GATE Problems in Probability

***Abstract*—These problems have been selected from GATE question papers and can be used for conducting tutorials in courses related to a first course in probability.**

1) Let X be a random variable with the following cumulative distribution function:

$$F(x) = \begin{cases} 0 & x < 0 \\ x^2 & 0 \le x < \frac{1}{2} \\ \frac{3}{4} & \frac{1}{2} \le x < 1 \\ 1 & x \ge 1 \end{cases} \tag{1}$$

Then $P\left(\frac{1}{4} < X < 1\right)$ is equal to
Solution:

$$P\left(\frac{1}{4} < X < 1\right) = F\left(1^-\right) - F\left(\frac{1}{4}\right) \tag{2}$$

$$= \frac{3}{4} - \left(\frac{1}{4}\right)^2 \tag{3}$$

$$= \frac{11}{16} \tag{4}$$

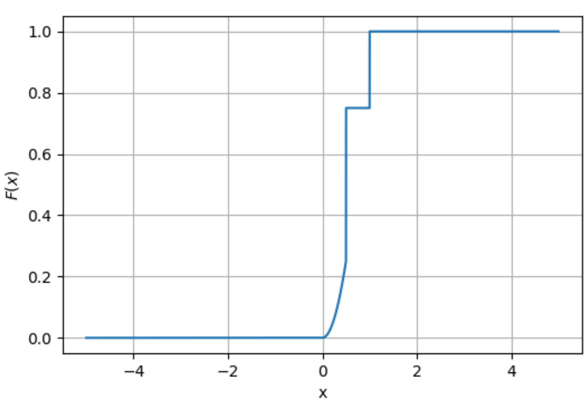

Fig. 1: The CDF of X

2) Let X and Y be continuous random variables with the joint probability density function

$$f(x, y) = \begin{cases} ae^{-2y} & 0 < x < y < \infty \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Then $E(X|Y = 2)$ is . . . **Solution:** Given X and Y are two continuous random variables with joint probability density function,

$$f(x, y) = \begin{cases} ae^{-2y} & 0 < x < y < \infty \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

We know that,
$0 < x < y < \infty \implies x < y < \infty$ for $0 < x < \infty$.
Then,

$$f_X(x) = \int f_{XY}(x, y)\, dy \tag{7}$$

$$= \int_x^{\infty} ae^{-2y} dy \tag{8}$$

$$= \left[\frac{ae^{-2y}}{(-2)}\right]_x^{\infty} \tag{9}$$

$$= \frac{-a}{2}\left[e^{-2y}\right]_x^{\infty} \tag{10}$$

$$= \frac{-a}{2}[0 - e^{-2x}] \tag{11}$$

$$\implies f_X(x) = \begin{cases} \frac{a}{2}e^{-2x} & 0 < x < \infty \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

Similarly,
$0 < x < y < \infty \implies 0 < x < y$ for $0 < y < \infty$
Then,

$$f_y(y) = \int f_{XY}(x, y)\, dx \tag{13}$$

$$= \int_0^{y} ae^{-2y} dx \tag{14}$$

$$= ae^{-2y}[x]_0^y \tag{15}$$

$$= aye^{-2y} \tag{16}$$

$$\implies f_Y(y) = \begin{cases} aye^{-2y} & 0 < y < \infty \\ 0 & \text{otherwise.} \end{cases} \tag{17}$$

Therefore ,

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} \tag{18}$$
$$= \frac{ae^{-2y}}{aye^{-2y}} \tag{19}$$
$$= \frac{1}{y} \tag{20}$$
$$\implies f_{X|Y}(x|y) = \begin{cases} \frac{1}{y} & \text{if } 0 < x < y < \infty \\ 0 & \text{otherwise} \end{cases} \tag{21}$$

Then,

$$E(X|Y=y) = \int_{-\infty}^{\infty} (x) f_{X|Y}(x|y)\, dx \tag{22}$$
$$= \int_0^y (x)\left(\frac{1}{y}\right) dx \tag{23}$$
$$= \frac{1}{y}\int_0^y (x)dx \tag{24}$$
$$= \frac{1}{y}\left[\frac{x^2}{2}\right]_0^y \tag{25}$$
$$= \frac{1}{y}\left(\frac{y^2}{2}\right) \tag{26}$$
$$= \frac{y}{2} \tag{27}$$
$$\implies E(X|Y=y) = \frac{y}{2} \tag{28}$$
$$\therefore E(X|Y=2) = 1 \tag{29}$$

3) A continuous random variable X has the probability density function

$$f(x) = \begin{cases} \frac{3}{5}e^{-\frac{3}{5}x} & x > 0 \\ 0 & x \le 0 \end{cases}$$

The probability density function of $Y = 3X + 2$ is

a)

$$f(y) = \begin{cases} \frac{1}{5}e^{-\frac{1}{5}(y-2)} & y > 2 \\ 0 & y \le 2 \end{cases}$$

b)

$$f(y) = \begin{cases} \frac{2}{5}e^{-\frac{2}{5}(y-2)} & y > 2 \\ 0 & y \le 2 \end{cases}$$

c)

$$f(y) = \begin{cases} \frac{3}{5}e^{-\frac{3}{5}(y-2)} & y > 2 \\ 0 & y \le 2 \end{cases}$$

d)

$$f(y) = \begin{cases} \frac{4}{5}e^{-\frac{4}{5}(y-2)} & y > 2 \\ 0 & y \le 2 \end{cases}$$

