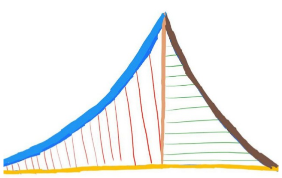
GEOMETRY
Through Equations



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CONTENTS

1 The Baudhayana Theorem

Use Fig. 1.1 for all problems in this section.

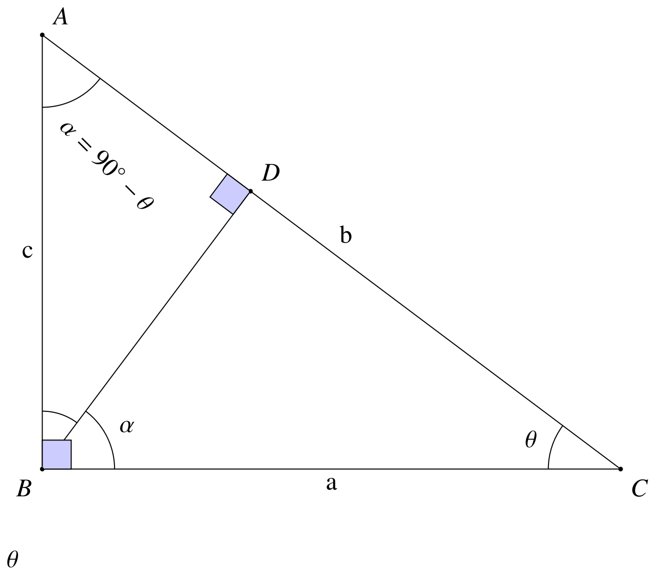

Fig. 1.1: Baudhayana Theorem

1.1. Show that
$$b = a\cos\theta + c\sin\theta \tag{1.1.1}$$
Solution: We observe that
$$CD = a\cos\theta \tag{1.1.2}$$
$$AD = c\cos\alpha = c\sin\theta \quad (\text{From} \quad (A.2.2)) \tag{1.1.3}$$
Thus,
$$CD + AD = b = a\cos\theta + c\sin\theta \tag{1.1.4}$$

1.2. From (1.1.1), show that
$$\sin^2\theta + \cos^2\theta = 1 \tag{1.2.1}$$
Solution: Dividing both sides of (1.1.1) by b,
$$1 = \frac{a}{b}\cos\theta + \frac{c}{b}\sin\theta \tag{1.2.2}$$
$$\Rightarrow \sin^2\theta + \cos^2\theta = 1 \quad (\text{from} \quad (A.1.1)) \tag{1.2.3}$$

1.3. In a right angled triangle, the hypotenuse is the longest side.
Solution: From (1.2.1),
$$0 \le \sin\theta, \cos\theta \le 1 \tag{1.3.1}$$
Hence,
$$b\sin\theta \le b \implies c \le b \tag{1.3.2}$$
Similalry,
$$a \le b \tag{1.3.3}$$

1.4. Using (1.1.1), show that
$$b^2 = a^2 + c^2 \tag{1.4.1}$$
(1.4.1) is known as the Baudhayana theorem. It is also known as the Pythagoras theorem.
Solution: From (1.1.1),
$$b = a\frac{a}{b} + c\frac{c}{b} \quad (\text{from} \quad (A.1.1)) \tag{1.4.2}$$
$$\implies b^2 = a^2 + c^2 \tag{1.4.3}$$

2 Area of a Triangle

2.1. Show that the area of ΔABC in Fig. 2.1.1 is $\frac{1}{2}ab\sin C$.

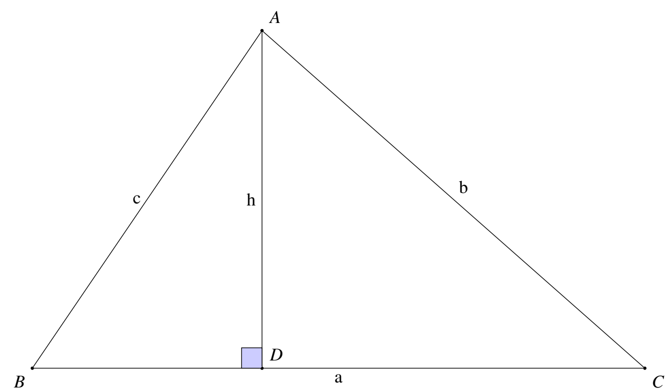

Fig. 2.1.1: Area of a Triangle

Solution: We have
$$ar(\Delta ABC) = \frac{1}{2}ah = \frac{1}{2}ab\sin C \quad (\because \quad h = b\sin C). \tag{2.1.1}$$

2.2. Show that
$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \tag{2.2.1}$$
Solution: Fig. 2.1.1 can be suitably modified to obtain
$$ar(\Delta ABC) = \frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B \tag{2.2.2}$$
Dividing the above by abc, we obtain
$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \tag{2.2.3}$$
This is known as the sine formula.

2.3. Show that
$$\alpha > \beta \implies \sin\alpha > \sin\beta \tag{2.3.1}$$
Solution: In Fig. 2.3.1,
$$ar(\triangle ABD) < ar(\triangle ABC) \tag{2.3.2}$$
$$\implies \frac{1}{2}lc\sin\theta_1 < \frac{1}{2}ac\sin(\theta_1 + \theta_2) \tag{2.3.3}$$
$$\implies \frac{l}{a} < \frac{\sin(\theta_1 + \theta_2)}{\sin\theta_1} \tag{2.3.4}$$
$$\text{or, } 1 < \frac{l}{a} < \frac{\sin(\theta_1 + \theta_2)}{\sin\theta_1} \tag{2.3.5}$$
from Theorem 1.3, yielding
$$\implies \frac{\sin(\theta_1 + \theta_2)}{\sin\theta_1} > 1. \tag{2.3.6}$$
This proves (2.3.1).

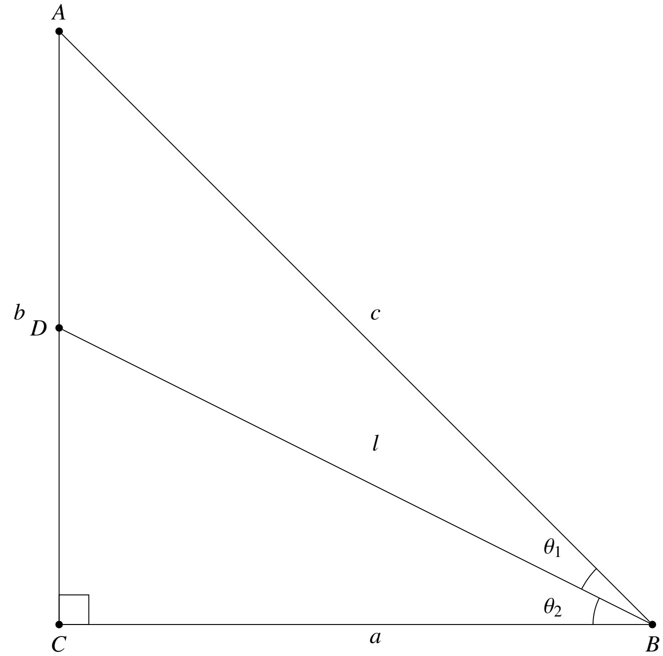

Fig. 2.3.1

2.4. Using Fig. 2.3.1, show that

$$\sin\theta_1 = \sin(\theta_1+\theta_2)\cos\theta_2 - \cos(\theta_1+\theta_2)\sin\theta_2 \quad (2.4.1)$$

Solution: The following equations can be obtained from the figure using the forumula for the area of a triangle

$$ar(\Delta ABC) = \frac{1}{2}ac\sin(\theta_1+\theta_2) \quad (2.4.2)$$
$$= ar(\Delta BDC) + ar(\Delta ADB) \quad (2.4.3)$$
$$= \frac{1}{2}cl\sin\theta_1 + \frac{1}{2}al\sin\theta_2 \quad (2.4.4)$$
$$= \frac{1}{2}ac\sin\theta_1\sec\theta_2 + \frac{1}{2}a^2\tan\theta_2 \quad (2.4.5)$$

($\because l = a\sec\theta_2$). From the above,

$$\sin(\theta_1+\theta_2) = \sin\theta_1\sec\theta_2 + \frac{a}{c}\tan\theta_2 \quad (2.4.6)$$
$$= \sin\theta_1\sec\theta_2 + \cos(\theta_1+\theta_2)\tan\theta_2 \quad (2.4.7)$$

Multiplying both sides by $\cos\theta_2$,

$$\sin(\theta_1+\theta_2)\cos\theta_2 = \sin\theta_1 + \cos(\theta_1+\theta_2)\sin\theta_2 \quad (2.4.8)$$

resulting in (2.4.1).

