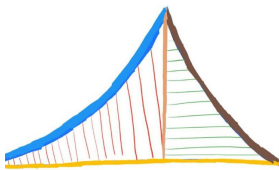


GEOMETRY

Through Equations



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1 THE BAUDHAYANA THEOREM

Use Fig. 1.1 for all problems in this section.

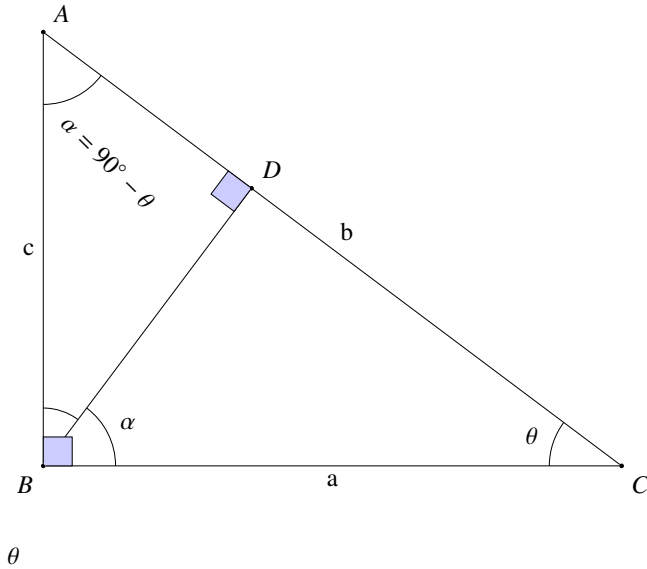


Fig. 1.1: Baudhayana Theorem

- 1.1. For simplicity, let the greek letter $\theta = \angle C$. We have the following definitions.

$$\begin{aligned} \sin \theta &= \frac{c}{b} & \cos \theta &= \frac{a}{b} \\ \tan \theta &= \frac{c}{a} & \cot \theta &= \frac{a}{c} \\ \csc \theta &= \frac{b}{c} & \sec \theta &= \frac{b}{a} \end{aligned} \quad (1.1.1)$$

- 1.2. Show that

$$\cos \theta = \sin(90^\circ - \theta) \quad (1.2.1)$$

Solution: From (1.1.1),

$$\cos \angle BAC = \cos \alpha = \cos(90^\circ - \theta) = \frac{c}{b} = \sin \angle ABC = \sin \theta \quad (1.2.2)$$

- 1.3. Show that

$$b = a \cos \theta + c \sin \theta \quad (1.3.1)$$

Solution: We observe that

$$CD = a \cos \theta \quad (1.3.2)$$

$$AD = c \cos \alpha = c \sin \theta \quad (\text{From (1.2.2)}) \quad (1.3.3)$$

Thus,

$$CD + AD = b = a \cos \theta + c \sin \theta \quad (1.3.4)$$

- 1.4. From (1.3.1), show that

$$\sin^2 \theta + \cos^2 \theta = 1 \quad (1.4.1)$$

Solution: Dividing both sides of (1.3.1) by b ,

$$1 = \frac{a}{b} \cos \theta + \frac{c}{b} \sin \theta \quad (1.4.2)$$

$$\Rightarrow \sin^2 \theta + \cos^2 \theta = 1 \quad (\text{from (1.1.1)}) \quad (1.4.3)$$

- 1.5. In a right angled triangle, the hypotenuse is the longest side.

Solution: From (1.4.1),

$$0 \leq \sin \theta, \cos \theta \leq 1 \quad (1.5.1)$$

Hence,

$$b \sin \theta \leq b \Rightarrow c \leq b \quad (1.5.2)$$

Similarly,

$$a \leq b \quad (1.5.3)$$

- 1.6. Using (1.3.1), show that

$$b^2 = a^2 + c^2 \quad (1.6.1)$$

(1.6.1) is known as the Baudhayana theorem. It is also known as the Pythagoras theorem.

Solution: From (1.3.1),

$$b = a \frac{a}{b} + c \frac{c}{b} \quad (\text{from (1.1.1)}) \quad (1.6.2)$$

$$\Rightarrow b^2 = a^2 + c^2 \quad (1.6.3)$$

2 AREA OF A TRIANGLE

- 2.1. Show that the area of $\triangle ABC$ in Fig. 2.1.1 is $\frac{1}{2}ab \sin C$.

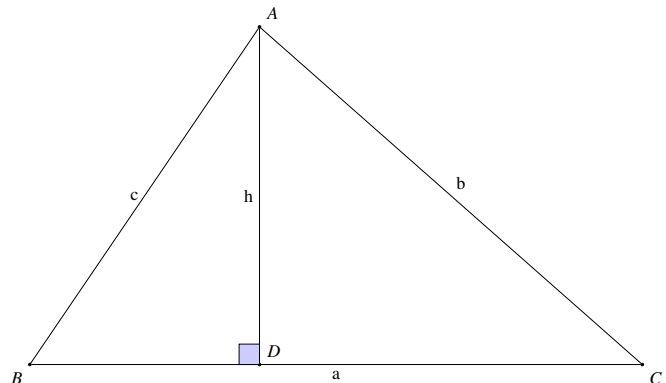


Fig. 2.1.1: Area of a Triangle

Solution: We have

$$\text{ar}(\triangle ABC) = \frac{1}{2}ah = \frac{1}{2}ab \sin C \quad (\because h = b \sin C). \quad (2.1.1)$$

- 2.2. Show that

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \quad (2.2.1)$$

Solution: Fig. 2.1.1 can be suitably modified to obtain

$$\text{ar}(\triangle ABC) = \frac{1}{2}ab \sin C = \frac{1}{2}bc \sin A = \frac{1}{2}ca \sin B \quad (2.2.2)$$

Dividing the above by abc , we obtain

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \quad (2.2.3)$$

This is known as the sine formula.

- 2.3. Show that

$$\alpha > \beta \Rightarrow \sin \alpha > \sin \beta \quad (2.3.1)$$

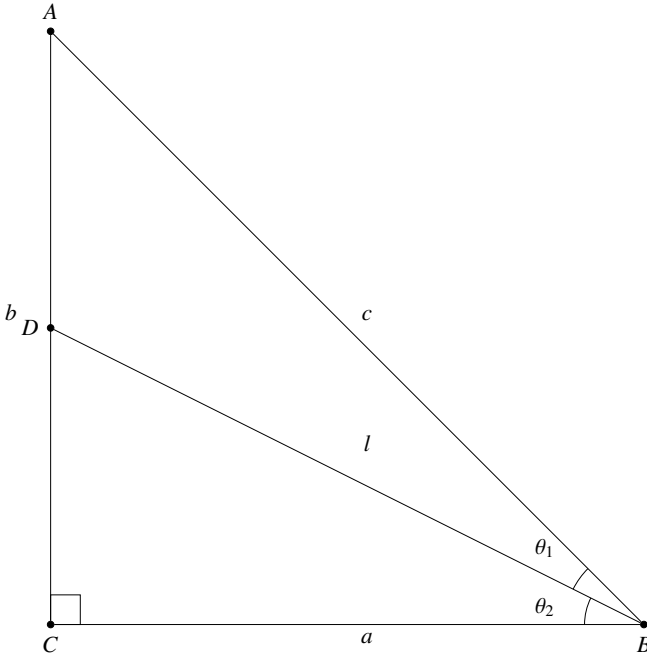


Fig. 2.3.1

Solution: In Fig. 2.3.1,

$$ar(\triangle ABD) < ar(\triangle ABC) \quad (2.3.2)$$

$$\Rightarrow \frac{1}{2}lc \sin \theta_1 < \frac{1}{2}ac \sin(\theta_1 + \theta_2) \quad (2.3.3)$$

$$\Rightarrow \frac{l}{a} < \frac{\sin(\theta_1 + \theta_2)}{\sin \theta_1} \quad (2.3.4)$$

$$\text{or, } 1 < \frac{l}{a} < \frac{\sin(\theta_1 + \theta_2)}{\sin \theta_1} \quad (2.3.5)$$

from Theorem 1.5, yielding

$$\Rightarrow \frac{\sin(\theta_1 + \theta_2)}{\sin \theta_1} > 1. \quad (2.3.6)$$

This proves (2.3.1).