Solution: Given $Y = 3X + 2$
CDF of Y,

$$F_y(Y) = \Pr(Y \le y)$$
$$= \Pr\left(X \le \frac{y-2}{3}\right)$$
$$= F_x\left(\frac{y-2}{3}\right)$$

Thus, pdf of Y ,

$$f_Y(y) = \frac{1}{3} f_X\left(\frac{y-2}{3}\right)$$
$$= \frac{1}{3} \times \begin{cases} \frac{3}{5}e^{-\frac{3}{5}\left(\frac{y-2}{3}\right)} & y > 2 \\ 0 & y \le 2 \end{cases}$$
$$= \begin{cases} \frac{1}{5}e^{-\frac{1}{5}(y-2)} & y > 2 \\ 0 & y \le 2 \end{cases}$$

Hence, correct option is 1.

4) Let the probability density function of a random variable X be

$$f(x) = \begin{cases} x & 0 \leqslant x < \frac{1}{2} \\ c(2x-1)^2 & \frac{1}{2} \leqslant x < 1 \\ 0 & \text{Otherwise} \end{cases}$$

Then value of c is equal to ...

Solution: We know that,

$$\int_{-\infty}^{\infty} f_x(x)\,dx = 1 \quad (30)$$

$$\int_{-\infty}^{0} f_x(x)\,dx + \int_{0}^{\frac{1}{2}} f_x(x)\,dx + \int_{\frac{1}{2}}^{1} f_x(x)\,dx + \int_{1}^{\infty} f_x(x)\,dx = 1 \quad (31)$$

$$\int_{0}^{\frac{1}{2}} x\,dx + \int_{\frac{1}{2}}^{1} c(2x-1)^2\,dx = 1 \quad (32)$$

$$\left[\frac{x^2}{2}\right]_0^{\frac{1}{2}} + c\left[\frac{(2x-1)^3}{6}\right]_{\frac{1}{2}}^{1} = 1 \quad (33)$$

$$\frac{1}{8} + \frac{c}{6} = 1 \quad (34)$$

$$c = \frac{21}{4} \quad (35)$$

$$\therefore \text{Required value of } c = \frac{21}{4}$$

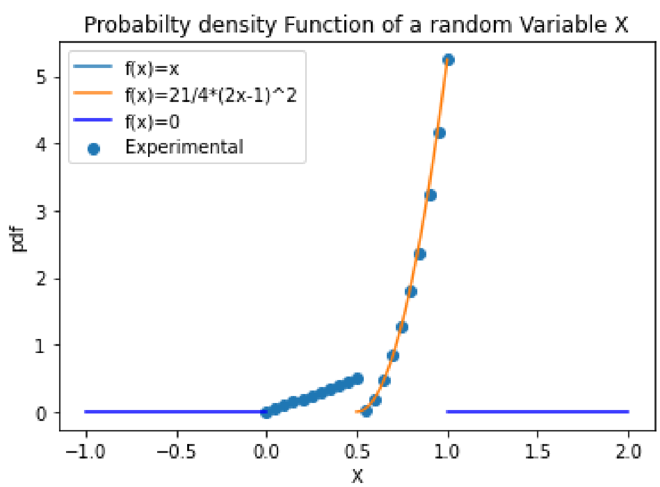

Fig. 2: Experimental and Theoritical pdf of X

5) Let $A_1, A_2, ...A_n$ be n independent events in which the Probability of occurence of the event A_i is given by P(A_i) = 1 - $\frac{1}{\alpha^i}$, $\alpha > 1$, i = 1,2,3,....n.Then the probability that atleast one of the events occurs is

a) 1 - $\frac{1}{\alpha^{\frac{n(n+1)}{2}}}$

b) $\frac{1}{\alpha^{\frac{n(n+1)}{2}}}$

c) $\frac{1}{\alpha^n}$

d) 1 - $\frac{1}{\alpha^n}$

Solution: Let $A_1 + A_2 + A_3.... + A_n$ = S,
$\Pr(S)$ = Probability of atleast one event occuring De morgan's law states that $(A+B)' = A'B'$

$$\implies \Pr(S) = 1 - \Pr(S') \quad (36)$$

$$1 - \Pr(S') = 1 - \Pr(A_1' A_2' A_3' .... A_n') \quad (37)$$

$\forall$ i $\in$ 1,2,....n
Since, A_i are independent.
$\therefore$ Complements of A_i are also independent.
$\implies$

$$\Pr(A_1' A_2' A_3' .... A_n') = \prod_{i=1}^{n} \Pr(A_i') \quad (38)$$

$$\Pr(A_i) = 1 - \frac{1}{\alpha^i} \implies \Pr(A_i') = \frac{1}{\alpha^i} \quad (39)$$

substituting (39) in (38),

$$\Pr(A_1' A_2' A_3' .... A_n') = \prod_{i=1}^{n} \frac{1}{\alpha^i} \quad (40)$$

$$\prod_{i=1}^{n} \frac{1}{\alpha^i} = \frac{1}{\alpha^{\sum_i^n i}} = \frac{1}{\alpha^{\frac{n(n+1)}{2}}} \quad (41)$$

$$\therefore \Pr(A_1' A_2' A_3' .... A_n') = \Pr(S') = \frac{1}{\alpha^{\frac{n(n+1)}{2}}} \quad (42)$$

from equations (37) and (42)

$$\implies \Pr(S) = 1 - \Pr(S') = 1 - \frac{1}{\alpha^{\frac{n(n+1)}{2}}} \quad (43)$$

