



Linear Algebra



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Abstract—This book provides solved examples on Linear Algebra.

1

1.1. Consider the vector space \mathbb{P}_n of real polynomials in x of degree $\leq n$. Define

$$T : \mathbb{P}_2 \rightarrow \mathbb{P}_3 \quad (1.1.1)$$

by

$$(Tf)(x) = \int_0^x f(t) dt + f'(x). \quad (1.1.2)$$

Then find the matrix representation of T with respect to the bases

$$\{1, x, x^2\} \text{ and } \{1, x, x^2, x^3\} \quad (1.1.3)$$

1.2. Let $P_A(x)$ denote the characteristic polynomial of a matrix A . Then for which of the following matrices is

$$P_A(x) - P_{A^{-1}}(x) \quad (1.2.1)$$

a constant?

a) $\begin{pmatrix} 3 & 3 \\ 2 & 4 \end{pmatrix}$
b) $\begin{pmatrix} 4 & 3 \\ 2 & 3 \end{pmatrix}$

c) $\begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}$
d) $\begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$

1.3. Which of the following matrices is not diagonalizable over \mathbb{R} ?

a) $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$
b) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

c) $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$
d) $\begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$

1.4. What is the rank of the following matrix?

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} \quad (1.4.1)$$

1.5. Let V denote the vector space of real valued continuous functions on the close interval $[0, 1]$. Let W be the subspace of V spanned by $\{\sin x, \cos x, \tan x\}$. Find the dimension of W over \mathbb{R} .

1.6. Let V be the vector space of polynomials in the variable t of degree at most 2 over \mathbb{R} . An

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inner product on V is defined by

$$f^T g = \int_0^1 f(t)g(t) dt, \quad f, g \in V. \quad (1.6.1) \quad 1.10.$$

Let

$$W = \text{span}\{1 - t^2, 1 + t^2\} \quad (1.6.2)$$

and W^\perp be the orthogonal complement of W in V . Which of the following conditions is satisfied for all $h \in W^\perp$?

- a) h is an even function
- b) h is an odd function
- c) $h(t) = 0$ has a real solution
- d) $h(0) = 0$

1.7. Consider solving the following system by Jacobi iteration scheme

$$\begin{pmatrix} 1 & 2m & -2m \\ n & 1 & n \\ 2m & 2m & 1 \end{pmatrix} (x) = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \quad (1.7.1)$$

where $m, n \in \mathbb{Z}$. With any initial vector, the scheme converges provided m, n satisfy

- a) $m + n = 3$
- b) $m > n$
- c) $m < n$
- d) $m = n$

1.8. Consider a Markov Chain with state space $\{0, 1, 2, 3, 4\}$ and transition matrix

$$P = \begin{matrix} & \begin{matrix} 0 & 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{matrix} \quad (1.8.1)$$

Then find

$$\lim_{n \rightarrow \infty} p_{23}^{(n)} \quad (1.8.2)$$

1.9. Let $L(\mathbb{R})^n$ be the space of \mathbb{R} -linear maps from \mathbb{R}^n to \mathbb{R}^n . If $\text{Ker}(T)$ denotes the kernel of T then which of the following are true?

- a) There exists $T \in L(\mathbb{R}^5) \setminus \{0\}$ such that $\text{Range}(T) = \text{Ker}(T)$
- b) There does not exist $T \in L(\mathbb{R}^5) \setminus \{0\}$ such that $\text{Range}(T) = \text{Ker}(T)$
- c) There exists $T \in L(\mathbb{R}^6) \setminus \{0\}$ such that $\text{Range}(T) = \text{Ker}(T)$

d) There does not exist $T \in L(\mathbb{R}^6) \setminus \{0\}$ such that $\text{Range}(T) = \text{Ker}(T)$

1.10. Let V be a finite dimensional vector space over \mathbb{R} and $T : V \rightarrow V$ be a linear map. Can you always write $T = T_2 \circ T_1$ for some linear maps

$$T_1 : V \rightarrow W, T : W \rightarrow V, \quad (1.10.1)$$

where W is some finite dimensional vector space such that

- a) both T_1 and T_2 are onto
- b) both T_1 and T_2 are one to one
- c) T_1 is onto, T_2 is one to one
- d) T_1 is one to one, T_2 is onto

