Solution Gate EC 29.2022

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Question 29 Let H(X) denote the entropy of a discrete random variable X taking K possible distinct real values. Which of the following statements is/are necessarily true?

(A) $H(X) \leq \log_2 K$ bits

(B) $H(X) \leq H(2X)$

(C) $H(X) \leq H\left(X^2\right)$

(D) $H(X) \leq H\left(2^X\right)$

Solution:

Random independent variable	value of R.V	Description
X	$X \in (x_1, x_2..., x_K)$	Value of the discrete variable X

1) For Option(A) we will find
We know that :

$$\max_{p_X(k)} \quad H(X)$$
$$\text{s.t.} \quad \sum_{k=0}^{K} p_X(k) = 1$$

$\implies$

$$\max_{p_X(k)} \quad -\sum_{k=0}^{K} p_X(k) \log_2 p_X(k)$$
$$\text{s.t.} \quad \sum_{k=0}^{K} p_X(k) = 1$$

Now,we use lagranges multiplier to find the maximum entropy subject to the lagranges multiplier constant λ and $p_X(k)$

$$L(p_X(k), \lambda) = -\sum_{k=0}^{K} p_X(k) \log_2 p_X(k) + \lambda\left(\sum_{k=0}^{k} p_X(k) - 1\right) \tag{1}$$

$$\frac{\partial L}{\partial p_X(k)} = -\log_2 p_X(k) - 1 + \lambda \tag{2}$$

Now, we take the derivative of L with respect to each $p_X(k)$ equal to zero for $H(X)\, max$

$$\lambda = \log_2 \frac{2}{k} \tag{3}$$

$$p_X(k) = 1/K \tag{4}$$

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On solving, we get the value of

$$H(X)_{max} = \log_2 K \tag{5}$$

$$H(X) \leq \log_2 K \tag{6}$$

Hence, Option(A) is correct

2) Let's consider the discrete variable as follows

$X \in x_i$	$p_X(k)$
-1	$\frac{1}{4}$
0	$\frac{1}{2}$
1	$\frac{1}{4}$

$$H(X) = \frac{1}{4}\log_2 4 + \frac{1}{2}\log_2 2 + \frac{1}{4}\log_2 4 \tag{7}$$

$$H(X) = 1.5units \tag{8}$$

Now $Y = 2X$

$Y \in y_i$	$p_Y(k)$
-2	$\frac{1}{4}$
0	$\frac{1}{2}$
2	$\frac{1}{4}$

$$H(Y) = \sum_{i=0}^{2} p_Y(k)\log_2 \frac{1}{p_Y(k)} \tag{9}$$

$$H(Y) = 1.5units \tag{10}$$

$$H(Y) = H(2X) = H(X) \tag{11}$$

Hence, Option(B) is correct

3) Similarly on substituting $Y = X^2$

$Y \in y_i$	$p_Y(k)$
0	$\frac{1}{2}$
1	$\frac{1}{2}$

$$H(Y) = \sum_{i=0}^{1} p_Y(k)\log_2 \frac{1}{p_Y(k)} \tag{12}$$

$$H(Y) = 1units \tag{13}$$

$$H(Y) = H\left(X^2\right) \leq H(X) \tag{14}$$

Hence, Option(C) is incorrect

4) Now for $Y = 2^X$

$Y \in y_i$	$p_Y(k)$
$2^{-1} = \frac{1}{2}$	$\frac{1}{4}$
$2^0 = 1$	$\frac{1}{2}$
$2^1 = 2$	$\frac{1}{4}$

$$H(Y) = \sum_{i=0}^{2} p_Y(k) \log_2 \frac{1}{p_Y(k)} \tag{15}$$

$$H(Y) = 1.5 units \tag{16}$$

$$H(Y) = H\left(2^X\right) = H(X) \tag{17}$$

Hence, Option(D) is correct
The ans is (A), (B), (D)
These options are correct for the particular example.