Linear Algebra

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***Abstract*—This book provides solved examples on Linear Algebra.**

1 June 2019

1.1. Consider the vector space $\mathbb{P}_n$ of real polynomials in x of degree $\le n$. Define

$$T : \mathbb{P}_2 \to \mathbb{P}_3 \tag{1.1.1}$$

by

$$(Tf)(x) = \int_0^x f(t)\,dt + f'(x). \tag{1.1.2}$$

Then find the matrix representation of T with respect to the bases

$$\left\{1, x, x^2\right\} \text{ and } \left\{1, x, x^2, x^3\right\} \tag{1.1.3}$$

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1.2. Let $P_A(x)$ denote the characteristic polynomial of a matrix A. Then for which of the following matrices is

$$P_A(x) - P_{A^{-1}}(x) \tag{1.2.1}$$

a constant?

a) $\begin{pmatrix} 3 & 3 \\ 2 & 4 \end{pmatrix}$
b) $\begin{pmatrix} 4 & 3 \\ 2 & 3 \end{pmatrix}$
c) $\begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}$
d) $\begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$

1.3. Which of the following matrices is not diagonalizable over $\mathbb{R}$?

a) $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$
b) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$
c) $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$
d) $\begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$

1.4. What is the rank of the following matrix?

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} \tag{1.4.1}$$

1.5. Let V denote the vector space of real valued continuous functions on the close interval $[0, 1]$. Let W be the subspace of V spanned by $\{\sin x, \cos x, \tan x\}$. Find the dimension of W over $\mathbb{R}$.

1.6. Let V be the vector space of polynomials in the variable t of degree at most 2 over $\mathbb{R}$. An inner product on V is defined by

$$f^T g = \int_0^1 f(t)g(t)\,dt, \quad f, g \in V. \tag{1.6.1}$$

Let

$$W = span\left\{1 - t^2, 1 + t^2\right\} \tag{1.6.2}$$

and $W^\perp$ be the orthogonal complement of W in V. Which of the following conditions is satisfied for all $h \in W^\perp$?

a) h is an even function
b) h is an odd function
c) $h(t) = 0$ has a real solution
d) $h(0) = 0$

1.7. Consider solving the following system by Jacobi iteration scheme

$$\begin{pmatrix} 1 & 2m & -2m \\ n & 1 & n \\ 2m & 2m & 1 \end{pmatrix}\left(x\right) = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \tag{1.7.1}$$

where $m, n \in \mathbb{Z}$. With any initial vector, the scheme converges provided m, n satisfy

a) $m + n = 3$
b) $m > n$
c) $m < n$
d) $m = n$

1.8. Consider a Markov Chain with state space $\{0, 1, 2, 3, 4\}$ and transition matrix

$$P = \begin{array}{c c} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \tag{1.8.1}$$

Then find

$$\lim_{n\to\infty} p_{23}^{(n)} \tag{1.8.2}$$

1.9. Let $L(\mathbb{R})^n$ be the space of $\mathbb{R}-$linear maps from $\mathbb{R}^n$ to $\mathbb{R}^n$. If $Ker(T)$ denotes the kernel of T then which of the following are true?

a) There exists $T \in L\left(\mathbb{R}^5\right)$ $\{0\}$ such that $Range(T) = Ker(T)$
b) There does not exist $T \in L\left(\mathbb{R}^5\right)$ $\{0\}$ such that $Range(T) = Ker(T)$
c) There exists $T \in L\left(\mathbb{R}^6\right)$ $\{0\}$ such that $Range(T) = Ker(T)$
d) There does not exist $T \in L\left(\mathbb{R}^6\right)$ $\{0\}$ such that $Range(T) = Ker(T)$

1.10. Let V be a finite dimensional vector space over $\mathbb{R}$ and $T : V \to V$ be a linear map. Can you always write $T = T_2 \circ T_1$ for some linear maps

$$T_1 : V \to W, T : W \to V, \tag{1.10.1}$$

where W is some finite dimensional vector space such that

a) both T_1 and T_2 are onto
b) both T_1 and T_2 are one to one
c) T_1 is onto, T_2 is one to one
d) T_1 is one to one, T_2 is onto

1.11. Let $A = \left[a_{ij}\right]$ be a 3×3 complex matrix. Identify the correct statements

a) $det\left[(-1)^{i+j} a_{ij}\right] = det(A)$
b) $det\left[(-1)^{i+j} a_{ij}\right] = -det(A)$
c) $det\left[\left(\sqrt{-1}\right)^{i+j} a_{ij}\right] = det(A)$
d) $det\left[\left(\sqrt{-1}\right)^{i+j} a_{ij}\right] = -det(A)$

1.12. Let

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n \tag{1.12.1}$$

be a non-constant polynomial of degree $n \geq 1$. Consider the polynomial

$$q(x) = \int_0^x p(t)\,dt, r(x) = \frac{d}{dx}p(x) \tag{1.12.2}$$

Let V denote the real vector space of all polynomials in x. Then which of the following are true?

a) q and r are linearly independent in V
b) q and r are linearly dependent in V
c) x^n belongs to the linear span of q and r
d) x^{n+1} belongs to the linear span of q and r.

