



# Solutions: Linear Algebra by Hoffman and Kunze



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## CONTENTS

**Abstract—**This book provides solutions to the Linear Algebra book by Hoffman and Kunze.

## 1 LINEAR EQUATIONS

### 1.1 Fields and Linear Equations

1.1.1. Verify that the set of complex numbers numbers described in the form of  $c$  where  $x$  and  $y$  are rational is a sub-field of  $\mathbb{C}$ .

**Solution:** Lets consider the set  $S = \{x + y\sqrt{2}, x, y \in \mathbb{Q}\}$ ,  $S \subset \mathbb{C}$  We must verify that  $S$  meets the following two conditions:

$$0, 1 \in S \quad (1.1.1.1)$$

$$a, b \in S, a + b, -a, ab, a^{-1} \in S \quad (1.1.1.2)$$

Throughout let

$$a = x + y\sqrt{2}, b = w + z\sqrt{2} \quad (1.1.1.3)$$

If

a)

$$x = 0, y = 0 \in \mathbb{Q}, a = 0 + \sqrt{2}.0 = 0, 0 \in S \quad (1.1.1.4)$$

b)

$$x = 1, y = 0, a = 1 + \sqrt{2}.0 = 1, 1 \in S \quad (1.1.1.5)$$

c)

$$a + b = x + y\sqrt{2} + w + z\sqrt{2} = b + a \quad (1.1.1.6)$$

d)

$$-a = -x - y\sqrt{2}, x, y \in \mathbb{Q} \text{ so } -x, -y \in \mathbb{Q}, a \in S \quad (1.1.1.7)$$

e)

$$ab = (x + y\sqrt{2})(w + z\sqrt{2}) = ba, ab \in S \quad (1.1.1.8)$$

f)

$$a^{-1}a = (x + y\sqrt{2})^{-1}(x + y\sqrt{2}) = 1, a^{-1} \in S \quad (1.1.1.9)$$

Hence (1.1.1.1), (1.1.1.2) is verified. Therefore by considering the (1.1.1.1) and (1.1.1.2) we can say set complex numbers of given form  $x + y\sqrt{2}$  is subfield of  $\mathbb{C}$ .

1.1.2. Let  $\mathbb{F}$  be a set which contains exactly two elements, 0 and 1. Define an addition and multiplication by tables. Verify that the set  $\mathbb{F}$ , together with these two operations, is a field.

**Solution:**

To prove that  $(\mathbb{F}, +, \cdot)$  is a field we need to

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+	0	1
0	0	1
1	1	0

  

·	0	1
0	0	0
1	0	1

satisfy the following,

- a) + and · should be closed
  - For any a and b in  $\mathbb{F}$ ,  $a+b \in \mathbb{F}$  and  $a \cdot b \in \mathbb{F}$ . For example  $0+0=0$  and  $0 \cdot 0=0$ .
- b) + and · should be commutative
  - For any a and b in  $\mathbb{F}$ ,  $a+b = b+a$  and  $a \cdot b = b \cdot a$ . For example  $0+1=1+0$  and  $0 \cdot 1=1 \cdot 0$ .
- c) + and · should be associative
  - For any a and b in  $\mathbb{F}$ ,  $a+(b+c) = (a+b)+c$  and  $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ . For example  $0+(1+0)=(0+1)+0$  and  $0 \cdot (1 \cdot 0)=(0 \cdot 1) \cdot 0$ .
- d) + and · operations should have an identity element
  - If we perform  $a + 0$  then for any value of a from  $\mathbb{F}$  the result will be a itself. Hence 0 is an identity element of + operation. If we perform  $a \cdot 1$  then for any value of a from  $\mathbb{F}$  the result will be a itself. Hence 1 is an identity element of · operation.
- e)  $\forall a \in \mathbb{F}$  there exists an additive inverse
  - For additive inverse to exist,  $\forall a$  in  $\mathbb{F}$   $a+(-a)=0$ . For example.  $1-1=0$  and  $0-0=0$ .
- f)  $\forall a \in \mathbb{F}$  such that a is non zero there exists a multiplicative inverse
  - For multiplicative inverse to exist,  $\forall a$  such that a is non zero in  $\mathbb{F}$ ,  $a \cdot a^{-1}=1$ . For example  $1 \cdot 1^{-1} = 1$ .
- g) + and · should hold distributive property
  - For any a,b and c in  $\mathbb{F}$  the property  $a \cdot (b+c)=a \cdot b+a \cdot c$  should always hold true. For example  $0 \cdot (1+1)=0 \cdot 1+0 \cdot 1$ .

