Solutions: Linear Algebra by Hoffman and Kunze

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Contents

***Abstract*—This book provides solutions to the Linear Algebra book by Hoffman and Kunze.**

1 Linear Equations

1.1 Fields and Linear Equations

1.1.1. Verify that the set of complex numbers numbers described in the form of c where x and y are rational is a sub-field of C.

Solution: Lets consider the set S = $\{x + y\sqrt{2}, x, y \in Q\}$, $S \subset C$ We must verify that S meets the following two conditions:

$$0, 1 \in S \tag{1.1.1.1}$$

$$a, b \in S, a + b, -a, ab, a^{-1} \in S \tag{1.1.1.2}$$

Throughout let

$$a = x + y\sqrt{2}, b = w + z\sqrt{2} \tag{1.1.1.3}$$

If

a)

$$x = 0, y = 0 \in Q, a = 0 + \sqrt{2}.0 = 0, 0 \in S \tag{1.1.1.4}$$

b)

$$x = 1, y = 0, a = 1 + \sqrt{2}.0 = 1, 1 \in S \tag{1.1.1.5}$$

c)

$$a + b = x + y\sqrt{2} + w + z\sqrt{2} = b + a \tag{1.1.1.6}$$

d)

$$-a = -x - y\sqrt{2}, x, y \in Qso - x, -y \in Q, a \in S \tag{1.1.1.7}$$

e)

$$ab = (x + y\sqrt{2})(w + z\sqrt{2}) = ba, ab \in S \tag{1.1.1.8}$$

f)

$$a^{-1}a = (x + y\sqrt{2})^{-1}(x + y\sqrt{2}) = 1, a^{-1} \in S \tag{1.1.1.9}$$

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Hence (1.1.1.1) ,(1.1.1.2) is verfied . Therefore by considering the (1.1.1.1) and (1.1.1.2) we can say set complex numbers of given form $x + y\sqrt{2}$ is subfield of C .

1.1.2. Let $\mathbb{F}$ be the field of complex numbers. Are the following two systems of linear equations equivalent? If so, express each equation in each system as a linear combination of the equations in the other system.

$$x_1 - x_2 = 0$$
$$2x_1 + x_2 = 0$$

and

$$3x_1 + x_2 = 0$$
$$x_1 + x_2 = 0$$

Solution: The given system of linear equations can be written as,

$$\mathbf{Ax} = 0 \tag{1.1.2.1}$$
$$\implies \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}\mathbf{x} = 0 \tag{1.1.2.2}$$
$$\mathbf{Bx} = 0 \tag{1.1.2.3}$$
$$\implies \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}\mathbf{x} = 0 \tag{1.1.2.4}$$

Now we can obtain **B** from matrix **A** by performing elementary row operations given as,

$$\mathbf{B} = \mathbf{CA} \tag{1.1.2.5}$$
$$\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} = \mathbf{C}\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \tag{1.1.2.6}$$

where **C** is product of elementary matrices given as,

$$\mathbf{C} = (\mathbf{E_7E_6E_5E_4E_3E_2E_1})$$
$$= \begin{pmatrix} 1 & 0 \\ \frac{1}{3} & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \frac{2}{3} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \frac{1}{3} & \frac{4}{3} \\ \frac{-1}{3} & \frac{2}{3} \end{pmatrix} \tag{1.1.2.7}$$

Now, performing elementary operations on the right side of **A** we obtain matrix **B** given as,

$$\mathbf{B} = \mathbf{AP} \tag{1.1.2.8}$$
$$\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}\mathbf{P} \tag{1.1.2.9}$$

where, **P** is product of elementary matrices given by,

$$\mathbf{P} = (\mathbf{E_1E_2E_3E_4E_5})$$
$$= \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \frac{4}{3} & \frac{2}{3} \\ \frac{-5}{3} & \frac{-1}{3} \end{pmatrix} \tag{1.1.2.10}$$

Similarly, **A** can be obtained from matrix **B** from (1.1.2.5) as,

$$\mathbf{A} = \mathbf{C}^{-1}\mathbf{B} \tag{1.1.2.11}$$
$$\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} = \mathbf{C}^{-1}\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \tag{1.1.2.12}$$

Matrix **C** is product of elementary matrices and hence invertible and is given as,

$$\mathbf{C}^{-1} = \left(\mathbf{E_1}^{-1}\mathbf{E_2}^{-1}\mathbf{E_3}^{-1}\mathbf{E_4}^{-1}\mathbf{E_5}^{-1}\mathbf{E_6}^{-1}\mathbf{E_7}^{-1}\right)$$
$$= \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \frac{3}{2} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{-1}{3} & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & -2 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \tag{1.1.2.13}$$

Matrix **A** can also be obtained from (1.1.2.8) given as,

$$\mathbf{A} = \mathbf{BP}^{-1} \tag{1.1.2.14}$$
$$\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}\mathbf{P}^{-1} \tag{1.1.2.15}$$

where,

$$\mathbf{P}^{-1} = \left(\mathbf{E_5}^{-1}\mathbf{E_4}^{-1}\mathbf{E_3}^{-1}\mathbf{E_2}^{-1}\mathbf{E_1}^{-1}\right)$$
$$= \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \frac{-1}{2} & -1 \\ \frac{5}{2} & 2 \end{pmatrix} \tag{1.1.2.16}$$

Thus (1.1.2.4) can be obtained from (1.1.2.2) by multiplying it with matrix **C**, and by inverse row operations (1.1.2.2) can be obtained back from (1.1.2.4) since **C** is product of elementary matrices and hence invertible.
Thus the two given homogeneous systems are row equivalent.
Now writing equations in matrix-vector form

as,

$$3x_1 + x_2 = \begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} \tag{1.1.2.17}$$

$$\implies \begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} = \frac{1}{3} \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} + \frac{4}{3} \begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} \tag{1.1.2.18}$$

$$x_1 + x_2 = \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} \tag{1.1.2.19}$$

$$\implies \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = \frac{-1}{3} \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} + \frac{2}{3} \begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} \tag{1.1.2.20}$$

(1.1.2.18), (1.1.2.20) is same as multiplying **C** with **A** as it takes the linear combination of each rows of matrix **A** i.e, (1.1.2.6)

$$x_1 - x_2 = \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} \tag{1.1.2.21}$$

$$\implies \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = (1) \begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} + (-2) \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} \tag{1.1.2.22}$$

$$2x_1 + x_2 = \begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} \tag{1.1.2.23}$$

$$\implies \begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = \frac{1}{2} \begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} + \frac{1}{2} \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} \tag{1.1.2.24}$$

(1.1.2.22), (1.1.2.24) is same as multiplying $\mathbf{C}^{-1}$ with **B** as it takes the linear combination of each rows of matrix **B** i.e, (1.1.2.12)
Thus each equation in each system can be expressed as a linear combination of the equations in the other system when they are equivalent.

1.1.3. Are the following two systems of linear equations equivalent?

$$\begin{aligned} -x_1 + x_2 + 4x_3 &= 0 \\ x_1 + 3x_2 + 8x_3 &= 0 \\ \frac{1}{2}x_1 + x_2 + \frac{5}{2}x_3 &= 0 \end{aligned} \tag{1.1.3.1}$$

Solution:

$$\begin{aligned} x_1 - x_3 &= 0 \\ x_2 + 3x_3 &= 0 \end{aligned} \tag{1.1.3.2}$$

System of linear equations in (1.1.3.1) can be expressed in matrix form as,

$$\mathbf{A}\mathbf{x} = 0 \tag{1.1.3.3}$$

$$\begin{pmatrix} -1 & 1 & 4 \\ 1 & 3 & 8 \\ \frac{1}{2} & 1 & \frac{5}{2} \end{pmatrix} \mathbf{x} = 0 \tag{1.1.3.4}$$

System of linear equations in (1.1.3.2) can be expressed in matrix form as,

$$\mathbf{B}\mathbf{x} = 0 \tag{1.1.3.5}$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \end{pmatrix} \mathbf{x} = 0 \tag{1.1.3.6}$$

Two system of linear equations are equivalent if one system can be expressed as a linear combination of other system.
Matrix **B** can be obtained from matrix **A** as,

$$\mathbf{B} = \mathbf{C}\mathbf{A} \tag{1.1.3.7}$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \end{pmatrix} = \mathbf{C} \begin{pmatrix} -1 & 1 & 4 \\ 1 & 3 & 8 \\ \frac{1}{2} & 1 & \frac{5}{2} \end{pmatrix} \tag{1.1.3.8}$$

$$\mathbf{C} = \begin{pmatrix} -1 & 1 & -2 \\ \frac{1}{2} & -\frac{1}{2} & 2 \end{pmatrix} \tag{1.1.3.9}$$

Now, writing equations in matrix-vector form,

$$x_1 - x_3 = \begin{pmatrix} 1 & 0 & -1 \end{pmatrix} \mathbf{x}$$

$$\implies \begin{pmatrix} 1 & 0 & -1 \end{pmatrix} \mathbf{x} = -1 \begin{pmatrix} -1 & 1 & 4 \end{pmatrix} \mathbf{x} + 1 \begin{pmatrix} 1 & 3 & 8 \end{pmatrix} \mathbf{x} - 2 \begin{pmatrix} \frac{1}{2} & 1 & \frac{5}{2} \end{pmatrix} \mathbf{x} \tag{1.1.3.10}$$

$$x_2 + 3x_3 = \begin{pmatrix} 0 & 1 & 3 \end{pmatrix} \mathbf{x}$$

$$\implies \begin{pmatrix} 0 & 1 & 3 \end{pmatrix} \mathbf{x} = \frac{1}{2} \begin{pmatrix} -1 & 1 & 4 \end{pmatrix} \mathbf{x} - \frac{1}{2} \begin{pmatrix} 1 & 3 & 8 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} \frac{1}{2} & 1 & \frac{5}{2} \end{pmatrix} \mathbf{x} \tag{1.1.3.11}$$

Equations (1.1.3.10) and (1.1.3.11) is same as multiplying **C** with **A** which is the linear combination of rows of matrix **A**.
Thus each equation in second system can be expressed as linear combination of the equations in first system.
Therefore, the two system of linear equations are equivalent.

1.1.4. Let $\mathbb{F}$ be the field of complex numbers. Are the following two systems of linear equations equivalent? If so, express each equation in each system as a linear combination of equations in other system. First system of equations:

$$2x_1 + (-1+i)x_2 + x_4 = 0 \tag{1.1.4.1}$$
$$3x_2 - 2ix_3 + 5x_4 = 0 \tag{1.1.4.2}$$

The second system of equations:

$$(1+\frac{i}{2})x_1 + 8x_2 - ix_3 - x_4 = 0 \tag{1.1.4.3}$$
$$\frac{2}{3}x_1 - \frac{1}{2}x_2 + x_3 + 7x_4 = 0 \tag{1.1.4.4}$$

Solution: Let $\mathbf{R_1}$ and $\mathbf{R_2}$ be the reduced row echelon forms of the augumented matrices of the following systems of homogeneous equations respectively.

$$\mathbf{AX} = \mathbf{0} \tag{1.1.4.5}$$
$$\mathbf{BX} = \mathbf{0} \tag{1.1.4.6}$$

Where $\mathbf{A}$ and $\mathbf{B}$ as follows

$$\mathbf{A} = \begin{pmatrix} 2 & -1+i & 0 & 1 \\ 0 & 3 & -2i & 5 \end{pmatrix} \tag{1.1.4.7}$$
$$\mathbf{B} = \begin{pmatrix} 1+\frac{i}{2} & 8 & -i & -1 \\ \frac{2}{3} & \frac{-1}{2} & 1 & 7 \end{pmatrix} \tag{1.1.4.8}$$

On performing elementary row operations on (1.1.4.7),

$$\mathbf{R_1} = \mathbf{CA} \tag{1.1.4.9}$$

where $\mathbf{C}$ is the product of all elementary matrices. Reducing the first system of linear equations, we get,

$$\mathbf{C} = \begin{pmatrix} 1 & \frac{1-i}{2} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} \tag{1.1.4.10}$$
$$\mathbf{R_1} = \begin{pmatrix} 1 & 0 & \frac{-1-i}{3} & \frac{4}{3} - \frac{5i}{6} \\ 0 & 1 & \frac{-2i}{3} & \frac{5}{3} \end{pmatrix} \tag{1.1.4.11}$$

On performing elementary row operations on (1.1.4.8),

$$\mathbf{R_2} = \mathbf{DA} \tag{1.1.4.12}$$

where $\mathbf{D}$ is the product of all elementary matrices. Reducing the second system of linear equations, we get,

$$\mathbf{D} = \begin{pmatrix} \frac{4}{5}(1-\frac{i}{2}) & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{-2}{3} & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \frac{-6(143+43i)}{4909} \end{pmatrix}\begin{pmatrix} 1 & \frac{16(-2+i)}{5} \\ 0 & 1 \end{pmatrix} \tag{1.1.4.13}$$

$$\mathbf{R_2} = \begin{pmatrix} 1 & 0 & \frac{6702}{4909} - \frac{708i}{4909} & \frac{46620}{4909} - \frac{1998i}{4909} \\ 0 & 1 & \frac{-2(441+472i)}{4909} & \frac{-2(3283+1332i)}{4909} \end{pmatrix} \tag{1.1.4.14}$$

From the equations (1.1.4.11) and (1.1.4.14), we can say that

$$\mathbf{R_1} \neq \mathbf{R_2} \tag{1.1.4.15}$$

Hence the given systems of linear equations are not equivalent.

1.1.5. Let $\mathbb{F}$ be a set which contains exactly two elements,0 and 1.Define an addition and multiplication by tables. Verify that the set $\mathbb{F}$,

+	0	1
0	0	1
1	1	0

·	0	1
0	0	0
1	0	1

together with these two operations, is a field.
Solution:
To prove that ($\mathbb{F}$,+,·) is a field we need to satisfy the following,

a) + and · should be closed
   - For any a and b in $\mathbb{F}$, a+b ∈ $\mathbb{F}$ and a · b ∈ $\mathbb{F}$.For example 0+0=0 and 0·0=0.

b) + and · should be commutative
   - For any a and b in $\mathbb{F}$, a+b = b+a and a · b = b· a. For example 0+1=1+0 and 0· 1=1· 0.

c) + and · should be associative
   - For any a and b in $\mathbb{F}$, a+(b+c) = (a+b)+c and a· (b · c) = (a · b) · c. For example 0+(1+0)=(0+1)+0 and 0·(1·0)=(0·1)·0.

d) + and · operations should have an identity element
   - If we perform a + 0 then for any value of a from $\mathbb{F}$ the result will be a itself. Hence 0 is an identity element of + operation.If we perform a · 1 then for any value of a from $\mathbb{F}$ the result will be a itself. Hence 1 is an identity element of · operation.

e) ∀ a ∈ $\mathbb{F}$ there exists an additive inverse
   - For additive inverse to exist, ∀ a in $\mathbb{F}$ a+(-a)=0. For example. 1-1=0 and 0-0=0.

f) $\forall$ a $\in \mathbb{F}$ such that a is non zero there exists a multiplicative inverse
   - For multiplicative inverse to exist, $\forall$ a such that a is non zero in $\mathbb{F}$, $a \cdot a^{-1}$=1.For example $1 \cdot 1^{-1} = 1$.

g) + and $\cdot$ should hold distributive property
   - For any a,b and c in $\mathbb{F}$ the property a$\cdot$(b+c)=a$\cdot$b+a$\cdot$c should always hold true.For example 0$\cdot$(1+1)=0$\cdot$1+0$\cdot$1.

Since the above properties are satisfied we can say that ($\mathbb{F}$,+,$\cdot$) is a field.

1.1.6. Prove that if two homogenous systems of linear equations in two unknowns have the same solutions, then they are equivalent.

