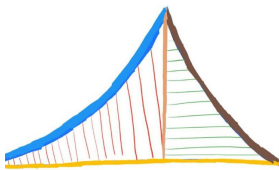


GEOMETRY

Through Equations



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CONTENTS

1	The Baudhayana Theorem	3
2	Area of a Triangle	3
3	Altitudes of a Triangle	4
4	Median	5
5	Angle Bisectors	6
6	Perpendicular Bisectors	6
7	Circumcircle: Circle Equation	7
8	Tangent	8
	Appendix A: Ratios	9
	Appendix B: Vectors	10
	Appendix C: Matrices: Cosine Formula	11
	Appendix D: Collinear Points	12
	Appendix E: Identities	12

1 THE BAUDHAYANA THEOREM

Use Fig. 1.1 for all problems in this section.

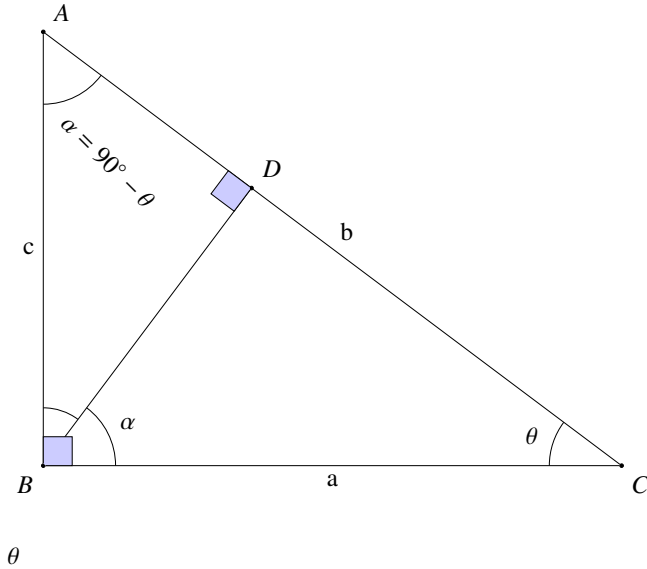


Fig. 1.1: Baudhayana Theorem

1.1. Show that

$$b = a \cos \theta + c \sin \theta \quad (1.1.1)$$

Solution: We observe that

$$CD = a \cos \theta \quad (1.1.2)$$

$$AD = c \cos \alpha = c \sin \theta \quad (\text{From (A.2.2)}) \quad (1.1.3)$$

Thus,

$$CD + AD = b = a \cos \theta + c \sin \theta \quad (1.1.4)$$

1.2. From (1.1.1), show that

$$\sin^2 \theta + \cos^2 \theta = 1 \quad (1.2.1)$$

Solution: Dividing both sides of (1.1.1) by b ,

$$1 = \frac{a}{b} \cos \theta + \frac{c}{b} \sin \theta \quad (1.2.2)$$

$$\Rightarrow \sin^2 \theta + \cos^2 \theta = 1 \quad (\text{from (A.1.1)}) \quad (1.2.3)$$

1.3. In a right angled triangle, the hypotenuse is the longest side.

Solution: From (1.2.1),

$$0 \leq \sin \theta, \cos \theta \leq 1 \quad (1.3.1)$$

Hence,

$$b \sin \theta \leq b \Rightarrow c \leq b \quad (1.3.2)$$

Similalry,

$$a \leq b \quad (1.3.3)$$

1.4. Using (1.1.1), show that

$$b^2 = a^2 + c^2 \quad (1.4.1)$$

(1.4.1) is known as the Baudhayana theorem. It is also known as the Pythagoras theorem.

Solution: From (1.1.1),

$$b = a \frac{a}{b} + c \frac{c}{b} \quad (\text{from (A.1.1)}) \quad (1.4.2)$$

$$\Rightarrow b^2 = a^2 + c^2 \quad (1.4.3)$$

2 AREA OF A TRIANGLE

2.1. Show that the area of $\triangle ABC$ in Fig. 2.1.1 is $\frac{1}{2}ab \sin C$.

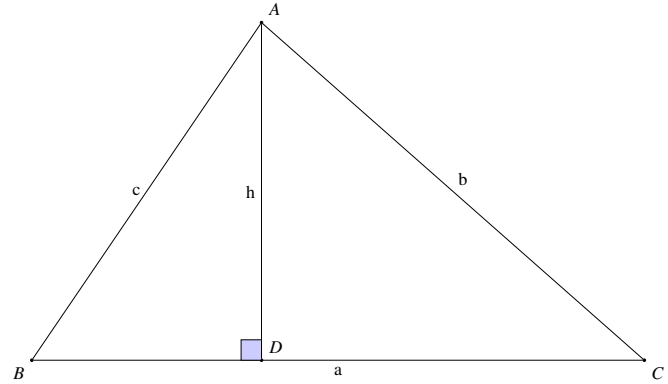


Fig. 2.1.1: Area of a Triangle

Solution: We have

$$ar(\triangle ABC) = \frac{1}{2}ah = \frac{1}{2}ab \sin C \quad (\because h = b \sin C). \quad (2.1.1)$$

2.2. Show that

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \quad (2.2.1)$$

Solution: Fig. 2.1.1 can be suitably modified to obtain

$$ar(\triangle ABC) = \frac{1}{2}ab \sin C = \frac{1}{2}bc \sin A = \frac{1}{2}ca \sin B \quad (2.2.2)$$

Dividing the above by abc , we obtain

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \quad (2.2.3)$$

This is known as the sine formula.

2.3. Show that

$$\alpha > \beta \Rightarrow \sin \alpha > \sin \beta \quad (2.3.1)$$

Solution: In Fig. 2.3.1,

$$ar(\triangle ABD) < ar(\triangle ABC) \quad (2.3.2)$$

$$\Rightarrow \frac{1}{2}lc \sin \theta_1 < \frac{1}{2}ac \sin(\theta_1 + \theta_2) \quad (2.3.3)$$

$$\Rightarrow \frac{l}{a} < \frac{\sin(\theta_1 + \theta_2)}{\sin \theta_1} \quad (2.3.4)$$

$$\text{or, } 1 < \frac{l}{a} < \frac{\sin(\theta_1 + \theta_2)}{\sin \theta_1} \quad (2.3.5)$$

from Theorem 1.3, yielding

$$\Rightarrow \frac{\sin(\theta_1 + \theta_2)}{\sin \theta_1} > 1. \quad (2.3.6)$$

This proves (2.3.1).

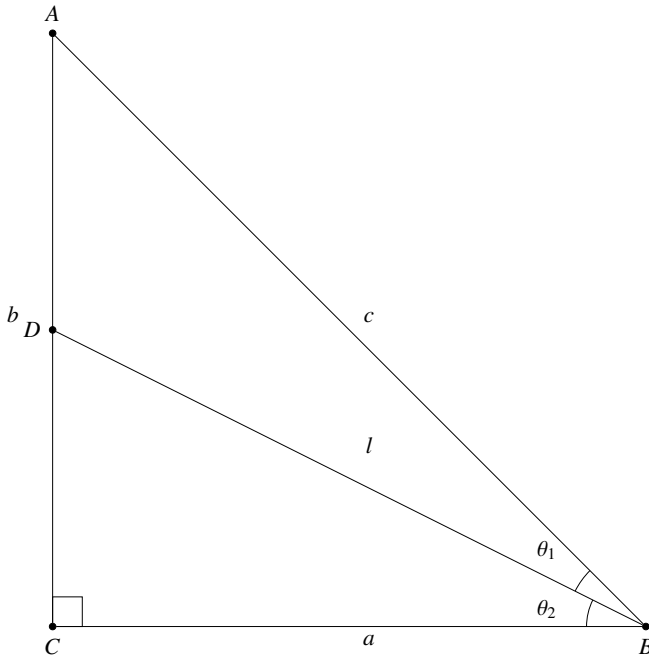


Fig. 2.3.1

2.4. Using Fig. 2.3.1, show that

$$\sin \theta_1 = \sin (\theta_1 + \theta_2) \cos \theta_2 - \cos (\theta_1 + \theta_2) \sin \theta_2 \quad (2.4.1)$$

Solution: The following equations can be obtained from the figure using the formula for the area of a triangle

$$ar(\triangle ABC) = \frac{1}{2} ac \sin (\theta_1 + \theta_2) \quad (2.4.2)$$

$$= ar(\triangle BDC) + ar(\triangle ADB) \quad (2.4.3)$$

$$= \frac{1}{2} cl \sin \theta_1 + \frac{1}{2} al \sin \theta_2 \quad (2.4.4)$$

$$= \frac{1}{2} ac \sin \theta_1 \sec \theta_2 + \frac{1}{2} a^2 \tan \theta_2 \quad (2.4.5)$$

($\because l = a \sec \theta_2$). From the above,

$$\sin (\theta_1 + \theta_2) = \sin \theta_1 \sec \theta_2 + \frac{a}{c} \tan \theta_2 \quad (2.4.6)$$

$$= \sin \theta_1 \sec \theta_2 + \cos (\theta_1 + \theta_2) \tan \theta_2 \quad (2.4.7)$$

Multiplying both sides by $\cos \theta_2$,

$$\sin (\theta_1 + \theta_2) \cos \theta_2 = \sin \theta_1 + \cos (\theta_1 + \theta_2) \sin \theta_2 \quad (2.4.8)$$

resulting in (2.4.1).

