GATE Problems in Probability

***Abstract*—These problems have been selected from GATE question papers and can be used for conducting tutorials in courses related to a first course in probability.**

1) Let X be a random variable with the following cumulative distribution function:

$$F(x) = \begin{cases} 0 & x < 0 \\ x^2 & 0 \le x < \frac{1}{2} \\ \frac{3}{4} & \frac{1}{2} \le x < 1 \\ 1 & x \ge 1 \end{cases} \tag{1}$$

Then $P\left(\frac{1}{4} < X < 1\right)$ is equal to
Solution:

$$P\left(\frac{1}{4} < X < 1\right) = F\left(1^-\right) - F\left(\frac{1}{4}\right) \tag{2}$$

$$= \frac{3}{4} - \left(\frac{1}{4}\right)^2 \tag{3}$$

$$= \frac{11}{16} \tag{4}$$

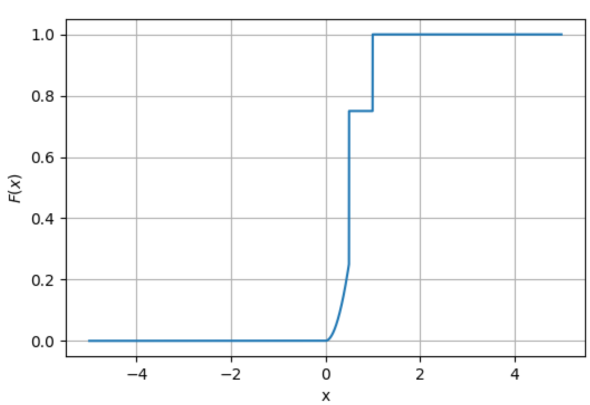

Fig. 1: The CDF of X

2) Let X and Y be continuous random variables with the joint probability density function

$$f(x, y) = \begin{cases} ae^{-2y} & 0 < x < y < \infty \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Then $E(X|Y = 2)$ is ... **Solution:** Given X and Y are two continuous random variables with joint probability density function,

$$f(x, y) = \begin{cases} ae^{-2y} & 0 < x < y < \infty \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

We know that,
$0 < x < y < \infty \implies x < y < \infty$ for $0 < x < \infty$.
Then,

$$f_X(x) = \int f_{XY}(x, y)\, dy \tag{7}$$

$$= \int_x^\infty ae^{-2y} dy \tag{8}$$

$$= \left[\frac{ae^{-2y}}{(-2)}\right]_x^\infty \tag{9}$$

$$= \frac{-a}{2}\left[e^{-2y}\right]_x^\infty \tag{10}$$

$$= \frac{-a}{2}\left[0 - e^{-2x}\right] \tag{11}$$

$$\implies f_X(x) = \begin{cases} \frac{a}{2}e^{-2x} & 0 < x < \infty \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

Similarly,
$0 < x < y < \infty \implies 0 < x < y$ for $0 < y < \infty$
Then,

$$f_y(y) = \int f_{XY}(x, y)\, dx \tag{13}$$

$$= \int_0^y ae^{-2y} dx \tag{14}$$

$$= ae^{-2y}[x]_0^y \tag{15}$$

$$= aye^{-2y} \tag{16}$$

$$\implies f_Y(y) = \begin{cases} aye^{-2y} & 0 < y < \infty \\ 0 & \text{otherwise.} \end{cases} \tag{17}$$

Therefore ,

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} \tag{18}$$
$$= \frac{ae^{-2y}}{aye^{-2y}} \tag{19}$$
$$= \frac{1}{y} \tag{20}$$
$$\implies f_{X|Y}(x|y) = \begin{cases} \frac{1}{y} & \text{if } 0 < x < y < \infty \\ 0 & \text{otherwise} \end{cases} \tag{21}$$

Then,

$$E(X|Y=y) = \int_{-\infty}^{\infty} (x) f_{X|Y}(x|y)\, dx \tag{22}$$
$$= \int_0^y (x)\left(\frac{1}{y}\right) dx \tag{23}$$
$$= \frac{1}{y}\int_0^y (x) dx \tag{24}$$
$$= \frac{1}{y}\left[\frac{x^2}{2}\right]_0^y \tag{25}$$
$$= \frac{1}{y}\left(\frac{y^2}{2}\right) \tag{26}$$
$$= \frac{y}{2} \tag{27}$$
$$\implies E(X|Y=y) = \frac{y}{2} \tag{28}$$
$$\therefore E(X|Y=2) = 1 \tag{29}$$

3) A continuous random variable X has the probability density function

$$f(x) = \begin{cases} \frac{3}{5}e^{-\frac{3}{5}x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

The probability density function of $Y = 3X + 2$ is

a)

$$f(y) = \begin{cases} \frac{1}{5}e^{-\frac{1}{5}(y-2)} & y > 2 \\ 0 & y \leq 2 \end{cases}$$

b)

$$f(y) = \begin{cases} \frac{2}{5}e^{-\frac{2}{5}(y-2)} & y > 2 \\ 0 & y \leq 2 \end{cases}$$

c)

$$f(y) = \begin{cases} \frac{3}{5}e^{-\frac{3}{5}(y-2)} & y > 2 \\ 0 & y \leq 2 \end{cases}$$

d)

$$f(y) = \begin{cases} \frac{4}{5}e^{-\frac{4}{5}(y-2)} & y > 2 \\ 0 & y \leq 2 \end{cases}$$