Solution: Let the two systems of homogenous equations be

$$\mathbf{Ax} = \mathbf{0} \tag{1.1.6.1}$$
$$\mathbf{By} = \mathbf{0} \tag{1.1.6.2}$$

We can write

$$\mathbf{CAx} = \mathbf{0} \tag{1.1.6.3}$$
$$\mathbf{DBy} = \mathbf{0} \tag{1.1.6.4}$$

where $\mathbf{C}$ and $\mathbf{D}$ are product of elementary matrices that reduce $\mathbf{A}$ and $\mathbf{B}$ into their reduced row echelon forms $\mathbf{R_1}$ and $\mathbf{R_2}$
(1.1.6.3) and (1.1.6.4) imply

$$\mathbf{R_1x} = 0 \tag{1.1.6.5}$$
$$\mathbf{R_2y} = 0 \tag{1.1.6.6}$$

Given that they have same solution, we can write

$$\mathbf{R_1x} = 0 \tag{1.1.6.7}$$
$$\mathbf{R_2x} = 0 \tag{1.1.6.8}$$
$$\implies (\mathbf{R_1} - \mathbf{R_2})\mathbf{x} = 0 \tag{1.1.6.9}$$

Note that for a solution to exist, $\mathbf{R_1}$ and $\mathbf{R_2}$ can be either of matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{or} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.1.6.10}$$

Case 1 Let us assume that the solution is unique. The unique solution is

$$\mathbf{x} = \mathbf{0} \tag{1.1.6.11}$$

Since they have the same solution, both $\mathbf{R_1}, \mathbf{R_2}$ must have their rank as 2.
So,

$$\mathbf{R_1} = \mathbf{R_2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.1.6.12}$$

Case 2 Let us assume that (1.1.6.3),(1.1.6.4) have infinitely many solutions
So,

$$\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{B}) = 1 \tag{1.1.6.13}$$

equation (1.1.6.9) for solutions other than zero solution implies

$$\mathbf{R_1} = \mathbf{R_2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{1.1.6.14}$$

So, in both the cases, we have

$$\mathbf{R_1} = \mathbf{R_2} \tag{1.1.6.15}$$
$$\implies \mathbf{CA} = \mathbf{DB} \tag{1.1.6.16}$$

Since $\mathbf{C}, \mathbf{D}$ are product of elementary matrices, they are invertible.

$$\implies \mathbf{A} = \mathbf{C^{-1}DB} \tag{1.1.6.17}$$
$$\mathbf{B} = \mathbf{D^{-1}CA} \tag{1.1.6.18}$$
$$\text{Let } \mathbf{C^{-1}D} = \mathbf{E} \tag{1.1.6.19}$$

where $\mathbf{E}$ is also a product of elementary matrices
(1.1.6.17) and (1.1.6.18) hence become

$$\mathbf{A} = \mathbf{EB} \tag{1.1.6.20}$$
$$\mathbf{B} = \mathbf{E^{-1}A} \tag{1.1.6.21}$$

Hence the two systems of equations are equivalent.

1.1.7. Prove that each subfield of the field of complex number contains every rational number

Solution:
Complex Numbers: A complex number is a number that can be expressed in the form $a + bi$, where a and b are real numbers, and i represents the imaginary unit, satisfying the equation $i^2 = -1$.The set of complex numbers is denoted by $\mathbb{C}$

$$\mathbb{C} = \{(a, b) : a, b \in \mathbb{R}\} \tag{1.1.7.1}$$

Rational Numbers: A number in the form $\frac{p}{q}$, where both p and q(non-zero) are integers, is called a rational number.The set of rational numbers is dentoed by $\mathbb{Q}$ Let $\mathbb{Q}$ be the set of rational numbers.

$$\mathbb{Q} = \left\{ \frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{Z}_{\neq 0} \right\} \tag{1.1.7.2}$$

Let $\mathbb{C}$ be the field of complex numbers and given $\mathbb{F}$ be the subfield of field of complex numbers $\mathbb{C}$ Since $\mathbb{F}$ is the subfield , we could say that

$$0 \in \mathbb{F} \tag{1.1.7.3}$$
$$1 \in \mathbb{F} \tag{1.1.7.4}$$

Closed under addition: Here $\mathbb{F}$ is closed under addition since it is subfield

$$1 + 1 = 2 \in \mathbb{F} \tag{1.1.7.5}$$
$$1 + 1 + 1 = 3 \in \mathbb{F} \tag{1.1.7.6}$$
$$\vdots$$
$$1 + 1 + \cdots + 1(\text{p times}) = p \in \mathbb{F} \tag{1.1.7.7}$$
$$1 + 1 + \cdots + 1(\text{q times}) = q \in \mathbb{F} \tag{1.1.7.8}$$

By using the above property we could say that zero and other positive integers belongs to $\mathbb{F}$.Since p and q are integers we say,

$$p \in \mathbb{Z} \tag{1.1.7.9}$$
$$q \in \mathbb{Z} \tag{1.1.7.10}$$

Additive Inverse: Let x be the positive integer belong $\mathbb{F}$ and by additive inverse we could say,

$$\forall x \in \mathbb{F} \tag{1.1.7.11}$$
$$(-x) \in \mathbb{F} \tag{1.1.7.12}$$

Therefore field $\mathbb{F}$ contains every integers. Let n be a integer then,

$$n \in \mathbb{Z} \implies n \in \mathbb{F} \tag{1.1.7.13}$$
$$\mathbb{Z} \subseteq \mathbb{F} \tag{1.1.7.14}$$

Where $\mathbb{Z}$ is subset of $\mathbb{F}$ Multiplicative Inverse: Every element except zero in the subfield $\mathbb{F}$ has an multiplicative inverse. From equation (1.1.7.8), since q $\in \mathbb{F}$ we could say ,

$$\frac{1}{q} \in \mathbb{F} \quad \text{and } q \neq 0 \tag{1.1.7.15}$$

Closed under multiplication: Also, $\mathbb{F}$ is closed under multiplication and thus, from equation (1.1.7.7) and (1.1.7.15) we get ,

$$p \cdot \frac{1}{q} \in \mathbb{F} \tag{1.1.7.16}$$
$$\implies \frac{p}{q} \in \mathbb{F} \tag{1.1.7.17}$$

where , $p \in \mathbb{Z}$ and $q \in \mathbb{Z}_{\neq 0}$ (from equation (1.1.7.3) and (1.1.7.15)) Conclusion From (1.1.7.2) and (1.1.7.17) we could say ,

$$\mathbb{Q} \subseteq \mathbb{F} \tag{1.1.7.18}$$

From equation (1.1.7.18) we could say that each subfield of the field of complex number contains every rational number

Hence Proved

1.1.8. Prove that, each field of the characteristic zero contains a copy of the rational number field.
Solution: The characteristic of a field is defined to be the smallest number of times one must use the field's multiplicative identity (1) in a sum to get the additive identity. If this sum never reaches the additive identity (0), then the field is said to have characteristic zero.
Let $\mathbb{Q}$ be the rational number field. Hence,

$$0 \in \mathbb{Q} \quad [\text{Additive Identity}] \tag{1.1.8.1}$$
$$1 \in \mathbb{Q} \quad [\text{Multiplicative Identity}] \tag{1.1.8.2}$$

As addition is defined on $\mathbb{Q}$ hence we have,

$$1 \neq 0 \tag{1.1.8.3}$$
$$1 + 1 = 2 \neq 0 \tag{1.1.8.4}$$

And so on,

$$1 + 1 + \cdots + 1 = n \neq 0 \tag{1.1.8.5}$$

From the definition of characteristic of a field and from (1.1.8.3), (1.1.8.4) and so on upto (1.1.8.5), the rational number field, $\mathbb{Q}$ has characteristic 0.

1.2 Matrices and Elementary Row Operations

1.2.1. Find all solutions to the system of equations

$$\begin{aligned} (1 - i) x_1 - i x_2 &= 0 \\ 2x_1 + (1 - i) x_2 &= 0 \end{aligned} \tag{1.2.1.1}$$

Solution: System of Linear Equations (1.2.1.1) can be expressed in matrix form as,

$$\mathbf{A}\mathbf{x} = 0 \tag{1.2.1.2}$$

$$\begin{pmatrix} 1-i & -i \\ 2 & 1-i \end{pmatrix}\mathbf{x} = 0 \tag{1.2.1.3}$$

By row reduction ,

$$\begin{pmatrix} 1-i & -i \\ 2 & 1-i \end{pmatrix} \xleftrightarrow[R_1\leftarrow R_1/2]{R_1\leftrightarrow R_2} \begin{pmatrix} 1 & \frac{1-i}{2} \\ 1-i & -i \end{pmatrix} \tag{1.2.1.4}$$

$$\xleftrightarrow{R_2\leftarrow R_2-(1-i)R_1} \begin{pmatrix} 1 & \frac{1-i}{2} \\ 0 & 0 \end{pmatrix} \tag{1.2.1.5}$$

$$\begin{pmatrix} 1 & \frac{1-i}{2} \end{pmatrix}\mathbf{x} = 0 \tag{1.2.1.6}$$

$$\begin{pmatrix} 1 & \frac{1-i}{2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \tag{1.2.1.7}$$

$$x_1 = -\frac{1-i}{2}x_2 \tag{1.2.1.8}$$

$$\mathbf{x} = \begin{pmatrix} -\frac{1-i}{2}x_2 \\ x_2 \end{pmatrix} \tag{1.2.1.9}$$

$$\implies \mathbf{x} = x_2\begin{pmatrix} -\frac{1-i}{2} \\ 1 \end{pmatrix} \tag{1.2.1.10}$$

1.2.2.

$$\mathbf{A} = \begin{pmatrix} 6 & -4 & 0 \\ 4 & -2 & 0 \\ -1 & 0 & 3 \end{pmatrix} \tag{1.2.2.1}$$

Find all solutions of $\mathbf{AX} = 2\mathbf{X}$and all solutions of $\mathbf{AX} = 3\mathbf{X}$.The symbol c$\mathbf{X}$ denotes the matrix each entry of which is c times corresponding entry.

Solution:

$$\mathbf{A} = \begin{pmatrix} 6 & -4 & 0 \\ 4 & -2 & 0 \\ -1 & 0 & 3 \end{pmatrix} \tag{1.2.2.2}$$

To calculate solution of $\mathbf{AX} = 2\mathbf{X}$and all solutions of $\mathbf{AX} = 3\mathbf{X}$we calculate eigen values of $\mathbf{A}$:

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{X} = 0 \tag{1.2.2.3}$$

Substituting values in (1.2.2.3),

$$\begin{pmatrix} 6-\lambda & -4 & 0 \\ 4 & -2-\lambda & 0 \\ -1 & 0 & 3-\lambda \end{pmatrix}\mathbf{X} = 0 \tag{1.2.2.4}$$

Simplifying:

$$\begin{pmatrix} 6-\lambda & -4 & 0 \\ 4 & -2-\lambda & 0 \\ -1 & 0 & 3-\lambda \end{pmatrix} \xleftrightarrow{R_1\leftarrow R_1-R_2}$$

$$\begin{pmatrix} 2-\lambda & -2+\lambda & 0 \\ 4 & -2-\lambda & 0 \\ -1 & 0 & 3-\lambda \end{pmatrix} \tag{1.2.2.5}$$

Taking $(3-\lambda)$ and $(2-\lambda)$common from C_3 and R_1

$$(3-\lambda)(2-\lambda)\begin{pmatrix} 1 & -1 & 0 \\ 4 & -2-\lambda & 0 \\ -1 & 0 & 1 \end{pmatrix} \tag{1.2.2.6}$$

$$\begin{pmatrix} 1 & -1 & 0 \\ 4 & -2-\lambda & 0 \\ -1 & 0 & 1 \end{pmatrix} \xleftrightarrow{R_2\leftarrow R_2-4R_1} \begin{pmatrix} 1 & -1 & 0 \\ 0 & -\lambda+2 & 0 \\ -1 & 0 & 1 \end{pmatrix} \tag{1.2.2.7}$$

Taking $(2-\lambda)$ common from R_2:

$$(2-\lambda)^2(3-\lambda)\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \tag{1.2.2.8}$$

Eigen values are:

$$\lambda_1 = 2 \tag{1.2.2.9}$$

$$\lambda_2 = 3 \tag{1.2.2.10}$$

solution to $\mathbf{AX} = 2\mathbf{X}$ is eigen vector corresponding to $\lambda = 2$

$$(\mathbf{A} - 2\mathbf{I})\mathbf{X} = 0 \tag{1.2.2.11}$$

Substituting values:

$$\begin{pmatrix} 4 & -4 & 0 & 0 \\ 4 & -4 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix} \xleftrightarrow{R_1\leftarrow \frac{R_1}{4}} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 4 & -4 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix} \xleftrightarrow{R_2\leftarrow R_2-4R_1}$$

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix} \xleftrightarrow{R_3\leftarrow R_3-R_1} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{pmatrix} \xleftrightarrow{R_3\longleftrightarrow R_2}$$

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xleftrightarrow{R_2\leftarrow -R_2} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xleftrightarrow{R_1\leftarrow R_1+R_2}$$

$$\begin{pmatrix}1 & 0 & -1 & 0\\0 & 1 & -1 & 0\\0 & 0 & 0 & 0\end{pmatrix} \tag{1.2.2.12}$$

So, x_3 is a free variable: Let $x_3 = c$.

$$x_2 - x_3 = 0 \implies x_2 = x_3 = c \tag{1.2.2.13}$$
$$x_1 - x_3 = 0 \implies x_1 = x_3 = c \tag{1.2.2.14}$$

So,the solution to $\mathbf{AX} = 2\mathbf{X}$is

$$\mathbf{X} = c\begin{pmatrix}1\\1\\1\end{pmatrix} \tag{1.2.2.15}$$

solution of $\mathbf{AX} = 3\mathbf{X}$ is eigen vector corresponding to $\lambda = 3$

$$(\mathbf{A} - 3\mathbf{I})\mathbf{X} = 0 \tag{1.2.2.16}$$

substituting we have:

$$\begin{pmatrix}3 & -4 & 0 & 0\\1 & -2 & 0 & 0\\-1 & 0 & 0 & 0\end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{R_1}{3}} \begin{pmatrix}1 & -\frac{4}{3} & 0 & 0\\4 & -5 & 0 & 0\\-1 & 0 & 0 & 0\end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - 4R_1}$$

$$\begin{pmatrix}1 & -\frac{4}{3} & 0 & 0\\0 & \frac{1}{3} & 0 & 0\\-1 & 0 & 0 & 0\end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 + R_1} \begin{pmatrix}1 & -\frac{4}{3} & 0 & 0\\0 & \frac{1}{3} & 0 & 0\\0 & -\frac{4}{3} & 0 & 0\end{pmatrix} \xleftrightarrow{R_2 \leftarrow \frac{R_2}{3}}$$

$$\begin{pmatrix}1 & \frac{-4}{3} & 0 & 0\\0 & 1 & 0 & 0\\0 & -\frac{4}{3} & 0 & 0\end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 - \frac{4}{3}R_2} \begin{pmatrix}1 & \frac{4}{3} & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 0\end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_1 + \frac{4}{3}R_2}$$

$$\begin{pmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 0\end{pmatrix} \tag{1.2.2.17}$$

So x_3 is a free variable:

$$x_1 = 0 \tag{1.2.2.18}$$
$$x_2 = 0 \tag{1.2.2.19}$$
$$x_3 = c \tag{1.2.2.20}$$

So,the solution to $\mathbf{AX}$ =3$\mathbf{X}$ is,

$$\mathbf{X} = c\begin{pmatrix}0\\0\\1\end{pmatrix} \tag{1.2.2.21}$$

1.2.3. Find a row-reduced matrix which is row equivalent to,

$$\mathbf{A} = \begin{pmatrix}i & -(1+i) & 0\\1 & -2 & 1\\1 & 2i & -1\end{pmatrix} \tag{1.2.3.1}$$

