GEOMETRY
Through Equations

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Contents

1 The Baudhayana Theorem

Use Fig. 1.1 for all problems in this section.

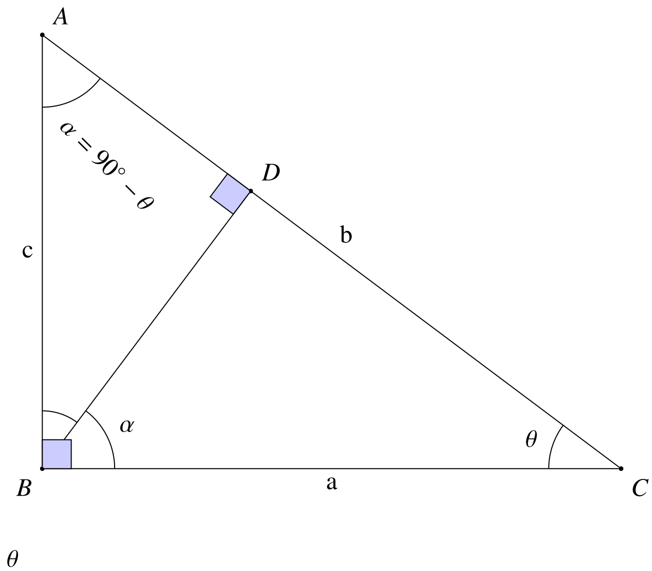

Fig. 1.1: Baudhayana Theorem

1.1. Show that

$$b = a\cos\theta + c\sin\theta \tag{1.1.1}$$

Solution: We observe that

$$CD = a\cos\theta \tag{1.1.2}$$
$$AD = c\cos\alpha = c\sin\theta \quad \text{(From} \quad \text{(A.2.2))} \tag{1.1.3}$$

Thus,

$$CD + AD = b = a\cos\theta + c\sin\theta \tag{1.1.4}$$

1.2. From (1.1.1), show that

$$\sin^2\theta + \cos^2\theta = 1 \tag{1.2.1}$$

Solution: Dividing both sides of (1.1.1) by b,

$$1 = \frac{a}{b}\cos\theta + \frac{c}{b}\sin\theta \tag{1.2.2}$$
$$\Rightarrow \sin^2\theta + \cos^2\theta = 1 \quad \text{(from} \quad \text{(A.1.1))} \tag{1.2.3}$$

1.3. In a right angled triangle, the hypotenuse is the longest side.
Solution: From (1.2.1),

$$0 \le \sin\theta, \cos\theta \le 1 \tag{1.3.1}$$

Hence,

$$b\sin\theta \le b \implies c \le b \tag{1.3.2}$$

Similalry,

$$a \le b \tag{1.3.3}$$

1.4. Using (1.1.1), show that

$$b^2 = a^2 + c^2 \tag{1.4.1}$$

(1.4.1) is known as the Baudhayana theorem. It is also known as the Pythagoras theorem.
Solution: From (1.1.1),

$$b = a\frac{a}{b} + c\frac{c}{b} \quad \text{(from} \quad \text{(A.1.1))} \tag{1.4.2}$$
$$\implies b^2 = a^2 + c^2 \tag{1.4.3}$$

2 Area of a Triangle

2.1. Show that the area of ΔABC in Fig. 2.1.1 is $\frac{1}{2}ab\sin C$.

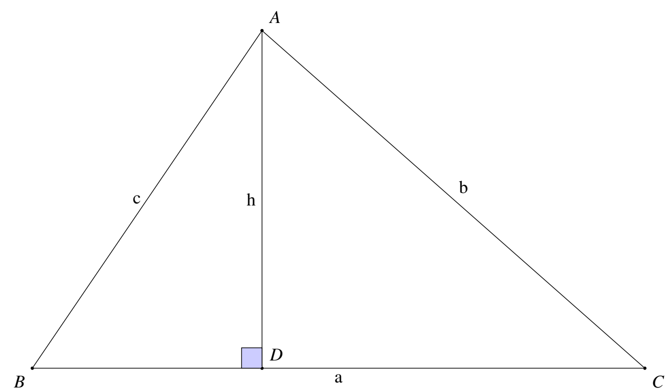

Fig. 2.1.1: Area of a Triangle

Solution: We have

$$ar(\Delta ABC) = \frac{1}{2}ah = \frac{1}{2}ab\sin C \quad (\because \quad h = b\sin C). \tag{2.1.1}$$

2.2. Show that

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \tag{2.2.1}$$

Solution: Fig. 2.1.1 can be suitably modified to obtain

$$ar(\Delta ABC) = \frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B \tag{2.2.2}$$

Dividing the above by abc, we obtain

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \tag{2.2.3}$$

This is known as the sine formula.

2.3. Show that

$$\alpha > \beta \implies \sin\alpha > \sin\beta \tag{2.3.1}$$

Solution: In Fig. 2.3.1,

$$ar(\triangle ABD) < ar(\triangle ABC) \tag{2.3.2}$$
$$\implies \frac{1}{2}lc\sin\theta_1 < \frac{1}{2}ac\sin(\theta_1 + \theta_2) \tag{2.3.3}$$
$$\implies \frac{l}{a} < \frac{\sin(\theta_1 + \theta_2)}{\sin\theta_1} \tag{2.3.4}$$
$$\text{or, } 1 < \frac{l}{a} < \frac{\sin(\theta_1 + \theta_2)}{\sin\theta_1} \tag{2.3.5}$$

from Theorem 1.3, yielding

$$\implies \frac{\sin(\theta_1 + \theta_2)}{\sin\theta_1} > 1. \tag{2.3.6}$$

This proves (2.3.1).

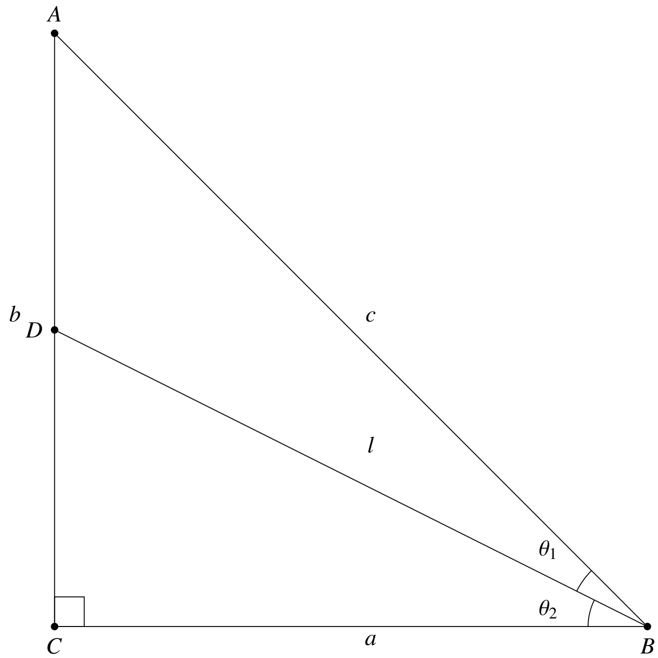

Fig. 2.3.1

2.4. Using Fig. 2.3.1, show that

$$\sin\theta_1 = \sin(\theta_1+\theta_2)\cos\theta_2 - \cos(\theta_1+\theta_2)\sin\theta_2 \quad (2.4.1)$$

Solution: The following equations can be obtained from the figure using the forumula for the area of a triangle

$$\begin{aligned} ar(\Delta ABC) &= \frac{1}{2}ac\sin(\theta_1+\theta_2) && (2.4.2)\\ &= ar(\Delta BDC) + ar(\Delta ADB) && (2.4.3)\\ &= \frac{1}{2}cl\sin\theta_1 + \frac{1}{2}al\sin\theta_2 && (2.4.4)\\ &= \frac{1}{2}ac\sin\theta_1\sec\theta_2 + \frac{1}{2}a^2\tan\theta_2 && (2.4.5) \end{aligned}$$

($\because l = a\sec\theta_2$). From the above,

$$\begin{aligned} \sin(\theta_1+\theta_2) &= \sin\theta_1\sec\theta_2 + \frac{a}{c}\tan\theta_2 && (2.4.6)\\ &= \sin\theta_1\sec\theta_2 + \cos(\theta_1+\theta_2)\tan\theta_2 && (2.4.7) \end{aligned}$$

Multiplying both sides by $\cos\theta_2$,

$$\sin(\theta_1+\theta_2)\cos\theta_2 = \sin\theta_1 + \cos(\theta_1+\theta_2)\sin\theta_2 \quad (2.4.8)$$

resulting in (2.4.1).