2.5. Find Hero's formula for the area of a triangle.

Solution: From (2.1), the area of $\triangle ABC$ is

$$\frac{1}{2}ab\sin C = \frac{1}{2}ab\sqrt{1-\cos^2 C} \quad \text{(from (1.2.1))} \quad (2.5.1)$$
$$= \frac{1}{2}ab\sqrt{1-\left(\frac{a^2+b^2-c^2}{2ab}\right)^2} \text{(from (C.2.1))} \quad (2.5.2)$$
$$= \frac{1}{4}\sqrt{(2ab)^2-(a^2+b^2-c^2)} \quad (2.5.3)$$
$$= \frac{1}{4}\sqrt{(2ab+a^2+b^2-c^2)(2ab-a^2-b^2+c^2)} \quad (2.5.4)$$
$$= \frac{1}{4}\sqrt{\left\{(a+b)^2-c^2\right\}\left\{c^2-(a-b)^2\right\}} \quad (2.5.5)$$
$$= \frac{1}{4}\sqrt{(a+b+c)(a+b-c)(a+c-b)(b+c-a)} \quad (2.5.6)$$

Substituting

$$s = \frac{a+b+c}{2} \quad (2.5.7)$$

in (2.5.6), the area of $\triangle ABC$ is

$$\sqrt{s(s-a)(s-b)(s-c)} \quad (2.5.8)$$

This is known as Hero's formula.

2.6. The area of the triangle with vertices $\mathbf{A},\mathbf{B},\mathbf{C}$ is given by

$$\frac{1}{2}\|(\mathbf{A}-\mathbf{B})\times(\mathbf{A}-\mathbf{C})\| = \frac{1}{2}\|\mathbf{A}\times\mathbf{B}+\mathbf{B}\times\mathbf{C}+\mathbf{C}\times\mathbf{A}\| \quad (2.6.1)$$

where the *cross product* or *vector product* defined as $\mathbf{A}\times\mathbf{B}$ is given by

$$\begin{vmatrix}\mathbf{A} & \mathbf{B}\end{vmatrix} = \begin{vmatrix}a_1 & b_1\\ a_2 & b_2\end{vmatrix} = a_1b_2 - a_2b_1 \quad (2.6.2)$$

2.7. If

$$\|\mathbf{A}\times\mathbf{B}\| = \|\mathbf{C}\times\mathbf{D}\|, \quad \text{then} \quad (2.7.1)$$
$$\mathbf{A}\times\mathbf{B} = \pm(\mathbf{C}\times\mathbf{D}) \quad (2.7.2)$$

where the sign depends on the orientation of the vectors.

3 Altitudes of a Triangle

3.1. In Fig. 3.2.1, $AD \perp BC$ and $BE \perp AC$ are defined to be the altitudes of $\triangle ABC$.

3.2. Let $\mathbf{H}$ be the intersection of the altitudes AD and BE as shown in Fig. 3.2.1. CH is extended to meet AB at $\mathbf{F}$. Show that $CF \perp AB$.

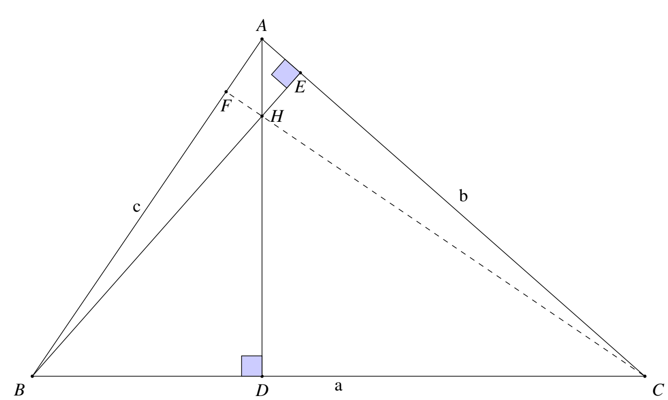

Fig. 3.2.1: Altitudes of a triangle meet at the orthocentre H

Solution: From (C.6.2),

$$(\mathbf{B}-\mathbf{C})^\top(\mathbf{H}-\mathbf{A})=0 \tag{3.2.1}$$
$$(\mathbf{C}-\mathbf{A})^\top(\mathbf{H}-\mathbf{B})=0 \tag{3.2.2}$$

Adding both the above and simplifying,

$$(\mathbf{B}-\mathbf{A})^\top(\mathbf{H}-\mathbf{C})=0 \tag{3.2.3}$$

$\implies CH \perp AB$, or $CF \perp AB$.

3.3. Altitudes of a $\triangle$ meet at the *orthocentre* H.

4 Median

4.1. In Fig. 4.3.1

$$AF = BF, AE = BE, \tag{4.1.1}$$

and the medians BE and CF meet at $\mathbf{G}$. Show that

$$ar(BEC) = ar(BFC) = \frac{1}{2}ar(ABC) \tag{4.1.2}$$

Solution: From (2.2.2),

$$ar(BEC) = \frac{1}{2}a\left(\frac{b}{2}\right)\sin C \tag{4.1.3}$$
$$ar(BFC) = \frac{1}{2}a\left(\frac{c}{2}\right)\sin B \tag{4.1.4}$$

yielding (4.1.2).

4.2. Show that

$$ar(CGE) = ar(BGF) \tag{4.2.1}$$

Solution: From Fig. 4.3.1 and (4.1.2),

$$ar(BGF) + ar(BGC) = ar(CGE) + ar(BGC) \tag{4.2.2}$$

yielding (4.2.1).

4.3. If $\mathbf{G}$ divides BE and CF in the ratios k_1 and k_2 respectively, show that

$$k_1 = k_2 \tag{4.3.1}$$

Solution: Let

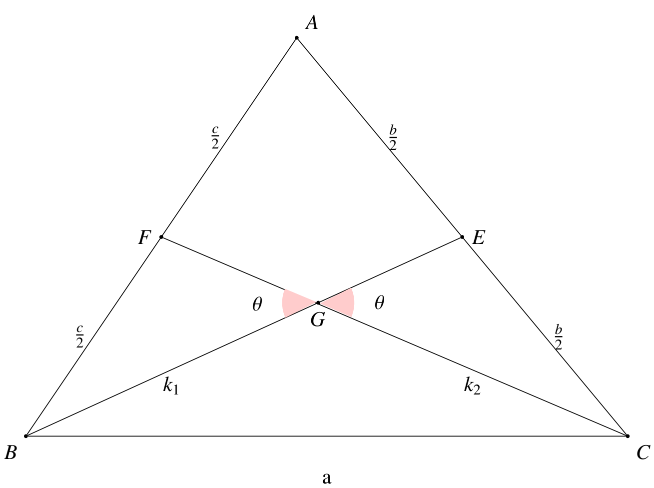

Fig. 4.3.1: $k_1 = k_2$.

$$GE = l_1, GF = l_2 \tag{4.3.2}$$

From (2.2.2) and (4.2.1),

$$\frac{1}{2}l_1(k_2 l_2)\sin\theta = \frac{1}{2}l_2(k_1 l_1)\sin\theta \tag{4.3.3}$$

yielding (4.3.1).