Solution: Step 1: Performing scaling operation to matrix $\mathbf{A}$ as $R_1 \leftarrow \frac{1}{i}R_1$ by scaling matrix D_1 given as,

$$\mathbf{D_1} = \begin{pmatrix}\frac{1}{i} & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix} \tag{1.2.3.2}$$

$$\mathbf{D_1A} = \begin{pmatrix}\frac{1}{i} & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}i & -(1+i) & 0\\1 & -2 & 1\\1 & 2i & -1\end{pmatrix} \tag{1.2.3.3}$$

$$\implies \mathbf{D_1A} = \begin{pmatrix}1 & -1+i & 0\\1 & -2 & 1\\1 & 2i & -1\end{pmatrix} \tag{1.2.3.4}$$

Step 2: Performing $R_2 \leftarrow R_2 - R_1$ and $R_3 \leftarrow R_3 - R_1$ given by elementary matrix $\mathbf{E_{31}E_{21}}$ on equation (1.2.3.4),

$$\mathbf{E_{31}E_{21}} = \begin{pmatrix}1 & 0 & 0\\-1 & 1 & 0\\-1 & 0 & 1\end{pmatrix} \tag{1.2.3.5}$$

$$\mathbf{E_{31}E_{21}D_1A} = \begin{pmatrix}1 & 0 & 0\\-1 & 1 & 0\\-1 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & -1+i & 0\\1 & -2 & 1\\1 & 2i & -1\end{pmatrix} \tag{1.2.3.6}$$

$$\implies \mathbf{A_1} = \mathbf{E_{31}E_{21}D_1A} = \begin{pmatrix}1 & -1+i & 0\\0 & -1-i & 1\\0 & 1+i & -1\end{pmatrix} \tag{1.2.3.7}$$

Step 3: Performing $R_2 \leftarrow \frac{-1}{1+i}R_2$ given by $\mathbf{D_2}$

on equation (1.2.3.7),

$$\mathbf{D_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2}(-1+i) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.2.3.8}$$

$$\mathbf{D_2A_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2}(-1+i) & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1+i & 0 \\ 0 & -1-i & 1 \\ 0 & 1+i & -1 \end{pmatrix} \tag{1.2.3.9}$$

$$\implies \mathbf{A_2} = \mathbf{D_2A_1} = \begin{pmatrix} 1 & -1+i & 0 \\ 0 & 1 & \frac{1}{2}(-1+i) \\ 0 & 1+i & -1 \end{pmatrix} \tag{1.2.3.10}$$

Step 4: Performing $R_3 \leftarrow R_3 - (1+i)R_2$ given by $\mathbf{E_{32}}$ on equation (1.2.3.10),

$$\mathbf{E_{32}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -(1+i) & 1 \end{pmatrix} \tag{1.2.3.11}$$

$$\mathbf{E_{32}A_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1-i & 1 \end{pmatrix}\begin{pmatrix} 1 & -1+i & 0 \\ 0 & 1 & \frac{-1+i}{2} \\ 0 & 1+i & -1 \end{pmatrix} \tag{1.2.3.12}$$

$$\implies \mathbf{A_3} = \mathbf{E_{32}A_2} = \begin{pmatrix} 1 & -1+i & 0 \\ 0 & 1 & \frac{-1+i}{2} \\ 0 & 0 & 1 \end{pmatrix} \tag{1.2.3.13}$$

Step 5: Performing $R_1 \leftarrow R_1 - (-1+i)R_2$ given by $\mathbf{E_{12}}$ on equation (1.2.3.13),

$$\mathbf{E_{12}} = \begin{pmatrix} 1 & 1-i & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.2.3.14}$$

$$\mathbf{E_{12}A_3} = \begin{pmatrix} 1 & 1-i & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1+i & 0 \\ 0 & 1 & \frac{-1+i}{2} \\ 0 & 0 & 1 \end{pmatrix} \tag{1.2.3.15}$$

$$\implies \mathbf{A_4} = \mathbf{E_{12}A_3} = \begin{pmatrix} 1 & 0 & i \\ 0 & 1 & \frac{-1+i}{2} \\ 0 & 0 & 1 \end{pmatrix} \tag{1.2.3.16}$$

Step 6: Performing $R_1 \leftarrow R_1 - iR_3$ and $R_2 \leftarrow R_2 - \frac{-1+i}{2}R_3$ given by $\mathbf{E_{13}E_{23}}$ on equation (1.2.3.16),

$$\mathbf{E_{13}E_{23}} = \begin{pmatrix} 1 & 0 & -i \\ 0 & 1 & -\left(\frac{-1+i}{2}\right) \\ 0 & 0 & 1 \end{pmatrix} \tag{1.2.3.17}$$

$$\mathbf{E_{13}E_{23}A_4} = \begin{pmatrix} 1 & 0 & -i \\ 0 & 1 & -\left(\frac{-1+i}{2}\right) \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & i \\ 0 & 1 & \frac{-1+i}{2} \\ 0 & 0 & 1 \end{pmatrix} \tag{1.2.3.18}$$

$$\implies \mathbf{A_5} = \mathbf{E_{13}E_{23}A_4} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.2.3.19}$$

$\therefore$ Row-reduced matrix of $\mathbf{A}$ given by equation (1.2.3.1) is,

$$\mathbf{A} = \begin{pmatrix} i & -1-i & 0 \\ 1 & -2 & 1 \\ 1 & 2i & -1 \end{pmatrix} \xleftrightarrow{RREF} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I} \tag{1.2.3.20}$$

1.2.4. Prove that the following two matrices are not row equivalent

$$\begin{pmatrix} 2 & 0 & 0 \\ a & -1 & 0 \\ b & c & 3 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 2 \\ -2 & 0 & -1 \\ 1 & 3 & 5 \end{pmatrix} \tag{1.2.4.1}$$

Solution: Call the first matrix $\mathbf{A}$ and the second matrix $\mathbf{B}$.

$$\mathbf{A}^T = \begin{pmatrix} 2 & a & b \\ 0 & -1 & c \\ 0 & 0 & 3 \end{pmatrix} \tag{1.2.4.2}$$

$\mathbf{A}^T$ is a upper triangular matrix with non-zero diagonal. Hence it has full rank = 3.

$$\mathbf{B}^T = \begin{pmatrix} 1 & -2 & 1 \\ 1 & 0 & 3 \\ 2 & -1 & 5 \end{pmatrix} \xleftrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - R_1} \begin{pmatrix} 1 & -2 & 1 \\ 0 & 2 & 2 \\ 0 & 3 & 3 \end{pmatrix} \tag{1.2.4.3}$$

$$\xleftrightarrow[R_2 \leftarrow R_2/2]{R_3 \leftarrow R_3/3} \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \tag{1.2.4.4}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - R_2} \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.2.4.5}$$

$\mathbf{B}^T$ is a upper triangular matrix with zero diagonal. Hence it doesn't have full rank. Therefore both matrices have different rank , so it cannot be row equivalent.

1.2.5. Let

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{1.2.5.1}$$

be a 2×2 matrix with complex entries. Suppose A is row-reduced and also that $a+b+c+d = 0$. Prove that there are exactly three such matrices.

Solution: A matrix is in row echelon form if it follows the following conditions
1. All nonzero rows are above any rows of all zeros.
2. Each leading entry (i.e. left most nonzero entry) of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zero Row Reduced Echelon Form A matrix is in row reduced echelon form if it follows the following conditions
1. The matrix should be row echelon form
2. The leading entry in each nonzero row is 1.
3. Each leading 1 is the only nonzero entry in its column. Proof
Given ,

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{1.2.5.2}$$

Condition 1 : Matrix **A** should be in row-reduced echelon form
Condition 2 : $a + b + c + d = 0$ where a,b,c and d are the elements of the matrix **A**

Reducing the matrix **A** from equation (1.2.5.2)

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \xleftrightarrow{R_1 = \frac{1}{a} R_1} \begin{pmatrix} 1 & \frac{b}{a} \\ c & d \end{pmatrix} \tag{1.2.5.3}$$

$$\xleftrightarrow{R_2 = R_2 - cR_1} \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & \frac{ad-bc}{a} \end{pmatrix} \tag{1.2.5.4}$$

$$\xleftrightarrow{R_2 = \frac{a}{ad-bc} R_2} \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix} \tag{1.2.5.5}$$

$$\xleftrightarrow{R_1 = R_1 - \frac{b}{a} R_2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.2.5.6}$$

Case 1: Matrix **A** of Rank 2
From the equation (1.2.5.4),for the matrix to be in row reduced echelon form,

$$b = 0$$
$$a \neq 0$$
$$d = 1$$
$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.2.5.7}$$

For the condition 2 to get satisfied,

$$a + 0 + c + 1 = 0 \tag{1.2.5.8}$$
$$\implies a = -(c+1) \tag{1.2.5.9}$$
$$\implies c \neq -1 \tag{1.2.5.10}$$

Both the condition gets satisfied and so exactly one matrix **A** can be formed of Rank 2 with given conditions

Case 2: Matrix **A** of Rank 1
From the equation (1.2.5.4),for the matrix to be in row reduced echelon form,

$$a \neq 0$$
$$d = 0$$
$$c = 0$$

For the condition 2 to get satisfied,

$$a + b + 0 + 0 = 0 \tag{1.2.5.11}$$
$$\implies b = -a \tag{1.2.5.12}$$

$$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \tag{1.2.5.13}$$

Both the condition gets satisfied and so exactly one matrix **A** can be formed of Rank 1 with given conditions

Case 3: Matrix **A** of Rank 0
From equation (1.2.5.2),for the matrix to be in

row reduced echelon form,

$$a = 0$$
$$b = 0$$
$$c = 0$$
$$d = 0$$
$$\mathbf{A} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{1.2.5.14}$$

Both the condition gets satisfied and so exactly one matrix **A** can be formed of Rank 0 with given conditions

Therefore matrix **A** shown in equation (1.2.5.7),(1.2.5.13) and (1.2.5.14) are the exactly three such matrices that can be formed with given conditions.

1.2.6. Prove that the interchange of two rows of a matrix can be accomplished by a finite sequence of elementary row operations of the other two types.

Solution: Let **A** be a 3×3 matrix with having row vectors $\mathbf{a}_1$,$\mathbf{a}_2$ and $\mathbf{a}_3$.

$$\mathbf{A} = \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} \tag{1.2.6.1}$$

Let's exchange row $\mathbf{a}_1$ and $\mathbf{a}_2$. Let's call this elementary operation $\mathbf{E_1}$.

$$\mathbf{E}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.2.6.2}$$

$$\tag{1.2.6.3}$$

Now performing operation $\mathbf{E}_1$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{a}_2 \\ \mathbf{a}_1 \\ \mathbf{a}_3 \end{pmatrix} \tag{1.2.6.4}$$

Now, to prove that same matrix can be obtained by elementary operations let's call them $\mathbf{E_2}$ and $\mathbf{E_3}$.Now performing operation $\mathbf{E}_2$ by adding row 2 to row 1.

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{a}_1 + \mathbf{a}_2 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} \tag{1.2.6.5}$$

Using elementary operation $\mathbf{E_2}$ we will subtract row 1 from row 2.

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}_1 + \mathbf{a}_2 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{a}_1 + \mathbf{a}_2 \\ -\mathbf{a}_1 \\ \mathbf{a}_3 \end{pmatrix} \tag{1.2.6.6}$$

Using elementary operation $\mathbf{E_2}$ we will add row 2 to row 1.

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}_1 + \mathbf{a}_2 \\ -\mathbf{a}_1 \\ \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{a}_2 \\ -\mathbf{a}_1 \\ \mathbf{a}_3 \end{pmatrix} \tag{1.2.6.7}$$

Using elementary operation $\mathbf{E_3}$ we will multiply row 2 by -1.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}_2 \\ -\mathbf{a}_1 \\ \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{a}_2 \\ \mathbf{a}_1 \\ \mathbf{a}_3 \end{pmatrix} \tag{1.2.6.8}$$

Hence, we can say that,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} \tag{1.2.6.9}$$

Let us assume a matrix **A**

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \tag{1.2.6.10}$$

Let's exchange row $\mathbf{a}_1$ and $\mathbf{a}_2$ by applying operation $\mathbf{E_1}$.

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & 3 \\ 1 & 1 & 0 \end{pmatrix} \tag{1.2.6.11}$$

Now, to prove that same matrix can be obtained by other two elementary operations.We will first perform elementary operation $\mathbf{E_2}$ by adding row 2 to row 1.

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \tag{1.2.6.12}$$

Using elementary operation $\mathbf{E_2}$ we will subtract row 1 from row 2.

$$\begin{pmatrix}1&0&0\\-1&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&3&3\\0&1&0\\1&1&0\end{pmatrix}=\begin{pmatrix}1&3&3\\-1&-2&-3\\1&1&0\end{pmatrix} \tag{1.2.6.13}$$

Using elementary operation $\mathbf{E_2}$ we will add row 2 to row 1.

$$\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&3&3\\-1&-2&-3\\1&1&0\end{pmatrix}=\begin{pmatrix}0&1&0\\-1&-2&-3\\1&1&0\end{pmatrix} \tag{1.2.6.14}$$

Using elementary operation $\mathbf{E_3}$ we will multiply row 2 by -1.

$$\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}\begin{pmatrix}0&1&0\\-1&-2&-3\\1&1&0\end{pmatrix}=\begin{pmatrix}0&1&0\\1&2&3\\1&1&0\end{pmatrix} \tag{1.2.6.15}$$

Hence,we can say that,

$$\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\-1&1&0\\0&0&1\end{pmatrix}$$
$$\times\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&2&3\\0&1&0\\1&1&0\end{pmatrix}=\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&2&3\\0&1&0\\1&1&0\end{pmatrix} \tag{1.2.6.16}$$

1.2.7. Consider the system of equations $\mathbf{AX}=0$ where

$$\mathbf{A}=\begin{pmatrix}a&b\\c&d\end{pmatrix}$$

is a 2×2 matrix over the field F. Prove the following -

- If every entry of $\mathbf{A}$ is 0, then every pair x_1 and x_2 is a solution of $\mathbf{AX}=0$.
- If $ad-bc\neq 0$, then the system $\mathbf{AX}=0$ has only the trivial solution $x_1=x_2=0$
- If $ad-bc=0$ and some entry of $\mathbf{A}$ is different from 0, then there is a solution x_1^0 and x_2^0 such that x_1 and x_2 is a solution if and only if there is a scalar y such that $x_1=yx_1^0$ and $x_2=yx_2^0$

Solution: Solution 1 If every entry of $\mathbf{A}$ is 0 then the equation $\mathbf{AX}=0$ becomes,

$$\begin{pmatrix}0&0\\0&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=0 \tag{1.2.7.1}$$

$$\implies 0.x_1+0.x_2=0 \qquad \forall x_1,x_2\in F \tag{1.2.7.2}$$

Hence proved, every pair x_1 and x_2 is a solution for the equation $\mathbf{AX}=0$. Solution 2 **Case 1:** Let $a=0$. Since $ad-bc\neq 0$. As $bc\neq 0$ therefore $b\neq 0$ and $c\neq 0$. Hence, we can perform row reduction on the augmented matrix of equation $\mathbf{AX}$=0 as follows,

$$\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&b&0\\c&d&0\end{pmatrix}=\begin{pmatrix}\frac{1}{c}&0\\0&1\end{pmatrix}\begin{pmatrix}c&d&0\\0&b&0\end{pmatrix} \tag{1.2.7.3}$$

$$=\begin{pmatrix}1&0\\0&\frac{1}{b}\end{pmatrix}\begin{pmatrix}1&\frac{d}{c}&0\\0&b&0\end{pmatrix} \tag{1.2.7.4}$$

$$=\begin{pmatrix}1&-\frac{d}{c}\\0&1\end{pmatrix}\begin{pmatrix}1&\frac{d}{c}&0\\0&1&0\end{pmatrix} \tag{1.2.7.5}$$

$$=\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix} \tag{1.2.7.6}$$

Case 2: Let $a,b,c,d\neq 0$. Considering the following case,

$$\mathbf{AX}=\mathbf{u} \tag{1.2.7.7}$$

$$\implies \begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}u_1\\u_2\end{pmatrix} \tag{1.2.7.8}$$