Since the above properties are satisfied we can say that  $(\mathbb{F}, +, \cdot)$  is a field.

1.1.3. Prove that each subfield of the field of complex number contains every rational number

**Solution:**

Complex Numbers: A complex number is a number that can be expressed in the form

$a + bi$ , where a and b are real numbers, and i represents the imaginary unit, satisfying the equation  $i^2 = -1$ . The set of complex numbers is denoted by  $\mathbb{C}$

$$\mathbb{C} = \{(a, b) : a, b \in \mathbb{R}\} \quad (1.1.3.1)$$

Rational Numbers: A number in the form  $\frac{p}{q}$ , where both p and q (non-zero) are integers, is called a rational number. The set of rational numbers is denoted by  $\mathbb{Q}$ . Let  $\mathbb{Q}$  be the set of rational numbers.

$$\mathbb{Q} = \left\{ \frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{Z}_{\neq 0} \right\} \quad (1.1.3.2)$$

Let  $\mathbb{C}$  be the field of complex numbers and given  $\mathbb{F}$  be the subfield of field of complex numbers  $\mathbb{C}$ . Since  $\mathbb{F}$  is the subfield, we could say that

$$0 \in \mathbb{F} \quad (1.1.3.3)$$

$$1 \in \mathbb{F} \quad (1.1.3.4)$$

Closed under addition: Here  $\mathbb{F}$  is closed under addition since it is subfield

$$1 + 1 = 2 \in \mathbb{F} \quad (1.1.3.5)$$

$$1 + 1 + 1 = 3 \in \mathbb{F} \quad (1.1.3.6)$$

⋮

$$1 + 1 + \dots + 1 (p \text{ times}) = p \in \mathbb{F} \quad (1.1.3.7)$$

$$1 + 1 + \dots + 1 (q \text{ times}) = q \in \mathbb{F} \quad (1.1.3.8)$$

By using the above property we could say that zero and other positive integers belong to  $\mathbb{F}$ . Since p and q are integers we say,

$$p \in \mathbb{Z} \quad (1.1.3.9)$$

$$q \in \mathbb{Z} \quad (1.1.3.10)$$

Additive Inverse: Let x be the positive integer belong  $\mathbb{F}$  and by additive inverse we could say,

$$\forall x \in \mathbb{F} \quad (1.1.3.11)$$

$$(-x) \in \mathbb{F} \quad (1.1.3.12)$$

Therefore field  $\mathbb{F}$  contains every integers. Let n be a integer then,

$$n \in \mathbb{Z} \implies n \in \mathbb{F} \quad (1.1.3.13)$$

$$\mathbb{Z} \subseteq \mathbb{F} \quad (1.1.3.14)$$

Where  $\mathbb{Z}$  is subset of  $\mathbb{F}$  Multiplicative Inverse:

Every element except zero in the subfield  $\mathbb{F}$  has an multiplicative inverse. From equation (1.1.3.8), since  $q \in \mathbb{F}$  we could say ,

$$\frac{1}{q} \in \mathbb{F} \quad \text{and } q \neq 0 \quad (1.1.3.15)$$

Closed under multiplication: Also,  $\mathbb{F}$  is closed under multiplication and thus, from equation (1.1.3.7) and (1.1.3.15) we get ,

$$p \cdot \frac{1}{q} \in \mathbb{F} \quad (1.1.3.16)$$

$$\Rightarrow \frac{p}{q} \in \mathbb{F} \quad (1.1.3.17)$$

where ,  $p \in \mathbb{Z}$  and  $q \in \mathbb{Z}_{\neq 0}$  (from equation (1.1.3.3) and (1.1.3.15)) Conclusion From (1.1.3.2) and (1.1.3.17) we could say ,

$$\mathbb{Q} \subseteq \mathbb{F} \quad (1.1.3.18)$$

From equation (1.1.3.18) we could say that each subfield of the field of complex number contains every rational number

Hence Proved

1.1.4. Prove that, each field of the characteristic zero contains a copy of the rational number field.

**Solution:** The characteristic of a field is defined to be the smallest number of times one must use the field's multiplicative identity (1) in a sum to get the additive identity. If this sum never reaches the additive identity (0), then the field is said to have characteristic zero.

Let  $\mathbb{Q}$  be the rational number field. Hence,

$$0 \in \mathbb{Q} \quad [\text{Additive Identity}] \quad (1.1.4.1)$$

$$1 \in \mathbb{Q} \quad [\text{Multiplicative Identity}] \quad (1.1.4.2)$$

As addition is defined on  $\mathbb{Q}$  hence we have,

$$1 \neq 0 \quad (1.1.4.3)$$

$$1 + 1 = 2 \neq 0 \quad (1.1.4.4)$$

And so on,

$$1 + 1 + \cdots + 1 = n \neq 0 \quad (1.1.4.5)$$

From the definition of characteristic of a field and from (1.1.4.3), (1.1.4.4) and so on up to (1.1.4.5), the rational number field,  $\mathbb{Q}$  has characteristic 0.

## 1.2 Matrices and Elementary Row Operations

1.2.1. Find a row-reduced matrix which is row equivalent to,

$$\mathbf{A} = \begin{pmatrix} i & -(1+i) & 0 \\ 1 & -2 & 1 \\ 1 & 2i & -1 \end{pmatrix} \quad (1.2.1.1)$$

**Solution: Step 1:** Performing scaling operation to matrix  $\mathbf{A}$  as  $R_1 \leftarrow \frac{1}{i}R_1$  by scaling matrix  $\mathbf{D}_1$  given as,

$$\mathbf{D}_1 = \begin{pmatrix} \frac{1}{i} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (1.2.1.2)$$

$$\mathbf{D}_1\mathbf{A} = \begin{pmatrix} \frac{1}{i} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} i & -(1+i) & 0 \\ 1 & -2 & 1 \\ 1 & 2i & -1 \end{pmatrix} \quad (1.2.1.3)$$

$$\Rightarrow \mathbf{D}_1\mathbf{A} = \begin{pmatrix} 1 & -1+i & 0 \\ 1 & -2 & 1 \\ 1 & 2i & -1 \end{pmatrix} \quad (1.2.1.4)$$

**Step 2:** Performing  $R_2 \leftarrow R_2 - R_1$  and  $R_3 \leftarrow R_3 - R_1$  given by elementary matrix  $\mathbf{E}_{31}\mathbf{E}_{21}$  on equation (1.2.1.4),

$$\mathbf{E}_{31}\mathbf{E}_{21} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \quad (1.2.1.5)$$

$$\mathbf{E}_{31}\mathbf{E}_{21}\mathbf{D}_1\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1+i & 0 \\ 1 & -2 & 1 \\ 1 & 2i & -1 \end{pmatrix} \quad (1.2.1.6)$$

$$\Rightarrow \mathbf{A}_1 = \mathbf{E}_{31}\mathbf{E}_{21}\mathbf{D}_1\mathbf{A} = \begin{pmatrix} 1 & -1+i & 0 \\ 0 & -1-i & 1 \\ 0 & 1+i & -1 \end{pmatrix} \quad (1.2.1.7)$$