Solution: Given $Y = 3X + 2$
CDF of Y,

$$F_y(Y) = \Pr(Y \leq y)$$
$$= \Pr\left(X \leq \frac{y-2}{3}\right)$$
$$= F_x\left(\frac{y-2}{3}\right)$$

Thus, pdf of Y ,

$$f_Y(y) = \frac{1}{3} f_X\left(\frac{y-2}{3}\right)$$
$$= \frac{1}{3} \times \begin{cases} \frac{3}{5}e^{-\frac{3}{5}\left(\frac{y-2}{3}\right)} & y > 2 \\ 0 & y \leq 2 \end{cases}$$
$$= \begin{cases} \frac{1}{5}e^{-\frac{1}{5}(y-2)} & y > 2 \\ 0 & y \leq 2 \end{cases}$$

Hence, correct option is 1.

4) Let the probability density function of a random variable X be

$$f(x) = \begin{cases} x & 0 \leqslant x < \frac{1}{2} \\ c(2x-1)^2 & \frac{1}{2} \leqslant x < 1 \\ 0 & \text{Otherwise} \end{cases}$$

Then value of c is equal to ...

Solution: We know that,

$$\int_{-\infty}^{\infty} f_x(x)\,dx = 1 \tag{30}$$

$$\int_{-\infty}^{0} f_x(x)\,dx + \int_{0}^{\frac{1}{2}} f_x(x)\,dx$$
$$+ \int_{\frac{1}{2}}^{1} f_x(x)\,dx + \int_{1}^{\infty} f_x(x)\,dx = 1 \tag{31}$$

$$\int_{0}^{\frac{1}{2}} x\,dx + \int_{\frac{1}{2}}^{1} c(2x-1)^2\,dx = 1 \tag{32}$$

$$\left[\frac{x^2}{2}\right]_0^{\frac{1}{2}} + c\left[\frac{(2x-1)^3}{6}\right]_{\frac{1}{2}}^{1} = 1 \tag{33}$$

$$\frac{1}{8} + \frac{c}{6} = 1 \tag{34}$$

$$c = \frac{21}{4} \tag{35}$$

$$\therefore \text{Required value of } c = \frac{21}{4}$$

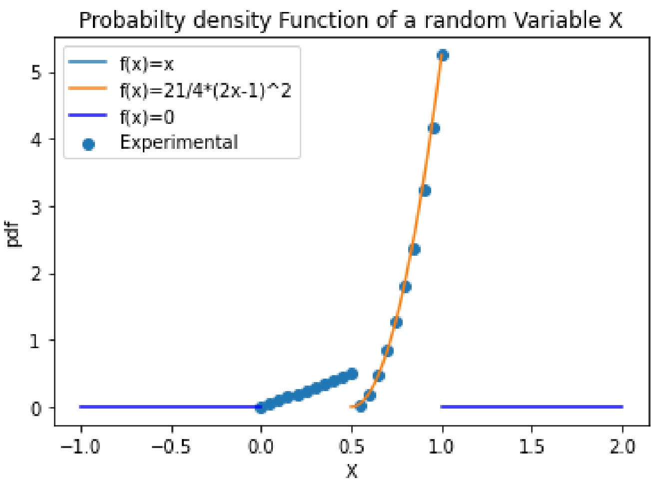

Fig. 2: Experimental and Theoritical pdf of X

5) Let $A_1, A_2, ...A_n$ be n independent events in which the Probability of occurence of the event A_i is given by P(A_i) = 1 - $\frac{1}{\alpha^i}$, $\alpha > 1$, i = 1,2,3,....n.Then the probability that atleast one of the events occurs is

a) 1 - $\frac{1}{\alpha^{\frac{n(n+1)}{2}}}$

b) $\frac{1}{\alpha^{\frac{n(n+1)}{2}}}$

c) $\frac{1}{\alpha^n}$

d) 1 - $\frac{1}{\alpha^n}$

Solution: Let $A_1 + A_2 + A_3.... + A_n$ = S,
$\Pr(S)$ = Probability of atleast one event occuring De morgan's law states that $(A+B)' = A'B'$

$$\implies \Pr(S) = 1 - \Pr(S') \tag{36}$$

$$1 - \Pr(S') = 1 - \Pr(A_1'A_2'A_3'....A_n') \tag{37}$$

$\forall$ i $\in$ 1,2,....n
Since, A_i are independent.
$\therefore$ Complements of A_i are also independent.
$\implies$

$$\Pr(A_1'A_2'A_3'....A_n') = \prod_{i=1}^{n} \Pr(A_i') \tag{38}$$

$$\Pr(A_i) = 1 - \frac{1}{\alpha^i} \implies \Pr(A_i') = \frac{1}{\alpha^i} \tag{39}$$

substituting (39) in (38),

$$\Pr(A_1'A_2'A_3'....A_n') = \prod_{i=1}^{n} \frac{1}{\alpha^i} \tag{40}$$

$$\prod_{i=1}^{n} \frac{1}{\alpha^i} = \frac{1}{\alpha^{\sum_i^n i}} = \frac{1}{\alpha^{\frac{n(n+1)}{2}}} \tag{41}$$

$$\therefore \Pr(A_1'A_2'A_3....A_n') = \Pr(S') = \frac{1}{\alpha^{\frac{n(n+1)}{2}}} \tag{42}$$

from equations (37) and (42)

$$\implies \Pr(S) = 1 - \Pr(S') = 1 - \frac{1}{\alpha^{\frac{n(n+1)}{2}}} \tag{43}$$

