Solutions: Linear Algebra by Hoffman and Kunze

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***Abstract*—This book provides solutions to the Linear Algebra book by Hoffman and Kunze.**

1 Linear Equations

1.1 Fields and Linear Equations

1.1.1. Verify that the set of complex numbers numbers described in the form of c where x and y are rational is a sub-field of C.

Solution: Lets consider the set S = $\{x + y\sqrt{2}, x, y \in Q\}$, $S \subset C$ We must verify that S meets the following two conditions:

$$0, 1 \in S \tag{1.1.1.1}$$

$$a, b \in S, a + b, -a, ab, a^{-1} \in S \tag{1.1.1.2}$$

Throughout let

$$a = x + y\sqrt{2}, b = w + z\sqrt{2} \tag{1.1.1.3}$$

If

a)

$$x = 0, y = 0 \in Q, a = 0 + \sqrt{2}.0 = 0, 0 \in S \tag{1.1.1.4}$$

b)

$$x = 1, y = 0, a = 1 + \sqrt{2}.0 = 1, 1 \in S \tag{1.1.1.5}$$

c)

$$a + b = x + y\sqrt{2} + w + z\sqrt{2} = b + a \tag{1.1.1.6}$$

d)

$$-a = -x - y\sqrt{2}, x, y \in Q so - x, -y \in Q, a \in S \tag{1.1.1.7}$$

e)

$$ab = (x + y\sqrt{2})(w + z\sqrt{2}) = ba, ab \in S \tag{1.1.1.8}$$

f)

$$a^{-1}a = (x + y\sqrt{2})^{-1}(x + y\sqrt{2}) = 1, a^{-1} \in S \tag{1.1.1.9}$$

Hence (1.1.1.1) ,(1.1.1.2) is verfied . Therefore by considering the (1.1.1.1) and (1.1.1.2) we can say set complex numbers of given form $x + y\sqrt{2}$ is subfield of C .

1.1.2. Let $\mathbb{F}$ be the field of complex numbers. Are the following two systems of linear equations equivalent? If so, express each equation in each system as a linear combination of the equations in the other system.

$$x_1 - x_2 = 0$$
$$2x_1 + x_2 = 0$$

and

$$3x_1 + x_2 = 0$$
$$x_1 + x_2 = 0$$

Solution: The given system of linear equations can be written as,

$$\mathbf{Ax} = 0 \tag{1.1.2.1}$$

$$\implies \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{1.1.2.2}$$

$$\mathbf{Bx} = 0 \tag{1.1.2.3}$$

$$\implies \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{1.1.2.4}$$

Now we can obtain **B** from matrix **A** by performing elementary row operations given as,

$$\mathbf{B} = \mathbf{CA} \tag{1.1.2.5}$$

$$\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} = \mathbf{C} \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \tag{1.1.2.6}$$

where **C** is product of elementary matrices given as,

$$\begin{aligned} \mathbf{C} &= (\mathbf{E_7 E_6 E_5 E_4 E_3 E_2 E_1}) \\ &= \begin{pmatrix} 1 & 0 \\ \frac{1}{3} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{3} & \frac{4}{3} \\ \frac{-1}{3} & \frac{2}{3} \end{pmatrix} \end{aligned} \tag{1.1.2.7}$$

Now, performing elementary operations on the right side of **A** we obtain matrix **B** given as,

$$\mathbf{B} = \mathbf{AP} \tag{1.1.2.8}$$

$$\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \mathbf{P} \tag{1.1.2.9}$$

where, **P** is product of elementary matrices given by,

$$\mathbf{P} = (\mathbf{E_1 E_2 E_3 E_4 E_5}) = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{4}{3} & \frac{2}{3} \\ \frac{-5}{3} & \frac{-1}{3} \end{pmatrix} \quad (1.1.2.10)$$

Similarly, $\mathbf{A}$ can be obtained from matrix $\mathbf{B}$ from (1.1.2.5) as,

$$\mathbf{A} = \mathbf{C}^{-1}\mathbf{B} \quad (1.1.2.11)$$

$$\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} = \mathbf{C}^{-1}\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \quad (1.1.2.12)$$

Matrix $\mathbf{C}$ is product of elementary matrices and hence invertible and is given as,

$$\mathbf{C}^{-1} = \left(\mathbf{E_1}^{-1}\mathbf{E_2}^{-1}\mathbf{E_3}^{-1}\mathbf{E_4}^{-1}\mathbf{E_5}^{-1}\mathbf{E_6}^{-1}\mathbf{E_7}^{-1}\right) = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \frac{3}{2} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{-1}{3} & 1 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad (1.1.2.13)$$

Matrix $\mathbf{A}$ can also be obtained from (1.1.2.8) given as,

$$\mathbf{A} = \mathbf{B}\mathbf{P}^{-1} \quad (1.1.2.14)$$

$$\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}\mathbf{P}^{-1} \quad (1.1.2.15)$$

where,

$$\mathbf{P}^{-1} = \left(\mathbf{E_5}^{-1}\mathbf{E_4}^{-1}\mathbf{E_3}^{-1}\mathbf{E_2}^{-1}\mathbf{E_1}^{-1}\right) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} \frac{-1}{2} & -1 \\ \frac{5}{2} & 2 \end{pmatrix} \quad (1.1.2.16)$$

Thus (1.1.2.4) can be obtained from (1.1.2.2) by multiplying it with matrix $\mathbf{C}$, and by inverse row operations (1.1.2.2) can be obtained back from (1.1.2.4) since $\mathbf{C}$ is product of elementary matrices and hence invertible.
Thus the two given homogeneous systems are row equivalent.
Now writing equations in matrix-vector form as,

$$3x_1 + x_2 = \begin{pmatrix} 3 & 1 \end{pmatrix}\mathbf{x} \quad (1.1.2.17)$$

$$\implies \begin{pmatrix} 3 & 1 \end{pmatrix}\mathbf{x} = \frac{1}{3}\begin{pmatrix} 1 & -1 \end{pmatrix}\mathbf{x} + \frac{4}{3}\begin{pmatrix} 2 & 1 \end{pmatrix}\mathbf{x} \quad (1.1.2.18)$$

$$x_1 + x_2 = \begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} \quad (1.1.2.19)$$

$$\implies \begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} = \frac{-1}{3}\begin{pmatrix} 1 & -1 \end{pmatrix}\mathbf{x} + \frac{2}{3}\begin{pmatrix} 2 & 1 \end{pmatrix}\mathbf{x} \quad (1.1.2.20)$$

(1.1.2.18), (1.1.2.20) is same as multiplying $\mathbf{C}$ with $\mathbf{A}$ as it takes the linear combination of each rows of matrix $\mathbf{A}$ i.e, (1.1.2.6)

$$x_1 - x_2 = \begin{pmatrix} 1 & -1 \end{pmatrix}\mathbf{x} \quad (1.1.2.21)$$

$$\implies \begin{pmatrix} 1 & -1 \end{pmatrix}\mathbf{x} = (1)\begin{pmatrix} 3 & 1 \end{pmatrix}\mathbf{x} + (-2)\begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} \quad (1.1.2.22)$$

$$2x_1 + x_2 = \begin{pmatrix} 2 & 1 \end{pmatrix}\mathbf{x} \quad (1.1.2.23)$$

$$\implies \begin{pmatrix} 2 & 1 \end{pmatrix}\mathbf{x} = \frac{1}{2}\begin{pmatrix} 3 & 1 \end{pmatrix}\mathbf{x} + \frac{1}{2}\begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} \quad (1.1.2.24)$$

(1.1.2.22), (1.1.2.24) is same as multiplying $\mathbf{C}^{-1}$ with $\mathbf{B}$ as it takes the linear combination of each rows of matrix $\mathbf{B}$ i.e, (1.1.2.12)
Thus each equation in each system can be expressed as a linear combination of the equations in the other system when they are equivalent.

1.1.3. Are the following two systems of linear equations equivalent?

$$\begin{aligned} -x_1 + x_2 + 4x_3 &= 0 \\ x_1 + 3x_2 + 8x_3 &= 0 \\ \frac{1}{2}x_1 + x_2 + \frac{5}{2}x_3 &= 0 \end{aligned} \quad (1.1.3.1)$$

Solution:

$$\begin{aligned} x_1 - x_3 &= 0 \\ x_2 + 3x_3 &= 0 \end{aligned} \quad (1.1.3.2)$$

System of linear equations in (1.1.3.1) can be expressed in matrix form as,

$$\mathbf{A}\mathbf{x} = 0 \tag{1.1.3.3}$$

$$\begin{pmatrix} -1 & 1 & 4 \\ 1 & 3 & 8 \\ \frac{1}{2} & 1 & \frac{5}{2} \end{pmatrix} \mathbf{x} = 0 \tag{1.1.3.4}$$

System of linear equations in (1.1.3.2) can be expressed in matrix form as,

$$\mathbf{B}\mathbf{x} = 0 \tag{1.1.3.5}$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \end{pmatrix} \mathbf{x} = 0 \tag{1.1.3.6}$$

Two system of linear equations are equivalent if one system can be expressed as a linear combination of other system.
Matrix $\mathbf{B}$ can be obtained from matrix $\mathbf{A}$ as,

$$\mathbf{B} = \mathbf{C}\mathbf{A} \tag{1.1.3.7}$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \end{pmatrix} = \mathbf{C} \begin{pmatrix} -1 & 1 & 4 \\ 1 & 3 & 8 \\ \frac{1}{2} & 1 & \frac{5}{2} \end{pmatrix} \tag{1.1.3.8}$$

$$\mathbf{C} = \begin{pmatrix} -1 & 1 & -2 \\ \frac{1}{2} & -\frac{1}{2} & 2 \end{pmatrix} \tag{1.1.3.9}$$

Now, writing equations in matrix-vector form,

$$x_1 - x_3 = \begin{pmatrix} 1 & 0 & -1 \end{pmatrix} \mathbf{x}$$

$$\implies \begin{pmatrix} 1 & 0 & -1 \end{pmatrix} \mathbf{x} = -1 \begin{pmatrix} -1 & 1 & 4 \end{pmatrix} \mathbf{x} \\ + 1 \begin{pmatrix} 1 & 3 & 8 \end{pmatrix} \mathbf{x} - 2 \begin{pmatrix} \frac{1}{2} & 1 & \frac{5}{2} \end{pmatrix} \mathbf{x} \tag{1.1.3.10}$$

$$x_2 + 3x_3 = \begin{pmatrix} 0 & 1 & 3 \end{pmatrix} \mathbf{x}$$

$$\implies \begin{pmatrix} 0 & 1 & 3 \end{pmatrix} \mathbf{x} = \frac{1}{2} \begin{pmatrix} -1 & 1 & 4 \end{pmatrix} \mathbf{x} \\ - \frac{1}{2} \begin{pmatrix} 1 & 3 & 8 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} \frac{1}{2} & 1 & \frac{5}{2} \end{pmatrix} \mathbf{x} \tag{1.1.3.11}$$

Equations (1.1.3.10) and (1.1.3.11) is same as multiplying $\mathbf{C}$ with $\mathbf{A}$ which is the linear combination of rows of matrix $\mathbf{A}$.
Thus each equation in second system can be expressed as linear combination of the equations in first system.
Therefore, the two system of linear equations are equivalent.

1.1.4. Let $\mathbb{F}$ be the field of complex numbers. Are the following two systems of linear equations equivalent? If so, express each equation in each system as a linear combination of equations in other system. First system of equations:

$$2x_1 + (-1 + i)x_2 + x_4 = 0 \tag{1.1.4.1}$$

$$3x_2 - 2ix_3 + 5x_4 = 0 \tag{1.1.4.2}$$

The second system of equations:

$$(1 + \frac{i}{2})x_1 + 8x_2 - ix_3 - x_4 = 0 \tag{1.1.4.3}$$

$$\frac{2}{3}x_1 - \frac{1}{2}x_2 + x_3 + 7x_4 = 0 \tag{1.1.4.4}$$

Solution: Let $\mathbf{R_1}$ and $\mathbf{R_2}$ be the reduced row echelon forms of the augumented matrices of the following systems of homogeneous equations respectively.

$$\mathbf{A}\mathbf{X} = \mathbf{0} \tag{1.1.4.5}$$

$$\mathbf{B}\mathbf{X} = \mathbf{0} \tag{1.1.4.6}$$

Where $\mathbf{A}$ and $\mathbf{B}$ as follows

$$\mathbf{A} = \begin{pmatrix} 2 & -1 + i & 0 & 1 \\ 0 & 3 & -2i & 5 \end{pmatrix} \tag{1.1.4.7}$$

$$\mathbf{B} = \begin{pmatrix} 1 + \frac{i}{2} & 8 & -i & -1 \\ \frac{2}{3} & \frac{-1}{2} & 1 & 7 \end{pmatrix} \tag{1.1.4.8}$$

On performing elementary row operations on (1.1.4.7),

$$\mathbf{R_1} = \mathbf{C}\mathbf{A} \tag{1.1.4.9}$$

where $\mathbf{C}$ is the product of all elementary matrices. Reducing the first system of linear equations, we get,

$$\mathbf{C} = \begin{pmatrix} 1 & \frac{1-i}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} \tag{1.1.4.10}$$

$$\mathbf{R_1} = \begin{pmatrix} 1 & 0 & \frac{-1-i}{3} & \frac{4}{3} - \frac{5i}{6} \\ 0 & 1 & \frac{-2i}{3} & \frac{5}{3} \end{pmatrix} \tag{1.1.4.11}$$

On performing elementary row operations on (1.1.4.8),

$$\mathbf{R_2} = \mathbf{D}\mathbf{A} \tag{1.1.4.12}$$

where $\mathbf{D}$ is the product of all elementary matrices. Reducing the second system of linear equations, we get,

$$\mathbf{D} = \begin{pmatrix} \frac{4}{5}(1 - \frac{i}{2}) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{-2}{3} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{-6(143+43i)}{4909} \end{pmatrix} \begin{pmatrix} 1 & \frac{16(-2+i)}{5} \\ 0 & 1 \end{pmatrix} \tag{1.1.4.13}$$

$$\mathbf{R_2} = \begin{pmatrix} 1 & 0 & \frac{6702}{4909} - \frac{708i}{4909} & \frac{46620}{4909} - \frac{1998i}{4909} \\ 0 & 1 & \frac{-2(441+472i)}{4909} & \frac{-2(3283+1332i)}{4909} \end{pmatrix} \quad (1.1.4.14)$$

From the equations (1.1.4.11) and (1.1.4.14), we can say that

$$\mathbf{R_1} \neq \mathbf{R_2} \quad (1.1.4.15)$$

Hence the given systems of linear equations are not equivalent.

1.1.5. Let $\mathbb{F}$ be a set which contains exactly two elements,0 and 1.Define an addition and multiplication by tables. Verify that the set $\mathbb{F}$,

+	0	1
0	0	1
1	1	0

·	0	1
0	0	0
1	0	1

together with these two operations, is a field.
Solution:
To prove that ($\mathbb{F}$,+,·) is a field we need to satisfy the following,

a) + and · should be closed
- For any a and b in $\mathbb{F}$, a+b ∈ $\mathbb{F}$ and a · b ∈ $\mathbb{F}$.For example 0+0=0 and 0·0=0.

b) + and · should be commutative
- For any a and b in $\mathbb{F}$, a+b = b+a and a · b = b· a. For example 0+1=1+0 and 0· 1=1· 0.

c) + and · should be associative
- For any a and b in $\mathbb{F}$, a+(b+c) = (a+b)+c and a· (b · c) = (a · b) · c. For example 0+(1+0)=(0+1)+0 and 0·(1·0)=(0·1)·0.

d) + and · operations should have an identity element
- If we perform a + 0 then for any value of a from $\mathbb{F}$ the result will be a itself. Hence 0 is an identity element of + operation.If we perform a · 1 then for any value of a from $\mathbb{F}$ the result will be a itself. Hence 1 is an identity element of · operation.

e) ∀ a ∈ $\mathbb{F}$ there exists an additive inverse
- For additive inverse to exist, ∀ a in $\mathbb{F}$ a+(-a)=0. For example. 1-1=0 and 0-0=0.

f) ∀ a ∈ $\mathbb{F}$ such that a is non zero there exists a multiplicative inverse
- For multiplicative inverse to exist, ∀ a such that a is non zero in $\mathbb{F}, a \cdot a^{-1}$=1.For example $1 \cdot 1^{-1} = 1$.

g) + and · should hold distributive property
- For any a,b and c in $\mathbb{F}$ the property a·(b+c)=a·b+a·c should always hold true.For example 0·(1+1)=0·1+0·1.

Since the above properties are satisfied we can say that ($\mathbb{F}$,+,·) is a field.

1.1.6. Prove that if two homogenous systems of linear equations in two unknowns have the same solutions, then they are equivalent.
Solution: Let the two systems of homogenous equations be

$$\mathbf{Ax} = \mathbf{0} \quad (1.1.6.1)$$
$$\mathbf{By} = \mathbf{0} \quad (1.1.6.2)$$

We can write

$$\mathbf{CAx} = \mathbf{0} \quad (1.1.6.3)$$
$$\mathbf{DBy} = \mathbf{0} \quad (1.1.6.4)$$

where **C** and **D** are product of elementary matrices that reduce **A** and **B** into their reduced row echelon forms $\mathbf{R_1}$ and $\mathbf{R_2}$
(1.1.6.3) and (1.1.6.4) imply

$$\mathbf{R_1x} = 0 \quad (1.1.6.5)$$
$$\mathbf{R_2y} = 0 \quad (1.1.6.6)$$

Given that they have same solution, we can write

$$\mathbf{R_1x} = 0 \quad (1.1.6.7)$$
$$\mathbf{R_2x} = 0 \quad (1.1.6.8)$$
$$\implies (\mathbf{R_1} - \mathbf{R_2})\mathbf{x} = 0 \quad (1.1.6.9)$$

Note that for a solution to exist, $\mathbf{R_1}$ and $\mathbf{R_2}$ can be either of matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{or} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (1.1.6.10)$$

Case 1 Let us assume that the solution is unique. The unique solution is

$$\mathbf{x} = \mathbf{0} \quad (1.1.6.11)$$

Since they have the same solution, both $\mathbf{R_1}, \mathbf{R_2}$ must have their rank as 2.

So,

$$\mathbf{R_1} = \mathbf{R_2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.1.6.12}$$

Case 2 Let us assume that (1.1.6.3),(1.1.6.4) have infinitely many solutions
So,

$$\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{B}) = 1 \tag{1.1.6.13}$$

equation (1.1.6.9) for solutions other than zero solution implies

$$\mathbf{R_1} = \mathbf{R_2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{1.1.6.14}$$

So, in both the cases, we have

$$\mathbf{R_1} = \mathbf{R_2} \tag{1.1.6.15}$$

$$\implies \mathbf{CA} = \mathbf{DB} \tag{1.1.6.16}$$

Since $\mathbf{C}, \mathbf{D}$ are product of elementary matrices, they are invertible.

$$\implies \mathbf{A} = \mathbf{C^{-1}DB} \tag{1.1.6.17}$$

$$\mathbf{B} = \mathbf{D^{-1}CA} \tag{1.1.6.18}$$

$$\text{Let } \mathbf{C^{-1}D} = \mathbf{E} \tag{1.1.6.19}$$

where $\mathbf{E}$ is also a product of elementary matrices
(1.1.6.17) and (1.1.6.18) hence become

$$\mathbf{A} = \mathbf{EB} \tag{1.1.6.20}$$

$$\mathbf{B} = \mathbf{E^{-1}A} \tag{1.1.6.21}$$

Hence the two systems of equations are equivalent.

1.1.7. Prove that each subfield of the field of complex number contains every rational number

Solution:

Complex Numbers: A complex number is a number that can be expressed in the form $a + bi$, where a and b are real numbers, and i represents the imaginary unit, satisfying the equation $i^2 = -1$.The set of complex numbers is denoted by $\mathbb{C}$

$$\mathbb{C} = \{(a, b) : a, b \in \mathbb{R}\} \tag{1.1.7.1}$$

Rational Numbers: A number in the form $\frac{p}{q}$, where both p and q(non-zero) are integers, is called a rational number.The set of rational numbers is dentoed by $\mathbb{Q}$ Let $\mathbb{Q}$ be the set of rational numbers.

$$\mathbb{Q} = \left\{\frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{Z}_{\neq 0}\right\} \tag{1.1.7.2}$$

Let $\mathbb{C}$ be the field of complex numbers and given $\mathbb{F}$ be the subfield of field of complex numbers $\mathbb{C}$ Since $\mathbb{F}$ is the subfield , we could say that

$$0 \in \mathbb{F} \tag{1.1.7.3}$$

$$1 \in \mathbb{F} \tag{1.1.7.4}$$

Closed under addition: Here $\mathbb{F}$ is closed under addition since it is subfield

$$1 + 1 = 2 \in \mathbb{F} \tag{1.1.7.5}$$

$$1 + 1 + 1 = 3 \in \mathbb{F} \tag{1.1.7.6}$$

$$\vdots$$

$$1 + 1 + \cdots + 1(\text{p times}) = p \in \mathbb{F} \tag{1.1.7.7}$$

$$1 + 1 + \cdots + 1(\text{q times}) = q \in \mathbb{F} \tag{1.1.7.8}$$

By using the above property we could say that zero and other positive integers belongs to $\mathbb{F}$.Since p and q are integers we say,

$$p \in \mathbb{Z} \tag{1.1.7.9}$$

$$q \in \mathbb{Z} \tag{1.1.7.10}$$

Additive Inverse: Let x be the positive integer belong $\mathbb{F}$ and by additive inverse we could say,

$$\forall x \in \mathbb{F} \tag{1.1.7.11}$$

$$(-x) \in \mathbb{F} \tag{1.1.7.12}$$

Therefore field $\mathbb{F}$ contains every integers. Let n be a integer then,

$$n \in \mathbb{Z} \implies n \in \mathbb{F} \tag{1.1.7.13}$$

$$\mathbb{Z} \subseteq \mathbb{F} \tag{1.1.7.14}$$

Where $\mathbb{Z}$ is subset of $\mathbb{F}$ Multiplicative Inverse: Every element except zero in the subfield $\mathbb{F}$ has an multiplicative inverse. From equation (1.1.7.8), since q $\in \mathbb{F}$ we could say ,

$$\frac{1}{q} \in \mathbb{F} \quad \text{and } q \neq 0 \tag{1.1.7.15}$$

Closed under multiplication: Also, $\mathbb{F}$ is closed under multiplication and thus, from equation

(1.1.7.7) and (1.1.7.15) we get ,

$$p \cdot \frac{1}{q} \in \mathbb{F} \tag{1.1.7.16}$$

$$\implies \frac{p}{q} \in \mathbb{F} \tag{1.1.7.17}$$

where , $p \in \mathbb{Z}$ and $q \in \mathbb{Z}_{\neq 0}$ (from equation (1.1.7.3) and (1.1.7.15)) Conclusion From (1.1.7.2) and (1.1.7.17) we could say ,

$$\mathbb{Q} \subseteq \mathbb{F} \tag{1.1.7.18}$$

From equation (1.1.7.18) we could say that each subfield of the field of complex number contains every rational number

Hence Proved

1.1.8. Prove that, each field of the characteristic zero contains a copy of the rational number field.
Solution: The characteristic of a field is defined to be the smallest number of times one must use the field's multiplicative identity (1) in a sum to get the additive identity. If this sum never reaches the additive identity (0), then the field is said to have characteristic zero.
Let $\mathbb{Q}$ be the rational number field. Hence,

$$0 \in \mathbb{Q} \qquad \text{[Additive Identity]} \tag{1.1.8.1}$$

$$1 \in \mathbb{Q} \qquad \text{[Multiplicative Identity]} \tag{1.1.8.2}$$

As addition is defined on $\mathbb{Q}$ hence we have,

$$1 \neq 0 \tag{1.1.8.3}$$

$$1 + 1 = 2 \neq 0 \tag{1.1.8.4}$$

And so on,

$$1 + 1 + \cdots + 1 = n \neq 0 \tag{1.1.8.5}$$

From the definition of characteristic of a field and from (1.1.8.3), (1.1.8.4) and so on upto (1.1.8.5), the rational number field, $\mathbb{Q}$ has characteristic 0.

1.2 Matrices and Elementary Row Operations

1.2.1. Find all solutions to the system of equations

$$\begin{aligned} (1-i)\,x_1 - ix_2 &= 0 \\ 2x_1 + (1-i)\,x_2 &= 0 \end{aligned} \tag{1.2.1.1}$$

Solution: System of Linear Equations (1.2.1.1) can be expressed in matrix form as,

$$\mathbf{Ax} = 0 \tag{1.2.1.2}$$

$$\begin{pmatrix} 1-i & -i \\ 2 & 1-i \end{pmatrix}\mathbf{x} = 0 \tag{1.2.1.3}$$

By row reduction ,

$$\begin{pmatrix} 1-i & -i \\ 2 & 1-i \end{pmatrix} \xleftrightarrow[R_1 \leftarrow R_1/2]{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & \frac{1-i}{2} \\ 1-i & -i \end{pmatrix} \tag{1.2.1.4}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - (1-i)R_1} \begin{pmatrix} 1 & \frac{1-i}{2} \\ 0 & 0 \end{pmatrix} \tag{1.2.1.5}$$

$$\begin{pmatrix} 1 & \frac{1-i}{2} \end{pmatrix}\mathbf{x} = 0 \tag{1.2.1.6}$$

$$\begin{pmatrix} 1 & \frac{1-i}{2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \tag{1.2.1.7}$$

$$x_1 = -\frac{1-i}{2}x_2 \tag{1.2.1.8}$$

$$\mathbf{x} = \begin{pmatrix} -\frac{1-i}{2}x_2 \\ x_2 \end{pmatrix} \tag{1.2.1.9}$$

$$\implies \mathbf{x} = x_2\begin{pmatrix} -\frac{1-i}{2} \\ 1 \end{pmatrix} \tag{1.2.1.10}$$

1.2.2. If

$$A = \begin{pmatrix} 3 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 0 \end{pmatrix} \tag{1.2.2.1}$$

Find all solutions of $AX = 0$ by row reducing A.
Solution: For the given equation $AX = 0$ can be defined as follows:

$$\begin{pmatrix} 3 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{1.2.2.2}$$

Now, we can apply Row Reduction Methodology of matrix A :

$$\begin{pmatrix}3 & -1 & 2 & 0\\ 2 & 1 & 1 & 0\\ 1 & -3 & 0 & 0\end{pmatrix} \xleftrightarrow{R_1=R_1+R_2} \begin{pmatrix}5 & 0 & 3 & 0\\ 2 & 1 & 1 & 0\\ 1 & -3 & 0 & 0\end{pmatrix} \tag{1.2.2.3}$$

$$\xleftrightarrow{R_2=R_2-2R_3} \begin{pmatrix}5 & 0 & 3 & 0\\ 0 & 7 & 1 & 0\\ 1 & -3 & 0 & 0\end{pmatrix} \tag{1.2.2.4}$$

$$\xleftrightarrow{R_3=R_3-\frac{1}{5}R_1} \begin{pmatrix}5 & 0 & 3 & 0\\ 0 & 7 & 1 & 0\\ 0 & -3 & -\frac{3}{5} & 0\end{pmatrix} \tag{1.2.2.5}$$

$$\xleftrightarrow{R_1=\frac{1}{5}R_1} \begin{pmatrix}1 & 0 & \frac{3}{5} & 0\\ 0 & 7 & 1 & 0\\ 0 & -3 & -\frac{3}{5} & 0\end{pmatrix} \tag{1.2.2.6}$$

$$\xleftrightarrow{R_2=\frac{1}{7}R_2} \begin{pmatrix}1 & 0 & \frac{3}{5} & 0\\ 0 & 1 & \frac{1}{7} & 0\\ 0 & -3 & -\frac{3}{5} & 0\end{pmatrix} \tag{1.2.2.7}$$

$$\xleftrightarrow{R_3=R_3+3R_2} \begin{pmatrix}1 & 0 & \frac{3}{5} & 0\\ 0 & 1 & \frac{1}{7} & 0\\ 0 & 0 & -\frac{6}{35} & 0\end{pmatrix} \tag{1.2.2.8}$$

$$\xleftrightarrow{R_3=-\frac{35}{6}R_3} \begin{pmatrix}1 & 0 & \frac{3}{5} & 0\\ 0 & 1 & \frac{1}{7} & 0\\ 0 & 0 & 1 & 0\end{pmatrix} \tag{1.2.2.9}$$

$$\xleftrightarrow{R_2=R_2-\frac{1}{7}R_3} \begin{pmatrix}1 & 0 & \frac{3}{5} & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{pmatrix} \tag{1.2.2.10}$$

$$\xleftrightarrow{R_1=R_1-\frac{3}{5}R_3} \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{pmatrix} \tag{1.2.2.11}$$

So, as we can see the only solution we got after row reducing of matrix A is zero vector. Thus, the solution is :

$$\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 0\end{pmatrix} \tag{1.2.2.12}$$

1.2.3.

$$\mathbf{A} = \begin{pmatrix}6 & -4 & 0\\ 4 & -2 & 0\\ -1 & 0 & 3\end{pmatrix} \tag{1.2.3.1}$$

Find all solutions of $\mathbf{AX} = 2\mathbf{X}$and all solutions of $\mathbf{AX} = 3\mathbf{X}$.The symbol c$\mathbf{X}$ denotes the matrix each entry of which is c times corresponding entry.

Solution:

$$\mathbf{A} = \begin{pmatrix}6 & -4 & 0\\ 4 & -2 & 0\\ -1 & 0 & 3\end{pmatrix} \tag{1.2.3.2}$$

To calculate solution of $\mathbf{AX} = 2\mathbf{X}$and all solutions of $\mathbf{AX} = 3\mathbf{X}$we calculate eigen values of $\mathbf{A}$:

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{X} = 0 \tag{1.2.3.3}$$

Substituting values in (1.2.3.3),

$$\begin{pmatrix}6-\lambda & -4 & 0\\ 4 & -2-\lambda & 0\\ -1 & 0 & 3-\lambda\end{pmatrix}\mathbf{X} = 0 \tag{1.2.3.4}$$

Simplifying:

$$\begin{pmatrix}6-\lambda & -4 & 0\\ 4 & -2-\lambda & 0\\ -1 & 0 & 3-\lambda\end{pmatrix} \xleftrightarrow{R_1\leftarrow R_1-R_2}$$

$$\begin{pmatrix}2-\lambda & -2+\lambda & 0\\ 4 & -2-\lambda & 0\\ -1 & 0 & 3-\lambda\end{pmatrix} \tag{1.2.3.5}$$

Taking $(3-\lambda)$ and $(2-\lambda)$common from C_3 and R_1

$$(3-\lambda)(2-\lambda)\begin{pmatrix}1 & -1 & 0\\ 4 & -2-\lambda & 0\\ -1 & 0 & 1\end{pmatrix} \tag{1.2.3.6}$$

$$\begin{pmatrix}1 & -1 & 0\\ 4 & -2-\lambda & 0\\ -1 & 0 & 1\end{pmatrix} \xleftrightarrow{R_2\leftarrow R_2-4R_1} \begin{pmatrix}1 & -1 & 0\\ 0 & -\lambda+2 & 0\\ -1 & 0 & 1\end{pmatrix} \tag{1.2.3.7}$$

Taking $(2-\lambda)$ common from R_2:

$$(2-\lambda)^2(3-\lambda)\begin{pmatrix}1 & -1 & 0\\ 0 & 1 & 0\\ -1 & 0 & 1\end{pmatrix} \tag{1.2.3.8}$$

Eigen values are:

$$\lambda_1 = 2 \tag{1.2.3.9}$$
$$\lambda_2 = 3 \tag{1.2.3.10}$$

solution to $\mathbf{AX} = 2\mathbf{X}$ is eigen vector corresponding to $\lambda = 2$

$$(\mathbf{A} - 2\mathbf{I})\mathbf{X} = 0 \tag{1.2.3.11}$$

Substituting values:

$$\begin{pmatrix} 4 & -4 & 0 & 0 \\ 4 & -4 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{R_1}{4}} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 4 & -4 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - 4R_1}$$

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 - R_1} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{pmatrix} \xleftrightarrow{R_3 \longleftrightarrow R_2}$$

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow -R_2} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 + R_2}$$

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1.2.3.12}$$

So, x_3 is a free variable: Let $x_3 = c$.

$$x_2 - x_3 = 0 \implies x_2 = x_3 = c \tag{1.2.3.13}$$
$$x_1 - x_3 = 0 \implies x_1 = x_3 = c \tag{1.2.3.14}$$

So,the solution to $\mathbf{AX} = 2\mathbf{X}$is

$$\mathbf{X} = c\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{1.2.3.15}$$

solution of $\mathbf{AX} = 3\mathbf{X}$ is eigen vector corresponding to $\lambda = 3$

$$(\mathbf{A} - 3\mathbf{I})\mathbf{X} = 0 \tag{1.2.3.16}$$

substituting we have:

$$\begin{pmatrix} 3 & -4 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{R_1}{3}} \begin{pmatrix} 1 & -\frac{4}{3} & 0 & 0 \\ 4 & -5 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - 4R_1}$$

$$\begin{pmatrix} 1 & -\frac{4}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 + R_1} \begin{pmatrix} 1 & -\frac{4}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & -\frac{4}{3} & 0 & 0 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow \frac{R_2}{3}}$$

$$\begin{pmatrix} 1 & \frac{-4}{3} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\frac{4}{3} & 0 & 0 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 - \frac{4}{3}R_2} \begin{pmatrix} 1 & \frac{4}{3} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_1 + \frac{4}{3}R_2}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1.2.3.17}$$

So x_3 is a free variable:

$$x_1 = 0 \tag{1.2.3.18}$$
$$x_2 = 0 \tag{1.2.3.19}$$
$$x_3 = c \tag{1.2.3.20}$$

So,the solution to $\mathbf{AX}$ =3$\mathbf{X}$ is,

$$\mathbf{X} = c\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{1.2.3.21}$$

1.2.4. Find a row-reduced matrix which is row equivalent to,

$$\mathbf{A} = \begin{pmatrix} i & -(1+i) & 0 \\ 1 & -2 & 1 \\ 1 & 2i & -1 \end{pmatrix} \tag{1.2.4.1}$$

Solution: Step 1: Performing scaling operation to matrix $\mathbf{A}$ as $R_1 \leftarrow \frac{1}{i}R_1$ by scaling matrix D_1 given as,

$$\mathbf{D_1} = \begin{pmatrix} \frac{1}{i} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.2.4.2}$$

$$\mathbf{D_1A} = \begin{pmatrix} \frac{1}{i} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} i & -(1+i) & 0 \\ 1 & -2 & 1 \\ 1 & 2i & -1 \end{pmatrix} \tag{1.2.4.3}$$

$$\implies \mathbf{D_1A} = \begin{pmatrix} 1 & -1+i & 0 \\ 1 & -2 & 1 \\ 1 & 2i & -1 \end{pmatrix} \tag{1.2.4.4}$$

Step 2: Performing $R_2 \leftarrow R_2 - R_1$ and $R_3 \leftarrow R_3 - R_1$ given by elementary matrix $\mathbf{E_{31}E_{21}}$ on equation (1.2.4.4),

$$\mathbf{E_{31}E_{21}} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \tag{1.2.4.5}$$

$$\mathbf{E_{31}E_{21}D_1A} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1+i & 0 \\ 1 & -2 & 1 \\ 1 & 2i & -1 \end{pmatrix} \tag{1.2.4.6}$$

$$\implies \mathbf{A_1} = \mathbf{E_{31}E_{21}D_1A} = \begin{pmatrix} 1 & -1+i & 0 \\ 0 & -1-i & 1 \\ 0 & 1+i & -1 \end{pmatrix} \tag{1.2.4.7}$$

Step 3: Performing $R_2 \leftarrow \frac{-1}{1+i}R_2$ given by $\mathbf{D_2}$ on equation (1.2.4.7),

$$\mathbf{D_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2}(-1+i) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.2.4.8}$$

$$\mathbf{D_2A_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2}(-1+i) & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1+i & 0 \\ 0 & -1-i & 1 \\ 0 & 1+i & -1 \end{pmatrix} \tag{1.2.4.9}$$

$$\implies \mathbf{A_2} = \mathbf{D_2A_1} = \begin{pmatrix} 1 & -1+i & 0 \\ 0 & 1 & \frac{1}{2}(-1+i) \\ 0 & 1+i & -1 \end{pmatrix} \tag{1.2.4.10}$$

Step 4: Performing $R_3 \leftarrow R_3 - (1+i)R_2$ given by $\mathbf{E_{32}}$ on equation (1.2.4.10),

$$\mathbf{E_{32}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -(1+i) & 1 \end{pmatrix} \tag{1.2.4.11}$$

$$\mathbf{E_{32}A_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1-i & 1 \end{pmatrix}\begin{pmatrix} 1 & -1+i & 0 \\ 0 & 1 & \frac{-1+i}{2} \\ 0 & 1+i & -1 \end{pmatrix} \tag{1.2.4.12}$$

$$\implies \mathbf{A_3} = \mathbf{E_{32}A_2} = \begin{pmatrix} 1 & -1+i & 0 \\ 0 & 1 & \frac{-1+i}{2} \\ 0 & 0 & 1 \end{pmatrix} \tag{1.2.4.13}$$

Step 5: Performing $R_1 \leftarrow R_1 - (-1+i)R_2$ given by $\mathbf{E_{12}}$ on equation (1.2.4.13),

$$\mathbf{E_{12}} = \begin{pmatrix} 1 & 1-i & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.2.4.14}$$

$$\mathbf{E_{12}A_3} = \begin{pmatrix} 1 & 1-i & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1+i & 0 \\ 0 & 1 & \frac{-1+i}{2} \\ 0 & 0 & 1 \end{pmatrix} \tag{1.2.4.15}$$

$$\implies \mathbf{A_4} = \mathbf{E_{12}A_3} = \begin{pmatrix} 1 & 0 & i \\ 0 & 1 & \frac{-1+i}{2} \\ 0 & 0 & 1 \end{pmatrix} \tag{1.2.4.16}$$

Step 6: Performing $R_1 \leftarrow R_1 - iR_3$ and $R_2 \leftarrow R_2 - \frac{-1+i}{2}R_3$ given by $\mathbf{E_{13}E_{23}}$ on equation (1.2.4.16),

$$\mathbf{E_{13}E_{23}} = \begin{pmatrix} 1 & 0 & -i \\ 0 & 1 & -\left(\frac{-1+i}{2}\right) \\ 0 & 0 & 1 \end{pmatrix} \tag{1.2.4.17}$$

$$\mathbf{E_{13}E_{23}A_4} = \begin{pmatrix} 1 & 0 & -i \\ 0 & 1 & -\left(\frac{-1+i}{2}\right) \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & i \\ 0 & 1 & \frac{-1+i}{2} \\ 0 & 0 & 1 \end{pmatrix} \tag{1.2.4.18}$$

$$\implies \mathbf{A_5} = \mathbf{E_{13}E_{23}A_4} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.2.4.19}$$

$\therefore$ Row-reduced matrix of $\mathbf{A}$ given by equation (1.2.4.1) is,

$$\mathbf{A} = \begin{pmatrix} i & -1-i & 0 \\ 1 & -2 & 1 \\ 1 & 2i & -1 \end{pmatrix} \xleftrightarrow{RREF} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I} \tag{1.2.4.20}$$

1.2.5. Prove that the following two matrices are not row equivalent

$$\begin{pmatrix} 2 & 0 & 0 \\ a & -1 & 0 \\ b & c & 3 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 2 \\ -2 & 0 & -1 \\ 1 & 3 & 5 \end{pmatrix} \tag{1.2.5.1}$$

Solution: Call the first matrix $\mathbf{A}$ and the second matrix $\mathbf{B}$.

$$\mathbf{A}^T = \begin{pmatrix} 2 & a & b \\ 0 & -1 & c \\ 0 & 0 & 3 \end{pmatrix} \tag{1.2.5.2}$$

$\mathbf{A}^T$ is a upper triangular matrix with non-zero diagonal. Hence it has full rank = 3.

$$\mathbf{B}^T = \begin{pmatrix} 1 & -2 & 1 \\ 1 & 0 & 3 \\ 2 & -1 & 5 \end{pmatrix} \xleftrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - R_1} \begin{pmatrix} 1 & -2 & 1 \\ 0 & 2 & 2 \\ 0 & 3 & 3 \end{pmatrix} \tag{1.2.5.3}$$

$$\xleftrightarrow[R_2 \leftarrow R_2/2]{R_3 \leftarrow R_3/3} \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \tag{1.2.5.4}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - R_2} \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.2.5.5}$$

$\mathbf{B}^T$ is a upper triangular matrix with zero diagonal. Hence it doesn't have full rank. Therefore both matrices have different rank , so it cannot be row equivalent.

1.2.6. Let

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{1.2.6.1}$$

be a 2×2 matrix with complex entries. Suppose A is row-reduced and also that $a+b+c+d = 0$. Prove that there are exactly three such matrices.

Solution: A matrix is in row echelon form if it follows the following conditions

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry (i.e. left most nonzero entry) of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zero Row Reduced Echelon Form A matrix is in row reduced echelon form if it follows the following conditions
1. The matrix should be row echelon form
2. The leading entry in each nonzero row is 1.
3. Each leading 1 is the only nonzero entry in its column. Proof

Given ,

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{1.2.6.2}$$

Condition 1 : Matrix **A** should be in row-reduced echelon form
Condition 2 : $a + b + c + d = 0$ where a,b,c and d are the elements of the matrix **A**
Reducing the matrix **A** from equation (1.2.6.2)

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \xleftrightarrow{R_1 = \frac{1}{a} R_1} \begin{pmatrix} 1 & \frac{b}{a} \\ c & d \end{pmatrix} \tag{1.2.6.3}$$

$$\xleftrightarrow{R_2 = R_2 - cR_1} \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & \frac{ad-bc}{a} \end{pmatrix} \tag{1.2.6.4}$$

$$\xleftrightarrow{R_2 = \frac{a}{ad-bc} R_2} \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix} \tag{1.2.6.5}$$

$$\xleftrightarrow{R_1 = R_1 - \frac{b}{a} R_2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.2.6.6}$$

Case 1: Matrix **A** of Rank 2
From the equation (1.2.6.4),for the matrix to be in row reduced echelon form,

$$b = 0$$
$$a \neq 0$$
$$d = 1$$
$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.2.6.7}$$

For the condition 2 to get satisfied,

$$a + 0 + c + 1 = 0 \tag{1.2.6.8}$$
$$\implies a = -(c+1) \tag{1.2.6.9}$$
$$\implies c \neq -1 \tag{1.2.6.10}$$

Both the condition gets satisfied and so exactly one matrix **A** can be formed of Rank 2 with given conditions
Case 2: Matrix **A** of Rank 1
From the equation (1.2.6.4),for the matrix to be in row reduced echelon form,

$$a \neq 0$$
$$d = 0$$
$$c = 0$$

For the condition 2 to get satisfied,

$$a + b + 0 + 0 = 0 \tag{1.2.6.11}$$
$$\implies b = -a \tag{1.2.6.12}$$

$$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \tag{1.2.6.13}$$

Both the condition gets satisfied and so exactly one matrix $\mathbf{A}$ can be formed of Rank 1 with given conditions

Case 3: Matrix $\mathbf{A}$ of Rank 0

From equation (1.2.6.2),for the matrix to be in row reduced echelon form,

$$a = 0$$
$$b = 0$$
$$c = 0$$
$$d = 0$$
$$\mathbf{A} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{1.2.6.14}$$

Both the condition gets satisfied and so exactly one matrix $\mathbf{A}$ can be formed of Rank 0 with given conditions

Therefore matrix $\mathbf{A}$ shown in equation (1.2.6.7),(1.2.6.13) and (1.2.6.14) are the exactly three such matrices that can be formed with given conditions.

1.2.7. Prove that the interchange of two rows of a matrix can be accomplished by a finite sequence of elementary row operations of the other two types.

Solution: Let $\mathbf{A}$ be a 3×3 matrix with having row vectors $\mathbf{a}_1$,$\mathbf{a}_2$ and $\mathbf{a}_3$.

$$\mathbf{A} = \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} \tag{1.2.7.1}$$

Let's exchange row $\mathbf{a}_1$ and $\mathbf{a}_2$. Let's call this elementary operation $\mathbf{E_1}$.

$$\mathbf{E}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.2.7.2}$$

$$\tag{1.2.7.3}$$

Now performing operation $\mathbf{E}_1$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{a}_2 \\ \mathbf{a}_1 \\ \mathbf{a}_3 \end{pmatrix} \tag{1.2.7.4}$$

Now, to prove that same matrix can be obtained by elementary operations let's call them $\mathbf{E_2}$ and $\mathbf{E_3}$.Now performing operation $\mathbf{E}_2$ by adding row 2 to row 1.

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{a}_1 + \mathbf{a}_2 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} \tag{1.2.7.5}$$

Using elementary operation $\mathbf{E_2}$ we will subtract row 1 from row 2.

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}_1 + \mathbf{a}_2 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{a}_1 + \mathbf{a}_2 \\ -\mathbf{a}_1 \\ \mathbf{a}_3 \end{pmatrix} \tag{1.2.7.6}$$

Using elementary operation $\mathbf{E_2}$ we will add row 2 to row 1.

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}_1 + \mathbf{a}_2 \\ -\mathbf{a}_1 \\ \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{a}_2 \\ -\mathbf{a}_1 \\ \mathbf{a}_3 \end{pmatrix} \tag{1.2.7.7}$$

Using elementary operation $\mathbf{E_3}$ we will multiply row 2 by -1.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}_2 \\ -\mathbf{a}_1 \\ \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{a}_2 \\ \mathbf{a}_1 \\ \mathbf{a}_3 \end{pmatrix} \tag{1.2.7.8}$$

Hence, we can say that,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} \tag{1.2.7.9}$$

Let us assume a matrix $\mathbf{A}$

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \tag{1.2.7.10}$$

Let's exchange row $\mathbf{a}_1$ and $\mathbf{a}_2$ by applying operation $\mathbf{E_1}$.

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & 3 \\ 1 & 1 & 0 \end{pmatrix} \tag{1.2.7.11}$$

Now, to prove that same matrix can be obtained by other two elementary operations.We will first perform elementary operation $\mathbf{E_2}$ by adding row 2 to row 1.

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \tag{1.2.7.12}$$

Using elementary operation $\mathbf{E_2}$ we will subtract row 1 from row 2.

$$\begin{pmatrix}1&0&0\\-1&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&3&3\\0&1&0\\1&1&0\end{pmatrix}=\begin{pmatrix}1&3&3\\-1&-2&-3\\1&1&0\end{pmatrix} \quad (1.2.7.13)$$

Using elementary operation $\mathbf{E_2}$ we will add row 2 to row 1.

$$\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&3&3\\-1&-2&-3\\1&1&0\end{pmatrix}=\begin{pmatrix}0&1&0\\-1&-2&-3\\1&1&0\end{pmatrix} \quad (1.2.7.14)$$

Using elementary operation $\mathbf{E_3}$ we will multiply row 2 by -1.

$$\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}\begin{pmatrix}0&1&0\\-1&-2&-3\\1&1&0\end{pmatrix}=\begin{pmatrix}0&1&0\\1&2&3\\1&1&0\end{pmatrix} \quad (1.2.7.15)$$

Hence,we can say that,

$$\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\-1&1&0\\0&0&1\end{pmatrix}$$
$$\times\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&2&3\\0&1&0\\1&1&0\end{pmatrix}=\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&2&3\\0&1&0\\1&1&0\end{pmatrix} \quad (1.2.7.16)$$

1.2.8. Consider the system of equations $\mathbf{AX} = 0$ where

$$\mathbf{A}=\begin{pmatrix}a&b\\c&d\end{pmatrix}$$

is a 2×2 matrix over the field F. Prove the following -

- If every entry of $\mathbf{A}$ is 0, then every pair x_1 and x_2 is a solution of $\mathbf{AX} = 0$.
- If $ad-bc \neq 0$, then the system $\mathbf{AX} = 0$ has only the trivial solution $x_1 = x_2 = 0$
- If $ad - bc = 0$ and some entry of $\mathbf{A}$ is different from 0, then there is a solution x_1^0 and x_2^0 such that x_1 and x_2 is a solution if and only if there is a scalar y such that $x_1 = yx_1^0$ and $x_2 = yx_2^0$

Solution: Solution 1 If every entry of $\mathbf{A}$ is 0 then the equation $\mathbf{AX} = 0$ becomes,

$$\begin{pmatrix}0&0\\0&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=0 \quad (1.2.8.1)$$

$$\implies 0.x_1+0.x_2=0 \qquad \forall x_1,x_2\in F \quad (1.2.8.2)$$

Hence proved, every pair x_1 and x_2 is a solution for the equation $\mathbf{AX} = 0$. Solution 2 **Case 1:** Let $a = 0$. Since $ad - bc \neq 0$. As $bc \neq 0$ therefore $b \neq 0$ and $c \neq 0$. Hence, we can perform row reduction on the augmented matrix of equation $\mathbf{AX}$=0 as follows,

$$\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&b&0\\c&d&0\end{pmatrix}=\begin{pmatrix}\frac{1}{c}&0\\0&1\end{pmatrix}\begin{pmatrix}c&d&0\\0&b&0\end{pmatrix} \quad (1.2.8.3)$$

$$=\begin{pmatrix}1&0\\0&\frac{1}{b}\end{pmatrix}\begin{pmatrix}1&\frac{d}{c}&0\\0&b&0\end{pmatrix} \quad (1.2.8.4)$$

$$=\begin{pmatrix}1&-\frac{d}{c}\\0&1\end{pmatrix}\begin{pmatrix}1&\frac{d}{c}&0\\0&1&0\end{pmatrix} \quad (1.2.8.5)$$

$$=\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix} \quad (1.2.8.6)$$

Case 2: Let $a,b,c,d \neq 0$. Considering the following case,

$$\mathbf{AX}=\mathbf{u} \quad (1.2.8.7)$$

$$\implies \begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}u_1\\u_2\end{pmatrix} \quad (1.2.8.8)$$

Row Reducing the augmented matrix of (1.2.8.8) we get,

$$\begin{pmatrix}\frac{1}{a}&0\\0&1\end{pmatrix}\begin{pmatrix}a&b&u_1\\c&d&u_2\end{pmatrix}=\begin{pmatrix}1&0\\-c&1\end{pmatrix}\begin{pmatrix}1&\frac{b}{a}&\frac{u_1}{a}\\c&d&u_2\end{pmatrix} \quad (1.2.8.9)$$

$$=\begin{pmatrix}1&0\\0&\frac{a}{ad-bc}\end{pmatrix}\begin{pmatrix}1&\frac{b}{a}&\frac{u_1}{a}\\0&\frac{ad-bc}{a}&\frac{au_2-cu_1}{a}\end{pmatrix} \quad (1.2.8.10)$$

$$=\begin{pmatrix}1&-\frac{b}{a}\\0&1\end{pmatrix}\begin{pmatrix}1&\frac{b}{a}&\frac{u_1}{a}\\0&1&\frac{au_2-cu_1}{ad-bc}\end{pmatrix} \quad (1.2.8.11)$$

$$=\begin{pmatrix}1&0&\frac{du_1-bu_2}{ad-bc}\\0&1&\frac{au_2-cu_1}{ad-bc}\end{pmatrix} \quad (1.2.8.12)$$

From (1.2.8.12) we get,

$$x_1 = \frac{du_1 - bu_2}{ad - bc} \tag{1.2.8.13}$$

$$x_2 = \frac{au_2 - cu_1}{ad - bc} \tag{1.2.8.14}$$

Since $u_1 = 0$ and $u_2 = 0$ then from (1.2.8.13) and (1.2.8.14),

$$x_1 = 0 \tag{1.2.8.15}$$

$$x_2 = 0 \tag{1.2.8.16}$$

Hence we get,

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1.2.8.17}$$

In (1.2.8.6) and (1.2.8.17), we can see that $\mathbf{AX} = 0$ has only one trivial solution i.e $x_1 = x_2 = 0$ in all cases. Hence proved, the equation $\mathbf{AX}$=0 has only one trivial solution $x_1 = x_2 = 0$ Solution 3 **Case 1:** Let, $a \neq 0$ for $\mathbf{A}$. Given $ad - bc = 0$, we can perform row reduction on augmented matrix of equation $\mathbf{AX} = 0$ as follows,

$$\begin{pmatrix} \frac{1}{a} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b & 0 \\ c & d & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -c & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{b}{a} & 0 \\ c & d & 0 \end{pmatrix} \tag{1.2.8.18}$$

$$= \begin{pmatrix} 1 & \frac{b}{a} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad [\because ad - bc = 0] \tag{1.2.8.19}$$

Hence from (1.2.8.19), $\mathbf{AX} = 0$ if and only if

$$x_1 = -\frac{b}{a}x_2 \qquad [a \neq 0] \tag{1.2.8.20}$$

Letting $x_1^0 = -\frac{b}{a}$ and $x_2^0 = 1$ we get for $y = 1$,

$$x_1 = yx_1^0 \tag{1.2.8.21}$$

$$x_2 = yx_2^0 \tag{1.2.8.22}$$

which is a solution of the equation $\mathbf{AX} = 0$.
Case 2: Let, $b \neq 0$ for $\mathbf{A}$. Given $ad - bc = 0$, at first we multiply by elementary matrix to change the columns and the we can perform row reduction on augmented matrix of equation $\mathbf{AX} = 0$ as follows,

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} b & a & 0 \\ d & c & 0 \end{pmatrix} \tag{1.2.8.23}$$

Hence using the result obtained from (1.2.8.19) we can conclude for (1.2.8.23), $\mathbf{AX} = 0$ if and only if

$$x_2 = -\frac{a}{b}x_1 \qquad [b \neq 0] \tag{1.2.8.24}$$

Letting $x_2^0 = -\frac{a}{b}$ and $x_1^0 = 1$ we get for $y = 1$,

$$x_1 = yx_1^0 \tag{1.2.8.25}$$

$$x_2 = yx_2^0 \tag{1.2.8.26}$$

which is a solution of the equation $\mathbf{AX} = 0$.
Case 3: Let, $c \neq 0$ for $\mathbf{A}$. Given $ad - bc = 0$, we can perform row reduction on augmented matrix of equation $\mathbf{AX} = 0$ as follows,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b & 0 \\ c & d & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{c} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c & d & 0 \\ a & b & 0 \end{pmatrix} \tag{1.2.8.27}$$

$$= \begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{d}{c} & 0 \\ a & b & 0 \end{pmatrix} \tag{1.2.8.28}$$

$$= \begin{pmatrix} 1 & \frac{d}{c} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad [\because ad - bc = 0] \tag{1.2.8.29}$$

Hence from (1.2.8.29), $\mathbf{AX} = 0$ if and only if

$$x_1 = -\frac{d}{c}x_2 \qquad [a \neq 0] \tag{1.2.8.30}$$

Letting $x_1^0 = -\frac{d}{c}$ and $x_2^0 = 1$ we get for $y = 1$,

$$x_1 = yx_1^0 \tag{1.2.8.31}$$

$$x_2 = yx_2^0 \tag{1.2.8.32}$$

which is a solution of the equation $\mathbf{AX} = 0$.
Case 4: Let, $d \neq 0$ for $\mathbf{A}$. Given $ad - bc = 0$, at first we multiply by elementary matrix to change the columns and then we can perform row reduction on augmented matrix of equation $\mathbf{AX} = 0$ as follows,

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} b & a & 0 \\ d & c & 0 \end{pmatrix} \tag{1.2.8.33}$$

$$= \begin{pmatrix} d & c & 0 \\ b & a & 0 \end{pmatrix} \tag{1.2.8.34}$$

Hence using the result from (1.2.8.29) we can conclude for (1.2.8.34), $\mathbf{AX} = 0$ if and only if

$$x_2 = -\frac{c}{d}x_1 \qquad [a \neq 0] \tag{1.2.8.35}$$

Letting $x_2^0 = -\frac{c}{d}$ and $x_1^0 = 1$ we get for $y = 1$,

$$x_1 = yx_1^0 \tag{1.2.8.36}$$

$$x_2 = yx_2^0 \tag{1.2.8.37}$$

which is a solution of the equation $\mathbf{AX} = 0$.

1.3 Row Reduced Echelon Matrices

1.3.1. Find all solutions to the following system of equations by row-reducing the co-efficient matrix:

$$\frac{1}{3}x_1 + 2x_2 - 6x_3 = 0 \tag{1.3.1.1}$$

$$-4x_1 \qquad + 5x_3 = 0 \tag{1.3.1.2}$$

$$-3x_1 + 6x_2 - 13x_3 = 0 \tag{1.3.1.3}$$

$$-\frac{7}{3}x_1 + 2x_2 - \frac{8}{3}x_3 = 0 \tag{1.3.1.4}$$

Solution: The coefficient matrix is:

$$A = \begin{pmatrix} \frac{1}{3} & 2 & -6 \\ -4 & 0 & 5 \\ -3 & 6 & -13 \\ -\frac{7}{3} & 2 & -\frac{8}{3} \end{pmatrix} \tag{1.3.1.5}$$

The number of rows of this coefficient matrix is $m = 4$ and the number of columns is n= 3, So in this case, $n < m$. Now the row operations are:

$$\begin{pmatrix} \frac{1}{3} & 2 & -6 \\ -4 & 0 & 5 \\ -3 & 6 & -13 \\ -\frac{7}{3} & 2 & -\frac{8}{3} \end{pmatrix} \xleftrightarrow[R_1 \leftarrow R_1 \times 3]{R_4 \leftarrow R_4 \times 3} \begin{pmatrix} 1 & 6 & -18 \\ -4 & 0 & 5 \\ -3 & 6 & -13 \\ -7 & 6 & -8 \end{pmatrix} \tag{1.3.1.6}$$

$$\xleftrightarrow{R_3 \leftarrow R_2 + R_3} \begin{pmatrix} 1 & 6 & -18 \\ -4 & 0 & 5 \\ -7 & 6 & -8 \\ -7 & 6 & -8 \end{pmatrix} \xleftrightarrow{R_4 \leftarrow R_4 - R_3} \tag{1.3.1.7}$$

$$\begin{pmatrix} 1 & 6 & -18 \\ -4 & 0 & 5 \\ -7 & 6 & -8 \\ 0 & 0 & 0 \end{pmatrix} \xleftrightarrow[R_3 \leftarrow R_3 + 7R_1]{R_2 \leftarrow R_2 + 4R_1} \begin{pmatrix} 1 & 6 & -18 \\ 0 & 24 & -67 \\ 0 & 48 & -138 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.3.1.8}$$

$$\xleftrightarrow{R_3 \leftarrow R_3/2} \begin{pmatrix} 1 & 6 & -18 \\ 0 & 24 & -67 \\ 0 & 24 & -69 \\ 0 & 0 & 0 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 - R_2} \tag{1.3.1.9}$$

$$\begin{pmatrix} 1 & 6 & -18 \\ 0 & 24 & -67 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow \frac{R_3}{(-2)}} \begin{pmatrix} 1 & 6 & -18 \\ 0 & 24 & -67 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.3.1.10}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{R_2}{4}} \begin{pmatrix} 1 & 6 & -18 \\ 0 & 6 & -\frac{67}{4} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 - R_2} \tag{1.3.1.11}$$

$$\begin{pmatrix} 1 & 0 & -\frac{5}{4} \\ 0 & 6 & -\frac{67}{4} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow \frac{R_2}{6}} \begin{pmatrix} 1 & 0 & -\frac{5}{4} \\ 0 & 1 & -\frac{67}{24} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.3.1.12}$$

$$\xleftrightarrow[R_1 \leftarrow R_1 + \frac{5R_3}{4}]{R_2 \leftarrow R_2 + \frac{67R_3}{24}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.3.1.13}$$

Now,

$$A\mathbf{x} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \tag{1.3.1.14}$$

So,

$$\mathbf{I_3x} = 0 \tag{1.3.1.15}$$

$$\implies \mathbf{x} = 0 \tag{1.3.1.16}$$

1.3.2. Find a row-reduced matrix which is row equivalent to A.What are the solutions of $\mathbf{Ax} = \mathbf{0}$?

$$\mathbf{A} = \begin{pmatrix} 1 & -i \\ 2 & 2 \\ i & 1+i \end{pmatrix} \tag{1.3.2.1}$$

Solution: Let R be a row-reduced echelon matrix which is row equivalent to A. Then the systems

$$A\mathbf{x} = \mathbf{0}, R\mathbf{x} = \mathbf{0} \tag{1.3.2.2}$$

have the same solutions. On performing elementary row operations on (1.3.2.1),

$$\mathbf{R} = \mathbf{BA} \tag{1.3.2.3}$$

where $\mathbf{B}$ is the product of all elementary matrices. Reducing the given matrix, we get

$$\begin{aligned} \mathbf{B} &= (\mathbf{E_5E_4E_3E_2E_1}) \\ &= \begin{pmatrix} 1 & i & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -i & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4}(1-i) & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &\quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -i & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2}(1-i) & \frac{1}{4}(1+i) & 0 \\ \frac{1}{2}(-1+i) & \frac{1}{4}(1-i) & 0 \\ \frac{1}{2}(1-i) & \frac{1}{4}(-1-i) & 1 \end{pmatrix} \end{aligned} \tag{1.3.2.4}$$

$$\mathbf{R} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{1.3.2.5}$$

$\therefore$ Row-reduced matrix of $\mathbf{A}$ is,

$$\mathbf{A} = \begin{pmatrix} 1 & -i \\ 2 & 2 \\ i & 1+i \end{pmatrix} \xleftrightarrow{RREF} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{1.3.2.6}$$

From(1.3.2.2) and (1.3.2.6),

$$A\mathbf{x} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \tag{1.3.2.7}$$

The solution of $\mathbf{Ax} = \mathbf{0}$ is,

$$\mathbf{I_2x} = 0 \tag{1.3.2.8}$$

$$\implies \mathbf{x} = 0 \tag{1.3.2.9}$$

As $\mathbf{I_2}$ is invertible.

1.3.3. Describe explicitly all 2x2 row-reduced echelon matrices.

Solution:

2x2 matrices which are row-reduced echelon matrix can be represented as a linear combination of three matrices:-

$$c_1\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + c_2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c_3\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.3.3.1}$$

1.3.4. Consider the system of the equations

$$x_1 - x_2 + 2x_3 = 1 \tag{1.3.4.1}$$

$$x_1 - 0x_2 + 2x_3 = 1 \tag{1.3.4.2}$$

$$x_1 - 3x_2 + 4x_3 = 2 \tag{1.3.4.3}$$

Does this system have a solution ? If so describe explicitly all solutions.

Solution: Let $\mathbf{V}$ is the set of all $\begin{pmatrix} x_1, & x_2, & x_3 \end{pmatrix} \in \mathbb{R}^3$ which satisfy the (1.3.4.1), (1.3.4.2) and (1.3.4.3)

From equation (1.3.4.1) to (1.3.4.3) we can write,

$$\begin{pmatrix} 1 & -1 & 2 \\ 1 & 0 & 2 \\ 1 & -3 & 4 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \tag{1.3.4.4}$$

$$\implies \mathbf{Ax} = \mathbf{b} \tag{1.3.4.5}$$

Where,

$$\tag{1.3.4.6}$$

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 2 \\ 1 & 0 & 2 \\ 1 & -3 & 4 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{1.3.4.7}$$

Solving the matrix **A** for rank we get,

$$\begin{pmatrix}1 & -1 & 2\\2 & 0 & 2\\1 & -3 & 4\end{pmatrix}\xleftrightarrow{R_2=R_1-2R_1}\begin{pmatrix}1 & -1 & 2\\0 & 2 & -2\\1 & -3 & 4\end{pmatrix} \tag{1.3.4.8}$$

$$\xleftrightarrow{R_3=R_3-R_1}\begin{pmatrix}1 & -1 & 2\\0 & 2 & -2\\0 & -2 & 2\end{pmatrix} \tag{1.3.4.9}$$

$$\xleftrightarrow{R_3=R_3+R_2}\begin{pmatrix}1 & -1 & 2\\0 & 2 & -2\\0 & 0 & 0\end{pmatrix} \tag{1.3.4.10}$$

Hence, rank $(\mathbf{A}) = 2$. Now solving the augmented matrix of (1.3.4.5) we get,

$$\begin{pmatrix}1 & -1 & 2 & 1\\2 & 0 & 2 & 1\\1 & -3 & 4 & 2\end{pmatrix}\xleftrightarrow{R_2=R_1-2R_1}\begin{pmatrix}1 & -1 & 2 & 1\\0 & 2 & -2 & -1\\1 & -3 & 4 & 2\end{pmatrix} \tag{1.3.4.11}$$

$$\xleftrightarrow{R_3=R_3-R_1}\begin{pmatrix}1 & -1 & 2 & 1\\0 & 2 & -2 & -1\\0 & -2 & 2 & 1\end{pmatrix} \tag{1.3.4.12}$$

$$\xleftrightarrow{R_3=R_3+R_2}\begin{pmatrix}1 & -1 & 2 & 1\\0 & 2 & -2 & -1\\0 & 0 & 0 & 0\end{pmatrix} \tag{1.3.4.13}$$

We have rank $(\mathbf{A})$ = rank $(\mathbf{A} : \mathbf{b}) = 2 < n$, where $n = 3$. Hence we have infinite no of solutions for given system of equations.
Using Gauss - Jordan elimination method to getting the solution,

$$\begin{pmatrix}1 & -1 & 2 & 1\\2 & 0 & 2 & 1\\1 & -3 & 4 & 2\end{pmatrix}\xleftrightarrow{R_2=R_1-2R_1}\begin{pmatrix}1 & -1 & 2 & 1\\0 & 2 & -2 & -1\\1 & -3 & 4 & 2\end{pmatrix} \tag{1.3.4.14}$$

$$\xleftrightarrow{R_3=R_3-R_1}\begin{pmatrix}1 & -1 & 2 & 1\\0 & 2 & -2 & -1\\0 & -2 & 2 & 1\end{pmatrix} \tag{1.3.4.15}$$

$$\xleftrightarrow{R_2=\frac{R_2}{2}}\begin{pmatrix}1 & -1 & 2 & 1\\0 & 1 & -1 & -\frac{1}{2}\\0 & -2 & 2 & 1\end{pmatrix} \tag{1.3.4.16}$$

$$\xleftrightarrow{R_3=R_3+2R_2}\begin{pmatrix}1 & -1 & 2 & 1\\0 & 1 & -1 & -\frac{1}{2}\\0 & 0 & 0 & 0\end{pmatrix} \tag{1.3.4.17}$$

$$\xleftrightarrow{R_1=R_1+R_2}\begin{pmatrix}1 & 0 & 1 & \frac{1}{2}\\0 & 1 & -1 & -\frac{1}{2}\\0 & 0 & 0 & 0\end{pmatrix} \tag{1.3.4.18}$$

$$\implies x_1 + x_3 = \frac{1}{2}, x_2 - x_3 = -\frac{1}{2} \tag{1.3.4.19}$$

$$\implies x_2 = -\frac{1}{2} + x_3, x_1 = \frac{1}{2} - x_3 \tag{1.3.4.20}$$

From equation (1.3.4.19) and (1.3.4.20)

$$\mathbf{x} = \begin{pmatrix}\frac{1}{2} - x_3\\-\frac{1}{2} + x_3\\x_3\end{pmatrix} \tag{1.3.4.21}$$

which can be written as,

$$\mathbf{x} = x_3\begin{pmatrix}-1\\1\\1\end{pmatrix} + \begin{pmatrix}-\frac{1}{2}\\-\frac{1}{2}\\0\end{pmatrix} \tag{1.3.4.22}$$

from 1.3.4.22 we can say that for any value x_3, **V** will no be gives zero vector. Hence the given solution space will not span of the vector space **V**

1.3.5. Give an example of a system of two linear equations in two unknowns which has no solution.

Solution: Let us assume two equations as given below $\begin{pmatrix}5 & 2\end{pmatrix}\mathbf{x} = 7$ and $\begin{pmatrix}10 & 4\end{pmatrix}\mathbf{x} = -3$

Let the coefficient matrix be given as

$$\mathbf{A} = \begin{pmatrix}5 & 2\\10 & 4\end{pmatrix}, \mathbf{b} = \begin{pmatrix}7\\-3\end{pmatrix} \tag{1.3.5.1}$$

the augmented matrix be given as matrix be given as

$$\mathbf{A}|\mathbf{B} = \begin{pmatrix}5 & 2 & 7\\10 & 4 & -3\end{pmatrix} \tag{1.3.5.2}$$

Applying row reduction

$$\begin{pmatrix}5 & 2 & 7\\ 10 & 4 & -3\end{pmatrix} \xleftrightarrow{R_2=R_2-2R_1} \begin{pmatrix}5 & 2 & 7\\ 0 & 0 & -17\end{pmatrix} \tag{1.3.5.3}$$

$$\xleftrightarrow{R_1=\frac{R_1}{5}} \begin{pmatrix}1 & \frac{2}{5} & \frac{7}{5}\\ 0 & 0 & -17\end{pmatrix} \tag{1.3.5.4}$$

$$\xleftrightarrow{R_2=\frac{R_2}{-17}} \begin{pmatrix}1 & \frac{2}{5} & \frac{7}{5}\\ 0 & 0 & 1\end{pmatrix} \tag{1.3.5.5}$$

$$\xleftrightarrow{R_1=R_1-\frac{7}{5}R_2} \begin{pmatrix}1 & \frac{2}{5} & 0\\ 0 & 0 & 1\end{pmatrix} \tag{1.3.5.6}$$

$$\tag{1.3.5.7}$$

Clearly,On comparing the ranks of matrix **A** and **A|B**, we find that rank of matrix **A|B** ≠ **A** Hence the system of linear equation have no solutions. Consider the system **Ax** = **b**, with coefficient matrix **A** and augmented matrix **A|B**.

As above, the sizes of **b**, **A**, and **A|B** are m × 1, m × n, and m × (n + 1), respectively; in addition, the number of unknowns is n.

Ax is inconsistent (i.e., no solution exists) if and only if rank **A** < rank **A|B**.

1.3.6. Find all solutions of

$$x_1 - 2x_2 + x_3 + 2x_4 = 1$$
$$x_1 + x_2 - x_3 + x_4 + x_5 = 2$$
$$x_1 + 7x_2 - 5x_3 - x_4 = 3$$

Solution: The given equations can be written as,

$$\mathbf{Ax} = B \tag{1.3.6.1}$$

$$\begin{pmatrix}1 & -2 & 1 & 2\\ 1 & 1 & -1 & 1\\ 1 & 7 & -5 & -1\end{pmatrix}\mathbf{x} = \begin{pmatrix}1\\2\\3\end{pmatrix} \tag{1.3.6.2}$$

Now, we form the augmented matrix and perform Row reduction,

$$\left(\begin{array}{cccc|c}1 & -2 & 1 & 2 & 1\\ 1 & 1 & -1 & 1 & 2\\ 1 & 7 & -5 & -1 & 3\end{array}\right) \tag{1.3.6.3}$$

$$\xleftrightarrow{R_2=R_2-R_1, R_3=R_3-R_1} \left(\begin{array}{cccc|c}1 & -2 & 1 & 2 & 1\\ 0 & 3 & -2 & -1 & 1\\ 0 & 9 & -6 & -3 & 2\end{array}\right) \tag{1.3.6.4}$$

$$\xleftrightarrow{R_2=\frac{1}{3}R_2} \left(\begin{array}{cccc|c}1 & -2 & 1 & 2 & 1\\ 0 & 1 & \frac{-2}{3} & \frac{-1}{3} & \frac{1}{3}\\ 0 & 9 & -6 & -3 & 2\end{array}\right) \tag{1.3.6.5}$$

$$\xleftrightarrow{R_3=R_3-9R_1} \left(\begin{array}{cccc|c}1 & -2 & 1 & 2 & 1\\ 0 & 1 & \frac{-2}{3} & \frac{-1}{3} & \frac{1}{3}\\ 0 & 0 & 0 & 0 & -1\end{array}\right) \tag{1.3.6.6}$$

Rank of **A** is less than rank of the augmented matrix. Hence, the given system has no solution.

1.3.7. Find all solutions of

$$2x_1 - 3x_2 - 7x_3 + 5x_4 + 2x_5 = -2 \tag{1.3.7.1}$$
$$x_1 - 2x_2 - 4x_3 + 3x_4 + x_5 = -2 \tag{1.3.7.2}$$
$$2x_1 - 4x_3 + 2x_4 + x_5 = 3 \tag{1.3.7.3}$$
$$x_1 - 5x_2 - 7x_3 + 6x_4 + 2x_5 = -7 \tag{1.3.7.4}$$

Solution: The given equations can be written as,

$$\begin{pmatrix}2 & -3 & -7 & 5 & 2\\ 1 & -2 & -4 & 3 & 1\\ 2 & 0 & -4 & 2 & 1\\ 1 & -5 & -7 & 6 & 2\end{pmatrix}\mathbf{x} = \begin{pmatrix}-2\\-2\\3\\7\end{pmatrix} \tag{1.3.7.5}$$

Now, we form the augmented matrix and perform Row reduction,

$$\left(\begin{array}{ccccc|c} 2 & -3 & -7 & 5 & 2 & -2 \\ 1 & -2 & -4 & 3 & 1 & -2 \\ 2 & 0 & -4 & 2 & 1 & 3 \\ 1 & -5 & -7 & 6 & 2 & 7 \end{array}\right) \tag{1.3.7.6}$$

$$\xleftrightarrow{R_3=R_3-R_1} \left(\begin{array}{ccccc|c} 2 & -3 & -7 & 5 & 2 & -2 \\ 1 & -2 & -4 & 3 & 1 & -2 \\ 0 & 3 & 3 & -3 & -1 & 5 \\ 1 & -5 & -7 & 6 & 2 & 7 \end{array}\right) \tag{1.3.7.7}$$

$$\xleftrightarrow{R_1=\frac{1}{2}R_1} \left(\begin{array}{ccccc|c} 1 & \frac{-3}{2} & \frac{-7}{2} & \frac{5}{2} & 1 & -1 \\ 1 & -2 & -4 & 3 & 1 & -2 \\ 0 & 3 & 3 & -3 & -1 & 5 \\ 1 & -5 & -7 & 6 & 2 & 7 \end{array}\right) \tag{1.3.7.8}$$

$$\xleftrightarrow{R_2=R_2-R_1, R_4=R_4-R_1} \left(\begin{array}{ccccc|c} 1 & \frac{-3}{2} & \frac{-7}{2} & \frac{5}{2} & 1 & -1 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & -1 \\ 0 & 3 & 3 & -3 & -1 & 5 \\ 0 & -\frac{7}{2} & -\frac{7}{2} & \frac{7}{2} & 1 & -6 \end{array}\right) \tag{1.3.7.9}$$

$$\xleftrightarrow{R_1=R_1-3R_2} \left(\begin{array}{ccccc|c} 1 & 0 & -2 & 1 & 1 & 2 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & -1 \\ 0 & 3 & 3 & -3 & -1 & 5 \\ 0 & -\frac{7}{2} & -\frac{7}{2} & \frac{7}{2} & 1 & -6 \end{array}\right) \tag{1.3.7.10}$$

$$\xleftrightarrow{R_3=R_3+6R_2, R_4=R_4-7R_2} \left(\begin{array}{ccccc|c} 1 & 0 & -2 & 1 & 1 & 2 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right) \tag{1.3.7.11}$$

$$\xleftrightarrow{R_2=-2R_2} \left(\begin{array}{ccccc|c} 1 & 0 & -2 & 1 & 1 & 2 \\ 0 & 1 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right) \tag{1.3.7.12}$$

$$\xleftrightarrow{R_1=R_1+R_3, R_4=R_4+R_3, R_3=-R_3} \left(\begin{array}{ccccc|c} 1 & 0 & -2 & 1 & 0 & 1 \\ 0 & 1 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \tag{1.3.7.13}$$

So,

$$x_1 - 2x_3 + x_4 = 1 \tag{1.3.7.14}$$

$$x_2 + x_3 - x_4 = 2 \tag{1.3.7.15}$$

$$x_5 = 1 \tag{1.3.7.16}$$

Solving the equations we get,

$$x_1 = 1 + 2x_3 - x_4 \tag{1.3.7.17}$$

$$x_2 = 2 - x_3 + x_4 \tag{1.3.7.18}$$

$$x_5 = 1 \tag{1.3.7.19}$$

which can be written as,

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \tag{1.3.7.20}$$

$$\implies \mathbf{x} = \begin{pmatrix} 1 + 2x_3 - x_4 \\ 2 - x_3 + x_4 \\ x_3 \\ x_4 \\ 1 \end{pmatrix} \tag{1.3.7.21}$$

We can express (1.3.7.21) as a sum of linear combination of vectors,

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \mathbf{x_3} + \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \mathbf{x_4} \tag{1.3.7.22}$$

where $x_3,\ x_4 \in \mathbb{R}$.

Note that the above solution space is not closed on vector addition and scalar multiplication.As $x_5 = 1$, the zero vector is not included in the solution space. Hence, $\mathbf{x}$ is not a vector space. Since, $\mathbf{x}$ is not a vector space, it cannot be expressed in the form of linear combination of basis vectors.

1.3.8. Let

$$\mathbf{A} = \begin{pmatrix} 3 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 0 \end{pmatrix} \tag{1.3.8.1}$$

For which triples (y_1, y_2, y_3) does the system $\mathbf{AX} = \mathbf{Y}$ have a solution ?

Solution:

Given ,

$$\mathbf{AX} = \mathbf{Y} \tag{1.3.8.2}$$

$$\begin{pmatrix} 3 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 0 \end{pmatrix} \mathbf{X} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \tag{1.3.8.3}$$

Now we try to find the matrix $\mathbf{B}$ such that $\mathbf{BA}$ gives the row echelon form of matrix $\mathbf{A}$.

Here, **B** is given by ,

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ -\frac{7}{5} & \frac{8}{5} & 1 \end{pmatrix} \tag{1.3.8.4}$$

$$\implies \mathbf{BA} = \begin{pmatrix} 3 & -1 & 2 \\ 0 & \frac{5}{3} & -\frac{1}{3} \\ 0 & 0 & -\frac{6}{5} \end{pmatrix} \tag{1.3.8.5}$$

Therefore, from (1.3.8.5) rank of matrix **A** is 3 and it is a full rank matrix.
Hence the columns of **A** are linearly independent.
Therefore, the triples (y_1, y_2, y_3) are linear combination of columns of matrix **A**.

$$\implies \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = a\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + b\begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix} + c\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \tag{1.3.8.6}$$

where a,b,c can be any real value.

1.3.9. Let

$$\mathbf{A} = \begin{pmatrix} 3 & -6 & 2 & -1 \\ -2 & 4 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 1 & -2 & 1 & 0 \end{pmatrix} \tag{1.3.9.1}$$

For which (y_1, y_2, y_3, y_4) does the system of equations $\mathbf{AX} = \mathbf{Y}$ have a solution ? **Solution:** Given ,

$$\mathbf{AX} = \mathbf{Y} \tag{1.3.9.2}$$

$$\begin{pmatrix} 3 & -6 & 2 & -1 \\ -2 & 4 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 1 & -2 & 1 & 0 \end{pmatrix}\mathbf{X} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} \tag{1.3.9.3}$$

Now we try to find the matrix **B** such that **BA** gives the row echelon form of matrix **A** Here,**B** is given by ,

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{2}{3} & 1 & 0 & 0 \\ -\frac{2}{7} & -\frac{3}{7} & 1 & 0 \\ 0 & \frac{1}{2} & -\frac{3}{2} & 1 \end{pmatrix} \tag{1.3.9.4}$$

$$\mathbf{BA} = \begin{pmatrix} 3 & -6 & 2 & -1 \\ 0 & 0 & \frac{7}{3} & \frac{7}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1.3.9.5}$$

Therefore, rank of matrix **A** is 2 Now **B** is expressed in terms of two block matrices

$$\mathbf{B} = \begin{pmatrix} \mathbf{B_1} \\ \mathbf{B_2} \end{pmatrix} \tag{1.3.9.6}$$

$$\mathbf{B_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{2}{3} & 1 & 0 & 0 \end{pmatrix} \tag{1.3.9.7}$$

$$\mathbf{B_2} = \begin{pmatrix} -\frac{2}{7} & -\frac{3}{7} & 1 & 0 \\ 0 & \frac{1}{2} & -\frac{3}{2} & 1 \end{pmatrix} \tag{1.3.9.8}$$

Multiplying matrix **B** to both sides on the equation (1.3.9.2), we get ,

$$\begin{pmatrix} \mathbf{B_1} \\ \mathbf{B_2} \end{pmatrix}\mathbf{AX} = \begin{pmatrix} \mathbf{B_1} \\ \mathbf{B_2} \end{pmatrix}\mathbf{Y} \tag{1.3.9.9}$$

We know that , matrix **A** is of rank 2 The augumented matrix of (1.3.9.9) is given by

$$\left(\begin{array}{c|c} \mathbf{B_1A} & \mathbf{B_1Y} \\ \mathbf{B_2A} & \mathbf{B_2Y} \end{array}\right) \tag{1.3.9.10}$$

$$\mathbf{B_1A} = \begin{pmatrix} 3 & -6 & 2 & -1 \\ 0 & 0 & \frac{7}{3} & \frac{7}{3} \end{pmatrix} \tag{1.3.9.11}$$

$$\mathbf{B_2A} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1.3.9.12}$$

Since $\mathbf{B_2A}$ is zero matrix and for the given system $\mathbf{AX} = \mathbf{Y}$ to have a solution,

$$\mathbf{B_2Y} = 0 \tag{1.3.9.13}$$

$$\begin{pmatrix} -\frac{2}{7} & -\frac{3}{7} & 1 & 0 \\ 0 & \frac{1}{2} & -\frac{3}{2} & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = 0 \tag{1.3.9.14}$$

The augumented matrix of (1.3.9.14) is given by,

$$\left(\begin{array}{cccc|c} -\frac{2}{7} & -\frac{3}{7} & 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{3}{2} & 1 & 0 \end{array}\right) \tag{1.3.9.15}$$

By row reduction technique,

$$\xleftrightarrow{R_1=-\frac{7}{2}R_1}\left(\begin{array}{cccc|c} 1 & \frac{3}{2} & -\frac{7}{2} & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{3}{2} & 1 & 0 \end{array}\right) \tag{1.3.9.16}$$

$$\xleftrightarrow{R_2=2R_2}\left(\begin{array}{cccc|c} 1 & \frac{3}{2} & -\frac{7}{2} & 0 & 0 \\ 0 & 1 & -3 & 2 & 0 \end{array}\right) \tag{1.3.9.17}$$

$$\xleftrightarrow{R_1=R_1-\frac{3}{2}R_2}\left(\begin{array}{cccc|c} 1 & 0 & 1 & -3 & 0 \\ 0 & 1 & -3 & 2 & 0 \end{array}\right) \tag{1.3.9.18}$$

Equation (1.3.9.14) can be modified as ,

$$\begin{pmatrix}1 & 0 & 1 & -3\\0 & 1 & -3 & 2\end{pmatrix}\begin{pmatrix}y_1\\y_2\\y_3\\y_4\end{pmatrix} = 0 \tag{1.3.9.19}$$

Here y_3 and y_4 are free variables
If $y_3 = a$ and $y_4 = b$,then the solution to the system of equation $\mathbf{AX} = \mathbf{Y}$ is given by,

$$\begin{pmatrix}y_1\\y_2\\y_3\\y_4\end{pmatrix} = a\begin{pmatrix}-1\\3\\1\\0\end{pmatrix} + b\begin{pmatrix}3\\-2\\0\\1\end{pmatrix} \tag{1.3.9.20}$$

One of the solution when $a = 1$ and $b = 2$ is given by ,

$$\begin{pmatrix}y_1\\y_2\\y_3\\y_4\end{pmatrix} = \begin{pmatrix}-1\\3\\1\\0\end{pmatrix} + 2\begin{pmatrix}3\\-2\\0\\1\end{pmatrix} \tag{1.3.9.21}$$

1.3.10. Suppose $\mathbf{R}$ and $\mathbf{R}'$ are 2×3 row-reduced echelon matrices and that the system $\mathbf{RX}$=0 and $\mathbf{R}'\mathbf{X}$=0 have exactly the same solutions. Prove that $\mathbf{R} = \mathbf{R}'$.
Solution:
Since $\mathbf{R}$ and $\mathbf{R}'$ are 2×3 row-reduced echelon matrices they can be of following three types:-

a) Suppose matrix $\mathbf{R}$ has one non-zero row then $\mathbf{RX}$=0 will have two free variables.Since $\mathbf{R}'\mathbf{X}$=0 will have the exact same solution as $\mathbf{RX} = 0, \mathbf{R}'\mathbf{X}$=0 will also have two free variables. Thus $\mathbf{R}'$ have one non zero row. Now let's consider a matrix $\mathbf{A}$ with the first row as the non-zero row $\mathbf{R}$ and second row as the second row of $\mathbf{R}'$.

$$\mathbf{R} = \begin{pmatrix}1 & a & b\\0 & 0 & 0\end{pmatrix} \tag{1.3.10.1}$$

$$\mathbf{R}' = \begin{pmatrix}1 & c & d\\0 & 0 & 0\end{pmatrix} \tag{1.3.10.2}$$

$$\tag{1.3.10.3}$$

Let $\mathbf{X}$ satisfy

$$\mathbf{RX} = 0 \tag{1.3.10.4}$$

$$\begin{pmatrix}1 & \mathbf{a}^T\end{pmatrix}\begin{pmatrix}x\\\mathbf{y}\end{pmatrix} = 0 \tag{1.3.10.5}$$

$$x + \mathbf{a}^T\mathbf{y} = 0 \tag{1.3.10.6}$$

where

$$\mathbf{a} = \begin{pmatrix}a\\b\end{pmatrix} \tag{1.3.10.7}$$

$$\mathbf{R}'\mathbf{X} = 0 \tag{1.3.10.8}$$

$$\begin{pmatrix}1 & \mathbf{b}^T\end{pmatrix}\begin{pmatrix}x\\\mathbf{y}\end{pmatrix} = 0 \tag{1.3.10.9}$$

$$x + \mathbf{b}^T\mathbf{y} = 0 \tag{1.3.10.10}$$

where

$$\mathbf{b} = \begin{pmatrix}c\\d\end{pmatrix} \tag{1.3.10.11}$$

Subtracting (1.3.10.10) from (1.3.10.6),

$$x + \mathbf{a}^T\mathbf{y} - x - \mathbf{b}^T\mathbf{y} = 0 \tag{1.3.10.12}$$

$$(\mathbf{a}^T - \mathbf{b}^T)\mathbf{y} = 0 \tag{1.3.10.13}$$

Since $\mathbf{y}$ is a 2×1 vector,

$$\implies y_1\mathbf{a} - y_2\mathbf{b} = 0 \tag{1.3.10.14}$$

Which can be written as,

$$\mathbf{a} = k\mathbf{b} \tag{1.3.10.15}$$

where,k=$\frac{\mathbf{y_2}}{\mathbf{y_1}}$ assuming $\mathbf{y}_1$ $\neq$0.Now,Substituting (1.3.10.15) in (1.3.10.6)

$$x + k\mathbf{b}^T\mathbf{y} = 0 \tag{1.3.10.16}$$

Comparing (1.3.10.16) with (1.3.10.10)

$$x + \mathbf{b}^T\mathbf{y} = 0 \tag{1.3.10.17}$$

$$x + k\mathbf{b}^T\mathbf{y} = 0 \tag{1.3.10.18}$$

Hence k=1 which means y_1=y_2 and from this we can say that $\mathbf{a}$=$\mathbf{b}$.If in the above case we take y_1=0 then

$$y_1\mathbf{a} - y_2\mathbf{b} = 0 \tag{1.3.10.19}$$

$$y_2\mathbf{b} = 0 \tag{1.3.10.20}$$

Hence for the (1.3.10.20) to be always true $\mathbf{b}$ should be zero.Now from (1.3.10.15) we will see that $\mathbf{a}$ will also be 0. Hence, $\mathbf{R}$=$\mathbf{R}'$

b) Let $\mathbf{R}$ and $\mathbf{R}'$ have all rows as non zero.

$$\mathbf{R} = \begin{pmatrix}1 & 0 & b\\0 & 1 & c\end{pmatrix} \tag{1.3.10.21}$$

$$\mathbf{R}' = \begin{pmatrix}1 & 0 & e\\0 & 1 & f\end{pmatrix} \tag{1.3.10.22}$$

Let $\mathbf{X}$ satisfy

$$\mathbf{RX} = 0 \quad (1.3.10.23)$$
$$\mathbf{X}^T\mathbf{R}^T = 0 \quad (1.3.10.24)$$

Here,

$$\mathbf{R} = \begin{pmatrix}\mathbf{I} & \mathbf{a}\end{pmatrix} \quad (1.3.10.25)$$
$$\mathbf{a} = \begin{pmatrix} b \\ c \end{pmatrix} \quad (1.3.10.26)$$
$$\mathbf{R}^T = \begin{pmatrix}\mathbf{I} \\ \mathbf{a}^T\end{pmatrix} \quad (1.3.10.27)$$

Let,

$$\mathbf{X}^T = \begin{pmatrix}\mathbf{y}^T & z\end{pmatrix} \quad (1.3.10.28)$$

where z is a scalar constant. Now,substituting (1.3.10.28) and (1.3.10.25) in (1.3.10.24)

$$\begin{pmatrix}\mathbf{y}^T & z\end{pmatrix}\begin{pmatrix}\mathbf{I} \\ \mathbf{a}^T\end{pmatrix} = 0 \quad (1.3.10.29)$$
$$\mathbf{y}^T + z\mathbf{a}^T = 0 \quad (1.3.10.30)$$

Now for,

$$\mathbf{R}'\mathbf{X} = 0 \quad (1.3.10.31)$$
$$\mathbf{X}^T\mathbf{R}'^T = 0 \quad (1.3.10.32)$$

Here,

$$\mathbf{R}' = \begin{pmatrix}\mathbf{I} & \mathbf{b}\end{pmatrix} \quad (1.3.10.33)$$
$$\mathbf{b} = \begin{pmatrix} e \\ f \end{pmatrix} \quad (1.3.10.34)$$

Let,

$$\mathbf{X}^T = \begin{pmatrix}\mathbf{y}^T & z\end{pmatrix} \quad (1.3.10.35)$$

where z is a scalar constant.Now,substituting (1.3.10.35) and (1.3.10.33) in (1.3.10.32)

$$\begin{pmatrix}\mathbf{y}^T & z\end{pmatrix}\begin{pmatrix}\mathbf{I} \\ \mathbf{b}^T\end{pmatrix} = 0 \quad (1.3.10.36)$$
$$\mathbf{y}^T + z\mathbf{b}^T = 0 \quad (1.3.10.37)$$

Subtracting (1.3.10.37) from (1.3.10.30)

$$\mathbf{y}^T + z\mathbf{a}^T - \mathbf{y}^T - z\mathbf{b}^T = 0 \quad (1.3.10.38)$$
$$(\mathbf{a}^T - \mathbf{b}^T)z = 0 \quad (1.3.10.39)$$
$$\mathbf{a}^T = \mathbf{b}^T \quad (1.3.10.40)$$

c) Suppose matrix $\mathbf{R}$ have all the rows as zero then $\mathbf{RX}$=0 will be satisfied for all values of $\mathbf{X}$. We know that $\mathbf{R}'\mathbf{X}$=0 will have the exact same solution as $\mathbf{RX}$=0 then we can say that for all values of $\mathbf{X}$=0 equation $\mathbf{R}'\mathbf{X}$=0 will be satisfied.Hence, $\mathbf{R}'$=$\mathbf{R}$=0.

1.4 Matrix Multiplication

1.4.1. Let

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 2 & 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 1 & -1 \end{pmatrix} \quad (1.4.1.1)$$

Compute $\mathbf{ABC}$ and $\mathbf{CAB}$.
Solution: Given,

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 2 & 1 \end{pmatrix} \quad (1.4.1.2)$$
$$\mathbf{B} = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} \quad (1.4.1.3)$$
$$\mathbf{C} = \begin{pmatrix} 1 & -1 \end{pmatrix} \quad (1.4.1.4)$$

Take, $\mathbf{ABC} = (\mathbf{AB})\ \mathbf{C}$

$$\mathbf{AB} = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} \quad (1.4.1.5)$$
$$\mathbf{AB} = \begin{pmatrix} 6-1-1 \\ 3+2-1 \end{pmatrix} \quad (1.4.1.6)$$
$$\mathbf{AB} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \quad (1.4.1.7)$$

Now,

$$\mathbf{ABC} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}\begin{pmatrix} 1 & -1 \end{pmatrix} \quad (1.4.1.8)$$
$$\mathbf{ABC} = \begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix} \quad (1.4.1.9)$$

similarly, $\mathbf{CAB} = \mathbf{C}(\mathbf{AB})$

$$\mathbf{CAB} = \begin{pmatrix} 1 & -1 \end{pmatrix}\begin{pmatrix} 4 \\ 4 \end{pmatrix} \quad (1.4.1.10)$$
$$\implies \mathbf{CAB} = 0 \quad (1.4.1.11)$$

therefore,

$$\mathbf{ABC} = \begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix} \quad (1.4.1.12)$$
$$\mathbf{CAB} = 0 \quad (1.4.1.13)$$

1.4.2. Let

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 2 & -2 \\ 1 & 3 \\ 4 & 4 \end{pmatrix} \tag{1.4.2.1}$$

Verify directly that $A(AB) = A^2B$ **Solution:**

$$A^2 = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{pmatrix} \tag{1.4.2.2}$$

$$A^2 = \begin{pmatrix} 2 & -1 & 1 \\ 5 & -2 & 3 \\ 6 & -3 & 4 \end{pmatrix} \tag{1.4.2.3}$$

and

$$AB = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & -2 \\ 1 & 3 \\ 4 & 4 \end{pmatrix} \tag{1.4.2.4}$$

$$AB = \begin{pmatrix} 5 & -1 \\ 8 & 0 \\ 10 & -2 \end{pmatrix} \tag{1.4.2.5}$$

Now RHS is

$$A^2B = \begin{pmatrix} 2 & -1 & 1 \\ 5 & -2 & 3 \\ 6 & -3 & 4 \end{pmatrix}\begin{pmatrix} 2 & -2 \\ 1 & 3 \\ 4 & 4 \end{pmatrix} \tag{1.4.2.6}$$

$$A^2B = \begin{pmatrix} 7 & -3 \\ 20 & -4 \\ 25 & -5 \end{pmatrix} \tag{1.4.2.7}$$

Now LHS is

$$A(AB) = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{pmatrix}\begin{pmatrix} 5 & -1 \\ 8 & 0 \\ 10 & -2 \end{pmatrix} \tag{1.4.2.8}$$

$$A(AB) = \begin{pmatrix} 7 & -3 \\ 20 & -4 \\ 25 & -5 \end{pmatrix} \tag{1.4.2.9}$$

Hence verified.

1.4.3. Find two different 2×2 matrices $\mathbf{A}$ such that $\mathbf{A}^2 = 0$ but $\mathbf{A} \neq 0$

Solution: The matrix $\mathbf{A}$ can be given by,

$$\mathbf{A} = \begin{pmatrix} \mathbf{m} & \mathbf{n} \end{pmatrix} \tag{1.4.3.1}$$

$$\mathbf{m} = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, \mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \tag{1.4.3.2}$$

Now,

$$\mathbf{A}^2 = \mathbf{A}\mathbf{A} = \mathbf{0} \tag{1.4.3.3}$$

$$\implies \mathbf{A}^2 = \begin{pmatrix} \mathbf{Am} & \mathbf{An} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \end{pmatrix} \tag{1.4.3.4}$$

From (1.4.3.4), we say that the the null space of $\mathbf{A}$ contains columns of matrix $\mathbf{A}$. Also atleast one of the columns must be non-zero since given $\mathbf{A} \neq 0$. Thus, the null space of $\mathbf{A}$ contains non zero vectors, $rank(\mathbf{A}) < 2$. Hence, $\mathbf{A}$ is a singular matrix. This implies that the columns of $\mathbf{A}$ are linearly dependent.

$$\mathbf{Ax} = 0 \tag{1.4.3.5}$$

$$\begin{pmatrix} \mathbf{m} & \mathbf{n} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \tag{1.4.3.6}$$

$$x_1\mathbf{m} + x_2\mathbf{n} = 0 \tag{1.4.3.7}$$

$$\mathbf{n} = \frac{-x_1}{x_2}\mathbf{m} \tag{1.4.3.8}$$

$$\implies \mathbf{n} = k\mathbf{m} \tag{1.4.3.9}$$

where $\mathbf{m} \neq 0$ as $\mathbf{A} \neq 0$
Now from (1.4.3.4),

$$\mathbf{Am} = 0 \tag{1.4.3.10}$$

$$m_1\mathbf{m} + m_2\mathbf{n} = 0 \tag{1.4.3.11}$$

$$(m_1 + km_2)\,\mathbf{m} = 0 \tag{1.4.3.12}$$

Thus we get, $m_1 = -km_2$

$$\mathbf{A} = \begin{pmatrix} -km_2 & -k^2m_2 \\ m_2 & km_2 \end{pmatrix}; m_2 \neq 0 \tag{1.4.3.13}$$

(1.4.3.9) can be written as,

$$\implies \mathbf{m} = \frac{1}{k}\mathbf{n} \tag{1.4.3.14}$$

$$\implies \mathbf{m} = c\mathbf{n} \tag{1.4.3.15}$$

where $\mathbf{n} \neq 0$ as $\mathbf{A} \neq 0$
From (1.4.3.4),

$$\mathbf{An} = 0 \tag{1.4.3.16}$$

$$n_1\mathbf{m} + n_2\mathbf{n} = 0 \tag{1.4.3.17}$$

$$(cn_1 + n_2)\,\mathbf{n} = 0 \tag{1.4.3.18}$$

Thus we get, $n_2 = -cn_1$

$$\mathbf{A} = \begin{pmatrix} cn_1 & n_1 \\ -c^2n_1 & -cn_1 \end{pmatrix}; n_1 \neq 0 \tag{1.4.3.19}$$

From (1.4.3.13), (1.4.3.19) two different 2×2 matrices **A** can be given as,

$$\mathbf{A} = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \tag{1.4.3.20}$$

$$\mathbf{A} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \tag{1.4.3.21}$$

1.4.4. For the matrix $\mathbf{A} = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{pmatrix}$, find elementary matrices $\mathbf{E_1}, \mathbf{E_2}, ..., \mathbf{E_k}$ such that

$$\mathbf{E_k}...\mathbf{E_2}\mathbf{E_1}\mathbf{A} = \mathbf{I} \tag{1.4.4.1}$$

Solution: Given,

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{pmatrix} \tag{1.4.4.2}$$

Take,

$$\mathbf{E_1} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.4.4.3}$$

$$\mathbf{E_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix} \tag{1.4.4.4}$$

$$\mathbf{E_3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.4.4.5}$$

$$\mathbf{E_4} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.4.4.6}$$

$$\mathbf{E_5} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix} \tag{1.4.4.7}$$

$$\mathbf{E_6} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{2}{7} \end{pmatrix} \tag{1.4.4.8}$$

$$\mathbf{E_7} = \begin{pmatrix} 1 & 0 & \frac{-1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.4.4.9}$$

$$\mathbf{E_8} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix} \tag{1.4.4.10}$$

Now, we calculate

$$\mathbf{E_8E_7E_6E_5E_4E_3E_2E_1} = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 0 & 3 & -2 \end{pmatrix} \tag{1.4.4.11}$$

Hence,

$$(\mathbf{E_8E_7E_6E_5E_4E_3E_2E_1})\,\mathbf{A} = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 0 & 3 & -2 \end{pmatrix}\begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.4.4.12}$$

1.4.5. Let $A = \begin{pmatrix} 1 & -1 \\ 2 & 2 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 1 \\ -4 & 4 \end{pmatrix}$ Is there any matrix C such that $CA = B$?

Solution: The matrix B is obtained by multiplying the matrix A with matrix C. B is a 2×2 matrix and A is a 3×2 matrix. so matrix C must be a 2×3 matrix. Let the matrix C is:

$$C = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix} \tag{1.4.5.1}$$

$$\implies C^T = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \end{pmatrix} \tag{1.4.5.2}$$

So, after multiplying with A matrix we get,

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 2 & 2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a_1 + 2b_1 + c_1 & -a_1 + 2b_1 \\ a_2 + 2b_2 + c_2 & -a_2 + 2b_2 \end{pmatrix} \tag{1.4.5.3}$$

Matrix A is a rectangular matrix. Now, Considering $CA = B$ and by transposing both side,

$$(CA)^T = B^T \tag{1.4.5.4}$$

$$\implies A^T C^T = B^T \tag{1.4.5.5}$$

$$\implies \begin{pmatrix} 1 & 2 & 1 \\ -1 & 2 & 0 \end{pmatrix}\begin{pmatrix} \mathbf{c_1} & \mathbf{c_2} \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 1 & 4 \end{pmatrix} \tag{1.4.5.6}$$

We can represent it like this:

$$\begin{pmatrix} 1 & 2 & 1 \\ -1 & 2 & 0 \end{pmatrix} \mathbf{c_1} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \tag{1.4.5.7}$$

$$\tag{1.4.5.8}$$

Now the augmented matrix is:

$$\begin{pmatrix} 1 & 2 & 1 & 3 \\ -1 & 2 & 0 & 1 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_1 + R_2} \begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 4 & 1 & 4 \end{pmatrix}$$
$$\xleftrightarrow{R_2 \leftarrow R_2/2} \begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 2 & \frac{1}{2} & 2 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 - R_2}$$
$$\begin{pmatrix} 1 & 0 & \frac{1}{2} & 1 \\ 0 & 2 & \frac{1}{2} & 2 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2/2} \begin{pmatrix} 1 & 0 & \frac{1}{2} & 1 \\ 0 & 1 & \frac{1}{4} & 1 \end{pmatrix} \tag{1.4.5.9}$$

Similarly,

$$\begin{pmatrix} 1 & 2 & 1 \\ -1 & 2 & 0 \end{pmatrix} \mathbf{c_2} = \begin{pmatrix} -4 \\ 4 \end{pmatrix} \tag{1.4.5.10}$$

$$\tag{1.4.5.11}$$

Now the augmented matrix is:

$$\begin{pmatrix} 1 & 2 & 1 & -4 \\ -1 & 2 & 0 & 4 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_1 + R_2} \begin{pmatrix} 1 & 2 & 1 & -4 \\ 0 & 4 & 1 & 0 \end{pmatrix}$$
$$\xleftrightarrow{R_2 \leftarrow R_2/2} \begin{pmatrix} 1 & 2 & 1 & -4 \\ 0 & 2 & \frac{1}{2} & 0 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 - R_2}$$
$$\begin{pmatrix} 1 & 0 & \frac{1}{2} & -4 \\ 0 & 2 & \frac{1}{2} & 0 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2/2} \begin{pmatrix} 1 & 0 & \frac{1}{2} & -4 \\ 0 & 1 & \frac{1}{4} & 0 \end{pmatrix} \tag{1.4.5.12}$$

From equations 1.4.5.9 and 1.4.5.12, it can be observed that solutions exist and there is a matrix C such that $CA = B$. Now,

$$\mathbf{c_1} = \begin{pmatrix} 1 - \frac{c_1}{2} \\ 1 - \frac{c_1}{4} \\ c_1 \end{pmatrix} \tag{1.4.5.13}$$

$$\implies \mathbf{c_1} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_1 \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{4} \\ 1 \end{pmatrix} \tag{1.4.5.14}$$

$$\mathbf{c_2} = \begin{pmatrix} -4 - \frac{c_2}{2} \\ -\frac{c_2}{4} \\ c_2 \end{pmatrix} \tag{1.4.5.15}$$

$$\implies \mathbf{c_2} = \begin{pmatrix} -4 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{4} \\ 1 \end{pmatrix} \tag{1.4.5.16}$$

Now,

$$C^T = \begin{pmatrix} 1 & -4 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} + c_1 \begin{pmatrix} -\frac{1}{2} & 0 \\ -\frac{1}{4} & 0 \\ 1 & 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 & -\frac{1}{2} \\ 0 & -\frac{1}{4} \\ 0 & 1 \end{pmatrix}$$
$$\implies C = \begin{pmatrix} 1 & 1 & 0 \\ -4 & 0 & 0 \end{pmatrix} + c_1 \begin{pmatrix} -\frac{1}{2} & -\frac{1}{4} & 1 \\ 0 & 0 & 0 \end{pmatrix}$$
$$+ c_2 \begin{pmatrix} 0 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{4} & 1 \end{pmatrix} \tag{1.4.5.17}$$

Now,

$$CA = \begin{pmatrix} 1 & 1 & 0 \\ -4 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 2 & 2 \\ 1 & 0 \end{pmatrix}$$
$$+ c_1 \begin{pmatrix} -\frac{1}{2} & -\frac{1}{4} & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 2 & 2 \\ 1 & 0 \end{pmatrix}$$
$$+ c_2 \begin{pmatrix} 0 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{4} & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 2 & 2 \\ 1 & 0 \end{pmatrix}$$
$$\implies CA = \begin{pmatrix} 3 & 1 \\ -4 & 4 \end{pmatrix} + c_1 \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$
$$\implies CA = B \tag{1.4.5.18}$$

Hence, it is proved that there there exist a matrix C such that $CA = B$.

