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A Matrix Approach to Conic Sections

ARTICLE HISTORY
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ABSTRACT
Proofs of some theorems related to cyclic quadrilaterals are provided using coordinate geometry and trigonometry. Through this approach, constructions and proofs using contradiction are avoided.

KEYWORDS
cyclic quadrilateral, coordinate geometry, trigonometry

1. Introduction

2. Preliminaries

Definition 2.1. The equation of a line passing through $\mathbf{x}_0$ in $\mathbb{R}^2$ is given by

$$\begin{aligned} \mathbf{n}^\top(\mathbf{x}-\mathbf{x}_0) = 0 \qquad & \text{(normal form)} \\ \text{or, } \mathbf{x} = \mathbf{x}_0 + \lambda\mathbf{m} \quad & \text{(parametric form)} \end{aligned} \tag{1}$$

where

$$\mathbf{m} = \begin{pmatrix} 1 \\ m \end{pmatrix} \tag{2}$$

is defined to be the *direction vector* of the line, m being the slope and $\mathbf{n}$ defined by

$$\mathbf{m}^\top\mathbf{n} = 0 \tag{3}$$

is the *normal vector*.

Lemma 2.2. *The distance of a point* $\mathbf{P}$ *from the line*

$$\mathbf{n}^\top\mathbf{x} = c \tag{4}$$

is given by

$$d = \frac{\left|\mathbf{n}^\top\mathbf{P} - c\right|}{\|\mathbf{n}\|} \tag{5}$$

3. Vector Equation of a Conic Section

Definition 3.1. Let $\mathbf{P}$ be a point such that the ratio of its distance from a fixed point $\mathbf{F}$ and the distance (d) from a fixed line $L : \mathbf{n}^\top \mathbf{x} = c$ is constant, given by

$$\frac{\|\mathbf{P} - \mathbf{F}\|}{d} = e \tag{6}$$

The locus of $\mathbf{P}$ such is known as a conic section. The line L is known as the directrix and the point $\mathbf{F}$ is the focus. e is defined to be the eccentricity of the conic.

(1) For $e = 1$, the conic is a parabola
(2) For $e < 1$, the conic is an ellipse
(3) For $e > 1$, the conic is a hyperbola

Theorem 3.2. *The equation of a conic with directrix $\mathbf{n}^\top \mathbf{x} = c$, eccentricity e and focus $\mathbf{F}$ is given by*

$$\mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{7}$$

where

$$\mathbf{V} = \|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{n}\mathbf{n}^\top, \tag{8}$$
$$\mathbf{u} = ce^2 \mathbf{n} - \|\mathbf{n}\|^2 \mathbf{F}, \tag{9}$$
$$f = \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2 e^2 \tag{10}$$

Proof. Using Definition 3.1 and Lemma 2.2, for any point $\mathbf{x}$ on the conic,

$$\|\mathbf{x} - \mathbf{F}\|^2 = e^2 \frac{\left(\mathbf{n}^\top \mathbf{x} - c\right)^2}{\|\mathbf{n}\|^2} \tag{11}$$
$$\implies \|\mathbf{n}\|^2 (\mathbf{x} - \mathbf{F})^\top (\mathbf{x} - \mathbf{F}) = e^2 \left(\mathbf{n}^\top \mathbf{x} - c\right)^2 \tag{12}$$
$$\implies \|\mathbf{n}\|^2 \left(\mathbf{x}^\top \mathbf{x} - 2\mathbf{F}^\top \mathbf{x} + \|\mathbf{F}\|^2\right) = e^2 \left(c^2 + \left(\mathbf{n}^\top \mathbf{x}\right)^2 - 2c\mathbf{n}^\top \mathbf{x}\right) \tag{13}$$
$$= e^2 \left(c^2 + \left(\mathbf{x}^\top \mathbf{n}\mathbf{n}^\top \mathbf{x}\right) - 2c\mathbf{n}^\top \mathbf{x}\right) \tag{14}$$

which can be expressed as (7) after simplification.

□

Lemma 3.3. (7) *represents a parabola for $\left|V\right| = 0$, ellipse for $\left|V\right| > 0$ and hyperbola for $\left|V\right| < 0$. In general* (7) *represents a conic if and only if*

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^\top & f \end{vmatrix} \neq 0 \tag{15}$$

else, it represents a pair of straight lines.