**Step 3:** Performing  $R_2 \leftarrow \frac{-1}{1+i}R_2$  given by  $\mathbf{D}_2$  on equation (1.2.1.7),

$$\mathbf{D}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2}(-1+i) & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (1.2.1.8)$$

$$\mathbf{D}_2\mathbf{A}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2}(-1+i) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1+i & 0 \\ 0 & -1-i & 1 \\ 0 & 1+i & -1 \end{pmatrix} \quad (1.2.1.9)$$

$$\Rightarrow \mathbf{A}_2 = \mathbf{D}_2\mathbf{A}_1 = \begin{pmatrix} 1 & -1+i & 0 \\ 0 & 1 & \frac{1}{2}(-1+i) \\ 0 & 1+i & -1 \end{pmatrix} \quad (1.2.1.10)$$

**Step 4:** Performing  $R_3 \leftarrow R_3 - (1+i)R_2$  given by  $\mathbf{E}_{32}$  on equation (1.2.1.10),

$$\mathbf{E}_{32} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -(1+i) & 1 \end{pmatrix} \quad (1.2.1.11)$$

$$\mathbf{E}_{32}\mathbf{A}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1-i & 1 \end{pmatrix} \begin{pmatrix} 1 & -1+i & 0 \\ 0 & 1 & \frac{-1+i}{2} \\ 0 & 1+i & -1 \end{pmatrix} \quad (1.2.1.12)$$

$$\Rightarrow \mathbf{A}_3 = \mathbf{E}_{32}\mathbf{A}_2 = \begin{pmatrix} 1 & -1+i & 0 \\ 0 & 1 & \frac{-1+i}{2} \\ 0 & 0 & 1 \end{pmatrix} \quad (1.2.1.13)$$

**Step 5:** Performing  $R_1 \leftarrow R_1 - (-1+i)R_2$  given by  $\mathbf{E}_{12}$  on equation (1.2.1.13),

$$\mathbf{E}_{12} = \begin{pmatrix} 1 & 1-i & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (1.2.1.14)$$

$$\mathbf{E}_{12}\mathbf{A}_3 = \begin{pmatrix} 1 & 1-i & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1+i & 0 \\ 0 & 1 & \frac{-1+i}{2} \\ 0 & 0 & 1 \end{pmatrix} \quad (1.2.1.15)$$

$$\Rightarrow \mathbf{A}_4 = \mathbf{E}_{12}\mathbf{A}_3 = \begin{pmatrix} 1 & 0 & i \\ 0 & 1 & \frac{-1+i}{2} \\ 0 & 0 & 1 \end{pmatrix} \quad (1.2.1.16)$$

**Step 6:** Performing  $R_1 \leftarrow R_1 - iR_3$  and  $R_2 \leftarrow R_2 - \frac{-1+i}{2}R_3$  given by  $\mathbf{E}_{13}\mathbf{E}_{23}$  on equation

(1.2.1.16),

$$\mathbf{E}_{13}\mathbf{E}_{23} = \begin{pmatrix} 1 & 0 & -i \\ 0 & 1 & -\left(\frac{-1+i}{2}\right) \\ 0 & 0 & 1 \end{pmatrix} \quad (1.2.1.17)$$

$$\mathbf{E}_{13}\mathbf{E}_{23}\mathbf{A}_4 = \begin{pmatrix} 1 & 0 & -i \\ 0 & 1 & -\left(\frac{-1+i}{2}\right) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & i \\ 0 & 1 & \frac{-1+i}{2} \\ 0 & 0 & 1 \end{pmatrix} \quad (1.2.1.18)$$

$$\Rightarrow \mathbf{A}_5 = \mathbf{E}_{13}\mathbf{E}_{23}\mathbf{A}_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (1.2.1.19)$$

$\therefore$  Row-reduced matrix of  $\mathbf{A}$  given by equation (1.2.1.1) is,

$$\mathbf{A} = \begin{pmatrix} i & -1-i & 0 \\ 1 & -2 & 1 \\ 1 & 2i & -1 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I} \quad (1.2.1.20)$$

1.2.2. Let

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (1.2.2.1)$$

be a  $2 \times 2$  matrix with complex entries. Suppose  $\mathbf{A}$  is row-reduced and also that  $a+b+c+d=0$ . Prove that there are exactly three such matrices.