1.4.6. Let $\mathbf{A}$ be an $m \times n$ matrix and $\mathbf{B}$ be an $n \times k$ matrix.Show that the columns of $\mathbf{C} = \mathbf{AB}$ are linear combinations of columns of $\mathbf{A}$.If $\alpha_1, \alpha_2, \dots, \alpha_n$ are the columns of $\mathbf{A}$ and $\gamma_1, \gamma_2, \dots, \gamma_k$ are the columns of $\mathbf{C}$ then,

$$\gamma_j = \sum_{r=1}^{n} B_{rj} \alpha_r \tag{1.4.6.1}$$

Solution:

$$\mathbf{C} = \mathbf{AB} \tag{1.4.6.2}$$

$$\mathbf{C} = \begin{pmatrix} \gamma_1 & \gamma_2 & \dots & \gamma_k \end{pmatrix} \tag{1.4.6.3}$$

$$\mathbf{A} = \begin{pmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_n \end{pmatrix} \tag{1.4.6.4}$$

$$\mathbf{B} = \begin{pmatrix} \beta_1 & \beta_2 & \dots & \beta_k \end{pmatrix} \tag{1.4.6.5}$$

$$= \begin{pmatrix} B_{11} & B_{12} & \dots & B_{1k} \\ B_{21} & B_{22} & \dots & B_{2k} \\ \vdots & \vdots & \dots & \vdots \\ B_{n1} & B_{n2} & \dots & B_{nk} \end{pmatrix} \tag{1.4.6.6}$$

By matrix multiplication, we can write

$$\begin{pmatrix}\gamma_\mathbf{1} & \gamma_\mathbf{2} & \dots & \gamma_\mathbf{k}\end{pmatrix} = \begin{pmatrix}\mathbf{A}\beta_\mathbf{1} & \mathbf{A}\beta_\mathbf{2} & \dots & \mathbf{A}\beta_\mathbf{k}\end{pmatrix} \tag{1.4.6.7}$$

Consider $\gamma_\mathbf{1}$

$$\gamma_\mathbf{1} = \mathbf{A}\beta_\mathbf{1} \tag{1.4.6.8}$$

$$= \begin{pmatrix}\alpha_\mathbf{1} & \alpha_\mathbf{2} & \dots & \alpha_\mathbf{n}\end{pmatrix}\begin{pmatrix}B_{11}\\B_{21}\\\vdots\\B_{n1}\end{pmatrix} \tag{1.4.6.9}$$

$$= B_{11}\alpha_\mathbf{1} + B_{21}\alpha_\mathbf{2} + \dots + B_{n1}\alpha_\mathbf{n} \tag{1.4.6.10}$$

Similarly, considering j^{th} column of $\mathbf{C}$

$$\gamma_\mathbf{j} = \begin{pmatrix}\alpha_\mathbf{1} & \alpha_\mathbf{2} & \dots & \alpha_\mathbf{n}\end{pmatrix}\begin{pmatrix}B_{1j}\\B_{2j}\\\vdots\\B_{nj}\end{pmatrix} \tag{1.4.6.11}$$

$$= B_{1j}\alpha_\mathbf{1} + B_{2j}\alpha_\mathbf{2} + \dots + B_{nj}\alpha_\mathbf{n} \tag{1.4.6.12}$$

$$\implies \gamma_\mathbf{j} = \sum_{r=1}^{n} B_{rj}\alpha_\mathbf{r} \tag{1.4.6.13}$$

which proves that columns of $\mathbf{C}$ are linear combinations of columns of $\mathbf{A}$

1.4.7. Let $\mathbf{A}$ and $\mathbf{B}$ be $n{\times}n$ matrices such that $\mathbf{AB} = \mathbf{I}$. Prove that $\mathbf{BA} = \mathbf{I}$. **Solution:** Let $\mathbf{BX} = 0$ be a system of linear equation with n unknowns and n equations as $\mathbf{B}$ is $n \times n$ matrix. Hence,

$$\mathbf{BX} = 0 \tag{1.4.7.1}$$

$$\implies \mathbf{A}(\mathbf{BX}) = 0 \tag{1.4.7.2}$$

$$\implies (\mathbf{AB})\mathbf{X} = 0 \tag{1.4.7.3}$$

$$\implies \mathbf{IX} = 0 \quad [\because \mathbf{AB} = \mathbf{I}] \tag{1.4.7.4}$$

$$\implies \mathbf{X} = 0 \tag{1.4.7.5}$$

From (1.4.7.5) since $\mathbf{X} = 0$ is the only solution of (1.4.7.1), hence $rank(\mathbf{B}) = n$. Which implies all columns of $\mathbf{B}$ are linearly independent. Hence $\mathbf{B}$ is invertible. Therefore, every left inverse of $\mathbf{B}$ is also a right inverse of $\mathbf{B}$. Hence there exists a $n \times n$ matrix $\mathbf{C}$ such that,

$$\mathbf{BC} = \mathbf{CB} = \mathbf{I} \tag{1.4.7.6}$$

Again given that $\mathbf{AB} = \mathbf{I}$. Hence,

$$\mathbf{AB} = \mathbf{I} \tag{1.4.7.7}$$

$$\implies \mathbf{ABC} = \mathbf{C} \tag{1.4.7.8}$$

$$\implies \mathbf{A}(\mathbf{BC}) = \mathbf{C} \tag{1.4.7.9}$$

$$\implies \mathbf{A} = \mathbf{C} \qquad [\because \mathbf{BC} = \mathbf{I}] \tag{1.4.7.10}$$

Hence using (1.4.7.10) and (1.4.7.6) we can write,

$$\mathbf{BA} = \mathbf{I} \tag{1.4.7.11}$$

Hence Proved.

1.4.8. Let,

$$\mathbf{C} = \begin{pmatrix}C_{11} & C_{12}\\C_{21} & C_{22}\end{pmatrix} \tag{1.4.8.1}$$

be a 2×2 matrix. We inquire when it is possible to find 2×2 matrices $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{C}=\mathbf{AB}-\mathbf{BA}$. Prove that such matrices can be found if and only if $C_{11}+C_{22}$=0. **Solution:** We have to find,

$$tr(\mathbf{C}) = C_{11} + C_{22} = tr(\mathbf{AB} - \mathbf{BA}) \tag{1.4.8.2}$$

$$\implies tr(\mathbf{C}) = tr(\mathbf{AB}) - tr(\mathbf{BA}) \tag{1.4.8.3}$$

We know that,

$$tr(\mathbf{AB}) = \sum_{i=1}^{2}(\mathbf{AB})_{ii} \tag{1.4.8.4}$$

$$\implies \sum_{i=1}^{2}\sum_{j=1}^{2} a_{ij}b_{ji} \tag{1.4.8.5}$$

$$\implies \sum_{j=1}^{2}\sum_{i=1}^{2} b_{ji}a_{ij} \tag{1.4.8.6}$$

$$\implies tr(\mathbf{AB}) = \sum_{j=1}^{2}\mathbf{BA}_{jj} \tag{1.4.8.7}$$

$$\implies tr(\mathbf{AB}) = tr(\mathbf{BA}) \tag{1.4.8.8}$$

Substituting equation (1.4.8.8) to (1.4.8.3) we get

$$\implies tr(\mathbf{C}) = tr(\mathbf{AB}) - tr(\mathbf{BA}) = 0 \tag{1.4.8.9}$$

1.5 Invertible Matrices

1.5.1. Let

$$\mathbf{A} = \begin{pmatrix}1 & 2 & 1 & 0\\-1 & 0 & 3 & 5\\1 & -2 & 1 & 1\end{pmatrix} \tag{1.5.1.1}$$

Find a row-reduced echelon matrix **R** which is row-equivalent to **A** and an invertible 3x3 matrix **P** such that **R** = **P A**. **Solution:** Given

$$\mathbf{A}=\begin{pmatrix}1&2&1&0\\-1&0&3&5\\1&-2&1&1\end{pmatrix} \tag{1.5.1.2}$$

Row reduce **A** by applying the elementary row operations and equivalently at each operations find the elementary matrix **E**

$$\mathbf{A|I}=\left(\begin{array}{cccc|ccc}1&2&1&0&1&0&0\\-1&0&3&5&0&1&0\\1&-2&1&1&0&0&1\end{array}\right) \tag{1.5.1.3}$$

$$\xleftrightarrow{R_2=R_2+R_1}\left(\begin{array}{cccc|ccc}1&2&1&0&1&0&0\\0&2&4&5&1&1&0\\1&-2&1&1&0&0&1\end{array}\right) \tag{1.5.1.4}$$

$$\xleftrightarrow{R_3=R_3-R_1}\left(\begin{array}{cccc|ccc}1&2&1&0&1&0&0\\0&2&4&5&1&1&0\\0&-4&0&1&-1&0&1\end{array}\right) \tag{1.5.1.5}$$

$$\xleftrightarrow{R_1=R_1-R_2}\left(\begin{array}{cccc|ccc}1&0&-3&-5&0&-1&0\\0&2&4&5&1&1&0\\0&-4&0&1&-1&0&1\end{array}\right) \tag{1.5.1.6}$$

$$\xleftrightarrow{R_3=R_3+2R_2}\left(\begin{array}{cccc|ccc}1&0&-3&-5&0&-1&0\\0&2&4&5&1&1&0\\0&0&8&11&1&2&1\end{array}\right) \tag{1.5.1.7}$$

$$\xleftrightarrow{R_2=\frac{R_2}{2}}\left(\begin{array}{cccc|ccc}1&0&-3&-5&0&-1&0\\0&1&2&\frac{5}{2}&\frac{1}{2}&\frac{1}{2}&0\\0&0&8&11&1&2&1\end{array}\right) \tag{1.5.1.8}$$

$$\xleftrightarrow{R_3=\frac{R_3}{8}}\left(\begin{array}{cccc|ccc}1&0&-3&-5&0&-1&0\\0&1&2&\frac{5}{2}&\frac{1}{2}&\frac{1}{2}&0\\0&0&1&\frac{11}{8}&\frac{1}{8}&\frac{1}{4}&\frac{1}{8}\end{array}\right) \tag{1.5.1.9}$$

$$\xleftrightarrow{R_1=R_1+3R_3}\left(\begin{array}{cccc|ccc}1&0&0&-\frac{7}{8}&\frac{3}{8}&-\frac{1}{4}&\frac{3}{8}\\0&1&2&\frac{5}{2}&\frac{1}{2}&\frac{1}{2}&0\\0&0&1&\frac{11}{8}&\frac{1}{8}&\frac{1}{4}&\frac{1}{8}\end{array}\right) \tag{1.5.1.10}$$

$$\xleftrightarrow{R_2=R_2-2R_3}\left(\begin{array}{cccc|ccc}1&0&0&-\frac{7}{8}&\frac{3}{8}&-\frac{1}{4}&\frac{3}{8}\\0&1&0&-\frac{1}{4}&\frac{1}{4}&0&-\frac{1}{4}\\0&0&1&\frac{11}{8}&\frac{1}{8}&\frac{1}{4}&\frac{1}{8}\end{array}\right) \tag{1.5.1.11}$$

Hence,row reduced echelon matrix that is row equivalent to **A** is

$$\mathbf{R}=\begin{pmatrix}1&0&0&-\frac{7}{8}\\0&1&0&-\frac{1}{4}\\0&0&1&\frac{11}{8}\end{pmatrix} \tag{1.5.1.12}$$

where **E** is the elementary matrices that transform **A** to **R** Thus:-

$$\mathbf{EA}=\mathbf{R} \tag{1.5.1.13}$$

Since elementary matrices is invertible

$$\mathbf{P}=\mathbf{E} \tag{1.5.1.14}$$

is invertible.
From (1.5.1.11)

$$\mathbf{P}=\begin{pmatrix}\frac{3}{8}&-\frac{1}{4}&\frac{3}{8}\\\frac{1}{4}&0&-\frac{1}{4}\\\frac{1}{8}&\frac{1}{4}&\frac{1}{8}\end{pmatrix} \tag{1.5.1.15}$$

$$\mathbf{R}=\begin{pmatrix}1&0&0&-\frac{7}{8}\\0&1&0&-\frac{1}{4}\\0&0&1&\frac{11}{8}\end{pmatrix} \tag{1.5.1.16}$$

such that **R** = **PA**.

1.5.2. Let **A** = $\begin{pmatrix}2&0&i\\1&-3&-i\\i&1&1\end{pmatrix}$, find a row-reduced echelon matrix **R** which is row-equivalent to **A** and an invertible 3x3 matrix **P** such that **R** = **P A**. **Solution:** Given,

$$\mathbf{A}=\begin{pmatrix}2&0&i\\1&-3&-i\\i&1&1\end{pmatrix} \tag{1.5.2.1}$$

Row reduce **A** by applying the elementary row operations and equivalently at each operations find the elementary matrix **E**

$$[\mathbf{A\ I}]=\begin{pmatrix}2&0&i&|&1&0&0\\1&-3&-i&|&0&1&0\\i&1&1&|&0&0&1\end{pmatrix} \tag{1.5.2.2}$$

$$\xleftrightarrow{R_1\leftrightarrow R_2}\begin{pmatrix}1&-3&-i&|&0&1&0\\2&0&i&|&1&0&0\\i&1&1&|&0&0&1\end{pmatrix} \tag{1.5.2.3}$$

$$\xleftrightarrow{R_2\leftarrow R_2-2R_1} \left(\begin{array}{ccc|ccc} 1 & -3 & -i & 0 & 1 & 0 \\ 0 & 6 & 3i & 1 & -2 & 0 \\ i & 1 & 1 & 0 & 0 & 1 \end{array}\right) \tag{1.5.2.4}$$

$$\xleftrightarrow{R_3\leftarrow R_3-iR_1} \left(\begin{array}{ccc|ccc} 1 & -3 & -i & 0 & 1 & 0 \\ 0 & 6 & 3i & 1 & -2 & 0 \\ 0 & 1+3i & 0 & 0 & -i & 1 \end{array}\right) \tag{1.5.2.5}$$

$$\xleftrightarrow{R_2\leftarrow \frac{R_2}{6}} \left(\begin{array}{ccc|ccc} 1 & -3 & -i & 0 & 1 & 0 \\ 0 & 1 & \frac{i}{2} & \frac{1}{6} & -\frac{1}{3} & 0 \\ 0 & 1+3i & 0 & 0 & -i & 1 \end{array}\right) \tag{1.5.2.6}$$

$$\xleftrightarrow{R_1\leftarrow\ R_1+3R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{i}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{i}{2} & \frac{1}{6} & -\frac{1}{3} & 0 \\ 0 & 1+3i & 0 & 0 & -i & 1 \end{array}\right) \tag{1.5.2.7}$$

$$\xleftrightarrow{R_3\leftarrow\ R_3/(3-i)/2} \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{i}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{i}{2} & \frac{1}{6} & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & -\frac{(i)}{3} & \frac{3+i}{15} & \frac{3+i}{5} \end{array}\right) \tag{1.5.2.8}$$

$$\xleftrightarrow{R_1\leftarrow R_1-\frac{i}{2}R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & \frac{1-3i}{30} & \frac{1-3i}{10} \\ 0 & 1 & \frac{i}{2} & \frac{1}{6} & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & -\frac{(i)}{3} & \frac{3+i}{15} & \frac{3+i}{5} \end{array}\right) \tag{1.5.2.9}$$

$$\xleftrightarrow{R_2\leftarrow R_2-\frac{i}{2}R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & \frac{1-3i}{30} & \frac{1-3i}{10} \\ 0 & 1 & 0 & 0 & -\frac{3+i}{10} & \frac{1-3i}{10} \\ 0 & 0 & 1 & -\frac{i}{3} & \frac{3+i}{15} & \frac{3+i}{5} \end{array}\right) \tag{1.5.2.10}$$

$$= [\mathbf{I}\ \mathbf{E}]$$

Hence, the row reduced matrix that is row equivalent to $\mathbf{A}$ is

$$\mathbf{R} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I} \tag{1.5.2.11}$$

Using Gauss-Jordan Elimination, if there exists an elimentary matrix $\mathbf{E}$ such that $\mathbf{E}[\mathbf{A}\ \mathbf{I}] = [\mathbf{I}\ \mathbf{E}]$ then $\mathbf{E}$ is the inverse of A i.e

$$\mathbf{E} = \mathbf{A}^{-1}$$

$$\mathbf{E} = \mathbf{A}^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{1-3i}{30} & \frac{1-3i}{10} \\ 0 & -\frac{3+i}{10} & \frac{1-3i}{10} \\ -\frac{i}{3} & \frac{3+i}{15} & \frac{3+i}{5} \end{pmatrix} \tag{1.5.2.12}$$

Since,

$$\mathbf{R} = \mathbf{PA} \implies \mathbf{P} = \mathbf{A}^{-1}\mathbf{R} \tag{1.5.2.13}$$

$$\mathbf{P} = \begin{pmatrix} \frac{1}{3} & \frac{1-3i}{30} & \frac{1-3i}{10} \\ 0 & -\frac{3+i}{10} & \frac{1-3i}{10} \\ -\frac{i}{3} & \frac{3+i}{15} & \frac{3+i}{5} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.5.2.14}$$

Thus,

$$\mathbf{P} = \begin{pmatrix} \frac{1}{3} & \frac{1-3i}{30} & \frac{1-3i}{10} \\ 0 & -\frac{3+i}{10} & \frac{1-3i}{10} \\ -\frac{i}{3} & \frac{3+i}{15} & \frac{3+i}{5} \end{pmatrix}$$

$$\mathbf{R} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

1.5.3. For each of the two matrices use elementary row operations to discover whether it is invertible, and to find the inverse in case it is invertible.

$$\mathbf{A} = \begin{pmatrix} 2 & 5 & -1 \\ 4 & -1 & 2 \\ 6 & 4 & 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 2 & 4 \\ 0 & 1 & -2 \end{pmatrix}$$

Solution: Given

$$\mathbf{A} = \begin{pmatrix} 2 & 5 & -1 \\ 4 & -1 & 2 \\ 6 & 4 & 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 2 & 4 \\ 0 & 1 & -2 \end{pmatrix} \tag{1.5.3.1}$$

By applying row reductions on $\mathbf{A}$

$$\begin{pmatrix} 2 & 5 & -1 \\ 4 & -1 & 2 \\ 6 & 4 & 1 \end{pmatrix} \xleftrightarrow{R_2=R_2-2R_1} \mathbf{A} = \begin{pmatrix} 2 & 5 & -1 \\ 0 & -11 & 4 \\ 6 & 4 & 1 \end{pmatrix} \tag{1.5.3.2}$$

$$\xleftrightarrow{R_3=R_3-3R_1} \begin{pmatrix} 2 & 5 & -1 \\ 0 & -11 & 4 \\ 0 & -11 & 4 \end{pmatrix} \tag{1.5.3.3}$$

$$\xleftrightarrow{R_1=\frac{R_1}{2}} \begin{pmatrix} 1 & \frac{5}{2} & \frac{-1}{2} \\ 0 & -11 & 4 \\ 0 & -11 & 4 \end{pmatrix} \tag{1.5.3.4}$$

$$\xleftrightarrow{R_3=R_3-R_2}\begin{pmatrix}1 & \frac{5}{2} & \frac{-1}{2}\\ 0 & -11 & 4\\ 0 & 0 & 0\end{pmatrix} \tag{1.5.3.5}$$

$$\xleftrightarrow{R_2=\frac{-R_2}{11}}\begin{pmatrix}1 & \frac{5}{2} & \frac{-1}{2}\\ 0 & 1 & \frac{-4}{11}\\ 0 & 0 & 0\end{pmatrix} \tag{1.5.3.6}$$

$$\xleftrightarrow{R_1=R_1-\frac{5}{2}R_2}\begin{pmatrix}1 & 0 & \frac{9}{22}\\ 0 & 1 & \frac{-4}{11}\\ 0 & 0 & 0\end{pmatrix} \tag{1.5.3.7}$$

For a matrix to be invertible, it has to be a matrix of full rank. However the matrix **A** is not of full rank ($Rank(\mathbf{A}) < 3$).Therefore **A** is not invertible.

Let us now consider augmented matrix **B|I**, By applying row reductions on **B|I**

$$\left(\begin{array}{ccc|ccc}1 & -1 & 2 & 1 & 0 & 0\\ 3 & 2 & 4 & 0 & 1 & 0\\ 0 & 1 & -2 & 0 & 0 & 1\end{array}\right)\xleftrightarrow{R_2=R_2-3R_1}$$

$$\left(\begin{array}{ccc|ccc}1 & -1 & 2 & 1 & 0 & 0\\ 0 & 5 & -2 & -3 & 1 & 0\\ 0 & 1 & -2 & 0 & 0 & 1\end{array}\right) \tag{1.5.3.8}$$

$$\xleftrightarrow{R_2=\frac{R_2}{5}}\left(\begin{array}{ccc|ccc}1 & -1 & 2 & 1 & 0 & 0\\ 0 & 1 & \frac{-2}{5} & \frac{-3}{5} & \frac{1}{5} & 0\\ 0 & 1 & -2 & 0 & 0 & 1\end{array}\right) \tag{1.5.3.9}$$

$$\xleftrightarrow{R_1=R_1+R_2}\left(\begin{array}{ccc|ccc}1 & 0 & \frac{8}{5} & \frac{2}{5} & \frac{1}{5} & 0\\ 0 & 1 & \frac{-2}{5} & \frac{-3}{5} & \frac{1}{5} & 0\\ 0 & 1 & -2 & 0 & 0 & 1\end{array}\right) \tag{1.5.3.10}$$

$$\xleftrightarrow{R_3=R_3-R_2}\left(\begin{array}{ccc|ccc}1 & 0 & \frac{8}{5} & \frac{2}{5} & \frac{1}{5} & 0\\ 0 & 1 & \frac{-2}{5} & \frac{-3}{5} & \frac{1}{5} & 0\\ 0 & 0 & \frac{-8}{5} & \frac{3}{5} & \frac{-1}{5} & 1\end{array}\right) \tag{1.5.3.11}$$

$$\xleftrightarrow{R_1=R_1+R_3}\left(\begin{array}{ccc|ccc}1 & 0 & 0 & 1 & 0 & 1\\ 0 & 1 & \frac{-2}{5} & \frac{-3}{5} & \frac{1}{5} & 0\\ 0 & 0 & \frac{-8}{5} & \frac{3}{5} & \frac{-1}{5} & 1\end{array}\right) \tag{1.5.3.12}$$

$$\xleftrightarrow{R_3=\frac{-5}{8}R_3}\left(\begin{array}{ccc|ccc}1 & 0 & 0 & 1 & 0 & 1\\ 0 & 1 & \frac{-2}{5} & \frac{-3}{5} & \frac{1}{5} & 0\\ 0 & 0 & 1 & \frac{-3}{8} & \frac{1}{8} & \frac{-5}{8}\end{array}\right) \tag{1.5.3.13}$$

$$\xleftrightarrow{R_3=R_2+\frac{2}{5}R_3}\left(\begin{array}{ccc|ccc}1 & 0 & 0 & 1 & 0 & 1\\ 0 & 1 & 0 & \frac{-3}{4} & \frac{1}{4} & \frac{-1}{4}\\ 0 & 0 & 1 & \frac{-3}{8} & \frac{1}{8} & \frac{-5}{8}\end{array}\right) \tag{1.5.3.14}$$

For a matrix to be invertible, it has to be a matrix of full rank. Here, the matrix **B** is of full rank ($Rank(\mathbf{B}) = 3$). Therefore **B** is invertible and the inverse matrix $\mathbf{B}^{-1}$ can be written from (1.5.3.14):

$$\mathbf{B}^{-1} = \begin{pmatrix}1 & 0 & 1\\ \frac{-3}{4} & \frac{1}{4} & \frac{-1}{4}\\ \frac{-3}{8} & \frac{1}{8} & \frac{-5}{8}\end{pmatrix} \tag{1.5.3.15}$$

1.5.4. Let

$$\mathbf{A} = \begin{pmatrix}5 & 0 & 0\\ 1 & 5 & 0\\ 0 & 1 & 5\end{pmatrix} \tag{1.5.4.1}$$

For which **X** does there exist a scalar c such that $\mathbf{AX} = c\mathbf{X}$

Solution: Given

$$\mathbf{A} = \begin{pmatrix}5 & 0 & 0\\ 1 & 5 & 0\\ 0 & 1 & 5\end{pmatrix} \tag{1.5.4.2}$$

The given matrix has single eigenvalue as it is the lower triangular matrix and has equal diagonal elements. Hence $c_1 = c_2 = c_3 = 5$. To find the corresponding eigenvector, consider the following

$$(\mathbf{A} - c\mathbf{I})\mathbf{X} = 0 \tag{1.5.4.3}$$

$$\implies \begin{pmatrix}0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 0\end{pmatrix} \tag{1.5.4.4}$$

Solving the homogeneous system of linear equations by performing rref, we get

$$\begin{pmatrix}30 & 0 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\xleftrightarrow[R_3\leftrightarrow R_2]{R_2\leftrightarrow R_1}\begin{pmatrix}31 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix} \tag{1.5.4.5}$$

Hence we get,

$$x_1 = 0, x_2 = 0, x_3 = t \tag{1.5.4.6}$$

where, x_3 is arbitrary. Therefore,

$$\mathbf{X} = \begin{pmatrix}0\\ 0\\ 1\end{pmatrix}t \tag{1.5.4.7}$$

Hence, the given matrix has single eigenvector and is not diagonalizable.

1.5.5. Discover whether

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{pmatrix} \tag{1.5.5.1}$$

is invertible, and find $\mathbf{A^{-1}}$ if it exists.

Solution: The matrix $\mathbf{A}$ is in row reduced echolon form with four pivot elements. Therefore the rank($\mathbf{A}$) is 4. Hence the rows of matrix $\mathbf{A}$ constitute of 4 linearly independent vectors. Thus it can be concluded that matrix $\mathbf{A}$ is invertible. Using Gauss-Jordan Elimination, if there exists an elimentary matrix $\mathbf{E}$ such that $\mathbf{E}[\mathbf{A}\ \mathbf{I}] = [\mathbf{I}\ \mathbf{E}]$ then $\mathbf{E}$ is the inverse of A i.e $\mathbf{E} = \mathbf{A^{-1}}$.

$$[\mathbf{A}\ \mathbf{I}] = \left(\begin{array}{cccc|cccc} 1 & 2 & 3 & 4 & 1 & 0 & 0 & 0 \\ 0 & 2 & 3 & 4 & 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 1 \end{array}\right) \tag{1.5.5.2}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - R_2} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 2 & 3 & 4 & 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 1 \end{array}\right) \tag{1.5.5.3}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - R_3} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & 4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 1 \end{array}\right) \tag{1.5.5.4}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - R_4} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 1 \end{array}\right) \tag{1.5.5.5}$$

$$\xleftrightarrow[R_2 \leftarrow \frac{R_2}{2}\ R_3 \leftarrow \frac{R_3}{3}]{R_4 \leftarrow \frac{R_4}{4}} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{4} \end{array}\right) = [\mathbf{I}\ \mathbf{E}] \tag{1.5.5.6}$$

Therefore, for the given problem,

$$\mathbf{A^{-1}} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix} \tag{1.5.5.7}$$

Generalization of above result to a matrix of any arbitrary size: Let

$$\mathbf{A} = \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_N \\ 0 & a_2 & a_3 & \dots & a_N \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & \dots & \dots & a_N \end{pmatrix} \tag{1.5.5.8}$$

Then

$$\mathbf{E}_1\mathbf{A} = \begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & \dots & \dots & 1 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_N \\ 0 & a_2 & a_3 & \dots & a_N \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & \dots & \dots & a_N \end{pmatrix} \tag{1.5.5.9}$$

$$= \begin{pmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & a_3 & \dots & a_N \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & \dots & \dots & a_N \end{pmatrix} \tag{1.5.5.10}$$

$$\mathbf{E}_2\mathbf{E}_1\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & -1 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & \dots & \dots & 1 \end{pmatrix} \begin{pmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & a_3 & \dots & a_N \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & \dots & \dots & a_N \end{pmatrix} \tag{1.5.5.11}$$

$$= \begin{pmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & \dots & \dots & a_N \end{pmatrix} \tag{1.5.5.12}$$

Proceeding in similar manner, we get

$$\mathbf{E}_N\mathbf{E}_{N-1}\dots\mathbf{E}_2\mathbf{E}_1\mathbf{A} = \mathbf{U} = \begin{pmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & 0 & \dots & 0 \\ 0 & 0 & a_3 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & \dots & \dots & a_N \end{pmatrix} \tag{1.5.5.13}$$

$$= \text{diag}\begin{pmatrix} a_1 & a_2 & \dots & a_N \end{pmatrix} \tag{1.5.5.14}$$

$$\implies \quad \mathbf{A} = \mathbf{LU} \tag{1.5.5.15}$$

where $\quad \mathbf{L} = \mathbf{E}_1^{-1}\mathbf{E}_2^{-1}\dots\mathbf{E}_N^{-1}$

$$\implies \quad \mathbf{A}^{-1} = \mathbf{U}^{-1}\mathbf{L}^{-1} \tag{1.5.5.16}$$

$$\implies \quad \mathbf{A}^{-1} = \begin{pmatrix} \frac{1}{a_1} & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{a_2} & 0 & \dots & 0 \\ 0 & 0 & \frac{1}{a_3} & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & \dots & \dots & \frac{1}{a_N} \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & -1 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & \dots & \dots & 1 \end{pmatrix} \tag{1.5.5.17}$$

Therefore

$$\mathbf{A}^{-1} = \begin{pmatrix} \frac{1}{a_1} & -\frac{1}{a_1} & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{a_2} & -\frac{1}{a_2} & 0 & \dots & 0 \\ 0 & 0 & \frac{1}{a_3} & -\frac{1}{a_3} & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & \frac{1}{a_N} \end{pmatrix} \tag{1.5.5.18}$$

From (1.5.5.18) for the above problem

$$\mathbf{A}^{-1} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix} \tag{1.5.5.19}$$

1.5.6. Suppose **A** is a 2×1 matrix and **B** is 1×2 matrix.Prove that **C**=**AB** is non invertible.
Solution: Let's take **A** and **B** to be non zero vectors. Now,we know that for **C** to be non invertible **Cx** = 0 should have a non trivial solution.So,

$$\mathbf{C}\mathbf{x} = 0 \tag{1.5.6.1}$$
$$\implies \mathbf{A}\mathbf{B}\mathbf{x} = 0 \tag{1.5.6.2}$$

Here, we know that **B** is 1 × 2 matrix and **x** is 2 × 1 matrix then **Bx** will result to a scalar constant k.

$$\implies \mathbf{A}k = 0 \tag{1.5.6.3}$$

For (1.5.6.3) to be true k should be zero. We also know that **B** is 1 × 2 matrix i.e. rows are less than column hence,

$$\mathbf{B}\mathbf{x} = 0 \tag{1.5.6.4}$$

will have a non trivial solution. Hence, using (1.5.6.3) and (1.5.6.4) we can say,

$$\mathbf{A}\mathbf{B}\mathbf{x} = 0 \tag{1.5.6.5}$$

will have a non trivial solution so, **C** is non invertible.

1.5.7. Let **A** be an $n \times n$ (square) matrix, Prove the following two statements:

a) If **A** is invertible and **AB** = 0 for some $n \times n$ matrix **B**, then **B** = 0.
b) If **A** is not invertible, then there exists an $n \times n$ matrix **B** such that **AB** = 0 but **B** ≠ 0.

Solution:

a) Given **A** is an invertible matrix and **AB** = 0 then,

$$\mathbf{A}\mathbf{B} = 0 \tag{1.5.7.1}$$
$$\implies \mathbf{A}^{-1}(\mathbf{A}\mathbf{B}) = 0 \tag{1.5.7.2}$$
$$\implies (\mathbf{A}^{-1}\mathbf{A})\mathbf{B} = 0 \tag{1.5.7.3}$$
$$\implies \mathbf{I}\mathbf{B} = 0 \quad [\because \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}] \tag{1.5.7.4}$$
$$\implies \mathbf{B} = 0 \tag{1.5.7.5}$$

b) If **A** is not invertible, then there exists an $n \times n$ matrix **B** such that **AB** = 0 but **B** ≠ 0. Since **A** is not invertible, **AX** = 0 must have a non-trivial solution. Let the non-trivial solution be,

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \tag{1.5.7.6}$$

Let **B** which is an $n \times n$ matrix have all its columns as **y**.

$$\mathbf{B} = \begin{pmatrix} \mathbf{y} & \mathbf{y} & \cdots & \mathbf{y} \end{pmatrix} \tag{1.5.7.7}$$

From equation (1.5.7.7), we can say that **B** ≠ 0 but **AB** = 0

1.5.8. Let

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{1.5.8.1}$$

Prove, using elementary row operations that A is invertible if and only if $(ad - bc) \neq 0$
Solution:
The goal is to effect the transformation $(\mathbf{A}|\mathbf{I}) \rightarrow \left(\mathbf{I}|\mathbf{A}^{-1}\right)$. Augmenting **A** with the 2×2 identity matrix, we get:

$$\begin{pmatrix} a & b & 1 & 0 \\ c & d & 0 & 1 \end{pmatrix} \tag{1.5.8.2}$$

Now, if $a = 0$, switch the rows. If c is also

0, then the process of reducing **A** to **I** cannot even begin. So, one necessary condition for **A** to be invertible is that the entries a and c are not both 0.

a) Assume that $a \neq 0$, Then :

$$\begin{pmatrix} a & b & 1 & 0 \\ c & d & 0 & 1 \end{pmatrix} \xleftrightarrow{R_1=R_1/a} \begin{pmatrix} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ c & d & 0 & 1 \end{pmatrix} \tag{1.5.8.3}$$

$$\xleftrightarrow{R_2=R_2-cR_1} \begin{pmatrix} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & \frac{ad-bc}{a} & \frac{-c}{a} & 1 \end{pmatrix}$$

Next, assuming that $ad - bc \neq 0$, we get:

$$\xleftrightarrow{R_1=R_1-\frac{b}{ad-bc}R_2} \begin{pmatrix} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & \frac{ad-bc}{a} & \frac{-c}{a} & 1 \end{pmatrix}$$

$$\xleftrightarrow{R_2=R_2\frac{a}{ad-bc}} \begin{pmatrix} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$$

Therefore, if $ad - bc \neq 0$, then the matrix is invertible and it's inverse is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \tag{1.5.8.4}$$

b) In (1.5.8.3), we have assumed that $a \neq 0$. Now consider $a = 0$, then, as we have seen before, it is mandatory that $c \neq 0$:

$$\begin{pmatrix} 0 & b & 1 & 0 \\ c & d & 0 & 1 \end{pmatrix} \xleftrightarrow{R_1\leftrightarrow R_2} \begin{pmatrix} c & d & 0 & 1 \\ 0 & b & 1 & 0 \end{pmatrix} \tag{1.5.8.5}$$

$$\xleftrightarrow{R_1=R_1/c} \begin{pmatrix} 1 & \frac{d}{c} & 0 & \frac{1}{c} \\ 0 & b & 1 & 0 \end{pmatrix}$$

$$\xleftrightarrow{R_1=R_1-R_2\times\frac{d}{bc}} \begin{pmatrix} 1 & 0 & -\frac{d}{bc} & \frac{1}{c} \\ 0 & b & 1 & 0 \end{pmatrix}$$

$$\xleftrightarrow{R_2=R_2/b} \begin{pmatrix} 1 & 0 & -\frac{d}{bc} & \frac{1}{c} \\ 0 & 1 & \frac{1}{b} & 0 \end{pmatrix}$$

Therefore, When we consider $a = 0$ the matrix is invertible if $bc \neq 0$, which is included in the condition $ad - bc \neq 0$.

c) Similarly, consider $c = 0$, then, as we have seen before, it is mandatory that $a \neq 0$:

$$\begin{pmatrix} a & b & 1 & 0 \\ 0 & d & 0 & 1 \end{pmatrix} \xleftrightarrow{R_1=R_1/a} \begin{pmatrix} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & d & 0 & 1 \end{pmatrix} \tag{1.5.8.6}$$

$$\xleftrightarrow{R_1=R_1-R_2\times\frac{b}{ad}} \begin{pmatrix} 1 & 0 & \frac{1}{a} & -\frac{b}{ad} \\ 0 & d & 0 & 1 \end{pmatrix}$$

$$\xleftrightarrow{R_2=R_2/d} \begin{pmatrix} 1 & 0 & \frac{1}{a} & -\frac{b}{ad} \\ 0 & 1 & 0 & \frac{1}{d} \end{pmatrix}$$

Therefore, When we consider $c = 0$, the matrix is invertible if $ad \neq 0$, which is included in the condition $ad - bc \neq 0$.

Hence, it is proved from above three cases that the given matrix is invertible iff $ad - bc \neq 0$.

1.5.9. An $n \times n$ matrix **A** is called upper-triangular if $\mathbf{A_{ij}} = 0$ for $i > j$, that is, if every entry below the main diagonal is 0. Prove that an upper-triangular (square) matrix is invertible if and only if every entry on its main diagonal is different from 0. **Solution:** An $n \times n$ matrix **A** is called upper-triangular if $\mathbf{A_{ij}} = 0$ for $i > j$, that is, if every entry below the main diagonal is 0. Prove that an upper-triangular (square) matrix is invertible if and only if every entry on its main diagonal is different from 0. Considering **A**, an upper triangular matrix. Using the property that determinant of upper triangular matrix is the product of diagonal elements,

$$\left|\mathbf{A}\right| = \prod_{i=1}^{n} a_{i,i} \tag{1.5.9.1}$$

If **A** be invertible then $\left|\mathbf{A}\right| \neq 0$. Hence from (1.5.9.1) we get,

$$\prod_{i=1}^{n} a_{i,i} \neq 0 \tag{1.5.9.2}$$

if any diagonal element is 0 then (1.5.9.2) won't be right hence no diagonal elements should be 0. Hence Proved.

1.5.10. Let A be a $m \times n$ matrix. Show that by a finite number of elementary row and/or column operations one can pass from A to a matrix R which is both row-reduced echelon and column-reduced echelon , i.e., $R_{ij} = 0$ if $i \neq j$, $R_{ii} = 1$, $1 \leq i \leq r$, $R_{ii} = 0$, if $i > r$. Show that $R = PAQ$, where P is an invertible $m \times m$ matrix and Q is an invertible $n \times n$ matrix.

Solution:
Lemma Every elementary matrix is invertible and the inverse is again an elementary matrix. If an elementary matrix E is obtained from I by using a certain row or column operation q, then E^{-1} is obtained from I by the "inverse" operation q^{-1}.
Solution Given **A** is a $m\times n$ matrix. Converting **A** into row reduced echelon form by performing a series of elementary row operations **P**. Let $\mathbf{R}'$ be the row reduced echelon matrix. Also, by using the lemma we can tell that **P** is invertible and order $m \times m$.

$$\mathbf{R}' = \mathbf{PA} \tag{1.5.10.1}$$

where,

$$\mathbf{R}' = \begin{pmatrix} \mathbf{I} & \mathbf{F} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$$

I is an identity matrix, **F** is Free variables matrix and **0** represents a block of zeroes

$\mathbf{R}'$ is in row-reduced echelon form. To perform column operations, elementary matrices should be multiplied on the right side in order to convert the $\mathbf{R}'$ into column-reduced echelon form

$$\mathbf{R} = \mathbf{R}'\mathbf{Q} \tag{1.5.10.2}$$

But performing column operations on a matrix is equivalent to performing row operations on the transposed matrix.

$$\mathbf{R}^T = (\mathbf{R}'\mathbf{Q})^T$$
$$\implies \mathbf{R}^T = \mathbf{Q}^T\mathbf{R}'^T \tag{1.5.10.3}$$

Hence, by using lemma it can be observed that $\mathbf{Q}^T$ is invertible and of the order $n\times n$. Converting $\mathbf{R}^T$ to row-reduced echelon is equivalent to converting **R** to column-reduced echelon.

$$\mathbf{R} = \mathbf{PAQ} \tag{1.5.10.4}$$

where,

$$\mathbf{R} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \tag{1.5.10.5}$$

I is an identity matrix and **0** represents a block of zeroes. **Q** is a upper triangular matrix. **R** in (1.5.10.4) is in both row and column reduced echelon form. Hence proved.
Example Let,

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 5 & 7 \\ 1 & 2 & 3 & 4 \end{pmatrix} \tag{1.5.10.6}$$

To convert (1.5.10.6) into row reduced echelon form, **A** has to be multiplied by **P**

$$\mathbf{P} = \begin{pmatrix} -5 & 3 & 0 \\ 2 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \tag{1.5.10.7}$$

$$\mathbf{R}' = \mathbf{PA} = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1.5.10.8}$$

$\mathbf{R}'$ is in row reduced echelon form. To convert (1.5.10.8) into column-reduced echelon form, elementary operations have to be performed on $\mathbf{R}'^T$. By multiplying all the elementary matrices,

$$\mathbf{Q}^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & 1 & 0 & 0 \\ -1 & 0 & -1 & 1 \end{pmatrix} \tag{1.5.10.9}$$

$$\implies \mathbf{Q} = \begin{pmatrix} 1 & 0 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.5.10.10}$$

So **PAQ** is in both row-reduced and column-reduced echelon form.

$$\mathbf{R} = \mathbf{PAQ} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1.5.10.11}$$

The inverses of **P** and **Q** are,

$$\mathbf{P}^{-1} = \begin{pmatrix} 1 & 3 & 0 \\ 2 & 5 & 0 \\ 1 & 3 & 1 \end{pmatrix}; \quad \mathbf{Q}^{-1} = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.5.10.12}$$

1.5.11. Prove that the given matrix is invertible and

$\mathbf{A}^{-1}$ has integer values.

$$\mathbf{A} = \begin{pmatrix} 1 & \frac{1}{2} & . & . & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & . & . & \frac{1}{n+1} \\ . & . & . & . & . \\ . & . & . & . & . \\ \frac{1}{n} & \frac{1}{n+1} & . & . & \frac{1}{2n-1} \end{pmatrix} \tag{1.5.11.1}$$

Solution: <u>Proof that A is Invertible.</u>
The elements of the given matrix is of the form:

$$A_{ij} = \frac{1}{i+j-1} = \int_0^1 t^{i+j-2} dt = \int_0^1 t^{i-1} t^{j-1} dt \tag{1.5.11.2}$$

We will prove that the matrix $\mathbf{A}$ is positive definite:
$\mathbf{A}$ is Positive definite, if $\mathbf{XAX}^T > 0$

$$\text{Let } \mathbf{X} = (x_i)_{1\le i\le n} \in \mathbb{R}^N \tag{1.5.11.3}$$

$$\mathbf{XAX}^T = \sum_{1\le i,j\le n} \frac{x_i x_j}{i+j-1} \tag{1.5.11.4}$$

From (1.5.11.2),

$$\mathbf{XAX}^T = \sum_{1\le i,j\le n} x_i x_j \int_0^1 t^{i+j-2} dt \tag{1.5.11.5}$$

$$= \int_0^1 \left(\sum_{i=1}^n x_i t^{i-1}\right)\left(\sum_{j=1}^n x_j t^{j-1}\right) dt \tag{1.5.11.6}$$

$$\implies \mathbf{XAX}^T = \int_0^1 \left(\sum_{i=1}^n x_i t^{i-1}\right)^2 dt > 0 \tag{1.5.11.7}$$

Thus, Matrix $\mathbf{A}$ is Positive definite.
Now, let's say λ is an eigen value of $\mathbf{A}$. Then, for the corresponding eigen vector $\mathbf{X} = (x_1, x_2, ..., x_n)$, we can write:

$$\mathbf{XAX}^T = \mathbf{X}\lambda\mathbf{X}^T \quad [\because \mathbf{AX}^T = \lambda\mathbf{X}^T] \tag{1.5.11.8}$$

$$= \|\mathbf{X}\|^2 \lambda \tag{1.5.11.9}$$

$$\implies \lambda = \frac{\mathbf{XAX}^T}{\|\mathbf{X}\|^2} > 0 \tag{1.5.11.10}$$

So, all of the eigenvalues belonging to $\mathbf{A}$ must be positive. The product of the eigenvalues of a matrix equals the determinant.

$$\boxed{\therefore \det(\mathbf{A}) > 0} \tag{1.5.11.11}$$

Thus, the given matrix $\mathbf{A}$ is non-singular and its inverse exist (Invertible).

<u>Proof that $\mathbf{A}^{-1}$ has integer values.</u>
Let us consider a set of shifted legendre polynomials as follow:

$$P_i(x) = \begin{cases} P_1(x) = P_{11} \\ P_2(x) = P_{21} + P_{22}x \\ P_3(x) = P_{31} + P_{32}x + p_{33}x^2 \\ . \\ . \\ P_n(x) = P_{n1} + P_{n2}x + P_{n3}x^2 + .. + P_{nn}x^{n-1} \end{cases} \tag{1.5.11.12}$$

Where, the coefficients P_{ij} are given as:

$$P_{ij} = (-1)^{i+j-1}\binom{j-1}{i-1}\binom{i+j-2}{i-1} \tag{1.5.11.13}$$

The shifted legendre polynomials are analogous to legendre polynomials, but defined on the interval $[0, 1]$ (whereas the interval is $[-1, 1]$ for legendre polynomial).

A set of shifted legendre polynomials obey the following orthogonal relationship:

$$\int_0^1 P_i(x)P_j(x)dx = 0 \text{ for } i \neq j \tag{1.5.11.14}$$

$$\int_0^1 P_i(x)P_j(x)dx = \frac{1}{2i+1} \text{ for } i = j \tag{1.5.11.15}$$

Forming a matrix $\mathbf{P}$ whose elements are the coefficients of polynomials in (1.5.11.12)

$$\mathbf{P} = \begin{pmatrix} P_{11} & 0 & . & . & 0 \\ P_{21} & P_{22} & . & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ P_{n1} & P_{n2} & . & . & P_{nn} \end{pmatrix} \tag{1.5.11.16}$$

Forming a matrix $\mathbf{PAP}^T$, the elements of the matrix $\mathbf{PAP}^T$ can be written as:

$$\mathbf{PAP}^T_{ij} = \sum_{s=1}^{N}\sum_{r=1}^{N} P_{ir}P_{js}A_{rs} \tag{1.5.11.17}$$

From (1.5.11.2) with suitable change in variable notations, we can write:

$$A_{rs} = \int_0^1 x^{r-1}x^{s-1}dx \tag{1.5.11.18}$$

From (1.5.11.17) and (1.5.11.18),

$$\mathbf{PAP}^T_{ij} = \int_0^1 \sum_{s=1}^{N}\sum_{r=1}^{N} P_{ir}P_{js}x^{r-1}x^{s-1}dx \tag{1.5.11.19}$$

$$= \int_0^1 \sum_{s=1}^{N} P_{ir}x^{r-1} \sum_{r=1}^{N} P_{js}x^{s-1}dx \tag{1.5.11.20}$$

$$= \int_0^1 P_i(x)P_j(x)dx \tag{1.5.11.21}$$

From (1.5.11.15)

$$\mathbf{PAP}^T_{ij} = \begin{cases} 0 & i \neq j \\ \frac{1}{2i+1} & i = j \end{cases} \tag{1.5.11.22}$$

Thus, Matrix $\mathbf{PAP}^T$ is diagonal matrix:

$$\mathbf{PAP}^T = \mathbf{D} = \begin{pmatrix} \frac{1}{3} & 0 & . & . & 0 \\ 0 & \frac{1}{5} & . & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & . & . & \frac{1}{2n+1} \end{pmatrix} \tag{1.5.11.23}$$

From (1.5.11.23), the inverse of matrix $\mathbf{A}$ can be written as:

$$\mathbf{A} = \mathbf{P}^{-1}\mathbf{D}(\mathbf{P}^T)^{-1} \tag{1.5.11.24}$$

$$\implies \mathbf{A}^{-1} = \mathbf{P}^T\mathbf{D}^{-1}\mathbf{P} \tag{1.5.11.25}$$

From (1.5.11.13) and (1.5.11.23), It can be clearly observed that the elements of matrix $\mathbf{P}$, $\mathbf{P}^T$ and $\mathbf{D}^{-1}$ are all integers given as:

$$\mathbf{P}_{ij} = (-1)^{i+j-1}\binom{j-1}{i-1}\binom{i+j-2}{i-1} \tag{1.5.11.26}$$

$$\mathbf{D}^{-1}_{ij} = \frac{1}{\mathbf{D}_{ij}} = \begin{cases} 0 & i \neq j \\ 2i+1 & i = j \end{cases} \tag{1.5.11.27}$$

Since, matrix $\mathbf{P}$, $\mathbf{P}^T$ and $\mathbf{D}^{-1}$ are integer matrices, therefore $\mathbf{A}^{-1}$ is also an integer matrix.
Hence proved.

Solution 2

https://github.com/Arko98/EE5609/blob/master/Assignment_14

Let $\mathbf{A_3}$ be 3×3 matrix i.e

$$\mathbf{A_3} = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{pmatrix} \tag{1.5.11.28}$$

Now we find the inverse of the matrix $\mathbf{A_3}$ as follows,

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & 0 & 0 & 1 \end{pmatrix} \tag{1.5.11.29}$$

$$\xleftrightarrow[R_3=R_3-\frac{1}{3}R_1]{R_2=R_2-\frac{1}{2}R_1} \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & \frac{1}{12} & \frac{1}{12} & -\frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{12} & \frac{4}{45} & -\frac{1}{3} & 0 & 1 \end{pmatrix} \tag{1.5.11.30}$$

$$\xleftrightarrow{R_3=R_3-R_2} \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & \frac{1}{12} & \frac{1}{12} & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & \frac{1}{180} & \frac{1}{6} & -1 & 1 \end{pmatrix} \tag{1.5.11.31}$$

$$\xleftrightarrow[R_3=180R_3]{R_2=12R_2} \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & 1 & 1 & -6 & 12 & 0 \\ 0 & 0 & 1 & 30 & -180 & 180 \end{pmatrix} \tag{1.5.11.32}$$

$$\xleftrightarrow[R_1=R_1-R_3]{R_2=R_2-R3} \begin{pmatrix} 1 & \frac{1}{2} & 0 & -9 & 60 & -60 \\ 0 & 1 & 0 & -36 & 192 & -180 \\ 0 & 0 & 1 & 30 & -180 & 180 \end{pmatrix} \tag{1.5.11.33}$$

$$\xleftrightarrow{R_1=R_1-\frac{1}{2}R_2} \begin{pmatrix} 1 & 0 & 0 & 9 & -36 & 30 \\ 0 & 1 & 0 & -36 & 192 & -180 \\ 0 & 0 & 1 & 30 & -180 & 180 \end{pmatrix} \tag{1.5.11.34}$$

Hence we see that $\mathbf{A_3}$ is invertible and the inverse contains integer entries and $\mathbf{A_3^{-1}}$ is given by,

$$\mathbf{A_3^{-1}} = \begin{pmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{pmatrix} \tag{1.5.11.35}$$

Let, $\mathbf{A_4}$ be 4×4 matrix as follows,

$$\mathbf{A_4} = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{pmatrix} \tag{1.5.11.36}$$

Now, expressing $\mathbf{A_4}$ using $\mathbf{A_3}$ we get,

$$\mathbf{A_4} = \begin{pmatrix} \mathbf{A_3} & \mathbf{u} \\ \mathbf{u^T} & d \end{pmatrix} \tag{1.5.11.37}$$

where,

$$\mathbf{u} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{5} \\ \frac{1}{6} \end{pmatrix} \tag{1.5.11.38}$$

$$d = \frac{1}{7} \tag{1.5.11.39}$$

Now assuming $\mathbf{A_4}$ has an inverse, then from (1.5.11.37), the inverse of $\mathbf{A_4}$ can be written using block matrix inversion,

Block matrix inversion

If a matrix is partitioned into four blocks, it can be inverted blockwise as follows:

$$\text{If } \mathbf{M} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} \text{ then,}$$

$$\mathbf{M}^{-1} = \begin{pmatrix} \mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{B}\left(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}\right)^{-1}\mathbf{C}\mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{B}\left(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}\right)^{-1} \\ -\left(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}\right)^{-1}\mathbf{C}\mathbf{A}^{-1} & \left(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}\right)^{-1} \end{pmatrix} \tag{1.5.11.40}$$

$$\therefore \mathbf{A_4^{-1}} = \begin{pmatrix} \mathbf{A_3^{-1}} + \mathbf{A_3^{-1}}\mathbf{u}x_4^{-1}\mathbf{u^T}\mathbf{A_3^{-1}} & -\mathbf{A_3^{-1}}\mathbf{u}x_4^{-1} \\ -x_4^{-1}\mathbf{u^T}\mathbf{A_3^{-1}} & x_4^{-1} \end{pmatrix} \tag{1.5.11.41}$$

$$= x_4^{-1}\begin{pmatrix} \mathbf{A_3^{-1}}x_4 + \mathbf{A_3^{-1}}\mathbf{u}\mathbf{u^T}\mathbf{A_3^{-1}} & -\mathbf{A_3^{-1}}\mathbf{u} \\ \mathbf{u^T}\mathbf{A_3^{-1}} & 1 \end{pmatrix} \tag{1.5.11.42}$$

$$\text{where, } x_4 = d - \mathbf{u^T}\mathbf{A_3^{-1}}\mathbf{u} \tag{1.5.11.43}$$

Now, the assumption of $\mathbf{A_4}$ being invertible will hold if and only if $\mathbf{A_3}$ is invertible, which has been proved in (1.5.11.35) and x_4 from (1.5.11.43) is invertible or x_4 is a nonzero scalar. We now prove that x_4 is invertible as follows,

$$x_4 = \frac{1}{7} - \begin{pmatrix} \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{pmatrix} \begin{pmatrix} \frac{1}{4} \\ \frac{1}{5} \\ \frac{1}{6} \end{pmatrix} \tag{1.5.11.44}$$

$$\implies x_4 = \frac{1}{2800} \tag{1.5.11.45}$$

Hence, x_4 is a scalar, hence x_4^{-1} exists and is given by,

$$x_4^{-1} = 2800 \tag{1.5.11.46}$$

Hence, $\mathbf{A_4}$ is invertible. Now putting the values of $\mathbf{A_3^{-1}}$, x_4^{-1} and $\mathbf{u}$ we get,

$$\mathbf{A_3^{-1}} + \mathbf{A_3^{-1}}\mathbf{u}x_4^{-1}\mathbf{u^T}\mathbf{A_3^{-1}} = \begin{pmatrix} 16 & -120 & 240 \\ -120 & 1200 & -2700 \\ 240 & -2700 & 6480 \end{pmatrix} \tag{1.5.11.47}$$

$$-\mathbf{A_3^{-1}}\mathbf{u}x_4^{-1} = \begin{pmatrix} -140 \\ 1680 \\ -4200 \end{pmatrix} \tag{1.5.11.48}$$

$$x_4^{-1}\mathbf{u^T}\mathbf{A_3^{-1}} = \begin{pmatrix} -140 & 1680 & -4200 \end{pmatrix} \tag{1.5.11.49}$$

$$x_4^{-1} = 2800 \tag{1.5.11.50}$$

Putting values from (1.5.11.47), (1.5.11.48), (1.5.11.49) and (1.5.11.50) into (1.5.11.41) we get,

$$\mathbf{A_4^{-1}} = \begin{pmatrix} 16 & -120 & 240 & -140 \\ -120 & 1200 & -2700 & 1680 \\ 240 & -2700 & 6480 & -4200 \\ -140 & 1680 & -4200 & 2800 \end{pmatrix} \tag{1.5.11.51}$$

Hence, from (1.5.11.51) we proved that, $\mathbf{A_4}$ is invertible and has integer entries.

By successively repeating this method, we can prove that $\mathbf{A_5}$, $\mathbf{A_6}$,$\mathbf{A_7}$,.... and so on, are invertible and have integer values. Thus, we can say, $\mathbf{A_{n-1}}$ will be invertible with integer entries. Then we can represent $\mathbf{A_n}$ as follows,

$$\mathbf{A_n} = \begin{pmatrix} \mathbf{A_{n-1}} & \mathbf{u} \\ \mathbf{u^T} & d \end{pmatrix} \tag{1.5.11.52}$$

where,

$$\mathbf{u} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{5} \\ \vdots \\ \frac{1}{2n-2} \end{pmatrix} \tag{1.5.11.53}$$

$$d = \frac{1}{2n-1} \tag{1.5.11.54}$$

Now assuming $\mathbf{A_n}$ has an inverse, then from (1.5.11.52), the inverse of $\mathbf{A_n}$ can be written using block matrix inversion as follows,

$$\mathbf{A_n}^{-1} = \begin{pmatrix} \mathbf{A_{n-1}^{-1}} + \mathbf{A_{n-1}^{-1}}\mathbf{u}x_n^{-1}\mathbf{u^T}\mathbf{A_{n-1}^{-1}} & -\mathbf{A_{n-1}^{-1}}\mathbf{u}x_n^{-1} \\ -x_n^{-1}\mathbf{u^T}\mathbf{A_{n-1}^{-1}} & x_n^{-1} \end{pmatrix} \tag{1.5.11.55}$$

$$= x_n^{-1}\begin{pmatrix} x_n\mathbf{A_{n-1}^{-1}} + \mathbf{A_{n-1}^{-1}}\mathbf{u}\mathbf{u^T}\mathbf{A_{n-1}^{-1}} & -\mathbf{A_{n-1}^{-1}}\mathbf{u} \\ -\mathbf{u^T}\mathbf{A_{n-1}^{-1}} & 1 \end{pmatrix} \tag{1.5.11.56}$$

where,

$$x_n = d - \mathbf{u^T}\mathbf{A_{n-1}^{-1}}\mathbf{u} \tag{1.5.11.57}$$

Now, the assumption of $\mathbf{A_n}$ being invertible will hold if and only if $\mathbf{A_{n-1}}$ is invertible, which is intuitively proved and x from (1.5.11.57) is invertible or x_n is a nonzero scalar. We now prove that x_n is invertible as follows,

$$x_n = \frac{1}{2n-1} - \begin{pmatrix} \frac{1}{4} & \frac{1}{5} & \dots & \frac{1}{2n-2} \end{pmatrix} \mathbf{A_{n-1}^{-1}} \begin{pmatrix} \frac{1}{4} \\ \frac{1}{5} \\ \dots \\ \frac{1}{2n-2} \end{pmatrix} \tag{1.5.11.58}$$

In equation (1.5.11.58) $\mathbf{u}$ contains no negative or zero entries, again $\mathbf{A_{n-1}^{-1}}$ has non zero integer entries, hence $\mathbf{u^T}\mathbf{A_{n-1}^{-1}}\mathbf{u}$ is a non zero scalar. Moreover d is not equal to $\mathbf{u^T}\mathbf{A_{n-1}^{-1}}\mathbf{u}$ hence in (1.5.11.58) x is non-zero scalar and invertible and hence it has an inverse. Hence $\mathbf{A_n}$ is invertible, proved.

Proof for $\mathbf{A}_{n+1}$:

Expressing $\mathbf{A}_{n+1}$ using $\mathbf{A}_n$ we get:

$$\mathbf{A_{n+1}} = \begin{pmatrix} \mathbf{A_n} & \mathbf{u} \\ \mathbf{u^T} & d \end{pmatrix} \tag{1.5.11.59}$$

where,

$$\mathbf{u} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{5} \\ \vdots \\ \frac{1}{2n-1} \end{pmatrix} \tag{1.5.11.60}$$

$$d = \frac{1}{2n} \tag{1.5.11.61}$$

Now assuming $\mathbf{A_{n+1}}$ has an inverse, then from (1.5.11.59), the inverse of $\mathbf{A_{n+1}}$ can be written using block matrix inversion as follows,

$$\mathbf{A_{n+1}}^{-1} = \begin{pmatrix} \mathbf{A_n^{-1}} + \mathbf{A_n^{-1}} \mathbf{u} x_{n+1}^{-1} \mathbf{u^T} \mathbf{A_n^{-1}} & -\mathbf{A_n^{-1}} \mathbf{u} x_{n+1}^{-1} \\ -x_{n+1}^{-1} \mathbf{u^T} \mathbf{A_n^{-1}} & x_{n+1}^{-1} \end{pmatrix} \tag{1.5.11.62}$$

$$= x_{n+1}^{-1} \begin{pmatrix} x_{n+1} \mathbf{A_n^{-1}} + \mathbf{A_n^{-1}} \mathbf{u^T} \mathbf{A_n^{-1}} & -\mathbf{A_n^{-1}} \mathbf{u} \\ -\mathbf{u^T} \mathbf{A_n^{-1}} & 1 \end{pmatrix} \tag{1.5.11.63}$$

where,

$$x_{n+1} = d - \mathbf{u^T} \mathbf{A_n^{-1}} \mathbf{u} \tag{1.5.11.64}$$

Now, the assumption of $\mathbf{A_{n+1}}$ being invertible will hold if and only if $\mathbf{A_n}$ is invertible, which is proved and x from (1.5.11.64) is invertible or x_{n+1} is a nonzero scalar. We now prove that x_{n+1} is invertible as follows,

$$x_{n+1} = \frac{1}{2n} - \begin{pmatrix} \frac{1}{4} & \frac{1}{5} & \dots & \frac{1}{2n-1} \end{pmatrix} \mathbf{A_n^{-1}} \begin{pmatrix} \frac{1}{4} \\ \frac{1}{5} \\ \dots \\ \frac{1}{2n-1} \end{pmatrix} \tag{1.5.11.65}$$

In equation (1.5.11.65) $\mathbf{u}$ contains no negative or zero entries, again $\mathbf{A_n^{-1}}$ has non zero integer entries, hence $\mathbf{u^T} \mathbf{A_n^{-1}} \mathbf{u}$ is a non zero scalar. Moreover d is not equal to $\mathbf{u^T} \mathbf{A_n^{-1}} \mathbf{u}$ hence in (1.5.11.65) x is non-zero scalar and invertible and hence it has an inverse. Hence $\mathbf{A_{n+1}}$ is also invertible.

Problem statement: If $\mathbf{A_{n-1}^{-1}}$ is invertible and has integer values, Then $\mathbf{A_n^{-1}}$ also has integer values.

Proof:
The matrix $\mathbf{A_n^{-1}}$ can be expressed as:

$$\mathbf{A_n} = \begin{pmatrix} \mathbf{A_{n-1}} & \mathbf{u} \\ \mathbf{u^T} & d \end{pmatrix} \tag{1.5.11.66}$$

where,

$$\mathbf{u} = \begin{pmatrix} \frac{1}{n} \\ \frac{1}{n+1} \\ \vdots \\ \frac{1}{2n-2} \end{pmatrix} \tag{1.5.11.67}$$

$$d = \frac{1}{2n-1} \tag{1.5.11.68}$$

The inverse of $\mathbf{A_n}$ can be written using block matrix inversion,

Block matrix inversion

If a matrix is partitioned into four blocks, it can be inverted blockwise as follows:

$$\text{If } \mathbf{M} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} \text{ then,}$$

$$\mathbf{M}^{-1} = \begin{pmatrix} \mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{B}\left(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}\right)^{-1}\mathbf{C}\mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{B}\left(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}\right)^{-1} \\ -\left(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}\right)^{-1}\mathbf{C}\mathbf{A}^{-1} & \left(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}\right)^{-1} \end{pmatrix} \tag{1.5.11.69}$$

$$\mathbf{A_n}^{-1} = \begin{pmatrix} \mathbf{A_{n-1}^{-1}} + \mathbf{A_{n-1}^{-1}} \mathbf{u} x_n^{-1} \mathbf{u^T} \mathbf{A_{n-1}^{-1}} & -\mathbf{A_{n-1}^{-1}} \mathbf{u} x_n^{-1} \\ -x_n^{-1} \mathbf{u^T} \mathbf{A_{n-1}^{-1}} & x_n^{-1} \end{pmatrix} \tag{1.5.11.70}$$

where,

$$x_n = d - \mathbf{u^T} \mathbf{A_{n-1}^{-1}} \mathbf{u} \tag{1.5.11.71}$$

For $\mathbf{A_n^{-1}}$ to have integer values, each of the four blocks in (1.5.11.70) should have integer values.
Let us first prove that x_n^{-1} have integer values as follow:

$$x_n = d - \mathbf{u^T} \mathbf{A_{n-1}^{-1}} \mathbf{u} \tag{1.5.11.72}$$

$$= \frac{1}{2n-1} - \sum_{i}^{n-1} \sum_{j}^{n-1} u_i u_j (A_{n-1}^{-1})_{ij} \tag{1.5.11.73}$$

$$= \frac{1}{2n-1} - \sum_{i}^{n-1} \sum_{j}^{n-1} \frac{1}{n+i-1} \frac{1}{n+j-1} (A_{n-1}^{-1})_{ij} \tag{1.5.11.74}$$

Thus, x_n is a scalar of the form $\frac{p}{q}$. Also, by calculation, $x_2 = \frac{1}{12}$, $x_3 = \frac{1}{180}$ and $x_3 = \frac{1}{2800}$. Thus, by induction, we can say $p = 1$ for any value of n. Thus, x_n^{-1} can be written as:

$$x_n^{-1} = \frac{1}{x_n} = \frac{q}{p} \tag{1.5.11.75}$$

Since, $p = 1$ for any value of n, Hence $x_n^{-1} = q$ is always integer.

Now, let us consider the block $-\mathbf{A_{n-1}^{-1}}\mathbf{u}x_n^{-1}$, we can write:

$$(-\mathbf{A_{n-1}^{-1}}\mathbf{u}x_n^{-1})_{i,1} = -q\sum_{j}^{n-1}\frac{1}{n+j-1}(A_{n-1}^{-1})_{i,j} \tag{1.5.11.76}$$

Here, the considered block is a $(n-1)\times 1$ matrix, with each element of rational form $\frac{p_1}{q_1}$. For $n = 2, 3, 4$ the value of q_1 comes as 1. Thus by induction, the value of q_1 is always 1 for any value of n. Hence, this matrix always has integer values.

Now, let us consider the block $-x_n^{-1}\mathbf{u}^T\mathbf{A_{n-1}^{-1}}$, we can write:

$$(-x_n^{-1}\mathbf{u}^T\mathbf{A_{n-1}^{-1}})_{1,j} = -q\sum_{i}^{n-1}\frac{1}{n+i-1}(A_{n-1}^{-1})_{i,j} \tag{1.5.11.77}$$

Here, the considered block is a $1\times(n-1)$ matrix, with each element of rational form $\frac{p_2}{q_2}$. For $n = 2, 3, 4$ the value of q_2 comes as 1. Thus by induction, the value of q_2 is always 1 for any value of n. Hence, this matrix always has integer values.

Considering the block $\mathbf{A_{n-1}^{-1}} + \mathbf{A_{n-1}^{-1}}\mathbf{u}x_n^{-1}\mathbf{u^T}\mathbf{A_{n-1}^{-1}}$, let denote it as V_{n-1}^{-1}:

$$V_{n-1}^{-1} = \mathbf{A_{n-1}^{-1}} + \mathbf{A_{n-1}^{-1}}\mathbf{u}x_n^{-1}\mathbf{u^T}\mathbf{A_{n-1}^{-1}} \tag{1.5.11.78}$$

$$= \mathbf{A_{n-1}^{-1}} + \mathbf{A_{n-1}^{-1}}\mathbf{u}\left(d - \mathbf{u^T}\mathbf{A_{n-1}^{-1}}\mathbf{u}\right)^{-1}\mathbf{u^T}\mathbf{A_{n-1}^{-1}} \tag{1.5.11.79}$$

The Woodbury matrix identity is given by:

$$\boxed{(A + UCV)^{-1} = A^{-1} - A^{-1}U\left(C^{-1} + VA^{-1}U\right)^{-1} VA^{-1}} \tag{1.5.11.80}$$

Comparing (1.5.11.79) and (1.5.11.80), we can write:

$$V_{n-1}^{-1} = \left(A_{n-1} - (2n-1)\mathbf{uu}^T\right)^{-1} \tag{1.5.11.81}$$

$$\therefore V_{n-1} = A_{n-1} - (2n-1)\mathbf{uu}^T \tag{1.5.11.82}$$

Here, V_{n-1} can be expanded as:

$$V_{n-1} = A_{n-1} - (2n-1)\mathbf{uu}^T \tag{1.5.11.83}$$

$$= \frac{1}{i+j-1} - \frac{(2n-1)}{(n+i-1)(n+j-1)} \tag{1.5.11.84}$$

$$= \frac{n^2 - ni - nj - ij}{(i+j-1)(n+i-1)(n+j-1)} \tag{1.5.11.85}$$

$$= \frac{(n-i)(n-j)}{(i+j-1)(n+i-1)(n+j-1)} \tag{1.5.11.86}$$

$$\therefore (V_{n-1})_{ij} = \left((A_{n-1})_{ij}\frac{(n-i)(n-j)}{(n+i-1)(n+j-1)}\right) \tag{1.5.11.87}$$

Using (1.5.11.87), The inverse V_{n-1}^{-1} can be written as:

$$(V_{n-1}^{-1})_{ij} = \left((A_{n-1}^{-1})_{ij}\frac{(n+i-1)(n+j-1)}{(n-i)(n-j)}\right) \tag{1.5.11.88}$$

Here, the considered block is a $(n-1)\times(n-1)$ matrix, with each element of rational form $\frac{p_3}{q_3}$. For $n = 2, 3, 4$ the value of q_3 comes as 1. Thus by induction, the value of q_3 is always 1 for any value of n. Hence, this matrix always has integer values.
Since, all the four blocks has integer values, the inverse A_n^{-1} has integer values. **Observations:**

a) The given matrix is a $n\times n$ Hilbert matrix. Which is always invertible with its inverse having integer values.
b) The Hilbert matrix is symmetric and positive definite.

2 Vector Spaces

2.1 *Vector Spaces*

2.1.1. If $\mathbf{F}$ is a field, verify that vector space of all ordered n-tuples $\mathbf{F}^n$ is a vector space over the field $\mathbf{F}$.
Solution: Let $\mathbf{F}^n$ be a set of all ordered n-tuples

over $\mathbf{F}$ i.e

$$\mathbf{F}^n = \left\{ \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} : a_1, a_2, \ldots, a_n \in \mathbf{F} \right\} \quad (2.1.1.1)$$

For $\mathbf{F}^n$ to be a vector space over $\mathbf{F}$ it must satisfy the closure property of vector addition and scalar multiplication.

Vector Addition in $\mathbf{F}^n$:
Let $\alpha = \left(a_i\right)$ and $\beta = \left(b_i\right)$ $\forall\ i = 1, 2, \cdots, n \in \mathbf{F}^n$ then

$$\alpha + \beta = \left(a_i\right) + \left(b_i\right) \quad (2.1.1.2)$$
$$= \left(a_i + b_i\right) \quad (2.1.1.3)$$

Since

$$a_i + b_i \in \mathbf{F}\ \forall\ i = 1, 2, \cdots, n \quad (2.1.1.4)$$
$$\implies \alpha + \beta \in \mathbf{F}^n \quad (2.1.1.5)$$

Scalar multiplication in $\mathbf{F}^n$ over $\mathbf{F}$:
Let $\alpha = \left(a_i\right)$ $\forall\ i = 1, 2, \cdots, n \ \in \mathbf{F}^n$ and $a \in \mathbf{F}$ then

$$a\alpha = \left(aa_i\right) \quad (2.1.1.6)$$

Since

$$aa_i \in \mathbf{F}\ \forall\ i = 1, 2 \cdots, n \quad (2.1.1.7)$$
$$\implies a\alpha \in \mathbf{F}^n \quad (2.1.1.8)$$

Associativity of addition in $\mathbf{F}^n$:
Let $\alpha = \left(a_i\right)$, $\beta = \left(b_i\right)$, $\gamma = \left(g_i\right)$ $\forall\ i = 1, 2, \cdots, n \in \mathbf{F}^n$ then

$$\alpha + (\beta + \gamma) = \left(a_i\right) + \left(b_i + g_i\right) \quad (2.1.1.9)$$
$$= \left(a_i + b_i + g_i\right) \quad (2.1.1.10)$$
$$= \left(a_i + b_i\right) + \left(g_i\right) \quad (2.1.1.11)$$
$$= (\alpha + \beta) + \gamma \quad (2.1.1.12)$$

Existence of additive identity in $\mathbf{F}^n$:

We have $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbf{F}^n$ and $\alpha = \left(a_i\right)$ $\forall\ i = 1, 2, \cdots, n \in \mathbf{F}^n$ then

$$\left(a_i\right) + \left(0\right) = \left(a_i + 0\right) \quad (2.1.1.13)$$
$$= \left(a_i\right) \quad (2.1.1.14)$$

Therefore $\mathbf{0}$ is the additive identity in $\mathbf{F}^n$.

Existence of additive inverse of each element of $\mathbf{F}^n$:
If $\alpha = \left(a_i\right)$ $\forall\ i = 1, 2, \cdots, n \in \mathbf{F}^n$ then $\left(-a_i\right) \in \mathbf{F}^n$. Also we have

$$\left(-a_i\right) + \left(a_i\right) = \mathbf{0} \quad (2.1.1.15)$$

Therefore $-\alpha = \left(-a_i\right)$ is the additive inverse of α. Thus $\mathbf{F}^n$ is an abelian group with respect to addition.

Futher we observe that

a) If $a \in \mathbf{F}$ and $\alpha = \left(a_i\right)$, $\beta = \left(b_i\right)$ $\forall\ i = 1, 2, \cdots, n \in \mathbf{F}^n$ then

$$a(\alpha + \beta) = a\left(a_i + b_i\right) \quad (2.1.1.16)$$

$$= \left(a[a_i + b_i]\right) \quad (2.1.1.17)$$
$$= \left(aa_i + ab_i\right) \quad (2.1.1.18)$$
$$\left(aa_i\right) + \left(ab_i\right) \quad (2.1.1.19)$$
$$= a\left(a_i\right) + a\left(b_i\right) \quad (2.1.1.20)$$
$$= a\alpha + a\beta \quad (2.1.1.21)$$

b) If $a,b \in \mathbf{F}$ and $\alpha = \left(a_i\right)$ $\forall\ i = 1, 2, \cdots, n \in \mathbf{F}^n$ then

$$(a + b)\alpha = \left([a + b]a_i\right) \quad (2.1.1.22)$$

$$= \left(aa_i + ba_i\right) \quad (2.1.1.23)$$
$$= \left(aa_i\right) + \left(ba_i\right) \quad (2.1.1.24)$$
$$= a\left(a_i\right) + b\left(a_i\right) \quad (2.1.1.25)$$
$$= a\alpha + b\alpha \quad (2.1.1.26)$$

c) If $a,b \in \mathbf{F}$ and $\alpha = \left(a_i\right)$ $\forall\ i = 1, 2, \cdots, n \in \mathbf{F}^n$

then

$$(ab)\alpha = \left([ab]a_i\right) \quad (2.1.1.27)$$
$$= \left(a[ba_i]\right) \quad (2.1.1.28)$$
$$= a\left(ba_i\right) \quad (2.1.1.29)$$
$$= a(b\alpha) \quad (2.1.1.30)$$

d) If 1 is the unity element of **F** and $\alpha = \left(a_i\right)\ \forall\ i = 1, 2, \cdots, n \in \mathbf{F}^n$ then

$$1\alpha = \left(1a_i\right) \quad (2.1.1.31)$$
$$= \left(a_i\right) \quad (2.1.1.32)$$
$$= \alpha \quad (2.1.1.33)$$

Hence $\mathbf{F}^n$ is a vector space over **F**.

2.1.2. If **V** is a vector space over field **F**,verify that:

$$(\alpha_1 + \alpha_2) + (\alpha_3 + \alpha_4) = [\alpha_2 + (\alpha_3 + \alpha_1)] + \alpha_4 \quad (2.1.2.1)$$

Solution: Using property of commutativity of (+) in **V**

$$(\alpha_1 + \alpha_2) + (\alpha_3 + \alpha_4) = (\alpha_2 + \alpha_1) + (\alpha_3 + \alpha_4) \quad (2.1.2.2)$$

Using property of associativity of (+) in **V**

$$(\alpha_2 + \alpha_1) + (\alpha_3 + \alpha_4) = \alpha_2 + [\alpha_1 + (\alpha_3 + \alpha_4)] \quad (2.1.2.3)$$

Using property of commutativity of (+) in **V**

$$\alpha_2 + [\alpha_1 + (\alpha_3 + \alpha_4)] = \alpha_2 + (\alpha_3 + \alpha_1) + \alpha_4 \quad (2.1.2.4)$$

Using property of associativity of (+) in **V**

$$\alpha_2 + (\alpha_3 + \alpha_1) + \alpha_4 = [\alpha_2 + (\alpha_3 + \alpha_1] + \alpha_4 \quad (2.1.2.5)$$

2.1.3. If $\mathbb{C}$ is the field of complex numbers, which vectors in $\mathbf{C}^3$ are linear combinations of $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$?

Solution: Expressing the given vectors as the columns of a matrix,

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \quad (2.1.3.1)$$

The row reduced echelon form of the matrix on performing elementary row operations can be given as,

$$\mathbf{R} = \mathbf{CA} \quad (2.1.3.2)$$

where **C** is the product of elementary matrices,

$$\mathbf{C} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} \quad (2.1.3.3)$$

Thus we get,

$$\mathbf{R} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (2.1.3.4)$$

From (2.1.3.4), $rank(\mathbf{A}) = 3$. Thus **A** is a full rank matrix. Hence the columns of **A** are linearly independent i.e., the given vectors are linearly independent and forms the basis for $\mathbf{C}^3$.
Hence any vector $\mathbf{Y} \in \mathbf{C}^3$ can be written as the linear combinations of $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

2.1.4. Let V be the set of all pairs (x,y) of real numbers and let F be the field of real numbers. Define

$$(x, y) + (x_1, y_1) = (x + x_1, y + y_1) \quad (2.1.4.1)$$
$$c(x, y) = (cx, y) \quad (2.1.4.2)$$

Is V with these operations , a vector space over the field of real numbers ?

Solution: V={(x,y) $|x, y \in R\}$, consider $u = (x_1, y_1) \in V, a, b, c \in R$. Axioms with respect to addition and scalar multiplication.

a)

$$(a + b)u = (a + b)(x_1, y_1) \quad (2.1.4.3)$$
$$= ((a + b)x_1, y_1) \neq au + bu \quad (2.1.4.4)$$

Since V with the given operations the equation (2.1.4.4) contradicts the axioms of scalar multiplication . Hence it is not vector space over real number with these operations.

2.1.5. On $\mathbb{R}^n$ define two operations

$$\alpha \oplus \beta = \alpha - \beta \quad (2.1.5.1)$$
$$c \cdot \alpha = -c\alpha \quad (2.1.5.2)$$

The operations on the right are usual ones.

Which of the axioms for a vector space are satisfied by $(\mathbb{R}^n, \oplus, \cdot)$?
Solution: Let $(\alpha,\beta,\gamma) \in \mathbb{R}^n$ and c, c_1, c_2 are scalars taken from the field $\mathbb{R}$ where the vector space is defined on. Table 2.1.5 lists the axioms satisfied and not satisfied for $(\mathbb{R}^n, \oplus, \cdot)$.

2.1.6. Let **V** be the set of pairs (x, y) of real numbers and let **F** be the field of real numbers. Define

$$(x, y) + (x_1, y_1) = (x + x_1, 0) \tag{2.1.6.1}$$

$$c(x, y) = (cx, 0) \tag{2.1.6.2}$$

Is **V**, with these operations, a vector space?
Solution: **V** is a vector space if it satisfies all properties of the vector space. Let us consider the property of Existence of additive identity. According to Existence of additive identity, there is a unique vector **0** in **V** called the zero vector, such that $\alpha + \mathbf{0} = \alpha$ for all α in **V**.
Let $u = (x_1, y_1) \in \mathbf{V}$

$$\begin{aligned} u + \mathbf{0} &= (x_1, y_1) + (0, 0) \\ &= (x_1 + 0, 0) \\ &= (x_1, 0) \\ &\neq u \end{aligned} \tag{2.1.6.3}$$

From (2.1.6.3), there does not exist an additive identity for **V**.
Hence **V** is not a vector space.

2.1.7. Let $\mathbb{V}$ be the set of all complex-valued functions f on the real line such that

$$f(-t) = \overline{f(t)} \tag{2.1.7.1}$$

The bar denotes complex conjugation.Show that $\mathbb{V}$,with the operations

$$(f + g)(t) = f(t) + g(t) \tag{2.1.7.2}$$

$$(cf)(t) = cf(t) \tag{2.1.7.3}$$

is a vector space over the field of real numbers. Give an example of a function in V which is not real valued.
Solution: To prove that $\mathbb{V}$ with the given operations is a vector space over the field of real numbers, we have to start by proving that additivity and homogeneity both hold true. So, we have to prove that (cf+g)(t) is equal to cf(t)+g(t).

$$(cf + g)(t) \tag{2.1.7.4}$$

$$= (cf)(t) + g(t) \tag{2.1.7.5}$$

$$= cf(t) + g(t) \tag{2.1.7.6}$$

Now, we know that f(-t)= $\overline{f(-t)}$ and so (cf+g)(t) should also satisfy the property,

$$(cf + g)(-t) \tag{2.1.7.7}$$

$$= cf(-t) + g(-t) \tag{2.1.7.8}$$

$$= c\overline{f(t)} + \overline{g(t)} \tag{2.1.7.9}$$

$$= \overline{cf(t) + g(t)} \tag{2.1.7.10}$$

$$= \overline{(cf + g)(t)} \tag{2.1.7.11}$$

Example Let's take f(x)=a+ix

$$f(1) = a + i \tag{2.1.7.12}$$

Hence, f(x) is not real valued.Now,

$$f(x) = a + ix \tag{2.1.7.13}$$

$$f(-x) = a - ix \tag{2.1.7.14}$$

$$f(-x) = \overline{f(x)} \tag{2.1.7.15}$$

Since a and x $\in \mathbb{R}$, so f $\in \mathbb{V}$

2.2 Subspaces

2.2.1. Which of the following set of vectors

$$\alpha = (a_1, a_2..........a_n)$$

in $\mathbf{R^n}$ are subspace of $\mathbf{R^n}$ $(n \geq 3)$?

a) All α such that $a_1 \geq 0$

b) All α such that $a_1 + 3a_2 = a_3$

c) All α such that$a_2 = a_1^2$

d) All α such that $a_1 a_2 = 0$

e) All α such that a_2 is rational

Solution: Table 2.2.1 lists the summary of which set of vectors in $\mathbf{R}^n$ are subspace of $\mathbf{R}^n$ $(n \geq 3)$.

2.2.2. Let **V** be the (real) vector space of all functions f from **R** into **R**.

a) Is $f(0) = f(1)$ a subspace of **V** ?
Solution: A non-empty subset **W** of **V** is a

UNSATISTIFD	SATISFIED
Associativity of addition $\alpha \oplus (\beta \oplus \gamma) = \alpha - \beta + \gamma$ $(\alpha \oplus \beta) \oplus \gamma = \alpha - \beta - \gamma$ $\alpha \oplus (\beta \oplus \gamma) \neq (\alpha \oplus \beta) \oplus \gamma$	**Additive identity** $\alpha \oplus \beta = \alpha - \beta = \alpha$ Additive identity is β unique $\beta = (0, 0, ...0)$
Commutativity of addition $\alpha \oplus \beta = \alpha - \beta$ $\beta \oplus \alpha = \beta - \alpha$ $\alpha \oplus \beta \neq \beta \oplus \alpha$	**Additive inverse** $\alpha \oplus \alpha = \alpha - \alpha = 0$ Additive inverse is α
Scalar multiplication with field multiplication $(c_1 c_2) \cdot \alpha = (-c_1 c_2)\alpha$ $c_1 \cdot (c_2 \cdot \alpha) = c_1 c_2 \alpha$ $(c_1 c_2) \cdot \alpha \neq c_1 \cdot (c_2 \cdot \alpha)$	
Identity element of scalar multiplication $1 \cdot \alpha = -\alpha = \alpha$ for $\alpha = (0, 0, ..., 0)$ $1 \cdot \alpha = -\alpha \neq \alpha \quad \forall \quad \alpha \neq (0, 0, ..., 0)$	
Distributivity of scalar multiplication w.r.t vector addition $c \cdot (\alpha \oplus \beta) = -c(\alpha - \beta)$ $c \cdot \alpha \oplus c \cdot \beta = -c\alpha - (-c\beta)$ $c \cdot (\alpha \oplus \beta) \neq c \cdot \alpha \oplus c \cdot \beta$	
Distributivity of scalar multiplication w.r.t field addition $(c_1 + c_2) \cdot \alpha = -(c_1 + c_2)\alpha$ $c_1 \cdot \alpha \oplus c_2 \cdot \beta = -c_1\alpha - (-c_2\beta)$ $(c_1 + c_2) \cdot \alpha \neq c_1 \cdot \alpha \oplus c_2 \cdot \beta$	

TABLE 2.1.5: Axioms of vector space $(\mathbb{R}^n, \oplus, \cdot)$

$\alpha = (a_1, a_2..........a_n)$	
Vector space	**Subspace summary**
$\alpha = (a_1, a_2, a_3, a_4.........., a_n); \quad a_1 \geq 0$	Not a subspace. Scalar multiplication is not satisfied. $-1(\alpha) \neq \alpha$
$\alpha = (a_1, a_2, a_3, a_4.........., a_n); \quad a_1 + 3a_2 = a_3$	It is a subspace
$\alpha = (a_1, a_2, a_3, a_4.........., a_n); \quad a_2 = a_1^2$	Not a subspace. Addition is not satisfied. $(a_1 + b_1)^2 \neq a_1^2 + b_1^2$
$\alpha = (a_1, a_2, a_3, a_4.........., a_n); \quad a_1 a_2 = 0$	Not a subspace. Addition is not satisfied. $a_1 b_1 \neq 0$
$\alpha = (a_1, a_2, a_3, a_4.........., a_n); \quad a_2$ is rational	Not a subspace. Scalar multiplication is not satisfied. $a_2 \neq \sqrt{2}a_1$

TABLE 2.2.1: Summary

subspace of **V** if and only if for each pair of vectors α β in **W** and each scalar c in **R** the vector $c\alpha + \beta$ is again in **W**. For each of the function to be a subspace, it must be closed with respect to addition and scalar multiplication in **V** defined as, for f g ϵ **W** and c ϵ R

Then,

$$h = cf + g \quad (2.2.2.1)$$
$$h(0) = cf(0) + g(0) \quad (2.2.2.2)$$
$$= cf(1) + g(1) \quad (2.2.2.3)$$
$$= h(1) \quad (2.2.2.4)$$

Thus, $h(0) = h(1)$. Therefore,**W** is a subset of **V** and also a vector space. Therefore **W** is a subspace of **V**.
Hence, $f(0) = f(1)$ is a subspace of **V**.

b) Let **V** be the (real) vector space of all functions f from **R** into **R**. Is $f(3) = 1 + f(-5)$ a subspace of **V**
Solution: For each of the function to be a subspace, it must be closed with respect to addition and scalar multiplication in **V** defined as, for $f\ g \in$ **W**
Then,

$$(f + g)(3) = f(3) + g(3) \quad (2.2.2.5)$$
$$= 1 + f(-5) + 1 + g(-5) \quad (2.2.2.6)$$
$$= 2 + f(-5) + g(-5) \quad (2.2.2.7)$$
$$= 2 + (f + g)(-5) \quad (2.2.2.8)$$
$$\neq 1 + (f + g)(-5) \quad (2.2.2.9)$$

Since **W** is not closed with respect to addition $\therefore$ It is not a subspace of **V**.

c) Let **V** be the (real) vector space of all functions f from **R** into **R**. Is $f(-1) = 0$ a subspace of **V**
Solution: A non-empty subset **W** of **V** is a subspace of **V** if and only if for each pair of vectors $\alpha\ \beta$ in **W** and each scalar c in $\mathbb{R}$ the vector $c\alpha + \beta$ is again in **W**.
For each of the function to be a subspace, it must be closed with respect to addition and scalar multiplication in **V** defined as, for $f\ g \in$ **W** and $c \in \mathbb{R}$

Then,

$$(\mathbf{cf} + \mathbf{g})(-1) = \mathbf{cf}(-1) + \mathbf{g}(-1) \quad (2.2.2.10)$$
$$= \mathbf{c}(0) + 0 \quad (2.2.2.11)$$
$$= 0 \quad (2.2.2.12)$$

Thus, $(\mathbf{cf} + \mathbf{g}(-1)) = 0$. Therefore,**W** is a subset of **V** and also a vector space. Therefore **W** is a subspace of **V**.
Hence, $f(-1) = 0$ is a subspace of **V**.

d) Whether the set containing all functions which are continuous is subspace of V
Solution:
If V is a vector Space over field F. A subspace of V is a subset W of V which is itself a vector space over F with the operations of vector addition and scalar multiplication on V
Let f and g be any continuous functions from $R \to R$
and let c be any scalar $\in R$
From real analysis we know that sum and product of continuous functions is continuous. So $cf + g$ is also a continuous function.
Proof: f and g are continuous at a, condition for continuity will be satisfied

$$\lim_{x\to a} f(x) = k_1 \quad (2.2.2.13)$$

$$\lim_{x\to a} g(x) = k_2 \quad (2.2.2.14)$$

Applying limits to $cf + g$

$$\lim_{x\to a} cf(x) + g(x) = \lim_{x\to a} g(x) + \lim_{x\to a} f(x) = k_1 + k_2 \quad (2.2.2.15)$$

So $cf + g$ is also continuous at a We know from theorem that any non-empty subset W of V is a subspace of V if and only if for each pair of vectors α, β in W and each scalar c in F the vector $c\alpha + \beta$ is again in W. So given vector space is a subspace.

2.2.3. Is the vector $\begin{pmatrix} 3 \\ -1 \\ 0 \\ -1 \end{pmatrix}$ in the subspace of $\mathbf{R}^4$ spanned by the vectors $\begin{pmatrix} 2 \\ -1 \\ 3 \\ 2 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 1 \\ 1 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 9 \\ -5 \end{pmatrix}$?

Solution: Expressing the given three vectors as columns of a matrix,

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 1 & 1 \\ 3 & 1 & 9 \\ 2 & -3 & -5 \end{pmatrix} \quad (2.2.3.1)$$

and

$$\mathbf{b} = \begin{pmatrix} 3 \\ -1 \\ 0 \\ -1 \end{pmatrix} \tag{2.2.3.2}$$

For the vector **b** to be in the subspace of $\mathbf{R}^4$ spanned by the three vectors.

$$\mathbf{Ax} = \mathbf{b} \tag{2.2.3.3}$$

must have a solution.

$$\begin{pmatrix} 2 & -1 & 1 \\ -1 & 1 & 1 \\ 3 & 1 & 9 \\ 2 & -3 & -5 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 3 \\ -1 \\ 0 \\ -1 \end{pmatrix} \tag{2.2.3.4}$$

Forming the augmented matrix and row reducing it by elementary row operations,

$$\begin{pmatrix} 2 & -1 & 1 & 3 \\ -1 & 1 & 1 & -1 \\ 3 & 1 & 9 & 0 \\ 2 & -3 & -5 & -1 \end{pmatrix} \xleftrightarrow[R_4 \leftarrow R_4 - R_1]{R_2 \leftarrow 2R_2 + R_1, R_3 \leftarrow R_3 - \frac{3}{2}R_1} \tag{2.2.3.5}$$

$$\begin{pmatrix} 2 & -1 & 1 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & \frac{5}{2} & \frac{15}{2} & \frac{-9}{2} \\ 0 & -2 & -6 & -4 \end{pmatrix} \xleftrightarrow[R_4 \leftarrow R_4 + 2R_2]{R_3 \leftarrow 2R_3 - 5R_2} \begin{pmatrix} 2 & -1 & 1 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & -14 \\ 0 & 0 & 0 & -2 \end{pmatrix} \tag{2.2.3.6}$$

From (2.2.3.6), it is clear that the system does not have a solution. Hence the vector $\begin{pmatrix} 3 \\ -1 \\ 0 \\ -1 \end{pmatrix}$ does not lie in the subspace of $\mathbf{R}^4$ spanned by the given three vectors.

2.2.4. Let **W** be the set of all $(x_1, x_2, x_3, x_4, x_5)$ in $\mathbb{R}^5$ which satisfy

$$2x_1 - x_2 + \frac{4}{3}x_3 - x_4 = 0 \tag{2.2.4.1}$$

$$x_1 + \frac{2}{3}x_3 - x_5 = 0 \tag{2.2.4.2}$$

$$9x_1 - 3x_2 + 6x_3 - 3x_4 - 3x_5 = 0 \tag{2.2.4.3}$$

Find a finite set of vectors which spans **W**.

Solution: The given equations are,

$$2x_1 - x_2 + \frac{4}{3}x_3 - x_4 = 0 \tag{2.2.4.4}$$

$$x_1 + \frac{2}{3}x_3 - x_5 = 0 \tag{2.2.4.5}$$

$$9x_1 - 3x_2 + 6x_3 - 3x_4 - 3x_5 = 0 \tag{2.2.4.6}$$

which can be written as,

$$\begin{pmatrix} 2 & -1 & \frac{4}{3} & -1 & 0 \\ 1 & 0 & \frac{2}{3} & 0 & -1 \\ 9 & -3 & 6 & -3 & -3 \end{pmatrix} \mathbf{x} = 0 \tag{2.2.4.7}$$

Now, the augmented matrix,

$$\left(\begin{array}{ccccc|c} 2 & -1 & \frac{4}{3} & -1 & 0 & 0 \\ 1 & 0 & \frac{2}{3} & 0 & -1 & 0 \\ 9 & -3 & 6 & -3 & -3 & 0 \end{array}\right) \tag{2.2.4.8}$$

$$\xleftrightarrow{R_3 = R_3 - 3R_1 - 3R_2} \left(\begin{array}{ccccc|c} 2 & -1 & \frac{4}{3} & -1 & 0 & 0 \\ 1 & 0 & \frac{2}{3} & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \tag{2.2.4.9}$$

$$\xleftrightarrow{R_2 = R_2 - \frac{1}{2}R_1} \left(\begin{array}{ccccc|c} 2 & -1 & \frac{4}{3} & -1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \tag{2.2.4.10}$$

$$\xleftrightarrow{R_2 = 2R_2} \left(\begin{array}{ccccc|c} 2 & -1 & \frac{4}{3} & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \tag{2.2.4.11}$$

$$\xleftrightarrow{R_1 = R_1 + R_2} \left(\begin{array}{ccccc|c} 2 & 0 & \frac{4}{3} & 0 & -2 & 0 \\ 0 & 1 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \tag{2.2.4.12}$$

So,

$$2x_1 + \frac{4}{3}x_3 - 2x_5 = 0 \tag{2.2.4.13}$$

$$x_2 + x_4 - 2x_5 = 0 \tag{2.2.4.14}$$

Solving the equations we get,

$$x_1 = -\frac{2}{3}x_3 + x_5 \tag{2.2.4.15}$$

$$x_2 = -x_4 + 2x_5 \tag{2.2.4.16}$$

which can be written as,

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \tag{2.2.4.17}$$

$$= \begin{pmatrix} -\frac{2}{3}x_3 + x_5 \\ -x_4 + 2x_5 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \tag{2.2.4.18}$$

$$= x_3 \begin{pmatrix} -\frac{2}{3} \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{2.2.4.19}$$

where x_3,x_4 and $x_5 \in \mathbb{R}$.Hence, the vectors $\begin{pmatrix} -\frac{2}{3} \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ will span **W**

2.2.5. Let **F** be a field and let n be a positive integer (n≥2). Let **V** be the vector space of all n×n matrices over **F**. Which of the following set of matrices **A** in **V** are subspaces of **V**?

a) all invertible **A**;
b) all non-invertible **A**;
c) all **A** such that **AB** = **BA**, where **B** is some fixed matrix in **V**;
d) all **A** such that $\mathbf{A}^2 = \mathbf{A}$.

Solution:

a) Let the matrices **A** and **B** $\in$ **V**, be set of invertible matrix. For them to be a subspace they need to be closed under addition. Let,

$$\mathbf{A} = \mathbf{I} \tag{2.2.5.1}$$

$$\mathbf{B} = -\mathbf{I} \tag{2.2.5.2}$$

It could be easily proven that both matrices **A** and **B** are invertible as,

$$rank(\mathbf{I}_{nxn}) = rank\left(\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}_{nxn}\right) \tag{2.2.5.3}$$

$$\implies rank(-\mathbf{I}_{nxn}) = rank(\mathbf{I}_{nxn}) = n \tag{2.2.5.4}$$

or it is a full rank matrix as there are n pivots.

$$\therefore \mathbf{A} + \mathbf{B} = \mathbf{0}. \tag{2.2.5.5}$$

But the zero matrix **0** is non-invertible as,

$$rank(\mathbf{0}_{nxn}) = rank\left(\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}_{nxn}\right) \tag{2.2.5.6}$$

$$\implies rank(\mathbf{0}_{nxn}) = 0 \tag{2.2.5.7}$$

$\therefore$ **the set of invertible matrices are not closed under addition. Hence not a subspace of V.**

b) Let the matrices $\mathbf{A_1}$, $\mathbf{A_2}, \cdots, \mathbf{A_n} \in$ **V**, be set of non-invertible matrix. For them to be a subspace they need to be closed under addition. Let,

$$\mathbf{A_1} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}_{nxn} \tag{2.2.5.8}$$

$$\mathbf{A_2} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}_{nxn} \tag{2.2.5.9}$$

$$\mathbf{A_n} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}_{nxn} \tag{2.2.5.10}$$

(2.2.5.11)

It could be proven that matrices $\mathbf{A_1}$,

$\mathbf{A_2}, \cdots, \mathbf{A_n}$ are non-invertible as,

$$rank(\mathbf{A_1}) = rank\left(\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}\right) \tag{2.2.5.12}$$

$$\implies rank(\mathbf{A_1}) = 1 \tag{2.2.5.13}$$

or there is only one pivot hence rank is 1.

$$\implies \mathbf{A_1} + \mathbf{A_2} + \mathbf{A_3} + \cdots \mathbf{A_n} = \mathbf{I}_{nxn} \tag{2.2.5.14}$$

Now the identity matrix $\mathbf{I}$ is invertible as shown in equation (2.2.5.4). $\therefore$ **the set of non-invertible matrices are not closed under addition. Hence not a subspace of V.**

c) **Theorem 1:**. A non-empty subset W of V is a subspace of V if and only if for each pair of vectors α, β in W and each scalar c $\in$ F, the vector $c\alpha + \beta \in$ W.
Let the matrices $\mathbf{A_1}$ and $\mathbf{A_2}$ satisfy,

$$\mathbf{A_1 B} = \mathbf{B A_1} \tag{2.2.5.15}$$

$$\mathbf{A_2 B} = \mathbf{B A_2} \tag{2.2.5.16}$$

Let, c$\in$ $\mathbf{F}$ be any constant.

$$\therefore (c\mathbf{A_1} + \mathbf{A_2})\,\mathbf{B} = c\mathbf{A_1 B} + \mathbf{A_2 B} \tag{2.2.5.17}$$

Substituting from equations (2.2.5.15) and (2.2.5.16) to (2.2.5.17),

$$\implies (c\mathbf{A_1} + \mathbf{A_2})\,\mathbf{B} = c\mathbf{B A_1} + \mathbf{B A_2} \tag{2.2.5.18}$$

$$\implies \mathbf{B}c\mathbf{A_1} + \mathbf{B A_2} \tag{2.2.5.19}$$

$$\implies \mathbf{B}\,(c\mathbf{A_1} + \mathbf{A_2}) \tag{2.2.5.20}$$

Thus, $(c\mathbf{A_1} + \mathbf{A_2})$ satisfy the criteria and from Theorem-1 it can be seen that the set is a subspace of V.

d) Let $\mathbf{A}$ and $\mathbf{B}$ $\in$ $\mathbf{V}$ be set of matrices such that,

$$\mathbf{A^2} = \mathbf{A} \tag{2.2.5.21}$$

$$\mathbf{B^2} = \mathbf{B} \tag{2.2.5.22}$$

Now for them to be closed under addition,

$$(\mathbf{A} + \mathbf{B})^2 = \mathbf{A} + \mathbf{B} \tag{2.2.5.23}$$

Which is not always same. Example let,

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \tag{2.2.5.24}$$

$$\mathbf{B} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.2.5.25}$$

Clearly,

$$\mathbf{A}^2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \mathbf{A} \tag{2.2.5.26}$$

$$\mathbf{B}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{B} \tag{2.2.5.27}$$

Now,

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{2.2.5.28}$$

$$\implies (\mathbf{A} + \mathbf{B})^2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \tag{2.2.5.29}$$

Hence, clearly from equations (2.2.5.28) and (2.2.5.29),

$$(\mathbf{A} + \mathbf{B})^2 \neq \mathbf{A} + \mathbf{B} \tag{2.2.5.30}$$

$\therefore$ **the set of all A such that $\mathbf{A}^2 = \mathbf{A}$ is not closed under addition. Hence, not a subspace of V.**

2.2.6. a. Prove that only subspace of $\mathbb{R}^1$ are $\mathbb{R}^1$ and the zero subspace

b. Prove that a subspace of $\mathbb{R}^2$ is $\mathbb{R}^2$, or the zero subspace, or consists of all scalar multiples of some fixed vector in $\mathbb{R}^2$. (The last type of subspace is, intuitively, a straight line through the origin.)

c. Can you describe the subspaces of $\mathbb{R}^3$?

Solution:

a. Let $W \neq 0$ be subspace of $\mathbb{R}^1$. Then W is a nonempty subset of $\mathbb{R}^1$ and there exist $w \in W$ such that $w \neq 0$ which gives us that there exist w^{-1}.

Let $x \in \mathbb{R}^1$. Since W is in $\mathbb{R}^1$ we have that it is closed under scalar multiplication which gives us that $(xw^{-1})w = x(w^{-1}w) = x.1 = x \in W$

Hence $\mathbb{R}^1 \subset W$ and therefore $W = \mathbb{R}^1$

Thus the only subspace of $\mathbb{R}^1$ distinct of 0 is $\mathbb{R}^1$ and therefore only subspaces of $\mathbb{R}^1$ are 0 and $\mathbb{R}^1$.

b. Clearly, 0 and $\mathbb{R}^2$ itself are subspaces of $\mathbb{R}^2$. If $u \neq 0$ and $u \in \mathbb{R}^2$ then span$\{\mathbf{u}\}$ = $c\mathbf{u} : c \in \mathbb{R}$ = set of all scalar multiples of $\mathbf{u}$ is a subspace of $\mathbb{R}^2$.

To show that these are the only subspaces of $\mathbb{R}^2$, assume that $W \subset \mathbb{R}^2$ is any subspace of $\mathbb{R}^2$. Since $W \subset \mathbb{R}^2$ is a subspace of $\mathbb{R}^2$, we have that $\mathbf{0} \in W$. If $W \neq \mathbf{0}$ then there is a vector $\mathbf{u} \neq 0$ and $\mathbf{u} \in W$, and hence W contains $c\mathbf{u}$ for every $c \in \mathbb{R}$. If $W \neq span\{\mathbf{u}\}$, then there is a vector $v \in W$ so that $\mathbf{v} \neq k\mathbf{u}$ for any $k \in \mathbb{R}$.

Then $\mathbf{z} = c\mathbf{u} + d\mathbf{v} \in span\{\mathbf{u},\mathbf{v}\}$ for any $c, d \in \mathbb{R}$. Since W is a subspace $c\mathbf{u}$ and $d\mathbf{v} \in W$ for any $c, d \in \mathbb{R}$, and hence so does $\mathbf{z} = c\mathbf{u} + d\mathbf{v}$. Thus $\mathbf{z} \in span\{\mathbf{u},\mathbf{v}\} \implies z \in W$, and so $span\{\mathbf{u},\mathbf{v}\} \subset W \subset \mathbb{R}^2$.

Let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$ be any vector in $\mathbb{R}^2$, and let $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and let $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. We show that there are real numbers c and d so that $c\mathbf{u} + d\mathbf{v} = \mathbf{x}$

$$\begin{pmatrix} cu_1 \\ cu_2 \end{pmatrix} + \begin{pmatrix} dv_1 \\ dv_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{2.2.6.1}$$

$$\begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{2.2.6.2}$$

Since $\mathbf{v} \neq k\mathbf{u}$ for any $k \in \mathbb{R}$ and since $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ assume that $u_1 \neq 0$, and since $k\mathbf{u} \neq \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ assume that $v_2 \neq 0$. Then

$$A = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.2.6.3}$$

Hence A is row equivalent to I_2 and so A is invertible and so (2.2.6.2) has unique solution for c and d. Thus for any $\mathbf{x} \in \mathbb{R}^2$ we can find real numbers c and d such that $\mathbf{x} = c\mathbf{u} + d\mathbf{v}$. Hence $\mathbf{x} \in \mathbb{R}^2 \implies x \in span\{\mathbf{u}, \mathbf{v}\}$. Thus $\mathbb{R}^2 \subset span\{\mathbf{u}, \mathbf{v}\} \subset W \subset \mathbb{R}^2$.

Hence $span\{\mathbf{u},\mathbf{v}\}$ = W = $\mathbb{R}^2$, and so the only subspace of $\mathbb{R}^2$ are $\mathbf{0}$, $\mathbb{R}^2$, and $L = c\mathbf{u} : \mathbf{u} \neq 0, c \in \mathbb{R}$.

c. The following are the subspaces of $\mathbb{R}^3$:
   1. Origin is a trivial subspace of $\mathbb{R}^3$.

   2. $\mathbb{R}^3$ itself is a trivial subspace of $\mathbb{R}^3$.

   3. Every line through origin is subspace of $\mathbb{R}^3$.

   4. Every plane in $\mathbb{R}^3$ passing through origin is a subspace $\mathbb{R}^3$.
   Proof : Let W be a plane passing through origin. We need $\mathbf{0} \in W$, but we have that since we're only considering planes that contain origin. Next, we need W is closed under vector addition. If $\mathbf{w_1}$ and $\mathbf{w_2}$ both belong to W, then so does $\mathbf{w_1} + \mathbf{w_2}$ because it's found by constructing a parallelogram, and the whole parallelogram lies in the plane W. Finally, we need W is closed under scalar products, but it is since scalar multiples lie in a straight line through the origin, and that line lies in W. Thus, each plane W passing through the origin is a subspace of $\mathbb{R}^3$.

   5. The intersection of any of the above subspaces will also be a subspace of $\mathbb{R}^3$. Because intersection of subspaces of a vector space is also a subspace of vector space.
   Proof : Let W be a collection of subspaces of V, and let $W = \cap W_i$ be their intersection. Since each W_i is a subspace, each of it contains the zero vector. Thus the zero vector is in the

intersection W, and W is non-empty. Let α and β be vectors in W and let c be a scalar. By definition of W, both α and β belong to each W_i, and because each W_i is a subspace, the vector $(c\alpha + \beta)$ is again in W. Hence by definition of subspace, W is a subspace of V.

