Assignment 15

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***Abstract*—This document solves a problem involving linear transformations.**

Download latex-tikz codes from

https://github.com/Vaibhav11002/EE5609/tree/master/Assignment_15

1 Problem

Let $\mathbf{V}$ be a vector space over the field $\mathbf{F}$ and $\mathbf{T}$ is a linear operator on $\mathbf{V}$. If $\mathbf{T}^2 = 0$, what can you say about the relation of the range of $\mathbf{T}$ to the null space of $\mathbf{T}$? Give an example of linear operator $\mathbf{T}$ on $\mathbf{R}^2$ such that $\mathbf{T}^2 = 0$ but $\mathbf{T} \neq 0$.

2 Solution

Given,

$$\mathbf{T} : \mathbf{V} \to \mathbf{V} \tag{2.0.1}$$

Now, $\mathbf{T}^2$ is also a linear operator as,

$$\mathbf{T}^2(c\alpha) = \mathbf{T}(\mathbf{T}(c\alpha)) = \mathbf{T}(c\mathbf{T}(\alpha)) \tag{2.0.2}$$

$$= c\mathbf{T}(\mathbf{T}(\alpha)) = c\mathbf{T}^2(\alpha) \tag{2.0.3}$$

Let some vector $\mathbf{y} \in \text{Range}(\mathbf{T})$ then there exists $\mathbf{x} \in \mathbf{V}$ such that,

$$\mathbf{T}(\mathbf{x}) = \mathbf{y} \tag{2.0.4}$$

Now given that,

$$\mathbf{T}^2(\mathbf{x}) = \mathbf{0} \tag{2.0.5}$$

$$\implies \mathbf{T}(\mathbf{T}(\mathbf{x})) = \mathbf{0} \tag{2.0.6}$$

$$\mathbf{T}(\mathbf{y}) = \mathbf{0} \tag{2.0.7}$$

$\therefore$ $\mathbf{y}$ lies in the Null space of $\mathbf{T}$. Hence $\mathbf{T}$ is singular. Thus, the range of $\mathbf{T}$ must be contained in Null space of $\mathbf{T}$ i.e., $\text{Range}(\mathbf{T}) \subseteq \text{NullSpace}(\mathbf{T})$

2.1 Example

$$\mathbf{T} : \mathbf{R}^2 \to \mathbf{R}^2 \tag{2.1.1}$$

Consider,

$$\mathbf{T}(\mathbf{x}) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mathbf{x} \tag{2.1.2}$$

$$\implies \mathbf{T} \neq 0 \tag{2.1.3}$$

Now,

$$\mathbf{T}^2 : \mathbf{R}^2 \to \mathbf{R}^2 \tag{2.1.4}$$

$$\mathbf{T}^2(\mathbf{x}) = \mathbf{T}(\mathbf{T}(\mathbf{x})) \tag{2.1.5}$$

$$= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mathbf{x} = \mathbf{0} \tag{2.1.6}$$

$$\implies \mathbf{T}^2(\mathbf{x}) = \mathbf{0} \tag{2.1.7}$$

Thus $\mathbf{T}^2$ is a zero transformation, $\mathbf{T}^2 = \mathbf{0}$.

Now, Kernel of $\mathbf{T}$ is given by,

$$\mathbf{T}(\mathbf{x}) = \mathbf{0} \tag{2.1.8}$$

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.1.9}$$

$$\implies x = 0 \tag{2.1.10}$$

Thus,

$$\mathbf{Ker}(\mathbf{T}) = y \begin{pmatrix} 0 \\ 1 \end{pmatrix}; y \in \mathbf{R} \tag{2.1.11}$$

Now,

$$\mathbf{Range}(\mathbf{T}) = \mathbf{ColumnSpace} \left\{ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\} \tag{2.1.12}$$

$$= k \begin{pmatrix} 0 \\ 1 \end{pmatrix}; k \in \mathbf{R} \tag{2.1.13}$$

Thus for the example, $\text{Range}(\mathbf{T}) = \text{Kernel}(\mathbf{T})$ and from (2.1.3), (2.1.7) it is clear that $\mathbf{T}^2 = 0$ but $\mathbf{T} \neq 0$.