1.13. Let $M_n(\mathbb{R})$ be the ring of $n \times n$ matrices over $\mathbb{R}$. Which of the following are true for every $n \geq 2$?

a) there exist matrices $A, B \in M_n(\mathbb{R})$ such that $AB - BA = I_n$, where I_n denotes the identity matrix.
b) If $A, B \in M_n(\mathbb{R})$ and $AB = BA$, then A is diagonalisable over $\mathbb{R}$ if and only if B is diagonalisable over $\mathbb{R}$.
c) If $A, B \in M_n(\mathbb{R})$, then AB and BA have the

same minimal polynomial.

d) If $A, B \in M_n(\mathbb{R})$, then AB and BA have the same eigenvalues in $\mathbb{R}$.

1.14. Consider a matrix

$$A = \left[a_{ij}\right], 1 \le i, j \le 5 \tag{1.14.1}$$

such that

$$a_{ij} = \frac{1}{n_i + n_j + 1}, \quad n_i, n_j \in \mathbb{N} \tag{1.14.2}$$

Then in which of the following cases A is a positive definite matrix?

a) $n_i = 1 \forall i = 1, 2, 3, 4, 5$.
b) $n_1 < n_2 < \cdots < n_5$.
c) $n_1 = n_2 = \cdots = n_5$.
d) $n_1 > n_2 > \cdots > n_5$.

1.15. For a nonzero $w \in \mathbb{R}^n$, define

$$T_w : \mathbb{R}^n \to \mathbb{R}^n \tag{1.15.1}$$

by

$$T_w = v - \frac{2v^T w}{w^T w} w, \quad v \in \mathbb{R}^n \tag{1.15.2}$$

Which of the following are true?

a) $det(T_w) = 1$
b) $T_w(v_1)_w^T(v_2) = v_1^T v_2 \forall v_1, v_2 \in \mathbb{R}^n$
c) $T_w = T_w^{-1}$
d) $T_{2w} = 2T_w$

1.16. Consider the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{1.16.1}$$

over the field $\mathbb{Q}$ of rationals. Which of the following matrices are of the form $P^T AP$ for suitable 2×2 invertible matrix P over $\mathbb{Q}$?

a) $\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$
c) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
b) $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$
d) $\begin{pmatrix} 3 & 4 \\ 4 & 5 \end{pmatrix}$

1.17. Consider a Markov Chain with state space $\{0, 1, 2\}$ and transition matrix

$$P = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{pmatrix} \frac{1}{4} & \frac{5}{8} & \frac{1}{8} \\ \frac{1}{4} & 0 & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{8} & \frac{1}{8} \end{pmatrix} \end{array} \tag{1.17.1}$$

Then which of the following are true?

a) $\lim_{n\to\infty} p_{12}^{(n)} = 0$
b) $\lim_{n\to\infty} p_{12}^{(n)} = \lim_{n\to\infty} p_{21}^{(n)}$
c) $\lim_{n\to\infty} p_{22}^{(n)} = \frac{1}{8}$
d) $\lim_{n\to\infty} p_{21}^{(n)} = \frac{1}{3}$

2 December 2018

2.1. Consider the subspaces W_1 and W_2 of $\mathbb{R}^3$ given by

$$W_1 = \left\{\mathbf{x} \in \mathbb{R}^3 : \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \mathbf{x} = 0\right\} \tag{2.1.1}$$

$$W_2 = \left\{\mathbf{x} \in \mathbb{R}^3 : \begin{pmatrix} 1 & -1 & 1 \end{pmatrix} \mathbf{x} = 0\right\}. \tag{2.1.2}$$

If $W \subseteq \mathbb{R}^3$, such that

a) $W \cap W_2 = \text{span}\left\{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\}$
b) $\{W \cap W_1\} \perp \{W \cap W_2\}$,

then

a) $W = \text{span}\left\{\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\}$
b) $W = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}\right\}$
c) $W = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\}$
d) $W = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right\}$

2.2. Let

$$C = \left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right\} \tag{2.2.1}$$

be a basis of $\mathbb{R}^2$ and

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y \\ x - 2y \end{pmatrix}. \tag{2.2.2}$$

If $T[C]$ represents the matrix of T with respect to the basis C then which among the following is true?

a) $T[C] = \begin{pmatrix} -3 & -2 \\ 3 & 1 \end{pmatrix}$
b) $T[C] = \begin{pmatrix} 3 & -2 \\ -3 & 1 \end{pmatrix}$
c) $T[C] = \begin{pmatrix} -3 & -1 \\ 3 & 2 \end{pmatrix}$
d) $T[C] = \begin{pmatrix} 3 & -1 \\ -3 & 2 \end{pmatrix}$

2.3. Let $W_1 = \left\{\mathbf{x} \in \mathbb{R}^4 :\right\}$

$$\begin{pmatrix}1 & 1 & 1 & 0\end{pmatrix}\mathbf{x} = 0 \tag{2.3.1}$$
$$\begin{pmatrix}0 & 2 & 0 & 1\end{pmatrix}\mathbf{x} = 0 \tag{2.3.2}$$
$$\begin{pmatrix}2 & 0 & 2 & -1\end{pmatrix}\mathbf{x} = 0 \tag{2.3.3}$$

and $W_2 = \left\{\mathbf{x} \in \mathbb{R}^4 :\right\}$

$$\begin{pmatrix}1 & 1 & 0 & 1\end{pmatrix}\mathbf{x} = 0 \tag{2.3.4}$$
$$\begin{pmatrix}1 & 0 & 1 & -2\end{pmatrix}\mathbf{x} = 0 \tag{2.3.5}$$
$$\begin{pmatrix}0 & 1 & 0 & -1\end{pmatrix}\mathbf{x} = 0. \tag{2.3.6}$$

Then which among the following is true?