$\therefore$ The correct option is **(a)**

6) Let the random variable X have the distribution function
$$F(x) = \begin{cases} 0 & if\ x < 0 \\ \frac{x}{2} & if\ 0 \le x < 1 \\ \frac{3}{5} & if\ 1 \le x < 2 \\ \frac{1}{2} + \frac{x}{8} & if\ 2 \le x < 3 \\ 1 & if\ x \ge 3 \end{cases}$$
Then $\Pr(2 \le x < 4)$ is equal to
Solution:

Given,

$$F(x) = \begin{cases} 0 & if\ x < 0 \\ \frac{x}{2} & if\ 0 \le x < 1 \\ \frac{3}{5} & if\ 1 \le x < 2 \\ \frac{1}{2} + \frac{x}{8} & if\ 2 \le x < 3 \\ 1 & if\ x \ge 3 \end{cases} \quad (44)$$

We need to find $\Pr(2 \le x < 4)$,which is also can be written as

$$\Pr(2 \le x < 4) = \Pr(x < 4) - \Pr(x < 2) \quad (45)$$
$$= F(X = 4^-) - F(X = 2^-) \quad (46)$$

Using (44) in (46),

$$\Pr(2 \le x < 4) = 1 - \frac{3}{5} \quad (47)$$
$$= \frac{2}{5} \quad (48)$$
$$= 0.4 \quad (49)$$

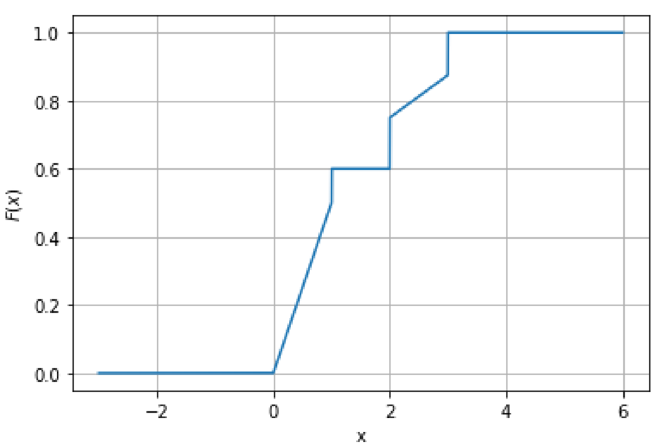

Fig. 3: cdf of random variable X

7) Let Z be the vertical coordinate, between -1 and 1, of a point chosen uniformly at random on the surface of a unit sphere in R^3. Then, $\Pr\left(\frac{-1}{2} \le Z \le \frac{1}{2}\right)$ is
Solution: The equation of the sphere can be written as : $x^2 + y^2 + z^2 = 1$. Now,

$$\Pr\left(\frac{-1}{2} \le z \le 0\right) = \Pr\left(0 \le z^2 \le \frac{1}{4}\right) \quad (50)$$
$$\Pr\left(0 \le z \le \frac{1}{2}\right) = \Pr\left(0 \le z^2 \le \frac{1}{4}\right) \quad (51)$$
$$\therefore \Pr\left(\frac{-1}{2} \le z \le \frac{1}{2}\right) = 2 \times \Pr\left(0 \le z^2 \le \frac{1}{4}\right) \quad (52)$$

$$\Pr\left(0 \le z^2 \le \frac{1}{4}\right) = \Pr\left(\frac{3}{4} \le x^2 + y^2 \le 1\right) \quad (53)$$

Taking, $x^2 + y^2 = r^2$. (54)

$$\Pr\left(\frac{3}{4} \le r^2 \le 1\right) = \frac{1}{4} \quad (55)$$

$\left(\text{Since, } r^2 \text{ is uniform between 0 and 1}\right)$

$$\therefore \Pr\left(\frac{-1}{2} \le Z \le \frac{1}{2}\right) = 2 \times \frac{1}{4} = \frac{1}{2} \quad (56)$$

8) Let X_1 and X_2 be independent geometric random variables with the same probability mass function given by $\Pr(X = k) = p(1-p)^{k-1}$, $k = 1, 2, \ldots$ Then the value of $\Pr(X_1 = 2 | X_1 + X_2 = 4)$ correct up to three decimal places is
Solution: Let

$$p_{X_i}(k) = \Pr(X_i = k) = \begin{cases} p(1-p)^{k-1} & n = 1, 2, ... \\ 0 & otherwise \end{cases} \quad (57)$$

where i=1,2

$$\Pr(A|B) = \frac{\Pr(AB)}{\Pr(B)} \quad (58)$$

$$(X_1 = 2) \cap (X_1 + X_2 = 4) = (X_1 = 2, X_2 = 2) \quad (59)$$

Thus,

$$\Pr(X_1 = 2 | X_1 + X_2 = 4) = \frac{\Pr(X_1 = 2, X_2 = 2)}{\Pr(X_1 + X_2 = 4)} \quad (60)$$

Since the two events are independent,

$$\Pr(X_1 = 2|X_1 + X_2 = 4) = \frac{\Pr(X_1 = 2)\Pr(X_2 = 2)}{\Pr(X_1 + X_2 = 4)} \tag{61}$$

