Linear Algebra

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Contents

***Abstract*—This book provides solved examples on Linear Algebra.**

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1.1. Consider the vector space $\mathbb{P}_n$ of real polynomials in x of degree $\leq n$. Define

$$T : \mathbb{P}_2 \to \mathbb{P}_3 \tag{1.1.1}$$

by

$$(T f)(x) = \int_0^x f(t)\, dt + f'(x). \tag{1.1.2}$$

Then find the matrix representation of T with respect to the bases

$$\left\{1, x, x^2\right\} \text{ and } \left\{1, x, x^2, x^3\right\} \tag{1.1.3}$$

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1.2. Let $P_A(x)$ denote the characteristic polynomial of a matrix A. Then for which of the following matrices is

$$P_A(x) - P_{A^{-1}}(x) \tag{1.2.1}$$

a constant?

a) $\begin{pmatrix} 3 & 3 \\ 2 & 4 \end{pmatrix}$

b) $\begin{pmatrix} 4 & 3 \\ 2 & 3 \end{pmatrix}$

c) $\begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}$

d) $\begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$

1.3. Which of the following matrices is not diagonalizable over $\mathbb{R}$?

a) $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

c) $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

d) $\begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$

1.4. What is the rank of the following matrix?

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} \tag{1.4.1}$$

1.5. Let V denote the vector space of real valued continuous functions on the close interval $[0, 1]$. Let W be the subspace of V spanned by $\{\sin x, \cos x, \tan x\}$. Find the dimension of W over $\mathbb{R}$.

1.6. Let V be the vector space of polynomials in the variable t of degree at most 2 over $\mathbb{R}$. An inner product on V is defined by

$$f^T g = \int_0^1 f(t)g(t)\,dt, \quad f, g \in V. \tag{1.6.1}$$

Let

$$W = span\left\{1 - t^2, 1 + t^2\right\} \tag{1.6.2}$$

and $W^\perp$ be the orthogonal complement of W in V. Which of the following conditions is satisfied for all $h \in W^\perp$?

a) h is an even function
b) h is an odd function
c) $h(t) = 0$ has a real solution
d) $h(0) = 0$

1.7. Consider solving the following system by Jacobi iteration scheme

$$\begin{pmatrix} 1 & 2m & -2m \\ n & 1 & n \\ 2m & 2m & 1 \end{pmatrix}\begin{pmatrix} x \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \tag{1.7.1}$$

where $m, n \in \mathbb{Z}$. With any initial vector, the scheme converges provided m, n satisfy

a) $m + n = 3$
b) $m > n$
c) $m < n$
d) $m = n$

1.8. Consider a Markov Chain with state space $\{0, 1, 2, 3, 4\}$ and transition matrix

$$P = \begin{array}{c c} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \tag{1.8.1}$$

Then find

$$\lim_{n\to\infty} p_{23}^{(n)} \tag{1.8.2}$$

1.9. Let $L(\mathbb{R})^n$ be the space of $\mathbb{R}-$linear maps from $\mathbb{R}^n$ to $\mathbb{R}^n$. If $Ker(T)$ denotes the kernel of T then which of the following are true?

a) There exists $T \in L\left(\mathbb{R}^5\right) \{0\}$ such that $Range(T) = Ker(T)$
b) There does not exist $T \in L\left(\mathbb{R}^5\right) \{0\}$ such that $Range(T) = Ker(T)$
c) There exists $T \in L\left(\mathbb{R}^6\right) \{0\}$ such that $Range(T) = Ker(T)$
d) There does not exist $T \in L\left(\mathbb{R}^6\right) \{0\}$ such that $Range(T) = Ker(T)$

1.10. Let V be a finite dimensional vector space over $\mathbb{R}$ and $T : V \to V$ be a linear map. Can you always write $T = T_2 \circ T_1$ for some linear maps

$$T_1 : V \to W, T : W \to V, \tag{1.10.1}$$

where W is some finite dimensional vector space such that

a) both T_1 and T_2 are onto
b) both T_1 and T_2 are one to one
c) T_1 is onto, T_2 is one to one
d) T_1 is one to one, T_2 is onto

1.11. Let $A = \left[a_{ij}\right]$ be a 3×3 complex matrix. Identify the correct statements

a) $det\left[(-1)^{i+j} a_{ij}\right] = det(A)$
b) $det\left[(-1)^{i+j} a_{ij}\right] = -det(A)$
c) $det\left[\left(\sqrt{-1}\right)^{i+j} a_{ij}\right] = det(A)$
d) $det\left[\left(\sqrt{-1}\right)^{i+j} a_{ij}\right] = -det(A)$

1.12. Let

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n \tag{1.12.1}$$

be a non-constant polynomial of degree $n \geq 1$. Consider the polynomial

$$q(x) = \int_0^x p(t)\,dt, r(x) = \frac{d}{dx}p(x) \tag{1.12.2}$$

Let V denote the real vector space of all polynomials in x. Then which of the following are true?

a) q and r are linearly independent in V
b) q and r are linearly dependent in V
c) x^n belongs to the linear span of q and r
d) x^{n+1} belongs to the linear span of q and r.

1.13. Let $M_n(\mathbb{R})$ be the ring of $n \times n$ matrices over $\mathbb{R}$. Which of the following are true for every $n \geq 2$?

a) there exist matrices $A, B \in M_n(\mathbb{R})$ such that $AB - BA = I_n$, where I_n denotes the identity matrix.
b) If $A, B \in M_n(\mathbb{R})$ and $AB = BA$, then A is diagonalisable over $\mathbb{R}$ if and only if B is diagonalisable over $\mathbb{R}$.
c) If $A, B \in M_n(\mathbb{R})$, then AB and BA have the

same minimal polynomial.

d) If $A, B \in M_n(\mathbb{R})$, then AB and BA have the same eigenvalues in $\mathbb{R}$.

1.14. Consider a matrix

$$A = \left[a_{ij}\right], 1 \le i, j \le 5 \tag{1.14.1}$$

such that

$$a_{ij} = \frac{1}{n_i + n_j + 1}, \quad n_i, n_j \in \mathbb{N} \tag{1.14.2}$$

Then in which of the following cases A is a positive definite matrix?

a) $n_i = 1 \forall i = 1, 2, 3, 4, 5$.
b) $n_1 < n_2 < \cdots < n_5$.
c) $n_1 = n_2 = \cdots = n_5$.
d) $n_1 > n_2 > \cdots > n_5$.

1.15. For a nonzero $w \in \mathbb{R}^n$, define

$$T_w : \mathbb{R}^n \to \mathbb{R}^n \tag{1.15.1}$$

by

$$T_w = v - \frac{2v^T w}{w^T w} w, \quad v \in \mathbb{R}^n \tag{1.15.2}$$

Which of the following are true?

a) $det(T_w) = 1$
b) $T_w(v_1)_w^T(v_2) = v_1^T v_2 \forall v_1, v_2 \in \mathbb{R}^n$
c) $T_w = T_w^{-1}$
d) $T_{2w} = 2T_w$

1.16. Consider the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{1.16.1}$$

over the field $\mathbb{Q}$ of rationals. Which of the following matrices are of the form $P^T AP$ for suitable 2×2 invertible matrix P over $\mathbb{Q}$?

a) $\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$ c) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

b) $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ d) $\begin{pmatrix} 3 & 4 \\ 4 & 5 \end{pmatrix}$

1.17. Consider a Markov Chain with state space $\{0, 1, 2\}$ and transition matrix

$$P = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{pmatrix} \frac{1}{4} & \frac{5}{8} & \frac{1}{8} \\ \frac{1}{4} & 0 & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{8} & \frac{1}{8} \end{pmatrix} \end{array} \tag{1.17.1}$$

Then which of the following are true?

a) $\lim_{n\to\infty} p_{12}^{(n)} = 0$
b) $\lim_{n\to\infty} p_{12}^{(n)} = \lim_{n\to\infty} p_{21}^{(n)}$
c) $\lim_{n\to\infty} p_{22}^{(n)} = \frac{1}{8}$
d) $\lim_{n\to\infty} p_{21}^{(n)} = \frac{1}{3}$

2 December 2018

2.1. Consider the subspaces W_1 and W_2 of $\mathbb{R}^3$ given by

$$W_1 = \left\{\mathbf{x} \in \mathbb{R}^3 : \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \mathbf{x} = 0\right\} \tag{2.1.1}$$

$$W_2 = \left\{\mathbf{x} \in \mathbb{R}^3 : \begin{pmatrix} 1 & -1 & 1 \end{pmatrix} \mathbf{x} = 0\right\}. \tag{2.1.2}$$

If $W \subseteq \mathbb{R}^3$, such that

a) $W \cap W_2 = \text{span}\left\{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\}$

b) $\{W \cap W_1\} \perp \{W \cap W_2\}$,

then

a) $W = \text{span}\left\{\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\}$

b) $W = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}\right\}$

c) $W = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\}$

d) $W = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right\}$

2.2. Let

$$C = \left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right\} \tag{2.2.1}$$

be a basis of $\mathbb{R}^2$ and

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y \\ x - 2y \end{pmatrix}. \tag{2.2.2}$$

If $T[C]$ represents the matrix of T with respect to the basis C then which among the following is true?