$\therefore$ The correct option is **(a)**

6) Let the random variable X have the distribution function
$$F(x) = \begin{cases} 0 & if\ x < 0 \\ \frac{x}{2} & if\ 0 \le x < 1 \\ \frac{3}{5} & if\ 1 \le x < 2 \\ \frac{1}{2} + \frac{x}{8} & if\ 2 \le x < 3 \\ 1 & if\ x \ge 3 \end{cases}$$
Then $\Pr(2 \le x < 4)$ is equal to
Solution:

Given,

$$F(x) = \begin{cases} 0 & if\ x < 0 \\ \frac{x}{2} & if\ 0 \le x < 1 \\ \frac{3}{5} & if\ 1 \le x < 2 \\ \frac{1}{2} + \frac{x}{8} & if\ 2 \le x < 3 \\ 1 & if\ x \ge 3 \end{cases} \quad (44)$$

We need to find $\Pr(2 \le x < 4)$,which is also can be written as

$$\Pr(2 \le x < 4) = \Pr(x < 4) - \Pr(x < 2) \quad (45)$$
$$= F(X = 4^-) - F(X = 2^-) \quad (46)$$

Using (44) in (46),

$$\Pr(2 \le x < 4) = 1 - \frac{3}{5} \quad (47)$$
$$= \frac{2}{5} \quad (48)$$
$$= 0.4 \quad (49)$$

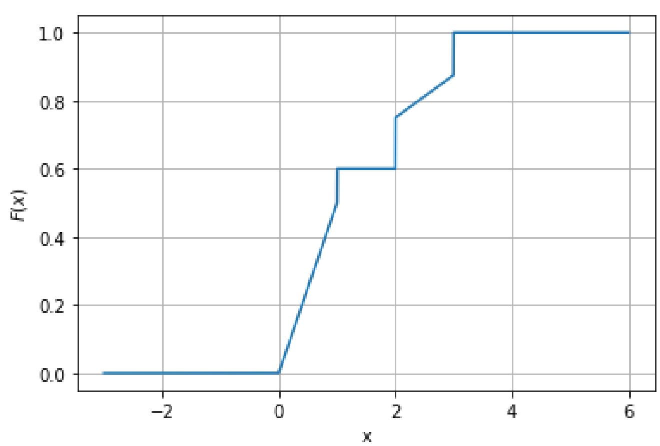

Fig. 3: cdf of random variable X

7) Let Z be the vertical coordinate, between -1 and 1, of a point chosen uniformly at random on the surface of a unit sphere in R^3. Then, $\Pr\left(\frac{-1}{2} \le Z \le \frac{1}{2}\right)$ is
Solution: The equation of the sphere can be written as : $x^2 + y^2 + z^2 = 1$. Now,

$$\Pr\left(\frac{-1}{2} \le z \le 0\right) = \Pr\left(0 \le z^2 \le \frac{1}{4}\right) \quad (50)$$
$$\Pr\left(0 \le z \le \frac{1}{2}\right) = \Pr\left(0 \le z^2 \le \frac{1}{4}\right) \quad (51)$$
$$\therefore \Pr\left(\frac{-1}{2} \le z \le \frac{1}{2}\right) = 2 \times \Pr\left(0 \le z^2 \le \frac{1}{4}\right) \quad (52)$$

$$\Pr\left(0 \le z^2 \le \frac{1}{4}\right) = \Pr\left(\frac{3}{4} \le x^2 + y^2 \le 1\right) \quad (53)$$

Taking, $x^2 + y^2 = r^2$. (54)

$$\Pr\left(\frac{3}{4} \le r^2 \le 1\right) = \frac{1}{4} \quad (55)$$

$\left(\text{Since, } r^2 \text{ is uniform between 0 and 1}\right)$

$$\therefore \Pr\left(\frac{-1}{2} \le Z \le \frac{1}{2}\right) = 2 \times \frac{1}{4} = \frac{1}{2} \quad (56)$$

8) Let X_1 and X_2 be independent geometric random variables with the same probability mass function given by $\Pr(X = k) = p(1-p)^{k-1}$, $k = 1, 2, \ldots$ Then the value of $\Pr(X_1 = 2 | X_1 + X_2 = 4)$ correct up to three decimal places is
Solution: Let

$$p_{X_i}(k) = \Pr(X_i = k) = \begin{cases} p(1-p)^{k-1} & n = 1, 2, ... \\ 0 & otherwise \end{cases} \quad (57)$$

where i=1,2

$$\Pr(A|B) = \frac{\Pr(AB)}{\Pr(B)} \quad (58)$$

$$(X_1 = 2) \cap (X_1 + X_2 = 4) = (X_1 = 2, X_2 = 2) \quad (59)$$

Thus,

$$\Pr(X_1 = 2 | X_1 + X_2 = 4) = \frac{\Pr(X_1 = 2, X_2 = 2)}{\Pr(X_1 + X_2 = 4)} \quad (60)$$