**Solution:** A matrix is in row echelon form if it follows the following conditions

1. All nonzero rows are above any rows of all zeros.
  2. Each leading entry (i.e. left most nonzero entry) of a row is in a column to the right of the leading entry of the row above it.
  3. All entries in a column below a leading entry are zero
- Row Reduced Echelon Form A matrix is in row reduced echelon form if it follows the following conditions

1. The matrix should be row echelon form
2. The leading entry in each nonzero row is 1.
3. Each leading 1 is the only nonzero entry in its column.

Proof

Given ,

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (1.2.2.2)$$

**Condition 1 :** Matrix  $\mathbf{A}$  should be in row-reduced echelon form

**Condition 2 :**  $a + b + c + d = 0$  where  $a, b, c$  and  $d$  are the elements of the matrix  $\mathbf{A}$   
Reducing the matrix  $\mathbf{A}$  from equation (1.2.2.2)

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \xrightarrow{R_1 = \frac{1}{a}R_1} \begin{pmatrix} 1 & \frac{b}{a} \\ c & d \end{pmatrix} \quad (1.2.2.3)$$

$$\xrightarrow{R_2 = R_2 - cR_1} \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & \frac{ad-bc}{a} \end{pmatrix} \quad (1.2.2.4)$$

$$\xrightarrow{R_2 = \frac{a}{ad-bc}R_2} \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix} \quad (1.2.2.5)$$

$$\xrightarrow{R_1 = R_1 - \frac{b}{a}R_2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (1.2.2.6)$$

**Case 1:** Matrix  $\mathbf{A}$  of Rank 2

From the equation (1.2.2.4), for the matrix to be in row reduced echelon form,

$$b = 0$$

$$a \neq 0$$

$$d = 1$$

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (1.2.2.7)$$

For the condition 2 to get satisfied,

$$a + 0 + c + 1 = 0 \quad (1.2.2.8)$$

$$\Rightarrow a = -(c + 1) \quad (1.2.2.9)$$

$$\Rightarrow c \neq -1 \quad (1.2.2.10)$$

Both the condition gets satisfied and so exactly one matrix  $\mathbf{A}$  can be formed of Rank 2 with given conditions

**Case 2:** Matrix  $\mathbf{A}$  of Rank 1

From the equation (1.2.2.4), for the matrix to be in row reduced echelon form,

$$a \neq 0$$

$$d = 0$$

$$c = 0$$

For the condition 2 to get satisfied,

$$a + b + 0 + 0 = 0 \quad (1.2.2.11)$$

$$\Rightarrow b = -a \quad (1.2.2.12)$$

$$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \quad (1.2.2.13)$$

Both the condition gets satisfied and so exactly one matrix  $\mathbf{A}$  can be formed of Rank 1 with given conditions

**Case 3:** Matrix  $\mathbf{A}$  of Rank 0

From equation (1.2.2.2), for the matrix to be in row reduced echelon form,

$$a = 0$$

$$b = 0$$

$$c = 0$$

$$d = 0$$

$$\mathbf{A} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad (1.2.2.14)$$

Both the condition gets satisfied and so exactly one matrix  $\mathbf{A}$  can be formed of Rank 0 with given conditions

Therefore matrix  $\mathbf{A}$  shown in equation (1.2.2.7), (1.2.2.13) and (1.2.2.14) are the exactly three such matrices that can be formed with given conditions.

1.2.3. Prove that the interchange of two rows of a matrix can be accomplished by a finite sequence of elementary row operations of the other two types.