a) $T[C] = \begin{pmatrix} -3 & -2 \\ 3 & 1 \end{pmatrix}$

b) $T[C] = \begin{pmatrix} 3 & -2 \\ -3 & 1 \end{pmatrix}$

c) $T[C] = \begin{pmatrix} -3 & -1 \\ 3 & 2 \end{pmatrix}$

d) $T[C] = \begin{pmatrix} 3 & -1 \\ -3 & 2 \end{pmatrix}$

2.3. Let $W_1 = \left\{\mathbf{x} \in \mathbb{R}^4 :\right\}$

$$\begin{pmatrix}1 & 1 & 1 & 0\end{pmatrix}\mathbf{x} = 0 \quad (2.3.1)$$
$$\begin{pmatrix}0 & 2 & 0 & 1\end{pmatrix}\mathbf{x} = 0 \quad (2.3.2)$$
$$\begin{pmatrix}2 & 0 & 2 & -1\end{pmatrix}\mathbf{x} = 0 \quad (2.3.3)$$

and $W_2 = \left\{\mathbf{x} \in \mathbb{R}^4 :\right\}$

$$\begin{pmatrix}1 & 1 & 0 & 1\end{pmatrix}\mathbf{x} = 0 \quad (2.3.4)$$
$$\begin{pmatrix}1 & 0 & 1 & -2\end{pmatrix}\mathbf{x} = 0 \quad (2.3.5)$$
$$\begin{pmatrix}0 & 1 & 0 & -1\end{pmatrix}\mathbf{x} = 0. \quad (2.3.6)$$

Then which among the following is true?

a) $\dim(W_1) = 1$
b) $\dim(W_2) = 2$
c) $\dim(W_1 \cap W_2) = 1$
d) $\dim(W_1 + W_2) = 3$

2.4. Let A be an $n \times n$ complex matrix. Assume that A is self-adjoint and let B denote the inverse of $A + \jmath I$. Then all eigenvalues of $(A - \jmath I)B$ are

a) purely imaginary
b) of modulus one
c) real
d) of modulus less than one

2.5. Let $\{u_1, u_2, \ldots, u_n\}$ be an orthonormal basis of $\mathbb{C}^n$ as column vectors.Let

$$\mathbf{M} = \begin{pmatrix}\mathbf{u}_1 & \mathbf{u}_2 & \ldots & \mathbf{u}_k\end{pmatrix}, \quad (2.5.1)$$
$$\mathbf{N} = \begin{pmatrix}\mathbf{u}_{k+1} & \mathbf{u}_{k+2} & \ldots & \mathbf{u}_n\end{pmatrix} \quad (2.5.2)$$

and $\mathbf{P}$ be the diagonal $k \times k$ matrix with diagonal entries $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R}$. Then which of the following is true?

a) rank($\mathbf{MPM}^*$) $= k$ whenever $\alpha_i \neq \alpha_j, 1 \leq i, j \leq k$.
b) tr($\mathbf{MPM}^*$) $= \sum_{i=1}^{k} \alpha_i$
c) rank($\mathbf{M}^*\mathbf{N}$) $= \min(k, n-k)$
d) rank($\mathbf{MM}^* + \mathbf{NN}^*$) $< n$.

2.6. Let $B : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the function

$$B(a, b) = ab \quad (2.6.1)$$

Which of the following is true?

a) B is a linear transformation
b) B is a positive definite bilinear form
c) B is symmetric but not positive definite
d) B is neither linear nor bilinear

2.7. Let $\mathbf{A}$ be an invertible real $n \times n$ matrix. Define a function

$$F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \quad (2.7.1)$$

by

$$F(\mathbf{x}, \mathbf{y}) = (F\mathbf{x})^T \mathbf{y} \quad (2.7.2)$$

Let $DF(\mathbf{x}, \mathbf{y})$ denote the derivate of F at $(\mathbf{x}, \mathbf{y})$ which is a linear transformation from

$$\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \quad (2.7.3)$$

Then, if

a) $\mathbf{x} \neq 0, DF(\mathbf{x}, \mathbf{0}) \neq 0$
b) $\mathbf{y} \neq 0, DF(\mathbf{0}, \mathbf{y}) \neq 0$
c) $(\mathbf{x}, \mathbf{y}) \neq (\mathbf{0}, \mathbf{0}), DF(\mathbf{x}, \mathbf{0}) \neq 0$
d) $\mathbf{x} = 0$ or $\mathbf{y} = 0, DF(\mathbf{x}, \mathbf{y}) = 0$

2.8. Let

$$T : \mathbb{R}^n \to \mathbb{R}^n \quad (2.8.1)$$

be a linear map that satisfies

$$T^2 = T - I. \quad (2.8.2)$$

Then which of the following is true?

a) T is invertible.
b) $T - I$ is not invertible.
c) T has a real eigenvalue.
d) $T^3 = -I$.

2.9. Let

$$\mathbf{M} = \begin{pmatrix}2 & 0 & 3 & 2 & 0 & -2\\ 0 & 1 & 0 & -1 & 3 & 4\\ 0 & 0 & 1 & 0 & 4 & 4\\ 1 & 1 & 1 & 0 & 1 & 1\end{pmatrix} \quad (2.9.1)$$

$$\mathbf{b}_1 = \begin{pmatrix}5\\1\\1\\4\end{pmatrix}, \mathbf{b}_2 = \begin{pmatrix}5\\1\\3\\3\end{pmatrix}. \quad (2.9.2)$$

Then which of the following are true?

a) both systems $\mathbf{Mx} = \mathbf{b}_1$ and $\mathbf{Mx} = \mathbf{b}_2$ are inconsistent.
b) both systems $\mathbf{Mx} = \mathbf{b}_1$ and $\mathbf{Mx} = \mathbf{b}_2$ are consistent.
c) the system $\mathbf{Mx} = \mathbf{b}_1 - \mathbf{b}_2$ is consistent.
d) the system $\mathbf{Mx} = \mathbf{b}_1 - \mathbf{b}_2$ is inconsistent.

2.10. Let

$$\mathbf{M} = \begin{pmatrix}1 & -1 & 1\\ 2 & 1 & 4\\ -2 & 1 & -4\end{pmatrix}. \quad (2.10.1)$$

Given that 1 is an eigenvalue of $\mathbf{M}$, then which among the following are correct?

a) The minimal polynomial of $\mathbf{M}$ is $(x-1)(x+4)$
b) The minimal polynomial of $\mathbf{M}$ is $(x-1)^2(x+4)$
c) $\mathbf{M}$ is not diagonalizable.
d) $\mathbf{M}^{-1} = \frac{1}{4}(\mathbf{M}+3\mathbf{I})$.

2.11. Let $\mathbf{A}$ be a real matrix with characteristic polynomial $(x-1)^3$. Pick the correct statements from below:

a) $\mathbf{A}$ is necessarily diagonalizable.
b) If the minimal polynomial of $\mathbf{A}$ is $(x-1)^3$, then $\mathbf{A}$ is diagonalizable.
c) The characteristic polynomial of $\mathbf{A}^2$ is $(x-1)^3$
d) If $\mathbf{A}$ has exactly two Jordan blocks, then $(\mathbf{A}-\mathbf{I})^2$ is diagonalizable.

2.12. Let P_3 be the vector space of polynomails with real coefficients and of degree at most 3. Consider the linear map

$$T : P_3 \to P_3 \tag{2.12.1}$$

defined by

$$T(p(x)) = p(x-1) + p(x+1) \tag{2.12.2}$$

Which of the following properties does the matrix of T with respect to the standard basis $B = \left\{1, x, x^2, x^3\right\}$ of P_3 satisfy?

a) $detT = 0$.
b) $(T-2I)^4 = 0$ but $(T-2I)^3 \neq 0$.
c) $(T-2I)^3 = 0$ but $(T-2I)^2 \neq 0$.
d) 2 is an eigenvalue with multiplicity 4.

2.13. Let $\mathbf{M}$ be an $n \times n$ Hermitian matrix of rank $k, k \neq n$. If $\lambda \neq= 0$is an eigenvalue of $\mathbf{M}$ with corresponding unit column vector $\mathbf{u}$, then which of the following are true?

a) $\text{rank}(\mathbf{M} - \lambda\mathbf{u}\mathbf{u}^*) = k-1$.
b) $\text{rank}(\mathbf{M} - \lambda\mathbf{u}\mathbf{u}^*) = k$.
c) $\text{rank}(\mathbf{M} - \lambda\mathbf{u}\mathbf{u}^*) = k+1$.
d) $(\mathbf{M} - \lambda\mathbf{u}\mathbf{u}^*)^n = \mathbf{M}^n - \lambda^n\mathbf{u}\mathbf{u}^*$.

2.14. Define a real valued function B on $\mathbb{R}^2 \times \mathbb{R}^2$ as

$$B(\mathbf{x}, \mathbf{y}) = x_1y_1 - x_1y_2 - x_2y_1 + 4x_2y_2 \tag{2.14.1}$$

Let $\mathbf{v}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and

$$W = \left\{\mathbf{v} \in \mathbb{R}^2 : B(\mathbf{v}_0, \mathbf{v}) = 0\right\} \tag{2.14.2}$$

Then W

a) is not a subspace of $\mathbb{R}^2$.
b) equals $\mathbf{0}$.
c) is the y axis
d) is the line passing through $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

2.15. Consider the Quadratic forms

$$Q_1(x, y) = xy \tag{2.15.1}$$

$$Q_2(x, y) = x^2 + 2xy + y^2 \tag{2.15.2}$$

$$Q_3(x, y) = x^2 + 3xy + 2y^2 \tag{2.15.3}$$

on $\mathbb{R}^2$. Choose the correct statements from below

a) Q_1 and Q_2 are equivalent.
b) Q_1 and Q_3 are equivalent.
c) Q_2 and Q_3 are equivalent.
d) all are equivalent.