2.5. Find Hero's formula for the area of a triangle.

Solution: From (2.1), the area of $\triangle ABC$ is

$$\frac{1}{2} ab \sin C = \frac{1}{2} ab \sqrt{1 - \cos^2 C} \quad (\text{from (1.2.1)}) \quad (2.5.1)$$

$$= \frac{1}{2} ab \sqrt{1 - \left(\frac{a^2 + b^2 - c^2}{2ab} \right)^2} \quad (\text{from (C.3.1)}) \quad (2.5.2)$$

$$= \frac{1}{4} \sqrt{(2ab)^2 - (a^2 + b^2 - c^2)^2} \quad (2.5.3)$$

$$= \frac{1}{4} \sqrt{(2ab + a^2 + b^2 - c^2)(2ab - a^2 - b^2 + c^2)} \quad (2.5.4)$$

$$= \frac{1}{4} \sqrt{\{(a+b)^2 - c^2\} \{c^2 - (a-b)^2\}} \quad (2.5.5)$$

$$= \frac{1}{4} \sqrt{(a+b+c)(a+b-c)(a+c-b)(b+c-a)} \quad (2.5.6)$$

Substituting

$$s = \frac{a+b+c}{2} \quad (2.5.7)$$

in (2.5.6), the area of $\triangle ABC$ is

$$\sqrt{s(s-a)(s-b)(s-c)} \quad (2.5.8)$$

This is known as Hero's formula.

2.6. The area of the triangle with vertices **A, B, C** is given by

$$\frac{1}{2} \|(\mathbf{A} - \mathbf{B}) \times (\mathbf{A} - \mathbf{C})\| = \frac{1}{2} \|\mathbf{A} \times \mathbf{B} + \mathbf{B} \times \mathbf{C} + \mathbf{C} \times \mathbf{A}\| \quad (2.6.1)$$

where the *cross product* or *vector product* defined as $\mathbf{A} \times \mathbf{B}$ is given by (C.1.2) for 2×1 vectors.

2.7. If

$$\|\mathbf{A} \times \mathbf{B}\| = \|\mathbf{C} \times \mathbf{D}\|, \quad \text{then} \quad (2.7.1)$$

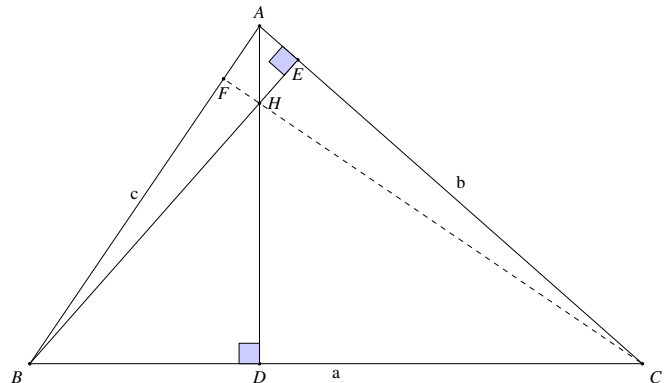
$$\mathbf{A} \times \mathbf{B} = \pm (\mathbf{C} \times \mathbf{D}) \quad (2.7.2)$$

where the sign depends on the orientation of the vectors.

3 ALTITUDES OF A TRIANGLE

3.1. In Fig. 3.2.1, $AD \perp BC$ and $BE \perp AC$ are defined to be the altitudes of $\triangle ABC$.

3.2. Let **H** be the intersection of the altitudes AD and BE as shown in Fig. 3.2.1. CH is extended to meet AB at **F**. Show that $CF \perp AB$.

Fig. 3.2.1: Altitudes of a triangle meet at the orthocentre **H**

Solution: From (C.5.2),

$$(\mathbf{B} - \mathbf{C})^\top (\mathbf{H} - \mathbf{A}) = 0 \quad (3.2.1)$$

$$(\mathbf{C} - \mathbf{A})^\top (\mathbf{H} - \mathbf{B}) = 0 \quad (3.2.2)$$

Adding both the above and simplifying,

$$(\mathbf{B} - \mathbf{A})^\top (\mathbf{H} - \mathbf{C}) = 0 \quad (3.2.3)$$

$$\Rightarrow CH \perp AB, \text{ or } CF \perp AB.$$

3.3. Altitudes of a \triangle meet at the *orthocentre* H .

4 MEDIAN

4.1. In Fig. 4.3.1

$$AF = BF, AE = BE, \quad (4.1.1)$$

and the medians BE and CF meet at \mathbf{G} . Show that

$$ar(BEC) = ar(BFC) = \frac{1}{2} ar(ABC) \quad (4.1.2)$$

Solution: From (2.2.2),

$$ar(BEC) = \frac{1}{2} a \left(\frac{b}{2} \right) \sin C \quad (4.1.3)$$

$$ar(BFC) = \frac{1}{2} a \left(\frac{c}{2} \right) \sin B \quad (4.1.4)$$

yielding (4.1.2).

4.2. Show that

$$ar(CGE) = ar(BGF) \quad (4.2.1)$$

Solution: From Fig. 4.3.1 and (4.1.2),

$$ar(BGF) + ar(BGC) = ar(CGE) + ar(BGC) \quad (4.2.2)$$

yielding (4.2.1).

4.3. If \mathbf{G} divides BE and CF in the ratios k_1 and k_2 respectively, show that

$$k_1 = k_2 \quad (4.3.1)$$

Solution: Let

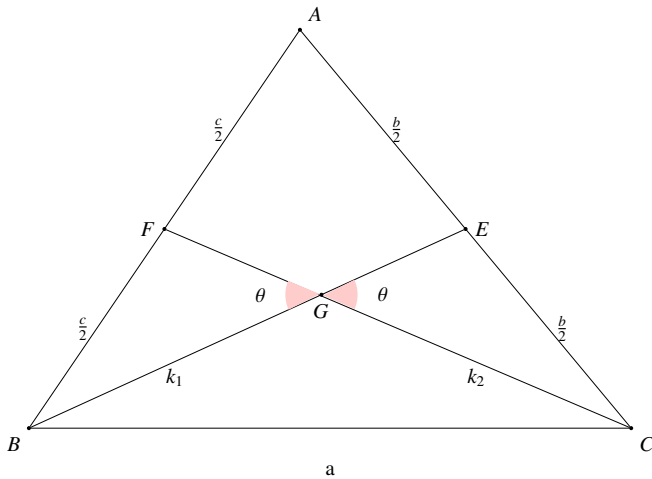


Fig. 4.3.1: $k_1 = k_2$.

$$GE = l_1, GF = l_2 \quad (4.3.2)$$

From (2.2.2) and (4.2.1),

$$\frac{1}{2} l_1 (k_2 l_2) \sin \theta = \frac{1}{2} l_2 (k_1 l_1) \sin \theta \quad (4.3.3)$$

yielding (4.3.1).