4.4. Show that

$$k_1 = k_2 = 2 \tag{4.4.1}$$

Solution: Let

$$k_1 = k_2 = k \tag{4.4.2}$$

Using (B.6.3),

$$\mathbf{G} = \frac{k\mathbf{E}+\mathbf{B}}{k+1} = \frac{k\mathbf{F}+\mathbf{C}}{k+1} \tag{4.4.3}$$
$$\implies k\left(\frac{\mathbf{A}+\mathbf{C}}{2}\right)+\mathbf{B} = k\left(\frac{\mathbf{A}+\mathbf{B}}{2}\right)+\mathbf{C} \tag{4.4.4}$$
$$\implies k(\mathbf{B}-\mathbf{C}) = 2(\mathbf{B}-\mathbf{C}) \tag{4.4.5}$$

resulting in (4.4.2).

4.5. Substituting $k = 2$ in (4.4.4),

$$\mathbf{G} = \frac{\mathbf{A}+\mathbf{B}+\mathbf{C}}{3} \tag{4.5.1}$$

4.6. In Fig. 4.6.1, AG is extended to join BC at $\mathbf{D}$. Show that AD is also a median.

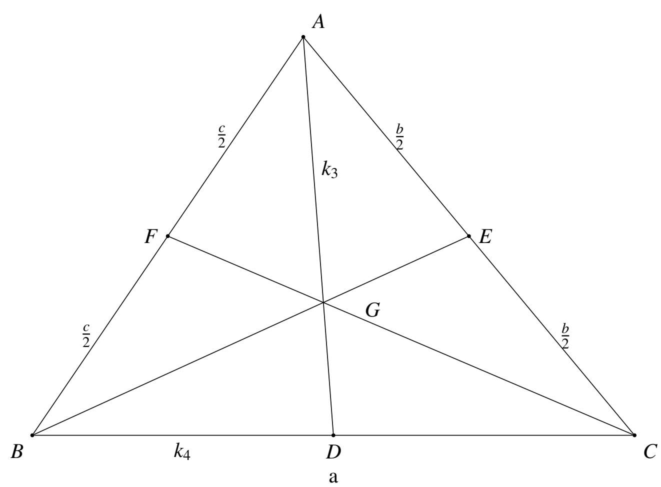

Fig. 4.6.1: $k_3 = 2, k_4 = 1$

Solution: Considering the ratios in Fig. 4.6.1,

$$\mathbf{G} = \frac{k_3\mathbf{D}+\mathbf{A}}{k_3+1} \tag{4.6.1}$$
$$\mathbf{D} = \frac{k_4\mathbf{C}+\mathbf{B}}{k_4+1} \tag{4.6.2}$$

Substituting from (4.5.1) in the above,

$$(k_3+1)\left(\frac{\mathbf{A}+\mathbf{B}+\mathbf{C}}{3}\right) = k_3\left(\frac{k_4\mathbf{C}+\mathbf{B}}{k_4+1}\right)+\mathbf{A} \tag{4.6.3}$$

which can be expressed as

$$(k_3+1)(k_4+1)(\mathbf{A}+\mathbf{B}+\mathbf{C}) = 3\{k_3(k_4\mathbf{C}+\mathbf{B})+(k_4+1)\mathbf{A}\} \tag{4.6.4}$$

which can be expressed as

$$(k_3k_4 + k_3 - 2k_4 - 2)\,\mathbf{A} - (-k_3k_4 - k_4 + 2k_3 - 1)\,\mathbf{B} - (-k_3 - k_4 - 1 + 2k_3k_4)\,\mathbf{C} = \mathbf{0} \quad (4.6.5)$$

Comparing the above with (B.7.3),

$$p = -k_3k_4 - k_4 + 2k_3 - 1, q = -k_3 - k_4 - 1 + 2k_3k_4 \quad (4.6.6)$$

yielding

$$-k_3k_4 - k_4 + 2k_3 - 1 = 0 \quad (4.6.7)$$
$$-k_3 - k_4 - 1 + 2k_3k_4 = 0 \quad (4.6.8)$$

Subtracting (4.6.7) from (4.6.8),

$$3k_3\,(k_4 - 1) = 0 \quad (4.6.9)$$
$$\implies k_4 = 1 \quad (4.6.10)$$

which upon substituting in (4.6.7) yields

$$k_3 = 2 \quad (4.6.11)$$

5 Angle Bisectors

5.1. In Fig. 5.1.1, the bisectors of $\angle B$ and $\angle C$ meet at $\mathbf{I}$. Show that IA bisects $\angle A$.

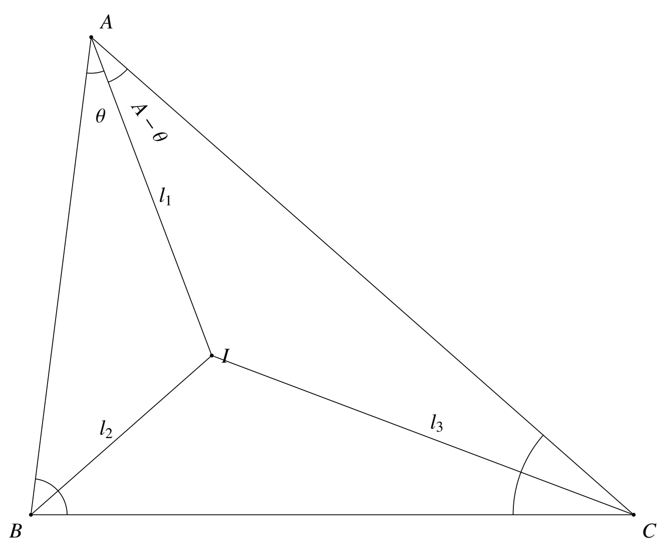

Fig. 5.1.1: Incentre I of $\triangle ABC$

Solution: Using sine formula in (2.2.3)

$$\frac{l_1}{\sin\frac{C}{2}} = \frac{l_3}{\sin(A-\theta)} \quad (5.1.1)$$
$$\frac{l_3}{\sin\frac{B}{2}} = \frac{l_2}{\sin\frac{C}{2}} \quad (5.1.2)$$
$$\frac{l_2}{\sin\theta} = \frac{l_1}{\sin\frac{B}{2}} \quad (5.1.3)$$

Multiplying the above equations,

$$\sin\theta = \sin(A-\theta) \quad (5.1.4)$$
$$\implies \theta = A - \theta \quad (5.1.5)$$
$$\text{or, } \theta = \frac{A}{2} \quad (5.1.6)$$