Let

$$X = X_1 + X_2 \tag{62}$$

From (62),

$$p_X(n) = \Pr(X_1 + X_2 = n) = \Pr(X_1 = n - X_2) \tag{63}$$

$$= \sum_k \Pr(X_1 = n - k|X_2 = k)\, p_{X_2}(k) \tag{64}$$

after unconditioning. $\because$ X_1 and X_2 are independent,

$$\Pr(X_1 = n - k|X_2 = k) = \Pr(X_1 = n - k) = p_{X_1}(n - k) \tag{65}$$

From (64) and (65),

$$p_X(n) = \sum_k p_{X_1}(n - k)p_{X_2}(k) = p_{X_1}(n) * p_{X_2}(n) \tag{66}$$

where $*$ denotes the convolution operation. Substituting from (57) in (66),

$$p_X(n) = \sum_{k=1}^{n-1} p_{X_1}(n - k)p_{X_2}(k) \tag{67}$$

$$= \sum_{k=1}^{n-1} (1 - p)^{k-1} p \cdot (1 - p)^{n-k-1} p \tag{68}$$

$$= (1 - p)^{n-2} p^2 \sum_{k=1}^{n-1} 1 \tag{69}$$

$$= (n - 1)(1 - p)^{n-2} p^2 \tag{70}$$

From (70) and (57) we have

$$\Pr(X_1 = 2) = \Pr(X_2 = 2) = p(1 - p) \tag{71}$$

$$\Pr(X_1 + X_2 = 4) = 3(1 - p)^2 p^2 \tag{72}$$

Substituting in (61)

$$\Pr(X_1 = 2|X_1 + X_2 = 4) = \frac{(1 - p)^2 p^2}{3(1 - p)^2 p^2} \tag{73}$$

$$= \frac{1}{3} \tag{74}$$

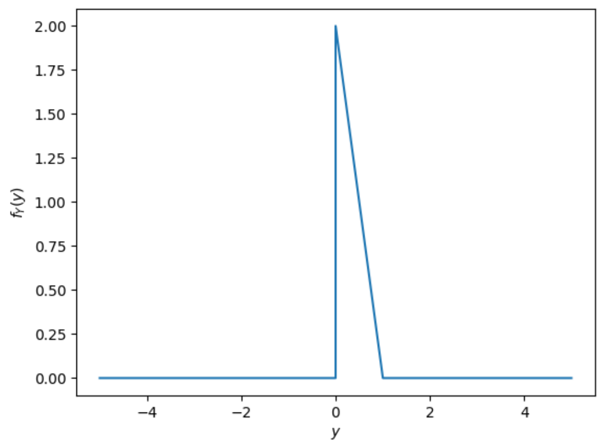

Fig. 4: Marginal PDF

9) Let X and Y have joint probability function given by

$$f_{X,Y}(x, y) = \begin{cases} 2 & 0 \le x \le 1 - y, 0 \le y \le 1 \\ 0 & otherwise \end{cases}$$

If f_Y denotes the marginal probability density function of Y, then $f_Y(1/2) =$?

Solution:

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y).dx \tag{23.1}$$

$$\implies f_Y(y) = \begin{cases} 0 + \int_0^{1-y} 2.dx & 0 \le y \le 1 \\ 0 & otherwise \end{cases} \tag{23.2}$$

$$\implies f_Y(y) = \begin{cases} 2(1 - y) & 0 \le y \le 1 \\ 0 & otherwise \end{cases} \tag{23.3}$$

$$\therefore f_Y(1/2) = 1 \tag{23.4}$$

10) Let X be a standard normal random variable. Then $\Pr(X < 0|\ \|\lfloor X \rfloor\| = 1)$ is equal to

a) $\dfrac{\Phi(1) - \frac{1}{2}}{\Phi(2) - \frac{1}{2}}$

b) $\dfrac{\Phi(1) + \frac{1}{2}}{\Phi(2) + \frac{1}{2}}$

c) $\dfrac{\Phi(1)-\frac{1}{2}}{\Phi(2)+\frac{1}{2}}$

d) $\dfrac{\Phi(1)+1}{\Phi(2)+1}$

Solution:

$$|\lfloor X\rfloor| = 1 \tag{75}$$
$$\implies \lfloor X\rfloor = 1 \text{ } or \text{ } -1 \tag{76}$$
$$\implies X \in [1,2) \cup [-1,0) \tag{77}$$

Here

$\lfloor X\rfloor = greatest\ integer\ less\ than\ or\ equal\ to\ X$

Thus required probability

$$= \frac{\Pr(X \in [-1,0))}{\Pr(X \in [1,2) \cup [-1,0))} \tag{78}$$

Using symmetry of standard normal random variable about y = 0, we have required probability

$$= \frac{\Pr(X \in (0,1])}{\Pr(X \in [1,2) \cup (0,1]))} \tag{79}$$
$$= \frac{\Pr(X \in (0,1])}{\Pr(X \in (0,2))} \tag{80}$$
$$= \frac{\Pr(X < 1) - \Pr(X < 0)}{\Pr(X < 2) - \Pr(X < 0)} \tag{81}$$
$$= \frac{\Phi(1) - \Phi(0)}{\Phi(2) - \Phi(0)} \tag{82}$$
$$= \frac{\Phi(1) - \frac{1}{2}}{\Phi(2) - \frac{1}{2}} \tag{83}$$
$$= \frac{0.841 - 0.5}{0.977 - 0.5} \tag{84}$$
$$= 0.715 \tag{85}$$