Example 3.4. The focus and directrix of a parabola are given to be

$$F = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \tag{16}$$

$$\begin{pmatrix} 1 & -4 \end{pmatrix} \mathbf{x} = -3 \tag{17}$$

From the given information,

$$\mathbf{n} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}, c = -3, e = 1 \tag{18}$$

Substituting in (8), (9) and (10),

$$\mathbf{V} = (17) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 \\ -4 \end{pmatrix} \begin{pmatrix} 1 & -4 \end{pmatrix} = \begin{pmatrix} 16 & 4 \\ 4 & 1 \end{pmatrix} \tag{19}$$

$$\mathbf{u} = (-3) \begin{pmatrix} 1 \\ -4 \end{pmatrix} - (17) \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -37 \\ -39 \end{pmatrix} \tag{20}$$

$$f = (17)(13) - (9) = 212 \tag{21}$$

which is exactly the same as [1, Article 199, p. 176]

Example 3.5. The focus, directrix and eccentricity of an ellipse are given by

$$F = \begin{pmatrix} -2 \\ 3 \end{pmatrix} \tag{22}$$

$$\begin{pmatrix} 2 & 3 \end{pmatrix} \mathbf{x} = -4 \tag{23}$$

$$e = \frac{4}{5} \tag{24}$$

Substituting in (8), (9) and (10),

$$\mathbf{V} = (13) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \left(\frac{4}{5}\right)^2 \begin{pmatrix} 2 \\ 3 \end{pmatrix} \begin{pmatrix} 2 & 3 \end{pmatrix} = \frac{1}{25} \begin{pmatrix} 261 & -96 \\ -96 & 181 \end{pmatrix} \tag{25}$$

$$\mathbf{u} = (-4) \left(\frac{4}{5}\right)^2 \begin{pmatrix} 2 \\ 3 \end{pmatrix} - (13) \begin{pmatrix} -2 \\ 3 \end{pmatrix} = \frac{1}{25} \begin{pmatrix} 522 \\ -1167 \end{pmatrix} \tag{26}$$

$$f = (13)(13) - (16) \left(\frac{4}{5}\right)^2 = \frac{3969}{25} \tag{27}$$

which matches [1, Article 249, p. 227]

4. Conic Parameters

4.1. Through Eigenvalue Decomposition

Theorem 4.1. *The conic in* (7) *can be expressed in standard form (centre/vertex at the origin, major axis - x axis) as*

$$\mathbf{y}^\top \mathbf{D}\mathbf{y} = \mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f \qquad |V| \neq 0 \tag{28}$$

$$\mathbf{y}^\top \mathbf{D}\mathbf{y} = -2\eta \mathbf{e}_1^\top \mathbf{y} \qquad |V| = 0 \tag{29}$$

where

$$\mathbf{P}^\top \mathbf{V}\mathbf{P} = \mathbf{D}. \quad \textit{(Eigenvalue Decomposition)} \tag{30}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \tag{31}$$

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix}, \quad \mathbf{P}^\top = \mathbf{P}^{-1}, \tag{32}$$

$$\eta = \mathbf{u}^\top \mathbf{p}_1 \tag{33}$$

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{34}$$

Proof. See Appendix A. □

Corollary 4.2. *The centre/vertex of the conic is given by*

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \qquad |V| \neq 0 \tag{35}$$

$$\begin{pmatrix} \mathbf{u}^\top + \eta \mathbf{p}_1^\top \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \eta \mathbf{p}_1 - \mathbf{u} \end{pmatrix} \qquad |V| = 0 \tag{36}$$

Proof. From (A1),

$$\mathbf{y} = \mathbf{P}^\top (\mathbf{x} - \mathbf{c}) \tag{37}$$

For the standard conic, $\mathbf{y} = \mathbf{0}$ is the centre/vertex and in (37),

$$\mathbf{y} = \mathbf{0} \implies \mathbf{x} = \mathbf{c} \tag{38}$$

□

In the following, various parameters are directly obtained from the standard forms in (28) and (29) and the results in [2]

Corollary 4.3. *The focal length of the parabola in* (29) *is given by*

$$\left| \frac{2\eta}{\lambda_2} \right| \tag{39}$$

where λ_2 is the nonzero eigenvalue of $\mathbf{V}$ and η is defined in (33).

Proof. From (A11), by comparing the coefficients of y_2^2 and y_1, the focal length of the parabola is obtained as (39) □

Corollary 4.4. *For $\left|V\right| \neq 0$, the lengths of the semi-major and semi-minor axes of the conic in* (7) *are given by*

$$\sqrt{\frac{\mathbf{u}^\top\mathbf{V}^{-1}\mathbf{u}-f}{\lambda_1}}, \sqrt{\frac{\mathbf{u}^\top\mathbf{V}^{-1}\mathbf{u}-f}{\lambda_2}}. \quad (\textit{ellipse}) \tag{40}$$

$$\sqrt{\frac{\mathbf{u}^\top\mathbf{V}^{-1}\mathbf{u}-f}{\lambda_1}}, \sqrt{\frac{f-\mathbf{u}^\top\mathbf{V}^{-1}\mathbf{u}}{\lambda_2}}, \quad (\textit{hyperbola}\ ) \tag{41}$$

Proof. For

$$\left|\mathbf{V}\right| > 0, \quad \text{or, } \lambda_1 > 0, \lambda_2 > 0 \tag{42}$$

and (28) becomes

$$\lambda_1 y_1^2 + \lambda_2 y_2^2 = \mathbf{u}^\top\mathbf{V}^{-1}\mathbf{u} - f \tag{43}$$

yielding (40). Similarly, (41) can be obtained for

$$\left|\mathbf{V}\right| < 0, \quad \text{or, } \lambda_1 > 0, \lambda_2 < 0 \tag{44}$$