a) $\dim(W_1) = 1$
b) $\dim(W_2) = 2$
c) $\dim(W_1 \cap W_2) = 1$
d) $\dim(W_1 + W_2) = 3$

2.4. Let A be an $n\times n$ complex matrix. Assume that A is self-adjoint and let B denote the inverse of $A + \jmath I$. Then all eigenvalues of $(A - \jmath I)\,B$ are

a) purely imaginary
b) of modulus one
c) real
d) of modulus less than one

2.5. Let $\{u_1, u_2, \ldots, u_n\}$ be an orthonormal basis of $\mathbb{C}^n$ as column vectors.Let

$$\mathbf{M} = \begin{pmatrix}\mathbf{u}_1 & \mathbf{u}_2 & \ldots & \mathbf{u}_k\end{pmatrix}, \tag{2.5.1}$$
$$\mathbf{N} = \begin{pmatrix}\mathbf{u}_{k+1} & \mathbf{u}_{k+2} & \ldots & \mathbf{u}_n\end{pmatrix} \tag{2.5.2}$$

and $\mathbf{P}$ be the diagonal $k\times k$ matrix with diagonal entries $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R}$. Then which of the following is true?

a) rank($\mathbf{MPM}^*$) $= k$ whenever $\alpha_i \neq \alpha_j, 1 \leq i, j \leq k$.
b) tr($\mathbf{MPM}^*$) $= \sum_{i=1}^{k} \alpha_i$
c) rank($\mathbf{M}^*\mathbf{N}$) $= \min(k, n-k)$
d) rank($\mathbf{MM}^* + \mathbf{NN}^*$) $< n$.

2.6. Let $B : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the function

$$B(a, b) = ab \tag{2.6.1}$$

Which of the following is true?

a) B is a linear transformation
b) B is a positive definite bilinear form
c) B is symmetric but not positive definite
d) B is neither linear nor bilinear

2.7. Let $\mathbf{A}$ be an invertible real $n\times n$ matrix. Define a function

$$F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \tag{2.7.1}$$

by

$$F(\mathbf{x}, \mathbf{y}) = (F\mathbf{x})^T\,\mathbf{y} \tag{2.7.2}$$

Let $DF(\mathbf{x}, \mathbf{y})$ denote the derivate of F at $(\mathbf{x}, \mathbf{y})$ which is a linear transformation from

$$\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \tag{2.7.3}$$

Then, if

a) $\mathbf{x} \neq 0, DF(\mathbf{x}, \mathbf{0}) \neq 0$
b) $\mathbf{y} \neq 0, DF(\mathbf{0}, \mathbf{y}) \neq 0$
c) $(\mathbf{x}, \mathbf{y}) \neq (\mathbf{0}, \mathbf{0}), DF(\mathbf{x}, \mathbf{0}) \neq 0$
d) $\mathbf{x} = 0$ or $\mathbf{y} = 0, DF(\mathbf{x}, \mathbf{y}) = 0$

2.8. Let

$$T : \mathbb{R}^n \to \mathbb{R}^n \tag{2.8.1}$$

be a linear map that satisfies

$$T^2 = T - I. \tag{2.8.2}$$

Then which of the following is true?

a) T is invertible.
b) $T - I$ is not invertible.
c) T has a real eigenvalue.
d) $T^3 = -I$.

2.9. Let

$$\mathbf{M} = \begin{pmatrix}2 & 0 & 3 & 2 & 0 & -2\\ 0 & 1 & 0 & -1 & 3 & 4\\ 0 & 0 & 1 & 0 & 4 & 4\\ 1 & 1 & 1 & 0 & 1 & 1\end{pmatrix} \tag{2.9.1}$$

$$\mathbf{b}_1 = \begin{pmatrix}5\\1\\1\\4\end{pmatrix}, \mathbf{b}_2 = \begin{pmatrix}5\\1\\3\\3\end{pmatrix}. \tag{2.9.2}$$

Then which of the following are true?

a) both systems $\mathbf{Mx} = \mathbf{b}_1$ and $\mathbf{Mx} = \mathbf{b}_2$ are inconsistent.
b) both systems $\mathbf{Mx} = \mathbf{b}_1$ and $\mathbf{Mx} = \mathbf{b}_2$ are consistent.
c) the system $\mathbf{Mx} = \mathbf{b}_1 - \mathbf{b}_2$ is consistent.
d) the system $\mathbf{Mx} = \mathbf{b}_1 - \mathbf{b}_2$ is inconsistent.

2.10. Let

$$\mathbf{M} = \begin{pmatrix}1 & -1 & 1\\ 2 & 1 & 4\\ -2 & 1 & -4\end{pmatrix}. \tag{2.10.1}$$

Given that 1 is an eigenvalue of $\mathbf{M}$, then which among the following are correct?

a) The minimal polynomial of $\mathbf{M}$ is $(x-1)(x+4)$
b) The minimal polynomial of $\mathbf{M}$ is $(x-1)^2(x+4)$
c) $\mathbf{M}$ is not diagonalizable.
d) $\mathbf{M}^{-1} = \frac{1}{4}(\mathbf{M}+3\mathbf{I})$.

2.11. Let $\mathbf{A}$ be a real matrix with characteristic polynomial $(x-1)^3$. Pick the correct statements from below:

a) $\mathbf{A}$ is necessarily diagonalizable.
b) If the minimal polynomial of $\mathbf{A}$ is $(x-1)^3$, then $\mathbf{A}$ is diagonalizable.
c) The characteristic polynomial of $\mathbf{A}^2$ is $(x-1)^3$
d) If $\mathbf{A}$ has exactly two Jordan blocks, then $(\mathbf{A}-\mathbf{I})^2$ is diagonalizable.

