | Example  |
|---|---|--|
| characteristic polynomial or Algebraic multiplicity | The multiplicity of $\lambda$ in the characteristic polynomial determines the size of the Jordan block for that eigenvalue.<br>$A_M = \text{Size of Jordan block for } \lambda$ | Let, $f = (x - 2)^4$ be characteristic polynomial<br>$\mathbf{J} = \begin{pmatrix} 2 & * & 0 & 0 \\ 0 & 2 & * & 0 \\ 0 & 0 & 2 & * \\ 0 & 0 & 0 & 2 \end{pmatrix}$ where * can be either 1 or 0  |
| Geometric multiplicity                              | The geometric multiplicity determines the total number of Jordan sub blocks for $\lambda$   | If $A_M = 4$ ; $G_M = 2$ ; $\lambda = 2$<br>$\Rightarrow$ There should be 2 Jordan sub blocks for $\lambda = 2$ . So, $\mathbf{J}$ has 2 possibilities<br>$\mathbf{J} = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}; \text{ or } \mathbf{J} = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$ |
| minimal polynomial                                  | The multiplicity of $\lambda$ in the minimal polynomial determines the size of the largest sub-block<br><br>(Elementary Jordan block).  | Let $p = (x - 2)^3$ be minimal polynomial<br>$\Rightarrow$ Size of largest sub-block is 3<br>Hence, one sub-block of size 3 and one sub-block of size 1<br>$\mathbf{J} = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$   |

TABLE 7.2.1.1: Properties of Jordan blocks and Jordan canonical form

| Parameter                    | Description   |
|------------------------------|---|
| $A_M$                        | Algebraic multiplicity of characteristic value $\lambda$ in the characteristic polynomial, also equal to the size of Jordan block for that eigen value                |
| $G_M$                        | Geometric multiplicity determines the number of Jordan sub-blocks in a Jordan block for $\lambda$ .   |
| $\mathbf{J}_{(x-\lambda)^k}$ | Jordan block corresponding to the eigen value $\lambda$ and k is the multiplicity of $\lambda$ in the minimal polynomial determines size of largest Jordan sub-block. |

TABLE 7.2.2.1: Parameters

Again we observe that for  $k = 4$  we have,

$$\mathbf{A}^4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (7.2.3.6)$$

$$\Rightarrow f(\mathbf{A}) = 0 \quad (7.2.3.7)$$

And for  $k = 3$  we also have,

$$\mathbf{A}^3 = \begin{pmatrix} 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (7.2.3.8)$$

$$\Rightarrow \mathbf{A}^3 \neq 0 \quad (7.2.3.9)$$

From (7.2.3.7) and (7.2.3.9) we conclude that

| Feature   | Explanation  |
|---|--|
| Characteristic Polynomial                         | $(x+2)^4(x-1)^2$ (7.2.2.1)   |
| Algebraic Multiplicity, $A_M$                     | For $\lambda = -2, A_M = 4$ (7.2.2.2)  |
|   | For $\lambda = 1, A_M = 2$ (7.2.2.3)   |
| Minimal Polynomial                                | $p = (x+2)^a(x-1)^b, a \leq 4, b \leq 2$ (7.2.2.4)   |
| Possibilities of minimal polynomial               | From equation (7.2.2.4), there are 8 different minimal polynomials possible.   |
| Jordan Form                                       | <p>Jordan form built from Jordan blocks listed in table 7.2.2.3 will have the form,</p> $\mathbf{J} = \begin{pmatrix} -2 & * & 0 & 0 & 0 & 0 \\ 0 & -2 & * & 0 & 0 & 0 \\ 0 & 0 & -2 & * & 0 & 0 \\ 0 & 0 & 0 & -2 & * & 0 \\ 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (7.2.2.5)$ <p>where * can be either 1 or 0</p>  |
| Number of possibilities of Jordan canonical forms | From Table 7.2.2.3, there are $5 \times 2 = 10$ different Jordan forms possible.   |
| Jordan Form corresponding to $p = (x+2)^2(x-1)$   | One minimal polynomial can correspond to more than one Jordan forms. For example, minimal polynomial $p = (x+2)^2(x-1)$ can correspond to two different Jordan forms namely,   |
|   | $\mathbf{J} = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{J} = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (7.2.2.6)$ |

TABLE 7.2.2.2: Parameters

| Factor  | Possible Jordan blocks  | $G_M$ |
|---------|---|-------|
| $(x+2)$ | $\mathbf{J}_{(x+2)} = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$   | 4     |
|         | $\mathbf{J}_{(x+2)^2} = \begin{pmatrix} -2 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$ | 3     |
|         | $\mathbf{J}_{(x+2)^2} = \begin{pmatrix} -2 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & -2 \end{pmatrix}$ | 2     |
|         | $\mathbf{J}_{(x+2)^3} = \begin{pmatrix} -2 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$ | 2     |
|         | $\mathbf{J}_{(x+2)^4} = \begin{pmatrix} -2 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & -2 \end{pmatrix}$ | 1     |
| $(x-1)$ | $\mathbf{J}_{(x-1)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$   | 2     |
|         | $\mathbf{J}_{(x-1)^2} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$   | 1     |

TABLE 7.2.2.3: Possible Jordan Blocks

the minimal polynomial of  $\mathbf{A}$  is,

$$g(\lambda) = \lambda^4 \quad (7.2.3.10)$$

Hence, the Jordan form of  $\mathbf{A}$  is a  $4 \times 4$  matrix consisting of only one block with principal

diagonal values as  $\lambda_1 = 0$  and super diagonal of the matrix (i.e the set of elements that lies directly above the elements comprising the principal diagonal) contains 1. Hence the

required Jordan form of  $\mathbf{A}$  is,

$$\mathbf{J} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (7.2.3.11)$$

(7.2.3.11) is the required Jordan form of  $\mathbf{A}$ .