2.4. Using Fig. 2.3.1, show that

$$\sin \theta_1 = \sin(\theta_1 + \theta_2) \cos \theta_2 - \cos(\theta_1 + \theta_2) \sin \theta_2 \quad (2.4.1)$$

Solution: The following equations can be obtained from the figure using the formula for the area of a triangle

$$ar(\triangle ABC) = \frac{1}{2}ac \sin(\theta_1 + \theta_2) \quad (2.4.2)$$

$$= ar(\triangle BDC) + ar(\triangle ADB) \quad (2.4.3)$$

$$= \frac{1}{2}cl \sin \theta_1 + \frac{1}{2}al \sin \theta_2 \quad (2.4.4)$$

$$= \frac{1}{2}ac \sin \theta_1 \sec \theta_2 + \frac{1}{2}a^2 \tan \theta_2 \quad (2.4.5)$$

($\because l = a \sec \theta_2$). From the above,

$$\sin(\theta_1 + \theta_2) = \sin \theta_1 \sec \theta_2 + \frac{a}{c} \tan \theta_2 \quad (2.4.6)$$

$$= \sin \theta_1 \sec \theta_2 + \cos(\theta_1 + \theta_2) \tan \theta_2 \quad (2.4.7)$$

Multiplying both sides by $\cos \theta_2$,

$$\sin(\theta_1 + \theta_2) \cos \theta_2 = \sin \theta_1 + \cos(\theta_1 + \theta_2) \sin \theta_2 \quad (2.4.8)$$

resulting in (2.4.1).

2.5. Find Hero's formula for the area of a triangle.

Solution: From (2.1), the area of $\triangle ABC$ is

$$\frac{1}{2}ab \sin C = \frac{1}{2}ab \sqrt{1 - \cos^2 C} \quad (\text{from (1.4.1)}) \quad (2.5.1)$$

$$= \frac{1}{2}ab \sqrt{1 - \left(\frac{a^2 + b^2 - c^2}{2ab}\right)^2} \quad (\text{from (B.2.1)}) \quad (2.5.2)$$

$$= \frac{1}{4} \sqrt{(2ab)^2 - (a^2 + b^2 - c^2)^2} \quad (2.5.3)$$

$$= \frac{1}{4} \sqrt{(2ab + a^2 + b^2 - c^2)(2ab - a^2 - b^2 + c^2)} \quad (2.5.4)$$

$$= \frac{1}{4} \sqrt{\{(a+b)^2 - c^2\} \{c^2 - (a-b)^2\}} \quad (2.5.5)$$

$$= \frac{1}{4} \sqrt{(a+b+c)(a+b-c)(a+c-b)(b+c-a)} \quad (2.5.6)$$

Substituting

$$s = \frac{a+b+c}{2} \quad (2.5.7)$$

in (2.5.6), the area of $\triangle ABC$ is

$$\sqrt{s(s-a)(s-b)(s-c)} \quad (2.5.8)$$

This is known as Hero's formula.

2.6. The area of the triangle with vertices **A, B, C** is given by

$$\frac{1}{2} \|(\mathbf{A} - \mathbf{B}) \times (\mathbf{A} - \mathbf{C})\| = \frac{1}{2} \|\mathbf{A} \times \mathbf{B} + \mathbf{B} \times \mathbf{C} + \mathbf{C} \times \mathbf{A}\| \quad (2.6.1)$$

where the *cross product* or *vector product* defined as $\mathbf{A} \times \mathbf{B}$ is given by

$$|\mathbf{A} \times \mathbf{B}| = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1 \quad (2.6.2)$$

2.7. If

$$\|\mathbf{A} \times \mathbf{B}\| = \|\mathbf{C} \times \mathbf{D}\|, \quad \text{then} \quad (2.7.1)$$

$$\mathbf{A} \times \mathbf{B} = \pm (\mathbf{C} \times \mathbf{D}) \quad (2.7.2)$$

where the sign depends on the orientation of the vectors.

3 ALTITUDES OF A TRIANGLE

3.1. In Fig. 3.2.1, $AD \perp BC$ and $BE \perp AC$ are defined to be the altitudes of $\triangle ABC$.

3.2. Let **H** be the intersection of the altitudes AD and BE as shown in Fig. 3.2.1. CH is extended to meet AB at **F**. Show that $CF \perp AB$.

Solution: From (B.6.2),

$$(\mathbf{B} - \mathbf{C})^\top (\mathbf{H} - \mathbf{A}) = 0 \quad (3.2.1)$$

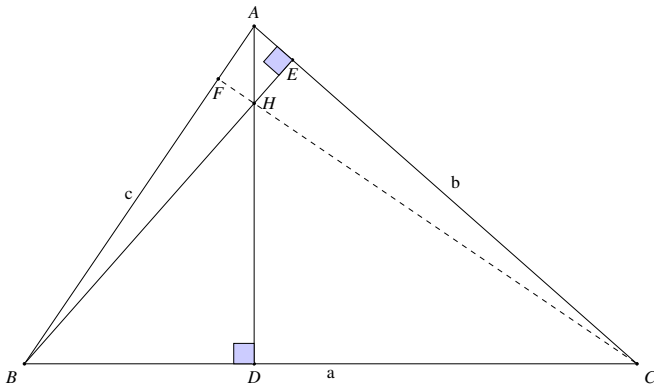
$$(\mathbf{C} - \mathbf{A})^\top (\mathbf{H} - \mathbf{B}) = 0 \quad (3.2.2)$$

Adding both the above and simplifying,

$$(\mathbf{B} - \mathbf{A})^\top (\mathbf{H} - \mathbf{C}) = 0 \quad (3.2.3)$$

$$\Rightarrow CH \perp AB, \text{ or } CF \perp AB.$$

3.3. Altitudes of a \triangle meet at the *orthocentre* **H**.

Fig. 3.2.1: Altitudes of a triangle meet at the orthocentre H

4 MEDIAN

4.1. In Fig. 4.3.1

$$AF = BF, AE = BE, \quad (4.1.1)$$

and the medians BE and CF meet at G . Show that

$$ar(BEC) = ar(BFC) = \frac{1}{2}ar(ABC) \quad (4.1.2)$$

Solution: From (2.2.2),

$$ar(BEC) = \frac{1}{2}a\left(\frac{b}{2}\right)\sin C \quad (4.1.3)$$

$$ar(BFC) = \frac{1}{2}a\left(\frac{c}{2}\right)\sin B \quad (4.1.4)$$

yielding (4.1.2).