Since the two events are independent,

$$\Pr(X_1 = 2 | X_1 + X_2 = 4) = \frac{\Pr(X_1 = 2)\Pr(X_2 = 2)}{\Pr(X_1 + X_2 = 4)} \tag{61}$$

Let

$$X = X_1 + X_2 \tag{62}$$

From (62),

$$p_X(n) = \Pr(X_1 + X_2 = n) = \Pr(X_1 = n - X_2) \tag{63}$$

$$= \sum_k \Pr(X_1 = n - k | X_2 = k)\, p_{X_2}(k) \tag{64}$$

after unconditioning. $\because X_1$ and X_2 are independent,

$$\Pr(X_1 = n - k | X_2 = k) = \Pr(X_1 = n - k) = p_{X_1}(n - k) \tag{65}$$

From (64) and (65),

$$p_X(n) = \sum_k p_{X_1}(n-k) p_{X_2}(k) = p_{X_1}(n) * p_{X_2}(n) \tag{66}$$

where $*$ denotes the convolution operation. Substituting from (57) in (66),

$$p_X(n) = \sum_{k=1}^{n-1} p_{X_1}(n-k) p_{X_2}(k) \tag{67}$$

$$= \sum_{k=1}^{n-1} (1-p)^{k-1} p \cdot (1-p)^{n-k-1} p \tag{68}$$

$$= (1-p)^{n-2} p^2 \sum_{k=1}^{n-1} 1 \tag{69}$$

$$= (n-1)(1-p)^{n-2} p^2 \tag{70}$$

From (70) and (57) we have

$$\Pr(X_1 = 2) = \Pr(X_2 = 2) = p(1-p) \tag{71}$$

$$\Pr(X_1 + X_2 = 4) = 3(1-p)^2 p^2 \tag{72}$$

Substituting in (61)

$$\Pr(X_1 = 2 | X_1 + X_2 = 4) = \frac{(1-p)^2 p^2}{3(1-p)^2 p^2} \tag{73}$$

$$= \frac{1}{3} \tag{74}$$

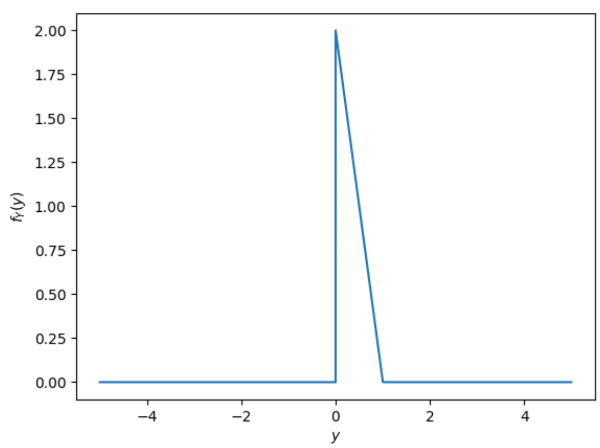

Fig. 4: Marginal PDF

9) Let X and Y have joint probability function given by

$$f_{X,Y}(x,y) = \begin{cases} 2 & 0 \le x \le 1-y, 0 \le y \le 1 \\ 0 & otherwise \end{cases}$$

If f_Y denotes the marginal probability density function of Y, then $f_Y(1/2) = ?$

Solution:

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y).dx \tag{23.1}$$

$$\implies f_Y(y) = \begin{cases} 0 + \int_0^{1-y} 2.dx & 0 \le y \le 1 \\ 0 & otherwise \end{cases} \tag{23.2}$$

$$\implies f_Y(y) = \begin{cases} 2(1-y) & 0 \le y \le 1 \\ 0 & otherwise \end{cases} \tag{23.3}$$

$$\therefore f_Y(1/2) = 1 \tag{23.4}$$

10) Let X be a standard normal random variable. Then $\Pr(X < 0 | \ \lfloor X \rfloor = 1)$ is equal to

a) $\frac{\Phi(1) - \frac{1}{2}}{\Phi(2) - \frac{1}{2}}$

b) $\frac{\Phi(1) + \frac{1}{2}}{\Phi(2) + \frac{1}{2}}$

c) $\dfrac{\Phi(1)-\frac{1}{2}}{\Phi(2)+\frac{1}{2}}$

d) $\dfrac{\Phi(1)+1}{\Phi(2)+1}$

Solution:

$$|\lfloor X\rfloor| = 1 \quad (75)$$
$$\implies \lfloor X\rfloor = 1 \text{ or } -1 \quad (76)$$
$$\implies X \in [1,2)\cup[-1,0) \quad (77)$$

Here

$\lfloor X\rfloor = \textit{greatest integer less than or equal to } X$

Thus required probability

$$= \frac{\Pr(X\in[-1,0))}{\Pr(X\in[1,2)\cup[-1,0))} \quad (78)$$

Using symmetry of standard normal random variable about y = 0, we have required probability

$$= \frac{\Pr(X\in(0,1])}{\Pr(X\in[1,2)\cup(0,1]))} \quad (79)$$
$$= \frac{\Pr(X\in(0,1])}{\Pr(X\in(0,2))} \quad (80)$$
$$= \frac{\Pr(X<1)-\Pr(X<0)}{\Pr(X<2)-\Pr(X<0)} \quad (81)$$
$$= \frac{\Phi(1)-\Phi(0)}{\Phi(2)-\Phi(0)} \quad (82)$$
$$= \frac{\Phi(1)-\frac{1}{2}}{\Phi(2)-\frac{1}{2}} \quad (83)$$
$$= \frac{0.841-0.5}{0.977-0.5} \quad (84)$$
$$= 0.715 \quad (85)$$