□

Corollary 4.5. *The equation of the minor and major axes are respectively given by*

$$\mathbf{p}_i^\top\left(\mathbf{x}-\mathbf{c}\right) = 0, i = 1, 2 \tag{45}$$

4.2. Through Vector Equation

Theorem 4.6. *The eccentricity, directrices and foci of* (7) *are given by*

$$e = \sqrt{1-\frac{\lambda_1}{\lambda_2}} \tag{46}$$

$$\mathbf{n} = \sqrt{\lambda_2}\mathbf{p}_1, c = \begin{cases} \frac{e\mathbf{u}^\top\mathbf{n} \pm \sqrt{e^2(\mathbf{u}^\top\mathbf{n})^2 - \lambda_2(e^2-1)\left(\left\|\mathbf{u}\right\|^2-\lambda_2 f\right)}}{\lambda_2 e(e^2-1)} & e \neq 1 \\ \frac{\left\|\mathbf{u}\right\|^2-\lambda_2 f}{2e^2\mathbf{u}^\top\mathbf{n}} & e = 1 \end{cases} \tag{47}$$

$$\mathbf{F} = \frac{ce^2\mathbf{n}-\mathbf{u}}{\lambda_2} \tag{48}$$

Proof. From (8),

$$\mathbf{V}^\top\mathbf{V} = \left(\left\|\mathbf{n}\right\|^2\mathbf{I} - e^2\mathbf{n}\mathbf{n}^\top\right)^\top\left(\left\|\mathbf{n}\right\|^2\mathbf{I} - e^2\mathbf{n}\mathbf{n}^\top\right) \tag{49}$$

$$\implies \mathbf{V}^2 = \left\|\mathbf{n}\right\|^4\mathbf{I} + e^4\mathbf{n}\mathbf{n}^\top\mathbf{n}\mathbf{n}^\top - 2e^2\left\|\mathbf{n}\right\|^2\mathbf{n}\mathbf{n}^\top \tag{50}$$

$$= \left\|\mathbf{n}\right\|^4\mathbf{I} + e^4\left\|\mathbf{n}\right\|^2\mathbf{n}\mathbf{n}^\top - 2e^2\left\|\mathbf{n}\right\|^2\mathbf{n}\mathbf{n}^\top \tag{51}$$

$$= \left\|\mathbf{n}\right\|^4\mathbf{I} + e^2\left(e^2-2\right)\left\|\mathbf{n}\right\|^2\mathbf{n}\mathbf{n}^\top \tag{52}$$

$$= \left\|\mathbf{n}\right\|^4\mathbf{I} + \left(e^2-2\right)\left\|\mathbf{n}\right\|^2\left(\left\|\mathbf{n}\right\|^2\mathbf{I} - \mathbf{V}\right) \tag{53}$$

which can be expressed as

$$\mathbf{V}^2 + \left(e^2 - 2\right) \|\mathbf{n}\|^2 \mathbf{V} - \left(e^2 - 1\right) \|\mathbf{n}\|^4 \mathbf{I} = 0 \tag{54}$$

Using the Cayley-Hamilton theorem, (54) results in the characteristic equation,

$$\lambda^2 - \left(2 - e^2\right) \|\mathbf{n}\|^2 \lambda + \left(1 - e^2\right) \|\mathbf{n}\|^4 = 0 \tag{55}$$

$$\implies \left(\frac{\lambda}{\|\mathbf{n}\|^2}\right)^2 - \left(2 - e^2\right)\left(\frac{\lambda}{\|\mathbf{n}\|^2}\right) + \left(1 - e^2\right) = 0 \tag{56}$$

$$\implies \frac{\lambda}{\|\mathbf{n}\|^2} = 1 - e^2, 1 \tag{57}$$

$$\text{or, } \lambda_2 = \|\mathbf{n}\|^2, \lambda_1 = \left(1 - e^2\right)\lambda_2 \tag{58}$$

From (58), the eccentricity of (7) is given by (46). Multiplying both sides of (8) by $\mathbf{n}$,

$$\mathbf{V}\mathbf{n} = \|\mathbf{n}\|^2 \mathbf{n} - e^2 \mathbf{n}\mathbf{n}^\top \mathbf{n} \tag{59}$$

$$= \|\mathbf{n}\|^2 \left(1 - e^2\right) \mathbf{n} \tag{60}$$

$$= \lambda_1 \mathbf{n} \tag{61}$$

from (58) Thus, λ_1 is the corresponding eigenvalue for $\mathbf{n}$. From (32), (58) and (61),

$$\mathbf{n} = \|\mathbf{n}\| \mathbf{p}_1 = \sqrt{\lambda_2}\mathbf{p}_1 \tag{62}$$