4.2. Show that

$$ar(CGE) = ar(BGF) \quad (4.2.1)$$

Solution: From Fig. 4.3.1 and (4.1.2),

$$ar(BGF) + ar(BGC) = ar(CGE) + ar(BGC) \quad (4.2.2)$$

yielding (4.2.1).

4.3. If G divides BE and CF in the ratios k_1 and k_2 respectively, show that

$$k_1 = k_2 \quad (4.3.1)$$

Solution: Let

$$GE = l_1, GF = l_2 \quad (4.3.2)$$

From (2.2.2) and (4.2.1),

$$\frac{1}{2}l_1(k_2l_2)\sin\theta = \frac{1}{2}l_2(k_1l_1)\sin\theta \quad (4.3.3)$$

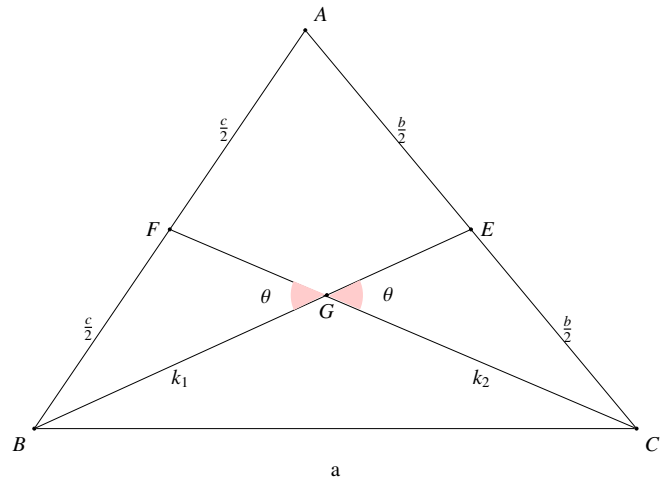
yielding (4.3.1).

4.4. Show that

$$k_1 = k_2 = 2 \quad (4.4.1)$$

Solution: Let

$$k_1 = k_2 = k \quad (4.4.2)$$

Fig. 4.3.1: $k_1 = k_2$.

Using (A.6.3),

$$\mathbf{G} = \frac{k\mathbf{E} + \mathbf{B}}{k+1} = \frac{k\mathbf{F} + \mathbf{C}}{k+1} \quad (4.4.3)$$

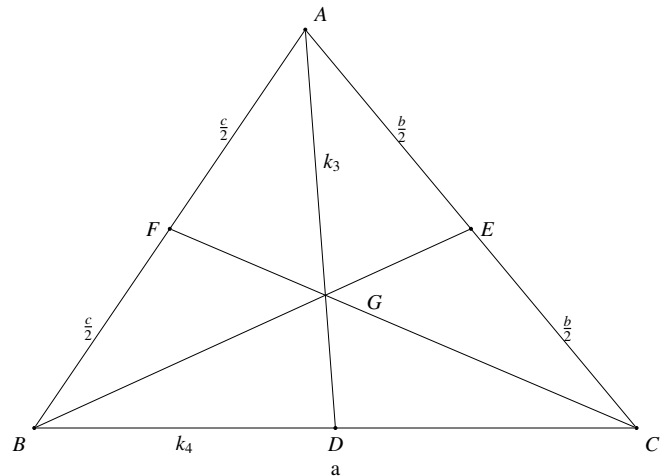
$$\Rightarrow k\left(\frac{\mathbf{A} + \mathbf{C}}{2}\right) + \mathbf{B} = k\left(\frac{\mathbf{A} + \mathbf{B}}{2}\right) + \mathbf{C} \quad (4.4.4)$$

$$\Rightarrow k(\mathbf{B} - \mathbf{C}) = 2(\mathbf{B} - \mathbf{C}) \quad (4.4.5)$$

resulting in (4.4.2).

4.5. Substituting $k = 2$ in (4.4.4),

$$\mathbf{G} = \frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} \quad (4.5.1)$$

4.6. In Fig. 4.6.1, AG is extended to join BC at D . Show that AD is also a median.Fig. 4.6.1: $k_3 = 2, k_4 = 1$ **Solution:** Considering the ratios in Fig. 4.6.1,

$$\mathbf{G} = \frac{k_3\mathbf{D} + \mathbf{A}}{k_3 + 1} \quad (4.6.1)$$

$$\mathbf{D} = \frac{k_4\mathbf{C} + \mathbf{B}}{k_4 + 1} \quad (4.6.2)$$

Substituting from (4.5.1) in the above,

$$(k_3 + 1) \left(\frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} \right) = k_3 \left(\frac{k_4 \mathbf{C} + \mathbf{B}}{k_4 + 1} \right) + \mathbf{A} \quad (4.6.3)$$

which can be expressed as

$$(k_3 + 1)(k_4 + 1)(\mathbf{A} + \mathbf{B} + \mathbf{C}) = 3 \{ k_3 (k_4 \mathbf{C} + \mathbf{B}) + (k_4 + 1) \mathbf{A} \} \quad (4.6.4)$$

which can be expressed as

$$\begin{aligned} (k_3 k_4 + k_3 - 2k_4 - 2) \mathbf{A} \\ - (-k_3 k_4 - k_4 + 2k_3 - 1) \mathbf{B} \\ - (-k_3 - k_4 - 1 + 2k_3 k_4) \mathbf{C} = \mathbf{0} \end{aligned} \quad (4.6.5)$$

Comparing the above with (A.7.3),

$$p = -k_3 k_4 - k_4 + 2k_3 - 1, q = -k_3 - k_4 - 1 + 2k_3 k_4 \quad (4.6.6)$$

yielding

$$-k_3 k_4 - k_4 + 2k_3 - 1 = 0 \quad (4.6.7)$$

$$-k_3 - k_4 - 1 + 2k_3 k_4 = 0 \quad (4.6.8)$$

Subtracting (4.6.7) from (4.6.8),

$$3k_3(k_4 - 1) = 0 \quad (4.6.9)$$

$$\Rightarrow k_4 = 1 \quad (4.6.10)$$

which upon substituting in (4.6.7) yields

$$k_3 = 2 \quad (4.6.11)$$

5 ANGLE BISECTORS

5.1. In Fig. 5.1.1, the bisectors of $\angle B$ and $\angle C$ meet at I . Show that IA bisects $\angle A$.

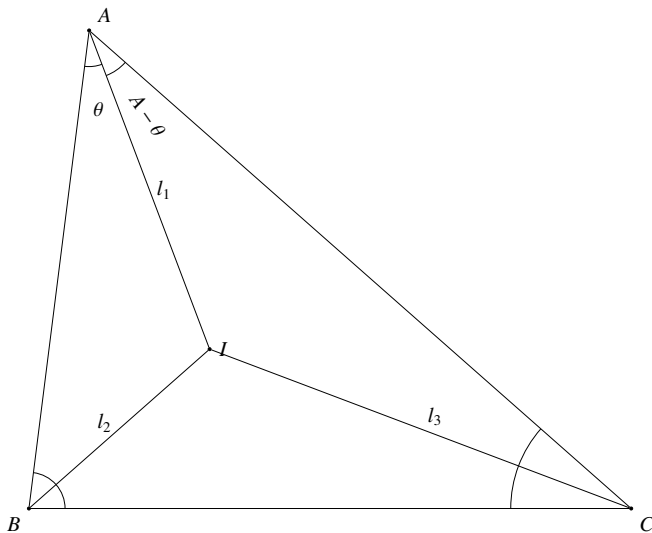


Fig. 5.1.1: Incentre I of $\triangle ABC$

Solution: Using sine formula in (2.2.3)

