Solutions: Linear Algebra by Hoffman and Kunze

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Contents

Abstract—**This book provides solutions to the Linear Algebra book by Hoffman and Kunze.**

1 Linear Equations

1.1 Fields and Linear Equations

1.1.1. Verify that the set of complex numbers numbers described in the form of c where x and y are rational is a sub-field of C.

Solution: Lets consider the set S = $\{x + y\sqrt{2}, x, y \in Q\}$, $S \subset C$ We must verify that S meets the following two conditions:

$$0, 1 \in S \tag{1.1.1.1}$$

$$a, b \in S, a + b, -a, ab, a^{-1} \in S \tag{1.1.1.2}$$

Throughout let

$$a = x + y\sqrt{2}, b = w + z\sqrt{2} \tag{1.1.1.3}$$

If

a)
$$x = 0, y = 0 \in Q, a = 0 + \sqrt{2}.0 = 0, 0 \in S \tag{1.1.1.4}$$

b)
$$x = 1, y = 0, a = 1 + \sqrt{2}.0 = 1, 1 \in S \tag{1.1.1.5}$$

c)
$$a + b = x + y\sqrt{2} + w + z\sqrt{2} = b + a \tag{1.1.1.6}$$

d)
$$-a = -x - y\sqrt{2}, x, y \in Qso - x, -y \in Q, a \in S \tag{1.1.1.7}$$

e)
$$ab = (x + y\sqrt{2})(w + z\sqrt{2}) = ba, ab \in S \tag{1.1.1.8}$$

f)
$$a^{-1}a = (x + y\sqrt{2})^{-1}(x + y\sqrt{2}) = 1, a^{-1} \in S \tag{1.1.1.9}$$

Hence (1.1.1.1) ,(1.1.1.2) is verfied . Therefore by considering the (1.1.1.1) and (1.1.1.2) we can say set complex numbers of given form $x + y\sqrt{2}$ is subfield of C .

1.1.2. Let $\mathbb{F}$ be the field of complex numbers. Are

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the following two systems of linear equations equivalent? If so, express each equation in each system as a linear combination of the equations in the other system.

$$\begin{aligned} x_1 - x_2 &= 0 \\ 2x_1 + x_2 &= 0 \end{aligned}$$

and

$$\begin{aligned} 3x_1 + x_2 &= 0 \\ x_1 + x_2 &= 0 \end{aligned}$$

Solution: The given system of linear equations can be written as,

$$\mathbf{Ax} = 0 \tag{1.1.2.1}$$

$$\implies \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{1.1.2.2}$$

$$\mathbf{Bx} = 0 \tag{1.1.2.3}$$

$$\implies \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{1.1.2.4}$$

Now we can obtain $\mathbf{B}$ from matrix $\mathbf{A}$ by performing elementary row operations given as,

$$\mathbf{B} = \mathbf{CA} \tag{1.1.2.5}$$

$$\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} = \mathbf{C} \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \tag{1.1.2.6}$$

where $\mathbf{C}$ is product of elementary matrices given as,

$$\begin{aligned} \mathbf{C} &= (\mathbf{E_7 E_6 E_5 E_4 E_3 E_2 E_1}) \\ &= \begin{pmatrix} 1 & 0 \\ \frac{1}{3} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{3} & \frac{4}{3} \\ \frac{-1}{3} & \frac{2}{3} \end{pmatrix} \end{aligned} \tag{1.1.2.7}$$

Now, performing elementary operations on the right side of $\mathbf{A}$ we obtain matrix $\mathbf{B}$ given as,

$$\mathbf{B} = \mathbf{AP} \tag{1.1.2.8}$$

$$\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \mathbf{P} \tag{1.1.2.9}$$

where, $\mathbf{P}$ is product of elementary matrices given by,

$$\begin{aligned} \mathbf{P} &= (\mathbf{E_1 E_2 E_3 E_4 E_5}) \\ &= \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{4}{3} & \frac{2}{3} \\ \frac{-5}{3} & \frac{-1}{3} \end{pmatrix} \end{aligned} \tag{1.1.2.10}$$

Similarly, $\mathbf{A}$ can be obtained from matrix $\mathbf{B}$ from (1.1.2.5) as,

$$\mathbf{A} = \mathbf{C}^{-1}\mathbf{B} \tag{1.1.2.11}$$

$$\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} = \mathbf{C}^{-1} \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \tag{1.1.2.12}$$

Matrix $\mathbf{C}$ is product of elementary matrices and hence invertible and is given as,

$$\begin{aligned} \mathbf{C}^{-1} &= \left(\mathbf{E_1}^{-1}\mathbf{E_2}^{-1}\mathbf{E_3}^{-1}\mathbf{E_4}^{-1}\mathbf{E_5}^{-1}\mathbf{E_6}^{-1}\mathbf{E_7}^{-1}\right) \\ &= \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{pmatrix} \\ &\quad \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{3}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{-1}{3} & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -2 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \end{aligned} \tag{1.1.2.13}$$

Matrix $\mathbf{A}$ can also be obtained from (1.1.2.8) given as,

$$\mathbf{A} = \mathbf{BP}^{-1} \tag{1.1.2.14}$$

$$\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \mathbf{P}^{-1} \tag{1.1.2.15}$$

where,

$$\begin{aligned} \mathbf{P}^{-1} &= \left(\mathbf{E_5}^{-1}\mathbf{E_4}^{-1}\mathbf{E_3}^{-1}\mathbf{E_2}^{-1}\mathbf{E_1}^{-1}\right) \\ &= \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{-1}{2} & -1 \\ \frac{5}{2} & 2 \end{pmatrix} \end{aligned} \tag{1.1.2.16}$$

Thus (1.1.2.4) can be obtained from (1.1.2.2) by multiplying it with matrix $\mathbf{C}$, and by inverse row operations (1.1.2.2) can be obtained back from (1.1.2.4) since $\mathbf{C}$ is product of elementary matrices and hence invertible.

Thus the two given homogeneous systems are row equivalent.

Now writing equations in matrix-vector form as,

$$3x_1 + x_2 = \begin{pmatrix}3 & 1\end{pmatrix}\mathbf{x} \tag{1.1.2.17}$$

$$\implies \begin{pmatrix}3 & 1\end{pmatrix}\mathbf{x} = \frac{1}{3}\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} + \frac{4}{3}\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} \tag{1.1.2.18}$$

$$x_1 + x_2 = \begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} \tag{1.1.2.19}$$

$$\implies \begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = \frac{-1}{3}\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} + \frac{2}{3}\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} \tag{1.1.2.20}$$

(1.1.2.18), (1.1.2.20) is same as multiplying $\mathbf{C}$ with $\mathbf{A}$ as it takes the linear combination of each rows of matrix $\mathbf{A}$ i.e, (1.1.2.6)

$$x_1 - x_2 = \begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} \tag{1.1.2.21}$$

$$\implies \begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = (1)\begin{pmatrix}3 & 1\end{pmatrix}\mathbf{x} + (-2)\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} \tag{1.1.2.22}$$

$$2x_1 + x_2 = \begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} \tag{1.1.2.23}$$

$$\implies \begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} = \frac{1}{2}\begin{pmatrix}3 & 1\end{pmatrix}\mathbf{x} + \frac{1}{2}\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} \tag{1.1.2.24}$$

(1.1.2.22), (1.1.2.24) is same as multiplying $\mathbf{C}^{-1}$ with $\mathbf{B}$ as it takes the linear combination of each rows of matrix $\mathbf{B}$ i.e, (1.1.2.12)
Thus each equation in each system can be expressed as a linear combination of the equations in the other system when they are equivalent.

