GATE Problems in Probability

***Abstract*—These problems have been selected from GATE question papers and can be used for conducting tutorials in courses related to a first course in probability.**

1) Let X be a random variable with the following cumulative distribution function:

$$F(x) = \begin{cases} 0 & x < 0 \\ x^2 & 0 \le x < \frac{1}{2} \\ \frac{3}{4} & \frac{1}{2} \le x < 1 \\ 1 & x \ge 1 \end{cases} \tag{1}$$

Then $P\left(\frac{1}{4} < X < 1\right)$ is equal to
Solution:

$$P\left(\frac{1}{4} < X < 1\right) = F\left(1^-\right) - F\left(\frac{1}{4}\right) \tag{2}$$
$$= \frac{3}{4} - \left(\frac{1}{4}\right)^2 \tag{3}$$
$$= \frac{11}{16} \tag{4}$$

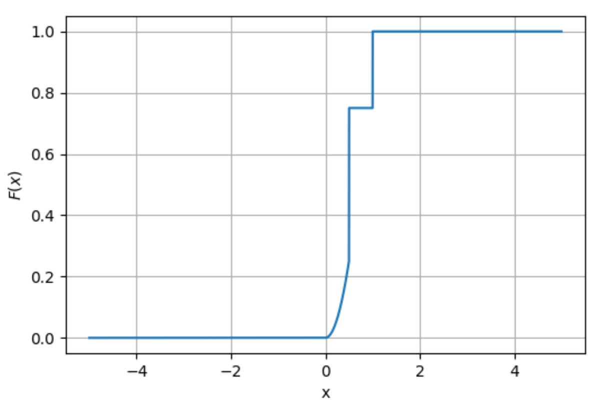

Fig. 1: The CDF of X

2) Let X and Y be continuous random variables with the joint probability density function

$$f(x, y) = \begin{cases} ae^{-2y} & 0 < x < y < \infty \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Then $E(X|Y = 2)$ is . . . **Solution:** Given X and Y are two continuous random variables with joint probability density function,

$$f(x, y) = \begin{cases} ae^{-2y} & 0 < x < y < \infty \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

We know that,
$0 < x < y < \infty \implies x < y < \infty$ for $0 < x < \infty$.
Then,

$$f_X(x) = \int f_{XY}(x, y)\, dy \tag{7}$$
$$= \int_x^\infty ae^{-2y} dy \tag{8}$$
$$= \left[\frac{ae^{-2y}}{(-2)}\right]_x^\infty \tag{9}$$
$$= \frac{-a}{2}\left[e^{-2y}\right]_x^\infty \tag{10}$$
$$= \frac{-a}{2}[0 - e^{-2x}] \tag{11}$$
$$\implies f_X(x) = \begin{cases} \frac{a}{2}e^{-2x} & 0 < x < \infty \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

Similarly,
$0 < x < y < \infty \implies 0 < x < y$ for $0 < y < \infty$
Then,

$$f_y(y) = \int f_{XY}(x, y)\, dx \tag{13}$$
$$= \int_0^y ae^{-2y} dx \tag{14}$$
$$= ae^{-2y}[x]_0^y \tag{15}$$
$$= aye^{-2y} \tag{16}$$
$$\implies f_Y(y) = \begin{cases} aye^{-2y} & 0 < y < \infty \\ 0 & \text{otherwise.} \end{cases} \tag{17}$$

Therefore ,

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} \tag{18}$$
$$= \frac{ae^{-2y}}{aye^{-2y}} \tag{19}$$
$$= \frac{1}{y} \tag{20}$$
$$\implies f_{X|Y}(x|y) = \begin{cases} \frac{1}{y} & \text{if } 0 < x < y < \infty \\ 0 & \text{otherwise} \end{cases} \tag{21}$$

Then,

$$E(X|Y=y) = \int_{-\infty}^{\infty} (x) f_{X|Y}(x|y)\, dx \tag{22}$$
$$= \int_0^y (x)\left(\frac{1}{y}\right) dx \tag{23}$$
$$= \frac{1}{y}\int_0^y (x) dx \tag{24}$$
$$= \frac{1}{y}\left[\frac{x^2}{2}\right]_0^y \tag{25}$$
$$= \frac{1}{y}\left(\frac{y^2}{2}\right) \tag{26}$$
$$= \frac{y}{2} \tag{27}$$
$$\implies E(X|Y=y) = \frac{y}{2} \tag{28}$$
$$\therefore E(X|Y=2) = 1 \tag{29}$$