2.5. Find Hero's formula for the area of a triangle.

Solution: From (2.1), the area of $\triangle ABC$ is

$$\begin{aligned} \frac{1}{2}ab\sin C &= \frac{1}{2}ab\sqrt{1-\cos^2 C} \quad \text{(from (1.2.1))} && (2.5.1)\\ &= \frac{1}{2}ab\sqrt{1-\left(\frac{a^2+b^2-c^2}{2ab}\right)^2} \text{(from (C.3.1))} && (2.5.2)\\ &= \frac{1}{4}\sqrt{(2ab)^2-(a^2+b^2-c^2)} && (2.5.3)\\ &= \frac{1}{4}\sqrt{(2ab+a^2+b^2-c^2)(2ab-a^2-b^2+c^2)} && (2.5.4)\\ &= \frac{1}{4}\sqrt{\left\{(a+b)^2-c^2\right\}\left\{c^2-(a-b)^2\right\}} && (2.5.5)\\ &= \frac{1}{4}\sqrt{(a+b+c)(a+b-c)(a+c-b)(b+c-a)} && (2.5.6) \end{aligned}$$

Substituting

$$s = \frac{a+b+c}{2} \quad (2.5.7)$$

in (2.5.6), the area of $\triangle ABC$ is

$$\sqrt{s(s-a)(s-b)(s-c)} \quad (2.5.8)$$

This is known as Hero's formula.

2.6. The area of the triangle with vertices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ is given by

$$\frac{1}{2}\|(\mathbf{A}-\mathbf{B})\times(\mathbf{A}-\mathbf{C})\| = \frac{1}{2}\|\mathbf{A}\times\mathbf{B}+\mathbf{B}\times\mathbf{C}+\mathbf{C}\times\mathbf{A}\| \quad (2.6.1)$$

where the *cross product* or *vector product* defined as $\mathbf{A}\times\mathbf{B}$ is given by (C.1.2) for 2×1 vectors.

2.7. If

$$\begin{aligned} \|\mathbf{A}\times\mathbf{B}\| &= \|\mathbf{C}\times\mathbf{D}\|, \quad \text{then} && (2.7.1)\\ \mathbf{A}\times\mathbf{B} &= \pm(\mathbf{C}\times\mathbf{D}) && (2.7.2) \end{aligned}$$

where the sign depends on the orientation of the vectors.

3 Altitudes of a Triangle

3.1. In Fig. 3.2.1, $AD \perp BC$ and $BE \perp AC$ are defined to be the altitudes of $\triangle ABC$.

3.2. Let $\mathbf{H}$ be the intersection of the altitudes AD and BE as shown in Fig. 3.2.1. CH is extended to meet AB at $\mathbf{F}$. Show that $CF \perp AB$.

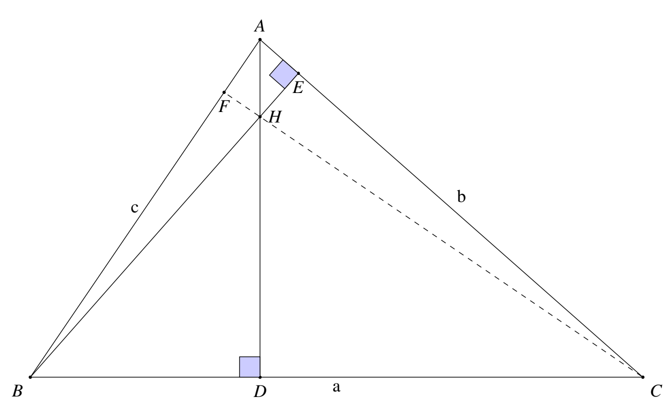

Fig. 3.2.1: Altitudes of a triangle meet at the orthocentre H

Solution: From (C.5.2),

$$(\mathbf{B}-\mathbf{C})^\top(\mathbf{H}-\mathbf{A})=0 \tag{3.2.1}$$
$$(\mathbf{C}-\mathbf{A})^\top(\mathbf{H}-\mathbf{B})=0 \tag{3.2.2}$$

Adding both the above and simplifying,

$$(\mathbf{B}-\mathbf{A})^\top(\mathbf{H}-\mathbf{C})=0 \tag{3.2.3}$$

$\implies CH \perp AB$, or $CF \perp AB$.

3.3. Altitudes of a $\triangle$ meet at the *orthocentre* H.

4 Median

4.1. In Fig. 4.3.1

$$AF=BF,\ AE=BE, \tag{4.1.1}$$

and the medians BE and CF meet at $\mathbf{G}$. Show that

$$ar(BEC)=ar(BFC)=\frac{1}{2}ar(ABC) \tag{4.1.2}$$

Solution: From (2.2.2),

$$ar(BEC)=\frac{1}{2}a\left(\frac{b}{2}\right)\sin C \tag{4.1.3}$$
$$ar(BFC)=\frac{1}{2}a\left(\frac{c}{2}\right)\sin B \tag{4.1.4}$$

yielding (4.1.2).

4.2. Show that

$$ar(CGE)=ar(BGF) \tag{4.2.1}$$

Solution: From Fig. 4.3.1 and (4.1.2),

$$ar(BGF)+ar(BGC)=ar(CGE)+ar(BGC) \tag{4.2.2}$$

yielding (4.2.1).

4.3. If $\mathbf{G}$ divides BE and CF in the ratios k_1 and k_2 respectively, show that

$$k_1=k_2 \tag{4.3.1}$$

Solution: Let

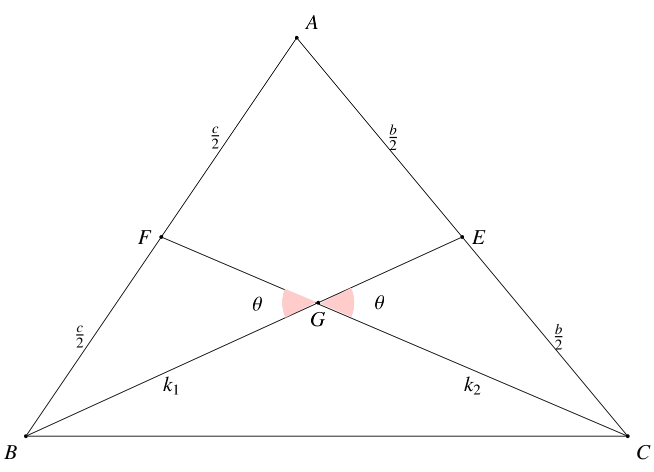

Fig. 4.3.1: $k_1=k_2$.

$$GE=l_1, GF=l_2 \tag{4.3.2}$$

From (2.2.2) and (4.2.1),

$$\frac{1}{2}l_1(k_2l_2)\sin\theta=\frac{1}{2}l_2(k_1l_1)\sin\theta \tag{4.3.3}$$

yielding (4.3.1).

