Solutions: Linear Algebra by Hoffman and Kunze

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Contents

***Abstract*—This book provides solutions to the Linear Algebra book by Hoffman and Kunze.**

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1 Linear Equations

1.1 Fields and Linear Equations

1.1.1. Verify that the set of complex numbers numbers described in the form of c where x and y are rational is a sub-field of C.
Solution: Lets consider the set S = $\{x + y\sqrt{2}, x, y \in Q\}$, $S \subset C$ We must verify that S meets the following two conditions:

$$0, 1 \in S \tag{1.1.1.1}$$

$$a, b \in S, a + b, -a, ab, a^{-1} \in S \tag{1.1.1.2}$$

Throughout let

$$a = x + y\sqrt{2}, b = w + z\sqrt{2} \tag{1.1.1.3}$$

If

a)
$$x = 0, y = 0 \in Q, a = 0 + \sqrt{2}.0 = 0, 0 \in S \tag{1.1.1.4}$$

b)
$$x = 1, y = 0, a = 1 + \sqrt{2}.0 = 1, 1 \in S \tag{1.1.1.5}$$

c)
$$a + b = x + y\sqrt{2} + w + z\sqrt{2} = b + a \tag{1.1.1.6}$$

d)

$$-a = -x - y\sqrt{2}, x, y \in Q so - x, -y \in Q, a \in S \quad (1.1.1.7)$$

e)

$$ab = (x + y\sqrt{2})(w + z\sqrt{2}) = ba, ab \in S \quad (1.1.1.8)$$

f)

$$a^{-1}a = (x + y\sqrt{2})^{-1}(x + y\sqrt{2}) = 1, a^{-1} \in S \quad (1.1.1.9)$$

Hence (1.1.1.1) ,(1.1.1.2) is verfied . Therefore by considering the (1.1.1.1) and (1.1.1.2) we can say set complex numbers of given form $x + y\sqrt{2}$ is subfield of **C** .

1.1.2. Let $\mathbb{F}$ be the field of complex numbers. Are the following two systems of linear equations equivalent? If so, express each equation in each system as a linear combination of the equations in the other system.

$$x_1 - x_2 = 0$$
$$2x_1 + x_2 = 0$$

and

$$3x_1 + x_2 = 0$$
$$x_1 + x_2 = 0$$

Solution: The given system of linear equations can be written as,

$$\mathbf{A}\mathbf{x} = 0 \quad (1.1.2.1)$$

$$\implies \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}\mathbf{x} = 0 \quad (1.1.2.2)$$

$$\mathbf{B}\mathbf{x} = 0 \quad (1.1.2.3)$$

$$\implies \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}\mathbf{x} = 0 \quad (1.1.2.4)$$

Now we can obtain **B** from matrix **A** by performing elementary row operations given as,

$$\mathbf{B} = \mathbf{C}\mathbf{A} \quad (1.1.2.5)$$

$$\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} = \mathbf{C}\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \quad (1.1.2.6)$$

where **C** is product of elementary matrices given as,

$$\mathbf{C} = (\mathbf{E_7}\mathbf{E_6}\mathbf{E_5}\mathbf{E_4}\mathbf{E_3}\mathbf{E_2}\mathbf{E_1})$$
$$= \begin{pmatrix} 1 & 0 \\ \frac{1}{3} & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \frac{2}{3} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \frac{1}{3} & \frac{4}{3} \\ \frac{-1}{3} & \frac{2}{3} \end{pmatrix} \quad (1.1.2.7)$$

Now, performing elementary operations on the right side of **A** we obtain matrix **B** given as,

$$\mathbf{B} = \mathbf{A}\mathbf{P} \quad (1.1.2.8)$$

$$\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}\mathbf{P} \quad (1.1.2.9)$$

where, **P** is product of elementary matrices given by,

$$\mathbf{P} = (\mathbf{E_1}\mathbf{E_2}\mathbf{E_3}\mathbf{E_4}\mathbf{E_5})$$
$$= \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \frac{4}{3} & \frac{2}{3} \\ \frac{-5}{3} & \frac{-1}{3} \end{pmatrix} \quad (1.1.2.10)$$

Similarly, **A** can be obtained from matrix **B** from (1.1.2.5) as,

$$\mathbf{A} = \mathbf{C}^{-1}\mathbf{B} \quad (1.1.2.11)$$

$$\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} = \mathbf{C}^{-1}\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \quad (1.1.2.12)$$

Matrix **C** is product of elementary matrices and hence invertible and is given as,

$$\mathbf{C}^{-1} = \left(\mathbf{E_1}^{-1}\mathbf{E_2}^{-1}\mathbf{E_3}^{-1}\mathbf{E_4}^{-1}\mathbf{E_5}^{-1}\mathbf{E_6}^{-1}\mathbf{E_7}^{-1}\right)$$
$$= \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \frac{3}{2} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{-1}{3} & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & -2 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad (1.1.2.13)$$

Matrix **A** can also be obtained from (1.1.2.8) given as,

$$\mathbf{A} = \mathbf{B}\mathbf{P}^{-1} \quad (1.1.2.14)$$

$$\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}\mathbf{P}^{-1} \quad (1.1.2.15)$$

where,

$$\mathbf{P}^{-1} = \left(\mathbf{E_5}^{-1}\mathbf{E_4}^{-1}\mathbf{E_3}^{-1}\mathbf{E_2}^{-1}\mathbf{E_1}^{-1}\right)$$
$$= \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \frac{-1}{2} & -1 \\ \frac{5}{2} & 2 \end{pmatrix} \quad (1.1.2.16)$$

Thus (1.1.2.4) can be obtained from (1.1.2.2) by multiplying it with matrix **C**, and by inverse row operations (1.1.2.2) can be obtained back from (1.1.2.4) since **C** is product of elementary matrices and hence invertible.

Thus the two given homogeneous systems are row equivalent.

Now writing equations in matrix-vector form as,

$$3x_1 + x_2 = \begin{pmatrix} 3 & 1 \end{pmatrix}\mathbf{x} \quad (1.1.2.17)$$
$$\implies \begin{pmatrix} 3 & 1 \end{pmatrix}\mathbf{x} = \frac{1}{3}\begin{pmatrix} 1 & -1 \end{pmatrix}\mathbf{x} + \frac{4}{3}\begin{pmatrix} 2 & 1 \end{pmatrix}\mathbf{x} \quad (1.1.2.18)$$
$$x_1 + x_2 = \begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} \quad (1.1.2.19)$$
$$\implies \begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} = \frac{-1}{3}\begin{pmatrix} 1 & -1 \end{pmatrix}\mathbf{x} + \frac{2}{3}\begin{pmatrix} 2 & 1 \end{pmatrix}\mathbf{x} \quad (1.1.2.20)$$

(1.1.2.18), (1.1.2.20) is same as multiplying **C** with **A** as it takes the linear combination of each rows of matrix **A** i.e, (1.1.2.6)

$$x_1 - x_2 = \begin{pmatrix} 1 & -1 \end{pmatrix}\mathbf{x} \quad (1.1.2.21)$$
$$\implies \begin{pmatrix} 1 & -1 \end{pmatrix}\mathbf{x} = (1)\begin{pmatrix} 3 & 1 \end{pmatrix}\mathbf{x} + (-2)\begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} \quad (1.1.2.22)$$
$$2x_1 + x_2 = \begin{pmatrix} 2 & 1 \end{pmatrix}\mathbf{x} \quad (1.1.2.23)$$
$$\implies \begin{pmatrix} 2 & 1 \end{pmatrix}\mathbf{x} = \frac{1}{2}\begin{pmatrix} 3 & 1 \end{pmatrix}\mathbf{x} + \frac{1}{2}\begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} \quad (1.1.2.24)$$

(1.1.2.22), (1.1.2.24) is same as multiplying $\mathbf{C}^{-1}$ with **B** as it takes the linear combination of each rows of matrix **B** i.e, (1.1.2.12)

Thus each equation in each system can be expressed as a linear combination of the equations in the other system when they are equivalent.

1.1.3. Are the following two systems of linear equations equivalent?

$$\begin{aligned} -x_1 + x_2 + 4x_3 &= 0 \\ x_1 + 3x_2 + 8x_3 &= 0 \\ \frac{1}{2}x_1 + x_2 + \frac{5}{2}x_3 &= 0 \end{aligned} \quad (1.1.3.1)$$

Solution:

$$\begin{aligned} x_1 - x_3 &= 0 \\ x_2 + 3x_3 &= 0 \end{aligned} \quad (1.1.3.2)$$

System of linear equations in (1.1.3.1) can be expressed in matrix form as,

$$\mathbf{A}\mathbf{x} = 0 \quad (1.1.3.3)$$
$$\begin{pmatrix} -1 & 1 & 4 \\ 1 & 3 & 8 \\ \frac{1}{2} & 1 & \frac{5}{2} \end{pmatrix}\mathbf{x} = 0 \quad (1.1.3.4)$$

System of linear equations in (1.1.3.2) can be expressed in matrix form as,

$$\mathbf{B}\mathbf{x} = 0 \quad (1.1.3.5)$$
$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \end{pmatrix}\mathbf{x} = 0 \quad (1.1.3.6)$$

Two system of linear equations are equivalent if one system can be expressed as a linear combination of other system.

Matrix **B** can be obtained from matrix **A** as,

$$\mathbf{B} = \mathbf{C}\mathbf{A} \quad (1.1.3.7)$$
$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \end{pmatrix} = \mathbf{C}\begin{pmatrix} -1 & 1 & 4 \\ 1 & 3 & 8 \\ \frac{1}{2} & 1 & \frac{5}{2} \end{pmatrix} \quad (1.1.3.8)$$
$$\mathbf{C} = \begin{pmatrix} -1 & 1 & -2 \\ \frac{1}{2} & -\frac{1}{2} & 2 \end{pmatrix} \quad (1.1.3.9)$$

Now, writing equations in matrix-vector form,

$$x_1 - x_3 = \begin{pmatrix} 1 & 0 & -1 \end{pmatrix}\mathbf{x}$$

$$\implies \begin{pmatrix} 1 & 0 & -1 \end{pmatrix}\mathbf{x} = -1\begin{pmatrix} -1 & 1 & 4 \end{pmatrix}\mathbf{x}$$
$$+ 1\begin{pmatrix} 1 & 3 & 8 \end{pmatrix}\mathbf{x} - 2\begin{pmatrix} \frac{1}{2} & 1 & \frac{5}{2} \end{pmatrix}\mathbf{x} \quad (1.1.3.10)$$

$$x_2 + 3x_3 = \begin{pmatrix} 0 & 1 & 3 \end{pmatrix}\mathbf{x}$$

$$\implies \begin{pmatrix}0 & 1 & 3\end{pmatrix}\mathbf{x} = \frac{1}{2}\begin{pmatrix}-1 & 1 & 4\end{pmatrix}\mathbf{x}$$
$$-\frac{1}{2}\begin{pmatrix}1 & 3 & 8\end{pmatrix}\mathbf{x} + 2\begin{pmatrix}\frac{1}{2} & 1 & \frac{5}{2}\end{pmatrix}\mathbf{x} \quad (1.1.3.11)$$

Equations (1.1.3.10) and (1.1.3.11) is same as multiplying **C** with **A** which is the linear combination of rows of matrix **A**.
Thus each equation in second system can be expressed as linear combination of the equations in first system.
Therefore, the two system of linear equations are equivalent.

1.1.4. Let $\mathbb{F}$ be the field of complex numbers. Are the following two systems of linear equations equivalent? If so, express each equation in each system as a linear combination of equations in other system. First system of equations:

$$2x_1 + (-1+i)x_2 + x_4 = 0 \quad (1.1.4.1)$$
$$3x_2 - 2ix_3 + 5x_4 = 0 \quad (1.1.4.2)$$

The second system of equations:

$$(1+\frac{i}{2})x_1 + 8x_2 - ix_3 - x_4 = 0 \quad (1.1.4.3)$$
$$\frac{2}{3}x_1 - \frac{1}{2}x_2 + x_3 + 7x_4 = 0 \quad (1.1.4.4)$$

Solution: Let $\mathbf{R_1}$ and $\mathbf{R_2}$ be the reduced row echelon forms of the augumented matrices of the following systems of homogeneous equations respectively.

$$\mathbf{AX} = \mathbf{0} \quad (1.1.4.5)$$
$$\mathbf{BX} = \mathbf{0} \quad (1.1.4.6)$$

Where **A** and **B** as follows

$$\mathbf{A} = \begin{pmatrix}2 & -1+i & 0 & 1\\ 0 & 3 & -2i & 5\end{pmatrix} \quad (1.1.4.7)$$
$$\mathbf{B} = \begin{pmatrix}1+\frac{i}{2} & 8 & -i & -1\\ \frac{2}{3} & \frac{-1}{2} & 1 & 7\end{pmatrix} \quad (1.1.4.8)$$

On performing elementary row operations on (1.1.4.7),

$$\mathbf{R_1} = \mathbf{CA} \quad (1.1.4.9)$$

where **C** is the product of all elementary matrices. Reducing the first system of linear equations, we get,

$$\mathbf{C} = \begin{pmatrix}1 & \frac{1-i}{2}\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & \frac{1}{3}\end{pmatrix}\begin{pmatrix}\frac{1}{2} & 0\\ 0 & 1\end{pmatrix} \quad (1.1.4.10)$$
$$\mathbf{R_1} = \begin{pmatrix}1 & 0 & \frac{-1-i}{3} & \frac{4}{3}-\frac{5i}{6}\\ 0 & 1 & \frac{-2i}{3} & \frac{5}{3}\end{pmatrix} \quad (1.1.4.11)$$

On performing elementary row operations on (1.1.4.8),

$$\mathbf{R_2} = \mathbf{DA} \quad (1.1.4.12)$$

where **D** is the product of all elementary matrices. Reducing the second system of linear equations, we get,

$$\mathbf{D} = \begin{pmatrix}\frac{4}{5}(1-\frac{i}{2}) & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \frac{-2}{3} & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & \frac{-6(143+43i)}{4909}\end{pmatrix}\begin{pmatrix}1 & \frac{16(-2+i)}{5}\\ 0 & 1\end{pmatrix} \quad (1.1.4.13)$$

$$\mathbf{R_2} = \begin{pmatrix}1 & 0 & \frac{6702}{4909}-\frac{708i}{4909} & \frac{46620}{4909}-\frac{1998i}{4909}\\ 0 & 1 & \frac{-2(441+472i)}{4909} & \frac{-2(3283+1332i)}{4909}\end{pmatrix} \quad (1.1.4.14)$$

From the equations (1.1.4.11) and (1.1.4.14), we can say that

$$\mathbf{R_1} \neq \mathbf{R_2} \quad (1.1.4.15)$$

Hence the given systems of linear equations are not equivalent.

1.1.5. Let $\mathbb{F}$ be a set which contains exactly two elements,0 and 1.Define an addition and multiplication by tables. Verify that the set $\mathbb{F}$,

+	0	1
0	0	1
1	1	0

·	0	1
0	0	0
1	0	1

together with these two operations, is a field.
Solution:
To prove that ($\mathbb{F}$,+,·) is a field we need to satisfy the following,

a) + and · should be closed
   - For any a and b in $\mathbb{F}$, a+b $\in \mathbb{F}$ and a · b $\in \mathbb{F}$.For example 0+0=0 and 0·0=0.

b) + and · should be commutative

- For any a and b in $\mathbb{F}$, a+b = b+a and a $\cdot$ b = b$\cdot$ a. For example 0+1=1+0 and 0$\cdot$ 1=1$\cdot$ 0.

c) + and $\cdot$ should be associative

- For any a and b in $\mathbb{F}$, a+(b+c) = (a+b)+c and a$\cdot$ (b $\cdot$ c) = (a $\cdot$ b) $\cdot$ c. For example 0+(1+0)=(0+1)+0 and 0$\cdot$(1$\cdot$0)=(0$\cdot$1)$\cdot$0.

d) + and $\cdot$ operations should have an identity element

- If we perform a + 0 then for any value of a from $\mathbb{F}$ the result will be a itself. Hence 0 is an identity element of + operation.If we perform a $\cdot$ 1 then for any value of a from $\mathbb{F}$ the result will be a itself. Hence 1 is an identity element of $\cdot$ operation.

e) $\forall$ a $\in \mathbb{F}$ there exists an additive inverse

- For additive inverse to exist, $\forall$ a in $\mathbb{F}$ a+(-a)=0. For example. 1-1=0 and 0-0=0.

f) $\forall$ a $\in \mathbb{F}$ such that a is non zero there exists a multiplicative inverse

- For multiplicative inverse to exist, $\forall$ a such that a is non zero in $\mathbb{F}$, $a \cdot a^{-1}$=1.For example $1 \cdot 1^{-1} = 1$.

g) + and $\cdot$ should hold distributive property

- For any a,b and c in $\mathbb{F}$ the property a$\cdot$(b+c)=a$\cdot$b+a$\cdot$c should always hold true.For example 0$\cdot$(1+1)=0$\cdot$1+0$\cdot$1.

Since the above properties are satisfied we can say that ($\mathbb{F}$,+,$\cdot$) is a field.

1.1.6. Prove that if two homogenous systems of linear equations in two unknowns have the same solutions, then they are equivalent.

Solution: Let the two systems of homogenous equations be

$$\mathbf{A}\mathbf{x} = \mathbf{0} \tag{1.1.6.1}$$
$$\mathbf{B}\mathbf{y} = \mathbf{0} \tag{1.1.6.2}$$

We can write

$$\mathbf{C}\mathbf{A}\mathbf{x} = \mathbf{0} \tag{1.1.6.3}$$
$$\mathbf{D}\mathbf{B}\mathbf{y} = \mathbf{0} \tag{1.1.6.4}$$

where $\mathbf{C}$ and $\mathbf{D}$ are product of elementary matrices that reduce $\mathbf{A}$ and $\mathbf{B}$ into their reduced row echelon forms $\mathbf{R_1}$ and $\mathbf{R_2}$

(1.1.6.3) and (1.1.6.4) imply

$$\mathbf{R_1}\mathbf{x} = 0 \tag{1.1.6.5}$$
$$\mathbf{R_2}\mathbf{y} = 0 \tag{1.1.6.6}$$

Given that they have same solution, we can write

$$\mathbf{R_1}\mathbf{x} = 0 \tag{1.1.6.7}$$
$$\mathbf{R_2}\mathbf{x} = 0 \tag{1.1.6.8}$$
$$\implies (\mathbf{R_1} - \mathbf{R_2})\mathbf{x} = 0 \tag{1.1.6.9}$$

Note that for a solution to exist, $\mathbf{R_1}$ and $\mathbf{R_2}$ can be either of matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{or} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.1.6.10}$$

Case 1 Let us assume that the solution is unique. The unique solution is

$$\mathbf{x} = \mathbf{0} \tag{1.1.6.11}$$

Since they have the same solution, both $\mathbf{R_1}, \mathbf{R_2}$ must have their rank as 2.
So,

$$\mathbf{R_1} = \mathbf{R_2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.1.6.12}$$

Case 2 Let us assume that (1.1.6.3),(1.1.6.4) have infinitely many solutions
So,

$$\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{B}) = 1 \tag{1.1.6.13}$$

equation (1.1.6.9) for solutions other than zero solution implies

$$\mathbf{R_1} = \mathbf{R_2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{1.1.6.14}$$

So, in both the cases, we have

$$\mathbf{R_1} = \mathbf{R_2} \tag{1.1.6.15}$$
$$\implies \mathbf{C}\mathbf{A} = \mathbf{D}\mathbf{B} \tag{1.1.6.16}$$

Since $\mathbf{C}, \mathbf{D}$ are product of elementary matrices, they are invertible.

$$\implies \mathbf{A} = \mathbf{C}^{-1}\mathbf{D}\mathbf{B} \tag{1.1.6.17}$$
$$\mathbf{B} = \mathbf{D}^{-1}\mathbf{C}\mathbf{A} \tag{1.1.6.18}$$
$$\text{Let } \mathbf{C}^{-1}\mathbf{D} = \mathbf{E} \tag{1.1.6.19}$$

where $\mathbf{E}$ is also a product of elementary matrices

(1.1.6.17) and (1.1.6.18) hence become

$$\mathbf{A} = \mathbf{E}\mathbf{B} \tag{1.1.6.20}$$

$$\mathbf{B} = \mathbf{E}^{-1}\mathbf{A} \tag{1.1.6.21}$$

Hence the two systems of equations are equivalent.

1.1.7. Prove that each subfield of the field of complex number contains every rational number

Solution:

Complex Numbers: A complex number is a number that can be expressed in the form $a + bi$, where a and b are real numbers, and i represents the imaginary unit, satisfying the equation $i^2 = -1$.The set of complex numbers is denoted by $\mathbb{C}$

$$\mathbb{C} = \{(a, b) : a, b \in \mathbb{R}\} \tag{1.1.7.1}$$

Rational Numbers: A number in the form $\frac{p}{q}$, where both p and q(non-zero) are integers, is called a rational number.The set of rational numbers is dentoed by $\mathbb{Q}$ Let $\mathbb{Q}$ be the set of rational numbers.

$$\mathbb{Q} = \left\{\frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{Z}_{\neq 0}\right\} \tag{1.1.7.2}$$

Let $\mathbb{C}$ be the field of complex numbers and given $\mathbb{F}$ be the subfield of field of complex numbers $\mathbb{C}$ Since $\mathbb{F}$ is the subfield , we could say that

$$0 \in \mathbb{F} \tag{1.1.7.3}$$

$$1 \in \mathbb{F} \tag{1.1.7.4}$$

Closed under addition: Here $\mathbb{F}$ is closed under addition since it is subfield

$$1 + 1 = 2 \in \mathbb{F} \tag{1.1.7.5}$$

$$1 + 1 + 1 = 3 \in \mathbb{F} \tag{1.1.7.6}$$

$$\vdots$$

$$1 + 1 + \cdots + 1(\text{p times}) = p \in \mathbb{F} \tag{1.1.7.7}$$

$$1 + 1 + \cdots + 1(\text{q times}) = q \in \mathbb{F} \tag{1.1.7.8}$$

By using the above property we could say that zero and other positive integers belongs to $\mathbb{F}$.Since p and q are integers we say,

$$p \in \mathbb{Z} \tag{1.1.7.9}$$

$$q \in \mathbb{Z} \tag{1.1.7.10}$$

Additive Inverse: Let x be the positive integer belong $\mathbb{F}$ and by additive inverse we could say,

$$\forall x \in \mathbb{F} \tag{1.1.7.11}$$

$$(-x) \in \mathbb{F} \tag{1.1.7.12}$$

Therefore field $\mathbb{F}$ contains every integers. Let n be a integer then,

$$n \in \mathbb{Z} \implies n \in \mathbb{F} \tag{1.1.7.13}$$

$$\mathbb{Z} \subseteq \mathbb{F} \tag{1.1.7.14}$$

Where $\mathbb{Z}$ is subset of $\mathbb{F}$ Multiplicative Inverse: Every element except zero in the subfield $\mathbb{F}$ has an multiplicative inverse. From equation (1.1.7.8), since q $\in \mathbb{F}$ we could say ,

$$\frac{1}{q} \in \mathbb{F} \quad \text{and } q \neq 0 \tag{1.1.7.15}$$

Closed under multiplication: Also, $\mathbb{F}$ is closed under multiplication and thus, from equation (1.1.7.7) and (1.1.7.15) we get ,

$$p \cdot \frac{1}{q} \in \mathbb{F} \tag{1.1.7.16}$$

$$\implies \frac{p}{q} \in \mathbb{F} \tag{1.1.7.17}$$

where , $p \in \mathbb{Z}$ and $q \in \mathbb{Z}_{\neq 0}$ (from equation (1.1.7.3) and (1.1.7.15)) Conclusion From (1.1.7.2) and (1.1.7.17) we could say ,

$$\mathbb{Q} \subseteq \mathbb{F} \tag{1.1.7.18}$$

From equation (1.1.7.18) we could say that each subfield of the field of complex number contains every rational number

Hence Proved

1.1.8. Prove that, each field of the characteristic zero contains a copy of the rational number field.

Solution: The characteristic of a field is defined to be the smallest number of times one must use the field's multiplicative identity (1) in a sum to get the additive identity. If this sum never reaches the additive identity (0), then the field is said to have characteristic zero.

Let $\mathbb{Q}$ be the rational number field. Hence,

$$0 \in \mathbb{Q} \qquad \text{[Additive Identity]} \tag{1.1.8.1}$$

$$1 \in \mathbb{Q} \qquad \text{[Multiplicative Identity]} \tag{1.1.8.2}$$

As addition is defined on $\mathbb{Q}$ hence we have,

$$1 \neq 0 \tag{1.1.8.3}$$

$$1 + 1 = 2 \neq 0 \tag{1.1.8.4}$$

And so on,

$$1 + 1 + \cdots + 1 = n \neq 0 \tag{1.1.8.5}$$

From the definition of characteristic of a field and from (1.1.8.3), (1.1.8.4) and so on upto (1.1.8.5), the rational number field, $\mathbb{Q}$ has characteristic 0.

1.2 Matrices and Elementary Row Operations

1.2.1. Find all solutions to the system of equations

$$\begin{aligned}(1-i)\,x_1 - ix_2 &= 0\\ 2x_1 + (1-i)\,x_2 &= 0\end{aligned} \tag{1.2.1.1}$$

Solution: System of Linear Equations (1.2.1.1) can be expressed in matrix form as,

$$\mathbf{Ax} = 0 \tag{1.2.1.2}$$

$$\begin{pmatrix}1-i & -i\\ 2 & 1-i\end{pmatrix}\mathbf{x} = 0 \tag{1.2.1.3}$$

By row reduction ,

$$\begin{pmatrix}1-i & -i\\ 2 & 1-i\end{pmatrix} \xleftrightarrow[R_1\leftarrow R_1/2]{R_1\leftrightarrow R_2} \begin{pmatrix}1 & \frac{1-i}{2}\\ 1-i & -i\end{pmatrix} \tag{1.2.1.4}$$

$$\xleftrightarrow{R_2\leftarrow R_2-(1-i)R_1} \begin{pmatrix}1 & \frac{1-i}{2}\\ 0 & 0\end{pmatrix} \tag{1.2.1.5}$$

$$\begin{pmatrix}1 & \frac{1-i}{2}\end{pmatrix}\mathbf{x} = 0 \tag{1.2.1.6}$$

$$\begin{pmatrix}1 & \frac{1-i}{2}\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} = 0 \tag{1.2.1.7}$$

$$x_1 = -\frac{1-i}{2}x_2 \tag{1.2.1.8}$$

$$\mathbf{x} = \begin{pmatrix}-\frac{1-i}{2}x_2\\ x_2\end{pmatrix} \tag{1.2.1.9}$$

$$\implies \mathbf{x} = x_2\begin{pmatrix}-\frac{1-i}{2}\\ 1\end{pmatrix} \tag{1.2.1.10}$$

1.2.2. If

$$A = \begin{pmatrix}3 & -1 & 2\\ 2 & 1 & 1\\ 1 & -3 & 0\end{pmatrix} \tag{1.2.2.1}$$

Find all solutions of $AX = 0$ by row reducing A.

Solution: For the given equation $AX = 0$ can be defined as follows:

$$\begin{pmatrix}3 & -1 & 2\\ 2 & 1 & 1\\ 1 & -3 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 0\end{pmatrix} \tag{1.2.2.2}$$

Now, we can apply Row Reduction Methodology of matrix A :

$$\begin{pmatrix}3 & -1 & 2 & 0\\ 2 & 1 & 1 & 0\\ 1 & -3 & 0 & 0\end{pmatrix} \xleftrightarrow{R_1=R_1+R_2} \begin{pmatrix}5 & 0 & 3 & 0\\ 2 & 1 & 1 & 0\\ 1 & -3 & 0 & 0\end{pmatrix} \tag{1.2.2.3}$$

$$\xleftrightarrow{R_2=R_2-2R_3} \begin{pmatrix}5 & 0 & 3 & 0\\ 0 & 7 & 1 & 0\\ 1 & -3 & 0 & 0\end{pmatrix} \tag{1.2.2.4}$$

$$\xleftrightarrow{R_3=R_3-\frac{1}{5}R_1} \begin{pmatrix}5 & 0 & 3 & 0\\ 0 & 7 & 1 & 0\\ 0 & -3 & -\frac{3}{5} & 0\end{pmatrix} \tag{1.2.2.5}$$

$$\xleftrightarrow{R_1=\frac{1}{5}R_1} \begin{pmatrix}1 & 0 & \frac{3}{5} & 0\\ 0 & 7 & 1 & 0\\ 0 & -3 & -\frac{3}{5} & 0\end{pmatrix} \tag{1.2.2.6}$$

$$\xleftrightarrow{R_2=\frac{1}{7}R_2} \begin{pmatrix}1 & 0 & \frac{3}{5} & 0\\ 0 & 1 & \frac{1}{7} & 0\\ 0 & -3 & -\frac{3}{5} & 0\end{pmatrix} \tag{1.2.2.7}$$

$$\xleftrightarrow{R_3=R_3+3R_2} \begin{pmatrix}1 & 0 & \frac{3}{5} & 0\\ 0 & 1 & \frac{1}{7} & 0\\ 0 & 0 & -\frac{6}{35} & 0\end{pmatrix} \tag{1.2.2.8}$$

$$\xleftrightarrow{R_3=-\frac{35}{6}R_3} \begin{pmatrix}1 & 0 & \frac{3}{5} & 0\\ 0 & 1 & \frac{1}{7} & 0\\ 0 & 0 & 1 & 0\end{pmatrix} \tag{1.2.2.9}$$

$$\xleftrightarrow{R_2=R_2-\frac{1}{7}R_3} \begin{pmatrix}1 & 0 & \frac{3}{5} & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{pmatrix} \tag{1.2.2.10}$$

$$\xleftrightarrow{R_1=R_1-\frac{3}{5}R_3} \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{pmatrix} \tag{1.2.2.11}$$

So, as we can see the only solution we got after row reducing of matrix A is zero vector. Thus, the solution is :

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{1.2.2.12}$$

1.2.3.

$$\mathbf{A} = \begin{pmatrix} 6 & -4 & 0 \\ 4 & -2 & 0 \\ -1 & 0 & 3 \end{pmatrix} \tag{1.2.3.1}$$

Find all solutions of $\mathbf{AX} = 2\mathbf{X}$and all solutions of $\mathbf{AX} = 3\mathbf{X}$.The symbol c$\mathbf{X}$ denotes the matrix each entry of which is c times corresponding entry.

Solution:

$$\mathbf{A} = \begin{pmatrix} 6 & -4 & 0 \\ 4 & -2 & 0 \\ -1 & 0 & 3 \end{pmatrix} \tag{1.2.3.2}$$

To calculate solution of $\mathbf{AX} = 2\mathbf{X}$and all solutions of $\mathbf{AX} = 3\mathbf{X}$we calculate eigen values of $\mathbf{A}$:

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{X} = 0 \tag{1.2.3.3}$$

Substituting values in (1.2.3.3),

$$\begin{pmatrix} 6-\lambda & -4 & 0 \\ 4 & -2-\lambda & 0 \\ -1 & 0 & 3-\lambda \end{pmatrix}\mathbf{X} = 0 \tag{1.2.3.4}$$

Simplifying:

$$\begin{pmatrix} 6-\lambda & -4 & 0 \\ 4 & -2-\lambda & 0 \\ -1 & 0 & 3-\lambda \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 - R_2}$$

$$\begin{pmatrix} 2-\lambda & -2+\lambda & 0 \\ 4 & -2-\lambda & 0 \\ -1 & 0 & 3-\lambda \end{pmatrix} \tag{1.2.3.5}$$

Taking $(3-\lambda)$ and $(2-\lambda)$common from C_3 and R_1

$$(3-\lambda)(2-\lambda)\begin{pmatrix} 1 & -1 & 0 \\ 4 & -2-\lambda & 0 \\ -1 & 0 & 1 \end{pmatrix} \tag{1.2.3.6}$$

$$\begin{pmatrix} 1 & -1 & 0 \\ 4 & -2-\lambda & 0 \\ -1 & 0 & 1 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - 4R_1} \begin{pmatrix} 1 & -1 & 0 \\ 0 & -\lambda+2 & 0 \\ -1 & 0 & 1 \end{pmatrix} \tag{1.2.3.7}$$

Taking $(2-\lambda)$ common from R_2:

$$(2-\lambda)^2(3-\lambda)\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \tag{1.2.3.8}$$

Eigen values are:

$$\lambda_1 = 2 \tag{1.2.3.9}$$

$$\lambda_2 = 3 \tag{1.2.3.10}$$

solution to $\mathbf{AX} = 2\mathbf{X}$ is eigen vector corresponding to $\lambda = 2$

$$(\mathbf{A} - 2\mathbf{I})\mathbf{X} = 0 \tag{1.2.3.11}$$

Substituting values:

$$\begin{pmatrix} 4 & -4 & 0 & 0 \\ 4 & -4 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{R_1}{4}} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 4 & -4 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - 4R_1}$$

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 - R_1} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{pmatrix} \xleftrightarrow{R_3 \leftrightarrow R_2}$$

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow -R_2} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 + R_2}$$

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1.2.3.12}$$

So, x_3 is a free variable: Let $x_3 = c$.

$$x_2 - x_3 = 0 \implies x_2 = x_3 = c \tag{1.2.3.13}$$

$$x_1 - x_3 = 0 \implies x_1 = x_3 = c \tag{1.2.3.14}$$

So,the solution to $\mathbf{AX} = 2\mathbf{X}$is

$$\mathbf{X} = c\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{1.2.3.15}$$

solution of $\mathbf{AX} = 3\mathbf{X}$ is eigen vector corresponding to $\lambda = 3$

$$(\mathbf{A} - 3\mathbf{I})\mathbf{X} = 0 \tag{1.2.3.16}$$

substituting we have:

$$\begin{pmatrix} 3 & -4 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{R_1}{3}} \begin{pmatrix} 1 & -\frac{4}{3} & 0 & 0 \\ 4 & -5 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - 4R_1}$$

$$\begin{pmatrix} 1 & -\frac{4}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 + R_1} \begin{pmatrix} 1 & -\frac{4}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & -\frac{4}{3} & 0 & 0 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow \frac{R_2}{3}}$$

$$\begin{pmatrix} 1 & \frac{-4}{3} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\frac{4}{3} & 0 & 0 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 - \frac{4}{3}R_2} \begin{pmatrix} 1 & \frac{4}{3} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_1 + \frac{4}{3}R_2}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1.2.3.17}$$

So x_3 is a free variable:

$$x_1 = 0 \tag{1.2.3.18}$$
$$x_2 = 0 \tag{1.2.3.19}$$
$$x_3 = c \tag{1.2.3.20}$$

So,the solution to $\mathbf{AX} = 3\mathbf{X}$ is,

$$\mathbf{X} = c\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{1.2.3.21}$$

1.2.4. Find a row-reduced matrix which is row equivalent to,

$$\mathbf{A} = \begin{pmatrix} i & -(1+i) & 0 \\ 1 & -2 & 1 \\ 1 & 2i & -1 \end{pmatrix} \tag{1.2.4.1}$$

Solution: Step 1: Performing scaling operation to matrix **A** as $R_1 \leftarrow \frac{1}{i}R_1$ by scaling matrix D_1 given as,

$$\mathbf{D_1} = \begin{pmatrix} \frac{1}{i} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.2.4.2}$$

$$\mathbf{D_1A} = \begin{pmatrix} \frac{1}{i} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} i & -(1+i) & 0 \\ 1 & -2 & 1 \\ 1 & 2i & -1 \end{pmatrix} \tag{1.2.4.3}$$

$$\implies \mathbf{D_1A} = \begin{pmatrix} 1 & -1+i & 0 \\ 1 & -2 & 1 \\ 1 & 2i & -1 \end{pmatrix} \tag{1.2.4.4}$$

Step 2: Performing $R_2 \leftarrow R_2 - R_1$ and $R_3 \leftarrow R_3 - R_1$ given by elementary matrix $\mathbf{E_{31}E_{21}}$ on equation (1.2.4.4),

$$\mathbf{E_{31}E_{21}} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \tag{1.2.4.5}$$

$$\mathbf{E_{31}E_{21}D_1A} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1+i & 0 \\ 1 & -2 & 1 \\ 1 & 2i & -1 \end{pmatrix} \tag{1.2.4.6}$$

$$\implies \mathbf{A_1} = \mathbf{E_{31}E_{21}D_1A} = \begin{pmatrix} 1 & -1+i & 0 \\ 0 & -1-i & 1 \\ 0 & 1+i & -1 \end{pmatrix} \tag{1.2.4.7}$$

Step 3: Performing $R_2 \leftarrow \frac{-1}{1+i}R_2$ given by $\mathbf{D_2}$ on equation (1.2.4.7),

$$\mathbf{D_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2}(-1+i) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.2.4.8}$$

$$\mathbf{D_2A_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2}(-1+i) & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1+i & 0 \\ 0 & -1-i & 1 \\ 0 & 1+i & -1 \end{pmatrix} \tag{1.2.4.9}$$

$$\implies \mathbf{A_2} = \mathbf{D_2A_1} = \begin{pmatrix} 1 & -1+i & 0 \\ 0 & 1 & \frac{1}{2}(-1+i) \\ 0 & 1+i & -1 \end{pmatrix} \tag{1.2.4.10}$$

Step 4: Performing $R_3 \leftarrow R_3 - (1+i)R_2$ given by $\mathbf{E_{32}}$ on equation (1.2.4.10),

$$\mathbf{E_{32}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -(1+i) & 1 \end{pmatrix} \tag{1.2.4.11}$$

$$\mathbf{E_{32}A_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1-i & 1 \end{pmatrix}\begin{pmatrix} 1 & -1+i & 0 \\ 0 & 1 & \frac{-1+i}{2} \\ 0 & 1+i & -1 \end{pmatrix} \tag{1.2.4.12}$$

$$\implies \mathbf{A_3} = \mathbf{E_{32}A_2} = \begin{pmatrix} 1 & -1+i & 0 \\ 0 & 1 & \frac{-1+i}{2} \\ 0 & 0 & 1 \end{pmatrix} \tag{1.2.4.13}$$

Step 5: Performing $R_1 \leftarrow R_1 - (-1+i)R_2$ given by $\mathbf{E_{12}}$ on equation (1.2.4.13),

$$\mathbf{E_{12}} = \begin{pmatrix} 1 & 1-i & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.2.4.14}$$

$$\mathbf{E_{12}A_3} = \begin{pmatrix} 1 & 1-i & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1+i & 0 \\ 0 & 1 & \frac{-1+i}{2} \\ 0 & 0 & 1 \end{pmatrix} \tag{1.2.4.15}$$

$$\implies \mathbf{A_4} = \mathbf{E_{12}A_3} = \begin{pmatrix} 1 & 0 & i \\ 0 & 1 & \frac{-1+i}{2} \\ 0 & 0 & 1 \end{pmatrix} \tag{1.2.4.16}$$

Step 6: Performing $R_1 \leftarrow R_1 - iR_3$ and $R_2 \leftarrow R_2 - \frac{-1+i}{2}R_3$ given by $\mathbf{E_{13}E_{23}}$ on equation (1.2.4.16),

$$\mathbf{E_{13}E_{23}} = \begin{pmatrix} 1 & 0 & -i \\ 0 & 1 & -\left(\frac{-1+i}{2}\right) \\ 0 & 0 & 1 \end{pmatrix} \tag{1.2.4.17}$$

$$\mathbf{E_{13}E_{23}A_4} = \begin{pmatrix} 1 & 0 & -i \\ 0 & 1 & -\left(\frac{-1+i}{2}\right) \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & i \\ 0 & 1 & \frac{-1+i}{2} \\ 0 & 0 & 1 \end{pmatrix} \tag{1.2.4.18}$$

$$\implies \mathbf{A_5} = \mathbf{E_{13}E_{23}A_4} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.2.4.19}$$

$\therefore$ Row-reduced matrix of $\mathbf{A}$ given by equation (1.2.4.1) is,

$$\mathbf{A} = \begin{pmatrix} i & -1-i & 0 \\ 1 & -2 & 1 \\ 1 & 2i & -1 \end{pmatrix} \xleftrightarrow{RREF} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I} \tag{1.2.4.20}$$

1.2.5. Prove that the following two matrices are not row equivalent

$$\begin{pmatrix} 2 & 0 & 0 \\ a & -1 & 0 \\ b & c & 3 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 2 \\ -2 & 0 & -1 \\ 1 & 3 & 5 \end{pmatrix} \tag{1.2.5.1}$$

Solution: Call the first matrix $\mathbf{A}$ and the second matrix $\mathbf{B}$.

$$\mathbf{A}^T = \begin{pmatrix} 2 & a & b \\ 0 & -1 & c \\ 0 & 0 & 3 \end{pmatrix} \tag{1.2.5.2}$$

$\mathbf{A}^T$ is a upper triangular matrix with non-zero diagonal. Hence it has full rank = 3.

$$\mathbf{B}^T = \begin{pmatrix} 1 & -2 & 1 \\ 1 & 0 & 3 \\ 2 & -1 & 5 \end{pmatrix} \xleftrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - R_1} \begin{pmatrix} 1 & -2 & 1 \\ 0 & 2 & 2 \\ 0 & 3 & 3 \end{pmatrix} \tag{1.2.5.3}$$

$$\xleftrightarrow[R_2 \leftarrow R_2/2]{R_3 \leftarrow R_3/3} \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \tag{1.2.5.4}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - R_2} \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.2.5.5}$$

$\mathbf{B}^T$ is a upper triangular matrix with zero diagonal. Hence it doesn't have full rank. Therefore both matrices have different rank , so it cannot be row equivalent.

1.2.6. Let

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{1.2.6.1}$$

be a 2×2 matrix with complex entries. Suppose A is row-reduced and also that $a+b+c+d = 0$. Prove that there are exactly three such matrices.

Solution: A matrix is in row echelon form if it follows the following conditions

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry (i.e. left most nonzero entry) of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zero Row Reduced Echelon Form A matrix is in row reduced echelon form if it follows the following conditions
1. The matrix should be row echelon form
2. The leading entry in each nonzero row is 1.
3. Each leading 1 is the only nonzero entry in its column. Proof

Given ,

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (1.2.6.2)$$

Condition 1 : Matrix **A** should be in row-reduced echelon form

Condition 2 : $a + b + c + d = 0$ where a,b,c and d are the elements of the matrix **A**

Reducing the matrix **A** from equation (1.2.6.2)

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \xleftrightarrow{R_1=\frac{1}{a}R_1} \begin{pmatrix} 1 & \frac{b}{a} \\ c & d \end{pmatrix} \quad (1.2.6.3)$$

$$\xleftrightarrow{R_2=R_2-cR_1} \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & \frac{ad-bc}{a} \end{pmatrix} \quad (1.2.6.4)$$

$$\xleftrightarrow{R_2=\frac{a}{ad-bc}R_2} \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix} \quad (1.2.6.5)$$

$$\xleftrightarrow{R_1=R_1-\frac{b}{a}R_2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (1.2.6.6)$$

Case 1: Matrix **A** of Rank 2

From the equation (1.2.6.4),for the matrix to be in row reduced echelon form,

$$b = 0$$
$$a \neq 0$$
$$d = 1$$
$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (1.2.6.7)$$

For the condition 2 to get satisfied,

$$a + 0 + c + 1 = 0 \quad (1.2.6.8)$$
$$\implies a = -(c + 1) \quad (1.2.6.9)$$
$$\implies c \neq -1 \quad (1.2.6.10)$$

Both the condition gets satisfied and so exactly one matrix **A** can be formed of Rank 2 with given conditions

Case 2: Matrix **A** of Rank 1

From the equation (1.2.6.4),for the matrix to be in row reduced echelon form,

$$a \neq 0$$
$$d = 0$$
$$c = 0$$

For the condition 2 to get satisfied,

$$a + b + 0 + 0 = 0 \quad (1.2.6.11)$$
$$\implies b = -a \quad (1.2.6.12)$$

$$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \quad (1.2.6.13)$$

Both the condition gets satisfied and so exactly one matrix **A** can be formed of Rank 1 with given conditions

Case 3: Matrix **A** of Rank 0

From equation (1.2.6.2),for the matrix to be in row reduced echelon form,

$$a = 0$$
$$b = 0$$
$$c = 0$$
$$d = 0$$
$$\mathbf{A} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad (1.2.6.14)$$

Both the condition gets satisfied and so exactly one matrix **A** can be formed of Rank 0 with given conditions

Therefore matrix **A** shown in equation (1.2.6.7),(1.2.6.13) and (1.2.6.14) are the exactly three such matrices that can be formed with given conditions.

1.2.7. Prove that the interchange of two rows of a matrix can be accomplished by a finite sequence of elementary row operations of the other two types.