Row Reducing the augmented matrix of (1.2.7.8) we get,

$$\begin{pmatrix}\frac{1}{a}&0\\0&1\end{pmatrix}\begin{pmatrix}a&b&u_1\\c&d&u_2\end{pmatrix}=\begin{pmatrix}1&0\\-c&1\end{pmatrix}\begin{pmatrix}1&\frac{b}{a}&\frac{u_1}{a}\\c&d&u_2\end{pmatrix} \tag{1.2.7.9}$$

$$=\begin{pmatrix}1&0\\0&\frac{a}{ad-bc}\end{pmatrix}\begin{pmatrix}1&\frac{b}{a}&\frac{u_1}{a}\\0&\frac{ad-bc}{a}&\frac{au_2-cu_1}{a}\end{pmatrix} \tag{1.2.7.10}$$

$$=\begin{pmatrix}1&-\frac{b}{a}\\0&1\end{pmatrix}\begin{pmatrix}1&\frac{b}{a}&\frac{u_1}{a}\\0&1&\frac{au_2-cu_1}{ad-bc}\end{pmatrix} \tag{1.2.7.11}$$

$$=\begin{pmatrix}1&0&\frac{du_1-bu_2}{ad-bc}\\0&1&\frac{au_2-cu_1}{ad-bc}\end{pmatrix} \tag{1.2.7.12}$$

From (1.2.7.12) we get,

$$x_1 = \frac{du_1 - bu_2}{ad - bc} \tag{1.2.7.13}$$

$$x_2 = \frac{au_2 - cu_1}{ad - bc} \tag{1.2.7.14}$$

Since $u_1 = 0$ and $u_2 = 0$ then from (1.2.7.13) and (1.2.7.14),

$$x_1 = 0 \tag{1.2.7.15}$$

$$x_2 = 0 \tag{1.2.7.16}$$

Hence we get,

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1.2.7.17}$$

In (1.2.7.6) and (1.2.7.17), we can see that $\mathbf{AX} = 0$ has only one trivial solution i.e $x_1 = x_2 = 0$ in all cases. Hence proved, the equation $\mathbf{AX}$=0 has only one trivial solution $x_1 = x_2 = 0$ Solution 3 **Case 1:** Let, $a \neq 0$ for $\mathbf{A}$. Given $ad - bc = 0$, we can perform row reduction on augmented matrix of equation $\mathbf{AX} = 0$ as follows,

$$\begin{pmatrix} \frac{1}{a} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b & 0 \\ c & d & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -c & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{b}{a} & 0 \\ c & d & 0 \end{pmatrix} \tag{1.2.7.18}$$

$$= \begin{pmatrix} 1 & \frac{b}{a} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad [\because ad - bc = 0] \tag{1.2.7.19}$$

Hence from (1.2.7.19), $\mathbf{AX} = 0$ if and only if

$$x_1 = -\frac{b}{a} x_2 \qquad [a \neq 0] \tag{1.2.7.20}$$

Letting $x_1^0 = -\frac{b}{a}$ and $x_2^0 = 1$ we get for $y = 1$,

$$x_1 = y x_1^0 \tag{1.2.7.21}$$

$$x_2 = y x_2^0 \tag{1.2.7.22}$$

which is a solution of the equation $\mathbf{AX} = 0$.
Case 2: Let, $b \neq 0$ for $\mathbf{A}$. Given $ad - bc = 0$, at first we multiply by elementary matrix to change the columns and the we can perform row reduction on augmented matrix of equation $\mathbf{AX} = 0$ as follows,

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} b & a & 0 \\ d & c & 0 \end{pmatrix} \tag{1.2.7.23}$$

Hence using the result obtained from (1.2.7.19) we can conclude for (1.2.7.23), $\mathbf{AX} = 0$ if and only if

$$x_2 = -\frac{a}{b} x_1 \qquad [b \neq 0] \tag{1.2.7.24}$$

Letting $x_2^0 = -\frac{a}{b}$ and $x_1^0 = 1$ we get for $y = 1$,

$$x_1 = y x_1^0 \tag{1.2.7.25}$$

$$x_2 = y x_2^0 \tag{1.2.7.26}$$

which is a solution of the equation $\mathbf{AX} = 0$.
Case 3: Let, $c \neq 0$ for $\mathbf{A}$. Given $ad - bc = 0$, we can perform row reduction on augmented matrix of equation $\mathbf{AX} = 0$ as follows,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b & 0 \\ c & d & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{c} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c & d & 0 \\ a & b & 0 \end{pmatrix} \tag{1.2.7.27}$$

$$= \begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{d}{c} & 0 \\ a & b & 0 \end{pmatrix} \tag{1.2.7.28}$$

$$= \begin{pmatrix} 1 & \frac{d}{c} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad [\because ad - bc = 0] \tag{1.2.7.29}$$

Hence from (1.2.7.29), $\mathbf{AX} = 0$ if and only if

$$x_1 = -\frac{d}{c} x_2 \qquad [a \neq 0] \tag{1.2.7.30}$$

Letting $x_1^0 = -\frac{d}{c}$ and $x_2^0 = 1$ we get for $y = 1$,

$$x_1 = y x_1^0 \tag{1.2.7.31}$$

$$x_2 = y x_2^0 \tag{1.2.7.32}$$

which is a solution of the equation $\mathbf{AX} = 0$.
Case 4: Let, $d \neq 0$ for $\mathbf{A}$. Given $ad - bc = 0$, at first we multiply by elementary matrix to change the columns and then we can perform row reduction on augmented matrix of equation $\mathbf{AX} = 0$ as follows,

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b & a & 0 \\ d & c & 0 \end{pmatrix} \tag{1.2.7.33}$$

$$= \begin{pmatrix} d & c & 0 \\ b & a & 0 \end{pmatrix} \tag{1.2.7.34}$$

Hence using the result from (1.2.7.29) we can conclude for (1.2.7.34), $\mathbf{AX} = 0$ if and only if

$$x_2 = -\frac{c}{d} x_1 \qquad [a \neq 0] \tag{1.2.7.35}$$

Letting $x_2^0 = -\frac{c}{d}$ and $x_1^0 = 1$ we get for $y = 1$,

$$x_1 = yx_1^0 \tag{1.2.7.36}$$

$$x_2 = yx_2^0 \tag{1.2.7.37}$$

which is a solution of the equation $\mathbf{AX} = 0$.

1.3 Row Reduced Echelon Matrices

1.3.1. Find all solutions to the following system of equations by row-reducing the co-efficient matrix:

$$\frac{1}{3}x_1 + 2x_2 - 6x_3 = 0 \tag{1.3.1.1}$$

$$-4x_1 \qquad + 5x_3 = 0 \tag{1.3.1.2}$$

$$-3x_1 + 6x_2 - 13x_3 = 0 \tag{1.3.1.3}$$

$$-\frac{7}{3}x_1 + 2x_2 - \frac{8}{3}x_3 = 0 \tag{1.3.1.4}$$

Solution: The coefficient matrix is:

$$A = \begin{pmatrix} \frac{1}{3} & 2 & -6 \\ -4 & 0 & 5 \\ -3 & 6 & -13 \\ -\frac{7}{3} & 2 & -\frac{8}{3} \end{pmatrix} \tag{1.3.1.5}$$

The number of rows of this coefficient matrix is $m = 4$ and the number of columns is n= 3, So in this case, $n < m$. Now the row operations are:

$$\begin{pmatrix} \frac{1}{3} & 2 & -6 \\ -4 & 0 & 5 \\ -3 & 6 & -13 \\ -\frac{7}{3} & 2 & -\frac{8}{3} \end{pmatrix} \xleftrightarrow[R_1 \leftarrow R_1 \times 3]{R_4 \leftarrow R_4 \times 3} \begin{pmatrix} 1 & 6 & -18 \\ -4 & 0 & 5 \\ -3 & 6 & -13 \\ -7 & 6 & -8 \end{pmatrix} \tag{1.3.1.6}$$

$$\xleftrightarrow{R_3 \leftarrow R_2 + R_3} \begin{pmatrix} 1 & 6 & -18 \\ -4 & 0 & 5 \\ -7 & 6 & -8 \\ -7 & 6 & -8 \end{pmatrix} \xleftrightarrow{R_4 \leftarrow R_4 - R_3} \tag{1.3.1.7}$$

$$\begin{pmatrix} 1 & 6 & -18 \\ -4 & 0 & 5 \\ -7 & 6 & -8 \\ 0 & 0 & 0 \end{pmatrix} \xleftrightarrow[R_3 \leftarrow R_3 + 7R_1]{R_2 \leftarrow R_2 + 4R_1} \begin{pmatrix} 1 & 6 & -18 \\ 0 & 24 & -67 \\ 0 & 48 & -138 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.3.1.8}$$

$$\xleftrightarrow{R_3 \leftarrow R_3/2} \begin{pmatrix} 1 & 6 & -18 \\ 0 & 24 & -67 \\ 0 & 24 & -69 \\ 0 & 0 & 0 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 - R_2} \tag{1.3.1.9}$$

$$\begin{pmatrix} 1 & 6 & -18 \\ 0 & 24 & -67 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow \frac{R_3}{(-2)}} \begin{pmatrix} 1 & 6 & -18 \\ 0 & 24 & -67 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.3.1.10}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{R_2}{4}} \begin{pmatrix} 1 & 6 & -18 \\ 0 & 6 & -\frac{67}{4} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 - R_2} \tag{1.3.1.11}$$

$$\begin{pmatrix} 1 & 0 & -\frac{5}{4} \\ 0 & 6 & -\frac{67}{4} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow \frac{R_2}{6}} \begin{pmatrix} 1 & 0 & -\frac{5}{4} \\ 0 & 1 & -\frac{67}{24} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.3.1.12}$$

$$\xleftrightarrow[R_1 \leftarrow R_1 + \frac{5R_3}{4}]{R_2 \leftarrow R_2 + \frac{67R_3}{24}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.3.1.13}$$

Now,

$$A\mathbf{x} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \tag{1.3.1.14}$$

So,

$$\mathbf{I_3x} = 0 \tag{1.3.1.15}$$

$$\implies \mathbf{x} = 0 \tag{1.3.1.16}$$

1.3.2. Let

$$\mathbf{A} = \begin{pmatrix} 3 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 0 \end{pmatrix} \tag{1.3.2.1}$$

For which triples (y_1, y_2, y_3) does the system $\mathbf{AX} = \mathbf{Y}$ have a solution ?
Solution:
Given ,

$$\mathbf{AX} = \mathbf{Y} \tag{1.3.2.2}$$

$$\begin{pmatrix} 3 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 0 \end{pmatrix} \mathbf{X} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \tag{1.3.2.3}$$

Now we try to find the matrix $\mathbf{B}$ such that $\mathbf{BA}$ gives the row echelon form of matrix $\mathbf{A}$.
Here, $\mathbf{B}$ is given by ,

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ -\frac{7}{5} & \frac{8}{5} & 1 \end{pmatrix} \tag{1.3.2.4}$$

$$\implies \mathbf{BA} = \begin{pmatrix} 3 & -1 & 2 \\ 0 & \frac{5}{3} & -\frac{1}{3} \\ 0 & 0 & -\frac{6}{5} \end{pmatrix} \tag{1.3.2.5}$$

Therefore, from (1.3.2.5) rank of matrix $\mathbf{A}$ is 3 and it is a full rank matrix.
Hence the columns of $\mathbf{A}$ are linearly independent.
Therefore, the triples (y_1, y_2, y_3) are linear combination of columns of matrix $\mathbf{A}$.

$$\implies \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = a \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + b \begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix} + c \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \tag{1.3.2.6}$$

where a,b,c can be any real value.

1.3.3. Let

$$\mathbf{A} = \begin{pmatrix} 3 & -6 & 2 & -1 \\ -2 & 4 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 1 & -2 & 1 & 0 \end{pmatrix} \tag{1.3.3.1}$$

For which (y_1, y_2, y_3, y_4) does the system of equations $\mathbf{AX} = \mathbf{Y}$ have a solution ? **Solution:**
Given ,

$$\mathbf{AX} = \mathbf{Y} \tag{1.3.3.2}$$

$$\begin{pmatrix} 3 & -6 & 2 & -1 \\ -2 & 4 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 1 & -2 & 1 & 0 \end{pmatrix} \mathbf{X} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} \tag{1.3.3.3}$$

Now we try to find the matrix $\mathbf{B}$ such that $\mathbf{BA}$ gives the row echelon form of matrix $\mathbf{A}$ Here,$\mathbf{B}$ is given by ,

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{2}{3} & 1 & 0 & 0 \\ -\frac{2}{7} & -\frac{3}{7} & 1 & 0 \\ 0 & \frac{1}{2} & -\frac{3}{2} & 1 \end{pmatrix} \tag{1.3.3.4}$$

$$\mathbf{BA} = \begin{pmatrix} 3 & -6 & 2 & -1 \\ 0 & 0 & \frac{7}{3} & \frac{7}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1.3.3.5}$$

Therefore, rank of matrix $\mathbf{A}$ is 2 Now $\mathbf{B}$ is expressed in terms of two block matrices

$$\mathbf{B} = \begin{pmatrix} \mathbf{B_1} \\ \mathbf{B_2} \end{pmatrix} \tag{1.3.3.6}$$

$$\mathbf{B_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{2}{3} & 1 & 0 & 0 \end{pmatrix} \tag{1.3.3.7}$$

$$\mathbf{B_2} = \begin{pmatrix} -\frac{2}{7} & -\frac{3}{7} & 1 & 0 \\ 0 & \frac{1}{2} & -\frac{3}{2} & 1 \end{pmatrix} \tag{1.3.3.8}$$

Multiplying matrix $\mathbf{B}$ to both sides on the equation (1.3.3.2), we get ,

$$\begin{pmatrix} \mathbf{B_1} \\ \mathbf{B_2} \end{pmatrix} \mathbf{AX} = \begin{pmatrix} \mathbf{B_1} \\ \mathbf{B_2} \end{pmatrix} \mathbf{Y} \tag{1.3.3.9}$$

We know that , matrix $\mathbf{A}$ is of rank 2 The augumented matrix of (1.3.3.9) is given by

$$\left(\begin{array}{c|c} \mathbf{B_1A} & \mathbf{B_1Y} \\ \mathbf{B_2A} & \mathbf{B_2Y} \end{array}\right) \tag{1.3.3.10}$$

$$\mathbf{B_1A} = \begin{pmatrix} 3 & -6 & 2 & -1 \\ 0 & 0 & \frac{7}{3} & \frac{7}{3} \end{pmatrix} \tag{1.3.3.11}$$

$$\mathbf{B_2A} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1.3.3.12}$$

Since $\mathbf{B_2A}$ is zero matrix and for the given system $\mathbf{AX} = \mathbf{Y}$ to have a solution,

$$\mathbf{B_2Y} = 0 \tag{1.3.3.13}$$

$$\begin{pmatrix} -\frac{2}{7} & -\frac{3}{7} & 1 & 0 \\ 0 & \frac{1}{2} & -\frac{3}{2} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = 0 \tag{1.3.3.14}$$