These 5 are only subspaces of $\mathbb{R}^3$ possible. Because dimension of vector space $\mathbb{R}^3$ is 3. Any subspace of $\mathbb{R}^3$ should have dimension less than or equal to it's dimension. Hence possible dimensions of subspaces are 0,1,2,3. Only subspace with 0 dimension is origin. Subspaces of dimension 1 with zero vector are lines passing through origin. Subspaces of dimension 2 with zero vector are plane passing through origin. Subspace of dimension 3 are all of $\mathbb{R}^3$ itself.

2.2.7. Let $\mathbf{W}_1$ and $\mathbf{W}_2$ be subspaces of a vector space $\mathbf{V}$ such that the set-theoretic union of $\mathbf{W}_1$ and $\mathbf{W}_2$ is also a subspace. Prove that one of the spaces $\mathbf{W}_i$ is contained in the other. **Solution:** Given $\mathbf{W}_1 \cup \mathbf{W}_2$ is a subspace, we need to prove that

$$\mathbf{W}_1 \subseteq \mathbf{W}_2 \quad or \quad \mathbf{W}_2 \subseteq \mathbf{W}_1 \tag{2.2.7.1}$$

Let us assume that

$$\mathbf{W}_1 \nsubseteq \mathbf{W}_2 \tag{2.2.7.2}$$

We need to show that

$$\mathbf{W}_2 \subseteq \mathbf{W}_1 \tag{2.2.7.3}$$

i.e., the generators of $\mathbf{W}_2$ are in $\mathbf{W}_1$. Consider a vector, $\mathbf{w}_1 \in \mathbf{W}_1 \backslash \mathbf{W}_2$ and a vector $\mathbf{w}_2 \in \mathbf{W}_2$. Since $\mathbf{W}_1 \cup \mathbf{W}_2$ is a subspace,

$$\mathbf{w}_1 + \mathbf{w}_2 \in \mathbf{W}_1 \cup \mathbf{W}_2 \tag{2.2.7.4}$$
$$\implies \mathbf{w}_1 + \mathbf{w}_2 \in \mathbf{W}_1 \quad or \tag{2.2.7.5}$$
$$\mathbf{w}_1 + \mathbf{w}_2 \in \mathbf{W}_2 \tag{2.2.7.6}$$

But, $\mathbf{w}_1 + \mathbf{w}_2 \notin \mathbf{W}_2$ because for some vector $-\mathbf{w}_2 \in \mathbf{W}_2$,

$$(\mathbf{w}_1 + \mathbf{w}_2) - \mathbf{w}_2 = \mathbf{w}_1 \notin \mathbf{W}_2 \tag{2.2.7.7}$$

Hence it must be that, $\mathbf{w}_1 + \mathbf{w}_2 \in \mathbf{W}_1$ because for some vector $-\mathbf{w}_1 \in \mathbf{W}_1$,

$$(\mathbf{w}_1 + \mathbf{w}_2) - \mathbf{w}_1 = w_2 \in \mathbf{W}_1 \tag{2.2.7.8}$$

Thus, we have shown that every vector $\mathbf{w}_2$ in $\mathbf{W}_2$ is also in $\mathbf{W}_1$. Hence, $\mathbf{W}_2 \subseteq \mathbf{W}_1$

2.2.8. Let $\mathbf{V}$ be the vector space of all functions from $\mathbf{R}$ into $\mathbf{R}$; let $\mathbf{V_e}$ be the subset of even functions, $f(-x) = f(x)$; let $\mathbf{V_o}$ be the subset of odd functions, $f(-x) = -f(x)$.

a) Prove that $\mathbf{V_e}$ and $\mathbf{V_o}$ are subspaces of $\mathbf{V}$
b) Prove that $\mathbf{V_e} + \mathbf{V_o} = \mathbf{V}$
c) Prove that $\mathbf{V_e} \cap \mathbf{V_o} = \{0\}$

Solution:

a) Prove that $\mathbf{V_e}$ and $\mathbf{V_o}$ are subspaces of $\mathbf{V}$.

A non-empty subset $\mathbf{W}$ of $\mathbf{V}$ is a subspace of $\mathbf{V}$ if and only if for each pair of vectors α, β in $\mathbf{W}$ and each scalar c in $\mathbf{F}$ the vector $c\alpha + \beta$ is again in $\mathbf{W}$.

Let $\mathbf{u}$, $\mathbf{v} \in \mathbf{V_e}$ and $c \in \mathbf{R}$ and let $\mathbf{h} = c\mathbf{u} + \mathbf{v}$. Then,

$$\begin{aligned} \mathbf{h}(-x) &= c\mathbf{u}(-x) + \mathbf{v}(-x) \\ &= c\mathbf{u}(x) + \mathbf{v}(x) \\ &= \mathbf{h}(x) \end{aligned} \tag{2.2.8.1}$$

From (2.2.8.1)

$$\implies \mathbf{h}(-x) = \mathbf{h}(x) \tag{2.2.8.2}$$
$$\implies \mathbf{h} \in \mathbf{V_e} \tag{2.2.8.3}$$

Let $\mathbf{u}$, $\mathbf{v} \in \mathbf{V_o}$ and $c \in \mathbf{R}$ and let $\mathbf{h} = c\mathbf{u} + \mathbf{v}$. Then,

$$\begin{aligned} \mathbf{h}(-x) &= c\mathbf{u}(-x) + \mathbf{v}(-x) \\ &= -c\mathbf{u}(x) - \mathbf{v}(x) \\ &= -\mathbf{h}(x) \end{aligned} \tag{2.2.8.4}$$

From (2.2.8.4)

$$\implies \mathbf{h}(-x) = -\mathbf{h}(x) \tag{2.2.8.5}$$
$$\implies \mathbf{h} \in \mathbf{V_o} \tag{2.2.8.6}$$

From (2.2.8.3) and (2.2.8.6), $\mathbf{V_e}$ and $\mathbf{V_o}$ are subspaces of $\mathbf{V}$.

a) Prove that $\mathbf{V_e} + \mathbf{V_o} = \mathbf{V}$.

Let $\mathbf{u} \in \mathbf{V}$

$$\mathbf{u_e}(x) = \frac{\mathbf{u}(x) + \mathbf{u}(-x)}{2} \tag{2.2.1.7}$$
$$\mathbf{u_o}(x) = \frac{\mathbf{u}(x) - \mathbf{u}(-x)}{2} \tag{2.2.1.8}$$

Equation equation (2.2.1.7) and (2.2.1.8), $\mathbf{u_e}$ is even and $\mathbf{u_o}$ is odd. Adding both the equations,

$$\mathbf{u} = \mathbf{u_e} + \mathbf{u_o} \tag{2.2.1.9}$$

a) Prove that $\mathbf{V_e} \cap \mathbf{V_o} = \{0\}$.
Let $\mathbf{u} \in \mathbf{V_e} \cap \mathbf{V_o}$

$$\mathbf{u} \in \mathbf{V_e} \implies \mathbf{u}(-x) = \mathbf{u}(x) \tag{2.2.2.10}$$
$$\mathbf{u} \in \mathbf{V_o} \implies \mathbf{u}(-x) = -\mathbf{u}(x) \tag{2.2.2.11}$$

Equating (2.2.2.10) and (2.2.2.11),

$$\mathbf{u}(x) = -\mathbf{u}(x) \tag{2.2.2.12}$$
$$\implies 2\mathbf{u}(x) = 0 \tag{2.2.2.13}$$
$$\implies \mathbf{u} = 0 \tag{2.2.2.14}$$

Equations (2.2.8.3), (2.2.8.6), (2.2.1.9), (2.2.2.14) proves 1, 2 and 3.

2.2.3. Let $\mathbf{W_1}$ and $\mathbf{W_2}$ be subspaces of a vector space $\mathbf{V}$ such that

$$\mathbf{W_1} + \mathbf{W_2} = \mathbf{V} \tag{2.2.3.1}$$
$$\text{and } \mathbf{W_1} \cap \mathbf{W_2} = \mathbf{0} \tag{2.2.3.2}$$

Prove that for each vector α in $\mathbf{V}$ there are unique vectors α_1 in $\mathbf{W_1}$ and α_2 in $\mathbf{W_2}$ such that

$$\alpha = \alpha_1 + \alpha_2 \tag{2.2.3.3}$$

Solution: Suppose, vectors α_1 and α_2 are not unique.
Consider

$$\alpha_1' \in \mathbf{W_1}, \tag{2.2.3.4}$$
$$\alpha_2' \in \mathbf{W_2} \tag{2.2.3.5}$$
$$\text{such that } \alpha = \alpha_1' + \alpha_2' \tag{2.2.3.6}$$

(2.2.3.3) and (2.2.3.6) indicate

$$\alpha_1 + \alpha_2 = \alpha_1' + \alpha_2' \tag{2.2.3.7}$$
$$\implies \alpha_1 - \alpha_1' = \alpha_2' - \alpha_2 \tag{2.2.3.8}$$

For α_1 and α_1' lying in subspace $\mathbf{W_1}$, defined on field $\mathbb{F}$, the following holds

$$\alpha_1 + c\alpha_1' \in \mathbf{W_1}, c \in \mathbb{F} \tag{2.2.3.9}$$
$$c = -1 \implies \alpha_1 - \alpha_1' \in \mathbf{W_1} \tag{2.2.3.10}$$
$$\text{Similarly, } \alpha_2' - \alpha_2 \in \mathbf{W_2} \tag{2.2.3.11}$$
$$(2.2.3.8) \implies \alpha_1 - \alpha_1' \in \mathbf{W_2} \tag{2.2.3.12}$$

(2.2.3.2),(2.2.3.10),(2.2.3.12) indicate

$$\alpha_1 - \alpha_1' = \alpha_2' - \alpha_2 = \mathbf{0} \tag{2.2.3.13}$$
$$\implies \alpha_1 = \alpha_1' \tag{2.2.3.14}$$
$$\alpha_2 = \alpha_2' \tag{2.2.3.15}$$

So, there exists a unique $\alpha_1 \in \mathbf{W_1}$ and $\alpha_2 \in \mathbf{W_2}$ such that

$$\alpha = \alpha_1 + \alpha_2 \tag{2.2.3.16}$$

where $\alpha \in \mathbf{V}$

2.2.4. Let $\mathbf{V}$ be the set of all 2 X 2 matrices $\mathbf{A}$ with complex entries which satisfy $\mathbf{A_{11}} + \mathbf{A_{22}} = 0$ and Show that $\mathbf{V}$ is a vector space over the field of real numbers, with the usual operations of matrix addition and multiplication of a matrix by a scalar.
Solution: A non-empty subset $\mathbf{W}$ of $\mathbf{V}$ is a subspace of $\mathbf{V}$ if and only if for each pair of vectors $\mathbf{a}$, $\mathbf{b}$ in $\mathbf{W}$ and each scalar c in $\mathbf{F}$ the vector c$\mathbf{a}$ + $\mathbf{b}$ is again in $\mathbf{W}$
Let $\mathbf{M}$ be the vector space of all 2 x 2 matrices over $\mathbb{C}$ i.e set of complex numbers and let $\mathbb{R}$ be set of real numbers.
Consider $\mathbf{A}$,$\mathbf{B}$ ϵ $\mathbf{V}$ and c ϵ $\mathbb{C}$

$$\text{Let } \mathbf{A} = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \tag{2.2.4.1}$$
$$\text{and } \mathbf{B} = \begin{pmatrix} x' & y' \\ z' & w' \end{pmatrix} \tag{2.2.4.2}$$
$$x + w = x' + w' = 0 \tag{2.2.4.3}$$
$$c\mathbf{A} + \mathbf{B} = \begin{pmatrix} cx + x' & cy + y' \\ cz + z' & cw + w' \end{pmatrix} \tag{2.2.4.4}$$
$$\therefore (cx + x') + (cw + w') = c(x + w) + (x' + w') = 0 \tag{2.2.4.5}$$

Since $\mathbf{V}$ is a subset of $\mathbf{M}$.
Therefore using above Theorem,

$$c\mathbf{A} + \mathbf{B} \;\epsilon\; \mathbf{V} \tag{2.2.4.6}$$

Hence $\mathbf{V}$ is a subspace of $\mathbf{M}$ as a vector space over $\mathbb{C}$.Hence $\mathbf{V}$ is a vector space over $\mathbb{R}$.

2.3 Bases and Dimension

2.3.1. Prove that if two vectors are linearly dependent, one of them is a scalar multiple of the other.

Solution: consider the row reduced matrix

$$\begin{pmatrix}1 & 1 & 2 & 4\\ 2 & -1 & -5 & 2\\ 1 & -1 & -4 & 0\\ 2 & 1 & 1 & 6\end{pmatrix} \xleftrightarrow[R_3\to R_3-R_1]{R_2\to R_2-2R_1} \begin{pmatrix}1 & 1 & 2 & 4\\ 0 & -3 & -9 & -6\\ 0 & -2 & -6 & -4\\ 2 & 1 & 1 & 6\end{pmatrix} \tag{2.3.1.1}$$

$$\xleftrightarrow[R_2\leftarrow R_4]{R_4\leftarrow R_4-2R_1} \begin{pmatrix}1 & 1 & 2 & 4\\ 0 & -1 & -3 & -2\\ 0 & -2 & -6 & -4\\ 0 & -3 & -9 & -6\end{pmatrix} \tag{2.3.1.2}$$

$$\xleftrightarrow[R_2\leftarrow -R_2]{R_4\leftarrow R_2} \begin{pmatrix}1 & 1 & 2 & 4\\ 0 & 1 & 3 & 2\\ 0 & -3 & -9 & -6\\ 0 & -2 & -6 & -4\end{pmatrix} \tag{2.3.1.3}$$

$$\xleftrightarrow[R_4\leftarrow R_4+2R_2]{R_3\leftarrow R_3+3R_2} \begin{pmatrix}1 & 1 & 2 & 4\\ 0 & 1 & 3 & 2\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix} \tag{2.3.1.4}$$

Therefore the rank = no. of pivot columns = 2 (less than no. of columns).Thus the fours vectors are not linearly independent.

2.3.2. Are the vectors

$$\alpha_1 = \left(1, 1, 2, 4\right), \alpha_2 = \left(2, -1, -5, 2\right) \tag{2.3.2.1}$$

$$\alpha_3 = \left(1, -1, -4, 0\right), \alpha_4 = \left(2, 1, 1, 6\right) \tag{2.3.2.2}$$

linearly independent in R^4

Solution: consider the row reduced matrix

$$\begin{pmatrix}1 & 1 & 2 & 4\\ 2 & -1 & -5 & 2\\ 1 & -1 & -4 & 0\\ 2 & 1 & 1 & 6\end{pmatrix} \xleftrightarrow[R_3\to R_3-R_1]{R_2\to R_2-2R_1} \begin{pmatrix}1 & 1 & 2 & 4\\ 0 & -3 & -9 & -6\\ 0 & -2 & -6 & -4\\ 2 & 1 & 1 & 6\end{pmatrix} \tag{2.3.2.3}$$

$$\xleftrightarrow[R_2\leftarrow R_4]{R_4\leftarrow R_4-2R_1} \begin{pmatrix}1 & 1 & 2 & 4\\ 0 & -1 & -3 & -2\\ 0 & -2 & -6 & -4\\ 0 & -3 & -9 & -6\end{pmatrix} \tag{2.3.2.4}$$

$$\xleftrightarrow[R_2\leftarrow -R_2]{R_4\leftarrow R_2} \begin{pmatrix}1 & 1 & 2 & 4\\ 0 & 1 & 3 & 2\\ 0 & -3 & -9 & -6\\ 0 & -2 & -6 & -4\end{pmatrix} \tag{2.3.2.5}$$

$$\xleftrightarrow[R_4\leftarrow R_4+2R_2]{R_3\leftarrow R_3+3R_2} \begin{pmatrix}1 & 1 & 2 & 4\\ 0 & 1 & 3 & 2\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix} \tag{2.3.2.6}$$

Therefore the rank = no. of pivot columns = 2 (less than no. of columns).Thus the fours vectors are not linearly independent.

2.3.3. Find a basis for the subspace of $\mathbb{R}^4$ spanned by the four vectors

$$\alpha_{\mathbf{1}} = \begin{pmatrix}1 & 1 & 2 & 4\end{pmatrix} \tag{2.3.3.1}$$

$$\alpha_{\mathbf{2}} = \begin{pmatrix}2 & -1 & -5 & 2\end{pmatrix} \tag{2.3.3.2}$$

$$\alpha_{\mathbf{3}} = \begin{pmatrix}1 & -1 & -4 & 0\end{pmatrix} \tag{2.3.3.3}$$

$$\alpha_{\mathbf{4}} = \begin{pmatrix}2 & 1 & 1 & 6\end{pmatrix} \tag{2.3.3.4}$$

Solution: The basis of the given four vectors is equivalent to finding the basis of column-space $C(\mathbf{A})$ of a matrix $\mathbf{A}$ defined as follows,

$$\mathbf{A} = \begin{pmatrix}1 & 2 & 1 & 2\\ 1 & -1 & -1 & 1\\ 2 & -5 & -4 & 1\\ 4 & 2 & 0 & 6\end{pmatrix} \tag{2.3.3.5}$$

Now we calculate the row echelon form of $\mathbf{A}$

as follows,

$$\begin{pmatrix}1 & 2 & 1 & 2\\1 & -1 & -1 & 1\\2 & -5 & -4 & 1\\4 & 2 & 0 & 6\end{pmatrix}\xleftrightarrow[R_2=R_2-R_1]{R_2=R_2-R_1}\begin{pmatrix}1 & 2 & 1 & 2\\0 & -3 & -2 & -1\\0 & -9 & -6 & -3\\4 & 2 & 0 & 6\end{pmatrix} \tag{2.3.3.6}$$

$$\xleftrightarrow{R_4=R_4-R_1}\begin{pmatrix}1 & 2 & 1 & 2\\0 & -3 & -2 & -1\\0 & -9 & -6 & -3\\0 & -6 & -4 & -2\end{pmatrix} \tag{2.3.3.7}$$

$$\xleftrightarrow{R_2=-\frac{1}{3}R_2}\begin{pmatrix}1 & 2 & 1 & 2\\0 & 1 & -\frac{2}{3} & \frac{1}{3}\\0 & -9 & -6 & -3\\0 & -6 & -4 & -2\end{pmatrix} \tag{2.3.3.8}$$

$$\xleftrightarrow{R_3=R_3-9R_2}\begin{pmatrix}1 & 2 & 1 & 2\\0 & 1 & -\frac{2}{3} & \frac{1}{3}\\0 & 0 & 0 & 0\\0 & -6 & -4 & -2\end{pmatrix} \tag{2.3.3.9}$$

$$\xleftrightarrow{R_4=R_4+6R_2}\begin{pmatrix}1 & 2 & 1 & 2\\0 & 1 & -\frac{2}{3} & \frac{1}{3}\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{pmatrix} \tag{2.3.3.10}$$

From (2.3.3.10) we can see that the first column and second column of $\mathbf{A}$ contains pivot values. Hence the column 1 and column 2 are the basis of the subspace of $\mathbb{R}^4$ spanned by the given vectors α_1, α_2, α_3, α_4
Hence the required basis vectors are,

$$\mathbf{a_1} = \begin{pmatrix}1 & 1 & 2 & 4\end{pmatrix} \tag{2.3.3.11}$$

$$\mathbf{a_2} = \begin{pmatrix}2 & -1 & -5 & 2\end{pmatrix} \tag{2.3.3.12}$$

2.3.4. Show that the vectors

$$\alpha_1 = \begin{pmatrix}1 & 0 & -1\end{pmatrix} \quad \alpha_2 = \begin{pmatrix}1 & 2 & 1\end{pmatrix} \tag{2.3.4.1}$$

$$\alpha_3 = \begin{pmatrix}0 & -3 & 2\end{pmatrix} \tag{2.3.4.2}$$

form a basis for $\mathbb{R}^3$. Express each of the standard basis vectors as linear combinations of $\begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_3\end{pmatrix}$
Solution:

Theorem 2.1. *Let* $\mathbf{V}$ *be an n-dimensional vector space over the field* $\mathbf{F}$, *and let* β *and* β' *be two ordered basis of* $\mathbf{V}$. *Then, there is a unique, necessarily invertible,* $n \times n$ *matrix* $\mathbf{P}$ *with entries in* $\mathbf{F}$ *such that*
a) $\left[\alpha\right]_\beta = \mathbf{P}\left[\alpha\right]_{\beta'}$
b) $\left[\alpha\right]_{\beta'} = \mathbf{P^{-1}}\left[\alpha\right]_\beta$
for every vector α *in* $\mathbf{V}$. *The columns of* $\mathbf{P}$ *are given by*

$$\mathbf{P_j} = \left[\alpha_j\right]_\beta \qquad j = 1, 2, ..., n \tag{2.3.4.3}$$

In order to show that the set of vectors α_1, α_2,and α_3 are basis for $\mathbb{R}^3$. We first show that α_1, α_2,and α_3 a are linearly independent in $\mathbb{R}^3$ and also they span $\mathbb{R}^3$. Consider,

$$\mathbf{A} = \begin{pmatrix}\alpha_1^T & \alpha_2^T & \alpha_3^T\end{pmatrix} \tag{2.3.4.4}$$

$$\mathbf{A} = \begin{pmatrix}1 & 1 & 0\\0 & 2 & -3\\-1 & 1 & 2\end{pmatrix} \tag{2.3.4.5}$$

Now,by row reduction

$$\begin{pmatrix}1 & 1 & 0\\0 & 2 & -3\\-1 & 1 & 2\end{pmatrix}\xleftrightarrow{R_3=R_3+R_1}\begin{pmatrix}1 & 1 & 0\\0 & 2 & -3\\0 & 2 & 2\end{pmatrix} \tag{2.3.4.6}$$

$$\xleftrightarrow{R_3=R_3-R_2}\begin{pmatrix}1 & 1 & 0\\0 & 2 & -3\\0 & 0 & 5\end{pmatrix} \tag{2.3.4.7}$$

$$\xleftrightarrow{R_2=\frac{R_2}{2}}\begin{pmatrix}1 & 1 & 0\\0 & 1 & \frac{-3}{2}\\0 & 0 & 5\end{pmatrix} \tag{2.3.4.8}$$

$$\xleftrightarrow{R_1=R_1-R_2}\begin{pmatrix}1 & 0 & \frac{3}{2}\\0 & 1 & \frac{-3}{2}\\0 & 0 & 5\end{pmatrix} \tag{2.3.4.9}$$

$$\xleftrightarrow{R_3=\frac{R_3}{5}}\begin{pmatrix}1 & 0 & \frac{3}{2}\\0 & 1 & \frac{-3}{2}\\0 & 0 & 1\end{pmatrix} \tag{2.3.4.10}$$

$$\xleftrightarrow{R_1=R_1-\frac{3}{2}R_3}\begin{pmatrix}1 & 0 & 0\\0 & 1 & \frac{-3}{2}\\0 & 0 & 1\end{pmatrix} \tag{2.3.4.11}$$

$$\xleftrightarrow{R_2=R_2+\frac{3}{2}R_3}\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix} \tag{2.3.4.12}$$

(2.3.4.12) is the row reduced echelon form of $\mathbf{A}$ and since it is identity matrix of order 3, we say that vectors α_1, α_2, and α_3 are linearly independent and their column space is $\mathbb{R}^3$ which means vectors α_1, α_2, and α_3 span $\mathbb{R}^3$. Hence, vectors α_1, α_2, and α_3 form a basis for

$\mathbb{R}^3$.

Now,use theorem (2.1), and calculate the inverse of (2.3.4.5) then the columns of $\mathbf{A^{-1}}$ will give the coefficients to write the standard basis vectors in terms of $\alpha_i's$. We try to find the inverse of $\mathbf{A}$ by row-reducing the augmented matrix.$\mathbf{A|I}$

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & -3 \\ -1 & 1 & 2 \end{pmatrix} \tag{2.3.4.13}$$

Now, by row reducing $\mathbf{A|I}$ as follows

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & -3 & 0 & 1 & 0 \\ -1 & 1 & 2 & 0 & 0 & 1 \end{pmatrix} \tag{2.3.4.14}$$

$$\xleftrightarrow{R_3=R_3+R_1} \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & -3 & 0 & 1 & 0 \\ 0 & 2 & 2 & 1 & 0 & 1 \end{pmatrix} \tag{2.3.4.15}$$

$$\xleftrightarrow{R_3=R_3-R_2} \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & -3 & 0 & 1 & 0 \\ 0 & 0 & 5 & 1 & -1 & 1 \end{pmatrix} \tag{2.3.4.16}$$

$$\xleftrightarrow{R_2=\frac{R_2}{2}} \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & \frac{-3}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 5 & 1 & -1 & 1 \end{pmatrix} \tag{2.3.4.17}$$

$$\xleftrightarrow{R_1=R_1-R_2} \begin{pmatrix} 1 & 0 & \frac{3}{2} & 1 & \frac{-1}{2} & 0 \\ 0 & 1 & \frac{-3}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 5 & 1 & -1 & 1 \end{pmatrix} \tag{2.3.4.18}$$

$$\xleftrightarrow{R_3=\frac{R_3}{5}} \begin{pmatrix} 1 & 0 & \frac{3}{2} & 1 & \frac{-1}{2} & 0 \\ 0 & 1 & \frac{-3}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{5} & \frac{-1}{5} & \frac{1}{5} \end{pmatrix} \tag{2.3.4.19}$$

$$\xleftrightarrow{R_1=R_1-\frac{3R_3}{2}} \begin{pmatrix} 1 & 0 & 0 & \frac{7}{10} & \frac{-1}{5} & \frac{-3}{10} \\ 0 & 1 & \frac{-3}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{5} & \frac{-1}{5} & \frac{1}{5} \end{pmatrix} \tag{2.3.4.20}$$

$$\xleftrightarrow{R_2=R_2+\frac{3R_3}{2}} \begin{pmatrix} 1 & 0 & 0 & \frac{7}{10} & \frac{-1}{5} & \frac{-3}{10} \\ 0 & 1 & 0 & \frac{3}{10} & \frac{1}{5} & \frac{3}{10} \\ 0 & 0 & 1 & \frac{1}{5} & \frac{-1}{5} & \frac{1}{5} \end{pmatrix} \tag{2.3.4.21}$$

Thus, by (2.3.4.21), we have

$$\mathbf{A^{-1}} = \begin{pmatrix} \frac{7}{10} & \frac{-1}{5} & \frac{-3}{10} \\ \frac{3}{10} & \frac{1}{5} & \frac{3}{10} \\ \frac{1}{5} & \frac{-1}{5} & \frac{1}{5} \end{pmatrix} \tag{2.3.4.22}$$

Now, let $\mathbf{e_1} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$, $\mathbf{e_2} = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}$, and $\mathbf{e_3} = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$ be the standard basis for $\mathbb{R}^3$. Hence,each of the standard basis vectors as linear combinations of $\alpha_1, \alpha_2, \alpha_3$ is as under

$$\mathbf{e_1} = \frac{7}{10}\alpha_1 + \frac{3}{10}\alpha_2 + \frac{1}{5}\alpha_3 \tag{2.3.4.23}$$

$$\mathbf{e_2} = -\frac{1}{5}\alpha_1 + \frac{1}{5}\alpha_2 - \frac{1}{5}\alpha_3 \tag{2.3.4.24}$$

$$\mathbf{e_3} = \frac{-3}{10}\alpha_1 + \frac{3}{10}\alpha_2 + \frac{1}{5}\alpha_3 \tag{2.3.4.25}$$

2.3.5. Find three vectors in R^3 which are linearly dependent and are such that any two of them are linearly independent

Solution: Let

$$\mathbf{v_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{2.3.5.1}$$

$$\mathbf{v_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \tag{2.3.5.2}$$

$$\mathbf{v_3} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \tag{2.3.5.3}$$

Then,

$$\mathbf{v_1} + \mathbf{v_2} - \mathbf{v_3} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{2.3.5.4}$$

Thus, they are linearly dependent. As We know that $\mathbf{v_1} and \mathbf{v_2}$ are linearly independent as they form the standard basis vectors.

Now, Suppose

$$\mathbf{a}\mathbf{v_1} + \mathbf{b}\mathbf{v_2} = 0 \tag{2.3.5.5}$$

Then

$$\begin{pmatrix} a+b \\ b \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{2.3.5.6}$$

Hence, the second co-ordinate implies $\mathbf{b} = 0$ and the first coordinate in turn implies $\mathbf{a} = 0$. Thus $\mathbf{v_1}$ and $\mathbf{v_2}$ are linearly independent. Similarly, $\mathbf{v_2}$ and $\mathbf{v_3}$ are linearly independent

2.3.6. Let $\mathbf{V}$ be the vector space of all 2×2 matrices over the field $\mathbf{F}$. Prove that $\mathbf{V}$ has dimension 4 by exhibiting a basis for $\mathbf{V}$ which has four elements.

Solution:

Let

$$v_{11} = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \quad v_{12} = \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} \tag{2.3.6.1}$$

$$v_{21} = \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}, \quad v_{22} = \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} \tag{2.3.6.2}$$

Suppose $av_{11}+bv_{12}+cv_{21}+dv_{22} = \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}$. Then

$$\begin{pmatrix}a & b\\c & d\end{pmatrix} = \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix} \tag{2.3.6.3}$$

The only values of a, b, c, d which makes the equation (2.3.6.3) satisfied is, when $a = b = c = d = 0$. Thus v_1, v_2, v_3, v_4 are linearly independent.

Now, let $\begin{pmatrix}a & b\\c & d\end{pmatrix}$ be any 2×2 matrix. Then $\begin{pmatrix}a & b\\c & d\end{pmatrix} = av_{11} + bv_{12} + cv_{21} + dv_{22}$. Thus $v_{11}, v_{12}, v_{21}, v_{22}$ span the space of 2×2 matrix.

Thus $v_{11}, v_{12}, v_{21}, v_{22}$ are both linearly independent and they span the span of all 2×2 matrices. So, $v_{11}, v_{12}, v_{21}, v_{22}$ constitute a basis for the space of all 2×2 matrices.

We know that, the dimension of a vector space **V**, denoted by $dim(\mathbf{V})$, is the number of basis for **V**. Therefore, $dim(\mathbf{V}) = 4$.

2.3.7. Let V be the vector space of all 2×2 matrices over the field $\mathbb{F}$. Let W_1 be the set of matrices of the form

$$\begin{pmatrix}x & -x\\y & z\end{pmatrix} \tag{2.3.7.1}$$

and let W_2 be the set of matrices of the form

$$\begin{pmatrix}a & b\\-a & c\end{pmatrix} \tag{2.3.7.2}$$

a) Prove that W_1 and W_2 are subspaces of V.
b) Find the dimension of $W_1, W_2, W_1 + W_2$ and $W_1 \cap W_2$.

Solution: A non-empty subset W of V is a subspace of V if and only if for each pair of vectors α, β in W and each scalar c $\in$ F, the vector $c\alpha + \beta \in$ W.

a) Let $A_1, A_2 \in W_1$ where,

$$A_1 = \begin{pmatrix}x_1 & -x_1\\y_1 & z_1\end{pmatrix}, A_2 = \begin{pmatrix}x_2 & -x_2\\y_2 & z_2\end{pmatrix} \tag{2.3.7.3}$$

Let $c \in F$ then,

$$cA_1 + A_2 = \begin{pmatrix}cx_1 + x_2 & -cx_1 - x_2\\cy_1 + y_2 & cz_1 + z_2\end{pmatrix} = \begin{pmatrix}u & -u\\v & w\end{pmatrix} \tag{2.3.7.4}$$

Thus $cA_1 + A_2 \in W_1$. **Hence W_1 is a subspace.** Similarly, let $A_1, A_2 \in W_2$ where,

$$A_1 = \begin{pmatrix}a_1 & b_1\\-a_1 & c_1\end{pmatrix}, A_2 = \begin{pmatrix}a_2 & b_2\\-a_2 & c_2\end{pmatrix} \tag{2.3.7.5}$$

Let $c \in F$ then,

$$cA_1 + A_2 = \begin{pmatrix}ca_1 + a_2 & cb_1 + b_2\\-ca_1 - a_2 & cc_1 + c_2\end{pmatrix} = \begin{pmatrix}u & v\\-u & w\end{pmatrix} \tag{2.3.7.6}$$

Thus $cA_1 + A_2 \in W_2$. **Hence W_2 is a subspace.**

b) The subspace W_1 can be given as,

$$\begin{pmatrix}x & -x\\y & z\end{pmatrix} = x\begin{pmatrix}1 & -1\\0 & 0\end{pmatrix} + y\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix} + z\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} \tag{2.3.7.7}$$

$$= xA_1 + yA_2 + zA_2 \tag{2.3.7.8}$$

Now,

$$x\begin{pmatrix}1 & -1\\0 & 0\end{pmatrix} + y\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix} + z\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} = \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix} \tag{2.3.7.9}$$

$$\implies x = y = z = 0 \tag{2.3.7.10}$$

A_1, A_2, A_3 are linearly independent and spans W_1. Thus $\{A_1, A_2, A_3\}$ forms basis for W_1.
$\therefore$ **dimension of W_1 is 3.**
The subspace W_2 can be given as,

$$\begin{pmatrix}a & b\\-a & c\end{pmatrix} = a\begin{pmatrix}1 & 0\\-1 & 0\end{pmatrix} + b\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} + c\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} \tag{2.3.7.11}$$

$$= aA_1 + bA_2 + cA_2 \tag{2.3.7.12}$$

Now,

$$a\begin{pmatrix}1 & 0\\-1 & 0\end{pmatrix} + b\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} + c\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} = \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix} \tag{2.3.7.13}$$

$$\implies a = b = c = 0 \tag{2.3.7.14}$$

A_1, A_2, A_3 are linearly independent and spans W_2. Thus $\{A_1, A_2, A_3\}$ forms basis for W_2.
$\therefore$ **dimension of W_2 is 3.**
Subspace $W_1 + W_2$ is given by,

$$\begin{pmatrix} x+a & -x+b \\ y-a & z+c \end{pmatrix} \tag{2.3.7.15}$$

For $x + a \neq -x + b \neq y - a \neq z + c$,

$$\begin{pmatrix} x+a & -x+b \\ y-a & z+c \end{pmatrix} = \begin{pmatrix} j & k \\ l & m \end{pmatrix} \tag{2.3.7.16}$$

$$= j\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + k\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + l\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + m\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.3.7.17}$$

$$= jA_1 + kA_2 + lA_3 + mA_4 \tag{2.3.7.18}$$

Now,

$$jA_1 + kA_2 + lA_3 + mA_4 = 0 \tag{2.3.7.19}$$

$$\implies j = k = l = m = 0 \tag{2.3.7.20}$$

A_1, A_2, A_3, A_4 are linearly independent and spans $W_1 + W_2$. Thus $\{A_1, A_2, A_3, A_4\}$ forms a basis.
$\therefore$ **dimension of $W_1 + W_2$ is 4.**
The subspace $W_1 \cap W_2$ is given as,

$$\begin{pmatrix} x & -x \\ -x & y \end{pmatrix} = x\begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix} + y\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.3.7.21}$$

$$= xA_1 + yA_2 \tag{2.3.7.22}$$

Now,

$$x\begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix} + y\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{2.3.7.23}$$

$$\implies x = y = 0 \tag{2.3.7.24}$$

A_1, A_2 are linearly independent and spans $W_1 \cap W_2$. Thus, $\{A_1, A_2\}$ forms a basis.
$\therefore$ **dimension of $W_1 \cap W_2$ is 2.**

2.3.8. Let $\mathbf{V}$ be the space of 2×2 matrices over $\mathbf{F}$. Find a basis $\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4\}$ for $\mathbf{V}$ such that $\mathbf{A}_j^2 = \mathbf{A}_j$ for each j
Solution: Every 2×2 matrix may be written as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.3.8.1}$$

This shows that

$$\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4\} = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\} \tag{2.3.8.2}$$

can be the basis for the space $\mathbf{V}$ of all 2×2 matrices. However $\mathbf{A}_2$ and $\mathbf{A}_3$ doesn't satisfy the property of $\mathbf{A}^2 = \mathbf{A}$. Consider b = 0 and c = 0, then the matrix

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \tag{2.3.8.3}$$

can't be a basis as it is the linear combination of $\mathbf{A}_1$ and $\mathbf{A}_4$. Hence either b or c or both must be non zero. Hence,

$$\mathbf{A}_2 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \tag{2.3.8.4}$$

$$\mathbf{A}_3 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \tag{2.3.8.5}$$

Here, $\mathbf{A}_2^2 = \mathbf{A}_2$ and $\mathbf{A}_3^2 = \mathbf{A}_3$. Therefore the basis can be

$$\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4\} = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\} \tag{2.3.8.6}$$

$\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4\}$ forms the basis, iff they are linearly independent and the linear combination of them span the space $\mathbf{V}$. To show that they are linearly independent, we show that the equation has a trivial solution.

$$a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{2.3.8.7}$$

$$\implies a + b = 0 \tag{2.3.8.8}$$

$$b = 0 \tag{2.3.8.9}$$

$$c = 0 \tag{2.3.8.10}$$

$$c + d = 0 \tag{2.3.8.11}$$

The corresponding matrix form is $\mathbf{Ax} = 0$

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{2.3.8.12}$$

Row reducing the augmented matrix,

$$\begin{pmatrix}1&1&0&0&0\\0&0&1&0&0\\0&1&0&0&0\\0&0&1&1&0\end{pmatrix}\xleftrightarrow[R_4\leftarrow R_4-R_3]{R_2\leftrightarrow R_3}\begin{pmatrix}1&1&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\end{pmatrix} \quad (2.3.8.13)$$

$$\xleftrightarrow{R_1\leftarrow R_1-R_2}\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\end{pmatrix} \quad (2.3.8.14)$$

Therefore, a = b = c = d = 0. Hence the matrices are linearly independent. To show that the linear combination of $\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4\}$ span the space $\mathbf{V}$, consider an arbitrary matrix,

$$\begin{pmatrix}w&x\\y&z\end{pmatrix} \quad (2.3.8.15)$$

Compute a, b, c, d such that

$$\begin{pmatrix}w&x\\y&z\end{pmatrix}=a\begin{pmatrix}1&0\\0&0\end{pmatrix}+b\begin{pmatrix}1&0\\1&0\end{pmatrix}+c\begin{pmatrix}0&1\\0&1\end{pmatrix}+d\begin{pmatrix}0&0\\0&1\end{pmatrix} \quad (2.3.8.16)$$

$$=\begin{pmatrix}a+b&c\\b&c+d\end{pmatrix} \quad (2.3.8.17)$$

Equating the entries, this produces system of linear equations,

$$a+b=w, y=b, x=c, z=c+d \quad (2.3.8.18)$$
$$\implies a=w-y \quad (2.3.8.19)$$
$$b=y \quad (2.3.8.20)$$
$$c=x \quad (2.3.8.21)$$
$$d=z-x \quad (2.3.8.22)$$

In particular, there exists atleast one solution regardless of the values of w, x, y, z. For example, consider the following matrix,

$$\begin{pmatrix}w&x\\y&z\end{pmatrix}=\begin{pmatrix}3&4\\-2&7\end{pmatrix} \quad (2.3.8.23)$$

Here, $a=5, b=-2, c=4, d=3$. Using (2.3.8.16), we get

$$5\begin{pmatrix}1&0\\0&0\end{pmatrix}-2\begin{pmatrix}1&0\\1&0\end{pmatrix}+4\begin{pmatrix}0&1\\0&1\end{pmatrix}+3\begin{pmatrix}0&0\\0&1\end{pmatrix}=\begin{pmatrix}3&4\\-2&7\end{pmatrix} \quad (2.3.8.24)$$

Hence $\left\{\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}1&0\\1&0\end{pmatrix},\begin{pmatrix}0&1\\0&1\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}$ forms the basis for the given space $\mathbf{V}$.

2.3.9. Let $\mathbf{V}$ be a vector space over a subfield $\mathbf{F}$ of complex numbers. Suppose α, β and γ are linearly independent vectors in $\mathbf{V}$. Prove that $(\alpha+\beta)$,$(\beta+\gamma)$ and $(\gamma+\alpha)$ are linearly independent.

Solution: Let α, β and γ be three n× 1 dimensional vectors. We need to prove that,

$$\begin{pmatrix}\alpha+\beta & \beta+\gamma & \gamma+\alpha\end{pmatrix}\mathbf{x}=0 \quad (2.3.9.1)$$

will only have a trivial solution.The above equation can be written as

$$\begin{pmatrix}\alpha & \beta & \gamma\end{pmatrix}\begin{pmatrix}1&0&1\\1&1&0\\0&1&1\end{pmatrix}\mathbf{x}=0 \quad (2.3.9.2)$$

$$\mathbf{x}^T\begin{pmatrix}1&1&0\\0&1&1\\1&0&1\end{pmatrix}\begin{pmatrix}\alpha^T\\\beta^T\\\gamma^T\end{pmatrix}=0 \quad (2.3.9.3)$$

Since, α, β and γ are independent.

$$\mathbf{x}^T\begin{pmatrix}1&1&0\\0&1&1\\1&0&1\end{pmatrix}=0 \quad (2.3.9.4)$$

In the above equation we can see that the 3×3 matrix has linearly independent rows and hence will have a trivial solution.So,$\mathbf{x}$ is a zero vector.Hence, $(\alpha+\beta)$,$(\beta+\gamma)$ and $(\gamma+\alpha)$ are linearly independent.

2.3.10. Prove that the space of all $\mathbf{m}x\mathbf{n}$ matrices over the field $\mathbf{F}$ has dimension mn, by exhibiting a basis for this space.

Solution: Let $\mathbf{M}$ be the space of all $\mathbf{m}\times\mathbf{n}$ matrices. Let, $\mathbf{M}_{ij}\in\mathbf{M}$ be,

$$\mathbf{M}_{ij}=\begin{cases}0 & m\neq i, n\neq j\\1 & m=i, n=j\end{cases} \quad (2.3.10.1)$$

For example,

$$\mathbf{M}_{12}=\begin{pmatrix}0&1&0&\cdots&0\\0&0&0&\cdots&0\\\vdots&\vdots&\vdots&\vdots&\vdots\\0&0&0&\cdots&0\end{pmatrix}_{mxn} \quad (2.3.10.2)$$

$$(2.3.10.3)$$

Let $\mathbf{A} \in \mathbf{M}$ given as,

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}_{mxn} \tag{2.3.10.4}$$

Now clearly,

$$\mathbf{a}_{11} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \tag{2.3.10.5}$$

$$\implies \mathbf{a}_{11} = \mathbf{A}\mathbf{M}_{11} \tag{2.3.10.6}$$

$$\therefore \mathbf{A} = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} M_{ij} \tag{2.3.10.7}$$

$\implies$ $\mathbf{M}_{ij}$ span $\mathbf{M}$. Also from the above equation $\mathbf{A}$= 0 if and only if all elements are zero, that is,

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \tag{2.3.10.8}$$

$$\implies a_{ij} = 0 \tag{2.3.10.9}$$

Hence, $\mathbf{M}_{ij}$ are linearly independent as well. Hence, $\mathbf{M}_{ij}$ constitutes a basis for $\mathbf{M}$. and number of elememts in basis are mn. Hence dimension of space of all mxn matrices $\mathbf{M}$ is mn.

2.3.11. Let V be a vector space over the field $F = \{0, 1\}$. Suppose α, β and γ are linearly independent vectors in V. Comment on $(\alpha+\beta)$, $(\beta+\gamma)$ and $(\gamma+\alpha)$

Solution: The addition of elements in the field $\mathbf{F}$ is defined as,

$$\begin{aligned} 0 + 0 &= 0 \\ 1 + 1 &= 0 \end{aligned} \tag{2.3.11.1}$$

A set are vectors $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ are linearly independent if

$$a\mathbf{v_1} + b\mathbf{v_2} + c\mathbf{v_3} = 0 \tag{2.3.11.2}$$

has only one trivial solution

$$a = b = c = 0 \tag{2.3.11.3}$$

Now,

$$a(\alpha+\beta) + b(\beta+\gamma) + c(\gamma+\alpha) = 0 \tag{2.3.11.4}$$

$$\implies (a+c)\alpha + (a+b)\beta + (b+c)\gamma = 0 \tag{2.3.11.5}$$

Writing (2.3.11.5) in matrix form,

$$\begin{pmatrix} \alpha & \beta & \gamma \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{2.3.11.6}$$

where,

$$\mathbf{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\mathbf{x}^T \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = 0 \tag{2.3.11.7}$$

Since α, β and γ are linearly independent vectors,

$$\mathbf{x}^T \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \tag{2.3.11.8}$$

Transposing on both sides,

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{2.3.11.9}$$

By using the properties from (2.3.11.1) and reducing (2.3.11.9) to row echelon form,

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_1 + R_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_2 + R_3} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.3.11.10}$$

Expressing (2.3.11.10) as a linear combination of vectors,

$$a\begin{pmatrix}1\\0\\0\end{pmatrix}+b\begin{pmatrix}0\\1\\0\end{pmatrix}+c\begin{pmatrix}1\\1\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\implies \begin{pmatrix}a+c\\b+c\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\implies a+c=0;\quad b+c=0 \qquad (2.3.11.11)$$

The solutions to (2.3.11.11) are,

$$a=b=c=0;\quad a=b=c=1 \qquad (2.3.11.12)$$

Since there is no trivial solution, $(\alpha+\beta)$, $(\beta+\gamma)$ and $(\gamma+\alpha)$ are linearly dependent

2.3.12. Let **V** be the set of real numbers.Regard **V** as a vector space over the field of rational numbers,with usual operations.Prove that this vector space is not finite-dimensional.

Solution: Given **V** is a vector space over field $\mathbb{Q}$ (rational numbers)
It is finite dimensional with dimensionality n if every vector **v** in **V** can be written as

$$\mathbf{v}=\sum_{i=0}^{n-1}c_i\alpha_i \qquad (2.3.12.1)$$

$$\text{where } c_i\in\mathbb{Q} \qquad (2.3.12.2)$$

$$\text{and } \mathbf{B}=\{\alpha_0,\alpha_1,\dots,\alpha_{n-1}\} \qquad (2.3.12.3)$$

is the basis with linearly independent α_i
that is, basis is the largest set with linearly independent vectors from **V**
Consider the set of vectors $\{1,x\}$, where x is irrational.
Assume there exists non zero $\beta_0,\beta_1\in\mathbb{Q}$ such that

$$\beta_0+\beta_1x=0 \qquad (2.3.12.4)$$

$$\implies x=-\frac{\beta_0}{\beta_1} \qquad (2.3.12.5)$$

But x is irrational and $-\frac{\beta_0}{\beta_1}$ is rational so (2.3.12.5) can't be possible so $\beta_0,\beta_1=0$
Hence $\{1,x\}$ are independent.
Similarly for the set $\left\{1,x,x^2\right\}$
for $\beta_0,\beta_1,\beta_2\in\mathbb{Q}$

$$\beta_0+\beta_1x+\beta_2x^2=0 \qquad (2.3.12.6)$$

$\beta_1x+\beta_2x^2$ is irrational and β_0 is rational. Therefore

$$\beta_0=0 \qquad (2.3.12.7)$$

$$\text{and } \beta_1x+\beta_2x^2=0,(x\neq 0) \qquad (2.3.12.8)$$

$$\implies \beta_1+\beta_2x=0 \qquad (2.3.12.9)$$

$$\implies \beta_1,\beta_2=0 \qquad (2.3.12.10)$$

$$\therefore \beta_0+\beta_1x+\beta_2x^2=0 \qquad (2.3.12.11)$$

$$\iff \beta_0,\beta_1,\beta_2=0 \qquad (2.3.12.12)$$

Hence $\left\{1,x,x^2\right\}$ are independent
By induction, let us say the set $\left\{1,x,x^2,\dots,x^n\right\}$ is independent

$$\text{for } \beta_0,\beta_1,\beta_2,\dots,\beta_n\in\mathbb{Q} \qquad (2.3.12.13)$$

$$\beta_0+\beta_1x+\beta_2x^2+\dots+\beta_nx^n=0 \qquad (2.3.12.14)$$

$$\iff \beta_0,\beta_1,\beta_2,\dots,\beta_n=0 \qquad (2.3.12.15)$$

To prove this for the set **A**=$\left\{1,x,x^2,\dots,x^{n+1}\right\}$

$$\text{for } \beta_0,\beta_1,\beta_2,\dots,\beta_n,\beta_{n+1}\in\mathbb{Q} \qquad (2.3.12.16)$$

$$\beta_0+\beta_1x+\dots+\beta_nx^n+\beta_{n+1}x^{n+1}=0 \qquad (2.3.12.17)$$

Comparing to (2.3.12.7) and (2.3.12.8)

$$\beta_0=0 \qquad (2.3.12.18)$$

$$\beta_1+\beta_2x+\dots+\beta_{n+1}x^n=0 \qquad (2.3.12.19)$$

Comparing with (2.3.12.14),we have $\beta_1,\beta_2,\dots,\beta_{n+1}=0$

$$\therefore \beta_0+\beta_1x+\dots+\beta_nx^n+\beta_{n+1}x^{n+1}=0 \qquad (2.3.12.20)$$

$$\iff \beta_0,\beta_1,\beta_2,\dots,\beta_n,\beta_{n+1}=0 \qquad (2.3.12.21)$$

Hence **A** has linearly independent vectors
Let the set $\mathbf{B}=\left\{1,x,x^2,\dots,x^m\right\}$ be the largest linearly independent set in **V** and hence can form the basis leading to dimensionality $m+1$
But from induction, we have proved that $\left\{1,x,x^2,\dots,x^m,x^{m+1}\right\}$ is also independent which is a contradiction to dimensionality being $m+1$
Hence we deduce that the vector space **V** is not finite dimensional over the field $\mathbb{Q}$

2.4.1. Show that the vectors

$$\alpha_{\mathbf{1}} = \begin{pmatrix}1 & 1 & 0 & 0\end{pmatrix} \quad \alpha_{\mathbf{2}} = \begin{pmatrix}0 & 0 & 1 & 1\end{pmatrix} \tag{2.4.1.1}$$

$$\alpha_{\mathbf{3}} = \begin{pmatrix}1 & 0 & 0 & 4\end{pmatrix} \quad \alpha_{\mathbf{4}} = \begin{pmatrix}0 & 0 & 0 & 2\end{pmatrix} \tag{2.4.1.2}$$

form a basis for $\mathfrak{R}^4$. Find the coordinates of each of the standard basis vectors in the ordered basis $\begin{pmatrix}\alpha_{\mathbf{1}} & \alpha_{\mathbf{2}} & \alpha_{\mathbf{3}} & \alpha_{\mathbf{4}}\end{pmatrix}$

Solution:

Theorem 2.2. *Let* **V** *be an n-dimensional vector space over the field* **F**, *and let* β *and* β' *be two ordered basis of* **V**. *Then, there is a unique, necessarily invertible,* $n \times n$ *matrix* **P** *with entries in* **F** *such that*

a) $\left[\alpha\right]_{\beta} = \mathbf{P}\left[\alpha\right]_{\beta'}$

b) $\left[\alpha\right]_{\beta'} = \mathbf{P^{-1}}\left[\alpha\right]_{\beta}$

for every vector α *in* **V**. *The columns of* **P** *are given by*

$$\mathbf{P_j} = \left[\alpha_{\mathbf{j}}\right]_{\beta} \qquad j = 1, 2, ..., n \tag{2.4.1.3}$$

Firt, we need to show that the set of vectors α_1, α_2, α_3 and α_4 are basis for $\mathfrak{R}^4$. For, this we first show that α_1, α_2, α_3 and α_4 are linearly independent in $\mathfrak{R}^4$ and also they span $\mathfrak{R}^4$. Consider,

$$\mathbf{A} = \begin{pmatrix}\alpha_{\mathbf{1}}^T & \alpha_{\mathbf{2}}^T & \alpha_{\mathbf{3}}^T & \alpha_{\mathbf{4}}^T\end{pmatrix} \tag{2.4.1.4}$$

$$\mathbf{A} = \begin{pmatrix}1 & 0 & 1 & 0\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 1 & 4 & 2\end{pmatrix} \tag{2.4.1.5}$$

Now,

$$\begin{pmatrix}1 & 0 & 1 & 0\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 1 & 4 & 2\end{pmatrix} \xleftrightarrow{r_2 = r_2 - r_1} \begin{pmatrix}1 & 0 & 1 & 0\\ 0 & 0 & -1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 1 & 4 & 2\end{pmatrix} \tag{2.4.1.6}$$

$$\begin{pmatrix}1 & 0 & 1 & 0\\ 0 & 0 & -1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 1 & 4 & 2\end{pmatrix} \xleftrightarrow{r_2 \leftrightarrow r_3} \begin{pmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 1 & 4 & 2\end{pmatrix} \tag{2.4.1.7}$$

$$\begin{pmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 1 & 4 & 2\end{pmatrix} \xleftrightarrow{r_4 = r_4 - r_2} \begin{pmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 4 & 2\end{pmatrix} \tag{2.4.1.8}$$

$$\begin{pmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 4 & 2\end{pmatrix} \xleftrightarrow{r_3 = -r_3} \begin{pmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 4 & 2\end{pmatrix} \tag{2.4.1.9}$$

$$\begin{pmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 4 & 2\end{pmatrix} \xleftrightarrow{r_4 = r_4 - 4r_3} \begin{pmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 2\end{pmatrix} \tag{2.4.1.10}$$

$$\begin{pmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 2\end{pmatrix} \xleftrightarrow{r_1 = r_1 - r_3} \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 2\end{pmatrix} \tag{2.4.1.11}$$

$$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 2\end{pmatrix} \xleftrightarrow{r_4 = \frac{r_4}{2}} \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} \tag{2.4.1.12}$$

(2.4.1.12) is the row reduced echelon form of **A** and since it is identity matrix of order 4, we say that vectors $\alpha_{\mathbf{1}}$, $\alpha_{\mathbf{2}}$, $\alpha_{\mathbf{3}}$ and $\alpha_{\mathbf{4}}$ are linearly independent and their column space is $\mathfrak{R}^4$ which means vectors $\alpha_{\mathbf{1}}$, $\alpha_{\mathbf{2}}$, $\alpha_{\mathbf{3}}$ and $\alpha_{\mathbf{4}}$ span $\mathfrak{R}^4$. Hence, vectors $\alpha_{\mathbf{1}}$, $\alpha_{\mathbf{2}}$, $\alpha_{\mathbf{3}}$ and $\alpha_{\mathbf{4}}$ form a basis for $\mathfrak{R}^4$.

Now, we use theorem (2.2), and if we calculate the inverse of

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 2 \end{pmatrix} \tag{2.4.1.13}$$

then the columns of $\mathbf{A^{-1}}$ will give the coeffients to write the standard basis vectors in terms of $\alpha_i' s$. We try to find the inverse of $\mathbf{A}$ by row-reducing the augumented matrix.

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 4 & 2 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.4.1.14}$$

Now, we solve for $\mathbf{A^{-1}}$ as follows

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 4 & 2 & 0 & 0 & 0 & 1 \end{pmatrix} \xleftrightarrow{r_2 = r_2 - r_1} \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 4 & 2 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.4.1.15}$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 4 & 2 & 0 & 0 & 0 & 1 \end{pmatrix} \xleftrightarrow{r_2 \leftrightarrow r_3} \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 4 & 2 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.4.1.16}$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 4 & 2 & 0 & 0 & 0 & 1 \end{pmatrix} \xleftrightarrow{r_4 = r_4 - r_2} \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 & 0 & -1 & 1 \end{pmatrix} \tag{2.4.1.17}$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 & 0 & -1 & 1 \end{pmatrix} \xleftrightarrow{r_3 = -r_3} \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 & 0 & -1 & 1 \end{pmatrix} \tag{2.4.1.18}$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 & 0 & -1 & 1 \end{pmatrix} \xleftrightarrow{r_4 = r_4 - 4r_3} \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & -4 & 4 & -1 & 1 \end{pmatrix} \tag{2.4.1.19}$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & -4 & 4 & -1 & 1 \end{pmatrix} \xleftrightarrow{r_1 = r_1 - r_3} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & -4 & 4 & -1 & 1 \end{pmatrix} \tag{2.4.1.20}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & -4 & 4 & -1 & 1 \end{pmatrix} \xleftrightarrow{r_4 = \frac{r_4}{2}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 2 & \frac{-1}{2} & \frac{1}{2} \end{pmatrix} \tag{2.4.1.21}$$

Thus, by (2.4.1.21), we have

$$\mathbf{A^{-1}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ -2 & 2 & \frac{-1}{2} & \frac{1}{2} \end{pmatrix} \tag{2.4.1.22}$$

Now, let $\mathbf{e_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}$, $\mathbf{e_2} = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}$, $\mathbf{e_3} = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}$ and $\mathbf{e_4} = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}$ be the standard basis for $\mathfrak{R}^4$. Hence,

$$\mathbf{e_1} = \alpha_3 - 2\alpha_4 \tag{2.4.1.23}$$
$$\mathbf{e_2} = \alpha_1 - \alpha_3 + 2\alpha_4 \tag{2.4.1.24}$$
$$\mathbf{e_3} = \alpha_2 - \frac{1}{2}\alpha_4 \tag{2.4.1.25}$$
$$\mathbf{e_4} = \frac{1}{2}\alpha_4 \tag{2.4.1.26}$$

2.4.2. Find the coordinate matrix of the vector $\begin{pmatrix}1 & 0 & 1\end{pmatrix}$ in the basis of C^3 consisting of the vectors $\begin{pmatrix}2i & 1 & 0\end{pmatrix}$, $\begin{pmatrix}2 & -1 & 1\end{pmatrix}$, $\begin{pmatrix}0 & 1+i & 1-i\end{pmatrix}$ in that order.

Solution:

$$\begin{pmatrix}1 & 0 & 1\end{pmatrix} = \alpha_1\begin{pmatrix}2i & 1 & 0\end{pmatrix} + \alpha_2\begin{pmatrix}2 & -1 & 1\end{pmatrix} + \alpha_3\begin{pmatrix}0 & 1+i & 1-i\end{pmatrix} \tag{2.4.2.1}$$

$$\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}2i & 2 & 0\\ 1 & -1 & 1+i\\ 0 & 1 & 1-i\end{pmatrix}\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix} \tag{2.4.2.2}$$

Now we find α_i by row reducing augmented matrix.

$$\begin{pmatrix}2i & 2 & 0 & 1\\ 1 & -1 & 1+i & 0\\ 0 & 1 & 1-i & 1\end{pmatrix} \xleftrightarrow[R_2\to R_2-2iR_3]{R_1\to R_2} \begin{pmatrix}1 & -1 & 1+i & 0\\ 0 & 2+2i & 2-2i & 1\\ 0 & 1 & 1-i & 1\end{pmatrix} \tag{2.4.2.3}$$

$$\xleftrightarrow[R_3\leftarrow R_3-R_2]{R_2\leftarrow R_2/2} \begin{pmatrix}1 & -1 & 1+i & 0\\ 0 & 1+i & 1-i & \frac{1}{2}\\ 0 & -i & 0 & \frac{1}{2}\end{pmatrix} \tag{2.4.2.4}$$

Therefore the coordinate matrix of the vector is

$$\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix} = \begin{pmatrix}\frac{-1-i}{2}\\ \frac{i}{2}\\ \frac{3+i}{4}\end{pmatrix} \tag{2.4.2.5}$$

2.4.3. Let $\mathbf{B} = \begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_3\end{pmatrix}$ be the ordered basis for R^3 consisting of

$\alpha_1 = \begin{pmatrix}1 & 0 & -1\end{pmatrix}, \alpha_2 = \begin{pmatrix}1 & 1 & 1\end{pmatrix}, \alpha_3 = \begin{pmatrix}1 & 0 & 0\end{pmatrix}.$

What are the coordinates of vector $\begin{pmatrix}a & b & c\end{pmatrix}$ in the ordered basis $\mathbf{B}$?

Solution: Given

$$\mathbf{B} = \begin{pmatrix}1 & 1 & 1\\ 0 & 1 & 0\\ -1 & 1 & 0\end{pmatrix} \tag{2.4.3.1}$$

be the ordered basis for R^3,then the coordinates of vector,

$$\alpha = \begin{pmatrix}a\\b\\c\end{pmatrix} \tag{2.4.3.2}$$

in the ordered basis R^3 is the vector,

$$[\alpha]_{\mathbf{B}} = \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} \tag{2.4.3.3}$$

hence

$$x_1\alpha_1 + x_2\alpha_2 + x_3\alpha_3 = \alpha \tag{2.4.3.4}$$

substituting (2.4.3.1) and (2.4.3.2) in (2.4.3.4)

$$\begin{pmatrix}1 & 1 & 1\\ 0 & 1 & 0\\ -1 & 1 & 0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}a\\b\\c\end{pmatrix} \tag{2.4.3.5}$$

augmented matrix form

$$\begin{pmatrix}1 & 1 & 1 & a\\ 0 & 1 & 0 & b\\ -1 & 1 & 0 & c\end{pmatrix} \tag{2.4.3.6}$$

converting above matrix into row reduced echelon form

$$\begin{pmatrix}1 & 1 & 1 & a\\ 0 & 1 & 0 & b\\ -1 & 1 & 0 & c\end{pmatrix} \xleftrightarrow{R_3\leftarrow R_3+R_1} \begin{pmatrix}1 & 1 & 1 & a\\ 0 & 1 & 0 & b\\ 0 & 2 & 1 & c+a\end{pmatrix} \tag{2.4.3.7}$$

$$\xleftrightarrow{R_3\leftarrow R_3-2R_2} \begin{pmatrix}1 & 1 & 1 & a\\ 0 & 1 & 0 & b\\ 0 & 0 & 1 & a-2b+c\end{pmatrix} \tag{2.4.3.8}$$

$$\xleftrightarrow{R_1\leftarrow R_1-R_2} \begin{pmatrix}1 & 0 & 1 & a-b\\ 0 & 1 & 0 & b\\ 0 & 0 & 1 & a-2b+c\end{pmatrix} \tag{2.4.3.9}$$

$$\xleftrightarrow{R_1\leftarrow R_1-R_3} \begin{pmatrix}1 & 0 & 0 & b-c\\ 0 & 1 & 0 & b\\ 0 & 0 & 1 & a-2b+c\end{pmatrix} \tag{2.4.3.10}$$

$\therefore$ The coordinates of α w.r.t $\mathbf{B}$ is

$$[\alpha]_{\mathbf{B}} = \begin{pmatrix} b-c \\ b \\ a-2b+c \end{pmatrix} \tag{2.4.3.11}$$

2.4.4. Let $\mathbf{W}$ be the subspace of $\mathbf{C}^3$ spanned by $\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}$ and $\alpha_2 = \begin{pmatrix} 1 \\ i \\ 1+i \end{pmatrix}$.

a) Show that α_1 and α_2 form a basis for $\mathbf{W}$.

b) Show that the vectors $\beta_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\beta_2 = \begin{pmatrix} 1 \\ i \\ 1+i \end{pmatrix}$ are in $\mathbf{W}$ and form another basis for $\mathbf{W}$.

c) What are the coordinates of α_1 and α_2 in the ordered basis $\{\beta_1, \beta_2\}$ for $\mathbf{W}$.

Solution:

a) It is given that α_1 and α_2 span $\mathbf{W}$. For α_1 and α_2 to be the basis for $\mathbf{W}$ they must be linearly independent. Let

$$S_1 = \{\alpha_1, \alpha_2\} = \left\{ \begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}, \begin{pmatrix} 1+i \\ 1 \\ -1 \end{pmatrix} \right\} \tag{2.4.4.1}$$

Using row reduction on matrix $\mathbf{A} = \begin{pmatrix} \alpha_1 & \alpha_2 \end{pmatrix}$

$$\begin{pmatrix} 1 & 1+i \\ 0 & 1 \\ i & -1 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 - iR_1} \begin{pmatrix} 1 & 1+i \\ 0 & 1 \\ 0 & -i \end{pmatrix} \tag{2.4.4.2}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 + iR_2} \begin{pmatrix} 1 & 1+i \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{2.4.4.3}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - (i+1)R_2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{2.4.4.4}$$

Since $\mathbf{A}$ is a full-rank matrix the column vectors are linearly independent. Therefore $S_1 = \{\alpha_1, \alpha_2\}$ is a basis set for $\mathbf{W}$.

b)

$$\beta_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \tag{2.4.4.5}$$

$$\beta_2 = \begin{pmatrix} 1 \\ i \\ 1+i \end{pmatrix} \tag{2.4.4.6}$$

Since column vectors of $\mathbf{A} = \begin{pmatrix} \alpha_1 & \alpha_2 \end{pmatrix}$ are basis for $\mathbf{W}$ and if β_1 and $\beta_2 \in \mathbf{W}$ there exist a unique solution $\mathbf{x}$ such that

$$\begin{pmatrix} \alpha_1 & \alpha_2 \end{pmatrix} \mathbf{x} = \begin{pmatrix} \beta_1 & \beta_2 \end{pmatrix} \tag{2.4.4.7}$$

Using row reduction on augmented matrix

$$\begin{pmatrix} 1 & 1+i & | & 1 & 1 \\ 0 & 1 & | & 1 & i \\ i & -1 & | & 0 & 1+i \end{pmatrix} \tag{2.4.4.8}$$

$$\xleftrightarrow{R3 \leftarrow R_3 - iR-1} \begin{pmatrix} 1 & 1+i & | & 1 & 1 \\ 0 & 1 & | & 1 & i \\ 0 & -i & | & -i & 1 \end{pmatrix} \tag{2.4.4.9}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 + iR_2} \begin{pmatrix} 1 & 1+i & | & 1 & 1 \\ 0 & 1 & | & 1 & i \\ 0 & 0 & | & 0 & 0 \end{pmatrix} \tag{2.4.4.10}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - (i+1)R_2} \begin{pmatrix} 1 & 0 & | & -i & 2-i \\ 0 & 1 & | & 1 & i \\ 0 & 0 & | & 0 & 0 \end{pmatrix} \tag{2.4.4.11}$$

$$\implies \mathbf{x} = \begin{pmatrix} -i & 2-i \\ 1 & i \end{pmatrix} \tag{2.4.4.12}$$

Therefore β_1 and $\beta_2 \in \mathbf{W}$.
Consider

$$S_2 = \{\beta_1, \beta_2\} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ i \\ 1+i \end{pmatrix} \right\} \tag{2.4.4.13}$$

and also let

$$\mathbf{B} = \begin{pmatrix} 1 & 1 \\ 1 & i \\ 0 & 1+i \end{pmatrix} \tag{2.4.4.14}$$

Using row reduction on matrix $\mathbf{B}$

$$\begin{pmatrix}1 & 1\\ 1 & i\\ 0 & 1+i\end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix}1 & 1\\ 0 & i-1\\ 0 & 1+i\end{pmatrix} \quad (2.4.4.15)$$

$$\xleftrightarrow{R_2 \leftarrow \frac{R_2}{i-1}} \begin{pmatrix}1 & 1\\ 0 & 1\\ 0 & 1+i\end{pmatrix} \quad (2.4.4.16)$$

$$\xleftrightarrow[R_3 \leftarrow R_3 - (i-1)R_2]{R_1 \leftarrow R_1 - R_2} \begin{pmatrix}1 & 0\\ 0 & 1\\ 0 & 0\end{pmatrix} \quad (2.4.4.17)$$

Since $\mathbf{B}$ is a full rank matrix the column vectors are linearly independent.
Let α be any vector in the subspace $\mathbf{W}$, then it can be expressed as span $\{\alpha_1, \alpha_2\}$ i.e

$$\alpha = \begin{pmatrix}\alpha_1 & \alpha_2\end{pmatrix} \mathbf{x_1} = \mathbf{A}\mathbf{x_1} \quad (2.4.4.18)$$

$S_2 = \{\beta_1, \beta_2\}$ spans $\mathbf{W}$ if any vector $\alpha \in \mathbf{W}$ can be expressed as

$$\alpha = \begin{pmatrix}\beta_1, \beta_2\end{pmatrix} \mathbf{x_2} = \mathbf{B}\mathbf{x_2} \quad (2.4.4.19)$$

From (2.4.4.18) and (2.4.4.19) we conclude

$$\mathbf{B}\mathbf{x_2} = \mathbf{A}\mathbf{x_1} \quad (2.4.4.20)$$
$$\implies \mathbf{x_2} = \mathbf{B}^{-1}\mathbf{A}\mathbf{x_1} \quad (2.4.4.21)$$

Therefore from (2.4.4.21) $\mathbf{x_2}$ exists if $\mathbf{B}$ is invertible. From (2.4.4.17) we conclude $\mathbf{x_2}$ exists and hence any vector $\alpha \in \mathbf{W}$ can be expressed as span$\{\beta_1, \beta_2\}$. Therefore $\{\beta_1, \beta_2\}$ is basis for $\mathbf{W}$.

c) Since $\alpha_1, \alpha_2 \in \mathbf{W}$ and $\{\beta_1, \beta_2\}$ are ordered basis for $\mathbf{W}$ there must exist unique value of $\mathbf{x}$ such that

$$\begin{pmatrix}\beta_1 & \beta_2\end{pmatrix} \mathbf{x} = \begin{pmatrix}\alpha_1 & \alpha_2\end{pmatrix} \quad (2.4.4.22)$$

Using row reduction on (2.4.4.22) we get,

$$\left(\begin{array}{cc|cc}1 & 1 & 1 & 1+i\\ 1 & i & 0 & 1\\ 0 & 1+i & i & -1\end{array}\right) \quad (2.4.4.23)$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - R_1} \left(\begin{array}{cc|cc}1 & 1 & 1 & 1+i\\ 0 & i-1 & -1 & -i\\ 0 & 1+i & i & -1\end{array}\right) \quad (2.4.4.24)$$

$$\xleftrightarrow{R_2 \leftarrow \frac{R_2}{i-1}} \left(\begin{array}{cc|cc}1 & 1 & 1 & 1+i\\ 0 & 1 & \frac{1+i}{2} & \frac{-1+i}{2}\\ 0 & 1+i & i & -1\end{array}\right) \quad (2.4.4.25)$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - (i+1)R_2} \left(\begin{array}{cc|cc}1 & 1 & 1 & 1+i\\ 0 & 1 & \frac{1+i}{2} & \frac{-1+i}{2}\\ 0 & 0 & 0 & 0\end{array}\right) \quad (2.4.4.26)$$

$$\xleftrightarrow{R_1 \leftarrow R_2 - R_1} \left(\begin{array}{cc|cc}1 & 0 & \frac{1-i}{2} & \frac{3+i}{2}\\ 0 & 1 & \frac{1+i}{2} & \frac{-1+i}{2}\\ 0 & 0 & 0 & 0\end{array}\right) \quad (2.4.4.27)$$

$$\implies \mathbf{x} = \frac{1}{2}\begin{pmatrix}1-i & 3+i\\ 1+i & -1+i\end{pmatrix} \quad (2.4.4.28)$$

Thus the column vectors of (2.4.4.28) are corresponding coordinates of α_1 and α_2 in ordered basis $\{\beta_1, \beta_2\}$.

2.4.5. let $\alpha = \begin{pmatrix}x_1\\ x_2\end{pmatrix}$ and $\beta = \begin{pmatrix}y_1\\ y_2\end{pmatrix}$ be vectors in $\mathbb{R}^2$ such that
$x_1y_1 + x_2y_2 = 0; \quad x_1^2 + x_2^2 = y_1^2 + y_2^2 = 1.$
Proove that ß $= \{\alpha, \beta\}$ is a basis of $\mathbb{R}^2$. Find the coordinates of the vector $\begin{pmatrix}a, b\end{pmatrix}$ in the ordered basis ß $= \{\alpha, \beta\}$. (The conditions on α and β say, geometrically, that α and β are perpendicular and each has length 1).
Solution: we need to show that α and β are linearly independent in order to proove that ß $= \{\alpha, \beta\}$ is a basis of $\mathbb{R}^2$.

Given in the question are:

$$\alpha^T \beta = 0 \quad (2.4.5.1)$$
$$\|\alpha\|^2 = \|\beta\|^2 = 1 \quad (2.4.5.2)$$

Let,

$$\mathbf{A} = \begin{pmatrix}\alpha & \beta\end{pmatrix} = \begin{pmatrix}x_1 & y_1 \\ x_2 & y_2\end{pmatrix} \tag{2.4.5.3}$$

then,

$$\mathbf{A}^T\mathbf{A} = \begin{pmatrix}\|\alpha\|^2 & \alpha^T\beta \\ \alpha^T\beta & \|\beta\|^2\end{pmatrix} \tag{2.4.5.4}$$

$$= \begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix} \tag{2.4.5.5}$$

$$\therefore \mathbf{A}^T\mathbf{A} = \mathbf{I} \tag{2.4.5.6}$$

Inverse of $\mathbf{A}$ exist. $\mathbf{A}^T$ is the inverse of $\mathbf{A}$. Thus, the columns of $\mathbf{A}$ are linearly independent i.e, α and β are linearly independent.
Hence, ß = $\{\alpha, \beta\}$ is a basis of $\mathbb{R}^2$.

To, find the coordinates of the vector $\begin{pmatrix}a, b\end{pmatrix}$ in the ordered basis ß = $\{\alpha, \beta\}$.

$$\begin{pmatrix}\alpha & \beta\end{pmatrix}\begin{pmatrix}c_1 \\ c_2\end{pmatrix} = \begin{pmatrix}a \\ b\end{pmatrix} \tag{2.4.5.7}$$

$$\mathbf{A}^T\mathbf{A}\begin{pmatrix}c_1 \\ c_2\end{pmatrix} = \mathbf{A}^T\begin{pmatrix}a \\ b\end{pmatrix} \tag{2.4.5.8}$$

$$\begin{pmatrix}c_1 \\ c_2\end{pmatrix} = \mathbf{A}^T\begin{pmatrix}a \\ b\end{pmatrix} \tag{2.4.5.9}$$

$$\begin{pmatrix}c_1 \\ c_2\end{pmatrix} = \begin{pmatrix}x_1 & x_2 \\ y_1 & y_2\end{pmatrix}\begin{pmatrix}a \\ b\end{pmatrix} \tag{2.4.5.10}$$

2.4.6. Let $\mathbf{V}$ be the vector space over the complex numbers of all functions from $\mathbf{R}$ into $\mathbf{C}$, i.e. the space of all complex-valued functions on the real line. Let $f_1(x) = 1$, $f_2(x) = e^{ix}$, $f_3(x) = e^{-ix}$.

(a) Prove that f_1, f_2, and f_3 are linearly dependent.

(b) Let $g_1(x) = 1$, $g_2(x) = \cos(x)$, $g_3(x) = \sin(x)$. Find an invertible 3×3 matrix P such that $g_i = \sum_{i=1}^{3} P_{ii} f_i$

Solution:

(a) To check for independence, lets represent the function in a polynomial format as

$$\alpha f_1 + \beta f_2 + \gamma f_3 = 0 \tag{2.4.6.1}$$

$$\alpha + \beta e^{ix} + \gamma e^{-ix} = 0 \tag{2.4.6.2}$$

Multiply the whole equation with e^{ix}

$$\beta(e^{ix})^2 + \alpha e^{ix} + \gamma = 0 \tag{2.4.6.3}$$

Let $y = e^{ix}$

$$\beta y^2 + \alpha y + \gamma = 0 \tag{2.4.6.4}$$

The above quadratic polynomial in y can be zero for atmost two values of y. But,

$$y = e^{ix} \quad x \in \mathbf{R} \tag{2.4.6.5}$$

So (2.4.6.4) cannot be zero for all $y = e^{ix}$. Which implies there is a contradiction.

Then, the only case when (2.4.6.4) gets satisfied is

$$\alpha = \beta = \gamma = 0 \tag{2.4.6.6}$$

Therefore, f_1, f_2, f_3 are linearly independent.

(b) We need to find the coordinates of vectors g_i where $i = 1, 2, 3$ in ordered basis

$$B = \begin{pmatrix}f_1 & f_2 & f_3\end{pmatrix} \tag{2.4.6.7}$$

It is given that $g_1 = 1$, which can be written as

$$g_1 = f_1 \tag{2.4.6.8}$$

$$\implies g_1 = \begin{pmatrix}f_1 & f_2 & f_3\end{pmatrix}\begin{pmatrix}1 \\ 0 \\ 0\end{pmatrix} \tag{2.4.6.9}$$

We can use the following identities:-

$$\cos(x) = \frac{1}{2}e^{ix} + \frac{1}{2}e^{-ix} \tag{2.4.6.10}$$

$$\sin(x) = \frac{1}{2i}e^{ix} - \frac{1}{2i}e^{-ix} \tag{2.4.6.11}$$

Comparing equations (2.4.6.10) and (2.4.6.11) with f_2, f_3, we can write g_2 and g_3 as

$$g_2 = \frac{1}{2}f_2 + \frac{1}{2}f_3 \tag{2.4.6.12}$$

$$\implies g_2 = \begin{pmatrix}f_1 & f_2 & f_3\end{pmatrix}\begin{pmatrix}0 \\ \frac{1}{2} \\ \frac{1}{2}\end{pmatrix} \tag{2.4.6.13}$$

$$g_3 = \frac{1}{2i}f_2 - \frac{1}{2i}f_3 \tag{2.4.6.14}$$

$$\implies g_3 = \begin{pmatrix} f_1 & f_2 & f_3 \end{pmatrix}\begin{pmatrix} 0 \\ \frac{1}{2i} \\ \frac{-1}{2i} \end{pmatrix} \tag{2.4.6.15}$$

Therefore, the required matrix P is

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2i} \\ 0 & \frac{1}{2} & \frac{-1}{2i} \end{pmatrix} \tag{2.4.6.16}$$

2.4.7. Let **V** be the real vector space of all polynomial functions from $\mathbb{R}$ to $\mathbb{R}$ of degree 2 or less, i.e, the space of all functions f of the form,

$$f(x) = c_0 + c_1 x + c_2 x^2$$

Let t be a fixed real number and define

$$g_1(x) = 1, g_2(x) = x + t, g_3(x) = (x + t)^2$$

Prove that β={g1,g2,g3} is a basis for V. If

$$f(x) = c_0 + c_1 x + c_2 x^2$$

what are the coordinates of f in the ordered basis β

Solution: We start by taking,

$$\mathbf{f} = \begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix} \tag{2.4.7.1}$$

Let's start by proving that **g** is linearly independent.

$$\mathbf{g} = \mathbf{Bf} \tag{2.4.7.2}$$

where,

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ t^2 & 2t & 1 \end{pmatrix} \tag{2.4.7.3}$$

Now,

$$\mathbf{v}^T\mathbf{g} = 0 \tag{2.4.7.4}$$

$$\implies \mathbf{v}^T\mathbf{Bf} = 0 \tag{2.4.7.5}$$

Since **f** is linearly independent,

$$\mathbf{v}^T\mathbf{B} = 0 \tag{2.4.7.6}$$

$$\mathbf{B}^T\mathbf{v} = 0 \tag{2.4.7.7}$$

Since $\mathbf{B}^T$ is an upper triangular matrix with non zero values in principal diagonal , it is invertible matrix and hence **v** will be zero vector.Now, Finding the inverse of $\mathbf{B}^T$

$$\left(\begin{array}{ccc|ccc} 1 & t & t^2 & 1 & 0 & 0 \\ 0 & 1 & 2t & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \tag{2.4.7.8}$$

$$\xleftrightarrow{R_1=R_1-tR_2} \left(\begin{array}{ccc|ccc} 1 & 0 & -t^2 & 1 & -t & 0 \\ 0 & 1 & 2t & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \tag{2.4.7.9}$$

$$\xleftrightarrow{R_1=R_1+t^2R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -t & t^2 \\ 0 & 1 & 2t & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \tag{2.4.7.10}$$

$$\xleftrightarrow{R_2=R_2-2tR_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -t & t^2 \\ 0 & 1 & 0 & 0 & 1 & -2t \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \tag{2.4.7.11}$$

So,

$$(\mathbf{B}^T)^{-1} = \begin{pmatrix} 1 & -t & t^2 \\ 0 & 1 & -2t \\ 0 & 0 & 1 \end{pmatrix} \tag{2.4.7.12}$$

Now, to find the coordinates,

$$f(x) = \mathbf{w}^T\mathbf{g} \tag{2.4.7.13}$$

So,

$$\mathbf{c}^T\mathbf{f} = \mathbf{w}^T\mathbf{g} \tag{2.4.7.14}$$

$$\mathbf{c}^T\mathbf{f} = \mathbf{w}^T\mathbf{Bf} \tag{2.4.7.15}$$

$$(\mathbf{c}^T - \mathbf{w}^T\mathbf{B})\mathbf{f} = 0 \tag{2.4.7.16}$$

Since, **f** is linearly independent,

$$\mathbf{c}^T - \mathbf{w}^T\mathbf{B} = 0 \tag{2.4.7.17}$$

$$\mathbf{c}^T = \mathbf{w}^T\mathbf{B} \tag{2.4.7.18}$$

$$\mathbf{c}^T\mathbf{B}^{-1} = \mathbf{w}^T \tag{2.4.7.19}$$

$$(\mathbf{B}^{-1})^T\mathbf{c} = \mathbf{w} \tag{2.4.7.20}$$

Using (2.4.7.12) in (2.4.7.20)

$$\mathbf{w} = \begin{pmatrix} 1 & -t & t^2 \\ 0 & 1 & -2t \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} \tag{2.4.7.21}$$

2.5 Summary of Row Equivalence

2.5.1. Consider the vectors in $\mathbb{R}^4$ defined by:

$$\alpha_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 2 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 3 \\ 4 \\ -2 \\ 5 \end{pmatrix} \text{ and } \alpha_3 = \begin{pmatrix} 1 \\ 4 \\ 0 \\ 9 \end{pmatrix}.$$

Find a system of homogeneous linear equations for which the space of solutions is exactly the subspace of $\mathbb{R}^4$ spanned by the given three vectors.

Solution: A system of linear equations is homogeneous if all of the constant terms are zero. It can be represented as,

$$\mathbf{AX} = 0 \tag{2.5.1.1}$$

Let **R** be a echelon matrix which is reduced to A. Then the systems $\mathbf{AX} = 0$ and $\mathbf{RX} = 0$ have the same solutions. Here,

$$\mathbf{A} = \begin{pmatrix} -1 & 0 & 1 & 2 \\ 3 & 4 & -2 & 5 \\ 1 & 4 & 0 & 9 \end{pmatrix} \tag{2.5.1.2}$$

By row reducing on **A**, we get:

$$\mathbf{A} = \begin{pmatrix} -1 & 0 & 1 & 2 \\ 3 & 4 & -2 & 5 \\ 1 & 4 & 0 & 9 \end{pmatrix} \xleftrightarrow{R_3=R_3-2R_1-R_2} \begin{pmatrix} -1 & 0 & 1 & 2 \\ 3 & 4 & -2 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.5.1.3}$$

$$\xleftrightarrow{R_2=R_2+3R_1} \begin{pmatrix} -1 & 0 & 1 & 2 \\ 0 & 4 & 1 & 11 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.5.1.4}$$

$$\xleftrightarrow{R_1=-R_1} \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 4 & 1 & 11 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.5.1.5}$$

The bais vector is non zero vector which are given from 2.5.1.5,

$$\rho_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ -2 \end{pmatrix}, \rho_2 = \begin{pmatrix} 0 \\ 4 \\ 1 \\ 11 \end{pmatrix} \tag{2.5.1.6}$$

ρ_1, ρ_2 forms the basis of the solution space. The subspace spanned by b_1 and b_2 is given as:

$$\begin{pmatrix} \rho_1 & \rho_2 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \mathbf{X} \tag{2.5.1.7}$$

Using 2.5.1.7, we can write the augmented matrix as:

$$\begin{pmatrix} 1 & 0 & x_1 \\ 0 & 4 & x_2 \\ -1 & 1 & x_3 \\ -2 & 11 & x_4 \end{pmatrix} \xleftrightarrow{R_3=4R_3+4R_1-R_2} \tag{2.5.1.8}$$

$$\begin{pmatrix} 1 & 0 & x_1 \\ 0 & 4 & x_2 \\ 0 & 0 & 4x_1 - x_2 + 4x_3 \\ -2 & 11 & x_4 \end{pmatrix} \xleftrightarrow{R_4=4R_4+8R_1-11R_2} \tag{2.5.1.9}$$

$$\begin{pmatrix} 1 & 0 & x_1 \\ 0 & 4 & x_2 \\ 0 & 0 & 4x_1 - x_2 + 4x_3 \\ 0 & 0 & 8x_1 - 11x_2 + 4x_4 \end{pmatrix} \tag{2.5.1.10}$$

Using 2.5.1.10, The required homogeneous equation is given as:

$$\begin{pmatrix} 4 & -1 & 4 & 0 \\ 8 & -11 & 0 & 4 \end{pmatrix} \mathbf{X} = 0 \tag{2.5.1.11}$$

2.5.2. Let $s<n$ and A an $s \times n$ matrix with entries in the field $\mathbb{F}$. Use Theorem 4 to show that there is a non-zero **x** in $\mathbb{F}^{n\times 1}$ such that $\mathbf{Ax} = \mathbf{0}$.
Solution: Theorem 4:*Let $\mathbb{V}$ be a vector space which is spanned by a finite set of vectors $\beta_1, \beta_2, ..., \beta_m$. Then any independent set of vectors in $\mathbb{V}$ is finite and contains no more than m elements.* Let $\mathbb{V}$ be a vector space spanned by $a_1, a_2, \ldots, a_n$, where a_i, i=1,2,...,n are columns of matrix $\mathbf{A}_{s\times n}$.

$$\mathbf{A} = \begin{pmatrix} a_1 & a_2 & \ldots & a_n \end{pmatrix} \tag{2.5.2.1}$$

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & & & \vdots \\ a_{s1} & a_{s2} & \ldots & a_{sn} \end{pmatrix} \tag{2.5.2.2}$$

Let us take a_i,i=1,2,...,n as standard $s \times 1$ bases.

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & \dots & 0 & a_{1,s+1} & \dots & a_{1n} \\ 0 & 1 & \dots & 0 & a_{2,s+1} & \dots & a_{2n} \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & 0 & \dots & 1 & a_{s,s+1} & \dots & a_{sn} \end{pmatrix} \tag{2.5.2.3}$$

From (2.5.2.3), it is clear that

$$dim(col(A)) \leq s \tag{2.5.2.4}$$
$$\implies rank(A) \leq s \tag{2.5.2.5}$$

Now, from rank-nullity theorem,

$$rank(A) + nullity(A) = n \tag{2.5.2.6}$$
$$nullity(A) = n - rank(A) \tag{2.5.2.7}$$
$$\implies nullity(A) > 0 \tag{2.5.2.8}$$

From equation (2.5.2.8) it is clear that there will be a non zero $\mathbf{x}$ such that $\mathbf{A}\mathbf{x} = \mathbf{0}$

2.5.3. Let

$$\alpha_1 = \begin{pmatrix} 1 & 1 & -2 & 1 \end{pmatrix}^T \tag{2.5.3.1}$$
$$\alpha_2 = \begin{pmatrix} 3 & 0 & 4 & -1 \end{pmatrix}^T \tag{2.5.3.2}$$
$$\alpha_3 = \begin{pmatrix} -1 & 2 & 5 & 2 \end{pmatrix}^T \tag{2.5.3.3}$$

Let

$$\alpha = \begin{pmatrix} 4 & -5 & 9 & -7 \end{pmatrix}^T \tag{2.5.3.4}$$
$$\beta = \begin{pmatrix} 3 & 1 & -4 & 4 \end{pmatrix}^T \tag{2.5.3.5}$$
$$\gamma = \begin{pmatrix} -1 & 1 & 0 & 1 \end{pmatrix}^T \tag{2.5.3.6}$$

a) Which of the vectors α, β, γ are in the subspace of $\mathbb{R}^4$ spanned by α_i?
b) Which of the vectors α, β, γ are in the subspace of $\mathbb{C}^4$ spanned by α_i?
c) Does this suggest a theorem?

Solution:

a) The linear combination of α_i for $i = 1, 2, 3$ spans subspace S. We can write,

$$c_1 \begin{pmatrix} 1 \\ 1 \\ -2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 0 \\ 4 \\ -1 \end{pmatrix} + c_3 \begin{pmatrix} -1 \\ 2 \\ 5 \\ 2 \end{pmatrix} = \text{span(S)} \tag{2.5.3.7}$$

where c_1,c_2,c_3 are scalars. Vectors in matrix form is given by

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & -1 \\ 1 & 0 & 2 \\ -2 & 4 & 5 \\ 1 & -1 & 2 \end{pmatrix} \tag{2.5.3.8}$$

We can observe that the columns of matrix $\mathbf{A}$ formed by vectors α_i are independent as the rank of matrix is 3. Hence α_i forms basis for subspace S.

i) **Checking for** α: To check if a solution exists for $\mathbf{AX} = \alpha$. The corresponding agumented matrix can be written as,

$$\begin{pmatrix} \mathbf{A} & \alpha \end{pmatrix} = \begin{pmatrix} 1 & 3 & -1 & 4 \\ 1 & 0 & 2 & -5 \\ -2 & 4 & 5 & 9 \\ 1 & -1 & 2 & -7 \end{pmatrix} \tag{2.5.3.9}$$

On performing row-reduction on (2.5.3.9),

$$\begin{pmatrix} \mathbf{A} & \alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.5.3.10}$$

As Rank($\begin{pmatrix} \mathbf{A} & \alpha \end{pmatrix}$)=Rank($\mathbf{A}$)=3, the vector α can be represented as linear combination of α_i. From equation (2.5.3.10), we can write

$$-3 \begin{pmatrix} 1 \\ 1 \\ -2 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} 3 \\ 0 \\ 4 \\ -1 \end{pmatrix} - 1 \begin{pmatrix} -1 \\ 2 \\ 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ -5 \\ 9 \\ -7 \end{pmatrix} \tag{2.5.3.11}$$

Hence α is in the subspace S.

ii) **Checking for** β: To check if a solution exists for $\mathbf{AX} = \beta$. The corresponding agumented matrix can be written as,

$$\begin{pmatrix} \mathbf{A} & \beta \end{pmatrix} = \begin{pmatrix} 1 & 3 & -1 & 3 \\ 1 & 0 & 2 & 1 \\ -2 & 4 & 5 & -4 \\ 1 & -1 & 2 & 4 \end{pmatrix} \tag{2.5.3.12}$$

On performing row-reduction on

(2.5.3.12),

$$\begin{pmatrix}\mathbf{A} & \beta\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} \qquad (2.5.3.13)$$

As Rank($\begin{pmatrix}\mathbf{A} & \beta\end{pmatrix}$)=4 and Rank($\mathbf{A}$)=3, Solution doesn't exist for $AX = \beta$ and hence β is not in the subspace S.

iii) **Checking for** γ: To check if a solution exists for $\mathbf{AX} = \gamma$. The corresponding agumented matrix can be written as,

$$\begin{pmatrix}\mathbf{A} & \gamma\end{pmatrix} = \begin{pmatrix}1 & 3 & -1 & -1\\ 1 & 0 & 2 & 1\\ -2 & 4 & 5 & 0\\ 1 & -1 & 2 & 1\end{pmatrix} \qquad (2.5.3.14)$$

On performing row-reduction on (2.5.3.14),

$$\begin{pmatrix}\mathbf{A} & \gamma\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} \qquad (2.5.3.15)$$

As Rank($\begin{pmatrix}\mathbf{A} & \gamma\end{pmatrix}$)=4 and Rank($\mathbf{A}$)=3, Solution doesn't exist for $AX = \gamma$ and hence γ is not in the subspace S.

b) In part 1, we haven't considered the field to be either $\mathbb{R}$ or $\mathbb{C}$. The above equations solved holds for field $\mathbb{C}$ and that implies, they hold for field $\mathbb{R}$ also. Hence α is in the subspace and β and γ are not in the subspace.

c) **Theorem suggested:** Let $\mathbb{F}_1$ and $\mathbb{F}_2$ are two fields where $\mathbb{F}_2$ is subfield of $\mathbb{F}_1$. Let α_i, i=1,2,3...,n forms basis for subspace of $\mathbb{F}_2^n$ and a vector $\alpha \in \mathbb{F}_2^n$. Then α is in the subspace of $\mathbb{F}_2^n$ spanned by α_i, i=1,2,3...,n if only if α is in the subspace of $\mathbb{F}_1^n$ spanned by α_i, i=1,2,3...,n.

2.5.4. In C^3, let $\boldsymbol{\alpha_1} = (1, 0, -i)$, $\boldsymbol{\alpha_2} = (1 + i, 1 - i, 1)$, $\boldsymbol{\alpha_3} = (i, i, i)$. Prove that these vectors form a basis for C^3. What are the coordinates of the vector (a,b,c) in the basis?

Solution: Now,

$$C_1\boldsymbol{\alpha_1} + C_2\boldsymbol{\alpha_2} + C_3\boldsymbol{\alpha_3} = \mathbf{0} \qquad (2.5.4.1)$$

$$\implies C_1\begin{pmatrix}1\\0\\-i\end{pmatrix} + C_2\begin{pmatrix}1+i\\1-i\\1\end{pmatrix} + C_3\begin{pmatrix}i\\i\\i\end{pmatrix} = \mathbf{0} \qquad (2.5.4.2)$$

So,

$$\begin{pmatrix}1 & 1+i & i\\ 0 & 1-i & i\\ -i & 1 & i\end{pmatrix}\begin{pmatrix}C_1\\C_2\\C_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \qquad (2.5.4.3)$$

Considering the co-efficient matrix A:

$$\begin{pmatrix}1 & 1+i & i\\ 0 & 1-i & i\\ -i & 1 & i\end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 + iR_1} \begin{pmatrix}1 & 1+i & i\\ 0 & 1-i & i\\ 0 & i & i-1\end{pmatrix}$$

$$\xleftrightarrow{R_3 \leftarrow R_3/i} \begin{pmatrix}1 & 1+i & i\\ 0 & 1-i & i\\ 0 & 1 & 1+i\end{pmatrix} \xleftrightarrow{R_3 \leftarrow (1-i)R_3}$$

$$\begin{pmatrix}1 & 1+i & i\\ 0 & 1-i & i\\ 0 & 1-i & 2\end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 - R_2} \begin{pmatrix}1 & 1+i & i\\ 0 & 1-i & i\\ 0 & 0 & 2-i\end{pmatrix}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{1+i}{1-i}R_2} \begin{pmatrix}1 & 1+i & i\\ 0 & 1+i & -1\\ 0 & 0 & 2-i\end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 - R_2}$$

$$\begin{pmatrix}1 & 0 & i+1\\ 0 & 1+i & -1\\ 0 & 0 & 2-i\end{pmatrix} \qquad (2.5.4.4)$$

Now let

$$R = \begin{pmatrix}1 & 0 & i+1\\ 0 & 1+i & -1\\ 0 & 0 & 2-i\end{pmatrix} \qquad (2.5.4.5)$$

Where R is the row reduced form of matrix A. So $\boldsymbol{\alpha_1}$,$\boldsymbol{\alpha_2}$ and $\boldsymbol{\alpha_3}$ are linearly independent which implies that these 3 vectors form a basis of vector space C^3.

Now, consider a vector $\beta = \begin{pmatrix}a\\b\\c\end{pmatrix}$ and let the coordinates are $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ such that

$$Ax = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{2.5.4.6}$$

$$\implies x = A^{-1}\begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{2.5.4.7}$$

Let us consider a matrix $(A|I)$ where I is a 3x3 identity matrix. Now, applying the Gauss-Jordon theorem we can get A^{-1}

$$\begin{pmatrix} 1 & 1+i & i & 1 & 0 & 0 \\ 0 & 1-i & i & 0 & 1 & 0 \\ -i & 1 & i & 0 & 0 & 1 \end{pmatrix}
\xleftrightarrow{R_3 \leftarrow R_3 + iR_1}
\begin{pmatrix} 1 & 1+i & i & 1 & 0 & 0 \\ 0 & 1-i & i & 0 & 1 & 0 \\ 0 & i & i-1 & i & 0 & 1 \end{pmatrix}
\xleftrightarrow{R_3 \leftarrow R_3/i}
\begin{pmatrix} 1 & 1+i & i & 1 & 0 & 0 \\ 0 & 1-i & i & 0 & 1 & 0 \\ 0 & 1 & 1+i & 1 & 0 & -i \end{pmatrix}
\xleftrightarrow{R_3 \leftarrow (1-i)R_3}
\begin{pmatrix} 1 & 1+i & i & 1 & 0 & 0 \\ 0 & 1-i & i & 0 & 1 & 0 \\ 0 & 1-i & 2 & 1-i & 0 & -i-1 \end{pmatrix}
\xleftrightarrow{R_3 \leftarrow R_3 - R_2}
\begin{pmatrix} 1 & 1+i & i & 1 & 0 & 0 \\ 0 & 1-i & i & 0 & 1 & 0 \\ 0 & 0 & 2-i & 1-i & -1 & -i-1 \end{pmatrix}
\xleftrightarrow{R_2 \leftarrow \frac{1+i}{1-i}R_2}
\begin{pmatrix} 1 & 1+i & i & 1 & 0 & 0 \\ 0 & 1+i & -1 & 0 & i & 0 \\ 0 & 0 & 2-i & 1-i & -1 & -i-1 \end{pmatrix}
\xleftrightarrow{R_1 \leftarrow R_1 - R_2}
\begin{pmatrix} 1 & 0 & i+1 & 1 & -i & 0 \\ 0 & 1+i & -1 & 0 & i & 0 \\ 0 & 0 & 2-i & 1-i & -1 & -i-1 \end{pmatrix}
\xleftrightarrow[R_1 \leftarrow R_1 + R_3]{R_3 \leftarrow \frac{R_3}{2-i}}
\begin{pmatrix} 1 & 0 & 3 & 2-i & -i-1 & -i-1 \\ 0 & 1+i & -1 & 0 & i & 0 \\ 0 & 0 & 1 & \frac{3-i}{5} & -\frac{2+i}{5} & -\frac{3i+1}{5} \end{pmatrix}
\xleftrightarrow{R_2 \leftarrow R_2 + R_3}
\begin{pmatrix} 1 & 0 & 3 & 2-i & -i-1 & -i-1 \\ 0 & 1+i & 0 & \frac{3-i}{5} & \frac{-2+4i}{5} & -\frac{3i+1}{5} \\ 0 & 0 & 1 & \frac{3-i}{5} & -\frac{2+i}{5} & -\frac{3i+1}{5} \end{pmatrix}
\xleftrightarrow{R_2 \leftarrow \frac{R_2}{1+i}}
\begin{pmatrix} 1 & 0 & 3 & 2-i & -i-1 & -i-1 \\ 0 & 1 & 0 & \frac{3-i}{5(1+i)} & \frac{-2+4i}{5(1+i)} & -\frac{3i+1}{5(1+i)} \\ 0 & 0 & 1 & \frac{3-i}{5} & -\frac{2+i}{5} & -\frac{3i+1}{5} \end{pmatrix}
\xleftrightarrow{R_1 \leftarrow R_1 - 3R_3} \tag{2.5.4.8}$$

$$\begin{pmatrix} 1 & 0 & 0 & \frac{1-4i}{5} & \frac{1-2i}{5} & \frac{-2+4i}{5} \\ 0 & 1 & 0 & \frac{1-2i}{5} & \frac{1+3i}{5} & -\frac{2+i}{5} \\ 0 & 0 & 1 & \frac{3-i}{5} & -\frac{2+i}{5} & -\frac{3i+1}{5} \end{pmatrix} \tag{2.5.4.9}$$

$$\implies (I|A^{-1}) = \begin{pmatrix} 1 & 0 & 0 & \frac{1-4i}{5} & \frac{1-2i}{5} & \frac{-2+4i}{5} \\ 0 & 1 & 0 & \frac{1-2i}{5} & \frac{1+3i}{5} & -\frac{2+i}{5} \\ 0 & 0 & 1 & \frac{3-i}{5} & -\frac{2+i}{5} & -\frac{3i+1}{5} \end{pmatrix} \tag{2.5.4.10}$$

$$\implies A^{-1} = \begin{pmatrix} \frac{1-4i}{5} & \frac{1-2i}{5} & \frac{-2+4i}{5} \\ \frac{1-2i}{5} & \frac{1+3i}{5} & -\frac{2+i}{5} \\ \frac{3-i}{5} & -\frac{2+i}{5} & -\frac{3i+1}{5} \end{pmatrix} \tag{2.5.4.11}$$

So,

$$x = A^{-1}\begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{2.5.4.12}$$

$$\implies x = \begin{pmatrix} \frac{1-4i}{5} & \frac{1-2i}{5} & \frac{-2+4i}{5} \\ \frac{1-2i}{5} & \frac{1+3i}{5} & -\frac{2+i}{5} \\ \frac{3-i}{5} & -\frac{2+i}{5} & -\frac{3i+1}{5} \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{2.5.4.13}$$

$$\implies \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1-4i}{5} & \frac{1-2i}{5} & \frac{-2+4i}{5} \\ \frac{1-2i}{5} & \frac{1+3i}{5} & -\frac{2+i}{5} \\ \frac{3-i}{5} & -\frac{2+i}{5} & -\frac{3i+1}{5} \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{2.5.4.14}$$

2.5.5. Give an explicit description of the type $b_j = \sum_{i=1}^{r} b_{ki}R_{ij}$ for the vectors

$$\beta = (b_1, b_2, b_3, b_4, b_5)$$

in R^5 which are linear combinations of the vectors

$$\alpha_1 = (1, 0, 2, 1, -1), \tag{2.5.5.1}$$

$$\alpha_2 = (-1, 2, -4, 2, 0), \tag{2.5.5.2}$$

$$\alpha_3 = (2, -1, 5, 2, 1), \tag{2.5.5.3}$$

$$\alpha_4 = (2, 1, 3, 5, 2) \tag{2.5.5.4}$$

Solution: Above matrix represented as: $Ax = \beta$

$$\begin{pmatrix} 1 & -1 & 2 & 2 \\ 0 & 2 & -1 & 1 \\ 2 & -4 & 5 & 3 \\ 1 & 2 & 2 & 5 \\ -1 & 0 & 1 & 2 \end{pmatrix} x = \beta \tag{2.5.5.5}$$

$$\begin{pmatrix} 1 & -1 & 2 & 2 \\ 0 & 2 & -1 & 1 \\ 2 & -4 & 5 & 3 \\ 1 & 2 & 2 & 5 \\ -1 & 0 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{pmatrix} \tag{2.5.5.6}$$

$AX^T = Y^T$, where A, X and Y matrices have shown below:

$$A = \begin{pmatrix} 1 & -1 & 2 & 2 \\ 0 & 2 & -1 & 1 \\ 2 & -4 & 5 & 3 \\ 1 & 2 & 2 & 5 \\ -1 & 0 & 1 & 2 \end{pmatrix} \tag{2.5.5.7}$$

$$X = \begin{pmatrix} \frac{67}{100} & -\frac{167}{100} & -2 & \frac{133}{100} \\ \frac{1}{2} & -\frac{3}{2} & -\frac{5}{2} & \frac{3}{2} \\ -\frac{4}{25} & \frac{117}{100} & \frac{3}{2} & -\frac{83}{100} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \tag{2.5.5.8}$$

$$Y = \begin{pmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.5.5.9}$$

Now since the Y^T matrix is full rank thus we can say that columns of matrix A are linearly independent and b is described by (2.5.5.6)

2.5.6. Let $\mathbb{V}$ be a vector space which is spanned by the rows of matrix

$$\mathbf{A} = \begin{pmatrix} 3 & 21 & 0 & 9 & 0 \\ 1 & 7 & -1 & -2 & -1 \\ 2 & 14 & 0 & 6 & 1 \\ 6 & 42 & -1 & 13 & 0 \end{pmatrix} \tag{2.5.6.1}$$

a. Find a basis for $\mathbb{V}$

b. Tell which vectors $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$ are elements of $\mathbb{V}$

c. If $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$ is in $\mathbb{V}$,what are its coordinates in the basis chosen in part(a)?