3) A continuous random variable X has the probability density function

$$f(x) = \begin{cases} \frac{3}{5}e^{-\frac{3}{5}x} & x > 0 \\ 0 & x \le 0 \end{cases}$$

The probability density function of $Y = 3X + 2$ is

a)

$$f(y) = \begin{cases} \frac{1}{5}e^{-\frac{1}{5}(y-2)} & y > 2 \\ 0 & y \le 2 \end{cases}$$

b)

$$f(y) = \begin{cases} \frac{2}{5}e^{-\frac{2}{5}(y-2)} & y > 2 \\ 0 & y \le 2 \end{cases}$$

c)

$$f(y) = \begin{cases} \frac{3}{5}e^{-\frac{3}{5}(y-2)} & y > 2 \\ 0 & y \le 2 \end{cases}$$

d)

$$f(y) = \begin{cases} \frac{4}{5}e^{-\frac{4}{5}(y-2)} & y > 2 \\ 0 & y \le 2 \end{cases}$$

Solution: Given $Y = 3X + 2$
CDF of Y,

$$F_y(Y) = \Pr(Y \le y)$$
$$= \Pr\left(X \le \frac{y-2}{3}\right)$$
$$= F_x\left(\frac{y-2}{3}\right)$$

Thus, pdf of Y ,

$$f_Y(y) = \frac{1}{3} f_X\left(\frac{y-2}{3}\right)$$
$$= \frac{1}{3} \times \begin{cases} \frac{3}{5}e^{-\frac{3}{5}\left(\frac{y-2}{3}\right)} & y > 2 \\ 0 & y \le 2 \end{cases}$$
$$= \begin{cases} \frac{1}{5}e^{-\frac{1}{5}(y-2)} & y > 2 \\ 0 & y \le 2 \end{cases}$$

Hence, correct option is 1.

4) Let the probability density function of a random variable X be

$$f(x) = \begin{cases} x & 0 \leqslant x < \frac{1}{2} \\ c(2x-1)^2 & \frac{1}{2} \leqslant x < 1 \\ 0 & \text{Otherwise} \end{cases}$$

Then value of c is equal to ...

Solution: We know that,

$$\int_{-\infty}^{\infty} f_x(x)\,dx = 1 \tag{30}$$

$$\int_{-\infty}^{0} f_x(x)\,dx + \int_{0}^{\frac{1}{2}} f_x(x)\,dx$$
$$+ \int_{\frac{1}{2}}^{1} f_x(x)\,dx + \int_{1}^{\infty} f_x(x)\,dx = 1 \tag{31}$$

$$\int_{0}^{\frac{1}{2}} x\,dx + \int_{\frac{1}{2}}^{1} c(2x-1)^2\,dx = 1 \tag{32}$$

$$\left[\frac{x^2}{2}\right]_0^{\frac{1}{2}} + c\left[\frac{(2x-1)^3}{6}\right]_{\frac{1}{2}}^{1} = 1 \tag{33}$$