$$\frac{l_1}{\sin \frac{C}{2}} = \frac{l_3}{\sin(A - \theta)} \quad (5.1.1)$$

$$\frac{l_3}{\sin \frac{B}{2}} = \frac{l_2}{\sin \frac{C}{2}} \quad (5.1.2)$$

$$\frac{l_2}{\sin \theta} = \frac{l_1}{\sin \frac{B}{2}} \quad (5.1.3)$$

Multiplying the above equations,

$$\sin \theta = \sin(A - \theta) \quad (5.1.4)$$

$$\Rightarrow \theta = A - \theta \quad (5.1.5)$$

$$\text{or, } \theta = \frac{A}{2} \quad (5.1.6)$$

5.2. In Fig. 5.2.1,

$$ID \perp BC, IE \perp AC, IF \perp AB. \quad (5.2.1)$$

Show that

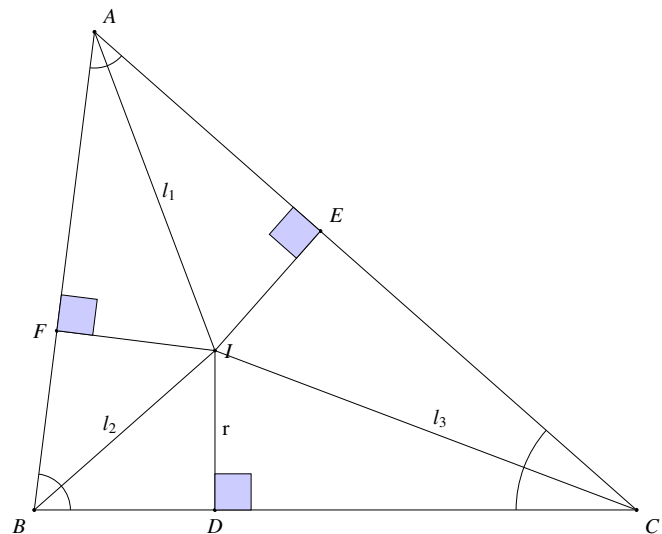


Fig. 5.2.1: Inradius r of $\triangle ABC$

$$ID = IE = IF = r \quad (5.2.2)$$

Solution: In $\triangle IDC$ and IEC ,

$$ID = IE = \frac{l_3}{\sin \frac{C}{2}} \quad (5.2.3)$$

Similarly, in $\triangle IEA$ and IFA ,

$$IF = IE = \frac{l_1}{\sin \frac{A}{2}} \quad (5.2.4)$$

yielding (5.2.2)

5.3. In Fig. 5.2.1, show that

$$BD = BF, AE = AF, CD = CE \quad (5.3.1)$$

Solution: From Fig. 5.2.1, in $\triangle IBD$ and IBF ,

$$x = BD = BF = r \cot \frac{B}{2} \quad (5.3.2)$$

Similarly, other results can be obtained.

5.4. The circle with centre **I** and radius r in Fig. 5.4.1 is known as the *incircle*.

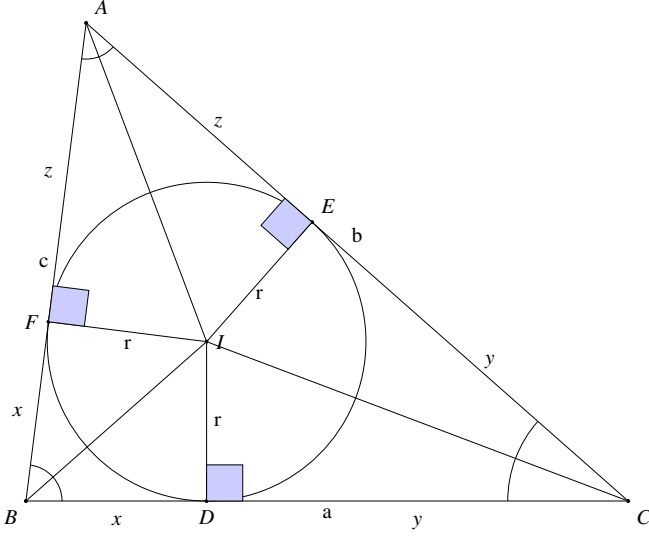


Fig. 5.4.1: Incircle of $\triangle ABC$

6 PERPENDICULAR BISECTORS

6.1. In Fig. 6.1.1,

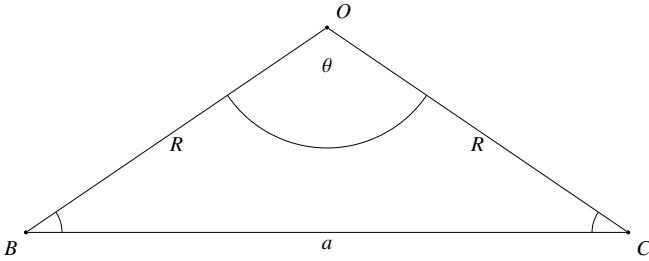


Fig. 6.1.1: Isosceles Triangle

$$OB = OC = R \quad (6.1.1)$$

Such a triangle is known as an isosceles triangle. Show that

$$\angle B = \angle C \quad (6.1.2)$$

Solution: Using (2.2.3),

$$\frac{\sin B}{R} = \frac{\sin C}{R} \quad (6.1.3)$$

$$\Rightarrow \sin B = \sin C \quad (6.1.4)$$

$$\text{or, } \angle B = \angle C. \quad (6.1.5)$$

6.2. In Fig. 6.1.1, show that

$$a = 2R \sin \frac{\theta}{2} \quad (6.2.1)$$

Solution: In $\triangle OBC$, using the cosine formula from (B.2.1),

$$\cos \theta = \frac{R^2 + R^2 - a^2}{2R^2} = 1 - \frac{a^2}{2R^2} \quad (6.2.2)$$

$$\Rightarrow \frac{a^2}{2R^2} = 2 \sin^2 \frac{\theta}{2} \quad (6.2.3)$$

yielding (6.2.1).

6.3. In Fig. 7.1.1, show that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R. \quad (6.3.1)$$

Solution: From (7.4.1) and (6.2.1)

$$a = 2R \sin A \quad (6.3.2)$$

6.4. In Fig. 6.4.1,

$$OB = OC = R, BD = DC. \quad (6.4.1)$$

Show that $OD \perp BC$.

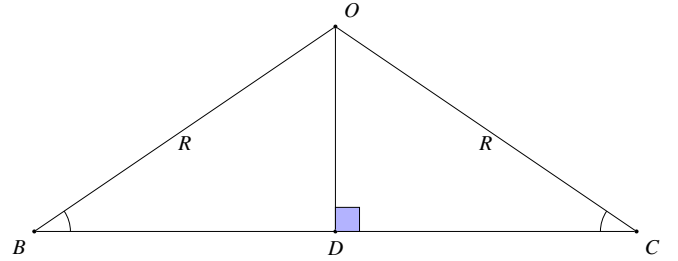


Fig. 6.4.1: Perpendicular bisector.