1.1.3. Are the following two systems of linear equations equivalent?

$$\begin{aligned} -x_1 + x_2 + 4x_3 &= 0 \\ x_1 + 3x_2 + 8x_3 &= 0 \\ \frac{1}{2}x_1 + x_2 + \frac{5}{2}x_3 &= 0 \end{aligned} \tag{1.1.3.1}$$

Solution:

$$\begin{aligned} x_1 - x_3 &= 0 \\ x_2 + 3x_3 &= 0 \end{aligned} \tag{1.1.3.2}$$

System of linear equations in (1.1.3.1) can be expressed in matrix form as,

$$\mathbf{A}\mathbf{x} = 0 \tag{1.1.3.3}$$

$$\begin{pmatrix}-1 & 1 & 4 \\ 1 & 3 & 8 \\ \frac{1}{2} & 1 & \frac{5}{2}\end{pmatrix}\mathbf{x} = 0 \tag{1.1.3.4}$$

System of linear equations in (1.1.3.2) can be expressed in matrix form as,

$$\mathbf{B}\mathbf{x} = 0 \tag{1.1.3.5}$$

$$\begin{pmatrix}1 & 0 & -1 \\ 0 & 1 & 3\end{pmatrix}\mathbf{x} = 0 \tag{1.1.3.6}$$

Two system of linear equations are equivalent if one system can be expressed as a linear combination of other system.
Matrix $\mathbf{B}$ can be obtained from matrix $\mathbf{A}$ as,

$$\mathbf{B} = \mathbf{C}\mathbf{A} \tag{1.1.3.7}$$

$$\begin{pmatrix}1 & 0 & -1 \\ 0 & 1 & 3\end{pmatrix} = \mathbf{C}\begin{pmatrix}-1 & 1 & 4 \\ 1 & 3 & 8 \\ \frac{1}{2} & 1 & \frac{5}{2}\end{pmatrix} \tag{1.1.3.8}$$

$$\mathbf{C} = \begin{pmatrix}-1 & 1 & -2 \\ \frac{1}{2} & -\frac{1}{2} & 2\end{pmatrix} \tag{1.1.3.9}$$

Now, writing equations in matrix-vector form,

$$x_1 - x_3 = \begin{pmatrix}1 & 0 & -1\end{pmatrix}\mathbf{x}$$

$$\implies \begin{pmatrix}1 & 0 & -1\end{pmatrix}\mathbf{x} = -1\begin{pmatrix}-1 & 1 & 4\end{pmatrix}\mathbf{x} + 1\begin{pmatrix}1 & 3 & 8\end{pmatrix}\mathbf{x} - 2\begin{pmatrix}\frac{1}{2} & 1 & \frac{5}{2}\end{pmatrix}\mathbf{x} \tag{1.1.3.10}$$

$$x_2 + 3x_3 = \begin{pmatrix}0 & 1 & 3\end{pmatrix}\mathbf{x}$$

$$\implies \begin{pmatrix}0 & 1 & 3\end{pmatrix}\mathbf{x} = \frac{1}{2}\begin{pmatrix}-1 & 1 & 4\end{pmatrix}\mathbf{x} - \frac{1}{2}\begin{pmatrix}1 & 3 & 8\end{pmatrix}\mathbf{x} + 2\begin{pmatrix}\frac{1}{2} & 1 & \frac{5}{2}\end{pmatrix}\mathbf{x} \tag{1.1.3.11}$$

Equations (1.1.3.10) and (1.1.3.11) is same as multiplying $\mathbf{C}$ with $\mathbf{A}$ which is the linear combination of rows of matrix $\mathbf{A}$.
Thus each equation in second system can be expressed as linear combination of the equations in first system.
Therefore, the two system of linear equations are equivalent.

1.1.4. Let $\mathbb{F}$ be a set which contains exactly two elements,0 and 1.Define an addition and multiplication by tables. Verify that the set $\mathbb{F}$,

+	0	1
0	0	1
1	1	0

·	0	1
0	0	0
1	0	1

together with these two operations, is a field.
Solution:
To prove that ($\mathbb{F}$,+,·) is a field we need to satisfy the following,

a) + and · should be closed
   - For any a and b in $\mathbb{F}$, a+b $\in \mathbb{F}$ and a · b $\in \mathbb{F}$.For example 0+0=0 and 0·0=0.

b) + and · should be commutative
   - For any a and b in $\mathbb{F}$, a+b = b+a and a · b = b· a. For example 0+1=1+0 and 0· 1=1· 0.

c) + and · should be associative
   - For any a and b in $\mathbb{F}$, a+(b+c) = (a+b)+c and a· (b · c) = (a · b) · c. For example 0+(1+0)=(0+1)+0 and 0·(1·0)=(0·1)·0.

d) + and · operations should have an identity element
   - If we perform a + 0 then for any value of a from $\mathbb{F}$ the result will be a itself. Hence 0 is an identity element of + operation.If we perform a · 1 then for any value of a from $\mathbb{F}$ the result will be a itself. Hence 1 is an identity element of · operation.

e) $\forall$ a $\in \mathbb{F}$ there exists an additive inverse
   - For additive inverse to exist, $\forall$ a in $\mathbb{F}$ a+(-a)=0. For example. 1-1=0 and 0-0=0.

f) $\forall$ a $\in \mathbb{F}$ such that a is non zero there exists a multiplicative inverse
   - For multiplicative inverse to exist, $\forall$ a such that a is non zero in $\mathbb{F}$, $a \cdot a^{-1}$=1.For example $1 \cdot 1^{-1} = 1$.

g) + and · should hold distributive property
   - For any a,b and c in $\mathbb{F}$ the property a·(b+c)=a·b+a·c should always hold true.For example 0·(1+1)=0·1+0·1.

Since the above properties are satisfied we can say that ($\mathbb{F}$,+,·) is a field.

1.1.5. Prove that if two homogenous systems of linear equations in two unknowns have the same solutions, then they are equivalent.
Solution: Let the two systems of homogenous equations be

$$\mathbf{Ax} = \mathbf{0} \tag{1.1.5.1}$$
$$\mathbf{By} = \mathbf{0} \tag{1.1.5.2}$$

We can write

$$\mathbf{CAx} = \mathbf{0} \tag{1.1.5.3}$$
$$\mathbf{DBy} = \mathbf{0} \tag{1.1.5.4}$$

where **C** and **D** are product of elementary matrices that reduce **A** and **B** into their reduced row echelon forms $\mathbf{R_1}$ and $\mathbf{R_2}$
(1.1.5.3) and (1.1.5.4) imply

$$\mathbf{R_1x} = 0 \tag{1.1.5.5}$$
$$\mathbf{R_2y} = 0 \tag{1.1.5.6}$$

Given that they have same solution, we can write

$$\mathbf{R_1x} = 0 \tag{1.1.5.7}$$
$$\mathbf{R_2x} = 0 \tag{1.1.5.8}$$
$$\implies (\mathbf{R_1} - \mathbf{R_2})\mathbf{x} = 0 \tag{1.1.5.9}$$

Note that for a solution to exist, $\mathbf{R_1}$ and $\mathbf{R_2}$ can be either of matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{or} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.1.5.10}$$

Case 1 Let us assume that the solution is unique. The unique solution is

$$\mathbf{x} = \mathbf{0} \tag{1.1.5.11}$$

Since they have the same solution, both $\mathbf{R_1}, \mathbf{R_2}$ must have their rank as 2.
So,

$$\mathbf{R_1} = \mathbf{R_2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.1.5.12}$$

Case 2 Let us assume that (1.1.5.3),(1.1.5.4) have infinitely many solutions
So,

$$\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{B}) = 1 \tag{1.1.5.13}$$

equation (1.1.5.9) for solutions other than zero solution implies

$$\mathbf{R_1} = \mathbf{R_2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{1.1.5.14}$$

So, in both the cases, we have

$$\mathbf{R_1} = \mathbf{R_2} \tag{1.1.5.15}$$

$$\implies \mathbf{CA} = \mathbf{DB} \tag{1.1.5.16}$$

Since $\mathbf{C}, \mathbf{D}$ are product of elementary matrices, they are invertible.

$$\implies \mathbf{A} = \mathbf{C}^{-1}\mathbf{DB} \tag{1.1.5.17}$$

$$\mathbf{B} = \mathbf{D}^{-1}\mathbf{CA} \tag{1.1.5.18}$$

$$\text{Let } \mathbf{C}^{-1}\mathbf{D} = \mathbf{E} \tag{1.1.5.19}$$

where $\mathbf{E}$ is also a product of elementary matrices
(1.1.5.17) and (1.1.5.18) hence become

$$\mathbf{A} = \mathbf{EB} \tag{1.1.5.20}$$

$$\mathbf{B} = \mathbf{E}^{-1}\mathbf{A} \tag{1.1.5.21}$$

Hence the two systems of equations are equivalent.