The augumented matrix of (1.3.3.14) is given by,

$$\left(\begin{array}{cccc|c} -\frac{2}{7} & -\frac{3}{7} & 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{3}{2} & 1 & 0 \end{array}\right) \tag{1.3.3.15}$$

By row reduction technique,

$$\xleftrightarrow{R_1=-\frac{7}{2}R_1} \left(\begin{array}{cccc|c} 1 & \frac{3}{2} & -\frac{7}{2} & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{3}{2} & 1 & 0 \end{array}\right) \tag{1.3.3.16}$$

$$\xleftrightarrow{R_2=2R_2} \left(\begin{array}{cccc|c} 1 & \frac{3}{2} & -\frac{7}{2} & 0 & 0 \\ 0 & 1 & -3 & 2 & 0 \end{array}\right) \tag{1.3.3.17}$$

$$\xleftrightarrow{R_1=R_1-\frac{3}{2}R_2} \left(\begin{array}{cccc|c} 1 & 0 & 1 & -3 & 0 \\ 0 & 1 & -3 & 2 & 0 \end{array}\right) \tag{1.3.3.18}$$

Equation (1.3.3.14) can be modified as ,

$$\begin{pmatrix} 1 & 0 & 1 & -3 \\ 0 & 1 & -3 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = 0 \tag{1.3.3.19}$$

Here y_3 and y_4 are free variables
If $y_3 = a$ and $y_4 = b$,then the solution to the system of equation $\mathbf{AX} = \mathbf{Y}$ is given by,

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = a \begin{pmatrix} -1 \\ 3 \\ 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 3 \\ -2 \\ 0 \\ 1 \end{pmatrix} \tag{1.3.3.20}$$

One of the solution when $a = 1$ and $b = 2$ is given by ,

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 1 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 3 \\ -2 \\ 0 \\ 1 \end{pmatrix} \tag{1.3.3.21}$$

1.3.4. Suppose $\mathbf{R}$ and $\mathbf{R}'$ are 2×3 row-reduced echelon matrices and that the system $\mathbf{RX}$=0 and $\mathbf{R}'\mathbf{X}$=0 have exactly the same solutions. Prove that $\mathbf{R} = \mathbf{R}'$.
Solution:
Since $\mathbf{R}$ and $\mathbf{R}'$ are 2×3 row-reduced echelon matrices they can be of following three types:-

a) Suppose matrix $\mathbf{R}$ has one non-zero row then $\mathbf{RX}$=0 will have two free variables.Since $\mathbf{R}'\mathbf{X}$=0 will have the exact same solution as $\mathbf{RX} = 0, \mathbf{R}'\mathbf{X}$=0 will also have two free variables. Thus $\mathbf{R}'$ have one non zero row. Now let's consider a matrix $\mathbf{A}$ with the first row as the non-zero row $\mathbf{R}$ and second row as the second row of $\mathbf{R}'$.

$$\mathbf{R} = \begin{pmatrix} 1 & a & b \\ 0 & 0 & 0 \end{pmatrix} \tag{1.3.4.1}$$

$$\mathbf{R}' = \begin{pmatrix} 1 & c & d \\ 0 & 0 & 0 \end{pmatrix} \tag{1.3.4.2}$$

$$\tag{1.3.4.3}$$

Let $\mathbf{X}$ satisfy

$$\mathbf{RX} = 0 \tag{1.3.4.4}$$

$$\begin{pmatrix} 1 & \mathbf{a}^T \end{pmatrix} \begin{pmatrix} x \\ \mathbf{y} \end{pmatrix} = 0 \tag{1.3.4.5}$$

$$x + \mathbf{a}^T\mathbf{y} = 0 \tag{1.3.4.6}$$

where

$$\mathbf{a} = \begin{pmatrix} a \\ b \end{pmatrix} \tag{1.3.4.7}$$

$$\mathbf{R}'\mathbf{X} = 0 \tag{1.3.4.8}$$

$$\begin{pmatrix} 1 & \mathbf{b}^T \end{pmatrix} \begin{pmatrix} x \\ \mathbf{y} \end{pmatrix} = 0 \tag{1.3.4.9}$$

$$x + \mathbf{b}^T\mathbf{y} = 0 \tag{1.3.4.10}$$

where

$$\mathbf{b} = \begin{pmatrix} c \\ d \end{pmatrix} \tag{1.3.4.11}$$

Subtracting (1.3.4.10) from (1.3.4.6),

$$x + \mathbf{a}^T\mathbf{y} - x - \mathbf{b}^T\mathbf{y} = 0 \tag{1.3.4.12}$$

$$(\mathbf{a}^T - \mathbf{b}^T)\mathbf{y} = 0 \tag{1.3.4.13}$$

Since $\mathbf{y}$ is a 2×1 vector,

$$\implies y_1\mathbf{a} - y_2\mathbf{b} = 0 \tag{1.3.4.14}$$

Which can be written as,

$$\mathbf{a} = k\mathbf{b} \tag{1.3.4.15}$$

where,k=$\frac{\mathbf{y_2}}{\mathbf{y_1}}$ assuming $\mathbf{y}_1$ $\neq$0.Now,Substituting (1.3.4.15) in

(1.3.4.6)

$$x + k\mathbf{b}^T\mathbf{y} = 0 \tag{1.3.4.16}$$

Comparing (1.3.4.16) with (1.3.4.10)

$$x + \mathbf{b}^T\mathbf{y} = 0 \tag{1.3.4.17}$$
$$x + k\mathbf{b}^T\mathbf{y} = 0 \tag{1.3.4.18}$$

Hence k=1 which means y_1=y_2 and from this we can say that **a**=**b**.If in the above case we take y_1=0 then

$$y_1\mathbf{a} - y_2\mathbf{b} = 0 \tag{1.3.4.19}$$
$$y_2\mathbf{b} = 0 \tag{1.3.4.20}$$

Hence for the (1.3.4.20) to be always true **b** should be zero.Now from (1.3.4.15) we will see that **a** will also be 0. Hence, **R**=**R′**

b) Let **R** and $\mathbf{R}'$ have all rows as non zero.

$$\mathbf{R} = \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & c \end{pmatrix} \tag{1.3.4.21}$$
$$\mathbf{R}' = \begin{pmatrix} 1 & 0 & e \\ 0 & 1 & f \end{pmatrix} \tag{1.3.4.22}$$

Let **X** satisfy

$$\mathbf{R}\mathbf{X} = 0 \tag{1.3.4.23}$$
$$\mathbf{X}^T\mathbf{R}^T = 0 \tag{1.3.4.24}$$

Here,

$$\mathbf{R} = \begin{pmatrix} \mathbf{I} & \mathbf{a} \end{pmatrix} \tag{1.3.4.25}$$
$$\mathbf{a} = \begin{pmatrix} b \\ c \end{pmatrix} \tag{1.3.4.26}$$
$$\mathbf{R}^T = \begin{pmatrix} \mathbf{I} \\ \mathbf{a}^T \end{pmatrix} \tag{1.3.4.27}$$

Let,

$$\mathbf{X}^T = \begin{pmatrix} \mathbf{y}^T & z \end{pmatrix} \tag{1.3.4.28}$$

where z is a scalar constant. Now,substituting (1.3.4.28) and (1.3.4.25) in (1.3.4.24)

$$\begin{pmatrix} \mathbf{y}^T & z \end{pmatrix}\begin{pmatrix} \mathbf{I} \\ \mathbf{a}^T \end{pmatrix} = 0 \tag{1.3.4.29}$$
$$\mathbf{y}^T + z\mathbf{a}^T = 0 \tag{1.3.4.30}$$

Now for,

$$\mathbf{R}'\mathbf{X} = 0 \tag{1.3.4.31}$$
$$\mathbf{X}^T\mathbf{R}'^T = 0 \tag{1.3.4.32}$$

Here,

$$\mathbf{R}' = \begin{pmatrix} \mathbf{I} & \mathbf{b} \end{pmatrix} \tag{1.3.4.33}$$
$$\mathbf{b} = \begin{pmatrix} e \\ f \end{pmatrix} \tag{1.3.4.34}$$

Let,

$$\mathbf{X}^T = \begin{pmatrix} \mathbf{y}^T & z \end{pmatrix} \tag{1.3.4.35}$$

where z is a scalar constant.Now,substituting (1.3.4.35) and (1.3.4.33) in (1.3.4.32)

$$\begin{pmatrix} \mathbf{y}^T & z \end{pmatrix}\begin{pmatrix} \mathbf{I} \\ \mathbf{b}^T \end{pmatrix} = 0 \tag{1.3.4.36}$$
$$\mathbf{y}^T + z\mathbf{b}^T = 0 \tag{1.3.4.37}$$

Subtracting (1.3.4.37) from (1.3.4.30)

$$\mathbf{y}^T + z\mathbf{a}^T - \mathbf{y}^T - z\mathbf{b}^T = 0 \tag{1.3.4.38}$$
$$(\mathbf{a}^T - \mathbf{b}^T)z = 0 \tag{1.3.4.39}$$
$$\mathbf{a}^T = \mathbf{b}^T \tag{1.3.4.40}$$

c) Suppose matrix **R** have all the rows as zero then **RX**=0 will be satisfied for all values of **X**. We know that $\mathbf{R}'\mathbf{X}$=0 will have the exact same solution as **RX**=0 then we can say that for all values of **X**=0 equation $\mathbf{R}'\mathbf{X}$=0 will be satisfied.Hence, $\mathbf{R}'$=**R**=0.

1.4 Matrix Multiplication

1.4.1. Let

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 2 & -2 \\ 1 & 3 \\ 4 & 4 \end{pmatrix} \tag{1.4.1.1}$$

Verify directly that $A(AB) = A^2B$ **Solution:**

$$A^2 = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{pmatrix} \tag{1.4.1.2}$$

$$A^2 = \begin{pmatrix} 2 & -1 & 1 \\ 5 & -2 & 3 \\ 6 & -3 & 4 \end{pmatrix} \tag{1.4.1.3}$$

and

$$AB = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & -2 \\ 1 & 3 \\ 4 & 4 \end{pmatrix} \tag{1.4.1.4}$$

$$AB = \begin{pmatrix} 5 & -1 \\ 8 & 0 \\ 10 & -2 \end{pmatrix} \tag{1.4.1.5}$$

Now RHS is

$$A^2B = \begin{pmatrix} 2 & -1 & 1 \\ 5 & -2 & 3 \\ 6 & -3 & 4 \end{pmatrix} \begin{pmatrix} 2 & -2 \\ 1 & 3 \\ 4 & 4 \end{pmatrix} \tag{1.4.1.6}$$

$$A^2B = \begin{pmatrix} 7 & -3 \\ 20 & -4 \\ 25 & -5 \end{pmatrix} \tag{1.4.1.7}$$

Now LHS is

$$A(AB) = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 5 & -1 \\ 8 & 0 \\ 10 & -2 \end{pmatrix} \tag{1.4.1.8}$$

$$A(AB) = \begin{pmatrix} 7 & -3 \\ 20 & -4 \\ 25 & -5 \end{pmatrix} \tag{1.4.1.9}$$

Hence verified.

1.4.2. Find two different 2×2 matrices $\mathbf{A}$ such that $\mathbf{A}^2 = 0$ but $\mathbf{A} \neq 0$

Solution: The matrix $\mathbf{A}$ can be given by,

$$\mathbf{A} = \begin{pmatrix} \mathbf{m} & \mathbf{n} \end{pmatrix} \tag{1.4.2.1}$$

$$\mathbf{m} = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, \mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \tag{1.4.2.2}$$

Now,

$$\mathbf{A}^2 = \mathbf{A}\mathbf{A} = \mathbf{0} \tag{1.4.2.3}$$

$$\implies \mathbf{A}^2 = \begin{pmatrix} \mathbf{Am} & \mathbf{An} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \end{pmatrix} \tag{1.4.2.4}$$

From (1.4.2.4), we say that the the null space of $\mathbf{A}$ contains columns of matrix $\mathbf{A}$. Also atleast one of the columns must be non-zero since given $\mathbf{A} \neq 0$. Thus, the null space of $\mathbf{A}$ contains non zero vectors, $rank(\mathbf{A}) < 2$. Hence, $\mathbf{A}$ is a singular matrix. This implies that the columns of $\mathbf{A}$ are linearly dependent.

$$\mathbf{Ax} = 0 \tag{1.4.2.5}$$

$$\begin{pmatrix} \mathbf{m} & \mathbf{n} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \tag{1.4.2.6}$$

$$x_1\mathbf{m} + x_2\mathbf{n} = 0 \tag{1.4.2.7}$$

$$\mathbf{n} = \frac{-x_1}{x_2}\mathbf{m} \tag{1.4.2.8}$$

$$\implies \mathbf{n} = k\mathbf{m} \tag{1.4.2.9}$$

where $\mathbf{m} \neq 0$ as $\mathbf{A} \neq 0$
Now from (1.4.2.4),

$$\mathbf{Am} = 0 \tag{1.4.2.10}$$

$$m_1\mathbf{m} + m_2\mathbf{n} = 0 \tag{1.4.2.11}$$

$$(m_1 + km_2)\,\mathbf{m} = 0 \tag{1.4.2.12}$$

Thus we get, $m_1 = -km_2$

$$\mathbf{A} = \begin{pmatrix} -km_2 & -k^2m_2 \\ m_2 & km_2 \end{pmatrix}; m_2 \neq 0 \tag{1.4.2.13}$$

(1.4.2.9) can be written as,

$$\implies \mathbf{m} = \frac{1}{k}\mathbf{n} \tag{1.4.2.14}$$

$$\implies \mathbf{m} = c\mathbf{n} \tag{1.4.2.15}$$

where $\mathbf{n} \neq 0$ as $\mathbf{A} \neq 0$
From (1.4.2.4),

$$\mathbf{An} = 0 \tag{1.4.2.16}$$

$$n_1\mathbf{m} + n_2\mathbf{n} = 0 \tag{1.4.2.17}$$

$$(cn_1 + n_2)\,\mathbf{n} = 0 \tag{1.4.2.18}$$

Thus we get, $n_2 = -cn_1$

$$\mathbf{A} = \begin{pmatrix} cn_1 & n_1 \\ -c^2n_1 & -cn_1 \end{pmatrix}; n_1 \neq 0 \tag{1.4.2.19}$$

From (1.4.2.13), (1.4.2.19) two different 2×2 matrices $\mathbf{A}$ can be given as,

$$\mathbf{A} = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \tag{1.4.2.20}$$

$$\mathbf{A} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \tag{1.4.2.21}$$

1.4.3. Let $\mathbf{A}$ be an $m \times n$ matrix and $\mathbf{B}$ be an $n \times k$ matrix.Show that the columns of $\mathbf{C} = \mathbf{AB}$ are linear combinations of columns of $\mathbf{A}$.If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the columns of $\mathbf{A}$ and

$\gamma_{\mathbf{1}}, \gamma_{\mathbf{2}}, \ldots, \gamma_{\mathbf{k}}$ are the columns of $\mathbf{C}$ then,

$$\gamma_{\mathbf{j}} = \sum_{r=1}^{n} B_{rj}\alpha_{\mathbf{r}} \tag{1.4.3.1}$$

Solution:

$$\mathbf{C} = \mathbf{AB} \tag{1.4.3.2}$$

$$\mathbf{C} = \begin{pmatrix}\gamma_{\mathbf{1}} & \gamma_{\mathbf{2}} & \ldots & \gamma_{\mathbf{k}}\end{pmatrix} \tag{1.4.3.3}$$

$$\mathbf{A} = \begin{pmatrix}\alpha_{\mathbf{1}} & \alpha_{\mathbf{2}} & \ldots & \alpha_{\mathbf{n}}\end{pmatrix} \tag{1.4.3.4}$$

$$\mathbf{B} = \begin{pmatrix}\beta_{\mathbf{1}} & \beta_{\mathbf{2}} & \ldots & \beta_{\mathbf{k}}\end{pmatrix} \tag{1.4.3.5}$$

$$= \begin{pmatrix} B_{11} & B_{12} & \ldots & B_{1k} \\ B_{21} & B_{22} & \ldots & B_{2k} \\ \vdots & \vdots & \ldots & \vdots \\ B_{n1} & B_{n2} & \ldots & B_{nk} \end{pmatrix} \tag{1.4.3.6}$$

