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ENTRANCE EXAMINATION, 2018

Ph.D. IN MATHEMATICAL SCIENCES

[Field of Study Code : MATH (897)]

Time Allowed : 3 hours

Maximum Marks : 100

INSTRUCTIONS FOR CANDIDATES

(i) All questions are compulsory.

(ii) There are 10 questions in **Section—A** and 8 questions in each of **Sections B** and **C**.

(iii) Answers must be written in the space provided after each question. Answers that are written elsewhere will not be evaluated.

(iv) A question in **Section—A** may have more than one true option. Your answer will be considered correct provided it consists only of all the true options. A correct answer will be awarded **+2** marks and an incorrect/incomplete answer or an unattempted question will get **0** mark.

(v) Each question in **Section—B** carries 4 marks; each question in **Section—C** carries 6 marks. Answers to all the questions in **Sections B** and **C** must be **justified with mathematical reasoning**.

(vi) In the following, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ denote, respectively, the set of natural numbers, the set of integers, the set of rational numbers, the set of real numbers and the set of complex numbers.

(vii) $\mathbb{C}^n$, $\mathbb{R}^n$ and its subsets are assumed to have the usual topology arising from the usual Euclidean norm unless mentioned otherwise.

(viii) For sets A and B, let $A \setminus B := \{x \in A \mid x \notin B\}$. For $a, b \in \mathbb{R}$, let $[a, b] := \{x \in \mathbb{R} \mid a \le x \le b\}$, $(a, b] := [a, b] \setminus \{a\}$ and $(a, b) := [a, b] \setminus \{a, b\}$.

(ix) Extra pages are attached at the end of the question paper for Rough Work.

(x) **Use of calculator, smart phone, mobile phone or any other digital device is strictly not permitted.**

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Ph.D. IN MATHEMATICAL SCIENCES

SUBJECT . . . .
(Field of Study/Language)

FIELD OF STUDY CODE . . . .

NAME OF THE CANDIDATE . . . .

. . . .

REGISTRATION NO.

CENTRE OF EXAMINATION . . . .

. . . .

DATE . . . .

(Signature of Candidate)

(Signature of Invigilator)

(Signature and Seal of Presiding Officer)

For Official Use Only

Not to be filled in by the candidate

Final Grading Table

Total of **Section—A**	
Total of **Section—B**	
Total of **Section—C**	
Grand Total	

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Not to be filled in by the candidate

Grading Table for Section—A

Question Nos.	**Marks Awarded**	**Question Nos.**	**Marks Awarded**
A1		**A2**	
A3		**A4**	
A5		**A6**	
A7		**A8**	
A9		**A10**	
		Total (Out of 20)	

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Grading Table for Sections B and C

Question Nos.	**Marks Awarded**	**Question Nos.**	**Marks Awarded**
B1		**C1**	
B2		**C2**	
B3		**C3**	
B4		**C4**	
B5		**C5**	
B6		**C6**	
B7		**C7**	
B8		**C8**	
Total (Out of 32)		**Total** (Out of 48)	

SECTION—A

Each question carries 2 marks.

A1. What is the number of zeroes at the end of (2017)!?

Answer :

A2. Five friends went to A to Z ice-cream shop that was selling 26 flavours of ice cream to have one scoop of ice cream each. One friend asked everyone which flavour of ice-cream they wanted and went to the counter to place the order. How many distinct orders of ice creams can be placed?

Answer :

A3. Let A, B, C and D be finite sets and $\varphi : A \to B$, $\psi : B \to D$, $\theta : A \to C$ and $\eta : C \to D$ be maps such that $\psi \circ \varphi = \eta \circ \theta$. Consider the following statements :

(a) φ is surjective

(b) B and D have the same number of elements

(c) θ is injective

(d) A has at least as many elements as in Image (ψ)

Which of the above statements are correct?

Answer :

A4. Is $\mathbb{Z}/899\mathbb{Z}$ an integral domain? Provide your answer in **Yes** or **No** below.