Here $\Phi(x)$ represents the standard normal cumulative density function. Thus

$$X \sim \mathcal{N}1 \tag{86}$$

and

$$\Phi(x) = \int_{-\infty}^{x} f_X(x)dx \tag{87}$$

It can easily be seen that $\Phi(0) = \frac{1}{2}$, which has been used to obtain (83). (84) was obtained by consulting tables for $\Phi(x)$

11) Let X be a random variable with probability mass function $p(n) = \left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^{n-1}$ $n = 1,2\ldots$ Then $E[X-3|X>3]$ is ...
Solution:
Given

$$\Pr(X = n) = \begin{cases} \left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^{n-1} & n = 1,2\ldots \\ 0 & otherwise \end{cases} \tag{88}$$

Using the linearity of the expectation operator:

$$E[X-3 \mid X>3] = E[X \mid X>3] - 3 \tag{89}$$

Now ,

$$E[X \mid X>3] = \sum_{x=1}^{\infty} x \Pr(X = x \mid X > 3) \tag{90}$$
$$= \sum_{x=1}^{\infty} x \frac{\Pr(X = x, X > 3)}{\Pr(X > 3)} \tag{91}$$

Calculating $\Pr(X > 3)$

$$\Pr(X > 3) = 1 - \Pr(X \le 3) \tag{92}$$
$$= 1 - \sum_{x'=1}^{3} \Pr(X = x') \tag{93}$$
$$= 1 - \sum_{x'=1}^{3} \left(\frac{3}{4}\right)^{x'-1} \left(\frac{1}{4}\right) \tag{94}$$
$$= \frac{27}{64} \tag{95}$$

Also,

$$\Pr(X = x, X > 3) = \begin{cases} \Pr(X = x) & x > 3 \\ 0 & x \le 3 \end{cases} \tag{96}$$

Substituting (95) and (96) in (91) we get

$$E[X \mid X>3] = \sum_{x=1}^{3} 0 + \sum_{x=4}^{\infty} \left[x \frac{\Pr(X = x)}{\frac{27}{64}} \right] \tag{97}$$
$$= \frac{64}{27} \sum_{x=4}^{\infty} \left[x \left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^{x-1} \right] \tag{98}$$
$$= \frac{16}{27} \sum_{x=4}^{\infty} \left[x \left(\frac{3}{4}\right)^{x-1} \right] \tag{99}$$

Let

$$S = \sum_{x=4}^{\infty}\left[x\left(\frac{3}{4}\right)^{x-1}\right] \tag{100}$$

Multiplying ((100)) with $\frac{3}{4}$ on both sides gives

$$\frac{3}{4}S = \sum_{x=4}^{\infty} x\frac{1}{4}\left(\frac{3}{4}\right)^{x} \tag{101}$$

From (101) and(100)

$$S = 4\left(\frac{3}{4}\right)^3 + 5\left(\frac{3}{4}\right)^4 + 6\left(\frac{3}{4}\right)^5 \dots \tag{102}$$

$$\frac{3}{4}S = 0\left(\frac{3}{4}\right)^3 + 4\left(\frac{3}{4}\right)^4 + 5\left(\frac{3}{4}\right)^5 + \dots \tag{103}$$

subtracting (101) from (100) we get

$$\frac{S}{4} = 4\left(\frac{3}{4}\right)^3 + \left(\frac{3}{4}\right)^4 + \left(\frac{3}{4}\right)^5 + \left(\frac{3}{4}\right)^6 + \dots \tag{104}$$

$$= 4\left(\frac{3}{4}\right)^3 + \sum_{x=4}^{\infty}\left(\frac{3}{4}\right)^x \tag{105}$$

$$= \frac{189}{64} \tag{106}$$

Substituting vale of S in (99) we get

$$E[X|X > 3] = 7 \tag{107}$$

Thus putting this in (89)

$$E[X - 3|X > 3] = 4 \tag{108}$$

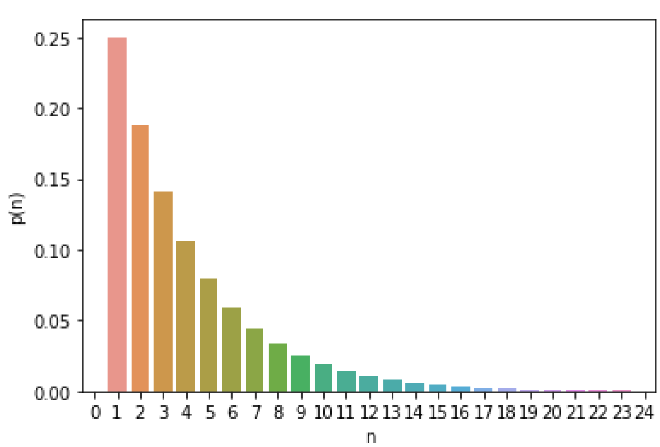

Fig. 5: PMF of X

12) Let (X,Y) be a random vector such that, for any $y > 0$, the conditional probability density function of X given $Y = y$ is

$$f_{X|Y=y}(x) = ye^{-yx}, x > 0.$$

If the marginal probability density function of Y is

$$g(y) = ye^{-y}, y > 0$$

then $E(Y|x = 1) =$

Solution: Given, the conditional probability density function of X given $Y = y$,