4.4. Show that

$$k_1 = k_2 = 2 \quad (4.4.1)$$

Solution: Let

$$k_1 = k_2 = k \quad (4.4.2)$$

Using (B.6.3),

$$\mathbf{G} = \frac{k\mathbf{E} + \mathbf{B}}{k+1} = \frac{k\mathbf{F} + \mathbf{C}}{k+1} \quad (4.4.3)$$

$$\Rightarrow k \left(\frac{\mathbf{A} + \mathbf{C}}{2} \right) + \mathbf{B} = k \left(\frac{\mathbf{A} + \mathbf{B}}{2} \right) + \mathbf{C} \quad (4.4.4)$$

$$\Rightarrow k(\mathbf{B} - \mathbf{C}) = 2(\mathbf{B} - \mathbf{C}) \quad (4.4.5)$$

resulting in (4.4.2).

4.5. Substituting $k = 2$ in (4.4.4),

$$\mathbf{G} = \frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} \quad (4.5.1)$$

4.6. In Fig. 4.6.1, AG is extended to join BC at \mathbf{D} . Show that AD is also a median.

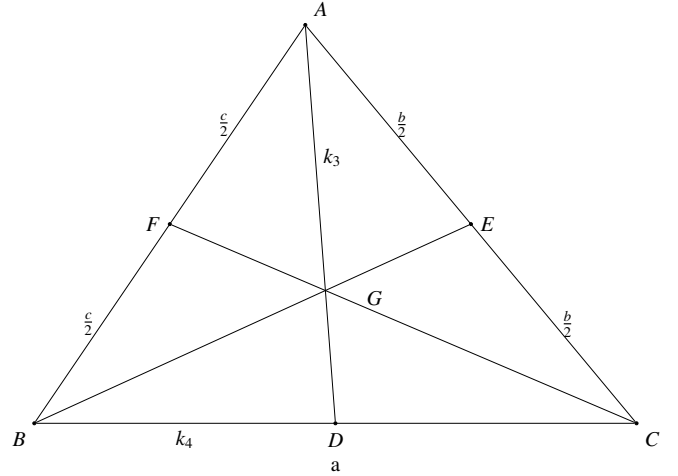


Fig. 4.6.1: $k_3 = 2, k_4 = 1$

Solution: Considering the ratios in Fig. 4.6.1,

$$\mathbf{G} = \frac{k_3 \mathbf{D} + \mathbf{A}}{k_3 + 1} \quad (4.6.1)$$

$$\mathbf{D} = \frac{k_4 \mathbf{C} + \mathbf{B}}{k_4 + 1} \quad (4.6.2)$$

Substituting from (4.5.1) in the above,

$$(k_3 + 1) \left(\frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} \right) = k_3 \left(\frac{k_4 \mathbf{C} + \mathbf{B}}{k_4 + 1} \right) + \mathbf{A} \quad (4.6.3)$$

which can be expressed as

$$(k_3 + 1)(k_4 + 1)(\mathbf{A} + \mathbf{B} + \mathbf{C}) = 3 \{ k_3 (k_4 \mathbf{C} + \mathbf{B}) + (k_4 + 1) \mathbf{A} \} \quad (4.6.4)$$

which can be expressed as

$$\begin{aligned} (k_3k_4 + k_3 - 2k_4 - 2)\mathbf{A} \\ - (-k_3k_4 - k_4 + 2k_3 - 1)\mathbf{B} \\ - (-k_3 - k_4 - 1 + 2k_3k_4)\mathbf{C} = \mathbf{0} \end{aligned} \quad (4.6.5)$$

Comparing the above with (D.4.3),

$$p = -k_3k_4 - k_4 + 2k_3 - 1, q = -k_3 - k_4 - 1 + 2k_3k_4 \quad (4.6.6)$$

yielding

$$-k_3k_4 - k_4 + 2k_3 - 1 = 0 \quad (4.6.7)$$

$$-k_3 - k_4 - 1 + 2k_3k_4 = 0 \quad (4.6.8)$$

Subtracting (4.6.7) from (4.6.8),

$$3k_3(k_4 - 1) = 0 \quad (4.6.9)$$

$$\Rightarrow k_4 = 1 \quad (4.6.10)$$

which upon substituting in (4.6.7) yields

$$k_3 = 2 \quad (4.6.11)$$

5 ANGLE BISECTORS

5.1. In Fig. 5.1.1, the bisectors of $\angle B$ and $\angle C$ meet at I . Show that IA bisects $\angle A$.

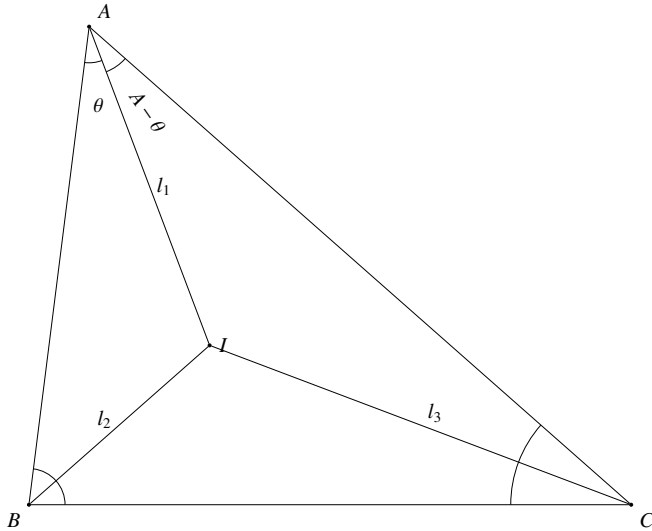


Fig. 5.1.1: Incentre I of $\triangle ABC$

Solution: Using sine formula in (2.2.3)

$$\frac{l_1}{\sin \frac{C}{2}} = \frac{l_3}{\sin(A - \theta)} \quad (5.1.1)$$

$$\frac{l_3}{\sin \frac{B}{2}} = \frac{l_2}{\sin \frac{C}{2}} \quad (5.1.2)$$

$$\frac{l_2}{\sin \theta} = \frac{l_1}{\sin \frac{B}{2}} \quad (5.1.3)$$

Multiplying the above equations,

$$\sin \theta = \sin(A - \theta) \quad (5.1.4)$$

$$\Rightarrow \theta = A - \theta \quad (5.1.5)$$

$$\text{or, } \theta = \frac{A}{2} \quad (5.1.6)$$

5.2. In Fig. 5.2.1,

$$ID \perp BC, IE \perp AC, IF \perp AB. \quad (5.2.1)$$

Show that

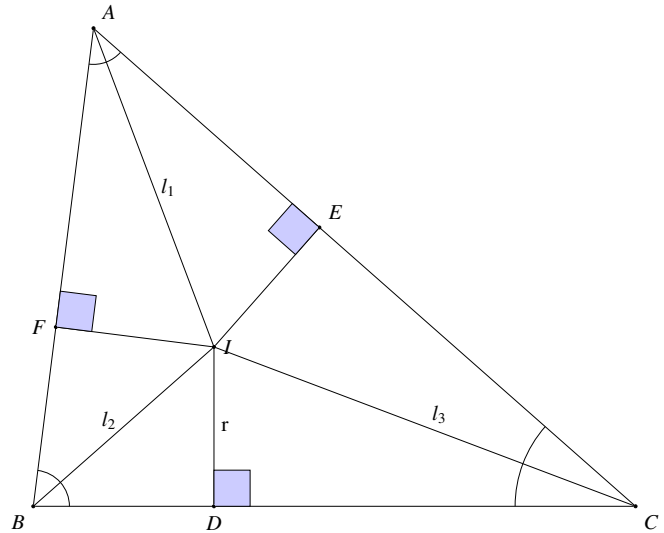


Fig. 5.2.1: Inradius r of $\triangle ABC$

$$ID = IE = IF = r \quad (5.2.2)$$

Solution: In $\triangle IDC$ and IEC ,

$$ID = IE = \frac{l_3}{\sin \frac{C}{2}} \quad (5.2.3)$$

Similarly, in $\triangle IEA$ and IFA ,

$$IF = IE = \frac{l_1}{\sin \frac{A}{2}} \quad (5.2.4)$$

yielding (5.2.2)

5.3. In Fig. 5.2.1, show that

$$BD = BF, AE = AF, CD = CE \quad (5.3.1)$$

Solution: From Fig. 5.2.1, in $\triangle IBD$ and IBF ,

$$x = BD = BF = r \cot \frac{B}{2} \quad (5.3.2)$$

Similarly, other results can be obtained.