From (9) and (58),

$$\mathbf{F} = \frac{ce^2\mathbf{n} - \mathbf{u}}{\lambda_2} \tag{63}$$

$$\implies \|\mathbf{F}\|^2 = \frac{\left(ce^2\mathbf{n} - \mathbf{u}\right)^\top \left(ce^2\mathbf{n} - \mathbf{u}\right)}{\lambda_2^2} \tag{64}$$

$$\implies \lambda_2^2 \|\mathbf{F}\|^2 = c^2 e^4 \lambda_2 - 2ce^2 \mathbf{u}^\top \mathbf{n} + \|\mathbf{u}\|^2 \tag{65}$$

Also, (10) can be expressed as

$$\lambda_2 \|\mathbf{F}\|^2 = f + c^2 e^2 \tag{66}$$

From (65) and (66),

$$c^2 e^4 \lambda_2 - 2ce^2 \mathbf{u}^\top \mathbf{n} + \|\mathbf{u}\|^2 = \lambda_2 \left(f + c^2 e^2\right) \tag{67}$$

$$\implies \lambda_2 e^2 \left(e^2 - 1\right) c^2 - 2ce^2 \mathbf{u}^\top \mathbf{n} + \|\mathbf{u}\|^2 - \lambda_2 f = 0 \tag{68}$$

$$\text{or, } c = \begin{cases} \frac{e\mathbf{u}^\top \mathbf{n} \pm \sqrt{e^2\left(\mathbf{u}^\top \mathbf{n}\right)^2 - \lambda_2\left(e^2-1\right)\left(\|\mathbf{u}\|^2 - \lambda_2 f\right)}}{\lambda_2 e\left(e^2-1\right)} & e \neq 1 \\ \frac{\|\mathbf{u}\|^2 - \lambda_2 f}{2e^2\mathbf{u}^\top \mathbf{n}} & e = 1 \end{cases} \tag{69}$$

□

Example 4.7. In Example 3.4, starting with the equation of the conic

$$\mathbf{x}^\top \begin{pmatrix} 16 & 4 \\ 4 & 1 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} -37 \\ -39 \end{pmatrix} \mathbf{x} + 212 = 0 \tag{70}$$

the eigenvector decomposition of

$$\mathbf{V} = \begin{pmatrix} 16 & 4 \\ 4 & 1 \end{pmatrix} \tag{71}$$

is given by

$$\mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & 17 \end{pmatrix} \implies \lambda_1 = 0, \lambda_2 = 17, \tag{72}$$

$$\mathbf{P} = \frac{1}{\sqrt{17}} \begin{pmatrix} -1 & 4 \\ 4 & 1 \end{pmatrix} \implies \mathbf{p}_1 = \frac{1}{\sqrt{17}} \begin{pmatrix} -1 \\ 4 \end{pmatrix}, \mathbf{p}_2 = \frac{1}{\sqrt{17}} \begin{pmatrix} 4 \\ 1 \end{pmatrix} \tag{73}$$

From (46) and (72), the eccentricity

$$e = 1. \tag{74}$$

From (47) (20) and (21), substituting $e = 1$,

$$\mathbf{n} = \sqrt{17} \frac{1}{\sqrt{17}} \begin{pmatrix} -1 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix} \tag{75}$$

$$c = \frac{\left(37^2 + 39^2\right)^2 - (17)(212)}{2 \begin{pmatrix} -37 & -39 \end{pmatrix} \begin{pmatrix} -1 \\ 4 \end{pmatrix}} = 3 \tag{76}$$

which are the parameters of the directrix in (17). Substituting the above values in (48),

$$\mathbf{F} = \frac{3 \begin{pmatrix} -1 \\ 4 \end{pmatrix} + \begin{pmatrix} 37 \\ 39 \end{pmatrix}}{17} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \tag{77}$$

which matches (16). Thus, Theorem 4.6 is verified for the parabola in (70).

Example 4.8. In Example 3.5, starting with the equation of the conic

$$\mathbf{x}^\top \begin{pmatrix} 261 & -96 \\ -96 & 181 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} 522 \\ -1167 \end{pmatrix} \mathbf{x} + 3969 = 0 \tag{78}$$

the eigenvector decomposition of

$$\mathbf{V} = \begin{pmatrix} 261 & -96 \\ -96 & 181 \end{pmatrix} \tag{79}$$

is given by

$$\mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & 17 \end{pmatrix} \implies \lambda_1 = 0, \lambda_2 = 17, \tag{80}$$

$$\mathbf{P} = \frac{1}{\sqrt{17}} \begin{pmatrix} -1 & 4 \\ 4 & 1 \end{pmatrix} \implies \mathbf{p}_1 = \frac{1}{\sqrt{17}} \begin{pmatrix} -1 \\ 4 \end{pmatrix}, \mathbf{p}_2 = \frac{1}{\sqrt{17}} \begin{pmatrix} 4 \\ 1 \end{pmatrix} \tag{81}$$

From (46) and (80), the eccentricity

$$e = 1. \tag{82}$$

From (47) (26) and (27), substituting $e = 1$,

$$\mathbf{n} = \sqrt{17} \frac{1}{\sqrt{17}} \begin{pmatrix} -1 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix} \tag{83}$$

$$c = \frac{\left(37^2 + 39^2\right)^2 - (17)(212)}{2 \begin{pmatrix} -37 & -39 \end{pmatrix} \begin{pmatrix} -1 \\ 4 \end{pmatrix}} = 3 \tag{84}$$

which are the parameters of the directrix in (23).