5.2. In Fig. 5.2.1,

$$ID \perp BC, IE \perp AC, IF \perp AB. \quad (5.2.1)$$

Show that

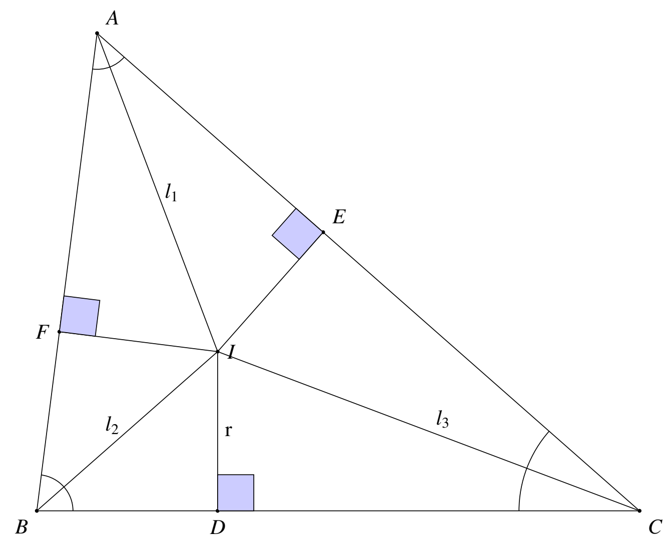

Fig. 5.2.1: Inradius r of $\triangle ABC$

$$ID = IE = IF = r \quad (5.2.2)$$

Solution: In $\triangle$s IDC and IEC,

$$ID = IE = \frac{l_3}{\sin\frac{C}{2}} \quad (5.2.3)$$

Similarly, in $\triangle$s IEA and IFA,

$$IF = IE = \frac{l_1}{\sin\frac{A}{2}} \quad (5.2.4)$$

yielding (5.2.2)

5.3. In Fig. 5.2.1, show that

$$BD = BF, AE = AF, CD = CE \quad (5.3.1)$$

Solution: From Fig. 5.2.1, in $\triangle$s IBD and IBF,

$$x = BD = BF = r\cot\frac{B}{2} \quad (5.3.2)$$

Similarly, other results can be obtained.

5.4. The circle with centre $\mathbf{I}$ and radius r in Fig. 5.4.1 is known as the *incircle*.

6 Perpendicular Bisectors

6.1. In Fig. 6.1.1,

$$OB = OC = R \quad (6.1.1)$$

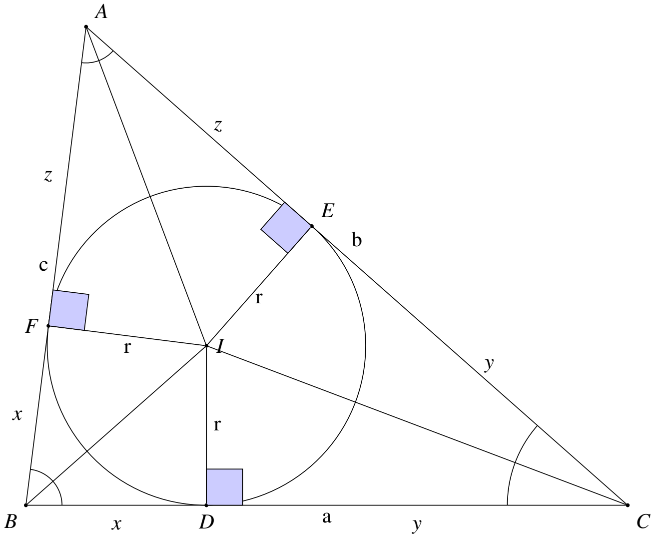

Fig. 5.4.1: Incircle of $\triangle ABC$

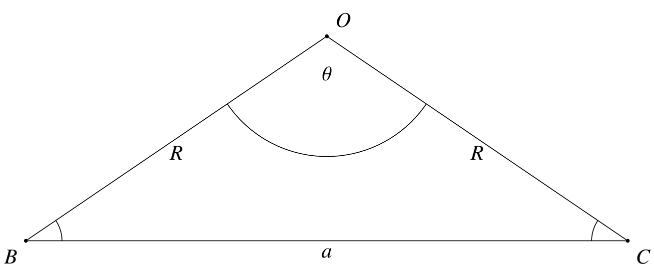

Fig. 6.1.1: Isosceles Triangle

Such a triangle is known as an isosceles triangle. Show that

$$\angle B = \angle C \tag{6.1.2}$$

Solution: Using (2.2.3),

$$\frac{\sin B}{R} = \frac{\sin C}{R} \tag{6.1.3}$$

$$\implies \sin B = \sin C \tag{6.1.4}$$

$$\text{or, } \angle B = \angle C. \tag{6.1.5}$$

6.2. In Fig. 6.1.1, show that

$$a = 2R \sin \frac{\theta}{2} \tag{6.2.1}$$

Solution: In $\triangle OBC$, using the cosine formula from (C.2.1),

$$\cos\theta = \frac{R^2 + R^2 - a^2}{2R^2} = 1 - \frac{a^2}{2R^2} \tag{6.2.2}$$

$$\implies \frac{a^2}{2R^2} = 2\sin^2\frac{\theta}{2} \tag{6.2.3}$$

yielding (6.2.1).

6.3. In Fig. 7.1.1, show that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R. \tag{6.3.1}$$

Solution: From (7.4.1) and (6.2.1)

$$a = 2R\sin A \tag{6.3.2}$$

6.4. In Fig. 6.4.1,

$$OB = OC = R, BD = DC. \tag{6.4.1}$$

Show that $OD \perp BC$.

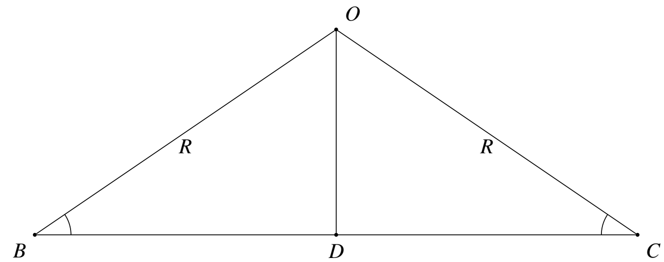

Fig. 6.4.1: Perpendicular bisector.

Solution:

$$\|\mathbf{O} - \mathbf{C}\| = \|\mathbf{O} - \mathbf{B}\| = R \tag{6.4.2}$$

$$\implies \|\mathbf{O} - \mathbf{C}\|^2 = \|\mathbf{O} - \mathbf{B}\|^2 \tag{6.4.3}$$

which can be expressed as

$$(\mathbf{O} - \mathbf{C})^\top (\mathbf{O} - \mathbf{C}) = (\mathbf{O} - \mathbf{B})^\top (\mathbf{O} - \mathbf{B}) \tag{6.4.4}$$

$$\|\mathbf{O}\|^2 - 2\mathbf{O}^\top\mathbf{C} + \|\mathbf{C}\|^2 = \|\mathbf{O}\|^2 - 2\mathbf{O}^\top\mathbf{B} + \|\mathbf{B}\|^2 \tag{6.4.5}$$

$$\implies (\mathbf{B} - \mathbf{C})^\top \mathbf{O} = \frac{\|\mathbf{B}\|^2 - \|\mathbf{C}\|^2}{2} \tag{6.4.6}$$

which can be simplified to obtain

$$(\mathbf{B} - \mathbf{C})^\top \left\{\mathbf{O} - \left(\frac{\mathbf{B} + \mathbf{C}}{2}\right)\right\} = 0 \tag{6.4.7}$$

$$\text{or, } (\mathbf{B} - \mathbf{C})^\top \{\mathbf{O} - \mathbf{D}\} = 0 \tag{6.4.8}$$

which proves the give result using (B.6.3) and (C.6.2).

6.5. In Fig. 6.5.1, OD and OE are the perpendicular bisectors of sides BC and AC respectively. Show that $OA = R$.
Solution: Tracing (6.4.8) backwards yields

$$OB = OC, OC = OA = R. \tag{6.5.1}$$

7 Circumcircle: Circle Equation

7.1. The equation of the circle in Fig. 7.1.1, is

$$\|\mathbf{x} - \mathbf{O}\| = R \tag{7.1.1}$$

This is known as the *circumcircle* of $\triangle ABC$.