**Solution:** Let  $\mathbf{A}$  be a  $3 \times 3$  matrix with having row vectors  $\mathbf{a}_1, \mathbf{a}_2$  and  $\mathbf{a}_3$ .

$$\mathbf{A} = \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} \quad (1.2.3.1)$$

Let's exchange row  $\mathbf{a}_1$  and  $\mathbf{a}_2$ . Let's call this elementary operation  $\mathbf{E}_1$ .

$$\mathbf{E}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (1.2.3.2)$$

$$(1.2.3.3)$$

Now performing operation  $\mathbf{E}_1$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{a}_2 \\ \mathbf{a}_1 \\ \mathbf{a}_3 \end{pmatrix} \quad (1.2.3.4)$$

Now, to prove that same matrix can be obtained by elementary operations let's call them  $\mathbf{E}_2$  and  $\mathbf{E}_3$ . Now performing operation  $\mathbf{E}_2$  by adding row 2 to row 1.

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{a}_1 + \mathbf{a}_2 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} \quad (1.2.3.5)$$

Using elementary operation  $\mathbf{E}_2$  we will subtract

row 1 from row 2.

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}_1 + \mathbf{a}_2 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{a}_1 + \mathbf{a}_2 \\ -\mathbf{a}_1 \\ \mathbf{a}_3 \end{pmatrix} \quad (1.2.3.6)$$

Using elementary operation  $\mathbf{E}_2$  we will add row 2 to row 1.

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}_1 + \mathbf{a}_2 \\ -\mathbf{a}_1 \\ \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{a}_2 \\ -\mathbf{a}_1 \\ \mathbf{a}_3 \end{pmatrix} \quad (1.2.3.7)$$

Using elementary operation  $\mathbf{E}_3$  we will multiply row 2 by -1.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}_2 \\ -\mathbf{a}_1 \\ \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{a}_2 \\ \mathbf{a}_1 \\ \mathbf{a}_3 \end{pmatrix} \quad (1.2.3.8)$$

Hence, we can say that,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} \quad (1.2.3.9)$$

Let us assume a matrix  $\mathbf{A}$

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \quad (1.2.3.10)$$

Let's exchange row  $\mathbf{a}_1$  and  $\mathbf{a}_2$  by applying operation  $\mathbf{E}_1$ .

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & 3 \\ 1 & 1 & 0 \end{pmatrix} \quad (1.2.3.11)$$

Now, to prove that same matrix can be obtained by other two elementary operations. We will first perform elementary operation  $\mathbf{E}_2$  by adding row 2 to row 1.

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \quad (1.2.3.12)$$

Using elementary operation  $\mathbf{E}_2$  we will subtract

row 1 from row 2.

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 3 \\ -1 & -2 & -3 \\ 1 & 1 & 0 \end{pmatrix} \quad (1.2.3.13)$$

Using elementary operation  $\mathbf{E}_2$  we will add row 2 to row 1.

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 3 \\ -1 & -2 & -3 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & -2 & -3 \\ 1 & 1 & 0 \end{pmatrix} \quad (1.2.3.14)$$

Using elementary operation  $\mathbf{E}_3$  we will multiply row 2 by -1.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & -2 & -3 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & 3 \\ 1 & 1 & 0 \end{pmatrix} \quad (1.2.3.15)$$

Hence, we can say that,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} \quad (1.2.3.16)$$

1.2.4. Consider the system of equations  $\mathbf{A}\mathbf{X} = 0$  where

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a  $2 \times 2$  matrix over the field  $F$ . Prove the following -

- If every entry of  $\mathbf{A}$  is 0, then every pair  $x_1$  and  $x_2$  is a solution of  $\mathbf{A}\mathbf{X} = 0$ .
- If  $ad - bc \neq 0$ , then the system  $\mathbf{A}\mathbf{X} = 0$  has only the trivial solution  $x_1 = x_2 = 0$ .
- If  $ad - bc = 0$  and some entry of  $\mathbf{A}$  is different from 0, then there is a solution  $x_1^0$  and  $x_2^0$  such that  $x_1$  and  $x_2$  is a solution if and only if there is a scalar  $y$  such that  $x_1 = yx_1^0$  and  $x_2 = yx_2^0$ .