Solution: Let **A** be a 3×3 matrix with having row vectors $\mathbf{a}_1$,$\mathbf{a}_2$ and $\mathbf{a}_3$.

$$\mathbf{A} = \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} \quad (1.2.7.1)$$

Let's exchange row $\mathbf{a}_1$ and $\mathbf{a}_2$. Let's call this elementary operation $\mathbf{E_1}$.

$$\mathbf{E}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (1.2.7.2)$$

$$(1.2.7.3)$$

Now performing operation $\mathbf{E}_1$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{a}_2 \\ \mathbf{a}_1 \\ \mathbf{a}_3 \end{pmatrix} \quad (1.2.7.4)$$

Now, to prove that same matrix can be obtained by elementary operations let's call them $\mathbf{E_2}$ and $\mathbf{E_3}$.Now performing operation $\mathbf{E}_2$ by adding row 2 to row 1.

$$\begin{pmatrix}1 & 1 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}\mathbf{a}_1\\\mathbf{a}_2\\\mathbf{a}_3\end{pmatrix} = \begin{pmatrix}\mathbf{a}_1+\mathbf{a}_2\\\mathbf{a}_2\\\mathbf{a}_3\end{pmatrix} \tag{1.2.7.5}$$

Using elementary operation $\mathbf{E_2}$ we will subtract row 1 from row 2.

$$\begin{pmatrix}1 & 0 & 0\\-1 & 1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}\mathbf{a}_1+\mathbf{a}_2\\\mathbf{a}_2\\\mathbf{a}_3\end{pmatrix} = \begin{pmatrix}\mathbf{a}_1+\mathbf{a}_2\\-\mathbf{a}_1\\\mathbf{a}_3\end{pmatrix} \tag{1.2.7.6}$$

Using elementary operation $\mathbf{E_2}$ we will add row 2 to row 1.

$$\begin{pmatrix}1 & 1 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}\mathbf{a}_1+\mathbf{a}_2\\-\mathbf{a}_1\\\mathbf{a}_3\end{pmatrix} = \begin{pmatrix}\mathbf{a}_2\\-\mathbf{a}_1\\\mathbf{a}_3\end{pmatrix} \tag{1.2.7.7}$$

Using elementary operation $\mathbf{E_3}$ we will multiply row 2 by -1.

$$\begin{pmatrix}1 & 0 & 0\\0 & -1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}\mathbf{a}_2\\-\mathbf{a}_1\\\mathbf{a}_3\end{pmatrix} = \begin{pmatrix}\mathbf{a}_2\\\mathbf{a}_1\\\mathbf{a}_3\end{pmatrix} \tag{1.2.7.8}$$

Hence, we can say that,

$$\begin{pmatrix}1 & 0 & 0\\0 & -1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 1 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\-1 & 1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 1 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}\mathbf{a}_1\\\mathbf{a}_2\\\mathbf{a}_3\end{pmatrix} = \begin{pmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}\mathbf{a}_1\\\mathbf{a}_2\\\mathbf{a}_3\end{pmatrix} \tag{1.2.7.9}$$

Let us assume a matrix $\mathbf{A}$

$$\mathbf{A} = \begin{pmatrix}1 & 2 & 3\\0 & 1 & 0\\1 & 1 & 0\end{pmatrix} \tag{1.2.7.10}$$

Let's exchange row $\mathbf{a}_1$ and $\mathbf{a}_2$ by applying operation $\mathbf{E_1}$.

$$\begin{pmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\0 & 1 & 0\\1 & 1 & 0\end{pmatrix} = \begin{pmatrix}0 & 1 & 0\\1 & 2 & 3\\1 & 1 & 0\end{pmatrix} \tag{1.2.7.11}$$

Now, to prove that same matrix can be obtained by other two elementary operations.We will first perform elementary operation $\mathbf{E_2}$ by adding row 2 to row 1.

$$\begin{pmatrix}1 & 1 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\0 & 1 & 0\\1 & 1 & 0\end{pmatrix} = \begin{pmatrix}1 & 3 & 3\\0 & 1 & 0\\1 & 1 & 0\end{pmatrix} \tag{1.2.7.12}$$

Using elementary operation $\mathbf{E_2}$ we will subtract row 1 from row 2.

$$\begin{pmatrix}1 & 0 & 0\\-1 & 1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 3 & 3\\0 & 1 & 0\\1 & 1 & 0\end{pmatrix} = \begin{pmatrix}1 & 3 & 3\\-1 & -2 & -3\\1 & 1 & 0\end{pmatrix} \tag{1.2.7.13}$$

Using elementary operation $\mathbf{E_2}$ we will add row 2 to row 1.

$$\begin{pmatrix}1 & 1 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 3 & 3\\-1 & -2 & -3\\1 & 1 & 0\end{pmatrix} = \begin{pmatrix}0 & 1 & 0\\-1 & -2 & -3\\1 & 1 & 0\end{pmatrix} \tag{1.2.7.14}$$

Using elementary operation $\mathbf{E_3}$ we will multiply row 2 by -1.

$$\begin{pmatrix}1 & 0 & 0\\0 & -1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1 & 0\\-1 & -2 & -3\\1 & 1 & 0\end{pmatrix} = \begin{pmatrix}0 & 1 & 0\\1 & 2 & 3\\1 & 1 & 0\end{pmatrix} \tag{1.2.7.15}$$

Hence,we can say that,

$$\begin{pmatrix}1 & 0 & 0\\0 & -1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 1 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\-1 & 1 & 0\\0 & 0 & 1\end{pmatrix} \times\begin{pmatrix}1 & 1 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\0 & 1 & 0\\1 & 1 & 0\end{pmatrix} = \begin{pmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\0 & 1 & 0\\1 & 1 & 0\end{pmatrix} \tag{1.2.7.16}$$

1.2.8. Consider the system of equations $\mathbf{AX} = 0$ where

$$\mathbf{A} = \begin{pmatrix}a & b\\c & d\end{pmatrix}$$

is a 2×2 matrix over the field F. Prove the following -

- If every entry of $\mathbf{A}$ is 0, then every pair x_1 and x_2 is a solution of $\mathbf{AX} = 0$.
- If $ad - bc \neq 0$, then the system $\mathbf{AX} = 0$ has only the trivial solution $x_1 = x_2 = 0$
- If $ad - bc = 0$ and some entry of $\mathbf{A}$ is different from 0, then there is a solution x_1^0 and x_2^0 such that x_1 and x_2 is a solution if and only if there is a scalar y such that $x_1 = yx_1^0$ and $x_2 = yx_2^0$

Solution: Solution 1 If every entry of $\mathbf{A}$ is 0 then the equation $\mathbf{AX} = 0$ becomes,

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \tag{1.2.8.1}$$

$$\implies 0.x_1 + 0.x_2 = 0 \qquad \forall x_1, x_2 \in F \tag{1.2.8.2}$$

Hence proved, every pair x_1 and x_2 is a solution for the equation $\mathbf{AX} = 0$. Solution 2 **Case 1:** Let $a = 0$. Since $ad - bc \neq 0$. As $bc \neq 0$ therefore $b \neq 0$ and $c \neq 0$. Hence, we can perform row reduction on the augmented matrix of equation $\mathbf{AX}$=0 as follows,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & b & 0 \\ c & d & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{c} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c & d & 0 \\ 0 & b & 0 \end{pmatrix} \tag{1.2.8.3}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{b} \end{pmatrix}\begin{pmatrix} 1 & \frac{d}{c} & 0 \\ 0 & b & 0 \end{pmatrix} \tag{1.2.8.4}$$

$$= \begin{pmatrix} 1 & -\frac{d}{c} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{d}{c} & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{1.2.8.5}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{1.2.8.6}$$

Case 2: Let $a, b, c, d \neq 0$. Considering the following case,

$$\mathbf{AX} = \mathbf{u} \tag{1.2.8.7}$$

$$\implies \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{1.2.8.8}$$

Row Reducing the augmented matrix of (1.2.8.8) we get,

$$\begin{pmatrix} \frac{1}{a} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b & u_1 \\ c & d & u_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -c & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{b}{a} & \frac{u_1}{a} \\ c & d & u_2 \end{pmatrix} \tag{1.2.8.9}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & \frac{a}{ad-bc} \end{pmatrix}\begin{pmatrix} 1 & \frac{b}{a} & \frac{u_1}{a} \\ 0 & \frac{ad-bc}{a} & \frac{au_2-cu_1}{a} \end{pmatrix} \tag{1.2.8.10}$$

$$= \begin{pmatrix} 1 & -\frac{b}{a} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{b}{a} & \frac{u_1}{a} \\ 0 & 1 & \frac{au_2-cu_1}{ad-bc} \end{pmatrix} \tag{1.2.8.11}$$

$$= \begin{pmatrix} 1 & 0 & \frac{du_1-bu_2}{ad-bc} \\ 0 & 1 & \frac{au_2-cu_1}{ad-bc} \end{pmatrix} \tag{1.2.8.12}$$

From (1.2.8.12) we get,

$$x_1 = \frac{du_1 - bu_2}{ad - bc} \tag{1.2.8.13}$$

$$x_2 = \frac{au_2 - cu_1}{ad - bc} \tag{1.2.8.14}$$

Since $u_1 = 0$ and $u_2 = 0$ then from (1.2.8.13) and (1.2.8.14),

$$x_1 = 0 \tag{1.2.8.15}$$

$$x_2 = 0 \tag{1.2.8.16}$$

Hence we get,

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1.2.8.17}$$

In (1.2.8.6) and (1.2.8.17), we can see that $\mathbf{AX} = 0$ has only one trivial solution i.e $x_1 = x_2 = 0$ in all cases. Hence proved, the equation $\mathbf{AX}$=0 has only one trivial solution $x_1 = x_2 = 0$ Solution 3 **Case 1:** Let, $a \neq 0$ for $\mathbf{A}$. Given $ad - bc = 0$, we can perform row reduction on augmented matrix of equation $\mathbf{AX} = 0$ as follows,

$$\begin{pmatrix} \frac{1}{a} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b & 0 \\ c & d & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -c & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{b}{a} & 0 \\ c & d & 0 \end{pmatrix} \tag{1.2.8.18}$$

$$= \begin{pmatrix} 1 & \frac{b}{a} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad [\because ad - bc = 0] \tag{1.2.8.19}$$

Hence from (1.2.8.19), $\mathbf{AX} = 0$ if and only if

$$x_1 = -\frac{b}{a}x_2 \qquad [a \neq 0] \tag{1.2.8.20}$$

Letting $x_1^0 = -\frac{b}{a}$ and $x_2^0 = 1$ we get for $y = 1$,

$$x_1 = yx_1^0 \tag{1.2.8.21}$$

$$x_2 = yx_2^0 \tag{1.2.8.22}$$

which is a solution of the equation $\mathbf{AX} = 0$.
Case 2: Let, $b \neq 0$ for $\mathbf{A}$. Given $ad - bc = 0$, at first we multiply by elementary matrix to change the columns and the we can perform row reduction on augmented matrix of equation $\mathbf{AX} = 0$ as follows,

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} b & a & 0 \\ d & c & 0 \end{pmatrix} \tag{1.2.8.23}$$

Hence using the result obtained from (1.2.8.19) we can conclude for (1.2.8.23), $\mathbf{AX} = 0$ if and only if

$$x_2 = -\frac{a}{b}x_1 \qquad [b \neq 0] \qquad (1.2.8.24)$$

Letting $x_2^0 = -\frac{a}{b}$ and $x_1^0 = 1$ we get for $y = 1$,

$$x_1 = yx_1^0 \qquad (1.2.8.25)$$
$$x_2 = yx_2^0 \qquad (1.2.8.26)$$

which is a solution of the equation $\mathbf{AX} = 0$.

Case 3: Let, $c \neq 0$ for $\mathbf{A}$. Given $ad - bc = 0$, we can perform row reduction on augmented matrix of equation $\mathbf{AX} = 0$ as follows,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b & 0 \\ c & d & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{c} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c & d & 0 \\ a & b & 0 \end{pmatrix} \qquad (1.2.8.27)$$

$$= \begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{d}{c} & 0 \\ a & b & 0 \end{pmatrix} \qquad (1.2.8.28)$$

$$= \begin{pmatrix} 1 & \frac{d}{c} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad [\because ad - bc = 0] \qquad (1.2.8.29)$$

Hence from (1.2.8.29), $\mathbf{AX} = 0$ if and only if

$$x_1 = -\frac{d}{c}x_2 \qquad [a \neq 0] \qquad (1.2.8.30)$$

Letting $x_1^0 = -\frac{d}{c}$ and $x_2^0 = 1$ we get for $y = 1$,

$$x_1 = yx_1^0 \qquad (1.2.8.31)$$
$$x_2 = yx_2^0 \qquad (1.2.8.32)$$

which is a solution of the equation $\mathbf{AX} = 0$.

Case 4: Let, $d \neq 0$ for $\mathbf{A}$. Given $ad - bc = 0$, at first we multiply by elementary matrix to change the columns and then we can perform row reduction on augmented matrix of equation $\mathbf{AX} = 0$ as follows,

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} b & a & 0 \\ d & c & 0 \end{pmatrix} \qquad (1.2.8.33)$$

$$= \begin{pmatrix} d & c & 0 \\ b & a & 0 \end{pmatrix} \qquad (1.2.8.34)$$

Hence using the result from (1.2.8.29) we can conclude for (1.2.8.34), $\mathbf{AX} = 0$ if and only if

$$x_2 = -\frac{c}{d}x_1 \qquad [a \neq 0] \qquad (1.2.8.35)$$

Letting $x_2^0 = -\frac{c}{d}$ and $x_1^0 = 1$ we get for $y = 1$,

$$x_1 = yx_1^0 \qquad (1.2.8.36)$$
$$x_2 = yx_2^0 \qquad (1.2.8.37)$$

which is a solution of the equation $\mathbf{AX} = 0$.

1.3 Row Reduced Echelon Matrices

1.3.1. Find all solutions to the following system of equations by row-reducing the co-efficient matrix:

$$\frac{1}{3}x_1 + 2x_2 - 6x_3 = 0 \qquad (1.3.1.1)$$
$$-4x_1 \qquad + 5x_3 = 0 \qquad (1.3.1.2)$$
$$-3x_1 + 6x_2 - 13x_3 = 0 \qquad (1.3.1.3)$$
$$-\frac{7}{3}x_1 + 2x_2 - \frac{8}{3}x_3 = 0 \qquad (1.3.1.4)$$

Solution: The coefficient matrix is:

$$A = \begin{pmatrix} \frac{1}{3} & 2 & -6 \\ -4 & 0 & 5 \\ -3 & 6 & -13 \\ -\frac{7}{3} & 2 & -\frac{8}{3} \end{pmatrix} \qquad (1.3.1.5)$$

The number of rows of this coefficient matrix is $m = 4$ and the number of columns is n= 3, So in this case, $n < m$. Now the row operations are:

$$\begin{pmatrix} \frac{1}{3} & 2 & -6 \\ -4 & 0 & 5 \\ -3 & 6 & -13 \\ -\frac{7}{3} & 2 & -\frac{8}{3} \end{pmatrix} \xleftrightarrow[R_1 \leftarrow R_1 \times 3]{R_4 \leftarrow R_4 \times 3} \begin{pmatrix} 1 & 6 & -18 \\ -4 & 0 & 5 \\ -3 & 6 & -13 \\ -7 & 6 & -8 \end{pmatrix} \tag{1.3.1.6}$$

$$\xleftrightarrow{R_3 \leftarrow R_2 + R_3} \begin{pmatrix} 1 & 6 & -18 \\ -4 & 0 & 5 \\ -7 & 6 & -8 \\ -7 & 6 & -8 \end{pmatrix} \xleftrightarrow{R_4 \leftarrow R_4 - R_3} \tag{1.3.1.7}$$

$$\begin{pmatrix} 1 & 6 & -18 \\ -4 & 0 & 5 \\ -7 & 6 & -8 \\ 0 & 0 & 0 \end{pmatrix} \xleftrightarrow[R_3 \leftarrow R_3 + 7R_1]{R_2 \leftarrow R_2 + 4R_1} \begin{pmatrix} 1 & 6 & -18 \\ 0 & 24 & -67 \\ 0 & 48 & -138 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.3.1.8}$$

$$\xleftrightarrow{R_3 \leftarrow R_3/2} \begin{pmatrix} 1 & 6 & -18 \\ 0 & 24 & -67 \\ 0 & 24 & -69 \\ 0 & 0 & 0 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 - R_2} \tag{1.3.1.9}$$

$$\begin{pmatrix} 1 & 6 & -18 \\ 0 & 24 & -67 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow \frac{R_3}{(-2)}} \begin{pmatrix} 1 & 6 & -18 \\ 0 & 24 & -67 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.3.1.10}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{R_2}{4}} \begin{pmatrix} 1 & 6 & -18 \\ 0 & 6 & -\frac{67}{4} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 - R_2} \tag{1.3.1.11}$$

$$\begin{pmatrix} 1 & 0 & -\frac{5}{4} \\ 0 & 6 & -\frac{67}{4} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow \frac{R_2}{6}} \begin{pmatrix} 1 & 0 & -\frac{5}{4} \\ 0 & 1 & -\frac{67}{24} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.3.1.12}$$

$$\xleftrightarrow[R_1 \leftarrow R_1 + \frac{5R_3}{4}]{R_2 \leftarrow R_2 + \frac{67R_3}{24}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.3.1.13}$$

Now,

$$A\mathbf{x} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \tag{1.3.1.14}$$

So,

$$\mathbf{I_3 x} = 0 \tag{1.3.1.15}$$

$$\implies \mathbf{x} = 0 \tag{1.3.1.16}$$

1.3.2. Find a row-reduced matrix which is row equivalent to A.What are the solutions of $\mathbf{Ax} = \mathbf{0}$?

$$\mathbf{A} = \begin{pmatrix} 1 & -i \\ 2 & 2 \\ i & 1+i \end{pmatrix} \tag{1.3.2.1}$$

Solution: Let R be a row-reduced echelon matrix which is row equivalent to A. Then the systems

$$A\mathbf{x} = \mathbf{0}, R\mathbf{x} = \mathbf{0} \tag{1.3.2.2}$$

have the same solutions. On performing elementary row operations on (1.3.2.1),

$$\mathbf{R} = \mathbf{BA} \tag{1.3.2.3}$$

where $\mathbf{B}$ is the product of all elementary matrices. Reducing the given matrix, we get

$$\begin{aligned} \mathbf{B} &= (\mathbf{E_5 E_4 E_3 E_2 E_1}) \\ &= \begin{pmatrix} 1 & i & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -i & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4}(1-i) & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &\quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -i & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2}(1-i) & \frac{1}{4}(1+i) & 0 \\ \frac{1}{2}(-1+i) & \frac{1}{4}(1-i) & 0 \\ \frac{1}{2}(1-i) & \frac{1}{4}(-1-i) & 1 \end{pmatrix} \end{aligned} \tag{1.3.2.4}$$

$$\mathbf{R} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{1.3.2.5}$$

$\therefore$ Row-reduced matrix of $\mathbf{A}$ is,

$$\mathbf{A} = \begin{pmatrix} 1 & -i \\ 2 & 2 \\ i & 1+i \end{pmatrix} \xleftrightarrow{RREF} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{1.3.2.6}$$

From(1.3.2.2) and (1.3.2.6),

$$A\mathbf{x} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \tag{1.3.2.7}$$

The solution of $\mathbf{Ax} = \mathbf{0}$ is,

$$\mathbf{I_2x} = 0 \quad (1.3.2.8)$$
$$\implies \mathbf{x} = 0 \quad (1.3.2.9)$$

As $\mathbf{I_2}$ is invertible.

1.3.3. Describe explicitly all 2x2 row-reduced echelon matrices.

Solution:

2x2 matrices which are row-reduced echelon matrix can be represented as a linear combination of three matrices:-

$$c_1\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} + c_2\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} + c_3\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} \quad (1.3.3.1)$$

1.3.4. Consider the system of the equations

$$x_1 - x_2 + 2x_3 = 1 \quad (1.3.4.1)$$
$$x_1 - 0x_2 + 2x_3 = 1 \quad (1.3.4.2)$$
$$x_1 - 3x_2 + 4x_3 = 2 \quad (1.3.4.3)$$

Does this system have a solution ? If so describe explicitly all solutions.

Solution: Let $\mathbf{V}$ is the set of all $\begin{pmatrix}x_1, & x_2, & x_3\end{pmatrix} \in \mathbb{R}^3$ which satisfy the (1.3.4.1), (1.3.4.2) and (1.3.4.3)

From equation (1.3.4.1) to (1.3.4.3) we can write,

$$\begin{pmatrix}1 & -1 & 2\\1 & 0 & 2\\1 & -3 & 4\end{pmatrix}\mathbf{x} = \begin{pmatrix}1\\1\\2\end{pmatrix} \quad (1.3.4.4)$$
$$\implies \mathbf{Ax} = \mathbf{b} \quad (1.3.4.5)$$

Where,

$$(1.3.4.6)$$

$$\mathbf{A} = \begin{pmatrix}1 & -1 & 2\\1 & 0 & 2\\1 & -3 & 4\end{pmatrix}, \mathbf{b} = \begin{pmatrix}1\\1\\2\end{pmatrix}, \mathbf{x} = \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} \quad (1.3.4.7)$$

Solving the matrix $\mathbf{A}$ for rank we get,

$$\begin{pmatrix}1 & -1 & 2\\2 & 0 & 2\\1 & -3 & 4\end{pmatrix} \xleftrightarrow{R_2=R_1-2R_1} \begin{pmatrix}1 & -1 & 2\\0 & 2 & -2\\1 & -3 & 4\end{pmatrix} \quad (1.3.4.8)$$

$$\xleftrightarrow{R_3=R_3-R_1} \begin{pmatrix}1 & -1 & 2\\0 & 2 & -2\\0 & -2 & 2\end{pmatrix} \quad (1.3.4.9)$$

$$\xleftrightarrow{R_3=R_3+R_2} \begin{pmatrix}1 & -1 & 2\\0 & 2 & -2\\0 & 0 & 0\end{pmatrix} \quad (1.3.4.10)$$

Hence, rank $(\mathbf{A}) = 2$. Now solving the augmented matrix of (1.3.4.5) we get,

$$\begin{pmatrix}1 & -1 & 2 & 1\\2 & 0 & 2 & 1\\1 & -3 & 4 & 2\end{pmatrix} \xleftrightarrow{R_2=R_1-2R_1} \begin{pmatrix}1 & -1 & 2 & 1\\0 & 2 & -2 & -1\\1 & -3 & 4 & 2\end{pmatrix} \quad (1.3.4.11)$$

$$\xleftrightarrow{R_3=R_3-R_1} \begin{pmatrix}1 & -1 & 2 & 1\\0 & 2 & -2 & -1\\0 & -2 & 2 & 1\end{pmatrix} \quad (1.3.4.12)$$

$$\xleftrightarrow{R_3=R_3+R_2} \begin{pmatrix}1 & -1 & 2 & 1\\0 & 2 & -2 & -1\\0 & 0 & 0 & 0\end{pmatrix} \quad (1.3.4.13)$$

We have rank $(\mathbf{A}) = $ rank $(\mathbf{A} : \mathbf{b}) = 2 < n$, where $n = 3$. Hence we have infinite no of solutions for given system of equations.

Using Gauss - Jordan elimination method to getting the solution,

$$\begin{pmatrix}1 & -1 & 2 & 1\\2 & 0 & 2 & 1\\1 & -3 & 4 & 2\end{pmatrix} \xleftrightarrow{R_2=R_1-2R_1} \begin{pmatrix}1 & -1 & 2 & 1\\0 & 2 & -2 & -1\\1 & -3 & 4 & 2\end{pmatrix} \quad (1.3.4.14)$$

$$\xleftrightarrow{R_3=R_3-R_1} \begin{pmatrix}1 & -1 & 2 & 1\\0 & 2 & -2 & -1\\0 & -2 & 2 & 1\end{pmatrix} \quad (1.3.4.15)$$

$$\xleftrightarrow{R_2=\frac{R_2}{2}} \begin{pmatrix}1 & -1 & 2 & 1\\0 & 1 & -1 & -\frac{1}{2}\\0 & -2 & 2 & 1\end{pmatrix} \quad (1.3.4.16)$$

$$\xleftrightarrow{R_3=R_3+2R_2}\begin{pmatrix}1 & -1 & 2 & 1\\0 & 1 & -1 & -\frac{1}{2}\\0 & 0 & 0 & 0\end{pmatrix} \quad (1.3.4.17)$$

$$\xleftrightarrow{R_1=R_1+R_2}\begin{pmatrix}1 & 0 & 1 & \frac{1}{2}\\0 & 1 & -1 & -\frac{1}{2}\\0 & 0 & 0 & 0\end{pmatrix} \quad (1.3.4.18)$$

$$\implies x_1 + x_3 = \frac{1}{2}, x_2 - x_3 = -\frac{1}{2} \quad (1.3.4.19)$$

$$\implies x_2 = -\frac{1}{2} + x_3, x_1 = \frac{1}{2} - x_3 \quad (1.3.4.20)$$

From equation (1.3.4.19) and (1.3.4.20)

$$\mathbf{x} = \begin{pmatrix}\frac{1}{2} - x_3\\-\frac{1}{2} + x_3\\x_3\end{pmatrix} \quad (1.3.4.21)$$

which can be written as,

$$\mathbf{x} = x_3\begin{pmatrix}-1\\1\\1\end{pmatrix} + \begin{pmatrix}-\frac{1}{2}\\-\frac{1}{2}\\0\end{pmatrix} \quad (1.3.4.22)$$

from 1.3.4.22 we can say that for any value x_3, $\mathbf{V}$ will no be gives zero vector. Hence the given solution space will not span of the vector space $\mathbf{V}$

1.3.5. Give an example of a system of two linear equations in two unknowns which has no solution.

Solution: Let us assume two equations as given below $\begin{pmatrix}5 & 2\end{pmatrix}\mathbf{x} = 7$ and $\begin{pmatrix}10 & 4\end{pmatrix}\mathbf{x} = -3$

Let the coefficient matrix be given as

$$\mathbf{A} = \begin{pmatrix}5 & 2\\10 & 4\end{pmatrix}, \mathbf{b} = \begin{pmatrix}7\\-3\end{pmatrix} \quad (1.3.5.1)$$

the augmented matrix be given as matrix be given as

$$\mathbf{A|B} = \begin{pmatrix}5 & 2 & 7\\10 & 4 & -3\end{pmatrix} \quad (1.3.5.2)$$

Applying row reduction

$$\begin{pmatrix}5 & 2 & 7\\10 & 4 & -3\end{pmatrix}\xleftrightarrow{R_2=R_2-2R_1}\begin{pmatrix}5 & 2 & 7\\0 & 0 & -17\end{pmatrix} \quad (1.3.5.3)$$

$$\xleftrightarrow{R_1=\frac{R_1}{5}}\begin{pmatrix}1 & \frac{2}{5} & \frac{7}{5}\\0 & 0 & -17\end{pmatrix} \quad (1.3.5.4)$$

$$\xleftrightarrow{R_2=\frac{R_2}{-17}}\begin{pmatrix}1 & \frac{2}{5} & \frac{7}{5}\\0 & 0 & 1\end{pmatrix} \quad (1.3.5.5)$$

$$\xleftrightarrow{R_1=R_1-\frac{7}{5}R_2}\begin{pmatrix}1 & \frac{2}{5} & 0\\0 & 0 & 1\end{pmatrix} \quad (1.3.5.6)$$

$$(1.3.5.7)$$

Clearly,On comparing the ranks of matrix $\mathbf{A}$ and $\mathbf{A|B}$, we find that rank of matrix $\mathbf{A|B} \neq \mathbf{A}$ Hence the system of linear equation have no solutions. Consider the system $\mathbf{Ax} = \mathbf{b}$, with coefficient matrix $\mathbf{A}$ and augmented matrix $\mathbf{A|B}$.

As above, the sizes of $\mathbf{b}$, $\mathbf{A}$, and $\mathbf{A|B}$ are m $\times$ 1, m $\times$ n, and m $\times$ (n + 1), respectively; in addition, the number of unknowns is n.

$\mathbf{Ax}$ is inconsistent (i.e., no solution exists) if and only if rank $\mathbf{A}$ < rank $\mathbf{A|B}$.

1.3.6. Find all solutions of

$$x_1 - 2x_2 + x_3 + 2x_4 = 1$$
$$x_1 + x_2 - x_3 + x_4 + x_5 = 2$$
$$x_1 + 7x_2 - 5x_3 - x_4 = 3$$

Solution: The given equations can be written as,

$$\mathbf{Ax} = B \quad (1.3.6.1)$$

$$\begin{pmatrix}1 & -2 & 1 & 2\\1 & 1 & -1 & 1\\1 & 7 & -5 & -1\end{pmatrix}\mathbf{x} = \begin{pmatrix}1\\2\\3\end{pmatrix} \quad (1.3.6.2)$$

Now, we form the augmented matrix and perform Row reduction,

$$\left(\begin{array}{cccc|c} 1 & -2 & 1 & 2 & 1 \\ 1 & 1 & -1 & 1 & 2 \\ 1 & 7 & -5 & -1 & 3 \end{array}\right) \tag{1.3.6.3}$$

$$\xleftrightarrow{R_2=R_2-R_1, R_3=R_3-R_1} \left(\begin{array}{cccc|c} 1 & -2 & 1 & 2 & 1 \\ 0 & 3 & -2 & -1 & 1 \\ 0 & 9 & -6 & -3 & 2 \end{array}\right) \tag{1.3.6.4}$$

$$\xleftrightarrow{R_2=\frac{1}{3}R_2} \left(\begin{array}{cccc|c} 1 & -2 & 1 & 2 & 1 \\ 0 & 1 & \frac{-2}{3} & \frac{-1}{3} & \frac{1}{3} \\ 0 & 9 & -6 & -3 & 2 \end{array}\right) \tag{1.3.6.5}$$

$$\xleftrightarrow{R_3=R_3-9R_1} \left(\begin{array}{cccc|c} 1 & -2 & 1 & 2 & 1 \\ 0 & 1 & \frac{-2}{3} & \frac{-1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & -1 \end{array}\right) \tag{1.3.6.6}$$

Rank of $\mathbf{A}$ is less than rank of the augmented matrix. Hence, the given system has no solution.

1.3.7. Find all solutions of

$$2x_1 - 3x_2 - 7x_3 + 5x_4 + 2x_5 = -2 \tag{1.3.7.1}$$
$$x_1 - 2x_2 - 4x_3 + 3x_4 + x_5 = -2 \tag{1.3.7.2}$$
$$2x_1 - 4x_3 + 2x_4 + x_5 = 3 \tag{1.3.7.3}$$
$$x_1 - 5x_2 - 7x_3 + 6x_4 + 2x_5 = -7 \tag{1.3.7.4}$$

Solution: The given equations can be written as,

$$\begin{pmatrix} 2 & -3 & -7 & 5 & 2 \\ 1 & -2 & -4 & 3 & 1 \\ 2 & 0 & -4 & 2 & 1 \\ 1 & -5 & -7 & 6 & 2 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -2 \\ -2 \\ 3 \\ 7 \end{pmatrix} \tag{1.3.7.5}$$

Now, we form the augmented matrix and perform Row reduction,

$$\left(\begin{array}{ccccc|c} 2 & -3 & -7 & 5 & 2 & -2 \\ 1 & -2 & -4 & 3 & 1 & -2 \\ 2 & 0 & -4 & 2 & 1 & 3 \\ 1 & -5 & -7 & 6 & 2 & 7 \end{array}\right) \tag{1.3.7.6}$$

$$\xleftrightarrow{R_3=R_3-R_1} \left(\begin{array}{ccccc|c} 2 & -3 & -7 & 5 & 2 & -2 \\ 1 & -2 & -4 & 3 & 1 & -2 \\ 0 & 3 & 3 & -3 & -1 & 5 \\ 1 & -5 & -7 & 6 & 2 & 7 \end{array}\right) \tag{1.3.7.7}$$

$$\xleftrightarrow{R_1=\frac{1}{2}R_1} \left(\begin{array}{ccccc|c} 1 & \frac{-3}{2} & \frac{-7}{2} & \frac{5}{2} & 1 & -1 \\ 1 & -2 & -4 & 3 & 1 & -2 \\ 0 & 3 & 3 & -3 & -1 & 5 \\ 1 & -5 & -7 & 6 & 2 & 7 \end{array}\right) \tag{1.3.7.8}$$

$$\xleftrightarrow{R_2=R_2-R_1, R_4=R_4-R_1} \left(\begin{array}{ccccc|c} 1 & \frac{-3}{2} & \frac{-7}{2} & \frac{5}{2} & 1 & -1 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & -1 \\ 0 & 3 & 3 & -3 & -1 & 5 \\ 0 & -\frac{7}{2} & -\frac{7}{2} & \frac{7}{2} & 1 & -6 \end{array}\right) \tag{1.3.7.9}$$

$$\xleftrightarrow{R_1=R_1-3R_2} \left(\begin{array}{ccccc|c} 1 & 0 & -2 & 1 & 1 & 2 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & -1 \\ 0 & 3 & 3 & -3 & -1 & 5 \\ 0 & -\frac{7}{2} & -\frac{7}{2} & \frac{7}{2} & 1 & -6 \end{array}\right) \tag{1.3.7.10}$$

$$\xleftrightarrow{R_3=R_3+6R_2, R_4=R_4-7R_2} \left(\begin{array}{ccccc|c} 1 & 0 & -2 & 1 & 1 & 2 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right) \tag{1.3.7.11}$$

$$\xleftrightarrow{R_2=-2R_2} \left(\begin{array}{ccccc|c} 1 & 0 & -2 & 1 & 1 & 2 \\ 0 & 1 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right) \tag{1.3.7.12}$$

$$\xleftrightarrow{R_1=R_1+R_3, R_4=R_4+R_3, R_3=-R_3} \left(\begin{array}{ccccc|c} 1 & 0 & -2 & 1 & 0 & 1 \\ 0 & 1 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \tag{1.3.7.13}$$

So,

$$x_1 - 2x_3 + x_4 = 1 \tag{1.3.7.14}$$
$$x_2 + x_3 - x_4 = 2 \tag{1.3.7.15}$$
$$x_5 = 1 \tag{1.3.7.16}$$

Solving the equations we get,

$$x_1 = 1 + 2x_3 - x_4 \tag{1.3.7.17}$$
$$x_2 = 2 - x_3 + x_4 \tag{1.3.7.18}$$
$$x_5 = 1 \tag{1.3.7.19}$$

which can be written as,

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \tag{1.3.7.20}$$

$$\implies \mathbf{x} = \begin{pmatrix} 1 + 2x_3 - x_4 \\ 2 - x_3 + x_4 \\ x_3 \\ x_4 \\ 1 \end{pmatrix} \tag{1.3.7.21}$$

We can express (1.3.7.21) as a sum of linear combination of vectors,

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \mathbf{x_3} + \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \mathbf{x_4} \tag{1.3.7.22}$$

where $x_3, x_4 \in \mathbb{R}$.
Note that the above solution space is not closed on vector addition and scalar multiplication.As $x_5 = 1$, the zero vector is not included in the solution space. Hence, **x** is not a vector space. Since, **x** is not a vector space, it cannot be expressed in the form of linear combination of basis vectors.

1.3.8. Let

$$\mathbf{A} = \begin{pmatrix} 3 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 0 \end{pmatrix} \tag{1.3.8.1}$$

For which triples (y_1, y_2, y_3) does the system $\mathbf{AX} = \mathbf{Y}$ have a solution ?
Solution:
Given ,

$$\mathbf{AX} = \mathbf{Y} \tag{1.3.8.2}$$

$$\begin{pmatrix} 3 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 0 \end{pmatrix} \mathbf{X} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \tag{1.3.8.3}$$

Now we try to find the matrix **B** such that **BA** gives the row echelon form of matrix **A**.
Here, **B** is given by ,

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ -\frac{7}{5} & \frac{8}{5} & 1 \end{pmatrix} \tag{1.3.8.4}$$

$$\implies \mathbf{BA} = \begin{pmatrix} 3 & -1 & 2 \\ 0 & \frac{5}{3} & -\frac{1}{3} \\ 0 & 0 & -\frac{6}{5} \end{pmatrix} \tag{1.3.8.5}$$

Therefore, from (1.3.8.5) rank of matrix **A** is 3 and it is a full rank matrix.
Hence the columns of **A** are linearly independent.
Therefore, the triples (y_1, y_2, y_3) are linear combination of columns of matrix **A**.

$$\implies \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = a \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + b \begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix} + c \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \tag{1.3.8.6}$$

where a,b,c can be any real value.

1.3.9. Let

$$\mathbf{A} = \begin{pmatrix} 3 & -6 & 2 & -1 \\ -2 & 4 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 1 & -2 & 1 & 0 \end{pmatrix} \tag{1.3.9.1}$$

For which (y_1, y_2, y_3, y_4) does the system of equations $\mathbf{AX} = \mathbf{Y}$ have a solution ? **Solution:**
Given ,

$$\mathbf{AX} = \mathbf{Y} \tag{1.3.9.2}$$

$$\begin{pmatrix} 3 & -6 & 2 & -1 \\ -2 & 4 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 1 & -2 & 1 & 0 \end{pmatrix} \mathbf{X} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} \tag{1.3.9.3}$$

Now we try to find the matrix **B** such that **BA** gives the row echelon form of matrix **A** Here,**B** is given by ,

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{2}{3} & 1 & 0 & 0 \\ -\frac{2}{7} & -\frac{3}{7} & 1 & 0 \\ 0 & \frac{1}{2} & -\frac{3}{2} & 1 \end{pmatrix} \tag{1.3.9.4}$$

$$\mathbf{BA} = \begin{pmatrix} 3 & -6 & 2 & -1 \\ 0 & 0 & \frac{7}{3} & \frac{7}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1.3.9.5}$$

Therefore, rank of matrix **A** is 2 Now **B** is expressed in terms of two block matrices

$$\mathbf{B} = \begin{pmatrix} \mathbf{B_1} \\ \mathbf{B_2} \end{pmatrix} \tag{1.3.9.6}$$

$$\mathbf{B_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{2}{3} & 1 & 0 & 0 \end{pmatrix} \tag{1.3.9.7}$$

$$\mathbf{B_2} = \begin{pmatrix} -\frac{2}{7} & -\frac{3}{7} & 1 & 0 \\ 0 & \frac{1}{2} & -\frac{3}{2} & 1 \end{pmatrix} \tag{1.3.9.8}$$

Multiplying matrix **B** to both sides on the equation (1.3.9.2), we get ,

$$\begin{pmatrix} \mathbf{B_1} \\ \mathbf{B_2} \end{pmatrix} \mathbf{AX} = \begin{pmatrix} \mathbf{B_1} \\ \mathbf{B_2} \end{pmatrix} \mathbf{Y} \tag{1.3.9.9}$$

We know that , matrix **A** is of rank 2 The augumented matrix of (1.3.9.9) is given by

$$\left(\begin{array}{c|c} \mathbf{B_1A} & \mathbf{B_1Y} \\ \mathbf{B_2A} & \mathbf{B_2Y} \end{array}\right) \tag{1.3.9.10}$$

$$\mathbf{B_1A} = \begin{pmatrix} 3 & -6 & 2 & -1 \\ 0 & 0 & \frac{7}{3} & \frac{7}{3} \end{pmatrix} \tag{1.3.9.11}$$

$$\mathbf{B_2A} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1.3.9.12}$$

Since $\mathbf{B_2A}$ is zero matrix and for the given system $\mathbf{AX} = \mathbf{Y}$ to have a solution,

$$\mathbf{B_2Y} = 0 \tag{1.3.9.13}$$

$$\begin{pmatrix} -\frac{2}{7} & -\frac{3}{7} & 1 & 0 \\ 0 & \frac{1}{2} & -\frac{3}{2} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = 0 \tag{1.3.9.14}$$

The augumented matrix of (1.3.9.14) is given by,

$$\left(\begin{array}{cccc|c} -\frac{2}{7} & -\frac{3}{7} & 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{3}{2} & 1 & 0 \end{array}\right) \tag{1.3.9.15}$$

By row reduction technique,

$$\xleftrightarrow{R_1=-\frac{7}{2}R_1} \left(\begin{array}{cccc|c} 1 & \frac{3}{2} & -\frac{7}{2} & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{3}{2} & 1 & 0 \end{array}\right) \tag{1.3.9.16}$$

$$\xleftrightarrow{R_2=2R_2} \left(\begin{array}{cccc|c} 1 & \frac{3}{2} & -\frac{7}{2} & 0 & 0 \\ 0 & 1 & -3 & 2 & 0 \end{array}\right) \tag{1.3.9.17}$$

$$\xleftrightarrow{R_1=R_1-\frac{3}{2}R_2} \left(\begin{array}{cccc|c} 1 & 0 & 1 & -3 & 0 \\ 0 & 1 & -3 & 2 & 0 \end{array}\right) \tag{1.3.9.18}$$

Equation (1.3.9.14) can be modified as ,

$$\begin{pmatrix} 1 & 0 & 1 & -3 \\ 0 & 1 & -3 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = 0 \tag{1.3.9.19}$$

Here y_3 and y_4 are free variables
If $y_3 = a$ and $y_4 = b$,then the solution to the system of equation $\mathbf{AX} = \mathbf{Y}$ is given by,

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = a \begin{pmatrix} -1 \\ 3 \\ 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 3 \\ -2 \\ 0 \\ 1 \end{pmatrix} \tag{1.3.9.20}$$

One of the solution when $a = 1$ and $b = 2$ is given by ,

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 1 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 3 \\ -2 \\ 0 \\ 1 \end{pmatrix} \tag{1.3.9.21}$$

1.3.10. Suppose **R** and $\mathbf{R}'$ are 2×3 row-reduced echelon matrices and that the system **RX**=0 and $\mathbf{R}'\mathbf{X}$=0 have exactly the same solutions. Prove that $\mathbf{R} = \mathbf{R}'$.

Solution:

Since **R** and $\mathbf{R}'$ are 2×3 row-reduced echelon matrices they can be of following three types:-

a) Suppose matrix **R** has one non-zero row then **RX**=0 will have two free variables.Since $\mathbf{R}'\mathbf{X}$=0 will have the exact same solution as $\mathbf{RX} = 0, \mathbf{R}'\mathbf{X}$=0 will also have two free variables. Thus $\mathbf{R}'$ have one non zero row. Now let's consider a matrix **A** with the first row as the non-zero row **R** and second row as the second row of $\mathbf{R}'$.

$$\mathbf{R} = \begin{pmatrix} 1 & a & b \\ 0 & 0 & 0 \end{pmatrix} \tag{1.3.10.1}$$

$$\mathbf{R}' = \begin{pmatrix} 1 & c & d \\ 0 & 0 & 0 \end{pmatrix} \tag{1.3.10.2}$$

$$\tag{1.3.10.3}$$

Let **X** satisfy

$$\mathbf{RX} = 0 \tag{1.3.10.4}$$

$$\begin{pmatrix} 1 & \mathbf{a}^T \end{pmatrix} \begin{pmatrix} x \\ \mathbf{y} \end{pmatrix} = 0 \tag{1.3.10.5}$$

$$x + \mathbf{a}^T \mathbf{y} = 0 \tag{1.3.10.6}$$

where

$$\mathbf{a} = \begin{pmatrix} a \\ b \end{pmatrix} \quad (1.3.10.7)$$

$$\mathbf{R}'\mathbf{X} = 0 \quad (1.3.10.8)$$

$$\begin{pmatrix} 1 & \mathbf{b}^T \end{pmatrix} \begin{pmatrix} x \\ \mathbf{y} \end{pmatrix} = 0 \quad (1.3.10.9)$$

$$x + \mathbf{b}^T \mathbf{y} = 0 \quad (1.3.10.10)$$

where

$$\mathbf{b} = \begin{pmatrix} c \\ d \end{pmatrix} \quad (1.3.10.11)$$

Subtracting (1.3.10.10) from (1.3.10.6),

$$x + \mathbf{a}^T \mathbf{y} - x - \mathbf{b}^T \mathbf{y} = 0 \quad (1.3.10.12)$$

$$(\mathbf{a}^T - \mathbf{b}^T)\mathbf{y} = 0 \quad (1.3.10.13)$$

Since $\mathbf{y}$ is a 2×1 vector,

$$\implies y_1\mathbf{a} - y_2\mathbf{b} = 0 \quad (1.3.10.14)$$

Which can be written as,

$$\mathbf{a} = k\mathbf{b} \quad (1.3.10.15)$$

where,k=$\frac{\mathbf{y}_2}{\mathbf{y}_1}$ assuming $\mathbf{y}_1 \neq 0$.Now,Substituting (1.3.10.15) in (1.3.10.6)

$$x + k\mathbf{b}^T \mathbf{y} = 0 \quad (1.3.10.16)$$

Comparing (1.3.10.16) with (1.3.10.10)

$$x + \mathbf{b}^T \mathbf{y} = 0 \quad (1.3.10.17)$$

$$x + k\mathbf{b}^T \mathbf{y} = 0 \quad (1.3.10.18)$$

Hence k=1 which means y_1=y_2 and from this we can say that **a**=**b**.If in the above case we take y_1=0 then

$$y_1\mathbf{a} - y_2\mathbf{b} = 0 \quad (1.3.10.19)$$

$$y_2\mathbf{b} = 0 \quad (1.3.10.20)$$

Hence for the (1.3.10.20) to be always true **b** should be zero.Now from (1.3.10.15) we will see that **a** will also be 0. Hence, **R**=**R′**

b) Let **R** and $\mathbf{R}'$ have all rows as non zero.

$$\mathbf{R} = \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & c \end{pmatrix} \quad (1.3.10.21)$$

$$\mathbf{R}' = \begin{pmatrix} 1 & 0 & e \\ 0 & 1 & f \end{pmatrix} \quad (1.3.10.22)$$

Let **X** satisfy

$$\mathbf{R}\mathbf{X} = 0 \quad (1.3.10.23)$$

$$\mathbf{X}^T\mathbf{R}^T = 0 \quad (1.3.10.24)$$

Here,

$$\mathbf{R} = \begin{pmatrix} \mathbf{I} & \mathbf{a} \end{pmatrix} \quad (1.3.10.25)$$

$$\mathbf{a} = \begin{pmatrix} b \\ c \end{pmatrix} \quad (1.3.10.26)$$

$$\mathbf{R}^T = \begin{pmatrix} \mathbf{I} \\ \mathbf{a}^T \end{pmatrix} \quad (1.3.10.27)$$

Let,

$$\mathbf{X}^T = \begin{pmatrix} \mathbf{y}^T & z \end{pmatrix} \quad (1.3.10.28)$$

where z is a scalar constant. Now,substituting (1.3.10.28) and (1.3.10.25) in (1.3.10.24)

$$\begin{pmatrix} \mathbf{y}^T & z \end{pmatrix} \begin{pmatrix} \mathbf{I} \\ \mathbf{a}^T \end{pmatrix} = 0 \quad (1.3.10.29)$$

$$\mathbf{y}^T + z\mathbf{a}^T = 0 \quad (1.3.10.30)$$

Now for,

$$\mathbf{R}'\mathbf{X} = 0 \quad (1.3.10.31)$$

$$\mathbf{X}^T\mathbf{R}'^T = 0 \quad (1.3.10.32)$$

Here,

$$\mathbf{R}' = \begin{pmatrix} \mathbf{I} & \mathbf{b} \end{pmatrix} \quad (1.3.10.33)$$

$$\mathbf{b} = \begin{pmatrix} e \\ f \end{pmatrix} \quad (1.3.10.34)$$

Let,

$$\mathbf{X}^T = \begin{pmatrix} \mathbf{y}^T & z \end{pmatrix} \quad (1.3.10.35)$$

where z is a scalar constant.Now,substituting (1.3.10.35) and (1.3.10.33) in (1.3.10.32)

$$\begin{pmatrix} \mathbf{y}^T & z \end{pmatrix} \begin{pmatrix} \mathbf{I} \\ \mathbf{b}^T \end{pmatrix} = 0 \quad (1.3.10.36)$$

$$\mathbf{y}^T + z\mathbf{b}^T = 0 \quad (1.3.10.37)$$

Subtracting (1.3.10.37) from (1.3.10.30)

$$\mathbf{y}^T + z\mathbf{a}^T - \mathbf{y}^T - z\mathbf{b}^T = 0 \quad (1.3.10.38)$$

$$(\mathbf{a}^T - \mathbf{b}^T)z = 0 \quad (1.3.10.39)$$

$$\mathbf{a}^T = \mathbf{b}^T \quad (1.3.10.40)$$

c) Suppose matrix **R** have all the rows as zero then **RX**=0 will be satisfied for all values of **X**. We know that $\mathbf{R}'\mathbf{X}$=0 will have the exact same solution as **RX**=0 then we can say that for all values of **X**=0 equation $\mathbf{R}'\mathbf{X}$=0 will be satisfied.Hence, $\mathbf{R}'$=**R**=0.

1.4 Matrix Multiplication

1.4.1. Let

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 2 & 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 1 & -1 \end{pmatrix} \tag{1.4.1.1}$$

Compute **ABC** and **CAB**.
Solution: Given,

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 2 & 1 \end{pmatrix} \tag{1.4.1.2}$$

$$\mathbf{B} = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} \tag{1.4.1.3}$$

$$\mathbf{C} = \begin{pmatrix} 1 & -1 \end{pmatrix} \tag{1.4.1.4}$$

Take, **ABC** = (**AB**) **C**

$$\mathbf{AB} = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} \tag{1.4.1.5}$$

$$\mathbf{AB} = \begin{pmatrix} 6 - 1 - 1 \\ 3 + 2 - 1 \end{pmatrix} \tag{1.4.1.6}$$

$$\mathbf{AB} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \tag{1.4.1.7}$$

Now,

$$\mathbf{ABC} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \begin{pmatrix} 1 & -1 \end{pmatrix} \tag{1.4.1.8}$$

$$\mathbf{ABC} = \begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix} \tag{1.4.1.9}$$

similarly, **CAB** = **C**(**AB**)

$$\mathbf{CAB} = \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 4 \\ 4 \end{pmatrix} \tag{1.4.1.10}$$

$$\implies \mathbf{CAB} = 0 \tag{1.4.1.11}$$

therefore,

$$\mathbf{ABC} = \begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix} \tag{1.4.1.12}$$

$$\mathbf{CAB} = 0 \tag{1.4.1.13}$$

1.4.2. Let

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 2 & -2 \\ 1 & 3 \\ 4 & 4 \end{pmatrix} \tag{1.4.2.1}$$

Verify directly that $A(AB) = A^2B$ **Solution:**

$$A^2 = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{pmatrix} \tag{1.4.2.2}$$

$$A^2 = \begin{pmatrix} 2 & -1 & 1 \\ 5 & -2 & 3 \\ 6 & -3 & 4 \end{pmatrix} \tag{1.4.2.3}$$

and

$$AB = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & -2 \\ 1 & 3 \\ 4 & 4 \end{pmatrix} \tag{1.4.2.4}$$

$$AB = \begin{pmatrix} 5 & -1 \\ 8 & 0 \\ 10 & -2 \end{pmatrix} \tag{1.4.2.5}$$

Now RHS is

$$A^2B = \begin{pmatrix} 2 & -1 & 1 \\ 5 & -2 & 3 \\ 6 & -3 & 4 \end{pmatrix} \begin{pmatrix} 2 & -2 \\ 1 & 3 \\ 4 & 4 \end{pmatrix} \tag{1.4.2.6}$$

$$A^2B = \begin{pmatrix} 7 & -3 \\ 20 & -4 \\ 25 & -5 \end{pmatrix} \tag{1.4.2.7}$$

Now LHS is

$$A(AB) = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 5 & -1 \\ 8 & 0 \\ 10 & -2 \end{pmatrix} \tag{1.4.2.8}$$

$$A(AB) = \begin{pmatrix} 7 & -3 \\ 20 & -4 \\ 25 & -5 \end{pmatrix} \tag{1.4.2.9}$$

Hence verified.