- 7.2.4. Let  $\mathbf{N}_1$  and  $\mathbf{N}_2$  be  $6 \times 6$  nilpotent matrices over the field  $\mathbf{F}$ . Suppose that  $\mathbf{N}_1$  and  $\mathbf{N}_2$  have the same minimal polynomial and the same nullity. Prove that  $\mathbf{N}_1$  and  $\mathbf{N}_2$  are similar. Show that this is not true for  $7 \times 7$  nilpotent matrices.

**Solution:** See Table 7.2.4.1

| Statement   | Derivation  |
|---|---|
| Given   | <p><math>\mathbf{N}_1</math> and <math>\mathbf{N}_2</math> be <math>6 \times 6</math> nilpotent matrices.<br/> <math>\mathbf{N}_1</math> and <math>\mathbf{N}_2</math> have the same minimal polynomial and the same nullity.<br/> To prove <math>\mathbf{N}_1</math> and <math>\mathbf{N}_2</math> are similar.</p>  |
| From given statement                                    | <p>Two matrices are similar if they have the same Jordan Canonical form.</p> <ol style="list-style-type: none"> <li>1. As <math>\mathbf{N}_1</math> and <math>\mathbf{N}_2</math> are nilpotent matrices, 0 is the only eigen value.</li> <li>2. As minimal polynomial is same, <math>p\mathbf{N}_1 = p\mathbf{N}_2</math>, the two matrices should have the same maximum block size.</li> <li>3. As they have same nullity, they will have same total number of blocks.</li> </ol> <p>If <math>\mathbf{J}_1</math> and <math>\mathbf{J}_2</math> are similar, then <math>\mathbf{N}_1</math> and <math>\mathbf{N}_2</math> are similar.</p> <p>Let us consider all the possibilities for the dimensions of the block matrices for both Jordan forms.</p>                   |
| Matrix size - 6 and Jordan size - 6                     | <p>If Jordan form <math>\mathbf{J}_1</math> consists of one block of dimension 6, then by (3) above <math>\mathbf{J}_2</math> also has one block of dimension 6.</p> $\mathbf{J}_1 = (\mathbf{J}_{11}) \quad \mathbf{J}_2 = (\mathbf{J}_{21})$ $\mathbf{J}_{11} : 6 \times 6 \quad \mathbf{J}_{21} : 6 \times 6$ $\mathbf{J}_{11}, \mathbf{J}_{21} \text{ are similar ,}$ $\mathbf{J}_1, \mathbf{J}_2 \text{ are similar}$ $\Rightarrow \mathbf{N}_1, \mathbf{N}_2 \text{ are similar}$   |
| Matrix size - 6 and Jordan size - 5 + 1                 | <p>If Jordan form <math>\mathbf{J}_1</math> consists of one block of dimension 5 and other 1, then by (2), <math>\mathbf{J}_2</math> also has same maximum block of dimension 5 and by (3) have other block of size 1.</p> $\mathbf{J}_1 = \begin{pmatrix} \mathbf{J}_{11} & 0 \\ 0 & \mathbf{J}_{12} \end{pmatrix} \quad \mathbf{J}_2 = \begin{pmatrix} \mathbf{J}_{21} & 0 \\ 0 & \mathbf{J}_{22} \end{pmatrix}$ $\mathbf{J}_{11} : 5 \times 5 \quad \mathbf{J}_{21} : 5 \times 5$ $\mathbf{J}_{12} : 1 \times 1 \quad \mathbf{J}_{22} : 1 \times 1$ $\mathbf{J}_{11}, \mathbf{J}_{21} \text{ and } \mathbf{J}_{12}, \mathbf{J}_{22} \text{ are similar ,}$ $\mathbf{J}_1, \mathbf{J}_2 \text{ are similar}$ $\Rightarrow \mathbf{N}_1, \mathbf{N}_2 \text{ are similar}$ |
| Matrix size - 6, Jordan size - 4+2, Jordan size - 4+1+1 | <p>Although there are two different possibilities for Jordan blocks,</p> <p>From (2) <math>\mathbf{J}_{11}, \mathbf{J}_{21}</math> are of dimension 4,<br/> From (3) <math>\mathbf{J}</math> have same number of Jordan blocks<br/> Case 1:</p>   |

$$\mathbf{J}_1 = \begin{pmatrix} \mathbf{J}_{11} & 0 \\ 0 & \mathbf{J}_{12} \end{pmatrix} \quad \mathbf{J}_2 = \begin{pmatrix} \mathbf{J}_{21} & 0 \\ 0 & \mathbf{J}_{22} \end{pmatrix}$$

$$\mathbf{J}_{11} : 4 \times 4 \quad \mathbf{J}_{21} : 4 \times 4$$

$$\mathbf{J}_{12} : 2 \times 2 \quad \mathbf{J}_{22} : 2 \times 2$$

$$\mathbf{J}_{11}, \mathbf{J}_{21} \text{ and } \mathbf{J}_{12}, \mathbf{J}_{22} \text{ are similar ,}$$

$$\mathbf{J}_1, \mathbf{J}_2 \text{ are similar}$$

$$\Rightarrow \mathbf{N}_1, \mathbf{N}_2 \text{ are similar}$$

Case 2:

$$\mathbf{J}_1 = \begin{pmatrix} \mathbf{J}_{11} & 0 & 0 \\ 0 & \mathbf{J}_{12} & 0 \\ 0 & 0 & \mathbf{J}_{13} \end{pmatrix} \quad \mathbf{J}_2 = \begin{pmatrix} \mathbf{J}_{21} & 0 & 0 \\ 0 & \mathbf{J}_{22} & 0 \\ 0 & 0 & \mathbf{J}_{23} \end{pmatrix}$$

$$\mathbf{J}_{11} : 4 \times 4 \quad \mathbf{J}_{21} : 4 \times 4$$

$$\mathbf{J}_{12} : 1 \times 1 \quad \mathbf{J}_{22} : 1 \times 1$$

$$\mathbf{J}_{13} : 1 \times 1 \quad \mathbf{J}_{23} : 1 \times 1$$

$$\mathbf{J}_{11}, \mathbf{J}_{21} \quad \mathbf{J}_{12}, \mathbf{J}_{22} \text{ and } \mathbf{J}_{13}, \mathbf{J}_{23} \text{ are similar ,}$$

$$\mathbf{J}_1, \mathbf{J}_2 \text{ are similar}$$

$$\Rightarrow \mathbf{N}_1, \mathbf{N}_2 \text{ are similar}$$

Matrix size - 6, Jordan size  
- 3+3, Jordan size - 3+2+1,  
Jordan size - 3+1+1+1