Here $\Phi(x)$ represents the standard normal cumulative density function. Thus

$$X \sim \mathcal{N}1 \quad (86)$$

and

$$\Phi(x) = \int_{-\infty}^{x} f_X(x)dx \quad (87)$$

It can easily be seen that $\Phi(0)=\frac{1}{2}$, which has been used to obtain (83). (84) was obtained by consulting tables for $\Phi(x)$

11) Let X be a random variable with probability mass function $p(n) = \left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^{n-1}$ $n = 1,2\dots$ Then $E[X-3|X>3]$ is ...

Solution:
Given

$$\Pr(X=n) = \begin{cases}\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^{n-1} & n=1,2\dots\\ 0 & \textit{otherwise}\end{cases} \quad (88)$$

Using the linearity of the expectation operator:

$$E[X-3 \mid X>3] = E[X \mid X>3]-3 \quad (89)$$

Now ,

$$E[X \mid X>3] = \sum_{x=1}^{\infty} x\Pr(X=x \mid X>3) \quad (90)$$
$$= \sum_{x=1}^{\infty} x\frac{\Pr(X=x, X>3)}{\Pr(X>3)} \quad (91)$$

Calculating $\Pr(X>3)$

$$\Pr(X>3) = 1-\Pr(X\le 3) \quad (92)$$
$$= 1-\sum_{x'=1}^{3}\Pr(X=x') \quad (93)$$
$$= 1-\sum_{x'=1}^{3}\left(\frac{3}{4}\right)^{x'-1}\left(\frac{1}{4}\right) \quad (94)$$
$$= \frac{27}{64} \quad (95)$$

Also,

$$\Pr(X=x, X>3) = \begin{cases}\Pr(X=x) & x>3\\ 0 & x\le 3\end{cases} \quad (96)$$

Substituting (95) and (96) in (91) we get

$$E[X \mid X>3] = \sum_{x=1}^{3}0+\sum_{x=4}^{\infty}\left[x\frac{\Pr(X=x)}{\frac{27}{64}}\right] \quad (97)$$
$$= \frac{64}{27}\sum_{x=4}^{\infty}\left[x\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^{x-1}\right] \quad (98)$$
$$= \frac{16}{27}\sum_{x=4}^{\infty}\left[x\left(\frac{3}{4}\right)^{x-1}\right] \quad (99)$$

Let

$$S = \sum_{x=4}^{\infty} \left[x \left(\frac{3}{4} \right)^{x-1} \right] \tag{100}$$

Multiplying ((100)) with $\frac{3}{4}$ on both sides gives

$$\frac{3}{4} S = \sum_{x=4}^{\infty} x \frac{1}{4} \left(\frac{3}{4} \right)^{x} \tag{101}$$

From (101) and(100)

$$S = 4 \left(\frac{3}{4} \right)^{3} + 5 \left(\frac{3}{4} \right)^{4} + 6 \left(\frac{3}{4} \right)^{5} \ldots \tag{102}$$

$$\frac{3}{4} S = 0 \left(\frac{3}{4} \right)^{3} + 4 \left(\frac{3}{4} \right)^{4} + 5 \left(\frac{3}{4} \right)^{5} + \ldots \tag{103}$$

subtracting (101) from (100) we get

$$\frac{S}{4} = 4 \left(\frac{3}{4} \right)^{3} + \left(\frac{3}{4} \right)^{4} + \left(\frac{3}{4} \right)^{5} + \left(\frac{3}{4} \right)^{6} + \ldots \tag{104}$$

$$= 4 \left(\frac{3}{4} \right)^{3} + \sum_{x=4}^{\infty} \left(\frac{3}{4} \right)^{x} \tag{105}$$

$$= \frac{189}{64} \tag{106}$$

Substituting vale of S in (99) we get

$$E[X|X > 3] = 7 \tag{107}$$

Thus putting this in (89)

$$E[X - 3|X > 3] = 4 \tag{108}$$

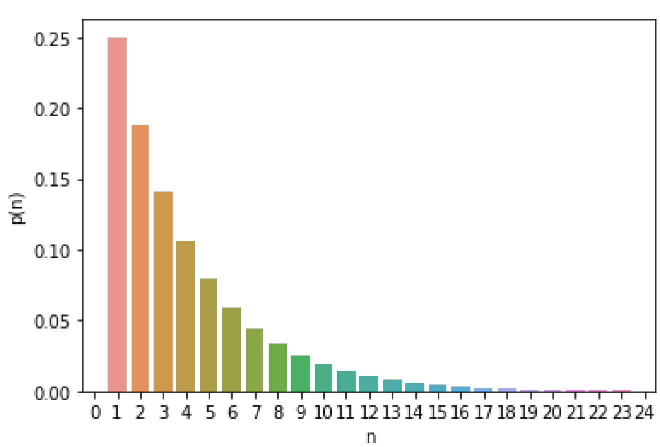

Fig. 5: PMF of X

12) Let (X,Y) be a random vector such that, for any $y > 0$, the conditional probability density function of X given $Y = y$ is