1.4.3. Find two different 2×2 matrices **A** such that $\mathbf{A}^2 = 0$ but $\mathbf{A} \neq 0$
Solution: The matrix **A** can be given by,

$$\mathbf{A} = \begin{pmatrix} \mathbf{m} & \mathbf{n} \end{pmatrix} \tag{1.4.3.1}$$

$$\mathbf{m} = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, \mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \tag{1.4.3.2}$$

Now,

$$\mathbf{A}^2 = \mathbf{AA} = \mathbf{0} \tag{1.4.3.3}$$

$$\implies \mathbf{A}^2 = \begin{pmatrix}\mathbf{Am} & \mathbf{An}\end{pmatrix} = \begin{pmatrix}\mathbf{0} & \mathbf{0}\end{pmatrix} \tag{1.4.3.4}$$

From (1.4.3.4), we say that the the null space of **A** contains columns of matrix **A**. Also atleast one of the columns must be non-zero since given $\mathbf{A} \neq 0$. Thus, the null space of **A** contains non zero vectors, $rank(\mathbf{A}) < 2$. Hence, **A** is a singular matrix. This implies that the columns of **A** are linearly dependent.

$$\mathbf{Ax} = 0 \tag{1.4.3.5}$$

$$\begin{pmatrix}\mathbf{m} & \mathbf{n}\end{pmatrix}\begin{pmatrix}x_1 \\ x_2\end{pmatrix} = 0 \tag{1.4.3.6}$$

$$x_1\mathbf{m} + x_2\mathbf{n} = 0 \tag{1.4.3.7}$$

$$\mathbf{n} = \frac{-x_1}{x_2}\mathbf{m} \tag{1.4.3.8}$$

$$\implies \mathbf{n} = k\mathbf{m} \tag{1.4.3.9}$$

where $\mathbf{m} \neq 0$ as $\mathbf{A} \neq 0$
Now from (1.4.3.4),

$$\mathbf{Am} = 0 \tag{1.4.3.10}$$

$$m_1\mathbf{m} + m_2\mathbf{n} = 0 \tag{1.4.3.11}$$

$$(m_1 + km_2)\,\mathbf{m} = 0 \tag{1.4.3.12}$$

Thus we get, $m_1 = -km_2$

$$\mathbf{A} = \begin{pmatrix}-km_2 & -k^2m_2 \\ m_2 & km_2\end{pmatrix}; m_2 \neq 0 \tag{1.4.3.13}$$

(1.4.3.9) can be written as,

$$\implies \mathbf{m} = \frac{1}{k}\mathbf{n} \tag{1.4.3.14}$$

$$\implies \mathbf{m} = c\mathbf{n} \tag{1.4.3.15}$$

where $\mathbf{n} \neq 0$ as $\mathbf{A} \neq 0$
From (1.4.3.4),

$$\mathbf{An} = 0 \tag{1.4.3.16}$$

$$n_1\mathbf{m} + n_2\mathbf{n} = 0 \tag{1.4.3.17}$$

$$(cn_1 + n_2)\,\mathbf{n} = 0 \tag{1.4.3.18}$$

Thus we get, $n_2 = -cn_1$

$$\mathbf{A} = \begin{pmatrix}cn_1 & n_1 \\ -c^2n_1 & -cn_1\end{pmatrix}; n_1 \neq 0 \tag{1.4.3.19}$$

From (1.4.3.13), (1.4.3.19) two different 2×2 matrices **A** can be given as,

$$\mathbf{A} = \begin{pmatrix}0 & 0 \\ 2 & 0\end{pmatrix} \tag{1.4.3.20}$$

$$\mathbf{A} = \begin{pmatrix}0 & 2 \\ 0 & 0\end{pmatrix} \tag{1.4.3.21}$$

1.4.4. For the matrix $\mathbf{A} = \begin{pmatrix}1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1\end{pmatrix}$, find elementary matrices $\mathbf{E_1},\mathbf{E_2},...,\mathbf{E_k}$ such that

$$\mathbf{E_k}...\mathbf{E_2}\mathbf{E_1}\mathbf{A} = \mathbf{I} \tag{1.4.4.1}$$

Solution: Given,

$$\mathbf{A} = \begin{pmatrix}1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1\end{pmatrix} \tag{1.4.4.2}$$

Take,

$$\mathbf{E_1} = \begin{pmatrix}1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1\end{pmatrix} \tag{1.4.4.3}$$

$$\mathbf{E_2} = \begin{pmatrix}1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1\end{pmatrix} \tag{1.4.4.4}$$

$$\mathbf{E_3} = \begin{pmatrix}1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1\end{pmatrix} \tag{1.4.4.5}$$

$$\mathbf{E_4} = \begin{pmatrix}1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{pmatrix} \tag{1.4.4.6}$$

$$\mathbf{E_5} = \begin{pmatrix}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1\end{pmatrix} \tag{1.4.4.7}$$

$$\mathbf{E_6} = \begin{pmatrix}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{2}{7}\end{pmatrix} \tag{1.4.4.8}$$

$$\mathbf{E_7} = \begin{pmatrix}1 & 0 & \frac{-1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{pmatrix} \tag{1.4.4.9}$$

$$\mathbf{E_8} = \begin{pmatrix}1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1\end{pmatrix} \tag{1.4.4.10}$$

Now, we calculate

$$\mathbf{E_8E_7E_6E_5E_4E_3E_2E_1} = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 0 & 3 & -2 \end{pmatrix} \tag{1.4.4.11}$$

Hence,

$$\left(\mathbf{E_8E_7E_6E_5E_4E_3E_2E_1}\right)\mathbf{A} = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 0 & 3 & -2 \end{pmatrix}\begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.4.4.12}$$

1.4.5. Let $A = \begin{pmatrix} 1 & -1 \\ 2 & 2 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 1 \\ -4 & 4 \end{pmatrix}$ Is there any matrix C such that $CA = B$?

Solution: The matrix B is obtained by multiplying the matrix A with matrix C. B is a 2×2 matrix and A is a 3×2 matrix. so matrix C must be a 2×3 matrix. Let the matrix C is:

$$C = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix} \tag{1.4.5.1}$$

$$\implies C^T = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \end{pmatrix} \tag{1.4.5.2}$$

So, after multiplying with A matrix we get,

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 2 & 2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a_1 + 2b_1 + c_1 & -a_1 + 2b_1 \\ a_2 + 2b_2 + c_2 & -a_2 + 2b_2 \end{pmatrix} \tag{1.4.5.3}$$

Matrix A is a rectangular matrix. Now, Considering $CA = B$ and by transposing both side,

$$(CA)^T = B^T \tag{1.4.5.4}$$

$$\implies A^TC^T = B^T \tag{1.4.5.5}$$

$$\implies \begin{pmatrix} 1 & 2 & 1 \\ -1 & 2 & 0 \end{pmatrix}\begin{pmatrix} \mathbf{c_1} & \mathbf{c_2} \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 1 & 4 \end{pmatrix} \tag{1.4.5.6}$$

We can represent it like this:

$$\begin{pmatrix} 1 & 2 & 1 \\ -1 & 2 & 0 \end{pmatrix}\mathbf{c_1} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \tag{1.4.5.7}$$

$$\tag{1.4.5.8}$$

Now the augmented matrix is:

$$\begin{pmatrix} 1 & 2 & 1 & 3 \\ -1 & 2 & 0 & 1 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_1 + R_2} \begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 4 & 1 & 4 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2/2} \begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 2 & \frac{1}{2} & 2 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 - R_2} \begin{pmatrix} 1 & 0 & \frac{1}{2} & 1 \\ 0 & 2 & \frac{1}{2} & 2 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2/2} \begin{pmatrix} 1 & 0 & \frac{1}{2} & 1 \\ 0 & 1 & \frac{1}{4} & 1 \end{pmatrix} \tag{1.4.5.9}$$

Similarly,

$$\begin{pmatrix} 1 & 2 & 1 \\ -1 & 2 & 0 \end{pmatrix}\mathbf{c_2} = \begin{pmatrix} -4 \\ 4 \end{pmatrix} \tag{1.4.5.10}$$

$$\tag{1.4.5.11}$$

Now the augmented matrix is:

$$\begin{pmatrix} 1 & 2 & 1 & -4 \\ -1 & 2 & 0 & 4 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_1 + R_2} \begin{pmatrix} 1 & 2 & 1 & -4 \\ 0 & 4 & 1 & 0 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2/2} \begin{pmatrix} 1 & 2 & 1 & -4 \\ 0 & 2 & \frac{1}{2} & 0 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 - R_2} \begin{pmatrix} 1 & 0 & \frac{1}{2} & -4 \\ 0 & 2 & \frac{1}{2} & 0 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2/2} \begin{pmatrix} 1 & 0 & \frac{1}{2} & -4 \\ 0 & 1 & \frac{1}{4} & 0 \end{pmatrix} \tag{1.4.5.12}$$

From equations 1.4.5.9 and 1.4.5.12, it can be observed that solutions exist and there is a matrix C such that $CA = B$. Now,

$$\mathbf{c_1} = \begin{pmatrix} 1 - \frac{c_1}{2} \\ 1 - \frac{c_1}{4} \\ c_1 \end{pmatrix} \tag{1.4.5.13}$$

$$\implies \mathbf{c_1} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_1\begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{4} \\ 1 \end{pmatrix} \tag{1.4.5.14}$$

$$\mathbf{c_2} = \begin{pmatrix} -4 - \frac{c_2}{2} \\ -\frac{c_2}{4} \\ c_2 \end{pmatrix} \tag{1.4.5.15}$$

$$\implies \mathbf{c_2} = \begin{pmatrix} -4 \\ 0 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{4} \\ 1 \end{pmatrix} \tag{1.4.5.16}$$

Now,

$$C^T = \begin{pmatrix} 1 & -4 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} + c_1 \begin{pmatrix} -\frac{1}{2} & 0 \\ -\frac{1}{4} & 0 \\ 1 & 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 & -\frac{1}{2} \\ 0 & -\frac{1}{4} \\ 0 & 1 \end{pmatrix}$$

$$\implies C = \begin{pmatrix} 1 & 1 & 0 \\ -4 & 0 & 0 \end{pmatrix} + c_1 \begin{pmatrix} -\frac{1}{2} & -\frac{1}{4} & 1 \\ 0 & 0 & 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{4} & 1 \end{pmatrix} \quad (1.4.5.17)$$

Now,

$$CA = \begin{pmatrix} 1 & 1 & 0 \\ -4 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 2 & 2 \\ 1 & 0 \end{pmatrix} + c_1 \begin{pmatrix} -\frac{1}{2} & -\frac{1}{4} & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 2 & 2 \\ 1 & 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{4} & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 2 & 2 \\ 1 & 0 \end{pmatrix}$$

$$\implies CA = \begin{pmatrix} 3 & 1 \\ -4 & 4 \end{pmatrix} + c_1 \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\implies CA = B \quad (1.4.5.18)$$

Hence, it is proved that there there exist a matrix C such that $CA = B$.

1.4.6. Let $\mathbf{A}$ be an $m \times n$ matrix and $\mathbf{B}$ be an $n \times k$ matrix.Show that the columns of $\mathbf{C} = \mathbf{AB}$ are linear combinations of columns of $\mathbf{A}$.If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the columns of $\mathbf{A}$ and $\gamma_1, \gamma_2, \ldots, \gamma_k$ are the columns of $\mathbf{C}$ then,

$$\gamma_j = \sum_{r=1}^{n} B_{rj} \alpha_r \quad (1.4.6.1)$$

Solution:

$$\mathbf{C} = \mathbf{AB} \quad (1.4.6.2)$$

$$\mathbf{C} = \begin{pmatrix} \gamma_1 & \gamma_2 & \ldots & \gamma_k \end{pmatrix} \quad (1.4.6.3)$$

$$\mathbf{A} = \begin{pmatrix} \alpha_1 & \alpha_2 & \ldots & \alpha_n \end{pmatrix} \quad (1.4.6.4)$$

$$\mathbf{B} = \begin{pmatrix} \beta_1 & \beta_2 & \ldots & \beta_k \end{pmatrix} \quad (1.4.6.5)$$

$$= \begin{pmatrix} B_{11} & B_{12} & \ldots & B_{1k} \\ B_{21} & B_{22} & \ldots & B_{2k} \\ \vdots & \vdots & \ldots & \vdots \\ B_{n1} & B_{n2} & \ldots & B_{nk} \end{pmatrix} \quad (1.4.6.6)$$

By matrix multiplication, we can write

$$\begin{pmatrix} \gamma_1 & \gamma_2 & \ldots & \gamma_k \end{pmatrix} = \begin{pmatrix} \mathbf{A}\beta_1 & \mathbf{A}\beta_2 & \ldots & \mathbf{A}\beta_k \end{pmatrix} \quad (1.4.6.7)$$

Consider γ_1

$$\gamma_1 = \mathbf{A}\beta_1 \quad (1.4.6.8)$$

$$= \begin{pmatrix} \alpha_1 & \alpha_2 & \ldots & \alpha_n \end{pmatrix} \begin{pmatrix} B_{11} \\ B_{21} \\ \vdots \\ B_{n1} \end{pmatrix} \quad (1.4.6.9)$$

$$= B_{11}\alpha_1 + B_{21}\alpha_2 + \ldots + B_{n1}\alpha_n \quad (1.4.6.10)$$

Similarly, considering j^{th} column of $\mathbf{C}$

$$\gamma_j = \begin{pmatrix} \alpha_1 & \alpha_2 & \ldots & \alpha_n \end{pmatrix} \begin{pmatrix} B_{1j} \\ B_{2j} \\ \vdots \\ B_{nj} \end{pmatrix} \quad (1.4.6.11)$$

$$= B_{1j}\alpha_1 + B_{2j}\alpha_2 + \ldots + B_{nj}\alpha_n \quad (1.4.6.12)$$

$$\implies \gamma_j = \sum_{r=1}^{n} B_{rj}\alpha_r \quad (1.4.6.13)$$

which proves that columns of $\mathbf{C}$ are linear combinations of columns of $\mathbf{A}$

1.4.7. Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ matrices such that $\mathbf{AB} = \mathbf{I}$. Prove that $\mathbf{BA} = \mathbf{I}$. **Solution:** Let $\mathbf{BX} = 0$ be a system of linear equation with n unknowns and n equations as $\mathbf{B}$ is $n \times n$ matrix. Hence,

$$\mathbf{BX} = 0 \quad (1.4.7.1)$$

$$\implies \mathbf{A}(\mathbf{BX}) = 0 \quad (1.4.7.2)$$

$$\implies (\mathbf{AB})\mathbf{X} = 0 \quad (1.4.7.3)$$

$$\implies \mathbf{IX} = 0 \quad [\because \mathbf{AB} = \mathbf{I}] \quad (1.4.7.4)$$

$$\implies \mathbf{X} = 0 \quad (1.4.7.5)$$

From (1.4.7.5) since $\mathbf{X} = 0$ is the only solution of (1.4.7.1), hence $rank(\mathbf{B}) = n$. Which implies all columns of $\mathbf{B}$ are linearly independent. Hence $\mathbf{B}$ is invertible. Therefore, every left inverse of $\mathbf{B}$ is also a right inverse of $\mathbf{B}$. Hence there exists a $n \times n$ matrix $\mathbf{C}$ such that,

$$\mathbf{BC} = \mathbf{CB} = \mathbf{I} \quad (1.4.7.6)$$

Again given that **AB** = **I**. Hence,

$$\mathbf{AB} = \mathbf{I} \tag{1.4.7.7}$$
$$\implies \mathbf{ABC} = \mathbf{C} \tag{1.4.7.8}$$
$$\implies \mathbf{A(BC)} = \mathbf{C} \tag{1.4.7.9}$$
$$\implies \mathbf{A} = \mathbf{C} \qquad [\because \mathbf{BC} = \mathbf{I}] \tag{1.4.7.10}$$

Hence using (1.4.7.10) and (1.4.7.6) we can write,

$$\mathbf{BA} = \mathbf{I} \tag{1.4.7.11}$$

Hence Proved.

1.4.8. Let,

$$\mathbf{C} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \tag{1.4.8.1}$$

be a 2×2 matrix. We inquire when it is possible to find 2×2 matrices **A** and **B** such that **C**=**AB** − **BA**. Prove that such matrices can be found if and only if $C_{11}+C_{22}$=0. **Solution:** We have to find,

$$tr(\mathbf{C}) = C_{11} + C_{22} = tr(\mathbf{AB} - \mathbf{BA}) \tag{1.4.8.2}$$
$$\implies tr(\mathbf{C}) = tr(\mathbf{AB}) - tr(\mathbf{BA}) \tag{1.4.8.3}$$

We know that,

$$tr(\mathbf{AB}) = \sum_{i=1}^{2} (\mathbf{AB})_{ii} \tag{1.4.8.4}$$
$$\implies \sum_{i=1}^{2}\sum_{j=1}^{2} a_{ij}b_{ji} \tag{1.4.8.5}$$
$$\implies \sum_{j=1}^{2}\sum_{i=1}^{2} b_{ji}a_{ij} \tag{1.4.8.6}$$
$$\implies tr(\mathbf{AB}) = \sum_{j=1}^{2} \mathbf{BA}_{jj} \tag{1.4.8.7}$$
$$\implies tr(\mathbf{AB}) = tr(\mathbf{BA}) \tag{1.4.8.8}$$

Substituting equation (1.4.8.8) to (1.4.8.3) we get

$$\implies tr(\mathbf{C}) = tr(\mathbf{AB}) - tr(\mathbf{BA}) = 0 \tag{1.4.8.9}$$

1.5 Invertible Matrices

1.5.1. Let

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 & 0 \\ -1 & 0 & 3 & 5 \\ 1 & -2 & 1 & 1 \end{pmatrix} \tag{1.5.1.1}$$

Find a row-reduced echelon matrix **R** which is row-equivalent to **A** and an invertible 3x3 matrix **P** such that **R** = **P A**. **Solution:** Given

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 & 0 \\ -1 & 0 & 3 & 5 \\ 1 & -2 & 1 & 1 \end{pmatrix} \tag{1.5.1.2}$$

Row reduce **A** by applying the elementary row operations and equivalently at each operations find the elementary matrix **E**

$$\mathbf{A}|\mathbf{I} = \left(\begin{array}{cccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 3 & 5 & 0 & 1 & 0 \\ 1 & -2 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \tag{1.5.1.3}$$

$$\xleftrightarrow{R_2=R_2+R_1} \left(\begin{array}{cccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 4 & 5 & 1 & 1 & 0 \\ 1 & -2 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \tag{1.5.1.4}$$

$$\xleftrightarrow{R_3=R_3-R_1} \left(\begin{array}{cccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 4 & 5 & 1 & 1 & 0 \\ 0 & -4 & 0 & 1 & -1 & 0 & 1 \end{array}\right) \tag{1.5.1.5}$$

$$\xleftrightarrow{R_1=R_1-R_2} \left(\begin{array}{cccc|ccc} 1 & 0 & -3 & -5 & 0 & -1 & 0 \\ 0 & 2 & 4 & 5 & 1 & 1 & 0 \\ 0 & -4 & 0 & 1 & -1 & 0 & 1 \end{array}\right) \tag{1.5.1.6}$$

$$\xleftrightarrow{R_3=R_3+2R_2} \left(\begin{array}{cccc|ccc} 1 & 0 & -3 & -5 & 0 & -1 & 0 \\ 0 & 2 & 4 & 5 & 1 & 1 & 0 \\ 0 & 0 & 8 & 11 & 1 & 2 & 1 \end{array}\right) \tag{1.5.1.7}$$

$$\xleftrightarrow{R_2=\frac{R_2}{2}} \left(\begin{array}{cccc|ccc} 1 & 0 & -3 & -5 & 0 & -1 & 0 \\ 0 & 1 & 2 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 8 & 11 & 1 & 2 & 1 \end{array}\right) \tag{1.5.1.8}$$

$$\xleftrightarrow{R_3=\frac{R_3}{8}} \left(\begin{array}{cccc|ccc} 1 & 0 & -3 & -5 & 0 & -1 & 0 \\ 0 & 1 & 2 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{11}{8} & \frac{1}{8} & \frac{1}{4} & \frac{1}{8} \end{array}\right) \tag{1.5.1.9}$$

$$\xleftrightarrow{R_1=R_1+3R_3} \left(\begin{array}{cccc|ccc} 1 & 0 & 0 & -\frac{7}{8} & \frac{3}{8} & -\frac{1}{4} & \frac{3}{8} \\ 0 & 1 & 2 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{11}{8} & \frac{1}{8} & \frac{1}{4} & \frac{1}{8} \end{array}\right) \tag{1.5.1.10}$$

$$\xleftrightarrow{R_2=R_2-2R_3}\left(\begin{array}{cccc|ccc}1 & 0 & 0 & -\frac{7}{8} & \frac{3}{8} & -\frac{1}{4} & \frac{3}{8}\\ 0 & 1 & 0 & -\frac{1}{4} & \frac{1}{4} & 0 & -\frac{1}{4}\\ 0 & 0 & 1 & \frac{11}{8} & \frac{1}{8} & \frac{1}{4} & \frac{1}{8}\end{array}\right) \tag{1.5.1.11}$$

Hence,row reduced echelon matrix that is row equivalent to **A** is

$$\mathbf{R}=\begin{pmatrix}1 & 0 & 0 & -\frac{7}{8}\\ 0 & 1 & 0 & -\frac{1}{4}\\ 0 & 0 & 1 & \frac{11}{8}\end{pmatrix} \tag{1.5.1.12}$$

where **E** is the elementary matrices that transform **A** to **R** Thus:-

$$\mathbf{EA}=\mathbf{R} \tag{1.5.1.13}$$

Since elementary matrices is invertible

$$\mathbf{P}=\mathbf{E} \tag{1.5.1.14}$$

is invertible.
From (1.5.1.11)

$$\mathbf{P}=\begin{pmatrix}\frac{3}{8} & -\frac{1}{4} & \frac{3}{8}\\ \frac{1}{4} & 0 & -\frac{1}{4}\\ \frac{1}{8} & \frac{1}{4} & \frac{1}{8}\end{pmatrix} \tag{1.5.1.15}$$

$$\mathbf{R}=\begin{pmatrix}1 & 0 & 0 & -\frac{7}{8}\\ 0 & 1 & 0 & -\frac{1}{4}\\ 0 & 0 & 1 & \frac{11}{8}\end{pmatrix} \tag{1.5.1.16}$$

such that $\mathbf{R}=\mathbf{PA}$.

1.5.2. Let $\mathbf{A}=\begin{pmatrix}2 & 0 & i\\ 1 & -3 & -i\\ i & 1 & 1\end{pmatrix}$, find a row-reduced echelon matrix **R** which is row-equivalent to **A** and an invertible 3x3 matrix **P** such that **R** = **P A**. **Solution:** Given,

$$\mathbf{A}=\begin{pmatrix}2 & 0 & i\\ 1 & -3 & -i\\ i & 1 & 1\end{pmatrix} \tag{1.5.2.1}$$

Row reduce **A** by applying the elementary row operations and equivalently at each operations find the elementary matrix **E**

$$[\mathbf{A}\ \mathbf{I}]=\left(\begin{array}{ccc|ccc}2 & 0 & i & 1 & 0 & 0\\ 1 & -3 & -i & 0 & 1 & 0\\ i & 1 & 1 & 0 & 0 & 1\end{array}\right) \tag{1.5.2.2}$$

$$\xleftrightarrow{R_1\leftrightarrow R_2}\left(\begin{array}{ccc|ccc}1 & -3 & -i & 0 & 1 & 0\\ 2 & 0 & i & 1 & 0 & 0\\ i & 1 & 1 & 0 & 0 & 1\end{array}\right) \tag{1.5.2.3}$$

$$\xleftrightarrow{R_2\leftarrow R_2-2R_1}\left(\begin{array}{ccc|ccc}1 & -3 & -i & 0 & 1 & 0\\ 0 & 6 & 3i & 1 & -2 & 0\\ i & 1 & 1 & 0 & 0 & 1\end{array}\right) \tag{1.5.2.4}$$

$$\xleftrightarrow{R_3\leftarrow R_3-iR_1}\left(\begin{array}{ccc|ccc}1 & -3 & -i & 0 & 1 & 0\\ 0 & 6 & 3i & 1 & -2 & 0\\ 0 & 1+3i & 0 & 0 & -i & 1\end{array}\right) \tag{1.5.2.5}$$

$$\xleftrightarrow{R_2\leftarrow \frac{R_2}{6}}\left(\begin{array}{ccc|ccc}1 & -3 & -i & 0 & 1 & 0\\ 0 & 1 & \frac{i}{2} & \frac{1}{6} & -\frac{1}{3} & 0\\ 0 & 1+3i & 0 & 0 & -i & 1\end{array}\right) \tag{1.5.2.6}$$

$$\xleftrightarrow{R_1\leftarrow\ R_1+3R_2}\left(\begin{array}{ccc|ccc}1 & 0 & \frac{i}{2} & \frac{1}{2} & 0 & 0\\ 0 & 1 & \frac{i}{2} & \frac{1}{6} & -\frac{1}{3} & 0\\ 0 & 1+3i & 0 & 0 & -i & 1\end{array}\right) \tag{1.5.2.7}$$

$$\xleftrightarrow{R_3\leftarrow\ R_3/(3-i)/2}\left(\begin{array}{ccc|ccc}1 & 0 & \frac{i}{2} & \frac{1}{2} & 0 & 0\\ 0 & 1 & \frac{i}{2} & \frac{1}{6} & -\frac{1}{3} & 0\\ 0 & 0 & 1 & -\frac{(i)}{3} & \frac{3+i}{15} & \frac{3+i}{5}\end{array}\right) \tag{1.5.2.8}$$

$$\xleftrightarrow{R_1\leftarrow R_1-\frac{i}{2}R_3}\left(\begin{array}{ccc|ccc}1 & 0 & 0 & \frac{1}{3} & \frac{1-3i}{30} & \frac{1-3i}{10}\\ 0 & 1 & \frac{i}{2} & \frac{1}{6} & -\frac{1}{3} & 0\\ 0 & 0 & 1 & -\frac{(i)}{3} & \frac{3+i}{15} & \frac{3+i}{5}\end{array}\right) \tag{1.5.2.9}$$

$$\xleftrightarrow{R_2\leftarrow R_2-\frac{i}{2}R_3}\left(\begin{array}{ccc|ccc}1 & 0 & 0 & \frac{1}{3} & \frac{1-3i}{30} & \frac{1-3i}{10}\\ 0 & 1 & 0 & 0 & -\frac{3+i}{10} & \frac{1-3i}{10}\\ 0 & 0 & 1 & -\frac{i}{3} & \frac{3+i}{15} & \frac{3+i}{5}\end{array}\right) \tag{1.5.2.10}$$

$$=[\mathbf{I}\ \mathbf{E}]$$

Hence, the row reduced matrix that is row equivalent to **A** is

$$\mathbf{R}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}=\mathbf{I} \tag{1.5.2.11}$$

Using Gauss-Jordan Elimination, if there exists an elimentary matrix **E** such that **E**[**A** **I**] = [**I** **E**] then **E** is the inverse of A i.e

$\mathbf{E} = \mathbf{A^{-1}}$

$$\mathbf{E} = \mathbf{A^{-1}} = \begin{pmatrix} \frac{1}{3} & \frac{1-3i}{30} & \frac{1-3i}{10} \\ 0 & -\frac{3+i}{10} & \frac{1-3i}{10} \\ -\frac{i}{3} & \frac{3+i}{15} & \frac{3+i}{5} \end{pmatrix} \quad (1.5.2.12)$$

Since,

$$\mathbf{R} = \mathbf{PA} \implies \mathbf{P} = \mathbf{A^{-1}R} \quad (1.5.2.13)$$

$$\mathbf{P} = \begin{pmatrix} \frac{1}{3} & \frac{1-3i}{30} & \frac{1-3i}{10} \\ 0 & -\frac{3+i}{10} & \frac{1-3i}{10} \\ -\frac{i}{3} & \frac{3+i}{15} & \frac{3+i}{5} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (1.5.2.14)$$

Thus,

$$\mathbf{P} = \begin{pmatrix} \frac{1}{3} & \frac{1-3i}{30} & \frac{1-3i}{10} \\ 0 & -\frac{3+i}{10} & \frac{1-3i}{10} \\ -\frac{i}{3} & \frac{3+i}{15} & \frac{3+i}{5} \end{pmatrix}$$

$$\mathbf{R} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

1.5.3. For each of the two matrices use elementary row operations to discover whether it is invertible, and to find the inverse in case it is invertible.

$$\mathbf{A} = \begin{pmatrix} 2 & 5 & -1 \\ 4 & -1 & 2 \\ 6 & 4 & 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 2 & 4 \\ 0 & 1 & -2 \end{pmatrix}$$

Solution: Given

$$\mathbf{A} = \begin{pmatrix} 2 & 5 & -1 \\ 4 & -1 & 2 \\ 6 & 4 & 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 2 & 4 \\ 0 & 1 & -2 \end{pmatrix} \quad (1.5.3.1)$$

By applying row reductions on $\mathbf{A}$

$$\begin{pmatrix} 2 & 5 & -1 \\ 4 & -1 & 2 \\ 6 & 4 & 1 \end{pmatrix} \xleftrightarrow{R_2=R_2-2R_1} \mathbf{A} = \begin{pmatrix} 2 & 5 & -1 \\ 0 & -11 & 4 \\ 6 & 4 & 1 \end{pmatrix} \quad (1.5.3.2)$$

$$\xleftrightarrow{R_3=R_3-3R_1} \begin{pmatrix} 2 & 5 & -1 \\ 0 & -11 & 4 \\ 0 & -11 & 4 \end{pmatrix} \quad (1.5.3.3)$$

$$\xleftrightarrow{R_1=\frac{R_1}{2}} \begin{pmatrix} 1 & \frac{5}{2} & \frac{-1}{2} \\ 0 & -11 & 4 \\ 0 & -11 & 4 \end{pmatrix} \quad (1.5.3.4)$$

$$\xleftrightarrow{R_3=R_3-R_2} \begin{pmatrix} 1 & \frac{5}{2} & \frac{-1}{2} \\ 0 & -11 & 4 \\ 0 & 0 & 0 \end{pmatrix} \quad (1.5.3.5)$$

$$\xleftrightarrow{R_2=\frac{-R_2}{11}} \begin{pmatrix} 1 & \frac{5}{2} & \frac{-1}{2} \\ 0 & 1 & \frac{-4}{11} \\ 0 & 0 & 0 \end{pmatrix} \quad (1.5.3.6)$$

$$\xleftrightarrow{R_1=R_1-\frac{5}{2}R_2} \begin{pmatrix} 1 & 0 & \frac{9}{22} \\ 0 & 1 & \frac{-4}{11} \\ 0 & 0 & 0 \end{pmatrix} \quad (1.5.3.7)$$

For a matrix to be invertible, it has to be a matrix of full rank. However the matrix $\mathbf{A}$ is not of full rank ($Rank(\mathbf{A}) < 3$).Therefore $\mathbf{A}$ is not invertible.
Let us now consider augmented matrix $\mathbf{B}|\mathbf{I}$, By applying row reductions on $\mathbf{B}|\mathbf{I}$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 3 & 2 & 4 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 \end{array}\right) \xleftrightarrow{R_2=R_2-3R_1}$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 5 & -2 & -3 & 1 & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 \end{array}\right) \quad (1.5.3.8)$$

$$\xleftrightarrow{R_2=\frac{R_2}{5}} \left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{-2}{5} & \frac{-3}{5} & \frac{1}{5} & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 \end{array}\right) \quad (1.5.3.9)$$

$$\xleftrightarrow{R_1=R_1+R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{8}{5} & \frac{2}{5} & \frac{1}{5} & 0 \\ 0 & 1 & \frac{-2}{5} & \frac{-3}{5} & \frac{1}{5} & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 \end{array}\right) \quad (1.5.3.10)$$

$$\xleftrightarrow{R_3=R_3-R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{8}{5} & \frac{2}{5} & \frac{1}{5} & 0 \\ 0 & 1 & \frac{-2}{5} & \frac{-3}{5} & \frac{1}{5} & 0 \\ 0 & 0 & \frac{-8}{5} & \frac{3}{5} & \frac{-1}{5} & 1 \end{array}\right) \quad (1.5.3.11)$$

$$\xleftrightarrow{R_1=R_1+R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & \frac{-2}{5} & \frac{-3}{5} & \frac{1}{5} & 0 \\ 0 & 0 & \frac{-8}{5} & \frac{3}{5} & \frac{-1}{5} & 1 \end{array}\right) \quad (1.5.3.12)$$

$$\xleftrightarrow{R_3=\frac{-5}{8}R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & \frac{-2}{5} & \frac{-3}{5} & \frac{1}{5} & 0 \\ 0 & 0 & 1 & \frac{-3}{8} & \frac{1}{8} & \frac{-5}{8} \end{array}\right) \quad (1.5.3.13)$$

$$\xleftrightarrow{R_3=R_2+\frac{2}{5}R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & \frac{-3}{4} & \frac{1}{4} & \frac{-1}{4} \\ 0 & 0 & 1 & \frac{-3}{8} & \frac{1}{8} & \frac{-5}{8} \end{array}\right) \quad (1.5.3.14)$$

For a matrix to be invertible, it has to be a matrix of full rank. Here, the matrix $\mathbf{B}$ is of full rank ($Rank(\mathbf{B}) = 3$). Therefore $\mathbf{B}$ is invertible and the inverse matrix $\mathbf{B}^{-1}$ can be written from (1.5.3.14):

$$\mathbf{B}^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ \frac{-3}{4} & \frac{1}{4} & \frac{-1}{4} \\ \frac{-3}{8} & \frac{1}{8} & \frac{-5}{8} \end{pmatrix} \tag{1.5.3.15}$$

1.5.4. Let

$$\mathbf{A} = \begin{pmatrix} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 1 & 5 \end{pmatrix} \tag{1.5.4.1}$$

For which $\mathbf{X}$ does there exist a scalar c such that $\mathbf{AX} = c\mathbf{X}$

Solution: Given

$$\mathbf{A} = \begin{pmatrix} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 1 & 5 \end{pmatrix} \tag{1.5.4.2}$$

The given matrix has single eigenvalue as it is the lower triangular matrix and has equal diagonal elements. Hence $c_1 = c_2 = c_3 = 5$. To find the corresponding eigenvector, consider the following

$$(\mathbf{A} - c\mathbf{I})\mathbf{X} = 0 \tag{1.5.4.3}$$

$$\implies \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{1.5.4.4}$$

Solving the homogeneous system of linear equations by performing rref, we get

$$\begin{pmatrix} 30 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xleftrightarrow[R_3 \leftrightarrow R_2]{R_2 \leftrightarrow R_1} \begin{pmatrix} 31 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1.5.4.5}$$

Hence we get,

$$x_1 = 0, x_2 = 0, x_3 = t \tag{1.5.4.6}$$

where, x_3 is arbitrary. Therefore,

$$\mathbf{X} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} t \tag{1.5.4.7}$$

Hence, the given matrix has single eigenvector and is not diagonalizable.

1.5.5. Discover whether

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{pmatrix} \tag{1.5.5.1}$$

is invertible, and find $\mathbf{A^{-1}}$ if it exists.

Solution: The matrix $\mathbf{A}$ is in row reduced echolon form with four pivot elements. Therefore the rank($\mathbf{A}$) is 4. Hence the rows of matrix $\mathbf{A}$ constitute of 4 linearly independent vectors. Thus it can be concluded that matrix $\mathbf{A}$ is invertible. Using Gauss-Jordan Elimination, if there exists an elimentary matrix $\mathbf{E}$ such that $\mathbf{E}[\mathbf{A}\ \mathbf{I}] = [\mathbf{I}\ \mathbf{E}]$ then $\mathbf{E}$ is the inverse of A i.e $\mathbf{E} = \mathbf{A^{-1}}$.

$$[\mathbf{A}\ \mathbf{I}] = \left(\begin{array}{cccc|cccc} 1 & 2 & 3 & 4 & 1 & 0 & 0 & 0 \\ 0 & 2 & 3 & 4 & 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 1 \end{array}\right) \tag{1.5.5.2}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - R_2} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 2 & 3 & 4 & 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 1 \end{array}\right) \tag{1.5.5.3}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - R_3} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & 4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 1 \end{array}\right) \tag{1.5.5.4}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - R_4} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 1 \end{array}\right) \tag{1.5.5.5}$$

$$\xleftrightarrow[R_2 \leftarrow \frac{R_2}{2}\ R_3 \leftarrow \frac{R_3}{3}]{R_4 \leftarrow \frac{R_4}{4}} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{4} \end{array}\right) = [\mathbf{I}\ \mathbf{E}] \tag{1.5.5.6}$$

Therefore, for the given problem,

$$\mathbf{A^{-1}} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix} \tag{1.5.5.7}$$

Generalization of above result to a matrix of any arbitrary size: Let

$$\mathbf{A} = \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_N \\ 0 & a_2 & a_3 & \dots & a_N \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & \dots & \dots & a_N \end{pmatrix} \tag{1.5.5.8}$$

Then

$$\mathbf{E}_1\mathbf{A} = \begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & \dots & \dots & 1 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_N \\ 0 & a_2 & a_3 & \dots & a_N \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & \dots & \dots & a_N \end{pmatrix} \tag{1.5.5.9}$$

$$= \begin{pmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & a_3 & \dots & a_N \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & \dots & \dots & a_N \end{pmatrix} \tag{1.5.5.10}$$

$$\mathbf{E}_2\mathbf{E}_1\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & -1 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & \dots & \dots & 1 \end{pmatrix} \begin{pmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & a_3 & \dots & a_N \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & \dots & \dots & a_N \end{pmatrix} \tag{1.5.5.11}$$

$$= \begin{pmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & \dots & \dots & a_N \end{pmatrix} \tag{1.5.5.12}$$

Proceeding in similar manner, we get

$$\mathbf{E}_N\mathbf{E}_{N-1}\dots\mathbf{E}_2\mathbf{E}_1\mathbf{A} = \mathbf{U} = \begin{pmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & 0 & \dots & 0 \\ 0 & 0 & a_3 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & \dots & \dots & a_N \end{pmatrix} \tag{1.5.5.13}$$

$$= \operatorname{diag}\begin{pmatrix} a_1 & a_2 & \dots & a_N \end{pmatrix} \tag{1.5.5.14}$$

$$\implies \quad \mathbf{A} = \mathbf{LU} \tag{1.5.5.15}$$

where $\quad \mathbf{L} = \mathbf{E}_1^{-1}\mathbf{E}_2^{-1}\dots\mathbf{E}_N^{-1}$

$$\implies \quad \mathbf{A}^{-1} = \mathbf{U}^{-1}\mathbf{L}^{-1} \tag{1.5.5.16}$$

$$\implies \quad \mathbf{A}^{-1} = \begin{pmatrix} \frac{1}{a_1} & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{a_2} & 0 & \dots & 0 \\ 0 & 0 & \frac{1}{a_3} & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & \dots & \dots & \frac{1}{a_N} \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & -1 & \dots & 0 \\ \vdots & & \vdots & & \\ 0 & \dots & \dots & \dots & 1 \end{pmatrix} \tag{1.5.5.17}$$

Therefore

$$\mathbf{A^{-1}} = \begin{pmatrix} \frac{1}{a_1} & -\frac{1}{a_1} & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{a_2} & -\frac{1}{a_2} & 0 & \dots & 0 \\ 0 & 0 & \frac{1}{a_3} & -\frac{1}{a_3} & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & \frac{1}{a_N} \end{pmatrix} \tag{1.5.5.18}$$

From (1.5.5.18) for the above problem

$$\mathbf{A^{-1}} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix} \tag{1.5.5.19}$$

1.5.6. Suppose **A** is a 2×1 matrix and **B** is 1×2 matrix.Prove that **C**=**AB** is non invertible.

Solution: Let's take **A** and **B** to be non zero vectors. Now,we know that for **C** to be non invertible **Cx** = 0 should have a non trivial solution.So,

$$\mathbf{Cx} = 0 \tag{1.5.6.1}$$

$$\implies \mathbf{ABx} = 0 \tag{1.5.6.2}$$

Here, we know that **B** is 1 × 2 matrix and **x** is 2 × 1 matrix then **Bx** will result to a scalar constant k.

$$\implies \mathbf{A}k = 0 \tag{1.5.6.3}$$

For (1.5.6.3) to be true k should be zero. We also know that **B** is 1 × 2 matrix i.e. rows are less than column hence,

$$\mathbf{Bx} = 0 \tag{1.5.6.4}$$

will have a non trivial solution. Hence, using (1.5.6.3) and (1.5.6.4) we can say,

$$\mathbf{ABx} = 0 \tag{1.5.6.5}$$

will have a non trivial solution so, **C** is non invertible.

1.5.7. Let $\mathbf{A}$ be an $n \times n$ (square) matrix, Prove the following two statements:

a) If $\mathbf{A}$ is invertible and $\mathbf{AB} = 0$ for some $n \times n$ matrix $\mathbf{B}$, then $\mathbf{B} = 0$.

b) If $\mathbf{A}$ is not invertible, then there exists an $n \times n$ matrix $\mathbf{B}$ such that $\mathbf{AB} = 0$ but $\mathbf{B} \neq 0$.

Solution:

a) Given $\mathbf{A}$ is an invertible matrix and $\mathbf{AB} = 0$ then,

$$\mathbf{AB} = 0 \tag{1.5.7.1}$$

$$\implies \mathbf{A}^{-1}(\mathbf{AB}) = 0 \tag{1.5.7.2}$$

$$\implies (\mathbf{A}^{-1}\mathbf{A})\mathbf{B} = 0 \tag{1.5.7.3}$$

$$\implies \mathbf{IB} = 0 \quad [\because \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}] \tag{1.5.7.4}$$

$$\implies \mathbf{B} = 0 \tag{1.5.7.5}$$

b) If $\mathbf{A}$ is not invertible, then there exists an $n \times n$ matrix $\mathbf{B}$ such that $\mathbf{AB} = 0$ but $\mathbf{B} \neq 0$. Since $\mathbf{A}$ is not invertible, $\mathbf{AX} = 0$ must have a non-trivial solution. Let the non-trivial solution be,

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \tag{1.5.7.6}$$

Let $\mathbf{B}$ which is an $n \times n$ matrix have all its columns as $\mathbf{y}$.

$$\mathbf{B} = \begin{pmatrix} \mathbf{y} & \mathbf{y} & \cdots & \mathbf{y} \end{pmatrix} \tag{1.5.7.7}$$

From equation (1.5.7.7), we can say that $\mathbf{B} \neq 0$ but $\mathbf{AB} = 0$

1.5.8. Let

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{1.5.8.1}$$

Prove, using elementary row operations that A is invertible if and only if $(ad - bc) \neq 0$

Solution:

The goal is to effect the transformation $(\mathbf{A}|\mathbf{I}) \to \left(\mathbf{I}|\mathbf{A}^{-1}\right)$. Augmenting $\mathbf{A}$ with the 2×2 identity matrix, we get:

$$\begin{pmatrix} a & b & 1 & 0 \\ c & d & 0 & 1 \end{pmatrix} \tag{1.5.8.2}$$

Now, if $a = 0$, switch the rows. If c is also 0, then the process of reducing $\mathbf{A}$ to $\mathbf{I}$ cannot even begin. So, one necessary condition for $\mathbf{A}$ to be invertible is that the entries a and c are not both 0.

a) Assume that $a \neq 0$, Then :

$$\begin{pmatrix} a & b & 1 & 0 \\ c & d & 0 & 1 \end{pmatrix} \xleftrightarrow{R_1 = R_1/a} \begin{pmatrix} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ c & d & 0 & 1 \end{pmatrix} \tag{1.5.8.3}$$

$$\xleftrightarrow{R_2 = R_2 - cR_1} \begin{pmatrix} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & \frac{ad-bc}{a} & \frac{-c}{a} & 1 \end{pmatrix}$$

Next, assuming that $ad - bc \neq 0$, we get:

$$\xleftrightarrow{R_1 = R_1 - \frac{b}{ad-bc}R_2} \begin{pmatrix} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & \frac{ad-bc}{a} & \frac{-c}{a} & 1 \end{pmatrix}$$

$$\xleftrightarrow{R_2 = R_2 \frac{a}{ad-bc}} \begin{pmatrix} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$$

Therefore, if $ad - bc \neq 0$, then the matrix is invertible and it's inverse is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \tag{1.5.8.4}$$

b) In (1.5.8.3), we have assumed that $a \neq 0$. Now consider $a = 0$, then, as we have seen before, it is mandatory that $c \neq 0$:

$$\begin{pmatrix} 0 & b & 1 & 0 \\ c & d & 0 & 1 \end{pmatrix} \xleftrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} c & d & 0 & 1 \\ 0 & b & 1 & 0 \end{pmatrix} \tag{1.5.8.5}$$

$$\xleftrightarrow{R_1 = R_1/c} \begin{pmatrix} 1 & \frac{d}{c} & 0 & \frac{1}{c} \\ 0 & b & 1 & 0 \end{pmatrix}$$

$$\xleftrightarrow{R_1 = R_1 - R_2 \times \frac{d}{bc}} \begin{pmatrix} 1 & 0 & -\frac{d}{bc} & \frac{1}{c} \\ 0 & b & 1 & 0 \end{pmatrix}$$

$$\xleftrightarrow{R_2 = R_2/b} \begin{pmatrix} 1 & 0 & -\frac{d}{bc} & \frac{1}{c} \\ 0 & 1 & \frac{1}{b} & 0 \end{pmatrix}$$

Therefore, When we consider $a = 0$ the matrix is invertible if $bc \neq 0$, which is included in the condition $ad - bc \neq 0$.

c) Similarly, consider $c = 0$, then, as we have seen before, it is mandatory that $a \neq 0$:

$$\begin{pmatrix} a & b & 1 & 0 \\ 0 & d & 0 & 1 \end{pmatrix} \xleftrightarrow{R_1=R_1/a} \begin{pmatrix} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & d & 0 & 1 \end{pmatrix} \tag{1.5.8.6}$$

$$\xleftrightarrow{R_1=R_1-R_2\times\frac{b}{ad}} \begin{pmatrix} 1 & 0 & \frac{1}{a} & -\frac{b}{ad} \\ 0 & d & 0 & 1 \end{pmatrix}$$

$$\xleftrightarrow{R_2=R_2/d} \begin{pmatrix} 1 & 0 & \frac{1}{a} & -\frac{b}{ad} \\ 0 & 1 & 0 & \frac{1}{d} \end{pmatrix}$$

Therefore, When we consider $c = 0$, the matrix is invertible if $ad \neq 0$, which is included in the condition $ad - bc \neq 0$.

Hence, it is proved from above three cases that the given matrix is invertible iff $ad - bc \neq 0$.

1.5.9. An $n \times n$ matrix $\mathbf{A}$ is called upper-triangular if $\mathbf{A_{ij}} = 0$ for $i > j$, that is, if every entry below the main diagonal is 0. Prove that an upper-triangular (square) matrix is invertible if and only if every entry on its main diagonal is different from 0. **Solution:** An $n \times n$ matrix $\mathbf{A}$ is called upper-triangular if $\mathbf{A_{ij}} = 0$ for $i > j$, that is, if every entry below the main diagonal is 0. Prove that an upper-triangular (square) matrix is invertible if and only if every entry on its main diagonal is different from 0. Considering $\mathbf{A}$, an upper triangular matrix. Using the property that determinant of upper triangular matrix is the product of diagonal elements,

$$\left|\mathbf{A}\right| = \prod_{i=1}^{n} a_{i,i} \tag{1.5.9.1}$$

If $\mathbf{A}$ be invertible then $\left|\mathbf{A}\right| \neq 0$. Hence from (1.5.9.1) we get,

$$\prod_{i=1}^{n} a_{i,i} \neq 0 \tag{1.5.9.2}$$

if any diagonal element is 0 then (1.5.9.2) won't be right hence no diagonal elements should be 0. Hence Proved.

1.5.10. Let A be a $m \times n$ matrix. Show that by a finite number of elementary row and/or column operations one can pass from A to a matrix R which is both row-reduced echelon and column-reduced echelon , i.e., $R_{ij} = 0$ if $i \neq j$, $R_{ii} = 1$, $1 \leq i \leq r$, $R_{ii} = 0$, if $i > r$. Show that $R = PAQ$, where P is an invertible $m \times m$ matrix and Q is an invertible $n \times n$ matrix.

Solution:

Lemma Every elementary matrix is invertible and the inverse is again an elementary matrix. If an elementary matrix E is obtained from I by using a certain row or column operation q, then E^{-1} is obtained from I by the "inverse" operation q^{-1}.

Solution Given $\mathbf{A}$ is a $m\times n$ matrix. Converting $\mathbf{A}$ into row reduced echelon form by performing a series of elementary row operations $\mathbf{P}$. Let $\mathbf{R}'$ be the row reduced echelon matrix. Also, by using the lemma we can tell that $\mathbf{P}$ is invertible and order $m \times m$.

$$\mathbf{R}' = \mathbf{PA} \tag{1.5.10.1}$$

where,

$$\mathbf{R}' = \begin{pmatrix} \mathbf{I} & \mathbf{F} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$$

$\mathbf{I}$ is an identity matrix, $\mathbf{F}$ is Free variables matrix and $\mathbf{0}$ represents a block of zeroes

$\mathbf{R}'$ is in row-reduced echelon form. To perform column operations, elementary matrices should be multiplied on the right side in order to convert the $\mathbf{R}'$ into column-reduced echelon form

$$\mathbf{R} = \mathbf{R}'\mathbf{Q} \tag{1.5.10.2}$$

But performing column operations on a matrix is equivalent to performing row operations on the transposed matrix.

$$\mathbf{R}^T = (\mathbf{R}'\mathbf{Q})^T$$
$$\implies \mathbf{R}^T = \mathbf{Q}^T\mathbf{R}'^T \tag{1.5.10.3}$$

Hence, by using lemma it can be observed that $\mathbf{Q}^T$ is invertible and of the order $n\times n$. Converting $\mathbf{R}^T$ to row-reduced echelon is equivalent to converting $\mathbf{R}$ to column-reduced echelon.

$$\mathbf{R} = \mathbf{PAQ} \tag{1.5.10.4}$$

where,

$$\mathbf{R} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \tag{1.5.10.5}$$

$\mathbf{I}$ is an identity matrix and $\mathbf{0}$ represents a block

of zeroes. $\mathbf{Q}$ is a upper triangular matrix. $\mathbf{R}$ in (1.5.10.4) is in both row and column reduced echelon form. Hence proved.
Example Let,

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 5 & 7 \\ 1 & 2 & 3 & 4 \end{pmatrix} \tag{1.5.10.6}$$

To convert (1.5.10.6) into row reduced echelon form, $\mathbf{A}$ has to be multiplied by $\mathbf{P}$

$$\mathbf{P} = \begin{pmatrix} -5 & 3 & 0 \\ 2 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \tag{1.5.10.7}$$

$$\mathbf{R}' = \mathbf{PA} = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1.5.10.8}$$

$\mathbf{R}'$ is in row reduced echelon form. To convert (1.5.10.8) into column-reduced echelon form, elementary operations have to be performed on $\mathbf{R}'^T$. By multiplying all the elementary matrices,

$$\mathbf{Q}^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & 1 & 0 & 0 \\ -1 & 0 & -1 & 1 \end{pmatrix} \tag{1.5.10.9}$$

$$\implies \mathbf{Q} = \begin{pmatrix} 1 & 0 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.5.10.10}$$

So $\mathbf{PAQ}$ is in both row-reduced and column-reduced echelon form.

$$\mathbf{R} = \mathbf{PAQ} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1.5.10.11}$$

The inverses of $\mathbf{P}$ and $\mathbf{Q}$ are,

$$\mathbf{P}^{-1} = \begin{pmatrix} 1 & 3 & 0 \\ 2 & 5 & 0 \\ 1 & 3 & 1 \end{pmatrix}; \quad \mathbf{Q}^{-1} = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.5.10.12}$$

2 Vector Spaces

2.1 Vector Spaces

2.1.1. If $\mathbf{F}$ is a field, verify that vector space of all ordered n-tuples $\mathbf{F}^n$ is a vector space over the field $\mathbf{F}$.

