2.8.16

EE25BTECH11033 - Kavin

Question:

Prove that the lines $x = py + q, z = ry + s$ and $x = p'y + q', z = r'y + s'$ are perpendicular if $pp' + rr' + 1 = 0$.

Solution:

Let line L_1 be the intersection of the planes,

$$x - py - q = 0 \ , \ z - ry - s = 0$$

Let line L_2 be the intersection of the planes,

$$x - p'y - q' = 0 \ , \ z - r'y - s' = 0$$

To find the direction vectors, the equations of the lines can be expressed as,

$$\mathbf{x} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \kappa \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{1}$$

where $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is the direction vector.

The equations for the line L_1 are:

$$x = py + q \tag{2}$$

$$z = ry + s \tag{3}$$

We can rearrange these to isolate y:

$$x - q = py \implies y = \frac{x - q}{p} \tag{4}$$

$$z - s = ry \implies y = \frac{z - s}{r} \tag{5}$$

By equating these expressions for y, and including the term for y itself, the line equation of L_1 to be:

$$\mathbf{x} = \begin{pmatrix} q \\ 0 \\ s \end{pmatrix} + \kappa_1 \begin{pmatrix} p \\ 1 \\ r \end{pmatrix} \tag{6}$$

From this, the direction vectors of the line L_1 are $\begin{pmatrix} p \\ 1 \\ r \end{pmatrix}$.

The equations for the line L_2 are:

$$x = p'y + q' \tag{7}$$
$$z = r'y + s' \tag{8}$$

Similarly, we rearrange to isolate y:

$$x - q' = p'y \implies y = \frac{x - q'}{p'} \tag{9}$$
$$z - s' = r'y \implies y = \frac{z - s'}{r'} \tag{10}$$

By equating these expressions for y, and including the term for y itself, the line equation of L_2 to be:

$$\mathbf{x} = \begin{pmatrix} q' \\ 0 \\ s' \end{pmatrix} + \kappa_2 \begin{pmatrix} p' \\ 1 \\ r' \end{pmatrix} \tag{11}$$

From this, the direction vectors of the line L_2 are $\begin{pmatrix} p' \\ 1 \\ r' \end{pmatrix}$.

If the lines are perpendicular, then their dot product of direction vectors must be zero.

$$\implies (direction\ vector\ of\ L_1)^{\top} (direction\ vector\ of\ L_2) = 0 \tag{12}$$

$$\implies \begin{pmatrix} p & 1 & r \end{pmatrix} \begin{pmatrix} p' \\ 1 \\ r' \end{pmatrix} = 0 \tag{13}$$

$$\implies pp' + rr' + 1 = 0 \tag{14}$$

$\therefore$ The lines $x = py + q, z = ry + s$ and $x = p'y + q', z = r'y + s'$ are perpendicular if

$pp' + rr' + 1 = 0$

Hence proved.