**Solution:** Solution 1 If every entry of  $\mathbf{A}$  is 0

then the equation  $\mathbf{AX} = 0$  becomes,

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \quad (1.2.4.1)$$

$$\implies 0.x_1 + 0.x_2 = 0 \quad \forall x_1, x_2 \in F \quad (1.2.4.2)$$

Hence proved, every pair  $x_1$  and  $x_2$  is a solution for the equation  $\mathbf{AX} = 0$ . **Solution 2 Case 1:** Let  $a = 0$ . Since  $ad - bc \neq 0$ . As  $bc \neq 0$  therefore  $b \neq 0$  and  $c \neq 0$ . Hence, we can perform row reduction on the augmented matrix of equation  $\mathbf{AX}=0$  as follows,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & b & 0 \\ c & d & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{c} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c & d & 0 \\ 0 & b & 0 \end{pmatrix} \quad (1.2.4.3)$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{b} \end{pmatrix} \begin{pmatrix} 1 & \frac{d}{b} & 0 \\ 0 & b & 0 \end{pmatrix} \quad (1.2.4.4)$$

$$= \begin{pmatrix} 1 & -\frac{d}{c} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{d}{c} & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (1.2.4.5)$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (1.2.4.6)$$

**Case 2:** Let  $a, b, c, d \neq 0$ . Considering the following case,

$$\mathbf{AX} = \mathbf{u} \quad (1.2.4.7)$$

$$\implies \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad (1.2.4.8)$$

Row Reducing the augmented matrix of (1.2.4.8) we get,

$$\begin{pmatrix} \frac{1}{a} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b & u_1 \\ c & d & u_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -c & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{b}{a} & \frac{u_1}{a} \\ c & d & u_2 \end{pmatrix} \quad (1.2.4.9)$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & \frac{a}{ad-bc} \end{pmatrix} \begin{pmatrix} 1 & \frac{b}{a} & \frac{u_1}{a} \\ 0 & \frac{ad-bc}{a} & \frac{au_2-cu_1}{a} \end{pmatrix} \quad (1.2.4.10)$$

$$= \begin{pmatrix} 1 & -\frac{b}{a} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{b}{a} & \frac{u_1}{a} \\ 0 & 1 & \frac{au_2-cu_1}{ad-bc} \end{pmatrix} \quad (1.2.4.11)$$

$$= \begin{pmatrix} 1 & 0 & \frac{du_1-bu_2}{ad-bc} \\ 0 & 1 & \frac{au_2-cu_1}{ad-bc} \end{pmatrix} \quad (1.2.4.12)$$

From (1.2.4.12) we get,

$$x_1 = \frac{du_1 - bu_2}{ad - bc} \quad (1.2.4.13)$$

$$x_2 = \frac{au_2 - cu_1}{ad - bc} \quad (1.2.4.14)$$

Since  $u_1 = 0$  and  $u_2 = 0$  then from (1.2.4.13) and (1.2.4.14),

$$x_1 = 0 \quad (1.2.4.15)$$

$$x_2 = 0 \quad (1.2.4.16)$$

Hence we get,

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (1.2.4.17)$$

In (1.2.4.6) and (1.2.4.17), we can see that  $\mathbf{AX} = 0$  has only one trivial solution i.e  $x_1 = x_2 = 0$  in all cases. Hence proved, the equation  $\mathbf{AX}=0$  has only one trivial solution  $x_1 = x_2 = 0$ . **Solution 3 Case 1:** Let,  $a \neq 0$  for  $\mathbf{A}$ . Given  $ad - bc = 0$ , we can perform row reduction on augmented matrix of equation  $\mathbf{AX} = 0$  as follows,

$$\begin{pmatrix} \frac{1}{a} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b & 0 \\ c & d & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -c & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{b}{a} & 0 \\ c & d & 0 \end{pmatrix} \quad (1.2.4.18)$$

$$= \begin{pmatrix} 1 & \frac{b}{a} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad [\because ad - bc = 0] \quad (1.2.4.19)$$