Solution: Let $\mathbf{F}^n$ be a set of all ordered n-tuples over $\mathbf{F}$ i.e

$$\mathbf{F}^n = \left\{ \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} : a_1, a_2, \dots, a_n \in \mathbf{F} \right\} \tag{2.1.1.1}$$

For $\mathbf{F}^n$ to be a vector space over $\mathbf{F}$ it must satisfy the closure property of vector addition and scalar multiplication.

Vector Addition in $\mathbf{F}^n$:
Let $\alpha = \begin{pmatrix} a_i \end{pmatrix}$ and $\beta = \begin{pmatrix} b_i \end{pmatrix}$ $\forall\ i = 1, 2, \cdots, n \in \mathbf{F}^n$ then

$$\alpha + \beta = \begin{pmatrix} a_i \end{pmatrix} + \begin{pmatrix} b_i \end{pmatrix} \tag{2.1.1.2}$$
$$= \begin{pmatrix} a_i + b_i \end{pmatrix} \tag{2.1.1.3}$$

Since

$$a_i + b_i \in \mathbf{F}\ \forall\ i = 1, 2, \cdots, n \tag{2.1.1.4}$$
$$\implies \alpha + \beta \in \mathbf{F}^n \tag{2.1.1.5}$$

Scalar multiplication in $\mathbf{F}^n$ over $\mathbf{F}$:
Let $\alpha = \begin{pmatrix} a_i \end{pmatrix}$ $\forall\ i = 1, 2, \cdots, n$ $\in \mathbf{F}^n$ and $a \in \mathbf{F}$ then

$$a\alpha = \begin{pmatrix} aa_i \end{pmatrix} \tag{2.1.1.6}$$

Since

$$aa_i \in \mathbf{F}\ \forall\ i = 1, 2 \cdots, n \tag{2.1.1.7}$$
$$\implies a\alpha \in \mathbf{F}^n \tag{2.1.1.8}$$

Associativity of addition in $\mathbf{F}^n$:
Let $\alpha = \begin{pmatrix} a_i \end{pmatrix}$, $\beta = \begin{pmatrix} b_i \end{pmatrix}$, $\gamma = \begin{pmatrix} g_i \end{pmatrix}$ $\forall\ i = 1, 2, \cdots, n \in \mathbf{F}^n$ then

$$\alpha + (\beta + \gamma) = \begin{pmatrix} a_i \end{pmatrix} + \begin{pmatrix} b_i + g_i \end{pmatrix} \tag{2.1.1.9}$$
$$= \begin{pmatrix} a_i + b_i + g_i \end{pmatrix} \tag{2.1.1.10}$$
$$= \begin{pmatrix} a_i + b_i \end{pmatrix} + \begin{pmatrix} g_i \end{pmatrix} \tag{2.1.1.11}$$
$$= (\alpha + \beta) + \gamma \tag{2.1.1.12}$$

Existence of additive identity in $\mathbf{F}^n$:

We have $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbf{F}^n$ and $\alpha = \begin{pmatrix} a_i \end{pmatrix}$ $\forall\ i =$

$1,2,\cdots,n \in \mathbf{F}^n$ then

$$\left(a_i\right)+\left(0\right)=\left(a_i+0\right) \tag{2.1.1.13}$$
$$=\left(a_i\right) \tag{2.1.1.14}$$

Therefore $\mathbf{0}$ is the additive identity in $\mathbf{F}^n$.

Existence of additive inverse of each element of $\mathbf{F}^n$:
If $\alpha = \left(a_i\right)$ $\forall$ $i = 1,2,\cdots,n \in \mathbf{F}^n$ then $\left(-a_i\right) \in \mathbf{F}^n$. Also we have

$$\left(-a_i\right)+\left(a_i\right)=\mathbf{0} \tag{2.1.1.15}$$

Therefore $-\alpha = \left(-a_i\right)$ is the additive inverse of α. Thus $\mathbf{F}^n$ is an abelian group with respect to addition.

Futher we observe that

a) If $a \in \mathbf{F}$ and $\alpha = \left(a_i\right)$, $\beta = \left(b_i\right)$ $\forall$ $i = 1,2,\cdots,n \in \mathbf{F}^n$ then

$$a(\alpha+\beta)=a\left(a_i+b_i\right) \tag{2.1.1.16}$$
$$=\left(a[a_i+b_i]\right) \tag{2.1.1.17}$$
$$=\left(aa_i+ab_i\right) \tag{2.1.1.18}$$
$$\left(aa_i\right)+\left(ab_i\right) \tag{2.1.1.19}$$
$$=a\left(a_i\right)+a\left(b_i\right) \tag{2.1.1.20}$$
$$=a\alpha+a\beta \tag{2.1.1.21}$$

b) If $a,b \in \mathbf{F}$ and $\alpha = \left(a_i\right)$ $\forall$ $i = 1,2,\cdots,n \in \mathbf{F}^n$ then

$$(a+b)\alpha=\left([a+b]a_i\right) \tag{2.1.1.22}$$
$$=\left(aa_i+ba_i\right) \tag{2.1.1.23}$$
$$=\left(aa_i\right)+\left(ba_i\right) \tag{2.1.1.24}$$
$$=a\left(a_i\right)+b\left(a_i\right) \tag{2.1.1.25}$$
$$=a\alpha+b\alpha \tag{2.1.1.26}$$

c) If $a,b \in \mathbf{F}$ and $\alpha = \left(a_i\right)$ $\forall$ $i = 1,2,\cdots,n \in \mathbf{F}^n$ then

$$(ab)\alpha=\left([ab]a_i\right) \tag{2.1.1.27}$$
$$=\left(a[ba_i]\right) \tag{2.1.1.28}$$
$$=a\left(ba_i\right) \tag{2.1.1.29}$$
$$=a(b\alpha) \tag{2.1.1.30}$$

d) If 1 is the unity element of $\mathbf{F}$ and $\alpha = \left(a_i\right)$ $\forall$ $i = 1,2,\cdots,n \in \mathbf{F}^n$ then

$$1\alpha=\left(1a_i\right) \tag{2.1.1.31}$$
$$=\left(a_i\right) \tag{2.1.1.32}$$
$$=\alpha \tag{2.1.1.33}$$

Hence $\mathbf{F}^n$ is a vector space over $\mathbf{F}$.

2.1.2. If $\mathbf{V}$ is a vector space over field $\mathbf{F}$,verify that:

$$(\alpha_1+\alpha_2)+(\alpha_3+\alpha_4)=[\alpha_2+(\alpha_3+\alpha_1)]+\alpha_4 \tag{2.1.2.1}$$

Solution: Using property of commutativity of (+) in $\mathbf{V}$

$$(\alpha_1+\alpha_2)+(\alpha_3+\alpha_4)=(\alpha_2+\alpha_1)+(\alpha_3+\alpha_4) \tag{2.1.2.2}$$

Using property of associativity of (+) in $\mathbf{V}$

$$(\alpha_2+\alpha_1)+(\alpha_3+\alpha_4)=\alpha_2+[\alpha_1+(\alpha_3+\alpha_4)] \tag{2.1.2.3}$$

Using property of commutativity of (+) in $\mathbf{V}$

$$\alpha_2+[\alpha_1+(\alpha_3+\alpha_4)]=\alpha_2+(\alpha_3+\alpha_1)+\alpha_4 \tag{2.1.2.4}$$

Using property of associativity of (+) in $\mathbf{V}$

$$\alpha_2+(\alpha_3+\alpha_1)+\alpha_4=[\alpha_2+(\alpha_3+\alpha_1]+\alpha_4 \tag{2.1.2.5}$$

2.1.3. If $\mathbb{C}$ is the field of complex numbers, which vectors in $\mathbf{C}^3$ are linear combinations of $\begin{pmatrix}1\\0\\-1\end{pmatrix}$, $\begin{pmatrix}0\\1\\1\end{pmatrix}$ and $\begin{pmatrix}1\\1\\1\end{pmatrix}$?

Solution: Expressing the given vectors as the columns of a matrix,

$$\mathbf{A}=\begin{pmatrix}1&0&1\\0&1&1\\-1&1&1\end{pmatrix} \tag{2.1.3.1}$$

The row reduced echelon form of the matrix on performing elementary row operations can be given as,

$$\mathbf{R} = \mathbf{C}\mathbf{A} \tag{2.1.3.2}$$

where $\mathbf{C}$ is the product of elementary matrices,

$$\mathbf{C} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} \tag{2.1.3.3}$$

Thus we get,

$$\mathbf{R} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.1.3.4}$$

From (2.1.3.4), $rank(\mathbf{A}) = 3$. Thus $\mathbf{A}$ is a full rank matrix. Hence the columns of $\mathbf{A}$ are linearly independent i.e., the given vectors are linearly independent and forms the basis for $\mathbf{C}^3$.
Hence any vector $\mathbf{Y} \in \mathbf{C}^3$ can be written as the linear combinations of $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

2.1.4. Let V be the set of all pairs (x,y) of real numbers and let F be the field of real numbers. Define

$$(x, y) + (x_1, y_1) = (x + x_1, y + y_1) \tag{2.1.4.1}$$
$$c(x, y) = (cx, y) \tag{2.1.4.2}$$

Is V with these operations , a vector space over the field of real numbers ?
Solution: V={(x,y) $|x, y \in R$} , consider $u = (x_1, y_1) \in V, a, b, c \in R$. Axioms with respect to addition and scalar multiplication.

a)

$$(a + b)u = (a + b)(x_1, y_1) \tag{2.1.4.3}$$
$$= ((a + b)x_1, y_1) \neq au + bu \tag{2.1.4.4}$$

Since V with the given operations the equation (2.1.4.4) contradicts the axioms of scalar multiplication . Hence it is not vector space over real number with these operations.

2.1.5. On $\mathbb{R}^n$ define two operations

$$\alpha \oplus \beta = \alpha - \beta \tag{2.1.5.1}$$
$$c \cdot \alpha = -c\alpha \tag{2.1.5.2}$$

The operations on the right are usual ones. Which of the axioms for a vector space are satisfied by $(\mathbb{R}^n, \oplus, \cdot)$?
Solution: Let $(\alpha, \beta, \gamma) \in \mathbb{R}^n$ and c, c_1, c_2 are scalars taken from the field $\mathbb{R}$ where the vector space is defined on. Table 2.1.5 lists the axioms satisfied and not satisfied for $(\mathbb{R}^n, \oplus, \cdot)$.

2.1.6. Let $\mathbf{V}$ be the set of pairs (x, y) of real numbers and let $\mathbf{F}$ be the field of real numbers. Define

$$(x, y) + (x_1, y_1) = (x + x_1, 0) \tag{2.1.6.1}$$
$$c(x, y) = (cx, 0) \tag{2.1.6.2}$$

Is $\mathbf{V}$, with these operations, a vector space?
Solution: $\mathbf{V}$ is a vector space if it satisfies all properties of the vector space. Let us consider the property of Existence of additive identity. According to Existence of additive identity, there is a unique vector $\mathbf{0}$ in $\mathbf{V}$ called the zero vector, such that $\alpha + \mathbf{0} = \alpha$ for all α in $\mathbf{V}$.
Let $u = (x_1, y_1) \in \mathbf{V}$

$$\begin{aligned} u + \mathbf{0} &= (x_1, y_1) + (0, 0) \\ &= (x_1 + 0, 0) \\ &= (x_1, 0) \\ &\neq u \end{aligned} \tag{2.1.6.3}$$

From (2.1.6.3), there does not exist an additive identity for $\mathbf{V}$.
Hence $\mathbf{V}$ is not a vector space.

2.1.7. Let $\mathbb{V}$ be the set of all complex-valued functions f on the real line such that

$$f(-t) = \overline{f(t)} \tag{2.1.7.1}$$

The bar denotes complex conjugation.Show that $\mathbb{V}$,with the operations

$$(f + g)(t) = f(t) + g(t) \tag{2.1.7.2}$$
$$(cf)(t) = cf(t) \tag{2.1.7.3}$$

is a vector space over the field of real numbers. Give an example of a function in V which is not real valued.
Solution: To prove that $\mathbb{V}$ with the given operations is a vector space over the field of real numbers, we have to start by proving that additivity and homogeneity both hold true. So, we have to prove that (cf+g)(t) is equal to

UNSATISTIFD	SATISFIED
Associativity of addition $\alpha \oplus (\beta \oplus \gamma) = \alpha - \beta + \gamma$ $(\alpha \oplus \beta) \oplus \gamma = \alpha - \beta - \gamma$ $\alpha \oplus (\beta \oplus \gamma) \neq (\alpha \oplus \beta) \oplus \gamma$	**Additive identity** $\alpha \oplus \beta = \alpha - \beta = \alpha$ Additive identity is β unique $\beta = (0, 0, ...0)$
Commutativity of addition $\alpha \oplus \beta = \alpha - \beta$ $\beta \oplus \alpha = \beta - \alpha$ $\alpha \oplus \beta \neq \beta \oplus \alpha$	**Additive inverse** $\alpha \oplus \alpha = \alpha - \alpha = 0$ Additive inverse is α
Scalar multiplication with field multiplication $(c_1c_2) \cdot \alpha = (-c_1c_2)\alpha$ $c_1 \cdot (c_2 \cdot \alpha) = c_1c_2\alpha$ $(c_1c_2) \cdot \alpha \neq c_1 \cdot (c_2 \cdot \alpha)$	
Identity element of scalar multiplication $1 \cdot \alpha = -\alpha = \alpha$ for $\alpha = (0, 0, ..., 0)$ $1 \cdot \alpha = -\alpha \neq \alpha \quad \forall \quad \alpha \neq (0, 0, ..., 0)$	
Distributivity of scalar multiplication w.r.t vector addition $c \cdot (\alpha \oplus \beta) = -c(\alpha - \beta)$ $c \cdot \alpha \oplus c \cdot \beta = -c\alpha - (-c\beta)$ $c \cdot (\alpha \oplus \beta) \neq c \cdot \alpha \oplus c \cdot \beta$	
Distributivity of scalar multiplication w.r.t field addition $(c_1 + c_2) \cdot \alpha = -(c_1 + c_2)\alpha$ $c_1 \cdot \alpha \oplus c_2 \cdot \beta = -c_1\alpha - (-c_2\beta)$ $(c_1 + c_2) \cdot \alpha \neq c_1 \cdot \alpha \oplus c_2 \cdot \beta$	

TABLE 2.1.5: Axioms of vector space $(\mathbb{R}^n, \oplus, \cdot)$

cf(t)+g(t).

$$(cf + g)(t) \tag{2.1.7.4}$$
$$= (cf)(t) + g(t) \tag{2.1.7.5}$$
$$= cf(t) + g(t) \tag{2.1.7.6}$$

Now, we know that f(-t)= $\overline{f(-t)}$ and so (cf+g)(t) should also satisfy the property,

$$(cf + g)(-t) \tag{2.1.7.7}$$
$$= cf(-t) + g(-t) \tag{2.1.7.8}$$
$$= c\overline{f(t)} + \overline{g(t)} \tag{2.1.7.9}$$
$$= \overline{cf(t) + g(t)} \tag{2.1.7.10}$$
$$= \overline{(cf + g)(t)} \tag{2.1.7.11}$$

Example Let's take f(x)=a+ix

$$f(1) = a + i \tag{2.1.7.12}$$

Hence, f(x) is not real valued.Now,

$$f(x) = a + ix \tag{2.1.7.13}$$
$$f(-x) = a - ix \tag{2.1.7.14}$$
$$f(-x) = \overline{f(x)} \tag{2.1.7.15}$$

Since a and x $\in \mathbb{R}$, so f $\in \mathbb{V}$

2.2 *Subspaces*

2.2.1. Which of the following set of vectors

$$\alpha = (a_1, a_2..........a_n)$$

in $\mathbf{R^n}$ are subspace of $\mathbf{R^n}$ $(n \geq 3)$?

a) All α such that $a_1 \geq 0$

b) All α such that $a_1 + 3a_2 = a_3$

c) All α such that$a_2 = a_1^2$

$\alpha = (a_1, a_2..........a_n)$	
Vector space	**Subspace summary**
$\alpha = (a_1, a_2, a_3, a_4.........., a_n); \quad a_1 \geq 0$	Not a subspace. Scalar multiplication is not satisfied. $-1(\alpha) \neq \alpha$
$\alpha = (a_1, a_2, a_3, a_4.........., a_n); \quad a_1 + 3a_2 = a_3$	It is a subspace
$\alpha = (a_1, a_2, a_3, a_4.........., a_n); \quad a_2 = a_1^2$	Not a subspace. Addition is not satisfied. $(a_1 + b_1)^2 \neq a_1^2 + b_1^2$
$\alpha = (a_1, a_2, a_3, a_4.........., a_n); \quad a_1 a_2 = 0$	Not a subspace. Addition is not satisfied. $a_1 b_1 \neq 0$
$\alpha = (a_1, a_2, a_3, a_4.........., a_n); \quad a_2$ is rational	Not a subspace. Scalar multiplication is not satisfied. $a_2 \neq \sqrt{2} a_1$

TABLE 2.2.1: Summary

d) All α such that $a_1 a_2 = 0$

e) All α such that a_2 is rational

Solution: Table 2.2.1 lists the summary of which set of vectors in $\mathbf{R}^n$ are subspace of $\mathbf{R}^n$ $(n \geq 3)$.

2.2.2. Is the vector $\begin{pmatrix} 3 \\ -1 \\ 0 \\ -1 \end{pmatrix}$ in the subspace of $\mathbf{R}^4$ spanned by the vectors $\begin{pmatrix} 2 \\ -1 \\ 3 \\ 2 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 1 \\ 1 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 9 \\ -5 \end{pmatrix}$? **Solution:** Expressing the given three vectors as columns of a matrix,

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 1 & 1 \\ 3 & 1 & 9 \\ 2 & -3 & -5 \end{pmatrix} \tag{2.2.2.1}$$

and

$$\mathbf{b} = \begin{pmatrix} 3 \\ -1 \\ 0 \\ -1 \end{pmatrix} \tag{2.2.2.2}$$

For the vector $\mathbf{b}$ to be in the subspace of $\mathbf{R}^4$ spanned by the three vectors.

$$\mathbf{Ax} = \mathbf{b} \tag{2.2.2.3}$$

must have a solution.

$$\begin{pmatrix} 2 & -1 & 1 \\ -1 & 1 & 1 \\ 3 & 1 & 9 \\ 2 & -3 & -5 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 3 \\ -1 \\ 0 \\ -1 \end{pmatrix} \tag{2.2.2.4}$$

Forming the augmented matrix and row reducing it by elementary row operations,

$$\begin{pmatrix} 2 & -1 & 1 & 3 \\ -1 & 1 & 1 & -1 \\ 3 & 1 & 9 & 0 \\ 2 & -3 & -5 & -1 \end{pmatrix} \xleftrightarrow[R_4 \leftarrow R_4 - R_1]{R_2 \leftarrow 2R_2 + R_1, R_3 \leftarrow R_3 - \frac{3}{2}R_1} \tag{2.2.2.5}$$

$$\begin{pmatrix} 2 & -1 & 1 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & \frac{5}{2} & \frac{15}{2} & \frac{-9}{2} \\ 0 & -2 & -6 & -4 \end{pmatrix} \xleftrightarrow[R_4 \leftarrow R_4 + 2R_2]{R_3 \leftarrow 2R_3 - 5R_2} \begin{pmatrix} 2 & -1 & 1 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & -14 \\ 0 & 0 & 0 & -2 \end{pmatrix} \tag{2.2.2.6}$$

From (2.2.2.6), it is clear that the system does not have a solution. Hence the vector $\begin{pmatrix} 3 \\ -1 \\ 0 \\ -1 \end{pmatrix}$ does not lie in the subspace of $\mathbf{R}^4$ spanned by the given three vectors.

2.2.3. Let $\mathbf{W}$ be the set of all $(x_1, x_2, x_3, x_4, x_5)$ in $\mathbb{R}^5$ which satisfy

$$2x_1 - x_2 + \frac{4}{3}x_3 - x_4 = 0 \tag{2.2.3.1}$$

$$x_1 + \frac{2}{3}x_3 - x_5 = 0 \tag{2.2.3.2}$$

$$9x_1 - 3x_2 + 6x_3 - 3x_4 - 3x_5 = 0 \tag{2.2.3.3}$$

Find a finite set of vectors which spans **W**.
Solution: The given equations are,

$$2x_1 - x_2 + \frac{4}{3}x_3 - x_4 = 0 \qquad (2.2.3.4)$$

$$x_1 + \frac{2}{3}x_3 - x_5 = 0 \qquad (2.2.3.5)$$

$$9x_1 - 3x_2 + 6x_3 - 3x_4 - 3x_5 = 0 \qquad (2.2.3.6)$$

which can be written as,

$$\begin{pmatrix} 2 & -1 & \frac{4}{3} & -1 & 0 \\ 1 & 0 & \frac{2}{3} & 0 & -1 \\ 9 & -3 & 6 & -3 & -3 \end{pmatrix} \mathbf{x} = 0 \qquad (2.2.3.7)$$

Now, the augmented matrix,

$$\left(\begin{array}{ccccc|c} 2 & -1 & \frac{4}{3} & -1 & 0 & 0 \\ 1 & 0 & \frac{2}{3} & 0 & -1 & 0 \\ 9 & -3 & 6 & -3 & -3 & 0 \end{array}\right) \qquad (2.2.3.8)$$

$$\xleftrightarrow{R_3=R_3-3R_1-3R_2} \left(\begin{array}{ccccc|c} 2 & -1 & \frac{4}{3} & -1 & 0 & 0 \\ 1 & 0 & \frac{2}{3} & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \qquad (2.2.3.9)$$

$$\xleftrightarrow{R_2=R_2-\frac{1}{2}R_1} \left(\begin{array}{ccccc|c} 2 & -1 & \frac{4}{3} & -1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \qquad (2.2.3.10)$$

$$\xleftrightarrow{R_2=2R_2} \left(\begin{array}{ccccc|c} 2 & -1 & \frac{4}{3} & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \qquad (2.2.3.11)$$

$$\xleftrightarrow{R_1=R_1+R_2} \left(\begin{array}{ccccc|c} 2 & 0 & \frac{4}{3} & 0 & -2 & 0 \\ 0 & 1 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \qquad (2.2.3.12)$$

So,

$$2x_1 + \frac{4}{3}x_3 - 2x_5 = 0 \qquad (2.2.3.13)$$

$$x_2 + x_4 - 2x_5 = 0 \qquad (2.2.3.14)$$

Solving the equations we get,

$$x_1 = -\frac{2}{3}x_3 + x_5 \qquad (2.2.3.15)$$

$$x_2 = -x_4 + 2x_5 \qquad (2.2.3.16)$$

which can be written as,

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \qquad (2.2.3.17)$$

$$= \begin{pmatrix} -\frac{2}{3}x_3 + x_5 \\ -x_4 + 2x_5 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \qquad (2.2.3.18)$$

$$= x_3 \begin{pmatrix} -\frac{2}{3} \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad (2.2.3.19)$$

where x_3,x_4 and $x_5 \in \mathbb{R}$.Hence, the vectors $\begin{pmatrix} -\frac{2}{3} \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ will span **W**

2.2.4. Let **F** be a field and let n be a positive integer (n≥2). Let **V** be the vector space of all n×n matrices over **F**. Which of the following set of matrices **A** in **V** are subspaces of **V**?

a) all invertible **A**;
b) all non-invertible **A**;
c) all **A** such that **AB** = **BA**, where **B** is some fixed matrix in **V**;
d) all **A** such that $\mathbf{A}^2 = \mathbf{A}$.

Solution:

a) Let the matrices **A** and **B** ∈ **V**, be set of invertible matrix. For them to be a subspace they need to be closed under addition. Let,

$$\mathbf{A} = \mathbf{I} \qquad (2.2.4.1)$$

$$\mathbf{B} = -\mathbf{I} \qquad (2.2.4.2)$$

It could be easily proven that both matrices

$\mathbf{A}$ and $\mathbf{B}$ are invertible as,

$$rank(\mathbf{I}_{nxn}) = rank\left(\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}_{nxn}\right) \tag{2.2.4.3}$$

$$\implies rank(-\mathbf{I}_{nxn}) = rank(\mathbf{I}_{nxn}) = n \tag{2.2.4.4}$$

or it is a full rank matrix as there are n pivots.

$$\therefore \mathbf{A} + \mathbf{B} = \mathbf{0}. \tag{2.2.4.5}$$

But the zero matrix $\mathbf{0}$ is non-invertible as,

$$rank(\mathbf{0}_{nxn}) = rank\left(\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}_{nxn}\right) \tag{2.2.4.6}$$

$$\implies rank(\mathbf{0}_{nxn}) = 0 \tag{2.2.4.7}$$

$\therefore$ **the set of invertible matrices are not closed under addition. Hence not a subspace of V.**

b) Let the matrices $\mathbf{A_1}$, $\mathbf{A_2}, \cdots, \mathbf{A_n} \in \mathbf{V}$, be set of non-invertible matrix. For them to be a subspace they need to be closed under addition. Let,

$$\mathbf{A_1} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}_{nxn} \tag{2.2.4.8}$$

$$\mathbf{A_2} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}_{nxn} \tag{2.2.4.9}$$

$$\mathbf{A_n} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}_{nxn} \tag{2.2.4.10}$$

$$\tag{2.2.4.11}$$

It could be proven that matrices $\mathbf{A_1}$, $\mathbf{A_2}, \cdots, \mathbf{A_n}$ are non-invertible as,

$$rank(\mathbf{A_1}) = rank\left(\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}\right) \tag{2.2.4.12}$$

$$\implies rank(\mathbf{A_1}) = 1 \tag{2.2.4.13}$$

or there is only one pivot hence rank is 1.

$$\implies \mathbf{A_1} + \mathbf{A_2} + \mathbf{A_3} + \cdots \mathbf{A_n} = \mathbf{I}_{nxn} \tag{2.2.4.14}$$

Now the identity matrix $\mathbf{I}$ is invertible as shown in equation (2.2.4.4). $\therefore$ **the set of non-invertible matrices are not closed under addition. Hence not a subspace of V.**

c) **Theorem 1:**. A non-empty subset W of V is a subspace of V if and only if for each pair of vectors α, β in W and each scalar c $\in$ F, the vector $c\alpha + \beta \in$ W.
Let the matrices $\mathbf{A_1}$ and $\mathbf{A_2}$ satisfy,

$$\mathbf{A_1B} = \mathbf{BA_1} \tag{2.2.4.15}$$

$$\mathbf{A_2B} = \mathbf{BA_2} \tag{2.2.4.16}$$

Let, c$\in$ $\mathbf{F}$ be any constant.

$$\therefore (c\mathbf{A_1} + \mathbf{A_2})\,\mathbf{B} = c\mathbf{A_1B} + \mathbf{A_2B} \tag{2.2.4.17}$$

Substituting from equations (2.2.4.15) and (2.2.4.16) to (2.2.4.17),

$$\implies (c\mathbf{A_1} + \mathbf{A_2})\,\mathbf{B} = c\mathbf{BA_1} + \mathbf{BA_2} \tag{2.2.4.18}$$

$$\implies \mathbf{B}c\mathbf{A_1} + \mathbf{BA_2} \tag{2.2.4.19}$$

$$\implies \mathbf{B}\,(c\mathbf{A_1} + \mathbf{A_2}) \tag{2.2.4.20}$$

Thus, $(c\mathbf{A_1} + \mathbf{A_2})$ satisfy the criteria and from Theorem-1 it can be seen that the set is a subspace of V.

d) Let $\mathbf{A}$ and $\mathbf{B} \in \mathbf{V}$ be set of matrices such that,

$$\mathbf{A^2} = \mathbf{A} \tag{2.2.4.21}$$

$$\mathbf{B^2} = \mathbf{B} \tag{2.2.4.22}$$

Now for them to be closed under addition,

$$(\mathbf{A}+\mathbf{B})^2 = \mathbf{A}+\mathbf{B} \tag{2.2.4.23}$$

Which is not always same. Example let,

$$\mathbf{A} = \begin{pmatrix}1 & 1\\0 & 0\end{pmatrix} \tag{2.2.4.24}$$

$$\mathbf{B} = \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} \tag{2.2.4.25}$$

Clearly,

$$\mathbf{A}^2 = \begin{pmatrix}1 & 1\\0 & 0\end{pmatrix}\begin{pmatrix}1 & 1\\0 & 0\end{pmatrix} = \begin{pmatrix}1 & 1\\0 & 0\end{pmatrix} = \mathbf{A} \tag{2.2.4.26}$$

$$\mathbf{B}^2 = \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} = \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} = \mathbf{B} \tag{2.2.4.27}$$

Now,

$$\mathbf{A}+\mathbf{B} = \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix} \tag{2.2.4.28}$$

$$\implies (\mathbf{A}+\mathbf{B})^2 = \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix} = \begin{pmatrix}1 & 2\\0 & 1\end{pmatrix} \tag{2.2.4.29}$$

Hence, clearly from equations (2.2.4.28) and (2.2.4.29),

$$(\mathbf{A}+\mathbf{B})^2 \neq \mathbf{A}+\mathbf{B} \tag{2.2.4.30}$$

$\therefore$ the set of all A such that $\mathbf{A}^2 = \mathbf{A}$ is not closed under addition. Hence, not a subspace of V.

2.2.5. a. Prove that only subspace of $\mathbb{R}^1$ are $\mathbb{R}^1$ and the zero subspace

b. Prove that a subspace of $\mathbb{R}^2$ is $\mathbb{R}^2$, or the zero subspace, or consists of all scalar multiples of some fixed vector in $\mathbb{R}^2$. (The last type of subspace is, intuitively, a straight line through the origin.)

c. Can you describe the subspaces of $\mathbb{R}^3$?

Solution:

a. Let $W \neq 0$ be subspace of $\mathbb{R}^1$. Then W is a nonempty subset of $\mathbb{R}^1$ and there exist $w \in W$ such that $w \neq 0$ which gives us that there exist w^{-1}.

Let $x \in \mathbb{R}^1$. Since W is in $\mathbb{R}^1$ we have that it is closed under scalar multiplication which gives us that $(xw^{-1})w = x(w^{-1}w) = x.1 = x \in W$

Hence $\mathbb{R}^1 \subset W$ and therefore $W = \mathbb{R}^1$

Thus the only subspace of $\mathbb{R}^1$ distinct of 0 is $\mathbb{R}^1$ and therefore only subspaces of $\mathbb{R}^1$ are 0 and $\mathbb{R}^1$.

b. Clearly, 0 and $\mathbb{R}^2$ itself are subspaces of $\mathbb{R}^2$. If $u \neq 0$ and $u \in \mathbb{R}^2$ then span$\{\mathbf{u}\}$ = $c\mathbf{u} : c \in \mathbb{R}$ = set of all scalar multiples of $\mathbf{u}$ is a subspace of $\mathbb{R}^2$.

To show that these are the only subspaces of $\mathbb{R}^2$, assume that $W \subset \mathbb{R}^2$ is any subspace of $\mathbb{R}^2$. Since $W \subset \mathbb{R}^2$ is a subspace of $\mathbb{R}^2$, we have that $\mathbf{0} \in W$. If $W \neq \mathbf{0}$ then there is a vector $\mathbf{u} \neq 0$ and $\mathbf{u} \in W$, and hence W contains $c\mathbf{u}$ for every $c \in \mathbb{R}$. If $W \neq span\{\mathbf{u}\}$, then there is a vector $v \in W$ so that $\mathbf{v} \neq k\mathbf{u}$ for any $k \in \mathbb{R}$.

Then $\mathbf{z} = c\mathbf{u} + d\mathbf{v} \in span\{\mathbf{u},\mathbf{v}\}$ for any $c, d \in \mathbb{R}$. Since W is a subspace $c\mathbf{u}$ and $d\mathbf{v} \in W$ for any $c, d \in \mathbb{R}$, and hence so does $\mathbf{z} = c\mathbf{u}+d\mathbf{v}$. Thus $\mathbf{z} \in span\{\mathbf{u},\mathbf{v}\} \implies z \in W$, and so $span\{\mathbf{u},\mathbf{v}\} \subset W \subset \mathbb{R}^2$.

Let $\mathbf{x} = \begin{pmatrix}x_1\\x_2\end{pmatrix} \in \mathbb{R}^2$ be any vector in $\mathbb{R}^2$, and let $\mathbf{u} = \begin{pmatrix}u_1\\u_2\end{pmatrix} \neq \begin{pmatrix}0\\0\end{pmatrix}$ and let $\mathbf{v} = \begin{pmatrix}v_1\\v_2\end{pmatrix} \neq \begin{pmatrix}0\\0\end{pmatrix}$. We show that there are real numbers c and d so that $c\mathbf{u} + d\mathbf{v} = \mathbf{x}$

$$\begin{pmatrix}cu_1\\cu_2\end{pmatrix} + \begin{pmatrix}dv_1\\dv_2\end{pmatrix} = \begin{pmatrix}x_1\\x_2\end{pmatrix} \tag{2.2.5.1}$$

$$\begin{pmatrix}u_1 & v_1\\u_2 & v_2\end{pmatrix}\begin{pmatrix}c\\d\end{pmatrix} = \begin{pmatrix}x_1\\x_2\end{pmatrix} \tag{2.2.5.2}$$

Since $\mathbf{v} \neq k\mathbf{u}$ for any $k \in \mathbb{R}$ and since $\mathbf{u} = \begin{pmatrix}u_1\\u_2\end{pmatrix} \neq \begin{pmatrix}0\\0\end{pmatrix}$ assume that $u_1 \neq 0$, and since $k\mathbf{u} \neq \mathbf{v} = \begin{pmatrix}v_1\\v_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$ assume that $v_2 \neq 0$. Then

$$A = \begin{pmatrix}u_1 & v_1\\u_2 & v_2\end{pmatrix} \to \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \tag{2.2.5.3}$$

Hence A is row equivalent to I_2 and so A is invertible and so (2.2.5.2) has unique solution for c and d. Thus for any $\mathbf{x} \in \mathbb{R}^2$ we can find real numbers c and d such that $\mathbf{x} = c\mathbf{u} + d\mathbf{v}$. Hence $\mathbf{x} \in \mathbb{R}^2 \implies x \in span\{\mathbf{u}, \mathbf{v}\}$. Thus $\mathbb{R}^2 \subset span\{\mathbf{u}, \mathbf{v}\} \subset W \subset \mathbb{R}^2$.

Hence $span\{\mathbf{u},\mathbf{v}\} = \mathrm{W} = \mathbb{R}^2$, and so the only subspace of $\mathbb{R}^2$ are $\mathbf{0}$, $\mathbb{R}^2$, and $L = c\mathbf{u} : \mathbf{u} \neq 0, c \in \mathbb{R}$.

c. The following are the subspaces of $\mathbb{R}^3$:

1. Origin is a trivial subspace of $\mathbb{R}^3$.

2. $\mathbb{R}^3$ itself is a trivial subspace of $\mathbb{R}^3$.

3. Every line through origin is subspace of $\mathbb{R}^3$.

4. Every plane in $\mathbb{R}^3$ passing through origin is a subspace $\mathbb{R}^3$.
   Proof : Let W be a plane passing through origin. We need $\mathbf{0} \in W$, but we have that since we're only considering planes that contain origin. Next, we need W is closed under vector addition. If $\mathbf{w_1}$ and $\mathbf{w_2}$ both belong to W, then so does $\mathbf{w_1} + \mathbf{w_2}$ because it's found by constructing a parallelogram, and the whole parallelogram lies in the plane W. Finally, we need W is closed under scalar products, but it is since scalar multiples lie in a straight line through the origin, and that line lies in W. Thus, each plane W passing through the origin is a subspace of $\mathbb{R}^3$.

5. The intersection of any of the above subspaces will also be a subspace of $\mathbb{R}^3$. Because intersection of subspaces of a vector space is also a subspace of vector space.
   Proof : Let W be a collection of subspaces of V, and let $W = \cap W_i$ be their intersection. Since each W_i is a subspace, each of it contains the zero vector. Thus the zero vector is in the intersection W, and W is non-empty. Let α and β be vectors in W and let c be a scalar. By definition of W, both α and β belong to each W_i, and because each W_i is a subspace, the vector $(c\alpha + \beta)$ is again in W. Hence by definition of subspace, W is a subspace of V.

These 5 are only subspaces of $\mathbb{R}^3$ possible. Because dimension of vector space $\mathbb{R}^3$ is 3. Any subspace of $\mathbb{R}^3$ should have dimension less than or equal to it's dimension. Hence possible dimensions of subspaces are 0,1,2,3. Only subspace with 0 dimension is origin. Subspaces of dimension 1 with zero vector are lines passing through origin. Subspaces of dimension 2 with zero vector are plane passing through origin. Subspace of dimension 3 are all of $\mathbb{R}^3$ itself.

2.2.6. Let $\mathbf{W}_1$ and $\mathbf{W}_2$ be subspaces of a vector space $\mathbf{V}$ such that the set-theoretic union of $\mathbf{W}_1$ and $\mathbf{W}_2$ is also a subspace. Prove that one of the spaces $\mathbf{W}_i$ is contained in the other. **Solution:** Given $\mathbf{W}_1 \cup \mathbf{W}_2$ is a subspace, we need to prove that

$$\mathbf{W}_1 \subseteq \mathbf{W}_2 \quad or \quad \mathbf{W}_2 \subseteq \mathbf{W}_1 \tag{2.2.6.1}$$

Let us assume that

$$\mathbf{W}_1 \nsubseteq \mathbf{W}_2 \tag{2.2.6.2}$$

We need to show that

$$\mathbf{W}_2 \subseteq \mathbf{W}_1 \tag{2.2.6.3}$$

i.e., the generators of $\mathbf{W}_2$ are in $\mathbf{W}_1$. Consider a vector, $\mathbf{w}_1 \in \mathbf{W}_1 \backslash \mathbf{W}_2$ and a vector $\mathbf{w}_2 \in \mathbf{W}_2$. Since $\mathbf{W}_1 \cup \mathbf{W}_2$ is a subspace,

$$\mathbf{w}_1 + \mathbf{w}_2 \in \mathbf{W}_1 \cup \mathbf{W}_2 \tag{2.2.6.4}$$

$$\implies \mathbf{w}_1 + \mathbf{w}_2 \in \mathbf{W}_1 \quad or \tag{2.2.6.5}$$

$$\mathbf{w}_1 + \mathbf{w}_2 \in \mathbf{W}_2 \tag{2.2.6.6}$$

But, $\mathbf{w}_1 + \mathbf{w}_2 \notin \mathbf{W}_2$ because for some vector $-\mathbf{w}_2 \in \mathbf{W}_2$,

$$(\mathbf{w}_1 + \mathbf{w}_2) - \mathbf{w}_2 = \mathbf{w}_1 \notin \mathbf{W}_2 \tag{2.2.6.7}$$

Hence it must be that, $\mathbf{w}_1 + \mathbf{w}_2 \in \mathbf{W}_1$ because for some vector $-\mathbf{w}_1 \in \mathbf{W}_1$,

$$(\mathbf{w}_1 + \mathbf{w}_2) - \mathbf{w}_1 = w_2 \in \mathbf{W}_1 \tag{2.2.6.8}$$

Thus, we have shown that every vector $\mathbf{w}_2$ in $\mathbf{W}_2$ is also in $\mathbf{W}_1$. Hence, $\mathbf{W}_2 \subseteq \mathbf{W}_1$

2.2.7. Let $\mathbf{V}$ be the vector space of all functions from $\mathbf{R}$ into $\mathbf{R}$; let $\mathbf{V_e}$ be the subset of even functions, $f(-x) = f(x)$; let $\mathbf{V_o}$ be the subset of odd functions, $f(-x) = -f(x)$.

a) Prove that $\mathbf{V_e}$ and $\mathbf{V_o}$ are subspaces of $\mathbf{V}$
b) Prove that $\mathbf{V_e} + \mathbf{V_o} = \mathbf{V}$
c) Prove that $\mathbf{V_e} \cap \mathbf{V_o} = \{0\}$

Solution:

a) Prove that $\mathbf{V_e}$ and $\mathbf{V_o}$ are subspaces of $\mathbf{V}$.

A non-empty subset $\mathbf{W}$ of $\mathbf{V}$ is a subspace of $\mathbf{V}$ if and only if for each pair of vectors α, β in $\mathbf{W}$ and each scalar c in $\mathbf{F}$ the vector $c\alpha + \beta$ is again in $\mathbf{W}$.

Let $\mathbf{u}, \mathbf{v} \in \mathbf{V_e}$ and $c \in \mathbf{R}$ and let $\mathbf{h} = c\mathbf{u} + \mathbf{v}$. Then,

$$\begin{aligned}\mathbf{h}(-x) &= c\mathbf{u}(-x) + \mathbf{v}(-x)\\ &= c\mathbf{u}(x) + \mathbf{v}(x) \qquad (2.2.7.1)\\ &= \mathbf{h}(x)\end{aligned}$$

From (2.2.7.1)

$$\implies \mathbf{h}(-x) = \mathbf{h}(x) \qquad (2.2.7.2)$$
$$\implies \mathbf{h} \in \mathbf{V_e} \qquad (2.2.7.3)$$

Let $\mathbf{u}, \mathbf{v} \in \mathbf{V_o}$ and $c \in \mathbf{R}$ and let $\mathbf{h} = c\mathbf{u} + \mathbf{v}$. Then,

$$\begin{aligned}\mathbf{h}(-x) &= c\mathbf{u}(-x) + \mathbf{v}(-x)\\ &= -c\mathbf{u}(x) - \mathbf{v}(x) \qquad (2.2.7.4)\\ &= -\mathbf{h}(x)\end{aligned}$$

From (2.2.7.4)

$$\implies \mathbf{h}(-x) = -\mathbf{h}(x) \qquad (2.2.7.5)$$
$$\implies \mathbf{h} \in \mathbf{V_o} \qquad (2.2.7.6)$$

From (2.2.7.3) and (2.2.7.6), $\mathbf{V_e}$ and $\mathbf{V_o}$ are subspaces of $\mathbf{V}$.

a) Prove that $\mathbf{V_e} + \mathbf{V_o} = \mathbf{V}$.

Let $\mathbf{u} \in \mathbf{V}$

$$\mathbf{u_e}(x) = \frac{\mathbf{u}(x) + \mathbf{u}(-x)}{2} \qquad (2.2.1.7)$$
$$\mathbf{u_o}(x) = \frac{\mathbf{u}(x) - \mathbf{u}(-x)}{2} \qquad (2.2.1.8)$$

Equation equation (2.2.1.7) and (2.2.1.8), $\mathbf{u_e}$ is even and $\mathbf{u_o}$ is odd. Adding both the equations,

$$\mathbf{u} = \mathbf{u_e} + \mathbf{u_o} \qquad (2.2.1.9)$$

a) Prove that $\mathbf{V_e} \cap \mathbf{V_o} = \{0\}$.

Let $\mathbf{u} \in \mathbf{V_e} \cap \mathbf{V_o}$

$$\mathbf{u} \in \mathbf{V_e} \implies \mathbf{u}(-x) = \mathbf{u}(x) \qquad (2.2.2.10)$$
$$\mathbf{u} \in \mathbf{V_o} \implies \mathbf{u}(-x) = -\mathbf{u}(x) \qquad (2.2.2.11)$$

Equating (2.2.2.10) and (2.2.2.11),

$$\mathbf{u}(x) = -\mathbf{u}(x) \qquad (2.2.2.12)$$
$$\implies 2\mathbf{u}(x) = 0 \qquad (2.2.2.13)$$
$$\implies \mathbf{u} = 0 \qquad (2.2.2.14)$$

Equations (2.2.7.3), (2.2.7.6), (2.2.1.9), (2.2.2.14) proves 1, 2 and 3.

2.2.3. Let $\mathbf{W_1}$ and $\mathbf{W_2}$ be subspaces of a vector space $\mathbf{V}$ such that

$$\mathbf{W_1} + \mathbf{W_2} = \mathbf{V} \qquad (2.2.3.1)$$
$$\text{and } \mathbf{W_1} \cap \mathbf{W_2} = \mathbf{0} \qquad (2.2.3.2)$$

Prove that for each vector α in $\mathbf{V}$ there are unique vectors α_1 in $\mathbf{W_1}$ and α_2 in $\mathbf{W_2}$ such that

$$\alpha = \alpha_1 + \alpha_2 \qquad (2.2.3.3)$$

Solution: Suppose, vectors α_1 and α_2 are not unique.

Consider

$$\alpha_1' \in \mathbf{W_1}, \qquad (2.2.3.4)$$
$$\alpha_2' \in \mathbf{W_2} \qquad (2.2.3.5)$$
$$\text{such that } \alpha = \alpha_1' + \alpha_2' \qquad (2.2.3.6)$$

(2.2.3.3) and (2.2.3.6) indicate

$$\alpha_1 + \alpha_2 = \alpha_1' + \alpha_2' \qquad (2.2.3.7)$$
$$\implies \alpha_1 - \alpha_1' = \alpha_2' - \alpha_2 \qquad (2.2.3.8)$$

For α_1 and α_1' lying in subspace $\mathbf{W_1}$, defined on field $\mathbb{F}$, the following holds

$$\alpha_1 + c\alpha_1' \in \mathbf{W_1}, c \in \mathbb{F} \qquad (2.2.3.9)$$
$$c = -1 \implies \alpha_1 - \alpha_1' \in \mathbf{W_1} \qquad (2.2.3.10)$$
$$\text{Similarly, } \alpha_2' - \alpha_2 \in \mathbf{W_2} \qquad (2.2.3.11)$$
$$(2.2.3.8) \implies \alpha_1 - \alpha_1' \in \mathbf{W_2} \qquad (2.2.3.12)$$

(2.2.3.2),(2.2.3.10),(2.2.3.12) indicate

$$\alpha_1 - \alpha_1' = \alpha_2' - \alpha_2 = \mathbf{0} \tag{2.2.3.13}$$

$$\implies \alpha_1 = \alpha_1' \tag{2.2.3.14}$$

$$\alpha_2 = \alpha_2' \tag{2.2.3.15}$$

So, there exists a unique $\alpha_1 \in \mathbf{W_1}$ and $\alpha_2 \in \mathbf{W_2}$ such that

$$\alpha = \alpha_1 + \alpha_2 \tag{2.2.3.16}$$

where $\alpha \in \mathbf{V}$

2.3 Bases and Dimension

2.3.1. Prove that if two vectors are linearly dependent, one of them is a scalar multiple of the other.

Solution: consider the row reduced matrix

$$\begin{pmatrix} 1 & 1 & 2 & 4 \\ 2 & -1 & -5 & 2 \\ 1 & -1 & -4 & 0 \\ 2 & 1 & 1 & 6 \end{pmatrix} \xleftrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \begin{pmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -9 & -6 \\ 0 & -2 & -6 & -4 \\ 2 & 1 & 1 & 6 \end{pmatrix} \tag{2.3.1.1}$$

$$\xleftrightarrow[R_2 \leftarrow R_4]{R_4 \leftarrow R_4 - 2R_1} \begin{pmatrix} 1 & 1 & 2 & 4 \\ 0 & -1 & -3 & -2 \\ 0 & -2 & -6 & -4 \\ 0 & -3 & -9 & -6 \end{pmatrix} \tag{2.3.1.2}$$

$$\xleftrightarrow[R_2 \leftarrow -R_2]{R_4 \leftarrow R_2} \begin{pmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & -3 & -9 & -6 \\ 0 & -2 & -6 & -4 \end{pmatrix} \tag{2.3.1.3}$$

$$\xleftrightarrow[R_4 \leftarrow R_4 + 2R_2]{R_3 \leftarrow R_3 + 3R_2} \begin{pmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.3.1.4}$$

Therefore the rank = no. of pivot columns = 2 (less than no. of columns).Thus the fours vectors are not linearly independent.

2.3.2. Are the vectors

$$\alpha_1 = \left(1, 1, 2, 4\right), \alpha_2 = \left(2, -1, -5, 2\right) \tag{2.3.2.1}$$

$$\alpha_3 = \left(1, -1, -4, 0\right), \alpha_4 = \left(2, 1, 1, 6\right) \tag{2.3.2.2}$$

linearly independent in R^4

Solution: consider the row reduced matrix

$$\begin{pmatrix} 1 & 1 & 2 & 4 \\ 2 & -1 & -5 & 2 \\ 1 & -1 & -4 & 0 \\ 2 & 1 & 1 & 6 \end{pmatrix} \xleftrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \begin{pmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -9 & -6 \\ 0 & -2 & -6 & -4 \\ 2 & 1 & 1 & 6 \end{pmatrix} \tag{2.3.2.3}$$

$$\xleftrightarrow[R_2 \leftarrow R_4]{R_4 \leftarrow R_4 - 2R_1} \begin{pmatrix} 1 & 1 & 2 & 4 \\ 0 & -1 & -3 & -2 \\ 0 & -2 & -6 & -4 \\ 0 & -3 & -9 & -6 \end{pmatrix} \tag{2.3.2.4}$$

$$\xleftrightarrow[R_2 \leftarrow -R_2]{R_4 \leftarrow R_2} \begin{pmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & -3 & -9 & -6 \\ 0 & -2 & -6 & -4 \end{pmatrix} \tag{2.3.2.5}$$

$$\xleftrightarrow[R_4 \leftarrow R_4 + 2R_2]{R_3 \leftarrow R_3 + 3R_2} \begin{pmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.3.2.6}$$

Therefore the rank = no. of pivot columns = 2 (less than no. of columns).Thus the fours vectors are not linearly independent.

