2.9.7

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Question :

$$\mathbf{a} = 2\hat{i} + \hat{j} + 3\hat{k},\ \ \mathbf{b} = -\hat{i} + 2\hat{j} + \hat{k},\ \ \mathbf{c} = 3\hat{i} + \hat{j} + 2\hat{k}$$

$$\text{then find } \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$$

Symbol	**Value**	**Description**
$\mathbf{a}$	$\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$	vector
$\mathbf{b}$	$\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$	vector
$\mathbf{c}$	$\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$	vector

Table : vectors

Solution :
The Gram matrix $\mathbf{G}$ for the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is:

$$\mathbf{G} = \begin{pmatrix} \mathbf{a}^\top\mathbf{a} & \mathbf{a}^\top\mathbf{b} & \mathbf{a}^\top\mathbf{c} \\ \mathbf{b}^\top\mathbf{a} & \mathbf{b}^\top\mathbf{b} & \mathbf{b}^\top\mathbf{c} \\ \mathbf{c}^\top\mathbf{a} & \mathbf{c}^\top\mathbf{b} & \mathbf{c}^\top\mathbf{c} \end{pmatrix} \tag{1}$$

Now, calculate the dot products:

$$\mathbf{a}^\top\mathbf{a} = 2^2 + 1^2 + 3^2 = 4 + 1 + 9 = 14 \tag{2}$$

$$\mathbf{a}^\top\mathbf{b} = (2)(-1) + (1)(2) + (3)(1) = -2 + 2 + 3 = 3 \tag{3}$$

$$\mathbf{a}^\top\mathbf{c} = (2)(3) + (1)(1) + (3)(2) = 6 + 1 + 6 = 13 \tag{4}$$

$$\mathbf{b}^\top\mathbf{a} = \mathbf{a}^\top\mathbf{b} = 3 \tag{5}$$

$$\mathbf{b}^\top\mathbf{b} = (-1)^2 + 2^2 + 1^2 = 1 + 4 + 1 = 6 \tag{6}$$

$$\mathbf{b}^\top\mathbf{c} = (-1)(3) + (2)(1) + (1)(2) = -3 + 2 + 2 = 1 \tag{7}$$

$$\mathbf{c}^\top\mathbf{a} = \mathbf{a}^\top\mathbf{c} = 13 \tag{8}$$

$$\mathbf{c}^\top \mathbf{b} = \mathbf{b}^\top \mathbf{c} = 1 \tag{9}$$

$$\mathbf{c}^\top \mathbf{c} = 3^2 + 1^2 + 2^2 = 9 + 1 + 4 = 14 \tag{10}$$

Thus, the Gram matrix $\mathbf{G}$ is:

$$\mathbf{G} = \begin{pmatrix} 14 & 3 & 13 \\ 3 & 6 & 1 \\ 13 & 1 & 14 \end{pmatrix} \tag{11}$$

The characteristic equation is obtained by solving the determinant equation $\left|\mathbf{G} - \lambda \mathbf{I}\right| = 0$. The characteristic polynomial for the matrix is:

$$\lambda^3 - 34\lambda^2 + 185\lambda - 100 = 0 \tag{12}$$

To find the eigenvalues, we solve the cubic equation:

$$\lambda^3 - 34\lambda^2 + 185\lambda - 100 = 0$$

By solving this equation , we obtain the eigenvalues:

$$\lambda_1 \approx 27.38, \quad \lambda_2 \approx 6.02, \quad \lambda_3 \approx 0.61. \tag{13}$$

The determinant of $\mathbf{G}$ is the product of its eigenvalues:

$$\left|\mathbf{G}\right| = \lambda_1 \lambda_2 \lambda_3 = 100. \tag{14}$$

The box product (scalar triple product) is the square root of the determinant of $\mathbf{G}$:

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \sqrt{\left|\mathbf{G}\right|} = \sqrt{100} = 10 \tag{15}$$

As the three vectors form a left-handed system , the box product is negative. Hence, the negative value should be considered.

Final Answer : The value of $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ = -10

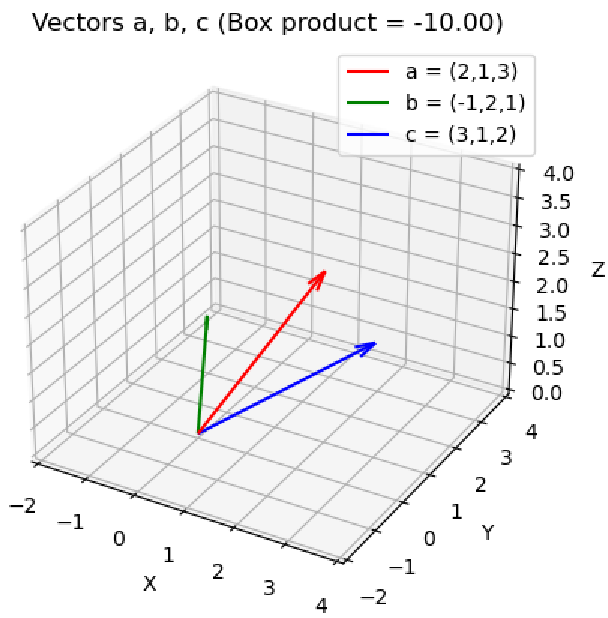
Vectors a, b, c (Box product = -10.00)
a = (2,1,3)
b = (-1,2,1)
c = (3,1,2)
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
Z
−2
−1
0
1
2
3
4
X
−2
−1
0
1
2
3
4
Y

Fig : Vectors