4.4. Show that

$$k_1=k_2=2 \tag{4.4.1}$$

Solution: Let

$$k_1=k_2=k \tag{4.4.2}$$

Using (D.3.1),

$$\mathbf{G}=\frac{k\mathbf{E}+\mathbf{B}}{k+1}=\frac{k\mathbf{F}+\mathbf{C}}{k+1} \tag{4.4.3}$$
$$\implies k\left(\frac{\mathbf{A}+\mathbf{C}}{2}\right)+\mathbf{B}=k\left(\frac{\mathbf{A}+\mathbf{B}}{2}\right)+\mathbf{C} \tag{4.4.4}$$
$$\implies k(\mathbf{B}-\mathbf{C})=2(\mathbf{B}-\mathbf{C}) \tag{4.4.5}$$

resulting in (4.4.2).

4.5. Substituting $k=2$ in (4.4.4),

$$\mathbf{G}=\frac{\mathbf{A}+\mathbf{B}+\mathbf{C}}{3} \tag{4.5.1}$$

4.6. In Fig. 4.6.1, AG is extended to join BC at $\mathbf{D}$. Show that AD is also a median.

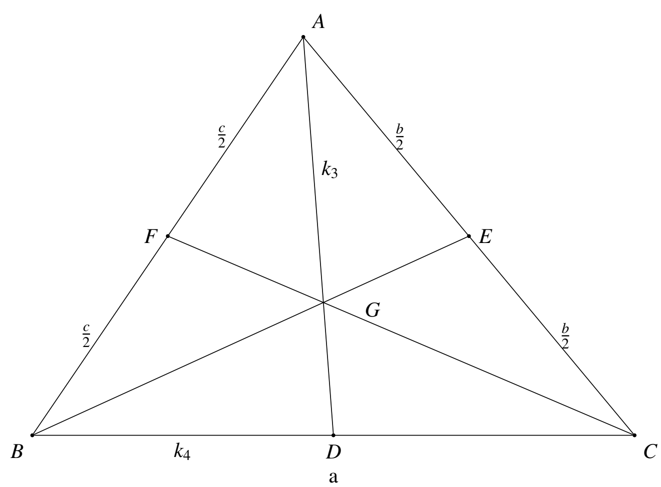

Fig. 4.6.1: $k_3=2, k_4=1$

Solution: Considering the ratios in Fig. 4.6.1,

$$\mathbf{G}=\frac{k_3\mathbf{D}+\mathbf{A}}{k_3+1} \tag{4.6.1}$$
$$\mathbf{D}=\frac{k_4\mathbf{C}+\mathbf{B}}{k_4+1} \tag{4.6.2}$$

Substituting from (4.5.1) in the above,

$$(k_3+1)\left(\frac{\mathbf{A}+\mathbf{B}+\mathbf{C}}{3}\right)=k_3\left(\frac{k_4\mathbf{C}+\mathbf{B}}{k_4+1}\right)+\mathbf{A} \tag{4.6.3}$$

which can be expressed as

$$(k_3+1)(k_4+1)(\mathbf{A}+\mathbf{B}+\mathbf{C})= 3\left\{k_3(k_4\mathbf{C}+\mathbf{B})+(k_4+1)\mathbf{A}\right\} \tag{4.6.4}$$

which can be expressed as

$$(k_3k_4 + k_3 - 2k_4 - 2)\,\mathbf{A} \\ - (-k_3k_4 - k_4 + 2k_3 - 1)\,\mathbf{B} \\ - (-k_3 - k_4 - 1 + 2k_3k_4)\,\mathbf{C} = \mathbf{0} \quad (4.6.5)$$

Comparing the above with (D.4.3),

$$p = -k_3k_4 - k_4 + 2k_3 - 1, q = -k_3 - k_4 - 1 + 2k_3k_4 \quad (4.6.6)$$

yielding

$$-k_3k_4 - k_4 + 2k_3 - 1 = 0 \quad (4.6.7)$$
$$-k_3 - k_4 - 1 + 2k_3k_4 = 0 \quad (4.6.8)$$

Subtracting (4.6.7) from (4.6.8),

$$3k_3\,(k_4 - 1) = 0 \quad (4.6.9)$$
$$\implies k_4 = 1 \quad (4.6.10)$$

which upon substituting in (4.6.7) yields

$$k_3 = 2 \quad (4.6.11)$$

5 Angle Bisectors

5.1. In Fig. 5.1.1, the bisectors of $\angle B$ and $\angle C$ meet at $\mathbf{I}$. Show that IA bisects $\angle A$.

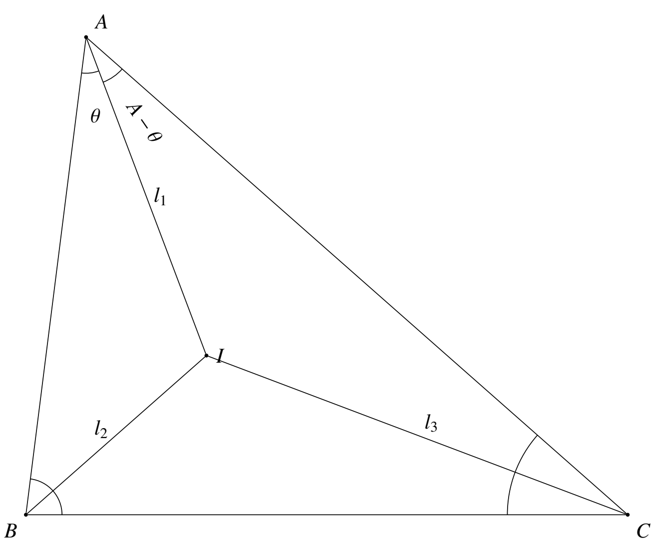

Fig. 5.1.1: Incentre I of $\triangle ABC$

Solution: Using sine formula in (2.2.3)

$$\frac{l_1}{\sin\frac{C}{2}} = \frac{l_3}{\sin(A-\theta)} \quad (5.1.1)$$
$$\frac{l_3}{\sin\frac{B}{2}} = \frac{l_2}{\sin\frac{C}{2}} \quad (5.1.2)$$
$$\frac{l_2}{\sin\theta} = \frac{l_1}{\sin\frac{B}{2}} \quad (5.1.3)$$

Multiplying the above equations,

$$\sin\theta = \sin(A-\theta) \quad (5.1.4)$$
$$\implies \theta = A - \theta \quad (5.1.5)$$
$$\text{or, } \theta = \frac{A}{2} \quad (5.1.6)$$

5.2. In Fig. 5.2.1,

$$ID \perp BC, IE \perp AC, IF \perp AB. \quad (5.2.1)$$

Show that

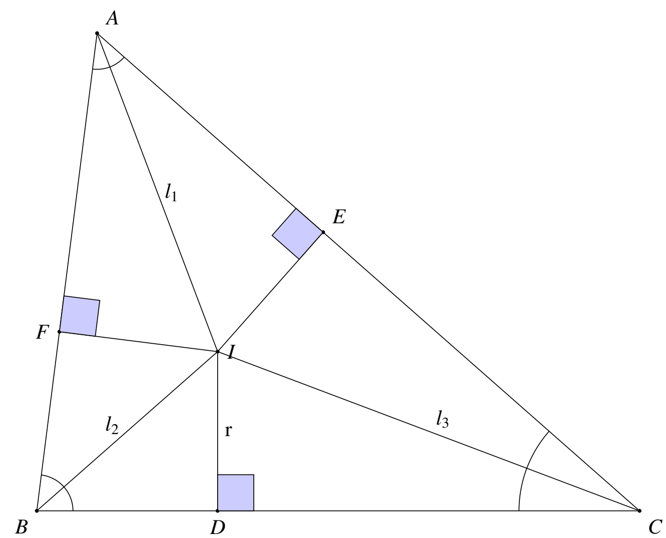

Fig. 5.2.1: Inradius r of $\triangle ABC$

$$ID = IE = IF = r \quad (5.2.2)$$

Solution: In $\triangle$s IDC and IEC,

$$ID = IE = \frac{l_3}{\sin\frac{C}{2}} \quad (5.2.3)$$

Similarly, in $\triangle$s IEA and IFA,

$$IF = IE = \frac{l_1}{\sin\frac{A}{2}} \quad (5.2.4)$$

yielding (5.2.2)

5.3. In Fig. 5.2.1, show that

$$BD = BF, AE = AF, CD = CE \quad (5.3.1)$$

Solution: From Fig. 5.2.1, in $\triangle$s IBD and IBF,

$$x = BD = BF = r\cot\frac{B}{2} \quad (5.3.2)$$

Similarly, other results can be obtained.