2.3.3. Find a basis for the subspace of $\mathbb{R}^4$ spanned by the four vectors

$$\alpha_1 = \begin{pmatrix} 1 & 1 & 2 & 4 \end{pmatrix} \tag{2.3.3.1}$$

$$\alpha_2 = \begin{pmatrix} 2 & -1 & -5 & 2 \end{pmatrix} \tag{2.3.3.2}$$

$$\alpha_3 = \begin{pmatrix} 1 & -1 & -4 & 0 \end{pmatrix} \tag{2.3.3.3}$$

$$\alpha_4 = \begin{pmatrix} 2 & 1 & 1 & 6 \end{pmatrix} \tag{2.3.3.4}$$

Solution: The basis of the given four vectors is equivalent to finding the basis of column-space $C(\mathbf{A})$ of a matrix $\mathbf{A}$ defined as follows,

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 & 2 \\ 1 & -1 & -1 & 1 \\ 2 & -5 & -4 & 1 \\ 4 & 2 & 0 & 6 \end{pmatrix} \tag{2.3.3.5}$$

Now we calculate the row echelon form of $\mathbf{A}$

as follows,

$$\begin{pmatrix}1 & 2 & 1 & 2\\1 & -1 & -1 & 1\\2 & -5 & -4 & 1\\4 & 2 & 0 & 6\end{pmatrix} \xleftrightarrow[R_2=R_2-R_1]{R_2=R_2-R_1} \begin{pmatrix}1 & 2 & 1 & 2\\0 & -3 & -2 & -1\\0 & -9 & -6 & -3\\4 & 2 & 0 & 6\end{pmatrix} \tag{2.3.3.6}$$

$$\xleftrightarrow{R_4=R_4-R_1} \begin{pmatrix}1 & 2 & 1 & 2\\0 & -3 & -2 & -1\\0 & -9 & -6 & -3\\0 & -6 & -4 & -2\end{pmatrix} \tag{2.3.3.7}$$

$$\xleftrightarrow{R_2=-\frac{1}{3}R_2} \begin{pmatrix}1 & 2 & 1 & 2\\0 & 1 & -\frac{2}{3} & \frac{1}{3}\\0 & -9 & -6 & -3\\0 & -6 & -4 & -2\end{pmatrix} \tag{2.3.3.8}$$

$$\xleftrightarrow{R_3=R_3-9R_2} \begin{pmatrix}1 & 2 & 1 & 2\\0 & 1 & -\frac{2}{3} & \frac{1}{3}\\0 & 0 & 0 & 0\\0 & -6 & -4 & -2\end{pmatrix} \tag{2.3.3.9}$$

$$\xleftrightarrow{R_4=R_4+6R_2} \begin{pmatrix}1 & 2 & 1 & 2\\0 & 1 & -\frac{2}{3} & \frac{1}{3}\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{pmatrix} \tag{2.3.3.10}$$

From (2.3.3.10) we can see that the first column and second column of **A** contains pivot values. Hence the column 1 and column 2 are the basis of the subspace of $\mathbb{R}^4$ spanned by the given vectors α_1, α_2, α_3, α_4
Hence the required basis vectors are,

$$\mathbf{a_1} = \begin{pmatrix}1 & 1 & 2 & 4\end{pmatrix} \tag{2.3.3.11}$$

$$\mathbf{a_2} = \begin{pmatrix}2 & -1 & -5 & 2\end{pmatrix} \tag{2.3.3.12}$$

2.3.4. Let V be the vector space of all 2×2 matrices over the field $\mathbb{F}$. Let W_1 be the set of matrices of the form

$$\begin{pmatrix}x & -x\\y & z\end{pmatrix} \tag{2.3.4.1}$$

and let W_2 be the set of matrices of the form

$$\begin{pmatrix}a & b\\-a & c\end{pmatrix} \tag{2.3.4.2}$$

a) Prove that W_1 and W_2 are subspaces of V.
b) Find the dimension of $W_1, W_2, W_1 + W_2$ and $W_1 \cap W_2$.

Solution: A non-empty subset W of V is a subspace of V if and only if for each pair of vectors α, β in W and each scalar c $\in$ F, the vector $c\alpha + \beta \in$ W.

a) Let $A_1, A_2 \in W_1$ where,

$$A_1 = \begin{pmatrix}x_1 & -x_1\\y_1 & z_1\end{pmatrix}, A_2 = \begin{pmatrix}x_2 & -x_2\\y_2 & z_2\end{pmatrix} \tag{2.3.4.3}$$

Let $c \in F$ then,

$$cA_1 + A_2 = \begin{pmatrix}cx_1 + x_2 & -cx_1 - x_2\\cy_1 + y_2 & cz_1 + z_2\end{pmatrix} = \begin{pmatrix}u & -u\\v & w\end{pmatrix} \tag{2.3.4.4}$$

Thus $cA_1 + A_2 \in W_1$. **Hence W_1 is a subspace.** Similarly, let $A_1, A_2 \in W_2$ where,

$$A_1 = \begin{pmatrix}a_1 & b_1\\-a_1 & c_1\end{pmatrix}, A_2 = \begin{pmatrix}a_2 & b_2\\-a_2 & c_2\end{pmatrix} \tag{2.3.4.5}$$

Let $c \in F$ then,

$$cA_1 + A_2 = \begin{pmatrix}ca_1 + a_2 & cb_1 + b_2\\-ca_1 - a_2 & cc_1 + c_2\end{pmatrix} = \begin{pmatrix}u & v\\-u & w\end{pmatrix} \tag{2.3.4.6}$$

Thus $cA_1 + A_2 \in W_2$. **Hence W_2 is a subspace.**

b) The subspace W_1 can be given as,

$$\begin{pmatrix}x & -x\\y & z\end{pmatrix} = x\begin{pmatrix}1 & -1\\0 & 0\end{pmatrix} + y\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix} + z\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} \tag{2.3.4.7}$$

$$= xA_1 + yA_2 + zA_2 \tag{2.3.4.8}$$

Now,

$$x\begin{pmatrix}1 & -1\\0 & 0\end{pmatrix} + y\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix} + z\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} = \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix} \tag{2.3.4.9}$$

$$\implies x = y = z = 0 \tag{2.3.4.10}$$

A_1, A_2, A_3 are linearly independent and spans W_1. Thus $\{A_1, A_2, A_3\}$ forms basis for W_1.
$\therefore$ **dimension of W_1 is 3.**
The subspace W_2 can be given as,

$$\begin{pmatrix}a & b\\-a & c\end{pmatrix} = a\begin{pmatrix}1 & 0\\-1 & 0\end{pmatrix} + b\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} + c\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} \tag{2.3.4.11}$$

$$= aA_1 + bA_2 + cA_2 \tag{2.3.4.12}$$

Now,

$$a\begin{pmatrix}1 & 0\\ -1 & 0\end{pmatrix}+b\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}+c\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} \tag{2.3.4.13}$$

$$\implies a=b=c=0 \tag{2.3.4.14}$$

A_1, A_2, A_3 are linearly independent and spans W_2. Thus $\{A_1, A_2, A_3\}$ forms basis for W_2.
$\therefore$ **dimension of W_2 is 3.**
Subspace $W_1 + W_2$ is given by,

$$\begin{pmatrix}x+a & -x+b\\ y-a & z+c\end{pmatrix} \tag{2.3.4.15}$$

For $x+a \neq -x+b \neq y-a \neq z+c$,

$$\begin{pmatrix}x+a & -x+b\\ y-a & z+c\end{pmatrix}=\begin{pmatrix}j & k\\ l & m\end{pmatrix} \tag{2.3.4.16}$$

$$= j\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}+k\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}+l\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}+m\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} \tag{2.3.4.17}$$

$$= jA_1+kA_2+lA_3+mA_4 \tag{2.3.4.18}$$

Now,

$$jA_1+kA_2+lA_3+mA_4=0 \tag{2.3.4.19}$$

$$\implies j=k=l=m=0 \tag{2.3.4.20}$$

A_1, A_2, A_3, A_4 are linearly independent and spans $W_1 + W_2$. Thus $\{A_1, A_2, A_3, A_4\}$ forms a basis.
$\therefore$ **dimension of $W_1 + W_2$ is 4.**
The subspace $W_1 \cap W_2$ is given as,

$$\begin{pmatrix}x & -x\\ -x & y\end{pmatrix}=x\begin{pmatrix}1 & -1\\ -1 & 0\end{pmatrix}+y\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} \tag{2.3.4.21}$$

$$= xA_1+yA_2 \tag{2.3.4.22}$$

Now,

$$x\begin{pmatrix}1 & -1\\ -1 & 0\end{pmatrix}+y\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} \tag{2.3.4.23}$$

$$\implies x=y=0 \tag{2.3.4.24}$$

A_1, A_2 are linearly independent and spans $W_1 \cap W_2$. Thus, $\{A_1, A_2\}$ forms a basis.
$\therefore$ **dimension of $W_1 \cap W_2$ is 2.**

2.3.5. Let $\mathbf{V}$ be the space of 2×2 matrices over $\mathbf{F}$. Find a basis $\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4\}$ for $\mathbf{V}$ such that $\mathbf{A}_j^2 = \mathbf{A}_j$ for each j

Solution: Every 2×2 matrix may be written as

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}=a\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}+b\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}+c\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}+d\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} \tag{2.3.5.1}$$

This shows that

$$\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4\}=\left\{\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix},\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix},\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix},\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\right\} \tag{2.3.5.2}$$

can be the basis for the space $\mathbf{V}$ of all 2×2 matrices. However $\mathbf{A}_2$ and $\mathbf{A}_3$ doesn't satisfy the property of $\mathbf{A}^2 = \mathbf{A}$. Consider b = 0 and c = 0, then the matrix

$$\begin{pmatrix}a & 0\\ 0 & d\end{pmatrix} \tag{2.3.5.3}$$

can't be a basis as it is the linear combination of $\mathbf{A}_1$ and $\mathbf{A}_4$. Hence either b or c or both must be non zero. Hence,

$$\mathbf{A}_2=\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix} \tag{2.3.5.4}$$

$$\mathbf{A}_3=\begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix} \tag{2.3.5.5}$$

Here, $\mathbf{A}_2^2 = \mathbf{A}_2$ and $\mathbf{A}_3^2 = \mathbf{A}_3$. Therefore the basis can be

$$\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4\}=\left\{\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix},\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix},\begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix},\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\right\} \tag{2.3.5.6}$$

$\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4\}$ forms the basis, iff they are linearly independent and the linear combination of them span the space $\mathbf{V}$. To show that they are linearly independent, we show that the equation has a trivial solution.

$$a\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}+b\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}+c\begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix}+d\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} \tag{2.3.5.7}$$

$$\implies a+b=0 \tag{2.3.5.8}$$

$$b=0 \tag{2.3.5.9}$$

$$c=0 \tag{2.3.5.10}$$

$$c+d=0 \tag{2.3.5.11}$$

The corresponding matrix form is $\mathbf{A}\mathbf{x} = 0$

$$\begin{pmatrix}1&1&0&0\\0&0&1&0\\0&1&0&0\\0&0&1&1\end{pmatrix}\begin{pmatrix}a\\b\\c\\d\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix} \quad (2.3.5.12)$$

Row reducing the augmented matrix,

$$\begin{pmatrix}1&1&0&0&0\\0&0&1&0&0\\0&1&0&0&0\\0&0&1&1&0\end{pmatrix}\xleftrightarrow[R_4\leftarrow R_4-R_3]{R_2\leftrightarrow R_3}\begin{pmatrix}1&1&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\end{pmatrix} \quad (2.3.5.13)$$

$$\xleftrightarrow{R_1\leftarrow R_1-R_2}\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\end{pmatrix} \quad (2.3.5.14)$$

Therefore, a = b = c = d = 0. Hence the matrices are linearly independent. To show that the linear combination of $\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4\}$ span the space $\mathbf{V}$, consider an arbitrary matrix,

$$\begin{pmatrix}w&x\\y&z\end{pmatrix} \quad (2.3.5.15)$$

Compute a, b, c, d such that

$$\begin{pmatrix}w&x\\y&z\end{pmatrix}=a\begin{pmatrix}1&0\\0&0\end{pmatrix}+b\begin{pmatrix}1&0\\1&0\end{pmatrix}+c\begin{pmatrix}0&1\\0&1\end{pmatrix}+d\begin{pmatrix}0&0\\0&1\end{pmatrix} \quad (2.3.5.16)$$

$$=\begin{pmatrix}a+b&c\\b&c+d\end{pmatrix} \quad (2.3.5.17)$$

Equating the entries, this produces system of linear equations,

$$a+b=w, y=b, x=c, z=c+d \quad (2.3.5.18)$$
$$\implies a = w-y \quad (2.3.5.19)$$
$$b = y \quad (2.3.5.20)$$
$$c = x \quad (2.3.5.21)$$
$$d = z-x \quad (2.3.5.22)$$

In particular, there exists atleast one solution regardless of the values of w, x, y, z. For example, consider the following matrix,

$$\begin{pmatrix}w&x\\y&z\end{pmatrix}=\begin{pmatrix}3&4\\-2&7\end{pmatrix} \quad (2.3.5.23)$$

Here, $a = 5, b = -2, c = 4, d = 3$. Using (2.3.5.16), we get

$$5\begin{pmatrix}1&0\\0&0\end{pmatrix}-2\begin{pmatrix}1&0\\1&0\end{pmatrix}+4\begin{pmatrix}0&1\\0&1\end{pmatrix}+3\begin{pmatrix}0&0\\0&1\end{pmatrix}=\begin{pmatrix}3&4\\-2&7\end{pmatrix} \quad (2.3.5.24)$$

Hence $\left\{\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}1&0\\1&0\end{pmatrix},\begin{pmatrix}0&1\\0&1\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}$ forms the basis for the given space $\mathbf{V}$.

2.3.6. Let $\mathbf{V}$ be a vector space over a subfield $\mathbf{F}$ of complex numbers. Suppose α, β and γ are linearly independent vectors in $\mathbf{V}$. Prove that $(\alpha+\beta)$,$(\beta+\gamma)$ and $(\gamma+\alpha)$ are linearly independent.

Solution: Let α, β and γ be three n× 1 dimensional vectors. We need to prove that,

$$\begin{pmatrix}\alpha+\beta & \beta+\gamma & \gamma+\alpha\end{pmatrix}\mathbf{x}=0 \quad (2.3.6.1)$$

will only have a trivial solution.The above equation can be written as

$$\begin{pmatrix}\alpha & \beta & \gamma\end{pmatrix}\begin{pmatrix}1&0&1\\1&1&0\\0&1&1\end{pmatrix}\mathbf{x}=0 \quad (2.3.6.2)$$

$$\mathbf{x}^T\begin{pmatrix}1&1&0\\0&1&1\\1&0&1\end{pmatrix}\begin{pmatrix}\alpha^T\\\beta^T\\\gamma^T\end{pmatrix}=0 \quad (2.3.6.3)$$

Since, α, β and γ are independent.

$$\mathbf{x}^T\begin{pmatrix}1&1&0\\0&1&1\\1&0&1\end{pmatrix}=0 \quad (2.3.6.4)$$

In the above equation we can see that the 3×3 matrix has linearly independent rows and hence will have a trivial solution.So,$\mathbf{x}$ is a zero vector.Hence, $(\alpha+\beta)$,$(\beta+\gamma)$ and $(\gamma+\alpha)$ are linearly independent.

2.3.7. Prove that the space of all $\mathbf{m}x\mathbf{n}$ matrices over the field $\mathbf{F}$ has dimension mn, by exhibiting a basis for this space.

Solution: Let $\mathbf{M}$ be the space of all $\mathbf{m} \times \mathbf{n}$ matrices. Let, $\mathbf{M}_{ij} \in \mathbf{M}$ be,

$$\mathbf{M}_{ij}=\begin{cases}0 & m\neq i, n\neq j\\ 1 & m=i, n=j\end{cases} \quad (2.3.7.1)$$

For example,

$$\mathbf{M}_{12} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{mxn} \tag{2.3.7.2}$$

$$\tag{2.3.7.3}$$

Let $\mathbf{A} \in \mathbf{M}$ given as,

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}_{mxn} \tag{2.3.7.4}$$

Now clearly,

$$\mathbf{a}_{11} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \tag{2.3.7.5}$$

$$\implies \mathbf{a}_{11} = \mathbf{A}\mathbf{M}_{11} \tag{2.3.7.6}$$

$$\therefore \mathbf{A} = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} M_{ij} \tag{2.3.7.7}$$

$\implies$ $\mathbf{M}_{ij}$ span $\mathbf{M}$. Also from the above equation $\mathbf{A}= 0$ if and only if all elements are zero, that is,

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \tag{2.3.7.8}$$

$$\implies a_{ij} = 0 \tag{2.3.7.9}$$

Hence, $\mathbf{M}_{ij}$ are linearly independent as well. Hence, $\mathbf{M}_{ij}$ constitutes a basis for $\mathbf{M}$. and number of elememts in basis are mn. Hence dimension of space of all mxn matrices $\mathbf{M}$ is mn.

2.3.8. Let V be a vector space over the field $F = \{0, 1\}$. Suppose α, β and γ are linearly independent vectors in V. Comment on $(\alpha+\beta)$, $(\beta+\gamma)$ and $(\gamma+\alpha)$

Solution: The addition of elements in the field $\mathbf{F}$ is defined as,

$$\begin{aligned} 0 + 0 &= 0 \\ 1 + 1 &= 0 \end{aligned} \tag{2.3.8.1}$$

A set are vectors $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ are linearly independent if

$$a\mathbf{v_1} + b\mathbf{v_2} + c\mathbf{v_3} = 0 \tag{2.3.8.2}$$

has only one trivial solution

$$a = b = c = 0 \tag{2.3.8.3}$$

Now,

$$a(\alpha+\beta) + b(\beta+\gamma) + c(\gamma+\alpha) = 0 \tag{2.3.8.4}$$

$$\implies (a+c)\alpha + (a+b)\beta + (b+c)\gamma = 0 \tag{2.3.8.5}$$

Writing (2.3.8.5) in matrix form,

$$\begin{pmatrix} \alpha & \beta & \gamma \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{2.3.8.6}$$

where,

$$\mathbf{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\mathbf{x}^T \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = 0 \tag{2.3.8.7}$$

Since α, β and γ are linearly independent vectors,

$$\mathbf{x}^T \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \tag{2.3.8.8}$$

Transposing on both sides,

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{2.3.8.9}$$

By using the properties from (2.3.8.1) and reducing (2.3.8.9) to row echelon form,

$$\begin{pmatrix}1&0&1\\1&1&0\\0&1&1\end{pmatrix} \xleftrightarrow{R_2\leftarrow R_1+R_2} \begin{pmatrix}1&0&1\\0&1&1\\0&1&1\end{pmatrix}$$

$$\xleftrightarrow{R_3\leftarrow R_2+R_3} \begin{pmatrix}1&0&1\\0&1&1\\0&0&0\end{pmatrix} \quad (2.3.8.10)$$

Expressing (2.3.8.10) as a linear combination of vectors,

$$a\begin{pmatrix}1\\0\\0\end{pmatrix} + b\begin{pmatrix}0\\1\\0\end{pmatrix} + c\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\implies \begin{pmatrix}a+c\\b+c\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\implies a+c=0; \quad b+c=0 \quad (2.3.8.11)$$

The solutions to (2.3.8.11) are,

$$a=b=c=0; \quad a=b=c=1 \quad (2.3.8.12)$$

Since there is no trivial solution, $(\alpha+\beta)$, $(\beta+\gamma)$ and $(\gamma+\alpha)$ are linearly dependent

2.3.9. Let **V** be the set of real numbers.Regard **V** as a vector space over the field of rational numbers,with usual operations.Prove that this vector space is not finite-dimensional.

Solution: Given **V** is a vector space over field $\mathbb{Q}$ (rational numbers)

It is finite dimensional with dimensionality n if every vector $\mathbf{v}$ in **V** can be written as

$$\mathbf{v} = \sum_{i=0}^{n-1} c_i\alpha_i \quad (2.3.9.1)$$

$$\text{where } c_i \in \mathbb{Q} \quad (2.3.9.2)$$

$$\text{and } \mathbf{B} = \{\alpha_0, \alpha_1, \dots, \alpha_{n-1}\} \quad (2.3.9.3)$$

is the basis with linearly independent α_i

that is, basis is the largest set with linearly independent vectors from **V**

Consider the set of vectors $\{1, x\}$, where x is irrational.

Assume there exists non zero $\beta_0,\beta_1 \in \mathbb{Q}$ such that

$$\beta_0 + \beta_1 x = 0 \quad (2.3.9.4)$$

$$\implies x = -\frac{\beta_0}{\beta_1} \quad (2.3.9.5)$$

But x is irrational and $-\frac{\beta_0}{\beta_1}$ is rational so (2.3.9.5) can't be possible so $\beta_0,\beta_1 = 0$

Hence $\{1, x\}$ are independent.

Similarly for the set $\left\{1, x, x^2\right\}$

for $\beta_0,\beta_1,\beta_2 \in \mathbb{Q}$

$$\beta_0 + \beta_1 x + \beta_2 x^2 = 0 \quad (2.3.9.6)$$

$\beta_1 x + \beta_2 x^2$ is irrational and β_0 is rational. Therefore

$$\beta_0 = 0 \quad (2.3.9.7)$$

$$\text{and } \beta_1 x + \beta_2 x^2 = 0, (x \neq 0) \quad (2.3.9.8)$$

$$\implies \beta_1 + \beta_2 x = 0 \quad (2.3.9.9)$$

$$\implies \beta_1,\beta_2 = 0 \quad (2.3.9.10)$$

$$\therefore \beta_0 + \beta_1 x + \beta_2 x^2 = 0 \quad (2.3.9.11)$$

$$\iff \beta_0,\beta_1,\beta_2 = 0 \quad (2.3.9.12)$$

Hence $\left\{1, x, x^2\right\}$ are independent

By induction, let us say the set $\left\{1, x, x^2, \dots, x^n\right\}$ is independent

$$\text{for } \beta_0,\beta_1,\beta_2,\dots,\beta_n \in \mathbb{Q} \quad (2.3.9.13)$$

$$\beta_0 + \beta_1 x + \beta_2 x^2 + \dots + \beta_n x^n = 0 \quad (2.3.9.14)$$

$$\iff \beta_0,\beta_1,\beta_2,\dots,\beta_n = 0 \quad (2.3.9.15)$$

To prove this for the set $\mathbf{A}=\left\{1, x, x^2, \dots, x^{n+1}\right\}$

$$\text{for } \beta_0,\beta_1,\beta_2,\dots,\beta_n,\beta_{n+1} \in \mathbb{Q} \quad (2.3.9.16)$$

$$\beta_0 + \beta_1 x + \dots + \beta_n x^n + \beta_{n+1} x^{n+1} = 0 \quad (2.3.9.17)$$

Comparing to (2.3.9.7) and (2.3.9.8)

$$\beta_0 = 0 \quad (2.3.9.18)$$

$$\beta_1 + \beta_2 x + \dots + \beta_{n+1} x^n = 0 \quad (2.3.9.19)$$

Comparing with (2.3.9.14),we have $\beta_1,\beta_2,\dots,\beta_{n+1} = 0$

$$\therefore \beta_0 + \beta_1 x + \dots + \beta_n x^n + \beta_{n+1} x^{n+1} = 0 \quad (2.3.9.20)$$

$$\iff \beta_0,\beta_1,\beta_2,\dots,\beta_n,\beta_{n+1} = 0 \quad (2.3.9.21)$$

Hence **A** has linearly independent vectors

Let the set $\mathbf{B} = \left\{1, x, x^2, \dots, x^m\right\}$ be the largest linearly independent set in **V** and hence can form the basis leading to dimensionality $m+1$

But from induction, we have proved that

$\left\{1, x, x^2, \ldots, x^m, x^{m+1}\right\}$ is also independent which is a contradiction to dimensionality being $m+1$
Hence we deduce that the vector space $\mathbf{V}$ is not finite dimensional over the field $\mathbb{Q}$

2.4 Coordinates

2.4.1. Show that the vectors

$$\alpha_{\mathbf{1}} = \begin{pmatrix}1 & 1 & 0 & 0\end{pmatrix} \quad \alpha_{\mathbf{2}} = \begin{pmatrix}0 & 0 & 1 & 1\end{pmatrix} \tag{2.4.1.1}$$

$$\alpha_{\mathbf{3}} = \begin{pmatrix}1 & 0 & 0 & 4\end{pmatrix} \quad \alpha_{\mathbf{4}} = \begin{pmatrix}0 & 0 & 0 & 2\end{pmatrix} \tag{2.4.1.2}$$

form a basis for $\Re^4$. Find the coordinates of each of the standard basis vectors in the ordered basis $\begin{pmatrix}\alpha_{\mathbf{1}} & \alpha_{\mathbf{2}} & \alpha_{\mathbf{3}} & \alpha_{\mathbf{4}}\end{pmatrix}$
Solution:

Theorem 2.1. *Let* $\mathbf{V}$ *be an n-dimensional vector space over the field* $\mathbf{F}$*, and let* β *and* β' *be two ordered basis of* $\mathbf{V}$*. Then, there is a unique, necessarily invertible,* $n \times n$ *matrix* $\mathbf{P}$ *with entries in* $\mathbf{F}$ *such that*
a) $\left[\alpha\right]_{\beta} = \mathbf{P}\left[\alpha\right]_{\beta'}$
b) $\left[\alpha\right]_{\beta'} = \mathbf{P^{-1}}\left[\alpha\right]_{\beta}$
for every vector α *in* $\mathbf{V}$*. The columns of* $\mathbf{P}$ *are given by*

$$\mathbf{P_j} = \left[\alpha_{\mathbf{j}}\right]_{\beta} \qquad j = 1, 2, ..., n \tag{2.4.1.3}$$

Firt, we need to show that the set of vectors α_1, α_2, α_3 and α_4 are basis for $\Re^4$. For, this we first show that α_1, α_2, α_3 and α_4 are linearly independent in $\Re^4$ and also they span $\Re^4$. Consider,

$$\mathbf{A} = \begin{pmatrix}\alpha_{\mathbf{1}}^T & \alpha_{\mathbf{2}}^T & \alpha_{\mathbf{3}}^T & \alpha_{\mathbf{4}}^T\end{pmatrix} \tag{2.4.1.4}$$

$$\mathbf{A} = \begin{pmatrix}1 & 0 & 1 & 0\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 1 & 4 & 2\end{pmatrix} \tag{2.4.1.5}$$

Now,

$$\begin{pmatrix}1 & 0 & 1 & 0\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 1 & 4 & 2\end{pmatrix} \xleftrightarrow{r_2 = r_2 - r_1} \begin{pmatrix}1 & 0 & 1 & 0\\ 0 & 0 & -1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 1 & 4 & 2\end{pmatrix} \tag{2.4.1.6}$$

$$\begin{pmatrix}1 & 0 & 1 & 0\\ 0 & 0 & -1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 1 & 4 & 2\end{pmatrix} \xleftrightarrow{r_2 \leftrightarrow r_3} \begin{pmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 1 & 4 & 2\end{pmatrix} \tag{2.4.1.7}$$

$$\begin{pmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 1 & 4 & 2\end{pmatrix} \xleftrightarrow{r_4 = r_4 - r_2} \begin{pmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 4 & 2\end{pmatrix} \tag{2.4.1.8}$$

$$\begin{pmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 4 & 2\end{pmatrix} \xleftrightarrow{r_3 = -r_3} \begin{pmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 4 & 2\end{pmatrix} \tag{2.4.1.9}$$

$$\begin{pmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 4 & 2\end{pmatrix} \xleftrightarrow{r_4 = r_4 - 4r_3} \begin{pmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 2\end{pmatrix} \tag{2.4.1.10}$$

$$\begin{pmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 2\end{pmatrix} \xleftrightarrow{r_1 = r_1 - r_3} \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 2\end{pmatrix} \tag{2.4.1.11}$$

$$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 2\end{pmatrix} \xleftrightarrow{r_4 = \frac{r_4}{2}} \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} \tag{2.4.1.12}$$

(2.4.1.12) is the row reduced echelon form of $\mathbf{A}$ and since it is identity matrix of order 4, we say that vectors $\alpha_{\mathbf{1}}$, $\alpha_{\mathbf{2}}$, $\alpha_{\mathbf{3}}$ and $\alpha_{\mathbf{4}}$ are linearly independent and their column space is $\Re^4$ which means vectors $\alpha_{\mathbf{1}}$, $\alpha_{\mathbf{2}}$, $\alpha_{\mathbf{3}}$ and $\alpha_{\mathbf{4}}$ span $\Re^4$. Hence, vectors $\alpha_{\mathbf{1}}$, $\alpha_{\mathbf{2}}$, $\alpha_{\mathbf{3}}$ and $\alpha_{\mathbf{4}}$ form a basis for $\Re^4$.
Now, we use theorem (2.1), and if we calculate

the inverse of

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 2 \end{pmatrix} \tag{2.4.1.13}$$

then the columns of $\mathbf{A^{-1}}$ will give the coeffients to write the standard basis vectors in terms of $\alpha_i' s$. We try to find the inverse of $\mathbf{A}$ by row-reducing the augumented matrix.

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 4 & 2 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.4.1.14}$$

Now, we solve for $\mathbf{A^{-1}}$ as follows

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 4 & 2 & 0 & 0 & 0 & 1 \end{pmatrix} \xleftrightarrow{r_2=r_2-r_1}$$
$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 4 & 2 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.4.1.15}$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 4 & 2 & 0 & 0 & 0 & 1 \end{pmatrix} \xleftrightarrow{r_2 \leftrightarrow r_3}$$
$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 4 & 2 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.4.1.16}$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 4 & 2 & 0 & 0 & 0 & 1 \end{pmatrix} \xleftrightarrow{r_4=r_4-r_2}$$
$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 & 0 & -1 & 1 \end{pmatrix} \tag{2.4.1.17}$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 & 0 & -1 & 1 \end{pmatrix} \xleftrightarrow{r_3=-r_3}$$
$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 & 0 & -1 & 1 \end{pmatrix} \tag{2.4.1.18}$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 & 0 & -1 & 1 \end{pmatrix} \xleftrightarrow{r_4=r_4-4r_3}$$
$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & -4 & 4 & -1 & 1 \end{pmatrix} \tag{2.4.1.19}$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & -4 & 4 & -1 & 1 \end{pmatrix} \xleftrightarrow{r_1=r_1-r_3}$$
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & -4 & 4 & -1 & 1 \end{pmatrix} \tag{2.4.1.20}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & -4 & 4 & -1 & 1 \end{pmatrix} \xleftrightarrow{r_4=\frac{r_4}{2}}$$
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 2 & \frac{-1}{2} & \frac{1}{2} \end{pmatrix} \tag{2.4.1.21}$$

Thus, by (2.4.1.21), we have

$$\mathbf{A^{-1}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ -2 & 2 & \frac{-1}{2} & \frac{1}{2} \end{pmatrix} \tag{2.4.1.22}$$

Now, let $\mathbf{e_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}$, $\mathbf{e_2} = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}$, $\mathbf{e_3} = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}$ and $\mathbf{e_4} = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}$ be the standard basis for $\mathfrak{R}^4$. Hence,

$$\mathbf{e_1} = \alpha_3 - 2\alpha_4 \tag{2.4.1.23}$$

$$\mathbf{e_2} = \alpha_1 - \alpha_3 + 2\alpha_4 \tag{2.4.1.24}$$

$$\mathbf{e_3} = \alpha_2 - \frac{1}{2}\alpha_4 \tag{2.4.1.25}$$

$$\mathbf{e_4} = \frac{1}{2}\alpha_4 \tag{2.4.1.26}$$

2.4.2. Find the coordinate matrix of the vector $\begin{pmatrix}1 & 0 & 1\end{pmatrix}$ in the basis of C^3 consisting of the vectors $\begin{pmatrix}2i & 1 & 0\end{pmatrix}$, $\begin{pmatrix}2 & -1 & 1\end{pmatrix}$, $\begin{pmatrix}0 & 1+i & 1-i\end{pmatrix}$ in that order.
Solution:

$$\begin{pmatrix}1 & 0 & 1\end{pmatrix} = \alpha_1\begin{pmatrix}2i & 1 & 0\end{pmatrix} + \alpha_2\begin{pmatrix}2 & -1 & 1\end{pmatrix} + \alpha_3\begin{pmatrix}0 & 1+i & 1-i\end{pmatrix} \tag{2.4.2.1}$$

$$\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}2i & 2 & 0\\1 & -1 & 1+i\\0 & 1 & 1-i\end{pmatrix}\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix} \tag{2.4.2.2}$$

Now we find α_i by row reducing augmented matrix.

$$\begin{pmatrix}2i & 2 & 0 & 1\\1 & -1 & 1+i & 0\\0 & 1 & 1-i & 1\end{pmatrix} \xleftrightarrow[R_2 \to R_2 - 2iR_3]{R_1 \to R_2} \begin{pmatrix}1 & -1 & 1+i & 0\\0 & 2+2i & 2-2i & 1\\0 & 1 & 1-i & 1\end{pmatrix} \tag{2.4.2.3}$$

$$\xleftrightarrow[R_3 \leftarrow R_3 - R_2]{R_2 \leftarrow R_2/2} \begin{pmatrix}1 & -1 & 1+i & 0\\0 & 1+i & 1-i & \frac{1}{2}\\0 & -i & 0 & \frac{1}{2}\end{pmatrix} \tag{2.4.2.4}$$

Therefore the coordinate matrix of the vector is

$$\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix} = \begin{pmatrix}\frac{-1-i}{2}\\\frac{i}{2}\\\frac{3+i}{4}\end{pmatrix} \tag{2.4.2.5}$$

2.4.3. Let $\mathbf{B} = \begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_3\end{pmatrix}$ be the ordered basis for R^3 consisting of

$\alpha_\mathbf{1} = \begin{pmatrix}1 & 0 & -1\end{pmatrix}, \alpha_\mathbf{2} = \begin{pmatrix}1 & 1 & 1\end{pmatrix}, \alpha_\mathbf{3} = \begin{pmatrix}1 & 0 & 0\end{pmatrix}.$

What are the coordinates of vector $\begin{pmatrix}a & b & c\end{pmatrix}$ in the ordered basis $\mathbf{B}$?

Solution: Given

$$\mathbf{B} = \begin{pmatrix}1 & 1 & 1\\0 & 1 & 0\\-1 & 1 & 0\end{pmatrix} \tag{2.4.3.1}$$

be the ordered basis for R^3,then the coordinates of vector,

$$\alpha = \begin{pmatrix}a\\b\\c\end{pmatrix} \tag{2.4.3.2}$$

in the ordered basis R^3 is the vector,

$$[\alpha]_\mathbf{B} = \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} \tag{2.4.3.3}$$

hence

$$x_1\alpha_1 + x_2\alpha_2 + x_3\alpha_3 = \alpha \tag{2.4.3.4}$$

substituting (2.4.3.1) and (2.4.3.2) in (2.4.3.4)

$$\begin{pmatrix}1 & 1 & 1\\0 & 1 & 0\\-1 & 1 & 0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}a\\b\\c\end{pmatrix} \tag{2.4.3.5}$$

augmented matrix form

$$\begin{pmatrix}1 & 1 & 1 & a\\0 & 1 & 0 & b\\-1 & 1 & 0 & c\end{pmatrix} \tag{2.4.3.6}$$

converting above matrix into row reduced echelon form

$$\begin{pmatrix}1 & 1 & 1 & a\\0 & 1 & 0 & b\\-1 & 1 & 0 & c\end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 + R_1} \begin{pmatrix}1 & 1 & 1 & a\\0 & 1 & 0 & b\\0 & 2 & 1 & c+a\end{pmatrix} \tag{2.4.3.7}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - 2R_2} \begin{pmatrix}1 & 1 & 1 & a\\0 & 1 & 0 & b\\0 & 0 & 1 & a-2b+c\end{pmatrix} \tag{2.4.3.8}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - R_2} \begin{pmatrix}1 & 0 & 1 & a-b\\0 & 1 & 0 & b\\0 & 0 & 1 & a-2b+c\end{pmatrix} \tag{2.4.3.9}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - R_3} \begin{pmatrix}1 & 0 & 0 & b-c\\0 & 1 & 0 & b\\0 & 0 & 1 & a-2b+c\end{pmatrix} \tag{2.4.3.10}$$

$\therefore$ The coordinates of α w.r.t $\mathbf{B}$ is

$$[\alpha]_{\mathbf{B}} = \begin{pmatrix} b-c \\ b \\ a-2b+c \end{pmatrix} \tag{2.4.3.11}$$

2.4.4. Let $\mathbf{W}$ be the subspace of $\mathbf{C}^3$ spanned by $\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}$ and $\alpha_2 = \begin{pmatrix} 1 \\ i \\ 1+i \end{pmatrix}$.

a) Show that α_1 and α_2 form a basis for $\mathbf{W}$.

b) Show that the vectors $\beta_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\beta_2 = \begin{pmatrix} 1 \\ i \\ 1+i \end{pmatrix}$ are in $\mathbf{W}$ and form another basis for $\mathbf{W}$.

c) What are the coordinates of α_1 and α_2 in the ordered basis $\{\beta_1, \beta_2\}$ for $\mathbf{W}$.

Solution:

a) It is given that α_1 and α_2 span $\mathbf{W}$. For α_1 and α_2 to be the basis for $\mathbf{W}$ they must be linearly independent. Let

$$S_1 = \{\alpha_1, \alpha_2\} = \left\{ \begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}, \begin{pmatrix} 1+i \\ 1 \\ -1 \end{pmatrix} \right\} \tag{2.4.4.1}$$

Using row reduction on matrix $\mathbf{A} = \begin{pmatrix} \alpha_1 & \alpha_2 \end{pmatrix}$

$$\begin{pmatrix} 1 & 1+i \\ 0 & 1 \\ i & -1 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 - iR_1} \begin{pmatrix} 1 & 1+i \\ 0 & 1 \\ 0 & -i \end{pmatrix} \tag{2.4.4.2}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 + iR_2} \begin{pmatrix} 1 & 1+i \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{2.4.4.3}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - (i+1)R_2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{2.4.4.4}$$

Since $\mathbf{A}$ is a full-rank matrix the column vectors are linearly independent. Therefore $S_1 = \{\alpha_1, \alpha_2\}$ is a basis set for $\mathbf{W}$.

b)

$$\beta_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \tag{2.4.4.5}$$

$$\beta_2 = \begin{pmatrix} 1 \\ i \\ 1+i \end{pmatrix} \tag{2.4.4.6}$$

Since column vectors of $\mathbf{A} = \begin{pmatrix} \alpha_1 & \alpha_2 \end{pmatrix}$ are basis for $\mathbf{W}$ and if β_1 and $\beta_2 \in \mathbf{W}$ there exist a unique solution $\mathbf{x}$ such that

$$\begin{pmatrix} \alpha_1 & \alpha_2 \end{pmatrix} \mathbf{x} = \begin{pmatrix} \beta_1 & \beta_2 \end{pmatrix} \tag{2.4.4.7}$$

Using row reduction on augmented matrix

$$\left(\begin{array}{cc|cc} 1 & 1+i & 1 & 1 \\ 0 & 1 & 1 & i \\ i & -1 & 0 & 1+i \end{array}\right) \tag{2.4.4.8}$$

$$\xleftrightarrow{R3 \leftarrow R_3 - iR-1} \left(\begin{array}{cc|cc} 1 & 1+i & 1 & 1 \\ 0 & 1 & 1 & i \\ 0 & -i & -i & 1 \end{array}\right) \tag{2.4.4.9}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 + iR_2} \left(\begin{array}{cc|cc} 1 & 1+i & 1 & 1 \\ 0 & 1 & 1 & i \\ 0 & 0 & 0 & 0 \end{array}\right) \tag{2.4.4.10}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - (i+1)R_2} \left(\begin{array}{cc|cc} 1 & 0 & -i & 2-i \\ 0 & 1 & 1 & i \\ 0 & 0 & 0 & 0 \end{array}\right) \tag{2.4.4.11}$$

$$\implies \mathbf{x} = \begin{pmatrix} -i & 2-i \\ 1 & i \end{pmatrix} \tag{2.4.4.12}$$

Therefore β_1 and $\beta_2 \in \mathbf{W}$.
Consider

$$S_2 = \{\beta_1, \beta_2\} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ i \\ 1+i \end{pmatrix} \right\} \tag{2.4.4.13}$$

and also let

$$\mathbf{B} = \begin{pmatrix} 1 & 1 \\ 1 & i \\ 0 & 1+i \end{pmatrix} \tag{2.4.4.14}$$

Using row reduction on matrix **B**

$$\begin{pmatrix}1 & 1\\1 & i\\0 & 1+i\end{pmatrix} \xleftrightarrow{R_2\leftarrow R_2-R_1} \begin{pmatrix}1 & 1\\0 & i-1\\0 & 1+i\end{pmatrix} \tag{2.4.4.15}$$

$$\xleftrightarrow{R_2\leftarrow \frac{R_2}{i-1}} \begin{pmatrix}1 & 1\\0 & 1\\0 & 1+i\end{pmatrix} \tag{2.4.4.16}$$

$$\xleftrightarrow[R_3\leftarrow R_3-(i-1)R_2]{R_1\leftarrow R_1-R_2} \begin{pmatrix}1 & 0\\0 & 1\\0 & 0\end{pmatrix} \tag{2.4.4.17}$$

Since **B** is a full rank matrix the column vectors are linearly independent.
Let α be any vector in the subspace **W**, then it can be expressed as span $\{\alpha_1, \alpha_2\}$ i.e

$$\alpha = \begin{pmatrix}\alpha_1 & \alpha_2\end{pmatrix}\mathbf{x_1} = \mathbf{Ax_1} \tag{2.4.4.18}$$

$S_2 = \{\beta_1, \beta_2\}$ spans **W** if any vector $\alpha \in$ **W** can be expressed as

$$\alpha = \begin{pmatrix}\beta_1, \beta_2\end{pmatrix}\mathbf{x_2} = \mathbf{Bx_2} \tag{2.4.4.19}$$

From (2.4.4.18) and (2.4.4.19) we conclude

$$\mathbf{Bx_2} = \mathbf{Ax_1} \tag{2.4.4.20}$$

$$\implies \mathbf{x_2} = \mathbf{B}^{-1}\mathbf{Ax_1} \tag{2.4.4.21}$$

Therefore from (2.4.4.21) $\mathbf{x_2}$ exists if **B** is invertible. From (2.4.4.17) we conclude $\mathbf{x_2}$ exists and hence any vector $\alpha \in$ **W** can be expressed as span$\{\beta_1, \beta_2\}$. Therefore $\{\beta_1, \beta_2\}$ is basis for **W**.

c) Since $\alpha_1, \alpha_2 \in$ **W** and $\{\beta_1, \beta_2\}$ are ordered basis for **W** there must exist unique value of **x** such that

$$\begin{pmatrix}\beta_1 & \beta_2\end{pmatrix}\mathbf{x} = \begin{pmatrix}\alpha_1 & \alpha_2\end{pmatrix} \tag{2.4.4.22}$$

Using row reduction on (2.4.4.22) we get,

$$\left(\begin{array}{cc|cc}1 & 1 & 1 & 1+i\\1 & i & 0 & 1\\0 & 1+i & i & -1\end{array}\right) \tag{2.4.4.23}$$

$$\xleftrightarrow{R_2\leftarrow R_2-R_1} \left(\begin{array}{cc|cc}1 & 1 & 1 & 1+i\\0 & i-1 & -1 & -i\\0 & 1+i & i & -1\end{array}\right) \tag{2.4.4.24}$$

$$\xleftrightarrow{R_2\leftarrow \frac{R_2}{i-1}} \left(\begin{array}{cc|cc}1 & 1 & 1 & 1+i\\0 & 1 & \frac{1+i}{2} & \frac{-1+i}{2}\\0 & 1+i & i & -1\end{array}\right) \tag{2.4.4.25}$$

$$\xleftrightarrow{R_3\leftarrow R_3-(i+1)R_2} \left(\begin{array}{cc|cc}1 & 1 & 1 & 1+i\\0 & 1 & \frac{1+i}{2} & \frac{-1+i}{2}\\0 & 0 & 0 & 0\end{array}\right) \tag{2.4.4.26}$$

$$\xleftrightarrow{R_1\leftarrow R_2-R_1} \left(\begin{array}{cc|cc}1 & 0 & \frac{1-i}{2} & \frac{3+i}{2}\\0 & 1 & \frac{1+i}{2} & \frac{-1+i}{2}\\0 & 0 & 0 & 0\end{array}\right) \tag{2.4.4.27}$$

$$\implies \mathbf{x} = \frac{1}{2}\begin{pmatrix}1-i & 3+i\\1+i & -1+i\end{pmatrix} \tag{2.4.4.28}$$

Thus the column vectors of (2.4.4.28) are corresponding coordinates of α_1 and α_2 in ordered basis $\{\beta_1, \beta_2\}$.

2.4.5. let $\alpha = \begin{pmatrix}x_1\\x_2\end{pmatrix}$ and $\beta = \begin{pmatrix}y_1\\y_2\end{pmatrix}$ be vectors in $\mathbb{R}^2$ such that
$x_1y_1 + x_2y_2 = 0; \quad x_1^2 + x_2^2 = y_1^2 + y_2^2 = 1.$
Proove that ß $= \{\alpha, \beta\}$ is a basis of $\mathbb{R}^2$. Find the coordinates of the vector $\begin{pmatrix}a, b\end{pmatrix}$ in the ordered basis ß $= \{\alpha, \beta\}$. (The conditions on α and β say, geometrically, that α and β are perpendicular and each has length 1).
Solution: we need to show that α and β are linearly independent in order to proove that ß $= \{\alpha, \beta\}$ is a basis of $\mathbb{R}^2$.

Given in the question are:

$$\alpha^T\beta = 0 \tag{2.4.5.1}$$

$$\|\alpha\|^2 = \|\beta\|^2 = 1 \tag{2.4.5.2}$$

Let,

$$\mathbf{A} = \begin{pmatrix} \alpha & \beta \end{pmatrix} = \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} \tag{2.4.5.3}$$

then,

$$\mathbf{A}^T\mathbf{A} = \begin{pmatrix} ||\alpha||^2 & \alpha^T\beta \\ \alpha^T\beta & ||\beta||^2 \end{pmatrix} \tag{2.4.5.4}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.4.5.5}$$

$$\therefore \mathbf{A}^T\mathbf{A} = \mathbf{I} \tag{2.4.5.6}$$

Inverse of $\mathbf{A}$ exist. $\mathbf{A}^T$ is the inverse of $\mathbf{A}$. Thus, the columns of $\mathbf{A}$ are linearly independent i.e, α and β are linearly independent.
Hence, ß = $\{\alpha, \beta\}$ is a basis of $\mathbb{R}^2$.

To, find the coordinates of the vector $\begin{pmatrix} a, b \end{pmatrix}$ in the ordered basis ß = $\{\alpha, \beta\}$.

$$\begin{pmatrix} \alpha & \beta \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \tag{2.4.5.7}$$

$$\mathbf{A}^T\mathbf{A}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \mathbf{A}^T\begin{pmatrix} a \\ b \end{pmatrix} \tag{2.4.5.8}$$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \mathbf{A}^T\begin{pmatrix} a \\ b \end{pmatrix} \tag{2.4.5.9}$$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} \tag{2.4.5.10}$$

2.4.6. Let **V** be the real vector space of all polynomial functions from $\mathbb{R}$ to $\mathbb{R}$ of degree 2 or less, i.e, the space of all functions f of the form,

$$f(x) = c_0 + c_1x + c_2x^2$$

Let t be a fixed real number and define

$$g_1(x) = 1, g_2(x) = x + t, g_3(x) = (x + t)^2$$

Prove that β={g1,g2,g3} is a basis for V. If

$$f(x) = c_0 + c_1x + c_2x^2$$

what are the coordinates of f in the ordered basis β
Solution: We start by taking,

$$\mathbf{f} = \begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix} \tag{2.4.6.1}$$

Let's start by proving that **g** is linearly independent.

$$\mathbf{g} = \mathbf{Bf} \tag{2.4.6.2}$$

where,

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ t^2 & 2t & 1 \end{pmatrix} \tag{2.4.6.3}$$

Now,

$$\mathbf{v}^T\mathbf{g} = 0 \tag{2.4.6.4}$$

$$\implies \mathbf{v}^T\mathbf{Bf} = 0 \tag{2.4.6.5}$$

Since **f** is linearly independent,

$$\mathbf{v}^T\mathbf{B} = 0 \tag{2.4.6.6}$$

$$\mathbf{B}^T\mathbf{v} = 0 \tag{2.4.6.7}$$

Since $\mathbf{B}^T$ is an upper triangular matrix with non zero values in principal diagonal , it is invertible matrix and hence **v** will be zero vector.Now, Finding the inverse of $\mathbf{B}^T$

$$\left(\begin{array}{ccc|ccc} 1 & t & t^2 & 1 & 0 & 0 \\ 0 & 1 & 2t & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \tag{2.4.6.8}$$

$$\xleftrightarrow{R_1=R_1-tR_2}\left(\begin{array}{ccc|ccc} 1 & 0 & -t^2 & 1 & -t & 0 \\ 0 & 1 & 2t & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \tag{2.4.6.9}$$

$$\xleftrightarrow{R_1=R_1+t^2R_3}\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -t & t^2 \\ 0 & 1 & 2t & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \tag{2.4.6.10}$$

$$\xleftrightarrow{R_2=R_2-2tR_3}\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -t & t^2 \\ 0 & 1 & 0 & 0 & 1 & -2t \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \tag{2.4.6.11}$$

So,

$$(\mathbf{B}^T)^{-1} = \begin{pmatrix} 1 & -t & t^2 \\ 0 & 1 & -2t \\ 0 & 0 & 1 \end{pmatrix} \tag{2.4.6.12}$$

Now, to find the coordinates,

$$f(x) = \mathbf{w}^T\mathbf{g} \tag{2.4.6.13}$$

So,

$$\mathbf{c}^T\mathbf{f} = \mathbf{w}^T\mathbf{g} \tag{2.4.6.14}$$
$$\mathbf{c}^T\mathbf{f} = \mathbf{w}^T\mathbf{B}\mathbf{f} \tag{2.4.6.15}$$
$$(\mathbf{c}^T - \mathbf{w}^T\mathbf{B})\mathbf{f} = 0 \tag{2.4.6.16}$$

Since, $\mathbf{f}$ is linearly independent,

$$\mathbf{c}^T - \mathbf{w}^T\mathbf{B} = 0 \tag{2.4.6.17}$$
$$\mathbf{c}^T = \mathbf{w}^T\mathbf{B} \tag{2.4.6.18}$$
$$\mathbf{c}^T\mathbf{B}^{-1} = \mathbf{w}^T \tag{2.4.6.19}$$
$$(\mathbf{B}^{-1})^T\mathbf{c} = \mathbf{w} \tag{2.4.6.20}$$

Using (2.4.6.12) in (2.4.6.20)

$$\mathbf{w} = \begin{pmatrix} 1 & -t & t^2 \\ 0 & 1 & -2t \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} \tag{2.4.6.21}$$

2.5 Summary of Row Equivalence

2.5.1. Consider the vectors in $\mathbb{R}^4$ defined by:
$$\alpha_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 2 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 3 \\ 4 \\ -2 \\ 5 \end{pmatrix} \text{ and } \alpha_3 = \begin{pmatrix} 1 \\ 4 \\ 0 \\ 9 \end{pmatrix}.$$
Find a system of homogeneous linear equations for which the space of solutions is exactly the subspace of $\mathbb{R}^4$ spanned by the given three vectors.