5. Tangent and Normal

Theorem 5.1. *The points of intersection of the line*

$$L: \quad \mathbf{x} = \mathbf{q} + \mu \mathbf{m} \quad \mu \in \mathbb{R} \tag{85}$$

with the conic section in (7) *are given by*

$$\mathbf{x}_i = \mathbf{q} + \mu_i \mathbf{m} \tag{86}$$

where

$$\mu_i = \frac{1}{\mathbf{m}^T \mathbf{V} \mathbf{m}} \left(-\mathbf{m}^T \left(\mathbf{V}\mathbf{q} + \mathbf{u} \right) \right. \\ \left. \pm \sqrt{\left[\mathbf{m}^T \left(\mathbf{V}\mathbf{q} + \mathbf{u} \right) \right]^2 - \left(\mathbf{q}^T \mathbf{V} \mathbf{q} + 2\mathbf{u}^T \mathbf{q} + f \right) \left(\mathbf{m}^T \mathbf{V} \mathbf{m} \right)} \right) \tag{87}$$

Proof. Substituting (85) in (7),

$$\left(\mathbf{q} + \mu \mathbf{m} \right)^T \mathbf{V} \left(\mathbf{q} + \mu \mathbf{m} \right) + 2\mathbf{u}^T \left(\mathbf{q} + \mu \mathbf{m} \right) + f = 0 \tag{88}$$

$$\implies \mu^2 \mathbf{m}^T \mathbf{V} \mathbf{m} + 2\mu \mathbf{m}^T \left(\mathbf{V}\mathbf{q} + \mathbf{u} \right) + \mathbf{q}^T \mathbf{V} \mathbf{q} + 2\mathbf{u}^T \mathbf{q} + f = 0 \tag{89}$$

Solving the above quadratic in (89) yields (87). □

Corollary 5.2. *If L in* (85) *touches* (7) *at exactly one point* $\mathbf{q}$,

$$\mathbf{m}^T \left(\mathbf{V}\mathbf{q} + \mathbf{u} \right) = 0 \tag{90}$$

Proof. In this case, (89) has exactly one root. Hence, in (87)

$$\left[\mathbf{m}^T(\mathbf{Vq}+\mathbf{u})\right]^2-\left(\mathbf{m}^T\mathbf{Vm}\right)\left(\mathbf{q}^T\mathbf{Vq}+2\mathbf{u}^T\mathbf{q}+f\right)=0 \tag{91}$$

$\because$ $\mathbf{q}$ is the point of contact, $\mathbf{q}$ satisfies (7) and

$$\mathbf{q}^T\mathbf{Vq}+2\mathbf{u}^T\mathbf{q}+f=0 \tag{92}$$

Substituting (92) in (91) and simplifying, we obtain (90). □

Theorem 5.3 (Tangent)**.** *Given the point of contact* $\mathbf{q}$*, the equation of a tangent to* (7) *is*

$$(\mathbf{Vq}+\mathbf{u})^T\mathbf{x}+\mathbf{u}^T\mathbf{q}+f=0 \tag{93}$$

Proof. The normal vector is obtained from (90) and (3) as

$$\mathbf{n}=\mathbf{Vq}+\mathbf{u} \tag{94}$$

From (94) and (1), the equation of the tangent is

$$(\mathbf{Vq}+\mathbf{u})^T(\mathbf{x}-\mathbf{q})=0 \tag{95}$$

$$\implies (\mathbf{Vq}+\mathbf{u})^T\mathbf{x}-\mathbf{q}^T\mathbf{Vq}-\mathbf{u}^T\mathbf{q}=0 \tag{96}$$

which, upon substituting from (92) and simplifying yields (85). □

Theorem 5.4. *If* $\mathbf{V}^{-1}$ *exists, given the normal vector* $\mathbf{n}$*, the tangent points of contact to* (7) *are given by*

$$\begin{aligned}\mathbf{q}_i&=\mathbf{V}^{-1}(\kappa_i\mathbf{n}-\mathbf{u}),i=1,2\\ \text{where } \kappa_i&=\pm\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}-f}{\mathbf{n}^T\mathbf{V}^{-1}\mathbf{n}}}\end{aligned} \tag{97}$$

Proof. From (94),

$$\mathbf{q}=\mathbf{V}^{-1}(\kappa\mathbf{n}-\mathbf{u}),\quad \kappa\in\mathbb{R} \tag{98}$$