2.16. Consider a Markov Chain with state space $\{0, 1, 2\}$ and transition matrix

$$P = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{3}{4} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \end{array} \tag{2.16.1}$$

For any two states i and j, let $p_{ij}^{(n)}$ denote the n-step transition probability of going from i to j. Identify correct statements.

a) $\lim_{n\to\infty} p_{11}^{(n)} = \frac{2}{9}$
b) $\lim_{n\to\infty} p_{21}^{(n)} = 0$
c) $\lim_{n\to\infty} p_{32}^{(n)} = \frac{1}{3}$
d) $\lim_{n\to\infty} p_{13}^{(n)} = \frac{1}{3}$

3 June 2018

3.1. Let $\mathbf{A}$ be a $(m \times n)$ matrix and $\mathbf{B}$ be a $(n \times m)$ matrix over real numbers with $m < n$. Then

a) $\mathbf{AB}$ is always nonsingular.
b) $\mathbf{AB}$ is always singular.
c) $\mathbf{BA}$ is always nonsingular.
d) $\mathbf{BA}$ is always singular.

3.2. If $\mathbf{A}$ is a (2×2) matrix over $\mathbb{R}$ with $det(\mathbf{A}+\mathbf{I}) = 1 + det(\mathbf{A})$. Then we can conclude that

a) $det(\mathbf{A}) = 0$.
b) $\mathbf{A} = 0$.
c) $tr(\mathbf{A}) = 0$.
d) $\mathbf{A}$ is nonsingular.

3.3. The system of equations

$$x + 2x^2 + 3xy = 6 \tag{3.3.1}$$
$$x + x^2 + 3xy + y = 5 \tag{3.3.2}$$
$$x - x^2 + y = 7 \tag{3.3.3}$$

a) has solutions in rational numbers.
b) has solutions in real numbers.
c) has solutions in complex numbers.
d) has no solutions.

3.4. The trace of the matrix

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}^{20} \tag{3.4.1}$$

is

a) 7^{20}.
b) $2^{20} + 3^{20}$.
c) $2^{21} + 3^{20}$.
d) $2^{20} + 3^{20} + 1$.

3.5. Given that there are real constants a, b, c, d such that the identity

$$\lambda x^2 + 2xy + y^2 = (ax + by)^2 + (cx + dy)^2, \quad \forall x, y \in \mathbb{R} \tag{3.5.1}$$

This implies that

a) $\lambda = -5$
b) $\lambda \geq 1$
c) $0 < \lambda < 1$
d) There is no such $\lambda \in \mathbb{R}$

3.6. Let $\mathbb{R}, n \geq 2$, be equipped with the standard inner product. Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ be n column vectors forming an orthonormal basis of $\mathbb{R}^n$. Let A be the $n \times n$ matrix formed by the column vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$. Then

a) $\mathbf{A} = \mathbf{A}^{-1}$
b) $\mathbf{A} = \mathbf{A}^\top$
c) $\mathbf{A}^{-1} = \mathbf{A}^\top$
d) $det(\mathbf{A}) = 1$

3.7. Consider a Markov Chain with state space $\{1, 2, 3, 4\}$ and transition matrix

$$P = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix} \end{array} \tag{3.7.1}$$

Then,

a) $\lim_{n\to\infty} p_{22}^{(n)} = 0, \sum_{n=0}^{\infty} p_{22}^{(n)} = \infty$
b) $\lim_{n\to\infty} p_{22}^{(n)} = 0, \sum_{n=0}^{\infty} p_{22}^{(n)} < \infty$
c) $\lim_{n\to\infty} p_{22}^{(n)} = 1, \sum_{n=0}^{\infty} p_{22}^{(n)} = \infty$
d) $\lim_{n\to\infty} p_{22}^{(n)} = 1, \sum_{n=0}^{\infty} p_{22}^{(n)} < \infty$

3.8. Let V denote the vector space of all sequences $\mathbf{a} = (a_1, a_2, \dots)$ of real numbers such that

$$\sum_n 2^n \left|a\right|_n \tag{3.8.1}$$

converges. Define

$$\|\cdot\| : V \to \mathbb{R} \tag{3.8.2}$$

by

$$\|\mathbf{a}\| = \sum_n 2^n \left|a\right|_n. \tag{3.8.3}$$

Which of the following are true?

a) V contains onlythe sequence $(0, 0, \dots)$
b) V is finite dimensional
c) V has a countable linear basis
d) V is a complete normed space

3.9. Let V be a vector space over $\mathbb{C}$ with dimension n. Let $T : V \to V$ be a linear transformation with only1 as eigenvalue. Then which of the following must be true?

a) $T - I = 0$
b) $(T - I)^{n-1} = 0$
c) $(T - I)^n = 0$
d) $(T - I)^{2n} = 0$

3.10. If $\mathbf{A}$ is a 5×5 matrix and the dimension of the solution space of $\mathbf{Ax} = 0$ is at least two, then

a) $\text{rank}\left(\mathbf{A}^2\right) \leq 3$
b) $\text{rank}\left(\mathbf{A}^2\right) \geq 3$
c) $\text{rank}\left(\mathbf{A}^2\right) = 3$
d) $\det\left(\mathbf{A}^2\right) = 0$

3.11. Let $\mathbf{A} \in M_3(\mathbb{R})$ be such that $\mathbf{A}^3 = \mathbf{I}_{3\times3}$. Then

a) minimal polynomial of $\mathbf{A}$ can only be of degree 2
b) minimal polynomial of $\mathbf{A}$ can only be of degree 3
c) either $\mathbf{A} = \mathbf{I}$ or $\mathbf{A} = -\mathbf{I}$
d) there can be uncountably many $\mathbf{A}$ satisfying the above.

3.12. Let $\mathbf{A}$ be an $n \times n, n > 1$ matrix satisfying

$$\mathbf{A}^2 - 7\mathbf{A} + 12\mathbf{I} = \mathbf{0} \tag{3.12.1}$$

Then which of the following statements is true?

a) $\mathbf{A}$ is invertible
b) $t^2 - 7t + 12n = 0$ where $t = tr(\mathbf{A})$
c) $d^2 - 7d + 12 = 0$ where $d = det(\mathbf{A})$
d) $\lambda^2 - 7\lambda + 12 = 0$ where λ is an eigenvalue of $\mathbf{A}$

3.13. Let $\mathbf{A}$ be a 6×6 matrix over $\mathbb{R}$ with characteristic polynomial

$$(x-3)^2 (x-2)^4 \tag{3.13.1}$$

and minimal polynomial

$$(x-3)(x-2)^2 \tag{3.13.2}$$

Then the Jordan canonical form of $\mathbf{A}$ can be

a) $\begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$

b) $\begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$

c) $\begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$

d) $\begin{pmatrix} 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$

3.14. Let V be an inner product space and S be a subset of V. Let $\bar{S}$ denote the closure of S in V with respect to the topology induced by the metric given by the inner product. Which of the following statements is true?

a) $S = (S^{\perp})^{\perp}$
b) $\bar{S} = (S^{\perp})^{\perp}$
c) $\overline{span(S)} = (S^{\perp})^{\perp}$
d) $S^{\perp} = \left((S^{\perp})^{\perp}\right)^{\perp}$

3.15. Let

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.15.1}$$

and

$$Q(\mathbf{x}) = \mathbf{x}^T \mathbf{A} \mathbf{x} \tag{3.15.2}$$

Which of the following statements is true?

a) The matrix of second order partial derivatives of the quadratic form Q is $2\mathbf{A}$
b) The rank of the quadratic form Q is 2
c) The signature of the quadratic form Q is $++0$
d) The quadratic form Q take the value 0 for some non-zero vector $\mathbf{x}$

3.16. Assume that a non-singular matrix

$$\mathbf{A} = \mathbf{L} + \mathbf{D} + \mathbf{U} \tag{3.16.1}$$

where $\mathbf{L}$ and $\mathbf{U}$ are lower and upper triangular matrices respectively with all diagonal entries are zero, and $\mathbf{D}$ si a diagonal matrix. Let $\mathbf{x}^*$ be the solution of $\mathbf{A}\mathbf{x} = \mathbf{b}$. Then the Gauss-Seidel iteration method

$$\mathbf{x}_{k+1} = \mathbf{H}\mathbf{x}_k + \mathbf{c}, k = 0, 1, 2, \dots \tag{3.16.2}$$

with $\|\mathbf{H}\| < 1$ converges to $\mathbf{x}^*$ provided $\mathbf{H}$ is equal to

a) $-\mathbf{D}^{-1}(\mathbf{L} + \mathbf{U})$
b) $-(\mathbf{D} + \mathbf{L})^{-1}\mathbf{U}$
c) $-\mathbf{D}(\mathbf{L} + \mathbf{U})^{-1}$
d) $-(\mathbf{L} - \mathbf{D})^{-1}\mathbf{U}$

3.17. Consider a Markov Chain with state space $S = \{1, 2, 3\}$ and transition matrix

$$P = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \end{array} \tag{3.17.1}$$

Let π be a stationary distribution of the Markov chain and $d(1)$ denote the period of state 1. Which of the following statements are correct?

a) $d(1) = 1$
b) $d(1) = 2$
c) $\pi_1 = \frac{1}{2}$
d) $\pi_1 = \frac{1}{3}$

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4.1. Let $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$. Then the smallest positive integer n such that $\mathbf{A}^n = \mathbf{I}$ is

Solution: *Property of eigen values of A:* Let $\mathbf{A}$ be an arbitary $n \times n$ matrix of complex numbers with eigen values $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then the eigen values of k$^{\text{th}}$ power of $\mathbf{A}$, that is the eigen values of $\mathbf{A}^k$, for any positive integer k are $\lambda_1^k, \lambda_2^k, \ldots, \lambda_n^k$. Let us calculate the eigen values of $\mathbf{A}$.