1.1.6. Prove that each subfield of the field of complex number contains every rational number

Solution:

Complex Numbers: A complex number is a number that can be expressed in the form $a + bi$, where a and b are real numbers, and i represents the imaginary unit, satisfying the equation $i^2 = -1$.The set of complex numbers is denoted by $\mathbb{C}$

$$\mathbb{C} = \{(a, b) : a, b \in \mathbb{R}\} \tag{1.1.6.1}$$

Rational Numbers: A number in the form $\frac{p}{q}$, where both p and q(non-zero) are integers, is called a rational number.The set of rational numbers is dentoed by $\mathbb{Q}$ Let $\mathbb{Q}$ be the set of rational numbers.

$$\mathbb{Q} = \left\{ \frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{Z}_{\neq 0} \right\} \tag{1.1.6.2}$$

Let $\mathbb{C}$ be the field of complex numbers and given $\mathbb{F}$ be the subfield of field of complex numbers $\mathbb{C}$ Since $\mathbb{F}$ is the subfield , we could say that

$$0 \in \mathbb{F} \tag{1.1.6.3}$$

$$1 \in \mathbb{F} \tag{1.1.6.4}$$

Closed under addition: Here $\mathbb{F}$ is closed under addition since it is subfield

$$1 + 1 = 2 \in \mathbb{F} \tag{1.1.6.5}$$

$$1 + 1 + 1 = 3 \in \mathbb{F} \tag{1.1.6.6}$$

$$\vdots$$

$$1 + 1 + \cdots + 1(\text{p times}) = p \in \mathbb{F} \tag{1.1.6.7}$$

$$1 + 1 + \cdots + 1(\text{q times}) = q \in \mathbb{F} \tag{1.1.6.8}$$

By using the above property we could say that zero and other positive integers belongs to $\mathbb{F}$.Since p and q are integers we say,

$$p \in \mathbb{Z} \tag{1.1.6.9}$$

$$q \in \mathbb{Z} \tag{1.1.6.10}$$

Additive Inverse: Let x be the positive integer belong $\mathbb{F}$ and by additive inverse we could say,

$$\forall x \in \mathbb{F} \tag{1.1.6.11}$$

$$(-x) \in \mathbb{F} \tag{1.1.6.12}$$

Therefore field $\mathbb{F}$ contains every integers. Let n be a integer then,

$$n \in \mathbb{Z} \implies n \in \mathbb{F} \tag{1.1.6.13}$$

$$\mathbb{Z} \subseteq \mathbb{F} \tag{1.1.6.14}$$

Where $\mathbb{Z}$ is subset of $\mathbb{F}$ Multiplicative Inverse: Every element except zero in the subfield $\mathbb{F}$ has an multiplicative inverse. From equation (1.1.6.8), since q $\in \mathbb{F}$ we could say ,

$$\frac{1}{q} \in \mathbb{F} \quad \text{and } q \neq 0 \tag{1.1.6.15}$$

Closed under multiplication: Also, $\mathbb{F}$ is closed under multiplication and thus, from equation (1.1.6.7) and (1.1.6.15) we get ,

$$p \cdot \frac{1}{q} \in \mathbb{F} \tag{1.1.6.16}$$

$$\implies \frac{p}{q} \in \mathbb{F} \tag{1.1.6.17}$$

where , $p \in \mathbb{Z}$ and $q \in \mathbb{Z}_{\neq 0}$ (from equation (1.1.6.3) and (1.1.6.15)) Conclusion From (1.1.6.2) and (1.1.6.17) we could say ,

$$\mathbb{Q} \subseteq \mathbb{F} \tag{1.1.6.18}$$

From equation (1.1.6.18) we could say that each subfield of the field of complex number contains every rational number

Hence Proved

1.1.7. Prove that, each field of the characteristic zero contains a copy of the rational number field.
Solution: The characteristic of a field is defined to be the smallest number of times one must use the field's multiplicative identity (1) in a sum to get the additive identity. If this sum never reaches the additive identity (0), then the field is said to have characteristic zero.
Let $\mathbb{Q}$ be the rational number field. Hence,

$$0 \in \mathbb{Q} \qquad \text{[Additive Identity]} \tag{1.1.7.1}$$

$$1 \in \mathbb{Q} \qquad \text{[Multiplicative Identity]} \tag{1.1.7.2}$$

As addition is defined on $\mathbb{Q}$ hence we have,

$$1 \neq 0 \tag{1.1.7.3}$$

$$1 + 1 = 2 \neq 0 \tag{1.1.7.4}$$

And so on,

$$1 + 1 + \cdots + 1 = n \neq 0 \tag{1.1.7.5}$$

From the definition of characteristic of a field and from (1.1.7.3), (1.1.7.4) and so on upto (1.1.7.5), the rational number field, $\mathbb{Q}$ has characteristic 0.

1.2 Matrices and Elementary Row Operations

1.2.1. Find a row-reduced matrix which is row equivalent to,

$$\mathbf{A} = \begin{pmatrix} i & -(1+i) & 0 \\ 1 & -2 & 1 \\ 1 & 2i & -1 \end{pmatrix} \tag{1.2.1.1}$$

Solution: Step 1: Performing scaling operation to matrix **A** as $R_1 \leftarrow \frac{1}{i}R_1$ by scaling matrix D_1 given as,

$$\mathbf{D_1} = \begin{pmatrix} \frac{1}{i} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.2.1.2}$$

$$\mathbf{D_1A} = \begin{pmatrix} \frac{1}{i} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} i & -(1+i) & 0 \\ 1 & -2 & 1 \\ 1 & 2i & -1 \end{pmatrix} \tag{1.2.1.3}$$

$$\implies \mathbf{D_1A} = \begin{pmatrix} 1 & -1+i & 0 \\ 1 & -2 & 1 \\ 1 & 2i & -1 \end{pmatrix} \tag{1.2.1.4}$$

Step 2: Performing $R_2 \leftarrow R_2 - R_1$ and $R_3 \leftarrow R_3 - R_1$ given by elementary matrix $\mathbf{E_{31}E_{21}}$ on equation (1.2.1.4),

$$\mathbf{E_{31}E_{21}} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \tag{1.2.1.5}$$

$$\mathbf{E_{31}E_{21}D_1A} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1+i & 0 \\ 1 & -2 & 1 \\ 1 & 2i & -1 \end{pmatrix} \tag{1.2.1.6}$$

$$\implies \mathbf{A_1} = \mathbf{E_{31}E_{21}D_1A} = \begin{pmatrix} 1 & -1+i & 0 \\ 0 & -1-i & 1 \\ 0 & 1+i & -1 \end{pmatrix} \tag{1.2.1.7}$$

Step 3: Performing $R_2 \leftarrow \frac{-1}{1+i}R_2$ given by $\mathbf{D_2}$ on equation (1.2.1.7),

$$\mathbf{D_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2}(-1+i) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.2.1.8}$$

$$\mathbf{D_2A_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2}(-1+i) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1+i & 0 \\ 0 & -1-i & 1 \\ 0 & 1+i & -1 \end{pmatrix} \tag{1.2.1.9}$$

$$\implies \mathbf{A_2} = \mathbf{D_2A_1} = \begin{pmatrix} 1 & -1+i & 0 \\ 0 & 1 & \frac{1}{2}(-1+i) \\ 0 & 1+i & -1 \end{pmatrix} \tag{1.2.1.10}$$