5.4. The circle with centre I and radius r in Fig. 5.4.1 is known as the *incircle*.

6 PERPENDICULAR BISECTORS

6.1. In Fig. 6.1.1,

$$OB = OC = R \quad (6.1.1)$$

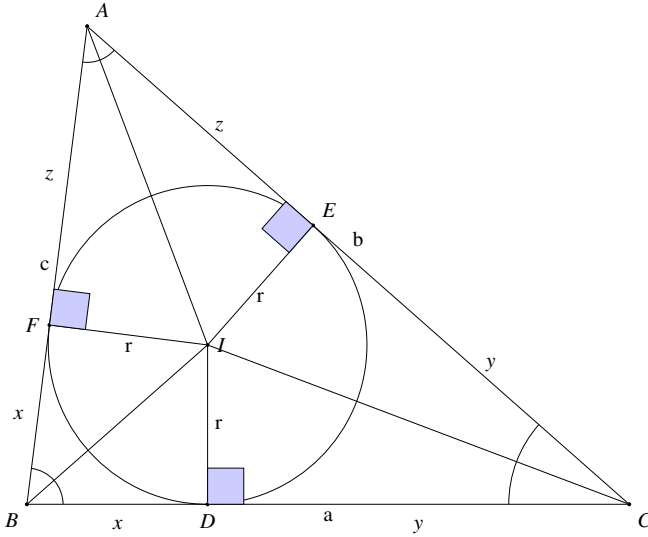
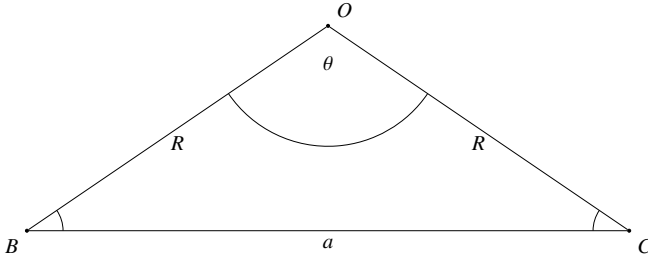
Fig. 5.4.1: Incircle of $\triangle ABC$ 

Fig. 6.1.1: Isosceles Triangle

Such a triangle is known as an isosceles triangle. Show that

$$\angle B = \angle C \quad (6.1.2)$$

Solution: Using (2.2.3),

$$\frac{\sin B}{R} = \frac{\sin C}{R} \quad (6.1.3)$$

$$\Rightarrow \sin B = \sin C \quad (6.1.4)$$

$$\text{or, } \angle B = \angle C. \quad (6.1.5)$$

6.2. In Fig. 6.1.1, show that

$$a = 2R \sin \frac{\theta}{2} \quad (6.2.1)$$

Solution: In $\triangle OBC$, using the cosine formula from (C.3.1),

$$\cos \theta = \frac{R^2 + R^2 - a^2}{2R^2} = 1 - \frac{a^2}{2R^2} \quad (6.2.2)$$

$$\Rightarrow \frac{a^2}{2R^2} = 2 \sin^2 \frac{\theta}{2} \quad (6.2.3)$$

yielding (6.2.1).

6.3. In Fig. 7.1.1, show that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R. \quad (6.3.1)$$

Solution: From (7.4.1) and (6.2.1)

$$a = 2R \sin A \quad (6.3.2)$$

6.4. In Fig. 6.4.1,

$$OB = OC = R, BD = DC. \quad (6.4.1)$$

Show that $OD \perp BC$.

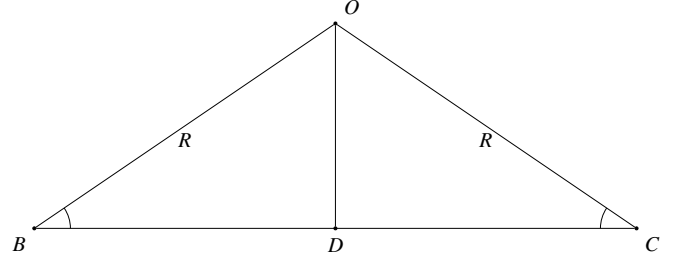


Fig. 6.4.1: Perpendicular bisector.

Solution:

$$\|\mathbf{O} - \mathbf{C}\| = \|\mathbf{O} - \mathbf{B}\| = R \quad (6.4.2)$$

$$\Rightarrow \|\mathbf{O} - \mathbf{C}\|^2 = \|\mathbf{O} - \mathbf{B}\|^2 \quad (6.4.3)$$

which can be expressed as

$$(\mathbf{O} - \mathbf{C})^\top (\mathbf{O} - \mathbf{C}) = (\mathbf{O} - \mathbf{B})^\top (\mathbf{O} - \mathbf{B}) \quad (6.4.4)$$

$$\|\mathbf{O}\|^2 - 2\mathbf{O}^\top \mathbf{C} + \|\mathbf{C}\|^2 = \|\mathbf{O}\|^2 - 2\mathbf{O}^\top \mathbf{B} + \|\mathbf{B}\|^2 \quad (6.4.5)$$

$$\Rightarrow (\mathbf{B} - \mathbf{C})^\top \mathbf{O} = \frac{\|\mathbf{B}\|^2 - \|\mathbf{C}\|^2}{2} \quad (6.4.6)$$

which can be simplified to obtain

$$(\mathbf{B} - \mathbf{C})^\top \left\{ \mathbf{O} - \left(\frac{\mathbf{B} + \mathbf{C}}{2} \right) \right\} = 0 \quad (6.4.7)$$

$$\text{or, } (\mathbf{B} - \mathbf{C})^\top \{\mathbf{O} - \mathbf{D}\} = 0 \quad (6.4.8)$$

which proves the give result using (B.6.3) and (C.5.2).

6.5. In Fig. 6.5.1, OD and OE are the perpendicular bisectors of sides BC and AC respectively. Show that $OA = R$.

Solution: Tracing (6.4.8) backwards yields

$$OB = OC, OC = OA = R. \quad (6.5.1)$$

7 CIRCUMCIRCLE: CIRCLE EQUATION

7.1. The equation of the circle in Fig. 7.1.1, is

$$\|\mathbf{x} - \mathbf{O}\| = R \quad (7.1.1)$$

This is known as the *circumcircle* of $\triangle ABC$.

7.2. Any point on the circle can be expressed as

$$\mathbf{x} = \mathbf{O} + R \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \quad 0 \in [0, 2\pi]. \quad (7.2.1)$$

7.3. Let

$$R = 1, \mathbf{O} = \mathbf{0}, \mathbf{A} = \begin{pmatrix} \cos \theta_1 \\ \sin \theta_1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} \cos \theta_2 \\ \sin \theta_2 \end{pmatrix}, \quad (7.3.1)$$

Show that

$$\|\mathbf{A} - \mathbf{B}\| = 2 \sin \left(\frac{\theta_1 - \theta_2}{2} \right) \quad (7.3.2)$$

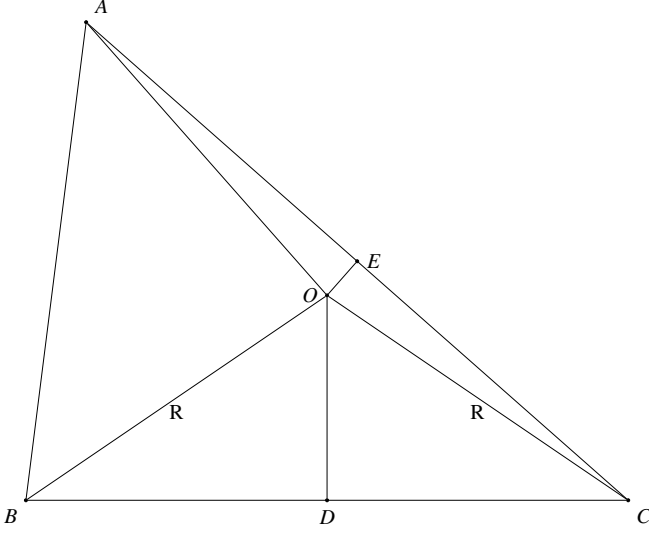


Fig. 6.5.1: Perpendicular bisectors of $\triangle ABC$ meet at O .