Solution: Row reducing (2.5.6.1)

$$\begin{pmatrix}3 & 21 & 0 & 9 & 0\\ 1 & 7 & -1 & -2 & -1\\ 2 & 14 & 0 & 6 & 1\\ 6 & 42 & -1 & 13 & 0\end{pmatrix}$$

$$\xleftrightarrow{R_1 \leftarrow \frac{R_1}{3}} \begin{pmatrix}1 & 7 & 0 & 3 & 0\\ 1 & 7 & -1 & -2 & -1\\ 2 & 14 & 0 & 6 & 1\\ 6 & 42 & -1 & 13 & 0\end{pmatrix}$$

$$\xleftrightarrow[R_2 \leftarrow R_2 - R_1]{R_3 \leftarrow R_3 - 2R_1} \begin{pmatrix}1 & 7 & 0 & 3 & 0\\ 0 & 0 & -1 & -5 & -1\\ 0 & 0 & 0 & 0 & 1\\ 0 & 0 & -1 & -5 & 0\end{pmatrix}$$

$$\xleftrightarrow{R_4 \leftarrow R_4 - R_2} \begin{pmatrix}1 & 7 & 0 & 3 & 0\\ 0 & 0 & -1 & -5 & -1\\ 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 1\end{pmatrix}$$

$$\xleftrightarrow{R_2 \leftarrow -R_2} \begin{pmatrix}1 & 7 & 0 & 3 & 0\\ 0 & 0 & 1 & 5 & 1\\ 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 1\end{pmatrix}$$

$$\xleftrightarrow[R_2 \leftarrow R_2 - R_3]{R_4 \leftarrow R_4 - R_3} \begin{pmatrix}1 & 7 & 0 & 3 & 0\\ 0 & 0 & 1 & 5 & 0\\ 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0\end{pmatrix} \tag{2.5.6.2}$$

a. For the basis of $\mathbb{V}$, we can take the non zero rows of (2.5.6.2)

$$\rho_1 = \begin{pmatrix}1\\7\\0\\3\\0\end{pmatrix} \rho_2 = \begin{pmatrix}0\\0\\1\\5\\0\end{pmatrix} \rho_3 = \begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix} \tag{2.5.6.3}$$

b. Vectors which are elements of $\mathbb{V}$ are of the form:

$$c_1\rho_1 + c_2\rho_2 + c_3\rho_3 = \begin{pmatrix}c_1\\ 7c_1\\ c_2\\ 3c_1 + 5c_2\\ c_3\end{pmatrix} \tag{2.5.6.4}$$

where c_1,c_2,c_3 are scalars.

c. Expressing (2.5.6.4) in matrix form,if $\begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix}$ is in $\mathbb{V}$,it must be of the form

$$\begin{pmatrix}1 & 0 & 0\\ 7 & 0 & 0\\ 0 & 1 & 0\\ 3 & 5 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix} = \begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix} \tag{2.5.6.5}$$

The augmented matrix form

$$\begin{pmatrix}1 & 0 & 0 & x_1\\ 7 & 0 & 0 & x_2\\ 0 & 1 & 0 & x_3\\ 3 & 5 & 0 & x_4\\ 0 & 0 & 1 & x_5\end{pmatrix} \tag{2.5.6.6}$$

Converting the above matrix into row reduced echelon form

$$\begin{pmatrix}1 & 0 & 0 & x_1\\ 7 & 0 & 0 & x_2\\ 0 & 1 & 0 & x_3\\ 3 & 5 & 0 & x_4\\ 0 & 0 & 1 & x_5\end{pmatrix} \xleftrightarrow[R_2 \leftarrow R_2 - 7R_1]{R_4 \leftarrow R_4 - 3R_1} \begin{pmatrix}1 & 0 & 0 & x_1\\ 0 & 0 & 0 & x_2 - 7x_1\\ 0 & 1 & 0 & x_3\\ 0 & 5 & 0 & x_4 - 3x_1\\ 0 & 0 & 1 & x_5\end{pmatrix}$$

$$\xleftrightarrow{R_2 \leftarrow R_3} \begin{pmatrix}1 & 0 & 0 & x_1\\ 0 & 1 & 0 & x_3\\ 0 & 0 & 0 & x_2 - 7x_1\\ 0 & 5 & 0 & x_4 - 3x_1\\ 0 & 0 & 1 & x_5\end{pmatrix}$$

$$\xleftrightarrow{R_4 \leftarrow R_4 - 5R_2} \begin{pmatrix}1 & 0 & 0 & x_1\\ 0 & 1 & 0 & x_3\\ 0 & 0 & 0 & x_2 - 7x_1\\ 0 & 0 & 0 & x_4 - 3x_1 - 5x_3\\ 0 & 0 & 1 & x_5\end{pmatrix}$$

$$\xleftrightarrow{R_5 \leftarrow R_3} \begin{pmatrix}1 & 0 & 0 & x_1\\ 0 & 1 & 0 & x_3\\ 0 & 0 & 1 & x_5\\ 0 & 0 & 0 & x_4 - 3x_1 - 5x_3\\ 0 & 0 & 0 & x_2 - 7x_1\end{pmatrix} \tag{2.5.6.7}$$

From (2.5.6.7),the coordinates of $\begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix}$ in the basis is

$$\begin{pmatrix} x_1 \\ x_3 \\ x_5 \end{pmatrix} \tag{2.5.6.8}$$

2.5.7. Let **A** be an m x n matrix over the field **F**, and consider the system of equations **AX** = **Y**. Prove that this system of equations has a solutions if and only if the row rank of **A** is equal to the row rank of augmented matrix of the system. **Solution:** Consider **A** as,

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}_{mxn} \tag{2.5.7.1}$$

Now to solve the equation **AX** = **Y** for **X**, we first write its augmented matrix $\mathbf{A_Y}$ and then convert it into row reduced echelon form given as $\mathbf{R_Y}$. Here **R** is the row reduced form of matrix **A**. Also we know that, for **AX** = **Y** to have a solution, **Y** should be in column space of **A**.

$$\mathbf{A_Y} = \begin{pmatrix} \mathbf{A} & | & \mathbf{Y} \end{pmatrix} \tag{2.5.7.2}$$

Assume that the last **k** rows of **R** are zero rows.This implies that we have **k** number of linear dependent rows in matrix **A**. Hence the row rank of matrix **A** is **r** = m-k. As there are m-k number of non-zero vectors in the row of **R**. Now,

$$\mathbf{R}_Y = \mathbf{E}\mathbf{A}_Y \tag{2.5.7.3}$$

$$\implies \mathbf{C}\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & y_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & y_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & y_m \end{array}\right) \tag{2.5.7.4}$$

Here, **E** is product of matrix responsible for performing operation on matrix such as scaling, Row exchange etc, given as,

$$\mathbf{E} = E_i E_{i-1} \cdots E_1 \tag{2.5.7.5}$$

Hence row reduced echelon form, $\mathbf{R_Y}$ is given as,

$$\mathbf{R_Y} = E\mathbf{A_Y} = \begin{pmatrix} E\mathbf{A} & | & E\mathbf{Y} \end{pmatrix} \tag{2.5.7.6}$$

Now RREF can be represented in Block matrix as,

$$\implies \mathbf{R_Y} = \left(\begin{array}{cc|c} I & F & Y \\ 0 & 0 & \end{array}\right) \tag{2.5.7.7}$$

Also from equation (2.5.7.7) it can be observed that for **AX** = **Y** to have a solution,

$$y'_{m-k} = y'_{m-k-1} = \cdots = y'_{m-1} = y'_m = 0 \tag{2.5.7.8}$$

Hence, the rank of $\mathbf{R'_Y}$ is, also **r**. This implies rank of augmented matrix $\mathbf{A_Y}$ is also **r**.

Example: Let **A** be a 3×4 matrix given as,

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 5 & 7 \\ 2 & 9 & 3 & 6 \\ 1 & 7 & 2 & 9 \end{pmatrix} \tag{2.5.7.9}$$

$$\mathbf{Y} = \begin{pmatrix} 4 \\ 8 \\ 14 \end{pmatrix} \tag{2.5.7.10}$$

For the time being it could be concluded that the following set of linear equation have a solution as $\mathbf{Y} \in$ Column-space(**A**) Hence to solve **AX** = **Y** first the augmented matrix $\mathbf{A_Y}$ is given as,

$$\mathbf{A_Y} = \left(\begin{array}{cccc|c} 1 & 2 & 5 & 7 & 4 \\ 2 & 9 & 3 & 6 & 8 \\ 1 & 7 & 2 & 9 & 14 \end{array}\right) \tag{2.5.7.11}$$

Now reducing in RREF we get,

$$\mathbf{R_Y} = E\mathbf{A_Y} \tag{2.5.7.12}$$

Here E is elementary matrix given as,

$$E = \begin{pmatrix} -3/20 & 31/20 & -39/20 \\ -1/20 & -3/20 & 7/20 \\ 1/4 & -1/4 & 1/4 \end{pmatrix} \tag{2.5.7.13}$$

$$\implies \mathbf{R_Y} = E\mathbf{A_Y} = \begin{pmatrix} E\mathbf{A} & | & E\mathbf{Y} \end{pmatrix} \tag{2.5.7.14}$$

$$\implies \mathbf{R_Y} = \begin{pmatrix} I & F & | & Y \end{pmatrix} \tag{2.5.7.15}$$

$$\mathbf{I}_{3x3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.5.7.16}$$

$$\mathbf{F}_{3x1} = \begin{pmatrix} -93/10 \\ 19/10 \\ 5/2 \end{pmatrix} \tag{2.5.7.17}$$

$$\mathbf{Y}_{3x1} = \begin{pmatrix} -31/2 \\ 7/2 \\ 5/2 \end{pmatrix} \tag{2.5.7.18}$$

Hence, rank = 3 as there are 0 non zero rows in row reduced echelon form. Also it could be observed that both $\mathbf{R}$ and $\mathbf{R_Y}$ have non zero rows hence we have a solution and same rank.

3 Linear Transformations

3.1 Linear Transformations

3.1.1. Find weather given functions $\mathbf{T}$ from $\mathbb{R}^2$ into $\mathbb{R}^2$ are linear transformations or not

a)

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 + x_1 \\ x_2 \end{pmatrix} \tag{3.1.1.1}$$

Solution: Counter example can be given as follows:-

$$x_1 = x_2 = 0 \tag{3.1.1.2}$$

Substituting (3.1.1.2) in (3.1.1.1) we get,

$$T\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.1.1.3}$$

(3.1.1.3) is clearly false because linear transformation on $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ will always be equal to $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

b)

$$\mathbf{T}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix} \tag{3.1.1.4}$$

Does function $\mathbf{T}$ from $\mathbb{R}^2$ into $\mathbb{R}^2$ is Linear Transformation.

Solution: Let,

$$\mathbf{x}, \mathbf{y} \in \mathbb{R}^2 \tag{3.1.1.5}$$

Using transformation on $\mathbf{T}$,

$$\mathbf{T}(\mathbf{x}) = \mathbf{A}\mathbf{x} \tag{3.1.1.6}$$

From (3.1.1.4) we get,

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.1.1.7}$$

With c being a scalar,

$$\mathbf{T}(c\mathbf{x} + \mathbf{y}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}(c\mathbf{x} + \mathbf{y}) \tag{3.1.1.8}$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} c\mathbf{x} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{y} \tag{3.1.1.9}$$

$$= \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{y} \tag{3.1.1.10}$$

$$= c\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{y} \tag{3.1.1.11}$$

From (3.1.1.11) we can say,

$$\mathbf{T}(c\mathbf{x} + \mathbf{y}) = c\mathbf{T}(\mathbf{x}) + \mathbf{T}(\mathbf{y}) \tag{3.1.1.12}$$

Hence from (3.1.1.12) we can say $\mathbf{T}$ is a Linear Transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$

c)

$$\mathbf{T}(x_1, x_2) = ({x_1}^2, x_2) \tag{3.1.1.13}$$

Solution: If $\mathbf{T}$ were a linear transformation then we would have

$$\mathbf{T}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.1.1.14}$$

$$\implies \mathbf{T}\left(-1\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = -1.\mathbf{T}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.1.1.15}$$

$$\implies -1.\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \tag{3.1.1.16}$$

which is a contradiction, since

$$\mathbf{T}\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.1.1.17}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \neq \begin{pmatrix} -1 \\ 0 \end{pmatrix}. \tag{3.1.1.18}$$

Hence non-linear transformation.

d) Is the following function $\mathbf{T}$ from $\mathbb{R}^2$ into $\mathbb{R}^2$ is linear transformation?

$$\mathbf{T}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \sin(x_1) \\ x_2 \end{pmatrix}$$

Solution: Let,

$$\mathbf{x} = \begin{pmatrix} \pi \\ 0 \end{pmatrix}; \quad \mathbf{y} = \begin{pmatrix} \frac{\pi}{2} \\ 0 \end{pmatrix}$$

$$\mathbf{T}(\mathbf{x}+\mathbf{y}) = \mathbf{T}\begin{pmatrix}\frac{3\pi}{2}\\0\end{pmatrix} = \begin{pmatrix}-1\\0\end{pmatrix} \tag{3.1.1.19}$$

$$\mathbf{T}(\mathbf{x}) + \mathbf{T}(\mathbf{y}) = \mathbf{T}\begin{pmatrix}\pi\\0\end{pmatrix} + \mathbf{T}\begin{pmatrix}\frac{\pi}{2}\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} \tag{3.1.1.20}$$

From (3.1.1.19) and (3.1.1.20) additive transformation property is not satisfied. Hence not a linear transformation.

e) Verify whether $T(x_1, x_2) = (x_1 - x_2, 0)$ is a linear transformation or not. **Solution:** Let **V** and **W** be the vector spaces. The function $T : \mathbf{V} \to \mathbf{W}$ is called a linear transformation of **V** into **W** if for all **u** and **v** in **V** and for any scalar k in field **F**,

$$T(k\mathbf{u}+\mathbf{v}) = kT(\mathbf{u}) + T(\mathbf{v}) \tag{3.1.1.21}$$

Given,

$$T\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}1 & -1\\0 & 0\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} \tag{3.1.1.22}$$

Consider,

$$T(k\mathbf{x}+\mathbf{y}) = \begin{pmatrix}1 & -1\\0 & 0\end{pmatrix}(k\mathbf{x}+\mathbf{y}) \tag{3.1.1.23}$$

$$= \begin{pmatrix}k & -k\\0 & 0\end{pmatrix}\mathbf{x} + \begin{pmatrix}1 & -1\\0 & 0\end{pmatrix}\mathbf{y} \tag{3.1.1.24}$$

$$= k\begin{pmatrix}1 & -1\\0 & 0\end{pmatrix}\mathbf{x} + \begin{pmatrix}1 & -1\\0 & 0\end{pmatrix}\mathbf{y} \tag{3.1.1.25}$$

$$T(\mathbf{x}+\mathbf{y}) = kT(\mathbf{x}) + T(\mathbf{y}) \tag{3.1.1.26}$$

Therefore, the given function T is a linear transformation.

3.1.2. Find the range, rank, null space, and nullity for the zero transformation and the identity transformation on a finite-dimensional space V.
Solution:
Suppose vector space **V** has dim(**V**) = n. Table 3.1.2 provides the properties of range, rank, null space and nullity of zero and identity transformation on a vector space **V**

3.1.3. *a)* Let **F** be a field and let **V** be the space of polynomial functions f from **F** into **F**, given by

$$f(x) = c_0 + c_1 x + \cdots + c_n x^n$$

Let **D** be a linear differentiation transformation defined as

$$(\mathbf{D}f)(x) = \frac{df(x)}{dx}$$

Then find the range and null space of **D**.

b) Let **R** be the field of real numbers and let **V** be the space of all functions from **R** into **R** which are continous. Let **T** be linear transformation defined by

$$(\mathbf{T}f)(x) = \int_0^x f(t)\,dt$$

Find the range and null space of **T**.

Solution: Let the vector space of n-dimension be deined as

$$\mathbf{V} = \left\{f : \mathbf{F} \to \mathbf{F} : f(x) = \sum_{k=0}^{n} c_k x^k,\ c_k \in \mathbf{F}\right\} \tag{3.1.3.1}$$

The corresponding standard basis for **V** is

$$\left\{\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}, \begin{pmatrix}0\\x\\\vdots\\0\end{pmatrix}, \cdots, \begin{pmatrix}0\\0\\\vdots\\x^{n-1}\end{pmatrix}\right\} \tag{3.1.3.2}$$

a) Let f and $g \in \mathbf{V}$ and let α and $\beta \in \mathbf{F}$ then

$$\mathbf{D}(\alpha f + \beta g) = \frac{d(\alpha f(x) + \beta g(x))}{dx} \tag{3.1.3.3}$$

$$= \alpha\frac{df(x)}{dx} + \beta\frac{dg(x)}{dx} \tag{3.1.3.4}$$

$$= \alpha(\mathbf{D}f) + \beta(\mathbf{D}g) \tag{3.1.3.5}$$

Therefore **D** is a linear transformation.
The **D** transformation maps the k^{th} basis vector as follows

$$\mathbf{D}\begin{pmatrix}0\\\vdots\\0\\x^k\\\vdots\\0\end{pmatrix} = \begin{pmatrix}0\\\vdots\\kx^{k-1}\\0\\\vdots\\0\end{pmatrix} \tag{3.1.3.6}$$

Properties	**Zero Transformation**	**Identity Transformation**
Transformation	$T_0(\mathbf{v}) = \mathbf{0}$	$T_I(\mathbf{v}) = \mathbf{v}$
Range	Zero subspace {0}	**V**
Rank	dim(**0**) = 0	dim(**V**) = n
Null space	**V**	Zero subspace {0}
Nullity	dim(**V**) = n	dim(**0**) = 0

TABLE 3.1.2: Properties of Zero and Identity transformation

Since the transformed vector

$$\begin{pmatrix} 0 \\ \vdots \\ kx^{k-1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbf{V} \tag{3.1.3.7}$$

the range of **D** is the vector space **V**. Thus the transformation is defined as $\mathbf{D} : \mathbf{V} \to \mathbf{V}$. Therefore the **D** Transformation on the basis vector set is

$$\mathbf{D}\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \tag{3.1.3.8}$$

$$= \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & n-2 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \tag{3.1.3.9}$$

Thus the **D** transformation coefficient matrix is

$$D = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & n-2 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \tag{3.1.3.10}$$

Since D contains a zero row hence $\left|D\right| = 0$. Therefore **D** transformation matrix is singular. The nullspace for differentiation transformation is defined as

$$\mathbf{N} = \{f \in \mathbf{V} : \mathbf{D}f = 0\} \tag{3.1.3.11}$$

Let the coefficient matrix of $f \in \mathbf{V}$ be

$$\mathbf{f} = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{pmatrix} \tag{3.1.3.12}$$

then

$$\mathbf{D}f = 0 \tag{3.1.3.13}$$

$$\implies \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & n-2 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}\begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{pmatrix} = \mathbf{0} \tag{3.1.3.14}$$

Since D is in row reduced echolon form and $rank(D) = n-1$ the solution of (3.1.3.14) is

$$\mathbf{f} = \begin{pmatrix} k \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{3.1.3.15}$$

where $k \in \mathbf{R}$. Therefore the nullspace for $\mathbf{D} : \mathbf{V} \to \mathbf{V}$ is

$$\mathbf{N} = \left\{\begin{pmatrix} k \\ 0 \\ \vdots \\ 0 \end{pmatrix} : k \in \mathbf{R}\right\} \tag{3.1.3.16}$$

b) Let f and $g \in \mathbf{V}$ and let α and $\beta \in \mathbf{F}$ then

$$\mathbf{T}(\alpha f + \beta g) = \int_0^x (\alpha f(t) + \beta g(t))\, dt \tag{3.1.3.17}$$

$$= \alpha \int_0^x f(t)\, dt + \beta \int_0^x g(t)\, dt \tag{3.1.3.18}$$

$$= \alpha(\mathbf{T}f) + \beta(\mathbf{T}g) \tag{3.1.3.19}$$

Therefore $\mathbf{T}$ is a linear transformation.
The $\mathbf{T}$ transformation maps the k^{th} basis vector as follows

$$\mathbf{T}\begin{pmatrix} 0 \\ \vdots \\ x^k \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{x^{k+1}}{k+1} \\ \vdots \\ 0 \end{pmatrix} \tag{3.1.3.20}$$

Since the transformed vector

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{x^{k+1}}{k+1} \\ \vdots \\ 0 \end{pmatrix} \in \mathbf{V} \tag{3.1.3.21}$$

the range of $\mathbf{T}$ is the vector space $\mathbf{V}$. Thus the transformation is defined as $\mathbf{T} : \mathbf{V} \to \mathbf{V}$. Therefore the $\mathbf{T}$ Transformation on the basis vector set is

$$\mathbf{T}\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \tag{3.1.3.22}$$

$$= \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & \frac{1}{n-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{n} \end{pmatrix} \tag{3.1.3.23}$$

Thus the $\mathbf{T}$ transformation coefficient matrix is

$$T = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & \frac{1}{n-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{n} \end{pmatrix} \tag{3.1.3.24}$$

Since T contains a zero row hence $\left|T\right| = 0$. Therefore $\mathbf{T}$ transformation matrix is singular. The nullspace for integration transformation is defined as

$$\mathbf{N} = \{f \in \mathbf{V} : \mathbf{T}f = 0\} \tag{3.1.3.25}$$

Let the coefficient matrix of $f \in \mathbf{V}$ be

$$\mathbf{f} = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{pmatrix} \tag{3.1.3.26}$$

then

$$\mathbf{T}f = 0 \tag{3.1.3.27}$$

$$\implies \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & \frac{1}{n-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{n} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{pmatrix} = \mathbf{0} \tag{3.1.3.28}$$

Since T is in row reduced echolon form and $rank(T) = n$ the solution of (3.1.3.28) is

$$\mathbf{f} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{3.1.3.29}$$

where $k \in \mathbf{R}$. Therefore the nullspace for $\mathbf{T} : \mathbf{V} \to \mathbf{V}$ is

$$\mathbf{N} = \left\{ \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} : k \in \mathbf{R} \right\} \tag{3.1.3.30}$$

3.1.4. Is there a linear transformation $\mathbf{T}$ from $\mathbb{R}^3$ into

$\mathbb{R}^2$ such that,

$$\mathbf{T}\begin{pmatrix}1\\-1\\1\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}\tag{3.1.4.1}$$

$$\mathbf{T}\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}\tag{3.1.4.2}$$

Solution: *Linear Transformation* A linear transformation is a function $\mathbf{T}:\mathbb{R}^n\to\mathbb{R}^m$ which satisfies:
1. $\forall\mathbf{x},\mathbf{y}\in\mathbb{R}^n$,

$$\mathbf{T}\left(\mathbf{x}+\mathbf{y}\right)=\mathbf{T}\left(\mathbf{x}\right)+\mathbf{T}\left(\mathbf{y}\right)\tag{3.1.4.3}$$

2. $\forall\mathbf{x}\in\mathbb{R}^n$ and c $\in\mathbb{R}$,

$$\mathbf{T}\left(c\mathbf{x}\right)=c\mathbf{T}\left(\mathbf{x}\right)\tag{3.1.4.4}$$

Matrix of the Linear Transformation Let, $\mathbf{T}:\mathbb{R}^n\to\mathbb{R}^m$ is a linear transformation and $\mathbf{x}\in\mathbb{R}^n$ is given by ,

$$\mathbf{x}=\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}\tag{3.1.4.5}$$

$$=x_1\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}+x_2\begin{pmatrix}0\\1\\\vdots\\0\end{pmatrix}+\cdots+x_n\begin{pmatrix}0\\0\\\vdots\\1\end{pmatrix}\tag{3.1.4.6}$$

Let $\mathbf{e_1},\mathbf{e_2},\ldots,\mathbf{e_n}$ be the standard basis of $\mathbb{R}^n$ and the equation (3.1.4.6) can be rewritten as,

$$\mathbf{x}=\sum_{i=1}^{n}x_i\mathbf{e_i}\tag{3.1.4.7}$$

$$\mathbf{T}\left(\mathbf{x}\right)=\sum_{i=1}^{n}x_i\mathbf{T}\left(\mathbf{e_i}\right)\tag{3.1.4.8}$$

$$=\begin{pmatrix}\mathbf{T}\left(\mathbf{e_1}\right)&\mathbf{T}\left(\mathbf{e_2}\right)&\ldots&\mathbf{T}\left(\mathbf{e_n}\right)\end{pmatrix}\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}\tag{3.1.4.9}$$

$$\mathbf{T}\left(\mathbf{x}\right)=\mathbf{A}\mathbf{x}\tag{3.1.4.10}$$

Where,

$$\mathbf{A}=\begin{pmatrix}\mathbf{T}\left(\mathbf{e_1}\right)&\mathbf{T}\left(\mathbf{e_n}\right)&\ldots&\mathbf{T}\left(\mathbf{e_n}\right)\end{pmatrix}\tag{3.1.4.11}$$

If $\mathbf{T}$ is any linear transformation which maps $\mathbb{R}^n\to\mathbb{R}^m$ there is always an $m\times n$ matrix $\mathbf{A}$ with the property that

$$\mathbf{T}\left(\mathbf{x}\right)=\mathbf{A}\mathbf{x}$$

where , $\mathbf{x}\in\mathbb{R}^n$ *Solution* Let,

$$\mathbf{v}=\begin{pmatrix}1\\-1\\1\end{pmatrix}\tag{3.1.4.12}$$

$$\mathbf{u}=\begin{pmatrix}1\\1\\1\end{pmatrix}\tag{3.1.4.13}$$

Given,

$$\mathbf{T}\left(\mathbf{v}\right)=\begin{pmatrix}1\\0\end{pmatrix}\tag{3.1.4.14}$$

$$\mathbf{T}\left(\mathbf{u}\right)=\begin{pmatrix}0\\1\end{pmatrix}\tag{3.1.4.15}$$

Let the standard basis vectors is denoted as,

$$\mathbf{e_1}=\begin{pmatrix}1\\0\\0\end{pmatrix}\tag{3.1.4.16}$$

$$\mathbf{e_2}=\begin{pmatrix}0\\1\\0\end{pmatrix}\tag{3.1.4.17}$$

$$\mathbf{e_3}=\begin{pmatrix}0\\0\\1\end{pmatrix}\tag{3.1.4.18}$$

Let, $\mathbf{T}:\mathbb{R}^3\to\mathbb{R}^2$ be a linear transformation. Then the function $\mathbf{T}$ is just matrix-vector multiplication $\mathbf{T}(\mathbf{x})=\mathbf{A}\mathbf{x}$ for some matrix $\mathbf{A}$ as shown in equation (3.1.4.10)
Matrix $\mathbf{A}$ of order 2×3 is given by,

$$\mathbf{A}=\begin{pmatrix}\mathbf{T}\left(\mathbf{e_1}\right)&\mathbf{T}\left(\mathbf{e_2}\right)&\mathbf{T}\left(\mathbf{e_3}\right)\end{pmatrix}\tag{3.1.4.19}$$

Consider the vector $\mathbf{b}\in\mathbb{R}^3$ which is the linear combinations of the vectors $\mathbf{v}$ and $\mathbf{u}$.
For $x_1,x_2\in\mathbb{R}$,

$$\mathbf{b}=\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}=x_1\begin{pmatrix}1\\-1\\1\end{pmatrix}+x_2\begin{pmatrix}1\\1\\1\end{pmatrix}\tag{3.1.4.20}$$

$$\mathbf{T}\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}=x_1\mathbf{T}\begin{pmatrix}1\\-1\\1\end{pmatrix}+x_2\mathbf{T}\begin{pmatrix}1\\1\\1\end{pmatrix}\tag{3.1.4.21}$$

To find x_1, x_2, we solve the linear system,$\mathbf{M}\mathbf{x} = \mathbf{b}$ where $\mathbf{M}$ is the 3×2 matrix obtained by stacking the given vectors $\mathbf{v}$ and $\mathbf{u}$ as columns

$$\mathbf{M} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 1 \end{pmatrix} \tag{3.1.4.22}$$

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \tag{3.1.4.23}$$

The augumented matrix of the equation (3.1.4.23) is given by ,

$$\left(\begin{array}{cc|c} 1 & 1 & b_1 \\ -1 & 1 & b_2 \\ 1 & 1 & b_3 \end{array}\right) \tag{3.1.4.24}$$

By row reducing the above equation (3.1.4.24),

$$\left(\begin{array}{cc|c} 1 & 1 & b_1 \\ -1 & 1 & b_2 \\ 1 & 1 & b_3 \end{array}\right) \xleftrightarrow{R_2=R_2+R_1} \left(\begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & 2 & b_2+b_1 \\ 1 & 1 & b_3 \end{array}\right) \tag{3.1.4.25}$$

$$\left(\begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & 2 & b_2+b_1 \\ 1 & 1 & b_3 \end{array}\right) \xleftrightarrow{R_3=R_3-R_1} \left(\begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & 2 & b_2+b_1 \\ 0 & 0 & b_3-b_1 \end{array}\right) \tag{3.1.4.26}$$

$$\left(\begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & 2 & b_2+b_1 \\ 0 & 0 & b_3-b_1 \end{array}\right) \xleftrightarrow{R_2=\frac{R_2}{2}} \left(\begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & 1 & \frac{b_2+b_1}{2} \\ 0 & 0 & b_3-b_1 \end{array}\right) \tag{3.1.4.27}$$

$$\left(\begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & 1 & \frac{b_2+b_1}{2} \\ 0 & 0 & b_3-b_1 \end{array}\right) \xleftrightarrow{R_1=R_1-R_2} \left(\begin{array}{cc|c} 1 & 0 & \frac{b_1-b_2}{2} \\ 0 & 1 & \frac{b_2+b_1}{2} \\ 0 & 0 & b_3-b_1 \end{array}\right) \tag{3.1.4.28}$$

Now equation (3.1.4.23) can be written as,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{b_1-b_2}{2} \\ \frac{b_2+b_1}{2} \\ b_3-b_1 \end{pmatrix} \tag{3.1.4.29}$$

Solving the above equation we get ,

$$x_1 = \frac{b_1 - b_2}{2} \tag{3.1.4.30}$$

$$x_2 = \frac{b_1 + b_2}{2} \tag{3.1.4.31}$$

Substituting the above equations (3.1.4.30),(3.1.4.31) in equation (3.1.4.21), we get,

$$\mathbf{T}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \left(\frac{b_1 - b_2}{2}\right)\mathbf{T}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \left(\frac{b_1 + b_2}{2}\right)\mathbf{T}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{3.1.4.32}$$

Substituting the equations (3.1.4.1) and (3.1.4.2) in equation (3.1.4.32) we get,

$$\mathbf{T}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \left(\frac{b_1 - b_2}{2}\right)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \left(\frac{b_1 + b_2}{2}\right)\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.1.4.33}$$

$$\mathbf{T}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} \frac{b_1-b_2}{2} \\ \frac{b_1+b_2}{2} \end{pmatrix} \tag{3.1.4.34}$$

Using the above equation (3.1.4.34) we compute,

$$\mathbf{T}\left(\mathbf{e_1}\right) = \mathbf{T}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \tag{3.1.4.35}$$

$$\mathbf{T}\left(\mathbf{e_2}\right) = \mathbf{T}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{-1}{2} \\ \frac{1}{2} \end{pmatrix} \tag{3.1.4.36}$$

$$\mathbf{T}\left(\mathbf{e_3}\right) = \mathbf{T}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.1.4.37}$$

Substituting the equations (3.1.4.35),(3.1.4.36) and (3.1.4.37) in equation (3.1.4.19) we get,

$$\mathbf{A} = \begin{pmatrix} \frac{1}{2} & \frac{-1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \tag{3.1.4.38}$$

$$\implies \mathbf{A} = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \tag{3.1.4.39}$$

Therefore from the above matrix $\mathbf{A}$ we can say that there is a linear transformation $\mathbf{T}$ from $\mathbb{R}^3$ into $\mathbb{R}^2$ which satisfies the given conditions (3.1.4.1) and (3.1.4.2).

3.1.5. Describe explicitly the linear transformation T from $\mathbb{F}^2$ into $\mathbb{F}^2$ such that $T(\epsilon_1) = (a, b), T(\epsilon_2) = (c, d)$. **Solution:** We are given a linear transformation,

$$T : \mathbb{F}^2 \to \mathbb{F}^2 \tag{3.1.5.1}$$

The transformation for $\in_1$ and $\in_2$ can be written as,

$$T(\in_1) = \begin{pmatrix} a \\ b \end{pmatrix} \tag{3.1.5.2}$$

$$T(\in_2) = \begin{pmatrix} c \\ d \end{pmatrix} \tag{3.1.5.3}$$

Now,let's assume $\in_1$ and $\in_2$ as linearly independent. So the linear transformation T for any vector **v** in two dimensional space will be,

$$T(\mathbf{v}) = \begin{pmatrix} T(\in_\mathbf{1}) & T(\in_\mathbf{2}) \end{pmatrix} \mathbf{v} \tag{3.1.5.4}$$

$$= \begin{pmatrix} a & c \\ b & d \end{pmatrix} \mathbf{v} \tag{3.1.5.5}$$

Now, there can be two cases here, transformation of linearly independent vector can be independent or it can be dependent.Considering the first case and (3.1.5.5) we can say that,

$$Range(T) = \text{columnspace of} \begin{pmatrix} a & c \\ b & d \end{pmatrix} \tag{3.1.5.6}$$

Now, considering the case when linear transformation will be linearly dependent,

$$Range(T) = \text{columnspace of} \begin{pmatrix} a \\ b \end{pmatrix} \tag{3.1.5.7}$$

Now, considering that vectors $\in_1$ and $\in_2$ itself are linearly dependent.Let $\mathbf{v}=\in_1 + \in_2$

$$T(\mathbf{v}) = T(\in_1) + T(\in_2) \tag{3.1.5.8}$$

$$= T(\in_1) + T(k \in_1) \tag{3.1.5.9}$$

$$= (k+1)T(\in_1) \tag{3.1.5.10}$$

$$= (k+1)\begin{pmatrix} a \\ b \end{pmatrix} \tag{3.1.5.11}$$

We can see from above equation that when $\in_1$ and $\in_2$ as linearly dependent then the transformation T will be along the line only.

Vectors Independent	**Vectors Dependent**
$T(\mathbf{v})=\begin{pmatrix} a & c \\ b & d \end{pmatrix}\mathbf{v}$	$T(\mathbf{v})=(\text{k}+1)\begin{pmatrix} a \\ b \end{pmatrix}$
Output: On the plane	Output: On the line

TABLE 3.1.5

3.1.6. Let $\mathbb{F}$ be a subfield of the complex numbers and let $\mathbb{T}$ be the function from $\mathbb{F}^3$ into $\mathbb{F}^3$ defined by

$$\mathbb{T}(x_1, x_2, x_3) = \tag{3.1.6.1}$$

$$(x_1 - x_2 + 2x_3, 2x_1 + x_2, -x_1 - 2x_2 + 2x_3) \tag{3.1.6.2}$$

a) Verify that $\mathbb{T}$ is a linear transformation.

b) If (a, b, c) is a vector in $\mathbb{F}^3$, what are the conditions on a, b, c that the vector be in the range of $\mathbb{T}$?What is the rank of $\mathbb{T}$?

c) What are the conditions on a, b, c that (a, b, c) be in the null space of $\mathbb{T}$? What is the nullity of $\mathbb{T}$?

Solution: Representing the transformation in matrix form

$$\mathbb{T}(x_1, x_2, x_3) = \mathbf{Tx} \tag{3.1.6.3}$$

$$\mathbf{T} = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ -1 & -2 & 2 \end{pmatrix} \tag{3.1.6.4}$$

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{3.1.6.5}$$

Part (a) Consider the matrices $\mathbf{x}, \mathbf{y} \in \mathbb{F}^3$ and the scalar $c \in \mathbb{F}$
By the associativity of matrix multiplications, we can write

$$\mathbf{T}(c\mathbf{x} + \mathbf{y}) = \mathbf{T}(c\mathbf{x}) + \mathbf{Ty} \tag{3.1.6.6}$$

$$= c\mathbf{Tx} + \mathbf{Ty} \tag{3.1.6.7}$$

So, **T** is a linear transformation. Part (b)

$$range(\mathbf{T}) = \left\{\mathbf{y} : \mathbf{Tx} = \mathbf{y} \text{ where } \mathbf{x}, \mathbf{y} \in \mathbb{F}^3\right\} \tag{3.1.6.8}$$

$$\mathbf{y} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad (3.1.6.9)$$

$$\mathbf{Tx} = \mathbf{y} \quad (3.1.6.10)$$

$$\implies \mathbf{BTx} = \mathbf{By} \quad (3.1.6.11)$$

$$\implies \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{-2}{3} & \frac{1}{3} & 0 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ -1 & -2 & 2 \end{pmatrix} \mathbf{x} = \quad (3.1.6.12)$$

$$\begin{pmatrix} \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{-2}{3} & \frac{1}{3} & 0 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad (3.1.6.13)$$

$$\begin{pmatrix} 1 & 0 & \frac{2}{3} \\ 0 & 1 & \frac{-4}{3} \\ 0 & 0 & 0 \end{pmatrix} \mathbf{x} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{-2}{3} & \frac{1}{3} & 0 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad (3.1.6.14)$$

So, rank($\mathbf{T}$)=2 and comparing the third row element in LHS and RHS of (3.1.6.14)

$$-a + b + c = 0 \quad (3.1.6.15)$$

All vectors $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{F}^3$ that satisfy (3.1.6.15) lie in the range of $\mathbf{T}$ Part (c)

$$nullspace(\mathbf{T}) = \left\{ \mathbf{x} : \mathbf{Tx} = \mathbf{0} \text{ where } \mathbf{x} \in \mathbb{F}^3 \right\} \quad (3.1.6.16)$$

$$\mathbf{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad (3.1.6.17)$$

$$\mathbf{Tx} = \mathbf{0} \quad (3.1.6.18)$$

$$\mathbf{BTx} = \mathbf{0} \quad (3.1.6.19)$$

where $\mathbf{BT}$ is in reduced row echelon form

$$\begin{pmatrix} \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{-2}{3} & \frac{1}{3} & 0 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ -1 & -2 & 2 \end{pmatrix} \mathbf{x} = \mathbf{0} \quad (3.1.6.20)$$

$$\implies \begin{pmatrix} 1 & 0 & \frac{2}{3} \\ 0 & 1 & \frac{-4}{3} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (3.1.6.21)$$

$$\implies a + \frac{2}{3}c = 0 \quad (3.1.6.22)$$

$$b - \frac{4}{3}c = 0 \quad (3.1.6.23)$$

The number of free variables in the reduced row echelon form of $\mathbf{T}$ is 1 hence nullity($\mathbf{T}$) =1

So, the null space of $\mathbf{T}$ is set of all vectors $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{F}^3$ that satisfy (3.1.6.22) and (3.1.6.23)

<u>Note</u>

rank($\mathbf{T}$)+nullity($\mathbf{T}$)=2+1=dim($\mathbb{F}^3$)

3.1.7. Describe explicitly a linear transformation from R^3 into R^3 which has as its range the subspace spanned by $\begin{pmatrix} 1 & 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 & 2 \end{pmatrix}$.

Solution: Transformation T from R^3 to R^3 range gives the column space. Hence,

$$T(\mathbf{x}) = \mathbf{Ax} \quad (3.1.7.1)$$

$$T(\mathbf{x}) = \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ -1 & 2 \end{pmatrix} \mathbf{x} \quad (3.1.7.2)$$

3.1.8. Let $\mathbf{V}$ be the vector space of all $n \times n$ matrices over the field $\mathbb{F}$, and let $\mathbf{B}$ be a fixed $n \times n$ matrix. If a transformation T defined as follows,

$$T(\mathbf{A}) = \mathbf{AB} - \mathbf{BA}$$

Prove that T is a linear transformation from $\mathbf{V}$ into $\mathbf{V}$ **Solution:** Let,

$$\mathbf{A_1} \in \mathbf{V} \quad (3.1.8.1)$$

$$\mathbf{A_2} \in \mathbf{V} \quad (3.1.8.2)$$

If c be any scalar of the field $\mathbb{F}$ we get,

$$c\mathbf{A_1} + \mathbf{A_2} \in \mathbf{V} \quad (3.1.8.3)$$

Applying transformation T on $(c\mathbf{A_1} + \mathbf{A_2})$ we get,

$$T(c\mathbf{A_1} + \mathbf{A_2}) = (c\mathbf{A_1} + \mathbf{A_2})\mathbf{B} - \mathbf{B}(c\mathbf{A_1} + \mathbf{A_2}) \quad (3.1.8.4)$$

$$= c\mathbf{A_1B} + \mathbf{A_2B} - c\mathbf{BA_1} - \mathbf{BA_2} \quad (3.1.8.5)$$

$$= c(\mathbf{A_1B} - \mathbf{BA_1}) + (\mathbf{A_2B} - \mathbf{BA_2}) \quad (3.1.8.6)$$

$$= cT(\mathbf{A_1}) + T(\mathbf{A_2}) \quad (3.1.8.7)$$

From (3.1.8.7) we conclude that T is a linear transformation from vector space $\mathbf{V}$ to $\mathbf{V}$.

3.1.9. Let V be the set of all complex numbers regarded as a vector space over the field of real numbers(usual operations). Find a function from V into V which is a linear transformation on the above vector space, but which is not a linear transformation on $\mathbb{C}$ i.e., which is not complex linear.

Solution: Let

$$T : V \to V \tag{3.1.9.1}$$

be a function such that,

$$T(x+iy) = Re(x+iy) = x \tag{3.1.9.2}$$

$$\implies T : x+iy \to x \tag{3.1.9.3}$$

where $x, y \in \mathbb{R}$.
Let, $\alpha = a+ib$, $\beta = c+id$.

$$\implies T(k\alpha+\beta) = T(ka+ikb+c+id) \tag{3.1.9.4}$$

$$= T(ka+c+i(kb+d)) \tag{3.1.9.5}$$

$$= ka+c \tag{3.1.9.6}$$

$$= kT(\alpha)+T(\beta) \tag{3.1.9.7}$$

Now, let $z \in V$ such that,

$$z = i \tag{3.1.9.8}$$

$$\implies T(z) = T(i) = 0 \tag{3.1.9.9}$$

We can also write,

$$T(i) = T(i(1)) = iT(1) = i \neq 0 \tag{3.1.9.10}$$

Thus from (3.1.9.7), T is real linear transformation and from (3.1.9.10), T is not complex linear.

3.1.10. Let $\mathbf{V}$ be the space of $n \times 1$ matrices over F and let $\mathbf{W}$ be the space of $m \times 1$ matrices over F. Let $\mathbf{A}$ be a fixed $m \times n$ matrix over F and let T be the linear transformation from $\mathbf{V}$ into $\mathbf{W}$ defined by $T(\mathbf{X}) = \mathbf{AX}$. Prove that T is the zero transformation if and only if $\mathbf{A}$ is the zero matrix. **Solution:** If $\mathbf{A}_{m\times n}$ is a zero transformation and $\mathbf{X}_{n\times 1}$ is a vector, then

$$\mathbf{AX} = \mathbf{0}_{m\times 1} \tag{3.1.10.1}$$

Let,

$$\mathbf{A} = \begin{pmatrix}\mathbf{A_1} & \dots & \mathbf{A_j} & \dots & \mathbf{A_n}\end{pmatrix}_{1\times n} \text{ and} \tag{3.1.10.2}$$

$$\mathbf{X_j} = \begin{pmatrix} x_1 \\ \vdots \\ x_j \\ \vdots \\ x_n \end{pmatrix}, \text{ where } x_i = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \tag{3.1.10.3}$$

If $\mathbf{A}_{m\times n}$ is zero transformation, then for any vector $\mathbf{X}_{n\times 1}$, $\mathbf{AX} = \mathbf{0}$.
Consider ,

$$\mathbf{AX_j} = \mathbf{0}_{m\times 1} \tag{3.1.10.4}$$

$$\begin{pmatrix}\mathbf{A_1} \dots \mathbf{A_j} \dots \mathbf{A_n}\end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_j \\ \vdots \\ x_n \end{pmatrix} = \mathbf{0}_{m\times 1} \tag{3.1.10.5}$$

From (3.1.10.3) and (3.1.10.5)

$$\mathbf{A_j} = \mathbf{0}_{m\times 1} \text{ for } j = 1, 2, \dots n \tag{3.1.10.6}$$

Substitute (3.1.10.6) in (3.1.10.2)

$$\mathbf{A} = \begin{pmatrix}\mathbf{0}_{m\times 1} & \mathbf{0}_{m\times 1} & \dots & \mathbf{0}_{m\times 1}\end{pmatrix}_{1\times n} \tag{3.1.10.7}$$

$$\therefore \mathbf{A} = \mathbf{0}_{m\times n} \tag{3.1.10.8}$$

Hence $\mathbf{A}$ is zero matrix.

Let us assume $\mathbf{A}_{m\times n}$ is a zero matrix

$$\mathbf{A} = \mathbf{0}_{m\times n} \tag{3.1.10.9}$$

Then,

$$T(\mathbf{X}) = \mathbf{AX} \tag{3.1.10.10}$$

$$= \mathbf{0}.\mathbf{X} \tag{3.1.10.11}$$

$$= \mathbf{0}_{m\times 1} \ , \forall \mathbf{X} \in F \tag{3.1.10.12}$$

Hence $T(\mathbf{X}) = \mathbf{AX}$ is the zero transformation.

From (3.1.10.8) and (3.1.10.12) it is proved that T is the zero transformation if and only if $\mathbf{A}$ is the zero matrix.

3.1.11. Let $\mathbf{V}$ be an n-dimensional vector space over the field **F** and let **T** be a linear transformation from **V** into **V** such that the range and null space of **T** are identical. Prove that n is even. (Can you give an example of such a linear transformation **T**)?

Solution: Let **V** and **W** be vector spaces over the field **F** and let **T** be a linear transformation from **V** into **W**. Then,

$$\text{rank}(\mathbf{T}) + \text{nullity}(\mathbf{T}) = \dim \mathbf{V} \tag{3.1.11.1}$$

It is given that range and null space of **T** are same, let us assume it to be m. Substituting in equation (3.1.11.1)

$$m + m = n \tag{3.1.11.2}$$

$$\implies n = 2m \tag{3.1.11.3}$$

From equation (3.1.11.3), we can say that n is even.

Example: Let us consider a vector space **V**, such that $\mathbf{V} \in \mathbb{R}^2$ and let us consider a linear transformation $\mathbf{T} : \mathbf{V} \to \mathbf{V}$ defined by $\mathbf{T}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ 0 \end{pmatrix}$ and is given by matrix **M**

$$\mathbf{T}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ 0 \end{pmatrix} \tag{3.1.11.4}$$

$$\mathbf{M} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{3.1.11.5}$$

Let us consider basis of $\mathbb{R}^2$ $\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$ and apply linear transformation on it.

$$\mathbf{T}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.1.11.6}$$

$$\mathbf{T}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.1.11.7}$$

From (3.1.11.5),
The range of matrix can be found from row reduced echelon form. But as matrix **M** is in RREF form,
the basis for range is given by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

The null space of matrix is,

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.1.11.8}$$

$$\implies x_1 = t \quad x_2 = 0 \tag{3.1.11.9}$$

$$\implies \mathbf{X} = \begin{pmatrix} t \\ 0 \end{pmatrix} = t\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.1.11.10}$$

The basis for null space is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

$$\text{rank}(\mathbf{T}) = 1 \quad \text{nullity}(\mathbf{T}) = 1 \tag{3.1.11.11}$$

$$\dim(\mathbf{V}) = 2 \tag{3.1.11.12}$$

Thus the range and null space are equal, and n is even.

3.1.12. Let $\mathbb{V}$ be a vector space and **T** a linear transformation from $\mathbb{V}$ into $\mathbb{V}$. Prove that the following two statements about **T** are equivalent.

(a) The intersection of the range of **T** and null space of **T** is the zero subspace of $\mathbb{V}$.

(b) If $\mathbf{T}(\mathbf{T}\alpha) = 0$, then $\mathbf{T}\alpha = 0$.

Solution:

Given	$\mathbf{T} : \mathbb{V} \to \mathbb{V}$
To prove	a) $range(\mathbf{T}) \cap nullspace(\mathbf{T}) = \{0\}$ b) If $\mathbf{T}(\mathbf{T}\alpha) = 0$, then $\mathbf{T}\alpha = 0$.

3.2 The Algebra of Linear Transformations

3.2.1. Let **T** and **U** be the linear operators on $\mathbb{R}^2$ defined by $\mathbf{T}(x_1, x_2) = (x_2, x_1)$ and $\mathbf{U}(x_1, x_2) = (x_1, 0)$.

a) Let**T** and **U** be the linear operators on $\mathbb{R}^2$ defined by

$$\mathbf{T}(x_1, x_2) = (x_2, x_1) \tag{3.2.1.1}$$

and

$$\mathbf{U}(x_1, x_2) = (x_1, 0) \tag{3.2.1.2}$$

How would you describe T and U geometrically ?

Solution: Geometrically, in the x-y plane, **T** is the reflection about the diagonal x = y

Proof(a)	Let $\mathbf{x} \in \mathbb{V}$ and $\mathbf{x} \in range(\mathbf{T}) \cap nullspace(\mathbf{T})$ then, $\mathbf{x} \in range(\mathbf{T})$ $\mathbf{x} \in nullspace(\mathbf{T})$ Consider $\mathbf{y} \in \mathbb{V}$ whose linear transformation into $\mathbb{V}$ is $\mathbf{x}$. $\implies \mathbf{T}(\mathbf{y}) = \mathbf{x}$ since $\mathbf{x} \in$ null space$(\mathbf{T})$ and the sub space is linearly independent $\mathbf{T}(\mathbf{x}) = 0$ from above equations $\mathbf{T}(\mathbf{T}(y)) = 0$ from the definition of linear transformation of independent vector space $\mathbf{T}(y) = 0$ $\implies \mathbf{x} = 0$ $\implies \{0\} \subseteq range(\mathbf{T}) \cap nullspace(\mathbf{T})$ $\therefore range(\mathbf{T}) \cap nullspace(\mathbf{T}) = \{0\}$ Hence Proved.
Proof(b)	If $\mathbf{T}(\mathbf{T}\alpha) = 0$ then, from the definition of linear transformation, $\mathbf{T}\alpha$ will be in the null space of linear transformation $\mathbf{T}$ and is linearly independent $\therefore \mathbf{T}\alpha = 0$

Eg:	Let $\alpha \in \mathbb{V}$ and $\alpha = \begin{pmatrix} 1 & 7 & -1 & -1 \\ -1 & 1 & 2 & 1 \\ 4 & -2 & 0 & -4 \\ 2 & 3 & 4 & -2 \end{pmatrix}$ linear transformation of α into $\mathbb{V}$ $\mathbf{T}(\alpha) = c\alpha$ then row reduced echelon form of $\mathbf{T}$ is $rref(\mathbf{T}) = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $\implies rank(\mathbf{T}) = 3,$ $nullity(\mathbf{T}) = 1$ $\implies range(\mathbf{T}) = \begin{pmatrix} 1 & 7 & -1 \\ -1 & 1 & 2 \\ 4 & -2 & 0 \\ 2 & 3 & 4 \end{pmatrix},$ $nullspace(\mathbf{T}) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ $\therefore range(\mathbf{T}) \cap nullspace(\mathbf{T}) = \{0\}$ Hence proved that the intersection of the range of $\mathbf{T}$ and null space of $\mathbf{T}$ is the zero subspace of $\mathbb{V}$.

and $\mathbf{U}$ is a projection onto the x-axis.

i) Reflection

Let Consider Matrix A as

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.2.1.3}$$

The matrix $\mathbf{A}$ is representation of the linear transformation T across the line y=x with respect to the standard basis.

Let suppose

$$\mathbf{x_1} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \mathbf{x_2} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \tag{3.2.1.4}$$

After applying linear operator $\mathbf{T}$ on it,

$$\mathbf{T}(x_1, x_2) = \mathbf{A}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{3.2.1.5}$$

$$\implies \mathbf{A}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{3.2.1.6}$$

Similarly

$$\mathbf{A}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \tag{3.2.1.7}$$

Hence after applying Operator $\mathbf{T}$ on $\mathbf{x_1}$ and $\mathbf{x_2}$

$$\mathbf{x_1} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \mathbf{x_2} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \tag{3.2.1.8}$$

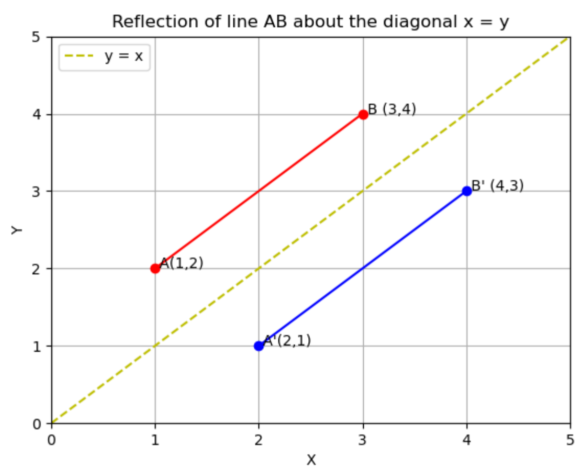

Fig. 3.2.1.1: Reflection of line AB about the x = y

ii) Projection

For projection let Consider Matrix B as

$$\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{3.2.1.9}$$

The matrix $\mathbf{B}$ is representation of the linear transformation $\mathbf{U}$ that is projection on x-axis.

Let suppose

$$\mathbf{x_1} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \mathbf{x_2} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \tag{3.2.1.10}$$

After applying linear operator $\mathbf{U}$ on $\mathbf{x_1}$ and x_2,

$$\mathbf{T}(x_1, x_2) = \mathbf{U}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{3.2.1.11}$$

$$\implies \mathbf{B}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.2.1.12}$$

Similarly

$$\mathbf{A}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \tag{3.2.1.13}$$

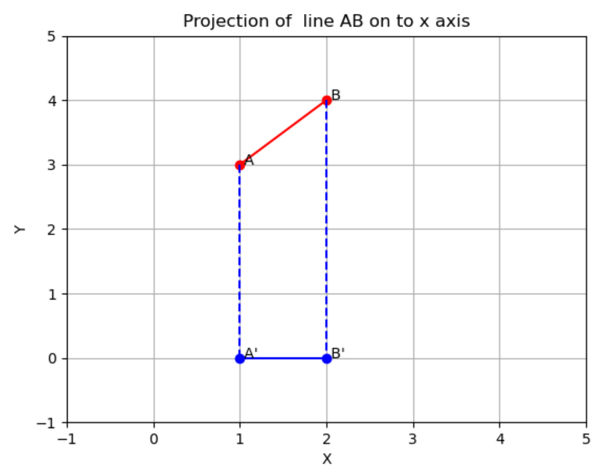

Fig. 3.2.1.2: Projection of AB onto x-axis

Hence after applying Operator $\mathbf{U}$ on $\mathbf{x_1}$ and $\mathbf{x_2}$

$$\mathbf{x_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{x_2} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \tag{3.2.1.14}$$

b) Give rules like the ones defining $\mathbf{T}$ and $\mathbf{U}$ for each of the transformations $\mathbf{U} + \mathbf{T}$, $\mathbf{UT}$, $\mathbf{TU}$, $\mathbf{T}^2$, $\mathbf{U}^2$. $\mathbb{R}^2$ into $\mathbb{R}^2$ is linear transformation?
Solution: Let $\mathbf{T}$ and $\mathbf{U}$ defined by matrices $\mathbf{A}$ and $\mathbf{B}$ such that ,

$$\mathbf{T}(\mathbf{x}) = \mathbf{Ax}; \quad \mathbf{U}(\mathbf{x}) = \mathbf{Bx} \tag{3.2.1.15}$$

Where,

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{3.2.1.16}$$

Table 3.2.1.1 lists the summary of each Transformations.

3.2.2. Let T be the unique linear operator on C^3 for

Transformations	Matrix	Vector
$\mathbf{U}+\mathbf{T}$	$(\mathbf{B}+\mathbf{A})=\begin{pmatrix}1&1\\1&0\end{pmatrix}$	$\begin{pmatrix}x_1+x_2\\x_1\end{pmatrix}$
$\mathbf{UT}$	$\mathbf{BA}=\begin{pmatrix}0&1\\0&0\end{pmatrix}$	$\begin{pmatrix}x_2\\0\end{pmatrix}$
$\mathbf{TU}$	$\mathbf{AB}=\begin{pmatrix}0&0\\1&0\end{pmatrix}$	$\begin{pmatrix}0\\x_1\end{pmatrix}$
$\mathbf{T}^2$	$\mathbf{A}^2=\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}x_1\\x_2\end{pmatrix}$
$\mathbf{U}^2$	$\mathbf{B}^2=\begin{pmatrix}1&0\\0&0\end{pmatrix}$	$\begin{pmatrix}x_1\\0\end{pmatrix}$

TABLE 3.2.1.1: Summary

which

$$T(\epsilon_1)=\begin{pmatrix}1&0&i\end{pmatrix}, T(\epsilon_2)=\begin{pmatrix}0&1&1\end{pmatrix}, T(\epsilon_3)=\begin{pmatrix}i&1&0\end{pmatrix} \tag{3.2.2.1}$$

Is T invertible ?

Solution: Let ϵ_i is basis for C^3 such that $T(\epsilon_i)$ is basis for C^3 T is said to be singular if

$$T(\epsilon)=0 \implies \epsilon\neq 0 \tag{3.2.2.2}$$

now,

$$\begin{pmatrix}1&0&i\\0&1&1\\i&1&0\end{pmatrix}\epsilon=0 \tag{3.2.2.3}$$

consider the row reduced matrix

$$\begin{pmatrix}1&0&i\\0&1&1\\i&1&0\end{pmatrix}\xleftrightarrow[R_3\to R_3-R_2]{R_3\to R_3-iR_1}\begin{pmatrix}1&0&i\\0&1&1\\0&0&0\end{pmatrix} \tag{3.2.2.4}$$

$$\epsilon=c\begin{pmatrix}-i\\-1\\1\end{pmatrix} \tag{3.2.2.5}$$

Hence it holds the condition of singularity therefore T is not invertible .

3.2.3. For the linear operator $\mathbf{T}$

$$\mathbf{T}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}3x_1\\x_1-x_2\\2x_1+x_2+x_3\end{pmatrix} \tag{3.2.3.1}$$

3.2.4. Let $\mathbf{T}$ be a linear operator on $\mathbf{R}^3$ defined by

$$\mathbf{T}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}3x_1\\x_1-x_2\\2x_1+x_2+x_3\end{pmatrix}$$

Is $\mathbf{T}$ invertible? If so, find a rule for $\mathbf{T}^{-1}$ like the one which defines T.

Solution: The transformed vector can be re-written by expanding the columns as follows

$$\begin{pmatrix}3x_1\\x_1-x_2\\2x_1+x_2+x_3\end{pmatrix}=\begin{pmatrix}3\\1\\2\end{pmatrix}x_1+\begin{pmatrix}0\\-1\\1\end{pmatrix}x_2+\begin{pmatrix}0\\0\\1\end{pmatrix}x_3 \tag{3.2.4.1}$$

$$=\begin{pmatrix}3&0&0\\1&-1&0\\2&1&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} \tag{3.2.4.2}$$

$$\implies \mathbf{T}=\begin{pmatrix}3&0&0\\1&-1&0\\2&1&1\end{pmatrix} \tag{3.2.4.3}$$

Using Gauss-Jordan Elimination to find the inverse of $\mathbf{T}$, if it exists

$$\left(\begin{array}{ccc|ccc}3&0&0&1&0&0\\1&-1&0&0&1&0\\2&1&1&0&0&1\end{array}\right) \tag{3.2.4.4}$$

$$\xleftrightarrow{R_1\leftarrow\frac{R_1}{3}}\left(\begin{array}{ccc|ccc}1&0&0&\frac{1}{3}&0&0\\1&-1&0&0&1&0\\2&1&1&0&0&1\end{array}\right) \tag{3.2.4.5}$$

$$\xleftrightarrow[R_3\leftarrow R_3-2R_1]{R_2\leftarrow R_2-R_1}\left(\begin{array}{ccc|ccc}1&0&0&\frac{1}{3}&0&0\\0&-1&0&-\frac{1}{3}&1&0\\0&1&1&-\frac{2}{3}&0&1\end{array}\right) \tag{3.2.4.6}$$

$$\xleftrightarrow{R_2\leftarrow -R_2}\left(\begin{array}{ccc|ccc}1&0&0&\frac{1}{3}&0&0\\0&1&0&\frac{1}{3}&-1&0\\0&1&1&-\frac{2}{3}&0&1\end{array}\right) \tag{3.2.4.7}$$

$$\xleftrightarrow{R_3\leftarrow R_3-R_2}\left(\begin{array}{ccc|ccc}1&0&0&\frac{1}{3}&0&0\\0&1&0&\frac{1}{3}&-1&0\\0&0&1&-1&1&1\end{array}\right) \tag{3.2.4.8}$$

Since $rank(\mathbf{T})=3$, $\mathbf{T}$ is invertible and the inverse is

$$\mathbf{T}^{-1} = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & -1 & 0 \\ -1 & 1 & 1 \end{pmatrix} \tag{3.2.4.9}$$

Now consider any vector $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbf{R}^3$, then

$$\mathbf{T}^{-1}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & -1 & 0 \\ -1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{3.2.4.10}$$

$$= \begin{pmatrix} \frac{x_1}{3} \\ \frac{x_1}{3} - x_2 \\ -x_1 + x_2 + x_3 \end{pmatrix} \tag{3.2.4.11}$$

Therefore the transformation $\mathbf{T}^{-1}$ is defined on $\mathbf{R}^3$ as

$$\mathbf{T}^{-1}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{x_1}{3} \\ \frac{x_1}{3} - x_2 \\ -x_1 + x_2 + x_3 \end{pmatrix} \tag{3.2.4.12}$$

Prove that

$$(\mathbf{T}^2 - I)(\mathbf{T} - 3I) = 0 \tag{3.2.4.13}$$

Solution: Expressing (3.2.3.1) in matrix form

$$\mathbf{T} = \begin{pmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & 1 & 1 \end{pmatrix} \tag{3.2.4.14}$$

The characteristic equation of $\mathbf{T}$ is given as follows,

$$\left|\mathbf{T} - \lambda\mathbf{I}\right| = \begin{vmatrix} 3-\lambda & 0 & 0 \\ 1 & -1-\lambda & 0 \\ 2 & 1 & 1-\lambda \end{vmatrix} = 0 \tag{3.2.4.15}$$

$$\implies (3-\lambda)(-1-\lambda)(1-\lambda) = 0$$
$$\implies (\lambda-3)(1+\lambda)(1-\lambda) = 0$$
$$\implies (\lambda-3)(1-\lambda^2) = 0$$
$$\implies (\lambda^2-1)(\lambda-3) = 0 \tag{3.2.4.16}$$

By the Cayley-Hamilton theorem,We can write (3.2.4.16) as

$$(\mathbf{T}^2 - I)(\mathbf{T} - 3I) = 0 \tag{3.2.4.17}$$

3.2.5. Let $\mathbb{C}$ be the complex vector space of 2×2 matrices with complex entries. Let

$$\mathbf{B} = \begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix} \tag{3.2.5.1}$$

and let $\mathbf{T}$ be the linear operator on $\mathbb{C}^{2\times2}$ defined by $\mathbf{T}(\mathbf{A}) = \mathbf{BA}$. What is the rank of $\mathbf{T}$? Can you describe $\mathbf{T^2}$?

Solution: An ordered basis for $\mathbb{C}^{2\times2}$ is given by

$$\mathbf{A_{11}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \mathbf{A_{12}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{3.2.5.2}$$

$$\mathbf{A_{21}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \mathbf{A_{22}} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.2.5.3}$$

Now, we compute

$$\mathbf{T}(\mathbf{A_{11}}) = \mathbf{BA_{11}} \tag{3.2.5.4}$$

$$= \begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{3.2.5.5}$$

$$= \begin{pmatrix} 1 & 0 \\ -4 & 0 \end{pmatrix} \tag{3.2.5.6}$$

from (3.2.5.6) we have

$$\mathbf{T}(\mathbf{A_{11}}) = \mathbf{A_{11}} - 4\mathbf{A_{21}} \tag{3.2.5.7}$$

$$\mathbf{T}(\mathbf{A_{12}}) = \mathbf{BA_{12}} \tag{3.2.5.8}$$

$$= \begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{3.2.5.9}$$

$$= \begin{pmatrix} 0 & 1 \\ 0 & -4 \end{pmatrix} \tag{3.2.5.10}$$

from (3.2.5.10), we have

$$\mathbf{T}(\mathbf{A_{12}}) = \mathbf{A_{12}} - 4\mathbf{A_{22}} \tag{3.2.5.11}$$

$$\mathbf{T}(\mathbf{A_{21}}) = \mathbf{BA_{21}} \tag{3.2.5.12}$$

$$= \begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{3.2.5.13}$$

$$= \begin{pmatrix} -1 & 0 \\ 4 & 0 \end{pmatrix} \tag{3.2.5.14}$$

from (3.2.5.14), we have

$$\mathbf{T}(\mathbf{A_{21}}) = -\mathbf{A_{11}} + 4\mathbf{A_{21}} \tag{3.2.5.15}$$

$$\mathbf{T}(\mathbf{A_{22}}) = \mathbf{BA_{22}} \tag{3.2.5.16}$$
$$= \begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.2.5.17}$$
$$= \begin{pmatrix} 0 & -1 \\ 0 & 4 \end{pmatrix} \tag{3.2.5.18}$$

from (3.2.5.18), we have

$$\mathbf{T}(\mathbf{A_{22}}) = -\mathbf{A_{12}} + 4\mathbf{A_{22}} \tag{3.2.5.19}$$

Now, by (3.2.5.7), (3.2.5.11), (3.2.5.15) and (3.2.5.19) we write matrix of the linear transformation as follows

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -4 & 0 & 4 & 0 \\ 0 & -4 & 0 & 4 \end{pmatrix} \tag{3.2.5.20}$$

Also, we know that the rank of a linear transformation is same as the rank of the matrix of the linear transformation. Thus, we find the rank of matrix **P**.

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -4 & 0 & 4 & 0 \\ 0 & -4 & 0 & 4 \end{pmatrix} \xleftrightarrow{r_3=r_3+4r_1} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 4 \end{pmatrix} \tag{3.2.5.21}$$

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 4 \end{pmatrix} \xleftrightarrow{r_4=r_4+4r_1} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.2.5.22}$$

from (3.2.5.22), we found out that rank($\mathbf{T}$) =2. Now, we compute

$$\mathbf{T^2}(\mathbf{A}) = \mathbf{T}(\mathbf{T}(\mathbf{A})) \tag{3.2.5.23}$$
$$= \mathbf{T}(\mathbf{BA}) \tag{3.2.5.24}$$
$$= \mathbf{B^2A} \tag{3.2.5.25}$$

where

$$\mathbf{B^2} = \begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix} \tag{3.2.5.26}$$
$$= \begin{pmatrix} 5 & -5 \\ -20 & 20 \end{pmatrix} \tag{3.2.5.27}$$

3.2.6. Let T be a linear transformation from $\mathbb{R}^3$ into $\mathbb{R}^2$, and let U be a linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^3$. Prove that the transformation UT is not invertible. Generalize the theorem.

Solution: Let $\mathbf{v}, \mathbf{x} \in \mathbb{R}^3$ and $\mathbf{w} \in \mathbb{R}^2$. Table 3.2.6.1 shows that maximum rank the transformation matrix $\mathbf{C}$ can have is 2.

$$Rank(\mathbf{C}) = 2 \tag{3.2.6.1}$$
$$dim(\mathbf{C}) = 3 \tag{3.2.6.2}$$
$$\implies Rank(\mathbf{C}) < dim(\mathbf{C}) \tag{3.2.6.3}$$

Therefore from the equation (3.2.6.3), we can say transformation UT is not invertible. Generalizing the proof, for $n > m$ and considering vectors $\mathbf{v}, \mathbf{x} \in \mathbb{R}^n$ and $\mathbf{w} \in \mathbb{R}^m$. From the Table 3.2.6.2,

$$Rank(\mathbf{C}) = m \tag{3.2.6.4}$$
$$dim(\mathbf{C}) = n \tag{3.2.6.5}$$
$$\implies Rank(\mathbf{C}) < dim(\mathbf{C}) \tag{3.2.6.6}$$

From equation (3.2.6.6)we can say that the transformation UT is not invertible. Let the vectors $\mathbf{v} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \in \mathbb{R}^3$ and $\mathbf{w} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} \in \mathbb{R}^2$.

a) Calculating transformation matrix **A**,

$$T(\mathbf{v}) = \mathbf{Av} \tag{3.2.6.7}$$

$$\begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \tag{3.2.6.8}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = rref(\mathbf{A}) \tag{3.2.6.9}$$

$$\implies Rank(\mathbf{A}) = 2 \tag{3.2.6.10}$$

b) Calculating transformation matrix **B**,

$$U(\mathbf{w}) = \mathbf{Bw} \tag{3.2.6.11}$$

$$\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & 2 \\ 1 & -1 \\ 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 4 \\ 2 \end{pmatrix} \tag{3.2.6.12}$$

$$\begin{pmatrix} \frac{3}{4} & 2 \\ 1 & -1 \\ 0 & \frac{1}{2} \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = rref(\mathbf{B}) \tag{3.2.6.13}$$

$$\implies Rank(\mathbf{B}) = 2 \tag{3.2.6.14}$$

c) Now for the transformation UT, calculating

Transformation	Matrix Representation	Dimension	Max Rank of transformation matrix
$T : \mathbb{R}^3 \to \mathbb{R}^2$	$T(\mathbf{v}) = \mathbf{Av}$	$\mathbf{A} : 2 \times 3$	$Rank(\mathbf{A}) = 2$
$U : \mathbb{R}^2 \to \mathbb{R}^3$	$U(\mathbf{w}) = \mathbf{Bw}$	$\mathbf{B} : 3 \times 2$	$Rank(\mathbf{B}) = 2$
$UT : \mathbb{R}^3 \to \mathbb{R}^3$	$UT(\mathbf{x}) = \mathbf{Cx}$ $U(T(\mathbf{x})) = \mathbf{B}(\mathbf{Ax})$ $\mathbf{C} = \mathbf{AB}$	$\mathbf{C} : 3 \times 3$	$Rank(\mathbf{C}) \leq min(Rank(\mathbf{B}), Rank(\mathbf{A}))$ $Rank(\mathbf{C}) = 2$

TABLE 3.2.6.1: Proof for non-invertibility of the transformation UT where $T : \mathbb{R}^3 \to \mathbb{R}^2$ and $U : \mathbb{R}^2 \to \mathbb{R}^3$

Transformation	Matrix Representation	Dimension	Max Rank of transformation matrix
$T : \mathbb{R}^n \to \mathbb{R}^m$	$T(\mathbf{v}) = \mathbf{Av}$	$\mathbf{A} : m \times n$	$Rank(\mathbf{A}) = m$
$U : \mathbb{R}^m \to \mathbb{R}^n$	$U(\mathbf{w}) = \mathbf{Bw}$	$\mathbf{B} : n \times m$	$Rank(\mathbf{B}) = m$
$UT : \mathbb{R}^n \to \mathbb{R}^n$	$UT(\mathbf{x}) = \mathbf{Cx}$ $U(T(\mathbf{x})) = \mathbf{B}(\mathbf{Ax})$ $\mathbf{C} = \mathbf{AB}$	$\mathbf{C} : n \times n$	$Rank(\mathbf{C}) \leq min(Rank(\mathbf{B}), Rank(\mathbf{A}))$ $Rank(\mathbf{C}) = m$

TABLE 3.2.6.2: Generalization of the proof

the transformation matrix $\mathbf{C}$,

$$UT : \mathbb{R}^3 \to \mathbb{R}^3 \tag{3.2.6.15}$$

$$\implies UT(\mathbf{x}) = \mathbf{Cx} \tag{3.2.6.16}$$

$$U(T(\mathbf{x})) = \mathbf{B}(\mathbf{Ax}) \tag{3.2.6.17}$$

$$\implies \mathbf{C} = \mathbf{BA} \tag{3.2.6.18}$$

$$\mathbf{C} = \begin{pmatrix} \frac{3}{4} & 2 \\ 1 & -1 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & 0 & \frac{3}{4} \\ 0 & 0 & 2 \\ \frac{1}{2} & 0 & \frac{-1}{2} \end{pmatrix} \tag{3.2.6.19}$$

$$\begin{pmatrix} \frac{3}{4} & 0 & \frac{3}{4} \\ 0 & 0 & 2 \\ \frac{1}{2} & 0 & \frac{-1}{2} \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = rref(\mathbf{C}) \tag{3.2.6.20}$$

$$\implies Rank(\mathbf{C}) = 2 \tag{3.2.6.21}$$

$$dim(\mathbf{C}) = 3 \tag{3.2.6.22}$$

As $Rank(\mathbf{C}) < dim(\mathbf{C})$, transformation UT is not invertible.

3.2.7. Find two linear operators $\mathbf{T}$ and $\mathbf{U}$ on $\mathbf{R}^2$ such that $\mathbf{TU} = 0$ but $\mathbf{UT} \neq 0$

Solution: Let,

$$\mathbf{x}, \mathbf{y} \in \mathbf{R^2} \tag{3.2.7.1}$$

Let $\mathbf{T}$ and $\mathbf{U}$ be given by the matrices

$$\mathbf{T}(\mathbf{x}) = \mathbf{Ax}; \quad \mathbf{U}(\mathbf{x}) = \mathbf{Bx} \tag{3.2.7.2}$$

where,

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \quad \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{3.2.7.3}$$

$$\mathbf{T}(a\mathbf{x} + \mathbf{y}) = a\mathbf{Tx} + \mathbf{Ty} \tag{3.2.7.4}$$

$$\mathbf{U}(a\mathbf{x} + \mathbf{y}) = a\mathbf{Ux} + \mathbf{Uy} \tag{3.2.7.5}$$

From (3.2.7.4) and (3.2.7.5), we can tell that $\mathbf{T}$ and $\mathbf{U}$ are linear operators. Now,

$$\mathbf{TU} = \mathbf{AB} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbf{0} \tag{3.2.7.6}$$

$$\mathbf{UT} = \mathbf{BA} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq \mathbf{0} \tag{3.2.7.7}$$

From (3.2.7.6) and (3.2.7.7) it can be observed that $\mathbf{TU} = 0$ but $\mathbf{UT} \neq 0$

3.2.8. Let $\mathbf{V}$ be a vector space over the field $\mathbf{F}$ and $\mathbf{T}$ is a linear operator on $\mathbf{V}$. If $\mathbf{T}^2 = 0$, what can you say about the relation of the range of $\mathbf{T}$ to the null space of $\mathbf{T}$? Give an example of linear operator $\mathbf{T}$ on $\mathbf{R}^2$ such that $\mathbf{T}^2 = 0$ but $\mathbf{T} \neq 0$.

Solution: Given,

$$\mathbf{T} : \mathbf{V} \to \mathbf{V} \tag{3.2.8.1}$$

Now, $\mathbf{T}^2$ is also a linear operator as,

$$\mathbf{T}^2(c\alpha) = \mathbf{T}(\mathbf{T}(c\alpha)) = \mathbf{T}(c\mathbf{T}(\alpha)) \tag{3.2.8.2}$$
$$= c\mathbf{T}(\mathbf{T}(\alpha)) = c\mathbf{T}^2(\alpha) \tag{3.2.8.3}$$

Let some vector $\mathbf{y} \in \text{Range}(\mathbf{T})$ then there exists $\mathbf{x} \in \mathbf{V}$ such that,

$$\mathbf{T}(\mathbf{x}) = \mathbf{y} \tag{3.2.8.4}$$

Now given that,

$$\mathbf{T}^2(\mathbf{x}) = \mathbf{0} \tag{3.2.8.5}$$
$$\implies \mathbf{T}(\mathbf{T}(\mathbf{x})) = \mathbf{0} \tag{3.2.8.6}$$
$$\mathbf{T}(\mathbf{y}) = \mathbf{0} \tag{3.2.8.7}$$

$\therefore$ $\mathbf{y}$ lies in the Null space of $\mathbf{T}$. Hence $\mathbf{T}$ is singular. Thus, the range of $\mathbf{T}$ must be contained in Null space of $\mathbf{T}$ i.e., $\text{Range}(\mathbf{T}) \subseteq \text{NullSpace}(\mathbf{T})$

Example:

$$\mathbf{T} : \mathbf{R}^2 \to \mathbf{R}^2 \tag{3.2.8.8}$$

Consider,

$$\mathbf{T}(\mathbf{x}) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mathbf{x} \tag{3.2.8.9}$$
$$\implies \mathbf{T} \neq 0 \tag{3.2.8.10}$$

Now,

$$\mathbf{T}^2 : \mathbf{R}^2 \to \mathbf{R}^2 \tag{3.2.8.11}$$

$$\mathbf{T}^2(\mathbf{x}) = \mathbf{T}(\mathbf{T}(\mathbf{x})) \tag{3.2.8.12}$$
$$= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mathbf{x} = \mathbf{0} \tag{3.2.8.13}$$
$$\implies \mathbf{T}^2(\mathbf{x}) = \mathbf{0} \tag{3.2.8.14}$$

Thus $\mathbf{T}^2$ is a zero transformation, $\mathbf{T}^2 = \mathbf{0}$.
Now, Kernel of $\mathbf{T}$ is given by,

$$\mathbf{T}(\mathbf{x}) = \mathbf{0} \tag{3.2.8.15}$$
$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.2.8.16}$$
$$\implies x = 0 \tag{3.2.8.17}$$

Thus,

$$\mathbf{Ker}(\mathbf{T}) = y \begin{pmatrix} 0 \\ 1 \end{pmatrix}; y \in \mathbf{R} \tag{3.2.8.18}$$

Now,

$$\mathbf{Range}(\mathbf{T}) = \mathbf{ColumnSpace} \left\{ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\} \tag{3.2.8.19}$$
$$= k \begin{pmatrix} 0 \\ 1 \end{pmatrix}; k \in \mathbf{R} \tag{3.2.8.20}$$

Thus for the example, $\text{Range}(\mathbf{T}) = \text{Kernel}(\mathbf{T})$ and from (3.2.8.10), (3.2.8.14) it is clear that $\mathbf{T}^2 = 0$ but $\mathbf{T} \neq 0$.

3.2.9. Let $\mathbf{T}$ be a linear operator on the finite-dimensional space $\mathbf{V}$. Support there is a linear operator $\mathbf{U}$ on $\mathbf{V}$ such that $\mathbf{TU} = \mathbf{I}$. Prove that $\mathbf{T}$ is invertible and $\mathbf{U} = \mathbf{T}^{-1}$. Give an example which shows that this is false when $\mathbf{V}$ is not finite-dimensional.

Solution: Theorem

Theorem 3.1. *Let f be a function from X into Y. We say that f is invertible if there is a function g from Y to X such that*

a) $g \circ f$ is the identity function on X i.e. $g \circ f = I$. Here, g will be onto and f will be one-one.
b) $f \circ g$ is the identity function on Y i.e. $f \circ g = I$. Here, f will be onto and g will be one-one.

Theorem 3.2. *Let V and W be finite dimensional vector spaces such that $\dim V = \dim W$. If T is a linear transformation from V into W, then the following are equivalent:*

a) T is non-singular
b) T is onto

If any of the above two condition is satisfied then T is invertible.

a) We are given $\mathbf{V}$ which is a finite dimensional vector space, with the following linear operators defined as:-

$$\mathbf{T} : \mathbf{V} \to \mathbf{V} \tag{3.2.9.1}$$
$$\mathbf{U} : \mathbf{V} \to \mathbf{V} \tag{3.2.9.2}$$

The linear operators also satifies the condition

$$\mathbf{TU} = \mathbf{I} \tag{3.2.9.3}$$

Where $\mathbf{I}$ is an Identity transformation. This identity transformation can be written as

$$\mathbf{I} : \mathbf{V} \to \mathbf{V} \tag{3.2.9.4}$$

$$\implies \mathbf{TU} : \mathbf{V} \to \mathbf{V} \tag{3.2.9.5}$$

$$\implies \mathbf{T}\left[\mathbf{U}\left(\mathbf{V}\right)\right] = \mathbf{V} \tag{3.2.9.6}$$

From theorem (3.1) we can say that $\mathbf{U}$ must be one-one and $\mathbf{V}$ must be onto.
From theorem (3.2) we can say that $\mathbf{T}$ is invertible.

Now we know that

$$\mathbf{TT}^{-1} = \mathbf{I} \tag{3.2.9.7}$$

Comparing (3.2.9.3) and (3.2.9.7) we get

$$\mathbf{TT}^{-1} = \mathbf{I} = \mathbf{TU} \tag{3.2.9.8}$$

Multiply both sides with $\mathbf{T}^{-1}$

$$\mathbf{T}^{-1}\left(\mathbf{TT}^{-1}\right) = \mathbf{T}^{-1}\left(\mathbf{TU}\right) \tag{3.2.9.9}$$

$$\mathbf{T}^{-1}\mathbf{I} = \left(\mathbf{T}^{-1}\mathbf{T}\right)\mathbf{U} \tag{3.2.9.10}$$

$$\mathbf{T}^{-1} = \mathbf{IU} \tag{3.2.9.11}$$

$$\therefore \mathbf{T}^{-1} = \mathbf{U} \tag{3.2.9.12}$$

b) Let $\mathbf{D}$ be a differential operator $\mathbf{D} : \mathbf{V} \to \mathbf{V}$, where $\mathbf{V}$ is a space of polynomial functions in one variable over $\mathbf{R}$.

$$\mathbf{D}\left(c_0 + c_1 x + ... + c_n x^n\right) = c_1 + c_2 x + ... + c_n x^{n-1} \tag{3.2.9.13}$$

And $\mathbf{U} : \mathbf{V} \to \mathbf{V}$ be linear operator such that

$$\mathbf{U}\left(c_0 + c_1 x + ... + c_n x^n\right) = c_0 x + \frac{c_1 x^2}{2} + ... + \frac{c_n x^{n+1}}{n+1} \tag{3.2.9.14}$$

Therefore, the linear operator $\mathbf{UD} : \mathbf{V} \to \mathbf{V}$ will be $\mathbf{UD}\left(c_0 + c_1 x + ... + c_n x^n\right)$

$$\begin{aligned}
&= \mathbf{U}\left[\mathbf{D}\left(c_0 + c_1 x + ... + c_n x^n\right)\right] \\
&= \mathbf{U}\left[c_1 + c_2 x + ... + c_n x^{n-1}\right] \\
&= c_1 x + \frac{c_2 x^2}{2} + ... + \frac{c_n x^n}{n} \\
&= c_1 x + c_2 x^2 + ... + c_n x^n \\
&\neq \mathbf{I}
\end{aligned} \tag{3.2.9.15}$$

Now, the linear operator $\mathbf{DU} : \mathbf{V} \to \mathbf{V}$ will be $\mathbf{DU}\left(c_0 + c_1 x + ... + c_n x^n\right)$

$$\begin{aligned}
&= \mathbf{D}\left[\mathbf{U}\left(c_0 + c_1 x + ... + c_n x^n\right)\right] \\
&= \mathbf{D}\left[c_0 x + \frac{c_1 x^2}{2} + ... + \frac{c_n x^{n+1}}{n+1}\right] \\
&= c_0 + \frac{2c_2 x}{2} + ... + \frac{\left(n+1\right) c_n x^n}{n+1} \\
&= c_0 + c_1 x + c_2 x^2 + ... + c_n x^n \\
&= \mathbf{I}
\end{aligned} \tag{3.2.9.16}$$

From (3.2.9.15) and (3.2.9.16) we see that $\mathbf{DU} = \mathbf{I}$, but $\mathbf{UD} \neq \mathbf{I}$.

3.2.10. Let $\mathbf{A}$ be an $m \times n$ matrix with entries in F and let T be the linear transformation from $F^{n\times 1}$ into $F^{m\times l}$ defined by $T(\mathbf{X}) = \mathbf{AX}$. Show that

a) if $m < n$ it may happen that T is onto without being non-singular

b) if $m > n$ we may have T non-singular but not onto.

Solution: Proof

a) $m < n$

$$\text{Let, } T : \mathbb{R}^3 \to \mathbb{R}^2 \tag{3.2.10.1}$$

$$T(\mathbf{X}) = \mathbf{AX} = \mathbf{b} \tag{3.2.10.2}$$

$$\text{Let, } \mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \tag{3.2.10.3}$$

$$\text{Consider, } \mathbf{X} = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} \tag{3.2.10.4}$$

$$\implies \mathbf{AX} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} \tag{3.2.10.5}$$

$$= \begin{pmatrix} 6 \\ 5 \end{pmatrix} \tag{3.2.10.6}$$

singular	A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is said to be singular if $\exists$ some non-zero $\mathbf{X} \in \mathbb{R}^n$ s.t $\mathbf{AX} = \mathbf{0}$ i.e $Nullity(A) \neq 0$. From rank-nullity theorem we can say $rank(A) < n$
non-singular	A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be non-singular if $\mathbf{AX} = \mathbf{0}$ implies $\mathbf{X} = \mathbf{0}$ i.e $Nullity(A) = 0$
onto	A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, $m \leq n$ is said to be onto if for every $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{AX} = \mathbf{b}$ has atleast one solution $\mathbf{X} \in \mathbb{R}^n$.. i.e $dim(Col(\mathbf{A})) = m$ or $Rank(\mathbf{A}) = m$ If $m > n$, then $\mathbf{AX} = \mathbf{b}$ has no solution because rank-nullity theorem is not satisfied.

TABLE 3.2.10.1

Hence T is onto.

$$\text{Consider, } \mathbf{X} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \tag{3.2.10.7}$$

$$\implies \mathbf{AX} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \tag{3.2.10.8}$$

$$= \mathbf{0} \tag{3.2.10.9}$$

Since $\exists$ $\mathbf{X} \neq \mathbf{0}$ such that $\mathbf{AX} = \mathbf{0}$, T is singular.

$\therefore$ T is both onto and singular.

Let $\mathbf{A}$ be an $m \times n$ matrix with entries in F and let T be the linear transformation from $F^{n\times 1}$ into $F^{m\times l}$ defined by $T(\mathbf{X}) = \mathbf{AX}$. If,		
	$m < n$	$m > n$
singular	Since $rank(\mathbf{A}) < n$, by definition T is singular	Consider an non-singular T such that $rank(\mathbf{A}) > n$
onto	Since $m < n$, by definition T can be onto	Since $m > n$, by definition T is not onto.

TABLE 3.2.10.2

Given	$\mathbf{T}:\mathbf{V}\to\mathbf{V}$ be a linear operator. Rank(T^2)=Rank(T)
Rank of T and T^2	Let $\left(e_1,e_2,...,e_n\right)$ be a basis for T,then Rank(T) is linearly independent vectors in the set $\left(Te_1,Te_2,...,Te_n\right)$ Let,Rank(T)=r=Rank(T^2)
Rank Nullity Theorem	If Rank(T)=r then $\left(Te_1,Te_2,...,Te_r\right)$ is the basis of range T. Similarly for T^2,$\left(T^2e_1,T^2e_2,...,T^2e_r\right)$ is the basis of range T^2
v∈ range(T) **v**∈ nullspace(T)	**v**=$c_1Te_1+c_2Te_2+....+c_rTe_r$ T(**v**)=0 T($c_1Te_1+c_2Te_2+....+c_rTe_r$)=0 $c_1T^2e_1+c_2T^2e_2+....+c_rT^2e_r$=0 But,$\left(T^2e_1,T^2e_2,...,T^2e_r\right)$ is the basis of range T^2 So,c_1=c_2=......=c_r=0 Substituting these in **v** we get **v**=0
Conclusion	Hence from above it can be seen that when **v** belongs to both range(T) and nullspace(T) then **v** is a zero vector. Hence,range(T) and nullspace(T) are disjoint.

TABLE 3.2.11.1

b) $m>n$

$$\text{Let, } T:\mathbb{R}^3\to\mathbb{R}^2 \tag{3.2.10.10}$$

$$T(\mathbf{X})=\mathbf{AX}=\mathbf{b} \tag{3.2.10.11}$$

$$\text{Let, }\mathbf{A}=\begin{pmatrix}1&0\\0&1\\1&0\end{pmatrix} \tag{3.2.10.12}$$

$$\text{Consider, }\mathbf{X}=\begin{pmatrix}-1\\2\end{pmatrix} \tag{3.2.10.13}$$

$$\implies \mathbf{AX}=\begin{pmatrix}1&0\\0&1\\1&0\end{pmatrix}\begin{pmatrix}-1\\2\end{pmatrix} \tag{3.2.10.14}$$

$$=\begin{pmatrix}-1\\2\\-1\end{pmatrix} \tag{3.2.10.15}$$

$$\tag{3.2.10.16}$$

∴ T is not onto, and is also non-singular.

3.2.11. Let V be a finite-dimensional vector space and let T be a linear operator on V. Suppose that rank (T^2) = rank (T). Prove that the range and null space of T are disjoint, i.e., have only the zero vector in common. **Solution:** See Table 3.2.11.1.

3.2.12. Let p,m,n be positive integers and $\mathbb{F}$ a field.Let **V** be the space of $m\times n$ matrices over $\mathbb{F}$ and **W** the space of $p\times n$ matrices over $\mathbb{F}$.Let **B** be a fixed $p\times m$ matrix and let $\mathbb{T}$ be the linear transformation from **V** into **W** defined by $\mathbb{T}(\mathbf{A})=\mathbf{BA}$.Prove that $\mathbb{T}$ is invertible if and only if $p=m$ and **B** is an invertible $m\times m$ matrix. **Solution:**

Parameter	**Description**
p,m,n	Positive integers
$\mathbb{F}$	Field
V	Space of $m\times n$ matrices over $\mathbb{F}$
W	Space of $p\times n$ matrices over $\mathbb{F}$
B	Fixed $p\times m$ matrix
Linear transformation $\mathbb{T}:\mathbf{V}\to\mathbf{W}$	$\mathbb{T}(\mathbf{A})=\mathbf{BA}$

TABLE 3.2.12.1: Input Parameters

$$\mathbb{T}(\mathbf{A}) = \mathbf{BA} \tag{3.2.12.1}$$

So, $\mathbf{B}$ is the transformation matrix.
$\mathbf{B}$ is invertible if

a) $\mathbb{T}$ is one to one mapping,that is

$$\mathbf{BA} = \mathbf{BA'} \tag{3.2.12.2}$$
$$\implies \mathbf{A} = \mathbf{A'} \tag{3.2.12.3}$$

b) $\mathbb{T}$ must be onto, that is range($\mathbf{B}$)=$\mathbf{W}$

Case 1: Let us assume that $\mathbb{T}$ is invertible with inverse transformation $\mathbb{T}_1$ from $\mathbf{W}$ to $\mathbf{V}$ that satisfies

$$\mathbb{T}(\mathbf{A}) = \mathbf{BA} \in \mathbf{W} \tag{3.2.12.4}$$
$$\implies \mathbb{T}_1(\mathbf{BA}) = \mathbf{A} \in \mathbf{V} \tag{3.2.12.5}$$
$$\dim(\mathbf{V}) = mn, \dim(\mathbf{W}) = pn \tag{3.2.12.6}$$

Since $\mathbb{T}$ is one-one mapping, the zero vector in $\mathbf{V}, \mathbf{0}_{m\times n}$ is uniquely mapped to

$$\mathbb{T}(\mathbf{0}_{m\times n}) = \mathbf{B0}_{m\times n} = \mathbf{0}_{p\times n} \tag{3.2.12.7}$$
$$\text{So, } \mathbf{BA} = \mathbf{0} \iff \mathbf{A} = \mathbf{0} \tag{3.2.12.8}$$

Let $\{\mathbf{V}_1, \mathbf{V}_2, \dots, \mathbf{V}_{mn}\}$ be the basis for $\mathbf{V}$

$$c_1\mathbf{V}_1 + c_2\mathbf{V}_2 + \dots + c_{mn}\mathbf{V}_{mn} = \mathbf{0} \tag{3.2.12.9}$$
$$\iff c_1, c_2, \dots, c_{mn} \in \mathbb{F} = 0 \tag{3.2.12.10}$$

Any matrix $\mathbf{A} \in \mathbf{V}$ can be written as

$$\mathbf{A} = \sum_{i=1}^{mn} \alpha_i \mathbf{V}_i \tag{3.2.12.11}$$

Since $\mathbb{T}$ is onto, any matrix $\mathbf{C} \in \mathbf{W}$ can be expressed as

$$\mathbf{C} = \mathbf{B}\left(\sum_{i=1}^{mn} \alpha_i \mathbf{V}_i\right) \tag{3.2.12.12}$$
$$= \sum_{i=1}^{mn} \alpha_i (\mathbf{BV}_i) \tag{3.2.12.13}$$

So, the set $\mathbf{S} = \{\mathbf{BV}_1, \mathbf{BV}_2, \dots, \mathbf{BV}_{mn}\}$ forms basis of $\mathbf{W}$ if all matrices in it are linearly independent.

$$c_1(\mathbf{BV}_1) + c_2(\mathbf{BV}_2) + \dots + c_{mn}(\mathbf{BV}_{mn}) = \mathbf{0} \tag{3.2.12.14}$$
$$\mathbf{B}(c_1\mathbf{V}_1 + c_2\mathbf{V}_2 + \dots + c_{mn}\mathbf{V}_{mn}) = \mathbf{0} \tag{3.2.12.15}$$
$$(3.2.12.8) \implies c_1\mathbf{V}_1 + \dots + c_{mn}\mathbf{V}_{mn} = 0 \tag{3.2.12.16}$$
$$\iff c_1, c_2, \dots, c_{mn} = 0 (\text{from } (3.2.12.10)) \tag{3.2.12.17}$$

So,the set $\mathbf{S}$ with cardinality mn is basis for $\mathbf{W}$

$$(3.2.12.6) \implies pn = mn \tag{3.2.12.18}$$
$$p = m \tag{3.2.12.19}$$

(3.2.12.8),(3.2.12.19) prove that $\mathbf{B}$ is invertible $m \times m$ matrix. Case 2: Consider $p = m$ and $\mathbf{B}$ is an invertible $m \times m$ matrix.
Verifying if $\mathbb{T}$ is onto,
Let the set of matrices $\{\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_{mn}\}$ be the basis for $\mathbf{V}$
Any matrix $\mathbf{A} \in \mathbf{V}$ can be written as

$$\mathbf{A} = \sum_{i=1}^{mn} \alpha_i \mathbf{A}_i \tag{3.2.12.20}$$

where $\alpha_i \in \mathbb{F}$
The set $\mathbf{M} = \{\mathbf{BA}_1, \mathbf{BA}_2, \dots, \mathbf{BA}_{mn}\}$ lie in $\mathbf{W}$

$$c_1(\mathbf{BA}_1) + c_2(\mathbf{BA}_2) + \dots + c_{mn}(\mathbf{BA}_{mn}) = \mathbf{0} \tag{3.2.12.21}$$
$$\implies \mathbf{B}(c_1\mathbf{A}_1 + c_2\mathbf{A}_2 + \dots + c_{mn}\mathbf{A}_{mn}) = \mathbf{0} \tag{3.2.12.22}$$

Since $\mathbf{B}$ is non-singular,

$$(c_1\mathbf{A}_1 + c_2\mathbf{A}_2 + \dots + c_{mn}\mathbf{A}_{mn}) = \mathbf{0} \tag{3.2.12.23}$$
$$\iff c_1, c_2, \dots, c_{mn} = 0 \tag{3.2.12.24}$$

because $\{\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_{mn}\}$ are linearly independent
So,$\mathbf{M}$ forms basis for $\mathbf{W}$
Any vector $\mathbf{C} \in \mathbf{W}$ can be written as

$$\mathbf{C} = \sum_{i=1}^{mn} \beta_i \mathbf{BA}_i \text{ where } \beta_i \in \mathbb{F} \tag{3.2.12.25}$$
$$= \mathbf{B}(\sum_{i=1}^{mn} \beta_i \mathbf{A}_i) \tag{3.2.12.26}$$
$$= \mathbf{BA} \text{ (from (3.2.12.20))} \tag{3.2.12.27}$$

So,range($\mathbf{B}$)=$\mathbf{W}$

Consider the matrix $\mathbf{A}, \mathbf{A}' \in \mathbf{V}$ such that

$$\mathbf{BA} = \mathbf{BA}' \tag{3.2.12.28}$$

$$\mathbf{B}^{-1}(\mathbf{BA}) = \mathbf{B}^{-1}(\mathbf{BA}') \tag{3.2.12.29}$$

$$(\mathbf{B}^{-1}\mathbf{B})\mathbf{A} = (\mathbf{B}^{-1}\mathbf{B})\mathbf{A}' \tag{3.2.12.30}$$

$$\implies \mathbf{A} = \mathbf{A}' \tag{3.2.12.31}$$

So, $\mathbb{T}$ is invertible. Conclusion: From case 1,case 2 $\mathbb{T}$ is invertible if and only if $p = m$ and $\mathbf{B}$ is an invertible $m \times m$ matrix. Example: Let $p = m = 3, n = 4$ Let $\mathbb{T} : \mathbf{V} \to \mathbf{W}$ adds row 2 to row 3 for a matrix $\mathbf{A} \in \mathbf{V}$
The elementary matrix that performs this is

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \tag{3.2.12.32}$$

$$\text{Let } \mathbf{A} = \begin{pmatrix} 1 & 2 & 2 & 5 \\ 1 & 3 & 6 & 7 \\ 4 & 9 & 2 & 6 \end{pmatrix} \tag{3.2.12.33}$$

$$\mathbb{T}(\mathbf{A}) = \mathbf{BA} \tag{3.2.12.34}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 2 & 5 \\ 1 & 3 & 6 & 7 \\ 4 & 9 & 2 & 6 \end{pmatrix} \tag{3.2.12.35}$$

$$= \begin{pmatrix} 1 & 2 & 2 & 5 \\ 1 & 3 & 6 & 7 \\ 5 & 12 & 8 & 13 \end{pmatrix} \tag{3.2.12.36}$$

$$= \mathbf{C} \in \mathbf{W} \tag{3.2.12.37}$$

Let transformation $\mathbb{T}_1 : \mathbf{W} \to \mathbf{V}$ subtracts row2 from row 3 for a matrix $\mathbf{C} \in \mathbf{W}$ and is performed by elementary matrix

$$\mathbf{U} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \tag{3.2.12.38}$$

$$\text{Let } \mathbf{C} = \begin{pmatrix} 1 & 2 & 2 & 5 \\ 1 & 3 & 6 & 7 \\ 5 & 12 & 8 & 13 \end{pmatrix} \tag{3.2.12.39}$$

$$\mathbb{T}_1(\mathbf{C}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 2 & 5 \\ 1 & 3 & 6 & 7 \\ 5 & 12 & 8 & 13 \end{pmatrix} \tag{3.2.12.40}$$

$$= \begin{pmatrix} 1 & 2 & 2 & 5 \\ 1 & 3 & 6 & 7 \\ 4 & 9 & 2 & 6 \end{pmatrix} \tag{3.2.12.41}$$

$$= \mathbf{A} \tag{3.2.12.42}$$

$$\implies \mathbb{T}_1(\mathbf{C}) = \mathbf{A} \tag{3.2.12.43}$$

$$\mathbb{T}_1(\mathbb{T}(\mathbf{A})) = \mathbf{A} \tag{3.2.12.44}$$

$$\text{and } \mathbb{T}(\mathbf{A}) = \mathbf{C} \tag{3.2.12.45}$$

$$\implies \mathbb{T}(\mathbb{T}_1(\mathbf{C})) = \mathbf{C} \tag{3.2.12.46}$$

So,$\mathbb{T}_1$ is the inverse transformation of $\mathbb{T}$ and

$$\mathbb{T}_1 = \mathbb{T}^{-1} \tag{3.2.12.47}$$

$$\mathbf{UB} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \tag{3.2.12.48}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.2.12.49}$$

$$\mathbf{BU} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \tag{3.2.12.50}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.2.12.51}$$

$$\implies \mathbf{B}^{-1} = \mathbf{U} \tag{3.2.12.52}$$

So, $\mathbb{T}$ is invertible and ,$\mathbf{B}$ is an invertible 3×3 matrix.

3.3.1. Let **V** be the set of complex numbers and let **F** be the field of real numbers. With the usual operations, **V** is a vector space over **F**. Describe explicitly an isomorphism of this space onto $\mathbb{R}^2$.

Solution: Let,

$$\mathbf{T} : \mathbf{V} \to \mathbb{R}^2 \tag{3.3.1.1}$$

$$\mathbf{T}(x + iy) = \begin{pmatrix} x \\ y \end{pmatrix} \tag{3.3.1.2}$$

$$x, y \in \mathbb{R} \quad i \in \mathbb{C} \tag{3.3.1.3}$$

Consider two vectors,

$$\mathbf{u}, \mathbf{v} \in \mathbf{V} \tag{3.3.1.4}$$

Using (3.3.1.2) if **T** is Linear Transformation.

$$\mathbf{T}(\mathbf{u} + c\mathbf{v}) = \mathbf{T}(\mathbf{u}) + \mathbf{T}(c\mathbf{v}) = \mathbf{T}(\mathbf{u}) + c\mathbf{T}(\mathbf{v}) \tag{3.3.1.5}$$

Hence this is a Linear transformation. Now, checking if **T** is one-one. let,

$$\mathbf{u} = 0 = 0 + j0 \tag{3.3.1.6}$$

$$\tag{3.3.1.7}$$

from (3.3.1.2),

$$\mathbf{T}(\mathbf{u}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.3.1.8}$$

From (3.3.1.8) we see **T** is one-one. Now, checking if **T** is onto. let,

$$\mathbf{T}(\mathbf{u}) = \begin{pmatrix} a \\ b \end{pmatrix} \tag{3.3.1.9}$$

where, a, b are scalars. Using (3.3.1.2), we see there exists a solution for (3.3.1.8) for all a, b. Hence **T** is onto. Therefore **T** is isomorphic over **V** and $\mathbb{R}^2$

3.3.2. Let V be a vector space over the field of complex numbers , and suppose there is an isomorphism T of V onto C^3. Let α_1 , α_2 , α_3 , α_4 be vectors in V such that

$$\begin{gathered} T(\alpha_1) = \begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}, T(\alpha_2) = \begin{pmatrix} -2 \\ 1+i \\ 0 \end{pmatrix}, \\ T(\alpha_3) = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, T(\alpha_4) = \begin{pmatrix} \sqrt{2} \\ i \\ 3 \end{pmatrix} \end{gathered} \tag{3.3.2.1}$$

a) Is α_1 in the subspace spanned by α_2 and α_3?
Solution: T : V → W is an isomorphism if
(1) T is one one.
(2) T is onto.

$$\begin{pmatrix} 1 & -2 & -1 & \sqrt{2} \\ 0 & 1+i & 1 & i \\ i & 0 & 1 & 3 \end{pmatrix} \xrightarrow{ref} \begin{pmatrix} 1 & 0 & -i & -3i \\ 0 & 2 & 1-i & i+1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.3.2.2}$$

T is one one over C^3 if

$$T(\alpha) = 0 \implies \alpha = 0 \tag{3.3.2.3}$$

now,

$$\begin{pmatrix} 1 & -2 & \sqrt{2} \\ 0 & 1+i & i \\ i & 0 & 3 \end{pmatrix} \alpha = 0 \tag{3.3.2.4}$$

consider the row reduced matrix

$$\begin{pmatrix} 1 & -2 & \sqrt{2} \\ 0 & 1+i & i \\ i & 0 & 3 \end{pmatrix} \xleftrightarrow[R_3 \to iR_3]{R_3 \to R_3 - iR_1} \begin{pmatrix} 1 & -2 & \sqrt{2} \\ 0 & 1+i & i \\ 0 & -2 & \sqrt{2}+3i \end{pmatrix} \tag{3.3.2.5}$$

$$\xleftrightarrow[R_3 \leftarrow R_3 + R_2]{R_2 \leftarrow (1-i)R_2} \begin{pmatrix} 1 & -2 & \sqrt{2} \\ 0 & 2 & i+1 \\ 0 & 0 & \sqrt{2}+4i+1 \end{pmatrix} \tag{3.3.2.6}$$

$$\alpha = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{3.3.2.7}$$

Therefore it holds the condition of one one and the rank = no. of pivot columns = 3 (equal to no. of columns).Thus the vectors are linearly independent hence it is onto . Since T is an isomorphoism onto C^3.

$$T(\alpha_1) = c_1 T(\alpha_2) + c_2 T(\alpha_3) \tag{3.3.2.8}$$

c_1 and c_2 are scalar.

$$\begin{pmatrix}1\\0\\i\end{pmatrix} = c_1\begin{pmatrix}-2\\1+i\\0\end{pmatrix} + c_2\begin{pmatrix}-1\\1\\1\end{pmatrix} \tag{3.3.2.9}$$

$$\begin{pmatrix}1\\0\\i\end{pmatrix} = \begin{pmatrix}-2 & -1\\1+i & 1\\0 & 1\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix} \tag{3.3.2.10}$$

Now we find c_i by row reducing augmented matrix.

$$\begin{pmatrix}-2 & -1 & 1\\1+i & 1 & 0\\0 & 1 & i\end{pmatrix} \xleftrightarrow[R_2\to R_3]{R_1\to -R_1/2} \begin{pmatrix}1 & \frac{1}{2} & -\frac{1}{2}\\0 & 1 & i\\1+i & 1 & 0\end{pmatrix} \tag{3.3.2.11}$$

$$\xleftrightarrow[R_3\leftarrow R_3-(1+i)R_1]{R_1\leftarrow R_1-R_2/2} \begin{pmatrix}1 & 0 & \frac{-1-i}{2}\\0 & 1 & i\\0 & \frac{1-i}{2} & \frac{1+i}{2}\end{pmatrix} \tag{3.3.2.12}$$

$$\xleftrightarrow{R_3\leftarrow R_3-(1-i)/2R_2} \begin{pmatrix}1 & 0 & \frac{-1-i}{2}\\0 & 1 & i\\0 & 0 & 0\end{pmatrix} \tag{3.3.2.13}$$

Therefore the coordinate matrix of the vector is

$$\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}\frac{-1-i}{2}\\i\end{pmatrix} \tag{3.3.2.14}$$

substituting the c_i in (3.3.2.8)

$$T(\alpha_1) = -\frac{1+i}{2}T(\alpha_2) + iT(\alpha_3) \tag{3.3.2.15}$$

Hence α_1 belongs to the subspace spanned by α_2 and α_3.

b) Let V be a vector space over the field of complex numbers , and suppose there is an isomorphism T of V onto C^3. Let α_1 , α_2 , α_3 , α_4 be vectors in V such that

$$\begin{gathered}T(\alpha_1) = \begin{pmatrix}1\\0\\i\end{pmatrix}, T(\alpha_2) = \begin{pmatrix}-2\\1+i\\0\end{pmatrix},\\ T(\alpha_3) = \begin{pmatrix}-1\\1\\1\end{pmatrix}, T(\alpha_4) = \begin{pmatrix}\sqrt{2}\\i\\3\end{pmatrix}\end{gathered} \tag{3.3.2.16}$$

Let W_1 be the subspace spanned by α_1 and α_2, and let W_2 be the subspace spanned by α_3 and α_4. What is the intersection of W_1 and W_2.