5.4. The circle with centre $\mathbf{I}$ and radius r in Fig. 5.4.1 is known as the *incircle*.

6 Perpendicular Bisectors

6.1. In Fig. 6.1.1,

$$OB = OC = R \quad (6.1.1)$$

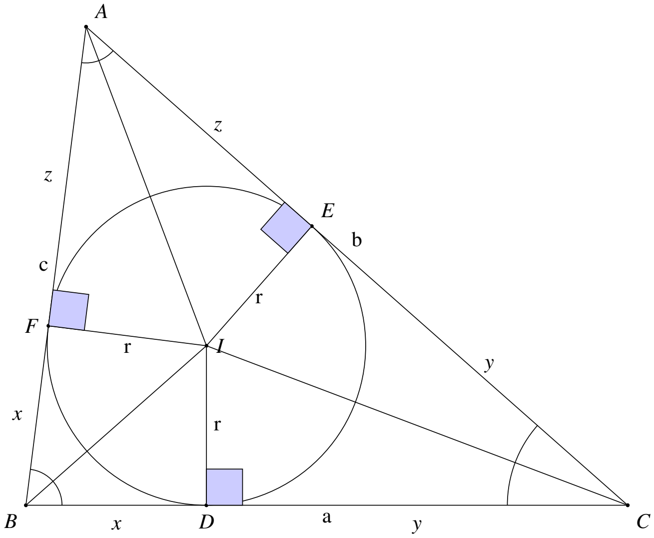

Fig. 5.4.1: Incircle of $\triangle ABC$

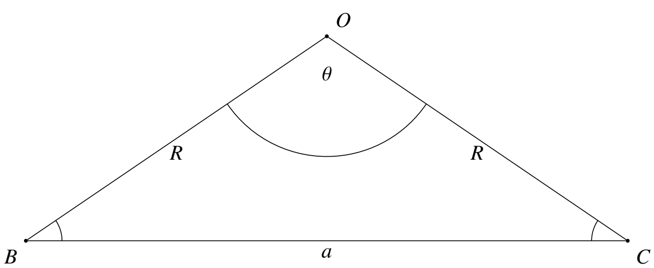

Fig. 6.1.1: Isosceles Triangle

Such a triangle is known as an isosceles triangle. Show that

$$\angle B = \angle C \tag{6.1.2}$$

Solution: Using (2.2.3),

$$\frac{\sin B}{R} = \frac{\sin C}{R} \tag{6.1.3}$$

$$\implies \sin B = \sin C \tag{6.1.4}$$

$$\text{or, } \angle B = \angle C. \tag{6.1.5}$$

6.2. In Fig. 6.1.1, show that

$$a = 2R\sin\frac{\theta}{2} \tag{6.2.1}$$

Solution: In $\triangle OBC$, using the cosine formula from (C.3.1),

$$\cos\theta = \frac{R^2 + R^2 - a^2}{2R^2} = 1 - \frac{a^2}{2R^2} \tag{6.2.2}$$

$$\implies \frac{a^2}{2R^2} = 2\sin^2\frac{\theta}{2} \tag{6.2.3}$$

yielding (6.2.1).

6.3. In Fig. 7.1.1, show that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R. \tag{6.3.1}$$

Solution: From (7.4.1) and (6.2.1)

$$a = 2R\sin A \tag{6.3.2}$$

6.4. In Fig. 6.4.1,

$$OB = OC = R, BD = DC. \tag{6.4.1}$$

Show that $OD \perp BC$.

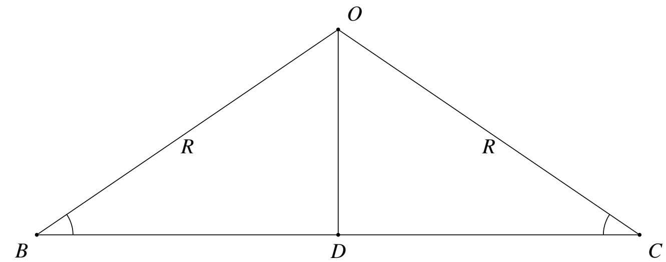

Fig. 6.4.1: Perpendicular bisector.

Solution:

$$\|\mathbf{O} - \mathbf{C}\| = \|\mathbf{O} - \mathbf{B}\| = R \tag{6.4.2}$$

$$\implies \|\mathbf{O} - \mathbf{C}\|^2 = \|\mathbf{O} - \mathbf{B}\|^2 \tag{6.4.3}$$

which can be expressed as

$$(\mathbf{O} - \mathbf{C})^\top (\mathbf{O} - \mathbf{C}) = (\mathbf{O} - \mathbf{B})^\top (\mathbf{O} - \mathbf{B}) \tag{6.4.4}$$

$$\|\mathbf{O}\|^2 - 2\mathbf{O}^\top\mathbf{C} + \|\mathbf{C}\|^2 = \|\mathbf{O}\|^2 - 2\mathbf{O}^\top\mathbf{B} + \|\mathbf{B}\|^2 \tag{6.4.5}$$

$$\implies (\mathbf{B} - \mathbf{C})^\top \mathbf{O} = \frac{\|\mathbf{B}\|^2 - \|\mathbf{C}\|^2}{2} \tag{6.4.6}$$

which can be simplified to obtain

$$(\mathbf{B} - \mathbf{C})^\top \left\{\mathbf{O} - \left(\frac{\mathbf{B} + \mathbf{C}}{2}\right)\right\} = 0 \tag{6.4.7}$$

$$\text{or, } (\mathbf{B} - \mathbf{C})^\top \{\mathbf{O} - \mathbf{D}\} = 0 \tag{6.4.8}$$

which proves the give result using (D.3.1) and (C.5.2).

6.5. In Fig. 6.5.1, OD and OE are the perpendicular bisectors of sides BC and AC respectively. Show that $OA = R$.
Solution: Tracing (6.4.8) backwards yields

$$OB = OC, OC = OA = R. \tag{6.5.1}$$

7 Circumcircle: Circle Equation

7.1. The equation of the circle in Fig. 7.1.1, is

$$\|\mathbf{x} - \mathbf{O}\| = R \tag{7.1.1}$$

This is known as the *circumcircle* of $\triangle ABC$.