Solution: A system of linear equations is homogeneous if all of the constant terms are zero. It can be represented as,

$$\mathbf{A}\mathbf{X} = 0 \tag{2.5.1.1}$$

Let $\mathbf{R}$ be a echelon matrix which is reduced to A. Then the systems $\mathbf{A}\mathbf{X} = 0$ and $\mathbf{R}\mathbf{X} = 0$ have the same solutions. Here,

$$\mathbf{A} = \begin{pmatrix} -1 & 0 & 1 & 2 \\ 3 & 4 & -2 & 5 \\ 1 & 4 & 0 & 9 \end{pmatrix} \tag{2.5.1.2}$$

By row reducing on $\mathbf{A}$, we get:

$$\mathbf{A} = \begin{pmatrix} -1 & 0 & 1 & 2 \\ 3 & 4 & -2 & 5 \\ 1 & 4 & 0 & 9 \end{pmatrix} \xleftrightarrow{R_3=R_3-2R_1-R_2} \begin{pmatrix} -1 & 0 & 1 & 2 \\ 3 & 4 & -2 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.5.1.3}$$

$$\xleftrightarrow{R_2=R_2+3R_1} \begin{pmatrix} -1 & 0 & 1 & 2 \\ 0 & 4 & 1 & 11 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.5.1.4}$$

$$\xleftrightarrow{R_1=-R_1} \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 4 & 1 & 11 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.5.1.5}$$

The bais vector is non zero vector which are given from 2.5.1.5,

$$\rho_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ -2 \end{pmatrix}, \rho_2 = \begin{pmatrix} 0 \\ 4 \\ 1 \\ 11 \end{pmatrix} \tag{2.5.1.6}$$

ρ_1, ρ_2 forms the basis of the solution space. The subspace spanned by b_1 and b_2 is given as:

$$\begin{pmatrix} \rho_1 & \rho_2 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \mathbf{X} \tag{2.5.1.7}$$

Using 2.5.1.7, we can write the augmented matrix as:

$$\begin{pmatrix} 1 & 0 & x_1 \\ 0 & 4 & x_2 \\ -1 & 1 & x_3 \\ -2 & 11 & x_4 \end{pmatrix} \xleftrightarrow{R_3=4R_3+4R_1-R_2} \tag{2.5.1.8}$$

$$\begin{pmatrix} 1 & 0 & x_1 \\ 0 & 4 & x_2 \\ 0 & 0 & 4x_1 - x_2 + 4x_3 \\ -2 & 11 & x_4 \end{pmatrix} \xleftrightarrow{R_4=4R_4+8R_1-11R_2} \tag{2.5.1.9}$$

$$\begin{pmatrix} 1 & 0 & x_1 \\ 0 & 4 & x_2 \\ 0 & 0 & 4x_1 - x_2 + 4x_3 \\ 0 & 0 & 8x_1 - 11x_2 + 4x_4 \end{pmatrix} \tag{2.5.1.10}$$

Using 2.5.1.10, The required homogeneous equation is given as:

$$\begin{pmatrix} 4 & -1 & 4 & 0 \\ 8 & -11 & 0 & 4 \end{pmatrix} \mathbf{X} = 0 \tag{2.5.1.11}$$

2.5.2. Let $s<n$ and A an $s \times n$ matrix with entries in the field $\mathbb{F}$. Use Theorem 4 to show that there is a non-zero $\mathbf{x}$ in $\mathbb{F}^{n\times 1}$ such that $\mathbf{Ax} = \mathbf{0}$.

Solution: Theorem 4: *Let $\mathbb{V}$ be a vector space which is spanned by a finite set of vectors $\beta_1, \beta_2, ..., \beta_m$. Then any independent set of vectors in $\mathbb{V}$ is finite and contains no more than m elements.* Let $\mathbb{V}$ be a vector space spanned by $a_1, a_2, \ldots, a_n$, where a_i, i=1,2,...,n are columns of matrix $\mathbf{A}_{s\times n}$.

$$\mathbf{A} = \begin{pmatrix} a_1 & a_2 & \ldots & a_n \end{pmatrix} \tag{2.5.2.1}$$

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & & & \vdots \\ a_{s1} & a_{s2} & \ldots & a_{sn} \end{pmatrix} \tag{2.5.2.2}$$

Let us take a_i,i=1,2,...,n as standard $s \times 1$ bases.

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & \ldots & 0 & a_{1,s+1} & \ldots & a_{1n} \\ 0 & 1 & \ldots & 0 & a_{2,s+1} & \ldots & a_{2n} \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & 0 & \ldots & 1 & a_{s,s+1} & \ldots & a_{sn} \end{pmatrix} \tag{2.5.2.3}$$

From (2.5.2.3), it is clear that

$$dim(col(A)) \leq s \tag{2.5.2.4}$$

$$\implies rank(A) \leq s \tag{2.5.2.5}$$

Now, from rank-nullity theorem,

$$rank(A) + nullity(A) = n \tag{2.5.2.6}$$

$$nullity(A) = n - rank(A) \tag{2.5.2.7}$$

$$\implies nullity(A) > 0 \tag{2.5.2.8}$$

From equation (2.5.2.8) it is clear that there will be a non zero $\mathbf{x}$ such that $\mathbf{Ax} = \mathbf{0}$

2.5.3. Let

$$\alpha_1 = \begin{pmatrix} 1 & 1 & -2 & 1 \end{pmatrix}^T \tag{2.5.3.1}$$

$$\alpha_2 = \begin{pmatrix} 3 & 0 & 4 & -1 \end{pmatrix}^T \tag{2.5.3.2}$$

$$\alpha_3 = \begin{pmatrix} -1 & 2 & 5 & 2 \end{pmatrix}^T \tag{2.5.3.3}$$

Let

$$\alpha = \begin{pmatrix} 4 & -5 & 9 & -7 \end{pmatrix}^T \tag{2.5.3.4}$$

$$\beta = \begin{pmatrix} 3 & 1 & -4 & 4 \end{pmatrix}^T \tag{2.5.3.5}$$

$$\gamma = \begin{pmatrix} -1 & 1 & 0 & 1 \end{pmatrix}^T \tag{2.5.3.6}$$

a) Which of the vectors α, β, γ are in the subspace of $\mathbb{R}^4$ spanned by α_i?
b) Which of the vectors α, β, γ are in the subspace of $\mathbb{C}^4$ spanned by α_i?
c) Does this suggest a theorem?

Solution:

a) The linear combination of α_i for $i = 1, 2, 3$ spans subspace S. We can write,

$$c_1 \begin{pmatrix} 1 \\ 1 \\ -2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 0 \\ 4 \\ -1 \end{pmatrix} + c_3 \begin{pmatrix} -1 \\ 2 \\ 5 \\ 2 \end{pmatrix} = \text{span(S)} \tag{2.5.3.7}$$

where c_1,c_2,c_3 are scalars. Vectors in matrix form is given by

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & -1 \\ 1 & 0 & 2 \\ -2 & 4 & 5 \\ 1 & -1 & 2 \end{pmatrix} \tag{2.5.3.8}$$

We can observe that the columns of matrix $\mathbf{A}$ formed by vectors α_i are independent as the rank of matrix is 3. Hence α_i forms basis for subspace S.

i) **Checking for** α: To check if a solution exists for $\mathbf{AX} = \alpha$. The corresponding agumented matrix can be written as,

$$\begin{pmatrix} \mathbf{A} & \alpha \end{pmatrix} = \begin{pmatrix} 1 & 3 & -1 & 4 \\ 1 & 0 & 2 & -5 \\ -2 & 4 & 5 & 9 \\ 1 & -1 & 2 & -7 \end{pmatrix} \tag{2.5.3.9}$$

On performing row-reduction on (2.5.3.9),

$$\begin{pmatrix} \mathbf{A} & \alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.5.3.10}$$

As Rank($\begin{pmatrix} \mathbf{A} & \alpha \end{pmatrix}$)=Rank($\mathbf{A}$)=3, the vector α can be represented as linear combination of α_i. From equation (2.5.3.10), we can write

$$-3\begin{pmatrix}1\\1\\-2\\1\end{pmatrix}+2\begin{pmatrix}3\\0\\4\\-1\end{pmatrix}-1\begin{pmatrix}-1\\2\\5\\2\end{pmatrix}=\begin{pmatrix}4\\-5\\9\\-7\end{pmatrix} \tag{2.5.3.11}$$

Hence α is in the subspace S.

ii) **Checking for** β: To check if a solution exists for $\mathbf{AX} = \beta$. The corresponding agumented matrix can be written as,

$$\begin{pmatrix}\mathbf{A} & \beta\end{pmatrix}=\begin{pmatrix}1&3&-1&3\\1&0&2&1\\-2&4&5&-4\\1&-1&2&4\end{pmatrix} \tag{2.5.3.12}$$

On performing row-reduction on (2.5.3.12),

$$\begin{pmatrix}\mathbf{A} & \beta\end{pmatrix}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} \tag{2.5.3.13}$$

As Rank($\begin{pmatrix}\mathbf{A} & \beta\end{pmatrix}$)=4 and Rank($\mathbf{A}$)=3, Solution doesn't exist for $AX = \beta$ and hence β is not in the subspace S.

iii) **Checking for** γ: To check if a solution exists for $\mathbf{AX} = \gamma$. The corresponding agumented matrix can be written as,

$$\begin{pmatrix}\mathbf{A} & \gamma\end{pmatrix}=\begin{pmatrix}1&3&-1&-1\\1&0&2&1\\-2&4&5&0\\1&-1&2&1\end{pmatrix} \tag{2.5.3.14}$$

On performing row-reduction on (2.5.3.14),

$$\begin{pmatrix}\mathbf{A} & \gamma\end{pmatrix}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} \tag{2.5.3.15}$$

As Rank($\begin{pmatrix}\mathbf{A} & \gamma\end{pmatrix}$)=4 and Rank($\mathbf{A}$)=3, Solution doesn't exist for $AX = \gamma$ and hence γ is not in the subspace S.

b) In part 1, we haven't considered the field to be either $\mathbb{R}$ or $\mathbb{C}$. The above equations solved holds for field $\mathbb{C}$ and that implies, they hold for field $\mathbb{R}$ also. Hence α is in the subspace and β and γ are not in the subspace.

c) **Theorem suggested:** Let $\mathbb{F}_1$ and $\mathbb{F}_2$ are two fields where $\mathbb{F}_2$ is subfield of $\mathbb{F}_1$. Let α_i, i=1,2,3...,n forms basis for subspace of $\mathbb{F}_2^n$ and a vector $\alpha \in \mathbb{F}_2^n$. Then α is in the subspace of $\mathbb{F}_2^n$ spanned by α_i, i=1,2,3...,n if only if α is in the subspace of $\mathbb{F}_1^n$ spanned by α_i, i=1,2,3...,n.

2.5.4. In C^3, let $\alpha_{\mathbf{1}} = (1, 0, -i)$, $\alpha_{\mathbf{2}} = (1+i, 1-i, 1)$, $\alpha_{\mathbf{3}} = (i, i, i)$. Prove that these vectors form a basis for C^3. What are the coordinates of the vector (a,b,c) in the basis?

Solution: Now,

$$C_1\alpha_{\mathbf{1}} + C_2\alpha_{\mathbf{2}} + C_3\alpha_{\mathbf{3}} = \mathbf{0} \tag{2.5.4.1}$$

$$\implies C_1\begin{pmatrix}1\\0\\-i\end{pmatrix}+C_2\begin{pmatrix}1+i\\1-i\\1\end{pmatrix}+C_3\begin{pmatrix}i\\i\\i\end{pmatrix}=\mathbf{0} \tag{2.5.4.2}$$

So,

$$\begin{pmatrix}1&1+i&i\\0&1-i&i\\-i&1&i\end{pmatrix}\begin{pmatrix}C_1\\C_2\\C_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix} \tag{2.5.4.3}$$

Considering the co-efficient matrix A:

$$\begin{pmatrix}1&1+i&i\\0&1-i&i\\-i&1&i\end{pmatrix}\xleftrightarrow{R_3\leftarrow R_3+iR_1}\begin{pmatrix}1&1+i&i\\0&1-i&i\\0&i&i-1\end{pmatrix}$$
$$\xleftrightarrow{R_3\leftarrow R_3/i}\begin{pmatrix}1&1+i&i\\0&1-i&i\\0&1&1+i\end{pmatrix}\xleftrightarrow{R_3\leftarrow(1-i)R_3}$$
$$\begin{pmatrix}1&1+i&i\\0&1-i&i\\0&1-i&2\end{pmatrix}\xleftrightarrow{R_3\leftarrow R_3-R_2}\begin{pmatrix}1&1+i&i\\0&1-i&i\\0&0&2-i\end{pmatrix}$$
$$\xleftrightarrow{R_2\leftarrow\frac{1+i}{1-i}R_2}\begin{pmatrix}1&1+i&i\\0&1+i&-1\\0&0&2-i\end{pmatrix}\xleftrightarrow{R_1\leftarrow R_1-R_2}$$
$$\begin{pmatrix}1&0&i+1\\0&1+i&-1\\0&0&2-i\end{pmatrix} \tag{2.5.4.4}$$

Now let

$$R = \begin{pmatrix} 1 & 0 & i+1 \\ 0 & 1+i & -1 \\ 0 & 0 & 2-i \end{pmatrix} \tag{2.5.4.5}$$

Where R is the row reduced form of matrix A. So α_1,α_2 and α_3 are linearly independent which implies that these 3 vectors form a basis of vector space C^3.

Now, consider a vector $\beta = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and let the coordinates are $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ such that

$$Ax = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{2.5.4.6}$$

$$\implies x = A^{-1} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{2.5.4.7}$$

Let us consider a matrix $(A|I)$ where I is a 3x3 identity matrix. Now, applying the Gauss-Jordon theorem we can get A^{-1}

$$\begin{pmatrix} 1 & 1+i & i & 1 & 0 & 0 \\ 0 & 1-i & i & 0 & 1 & 0 \\ -i & 1 & i & 0 & 0 & 1 \end{pmatrix}$$
$$\xleftrightarrow{R_3 \leftarrow R_3 + iR_1}$$
$$\begin{pmatrix} 1 & 1+i & i & 1 & 0 & 0 \\ 0 & 1-i & i & 0 & 1 & 0 \\ 0 & i & i-1 & i & 0 & 1 \end{pmatrix}$$
$$\xleftrightarrow{R_3 \leftarrow R_3 / i}$$
$$\begin{pmatrix} 1 & 1+i & i & 1 & 0 & 0 \\ 0 & 1-i & i & 0 & 1 & 0 \\ 0 & 1 & 1+i & 1 & 0 & -i \end{pmatrix}$$
$$\xleftrightarrow{R_3 \leftarrow (1-i) R_3}$$
$$\begin{pmatrix} 1 & 1+i & i & 1 & 0 & 0 \\ 0 & 1-i & i & 0 & 1 & 0 \\ 0 & 1-i & 2 & 1-i & 0 & -i-1 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 - R_2}$$
$$\begin{pmatrix} 1 & 1+i & i & 1 & 0 & 0 \\ 0 & 1-i & i & 0 & 1 & 0 \\ 0 & 0 & 2-i & 1-i & -1 & -i-1 \end{pmatrix}$$
$$\xleftrightarrow{R_2 \leftarrow \frac{1+i}{1-i} R_2}$$
$$\begin{pmatrix} 1 & 1+i & i & 1 & 0 & 0 \\ 0 & 1+i & -1 & 0 & i & 0 \\ 0 & 0 & 2-i & 1-i & -1 & -i-1 \end{pmatrix}$$
$$\xleftrightarrow{R_1 \leftarrow R_1 - R_2}$$
$$\begin{pmatrix} 1 & 0 & i+1 & 1 & -i & 0 \\ 0 & 1+i & -1 & 0 & i & 0 \\ 0 & 0 & 2-i & 1-i & -1 & -i-1 \end{pmatrix}$$
$$\xleftrightarrow[R_1 \leftarrow R_1 + R_3]{R_3 \leftarrow \frac{R_3}{2-i}}$$
$$\begin{pmatrix} 1 & 0 & 3 & 2-i & -i-1 & -i-1 \\ 0 & 1+i & -1 & 0 & i & 0 \\ 0 & 0 & 1 & \frac{3-i}{5} & -\frac{2+i}{5} & -\frac{3i+1}{5} \end{pmatrix}$$
$$\xleftrightarrow{R_2 \leftarrow R_2 + R_3}$$
$$\begin{pmatrix} 1 & 0 & 3 & 2-i & -i-1 & -i-1 \\ 0 & 1+i & 0 & \frac{3-i}{5} & \frac{-2+4i}{5} & -\frac{3i+1}{5} \\ 0 & 0 & 1 & \frac{3-i}{5} & -\frac{2+i}{5} & -\frac{3i+1}{5} \end{pmatrix}$$
$$\xleftrightarrow{R_2 \leftarrow \frac{R_2}{1+i}}$$
$$\begin{pmatrix} 1 & 0 & 3 & 2-i & -i-1 & -i-1 \\ 0 & 1 & 0 & \frac{3-i}{5(1+i)} & \frac{-2+4i}{5(1+i)} & -\frac{3i+1}{5(1+i)} \\ 0 & 0 & 1 & \frac{3-i}{5} & -\frac{2+i}{5} & -\frac{3i+1}{5} \end{pmatrix}$$
$$\xleftrightarrow{R_1 \leftarrow R_1 - 3R_3} \tag{2.5.4.8}$$

$$\begin{pmatrix} 1 & 0 & 0 & \frac{1-4i}{5} & \frac{1-2i}{5} & \frac{-2+4i}{5} \\ 0 & 1 & 0 & \frac{1-2i}{5} & \frac{1+3i}{5} & -\frac{2+i}{5} \\ 0 & 0 & 1 & \frac{3-i}{5} & -\frac{2+i}{5} & -\frac{3i+1}{5} \end{pmatrix} \tag{2.5.4.9}$$

$$\implies (I|A^{-1}) = \begin{pmatrix} 1 & 0 & 0 & \frac{1-4i}{5} & \frac{1-2i}{5} & \frac{-2+4i}{5} \\ 0 & 1 & 0 & \frac{1-2i}{5} & \frac{1+3i}{5} & -\frac{2+i}{5} \\ 0 & 0 & 1 & \frac{3-i}{5} & -\frac{2+i}{5} & -\frac{3i+1}{5} \end{pmatrix} \tag{2.5.4.10}$$

$$\implies A^{-1} = \begin{pmatrix} \frac{1-4i}{5} & \frac{1-2i}{5} & \frac{-2+4i}{5} \\ \frac{1-2i}{5} & \frac{1+3i}{5} & -\frac{2+i}{5} \\ \frac{3-i}{5} & -\frac{2+i}{5} & -\frac{3i+1}{5} \end{pmatrix} \tag{2.5.4.11}$$

So,

$$x = A^{-1}\begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{2.5.4.12}$$

$$\implies x = \begin{pmatrix} \frac{1-4i}{5} & \frac{1-2i}{5} & \frac{-2+4i}{5} \\ \frac{1-2i}{5} & \frac{1+3i}{5} & -\frac{2+i}{5} \\ \frac{3-i}{5} & -\frac{2+i}{5} & -\frac{3i+1}{5} \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{2.5.4.13}$$

$$\implies \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1-4i}{5} & \frac{1-2i}{5} & \frac{-2+4i}{5} \\ \frac{1-2i}{5} & \frac{1+3i}{5} & -\frac{2+i}{5} \\ \frac{3-i}{5} & -\frac{2+i}{5} & -\frac{3i+1}{5} \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{2.5.4.14}$$

2.5.5. Let $\mathbb{V}$ be a vector space which is spanned by the rows of matrix

$$\mathbf{A} = \begin{pmatrix} 3 & 21 & 0 & 9 & 0 \\ 1 & 7 & -1 & -2 & -1 \\ 2 & 14 & 0 & 6 & 1 \\ 6 & 42 & -1 & 13 & 0 \end{pmatrix} \tag{2.5.5.1}$$

a. Find a basis for $\mathbb{V}$

b. Tell which vectors $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$ are elements of $\mathbb{V}$

c. If $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$ is in $\mathbb{V}$,what are its coordinates in the basis chosen in part(a)?

Solution: Row reducing (2.5.5.1)

$$\begin{pmatrix} 3 & 21 & 0 & 9 & 0 \\ 1 & 7 & -1 & -2 & -1 \\ 2 & 14 & 0 & 6 & 1 \\ 6 & 42 & -1 & 13 & 0 \end{pmatrix}$$

$$\xleftrightarrow{R_1 \leftarrow \frac{R_1}{3}} \begin{pmatrix} 1 & 7 & 0 & 3 & 0 \\ 1 & 7 & -1 & -2 & -1 \\ 2 & 14 & 0 & 6 & 1 \\ 6 & 42 & -1 & 13 & 0 \end{pmatrix}$$

$$\xleftrightarrow[R_2 \leftarrow R_2 - R_1]{R_3 \leftarrow R_3 - 2R_1} \begin{pmatrix} 1 & 7 & 0 & 3 & 0 \\ 0 & 0 & -1 & -5 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -5 & 0 \end{pmatrix}$$

$$\xleftrightarrow{R_4 \leftarrow R_4 - R_2} \begin{pmatrix} 1 & 7 & 0 & 3 & 0 \\ 0 & 0 & -1 & -5 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\xleftrightarrow{R_2 \leftarrow -R_2} \begin{pmatrix} 1 & 7 & 0 & 3 & 0 \\ 0 & 0 & 1 & 5 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\xleftrightarrow[R_2 \leftarrow R_2 - R_3]{R_4 \leftarrow R_4 - R_3} \begin{pmatrix} 1 & 7 & 0 & 3 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.5.5.2}$$

a. For the basis of $\mathbb{V}$, we can take the non zero rows of (2.5.5.2)

$$\rho_1 = \begin{pmatrix} 1 \\ 7 \\ 0 \\ 3 \\ 0 \end{pmatrix} \rho_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 5 \\ 0 \end{pmatrix} \rho_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{2.5.5.3}$$

b. Vectors which are elements of $\mathbb{V}$ are of the form:

$$c_1\rho_1 + c_2\rho_2 + c_3\rho_3 = \begin{pmatrix} c_1 \\ 7c_1 \\ c_2 \\ 3c_1 + 5c_2 \\ c_3 \end{pmatrix} \tag{2.5.5.4}$$

where c_1,c_2,c_3 are scalars.

c. Expressing (2.5.5.4) in matrix form,if $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$ is in $\mathbb{V}$,it must be of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 7 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \tag{2.5.5.5}$$

The augmented matrix form

$$\begin{pmatrix} 1 & 0 & 0 & x_1 \\ 7 & 0 & 0 & x_2 \\ 0 & 1 & 0 & x_3 \\ 3 & 5 & 0 & x_4 \\ 0 & 0 & 1 & x_5 \end{pmatrix} \tag{2.5.5.6}$$

Converting the above matrix into row reduced echelon form

$$\begin{pmatrix} 1 & 0 & 0 & x_1 \\ 7 & 0 & 0 & x_2 \\ 0 & 1 & 0 & x_3 \\ 3 & 5 & 0 & x_4 \\ 0 & 0 & 1 & x_5 \end{pmatrix} \xleftrightarrow[R_2 \leftarrow R_2 - 7R_1]{R_4 \leftarrow R_4 - 3R_1} \begin{pmatrix} 1 & 0 & 0 & x_1 \\ 0 & 0 & 0 & x_2 - 7x_1 \\ 0 & 1 & 0 & x_3 \\ 0 & 5 & 0 & x_4 - 3x_1 \\ 0 & 0 & 1 & x_5 \end{pmatrix}$$
$$\xleftrightarrow{R_2 \leftarrow R_3} \begin{pmatrix} 1 & 0 & 0 & x_1 \\ 0 & 1 & 0 & x_3 \\ 0 & 0 & 0 & x_2 - 7x_1 \\ 0 & 5 & 0 & x_4 - 3x_1 \\ 0 & 0 & 1 & x_5 \end{pmatrix}$$
$$\xleftrightarrow{R_4 \leftarrow R_4 - 5R_2} \begin{pmatrix} 1 & 0 & 0 & x_1 \\ 0 & 1 & 0 & x_3 \\ 0 & 0 & 0 & x_2 - 7x_1 \\ 0 & 0 & 0 & x_4 - 3x_1 - 5x_3 \\ 0 & 0 & 1 & x_5 \end{pmatrix}$$
$$\xleftrightarrow{R_5 \leftarrow R_3} \begin{pmatrix} 1 & 0 & 0 & x_1 \\ 0 & 1 & 0 & x_3 \\ 0 & 0 & 1 & x_5 \\ 0 & 0 & 0 & x_4 - 3x_1 - 5x_3 \\ 0 & 0 & 0 & x_2 - 7x_1 \end{pmatrix} \tag{2.5.5.7}$$

From (2.5.5.7),the coordinates of $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$ in the basis is

$$\begin{pmatrix} x_1 \\ x_3 \\ x_5 \end{pmatrix} \tag{2.5.5.8}$$

3 Linear Transformations

3.1 Linear Transformations

3.1.1. Find weather given functions **T** from $\mathbb{R}^2$ into $\mathbb{R}^2$ are linear transformations or not

a)

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 + x_1 \\ x_2 \end{pmatrix} \tag{3.1.1.1}$$

Solution: Counter example can be given as follows:-

$$x_1 = x_2 = 0 \tag{3.1.1.2}$$

Substituting (3.1.1.2) in (3.1.1.1) we get,

$$T\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.1.1.3}$$

(3.1.1.3) is clearly false because linear transformation on $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ will always be equal to $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

b)

$$\mathbf{T}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix} \tag{3.1.1.4}$$

Does function **T** from $\mathbb{R}^2$ into $\mathbb{R}^2$ is Linear Transformation.
Solution: Let,

$$\mathbf{x}, \mathbf{y} \in \mathbb{R}^2 \tag{3.1.1.5}$$

Using transformation on **T**,

$$\mathbf{T}(\mathbf{x}) = \mathbf{A}\mathbf{x} \tag{3.1.1.6}$$

From (3.1.1.4) we get,

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.1.1.7}$$

With c being a scalar,

$$\mathbf{T}(c\mathbf{x}+\mathbf{y}) = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}(c\mathbf{x}+\mathbf{y}) \tag{3.1.1.8}$$

$$= \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}c\mathbf{x} + \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\mathbf{y} \tag{3.1.1.9}$$

$$= \begin{pmatrix}0 & c\\c & 0\end{pmatrix}\mathbf{x} + \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\mathbf{y} \tag{3.1.1.10}$$

$$= c\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\mathbf{x} + \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\mathbf{y} \tag{3.1.1.11}$$

From (3.1.1.11) we can say,

$$\mathbf{T}(c\mathbf{x}+\mathbf{y}) = c\mathbf{T}(\mathbf{x}) + \mathbf{T}(\mathbf{y}) \tag{3.1.1.12}$$

Hence from (3.1.1.12) we can say $\mathbf{T}$ is a Linear Transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$

c)

$$\mathbf{T}(x_1, x_2) = ({x_1}^2, x_2) \tag{3.1.1.13}$$

Solution: If $\mathbf{T}$ were a linear transformation then we would have

$$\mathbf{T}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} \tag{3.1.1.14}$$

$$\implies \mathbf{T}\left(-1\begin{pmatrix}1\\0\end{pmatrix}\right) = -1.\mathbf{T}\begin{pmatrix}1\\0\end{pmatrix} \tag{3.1.1.15}$$

$$\implies -1.\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}-1\\0\end{pmatrix} \tag{3.1.1.16}$$

which is a contradiction, since

$$\mathbf{T}\begin{pmatrix}-1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} \tag{3.1.1.17}$$

$$\begin{pmatrix}1\\0\end{pmatrix} \neq \begin{pmatrix}-1\\0\end{pmatrix}. \tag{3.1.1.18}$$

Hence non-linear transformation.

d) Is the following function $\mathbf{T}$ from $\mathbb{R}^2$ into $\mathbb{R}^2$ is linear transformation?

$$\mathbf{T}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}\sin(x_1)\\x_2\end{pmatrix}$$

Solution: Let,

$$\mathbf{x} = \begin{pmatrix}\pi\\0\end{pmatrix}; \quad \mathbf{y} = \begin{pmatrix}\frac{\pi}{2}\\0\end{pmatrix}$$

$$\mathbf{T}(\mathbf{x}+\mathbf{y}) = \mathbf{T}\begin{pmatrix}\frac{3\pi}{2}\\0\end{pmatrix} = \begin{pmatrix}-1\\0\end{pmatrix} \tag{3.1.1.19}$$

$$\mathbf{T}(\mathbf{x}) + \mathbf{T}(\mathbf{y}) = \mathbf{T}\begin{pmatrix}\pi\\0\end{pmatrix} + \mathbf{T}\begin{pmatrix}\frac{\pi}{2}\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} \tag{3.1.1.20}$$

From (3.1.1.19) and (3.1.1.20) additive transformation property is not satisfied. Hence not a linear transformation.

e) Verify whether $T(x_1, x_2) = (x_1 - x_2, 0)$ is a linear transformation or not. **Solution:** Let $\mathbf{V}$ and $\mathbf{W}$ be the vector spaces. The function $T : \mathbf{V} \to \mathbf{W}$ is called a linear transformation of $\mathbf{V}$ into $\mathbf{W}$ if for all $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{V}$ and for any scalar k in field $\mathbf{F}$,

$$T(k\mathbf{u}+\mathbf{v}) = kT(\mathbf{u}) + T(\mathbf{v}) \tag{3.1.1.21}$$

Given,

$$T\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}1 & -1\\0 & 0\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} \tag{3.1.1.22}$$

Consider,

$$T(k\mathbf{x}+\mathbf{y}) = \begin{pmatrix}1 & -1\\0 & 0\end{pmatrix}(k\mathbf{x}+\mathbf{y}) \tag{3.1.1.23}$$

$$= \begin{pmatrix}k & -k\\0 & 0\end{pmatrix}\mathbf{x} + \begin{pmatrix}1 & -1\\0 & 0\end{pmatrix}\mathbf{y} \tag{3.1.1.24}$$

$$= k\begin{pmatrix}1 & -1\\0 & 0\end{pmatrix}\mathbf{x} + \begin{pmatrix}1 & -1\\0 & 0\end{pmatrix}\mathbf{y} \tag{3.1.1.25}$$

$$T(\mathbf{x}+\mathbf{y}) = kT(\mathbf{x}) + T(\mathbf{y}) \tag{3.1.1.26}$$

Therefore, the given function T is a linear transformation.

3.1.2. Find the range, rank, null space, and nullity for the zero transformation and the identity transformation on a finite-dimensional space V.
Solution:
Suppose vector space $\mathbf{V}$ has dim($\mathbf{V}$) = n. Table 3.1.2 provides the properties of range, rank, null space and nullity of zero and identity transformation on a vector space $\mathbf{V}$

3.1.3. *a)* Let $\mathbf{F}$ be a field and let $\mathbf{V}$ be the space of polynomial functions f from $\mathbf{F}$ into $\mathbf{F}$, given by

$$f(x) = c_0 + c_1 x + \cdots + c_n x^n$$

Let $\mathbf{D}$ be a linear differentiation transforma-

Properties	Zero Transformation	Identity Transformation
Transformation	$T_0(\mathbf{v}) = \mathbf{0}$	$T_I(\mathbf{v}) = \mathbf{v}$
Range	Zero subspace {0}	**V**
Rank	dim(**0**) = 0	dim(**V**) = n
Null space	**V**	Zero subspace {0}
Nullity	dim(**V**) = n	dim(**0**) = 0

TABLE 3.1.2: Properties of Zero and Identity transformation

tion defined as

$$(\mathbf{D}f)(x) = \frac{df(x)}{dx}$$

Then find the range and null space of **D**.

b) Let **R** be the field of real numbers and let **V** be the space of all functions from **R** into **R** which are continous. Let **T** be linear transformation defined by

$$(\mathbf{T}f)(x) = \int_0^x f(t)\,dt$$

Find the range and null space of **T**.

Solution: Let the vector space of n-dimension be deined as

$$\mathbf{V} = \left\{ f : \mathbf{F} \to \mathbf{F} : f(x) = \sum_{k=0}^{n} c_k x^k,\ c_k \in \mathbf{F} \right\} \tag{3.1.3.1}$$

The corresponding standard basis for **V** is

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x \\ \vdots \\ 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ x^{n-1} \end{pmatrix} \right\} \tag{3.1.3.2}$$

a) Let f and $g \in \mathbf{V}$ and let α and $\beta \in \mathbf{F}$ then

$$\mathbf{D}(\alpha f + \beta g) = \frac{d(\alpha f(x) + \beta g(x))}{dx} \tag{3.1.3.3}$$

$$= \alpha \frac{df(x)}{dx} + \beta \frac{dg(x)}{dx} \tag{3.1.3.4}$$

$$= \alpha(\mathbf{D}f) + \beta(\mathbf{D}g) \tag{3.1.3.5}$$

Therefore **D** is a linear transformation.

The **D** transformation maps the k^{th} basis vector as follows

$$\mathbf{D} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ x^k \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ kx^{k-1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{3.1.3.6}$$

Since the transformed vector

$$\begin{pmatrix} 0 \\ \vdots \\ kx^{k-1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbf{V} \tag{3.1.3.7}$$

the range of **D** is the vector space **V**. Thus the transformation is defined as $\mathbf{D} : \mathbf{V} \to \mathbf{V}$. Therefore the **D** Transformation on the basis vector set is

$$\mathbf{D} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \tag{3.1.3.8}$$

$$= \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & n-2 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \tag{3.1.3.9}$$

Thus the **D** transformation coefficient matrix is

$$D = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & n-2 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \tag{3.1.3.10}$$

Since D contains a zero row hence $\left|D\right| = 0$. Therefore **D** transformation matrix is singular. The nullspace for differentiation transformation is defined as

$$\mathbf{N} = \{f \in \mathbf{V} : \mathbf{D}f = 0\} \tag{3.1.3.11}$$

Let the coefficient matrix of $f \in \mathbf{V}$ be

$$\mathbf{f} = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{pmatrix} \tag{3.1.3.12}$$

then

$$\mathbf{D}f = 0 \tag{3.1.3.13}$$

$$\implies \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & n-2 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{pmatrix} = \mathbf{0} \tag{3.1.3.14}$$

Since D is in row reduced echolon form and $rank(D) = n-1$ the solution of (3.1.3.14) is

$$\mathbf{f} = \begin{pmatrix} k \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{3.1.3.15}$$

where $k \in \mathbf{R}$. Therefore the nullspace for $\mathbf{D} : \mathbf{V} \to \mathbf{V}$ is

$$\mathbf{N} = \left\{ \begin{pmatrix} k \\ 0 \\ \vdots \\ 0 \end{pmatrix} : k \in \mathbf{R} \right\} \tag{3.1.3.16}$$

b) Let f and $g \in \mathbf{V}$ and let α and $\beta \in \mathbf{F}$ then

$$\mathbf{T}(\alpha f + \beta g) = \int_0^x (\alpha f(t) + \beta g(t))\, dt \tag{3.1.3.17}$$

$$= \alpha \int_0^x f(t)\, dt + \beta \int_0^x g(t)\, dt \tag{3.1.3.18}$$

$$= \alpha(\mathbf{T}f) + \beta(\mathbf{T}g) \tag{3.1.3.19}$$

Therefore **T** is a linear transformation.
The **T** transformation maps the k^{th} basis vector as follows

$$\mathbf{T} \begin{pmatrix} 0 \\ \vdots \\ x^k \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{x^{k+1}}{k+1} \\ \vdots \\ 0 \end{pmatrix} \tag{3.1.3.20}$$

Since the transformed vector

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{x^{k+1}}{k+1} \\ \vdots \\ 0 \end{pmatrix} \in \mathbf{V} \tag{3.1.3.21}$$

the range of **T** is the vector space **V**. Thus the transformation is defined as $\mathbf{T} : \mathbf{V} \to \mathbf{V}$. Therefore the **T** Transformation on the basis vector set is

$$\mathbf{T} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \tag{3.1.3.22}$$

$$= \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & \frac{1}{n-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{n} \end{pmatrix} \tag{3.1.3.23}$$

Thus the **T** transformation coefficient matrix is

$$T = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & \frac{1}{n-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{n} \end{pmatrix} \tag{3.1.3.24}$$

Since T contains a zero row hence $\left|T\right| = 0$. Therefore $\mathbf{T}$ transformation matrix is singular. The nullspace for integration transformation is defined as

$$\mathbf{N} = \{f \in \mathbf{V} : \mathbf{T}f = 0\} \tag{3.1.3.25}$$

Let the coefficient matrix of $f \in \mathbf{V}$ be

$$\mathbf{f} = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{pmatrix} \tag{3.1.3.26}$$

then

$$\mathbf{T}f = 0 \tag{3.1.3.27}$$

$$\implies \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & \frac{1}{n-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{n} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{pmatrix} = \mathbf{0} \tag{3.1.3.28}$$

Since T is in row reduced echolon form and $rank(T) = n$ the solution of (3.1.3.28) is

$$\mathbf{f} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{3.1.3.29}$$

where $k \in \mathbf{R}$. Therefore the nullspace for $\mathbf{T} : \mathbf{V} \to \mathbf{V}$ is

$$\mathbf{N} = \left\{ \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} : k \in \mathbf{R} \right\} \tag{3.1.3.30}$$

3.1.4. Describe explicitly the linear transformation T from $\mathbb{F}^2$ into $\mathbb{F}^2$ such that $T(\epsilon_1) = (a, b), T(\epsilon_2) = (c, d)$. **Solution:** We are given a linear transformation,

$$T : \mathbb{F}^2 \to \mathbb{F}^2 \tag{3.1.4.1}$$

The transformation for $\in_1$ and $\in_2$ can be written as,

$$T(\in_1) = \begin{pmatrix} a \\ b \end{pmatrix} \tag{3.1.4.2}$$

$$T(\in_2) = \begin{pmatrix} c \\ d \end{pmatrix} \tag{3.1.4.3}$$

Now,let's assume $\in_1$ and $\in_2$ as linearly independent. So the linear transformation T for any vector $\mathbf{v}$ in two dimensional space will be,

$$T(\mathbf{v}) = \begin{pmatrix} T(\in_\mathbf{1}) & T(\in_\mathbf{2}) \end{pmatrix} \mathbf{v} \tag{3.1.4.4}$$

$$= \begin{pmatrix} a & c \\ b & d \end{pmatrix} \mathbf{v} \tag{3.1.4.5}$$

Now, there can be two cases here, transformation of linearly independent vector can be independent or it can be dependent.Considering the first case and (3.1.4.5) we can say that,

$$Range(T) = \text{columnspace of} \begin{pmatrix} a & c \\ b & d \end{pmatrix} \tag{3.1.4.6}$$

Now, considering the case when linear transformation will be linearly dependent,

$$Range(T) = \text{columnspace of} \begin{pmatrix} a \\ b \end{pmatrix} \tag{3.1.4.7}$$

Now, considering that vectors $\in_1$ and $\in_2$ itself are linearly dependent.Let $\mathbf{v}{=}\in_1 + \in_2$

$$T(\mathbf{v}) = T(\in_1) + T(\in_2) \tag{3.1.4.8}$$

$$= T(\in_1) + T(k \in_1) \tag{3.1.4.9}$$

$$= (k + 1)T(\in_1) \tag{3.1.4.10}$$

$$= (k + 1) \begin{pmatrix} a \\ b \end{pmatrix} \tag{3.1.4.11}$$

We can see from above equation that when $\in_1$ and $\in_2$ as linearly dependent then the transformation T will be along the line only.

3.1.5. Let $\mathbb{F}$ be a subfield of the complex numbers and let $\mathbb{T}$ be the function from $\mathbb{F}^3$ into $\mathbb{F}^3$ defined

Vectors Independent	Vectors Dependent
$T(\mathbf{v})=\begin{pmatrix} a & c \\ b & d \end{pmatrix}\mathbf{v}$	$T(\mathbf{v})=(\mathrm{k}+1)\begin{pmatrix} a \\ b \end{pmatrix}$
Output: On the plane	Output: On the line

TABLE 3.1.4

by

$$\mathbb{T}(x_1, x_2, x_3) = \tag{3.1.5.1}$$

$$(x_1 - x_2 + 2x_3, 2x_1 + x_2, -x_1 - 2x_2 + 2x_3) \tag{3.1.5.2}$$

a) Verify that $\mathbb{T}$ is a linear transformation.

b) If (a, b, c) is a vector in $\mathbb{F}^3$, what are the conditions on a, b, c that the vector be in the range of $\mathbb{T}$?What is the rank of $\mathbb{T}$?

c) What are the conditions on a, b, c that (a, b, c) be in the null space of $\mathbb{T}$? What is the nullity of $\mathbb{T}$?

Solution: Representing the transformation in matrix form

$$\mathbb{T}(x_1, x_2, x_3) = \mathbf{Tx} \tag{3.1.5.3}$$

$$\mathbf{T} = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ -1 & -2 & 2 \end{pmatrix} \tag{3.1.5.4}$$

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{3.1.5.5}$$

Part (a) Consider the matrices $\mathbf{x}, \mathbf{y} \in \mathbb{F}^3$ and the scalar $c \in \mathbb{F}$
By the associativity of matrix multiplications, we can write

$$\mathbf{T}(c\mathbf{x} + \mathbf{y}) = \mathbf{T}(c\mathbf{x}) + \mathbf{Ty} \tag{3.1.5.6}$$

$$= c\mathbf{Tx} + \mathbf{Ty} \tag{3.1.5.7}$$

So, $\mathbf{T}$ is a linear transformation. Part (b)

$$range(\mathbf{T}) = \left\{\mathbf{y} : \mathbf{Tx} = \mathbf{y} \text{ where } \mathbf{x}, \mathbf{y} \in \mathbb{F}^3\right\} \tag{3.1.5.8}$$

$$\mathbf{y} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{3.1.5.9}$$

$$\mathbf{Tx} = \mathbf{y} \tag{3.1.5.10}$$

$$\implies \mathbf{BTx} = \mathbf{By} \tag{3.1.5.11}$$

$$\implies \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{-2}{3} & \frac{1}{3} & 0 \\ -1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ -1 & -2 & 2 \end{pmatrix}\mathbf{x} = \tag{3.1.5.12}$$

$$\begin{pmatrix} \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{-2}{3} & \frac{1}{3} & 0 \\ -1 & 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{3.1.5.13}$$

$$\begin{pmatrix} 1 & 0 & \frac{2}{3} \\ 0 & 1 & \frac{-4}{3} \\ 0 & 0 & 0 \end{pmatrix}\mathbf{x} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{-2}{3} & \frac{1}{3} & 0 \\ -1 & 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{3.1.5.14}$$

So, rank($\mathbf{T}$)=2 and comparing the third row element in LHS and RHS of (3.1.5.14)

$$-a + b + c = 0 \tag{3.1.5.15}$$

All vectors $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{F}^3$ that satisfy (3.1.5.15) lie in the range of $\mathbf{T}$ Part (c)

$$nullspace(\mathbf{T}) = \left\{\mathbf{x} : \mathbf{Tx} = \mathbf{0} \text{ where } \mathbf{x} \in \mathbb{F}^3\right\} \tag{3.1.5.16}$$

$$\mathbf{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{3.1.5.17}$$

$$\mathbf{Tx} = \mathbf{0} \tag{3.1.5.18}$$

$$\mathbf{BTx} = \mathbf{0} \tag{3.1.5.19}$$

where $\mathbf{BT}$ is in reduced row echelon form

$$\begin{pmatrix} \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{-2}{3} & \frac{1}{3} & 0 \\ -1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ -1 & -2 & 2 \end{pmatrix}\mathbf{x} = \mathbf{0} \tag{3.1.5.20}$$

$$\implies \begin{pmatrix} 1 & 0 & \frac{2}{3} \\ 0 & 1 & \frac{-4}{3} \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{3.1.5.21}$$

$$\implies a + \frac{2}{3}c = 0 \tag{3.1.5.22}$$

$$b - \frac{4}{3}c = 0 \tag{3.1.5.23}$$

The number of free variables in the reduced row echelon form of $\mathbf{T}$ is 1 hence nullity($\mathbf{T}$) =1
So, the null space of $\mathbf{T}$ is set of all vectors

$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{F}^3$ that satisfy (3.1.5.22) and (3.1.5.23)

Note

rank($\mathbf{T}$)+nullity($\mathbf{T}$)=2+1=dim($\mathbb{F}^3$)

3.1.6. Describe explicitly a linear transformation from R^3 into R^3 which has as its range the subspace spanned by $\begin{pmatrix} 1 & 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 & 2 \end{pmatrix}$.

Solution: Transformation T from R^3 to R^3 range gives the column space. Hence,

$$T(\mathbf{x}) = \mathbf{A}\mathbf{x} \tag{3.1.6.1}$$

$$T(\mathbf{x}) = \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ -1 & 2 \end{pmatrix} \mathbf{x} \tag{3.1.6.2}$$

3.1.7. Let $\mathbf{V}$ be the vector space of all $n \times n$ matrices over the field $\mathbb{F}$, and let $\mathbf{B}$ be a fixed $n \times n$ matrix. If a transformation T defined as follows,

$$T(\mathbf{A}) = \mathbf{AB} - \mathbf{BA}$$

Prove that T is a linear transformation from $\mathbf{V}$ into $\mathbf{V}$ **Solution:** Let,

$$\mathbf{A_1} \in \mathbf{V} \tag{3.1.7.1}$$

$$\mathbf{A_2} \in \mathbf{V} \tag{3.1.7.2}$$

If c be any scalar of the field $\mathbb{F}$ we get,

$$c\mathbf{A_1} + \mathbf{A_2} \in \mathbf{V} \tag{3.1.7.3}$$

Applying transformation T on $(c\mathbf{A_1} + \mathbf{A_2})$ we get,

$$T(c\mathbf{A_1} + \mathbf{A_2}) = (c\mathbf{A_1} + \mathbf{A_2})\mathbf{B} - \mathbf{B}(c\mathbf{A_1} + \mathbf{A_2}) \tag{3.1.7.4}$$

$$= c\mathbf{A_1B} + \mathbf{A_2B} - c\mathbf{BA_1} - \mathbf{BA_2} \tag{3.1.7.5}$$

$$= c(\mathbf{A_1B} - \mathbf{BA_1}) + (\mathbf{A_2B} - \mathbf{BA_2}) \tag{3.1.7.6}$$

$$= cT(\mathbf{A_1}) + T(\mathbf{A_2}) \tag{3.1.7.7}$$

From (3.1.7.7) we conclude that T is a linear transformation from vector space $\mathbf{V}$ to $\mathbf{V}$.

3.1.8. Let V be the set of all complex numbers regarded as a vector space over the field of real numbers(usual operations). Find a function from V into V which is a linear transformation on the above vector space, but which is not a linear transformation on $\mathbb{C}$ i.e., which is not complex linear.