Solution: T : V → W is an isomorphism if
(1) T is one one.
(2) T is onto. If W_1 and W_2 are finite-dimensional subspaces of a vector space V, then $W_1 + W_2$ is finite-dimensional and

$$dim(W_1) + dim(W_2) = dim(W_1 \cap W_2) + dim(W_1 + W_2) \tag{3.3.2.17}$$

We have,

$$T = \begin{pmatrix}1 & -2 & -1 & \sqrt{2}\\0 & 1+i & 1 & i\\i & 0 & 1 & 3\end{pmatrix} \tag{3.3.2.18}$$

$$\begin{pmatrix}1 & -2 & -1 & \sqrt{2}\\0 & 1+i & 1 & i\\i & 0 & 1 & 3\end{pmatrix} \xrightarrow{rref} \begin{pmatrix}1 & 0 & -i & 0\\0 & 1 & \frac{1-i}{2} & 0\\0 & 0 & 0 & 1\end{pmatrix} \tag{3.3.2.19}$$

T is one one over C^3 if

$$T(\alpha) = 0 \implies \alpha = 0 \tag{3.3.2.20}$$

now,

$$\begin{pmatrix}1 & -2 & \sqrt{2}\\0 & 1+i & i\\i & 0 & 3\end{pmatrix}\alpha = 0 \tag{3.3.2.21}$$

consider the row reduced matrix

$$\begin{pmatrix}1 & -2 & \sqrt{2}\\0 & 1+i & i\\i & 0 & 3\end{pmatrix} \xleftrightarrow[R_3\to iR_3]{R_3\to R_3-iR_1} \begin{pmatrix}1 & -2 & \sqrt{2}\\0 & 1+i & i\\0 & -2 & \sqrt{2}+3i\end{pmatrix} \tag{3.3.2.22}$$

$$\xleftrightarrow[R_3\leftarrow R_3+R_2]{R_2\leftarrow(1-i)R_2} \begin{pmatrix}1 & -2 & \sqrt{2}\\0 & 2 & i+1\\0 & 0 & \sqrt{2}+4i+1\end{pmatrix} \tag{3.3.2.23}$$

$$\alpha = \begin{pmatrix}0\\0\\0\end{pmatrix} \tag{3.3.2.24}$$

Therefore it holds the condition of one one and the rank = no. of pivot columns = 3 (equal to no. of columns).Thus the vectors are linearly independent hence it is onto .

Hence, T is an isomorphism onto C^3.
Also, from (3.3.2.19), we observe,

$$T(\alpha_3) = -iT(\alpha_1) + \frac{i-1}{2}T(\alpha_2) \quad (3.3.2.25)$$

Hence $T(\alpha_3)$ belongs to the subspace spanned by $T(\alpha_1)$ and $T(\alpha_2)$.
Therefore, α_3 is in subspace spanned by α_1 and α_2. Therefore,

$$\alpha_3 \in W_1 \quad (3.3.2.26)$$

$$\implies \alpha_3 \in W_1 \cap W_2 \quad (3.3.2.27)$$

Since $T(\alpha_1)$ and $T(\alpha_2)$ are linearly independent, and $T(\alpha_3)$ and $T(\alpha_4)$ are linearly independent, we have,

$$dim(W_1) = dim(W_2) = 2 \quad (3.3.2.28)$$

From (3.3.2.17),

$$dim(W_1) + dim(W_2) = dim(W_1 \cap W_2) + dim(W_1 + W_2) \quad (3.3.2.29)$$

$$dim(W_1 + W_2) = 3 \quad (3.3.2.30)$$

$$\implies dim(W_1 \cap W_2) = 2 + 2 - 3 \quad (3.3.2.31)$$

$$\implies dim(W_1 \cap W_2) = 1 \quad (3.3.2.32)$$

Therefore,

$$W_1 \cap W_2 = c\alpha_3 \quad (3.3.2.33)$$

c) Find a basis for the subspace of V spanned by the 4 vectors α_i.
Solution: V is a vector space and V is isomorphic to C^3 via isomorphism T which implies that C^3 is also isomorphic to V via isomorphism T^{-1}.
As V is isomorphic to C^3, so

$$dim(V) = dim(C^3) = 3 \quad (3.3.2.34)$$

Now,

$$\begin{pmatrix} 1 & 0 & i \\ -2 & 1+i & 0 \\ -1 & 1 & 1 \\ \sqrt{2} & i & 3 \end{pmatrix} \xleftrightarrow[R_4 \leftarrow \sqrt{2}R_4]{R_3 \leftarrow R_3 + R_1} \begin{pmatrix} 1 & 0 & i \\ -2 & 1+i & 0 \\ 0 & 1 & 1+i \\ 2 & i\sqrt{2} & 3\sqrt{2} \end{pmatrix}$$

$$\xleftrightarrow{R_4 \leftarrow R_4 + R_2} \begin{pmatrix} 1 & 0 & i \\ -2 & 1+i & 0 \\ 0 & 1 & 1+i \\ 0 & 1+i(1+\sqrt{2}) & 3\sqrt{2} \end{pmatrix}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 + 2R_1} \begin{pmatrix} 1 & 0 & i \\ 0 & 1+i & 2i \\ 0 & 1 & 1+i \\ 0 & 1+i(1+\sqrt{2}) & 3\sqrt{2} \end{pmatrix}$$

$$\xleftrightarrow{R_3 \leftarrow (1+i)R_3} \begin{pmatrix} 1 & 0 & i \\ 0 & 1+i & 2i \\ 0 & 1+i & 2i \\ 0 & 1+i(1+\sqrt{2}) & 3\sqrt{2} \end{pmatrix}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - R_2} \begin{pmatrix} 1 & 0 & i \\ 0 & 1+i & 2i \\ 0 & 0 & 0 \\ 0 & 1+i(1+\sqrt{2}) & 3\sqrt{2} \end{pmatrix} \quad (3.3.2.35)$$

From here we can get that $T\alpha_3$ is dependent vector while $T\alpha_1$, $T\alpha_2$ and $T\alpha_4$ are independent vector. These $T\alpha_1$, $T\alpha_2$ and $T\alpha_4$ also span the vector space C^3, so these 3 vectors are the basis of C^3.
As $dim(V) = 3$, so it must have 3 basis and as V and C^3 are isomorphic so α_1, α_2 and α_4 are the basis of V.

3.3.3. Let $\mathbb{W}$ be the set of all 2×2 complex Hermitian matrices, that is the sset of 2×2 complex matrices $\mathbf{A}$ ssuch that $\mathbf{A}_{ij} = \overline{\mathbf{A_{ji}}}$ (the bar denoting complex conjugation). $\mathbb{W}$ is a vector space over the field of real numbers, under the usual operations. Verify that

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \to \begin{pmatrix} t+x & y+iz \\ y-iz & t-x \end{pmatrix} \quad (3.3.3.1)$$

is an isomorphism of $\mathbb{R}^4$ onto $\mathbb{W}$.
Solution:

a) **Check for linearity:** The transformation T is given by

$$T : \mathbb{R}^4 \to \mathbb{W} \tag{3.3.3.2}$$

$$T\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} t+x & y+iz \\ y-iz & t-x \end{pmatrix} \tag{3.3.3.3}$$

Let $\mathbf{x} = \begin{pmatrix} x \\ y \\ t \\ z \end{pmatrix}$. Expressing R.H.S of equation (3.3.3.3) using Kronecker Product,

$$T(\mathbf{x}) = \begin{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}\mathbf{x} & \begin{pmatrix} 0 & 1 & i & 0 \end{pmatrix}\mathbf{x} \\ \begin{pmatrix} 0 & 1 & -i & 0 \end{pmatrix}\mathbf{x} & \begin{pmatrix} -1 & 0 & 0 & 1 \end{pmatrix}\mathbf{x} \end{pmatrix} \tag{3.3.3.4}$$

$$= \begin{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -i & 0 \end{pmatrix}\mathbf{x} & \begin{pmatrix} 0 & 1 & i & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}\mathbf{x} \end{pmatrix} \tag{3.3.3.5}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & i & 0 \\ 0 & 1 & -i & 0 & -1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x & 0 \\ y & 0 \\ z & 0 \\ t & 0 \\ 0 & x \\ 0 & y \\ 0 & z \\ 0 & t \end{pmatrix} \tag{3.3.3.6}$$

$$\implies T(\mathbf{x}) = \begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} \begin{pmatrix} \mathbf{x} & \mathbf{0}_{4\times1} \\ \mathbf{0}_{4\times1} & \mathbf{x} \end{pmatrix} \tag{3.3.3.7}$$

Where $\mathbf{A}$ and $\mathbf{B}$ are block matrices.

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -i & 0 \end{pmatrix} \tag{3.3.3.8}$$

$$\mathbf{B} = \begin{pmatrix} 0 & 1 & i & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \tag{3.3.3.9}$$

The Kronecker Product of $\mathbf{I}_2$ and $\mathbf{x}$ gives the block matrix in equation (3.3.3.7).

$$\mathbf{I}_{2\times2} \otimes \mathbf{x}_{4\times1} = \begin{pmatrix} \mathbf{x} & \mathbf{0} \\ \mathbf{0} & \mathbf{x} \end{pmatrix}_{8\times2} \tag{3.3.3.10}$$

Hence we can write equation (3.3.3.7) as,

$$T(\mathbf{x}) = \begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} (\mathbf{I} \otimes \mathbf{x}) \tag{3.3.3.11}$$

Let $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^4$ and $\alpha, \beta \in \mathbb{R}$.

$$T(\alpha\mathbf{x}_1 + \beta\mathbf{x}_2) = \begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} (\mathbf{I} \otimes (\alpha\mathbf{x}_1 + \beta\mathbf{x}_2)) \tag{3.3.3.12}$$

$$= \alpha\begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} (\mathbf{I} \otimes \mathbf{x}_1) + \beta\begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} (\mathbf{I} \otimes \mathbf{x}_2) \tag{3.3.3.13}$$

$$= \alpha T\mathbf{x}_1 + \beta T\mathbf{x}_2 \tag{3.3.3.14}$$

Therefore from equation (3.3.3.14), we can say T is linear transformation.

b) **Check for one-one property:** For transformation T to be one-one, we can prove if $T(\mathbf{x}) = \mathbf{0}$, that implies $\mathbf{x} = \mathbf{0}$. From the equation (3.3.3.11),

$$T(\mathbf{x}) = \mathbf{0} \tag{3.3.3.15}$$

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} (\mathbf{I} \otimes \mathbf{x}) = \mathbf{0} \tag{3.3.3.16}$$

$$\implies \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & i & 0 \\ 0 & 1 & -i & 0 & -1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x & 0 \\ y & 0 \\ z & 0 \\ t & 0 \\ 0 & x \\ 0 & y \\ 0 & z \\ 0 & t \end{pmatrix} = \mathbf{0}_{2\times2} \tag{3.3.3.17}$$

From equation (3.3.3.3),

$$\begin{pmatrix} t+x & y+iz \\ y-iz & t-x \end{pmatrix} = \mathbf{0}_{2\times2} \tag{3.3.3.18}$$

$$\implies x = 0, y = 0, z = 0, t = 0 \tag{3.3.3.19}$$

$$\implies \mathbf{x} = \mathbf{0} \tag{3.3.3.20}$$

Hence from (3.3.3.15) and (3.3.3.20), T is one-one and that implies $T : \mathbb{R}^4 \to \mathbb{W}$ is isomorphism.

3.3.4. Show that $\mathbf{F}^{\mathbf{m}\times\mathbf{n}}$ is isomorphic to $\mathbf{F}^{\mathbf{mn}}$.

Solution: See Tables 3.3.4.1, 3.3.4.2 and 3.3.4.3.

$\mathbb{R}^{2\times2}$ is isomorphic to $\mathbb{R}^4$ ie, $\mathbb{R}^{2\times2} \cong \mathbb{R}^4$.

3.3.5. Let $\mathbf{V}$ be the set of complex numbers regarded as a vector space over the field of real numbers. We define a function T from $\mathbf{V}$ into the space of 2×2 real matrices, as follows. If $z = x + iy$

Invertible Linear Map	A linear map $\mathbf{T} \in \mathbf{L}(\mathbf{V}, \mathbf{W})$ is called invertible if there exists a linear map $\mathbf{S} \in \mathbf{L}(\mathbf{W}, \mathbf{V})$ such that $\mathbf{ST}$ equals the identity map on $\mathbf{V}$ and $\mathbf{TS}$ equals the identity map on $\mathbf{W}$. A linear map $\mathbf{S} \in \mathbf{L}(\mathbf{W}, \mathbf{V})$ satisfying $\mathbf{ST} = \mathbf{I_V}$ and $\mathbf{TS} = \mathbf{I_W}$ is called an inverse of $\mathbf{T}$.
Isomorphic Vector Spaces	Two vector spaces $\mathbf{V}$ and $\mathbf{W}$ are called isomorphic if there is an isomorphism from one vector space onto the other one. An isomorphism is an invertible linear map.
Rank Nullity Theorem	Let $\mathbf{V}$ and $\mathbf{W}$ be finite dimensional vector spaces. Let $\mathbf{T}$: $\mathbf{V} \to \mathbf{W}$ be a linear transformation Rank($\mathbf{T}$) + Nullity($\mathbf{T}$) = dim $\mathbf{V}$

TABLE 3.3.4.1: Definition

Result 1	The space of all $m \times n$ matrices over the field $\mathbf{F}$ has dimension mn.
Result 2	Let $\mathbf{V}$ and $\mathbf{W}$ be finite-dimensional vector spaces over the field $\mathbf{F}$ such that dim $\mathbf{V}$ = dim $\mathbf{W}$. If $\mathbf{T}$ is a linear transformation from $\mathbf{V}$ into $\mathbf{W}$, then the following are equivalent: (a). $\mathbf{T}$ is invertible. (b). $\mathbf{T}$ is non-singular. (c). $\mathbf{T}$ is onto, that is, range of $\mathbf{T}$ is $\mathbf{W}$.

TABLE 3.3.4.2: Results Used

with x and y real numbers, then

$$\mathbf{T}(z) = \begin{pmatrix} x + 7y & 5y \\ -10y & x - 7y \end{pmatrix}$$

a) Verify that T is a one-one (real) linear transformation of $\mathbf{V}$ into the space of 2×2 real matrices.
Solution: The kronecker product also called as matrix direct product is defined as

$$\mathbf{A} \otimes \mathbf{B} = \begin{pmatrix} a_{11}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ \vdots & \ddots & \vdots \\ a_{m1}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{pmatrix} \tag{3.3.5.1}$$

Also,

$$\mathbf{A} \otimes (\mathbf{B} + \mathbf{C}) = \mathbf{A} \otimes \mathbf{B} + \mathbf{A} \otimes \mathbf{C} \tag{3.3.5.2}$$

$$\mathbf{A} \otimes (k\mathbf{B}) = k(\mathbf{A} \otimes \mathbf{B}) \tag{3.3.5.3}$$

Given,

$$\mathbf{T} : \mathbf{C} \to \mathbf{R^{2\times2}}$$

$$\mathbf{T}(x + iy) = \begin{pmatrix} x + 7y & 5y \\ -10y & x - 7y \end{pmatrix} \tag{3.3.5.4}$$

Let,

$$z = x + iy; \quad w = a + ib; \quad z, w \in \mathbf{C}$$

Also the RHS of (3.3.5.4) can be expressed as,

$$\mathbf{T}(\mathbf{z}) = \left(\begin{pmatrix} 1 & 7 \\ 0 & -10 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \quad \begin{pmatrix} 0 & 5 \\ 1 & -7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\right)$$

$$= \begin{pmatrix} 1 & 7 & 0 & 5 \\ 0 & -10 & 1 & -7 \end{pmatrix}\begin{pmatrix} x & 0 \\ y & 0 \\ 0 & x \\ 0 & y \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix}\begin{pmatrix} \mathbf{x} & 0 \\ 0 & \mathbf{x} \end{pmatrix} \tag{3.3.5.5}$$

where $\mathbf{A}$ and $\mathbf{B}$ are block matrices and,

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$

The diagonal block matrix can be expressed as the kronecker product of $\mathbf{I}$ and $\mathbf{x}$

$$\mathbf{I} \otimes \mathbf{x} = \begin{pmatrix} \mathbf{x} & 0 \\ 0 & \mathbf{x} \end{pmatrix} \tag{3.3.5.6}$$

Defining Sets	We define set S and set T as $S = \{(a,b) : a,b \in \mathbb{N}, 1 \le a \le m, 1 \le b \le n\}, \quad T = \{1,2,...,mn\}$
Defining Bijection	We now define a bijection $\sigma : S \to T$ as $(a,b) \to (a-1)\text{n} + \text{b}$
Defining Function G	We now define a function G from $F^{m\times n}$ to F^{mn} as follows. Let $\mathbf{A} \in F^{m\times n}$. Then map $\mathbf{A}$ to the mn tupple that has $\mathbf{A_{ij}}$ in the $\sigma(i,j)$ position. In other words, $\mathbf{A} \to (\mathbf{A_{11}}, \mathbf{A_{12}}, ..., \mathbf{A_{1n}}, ..., \mathbf{A_{m1}}, \mathbf{A_{m2}}, ..., \mathbf{A_{mn}})$
Proving G to be Linear	Since, addition in $F^{m\times n}$ and in F^{mn} is performed component-wise, $G(\mathbf{A}+\mathbf{B}) = G(\mathbf{A}) + G(\mathbf{B})$ and scalar multiplication in $F^{m\times n}$ and in F^{mn} is also defined as $G(c\mathbf{A}) = cG(\mathbf{A})$.
Proving G to be One-One	$G(\mathbf{A}) = G(\mathbf{B})$ $\implies (\mathbf{A_{11}}, \mathbf{A_{12}}, ..., \mathbf{A_{1n}}, ..., \mathbf{A_{m1}}, \mathbf{A_{m2}}, ..., \mathbf{A_{mn}}) = (\mathbf{B_{11}}, \mathbf{B_{12}}, ..., \mathbf{B_{1n}}, ..., \mathbf{B_{m1}}, \mathbf{B_{m2}}, ..., \mathbf{B_{mn}})$ $\implies \mathbf{A_{i,j}} = \mathbf{B_{ij}} \quad \forall 1 \le i \le m, 1 \le j \le n$ $\implies \mathbf{A} = \mathbf{B}$
Proving G to be Onto	Since G is one to one, so Null(G) = 0. Thus, by Rank-Nullity Theorem dim(Range(G))= mn, proving G to be a surjective (onto) map as by Result 1 dimension of $F^{m\times n} = mn$
$F^{m\times n} \cong F^{mn}$	Since G has an inverse and is an isomorphism of $\mathbf{T}$. Thus, by Result 2 $F^{m\times n} \cong F^{mn}$

TABLE 3.3.4.3: Proof

Where $\mathbf{I}$ is an identity matrix. (3.3.5.5) can be rewritten as,

$$\mathbf{T}(\mathbf{z}) = \begin{pmatrix}\mathbf{A} & \mathbf{B}\end{pmatrix}(\mathbf{I} \otimes \mathbf{x}) \tag{3.3.5.7}$$

Consider,

$$\mathbf{T}(\alpha\mathbf{z} + \mathbf{w}) = \begin{pmatrix}\mathbf{A} & \mathbf{B}\end{pmatrix}(\mathbf{I} \otimes (\alpha\mathbf{z} + \mathbf{w}))$$

Using properties (3.3.5.2), (3.3.5.3), the above equation can be expressed as,

$$\begin{aligned}\mathbf{T}(\alpha\mathbf{z} + \mathbf{w}) &= \begin{pmatrix}\mathbf{A} & \mathbf{B}\end{pmatrix}(\mathbf{I} \otimes (\alpha\mathbf{z})) + \begin{pmatrix}\mathbf{A} & \mathbf{B}\end{pmatrix}(\mathbf{I} \otimes \mathbf{x}) \\ &= \alpha\begin{pmatrix}\mathbf{A} & \mathbf{B}\end{pmatrix}(\mathbf{I} \otimes \mathbf{z}) + \begin{pmatrix}\mathbf{A} & \mathbf{B}\end{pmatrix}(\mathbf{I} \otimes \mathbf{w}) \\ &= \alpha\mathbf{T}(\mathbf{z}) + \mathbf{T}(\mathbf{w}) \end{aligned} \tag{3.3.5.8}$$

From (3.3.5.8), it can be proved that $\mathbf{T}$ is a linear operator.

b) Verify that

$$\mathbf{T}(z_1 z_2) = \mathbf{T}(z_1)\mathbf{T}(z_2) \tag{3.3.5.9}$$

Solution: The product of two Kronecker products yields another Kronecker product:

$$(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = (\mathbf{AC}) \otimes (\mathbf{BD}) \tag{3.3.5.10}$$

Given,

$$\mathbf{T}(\mathbf{z}) = \begin{pmatrix} x + 7y & 5y \\ -10y & x - 7y \end{pmatrix} \tag{3.3.5.11}$$

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \tag{3.3.5.12}$$

$$\mathbf{T}(\mathbf{x}) = \begin{pmatrix} \begin{pmatrix}1 & 7\end{pmatrix}\mathbf{x} & \begin{pmatrix}0 & 5\end{pmatrix}\mathbf{x} \\ \begin{pmatrix}0 & -10\end{pmatrix}\mathbf{x} & \begin{pmatrix}1 & -7\end{pmatrix}\mathbf{x} \end{pmatrix} \tag{3.3.5.13}$$

$$= \left(\begin{pmatrix} 1 & 7 \\ 0 & -10 \end{pmatrix} \mathbf{x} \quad \begin{pmatrix} 0 & 5 \\ 1 & -7 \end{pmatrix} \mathbf{x} \right) \tag{3.3.5.14}$$

$$\text{Let } \mathbf{A} = \begin{pmatrix} 1 & 7 \\ 0 & -10 \end{pmatrix} \tag{3.3.5.15}$$

$$\mathbf{B} = \begin{pmatrix} 0 & 5 \\ 1 & -7 \end{pmatrix} \tag{3.3.5.16}$$

$$\implies \mathbf{T}(\mathbf{x}) = \begin{pmatrix} \mathbf{A}\mathbf{x} & \mathbf{B}\mathbf{x} \end{pmatrix} \tag{3.3.5.17}$$

$$\mathbf{T}(\mathbf{x}) = \begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} \begin{pmatrix} \mathbf{x} & \mathbf{0} \\ \mathbf{0} & \mathbf{x} \end{pmatrix} \tag{3.3.5.18}$$

The diagonal block matrix can be expressed as the kronecker product of $\mathbf{I}$ and $\mathbf{x}$

$$\mathbf{I} \otimes \mathbf{x} = \begin{pmatrix} \mathbf{x} & \mathbf{0} \\ \mathbf{0} & \mathbf{x} \end{pmatrix} \tag{3.3.5.19}$$

We can write (3.3.5.11) as

$$\mathbf{T}(\mathbf{x}) = \begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} (\mathbf{I} \otimes \mathbf{x}) \tag{3.3.5.20}$$

Starting with RHS of (3.3.5.10)

$$\mathbf{T}(z_1)\mathbf{T}(z_2) = \begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} (\mathbf{I} \otimes \mathbf{z_1}) \begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} (\mathbf{I} \otimes \mathbf{z_2}) \tag{3.3.5.21}$$

If

$$(\mathbf{I} \otimes \mathbf{z_1}) \begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} (\mathbf{I} \otimes \mathbf{z_2}) = (\mathbf{I} \otimes \mathbf{z_1 z_2}) \tag{3.3.5.22}$$

then,we can write (3.3.5.21) as

$$\mathbf{T}(z_1)\mathbf{T}(z_2) = \begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} (\mathbf{I} \otimes \mathbf{z_1 z_2}) = \mathbf{T}(z_1 z_2) \tag{3.3.5.23}$$

c) How would you describe the range of $\mathbb{T}$?
Solution:

$$\mathbb{T} : \mathbf{V} \to \mathbb{R}^{2\times2} \tag{3.3.5.24}$$

where $\mathbb{R}^{2\times2}$,is the space of all 2×2 real matrices

$$\mathbb{T}(z) = \begin{pmatrix} x + 7y & 5y \\ -10y & x - 7y \end{pmatrix} \tag{3.3.5.25}$$

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \tag{3.3.5.26}$$

$$\mathbb{T}(\mathbf{x}) = \begin{pmatrix} \begin{pmatrix} 1 & 7 \end{pmatrix} \mathbf{x} & \begin{pmatrix} 0 & 5 \end{pmatrix} \mathbf{x} \\ \begin{pmatrix} 0 & -10 \end{pmatrix} \mathbf{x} & \begin{pmatrix} 1 & -7 \end{pmatrix} \mathbf{x} \end{pmatrix} \tag{3.3.5.27}$$

$$= \left(\begin{pmatrix} 1 & 7 \\ 0 & -10 \end{pmatrix} \mathbf{x} \quad \begin{pmatrix} 0 & 5 \\ 1 & -7 \end{pmatrix} \mathbf{x} \right) \tag{3.3.5.28}$$

$$\text{Let } \mathbf{A} = \begin{pmatrix} 1 & 7 \\ 0 & -10 \end{pmatrix} \tag{3.3.5.29}$$

$$\mathbf{B} = \begin{pmatrix} 0 & 5 \\ 1 & -7 \end{pmatrix} \tag{3.3.5.30}$$

$$\implies \mathbb{T}(\mathbf{x}) = \begin{pmatrix} \mathbf{A}\mathbf{x} & \mathbf{B}\mathbf{x} \end{pmatrix} \tag{3.3.5.31}$$

$$\mathbb{T}(\mathbf{x}) = \begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} \begin{pmatrix} \mathbf{x} & \mathbf{0}_{2\times1} \\ \mathbf{0}_{2\times1} & \mathbf{x} \end{pmatrix} \tag{3.3.5.32}$$

The kronecker product of $\mathbf{I}, \mathbf{x}$ gives the block matrix

$$\mathbf{I}_{2\times2} \otimes \mathbf{x}_{2\times1} = \begin{pmatrix} \mathbf{x} & \mathbf{0} \\ \mathbf{0} & \mathbf{x} \end{pmatrix}_{4\times2} \tag{3.3.5.33}$$

$$(3.3.5.32) \implies \mathbb{T}(\mathbf{x}) = \begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} \mathbf{I} \otimes \mathbf{x} \tag{3.3.5.34}$$

$$= \begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} \mathbf{I} \otimes \left[\begin{pmatrix} x \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ y \end{pmatrix} \right] \tag{3.3.5.35}$$

$$= x \begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} \mathbf{I} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} \mathbf{I} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.3.5.36}$$

$$\mathbf{I} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{3.3.5.37}$$

$$\mathbf{I} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.3.5.38}$$

Kronecker product in the first term of (3.3.5.36) picks out 1st columns of $\mathbf{A}, \mathbf{B}$ and in the second term picks out 2nd columns of

$\mathbf{A}, \mathbf{B}$ so basis for range($\mathbb{T}$) is

$$\left\{\begin{pmatrix}\mathbf{A} & \mathbf{B}\end{pmatrix}\mathbf{I}\otimes\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}\mathbf{A} & \mathbf{B}\end{pmatrix}\mathbf{I}\otimes\begin{pmatrix}0\\1\end{pmatrix}\right\} = \tag{3.3.5.39}$$

$$\left\{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}7 & 5\\-10 & -7\end{pmatrix}\right\} \tag{3.3.5.40}$$

$$\text{range}(\mathbb{T}) = \tag{3.3.5.41}$$

$$\text{span of } \left\{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}7 & 5\\-10 & -7\end{pmatrix}\right\} \tag{3.3.5.42}$$

3.3.6. Let $\mathbf{V}$ and $\mathbf{W}$ be finite-dimensional vector spaces over the field $\mathbf{F}$. Prove that $\mathbf{V}$ and $\mathbf{W}$ are isomorphic if and only if $\dim \mathbf{V} = \dim \mathbf{W}$
Solution: Refer Table 3.3.6.1.
Therefore from equation (3.3.6.4) and (3.3.6.7) in Table 3.3.6.1, $\{T(\mathbf{v_1}), T(\mathbf{v_2}), \cdots, T(\mathbf{v_n})\}$ are linearly independent and span $\mathbf{W}$.

$$\implies \dim \mathbf{W} = \text{n} = \dim \mathbf{V}$$

$\mathbf{V}$ and $\mathbf{W}$ are isomorphic if and only if $\dim \mathbf{V} = \dim \mathbf{W}$

3.3.7. Let V and W be finite-dimensional vector spaces over the field F and let U be an isomorphism of V and W. Prove that $T \to UTU^{-1}$ is an isomorphism of $L(V, V)$ onto $L(W, W)$.
Solution:
Example Let

$$U = \begin{pmatrix}1 & 2\\3 & 4\end{pmatrix} \tag{3.3.7.1}$$

here U is an isomorphism from $\mathbb{R}^{2\times2}$ to $\mathbb{R}^{2\times2}$ since inverse of U exists and

$$U^{-1} = \begin{pmatrix}-2 & -\frac{3}{2}\\-1 & -\frac{1}{2}\end{pmatrix} \tag{3.3.7.2}$$

Consider

$$T = \begin{pmatrix}-1 & 2\\3 & 1\end{pmatrix} \in \mathbb{R}^{2\times2} \tag{3.3.7.3}$$

Now

$$UTU^{-1} = \begin{pmatrix}1 & 2\\3 & 4\end{pmatrix}\begin{pmatrix}-1 & 2\\3 & 1\end{pmatrix}\begin{pmatrix}-2 & -1\\-\frac{3}{2} & -\frac{1}{2}\end{pmatrix} \tag{3.3.7.4}$$

$$= \begin{pmatrix}-16 & -7\\-33 & -14\end{pmatrix} \in \mathbb{R}^{2\times2} \tag{3.3.7.5}$$

Also inverse exists for T

$$S = T^{-1} = \begin{pmatrix}-\frac{1}{2} & \frac{2}{7}\\\frac{3}{7} & \frac{1}{7}\end{pmatrix} \tag{3.3.7.6}$$

Since T inverse exists $\mathcal{T}(T) = UTU^{-1}$ is an isomorphism from $\mathbb{R}^{2\times2}$ onto $\mathbb{R}^{2\times2}$.

3.3.8. Let T be a linear operator on the finite-dimensional space $\mathbb{V}$. Suppose there is a linear operator U on $\mathbb{V}$ such that $TU = I$. Prove that T is invertible and $U = T^{-1}$. Give an example which shows that this is false when $\mathbb{V}$ is not finite-dimensional.
Solution: Let $T : \mathbb{V} \to \mathbb{V}$ be a linear operator, where $\mathbb{V}$ is a finite dimensional vectors space and $U : \mathbb{V} \to \mathbb{V}$ is also a linear operator such that,

$$TU = I \tag{3.3.8.1}$$

Where, I is an identity transformation. Now we know that linear transformations are functions. Hence,

$$TU = I \quad \text{is a function} \tag{3.3.8.2}$$

$$\implies I : \mathbb{V} \to \mathbb{V} \tag{3.3.8.3}$$

Such that $T(V) = V$. Defining $TU : \mathbb{V} \to \mathbb{V}$ to be a linear operator, we have,

$$T[U(V_i)] = V_i \qquad [V_i \in \mathbb{V}] \tag{3.3.8.4}$$

Now we show in the below Table that T is one-one and onto as follows,

Hence we get from Table 3.3.8.1 that, T is invertible. Hence we get the following,

$$TT^{-1} = I \tag{3.3.8.5}$$

Where T^{-1} is an inverse function of linear operator T. Hence,

$$TT^{-1} = I = TU \tag{3.3.8.6}$$

$$\implies T^{-1}(TT^{-1}) = T^{-1}(TU) \tag{3.3.8.7}$$

$$\implies T^{-1}(I) = IU \tag{3.3.8.8}$$

$$\implies T^{-1} = U \tag{3.3.8.9}$$

Hence from (3.3.8.9) it is proven that T is invertible and $T^{-1} = U$
Example: Let D be the differential operator $D : \mathbb{V} \to \mathbb{V}$ where $\mathbb{V}$ is a space of polynomial

<table>
<tr><td colspan="2">$T : \mathbf{V} \rightarrow \mathbf{W} \quad \mathbf{V} = \{\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_n}\} \quad \mathbf{W} = \{T(\mathbf{v_1}), T(\mathbf{v_2}), \cdots, T(\mathbf{v_n})\}$</td></tr>
<tr><td>Property Used</td><td>Derivation</td></tr>
<tr><td>T is one-one</td><td>Linear combination of vectors in W
$$\sum_{k=1}^{n} \alpha_k T(\mathbf{v_k}) = \mathbf{0} \tag{3.3.6.1}$$
$$\sum_{k=1}^{n} T(\alpha_k \mathbf{v_k}) = \mathbf{0} \tag{3.3.6.2}$$
$$\implies \sum_{k=1}^{n} \alpha_k \mathbf{v_k} = \mathbf{0} \tag{3.3.6.3}$$
$$\implies \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0 \tag{3.3.6.4}$$
From equation (3.3.6.1) and (3.3.6.4), the set of vectors $\{T(\mathbf{v_1}), T(\mathbf{v_2}), \cdots, T(\mathbf{v_n})\}$ are linearly independent</td></tr>
<tr><td>T is onto</td><td>For any $\mathbf{y} \in \mathbf{W}$ there exists an $\mathbf{x} \in \mathbf{V}$ such that $T(\mathbf{x}) = \mathbf{y}$.
$$\mathbf{x} = \sum_{k=1}^{n} \alpha_k \mathbf{v_k} \tag{3.3.6.5}$$
$$T(\mathbf{x}) = T(\sum_{k=1}^{n} \alpha_k \mathbf{v_k}) = \mathbf{y} \tag{3.3.6.6}$$
$$\sum_{k=1}^{n} \alpha_k T(\mathbf{v_k}) = \mathbf{y} \tag{3.3.6.7}$$
From equation (3.3.6.7), any vector in W can be represented as linear combination of $\{T(\mathbf{v_1}), T(\mathbf{v_2}), \cdots, T(\mathbf{v_n})\}$. That is it spans W.</td></tr>
</table>

TABLE 3.3.6.1: Derivation

functions in one variable x over $\mathbb{R}$ as follows,

$$D(c_0 + c_1 x + \cdots + c_n x^n) = c_1 + c_2' x + \cdots + c_n' x^{n-1} \tag{3.3.8.10}$$

We first prove that the vector space $\mathbb{V}$ is infinite dimensional.
Suppose to the contrary that $\mathbb{V}$ is finite dimensional vector space and is given by the span of k polynomials in $\mathbb{V}$ as follows,

$$span(\mathbb{V}) = \{p_1, p_2, \ldots, p_k\} \tag{3.3.8.11}$$

Also let m be the maximum of the degree of these k polynomials in (3.3.8.11). Now let an element of the vector space $\mathbb{V}$ be,

$$cx^{m+1} \in \mathbb{V} \tag{3.3.8.12}$$

As maximum degree of the basis of $\mathbb{V}$ is m hence cx^{m+1} cannot be represented by any linear combination of the basis of $\mathbb{V}$. If $\mathbb{F}$ is field corresponding to $\mathbb{V}$ then we have,

$$cx^{m+1} \neq \sum_{i=1}^{k} \alpha_i p_i \quad [\alpha_i \in \mathbb{F} \ \forall i] \tag{3.3.8.13}$$

Hence, cx^{m+1} is not in the span of $p_1, p_2, \ldots, p_k$. Hence, $\mathbb{V}$ is infinite dimensional vector space.
Next we prove that D is not one-one operator. Let, two different elements from the vector space $\mathbb{V}$ be as follows,

$$c_1 + x^m \in \mathbb{V} \tag{3.3.8.14}$$
$$c_2 + x^m \in \mathbb{V} \tag{3.3.8.15}$$

From definition (3.3.8.10) of operator D we

Given	$\mathcal{T}(T) : T \to UTU^{-1}$ U is isomorphism of V onto W that means U is $one-one$ $\mathcal{T} : L(V,V) \to L(W,W)$
To prove	$\mathcal{T}$ is isomorphism of $L(V,V)$ onto $L(W,W)$ It is same as proving $\mathcal{T}$ is invertible, because $isomorphim \implies one-one \implies invertible$ by definition
Proof	Consider inverse transformation $\mathcal{S} : L(W,W) \to L(V,V)$ $\mathcal{S} : S \to U^{-1}SU$ where $U^{-1}SU$ is a composition of 3 linear transformations $V \xrightarrow{U} W \xrightarrow{S} W \xrightarrow{U^{-1}} V$ Now consider $\mathcal{S}(UTU^{-1})$, $\mathcal{S}(UTU^{-1}) = U^{-1}(UTU^{-1})U = T$ Similarly consider $\mathcal{T}(U^{-1}SU)$, $\mathcal{T}(U^{-1}SU) = U(U^{-1}SU)U^{-1} = S$ $\implies TS = I$ and $ST = I$ we can say $\mathcal{T}$ is invertible since we have found an inverse $\mathcal{S}$ Hence $\mathcal{T}$ is one-one implies $\mathcal{T}$ isomorphism of V onto W

TABLE 3.3.7.1: Proof

have,

$$D(c_1 + x^m) = mx^{m-1} \tag{3.3.8.16}$$
$$D(c_2 + x^m) = mx^{m-1} \tag{3.3.8.17}$$

From (3.3.8.16) and (3.3.8.17),

$$c_1 + x^m \neq c_2 + x^m \tag{3.3.8.18}$$
$$D(c_1 + x^m) = D(c_2 + x^m) \tag{3.3.8.19}$$

Hence from (3.3.8.19) we see that D is not One-One operator.
And, $U : \mathbb{V} \to \mathbb{V}$ is another linear operator such that,

$$U(c_0 + c_1x + \cdots + c_nx^n) = c_0x + c_1\frac{x^2}{2} + \cdots + c_n\frac{x^{n+1}}{n+1} \tag{3.3.8.20}$$

Now, $DU : \mathbb{V} \to \mathbb{V}$ is a linear operator such that,

$$DU(c_0 + c_1x + \cdots + c_nx^n) \tag{3.3.8.21}$$
$$= D[U(c_0x + c_1\frac{x^2}{2} + \cdots + c_n\frac{x^{n+1}}{n+1})] \tag{3.3.8.22}$$
$$= D[c_0x + c_1\frac{x^2}{2} + \cdots + c_n\frac{x^{n+1}}{n+1}] \tag{3.3.8.23}$$
$$= c_0 + c_1\frac{2x}{2} + \cdots + c_n\frac{(n+1)x^n}{n+1} \tag{3.3.8.24}$$
$$= c_0 + c_1x + \cdots + c_nx^n \tag{3.3.8.25}$$

Hence, from (3.3.8.25),

$$DU = I \tag{3.3.8.26}$$

Again $UD : \mathbb{V} \to \mathbb{V}$ is a linear operator such that,

$$UD(c_0 + c_1x + \cdots + c_nx^n) \tag{3.3.8.27}$$
$$= U[D(c_0x + c_1\frac{x^2}{2} + \cdots + c_n\frac{x^{n+1}}{n+1})] \tag{3.3.8.28}$$
$$= U[c_1 + c_2'x + \cdots + c_n'x^{n-1}] \tag{3.3.8.29}$$
$$= c_1x + c_2\frac{x^2}{2} + \cdots + c_n\frac{x^n}{n} \tag{3.3.8.30}$$

Hence, from (3.3.8.30),

$$UD \neq I \tag{3.3.8.31}$$

Hence, from (3.3.8.26) and (3.3.8.31), D is not invertible.

3.4 Representation of Transformations by Matrices

3.4.1. Let T be the linear operator on $\mathbb{C}^2$ defined by $T(x_1, x_2) = \left(x_1, 0\right)$. Let β be the standard ordered basis for $\mathbb{C}^2$ and $\beta' = \{\alpha_1, \alpha_2\}$ be the ordered basis defined by $\alpha_1 = \left(1, i\right), \alpha_2 = \left(-i, 2\right)$.

linear transformation	Let V and W be vector spaces over field F. A **linear transformation** V into W is a function T from V into W such that $T(c\alpha + \beta) = c(T\alpha) + T\beta$ for all α and β in V and all scalars in c in F.
isomorphism	If V and W are vector spaces over the field F, any $one - one$ linear transformation $T : V \rightarrow W$ is called **isormorphism of** V **onto** W
one-one	A linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be **one-one** if for every $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{AX} = \mathbf{b}$ has atmost one solution in $\mathbb{R}^n$. Equivalently, if $T(\mathbf{u}) = T(\mathbf{v})$, then $u = v$. By definition, all *invertible* transformations are **one-one**
invertible	A linear transformation $T : V \rightarrow W$ is **invertible** if there exists another linear transformation $U : W \rightarrow V$ such that UT is the *identity* transformation on V and TU is the identity transformation on W. T is **invertible** if and only if T is $one - one$ and $onto$

TABLE 3.3.7.2: Definitions

Proof	Conclusion
Let $\mathbf{V_1}, \mathbf{V_2} \in \mathbb{V}$ then, If $\mathbf{V_1} \neq \mathbf{V_2}$ then, $T[U(\mathbf{V_1})] \neq T[U(\mathbf{V_2})]$	T is one-one function
T is linear operator on finite dimensional vector space	T is onto function

TABLE 3.3.8.1: Proof of Invertibility of transformation

a) What is the matrix of **T** relative to the pair β,β'?
Solution:

$$\beta = \{\epsilon_1, \epsilon_2\} \implies \epsilon_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \epsilon_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.4.1.1}$$

Hence,β as matrix

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.4.1.2}$$

$$\beta' = \{\alpha_1, \alpha_2\} \implies \alpha_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}, \alpha_2 = \begin{pmatrix} -i \\ 2 \end{pmatrix} \tag{3.4.1.3}$$

Hence, β' as matrix

$$\beta' = \begin{pmatrix} 1 & -i \\ i & 2 \end{pmatrix} \tag{3.4.1.4}$$

Geometrically, **T** is a projection onto the x-axis.

For projection, let Consider Matrix A as

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{3.4.1.5}$$

The matrix **A** is representation of the linear transformation **T** that is projection on x-axis. After applying linear operator **T** on it,

$$\mathbf{T}(\beta) = \mathbf{A}\beta = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{3.4.1.6}$$

$$\begin{pmatrix}1 & -i & 1 & 0\\ i & 2 & 0 & 0\end{pmatrix} \xleftrightarrow{R_2=R_2-iR_1} \begin{pmatrix}1 & -i & 1 & 0\\ 0 & 1 & -i & 0\end{pmatrix} \tag{3.4.1.7}$$

$$\xleftrightarrow{R_1=R_1+iR_2} \begin{pmatrix}1 & 0 & 2 & 0\\ 0 & 1 & -i & 0\end{pmatrix} \tag{3.4.1.8}$$

Therefore matrix of relative to the pair β , $\beta^{'}$

$$\mathbf{T}(\beta) = \begin{pmatrix}2 & 0\\ -i & 0\end{pmatrix}\beta^{'} \tag{3.4.1.9}$$

b) What is the matrix of T relative to the pair β , $\beta^{'}$?
Solution: Transformation T from C^2 to C^2. Let

$$\beta = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \tag{3.4.1.10}$$

$$\beta^{'} = \begin{pmatrix}1 & -i\\ i & 2\end{pmatrix} \tag{3.4.1.11}$$

T is defined by

$$T(\mathbf{x}) = \mathbf{A}\mathbf{x} \tag{3.4.1.12}$$

$$T(\beta) = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\beta \tag{3.4.1.13}$$

$$T(\beta) = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \tag{3.4.1.14}$$

To find relative matrix we will use row reduce augmented matrix.

$$\begin{pmatrix}1 & -i & 1 & 0\\ i & 2 & 0 & 0\end{pmatrix} \tag{3.4.1.15}$$

$$\begin{pmatrix}1 & -i & 1 & 0\\ i & 2 & 0 & 0\end{pmatrix} \xleftrightarrow[R_1\to R_1+iR_2]{R_2\to R_2-iR_1} \begin{pmatrix}1 & 0 & 2 & 0\\ 0 & 1 & -i & 0\end{pmatrix} \tag{3.4.1.16}$$

Hence the matrix of T in the order basis of $\beta^{'}$

$$\mathbf{B} = \begin{pmatrix}2 & 0\\ -i & 0\end{pmatrix} \tag{3.4.1.17}$$

Therefore matrix of relative to the pair β , $\beta^{'}$

$$T(\beta) = \mathbf{A}\beta = \mathbf{B}\beta^{'} = \begin{pmatrix}2 & 0\\ -i & 0\end{pmatrix}\beta^{'} \tag{3.4.1.18}$$

c) What is the matrix of **T** in the ordered basis β'
Solution:

Let

$$\beta = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \tag{3.4.1.19}$$

We have

$$\alpha_{\mathbf{1}} = \begin{pmatrix}1\\ i\end{pmatrix}, \alpha_{\mathbf{2}} = \begin{pmatrix}-i\\ 2\end{pmatrix} \tag{3.4.1.20}$$

So,

$$\beta' = \begin{pmatrix}1 & -i\\ i & 2\end{pmatrix} \tag{3.4.1.21}$$

Geometrically, **T** is a projection onto the x-axis.

For projection, let Consider Matrix A as

$$\mathbf{A} = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \tag{3.4.1.22}$$

The matrix **A** is representation of the linear transformation **T** that is projection on x-axis. After applying linear operator **T** on it,

$$\mathbf{T}(\beta') = \mathbf{A}\beta' = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}1 & -i\\ i & 2\end{pmatrix} = \begin{pmatrix}1 & -i\\ 0 & 0\end{pmatrix} \tag{3.4.1.23}$$

Now, for finding the matrix of **T** in the ordered basis β',we combine the 3.4.1.20 and 3.4.1.23 and use concept of row-reduction of the augmented matrix:

$$\begin{pmatrix}1 & -i & 1 & -i\\ i & 2 & 0 & 0\end{pmatrix} \xleftrightarrow{R_2=R_2-iR_1} \begin{pmatrix}1 & -i & 1 & -i\\ 0 & 1 & -i & -1\end{pmatrix} \tag{3.4.1.24}$$

$$\xleftrightarrow{R_1=R_1+iR_2} \begin{pmatrix}1 & 0 & 2 & -2i\\ 0 & 1 & -i & -1\end{pmatrix} \tag{3.4.1.25}$$

Hence, the matrix of **T** in the ordered basis

β' is

$$\mathbf{B} = \begin{pmatrix} 2 & 2i \\ -i & -1 \end{pmatrix} \tag{3.4.1.26}$$

And this can also be represented using β and β', which shows the relation between $\mathbf{T}$, β and β' .

$$\mathbf{T}(\beta) = \mathbf{A}\beta = \mathbf{B}\beta' \tag{3.4.1.27}$$

d) What is the matrix of T in the ordered basis $\{\alpha_2, \alpha_1\}$?
Solution: Transformation T from $\mathbb{C}^2$ to $\mathbb{C}^2$. Let

$$\mathbf{e_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{e_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.4.1.28}$$

$$\beta = \begin{pmatrix} \mathbf{e_1} & \mathbf{e_2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.4.1.29}$$

$$\beta' = \begin{pmatrix} \alpha_2 & \alpha_1 \end{pmatrix} = \begin{pmatrix} -i & 1 \\ 2 & i \end{pmatrix} \tag{3.4.1.30}$$

T in the ordered basis β is:

$$[\mathbf{T}]_\beta = \begin{pmatrix} -i & 1 \\ 2 & i \end{pmatrix} \tag{3.4.1.31}$$

T is defined by

$$T(\mathbf{x}) = \mathbf{A}\mathbf{x} \tag{3.4.1.32}$$

$$T(\mathbf{x}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\mathbf{x} \tag{3.4.1.33}$$

$$T(\alpha_2) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -i \\ 2 \end{pmatrix} = \begin{pmatrix} -i \\ 0 \end{pmatrix} \tag{3.4.1.34}$$

$$T(\alpha_1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.4.1.35}$$

$$[\alpha]_\beta = \begin{pmatrix} -i & 1 \\ 0 & 0 \end{pmatrix} \tag{3.4.1.36}$$

The matrix of T in the ordered basis $\{\alpha_2, \alpha_1\}$ is given as:

$$[\mathbf{T}_\alpha]_\beta = [\mathbf{T}]_\beta[\alpha]_\beta = \begin{pmatrix} -i & 1 \\ 2 & i \end{pmatrix}\begin{pmatrix} -i & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & -i \\ -2i & 2 \end{pmatrix} \tag{3.4.1.37}$$

3.4.2. Let $\mathbf{T}$ be the linear transformation from $\mathbb{R}^3$ into $\mathbb{R}^2$ defined by

$$\mathbf{T}(x_1, x_2, x_3) = (x_1 + x_2, 2x_3 - x_1) \tag{3.4.2.1}$$

If $\beta = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta' = (\beta_1, \beta_2)$ where

$$\alpha_\mathbf{1} = (1, 0, -1), \alpha_\mathbf{2} = (1, 1, 1), \alpha_\mathbf{3} = (1, 0, 0)$$

$$\beta_\mathbf{1} = (0, 1), \beta_\mathbf{2} = (1, 0)$$

a) What is the matrix of $\mathbf{T}$ relative to the pair β, β'
Solution: Let

$$\beta = (\alpha_1, \alpha_2, \alpha_3) \tag{3.4.2.2}$$

$$\implies \beta = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix} \tag{3.4.2.3}$$

and

$$\beta' = \{\beta_1, \beta_2\} \tag{3.4.2.4}$$

$$\implies \beta' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.4.2.5}$$

T is defined by

$$T(\mathbf{x}) = \mathbf{A}\mathbf{x} \tag{3.4.2.6}$$

using equation (3.4.2.1)

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ 2x_3 - x_1 \end{pmatrix} \tag{3.4.2.7}$$

R.H.S of the equation can be written as a product of 2×3 and 3×1 matrices,

$$= \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{3.4.2.8}$$

$$\implies \mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \end{pmatrix} \tag{3.4.2.9}$$

Now,

$$T(\beta) = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \end{pmatrix}\beta \tag{3.4.2.10}$$

$$T(\beta) = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ -3 & 1 & -1 \end{pmatrix} \tag{3.4.2.11}$$

To find relative matrix we will use row

reduce augmented matrix.

$$\left(\begin{array}{ccc|cc} 1 & 2 & 1 & 0 & 1 \\ -3 & 1 & -1 & 1 & 0 \end{array}\right) \quad (3.4.2.12)$$

$$\left(\begin{array}{ccc|cc} 1 & 2 & 1 & 0 & 1 \\ -3 & 1 & -1 & 1 & 0 \end{array}\right) \xleftrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|cc} -3 & 1 & -1 & 1 & 0 \\ 1 & 2 & 1 & 0 & 1 \end{array}\right) \quad (3.4.2.13)$$

Hence the matrix of **T** in the order basis of β'

$$\mathbf{B} = \begin{pmatrix} -3 & 1 & -1 \\ 1 & 2 & 1 \end{pmatrix} \quad (3.4.2.14)$$

Therefore matrix of relative to the pair β , β'

$$T(\beta) = \mathbf{A}\beta = \mathbf{B}\beta' = \begin{pmatrix} -3 & 1 & -1 \\ 1 & 2 & 1 \end{pmatrix}\beta' \quad (3.4.2.15)$$

3.4.3. Let T be the linear transformation from $\mathbb{R}^3$ into $\mathbb{R}^2$ defined by,

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ 2x_3 - x_1 \end{pmatrix} \quad (3.4.3.1)$$

a) If β is the standard ordered basis for $\mathbb{R}^3$ and β' is the standard ordered basis for $\mathbb{R}^2$, what is the matrix of T relative to the pair β, β'

3.4.4. Let T be a linear operator on $\mathbf{F^n}$, let **A** be the matrix of T in the standard ordered basis for $\mathbf{F^n}$, and let W be the subspace of $\mathbf{F^n}$ spanned by the column vectors of **A**. What does W have to do with T ?
Solution: See Table 3.4.4.1.

3.4.5. Let V be a two-dimensional vector space over the field F and let B be an ordered basis for V. If T is a linear operator on V and

$$[T]_B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (3.4.5.1)$$

Prove that

$$T^2 - (a + d)T + (ad - bc)I = 0 \quad (3.4.5.2)$$

Solution: Here T is a linear operator on V and B is an ordered basis of V. Let us consider $[T]_B = A$, so $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Now, the characteristic equation of A is:

$$\left|A - \lambda I\right| = 0 \quad (3.4.5.3)$$

$$\implies \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = 0 \quad (3.4.5.4)$$

$$\implies \lambda^2 - (a + d)\lambda + (ad - bc) = 0 \quad (3.4.5.5)$$

According to the Cayley-Hamilton's Theorem, every square matrix satisfies its own characteristic equation. Here A is a 2x2 square matrix, so it should also satisfy its characteristic equation. Now,

$$\lambda^2 - (a + d)\lambda + (ad - bc) = 0 \quad (3.4.5.6)$$

$$\implies A^2 - (a + d)A + (ad - bc)I = 0 \quad (3.4.5.7)$$

We can also write the equation 3.4.5.7 as:

$$[T]_B^2 - (a + d)[T]_B + (ad - bc)I = 0 \quad (3.4.5.8)$$

$$\text{or, } T^2 - (a + d)T + (ad - bc)I = 0 \quad (3.4.5.9)$$

3.4.6. Let T be a linear operator on $\mathbb{R}^3$, the matrix of which in the standard ordered basis is,

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 3 & 4 \end{pmatrix} \quad (3.4.6.1)$$

Find a basis for the range of T and a basis for the null-space of T.
Solution: The basis of the range of linear transformation T is the basis of the column-space of **A** or basis of $C(\mathbf{A})$. Hence the basis of the range of the linear transformation T is derived by reducing **A** into Reduced-Row Echelon form as follows,

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 3 & 4 \end{pmatrix} \xleftrightarrow[R_1 = R_1 - 2R_2]{R_3 = R_3 + R_1} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 5 & 5 \end{pmatrix} \quad (3.4.6.2)$$

$$\xleftrightarrow{R_3 = R_3 - 5R_2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad (3.4.6.3)$$

From (3.4.6.3) the basis of the range of linear operator T are as follows,

$$\mathbf{a_1} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \quad (3.4.6.4)$$

$$\mathbf{a_2} = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \quad (3.4.6.5)$$

Again, the basis for null-space of linear oper-

Let $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ be an ordered basis of $\mathbf{F^n}$	
Given	**Explanation**
T is a linear operator $\mathbf{F^n}$	As T is linear, $$T(\mathbf{x}) = \mathbf{Ax} \quad (3.4.4.1)$$ $$\mathbf{A} = \begin{pmatrix} T\epsilon_1 & T\epsilon_2 & \cdots & T\epsilon_n \end{pmatrix} \quad (3.4.4.2)$$ From equation (3.4.4.2), columns of $\mathbf{A}$ are the images of the standard basis elements of $\mathbf{F^n}$.
Range of T	$$\text{range}(T) = \{T\epsilon_1, T\epsilon_2, \cdots, T\epsilon_n\} \quad (3.4.4.3)$$ From equation (3.4.4.2) and (3.4.4.3), columns of $\mathbf{A}$ generate the range of T.
W spanned by column vectors $\mathbf{A}$	Since any generating set contains a basis for the generated space, we can say that the columns of $\mathbf{A}$ contains a basis of the range of T. As W is spanned by column vectors of $\mathbf{A}$, we can say that W contains a basis for the range of T.

TABLE 3.4.4.1: Expanation

ator T or $N(\mathbf{A})$ is a solution of the equation $\mathbf{Ax} = 0$. From (3.4.6.3) we have,

$$\mathbf{Ax} = 0 \quad (3.4.6.6)$$

$$\implies \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \quad (3.4.6.7)$$

Setting the value of the free variable $x_3 = 1$ we get the solution,

$$\mathbf{x} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad (3.4.6.8)$$

Hence, the basis of the null-space of the linear operator T is given by,

$$\mathbf{b} = \begin{pmatrix} 1 & -1 & 1 \end{pmatrix} \quad (3.4.6.9)$$

3.4.7. Let T be the linear operator on $\mathbb{R}^2$ defined by

$$T(x_1, x_2) = (-x_2, x_1) \quad (3.4.7.1)$$

a) What is the matrix of $\mathbf{T}$ in the standard ordered basis of $\mathbb{R}^2$?
Solution:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad (3.4.7.2)$$

$$\mathbf{T(x)} = \mathbf{Tx} \quad (3.4.7.3)$$

$$\implies \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} \quad (3.4.7.4)$$

The matrix of $\mathbf{T}$ in the standard ordered basis from (3.4.7.1) is

$$\mathbf{T} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (3.4.7.5)$$

b) Let T be the linear operator on $\mathbf{R}^2$ defined by

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} \quad (3.4.7.6)$$

What is the matrix of T in the ordered basis $\mathbf{B} = \{\alpha_1, \alpha_2\}$, where $\alpha_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\alpha_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$?
Solution:
Let $\mathbf{B}'$ be the standard ordered basis for $\mathbf{R}^2$. Then,

$$T(\alpha_1') = T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0\alpha_1 + 1\alpha_2 \quad (3.4.7.7)$$

$$T(\alpha_2') = T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} = -1\alpha_1 + 0\alpha_2 \quad (3.4.7.8)$$

Hence, the matrix of T in the standard ordered basis $\mathbf{B}'$ is

$$T_{\mathbf{B}'} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (3.4.7.9)$$

Given,

$$\alpha_1 = \begin{pmatrix}1\\2\end{pmatrix} \tag{3.4.7.10}$$

$$\alpha_2 = \begin{pmatrix}1\\-1\end{pmatrix} \tag{3.4.7.11}$$

Then,

$$\alpha_1' = 1\alpha_1 + 2\alpha_2 \tag{3.4.7.12}$$

$$\alpha_2' = 1\alpha_1 - 1\alpha_2 \tag{3.4.7.13}$$

So the **P** matrix is

$$\mathbf{P} = \begin{pmatrix}1 & 1\\2 & -1\end{pmatrix} \tag{3.4.7.14}$$

And

$$\mathbf{P}^{-1} = \begin{pmatrix}\frac{1}{3} & \frac{1}{3}\\\frac{2}{3} & -\frac{1}{3}\end{pmatrix} \tag{3.4.7.15}$$

Hence

$$[T]_{\mathbf{B}} = \mathbf{P}^{-1}[T]_{\mathbf{B}'}\mathbf{P} \tag{3.4.7.16}$$

$$= \begin{pmatrix}\frac{1}{3} & \frac{1}{3}\\\frac{2}{3} & -\frac{1}{3}\end{pmatrix}\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}\begin{pmatrix}1 & 1\\2 & -1\end{pmatrix} \tag{3.4.7.17}$$

$$= \begin{pmatrix}-\frac{1}{3} & \frac{1}{3}\\\frac{1}{3} & \frac{2}{3}\end{pmatrix}\begin{pmatrix}1 & 1\\2 & -1\end{pmatrix} \tag{3.4.7.18}$$

$$[T]_{\mathbf{B}} = \begin{pmatrix}\frac{1}{3} & -\frac{2}{3}\\\frac{5}{3} & -\frac{1}{3}\end{pmatrix} \tag{3.4.7.19}$$

Hence, $[T]_{\mathbf{B}}$ is the required matrix for the given ordered basis **B**.

c) Prove that for every real number c, the operator $(T - cI)$ is invertible.
Solution: From the equation (3.4.7.1), the matrix of T in standard order basis is,

$$\mathbf{T} = \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix} \tag{3.4.7.20}$$

To find the invertibility of the operator $(\mathbf{T} - c\mathbf{I})$ for every real number c, let us start with

$$(\mathbf{T} - c\mathbf{I})(\mathbf{T} + c\mathbf{I}) \tag{3.4.7.21}$$

$$= \mathbf{T}^2 - c^2\mathbf{I} \tag{3.4.7.22}$$

Consider $\mathbf{T}^2$

$$\mathbf{T}^2 = \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix} = \begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix} \implies \mathbf{T}^2 = -\mathbf{I} \tag{3.4.7.23}$$

Substituting equation (3.4.7.23) in (3.4.7.22),

$$(\mathbf{T} - c\mathbf{I})(\mathbf{T} + c\mathbf{I}) = -\left(1 + c^2\right)\mathbf{I} \tag{3.4.7.24}$$

As c is a real number, $c^2 \geq 0$ and hence factor $-\left(1 + c^2\right)$ is always non-zero. Therefore, from the equation (3.4.7.24),

$$(\mathbf{T} - c\mathbf{I})^{-1} = \frac{-1}{1 + c^2}(\mathbf{T} + c\mathbf{I}) \tag{3.4.7.25}$$

Hence the operator $(T - cI)$ is invertible and its inverse is given by the equation (3.4.7.25)

3.4.8. Let $\mathbb{T}$ be the linear operator on $\mathbb{R}^3$ defined by

$$\mathbb{T}(x_1, x_2, x_3) = \tag{3.4.8.1}$$

$$(3x_1 + x_3, -2x_1 + x_2, -x_1 + 2x_2 + 4x_3) \tag{3.4.8.2}$$

What is the matrix of $\mathbb{T}$ in the standard ordered basis of $\mathbb{R}^3$?
Solution:

$$\mathbf{x} = \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} \tag{3.4.8.3}$$

$$\mathbb{T}(\mathbf{x}) = \mathbf{T}\mathbf{x} \tag{3.4.8.4}$$

The matrix of $\mathbb{T}$ in the standard ordered basis from (3.4.8.2) is

$$\mathbf{T} = \begin{pmatrix}3 & 0 & 1\\-2 & 1 & 0\\-1 & 2 & 4\end{pmatrix} \tag{3.4.8.5}$$

3.4.9. The linear operator **T** on $\mathbf{R}^2$ defined by

$$\mathbf{T}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}x_1\\0\end{pmatrix} \tag{3.4.9.1}$$

is represented by the matrix

$$\mathbf{A} = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} \tag{3.4.9.2}$$

3.4.10. Let θ be a real number. Prove that the following two matrices are similar over the field of complex numbers:

$$\begin{pmatrix}\cos\theta & -\sin\theta\\\sin\theta & \cos\theta\end{pmatrix}, \quad \begin{pmatrix}e^{i\theta} & 0\\0 & e^{-i\theta}\end{pmatrix}$$

Solution: Two matrices **A** and **B** are said to be similar iff there exists a invertible matrix $\mathbf{P}$ such that:

$$\mathbf{B} = \mathbf{P^{-1}AP} \tag{3.4.10.1}$$

Let,

$$\mathbf{A} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}; \quad \mathbf{B} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \tag{3.4.10.2}$$

Finding the characteristic polynomial of $\mathbf{A}$,

$$\begin{aligned} \left|A - \lambda I\right| &= \begin{vmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{vmatrix} \\ &= (\cos\theta - \lambda)^2 + \sin^2\theta \\ &= 1 + \lambda^2 + 2\lambda\cos\theta \end{aligned} \tag{3.4.10.3}$$

The eigenvalues can be calculated by equating the characteristic polynomial to zero. The eigenvalues are,

$$\lambda_1 = \cos\theta + i\sin\theta; \ \lambda_2 = \cos\theta - i\sin\theta \tag{3.4.10.4}$$

The eigenvectors corresponding to (3.4.10.4) are,

$$\alpha_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}; \ \alpha_2 = \begin{pmatrix} -i \\ 1 \end{pmatrix} \tag{3.4.10.5}$$

$$\mathbf{P} = \begin{pmatrix} \alpha_1 & \alpha_2 \end{pmatrix} = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \tag{3.4.10.6}$$

Now,

$$\begin{aligned} \mathbf{P^{-1}AP} &= \frac{1}{2i}\begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta + i\sin\theta & 0 \\ 0 & \cos\theta - i\sin\theta \end{pmatrix} \\ &= \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} = \mathbf{B} \end{aligned} \tag{3.4.10.7}$$

Hence, from (3.4.10.7), $\mathbf{A}$ and $\mathbf{B}$ are similar matrices.

3.4.11. Let $\mathbb{V}$ be finite dimensional vector space over the field $\mathbb{F}$, and let $\mathbf{S}$ and $\mathbf{T}$ be linear operators on $\mathbb{V}$. When do there exist ordered bases $\mathcal{B}$ and $\mathcal{B}'$ for $\mathbb{V}$ such that $[S]_{\mathcal{B}} = [T]'_{\mathcal{B}}$? Prove that such bases exist if and only if there is an invertible linear operator $\mathbf{U}$ on $\mathbb{V}$ such that $\mathbf{T} = \mathbf{USU}^{-1}$

Solution: See Table 3.4.11.1.

3.4.12. Prove that if $\mathbf{S}$ is a linear operator on $\mathbf{R}^2$ such that $\mathbf{S}^2 = \mathbf{S}$, then $\mathbf{S} = \mathbf{0}$, or $\mathbf{S} = \mathbf{I}$, or there is an ordered basis $\mathbf{B}$ for $\mathbf{R}^2$ such that $[\mathbf{S}]_B = \mathbf{A}$.

Solution: If a linear operator $\mathbf{S}$ is defined on $\mathbf{R}^2$ such that $\mathbf{S}^2 = \mathbf{S}$, then

$$\mathbf{S}^2 - \mathbf{S} = \mathbf{0} \tag{3.4.12.1}$$

$$\mathbf{S}(\mathbf{S} - \mathbf{I}) = \mathbf{0} \tag{3.4.12.2}$$

$$\implies \mathbf{S} = \mathbf{0}, \mathbf{S} = \mathbf{I} \tag{3.4.12.3}$$

The transformation of a vector $\mathbf{x} \in \mathbf{R}^2$ can be represented as

$$\mathbf{Sx} = \mathbf{y} \tag{3.4.12.4}$$

$$\implies \mathbf{S}(\mathbf{Sx}) = \mathbf{Sy} \tag{3.4.12.5}$$

$$\implies \mathbf{S}^2\mathbf{x} = \mathbf{Sy} \tag{3.4.12.6}$$

$$\implies \mathbf{Sx} = \mathbf{Sy} \tag{3.4.12.7}$$

$$\implies \mathbf{x} = \mathbf{y} \tag{3.4.12.8}$$

Therefore the transformation of a vector $\mathbf{x} \in \mathbf{R}^2$ can be given as

$$\mathbf{Sx} = \mathbf{x} \ \forall \ \mathbf{x} \in \mathbf{R}^2 \tag{3.4.12.9}$$

Consider the ordered basis set

$$B = \{\epsilon_1, \epsilon_2\} \in \mathbf{R}^2 \tag{3.4.12.10}$$

and if

$$[\mathbf{S}]_{\mathbf{B}} = \mathbf{A} \tag{3.4.12.11}$$

$$\implies [\mathbf{S}]_{\mathbf{B}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{3.4.12.12}$$

Thus we can re-write the column vectors of $[\mathbf{S}]_{\mathbf{B}}$ using (3.4.12.9) as

$$\mathbf{S}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.4.12.13}$$

$$\mathbf{S}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.4.12.14}$$

Therefore, any vector $\mathbf{x}$ in column space of $[\mathbf{S}]_{\mathbf{B}}$ can be uniquely expressed by $\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$, hence it forms the basis for column space of $[\mathbf{S}]_{\mathbf{B}}$. Therefore one of the basis vector of B is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The other basis vector can be any vector which is linearly independent to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. One of the ordered basis set can be

$$B = \left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\} \tag{3.4.12.15}$$

Assume $[S]_{\mathcal{B}} = [T]_{\mathcal{B}'}$	Assume $\mathbf{T} = \mathbf{USU}^{-1}$
Given $\mathbb{V}$ is a finite dimensional vector space over field $\mathbb{F}$ $\mathbf{S}$ and $\mathbf{T}$ are linear operators on $\mathbb{V}$ $\mathcal{B}$ and $\mathcal{B}'$ are ordered bases for $\mathbb{V}$ $[S]_{\mathcal{B}} = [T]'_{\mathcal{B}}$	**Given** $\mathbb{V}$ is a finite dimensional vector space over field $\mathbb{F}$ $\mathbf{S}$ and $\mathbf{T}$ are linear operators on $\mathbb{V}$ $\mathcal{B}$ and $\mathcal{B}'$ are ordered bases for $\mathbb{V}$ There is an invertible linear operator $\mathbf{U}$ on $\mathbb{V}$ such that $\mathbf{T} = \mathbf{USU}^{-1}$
To prove There is an invertible linear operator $\mathbf{U}$ on $\mathbb{V}$ such that $\mathbf{T} = \mathbf{USU}^{-1}$	**To prove** $[S]_{\mathcal{B}} = [T]'_{\mathcal{B}}$
Assumptions Let $\mathbf{U}$ be the operator which carries $\mathcal{B}$ to $\mathcal{B}'$ $\mathcal{B} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ $\mathcal{B}' = \{\mathbf{x}'_1, \mathbf{x}'_2, \ldots, \mathbf{x}'_n\}$	**Assumptions** Let $\mathbf{U}$ be the operator which carries $\mathcal{B}$ to $\mathcal{B}'$ $\mathcal{B} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ $\mathcal{B}' = \{\mathbf{x}'_1, \mathbf{x}'_2, \ldots, \mathbf{x}'_n\}$
Proof For $\mathbf{v} \in \mathbb{V}$, expressed as a linear combination of the vectors of $\mathcal{B}$ $\mathbf{v} = a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \ldots + a_n\mathbf{x}_n$ $\mathbf{w} = \mathbf{U}(\mathbf{v}) = a_1\mathbf{x}'_1 + a_2\mathbf{x}'_2 + \ldots + a_n\mathbf{x}'_n$ $\because [S]_{\mathcal{B}} = [T]'_{\mathcal{B}}$ $\mathbf{S}(\mathbf{v}) = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \ldots + c_n\mathbf{x}_n$ and $\mathbf{T}(\mathbf{w}) = c_1\mathbf{x}'_1 + c_2\mathbf{x}'_2 + \ldots + c_n\mathbf{x}'_n$ $\mathbf{U}^{-1}\mathbf{TU}(\mathbf{v})$ $= \mathbf{U}^{-1}\mathbf{T}(\mathbf{w})$ $= \mathbf{U}^{-1}(c_1\mathbf{x}'_1 + c_2\mathbf{x}'_2 + \ldots + c_n\mathbf{x}'_n)$ $= c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \ldots + c_n\mathbf{x}_n$ $= \mathbf{S}(\mathbf{v})$ $\implies \mathbf{U}^{-1}\mathbf{TU} = \mathbf{S}$ $\implies \mathbf{T} = \mathbf{USU}^{-1}$	**Proof** For $\mathbf{v} \in \mathbb{V}$, expressed as a linear combination of the vectors of $\mathcal{B}'$ $\mathbf{v} = a_1\mathbf{x}'_1 + a_2\mathbf{x}'_2 + \ldots + a_n\mathbf{x}'_n$ $\mathbf{USU}^{-1}(\mathbf{v}) = \mathbf{T}(\mathbf{v})$ $= \mathbf{T}(a_1\mathbf{x}'_1 + a_2\mathbf{x}'_2 + \ldots + a_n\mathbf{x}'_n)$ $= c_1\mathbf{x}'_1 + c_2\mathbf{x}'_2 + \ldots + c_n\mathbf{x}'_n$ But we know that $\mathbf{USU}^{-1}(\mathbf{v}) = \mathbf{USU}^{-1}(a_1\mathbf{x}'_1 + a_2\mathbf{x}'_2 + \ldots + a_n\mathbf{x}'_n)$ $= \mathbf{US}(a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \ldots + a_n\mathbf{x}_n)$ So, $\mathbf{S}$ in basis $\mathcal{B}$ has the same entries as $\mathbf{T}$ in basis $\mathcal{B}'$ $\therefore [S]_{\mathcal{B}} = [T]_{\mathcal{B}'}$

TABLE 3.4.11.1

3.4.13. Let $\mathbf{W}$ be the space of all $n \times 1$ column matrices over a field $\mathbf{F}$. If $\mathbf{A}$ is an $n \times n$ matrix over $\mathbf{F}$, then $\mathbf{A}$ defines a linear operator $\mathbf{L_A}$ on $\mathbf{W}$ through left multiplication: $\mathbf{L_A}(\mathrm{X}) = \mathbf{A}\mathrm{X}$. Prove that every linear operator on $\mathbf{W}$ is left multiplication by some $n \times n$ matrix, i.e., is $\mathbf{L_A}$ for some $\mathbf{A}$.
Now suppose $\mathbf{V}$ is an n-dimensional vector

Defining Linear Map T	Let $\mathbf{T} : \mathbf{W} \to \mathbf{W}$ be a linear operator and $\left(e_1, e_2, ..., e_n\right)$ be a basis for $\mathbf{W}$. Now, $\mathbf{Te_1} = \alpha_{11}\mathbf{e_1} + \alpha_{12}\mathbf{e_2} + ... + \alpha_{1n}\mathbf{e_n}$ $\mathbf{Te_2} = \alpha_{21}\mathbf{e_1} + \alpha_{22}\mathbf{e_2} + ... + \alpha_{2n}\mathbf{e_n}$ $\vdots$ $\mathbf{Te_n} = \alpha_{n1}\mathbf{e_1} + \alpha_{n2}\mathbf{e_2} + ... + \alpha_{nn}\mathbf{e_n}$
Matrix of Linear Map T	Let $\mathbf{A}$ be matrix of linear transformation $\mathbf{T}$. Then $\mathbf{A}$ is $$\mathbf{A} = \begin{pmatrix} \alpha_{11} & \alpha_{21} & ... & \alpha_{n1} \\ \alpha_{12} & \alpha_{22} & ... & \alpha_{n2} \\ \vdots & \vdots & ... & \vdots \\ \alpha_{1n} & \alpha_{2n} & ... & \alpha_{nn} \end{pmatrix}$$
Proving every linear operator on $\mathbf{W}$ is left multiplication by some $n \times n$ matrix, i.e, is $\mathbf{L_A}$ for some $\mathbf{A}$	$\mathbf{Ae_1} = (\alpha_{11}, \alpha_{12}, ..., \alpha_{1n})$ $= \alpha_{11}\mathbf{e_1} + \alpha_{12}\mathbf{e_2} + ... + \alpha_{1n}\mathbf{e_n}$ $= \mathbf{Te_i}$ Hence, $\mathbf{Te_i} = \mathbf{Ae_i}$ Since T and A are linear. Then $\mathbf{Tx} = \mathbf{Ax}$ $\mathbf{Tx} = \mathbf{L_A x}$

TABLE 3.4.13.1

space over the field $\mathbf{F}$, and let β be an ordered basis for $\mathbf{V}$. For each α in $\mathbf{V}$, define $\mathbf{U}_\alpha = [\alpha]_\beta$. Prove that $\mathbf{U}$ is an isomorphism of $\mathbf{V}$ onto $\mathbf{W}$. If $\mathbf{T}$ is a linear operator on $\mathbf{V}$, then $\mathbf{UTU^{-1}}$ is a linear operator on $\mathbf{W}$. Accordingly, $\mathbf{UTU^{-1}}$ is left multiplication by some $n \times n$ matrix $\mathbf{A}$. What is A?

Solution: See Tables 3.4.13.1 and 3.4.13.2.

3.4.14. Let V be an n-dimensional vector space over the field F, and let $\mathscr{B} = \{\alpha_1, \ldots, \alpha_n\}$ be an ordered basis for V then there is a unique linear operator T on V such that

$$T\alpha_j = \alpha_{j+1}, j = 1, \ldots, n-1 \tag{3.4.14.1}$$

$$T\alpha_n = 0. \tag{3.4.14.2}$$

a) What is the matrix A of T in the ordered basis $\mathscr{B}$?

Solution: Given that,

$$T : V \to V \tag{3.4.14.3}$$

$$[T(\alpha)]_{\mathscr{B}} = A[\alpha]_{\mathscr{B}} \tag{3.4.14.4}$$

$$T\alpha_j = \alpha_{j+1} \tag{3.4.14.5}$$

$$T\alpha_n = 0 \tag{3.4.14.6}$$

where $j = 1, \ldots, n-1$. The matrix A of T in the ordered basis $\mathscr{B}$ is given by,

$$\implies A = \left([T\alpha_1]_{\mathscr{B}} \quad \cdots \quad [T\alpha_n]_{\mathscr{B}}\right) \tag{3.4.14.7}$$

For $j = 1, \ldots, n-1$ we have,

$$T\alpha_j = \alpha_{j+1} \tag{3.4.14.8}$$

we can write,

$$T\alpha_j = 0\alpha_1 + \ldots + 0\alpha_j + 1\alpha_{j+1} + \ldots + 0\alpha_n \tag{3.4.14.9}$$

$$\implies [T\alpha_j]_{\mathscr{B}} = (0, \ldots, 0, 1, 0, \ldots, 0)^T \tag{3.4.14.10}$$

Proving **U** as Linear and defining a linear map **T**	$\mathbf{U}(c\alpha_1 + \alpha_2) = [c\alpha_1 + \alpha_2]_\beta$ $= c[\alpha_1]_\beta + [\alpha_2]_\beta$ $= c\mathbf{U}(\alpha_1) + \mathbf{U}(\alpha_2)$ Suppose $\beta = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ be the ordered basis for **V**. Let **T** be the function from **W** to **V** as follows: $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \to a_1\alpha_1 + a_2\alpha_2 + ... + a_n\alpha_n$
Proving **U** to be an isomorphism	For isomorphism, we must show that **TU** is identity map on **V** and **UT** is an identity map on **W**.
$\mathbf{TU} = \mathbf{I_V}$	$\mathbf{TU}(\mathbf{x}) = \mathbf{TU}(a_1\alpha_1 + a_2\alpha_2 + ... + a_n\alpha_n)$ $= \mathbf{TU}(a_1\alpha_1 + a_2\alpha_2 + ... + a_n\alpha_n)$ $= a_1\mathbf{T}\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + a_2\mathbf{T}\begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + ... + a_n\mathbf{T}\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$ $= a_1\alpha_1 + a_2\alpha_2 + ... + a_n\alpha_n$ Hence, **TU** is identity map on **V**.
$\mathbf{UT} = \mathbf{I_W}$	$\mathbf{UT}(\mathbf{x}) = \mathbf{UT}(a_1e_1 + a_2e_2 + ... + a_ne_n)$ $= \mathbf{UT}((a_1e_1 + a_2e_2 + ... + a_ne_n)$ $= \mathbf{U}(a_1\mathbf{T}\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + a_2\mathbf{T}\begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + ... + a_n\mathbf{T}\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix})$ $= a_1\mathbf{U}(\alpha_1) + a_2\mathbf{U}(\alpha_2) + ... + a_n\mathbf{U}(\alpha_n)$ $= a_1e_1 + a_2e_2 + ... + a_ne_n$ Hence, **UT** is identity map on **W**.
Matrix of $\mathbf{UTU^{-1}}$	Now, we define the matrix of $\mathbf{UTU^{-1}}$. Since $\mathbf{U}\alpha_\mathbf{i}$ is the standard $n \times 1$ matrix with all zeros except in the ith place which equals one. Let β' be the standard basis for **W**. Then the matrix of **U** with respect to β and β' is the identity matrix. Likewise the matrix of $\mathbf{U}^{-1}$ with respect to β' and β is the identity matrix. Thus, $[\mathbf{UTU^{-1}}]_\beta = \mathbf{I}[\mathbf{T}]_\beta\mathbf{I^{-1}} = [T]_\beta$ Thus, the matrix **A** is simply $[\mathbf{T}]_\beta$, the matrix of **T** with respect to β.

TABLE 3.4.13.2

where 1 is in $(j+1)$th position. Now,

$$T\alpha_n = 0 \tag{3.4.14.11}$$
$$\implies [T\alpha_n]_{\mathcal{B}} = 0 \tag{3.4.14.12}$$

Thus from (3.4.14.7), (3.4.14.10) and (3.4.14.12) we get matrix A of T in the ordered basis $\mathcal{B}$ as,

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \tag{3.4.14.13}$$

3.4.15. Let V and W be finite-dimensional vector spaces over the field F and let T be a linear transformation from V into W. If

$$\mathcal{B} = \{\alpha_1, \dots, \alpha_n\} \text{ and } \mathcal{B}' = \{\beta_1, \dots, \beta_m\} \tag{3.4.15.1}$$

are ordered bases for V and W, respectively, define the linear transformation $E^{p,q}$ as in the proof of Theorem 5: $E^{p,q}(\alpha_i) = \delta_{iq}\beta_p$. Then the $E^{p,q}$, $1 \le p \le m$, $1 \le q \le n$, form a basis for $L(V, W)$ and so

$$T = \sum_{p=1}^{m} \sum_{q=1}^{n} A_{pq} E^{p,q} \tag{3.4.15.2}$$

for certain scalars A_{pq} (the coordinates of T in this basis for $L(V, W)$). Show that the matrix A with entries $A(p, q) = A_{pq}$ is precisely the matrix of T relative to the pair $\mathcal{B}, \mathcal{B}'$.

Solution: Given,

$$T = \sum_{p=1}^{m} \sum_{q=1}^{n} A_{pq} E^{p,q} \tag{3.4.15.3}$$

where

$$E^{p,q}(\alpha_i) = \begin{cases} \beta_p & p = i \\ 0 & \text{otherwise} \end{cases} \tag{3.4.15.4}$$
$$= \delta_{iq}\beta_p \tag{3.4.15.5}$$

where $\mathcal{B} = \{\alpha_1, \dots, \alpha_n\}$ is basis of V and $\mathcal{B}' = \{\beta_1, \dots, \beta_n\}$ is basis of W.

Consider a vector $\mathbf{x} = \{x_1, x_2, \dots, x_n\} \in V$,

$$\mathbf{x} = \sum_{q=1}^{n} x_q \alpha_q \tag{3.4.15.6}$$
$$\therefore E^{p,q}(\mathbf{x}) = \sum_{q=1}^{n} x_q E^{p,q}(\alpha_q) \tag{3.4.15.7}$$
$$= x_q \delta_{iq} \beta_p \tag{3.4.15.8}$$

Consider $T(\mathbf{x})$, from (3.4.15.3)

$$T(\mathbf{x}) = \sum_{p=1}^{m} \sum_{q=1}^{n} A_{pq} E^{p,q}(\mathbf{x}) \tag{3.4.15.9}$$

Substitute (3.4.15.8) in (3.4.15.9)

$$T(\mathbf{x}) = \sum_{p=1}^{m} \sum_{q=1}^{n} A_{pq} x_p \delta_{iq} \beta_q \tag{3.4.15.10}$$

From (3.4.15.5), $\delta_{iq}\beta_q$ is the transformation of basis of V to W. Hence $T : V \to W$ is

$$T = \begin{pmatrix} \sum_{p=1}^{n} A_{p1} x_p \\ \sum_{p=1}^{n} A_{p2} x_p \\ \vdots \\ \sum_{p=1}^{n} A_{pm} x_p \end{pmatrix} \tag{3.4.15.11}$$

$$T = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & \dots & \vdots \\ A_{m1} & A_{m2} & \dots & A_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \tag{3.4.15.12}$$

$\therefore$ We showed that the matrix A with entries $A(p, q) = A_{pq}$ is precisely the matrix of T relative to the pair $\mathcal{B}, \mathcal{B}'$.

3.5 Linear Functionals

3.5.1. In R^3, let $\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$ and $\alpha_3 = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$.

a) if f is a linear functional on R^3 such that $f(\alpha_1) = 1,\ f(\alpha_2) = -1,\ f(\alpha_3) = 3$,

And if $\alpha = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, find $f(\alpha)$.

Solution: Given, $\alpha = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$. Let,

$$\mathbf{A} = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \end{pmatrix} \tag{3.5.1.1}$$

$$\mathbf{AX} = \alpha \tag{3.5.1.2}$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & -2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{3.5.1.3}$$

$\mathbf{X} = A^{-1}\alpha$ will give solution of the equation.

$$\begin{pmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 1 & -2 & 0 & 0 & 0 & 1 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 - R_1} \tag{3.5.1.4}$$

$$\begin{pmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & -2 & 1 & -1 & 0 & 1 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 + 2R_2} \tag{3.5.1.5}$$

$$\begin{pmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 2 & 1 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3/(-1)} \tag{3.5.1.6}$$

$$\begin{pmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & -1 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 + R_3} \tag{3.5.1.7}$$

$$\begin{pmatrix} 1 & 0 & 0 & 2 & -2 & -1 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & -1 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 + R_3} \tag{3.5.1.8}$$

$$\begin{pmatrix} 1 & 0 & 0 & 2 & -2 & -1 \\ 0 & 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 & -2 & -1 \end{pmatrix} \tag{3.5.1.9}$$

Thus,

$$A^{-1} = \begin{pmatrix} 2 & -2 & -1 \\ 1 & -1 & -1 \\ 1 & -2 & -1 \end{pmatrix} \tag{3.5.1.10}$$

$$\mathbf{X} = A^{-1}\alpha = \begin{pmatrix} 2 & -2 & -1 \\ 1 & -1 & -1 \\ 1 & -2 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{3.5.1.11}$$

Given, f is a linear functional on R^3,

$$\alpha = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 \tag{3.5.1.12}$$

$$\implies f(\alpha) = \mathbf{X}^T \begin{pmatrix} f(\alpha_1) \\ f(\alpha_2) \\ f(\alpha_3) \end{pmatrix} \tag{3.5.1.13}$$

Given, $f(\alpha_1) = 1$, $f(\alpha_2) = -1$ and $f(\alpha_3) = 3$.

$$f(\alpha) = \mathbf{X}^T \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \tag{3.5.1.14}$$

$$\implies f(\alpha) = \begin{pmatrix} a \\ b \\ c \end{pmatrix}^T \begin{pmatrix} 2 & 1 & 1 \\ -2 & -1 & -2 \\ -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \tag{3.5.1.15}$$

$$f(\alpha) = \begin{pmatrix} a \\ b \\ c \end{pmatrix}^T \begin{pmatrix} 4 \\ -7 \\ -3 \end{pmatrix} \tag{3.5.1.16}$$

Hence,

$$f(\alpha) = 4a - 7b - 3c \tag{3.5.1.17}$$

b) Describe explicitly a linear functional f on R^3 such that $f(\alpha_1) = f(\alpha_2) = 0$ but $f(\alpha_3) \neq 0$.

Solution: Let us consider $\alpha = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ such that

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_2 x_2 = \alpha \tag{3.5.1.18}$$

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_2 x_2 = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{3.5.1.19}$$

The coefficient matrix is :

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & -2 & 0 \end{pmatrix} \tag{3.5.1.20}$$

So,

$$A\mathbf{x} = \alpha \tag{3.5.1.21}$$

$$\implies x = A^{-1}\alpha \tag{3.5.1.22}$$

$$\tag{3.5.1.23}$$

Now to get A^{-1}, we will consider Gauss-Jordon theorem. So, we will take $(A|I)$, where I is a 3×3 identity matrix.

$$\begin{pmatrix}1 & 0 & -1 & 1 & 0 & 0\\0 & 1 & -1 & 0 & 1 & 0\\1 & -2 & 0 & 0 & 0 & 1\end{pmatrix}\xleftrightarrow{R_3\leftarrow R_3-R_1}$$
$$\begin{pmatrix}1 & 0 & -1 & 1 & 0 & 0\\0 & 1 & -1 & 0 & 1 & 0\\0 & -2 & 1 & -1 & 0 & 1\end{pmatrix}\xleftrightarrow{R_3\leftarrow R_3+2R_2}$$
$$\begin{pmatrix}1 & 0 & -1 & 1 & 0 & 0\\0 & 1 & -1 & 0 & 1 & 0\\0 & 0 & -1 & -1 & 2 & 1\end{pmatrix}\xleftrightarrow{R_3\leftarrow R_3/(-1)}$$
$$\begin{pmatrix}1 & 0 & -1 & 1 & 0 & 0\\0 & 1 & -1 & 0 & 1 & 0\\0 & 0 & 1 & 1 & -2 & -1\end{pmatrix}\xleftrightarrow[R_1\leftarrow R_1+R_3]{R_2\leftarrow R_2+R_3}$$
$$\begin{pmatrix}1 & 0 & 0 & 2 & -2 & -1\\0 & 1 & 0 & 1 & -1 & -1\\0 & 0 & 1 & 1 & -2 & -1\end{pmatrix} \quad (3.5.1.24)$$

Now, we can say that

$$A^{-1}=\begin{pmatrix}2 & -2 & -1\\1 & -1 & -1\\1 & -2 & -1\end{pmatrix} \quad (3.5.1.25)$$

As

$$\mathbf{x}=A^{-1}\alpha \quad (3.5.1.26)$$
$$\implies \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=A^{-1}\begin{pmatrix}a\\b\\c\end{pmatrix} \quad (3.5.1.27)$$
$$\implies \mathbf{x}=\begin{pmatrix}2 & -2 & -1\\1 & -1 & -1\\1 & -2 & -1\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix} \quad (3.5.1.28)$$

Now, as f is a linear functional on R^3,

$$\alpha=\alpha_1x_1+\alpha_2x_2+\alpha_2x_2 \quad (3.5.1.29)$$
$$\implies f(\alpha)=f(\alpha_1x_1+\alpha_2x_2+\alpha_2x_2) \quad (3.5.1.30)$$
$$\implies f(\alpha)=x_1f(\alpha_1)+x_2f(\alpha_2)+x_3f(\alpha_3) \quad (3.5.1.31)$$
$$\implies f(\alpha)=\begin{pmatrix}x_1 & x_2 & x_3\end{pmatrix}\begin{pmatrix}f(\alpha_1)\\f(\alpha_2)\\f(\alpha_3)\end{pmatrix} \quad (3.5.1.32)$$
$$\implies f(\alpha)=\mathbf{x}^T\begin{pmatrix}f(\alpha_1)\\f(\alpha_2)\\f(\alpha_3)\end{pmatrix} \quad (3.5.1.33)$$

As mentioned in the problem statement, $f(\alpha_1)=f(\alpha_2)=0$ and $f(\alpha_3)\neq 0$.
Now,

$$f(\alpha)=\mathbf{x}^T\begin{pmatrix}f(\alpha_1)\\f(\alpha_2)\\f(\alpha_3)\end{pmatrix} \quad (3.5.1.34)$$
$$\implies f(\alpha)=\mathbf{x}^T\begin{pmatrix}0\\0\\f(\alpha_3)\end{pmatrix} \quad (3.5.1.35)$$
$$\implies f(\alpha)=\begin{pmatrix}a\\b\\c\end{pmatrix}^T\begin{pmatrix}2 & 1 & 1\\-2 & -1 & -2\\-1 & -1 & -1\end{pmatrix}\begin{pmatrix}0\\0\\f(\alpha_3)\end{pmatrix} \quad (3.5.1.36)$$
$$\implies f(\alpha)=\begin{pmatrix}a\\b\\c\end{pmatrix}^T\begin{pmatrix}f(\alpha_3)\\-2f(\alpha_3)\\-f(\alpha_3)\end{pmatrix} \quad (3.5.1.37)$$
$$\implies f(a,b,c)=f(\alpha_3)\begin{pmatrix}a\\b\\c\end{pmatrix}^T\begin{pmatrix}1\\-2\\-1\end{pmatrix} \quad (3.5.1.38)$$

So, the function can be defined as:

$$f(a,b,c)=f(\alpha_3)\begin{pmatrix}a\\b\\c\end{pmatrix}^T\begin{pmatrix}1\\-2\\-1\end{pmatrix} \quad (3.5.1.39)$$
$$\textbf{or,} f(\alpha)=f(\alpha_3)(a-2b-c) \quad (3.5.1.40)$$

3.5.2. Let $\mathbf{B} = \{\alpha_1, \alpha_2, \alpha_3\}$ be the basis for $\mathbf{C}^3$ defined by

$$\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} \tag{3.5.2.1}$$

Find the dual basis of $\mathbf{B}$.
Solution: Let $\{f_1, f_2, f_3\}$ be the dual basis of $\mathbf{B}$ such that,

$$f_i(\alpha_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{3.5.2.2}$$

and

$$f_i(\alpha_1, \alpha_2, \alpha_3) = \sum_{j=1}^{3} \delta_{ij}\alpha_j \tag{3.5.2.3}$$

Given,

$$\mathbf{B} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ -1 & 1 & 0 \end{pmatrix} \tag{3.5.2.4}$$

Then,

$$\mathbf{B}^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 1 \\ -\frac{1}{2} & 1 & -\frac{1}{2} \end{pmatrix} \tag{3.5.2.5}$$

So the elements of dual basis are rows of matrix $\mathbf{B}^{-1}$. Therefore we get,

$$(\delta_{11}, \delta_{12}, \delta_{13}) = (1, -1, 0) \tag{3.5.2.6}$$
$$(\delta_{21}, \delta_{22}, \delta_{23}) = (1, -1, 1) \tag{3.5.2.7}$$
$$(\delta_{31}, \delta_{32}, \delta_{33}) = (-\frac{1}{2}, 1, -\frac{1}{2}) \tag{3.5.2.8}$$

Using (3.5.2.3), we get

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 1 \\ -\frac{1}{2} & 1 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \tag{3.5.2.9}$$
$$= \begin{pmatrix} \alpha_1 - \alpha_2 \\ \alpha_1 - \alpha_2 + \alpha_3 \\ -\frac{1}{2}\alpha_1 + \alpha_2 - \frac{1}{2}\alpha_3 \end{pmatrix} \tag{3.5.2.10}$$

Hence, $\{f_1, f_2, f_3\}$ is the required dual basis for $\mathbf{B}$.

3.5.3. If $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ matrices over the field $\mathbf{F}$, show that trace($\mathbf{AB}$) = trace($\mathbf{BA}$). Now show that similar matrices have the same trace.
Solution:

$$trace(\mathbf{AB}) = \sum_{i=1}^{n} (\mathbf{AB})_{ii} \tag{3.5.3.1}$$
$$= \sum_{i=1}^{n} \sum_{k=1}^{n} \mathbf{A}_{ik}\mathbf{B}_{ki} \tag{3.5.3.2}$$
$$= \sum_{i=1}^{n} \sum_{k=1}^{n} \mathbf{B}_{ki}\mathbf{A}_{ik} \tag{3.5.3.3}$$
$$= \sum_{i=1}^{n} (\mathbf{BA})_{kk} \tag{3.5.3.4}$$
$$= trace(\mathbf{BA}) \tag{3.5.3.5}$$

Hence proved trace($\mathbf{AB}$) = trace($\mathbf{BA}$). Let $\mathbf{A}$ and $\mathbf{B}$ be similar matrices then $\exists$ $\mathbf{S}$ such that,

$$\mathbf{A} = \mathbf{S}^{-1}\mathbf{BS} \tag{3.5.3.6}$$

Taking trace of $\mathbf{A}$ we get,

$$trace(\mathbf{A}) = trace(\mathbf{S}^{-1}\mathbf{BS}) = trace(\mathbf{S}^{-1}(\mathbf{BS})) \tag{3.5.3.7}$$

Using (3.5.3.5) in (3.5.3.7)

$$trace(\mathbf{A}) = trace(\mathbf{S}^{-1}\mathbf{SB}) \tag{3.5.3.8}$$
$$= trace((\mathbf{S}^{-1}\mathbf{S})\mathbf{B}) \tag{3.5.3.9}$$
$$= trace(\mathbf{IB}) \tag{3.5.3.10}$$
$$= trace(\mathbf{B}) \tag{3.5.3.11}$$

Hence Proved

3.5.4. Let $\mathbf{V}$ be the vector space of all polynomial functions p from $\mathbf{R}$ into $\mathbf{R}$ which have degree 2 or less:

$$p(x) = c_0 + c_1x + c_2x^2$$

Define three linear functionals on $\mathbf{V}$ by

$$f_1(p) = \int_0^1 p(x)\,dx; \; f_2(p) = \int_0^2 p(x)\,dx;$$
$$f_3(p) = \int_0^{-1} p(x)\,dx$$

Show that $\{f_1, f_2, f_3\}$ is a basis for $\mathbf{V}^*$ by exhibiting the basis for $\mathbf{V}$ of which it is the dual.
Solution: Given the basis $\mathbf{F}$ and corresponding dual basis $\mathbf{G}$, the defining property of the dual

basis states that:

$$\mathbf{G}^T\mathbf{F} = \mathbf{I}$$
$$\implies \mathbf{G} = (\mathbf{F}^{-1})^T \tag{3.5.4.1}$$

Let the indexed vector sets,

$$\mathbf{V} = \{f_1, f_2, f_3\};\ \mathbf{V}^* = \{\alpha_1, \alpha_2, \alpha_3\}$$

1. Let,

$$\mathbf{p}' = \mathbf{c}^T\mathbf{x} \tag{3.5.4.2}$$

where,

$$\mathbf{c} = \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix};\quad \mathbf{x} = \begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix}$$

2. Representing the functionals as vector,

$$\mathbf{f} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \tag{3.5.4.3}$$

3. Representing the integrations as vector,

$$\mathbf{I} = \begin{pmatrix} \int_0^1 dx \\ \int_0^2 dx \\ \int_0^{-1} dx \end{pmatrix} \tag{3.5.4.4}$$

4. So,

$$\mathbf{f} = \mathbf{I}\mathbf{c}^T\mathbf{x} = \mathbf{I}\mathbf{p}' \tag{3.5.4.5}$$

(3.5.4.5) can written in matrix format as,

$$\mathbf{f} = \mathbf{P}\mathbf{c} \tag{3.5.4.6}$$

where,

$$\mathbf{P} = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 2 & 2 & \frac{8}{3} \\ -1 & \frac{1}{2} & \frac{-1}{3} \end{pmatrix} \tag{3.5.4.7}$$

5. **P** is one-one if it has a inverse. Calculating the determinant of **P**,

$$\implies \left|P\right| = -2 \tag{3.5.4.8}$$

From, (3.5.4.8), **P** is one-one. Also,

$$\mathbf{V} = \mathbf{f}^T = \begin{pmatrix} f_1 & f_2 & f_3 \end{pmatrix} \tag{3.5.4.9}$$

From (3.5.4.6), (3.5.4.8) and (3.5.4.9), the rows of **P** are isomorphic to **V**. So, finding the dual basis by performing matrix operations on $\mathbf{P}^T$

$$\begin{pmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ \frac{1}{2} & 2 & \frac{1}{2} & 0 & 1 & 0 \\ \frac{1}{3} & \frac{8}{3} & \frac{-1}{3} & 0 & 0 & 1 \end{pmatrix} \xleftrightarrow[R_2 \leftarrow R_2 - \frac{R_1}{2}]{R_3 \leftarrow R_3 - \frac{R_1}{3}}$$

$$\begin{pmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{-1}{2} & 1 & 0 \\ 0 & 2 & 0 & \frac{-1}{3} & 0 & 1 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow \frac{R_3 - 2R_2}{-2}}$$

$$\begin{pmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{-1}{2} & 1 & 0 \\ 0 & 0 & 1 & \frac{-1}{3} & 1 & \frac{-1}{2} \end{pmatrix} \xleftrightarrow[R_2 \leftarrow R_2 - R_3]{R_1 \leftarrow R_1 + R_3}$$

$$\begin{pmatrix} 1 & 2 & 0 & \frac{2}{3} & 1 & \frac{-1}{2} \\ 0 & 1 & 0 & \frac{-1}{6} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{-1}{3} & 1 & \frac{-1}{2} \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 - 2R_2}$$

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & \frac{-3}{2} \\ 0 & 1 & 0 & \frac{-1}{6} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{-1}{3} & 1 & \frac{-1}{2} \end{pmatrix} \tag{3.5.4.10}$$

From (3.5.4.10), the dual of $\mathbf{V} \implies \mathbf{V}^*$ can be written in matrix form as,

$$\mathbf{V}^* = \mathbf{A}\mathbf{x} \tag{3.5.4.11}$$

where,

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & \frac{-3}{2} \\ \frac{-1}{6} & 0 & \frac{1}{2} \\ \frac{-1}{3} & 1 & \frac{-1}{2} \end{pmatrix};\quad \mathbf{x} = \begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix}$$

3.5.5. If **A** and **B** are $n \times n$ matrices , show that

$$\mathbf{AB} - \mathbf{BA} = \mathbf{I} \tag{3.5.5.1}$$

is impossible.

Solution:

$$tr(\mathbf{AB}) = tr(\mathbf{BA}) \tag{3.5.5.2}$$

$$tr(\mathbf{I}) = \sum_{j=1}^{n} I_{jj} \tag{3.5.5.3}$$

$$\implies \sum_{j=1}^{n} 1 = n \tag{3.5.5.4}$$

Taking tr on both sides (3.5.5.1)

$$tr(\mathbf{AB} - \mathbf{BA}) = tr(\mathbf{I}) \tag{3.5.5.5}$$
$$\implies tr(\mathbf{AB}) - tr(\mathbf{BA}) = tr(\mathbf{I}) \tag{3.5.5.6}$$

From (3.5.5.2) and (3.5.5.4)

$$tr(\mathbf{AB}) - tr(\mathbf{BA}) = 0 \qquad (3.5.5.7)$$
$$tr(\mathbf{I}) = n \qquad (3.5.5.8)$$
$$= 0 \neq n \qquad (3.5.5.9)$$

Thus:-

$$tr(\mathbf{AB} - \mathbf{BA}) \neq tr(\mathbf{I}) \qquad (3.5.5.10)$$

hence no solution possible.

3.5.6. Let m and n be positive integers and field **F**.Let $f_1,\ldots,f_m$ be linear functions in F^n.For α in F^n define

$$T\alpha = (f_1(\alpha),\ldots,f_m(\alpha)) \qquad (3.5.6.1)$$

show that T is a linear transformation from F^n into F^m.Then show that every linear transformation from F^n into F^m is of the above form ,for some $f_1,\ldots,f_m$.

Solution: Let $\mathbf{b},\alpha \in F^n$ and a is a scalar

$$\begin{aligned}T(a\alpha + \mathbf{b}) &= (f_1(a\alpha + \mathbf{b}),\ldots,f_m(a\alpha + \mathbf{b}))\\ &= (af_1(\alpha) + f_1(\mathbf{b}),\ldots,af_m(\alpha) + f_m(\mathbf{b}))\\ &= a(f_1(\alpha),\ldots,f_m(\alpha)) + (f_1(\mathbf{b}),\ldots,f_m(\mathbf{b}))\end{aligned} \qquad (3.5.6.2)$$

The equation (3.5.6.2) can be written as

$$T(a\alpha + \mathbf{b}) = aT(\alpha) + T(\mathbf{b}) \qquad (3.5.6.3)$$

So, T is a linear transformation.
Let the matrix A of order $m \times n$ represent any linear transformation $\mathbf{X} \mapsto A\mathbf{X}$ from F^n into F^m. For i=1,…,m ,let

$$f_i(x_1,\ldots,x_n) = \sum_{j=1}^{n} A_{ij}x_j \qquad (3.5.6.4)$$

The transformation into F^m, $A\mathbf{X}$ can be written as

$$(f_1(\mathbf{X}),\ldots,f_m(\mathbf{X})) \qquad (3.5.6.5)$$

This is of the form (3.5.6.1)

3.5.7. Let $\alpha_1 = (1,0,-1,2)$ and $\alpha_2 = (2,3,1,1)$ and let **W** be the subspace of $\mathbb{R}^4$ spanned by α_1 and α_2. Which linear functionals **f** :

$$f(x_1,x_2,x_3,x_4) = c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 \qquad (3.5.7.1)$$

are in the annihilator of **W**?

Solution: (3.5.7.1), can be expressed as

Linear Functional	If **V** is a vector space over the field **F**, field **F** is also called a linear functional
Dual Space of **V**	If **V** is a vector space, the collection of It is the space $\mathbf{L}(\mathbf{V},\mathbf{F})$ which we denote
Annihilator of S	**V** is a vector space over the field **F** and S^o of linear functionals **f** on **V** such tha

$$\mathbf{f}(\mathbf{x}) = \mathbf{x}^T\mathbf{c} \qquad (3.5.7.2)$$

$$\text{where} \quad \mathbf{x} = \begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} \quad \text{and} \quad \mathbf{c} = \begin{pmatrix}c_1\\c_2\\c_3\\c_4\end{pmatrix} \qquad (3.5.7.3)$$

Given two vectors

$$\alpha_{\mathbf{1}} = \begin{pmatrix}1\\0\\-1\\2\end{pmatrix} \quad and \quad \alpha_{\mathbf{2}} = \begin{pmatrix}2\\3\\1\\1\end{pmatrix} \qquad (3.5.7.4)$$

Since, $\alpha_{\mathbf{1}}$ is not a scalar multiple of $\alpha_{\mathbf{2}}$. Thus, $\alpha_{\mathbf{1}}$ and $\alpha_{\mathbf{2}}$ are linearly independent. Also, given $\alpha_{\mathbf{1}}$ and $\alpha_{\mathbf{2}}$ span **W**, thus $\{\alpha_{\mathbf{1}}, \alpha_{\mathbf{2}}\}$ form basis for **W**. Hence, **W** has dimension 2.
Now, a functional **f** is in the annihilator of **W** if and only if $\mathbf{f}(\alpha_{\mathbf{1}}) = \mathbf{f}(\alpha_{\mathbf{2}}) = 0$. We find such **f** by solving the system

$$\mathbf{f}(\alpha_{\mathbf{1}}) = \mathbf{f}(\alpha_{\mathbf{2}}) = 0 \qquad (3.5.7.5)$$

From, (3.5.7.2) and (3.5.7.5), we have

$$\begin{pmatrix}1 & 0 & -1 & 2\\2 & 3 & 1 & 1\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\\c_4\end{pmatrix} = 0 \qquad (3.5.7.6)$$

Converting (3.5.7.6) into row reduced echelon form

$$\begin{pmatrix}1 & 0 & -1 & 2\\2 & 3 & 1 & 1\end{pmatrix} \xleftrightarrow{r_2=r_2-2r_1} \begin{pmatrix}1 & 0 & -1 & 2\\0 & 3 & 3 & -3\end{pmatrix} \qquad (3.5.7.7)$$

$$\begin{pmatrix}1 & 0 & -1 & 2\\0 & 3 & 3 & -3\end{pmatrix} \xleftrightarrow{r_2=\frac{r_2}{3}} \begin{pmatrix}1 & 0 & -1 & 2\\0 & 1 & 1 & -1\end{pmatrix} \qquad (3.5.7.8)$$

from (3.5.7.8), we have

$$\begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = 0 \tag{3.5.7.9}$$

The general element of $\mathbf{W}^o$ is therefore

$$\mathbf{f}(\mathbf{x}) = \mathbf{x}^T \mathbf{a} \tag{3.5.7.10}$$

where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \quad \textit{and} \quad \mathbf{a} = \begin{pmatrix} c_3 - 2c_4 \\ -c_3 + c_4 \\ c_3 \\ c_4 \end{pmatrix} \tag{3.5.7.11}$$

for arbitrary constants c_3 and c_4.
Also, $\mathbf{W}^o$ has dimension 2.

