Geometry through Linear Algebra

G V V Sharma*

Contents

***Abstract*—This book provides a vector approach to analytical geometry. The content and exercises are based on S L Loney's book on Plane Coordinate Geometry.**

1

1.1. Consider the vector space $\mathbb{P}_n$ of real polynomials in x of degree $\le n$. Define

$$T : \mathbb{P}_2 \to \mathbb{P}_3 \tag{1.1.1}$$

by

$$(Tf)(x) = \int_0^x f(t)\,dt + f'(x). \tag{1.1.2}$$

Then find the matrix representation of T with respect to the bases

$$\left\{1, x, x^2\right\} \text{ and } \left\{1, x, x^2, x^3\right\} \tag{1.1.3}$$

1.2. Let $P_A(x)$ denote the characteristic polynomial of a matrix A. Then for which of the following matrices is

$$P_A(x) - P_{A^{-1}}(x) \tag{1.2.1}$$

a constant?

a) $\begin{pmatrix} 3 & 3 \\ 2 & 4 \end{pmatrix}$

b) $\begin{pmatrix} 4 & 3 \\ 2 & 3 \end{pmatrix}$

c) $\begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}$

d) $\begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$

1.3. Which of the following matrices is not diagonalizable over $\mathbb{R}$?

a) $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

c) $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

d) $\begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$

1.4. What is the rank of the following matrix?

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} \tag{1.4.1}$$

1.5. Let V denote the vector space of real valued continuous functions on the close interval $[0, 1]$. Let W be the subspace of V spanned by $\{\sin x, \cos x, \tan x\}$. Find the dimension of W over $\mathbb{R}$.

1.6. Let V be the vector space of polynomials in the variable t of degree at most 2 over $\mathbb{R}$. An inner product on V is defined by

$$f^T g = \int_0^1 f(t)g(t)\,dt, \quad f, g \in V. \tag{1.6.1}$$

*The author is with the Department of Electrical Engineering, Indian Institute of Technology, Hyderabad 502285 India e-mail: gadepall@iith.ac.in.

Let

$$W = span\left\{1 - t^2, 1 + t^2\right\} \tag{1.6.2}$$

and $W^{\perp}$ be the orthogonal complement of W in V. Which of the following conditions is satisfied for all $h \in W^{\perp}$?

a) h is an even function
b) h is an odd function
c) $h(t) = 0$ has a real solution
d) $h(0) = 0$

1.7. Consider solving the following system by Jacobi iteration scheme

$$\begin{pmatrix} 1 & 2m & -2m \\ n & 1 & n \\ 2m & 2m & 1 \end{pmatrix} \begin{pmatrix} x \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \tag{1.7.1}$$

where $m, n \in \mathbb{Z}$. With any initial vector, the scheme converges provided m, n satisfy

a) $m + n = 3$
b) $m > n$
c) $m < n$
d) $m = n$

1.8. Consider a Markov Chain with state space $\{0, 1, 2, 3, 4\}$ and transition matrix

$$P = \begin{array}{c c} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \tag{1.8.1}$$

Then find

$$\lim_{n \to \infty} p_{23}^{(n)} \tag{1.8.2}$$

1.9. Let $L(\mathbb{R})^n$ be the space of $\mathbb{R}-$linear maps from $\mathbb{R}^n$ to $\mathbb{R}^n$. If $Ker(T)$ denotes the kernel of T then which of the following are true?

a) There exists $T \in L\left(\mathbb{R}^5\right)\ \{0\}$ such that $Range(T) = Ker(T)$
b) There does not exist $T \in L\left(\mathbb{R}^5\right)\ \{0\}$ such that $Range(T) = Ker(T)$
c) There exists $T \in L\left(\mathbb{R}^6\right)\ \{0\}$ such that $Range(T) = Ker(T)$
d) There does not exist $T \in L\left(\mathbb{R}^6\right)\ \{0\}$ such that $Range(T) = Ker(T)$

1.10. Let V be a finite dimensional vector space over $\mathbb{R}$ and $T : V \to V$ be a linear map. Can you always write $T = T_2 \circ T_1$ for some linear maps

$$T_1 : V \to W, T : W \to V, \tag{1.10.1}$$

where W is some finite dimensional vector space such that

a) both T_1 and T_2 are onto
b) both T_1 and T_2 are one to one
c) T_1 is onto, T_2 is one to one
d) T_1 is one to one, T_2 is onto