1.11. Let $A = [a_{ij}]$ be a 3×3 complex matrix. Identify the correct statements

- a) $\det[(-1)^{i+j} a_{ij}] = \det(A)$
- b) $\det[(-1)^{i+j} a_{ij}] = -\det(A)$
- c) $\det[(\sqrt{-1})^{i+j} a_{ij}] = \det(A)$
- d) $\det[(\sqrt{-1})^{i+j} a_{ij}] = -\det(A)$

1.12. Let

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n \quad (1.12.1)$$

be a non-constant polynomial of degree $n \geq 1$. Consider the polynomial

$$q(x) = \int_0^x p(t) dt, r(x) = \frac{d}{dx} p(x) \quad (1.12.2)$$

Let V denote the real vector space of all polynomials in x . Then which of the following are true?

- a) q and r are linearly independent in V
- b) q and r are linearly dependent in V
- c) x^n belongs to the linear span of q and r
- d) x^{n+1} belongs to the linear span of q and r .

1.13. Let $M_n(\mathbb{R})$ be the ring of $n \times n$ matrices over \mathbb{R} . Which of the following are true for every $n \geq 2$?

- a) there exist matrices $A, B \in M_n(\mathbb{R})$ such that $AB - BA = I_n$, where I_n denotes the identity matrix.
- b) If $A, B \in M_n(\mathbb{R})$ and $AB = BA$, then A is diagonalisable over \mathbb{R} if and only if B is diagonalisable over \mathbb{R} .
- c) If $A, B \in M_n(\mathbb{R})$, then AB and BA have the same minimal polynomial.
- d) If $A, B \in M_n(\mathbb{R})$, then AB and BA have the

same eigenvalues in \mathbb{R} .

1.14. Consider a matrix

$$A = [a_{ij}], 1 \leq i, j \leq 5 \quad (1.14.1)$$

such that

$$a_{ij} = \frac{1}{n_i + n_j + 1}, \quad n_i, n_j \in \mathbb{N} \quad (1.14.2)$$

Then in which of the following cases A is a positive definite matrix?

- a) $n_i = 1 \forall i = 1, 2, 3, 4, 5$.
- b) $n_1 < n_2 < \dots < n_5$.
- c) $n_1 = n_2 = \dots = n_5$.
- d) $n_1 > n_2 > \dots > n_5$.

1.15. For a nonzero $w \in \mathbb{R}^n$, define

$$T_w : \mathbb{R}^n \rightarrow \mathbb{R}^n \quad (1.15.1)$$

by

$$T_w v = v - \frac{2v^T w}{w^T w} w, \quad v \in \mathbb{R}^n \quad (1.15.2)$$

Which of the following are true?

- a) $\det(T_w) = 1$
- b) $T_w(v_1)^T_w(v_2) = v_1^T v_2 \forall v_1, v_2 \in \mathbb{R}^n$
- c) $T_w = T_w^{-1}$
- d) $T_{2w} = 2T_w$

1.16. Consider the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (1.16.1)$$

over the field \mathbb{Q} of rationals. Which of the following matrices are of the form $P^T A P$ for suitable 2×2 invertible matrix P over \mathbb{Q} ?

- a) $\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$
- b) $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$
- c) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
- d) $\begin{pmatrix} 3 & 4 \\ 4 & 5 \end{pmatrix}$

1.17. Consider a Markov Chain with state space $\{0, 1, 2\}$ and transition matrix

$$P = \begin{pmatrix} 0 & 1 & 2 \\ 0 & \frac{1}{4} & \frac{5}{8} & \frac{1}{8} \\ 1 & \frac{1}{4} & 0 & \frac{3}{4} \\ 2 & \frac{1}{2} & \frac{3}{8} & \frac{1}{8} \end{pmatrix} \quad (1.17.1)$$

Then which of the following are true?