$$\frac{1}{8} + \frac{c}{6} = 1 \tag{34}$$

$$c = \frac{21}{4} \tag{35}$$

$$\therefore \text{ Required value of } c = \frac{21}{4}$$

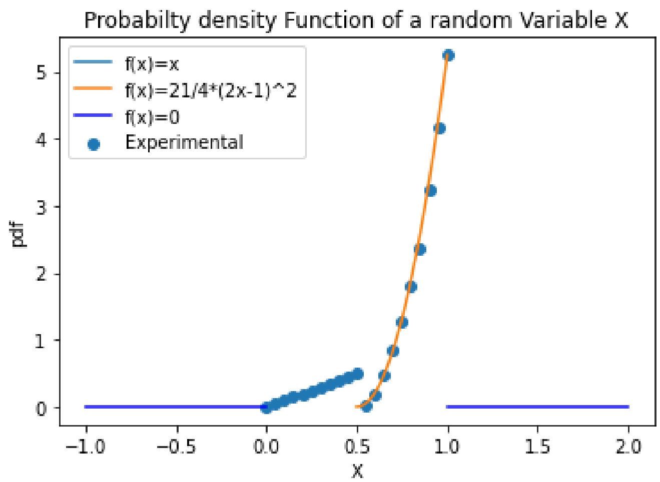

Fig. 2: Experimental and Theoritical pdf of X

5) Let $A_1, A_2, ...A_n$ be n independent events in which the Probability of occurence of the event A_i is given by P(A_i) = 1 - $\frac{1}{\alpha^i}$, $\alpha > 1$, i = 1,2,3,....n.Then the probability that atleast one of the events occurs is

a) 1 - $\frac{1}{\alpha^{\frac{n(n+1)}{2}}}$

b) $\frac{1}{\alpha^{\frac{n(n+1)}{2}}}$

c) $\frac{1}{\alpha^n}$

d) 1 - $\frac{1}{\alpha^n}$

Solution: Let $A_1 + A_2 + A_3.... + A_n$ = S,
$\Pr(S)$ = Probability of atleast one event occuring De morgan's law states that $(A+B)' = A'B'$

$$\implies \Pr(S) = 1 - \Pr(S') \tag{36}$$

$$1 - \Pr(S') = 1 - \Pr(A_1'A_2'A_3'....A_n') \tag{37}$$

$\forall$ i $\in$ 1,2,....n
Since, A_i are independent.
$\therefore$ Complements of A_i are also independent.
$\implies$

$$\Pr(A_1'A_2'A_3'....A_n') = \prod_{i=1}^{n} \Pr(A_i') \tag{38}$$

$$\Pr(A_i) = 1 - \frac{1}{\alpha^i} \implies \Pr(A_i') = \frac{1}{\alpha^i} \tag{39}$$

substituting (39) in (38),

$$\Pr(A_1'A_2'A_3'....A_n') = \prod_{i=1}^{n} \frac{1}{\alpha^i} \tag{40}$$

$$\prod_{i=1}^{n} \frac{1}{\alpha^i} = \frac{1}{\alpha^{\sum_i^n i}} = \frac{1}{\alpha^{\frac{n(n+1)}{2}}} \tag{41}$$

$$\therefore \Pr(A_1'A_2'A_3'....A_n') = \Pr(S') = \frac{1}{\alpha^{\frac{n(n+1)}{2}}} \tag{42}$$

from equations (37) and (42)

$$\implies \Pr(S) = 1 - \Pr(S') = 1 - \frac{1}{\alpha^{\frac{n(n+1)}{2}}} \tag{43}$$

$\therefore$ The correct option is **(a)**

6) Let the random variable X have the distribution function $F(x) = \begin{cases} 0 & if\ x < 0 \\ \frac{x}{2} & if\ 0 \le x < 1 \\ \frac{3}{5} & if\ 1 \le x < 2 \\ \frac{1}{2} + \frac{x}{8} & if\ 2 \le x < 3 \\ 1 & if\ x \ge 3 \end{cases}$
Then $\Pr(2 \le x < 4)$ is equal to
Solution:

Given,

$$F(x) = \begin{cases} 0 & if\ x < 0 \\ \frac{x}{2} & if\ 0 \le x < 1 \\ \frac{3}{5} & if\ 1 \le x < 2 \\ \frac{1}{2} + \frac{x}{8} & if\ 2 \le x < 3 \\ 1 & if\ x \ge 3 \end{cases} \quad (44)$$

We need to find $\Pr(2 \le x < 4)$,which is also can be written as

$$\Pr(2 \le x < 4) = \Pr(x < 4) - \Pr(x < 2) \quad (45)$$
$$= F(X = 4^-) - F(X = 2^-) \quad (46)$$

Using (44) in (46),

$$\Pr(2 \le x < 4) = 1 - \frac{3}{5} \quad (47)$$
$$= \frac{2}{5} \quad (48)$$
$$= 0.4 \quad (49)$$