Solution: Let

$$T : V \to V \tag{3.1.8.1}$$

be a function such that,

$$T(x + iy) = Re(x + iy) = x \tag{3.1.8.2}$$

$$\implies T : x + iy \to x \tag{3.1.8.3}$$

where $x, y \in \mathbb{R}$.
Let, $\alpha = a + ib$, $\beta = c + id$.

$$\implies T(k\alpha + \beta) = T(ka + ikb + c + id) \tag{3.1.8.4}$$

$$= T(ka + c + i(kb + d)) \tag{3.1.8.5}$$

$$= ka + c \tag{3.1.8.6}$$

$$= kT(\alpha) + T(\beta) \tag{3.1.8.7}$$

Now, let $z \in V$ such that,

$$z = i \tag{3.1.8.8}$$

$$\implies T(z) = T(i) = 0 \tag{3.1.8.9}$$

We can also write,

$$T(i) = T(i(1)) = iT(1) = i \neq 0 \tag{3.1.8.10}$$

Thus from (3.1.8.7), T is real linear transformation and from (3.1.8.10), T is not complex linear.

3.1.9. Let $\mathbf{V}$ be the space of $n \times 1$ matrices over F and let $\mathbf{W}$ be the space of $m \times 1$ matrices over F. Let $\mathbf{A}$ be a fixed $m \times n$ matrix over F and let T be the linear transformation from $\mathbf{V}$ into $\mathbf{W}$ defined by $T(\mathbf{X}) = \mathbf{AX}$. Prove that T is the zero transformation if and only if $\mathbf{A}$ is the zero matrix. **Solution:** If $\mathbf{A}_{m\times n}$ is a zero transformation and $\mathbf{X}_{n\times 1}$ is a vector, then

$$\mathbf{AX} = \mathbf{0}_{m\times 1} \tag{3.1.9.1}$$

Let,

$$\mathbf{A} = \begin{pmatrix}\mathbf{A_1} & \dots & \mathbf{A_j} & \dots & \mathbf{A_n}\end{pmatrix}_{1\times n} \text{ and}$$

$$(3.1.9.2)$$

$$\mathbf{X_j} = \begin{pmatrix} x_1 \\ \vdots \\ x_j \\ \vdots \\ x_n \end{pmatrix}, \text{ where } x_i = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \tag{3.1.9.3}$$

If $\mathbf{A}_{m\times n}$ is zero transformation, then for any vector $\mathbf{X}_{n\times 1}$, $\mathbf{AX} = \mathbf{0}$.
Consider ,

$$\mathbf{AX_j} = \mathbf{0}_{m\times 1} \tag{3.1.9.4}$$

$$\begin{pmatrix}\mathbf{A_1}\dots\mathbf{A_j}\dots\mathbf{A_n}\end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_j \\ \vdots \\ x_n \end{pmatrix} = \mathbf{0}_{m\times 1} \tag{3.1.9.5}$$

From (3.1.9.3) and (3.1.9.5)

$$\mathbf{A_j} = \mathbf{0}_{m\times 1} \text{ for } j = 1, 2, ...n \tag{3.1.9.6}$$

Substitute (3.1.9.6) in (3.1.9.2)

$$\mathbf{A} = \begin{pmatrix}\mathbf{0}_{m\times 1} & \mathbf{0}_{m\times 1} & \dots & \mathbf{0}_{m\times 1}\end{pmatrix}_{1\times n} \tag{3.1.9.7}$$

$$\therefore \mathbf{A} = \mathbf{0}_{m\times n} \tag{3.1.9.8}$$

Hence $\mathbf{A}$ is zero matrix.

Let us assume $\mathbf{A}_{m\times n}$ is a zero matrix

$$\mathbf{A} = \mathbf{0}_{m\times n} \tag{3.1.9.9}$$

Then,

$$T(\mathbf{X}) = \mathbf{AX} \tag{3.1.9.10}$$

$$= \mathbf{0}.\mathbf{X} \tag{3.1.9.11}$$

$$= \mathbf{0}_{m\times 1}\ ,\ \forall \mathbf{X} \in F \tag{3.1.9.12}$$

Hence $T(\mathbf{X}) = \mathbf{AX}$ is the zero transformation.

From (3.1.9.8) and (3.1.9.12) it is proved that T is the zero transformation if and only if $\mathbf{A}$ is the zero matrix.

3.1.10. Let $\mathbf{V}$ be an n-dimensional vector space over the field $\mathbf{F}$ and let $\mathbf{T}$ be a linear transformation from $\mathbf{V}$ into $\mathbf{V}$ such that the range and null space of $\mathbf{T}$ are identical. Prove that n is even. (Can you give an example of such a linear transformation $\mathbf{T}$)? **Solution:** Let $\mathbf{V}$ and $\mathbf{W}$ be vector spaces over the field $\mathbf{F}$ and let $\mathbf{T}$ be a linear transformation from $\mathbf{V}$ into $\mathbf{W}$. Then,

$$\text{rank}(\mathbf{T}) + \text{nullity}(\mathbf{T}) = \dim \mathbf{V} \tag{3.1.10.1}$$

It is given that range and null space of $\mathbf{T}$ are same, let us assume it to be m. Substituting in equation (3.1.10.1)

$$m + m = n \tag{3.1.10.2}$$

$$\implies n = 2m \tag{3.1.10.3}$$

From equation (3.1.10.3), we can say that n is even.

Example: Let us consider a vector space $\mathbf{V}$, such that $\mathbf{V} \in \mathbb{R}^2$ and let us consider a linear transformation $\mathbf{T} : \mathbf{V} \to \mathbf{V}$ defined by $\mathbf{T}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ 0 \end{pmatrix}$ and is given by matrix $\mathbf{M}$

$$\mathbf{T}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ 0 \end{pmatrix} \tag{3.1.10.4}$$

$$\mathbf{M} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{3.1.10.5}$$

Let us consider basis of $\mathbb{R}^2$ $\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$ and apply linear transformation on it.

$$\mathbf{T}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.1.10.6}$$

$$\mathbf{T}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.1.10.7}$$

From (3.1.10.5),
The range of matrix can be found from row reduced echelon form. But as matrix $\mathbf{M}$ is in RREF form,
the basis for range is given by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

The null space of matrix is,

$$\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \tag{3.1.10.8}$$

$$\implies x_1 = t \quad x_2 = 0 \tag{3.1.10.9}$$

$$\implies \mathbf{X} = \begin{pmatrix}t\\0\end{pmatrix} = t\begin{pmatrix}1\\0\end{pmatrix} \tag{3.1.10.10}$$

The basis for null space is $\begin{pmatrix}1\\0\end{pmatrix}$.

$$\text{rank}(\mathbf{T}) = 1 \quad \text{nullity}(\mathbf{T}) = 1 \tag{3.1.10.11}$$

$$\dim(\mathbf{V}) = 2 \tag{3.1.10.12}$$

Thus the range and null space are equal, and n is even.

3.2 The Algebra of Linear Transformations

3.2.1. Let $\mathbf{T}$ and $\mathbf{U}$ be the linear operators on $\mathbb{R}^2$ defined by $\mathbf{T}(x_1, x_2) = (x_2, x_1)$ and $\mathbf{U}(x_1, x_2) = (x_1, 0)$.

a) Let$\mathbf{T}$ and $\mathbf{U}$ be the linear operators on $\mathbb{R}^2$ defined by

$$\mathbf{T}(x_1, x_2) = (x_2, x_1) \tag{3.2.1.1}$$

and

$$\mathbf{U}(x_1, x_2) = (x_1, 0) \tag{3.2.1.2}$$

How would you describe T and U geometrically ?

Solution: Geometrically, in the x-y plane, $\mathbf{T}$ is the reflection about the diagonal x = y and $\mathbf{U}$ is a projection onto the x-axis.

i) Reflection

Let Consider Matrix A as

$$\mathbf{A} = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} \tag{3.2.1.3}$$

The matrix $\mathbf{A}$ is representation of the linear transformation T across the line y=x with respect to the standard basis.

Let suppose

$$\mathbf{x_1} = \begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}, \mathbf{x_2} = \begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}3\\4\end{pmatrix} \tag{3.2.1.4}$$

After applying linear operator $\mathbf{T}$ on it,

$$\mathbf{T}(x_1, x_2) = \mathbf{A}\begin{pmatrix}x_1\\x_2\end{pmatrix} \tag{3.2.1.5}$$

$$\implies \mathbf{A}\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix} \tag{3.2.1.6}$$

Similarly

$$\mathbf{A}\begin{pmatrix}3\\4\end{pmatrix} = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}3\\4\end{pmatrix} = \begin{pmatrix}4\\3\end{pmatrix} \tag{3.2.1.7}$$

Hence after applying Operator $\mathbf{T}$ on $\mathbf{x_1}$ and $\mathbf{x_2}$

$$\mathbf{x_1} = \begin{pmatrix}2\\1\end{pmatrix}, \mathbf{x_2} = \begin{pmatrix}4\\3\end{pmatrix} \tag{3.2.1.8}$$

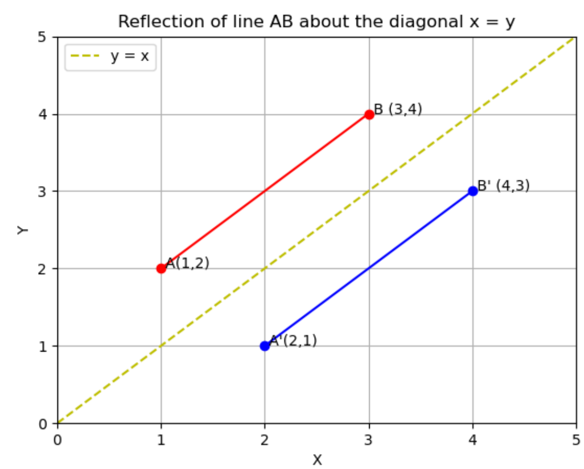

Fig. 3.2.1.1: Reflection of line AB about the x = y

ii) Projection

For projection let Consider Matrix B as

$$\mathbf{B} = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} \tag{3.2.1.9}$$

The matrix $\mathbf{B}$ is representation of the linear transformation $\mathbf{U}$ that is projection on x-axis.

Let suppose

$$\mathbf{x_1} = \begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}, \mathbf{x_2} = \begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}3\\4\end{pmatrix} \tag{3.2.1.10}$$

After applying linear operator $\mathbf{U}$ on $\mathbf{x_1}$ and

x_2,

$$\mathbf{T}(x_1, x_2) = \mathbf{U}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad (3.2.1.11)$$

$$\implies \mathbf{B}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.2.1.12)$$

Similarly

$$\mathbf{A}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \quad (3.2.1.13)$$

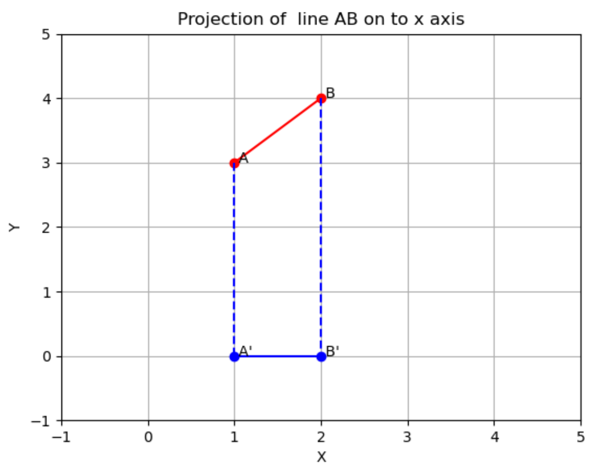

Fig. 3.2.1.2: Projection of AB onto x-axis

Hence after applying Operator **U** on $\mathbf{x_1}$ and $\mathbf{x_2}$

$$\mathbf{x_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{x_2} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \quad (3.2.1.14)$$

b) Give rules like the ones defining **T** and **U** for each of the transformations **U** + **T**, **UT**, **TU**, $\mathbf{T}^2$, $\mathbf{U}^2$. $\mathbb{R}^2$ into $\mathbb{R}^2$ is linear transformation?
Solution: Let **T** and **U** defined by matrices **A** and **B** such that ,

$$\mathbf{T}(\mathbf{x}) = \mathbf{A}\mathbf{x}; \quad \mathbf{U}(\mathbf{x}) = \mathbf{B}\mathbf{x} \quad (3.2.1.15)$$

Where,

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (3.2.1.16)$$

Table 3.2.1.1 lists the summary of each Transformations.

Transformations	**Matrix**	**Vector**
U + **T**	$(\mathbf{B} + \mathbf{A}) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} x_1 + x_2 \\ x_1 \end{pmatrix}$
UT	$\mathbf{BA} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} x_2 \\ 0 \end{pmatrix}$
TU	$\mathbf{AB} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ x_1 \end{pmatrix}$
$\mathbf{T}^2$	$\mathbf{A}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$
$\mathbf{U}^2$	$\mathbf{B}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} x_1 \\ 0 \end{pmatrix}$

TABLE 3.2.1.1: Summary

3.2.2. Let T be the unique linear operator on C^3 for which

$$T(\epsilon_1) = \begin{pmatrix} 1 & 0 & i \end{pmatrix}, T(\epsilon_2) = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix}, \quad T(\epsilon_3) = \begin{pmatrix} i & 1 & 0 \end{pmatrix} \quad (3.2.2.1)$$

Is T invertible ?
Solution: Let ϵ_i is basis for C^3 such that $T(\epsilon_i)$ is basis for C^3 T is said to be singular if

$$T(\epsilon) = 0 \implies \epsilon \neq 0 \quad (3.2.2.2)$$

now,

$$\begin{pmatrix} 1 & 0 & i \\ 0 & 1 & 1 \\ i & 1 & 0 \end{pmatrix}\epsilon = 0 \quad (3.2.2.3)$$

consider the row reduced matrix

$$\begin{pmatrix} 1 & 0 & i \\ 0 & 1 & 1 \\ i & 1 & 0 \end{pmatrix} \xleftrightarrow[R_3 \to R_3 - R_2]{R_3 \to R_3 - iR_1} \begin{pmatrix} 1 & 0 & i \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad (3.2.2.4)$$

$$\epsilon = c\begin{pmatrix} -i \\ -1 \\ 1 \end{pmatrix} \quad (3.2.2.5)$$

Hence it holds the condition of singularity therefore T is not invertible .

3.2.3. For the linear operator **T**

$$\mathbf{T}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3x_1 \\ x_1 - x_2 \\ 2x_1 + x_2 + x_3 \end{pmatrix} \quad (3.2.3.1)$$

3.2.4. Let $\mathbf{T}$ be a linear operator on $\mathbf{R}^3$ defined by

$$\mathbf{T}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}3x_1\\x_1-x_2\\2x_1+x_2+x_3\end{pmatrix}$$

Is $\mathbf{T}$ invertible? If so, find a rule for $\mathbf{T}^{-1}$ like the one which defines T.

Solution: The transformed vector can be re-written by expanding the columns as follows

$$\begin{pmatrix}3x_1\\x_1-x_2\\2x_1+x_2+x_3\end{pmatrix} = \begin{pmatrix}3\\1\\2\end{pmatrix}x_1 + \begin{pmatrix}0\\-1\\1\end{pmatrix}x_2 + \begin{pmatrix}0\\0\\1\end{pmatrix}x_3 \tag{3.2.4.1}$$

$$= \begin{pmatrix}3&0&0\\1&-1&0\\2&1&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} \tag{3.2.4.2}$$

$$\implies \mathbf{T} = \begin{pmatrix}3&0&0\\1&-1&0\\2&1&1\end{pmatrix} \tag{3.2.4.3}$$

Using Gauss-Jordan Elimination to find the inverse of $\mathbf{T}$, if it exists

$$\begin{pmatrix}3&0&0&|&1&0&0\\1&-1&0&|&0&1&0\\2&1&1&|&0&0&1\end{pmatrix} \tag{3.2.4.4}$$

$$\xleftrightarrow{R_1\leftarrow\frac{R_1}{3}} \begin{pmatrix}1&0&0&|&\frac{1}{3}&0&0\\1&-1&0&|&0&1&0\\2&1&1&|&0&0&1\end{pmatrix} \tag{3.2.4.5}$$

$$\xleftrightarrow[R_3\leftarrow R_3-2R_1]{R_2\leftarrow R_2-R_1} \begin{pmatrix}1&0&0&|&\frac{1}{3}&0&0\\0&-1&0&|&-\frac{1}{3}&1&0\\0&1&1&|&-\frac{2}{3}&0&1\end{pmatrix} \tag{3.2.4.6}$$

$$\xleftrightarrow{R_2\leftarrow\ -R_2} \begin{pmatrix}1&0&0&|&\frac{1}{3}&0&0\\0&1&0&|&\frac{1}{3}&-1&0\\0&1&1&|&-\frac{2}{3}&0&1\end{pmatrix} \tag{3.2.4.7}$$

$$\xleftrightarrow{R_3\leftarrow R_3-R_2} \begin{pmatrix}1&0&0&|&\frac{1}{3}&0&0\\0&1&0&|&\frac{1}{3}&-1&0\\0&0&1&|&-1&1&1\end{pmatrix} \tag{3.2.4.8}$$

Since $rank(\mathbf{T}) = 3$, $\mathbf{T}$ is invertible and the inverse is

$$\mathbf{T}^{-1} = \begin{pmatrix}\frac{1}{3}&0&0\\\frac{1}{3}&-1&0\\-1&1&1\end{pmatrix} \tag{3.2.4.9}$$

Now consider any vector $\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} \in \mathbf{R}^3$, then

$$\mathbf{T}^{-1}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}\frac{1}{3}&0&0\\\frac{1}{3}&-1&0\\-1&1&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} \tag{3.2.4.10}$$

$$= \begin{pmatrix}\frac{x_1}{3}\\\frac{x_1}{3}-x_2\\-x_1+x_2+x_3\end{pmatrix} \tag{3.2.4.11}$$

Therefore the transformation $\mathbf{T}^{-1}$ is defined on $\mathbf{R}^3$ as

$$\mathbf{T}^{-1}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}\frac{x_1}{3}\\\frac{x_1}{3}-x_2\\-x_1+x_2+x_3\end{pmatrix} \tag{3.2.4.12}$$

Prove that

$$(\mathbf{T}^2 - I)(\mathbf{T} - 3I) = 0 \tag{3.2.4.13}$$

Solution: Expressing (3.2.3.1) in matrix form

$$\mathbf{T} = \begin{pmatrix}3&0&0\\1&-1&0\\2&1&1\end{pmatrix} \tag{3.2.4.14}$$

The characteristic equation of $\mathbf{T}$ is given as follows,

$$\left|\mathbf{T} - \lambda\mathbf{I}\right| = \begin{vmatrix}3-\lambda&0&0\\1&-1-\lambda&0\\2&1&1-\lambda\end{vmatrix} = 0 \tag{3.2.4.15}$$

$$\implies (3-\lambda)(-1-\lambda)(1-\lambda) = 0$$
$$\implies (\lambda-3)(1+\lambda)(1-\lambda) = 0$$
$$\implies (\lambda-3)(1-\lambda^2) = 0$$
$$\implies (\lambda^2-1)(\lambda-3) = 0 \tag{3.2.4.16}$$

By the Cayley-Hamilton theorem,We can write (3.2.4.16) as

$$(\mathbf{T}^2 - I)(\mathbf{T} - 3I) = 0 \tag{3.2.4.17}$$

3.2.5. Let $\mathbb{C}$ be the complex vector space of 2×2

matrices with complex entries. Let

$$\mathbf{B} = \begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix} \tag{3.2.5.1}$$

and let $\mathbf{T}$ be the linear operator on $\mathbb{C}^{2\times 2}$ defined by $\mathbf{T}(\mathbf{A}) = \mathbf{BA}$. What is the rank of $\mathbf{T}$? Can you describe $\mathbf{T^2}$?

Solution: An ordered basis for $\mathbb{C}^{2\times 2}$ is given by

$$\mathbf{A_{11}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \mathbf{A_{12}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{3.2.5.2}$$

$$\mathbf{A_{21}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \mathbf{A_{22}} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.2.5.3}$$

Now, we compute

$$\mathbf{T}(\mathbf{A_{11}}) = \mathbf{BA_{11}} \tag{3.2.5.4}$$

$$= \begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{3.2.5.5}$$

$$= \begin{pmatrix} 1 & 0 \\ -4 & 0 \end{pmatrix} \tag{3.2.5.6}$$

from (3.2.5.6) we have

$$\mathbf{T}(\mathbf{A_{11}}) = \mathbf{A_{11}} - 4\mathbf{A_{21}} \tag{3.2.5.7}$$

$$\mathbf{T}(\mathbf{A_{12}}) = \mathbf{BA_{12}} \tag{3.2.5.8}$$

$$= \begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{3.2.5.9}$$

$$= \begin{pmatrix} 0 & 1 \\ 0 & -4 \end{pmatrix} \tag{3.2.5.10}$$

from (3.2.5.10), we have

$$\mathbf{T}(\mathbf{A_{12}}) = \mathbf{A_{12}} - 4\mathbf{A_{22}} \tag{3.2.5.11}$$

$$\mathbf{T}(\mathbf{A_{21}}) = \mathbf{BA_{21}} \tag{3.2.5.12}$$

$$= \begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{3.2.5.13}$$

$$= \begin{pmatrix} -1 & 0 \\ 4 & 0 \end{pmatrix} \tag{3.2.5.14}$$

from (3.2.5.14), we have

$$\mathbf{T}(\mathbf{A_{21}}) = -\mathbf{A_{11}} + 4\mathbf{A_{21}} \tag{3.2.5.15}$$

$$\mathbf{T}(\mathbf{A_{22}}) = \mathbf{BA_{22}} \tag{3.2.5.16}$$

$$= \begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.2.5.17}$$

$$= \begin{pmatrix} 0 & -1 \\ 0 & 4 \end{pmatrix} \tag{3.2.5.18}$$

from (3.2.5.18), we have

$$\mathbf{T}(\mathbf{A_{22}}) = -\mathbf{A_{12}} + 4\mathbf{A_{22}} \tag{3.2.5.19}$$

Now, by (3.2.5.7), (3.2.5.11), (3.2.5.15) and (3.2.5.19) we write matrix of the linear transformation as follows

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -4 & 0 & 4 & 0 \\ 0 & -4 & 0 & 4 \end{pmatrix} \tag{3.2.5.20}$$

Also, we know that the rank of a linear transformation is same as the rank of the matrix of the linear transformation. Thus, we find the rank of matrix $\mathbf{P}$.

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -4 & 0 & 4 & 0 \\ 0 & -4 & 0 & 4 \end{pmatrix} \xleftrightarrow{r_3=r_3+4r_1} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 4 \end{pmatrix} \tag{3.2.5.21}$$

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 4 \end{pmatrix} \xleftrightarrow{r_4=r_4+4r_1} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.2.5.22}$$

from (3.2.5.22), we found out that rank($\mathbf{T}$) =2. Now, we compute

$$\mathbf{T^2}(\mathbf{A}) = \mathbf{T}(\mathbf{T}(\mathbf{A})) \tag{3.2.5.23}$$

$$= \mathbf{T}(\mathbf{BA}) \tag{3.2.5.24}$$

$$= \mathbf{B^2A} \tag{3.2.5.25}$$

where

$$\mathbf{B^2} = \begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix} \tag{3.2.5.26}$$

$$= \begin{pmatrix} 5 & -5 \\ -20 & 20 \end{pmatrix} \tag{3.2.5.27}$$

3.2.6. Let T be a linear transformation from $\mathbb{R}^3$ into $\mathbb{R}^2$, and let U be a linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^3$. Prove that the transformation UT is not invertible. Generalize the theorem.

Solution: Let $\mathbf{v}, \mathbf{x} \in \mathbb{R}^3$ and $\mathbf{w} \in \mathbb{R}^2$. Table 3.2.6.1 shows that maximum rank the transformation matrix $\mathbf{C}$ can have is 2.

$$Rank(\mathbf{C}) = 2 \tag{3.2.6.1}$$
$$dim(\mathbf{C}) = 3 \tag{3.2.6.2}$$
$$\implies Rank(\mathbf{C}) < dim(\mathbf{C}) \tag{3.2.6.3}$$

Therefore from the equation (3.2.6.3), we can say transformation UT is not invertible. Generalizing the proof, for $n > m$ and considering vectors $\mathbf{v}, \mathbf{x} \in \mathbb{R}^n$ and $\mathbf{w} \in \mathbb{R}^m$. From the Table 3.2.6.2,

$$Rank(\mathbf{C}) = m \tag{3.2.6.4}$$
$$dim(\mathbf{C}) = n \tag{3.2.6.5}$$
$$\implies Rank(\mathbf{C}) < dim(\mathbf{C}) \tag{3.2.6.6}$$

From equation (3.2.6.6)we can say that the transformation UT is not invertible. Let the vectors $\mathbf{v} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \in \mathbb{R}^3$ and $\mathbf{w} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} \in \mathbb{R}^2$.

a) Calculating transformation matrix $\mathbf{A}$,

$$T(\mathbf{v}) = \mathbf{A}\mathbf{v} \tag{3.2.6.7}$$

$$\begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \tag{3.2.6.8}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = rref(\mathbf{A}) \tag{3.2.6.9}$$

$$\implies Rank(\mathbf{A}) = 2 \tag{3.2.6.10}$$

b) Calculating transformation matrix $\mathbf{B}$,

$$U(\mathbf{w}) = \mathbf{B}\mathbf{w} \tag{3.2.6.11}$$

$$\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & 2 \\ 1 & -1 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix} \tag{3.2.6.12}$$

$$\begin{pmatrix} \frac{3}{4} & 2 \\ 1 & -1 \\ 0 & \frac{1}{2} \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = rref(\mathbf{B}) \tag{3.2.6.13}$$

$$\implies Rank(\mathbf{B}) = 2 \tag{3.2.6.14}$$

c) Now for the transformation UT, calculating the transformation matrix $\mathbf{C}$,

$$UT : \mathbb{R}^3 \to \mathbb{R}^3 \tag{3.2.6.15}$$
$$\implies UT(\mathbf{x}) = \mathbf{C}\mathbf{x} \tag{3.2.6.16}$$
$$U(T(\mathbf{x})) = \mathbf{B}(\mathbf{A}\mathbf{x}) \tag{3.2.6.17}$$
$$\implies \mathbf{C} = \mathbf{B}\mathbf{A} \tag{3.2.6.18}$$

$$\mathbf{C} = \begin{pmatrix} \frac{3}{4} & 2 \\ 1 & -1 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & 0 & \frac{3}{4} \\ 0 & 0 & 2 \\ \frac{1}{2} & 0 & \frac{-1}{2} \end{pmatrix} \tag{3.2.6.19}$$

$$\begin{pmatrix} \frac{3}{4} & 0 & \frac{3}{4} \\ 0 & 0 & 2 \\ \frac{1}{2} & 0 & \frac{-1}{2} \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = rref(\mathbf{C}) \tag{3.2.6.20}$$

$$\implies Rank(\mathbf{C}) = 2 \tag{3.2.6.21}$$
$$dim(\mathbf{C}) = 3 \tag{3.2.6.22}$$

As $Rank(\mathbf{C}) < dim(\mathbf{C})$, transformation UT is not invertible.

3.2.7. Find two linear operators $\mathbf{T}$ and $\mathbf{U}$ on $\mathbf{R}^2$ such that $\mathbf{TU} = 0$ but $\mathbf{UT} \neq 0$

Solution: Let,

$$\mathbf{x}, \mathbf{y} \in \mathbf{R^2} \tag{3.2.7.1}$$

Let $\mathbf{T}$ and $\mathbf{U}$ be given by the matrices

$$\mathbf{T}(\mathbf{x}) = \mathbf{A}\mathbf{x}; \quad \mathbf{U}(\mathbf{x}) = \mathbf{B}\mathbf{x} \tag{3.2.7.2}$$

where,

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \quad \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{3.2.7.3}$$

$$\mathbf{T}(a\mathbf{x} + \mathbf{y}) = a\mathbf{T}\mathbf{x} + \mathbf{T}\mathbf{y} \tag{3.2.7.4}$$
$$\mathbf{U}(a\mathbf{x} + \mathbf{y}) = a\mathbf{U}\mathbf{x} + \mathbf{U}\mathbf{y} \tag{3.2.7.5}$$

From (3.2.7.4) and (3.2.7.5), we can tell that $\mathbf{T}$ and $\mathbf{U}$ are linear operators. Now,

$$\mathbf{TU} = \mathbf{AB} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbf{0} \tag{3.2.7.6}$$

$$\mathbf{UT} = \mathbf{BA} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq \mathbf{0} \tag{3.2.7.7}$$

From (3.2.7.6) and (3.2.7.7) it can be observed that $\mathbf{TU} = 0$ but $\mathbf{UT} \neq 0$

Transformation	Matrix Representation	Dimension	Max Rank of transformation matrix
$T:\mathbb{R}^3\to\mathbb{R}^2$	$T(\mathbf{v})=\mathbf{Av}$	$\mathbf{A}:2\times 3$	$Rank(\mathbf{A})=2$
$U:\mathbb{R}^2\to\mathbb{R}^3$	$U(\mathbf{w})=\mathbf{Bw}$	$\mathbf{B}:3\times 2$	$Rank(\mathbf{B})=2$
$UT:\mathbb{R}^3\to\mathbb{R}^3$	$UT(\mathbf{x})=\mathbf{Cx}$ $U(T(\mathbf{x}))=\mathbf{B}(\mathbf{Ax})$ $\mathbf{C}=\mathbf{AB}$	$\mathbf{C}:3\times 3$	$Rank(\mathbf{C})\leq min(Rank(\mathbf{B}),Rank(\mathbf{A}))$ $Rank(\mathbf{C})=2$

TABLE 3.2.6.1: Proof for non-invertibility of the transformation UT where $T:\mathbb{R}^3\to\mathbb{R}^2$ and $U:\mathbb{R}^2\to\mathbb{R}^3$

Transformation	Matrix Representation	Dimension	Max Rank of transformation matrix
$T:\mathbb{R}^n\to\mathbb{R}^m$	$T(\mathbf{v})=\mathbf{Av}$	$\mathbf{A}:m\times n$	$Rank(\mathbf{A})=m$
$U:\mathbb{R}^m\to\mathbb{R}^n$	$U(\mathbf{w})=\mathbf{Bw}$	$\mathbf{B}:n\times m$	$Rank(\mathbf{B})=m$
$UT:\mathbb{R}^n\to\mathbb{R}^n$	$UT(\mathbf{x})=\mathbf{Cx}$ $U(T(\mathbf{x}))=\mathbf{B}(\mathbf{Ax})$ $\mathbf{C}=\mathbf{AB}$	$\mathbf{C}:n\times n$	$Rank(\mathbf{C})\leq min(Rank(\mathbf{B}),Rank(\mathbf{A}))$ $Rank(\mathbf{C})=m$

TABLE 3.2.6.2: Generalization of the proof

3.2.8. Let $\mathbf{V}$ be a vector space over the field $\mathbf{F}$ and $\mathbf{T}$ is a linear operator on $\mathbf{V}$. If $\mathbf{T}^2=0$, what can you say about the relation of the range of $\mathbf{T}$ to the null space of $\mathbf{T}$? Give an example of linear operator $\mathbf{T}$ on $\mathbf{R}^2$ such that $\mathbf{T}^2=0$ but $\mathbf{T}\neq 0$.

Solution: Given,

$$\mathbf{T}:\mathbf{V}\to\mathbf{V} \tag{3.2.8.1}$$

Now, $\mathbf{T}^2$ is also a linear operator as,

$$\mathbf{T}^2(c\alpha)=\mathbf{T}(\mathbf{T}(c\alpha))=\mathbf{T}(c\mathbf{T}(\alpha)) \tag{3.2.8.2}$$
$$=c\mathbf{T}(\mathbf{T}(\alpha))=c\mathbf{T}^2(\alpha) \tag{3.2.8.3}$$

Let some vector $\mathbf{y}\in$ Range($\mathbf{T}$) then there exists $\mathbf{x}\in\mathbf{V}$ such that,

$$\mathbf{T}(\mathbf{x})=\mathbf{y} \tag{3.2.8.4}$$

Now given that,

$$\mathbf{T}^2(\mathbf{x})=\mathbf{0} \tag{3.2.8.5}$$
$$\implies \mathbf{T}(\mathbf{T}(\mathbf{x}))=\mathbf{0} \tag{3.2.8.6}$$
$$\mathbf{T}(\mathbf{y})=\mathbf{0} \tag{3.2.8.7}$$

$\therefore$ $\mathbf{y}$ lies in the Null space of $\mathbf{T}$. Hence $\mathbf{T}$ is singular. Thus, the range of $\mathbf{T}$ must be contained in Null space of $\mathbf{T}$ i.e., Range($\mathbf{T}$) $\subseteq$ NullSpace($\mathbf{T}$)

Example:

$$\mathbf{T}:\mathbf{R}^2\to\mathbf{R}^2 \tag{3.2.8.8}$$

Consider,

$$\mathbf{T}(\mathbf{x})=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\mathbf{x} \tag{3.2.8.9}$$
$$\implies \mathbf{T}\neq 0 \tag{3.2.8.10}$$

Now,

$$\mathbf{T}^2:\mathbf{R}^2\to\mathbf{R}^2 \tag{3.2.8.11}$$

$$\mathbf{T}^2(\mathbf{x})=\mathbf{T}(\mathbf{T}(\mathbf{x})) \tag{3.2.8.12}$$
$$=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\mathbf{x}=\mathbf{0} \tag{3.2.8.13}$$
$$\implies \mathbf{T}^2(\mathbf{x})=\mathbf{0} \tag{3.2.8.14}$$

Thus $\mathbf{T}^2$ is a zero transformation, $\mathbf{T}^2=\mathbf{0}$. Now, Kernel of $\mathbf{T}$ is given by,

$$\mathbf{T}(\mathbf{x})=\mathbf{0} \tag{3.2.8.15}$$
$$\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \tag{3.2.8.16}$$
$$\implies x=0 \tag{3.2.8.17}$$

Thus,

$$\mathbf{Ker(T)}=y\begin{pmatrix}0\\ 1\end{pmatrix};y\in\mathbf{R} \tag{3.2.8.18}$$

singular	A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is said to be singular if $\exists$ some non-zero $\mathbf{X} \in \mathbb{R}^n$ s.t $\mathbf{AX} = \mathbf{0}$ i.e $Nullity(A) \neq 0$. From rank-nullity theorem we can say $rank(A) < n$
non-singular	A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be non-singular if $\mathbf{AX} = \mathbf{0}$ implies $\mathbf{X} = \mathbf{0}$ i.e $Nullity(A) = 0$
onto	A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, $m \leq n$ is said to be onto if for every $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{AX} = \mathbf{b}$ has atleast one solution $\mathbf{X} \in \mathbb{R}^n$.. i.e $dim(Col(\mathbf{A})) = m$ or $Rank(\mathbf{A}) = m$ If $m > n$, then $\mathbf{AX} = \mathbf{b}$ has no solution because rank-nullity theorem is not satisfied.

TABLE 3.2.9.1

Now,

$$\mathbf{Range(T)} = \mathbf{ColumnSpace}\left\{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right\} \tag{3.2.8.19}$$

$$= k\begin{pmatrix} 0 \\ 1 \end{pmatrix}; k \in \mathbf{R} \tag{3.2.8.20}$$

Thus for the example, Range($\mathbf{T}$) = Kernel($\mathbf{T}$) and from (3.2.8.10), (3.2.8.14) it is clear that $\mathbf{T}^2 = 0$ but $\mathbf{T} \neq 0$.

3.2.9. Let $\mathbf{A}$ be an $m \times n$ matrix with entries in F and let T be the linear transformation from $F^{n\times 1}$ into $F^{m\times l}$ defined by $T(\mathbf{X}) = \mathbf{AX}$. Show that

a) if $m < n$ it may happen that T is onto without being non-singular
b) if $m > n$ we may have T non-singular but not onto.

Solution: Proof

a) $m < n$

$$\text{Let, } T : \mathbb{R}^3 \to \mathbb{R}^2 \tag{3.2.9.1}$$

$$T(\mathbf{X}) = \mathbf{AX} = \mathbf{b} \tag{3.2.9.2}$$

$$\text{Let, } \mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \tag{3.2.9.3}$$

$$\text{Consider, } \mathbf{X} = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} \tag{3.2.9.4}$$

$$\implies \mathbf{AX} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} \tag{3.2.9.5}$$

$$= \begin{pmatrix} 6 \\ 5 \end{pmatrix} \tag{3.2.9.6}$$

Hence T is onto.

$$\text{Consider, } \mathbf{X} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \tag{3.2.9.7}$$

$$\implies \mathbf{AX} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \tag{3.2.9.8}$$

$$= \mathbf{0} \tag{3.2.9.9}$$

Since $\exists$ $\mathbf{X} \neq \mathbf{0}$ such that $\mathbf{AX} = \mathbf{0}$, T is singular.

$\therefore$ T is both onto and singular.

b) $m > n$

$$\text{Let, } T : \mathbb{R}^3 \to \mathbb{R}^2 \tag{3.2.9.10}$$

$$T(\mathbf{X}) = \mathbf{AX} = \mathbf{b} \tag{3.2.9.11}$$

$$\text{Let, } \mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.2.9.12}$$

$$\text{Consider, } \mathbf{X} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \tag{3.2.9.13}$$

$$\implies \mathbf{AX} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} \tag{3.2.9.14}$$

$$= \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \tag{3.2.9.15}$$

$$\tag{3.2.9.16}$$

$\therefore$ T is not onto, and is also non-singular.

3.2.10. Let p, m, n be positive integers and $\mathbb{F}$ a field.Let $\mathbf{V}$ be the space of $m \times n$ matrices over $\mathbb{F}$ and $\mathbf{W}$ the space of $p \times n$ matrices over $\mathbb{F}$.Let $\mathbf{B}$ be a fixed $p \times m$ matrix and let $\mathbb{T}$ be the linear transformation from $\mathbf{V}$ into $\mathbf{W}$ defined by $\mathbb{T}(\mathbf{A}) = \mathbf{BA}$.Prove that $\mathbb{T}$ is invertible if and only if $p = m$ and $\mathbf{B}$ is an invertible $m \times m$ matrix. **Solution:**

$$\mathbb{T}(\mathbf{A}) = \mathbf{BA} \tag{3.2.10.1}$$

So, $\mathbf{B}$ is the transformation matrix.
$\mathbf{B}$ is invertible if

a) $\mathbb{T}$ is one to one mapping,that is

$$\mathbf{BA} = \mathbf{BA'} \tag{3.2.10.2}$$

$$\implies \mathbf{A} = \mathbf{A'} \tag{3.2.10.3}$$

b) $\mathbb{T}$ must be onto, that is range($\mathbf{B}$)=$\mathbf{W}$

Case 1: Let us assume that $\mathbb{T}$ is invertible with inverse transformation $\mathbb{T}_1$ from $\mathbf{W}$ to $\mathbf{V}$ that satisfies

$$\mathbb{T}(\mathbf{A}) = \mathbf{BA} \in \mathbf{W} \tag{3.2.10.4}$$

$$\implies \mathbb{T}_1(\mathbf{BA}) = \mathbf{A} \in \mathbf{V} \tag{3.2.10.5}$$

$$\dim(\mathbf{V}) = mn, \dim(\mathbf{W}) = pn \tag{3.2.10.6}$$

Since $\mathbb{T}$ is one-one mapping, the zero vector in $\mathbf{V}, \mathbf{0}_{m\times n}$ is uniquely mapped to

$$\mathbb{T}(\mathbf{0}_{m\times n}) = \mathbf{B0}_{m\times n} = \mathbf{0}_{p\times n} \tag{3.2.10.7}$$

$$\text{So, } \mathbf{BA} = \mathbf{0} \iff \mathbf{A} = \mathbf{0} \tag{3.2.10.8}$$

Let $\{\mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_{mn}\}$ be the basis for $\mathbf{V}$

$$c_1\mathbf{V}_1 + c_2\mathbf{V}_2 + \ldots + c_{mn}\mathbf{V}_{mn} = \mathbf{0} \tag{3.2.10.9}$$

$$\iff c_1, c_2, \ldots, c_{mn} \in \mathbb{F} = 0 \tag{3.2.10.10}$$

Any matrix $\mathbf{A} \in \mathbf{V}$ can be written as

$$\mathbf{A} = \sum_{i=1}^{mn} \alpha_i \mathbf{V}_i \tag{3.2.10.11}$$

Since $\mathbb{T}$ is onto, any matrix $\mathbf{C} \in \mathbf{W}$ can be expressed as

$$\mathbf{C} = \mathbf{B}\left(\sum_{i=1}^{mn} \alpha_i \mathbf{V}_i\right) \tag{3.2.10.12}$$

$$= \sum_{i=1}^{mn} \alpha_i (\mathbf{BV}_i) \tag{3.2.10.13}$$

So, the set $\mathbf{S} = \{\mathbf{BV}_1, \mathbf{BV}_2, \ldots, \mathbf{BV}_{mn}\}$ forms basis of $\mathbf{W}$ if all matrices in it are linearly

Let $\mathbf{A}$ be an $m \times n$ matrix with entries in F and let T be the linear transformation from $F^{n\times 1}$ into $F^{m\times l}$ defined by $T(\mathbf{X}) = \mathbf{AX}$. If,		
	$m < n$	$m > n$
singular	Since $rank(\mathbf{A}) < n$, by definition T is singular	Consider an non-singular T such that $rank(\mathbf{A}) > n$
onto	Since $m < n$, by definition T can be onto	Since $m > n$, by definition T is not onto.

TABLE 3.2.9.2

Parameter	Description
p, m, n	Positive integers
$\mathbb{F}$	Field
$\mathbf{V}$	Space of $m\times n$ matrices over $\mathbb{F}$
$\mathbf{W}$	Space of $p\times n$ matrices over $\mathbb{F}$
$\mathbf{B}$	Fixed $p \times m$ matrix
Linear transformation $\mathbb{T} : \mathbf{V} \to \mathbf{W}$	$\mathbb{T}(\mathbf{A}) = \mathbf{BA}$

TABLE 3.2.10.1: Input Parameters

independent.

$$c_1(\mathbf{BV}_1) + c_2(\mathbf{BV}_2) + \ldots + c_{mn}(\mathbf{BV}_{mn}) = \mathbf{0} \tag{3.2.10.14}$$

$$\mathbf{B}(c_1\mathbf{V}_1 + c_2\mathbf{V}_2 + \ldots + c_{mn}\mathbf{V}_{mn}) = \mathbf{0} \tag{3.2.10.15}$$

$$(3.2.10.8) \implies c_1\mathbf{V}_1 + \ldots + c_{mn}\mathbf{V}_{mn} = 0 \tag{3.2.10.16}$$

$$\iff c_1, c_2, \ldots, c_{mn} = 0 (\text{from } (3.2.10.10)) \tag{3.2.10.17}$$

So,the set $\mathbf{S}$ with cardinality mn is basis for $\mathbf{W}$

$$(3.2.10.6) \implies pn = mn \tag{3.2.10.18}$$

$$p = m \tag{3.2.10.19}$$

(3.2.10.8),(3.2.10.19) prove that $\mathbf{B}$ is invertible $m \times m$ matrix. Case 2: Consider $p = m$ and $\mathbf{B}$ is an invertible $m \times m$ matrix.
Verifying if $\mathbb{T}$ is onto,
Let the set of matrices $\{\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_{mn}\}$ be the basis for $\mathbf{V}$
Any matrix $\mathbf{A} \in \mathbf{V}$ can be written as

$$\mathbf{A} = \sum_{i=1}^{mn} \alpha_i \mathbf{A}_i \tag{3.2.10.20}$$

where $\alpha_i \in \mathbb{F}$
The set $\mathbf{M} = \{\mathbf{BA}_1, \mathbf{BA}_2, \ldots, \mathbf{BA}_{mn}\}$ lie in $\mathbf{W}$

$$c_1(\mathbf{BA}_1) + c_2(\mathbf{BA}_2) + \ldots + c_{mn}(\mathbf{BA}_{mn}) = \mathbf{0} \tag{3.2.10.21}$$

$$\implies \mathbf{B}(c_1\mathbf{A}_1 + c_2\mathbf{A}_2 + \ldots + c_{mn}\mathbf{A}_{mn}) = \mathbf{0} \tag{3.2.10.22}$$

Since $\mathbf{B}$ is non-singular,

$$(c_1\mathbf{A}_1 + c_2\mathbf{A}_2 + \ldots + c_{mn}\mathbf{A}_{mn}) = \mathbf{0} \tag{3.2.10.23}$$

$$\iff c_1, c_2, \ldots, c_{mn} = 0 \tag{3.2.10.24}$$

because $\{\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_{mn}\}$ are linearly independent
So,$\mathbf{M}$ forms basis for $\mathbf{W}$
Any vector $\mathbf{C} \in \mathbf{W}$ can be written as

$$\mathbf{C} = \sum_{i=1}^{mn} \beta_i \mathbf{BA}_i \text{ where } \beta_i \in \mathbb{F} \tag{3.2.10.25}$$

$$= \mathbf{B}(\sum_{i=1}^{mn} \beta_i \mathbf{A}_i) \tag{3.2.10.26}$$

$$= \mathbf{BA} \text{ (from (3.2.10.20))} \tag{3.2.10.27}$$

So,range($\mathbf{B}$)=$\mathbf{W}$
Consider the matrix $\mathbf{A}, \mathbf{A}' \in \mathbf{V}$ such that

$$\mathbf{BA} = \mathbf{BA}' \tag{3.2.10.28}$$

$$\mathbf{B}^{-1}(\mathbf{BA}) = \mathbf{B}^{-1}(\mathbf{BA}') \tag{3.2.10.29}$$

$$(\mathbf{B}^{-1}\mathbf{B})\mathbf{A} = (\mathbf{B}^{-1}\mathbf{B})\mathbf{A}' \tag{3.2.10.30}$$

$$\implies \mathbf{A} = \mathbf{A}' \tag{3.2.10.31}$$

So, $\mathbb{T}$ is invertible. Conclusion: From case 1,case 2 $\mathbb{T}$ is invertible if and only if $p = m$ and $\mathbf{B}$ is an invertible $m \times m$ matrix. Example: Let $p = m = 3, n = 4$ Let $\mathbb{T} : \mathbf{V} \to \mathbf{W}$ adds row 2 to row 3 for a matrix $\mathbf{A} \in \mathbf{V}$
The elementary matrix that performs this is

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \tag{3.2.10.32}$$

$$\text{Let } \mathbf{A} = \begin{pmatrix} 1 & 2 & 2 & 5 \\ 1 & 3 & 6 & 7 \\ 4 & 9 & 2 & 6 \end{pmatrix} \tag{3.2.10.33}$$

$$\mathbb{T}(\mathbf{A}) = \mathbf{BA} \tag{3.2.10.34}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 2 & 5 \\ 1 & 3 & 6 & 7 \\ 4 & 9 & 2 & 6 \end{pmatrix} \tag{3.2.10.35}$$

$$= \begin{pmatrix} 1 & 2 & 2 & 5 \\ 1 & 3 & 6 & 7 \\ 5 & 12 & 8 & 13 \end{pmatrix} \tag{3.2.10.36}$$

$$= \mathbf{C} \in \mathbf{W} \tag{3.2.10.37}$$

Let transformation $\mathbb{T}_1 : \mathbf{W} \to \mathbf{V}$ subtracts row2 from row 3 for a matrix $\mathbf{C} \in \mathbf{W}$ and is performed by elementary matrix

$$\mathbf{U} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \tag{3.2.10.38}$$

$$\text{Let } \mathbf{C} = \begin{pmatrix} 1 & 2 & 2 & 5 \\ 1 & 3 & 6 & 7 \\ 5 & 12 & 8 & 13 \end{pmatrix} \tag{3.2.10.39}$$

$$\mathbb{T}_1(\mathbf{C}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 2 & 5 \\ 1 & 3 & 6 & 7 \\ 5 & 12 & 8 & 13 \end{pmatrix} \tag{3.2.10.40}$$

$$= \begin{pmatrix} 1 & 2 & 2 & 5 \\ 1 & 3 & 6 & 7 \\ 4 & 9 & 2 & 6 \end{pmatrix} \tag{3.2.10.41}$$

$$= \mathbf{A} \tag{3.2.10.42}$$

$$\implies \mathbb{T}_1(\mathbf{C}) = \mathbf{A} \tag{3.2.10.43}$$

$$\mathbb{T}_1(\mathbb{T}(\mathbf{A})) = \mathbf{A} \tag{3.2.10.44}$$

$$\text{and } \mathbb{T}(\mathbf{A}) = \mathbf{C} \tag{3.2.10.45}$$

$$\implies \mathbb{T}(\mathbb{T}_1(\mathbf{C})) = \mathbf{C} \tag{3.2.10.46}$$

So,$\mathbb{T}_1$ is the inverse transformation of $\mathbb{T}$ and

$$\mathbb{T}_1 = \mathbb{T}^{-1} \tag{3.2.10.47}$$

$$\mathbf{UB} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \tag{3.2.10.48}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.2.10.49}$$

$$\mathbf{BU} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \tag{3.2.10.50}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.2.10.51}$$

$$\implies \mathbf{B}^{-1} = \mathbf{U} \tag{3.2.10.52}$$

So, $\mathbb{T}$ is invertible and ,$\mathbf{B}$ is an invertible 3×3 matrix.

