1.4.14

AI25BTECH11039-Harichandana Varanasi

Question: The points $A(-6, 10)$, $B(-4, 6)$ and $C(3, -8)$ are collinear. If $AB = \frac{2}{9}AC$, find the ratio $AB : BC$.

Solution:

$$\mathbf{A} = \begin{pmatrix} -6 \\ 10 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} -4 \\ 6 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 3 \\ -8 \end{pmatrix} \tag{0.1}$$

Let $\mathbf{B}$ divide $\mathbf{AC}$ in the ratio $k : 1$.

$$\mathbf{B} = \frac{k\mathbf{C} + \mathbf{A}}{k + 1} \tag{0.2}$$

Also,

$$k(\mathbf{B} - \mathbf{C}) = \mathbf{A} - \mathbf{B} \tag{0.3}$$

Hence,

$$k = \frac{(\mathbf{A} - \mathbf{B})^\top (\mathbf{B} - \mathbf{C})}{\|\mathbf{B} - \mathbf{C}\|^2} \tag{0.4}$$

Now,

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} -6 \\ 10 \end{pmatrix} - \begin{pmatrix} -4 \\ 6 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \end{pmatrix} \tag{0.5}$$

$$\mathbf{B} - \mathbf{C} = \begin{pmatrix} -4 \\ 6 \end{pmatrix} - \begin{pmatrix} 3 \\ -8 \end{pmatrix} = \begin{pmatrix} -7 \\ 14 \end{pmatrix} \tag{0.6}$$

$$\|\mathbf{B} - \mathbf{C}\|^2 = (-7)^2 + (14)^2 = 245 \tag{0.7}$$

$$k = \frac{\begin{pmatrix} -2 \\ 4 \end{pmatrix}^\top \begin{pmatrix} -7 \\ 14 \end{pmatrix}}{245} \tag{0.8}$$

$$= \frac{(-2)(-7) + (4)(14)}{245} \tag{0.9}$$

$$= \frac{14 + 56}{245} \tag{0.10}$$

$$= \frac{70}{245} = \frac{2}{7} \tag{0.11}$$

Therefore,

$$AB : BC = 2 : 7, \qquad \frac{AB}{AC} = \frac{2}{9}. \tag{0.12}$$

Verification:

$$\mathbf{B} = \frac{7\mathbf{A} + 2\mathbf{C}}{9} \tag{0.13}$$

$$= \frac{7\begin{pmatrix}-6\\10\end{pmatrix} + 2\begin{pmatrix}3\\-8\end{pmatrix}}{9} \tag{0.14}$$

$$= \frac{\begin{pmatrix}-42\\70\end{pmatrix} + \begin{pmatrix}6\\-16\end{pmatrix}}{9} \tag{0.15}$$

$$= \frac{\begin{pmatrix}-36\\54\end{pmatrix}}{9} \tag{0.16}$$

$$= \begin{pmatrix}-4\\6\end{pmatrix} \tag{0.17}$$

Thus, the given **B** lies on **AC** with the required ratio. **3D Diagram:**

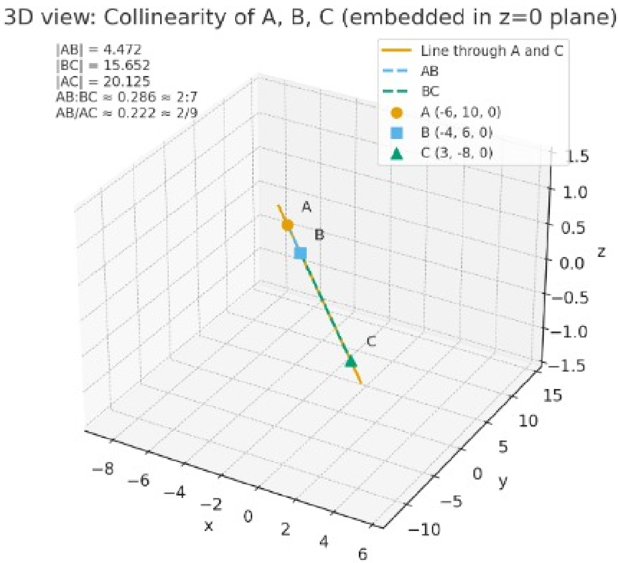

Fig. 0.1: 3D view (embedded in $z = 0$ plane) confirming collinearity