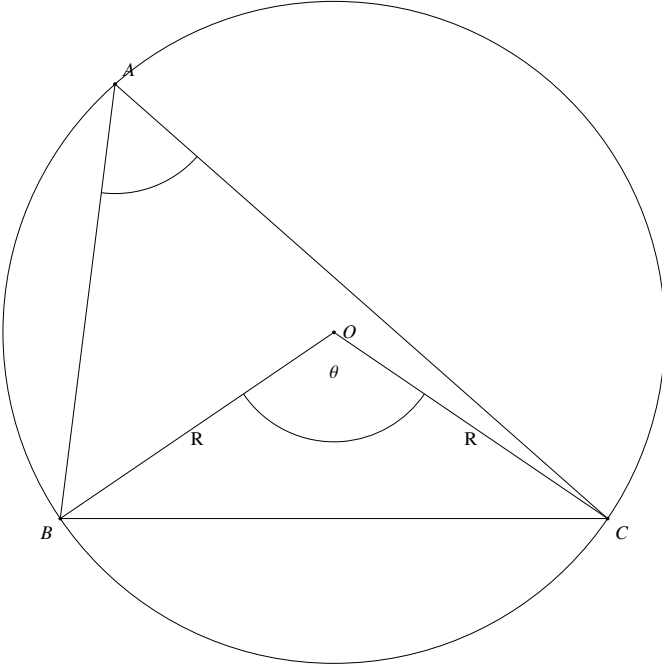


Fig. 7.1.1: Circumcircle of $\triangle ABC$

Solution: From (7.2.1).

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} \cos \theta_1 - \cos \theta_2 \\ \sin \theta_1 - \sin \theta_2 \end{pmatrix} \quad (7.3.3)$$

$$\Rightarrow \|\mathbf{A} - \mathbf{B}\|^2 = (\cos \theta_1 - \cos \theta_2)^2 + (\sin \theta_1 - \sin \theta_2)^2 \quad (7.3.4)$$

$$= 2\{1 - \cos(\theta_1 - \theta_2)\} = 4 \sin^2\left(\frac{\theta_1 - \theta_2}{2}\right) \quad (7.3.5)$$

yielding (7.3.2) from (E.6.3).

7.4. In Fig. 7.1.1, show that

$$\theta = 2A. \quad (7.4.1)$$

Solution: Let

$$\mathbf{C} = \begin{pmatrix} \cos \theta_3 \\ \sin \theta_3 \end{pmatrix} \quad (7.4.2)$$

Then, substituting from (7.3.2) in (C.4.2),

$$\cos A = \frac{4 \sin^2\left(\frac{\theta_1 - \theta_2}{2}\right) + 4 \sin^2\left(\frac{\theta_1 - \theta_3}{2}\right) - 4 \sin^2\left(\frac{\theta_2 - \theta_3}{2}\right)}{8 \sin\left(\frac{\theta_1 - \theta_2}{2}\right) \sin\left(\frac{\theta_1 - \theta_3}{2}\right)} \quad (7.4.3)$$

$$= \frac{2 \sin^2\left(\frac{\theta_1 - \theta_2}{2}\right) + \cos(\theta_2 - \theta_3) - \cos(\theta_1 - \theta_3)}{4 \sin\left(\frac{\theta_1 - \theta_2}{2}\right) \sin\left(\frac{\theta_1 - \theta_3}{2}\right)} \quad (7.4.4)$$

from (E.6.3). \therefore from (E.5.4),

$$\cos A = \frac{2 \sin^2\left(\frac{\theta_1 - \theta_2}{2}\right) + 2 \sin\left(\frac{\theta_1 - \theta_2}{2}\right) \sin\left(\frac{\theta_1 + \theta_2}{2} - \theta_3\right)}{4 \sin\left(\frac{\theta_1 - \theta_2}{2}\right) \sin\left(\frac{\theta_1 - \theta_3}{2}\right)} \quad (7.4.5)$$

$$= \frac{\sin\left(\frac{\theta_1 - \theta_2}{2}\right) + \sin\left(\frac{\theta_1 + \theta_2}{2} - \theta_3\right)}{2 \sin\left(\frac{\theta_1 - \theta_3}{2}\right)} \quad (7.4.6)$$

From (E.5.1), the above equation can be expressed as

$$\cos A = \frac{2 \sin\left(\frac{\theta_1 - \theta_3}{2}\right) \cos\left(\frac{\theta_1 - \theta_2}{2}\right)}{2 \sin\left(\frac{\theta_1 - \theta_3}{2}\right)} = \cos\left(\frac{\theta_2 - \theta_3}{2}\right) \quad (7.4.7)$$

$$\Rightarrow 2A = \theta_2 - \theta_3 \quad (7.4.8)$$

Similarly,

$$\cos \theta = \frac{1 + 1 - 4 \sin^2\left(\frac{\theta_2 - \theta_3}{2}\right)}{2} = \cos(\theta_2 - \theta_3) = \cos 2A \quad (7.4.9)$$

8 TANGENT

8.1. In Fig. 8.1.1, OC is the radius and PC touches the circle at C . Show that

$$OC \perp PC. \quad (8.1.1)$$

Solution: The equation of PC can be expressed as

$$\mathbf{x} = \mathbf{C} + \mu \mathbf{m} \quad (8.1.2)$$

and the equation of the circle is

$$\|\mathbf{x} - \mathbf{O}\| = R \quad (8.1.3)$$

Substituting (8.1.2) in (8.1.3),

$$\|\mathbf{C} + \mu \mathbf{m} - \mathbf{O}\|^2 = R^2 \quad (8.1.4)$$

$$\Rightarrow \mu^2 \|\mathbf{m}\|^2 + 2\mu \mathbf{m}^\top (\mathbf{C} - \mathbf{O}) + \|\mathbf{C} - \mathbf{O}\|^2 - R^2 = 0 \quad (8.1.5)$$

The above equation has only one root. Hence the discriminant of the above quadratic should be zero. So,

$$\{\mathbf{m}^\top (\mathbf{C} - \mathbf{O})\}^2 - \|\mathbf{m}\|^2 \{\|\mathbf{C} - \mathbf{O}\|^2 - R^2\} = 0 \quad (8.1.6)$$

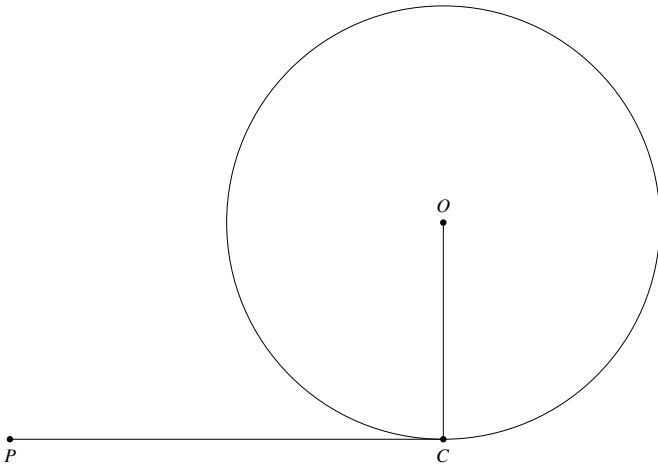


Fig. 8.1.1

Since \mathbf{C} is a point on the circle,

$$\|\mathbf{C} - \mathbf{O}\|^2 - R^2 = 0 \quad (8.1.7)$$

$$\Rightarrow \mathbf{m}^\top (\mathbf{C} - \mathbf{O}) = 0 \quad (8.1.8)$$

upon substituting in (8.1.6). Using the definition of the direction vector from (B.2.1)

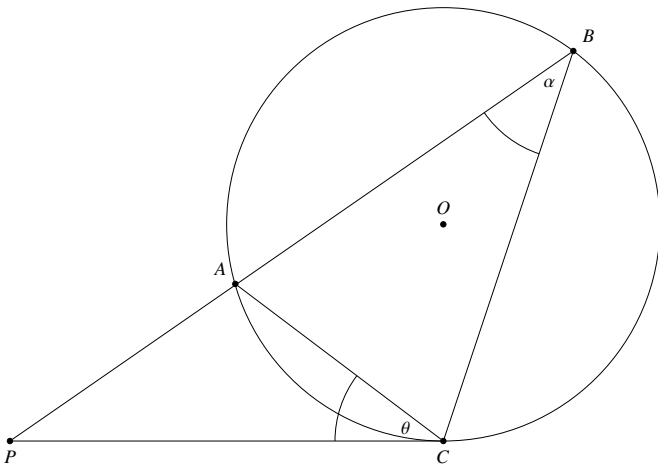
$$\mathbf{m} = \mathbf{P} - \mathbf{C} \quad (8.1.9)$$

$$\Rightarrow (\mathbf{P} - \mathbf{C})^\top (\mathbf{C} - \mathbf{O}) = 0 \quad (8.1.10)$$

which is equivalent to (8.1.1).