Solution:

$$\|O - C\| = \|O - B\| = R \quad (6.4.2)$$

$$\Rightarrow \|O - C\|^2 = \|O - B\|^2 \quad (6.4.3)$$

which can be expressed as

$$(O - C)^T (O - C) = (O - B)^T (O - B) \quad (6.4.4)$$

$$\|O\|^2 - 2O^T C + \|C\|^2 = \|O\|^2 - 2O^T B + \|B\|^2 \quad (6.4.5)$$

$$\Rightarrow (B - C)^T O = \frac{\|B\|^2 - \|C\|^2}{2} \quad (6.4.6)$$

which can be simplified to obtain

$$(B - C)^T \left\{ O - \left(\frac{B + C}{2} \right) \right\} = 0 \quad (6.4.7)$$

$$\text{or, } (B - C)^T \{O - D\} = 0 \quad (6.4.8)$$

which proves the give result using (A.6.3) and (B.6.2).

6.5. In Fig. 6.5.1, OD and OE are the perpendicular bisectors of sides BC and AC respectively. Show that $OA = R$.

Solution: Tracing (6.4.8) backwards yields

$$OB = OC, OC = OA = R. \quad (6.5.1)$$

7 CIRCUMCIRCLE: CIRCLE EQUATION

7.1. The equation of the circle in Fig. 7.1.1, is

$$\|x - O\| = R \quad (7.1.1)$$

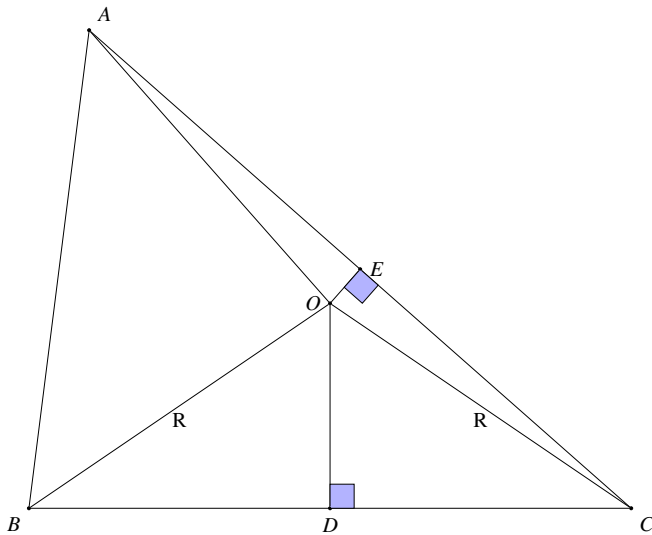


Fig. 6.5.1: Perpendicular bisectors of $\triangle ABC$ meet at \mathbf{O} .

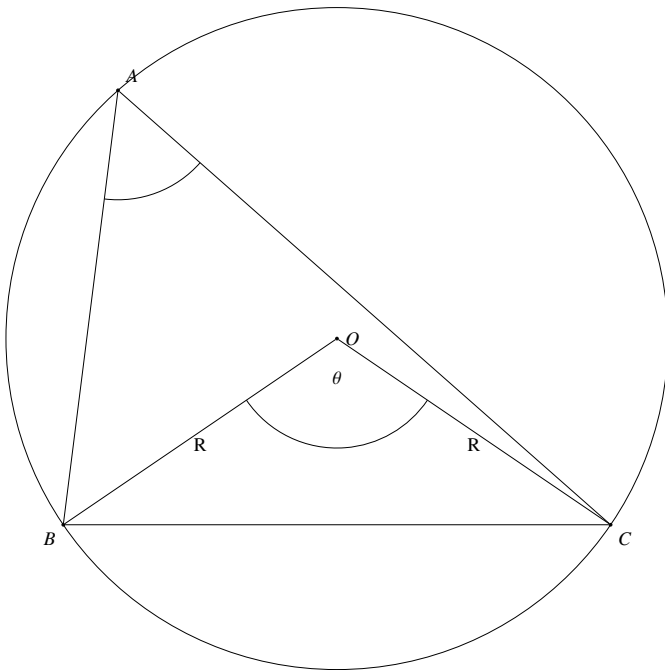


Fig. 7.1.1: Circumcircle of $\triangle ABC$

This is known as the *circumcircle* of $\triangle ABC$.

7.2. Any point on the circle can be expressed as

$$\mathbf{x} = \mathbf{O} + R \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \quad 0 \in [0, 2\pi]. \quad (7.2.1)$$

7.3. Let

$$R = 1, \mathbf{O} = \mathbf{0}, \mathbf{A} = \begin{pmatrix} \cos \theta_1 \\ \sin \theta_1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} \cos \theta_2 \\ \sin \theta_2 \end{pmatrix}, \quad (7.3.1)$$

Show that

$$\|\mathbf{A} - \mathbf{B}\| = 2 \sin \left(\frac{\theta_1 - \theta_2}{2} \right) \quad (7.3.2)$$

Solution: From (7.2.1).

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} \cos \theta_1 - \cos \theta_2 \\ \sin \theta_1 - \sin \theta_2 \end{pmatrix} \quad (7.3.3)$$

$$\Rightarrow \|\mathbf{A} - \mathbf{B}\|^2 = (\cos \theta_1 - \cos \theta_2)^2 + (\sin \theta_1 - \sin \theta_2)^2 \quad (7.3.4)$$

$$= 2 \{1 - \cos(\theta_1 - \theta_2)\} = 4 \sin^2 \left(\frac{\theta_1 - \theta_2}{2} \right) \quad (7.3.5)$$

yielding (7.3.2) from (C.4.3).

7.4. In Fig. 7.1.1, show that

$$\theta = 2A. \quad (7.4.1)$$

Solution: Let

$$\mathbf{C} = \begin{pmatrix} \cos \theta_3 \\ \sin \theta_3 \end{pmatrix} \quad (7.4.2)$$

Then, substituting from (7.3.2) in (B.5.2),

$$\cos A = \frac{4 \sin^2 \left(\frac{\theta_1 - \theta_2}{2} \right) + 4 \sin^2 \left(\frac{\theta_1 - \theta_3}{2} \right) - 4 \sin^2 \left(\frac{\theta_2 - \theta_3}{2} \right)}{8 \sin \left(\frac{\theta_1 - \theta_2}{2} \right) \sin \left(\frac{\theta_1 - \theta_3}{2} \right)} \quad (7.4.3)$$

$$= \frac{2 \sin^2 \left(\frac{\theta_1 - \theta_2}{2} \right) + \cos(\theta_2 - \theta_3) - \cos(\theta_1 - \theta_3)}{4 \sin \left(\frac{\theta_1 - \theta_2}{2} \right) \sin \left(\frac{\theta_1 - \theta_3}{2} \right)} \quad (7.4.4)$$

from (C.4.3). \therefore from (C.3.4),

$$\cos A = \frac{2 \sin^2 \left(\frac{\theta_1 - \theta_2}{2} \right) + 2 \sin \left(\frac{\theta_1 - \theta_2}{2} \right) \sin \left(\frac{\theta_1 + \theta_2}{2} - \theta_3 \right)}{4 \sin \left(\frac{\theta_1 - \theta_2}{2} \right) \sin \left(\frac{\theta_1 - \theta_3}{2} \right)} \quad (7.4.5)$$

$$= \frac{\sin \left(\frac{\theta_1 - \theta_2}{2} \right) + \sin \left(\frac{\theta_1 + \theta_2}{2} - \theta_3 \right)}{2 \sin \left(\frac{\theta_1 - \theta_3}{2} \right)} \quad (7.4.6)$$