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \tag{4.1.1}$$

$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0 \tag{4.1.2}$$

$$\begin{vmatrix} -\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = 0 \tag{4.1.3}$$

$$-\lambda(1-\lambda) + 1 = 0 \tag{4.1.4}$$

$$\lambda^2 - \lambda + 1 = 0 \tag{4.1.5}$$

$$\implies \lambda = \frac{-1 \pm \sqrt{3}i}{2} \tag{4.1.6}$$

From the above property, the eigen values of $\mathbf{A}^n$ are λ^n. Also as it is given that $\mathbf{A}^n = \mathbf{I}$,

$$\implies \lambda^n = 1 \tag{4.1.7}$$

$$\implies \left(\frac{-1 \pm \sqrt{3}i}{2}\right)^n = 1 \tag{4.1.8}$$

Clearly $n \neq 1$. For $n = 2$,

$$\left(\frac{-1 \pm \sqrt{3}i}{2}\right)^2 = \frac{-1 \mp \sqrt{3}i}{2} \tag{4.1.9}$$

For $n = 4$,

$$\left(\frac{-1 \pm \sqrt{3}i}{2}\right)^4 = \frac{-1 \pm \sqrt{3}i}{2} \tag{4.1.10}$$

For $n = 6$,

$$\left(\frac{-1 \pm \sqrt{3}i}{2}\right)^6 = 1 \tag{4.1.11}$$

Hence $n = 6$ is the smallest positive integer.

4.2. Let $\mathbf{A} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 2 & 3 & \alpha \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ \beta \end{pmatrix}$. Then the system $\mathbf{AX} = \mathbf{b}$ over the real numbers has

a) No solution when $\beta \neq 7$
b) Infinite number of solutions when $\alpha \neq 2$
c) Infinite number of solutions when $\alpha = 2$ and $\beta \neq 7$
d) A unique solution if $\alpha \neq 2$

Solution: First we derive the Row Reduced Echelon Form (RREF) of the augmented matrix of the system $\mathbf{AX} = \mathbf{b}$ as follows,

$$\begin{pmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 3 \\ 2 & 3 & \alpha & \beta \end{pmatrix} \xleftrightarrow[R_3 = R_3 - 2R_1]{R_2 = R_2 - R_1} \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 5 & \alpha - 2 & \beta - 2 \end{pmatrix} \tag{4.2.1}$$

$$\xleftrightarrow{R_2 = \frac{1}{2}R_2} \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 5 & \alpha - 2 & \beta - 2 \end{pmatrix} \tag{4.2.2}$$

$$\xleftrightarrow{R_1 = R_1 + R_2} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 5 & \alpha - 2 & \beta - 2 \end{pmatrix} \tag{4.2.3}$$

$$\xleftrightarrow{R_3 = R_3 - 5R_2} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & \alpha - 2 & \beta - 7 \end{pmatrix} \tag{4.2.4}$$

From the RREF of the augmented matrix of the system $\mathbf{AX} = \mathbf{b}$ in (4.2.4) we make the following observations for different values of α and β in Table 4.2.1. ,

Values	Observations
$\beta \neq 7$	Then the existence of solution and the number of solutions will entirely depend on value of α
$\alpha = 2$ $\beta \neq 7$	Then RREF in (4.2.4) will contain Zero Row in R_3. Moreover solvability condition will not satisfy. $\implies$ system will have Zero solutions
$\alpha \neq 2$	RREF in (4.2.4) will have all pivots $\implies$ RREF in (4.2.4) will be fullrank $\implies$ $\mathbf{AX} = \mathbf{b}$ have unique solution.

TABLE 4.2.1

Hence, if $\alpha \neq 2$ then the system $\mathbf{AX} = \mathbf{b}$ has unique solution.

4.3. Consider a Markov chain $\{X_n | n \geq 0\}$ with state space $\{1, 2, 3\}$ and transition matrix

$$\mathbf{P} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

Then, $P(X_3 = 1 \mid X_0 = 1)$ equals

Solution: The three step transitional probabilities are given as,

$$P(X_3 = j \mid X_0 = i) = P(X_{n+3} = j \mid X_n = i) = (\mathbf{P}^3)_{ij}\ for\ any\ n \tag{4.3.1}$$

$$\mathbf{P}^3 = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}^3 = \begin{pmatrix} \frac{1}{4} & \frac{3}{8} & \frac{3}{8} \\ \frac{3}{8} & \frac{1}{4} & \frac{3}{8} \\ \frac{3}{8} & \frac{3}{8} & \frac{1}{4} \end{pmatrix} \tag{4.3.2}$$

From (4.3.2),

$$P(X_3 = 1 \mid X_0 = 1) = (\mathbf{P}^3)_{11} = \frac{1}{4} \tag{4.3.3}$$

4.4. Let $\mathbf{A}$ be an $m \times n$ matrix with rank r. If the linear system $\mathbf{AX} = \mathbf{b}$ has a solution for each $\mathbf{b} \in \mathbf{R}^m$, then

a) $m = r$
b) the column space of $\mathbf{A}$ is a proper subspace of $\mathbf{R}^m$
c) the null space of $\mathbf{A}$ is a non-trivial subspace of $\mathbf{R}^n$ whenever $m = n$
d) $m \geq n$ implies $m = n$

Solution: *Theorem*

Theorem 4.1. *Consider the $m \times n$ system $Ax = b$, with either $b \neq 0$ or $b = 0$. We distinguish the following cases:*

a) ***Unique Solution****: If rank[A,b] = rank(A) = $n \leq m$, then and only then the system has a unique solution. In this case, indeed as many as $m - n$ equations are redundant. And the solution $\mathbf{X} = \mathbf{A}^{-1}\mathbf{b}$. This is called as* ***Exactly Determined****.*

b) ***No Solution****: If rank[A,b] > rank(A) which necessarily implies $\mathbf{b} \neq 0$ and $m >$ rank(A), then and only then the system has no solution. This is called as* ***Overdetermined****.*

See Table 4.4.1 If the columns of an $m \times n$ matrix $\mathbf{A}$ span $\mathbf{R}^m$ then the equation $\mathbf{Ax} = \mathbf{b}$ is consistent for each $\mathbf{b}$ in $\mathbf{R}^m$.

The **null space** of $\mathbf{A}$ is defined to be

$$Null(\mathbf{A}) = \{\mathbf{x} \in \mathbf{R}^n \mid \mathbf{Ax} = 0\} \tag{4.4.1}$$

$$\mathbf{A} = \begin{pmatrix} -3 & -2 & 4 \\ 14 & 8 & -18 \\ 4 & 2 & -4 \end{pmatrix} \tag{4.4.2}$$

Reduced Row Echelon form is

$$RREF\,(\mathbf{A}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.4.3}$$

$\therefore$ the only possible nullspace of the matrix $\mathbf{A}$ is $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

Let $\mathbf{B}$ be given as

$$\mathbf{B} = \begin{pmatrix} -3 & -2 & 4 \\ 14 & 8 & -18 \\ 4 & 2 & -4 \\ 28 & 16 & -36 \\ 8 & 4 & -8 \end{pmatrix} \tag{4.4.4}$$

Reduced Row Echelon form is

$$RREF\,(\mathbf{B}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.4.5}$$

$\therefore$ the rank of matrix $\mathbf{B} = 3$.

4.5. Let $\mathbf{M} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}\right.$ and eigen values of$\mathbf{A} \in \mathbb{Q}$

a) $\mathbf{M}$ is empty
b) $\mathbf{M} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \right\}$
c) If $\mathbf{A} \in \mathbf{M}$ then the eigen values of $\mathbf{A} \in \mathbb{Z}$
d) If $\mathbf{A}$,$\mathbf{B} \in \mathbf{M}$ such that $\mathbf{AB}$=$\mathbf{I}$ then $\left|\mathbf{A}\right| \in \{+1,-1\}$

Solution: See Table 4.5.1.

Options	Observations
$m = r$	The rank of any matrix **A** is the dimension of its column space. When the number of rows (m) is equal to the rank (r) of the matrix, then their linear combination gives us span of $\mathbf{R}^m$. $\therefore$ This statement is **True**.
the column space of **A** is a proper subspace of $\mathbf{R}^m$	Any subspace of a vector space **V** other than **V** itself is considered a proper subspace of **V**. Which means that linear combination of **A** will span less than m. That will make the resultant **b** span strictly less than m. But it is given that $\mathbf{b} \in \mathbf{R}^m$, which is contradicting. $\therefore$ This statement is **False**.
the null space of **A** is a non-trivial subsapce of $\mathbf{R}^n$ whenever $m = n$	From (4.4.2) we see that even when $m = n$ then also we are getting a trivial nullspace. $\therefore$ This statement is **False**.
$m \geq n$ implies $m = n$	It is given that the number of rows are greater than the column, and it is given that there exists a solution. If we refer to theorem (4.1) we see that the corresponding system will be **Exactly Determined** system. As an example, it will look like (4.4.4). $\therefore$ This statement is **True**.