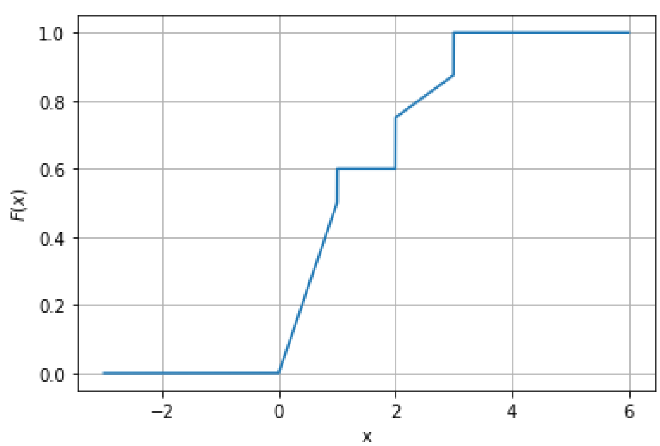

Fig. 3: cdf of random variable X

7) Let Z be the vertical coordinate, between -1 and 1, of a point chosen uniformly at random on the surface of a unit sphere in R^3. Then, $\Pr\left(\frac{-1}{2} \le Z \le \frac{1}{2}\right)$ is
Solution: The equation of the sphere can be written as : $x^2 + y^2 + z^2 = 1$. Now,

$$\Pr\left(\frac{-1}{2} \le z \le 0\right) = \Pr\left(0 \le z^2 \le \frac{1}{4}\right) \quad (50)$$
$$\Pr\left(0 \le z \le \frac{1}{2}\right) = \Pr\left(0 \le z^2 \le \frac{1}{4}\right) \quad (51)$$
$$\therefore \Pr\left(\frac{-1}{2} \le z \le \frac{1}{2}\right) = 2 \times \Pr\left(0 \le z^2 \le \frac{1}{4}\right) \quad (52)$$

$$\Pr\left(0 \le z^2 \le \frac{1}{4}\right) = \Pr\left(\frac{3}{4} \le x^2 + y^2 \le 1\right) \quad (53)$$

Taking, $x^2 + y^2 = r^2$. (54)

$$\Pr\left(\frac{3}{4} \le r^2 \le 1\right) = \frac{1}{4} \quad (55)$$

$\left(\text{Since, } r^2 \text{ is uniform between 0 and 1}\right)$

$$\therefore \Pr\left(\frac{-1}{2} \le Z \le \frac{1}{2}\right) = 2 \times \frac{1}{4} = \frac{1}{2} \quad (56)$$

8) Let X_1 and X_2 be independent geometric random variables with the same probability mass function given by $\Pr(X = k) = p(1 - p)^{k-1}$, $k = 1, 2, \dots$ Then the value of $\Pr(X_1 = 2 | X_1 + X_2 = 4)$ correct up to three decimal places is
Solution: Let

$$p_{X_i}(k) = \Pr(X_i = k) = \begin{cases} p(1-p)^{k-1} & n = 1, 2, ... \\ 0 & otherwise \end{cases} \quad (57)$$

where i=1,2

$$\Pr(A|B) = \frac{\Pr(AB)}{\Pr(B)} \quad (58)$$

$$(X_1 = 2) \cap (X_1 + X_2 = 4) = (X_1 = 2, X_2 = 2) \quad (59)$$

Thus,

$$\Pr(X_1 = 2 | X_1 + X_2 = 4) = \frac{\Pr(X_1 = 2, X_2 = 2)}{\Pr(X_1 + X_2 = 4)} \quad (60)$$

Since the two events are independent,

$$\Pr(X_1 = 2|X_1 + X_2 = 4) = \frac{\Pr(X_1 = 2)\Pr(X_2 = 2)}{\Pr(X_1 + X_2 = 4)} \tag{61}$$