From (C.3.1), the above equation can be expressed as

$$\cos A = \frac{2 \sin \left(\frac{\theta_1 - \theta_3}{2} \right) \cos \left(\frac{\theta_2 - \theta_3}{2} \right)}{2 \sin \left(\frac{\theta_1 - \theta_3}{2} \right)} = \cos \left(\frac{\theta_2 - \theta_3}{2} \right) \quad (7.4.7)$$

$$\Rightarrow 2A = \theta_2 - \theta_3 \quad (7.4.8)$$

Similarly,

$$\cos \theta = \frac{1 + 1 - 4 \sin^2 \left(\frac{\theta_2 - \theta_3}{2} \right)}{2} = \cos(\theta_2 - \theta_3) = \cos 2A \quad (7.4.9)$$

8 TANGENT

8.1. In Fig. 8.1.1, OC is the radius and PC touches the circle at C . Show that

$$OC \perp PC. \quad (8.1.1)$$

Solution: The equation of PC can be expressed as

$$\mathbf{x} = \mathbf{C} + \mu \mathbf{m} \quad (8.1.2)$$

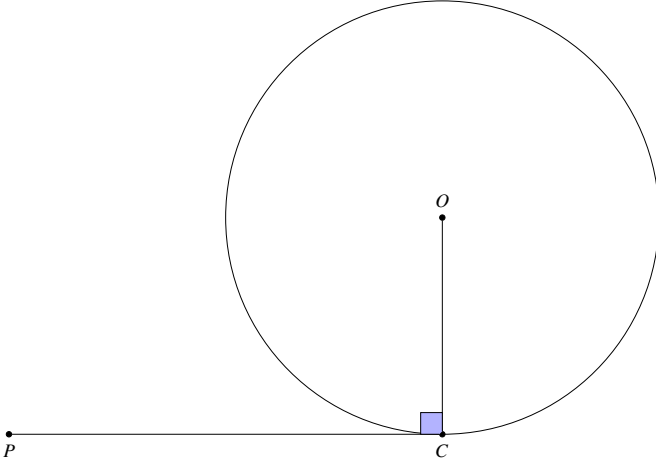


Fig. 8.1.1

and the equation of the circle is

$$\|\mathbf{x} - \mathbf{O}\| = R \quad (8.1.3)$$

Substituting (8.1.2) in (8.1.3),

$$\|\mathbf{C} + \mu\mathbf{m} - \mathbf{O}\|^2 = R^2 \quad (8.1.4)$$

$$\Rightarrow \mu^2 \|\mathbf{m}\|^2 + 2\mu\mathbf{m}^\top (\mathbf{C} - \mathbf{O}) + \|\mathbf{C} - \mathbf{O}\|^2 - R^2 = 0 \quad (8.1.5)$$

The above equation has only one root. Hence the discriminant of the above quadratic should be zero. So,

$$\{\mathbf{m}^\top (\mathbf{C} - \mathbf{O})\}^2 - \|\mathbf{m}\|^2 \{\|\mathbf{C} - \mathbf{O}\|^2 - R^2\} = 0 \quad (8.1.6)$$

Since \mathbf{C} is a point on the circle,

$$\|\mathbf{C} - \mathbf{O}\|^2 - R^2 = 0 \quad (8.1.7)$$

$$\Rightarrow \mathbf{m}^\top (\mathbf{C} - \mathbf{O}) = 0 \quad (8.1.8)$$

upon substituting in (8.1.6). Using the definition of the direction vector from (A.2.1)

$$\mathbf{m} = \mathbf{P} - \mathbf{C} \quad (8.1.9)$$

$$\Rightarrow (\mathbf{P} - \mathbf{C})^\top (\mathbf{C} - \mathbf{O}) = 0 \quad (8.1.10)$$

which is equivalent to (8.1.1).

8.2. In Fig. 8.2.1 show that

$$\theta = \alpha \quad (8.2.1)$$

Solution: Let Let

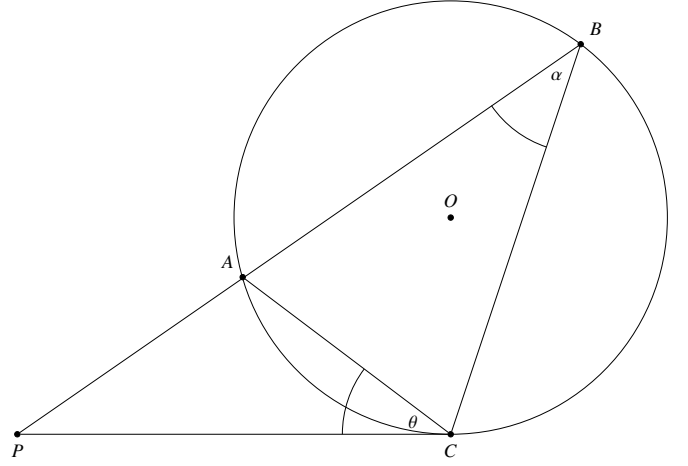
$$\mathbf{O} = \mathbf{OA} = \begin{pmatrix} \cos \theta_1 \\ \sin \theta_1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} \cos \theta_2 \\ \sin \theta_2 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} \cos \theta_3 \\ \sin \theta_3 \end{pmatrix} \quad (8.2.2)$$

Without loss of generality, let

$$\theta_3 = \frac{\pi}{2} \quad (8.2.3)$$

Then,

$$\mathbf{C} - \mathbf{O} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (8.2.4)$$

Fig. 8.2.1: $\theta = \alpha$.

From from (8.1.10),

$$\mathbf{C} - \mathbf{P} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (8.2.5)$$

From (B.5.1) and (8.2.5),

$$\begin{aligned} \cos \theta &= \frac{(\cos \theta_3 - \cos \theta_1 \quad \sin \theta_3 - \sin \theta_1) \begin{pmatrix} 1 \\ 0 \end{pmatrix}}{2 \sin \left(\frac{\theta_1 - \theta_3}{2} \right)} \quad (8.2.6) \\ &= \sin \left(\frac{\theta_1 + \theta_3}{2} \right) = \cos \left(\frac{\pi}{2} - \frac{\theta_1 + \theta_3}{2} \right) = \cos \left(\frac{\pi}{4} - \frac{\theta_1}{2} \right) \quad (8.2.7) \end{aligned}$$

upon substituting from (8.2.3). Similarly, from (7.4.7),

$$\cos \alpha = \cos \left(\frac{\theta_1 - \theta_3}{2} \right) = \cos \left(\frac{\pi}{4} - \frac{\theta_1}{2} \right) = \cos \theta \quad (8.2.8)$$

8.3. In Fig. 8.2.1, show that $PA \cdot PB = PC^2$.

Solution: In $\triangle APC$ and BPC , using (8.2.1),

$$\frac{AP}{\sin \theta} = \frac{AC}{\sin P} \quad (8.3.1)$$

$$\frac{PC}{\sin \theta} = \frac{BC}{\sin P} \quad (8.3.2)$$

$$\Rightarrow \frac{PC}{AP} = \frac{BC}{AC} \left(= \frac{BP}{CP} \right) \quad (8.3.3)$$

which gives the desired result. $\triangle APC$ and BPC are said to be *similar*.

