Matgeo Presentation - Problem 2.9.7

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Problem Statement

Question :

$$\mathbf{a} = 2\hat{i} + \hat{j} + 3\hat{k},\ \mathbf{b} = -\hat{i} + 2\hat{j} + \hat{k},\ \mathbf{c} = 3\hat{i} + \hat{j} + 2\hat{k}$$

then find $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$

Symbol	Value	Description
$\mathbf{a}$	$\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$	vector
$\mathbf{b}$	$\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$	vector
$\mathbf{c}$	$\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$	vector

Table : vectors

Solution

The Gram matrix $\mathbf{G}$ for the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is:

$$\mathbf{G} = \begin{pmatrix} \mathbf{a}^\top \mathbf{a} & \mathbf{a}^\top \mathbf{b} & \mathbf{a}^\top \mathbf{c} \\ \mathbf{b}^\top \mathbf{a} & \mathbf{b}^\top \mathbf{b} & \mathbf{b}^\top \mathbf{c} \\ \mathbf{c}^\top \mathbf{a} & \mathbf{c}^\top \mathbf{b} & \mathbf{c}^\top \mathbf{c} \end{pmatrix} \tag{0.1}$$

Now, calculate the dot products:

$$\mathbf{a}^\top \mathbf{a} = 2^2 + 1^2 + 3^2 = 4 + 1 + 9 = 14 \tag{0.2}$$

$$\mathbf{a}^\top \mathbf{b} = (2)(-1) + (1)(2) + (3)(1) = -2 + 2 + 3 = 3 \tag{0.3}$$

$$\mathbf{a}^\top \mathbf{c} = (2)(3) + (1)(1) + (3)(2) = 6 + 1 + 6 = 13 \tag{0.4}$$

$$\mathbf{b}^\top \mathbf{a} = \mathbf{a}^\top \mathbf{b} = 3 \tag{0.5}$$

$$\mathbf{b}^\top \mathbf{b} = (-1)^2 + 2^2 + 1^2 = 1 + 4 + 1 = 6 \tag{0.6}$$

$$\mathbf{b}^\top \mathbf{c} = (-1)(3) + (2)(1) + (1)(2) = -3 + 2 + 2 = 1 \tag{0.7}$$

$$\mathbf{c}^\top \mathbf{a} = \mathbf{a}^\top \mathbf{c} = 13 \tag{0.8}$$

$$\mathbf{c}^\top \mathbf{b} = \mathbf{b}^\top \mathbf{c} = 1 \tag{0.9}$$

$$\mathbf{c}^\top \mathbf{c} = 3^2 + 1^2 + 2^2 = 9 + 1 + 4 = 14 \tag{0.10}$$

Thus, the Gram matrix **G** is:

$$\mathbf{G} = \begin{pmatrix} 14 & 3 & 13 \\ 3 & 6 & 1 \\ 13 & 1 & 14 \end{pmatrix} \tag{0.11}$$

The characteristic equation is obtained by solving the determinant equation $\left|\mathbf{G} - \lambda \mathbf{I}\right| = 0$. The characteristic polynomial for the matrix is:

$$\lambda^3 - 34\lambda^2 + 185\lambda - 100 = 0 \tag{0.12}$$

To find the eigenvalues, we solve the cubic equation:

$$\lambda^3 - 34\lambda^2 + 185\lambda - 100 = 0$$

By solving this equation , we obtain the eigenvalues:

$$\lambda_1 \approx 27.38, \quad \lambda_2 \approx 6.02, \quad \lambda_3 \approx 0.61. \tag{0.13}$$

The determinant of $\mathbf{G}$ is the product of its eigenvalues:

$$\left|\mathbf{G}\right| = \lambda_1 \lambda_2 \lambda_3 = 100. \tag{0.14}$$

The box product (scalar triple product) is the square root of the determinant of $\mathbf{G}$:

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \sqrt{\left|\mathbf{G}\right|} = \sqrt{100} = 10 \tag{0.15}$$

As the three vectors form a left-handed system , the box product is negative. Hence, the negative value should be considered.
Final Answer : The value of $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) =$ -10