Step 4: Performing $R_3 \leftarrow R_3 - (1+i)R_2$ given by $\mathbf{E_{32}}$ on equation (1.2.1.10),

$$\mathbf{E_{32}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -(1+i) & 1 \end{pmatrix} \tag{1.2.1.11}$$

$$\mathbf{E_{32}A_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1-i & 1 \end{pmatrix} \begin{pmatrix} 1 & -1+i & 0 \\ 0 & 1 & \frac{-1+i}{2} \\ 0 & 1+i & -1 \end{pmatrix} \tag{1.2.1.12}$$

$$\implies \mathbf{A_3} = \mathbf{E_{32}A_2} = \begin{pmatrix} 1 & -1+i & 0 \\ 0 & 1 & \frac{-1+i}{2} \\ 0 & 0 & 1 \end{pmatrix} \tag{1.2.1.13}$$

Step 5: Performing $R_1 \leftarrow R_1 - (-1+i)R_2$ given by $\mathbf{E_{12}}$ on equation (1.2.1.13),

$$\mathbf{E_{12}} = \begin{pmatrix} 1 & 1-i & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.2.1.14}$$

$$\mathbf{E_{12}A_3} = \begin{pmatrix} 1 & 1-i & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1+i & 0 \\ 0 & 1 & \frac{-1+i}{2} \\ 0 & 0 & 1 \end{pmatrix} \tag{1.2.1.15}$$

$$\implies \mathbf{A_4} = \mathbf{E_{12}A_3} = \begin{pmatrix} 1 & 0 & i \\ 0 & 1 & \frac{-1+i}{2} \\ 0 & 0 & 1 \end{pmatrix} \tag{1.2.1.16}$$

Step 6: Performing $R_1 \leftarrow R_1 - iR_3$ and $R_2 \leftarrow R_2 - \frac{-1+i}{2}R_3$ given by $\mathbf{E_{13}E_{23}}$ on equation (1.2.1.16),

$$\mathbf{E_{13}E_{23}} = \begin{pmatrix} 1 & 0 & -i \\ 0 & 1 & -\left(\frac{-1+i}{2}\right) \\ 0 & 0 & 1 \end{pmatrix} \tag{1.2.1.17}$$

$$\mathbf{E_{13}E_{23}A_4} = \begin{pmatrix} 1 & 0 & -i \\ 0 & 1 & -\left(\frac{-1+i}{2}\right) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & i \\ 0 & 1 & \frac{-1+i}{2} \\ 0 & 0 & 1 \end{pmatrix} \tag{1.2.1.18}$$

$$\implies \mathbf{A_5} = \mathbf{E_{13}E_{23}A_4} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.2.1.19}$$

$\therefore$ Row-reduced matrix of $\mathbf{A}$ given by equation (1.2.1.1) is,

$$\mathbf{A} = \begin{pmatrix} i & -1-i & 0 \\ 1 & -2 & 1 \\ 1 & 2i & -1 \end{pmatrix} \xleftrightarrow{RREF} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I} \tag{1.2.1.20}$$

1.2.2. Prove that the following two matrices are not row equivalent

$$\begin{pmatrix} 2 & 0 & 0 \\ a & -1 & 0 \\ b & c & 3 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 2 \\ -2 & 0 & -1 \\ 1 & 3 & 5 \end{pmatrix} \tag{1.2.2.1}$$

Solution: Call the first matrix $\mathbf{A}$ and the second matrix $\mathbf{B}$.

$$\mathbf{A}^T = \begin{pmatrix} 2 & a & b \\ 0 & -1 & c \\ 0 & 0 & 3 \end{pmatrix} \tag{1.2.2.2}$$

$\mathbf{A}^T$ is a upper triangular matrix with non-zero diagonal. Hence it has full rank = 3.

$$\mathbf{B}^T = \begin{pmatrix} 1 & -2 & 1 \\ 1 & 0 & 3 \\ 2 & -1 & 5 \end{pmatrix} \xleftrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - R_1} \begin{pmatrix} 1 & -2 & 1 \\ 0 & 2 & 2 \\ 0 & 3 & 3 \end{pmatrix} \tag{1.2.2.3}$$

$$\xleftrightarrow[R_2 \leftarrow R_2/2]{R_3 \leftarrow R_3/3} \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \tag{1.2.2.4}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - R_2} \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.2.2.5}$$

$\mathbf{B}^T$ is a upper triangular matrix with zero diagonal. Hence it doesn't have full rank. Therefore both matrices have different rank , so it cannot be row equivalent.

1.2.3. Let

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{1.2.3.1}$$

be a 2×2 matrix with complex entries. Suppose A is row-reduced and also that $a+b+c+d = 0$. Prove that there are exactly three such matrices.

Solution: A matrix is in row echelon form if it follows the following conditions

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry (i.e. left most nonzero entry) of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zero Row Reduced Echelon Form A matrix is in row reduced echelon form if it follows the following conditions
1. The matrix should be row echelon form
2. The leading entry in each nonzero row is 1.
3. Each leading 1 is the only nonzero entry in its column. Proof

Given ,

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{1.2.3.2}$$

Condition 1 : Matrix **A** should be in row-reduced echelon form
Condition 2 : $a + b + c + d = 0$ where a,b,c and d are the elements of the matrix **A**
Reducing the matrix **A** from equation (1.2.3.2)

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \xleftrightarrow{R_1 = \frac{1}{a}R_1} \begin{pmatrix} 1 & \frac{b}{a} \\ c & d \end{pmatrix} \tag{1.2.3.3}$$

$$\xleftrightarrow{R_2 = R_2 - cR_1} \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & \frac{ad-bc}{a} \end{pmatrix} \tag{1.2.3.4}$$

$$\xleftrightarrow{R_2 = \frac{a}{ad-bc}R_2} \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix} \tag{1.2.3.5}$$

$$\xleftrightarrow{R_1 = R_1 - \frac{b}{a}R_2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.2.3.6}$$

Case 1: Matrix **A** of Rank 2
From the equation (1.2.3.4),for the matrix to be in row reduced echelon form,

$$b = 0$$
$$a \neq 0$$
$$d = 1$$
$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.2.3.7}$$

For the condition 2 to get satisfied,

$$a + 0 + c + 1 = 0 \tag{1.2.3.8}$$
$$\implies a = -(c + 1) \tag{1.2.3.9}$$
$$\implies c \neq -1 \tag{1.2.3.10}$$

Both the condition gets satisfied and so exactly one matrix **A** can be formed of Rank 2 with given conditions
Case 2: Matrix **A** of Rank 1
From the equation (1.2.3.4),for the matrix to be in row reduced echelon form,

$$a \neq 0$$
$$d = 0$$
$$c = 0$$

For the condition 2 to get satisfied,

$$a + b + 0 + 0 = 0 \tag{1.2.3.11}$$
$$\implies b = -a \tag{1.2.3.12}$$

$$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \tag{1.2.3.13}$$

Both the condition gets satisfied and so exactly one matrix **A** can be formed of Rank 1 with given conditions
Case 3: Matrix **A** of Rank 0
From equation (1.2.3.2),for the matrix to be in row reduced echelon form,

$$a = 0$$
$$b = 0$$
$$c = 0$$
$$d = 0$$
$$\mathbf{A} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{1.2.3.14}$$

Both the condition gets satisfied and so exactly one matrix **A** can be formed of Rank 0 with given conditions
Therefore matrix **A** shown in equation (1.2.3.7),(1.2.3.13) and (1.2.3.14) are the exactly three such matrices that can be formed with given conditions.