APPENDIX A COLLINEAR POINTS

A.1. In Fig. A.1.1,

$$a = y \cot \theta + x \quad (A.1.1)$$

$$\Rightarrow \mathbf{D} = \begin{pmatrix} -x \\ y \end{pmatrix} = \begin{pmatrix} -a + y \cot \theta \\ y \end{pmatrix} \quad (A.1.2)$$

$$= \begin{pmatrix} -a \\ 0 \end{pmatrix} + y \cot \theta \begin{pmatrix} 1 \\ \tan \theta \end{pmatrix} \quad (A.1.3)$$

$$\text{or, } \mathbf{D} \equiv \mathbf{B} + \kappa \mathbf{m} \quad (A.1.4)$$

The above equation can be generalized for any point on the line AB as

$$\mathbf{x} = \mathbf{B} + \kappa \mathbf{m} \quad (\text{A.1.5})$$

which is known as the *parametric* equation of a line. \mathbf{m} is defined to be the *direction vector* of AB and

$$m = \tan \theta \quad (\text{A.1.6})$$

is defined to be the *slope*.

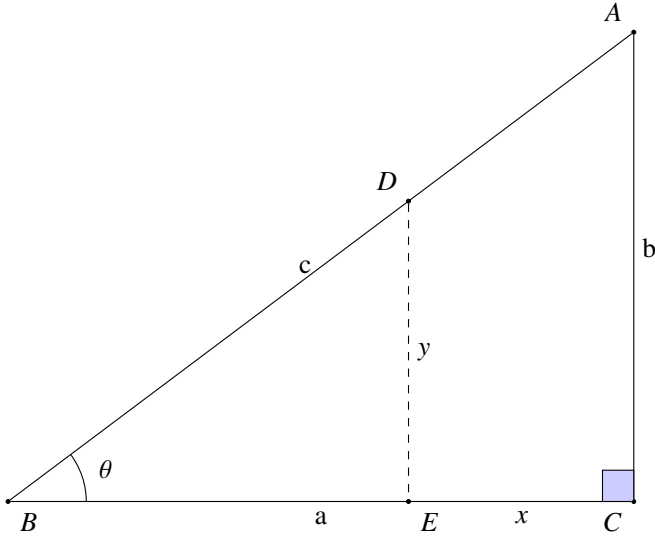


Fig. A.1.1: $k_1 = k_2 = 2$.

A.2. The direction vector of the line AB is

$$\mathbf{A} - \mathbf{B} \equiv \mathbf{B} - \mathbf{A} \equiv \kappa \begin{pmatrix} 1 \\ m \end{pmatrix}, \quad (\text{A.2.1})$$

A.3. (A.1.1) can also be expressed as

$$a = y \cot \theta + x \quad (\text{A.3.1})$$

$$\Rightarrow \begin{pmatrix} -\tan \theta & 1 \end{pmatrix} \begin{pmatrix} -x \\ y \end{pmatrix} = b \quad (\text{A.3.2})$$

$$\text{or, } \mathbf{n}^\top \mathbf{x} = b \quad (\text{A.3.3})$$

which is known as the *normal* equation of a line. Here,

$$\mathbf{n} = \begin{pmatrix} -m \\ 1 \end{pmatrix} \quad (\text{A.3.4})$$

is defined to be the *normal vector* of the line. The vector product in (A.3.2) is known as the *inner product* or *dot product*

A.4. It is easy to verify that

$$\mathbf{n}^\top \mathbf{m} = 0 \quad (\text{A.4.1})$$

and

A.5.

$$\mathbf{n} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{m} = \begin{pmatrix} \cos(\frac{\pi}{2}) & \sin(\frac{\pi}{2}) \\ \sin(\frac{\pi}{2}) & \cos(\frac{\pi}{2}) \end{pmatrix} \mathbf{m} \quad (\text{A.5.1})$$

The matrix

$$\mathbf{R}_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \quad (\text{A.5.2})$$

is defined to be the *rotation matrix*. (A.5.1) implies that \mathbf{n} can be obtained from \mathbf{m} through a 90° clockwise rotation.

A.6. From (A.1.5), since \mathbf{A}, \mathbf{D} and \mathbf{C} are on the same line,

$$\mathbf{D} = \mathbf{A} + q\mathbf{m} \quad (\text{A.6.1})$$

$$\mathbf{B} = \mathbf{D} + p\mathbf{m}$$

$$\Rightarrow p(\mathbf{D} - \mathbf{A}) + q(\mathbf{D} - \mathbf{B}) = 0, \quad p, q \neq 0 \quad (\text{A.6.2})$$

$$\Rightarrow \mathbf{D} = \frac{k\mathbf{A} + \mathbf{B}}{k+1}, \quad k = \frac{p}{q}. \quad (\text{A.6.3})$$

which is known as *section formula*. $(\mathbf{D} - \mathbf{A}), (\mathbf{D} - \mathbf{B})$ are then said to be *linearly dependent*.

A.7. Consequently, points \mathbf{A}, \mathbf{B} and \mathbf{C} form a triangle if

$$p(\mathbf{A} - \mathbf{B}) + q(\mathbf{C} - \mathbf{B}) \quad (\text{A.7.1})$$

$$= (p+q)\mathbf{B} - p\mathbf{A} - q\mathbf{C} = 0 \quad (\text{A.7.2})$$

$$\Rightarrow p = 0, q = 0 \quad (\text{A.7.3})$$

APPENDIX B COSINE FORMULA

B.1. In Fig. B.1.1, show that

$$\begin{pmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{pmatrix} \begin{pmatrix} \cos A \\ \cos B \\ \cos C \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad (\text{B.1.1})$$

Solution: From Fig. B.1.1,

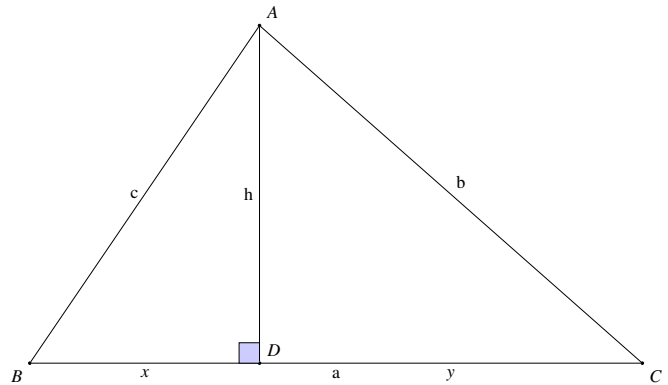


Fig. B.1.1: The cosine formula

$$a = x + y = b \cos C + c \cos B = \begin{pmatrix} \cos C & \cos B \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix} \quad (\text{B.1.2})$$

$$= \begin{pmatrix} 0 & b & c \end{pmatrix} \begin{pmatrix} \cos A \\ \cos C \\ \cos B \end{pmatrix} \quad (\text{B.1.3})$$

Similarly,

$$b = c \cos A + a \cos C = \begin{pmatrix} c & 0 & a \end{pmatrix} \begin{pmatrix} \cos A \\ \cos C \\ \cos B \end{pmatrix} \quad (\text{B.1.4})$$

$$c = b \cos A + a \cos B = \begin{pmatrix} b & a & 0 \end{pmatrix} \begin{pmatrix} \cos A \\ \cos C \\ \cos B \end{pmatrix} \quad (\text{B.1.5})$$

The above equations can be expressed in matrix form as (B.1.1).