7.2. Any point on the circle can be expressed as

$$\mathbf{x} = \mathbf{O} + R\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}, \quad 0 \in [0, 2\pi]. \tag{7.2.1}$$

7.3. Let

$$R = 1, \mathbf{O} = \mathbf{0}, \mathbf{A} = \begin{pmatrix}\cos\theta_1\\ \sin\theta_1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}\cos\theta_2\\ \sin\theta_2\end{pmatrix}, \tag{7.3.1}$$

Show that

$$\|\mathbf{A} - \mathbf{B}\| = 2\sin\left(\frac{\theta_1 - \theta_2}{2}\right) \tag{7.3.2}$$

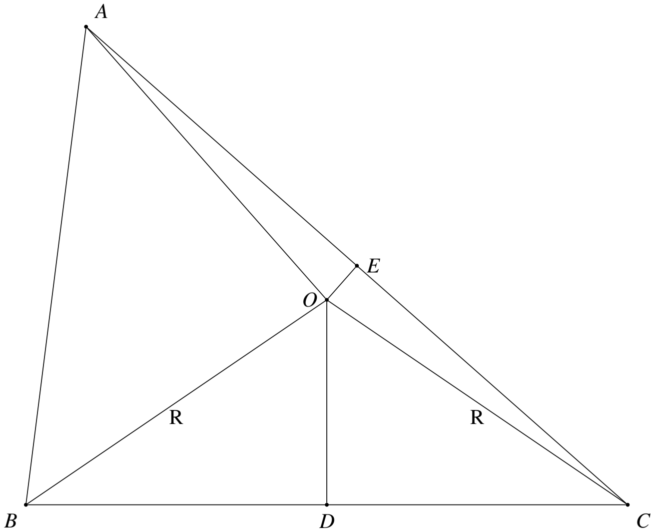

Fig. 6.5.1: Perpendicular bisectors of $\triangle ABC$ meet at $\mathbf{O}$.

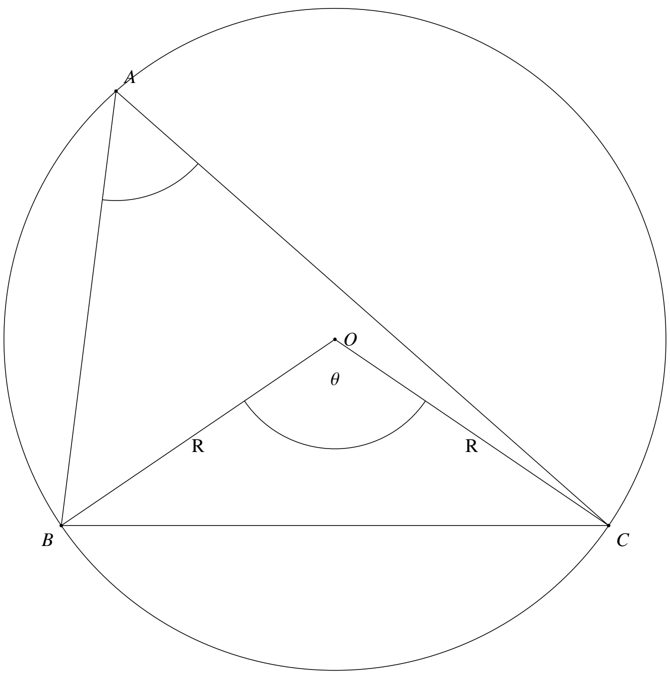

Fig. 7.1.1: Circumcircle of $\triangle ABC$

Solution: From (7.2.1).

$$\mathbf{A}-\mathbf{B}=\begin{pmatrix}\cos\theta_1-\cos\theta_2\\ \sin\theta_1-\sin\theta_2\end{pmatrix} \tag{7.3.3}$$

$$\implies \|\mathbf{A}-\mathbf{B}\|^2=(\cos\theta_1-\cos\theta_2)^2+(\sin\theta_1-\sin\theta_2)^2 \tag{7.3.4}$$

$$=2\left\{1-\cos(\theta_1-\theta_2)\right\}=4\sin^2\left(\frac{\theta_1-\theta_2}{2}\right) \tag{7.3.5}$$

yielding (7.3.2) from (E.6.3).

7.4. In Fig. 7.1.1, show that

$$\theta=2A. \tag{7.4.1}$$

Solution: Let

$$\mathbf{C}=\begin{pmatrix}\cos\theta_3\\ \sin\theta_3\end{pmatrix} \tag{7.4.2}$$

Then, substituting from (7.3.2) in (C.4.2),

$$\cos A=\frac{4\sin^2\left(\frac{\theta_1-\theta_2}{2}\right)+4\sin^2\left(\frac{\theta_1-\theta_3}{2}\right)-4\sin^2\left(\frac{\theta_2-\theta_3}{2}\right)}{8\sin\left(\frac{\theta_1-\theta_2}{2}\right)\sin\left(\frac{\theta_1-\theta_3}{2}\right)} \tag{7.4.3}$$

$$=\frac{2\sin^2\left(\frac{\theta_1-\theta_2}{2}\right)+\cos(\theta_2-\theta_3)-\cos(\theta_1-\theta_3)}{4\sin\left(\frac{\theta_1-\theta_2}{2}\right)\sin\left(\frac{\theta_1-\theta_3}{2}\right)} \tag{7.4.4}$$

from (E.6.3). $\therefore$ from (E.5.4),

$$\cos A=\frac{2\sin^2\left(\frac{\theta_1-\theta_2}{2}\right)+2\sin\left(\frac{\theta_1-\theta_2}{2}\right)\sin\left(\frac{\theta_1+\theta_2}{2}-\theta_3\right)}{4\sin\left(\frac{\theta_1-\theta_2}{2}\right)\sin\left(\frac{\theta_1-\theta_3}{2}\right)} \tag{7.4.5}$$

$$=\frac{\sin\left(\frac{\theta_1-\theta_2}{2}\right)+\sin\left(\frac{\theta_1+\theta_2}{2}-\theta_3\right)}{2\sin\left(\frac{\theta_1-\theta_3}{2}\right)} \tag{7.4.6}$$

From (E.5.1), the above equation can be expressed as

$$\cos A=\frac{2\sin\left(\frac{\theta_1-\theta_3}{2}\right)\cos\left(\frac{\theta_2-\theta_3}{2}\right)}{2\sin\left(\frac{\theta_1-\theta_3}{2}\right)}=\cos\left(\frac{\theta_2-\theta_3}{2}\right) \tag{7.4.7}$$

$$\implies 2A=\theta_2-\theta_3 \tag{7.4.8}$$

Similarly,

$$\cos\theta=\frac{1+1-4\sin^2\left(\frac{\theta_2-\theta_3}{2}\right)}{2}=\cos(\theta_2-\theta_3)=\cos 2A \tag{7.4.9}$$

8 Tangent

8.1. In Fig. 8.1.1, OC is the radius and PC touches the circle at C. Show that

$$OC\perp PC. \tag{8.1.1}$$

Solution: The equation of PC can be expressed as

$$\mathbf{x}=\mathbf{C}+\mu\mathbf{m} \tag{8.1.2}$$

and the equation of the circle is

$$\|\mathbf{x}-\mathbf{O}\|=R \tag{8.1.3}$$

Substituting (8.1.2) in (8.1.3),

$$\|\mathbf{C}+\mu\mathbf{m}-\mathbf{O}\|^2=R^2 \tag{8.1.4}$$

$$\implies \mu^2\|\mathbf{m}\|^2+2\mu\mathbf{m}^\top(\mathbf{C}-\mathbf{O})+\|\mathbf{C}-\mathbf{O}\|^2-R^2=0 \tag{8.1.5}$$

The above equation has only one root. Hence the discriminant of the above quadratic should be zero. So,

$$\left\{\mathbf{m}^\top(\mathbf{C}-\mathbf{O})\right\}^2-\|\mathbf{m}\|^2\left\{\|\mathbf{C}-\mathbf{O}\|^2-R^2\right\}=0 \tag{8.1.6}$$