1.2.4. Prove that the interchange of two rows of a matrix can be accomplished by a finite sequence of elementary row operations of the other two types.
Solution: Let **A** be a 3×3 matrix with having row vectors $\mathbf{a}_1$,$\mathbf{a}_2$ and $\mathbf{a}_3$.

$$\mathbf{A} = \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} \tag{1.2.4.1}$$

Let's exchange row $\mathbf{a}_1$ and $\mathbf{a}_2$. Let's call this elementary operation $\mathbf{E_1}$.

$$\mathbf{E}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.2.4.2}$$

$$\tag{1.2.4.3}$$

Now performing operation $\mathbf{E}_1$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{a}_2 \\ \mathbf{a}_1 \\ \mathbf{a}_3 \end{pmatrix} \tag{1.2.4.4}$$

Now, to prove that same matrix can be obtained by elementary operations let's call them $\mathbf{E_2}$ and $\mathbf{E_3}$.Now performing operation $\mathbf{E}_2$ by adding row 2 to row 1.

$$\begin{pmatrix}1 & 1 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}\mathbf{a}_1\\\mathbf{a}_2\\\mathbf{a}_3\end{pmatrix} = \begin{pmatrix}\mathbf{a}_1+\mathbf{a}_2\\\mathbf{a}_2\\\mathbf{a}_3\end{pmatrix} \quad (1.2.4.5)$$

Using elementary operation $\mathbf{E_2}$ we will subtract row 1 from row 2.

$$\begin{pmatrix}1 & 0 & 0\\-1 & 1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}\mathbf{a}_1+\mathbf{a}_2\\\mathbf{a}_2\\\mathbf{a}_3\end{pmatrix} = \begin{pmatrix}\mathbf{a}_1+\mathbf{a}_2\\-\mathbf{a}_1\\\mathbf{a}_3\end{pmatrix} \quad (1.2.4.6)$$

Using elementary operation $\mathbf{E_2}$ we will add row 2 to row 1.

$$\begin{pmatrix}1 & 1 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}\mathbf{a}_1+\mathbf{a}_2\\-\mathbf{a}_1\\\mathbf{a}_3\end{pmatrix} = \begin{pmatrix}\mathbf{a}_2\\-\mathbf{a}_1\\\mathbf{a}_3\end{pmatrix} \quad (1.2.4.7)$$

Using elementary operation $\mathbf{E_3}$ we will multiply row 2 by -1.

$$\begin{pmatrix}1 & 0 & 0\\0 & -1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}\mathbf{a}_2\\-\mathbf{a}_1\\\mathbf{a}_3\end{pmatrix} = \begin{pmatrix}\mathbf{a}_2\\\mathbf{a}_1\\\mathbf{a}_3\end{pmatrix} \quad (1.2.4.8)$$

Hence, we can say that,

$$\begin{pmatrix}1 & 0 & 0\\0 & -1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 1 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\-1 & 1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 1 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}\mathbf{a}_1\\\mathbf{a}_2\\\mathbf{a}_3\end{pmatrix} = \begin{pmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}\mathbf{a}_1\\\mathbf{a}_2\\\mathbf{a}_3\end{pmatrix} \quad (1.2.4.9)$$

Let us assume a matrix $\mathbf{A}$

$$\mathbf{A} = \begin{pmatrix}1 & 2 & 3\\0 & 1 & 0\\1 & 1 & 0\end{pmatrix} \quad (1.2.4.10)$$

Let's exchange row $\mathbf{a}_1$ and $\mathbf{a}_2$ by applying operation $\mathbf{E_1}$.

$$\begin{pmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\0 & 1 & 0\\1 & 1 & 0\end{pmatrix} = \begin{pmatrix}0 & 1 & 0\\1 & 2 & 3\\1 & 1 & 0\end{pmatrix} \quad (1.2.4.11)$$

Now, to prove that same matrix can be obtained by other two elementary operations.We will first perform elementary operation $\mathbf{E_2}$ by adding row 2 to row 1.

$$\begin{pmatrix}1 & 1 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\0 & 1 & 0\\1 & 1 & 0\end{pmatrix} = \begin{pmatrix}1 & 3 & 3\\0 & 1 & 0\\1 & 1 & 0\end{pmatrix} \quad (1.2.4.12)$$

Using elementary operation $\mathbf{E_2}$ we will subtract row 1 from row 2.

$$\begin{pmatrix}1 & 0 & 0\\-1 & 1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 3 & 3\\0 & 1 & 0\\1 & 1 & 0\end{pmatrix} = \begin{pmatrix}1 & 3 & 3\\-1 & -2 & -3\\1 & 1 & 0\end{pmatrix} \quad (1.2.4.13)$$

Using elementary operation $\mathbf{E_2}$ we will add row 2 to row 1.

$$\begin{pmatrix}1 & 1 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 3 & 3\\-1 & -2 & -3\\1 & 1 & 0\end{pmatrix} = \begin{pmatrix}0 & 1 & 0\\-1 & -2 & -3\\1 & 1 & 0\end{pmatrix} \quad (1.2.4.14)$$

Using elementary operation $\mathbf{E_3}$ we will multiply row 2 by -1.

$$\begin{pmatrix}1 & 0 & 0\\0 & -1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1 & 0\\-1 & -2 & -3\\1 & 1 & 0\end{pmatrix} = \begin{pmatrix}0 & 1 & 0\\1 & 2 & 3\\1 & 1 & 0\end{pmatrix} \quad (1.2.4.15)$$

Hence,we can say that,

$$\begin{pmatrix}1 & 0 & 0\\0 & -1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 1 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\-1 & 1 & 0\\0 & 0 & 1\end{pmatrix} \times\begin{pmatrix}1 & 1 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\0 & 1 & 0\\1 & 1 & 0\end{pmatrix} = \begin{pmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\0 & 1 & 0\\1 & 1 & 0\end{pmatrix} \quad (1.2.4.16)$$

1.2.5. Consider the system of equations $\mathbf{AX} = 0$ where

$$\mathbf{A} = \begin{pmatrix}a & b\\c & d\end{pmatrix}$$

is a 2×2 matrix over the field F. Prove the following -

- If every entry of $\mathbf{A}$ is 0, then every pair x_1 and x_2 is a solution of $\mathbf{AX} = 0$.
- If $ad - bc \neq 0$, then the system $\mathbf{AX} = 0$ has only the trivial solution $x_1 = x_2 = 0$
- If $ad - bc = 0$ and some entry of $\mathbf{A}$ is different from 0, then there is a solution x_1^0 and x_2^0 such that x_1 and x_2 is a solution if and only if there is a scalar y such that $x_1 = yx_1^0$ and $x_2 = yx_2^0$

Solution: Solution 1 If every entry of $\mathbf{A}$ is 0 then the equation $\mathbf{AX} = 0$ becomes,

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \tag{1.2.5.1}$$

$$\implies 0.x_1 + 0.x_2 = 0 \qquad \forall x_1, x_2 \in F \tag{1.2.5.2}$$

Hence proved, every pair x_1 and x_2 is a solution for the equation $\mathbf{AX} = 0$. Solution 2 **Case 1:** Let $a = 0$. Since $ad - bc \neq 0$. As $bc \neq 0$ therefore $b \neq 0$ and $c \neq 0$. Hence, we can perform row reduction on the augmented matrix of equation $\mathbf{AX}$=0 as follows,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & b & 0 \\ c & d & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{c} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c & d & 0 \\ 0 & b & 0 \end{pmatrix} \tag{1.2.5.3}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{b} \end{pmatrix} \begin{pmatrix} 1 & \frac{d}{c} & 0 \\ 0 & b & 0 \end{pmatrix} \tag{1.2.5.4}$$

$$= \begin{pmatrix} 1 & -\frac{d}{c} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{d}{c} & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{1.2.5.5}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{1.2.5.6}$$

Case 2: Let $a, b, c, d \neq 0$. Considering the following case,

$$\mathbf{AX} = \mathbf{u} \tag{1.2.5.7}$$

$$\implies \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{1.2.5.8}$$

Row Reducing the augmented matrix of (1.2.5.8) we get,

$$\begin{pmatrix} \frac{1}{a} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b & u_1 \\ c & d & u_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -c & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{b}{a} & \frac{u_1}{a} \\ c & d & u_2 \end{pmatrix} \tag{1.2.5.9}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & \frac{a}{ad-bc} \end{pmatrix} \begin{pmatrix} 1 & \frac{b}{a} & \frac{u_1}{a} \\ 0 & \frac{ad-bc}{a} & \frac{au_2-cu_1}{a} \end{pmatrix} \tag{1.2.5.10}$$

$$= \begin{pmatrix} 1 & -\frac{b}{a} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{b}{a} & \frac{u_1}{a} \\ 0 & 1 & \frac{au_2-cu_1}{ad-bc} \end{pmatrix} \tag{1.2.5.11}$$

$$= \begin{pmatrix} 1 & 0 & \frac{du_1-bu_2}{ad-bc} \\ 0 & 1 & \frac{au_2-cu_1}{ad-bc} \end{pmatrix} \tag{1.2.5.12}$$