$$f_{X|Y=y}(x) = ye^{-yx}, x > 0 \tag{109}$$

and, the marginal probability density function of Y,

$$g(y) = ye^{-y}, y > 0 \tag{110}$$

let the joint probability density function of (X,Y) be $f_{X,Y}(x, y)$. We know that,

$$f_{X|Y=y}(x) = \frac{f_{X,Y}(x, y)}{g(y)} \tag{111}$$

using (109) and (110) in (111),

$$f_{X,Y}(x, y) = y^2e^{-y(x+1)}, x, y > 0 \tag{112}$$

let the marginal probability density function of X be $f_X(x)$, as we know ,

$$f_X(x) = \int_0^{\infty} f_{X,Y}(x, y)\, dy \tag{113}$$

using (112) in (113),

$$f_X(x) = \int_0^{\infty} y^2e^{-y(x+1)}\, dy \tag{114}$$

$$= \frac{2}{(x + 1)^3}, x > 0 \tag{115}$$

The conditional probability density function of Y given $X = x$ is given by,

$$f_{Y|X=x}(y) = \frac{f_{X,Y}(x, y)}{f_X(x)} \tag{116}$$

using (112) and (115) in (116),

$$f_{Y|X=x}(y) = \frac{y^2e^{-y(x+1)}(x + 1)^3}{2}, x, y > 0 \tag{117}$$

The conditional probability density function of Y given $X = 1$ is given by,

$$f_{Y|X=1}(y) = 4y^2e^{-2y}, y > 0 \tag{118}$$

We need to find $E(Y|X=1)$ which is given by,

$$E(Y|X=1) = \int_0^\infty y f_{Y|X=1}(y)\, dy \tag{119}$$

using (118) in (119),

$$E(Y|X=1) = \int_0^\infty 4y^3 e^{-2y}\, dy \tag{120}$$

$$= \left[\frac{-e^{-2y}(8y^3+12y^2+12y+6)}{4}\right]_0^\infty \tag{121}$$

$$= \frac{3}{2} \tag{122}$$

13) Let X and Y be jointly distributed random variables having the joint probability density function

$$f(x,y) = \begin{cases} \frac{1}{\pi}, & \text{if} \quad x^2+y^2 \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Then $\Pr(Y > \max(X,-X))$ is
Solution:
The pdf of X and Y are:

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy \tag{123}$$

$$= \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi}dy \tag{124}$$

$$= \frac{2\sqrt{1-x^2}}{\pi} \tag{125}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx \tag{126}$$

$$= \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{\pi}dx \tag{127}$$

$$= \frac{2\sqrt{1-y^2}}{\pi} \tag{128}$$

The cdf of Y is:

$$F_Y(y) = \int_{-\infty}^{y} f_Y(y)dy \tag{129}$$

$$= \int_{-1}^{y} \frac{2\sqrt{1-y^2}}{\pi}dy \tag{130}$$

$$= \frac{2}{\pi}\left(\frac{\sin^{-1} y + y\sqrt{1-y^2}}{2} + \frac{\pi}{4}\right) \tag{131}$$

The value of $\Pr(-X < Y < X)$ is:

$$\Pr(-X < Y < X) = F_Y(X) - F_Y(-X) \tag{132}$$

$$= \frac{2}{\pi}\left(\sin^{-1} X + X\sqrt{1-X^2}\right) \tag{133}$$

Integrating our probability over all of X we get the value of $E[\Pr(-x < Y < x)]$:

$$= \int_{-\infty}^{\infty} f_X(x) \Pr(-x < Y < x)\, dx \tag{134}$$

$$= \left(\frac{2}{\pi}\right)^2 \int_0^1 \sqrt{1-x^2}\left(\sin^{-1} x + x\sqrt{1-x^2}\right) dx \tag{135}$$

Substituting

$$u = \sin^{-1} x + x\sqrt{1-x^2} \tag{136}$$

$$\frac{du}{dx} = 2\sqrt{1-x^2} \tag{137}$$

$$= \left(\frac{2}{\pi}\right)^2 \int_0^{\frac{\pi}{2}} \frac{u}{2}du \tag{138}$$

$$= \left(\frac{2}{\pi}\right)^2 \left(\left.\frac{u^2}{4}\right|_0^{\frac{\pi}{2}}\right) \tag{139}$$

$$= \left(\frac{2}{\pi}\right)^2 \left(\frac{\pi^2}{16} - 0\right) \tag{140}$$

$$= \frac{4\cdot\pi^2}{\pi^2\cdot 16} \tag{141}$$

$$= \frac{1}{4} \tag{142}$$

The probability for:

$$\Pr(Y > \max(X,-X)) = \frac{1}{4} \tag{143}$$

14) Let X and Y be two continuous random variables with the joint probability density function

$$f(x,y) = \begin{cases} 2, & 0 < x+y < 1, x > 0, y > 0, \\ 0, & elsewhere. \end{cases} \tag{144}$$