There are three different possibilities for Jordan blocks,

From (2)  $\mathbf{J}_{11}, \mathbf{J}_{21}$  are of dimension 3,

From (3)  $\mathbf{J}$  have same number of Jordan blocks

Case 1:

$$\mathbf{J}_1 = \begin{pmatrix} \mathbf{J}_{11} & 0 \\ 0 & \mathbf{J}_{12} \end{pmatrix} \quad \mathbf{J}_2 = \begin{pmatrix} \mathbf{J}_{21} & 0 \\ 0 & \mathbf{J}_{22} \end{pmatrix}$$

$$\mathbf{J}_{11} : 3 \times 3 \quad \mathbf{J}_{21} : 3 \times 3$$

$$\mathbf{J}_{12} : 3 \times 3 \quad \mathbf{J}_{22} : 3 \times 3$$

$$\mathbf{J}_{11}, \mathbf{J}_{21} \text{ and } \mathbf{J}_{12}, \mathbf{J}_{22} \text{ are similar ,}$$

$$\mathbf{J}_1, \mathbf{J}_2 \text{ are similar}$$

$$\Rightarrow \mathbf{N}_1, \mathbf{N}_2 \text{ are similar}$$

Case 2:

$$\mathbf{J}_1 = \begin{pmatrix} \mathbf{J}_{11} & 0 & 0 \\ 0 & \mathbf{J}_{12} & 0 \\ 0 & 0 & \mathbf{J}_{13} \end{pmatrix} \quad \mathbf{J}_2 = \begin{pmatrix} \mathbf{J}_{21} & 0 & 0 \\ 0 & \mathbf{J}_{22} & 0 \\ 0 & 0 & \mathbf{J}_{23} \end{pmatrix}$$

$$\mathbf{J}_{11} : 3 \times 3 \quad \mathbf{J}_{21} : 3 \times 3$$

$$\mathbf{J}_{12} : 2 \times 2 \quad \mathbf{J}_{22} : 2 \times 2$$

$$\mathbf{J}_{13} : 1 \times 1 \quad \mathbf{J}_{23} : 1 \times 1$$

$$\mathbf{J}_{11}, \mathbf{J}_{21} \quad \mathbf{J}_{12}, \mathbf{J}_{22} \text{ and } \mathbf{J}_{13}, \mathbf{J}_{23} \text{ are similar ,}$$

$$\mathbf{J}_1, \mathbf{J}_2 \text{ are similar}$$

$$\Rightarrow \mathbf{N}_1, \mathbf{N}_2 \text{ are similar}$$

Case 3:

$$\mathbf{J}_1 = \begin{pmatrix} \mathbf{J}_{11} & 0 & 0 & 0 \\ 0 & \mathbf{J}_{12} & 0 & 0 \\ 0 & 0 & \mathbf{J}_{13} & 0 \\ 0 & 0 & 0 & \mathbf{J}_{14} \end{pmatrix} \quad \mathbf{J}_2 = \begin{pmatrix} \mathbf{J}_{21} & 0 & 0 & 0 \\ 0 & \mathbf{J}_{22} & 0 & 0 \\ 0 & 0 & \mathbf{J}_{23} & 0 \\ 0 & 0 & 0 & \mathbf{J}_{24} \end{pmatrix}$$

$$\mathbf{J}_{11} : 3 \times 3 \quad \mathbf{J}_{21} : 3 \times 3$$

$$\mathbf{J}_{12} : 1 \times 1 \quad \mathbf{J}_{22} : 1 \times 1$$

$$\mathbf{J}_{13} : 1 \times 1 \quad \mathbf{J}_{23} : 1 \times 1$$

$$\mathbf{J}_{14} : 1 \times 1 \quad \mathbf{J}_{24} : 1 \times 1$$

$\mathbf{J}_{11}, \mathbf{J}_{21} \quad \mathbf{J}_{12}, \mathbf{J}_{22}$  and  $\mathbf{J}_{13}, \mathbf{J}_{23} \quad \mathbf{J}_{14}, \mathbf{J}_{24}$  are similar ,

$\mathbf{J}_1, \mathbf{J}_2$  are similar

$\Rightarrow \mathbf{N}_1, \mathbf{N}_2$  are similar

Matrix size - 6, Jordan size - 2+2+2, Jordan size - 2+2+1+1, Jordan size - 2+1+1+1+1

There are three different possibilities for Jordan blocks,

From (2)  $\mathbf{J}_{11}, \mathbf{J}_{21}$  are of dimension 2,

From (3)  $\mathbf{J}$  have same number of Jordan blocks

Case 1:

$$\mathbf{J}_1 = \begin{pmatrix} \mathbf{J}_{11} & 0 & 0 \\ 0 & \mathbf{J}_{12} & 0 \\ 0 & 0 & \mathbf{J}_{13} \end{pmatrix} \quad \mathbf{J}_2 = \begin{pmatrix} \mathbf{J}_{21} & 0 & 0 \\ 0 & \mathbf{J}_{22} & 0 \\ 0 & 0 & \mathbf{J}_{23} \end{pmatrix}$$

$$\mathbf{J}_{11} : 2 \times 2 \quad \mathbf{J}_{21} : 2 \times 2$$

$$\mathbf{J}_{12} : 2 \times 2 \quad \mathbf{J}_{22} : 2 \times 2$$

$$\mathbf{J}_{13} : 2 \times 2 \quad \mathbf{J}_{23} : 2 \times 2$$

$\mathbf{J}_{11}, \mathbf{J}_{21} \quad \mathbf{J}_{12}, \mathbf{J}_{22}$  and  $\mathbf{J}_{13}, \mathbf{J}_{23}$  are similar ,