3.5.8. Let $\mathbf{W}$ be the subspace of $\mathbb{R}^5$ which is spanned by the vectors

$$\begin{aligned} \alpha_1 &= \epsilon_1 + 2\epsilon_2 + \epsilon_3, \\ \alpha_2 &= \epsilon_2 + 3\epsilon_3 + 3\epsilon_4 + \epsilon_5, \\ \alpha_3 &= \epsilon_1 + 4\epsilon_2 + 6\epsilon_3 + 4\epsilon_4 + \epsilon_5 \end{aligned} \tag{3.5.8.1}$$

Find a basis for $\mathbf{W^0}$

Solution: If $\mathbf{V}$ is a vector space over the field $\mathbb{F}$ and $\mathbf{W}$ is a subset of $\mathbf{V}$, the annihilator of $\mathbf{W}$ is the set $\mathbf{W^0}$ of linear functionals $\mathbf{f}$ on $\mathbf{V}$ such that $\mathbf{f}(\alpha) = 0$ for every α in $\mathbf{W}$.
Properties of Annihilator: If f is a linear functional on R^n:

$$f(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{n} c_j x_j \tag{3.5.8.2}$$

Then f is in W^0 if and only if

$$\forall \alpha \in W: f(\alpha) = 0 \iff f = 0 \tag{3.5.8.3}$$

(3.5.8.2) can be expressed as:

$$\mathbf{f}(\mathbf{x}) = \mathbf{x}^T \mathbf{c} \tag{3.5.8.4}$$

$$\text{where } \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \text{ and } \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{pmatrix} \tag{3.5.8.5}$$

Given three vectors

$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 3 \\ 3 \\ 1 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 1 \\ 4 \\ 6 \\ 4 \\ 1 \end{pmatrix} \tag{3.5.8.6}$$

Let matrix A with column vectors $\alpha_1, \alpha_2, \alpha_3$:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 4 \\ 1 & 3 & 6 \\ 0 & 3 & 4 \\ 0 & 1 & 1 \end{pmatrix} \tag{3.5.8.7}$$

Given that $\mathbf{f}$ is a linear functional on $\mathbb{R}^5$, then $\mathbf{f}$ is in $\mathbf{W^0}$ if and only if,

$$f(\alpha_i) = 0, i = 1, 2, 3 \tag{3.5.8.8}$$

$$\implies \mathbf{A}^T \mathbf{c} = \mathbf{0} \tag{3.5.8.9}$$

Converting the (3.5.8.9) into system of equations, we have,

$$\begin{pmatrix} 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & 3 & 1 \\ 1 & 4 & 6 & 4 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{pmatrix} = 0 \tag{3.5.8.10}$$

Converting (3.5.8.10) into row reduced echelon form,

$$\begin{pmatrix} 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & 3 & 1 \\ 1 & 4 & 6 & 4 & 1 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & 0 & 0 & 4 & 3 \\ 0 & 1 & 0 & -3 & -2 \\ 0 & 0 & 1 & 2 & 1 \end{pmatrix} \tag{3.5.8.11}$$

From (3.5.8.11), we have,

$$c_1 = -(4c_4 + 3c_5) \tag{3.5.8.12}$$

$$c_2 = (3c_4 + 2c_5) \tag{3.5.8.13}$$

$$c_3 = -(2c_4 + c_5) \tag{3.5.8.14}$$

Therefore, general element of $\mathbf{W^0}$ is therefore,

$$\begin{aligned} f(x_1, \ldots, x_5) = -(4c_4 + 3c_5)x_1 + (3_4 + 2c_5)x_2 \\ -(2c_4 + c_5)x_3 + c_4x_4 + c_5x_5 \end{aligned} \tag{3.5.8.15}$$

Therefore, dimension of $\mathbf{W^0}$ is 2 and a basis $\{f_1, f_2\}$ can be obtained by putting $c_4 = 0, c_5 =$

1 and $c_4 = 1, c_5 = 0$ in (3.5.8.15)

$$f_1(x_1, \ldots, x_5) = -3x_1 + 2x_2 - x_3 + x_5 \tag{3.5.8.16}$$

$$f_2(x_1, \ldots, x_5) = -4x_1 + 3x_2 - 2x_3 + x_4 \tag{3.5.8.17}$$

3.5.9. Let $\mathbb{V}$ be the vector space of all 2×2 matrices over the field of real numbers, and let

$$\mathbf{B} = \begin{pmatrix} 2 & -2 \\ -1 & 1 \end{pmatrix} \tag{3.5.9.1}$$

Let $\mathbb{W}$ be the subspace of $\mathbb{V}$ consisting of all $\mathbf{A}$ such that $\mathbf{AB} = 0$. Let f be a linear functional on $\mathbb{V}$ which is in the annihilator of $\mathbb{W}$. Suppose that $f(\mathbf{I}) = 0$ and $f(\mathbf{C}) = 3$, where $\mathbf{I}$ is the 2×2 identity matrix and

$$\mathbf{C} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.5.9.2}$$

Find $f(\mathbf{B})$

Solution: The general Linear functional f on $\mathbf{V}$ is of the form,

$$f(\mathbf{A}) = aA_{11} + bA_{12} + cA_{21} + dA_{22} \tag{3.5.9.3}$$

for a,b,c,d $\in \mathbf{R}$ Let $\mathbf{A} \in \mathbf{W}$ be,

$$\mathbf{A} = \begin{pmatrix} p & q \\ q & s \end{pmatrix} \tag{3.5.9.4}$$

$$\because \mathbf{AB} = 0 \tag{3.5.9.5}$$

$$\implies \begin{pmatrix} p & q \\ q & s \end{pmatrix}\begin{pmatrix} 2 & -2 \\ -1 & 1 \end{pmatrix} = 0 \tag{3.5.9.6}$$

$$\implies \begin{pmatrix} 2p - q & -2p + q \\ 2q - s & -2q + s \end{pmatrix} = 0 \tag{3.5.9.7}$$

$\therefore$ q=2p and s=2q. Hence $\mathbf{W}$ consists of all matrices of the form

$$\begin{pmatrix} p & 2p \\ q & 2q \end{pmatrix} \tag{3.5.9.8}$$

Now V is an annihilator of W. Hence, f$\in \mathbf{W}^0$

$$\implies f\begin{pmatrix} p & 2p \\ q & 2q \end{pmatrix} = 0 \forall p, q \in \mathbf{R} \tag{3.5.9.9}$$

from equation (3.5.9.3)

$$\implies ap + 2bp + cq + 2dq = 0 \forall p, q \in \mathbf{R} \tag{3.5.9.10}$$

$$\implies (a + 2b)p + (c + 2d)q = 0, \forall p, q \in \mathbf{R} \tag{3.5.9.11}$$

Hence, b=$\frac{-1}{2}$a and d=$\frac{-1}{2}$c. Hence general $f \in \mathbf{W}^0$ is of the form,

$$f(\mathbf{A}) = aA_{11} - \frac{1}{2}aA_{12} + cA_{21} - \frac{1}{2}cA_{22} \tag{3.5.9.12}$$

Now, f($\mathbf{C}$) = 3 $\implies$ d =3 $\implies$ c=-6. Also given that, f($\mathbf{I}$) = 0 $\implies$ a-$\frac{1}{2}$c =0 $\implies$ a=-3. Substituting the above parameters in equation (3.5.9.12) we get,

$$\therefore f(\mathbf{A}) = -3A_{11} + \frac{3}{2}A_{12} - 6A_{21} + 3A_{22} \tag{3.5.9.13}$$

$$\text{Now, } f(\mathbf{B}) = f\begin{pmatrix} 2 & -2 \\ -1 & 1 \end{pmatrix} \tag{3.5.9.14}$$

$$\implies f(\mathbf{B}) = -3(2) + \frac{3}{2}(-2) - 6(-1) + 3(1) = 0 \tag{3.5.9.15}$$

3.5.10. Let F be a subfield of the complex numbers. We define n linear functionals on $F^n (n \geq 2)$ by

$$f_k(x_1, ..., x_n) = \sum_{j=1}^{n}(k - j)x_j, 1 \leq k \leq n. \tag{3.5.10.1}$$

What is the dimension of the subspace annihilated by $f_1, f_2, ..., f_n$?

Solution: See Tables 3.5.10.1 and 3.5.10.2.

Given	F be a subfield of the co Definition of n linear functionals on $F^n (n \geq 2)$ by
To find	The dimension of the subspace an
f_k	$f_k(x_1, ..., x_n) = \sum_{j=1}^{n}$ $f_k(x_1, ..., x_n) = k\sum_{j=1}^{n}$ All f_k are linear combinations of th

TABLE 3.5.10.1

3.5.11. Let W_1 and W_2 be subspaces of a finite-dimensional vector space $\mathbb{V}$. Prove that

a) $(W_1 + W_2)^0 = W_1^0 \cap W_2^0$
b) $(W_1 \cap W_2)^0 = W_1^0 + W_2^0$
Solution: See Table 3.5.11.1

Vector	$f_k(x_1, ..., x_n) = \sum_{j=1}^{n}(k - j)x_j$ $\mathbf{f} = \begin{pmatrix} f_1 \\ f_2 \\ . \\ . \\ f_n \end{pmatrix}$ $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{pmatrix}$ $f_1(\mathbf{x}) = 0.x_1 - x_2 - 2x_3 - ... - (n-1)x_n$ $f_2(\mathbf{x}) = x_1 + 0.x_2 - 1.x_3 - ... - (n-2)x_n$ . . $f_n(\mathbf{x}) = (n-1)x_1 + (n-2).x_2 + ... + (n-2)x_{n-1} + 0.x_n$ $A_{n*n} = \begin{pmatrix} 0 & -1 & -2 & . & . & -(n-1) \\ 1 & 0 & -1 & . & . & -(n-2) \\ . & . & . & . & . & . \\ (n-1) & (n-2) & . & . & (n-2) & 0 \end{pmatrix}$
Matrix	$AX = 0$ where the i^{th} row is defined by $A_i = (i-1, i-2, ..., i-n)$ $1 \leq i \leq n$ For the $n = 4$, matrix A is: $\begin{pmatrix} 0 & -1 & -2 & -3 \\ 1 & 0 & -1 & -2 \\ 2 & 1 & 0 & -1 \\ 3 & 2 & 1 & 0 \end{pmatrix}$ For $i \geq 3$, perform the following elementary operations of n linear functionals as defined below $(a)A_i \longrightarrow (1-i)A_2 + A_i$ $A_i = (0, i-2, 2(i-2), 3(i-2), ..., (n-1)(i-2))$ $(b)A_i \longrightarrow \frac{1}{i-2}A_i$ A_i = -A_1 $(c)A_i \longrightarrow A_i + A_1$ $A_i = 0$ Since, A_1 and A_2 are linearly independent Thus, the dimension of the subspace annihiliated $= n - 2$

TABLE 3.5.10.2

Given	W_1 and W_2 are subspaces of a finite dimensional vector space $\mathbb{V}$
1. To prove	$(W_1 + W_2)^0 = W_1^0 \cap W_2^0$
Proof of $(W_1 + W_2)^0 \subseteq (W_1^0 \cap W_2^0)$	Let $f \in (W_1 + W_2)^0$ $\forall \mathbf{v} \in (W_1 + W_2)$ $f(\mathbf{v}) = 0$ $\implies \forall \mathbf{w}_1 \in W_1, \mathbf{w}_2 \in W_2$ $\mathbf{w_1} + \mathbf{w_2} \in (W_1 + W_2)$ $\therefore f(\mathbf{w_1} + \mathbf{w_2}) = 0$ $\implies$ When $\mathbf{w}_2 = 0$, then $\forall \mathbf{w}_1 \in W_1$ $f(\mathbf{w_1}) = 0$ $\therefore f \in W_1^0$ And when $\mathbf{w_1} = 0, \forall \mathbf{w_2} \in W_2$ $f(\mathbf{w_2}) = 0$ $\therefore f \in W_2^0$ $\because f \in W_1^0, f \in W_2^0$ $\implies f \in (W_1^0 \cap W_2^0)$ $\therefore (W_1 + W_2)^0 \subseteq (W_1^0 \cap W_2^0)$
Proof of $(W_1^0 \cap W_2^0) \subseteq (W_1 + W_2)^0$	Let $f \in (W_1^0 \cap W_2^0)$ $\implies f \in W_1^0, f \in W_2^0$ $\implies \forall \mathbf{w}_1 \in W_1, \mathbf{w}_2 \in W_2$ $f(\mathbf{w}_1) = 0$ and $f(\mathbf{w}_2) = 0$ $\forall \mathbf{v} \in (W_1 + W_2)$,
	$\mathbf{v} = \mathbf{w}_1 + \mathbf{w}_2$ $\implies f(\mathbf{v}) = f(\mathbf{w}_1 + \mathbf{w}_2)$ $\implies f(\mathbf{v}) = f(\mathbf{w}_1) + f(\mathbf{w}_2)$ $\implies f(\mathbf{v}) = 0$ $\implies f \in (W_1 + W_2)^0$ $\therefore (W_1^0 \cap W_2^0) \subseteq (W_1 + W_2)^0$
	Hence, $(W_1 + W_2)^0 = (W_1^0 \cap W_2^0)$
2. To prove	$(W_1 \cap W_2)^0 = W_1^0 + W_2^0$

Proof of $(W_1^0 + W_2^0) \subseteq (W_1 \cap W_2)^0$	Let $f \in (W_1^0 + W_2^0)$ for some $f_1 \in W_1^0, f_2 \in W_2^0$, $f = f_1 + f_2$ Now, for $\mathbf{v} \in (W_1 \cap W_2)$ $f(\mathbf{v}) = (f_1 + f_2)(\mathbf{v})$ $\implies f(\mathbf{v}) = f_1(\mathbf{v}) + f_2(\mathbf{v})$ $\because \mathbf{v} \in (W_1 \cap W_2)$ $\implies \mathbf{v} \in W_1$, and $\mathbf{v} \in W_2$ So, $f_1(\mathbf{v}) = 0$, and $f_2(\mathbf{v}) = 0$ $\implies f(\mathbf{v}) = 0 + 0 = 0$ $\implies f \in (W_1 \cap W_2)^0$ $\therefore (W_1^0 + W_2^0) \subseteq (W_1 \cap W_2)^0$
Proof of $(W_1 \cap W_2)^0 \subseteq (W_1^0 + W_2^0)$	Let $f \in (W_1 \cap W_2)^0$ Assuming Basis of W_1 as $\{\alpha_1, \ldots, \alpha_{\mathbf{k}}, \beta_1, \ldots, \beta_l\}$ Basis of W_2 as $\{\alpha_1, \ldots, \alpha_{\mathbf{k}}, \gamma_1, \ldots, \gamma_m\}$ $\therefore$ Basis of $(W_1 \cap W_2)$ is $\{\alpha_1, \ldots, \alpha_{\mathbf{k}}\}$ and Basis of $W_1 + W_2$ is $\{\alpha_1, \ldots, \alpha_{\mathbf{k}}, \beta_1, \ldots, \beta_l, \gamma_1, \ldots, \gamma_m\}$ Now, for $\mathbf{v} \in (W_1 + W_2)$ $\mathbf{v} = \sum_{i=1}^{k} x_i\alpha_i + \sum_{i=1}^{l} y_i\beta_i + \sum_{i=1}^{m} z_i\gamma_i$ $f(\mathbf{v}) = \sum_{i=1}^{k} a_i x_i + \sum_{i=1}^{l} b_i y_i + \sum_{i=1}^{m} c_i z_i$ $\forall \mathbf{v} \in (W_1 \cap W_2)$ $\mathbf{v} = \sum_{i=1}^{k} x_i\alpha_i$
	$f(\mathbf{v}) = \sum_{i=1}^{k} a_i x_i$ But since $f \in (W_1 \cap W_2)^0$, thus $f(\mathbf{v}) = 0$ $\therefore a_1 = a_2 = \ldots = a_k = 0$ So, we can now write $f(\mathbf{v}) = \sum_{i=1}^{l} b_i y_i + \sum_{i=1}^{m} c_i z_i$ Now, $\forall \mathbf{v} \in W_1$, $\mathbf{v} = \sum_{i=1}^{k} x_i\alpha_i + \sum_{i=1}^{l} y_i\beta_i$ $\implies f_1(\mathbf{v}) = \sum_{i=1}^{k} a_i x_i + \sum_{i=1}^{l} b_i y_i$ Comparing this to the original equation, we can say $c_i = 0$ And $\forall \mathbf{v} \in W_2$, $\mathbf{v} = \sum_{i=1}^{k} x_i\alpha_i + \sum_{i=1}^{m} z_i\gamma_i$

	$\implies f_2(\mathbf{v}) = \sum_{i=1}^{k} a_i x_i + \sum_{i=1}^{m} c_i z_i$ Comparing this to the original equation, we can say $b_i = 0$ $f = f_1 + f_2$ $\because a_i = b_i = c_i = 0$, $f_1(\mathbf{v}) = 0$ $\implies f_1 \in W_1^0$ Also, $f_2(\mathbf{v}) = 0$ $\implies f_2 \in W_2^0$ So, $f_1 + f_2 \in (W_1^0 + W_2^0)$ $\implies f \in (W_1^0 + W_2^0)$ $\therefore (W_1 \cap W_2)^0 \subseteq (W_1^0 + W_2^0)$
	Hence, $(W_1 \cap W_2)^0 = (W_1^0 + W_2^0)$
Verification	1. Since the annihilator $(W_1 + W_2)^0$ is a complement of $(W_1 + W_2)$, Using Demorgan's Laws of the complement of union of two sets is the intersection of their complements, it is verified $(W_1 + W_2)^0 = W_1^0 \cap W_2^0$ 2. And using De Morgan's laws of the complement of intersection of two sets is the union of their complements, it is verified $(W_1 \cap W_2)^0 = W_1^0 + W_2^0$

TABLE 3.5.11.1: Proving properties of vectorspaces and subspaces

3.5.12. Let **F** be a subfield of the field of complex numbers and let **V** be any vector space over **F**. Suppose that f and g are linear functionals on **V** such that the function h defined by $h(\alpha) = f(\alpha)g(\alpha)$ is also a linear functional on **V**. Prove that either $f = 0$ or $g = 0$.
Solution: Refer Table 3.5.12.1.

3.5.13. Let $\mathbb{F}$ be a field of characteristic zero and let **V** be a finite dimensional vector space over $\mathbb{F}$.If $\alpha_1, \alpha_2, \ldots, \alpha_m$ are finitely many vectors in **V** , each different from the zero vector, prove that there is a linear functional f on **V** such that

$$f(\alpha_i) \neq 0, i = 1, 2, \ldots, m \tag{3.5.13.1}$$

Solution: We will make use of the following theorem
Let **V** be a finite dimensional vector space over the field $\mathbb{F}$ and let $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ be a basis for **V**.Then there exists a unique dual basis $\{f_1, f_2, \ldots, f_n\}$ for $\mathbf{V}^*$ such that

$$f_i(\mathbf{v}_j) = \delta_{ij} \tag{3.5.13.2}$$

and $\mathbf{V}^*$ is the space of all linear functionals on **V**

3.5.14. Similar matrices have the same trace. Thus we can define the trace of a linear operator on a finite-dimensional space to the trace of any matrix which represents the operator in a ordered basis. This is well-defined since all such representing matrices for one operator are similar.
Now let V be the space of all 2×2 matrices over the field F and let P be a fixed 2×2 matrix. Let T be the linear operator on V defined by $T(A) = PA$. Prove that $tr(T) = 2tr(P)$.
Solution: Given V is the space of all 2×2 matrices over field F. P is a 2×2 matrix,

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \tag{3.5.14.1}$$

Let $\mathcal{B} = \{e_{11}, e_{12}, e_{21}, e_{22}\}$ be the ordered basis of V where,

$$e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{3.5.14.2}$$

$$e_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.5.14.3}$$

Given, $T(A) = PA$,

$$T(e_{11}) = Pe_{11} = \begin{pmatrix} p_{11} & 0 \\ p_{21} & 0 \end{pmatrix} \tag{3.5.14.4}$$

$$= p_{11}e_{11} + p_{21}e_{21} \tag{3.5.14.5}$$

$$T(e_{12}) = Pe_{12} = \begin{pmatrix} 0 & p_{11} \\ 0 & p_{21} \end{pmatrix} \tag{3.5.14.6}$$

$$= p_{11}e_{12} + p_{21}e_{22} \tag{3.5.14.7}$$

$$T(e_{21}) = Pe_{21} = \begin{pmatrix} p_{12} & 0 \\ p_{22} & 0 \end{pmatrix} \tag{3.5.14.8}$$

$$= p_{12}e_{11} + p_{22}e_{21} \tag{3.5.14.9}$$

$$T(e_{22}) = Pe_{22} = \begin{pmatrix} 0 & p_{12} \\ 0 & p_{22} \end{pmatrix} \tag{3.5.14.10}$$

$$= p_{12}e_{12} + p_{22}e_{22} \tag{3.5.14.11}$$

The matrix representation of linear functional T in the ordered basis $\mathcal{B}$ is given as,

$$T = \begin{pmatrix} [T(e_{11})]_{\mathcal{B}} & [T(e_{12})]_{\mathcal{B}} & [T(e_{21})]_{\mathcal{B}} & [T(e_{22})]_{\mathcal{B}} \end{pmatrix} \tag{3.5.14.12}$$

$$T = \begin{pmatrix} p_{11} & 0 & p_{12} & 0 \\ 0 & p_{11} & 0 & p_{12} \\ p_{21} & 0 & p_{22} & 0 \\ 0 & p_{21} & 0 & p_{22} \end{pmatrix} \tag{3.5.14.13}$$

$$= \begin{pmatrix} p_{11}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & p_{12}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ p_{21}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & p_{22}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} \tag{3.5.14.14}$$

$$= \begin{pmatrix} p_{11}I & p_{12}I \\ p_{21}I & p_{22}I \end{pmatrix} = P \otimes I \tag{3.5.14.15}$$

Now by using the property of kronecker product we get,

$$tr(T) = tr(P \otimes I) = tr(P)tr(I) \tag{3.5.14.16}$$

$$= 2tr(P) \tag{3.5.14.17}$$

$$\therefore tr(T) = 2tr(P) \tag{3.5.14.18}$$

3.5.15. Show that the trace functional on $n \times n$ matrices is unique in the following sense. If W is the space of $n \times n$ matrices over the field F and if f is a linear functional on W such that $f(AB) = f(BA)$ for each A and B in W, then f is a scalar multiple of the trace function. If, in addition, $f(I) = n$, then f is the trace function.
Solution: See Tables 3.5.15.1 and 3.5.15.2

3.5.16. Let **W** be the space of $n \times n$ matrices over the field **F**, and let $\mathbf{W}_0$ be the subspace spanned by

Given	Derivation
f, g, h are linear functionals of $\mathbf{V}$	By contradiction, let us assume $f \neq 0$ and $g \neq 0$. For all $\mathbf{v} \in \mathbf{V}$ $$h(\mathbf{v}) = f(\mathbf{v})g(\mathbf{v}) \quad (3.5.12.1)$$ $$h(2\mathbf{v}) = f(2\mathbf{v})g(2\mathbf{v}) \quad (3.5.12.2)$$ $$= 2f(\mathbf{v})2g(\mathbf{v}) \quad (3.5.12.3)$$ $$= 4f(\mathbf{v})g(\mathbf{v}) \quad (3.5.12.4)$$ Similarly, $$h(2\mathbf{v}) = 2h(\mathbf{v}) \quad (3.5.12.5)$$ $$= 2f(\mathbf{v})g(\mathbf{v}) \quad (3.5.12.6)$$ From equation (3.5.12.4) and (3.5.12.6), $$\implies 4f(\mathbf{v})g(\mathbf{v}) = 2f(\mathbf{v})g(\mathbf{v}) \quad (3.5.12.7)$$ $$\implies f(\mathbf{v}).g(\mathbf{v}) = 0 \quad (3.5.12.8)$$
Choosing Basis	Let $\mathbf{B}$ be a basis for $\mathbf{V}$. Let, $$\mathbf{B_1} = \{\mathbf{b} \in \mathbf{B} \mid f(\mathbf{b}) = 0\}, \quad (3.5.12.9)$$ $$\mathbf{B_2} = \{\mathbf{b} \in \mathbf{B} \mid g(\mathbf{b}) = 0\} \quad (3.5.12.10)$$ Since, $$f(\mathbf{b}).g(\mathbf{b}) = 0 \quad \forall \mathbf{b} \in \mathbf{B} \quad (3.5.12.11)$$ $$\implies f(\mathbf{b}) = 0 \text{ or } g(\mathbf{b}) = 0 \quad (3.5.12.12)$$ $$\implies \mathbf{b} \in \mathbf{B_1} \text{ or } \mathbf{b} \in \mathbf{B_2} \quad (3.5.12.13)$$
Choosing $\mathbf{b_1}$ and $\mathbf{b_2}$ from basis	Let us choose $\mathbf{b_1} \in \mathbf{B_1} - \mathbf{B_2}$ and $\mathbf{b_2} \in \mathbf{B_2} - \mathbf{B_1}$ $\implies f(\mathbf{b_2}) \neq 0$ and $g(\mathbf{b_1}) \neq 0$ $$f(\mathbf{b_1}+\mathbf{b_2}).g(\mathbf{b_1}+\mathbf{b_2}) = (f(\mathbf{b_1}) + f(\mathbf{b_2})).(g(\mathbf{b_1}) + g(\mathbf{b_2})) \quad (3.5.12.14)$$ $$= f(\mathbf{b_1}).g(\mathbf{b_1}) + f(\mathbf{b_1}).g(\mathbf{b_2}) + f(\mathbf{b_2}).g(\mathbf{b_1}) + f(\mathbf{b_2}).g(\mathbf{b_2}) \quad (3.5.12.15)$$ $$= 0 + 0 + f(\mathbf{b_2}).g(\mathbf{b_1}) + 0 \quad (3.5.12.16)$$ $$= f(\mathbf{b_2}).g(\mathbf{b_1}) \neq 0 \quad (3.5.12.17)$$ Equation (3.5.12.17) is contradiction to the fact that $f(\mathbf{v}).g(\mathbf{v}) = 0$. $\implies \boxed{f = 0 \text{ or } g = 0}$

TABLE 3.5.12.1: Expanation

the matrices C of the form $C = AB - BA$. Prove that $\mathbf{W}_0$ is exactly the subspace of matrices which have trace zero.

Solution: Let there be two subspaces defined

PARAMETERS	DESCRIPTION
$\mathbb{F}$	Field
$\mathbf{V}$	Finite dimensional vector space over $\mathbb{F}$
$\alpha_1, \alpha_2, \ldots, \alpha_m$	non zero vectors in $\mathbf{V}$
$f : \mathbf{V} \to \mathbb{F}$	Linear functional on $\mathbf{V}$

TABLE 3.5.13.1: Input Parameters

as

$$\mathbf{W}_0 = \{A \in \mathbf{W} : trace(A) = 0\} \quad (3.5.16.1)$$

and

$$\mathbf{W}_1 = \{C \in \mathbf{W} : C = AB - BA\} \quad (3.5.16.2)$$

Consider $C \in \mathbf{W}$ such that $C = AB - BA$ where $A, B \in \mathbf{R}^{N\times N}$ and since

$$trace(AB) = trace(BA) \quad (3.5.16.3)$$
$$\implies trace(AB) - trace(BA) = 0 \quad (3.5.16.4)$$
$$\implies trace(C) = 0 \quad (3.5.16.5)$$

Thus any linear combination of matrices $C = AB - BA$ represents the subspace of all $N \times N$ matrices with trace equal to 0. Hence, the subspace $\mathbf{W}_0$ and $\mathbf{W}_1$ are equivalent.

3.6 The Double Dual

3.6.1. Let n be a positive integer and $\mathbf{F}$ a field. let $\mathbf{W}$ be the set of all vectors $\begin{pmatrix}\mathbf{x_1}, & \mathbf{x_2}, \cdots, \mathbf{x_n}\end{pmatrix} \in \mathbf{F}^n$ such that $\mathbf{x_1} + \mathbf{x_2} + \mathbf{x_3} + \cdots \mathbf{x_n} = 0$.

a) Prove that $\mathbf{W}^0$ consists of all linear functional f of the form

$$f\begin{pmatrix}\mathbf{x_1}, & \mathbf{x_2} & , \cdots & , \mathbf{x_n}\end{pmatrix} = c\sum_{j=1}^{n} \mathbf{x_j} \quad (3.6.1.1)$$

Solution: *Definition 1:* If the vector space $\mathbf{V}$ is finite dimensional (say dimension =n), the dimension of null-space $\mathbf{N}_f$ by rank nullity theorem is given by,

$$\left|\mathbf{N}_f\right| = |\mathbf{V}| - 1 = n - 1 \quad (3.6.1.2)$$

Definition 2: If $\mathbf{V}$ is a vector space over the field $\mathbf{F}$ and $\mathbf{S}$ is a subset of $\mathbf{V}$, the annihilator of $\mathbf{S}$ is the set $\mathbf{S}^0$ of linear functional f on $\mathbf{V}$ such that $f(\alpha) = 0$ for every α in $\mathbf{S}$. Solution

Let h be the functional,

$$h\begin{pmatrix}\mathbf{x_1}, & \mathbf{x_2}, \cdots, \mathbf{x_n}\end{pmatrix} = \mathbf{x_1} + \mathbf{x_2} + \cdots \mathbf{x_n} \quad (3.6.1.3)$$

$$\text{Let, } \mathbf{X} = \begin{pmatrix}\mathbf{x_1}\\ \mathbf{x_2}\\ \vdots\\ \mathbf{x_n}\end{pmatrix} \quad (3.6.1.4)$$

$$\implies h\begin{pmatrix}\mathbf{x_1}, & \mathbf{x_2}, \cdots, \mathbf{x_n}\end{pmatrix} = 0 \quad (3.6.1.5)$$

Then $\mathbf{W}$ is in null-space of h. Hence by definition-1, the dimension of $\mathbf{W}$ is,

$$|W| = n - 1 \quad (3.6.1.6)$$

Now let,

$$a_j = \epsilon_1 - \epsilon_{i+1}, \text{ for } i = (1, \cdots, n-1) \quad (3.6.1.7)$$

Hence clearly $\{a_1, a_2, \cdots, a_{n-1}\}$ are linearly independent. Hence from (3.6.1.6) and above statement we can conclude that $\{a_1, a_2, \cdots, a_{n-1}\}$ are all in $\mathbf{W}$ so they must form basis for $\mathbf{W}$. Now, it is given that f is linear functional hence,

$$f\begin{pmatrix}\mathbf{x_1}, & \mathbf{x_2}, \cdots, \mathbf{x_n}\end{pmatrix} = \sum_{j=1}^{n} c_j \mathbf{x_j} \quad (3.6.1.8)$$

$$\implies f\begin{pmatrix}\mathbf{x_1}, & \mathbf{x_2}, \cdots, \mathbf{x_n}\end{pmatrix} = C^T\mathbf{X} \quad (3.6.1.9)$$

Where,

$$C = \begin{pmatrix}c_1\\ \vdots\\ c_n\end{pmatrix} \quad (3.6.1.10)$$

Now $f \in \mathbf{W}^0$ from definition-2 is given as,

$$f(a_1) = f(a_2) = \cdots = f(a_n) = 0 \text{ for every } a_j \in \mathbf{W} \quad (3.6.1.11)$$

$$\implies c_1 - c_i = 0 \forall i = 2 \cdots n \qquad (3.6.1.12)$$

$$\implies c_i = c_1 = c \forall i \qquad (3.6.1.13)$$

$$\implies C = c\begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix} \qquad (3.6.1.14)$$

Hence,

$$f\left(\mathbf{x_1}, \quad \mathbf{x_2}, \cdots, \mathbf{x_n}\right) = C^T\mathbf{X} \qquad (3.6.1.15)$$

$$\implies f\left(\mathbf{x_1}, \quad \mathbf{x_2} \quad, \cdots \quad, \mathbf{x_n}\right) = c\sum_{j=1}^{n}\mathbf{x_j} \qquad (3.6.1.16)$$

Hence, proved

b) Show that the dual space $\mathbf{W}^*$ of $\mathbf{W}$ can be 'naturally' identified with the linear functionals

$$f(x_1, \ldots, x_n) = c_1x_1 + \ldots c_nx_n$$

on $\mathbf{F}^n$ which satisfy $c_1 + \ldots + c_n = 0$

Solution: See Table 3.6.1.1
Pictorial Representation

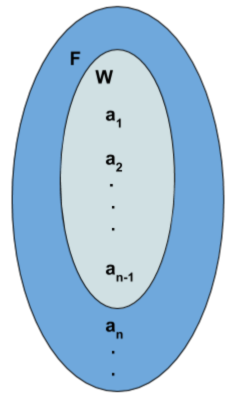

Fig. 3.6.1.1: $\mathbf{W}$ of dimension n-1, is the null space of $\mathbf{F}$, where $(a_1, \ldots, a_{n-1})$ are basis vectors for $\mathbf{W}$

PARAMETERS	MATRIX REPRESENTATION
Basis for $\mathbf{V}$	$\mathbf{B} = \begin{pmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_n\end{pmatrix}$
Basis for $\mathbf{V}^*$	$\mathbf{B}^* = \begin{pmatrix}\mathbf{f}_1 & \mathbf{f}_2 & \ldots & \mathbf{f}_n\end{pmatrix}$
$f_i(\mathbf{v}_j) = \delta_{ij}$	$(\mathbf{f}_i)^T\mathbf{v}_j = \delta_{ij}$
$\mathbf{B}, \mathbf{B}^*$ are dual basis	$(\mathbf{B}^*)^T\mathbf{B} = \mathbf{I}$
$\alpha_i = \sum_{k=1}^{n} a_k\mathbf{v}_k, i \in [1, m]$	$\alpha_i = \mathbf{Ba}$ where $\mathbf{a} = \begin{pmatrix}a_1\\a_2\\ \vdots \\a_n\end{pmatrix}, a_l \in \mathbb{F}, l \in [1, n]$
$\alpha_i \neq \mathbf{0}$	$\mathbf{a} \neq \mathbf{0}$
Any linear functional f over $\mathbf{V}$	$\mathbf{f} = \mathbf{B}^*\mathbf{c}$, where $\mathbf{c} = \begin{pmatrix}c_1\\c_2\\ \vdots \\c_n\end{pmatrix}, c_l \in \mathbb{F}, l \in [1, n]$
$f(\alpha_i)$	$\mathbf{f}^T\alpha_i$ $= (\mathbf{B}^*\mathbf{c})^T\alpha_i$ $= \mathbf{c}^T(\mathbf{B}^*)^T\mathbf{Ba}$ $= \mathbf{c}^T\mathbf{a}$
f with $c_l \neq 0 \forall l \in [1, n]$ lies in $\mathbf{V}^*, \mathbf{a} \neq \mathbf{0}$	$f(\alpha_i) = \mathbf{c}^T\mathbf{a} \neq 0, i = 1, 2, \ldots, m$

TABLE 3.5.13.2: Proof

Given	W is the space of $n \times n$ matrices over the field F f is a linear functional on W, that $f(AB) = f(BA)$
To prove	f is scalar multiple of trace $f(A) = c \times \text{trace}(A)$ and If $f(I) = n$, then f is the trace $f(A) = \text{trace}(A)$
Basis of W	Let $\{E^{pq}, 1 \le p, q \le n\}$ be the basis of W where E^{pq} is a $n \times n$ matrix $E^{pq} = \begin{cases} 1 & pq^{th}\text{-position} \\ 0 & \text{everywhere else} \end{cases}$

TABLE 3.5.15.1

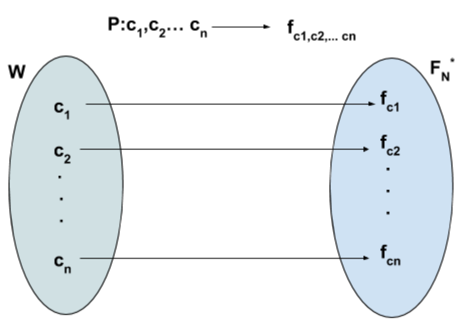

Fig. 3.6.1.2: Mapping from $\mathbf{W} \xrightarrow{P} \mathbf{F_N}^*$, where $P(c_1, \dots, c_n) = f_{c_1,\dots,c_n}$, $f_{c_1,\dots,c_n}(x_1, \dots, x_n) = c_1x_1 + \dots c_nx_n$

3.6.2. Use theorem 20 to prove the following.If $\mathbf{W}$ is a subspace of a finite dimensional vector space $\mathbf{V}$ and if $\{g_1, g_2, \dots, g_r\}$ is any basis for $\mathbf{W}^0$ then

$$\mathbf{W} = \bigcap_{i=1}^{r} \mathbf{N}_{g_i} \tag{3.6.2.1}$$

Solution:
Theorem 20: Let $g, f_1, f_2, \dots, f_r$ be linear functionals on vector space $\mathbf{V}$ with respective null spaces $\mathbf{N}, \mathbf{N}_1, \dots, \mathbf{N}_r$.

Find $f(E^{pq})$	If $p = q$, we know $I = \sum_{p,q=1}^{n} E^{pp}$ then
	$f(I) = \sum_{p,q=1}^{n} f(E^{pp}) \implies f(I) = nf(E^{pp})$
	$\implies f(E^{pp}) = \frac{f(I)}{n}$
	If $p \ne q$, then
	$f(E^{pq}) = f(E^{p1}E^{1q}) = f(E^{1q}E^{p1}) = \mathbf{0}$
	$\text{trace}(E^{pq}) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \ne j \end{cases}$
	$\therefore f(E^{pq}) = \frac{f(I)}{n}\text{trace}(E^{pq})$

Proof $(find\ \ f(A))$	Let $A \in W$, then $A = \sum_{p,q=1}^{n} c^{pq}E^{pq}$ Also, $\text{trace}(A) = \sum_{p,q=1}^{n} c^{pq}\text{trace}(E^{pq})$ Since f is linear, $f(A) = \sum_{p,q=1}^{n} c^{pq} f(E^{pq})$ $f(A) = \sum_{p,q=1}^{n} c^{pq}\left(\frac{f(I)}{n}\text{trace}(E^{pq})\right)$ $f(A) = \frac{f(I)}{n}\text{trace}(A)$ Hence f is a scalar multiple of trace. If $f(I) = n$, then $f(A) = \text{trace}(A)$ Hence f is the trace function

TABLE 3.5.15.2

Then g is a linear combination of $f_1, \dots, f_r$ if and only if $\mathbf{N}$ contains the intersection $\mathbf{N}_1 \cap \mathbf{N}_2 \cap \dots \cap \mathbf{N}_r$.
See Tables 3.6.2.1, 3.6.2.2 for the proof.

3.6.3. Let S be a set, F a field, and $V(S; F)$ the space of all functions from S into F:

$$(f + g)(x) = f(x) + g(x)$$
$$(cf)(x) = cf(x)$$

Let W be any n-dimensional subspace

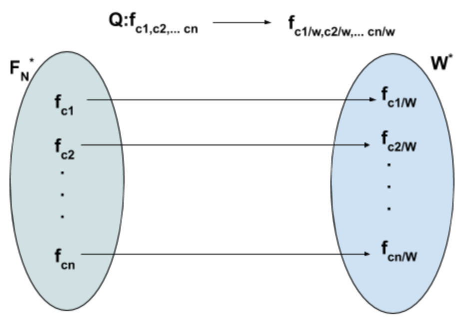

Fig. 3.6.1.3: Mapping from $\mathbf{F_N}^* \xrightarrow{Q} \mathbf{W}^*$, where $f_{c_1,\ldots,c_n}(x_1,\ldots,x_n) = c_1x_1 + \ldots c_nx_n$ is the linear functional with $c_1 + \ldots + c_n = 0$

of $V(S,F)$. Show that there exist points $x_1, x_2, \ldots, x_n$ in S and functions $f_1, \ldots, f_n$ in W such that $f_i(x_j) = \delta_{ij}$

Solution:

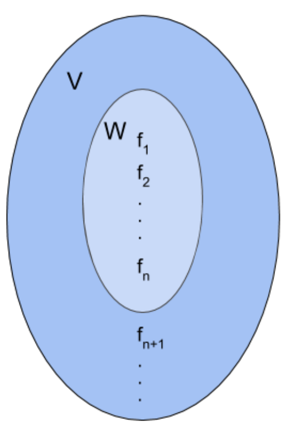

Fig. 3.6.3.1: Vector space of all function V and it's n-dimensional subspace W

Example:

Consider points $\{x_1, x_2\} \in S$, let

$$x_1 = 1 \tag{3.6.3.1}$$

$$x_2 = 0 \tag{3.6.3.2}$$

Also consider functions $\{f_1, f_2\} \in W$ where

$$f_1(x) = x \tag{3.6.3.3}$$

$$f_2(x) = 1 - x \tag{3.6.3.4}$$

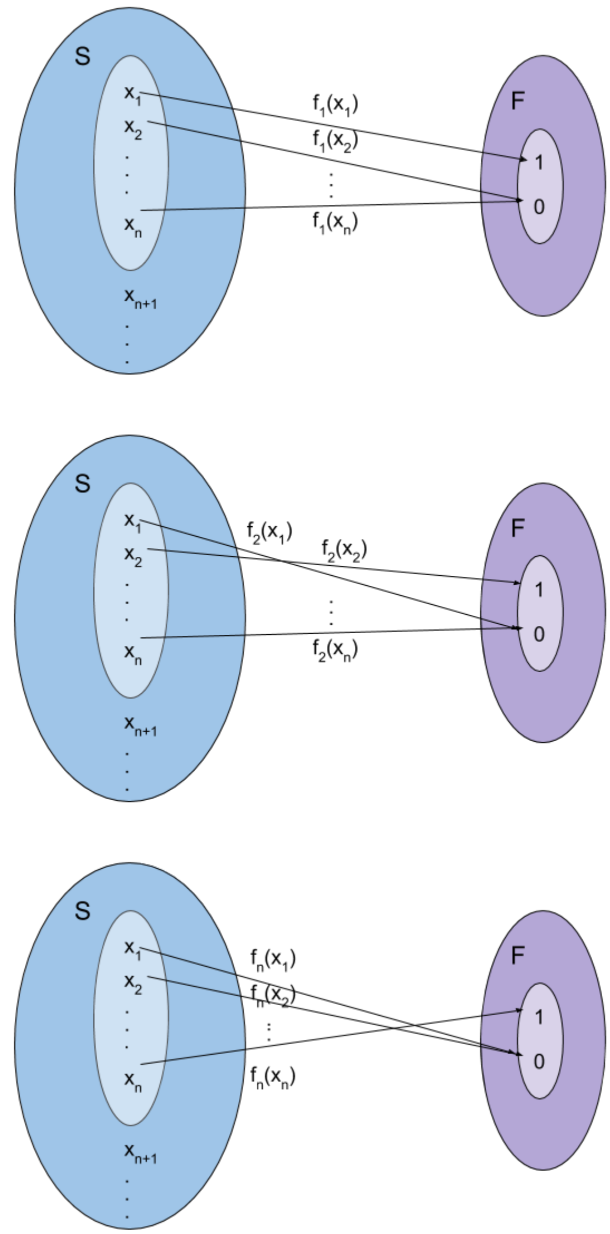

Fig. 3.6.3.2: Functions $f_1, f_2, \ldots, f_n$ where $f : S \to F$ and $f_i(x_j) = \delta_{ij}$ for $i, j = 1, \ldots, n$

Now, we have

$$f_1(x_1) = f_1(1) = 1 \tag{3.6.3.5}$$

$$f_1(x_2) = f_1(0) = 0 \tag{3.6.3.6}$$

Also,

$$f_2(x_1) = f_2(1) = 0 \tag{3.6.3.7}$$

$$f_2(x_2) = f_2(0) = 1 \tag{3.6.3.8}$$

Hence $f_i(x_j) = \delta_{ij}$.

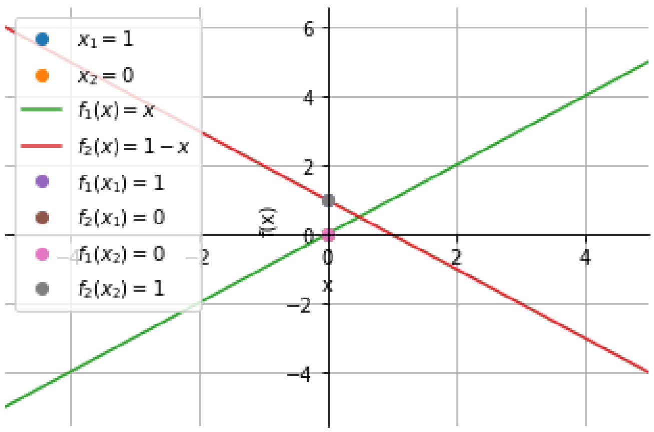

Fig. 3.6.3.3: Functions f_1, f_2 and points x_1, x_2

3.7 The Transpose of a Linear Transformation

3.7.1. Let $\mathbb{F}$ be a field and let f be the linear functional on $\mathbb{F}^2$ defined by,

$$f(x_1, x_2) = ax_1 + bx_2 \tag{3.7.1.1}$$

a) For the linear operator $T(x_1, x_2) = (x_1, 0)$ Let, $g = T^t y$ and find $g(x_1, x_2)$

Solution: The linear functional f on $\mathbb{F}^2$ is defined by,

$$f(x_1, x_2) = \mathbf{a}^{\mathbf{T}}\mathbf{x} \quad \forall (x_1, x_2) \in \mathbb{F}^2 \tag{3.7.1.2}$$

where,

$$\mathbf{a} = \begin{pmatrix} a \\ b \end{pmatrix} \tag{3.7.1.3}$$

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{3.7.1.4}$$

We use the following theorem,
Let $\mathbb{V}$ and $\mathbb{W}$ be vector spaces, over the field F. For each linear transformation $T : \mathbb{V} \to \mathbb{W}$, there is a unique linear transformation $T^t : \mathbb{W}^* \to \mathbb{V}^*$ such that,

$$(T^t g)(\alpha) = g(T\alpha) \tag{3.7.1.5}$$

The given linear operator T defined as,

$$T(x_1, x_2) = \mathbf{A}\mathbf{x} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \quad \forall (x_1, x_2) \in \mathbb{F}^2 \tag{3.7.1.6}$$

Where,

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{3.7.1.7}$$

$$\tag{3.7.1.8}$$

Consider the following mapping,

$$g = T^t f \tag{3.7.1.9}$$

Then, $\forall (x_1, x_2) \in \mathbb{F}^2$ we have,

$$g(x_1, x_2) = T^t f(x_1, x_2) \quad \text{[From (3.7.1.9)]} \tag{3.7.1.10}$$

$$= f(T(x_1, x_2)) \quad \text{[From (3.7.1.5)]} \tag{3.7.1.11}$$

$$= \mathbf{a}^{\mathbf{T}}\mathbf{A}\mathbf{x} \tag{3.7.1.12}$$

$$= \mathbf{a}^{\mathbf{T}} \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \quad \text{[From (3.7.1.6)]} \tag{3.7.1.13}$$

$$= ax_1 \quad \text{[From (3.7.1.2)]} \tag{3.7.1.14}$$

Hence, (3.7.1.14) is the required answer.

b) For given linear operator $\mathbf{T}$, such that

$$\mathbf{T}(x_1, x_2) = (-x_2, x_1) \tag{3.7.1.15}$$

Let

$$g = \mathbf{T}^t f \tag{3.7.1.16}$$

Then find $g(x_1, x_2)$

Solution:

The linear operator $\mathbf{T}$ can be represented as a matrix $\mathbf{A}$ as follows

$$\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{3.7.1.17}$$

$$\tag{3.7.1.18}$$

Let suppose,

$$\mathbf{X_1} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \mathbf{X_2} = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} \tag{3.7.1.19}$$

And,

$$\mathbf{U} = \begin{pmatrix} a & b \end{pmatrix} \tag{3.7.1.20}$$

$$\mathbf{T}(x_1, x_2) = \mathbf{AX_1} \quad (3.7.1.21)$$

$$\implies \mathbf{AX_1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{X_1} = \mathbf{X_2} \quad (3.7.1.22)$$

And (3.7.1.15) can be written as

$$f(x_1, x_2) = \mathbf{UX_1} \quad (3.7.1.23)$$

Now, we have given,

$$g = \mathbf{T}^t f \quad (3.7.1.24)$$

$$\implies g(x_1, x_2) = \mathbf{T}^t f(x_1, x_2) \quad (3.7.1.25)$$

We know that, if **V** and **W** be vector spaces over the field $\mathbb{F}$. For each linear transformation **T** from **V** into **W**, there is a unique linear transformation $\mathbf{T}^t$ from **W*** into **V*** such that,

$$(\mathbf{T}^t g)(\alpha) = g(\mathbf{T}\alpha) \quad (3.7.1.26)$$

Where for every g in **W*** and α in **V**.

Now using (3.7.1.26) in (3.7.1.25) we can write,

$$\mathbf{T}^t f(x_1, x_2) = f(\mathbf{T}(x_1, x_2)) \quad (3.7.1.27)$$

$$\implies f(\mathbf{T}(x_1, x_2)) = \mathbf{UAX_1} \quad (3.7.1.28)$$

$$f(\mathbf{T}(x_1, x_2)) = \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{UX_2} \quad (3.7.1.29)$$

$$\implies f(\mathbf{T}(x_1, x_2)) = -ax_2 + bx_1 \quad (3.7.1.30)$$

Hence,

$$f(\mathbf{T}(x_1, x_2)) = -ax_2 + bx_1 = g(x_1, x_2)$$
$$= \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} = \mathbf{UX_2} \quad (3.7.1.31)$$

c) For the linear operator $T(x_1, x_2) = (x_1 - x_2, x_1 + x_2)$
Let, $g = T^t y$ and find $g(x_1, x_2)$

Solution: Given, $\alpha = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$. Let,

$$\mathbf{A} = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \end{pmatrix} \quad (3.7.1.32)$$

$$\mathbf{AX} = \alpha \quad (3.7.1.33)$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & -2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad (3.7.1.34)$$

$\mathbf{X} = A^{-1}\alpha$ will give solution of the equation.

$$\begin{pmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 1 & -2 & 0 & 0 & 0 & 1 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 - R_1} \quad (3.7.1.35)$$

$$\begin{pmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & -2 & 1 & -1 & 0 & 1 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 + 2R_2} \quad (3.7.1.36)$$

$$\begin{pmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 2 & 1 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3/(-1)} \quad (3.7.1.37)$$

$$\begin{pmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & -1 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 + R_3} \quad (3.7.1.38)$$

$$\begin{pmatrix} 1 & 0 & 0 & 2 & -2 & -1 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & -1 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 + R_3} \quad (3.7.1.39)$$

$$\begin{pmatrix} 1 & 0 & 0 & 2 & -2 & -1 \\ 0 & 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 & -2 & -1 \end{pmatrix} \quad (3.7.1.40)$$

Thus,

$$A^{-1} = \begin{pmatrix} 2 & -2 & -1 \\ 1 & -1 & -1 \\ 1 & -2 & -1 \end{pmatrix} \quad (3.7.1.41)$$

$$\mathbf{X} = A^{-1}\alpha = \begin{pmatrix} 2 & -2 & -1 \\ 1 & -1 & -1 \\ 1 & -2 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad (3.7.1.42)$$

Given, f is a linear functional on R^3,

$$\alpha = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 \quad (3.7.1.43)$$

$$\implies f(\alpha) = \mathbf{X}^T \begin{pmatrix} f(\alpha_1) \\ f(\alpha_2) \\ f(\alpha_3) \end{pmatrix} \quad (3.7.1.44)$$

Given, $f(\alpha_1) = 0$, $f(\alpha_2) = 0$ and $f(\alpha_3) \neq 0$.

$$f(\alpha) = \mathbf{X}^T \begin{pmatrix} 0 \\ 0 \\ f(\alpha_3) \end{pmatrix} \quad (3.7.1.45)$$

$$\implies f(\alpha) = \begin{pmatrix} a \\ b \\ c \end{pmatrix}^T \begin{pmatrix} 2 & 1 & 1 \\ -2 & -1 & -2 \\ -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ f(\alpha_3) \end{pmatrix} \quad (3.7.1.46)$$

$$f(\alpha) = \begin{pmatrix} a \\ b \\ c \end{pmatrix}^T \begin{pmatrix} f(\alpha_3) \\ -2f(\alpha_3) \\ -f(\alpha_3) \end{pmatrix} \quad (3.7.1.47)$$

Hence,

$$f(\alpha) = f(\alpha_3)(a - 2b - c) \quad (3.7.1.48)$$

Given $\alpha = \begin{pmatrix} 2 & 3 & -1 \end{pmatrix}$ Substituting value of α in equation (3.7.1.48) we have,

$$f(\alpha) = -3f(\alpha_3) \neq 0 \quad (3.7.1.49)$$

Hence proved, $f(\alpha) \neq 0$

3.7.2. Let V be the vector space of all polynomial function over the field of real numbers. Let a and b be fixed real numbers and let f be the linear functional on V defined by

$$f(p) = \int_a^b p(x)\, dx \quad (3.7.2.1)$$

If D is the differentiation operator on V , what is $D^t f$?

Solution: Let

$$p(x) = c_0 + c_1 x + c_2 x^2 + \dots\dots\dots + c_n x^n \quad (3.7.2.2)$$

$$D^t f(p) = f(D(p)) \quad (3.7.2.3)$$

$$D^t f(p) = f\left(c_1 + 2c_2 x + 3c_3 x^2 + \dots\dots + nc_n x^{n-1}\right) \quad (3.7.2.4)$$

$$D^t f(p) = \int_a^b \left(c_1 + 2c_2 x + 3c_3 x^2 + \dots\dots + nc_n x^{n-1}\right) dx \quad (3.7.2.5)$$

$$D^t f(p) = c_1 x + c_2 x^2 + c_3 x^3 + \dots\dots + c_n x^n \Big|_a^b \quad (3.7.2.6)$$

$$D^t f(p) = p(b) - p(a) \quad (3.7.2.7)$$

3.7.3. Let, $\mathbf{V}$ be the space of all $n \times n$ matrices over a field $\mathbf{F}$ and let B be a fixed $n \times n$ matrix. If T is the linear operator on $\mathbf{V}$ defined by $\mathbf{T}(\mathbf{A}) = \mathbf{AB} - \mathbf{BA}$, and if f is the trace function, what is $\mathbf{T}^t f$?

Solution:

$$\mathbf{T}^t f(\mathbf{A}) = f(\mathbf{T}(\mathbf{A})) \quad (3.7.3.1)$$

$$= f(\mathbf{AB} - \mathbf{BA}) \quad (3.7.3.2)$$

$$= trace(\mathbf{AB} - \mathbf{BA}) \quad (3.7.3.3)$$

using $trace(\mathbf{AB}) = trace(\mathbf{BA})$ in (3.7.3.3),

$$\mathbf{T}^t f(\mathbf{A}) = trace(\mathbf{AB}) - trace(\mathbf{BA}) = 0 \quad (3.7.3.4)$$

Hence $\mathbf{T}^t f = 0$

3.7.4. Let $\mathbf{A}$ be an $m \times n$ matrix with real entries. Prove that $\mathbf{A} = \mathbf{0}$ is and only if $tr\left(\mathbf{A}^T\mathbf{A}\right) = 0$.

Solution: The proof is given in the table (3.7.4.2) and the properties used for the proof are listed in the table (3.7.4.1)

3.7.5. Let $\mathbf{V}$ be the vector space of all polynomial functions over the field of real numbers.Let a and b be fixed real numbers and let f be the linear functional on $\mathbf{V}$ defined by

$$f(p) = \int_a^b p(x)\, dx \quad (3.7.5.1)$$

If D is the differentiator operator on $\mathbf{V}$, what is $D^t f$?

Solution: See Tables 3.7.5.1 and 3.7.5.2

3.7.6. Let V be a finite-dimensional vector space over the field F. Show that $T \rightarrow T^t$ is an isomorphism of $L(V, V)$ onto $L(V^*, V^*)$.

Solution: See Tables 3.7.6.1 and 3.7.6.1

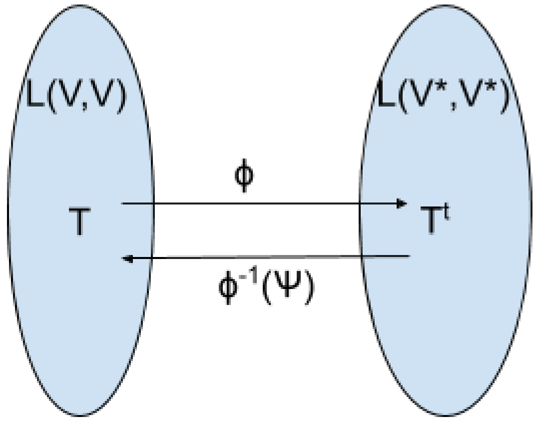

Fig. 3.7.6.1: $\phi : L(V, V) \to L(V^*, V^*)$

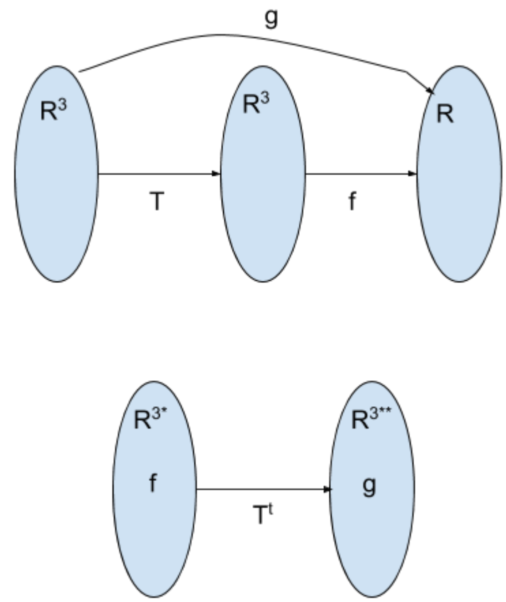

Fig. 3.7.6.2: $T : \mathbb{R}^3 \to \mathbb{R}^3$ and $T' : \mathbb{R}^{3^*} \to \mathbb{R}^{3^{**}}$

Example Consider, $T \in L(\mathbb{R}^3, \mathbb{R}^3)$

$$T = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \tag{3.7.6.1}$$

$$T' = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \tag{3.7.6.2}$$

If $T \in L(\mathbb{R}^3, \mathbb{R}^3)$ we can show that $T' \in L(\mathbb{R}^{3^*}, \mathbb{R}^{3^*})$ as follows.

Consider linear functional $f : \mathbb{R}^{3^*} \to F$

$$f(x, y, z) = 2x + 3y + 4z \tag{3.7.6.3}$$

Let $g : \mathbb{R}^3 \to F$. By definition of transpose,

$$g = T'f \tag{3.7.6.4}$$

$$= \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \\ 6 \end{pmatrix} \tag{3.7.6.5}$$

$$g(x, y, z) = 7x + 5y + 6z \tag{3.7.6.6}$$

Consider vector in $v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \in \mathbb{R}^3$,

$$Tv = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \\ 3 \end{pmatrix} \tag{3.7.6.7}$$

$$f(Tv) = f(7, 3, 3) \tag{3.7.6.8}$$

$$= 2 * 7 + 3 * 3 + 4 * 3 = 35 \tag{3.7.6.9}$$

$$g(1, 2, 3) = 7 * 1 + 5 * 2 + 6 * 3 = 35 \tag{3.7.6.10}$$

Hence verified $g = T'f$, and T' is transpose of T.

Now we will show that ϕ is invertible.

Given,

$$\phi T = T' \tag{3.7.6.11}$$
$$\phi = T'T^{-1} \tag{3.7.6.12}$$
$$= \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \tag{3.7.6.13}$$
$$= \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & -1 \\ -1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.7.6.14}$$
$$\phi = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.7.6.15}$$
$$\phi^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.7.6.16}$$

Since ϕ^{-1} exists, ϕ is isomorphism of $L(V, V)$ onto $L(V^*, V^*)$

3.7.7. Let V be the vector space of $n \times n$ matrices over the field F. If $\mathbf{B}$ is a fixed $n \times n$ matrix, define a function f_B on V by $f_B(\mathbf{A}) = tr(\mathbf{B}^t\mathbf{A})$.

a) Show that, f_B is a linear functional on V.
Solution:
See Table 3.7.7.1
Hence, it can be said that f_B is a linear functional on V.

b) Let $\mathbf{V}$ be the vector space of $n \times n$ matrices over the field $\mathbb{F}$. If $\mathbf{B}$ is a fixed $n \times n$ matrix, define a function f_B on $\mathbf{V}$ by $f_B(\mathbf{A}) = Tr(\mathbf{B}^T\mathbf{A})$. Show that every linear functional on $\mathbf{V}$ is of the form f_B for some $\mathbf{B}$.
Solution: For $\mathbf{A} = (a_{ij}) \in \mathbf{V}$, a linear functional $f_B : \mathbf{V} \longrightarrow \mathbb{F}$ is defined as:

$$f_B(\mathbf{A}) = \mathbf{A}^T\mathbf{c} \tag{3.7.7.1}$$

(3.7.7.1) can be written as,

$$f_B(\mathbf{A}) = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}a_{ij} \tag{3.7.7.2}$$

Now,

$$Tr(\mathbf{B}^t\mathbf{A}) = \sum_{i=1}^{n}(\mathbf{B}^t\mathbf{A})_{ii} \tag{3.7.7.3}$$
$$= \sum_{j=1}^{n}\sum_{i=1}^{n}(b_{ji})^T a_{ij} \tag{3.7.7.4}$$
$$= \sum_{j=1}^{n}\sum_{j=1}^{n} b_{ij}a_{ij} \tag{3.7.7.5}$$

Let $c_{ij} = b_{ij}$, from (3.7.7.2) and (3.7.7.5),

$$Tr(\mathbf{B}^T\mathbf{A}) = f_B \tag{3.7.7.6}$$

Hence, Proved.

4 Polynomials

4.1 The Algebra of Polynomials

4.1.1. Let F be a sub field of the complex numbers , and let A be the following 2 × 2 matrix over F.

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix} \tag{4.1.1.1}$$

Compute f($\mathbf{A}$) for the polynomial

a)

$$f(x) = x^2 - x + 2 \tag{4.1.1.2}$$

Solution: consider

$$\left|\mathbf{A} - \lambda\mathbf{I}\right| = 0 \tag{4.1.1.3}$$

$$\begin{pmatrix} 2-\lambda & 1 \\ -1 & 3-\lambda \end{pmatrix} = 0 \tag{4.1.1.4}$$

The characteristics equation of the matrix

$$\lambda^2 - 5\lambda + 7 = 0 \tag{4.1.1.5}$$

The characteristics equation will satisfy its own matrix

$$\mathbf{A}^2 - 5\mathbf{A} + 7 = 0 \tag{4.1.1.6}$$
$$\mathbf{A}^2 = 5\mathbf{A} - 7\mathbf{I} \tag{4.1.1.7}$$

The given polynomial

$$f(x) = x^2 - x + 2 \tag{4.1.1.8}$$
$$f(\mathbf{A}) = \mathbf{A}^2 - \mathbf{A} + 2\mathbf{I} \tag{4.1.1.9}$$

substituting (4.1.1.7) in (4.1.1.9) we get

$$f(\mathbf{A}) = 5\mathbf{A} - 7\mathbf{I} - \mathbf{A} + 2\mathbf{I} \tag{4.1.1.10}$$

$$f(\mathbf{A}) = 4\mathbf{A} - 5\mathbf{I} \tag{4.1.1.11}$$

$$f(\mathbf{A}) = 4\begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix} - 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4.1.1.12}$$

$$f(\mathbf{A}) = \begin{pmatrix} 3 & 4 \\ -4 & 7 \end{pmatrix} \tag{4.1.1.13}$$

b)

$$f = x^3 - 1 \tag{4.1.1.14}$$

Solution: We first find the eigen values of the **A**. We get the characteristic equation of **A** as follows,

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0 \tag{4.1.1.15}$$

$$\implies \lambda^2 - 5\lambda + 7 = 0 \tag{4.1.1.16}$$

Where **I** is 2×2 Identity matrix. Now using Cayley Hamilton Theorem we get from (4.1.1.16) the following,

$$\mathbf{A}^2 - 5\mathbf{A} + 7\mathbf{I} = 0 \tag{4.1.1.17}$$

Hence the polynomial $f(\mathbf{A})$ can be written using the characteristic function of **A** as follows,

$$f(\mathbf{A}) = \mathbf{A}^3 - \mathbf{I} \tag{4.1.1.18}$$

$$= (\mathbf{A} - \mathbf{I})(\mathbf{A}^2 + \mathbf{A} + \mathbf{I}) \tag{4.1.1.19}$$

$$= (\mathbf{A} - \mathbf{I})(\mathbf{A}^2 - 5\mathbf{A} + 7\mathbf{I} + 6\mathbf{A} - 6\mathbf{I}) \tag{4.1.1.20}$$

$$= 6(\mathbf{A} - \mathbf{I})^2 \quad \text{[From (4.1.1.17)]} \tag{4.1.1.21}$$

$$= 6\left[\begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]^2 \tag{4.1.1.22}$$

$$= \begin{pmatrix} 0 & 18 \\ -18 & 18 \end{pmatrix} \tag{4.1.1.23}$$

Here, (4.1.1.23) is the required answer.

c) $f(x) = x^2 - 5x + 7$

Solution: Characteristic equation of **A** can be written as:

$$\left|\mathbf{A} - \lambda\mathbf{I}\right| = 0 \tag{4.1.1.24}$$

$$\begin{pmatrix} 2-\lambda & 1 \\ -1 & 3-\lambda \end{pmatrix} = 0 \tag{4.1.1.25}$$

The characteristics equation of the matrix will be,

$$\lambda^2 - 5\lambda + 7 = 0 \tag{4.1.1.26}$$

The characteristics equation will satisfy its own matrix

$$\mathbf{A}^2 - 5\mathbf{A} + 7\mathbf{I} = \mathbf{0} \tag{4.1.1.27}$$

The given polynomial

$$f(x) = x^2 - 5x + 7 \tag{4.1.1.28}$$

$$f(\mathbf{A}) = \mathbf{A}^2 - 5\mathbf{A} + 7\mathbf{I} \tag{4.1.1.29}$$

substituting (4.1.1.27) in (4.1.1.29) we get

$$f(\mathbf{A}) = \mathbf{0} \tag{4.1.1.30}$$

4.1.2. Let T be the linear operator on $\mathbb{R}^3$ defined by

$$T(x_1, x_2, x_3) = (x_1, x_3, -2x_2 - x_3) \tag{4.1.2.1}$$

Let f be the polynomial over $\mathbb{R}$ defined by f= $-x^3 + 2$

Solution: The given transformation can be written as,

$$T(\mathbf{x}) = \mathbf{A}\mathbf{x} \tag{4.1.2.2}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -1 \end{pmatrix}\mathbf{x} \tag{4.1.2.3}$$

Hence,

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -1 \end{pmatrix} \tag{4.1.2.4}$$

Now the characteristic equation of **A** is given by,

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0 \tag{4.1.2.5}$$

$$= \begin{pmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ 0 & -2 & -1-\lambda \end{pmatrix} \tag{4.1.2.6}$$

$$\implies (1-\lambda)(\lambda^2 + \lambda + 2) = 0 \tag{4.1.2.7}$$

Now, simplifying the above equation,

$$(1-\lambda)(\lambda^2+\lambda+2)=0 \tag{4.1.2.8}$$
$$\lambda^2+\lambda+2-\lambda^3-\lambda^2-2\lambda=0 \tag{4.1.2.9}$$
$$\lambda^3=2-\lambda \tag{4.1.2.10}$$

Now using Cayley Hamilton Theorem we get,

$$\mathbf{A}^3=2\mathbf{I}-\mathbf{A} \tag{4.1.2.11}$$

Hence the polynomial $f(\mathbf{A})$ can be written using the characteristic function of $\mathbf{A}$ as follows,

$$f(\mathbf{A})=-\mathbf{A}^3+2\mathbf{I} \tag{4.1.2.12}$$
$$=2\mathbf{I}-\mathbf{A}+2\mathbf{I} \tag{4.1.2.13}$$
$$=\mathbf{A} \tag{4.1.2.14}$$

Hence,

$$f(T)(\mathbf{x})=\mathbf{A}\mathbf{x} \tag{4.1.2.15}$$
$$=\begin{pmatrix}1&0&0\\0&0&1\\0&-2&-1\end{pmatrix}\mathbf{x} \tag{4.1.2.16}$$

4.1.3. Let $\mathbf{A}$ be an $n\times n$ diagonal matrix over the field $\mathbf{F}$, i.e., a matrix satisfying $\mathbf{A}_{ij}=0$ for $i\neq j$. Let f be the polynomial over $\mathbf{F}$ defined by $f=(x-A_{11})...(x-A_{nn})$. What is the matrix $f(\mathbf{A})$?
Solution: The given transformation can be written as,

$$T(\mathbf{x})=\mathbf{A}\mathbf{x} \tag{4.1.3.1}$$
$$=\begin{pmatrix}1&0&0\\0&0&1\\0&-2&-1\end{pmatrix}\mathbf{x} \tag{4.1.3.2}$$

Hence,

$$\mathbf{A}=\begin{pmatrix}1&0&0\\0&0&1\\0&-2&-1\end{pmatrix} \tag{4.1.3.3}$$

Now the characteristic equation of $\mathbf{A}$ is given by,

$$\det(\mathbf{A}-\lambda\mathbf{I})=0 \tag{4.1.3.4}$$
$$=\begin{pmatrix}1-\lambda&0&0\\0&-\lambda&1\\0&-2&-1-\lambda\end{pmatrix} \tag{4.1.3.5}$$
$$\implies (1-\lambda)(\lambda^2+\lambda+2)=0 \tag{4.1.3.6}$$

Now, simplifying the above equation,

$$(1-\lambda)(\lambda^2+\lambda+2)=0 \tag{4.1.3.7}$$
$$\lambda^2+\lambda+2-\lambda^3-\lambda^2-2\lambda=0 \tag{4.1.3.8}$$
$$\lambda^3=2-\lambda \tag{4.1.3.9}$$

Now using Cayley Hamilton Theorem we get,

$$\mathbf{A}^3=2\mathbf{I}-\mathbf{A} \tag{4.1.3.10}$$

Hence the polynomial $f(\mathbf{A})$ can be written using the characteristic function of $\mathbf{A}$ as follows,

$$f(\mathbf{A})=-\mathbf{A}^3+2\mathbf{I} \tag{4.1.3.11}$$
$$=2\mathbf{I}-\mathbf{A}+2\mathbf{I} \tag{4.1.3.12}$$
$$=\mathbf{A} \tag{4.1.3.13}$$

Hence,

$$f(T)(\mathbf{x})=\mathbf{A}\mathbf{x} \tag{4.1.3.14}$$
$$=\begin{pmatrix}1&0&0\\0&0&1\\0&-2&-1\end{pmatrix}\mathbf{x} \tag{4.1.3.15}$$

4.1.4. If f and g are independent polynomials over a field **F**and h is a non-zero polynomial over **F**, show that fh and gh are independent.
Solution: Polynomials $(f_1, f_2, \ldots, f_m) \in k(x_1,\ldots,x_n)$are called algebraically independent over a field **F**, if there is no nonzero m-variate polynomial A $\in k[y_1,\ldots,y_m]$ such that $A(f_1,\ldots,f_m)=0$
Example: The smallest degree independent polynomials are: 1+x and 1-x.

$$a(1+x)+b(1-x)=0 \tag{4.1.4.1}$$

Simplifying,

$$a+b=0 \tag{4.1.4.2}$$
$$a-b=0 \tag{4.1.4.3}$$

solving,we get a=0 and b=0.So, polynomials are linearly independent. Given f and g are independent polynomials over a field **F**. Consider scalars a and b $\in$ **F** Hence,

$$af+bg=0 \tag{4.1.4.4}$$

Since f and g are independent Hence f and g $\neq 0$

$$\implies a,b=0. \tag{4.1.4.5}$$

Given h a non zero polynomial over **F**. Substituting in equation (4.1.4.4) we have,

$$a(fh) + b(gh) = 0 \tag{4.1.4.6}$$
$$(af)h + (bg)h = 0 \tag{4.1.4.7}$$
$$(af + bg)h = 0 \tag{4.1.4.8}$$
$$af + bg = 0 \tag{4.1.4.9}$$

f and g are independent polynomial.Also from equation (4.1.4.5) a=0 and b=0.
Hence proved fh and gh are independent.

4.1.5. Let **S** be a set of non-zero polynomials over a field F. If no two elements of **S** have the same degree, show that **S** is an independent set in **F**[**x**].
Solution: See Tables 4.1.5.1, 4.1.5.2 and 4.1.5.3
Hence, it is proved that **S** is an independent set in **F**[**x**].

4.1.6. If a and b are element of a filed $\mathbb{F}$ and $a \neq 0$, show that the ploynomial $1, ax + b, (ax+b)^2, (ax+b)^3, \dots$ form a basis of $\mathbb{F}[x]$.
Solution:
Let consider we have a set S such that,

$$S = \left\{1, ax + b, (ax+b)^2, (ax+b)^3, \dots\right\} \tag{4.1.6.1}$$

And let $\langle S \rangle$ be the subspace, that is spanned by S.

Since

$$1 \in S \tag{4.1.6.2}$$

and

$$ax + b \in S, \tag{4.1.6.3}$$
$$\implies b.1 + \frac{1}{a}(b + ax) \in \langle S \rangle \tag{4.1.6.4}$$

and hence, it follows

$$\implies x \in \langle S \rangle \tag{4.1.6.5}$$

Now to prove

$$x^2 \in \langle S \rangle \tag{4.1.6.6}$$

let consider another element form S which is

$$(ax+b)^2 \tag{4.1.6.7}$$

Subtracting $1.a^2 + 2.a.b.x$ from $(ax+b)^2$

$$\implies (ax+b)^2 - a^2 - 2.a.b.x = a^2.x^2 \tag{4.1.6.8}$$
$$\implies a^2.x^2 \in \langle S \rangle \tag{4.1.6.9}$$
$$\implies \frac{1}{a^2}.a^2.x^2 \in S. \tag{4.1.6.10}$$
$$\implies x^2 \in \langle S \rangle. \tag{4.1.6.11}$$

Now, Thus Hence using this concept with higher degree we can prove that,

$$x^n \in \langle S \rangle, \forall n \tag{4.1.6.12}$$

Consider,

$$S' = \left\{1, x, x^2, x^3, \dots\right\} \tag{4.1.6.13}$$

Hence we can say that, (4.1.6.13) span the space of all polynomials which form with the help of

$$(ax+b)^n \tag{4.1.6.14}$$

Hence we conclude that S spans the space of all polynomials. We can summarize our procedure step by step using Table 4.1.6.1.

Given	$x_1 + \ldots + x_n = 0$ $(x_1, \ldots, x_n) \in \mathbf{W}$ F is a field $\mathbf{W}^*$ is dual space of $\mathbf{W}$
To prove	$\mathbf{W} \to \mathbf{W}^*$ is a natural isomorphism $f(x_1, \ldots, x_n) = c_1x_1 + \ldots c_nx_n$ which satisfy $c_1 + \ldots + c_n = 0$
Proof	Let $\alpha_i = \epsilon_1 - \epsilon_{i+1}$ $i \in \{1, \ldots, n-1\}$ $\sum_{i=1}^{n-1} c_i\alpha_i = 0$ $\implies \left(\sum_{i=1}^{n-1} c_i\right)\epsilon_1 - \sum_{i=1}^{n-1} c_i\epsilon_{i+1} = 0$ $(\alpha_1, \ldots, \alpha_{n-1})$ are linearly independent and form a basis for $\mathbf{W}$ $\mathbf{W} \xrightarrow{P} (\mathbf{F}^n)^* \xrightarrow{Q} \mathbf{W}^*$ The function P is defined as $P(c_1, \ldots, c_n) = f_{c_1,\ldots,c_n}$; where, $f_{c_1,\ldots,c_n}(x_1, \ldots, x_n) = c_1x_1 + \ldots c_nx_n$ Let $Q \circ P(c_1, \ldots, c_n) = 0$; $(c_1, \ldots, c_n) \in \mathbf{W}$ $Q(f_{c_1,\ldots,c_n}) = 0 \implies f_{c_1,\ldots,c_n\|W} = 0$ $\implies f_{c_1,\ldots,c_n}(x_1, \ldots, x_n) = 0$ $f_{c_1,\ldots,c_n}(\alpha_i) = 0; \quad i = 1, \ldots, n-1$ $\implies c_1 = c_i; \quad i = 2, \ldots, n$ $\implies \sum_{i=2}^{n} c_i = (n-1)c_1$ since $(c_1, \ldots, c_n) \in \mathbf{W}$ $\sum_{i=1}^{n} c_i = 0$ $\implies c_1 = 0$ $\implies c_i = 0\ ; \quad i = 1, \ldots, n$ Hence, $f_{c_1,\ldots,c_n}$ is a zero function. Thus the mapping $\mathbf{W} \to \mathbf{W}^*$ is a natural isomorphism

TABLE 3.6.1.1

PARAMETERS	DESCRIPTION
$\mathbf{V}$	Finite dimensional vector space with $dim(\mathbf{V}) = n$
$\mathbf{W}$	Subspace of $\mathbf{V}$
$\mathbf{W}^0$	Annihilator of $\mathbf{W}$
$g \in \mathbf{W}^0$	$g(\mathbf{w}) = 0 \forall \mathbf{w} \in \mathbf{W}$
$\{g_1, g_2, \ldots, g_r\}$	basis for $\mathbf{W}^0$
$\mathbf{N}_{g_i}$	Null space of g_i

TABLE 3.6.2.1: Input Parameters

STATEMENT	DESCRIPTION
$g \in \mathbf{W}^0$	$$g = \sum_{i=1}^{r} c_i g_i \qquad (3.6.2.2)$$
If $\mathbf{N}$ is nullspace of g	From the theorem, $$\bigcap_{i=1}^{r} \mathbf{N}_{g_i} \subseteq \mathbf{N} \qquad (3.6.2.3)$$
$\forall \mathbf{w} \in \mathbf{W}$	$$g(\mathbf{w}) = 0 \qquad (3.6.2.4)$$
$(3.6.2.3), (3.6.2.4) \implies$	$$\mathbf{W} \subseteq \bigcap_{i=1}^{r} \mathbf{N}_{g_i} \qquad (3.6.2.5)$$
Starting with method of contradiction	$$\mathbf{W} \neq \bigcap_{i=1}^{r} \mathbf{N}_{g_i} \qquad (3.6.2.6)$$
From (3.6.2.5),(3.6.2.6) there exists a vector $\mathbf{e}$ such that	$$\mathbf{e} \in \bigcap_{i=1}^{r} \mathbf{N}_{g_i}, \mathbf{e} \notin \mathbf{W} \qquad (3.6.2.7)$$ $$\text{So } \forall g \in \mathbf{W}^0, g(\mathbf{e}) = 0 \qquad (3.6.2.8)$$
Since g is a linear functional on $\mathbf{V}$	$$(3.6.2.8) \implies \mathbf{e} \in \mathbf{V} \qquad (3.6.2.9)$$
f	A linear functional on $\mathbf{V}$ defined as $$f(\alpha) \begin{cases} = 0 & \text{for } \alpha \in \mathbf{W} \\ \neq 0 & \text{for } \alpha \in \mathbf{V}, \alpha \notin \mathbf{W} \end{cases} \qquad (3.6.2.10)$$ $$\therefore f \in \mathbf{W}^0 \text{ and } f(e) \neq 0 \qquad (3.6.2.11)$$
(3.6.2.8),(3.6.2.11) contradict each other	$$\mathbf{W} = \bigcap_{i=1}^{r} \mathbf{N}_{g_i} \qquad (3.6.2.12)$$

TABLE 3.6.2.2: Proof

Given	S is a set F is a field $V(S,F)$ is a linear functional such that W be n-dim subspace of $V(S,F)$. Also, $\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$
To prove	$f_i(x_j) = \delta_{ij}$ where $x_1, x_2, \ldots, x_n \in S$ and $f_1, f_2, \ldots, f_n \in W$
Proof	Let $\phi_x : W \to F$ Suppose $\phi_x(f) = 0\ \ \forall x \in S\ \ \&\ \ f \in W$ $\implies f(x) = 0$ If $\forall x,\ \ \phi_x(f) \neq 0$ for some $f \in W$ If $n > 0\ \exists \in S$ such that $\phi_x(f) \neq 0$ for some $f \in W$ $\implies f_1(x_1) \neq 0$ By scaling we can have $f_1(x_1) = 1$ Hence $f_i(x_j) = \delta_{ij}$

Properties Used	
SVD of matrix $\mathbf{A}$	$\mathbf{A} = \mathbf{U\Sigma V}^T$
$\mathbf{U}$ is unitary	$\mathbf{U}^T\mathbf{U} = \mathbf{I}$
$\mathbf{V}$ is unitary	$\mathbf{V}^T\mathbf{V} = \mathbf{I}$
$\mathbf{\Sigma}$ is diagonal	$\mathbf{\Sigma}^T\mathbf{\Sigma} = \mathbf{\Sigma}^2$
Cyclic property	$tr\left(\mathbf{ABC}\right) = tr\left(\mathbf{CAB}\right)$
$Rank\left(\mathbf{A}\right)$	$Rank\left(\mathbf{A}\right) =$ #non-zero singular values

TABLE 3.7.4.1: Properties

Statement	Proof
$tr\left(\mathbf{A}^T\mathbf{A}\right) = 0 \implies \mathbf{A} = \mathbf{0}$	$$tr\left(\mathbf{A}^T\mathbf{A}\right) = tr\left(\mathbf{V\Sigma}^T\mathbf{U}^T\mathbf{U\Sigma V}^T\right) \quad (3.7.4.1)$$ $$= tr\left(\mathbf{V\Sigma}^T\mathbf{\Sigma V}^T\right) \quad (3.7.4.2)$$ $$= tr\left(\mathbf{V\Sigma}^2\mathbf{V}^T\right) \quad (3.7.4.3)$$ $$= tr\left(\mathbf{V}^T\mathbf{V\Sigma}^2\right) \quad (3.7.4.4)$$ $$= tr\left(\mathbf{\Sigma}^2\right) \quad (3.7.4.5)$$ $$= \sum_{i=1}^{r} \sigma_i^2 \quad (3.7.4.6)$$ $$tr\left(\mathbf{A}^T\mathbf{A}\right) = 0 \quad (3.7.4.7)$$ $$\implies \sum_{i=1}^{r} \sigma_i^2 = 0 \ \forall i = 1, 2, \dots r \quad (3.7.4.8)$$ $$\implies \sigma_i = 0 \ \forall i = 1, 2, \dots r \quad (3.7.4.9)$$ $$\therefore Rank\left(\mathbf{A}\right) = \text{\#non-zero singular values} = 0 \quad (3.7.4.10)$$ $$\implies \mathbf{A} = \mathbf{0} \quad (3.7.4.11)$$
$\mathbf{A} = \mathbf{0} \implies tr\left(\mathbf{A}^T\mathbf{A}\right) = 0$	$$\mathbf{A} = \mathbf{0} \quad (3.7.4.12)$$ $$\implies \mathbf{A}^T\mathbf{A} = \mathbf{0} \quad (3.7.4.13)$$ $$\implies tr\left(\mathbf{A}^T\mathbf{A}\right) = 0 \quad (3.7.4.14)$$

TABLE 3.7.4.2: Proofs

PARAMETERS	**DESCRIPTION**
$\mathbb{R}$	Field of real numbers
$\mathbf{V}$	Vector space of all polynomials over $\mathbb{R}$
a, b	Fixed real numbers
f	Linear functional on $\mathbf{V}$
D	Differentiator operator on $\mathbf{V}$
D^t	Transpose of D

TABLE 3.7.5.1: Input Parameters

STATEMENTS	**DERIVATIONS**
D^t is transpose of D	$(D^t f)(p) = f[D(p)] \quad (3.7.5.2)$
A polynomial of degree n $p(x) = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n$	$p = \mathbf{c}^T \mathbf{x} \quad (3.7.5.3)$ $\mathbf{c} = \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}, \mathbf{x} = \begin{pmatrix} 1 \\ x \\ x^2 \\ \vdots \\ x^n \end{pmatrix} \quad (3.7.5.4)$
$f(p) = \int_a^b p(x)\, dx$	$f(p) = \mathbf{c}^T \mathbf{F} \quad (3.7.5.5)$ $\mathbf{F} = \begin{pmatrix} b-a \\ \frac{b^2-a^2}{2} \\ \frac{b^3-a^3}{3} \\ \vdots \\ \frac{b^{n+1}-a^{n+1}}{n+1} \end{pmatrix} \quad (3.7.5.6)$
$D(p) = c_1 + 2c_2 x + 3c_3 x^2 + \ldots + nc_n x^{n-1}$	$D(p) = \mathbf{c}^T \mathbf{D} \mathbf{x} \quad (3.7.5.7)$ $\mathbf{D} = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 2 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 3 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & n & 0 \end{pmatrix} \quad (3.7.5.8)$
$f[D(p)] = c_1(b-a) + c_2(b^2 - a^2) + \ldots + c_n(b^n - a^n)$	$\text{Let } D(p) = p' \quad (3.7.5.9)$ $\implies p' = \mathbf{c}'^T \mathbf{x} \quad (3.7.5.10)$ $\text{where } \mathbf{c}'^T = \begin{pmatrix} c_1 \\ 2c_2 \\ 3c_3 \\ \vdots \\ nc_n \\ 0 \end{pmatrix} \quad (3.7.5.11)$ $f[D(p)] = \mathbf{c}'^T \mathbf{F} \quad (3.7.5.12)$
$(D^t f)(p) = p(b) - p(a)$	$(D^t f)(p) = c_1(b-a) + c_2(b^2 - a^2) + \quad (3.7.5.13)$ $\ldots + c_n(b^n - a^n) + c_0 - c_0 \quad (3.7.5.14)$ $= (c_0 + c_1 b + c_2 b^2 + \ldots + c_n b^n) - \quad (3.7.5.15)$ $(c_0 + c_1 a + c_2 a^2 + \ldots + c_n a^n) \quad (3.7.5.16)$ $= p(b) - p(a) \quad (3.7.5.17)$

TABLE 3.7.5.2: Proof

iso-morphism	A linear transformation $T : V \to W$ is *isormorphism of V onto W* if T is $one - one$
one-one	A linear transfrmation $T : \mathbb{R}^n \to \mathbb{R}^m$ is *one-one* if $\forall \mathbf{b} \in \mathbb{R}^m$, $\mathbf{AX} = \mathbf{b}$ has at most one solution in $\mathbb{R}^n$ By definition, all *invertible* linear transformations are *one-one*
invertible	A linear transformation $T : V \to W$ is *invertible* if there exists another linear transformation $U : W \to V$ such that UT is identity transformation on V and TU is identity transformation on W

TABLE 3.7.6.1: Definitions

Given	$\phi(T) = T'$ where $\phi : L(V, V) \to L(V^*, V^*)$ $T \in L(V, V)$ and $T' \in L(V^*, V^*)$
To prove	ϕ is an *isomorphism* of $L(V, V)$ *onto* $L(V^*, V^*)$ *i.e* ϕ is $one - one \implies \phi$ is *invertible*
Proof	Consider a linear transformation $\psi : L(V^*, V^*) \to L((V^*)^*, (V^*)^*)$ We know, $L((V^*)^*, (V^*)^*) = L(V, V)$ Hence $\psi : L(V^*, V^*) \to L(V, V)$ We have $\phi \circ \psi = I$ and $\psi \circ \phi = I$ $\therefore$ ϕ is invertible with ψ its inverse. Hence ϕ is *isomorphic*.

TABLE 3.7.6.2: Solution

Given	V is a vector space of $n \times n$ matrices over the field F $\mathbf{B}$ is a fixed $n \times n$ matrix A function f_B on V is defined such that $f_B(\mathbf{A}) = tr(\mathbf{B}^t\mathbf{A})$
To prove	f_B is a linear functional on V
Properties of Trace of Matrix	1.$tr(\mathbf{M}+\mathbf{N}) = tr(\mathbf{M}) + tr(\mathbf{N})$ 2. $tr(c\mathbf{M}) = c(tr(\mathbf{M}))$ Where $\mathbf{M}$ and $\mathbf{N}$ are two square matrices
Proof	As $f_B(\mathbf{A}) = tr(\mathbf{B}^t\mathbf{A})$ $\Longrightarrow f_B(c\mathbf{A_1}+\mathbf{A_2})$ $= tr(\mathbf{B}^t(c\mathbf{A_1}+\mathbf{A_2}))$ $\Longrightarrow f_B(c\mathbf{A_1}+\mathbf{A_2}) = tr(c\mathbf{B}^t\mathbf{A_1})$ $+tr(\mathbf{B}^t\mathbf{A_2})$ $\Longrightarrow f_B(c\mathbf{A_1}+\mathbf{A_2}) = c(tr(\mathbf{B}^t\mathbf{A_1}))$ $+tr(\mathbf{B}^t\mathbf{A_2})$ $\Longrightarrow f_B(c\mathbf{A_1}+\mathbf{A_2})$ $= cf_B(\mathbf{A_1}) + f_B(\mathbf{A_2})$

TABLE 3.7.7.1

Given	$\mathbf{S}$ be a set of non-zero polynomial over a field F No two elements of $\mathbf{S}$ have the same degree
To prove	$\mathbf{S}$ is an independent set in $\mathbf{F}[\mathbf{x}]$
Linear Independency	Let $f_1, f_2, \ldots, f_n$ are the polynomials and they will be linearly independent if $a_1f_1 + a_2f_2 + \ldots + a_nf_n = \theta$ for $a_1 = a_2 = \ldots. = a_n = 0$ where $a_1, a_2, \ldots, a_n$ are scalars from field F

TABLE 4.1.5.1

Proof	Let the degrees of f_1,f_2,...,f_n are d_1,d_2,...,d_n respectively such that the degree of $f_i = d_i \neq d_j$ for j=1,2,....,n and $i \neq j$ and $d_1 < d_2 < .... < d_n$ so d_n is the largest degree
	Now, let $a_1 f_1 + a_2 f_2 + ... + a_n f_n = \theta$ where $f_1 = \sum_{i=0}^{d_1} k_{1i} x^i$ $f_2 = \sum_{i=0}^{d_2} k_{2i} x^i$ **or,**$f_2 = \sum_{i=0}^{d_1} k_{2i} x^i + \sum_{i=d_1+1}^{d_2} k_{2i} x^i$ Similarly, $f_{n-1} = \sum_{i=0}^{d_{n-1}} k_{(n-1)i} x^i$ $\implies f_{n-1} = \sum_{i=0}^{d_{n-2}} k_{(n-1)i} x^i + \sum_{i=d_{n-2}+1}^{d_{n-1}} k_{(n-1)i} x^i$ and $f_n = \sum_{i=0}^{d_{n-1}} k_{ni} x^i + \sum_{i=d_{n-1}+1}^{d_n} k_{ni} x^i$
	Now, $a_1 f_1 + a_2 f_2 + ... + a_n f_n$ $= a_1 \sum_{i=0}^{d_1} k_{1i} x^i + a_2(\sum_{i=0}^{d_1} k_{2i} x^i + \sum_{i=d_1+1}^{d_2} k_{2i} x^i)$ $+... + a_n(\sum_{i=0}^{d_1} k_{ni} x^i + \sum_{i=d_1+1}^{d_2} k_{ni} x^i + ..$ $.. + \sum_{i=d_{n-1}+1}^{d_n} k_{ni} x^i)$ $= \sum_{i=0}^{d_1}(a_1 k_{1i} + a_2 k_{2i} + .. + a_n k_{ni}) x^i$ $+ \sum_{i=d_1+1}^{d_2}(a_2 k_{2i} + .. + a_n k_{ni}) x^i$ $+.. + \sum_{i=d_{n-1}+1}^{d_n} a_n k_{ni} x^i$ Now, as $a_1 f_1 + a_2 f_2 + ... + a_n f_n = \theta$ for $d_{n-1} + 1 \leq i \leq d_n$, $k_{ni} \neq 0$ so a_n must be 0
	Now, discarding a_n associated term, we get $\sum_{i=0}^{d_1}(a_1 k_{1i} + a_2 k_{2i} + .. + a_n k_{ni}) x^i$ $+ \sum_{i=d_1+1}^{d_2}(a_2 k_{2i} + .. + a_n k_{ni}) x^i$ $+.. + \sum_{i=d_{n-2}+1}^{d_{n-1}} a_{n-1} k_{(n-1)i} x^i = 0$ so, for $d_{n-2} + 1 \leq i \leq d_{n-1}$, $k_{(n-1)i} \neq 0$ $\implies a_{n-1} = 0$

TABLE 4.1.5.2

Proof	Similarly, for $d_1 + 1 \leq i \leq d_2$ $\sum_{i=0}^{d_1}(a_1 k_{1i} + a_2 k_{2i}) x^i + \sum_{i=d_1+1}^{d_2} a_2 k_{2i} x^i$ $\implies a_2 = 0$ as $k_{2i} \neq 0$ $\implies \sum_{i=0}^{d_1} a_1 k_{1i} x^i = 0$ $\implies a_1 = 0$ as $k_{1i} \neq 0$
	In this way, it can be proved that $a_1 = a_2 = ... = a_{n-1} = a_n = 0$ for $a_1 f_1 + a_2 f_2 + ... + a_n f_n = \theta$ $\implies f_1$,f_2,....,f_n are linearly independent

TABLE 4.1.5.3

Sr. No	Description	Mathematical representation
1.	Consider a set S	$S = \{1, ax+b, \dots\}$
2.	Provide a proof that subset S span the subspace $\langle S \rangle$	Since $1 \in S$ and $ax+b \in S$, $\implies b.1 + \frac{1}{a}(b+ax) \in \langle S \rangle$ $\implies x \in \langle S \rangle$ Given element are $\in S$
3.	Repeat step 2 for the higher degree of polynomial also lie in the subspace and the also lie in the subset S.	Since $(ax+b)^2 \in S$ $\implies (ax+b)^2 - a^2 - 2.a.b.x = a^2.x^2$ $\implies a^2.x^2 \in \langle S \rangle$ $\implies \frac{1}{a^2}.a^2.x^2 \in S.$ $\implies x^2 \in \langle S \rangle$ Given element are $\in S$
4.	After providing proof for all element $\in S$ find the basis .	$S' = \left\{1, x, x^2, x^3, \dots\right\}$
5.	Show the element $\in S'$ are able to form all element S over $\mathbb{F}$.	Hence S form basis of $\mathbb{F}$

TABLE 4.1.6.1: Step of solution for given problem

PARAMETER	DESCRIPTION
$\mathbb{F}$	Field of complex numbers
$\mathbb{F}^{\infty}$	Vector space defined on the field $\mathbb{F}$
$\mathbb{F}[x]$	Subspace of $\mathbb{F}^{\infty}$ spanned by $\left\{1, x, x^2, x^3, \dots\right\}$
$T : \mathbb{F}[x] \to \mathbb{F}[x]$	Transformation T
$f = \sum_{i=0}^{n} c_i x^i$	Polynomial $f \in \mathbb{F}[x]$
$f' = \sum_{i=0}^{n} c_i' x^i$	Polynomial $f' \in \mathbb{F}[x]$
$c_i, c_i' \ \forall i = 0, 2, \dots n$	Scalars in $\mathbb{F}$ and coefficients of polynomials f and f'
$T(f) = g = \sum_{i=0}^{n} \frac{c_i}{i+1} x^{i+1}$	Transformed polynomial $g \in \mathbb{F}[x]$
$T(f') = g' = \sum_{i=0}^{n} \frac{c_i'}{i+1} x^{i+1}$	Transformed polynomial $g' \in \mathbb{F}[x]$
$\mathbf{M}_T$	Transformation matrix for T
$N(T)$	Null Space of T

TABLE 4.1.7.1: Parameters

4.1.7. Let $\mathbb{F}$ be a subfield of complex numbers and let T be the transformation on $\mathbb{F}(x)$ defined by

$$T\left(\sum_{i=0}^{n} c_i x^i\right) = \sum_{i=0}^{n} \frac{c_i}{i+1} x^{i+1} \qquad (4.1.7.1)$$

a) Show that T is a non-singular linear operator on $\mathbb{F}[x]$. Also show that T is not invertible.
Solution: The transformation T does integral of a polynomial. Table refeq:solutions/4/1/9/a/table:2 provides proof that the transformation T is a linear operator and non-singular. Table refeq:solutions/4/1/9/a/table:3 provides proof that T is not invertible, however there exists a left inverse. The parameters used in the proof are listed in the table refeq:solutions/4/1/9/a/table:1.

b) Let $\mathbf{F}$ be a sub-field of the complex numbers and let $\mathbf{D}$ be the transformation on $\mathbf{F}[x]$ defined by

$$\mathbf{D}\left(\sum_{i=0}^{n} c_i x^i\right) = \sum_{i=0}^{n} i c_i x^{i-1} \qquad (4.1.7.14)$$

$$\mathbf{D}\left(\mathbf{x}^{\mathbf{T}}\mathbf{c}\right) = \mathbf{y}^{\mathbf{T}}\mathbf{b} \qquad (4.1.7.15)$$

where

$$\mathbf{x} = \begin{pmatrix} 1 \\ x \\ \vdots \\ x^n \end{pmatrix} \quad and \quad \mathbf{c} = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{pmatrix} \qquad (4.1.7.16)$$

$$\mathbf{y} = \begin{pmatrix} 1 \\ x \\ \vdots \\ x^{n-1} \end{pmatrix} \quad and \quad \mathbf{b} = \begin{pmatrix} c_1 \\ 2c_2 \\ \vdots \\ nc_n \end{pmatrix} \qquad (4.1.7.17)$$

Show that $\mathbf{D}$ is a linear operator on $\mathbf{F}[x]$ and find its null space.
Solution: See Tables 4.1.7.4 and 4.1.7.5

4.2 Lagrange Interpolation

4.2.1. Use the Lagrange interpolation formula to find a polynomial f with real coefficients such that f has degree ≤ 3 and f(−1)=-6 , f(0)=2 , f(1)=-2 , f(2)=6.
Solution: Let the required polynomial be

$$f(x) = a + bx + cx^2 + dx^3 \qquad (4.2.1.1)$$

Testing points are

$$t_0 = -1, t_1 = 0, t_2 = 1, t_3 = 2 \qquad (4.2.1.2)$$

Statement	Derivation
$f = \sum\limits_{i=0}^{n} c_i x^i$	$f = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_n \end{pmatrix}^T_{(n+1)\times 1}$
$T[f] = \mathbf{M}_T f$	$T[f] = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{2} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{n+1} \end{pmatrix}_{(n+2)\times(n+1)} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}_{(n+1)\times 1} = \begin{pmatrix} 0 \\ c_0 \\ \frac{c_1}{2} \\ \frac{c_2}{3} \\ \vdots \\ \frac{c_n}{n+1} \end{pmatrix}_{(n+2)\times 1} = g \in \mathbb{F}[x]$
T is a linear operator	$T[\alpha f + f'] \quad (4.1.7.2)$ $= \mathbf{M}_T(\alpha f + f') \quad (4.1.7.3)$ $= \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{2} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{n+1} \end{pmatrix} \begin{pmatrix} \alpha c_0 + c_0' \\ \alpha c_1 + c_1' \\ \alpha c_2 + c_2' \\ \vdots \\ \alpha c_n + c_n' \end{pmatrix} \quad (4.1.7.4)$ $= \begin{pmatrix} 0 \\ \alpha c_0 + c_0' \\ \frac{\alpha c_1 + c_1'}{2} \\ \frac{\alpha c_2 + c_2'}{3} \\ \vdots \\ \frac{\alpha c_n + c_n'}{n+1} \end{pmatrix} = \alpha \begin{pmatrix} 0 \\ c_0 \\ \frac{c_1}{2} \\ \frac{c_2}{3} \\ \vdots \\ \frac{c_n}{n+1} \end{pmatrix} + \begin{pmatrix} 0 \\ c_0' \\ \frac{c_1'}{2} \\ \frac{c_2'}{3} \\ \vdots \\ \frac{c_n'}{n+1} \end{pmatrix} \quad (4.1.7.5)$ $= \alpha T[f] + T[f'] \quad (4.1.7.6)$ $= \alpha g + g' \quad (4.1.7.7)$ $\therefore T[\alpha f + f'] = \alpha T[f] + T[f'] \quad (4.1.7.8)$
T is non-singular	$T[f] = 0 \quad (4.1.7.9)$ $\implies \mathbf{M}_T f = \mathbf{0} \implies f = 0 \because \mathbf{M}_T \neq \mathbf{0} \quad (4.1.7.10)$ $\implies N(T) = \{0\} \quad (4.1.7.11)$

TABLE 4.1.7.2: Proof for Non-Singular and linear transformation T

Substituting the values of testing points in (4.2.1.1) we form matrix equation

$$\begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} -6 \\ 2 \\ -2 \\ 6 \end{pmatrix} \tag{4.2.1.3}$$

By row reducing augmented matrix we get

$$\begin{pmatrix} 1 & -1 & 1 & -1 & -6 \\ 1 & 0 & 0 & 0 & 2 \\ 1 & 1 & 1 & 1 & -2 \\ 1 & 2 & 4 & 8 & 6 \end{pmatrix} \tag{4.2.1.4}$$

Statement	**Derivation**
T is not invertible	As $\mathbf{M}_T$ is a non-square matrix with dimensions $(n+2) \times (n+1)$, the transformation T is not invertible
$\mathbf{M}_D$ is left inverse of $\mathbf{M}_T$	$$\mathbf{M}_D\mathbf{M}_T = \mathbf{I}_{n+1} \quad (4.1.7.12)$$ $$\implies \mathbf{M}_D = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & n+1 \end{pmatrix} \quad (4.1.7.13)$$

TABLE 4.1.7.3: Non-Invertibility of transformation T

Linear Transformation	A linear transformation from $\mathbf{V}$ into $\mathbf{W}$ is a function $\mathbf{T}$ from $\mathbf{V}$ into $\mathbf{W}$ such that $$\mathbf{T}(c\alpha + \beta) = c\mathbf{T}(\alpha) + \mathbf{T}(\beta)$$ $\forall$ α and β in $\mathbf{V}$ and $\forall$ scalars c in $\mathbf{F}$.
$\mathbf{F}[x]$	Let $\mathbf{F}[x]$ be the subspace of $\mathbf{F}^\infty$ spanned by the vectors $1, x, x^2, ...$ An element of $\mathbf{F}[x]$ is called a polynomial over $\mathbf{F}$.
Differentiation Transformation	Let $\mathbf{F}$ be a field and let $\mathbf{V}$ be the space of polynomial functions f from $\mathbf{F}$ into $\mathbf{F}$, given by $$f(x) = c_0 + c_1x + ... + c_kx^k$$ Then, $$\mathbf{D}f(x) = c_1 + 2c_2x + ... + kc_kx^{k-1}$$ is called Differentiation Transformation. The Differentiation Transformation is a Linear map because $$\begin{aligned}\mathbf{D}(cf+g)(x) &= \left(c.c_1 + c_1'\right) + 2\left(c.c_2 + c_2'\right)x + ... + k\left(c.c_k + c_k'\right)x^{k-1} \\ &= c.c_1 + 2c.c_2x + ... + kc.c_kx^{k-1} + c_1' + 2c_2'x + ... + kc_k'x^{k-1} \\ &= \mathrm{c}\mathbf{D}f(x) + \mathbf{D}g(x)\end{aligned}$$

TABLE 4.1.7.4

$$\xleftrightarrow[R_3\leftarrow R_3-R_1]{R_2\leftarrow R_2-R_1} \begin{pmatrix} 1 & -1 & 1 & -1 & -6 \\ 0 & 1 & -1 & 1 & 8 \\ 0 & 2 & 0 & 2 & 4 \\ 1 & 2 & 4 & 8 & 6 \end{pmatrix} \quad (4.2.1.5)$$

$$\xleftrightarrow[R_3\leftarrow R_3/2]{R_4\leftarrow R_4-R_1} \begin{pmatrix} 1 & -1 & 1 & -1 & -6 \\ 0 & 1 & -1 & 1 & 8 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 3 & 3 & 9 & 12 \end{pmatrix} \quad (4.2.1.6)$$

$$\xleftrightarrow[R_3\leftarrow R_3-R_2]{R_4\leftarrow R_4/3} \begin{pmatrix} 1 & -1 & 1 & -1 & -6 \\ 0 & 1 & -1 & 1 & 8 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 1 & 1 & 3 & 4 \end{pmatrix} \quad (4.2.1.7)$$

$$\xleftrightarrow[R_4\leftarrow R_4-R_2]{R_1\leftarrow R_1+R_2} \begin{pmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & -1 & 1 & 8 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 2 & 2 & -4 \end{pmatrix} \quad (4.2.1.8)$$

Proving **D** is Linear	From (4.1.7.14), clearly **D** is a function from **F**[**x**] to **F**[**x**]. We must show that **D** is linear. Clearly **D** is a Differentiation Transformation, and hence is linear. In other words Let $\mathbf{m} = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{pmatrix}$, $\mathbf{n} = \begin{pmatrix} c_0^{'} \\ c_1^{'} \\ \vdots \\ c_n^{'} \end{pmatrix}$ and α be a scalar. Then $$\mathbf{D}\left(\mathbf{x^T}(\alpha\mathbf{m}+\mathbf{n})\right) = \mathbf{y^T p}, \text{ where } \mathbf{p} = \begin{pmatrix} \alpha c_1 + c_1^{'} \\ 2\left(\alpha c_2 + c_2^{'}\right) \\ \vdots \\ n\left(\alpha c_n + c_n^{'}\right) \end{pmatrix} \text{ and } \alpha\mathbf{m}+\mathbf{n} = \begin{pmatrix} \alpha c_0 + c_0^{'} \\ \alpha c_1 + c_1^{'} \\ \vdots \\ \alpha c_n + c_n^{'} \end{pmatrix}$$ $$\mathbf{D}\left(\mathbf{x^T}(\alpha\mathbf{m}+\mathbf{n})\right) = \mathbf{y^T}(\mathbf{m}^{'}+\mathbf{n}^{'}), \text{ where } \mathbf{m}^{'} = \alpha\begin{pmatrix} c_1 \\ 2c_2 \\ \vdots \\ nc_n \end{pmatrix} \text{ and } \mathbf{n}^{'} = \begin{pmatrix} c_1^{'} \\ 2c_2^{'} \\ \vdots \\ nc_n^{'} \end{pmatrix}$$ Thus, $$\mathbf{D}\left(\mathbf{x^T}(\alpha\mathbf{m}+\mathbf{n})\right) = \mathbf{y^T m}^{'} + \mathbf{y^T n}^{'}$$ Now, $$\mathbf{D}\left(\mathbf{x^T}(\alpha\mathbf{m}+\mathbf{n})\right) = \alpha\mathbf{D}\left(\mathbf{x^T m}\right) + \mathbf{D}\left(\mathbf{x^T n}\right)$$ Hence, **D** is a linear transformation.
Null Space of **D**	Let **N**(**D**) denotes the nullspace of **D**. Then $$\mathbf{N}(\mathbf{D}) = \{f \in \mathbf{F}[\mathbf{x}] : \mathbf{D}f(x) = 0\}$$ A polynomial is zero if and only if its every coeficient is zero. Thus, it must be such that each $c_1 = c_2 = ... = 0$. Since, **D** is a Differentiation Transformation and we know that derivative of a constant polynomial is zero. Thus, the nullspace of **D** contains the constant polynomials. Hence, $$\mathbf{N}(\mathbf{D}) = \{f \in \mathbf{F}[\mathbf{x}] : f(x) = c\}$$

TABLE 4.1.7.5

$$\xleftrightarrow[R_4\leftarrow R_4-R_2]{R_1\leftarrow R_1+R_2} \begin{pmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & -1 & 1 & 8 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 2 & 2 & -4 \end{pmatrix} \tag{4.2.1.9}$$

$$\xleftrightarrow[R_2\leftarrow R_2+R_3]{R_4\leftarrow R_4/2} \begin{pmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 1 & -2 \end{pmatrix} \tag{4.2.1.10}$$

$$\xleftrightarrow[R_2\leftarrow R_2-R_4]{R_4\leftarrow R_4-R_3} \begin{pmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & 4 \end{pmatrix} \tag{4.2.1.11}$$

Therefore ,

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ -6 \\ 4 \end{pmatrix} \tag{4.2.1.12}$$

Hence required polynomial is

$$f(x) = 2 - 2x - 6x^2 + 4x^3 \tag{4.2.1.13}$$

4.2.2. Let T be the linear operator on R^3 defined by $T(x_1, x_2, x_3) = (x_1, x_3, -2x_2 - x_3)$
Let f be the polynomial over R defined by $f = -x^3 + 2$. Find $f(T)$.
Solution: The transformation is given as:

$$T(\mathbf{x}) = \begin{pmatrix} x_1 \\ x_3 \\ -2x_2 - x_3 \end{pmatrix} \tag{4.2.2.1}$$

$$T(\mathbf{x}) = \mathbf{TX} \tag{4.2.2.2}$$

$$T(\mathbf{x}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -1 \end{pmatrix} \mathbf{X} \tag{4.2.2.3}$$

Characteristic equation of $\mathbf{T}$ can be written as:

$$\left|\mathbf{T} - \lambda \mathbf{I}\right| = 0 \tag{4.2.2.4}$$

$$\begin{pmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ 0 & -2 & -1-\lambda \end{pmatrix} = 0 \tag{4.2.2.5}$$

The characteristics equation of the matrix will be,

$$-\lambda^3 - \lambda + 2 = 0 \tag{4.2.2.6}$$

$$-\lambda^3 + 2 = \lambda \tag{4.2.2.7}$$

Given, Polynomial

$$f(x) = -x^3 + 2 \tag{4.2.2.8}$$

$$f(\lambda) = -\lambda^3 + 2 \tag{4.2.2.9}$$

Using 4.2.2.7,

$$f(\lambda) = \lambda \tag{4.2.2.10}$$

Substituting $\mathbf{T}$ for λ in equation 4.2.2.10, We get

$$f(\mathbf{T}) = \mathbf{T} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -1 \end{pmatrix} \tag{4.2.2.11}$$

4.2.3. Let $\mathbf{F}$ be the field of real numbers,

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.2.3.1}$$

$$p = (x-2)(x-3)(x-1) \tag{4.2.3.2}$$

a) Let $\mathbb{F}$ be the field of real numbers,

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.2.3.3}$$

$$p = (x-2)(x-3)(x-1) \tag{4.2.3.4}$$

Solution:

i) Show that $p(\mathbf{A}) = 0$
ii) Let P_1, P_2, P_3 be the Lagrange polynomials for $t_1 = 2, t_2 = 3, t_3 = 1$. Compute $E_i = P_i(\mathbf{A})$, i=1,2,3

i) Since $\mathbf{A}$ is a diagonal matrix, It's characteristic polynomial is,

$$\det(\mathbf{A} - x\mathbf{I}) = 0 \tag{4.2.3.5}$$

$$f(x) = (x-2)^2(x-3)(x-1) = 0 \tag{4.2.3.6}$$

From, (4.2.3.6) and using Cayley Hamilton Theorem,

$$(\mathbf{A} - 2)^2(\mathbf{A} - 3)(\mathbf{A} - 1) = 0 \tag{4.2.3.7}$$

We can also see that (x-2)(x-3)(x-1) is a minimal polynomial for $\mathbf{A}$, Hence $p(\mathbf{A})$=0.

ii) Using Lagrange Interpolation,

$$P_1(x) = \frac{(x-3)(x-1)}{(2-3)(2-1)} \tag{4.2.3.8}$$

$$= -(x-3)(x-1) \tag{4.2.3.9}$$

$$P_2(x) = \frac{(x-2)(x-1)}{(3-2)(3-1)} \tag{4.2.3.10}$$

$$= \frac{(x-2)(x-1)}{2} \tag{4.2.3.11}$$

$$P_3(x) = \frac{(x-2)(x-3)}{(1-2)(1-3)} \tag{4.2.3.12}$$

$$= \frac{(x-2)(x-3)}{2} \tag{4.2.3.13}$$

Now,Substituting the value of **A**,

$$P_1(\mathbf{A}) = -\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.2.3.14}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.2.3.15}$$

$$P_2(\mathbf{A}) = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.2.3.16}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.2.3.17}$$

$$P_3(\mathbf{A}) = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} \tag{4.2.3.18}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.2.3.19}$$

b) Let P_1, P_2, P_3 be the Lagrange polynomials for $t_1 = 2$, $t_2 = 3$, $t_3 = 1$. Compute $E_i = P_i(\mathbf{A}) \quad i = 1, 2, 3$.
Solution: Lagrange polynomials are given by,

$$P_i = \prod_{i \neq j} \frac{x - t_j}{t_i - t_j} \tag{4.2.3.20}$$

$$P_1 = \frac{(x-3)(x-1)}{(2-3)(2-1)} = -(x-3)(x-1) \tag{4.2.3.21}$$

$$P_2 = \frac{(x-2)(x-1)}{(3-2)(3-1)} = \frac{1}{2}(x-2)(x-1) \tag{4.2.3.22}$$

$$P_3 = \frac{(x-2)(x-3)}{(1-2)(1-3)} = \frac{1}{2}(x-2)(x-3) \tag{4.2.3.23}$$

Now calculating E_i,

$$E_1 = P_1(\mathbf{A}) = -(\mathbf{A} - 3\mathbf{I})(\mathbf{A} - \mathbf{I}) \tag{4.2.3.24}$$

$$= -\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.2.3.25}$$

$$\implies E_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.2.3.26}$$

$$E_2 = P_2(\mathbf{A}) = \frac{1}{2}(\mathbf{A} - 2\mathbf{I})(\mathbf{A} - \mathbf{I}) \tag{4.2.3.27}$$

$$= \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.2.3.28}$$

$$\implies E_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.2.3.29}$$

$$E_3 = P_3(\mathbf{A}) = \frac{1}{2}(\mathbf{A} - 2\mathbf{I})(\mathbf{A} - 3\mathbf{I}) \tag{4.2.3.30}$$

$$= \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} \tag{4.2.3.31}$$

$$\implies E_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.2.3.32}$$

4.2.4. Let n be a positive integer and $\mathbb{F}$ be a field. Suppose $\mathbf{A}$ is an $n \times n$ matrix over field $\mathbb{F}$ and $\mathbf{P}$ is an invertible $n \times n$ matrix over field $\mathbb{F}$. If f is any polynomial over $\mathbb{F}$, prove that,

$$f(\mathbf{P}^{-1}\mathbf{A}\mathbf{P}) = \mathbf{P}^{-1}f(\mathbf{A})\mathbf{P} \tag{4.2.4.1}$$

Solution: First we observe the following,

$$(\mathbf{P}^{-1}\mathbf{A}\mathbf{P})^2 = (\mathbf{P}^{-1}\mathbf{A}\mathbf{P})(\mathbf{P}^{-1}\mathbf{A}\mathbf{P}) \tag{4.2.4.2}$$

$$= \mathbf{P}^{-1}\mathbf{A}^2\mathbf{P} \tag{4.2.4.3}$$

Let the (4.2.4.3) be true for a positive integer m i.e,

$$(\mathbf{P}^{-1}\mathbf{A}\mathbf{P})^m = \mathbf{P}^{-1}\mathbf{A}^m\mathbf{P} \tag{4.2.4.4}$$

Now for the integer $m+1$ we get from (4.2.4.4),

$$(\mathbf{P}^{-1}\mathbf{A}\mathbf{P})^{m+1} = (\mathbf{P}^{-1}\mathbf{A}^m\mathbf{P})(\mathbf{P}^{-1}\mathbf{A}\mathbf{P}) \tag{4.2.4.5}$$

$$= \mathbf{P}^{-1}\mathbf{A}^{m+1}\mathbf{P} \tag{4.2.4.6}$$

From (4.2.4.4) and (4.2.4.6) we get for any positive integer n,

$$(\mathbf{P}^{-1}\mathbf{A}\mathbf{P})^n = \mathbf{P}^{-1}\mathbf{A}^n\mathbf{P} \tag{4.2.4.7}$$

Again we have,

$$\mathbf{P}^{-1}\mathbf{P} = \mathbf{I} \tag{4.2.4.8}$$

The general form of polynomial $f(\mathbf{A})$ is defined as,

$$f(\mathbf{A}) = a_0 + a_1\mathbf{A} + a_2\mathbf{A}^2 + \cdots + a_n\mathbf{A}^n \tag{4.2.4.9}$$

Now we have,

$$f(\mathbf{P}^{-1}\mathbf{A}\mathbf{P}) = a_0 + a_1(\mathbf{P}^{-1}\mathbf{A}\mathbf{P}) + \cdots + a_n(\mathbf{P}^{-1}\mathbf{A}\mathbf{P})^n \tag{4.2.4.10}$$

$$= (\mathbf{P}^{-1}a_0\mathbf{P}) + (\mathbf{P}^{-1}a_1\mathbf{A}\mathbf{P}) + \cdots + (\mathbf{P}^{-1}a_n\mathbf{A}\mathbf{P})^n \tag{4.2.4.11}$$

$$= (\mathbf{P}^{-1}a_0\mathbf{P}) + (\mathbf{P}^{-1}a_1\mathbf{A}\mathbf{P}) + \cdots + (\mathbf{P}^{-1}a_n\mathbf{A}^n\mathbf{P}) \tag{4.2.4.12}$$

$$= \mathbf{P}^{-1}(a_0 + a_1\mathbf{A} + a_2\mathbf{A}^2 + \cdots + a_n\mathbf{A}^n)\mathbf{P} \tag{4.2.4.13}$$

$$= \mathbf{P}^{-1}f(\mathbf{A})\mathbf{P} \quad \text{[From (4.2.4.9)]} \tag{4.2.4.14}$$

Hence proved (4.2.4.14)

4.2.5. Let $\mathbf{F}$ be a field. We have considered certain special linear functionals on $\mathbf{F}[x]$ obtained via 'evaluation at t': $L(f) = f(t)$. Such functionals are not only linear but also have the property that L(fg) =L(f)L(g). Prove that if L is any linear functional on F[x] such that L(fg) =L(f)L(g)for all f and g, then either L = 0 or there is a t in F such that L(f) = f(t) for all f.
Solution: Let L be a non zero linear transformation.

$$f(x) = a_0 + a_1(x) + \cdots + a_n(x^n) \tag{4.2.5.1}$$

$$L(f) = L(a_0 + a_1(x) + \cdots + a_n(x^n)). \tag{4.2.5.2}$$

Given,L is any linear functional on f(x).
Hence,

$$L(f) = L(f.1) = L(f)L(1) \tag{4.2.5.3}$$

$\implies L(1) \neq 0.$
Similarly,

$$L(1) = L(1)L(1) \tag{4.2.5.4}$$

$$\implies L(1) = 1 \tag{4.2.5.5}$$

Similarly, $\forall a \in \mathbf{F}$.

$$L(a) = L(a.1) \implies aL(1) = a \tag{4.2.5.6}$$

Simplifying (4.2.5.2) we have,

$$L(f) = L(a_0) + L(a_1)L(x) + \cdots + L(a_n)L(x^n) \tag{4.2.5.7}$$

From (4.2.5.6) we have,$L(x) = t$.

$$L(f) = a_0 + a_1L(x) + \cdots a_nL(x)^n \tag{4.2.5.8}$$

$$\implies L(f) = a_0 + a_1(t) + \cdots a_n(t)^n \tag{4.2.5.9}$$

Hence proved,$L(f) = f(t)$.