$E\left(X \,\middle|\, Y = \frac{1}{2}\right)$ is

a) $1/4$

b) $1/2$

c) 1

d) 2

Solution:
The PDF of X and Y is,

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy \tag{145}$$

$$f_X(x) = \int_{0}^{1-x} 2dy \tag{146}$$

$$f_X(x) = 2 - 2x \tag{147}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx \tag{148}$$

$$f_Y(y) = \int_{0}^{1-y} 2dx \tag{149}$$

$$f_Y(y) = 2 - 2y \tag{150}$$

$$f_Y\left(\frac{1}{2}\right) = 1 \tag{151}$$

by using Bayes theorem,

$$f_{X|Y}\left(x \mid \frac{1}{2}\right) = \frac{f_{X,Y}\left(x, \frac{1}{2}\right)}{f_Y\left(\frac{1}{2}\right)} \tag{152}$$

$$f_{X|Y}\left(x \mid \frac{1}{2}\right) = \begin{cases} 2, & 0 < x < \frac{1}{2}, \\ 0, & elsewhere. \end{cases} \tag{153}$$

It is in the form of Bernoulli distribution, the expectation value is given by,

$$E\left(X \mid Y = \frac{1}{2}\right) = \sum_{-\infty}^{\infty} x f_{X|Y}\left(x \mid \frac{1}{2}\right) \tag{154}$$

$$E\left(X \mid Y = \frac{1}{2}\right) = \int_{-\infty}^{0} x(0)dx + \int_{0}^{\frac{1}{2}} x(2)dx + \int_{0}^{-\infty} x(0)dx \tag{155}$$

$$E\left(X \mid Y = \frac{1}{2}\right) = \int_{0}^{\frac{1}{2}} 2xdx \tag{156}$$

$$E\left(X \mid Y = \frac{1}{2}\right) = \frac{1}{4} \tag{157}$$

15) An urn contains four balls, each ball having equal probability of being white or black. Three black balls are added to the urn. The probability that five balls in the urn are black is
Solution:
The total number of black balls are 5
Number of black balls initially present + number of black balls added =5
So, the number of black balls initially in the urn is 5-3=2
Let X be the random variable denoting the number of black balls in the urn. So, by binomial distribution,

$$\Pr(X = 1) = p \tag{158}$$

$$\Pr(X = k) = \binom{n}{k} p^k (1-p)^{n-k} \tag{159}$$

$$k = 0, 1, 2, ..., n \tag{160}$$

For the given problem, $n = 4$ and $p = 0.5$, because there is equal probability for each ball of being white or black. For having exactly 2 black balls,
From (160),

$$\Pr(X = 2) = \binom{4}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 \tag{161}$$

$$= \frac{6}{16} \tag{162}$$

$$= \frac{3}{8} \tag{163}$$

16) There are five bags each containing identical sets of ten distinct chocolates. One chocolate is picked from each bag.
The probability that at least two chocolates are identical is
Solution:
Let $X \in \{0, 1, 2, 3, 4, 5\}$ represent the random variable, denoting the number of similar

chocolates in the picked chocolates
Here, we can neglect X=1 because there can't be one similar object.

$$\Pr(X \geq 2) + \Pr(X = 0) = 1 \tag{164}$$
$$\Pr(X = 0) = \frac{10.9.8.7.6}{10^5} \tag{165}$$
$$\Pr(X = 0) = 0.3024 \tag{166}$$
$$\Pr(X \geq 2) = 1 - \Pr(X = 0) \tag{167}$$
$$= 1 - 0.3024 \tag{168}$$
$$= 0.6976 \tag{169}$$

Consider the trinomial distribution with the probability mass function

$$\Pr(X = x, Y = y) = \left(\frac{7!}{x!y!(7-x-y)!}\right)(0.6)^x(0.2)^y(0.2)^{7-x-y}$$

where $x \geq 0, y \geq 0$ *and* $x + y \leq 7$. Then $E(Y|X = 3)$ is equal to
Solution: Probability mass function of a trinomial distribution is :

$$\Pr(X = x, Y = y) = \left(\frac{7!}{x!y!(7-x-y)!}\right)(0.6)^x(0.2)^y(0.2)^{7-x-y}$$
$$= \left(\frac{7!}{x!(7-x)!}\frac{(7-x)!}{y!(7-x-y)!}\right)(0.6)^x(0.2)^y(0.2)^{7-x-y}$$
$$\Pr(X = x, Y = y) = {}^7C_x {}^{7-x}C_y(0.6)^x(0.2)^y(0.2)^{7-x-y} \tag{170}$$

Using (170), $\Pr(X = x)$ is

$$\Pr(X = x) = \sum_{y=0}^{7-x} \Pr(X = x, Y = y)$$
$$= {}^7C_x(0.6)^x \sum_{y=0}^{7-x} {}^{7-x}C_y(0.2)^y(0.2)^{7-x-y}$$
$$= {}^7C_x(0.6)^x (0.4)^{7-x}$$
$$\Pr(X = x) = {}^7C_x(0.6)^x (0.4)^{7-x} \tag{171}$$

We have to find $E[Y|X = 3]$,

$$E[Y|X = 3] = \sum_{y=0}^{4} y \Pr(Y = y|X = 3) \tag{172}$$
$$E[Y|X = 3] = \sum_{y=0}^{4} y\left(\frac{\Pr(X = 3, Y = y)}{\Pr(X = 3)}\right) \tag{173}$$

By taking X=3 in (170) and (171) to use in (173),

$$E[Y|X = 3] = \sum_{y=0}^{4} y\left(\frac{\Pr(X = 3, Y = y)}{\Pr(X = 3)}\right)$$
$$= \sum_{y=0}^{4} y\left(\frac{{}^7C_3 {}^4C_y(0.6)^3(0.2)^y(0.2)^{4-y}}{{}^7C_3(0.6)^3 (0.4)^4}\right)$$
$$= \sum_{y=0}^{4} y\left(\frac{{}^4C_y(0.2)^4}{(0.4)^4}\right)$$
$$E[Y|X = 3] = \sum_{y=0}^{4} \frac{y({}^4C_y)}{16} \tag{174}$$