$\mathbf{J}_1, \mathbf{J}_2$  are similar

$\Rightarrow \mathbf{N}_1, \mathbf{N}_2$  are similar

Case 2:

$$\mathbf{J}_1 = \begin{pmatrix} \mathbf{J}_{11} & 0 & 0 & 0 \\ 0 & \mathbf{J}_{12} & 0 & 0 \\ 0 & 0 & \mathbf{J}_{13} & 0 \\ 0 & 0 & 0 & \mathbf{J}_{14} \end{pmatrix} \quad \mathbf{J}_2 = \begin{pmatrix} \mathbf{J}_{21} & 0 & 0 & 0 \\ 0 & \mathbf{J}_{22} & 0 & 0 \\ 0 & 0 & \mathbf{J}_{23} & 0 \\ 0 & 0 & 0 & \mathbf{J}_{24} \end{pmatrix}$$

$$\mathbf{J}_{11} : 2 \times 2 \quad \mathbf{J}_{21} : 2 \times 2$$

$$\mathbf{J}_{12} : 2 \times 2 \quad \mathbf{J}_{22} : 2 \times 2$$

$$\mathbf{J}_{13} : 1 \times 1 \quad \mathbf{J}_{23} : 1 \times 1$$

$$\mathbf{J}_{14} : 1 \times 1 \quad \mathbf{J}_{24} : 1 \times 1$$

$\mathbf{J}_{11}, \mathbf{J}_{21} \quad \mathbf{J}_{12}, \mathbf{J}_{22}$  and  $\mathbf{J}_{13}, \mathbf{J}_{23} \quad \mathbf{J}_{14}, \mathbf{J}_{24}$  are similar ,

$\mathbf{J}_1, \mathbf{J}_2$  are similar

$\Rightarrow \mathbf{N}_1, \mathbf{N}_2$  are similar

Case 3:



|  |  |
|--|--|
|  | $\mathbf{J}_1 = \begin{pmatrix} \mathbf{J}_{11} & 0 & 0 & 0 & 0 \\ 0 & \mathbf{J}_{12} & 0 & 0 & 0 \\ 0 & 0 & \mathbf{J}_{13} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{J}_{14} & 0 \\ 0 & 0 & 0 & 0 & \mathbf{J}_{15} \end{pmatrix} \quad \mathbf{J}_2 = \begin{pmatrix} \mathbf{J}_{21} & 0 & 0 & 0 & 0 \\ 0 & \mathbf{J}_{22} & 0 & 0 & 0 \\ 0 & 0 & \mathbf{J}_{23} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{J}_{24} & 0 \\ 0 & 0 & 0 & 0 & \mathbf{J}_{25} \end{pmatrix}$ $\begin{aligned} &\mathbf{J}_{11} : 2 \times 2 \quad \mathbf{J}_{21} : 2 \times 2 \\ &\mathbf{J}_{12} : 1 \times 1 \quad \mathbf{J}_{22} : 1 \times 1 \\ &\mathbf{J}_{13} : 1 \times 1 \quad \mathbf{J}_{23} : 1 \times 1 \\ &\mathbf{J}_{14} : 1 \times 1 \quad \mathbf{J}_{24} : 1 \times 1 \\ &\mathbf{J}_{15} : 1 \times 1 \quad \mathbf{J}_{25} : 1 \times 1 \end{aligned}$ <p><math>\mathbf{J}_{11}, \mathbf{J}_{21} \quad \mathbf{J}_{12}, \mathbf{J}_{22} \text{ and } \mathbf{J}_{13}, \mathbf{J}_{23} \quad \mathbf{J}_{14}, \mathbf{J}_{24} \quad \mathbf{J}_{15}, \mathbf{J}_{25} \text{ are similar ,}</math><br/> <math>\mathbf{J}_1, \mathbf{J}_2 \text{ are similar}</math><br/> <math>\Rightarrow \mathbf{N}_1, \mathbf{N}_2 \text{ are similar}</math></p> |
| Matrix size - 6, Jordan size-<br>1+1+1+1+1+1 | <p><math>\mathbf{J}_1</math> and <math>\mathbf{J}_2</math> will have same number of blocks because of (3).</p> $\mathbf{J}_1 = \begin{pmatrix} \mathbf{J}_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{J}_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{J}_{13} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{J}_{14} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{J}_{15} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{J}_{16} \end{pmatrix} \quad \mathbf{J}_2 = \begin{pmatrix} \mathbf{J}_{21} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{J}_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{J}_{23} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{J}_{24} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{J}_{25} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{J}_{26} \end{pmatrix}$ <p><math>\mathbf{J}_{11}, \mathbf{J}_{21} \quad \mathbf{J}_{12}, \mathbf{J}_{22} \quad \mathbf{J}_{13}, \mathbf{J}_{23} \quad \mathbf{J}_{14}, \mathbf{J}_{24} \quad \mathbf{J}_{15}, \mathbf{J}_{25} \text{ and } \mathbf{J}_{16}, \mathbf{J}_{26} \text{ are similar ,}</math><br/> <math>\mathbf{J}_1, \mathbf{J}_2 \text{ are similar}</math><br/> <math>\Rightarrow \mathbf{N}_1, \mathbf{N}_2 \text{ are similar}</math></p>  |
| 7 × 7 Nielpotent matrix                      | <p>Let us take a counter example,<br/> Matrix size - 7, Jordan block size - 3+3+1 and 3+2+2</p> $\mathbf{J}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \mathbf{J}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$ <p style="text-align: right;">(7.2.4.1)</p> <p>From above, <math>\mathbf{J}_1</math> and <math>\mathbf{J}_2</math> are not same.<br/> Hence <math>\mathbf{N}_1</math> and <math>\mathbf{N}_2</math> are not similar.</p>   |