From (1.2.5.12) we get,

$$x_1 = \frac{du_1 - bu_2}{ad - bc} \tag{1.2.5.13}$$

$$x_2 = \frac{au_2 - cu_1}{ad - bc} \tag{1.2.5.14}$$

Since $u_1 = 0$ and $u_2 = 0$ then from (1.2.5.13) and (1.2.5.14),

$$x_1 = 0 \tag{1.2.5.15}$$

$$x_2 = 0 \tag{1.2.5.16}$$

Hence we get,

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1.2.5.17}$$

In (1.2.5.6) and (1.2.5.17), we can see that $\mathbf{AX} = 0$ has only one trivial solution i.e $x_1 = x_2 = 0$ in all cases. Hence proved, the equation $\mathbf{AX}$=0 has only one trivial solution $x_1 = x_2 = 0$ Solution 3 **Case 1:** Let, $a \neq 0$ for $\mathbf{A}$. Given $ad - bc = 0$, we can perform row reduction on augmented matrix of equation $\mathbf{AX} = 0$ as follows,

$$\begin{pmatrix} \frac{1}{a} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b & 0 \\ c & d & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -c & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{b}{a} & 0 \\ c & d & 0 \end{pmatrix} \tag{1.2.5.18}$$

$$= \begin{pmatrix} 1 & \frac{b}{a} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad [\because ad - bc = 0] \tag{1.2.5.19}$$

Hence from (1.2.5.19), $\mathbf{AX} = 0$ if and only if

$$x_1 = -\frac{b}{a} x_2 \qquad [a \neq 0] \tag{1.2.5.20}$$

Letting $x_1^0 = -\frac{b}{a}$ and $x_2^0 = 1$ we get for $y = 1$,

$$x_1 = y x_1^0 \tag{1.2.5.21}$$

$$x_2 = y x_2^0 \tag{1.2.5.22}$$

which is a solution of the equation $\mathbf{AX} = 0$.
Case 2: Let, $b \neq 0$ for $\mathbf{A}$. Given $ad - bc = 0$, at first we multiply by elementary matrix to change the columns and the we can perform row reduction on augmented matrix of equation $\mathbf{AX} = 0$ as follows,

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} b & a & 0 \\ d & c & 0 \end{pmatrix} \tag{1.2.5.23}$$

Hence using the result obtained from (1.2.5.19) we can conclude for (1.2.5.23), $\mathbf{AX} = 0$ if and only if

$$x_2 = -\frac{a}{b}x_1 \qquad [b \neq 0] \tag{1.2.5.24}$$

Letting $x_2^0 = -\frac{a}{b}$ and $x_1^0 = 1$ we get for $y = 1$,

$$x_1 = yx_1^0 \tag{1.2.5.25}$$
$$x_2 = yx_2^0 \tag{1.2.5.26}$$

which is a solution of the equation $\mathbf{AX} = 0$.

Case 3: Let, $c \neq 0$ for $\mathbf{A}$. Given $ad - bc = 0$, we can perform row reduction on augmented matrix of equation $\mathbf{AX} = 0$ as follows,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b & 0 \\ c & d & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{c} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c & d & 0 \\ a & b & 0 \end{pmatrix} \tag{1.2.5.27}$$

$$= \begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{d}{c} & 0 \\ a & b & 0 \end{pmatrix} \tag{1.2.5.28}$$

$$= \begin{pmatrix} 1 & \frac{d}{c} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad [\because ad - bc = 0] \tag{1.2.5.29}$$

Hence from (1.2.5.29), $\mathbf{AX} = 0$ if and only if

$$x_1 = -\frac{d}{c}x_2 \qquad [a \neq 0] \tag{1.2.5.30}$$

Letting $x_1^0 = -\frac{d}{c}$ and $x_2^0 = 1$ we get for $y = 1$,

$$x_1 = yx_1^0 \tag{1.2.5.31}$$
$$x_2 = yx_2^0 \tag{1.2.5.32}$$

which is a solution of the equation $\mathbf{AX} = 0$.

Case 4: Let, $d \neq 0$ for $\mathbf{A}$. Given $ad - bc = 0$, at first we multiply by elementary matrix to change the columns and then we can perform row reduction on augmented matrix of equation $\mathbf{AX} = 0$ as follows,

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} b & a & 0 \\ d & c & 0 \end{pmatrix} \tag{1.2.5.33}$$

$$= \begin{pmatrix} d & c & 0 \\ b & a & 0 \end{pmatrix} \tag{1.2.5.34}$$

Hence using the result from (1.2.5.29) we can conclude for (1.2.5.34), $\mathbf{AX} = 0$ if and only if

$$x_2 = -\frac{c}{d}x_1 \qquad [a \neq 0] \tag{1.2.5.35}$$

Letting $x_2^0 = -\frac{c}{d}$ and $x_1^0 = 1$ we get for $y = 1$,

$$x_1 = yx_1^0 \tag{1.2.5.36}$$
$$x_2 = yx_2^0 \tag{1.2.5.37}$$

which is a solution of the equation $\mathbf{AX} = 0$.

1.2.6. Suppose $\mathbf{R}$ and $\mathbf{R}'$ are 2×3 row-reduced echelon matrices and that the system $\mathbf{RX}$=0 and $\mathbf{R}'\mathbf{X}$=0 have exactly the same solutions. Prove that $\mathbf{R} = \mathbf{R}'$.

Solution:

Since $\mathbf{R}$ and $\mathbf{R}'$ are 2×3 row-reduced echelon matrices they can be of following three types:-

a) Suppose matrix $\mathbf{R}$ has one non-zero row then $\mathbf{RX}$=0 will have two free variables.Since $\mathbf{R}'\mathbf{X}$=0 will have the exact same solution as $\mathbf{RX} = 0, \mathbf{R}'\mathbf{X}$=0 will also have two free variables. Thus $\mathbf{R}'$ have one non zero row. Now let's consider a matrix $\mathbf{A}$ with the first row as the non-zero row $\mathbf{R}$ and second row as the second row of $\mathbf{R}'$.

$$\mathbf{R} = \begin{pmatrix} 1 & a & b \\ 0 & 0 & 0 \end{pmatrix} \tag{1.2.6.1}$$
$$\mathbf{R}' = \begin{pmatrix} 1 & c & d \\ 0 & 0 & 0 \end{pmatrix} \tag{1.2.6.2}$$
$$\tag{1.2.6.3}$$

Let $\mathbf{X}$ satisfy

$$\mathbf{RX} = 0 \tag{1.2.6.4}$$
$$\begin{pmatrix} 1 & \mathbf{a}^T \end{pmatrix}\begin{pmatrix} x \\ \mathbf{y} \end{pmatrix} = 0 \tag{1.2.6.5}$$
$$x + \mathbf{a}^T\mathbf{y} = 0 \tag{1.2.6.6}$$

where

$$\mathbf{a} = \begin{pmatrix} a \\ b \end{pmatrix} \tag{1.2.6.7}$$

$$\mathbf{R}'\mathbf{X} = 0 \tag{1.2.6.8}$$
$$\begin{pmatrix} 1 & \mathbf{b}^T \end{pmatrix}\begin{pmatrix} x \\ \mathbf{y} \end{pmatrix} = 0 \tag{1.2.6.9}$$
$$x + \mathbf{b}^T\mathbf{y} = 0 \tag{1.2.6.10}$$

where

$$\mathbf{b} = \begin{pmatrix} c \\ d \end{pmatrix} \tag{1.2.6.11}$$

Subtracting (1.2.6.10) from (1.2.6.6),

$$x + \mathbf{a}^T\mathbf{y} - x - \mathbf{b}^T\mathbf{y} = 0 \tag{1.2.6.12}$$
$$(\mathbf{a}^T - \mathbf{b}^T)\mathbf{y} = 0 \tag{1.2.6.13}$$