7.2. Any point on the circle can be expressed as

$$\mathbf{x} = \mathbf{O} + R\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}, \quad 0 \in [0, 2\pi]. \tag{7.2.1}$$

7.3. Let

$$R = 1, \mathbf{O} = \mathbf{0}, \mathbf{A} = \begin{pmatrix}\cos\theta_1 \\ \sin\theta_1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}\cos\theta_2 \\ \sin\theta_2\end{pmatrix}, \tag{7.3.1}$$

Show that

$$\|\mathbf{A} - \mathbf{B}\| = 2\sin\left(\frac{\theta_1 - \theta_2}{2}\right) \tag{7.3.2}$$

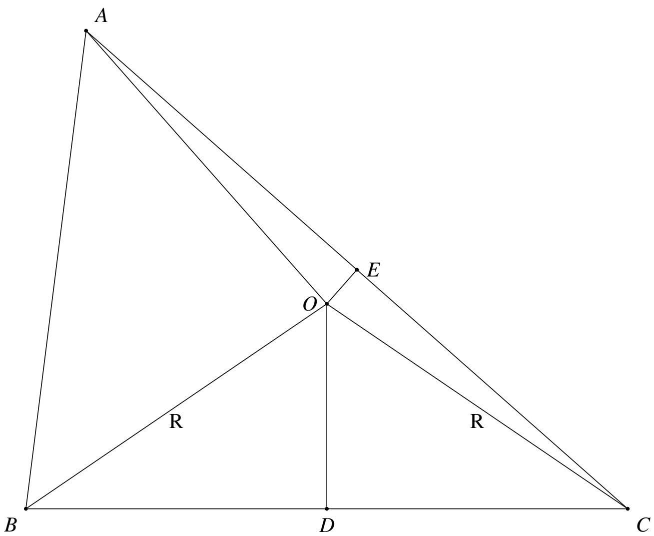

Fig. 6.5.1: Perpendicular bisectors of $\triangle ABC$ meet at $\mathbf{O}$.

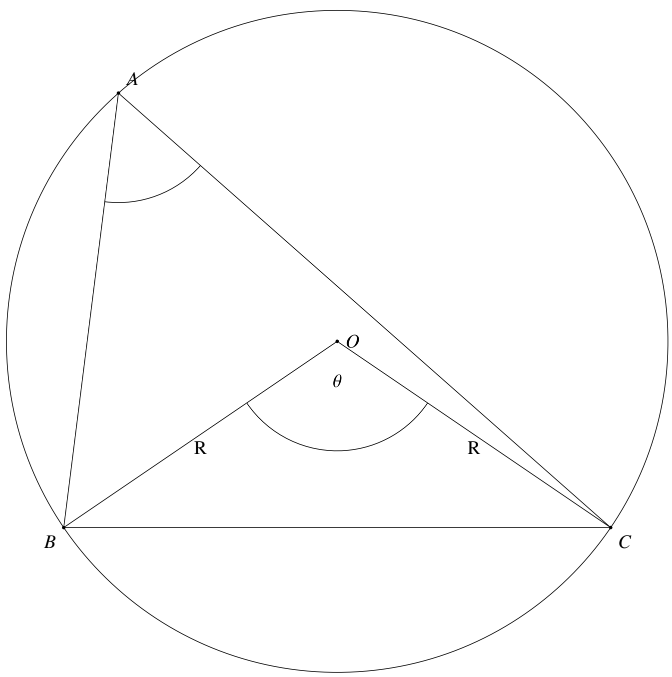

Fig. 7.1.1: Circumcircle of $\triangle ABC$

Solution: From (7.2.1).

$$\mathbf{A}-\mathbf{B} = \begin{pmatrix} \cos\theta_1 - \cos\theta_2 \\ \sin\theta_1 - \sin\theta_2 \end{pmatrix} \tag{7.3.3}$$

$$\implies \|\mathbf{A}-\mathbf{B}\|^2 = (\cos\theta_1 - \cos\theta_2)^2 + (\sin\theta_1 - \sin\theta_2)^2 \tag{7.3.4}$$

$$= 2\left\{1 - \cos(\theta_1 - \theta_2)\right\} = 4\sin^2\left(\frac{\theta_1-\theta_2}{2}\right) \tag{7.3.5}$$

yielding (7.3.2) from (D.4.3).

7.4. In Fig. 7.1.1, show that

$$\theta = 2A. \tag{7.4.1}$$

Solution: Let

$$\mathbf{C} = \begin{pmatrix} \cos\theta_3 \\ \sin\theta_3 \end{pmatrix} \tag{7.4.2}$$

Then, substituting from (7.3.2) in (C.5.2),

$$\cos A = \frac{4\sin^2\left(\frac{\theta_1-\theta_2}{2}\right) + 4\sin^2\left(\frac{\theta_1-\theta_3}{2}\right) - 4\sin^2\left(\frac{\theta_2-\theta_3}{2}\right)}{8\sin\left(\frac{\theta_1-\theta_2}{2}\right)\sin\left(\frac{\theta_1-\theta_3}{2}\right)} \tag{7.4.3}$$

$$= \frac{2\sin^2\left(\frac{\theta_1-\theta_2}{2}\right) + \cos(\theta_2-\theta_3) - \cos(\theta_1-\theta_3)}{4\sin\left(\frac{\theta_1-\theta_2}{2}\right)\sin\left(\frac{\theta_1-\theta_3}{2}\right)} \tag{7.4.4}$$

from (D.4.3). $\therefore$ from (D.3.4),

$$\cos A = \frac{2\sin^2\left(\frac{\theta_1-\theta_2}{2}\right) + 2\sin\left(\frac{\theta_1-\theta_2}{2}\right)\sin\left(\frac{\theta_1+\theta_2}{2}-\theta_3\right)}{4\sin\left(\frac{\theta_1-\theta_2}{2}\right)\sin\left(\frac{\theta_1-\theta_3}{2}\right)} \tag{7.4.5}$$

$$= \frac{\sin\left(\frac{\theta_1-\theta_2}{2}\right) + \sin\left(\frac{\theta_1+\theta_2}{2}-\theta_3\right)}{2\sin\left(\frac{\theta_1-\theta_3}{2}\right)} \tag{7.4.6}$$

From (D.3.1), the above equation can be expressed as

$$\cos A = \frac{2\sin\left(\frac{\theta_1-\theta_3}{2}\right)\cos\left(\frac{\theta_2-\theta_3}{2}\right)}{2\sin\left(\frac{\theta_1-\theta_3}{2}\right)} = \cos\left(\frac{\theta_2-\theta_3}{2}\right) \tag{7.4.7}$$

$$\implies 2A = \theta_2 - \theta_3 \tag{7.4.8}$$

Similarly,

$$\cos\theta = \frac{1+1-4\sin^2\left(\frac{\theta_2-\theta_3}{2}\right)}{2} = \cos(\theta_2-\theta_3) = \cos 2A \tag{7.4.9}$$

8 Tangent

8.1. In Fig. 8.1.1, OC is the radius and PC touches the circle at C. Show that

$$OC \perp PC. \tag{8.1.1}$$

Solution: The equation of PC can be expressed as

$$\mathbf{x} = \mathbf{C} + \mu\mathbf{m} \tag{8.1.2}$$

and the equation of the circle is

$$\|\mathbf{x} - \mathbf{O}\| = R \tag{8.1.3}$$

Substituting (8.1.2) in (8.1.3),

$$\|\mathbf{C} + \mu\mathbf{m} - \mathbf{O}\|^2 = R^2 \tag{8.1.4}$$

$$\implies \mu^2\|\mathbf{m}\|^2 + 2\mu\mathbf{m}^\top(\mathbf{C}-\mathbf{O}) + \|\mathbf{C}-\mathbf{O}\|^2 - R^2 = 0 \tag{8.1.5}$$