TABLE 7.2.4.1: Solution

7.2.5. If  $\mathbf{N}$  is a nilpotent  $3 \times 3$  matrix over  $\mathbb{C}$ , prove that  $\mathbf{A} = \mathbf{I} + \frac{1}{2}\mathbf{N} - \frac{1}{8}\mathbf{N}^2$  satisfies  $\mathbf{A}^2 = \mathbf{I} + \mathbf{N}$ , i.e.,  $\mathbf{A}$  is a square root of  $\mathbf{I} + \mathbf{N}$ . Use the binomial series for  $(1+t)^{\frac{1}{2}}$  to obtain a similar formula for a square root of  $\mathbf{I} + \mathbf{N}$ , where  $\mathbf{N}$  is any nilpotent  $n \times n$  matrix over  $\mathbb{C}$ .

**Solution:** We know that  $\mathbf{N}^3=0$  since the minimal polynomial of  $\mathbf{N}$  is  $x^3$ , So,

$$\mathbf{A}^2 = \left( \mathbf{I} + \frac{1}{2}\mathbf{N} - \frac{1}{8}\mathbf{N}^2 \right) \left( \mathbf{I} + \frac{1}{2}\mathbf{N} - \frac{1}{8}\mathbf{N}^2 \right) \quad (7.2.5.1)$$

$$= \mathbf{I} + \frac{1}{2}\mathbf{N} - \frac{1}{8}\mathbf{N}^2 + \frac{1}{4}\mathbf{N}^2 - \frac{1}{8}\mathbf{N}^2 \quad (7.2.5.2)$$

$$= \mathbf{I} + \mathbf{N} \quad (7.2.5.3)$$

Expanding  $(1+t)^{1/2}$ ,

$$(1+t)^{1/2} = \sum_{k=0}^{\infty} \binom{1/2}{k} t^k \quad (7.2.5.4)$$

Here,

$$\binom{1/2}{k} = \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)\cdots(\frac{1}{2}-k+1)}{k!} \quad (7.2.5.5)$$

$$= \frac{(-1)^{k-1}}{2^k k!} 1 \cdot 3 \cdot 5 \cdots (2k-3) \quad (7.2.5.6)$$

$$= \frac{(-1)^{k-1}}{2^k k!} \frac{(2k-2)!}{2^{k-1}(k-1)!} \quad (7.2.5.7)$$

$$= \frac{(-1)^{k-1}}{k 2^{2k-1}} \binom{2k-2}{k-1} \quad (7.2.5.8)$$

Thus,

$$\begin{aligned} (1+t)^{1/2} &= 1 - \sum_{k=1}^{\infty} \frac{2}{k} \binom{2k-2}{k-1} \left(-\frac{t}{4}\right)^k \quad (7.2.5.9) \\ &= 1 - \sum_{k=0}^{\infty} \frac{2}{k+1} \binom{2k}{k} \left(-\frac{t}{4}\right)^{k+1} \quad (7.2.5.10) \end{aligned}$$

So a square root for  $\mathbf{I}+\mathbf{N}$  where  $\mathbf{N}$  is a  $n \times n$  nilpotent matrix can be,

$$= \mathbf{I} - \sum_{k=0}^{n-1} \frac{2}{k+1} \binom{2k}{k} \left(-\frac{\mathbf{N}}{4}\right)^{k+1} \quad (7.2.5.11)$$

7.2.6. Use the result of Exercise 15 (that is, if  $\mathbf{A} = \mathbf{I} + \frac{1}{2}\mathbf{N} - \frac{1}{8}\mathbf{N}^2$  then  $\mathbf{A}^2 = \mathbf{I} + \mathbf{N}$ ) to prove that

if  $c$  is a non-zero complex number and  $\mathbf{N}$  is a nilpotent complex matrix, then  $(c\mathbf{I} + \mathbf{N})$  has a square root. Now use the Jordan form to prove that every non-singular complex  $n \times n$  matrix has a square root.