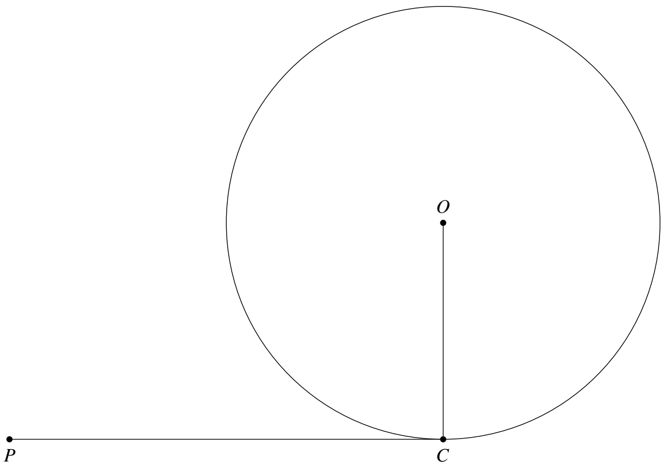

Fig. 8.1.1

Since $\mathbf{C}$ is a point on the circle,

$$\|\mathbf{C}-\mathbf{O}\|^2 - R^2 = 0 \tag{8.1.7}$$

$$\implies \mathbf{m}^\top(\mathbf{C}-\mathbf{O}) = 0 \tag{8.1.8}$$

upon substituting in (8.1.6). Using the definition of the direction vector from (D.1.1)

$$\mathbf{m} = \mathbf{P}-\mathbf{C} \tag{8.1.9}$$

$$\implies (\mathbf{P}-\mathbf{C})^\top(\mathbf{C}-\mathbf{O}) = 0 \tag{8.1.10}$$

which is equivalent to (8.1.1).

8.2. In Fig. 8.2.1 show that

$$\theta = \alpha \tag{8.2.1}$$

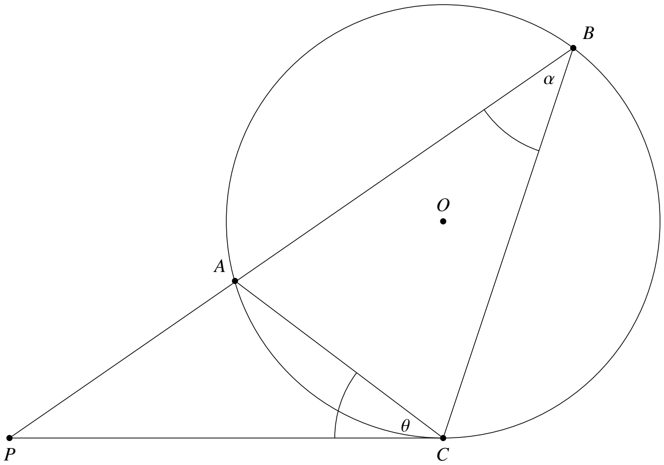

Fig. 8.2.1: $\theta = \alpha$.

Solution: Let Let

$$\mathbf{O} = \mathbf{0A} = \begin{pmatrix}\cos\theta_1\\ \sin\theta_1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}\cos\theta_2\\ \sin\theta_2\end{pmatrix}, \mathbf{C} = \begin{pmatrix}\cos\theta_3\\ \sin\theta_3\end{pmatrix} \tag{8.2.2}$$

Without loss of generality, let

$$\theta_3 = \frac{\pi}{2} \tag{8.2.3}$$

Then,

$$\mathbf{C}-\mathbf{O} = \begin{pmatrix}0\\1\end{pmatrix} \tag{8.2.4}$$

From from (8.1.10),

$$\mathbf{C}-\mathbf{P} = \begin{pmatrix}1\\0\end{pmatrix}, \tag{8.2.5}$$

From (C.4.1) and (8.2.5),

$$\cos\theta = \frac{\begin{pmatrix}\cos\theta_3-\cos\theta_1 & \sin\theta_3-\sin\theta_1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}}{2\sin\left(\frac{\theta_1-\theta_3}{2}\right)} \tag{8.2.6}$$

$$= \sin\left(\frac{\theta_1+\theta_3}{2}\right) = \cos\left(\frac{\pi}{2}-\frac{\theta_1+\theta_3}{2}\right) = \cos\left(\frac{\pi}{4}-\frac{\theta_1}{2}\right) \tag{8.2.7}$$

upon substituting from (8.2.3). Similarly, from (7.4.7),

$$\cos\alpha = \cos\left(\frac{\theta_1-\theta_3}{2}\right) = \cos\left(\frac{\pi}{4}-\frac{\theta_1}{2}\right) = \cos\theta \tag{8.2.8}$$

8.3. In Fig. 8.2.1, show that $PA.PB = PC^2$.
Solution: In $\triangle$s APC and BPC, using (8.2.1),

$$\frac{AP}{\sin\theta} = \frac{AC}{\sin P} \tag{8.3.1}$$

$$\frac{PC}{\sin\theta} = \frac{BC}{\sin P} \tag{8.3.2}$$

$$\implies \frac{PC}{AP} = \frac{BC}{AC}\left(=\frac{BP}{CP}\right) \tag{8.3.3}$$

which gives the desired result. $\triangle$s APC and BPC are said to be *similar*.

Appendix A
Ratios

A right angled triangle looks like Fig. 3.1. with angles

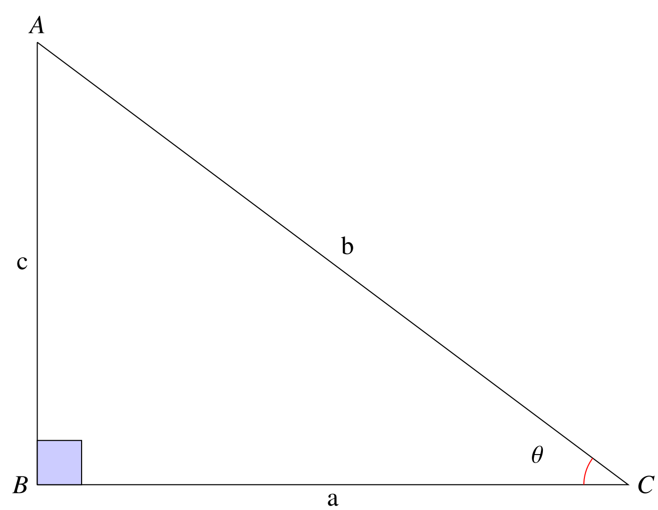

Fig. 3.1: Right Angled Triangle

$\angle A, \angle B$ and $\angle C$ and sides a, b and c. The unique feature of this triangle is $\angle B$ which is defined to be 90°.