Since $\mathbf{y}$ is a 2×1 vector,

$$\implies y_1\mathbf{a} - y_2\mathbf{b} = 0 \tag{1.2.6.14}$$

Which can be written as,

$$\mathbf{a} = k\mathbf{b} \tag{1.2.6.15}$$

where,k=$\frac{\mathbf{y}_2}{\mathbf{y}_1}$ assuming $\mathbf{y}_1 \neq 0$.Now,Substituting (1.2.6.15) in (1.2.6.6)

$$x + k\mathbf{b}^T\mathbf{y} = 0 \tag{1.2.6.16}$$

Comparing (1.2.6.16) with (1.2.6.10)

$$x + \mathbf{b}^T\mathbf{y} = 0 \tag{1.2.6.17}$$
$$x + k\mathbf{b}^T\mathbf{y} = 0 \tag{1.2.6.18}$$

Hence k=1 which means y_1=y_2 and from this we can say that $\mathbf{a}$=$\mathbf{b}$.If in the above case we take y_1=0 then

$$y_1\mathbf{a} - y_2\mathbf{b} = 0 \tag{1.2.6.19}$$
$$y_2\mathbf{b} = 0 \tag{1.2.6.20}$$

Hence for the (1.2.6.20) to be always true $\mathbf{b}$ should be zero.Now from (1.2.6.15) we will see that $\mathbf{a}$ will also be 0. Hence, $\mathbf{R}$=$\mathbf{R}'$

b) Let $\mathbf{R}$ and $\mathbf{R}'$ have all rows as non zero.

$$\mathbf{R} = \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & c \end{pmatrix} \tag{1.2.6.21}$$
$$\mathbf{R}' = \begin{pmatrix} 1 & 0 & e \\ 0 & 1 & f \end{pmatrix} \tag{1.2.6.22}$$

Let $\mathbf{X}$ satisfy

$$\mathbf{R}\mathbf{X} = 0 \tag{1.2.6.23}$$
$$\mathbf{X}^T\mathbf{R}^T = 0 \tag{1.2.6.24}$$

Here,

$$\mathbf{R} = \begin{pmatrix} \mathbf{I} & \mathbf{a} \end{pmatrix} \tag{1.2.6.25}$$
$$\mathbf{a} = \begin{pmatrix} b \\ c \end{pmatrix} \tag{1.2.6.26}$$
$$\mathbf{R}^T = \begin{pmatrix} \mathbf{I} \\ \mathbf{a}^T \end{pmatrix} \tag{1.2.6.27}$$

Let,

$$\mathbf{X}^T = \begin{pmatrix} \mathbf{y}^T & z \end{pmatrix} \tag{1.2.6.28}$$

where z is a scalar constant. Now,substituting (1.2.6.28) and (1.2.6.25) in (1.2.6.24)

$$\begin{pmatrix} \mathbf{y}^T & z \end{pmatrix}\begin{pmatrix} \mathbf{I} \\ \mathbf{a}^T \end{pmatrix} = 0 \tag{1.2.6.29}$$
$$\mathbf{y}^T + z\mathbf{a}^T = 0 \tag{1.2.6.30}$$

Now for,

$$\mathbf{R}'\mathbf{X} = 0 \tag{1.2.6.31}$$
$$\mathbf{X}^T\mathbf{R}'^T = 0 \tag{1.2.6.32}$$

Here,

$$\mathbf{R}' = \begin{pmatrix} \mathbf{I} & \mathbf{b} \end{pmatrix} \tag{1.2.6.33}$$
$$\mathbf{b} = \begin{pmatrix} e \\ f \end{pmatrix} \tag{1.2.6.34}$$

Let,

$$\mathbf{X}^T = \begin{pmatrix} \mathbf{y}^T & z \end{pmatrix} \tag{1.2.6.35}$$

where z is a scalar constant.Now,substituting (1.2.6.35) and (1.2.6.33) in (1.2.6.32)

$$\begin{pmatrix} \mathbf{y}^T & z \end{pmatrix}\begin{pmatrix} \mathbf{I} \\ \mathbf{b}^T \end{pmatrix} = 0 \tag{1.2.6.36}$$
$$\mathbf{y}^T + z\mathbf{b}^T = 0 \tag{1.2.6.37}$$

Subtracting (1.2.6.37) from (1.2.6.30)

$$\mathbf{y}^T + z\mathbf{a}^T - \mathbf{y}^T - z\mathbf{b}^T = 0 \tag{1.2.6.38}$$
$$(\mathbf{a}^T - \mathbf{b}^T)z = 0 \tag{1.2.6.39}$$
$$\mathbf{a}^T = \mathbf{b}^T \tag{1.2.6.40}$$

c) Suppose matrix $\mathbf{R}$ have all the rows as zero then $\mathbf{RX}$=0 will be satisfied for all values of $\mathbf{X}$. We know that $\mathbf{R}'\mathbf{X}$=0 will have the exact same solution as $\mathbf{RX}$=0 then we can say that for all values of $\mathbf{X}$=0 equation $\mathbf{R}'\mathbf{X}$=0 will be satisfied.Hence, $\mathbf{R}'$=$\mathbf{R}$=0.

1.3 Matrix Multiplication

1.3.1. Let

$$\mathbf{A} = \begin{pmatrix} 3 & -6 & 2 & -1 \\ -2 & 4 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 1 & -2 & 1 & 0 \end{pmatrix} \tag{1.3.1.1}$$

For which (y_1, y_2, y_3, y_4) does the system of equations $\mathbf{AX} = \mathbf{Y}$ have a solution ? **Solution:** Given ,

$$\mathbf{AX} = \mathbf{Y} \tag{1.3.1.2}$$

$$\begin{pmatrix} 3 & -6 & 2 & -1 \\ -2 & 4 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 1 & -2 & 1 & 0 \end{pmatrix} \mathbf{X} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} \tag{1.3.1.3}$$

Now we try to find the matrix $\mathbf{B}$ such that $\mathbf{BA}$ gives the row echelon form of matrix $\mathbf{A}$ Here,$\mathbf{B}$ is given by ,

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{2}{3} & 1 & 0 & 0 \\ -\frac{2}{7} & -\frac{3}{7} & 1 & 0 \\ 0 & \frac{1}{2} & -\frac{3}{2} & 1 \end{pmatrix} \tag{1.3.1.4}$$

$$\mathbf{BA} = \begin{pmatrix} 3 & -6 & 2 & -1 \\ 0 & 0 & \frac{7}{3} & \frac{7}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1.3.1.5}$$

Therefore, rank of matrix $\mathbf{A}$ is 2 Now $\mathbf{B}$ is expressed in terms of two block matrices

$$\mathbf{B} = \begin{pmatrix} \mathbf{B_1} \\ \mathbf{B_2} \end{pmatrix} \tag{1.3.1.6}$$

$$\mathbf{B_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{2}{3} & 1 & 0 & 0 \end{pmatrix} \tag{1.3.1.7}$$

$$\mathbf{B_2} = \begin{pmatrix} -\frac{2}{7} & -\frac{3}{7} & 1 & 0 \\ 0 & \frac{1}{2} & -\frac{3}{2} & 1 \end{pmatrix} \tag{1.3.1.8}$$

Multiplying matrix $\mathbf{B}$ to both sides on the equation (1.3.1.2), we get ,

$$\begin{pmatrix} \mathbf{B_1} \\ \mathbf{B_2} \end{pmatrix} \mathbf{AX} = \begin{pmatrix} \mathbf{B_1} \\ \mathbf{B_2} \end{pmatrix} \mathbf{Y} \tag{1.3.1.9}$$