8.2. In Fig. 8.2.1 show that

$$\theta = \alpha \quad (8.2.1)$$

Fig. 8.2.1: $\theta = \alpha$.

Solution: Let Let

$$\mathbf{O} = \mathbf{0A} = \begin{pmatrix} \cos \theta_1 \\ \sin \theta_1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} \cos \theta_2 \\ \sin \theta_2 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} \cos \theta_3 \\ \sin \theta_3 \end{pmatrix} \quad (8.2.2)$$

Without loss of generality, let

$$\theta_3 = \frac{\pi}{2} \quad (8.2.3)$$

Then,

$$\mathbf{C} - \mathbf{O} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (8.2.4)$$

From from (8.1.10),

$$\mathbf{C} - \mathbf{P} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (8.2.5)$$

From (C.4.1) and (8.2.5),

$$\cos \theta = \frac{(\cos \theta_3 - \cos \theta_1 \quad \sin \theta_3 - \sin \theta_1) \begin{pmatrix} 1 \\ 0 \end{pmatrix}}{2 \sin \left(\frac{\theta_1 - \theta_3}{2} \right)} \quad (8.2.6)$$

$$= \sin \left(\frac{\theta_1 + \theta_3}{2} \right) = \cos \left(\frac{\pi}{2} - \frac{\theta_1 + \theta_3}{2} \right) = \cos \left(\frac{\pi}{4} - \frac{\theta_1}{2} \right) \quad (8.2.7)$$

upon substituting from (8.2.3). Similarly, from (7.4.7),

$$\cos \alpha = \cos \left(\frac{\theta_1 - \theta_3}{2} \right) = \cos \left(\frac{\pi}{4} - \frac{\theta_1}{2} \right) = \cos \theta \quad (8.2.8)$$

8.3. In Fig. 8.2.1, show that $PA \cdot PB = PC^2$.

Solution: In $\triangle s$ APC and BPC , using (8.2.1),

$$\frac{AP}{\sin \theta} = \frac{AC}{\sin P} \quad (8.3.1)$$

$$\frac{PC}{\sin \theta} = \frac{BC}{\sin P} \quad (8.3.2)$$

$$\Rightarrow \frac{PC}{AP} = \frac{BC}{AC} \left(= \frac{BP}{CP} \right) \quad (8.3.3)$$

which gives the desired result. $\triangle s$ APC and BPC are said to be *similar*.

APPENDIX A RATIOS

A right angled triangle looks like Fig. 3.1. with angles

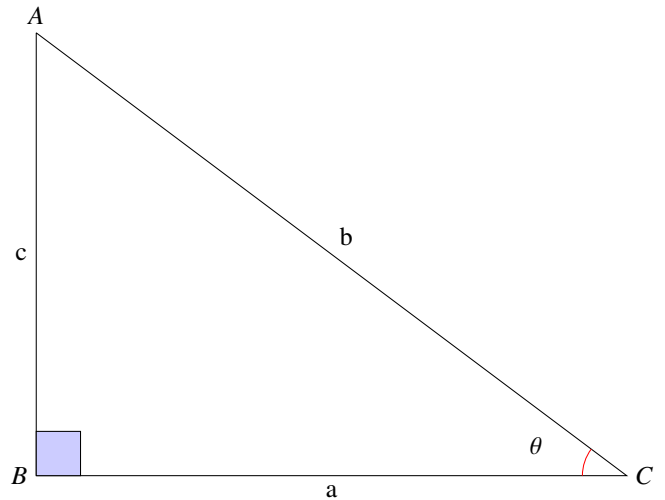


Fig. 3.1: Right Angled Triangle

$\angle A, \angle B$ and $\angle C$ and sides a, b and c . The unique feature of this triangle is $\angle B$ which is defined to be 90° .

A.1. For simplicity, let the greek letter $\theta = \angle C$. We have the following definitions.

$$\begin{aligned} \sin \theta &= \frac{c}{b} & \cos \theta &= \frac{a}{b} \\ \tan \theta &= \frac{c}{a} & \cot \theta &= \frac{a}{c} \\ \csc \theta &= \frac{b}{c} & \sec \theta &= \frac{b}{a} \end{aligned} \quad (\text{A.1.1})$$

A.2. Show that

$$\cos \theta = \sin(90^\circ - \theta) \quad (\text{A.2.1})$$

Solution: From (A.1.1),

$$\cos \angle BAC = \cos \alpha = \cos(90^\circ - \theta) = \frac{c}{b} = \sin \angle ABC = \sin \theta \quad (\text{A.2.2})$$

APPENDIX B VECTORS

B.1. In Fig. B.1.1,

$$a = y \cot \theta + x \quad (\text{B.1.1})$$

$$\Rightarrow \mathbf{D} = \begin{pmatrix} -x \\ y \end{pmatrix} = \begin{pmatrix} -a + y \cot \theta \\ y \end{pmatrix} \quad (\text{B.1.2})$$

$$= \begin{pmatrix} -a \\ 0 \end{pmatrix} + y \cot \theta \begin{pmatrix} 1 \\ \tan \theta \end{pmatrix} \quad (\text{B.1.3})$$

$$\text{or, } \mathbf{D} \equiv \mathbf{B} + \kappa \mathbf{m} \quad (\text{B.1.4})$$

The above equation can be generalized for any point on the line AB as

$$\mathbf{x} = \mathbf{B} + \kappa \mathbf{m} \quad (\text{B.1.5})$$

which is known as the *parametric equation* of a line. \mathbf{m} is defined to be the *direction vector* of AB and

$$m = \tan \theta \quad (\text{B.1.6})$$

is defined to be the *slope*.

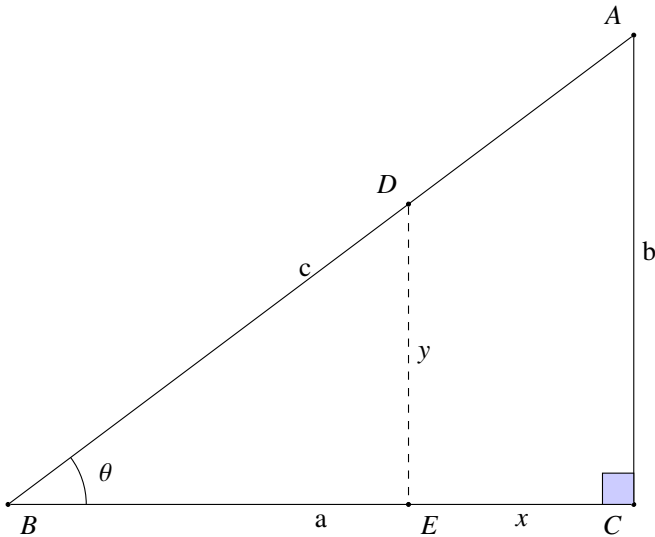


Fig. B.1.1: $k_1 = k_2 = 2$.

B.2. The direction vector of the line AB is

$$\mathbf{A} - \mathbf{B} \equiv \mathbf{B} - \mathbf{A} \equiv \kappa \begin{pmatrix} 1 \\ m \end{pmatrix}, \quad (\text{B.2.1})$$

B.3. (B.1.1) can also be expressed as

$$a = y \cot \theta + x \quad (\text{B.3.1})$$

$$\Rightarrow \begin{pmatrix} -\tan \theta & 1 \end{pmatrix} \begin{pmatrix} -x \\ y \end{pmatrix} = b \quad (\text{B.3.2})$$

$$\text{or, } \mathbf{n}^\top \mathbf{x} = b \quad (\text{B.3.3})$$

which is known as the *normal equation* of a line. Here,

$$\mathbf{n} = \begin{pmatrix} -m \\ 1 \end{pmatrix} \quad (\text{B.3.4})$$

is defined to be the *normal vector* of the line.