A.1. For simplicity, let the greek letter $\theta = \angle C$. We have the following definitions.

$$\begin{aligned} \sin\theta &= \tfrac{c}{b} & \cos\theta &= \tfrac{a}{b} \\ \tan\theta &= \tfrac{c}{a} & \cot\theta &= \tfrac{1}{\tan\theta} \\ \csc\theta &= \tfrac{1}{\sin\theta} & \sec\theta &= \tfrac{1}{\cos\theta} \end{aligned} \tag{A.1.1}$$

A.2. Show that

$$\cos\theta = \sin(90^\circ - \theta) \tag{A.2.1}$$

Solution: From (A.1.1),

$$\cos\angle BAC = \cos\alpha = \cos(90^\circ - \theta) = \frac{c}{b} = \sin\angle ABC = \sin\theta \tag{A.2.2}$$

Appendix B
Vectors

B.1. A *matrix* of the form

$$\mathbf{A} \triangleq \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \tag{B.1.1}$$

is defined be *column vector*, or simply, vector. In Fig. 3.1 the point vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}$ can be defined as

$$\mathbf{A} = \begin{pmatrix} 0 \\ c \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} a \\ 0 \end{pmatrix} \tag{B.1.2}$$

B.2.

$$\lambda\mathbf{A} \triangleq \begin{pmatrix} \lambda a_1 \\ \lambda a_2 \end{pmatrix} \tag{B.2.1}$$

B.3. For

$$\mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \tag{B.3.1}$$

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \end{pmatrix} \tag{B.3.2}$$

B.4. The transpose of $\mathbf{A}$ is the *row vector* defined as

$$\mathbf{A}^\top = \begin{pmatrix} a_1 & a_2 \end{pmatrix} \tag{B.4.1}$$

B.5. The *inner product* or *dot product* is defined as

$$\mathbf{A}^\top\mathbf{B} \equiv \mathbf{A}\cdot\mathbf{B} = \begin{pmatrix} a_1 & a_2 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = a_1b_1 + a_2b_2 \tag{B.5.1}$$

In Fig. 3.1,

$$\mathbf{A}^\top\mathbf{C} = 0 \tag{B.5.2}$$

B.6. The *norm* of $\mathbf{A}$ is defined as

$$\|\mathbf{A}\| = \sqrt{\mathbf{A}^\top\mathbf{A}} = \sqrt{a_1^2 + a_2^2} \tag{B.6.1}$$

B.7. In Fig. 3.1, it is easy to verify that

$$\|\mathbf{A} - \mathbf{C}\|^2 = \begin{pmatrix} -c & a \end{pmatrix}\begin{pmatrix} -c \\ a \end{pmatrix} = a^2 + c^2 = b^2 \tag{B.7.1}$$

from (1.4.1). Thus, the distance betwen any two points $\mathbf{A}$ and $\mathbf{B}$ is given by

$$\|\mathbf{A} - \mathbf{B}\| \tag{B.7.2}$$

B.8. Show that

$$\|\lambda\mathbf{A}\| = |\lambda|\,\|\mathbf{A}\| \tag{B.8.1}$$

B.9. Find the equation of the line BC.

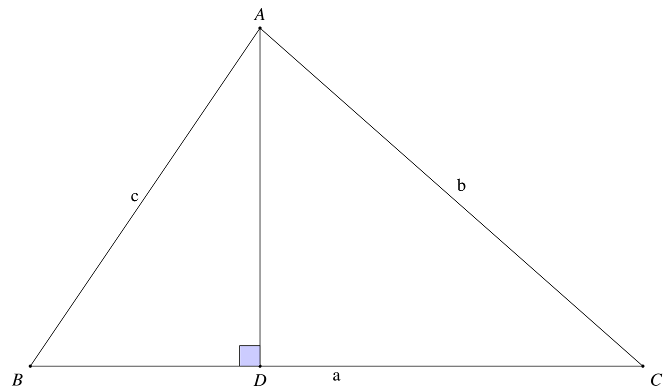

Fig. B.9.1: Drawing the altitude

Solution: Let $\mathbf{x}$ be any point on BC. Using section formula, for some k,

$$\mathbf{x} = \frac{k\mathbf{C} + \mathbf{B}}{k+1} = \frac{(k+1)\mathbf{C} + (\mathbf{B} - \mathbf{C})}{k+1} \tag{B.9.1}$$

$$\implies \mathbf{x} = \mathbf{C} + \lambda\mathbf{m} \tag{B.9.2}$$

where

$$\mathbf{m} = \frac{\mathbf{B} - \mathbf{C}}{k+1} \equiv \mathbf{B} - \mathbf{C} \tag{B.9.3}$$

B.10. The *normal vector* to $\mathbf{m}$ is defined as

$$\mathbf{n}^\top\mathbf{m} = 0 \tag{B.10.1}$$

$$\mathbf{n} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\mathbf{m} \tag{B.10.2}$$

B.11. From (B.10.1) and (B.9.2), it can be verified that

$$\mathbf{n}^\top\mathbf{x} = \mathbf{n}^\top\mathbf{C} + \lambda\mathbf{n}^\top\mathbf{m} \tag{B.11.1}$$

$$\implies \mathbf{n}^\top\mathbf{x} = \mathbf{n}^\top\mathbf{C} \tag{B.11.2}$$

(B.11.2) is defined to be the *normal form* of the line BC.

Appendix C
Matrices: Cosine Formula

C.1. The determinant of the 2×2 matrix

$$\mathbf{M} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \tag{C.1.1}$$

is defined as

$$\left|\mathbf{M}\right| = \left|\mathbf{A} \quad \mathbf{B}\right| \tag{C.1.2}$$

$$= \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1 \tag{C.1.3}$$

C.2. In Fig. C.2.1, show that

$$\begin{pmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{pmatrix}\begin{pmatrix} \cos A \\ \cos B \\ \cos C \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{C.2.1}$$

Solution: From Fig. C.2.1,

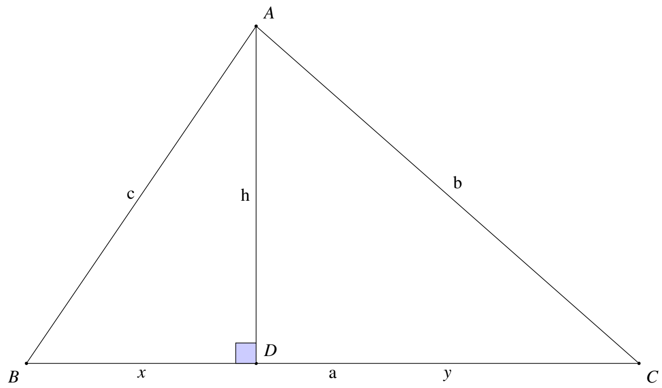

Fig. C.2.1: The cosine formula

$$a = x + y = b\cos C + c\cos B = \begin{pmatrix}\cos C & \cos B\end{pmatrix}\begin{pmatrix}b\\c\end{pmatrix} \tag{C.2.2}$$

$$= \begin{pmatrix}0 & b & c\end{pmatrix}\begin{pmatrix}\cos A\\ \cos C\\ \cos B\end{pmatrix} \tag{C.2.3}$$

Similarly,

$$b = c\cos A + a\cos C = \begin{pmatrix}c & 0 & a\end{pmatrix}\begin{pmatrix}\cos A\\ \cos C\\ \cos B\end{pmatrix} \tag{C.2.4}$$

$$c = b\cos A + a\cos B = \begin{pmatrix}b & a & 0\end{pmatrix}\begin{pmatrix}\cos A\\ \cos C\\ \cos B\end{pmatrix} \tag{C.2.5}$$

The above equations can be expressed in matrix form as (C.2.1).

C.3. Show that

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \tag{C.3.1}$$

Solution: Using the properties of determinants,

$$\cos A = \frac{\begin{vmatrix} a & c & b\\ b & 0 & a\\ c & a & 0\end{vmatrix}}{\begin{vmatrix} 0 & c & b\\ c & 0 & a\\ b & a & 0\end{vmatrix}} = \frac{ab^2 + ac^2 - a^3}{abc + abc} = \frac{b^2 + c^2 - a^2}{2abc} \tag{C.3.2}$$

C.4. In Fig. C.2.1 show that

$$\cos A = \frac{(\mathbf{A} - \mathbf{B})^\top (\mathbf{A} - \mathbf{C})}{\|\mathbf{A} - \mathbf{B}\| \|\mathbf{A} - \mathbf{C}\|} \tag{C.4.1}$$

Solution: From (C.3.1), using (B.7.2),

$$\cos A = \frac{\|\mathbf{A} - \mathbf{B}\|^2 + \|\mathbf{A} - \mathbf{C}\|^2 - \|\mathbf{B} - \mathbf{C}\|^2}{2\|\mathbf{A} - \mathbf{B}\| \|\mathbf{A} - \mathbf{C}\|} \tag{C.4.2}$$

$$= \frac{\|\mathbf{A}\|^2 - \mathbf{A}^\top\mathbf{B} - \mathbf{A}^\top\mathbf{C} + \mathbf{B}^\top\mathbf{C}}{\|\mathbf{A} - \mathbf{B}\| \|\mathbf{A} - \mathbf{C}\|} \tag{C.4.3}$$

which can be expressed as (C.4.1).