Substituting (98) in (92),

$$(\kappa\mathbf{n}-\mathbf{u})^T\mathbf{V}^{-1}(\kappa\mathbf{n}-\mathbf{u})+2\mathbf{u}^T\mathbf{V}^{-1}(\kappa\mathbf{n}-\mathbf{u})+f=0 \tag{99}$$

$$\implies \kappa^2\mathbf{n}^T\mathbf{V}^{-1}\mathbf{n}-\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}+f=0 \tag{100}$$

$$\text{or, }\kappa=\pm\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}-f}{\mathbf{n}^T\mathbf{V}^{-1}\mathbf{n}}} \tag{101}$$

Substituting (101) in (98) yields (97). □

Theorem 5.5. *If* $\mathbf{V}$ *is not invertible, given the normal vector* $\mathbf{n}$*, the point of contact to* (7) *is given by the matrix equation*

$$\begin{pmatrix} \mathbf{u} + \kappa \mathbf{n}^T \\ \mathbf{V} \end{pmatrix} \mathbf{q} = \begin{pmatrix} -f \\ \kappa \mathbf{n} - \mathbf{u} \end{pmatrix} \tag{102}$$

$$\text{where } \kappa = \frac{\mathbf{p}_1^T \mathbf{u}}{\mathbf{p}_1^T \mathbf{n}}, \quad \mathbf{V}\mathbf{p}_1 = 0 \tag{103}$$

Proof. If $\mathbf{V}$ is non-invertible, it has a zero eigenvalue. If the corresponding eigenvector is $\mathbf{p}_1$, then,

$$\mathbf{V}\mathbf{p}_1 = 0 \tag{104}$$

From (94),

$$\kappa \mathbf{n} = \mathbf{V}\mathbf{q} + \mathbf{u}, \quad \kappa \in \mathbb{R} \tag{105}$$

$$\implies \kappa \mathbf{p}_1^T \mathbf{n} = \mathbf{p}_1^T \mathbf{V}\mathbf{q} + \mathbf{p}_1^T \mathbf{u} \tag{106}$$

$$\text{or, } \kappa \mathbf{p}_1^T \mathbf{n} = \mathbf{p}_1^T \mathbf{u}, \quad \because \mathbf{p}_1^T \mathbf{V} = 0, \quad (\text{ from } (104)) \tag{107}$$

yielding κ in (103). From (105),

$$\kappa \mathbf{q}^T \mathbf{n} = \mathbf{q}^T \mathbf{V}\mathbf{q} + \mathbf{q}^T \mathbf{u} \tag{108}$$

$$\implies \kappa \mathbf{q}^T \mathbf{n} = -f - \mathbf{q}^T \mathbf{u} \quad \text{from (92)}, \tag{109}$$

$$\text{or, } (\kappa \mathbf{n} + \mathbf{u})\, \mathbf{q} = -f \tag{110}$$

(105) can be expressed as

$$\mathbf{V}\mathbf{q} = \kappa \mathbf{n} - \mathbf{u}. \tag{111}$$

(110) and (111) clubbed together result in (102). □

All the results related to conics are summarized in Table 1.

6. Asymptotes

Definition 6.1. When (7) is a hyperbola, its *asymptotes* are defined as the pair of intersecting straight lines

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + \mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} = 0, \quad |\mathbf{V}| < 0 \tag{112}$$

such that

Theorem 6.2. (112) *can be expressed as the lines*

$$\left(\sqrt{|\lambda_1|} \quad \pm\sqrt{|\lambda_2|} \right) \mathbf{P}^\top (\mathbf{x} - \mathbf{c}) = 0 \tag{113}$$

Conic	Property	Standard Form	Standard Parameters	Point(s) of Contact
Circle	$\mathbf{V} = \mathbf{I}$	$\frac{\mathbf{y}^T\mathbf{D}\mathbf{y}}{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}-f} = 1$ $\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ $\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T$ $\mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix}$	$\mathbf{c} = -\mathbf{u}$, $r = \sqrt{\mathbf{u}^T\mathbf{u} - f}$	$\mathbf{q} = \mathbf{V}^{-1}\left(\kappa\mathbf{n} - \mathbf{u}\right)$ $\kappa = \pm\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\mathbf{n}^T\mathbf{V}^{-1}\mathbf{n}}}$
Ellipse	$\lvert\mathbf{V}\rvert > 0$ $\lambda_1 > 0, \lambda_2 < 0$		$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u}$, $axes = \begin{cases} \sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}-f}{\lambda_1}} \\ \sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}-f}{\lambda_2}} \end{cases}$	
Hyperbola	$\lvert\mathbf{V}\rvert < 0$ $\lambda_1 > 0, \lambda_2 < 0$		$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u}$, $axes = \begin{cases} \sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}-f}{\lambda_1}} \\ \sqrt{\frac{f-\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}}{\lambda_2}} \end{cases}$	
Parabola	$\lvert\mathbf{V}\rvert = 0$ $\lambda_1 = 0$	$\mathbf{y}^T\mathbf{D}\mathbf{y} = -2\eta\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{y}$	focal length $= \left\lvert\frac{2\eta}{\lambda_2}\right\rvert$ $\begin{pmatrix} \mathbf{u}^T + \eta\mathbf{p}_1^T \\ \mathbf{V} \end{pmatrix}\mathbf{c} = \begin{pmatrix} -f \\ \eta\mathbf{p}_1 - \mathbf{u} \end{pmatrix}$ $\eta = \mathbf{p}_1^T\mathbf{u}$	$\begin{pmatrix} \mathbf{u} + \kappa\mathbf{n}^T \\ \mathbf{V} \end{pmatrix}\mathbf{q} = \begin{pmatrix} -f \\ \kappa\mathbf{n} - \mathbf{u} \end{pmatrix}$ $\kappa = \frac{\mathbf{p_1}^T\mathbf{u}}{\mathbf{p_1}^T\mathbf{n}}$