2.12. Let P_3 be the vector space of polynomails with real coefficients and of degree at most 3. Consider the linear map

$$T : P_3 \to P_3 \tag{2.12.1}$$

defined by

$$T(p(x)) = p(x-1) + p(x+1) \tag{2.12.2}$$

Which of the following properties does the matrix of T with respect to the standard basis $B = \left\{1, x, x^2, x^3\right\}$ of P_3 satisfy?

a) $detT = 0$.
b) $(T-2I)^4 = 0$ but $(T-2I)^3 \neq 0$.
c) $(T-2I)^3 = 0$ but $(T-2I)^2 \neq 0$.
d) 2 is an eigenvalue with multiplicity 4.

2.13. Let $\mathbf{M}$ be an $n \times n$ Hermitian matrix of rank $k, k \neq n$. If $\lambda \neq= 0$is an eigenvalue of $\mathbf{M}$ with corresponding unit column vector $\mathbf{u}$, then which of the following are true?

a) $\text{rank}(\mathbf{M} - \lambda\mathbf{u}\mathbf{u}^*) = k-1$.
b) $\text{rank}(\mathbf{M} - \lambda\mathbf{u}\mathbf{u}^*) = k$.
c) $\text{rank}(\mathbf{M} - \lambda\mathbf{u}\mathbf{u}^*) = k+1$.
d) $(\mathbf{M} - \lambda\mathbf{u}\mathbf{u}^*)^n = \mathbf{M}^n - \lambda^n\mathbf{u}\mathbf{u}^*$.

2.14. Define a real valued function B on $\mathbb{R}^2 \times \mathbb{R}^2$ as

$$B(\mathbf{x}, \mathbf{y}) = x_1y_1 - x_1y_2 - x_2y_1 + 4x_2y_2 \tag{2.14.1}$$

Let $\mathbf{v}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and

$$W = \left\{\mathbf{v} \in \mathbb{R}^2 : B(\mathbf{v}_0, \mathbf{v}) = 0\right\} \tag{2.14.2}$$

Then W

a) is not a subspace of $\mathbb{R}^2$.
b) equals $\mathbf{0}$.
c) is the y axis
d) is the line passing through $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

2.15. Consider the Quadratic forms

$$Q_1(x, y) = xy \tag{2.15.1}$$

$$Q_2(x, y) = x^2 + 2xy + y^2 \tag{2.15.2}$$

$$Q_3(x, y) = x^2 + 3xy + 2y^2 \tag{2.15.3}$$

on $\mathbb{R}^2$. Choose the correct statements from below

a) Q_1 and Q_2 are equivalent.
b) Q_1 and Q_3 are equivalent.
c) Q_2 and Q_3 are equivalent.
d) all are equivalent.

2.16. Consider a Markov Chain with state space $\{0, 1, 2\}$ and transition matrix

$$P = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & \frac{3}{4} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \end{array} \tag{2.16.1}$$

For any two states i and j, let $p_{ij}^{(n)}$ denote the n-step transition probability of going from i to j. Identify correct statements.

a) $\lim_{n\to\infty} p_{11}^{(n)} = \frac{2}{9}$
b) $\lim_{n\to\infty} p_{21}^{(n)} = 0$
c) $\lim_{n\to\infty} p_{32}^{(n)} = \frac{1}{3}$
d) $\lim_{n\to\infty} p_{13}^{(n)} = \frac{1}{3}$

3 June 2018

3.1. Let $\mathbf{A}$ be a $(m \times n)$ matrix and $\mathbf{B}$ be a $(n \times m)$ matrix over real numbers with $m < n$. Then

a) $\mathbf{AB}$ is always nonsingular.
b) $\mathbf{AB}$ is always singular.
c) $\mathbf{BA}$ is always nonsingular.
d) $\mathbf{BA}$ is always singular.

3.2. If $\mathbf{A}$ is a (2×2) matrix over $\mathbb{R}$ with $det(\mathbf{A}+\mathbf{I}) = 1 + det(\mathbf{A})$. Then we can conclude that

a) $det(\mathbf{A}) = 0$.
b) $\mathbf{A} = 0$.
c) $tr(\mathbf{A}) = 0$.
d) $\mathbf{A}$ is nonsingular.

3.3. The system of equations

$$x + 2x^2 + 3xy = 6 \tag{3.3.1}$$
$$x + x^2 + 3xy + y = 5 \tag{3.3.2}$$
$$x - x^2 + y = 7 \tag{3.3.3}$$

a) has solutions in rational numbers.
b) has solutions in real numbers.
c) has solutions in complex numbers.
d) has no solutions.

3.4. The trace of the matrix

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}^{20} \tag{3.4.1}$$

is

a) 7^{20}.
b) $2^{20} + 3^{20}$.
c) $2^{21} + 3^{20}$.
d) $2^{20} + 3^{20} + 1$.