The above equation has only one root. Hence the discriminant of the above quadratic should be zero. So,

$$\left\{\mathbf{m}^\top(\mathbf{C}-\mathbf{O})\right\}^2 - \|\mathbf{m}\|^2\left\{\|\mathbf{C}-\mathbf{O}\|^2 - R^2\right\} = 0 \tag{8.1.6}$$

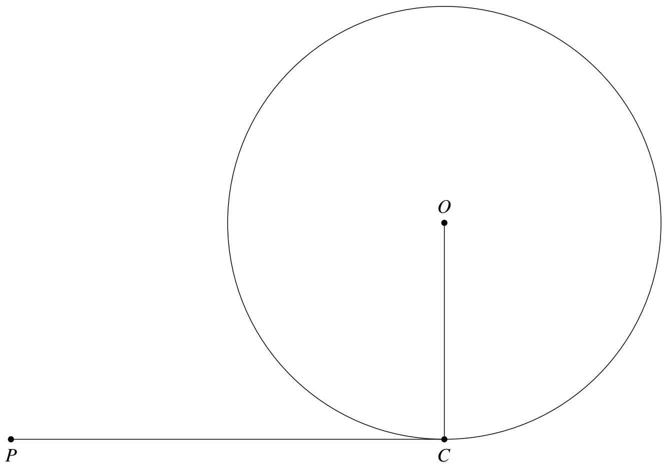

Fig. 8.1.1

Since $\mathbf{C}$ is a point on the circle,

$$\left\|\mathbf{C}-\mathbf{O}\right\|^2 - R^2 = 0 \tag{8.1.7}$$

$$\implies \mathbf{m}^\top\left(\mathbf{C}-\mathbf{O}\right) = 0 \tag{8.1.8}$$

upon substituting in (8.1.6). Using the definition of the direction vector from (B.2.1)

$$\mathbf{m} = \mathbf{P}-\mathbf{C} \tag{8.1.9}$$

$$\implies \left(\mathbf{P}-\mathbf{C}\right)^\top\left(\mathbf{C}-\mathbf{O}\right) = 0 \tag{8.1.10}$$

which is equivalent to (8.1.1).

8.2. In Fig. 8.2.1 show that

$$\theta = \alpha \tag{8.2.1}$$

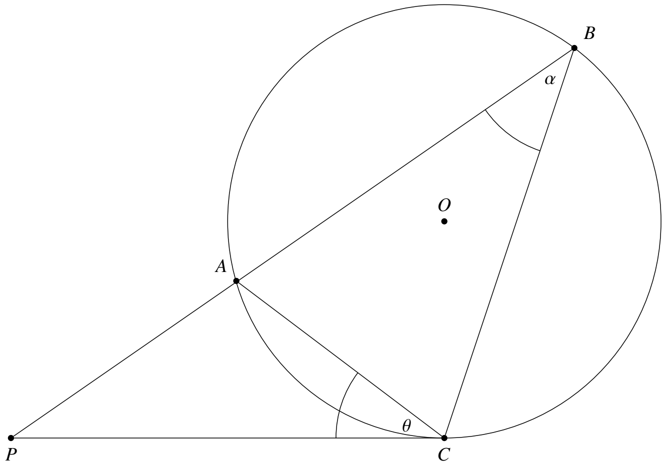

Fig. 8.2.1: $\theta = \alpha$.

Solution: Let Let

$$\mathbf{O} = \mathbf{0}\mathbf{A} = \begin{pmatrix}\cos\theta_1\\ \sin\theta_1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}\cos\theta_2\\ \sin\theta_2\end{pmatrix}, \mathbf{C} = \begin{pmatrix}\cos\theta_3\\ \sin\theta_3\end{pmatrix} \tag{8.2.2}$$

Without loss of generality, let

$$\theta_3 = \frac{\pi}{2} \tag{8.2.3}$$

Then,

$$\mathbf{C}-\mathbf{O} = \begin{pmatrix}0\\1\end{pmatrix} \tag{8.2.4}$$

From from (8.1.10),

$$\mathbf{C}-\mathbf{P} = \begin{pmatrix}1\\0\end{pmatrix}, \tag{8.2.5}$$

From (C.5.1) and (8.2.5),

$$\cos\theta = \frac{\begin{pmatrix}\cos\theta_3-\cos\theta_1 & \sin\theta_3-\sin\theta_1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}}{2\sin\left(\frac{\theta_1-\theta_3}{2}\right)} \tag{8.2.6}$$

$$= \sin\left(\frac{\theta_1+\theta_3}{2}\right) = \cos\left(\frac{\pi}{2}-\frac{\theta_1+\theta_3}{2}\right) = \cos\left(\frac{\pi}{4}-\frac{\theta_1}{2}\right) \tag{8.2.7}$$

upon substituting from (8.2.3). Similarly, from (7.4.7),

$$\cos\alpha = \cos\left(\frac{\theta_1-\theta_3}{2}\right) = \cos\left(\frac{\pi}{4}-\frac{\theta_1}{2}\right) = \cos\theta \tag{8.2.8}$$

8.3. In Fig. 8.2.1, show that $PA.PB = PC^2$.
Solution: In $\triangle$s APC and BPC, using (8.2.1),

$$\frac{AP}{\sin\theta} = \frac{AC}{\sin P} \tag{8.3.1}$$

$$\frac{PC}{\sin\theta} = \frac{BC}{\sin P} \tag{8.3.2}$$

$$\implies \frac{PC}{AP} = \frac{BC}{AC}\left(=\frac{BP}{CP}\right) \tag{8.3.3}$$

which gives the desired result. $\triangle$s APC and BPC are said to be *similar*.

Appendix A
Ratios

A right angled triangle looks like Fig. 3.1. with angles

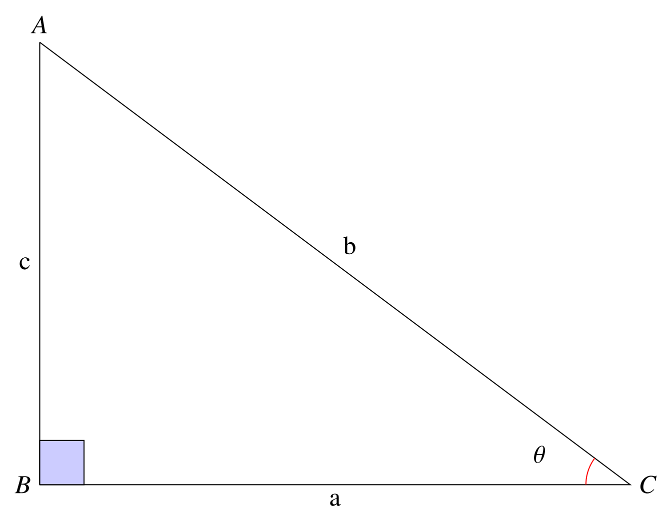

Fig. 3.1: Right Angled Triangle

$\angle A, \angle B$ and $\angle C$ and sides a, b and c. The unique feature of this triangle is $\angle B$ which is defined to be $90°$.