By matrix multiplication, we can write

$$\begin{pmatrix}\gamma_{\mathbf{1}} & \gamma_{\mathbf{2}} & \ldots & \gamma_{\mathbf{k}}\end{pmatrix} = \begin{pmatrix}\mathbf{A}\beta_{\mathbf{1}} & \mathbf{A}\beta_{\mathbf{2}} & \ldots & \mathbf{A}\beta_{\mathbf{k}}\end{pmatrix} \tag{1.4.3.7}$$

Consider $\gamma_{\mathbf{1}}$

$$\gamma_{\mathbf{1}} = \mathbf{A}\beta_{\mathbf{1}} \tag{1.4.3.8}$$

$$= \begin{pmatrix}\alpha_{\mathbf{1}} & \alpha_{\mathbf{2}} & \ldots & \alpha_{\mathbf{n}}\end{pmatrix} \begin{pmatrix} B_{11} \\ B_{21} \\ \vdots \\ B_{n1} \end{pmatrix} \tag{1.4.3.9}$$

$$= B_{11}\alpha_{\mathbf{1}} + B_{21}\alpha_{\mathbf{2}} + \ldots + B_{n1}\alpha_{\mathbf{n}} \tag{1.4.3.10}$$

Similarly, considering j^{th} column of $\mathbf{C}$

$$\gamma_{\mathbf{j}} = \begin{pmatrix}\alpha_{\mathbf{1}} & \alpha_{\mathbf{2}} & \ldots & \alpha_{\mathbf{n}}\end{pmatrix} \begin{pmatrix} B_{1j} \\ B_{2j} \\ \vdots \\ B_{nj} \end{pmatrix} \tag{1.4.3.11}$$

$$= B_{1j}\alpha_{\mathbf{1}} + B_{2j}\alpha_{\mathbf{2}} + \ldots + B_{nj}\alpha_{\mathbf{n}} \tag{1.4.3.12}$$

$$\implies \gamma_{\mathbf{j}} = \sum_{r=1}^{n} B_{rj}\alpha_{\mathbf{r}} \tag{1.4.3.13}$$

which proves that columns of $\mathbf{C}$ are linear combinations of columns of $\mathbf{A}$

1.4.4. Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ matrices such that $\mathbf{AB} = \mathbf{I}$. Prove that $\mathbf{BA} = \mathbf{I}$. **Solution:** Let $\mathbf{BX} = 0$ be a system of linear equation with n unknowns and n equations as $\mathbf{B}$ is $n \times n$ matrix. Hence,

$$\mathbf{BX} = 0 \tag{1.4.4.1}$$

$$\implies \mathbf{A}(\mathbf{BX}) = 0 \tag{1.4.4.2}$$

$$\implies (\mathbf{AB})\mathbf{X} = 0 \tag{1.4.4.3}$$

$$\implies \mathbf{IX} = 0 \quad [\because \mathbf{AB} = \mathbf{I}] \tag{1.4.4.4}$$

$$\implies \mathbf{X} = 0 \tag{1.4.4.5}$$

From (1.4.4.5) since $\mathbf{X} = 0$ is the only solution of (1.4.4.1), hence $rank(\mathbf{B}) = n$. Which implies all columns of $\mathbf{B}$ are linearly independent. Hence $\mathbf{B}$ is invertible. Therefore, every left inverse of $\mathbf{B}$ is also a right inverse of $\mathbf{B}$. Hence there exists a $n \times n$ matrix $\mathbf{C}$ such that,

$$\mathbf{BC} = \mathbf{CB} = \mathbf{I} \tag{1.4.4.6}$$

Again given that $\mathbf{AB} = \mathbf{I}$. Hence,

$$\mathbf{AB} = \mathbf{I} \tag{1.4.4.7}$$

$$\implies \mathbf{ABC} = \mathbf{C} \tag{1.4.4.8}$$

$$\implies \mathbf{A}(\mathbf{BC}) = \mathbf{C} \tag{1.4.4.9}$$

$$\implies \mathbf{A} = \mathbf{C} \quad [\because \mathbf{BC} = \mathbf{I}] \tag{1.4.4.10}$$

Hence using (1.4.4.10) and (1.4.4.6) we can write,

$$\mathbf{BA} = \mathbf{I} \tag{1.4.4.11}$$

Hence Proved.

1.4.5. Let,

$$\mathbf{C} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \tag{1.4.5.1}$$

be a 2×2 matrix. We inquire when it is possible to find 2×2 matrices $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{C}=\mathbf{AB} - \mathbf{BA}$. Prove that such matrices can be found if and only if $C_{11} + C_{22}=0$. **Solution:** We have to find,

$$tr(\mathbf{C}) = C_{11} + C_{22} = tr(\mathbf{AB} - \mathbf{BA}) \tag{1.4.5.2}$$

$$\implies tr(\mathbf{C}) = tr(\mathbf{AB}) - tr(\mathbf{BA}) \tag{1.4.5.3}$$

We know that,

$$tr(\mathbf{AB}) = \sum_{i=1}^{2} (\mathbf{AB})_{ii} \tag{1.4.5.4}$$

$$\implies \sum_{i=1}^{2}\sum_{j=1}^{2} a_{ij}b_{ji} \tag{1.4.5.5}$$

$$\implies \sum_{j=1}^{2}\sum_{i=1}^{2} b_{ji}a_{ij} \tag{1.4.5.6}$$

$$\implies tr(\mathbf{AB}) = \sum_{j=1}^{2} \mathbf{BA}_{jj} \tag{1.4.5.7}$$

$$\implies tr(\mathbf{AB}) = tr(\mathbf{BA}) \tag{1.4.5.8}$$

Substituting equation (1.4.5.8) to (1.4.5.3) we get

$$\implies tr(\mathbf{C}) = tr(\mathbf{AB}) - tr(\mathbf{BA}) = 0 \tag{1.4.5.9}$$

1.5 Invertible Matrices

1.5.1. Let

$$\mathbf{A} = \begin{pmatrix} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 1 & 5 \end{pmatrix} \tag{1.5.1.1}$$

For which $\mathbf{X}$ does there exist a scalar c such that $\mathbf{AX} = c\mathbf{X}$

Solution: Given

$$\mathbf{A} = \begin{pmatrix} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 1 & 5 \end{pmatrix} \tag{1.5.1.2}$$

The given matrix has single eigenvalue as it is the lower triangular matrix and has equal diagonal elements. Hence $c_1 = c_2 = c_3 = 5$. To find the corresponding eigenvector, consider the following

$$(\mathbf{A} - c\mathbf{I})\mathbf{X} = 0 \tag{1.5.1.3}$$

$$\implies \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{1.5.1.4}$$

Solving the homogeneous system of linear equations by performing rref, we get

$$\begin{pmatrix} 30 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xleftrightarrow[R_3 \leftrightarrow R_2]{R_2 \leftrightarrow R_1} \begin{pmatrix} 31 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1.5.1.5}$$

Hence we get,

$$x_1 = 0, x_2 = 0, x_3 = t \tag{1.5.1.6}$$

where, x_3 is arbitrary. Therefore,

$$\mathbf{X} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} t \tag{1.5.1.7}$$

Hence, the given matrix has single eigenvector and is not diagonalizable.

1.5.2. Discover whether

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{pmatrix} \tag{1.5.2.1}$$

is invertible, and find $\mathbf{A^{-1}}$ if it exists.

Solution: The matrix $\mathbf{A}$ is in row reduced echolon form with four pivot elements. Therefore the rank($\mathbf{A}$) is 4. Hence the rows of matrix $\mathbf{A}$ constitute of 4 linearly independent vectors. Thus it can be concluded that matrix $\mathbf{A}$ is invertible. Using Gauss-Jordan Elimination, if there exists an elimentary matrix $\mathbf{E}$ such that $\mathbf{E}[\mathbf{A}\ \mathbf{I}] = [\mathbf{I}\ \mathbf{E}]$ then $\mathbf{E}$ is the inverse of A i.e $\mathbf{E} = \mathbf{A^{-1}}$.

$$[\mathbf{A}\ \mathbf{I}] = \begin{pmatrix} 1 & 2 & 3 & 4 & | & 1 & 0 & 0 & 0 \\ 0 & 2 & 3 & 4 & | & 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 4 & | & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 & | & 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.5.2.2}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - R_2} \begin{pmatrix} 1 & 0 & 0 & 0 & | & 1 & -1 & 0 & 0 \\ 0 & 2 & 3 & 4 & | & 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 4 & | & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 & | & 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.5.2.3}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - R_3} \begin{pmatrix} 1 & 0 & 0 & 0 & | & 1 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 & | & 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & 4 & | & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 & | & 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.5.2.4}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - R_4} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 1 \end{array}\right) \tag{1.5.2.5}$$

$$\xleftrightarrow[R_2 \leftarrow \frac{R_2}{2} \; R_3 \leftarrow \frac{R_3}{3}]{R_4 \leftarrow \frac{R_4}{4}} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{4} \end{array}\right)$$

$$= [\mathbf{I} \ \mathbf{E}] \tag{1.5.2.6}$$

Therefore, for the given problem,

$$\mathbf{A^{-1}} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix} \tag{1.5.2.7}$$

Generalization of above result to a matrix of any arbitrary size: Let

$$\mathbf{A} = \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_N \\ 0 & a_2 & a_3 & \dots & a_N \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & \dots & \dots & a_N \end{pmatrix} \tag{1.5.2.8}$$

Then

$$\mathbf{E}_1\mathbf{A} = \begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & \dots & \dots & 1 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_N \\ 0 & a_2 & a_3 & \dots & a_N \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & \dots & \dots & a_N \end{pmatrix} \tag{1.5.2.9}$$

$$= \begin{pmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & a_3 & \dots & a_N \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & \dots & \dots & a_N \end{pmatrix} \tag{1.5.2.10}$$

$$\mathbf{E}_2\mathbf{E}_1\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & -1 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & \dots & \dots & 1 \end{pmatrix} \begin{pmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & a_3 & \dots & a_N \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & \dots & \dots & a_N \end{pmatrix} \tag{1.5.2.11}$$

$$= \begin{pmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & \dots & \dots & a_N \end{pmatrix} \tag{1.5.2.12}$$

Proceeding in similar manner, we get

$$\mathbf{E}_N\mathbf{E}_{N-1}\dots\mathbf{E}_2\mathbf{E}_1\mathbf{A} = \mathbf{U} = \begin{pmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & 0 & \dots & 0 \\ 0 & 0 & a_3 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & \dots & \dots & a_N \end{pmatrix} \tag{1.5.2.13}$$

$$= \operatorname{diag}\begin{pmatrix} a_1 & a_2 & \dots & a_N \end{pmatrix} \tag{1.5.2.14}$$

$$\implies \quad \mathbf{A} = \mathbf{L}\mathbf{U} \tag{1.5.2.15}$$

where $\mathbf{L} = \mathbf{E}_1^{-1}\mathbf{E}_2^{-1}\dots\mathbf{E}_N^{-1}$

$$\implies \quad \mathbf{A}^{-1} = \mathbf{U}^{-1}\mathbf{L}^{-1} \tag{1.5.2.16}$$

$$\implies \quad \mathbf{A}^{-1} = \begin{pmatrix} \frac{1}{a_1} & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{a_2} & 0 & \dots & 0 \\ 0 & 0 & \frac{1}{a_3} & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & \dots & \dots & \frac{1}{a_N} \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & -1 & \dots & 0 \\ \vdots & & \vdots & & \\ 0 & \dots & \dots & \dots & 1 \end{pmatrix} \tag{1.5.2.17}$$

Therefore

$$\mathbf{A^{-1}} = \begin{pmatrix} \frac{1}{a_1} & -\frac{1}{a_1} & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{a_2} & -\frac{1}{a_2} & 0 & \dots & 0 \\ 0 & 0 & \frac{1}{a_3} & -\frac{1}{a_3} & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & \frac{1}{a_N} \end{pmatrix} \tag{1.5.2.18}$$

From (1.5.2.18) for the above problem

$$\mathbf{A^{-1}} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix} \tag{1.5.2.19}$$

1.5.3. Suppose **A** is a 2×1 matrix and **B** is 1×2 matrix.Prove that **C=AB** is non invertible.

Solution: Let's take **A** and **B** to be non zero vectors. Now,we know that for **C** to be non invertible $\mathbf{Cx} = 0$ should have a non trivial solution.So,

$$\mathbf{Cx} = 0 \tag{1.5.3.1}$$

$$\implies \mathbf{ABx} = 0 \tag{1.5.3.2}$$

Here, we know that **B** is 1×2 matrix and **x** is 2×1 matrix then **Bx** will result to a scalar constant k.

$$\implies \mathbf{A}k = 0 \tag{1.5.3.3}$$

For (1.5.3.3) to be true k should be zero. We also know that $\mathbf{B}$ is 1×2 matrix i.e. rows are less than column hence,

$$\mathbf{Bx} = 0 \tag{1.5.3.4}$$

will have a non trivial solution. Hence, using (1.5.3.3) and (1.5.3.4) we can say,

$$\mathbf{ABx} = 0 \tag{1.5.3.5}$$

will have a non trivial solution so, $\mathbf{C}$ is non invertible.

1.5.4. Let $\mathbf{A}$ be an $n \times n$ (square) matrix, Prove the following two statements:

a) If $\mathbf{A}$ is invertible and $\mathbf{AB} = 0$ for some $n \times n$ matrix $\mathbf{B}$, then $\mathbf{B} = 0$.

b) If $\mathbf{A}$ is not invertible, then there exists an $n \times n$ matrix $\mathbf{B}$ such that $\mathbf{AB} = 0$ but $\mathbf{B} \neq 0$.

Solution:

a) Given $\mathbf{A}$ is an invertible matrix and $\mathbf{AB} = 0$ then,

$$\mathbf{AB} = 0 \tag{1.5.4.1}$$
$$\implies \mathbf{A}^{-1}(\mathbf{AB}) = 0 \tag{1.5.4.2}$$
$$\implies (\mathbf{A}^{-1}\mathbf{A})\mathbf{B} = 0 \tag{1.5.4.3}$$
$$\implies \mathbf{IB} = 0 \quad [\because \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}] \tag{1.5.4.4}$$
$$\implies \mathbf{B} = 0 \tag{1.5.4.5}$$

b) If $\mathbf{A}$ is not invertible, then there exists an $n \times n$ matrix $\mathbf{B}$ such that $\mathbf{AB} = 0$ but $\mathbf{B} \neq 0$. Since $\mathbf{A}$ is not invertible, $\mathbf{AX} = 0$ must have a non-trivial solution. Let the non-trivial solution be,

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \tag{1.5.4.6}$$

Let $\mathbf{B}$ which is an $n \times n$ matrix have all its columns as $\mathbf{y}$.

$$\mathbf{B} = \begin{pmatrix} \mathbf{y} & \mathbf{y} & \cdots & \mathbf{y} \end{pmatrix} \tag{1.5.4.7}$$

From equation (1.5.4.7), we can say that $\mathbf{B} \neq 0$ but $\mathbf{AB} = 0$

1.5.5. Let A be a $m \times n$ matrix. Show that by a finite number of elementary row and/or column operations one can pass from A to a matrix R which is both row-reduced echelon and column-reduced echelon , i.e., $R_{ij} = 0$ if $i \neq j$, $R_{ii} = 1$, $1 \leq i \leq r$, $R_{ii} = 0$, if $i > r$. Show that $R = PAQ$, where P is an invertible $m \times m$ matrix and Q is an invertible $n \times n$ matrix.