- a) $\lim_{n \rightarrow \infty} p_{12}^{(n)} = 0$
- b) $\lim_{n \rightarrow \infty} p_{12}^{(n)} = \lim_{n \rightarrow \infty} p_{21}^{(n)}$

- c) $\lim_{n \rightarrow \infty} p_{22}^{(n)} = \frac{1}{8}$
- d) $\lim_{n \rightarrow \infty} p_{21}^{(n)} = \frac{1}{3}$

2

2.1. Consider the subspaces W_1 and W_2 of \mathbb{R}^3 given by

$$W_1 = \{ \mathbf{x} \in \mathbb{R}^3 : \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \mathbf{x} = 0 \} \quad (2.1.1)$$

$$W_2 = \{ \mathbf{x} \in \mathbb{R}^3 : \begin{pmatrix} 1 & -1 & 1 \end{pmatrix} \mathbf{x} = 0 \}. \quad (2.1.2)$$

If $W \subseteq \mathbb{R}^3$, such that

$$\text{a) } W \cap W_2 = \text{span} \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$$\text{b) } \{W \cap W_1\} \perp \{W \cap W_2\}, \text{ then}$$

$$\text{a) } W = \text{span} \left\{ \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$$\text{b) } W = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$$

$$\text{c) } W = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$$\text{d) } W = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

2.2. Let

$$C = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\} \quad (2.2.1)$$

be a basis of \mathbb{R}^2 and

$$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y \\ x - 2y \end{pmatrix}. \quad (2.2.2)$$

If $T[C]$ represents the matrix of T with respect to the basis C then which among the following is true?

$$\text{a) } T[C] = \begin{pmatrix} -3 & -2 \\ 3 & 1 \end{pmatrix}$$

$$\text{b) } T[C] = \begin{pmatrix} 3 & -2 \\ -3 & 1 \end{pmatrix}$$

$$\text{c) } T[C] = \begin{pmatrix} -3 & -1 \\ 3 & 2 \end{pmatrix}$$

$$\text{d) } T[C] = \begin{pmatrix} 3 & -1 \\ -3 & 2 \end{pmatrix}$$

2.3. Let $W_1 = \{\mathbf{x} \in \mathbb{R}^4 : \}$

$$\begin{pmatrix} 1 & 1 & 1 & 0 \end{pmatrix} \mathbf{x} = 0 \quad (2.3.1)$$

$$\begin{pmatrix} 0 & 2 & 0 & 1 \end{pmatrix} \mathbf{x} = 0 \quad (2.3.2)$$

$$\begin{pmatrix} 2 & 0 & 2 & -1 \end{pmatrix} \mathbf{x} = 0 \quad (2.3.3)$$

and $W_2 = \{\mathbf{x} \in \mathbb{R}^4 : \}$

$$\begin{pmatrix} 1 & 1 & 0 & 1 \end{pmatrix} \mathbf{x} = 0 \quad (2.3.4)$$

$$\begin{pmatrix} 1 & 0 & 1 & -2 \end{pmatrix} \mathbf{x} = 0 \quad (2.3.5)$$

$$\begin{pmatrix} 0 & 1 & 0 & -1 \end{pmatrix} \mathbf{x} = 0. \quad (2.3.6)$$

Then which among the following is true?

- a) $\dim(W_1) = 1$
- b) $\dim(W_2) = 2$
- c) $\dim(W_1 \cap W_2) = 1$
- d) $\dim(W_1 + W_2) = 3$

2.4. Let A be an $n \times n$ complex matrix. Assume that A is self-adjoint and let B denote the inverse of $A + jI$. Then all eigenvalues of $(A - jI)B$ are

- a) purely imaginary
- b) of modulus one
- c) real
- d) of modulus less than one

2.5. Let $\{u_1, u_2, \dots, u_n\}$ be an orthonormal basis of \mathbb{C}^n as column vectors. Let

$$\mathbf{M} = (\mathbf{u}_1 \quad \mathbf{u}_2 \quad \dots \quad \mathbf{u}_k), \quad (2.5.1)$$

$$\mathbf{N} = (\mathbf{u}_{k+1} \quad \mathbf{u}_{k+2} \quad \dots \quad \mathbf{u}_n) \quad (2.5.2)$$

and \mathbf{P} be the diagonal $k \times k$ matrix with diagonal entries $\alpha_1, \alpha_2, \dots, \alpha_k \in \mathbb{R}$. Then which of the following is true?