TABLE 4.4.1: Solution

M is empty	Consider **A**=**I**=$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.The elements of **A** $\in \mathbb{Z}$ and it's eigen values $1 \in \mathbb{Q}$. So, **M** is not empty.
M $=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}\right\}$	Let **A**=$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ where elements of **A** $\in \mathbb{Z}$.The characteristic equation can be written as : $$\lambda^2 + 1 = 0 \implies \lambda = \pm i$$

	We see that $\lambda \in \mathbb{C}$ which is contradicting the main definition of $\mathbf{M}$ .So,this is not correct.
Eigen values of $\mathbf{A} \in \mathbb{Z}$	Given ,$\mathbf{A} \in \mathbf{M}$.Let λ_1,λ_2 be the eigen values of $\mathbf{A}$.The characteristic polynomial can be written as: $$\lambda^2 - tr(\mathbf{A})\lambda + \det \mathbf{A} = 0 \text{ where } tr(\mathbf{A}) = \lambda_1 + \lambda_2, \det \mathbf{A} = \lambda_1\lambda_2$$ Given the eigen values λ_1,$\lambda\_2 \in \mathbb{Q}$,For this to be possible the discriminant of above equation should $\in \mathbb{Z}$ $$\sqrt{(\lambda_1+\lambda_2)^2 - 4\lambda_1\lambda_2} \in \mathbb{Z}$$ $$\implies \sqrt{(\lambda_1-\lambda_2)^2} \in \mathbb{Z}$$ $$\implies \lambda_1 - \lambda_2 \in \mathbb{Z} \text{ This is possible when both } \lambda_1,\lambda_2 \in \mathbb{Z}.$$
If **AB**=**I** then $\left\|\mathbf{A}\right\| \in$ {+1,-1}	As $\mathbf{A},\mathbf{B} \in \mathbf{M}, \implies \left\|\mathbf{A}\right\|,\left\|\mathbf{B}\right\| \in \mathbb{Z}$ Given **AB**=**I** $\implies \left\|\mathbf{A}\right\|\left\|\mathbf{B}\right\|$=1 This is possible only when $\left\|\mathbf{A}\right\|$=$\left\|\mathbf{B}\right\|$= $\pm$ 1
Conclusion	options 3) and 4) are correct.

TABLE 4.5.1: Solution

4.6. Let A be a 3×3 matrix with real entries.Identify the correct statements.

a) A is necessarily diagonalizable over **R**
b) If A has distinct real eigen values than it is diagonalizable over**R**
c) If A has distinct eigen values than it is diagonalizable over **C**
d) If all eigen values are non zero than it is diagonalizable over **C**

Solution: See Table 4.6.1.

Statement 1.	A is necessarily diagonalizable over **R**
False statement Example:	Matrix A is diagonalizable if and only if there is a basis of $\mathbf{R}^3$consisting of eigenvectors of A. Consider a matrix $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{pmatrix} \quad (4.6.1)$ Eigen values are: $\begin{pmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 4-\lambda \end{pmatrix} = 0. \implies \lambda_1 = 1, \lambda_2 = 4 \quad (4.6.2)$ $\lambda_1 = 1$ has eigen vector $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and$\lambda_2 = 4$ has eigen vector $\begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix} \quad (4.6.3)$ We have found only two linearly independent eigenvectors for A,not diagonalisable
Statement 2.	If A has distinct real eigen values than it is diagonalizable over**R**
True statement	Distinct real eigenvalues implies linearly independent eigenvectors . and if a matrix has n linearly independent vectors than it is diagonalizable.
Proof 1: Proof 2:	**Distinct eigen values implies linearly independent vectors that spans entire space.** Consider 2 eigen vectors **v**,**w** with eigen values λ,μ respectively. such that $\lambda \neq \mu$ $\alpha(\mathbf{v}) + \beta(\mathbf{w}) = 0 \quad (4.6.4)$ $\alpha A(\mathbf{v}) + \beta A(\mathbf{w}) = 0 \quad (4.6.5)$ $\alpha\lambda\mathbf{v} + \beta\mu\mathbf{w} = 0 \quad (4.6.6)$ Multiplying (4.6.4)with -λ and subtracting from (4.6.6) we have, $\beta(\mu - \lambda)\mathbf{w} = 0 \quad (4.6.7)$ eigen values are distinct $(\mu - \lambda) \neq 0$. From equation(4.6.7) we have, $\beta = 0$ substituting $\beta = 0$ in equation (4.6.4)we have,$\alpha = 0$.As, $\mathbf{v} \neq 0$ **which proves that vectors are linearly independent**. **If a matrix has n linearly independent vectors than it is diagonalizable** If$\begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} & \cdots & \mathbf{p_n} \end{pmatrix}$are n independent eigen vectors then, $A\mathbf{p_1} = \lambda\mathbf{p_1}, \cdots, A\mathbf{p_n} = \lambda\mathbf{p_n}$ $D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} P = \begin{pmatrix} \mathbf{P_1} & \mathbf{P_2} & \cdots & \mathbf{P_n} \end{pmatrix} \quad (4.6.8)$ Now, $A\mathbf{P_i} = \lambda_i\mathbf{P_i} \implies AP = PD$

	so,$P^{-1}AP = D$ is a diagonal matrix.
Statement 3.	If A has distinct real eigen values than it is diagonalizable over$\mathbf{C}$
True statement . Proof: Example:	If A is an $N \times N$ complex matrix with n distinct eigenvalues, then any set of n corresponding eigenvectors form a basis for $\mathbf{C}^n$ It is sufficient to prove that the set of eigenvectors is linearly independent which is proved in statement 2. $$A = \begin{pmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{pmatrix} \quad (4.6.9)$$ Eigen values of A are: $$\lambda_1 = 2, \lambda_2 = 3, \lambda_3 = 6 \quad (4.6.10)$$
	Eigen vectors are: $$x_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, x_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, x_3 = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} \quad (4.6.11)$$ Matrix A is diagonalizable because there is a basis of $\mathbf{C}^3$consisting of eigenvectors of A.
Statement 4.	If all eigen values are non zero than it is diagonalizable over $\mathbf{C}$
False Statement:	Matrix would be diagonalizable if and only if it has linearly independent eigenvectors .
Example:	Consider a matrix $$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{pmatrix} \quad (4.6.12)$$ Eigen values are: $$\begin{pmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 4-\lambda \end{pmatrix} = 0. \implies \lambda_1 = 1, \lambda_2 = 4 \neq 0 \quad (4.6.13)$$ $$\lambda_1 = 1 \text{ has eigen vector} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{and} \lambda_2 = 4 \text{ has eigen vector} \begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix} \quad (4.6.14)$$ We have found only two linearly independent eigenvectors for A,not diagonalisable.

TABLE 4.6.1: Solution summary

Given	V be a vector space over C of all the polynomials in a variable X of degree atmost 3 $D : P_3 \rightarrow P_3$ $D : V \rightarrow V$ be the linear operator given by differentiation wrt X $D(P(x)) \rightarrow P'(x)$ A be the matrix of D wrt some basis for V Assume basis for V be $\{1, x, x^2, x^3\}$

TABLE 4.7.1

4.7. Let V be a vector space over C of all the polynomials in a variable X of degree atmost 3. Let $D : V \rightarrow V$ be the linear operator given by differentiation with respect to X. Let A be the matrix of D with respect to some basis for V. Which of the following are true?

a) A is nilpotent matrix

b) A is diagonalizable matrix

c) the rank of A is 2

d) the Jordan canonical form of A is

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Solution: See Tables 4.7.1, 4.7.2 and 4.7.3

4.8. For every 4×4 real symmetric non-singular matrix $\mathbf{A}$ there exists a positive integer p such that

a) $p\mathbf{I} + \mathbf{A}$ is positive definite
b) $\mathbf{A}^p$ is positive definite
c) $\mathbf{A}^{-p}$ is positive definite
d) $\exp(p\mathbf{A}) - \mathbf{I}$ is positive definite

Solution: A matrix is real symmetric implies its eigen values are real and eigen vectors are orthogonal,that is its eigen value decomposition is

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^T \tag{4.8.1}$$

$\mathbf{D}$ is the diagonal matrix containing the real eigen values of $\mathbf{A}$

$\mathbf{P}$ has the corresponding eigen vectors

$$\mathbf{P}\mathbf{P}^T = \mathbf{P}^T\mathbf{P} = \mathbf{I} \tag{4.8.2}$$

A real matrix is positive definite if

$$\mathbf{x}^T\mathbf{A}\mathbf{x} > 0 \tag{4.8.3}$$
$$\implies \mathbf{x}^T\lambda\mathbf{x} > 0 \tag{4.8.4}$$
$$\implies \lambda\mathbf{x}^T\mathbf{x} > 0 \tag{4.8.5}$$
$$\implies \lambda > 0 \tag{4.8.6}$$

In other words, all the eigen values of A are positive See Table 4.8.1
Let $\mathbf{A}$ be

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^T \tag{4.8.7}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix} \tag{4.8.8}$$

From the table,the choices would be option 1,2,3

4.9. Let $\mathbf{A}$ be an $m \times n$ matrix of rank m with $n > m$. If for some non-zero real number α, we have $\mathbf{x^T A A^T x} = \alpha \mathbf{x^T x}$, for all $x \in \mathbf{R^m}$, then $\mathbf{A^T A}$ has,

a) exactly two distinct eigenvalues.
b) 0 as an eigenvalue with multiplicity $n - m$.
c) α as a non-zero eigenvalue.
d) exactly two non-zero distinct eigenvalues.

Solution: Refer Table 4.9.1.
Refer Table 4.9.2.