Solution:

Lemma Every elementary matrix is invertible and the inverse is again an elementary matrix. If an elementary matrix E is obtained from I by using a certain row or column operation q, then E^{-1} is obtained from I by the "inverse" operation q^{-1}.

Solution Given $\mathbf{A}$ is a $m \times n$ matrix. Converting $\mathbf{A}$ into row reduced echelon form by performing a series of elementary row operations $\mathbf{P}$. Let $\mathbf{R}'$ be the row reduced echelon matrix. Also, by using the lemma we can tell that $\mathbf{P}$ is invertible and order $m \times m$.

$$\mathbf{R}' = \mathbf{PA} \tag{1.5.5.1}$$

where,

$$\mathbf{R}' = \begin{pmatrix} \mathbf{I} & \mathbf{F} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$$

$\mathbf{I}$ is an identity matrix, $\mathbf{F}$ is Free variables matrix and $\mathbf{0}$ represents a block of zeroes

$\mathbf{R}'$ is in row-reduced echelon form. To perform column operations, elementary matrices should be multiplied on the right side in order to convert the $\mathbf{R}'$ into column-reduced echelon form

$$\mathbf{R} = \mathbf{R}'\mathbf{Q} \tag{1.5.5.2}$$

But performing column operations on a matrix is equivalent to performing row operations on the transposed matrix.

$$\mathbf{R}^T = (\mathbf{R}'\mathbf{Q})^T$$
$$\implies \mathbf{R}^T = \mathbf{Q}^T\mathbf{R}'^T \tag{1.5.5.3}$$

Hence, by using lemma it can be observed that $\mathbf{Q}^T$ is invertible and of the order $n \times n$. Converting $\mathbf{R}^T$ to row-reduced echelon is equivalent to converting $\mathbf{R}$ to column-reduced echelon.

$$\mathbf{R} = \mathbf{PAQ} \tag{1.5.5.4}$$

where,

$$\mathbf{R} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \tag{1.5.5.5}$$

$\mathbf{I}$ is an identity matrix and $\mathbf{0}$ represents a block of zeroes. $\mathbf{Q}$ is a upper triangular matrix. $\mathbf{R}$ in (1.5.5.4) is in both row and column reduced echelon form. Hence proved.
Example Let,

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 5 & 7 \\ 1 & 2 & 3 & 4 \end{pmatrix} \tag{1.5.5.6}$$

To convert (1.5.5.6) into row reduced echelon form, $\mathbf{A}$ has to be multiplied by $\mathbf{P}$

$$\mathbf{P} = \begin{pmatrix} -5 & 3 & 0 \\ 2 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \tag{1.5.5.7}$$

$$\mathbf{R}' = \mathbf{PA} = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1.5.5.8}$$

$\mathbf{R}'$ is in row reduced echelon form. To convert (1.5.5.8) into column-reduced echelon form, elementary operations have to be performed on $\mathbf{R}'^T$. By multiplying all the elementary matrices,

$$\mathbf{Q}^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & 1 & 0 & 0 \\ -1 & 0 & -1 & 1 \end{pmatrix} \tag{1.5.5.9}$$

$$\implies \mathbf{Q} = \begin{pmatrix} 1 & 0 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.5.5.10}$$

So $\mathbf{PAQ}$ is in both row-reduced and column-reduced echelon form.

$$\mathbf{R} = \mathbf{PAQ} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1.5.5.11}$$

The inverses of $\mathbf{P}$ and $\mathbf{Q}$ are,

$$\mathbf{P}^{-1} = \begin{pmatrix} 1 & 3 & 0 \\ 2 & 5 & 0 \\ 1 & 3 & 1 \end{pmatrix}; \quad \mathbf{Q}^{-1} = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.5.5.12}$$

2 Vector Spaces

2.1 Vector Spaces

2.1.1. If $\mathbf{F}$ is a field, verify that vector space of all ordered n-tuples $\mathbf{F}^n$ is a vector space over the field $\mathbf{F}$.
Solution: Let $\mathbf{F}^n$ be a set of all ordered n-tuples over $\mathbf{F}$ i.e

$$\mathbf{F}^n = \left\{ \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} : a_1, a_2, \ldots, a_n \in \mathbf{F} \right\} \tag{2.1.1.1}$$

For $\mathbf{F}^n$ to be a vector space over $\mathbf{F}$ it must satisfy the closure property of vector addition and scalar multiplication.

Vector Addition in $\mathbf{F}^n$:
Let $\alpha = \left(a_i\right)$ and $\beta = \left(b_i\right)$ $\forall\ i = 1, 2, \cdots, n \in \mathbf{F}^n$ then

$$\alpha + \beta = \left(a_i\right) + \left(b_i\right) \tag{2.1.1.2}$$
$$= \left(a_i + b_i\right) \tag{2.1.1.3}$$

Since

$$a_i + b_i \in \mathbf{F}\ \forall\ i = 1, 2, \cdots, n \tag{2.1.1.4}$$
$$\implies \alpha + \beta \in \mathbf{F}^n \tag{2.1.1.5}$$

Scalar multiplication in $\mathbf{F}^n$ over $\mathbf{F}$:
Let $\alpha = \left(a_i\right)$ $\forall\ i = 1, 2, \cdots, n$ $\in \mathbf{F}^n$ and $a \in \mathbf{F}$ then

$$a\alpha = \left(aa_i\right) \tag{2.1.1.6}$$

Since

$$aa_i \in \mathbf{F}\ \forall\ i = 1, 2 \cdots, n \tag{2.1.1.7}$$
$$\implies a\alpha \in \mathbf{F}^n \tag{2.1.1.8}$$

Associativity of addition in $\mathbf{F}^n$:

Let $\alpha = \left(a_i\right)$, $\beta = \left(b_i\right)$, $\gamma = \left(g_i\right)$ $\forall\ i = 1, 2, \cdots, n \in \mathbf{F}^n$ then

$$\alpha + (\beta + \gamma) = \left(a_i\right) + \left(b_i + g_i\right) \tag{2.1.1.9}$$
$$= \left(a_i + b_i + g_i\right) \tag{2.1.1.10}$$
$$= \left(a_i + b_i\right) + \left(g_i\right) \tag{2.1.1.11}$$
$$= (\alpha + \beta) + \gamma \tag{2.1.1.12}$$

Existence of additive identity in $\mathbf{F}^n$:

We have $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbf{F}^n$ and $\alpha = \left(a_i\right)$ $\forall\ i = 1, 2, \cdots, n \in \mathbf{F}^n$ then

$$\left(a_i\right) + \left(0\right) = \left(a_i + 0\right) \tag{2.1.1.13}$$
$$= \left(a_i\right) \tag{2.1.1.14}$$

Therefore $\mathbf{0}$ is the additive identity in $\mathbf{F}^n$.

Existence of additive inverse of each element of $\mathbf{F}^n$:

If $\alpha = \left(a_i\right)$ $\forall\ i = 1, 2, \cdots, n \in \mathbf{F}^n$ then $\left(-a_i\right) \in \mathbf{F}^n$. Also we have

$$\left(-a_i\right) + \left(a_i\right) = \mathbf{0} \tag{2.1.1.15}$$

Therefore $-\alpha = \left(-a_i\right)$ is the additive inverse of α. Thus $\mathbf{F}^n$ is an abelian group with respect to addition.

Futher we observe that

a) If $a \in \mathbf{F}$ and $\alpha = \left(a_i\right)$, $\beta = \left(b_i\right)$ $\forall\ i = 1, 2, \cdots, n \in \mathbf{F}^n$ then

$$a(\alpha + \beta) = a\left(a_i + b_i\right) \tag{2.1.1.16}$$
$$= \left(a[a_i + b_i]\right) \tag{2.1.1.17}$$
$$= \left(aa_i + ab_i\right) \tag{2.1.1.18}$$
$$\left(aa_i\right) + \left(ab_i\right) \tag{2.1.1.19}$$
$$= a\left(a_i\right) + a\left(b_i\right) \tag{2.1.1.20}$$
$$= a\alpha + a\beta \tag{2.1.1.21}$$

b) If $a,b \in \mathbf{F}$ and $\alpha = \left(a_i\right)$ $\forall\ i = 1, 2, \cdots, n \in \mathbf{F}^n$ then

$$(a + b)\alpha = \left([a + b]a_i\right) \tag{2.1.1.22}$$
$$= \left(aa_i + ba_i\right) \tag{2.1.1.23}$$
$$= \left(aa_i\right) + \left(ba_i\right) \tag{2.1.1.24}$$
$$= a\left(a_i\right) + b\left(a_i\right) \tag{2.1.1.25}$$
$$= a\alpha + b\alpha \tag{2.1.1.26}$$

c) If $a,b \in \mathbf{F}$ and $\alpha = \left(a_i\right)$ $\forall\ i = 1, 2, \cdots, n \in \mathbf{F}^n$ then

$$(ab)\alpha = \left([ab]a_i\right) \tag{2.1.1.27}$$
$$= \left(a[ba_i]\right) \tag{2.1.1.28}$$
$$= a\left(ba_i\right) \tag{2.1.1.29}$$
$$= a(b\alpha) \tag{2.1.1.30}$$

d) If 1 is the unity element of $\mathbf{F}$ and $\alpha = \left(a_i\right)$ $\forall\ i = 1, 2, \cdots, n \in \mathbf{F}^n$ then

$$1\alpha = \left(1a_i\right) \tag{2.1.1.31}$$
$$= \left(a_i\right) \tag{2.1.1.32}$$
$$= \alpha \tag{2.1.1.33}$$

Hence $\mathbf{F}^n$ is a vector space over $\mathbf{F}$.

2.1.2. If $\mathbf{V}$ is a vector space over field $\mathbf{F}$,verify that:

$$(\alpha_1 + \alpha_2) + (\alpha_3 + \alpha_4) = [\alpha_2 + (\alpha_3 + \alpha_1)] + \alpha_4 \tag{2.1.2.1}$$

Solution: Using property of commutativity of (+) in $\mathbf{V}$

$$(\alpha_1 + \alpha_2) + (\alpha_3 + \alpha_4) = (\alpha_2 + \alpha_1) + (\alpha_3 + \alpha_4) \tag{2.1.2.2}$$

Using property of associativity of (+) in $\mathbf{V}$

$$(\alpha_2 + \alpha_1) + (\alpha_3 + \alpha_4) = \alpha_2 + [\alpha_1 + (\alpha_3 + \alpha_4)] \tag{2.1.2.3}$$

Using property of commutativity of (+) in $\mathbf{V}$

$$\alpha_2 + [\alpha_1 + (\alpha_3 + \alpha_4)] = \alpha_2 + (\alpha_3 + \alpha_1) + \alpha_4 \tag{2.1.2.4}$$

Using property of associativity of (+) in $\mathbf{V}$

$$\alpha_2 + (\alpha_3 + \alpha_1) + \alpha_4 = [\alpha_2 + (\alpha_3 + \alpha_1] + \alpha_4 \tag{2.1.2.5}$$

2.1.3. Let V be the set of all pairs (x,y) of real numbers and let F be the field of real numbers.

Define

$$(x, y) + (x_1, y_1) = (x + x_1, y + y_1) \quad (2.1.3.1)$$
$$c(x, y) = (cx, y) \quad (2.1.3.2)$$

Is V with these operations , a vector space over the field of real numbers ?

Solution: V={(x,y) $|x, y \in R\}$, consider $u = (x_1, y_1) \in V, a, b, c \in R$. Axioms with respect to addition and scalar multiplication.

a)

$$(a + b)u = (a + b)(x_1, y_1) \quad (2.1.3.3)$$
$$= ((a + b)x_1, y_1) \neq au + bu \quad (2.1.3.4)$$

Since V with the given operations the equation (2.1.3.4) contradicts the axioms of scalar multiplication . Hence it is not vector space over real number with these operations.

2.1.4. If $\mathbb{C}$ is the field of complex numbers, which vectors in $\mathbf{C}^3$ are linear combinations of $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$?

Solution: Expressing the given vectors as the columns of a matrix,

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \quad (2.1.4.1)$$

The row reduced echelon form of the matrix on performing elementary row operations can be given as,

$$\mathbf{R} = \mathbf{CA} \quad (2.1.4.2)$$

where $\mathbf{C}$ is the product of elementary matrices,

$$\mathbf{C} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} \quad (2.1.4.3)$$

Thus we get,

$$\mathbf{R} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (2.1.4.4)$$

From (2.1.4.4), $rank(\mathbf{A}) = 3$. Thus $\mathbf{A}$ is a full rank matrix. Hence the columns of $\mathbf{A}$ are linearly independent i.e., the given vectors are linearly independent and forms the basis for $\mathbf{C}^3$.

Hence any vector $\mathbf{Y} \in \mathbf{C}^3$ can be written as the linear combinations of $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

2.1.5. Let $\mathbf{V}$ be the set of pairs (x, y) of real numbers and let $\mathbf{F}$ be the field of real numbers. Define

$$(x, y) + (x_1, y_1) = (x + x_1, 0) \quad (2.1.5.1)$$
$$c(x, y) = (cx, 0) \quad (2.1.5.2)$$

Is $\mathbf{V}$, with these operations, a vector space?

Solution: $\mathbf{V}$ is a vector space if it satisfies all properties of the vector space. Let us consider the property of Existence of additive identity. According to Existence of additive identity, there is a unique vector $\mathbf{0}$ in $\mathbf{V}$ called the zero vector, such that $\alpha + \mathbf{0} = \alpha$ for all α in $\mathbf{V}$.

Let $u = (x_1, y_1) \in \mathbf{V}$

$$\begin{aligned} u + \mathbf{0} &= (x_1, y_1) + (0, 0) \\ &= (x_1 + 0, 0) \\ &= (x_1, 0) \\ &\neq u \end{aligned} \quad (2.1.5.3)$$

From (2.1.5.3), there does not exist an additive identity for $\mathbf{V}$.

Hence $\mathbf{V}$ is not a vector space.

2.1.6. Let $\mathbb{V}$ be the set of all complex-valued functions f on the real line such that

$$f(-t) = \overline{f(t)} \quad (2.1.6.1)$$

The bar denotes complex conjugation.Show that $\mathbb{V}$,with the operations

$$(f + g)(t) = f(t) + g(t) \quad (2.1.6.2)$$
$$(cf)(t) = cf(t) \quad (2.1.6.3)$$

is a vector space over the field of real numbers. Give an example of a function in V which is not real valued.