4.2.6. If F is a field and h is a polynomial over F of degree ≥ 1, show that the mapping $f \rightarrow f(h)$ is a one-one linear transformation of $F[x]$ into $F[x]$. Show that this transformation is an isomorphism of $F[x]$ onto $F[x]$ if and only if deg $h = 1$.

Solution: Here, $F[x]$ is a set of polynomials over field F, written as:

$$F[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \quad | \quad a_i \in F \right\} \tag{4.2.6.1}$$

Let,

$$G(f) = f(h) \tag{4.2.6.2}$$

Thus, $G(f)$ is clearly a function from $F[x]$ into $F[x]$.

Now, we need to show that the function G is one-one linear transformation. Let us first show that G is a linear transformation:

$$\text{Let, } f, g \in F[x] \text{ and } \alpha \in F$$

$$\begin{aligned} G(\alpha f + g) &= (\alpha f + g)(h) \\ &= (\alpha f)(h) + g(h) \\ &= \alpha f(h) + g(h) \\ &= \alpha G(f) + G(g) \end{aligned} \tag{4.2.6.3}$$

From (4.2.6.3), G is a linear transformation.

For G to be one-one linear transformation, it should map a set of linearly independent polynomials in $F(x)$ to another set of linearly independent polynomials in $F(x)$. let us consider the following basis set for $F(x)$:

$$S = \{f_0, f_1, f_2, f_3, f_4, \ldots\} \tag{4.2.6.4}$$

Where,

$$f_i = x^i \tag{4.2.6.5}$$

Since, the set S forms the basis for $F(x)$, the set S is a set of linearly independent polynomials. Let us apply the transformation G to set S, then we obtain another set S' as:

$$S' = \{f_0(h), f_1(h), f_2(h), f_3(h), f_4(h), \ldots\} \tag{4.2.6.6}$$

Where,

$$f_i = x^i \tag{4.2.6.7}$$

Here, The degree of each polynomial in set S' is distinct and given by $i{\cdot}\deg(h)$. Thus, set S' is also a set of linearly independent polynomials.

Conclusion: G will maps any arbitrary set S_a of linearly independent polynomials in $F(x)$ to another set S'_a of linearly independent polynomials in $F(x)$. (Since any arbitrary set S_a can be written in terms of basis set S). Hence, <u>G is one-one linear transformation</u>.

Now, Let us prove that G is an isomorphism of $F(x)$ onto $F(x)$ if and only if $\deg(h) = 1$.
Let $\deg(h) = 1$, then h can be written as:

$$h = a + bx, \quad \text{Where, } b \neq 0 \tag{4.2.6.8}$$

Let us define h' such that:

$$h' = \frac{1}{b}x - \frac{a}{b} \tag{4.2.6.9}$$

Let G' be the linear transformation from $F(x)$ to $F(x)$ given by:

$$G'(f) = f\left(\frac{1}{b}x - \frac{a}{b}\right) \tag{4.2.6.10}$$

It can be shown that G' is inverse of G as follow:

$$G(G'(f)) = G\left(f\left(\frac{1}{b}x - \frac{a}{b}\right)\right) \tag{4.2.6.11}$$

$$= f\left(a\left(\frac{1}{a}x - \frac{b}{a}\right) + b\right) \tag{4.2.6.12}$$

$$= f(x) \tag{4.2.6.13}$$

Similarly,

$$G'(G(f)) = G'\left(f\left(ax + b\right)\right) \tag{4.2.6.14}$$

$$= f\left(\frac{1}{a}\left(ax + b\right) - \frac{b}{a}\right) \tag{4.2.6.15}$$

$$= f(x) \tag{4.2.6.16}$$

Thus, G' is inverse of G. Therefore, G is isomorphism and we can say:

$$\boxed{\deg(h) = 1 \implies G \text{ is isomorphism.}} \tag{4.2.6.17}$$

Let $\deg(h) > 1$, then

$$\deg f(h) = \deg f \cdot \deg h \quad (4.2.6.18)$$
$$\implies \deg f(h) \geq 1 \quad (4.2.6.19)$$
$$\implies G(f) = f(h) \neq x \quad (4.2.6.20)$$

This means the image of G does not contain polynomials of degree one. Hence G is not onto and therefore G can not be an isomorphism. Thus we can write:

$$\boxed{\deg(h) > 1 \implies G \text{ is not isomorphism.}} \quad (4.2.6.21)$$

From (4.2.6.17) and (4.2.6.21), We can conclude:

$$\boxed{G \text{ is isomorphism.} \iff \deg(h) = 1} \quad (4.2.6.22)$$

4.3 Polynomial Ideals

4.3.1. Let $\mathbb{Q}$ be the field of rational numbers. Determine if the following subset of $\mathbb{Q}[x]$ is ideal or not.

a) The subset is defined by all f with degree ≥ 5
Solution: See Table 4.3.1.1

Example	The defined subset of $\mathbb{Q}[x]$ be $\mathbb{U}$, $f(x) = c_1x^5 + c_2x^4 \in \mathbb{U}$ $g(x) = -c_1x^5 + c_3x^4 \in \mathbb{U}$
Proof	If $\mathbb{U}$ is an ideal then, $\mathbb{U}$ must be a subset. $\mathbb{U}$ must be closed under addition. $f \in \mathbb{U}$ $g \in \mathbb{U}$ $\implies f + g \in \mathbb{U}$ But here, $f + g = c_1x^5 + c_2x^4 - c_1x^5 + c_3x^4$ $\implies f + g = (c_2 + c_3)x^4 \notin \mathbb{U}$
Observation	$f \in \mathbb{U}$ $g \in \mathbb{U}$ But, $f + g \notin \mathbb{U}$
Conclusion	$\mathbb{U}$ is not closed under addition $\implies \mathbb{U}$ is not a subset of $\mathbb{Q}[x]$ $\implies \mathbb{U}$ is not an ideal of $\mathbb{Q}[x]$

TABLE 4.3.1.1

4.3.2. a) Find the g.c.d of each of the following pairs of polynomials.

$$2x^5 - x^3 - 3x^2 - 6x + 4, x^4 + x^3 - x^2 - 2x - 2 \quad (4.3.2.1)$$

Solution: Refer Table 4.3.2.1.

b) Let Q be the field of rational numbers. Determine which of the following subsets of $Q[x]$ are ideals. When the set is an ideal, find its monic generator.
All f such that $f(2) = f(4) = 0$.
Solution: *Definition*: Let F be a field. An ideal in $F[x]$ is a subspace M of $F[x]$ such that fg belongs to M whenever f is in $F[x]$ and g is in M.
See Table 4.3.2.2

4.3.3. Let A be an $n \times n$ matrix over a field $\mathbf{F}$. Show that the set of all polynomials f in $\mathbf{F}[x]$ such that f(A) = 0 is an ideal.
Solution: Given a square matrix of order n and $f(A) = 0$. Let $\mathbf{I}$ be an ideal of $C[z1, \cdots, zn]$.

$$\mathbf{I} = \{\mathbf{F} \in \mathbf{F}(x) | f(A) = 0\} \quad (4.3.3.1)$$

Now,Consider the polynomials,

$$f = f_0 + f_1z_n + \cdots f_d{z_n}^d \quad (4.3.3.2)$$

$$g = g_0 + g_1z_n + \cdots g_e{z_n}^e \quad (4.3.3.3)$$

$$f, g \in \mathbf{I} \;\; \& \;\; k \in \mathbf{F}.$$

$$(kf + g)(A) = kf(A) + g(A) \quad (4.3.3.4)$$
$$\implies c.0 + 0 = 0 \quad (4.3.3.5)$$

From equation(4.3.3.5) $\mathbf{I}$ is a subspace of $\mathbf{F}(x)$
Now,f∈ $\mathbf{I}$ and $g \in F(x)$ then,

$$(gf)A = g(A)f(A) \quad (4.3.3.6)$$
$$\implies g(A).0 = 0 \quad (4.3.3.7)$$

Hence,proved $\mathbf{I}$ is ideal

4.3.4. Let $\mathbf{F}$ be a subfield of complex numbers, and let

$$\mathbf{A} = \begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix} \quad (4.3.4.1)$$

Find the monic generator of the ideal of all polynomials f in $F[z]$ such that $f(\mathbf{A}) = 0$.
Solution: let, ideal $I = f(x) \in F[x]$ such that

Let the field be rational numbers	
Steps	**Explanation**
Say f(x) and g(x)	$$f(x) = 2x^5 - x^3 - 3x^2 - 6x + 4 \quad (4.3.2.2)$$ $$g(x) = x^4 + x^3 - x^2 - 2x - 2 \quad (4.3.2.3)$$
Expanding f(x) in term of g(x)	$$2x^5 - x^3 - 3x^2 - 6x + 4 = \left(x^4 + x^3 - x^2 - 2x - 2\right)(2x - 2) + \left(3x^3 - x^2 - 6x\right) \quad (4.3.2.4)$$
Expanding degree four polynomial	$$x^4 + x^3 - x^2 - 2x - 2 = \left(3x^3 - x^2 - 6x\right)\left(\frac{1}{3}x + \frac{4}{9}\right) + \left(\frac{13}{9}x^2 + \frac{2}{3}x - 2\right) \quad (4.3.2.5)$$
Expanding degree three polynomial	$$3x^3 - x^2 - 6x = \left(\frac{13}{9}x^2 + \frac{2}{3}x - 2\right)\left(\frac{27}{13}x - \frac{279}{169}\right) + \left(-\frac{126}{169}x - \frac{558}{169}\right) \quad (4.3.2.6)$$
Expanding degree two polynomial	$$\frac{13}{9}x^2 + \frac{2}{3}x - 2 = \left(-\frac{126}{169}x - \frac{558}{169}\right)\left(-\frac{2197}{1134}x + \frac{61009}{7938}\right) + \left(\frac{10309}{441}\right) \quad (4.3.2.7)$$ Since it contains scalar polynomial hence the g.c.d of f(x) , g(x) is 1.

TABLE 4.3.2.1: Solution

$f(\mathbf{A}) = 0$. where,

$$f(x) = \sum_{i=0}^{n} a_i x^i \quad a_n = 1 \tag{4.3.4.2}$$

Computing $f(\mathbf{A})$ for $deg(f) \leq 1$ we get,

$$\mathbf{A}^0 = \mathbf{I} \tag{4.3.4.3}$$

$$\mathbf{A}^1 = \begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix} \tag{4.3.4.4}$$

$$f(\mathbf{A}) = a_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + a_1 \begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix} \tag{4.3.4.5}$$

As $\mathbf{I}$ and $\mathbf{A}$ are linearly independent so from (4.3.4.5) we see $f(\mathbf{A}) = 0$ only if $a_1 = 0$ and $a_0 = 0$ but this can't happen from (4.3.4.2). Hence for $deg(f) \leq 1$, $f(\mathbf{A}) \neq 0$. Hence for any $f(x) \in I$ such that $deg(f) = 2$ then f is Monic generator or minimal polymial. We can write,

$$f(x) = x^2 + a_1 x^1 + a_0 \tag{4.3.4.6}$$

Minimal polynomial or monic generator can be found using Characteristic equation,

$$\left|\mathbf{A} - \mathbf{I}\lambda\right| = 0 \tag{4.3.4.7}$$

$$\begin{vmatrix} 1-\lambda & -2 \\ 0 & 3-\lambda \end{vmatrix} = 0 \tag{4.3.4.8}$$

$$(1-\lambda)(3-\lambda) = 0 \tag{4.3.4.9}$$

$$\implies \lambda^2 - 4\lambda + 3 = 0 \tag{4.3.4.10}$$

comparing (4.3.4.6) and (4.3.4.10) we get,

$$f(x) = x^2 - 4x + 3 = 0 \tag{4.3.4.11}$$

using Cayley-Hamilton equation,

$$f(\mathbf{A}) = \mathbf{A}^2 - 4\mathbf{A} + 3 = 0 \tag{4.3.4.12}$$

Hence, $f = x^2 - 4x + 3$ is the monic generator.

Given	Q be the field of rational numbers. Subset: All f such that $f(2) = f(4) = 0$
To prove	Given subset is an Ideal. If Ideal then find its monotic generator.
Proof	Let, $M = \{f \in F[x] \| f(2) = f(4) = 0\}$. $g(2) = g(4) = 0$ and $f(2) = f(4) = 0$ $(df + g)(2) = df(2) + g(2) = 0$ $(df + g)(4) = df(4) + g(4) = 0$ $\implies M$ is a subspace of $F[x]$ Let suppose $f \in M$ and $g \in F[x]$. $(fg)(2) = f(2)g(2) = 0$ $(fg)(4) = f(4)g(4) = 0$ $\therefore fg \in M$ Hence, M is an ideal.
Representation of polynomial	$f(x) = (x - c)q(x) + r$ When $f(c) = 0$ then $r = 0$. $\implies f(x) = (x - 2)(x - 4)q(x)$ $\therefore f(x) = g(x)q(x)$ Hence, $f(x)$ is generated by $g(x)$.

TABLE 4.3.2.2

4.3.5. Let F be a field , show that the intersection of any numbers of ideals in F[x] is an ideal.
Solution: See Figs. 4.3.5.1, 4.3.5.2 and Table 4.3.5.1

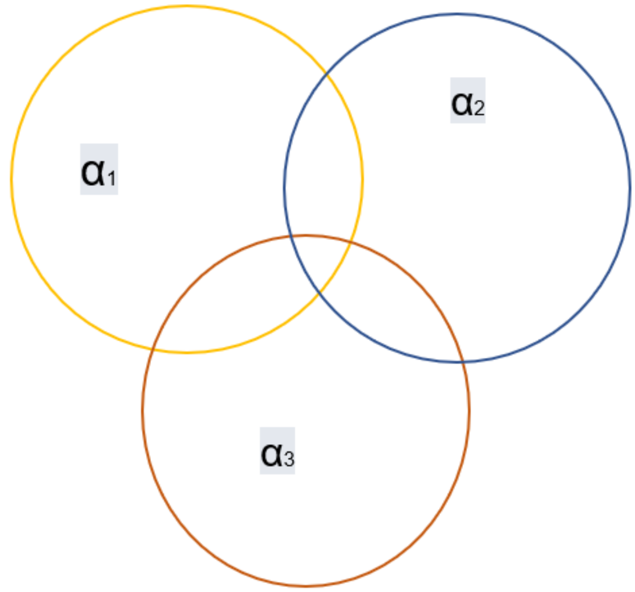

Fig. 4.3.5.1: Index set containing α

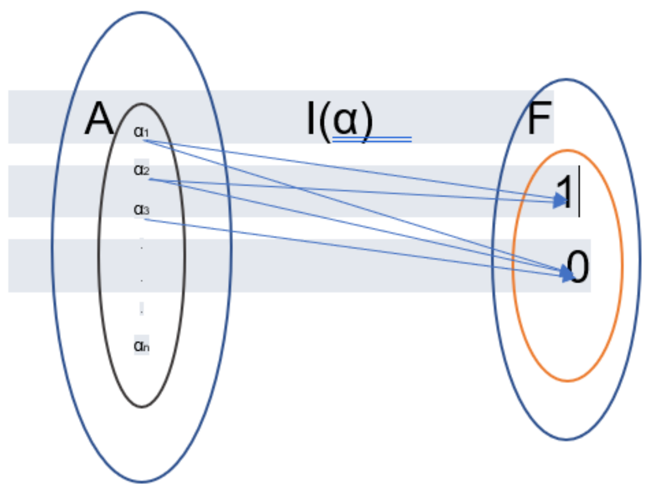

Fig. 4.3.5.2: ideal containing I_α

5 Determinants

6 Elementary Canonical Forms

6.1 Characteristic Values

6.1.1. In each of the following cases, let T be the linear operator on R^2 which is represented by matrix A in the standard ordered basis for R^2, and let U be the linear operator on C^2 represented by A in the standard ordered basis. Find the characteristic polynomial for T and that for U, find the characteristic value of each operator, and for each characteristic value c find a basis for the corresponding space of characteristic vectors.

$$\mathbf{A_1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{6.1.1.1}$$

$$\mathbf{A_2} = \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix} \tag{6.1.1.2}$$

$$\mathbf{A_3} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \tag{6.1.1.3}$$

Solution: See Table 6.1.1.1 and 6.1.1.2.

6.1.2. Let T be the linear operator on $\mathbf{R}^3$ which is represented in the standard ordered basis by

Given	F is a field
To prove	I=$\bigcap_{\alpha \in A} I_\alpha$ is an ideal
Proof	Let A be an index set and I_α be an in F[x] for each $\alpha \in$ A Obviously I is the subspace since I_α is a subspace of F[x] and arbitrary intersection of subspace is also a subspace. Let g$(x) \in$ F[x] and f$(x) \in$ I Since f$(x) \in$ I and I_α is an ideal follows that f(x) g$(x) \in I_\alpha \ \forall \alpha \in A$ Thus f(x) g$(x) \in$ I.

TABLE 4.3.5.1

the matrix

$$\begin{pmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{pmatrix} \tag{6.1.2.1}$$

Prove that T is diagonalizable by exhibiting a basis for $\mathbf{R}^3$, each vector of which is a characteristic vector of T.

Solution: Let $\mathbf{T}$ be a linear operator on a finite-dimensional space $\mathbf{V}$. Let $c_1, \cdots, c_k$ be the distinct characteristic values of $\mathbf{T}$ and let $\mathbf{W}_i$ be the null space of $\mathbf{T} - c_i\mathbf{I}$. The following are equivalent.

(i) $\mathbf{T}$ is diagonalizable.

(ii) The characteristic polynomial for $\mathbf{T}$ is,

$$f = (x - c_1)_1^d \cdots (x - c_k)^{d_k} \tag{6.1.2.2}$$

and $\dim W_i = d_i, i = 1, \cdots, k$

(iii) $\dim W_1 + \cdots + \dim W_k = \dim V$

Now let,

$$\mathbf{A} = \begin{pmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{pmatrix} \tag{6.1.2.3}$$

Solving $\left|\lambda\mathbf{I} - \mathbf{A}\right| = 0$

$$\left|\lambda\mathbf{I} - \mathbf{A}\right| = \begin{vmatrix} \lambda+9 & -4 & -4 \\ 8 & \lambda-3 & -4 \\ 16 & -8 & \lambda-7 \end{vmatrix} \tag{6.1.2.4}$$

$$\xleftrightarrow{C_2 \leftarrow C_2 - C_3} \begin{vmatrix} \lambda+9 & 0 & -4 \\ 8 & \lambda+1 & -4 \\ 16 & -\lambda-1 & \lambda-7 \end{vmatrix} \tag{6.1.2.5}$$

$$\therefore \left|\lambda\mathbf{I} - \mathbf{A}\right| = (\lambda+1) \begin{vmatrix} \lambda+9 & 0 & -4 \\ 8 & 1 & -4 \\ 16 & -1 & \lambda-7 \end{vmatrix} \tag{6.1.2.6}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 + R_2} (\lambda+1) \begin{vmatrix} \lambda+9 & 0 & -4 \\ 8 & 1 & -4 \\ 24 & 0 & \lambda-11 \end{vmatrix} \tag{6.1.2.7}$$

$$\implies (\lambda+1) \begin{vmatrix} \lambda+9 & -4 \\ 24 & \lambda-11 \end{vmatrix} \tag{6.1.2.8}$$

$$\implies \left|\lambda\mathbf{I} - \mathbf{A}\right| = (\lambda+1)^2(\lambda-3) = 0 \tag{6.1.2.9}$$

$$\implies \lambda_1 = -1, \lambda_2 = -1, \lambda_3 = 3 \tag{6.1.2.10}$$

Now at λ_1 and λ_3,

$$\mathbf{A} + \mathbf{I} = \begin{pmatrix} -8 & 4 & 4 \\ -8 & 4 & 4 \\ -16 & 8 & 8 \end{pmatrix} \tag{6.1.2.11}$$

$$\mathbf{A} - 3\mathbf{I} = \begin{pmatrix} -12 & 4 & 4 \\ -8 & 0 & 4 \\ -16 & 8 & 4 \end{pmatrix} \tag{6.1.2.12}$$

Now we know that $\mathbf{A} - 3\mathbf{I}$ is singular and rank$(\mathbf{A} - 3\mathbf{I}) \geq 2$. Therefore, rank$(\mathbf{A} - 3\mathbf{I}) = 2$. Hence from the theorem-1 (iii) it is evident that rank$(\mathbf{A} + \mathbf{I}) = 1$. Let X_1 and X_3 be the spaces of characteristic vectors associated with the characteristic values 1 and 3 respectively. We know from rank nullity theorem that $\dim X_1 = 2$ and $\dim X_3 = 1$. Hence by Theorem-2 (i) $\mathbf{T}$ is diagonalizable.

Given	T be the linear operator on R^2 which is represented by matrix A in the standard ordered basis for R^2 U be the linear operator on C^2 which is represented by matrix A in the standard ordered basis In all cases, denoting B_c the basis for the subspace corresponding to characteristic value c
To find	Characteristic value of each operator For each characteristic value c find a basis for the corresponding space of characteristic vectors

TABLE 6.1.1.1

The null-space of $\mathbf{T}+\mathbf{I}$ is spanned by the vectors,

$$\alpha_1 = \begin{pmatrix} 1 & 0 & 2 \end{pmatrix} \tag{6.1.2.13}$$

$$\alpha_2 = \begin{pmatrix} 1 & 2 & 0 \end{pmatrix} \tag{6.1.2.14}$$

As both α_1 and α_2 are independent, hence they form basis for X_1. The null-space of $\mathbf{T}-\mathbf{3I}$ is spanned by the vector,

$$\alpha_3 = \begin{pmatrix} 1 & 1 & 2 \end{pmatrix} \tag{6.1.2.15}$$

Here α_3 is a characteristic vector and a basis for $\mathbf{X_3}$. Also the matrix P which enables us to change coordinates from the basis β to the standard basis is the matrix which has transposes of α_1, α_2 and α_3 as it's column vectors:

$$\mathbf{P} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 2 & 0 & 2 \end{pmatrix} \tag{6.1.2.16}$$

6.1.3. Let

$$\mathbf{A} = \begin{pmatrix} 6 & -3 & -2 \\ 4 & -1 & -2 \\ 10 & -5 & -3 \end{pmatrix} \tag{6.1.3.1}$$

Is A similar over the field R to a diagonal matrix ?
Is A similar over the field C to a diagonal matrix?
Solution: See Tables 6.1.3.1 and 6.1.3.2

Matrix	Characteristic Polynomial	Basis
$\mathbf{A_1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	$\det(xI - A) = \begin{vmatrix} (x-1) & 0 \\ 0 & x \end{vmatrix}$ Characteristic Polynomial $= x(x-1)$ To find characteristic values of the operator $\det(xI - A) = 0$,which gives $c_1 = 0$ and $c_2 = 1$ Both c_1 and c_2 are the characteristic values Assume B_1 and B_2 are Basis for c_1 and c_2	Basis for characteristic value $c_1 = 0$ will be obtained by solving homogeneous equation,$(A - c_1 I)x = 0$ After solving, basis for characteristic value c_1 is $B_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ Similarly,we can find out the Basis for $c_2 = 1$ which is $B_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
$\mathbf{A_2} = \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix}$	$\det(xI - A) = \begin{vmatrix} (x-2) & -3 \\ 1 & (x-1) \end{vmatrix}$ Characteristic Polynomial $= x^2 - 3x + 5$ To find characteristic values of the operator $\det(xI - A) = 0$,which gives $c_1 = \frac{3+i\sqrt{11}}{2}$ and $c_2 = \frac{3-i\sqrt{11}}{2}$ Both c_1 and c_2 are the characteristic values Assume B_1 and B_2 are Basis for c_1 and c_2	Basis for characteristic value $c_1 = \frac{3+i\sqrt{11}}{2}$ will be obtained by solving homogeneous equation,$(A - c_1 I)x = 0$ After solving, basis for characteristic value c_1 is $B_1 = \begin{pmatrix} \frac{1+i\sqrt{11}}{2} \\ -1 \end{pmatrix}$ Similarly,we can find out the Basis for c_2 which is $B_2 = \begin{pmatrix} \frac{1-i\sqrt{11}}{2} \\ -1 \end{pmatrix}$
$\mathbf{A_3} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$	$\det(xI - A) = \begin{vmatrix} (x-1) & -1 \\ -1 & (x-1) \end{vmatrix}$ Characteristic Polynomial $= x(x-2)$ To find characteristic values of the operator $\det(xI - A) = 0$,which gives $c_1 = 0$ and $c_2 = 2$ Both c_1 and c_2 are the characteristic values Assume B_1 and B_2 are Basis for c_1 and c_2	Basis for characteristic value $c_1 = 0$ will be obtained by solving homogeneous equation,$(A - c_1 I)x = 0$ After solving, basis for characteristic value c_1 is $B_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ Similarly,we can find out the Basis for $c_2 = 2$ which is $B_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

TABLE 6.1.1.2: Finding of Characteristic Polynomial, Characteristic value and corresponding Basis

Theorem 2	Let T be the linear operator on a finite dimensional space V and $c_1,..,c_k$ be a distinct characteristic values of T and let W_i be the null space of $(T - c_i I)$ then 1)T is diagonalizable 2) characteristic polynomial for T is $f = (x - c_1)^{d_1}....(x - c_k)^{d_k}$ and 3) $dimW_i = d_i,\ i = 1, ...k$
Concept for diagonalization	A linear operator T on a n-dimensional space V is diagonalizable , if and only if T has n distinct characteristic vectors or null spaces corresponding to the characteristic values

TABLE 6.1.3.1: Illustration of theorem.

Given	Let the given matrix be $\mathbf{A}=\begin{pmatrix} 6 & -3 & -2 \\ 4 & -1 & -2 \\ 10 & -5 & -3 \end{pmatrix}$
Finding Characteristics polynomial	Characteristics polynomial of the matrix A is $det(xI - A)$ $\det(xI - A)=\begin{vmatrix} (x-6) & 3 & 2 \\ -4 & (x+1) & 2 \\ -10 & 5 & x+3 \end{vmatrix}$ Characteristic Polynomial $= (x - 2)(x^2 + 1) = (x - 2)(x - i)(x + i)$
Checking whether A similar over the field R to a diagonal matrix	As the characteristics polynomial is not product of linear factors over R . Therefore from Theorem 2, A is not diagonalizable over R
Checking whether A similar over the field C to a diagonal matrix	The Characteristic Polynomial can be wriiten as a product of linear factors over C i.e $\det(xI - A) = (x - 2)(x - i)(x + i)$ To find characteristic values of the operator $\det(xI - A) = 0$ which gives $c_1 = 2, c_2 = i, c_3 = -i$

	Thus over C matrix A has three distinct characteristic values. There will be atleast one characteristics vector i.e., one dimension with each characteristics value . From Theorem 2; $\sum_i W_i = n = 3$, which is equal to dim of A. Thus , A is diagonalizable over C.
Conclusion	1) A is not diagonalizable over R. 2) A is diagonalizable over C.

TABLE 6.1.3.2: Finding of characterisctics values and diagonalization over different field

6.1.4. Let $\mathbf{T}$ be the linear operator on $\mathbf{R}^4$ which is represented in the standard basis by the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \end{pmatrix} \tag{6.1.4.1}$$

Under what conditions on a, b and c in $\mathbf{T}$ is diagonalizable?

Solution:

Theorem 6.1. *A linear operator* $\mathbf{T}$ *on a* $n-$ *dimensional space* $\mathbf{V}$ *is diagonalizable, if and only if* $\mathbf{T}$ *has an* n *distinct characteristic vectors (or) null spaces corresponding to the characteristic values.*

Theorem 6.2. *Let* $\mathbf{T}$ *be a linear operator on a finite-dimensional space* $\mathbf{V}$. *Let* $c_1, c_2, ..., c_k$ *be the distinct characteristic values of* $\mathbf{T}$ *and let* $\mathbf{W_i}$ *be the null space of* $(\mathbf{T} - c_i\mathbf{I})$. *The following are equivalent:*

a) $\mathbf{T}$ *is diagonizable*

b) $dim\ \mathbf{W_1} + ... + dim\ \mathbf{W_k} = dim\ \mathbf{V}$

Let the given matrix be,

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \end{pmatrix} \tag{6.1.4.2}$$

As per theorem 6.1, we need to find the characteristic polynomial for the matrix $\mathbf{A}$. Characteristic equation is given by $det\,(x\mathbf{I} - \mathbf{A})$.

$$det\,(x\mathbf{I} - \mathbf{A}) = \begin{vmatrix} x-0 & 0 & 0 & 0 \\ -a & x-0 & 0 & 0 \\ 0 & -b & x-0 & 0 \\ 0 & 0 & -c & x-0 \end{vmatrix} \tag{6.1.4.3}$$

$$det\,(x\mathbf{I} - \mathbf{A}) = x^4 \tag{6.1.4.4}$$

The characteristic equation will be,

$$det\,(x\mathbf{I} - \mathbf{A}) = 0 \tag{6.1.4.5}$$

$$x^4 = 0 \tag{6.1.4.6}$$

From (6.1.4.6) we get the characteristic value as $c_1 = 0$ with a multiplicity of 4.

The basis for the characteristic value $c_1 = 0$ can be obtained by solving the equation

$$(\mathbf{A} - c_1\mathbf{I})\,\mathbf{x} = \mathbf{0} \tag{6.1.4.7}$$

i.e.

$$(\mathbf{A} - (0)\mathbf{I})\,\mathbf{x} = \mathbf{0} \tag{6.1.4.8}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \mathbf{0} \tag{6.1.4.9}$$

Solving the above equation we get

$$ax = 0, \ by = 0, \ cz = 0 \tag{6.1.4.10}$$

We know that the null space of $(\mathbf{A} - (0)\mathbf{I})$, is spanned by the vector $\mathbf{x}$, where the basis for the space $\mathbf{W_1}$ need to satisfy the condition of (6.1.4.10). If we assume that

$$a \neq 0\ ,\ b \neq 0\ ,\ c \neq 0 \tag{6.1.4.11}$$

This will correspond that the elements in the basis of the vector $\mathbf{x}$ will be

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ t \end{pmatrix} \tag{6.1.4.12}$$

Which implies that the $dim\ \mathbf{W_1} = 1$. From theorem 6.2, for $\mathbf{T}$ to be diagonalizable, the null space $\mathbf{W_1}$ of $\mathbf{A}$ must have the $dim\ \mathbf{W_1} = 4$, since $dim\ \mathbf{R^4} = 4$. So, there is a contradiction with (6.1.4.11).

$\therefore$ $\mathbf{A}$ is diagonalizable only if

$$a = b = c = 0 \tag{6.1.4.13}$$

i.e. $\mathbf{A}$ is a zero matrix.

6.1.5. Let **T** be the linear operator on a $n-$ dimensional vector space **V** and suppose that **T** has an n distinct characteristic values.Prove that **T** is diagonalizable.

Solution: See Table 6.1.5.2

Diagonalizable	A linear operator $\mathbf{T}$ on a finite-dimensional vector space $\mathbf{V}$ is diagonalizable if and only if there exists an basis of $\mathbf{V}$, consisting of eigen vectors of $\mathbf{T}$
Theorem	If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are eigenvectors of a linear operator $\mathbf{T}$ with distinct eigen values $\lambda_1, \lambda_2, \ldots, \lambda_k$,then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent. Let $S_k = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$.Let $P(k) : S_k$ is linearly independent.S_1 is linearly independent. So,$P(1)$ holds.Assume $P(k)$ holds for $1 \le k \le n$.Therefore,S_k is linearly independent. $$\text{Let} \sum_{i=1}^{k+1} a_i \mathbf{v}_i = 0 \quad (6.1.5.1)$$ Applying $\mathbf{T}$ on both sides , we get $$\implies \mathbf{T}(\sum_{i=1}^{k+1} a_i \mathbf{v}_i) = 0 \quad (6.1.5.2)$$ $$\implies \sum_{i=1}^{k+1} a_i \mathbf{T}(\mathbf{v}_i) = 0 \quad (6.1.5.3)$$ $$\implies \sum_{i=1}^{k+1} a_i \lambda_i \mathbf{v}_i = 0 \quad (6.1.5.4)$$ $$\implies \sum_{i=1}^{k} a_i \lambda_i \mathbf{v}_i + a_{k+1}\lambda_{k+1}\mathbf{v}_{k+1} = 0 \quad (6.1.5.5)$$ Multiplying (6.1.5.1) by λ_{k+1},we get $$\lambda_{k+1}(\sum_{i=1}^{k+1} a_i \mathbf{v}_i) = 0 \quad (6.1.5.6)$$ $$\implies \sum_{i=1}^{k+1} a_i \lambda_{k+1} \mathbf{v}_i = 0 \quad (6.1.5.7)$$
	$$\implies \sum_{i=1}^{k} a_i \lambda_{k+1} \mathbf{v}_i + a_{k+1}\lambda_{k+1}\mathbf{v}_{k+1} = 0 \quad (6.1.5.8)$$ Subtracting (6.1.5.5) and (6.1.5.8),we get $$\sum_{i=1}^{k} a_i(\lambda_i - \lambda_{k+1})\mathbf{v}_i = 0 \quad (6.1.5.9)$$ $$\text{As } \lambda_i \text{ are distinct } \forall i \le k, a_i = 0 \quad (6.1.5.10)$$ Substituting this in (6.1.5.1) $$\sum_{i=1}^{k+1} a_i \mathbf{v}_i = 0 \quad (6.1.5.11)$$ $$\implies a_{k+1}\mathbf{v}_{k+1} = 0 \quad (6.1.5.12)$$ As $\mathbf{v}_{k+1} \neq 0 \implies a_{k+1} = 0$

	Since $\forall i \leq k+1, a_i = 0. S_{k+1}$ is linearly independent By principle of mathematic induction, S_n is linearly independent.

TABLE 6.1.5.1: Definitions and theorem used

Given	**T** has an n distinct characteristic values and dim(**V**) = n
T is diagonalizable	Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be distinct eigen values of **T** and let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be the eigen vectors of **T**..From above results we can state that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is linearly independent.And also given that dim(**V**) = n .So,this set forms a basis of **V**. $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for **V** consisting of eigen vectors of **T**. So, **T** is diagonalizable.

TABLE 6.1.5.2: Solution

6.1.6. Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ matrices over the field F.Prove that if $(\mathbf{I} - \mathbf{AB})$ is invertibe

a) $(\mathbf{I} - \mathbf{BA})$ is invertible and

b) $(\mathbf{I} - \mathbf{BA})^{-1} = \mathbf{I} + \mathbf{B}\,(\mathbf{I} - \mathbf{AB})^{-1}\,\mathbf{A}$

Solution: See Table 6.1.6.1

6.1.7. Usig the result of the previous exercise, prove that, if A and B are $n \times n$ matrices over the field F, then AB and BA have precisely the same characteristic values. **Solution:** See 6.1.7.1

6.1.8. Suppose that $\mathbf{A}$ is a 2×2 matrix with real entries which is symmetric ($\mathbf{A}^t = \mathbf{A}$). Prove that $\mathbf{A}$ is similar over $\mathbb{R}$ to a diagonal matrix.

Solution: See Table 6.1.8.1

Invertible	A matrix $\mathbf{M}$ is invertible if it is non-singular i.e. the null space of $\mathbf{M}$ contains only zero vector.If $\mathbf{x}$ is a vector such that $\mathbf{Mx} = 0 \implies \mathbf{x} = 0$
Proof for 1	Consider a vecor $\mathbf{y}$ such that $(\mathbf{I} - \mathbf{BA})\,\mathbf{y} = 0$ $(\mathbf{I} - \mathbf{BA})\,\mathbf{y} = 0 \implies \mathbf{y} = \mathbf{BAy}$ $\mathbf{Ay} = \mathbf{ABAy} \implies (\mathbf{I} - \mathbf{AB})\,\mathbf{Ay} = 0$ since the matrix $(\mathbf{I} - \mathbf{AB})$ is invertible,$\mathbf{Ay} = 0$ $\mathbf{y} = \mathbf{B}\,(\mathbf{Ay}) \implies \mathbf{y} = 0$ Hence the matrix $(\mathbf{I} - \mathbf{BA})$ is invertible.
Observation	Let $\mathbf{C} = (\mathbf{I} - \mathbf{AB})^{-1}$,then $(\mathbf{I} - \mathbf{BA})\,(\mathbf{BCA}) = \mathbf{BCA} - \mathbf{BABCA} = \mathbf{B}\,(\mathbf{I} - \mathbf{AB})\,\mathbf{CA} = \mathbf{BC^{-1}CA} = \mathbf{BA}$ $(\mathbf{I} - \mathbf{BA})\,(\mathbf{BCA}) = \mathbf{BA}$
Proof for 2	Let us consider the product $(\mathbf{I} - \mathbf{BA})\,(\mathbf{BCA} + \mathbf{I})$ $(\mathbf{I} - \mathbf{BA})\,(\mathbf{BCA} + \mathbf{I}) = (\mathbf{I} - \mathbf{BA})\,(\mathbf{BCA}) + (\mathbf{I} - \mathbf{BA})$ $= \mathbf{BA} + \mathbf{I} - \mathbf{BA} = \mathbf{I}$ $(\mathbf{I} - \mathbf{BA})^{-1} = \mathbf{I} + \mathbf{B}\,(\mathbf{I} - \mathbf{AB})^{\mathbf{-1}}\,\mathbf{A}$ Hence proved.

TABLE 6.1.6.1: Proof

Given	A and B are $n \times n$ matrices over the field F. In Exercise 8, If $(I - AB)$ is invertible then $(I - BA)$ is invertible.
To prove	AB and BA have precisely the same characteristic values.
Observation	We have to show that if c is a characteristic value for AB then c is a characteristic value for BA. Conversely, This is equivalent to the statement if c is not a characteristic value for AB then it is not a characteristic value for BA.
Proof	Suppose that c is not a characteristic value for BA, this means that $\lvert cI - AB\rvert \neq 0$. $\implies c^n\lvert I - \frac{1}{c}AB\rvert \neq 0$ $\therefore (I - \frac{1}{c}AB)$ is invertible $\implies (I - \frac{1}{c}BA)$ is invertible. $\implies \lvert I - \frac{1}{c}AB\rvert \neq 0$ $\implies c^n\lvert I - \frac{1}{c}AB\rvert = c^n\lvert I - \frac{1}{c}AB\rvert \neq 0$ Hence, If c is not a characteristic value for AB then it is not a characteristic value for BA. Hence, AB and BA have precisely the same characteristic value.

TABLE 6.1.7.1

Given	$\mathbf{A}$ is a 2×2 matrix with real entries and $\mathbf{A}$ is symmetric ($\mathbf{A}^t = \mathbf{A}$)
To Prove	$\mathbf{A}$ is similar to diagonal matrix over $\mathbb{R}$
Theory	$\mathbf{A}$ is similar to diagonal matrix $\mathbf{\Lambda}$ if $\exists$ an invertible matrix $\mathbf{P}$ such that: $\mathbf{A} = \mathbf{P\Lambda P}^{-1}$
Proof	Let $\mathbf{A} = \begin{pmatrix} a & c \\ c & b \end{pmatrix}, a, b, c \in \mathbb{R}$ Characteristic polynomial: $p(t) = \left\|\mathbf{A} - \lambda\mathbf{I}\right\|$ $p(t) = \begin{vmatrix} a-t & c \\ c & b-t \end{vmatrix}$ $\implies p(t) = t^2 - (a+b)t + ab - c^2 = 0$ Roots of p(t) are eigenvalues of $\mathbf{A}$ Discriminant of $p(t)$ is given by $(a+b)^2 - 4(ab - c^2) = a^2 + b^2 - 2ab + c^2$ $= (a-b)^2 + 4c^2 > 0$ We observe that the above equation has positive discriminant, hence λ has real values
	Eigen vectors are obtained by: $(\mathbf{A} - \lambda\mathbf{I})\,\mathbf{X} = 0$ Let $\mathbf{v_1}$ and $\mathbf{v_2}$ be the eigen vectors corresponding to eigen values λ_1 and λ_2 $\implies \mathbf{Av_1} = \lambda_1\mathbf{v_1}$ and $\mathbf{Av_2} = \lambda_2\mathbf{v_2}$ Let linear combination of the two eigen vectors be, $c_1\mathbf{v_1} + c_2\mathbf{v_2} = \mathbf{0}$ Multiplying both sides by λ_1, we have, $\implies c_1\lambda_1\mathbf{v_1} + c_2\lambda_1\mathbf{v_2} = 0 \qquad \ldots(1)$ Consider, $\mathbf{A.0} = \mathbf{0}$ $\implies \mathbf{A}\,(c_1\mathbf{v_1} + c_2\mathbf{v_2}) = \mathbf{0}$ $\implies c_1\,(\mathbf{Av_1}) + c_2\,(\mathbf{Av_2}) = \mathbf{0}$ $\implies c_1\lambda_1\mathbf{v_1} + c_2\lambda_2\mathbf{v_2} = 0 \qquad \ldots(2)$

	Subtracting equation (1) and (2), we have, $c_2 (\lambda_1 - \lambda_2) \mathbf{v_2} = \mathbf{0}$ Since, λ_1 and λ_2 are real and distinct, $c_2 = 0$ Similarly, $c_1 = 0$ Therefore, eigen vectors $\mathbf{v_1}$ and $\mathbf{v_2}$ are linearly independent. Let $\mathbf{P} = \begin{pmatrix} \mathbf{v_1} & \mathbf{v_2} \end{pmatrix}$ $\implies \mathbf{AP} = \begin{pmatrix} \lambda_1 \mathbf{v_1} & \lambda_2 \mathbf{v_2} \end{pmatrix}$ $\implies \mathbf{AP} = \mathbf{P\Lambda}$, where $\mathbf{\Lambda} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ $\implies \mathbf{A} = \mathbf{P\Lambda P}^{-1}$ Therefore, $\mathbf{A}$ is similar to diagonal matrix $\mathbf{\Lambda}$ Hence, Proved.

TABLE 6.1.8.1: Proving that eigen vectors are linearly independent for real eigen values and symmetric matrix is similar to diagonal matrix

6.1.9. Let $\mathbf{N}$ be a 2×2 complex matrix such that $\mathbf{N}^2 = 0$. Prove that either $\mathbf{N} = 0$ or $\mathbf{N}$ is similar over $\mathbb{C}$ to

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{6.1.9.1}$$

Solution: See Table 6.1.9.1

Statement	Solution
Proof that $\mathbf{N} = 0$	Let $\mathbf{N} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ (6.1.9.2) Since $\mathbf{N}^2 = 0$ (6.1.9.3) If $\begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix}$ are linearly independent then $\mathbf{N}$ is diagonalizable to $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. If $\mathbf{PNP}^{-1} = 0$ (6.1.9.4) then $\mathbf{N} = \mathbf{P}^{-1}\mathbf{0P} = 0$ (6.1.9.5) So in this case $\mathbf{N}$ itself is the zero matrix. This contradicts the assumption that $\begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix}$ are linearly independent. $\therefore$ we can assume that $\begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix}$ are linearly dependent if both are equal to the zero vector then $\mathbf{N} = 0$. (6.1.9.6)
Assuming $\begin{pmatrix} b \\ d \end{pmatrix}$ as the zero vector	Therefore we can assume at least one vector is non-zero. Therefore $\mathbf{N} = \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix}$ So $\mathbf{N}^2 = 0$ (6.1.9.7) $\implies a^2 = 0$ (6.1.9.8) $\therefore a = 0$ (6.1.9.9) Thus $\mathbf{N} = \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix}$ (6.1.9.10) In this case $\mathbf{N}$ is similar to $\mathbf{N} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ via the matrix $\mathbf{P} = \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}$
Assuming $\begin{pmatrix} a \\ c \end{pmatrix}$ as the zero vector	Therefore $\mathbf{N} = \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix}$ Then $\mathbf{N}^2 = 0$ (6.1.9.11) $\implies d^2 = 0$ (6.1.9.12) $\therefore d = 0$ (6.1.9.13) Thus $\mathbf{N} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ (6.1.9.14)

	In this case $\mathbf{N}$ is similar to $\mathbf{N} = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}$ via the matrix $\mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, which is similar to $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ as above.
Hence	we can assume neither $\begin{pmatrix} a \\ c \end{pmatrix}$ or $\begin{pmatrix} b \\ d \end{pmatrix}$ is the zero vector.
Consequences of linear independence	Since they are linearly dependent we can assume, $$\begin{pmatrix} b \\ d \end{pmatrix} = x \begin{pmatrix} a \\ c \end{pmatrix} \quad (6.1.9.15)$$ $$\therefore \mathbf{N} = \begin{pmatrix} a & ax \\ c & cx \end{pmatrix} \quad (6.1.9.16)$$ $$\therefore \mathbf{N}^2 = 0 \quad (6.1.9.17)$$ $$\implies a(a + cx) = 0 \quad (6.1.9.18)$$ $$c(a + cx) = 0 \quad (6.1.9.19)$$ $$ax(a + cx) = 0 \quad (6.1.9.20)$$ $$cx(a + cx) = 0 \quad (6.1.9.21)$$
Proof that $\mathbf{N}$ is similar over $\mathbb{C}$ to $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$	We know that at least one of a or c is not zero. If a = 0 then c ≠ 0, it must be that x = 0. So in this case $\mathbf{N} = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}$ which is similar to $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ as before. $$\text{If } a \neq 0 \quad (6.1.9.22)$$ $$\text{then } x \neq 0 \quad (6.1.9.23)$$ $$\text{else } a(a + cx) = 0 \quad (6.1.9.24)$$ $$\implies a = 0 \quad (6.1.9.25)$$ $$\text{Thus } a + cx = 0 \quad (6.1.9.26)$$ $$\text{Hence } \mathbf{N} = \begin{pmatrix} a & ax \\ \frac{-a}{x} & -a \end{pmatrix} \quad (6.1.9.27)$$ This is similar to $\begin{pmatrix} a & a \\ -a & -a \end{pmatrix}$ via $\mathbf{P} = \begin{pmatrix} \sqrt{x} & 0 \\ 0 & \frac{1}{\sqrt{x}} \end{pmatrix}$. And $\begin{pmatrix} a & a \\ -a & -a \end{pmatrix}$ is similar to $\begin{pmatrix} 0 & 0 \\ -a & 0 \end{pmatrix}$ via $\mathbf{P} = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$ And this finally is similar to $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ as before.

Conclusion	Thus either $\mathbf{N} = 0$ or $\mathbf{N}$ is similar over $\mathbb{C}$ to $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

TABLE 6.1.9.1: Solution summary

6.1.10. Let A be the an $n \times n$ diagonal matrix with characteristic polynomial

$$(x-c_1)^{d_1}(x-c_2)^{d_2}\dots(x-c_k)^{d_k} \quad (6.1.10.1)$$

Where $c_1, c_2, \dots c_k$ are distinct. Let $\mathbf{V}$ the space of $n \times n$ matrices B such that

$$AB = BA \quad (6.1.10.2)$$

Prove that the dimension of $\mathbf{V}$ is,

$$d_1{}^2 + d_2{}^2 \dots + d_k{}^2 \quad (6.1.10.3)$$

Solution: Let consider we have a matrix A which is a diagonal matrix, which is given as

$$A = \begin{pmatrix} c_1I & 0 & 0 & \dots & 0 & 0 \\ 0 & c_2I & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & . & . \\ . & . & . & \dots & . & . \\ 0 & 0 & . & \dots & . & c_kI \end{pmatrix} \quad (6.1.10.4)$$

Consider B as :

$$B = \begin{pmatrix} B_{11} & B_{12} & . & \dots & . & B_{1k} \\ B_{21} & B_{22} & . & \dots & . & B_{2k} \\ . & . & . & \dots & . & . \\ . & . & . & \dots & . & . \\ B_{k1} & B_{k2} & . & \dots & . & B_{kk} \end{pmatrix} \quad (6.1.10.5)$$

Where B_{ij} has dimension $d_i \times d_j$. Since we have given ,

$$AB = BA \quad (6.1.10.6)$$

$$\implies \begin{pmatrix} c_1B_{11} & c_1B_{12} & . & \dots & . & c_1B_{1k} \\ c_2B_{21} & c_2B_{22} & . & \dots & . & c_2B_{2k} \\ . & . & . & \dots & . & . \\ . & . & . & \dots & . & . \\ c_kB_{k1} & c_kB_{k2} & . & \dots & . & c_kB_{kk} \end{pmatrix} =$$

$$\begin{pmatrix} c_1B_{11} & B_{12} & . & \dots & . & c_1B_{1k} \\ c_2B_{21} & B_{22} & . & \dots & . & c_2B_{2k} \\ . & . & . & \dots & . & . \\ . & . & . & \dots & . & . \\ c_kB_{k1} & c_kB_{k2} & . & \dots & . & c_kB_{kk} \end{pmatrix} \quad (6.1.10.7)$$

Hence, from above equation (6.1.10.7) we can conclude,

$$c_i \neq c_j, \forall i \neq j \quad (6.1.10.8)$$
$$\implies B_{ij} = 0, \forall i \neq j \quad (6.1.10.9)$$

We can have $B_{11}, B_{22} \dots$ any arbitrary matrices. From (6.1.10.7) we can have

$$D(B_{ij}) = d_i^2 \quad (6.1.10.10)$$

Where D represents dimension of matrix. Therefore the dimension of the space of all such B_{ij}'s matrices is given as :

$$d_1{}^2 + d_2{}^2 \dots + d_k{}^2 \quad (6.1.10.11)$$

Let suppose we have matrix A as :

$$A = \begin{pmatrix} c_1I & \mathbf{0} \\ \mathbf{0} & c_2I \end{pmatrix} \quad (6.1.10.12)$$

Where,

$$c_1I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, c_2I = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \quad (6.1.10.13)$$

$$\implies A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \quad (6.1.10.14)$$

and B as :

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

(6.1.10.15)

Where,

$$B_{11} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, B_{12} = \begin{pmatrix} b_{13} & b_{14} \\ b_{23} & b_{24} \end{pmatrix}$$
(6.1.10.16)
(6.1.10.17)

$$B_{21} = \begin{pmatrix} b_{31} & b_{32} \\ b_{41} & b_{42} \end{pmatrix}, B_{22} = \begin{pmatrix} b_{33} & b_{34} \\ b_{43} & b_{44} \end{pmatrix}$$
(6.1.10.18)
(6.1.10.19)

$$\implies B = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{pmatrix}$$
(6.1.10.20)

Consider,

$$C = AB \tag{6.1.10.21}$$

$$\implies C = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ 2b_{31} & 2b_{32} & 2b_{33} & 2b_{34} \\ 2b_{41} & 2b_{42} & 2b_{43} & 2b_{44} \end{pmatrix} \tag{6.1.10.22}$$

Let another matrix D as :

$$D = BA \tag{6.1.10.23}$$

$$B = \begin{pmatrix} b_{11} & b_{12} & 2b_{13} & 2b_{14} \\ b_{21} & b_{22} & 2b_{23} & 2b_{24} \\ b_{31} & b_{32} & 2b_{33} & 2b_{34} \\ b_{41} & b_{42} & 2b_{43} & 2b_{44} \end{pmatrix} \tag{6.1.10.24}$$

We have given as,

$$BA = AB \tag{6.1.10.25}$$

$$\implies C = D \tag{6.1.10.26}$$

it is possible only when,

$$b_{13} = b_{14} = b_{23} = b_{24} = 0 \tag{6.1.10.27}$$

And,

$$b_{31} = b_{32} = b_{41} = b_{42} = 0 \tag{6.1.10.28}$$

$$B_{12} = \begin{pmatrix} b_{13} & b_{14} \\ b_{23} & b_{24} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{6.1.10.29}$$

And,

$$B_{21} = \begin{pmatrix} b_{31} & b_{32} \\ b_{41} & b_{42} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{6.1.10.30}$$

Hence, therefore matrix B becomes,

$$B = \begin{pmatrix} b_{11} & b_{12} & 0 & 0 \\ b_{21} & b_{22} & 0 & 0 \\ 0 & 0 & 2b_{33} & 2b_{34} \\ 0 & 0 & 2b_{43} & 2b_{44} \end{pmatrix} \tag{6.1.10.31}$$

$$\implies B = \begin{pmatrix} c_1 B_{11} & \mathbf{0} \\ \mathbf{0} & c_2 B_{22} \end{pmatrix} \tag{6.1.10.32}$$

$$\implies B_{ij} = 0, \forall i \neq j \tag{6.1.10.33}$$

Now the basis of the $n \times n$ matrices for vector space of all $n \times n$ matrix B are,

$$\beta_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \beta_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{6.1.10.34}$$

$$\beta_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \beta_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{6.1.10.35}$$

$$\beta_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \beta_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{6.1.10.36}$$

$$\beta_7 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \beta_8 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{6.1.10.37}$$

Thus, Dimension of **V** (vector space of all $n \times n$ matrices B) = 8,

Also

$$d_1{}^2 + d_2{}^2 = 2^2 + 2^2 = 8. \tag{6.1.10.38}$$

Therefore, Dimension of **V** (vector space of all $n \times n$ matrix B is :

$$d_1^2 + d_2^2 \tag{6.1.10.39}$$

6.1.11. Let **V** be the space of $n \times n$ matrices over F. Let **A** be a fixed $n \times n$ matrix over F. Let **T** be the linear operator 'left multiplication by **A**' on **V**. Is it true that **A** and **T** have the same characteristic values?

Solution: See Table 6.1.11.1

Given	$\mathbf{V}$ is the space of $n \times n$ matrices over F $\mathbf{A}$ is a fixed $n \times n$ matrix over F $\mathbf{T}$ be the linear operator on $\mathbf{V}$ such that $\mathbf{T}(\mathbf{B}) = \mathbf{AB}$
To prove	$\mathbf{A}$ and $\mathbf{T}$ have the same characteristic values
Theorem	Let λ be a characteristic value of $\mathbf{T}$ and $\lambda \in F$ and $\mathbf{v}$ is the corresponding characteristic vector which is a $n \times n$ matrix, then if $\mathbf{T}$ is a linear operator on a finite dimensional space $\mathbf{V}$, it must be $\det(T - \lambda I) = 0$ and for $(\mathbf{T} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$, $\mathbf{v} \neq \mathbf{0}$
Proof	As per the problem statement, $\mathbf{Tv} = \lambda\mathbf{v}$ **or,** $\mathbf{Av} = \lambda\mathbf{v}$, as $\mathbf{Tv} = \mathbf{Av}$ **or,** $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$ From here 2 cases can be arrived.
Case 1:	$\det((\mathbf{A} - \lambda\mathbf{I})) = 0$, where $\mathbf{v} \neq \mathbf{0}$ $\implies$ λ is characteristic value of $\mathbf{A}$.
Case 2:	$\det((\mathbf{A} - \lambda\mathbf{I})) \neq 0$ $\implies$ $(\mathbf{A} - \lambda\mathbf{I})$ is invertible and $\mathbf{v} = \mathbf{0}$ so, for $(\mathbf{T} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$ and $(\mathbf{T} - \lambda\mathbf{I}) \neq \mathbf{0}$ $\implies$ $\mathbf{v}$ is not a charcteristic vector of $\mathbf{T}$ which is a contradiction. So, case 2 is not possible.
Conclusion	So, from the above 2 cases and from the theorem, it can be concluded that $\mathbf{A}$ and $\mathbf{T}$ have the same characteristic values

TABLE 6.1.11.1: **Solution summary**

6.2.1. Let a, b and c be the elements of a field **F**, and let **A** be the following 3× 3 matrix over **F**

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & c \\ 1 & 0 & b \\ 0 & 1 & a \end{pmatrix} \tag{6.2.1.1}$$

Prove that the characteristic polynomial for **A** is $x^3 - ax^2 - bx - c$ and that this is also minimal polynomial for **A**. **Solution:** Minimal polynomial of **A** is a polynomial which satisfies,

1) P(**A**) = 0
2) P(x) is monic.
3) It there is some other annihilating polynomial q(x) such that, q(**A**) = 0, then q does not divide p.

The characteristic polynomial is calculated by solving $\left|\mathbf{A} - \lambda\mathbf{I}\right| = 0$

$$\implies \left|\mathbf{A} - \lambda\mathbf{I}\right| = \begin{vmatrix} -\lambda & 0 & c \\ 1 & -\lambda & b \\ 0 & 1 & a-\lambda \end{vmatrix} \tag{6.2.1.2}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 + \lambda R_3} \begin{vmatrix} -\lambda & 0 & c \\ 1 & 0 & b + a\lambda - \lambda^2 \\ 0 & 1 & a - \lambda \end{vmatrix} \tag{6.2.1.3}$$

$$\implies \left|\mathbf{A} - \lambda\mathbf{I}\right| = 1 \begin{vmatrix} -\lambda & c \\ 1 & b + a\lambda - \lambda^2 \end{vmatrix} \tag{6.2.1.4}$$

$$\implies \left|\mathbf{A} - \lambda\mathbf{I}\right| = (-\lambda)(b + a\lambda - \lambda^2) - c \tag{6.2.1.5}$$

Hence the characteristic polynomial of **A** is,

$$\lambda^3 - a\lambda^2 - b\lambda - c \tag{6.2.1.6}$$

Now for any r,s ∈ **F** and considering the annihilating polynomial f with degree 2.

$$f(\mathbf{A}) = \mathbf{A}^2 + r\mathbf{A} + s \tag{6.2.1.7}$$

$$\implies \begin{pmatrix} 0 & c & ac \\ 0 & b & c+ba \\ 1 & a & b+a^2 \end{pmatrix} + \begin{pmatrix} 0 & 0 & rc \\ r & 0 & rb \\ 0 & r & ra \end{pmatrix} + \begin{pmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{pmatrix} \tag{6.2.1.8}$$

$$\therefore f(\mathbf{A}) = \mathbf{A}^2 + r\mathbf{A} + s = \begin{pmatrix} s & c & ac + rc \\ r & b+s & c + ba + br \\ 1 & a+r & b + a^2 + ra + s \end{pmatrix} \neq 0 \tag{6.2.1.9}$$

Element positioned at row-3 and column-1 is non-zero, hence for any $r, s \in \mathbf{F} \implies f(\mathbf{A}) \neq 0 \forall f \in \mathbf{F}$. Hence minimal polynomial cannot have degree 2. Hence degree of minimal polynomial is 3. Also $x^3 - ax^2 - bx - c$ divides f. Hence from definition-1 we can conclude that,

$$p(x) = x^3 - ax^2 - bx - c \tag{6.2.1.10}$$

is a minimal polynomial.
Example: Let a = 0,b = 0,c = 0 ∈ **F**. Hence,

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{6.2.1.11}$$

Now finding characteristic polynomial by substituting the values of a,b, and c in equation (6.2.1.6) we get,

$$\lambda^3 = 0 \tag{6.2.1.12}$$

Now let r=0, s=0 ∈ **F**, Hence f(**A**) is given by using the equation (6.2.1.9),

$$\implies f(\mathbf{A}) = \mathbf{A}^2 + 0.\mathbf{A} + 0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \neq 0 \tag{6.2.1.13}$$

Hence, $f(\mathbf{A}) \neq 0$, Hence degree of minimal polynomial is 3 and is equal to,

$$p(x) = x^3 \tag{6.2.1.14}$$

Verification by calculating p(**A**),

$$p(\mathbf{A}) = \mathbf{A}^3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{6.2.1.15}$$

$$\implies \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{6.2.1.16}$$

$$\implies f(\mathbf{A}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0 \tag{6.2.1.17}$$

Hence, from definition-1 it can be concluded that p(x) = x^3 is a minimal polynomial. For

Characteristic Polynomial	For an $n \times n$ matrix $\mathbf{A}$, characteristic polynomial is defined by, $$p(x) = \left\lvert x\mathbf{I} - \mathbf{A} \right\rvert$$
Cayley-Hamilton Theorem	If $p(x)$ is the characteristic polynomial of an $n \times n$ matrix $\mathbf{A}$, then, $$p(\mathbf{A}) = \mathbf{0}$$
Minimal Polynomial	Minimal polynomial $m(x)$ is he smallest factor of characteristic polynomial $p(x)$ such that, $$m(\mathbf{A}) = 0$$
Algebraic Multiplicity (A_M)	No. of times an eigen value appears in a characteristic equation.

TABLE 6.2.2.1: Definitions

the $\mathbf{A}$ given by equation (6.2.1.1) ,characteristic and minimal polynomial is given by,

$$x^3 - ax^2 - bx - c \tag{6.2.1.18}$$

6.2.2. Let $\mathbf{A}$ be the 4×4 real matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -2 & -2 & 2 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix} \tag{6.2.2.1}$$

Show that the characteristic polynomial for $\mathbf{A}$ is $x^2(x-1)^2$ and that it is also the minimal polynomial

Solution: See Tables 6.2.2.1 and 6.2.2.2

Characteristic polynomial	$\begin{aligned} p(x) &= \lvert x\mathbf{I} - \mathbf{A} \rvert \\ &= \begin{vmatrix} x-1 & -1 \\ 1 & x+1 \end{vmatrix} \begin{vmatrix} x-2 & -1 \\ 1 & x \end{vmatrix} \\ &= ((x-1)(x+1)+1)((x-2)x+1) \\ &= x^2\left(x^2-2x+1\right) \\ &= x^2(x-1)^2 \end{aligned}$
A_M	For $\lambda = 0$, $A_M = 2$ For $\lambda = 1$, $A_M = 2$
Minimal Polynomial	$p(x) = x^a(x-1)^b, \quad a \le 2, b \le 2$
$a = 1, b = 1$	$m(x) = x(x-1)$ $\implies m(\mathbf{A}) = \mathbf{A}(\mathbf{A} - \mathbf{I})$ $= \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -2 & -2 & 2 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 \\ -2 & -2 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix}$ $= \begin{pmatrix} -1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & -2 & -2 \end{pmatrix} \neq \mathbf{0}$ $\implies x(x-1)$ is not a minimal polynomial
$a = 2, b = 1$	$m(x) = x^2(x-1)$ $\implies m(\mathbf{A}) = \mathbf{A}^2(\mathbf{A} - \mathbf{I})$ $= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -3 & -3 & 3 & 2 \\ 2 & 2 & -2 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 \\ -2 & -2 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix}$ $= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix} \neq \mathbf{0}$ $\implies x^2(x-1)$ is not a minimal polynomial
$a = 1, b = 2$	

	$m(x) = x(x-1)^2$ $\implies m(\mathbf{A}) = \mathbf{A}(\mathbf{A}-\mathbf{I})^2$ $= \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -2 & -2 & 2 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -2 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $= \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq \mathbf{0}$ $\implies x^2(x-1)$ is not a minimal polynomial
$a = 2, b = 2$	$m(x) = x^2(x-1)^2$ $\implies m(\mathbf{A}) = \mathbf{A}^2(\mathbf{A}-\mathbf{I})^2$ $= p(\mathbf{A})$ $= \mathbf{0}$ (Cayley-Hamilton Theorem) $\implies x^2(x-1)^2$ is a minimal polynomial
Conclusion	For the given matrix $\mathbf{A}$, $x^2(x-1)^2$ is the characteristic polynomial as well as minimal polynomial.

TABLE 6.2.2.2: Checking for minimal polynomial

Characteristic Polynomial	For an $n \times n$ matrix $\mathbf{A}$, characteristic polynomial is defined by, $$p(x) = \lvert x\mathbf{I} - \mathbf{A} \rvert$$
Cayley-Hamilton Theorem	If $p(x)$ is the characteristic polynomial of an $n \times n$ matrix $\mathbf{A}$, then, $$p(\mathbf{A}) = \mathbf{0}$$
Minimal Polynomial	Minimal polynomial $m(x)$ is the smallest factor of characteristic polynomial $p(x)$ such that, $$m(\mathbf{A}) = 0$$ Every root of characteristic polynomial should be the root of minimal polynomial

TABLE 6.2.3.1: Definitions

6.2.3. Find a 3×3 matrix for which the minimal polynomial is x^2.
Solution: See Tables 6.2.3.1 and 6.2.3.2

Statement	Solution
Assuming matrix **A** as follows:	Let us Consider 3 × 3 upper triangular matrix, $$\mathbf{A} = \begin{pmatrix} e & a & b \\ 0 & f & c \\ 0 & 0 & d \end{pmatrix}$$
Characteristic polynomial of **A**	$$\lvert x\mathbf{I} - \mathbf{A} \rvert = \begin{pmatrix} x-e & -a & -b \\ 0 & x-f & -c \\ 0 & 0 & x-d \end{pmatrix}$$ $$= (x-e)(x-f)(x-d)$$
Given	The minimum polynomial is $$p(x) = x^2$$ Therefore p(x) must divide characteristic polynomial. This will be satisfied only if the values e,f,d are zeros.
Characteristic polynomial when e=0,f=0 and d=0	$$\lvert x\mathbf{I} - \mathbf{A} \rvert = x^3$$
Since $p(x) = x^2$ Hence $p(\mathbf{A}) = \mathbf{A}^2 = \mathbf{0}_{3\times3}$	Therefore calculating $p(\mathbf{A})$ as follows: $$\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \mathbf{0}_{3\times3}$$ $$\begin{pmatrix} 0 & 0 & ac \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathbf{0}_{3\times3}$$

With entries a=0,e=0,f=0,d=0 The matrix **A** will be: For b=1,c=1	For $\mathbf{A}^2$ to be a zero matrix, either a=0 or c=0 $$\mathbf{A} = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}$$ $$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$
Conclusion	Thus the matrix, $$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$ has the minimal polynomial as x^2.

TABLE 6.2.3.2: Solution summary

6.2.4. Let n be a positive integer, and let V be the space of polynomials over $\mathbb{R}$ which have degree at most n (throw in the 0-polynomial). Let **D** be the differentiation operator on V. What is the minimal polynomial for **D**?
Solution: See Tables 6.2.4.1 and 6.2.4.2

Characteristic Polynomial	For an $n \times n$ matrix $\mathbf{A}$, characteristic polynomial is defined by, $$p(x) = \left\lvert x\mathbf{I} - \mathbf{A} \right\rvert$$
Minimal Polynomial	Minimal polynomial $m(x)$ is the smallest factor of characteristic polynomial $p(x)$ such that, $$m(\mathbf{A}) = 0$$ Every root of characteristic polynomial should be the root of minimal polynomial

TABLE 6.2.4.1: Definitions and theorem used

Given	V is the space of polynomials over $\mathbb{R}$ which have degree at most n.
Matrix Representation	The basis for the space V is $$\mathcal{B} = \left\{1, x, x^2, \dots, x^n\right\} \tag{6.2.4.1}$$ Given that $\mathbf{D}$ is the differentiation operator.So, $$\mathbf{D}(1) = 0 \tag{6.2.4.2}$$ $$\mathbf{D}(x) = 1 \tag{6.2.4.3}$$ $$\vdots$$ $$\mathbf{D}(x^n) = nx^{n-1} \tag{6.2.4.4}$$ The vectors of differentiation operator with respect to basis $\mathcal{B}$ $$[\mathbf{D}(1)]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{(n+1)\times 1}, [\mathbf{D}(x)]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{(n+1)\times 1} \dots [\mathbf{D}(x^n)]_{\mathcal{B}} = \begin{pmatrix} 0 \\ \vdots \\ n \\ 0 \end{pmatrix}_{(n+1)\times 1} \tag{6.2.4.5}$$ The matrix representation can be written as: $$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & n \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \tag{6.2.4.6}$$

<table>
<tr><td>Characteristic polynomial</td><td>

$$p(x) = \left|x\mathbf{I} - \mathbf{A}\right| = \begin{vmatrix} x & -1 & 0 & \dots & 0 \\ 0 & x & -2 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & -n \\ 0 & 0 & 0 & \dots & x \end{vmatrix} \quad (6.2.4.7)$$

It is equal to the product of diagonal entries.

$$p(x) = x^{n+1} \quad (6.2.4.8)$$

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<tr><td>Minimal Polynomial</td><td>

The minimal polynomial of $\mathbf{A}$ can be any of $x, x^2, \dots, x^{n+1}$ such that,

$$m(\mathbf{A}) = 0 \quad (6.2.4.9)$$

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<tr><td>Explanation</td><td>

Let $P(n)$: Minimum polynomial of $\mathbf{D}$=x^{n+1} i.e $\mathbf{A}^{n+1} = 0$
For n=1

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (6.2.4.10)$$

$$\mathbf{A^2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad (6.2.4.11)$$

So,$P(1)$ is true.
Assume $P(k)$ holds for $1 \le k \le n$.

$$\mathbf{A}_k = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & k \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}_{(k+1 \times k+1)} \implies \mathbf{A}_k^{k+1} = \mathbf{0} \quad (6.2.4.12)$$

We need to show that $P(k+1)$ is true.

$$\mathbf{A}_{k+1} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & k & 0 \\ 0 & 0 & 0 & \dots & 0 & k+1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}_{(k+2 \times k+2)} \quad (6.2.4.13)$$

Expressing in terms of block matrices

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</table>

	$$\mathbf{A}_{k+1} = \begin{pmatrix} \mathbf{A}_k & \mathbf{x} \\ \mathbf{0}_{1\times k+1} & 0 \end{pmatrix}, \mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ k+1 \end{pmatrix}_{k+1\times 1} \quad (6.2.4.14)$$
Finding $\mathbf{A}_{k+1}^{k+2}$	$$\mathbf{A}_{k+1}^2 = \begin{pmatrix} \mathbf{A}_k & \mathbf{x} \\ \mathbf{0} & 0 \end{pmatrix}\begin{pmatrix} \mathbf{A}_k & \mathbf{x} \\ \mathbf{0} & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{A}_k^2 & \mathbf{A}_k\mathbf{x} \\ 0 & 0 \end{pmatrix} \quad (6.2.4.15)$$ $$\mathbf{A}_{k+1}^3 = \begin{pmatrix} \mathbf{A}_k^2 & \mathbf{A}_k\mathbf{x} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \mathbf{A}_k & \mathbf{x} \\ \mathbf{0} & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{A}_k^3 & \mathbf{A}_k^2\mathbf{x} \\ 0 & 0 \end{pmatrix} \quad (6.2.4.16)$$ $$\mathbf{A}_{k+1}^{k+2} = \begin{pmatrix} \mathbf{A}_k^{k+2} & \mathbf{A}_k^{k+1}\mathbf{x} \\ 0 & 0 \end{pmatrix} \quad (6.2.4.17)$$ From (6.2.4.12),We know that $\mathbf{A}_k^{k+1} = \mathbf{0}$ $$\implies \mathbf{A}_{k+1}^{k+2} = \mathbf{0} \quad (6.2.4.18)$$ So,$P(k+1)$ is true.
Conclusion	From above,by using the principle of induction we can say that the minimal polynomial is $$x^{n+1} \quad (6.2.4.19)$$

TABLE 6.2.4.2: Finding minimal polynomial

Characteristic Polynomial	For an $n \times n$ matrix $\mathbf{A}$, characteristic polynomial is defined by, $$p(x) = \lvert x\mathbf{I} - \mathbf{A} \rvert$$
Cayley-Hamilton Theorem	If $p(x)$ is the characteristic polynomial of an $n \times n$ matrix $\mathbf{A}$, then, $$p(\mathbf{A}) = \mathbf{0}$$
Minimal Polynomial	Minimal polynomial $m(x)$ is the smallest factor of characteristic polynomial $p(x)$ such that, $$m(\mathbf{A}) = 0$$ Every root of characteristic polynomial should be the root of minimal polynomial

TABLE 6.2.6.1: Definitions

6.2.5. Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ matrices over the field F.Prove that if $(\mathbf{I} - \mathbf{AB})$ is invertibe

a) $(\mathbf{I} - \mathbf{BA})$ is invertible and

b) $(\mathbf{I} - \mathbf{BA})^{-1} = \mathbf{I} + \mathbf{B}(\mathbf{I} - \mathbf{AB})^{-1}\mathbf{A}$

6.2.6. Let P be the operator on R^2 which projects each vector onto the x-axis, parallel to the y-axis: $p(x, y) = (x, 0)$. Show that P is linear. What is the minimal polynomial for P?
Solution: See Tables 6.2.6.1 and 6.2.6.2.

Proof of P is linear	Consider two vectors (x_1, y_1) and (x_2, y_2), then $P((x_1, y_1) + (x_2, y_2)) = P(x_1 + x_2, y_1 + y_2)$ $P((x_1, y_1) + (x_2, y_2)) = P(x_1 + x_2, 0)$ $P((x_1, y_1) + (x_2, y_2)) = P((x_1, 0), (x_2, 0))$ Now, consider some scalar k, then $P(k(x_1, y_1)) = P((kx_1, ky_1))$ $P(k(x_1, y_1)) = P((kx_1, 0))$ $P(k(x_1, y_1)) = kP(x_1, 0)$ Thus, using above observations we can conclude that P is linear.
Matrix form of Projection	For the projection $P(x, y) = (x, 0)$, the matrix of linear transform is, $P(x, y) = \mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = (x, 0)$ So, $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
Characteristic polynomial	$p(x) = \lvert x\mathbf{I} - \mathbf{A} \rvert$ $= \begin{vmatrix} x-1 & 0 \\ 0 & x-0 \end{vmatrix}$ $= x(x-1)$
Minimal Polynomial	$p(x) = x^a (x-1)^b, \quad a \leq 1, b \leq 1$
$a = 1, b = 1$	$m(x) = x(x-1)$ $\implies m(\mathbf{A}) = \mathbf{A}(\mathbf{A} - \mathbf{I}) = 0$ $\implies x(x-1)$ is a minimal polynomial
Conclusion	For the given matrix $\mathbf{A}$, $x(x-1)$ is the characteristic polynomial as well as minimal polynomial.

TABLE 6.2.6.2: Illustration of Proof and finding of minimal polynomial

6.2.7. Let **A** be an $n \times n$ matrix with characteristics polynomial

$$f = (x - c_1)^{d_1} \ldots. (x - c_k)^{d_k} \qquad (6.2.7.1)$$

Show that

$$c_1 d_1 + \ldots. + c_k d_k = tr(A) \qquad (6.2.7.2)$$

Solution: See Table 6.2.7.1.

Given	Let **A** be an $n \times n$ $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$ and Characteristics polynomial $f = (x - c_1)^{d_1} ....(x - c_k)^{d_k}$
To prove	$c_1 d_1 + ... + c_k d_k = tr(A)$
proof	Characteristics polynomial; $f = (x - c_1)^{d_1} ....(x - c_k)^{d_k}$ here, $c_1, ..., c_k$ are the distinct eigen values. and $d_1, ..., d_k$ denotes the repetition of eigen values Therefore, $\boxed{c_1 d_1 + c_2 d_2 + ... + c_k d_k = \sum_i \lambda_i}$ =Sum of all eigen values. As we know , Trace of a matrix is the sum of its eigen values. $\implies tr(A) = \sum_i \lambda_i$ therefore, $\boxed{\implies c_1 d_1 + c_2 d_2 + ... + c_k d_k = \sum_i \lambda_i = tr(A)}$ Hence, Proved.

TABLE 6.2.7.1: Solution Summary

6.2.8. Let **V** be the vector space of $n \times n$ matrices over field **F**. Let **A** be a fixed $n \times n$ matrix. Let **T** be the linear operator on **V** defined by

$$\mathbf{T}(\mathbf{B}) = \mathbf{AB} \qquad (6.2.8.1)$$

Show that the minimal polynomial for **T** is the minimal polynomial for **A**.

Solution: See Tables 6.2.8.1, 6.2.8.2 and 6.2.8.3.

Given	$\mathbf{A}$ is a fixed matrix from the vector space $\mathbf{V}$ of $n \times n$ matrices. A linear operator on the finite dimensional vector space $\mathbf{V}$, $\mathbf{T}$ is defined as $\mathbf{T}(\mathbf{B}) = \mathbf{AB}$.
Minimal polynomial	The minimal polynomial of a linear operator $\mathbf{T}$ is a monic polynomial which annihilates $\mathbf{T}$.
Matrix representation of $\mathbf{T}$	If we stack up the columns of the matrix $\mathbf{B}$,the linear operator $\mathbf{T}$ can be represented in the equivalent form as If $\mathbf{B} = \begin{pmatrix} b_1 & b_2 & . & . & b_n \end{pmatrix}$,then the linear transformation of $\mathbf{B}$ will be $\mathbf{T}(\mathbf{B}) = \begin{pmatrix} \mathbf{A}b_1 & \mathbf{A}b_2 & . & .\mathbf{A}b_n \end{pmatrix}$ $\mathbf{M_T}(\mathbf{B}) = \begin{pmatrix} \mathbf{T}(b_1) \\ \mathbf{T}(b_2) \\ . \\ . \\ \mathbf{T}(b_n) \end{pmatrix} = \begin{pmatrix} \mathbf{A} & & & \\ & \mathbf{A} & \mathbf{O} & \\ & & . & \\ & \mathbf{O} & . & \\ & & & \mathbf{A} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ . \\ . \\ b_n \end{pmatrix}$
	$\mathbf{M_T} = \begin{pmatrix} \mathbf{A} & & & \\ & \mathbf{A} & \mathbf{O} & \\ & & . & \\ & \mathbf{O} & . & \\ & & & \mathbf{A} \end{pmatrix}$

TABLE 6.2.8.1: Construction

Properties of minimal polynomial	The roots of the characteristic polynomial,eigen values and the minimal polynomial are same,except for multiplicities.The roots of the minimal polynomial of $\mathbf{A}$ are the roots of $\det(\mathbf{A} - \lambda\mathbf{I})$
The roots of minimal polynomial of $\mathbf{T}$	The roots of the minimal polynomial of $\mathbf{T}$ are the roots of $\det(\mathbf{T} - \lambda\mathbf{I})$ $\det(\mathbf{T} - \lambda\mathbf{I}) = \begin{vmatrix} (\mathbf{A} - \lambda\mathbf{I}) & & & \\ & (\mathbf{A} - \lambda\mathbf{I}) & \mathbf{O} & \\ & & . & \\ & \mathbf{O} & . & \\ & & & (\mathbf{A} - \lambda\mathbf{I}) \end{vmatrix}$ $=(\det(\mathbf{A} - \lambda\mathbf{I}))^n$ Therfore we can see that the eigen values of $\mathbf{A}$ are also the eigen values of the linear operator $\mathbf{T}$
Minimal polynomial of $\mathbf{T}$	The minimal polynomial of $\mathbf{A}$ divides the characteristic polynomial of $\mathbf{A}$ and $\mathbf{T}$. Let the minimal polynomial of $\mathbf{A}$ is of degree $p \leq n$ $f(x) = a_0 + a_1x + a_2x^2 ... a_px^p$ such that $f(\mathbf{A}) = 0$ $f(\mathbf{T}) = a_0\mathbf{I} + a_1\mathbf{T} + a_2\mathbf{T}^2 + .. + a_p\mathbf{T}^p$
	$f(\mathbf{T}) = \begin{pmatrix} f(\mathbf{A}) & & & \\ & f(\mathbf{A}) & \mathbf{O} & \\ & & . & \\ & \mathbf{O} & . & \\ & & & f(\mathbf{A}) \end{pmatrix} = \mathbf{O}_{n^2 \times n^2}$ Therefore the minimal polynomial for $\mathbf{T}$ is the minimal polynomial for $\mathbf{A}$.

TABLE 6.2.8.2: Proof

Assuming matrix **A** as follows:	Let us Consider 2×2 matrix, $$\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 0 & 2 \end{pmatrix}$$
Minimal polynomial of **A**	The eigen values of **A** are $1, 2$. So, the minimal polynomial is $f(x) = (x-1)(x-2)$
Matrix of linear operator	So, the matrix of the linear operator **T** with respect to the basis $\mathbf{e_1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{e_2} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{e_3} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{e_4} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ $\mathbf{T}(\mathbf{e_1}) = \begin{pmatrix} 1 & 4 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 1.\mathbf{e_1} + 0.\mathbf{e_2} + 0.\mathbf{e_3} + 0.\mathbf{e_4}$ $\mathbf{T}(\mathbf{e_2}) = \begin{pmatrix} 1 & 4 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 2 & 0 \end{pmatrix} = 4.\mathbf{e_1} + 2.\mathbf{e_2} + 0.\mathbf{e_3} + 0.\mathbf{e_4}$ $\mathbf{T}(\mathbf{e_3}) = \begin{pmatrix} 1 & 4 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0.\mathbf{e_1} + 0.\mathbf{e_2} + 1.\mathbf{e_3} + 0.\mathbf{e_4}$
	$\mathbf{T}(\mathbf{e_4}) = \begin{pmatrix} 1 & 4 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 4 \\ 0 & 2 \end{pmatrix} = 0.\mathbf{e_1} + 0.\mathbf{e_2} + 4.\mathbf{e_3} + 2.\mathbf{e_4}$ So the matrix of the linear operator will be $\mathbf{M_T} = \begin{pmatrix} 1 & 4 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} \mathbf{A} & \mathbf{O} \\ \mathbf{O} & \mathbf{A} \end{pmatrix}$
Characteristic equation of **T**	The characteristic equation of **T** is $(x-1)^2 (x-2)^2$ So the eigen values are $1, 1, 2, 2$
Minimal polynomial of **T**	$f(\mathbf{M_T}) = (\mathbf{T} - \mathbf{I})(\mathbf{T} - \mathbf{2I}) = \begin{pmatrix} \mathbf{A} - \mathbf{I} & \mathbf{O} \\ \mathbf{O} & \mathbf{A} - \mathbf{I} \end{pmatrix}\begin{pmatrix} \mathbf{A} - \mathbf{2I} & \mathbf{O} \\ \mathbf{O} & \mathbf{A} - \mathbf{2I} \end{pmatrix}$ $f(\mathbf{M_T}) = \begin{pmatrix} (\mathbf{A} - \mathbf{I})(\mathbf{A} - \mathbf{2I}) & \mathbf{O} \\ \mathbf{O} & (\mathbf{A} - \mathbf{I})(\mathbf{A} - \mathbf{2I}) \end{pmatrix} = \begin{pmatrix} f(\mathbf{A}) & \mathbf{O} \\ \mathbf{O} & f(\mathbf{A}) \end{pmatrix} = \mathbf{O}$ We know that eigen values of **T** should be roots of minimal polynomial of **T**, thus minimal polynomial should be of the form $(x-1)^p (x-2)^q$ where $p, q \in \mathbb{N}..1 \le p, q \le 2$ Therefore the minimal polynomial $f(\mathbf{A})$ of **A** annihilates **T**,thus we can conclude that $f(x)$ is the minimal polynomial of linear operator **T**

TABLE 6.2.8.3: Example

6.2.9. Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ matrices over the field $\mathbf{F}$. If the matrices $\mathbf{AB}$ and $\mathbf{BA}$ have the same characteristic values. Do they have the same characteristic polynomial? Do they have the same minimal polynomial?

Solution:

Theorem 6.3. *If* $\mathbf{T}$ *is a linear operator on a finite-dimensional space* $\mathbf{V}$ *and* c *is a characteristic value of* $\mathbf{T}$*, then the operator* $(c\mathbf{I} - \mathbf{T})$ *is singular, i.e.*

$$det\,(c\mathbf{I} - \mathbf{T}) = 0$$

Theorem 6.4. *If* $\mathbf{A}$ *and* $\mathbf{B}$ *are* $n \times n$ *matrix, then* $(\mathbf{I} - \mathbf{AB})$ *is invertible if and only if* $(\mathbf{I} - \mathbf{BA})$ *is invertible.*

To prove that $\mathbf{AB}$ and $\mathbf{BA}$ have the same characteristic values in $\mathbf{F}$, we can use theorem 6.3 and show

$$det\,(c\mathbf{I} - \mathbf{AB}) = det\,(c\mathbf{I} - \mathbf{BA}) = 0 \quad (6.2.9.1)$$

That is, if c is a characteristic value for $\mathbf{AB}$ then c is a characteristic value for $\mathbf{BA}$ as well. Which can be also said as, if c is not the characteristic value of $\mathbf{BA}$ then it won't be the characteristic value for $\mathbf{AB}$.

Let's say c is not a characteristic value of the matrix $\mathbf{BA}$, which would imply that

$$det(c\mathbf{I} - \mathbf{BA}) \neq 0 \quad (6.2.9.2)$$

There are two cases for this:

a) $\mathbf{c = 0}$

In this case $det(-(BA)) \neq 0$.

$$\begin{aligned} det(-(BA)) &= (-1)^n \; det(\mathbf{B}) \; det(\mathbf{A}) \\ &= (-1)^n \; det(\mathbf{A}) \; det(\mathbf{B}) \\ &= det(-\mathbf{AB}) \\ &= det(c\mathbf{I} - \mathbf{AB}) \end{aligned}$$

From (6.2.9.2) we can write this as

$$det(c\mathbf{I} - \mathbf{AB}) \neq 0 \quad (6.2.9.3)$$

b) $\mathbf{c \neq 0}$

In this case

$$c\mathbf{I} - \mathbf{BA} = c\left(\mathbf{I} - \frac{1}{c}\mathbf{BA}\right)$$
$$\implies det\left(c\left(\mathbf{I} - \frac{1}{c}\mathbf{BA}\right)\right) = c^n \; det\left(\mathbf{I} - \frac{1}{c}\mathbf{BA}\right)$$

From (6.2.9.2), we get

$$c^n \; det\left(\mathbf{I} - \frac{1}{c}\mathbf{BA}\right) \neq 0$$

Therefore, $\left(\mathbf{I} - \frac{1}{c}\mathbf{BA}\right)$ is invertible. From theorem 6.4 we can say $\left(\mathbf{I} - \frac{1}{c}\mathbf{AB}\right)$ is invertible.

$$\begin{aligned} &\implies det\left(\mathbf{I} - \frac{1}{c}\mathbf{AB}\right) \neq 0 \\ &\implies c^n \; det\left(\mathbf{I} - \frac{1}{c}\mathbf{AB}\right) \neq 0 \\ &\implies det\,(c\mathbf{I} - \mathbf{AB}) \neq 0 \quad (6.2.9.4) \end{aligned}$$

After proving both the cases of $c = 0$ and $c \neq 0$ we say that the characteristic value c must be the same.

$\therefore$ $\mathbf{AB}$ and $\mathbf{BA}$ have the same characteristic values.

After proving both the above statements, we can say that the two polynomials of degree n with exactly the same roots, will be equal in nature. So, the characteristic polynomials are equal.

Even though the characteristic polynomials are same, they may not necessarily be the same minimal polynomial.

Lets take the below examples

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (6.2.9.5)$$
$$\mathbf{B} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (6.2.9.6)$$

We get the matrices $\mathbf{AB}$ and $\mathbf{BA}$ as

Characteristic Polynomial	For an $n \times n$ matrix $\mathbf{A}$, characteristic polynomial is defined by, $$p(x) = \lvert x\mathbf{I} - \mathbf{A} \rvert$$
Theory	$\mathbf{A}$ is similar to triangular matrix $\mathbf{J}$ if $\exists$ an invertible matrix $\mathbf{P}$ such that $$\mathbf{A} = \mathbf{P}\mathbf{J}\mathbf{P}^{-1}$$

TABLE 6.3.1.1: Definitions

$$\mathbf{AB} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{6.2.9.7}$$

$$\mathbf{BA} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{6.2.9.8}$$

Then $\mathbf{AB} = \mathbf{A}$, whereas $\mathbf{BA}$ is the zero matrix. Since $\mathbf{A^2} = 0$ and $\mathbf{A} \neq 0$, the minimal polynomial of $\mathbf{AB}$ is x^2, whereas the minimal polynomial of $\mathbf{BA}$ is x.
$\therefore$, $\mathbf{AB}$ and $\mathbf{BA}$ have the same characteristic polynomial, but the minimal polynomials are not the same.

6.3 Invariant Subspaces

6.3.1. Let

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 2 & -2 & 2 \\ 2 & -3 & 2 \end{pmatrix} \tag{6.3.1.1}$$

Is $\mathbf{A}$ similar over the field of real numbers to a triangular matrix? If so, find such a triangular matrix.
Solution: See Tables 6.3.1.1 and 6.3.1.2

Characteristic polynomial	$p(x) = \lvert x\mathbf{I} - \mathbf{A} \rvert$ $= \begin{vmatrix} x & -1 & 0 \\ -2 & x+2 & -2 \\ -2 & 3 & x-2 \end{vmatrix}$ $= x((x+2)(x-2)+6) + (-2(x-2)+4) + 0$ $= x\left(x^2+2\right) - 2x$ $= x^3$ $\implies \lambda = 0$
$dim(Ker(\mathbf{A} - \lambda\mathbf{I}))$	$(\mathbf{A} - 0\mathbf{I})\,\mathbf{X} = \mathbf{0}$ $\implies \begin{pmatrix} 0 & 1 & 0 \\ 2 & -2 & 2 \\ 2 & -3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\implies x = -z, y = 0$ So, $\mathbf{v_1} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$
Find $\mathbf{v_2}$ such that $\mathbf{Av_2} = \mathbf{v_1}$	$\begin{pmatrix} 0 & 1 & 0 \\ 2 & -2 & 2 \\ 2 & -3 & 2 \end{pmatrix} \mathbf{v_2} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ $\implies \mathbf{v_2} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$
Find $\mathbf{v_3}$ such that $\mathbf{Av_3} = \mathbf{v_2}$	$\begin{pmatrix} 0 & 1 & 0 \\ 2 & -2 & 2 \\ 2 & -3 & 2 \end{pmatrix} \mathbf{v_3} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$ $\implies \mathbf{v_3} = \begin{pmatrix} -\frac{3}{2} \\ -1 \\ 0 \end{pmatrix}$
	Let $\mathbf{P} = \left(\mathbf{v_1} \quad \mathbf{v_2} \quad \mathbf{v_3}\right)$ Then, $\mathbf{AP} = \left(\mathbf{Av_1} \quad \mathbf{Av_2} \quad \mathbf{Av_3}\right)$

	$\implies \mathbf{AP} = \begin{pmatrix} 0 & \mathbf{v_1} & \mathbf{v_2} \end{pmatrix}$ $= \begin{pmatrix} \mathbf{v_1} & \mathbf{v_2} & \mathbf{v_3} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $= \mathbf{PJ}$ $\implies \mathbf{AP} = \mathbf{PJ}$ $\implies \mathbf{A} = \mathbf{PJP}^{-1}$
Conclusion	Therefore, **A** is similar to triangular matrix and the triangular matrix is, $\mathbf{J} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

TABLE 6.3.1.2: Checking triangularizability of **A**

Characteristic Polynomial	For an $n \times n$ matrix $\mathbf{A}$, characteristic polynomial is defined by, $$p(x) = \left\|x\mathbf{I} - \mathbf{A}\right\|$$
Cayley-Hamilton Theorem	If $p(x)$ is the characteristic polynomial of an $n \times n$ matrix $\mathbf{A}$, then, $$p(\mathbf{A}) = \mathbf{0}$$
Minimal Polynomial	Minimal polynomial $m(x)$ is the smallest factor of characteristic polynomial $p(x)$ such that, $$m(\mathbf{A}) = 0$$ Every root of characteristic polynomial should be the root of minimal polynomial
Theorem	Let $\mathbf{V}$ be a finite-dimensional vector space over the field $\mathbf{F}$ and let $\mathbf{T}$ be a linear operator on $\mathbf{V}$ Then $\mathbf{T}$ is diagonalizable if and only if the minimal polynomial for $\mathbf{T}$ has the form p = $(x - c_1)...(x - c_k)$,where $c_1, c_2, ..., c_k$ are distinct elements of F.

TABLE 6.3.2.1: Definitions

6.3.2. Every matrix A such that $A^2 = A$ is similar to a diagonal matrix.
Solution: See Tables 6.3.2.1 and 6.3.2.2

Proof	$A^2 = A$ (Given) $\implies A$ satisfies the polynomial $x^2 - x = x(x-1)$
Minimal Polynomial	$p(x) = x^a (x-1)^b, \quad a \leq 1, b \leq 1$ Minimal Polynomial $m_A(x)$, of A divides $x^2 - x$, that is $m_A(x)$=x or $m_A(x)$=$x - 1$ or $m_A(x)$=$x(x-1)$ If $m_A(x)$=x, then $A = 0$. If $m_A(x)$=$x - 1$, then $A = I$. If $m_A(x)$=$x(x-1)$ In all above three cases, the minimal polynomial factors into distinct linears. So, it follows A is diagonisable.
Conclusion	In all three cases, the minimal polynomial splits into distinct linear factors, so it follows that **A** is diagonalisable according to the Theorem mentioned in definitions section.