4.10. Consider a Markov chain with five states

$\{1, 2, 3, 4, 5\}$ and transition matrix

$$P = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{7} & 0 & 0 & \frac{6}{7} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} & 0 \\ 0 & \frac{5}{8} & 0 & 0 & \frac{3}{8} \end{pmatrix} \qquad (4.10.1)$$

Which of the following are true?

a) 3 and 1 are in the same communicating class
b) 1 and 4 are in the same communicating class
c) 4 and 2 are in the same communicating class
d) 2 and 5 are in the same communicating class

Solution: See Tables 4.10.1 and 4.10.2

Matrix	$D(1) = 0 = 0.1 + 0.x + 0.x^2 + 0.x^3$ $D(1) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $D(x) = 1 = 1.1 + 0.x + 0.x^2 + 0.x^3$ $D(x) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $D(x^2) = 2x = 0.1 + 2.x + 0.x^2 + 0.x^3$ $D(x^2) = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}$ $D(x^3) = 3x^2 = 0.1 + 0.x + 3.x^2 + 0.x^3$ $D(x^3) = \begin{pmatrix} 0 \\ 0 \\ 3 \\ 0 \end{pmatrix}$ Matrix $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
Inference	An $n \times n$ matrix with λ as diagonal elements, ones on the super diagonal and zeroes in all other entries is nilpotent with minimal polynomial $(A - \lambda I)^n$
Nilpotent	$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ All eigen values of matrix A is 0 Thus, above matrix is nilpotent matrix Thus, above statement is true

TABLE 4.7.2

Diagonalizable	$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $Rank(A) + nullity(A) =$ no of column $Rank(A) = 3$, no of column $= 4$ $nullity(A) = 4 - 3 = 1$ means there exists only one linearly independent eigen vector corresponding to 0 eigen values Thus,matrix A is not Diagonalizable. Thus, above statement is false
Rank	$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ Rank of matrix A is 3 Thus, above statement is false
Jordan CF	Assume characteristic polynomial of matrix A is $c_A(x)$ $c_A(x) = x^4$ Assume minimal polynomial of A is $m_A(x)$ $m_A(x)$ always divide $c_A(x)$ $m_A(x) = \{x, x^2, x^3, x^4\}$ Minimal polynomial always annihilates its matrix.Thus,we see that $m_A(A) = \{A = 0, A^2 = 0, A^3 = 0, A^4 = 0\}$ But we see that neither A is zero matrix nor A^2 and A^3 equal to zero but A^4 is equal to zero.Thus,x^4 is minimal polynomial. Algebraic Multiplicity $= a_M(\lambda = 0) = 4$ Geometric Multiplicity $= g_M(\lambda = 0) =$ $nullity(A - 0I) = nullity(A) = 1$ Hence, Jordan form of block size 4 Using Inference, $\mathbf{J} = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$ $\lambda = 0$ $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ which is same as given in the question. Thus,statement is true

OPTIONS	DERIVATIONS
Choice 1	$p\mathbf{I}+\mathbf{A} = \mathbf{P}(p\mathbf{I})\mathbf{P}^T + \mathbf{PDP}^T \quad (4.8.9)$ $= \mathbf{PD}_1\mathbf{P}^T \quad (4.8.10)$ $\mathbf{D}_1 = \begin{pmatrix} \lambda_1+p & 0 & 0 & 0 \\ 0 & \lambda_2+p & 0 & 0 \\ 0 & 0 & \lambda_3+p & 0 \\ 0 & 0 & 0 & \lambda_4+p \end{pmatrix} \quad (4.8.11)$ Some of the eigen values of $\mathbf{A}$ may be negative. All the eigen values in $\mathbf{D}_1$ are positive only if $p > \lvert\lambda_i\rvert \ \forall i \in [1,4] \quad (4.8.12)$
Choice 2	$\mathbf{A}^2 = \mathbf{AA} \quad (4.8.13)$ $= (\mathbf{PDP}^T)(\mathbf{PDP}^T) \quad (4.8.14)$ $= \mathbf{PD}^2\mathbf{P}^T \quad (4.8.15)$ Similarly, $\mathbf{A}^p = \mathbf{PD}^p\mathbf{P}^T \quad (4.8.16)$ $\mathbf{D}^p = \begin{pmatrix} \lambda_1^p & 0 & 0 & 0 \\ 0 & \lambda_2^p & 0 & 0 \\ 0 & 0 & \lambda_3^p & 0 \\ 0 & 0 & 0 & \lambda_4^p \end{pmatrix} \quad (4.8.17)$ $\mathbf{A}^p$ is positive definite only if p is even.
Choice 3	$\mathbf{A}^{-p} = \mathbf{PD}^{-p}\mathbf{P}^T \quad (4.8.18)$ $\mathbf{D}^{-p} = \begin{pmatrix} \lambda_1^{-p} & 0 & 0 & 0 \\ 0 & \lambda_2^{-p} & 0 & 0 \\ 0 & 0 & \lambda_3^{-p} & 0 \\ 0 & 0 & 0 & \lambda_4^{-p} \end{pmatrix} \quad (4.8.19)$ $\mathbf{A}^{-p}$ is positive definite only if p is even.
Choice 4	$\exp(p\mathbf{A}) = \sum_{k=0}^{\infty} \frac{(p\mathbf{A})^k}{k!} \quad (4.8.20)$ $\implies \exp(p\mathbf{A}) - \mathbf{I} = \mathbf{P}\exp(p\mathbf{D})\mathbf{P}^T - \mathbf{PIP}^T \quad (4.8.21)$ $= \mathbf{P}(\exp(p\mathbf{D}) - \mathbf{I})\mathbf{P}^T \quad (4.8.22)$ $\exp(p\mathbf{D}) - \mathbf{I} = \begin{pmatrix} e^{\lambda_1}-1 & 0 & 0 & 0 \\ 0 & e^{\lambda_2}-1 & 0 & 0 \\ 0 & 0 & e^{\lambda_3}-1 & 0 \\ 0 & 0 & 0 & e^{\lambda_4}-1 \end{pmatrix} \quad (4.8.23)$ $\mathbf{A}$ is non-singular $\implies \forall i \in [1,4], \lambda_i \neq 0 \quad (4.8.24)$ $e^{\lambda_i} < 1 \quad (4.8.25)$ So, $\exp(p\mathbf{A}) - \mathbf{I}$ is not positive definite.

TABLE 4.8.1: Solution

Given	**Derivation**
Given	$\mathbf{A}$ is a $m \times n$ matrix of rank m with $n > m$. A non-zero real number α. To find eigenvalues of $\mathbf{A^TA}$.
Eigenvalues of $\mathbf{AA^T}$	$\mathbf{AA^T}$ is a $m \times m$ matrix and $\mathbf{A^TA}$ is a $n \times n$ matrix. Let, λ be a non-zero eigen value of $\mathbf{A^TA}$. $\mathbf{A^TAv} = \lambda\mathbf{v} \quad \mathbf{v} \in \mathbf{R^n} \qquad (4.9.1)$ $\mathbf{AA^TAv} = \lambda\mathbf{Av} \qquad (4.9.2)$ $\text{Let,} \quad \mathbf{x} = \mathbf{Av} \quad \mathbf{x} \in \mathbf{R^m} \qquad (4.9.3)$ $\mathbf{AA^Tx} = \lambda\mathbf{x} \qquad (4.9.4)$ $\mathbf{x^TAA^Tx} = \lambda\mathbf{x^Tx} \qquad (4.9.5)$ $\text{Given,} \quad \mathbf{x^TAA^Tx} = \alpha\mathbf{x^Tx} \qquad (4.9.6)$ $\implies \alpha\mathbf{x^Tx} = \lambda\mathbf{x^Tx} \qquad (4.9.7)$ From equation (4.9.7), $\lambda = \alpha$ as $\|\mathbf{x}\| \neq 0$ As rank($\mathbf{A^TA}$) = rank($\mathbf{A}$) = m and equation (4.9.7) satisfies the condition in question. Therefore the only non-zero eigen value is α $\mathbf{A^TA}$ has an eigenvalue α with multiplicity m.
Eigenvalues of $\mathbf{A^TA}$	$\mathbf{A^TA}$ is a $n \times n$ matrix. Given $n > m$, We know that, $\mathbf{A^TA}$ and $\mathbf{AA^T}$ have same number of non-zero eigenvalues and if one of them has more number of eigenvalues than the other then these eigenvalues are zero. 1. From above, as α is non-zero, $\mathbf{A^TA}$ has α as its eigenvalue with multiplicity m 2. $\mathbf{A^TA}$ has 0 as its eigenvalue with multiplicity $n - m$ 3. Therefore, the two distinct eigenvalues of $\mathbf{A^TA}$ are α and 0.

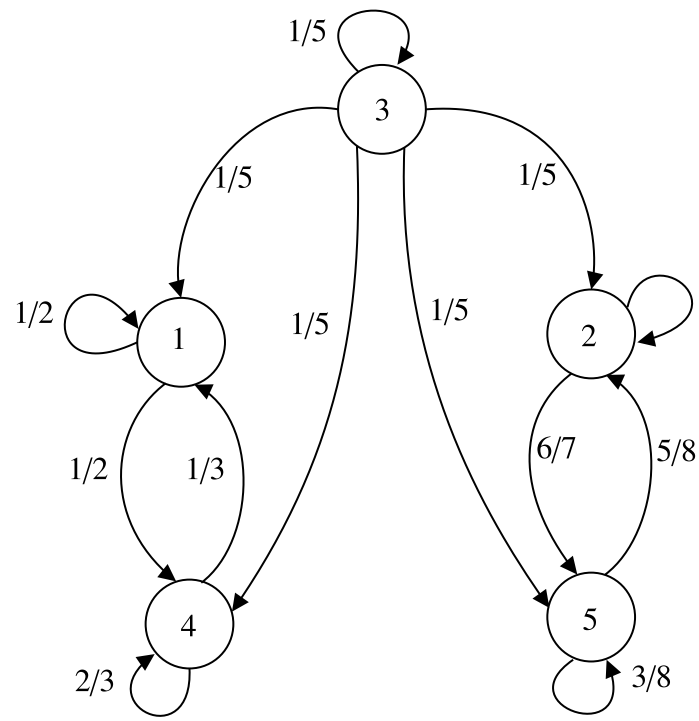
TABLE 4.9.1: Explanation

$\mathbf{A^TA}$ has exactly two distinct eigenvalues.	True statement
$\mathbf{A^TA}$ has 0 as an eigenvalue with multiplicity $n - m$	True statement
$\mathbf{A^TA}$ has α as a non-zero eigenvalue	True statement
$\mathbf{A^TA}$ has exactly two non-zero distinct eigenvalues.	False statement

TABLE 4.9.2: Solution

Accessibility of states in Markov's chain	We say that state j is accessible from state i, written as $i \to j$, if $p_{ij}^{(n)} > 0$ for some n. Every state is accessible from itself since $p_{ii}^{(0)} = 1$
Communication between states	Two states i and j are said to communicate, written as $i \leftrightarrow j$, if they are accessible from each other. In other words, $i \leftrightarrow j \text{ means } i \to j \text{ and } j \to i.$
Communicating class	For each Markov chain, there exists a unique decomposition of the state space S into a sequence of disjoint subsets $C_1, C_2, \ldots,$ $S = \bigcup_{i=1}^{\infty} C_i$ in which each subset has the property that all states within it communicate. Each such subset is called a communication class of the Markov chain.