Solution: To prove that $\mathbb{V}$ with the given operations is a vector space over the field of real numbers, we have to start by proving that additivity and homogeneity both hold true. So, we have to prove that (cf+g)(t) is equal to cf(t)+g(t).

$$(cf + g)(t) \quad (2.1.6.4)$$
$$= (cf)(t) + g(t) \quad (2.1.6.5)$$
$$= cf(t) + g(t) \quad (2.1.6.6)$$

Now, we know that f(-t)= $\overline{f(-t)}$ and so (cf+g)(t) should also satisfy the property,

$$(cf+g)(-t) \tag{2.1.6.7}$$
$$= cf(-t)+g(-t) \tag{2.1.6.8}$$
$$= c\overline{f(t)}+\overline{g(t)} \tag{2.1.6.9}$$
$$= \overline{cf(t)+g(t)} \tag{2.1.6.10}$$
$$= \overline{(cf+g)(t)} \tag{2.1.6.11}$$

Example Let's take f(x)=a+ix

$$f(1) = a+i \tag{2.1.6.12}$$

Hence, f(x) is not real valued.Now,

$$f(x) = a+ix \tag{2.1.6.13}$$
$$f(-x) = a-ix \tag{2.1.6.14}$$
$$f(-x) = \overline{f(x)} \tag{2.1.6.15}$$

Since a and x $\in \mathbb{R}$, so f $\in \mathbb{V}$

2.2 *Subspaces*

2.2.1. Is the vector $\begin{pmatrix}3\\-1\\0\\-1\end{pmatrix}$ in the subspace of $\mathbf{R}^4$ spanned by the vectors $\begin{pmatrix}2\\-1\\3\\2\end{pmatrix}$, $\begin{pmatrix}-1\\1\\1\\-3\end{pmatrix}$ and $\begin{pmatrix}1\\1\\9\\-5\end{pmatrix}$? **Solution:** Expressing the given three vectors as columns of a matrix,

$$\mathbf{A} = \begin{pmatrix}2&-1&1\\-1&1&1\\3&1&9\\2&-3&-5\end{pmatrix} \tag{2.2.1.1}$$

and

$$\mathbf{b} = \begin{pmatrix}3\\-1\\0\\-1\end{pmatrix} \tag{2.2.1.2}$$

For the vector **b** to be in the subspace of $\mathbf{R}^4$ spanned by the three vectors.

$$\mathbf{Ax} = \mathbf{b} \tag{2.2.1.3}$$

must have a solution.

$$\begin{pmatrix}2&-1&1\\-1&1&1\\3&1&9\\2&-3&-5\end{pmatrix}\mathbf{x} = \begin{pmatrix}3\\-1\\0\\-1\end{pmatrix} \tag{2.2.1.4}$$

Forming the augmented matrix and row reducing it by elementary row operations,

$$\begin{pmatrix}2&-1&1&3\\-1&1&1&-1\\3&1&9&0\\2&-3&-5&-1\end{pmatrix} \xleftrightarrow[R_4\leftarrow R_4-R_1]{R_2\leftarrow 2R_2+R_1, R_3\leftarrow R_3-\frac{3}{2}R_1} \tag{2.2.1.5}$$

$$\begin{pmatrix}2&-1&1&3\\0&1&3&1\\0&\frac{5}{2}&\frac{15}{2}&\frac{-9}{2}\\0&-2&-6&-4\end{pmatrix} \xleftrightarrow[R_4\leftarrow R_4+2R_2]{R_3\leftarrow 2R_3-5R_2} \begin{pmatrix}2&-1&1&3\\0&1&3&1\\0&0&0&-14\\0&0&0&-2\end{pmatrix} \tag{2.2.1.6}$$

From (2.2.1.6), it is clear that the system does not have a solution. Hence the vector $\begin{pmatrix}3\\-1\\0\\-1\end{pmatrix}$ does not lie in the subspace of $\mathbf{R}^4$ spanned by the given three vectors.

2.2.2. Let **W** be the set of all $(x_1, x_2, x_3, x_4, x_5)$ in $\mathbb{R}^5$ which satisfy

$$2x_1 - x_2 + \frac{4}{3}x_3 - x_4 = 0 \tag{2.2.2.1}$$
$$x_1 + \frac{2}{3}x_3 - x_5 = 0 \tag{2.2.2.2}$$
$$9x_1 - 3x_2 + 6x_3 - 3x_4 - 3x_5 = 0 \tag{2.2.2.3}$$

Find a finite set of vectors which spans **W**. **Solution:** The given equations are,

$$2x_1 - x_2 + \frac{4}{3}x_3 - x_4 = 0 \tag{2.2.2.4}$$
$$x_1 + \frac{2}{3}x_3 - x_5 = 0 \tag{2.2.2.5}$$
$$9x_1 - 3x_2 + 6x_3 - 3x_4 - 3x_5 = 0 \tag{2.2.2.6}$$

which can be written as,

$$\begin{pmatrix}2&-1&\frac{4}{3}&-1&0\\1&0&\frac{2}{3}&0&-1\\9&-3&6&-3&-3\end{pmatrix}\mathbf{x} = 0 \tag{2.2.2.7}$$

Now, the augmented matrix,

$$\left(\begin{array}{ccccc|c} 2 & -1 & \frac{4}{3} & -1 & 0 & 0 \\ 1 & 0 & \frac{2}{3} & 0 & -1 & 0 \\ 9 & -3 & 6 & -3 & -3 & 0 \end{array}\right) \tag{2.2.2.8}$$

$$\xleftrightarrow{R_3=R_3-3R_1-3R_2} \left(\begin{array}{ccccc|c} 2 & -1 & \frac{4}{3} & -1 & 0 & 0 \\ 1 & 0 & \frac{2}{3} & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \tag{2.2.2.9}$$

$$\xleftrightarrow{R_2=R_2-\frac{1}{2}R_1} \left(\begin{array}{ccccc|c} 2 & -1 & \frac{4}{3} & -1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \tag{2.2.2.10}$$

$$\xleftrightarrow{R_2=2R_2} \left(\begin{array}{ccccc|c} 2 & -1 & \frac{4}{3} & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \tag{2.2.2.11}$$

$$\xleftrightarrow{R_1=R_1+R_2} \left(\begin{array}{ccccc|c} 2 & 0 & \frac{4}{3} & 0 & -2 & 0 \\ 0 & 1 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \tag{2.2.2.12}$$

So,

$$2x_1 + \frac{4}{3}x_3 - 2x_5 = 0 \tag{2.2.2.13}$$

$$x_2 + x_4 - 2x_5 = 0 \tag{2.2.2.14}$$

Solving the equations we get,

$$x_1 = -\frac{2}{3}x_3 + x_5 \tag{2.2.2.15}$$

$$x_2 = -x_4 + 2x_5 \tag{2.2.2.16}$$

which can be written as,

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \tag{2.2.2.17}$$

$$= \begin{pmatrix} -\frac{2}{3}x_3 + x_5 \\ -x_4 + 2x_5 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \tag{2.2.2.18}$$

$$= x_3 \begin{pmatrix} -\frac{2}{3} \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{2.2.2.19}$$

where x_3,x_4 and $x_5 \in \mathbb{R}$.Hence, the vectors $\begin{pmatrix} -\frac{2}{3} \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ will span **W**

2.2.3. Let **F** be a field and let n be a positive integer (n≥2). Let **V** be the vector space of all n×n matrices over **F**. Which of the following set of matrices **A** in **V** are subspaces of **V**?

a) all invertible **A**;
b) all non-invertible **A**;
c) all **A** such that **AB** = **BA**, where **B** is some fixed matrix in **V**;
d) all **A** such that $\mathbf{A}^2 = \mathbf{A}$.

Solution:

a) Let the matrices **A** and **B** $\in$ **V**, be set of invertible matrix. For them to be a subspace they need to be closed under addition. Let,

$$\mathbf{A} = \mathbf{I} \tag{2.2.3.1}$$

$$\mathbf{B} = -\mathbf{I} \tag{2.2.3.2}$$

It could be easily proven that both matrices

$\mathbf{A}$ and $\mathbf{B}$ are invertible as,

$$rank(\mathbf{I}_{nxn}) = rank\left(\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}_{nxn}\right) \quad (2.2.3.3)$$

$$\implies rank(-\mathbf{I}_{nxn}) = rank(\mathbf{I}_{nxn}) = n \quad (2.2.3.4)$$

or it is a full rank matrix as there are n pivots.

$$\therefore \mathbf{A} + \mathbf{B} = \mathbf{0}. \quad (2.2.3.5)$$

But the zero matrix $\mathbf{0}$ is non-invertible as,

$$rank(\mathbf{0}_{nxn}) = rank\left(\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}_{nxn}\right) \quad (2.2.3.6)$$

$$\implies rank(\mathbf{0}_{nxn}) = 0 \quad (2.2.3.7)$$

$\therefore$ **the set of invertible matrices are not closed under addition. Hence not a subspace of V.**

b) Let the matrices $\mathbf{A_1}$, $\mathbf{A_2}, \cdots, \mathbf{A_n} \in \mathbf{V}$, be set of non-invertible matrix. For them to be a subspace they need to be closed under addition. Let,

$$\mathbf{A_1} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}_{nxn} \quad (2.2.3.8)$$

$$\mathbf{A_2} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}_{nxn} \quad (2.2.3.9)$$

$$\mathbf{A_n} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}_{nxn} \quad (2.2.3.10)$$

(2.2.3.11)

It could be proven that matrices $\mathbf{A_1}$, $\mathbf{A_2}, \cdots, \mathbf{A_n}$ are non-invertible as,

$$rank(\mathbf{A_1}) = rank\left(\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}\right) \quad (2.2.3.12)$$

$$\implies rank(\mathbf{A_1}) = 1 \quad (2.2.3.13)$$

or there is only one pivot hence rank is 1.

$$\implies \mathbf{A_1} + \mathbf{A_2} + \mathbf{A_3} + \cdots \mathbf{A_n} = \mathbf{I}_{nxn} \quad (2.2.3.14)$$

Now the identity matrix $\mathbf{I}$ is invertible as shown in equation (2.2.3.4). $\therefore$ **the set of non-invertible matrices are not closed under addition. Hence not a subspace of V.**

c) **Theorem 1:**. A non-empty subset W of V is a subspace of V if and only if for each pair of vectors α, β in W and each scalar c $\in$ F, the vector $c\alpha + \beta \in$ W.
Let the matrices $\mathbf{A_1}$ and $\mathbf{A_2}$ satisfy,

$$\mathbf{A_1 B} = \mathbf{B A_1} \quad (2.2.3.15)$$

$$\mathbf{A_2 B} = \mathbf{B A_2} \quad (2.2.3.16)$$

Let, c$\in$ $\mathbf{F}$ be any constant.

$$\therefore (c\mathbf{A_1} + \mathbf{A_2})\,\mathbf{B} = c\mathbf{A_1 B} + \mathbf{A_2 B} \quad (2.2.3.17)$$

Substituting from equations (2.2.3.15) and (2.2.3.16) to (2.2.3.17),

$$\implies (c\mathbf{A_1} + \mathbf{A_2})\,\mathbf{B} = c\mathbf{B A_1} + \mathbf{B A_2} \quad (2.2.3.18)$$

$$\implies \mathbf{B}c\mathbf{A_1} + \mathbf{B A_2} \quad (2.2.3.19)$$

$$\implies \mathbf{B}\,(c\mathbf{A_1} + \mathbf{A_2}) \quad (2.2.3.20)$$

Thus, $(c\mathbf{A_1} + \mathbf{A_2})$ **satisfy the criteria and from Theorem-1 it can be seen that the set is a subspace of V.**

d) Let $\mathbf{A}$ and $\mathbf{B} \in \mathbf{V}$ be set of matrices such that,

$$\mathbf{A}^2 = \mathbf{A} \quad (2.2.3.21)$$

$$\mathbf{B}^2 = \mathbf{B} \quad (2.2.3.22)$$

Now for them to be closed under addition,

$$(\mathbf{A}+\mathbf{B})^2 = \mathbf{A}+\mathbf{B} \tag{2.2.3.23}$$

Which is not always same. Example let,

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \tag{2.2.3.24}$$

$$\mathbf{B} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.2.3.25}$$

Clearly,

$$\mathbf{A}^2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \mathbf{A} \tag{2.2.3.26}$$

$$\mathbf{B}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{B} \tag{2.2.3.27}$$

Now,

$$\mathbf{A}+\mathbf{B} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{2.2.3.28}$$

$$\implies (\mathbf{A}+\mathbf{B})^2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \tag{2.2.3.29}$$

Hence, clearly from equations (2.2.3.28) and (2.2.3.29),

$$(\mathbf{A}+\mathbf{B})^2 \neq \mathbf{A}+\mathbf{B} \tag{2.2.3.30}$$

$\therefore$ **the set of all A such that $\mathbf{A}^2 = \mathbf{A}$ is not closed under addition. Hence, not a subspace of V.**

2.2.4. Let $\mathbf{W_1}$ and $\mathbf{W_2}$ be subspaces of a vector space $\mathbf{V}$ such that

$$\mathbf{W_1}+\mathbf{W_2} = \mathbf{V} \tag{2.2.4.1}$$

$$\text{and } \mathbf{W_1}\cap\mathbf{W_2} = \mathbf{0} \tag{2.2.4.2}$$

Prove that for each vector α in $\mathbf{V}$ there are unique vectors α_1 in $\mathbf{W_1}$ and α_2 in $\mathbf{W_2}$ such that

$$\alpha = \alpha_1 + \alpha_2 \tag{2.2.4.3}$$

Solution: Suppose, vectors α_1 and α_2 are not unique.

Consider

$$\alpha_1' \in \mathbf{W_1}, \tag{2.2.4.4}$$

$$\alpha_2' \in \mathbf{W_2} \tag{2.2.4.5}$$

$$\text{such that } \alpha = \alpha_1' + \alpha_2' \tag{2.2.4.6}$$

(2.2.4.3) and (2.2.4.6) indicate

$$\alpha_1 + \alpha_2 = \alpha_1' + \alpha_2' \tag{2.2.4.7}$$

$$\implies \alpha_1 - \alpha_1' = \alpha_2' - \alpha_2 \tag{2.2.4.8}$$

For α_1 and α_1' lying in subspace $\mathbf{W_1}$, defined on field $\mathbb{F}$, the following holds

$$\alpha_1 + c\alpha_1' \in \mathbf{W_1}, c \in \mathbb{F} \tag{2.2.4.9}$$

$$c = -1 \implies \alpha_1 - \alpha_1' \in \mathbf{W_1} \tag{2.2.4.10}$$

$$\text{Similarly, } \alpha_2' - \alpha_2 \in \mathbf{W_2} \tag{2.2.4.11}$$

$$(2.2.4.8) \implies \alpha_1 - \alpha_1' \in \mathbf{W_2} \tag{2.2.4.12}$$

(2.2.4.2),(2.2.4.10),(2.2.4.12) indicate

$$\alpha_1 - \alpha_1' = \alpha_2' - \alpha_2 = \mathbf{0} \tag{2.2.4.13}$$

$$\implies \alpha_1 = \alpha_1' \tag{2.2.4.14}$$

$$\alpha_2 = \alpha_2' \tag{2.2.4.15}$$

So, there exists a unique $\alpha_1 \in \mathbf{W_1}$ and $\alpha_2 \in \mathbf{W_2}$ such that

$$\alpha = \alpha_1 + \alpha_2 \tag{2.2.4.16}$$

where $\alpha \in \mathbf{V}$

2.3 *Bases and Dimension*

2.3.1. Let $\mathbf{V}$ be a vector space over a subfield $\mathbf{F}$ of complex numbers. Suppose α, β and γ are linearly independent vectors in $\mathbf{V}$. Prove that $(\alpha+\beta)$,$(\beta+\gamma)$ and $(\gamma+\alpha)$ are linearly independent.

Solution: Let α, β and γ be three n$\times$ 1 dimensional vectors. We need to prove that,

$$\begin{pmatrix} \alpha+\beta & \beta+\gamma & \gamma+\alpha \end{pmatrix}\mathbf{x} = 0 \tag{2.3.1.1}$$

will only have a trivial solution.The above equation can be written as

$$\begin{pmatrix} \alpha & \beta & \gamma \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}\mathbf{x} = 0 \tag{2.3.1.2}$$

$$\mathbf{x}^T\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha^T \\ \beta^T \\ \gamma^T \end{pmatrix} = 0 \tag{2.3.1.3}$$

Since, α, β and γ are independent.

$$\mathbf{x}^T \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} = 0 \tag{2.3.1.4}$$

In the above equation we can see that the 3×3 matrix has linearly independent rows and hence will have a trivial solution.So,$\mathbf{x}$ is a zero vector.Hence, $(\alpha+\beta)$,$(\beta+\gamma)$ and $(\gamma+\alpha)$ are linearly independent.