**Solution:** See Table 7.2.6.1

|                   |  |
|-------------------|--|
| <b>Given</b>      | <p>If <math>\mathbf{A} = \mathbf{I} + \frac{1}{2}\mathbf{N} - \frac{1}{8}\mathbf{N}^2</math><br/> then <math>\mathbf{A}^2 = \mathbf{I} + \mathbf{N}</math> that is, <math>\mathbf{I} + \mathbf{N}</math> has a<br/> square root, where <math>\mathbf{N}</math> is a nilpotent matrix.</p>  |
| <b>To prove</b>   |  |
| <b>1</b>          | <p><math>(c\mathbf{I} + \mathbf{N})</math> has a square root, that is <math>(c\mathbf{I} + \mathbf{N}) = \mathbf{X}^2</math><br/> where <math>c</math> is a non-zero complex number, <math>\mathbf{N}</math> is a nilpotent<br/> complex matrix and <math>\mathbf{X}</math> is any matrix.</p>   |
| <b>2</b>          | <p>Every non-singular complex <math>n \times n</math> matrix<br/> <math>(\mathbf{B})</math> has a square root, that is, <math>\mathbf{B} = \mathbf{Y}^2</math> where <math>\mathbf{Y}</math><br/> is any matrix.</p>   |
| <b>Proof 1</b>    | <p>Let us consider, <math>c \neq 0</math> and <math>c \in \mathbb{C}</math><br/> then, there exists <math>\frac{1}{c} \in \mathbb{C}</math><br/> Given <math>\mathbf{N}</math> is a nilpotent,<br/> <math>\Rightarrow \frac{1}{c}\mathbf{N}</math> is also nilpotent.<br/> From the problem statement, we get that <math>(\mathbf{I} + \frac{1}{c}\mathbf{N})</math><br/> has a square root, that is, <math>(\mathbf{I} + \frac{1}{c}\mathbf{N}) = \mathbf{M}^2</math><br/> where <math>\mathbf{M}^2</math> is any matrix.<br/> so, <math>(\mathbf{I} + \frac{1}{c}\mathbf{N}) = \mathbf{M}^2</math><br/> Multiplying both side with <math>c</math>, we get<br/> <math>c(\mathbf{I} + \frac{1}{c}\mathbf{N}) = c\mathbf{M}^2</math><br/> <math>(c\mathbf{I} + \mathbf{N}) = c\mathbf{M}^2</math><br/> <math>\Rightarrow (c\mathbf{I} + \mathbf{N}) = (\sqrt{c}\mathbf{M})^2</math><br/> <math>\Rightarrow (c\mathbf{I} + \mathbf{N}) = \mathbf{X}^2</math><br/> where <math>\mathbf{X} = (\sqrt{c}\mathbf{M})</math></p> |
| <b>Conclusion</b> | <p>Hence it is proved that <math>(c\mathbf{I} + \mathbf{N})</math><br/> has a square root</p>  |
| <b>Proof 2</b>    | <p>Let <math>\mathbf{B}</math> is any non-singular complex matrix.<br/> So, <math>\mathbf{B}\mathbf{B}^{-1} = \mathbf{I}</math><br/> As per the Jordon's Theorem, every square matrix <math>\mathbf{B}</math><br/> is similar to a Jordon matrix <math>\mathbf{J}</math>, that is, <math>\mathbf{B} = \mathbf{PJP}^{-1}</math><br/> Now, let consider a <math>n \times n</math> nilpotent shift matrix <math>\mathbf{N}</math><br/> <math>\mathbf{N} =</math><br/> <math display="block">\begin{pmatrix} 0 &amp; 1 &amp; 0 &amp; \dots &amp; 0 \\ 0 &amp; 0 &amp; 1 &amp; \dots &amp; 0 \\ \cdot &amp; \cdot &amp; \cdot &amp; \dots &amp; \cdot \\ \cdot &amp; \cdot &amp; \cdot &amp; \dots &amp; \cdot \\ \cdot &amp; \cdot &amp; \cdot &amp; \dots &amp; 1 \\ 0 &amp; 0 &amp; 0 &amp; \dots &amp; 0 \end{pmatrix}</math> <math>\mathbf{I}</math> is a <math>n \times n</math> identity matrix, so<br/> <math>c\mathbf{I} + \mathbf{N} =</math></p>   |

|                   |   |
|-------------------|---|
|                   | $\begin{pmatrix} c & 1 & 0 & \dots & 0 \\ 0 & c & 1 & \dots & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & 1 \\ 0 & 0 & 0 & \dots & c \end{pmatrix}$ <p>and this is a Jordan form.</p> <p>So, we can consider <math>\mathbf{J} = c\mathbf{I} + \mathbf{N}</math> and<br/> as <math>c\mathbf{I} + \mathbf{N} = \mathbf{X}^2 \implies \mathbf{J} = \mathbf{X}^2</math><br/> <math>\implies \mathbf{B} = \mathbf{P}\mathbf{X}^2\mathbf{P}^{-1}</math><br/> <math>\implies \mathbf{B} = \mathbf{P}\mathbf{X}\mathbf{P}^{-1}\mathbf{P}\mathbf{X}\mathbf{P}^{-1}</math><br/> <math>\implies \mathbf{B} = (\mathbf{P}\mathbf{X}\mathbf{P}^{-1})^2</math><br/> <math>\implies \mathbf{B} = \mathbf{Y}^2</math>, where <math>\mathbf{Y} = \mathbf{P}\mathbf{X}\mathbf{P}^{-1}</math></p> <p>This implies that <math>\mathbf{B}</math> has a square root.</p> |
| <b>Conclusion</b> | <p>Hence it is proved that every non-singular complex <math>n \times n</math> matrix has a square root.</p>   |

TABLE 7.2.6.1: **Solution summary**

### 7.3 Computation of Invariant Factors

- 7.3.1. Construct a linear operator  $T$  with minimal polynomial  $x^2(x-1)^2$  and characteristic polynomial  $x^3(x-1)^4$ . Describe the primary decomposition of the vector space under  $T$  and find the projections on the primary components. Find a basis in which the matrix  $T$  is in Jordan form. Also find an explicit direct sum decomposition of the space into  $T$ -cyclic subspaces as in theorem 3 and give the invariant factors.

**Solution:** See Table 7.3.1.1

| Statement  | Solution  |
|--|---|
| <b>Jordan Form</b>   |   |
| Given  | <p>Linear operator</p> $T : \mathbf{V} \rightarrow \mathbf{V} \quad (7.3.1.1)$ <p>Characteristic polynomial <math>f(x) = x^3(x-1)^4 \quad (7.3.1.2)</math></p> <p>Minimal polynomial <math>p(x) = x^2(x-1)^2 \quad (7.3.1.3)</math></p>   |
| The jordan block corresponding to eigen value 0                  | $\mathbf{J}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (7.3.1.4)$  |
| One of the possible jordan blocks corresponding to eigen value 1 | $\mathbf{J}_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (7.3.1.5)$   |
| The jordan form of transformation matrix $\mathbf{T}$            | $\mathbf{J} = \begin{pmatrix} \mathbf{J}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_2 \end{pmatrix} \quad (7.3.1.6)$ $= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (7.3.1.7)$                   |
| <b>Primary Decomposition, Finding the basis</b>                  |   |
| According to primary decomposition theorem                       | <p>If <math>p(x) = p_1(x)^{r_1} p_2(x)^{r_2}</math>, <math>(7.3.1.8)</math></p> <p><math>\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2 \quad (7.3.1.9)</math></p> <p><math>\mathbf{V}_i = \text{Null space of } (p_i(\mathbf{T}))_i^r \quad (7.3.1.10)</math></p> <p><math>p_1(x)^{r_1} = x^2 \quad (7.3.1.11)</math></p> <p><math>p_2(x)^{r_2} = (x-1)^2 \quad (7.3.1.12)</math></p>                |
| Null space of $\mathbf{J}^2$                                     | $\mathbf{J}^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (7.3.1.13)$ <p>Nullity of <math>\mathbf{J}^2 = 3 \quad (7.3.1.14)</math></p> <p>From (7.3.1.13), the basis for the nullspace is</p> |