TABLE 4.10.1: Definition and Result used

<table>
<tr>
<td>Drawing Transition diagram</td>
<td>
</td>
</tr>
<tr>
<td>Checking whether the states 3 and 1 are in the same communicating class</td>
<td>Here,
State 1 is accessible from the state 3.
But, State 3 is not accessible from the state 1
i.e. $3 \to 1,\ 1 \nrightarrow 3$
$\implies \boxed{3 \nleftrightarrow 1}$

Therefore, 3 and 1 are not in the same communicating class.</td>
</tr>
<tr>
<td>Checking whether the states 1 and 4 are in the same communicating class</td>
<td>Here,
State 1 is accessible from the state 4.
Also, State 4 is accessible from the state 1
i.e. $3 \to 1,\ 1 \to 3$
$\implies \boxed{3 \leftrightarrow 1}$

Therefore, 1 and 4 are in the same communicating class.</td>
</tr>
<tr>
<td>Checking whether the states 4 and 2 are in the same communicating class</td>
<td>Here,
State 2 is not accessible from the state 4.
Also, State 4 is not accessible from the state 2
i.e. $4 \nrightarrow 2,\ 2 \nrightarrow 4$</td>
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<td></td>
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</table>

	$\implies \boxed{4 \nleftrightarrow 2}$ Therefore, 4 and 2 are not in the same communicating class.
Checking whether the states 2 and 5 are in the same communicating class	Here, State 2 is accessible from the state 5. Also, State 5 is accessible from the state 2 i.e. $5 \to 2,\ 2 \to 5$ $\implies \boxed{2 \leftrightarrow 5}$ Therefore, 2 and 5 are in the same communicating class.
Conclusion	Communication classes are: $\boxed{S = \{1,4\} \cup \{3\} \cup \{2,5\}}$ Option 2) and 4) are true.

TABLE 4.10.2: Solution

5 June 2017

5.1. Let $\mathbf{A}$ be an n × n self-adjoint matrix with eigenvalues $\lambda_1, \cdots, \lambda_2$. Let,

$$\|\mathbf{X}\|_2 = \sqrt{|\mathbf{X}_1^2| + \cdots + |\mathbf{X}_n^2|} \tag{5.1.1}$$

for $\mathbf{X}=(\mathbf{X}_1, \cdots, \mathbf{X}_n) \in \mathbb{C}^n$. If

$$p(\mathbf{A}) = a_0\mathbf{I} + a_1\mathbf{A} + \cdots + a_n\mathbf{A}^n \tag{5.1.2}$$

then $sup_{\|\mathbf{X}\|_2=1} \|p(\mathbf{A})\mathbf{X}\|_2$ is equal to

Solution: We know that $\mathbf{A}$ is a self adjoint matrix and hence $\mathbf{A} = \mathbf{A}^*$ with eigen values $\lambda_1, \lambda_2, \cdots, \lambda_n$.Now as we are given,

$$p(\mathbf{A}) = a_0\mathbf{I} + a_1\mathbf{A} + \cdots + a_n\mathbf{A}^n \tag{5.1.3}$$

then,

$$(p(\mathbf{A}))^* = a_0\mathbf{I}^* + a_1\mathbf{A}^* + \cdots + a_n(\mathbf{A}^*)^n \tag{5.1.4}$$

Since, $\mathbf{A} = \mathbf{A}^*$ we can state that,

$$p(\mathbf{A})(p(\mathbf{A}))^* = (p(\mathbf{A}))^* p(\mathbf{A}) \tag{5.1.5}$$

Hence p($\mathbf{A}$) is a normal matrix. Now using spectral theorem for a normal matrix,

$$\|p(\mathbf{A})\|_2 = \rho(p(\mathbf{A})) \tag{5.1.6}$$

sup refers to the smallest element that is greater than or equal to every number in the set.Hence, sup of $\|p(\mathbf{A})\|_2$ will be,

$$= \max\{|\alpha| : \alpha \text{ is the eigen value of p(A)}\} \tag{5.1.7}$$

$$= \max\{|p(\lambda_j)| : j = 1, 2, \cdots n\} \tag{5.1.8}$$

$$= \max\{|a_0 + a_1\lambda_j + \cdots + a_n{\lambda_j}^n| : j = 1, 2, \cdots n\} \tag{5.1.9}$$

Now, to find $sup \|p(\mathbf{A})\mathbf{X}\|_2$,

$$= max\{|a_0 + a_1\lambda_j + \cdots + a_n{\lambda_j}^n| : j = 1, 2, \cdots n\} \|\mathbf{X}\|_2 \tag{5.1.10}$$

Since, we have to find $sup_{\|\mathbf{X}\|_2=1}$ i.e,

$$\|\mathbf{X}\|_2 = \sqrt{|\mathbf{X}_1^2| + \cdots + |\mathbf{X}_n^2|} = 1 \tag{5.1.11}$$

Hence the final answer will be,

$$= max\{|a_0 + a_1\lambda_j + \cdots + a_n{\lambda_j}^n| : j = 1, 2, \cdots n\} \tag{5.1.12}$$

5.2. Find the Eigenvalues of the matrix,

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & -2 & 5 \\ 2 & 5 & -3 \end{pmatrix} \tag{5.2.1}$$

a) -4, 3, -3
b) 4, 3, 1
c) 4, -4± $\sqrt{13}$
d) 4, -2± $\sqrt{7}$

Solution: Using the characteristic equation of the matrix can find the Eigenvalues,

$$\left|\lambda\mathbf{I} - \mathbf{A}\right| = 0 \tag{5.2.2}$$

$$\implies \begin{vmatrix} \lambda - 1 & -1 & -2 \\ -1 & \lambda + 2 & -5 \\ -2 & -5 & \lambda + 3 \end{vmatrix} = 0 \tag{5.2.3}$$

The expression that is obtained after expanding the determinant and simplifying it is,

$$(\lambda - 1)(\lambda^2 + 5\lambda - 19) - (5\lambda + 31) = 0 \tag{5.2.4}$$

Further simplifying this we obtain the cubic equation,

$$\lambda^3 + 4\lambda^2 - 29\lambda - 12 = 0 \tag{5.2.5}$$

Solving this equation, the Eigenvalues obtained are,

$$\lambda_1 = -7.605, \ \lambda_2 = -0.394 \ and \ \lambda_3 = 4 \tag{5.2.6}$$

Therefore, the Eigenvalues of the given matrix are 4, -4 ± $\sqrt{13}$ (Option 3)

5.3. Let $\mathbf{V}$ be the vector space of polynomials of degree at most 3 in a variable x with coefficients in $\mathbb{R}$. Let $\mathbf{T}$=d/dx be the linear transformation of $\mathbf{V}$ to itself given by differentiation.

Which of the following are correct?

a) $\mathbf{T}$ is invertible
b) 0 is an eigenvalue of $\mathbf{T}$
c) There is a basis with respect to which the matrix of $\mathbf{T}$ is nilpotent.
d) The matrix of $\mathbf{T}$ with respect to the basis $\left(1, 1 + x, 1 + x + x^2, 1 + x + x^2 + x^3\right)$ is diagonal.

Solution: See Tables 5.3.1 , 5.3.2 and 5.3.3.

Nilpotent Matrix	1. If all the eigen values of matrix is zero then it is said to nilpotent matrix 2. Determinant and trace of nilpotent matrix are always zero.
Invertible Matrix	A matrix is said to be invertible matrix if its determinant is non zero.
Diagonal matrix	diagonal matrix is a matrix in which the entries outside the main diagonal are all zero.

TABLE 5.3.1: Definition

Given	$T : P_3 \to P_3$ $T : V \to V$ be the linear operator given by differentiation wrt x $T(P(x)) \to P'(x)$ A be the matrix of T wrt some basis for V Assume basis for V be $\{1, x, x^2, x^3\}$

TABLE 5.3.2: Result used

Checking whether matrix of T is nilpotent	$T : V \to V$ $TP(x) = P'(x)$ Differentiating wrt x to find matrix A; $T(1) = 0 = a_1x + b_1x + c_1x^2 + d_1x^3$ $T(x) = 1 = a_2 + b_2x + c_2x^2 + d_2x^3$ $T(x^2) = 2x = a_3 + b_3x + c_3x^2 + d_3x^3$ $T(x^3) = 3x^2 = a_4 + b_4x + c_4x^2 + d_4x^3$ Representing A in matrix form ; $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ from the above matrix of T we can say it is nilpotent matrix.
Checking eigen value of matrix T	$A = \begin{pmatrix} 0-\lambda & 1 & 0 & 0 \\ 0 & 0-\lambda & 2 & 0 \\ 0 & 0 & 0-\lambda & 3 \\ 0 & 0 & 0 & 0-\lambda \end{pmatrix}$ $\implies \lambda = 0$
Checking whether matrix of T is invertible	Since $\det A = 0$. Therefore matrix of T is not invertible
Checking whether Matrix of T is diagonal matrix	Let basis be $B' = \{1, 1+x, 1+x+x^2, 1+x+x^2+x^3\}$ Differentiating wrt x ;

	$T(1) = 0 = a_1x + b_1(1 + x) + c_1(1 + x + x^2) + d_1(1 + x + x^2 + x^3)$ $T(1 + x) = 1 = a_2 + b_2(1 + x) + c_2(1 + x + x^2) + d_2(1 + x + x^2x^3)$ $T(1 + x + x^2) = 1 + 2x = a_3 + b_3(1 + x) + c_3(1 + x + x^2)$ $+d_3(1 + x + x^2 + x^3)$ $T(1 + x + x^2 + x^3) = 1 + 2x + 3x^2 = a_4 + b_4(1 + x) + c_4(1 + x + x^2)$ $+d_4(1 + x + x^2 + x^3)$ $B = \begin{pmatrix} 0 & 1 & -1 & -1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ above matrix is not a diagonal matrix
Conclusion	Thus we can conclude Option 2) and 3) are correct.