A.1. For simplicity, let the greek letter $\theta = \angle C$. We have the following definitions.

$$\begin{aligned} \sin\theta &= \tfrac{c}{b} & \cos\theta &= \tfrac{a}{b} \\ \tan\theta &= \tfrac{c}{q} & \cot\theta &= \tfrac{1}{\tan\theta} \\ \csc\theta &= \tfrac{1}{\sin\theta} & \sec\theta &= \tfrac{1}{\cos\theta} \end{aligned} \tag{A.1.1}$$

A.2. Show that

$$\cos\theta = \sin(90^\circ - \theta) \tag{A.2.1}$$

Solution: From (A.1.1),

$$\cos\angle BAC = \cos\alpha = \cos(90^\circ - \theta) = \frac{c}{b} = \sin\angle ABC = \sin\theta \tag{A.2.2}$$

Appendix B
Collinear Points

B.1. In Fig. B.1.1,

$$a = y\cot\theta + x \tag{B.1.1}$$

$$\implies \mathbf{D} = \begin{pmatrix} -x \\ y \end{pmatrix} = \begin{pmatrix} -a + y\cot\theta \\ y \end{pmatrix} \tag{B.1.2}$$

$$= \begin{pmatrix} -a \\ 0 \end{pmatrix} + y\cot\theta\begin{pmatrix} 1 \\ \tan\theta \end{pmatrix} \tag{B.1.3}$$

$$\text{or, } \mathbf{D} \equiv \mathbf{B} + \kappa\mathbf{m} \tag{B.1.4}$$

The above equation can be generalized for any point on the line AB as

$$\mathbf{x} = \mathbf{B} + \kappa\mathbf{m} \tag{B.1.5}$$

which is known as the *parametric* equation of a line. $\mathbf{m}$ is defined to be the *direction vector* of AB and

$$m = \tan\theta \tag{B.1.6}$$

is defined to be the *slope*.

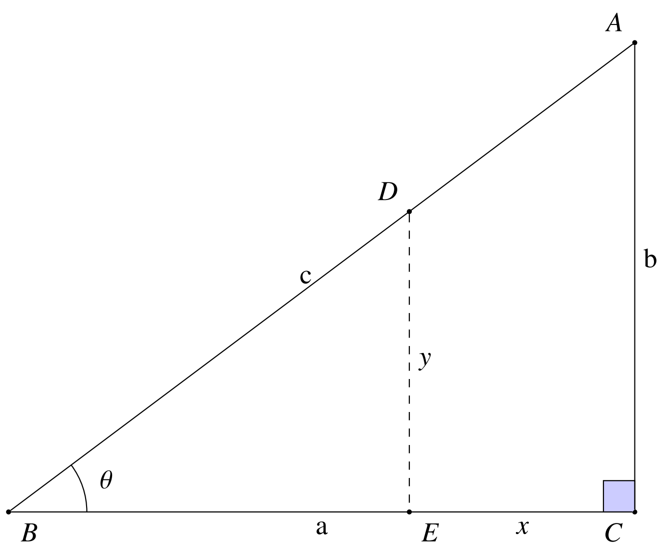

Fig. B.1.1: $k_1 = k_2 = 2$.

B.2. The direction vector of the line AB is

$$\mathbf{A} - \mathbf{B} \equiv \mathbf{B} - \mathbf{A} \equiv \kappa\begin{pmatrix} 1 \\ m \end{pmatrix}, \tag{B.2.1}$$

B.3. (B.1.1) can also be expressed as

$$a = y\cot\theta + x \tag{B.3.1}$$

$$\implies \begin{pmatrix} -\tan\theta & 1 \end{pmatrix}\begin{pmatrix} -x \\ y \end{pmatrix} = b \tag{B.3.2}$$

$$\text{or, } \mathbf{n}^\top\mathbf{x} = b \tag{B.3.3}$$

which is known as the *normal* equation of a line. Here,

$$\mathbf{n} = \begin{pmatrix} -m \\ 1 \end{pmatrix} \tag{B.3.4}$$

is defined to be the *normal vector* of the line. The vector product in (B.3.2) is known as the *inner product* or *dot product*

B.4. It is easy to verify that

$$\mathbf{n}^\top\mathbf{m} = 0 \tag{B.4.1}$$

and

B.5.

$$\mathbf{n} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\mathbf{m} = \begin{pmatrix} \cos\left(\frac{\pi}{2}\right) & \sin\left(\frac{\pi}{2}\right) \\ \sin\left(\frac{\pi}{2}\right) & \cos\left(\frac{\pi}{2}\right) \end{pmatrix}\mathbf{m} \tag{B.5.1}$$

The matrix

$$\mathbf{R}_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{B.5.2}$$

is defined to be the *rotation matrix*. (B.5.1) implies that $\mathbf{n}$ can be obtained from $\mathbf{m}$ through a 90° clockwise rotation.

B.6. From (B.1.5), since $\mathbf{A}, \mathbf{D}$ and $\mathbf{C}$ are on the same line,

$$\begin{aligned} \mathbf{D} &= \mathbf{A} + q\mathbf{m} \\ \mathbf{B} &= \mathbf{D} + p\mathbf{m} \end{aligned} \tag{B.6.1}$$

$$\implies p(\mathbf{D} - \mathbf{A}) + q(\mathbf{D} - \mathbf{B}) = 0, \quad p, q \neq 0 \tag{B.6.2}$$

$$\implies \mathbf{D} = \frac{k\mathbf{A} + \mathbf{B}}{k+1}, \quad k = \frac{p}{q}. \tag{B.6.3}$$

which is known as *section formula*. $(\mathbf{D} - \mathbf{A}), (\mathbf{D} - \mathbf{B})$ are then said to be *linearly dependent*.

B.7. Consequently, points $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ form a triangle if

$$p(\mathbf{A} - \mathbf{B}) + q(\mathbf{C} - \mathbf{B}) \tag{B.7.1}$$

$$= (p+q)\mathbf{B} - p\mathbf{A} - q\mathbf{C} = 0 \tag{B.7.2}$$

$$\implies p = 0, q = 0 \tag{B.7.3}$$

Appendix C
Cosine Formula

C.1. In Fig. C.1.1, show that

$$\begin{pmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{pmatrix}\begin{pmatrix} \cos A \\ \cos B \\ \cos C \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{C.1.1}$$

Solution: From Fig. C.1.1,

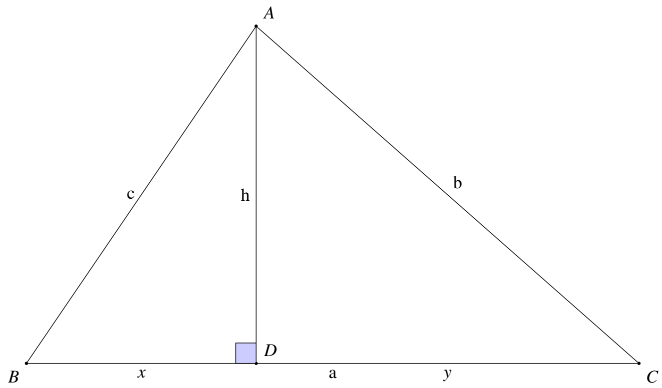

Fig. C.1.1: The cosine formula

$$a = x + y = b\cos C + c\cos B = \begin{pmatrix}\cos C & \cos B\end{pmatrix}\begin{pmatrix}b\\c\end{pmatrix} \tag{C.1.2}$$

$$= \begin{pmatrix}0 & b & c\end{pmatrix}\begin{pmatrix}\cos A\\ \cos C\\ \cos B\end{pmatrix} \tag{C.1.3}$$

Similarly,

$$b = c\cos A + a\cos C = \begin{pmatrix}c & 0 & a\end{pmatrix}\begin{pmatrix}\cos A\\ \cos C\\ \cos B\end{pmatrix} \tag{C.1.4}$$

$$c = b\cos A + a\cos B = \begin{pmatrix}b & a & 0\end{pmatrix}\begin{pmatrix}\cos A\\ \cos C\\ \cos B\end{pmatrix} \tag{C.1.5}$$

The above equations can be expressed in matrix form as (C.1.1).