3.5. Given that there are real constants a, b, c, d such that the identity

$$\lambda x^2 + 2xy + y^2 = (ax + by)^2 + (cx + dy)^2, \quad \forall x, y \in \mathbb{R} \tag{3.5.1}$$

This implies that

a) $\lambda = -5$
b) $\lambda \geq 1$
c) $0 < \lambda < 1$
d) There is no such $\lambda \in \mathbb{R}$

3.6. Let $\mathbb{R}, n \geq 2$, be equipped with the standard inner product. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be n column vectors forming an orthonormal basis of $\mathbb{R}^n$. Let A be the $n \times n$ matrix formed by the column vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. Then

a) $\mathbf{A} = \mathbf{A}^{-1}$
b) $\mathbf{A} = \mathbf{A}^{\top}$
c) $\mathbf{A}^{-1} = \mathbf{A}^{\top}$
d) $det(\mathbf{A}) = 1$

3.7. Consider a Markov Chain with state space $\{1, 2, 3, 4\}$ and transition matrix

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix} \end{array} \tag{3.7.1}$$

Then,

a) $\lim_{n\to\infty} p_{22}^{(n)} = 0, \sum_{n=0}^{\infty} p_{22}^{(n)} = \infty$
b) $\lim_{n\to\infty} p_{22}^{(n)} = 0, \sum_{n=0}^{\infty} p_{22}^{(n)} < \infty$
c) $\lim_{n\to\infty} p_{22}^{(n)} = 1, \sum_{n=0}^{\infty} p_{22}^{(n)} = \infty$
d) $\lim_{n\to\infty} p_{22}^{(n)} = 1, \sum_{n=0}^{\infty} p_{22}^{(n)} < \infty$

3.8. Let V denote the vector space of all sequences $\mathbf{a} = (a_1, a_2, \ldots)$ of real numbers such that

$$\sum_n 2^n |a|_n \tag{3.8.1}$$

converges. Define

$$\|\cdot\| : V \to \mathbb{R} \tag{3.8.2}$$

by

$$\|\mathbf{a}\| = \sum_n 2^n |a|_n . \tag{3.8.3}$$

Which of the following are true?

a) V contains onlythe sequence $(0, 0, \ldots)$
b) V is finite dimensional
c) V has a countable linear basis
d) V is a complete normed space

3.9. Let V be a vector space over $\mathbb{C}$ with dimension n. Let $T : V \to V$ be a linear transformation with only1 as eigenvalue. Then which of the following must be true?

a) $T - I = 0$
b) $(T - I)^{n-1} = 0$
c) $(T - I)^n = 0$
d) $(T - I)^{2n} = 0$

3.10. If $\mathbf{A}$ is a 5×5 matrix and the dimension of the solution space of $\mathbf{A}\mathbf{x} = 0$ is at least two, then

a) $\text{rank}\left(\mathbf{A}^2\right) \leq 3$
b) $\text{rank}\left(\mathbf{A}^2\right) \geq 3$
c) $\text{rank}\left(\mathbf{A}^2\right) = 3$
d) $\det\left(\mathbf{A}^2\right) = 0$

3.11. Let $\mathbf{A} \in M_3(\mathbb{R})$ be such that $\mathbf{A}^3 = \mathbf{I}_{3\times 3}$. Then

a) minimal polynomial of $\mathbf{A}$ can only be of degree 2
b) minimal polynomial of $\mathbf{A}$ can only be of degree 3
c) either $\mathbf{A} = \mathbf{I}$ or $\mathbf{A} = -\mathbf{I}$
d) there can be uncountably many $\mathbf{A}$ satisfying the above.

3.12. Let $\mathbf{A}$ be an $n \times n, n > 1$ matrix satisfying

$$\mathbf{A}^2 - 7\mathbf{A} + 12\mathbf{I} = \mathbf{0} \tag{3.12.1}$$

Then which of the following statements is true?

a) $\mathbf{A}$ is invertible
b) $t^2 - 7t + 12n = 0$ where $t = tr(\mathbf{A})$
c) $d^2 - 7d + 12 = 0$ where $d = det(\mathbf{A})$
d) $\lambda^2 - 7\lambda + 12 = 0$ where λ is an eigenvalue of $\mathbf{A}$

3.13. Let $\mathbf{A}$ be a 6×6 matrix over $\mathbb{R}$ with characteristic polynomial

$$(x-3)^2 (x-2)^4 \tag{3.13.1}$$

and minimal polynomial

$$(x-3)(x-2)^2 \tag{3.13.2}$$

Then the Jordan canonical form of $\mathbf{A}$ can be

a) $\begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$

b) $\begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$

c) $\begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$

d) $\begin{pmatrix} 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$

3.14. Let V be an inner product space and S be a subset of V. Let $\bar{S}$ denote the closure of S in V with respect to the topology induced by the metric given by the inner product. Which of the following statements is true?

a) $S = (S^{\perp})^{\perp}$
b) $\bar{S} = (S^{\perp})^{\perp}$
c) $\overline{span(S)} = (S^{\perp})^{\perp}$
d) $S^{\perp} = \left((S^{\perp})^{\perp}\right)^{\perp}$

3.15. Let

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.15.1}$$

and

$$Q(\mathbf{x}) = \mathbf{x}^T \mathbf{A} \mathbf{x} \tag{3.15.2}$$

Which of the following statements is true?

a) The matrix of second order partial derivatives of the quadratic form Q is $2\mathbf{A}$
b) The rank of the quadratic form Q is 2
c) The signature of the quadratic form Q is $++0$
d) The quadratic form Q take the value 0 for some non-zero vector $\mathbf{x}$