Let

$$X = X_1 + X_2 \tag{62}$$

From (62),

$$p_X(n) = \Pr(X_1 + X_2 = n) = \Pr(X_1 = n - X_2) \tag{63}$$

$$= \sum_k \Pr(X_1 = n - k|X_2 = k)\, p_{X_2}(k) \tag{64}$$

after unconditioning. $\because$ X_1 and X_2 are independent,

$$\Pr(X_1 = n - k|X_2 = k) = \Pr(X_1 = n - k) = p_{X_1}(n - k) \tag{65}$$

From (64) and (65),

$$p_X(n) = \sum_k p_{X_1}(n-k)p_{X_2}(k) = p_{X_1}(n) * p_{X_2}(n) \tag{66}$$

where $*$ denotes the convolution operation. Substituting from (57) in (66),

$$p_X(n) = \sum_{k=1}^{n-1} p_{X_1}(n-k)p_{X_2}(k) \tag{67}$$

$$= \sum_{k=1}^{n-1} (1-p)^{k-1}p \cdot (1-p)^{n-k-1}p \tag{68}$$

$$= (1-p)^{n-2}p^2 \sum_{k=1}^{n-1} 1 \tag{69}$$

$$= (n-1)(1-p)^{n-2}p^2 \tag{70}$$

From (70) and (57) we have

$$\Pr(X_1 = 2) = \Pr(X_2 = 2) = p(1-p) \tag{71}$$

$$\Pr(X_1 + X_2 = 4) = 3(1-p)^2p^2 \tag{72}$$

Substituting in (61)

$$\Pr(X_1 = 2|X_1 + X_2 = 4) = \frac{(1-p)^2p^2}{3(1-p)^2p^2} \tag{73}$$

$$= \frac{1}{3} \tag{74}$$

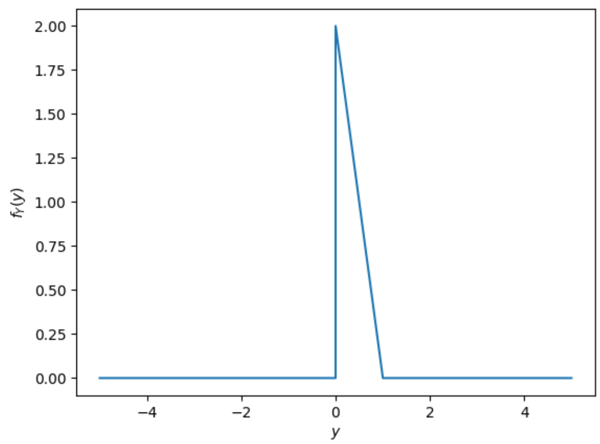

Fig. 4: Marginal PDF

9) Let X and Y have joint probability function given by

$$f_{X,Y}(x,y) = \begin{cases} 2 & 0 \le x \le 1-y, 0 \le y \le 1 \\ 0 & otherwise \end{cases}$$

If f_Y denotes the marginal probability density function of Y, then $f_Y(1/2) =$?

Solution:

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y).dx \tag{23.1}$$

$$\implies f_Y(y) = \begin{cases} 0 + \int_0^{1-y} 2.dx & 0 \le y \le 1 \\ 0 & otherwise \end{cases} \tag{23.2}$$

$$\implies f_Y(y) = \begin{cases} 2(1-y) & 0 \le y \le 1 \\ 0 & otherwise \end{cases} \tag{23.3}$$

$$\therefore f_Y(1/2) = 1 \tag{23.4}$$

10) Let X be a standard normal random variable. Then $\Pr(X < 0| \ \lfloor X \rfloor = 1)$ is equal to

a) $\dfrac{\Phi(1) - \frac{1}{2}}{\Phi(2) - \frac{1}{2}}$

b) $\dfrac{\Phi(1) + \frac{1}{2}}{\Phi(2) + \frac{1}{2}}$

c) $\dfrac{\Phi(1)-\frac{1}{2}}{\Phi(2)+\frac{1}{2}}$

d) $\dfrac{\Phi(1)+1}{\Phi(2)+1}$

Solution:

$$|\lfloor X \rfloor| = 1 \tag{75}$$
$$\implies \lfloor X \rfloor = 1 \textit{ or } -1 \tag{76}$$
$$\implies X \in [1,2) \cup [-1,0) \tag{77}$$

