# Assignment 6

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Download all python codes from

https://github.com/Ananthoju−Pranav−Sai/AI1103/tree/main/Assignment_6/Codes

and latex codes from

https://github.com/Ananthoju−Pranav−Sai/AI1103/blob/main/Assignment_6/main.tex

## GATE 1997 MA -Problem 8

Find the characteristic function of $Y = \sum_{r=1}^{n} a_r X_r$, where $a_1, a_2, ..., a_n$ are constants and $X_1, X_2, X_3, .., X_n$ are random variables, each of which takes the values -1 and 1 with probability $\frac{1}{2}$. Taking $a_r = 2^{-r}$ for each $r$, show that $Y$ converges in distribution to uniform distribution on (-1,1).

## Solution

Given,

$$Y = \sum_{r=1}^{n} a_r X_r \tag{0.0.1}$$

The characteristic function of a random variable Y is defined as

$$C_Y(t) = E[e^{itY}] \tag{0.0.2}$$

$$\implies C_Y(t) = E[e^{it\sum_{r=1}^{n} a_r X_r}] \tag{0.0.3}$$

$$\implies C_Y(t) = \prod_{r=1}^{n} E[e^{ita_r X_r}] \tag{0.0.4}$$

$$\implies C_Y(t) = \prod_{r=1}^{n} C_{X_r}(a_r t) \tag{0.0.5}$$

By taking $a_r = 2^{-r}$ we get,

$$C_Y(t) = \prod_{r=1}^{n} C_{X_r}\left(\frac{t}{2^r}\right) \tag{0.0.6}$$

As random variables $X_r's$ follow discrete uniform distribution with only two possible outcomes ($X_r = -1$ and $X_r = 1$) the characteristic equation of $X_r$ is

$$C_{X_r}(t) = \sum_{k} e^{ikt} \Pr(X_r = k) \tag{0.0.7}$$

$$\implies C_{X_r}(t) = \frac{e^{-it}}{2} + \frac{e^{it}}{2} \tag{0.0.8}$$

$$\implies C_{X_r}(t) = \frac{1 + e^{2it}}{2e^{it}} \tag{0.0.9}$$

$$\implies C_{X_r}\left(\frac{t}{2^r}\right) = \frac{1 + e^{2i\left(\frac{t}{2^r}\right)}}{2e^{i\left(\frac{t}{2^r}\right)}} \tag{0.0.10}$$

using (0.0.10) in (0.0.6)

$$C_Y(t) = \prod_{r=1}^{n} \left(\frac{1 + e^{2i\left(\frac{t}{2^r}\right)}}{2e^{i\left(\frac{t}{2^r}\right)}}\right) \tag{0.0.11}$$

$$\implies C_Y(t) = \frac{\left(1 + e^{it}\right)\left(1 + e^{i\left(\frac{t}{2}\right)}\right)....\left(1 + e^{i\left(\frac{t}{2^{n-1}}\right)}\right)}{2^n \left(e^{i\sum_{r=1}^{n}\frac{t}{2^r}}\right)} \tag{0.0.12}$$

$$\therefore C_Y(t) = \frac{\left(e^{2it} - 1\right)}{2^n e^{it\left(\frac{2^{n+1}-1}{2^{n+1}}\right)}\left(e^{\left(\frac{it}{2^{n-1}}\right)} - 1\right)} \tag{0.0.13}$$

Now consider

$$\lim_{n\to\infty} C_Y(t) \tag{0.0.14}$$

using (0.0.13) in (0.0.14)

$$\implies \lim_{n\to\infty} \frac{\left(e^{2it} - 1\right)}{2^n e^{it\left(\frac{2^{n+1}-1}{2^{n+1}}\right)}\left(e^{\left(\frac{it}{2^{n-1}}\right)} - 1\right)} \tag{0.0.15}$$

We know that,

$$\lim_{n\to\infty}\left(e^{\left(\frac{it}{2^{n-1}}\right)} - 1\right) = \frac{it}{2^{n-1}} \tag{0.0.16}$$

$$\lim_{n\to\infty} e^{it\left(\frac{2^{n+1}-1}{2^{n+1}}\right)} = e^{it} \tag{0.0.17}$$

Using (0.0.16) and (0.0.17) in (0.0.15)

$$\lim_{n\to\infty} C_Y(t) = \frac{\left(e^{2it} - 1\right)}{2ite^{it}} \tag{0.0.18}$$

Now, let's assume that if $Y$ follows uniform distribution on (-1,1) then it's pdf can be written as

$$f_Y(y) = \begin{cases} \frac{1}{2} & -1 \le y \le 1 \\ 0 & otherwise \end{cases} \tag{0.0.19}$$

And it's cdf would be

$$F_Y(y) = \begin{cases} 0 & y < -1 \\ \frac{1+y}{2} & -1 \le y \le 1 \\ 1 & otherwise \end{cases} \tag{0.0.20}$$

It's characteristic function would be

$$C_Y(t) = \int_{-\infty}^{\infty} e^{ity}.f_Y(y)\,dy \tag{0.0.21}$$

$$\implies C_Y(t) = \int_{-1}^{1} e^{ity}.\left(\frac{1}{2}\right) dy \tag{0.0.22}$$

$$\implies C_Y(t) = \frac{\left(e^{2it} - 1\right)}{2ite^{it}} \tag{0.0.23}$$

So from (0.0.17) and (0.0.23) we conclude that as $(n \to \infty)$ $Y$ converges in distribution to uniform distribution on (-1,1).

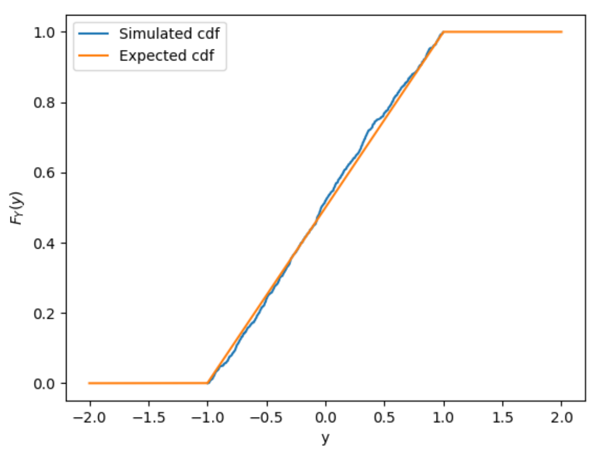


Fig. 0: Simulated vs expected cdf plot of random variable Y