3.16. Assume that a non-singular matrix

$$\mathbf{A} = \mathbf{L} + \mathbf{D} + \mathbf{U} \tag{3.16.1}$$

where $\mathbf{L}$ and $\mathbf{U}$ are lower and upper triangular matrices respectively with all diagonal entries are zero, and $\mathbf{D}$ si a diagonal matrix. Let $\mathbf{x}^*$ be the solution of $\mathbf{A}\mathbf{x} = \mathbf{b}$. Then the Gauss-Seidel iteration method

$$\mathbf{x}_{k+1} = \mathbf{H}\mathbf{x}_k + \mathbf{c}, k = 0, 1, 2, \ldots \tag{3.16.2}$$

with $\|\mathbf{H}\| < 1$ converges to $\mathbf{x}^*$ provided $\mathbf{H}$ is equal to

a) $-\mathbf{D}^{-1}(\mathbf{L} + \mathbf{U})$
b) $-(\mathbf{D} + \mathbf{L})^{-1}\mathbf{U}$
c) $-\mathbf{D}(\mathbf{L} + \mathbf{U})^{-1}$
d) $-(\mathbf{L} - \mathbf{D})^{-1}\mathbf{U}$

3.17. Consider a Markov Chain with state space $S = \{1, 2, 3\}$ and transition matrix

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \end{array} \tag{3.17.1}$$

Let π be a stationary distribution of the Markov chain and $d(1)$ denote the period of state 1. Which of the following statements are correct?

a) $d(1) = 1$
b) $d(1) = 2$
c) $\pi_1 = \frac{1}{2}$
d) $\pi_1 = \frac{1}{3}$

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4.1. Let $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$. Then the smallest positive integer n such that $\mathbf{A}^n = \mathbf{I}$ is

Solution: *Property of eigen values of A:* Let $\mathbf{A}$ be an arbitary $n \times n$ matrix of complex numbers with eigen values $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then the eigen values of k$^{\text{th}}$ power of $\mathbf{A}$, that is the eigen values of $\mathbf{A}^k$, for any positive integer k are $\lambda_1^k, \lambda_2^k, \ldots, \lambda_n^k$. Let us calculate the eigen values of $\mathbf{A}$.

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \tag{4.1.1}$$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0 \tag{4.1.2}$$

$$\begin{vmatrix} -\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = 0 \tag{4.1.3}$$

$$-\lambda(1-\lambda) + 1 = 0 \tag{4.1.4}$$

$$\lambda^2 - \lambda + 1 = 0 \tag{4.1.5}$$

$$\implies \lambda = \frac{-1 \pm \sqrt{3}i}{2} \tag{4.1.6}$$

From the above property, the eigen values of $\mathbf{A}^n$ are λ^n. Also as it is given that $\mathbf{A}^n = \mathbf{I}$,

$$\implies \lambda^n = 1 \tag{4.1.7}$$

$$\implies \left(\frac{-1 \pm \sqrt{3}i}{2}\right)^n = 1 \tag{4.1.8}$$

Clearly $n \neq 1$. For $n = 2$,

$$\left(\frac{-1 \pm \sqrt{3}i}{2}\right)^2 = \frac{-1 \mp \sqrt{3}i}{2} \tag{4.1.9}$$

For $n = 4$,

$$\left(\frac{-1 \pm \sqrt{3}i}{2}\right)^4 = \frac{-1 \pm \sqrt{3}i}{2} \tag{4.1.10}$$

For $n = 6$,

$$\left(\frac{-1 \pm \sqrt{3}i}{2}\right)^6 = 1 \tag{4.1.11}$$

Hence $n = 6$ is the smallest positive integer.

4.2. Let $\mathbf{A} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 2 & 3 & \alpha \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ \beta \end{pmatrix}$. Then the system $\mathbf{AX} = \mathbf{b}$ over the real numbers has

a) No solution when $\beta \neq 7$
b) Infinite number of solutions when $\alpha \neq 2$
c) Infinite number of solutions when $\alpha = 2$ and $\beta \neq 7$
d) A unique solution if $\alpha \neq 2$

Solution: First we derive the Row Reduced Echelon Form (RREF) of the augmented matrix of the system $\mathbf{AX} = \mathbf{b}$ as follows,

$$\begin{pmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 3 \\ 2 & 3 & \alpha & \beta \end{pmatrix} \xleftrightarrow[R_3 = R_3 - 2R_1]{R_2 = R_2 - R_1} \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 5 & \alpha - 2 & \beta - 2 \end{pmatrix} \tag{4.2.1}$$

$$\xleftrightarrow{R_2 = \frac{1}{2}R_2} \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 5 & \alpha - 2 & \beta - 2 \end{pmatrix} \tag{4.2.2}$$

$$\xleftrightarrow{R_1 = R_1 + R_2} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 5 & \alpha - 2 & \beta - 2 \end{pmatrix} \tag{4.2.3}$$

$$\xleftrightarrow{R_3 = R_3 - 5R_2} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & \alpha - 2 & \beta - 7 \end{pmatrix} \tag{4.2.4}$$

From the RREF of the augmented matrix of the system $\mathbf{AX} = \mathbf{b}$ in (4.2.4) we make the following observations for different values of α and β in Table 4.2.1. ,

Values	Observations
$\beta \neq 7$	Then the existence of solution and the number of solutions will entirely depend on value of α
$\alpha = 2$ $\beta \neq 7$	Then RREF in (4.2.4) will contain Zero Row in R_3. Moreover solvability condition will not satisfy. $\implies$ system will have Zero solutions
$\alpha \neq 2$	RREF in (4.2.4) will have all pivots $\implies$ RREF in (4.2.4) will be fullrank $\implies$ $\mathbf{AX} = \mathbf{b}$ have unique solution.

TABLE 4.2.1

Hence, if $\alpha \neq 2$ then the system $\mathbf{AX} = \mathbf{b}$ has unique solution.