Here

$\lfloor X \rfloor = \textit{greatest integer less than or equal to } X$

Thus required probability

$$= \frac{\Pr(X \in [-1,0))}{\Pr(X \in [1,2) \cup [-1,0))} \tag{78}$$

Using symmetry of standard normal random variable about y = 0, we have required probability

$$= \frac{\Pr(X \in (0,1])}{\Pr(X \in [1,2) \cup (0,1]))} \tag{79}$$
$$= \frac{\Pr(X \in (0,1])}{\Pr(X \in (0,2))} \tag{80}$$
$$= \frac{\Pr(X < 1) - \Pr(X < 0)}{\Pr(X < 2) - \Pr(X < 0)} \tag{81}$$
$$= \frac{\Phi(1) - \Phi(0)}{\Phi(2) - \Phi(0)} \tag{82}$$
$$= \frac{\Phi(1) - \frac{1}{2}}{\Phi(2) - \frac{1}{2}} \tag{83}$$
$$= \frac{0.841 - 0.5}{0.977 - 0.5} \tag{84}$$
$$= 0.715 \tag{85}$$

Here $\Phi(x)$ represents the standard normal cumulative density function. Thus

$$X \sim \mathcal{N}1 \tag{86}$$

and

$$\Phi(x) = \int_{-\infty}^{x} f_X(x)dx \tag{87}$$

It can easily be seen that $\Phi(0) = \frac{1}{2}$, which has been used to obtain (83). (84) was obtained by consulting tables for $\Phi(x)$

11) Let X be a random variable with probability mass function $p(n) = \left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^{n-1}$ $n = 1, 2\ldots$ Then $E[X-3|X>3]$ is ...

Solution:
Given

$$\Pr(X = n) = \begin{cases} \left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^{n-1} & n = 1, 2\ldots \\ 0 & otherwise \end{cases} \tag{88}$$

Using the linearity of the expectation operator:

$$E[X-3 \mid X>3] = E[X \mid X>3] - 3 \tag{89}$$

Now ,

$$E[X \mid X>3] = \sum_{x=1}^{\infty} x \Pr(X = x \mid X > 3) \tag{90}$$
$$= \sum_{x=1}^{\infty} x \frac{\Pr(X = x, X > 3)}{\Pr(X > 3)} \tag{91}$$

Calculating $\Pr(X > 3)$

$$\Pr(X > 3) = 1 - \Pr(X \le 3) \tag{92}$$
$$= 1 - \sum_{x'=1}^{3} \Pr(X = x') \tag{93}$$
$$= 1 - \sum_{x'=1}^{3} \left(\frac{3}{4}\right)^{x'-1} \left(\frac{1}{4}\right) \tag{94}$$
$$= \frac{27}{64} \tag{95}$$

Also,

$$\Pr(X = x, X > 3) = \begin{cases} \Pr(X = x) & x > 3 \\ 0 & x \le 3 \end{cases} \tag{96}$$

Substituting (95) and (96) in (91) we get

$$E[X \mid X > 3] = \sum_{x=1}^{3} 0 + \sum_{x=4}^{\infty} \left[x \frac{\Pr(X = x)}{\frac{27}{64}} \right] \tag{97}$$
$$= \frac{64}{27} \sum_{x=4}^{\infty} \left[x \left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^{x-1} \right] \tag{98}$$
$$= \frac{16}{27} \sum_{x=4}^{\infty} \left[x \left(\frac{3}{4}\right)^{x-1} \right] \tag{99}$$

Let

$$S = \sum_{x=4}^{\infty} \left[x \left(\frac{3}{4} \right)^{x-1} \right] \tag{100}$$

Multiplying ((100)) with $\frac{3}{4}$ on both sides gives

$$\frac{3}{4} S = \sum_{x=4}^{\infty} x \frac{1}{4} \left(\frac{3}{4} \right)^{x} \tag{101}$$