- a) $\text{rank}(\mathbf{M}\mathbf{P}\mathbf{M}^*) = k$ whenever $\alpha_i \neq \alpha_j, 1 \leq i, j \leq k$.
- b) $\text{tr}(\mathbf{M}\mathbf{P}\mathbf{M}^*) = \sum_{i=1}^k \alpha_i$
- c) $\text{rank}(\mathbf{M}^*\mathbf{N}) = \min(k, n - k)$
- d) $\text{rank}(\mathbf{M}\mathbf{M}^* + \mathbf{N}\mathbf{N}^*) < n$.

2.6. Let $B : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be the function

$$B(a, b) = ab \quad (2.6.1)$$

Which of the following is true?

- a) B is a linear transformation
- b) B is a positive definite bilinear form
- c) B is symmetric but not positive definite
- d) B is neither linear nor bilinear

2.7. Let \mathbf{A} be an invertible real $n \times n$ matrix. Define

a function

$$F : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} \quad (2.7.1)$$

by

$$F(\mathbf{x}, \mathbf{y}) = (F\mathbf{x})^T \mathbf{y} \quad (2.7.2)$$

Let $DF(\mathbf{x}, \mathbf{y})$ denote the derivate of F at (\mathbf{x}, \mathbf{y}) which is a linear transformation from

$$\mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} \quad (2.7.3)$$

Then, if

- a) $\mathbf{x} \neq 0, DF(\mathbf{x}, \mathbf{0}) \neq 0$
- b) $\mathbf{y} \neq 0, DF(\mathbf{0}, \mathbf{y}) \neq 0$
- c) $(\mathbf{x}, \mathbf{y}) \neq (\mathbf{0}, \mathbf{0}), DF(\mathbf{x}, \mathbf{0}) \neq 0$
- d) $\mathbf{x} = 0$ or $\mathbf{y} = 0, DF(\mathbf{x}, \mathbf{y}) = 0$

2.8. Let

$$T : \mathbb{R}^n \rightarrow \mathbb{R}^n \quad (2.8.1)$$

be a linear map that satisfies

$$T^2 = T - I. \quad (2.8.2)$$

Then which of the following is true?

- a) T is invertible.
- b) $T - I$ is not invertible.
- c) T has a real eigenvalue.
- d) $T^3 = -I$.

2.9. Let

$$\mathbf{M} = \begin{pmatrix} 2 & 0 & 3 & 2 & 0 & -2 \\ 0 & 1 & 0 & -1 & 3 & 4 \\ 0 & 0 & 1 & 0 & 4 & 4 \\ 1 & 1 & 1 & 0 & 1 & 1 \end{pmatrix} \quad (2.9.1)$$

$$\mathbf{b}_1 = \begin{pmatrix} 5 \\ 1 \\ 1 \\ 4 \end{pmatrix}, \mathbf{b}_2 = \begin{pmatrix} 5 \\ 1 \\ 3 \\ 3 \end{pmatrix}. \quad (2.9.2)$$

Then which of the following are true?

- a) both systems $\mathbf{M}\mathbf{x} = \mathbf{b}_1$ and $\mathbf{M}\mathbf{x} = \mathbf{b}_2$ are inconsistent.
- b) both systems $\mathbf{M}\mathbf{x} = \mathbf{b}_1$ and $\mathbf{M}\mathbf{x} = \mathbf{b}_2$ are consistent.
- c) the system $\mathbf{M}\mathbf{x} = \mathbf{b}_1 - \mathbf{b}_2$ is consistent.
- d) the system $\mathbf{M}\mathbf{x} = \mathbf{b}_1 - \mathbf{b}_2$ is inconsistent.

2.10. Let

$$\mathbf{M} = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 4 \\ -2 & 1 & -4 \end{pmatrix}. \quad (2.10.1)$$

Given that 1 is an eigenvalue of \mathbf{M} , then which among the following are correct?

- a) The minimal polynomial of \mathbf{M} is $(x-1)(x+4)$
- b) The minimal polynomial of \mathbf{M} is $(x-1)^2(x+4)$
- c) \mathbf{M} is not diagonalizable.
- d) $\mathbf{M}^{-1} = \frac{1}{4}(\mathbf{M} + 3\mathbf{I})$.