TABLE 6.3.2.2: Illustration of Proof

6.3.3. True or false? If the triangular matrix **A** is similar to a diagonal matrix, then **A** is already diagonal.
Solution: See Tables 6.3.3.1 and 6.3.3.2

Characteristic Polynomial	For an $n \times n$ matrix $\mathbf{A}$, characteristic polynomial is defined by, $$p(x) = \left\lvert x\mathbf{I} - \mathbf{A} \right\rvert$$
Minimal Polynomial	Minimal polynomial $m(x)$ is the smallest factor of characteristic polynomial $p(x)$ such that, $$m(\mathbf{A}) = 0$$
Theorem	Let $\mathbf{V}$ be a finite-dimensional vector space over the field $\mathbf{F}$ and let $\mathbf{T}$ be a linear operator on $\mathbf{V}$.Then $\mathbf{T}$ is diagonalizable if and only if the minimal polynomial for $\mathbf{T}$ has the form $$p = (x - c_1)\dots(x - c_k) \tag{6.3.3.1}$$ where $c_1, c_2, ..., c_k$ are distinct elements of F.
Diagonalizable	$\mathbf{A}$ is called diagonalizable if it is similar to diagnol matrix $\mathbf{B}$ i.e., if $\exists$ an invertible matrix $\mathbf{P}$ such that $$\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} \tag{6.3.3.2}$$

TABLE 6.3.3.1: Definitions and theorem used

Given	The triangular matrix $\mathbf{A}$ is similar to a diagonal matrix.
Example	Let $$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \tag{6.3.3.3}$$ We can see that $\mathbf{A}$ is triangular but not diagonal.
Characteristic polynomial	$$\left\lvert x\mathbf{I} - \mathbf{A} \right\rvert = \begin{vmatrix} x-1 & 2 \\ 0 & x-3 \end{vmatrix} \tag{6.3.3.4}$$ $$= (x-1)(x-3) \tag{6.3.3.5}$$
Minimal polynomial	As the eigen values are distinct,minimal polynomial

	$$m(x) = (x-1)(x-3) \tag{6.3.3.6}$$
Diagonalizable	From theorem (3.7.3.3),We can say that **A** diagonalizable i.e., it is similar to a diagnol matrix.
Finding matrix similar to **A**	The eigen values are $$\lambda_1 = 1, \lambda_2 = 3 \tag{6.3.3.7}$$ The eigen vectors are $$(\mathbf{A} - \lambda_i \mathbf{I})\mathbf{x_i} = 0 \tag{6.3.3.8}$$ $$\lambda_1 = 1 \implies \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix}\mathbf{x_1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies \mathbf{x_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{6.3.3.9}$$ $$\lambda_2 = 3 \implies \begin{pmatrix} -2 & 2 \\ 0 & 0 \end{pmatrix}\mathbf{x_2} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies \mathbf{x_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{6.3.3.10}$$ The invertible matrix $$\mathbf{P} = \begin{pmatrix} \mathbf{x_1} & \mathbf{x_2} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{6.3.3.11}$$ The diagnol matrix similar to **A** $$\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{6.3.3.12}$$ $$\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \tag{6.3.3.13}$$
Conclusion	From above,we can say that **A** need not be diagonal to satisfy given conditions.So, given statement is false.

TABLE 6.3.3.2: Finding minimal polynomial and similar matrix

Theorem	According to theorem , if a 2×2 matrix has two characteristics values then the $\mathbf{P}$ that diagonalize $\mathbf{A}$ will necessarily also diagonalize any $\mathbf{B}$ that commutes with $\mathbf{A}$.
Common Basis	Let there exist a $\mathbf{P}$ in basis $\beta = \{\mathbf{b}_1, ..., \mathbf{b}_n\}$ of $\mathbb{V}$ consisting of eigen vector which are common to both $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{A}\mathbf{b}_i = \lambda_i \mathbf{b}_i$ $\mathbf{B}\mathbf{b}_i = \mu_i \mathbf{b}_i$ $\Lambda_A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ $\Lambda_B = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}$ $\Lambda_A = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ $\Lambda_B = \mathbf{P}^{-1}\mathbf{B}\mathbf{P}$

TABLE 6.4.1.1: Theorems

6.4 Simultaneous Triangulation; Simultaneous Diagonalization

6.4.1. Find an invertible matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ and $\mathbf{P}^{-1}\mathbf{B}\mathbf{P}$ are both diagonal where $\mathbf{A}$ and $\mathbf{B}$ are real matrices.

a)

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 3 & -8 \\ 0 & -1 \end{pmatrix} \qquad (6.4.1.1)$$

Solution: See Tables 6.4.1.1 and 6.4.1.2

Operations	Matrix A	Matrix B
Characteristic Polynomial	$p(x) = \lvert x\mathbf{I} - \mathbf{A}\rvert = \begin{vmatrix} x-1 & -2 \\ 0 & x-2 \end{vmatrix} = (x-1)(x-2)$	$p(x) = \lvert x\mathbf{I} - \mathbf{B}\rvert = \begin{vmatrix} x-3 & 8 \\ 0 & x+1 \end{vmatrix} = (x-3)(x+1)$
Characteristic values	$p(x) = 0$ $(x-1)(x-2) = 0$ $\lambda_1 = 1$,$\lambda_2 = 2$	$p(x) = 0$ $(x-3)(x+1) = 0$ $\mu_1 = 3$, $\mu_2 = -1$
Basis for Characteristics Values	For $\lambda_1 = 1$ $(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{b_1} = 0$ $\left(\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\mathbf{b_1} = 0$ $\left(\begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix}\right)\mathbf{b_1} = 0$ $\mathbf{b_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ For $\lambda_2 = 2$ $(\mathbf{A} - \lambda_2\mathbf{I})\mathbf{b_2} = 0$ $\left(\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\mathbf{b_2} = 0$ $\left(\begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix}\right)\mathbf{b_2} = 0$ $\mathbf{b_2} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\mathbf{P} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$	For $\mu_1 = 3$ $(\mathbf{B} - \mu_1\mathbf{I})\mathbf{b_1} = 0$ $\left(\begin{pmatrix} 3 & -8 \\ 0 & -1 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\mathbf{b_1} = 0$ $\left(\begin{pmatrix} 0 & -8 \\ 0 & -4 \end{pmatrix}\right)\mathbf{b_1} = 0$ $\mathbf{b_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ For $\mu_2 = -1$ $(\mathbf{B} - \mu_2\mathbf{I})\mathbf{b_2} = 0$ $\left(\begin{pmatrix} 3 & -8 \\ 0 & -1 \end{pmatrix} - (-1)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\mathbf{b_2} = 0$ $\left(\begin{pmatrix} 4 & -8 \\ 0 & 0 \end{pmatrix}\right)\mathbf{b_2} = 0$ $\mathbf{b_2} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\mathbf{P} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$
Verification	$\Lambda_A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ $\Lambda_A = \mathbf{P^{-1}AP}$ $\Longrightarrow \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$	$\Lambda_B = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$ $\Lambda_B = \mathbf{P^{-1}BP}$ $\Longrightarrow \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & -8 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

	$= \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \Lambda_A$	$= \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} = \Lambda_B$
Answer.The Invertible Matrix **P**	$\mathbf{P} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$	$\mathbf{P} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

TABLE 6.4.1.2: Solution Table

Characteristic Polynomial	For an $n \times n$ matrix $\mathbf{A}$, characteristic polynomial is defined by, $$p(x) = \lvert x\mathbf{I} - \mathbf{A} \rvert$$
Theorem	According to theorem 8, if a 2×2 matrix has two characteristics values then the $\mathbf{P}$ that diagonalize $\mathbf{A}$ will necessarily also diagonalize any $\mathbf{B}$ that commutes with $\mathbf{A}$.
Basis	Let there exist a $\mathbf{P}$ in basis $\beta = \{\mathbf{b}_1, ..., \mathbf{b}_n\}$ of $\mathbb{V}$ consisting of eigen vector which are common to both $\mathbf{A}$ and $\mathbf{B}$ such that $$\mathbf{A}\mathbf{b}_i = \lambda_i \mathbf{b}_i \qquad \mathbf{B}\mathbf{b}_i = \mu_i \mathbf{b}_i$$ $$\Lambda_A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \qquad \Lambda_B = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}$$ $$\Lambda_A = \mathbf{P^{-1}AP} \qquad \Lambda_B = \mathbf{P^{-1}BP}$$

TABLE 6.4.1.3: Definitions and theorem used

b)

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \tag{6.4.1.2}$$

$$\mathbf{B} = \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix} \tag{6.4.1.3}$$

Solution: See Tables 6.4.1.3 and 6.4.1.4

Operations	Matrix A	Matrix B
Characteristic Polynomial	$p(x) = \lvert x\mathbf{I} - \mathbf{A} \rvert$ $= \begin{vmatrix} x-1 & -1 \\ -1 & x-1 \end{vmatrix}$ $= (x-1)(x-1) - 1$	$p(x) = \lvert x\mathbf{I} - \mathbf{B} \rvert$ $= \begin{vmatrix} x-1 & -a \\ -a & x-1 \end{vmatrix}$ $= (x-1)(x-1) - a^2$
Characteristic values	$p(x) = 0$ $x(x-1) = 0$ $\lambda_1 = 0\ , \lambda_2 = 2$	$p(x) = 0$ $(x-1)^2 - a^2 = 0$ $\mu_1 = (1-a)\ ,\ \mu_2 = (1+a)$
Basis for Characteristics Values	$(\mathbf{A} - \lambda_i \mathbf{I})\mathbf{X} = 0$ $\implies$ For $\lambda_1 = 0$ $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$ So, $\mathbf{b_1} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $\implies$ For $\lambda_2 = 2$ $\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$ So, $\mathbf{b_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$	$(\mathbf{B} - \mu_i \mathbf{I})\mathbf{X} = 0$ $\implies$ For $\mu_1 = (1-a)$ $\begin{pmatrix} a & a \\ a & a \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$ So, $\mathbf{b_1} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $\implies$ For $\mu_2 = (1+a)$ $\begin{pmatrix} -a & a \\ a & -a \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$ So, $\mathbf{b_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
Invertible matrix	Let $\mathbf{P} = \begin{pmatrix} \mathbf{b_1} & \mathbf{b_2} \end{pmatrix}$ Then, $\mathbf{P} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$	Let $\mathbf{P} = \begin{pmatrix} \mathbf{b_1} & \mathbf{b_2} \end{pmatrix}$ Then, $\implies \mathbf{P} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$
Verification	$\Lambda_A = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$	$\Lambda_B = \begin{pmatrix} 1-a & 0 \\ 0 & 1+a \end{pmatrix}$

	$\Lambda_A = \mathbf{P^{-1}AP}$ $\Lambda_A = \begin{pmatrix} \frac{-1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$ $= \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} = \Lambda_A$	$\Lambda_B = \mathbf{P^{-1}BP}$ $\Lambda_B = \begin{pmatrix} \frac{-1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$ $= \begin{pmatrix} 1-a & 0 \\ 0 & 1+a \end{pmatrix} = \Lambda_B$
Conclusion	$\mathbf{P} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$	$\mathbf{P} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$

TABLE 6.4.1.4: Finding an Invertible matrix

6.4.2. Let $\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}$ be $n\times n$ complex matrices which commute. Let $\mathbf{E}$ be the $2n\times 2n$ matrix

$$\mathbf{E} = \begin{pmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{pmatrix} \tag{6.4.2.1}$$

prove that $\det(\mathbf{E}) = \det(\mathbf{AD} - \mathbf{BC})$

Solution: Given matrices $\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}$ commute. Let $\mathbf{P}$ be an invertible matrix that can simultaneously diagonalize matrices $\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}$ as below

$$\mathbf{A} = \mathbf{P}\mathbf{\Lambda_a}\mathbf{P}^{-1} \tag{6.4.2.2}$$
$$\mathbf{B} = \mathbf{P}\mathbf{\Lambda_b}\mathbf{P}^{-1} \tag{6.4.2.3}$$
$$\mathbf{C} = \mathbf{P}\mathbf{\Lambda_c}\mathbf{P}^{-1} \tag{6.4.2.4}$$
$$\mathbf{D} = \mathbf{P}\mathbf{\Lambda_d}\mathbf{P}^{-1} \tag{6.4.2.5}$$

where $\mathbf{\Lambda_a},\mathbf{\Lambda_b},\mathbf{\Lambda_c},\mathbf{\Lambda_d}$ are diagonal matrices whose diagonal values are eigenvalues of matrices $\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}$ respectively and matrix $\mathbf{P}$ is formed by n-linearly independent eigen vectors.

Now (6.4.2.1) can be written as

$$\mathbf{E} = \begin{pmatrix}\mathbf{P}\mathbf{\Lambda_a}\mathbf{P}^{-1} & \mathbf{P}\mathbf{\Lambda_b}\mathbf{P}^{-1}\\ \mathbf{P}\mathbf{\Lambda_c}\mathbf{P}^{-1} & \mathbf{P}\mathbf{\Lambda_d}\mathbf{P}^{-1}\end{pmatrix} \tag{6.4.2.6}$$

Using block matrix multiplication, we get

$$\implies \mathbf{E} = \begin{pmatrix}\mathbf{P} & \mathbf{0}\\ \mathbf{0} & \mathbf{P}\end{pmatrix}\begin{pmatrix}\mathbf{\Lambda_a} & \mathbf{\Lambda_b}\\ \mathbf{\Lambda_c} & \mathbf{\Lambda_d}\end{pmatrix}\begin{pmatrix}\mathbf{P}^{-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{P}^{-1}\end{pmatrix} \tag{6.4.2.7}$$

$$\implies \mathbf{E} = \mathbf{M}\mathbf{D}\mathbf{M}^{-1} \tag{6.4.2.8}$$

where

$$\mathbf{M} = \begin{pmatrix}\mathbf{P} & \mathbf{0}\\ \mathbf{0} & \mathbf{P}\end{pmatrix} \tag{6.4.2.9}$$

$$\mathbf{D} = \begin{pmatrix}\mathbf{\Lambda_a} & \mathbf{\Lambda_b}\\ \mathbf{\Lambda_c} & \mathbf{\Lambda_d}\end{pmatrix} \tag{6.4.2.10}$$

Now we will calculate $\det(\mathbf{E})$,

$$\left|\mathbf{E}\right| = \left|\mathbf{M}\mathbf{D}\mathbf{M}^{-1}\right| \tag{6.4.2.11}$$
$$\implies \left|\mathbf{E}\right| = \left|\mathbf{M}\right|\left|\mathbf{D}\right|\left|\mathbf{M}^{-1}\right| \tag{6.4.2.12}$$
$$\implies \left|\mathbf{E}\right| = \left|\mathbf{M}\right|\left|\mathbf{D}\right|\left|\mathbf{M}\right|^{-1} \tag{6.4.2.13}$$
$$\implies \left|\mathbf{E}\right| = \left|\mathbf{D}\right| \tag{6.4.2.14}$$
$$\implies \left|\mathbf{E}\right| = \begin{vmatrix}\mathbf{\Lambda_a} & \mathbf{\Lambda_b}\\ \mathbf{\Lambda_c} & \mathbf{\Lambda_d}\end{vmatrix} \tag{6.4.2.15}$$

$$= \begin{vmatrix}\lambda_{1a} & 0 & \dots & 0 & \lambda_{1b} & 0 & \dots & 0\\ 0 & \lambda_{2a} & \dots & 0 & 0 & \lambda_{2b} & \dots & 0\\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots\\ 0 & 0 & \dots & \lambda_{na} & 0 & 0 & \dots & \lambda_{nb}\\ \lambda_{1c} & 0 & \dots & 0 & \lambda_{1d} & 0 & \dots & 0\\ 0 & \lambda_{2c} & \dots & 0 & 0 & \lambda_{2d} & \dots & 0\\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots\\ 0 & 0 & \dots & \lambda_{nc} & 0 & 0 & \dots & \lambda_{nd}\end{vmatrix} \tag{6.4.2.16}$$

Using row reduction,

$$\xleftrightarrow{R_{n+1}=R_{n+1}-\frac{\lambda_{1c}}{\lambda_{1a}}R_1}$$

$$\begin{vmatrix}\lambda_{1a} & 0 & \dots & 0 & \lambda_{1b} & 0 & \dots & 0\\ 0 & \lambda_{2a} & \dots & 0 & 0 & \lambda_{2b} & \dots & 0\\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots\\ 0 & 0 & \dots & \lambda_{na} & 0 & 0 & \dots & \lambda_{nb}\\ 0 & 0 & \dots & 0 & \lambda_{1d}-\frac{\lambda_{1c}\lambda_{1b}}{\lambda_{1a}} & 0 & \dots & 0\\ 0 & \lambda_{2c} & \dots & 0 & 0 & \lambda_{2d} & \dots & 0\\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots\\ 0 & 0 & \dots & \lambda_{nc} & 0 & 0 & \dots & \lambda_{nd}\end{vmatrix} \tag{6.4.2.17}$$

similarly doing elementary row operations for rows R_{n+2} to R_{2n}, we get

$$\left|\mathbf{E}\right| = \begin{vmatrix}\lambda_{1a} & \dots & 0 & \lambda_{1b} & \dots & 0\\ \vdots & \dots & \vdots & \vdots & \dots & \vdots\\ 0 & \dots & \lambda_{na} & 0 & \dots & \lambda_{nb}\\ 0 & \dots & 0 & \lambda_{1d}-\frac{\lambda_{1c}\lambda_{1b}}{\lambda_{1a}} & \dots & 0\\ \vdots & \dots & \vdots & \vdots & \dots & \vdots\\ 0 & \dots & 0 & 0 & \dots & \lambda_{nd}-\frac{\lambda_{nc}\lambda_{nb}}{\lambda_{na}}\end{vmatrix} \tag{6.4.2.18}$$

Since it is upper triangular matrix, then $\left|\mathbf{E}\right|$ will be multiplication of diagonal elements.

$$\implies \left|\mathbf{E}\right| = \lambda_{1a}\lambda_{2a}\ldots\lambda_{na} \times\left(\lambda_{1d} - \frac{\lambda_{1c}\lambda_{1b}}{\lambda_{1a}}\right)\ldots\left(\lambda_{nd} - \frac{\lambda_{nc}\lambda_{nb}}{\lambda_{na}}\right) \quad (6.4.2.19)$$

$$\implies \left|\mathbf{E}\right| = (\lambda_{1a}\lambda_{1d} - \lambda_{1c}\lambda_{1b}) \times(\lambda_{2a}\lambda_{2d} - \lambda_{2c}\lambda_{2b})\ldots(\lambda_{na}\lambda_{nd} - \lambda_{nc}\lambda_{nb}) \quad (6.4.2.20)$$

Now we will calculate det($\mathbf{AD} - \mathbf{BC}$) by substituting (6.4.2.2),(6.4.2.3),(6.4.2.4),(6.4.2.5)

$$\left|\mathbf{AD} - \mathbf{BC}\right| = \left|\mathbf{P\Lambda_a P^{-1} P\Lambda_d P^{-1}} - \mathbf{P\Lambda_b P^{-1} P\Lambda_c P^{-1}}\right| \quad (6.4.2.21)$$

$$= \left|\mathbf{P\Lambda_a\Lambda_d P^{-1}} - \mathbf{P\Lambda_b\Lambda_c P^{-1}}\right| \quad (6.4.2.22)$$

$$= \left|\mathbf{P(\Lambda_a\Lambda_d} - \mathbf{\Lambda_b\Lambda_c)P^{-1}}\right| \quad (6.4.2.23)$$

$$= \left|\mathbf{P}\right|\left|\mathbf{\Lambda_a\Lambda_d} - \mathbf{\Lambda_b\Lambda_c}\right|\left|\mathbf{P^{-1}}\right| \quad (6.4.2.24)$$

$$= \left|\mathbf{P}\right|\left|\mathbf{P}\right|^{-1}\left|\mathbf{\Lambda_a\Lambda_d} - \mathbf{\Lambda_b\Lambda_c}\right| \quad (6.4.2.25)$$

$$\left|\mathbf{AD} - \mathbf{BC}\right| = \left|\mathbf{\Lambda_a\Lambda_d} - \mathbf{\Lambda_b\Lambda_c}\right| \quad (6.4.2.26)$$

Since $\mathbf{\Lambda_a}$,$\mathbf{\Lambda_b}$,$\mathbf{\Lambda_c}$,$\mathbf{\Lambda_d}$ are diagonal matrices, we get

$$\mathbf{\Lambda_a\Lambda_d} = \begin{pmatrix} \lambda_{1a}\lambda_{1d} & 0 & \ldots & 0 \\ 0 & \lambda_{2a}\lambda_{2d} & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & \lambda_{na}\lambda_{nd} \end{pmatrix} \quad (6.4.2.27)$$

$$\mathbf{\Lambda_b\Lambda_c} = \begin{pmatrix} \lambda_{1b}\lambda_{1c} & 0 & \ldots & 0 \\ 0 & \lambda_{2b}\lambda_{2c} & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & \lambda_{nb}\lambda_{nc} \end{pmatrix} \quad (6.4.2.28)$$

$$(6.4.2.29)$$

Substitute (6.4.2.27) and (6.4.2.28) in (6.4.2.26), we get

$$\left|\mathbf{AD} - \mathbf{BC}\right| = \begin{vmatrix} \lambda_{1a}\lambda_{1d} - \lambda_{1b}\lambda_{1c} & \ldots & 0 \\ 0 & \ldots & 0 \\ \vdots & \ldots & \vdots \\ 0 & \ldots & \lambda_{na}\lambda_{nd} - \lambda_{nb}\lambda_{nc} \end{vmatrix} \quad (6.4.2.30)$$

$$\implies \left|\mathbf{AD} - \mathbf{BC}\right| = (\lambda_{1a}\lambda_{1d} - \lambda_{1b}\lambda_{1c}) \times(\lambda_{2a}\lambda_{2d} - \lambda_{2b}\lambda_{2c})\ldots(\lambda_{na}\lambda_{nd} - \lambda_{nb}\lambda_{nc}) \quad (6.4.2.31)$$

Comparing (6.4.2.20) and (6.4.2.31) we can say that

$$\left|\mathbf{E}\right| = \left|\mathbf{AD} - \mathbf{BC}\right| \quad (6.4.2.32)$$

Hence proved.

Example: Let us consider below matrices **A**,**B**,**C**,**D** be $n \times n$ complex matrices which commute.

$$\mathbf{A} = \begin{pmatrix} 1-i & -2 \\ 3 & -1-i \end{pmatrix} \quad (6.4.2.33)$$

$$\mathbf{B} = \begin{pmatrix} 3-3i & -6 \\ 9 & -3-3i \end{pmatrix} \quad (6.4.2.34)$$

$$\mathbf{C} = \begin{pmatrix} 7-7i & -14 \\ 21 & -7-7i \end{pmatrix} \quad (6.4.2.35)$$

$$\mathbf{D} = \begin{pmatrix} -2+2i & 4 \\ -6 & 2+2i \end{pmatrix} \quad (6.4.2.36)$$

Lets find eigenvalues of matrix **A**

$$\left|\mathbf{A} - \lambda\mathbf{I}\right| = 0 \quad (6.4.2.37)$$

$$\implies \begin{vmatrix} 1-i-\lambda & -2 \\ 3 & -1-i-\lambda \end{vmatrix} = 0 \quad (6.4.2.38)$$

$$\implies \lambda^2 + 2\lambda i + 4 = 0 \quad (6.4.2.39)$$

$$\implies \lambda_{1a} = -(1+\sqrt{5})i \quad \lambda_{2a} = (-1+\sqrt{5})i \quad (6.4.2.40)$$

(6.4.2.40) are eigenvalues of matrix **A**.
The eigenvectors of matrix **A** are

$$\mathbf{v_1} = \begin{pmatrix} \frac{-1+\sqrt{5}i}{3} \\ 1 \end{pmatrix} \quad \mathbf{v_2} = \begin{pmatrix} \frac{-1-\sqrt{5}i}{3} \\ 1 \end{pmatrix} \quad (6.4.2.41)$$

Since matrices **A**,**B**,**C**,**D** commute, from (6.4.2.33), (6.4.2.34), (6.4.2.35), (6.4.2.36) we can say that

$$\mathbf{B} = 3\mathbf{A} \tag{6.4.2.42}$$

$$\mathbf{C} = 7\mathbf{A} \tag{6.4.2.43}$$

$$\mathbf{D} = -2\mathbf{A} \tag{6.4.2.44}$$

Then the eigenvalues of matrices $\mathbf{B},\mathbf{C},\mathbf{D}$ are

$$\lambda_{1b} = -3(1+\sqrt{5})i \quad \lambda_{2b} = 3(-1+\sqrt{5})i \tag{6.4.2.45}$$

$$\lambda_{1c} = -7(1+\sqrt{5})i \quad \lambda_{2c} = 7(-1+\sqrt{5})i \tag{6.4.2.46}$$

$$\lambda_{1d} = 2(1+\sqrt{5}+1)i \quad \lambda_{2d} = -2(-1+\sqrt{5})i \tag{6.4.2.47}$$

But the eigenvectors of matrices $\mathbf{B},\mathbf{C},\mathbf{D}$ are same as of matrix $\mathbf{A}$.
The eigenvalue decomposition of matrices $\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}$ is done as in (6.4.2.2), (6.4.2.3), (6.4.2.4), (6.4.2.5). Here eigenvector matrix $\mathbf{P}$ and $\mathbf{\Lambda_a},\mathbf{\Lambda_b},\mathbf{\Lambda_c},\mathbf{\Lambda_d}$ are

$$\mathbf{P} = \begin{pmatrix} \frac{-1+\sqrt{5}i}{3} & \frac{-1-\sqrt{5}i}{3} \\ 1 & 1 \end{pmatrix} \tag{6.4.2.48}$$

$$\mathbf{\Lambda_a} = \begin{pmatrix} mi & 0 \\ 0 & ni \end{pmatrix} \tag{6.4.2.49}$$

$$\mathbf{\Lambda_b} = \begin{pmatrix} 3mi & 0 \\ 0 & 3ni \end{pmatrix} \tag{6.4.2.50}$$

$$\mathbf{\Lambda_c} = \begin{pmatrix} 7mi & 0 \\ 0 & 7ni \end{pmatrix} \tag{6.4.2.51}$$

$$\mathbf{\Lambda_d} = \begin{pmatrix} -2mi & 0 \\ 0 & -2ni \end{pmatrix} \tag{6.4.2.52}$$

where

$$m = -(1+\sqrt{5}) \quad n = (-1+\sqrt{5}) \tag{6.4.2.53}$$

From (6.4.2.8), we got

$$\mathbf{E} = \mathbf{M}\mathbf{D}\mathbf{M}^{-1} \tag{6.4.2.54}$$

where

$$\mathbf{M} = \begin{pmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{P} \end{pmatrix} \tag{6.4.2.55}$$

$$= \begin{pmatrix} \frac{-1+\sqrt{5}i}{3} & \frac{-1-\sqrt{5}i}{3} & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & \frac{-1+\sqrt{5}i}{3} & \frac{-1-\sqrt{5}i}{3} \\ 0 & 0 & 1 & 1 \end{pmatrix} \tag{6.4.2.56}$$

$$\mathbf{D} = \begin{pmatrix} \mathbf{\Lambda_a} & \mathbf{\Lambda_b} \\ \mathbf{\Lambda_c} & \mathbf{\Lambda_d} \end{pmatrix}$$

$$= \begin{pmatrix} mi & 0 & 3mi & 0 \\ 0 & ni & 0 & 3ni \\ 7mi & 0 & -2mi & 0 \\ 0 & 7ni & 0 & -2ni \end{pmatrix} \tag{6.4.2.57}$$

Now we will calculate $\det(E)$, from (6.4.2.14) we got

$$\left|\mathbf{E}\right| = \left|\mathbf{D}\right| \tag{6.4.2.58}$$

$$= \begin{vmatrix} mi & 0 & 3mi & 0 \\ 0 & ni & 0 & 3ni \\ 7mi & 0 & -2mi & 0 \\ 0 & 7ni & 0 & -2ni \end{vmatrix} \tag{6.4.2.59}$$

Using row reduction technique,

$$\begin{vmatrix} mi & 0 & 3mi & 0 \\ 0 & ni & 0 & 3ni \\ 7mi & 0 & -2mi & 0 \\ 0 & 7ni & 0 & -2ni \end{vmatrix} \tag{6.4.2.60}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - 7R_1} \begin{vmatrix} mi & 0 & 3mi & 0 \\ 0 & ni & 0 & 3ni \\ 0 & 0 & -23mi & 0 \\ 0 & 7ni & 0 & -2ni \end{vmatrix} \tag{6.4.2.61}$$

$$\xleftrightarrow{R_4 \leftarrow R_4 - 7R_2} \begin{vmatrix} mi & 0 & 3mi & 0 \\ 0 & ni & 0 & 3ni \\ 0 & 0 & -23mi & 0 \\ 0 & 0 & 0 & -23ni \end{vmatrix} \tag{6.4.2.62}$$

Now

$$\left|\mathbf{E}\right| = \begin{vmatrix} mi & 0 & 3mi & 0 \\ 0 & ni & 0 & 3ni \\ 0 & 0 & -23mi & 0 \\ 0 & 0 & 0 & -23ni \end{vmatrix} \tag{6.4.2.63}$$

$$\implies \left|\mathbf{E}\right| = 529m^2n^2 \tag{6.4.2.64}$$

Substitute (6.4.2.53) in (6.4.2.64), we get

$$\left|\mathbf{E}\right| = 529(1+\sqrt{5})^2(-1+\sqrt{5})^2 = 8464 \tag{6.4.2.65}$$

Now we will calculate $\det(\mathbf{AD} - \mathbf{BC})$, from (6.4.2.26) we got

$$\left|\mathbf{AD} - \mathbf{BC}\right| = \left|\mathbf{\Lambda_a}\mathbf{\Lambda_d} - \mathbf{\Lambda_b}\mathbf{\Lambda_c}\right| \tag{6.4.2.66}$$

Using (6.4.2.49), (6.4.2.50), (6.4.2.51), (6.4.2.52) we get

$$\mathbf{\Lambda_a \Lambda_d} = \begin{pmatrix} mi & 0 \\ 0 & ni \end{pmatrix} \begin{pmatrix} -2mi & 0 \\ 0 & -2ni \end{pmatrix} \tag{6.4.2.67}$$

$$\implies \mathbf{\Lambda_a \Lambda_d} = \begin{pmatrix} 2m^2 & 0 \\ 0 & 2n^2 \end{pmatrix} \tag{6.4.2.68}$$

$$\mathbf{\Lambda_b \Lambda_c} = \begin{pmatrix} 3mi & 0 \\ 0 & 3ni \end{pmatrix} \begin{pmatrix} 7mi & 0 \\ 0 & 7ni \end{pmatrix} \tag{6.4.2.69}$$

$$\implies \mathbf{\Lambda_b \Lambda_c} = \begin{pmatrix} -21m^2 & 0 \\ 0 & -21n^2 \end{pmatrix} \tag{6.4.2.70}$$

Substitute (6.4.2.68) and (6.4.2.70) in (6.4.2.66), we get

$$\left|\mathbf{AD} - \mathbf{BC}\right| = \left|\begin{pmatrix} 2m^2 & 0 \\ 0 & 2n^2 \end{pmatrix} - \begin{pmatrix} -21m^2 & 0 \\ 0 & -21n^2 \end{pmatrix}\right| \tag{6.4.2.71}$$

$$= \begin{vmatrix} 23m^2 & 0 \\ 0 & 23n^2 \end{vmatrix} \tag{6.4.2.72}$$

$$= 529m^2n^2 \tag{6.4.2.73}$$

Substitute (6.4.2.53) in (6.4.2.66), we get

$$\left|\mathbf{AD} - \mathbf{BC}\right| = 529(1 + \sqrt{5})^2(-1 + \sqrt{5})^2$$
$$= 8464 \tag{6.4.2.74}$$

Comparing (6.4.2.65) and (6.4.2.74), we get

$$\left|\mathbf{E}\right| = \left|\mathbf{AD} - \mathbf{BC}\right| \tag{6.4.2.75}$$

6.5 Direct Sum Decomposition

6.5.1. Let $\mathbb{V}$ be a finite-dimensional vector space and let $\mathbb{W}_1$ be any subspace of $\mathbb{V}$.Prove that there is a subspace $\mathbb{W}_2$ of $\mathbb{V}$ such that $\mathbb{V} = \mathbb{W}_1 \oplus \mathbb{W}_2$
Solution: See Table 6.5.1.1

Assumption and Claim	Let $\beta = \{\mathbf{u}_1, ..., \mathbf{u}_n\}$ be a basis for $\mathbb{W}_1$.Since $\mathbb{W}_1$ is the subspace of $\mathbb{V}$. therefore let us take $\alpha = \{\mathbf{u}_1, ..., \mathbf{u}_n, \mathbf{u}_{n+1}, ..., \mathbf{u}_m\}$ the basis of $\mathbb{V}$ So $\mathbb{W}_2$=span($\{\mathbf{u}_{n+1}, ..., \mathbf{u}_m\}$) Claim that $\mathbb{V} = \mathbb{W}_1 \oplus \mathbb{W}_2$.
Proof of $\mathbb{V} = \mathbb{W}_1 + \mathbb{W}_2$	if $\mathbf{v} \in \mathbb{V}$, then $\mathbf{v} = \sum_{i=1}^{m} a_i\mathbf{u}_i = \sum_{i=1}^{n} a_i\mathbf{u}_i + \sum_{i=n+1}^{m} a_i\mathbf{u}_i \in \mathbb{W}_1 + \mathbb{W}_2$ for scalar a_i , $i = 1, ..., m$ This implies that $\mathbb{V} \subseteq \mathbb{W}_1 + \mathbb{W}_2$ But by the defination of $\mathbb{W}_1 + \mathbb{W}_2$ we know that $\mathbb{W}_1 + \mathbb{W}_2 \subseteq \mathbb{V}$. Hence $\mathbb{V} = \mathbb{W}_1 + \mathbb{W}_2$
Proof of $\mathbb{W}_1 \cap \mathbb{W}_2 = \{0\}$	Let $\mathbf{u} \in \mathbb{W}_1 \cap \mathbb{W}_2$ Then $\mathbf{u} = \sum_{i=1}^{n} b_i\mathbf{u}_i = \sum_{i=n+1}^{m} c_i\mathbf{u}_i$ for some scalar $b_1, ...b_n, c_{n+1}, ..., c_m$ $\implies \sum_{i=1}^{n} b_i\mathbf{u}_i + \sum_{i=n+1}^{m} (-c_i)\, \mathbf{u}_i = 0$ But α is linearly independent ,since α is a basis. Hence $b_1 = ... = b_n = c_{n+1}..... = c_m = 0$.This implies $\mathbf{u} = 0$. Thus $\mathbb{W}_1 \cap \mathbb{W}_2 = \{0\}$
Combining Both the proof	$\mathbb{V} = \mathbb{W}_1 + \mathbb{W}_2$ $\mathbb{W}_1 \cap \mathbb{W}_2 = \{0\}$
	From the above two condition we can say that $\mathbb{V}$ is the direct sum of subspaces $\mathbb{W}_1$ and $\mathbb{W}_2$. Hence it is represented as $\mathbb{V} = \mathbb{W}_1 \oplus \mathbb{W}_2$ Hence Proved.

TABLE 6.5.1.1: Solution Table

6.5.2. Find a projection $\mathbf{E}$ which projects $\mathbb{R}^2$ onto the subspace spanned by $(1,-1)$ along the subspace spanned by $(1,2)$.
Solution: See Table 6.5.2.1

Given	Let $\mathbf{x} \in \mathbb{R}^2$ $$\mathbf{x} = a\begin{pmatrix}1\\-1\end{pmatrix} + b\begin{pmatrix}1\\2\end{pmatrix} \tag{6.5.2.1}$$ where $\begin{pmatrix}a\\b\end{pmatrix}$ is representation of $\mathbf{x}$ in new basis.
To find	$$\mathbf{E}(\mathbf{x}) = a\begin{pmatrix}1\\-1\end{pmatrix} \tag{6.5.2.2}$$
Finding a Projection $\mathbf{E}$	As, $\begin{pmatrix}1\\-1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}$ are linearly independent. Therefore,$\{\begin{pmatrix}1\\-1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}\}$ is a basis of $\mathbb{R}^2$ As $\mathbf{x} \in \mathbb{R}^2$ $$\implies \mathbf{x} = a\begin{pmatrix}1\\-1\end{pmatrix} + b\begin{pmatrix}1\\2\end{pmatrix} \tag{6.5.2.3}$$
	$$\implies \mathbf{x} = \begin{pmatrix}1 & 1\\-1 & 2\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} \tag{6.5.2.4}$$ $$\implies \begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}1 & 1\\-1 & 2\end{pmatrix}^{-1}\mathbf{x} \tag{6.5.2.5}$$ $$\implies \begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}\frac{2}{3} & \frac{-1}{3}\\\frac{1}{3} & \frac{1}{3}\end{pmatrix}\mathbf{x} \tag{6.5.2.6}$$ Therefore, $$a = \begin{pmatrix}\frac{2}{3} & \frac{-1}{3}\end{pmatrix}\mathbf{x} \tag{6.5.2.7}$$ Projection of $\mathbf{x}$ on subspace spanned by $\begin{pmatrix}1\\-1\end{pmatrix}$;

	$$\mathbf{E}(\mathbf{x}) = a\begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{6.5.2.8}$$ Substituting (6.5.2.7) in (6.5.2.8) $$\mathbf{E}(\mathbf{x}) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}\begin{pmatrix} \frac{2}{3} & \frac{-1}{3} \end{pmatrix}\mathbf{x} \tag{6.5.2.9}$$ $$\mathbf{E}(\mathbf{x}) = \begin{pmatrix} \frac{2}{3} & \frac{-1}{3} \\ \frac{-2}{3} & \frac{1}{3} \end{pmatrix}\mathbf{x} \tag{6.5.2.10}$$ $$\implies \mathbf{E} = \begin{pmatrix} \frac{2}{3} & \frac{-1}{3} \\ \frac{-2}{3} & \frac{1}{3} \end{pmatrix} \tag{6.5.2.11}$$
Verification	If $n \times n$ matrix $\mathbf{E}$ is projection matrix, then $\mathbf{E}^2 = \mathbf{E}$ $$\mathbf{E^2} = \begin{pmatrix} \frac{2}{3} & \frac{-1}{3} \\ \frac{-2}{3} & \frac{1}{3} \end{pmatrix}\begin{pmatrix} \frac{2}{3} & \frac{-1}{3} \\ \frac{-2}{3} & \frac{1}{3} \end{pmatrix} \implies \mathbf{E} = \mathbf{E^2} = \begin{pmatrix} \frac{2}{3} & \frac{-1}{3} \\ \frac{-2}{3} & \frac{1}{3} \end{pmatrix} \tag{6.5.2.12}$$ Hence, Verified.

TABLE 6.5.2.1: Finding Projection Matrix

6.5.3. If E_1 and E_2 are the projections of independent subspaces, then $E_1 + E_2$ is a projection.True or False?
Solution: See Table 6.5.3.1

Definition	If V is a vector space, a projection of V is a linear operator E on V such that $E^2 = E$
Proof	Assume projection matrices in $R^{2\times 2}$ Let $E_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ Let $E_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ Now, E_1 is a projection of R^2 onto its first coordinate And E_2 is a projection of R^2 onto its second coordinate $E = E_1 + E_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $E^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ $E \neq E^2$ Here, $E_1 + E_2$ is not equal to its square and hence its not a projection.
Conclusion	With the help of above proof we can say that the given statement is false.

TABLE 6.5.3.1: Illustration of Proof

6.5.4. If **E** is the projection and f is a polynomial,then $f(\mathbf{E}) = a\mathbf{I} + \mathbf{bE}$. What are a and **b** in terms of the coefficient vector of f?

Solution: Given ,

$$\mathbf{E} \text{ is the projection}$$
$$f \text{ is a polynomial}$$

$$f(\mathbf{E}) = a\mathbf{I} + \mathbf{bE} \tag{6.5.4.1}$$

Let ,

$$f(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n \tag{6.5.4.2}$$

Then,$f(\mathbf{E})$ can be written as,

$$f(\mathbf{E}) = c_0\mathbf{I} + c_1\mathbf{E} + c_2\mathbf{E}^2 + \cdots + c_n\mathbf{E}^n \tag{6.5.4.3}$$

Since **E** is the projection,

$$\mathbf{E}^2 = \mathbf{E} \tag{6.5.4.4}$$
$$\mathbf{E}^k = \mathbf{E} \quad \text{for any } k > 1 \tag{6.5.4.5}$$

Using equations (6.5.4.4) and (6.5.4.5), equation (6.5.4.3) can be modified as ,

$$f(\mathbf{E}) = c_0\mathbf{I} + c_1\mathbf{E} + c_2\mathbf{E} + \cdots + c_n\mathbf{E} \tag{6.5.4.6}$$
$$f(\mathbf{E}) = c_0\mathbf{I} + \mathbf{C}^T\mathbf{E} \tag{6.5.4.7}$$

where, **C** is the column vector and it is given as follows.

$$\mathbf{C} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \tag{6.5.4.8}$$

Comparing the equations (6.5.4.1) and (6.5.4.7) we get,

$$a = c_0 \tag{6.5.4.9}$$
$$\mathbf{b} = \mathbf{C}^T = \begin{pmatrix} c_1 & c_2 & \dots & c_n \end{pmatrix} \tag{6.5.4.10}$$

Here, a is the constant term of f and **b** is the co-efficent vector of projection **E** which contains the all co-efficients of polynomial other than constant term c_0

6.5.5. True or False? If a diagonalizable operator has only the characteristic values 0 and 1, it is a projection.

Solution: See Tables 6.5.5.1 and 6.5.5.2

Diagonalizable Operator	For a linear operator $\mathbf{T}\colon \mathbf{V} \longrightarrow \mathbf{V}$, $\mathbf{T}$ is a diagonalizable operator if $\exists$ some basis for $\mathbf{V}$ such that the matrix representing $\mathbf{T}$ is a diagonal matrix i.e. $$\mathbf{T}(\mathbf{X}) = \mathbf{A}\mathbf{X},$$ $\implies$ $\mathbf{A}$ is a diagonalizable matrix
Properties of Projection	If $n \times n$ matrix $\mathbf{A}$ is projecion matrix, then $$\mathbf{A}^2 = \mathbf{A}$$

TABLE 6.5.5.1: Definitions

Diagonalizability	Let $\mathbf{A}$ be $n \times n$ matrix. Given that $\mathbf{A}$ is diagonalizable, it can be expressed as, $$\begin{aligned} &\mathbf{A} = \mathbf{P\Lambda P}^{-1} \\ \implies &\mathbf{AP} = \mathbf{P\Lambda} \qquad \ldots(1) \end{aligned}$$ where, $\mathbf{\Lambda} = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{pmatrix}$
Eigen values	Given that $\mathbf{A}$ has eigen values 0 and 1 $\implies$ $\mathbf{\Lambda}$ has diagonal entries of $0s$ and $1s$ only $\implies \lambda_i = 0$ or $1, \qquad i = 0, 1, \ldots, n$ $\implies \lambda_i^2 = 0$ or 1 $\implies \lambda_i^2 = \lambda_i$ $$\implies \mathbf{\Lambda}^2 = \mathbf{\Lambda\Lambda} = \begin{pmatrix} \lambda_1^2 & 0 & \ldots & 0 \\ 0 & \lambda_2^2 & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & \lambda_n^2 \end{pmatrix} = \mathbf{\Lambda} \qquad \ldots(2)$$
Projection	$$\begin{aligned} &\mathbf{A} = \mathbf{P\Lambda P}^{-1} \\ \implies &\mathbf{AA} = \mathbf{AP\Lambda P}^{-1} \\ \implies &\mathbf{A}^2 = (\mathbf{AP})\,\mathbf{\Lambda P}^{-1} \end{aligned}$$ From (1), $$\begin{aligned} \implies \mathbf{A}^2 &= \mathbf{P\Lambda\Lambda P}^{-1} \\ &= \mathbf{P\Lambda}^2\mathbf{P}^{-1} \end{aligned}$$ From (2), $$\begin{aligned} \implies \mathbf{A}^2 &= \mathbf{P\Lambda P}^{-1} \\ &= \mathbf{A} \end{aligned}$$ Therefore, $$\mathbf{A}^2 = \mathbf{A}$$ Hence, $\mathbf{A}$ is a projection matrix
Example	Consider a 2×2 matrix $$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$ Characteristic polynomial,

	$$p(x) = \lvert x\mathbf{I} - \mathbf{A} \rvert$$ $$= \begin{vmatrix} x & -1 \\ 0 & x-1 \end{vmatrix}$$ $$= x(x-1)$$ $$\implies \lambda_1 = 0, \lambda_2 = 1$$ Also, $\mathbf{A}$ is diagonalizable, $$\mathbf{A} = \mathbf{PJP}^{-1}$$ where $$\mathbf{P} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{J} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$ Now we check if $\mathbf{A}$ is projection matrix. $$\mathbf{A}^2 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$ $$= \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$ $$= \mathbf{A}$$ Therefore, $$\mathbf{A}^2 = \mathbf{A}$$ Hence, if $\mathbf{A}$ is diagonalizable and has eigen values 0 and 1, then $\mathbf{A}$ is a projection matrix.
Conclusion	Given statement is True

TABLE 6.5.5.2: Checking for projection matrix

6.5.6. Prove that if $\mathbf{E}$ is the projection on $\mathbf{R}$ along $\mathbf{N}$, then $(\mathbf{I}-\mathbf{E})$ is the projection on $\mathbf{N}$ along $\mathbf{R}$

Solution:

Theorem 6.5. *If* $\mathbf{V}$ *is a vector space, a projection of* $\mathbf{V}$ *is a linear operator* $\mathbf{E}$ *on* $\mathbf{V}$ *such that* $\mathbf{E}^2=\mathbf{E}$. *Let* $\mathbf{R}$ *be the range and let* $\mathbf{N}$ *be the nullspace of* $\mathbf{E}$. *Then the vector space* $\mathbf{V}$ *can be written as* $\mathbf{V}=\mathbf{R}\bigoplus\mathbf{N}$. *This operator is called as projection on* $\mathbf{R}$ *along* $\mathbf{N}$.

It is given that $\mathbf{E}$ is the projection. From thorem 6.5, the linear operator $\mathbf{E}$ will satisfy $\mathbf{E}^2=\mathbf{E}$. Let's check whether $\mathbf{I}-\mathbf{E}$ is also a projection.

$$\begin{aligned}(\mathbf{I}-\mathbf{E})^2 &= \mathbf{I}^2+\mathbf{E}^2-2\mathbf{IE}\\ &= \mathbf{I}+\mathbf{E}-2\mathbf{E}\\ &= (\mathbf{I}-\mathbf{E}) \end{aligned} \tag{6.5.6.1}$$

From (6.5.6.1), we can say that $(\mathbf{I}-\mathbf{E})$ is also a projector. But $(\mathbf{I}-\mathbf{E})$ is called as the "Complementary Projector", i.e.

$$range(\mathbf{I}-\mathbf{E}) = null(\mathbf{E}) \tag{6.5.6.2}$$

$$null(\mathbf{I}-\mathbf{E}) = range(\mathbf{E}) \tag{6.5.6.3}$$

Lets take a vector $\mathbf{v}$ such that $\mathbf{Ev}=0$, where $\mathbf{v}$ is in the null space of $\mathbf{E}$. Then,

$$\begin{aligned}(\mathbf{I}-\mathbf{E})\mathbf{v} &= \mathbf{v}-\mathbf{vE}\\ &= \mathbf{v}\end{aligned} \tag{6.5.6.4}$$

In other words, any $\mathbf{v}$ in the nullspace of $\mathbf{E}$ is also in the range of $(\mathbf{I}-\mathbf{E})$. We know that any $\mathbf{x}\in range(\mathbf{I}-\mathbf{E})$ is characterized by

$$\begin{aligned}\mathbf{x} &= (\mathbf{I}-\mathbf{E})\mathbf{v} \quad , \textit{for some } \mathbf{v}\\ &= \mathbf{v}-\mathbf{Ev}\\ &= -\ (\mathbf{Ev}-\mathbf{v})\end{aligned} \tag{6.5.6.5}$$

Now we need to check if $\mathbf{x}$ is in the nullspace of $\mathbf{E}$. i.e. $\mathbf{Ex}=0$

$$\begin{aligned}\mathbf{E}(-\ (\mathbf{Ev}-\mathbf{v})) &= -(\mathbf{E}^2\mathbf{v}-\mathbf{Ev})\\ &= -(\mathbf{Ev}-\mathbf{Ev}) \quad (\because \mathbf{E} \textit{ is a projection})\\ &= 0\end{aligned} \tag{6.5.6.6}$$

Thus, if $\mathbf{x}\in range(\mathbf{I}-\mathbf{E})$, then $\mathbf{x}\in null(\mathbf{E})$.

Therefore, we can say that $null(\mathbf{E}) = range(\mathbf{I}-\mathbf{E})$.

We can use the same argument as above for proving (6.5.6.3), by taking $\mathbf{E}=\mathbf{I}-(\mathbf{I}-\mathbf{E})$.

$\therefore$ we can say that $(\mathbf{I}-\mathbf{E})$ is the projection on $\mathbf{N}$ along $\mathbf{R}$.

As an example, lets take the below matrix.

$$\mathbf{A}=\begin{pmatrix}\frac{2}{3} & \frac{1}{3} & \frac{1}{3}\\ \frac{1}{3} & \frac{2}{3} & \frac{-1}{3}\\ \frac{1}{3} & \frac{-1}{3} & \frac{2}{3}\end{pmatrix} \tag{6.5.6.7}$$

We can check that the matrix in (6.5.6.7) satisfies the condition $\mathbf{A}^2=\mathbf{A}$. Thus, $\mathbf{A}$ is a projection matrix.

$$(\mathbf{I}-\mathbf{A})=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}-\begin{pmatrix}\frac{2}{3} & \frac{1}{3} & \frac{1}{3}\\ \frac{1}{3} & \frac{2}{3} & \frac{-1}{3}\\ \frac{1}{3} & \frac{-1}{3} & \frac{2}{3}\end{pmatrix}$$

$$(\mathbf{I}-\mathbf{A})=\begin{pmatrix}\frac{1}{3} & \frac{-1}{3} & \frac{-1}{3}\\ \frac{-1}{3} & \frac{1}{3} & \frac{1}{3}\\ \frac{-1}{3} & \frac{1}{3} & \frac{1}{3}\end{pmatrix} \tag{6.5.6.8}$$

We can check that the matrix in (6.5.6.8) satisfies the condition $(\mathbf{I}-\mathbf{A})^2=(\mathbf{I}-\mathbf{A})$. Thus, $(\mathbf{I}-\mathbf{A})$ is a projection matrix.

Null space of $\mathbf{A}$ is given by

$$null(\mathbf{A})=b\begin{pmatrix}-1\\1\\1\end{pmatrix} \quad (\textit{'b' is any real number}) \tag{6.5.6.9}$$

Range of $(\mathbf{I}-\mathbf{A})$ is given by

$$\begin{aligned}range(\mathbf{I}-\mathbf{A}) &= a\begin{pmatrix}\frac{1}{3}\\ \frac{-1}{3}\\ \frac{-1}{3}\end{pmatrix} \quad (\textit{'a' is any real number})\\ &= a\left(\frac{-1}{3}\right)\begin{pmatrix}-1\\1\\1\end{pmatrix}\\ &= a\begin{pmatrix}-1\\1\\1\end{pmatrix}\end{aligned} \tag{6.5.6.10}$$

$\therefore$ from (6.5.6.9) and (6.5.6.10), we can say that $(\mathbf{I}-\mathbf{A})$ is a projection matrix on $\mathbf{N}$ along $\mathbf{R}$.

6.5.7. Let $E_1,...,E_k$ be linear operators on the space $\mathbf{V}$ such that $E_1+...+E_k=\mathbf{I}$.

a) Prove that if $E_iE_j=0$ for $i\neq j$,then $E_i^2=E_i$ for each i.

b) In the case $k = 2$, prove the converse of (a). That is, if $E_1 + E_2 = \mathbf{I}$ and $E_1^2 = E_1$, $E_2^2 = E_2$, then $E_1E_2 = 0$

Solution:

a) From the given,

$$E_1 + ... + E_k = \mathbf{I} \tag{6.5.7.1}$$

$$E_i = \mathbf{I} - E_1 - ... - E_{i-1} - E_{i+1} - ... - E_k \tag{6.5.7.2}$$

$$E_i^2 = E_i (\mathbf{I} - E_1 - ... - E_{i-1} - E_{i+1} - ... - E_k) \tag{6.5.7.3}$$

$$E_i^2 = E_i - E_iE_1.. - E_iE_{i-1} - E_iE_{i+1} - .. - E_iE_k \tag{6.5.7.4}$$

$$\implies E_i^2 = E_i - \sum_{i \neq j} E_iE_j \tag{6.5.7.5}$$

substituting $E_iE_j = 0$ for $i \neq j$ in the above equation we get,

$$E_i^2 = E_i - 0 \tag{6.5.7.6}$$

$$\implies E_i^2 = E_i \tag{6.5.7.7}$$

Hence proved if $E_iE_j = 0$ for $i \neq j$, then $E_i^2 = E_i$

b) Using,

$$E_1 + E_2 = \mathbf{I} \tag{6.5.7.8}$$

Multiplying both sides by E_1,

$$E_1(E_1 + E_2) = E_1 \tag{6.5.7.9}$$

$$\implies E_1^2 + E_1E_2 = E_1 \tag{6.5.7.10}$$

Substituting $E_1^2 = E_1$ in (6.5.7.8) we get,

$$\implies E_1 + E_1E_2 = E_1 \tag{6.5.7.11}$$

$$\implies E_1E_2 = 0 \tag{6.5.7.12}$$

Similarly multiplying on both sides of (6.5.7.8)and substituting $E_2^2 = E_2$ we get,

$$\implies E_1E_2 + E_2^2 = E_2 \tag{6.5.7.13}$$

$$\implies E_1E_2 + E_2 = E_2 \tag{6.5.7.14}$$

$$\implies E_1E_2 = 0 \tag{6.5.7.15}$$

Hence proved from (6.5.7.12) and (6.5.7.15) that if $E_1 + E_2 = \mathbf{I}$ and $E_1^2 = E_1$, $E_2^2 = E_2$, then $E_1E_2 = 0$

6.5.8. Let $\mathbf{V}$ be a real vector space and E an idempotent linear operator on $\mathbf{V}$, i.e., a projection. Prove that $\left(\mathbf{I} + \mathbf{E}\right)$ is invertible. Find $\left(\mathbf{I} + \mathbf{E}\right)^{-1}$.

Solution: we have $\mathbf{E}$ and it is idempotent. And we know that the eigen value of idempotent matrix is either 0 or 1. When we add the identity matrix in this:

$$\mathbf{I} + \mathbf{E} \tag{6.5.8.1}$$

Then eigen value will be either 1 or 2. Hence $\left(\mathbf{I} + \mathbf{E}\right)$ is invertible. Since $\mathbf{E}$ is an idempotent matrix, that is :

$$\mathbf{E}^2 = \mathbf{E} \tag{6.5.8.2}$$

Let,

$$\mathbf{A} = \mathbf{I} + \mathbf{E} \tag{6.5.8.3}$$

$$\implies \mathbf{E} = \mathbf{A} - \mathbf{I} \tag{6.5.8.4}$$

$$\implies \mathbf{E}^2 = (\mathbf{A} - \mathbf{I})(\mathbf{A} - \mathbf{I}) = \mathbf{A}^2 - 2\mathbf{A} + \mathbf{I}^2 \tag{6.5.8.5}$$

From 6.5.8.2,

$$\implies \mathbf{E} = \mathbf{A}^2 - 2\mathbf{A} + \mathbf{I} \tag{6.5.8.6}$$

Using (6.5.8.4) we have,

$$\implies \mathbf{A} - \mathbf{I} = \mathbf{A}^2 - 2\mathbf{A} + \mathbf{I} = \mathbf{A}^2 - 3\mathbf{A} + 2\mathbf{I} = 0 \tag{6.5.8.7}$$

$$\implies \mathbf{I} = \frac{3\mathbf{A} - \mathbf{A}^2}{2} \tag{6.5.8.8}$$

multiplying $\mathbf{A}^{-1}$ both side,

$$\mathbf{A}^{-1} = \frac{3\mathbf{I} - \mathbf{A}}{2} = \frac{3\mathbf{I} - (\mathbf{I} + \mathbf{E})}{2} \tag{6.5.8.9}$$

$$\implies \mathbf{A}^{-1} = \frac{2\mathbf{I} - \mathbf{E}}{2} \tag{6.5.8.10}$$

Using (6.5.8.3), we have,

$$(\mathbf{I} + \mathbf{E})^{-1} = \mathbf{I} - \frac{1}{2}\mathbf{E} \tag{6.5.8.11}$$

Let consider a matrix $\mathbf{E}$ as :

$$\mathbf{E} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{6.5.8.12}$$

$$\implies \mathbf{E}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{6.5.8.13}$$

$$\implies \mathbf{E}^2 = E. \tag{6.5.8.14}$$

Now,

$$\mathbf{I}+\mathbf{E}=\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}+\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} \tag{6.5.8.15}$$

$$\implies \mathbf{I}+\mathbf{E}=\begin{pmatrix}2 & 0\\0 & 1\end{pmatrix}=\begin{pmatrix}2 & 0\\0 & 1\end{pmatrix} \tag{6.5.8.16}$$

$$\tag{6.5.8.17}$$

Now let find the eigen value of matrix $\left(\mathbf{I}+\mathbf{E}\right)$:

$$\implies \begin{pmatrix}2-\lambda & 0\\0 & 1-\lambda\end{pmatrix}=0 \tag{6.5.8.18}$$

$$\implies (2-\lambda)(1-\lambda)=0 \tag{6.5.8.19}$$

$$\implies \lambda_1=2, \lambda_2=1 \tag{6.5.8.20}$$

The eigen values of the matrix $(\mathbf{I}+\mathbf{E})$ from (6.5.8.20) are 2 and 1. Since none of the eigen value is zero, hence matrix is invertible. Inverse of the matrix from (6.5.8.11) is :

$$\left(\mathbf{I}+\mathbf{E}\right)^{-1}=\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}+\begin{pmatrix}-\frac{1}{2} & 0\\0 & 1\end{pmatrix} \tag{6.5.8.21}$$

6.6 Invariant Direct Sums

6.6.1. Let E be a projection of V and let T be a linear operator on V. Prove that the range of E is invariant under T if and only if $ETE = TE$. Prove that both the range and null space of E are invariant under T if and only if $ET = TE$.
Solution: See Table 6.6.1.1

Proof of $ETE = TE$	Any projection E is represented by a matrix that is a part of an identity matrix Assume the Basis can be defined as follows: $B = \{\alpha_1, ..., \alpha_r, ..., \alpha_n\}$ such that $E_{ii} = 1$ for i ≤ r and 0 elsewhere $E_B = \begin{pmatrix} \mathrm{I} & 0 \\ 0 & 0 \end{pmatrix}$ where I is the $r \times r$ matrix Let $\alpha = (a_1, ..., a_n)$ $\implies T(E_\alpha) = T(a_1, ..., a_r, ...0) = \beta$ If we assume T to be invariant over the range W of E, then $\beta \in W$ $\beta = (\beta_1, ..., \beta_r, ..., 0)$, $E_\beta = \beta$ Therefore, $ETE = TE$
Proof of $ET = TE$	Consider the same assumption for basis B and projection E as defined above. $B = \{\alpha_1, ..., \alpha_r, ..., \alpha_n\}$ $E_B = \begin{pmatrix} \mathrm{I} & 0 \\ 0 & 0 \end{pmatrix}$ where I is the $r \times r$ matrix Let $TE \neq ET$, then there exists some vector V such that $T(a_1, ..., a_r, ...0) \neq (Ta_1, ..., Ta_r, ..., 0)$ But for this case, T is not an invariant of W. Assuming that T is an invariant of W,$T(a_1, ..., a_r, ...0) \in W$ for all $\alpha \in W$. Therefore, $T(a_1, ..., a_r, ...0) = (Ta_1, ..., Ta_r, ..., 0) \implies ET = TE$
Conclusion	Hence, it is proved that the range of E is invariant under T if and only if $ETE = TE$. And both the range and null space of E are invariant under T if and only if $TE = ET$.

TABLE 6.6.1.1: Illustration of Proof

6.6.2. Let T be the linear operator on R^2, the matrix of which in the standard ordered basis is

$$\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

Let W_1 be the subspace of R^2 spanned by the vector $\epsilon_1 = (1, 0)$

a) Proove that W_1 is invariant under T.

b) Prove that there is no subspace W_2 which is invariant under T and is complementary to W_1: $R^2 = W_1 \oplus W_2$
(Compare with exercise 1 of section 6.5.)

Solution: See Table 6.6.2.1

Statement	Solution
Given	T be the linear operator on R^2 the matrix of which in the standard ordered basis is $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ W_1 be the subspace of R^2 spanned by the vector $\epsilon_1 = (1, 0)$.
To Proove	a) W_1 is invariant under T. b) There is no subspace W_2 which is invariant under T and is complementary to W_1: $R^2 = W_1 \oplus W_2$ c) Compare with exercise 1 of section 6.5.
Proof (a)	$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \quad (6.6.2.1)$ $\lvert A - \lambda I \rvert = 0 \quad (6.6.2.2)$ $\implies \begin{pmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{pmatrix} \quad (6.6.2.3)$ $= (2-\lambda)^2 = 0 \quad (6.6.2.4)$ $\therefore \lambda = 2 \quad (6.6.2.5)$ for $\lambda = 2$, the corresponding vector is $(\mathbf{A} - \lambda I)X = 0 \quad (6.6.2.6)$ $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} X = 0 \quad (6.6.2.7)$ $\therefore X = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (6.6.2.8)$ Hence, W_1 be the subspace of R^2 spanned by the vector $\epsilon_1 = (1, 0)$ is invariant under T.
Proof (b)	Corresponding to $\lambda = 2$, Among, two eigen vectors only one is independent and other one is dependent. Thus, $P^{-}1$ does not exist and A can not be diagonalized. Hence, there is no subspace W_2 which is invariant under T and is complementary to W_1: $R^2 = W_1 \oplus W_2$
Observation	In exercise 1 of section 6.5, for 2×2 matrix there is 2 distinct characteristic value, corresponding to which there is a eigen vector. Hence, P^{-1} exists. $\therefore$ the given matrix is diagonalizable.

TABLE 6.6.2.1: Solution

Invariant Subspaces	Suppose $\mathbf{T} \in \mathbf{L}(\mathbf{V})$. A subspace $\mathbf{U}$ of $\mathbf{V}$ is called invariant under $\mathbf{T}$ if $\mathbf{u} \in \mathbf{U}$ implies $\mathbf{T}(\mathbf{u}) \in \mathbf{U}$. Suppose $\mathbf{T} \in \mathbf{L}(\mathbf{V})$, then null $\mathbf{T}$ and range $\mathbf{T}$ are invariant subspaces of $\mathbf{T}$.
Complementary $\mathbf{T}$ invariant subspace	Suppose we have a vector space $\mathbf{V}$, if $\mathbf{V}$ is written as direct sum of its subspaces $\mathbf{W}$ and $\mathbf{W}'$, i.e $\mathbf{V} = \mathbf{W} \bigoplus \mathbf{W}'$ and each of $\mathbf{W}$ and $\mathbf{W}'$ is invariant under $\mathbf{T}$, then we say $\mathbf{W}$ has a complementary $\mathbf{T}$ invariant subspace.

TABLE 7.1.1.1: Definition and Result used

7 The Rational and Jordan Forms

7.1 Cyclic Decompositions and the Rational Form

7.1.1. Let, $\mathbf{T}$ be the linear operator on $\mathbb{R}^3$ which is represented in the standard ordered basis by the matrix

$$\begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \tag{7.1.1.1}$$

Let $\mathbf{W}$ be the null space of $\mathbf{T} - 2\mathbf{I}$. Prove that $\mathbf{W}$ has no complementary $\mathbf{T}$-invariant subspace.
Solution: See Tables 7.1.1.1 and 7.1.1.2

7.2 The Jordan Form

7.2.1. If A is a complex 5×5 matrix with characteristic polynomial $f = (x-2)^3(x+7)^2$ and minimal polynomial $p = (x-2)^2(x+7)$, what is the Jordan form for A?
Solution: *Theory:* Table 7.2.1.1 gives the overview of properties of a Jordan block based on characteristic polynomial, minimal polynomial, algebraic multiplicity and geometric multiplicity.
From the properties stated in table 7.2.1.1, the Jordan blocks for eigenvalues of $\mathbf{A}$ can be written as,

$$\mathbf{J_1} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}; \quad \mathbf{J_2} = \begin{pmatrix} -7 & 0 \\ 0 & -7 \end{pmatrix} \tag{7.2.1.1}$$

Where $\mathbf{J_1}$ and $\mathbf{J_1}$ are the Jordan blocks corresponding to $\lambda_1 = 2$ and $\lambda_1 = -7$ respectively.

The Jordan form for $\mathbf{A}$ can be written as,

$$\mathbf{J} = \begin{pmatrix} \mathbf{J_1} & 0 \\ 0 & \mathbf{J_2} \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -7 & 0 \\ 0 & 0 & 0 & 0 & -7 \end{pmatrix} \tag{7.2.1.2}$$

Inference An $n \times n$ matrix with with λ as diagonal elements, ones on the super diagonal and zeroes in all other entries is nilpotent with minimal polynomial $(A - \lambda I)^n$ *Example* Let,

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{7.2.1.3}$$

(7.2.1.3) is nilpotent for minimal polynomial A^3

7.2.2. How many possible Jordan forms are there for a 6×6 complex matrix with characteristic polynomial $(x+2)^4(x-1)^2$?
Solution: See Tables 7.2.2.1, 7.2.2.2 and 7.2.2.3

7.2.3. The differentiation operator on the space of polynomials of degree less than or equal to 3 is represented in the natural ordered basis by the matrix,

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{7.2.3.1}$$

What is the Jordan form of this matrix? ($\mathbb{F}$ a subfield of the complex numbers.)
Solution: First, we find the characteristic poly-

Nullspace of $\mathbf{T}-2\mathbf{I}$	We Know that Nullspace of a linear operator $\mathbf{T}$ is the nullspace of its matrix representation of $\mathbf{T}$ w.r.t standard basis. Thus, Nullspace($\mathbf{W}$) = Nullspace($\mathbf{T}-2\mathbf{I}$). Now, Nullspace($\mathbf{T}-2\mathbf{I}$) = Nullspace $\begin{pmatrix}0&0&0\\1&0&0\\0&0&1\end{pmatrix}$ Hence, Nullspace($\mathbf{T}-2\mathbf{I}$) $= \{\begin{pmatrix}0\\k\\0\end{pmatrix} : k \in \mathbb{R}\}$ $= \{k\begin{pmatrix}0\\1\\0\end{pmatrix} : k \in \mathbb{R}\}$
Proof	Let $\beta = \begin{pmatrix}1\\0\\0\end{pmatrix}$. Then $(\mathbf{T}-2\mathbf{I})\beta = \begin{pmatrix}0&0&0\\1&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix} = \gamma \in \mathbf{W}$ Now, $(\mathbf{T}-2\mathbf{I})\gamma = \begin{pmatrix}0&0&0\\1&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \in \mathbf{W}.$ Now, we assume that $\mathbf{W}$ has a complementary $\mathbf{T}$-invariant subspace $\mathbf{S}$. Then β can be written as $\beta = s + w, s \in \mathbf{W}, w \in \mathbf{W}'$. Finally, we see that $\begin{pmatrix}0\\1\\0\end{pmatrix} = (\mathbf{T}-2\mathbf{I})\beta = (\mathbf{T}-2\mathbf{I})(s+w) = (\mathbf{T}-2\mathbf{I})w \in \mathbf{W}'$ as $\mathbf{W}'$ is invariant under $\mathbf{T}$ and s $\in$ Nullspace $\mathbf{W}$. Thus, we coclude that $\begin{pmatrix}0\\1\\0\end{pmatrix} \in \mathbf{W} \cap \mathbf{W}'$, which is a contradiction. Since, $\mathbf{V} = \mathbf{W} \bigoplus \mathbf{W}'$, thus $\mathbf{W} \cap \mathbf{W}' = \begin{pmatrix}0\\0\\0\end{pmatrix}$. Therefore, $\mathbf{W}$ has no complementary $\mathbf{T}$-invariant subspace.

TABLE 7.1.1.2: Solution

nomial of $\mathbf{A}$,

$$\det(\mathbf{A}-\lambda\mathbf{I}) = 0 \tag{7.2.3.2}$$

$$\implies \begin{vmatrix}-\lambda&1&0&0\\0&-\lambda&2&0\\0&0&-\lambda&3\\0&0&0&-\lambda\end{vmatrix} = 0 \tag{7.2.3.3}$$

$$\implies \lambda^4 = 0 \tag{7.2.3.4}$$

(7.2.3.4) is the required characteristic equation and $\lambda_1 = 0$ is the only eigen value of $\mathbf{A}$. Hence the characteristic polynomial of $\mathbf{A}$ is,

$$f(\lambda) = \lambda^4 \tag{7.2.3.5}$$

Feature	Effect on Jordan block	Example
characteristic polynomial or Algebraic multiplicity	The multiplicity of λ in the characteristic polynomial determines the size of the Jordan block for that eigenvalue. A_M = Size of Jordan block for λ	Let, $f = (x-2)^4$ be characteristic polynomial $\mathbf{J} = \begin{pmatrix} 2 & * & 0 & 0 \\ 0 & 2 & * & 0 \\ 0 & 0 & 2 & * \\ 0 & 0 & 0 & 2 \end{pmatrix}$ where $*$ can be either 1 or 0
Geometric multiplicity	The geometric multiplicity determines the total number of Jordan sub blocks for λ	If $A_M = 4$; $G_M = 2$; $\lambda = 2$ $\implies$ There should be 2 Jordan sub blocks for $\lambda = 2$. So, $\mathbf{J}$ has 2 possibilities $\mathbf{J} = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$; or $\mathbf{J} = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$
minimal polynomial	The multiplicity of λ in the minimal polynomial determines the size of the largest sub-block (Elementary Jordan block).	Let $p = (x-2)^3$ be minimal polynomial $\implies$ Size of largest sub-block is 3 Hence, one sub-block of size 3 and one sub-block of size 1 $\mathbf{J} = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

TABLE 7.2.1.1: Properties of Jordan blocks and Jordan canonical form

Parameter	Description
A_M	Algebraic multiplicity of characteristic value λ in the characteristic polynomial, also equal to the size of Jordan block for that eigen value
G_M	Geometric multiplicity determines the number of Jordan sub-blocks in a Jordan block for λ.
$\mathbf{J}_{(x-\lambda)^k}$	Jordan block corresponding to the eigen value λ and k is the multiplicity of λ in the minimal polynomial determines size of largest Jordan sub-block.

TABLE 7.2.2.1: Parameters

Again we observe that for $k = 4$ we have,

$$\mathbf{A}^4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{7.2.3.6}$$

$$\implies f(\mathbf{A}) = 0 \tag{7.2.3.7}$$

And for $k = 3$ we also have,

$$\mathbf{A}^3 = \begin{pmatrix} 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{7.2.3.8}$$

$$\implies \mathbf{A}^3 \neq 0 \tag{7.2.3.9}$$

From (7.2.3.7) and (7.2.3.9) we conclude that

Feature	Explanation
Characteristic Polynomial	$(x+2)^4 (x-1)^2 \quad (7.2.2.1)$
Algebraic Multiplicity, A_M	For $\lambda = -2, A_M = 4 \quad (7.2.2.2)$ For $\lambda = 1, A_M = 2 \quad (7.2.2.3)$
Minimal Polynomial	$p = (x+2)^a (x-1)^b \ , a \leq 4, b \leq 2 \quad (7.2.2.4)$
Possibilities of minimal polynomial	From equation (7.2.2.4), there are 8 different minimal polynomials possible.
Jordan Form	Jordan form built from Jordan blocks listed in table 7.2.2.3 will have the form, $\mathbf{J} = \begin{pmatrix} -2 & * & 0 & 0 & 0 & 0 \\ 0 & -2 & * & 0 & 0 & 0 \\ 0 & 0 & -2 & * & 0 & 0 \\ 0 & 0 & 0 & -2 & * & 0 \\ 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (7.2.2.5)$ where $*$ can be either 1 or 0
Number of possibilities of Jordan canonical forms	From Table 7.2.2.3, there are $5 \times 2 = 10$ different Jordan forms possible.
Jordan Form corresponding to $p = (x+2)^2 (x-1)$	One minimal polynomial can correspond to more than one Jordan forms. For example, minimal polynomial $p = (x+2)^2 (x-1)$ can correspond to two different Jordan forms namely, $\mathbf{J} = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{J} = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ (7.2.2.6)

TABLE 7.2.2.2: Parameters

Factor	Possible Jordan blocks	G_M
$(x+2)$	$\mathbf{J}_{(x+2)} = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$	4
	$\mathbf{J}_{(x+2)^2} = \begin{pmatrix} -2 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$	3
	$\mathbf{J}_{(x+2)^2} = \begin{pmatrix} -2 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & -2 \end{pmatrix}$	2
	$\mathbf{J}_{(x+2)^3} = \begin{pmatrix} -2 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$	2
	$\mathbf{J}_{(x+2)^4} = \begin{pmatrix} -2 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & -2 \end{pmatrix}$	1
$(x-1)$	$\mathbf{J}_{(x-1)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	2
	$\mathbf{J}_{(x-1)^2} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	1

TABLE 7.2.2.3: Possible Jordan Blocks

the minimal polynomial of $\mathbf{A}$ is,

$$g(\lambda) = \lambda^4 \tag{7.2.3.10}$$

Hence, the Jordan form of $\mathbf{A}$ is a 4×4 matrix consisting of only one block with principal diagonal values as $\lambda_1 = 0$ and super diagonal of the matrix (i.e the set of elements that lies directly above the elements comprising the principal diagonal) contains 1. Hence the required Jordan form of **A** is,

$$\mathbf{J} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad (7.2.3.11)$$

(7.2.3.11) is the required Jordan form of **A**.

7.2.4. Let $\mathbf{N_1}$ and $\mathbf{N_2}$ be 6×6 nilpotent matrices over the field **F**. Suppose that $\mathbf{N_1}$ and $\mathbf{N_2}$ have the same minimal polynomial and the same nullity. Prove that $\mathbf{N_1}$ and $\mathbf{N_2}$ are similar. Show that this is not true for 7×7 nilpotent matrices.
Solution: See Table 7.2.4.1

Statement	**Derivation**
Given	$\mathbf{N_1}$ and $\mathbf{N_2}$ be 6×6 nilpotent matrices. $\mathbf{N_1}$ and $\mathbf{N_2}$ have the same minimal polynomial and the same nullity. To prove $\mathbf{N_1}$ and $\mathbf{N_2}$ are similar.
From given statement	Two matrices are similar if they have the same Jordan Canonical form. 1. As $\mathbf{N_1}$ and $\mathbf{N_2}$ are nilpotent matrices, 0 is the only eigen value. 2. As minimal polynomial is same, $p\mathbf{N_1} = p\mathbf{N_2}$, the two matrices should have the same maximum block size. 3. As they have same nullity, they will have same total number of blocks. If $\mathbf{J_1}$ and $\mathbf{J_2}$ are similar, then $\mathbf{N_1}$ and $\mathbf{N_2}$ are similar. Let us consider all the possibilities for the dimensions of the block matrices for both Jordan forms.
Matrix size - 6 and Jordan size - 6	If Jordan form $\mathbf{J_1}$ consists of one block of dimension 6, then by (3) above $\mathbf{J_2}$ also has one block of dimension 6. $$\mathbf{J_1} = \begin{pmatrix}\mathbf{J}_{11}\end{pmatrix} \quad \mathbf{J_2} = \begin{pmatrix}\mathbf{J}_{21}\end{pmatrix}$$ $$\mathbf{J}_{11} : 6 \times 6 \quad \mathbf{J}_{21} : 6 \times 6$$ $$\mathbf{J}_{11}, \mathbf{J}_{21} \text{ are similar },$$ $$\mathbf{J}_1, \mathbf{J}_2 \text{ are similar}$$ $$\implies \mathbf{N}_1, \mathbf{N}_2 \text{ are similar}$$
Matrix size - 6 and Jordan size - 5 + 1	If Jordan form $\mathbf{J_1}$ consists of one block of dimension 5 and other 1, then by (2), $\mathbf{J_2}$ also has same maximum block of dimension 5 and by (3) have other block of size 1. $$\mathbf{J_1} = \begin{pmatrix}\mathbf{J}_{11} & 0 \\ 0 & \mathbf{J}_{12}\end{pmatrix} \quad \mathbf{J_2} = \begin{pmatrix}\mathbf{J}_{21} & 0 \\ 0 & \mathbf{J}_{22}\end{pmatrix}$$ $$\mathbf{J}_{11} : 5 \times 5 \quad \mathbf{J}_{21} : 5 \times 5$$ $$\mathbf{J}_{12} : 1 \times 1 \quad \mathbf{J}_{22} : 1 \times 1$$ $$\mathbf{J}_{11}, \mathbf{J}_{21} \text{ and } \mathbf{J}_{12}, \mathbf{J}_{22} \text{ are similar },$$ $$\mathbf{J}_1, \mathbf{J}_2 \text{ are similar}$$ $$\implies \mathbf{N}_1, \mathbf{N}_2 \text{ are similar}$$
Matrix size - 6, Jordan size - 4+2, Jordan size - 4+1+1	Although there are two different possibilities for Jordan blocks, From (2) $\mathbf{J}_{11}, \mathbf{J}_{21}$ are of dimension 4, From (3) $\mathbf{J}$ have same number of Jordan blocks Case 1:

	$\mathbf{J_1} = \begin{pmatrix} \mathbf{J}_{11} & 0 \\ 0 & \mathbf{J}_{12} \end{pmatrix} \quad \mathbf{J_2} = \begin{pmatrix} \mathbf{J}_{21} & 0 \\ 0 & \mathbf{J}_{22} \end{pmatrix}$ $\mathbf{J}_{11} : 4 \times 4 \quad \mathbf{J}_{21} : 4 \times 4$ $\mathbf{J}_{12} : 2 \times 2 \quad \mathbf{J}_{22} : 2 \times 2$ $\mathbf{J}_{11}, \mathbf{J}_{21}$ and $\mathbf{J}_{12}, \mathbf{J}_{22}$ are similar , $\mathbf{J}_1, \mathbf{J}_2$ are similar $\implies \mathbf{N}_1, \mathbf{N}_2$ are similar Case 2: $\mathbf{J_1} = \begin{pmatrix} \mathbf{J}_{11} & 0 & 0 \\ 0 & \mathbf{J}_{12} & 0 \\ 0 & 0 & \mathbf{J}_{13} \end{pmatrix} \quad \mathbf{J_2} = \begin{pmatrix} \mathbf{J}_{21} & 0 & 0 \\ 0 & \mathbf{J}_{22} & 0 \\ 0 & 0 & \mathbf{J}_{23} \end{pmatrix}$ $\mathbf{J}_{11} : 4 \times 4 \quad \mathbf{J}_{21} : 4 \times 4$ $\mathbf{J}_{12} : 1 \times 1 \quad \mathbf{J}_{22} : 1 \times 1$ $\mathbf{J}_{13} : 1 \times 1 \quad \mathbf{J}_{23} : 1 \times 1$ $\mathbf{J}_{11}, \mathbf{J}_{21} \quad \mathbf{J}_{12}, \mathbf{J}_{22}$ and $\mathbf{J}_{13}, \mathbf{J}_{23}$ are similar , $\mathbf{J}_1, \mathbf{J}_2$ are similar $\implies \mathbf{N}_1, \mathbf{N}_2$ are similar
Matrix size - 6, Jordan size - 3+3, Jordan size - 3+2+1, Jordan size - 3+1+1+1	There are three different possibilities for Jordan blocks, From (2) $\mathbf{J}_{11}, \mathbf{J}_{21}$ are of dimension 3, From (3) $\mathbf{J}$ have same number of Jordan blocks Case 1: $\mathbf{J_1} = \begin{pmatrix} \mathbf{J}_{11} & 0 \\ 0 & \mathbf{J}_{12} \end{pmatrix} \quad \mathbf{J_2} = \begin{pmatrix} \mathbf{J}_{21} & 0 \\ 0 & \mathbf{J}_{22} \end{pmatrix}$ $\mathbf{J}_{11} : 3 \times 3 \quad \mathbf{J}_{21} : 3 \times 3$ $\mathbf{J}_{12} : 3 \times 3 \quad \mathbf{J}_{22} : 3 \times 3$ $\mathbf{J}_{11}, \mathbf{J}_{21}$ and $\mathbf{J}_{12}, \mathbf{J}_{22}$ are similar , $\mathbf{J}_1, \mathbf{J}_2$ are similar $\implies \mathbf{N}_1, \mathbf{N}_2$ are similar Case 2: $\mathbf{J_1} = \begin{pmatrix} \mathbf{J}_{11} & 0 & 0 \\ 0 & \mathbf{J}_{12} & 0 \\ 0 & 0 & \mathbf{J}_{13} \end{pmatrix} \quad \mathbf{J_2} = \begin{pmatrix} \mathbf{J}_{21} & 0 & 0 \\ 0 & \mathbf{J}_{22} & 0 \\ 0 & 0 & \mathbf{J}_{23} \end{pmatrix}$ $\mathbf{J}_{11} : 3 \times 3 \quad \mathbf{J}_{21} : 3 \times 3$ $\mathbf{J}_{12} : 2 \times 2 \quad \mathbf{J}_{22} : 2 \times 2$ $\mathbf{J}_{13} : 1 \times 1 \quad \mathbf{J}_{23} : 1 \times 1$ $\mathbf{J}_{11}, \mathbf{J}_{21} \quad \mathbf{J}_{12}, \mathbf{J}_{22}$ and $\mathbf{J}_{13}, \mathbf{J}_{23}$ are similar , $\mathbf{J}_1, \mathbf{J}_2$ are similar $\implies \mathbf{N}_1, \mathbf{N}_2$ are similar

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Case 3:

$$\mathbf{J_1} = \begin{pmatrix} \mathbf{J}_{11} & 0 & 0 & 0 \\ 0 & \mathbf{J}_{12} & 0 & 0 \\ 0 & 0 & \mathbf{J}_{13} & 0 \\ 0 & 0 & 0 & \mathbf{J}_{14} \end{pmatrix} \quad \mathbf{J_2} = \begin{pmatrix} \mathbf{J}_{21} & 0 & 0 & 0 \\ 0 & \mathbf{J}_{22} & 0 & 0 \\ 0 & 0 & \mathbf{J}_{23} & 0 \\ 0 & 0 & 0 & \mathbf{J}_{24} \end{pmatrix}$$

$$\mathbf{J}_{11} : 3 \times 3 \quad \mathbf{J}_{21} : 3 \times 3$$
$$\mathbf{J}_{12} : 1 \times 1 \quad \mathbf{J}_{22} : 1 \times 1$$
$$\mathbf{J}_{13} : 1 \times 1 \quad \mathbf{J}_{23} : 1 \times 1$$
$$\mathbf{J}_{14} : 1 \times 1 \quad \mathbf{J}_{24} : 1 \times 1$$
$$\mathbf{J}_{11}, \mathbf{J}_{21} \quad \mathbf{J}_{12}, \mathbf{J}_{22} \text{ and } \mathbf{J}_{13}, \mathbf{J}_{23} \quad \mathbf{J}_{14}, \mathbf{J}_{24} \text{ are similar },$$
$$\mathbf{J}_1, \mathbf{J}_2 \text{ are similar}$$
$$\implies \mathbf{N}_1, \mathbf{N}_2 \text{ are similar}$$
</td></tr>
<tr><td>Matrix size - 6, Jordan size - 2+2+2, Jordan size - 2+2+1+1, Jordan size - 2+1+1+1+1</td><td>
There are three different possibilities for Jordan blocks,

From (2) $\mathbf{J}_{11}, \mathbf{J}_{21}$ are of dimension 2,
From (3) $\mathbf{J}$ have same number of Jordan blocks
Case 1:

$$\mathbf{J_1} = \begin{pmatrix} \mathbf{J}_{11} & 0 & 0 \\ 0 & \mathbf{J}_{12} & 0 \\ 0 & 0 & \mathbf{J_{13}} \end{pmatrix} \quad \mathbf{J_2} = \begin{pmatrix} \mathbf{J}_{21} & 0 & 0 \\ 0 & \mathbf{J}_{22} & 0 \\ 0 & 0 & \mathbf{J_{23}} \end{pmatrix}$$

$$\mathbf{J}_{11} : 2 \times 2 \quad \mathbf{J}_{21} : 2 \times 2$$
$$\mathbf{J}_{12} : 2 \times 2 \quad \mathbf{J}_{22} : 2 \times 2$$
$$\mathbf{J}_{13} : 2 \times 2 \quad \mathbf{J}_{23} : 2 \times 2$$
$$\mathbf{J}_{11}, \mathbf{J}_{21} \quad \mathbf{J}_{12}, \mathbf{J}_{22} \text{ and } \mathbf{J}_{13}, \mathbf{J}_{23} \text{ are similar },$$
$$\mathbf{J}_1, \mathbf{J}_2 \text{ are similar}$$
$$\implies \mathbf{N}_1, \mathbf{N}_2 \text{ are similar}$$

Case 2:

$$\mathbf{J_1} = \begin{pmatrix} \mathbf{J}_{11} & 0 & 0 & 0 \\ 0 & \mathbf{J}_{12} & 0 & 0 \\ 0 & 0 & \mathbf{J}_{13} & 0 \\ 0 & 0 & 0 & \mathbf{J}_{14} \end{pmatrix} \quad \mathbf{J_2} = \begin{pmatrix} \mathbf{J}_{21} & 0 & 0 & 0 \\ 0 & \mathbf{J}_{22} & 0 & 0 \\ 0 & 0 & \mathbf{J}_{23} & 0 \\ 0 & 0 & 0 & \mathbf{J}_{24} \end{pmatrix}$$

$$\mathbf{J}_{11} : 2 \times 2 \quad \mathbf{J}_{21} : 2 \times 2$$
$$\mathbf{J}_{12} : 2 \times 2 \quad \mathbf{J}_{22} : 2 \times 2$$
$$\mathbf{J}_{13} : 1 \times 1 \quad \mathbf{J}_{23} : 1 \times 1$$
$$\mathbf{J}_{14} : 1 \times 1 \quad \mathbf{J}_{24} : 1 \times 1$$
$$\mathbf{J}_{11}, \mathbf{J}_{21} \quad \mathbf{J}_{12}, \mathbf{J}_{22} \text{ and } \mathbf{J}_{13}, \mathbf{J}_{23} \quad \mathbf{J}_{14}, \mathbf{J}_{24} \text{ are similar },$$
$$\mathbf{J}_1, \mathbf{J}_2 \text{ are similar}$$
$$\implies \mathbf{N}_1, \mathbf{N}_2 \text{ are similar}$$

Case 3:
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	$\mathbf{J_1} = \begin{pmatrix} \mathbf{J}_{11} & 0 & 0 & 0 & 0 \\ 0 & \mathbf{J}_{12} & 0 & 0 & 0 \\ 0 & 0 & \mathbf{J}_{13} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{J}_{14} & 0 \\ 0 & 0 & 0 & 0 & \mathbf{J}_{15} \end{pmatrix} \quad \mathbf{J_2} = \begin{pmatrix} \mathbf{J}_{21} & 0 & 0 & 0 & 0 \\ 0 & \mathbf{J}_{22} & 0 & 0 & 0 \\ 0 & 0 & \mathbf{J}_{23} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{J}_{24} & 0 \\ 0 & 0 & 0 & 0 & \mathbf{J}_{25} \end{pmatrix}$ $\mathbf{J}_{11} : 2\times 2 \quad \mathbf{J}_{21} : 2\times 2$ $\mathbf{J}_{12} : 1\times 1 \quad \mathbf{J}_{22} : 1\times 1$ $\mathbf{J}_{13} : 1\times 1 \quad \mathbf{J}_{23} : 1\times 1$ $\mathbf{J}_{14} : 1\times 1 \quad \mathbf{J}_{24} : 1\times 1$ $\mathbf{J}_{15} : 1\times 1 \quad \mathbf{J}_{25} : 1\times 1$ $\mathbf{J}_{11},\mathbf{J}_{21} \quad \mathbf{J}_{12},\mathbf{J}_{22}$ and $\mathbf{J}_{13},\mathbf{J}_{23} \quad \mathbf{J}_{14},\mathbf{J}_{24} \quad \mathbf{J}_{15},\mathbf{J}_{25}$ are similar , $\mathbf{J}_1,\mathbf{J}_2$ are similar $\implies \mathbf{N}_1,\mathbf{N}_2$ are similar
Matrix size - 6, Jordan size-1+1+1+1+1+1	$\mathbf{J_1}$ and $\mathbf{J_2}$ will have same number of blocks because of (3). $\mathbf{J_1} = \begin{pmatrix} \mathbf{J}_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{J}_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{J}_{13} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{J}_{14} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{J}_{15} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{J}_{16} \end{pmatrix} \quad \mathbf{J_2} = \begin{pmatrix} \mathbf{J}_{21} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{J}_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{J}_{23} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{J}_{24} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{J}_{25} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{J}_{26} \end{pmatrix}$ $\mathbf{J}_{11},\mathbf{J}_{21} \quad \mathbf{J}_{12},\mathbf{J}_{22} \quad \mathbf{J}_{13},\mathbf{J}_{23} \quad \mathbf{J}_{14},\mathbf{J}_{24} \quad \mathbf{J}_{15},\mathbf{J}_{25}$ and $\mathbf{J}_{16},\mathbf{J}_{26}$ are similar , $\mathbf{J}_1,\mathbf{J}_2$ are similar $\implies \mathbf{N}_1,\mathbf{N}_2$ are similar
7×7 Nielpotent matrix	Let us take a counter example, Matrix size - 7, Jordan block size - 3+3+1 and 3+2+2 $\mathbf{J_1} = \begin{pmatrix} 0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0 \\ 0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0 \end{pmatrix} \quad \mathbf{J_2} = \begin{pmatrix} 0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0 \end{pmatrix} \qquad (7.2.4.1)$ From above, $\mathbf{J_1}$ and $\mathbf{J_2}$ are not same. Hence $\mathbf{N_1}$ and $\mathbf{N_2}$ are not similar.

TABLE 7.2.4.1: Solution

7.2.5. If $\mathbf{N}$ is a nilpotent 3×3 matrix over C, prove that $\mathbf{A} = \mathbf{I} + \frac{1}{2}\mathbf{N} - \frac{1}{8}\mathbf{N}^2$ satisfies $\mathbf{A}^2 = \mathbf{I} + \mathbf{N}$, i.e., $\mathbf{A}$ is a square root of $\mathbf{I} + \mathbf{N}$. Use the binomial series for $(1 + t)^{\frac{1}{2}}$ to obtain a similar formula for a square root of $\mathbf{I} + \mathbf{N}$, where $\mathbf{N}$ is any nilpotent $n \times n$ matrix over C.

Solution: We know that $\mathbf{N}^3=0$ since the minimal polynomial of $\mathbf{N}$ is x^3,So,

$$\mathbf{A}^2 = \left(\mathbf{I} + \frac{1}{2}\mathbf{N} - \frac{1}{8}\mathbf{N}^2\right)\left(\mathbf{I} + \frac{1}{2}\mathbf{N} - \frac{1}{8}\mathbf{N}^2\right) \tag{7.2.5.1}$$

$$= \mathbf{I} + \frac{1}{2}\mathbf{N} - \frac{1}{8}\mathbf{N}^2 + \frac{1}{4}\mathbf{N}^2 - \frac{1}{8}\mathbf{N}^2 \tag{7.2.5.2}$$

$$= \mathbf{I} + \mathbf{N} \tag{7.2.5.3}$$

Expanding $(1 + t)^{1/2}$,

$$(1 + t)^{1/2} = \sum_{k=0}^{\infty} \binom{1/2}{k} t^k \tag{7.2.5.4}$$

Here,

$$\binom{1/2}{k} = \frac{\frac{1}{2}(\frac{1}{2} - 1)(\frac{1}{2} - 2)\cdots(\frac{1}{2} - k + 1)}{k!} \tag{7.2.5.5}$$

$$= \frac{(-1)^{k-1}}{2^k k!} 1 \cdot 3 \cdot 5 \cdots (2k - 3) \tag{7.2.5.6}$$

$$= \frac{(-1)^{k-1}}{2^k k!} \frac{(2k - 2)!}{2^{k-1}(k - 1)!} \tag{7.2.5.7}$$

$$= \frac{(-1)^{k-1}}{k 2^{2k-1}} \binom{2k - 2}{k - 1} \tag{7.2.5.8}$$

Thus,

$$(1 + t)^{1/2} = 1 - \sum_{k=1}^{\infty} \frac{2}{k} \binom{2k - 2}{k - 1} \left(-\frac{t}{4}\right)^k \tag{7.2.5.9}$$

$$= 1 - \sum_{k=0}^{\infty} \frac{2}{k + 1} \binom{2k}{k} \left(-\frac{t}{4}\right)^{k+1} \tag{7.2.5.10}$$

So a square root for $\mathbf{I}+\mathbf{N}$ where $\mathbf{N}$ is a $n\times n$ nilpotent matrix can be,

$$= \mathbf{I} - \sum_{k=0}^{n-1} \frac{2}{k + 1} \binom{2k}{k} \left(-\frac{\mathbf{N}}{4}\right)^{k+1} \tag{7.2.5.11}$$

7.3 Computation of Invariant Factors

7.3.1. Construct a linear operator T with minimal polynomial $x^2(x-1)^2$ and characteristic polynomial $x^3(x-1)^4$.Describe the primary decomposition of the vector space under T and find the projections on the primary components.Find a basis in which the matrix T is in Jordan form.Also find an explicit direct sum decomposition of the space into Tcyclic subspaces as in theorem 3 and give the invariant factors.

Solution: See Table 7.3.1.1

Statement	Solution
	Jordan Form
Given	Linear operator $T : \mathbf{V} \to \mathbf{V}$ (7.3.1.1) Characteristic polynomial$f(x) = x^3(x-1)^4$ (7.3.1.2) Minimal polynomial$p(x) = x^2(x-1)^2$ (7.3.1.3)
The jordan block corresponding to eigen value 0	$\mathbf{J}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ (7.3.1.4)
One of the possible jordan blocks corresponding to eigen value 1	$\mathbf{J}_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ (7.3.1.5)
The jordan form of transformation matrix $\mathbf{T}$	$\mathbf{J} = \begin{pmatrix} \mathbf{J}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_2 \end{pmatrix}$ (7.3.1.6) $= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ (7.3.1.7)
	Primary Decomposition,Finding the basis
According to primary decomposition theorem	If $p(x) = p_1(x)^{r_1} p_2(x)^{r_2}$, (7.3.1.8) $\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2$ (7.3.1.9) $\mathbf{V}_i =$ Null space of$(p_i(\mathbf{T}))_i^r$ (7.3.1.10) $p_1(x)^{r_1} = x^2$ (7.3.1.11) $p_2(x)^{r_2} = (x-1)^2$ (7.3.1.12)
Null space of $\mathbf{J}^2$	$\mathbf{J}^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ (7.3.1.13) Nullity of $\mathbf{J}^2 = 3$ (7.3.1.14) From (7.3.1.13),the basis for the nullspace is

	$$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \quad (7.3.1.15)$$ $$\mathbf{v}_1 = \begin{pmatrix}1\\0\\0\\0\\0\\0\\0\end{pmatrix}, \mathbf{v_2} = \begin{pmatrix}0\\1\\0\\0\\0\\0\\0\end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix}0\\0\\1\\0\\0\\0\\0\end{pmatrix} \quad (7.3.1.16)$$
Nullspace of $(\mathbf{J}-\mathbf{I})^2$	$$(\mathbf{J}-\mathbf{I})^2 = \begin{pmatrix}1 & -2 & 0 & 0 & 0 & 0 & 0\\0 & 1 & 0 & 0 & 0 & 0 & 0\\0 & 0 & 1 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix} \quad (7.3.1.17)$$ $$\text{Nullity of } (\mathbf{J}-\mathbf{I})^2 = 4 \quad (7.3.1.18)$$ From (7.3.1.17),the basis for the nullspace is $$\{\mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6, \mathbf{v}_7\} \quad (7.3.1.19)$$ $$\mathbf{v}_4 = \begin{pmatrix}0\\0\\0\\1\\0\\0\\0\end{pmatrix}, \mathbf{v_5} = \begin{pmatrix}0\\0\\0\\0\\1\\0\\0\end{pmatrix}, \mathbf{v}_6 = \begin{pmatrix}0\\0\\0\\0\\0\\1\\0\end{pmatrix}, \mathbf{v}_7 = \begin{pmatrix}0\\0\\0\\0\\0\\0\\1\end{pmatrix} \quad (7.3.1.20)$$
$$\mathbf{T} = \mathbf{J} \quad (7.3.1.21)$$	$\mathbf{T}$ is similar to block diagonal jordan matrix $\mathbf{J}$ in the basis $$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6, \mathbf{v}_7\} \quad (7.3.1.22)$$ which is the standard ordered basis.
Finding the projections	
The projection matrices $\mathbf{E}_1, \mathbf{E}_2$ are such that	a) for $i \in [1, 2]$ $$\mathbf{E}_i(\mathbf{v}) = \begin{cases}\mathbf{v} & \text{for } \mathbf{v} \in \mathbf{V}_i\\ 0 & \text{for } \mathbf{v} \notin \mathbf{V}_i\end{cases} \quad (7.3.1.23)$$ b) $$(\mathbf{E}_i)^2 = \mathbf{E}_i \quad (7.3.1.24)$$ c) $$\mathbf{E}_1 + \mathbf{E}_2 = \mathbf{I} \quad (7.3.1.25)$$

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<tr><td>The projection matrices are</td><td>

$$\mathbf{E}_1 = \begin{pmatrix} \mathbf{I}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \tag{7.3.1.26}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{7.3.1.27}$$

$$\mathbf{E}_2 = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_4 \end{pmatrix} \tag{7.3.1.28}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{7.3.1.29}$$

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<tr><td colspan="2">Cyclic Decomposition</td></tr>
<tr><td>Cyclic decomposition theorem(Theorem 3)</td><td>

Let T be a linear operator on a finite-dimensional vector space $\mathbf{V}$ and let $\mathbf{W}_0$ be a proper T-admissible subspace of $\mathbf{V}$. There exists non zero vectors $\alpha_1, \alpha_2, \ldots, \alpha_r$ in $\mathbf{V}$ with respective T-annihilators $p_1, p_2, \ldots, p_r$ such that

a)

$$\mathbf{V} = \mathbf{W}_0 \oplus \mathbf{Z}(\alpha_1; T) \oplus \mathbf{Z}(\alpha_2; T) \oplus \tag{7.3.1.30}$$

$$\ldots \oplus \mathbf{Z}(\alpha_r; T) \tag{7.3.1.31}$$

b) p_k divides p_{k-1},$k = 2, \ldots, r$

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<tr><td>The T-cyclic subspace $\mathbf{Z}(\alpha_i; T)$ is defined as</td><td>

$$\text{degree of } p_i = k \tag{7.3.1.32}$$

$$\implies \text{basis of } \mathbf{Z}(\alpha_i; T) = \tag{7.3.1.33}$$

$$\left\{\alpha_i, \mathbf{T}\alpha_i, \ldots, \mathbf{T}^{k-1}\alpha_i\right\} \tag{7.3.1.34}$$

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<tr><td>Finding the cyclic subspaces</td><td>

Let us choose

$$\mathbf{W}_0 = \mathbf{0} \tag{7.3.1.35}$$

$$\tag{7.3.1.36}$$

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$$p_1 = x^2(x-1)^2 \tag{7.3.1.37}$$

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$$\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{7.3.1.38}$$

$$\mathbf{T}^2(\mathbf{T}-\mathbf{I})^2 = \mathbf{0}_{7\times 7} \tag{7.3.1.39}$$

$$\implies p_1(\mathbf{T})(\alpha_1) = 0 \tag{7.3.1.40}$$

$$\dim(\mathbf{Z}(\alpha_1; T)) = 4 \tag{7.3.1.41}$$

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$$p_2 = x(x-1)^2 \tag{7.3.1.42}$$

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Now to chose α_2 we need to chose a vector such that

$$\alpha_2 \notin \mathbf{Z}(\alpha_1; T), p_2(\mathbf{T})\alpha_2 = 0 \tag{7.3.1.43}$$

$$p_2 = x(x-1)^2 \tag{7.3.1.44}$$

$$\frac{p_1}{p_2} = x \implies p_2 \text{ divides } p_1 \tag{7.3.1.45}$$

$$p_2(\mathbf{T}) = \mathbf{T}(\mathbf{T}-\mathbf{I})^2 = \tag{7.3.1.46}$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{7.3.1.47}$$

One such vector that satisfies (7.3.1.43) is

$$\alpha_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{7.3.1.48}$$

$$\dim(\mathbf{Z}(\alpha_2; T)) = 3 \tag{7.3.1.49}$$

$$\dim\mathbf{Z}(\alpha_1; T) + \dim\mathbf{Z}(\alpha_2; T) = 7 \tag{7.3.1.50}$$

$$\implies \mathbf{V} = \mathbf{Z}(\alpha_1; T) \oplus \mathbf{Z}(\alpha_2; T) \tag{7.3.1.51}$$

is the cyclic decomposition.

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<tr>
<td colspan="2">Invariant Factors</td>
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<td>Invariant factors are</td>
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$$p_1 = x^2(x-1)^2 \tag{7.3.1.52}$$

$$p_2 = x(x-1)^2 \tag{7.3.1.53}$$

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TABLE 7.3.1.1: Solution

8 Inner Product Spaces

8.1 Inner Products

8.1.1. Let $(|)$ be the standard inner product on $\mathbb{R}^2$.

a) Let

$$\alpha = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \beta = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{8.1.1.1}$$

If γ is a vector such that $(\alpha^T \gamma) = -1$ and $(\beta^T \gamma) = 3$. Find γ

b) Show that for any α in $\mathbb{R}^2$ we have

$$\alpha = (\alpha^T \epsilon_1)\epsilon_1 + (\alpha^T \epsilon_2)\epsilon_2 \tag{8.1.1.2}$$

Solution:

(a) From $\alpha = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $(\alpha^T \gamma) = -1$ we get,

$$(\alpha^T \gamma) = -1 \tag{8.1.1.3}$$

$$\implies \begin{pmatrix} 1 \\ 2 \end{pmatrix}^T \gamma = -1 \tag{8.1.1.4}$$

from $\beta = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $(\beta^T \gamma) = 3$ we get,

$$(\beta^T \gamma) = 3 \tag{8.1.1.5}$$

$$\implies \begin{pmatrix} -1 \\ 1 \end{pmatrix}^T \gamma = 3 \tag{8.1.1.6}$$

using (8.1.1.4) and (8.1.1.6),

$$\begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \gamma \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix} \tag{8.1.1.7}$$

row reductions,

$$\begin{pmatrix} 1 & 2 & -1 \\ -1 & 1 & 3 \end{pmatrix} \xleftrightarrow{R_2 \to R_2 + R_1} \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & 2 \end{pmatrix} \tag{8.1.1.8}$$

$$\xleftrightarrow{R_2 \to \frac{1}{3} R_2} \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & \frac{2}{3} \end{pmatrix} \xleftrightarrow{R_1 \to R_1 - 2R_2} \begin{pmatrix} 1 & 0 & \frac{-7}{3} \\ 0 & 1 & \frac{2}{3} \end{pmatrix} \tag{8.1.1.9}$$

Hence $\gamma = \begin{pmatrix} \frac{-7}{3} \\ \frac{2}{3} \end{pmatrix}$

(b) Here ϵ_1, ϵ_2 are standard basis vector in $\mathbb{R}^2$. As $\alpha \in \mathbb{R}^2$ we can write it as,

$$\alpha = \alpha_1 \epsilon_1 + \alpha_2 \epsilon_2 = \alpha^T \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix} \tag{8.1.1.10}$$

using this we can write,

$$(\alpha^T \epsilon_1) = \left(\alpha^T \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix} \right)^T \epsilon_1 = \alpha^T \begin{pmatrix} \epsilon_1^T \\ \epsilon_2^T \end{pmatrix} \epsilon_1 \tag{8.1.1.11}$$

$$= \alpha^T \begin{pmatrix} \epsilon_1^T \epsilon_1 \\ \epsilon_2^T \epsilon_1 \end{pmatrix} = \alpha^T \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{8.1.1.12}$$

$$\implies (\alpha^T \epsilon_1)\epsilon_1 = \alpha^T \begin{pmatrix} \epsilon_1 \\ 0 \end{pmatrix} \tag{8.1.1.13}$$

$$(\alpha^T \epsilon_2) = \left(\alpha^T \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix} \right)^T \epsilon_2 = \alpha^T \begin{pmatrix} \epsilon_1^T \\ \epsilon_2^T \end{pmatrix} \epsilon_2 \tag{8.1.1.14}$$

$$= \alpha^T \begin{pmatrix} \epsilon_1^T \epsilon_2 \\ \epsilon_2^T \epsilon_2 \end{pmatrix} = \alpha^T \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{8.1.1.15}$$

$$\implies (\alpha^T \epsilon_2)\epsilon_2 = \alpha^T \begin{pmatrix} 0 \\ \epsilon_2 \end{pmatrix} \tag{8.1.1.16}$$

using (8.1.1.13) and (8.1.1.16) we can write,

$$(\alpha^T \epsilon_1)\epsilon_1 + (\alpha^T \epsilon_2)\epsilon_2 = \alpha^T \begin{pmatrix} \epsilon_1 \\ 0 \end{pmatrix} + \alpha^T \begin{pmatrix} 0 \\ \epsilon_2 \end{pmatrix} \tag{8.1.1.17}$$

$$\implies (\alpha^T \epsilon_1)\epsilon_1 + (\alpha^T \epsilon_2)\epsilon_2 = \alpha^T \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix} \tag{8.1.1.18}$$

hence using (8.1.1.10)and (8.1.1.18) we get,

$$\alpha = (\alpha^T \epsilon_1)\epsilon_1 + (\alpha^T \epsilon_2)\epsilon_2 \tag{8.1.1.19}$$

Hence proved