4.3. Consider a Markov chain $\{X_n | n \geq 0\}$ with state space $\{1, 2, 3\}$ and transition matrix

$$\mathbf{P} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

Then, $P(X_3 = 1 \mid X_0 = 1)$ equals
Solution:

5 Problem

The three step transitional probabilities are given as,

$$P(X_3 = j \mid X_0 = i) = P(X_{n+3} = j \mid X_n = i) = (\mathbf{P}^3)_{ij} \; for\; any\; n \tag{5.3.1}$$

$$\mathbf{P}^3 = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}^3 = \begin{pmatrix} \frac{1}{4} & \frac{3}{8} & \frac{3}{8} \\ \frac{3}{8} & \frac{1}{4} & \frac{3}{8} \\ \frac{3}{8} & \frac{3}{8} & \frac{1}{4} \end{pmatrix} \tag{5.3.2}$$

From (5.3.2),

$$P(X_3 = 1 \mid X_0 = 1) = (\mathbf{P}^3)_{11} = \frac{1}{4} \tag{5.3.3}$$

5.4. For every 4×4 real symmetric non-singular matrix $\mathbf{A}$ there exists a positive integer p such that

a) $p\mathbf{I} + \mathbf{A}$ is positive definite
b) $\mathbf{A}^p$ is positive definite
c) $\mathbf{A}^{-p}$ is positive definite
d) $\exp(p\mathbf{A}) - \mathbf{I}$ is positive definite

Solution: A matrix is real symmetric implies its eigen values are real and eigen vectors are orthogonal,that is its eigen value decomposition is

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^T \tag{5.4.1}$$

$\mathbf{D}$ is the diagonal matrix containing the real eigen values of $\mathbf{A}$
$\mathbf{P}$ has the corresponding eigen vectors

$$\mathbf{P}\mathbf{P}^T = \mathbf{P}^T\mathbf{P} = \mathbf{I} \tag{5.4.2}$$

A real matrix is positive definite if

$$\mathbf{x}^T\mathbf{A}\mathbf{x} > 0 \tag{5.4.3}$$
$$\implies \mathbf{x}^T\lambda\mathbf{x} > 0 \tag{5.4.4}$$
$$\implies \lambda\mathbf{x}^T\mathbf{x} > 0 \tag{5.4.5}$$
$$\implies \lambda > 0 \tag{5.4.6}$$

In other words, all the eigen values of A are positive See Table 5.4.1

Let $\mathbf{A}$ be

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^T \tag{5.4.7}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix} \tag{5.4.8}$$

From the table,the choices would be option 1,2,3

5.5. Consider a Markov chain with five states $\{1, 2, 3, 4, 5\}$ and transition matrix

$$P = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{7} & 0 & 0 & \frac{6}{7} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} & 0 \\ 0 & \frac{5}{8} & 0 & 0 & \frac{3}{8} \end{pmatrix} \tag{5.5.1}$$

Which of the following are true?

a) 3 and 1 are in the same communicating class
b) 1 and 4 are in the same communicating class
c) 4 and 2 are in the same communicating class
d) 2 and 5 are in the same communicating class