Table 1. $\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0$ can be expressed in the above standard form for various conics. $\mathbf{c}$ represents the centre/vertex of the conic. $\mathbf{q}$ is/are the point(s) of contact for the tangent(s).

Proof. Reducing (112) to standard form using the *affine transformation* yields

$$\lambda_1 y_1^2 - (-\lambda_2)\, y_1^2 = 0 \tag{114}$$

From (112), the equation of the asymptotes for (114) is

$$\left(\sqrt{|\lambda_1|} \quad \pm\sqrt{|\lambda_2|}\right)\mathbf{y} = 0 \tag{115}$$

from which (113) is obtained using (A1).

□

Corollary 6.3. *The angle between the asymptotes is then given by using the inner product*

$$\cos\theta = \frac{|\lambda_1| - |\lambda_2|}{|\lambda_1| + |\lambda_2|} \tag{116}$$

Proof. The normal vectors of the lines in (113) are

$$\begin{aligned} \mathbf{n}_1 &= \mathbf{P}\begin{pmatrix} \sqrt{|\lambda_1|} \\ \sqrt{|\lambda_2|} \end{pmatrix} \\ \mathbf{n}_2 &= \mathbf{P}\begin{pmatrix} \sqrt{|\lambda_1|} \\ -\sqrt{|\lambda_2|} \end{pmatrix} \end{aligned} \tag{117}$$

The angle between the asymptotes is given by

$$\cos\theta = \frac{\mathbf{n_1}^\top \mathbf{n_2}}{\|\mathbf{n_1}\| \, \|\mathbf{n_2}\|} \tag{118}$$

The orthogonal matrix $\mathbf{P}$ preserves the norm, i.e.

$$\|\mathbf{n_1}\| = \left\| \mathbf{P} \begin{pmatrix} \sqrt{|\lambda_1|} \\ \sqrt{|\lambda_2|} \end{pmatrix} \right\| \tag{119}$$

$$= \left\| \begin{pmatrix} \sqrt{|\lambda_1|} \\ \sqrt{|\lambda_2|} \end{pmatrix} \right\| = \sqrt{|\lambda_1| + |\lambda_2|} = \|\mathbf{n_2}\| \tag{120}$$

It is easy to verify that

$$\mathbf{n_1}^\top \mathbf{n_2} = |\lambda_1| - |\lambda_2| \tag{121}$$

Thus, the angle between the asymptotes is obtained from (118) as (116). □

Lemma 6.4 (Conjugate Hyperbola)**.** *Another hyperbola with the same asymptotes as* (113) *can be obtained from* (7) *and* (112) *as*

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + 2\mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f = 0 \tag{122}$$

7. Examples

Example 7.1. Find the coordinates of the foci, the vertices, the length of major axis, the minor axis, the eccentricity and the latus rectum of the ellipse

$$\mathbf{x}^\top \begin{pmatrix} \frac{1}{25} & 0 \\ 0 & \frac{1}{9} \end{pmatrix} \mathbf{x} = 1 \tag{123}$$

Solution: Comparing (123) with (7),

$$\mathbf{V} = \begin{pmatrix} \frac{1}{25} & 0 \\ 0 & \frac{1}{9} \end{pmatrix}, \mathbf{u} = 0, f = -1 \tag{124}$$

The eigenvectors of $\mathbf{V}$ are trivially obtained as $\mathbf{e}_1, \mathbf{e}_2$.

References

[1] S.L. Loney, *The Elements of Coordinate Geometry*, Macmillan and Co, 1895.
[2] NCERT, *Mathematics*, 1st ed., chap. Conic Sections, Textbook for Class XI, National Council of Educational Research and Training, New Delhi (2019).

8. Appendices

Appendix A.