$$f_{X|Y=y}(x) = ye^{-yx}\ , x > 0.$$

If the marginal probability density function of Y is

$$g(y) = ye^{-y}\ , y > 0$$

then $E(Y|x = 1) =$

Solution: Given, the conditional probability density function of X given $Y = y$,

$$f_{X|Y=y}(x) = ye^{-yx}\ , x > 0 \tag{109}$$

and, the marginal probability density function of Y,

$$g(y) = ye^{-y}\ , y > 0 \tag{110}$$

let the joint probability density function of (X,Y) be $f_{X,Y}(x, y)$. We know that,

$$f_{X|Y=y}(x) = \frac{f_{X,Y}(x, y)}{g(y)} \tag{111}$$

using (109) and (110) in (111),

$$f_{X,Y}(x, y) = y^2 e^{-y(x+1)}\ , x, y > 0 \tag{112}$$

let the marginal probability density function of X be $f_X(x)$, as we know ,

$$f_X(x) = \int_0^{\infty} f_{X,Y}(x, y)\, dy \tag{113}$$

using (112) in (113),

$$f_X(x) = \int_0^{\infty} y^2 e^{-y(x+1)}\, dy \tag{114}$$

$$= \frac{2}{(x + 1)^3}\ , x > 0 \tag{115}$$

The conditional probability density function of Y given $X = x$ is given by,

$$f_{Y|X=x}(y) = \frac{f_{X,Y}(x, y)}{f_X(x)} \tag{116}$$

using (112) and (115) in (116),

$$f_{Y|X=x}(y) = \frac{y^2 e^{-y(x+1)}(x + 1)^3}{2}\ , x, y > 0 \tag{117}$$

The conditional probability density function of Y given $X = 1$ is given by,

$$f_{Y|X=1}(y) = 4y^2 e^{-2y}\ , y > 0 \tag{118}$$

We need to find $E(Y|X=1)$ which is given by,

$$E(Y|X=1) = \int_0^\infty y f_{Y|X=1}(y)\,dy \tag{119}$$

using (118) in (119),

$$E(Y|X=1) = \int_0^\infty 4y^3 e^{-2y}\,dy \tag{120}$$

$$= \left[\frac{-e^{-2y}(8y^3+12y^2+12y+6)}{4}\right]_0^\infty \tag{121}$$

$$= \frac{3}{2} \tag{122}$$

13) Let X and Y be jointly distributed random variables having the joint probability density function

$$f(x,y) = \begin{cases} \frac{1}{\pi}, & \text{if} \quad x^2+y^2 \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Then $\Pr(Y > \max(X,-X))$ is
Solution:
The pdf of X and Y are:

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy \tag{123}$$

$$= \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi}dy \tag{124}$$

$$= \frac{2\sqrt{1-x^2}}{\pi} \tag{125}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx \tag{126}$$

$$= \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{\pi}dx \tag{127}$$

$$= \frac{2\sqrt{1-y^2}}{\pi} \tag{128}$$

The cdf of Y is:

$$F_Y(y) = \int_{-\infty}^{y} f_Y(y)dy \tag{129}$$

$$= \int_{-1}^{y} \frac{2\sqrt{1-y^2}}{\pi}dy \tag{130}$$

$$= \frac{2}{\pi}\left(\frac{\sin^{-1}y + y\sqrt{1-y^2}}{2} + \frac{\pi}{4}\right) \tag{131}$$

The value of $\Pr(-X < Y < X)$ is:

$$\Pr(-X < Y < X) = F_Y(X) - F_Y(-X) \tag{132}$$

$$= \frac{2}{\pi}\left(\sin^{-1}X + X\sqrt{1-X^2}\right) \tag{133}$$

Integrating our probability over all of X we get the value of $E[\Pr(-x < Y < x)]$:

$$= \int_{-\infty}^{\infty} f_X(x)\Pr(-x < Y < x)\,dx \tag{134}$$

$$= \left(\frac{2}{\pi}\right)^2 \int_0^1 \sqrt{1-x^2}\left(\sin^{-1}x + x\sqrt{1-x^2}\right)dx \tag{135}$$

Substituting

$$u = \sin^{-1}x + x\sqrt{1-x^2} \tag{136}$$

$$\frac{du}{dx} = 2\sqrt{1-x^2} \tag{137}$$

$$= \left(\frac{2}{\pi}\right)^2 \int_0^{\frac{\pi}{2}} \frac{u}{2}du \tag{138}$$

$$= \left(\frac{2}{\pi}\right)^2 \left(\left.\frac{u^2}{4}\right|_0^{\frac{\pi}{2}}\right) \tag{139}$$

$$= \left(\frac{2}{\pi}\right)^2 \left(\frac{\pi^2}{16} - 0\right) \tag{140}$$

$$= \frac{4\cdot\pi^2}{\pi^2\cdot 16} \tag{141}$$

$$= \frac{1}{4} \tag{142}$$

The probability for:

$$\Pr(Y > \max(X,-X)) = \frac{1}{4} \tag{143}$$

14) Let X and Y be two continuous random variables with the joint probability density function

$$f(x,y) = \begin{cases} 2, & 0 < x+y < 1, x > 0, y > 0, \\ 0, & elsewhere. \end{cases} \tag{144}$$