3.3 Isomorphism

3.3.1. Let V be a vector space over the field of complex numbers , and suppose there is an isomorphism T of V onto C^3. Let α_1 , α_2 , α_3 , α_4 be vectors in V such that

$$\begin{aligned} T(\alpha_1) &= \begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}, T(\alpha_2) = \begin{pmatrix} -2 \\ 1+i \\ 0 \end{pmatrix}, \\ T(\alpha_3) &= \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, T(\alpha_4) = \begin{pmatrix} \sqrt{2} \\ i \\ 3 \end{pmatrix} \end{aligned} \tag{3.3.1.1}$$

a) Is α_1 in the subspace spanned by α_2 and α_3?
Solution: T : V → W is an isomorphism if
(1) T is one one.
(2) T is onto.

$$\begin{pmatrix} 1 & -2 & -1 & \sqrt{2} \\ 0 & 1+i & 1 & i \\ i & 0 & 1 & 3 \end{pmatrix} \xrightarrow{ref} \begin{pmatrix} 1 & 0 & -i & -3i \\ 0 & 2 & 1-i & i+1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.3.1.2}$$

T is one one over C^3 if

$$T(\alpha) = 0 \implies \alpha = 0 \tag{3.3.1.3}$$

now,

$$\begin{pmatrix} 1 & -2 & \sqrt{2} \\ 0 & 1+i & i \\ i & 0 & 3 \end{pmatrix} \alpha = 0 \tag{3.3.1.4}$$

consider the row reduced matrix

$$\begin{pmatrix} 1 & -2 & \sqrt{2} \\ 0 & 1+i & i \\ i & 0 & 3 \end{pmatrix} \xleftrightarrow[R_3 \to iR_3]{R_3 \to R_3 - iR_1} \begin{pmatrix} 1 & -2 & \sqrt{2} \\ 0 & 1+i & i \\ 0 & -2 & \sqrt{2}+3i \end{pmatrix} \tag{3.3.1.5}$$

$$\xleftrightarrow[R_3 \leftarrow R_3 + R_2]{R_2 \leftarrow (1-i)R_2} \begin{pmatrix} 1 & -2 & \sqrt{2} \\ 0 & 2 & i+1 \\ 0 & 0 & \sqrt{2}+4i+1 \end{pmatrix} \tag{3.3.1.6}$$

$$\alpha = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{3.3.1.7}$$

Therefore it holds the condition of one one and the rank = no. of pivot columns = 3 (equal to no. of columns).Thus the vectors are linearly independent hence it is onto . Since T is an isomorphoism onto C^3.

$$T(\alpha_1) = c_1 T(\alpha_2) + c_2 T(\alpha_3) \qquad (3.3.1.8)$$

c_1 and c_2 are scalar.

$$\begin{pmatrix}1\\0\\i\end{pmatrix} = c_1\begin{pmatrix}-2\\1+i\\0\end{pmatrix} + c_2\begin{pmatrix}-1\\1\\1\end{pmatrix} \qquad (3.3.1.9)$$

$$\begin{pmatrix}1\\0\\i\end{pmatrix} = \begin{pmatrix}-2 & -1\\1+i & 1\\0 & 1\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix} \qquad (3.3.1.10)$$

Now we find c_i by row reducing augmented matrix.

$$\begin{pmatrix}-2 & -1 & 1\\1+i & 1 & 0\\0 & 1 & i\end{pmatrix} \xleftrightarrow[R_2\to R_3]{R_1\to -R_1/2} \begin{pmatrix}1 & \frac{1}{2} & -\frac{1}{2}\\0 & 1 & i\\1+i & 1 & 0\end{pmatrix} \qquad (3.3.1.11)$$

$$\xleftrightarrow[R_3\leftarrow R_3-(1+i)R_1]{R_1\leftarrow R_1-R_2/2} \begin{pmatrix}1 & 0 & \frac{-1-i}{2}\\0 & 1 & i\\0 & \frac{1-i}{2} & \frac{1+i}{2}\end{pmatrix} \qquad (3.3.1.12)$$

$$\xleftrightarrow{R_3\leftarrow R_3-(1-i)/2R_2} \begin{pmatrix}1 & 0 & \frac{-1-i}{2}\\0 & 1 & i\\0 & 0 & 0\end{pmatrix} \qquad (3.3.1.13)$$

Therefore the coordinate matrix of the vector is

$$\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}\frac{-1-i}{2}\\i\end{pmatrix} \qquad (3.3.1.14)$$

substituting the c_i in (3.3.1.8)

$$T(\alpha_1) = -\frac{1+i}{2}T(\alpha_2) + iT(\alpha_3) \qquad (3.3.1.15)$$

Hence α_1 belongs to the subspace spanned by α_2 and α_3.

b) Find a basis for the subspace of V spanned by the 4 vectors α_i.
Solution: V is a vector space and V is isomorphic to C^3 via isomorphism T which implies that C^3 is also isomorphic to V via isomorphism T^{-1}.
As V is isomorphic to C^3, so

$$dim(V) = dim(C^3) = 3 \qquad (3.3.1.16)$$

Now,

$$\begin{pmatrix}1 & 0 & i\\-2 & 1+i & 0\\-1 & 1 & 1\\\sqrt{2} & i & 3\end{pmatrix} \xleftrightarrow[R_4\leftarrow\sqrt{2}R_4]{R_3\leftarrow R_3+R_1} \begin{pmatrix}1 & 0 & i\\-2 & 1+i & 0\\0 & 1 & 1+i\\2 & i\sqrt{2} & 3\sqrt{2}\end{pmatrix}$$

$$\xleftrightarrow{R_4\leftarrow R_4+R_2} \begin{pmatrix}1 & 0 & i\\-2 & 1+i & 0\\0 & 1 & 1+i\\0 & 1+i(1+\sqrt{2}) & 3\sqrt{2}\end{pmatrix}$$

$$\xleftrightarrow{R_2\leftarrow R_2+2R_1} \begin{pmatrix}1 & 0 & i\\0 & 1+i & 2i\\0 & 1 & 1+i\\0 & 1+i(1+\sqrt{2}) & 3\sqrt{2}\end{pmatrix}$$

$$\xleftrightarrow{R_3\leftarrow(1+i)R_3} \begin{pmatrix}1 & 0 & i\\0 & 1+i & 2i\\0 & 1+i & 2i\\0 & 1+i(1+\sqrt{2}) & 3\sqrt{2}\end{pmatrix}$$

$$\xleftrightarrow{R_3\leftarrow R_3-R_2} \begin{pmatrix}1 & 0 & i\\0 & 1+i & 2i\\0 & 0 & 0\\0 & 1+i(1+\sqrt{2}) & 3\sqrt{2}\end{pmatrix} \qquad (3.3.1.17)$$

From here we can get that $T\alpha_3$ is dependent vector while $T\alpha_1$, $T\alpha_2$ and $T\alpha_4$ are independent vector. These $T\alpha_1$, $T\alpha_2$ and $T\alpha_4$ also span the vector space C^3, so these 3 vectors are the basis of C^3.
As $dim(V) = 3$, so it must have 3 basis and as V and C^3 are isomorphic so α_1, α_2 and α_4 are the basis of V.

3.3.2. Let $\mathbb{W}$ be the set of all 2×2 complex Hermitian matrices, that is the sset of 2×2 complex matrices $\mathbf{A}$ ssuch that $\mathbf{A}_{ij} = \overline{\mathbf{A}_{\mathbf{ji}}}$ (the bar denoting complex conjugation). $\mathbb{W}$ is a vector space over the field of real numbers, under the usual operations. Verify that

$$\begin{pmatrix}x\\y\\z\\t\end{pmatrix} \to \begin{pmatrix}t+x & y+iz\\y-iz & t-x\end{pmatrix} \qquad (3.3.2.1)$$

is an isomorphism of $\mathbb{R}^4$ onto $\mathbb{W}$.
Solution:

a) **Check for linearity:** The transformation T is given by

$$T : \mathbb{R}^4 \to \mathbb{W} \tag{3.3.2.2}$$

$$T\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} t+x & y+iz \\ y-iz & t-x \end{pmatrix} \tag{3.3.2.3}$$

Let $\mathbf{x} = \begin{pmatrix} x \\ y \\ t \\ z \end{pmatrix}$. Expressing R.H.S of equation (3.3.2.3) using Kronecker Product,

$$T(\mathbf{x}) = \begin{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}\mathbf{x} & \begin{pmatrix} 0 & 1 & i & 0 \end{pmatrix}\mathbf{x} \\ \begin{pmatrix} 0 & 1 & -i & 0 \end{pmatrix}\mathbf{x} & \begin{pmatrix} -1 & 0 & 0 & 1 \end{pmatrix}\mathbf{x} \end{pmatrix} \tag{3.3.2.4}$$

$$= \begin{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -i & 0 \end{pmatrix}\mathbf{x} & \begin{pmatrix} 0 & 1 & i & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}\mathbf{x} \end{pmatrix} \tag{3.3.2.5}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & i & 0 \\ 0 & 1 & -i & 0 & -1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x & 0 \\ y & 0 \\ z & 0 \\ t & 0 \\ 0 & x \\ 0 & y \\ 0 & z \\ 0 & t \end{pmatrix} \tag{3.3.2.6}$$

$$\implies T(\mathbf{x}) = \begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} \begin{pmatrix} \mathbf{x} & \mathbf{0}_{4\times1} \\ \mathbf{0}_{4\times1} & \mathbf{x} \end{pmatrix} \tag{3.3.2.7}$$

Where $\mathbf{A}$ and $\mathbf{B}$ are block matrices.

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -i & 0 \end{pmatrix} \tag{3.3.2.8}$$

$$\mathbf{B} = \begin{pmatrix} 0 & 1 & i & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \tag{3.3.2.9}$$

The Kronecker Product of $\mathbf{I}_2$ and $\mathbf{x}$ gives the block matrix in equation (3.3.2.7).

$$\mathbf{I}_{2\times2} \otimes \mathbf{x}_{4\times1} = \begin{pmatrix} \mathbf{x} & \mathbf{0} \\ \mathbf{0} & \mathbf{x} \end{pmatrix}_{8\times2} \tag{3.3.2.10}$$

Hence we can write equation (3.3.2.7) as,

$$T(\mathbf{x}) = \begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} (\mathbf{I} \otimes \mathbf{x}) \tag{3.3.2.11}$$

Let $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^4$ and $\alpha, \beta \in \mathbb{R}$.

$$T(\alpha\mathbf{x}_1 + \beta\mathbf{x}_2) = \begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} (\mathbf{I} \otimes (\alpha\mathbf{x}_1 + \beta\mathbf{x}_2)) \tag{3.3.2.12}$$

$$= \alpha\begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} (\mathbf{I} \otimes \mathbf{x}_1) + \beta\begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} (\mathbf{I} \otimes \mathbf{x}_2) \tag{3.3.2.13}$$

$$= \alpha T\mathbf{x}_1 + \beta T\mathbf{x}_2 \tag{3.3.2.14}$$

Therefore from equation (3.3.2.14), we can say T is linear transformation.

b) **Check for one-one property:** For transformation T to be one-one, we can prove if $T(\mathbf{x}) = \mathbf{0}$, that implies $\mathbf{x} = \mathbf{0}$. From the equation (3.3.2.11),

$$T(\mathbf{x}) = \mathbf{0} \tag{3.3.2.15}$$

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} (\mathbf{I} \otimes \mathbf{x}) = \mathbf{0} \tag{3.3.2.16}$$

$$\implies \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & i & 0 \\ 0 & 1 & -i & 0 & -1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x & 0 \\ y & 0 \\ z & 0 \\ t & 0 \\ 0 & x \\ 0 & y \\ 0 & z \\ 0 & t \end{pmatrix} = \mathbf{0}_{2\times2} \tag{3.3.2.17}$$

From equation (3.3.2.3),

$$\begin{pmatrix} t+x & y+iz \\ y-iz & t-x \end{pmatrix} = \mathbf{0}_{2\times2} \tag{3.3.2.18}$$

$$\implies x = 0, y = 0, z = 0, t = 0 \tag{3.3.2.19}$$

$$\implies \mathbf{x} = \mathbf{0} \tag{3.3.2.20}$$

Hence from (3.3.2.15) and (3.3.2.20), T is one-one and that implies $T : \mathbb{R}^4 \to \mathbb{W}$ is isomorphism.

3.3.3. Show that $\mathbf{F^{m\times n}}$ is isomorphic to $\mathbf{F^{mn}}$.
Solution: See Tables 3.3.3.1, 3.3.3.2 and 3.3.3.3.
$\mathbb{R}^{2\times2}$ is isomorphic to $\mathbb{R}^4$ ie, $\mathbb{R}^{2\times2} \cong \mathbb{R}^4$.

3.3.4. Let $\mathbf{V}$ be the set of complex numbers regarded as a vector space over the field of real numbers. We define a function T from $\mathbf{V}$ into the space of 2×2 real matrices, as follows. If $z = x + iy$

Invertible Linear Map	A linear map $\mathbf{T} \in \mathbf{L}(\mathbf{V}, \mathbf{W})$ is called invertible if there exists a linear map $\mathbf{S} \in \mathbf{L}(\mathbf{W}, \mathbf{V})$ such that $\mathbf{ST}$ equals the identity map on $\mathbf{V}$ and $\mathbf{TS}$ equals the identity map on $\mathbf{W}$. A linear map $\mathbf{S} \in \mathbf{L}(\mathbf{W}, \mathbf{V})$ satisfying $\mathbf{ST} = \mathbf{I_V}$ and $\mathbf{TS} = \mathbf{I_W}$ is called an inverse of $\mathbf{T}$.
Isomorphic Vector Spaces	Two vector spaces $\mathbf{V}$ and $\mathbf{W}$ are called isomorphic if there is an isomorphism from one vector space onto the other one. An isomorphism is an invertible linear map.
Rank Nullity Theorem	Let $\mathbf{V}$ and $\mathbf{W}$ be finite dimensional vector spaces. Let $\mathbf{T}$: $\mathbf{V} \to \mathbf{W}$ be a linear transformation $\text{Rank}(\mathbf{T}) + \text{Nullity}(\mathbf{T}) = \dim \mathbf{V}$

TABLE 3.3.3.1: Definition

Result 1	The space of all $m \times n$ matrices over the field $\mathbf{F}$ has dimension mn.
Result 2	Let $\mathbf{V}$ and $\mathbf{W}$ be finite-dimensional vector spaces over the field $\mathbf{F}$ such that $\dim \mathbf{V} = \dim \mathbf{W}$. If $\mathbf{T}$ is a linear transformation from $\mathbf{V}$ into $\mathbf{W}$, then the following are equivalent: (a). $\mathbf{T}$ is invertible. (b). $\mathbf{T}$ is non-singular. (c). $\mathbf{T}$ is onto, that is, range of $\mathbf{T}$ is $\mathbf{W}$.

TABLE 3.3.3.2: Results Used

with x and y real numbers, then

$$\mathbf{T}(z) = \begin{pmatrix} x + 7y & 5y \\ -10y & x - 7y \end{pmatrix}$$

a) Verify that T is a one-one (real) linear transformation of $\mathbf{V}$ into the space of 2×2 real matrices.
Solution: The kronecker product also called as matrix direct product is defined as

$$\mathbf{A} \otimes \mathbf{B} = \begin{pmatrix} a_{11}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ \vdots & \ddots & \vdots \\ a_{m1}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{pmatrix} \tag{3.3.4.1}$$

Also,

$$\mathbf{A} \otimes (\mathbf{B} + \mathbf{C}) = \mathbf{A} \otimes \mathbf{B} + \mathbf{A} \otimes \mathbf{C} \tag{3.3.4.2}$$

$$\mathbf{A} \otimes (k\mathbf{B}) = k(\mathbf{A} \otimes \mathbf{B}) \tag{3.3.4.3}$$

Given,

$$\mathbf{T} : \mathbf{C} \to \mathbf{R}^{2\times2}$$

$$\mathbf{T}(x + iy) = \begin{pmatrix} x + 7y & 5y \\ -10y & x - 7y \end{pmatrix} \tag{3.3.4.4}$$

Let,

$$z = x + iy; \quad w = a + ib; \quad z, w \in \mathbf{C}$$

Also the RHS of (3.3.4.4) can be expressed as,

$$\mathbf{T}(\mathbf{z}) = \left(\begin{pmatrix} 1 & 7 \\ 0 & -10 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad \begin{pmatrix} 0 & 5 \\ 1 & -7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right)$$

$$= \begin{pmatrix} 1 & 7 & 0 & 5 \\ 0 & -10 & 1 & -7 \end{pmatrix} \begin{pmatrix} x & 0 \\ y & 0 \\ 0 & x \\ 0 & y \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} \begin{pmatrix} \mathbf{x} & 0 \\ 0 & \mathbf{x} \end{pmatrix} \tag{3.3.4.5}$$

where $\mathbf{A}$ and $\mathbf{B}$ are block matrices and,

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$

The diagonal block matrix can be expressed as the kronecker product of $\mathbf{I}$ and $\mathbf{x}$

$$\mathbf{I} \otimes \mathbf{x} = \begin{pmatrix} \mathbf{x} & 0 \\ 0 & \mathbf{x} \end{pmatrix} \tag{3.3.4.6}$$

Defining Sets	We define set S and set T as $S = \{(a,b) : a,b \in \mathbb{N}, 1 \leq a \leq m, 1 \leq b \leq n\}, \quad T = \{1,2,...,mn\}$
Defining Bijection	We now define a bijection $\sigma : S \to T$ as $(a,b) \to (a-1)\text{n} + \text{b}$
Defining Function G	We now define a function G from $F^{m\times n}$ to F^{mn} as follows. Let $\mathbf{A} \in F^{m\times n}$. Then map $\mathbf{A}$ to the mn tupple that has $\mathbf{A_{ij}}$ in the $\sigma(i,j)$ position. In other words, $\mathbf{A} \to (\mathbf{A_{11}}, \mathbf{A_{12}}, ..., \mathbf{A_{1n}}, ..., \mathbf{A_{m1}}, \mathbf{A_{m2}}, ..., \mathbf{A_{mn}})$
Proving G to be Linear	Since, addition in $F^{m\times n}$ and in F^{mn} is performed component-wise, $G(\mathbf{A}+\mathbf{B}) = G(\mathbf{A}) + G(\mathbf{B})$ and scalar multiplication in $F^{m\times n}$ and in F^{mn} is also defined as $G(c\mathbf{A}) = cG(\mathbf{A})$.
Proving G to be One-One	$G(\mathbf{A}) = G(\mathbf{B})$ $\implies (\mathbf{A_{11}}, \mathbf{A_{12}}, ..., \mathbf{A_{1n}}, ..., \mathbf{A_{m1}}, \mathbf{A_{m2}}, ..., \mathbf{A_{mn}}) = (\mathbf{B_{11}}, \mathbf{B_{12}}, ..., \mathbf{B_{1n}}, ..., \mathbf{B_{m1}}, \mathbf{B_{m2}}, ..., \mathbf{B_{mn}})$ $\implies \mathbf{A_{i,j}} = \mathbf{B_{ij}} \quad \forall 1 \leq i \leq m, 1 \leq j \leq n$ $\implies \mathbf{A} = \mathbf{B}$
Proving G to be Onto	Since G is one to one, so Null(G) = 0. Thus, by Rank-Nullity Theorem dim(Range(G))= mn, proving G to be a surjective (onto) map as by Result 1 dimension of $F^{m\times n} = mn$
$F^{m\times n} \cong F^{mn}$	Since G has an inverse and is an isomorphism of $\mathbf{T}$. Thus, by Result 2 $F^{m\times n} \cong F^{mn}$

TABLE 3.3.3.3: Proof

Where $\mathbf{I}$ is an identity matrix. (3.3.4.5) can be rewritten as,

$$\mathbf{T}(\mathbf{z}) = \begin{pmatrix}\mathbf{A} & \mathbf{B}\end{pmatrix}(\mathbf{I} \otimes \mathbf{x}) \tag{3.3.4.7}$$

Consider,

$$\mathbf{T}(\alpha\mathbf{z} + \mathbf{w}) = \begin{pmatrix}\mathbf{A} & \mathbf{B}\end{pmatrix}(\mathbf{I} \otimes (\alpha\mathbf{z} + \mathbf{w}))$$

Using properties (3.3.4.2), (3.3.4.3), the above equation can be expressed as,

$$\begin{aligned}\mathbf{T}(\alpha\mathbf{z} + \mathbf{w}) &= \begin{pmatrix}\mathbf{A} & \mathbf{B}\end{pmatrix}(\mathbf{I} \otimes (\alpha\mathbf{z})) + \begin{pmatrix}\mathbf{A} & \mathbf{B}\end{pmatrix}(\mathbf{I} \otimes \mathbf{x}) \\ &= \alpha\begin{pmatrix}\mathbf{A} & \mathbf{B}\end{pmatrix}(\mathbf{I} \otimes \mathbf{z}) + \begin{pmatrix}\mathbf{A} & \mathbf{B}\end{pmatrix}(\mathbf{I} \otimes \mathbf{w}) \\ &= \alpha\mathbf{T}(\mathbf{z}) + \mathbf{T}(\mathbf{w}) \end{aligned} \tag{3.3.4.8}$$

From (3.3.4.8), it can be proved that $\mathbf{T}$ is a linear operator.

b) How would you describe the range of $\mathbb{T}$?
Solution:

$$\mathbb{T} : \mathbf{V} \to \mathbb{R}^{2\times2} \tag{3.3.4.9}$$

where $\mathbb{R}^{2\times2}$,is the space of all 2×2 real matrices

$$\mathbb{T}(z) = \begin{pmatrix} x+7y & 5y \\ -10y & x-7y \end{pmatrix} \tag{3.3.4.10}$$

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \tag{3.3.4.11}$$

$$\mathbb{T}(\mathbf{x}) = \begin{pmatrix} \begin{pmatrix} 1 & 7 \end{pmatrix}\mathbf{x} & \begin{pmatrix} 0 & 5 \end{pmatrix}\mathbf{x} \\ \begin{pmatrix} 0 & -10 \end{pmatrix}\mathbf{x} & \begin{pmatrix} 1 & -7 \end{pmatrix}\mathbf{x} \end{pmatrix} \tag{3.3.4.12}$$

$$= \begin{pmatrix} \begin{pmatrix} 1 & 7 \\ 0 & -10 \end{pmatrix}\mathbf{x} & \begin{pmatrix} 0 & 5 \\ 1 & -7 \end{pmatrix}\mathbf{x} \end{pmatrix} \tag{3.3.4.13}$$

$$\text{Let } \mathbf{A} = \begin{pmatrix} 1 & 7 \\ 0 & -10 \end{pmatrix} \tag{3.3.4.14}$$

$$\mathbf{B} = \begin{pmatrix} 0 & 5 \\ 1 & -7 \end{pmatrix} \tag{3.3.4.15}$$

$$\implies \mathbb{T}(\mathbf{x}) = \begin{pmatrix} \mathbf{A}\mathbf{x} & \mathbf{B}\mathbf{x} \end{pmatrix} \tag{3.3.4.16}$$

$$\mathbb{T}(\mathbf{x}) = \begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} \begin{pmatrix} \mathbf{x} & \mathbf{0}_{2\times1} \\ \mathbf{0}_{2\times1} & \mathbf{x} \end{pmatrix} \tag{3.3.4.17}$$

The kronecker product of $\mathbf{I}, \mathbf{x}$ gives the block matrix

$$\mathbf{I}_{2\times2} \otimes \mathbf{x}_{2\times1} = \begin{pmatrix} \mathbf{x} & \mathbf{0} \\ \mathbf{0} & \mathbf{x} \end{pmatrix}_{4\times2} \tag{3.3.4.18}$$

$$(3.3.4.17) \implies \mathbb{T}(\mathbf{x}) = \begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} \mathbf{I} \otimes \mathbf{x} \tag{3.3.4.19}$$

$$= \begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} \mathbf{I} \otimes \left[\begin{pmatrix} x \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ y \end{pmatrix} \right] \tag{3.3.4.20}$$

$$= x\begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} \mathbf{I} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} \mathbf{I} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.3.4.21}$$

$$\mathbf{I} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{3.3.4.22}$$

$$\mathbf{I} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.3.4.23}$$

Kronecker product in the first term of (3.3.4.21) picks out 1st columns of $\mathbf{A}, \mathbf{B}$ and in the second term picks out 2nd columns of $\mathbf{A}, \mathbf{B}$ so basis for range($\mathbb{T}$) is

$$\left\{ \begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} \mathbf{I} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} \mathbf{I} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} = \tag{3.3.4.24}$$

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 7 & 5 \\ -10 & -7 \end{pmatrix} \right\} \tag{3.3.4.25}$$

$$\text{range}(\mathbb{T}) = \tag{3.3.4.26}$$

$$\text{span of } \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 7 & 5 \\ -10 & -7 \end{pmatrix} \right\} \tag{3.3.4.27}$$

3.3.5. Let V and W be finite-dimensional vector spaces over the field F and let U be an isomorphism of V and W. Prove that $T \to UTU^{-1}$ is an isomorphism of $L(V, V)$ onto $L(W, W)$.

Solution:

Example Let

$$U = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \tag{3.3.5.1}$$

here U is an isomorphism from $\mathbb{R}^{2\times2}$ to $\mathbb{R}^{2\times2}$ since inverse of U exists and

$$U^{-1} = \begin{pmatrix} -2 & -\frac{3}{2} \\ -1 & -\frac{1}{2} \end{pmatrix} \tag{3.3.5.2}$$

Consider

$$T = \begin{pmatrix} -1 & 2 \\ 3 & 1 \end{pmatrix} \in \mathbb{R}^{2\times2} \tag{3.3.5.3}$$

Now

$$UTU^{-1} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} -2 & -1 \\ -\frac{3}{2} & -\frac{1}{2} \end{pmatrix} \tag{3.3.5.4}$$

$$= \begin{pmatrix} -16 & -7 \\ -33 & -14 \end{pmatrix} \in \mathbb{R}^{2\times2} \tag{3.3.5.5}$$

Also inverse exists for T

$$S = T^{-1} = \begin{pmatrix} -\frac{1}{2} & \frac{2}{7} \\ \frac{3}{7} & \frac{1}{7} \end{pmatrix} \tag{3.3.5.6}$$

Since T inverse exists $\mathcal{T}(T) = UTU^{-1}$ is an isomorphism from $\mathbb{R}^{2\times2}$ onto $\mathbb{R}^{2\times2}$.

3.3.6. Let T be a linear operator on the finite-dimensional space $\mathbb{V}$. Suppose there is a linear operator U on $\mathbb{V}$ such that $TU = I$. Prove that T is invertible and $U = T^{-1}$. Give an example which shows that this is false when $\mathbb{V}$ is not finite-dimensional.

Given	$\mathcal{T}(T) : T \to UTU^{-1}$ U is isomorphism of V onto W that means U is $one - one$ $\mathcal{T} : L(V, V) \to L(W, W)$
To prove	$\mathcal{T}$ is isomorphism of $L(V, V)$ onto $L(W, W)$ It is same as proving $\mathcal{T}$ is invertible, because $isomorphim \implies one - one \implies$ *invertible* by definition
Proof	Consider inverse transformation $\mathcal{S} : L(W, W) \to L(V, V)$ $\mathcal{S} : S \to U^{-1}SU$ where $U^{-1}SU$ is a composition of 3 linear transformations $V \xrightarrow{U} W \xrightarrow{S} W \xrightarrow{U^{-1}} V$ Now consider $\mathcal{S}(UTU^{-1})$, $\mathcal{S}(UTU^{-1}) = U^{-1}(UTU^{-1})U = T$ Similarly consider $\mathcal{T}(U^{-1}SU)$, $\mathcal{T}(U^{-1}SU) = U(U^{-1}SU)U^{-1} = S$ $\implies TS = I$ and $ST = I$ we can say $\mathcal{T}$ is invertible since we have found an inverse $\mathcal{S}$ Hence $\mathcal{T}$ is one-one implies $\mathcal{T}$ isomorphism of V onto W

TABLE 3.3.5.1: Proof

Solution: Let $T : \mathbb{V} \to \mathbb{V}$ be a linear operator, where $\mathbb{V}$ is a finite dimensional vectors space and $U : \mathbb{V} \to \mathbb{V}$ is also a linear operator such that,

$$TU = I \tag{3.3.6.1}$$

Where, I is an identity transformation. Now we know that linear transformations are functions. Hence,

$$TU = I \quad \text{is a function} \tag{3.3.6.2}$$

$$\implies I : \mathbb{V} \to \mathbb{V} \tag{3.3.6.3}$$

Such that $T(V) = V$. Defining $TU : \mathbb{V} \to \mathbb{V}$ to be a linear operator, we have,

$$T[U(V_i)] = V_i \qquad [V_i \in \mathbb{V}] \tag{3.3.6.4}$$

Now we show in the below Table that T is one-one and onto as follows,

Hence we get from Table 3.3.6.1 that, T is invertible. Hence we get the following,

$$TT^{-1} = I \tag{3.3.6.5}$$

Where T^{-1} is an inverse function of linear operator T. Hence,

$$TT^{-1} = I = TU \tag{3.3.6.6}$$

$$\implies T^{-1}(TT^{-1}) = T^{-1}(TU) \tag{3.3.6.7}$$

$$\implies T^{-1}(I) = IU \tag{3.3.6.8}$$

$$\implies T^{-1} = U \tag{3.3.6.9}$$

Hence from (3.3.6.9) it is proven that T is invertible and $T^{-1} = U$

Example: Let D be the differential operator $D : \mathbb{V} \to \mathbb{V}$ where $\mathbb{V}$ is a space of polynomial functions in one variable x over $\mathbb{R}$ as follows,

$$D(c_0 + c_1x + \cdots + c_nx^n) = c_1 + c_2'x + \cdots + c_n'x^{n-1} \tag{3.3.6.10}$$

We first prove that the vector space $\mathbb{V}$ is infinite dimensional.
Suppose to the contrary that $\mathbb{V}$ is finite dimensional vector space and is given by the span of k polynomials in $\mathbb{V}$ as follows,

$$span(\mathbb{V}) = \{p_1, p_2, \ldots, p_k\} \tag{3.3.6.11}$$

Also let m be the maximum of the degree of these k polynomials in (3.3.6.11). Now let an

linear transformation	Let V and W be vector spaces over field F. A **linear transformation** V into W is a function T from V into W such that $T(c\alpha + \beta) = c(T\alpha) + T\beta$ for all α and β in V and all scalars in c in F.
isomorphism	If V and W are vector spaces over the field F, any $one - one$ linear transformation $T : V \to W$ is called **isormorphism of** V **onto** W
one-one	A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **one-one** if for every $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{AX} = \mathbf{b}$ has atmost one solution in $\mathbb{R}^n$. Equivalently, if $T(\mathbf{u}) = T(\mathbf{v})$, then $u = v$. By definition, all *invertible* transformations are **one-one**
invertible	A linear transformation $T : V \to W$ is **invertible** if there exists another linear transformation $U : W \to V$ such that UT is the *identity* transformation on V and TU is the identity transformation on W. T is **invertible** if and only if T is $one - one$ and $onto$

TABLE 3.3.5.2: Definitions

Proof	**Conclusion**
Let $\mathbf{V_1}, \mathbf{V_2} \in \mathbb{V}$ then, If $\mathbf{V_1} \neq \mathbf{V_2}$ then, $T[U(\mathbf{V_1})] \neq T[U(\mathbf{V_2})]$	T is one-one function
T is linear operator on finite dimensional vector space	T is onto function

TABLE 3.3.6.1: Proof of Invertibility of transformation

element of the vector space $\mathbb{V}$ be,

$$cx^{m+1} \in \mathbb{V} \tag{3.3.6.12}$$

As maximum degree of the basis of $\mathbb{V}$ is m hence cx^{m+1} cannot be represented by any linear combination of the basis of $\mathbb{V}$. If $\mathbb{F}$ is field corresponding to $\mathbb{V}$ then we have,

$$cx^{m+1} \neq \sum_{i=1}^{k} \alpha_i p_i \quad [\alpha_i \in \mathbb{F}\ \forall i] \tag{3.3.6.13}$$

Hence, cx^{m+1} is not in the span of $p_1, p_2, \ldots, p_k$. Hence, $\mathbb{V}$ is infinite dimensional vector space.
Next we prove that D is not one-one operator.
Let, two different elements from the vector space $\mathbb{V}$ be as follows,

$$c_1 + x^m \in \mathbb{V} \tag{3.3.6.14}$$
$$c_2 + x^m \in \mathbb{V} \tag{3.3.6.15}$$

From definition (3.3.6.10) of operator D we have,

$$D(c_1 + x^m) = mx^{m-1} \tag{3.3.6.16}$$
$$D(c_2 + x^m) = mx^{m-1} \tag{3.3.6.17}$$

From (3.3.6.16) and (3.3.6.17),

$$c_1 + x^m \neq c_2 + x^m \tag{3.3.6.18}$$
$$D(c_1 + x^m) = D(c_2 + x^m) \tag{3.3.6.19}$$

Hence from (3.3.6.19) we see that D is not One-One operator.
And, $U : \mathbb{V} \to \mathbb{V}$ is another linear operator

such that,

$$U(c_0 + c_1x + \cdots + c_nx^n) = c_0x + c_1\frac{x^2}{2} + \cdots + c_n\frac{x^{n+1}}{n+1} \quad (3.3.6.20)$$

Now, $DU : \mathbb{V} \to \mathbb{V}$ is a linear operator such that,

$$DU(c_0 + c_1x + \cdots + c_nx^n) \quad (3.3.6.21)$$

$$= D[U(c_0x + c_1\frac{x^2}{2} + \cdots + c_n\frac{x^{n+1}}{n+1})] \quad (3.3.6.22)$$

$$= D[c_0x + c_1\frac{x^2}{2} + \cdots + c_n\frac{x^{n+1}}{n+1}] \quad (3.3.6.23)$$

$$= c_0 + c_1\frac{2x}{2} + \cdots + c_n\frac{(n+1)x^n}{n+1} \quad (3.3.6.24)$$

$$= c_0 + c_1x + \cdots + c_nx^n \quad (3.3.6.25)$$

Hence, from (3.3.6.25),

$$DU = I \quad (3.3.6.26)$$

Again $UD : \mathbb{V} \to \mathbb{V}$ is a linear operator such that,

$$UD(c_0 + c_1x + \cdots + c_nx^n) \quad (3.3.6.27)$$

$$= U[D(c_0x + c_1\frac{x^2}{2} + \cdots + c_n\frac{x^{n+1}}{n+1})] \quad (3.3.6.28)$$

$$= U[c_1 + c_2'x + \cdots + c_n'x^{n-1}] \quad (3.3.6.29)$$

$$= c_1x + c_2\frac{x^2}{2} + \cdots + c_n\frac{x^n}{n} \quad (3.3.6.30)$$

Hence, from (3.3.6.30),

$$UD \neq I \quad (3.3.6.31)$$

Hence, from (3.3.6.26) and (3.3.6.31), D is not invertible.

3.4 Representation of Transformations by Matrices

3.4.1. Let T be the linear operator on $\mathbb{C}^2$ defined by $T(x_1, x_2) = \left(x_1, 0\right)$. Let β be the standard ordered basis for $\mathbb{C}^2$ and $\beta' = \{\alpha_1, \alpha_2\}$ be the ordered basis defined by $\alpha_1 = \left(1, i\right), \alpha_2 = \left(-i, 2\right)$.

a) What is the matrix of T in the ordered basis $\{\alpha_2, \alpha_1\}$?

Solution: Transformation T from $\mathbb{C}^2$ to $\mathbb{C}^2$. Let

$$\mathbf{e_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{e_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (3.4.1.1)$$

$$\beta = \begin{pmatrix} \mathbf{e_1} & \mathbf{e_2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (3.4.1.2)$$

$$\beta' = \begin{pmatrix} \alpha_2 & \alpha_1 \end{pmatrix} = \begin{pmatrix} -i & 1 \\ 2 & i \end{pmatrix} \quad (3.4.1.3)$$

T in the ordered basis β is:

$$[\mathbf{T}]_\beta = \begin{pmatrix} -i & 1 \\ 2 & i \end{pmatrix} \quad (3.4.1.4)$$

T is defined by

$$T(\mathbf{x}) = \mathbf{A}\mathbf{x} \quad (3.4.1.5)$$

$$T(\mathbf{x}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\mathbf{x} \quad (3.4.1.6)$$

$$T(\alpha_2) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -i \\ 2 \end{pmatrix} = \begin{pmatrix} -i \\ 0 \end{pmatrix} \quad (3.4.1.7)$$

$$T(\alpha_1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.4.1.8)$$

$$[\alpha]_\beta = \begin{pmatrix} -i & 1 \\ 0 & 0 \end{pmatrix} \quad (3.4.1.9)$$

The matrix of T in the ordered basis $\{\alpha_2, \alpha_1\}$ is given as:

$$[\mathbf{T}_\alpha]_\beta = [\mathbf{T}]_\beta[\alpha]_\beta = \begin{pmatrix} -i & 1 \\ 2 & i \end{pmatrix}\begin{pmatrix} -i & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & -i \\ -2i & 2 \end{pmatrix} \quad (3.4.1.10)$$

3.4.2. Let V be a two-dimensional vector space over the field F and let B be an ordered basis for V. If T is a linear operator on V and

$$[T]_B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (3.4.2.1)$$

Prove that

$$T^2 - (a+d)T + (ad - bc)I = 0 \quad (3.4.2.2)$$

Solution: Here T is a linear operator on V and B is an ordered basis of V. Let us consider $[T]_B = A$, so $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Now, the characteristic equation of A is:

$$\left|A - \lambda I\right| = 0 \quad (3.4.2.3)$$

$$\implies \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0 \quad (3.4.2.4)$$

$$\implies \lambda^2 - (a+d)\lambda + (ad-bc) = 0 \quad (3.4.2.5)$$

According to the Cayley-Hamilton's Theorem, every square matrix satisfies its own characteristic equation. Here A is a 2x2 square matrix, so it should also satisfy its characteristic equation. Now,

$$\lambda^2 - (a+d)\lambda + (ad-bc) = 0 \quad (3.4.2.6)$$

$$\implies A^2 - (a+d)A + (ad-bc)I = 0 \quad (3.4.2.7)$$

We can also write the equation 3.4.2.7 as:

$$[T]_B^2 - (a+d)[T]_B + (ad-bc)I = 0 \quad (3.4.2.8)$$

$$\text{or, } T^2 - (a+d)T + (ad-bc)I = 0 \quad (3.4.2.9)$$

3.4.3. Let T be a linear operator on $\mathbb{R}^3$, the matrix of which in the standard ordered basis is,

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 3 & 4 \end{pmatrix} \quad (3.4.3.1)$$

Find a basis for the range of T and a basis for the null-space of T.

Solution: The basis of the range of linear transformation T is the basis of the column-space of $\mathbf{A}$ or basis of $C(\mathbf{A})$. Hence the basis of the range of the linear transformation T is derived by reducing $\mathbf{A}$ into Reduced-Row Echelon form as follows,

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 3 & 4 \end{pmatrix} \xleftrightarrow[R_1=R_1-2R_2]{R_3=R_3+R_1} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 5 & 5 \end{pmatrix} \quad (3.4.3.2)$$

$$\xleftrightarrow{R_3=R_3-5R_2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad (3.4.3.3)$$

From (3.4.3.3) the basis of the range of linear operator T are as follows,

$$\mathbf{a_1} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \quad (3.4.3.4)$$

$$\mathbf{a_2} = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \quad (3.4.3.5)$$

Again, the basis for null-space of linear operator T or $N(\mathbf{A})$ is a solution of the equation $\mathbf{Ax} = 0$. From (3.4.3.3) we have,

$$\mathbf{Ax} = 0 \quad (3.4.3.6)$$

$$\implies \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \quad (3.4.3.7)$$

Setting the value of the free variable $x_3 = 1$ we get the solution,

$$\mathbf{x} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad (3.4.3.8)$$

Hence, the basis of the null-space of the linear operator T is given by,

$$\mathbf{b} = \begin{pmatrix} 1 & -1 & 1 \end{pmatrix} \quad (3.4.3.9)$$

3.4.4. Let $\mathbb{T}$ be the linear operator on $\mathbb{R}^3$ defined by

$$\mathbb{T}(x_1, x_2, x_3) = \quad (3.4.4.1)$$

$$(3x_1 + x_3, -2x_1 + x_2, -x_1 + 2x_2 + 4x_3) \quad (3.4.4.2)$$

What is the matrix of $\mathbb{T}$ in the standard ordered basis of $\mathbb{R}^3$?

Solution:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad (3.4.4.3)$$

$$\mathbb{T}(\mathbf{x}) = \mathbf{Tx} \quad (3.4.4.4)$$

The matrix of $\mathbb{T}$ in the standard ordered basis from (3.4.4.2) is

$$\mathbf{T} = \begin{pmatrix} 3 & 0 & 1 \\ -2 & 1 & 0 \\ -1 & 2 & 4 \end{pmatrix} \quad (3.4.4.5)$$

3.4.5. The linear operator $\mathbf{T}$ on $\mathbf{R}^2$ defined by

$$\mathbf{T}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \quad (3.4.5.1)$$

is represented by the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (3.4.5.2)$$

3.4.6. Prove that if $\mathbf{S}$ is a linear operator on $\mathbf{R}^2$ such that $\mathbf{S}^2 = \mathbf{S}$, then $\mathbf{S} = \mathbf{0}$, or $\mathbf{S} = \mathbf{I}$, or there is

an ordered basis $\mathbf{B}$ for $\mathbf{R}^2$ such that $[\mathbf{S}]_B = \mathbf{A}$.
Solution: If a linear operator $\mathbf{S}$ is defined on $\mathbf{R}^2$ such that $\mathbf{S}^2 = \mathbf{S}$, then

$$\mathbf{S}^2 - \mathbf{S} = \mathbf{0} \tag{3.4.6.1}$$
$$\mathbf{S}(\mathbf{S} - \mathbf{I}) = \mathbf{0} \tag{3.4.6.2}$$
$$\implies \mathbf{S} = \mathbf{0}, \mathbf{S} = \mathbf{I} \tag{3.4.6.3}$$

The transformation of a vector $\mathbf{x} \in \mathbf{R}^2$ can be represented as

$$\mathbf{S}\mathbf{x} = \mathbf{y} \tag{3.4.6.4}$$
$$\implies \mathbf{S}(\mathbf{S}\mathbf{x}) = \mathbf{S}\mathbf{y} \tag{3.4.6.5}$$
$$\implies \mathbf{S}^2\mathbf{x} = \mathbf{S}\mathbf{y} \tag{3.4.6.6}$$
$$\implies \mathbf{S}\mathbf{x} = \mathbf{S}\mathbf{y} \tag{3.4.6.7}$$
$$\implies \mathbf{x} = \mathbf{y} \tag{3.4.6.8}$$

Therefore the transformation of a vector $\mathbf{x} \in \mathbf{R}^2$ can be given as

$$\mathbf{S}\mathbf{x} = \mathbf{x}\ \forall\ \mathbf{x} \in \mathbf{R}^2 \tag{3.4.6.9}$$

Consider the ordered basis set

$$B = \{\epsilon_1, \epsilon_2\} \in \mathbf{R}^2 \tag{3.4.6.10}$$

and if

$$[\mathbf{S}]_{\mathbf{B}} = \mathbf{A} \tag{3.4.6.11}$$
$$\implies [\mathbf{S}]_{\mathbf{B}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{3.4.6.12}$$

Thus we can re-write the column vectors of $[\mathbf{S}]_{\mathbf{B}}$ using (3.4.6.9) as

$$\mathbf{S}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.4.6.13}$$
$$\mathbf{S}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.4.6.14}$$

Therefore, any vector $\mathbf{x}$ in column space of $[\mathbf{S}]_{\mathbf{B}}$ can be uniquely expressed by $\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$, hence it forms the basis for column space of $[\mathbf{S}]_{\mathbf{B}}$. Therefore one of the basis vector of B is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The other basis vector can be any vector which is linearly independent to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. One of the ordered basis set can be

$$B = \left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\} \tag{3.4.6.15}$$

3.4.7. Let V be an n-dimensional vector space over the field F, and let $\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\}$ be an ordered basis for V then there is a unique linear operator T on V such that

$$T\alpha_j = \alpha_{j+1}, j = 1, \ldots, n-1 \tag{3.4.7.1}$$
$$T\alpha_n = 0. \tag{3.4.7.2}$$

a) What is the matrix A of T in the ordered basis $\mathcal{B}$?
Solution: Given that,

$$T : V \to V \tag{3.4.7.3}$$
$$[T(\alpha)]_{\mathcal{B}} = A[\alpha]_{\mathcal{B}} \tag{3.4.7.4}$$
$$T\alpha_j = \alpha_{j+1} \tag{3.4.7.5}$$
$$T\alpha_n = 0 \tag{3.4.7.6}$$

where $j = 1, \ldots, n-1$. The matrix A of T in the ordered basis $\mathcal{B}$ is given by,

$$\implies A = \begin{pmatrix} [T\alpha_1]_{\mathcal{B}} & \cdots & [T\alpha_n]_{\mathcal{B}} \end{pmatrix} \tag{3.4.7.7}$$

For $j = 1, \ldots, n-1$ we have,

$$T\alpha_j = \alpha_{j+1} \tag{3.4.7.8}$$

we can write,

$$T\alpha_j = 0\alpha_1 + \ldots + 0\alpha_j + 1\alpha_{j+1} + \ldots + 0\alpha_n \tag{3.4.7.9}$$
$$\implies [T\alpha_j]_{\mathcal{B}} = (0, \ldots, 0, 1, 0, \ldots, 0)^T \tag{3.4.7.10}$$

where 1 is in $(j+1)$th position. Now,

$$T\alpha_n = 0 \tag{3.4.7.11}$$
$$\implies [T\alpha_n]_{\mathcal{B}} = 0 \tag{3.4.7.12}$$

Thus from (3.4.7.7), (3.4.7.10) and (3.4.7.12) we get matrix A of T in the ordered basis $\mathcal{B}$ as,

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \tag{3.4.7.13}$$

3.4.8. Let V and W be finite-dimensional vector spaces over the field F and let T be a linear transformation from V into W. If

$$\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\} \text{ and } \mathcal{B}' = \{\beta_1, \ldots, \beta_m\} \tag{3.4.8.1}$$

are ordered bases for V and W, respectively, define the linear transformation $E^{p,q}$ as in the proof of Theorem 5: $E^{p,q}(\alpha_i) = \delta_{iq}\beta_p$. Then the $E^{p,q}$, $1 \leq p \leq m$, $1 \leq q \leq n$, form a basis for $L(V, W)$ and so

$$T = \sum_{p=1}^{m}\sum_{q=1}^{n} A_{pq}E^{p,q} \tag{3.4.8.2}$$

for certain scalars A_{pq} (the coordinates of T in this basis for $L(V, W)$). Show that the matrix A with entries $A(p, q) = A_{pq}$ is precisely the matrix of T relative to the pair $\mathcal{B}, \mathcal{B}'$.

Solution: Given,

$$T = \sum_{p=1}^{m}\sum_{q=1}^{n} A_{pq}E^{p,q} \tag{3.4.8.3}$$

where

$$E^{p,q}(\alpha_i) = \begin{cases} \beta_p & p = i \\ 0 & \text{otherwise} \end{cases} \tag{3.4.8.4}$$

$$= \delta_{iq}\beta_p \tag{3.4.8.5}$$

where $\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\}$ is basis of V and $\mathcal{B}' = \{\beta_1, \ldots, \beta_n\}$ is basis of W.

Consider a vector $\mathbf{x} = \{x_1, x_2, \ldots, x_n\} \in V$,

$$\mathbf{x} = \sum_{q=1}^{n} x_q\alpha_q \tag{3.4.8.6}$$

$$\therefore E^{p,q}(\mathbf{x}) = \sum_{q=1}^{n} x_qE^{p,q}(\alpha_q) \tag{3.4.8.7}$$

$$= x_q\delta_{iq}\beta_p \tag{3.4.8.8}$$

Consider $T(\mathbf{x})$, from (3.4.8.3)

$$T(\mathbf{x}) = \sum_{p=1}^{m}\sum_{q=1}^{n} A_{pq}E^{p,q}(\mathbf{x}) \tag{3.4.8.9}$$

Substitute (3.4.8.8) in (3.4.8.9)

$$T(\mathbf{x}) = \sum_{p=1}^{m}\sum_{q=1}^{n} A_{pq}x_p\delta_{iq}\beta_q \tag{3.4.8.10}$$

From (3.4.8.5), $\delta_{iq}\beta_q$ is the transformation of basis of V to W. Hence $T : V \to W$ is

$$T = \begin{pmatrix} \sum_{p=1}^{n} A_{p1}x_p \\ \sum_{p=1}^{n} A_{p2}x_p \\ \vdots \\ \sum_{p=1}^{n} A_{pm}x_p \end{pmatrix} \tag{3.4.8.11}$$

$$T = \begin{pmatrix} A_{11} & A_{12} & \ldots & A_{1n} \\ A_{21} & A_{22} & \ldots & A_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ A_{m1} & A_{m2} & \ldots & A_{mn} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \tag{3.4.8.12}$$

$\therefore$ We showed that the matrix A with entries $A(p, q) = A_{pq}$ is precisely the matrix of T relative to the pair $\mathcal{B}, \mathcal{B}'$.

3.5 Linear Functionals

3.5.1. Let $\mathbb{V}$ be the vector space of all 2×2 matrices over the field of real numbers, and let

$$\mathbf{B} = \begin{pmatrix} 2 & -2 \\ -1 & 1 \end{pmatrix} \tag{3.5.1.1}$$

Let $\mathbb{W}$ be the subspace of $\mathbb{V}$ consisting of all $\mathbf{A}$ such that $\mathbf{AB} = 0$. Let f be a linear functional on $\mathbb{V}$ which is in the annihilator of $\mathbb{W}$. Suppose that $f(\mathbf{I}) = 0$ and $f(\mathbf{C}) = 3$, where $\mathbf{I}$ is the 2×2 identity matrix and

$$\mathbf{C} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.5.1.2}$$

Find $f(\mathbf{B})$

Solution: