Solutions: Linear Algebra by Hoffman and Kunze

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Contents

1 Fields and Linear Equations 1

2 Matrices and Elementary Row Operations 3

***Abstract*—This book provides solutions to the Linear Algebra book by Hoffman and Kunze.**

1 Fields and Linear Equations

1.1. Let $\mathbb{F}$ be a set which contains exactly two elements,0 and 1.Define an addition and multiplication by tables. Verify that the set $\mathbb{F}$,

+	0	1
0	0	1
1	1	0

$\cdot$	0	1
0	0	0
1	0	1

together with these two operations, is a field.
Solution:

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To prove that ($\mathbb{F}$,+,$\cdot$) is a field we need to satisfy the following,

a) + and $\cdot$ should be closed
   - For any a and b in $\mathbb{F}$, a+b $\in \mathbb{F}$ and a $\cdot$ b $\in \mathbb{F}$.For example 0+0=0 and 0$\cdot$0=0.

b) + and $\cdot$ should be commutative
   - For any a and b in $\mathbb{F}$, a+b = b+a and a $\cdot$ b = b$\cdot$ a. For example 0+1=1+0 and 0$\cdot$ 1=1$\cdot$ 0.

c) + and $\cdot$ should be associative
   - For any a and b in $\mathbb{F}$, a+(b+c) = (a+b)+c and a$\cdot$ (b $\cdot$ c) = (a $\cdot$ b) $\cdot$ c. For example 0+(1+0)=(0+1)+0 and 0$\cdot$(1$\cdot$0)=(0$\cdot$1)$\cdot$0.

d) + and $\cdot$ operations should have an identity element
   - If we perform a + 0 then for any value of a from $\mathbb{F}$ the result will be a itself. Hence 0 is an identity element of + operation.If we perform a $\cdot$ 1 then for any value of a from $\mathbb{F}$ the result will be a itself. Hence 1 is an identity element of $\cdot$ operation.

e) $\forall$ a $\in \mathbb{F}$ there exists an additive inverse
   - For additive inverse to exist, $\forall$ a in $\mathbb{F}$ a+(-a)=0. For example. 1-1=0 and 0-0=0.

f) $\forall$ a $\in \mathbb{F}$ such that a is non zero there exists a multiplicative inverse
   - For multiplicative inverse to exist, $\forall$ a such that a is non zero in $\mathbb{F}, a \cdot a^{-1}$=1.For example $1 \cdot 1^{-1} = 1$.

g) + and · should hold distributive property
- For any a,b and c in $\mathbb{F}$ the property a·(b+c)=a·b+a·c should always hold true.For example 0·(1+1)=0·1+0·1.

Since the above properties are satisfied we can say that ($\mathbb{F}$,+,·) is a field.

1.2. Prove that each subfield of the field of complex number contains every rational number
Solution:
Complex Numbers: A complex number is a number that can be expressed in the form $a + bi$, where a and b are real numbers, and i represents the imaginary unit, satisfying the equation $i^2 = -1$.The set of complex numbers is denoted by $\mathbb{C}$

$$\mathbb{C} = \{(a, b) : a, b \in \mathbb{R}\} \tag{1.2.1}$$

Rational Numbers: A number in the form $\frac{p}{q}$, where both p and q(non-zero) are integers, is called a rational number.The set of rational numbers is dentoed by $\mathbb{Q}$ Let $\mathbb{Q}$ be the set of rational numbers.

$$\mathbb{Q} = \left\{\frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{Z}_{\neq 0}\right\} \tag{1.2.2}$$

Let $\mathbb{C}$ be the field of complex numbers and given $\mathbb{F}$ be the subfield of field of complex numbers $\mathbb{C}$ Since $\mathbb{F}$ is the subfield , we could say that

$$0 \in \mathbb{F} \tag{1.2.3}$$
$$1 \in \mathbb{F} \tag{1.2.4}$$

Closed under addition: Here $\mathbb{F}$ is closed under addition since it is subfield

$$1 + 1 = 2 \in \mathbb{F} \tag{1.2.5}$$
$$1 + 1 + 1 = 3 \in \mathbb{F} \tag{1.2.6}$$
$$\vdots$$
$$1 + 1 + \cdots + 1(\text{p times}) = p \in \mathbb{F} \tag{1.2.7}$$
$$1 + 1 + \cdots + 1(\text{q times}) = q \in \mathbb{F} \tag{1.2.8}$$

By using the above property we could say that zero and other positive integers belongs to $\mathbb{F}$.Since p and q are integers we say,

$$p \in \mathbb{Z} \tag{1.2.9}$$
$$q \in \mathbb{Z} \tag{1.2.10}$$

Additive Inverse: Let x be the positive integer belong $\mathbb{F}$ and by additive inverse we could say,

$$\forall x \in \mathbb{F} \tag{1.2.11}$$
$$(-x) \in \mathbb{F} \tag{1.2.12}$$

Therefore field $\mathbb{F}$ contains every integers. Let n be a integer then,

$$n \in \mathbb{Z} \implies n \in \mathbb{F} \tag{1.2.13}$$
$$\mathbb{Z} \subseteq \mathbb{F} \tag{1.2.14}$$

Where $\mathbb{Z}$ is subset of $\mathbb{F}$ Multiplicative Inverse: Every element except zero in the subfield $\mathbb{F}$ has an multiplicative inverse. From equation (1.2.8), since q $\in \mathbb{F}$ we could say ,

$$\frac{1}{q} \in \mathbb{F} \quad \text{and } q \neq 0 \tag{1.2.15}$$

Closed under multiplication: Also, $\mathbb{F}$ is closed under multiplication and thus, from equation (1.2.7) and (1.2.15) we get ,

$$p \cdot \frac{1}{q} \in \mathbb{F} \tag{1.2.16}$$
$$\implies \frac{p}{q} \in \mathbb{F} \tag{1.2.17}$$

where , $p \in \mathbb{Z}$ and $q \in \mathbb{Z}_{\neq 0}$ (from equation (1.2.10) and (1.2.15)) Conclusion From (1.2.2) and (1.2.17) we could say ,

$$\mathbb{Q} \subseteq \mathbb{F} \tag{1.2.18}$$

From equation (1.2.18) we could say that each subfield of the field of complex number contains every rational number

Hence Proved

1.3. Prove that, each field of the characteristic zero contains a copy of the rational number field.
Solution: The characteristic of a field is defined to be the smallest number of times one must use the field's multiplicative identity (1) in a sum to get the additive identity. If this sum never reaches the additive identity (0), then the field is said to have characteristic zero.
Let $\mathbb{Q}$ be the rational number field. Hence,

$$0 \in \mathbb{Q} \qquad [\text{Additive Identity}] \tag{1.3.1}$$
$$1 \in \mathbb{Q} \qquad [\text{Multiplicative Identity}] \tag{1.3.2}$$

As addition is defined on $\mathbb{Q}$ hence we have,

$$1 \neq 0 \tag{1.3.3}$$
$$1 + 1 = 2 \neq 0 \tag{1.3.4}$$

And so on,

$$1 + 1 + \cdots + 1 = n \neq 0 \tag{1.3.5}$$

From the definition of characteristic of a field and from (1.3.3), (1.3.4) and so on up-to (1.3.5), the rational number field, $\mathbb{Q}$ has characteristic 0.

2 Matrices and Elementary Row Operations

2.1. Prove that the interchange of two rows of a matrix can be accomplished by a finite sequence of elementary row operations of the other two types.
Solution: Let $\mathbf{A}$ be a 3×3 matrix with having row vectors $\mathbf{a}_1$,$\mathbf{a}_2$ and $\mathbf{a}_3$.

$$\mathbf{A} = \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} \tag{2.1.1}$$

Let's exchange row $\mathbf{a}_1$ and $\mathbf{a}_2$. Let's call this elementary operation $\mathbf{E_1}$.

$$\mathbf{E}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.1.2}$$
$$\tag{2.1.3}$$

Now performing operation $\mathbf{E}_1$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{a}_2 \\ \mathbf{a}_1 \\ \mathbf{a}_3 \end{pmatrix} \tag{2.1.4}$$

Now, to prove that same matrix can be obtained by elementary operations let's call them $\mathbf{E_2}$ and $\mathbf{E_3}$.Now performing operation $\mathbf{E}_2$ by adding row 2 to row 1.

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{a}_1 + \mathbf{a}_2 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} \tag{2.1.5}$$

Using elementary operation $\mathbf{E_2}$ we will subtract row 1 from row 2.

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}_1 + \mathbf{a}_2 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{a}_1 + \mathbf{a}_2 \\ -\mathbf{a}_1 \\ \mathbf{a}_3 \end{pmatrix} \tag{2.1.6}$$

Using elementary operation $\mathbf{E_2}$ we will add row 2 to row 1.

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}_1 + \mathbf{a}_2 \\ -\mathbf{a}_1 \\ \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{a}_2 \\ -\mathbf{a}_1 \\ \mathbf{a}_3 \end{pmatrix} \tag{2.1.7}$$

Using elementary operation $\mathbf{E_3}$ we will multiply row 2 by -1.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}_2 \\ -\mathbf{a}_1 \\ \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{a}_2 \\ \mathbf{a}_1 \\ \mathbf{a}_3 \end{pmatrix} \tag{2.1.8}$$

Hence, we can say that,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} \tag{2.1.9}$$

Let us assume a matrix $\mathbf{A}$

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \tag{2.1.10}$$

Let's exchange row $\mathbf{a}_1$ and $\mathbf{a}_2$ by applying operation $\mathbf{E_1}$.

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & 3 \\ 1 & 1 & 0 \end{pmatrix} \tag{2.1.11}$$

Now, to prove that same matrix can be obtained by other two elementary operations.We will first perform elementary operation $\mathbf{E_2}$ by adding row 2 to row 1.

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \tag{2.1.12}$$

Using elementary operation $\mathbf{E_2}$ we will subtract row 1 from row 2.

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 3 \\ -1 & -2 & -3 \\ 1 & 1 & 0 \end{pmatrix} \tag{2.1.13}$$

Using elementary operation $\mathbf{E_2}$ we will add row

2 to row 1.

$$\begin{pmatrix}1 & 1 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 3 & 3\\-1 & -2 & -3\\1 & 1 & 0\end{pmatrix}=\begin{pmatrix}0 & 1 & 0\\-1 & -2 & -3\\1 & 1 & 0\end{pmatrix} \tag{2.1.14}$$

Using elementary operation $\mathbf{E_3}$ we will multiply row 2 by -1.

$$\begin{pmatrix}1 & 0 & 0\\0 & -1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1 & 0\\-1 & -2 & -3\\1 & 1 & 0\end{pmatrix}=\begin{pmatrix}0 & 1 & 0\\1 & 2 & 3\\1 & 1 & 0\end{pmatrix} \tag{2.1.15}$$

Hence,we can say that,

$$\begin{pmatrix}1 & 0 & 0\\0 & -1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 1 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\-1 & 1 & 0\\0 & 0 & 1\end{pmatrix}$$
$$\times\begin{pmatrix}1 & 1 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\0 & 1 & 0\\1 & 1 & 0\end{pmatrix}=\begin{pmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\0 & 1 & 0\\1 & 1 & 0\end{pmatrix} \tag{2.1.16}$$