1.11. Let $A = \left[a_{ij}\right]$ be a 3×3 complex matrix. Identify the correct statements

a) $det\left[(-1)^{i+j} a_{ij}\right] = det(A)$
b) $det\left[(-1)^{i+j} a_{ij}\right] = -det(A)$
c) $det\left[\left(\sqrt{-1}\right)^{i+j} a_{ij}\right] = det(A)$
d) $det\left[\left(\sqrt{-1}\right)^{i+j} a_{ij}\right] = -det(A)$

1.12. Let

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n \tag{1.12.1}$$

be a non-constant polynomial of degree $n \geq 1$. Consider the polynomial

$$q(x) = \int_0^x p(t)\, dt, r(x) = \frac{d}{dx} p(x) \tag{1.12.2}$$

Let V denote the real vector space of all polynomials in x. Then which of the following are true?

a) q and r are linearly independent in V
b) q and r are linearly dependent in V
c) x^n belongs to the linear span of q and r
d) x^{n+1} belongs to the linear span of q and r.

1.13. Let $M_n(\mathbb{R})$ be the ring of $n \times n$ matrices over $\mathbb{R}$. Which of the following are true for every $n \geq 2$?

a) there exist matrices $A, B \in M_n(\mathbb{R})$ such that $AB - BA = I_n$, where I_n denotes the identity matrix.
b) If $A, B \in M_n(\mathbb{R})$ and $AB = BA$, then A is diagonalisable over $\mathbb{R}$ if and only if B is diagonalisable over $\mathbb{R}$.
c) If $A, B \in M_n(\mathbb{R})$, then AB and BA have the same minimal polynomial.
d) If $A, B \in M_n(\mathbb{R})$, then AB and BA have the same eigenvalues in $\mathbb{R}$.

1.14. Consider a matrix

$$A = \left[a_{ij}\right], 1 \leq i, j \leq 5 \tag{1.14.1}$$

such that

$$a_{ij} = \frac{1}{n_i + n_j + 1}, \quad n_i, n_j \in \mathbb{N} \qquad (1.14.2)$$

Then in which of the following cases A is a positive definite matrix?

a) $n_i = 1 \forall i = 1, 2, 3, 4, 5.$
b) $n_1 < n_2 < \cdots < n_5.$
c) $n_1 = n_2 = \cdots = n_5.$
d) $n_1 > n_2 > \cdots > n_5.$

1.15. For a nonzero $w \in \mathbb{R}^n$, define

$$T_w : \mathbb{R}^n \to \mathbb{R}^n \qquad (1.15.1)$$

by

$$T_w = v - \frac{2v^T w}{w^T w} w, \quad v \in \mathbb{R}^n \qquad (1.15.2)$$

Which of the following are true?

a) $det(T_w) = 1$
b) $T_w(v_1)_w^T(v_2) = v_1^T v_2 \forall v_1, v_2 \in \mathbb{R}^n$
c) $T_w = T_w^{-1}$
d) $T_{2w} = 2T_w$

1.16. Consider the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad (1.16.1)$$

over the field $\mathbb{Q}$ of rationals. Which of the following matrices are of the form $P^T AP$ for suitable 2×2 invertible matrix P over $\mathbb{Q}$?

a) $\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$
b) $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$
c) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
d) $\begin{pmatrix} 3 & 4 \\ 4 & 5 \end{pmatrix}$

1.17. Consider a Markov Chain with state space $\{0, 1, 2\}$ and transition matrix

$$P = \begin{array}{c|c} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{pmatrix} \frac{1}{4} & \frac{5}{8} & \frac{1}{8} & \\ \frac{1}{4} & 0 & \frac{3}{4} & \\ 0 & \frac{1}{2} & \frac{3}{8} & \frac{1}{8} \end{pmatrix} \end{array} \qquad (1.17.1)$$

Then which of the following are true?

a) $\lim_{n\to\infty} p_{12}^{(}n) = 0$
b) $\lim_{n\to\infty} p_{12}^{(}n) = \lim_{n\to\infty} p_{21}^{(}n)$
c) $\lim_{n\to\infty} p_{22}^{(}n) = \frac{1}{8}$
d) $\lim_{n\to\infty} p_{21}^{(}n) = \frac{1}{3}$