$E\left(X \mid Y = \frac{1}{2}\right)$ is

a) 1/4

b) 1/2

c) 1

d) 2

Solution:
The PDF of X and Y is,

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy \tag{145}$$

$$f_X(x) = \int_{0}^{1-x} 2dy \tag{146}$$

$$f_X(x) = 2 - 2x \tag{147}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx \tag{148}$$

$$f_Y(y) = \int_{0}^{1-y} 2dx \tag{149}$$

$$f_Y(y) = 2 - 2y \tag{150}$$

$$f_Y\left(\frac{1}{2}\right) = 1 \tag{151}$$

by using Bayes theorem,

$$f_{X|Y}\left(x \mid \frac{1}{2}\right) = \frac{f_{X,Y}\left(x, \frac{1}{2}\right)}{f_Y\left(\frac{1}{2}\right)} \tag{152}$$

$$f_{X|Y}\left(x \mid \frac{1}{2}\right) = \begin{cases} 2, & 0 < x < \frac{1}{2}, \\ 0, & elsewhere. \end{cases} \tag{153}$$

It is in the form of Bernoulli distribution, the expectation value is given by,

$$E\left(X \mid Y = \frac{1}{2}\right) = \sum_{-\infty}^{\infty} x f_{X|Y}\left(x \mid \frac{1}{2}\right) \tag{154}$$

$$E\left(X \mid Y = \frac{1}{2}\right) = \int_{-\infty}^{0} x(0)dx + \int_{0}^{\frac{1}{2}} x(2)dx + \int_{0}^{-\infty} x(0)dx \tag{155}$$

$$E\left(X \mid Y = \frac{1}{2}\right) = \int_{0}^{\frac{1}{2}} 2xdx \tag{156}$$

$$E\left(X \mid Y = \frac{1}{2}\right) = \frac{1}{4} \tag{157}$$

15) An urn contains four balls, each ball having equal probability of being white or black. Three black balls are added to the urn. The probability that five balls in the urn are black is
Solution:
The total number of black balls are 5
Number of black balls initially present + number of black balls added =5
So, the number of black balls initially in the urn is 5-3=2
Let X be the random variable denoting the number of black balls in the urn. So, by binomial distribution,

$$\Pr(X = 1) = p \tag{158}$$

$$\Pr(X = k) = \binom{n}{k} p^k (1-p)^{n-k} \tag{159}$$

$$k = 0, 1, 2, ..., n \tag{160}$$

For the given problem, $n = 4$ and $p = 0.5$, because there is equal probability for each ball of being white or black. For having exactly 2 black balls,
From (160),

$$\Pr(X = 2) = \binom{4}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 \tag{161}$$

$$= \frac{6}{16} \tag{162}$$

$$= \frac{3}{8} \tag{163}$$

16) There are five bags each containing identical sets of ten distinct chocolates. One chocolate is picked from each bag.
The probability that at least two chocolates are identical is
Solution:
Let $X \in \{0, 1, 2, 3, 4, 5\}$ represent the random variable, denoting the number of similar

chocolates in the picked chocolates
Here, we can neglect X=1 because there can't be one similar object.

$$\Pr(X \geq 2) + \Pr(X = 0) = 1 \tag{164}$$
$$\Pr(X = 0) = \frac{10.9.8.7.6}{10^5} \tag{165}$$
$$\Pr(X = 0) = 0.3024 \tag{166}$$
$$\Pr(X \geq 2) = 1 - \Pr(X = 0) \tag{167}$$
$$= 1 - 0.3024 \tag{168}$$
$$= 0.6976 \tag{169}$$

Solution:

17) Let E and F be any two events with $P(E \cup F) = 0.8$, $P(E) = 0.4$ and $P(E|F) = 0.3$ then P(F) is

a) $\frac{3}{7}$

b) $\frac{4}{7}$

c) $\frac{3}{5}$

d) $\frac{2}{5}$

Solution: Given,

$$\Pr(E) = 0.4 \tag{170}$$
$$\Pr(E + F) = 0.8 \tag{171}$$
$$\Pr(E|F) = 0.3 \tag{172}$$

By definition,

$$\Pr(E|F) = \frac{\Pr(EF)}{\Pr(F)} \tag{173}$$
$$\implies \Pr(EF) = \Pr(E|F) \times \Pr(F) \tag{174}$$
$$\implies \Pr(EF) = 0.3 \times \Pr(F) \tag{175}$$

Now using the identity,

$$\Pr(E + F) = \Pr(E) + \Pr(F) - \Pr(EF) \tag{176}$$

From (170),(171) and (175)

$$\implies 0.8 = 0.4 + \Pr(F) - (0.3 \times \Pr(F)) \tag{177}$$
$$\implies 0.4 = (1 - 0.3) \times \Pr(F) \tag{178}$$
$$\implies \Pr(F) = \frac{0.4}{0.7} \tag{179}$$
$$\boxed{\Pr(F) = \frac{4}{7}} \tag{180}$$