Answer :

A5. Suppose G is a group in which every proper subgroup is cyclic. Does this imply that G is cyclic? Provide your answer in **Yes** or **No** below.

Answer :

A6. Let $S \subset \mathbb{R}^2$ be defined by $S := \{(t\cos(t), t\sin(t)) \mid 0 < t \le 2\pi\}$. Consider the following statements :

(a) S is bounded

(b) S contains all its limit points

(c) S is connected

(d) S is not path-connected

Which of the above statements are correct?

Answer :

A7. Let X be a normed linear space. Which of the following statements are always true?

(a) X is locally compact with respect to the norm topology

(b) Every linear map $\varphi : X \to \mathbf{C}$ is continuous

(c) X is homeomorphic to its open unit ball $B := \{x \in X \mid \|x\| < 1\}$

(d) X is not homeomorphic to its closed unit ball $\overline{B} := \{x \in X \mid \|x\| \leq 1\}$

Answer :

A8. Let $E \subset \mathbb{R}$ be a Lebesgue measurable set and let m denote the Lebesgue measure. Which of the following statements are true?

(a) If $m(E) = \infty$, then E is dense in $\mathbb{R}$

(b) If $m(E) = 0$, then E has empty interior

(c) If $m(E) = 0$, then $E \subset \mathbb{Q}$

(d) If $m(E) = \infty$, then $E \cap \mathbb{Q}^c \neq \varnothing$

Answer :

A9. What is the value of $\lim\limits_{x \to 0+} \left(1+\dfrac{1}{x}\right)^x$?

Answer :

A10. The series

$$\sum_{n=1}^{\infty} \frac{z^n}{n}$$

(a) converges for all z with $|z| = 1$, except at $z = 1$

(b) converges for all z with $|z| = 1$, except at $z = 1, -1$

(c) diverges for all z with $|z| = 1$

(d) diverges for all z with $|z| = 1$, except at $z = -1$

Pick the correct options.

Answer :

SECTION—B

Each question carries 4 marks.

B1. Show that $\sqrt[3]{5\sqrt{2}+7}+\sqrt[3]{5\sqrt{2}-7}=2\sqrt{2}$. (*Hint* : First show that the given number is a root of the polynomial $x^3-3x-10\sqrt{2}$.)

B2. For $x \in (0, \infty)$, define $f(x) := x^{x+1}$ and $g(x) := (x+1)^x$. Determine the values of x for which $f(x) > g(x)$.

B3. Let $\mathcal{T}$ be a collection of subsets of $\mathbb{Z}$ containing $\varnothing$, $\mathbb{Z}$ and non-empty subsets U of $\mathbb{Z}$ which are unions of sets of the form

$$S(a, b) := \{an + b \mid n \in \mathbb{Z}\}$$

for $a, b \in \mathbb{Z}$ and $a \neq 0$. Show that $\mathcal{T}$ defines a topology on $\mathbb{Z}$. Prove or disprove that $\mathcal{T}$ is Hausdorff.

B4. Let X and Y be sets, $\varphi: X \to \mathbb{N}$ and $\psi: \mathbb{N} \to Y$ be maps such that their composite $\psi \circ \varphi: X \to Y$ is surjective. Prove or disprove the following assertion :

X and Y are both countable.

B5. Let $f : \mathbf{R}^3 \to \mathbf{R}^3$ be a map such that $\|f(x) - f(y)\| = \|x - y\|$ for all $x, \; y \in \mathbf{R}^3$. Prove or disprove the following assertion :

f maps every closed ball homeomorphically onto its image.

B6. Let X and Y be Banach spaces and suppose that X is finite dimensional. Let $T : X \to Y$ be a linear map. Prove or disprove the following assertion :

If B is a bounded set in X, then $\overline{T(B)}$ is compact in Y.