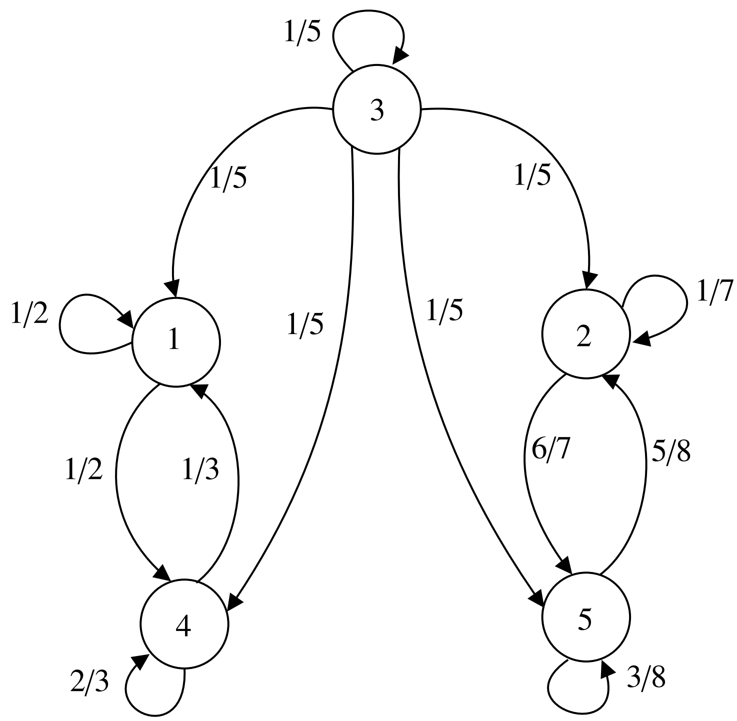
Solution: See Tables 5.5.1 and 5.5.2

OPTIONS	DERIVATIONS
Choice 1	$$p\mathbf{I}+\mathbf{A} = \mathbf{P}(p\mathbf{I})\mathbf{P}^T + \mathbf{P}\mathbf{D}\mathbf{P}^T \quad (5.4.9)$$ $$= \mathbf{P}\mathbf{D}_1\mathbf{P}^T \quad (5.4.10)$$ $$\mathbf{D}_1 = \begin{pmatrix} \lambda_1+p & 0 & 0 & 0 \\ 0 & \lambda_2+p & 0 & 0 \\ 0 & 0 & \lambda_3+p & 0 \\ 0 & 0 & 0 & \lambda_4+p \end{pmatrix} \quad (5.4.11)$$ Some of the eigen values of $\mathbf{A}$ may be negative. All the eigen values in $\mathbf{D}_1$ are positive only if $$p > \lvert\lambda_i\rvert \ \forall i \in [1,4] \quad (5.4.12)$$
Choice 2	$$\mathbf{A}^2 = \mathbf{A}\mathbf{A} \quad (5.4.13)$$ $$= (\mathbf{P}\mathbf{D}\mathbf{P}^T)(\mathbf{P}\mathbf{D}\mathbf{P}^T) \quad (5.4.14)$$ $$= \mathbf{P}\mathbf{D}^2\mathbf{P}^T \quad (5.4.15)$$ $$\text{Similarly, } \mathbf{A}^p = \mathbf{P}\mathbf{D}^p\mathbf{P}^T \quad (5.4.16)$$ $$\mathbf{D}^p = \begin{pmatrix} \lambda_1^p & 0 & 0 & 0 \\ 0 & \lambda_2^p & 0 & 0 \\ 0 & 0 & \lambda_3^p & 0 \\ 0 & 0 & 0 & \lambda_4^p \end{pmatrix} \quad (5.4.17)$$ $\mathbf{A}^p$ is positive definite only if p is even.
Choice 3	$$\mathbf{A}^{-p} = \mathbf{P}\mathbf{D}^{-p}\mathbf{P}^T \quad (5.4.18)$$ $$\mathbf{D}^{-p} = \begin{pmatrix} \lambda_1^{-p} & 0 & 0 & 0 \\ 0 & \lambda_2^{-p} & 0 & 0 \\ 0 & 0 & \lambda_3^{-p} & 0 \\ 0 & 0 & 0 & \lambda_4^{-p} \end{pmatrix} \quad (5.4.19)$$ $\mathbf{A}^{-p}$ is positive definite only if p is even.
Choice 4	$$\exp(p\mathbf{A}) = \sum_{k=0}^{\infty} \frac{(p\mathbf{A})^k}{k!} \quad (5.4.20)$$ $$\implies \exp(p\mathbf{A}) - \mathbf{I} = \mathbf{P}\exp(p\mathbf{D})\mathbf{P}^T - \mathbf{P}\mathbf{I}\mathbf{P}^T \quad (5.4.21)$$ $$= \mathbf{P}(\exp(p\mathbf{D}) - \mathbf{I})\mathbf{P}^T \quad (5.4.22)$$ $$\exp(p\mathbf{D}) - \mathbf{I} = \begin{pmatrix} e^{\lambda_1}-1 & 0 & 0 & 0 \\ 0 & e^{\lambda_2}-1 & 0 & 0 \\ 0 & 0 & e^{\lambda_3}-1 & 0 \\ 0 & 0 & 0 & e^{\lambda_4}-1 \end{pmatrix} \quad (5.4.23)$$ $\mathbf{A}$ is non-singular $$\implies \forall i \in [1,4], \lambda_i \neq 0 \quad (5.4.24)$$ $$e^{\lambda_i} < 1 \quad (5.4.25)$$ So, $\exp(p\mathbf{A}) - \mathbf{I}$ is not positive definite.

TABLE 5.4.1: Solution

Accessibility of states in Markov's chain	We say that state j is accessible from state i, written as $i \to j$, if $p_{ij}^{(n)} > 0$ for some n. Every state is accessible from itself since $p_{ii}^{(0)} = 1$
Communication between states	Two states i and j are said to communicate, written as $i \leftrightarrow j$, if they are accessible from each other. In other words, $i \leftrightarrow j$ means $i \to j$ and $j \to i$.
Communicating class	For each Markov chain, there exists a unique decomposition of the state space S into a sequence of disjoint subsets $C_1, C_2, ...,$ $S = \bigcup_{i=1}^{\infty} C_i$ in which each subset has the property that all states within it communicate. Each such subset is called a communication class of the Markov chain.

TABLE 5.5.1: Definition and Result used

Drawing Transition diagram	
Checking whether the states 3 and 1 are in the same communicating class	Here, State 1 is accessible from the state 3. But, State 3 is not accessible from the state 1 i.e. $3 \rightarrow 1$, $1 \nrightarrow 3$ $\implies \boxed{3 \nleftrightarrow 1}$ Therefore, 3 and 1 are not in the same communicating class.
Checking whether the states 1 and 4 are in the same communicating class	Here, State 1 is accessible from the state 4. Also, State 4 is accessible from the state 1 i.e. $3 \rightarrow 1$, $1 \rightarrow 3$ $\implies \boxed{3 \leftrightarrow 1}$ Therefore, 1 and 4 are in the same communicating class.
Checking whether the states 4 and 2 are in the same communicating class	Here, State 2 is not accessible from the state 4. Also, State 4 is not accessible from the state 2 i.e. $4 \nrightarrow 2$, $2 \nrightarrow 4$

	$\implies \boxed{4 \nleftrightarrow 2}$ Therefore, 4 and 2 are not in the same communicating class.
Checking whether the states 2 and 5 are in the same communicating class	Here, State 2 is accessible from the state 5. Also, State 5 is accessible from the state 2 i.e. $5 \to 2$, $2 \to 5$ $\implies \boxed{2 \leftrightarrow 5}$ Therefore, 2 and 5 are in the same communicating class.
Conclusion	Communication classes are: $\boxed{S = \{1,4\} \cup \{3\} \cup \{2,5\}}$ Option 2) and 4) are true.

TABLE 5.5.2: Solution