|  |  |
|--|--|
|  | $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \quad (7.3.1.15)$ $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (7.3.1.16)$  |
| Nullspace of $(\mathbf{J} - \mathbf{I})^2$                         | $(\mathbf{J} - \mathbf{I})^2 = \begin{pmatrix} 1 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (7.3.1.17)$ $\text{Nullity of } (\mathbf{J} - \mathbf{I})^2 = 4 \quad (7.3.1.18)$ <p>From (7.3.1.17), the basis for the nullspace is</p> $\{\mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6, \mathbf{v}_7\} \quad (7.3.1.19)$ $\mathbf{v}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{v}_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{v}_6 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v}_7 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad (7.3.1.20)$ |
| $\mathbf{T} = \mathbf{J} \quad (7.3.1.21)$                         | <p><math>\mathbf{T}</math> is similar to block diagonal jordan matrix <math>\mathbf{J}</math> in the basis</p> $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6, \mathbf{v}_7\} \quad (7.3.1.22)$ <p>which is the standard ordered basis.</p>   |
| <b>Finding the projections</b>                                     |  |
| The projection matrices $\mathbf{E}_1, \mathbf{E}_2$ are such that | <p>a) for <math>i \in [1, 2]</math></p> $\mathbf{E}_i(\mathbf{v}) = \begin{cases} \mathbf{v} & \text{for } \mathbf{v} \in \mathbf{V}_i \\ 0 & \text{for } \mathbf{v} \notin \mathbf{V}_i \end{cases} \quad (7.3.1.23)$ <p>b)</p> $(\mathbf{E}_i)^2 = \mathbf{E}_i \quad (7.3.1.24)$ <p>c)</p> $\mathbf{E}_1 + \mathbf{E}_2 = \mathbf{I} \quad (7.3.1.25)$  |

|  |   |
|--|---|
| The projection matrices are                                      | $\mathbf{E}_1 = \begin{pmatrix} \mathbf{I}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \quad (7.3.1.26)$ $= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (7.3.1.27)$ $\mathbf{E}_2 = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_4 \end{pmatrix} \quad (7.3.1.28)$ $= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (7.3.1.29)$ |
| <b>Cyclic Decomposition</b>                                      |   |
| Cyclic decomposition theorem(Theorem 3)                          | <p>Let <math>T</math> be a linear operator on a finite-dimensional vector space <math>\mathbf{V}</math> and let <math>\mathbf{W}_0</math> be a proper <math>T</math>-admissible subspace of <math>\mathbf{V}</math>. There exists non zero vectors <math>\alpha_1, \alpha_2, \dots, \alpha_r</math> in <math>\mathbf{V}</math> with respective <math>T</math>-annihilators <math>p_1, p_2, \dots, p_r</math> such that</p> <p>a)</p> $\mathbf{V} = \mathbf{W}_0 \oplus \mathbf{Z}(\alpha_1; T) \oplus \mathbf{Z}(\alpha_2; T) \oplus \dots \oplus \mathbf{Z}(\alpha_r; T) \quad (7.3.1.30)$ <p>b) <math>p_k</math> divides <math>p_{k-1}, k = 2, \dots, r</math></p>  |
| The $T$ -cyclic subspace $\mathbf{Z}(\alpha_i; T)$ is defined as | $\text{degree of } p_i = k \quad (7.3.1.32)$ $\implies \text{basis of } \mathbf{Z}(\alpha_i; T) = \quad (7.3.1.33)$ $\{\alpha_i, \mathbf{T}\alpha_i, \dots, \mathbf{T}^{k-1}\alpha_i\} \quad (7.3.1.34)$  |
| Finding the cyclic subspaces                                     | <p>Let us choose</p> $\mathbf{W}_0 = \mathbf{0} \quad (7.3.1.35)$ $\quad (7.3.1.36)$  |



|                                     |   |
|-------------------------------------|---|
| $p_1 = x^2(x-1)^2 \quad (7.3.1.37)$ | $\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (7.3.1.38)$ $\mathbf{T}^2(\mathbf{T} - \mathbf{I})^2 = \mathbf{0}_{7 \times 7} \quad (7.3.1.39)$ $\implies p_1(\mathbf{T})(\alpha_1) = 0 \quad (7.3.1.40)$ $\dim(\mathbf{Z}(\alpha_1; T)) = 4 \quad (7.3.1.41)$   |
| $p_2 = x(x-1)^2 \quad (7.3.1.42)$   | <p>Now to chose <math>\alpha_2</math> we need to chose a vector such that</p> $\alpha_2 \notin \mathbf{Z}(\alpha_1; T), p_2(\mathbf{T})\alpha_2 = 0 \quad (7.3.1.43)$ $p_2 = x(x-1)^2 \quad (7.3.1.44)$ $\frac{p_1}{p_2} = x \implies p_2 \text{ divides } p_1 \quad (7.3.1.45)$ $p_2(\mathbf{T}) = \mathbf{T}(\mathbf{T} - \mathbf{I})^2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (7.3.1.47)$ <p>One such vector that satisfies (7.3.1.43) is</p> $\alpha_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (7.3.1.48)$ $\dim(\mathbf{Z}(\alpha_2; T)) = 3 \quad (7.3.1.49)$ $\dim \mathbf{Z}(\alpha_1; T) + \dim \mathbf{Z}(\alpha_2; T) = 7 \quad (7.3.1.50)$ $\implies \mathbf{V} = \mathbf{Z}(\alpha_1; T) \oplus \mathbf{Z}(\alpha_2; T) \quad (7.3.1.51)$ <p>is the cyclic decomposition.</p> |
| <b>Invariant Factors</b>            |   |
| Invariant factors are               | $p_1 = x^2(x-1)^2 \quad (7.3.1.52)$ $p_2 = x(x-1)^2 \quad (7.3.1.53)$   |