Using

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c} \quad \text{(Affine Transformation)} \tag{A1}$$

(7) can be expressed as

$$(\mathbf{P}\mathbf{y} + \mathbf{c})^T \mathbf{V} (\mathbf{P}\mathbf{y} + \mathbf{c}) + 2\mathbf{u}^T (\mathbf{P}\mathbf{y} + \mathbf{c}) + f = 0, \tag{A2}$$

yielding

$$\mathbf{y}^T\mathbf{P}^T\mathbf{V}\mathbf{P}\mathbf{y} + 2(\mathbf{V}\mathbf{c} + \mathbf{u})^T \mathbf{P}\mathbf{y} + \mathbf{c}^T\mathbf{V}\mathbf{c} + 2\mathbf{u}^T\mathbf{c} + f = 0 \tag{A3}$$

From (A3) and (30),

$$\mathbf{y}^T\mathbf{D}\mathbf{y} + 2(\mathbf{V}\mathbf{c} + \mathbf{u})^T \mathbf{P}\mathbf{y} + \mathbf{c}^T(\mathbf{V}\mathbf{c} + \mathbf{u}) + \mathbf{u}^T\mathbf{c} + f = 0 \tag{A4}$$

When $\mathbf{V}^{-1}$ exists,

$$\mathbf{V}\mathbf{c} + \mathbf{u} = \mathbf{0}, \quad \text{or}, \mathbf{c} = -\mathbf{V}^{-1}\mathbf{u}, \tag{A5}$$

and substituting (A5) in (A4) yields (28). When $|\mathbf{V}| = 0, \lambda_1 = 0$ and

$$\mathbf{V}\mathbf{p}_1 = 0, \mathbf{V}\mathbf{p}_2 = \lambda_2\mathbf{p}_2. \tag{A6}$$

where $\mathbf{p}_1, \mathbf{p}_2$ are the eigenvectors of $\mathbf{V}$such that (30)

$$\mathbf{P} = \begin{pmatrix}\mathbf{p}_1 & \mathbf{p}_2\end{pmatrix}, \tag{A7}$$

Substituting (A7) in (A4),

$$\mathbf{y}^T\mathbf{D}\mathbf{y} + 2\left(\mathbf{c}^T\mathbf{V} + \mathbf{u}^T\right)\begin{pmatrix}\mathbf{p}_1 & \mathbf{p}_2\end{pmatrix}\mathbf{y} + \mathbf{c}^T(\mathbf{V}\mathbf{c} + \mathbf{u}) + \mathbf{u}^T\mathbf{c} + f = 0 \tag{A8}$$

$$\implies \mathbf{y}^T\mathbf{D}\mathbf{y} + 2\begin{pmatrix}\left(\mathbf{c}^T\mathbf{V} + \mathbf{u}^T\right)\mathbf{p}_1 & \left(\mathbf{c}^T\mathbf{V} + \mathbf{u}^T\right)\mathbf{p}_2\end{pmatrix}\mathbf{y} \tag{A9}$$

$$+\mathbf{c}^T(\mathbf{V}\mathbf{c} + \mathbf{u}) + \mathbf{u}^T\mathbf{c} + f = 0 \tag{A10}$$

$$\implies \mathbf{y}^T\mathbf{D}\mathbf{y} + 2\begin{pmatrix}\mathbf{u}^T\mathbf{p}_1 & \left(\lambda_2\mathbf{c}^T + \mathbf{u}^T\right)\mathbf{p}_2\end{pmatrix}\mathbf{y} + \mathbf{c}^T(\mathbf{V}\mathbf{c} + \mathbf{u}) + \mathbf{u}^T\mathbf{c} + f = 0 \text{ from (A6)}$$

$$\implies \lambda_2 y_2^2 + 2\left(\mathbf{u}^T\mathbf{p}_1\right)y_1 + 2y_2(\lambda_2\mathbf{c} + \mathbf{u})^T\mathbf{p}_2 + \mathbf{c}^T(\mathbf{V}\mathbf{c} + \mathbf{u}) + \mathbf{u}^T\mathbf{c} + f = 0 \tag{A11}$$

which is the equation of a parabola. Thus, (A11) can be expressed as (29) by choosing

$$\eta = \mathbf{u}^T\mathbf{p}_1 \tag{A12}$$

and $\mathbf{c}$ in (A4) such that

$$\mathbf{P}^T\left(\mathbf{V}\mathbf{c}+\mathbf{u}\right)=\eta\begin{pmatrix}1\\0\end{pmatrix} \tag{A13}$$

$$\mathbf{c}^T\left(\mathbf{V}\mathbf{c}+\mathbf{u}\right)+\mathbf{u}^T\mathbf{c}+f=0 \tag{A14}$$

Multiplying (A13) by $\mathbf{P}$ yields

$$\left(\mathbf{V}\mathbf{c}+\mathbf{u}\right)=\eta\mathbf{p}_1, \tag{A15}$$

which, upon substituting in (A14) results in

$$\eta\mathbf{c}^T\mathbf{p}_1+\mathbf{u}^T\mathbf{c}+f=0 \tag{A16}$$

(A15) and (A16) can be clubbed together to obtain (36).