We know that , matrix $\mathbf{A}$ is of rank 2 The augumented matrix of (1.3.1.9) is given by

$$\left(\begin{array}{c|c} \mathbf{B_1A} & \mathbf{B_1Y} \\ \mathbf{B_2A} & \mathbf{B_2Y} \end{array}\right) \tag{1.3.1.10}$$

$$\mathbf{B_1A} = \begin{pmatrix} 3 & -6 & 2 & -1 \\ 0 & 0 & \frac{7}{3} & \frac{7}{3} \end{pmatrix} \tag{1.3.1.11}$$

$$\mathbf{B_2A} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1.3.1.12}$$

Since $\mathbf{B_2A}$ is zero matrix and for the given system $\mathbf{AX} = \mathbf{Y}$ to have a solution,

$$\mathbf{B_2Y} = 0 \tag{1.3.1.13}$$

$$\begin{pmatrix} -\frac{2}{7} & -\frac{3}{7} & 1 & 0 \\ 0 & \frac{1}{2} & -\frac{3}{2} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = 0 \tag{1.3.1.14}$$

The augumented matrix of (1.3.1.14) is given by,

$$\left(\begin{array}{cccc|c} -\frac{2}{7} & -\frac{3}{7} & 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{3}{2} & 1 & 0 \end{array}\right) \tag{1.3.1.15}$$

By row reduction technique,

$$\xleftrightarrow{R_1 = -\frac{7}{2}R_1} \left(\begin{array}{cccc|c} 1 & \frac{3}{2} & -\frac{7}{2} & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{3}{2} & 1 & 0 \end{array}\right) \tag{1.3.1.16}$$

$$\xleftrightarrow{R_2 = 2R_2} \left(\begin{array}{cccc|c} 1 & \frac{3}{2} & -\frac{7}{2} & 0 & 0 \\ 0 & 1 & -3 & 2 & 0 \end{array}\right) \tag{1.3.1.17}$$

$$\xleftrightarrow{R_1 = R_1 - \frac{3}{2}R_2} \left(\begin{array}{cccc|c} 1 & 0 & 1 & -3 & 0 \\ 0 & 1 & -3 & 2 & 0 \end{array}\right) \tag{1.3.1.18}$$

Equation (1.3.1.14) can be modified as ,

$$\begin{pmatrix} 1 & 0 & 1 & -3 \\ 0 & 1 & -3 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = 0 \tag{1.3.1.19}$$

Here y_3 and y_4 are free variables
If $y_3 = a$ and $y_4 = b$,then the solution to the system of equation $\mathbf{AX} = \mathbf{Y}$ is given by,

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = a \begin{pmatrix} -1 \\ 3 \\ 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 3 \\ -2 \\ 0 \\ 1 \end{pmatrix} \tag{1.3.1.20}$$

One of the solution when $a = 1$ and $b = 2$ is given by ,

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 1 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 3 \\ -2 \\ 0 \\ 1 \end{pmatrix} \tag{1.3.1.21}$$

1.3.2. Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ matrices such that $\mathbf{AB} = \mathbf{I}$. Prove that $\mathbf{BA} = \mathbf{I}$. **Solution:** Let $\mathbf{BX} = 0$ be a system of linear equation with n unknowns and n equations as $\mathbf{B}$ is $n \times n$ matrix. Hence,

$$\mathbf{BX} = 0 \tag{1.3.2.1}$$
$$\implies \mathbf{A}(\mathbf{BX}) = 0 \tag{1.3.2.2}$$
$$\implies (\mathbf{AB})\mathbf{X} = 0 \tag{1.3.2.3}$$
$$\implies \mathbf{IX} = 0 \quad [\because \mathbf{AB} = \mathbf{I}] \tag{1.3.2.4}$$
$$\implies \mathbf{X} = 0 \tag{1.3.2.5}$$

From (1.3.2.5) since $\mathbf{X} = 0$ is the only solution of (1.3.2.1), hence $rank(\mathbf{B}) = n$. Which implies all columns of $\mathbf{B}$ are linearly independent. Hence $\mathbf{B}$ is invertible. Therefore, every left inverse of $\mathbf{B}$ is also a right inverse of $\mathbf{B}$. Hence there exists a $n \times n$ matrix $\mathbf{C}$ such that,

$$\mathbf{BC} = \mathbf{CB} = \mathbf{I} \tag{1.3.2.6}$$

Again given that $\mathbf{AB} = \mathbf{I}$. Hence,

$$\mathbf{AB} = \mathbf{I} \tag{1.3.2.7}$$
$$\implies \mathbf{ABC} = \mathbf{C} \tag{1.3.2.8}$$
$$\implies \mathbf{A}(\mathbf{BC}) = \mathbf{C} \tag{1.3.2.9}$$
$$\implies \mathbf{A} = \mathbf{C} \quad [\because \mathbf{BC} = \mathbf{I}] \tag{1.3.2.10}$$

Hence using (1.3.2.10) and (1.3.2.6) we can write,

$$\mathbf{BA} = \mathbf{I} \tag{1.3.2.11}$$

Hence Proved.

1.4 Invertible Matrices

1.4.1. Suppose $\mathbf{A}$ is a 2×1 matrix and $\mathbf{B}$ is 1×2 matrix.Prove that $\mathbf{C}$=$\mathbf{AB}$ is non invertible.
Solution: Let's take $\mathbf{A}$ and $\mathbf{B}$ to be non zero vectors. Now,we know that for $\mathbf{C}$ to be non invertible $\mathbf{Cx} = 0$ should have a non trivial solution.So,

$$\mathbf{Cx} = 0 \tag{1.4.1.1}$$
$$\implies \mathbf{ABx} = 0 \tag{1.4.1.2}$$

Here, we know that $\mathbf{B}$ is 1×2 matrix and $\mathbf{x}$ is 2×1 matrix then $\mathbf{Bx}$ will result to a scalar constant k.

$$\implies \mathbf{A}k = 0 \tag{1.4.1.3}$$

For (1.4.1.3) to be true k should be zero. We also know that $\mathbf{B}$ is 1×2 matrix i.e. rows are less than column hence,

$$\mathbf{Bx} = 0 \tag{1.4.1.4}$$

will have a non trivial solution. Hence, using (1.4.1.3) and (1.4.1.4) we can say,

$$\mathbf{ABx} = 0 \tag{1.4.1.5}$$

will have a non trivial solution so, $\mathbf{C}$ is non invertible.

1.4.2. Let $\mathbf{A}$ be an $n \times n$ (square) matrix, Prove the following two statements:

a) If $\mathbf{A}$ is invertible and $\mathbf{AB} = 0$ for some $n \times n$ matrix $\mathbf{B}$, then $\mathbf{B} = 0$.
b) If $\mathbf{A}$ is not invertible, then there exists an $n \times n$ matrix $\mathbf{B}$ such that $\mathbf{AB} = 0$ but $\mathbf{B} \neq 0$.

Solution:

a) Given $\mathbf{A}$ is an invertible matrix and $\mathbf{AB} = 0$ then,

$$\mathbf{AB} = 0 \tag{1.4.2.1}$$
$$\implies \mathbf{A}^{-1}(\mathbf{AB}) = 0 \tag{1.4.2.2}$$
$$\implies (\mathbf{A}^{-1}\mathbf{A})\mathbf{B} = 0 \tag{1.4.2.3}$$
$$\implies \mathbf{IB} = 0 \quad [\because \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}] \tag{1.4.2.4}$$
$$\implies \mathbf{B} = 0 \tag{1.4.2.5}$$

b) If $\mathbf{A}$ is not invertible, then there exists an $n \times n$ matrix $\mathbf{B}$ such that $\mathbf{AB} = 0$ but $\mathbf{B} \neq 0$. Since $\mathbf{A}$ is not invertible, $\mathbf{AX} = 0$ must have a non-trivial solution. Let the non-trivial solution be,

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \tag{1.4.2.6}$$

Let $\mathbf{B}$ which is an $n \times n$ matrix have all its columns as $\mathbf{y}$.

$$\mathbf{B} = \begin{pmatrix} \mathbf{y} & \mathbf{y} & \cdots & \mathbf{y} \end{pmatrix} \tag{1.4.2.7}$$

From equation (1.4.2.7), we can say that $\mathbf{B} \neq 0$ but $\mathbf{AB} = 0$