2.11. Let \mathbf{A} be a real matrix with characteristic polynomial $(x-1)^3$. Pick the correct statements from below:

- a) \mathbf{A} is necessarily diagonalizable.
- b) If the minimal polynomial of \mathbf{A} is $(x-1)^3$, then \mathbf{A} is diagonalizable.
- c) The characteristic polynomial of \mathbf{A}^2 is $(x-1)^3$
- d) If \mathbf{A} has exactly two Jordan blocks, then $(\mathbf{A} - \mathbf{I})^2$ is diagonalizable.

2.12. Let P_3 be the vector space of polynomials with real coefficients and of degree at most 3. Consider the linear map

$$T : P_3 \rightarrow P_3 \quad (2.12.1)$$

defined by

$$T(p(x)) = p(x-1) + p(x+1) \quad (2.12.2)$$

Which of the following properties does the matrix of T with respect to the standard basis $B = \{1, x, x^2, x^3\}$ of P_3 satisfy?

- a) $\det T = 0$.
- b) $(T - 2I)^4 = 0$ but $(T - 2I)^3 \neq 0$.
- c) $(T - 2I)^3 = 0$ but $(T - 2I)^2 \neq 0$.
- d) 2 is an eigenvalue with multiplicity 4.

2.13. Let \mathbf{M} be an $n \times n$ Hermitian matrix of rank $k, k \neq n$. If $\lambda \neq 0$ is an eigenvalue of \mathbf{M} with corresponding unit column vector \mathbf{u} , then which of the following are true?

- a) $\text{rank}(\mathbf{M} - \lambda \mathbf{u} \mathbf{u}^*) = k - 1$.
- b) $\text{rank}(\mathbf{M} - \lambda \mathbf{u} \mathbf{u}^*) = k$.
- c) $\text{rank}(\mathbf{M} - \lambda \mathbf{u} \mathbf{u}^*) = k + 1$.
- d) $(\mathbf{M} - \lambda \mathbf{u} \mathbf{u}^*)^n = \mathbf{M}^n - \lambda^n \mathbf{u} \mathbf{u}^*$.

2.14. Define a real valued function B on $\mathbb{R}^2 \times \mathbb{R}^2$ as

$$B(\mathbf{x}, \mathbf{y}) = x_1 y_1 - x_1 y_2 - x_2 y_1 + 4x_2 y_2 \quad (2.14.1)$$

Let $\mathbf{v}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and

$$W = \left\{ \mathbf{v} \in \mathbb{R}^2 : B(\mathbf{v}_0, \mathbf{v}) = 0 \right\} \quad (2.14.2)$$

Then W

- a) is not a subspace of \mathbb{R}^2 .
- b) equals $\mathbf{0}$.
- c) is the y axis
- d) is the line passing through $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

2.15. Consider the Quadratic forms

$$Q_1(x, y) = xy \quad (2.15.1)$$

$$Q_2(x, y) = x^2 + 2xy + y^2 \quad (2.15.2)$$

$$Q_3(x, y) = x^2 + 3xy + 2y^2 \quad (2.15.3)$$

on \mathbb{R}^2 . Choose the correct statements from below

- a) Q_1 and Q_2 are equivalent.
- b) Q_1 and Q_3 are equivalent.
- c) Q_2 and Q_3 are equivalent.
- d) all are equivalent.

2.16. Consider a Markov Chain with state space $\{0, 1, 2\}$ and transition matrix

$$P = \begin{pmatrix} 0 & 1 & 2 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{3}{4} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \quad (2.16.1)$$

For any two states i and j , let $p_{ij}^{(n)}$ denote the n -step transition probability of going from i to j . Identify correct statements.

- a) $\lim_{n \rightarrow \infty} p_{11}^{(n)} = \frac{2}{9}$
- b) $\lim_{n \rightarrow \infty} p_{21}^{(n)} = 0$
- c) $\lim_{n \rightarrow \infty} p_{32}^{(n)} = \frac{1}{3}$
- d) $\lim_{n \rightarrow \infty} p_{13}^{(n)} = \frac{1}{3}$