C.5. For $A = 90°$,

$$\cos A = 0 \tag{C.5.1}$$

$$\implies (\mathbf{A} - \mathbf{B})^\top (\mathbf{A} - \mathbf{C}) = 0 \tag{C.5.2}$$

from (C.4.1).

Appendix D
Collinear Points

D.1. The direction vector of the line AB is

$$\mathbf{A} - \mathbf{B} \equiv \mathbf{B} - \mathbf{A} \equiv \kappa\begin{pmatrix}1\\m\end{pmatrix}, \tag{D.1.1}$$

where m is defined to be the slope of AB. In Fig. 3.1,

$$\mathbf{A} - \mathbf{C} = \begin{pmatrix}-c\\a\end{pmatrix} \equiv \begin{pmatrix}1\\-\frac{a}{c}\end{pmatrix} = \begin{pmatrix}1\\-\tan\theta\end{pmatrix} \tag{D.1.2}$$

the slope of AC is $-\tan\theta$

D.2. Points $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ are on a line if they have the same direction vector, i.e.

$$p(\mathbf{B} - \mathbf{A}) + q(\mathbf{C} - \mathbf{B}) = 0 \implies p, q \neq 0. \tag{D.2.1}$$

$(\mathbf{A} - \mathbf{B}), (\mathbf{C} - \mathbf{B})$ are then said to be *linearly dependent*.

D.3. If points $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ are collinear,

$$\mathbf{B} = \frac{k\mathbf{A} + \mathbf{C}}{k + 1} \tag{D.3.1}$$

Solution: From (D.2.1),

$$p(\mathbf{A} - \mathbf{B}) + q(\mathbf{A} - \mathbf{C}) = 0 \implies \mathbf{B} = \frac{p\mathbf{A} + q\mathbf{C}}{p + q} \tag{D.3.2}$$

yielding (D.3.1) upon substituting

$$k = \frac{p}{q}. \tag{D.3.3}$$

This is known as *section formula*.

D.4. Consequently, points $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ form a triangle if

$$p(\mathbf{A} - \mathbf{B}) + q(\mathbf{C} - \mathbf{B}) \tag{D.4.1}$$

$$= (p + q)\mathbf{B} - p\mathbf{A} - q\mathbf{C} = 0 \tag{D.4.2}$$

$$\implies p = 0, q = 0 \tag{D.4.3}$$

Appendix E
Identities

E.1. Show that

$$\cos 90° = 0 \tag{E.1.1}$$

Solution: Using (C.3.1) in Fig. 3.1,

$$\cos 90° = \frac{a^2 + c^2 - b^2}{2ac} = 0 \tag{E.1.2}$$

upon substituting from (1.4.1).

E.2. Show that

$$\sin 90° = 1 \tag{E.2.1}$$

Solution: Trivial from (A.2.1).

E.3. Prove the following identities

a)

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta. \tag{E.3.1}$$

b)

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta. \tag{E.3.2}$$

Solution: In (2.4.1), let

$$\begin{aligned} \theta_1 + \theta_2 &= \alpha \\ \theta_2 &= \beta \end{aligned} \tag{E.3.3}$$

This gives (E.3.1). In (E.3.1), replace α by $90^\circ - \alpha$. This results in

$$\sin(90^\circ - \alpha - \beta) = \sin(90^\circ - \alpha)\cos\beta - \cos(90^\circ - \alpha)\sin\beta \tag{E.3.4}$$

$$\implies \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \tag{E.3.5}$$

E.4. Using (2.4.1) and (E.3.2), show that

$$\sin(\theta_1+\theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 \tag{E.4.1}$$

$$\cos(\theta_1-\theta_2) = \cos\theta_1\cos\theta_2\sin\theta_1\sin\theta_2 \tag{E.4.2}$$

Solution: From (2.4.1),

$$\sin(\theta_1+\theta_2)\cos\theta_2 = \sin\theta_1 + \cos(\theta_1+\theta_2)\sin\theta_2 \tag{E.4.3}$$

Using (E.3.2) in the above,

$$\sin(\theta_1+\theta_2)\cos\theta_2 = \sin\theta_1 + (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2)\sin\theta_2 \tag{E.4.4}$$

which can be expressed as

$$\sin(\theta_1+\theta_2)\cos\theta_2 = \sin\theta_1 + \cos\theta_1\cos\theta_2\sin\theta_2 - \sin\theta_1\sin^2\theta_2 \tag{E.4.5}$$

Since

$$\sin^2\theta_2 = 1 - \cos^2\theta_2, \tag{E.4.6}$$

we obtain

$$\sin(\theta_1+\theta_2)\cos\theta_2 = \cos\theta_1\cos\theta_2\sin\theta_2 + \sin\theta_1\cos^2\theta_2 \tag{E.4.7}$$

resulting in

$$\sin(\theta_1+\theta_2) = \cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2 \tag{E.4.8}$$

after factoring out $\cos\theta_2$. Using a similar approach, (E.4.2) can also be proved.

E.5. Show that

$$\sin\theta_1 + \sin\theta_2 = 2\sin\left(\frac{\theta_1+\theta_2}{2}\right)\cos\left(\frac{\theta_1-\theta_2}{2}\right) \tag{E.5.1}$$

$$\cos\theta_1 + \cos\theta_2 = 2\cos\left(\frac{\theta_1+\theta_2}{2}\right)\cos\left(\frac{\theta_1-\theta_2}{2}\right) \tag{E.5.2}$$

$$\sin\theta_1 - \sin\theta_2 = 2\sin\left(\frac{\theta_1-\theta_2}{2}\right)\cos\left(\frac{\theta_1+\theta_2}{2}\right) \tag{E.5.3}$$

$$\cos\theta_1 - \cos\theta_2 = 2\sin\left(\frac{\theta_1+\theta_2}{2}\right)\cos\left(\frac{\theta_2-\theta_1}{2}\right) \tag{E.5.4}$$

Solution: Let

$$\begin{aligned} \theta_1 &= \alpha + \beta \\ \theta_2 &= \alpha - \beta \end{aligned} \tag{E.5.5}$$

From (E.4.1),

$$\sin\theta_1 + \sin\theta_2 = \sin(\alpha+\beta) + \sin(\alpha-\beta) \tag{E.5.6}$$

$$= \sin\alpha\cos\beta + \cos\alpha\sin\beta \tag{E.5.7}$$

$$+ \sin\alpha\cos\beta - \cos\alpha\sin\beta \tag{E.5.8}$$

$$= 2\sin\alpha\cos\beta \tag{E.5.9}$$

resulting in (E.5.1)

$$\because \alpha = \frac{\theta_1+\theta_2}{2} \tag{E.5.10}$$

$$\beta = \frac{\theta_1-\theta_2}{2} \tag{E.5.11}$$

from (E.5.5). Other identities may be proved similarly.

E.6. Show that

$$\sin 2\theta = 2\sin\theta\cos\theta \tag{E.6.1}$$

$$\cos 2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1 \tag{E.6.2}$$

$$= \cos^2\theta - \sin^2\theta \tag{E.6.3}$$