C.2. Show that

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \tag{C.2.1}$$

Solution: Using the properties of determinants,

$$\cos A = \frac{\begin{vmatrix} a & c & b\\ b & 0 & a\\ c & a & 0\end{vmatrix}}{\begin{vmatrix} 0 & c & b\\ c & 0 & a\\ b & a & 0\end{vmatrix}} = \frac{ab^2 + ac^2 - a^3}{abc + abc} = \frac{b^2 + c^2 - a^2}{2abc} \tag{C.2.2}$$

C.3. The *norm* of $\mathbf{A}$ is defined as

$$\|\mathbf{A}\| = \sqrt{\mathbf{A}^\top\mathbf{A}} = \sqrt{a_1^2 + a_2^2} \tag{C.3.1}$$

C.4. In Fig. 3.1, it is easy to verify that

$$\|\mathbf{A} - \mathbf{C}\|^2 = \begin{pmatrix}-c & a\end{pmatrix}\begin{pmatrix}-c\\a\end{pmatrix} = a^2 + c^2 = b^2 \tag{C.4.1}$$

from (1.4.1). Thus, the distance betwen any two points $\mathbf{A}$ and $\mathbf{B}$ is given by

$$\|\mathbf{A} - \mathbf{B}\| \tag{C.4.2}$$

C.5. In Fig. C.1.1 show that

$$\cos A = \frac{(\mathbf{A} - \mathbf{B})^\top(\mathbf{A} - \mathbf{C})}{\|\mathbf{A} - \mathbf{B}\|\,\|\mathbf{A} - \mathbf{C}\|} \tag{C.5.1}$$

Solution: From (C.2.1), using (C.4.2),

$$\cos A = \frac{\|\mathbf{A} - \mathbf{B}\|^2 + \|\mathbf{A} - \mathbf{C}\|^2 - \|\mathbf{B} - \mathbf{C}\|^2}{2\,\|\mathbf{A} - \mathbf{B}\|\,\|\mathbf{A} - \mathbf{C}\|} \tag{C.5.2}$$

$$= \frac{\|\mathbf{A}\|^2 - \mathbf{A}^\top\mathbf{B} - \mathbf{A}^\top\mathbf{C} + \mathbf{B}^\top\mathbf{C}}{\|\mathbf{A} - \mathbf{B}\|\,\|\mathbf{A} - \mathbf{C}\|} \tag{C.5.3}$$

which can be expressed as (C.5.1).

C.6. For $A = 90°$,

$$\cos A = 0 \tag{C.6.1}$$

$$\implies (\mathbf{A} - \mathbf{B})^\top(\mathbf{A} - \mathbf{C}) = 0 \tag{C.6.2}$$

from (C.5.1).

Appendix D
Trigonometric Identities

D.1. Prove the following identities

a)

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta. \tag{D.1.1}$$

b)

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta. \tag{D.1.2}$$

Solution: In (2.4.1), let

$$\begin{aligned}\theta_1 + \theta_2 &= \alpha\\ \theta_2 &= \beta\end{aligned} \tag{D.1.3}$$

This gives (D.1.1). In (D.1.1), replace α by $90° - \alpha$. This results in

$$\sin(90° - \alpha - \beta) = \sin(90° - \alpha)\cos\beta - \cos(90° - \alpha)\sin\beta \tag{D.1.4}$$

$$\implies \cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \tag{D.1.5}$$

D.2. Using (2.4.1) and (D.1.2), show that

$$\sin(\theta_1 + \theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 \tag{D.2.1}$$

$$\cos(\theta_1 - \theta_2) = \cos\theta_1\cos\theta_2\sin\theta_1\sin\theta_2 \tag{D.2.2}$$

Solution: From (2.4.1),

$$\sin(\theta_1 + \theta_2)\cos\theta_2 = \sin\theta_1 + \cos(\theta_1 + \theta_2)\sin\theta_2 \tag{D.2.3}$$

Using (D.1.2) in the above,

$$\begin{aligned}\sin(\theta_1 + \theta_2)\cos\theta_2 = \sin\theta_1 &+ (\cos\theta_1\cos\theta_2\\ &- \sin\theta_1\sin\theta_2)\sin\theta_2\end{aligned} \tag{D.2.4}$$

which can be expressed as

$$\begin{aligned}\sin(\theta_1 + \theta_2)\cos\theta_2 &= \sin\theta_1\\ &+ \cos\theta_1\cos\theta_2\sin\theta_2 - \sin\theta_1\sin^2\theta_2\end{aligned} \tag{D.2.5}$$

Since

$$\sin^2\theta_2 = 1 - \cos^2\theta_2, \tag{D.2.6}$$

we obtain

$$\sin(\theta_1 + \theta_2)\cos\theta_2 = \cos\theta_1 \cos\theta_2 \sin\theta_2 + \sin\theta_1 \cos^2\theta_2 \tag{D.2.7}$$

resulting in

$$\sin(\theta_1 + \theta_2) = \cos\theta_1 \sin\theta_2 + \sin\theta_1 \cos\theta_2 \tag{D.2.8}$$

after factoring out $\cos\theta_2$. Using a similar approach, (D.2.2) can also be proved.

D.3. Show that

$$\sin\theta_1 + \sin\theta_2 = 2\sin\left(\frac{\theta_1 + \theta_2}{2}\right)\cos\left(\frac{\theta_1 - \theta_2}{2}\right) \tag{D.3.1}$$

$$\cos\theta_1 + \cos\theta_2 = 2\cos\left(\frac{\theta_1 + \theta_2}{2}\right)\cos\left(\frac{\theta_1 - \theta_2}{2}\right) \tag{D.3.2}$$

$$\sin\theta_1 - \sin\theta_2 = 2\sin\left(\frac{\theta_1 - \theta_2}{2}\right)\cos\left(\frac{\theta_1 + \theta_2}{2}\right) \tag{D.3.3}$$

$$\cos\theta_1 - \cos\theta_2 = 2\sin\left(\frac{\theta_1 + \theta_2}{2}\right)\cos\left(\frac{\theta_2 - \theta_1}{2}\right) \tag{D.3.4}$$

Solution: Let

$$\begin{aligned} \theta_1 &= \alpha + \beta \\ \theta_2 &= \alpha - \beta \end{aligned} \tag{D.3.5}$$

From (D.2.1),

$$\begin{aligned} \sin\theta_1 + \sin\theta_2 &= \sin(\alpha + \beta) + \sin(\alpha - \beta) && \text{(D.3.6)} \\ &= \sin\alpha\cos\beta + \cos\alpha\sin\beta && \text{(D.3.7)} \\ &+ \sin\alpha\cos\beta - \cos\alpha\sin\beta && \text{(D.3.8)} \\ &= 2\sin\alpha\cos\beta && \text{(D.3.9)} \end{aligned}$$

resulting in (D.3.1)

$$\because \alpha = \frac{\theta_1 + \theta_2}{2} \tag{D.3.10}$$

$$\beta = \frac{\theta_1 - \theta_2}{2} \tag{D.3.11}$$

from (D.3.5). Other identities may be proved similarly.

D.4. Show that

$$\sin 2\theta = 2\sin\theta\cos\theta \tag{D.4.1}$$

$$\cos 2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1 \tag{D.4.2}$$

$$= \cos^2\theta - \sin^2\theta \tag{D.4.3}$$