B.4. It is easy to verify that

$$\mathbf{n}^\top \mathbf{m} = 0 \quad (\text{B.4.1})$$

and

B.5.

$$\mathbf{n} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{m} = \begin{pmatrix} \cos(\frac{\pi}{2}) & \sin(\frac{\pi}{2}) \\ \sin(\frac{\pi}{2}) & \cos(\frac{\pi}{2}) \end{pmatrix} \mathbf{m} \quad (\text{B.5.1})$$

The matrix

$$\mathbf{R}_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \mathbf{m} \quad (\text{B.5.2})$$

B.6. From (B.1.5), since \mathbf{A}, \mathbf{D} and \mathbf{C} are on the same line,

$$\mathbf{D} = \mathbf{A} + q\mathbf{m}$$

$$\mathbf{B} = \mathbf{D} + p\mathbf{m} \quad (\text{B.6.1})$$

$$\Rightarrow p(\mathbf{D} - \mathbf{A}) + q(\mathbf{D} - \mathbf{B}) = 0, \quad p, q \neq 0 \quad (\text{B.6.2})$$

$$\Rightarrow \mathbf{D} = \frac{k\mathbf{A} + \mathbf{B}}{k+1}, \quad k = \frac{p}{q}. \quad (\text{B.6.3})$$

which is known as *section formula*. $(\mathbf{D} - \mathbf{A})$, $(\mathbf{D} - \mathbf{B})$ are then said to be *linearly dependent*. where m is defined to be the slope of AB .

B.7. The *inner product* or *dot product* is defined as

$$\mathbf{A}^\top \mathbf{B} \equiv \mathbf{A} \cdot \mathbf{B} = \begin{pmatrix} a_1 & a_2 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = a_1 b_1 + a_2 b_2 \quad (\text{B.7.1})$$

In Fig. 3.1,

$$\mathbf{A}^\top \mathbf{C} = 0 \quad (\text{B.7.2})$$

B.8. The *norm* of \mathbf{A} is defined as

$$\|\mathbf{A}\| = \sqrt{\mathbf{A}^\top \mathbf{A}} = \sqrt{a_1^2 + a_2^2} \quad (\text{B.8.1})$$

B.9. In Fig. 3.1, it is easy to verify that

$$\|\mathbf{A} - \mathbf{C}\|^2 = \begin{pmatrix} -c & a \end{pmatrix} \begin{pmatrix} -c \\ a \end{pmatrix} = a^2 + c^2 = b^2 \quad (\text{B.9.1})$$

from (1.4.1). Thus, the distance between any two points \mathbf{A} and \mathbf{B} is given by

$$\|\mathbf{A} - \mathbf{B}\| \quad (\text{B.9.2})$$

B.10. Show that

$$\|\lambda \mathbf{A}\| = |\lambda| \|\mathbf{A}\| \quad (\text{B.10.1})$$

B.11. Find the equation of the line BC .

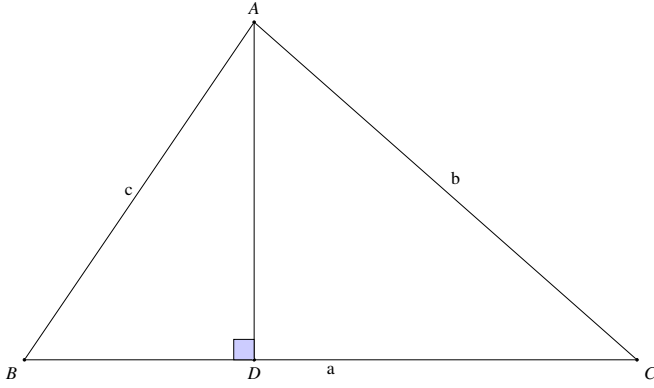


Fig. B.11.1: Drawing the altitude

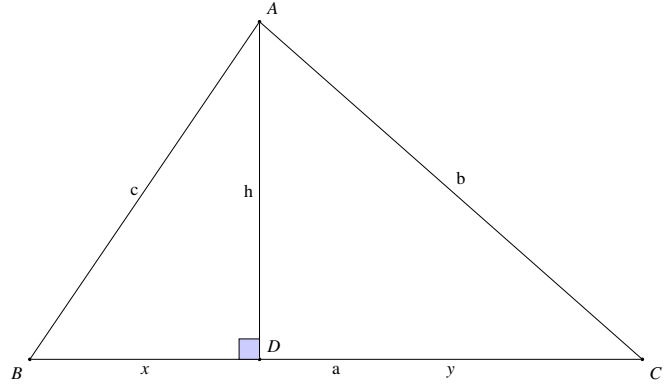


Fig. C.2.1: The cosine formula

Solution: Let \mathbf{x} be any point on BC . Using section formula, for some k ,

$$\mathbf{x} = \frac{k\mathbf{C} + \mathbf{B}}{k+1} = \frac{(k+1)\mathbf{C} + (\mathbf{B} - \mathbf{C})}{k+1} \quad (\text{B.11.1})$$

$$\Rightarrow \mathbf{x} = \mathbf{C} + \lambda \mathbf{m} \quad (\text{B.11.2})$$

where

$$\mathbf{m} = \frac{\mathbf{B} - \mathbf{C}}{k+1} \equiv \mathbf{B} - \mathbf{C} \quad (\text{B.11.3})$$

B.12. From (B.4.1) and (B.11.2), it can be verified that

$$\mathbf{n}^\top \mathbf{x} = \mathbf{n}^\top \mathbf{C} + \lambda \mathbf{n}^\top \mathbf{m} \quad (\text{B.12.1})$$

$$\Rightarrow \mathbf{n}^\top \mathbf{x} = \mathbf{n}^\top \mathbf{C} \quad (\text{B.12.2})$$

(B.12.2) is defined to be the *normal form* of the line BC .

APPENDIX C MATRICES: COSINE FORMULA

C.1. The determinant of the 2×2 matrix

$$\mathbf{M} = (\mathbf{A} \quad \mathbf{B}) = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \quad (\text{C.1.1})$$

is defined as

$$|\mathbf{M}| = |\mathbf{A} \quad \mathbf{B}| \quad (\text{C.1.2})$$

$$= \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1 \quad (\text{C.1.3})$$

C.2. In Fig. C.2.1, show that

$$\begin{pmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{pmatrix} \begin{pmatrix} \cos A \\ \cos B \\ \cos C \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad (\text{C.2.1})$$

Solution: From Fig. C.2.1,

$$a = x + y = b \cos C + c \cos B = (\cos C \quad \cos B) \begin{pmatrix} b \\ c \end{pmatrix} \quad (\text{C.2.2})$$

$$= (0 \quad b \quad c) \begin{pmatrix} \cos A \\ \cos C \\ \cos B \end{pmatrix} \quad (\text{C.2.3})$$

Similarly,

$$b = c \cos A + a \cos C = (c \quad 0 \quad a) \begin{pmatrix} \cos A \\ \cos C \\ \cos B \end{pmatrix} \quad (\text{C.2.4})$$

$$c = b \cos A + a \cos B = (b \quad a \quad 0) \begin{pmatrix} \cos A \\ \cos C \\ \cos B \end{pmatrix} \quad (\text{C.2.5})$$

The above equations can be expressed in matrix form as (C.2.1).

C.3. Show that

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \quad (\text{C.3.1})$$

Solution: Using the properties of determinants,

$$\cos A = \frac{\begin{vmatrix} a & c & b \\ b & 0 & a \\ c & a & 0 \end{vmatrix}}{\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}} = \frac{ab^2 + ac^2 - a^3}{abc + abc} = \frac{b^2 + c^2 - a^2}{2abc} \quad (\text{C.3.2})$$

C.4. In Fig. C.2.1 show that

$$\cos A = \frac{(\mathbf{A} - \mathbf{B})^\top (\mathbf{A} - \mathbf{C})}{\|\mathbf{A} - \mathbf{B}\| \|\mathbf{A} - \mathbf{C}\|} \quad (\text{C.4.1})$$

Solution: From (C.3.1), using (B.9.2),

$$\cos A = \frac{\|\mathbf{A} - \mathbf{B}\|^2 + \|\mathbf{A} - \mathbf{C}\|^2 - \|\mathbf{B} - \mathbf{C}\|^2}{2 \|\mathbf{A} - \mathbf{B}\| \|\mathbf{A} - \mathbf{C}\|} \quad (\text{C.4.2})$$

$$= \frac{\|\mathbf{A}\|^2 - \mathbf{A}^\top \mathbf{B} - \mathbf{A}^\top \mathbf{C} + \mathbf{B}^\top \mathbf{C}}{\|\mathbf{A} - \mathbf{B}\| \|\mathbf{A} - \mathbf{C}\|} \quad (\text{C.4.3})$$

which can be expressed as (C.4.1).