From (101) and(100)

$$S = 4\left(\frac{3}{4}\right)^3 + 5\left(\frac{3}{4}\right)^4 + 6\left(\frac{3}{4}\right)^5 \dots \tag{102}$$

$$\frac{3}{4} S = 0\left(\frac{3}{4}\right)^3 + 4\left(\frac{3}{4}\right)^4 + 5\left(\frac{3}{4}\right)^5 + \dots \tag{103}$$

subtracting (101) from (100) we get

$$\frac{S}{4} = 4\left(\frac{3}{4}\right)^3 + \left(\frac{3}{4}\right)^4 + \left(\frac{3}{4}\right)^5 + \left(\frac{3}{4}\right)^6 + \dots \tag{104}$$

$$= 4\left(\frac{3}{4}\right)^3 + \sum_{x=4}^{\infty} \left(\frac{3}{4}\right)^x \tag{105}$$

$$= \frac{189}{64} \tag{106}$$

Substituting vale of S in (99) we get

$$E[X|X > 3] = 7 \tag{107}$$

Thus putting this in (89)

$$E[X - 3|X > 3] = 4 \tag{108}$$

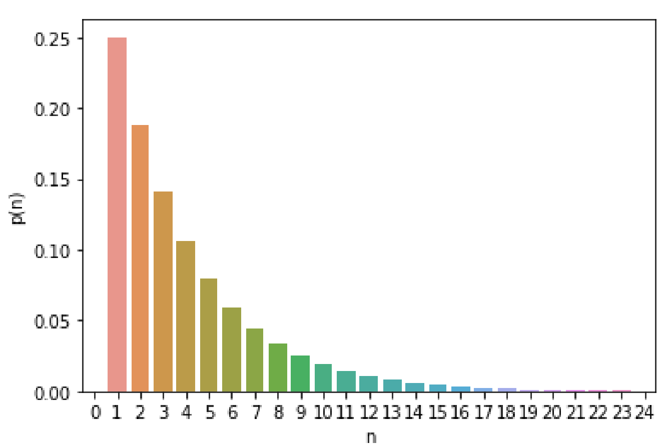

Fig. 5: PMF of X

12) Let (X,Y) be a random vector such that, for any $y > 0$, the conditional probability density function of X given $Y = y$ is

$$f_{X|Y=y}(x) = ye^{-yx}\,, x > 0.$$

If the marginal probability density function of Y is

$$g(y) = ye^{-y}\,, y > 0$$

then $E(Y|x = 1) =$

Solution: Given, the conditional probability density function of X given $Y = y$,

$$f_{X|Y=y}(x) = ye^{-yx}\,, x > 0 \tag{109}$$

and, the marginal probability density function of Y,

$$g(y) = ye^{-y}\,, y > 0 \tag{110}$$

let the joint probability density function of (X,Y) be $f_{X,Y}(x,y)$. We know that,

$$f_{X|Y=y}(x) = \frac{f_{X,Y}(x,y)}{g(y)} \tag{111}$$

using (109) and (110) in (111),

$$f_{X,Y}(x,y) = y^2 e^{-y(x+1)}\,, x, y > 0 \tag{112}$$

let the marginal probability density function of X be $f_X(x)$, as we know ,

$$f_X(x) = \int_0^{\infty} f_{X,Y}(x,y)\, dy \tag{113}$$

using (112) in (113),

$$f_X(x) = \int_0^{\infty} y^2 e^{-y(x+1)}\, dy \tag{114}$$

$$= \frac{2}{(x+1)^3}\,, x > 0 \tag{115}$$

The conditional probability density function of Y given $X = x$ is given by,

$$f_{Y|X=x}(y) = \frac{f_{X,Y}(x,y)}{f_X(x)} \tag{116}$$

using (112) and (115) in (116),

$$f_{Y|X=x}(y) = \frac{y^2 e^{-y(x+1)}(x+1)^3}{2}\,, x, y > 0 \tag{117}$$

The conditional probability density function of Y given $X = 1$ is given by,

$$f_{Y|X=1}(y) = 4y^2 e^{-2y}\,, y > 0 \tag{118}$$

We need to find $E(Y|X=1)$ which is given by,

$$E(Y|X=1) = \int_0^\infty y f_{Y|X=1}(y)\, dy \tag{119}$$

using (118) in (119),

$$\begin{aligned} E(Y|X=1) &= \int_0^\infty 4y^3 e^{-2y}\, dy && (120)\\ &= \left[\frac{-e^{-2y}(8y^3 + 12y^2 + 12y + 6)}{4}\right]_0^\infty && (121)\\ &= \frac{3}{2} && (122) \end{aligned}$$