B.2. Show that

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \quad (\text{B.2.1})$$

Solution: Using the properties of determinants,

$$\cos A = \frac{\begin{vmatrix} a & c & b \\ b & 0 & a \\ c & a & 0 \end{vmatrix}}{\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}} = \frac{ab^2 + ac^2 - a^3}{abc + abc} = \frac{b^2 + c^2 - a^2}{2abc} \quad (\text{B.2.2})$$

B.3. The *norm* of \mathbf{A} is defined as

$$\|\mathbf{A}\| = \sqrt{\mathbf{A}^\top \mathbf{A}} = \sqrt{a_1^2 + a_2^2} \quad (\text{B.3.1})$$

B.4. In Fig. 1.1 it is easy to verify that

$$\|\mathbf{A} - \mathbf{C}\|^2 = \begin{pmatrix} -a & c \end{pmatrix} \begin{pmatrix} -a \\ c \end{pmatrix} = a^2 + c^2 = b^2 \quad (\text{B.4.1})$$

from (1.6.1). Thus, the distance between any two points \mathbf{A} and \mathbf{B} is given by

$$\|\mathbf{A} - \mathbf{B}\| \quad (\text{B.4.2})$$

B.5. In Fig. B.1.1 show that

$$\cos A = \frac{(\mathbf{A} - \mathbf{B})^\top (\mathbf{A} - \mathbf{C})}{\|\mathbf{A} - \mathbf{B}\| \|\mathbf{A} - \mathbf{C}\|} \quad (\text{B.5.1})$$

Solution: From (B.2.1), using (B.4.2),

$$\cos A = \frac{\|\mathbf{A} - \mathbf{B}\|^2 + \|\mathbf{A} - \mathbf{C}\|^2 - \|\mathbf{B} - \mathbf{C}\|^2}{2 \|\mathbf{A} - \mathbf{B}\| \|\mathbf{A} - \mathbf{C}\|} \quad (\text{B.5.2})$$

$$= \frac{\|\mathbf{A}\|^2 - \mathbf{A}^\top \mathbf{B} - \mathbf{A}^\top \mathbf{C} + \mathbf{B}^\top \mathbf{C}}{\|\mathbf{A} - \mathbf{B}\| \|\mathbf{A} - \mathbf{C}\|} \quad (\text{B.5.3})$$

which can be expressed as (B.5.1).

B.6. For $A = 90^\circ$,

$$\cos A = 0 \quad (\text{B.6.1})$$

$$\Rightarrow (\mathbf{A} - \mathbf{B})^\top (\mathbf{A} - \mathbf{C}) = 0 \quad (\text{B.6.2})$$

from (B.5.1).

APPENDIX C TRIGONOMETRIC IDENTITIES

C.1. Prove the following identities

a)

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta. \quad (\text{C.1.1})$$

b)

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta. \quad (\text{C.1.2})$$

Solution: In (2.4.1), let

$$\begin{aligned} \theta_1 + \theta_2 &= \alpha \\ \theta_2 &= \beta \end{aligned} \quad (\text{C.1.3})$$

This gives (C.1.1). In (C.1.1), replace α by $90^\circ - \alpha$. This results in

$$\sin(90^\circ - \alpha - \beta) = \sin(90^\circ - \alpha) \cos \beta - \cos(90^\circ - \alpha) \sin \beta \quad (\text{C.1.4})$$

$$\Rightarrow \cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta \quad (\text{C.1.5})$$

C.2. Using (2.4.1) and (C.1.2), show that

$$\sin(\theta_1 + \theta_2) = \sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2 \quad (\text{C.2.1})$$

$$\cos(\theta_1 - \theta_2) = \cos \theta_1 \cos \theta_2 \sin \theta_1 \sin \theta_2 \quad (\text{C.2.2})$$

Solution: From (2.4.1),

$$\sin(\theta_1 + \theta_2) \cos \theta_2 = \sin \theta_1 + \cos(\theta_1 + \theta_2) \sin \theta_2 \quad (\text{C.2.3})$$

Using (C.1.2) in the above,

$$\begin{aligned} \sin(\theta_1 + \theta_2) \cos \theta_2 &= \sin \theta_1 + (\cos \theta_1 \cos \theta_2 \\ &\quad - \sin \theta_1 \sin \theta_2) \sin \theta_2 \end{aligned} \quad (\text{C.2.4})$$

which can be expressed as

$$\begin{aligned} \sin(\theta_1 + \theta_2) \cos \theta_2 &= \sin \theta_1 \\ &\quad + \cos \theta_1 \cos \theta_2 \sin \theta_2 - \sin \theta_1 \sin^2 \theta_2 \end{aligned} \quad (\text{C.2.5})$$

Since

$$\sin^2 \theta_2 = 1 - \cos^2 \theta_2, \quad (\text{C.2.6})$$

we obtain

$$\sin(\theta_1 + \theta_2) \cos \theta_2 = \cos \theta_1 \cos \theta_2 \sin \theta_2 + \sin \theta_1 \cos^2 \theta_2 \quad (\text{C.2.7})$$

resulting in

$$\sin(\theta_1 + \theta_2) = \cos \theta_1 \sin \theta_2 + \sin \theta_1 \cos \theta_2 \quad (\text{C.2.8})$$

after factoring out $\cos \theta_2$. Using a similar approach, (C.2.2) can also be proved.

C.3. Show that

$$\sin \theta_1 + \sin \theta_2 = 2 \sin\left(\frac{\theta_1 + \theta_2}{2}\right) \cos\left(\frac{\theta_1 - \theta_2}{2}\right) \quad (\text{C.3.1})$$

$$\cos \theta_1 + \cos \theta_2 = 2 \cos\left(\frac{\theta_1 + \theta_2}{2}\right) \cos\left(\frac{\theta_1 - \theta_2}{2}\right) \quad (\text{C.3.2})$$

$$\sin \theta_1 - \sin \theta_2 = 2 \sin\left(\frac{\theta_1 - \theta_2}{2}\right) \cos\left(\frac{\theta_1 + \theta_2}{2}\right) \quad (\text{C.3.3})$$

$$\cos \theta_1 - \cos \theta_2 = 2 \sin\left(\frac{\theta_1 + \theta_2}{2}\right) \cos\left(\frac{\theta_2 - \theta_1}{2}\right) \quad (\text{C.3.4})$$

Solution: Let

$$\begin{aligned} \theta_1 &= \alpha + \beta \\ \theta_2 &= \alpha - \beta \end{aligned} \quad (\text{C.3.5})$$

From (C.2.1),

$$\sin \theta_1 + \sin \theta_2 = \sin (\alpha + \beta) + \sin (\alpha - \beta) \quad (\text{C.3.6})$$

$$= \sin \alpha \cos \beta + \cos \alpha \sin \beta \quad (\text{C.3.7})$$

$$+ \sin \alpha \cos \beta - \cos \alpha \sin \beta \quad (\text{C.3.8})$$

$$= 2 \sin \alpha \cos \beta \quad (\text{C.3.9})$$

resulting in (C.3.1)

$$\therefore \alpha = \frac{\theta_1 + \theta_2}{2} \quad (\text{C.3.10})$$

$$\beta = \frac{\theta_1 - \theta_2}{2} \quad (\text{C.3.11})$$

from (C.3.5). Other identities may be proved similarly.

C.4. Show that

$$\sin 2\theta = 2 \sin \theta \cos \theta \quad (\text{C.4.1})$$

$$\cos 2\theta = 1 - 2 \sin^2 \theta = 2 \cos^2 \theta - 1 \quad (\text{C.4.2})$$

$$= \cos^2 \theta - \sin^2 \theta \quad (\text{C.4.3})$$