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TABLE 7.3.1.1: Solution

## 8 INNER PRODUCT SPACES

## 8.1 Inner Products

8.1.1. Let  $(\cdot)$  be the standard inner product on  $\mathbb{R}^2$ .

a) Let

$$\alpha = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \beta = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad (8.1.1.1)$$

If  $\gamma$  is a vector such that  $(\alpha^T \gamma) = -1$  and  $(\beta^T \gamma) = 3$ . Find  $\gamma$

b) Show that for any  $\alpha$  in  $\mathbb{R}^2$  we have

$$\alpha = (\alpha^T \epsilon_1) \epsilon_1 + (\alpha^T \epsilon_2) \epsilon_2 \quad (8.1.1.2)$$

**Solution:**

(a) From  $\alpha = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$  and  $(\alpha^T \gamma) = -1$  we get,

$$(\alpha^T \gamma) = -1 \quad (8.1.1.3)$$

$$\Rightarrow \begin{pmatrix} 1 \\ 2 \end{pmatrix}^T \gamma = -1 \quad (8.1.1.4)$$

from  $\beta = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$  and  $(\beta^T \gamma) = 3$  we get,

$$(\beta^T \gamma) = 3 \quad (8.1.1.5)$$

$$\Rightarrow \begin{pmatrix} -1 \\ 1 \end{pmatrix}^T \gamma = 3 \quad (8.1.1.6)$$

using (8.1.1.4) and (8.1.1.6),

$$\begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} (\gamma) = \begin{pmatrix} -1 \\ 3 \end{pmatrix} \quad (8.1.1.7)$$

row reductions,

$$\begin{pmatrix} 1 & 2 & -1 \\ -1 & 1 & 3 \end{pmatrix} \xrightarrow{R_2 \rightarrow R_2 + R_1} \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & 2 \end{pmatrix} \quad (8.1.1.8)$$

$$\xleftarrow{R_2 \rightarrow \frac{1}{3} R_2} \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & \frac{2}{3} \end{pmatrix} \xleftarrow{R_1 \rightarrow R_1 - 2R_2} \begin{pmatrix} 1 & 0 & -\frac{7}{3} \\ 0 & 1 & \frac{2}{3} \end{pmatrix} \quad (8.1.1.9)$$

Hence  $\gamma = \begin{pmatrix} -\frac{7}{3} \\ \frac{2}{3} \end{pmatrix}$

(b) Here  $\epsilon_1, \epsilon_2$  are standard basis vector in  $\mathbb{R}^2$ .  
As  $\alpha \in \mathbb{R}^2$  we can write it as,

$$\alpha = \alpha_1 \epsilon_1 + \alpha_2 \epsilon_2 = \alpha^T \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix} \quad (8.1.1.10)$$

using this we can write,

$$(\alpha^T \epsilon_1) = \left( \alpha^T \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix} \right)^T \epsilon_1 = \alpha^T \begin{pmatrix} \epsilon_1^T \\ \epsilon_2^T \end{pmatrix} \epsilon_1 \quad (8.1.1.11)$$

$$= \alpha^T \begin{pmatrix} \epsilon_1^T \epsilon_1 \\ \epsilon_2^T \epsilon_1 \end{pmatrix} = \alpha^T \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (8.1.1.12)$$

$$\Rightarrow (\alpha^T \epsilon_1) \epsilon_1 = \alpha^T \begin{pmatrix} \epsilon_1 \\ 0 \end{pmatrix} \quad (8.1.1.13)$$

$$(\alpha^T \epsilon_2) = \left( \alpha^T \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix} \right)^T \epsilon_2 = \alpha^T \begin{pmatrix} \epsilon_1^T \\ \epsilon_2^T \end{pmatrix} \epsilon_2 \quad (8.1.1.14)$$

$$= \alpha^T \begin{pmatrix} \epsilon_1^T \epsilon_2 \\ \epsilon_2^T \epsilon_2 \end{pmatrix} = \alpha^T \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (8.1.1.15)$$

$$\Rightarrow (\alpha^T \epsilon_2) \epsilon_2 = \alpha^T \begin{pmatrix} 0 \\ \epsilon_2 \end{pmatrix} \quad (8.1.1.16)$$

using (8.1.1.13) and (8.1.1.16) we can write,

$$(\alpha^T \epsilon_1) \epsilon_1 + (\alpha^T \epsilon_2) \epsilon_2 = \alpha^T \begin{pmatrix} \epsilon_1 \\ 0 \end{pmatrix} + \alpha^T \begin{pmatrix} 0 \\ \epsilon_2 \end{pmatrix} \quad (8.1.1.17)$$

$$\Rightarrow (\alpha^T \epsilon_1) \epsilon_1 + (\alpha^T \epsilon_2) \epsilon_2 = \alpha^T \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix} \quad (8.1.1.18)$$

hence using (8.1.1.10) and (8.1.1.18) we get,

$$\alpha = (\alpha^T \epsilon_1) \epsilon_1 + (\alpha^T \epsilon_2) \epsilon_2 \quad (8.1.1.19)$$

Hence proved