18) The number N of persons getting injured in a bomb blast at a busy market place is a random variable having a Poisson Distribution with parameter $\lambda(\geq 1)$. A person injured in the explosion may either suffer a minor injury requiring first aid or suffer a major injury requiring hospitalisation. Let the number of persons with minor injury be N_1 and the conditional distribution of N_1 given N is

$$\Pr(N_1 = i|N) = \frac{1}{N} \tag{181}$$

Find the expected number of persons requiring hospitalisation. **Solution:** We know,

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)} \tag{182}$$

Also, for a Poisson Distribution:

$$\Pr(N = x) = \frac{e^{-\lambda}\lambda^x}{x!} \tag{183}$$

where λ is the parameter
Let N_2 be the number of persons hospitalised. Let $N = a$, and $N_1 = i(i \leq a)$, then, $N_2 = a - i$
Then, from (181) and (183):

$$\Pr(N_2 = a - i) = \Pr(N_1 = i) \tag{184}$$
$$= \Pr(N_1 = i|N = a)\Pr(N = a) \tag{185}$$
$$= \frac{1}{a}\frac{e^{-\lambda}\lambda^a}{a!} \tag{186}$$

Thus,

$$E(N_2) = \sum_{a=0}^{\infty} \sum_{i=0}^{a} (a-i) \times \frac{1}{a} \frac{e^{-\lambda}\lambda^a}{a!} \tag{187}$$

$$= \sum_{a=0}^{\infty} \frac{e^{-\lambda}\lambda^a}{a!} \sum_{i=0}^{a} \frac{a-i}{a} \tag{188}$$

$$= \sum_{a=0}^{\infty} \frac{e^{-\lambda}\lambda^a}{a!} \left(a - \frac{(a+1)}{2}\right) \tag{189}$$

$$= \sum_{a=0}^{\infty} \frac{e^{-\lambda}\lambda^a}{a!} \frac{a-1}{2} \tag{190}$$

$$= \frac{e^{-\lambda}}{2} \left[\sum_{a=0}^{\infty} \frac{a\lambda^a}{a!} - \sum_{a=0}^{\infty} \frac{\lambda^a}{a!}\right] \tag{191}$$

$$= \frac{e^{-\lambda}}{2} \left[\lambda \sum_{a=1}^{\infty} \frac{\lambda^{a-1}}{(a-1)!} - \sum_{a=0}^{\infty} \frac{\lambda^a}{a!}\right] \tag{192}$$

$$= \frac{e^{-\lambda}}{2} \left[\lambda e^{\lambda} - e^{\lambda}\right] \tag{193}$$

$$= \frac{\lambda - 1}{2} \tag{194}$$

19) The time to failure, in months, of lights bulbs manufactured at two plants A and B obey the exponential distributions with means 6 and 2 months respectively. Plant B produces four times as many bulbs as plant A does. Bulbs from these two plants are indistinguishable. They are mixed and sold together. Given that a bulb purchased at random is working after 12 months, What is the probability that it was manufactured in plant A?

Solution:

This problem involves Bayes theorem and Exponential distribution

- Probability that bulb is from Plant A = $\Pr(A) = \frac{1}{5}$
- Probability that bulb is from Plant B = $\Pr(B) = \frac{4}{5}$

One can use exponential distribution to find out the probability that the bulbs work after 12 months

Let X be a variable representing the lifetime of a bulb in months.

So X has a Cumulative distribution Function:

$$F_X(x,\lambda) = \begin{cases} 1 - e^{-\lambda x} & if \quad x \geq 0 \\ 0 & if \quad x < 0 \end{cases} \tag{195}$$

Let us denote that the bulbs works after 12 months with the variable W.

$\frac{1}{\lambda}$	Mean of distribution
x	Time to failure (in months)
λ_A	$\frac{1}{6}$
λ_B	$\frac{1}{2}$
$\Pr(X \leq k)$	$F_X(X, \lambda)$

$$\Pr(W \mid A) = 1 - \Pr(\text{Fails within 12 months} \mid \text{A})$$

$$= 1 - F_X(12, \lambda_A) \tag{196}$$

$$= e^{-\lambda_A \times 12} \tag{197}$$

$$\Pr(W \mid B) = 1 - \Pr(\text{Fails within 12 months} \mid \text{B})$$

$$= 1 - F_X(12, \lambda_B) \tag{198}$$

$$= e^{-\lambda_B \times 12} \tag{199}$$

From Bayes theorem,

$$\Pr(A \mid W) = \frac{Pr(A) \times Pr(W \mid A)}{Pr(A) \times Pr(W \mid A) + Pr(B) \times Pr(W \mid B)} \tag{200}$$

$$= \frac{Pr(A) \times e^{-\lambda_A \times 12}}{Pr(A) \times e^{-\lambda_A \times 12} + Pr(B) \times e^{-\lambda_B \times 12}} \tag{201}$$

Substituting the known values, we get

$$\Pr(A \mid W) = \frac{\frac{1}{5} \times e^{-2}}{\frac{1}{5} \times e^{-2} + \frac{4}{5} \times e^{-6}} \tag{202}$$

$$= 0.93173845935 \tag{203}$$

So the probability that the Bulb is manufactured in Plant A given that it works after a year is 0.93173845935.