1.11.5

AI25BTECH11003 - Bhavesh Gaikwad

Question: The scalar product of vector $\overrightarrow{a} = \hat{i} + \hat{j} + \hat{k}$ with a unit vector along the sum of the vectors $\overrightarrow{b} = 2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\overrightarrow{c} = \lambda\hat{i} + 2\hat{j} + 3\hat{k}$ is equal to 1. Find the value of λ and hence find the unit vector along $\overrightarrow{b} + \overrightarrow{c}$.

Solution:

Given: $\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ 4 \\ -5 \end{pmatrix}$, $\mathbf{c} = \begin{pmatrix} \lambda \\ 2 \\ 3 \end{pmatrix}$.

Let $\mathbf{u}$ be the unit vector along $\mathbf{b} + \mathbf{c}$.

$$\mathbf{b} + \mathbf{c} = \begin{pmatrix} 2+\lambda \\ 4+2 \\ -5+3 \end{pmatrix} = \begin{pmatrix} 2+\lambda \\ 6 \\ -2 \end{pmatrix}.$$

$$\|\mathbf{b} + \mathbf{c}\| = \sqrt{(2+\lambda)^2 + 6^2 + (-2)^2} = \sqrt{\lambda^2 + 4\lambda + 44}.$$

$$\mathbf{u} = \frac{\mathbf{b} + \mathbf{c}}{\|\mathbf{b} + \mathbf{c}\|} = \frac{1}{\sqrt{\lambda^2 + 4\lambda + 44}} \begin{pmatrix} 2+\lambda \\ 6 \\ -2 \end{pmatrix}.$$

Given condition: $\mathbf{a} \cdot \mathbf{u} = 1$.

$$\mathbf{a} \cdot \mathbf{u} = \frac{\mathbf{a} \cdot (\mathbf{b} + \mathbf{c})}{\|\mathbf{b} + \mathbf{c}\|} = \frac{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2+\lambda \\ 6 \\ -2 \end{pmatrix}}{\sqrt{\lambda^2 + 4\lambda + 44}} = \frac{\lambda + 6}{\sqrt{\lambda^2 + 4\lambda + 44}} = 1.$$

$$\Rightarrow (\lambda + 6)^2 = \lambda^2 + 4\lambda + 44 \implies \lambda^2 + 12\lambda + 36 = \lambda^2 + 4\lambda + 44 \implies 8\lambda = 8$$

$$\implies \boxed{\lambda = 1}$$

Now, with $\lambda = 1$: $\quad \mathbf{b} + \mathbf{c} = \begin{pmatrix} 3 \\ 6 \\ -2 \end{pmatrix}, \quad \|\mathbf{b} + \mathbf{c}\| = \sqrt{3^2 + 6^2 + (-2)^2} = \sqrt{49} = 7.$

Unit vector along, $\mathbf{b} + \mathbf{c}$ is: $\quad \frac{1}{7}\begin{pmatrix} 3 \\ 6 \\ -2 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 3 \\ 6 \\ -2 \end{pmatrix}.$

$$\boxed{\lambda = 1} \quad \textit{and} \quad \boxed{\text{Unit vector along } \mathbf{b} + \mathbf{c} = \frac{1}{7}\begin{pmatrix} 3 \\ 6 \\ -2 \end{pmatrix}}. \tag{0.1}$$

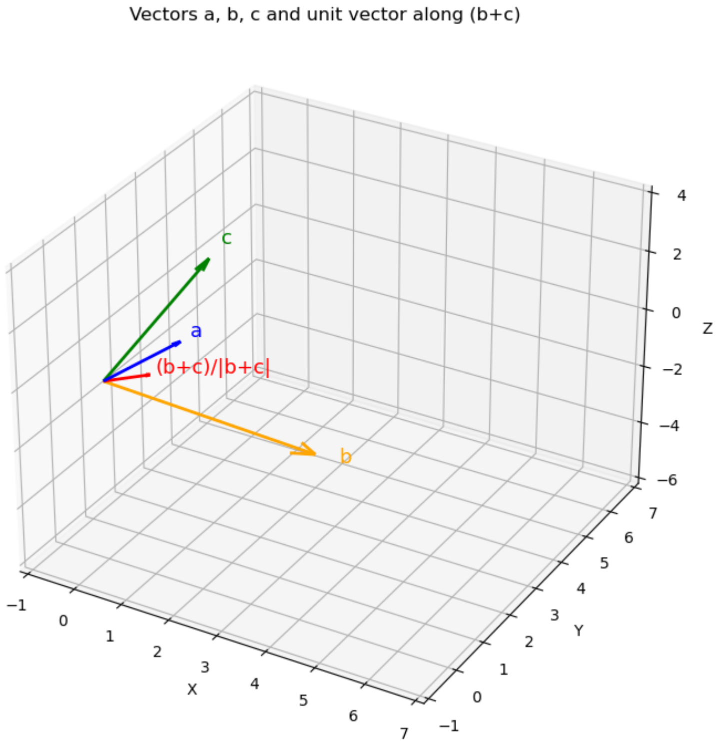

Fig. 0.1: Vector Representation