TABLE 5.3.3: Solution

5.4. For any $n \times n$ matrix B, let $N(B) = \{X \in \mathbb{R}^n : BX = 0\}$ be the null space of B. Let A be a 4×4 matrix with $dim(N(A - 4I)) = 2, dim(N(A - 2I)) = 1$ and $rank(A) = 3$ Which of the following are true?

a) 0,2 and 4 are eigenvalues of A
b) determinant(A)=0
c) A is not diagonalizable
d) trace(A)=8

Solution: See Table 5.4.1.

Given	A is a 4×4 matrix. $dim\,(N\,(A - 2I)) = 2,$ $dim\,(N\,(A - 4I)) = 1$, and $rank\,(A) = 3$
Eigenvalues of a matrix	The number λ is an eigenvalue of a matrix A if and only if$A - \lambda I$ is singular, i.e. $\lvert A - \lambda I \rvert = 0$ For $\lambda = 2$ Given, $dim\,(N\,(A - 2I)) = 2$ $\implies nullity(A - 2I) = 2$ $rank(A) + nullity(A) = n$ $\implies rank\,(A - 2I) = 4 - 2 = 2$ $\implies (A - 2I)$ is not a full rank matrix Therefore $\lvert A - 2I \rvert = 0$ Also, $\implies N\,(A - 2I) = \{X \in \mathbb{R}^4 : (A - 2I)X = 0\}$ $\implies (A - 2I)X = 0$ gives two eigen vectors

	$\implies$ 2 is an eigenvalue of A with multiplicity 2. Similarly, for $\lambda = 4$ Given, $dim\,(N\,(A - 4I)) = 1$ $\implies$ $rank\,(A - 4I) = 4 - 1 = 3$ $\implies$ $(A - 4I)$ is not a full rank matrix

	Therefore $\left\|A - 4I\right\| = 0$ $\implies$ 4 is an eigenvalue of A with multiplicity 1. For $\lambda = 0$ Given that $rank\,(A) = 3$ $\implies$ A is not a full rank matrix Therefore $\left\|A\right\| = 0$ $\implies$ 0 is an eigenvalue of A with multiplicity 1.
Determinant	Given that $rank\,(A) = 3$ $\implies$ A is not a full rank matrix Therefore $\left\|A\right\| = 0$
Diagonalizability	An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigen vectors. $rank\,(A) + nullity\,(A) = n$ $\implies$ for $\lambda = 0$, $nullity\,(A - \lambda I) = nullity\,(A) = 4 - 3 = 1$ $\implies$ There exists only one linearly independent eigen vector corresponding to 0 eigen value Thus, matrix A is not diagonalizable.
Trace	$Trace\,(A)$=sum of eigen values $\implies$ $Trace\,(A) = 0 + 2 + 2 + 4 = 8$
Conclusion	Option (1), (2) and (4) are correct

TABLE 5.4.1: Solution

Positive Semi Definite Matrix	A $n \times n$ symmetric real matrix $\mathbf{M}$ is said to be positive semi definite if $\mathbf{x^T M x} \geq 0$ for all non-zero $\mathbf{x}$ in $\mathbb{R}^n$. Formally $\mathbf{M}$ is positive semi-definite $\iff \mathbf{x^T M x} \geq 0 \; \forall \; \mathbf{x} \in \mathbb{R}^n \backslash \{0\}$
Theorem	For a symmetric $n \times n$ matrix $\mathbf{M} \in \mathbf{L(V)}$, following are equivalent. 1). $\mathbf{x^T M x} \geq 0 \; \forall \; \mathbf{x} \in \mathbf{V}$. 2). All the eigenvalues of $\mathbf{M}$ are non-negative.

TABLE 5.5.1: Definition and Result used

Calculating eigen values of $\mathbf{A}$	Given $\mathbf{A} = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ Calculating, eigen values of $\mathbf{A}$, ie $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ $\implies \left\lvert \begin{pmatrix} 3-\lambda & 1 & 2 \\ 1 & 2-\lambda & 3 \\ 2 & 3 & 1-\lambda \end{pmatrix} \right\rvert = 0$ $\implies (3-\lambda)\left((2-\lambda)(1-\lambda) - 9\right) - 1\,(1-\lambda-6) + 2\,(3 - 2(2-\lambda)) = 0$ $\implies \lambda^3 - 6\lambda^2 - 3\lambda + 18 = 0$ $\implies \lambda_1 = 6,\ \lambda_2 = \sqrt{3}$ and $\lambda_3 = -\sqrt{3}$ Hence, $\mathbf{A}$ has exactly two positive eigen values.
Proving $\mathbf{x^T A x} < 0$ for some $\mathbf{x} \in \mathbb{R}^3$ using contradiction	Suppose $\mathbf{x^T A x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^3$. Then, by theorem above in definition section, matrix $\mathbf{A}$ is positive semi definite. Hence, all the eigen values of $\mathbf{A}$ non-negative, but this is not the case as one of eigen value is $\lambda_3 = -\sqrt{3}$. So, $\mathbf{x^T A x} \geq 0$ is not true for all $\mathbf{x} \in \mathbb{R}^3$. Similarly, as $\lambda_i \leq 0, \forall i$ is also not true, so $\mathbf{x^T A x} \leq 0$ is not true for all $\mathbf{x} \in \mathbb{R}^3$. Thus, $\mathbf{x^T A x} < 0$ for some $\mathbf{x} \in \mathbb{R}^3$.
Correct Options	Hence, correct options are (1) and (4).

TABLE 5.5.2: Solution

5.5. Let $\mathbf{A} = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ and $\mathbf{Q(X)} = \mathbf{X^T A X}$ for $\mathbf{X} \in \mathbb{R}^3$. Then

a) $\mathbf{A}$ has exactly two positive eigen values.

b) all the eigen values of $\mathbf{A}$ are positive.

c) $\mathbf{Q(X)} \geq 0 \; \forall \; \mathbf{X} \in \mathbb{R}^3$

d) $\mathbf{Q(X)} < 0$ for some $\mathbf{X} \in \mathbb{R}^3$

Solution: See Tables 5.5.1 and 5.5.2

5.6. Consider the matrix

$$A(x) = \begin{pmatrix} 1+x^2 & 7 & 11 \\ 3x & 2x & 4 \\ 8x & 17 & 13 \end{pmatrix}; x \in \mathbf{R}. \tag{5.6.1}$$

Then,

a) A(x) has eigenvalue 0 for some $x \in \mathbf{R}$.

b) 0 is not an eigenvalue of A(x) for any $x \in \mathbf{R}$.

c) A(x) has eigenvalue 0 $\forall x \in \mathbf{R}$.

d) A(x) is invertible $\forall x \in \mathbf{R}$.

Solution: Let $\lambda = 0$ be an eigenvalue. Hence,

$$\left|A - \lambda I\right| = 0 \tag{5.6.2}$$

$$\implies \left|A\right| = 0 \tag{5.6.3}$$

$$\implies \left|A\right| = \begin{vmatrix} 1+x^2 & 7 & 11 \\ 3x & 2x & 4 \\ 8x & 17 & 13 \end{vmatrix} = 0 \tag{5.6.4}$$

Performing row reduction we get,

$$\begin{vmatrix} 1+x^2 & 7 & 11 \\ 0 & \frac{2x^3-19x}{1+x^2} & \frac{4x^2-33x+4}{1+x^2} \\ 0 & 0 & \frac{26x^3-244x^2+538x-68}{2x^3-19x} \end{vmatrix} = 0 \tag{5.6.5}$$

$$\implies 26x^3 - 244x^2 + 538x - 68 = 0 \tag{5.6.6}$$

$$\implies x_1 = 6.01, x_2 = 3.23, x_3 = 0.13 \tag{5.6.7}$$

See Table 5.6.1

OPTIONS	Explanation
Option (b)	At the Values of x given by (5.6.7), eigen value $\lambda = 0$. Hence option (b) can't be correct.
Option (c)	If one of the eigenvalue is 0 for A(x) then, $\lvert A(x) \rvert = 0 \forall x \in R$. But from (5.6.7) we have concluded that $\lvert A \rvert = 0$ only for, $x_1 = 6.01, x_2 = 3.23, x_3 = 0.13$. Hence, Option (c) is incorrect.
Option (d)	Now for the values of x given by (5.6.7), $\lvert A \rvert = 0$. Hence it is not invertible $\forall x \in \mathbf{R}$ Hence Option (d) is incorrect.
Option (a)	Now clearly from above arguments A(x) has eigenvalue 0 for some $x \in R$ Hence Option (a) is Correct.

TABLE 5.6.1