B7. Let $E \subseteq [0, 1]$ be Lebesgue measurable with $m(E) = 1$ and $f : [0, 1] \to \mathbb{R}$ be a continuous function. Prove or disprove the following assertion :

If $f(E)$ is a singleton, then so is $f([0, 1])$.

B8. Let $f : \mathbf{C} \to \mathbf{C}$ be an entire function such that $|f(z)| < |e^z|$ for all z and $f(0) = \frac{1}{2}$. Find the value of $f(\pi)$.

SECTION—C

Each question carries 6 marks.

C1. Let S be a finite subset of $\mathbb{R}^2$ of the form $A \times B$, where $A, B \subset \mathbb{R}$. Suppose W is an open set in $\mathbb{R}^2$ that contains S. Show that there exist open sets U and V in $\mathbb{R}$ such that $S \subset U \times V \subseteq W$.

C2. Let $f : \mathbb{R} \to [0, \infty)$ be a measurable function such that $0 < \int f \, dm < \infty$. Consider $\mu : \mathcal{L} \to [0, \infty]$ given by $\mu(E) := \int_E f \, dm$, where m is the Lebesgue measure on the σ-algebra $\mathcal{L}$ of Lebesgue measurable sets in $\mathbb{R}$.

(a) Show that μ is a finite measure on $\mathcal{L}$, i.e., $\mu(\mathbb{R}) < \infty$ and for any countable mutually disjoint collection $\{E_n\}$ in $\mathcal{L}$, $\mu(\cup_n E_n) = \Sigma_n \mu(E_n)$.

(b) If $E \in \mathcal{L}$ has $m(E) = 0$, then show that $\mu(E) = 0$.

C3. Let $X = C([0, 1])$ be the complex Banach space consisting of complex-valued continuous functions on $[0, 1]$ with the sup norm and Y be the subspace consisting of polynomials of degree less than or equal to 5. Show that for each coset $x + Y \in X/Y$, there exists an element $z \in x + Y$ such that $||z|| = \inf\{||x + y|| : y \in Y\}$.

C4. Consider the polynomial $f(x) = x^4 + 1$ over the field $\mathbb{Z}/5\mathbb{Z}$.

(a) Is it irreducible? Justify.

(b) Find the splitting field of the polynomial of f over $\mathbb{Z}/5\mathbb{Z}$.

C5. Consider the curve E given by the equation $y^2 = x^3 + Ax + B$, where A and B are integers. Let $P = (a, b)$ and $Q = (c, d)$ be points on E such that $a, b, c, d \in \mathbb{Q}$, with $a \neq c$. Let L be the line joining P and Q. Show that E and L intersect in some point $R = (e, f)$, where $e, f \in \mathbb{Q}$.

C6. (a) Exhibit a 4×4 matrix over $\mathbf{C}$ whose eigenvalues over $\mathbb{R}$ are ± 1 and whose eigenvalues over $\mathbf{C}$ are ± 1 and $\pm i$.

(b) Exhibit three non-conjugate matrices over $\mathbb{Z}/7\mathbb{Z}$ of size 3×3 for which -2 is the only eigenvalue.

C6. (a) Exhibit a 4×4 matrix over $\mathbf{C}$ whose eigenvalues over $\mathbb{R}$ are ± 1 and whose eigenvalues over $\mathbf{C}$ are ± 1 and $\pm i$.

(b) Exhibit three non-conjugate matrices over $\mathbb{Z}/7\mathbb{Z}$ of size 3×3 for which -2 is the only eigenvalue.

C7. Find and classify the extreme values, if any, of the function $f : \mathbf{R}^2 \to \mathbf{R}$ given by $f(x, y) = y^2 - x^3$.

C8. Find all the poles of the function $f(z) = \frac{z}{\sin z - 1}$ and compute $\int_C f(z)\,dz$, where $C : [0, 2\pi] \to \mathbf{C}$ is the curve defined as $C(\theta) = 2e^{-i\theta}$.