Hence from (1.2.4.19),  $\mathbf{AX} = 0$  if and only if

$$x_1 = -\frac{b}{a}x_2 \quad [a \neq 0] \quad (1.2.4.20)$$

Letting  $x_1^0 = -\frac{b}{a}$  and  $x_2^0 = 1$  we get for  $y = 1$ ,

$$x_1 = yx_1^0 \quad (1.2.4.21)$$

$$x_2 = yx_2^0 \quad (1.2.4.22)$$

which is a solution of the equation  $\mathbf{AX} = 0$ .

**Case 2:** Let,  $b \neq 0$  for  $\mathbf{A}$ . Given  $ad - bc = 0$ , at first we multiply by elementary matrix to change the columns and then we can perform row reduction on augmented matrix of equation  $\mathbf{AX} = 0$  as follows,

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} b & a & 0 \\ d & c & 0 \end{pmatrix} \quad (1.2.4.23)$$

Hence using the result obtained from (1.2.4.19)

we can conclude for (1.2.4.23),  $\mathbf{AX} = 0$  if and only if

$$x_2 = -\frac{a}{b}x_1 \quad [b \neq 0] \quad (1.2.4.24)$$

Letting  $x_2^0 = -\frac{a}{b}$  and  $x_1^0 = 1$  we get for  $y = 1$ ,

$$x_1 = yx_1^0 \quad (1.2.4.25)$$

$$x_2 = yx_2^0 \quad (1.2.4.26)$$

which is a solution of the equation  $\mathbf{AX} = 0$ .

**Case 3:** Let,  $c \neq 0$  for  $\mathbf{A}$ . Given  $ad - bc = 0$ , we can perform row reduction on augmented matrix of equation  $\mathbf{AX} = 0$  as follows,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b & 0 \\ c & d & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{c} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c & d & 0 \\ a & b & 0 \end{pmatrix} \quad (1.2.4.27)$$

$$= \begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{d}{c} & 0 \\ a & b & 0 \end{pmatrix} \quad (1.2.4.28)$$

$$= \begin{pmatrix} 1 & \frac{d}{c} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad [\because ad - bc = 0] \quad (1.2.4.29)$$

Hence from (1.2.4.29),  $\mathbf{AX} = 0$  if and only if

$$x_1 = -\frac{d}{c}x_2 \quad [a \neq 0] \quad (1.2.4.30)$$

Letting  $x_1^0 = -\frac{d}{c}$  and  $x_2^0 = 1$  we get for  $y = 1$ ,

$$x_1 = yx_1^0 \quad (1.2.4.31)$$

$$x_2 = yx_2^0 \quad (1.2.4.32)$$

which is a solution of the equation  $\mathbf{AX} = 0$ .

**Case 4:** Let,  $d \neq 0$  for  $\mathbf{A}$ . Given  $ad - bc = 0$ , at first we multiply by elementary matrix to change the columns and then we can perform row reduction on augmented matrix of equation  $\mathbf{AX} = 0$  as follows,

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b & a & 0 \\ d & c & 0 \end{pmatrix} \quad (1.2.4.33)$$

$$= \begin{pmatrix} d & c & 0 \\ b & a & 0 \end{pmatrix} \quad (1.2.4.34)$$

Hence using the result from (1.2.4.29) we can conclude for (1.2.4.34),  $\mathbf{AX} = 0$  if and only if

$$x_2 = -\frac{c}{d}x_1 \quad [a \neq 0] \quad (1.2.4.35)$$

Letting  $x_2^0 = -\frac{c}{d}$  and  $x_1^0 = 1$  we get for  $y = 1$ ,

$$x_1 = yx_1^0 \quad (1.2.4.36)$$

$$x_2 = yx_2^0 \quad (1.2.4.37)$$

which is a solution of the equation  $\mathbf{AX} = 0$ .