C.5. For $A = 90^\circ$,

$$\cos A = 0 \quad (\text{C.5.1})$$

$$\Rightarrow (\mathbf{A} - \mathbf{B})^\top (\mathbf{A} - \mathbf{C}) = 0 \quad (\text{C.5.2})$$

from (C.4.1).

APPENDIX D

COLLINEAR POINTS

D.1. The equation of a line is given by

$$y = mx + c \quad (\text{D.1.1})$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ mx + c \end{pmatrix} = \begin{pmatrix} 0 \\ c \end{pmatrix} + x \begin{pmatrix} 1 \\ m \end{pmatrix} \quad (\text{D.1.2})$$

$$\text{or, } \mathbf{x} = \mathbf{A} + \kappa \mathbf{m} \quad (\text{D.1.3})$$

In Fig. 3.1,

$$\mathbf{A} - \mathbf{C} = \begin{pmatrix} -c \\ a \end{pmatrix} \equiv \begin{pmatrix} 1 \\ -\frac{a}{c} \end{pmatrix} = \begin{pmatrix} 1 \\ -\tan \theta \end{pmatrix} \quad (\text{D.1.4})$$

the slope of AC is $-\tan \theta$

D.2. Points **A**, **B** and **C** are on a line if they have the same direction vector, i.e.

$$p(\mathbf{B} - \mathbf{A}) + q(\mathbf{C} - \mathbf{B}) = 0 \Rightarrow p, q \neq 0. \quad (\text{D.2.1})$$

$(\mathbf{A} - \mathbf{B}), (\mathbf{C} - \mathbf{B})$ are then said to be *linearly dependent*.

D.3. If points **A**, **B** and **C** are collinear,

$$\mathbf{B} = \frac{k\mathbf{A} + \mathbf{C}}{k + 1} \quad (\text{D.3.1})$$

Solution: From (D.2.1),

$$p(\mathbf{A} - \mathbf{B}) + q(\mathbf{A} - \mathbf{C}) = 0 \Rightarrow \mathbf{B} = \frac{p\mathbf{A} + q\mathbf{C}}{p + q} \quad (\text{D.3.2})$$

yielding (B.6.3) upon substituting

$$k = \frac{p}{q}. \quad (\text{D.3.3})$$

This is known as *section formula*.

D.4. Consequently, points **A**, **B** and **C** form a triangle if

$$p(\mathbf{A} - \mathbf{B}) + q(\mathbf{C} - \mathbf{B}) \quad (\text{D.4.1})$$

$$= (p + q)\mathbf{B} - p\mathbf{A} - q\mathbf{C} = 0 \quad (\text{D.4.2})$$

$$\Rightarrow p = 0, q = 0 \quad (\text{D.4.3})$$

APPENDIX E

IDENTITIES

E.1. Show that

$$\cos 90^\circ = 0 \quad (\text{E.1.1})$$

Solution: Using (C.3.1) in Fig. 3.1,

$$\cos 90^\circ = \frac{a^2 + c^2 - b^2}{2ac} = 0 \quad (\text{E.1.2})$$

upon substituting from (1.4.1).

E.2. Show that

$$\sin 90^\circ = 1 \quad (\text{E.2.1})$$

Solution: Trivial from (A.2.1).

E.3. Prove the following identities

a)

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta. \quad (\text{E.3.1})$$

b)

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta. \quad (\text{E.3.2})$$

Solution: In (2.4.1), let

$$\begin{aligned} \theta_1 + \theta_2 &= \alpha \\ \theta_2 &= \beta \end{aligned} \quad (\text{E.3.3})$$

This gives (E.3.1). In (E.3.1), replace α by $90^\circ - \alpha$. This results in

$$\sin(90^\circ - \alpha - \beta) = \sin(90^\circ - \alpha) \cos \beta - \cos(90^\circ - \alpha) \sin \beta \quad (\text{E.3.4})$$

$$\Rightarrow \cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta \quad (\text{E.3.5})$$

E.4. Using (2.4.1) and (E.3.2), show that

$$\sin(\theta_1 + \theta_2) = \sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2 \quad (\text{E.4.1})$$

$$\cos(\theta_1 - \theta_2) = \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \quad (\text{E.4.2})$$

Solution: From (2.4.1),

$$\sin(\theta_1 + \theta_2) \cos \theta_2 = \sin \theta_1 + \cos(\theta_1 + \theta_2) \sin \theta_2 \quad (\text{E.4.3})$$

Using (E.3.2) in the above,

$$\begin{aligned} \sin(\theta_1 + \theta_2) \cos \theta_2 &= \sin \theta_1 + (\cos \theta_1 \cos \theta_2 \\ &\quad - \sin \theta_1 \sin \theta_2) \sin \theta_2 \end{aligned} \quad (\text{E.4.4})$$

which can be expressed as

$$\begin{aligned} \sin(\theta_1 + \theta_2) \cos \theta_2 &= \sin \theta_1 \\ &\quad + \cos \theta_1 \cos \theta_2 \sin \theta_2 - \sin \theta_1 \sin^2 \theta_2 \end{aligned} \quad (\text{E.4.5})$$

Since

$$\sin^2 \theta_2 = 1 - \cos^2 \theta_2, \quad (\text{E.4.6})$$

we obtain

$$\sin(\theta_1 + \theta_2) \cos \theta_2 = \cos \theta_1 \cos \theta_2 \sin \theta_2 + \sin \theta_1 \cos^2 \theta_2 \quad (\text{E.4.7})$$

resulting in

$$\sin(\theta_1 + \theta_2) = \cos \theta_1 \sin \theta_2 + \sin \theta_1 \cos \theta_2 \quad (\text{E.4.8})$$

after factoring out $\cos \theta_2$. Using a similar approach, (E.4.2) can also be proved.

E.5. Show that

$$\sin \theta_1 + \sin \theta_2 = 2 \sin \left(\frac{\theta_1 + \theta_2}{2} \right) \cos \left(\frac{\theta_1 - \theta_2}{2} \right) \quad (\text{E.5.1})$$

$$\cos \theta_1 + \cos \theta_2 = 2 \cos \left(\frac{\theta_1 + \theta_2}{2} \right) \cos \left(\frac{\theta_1 - \theta_2}{2} \right) \quad (\text{E.5.2})$$

$$\sin \theta_1 - \sin \theta_2 = 2 \sin \left(\frac{\theta_1 - \theta_2}{2} \right) \cos \left(\frac{\theta_1 + \theta_2}{2} \right) \quad (\text{E.5.3})$$

$$\cos \theta_1 - \cos \theta_2 = 2 \sin \left(\frac{\theta_1 + \theta_2}{2} \right) \cos \left(\frac{\theta_2 - \theta_1}{2} \right) \quad (\text{E.5.4})$$

Solution: Let

$$\begin{aligned} \theta_1 &= \alpha + \beta \\ \theta_2 &= \alpha - \beta \end{aligned} \quad (\text{E.5.5})$$

From (E.4.1),

$$\sin \theta_1 + \sin \theta_2 = \sin (\alpha + \beta) + \sin (\alpha - \beta) \quad (\text{E.5.6})$$

$$= \sin \alpha \cos \beta + \cos \alpha \sin \beta \quad (\text{E.5.7})$$

$$+ \sin \alpha \cos \beta - \cos \alpha \sin \beta \quad (\text{E.5.8})$$

$$= 2 \sin \alpha \cos \beta \quad (\text{E.5.9})$$

resulting in (E.5.1)

$$\therefore \alpha = \frac{\theta_1 + \theta_2}{2} \quad (\text{E.5.10})$$

$$\beta = \frac{\theta_1 - \theta_2}{2} \quad (\text{E.5.11})$$

from (E.5.5). Other identities may be proved similarly.

E.6. Show that

$$\sin 2\theta = 2 \sin \theta \cos \theta \quad (\text{E.6.1})$$

$$\cos 2\theta = 1 - 2 \sin^2 \theta = 2 \cos^2 \theta - 1 \quad (\text{E.6.2})$$

$$= \cos^2 \theta - \sin^2 \theta \quad (\text{E.6.3})$$