GATE ST 2021 Q. 50

Dishank - AI20BTECH11011

Question

Consider an amusement park where visitors are arriving according to a Poisson process with rate 1. Upon arrival, a visitor spends a random amount of time in the park and then departs. The time spent by the visitors is independent of one another, as well as of the arrival process and have common probability density function

$$f(x) = \begin{cases} e^{-x}, & x > 0 \\ 0, & otherwise \end{cases} \tag{1}$$

If at a given point, there are 10 visitors in the park, and p is the probability that there will be exactly two more arrivals before the next departure, then $\frac{1}{p}$ equals.....

Solution

According to the question, we want the following events to occur in order:

1. First visitor, P_1, arrives while no one leaves
2. Second visitor P_2, arrives while no one leaves
3. One or more person leaves before the third visitor, P_3, arrives

Let the above events be E_1, E_2 and E_3 respectively. Thus the required probability

$$= \Pr(E_1 E_2 E_3) \tag{2}$$

$$= \Pr(E_1) \Pr(E_2|E_1) \Pr(E_3|E_1 E_2) \tag{3}$$

Solution Contd.

Symbol	Representation
X_1	Arrival time of P_1
$X_1 + X_2$	Arrival time of P_2
$X_1 + X_2 + X_3$	Arrival time of P_3
$Y_1, ..., Y_{10}$	Departure times of the 10 people in park currently
$X_1 + Y_{11}$	Departure time of P_1
$X_1 + X_2 + Y_{12}$	Departure time of P_2

Table: Notations

Solution Contd.

First we present the following result which shall be useful later. For $n > 0$,

$$\int_0^\infty xe^{-nx}dx = \frac{1}{n^2} \tag{4}$$

The above can be derived using integration by parts as follows

$$\int_0^\infty xe^{-nx}dx = -\frac{xe^{-nx}}{n}\bigg|_0^\infty + \frac{1}{n}\int_0^\infty e^{-nx}dx \tag{5}$$

$$= -\frac{e^{-nx}}{n^2}\bigg|_0^\infty \tag{6}$$

$$= \frac{1}{n^2} \tag{7}$$

Solution Contd.

Next we note that X_1, X_2 and X_3 are identical random variables having Poisson distribution with rate 1. Thus for $i \in \{1, 2, 3\}$,

$$\lambda = 1 * X_i = X_i \tag{8}$$

$$k = 1 \tag{9}$$

$$\implies f_{X_i}(x) = \begin{cases} \dfrac{x^1 e^{-x}}{1!} = xe^{-x} & x > 0 \\ 0 & otherwise \end{cases} \tag{10}$$

Also $Y_1, ..., Y_{12}$ are identical random variables. Thus for $i \in \{1, ..., 12\}$, as given in question,

$$f_{Y_i}(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & otherwise \end{cases} \tag{11}$$

$$\implies F_{Y_i}(x) = \begin{cases} 1 - e^{-x} & x > 0 \\ 0 & otherwise \end{cases} \tag{12}$$

Solution Contd.

Now we find $\Pr(E_1)$, $\Pr(E_2|E_1)$ and $\Pr(E_3|E_1E_2)$ in order to find the required probability from eq (3).

$$\Pr(E_1) = \Pr(Y_1, ..., Y_{10} > X_1) \tag{13}$$

$$= \int_{-\infty}^{\infty} \Pr(Y_1, ..., Y_{10} > x | X_1 = x) \tag{14}$$

$$= \int_{-\infty}^{\infty} (1 - F_{Y_1}(x))^{10} f_{X_1}(x) dx \tag{15}$$

$$= \int_{0}^{\infty} x e^{-11x} dx \tag{16}$$

$$= \frac{1}{121} \tag{17}$$

Solution Contd.

$$\Pr(E_2|E_1) = \Pr(Y_1, ..., Y_{10}, X_1 + Y_{11} > X_1 + X_2 | Y_1, ..., Y_{10} > X_1) \quad (18)$$

Using memoryless property of exponential random variable,

$$\begin{aligned}
\Pr(E_2|E_1) &= \Pr(Y_1, ..., Y_{11} > X_2) && (19)\\
&= \int_{-\infty}^{\infty} \Pr(Y_1, ... Y_{11} > x | X_2 = x) && (20)\\
&= \int_{-\infty}^{\infty} (1 - F_{Y_1}(x))^{11} f_{X_2}(x) dx && (21)\\
&= \int_{0}^{\infty} x e^{-12x} dx && (22)\\
&= \frac{1}{144} && (23)
\end{aligned}$$

Solution Contd.

$$\Pr\left(E_3|E_1E_2\right) =$$

$$\Pr\left(min(Y_1, ..., Y_{10}, X_1 + Y_{11}, X_1 + X_2 + Y_{12}) < X_1 + X_2 + X_3 | Y_1, ..., Y_{10}, X_1 + Y_{11} > X_1 + X_2\right) \quad (24)$$

$$\implies \Pr\left(E_3|E_1E_2\right) = 1 - \Pr\left(Y_1, ..., Y_{10}, X_1 + Y_{11}, X_1 + X_2 + Y_{12} > X_1 + X_2 + X_3 | Y_1, ..., Y_{10}, X_1 + Y_{11} > X_1 + X_2\right) \quad (25)$$

Solution Contd.

Using memoryless property of exponential random variable,

$$\Pr(E_3|E_1E_2) = 1 - \Pr(Y_1, ..., Y_{12} > X_3) \tag{26}$$

$$= 1 - \int_{-\infty}^{\infty} \Pr(Y_1, ... Y_{12} > x | X_3 = x) \tag{27}$$

$$= 1 - \int_{-\infty}^{\infty} (1 - F_{Y_1}(x))^{12} f_{X_3}(x) dx \tag{28}$$

$$= 1 - \int_{0}^{\infty} x e^{-13x} dx \tag{29}$$

$$= 1 - \frac{1}{169} \tag{30}$$

$$= \frac{168}{169} \tag{31}$$

Thus on substituting values in (3),

$$\Pr(E_1E_2E_3) = \frac{1}{121} \times \frac{1}{144} \times \frac{168}{169} = 5.7 \times 10^{-5} \tag{32}$$