Assignment 4

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Download all python codes from

https://github.com/Taha−Adeel/AI1103/tree/main/Assignment_4/Codes

and latex-tikz codes from

https://github.com/Taha−Adeel/AI1103/tree/main/Assignment_4

1 Problem (GATE 2021 (ST) Q.19)

Let $\{X_n\}_{n\geq 1}$ be a sequence of independent and identically distributed random variables each having uniform distribution on (0,2). For $n \geq 1$, let

$$Z_n = -\log_e\left(\prod_{i=1}^{n}(2-X_i)\right)^{\frac{1}{n}}.$$

Then, as $n \to \infty$, the sequence $\{Z_n\}_{n\geq 1}$ converges almost surely to ______ (Round of to 2 decimal places).

2 Solution (GATE 2021 (ST) Q.19)

Simplifying Z_n, we have

$$\begin{aligned} Z_n &= -\log_e\left(\prod_{i=1}^{n}(2-X_i)\right)^{\frac{1}{n}} && (2.0.1)\\ &= -\frac{1}{n}\cdot\log_e\left(\prod_{i=1}^{n}(2-X_i)\right) && (2.0.2)\\ &= \sum_{i=1}^{n}\left((-\log_e(2-X_i))\cdot\frac{1}{n}\right) && (2.0.3)\\ &= E\left(-\log_e(2-X_i)\right) && (2.0.4)\end{aligned}$$

Let X and Z be random variables. X follows a uniform distribution from 0 to 2.

$$X \sim \mathcal{U}[0,2], \quad (2.0.5)$$

$$\text{and let } Z = -\log_e(2-X) \quad (2.0.6)$$

The sequence X_n converges in distribution to X. i.e.

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x), \quad (2.0.7)$$

From **The Law of Large Numbers**, we have that for large n, $Z_n = E\left(-\log_e(2-X_i)\right)$ should be close to $E\left(-\log_e(2-X)\right) = E(Z)$. i.e.

$$\Pr\left(\lim_{n\to\infty} Z_n = E(Z)\right) = 1 \quad (2.0.8)$$

If $\Pr\left(\lim_{n\to\infty} Y_n = Y\right)$**=1, we say that** Y_n **almost surely converges to** Y**.** Therefore, by (2.0.8) as $n \to \infty$, Z_n almost surely converges to $E(Z)$.

The CDF of Z is defined as

$$\begin{aligned} F_Z(z) &= \Pr(Z \leq z) && (2.0.9)\\ &= \Pr(-\log_e(2-X) \leq z) && (2.0.10)\\ &= \Pr(\log_e(2-X) \geq -z) && (2.0.11)\\ &= \Pr(2-X \geq \exp(-z)) && (2.0.12)\\ &= \Pr(X \leq 2-\exp(-z)) && (2.0.13)\\ &= F_X(2-\exp(-z)) && (2.0.14)\end{aligned}$$

The CDF for X ($F_X(x)$), a uniform distribution on $(0,2)$ is given by

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \frac{x}{2} & 0 \leq x \leq 2 \\ 1 & x > 2 \end{cases} \quad (2.0.15)$$

Substituting the above in (2.0.14),

$$F_X(2-\exp(-z)) = \begin{cases} 0 & 2-\exp(-z) < 0 \\ 1-\frac{\exp(-z)}{2} & 0 \leq 2-\exp(-z) \leq 2 \\ 1 & 2-\exp(-z) > 2 \end{cases} \quad (2.0.16)$$

After some algebra, the above conditions yield

$$F_Z(z) = \begin{cases} 0 & z < -\log_e(2) \\ 1-\frac{\exp(-z)}{2} & z \geq -\log_e(2) \end{cases} \quad (2.0.17)$$

$$\implies f_Z(z) = \frac{\mathrm{d}(F_Z(z))}{\mathrm{d}z} = \begin{cases} 0 & z < -\log_e(2) \\ \frac{\exp(-z)}{2} & z \geq -\log_e(2) \end{cases} \quad (2.0.18)$$

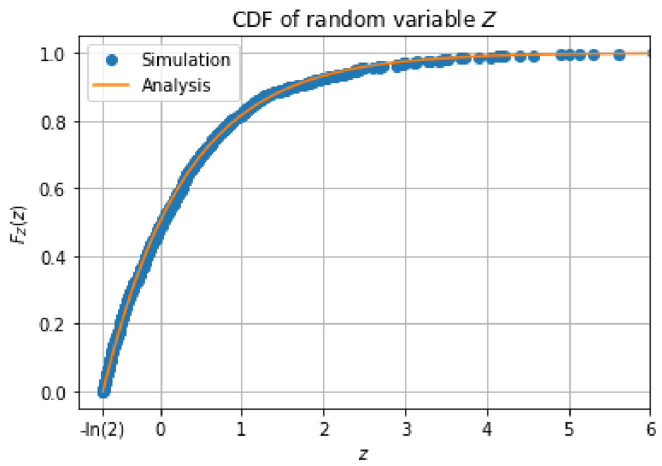

Fig. 0: $F_Z(z)$

Now calculating the expectation value for Z, we have

$$E(Z) = \int_{-\ln 2}^{\infty} z\, f_Z(z)\, dz \tag{2.0.19}$$

$$= \int_{-\ln 2}^{\infty} \frac{z\, e^{-z}}{2}\, dz \tag{2.0.20}$$

$$= \left[\frac{-(z+1)\, e^{-z}}{2} \right]_{-\ln 2}^{\infty} \tag{2.0.21}$$

$$= 1 - \ln(2) \tag{2.0.22}$$

$$\approx 0.3068 \tag{2.0.23}$$

From (2.0.8), we have as $n \to \infty$, Z_n almost surely converges to $E(Z) = 0.3068 \approx 0.31$(Rounded to 2 decimal places).

Ans: 0.31