Plot

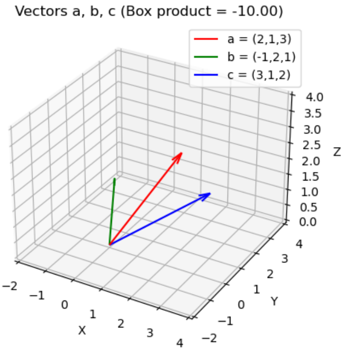

Fig : Vectors

C Code: points.c

```c
#include <math.h>
#include <stdio.h>

// Function to compute dot product
double dot(double u[3], double v[3]) {
  return u[0] * v[0] + u[1] * v[1] + u[2] * v[2];
}

// Function to compute determinant of 3x3 matrix
double det3(double M[3][3]) {
  return M[0][0] * (M[1][1] * M[2][2] - M[1][2] * M[2][1]) -
         M[0][1] * (M[1][0] * M[2][2] - M[1][2] * M[2][0]) +
         M[0][2] * (M[1][0] * M[2][1] - M[1][1] * M[2][0]);
}

// Function to compute box product using Gram matrix
double box_product() {
  double a[3] = {2, 1, 3};
  double b[3] = {-1, 2, 1};
  double c[3] = {3, 1, 2};

  double G[3][3] = {{dot(a, a), dot(a, b), dot(a, c)},
                   {dot(b, a), dot(b, b), dot(b, c)},
                   {dot(c, a), dot(c, b), dot(c, c)}};

  double detG = det3(G);

  // Negative since left-handed system
  double box = -sqrt(detG);

  return box;
}
```

Python: call_c.py

```
import sys
import ctypes
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D

# Load shared object
lib = ctypes.CDLL("./points.so")
lib.box_product.restype = ctypes.c_double

# Call the C function
box = lib.box_product()
print("Box␣product␣(from␣C)␣=", box)

# Define vectors
a = np.array([2,1,3])
b = np.array([-1,2,1])
c = np.array([3,1,2])

# 3D plot
fig = plt.figure()
ax = fig.add_subplot(111, projection='3d')

# Origin
origin = np.array([0,0,0])

# Plot vectors with arrowheads
ax.quiver(*origin, *a, color='r', arrow_length_ratio=0.1, label="a␣=␣(2,1,3)")
ax.quiver(*origin, *b, color='g', arrow_length_ratio=0.1, label="b␣=␣(-1,2,1)")
ax.quiver(*origin, *c, color='b', arrow_length_ratio=0.1, label="c␣=␣(3,1,2)")
```

Python: call_c.py

```
# Set limits
ax.set_xlim([-2,4])
ax.set_ylim([-2,4])
ax.set_zlim([0,4])

ax.set_xlabel("X")
ax.set_ylabel("Y")
ax.set_zlabel("Z")
ax.set_title(f"Vectors␣a,␣b,␣c␣(Box␣product␣=␣{box:.2f})")
ax.legend()

plt.savefig("vectors.png")
plt.show()
```

Python: plot.py

```
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D

# Vectors
a = np.array([2,1,3])
b = np.array([-1,2,1])
c = np.array([3,1,2])

# Gram matrix
G = np.array([
    [np.dot(a,a), np.dot(a,b), np.dot(a,c)],
    [np.dot(b,a), np.dot(b,b), np.dot(b,c)],
    [np.dot(c,a), np.dot(c,b), np.dot(c,c)]
])

# Determinant and box product
detG = np.linalg.det(G)
box = -np.sqrt(detG)

print("Determinant of Gram matrix =", detG)
print("Box product =", box)

# 3D plot
fig = plt.figure()
ax = fig.add_subplot(111, projection='3d')

origin = np.array([0,0,0])
```

Python: plot.py

```
# Plot vectors
ax.quiver(*origin, *a, color='r', arrow_length_ratio=0.1, label="a = (2,1,3)")
ax.quiver(*origin, *b, color='g', arrow_length_ratio=0.1, label="b = (-1,2,1)")
ax.quiver(*origin, *c, color='b', arrow_length_ratio=0.1, label="c = (3,1,2)")

ax.set_xlim([-2,4])
ax.set_ylim([-2,4])
ax.set_zlim([0,4])

ax.set_xlabel("X")
ax.set_ylabel("Y")
ax.set_zlabel("Z")
ax.set_title(f"Vectors a, b, c (Box product = {box:.2f})")
ax.legend()

plt.savefig("vectors.png")
plt.show()
```