We know that,

$${}^nC_r = \frac{n}{r}\left({}^{n-1}C_{r-1}\right) \tag{175}$$

Using (175) in (174),

$$E[Y|X = 3] = \frac{1}{16}\sum_{y=0}^{4} y({}^4C_y) \tag{176}$$
$$= \frac{1}{16}\sum_{y=1}^{4} y\left(\frac{4}{y}\right)({}^3C_{y-1}) \tag{177}$$
$$= \frac{1}{4}\sum_{k=0}^{3}({}^3C_k) \tag{178}$$
$$= \frac{1}{4}(1 + 1)^3 = \frac{1}{4}(8) \tag{179}$$
$$E[Y|X = 3] = 2 \tag{180}$$

Therefore the value of $E[Y|X = 3] = 2$.

17) Let E and F be any two events with $P(E \cup F) = 0.8$, $P(E) = 0.4$ and $P(E|F) = 0.3$ then P(F) is

a) $\frac{3}{7}$

b) $\frac{4}{7}$

c) $\frac{3}{5}$

d) $\frac{2}{5}$

Solution: Given,

$$\Pr(E) = 0.4 \tag{181}$$
$$\Pr(E+F) = 0.8 \tag{182}$$
$$\Pr(E|F) = 0.3 \tag{183}$$

By definition,

$$\Pr(E|F) = \frac{\Pr(EF)}{\Pr(F)} \tag{184}$$
$$\implies \Pr(EF) = \Pr(E|F) \times \Pr(F) \tag{185}$$
$$\implies \Pr(EF) = 0.3 \times \Pr(F) \tag{186}$$

Now using the identity,

$$\Pr(E+F) = \Pr(E) + \Pr(F) - \Pr(EF) \tag{187}$$

From (181),(182) and (186)

$$\implies 0.8 = 0.4 + \Pr(F) - (0.3 \times \Pr(F)) \tag{188}$$
$$\implies 0.4 = (1-0.3) \times \Pr(F) \tag{189}$$
$$\implies \Pr(F) = \frac{0.4}{0.7} \tag{190}$$
$$\boxed{\Pr(F) = \frac{4}{7}} \tag{191}$$

18) The number N of persons getting injured in a bomb blast at a busy market place is a random variable having a Poisson Distribution with parameter $\lambda (\geq 1)$. A person injured in the explosion may either suffer a minor injury requiring first aid or suffer a major injury requiring hospitalisation. Let the number of persons with minor injury be N_1 and the conditional distribution of N_1 given N is

$$\Pr(N_1 = i|N) = \frac{1}{N} \tag{192}$$

Find the expected number of persons requiring hospitalisation. **Solution:** We know,

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)} \tag{193}$$

Also, for a Poisson Distribution:

$$\Pr(N = x) = \frac{e^{-\lambda}\lambda^x}{x!} \tag{194}$$

where λ is the parameter
Let N_2 be the number of persons hospitalised.
Let $N = a$, and $N_1 = i (i \leq a)$, then, $N_2 = a - i$
Then, from (192) and (194):

$$\Pr(N_2 = a - i) = \Pr(N_1 = i) \tag{195}$$
$$= \Pr(N_1 = i|N = a)\Pr(N = a) \tag{196}$$
$$= \frac{1}{a}\frac{e^{-\lambda}\lambda^a}{a!} \tag{197}$$

Thus,

$$E(N_2) = \sum_{a=0}^{\infty}\sum_{i=0}^{a}(a-i) \times \frac{1}{a}\frac{e^{-\lambda}\lambda^a}{a!} \tag{198}$$
$$= \sum_{a=0}^{\infty}\frac{e^{-\lambda}\lambda^a}{a!}\sum_{i=0}^{a}\frac{a-i}{a} \tag{199}$$
$$= \sum_{a=0}^{\infty}\frac{e^{-\lambda}\lambda^a}{a!}\left(a - \frac{(a+1)}{2}\right) \tag{200}$$
$$= \sum_{a=0}^{\infty}\frac{e^{-\lambda}\lambda^a}{a!}\frac{a-1}{2} \tag{201}$$
$$= \frac{e^{-\lambda}}{2}\left[\sum_{a=0}^{\infty}\frac{a\lambda^a}{a!} - \sum_{a=0}^{\infty}\frac{\lambda^a}{a!}\right] \tag{202}$$
$$= \frac{e^{-\lambda}}{2}\left[\lambda\sum_{a=1}^{\infty}\frac{\lambda^{a-1}}{(a-1)!} - \sum_{a=0}^{\infty}\frac{\lambda^a}{a!}\right] \tag{203}$$
$$= \frac{e^{-\lambda}}{2}\left[\lambda e^{\lambda} - e^{\lambda}\right] \tag{204}$$
$$= \frac{\lambda - 1}{2} \tag{205}$$

19) The time to failure, in months, of lights bulbs manufactured at two plants A and B obey the exponential distributions with means 6 and 2 months respectively. Plant B produces four times as many bulbs as plant A does. Bulbs from these two plants are indistinguishable. They are mixed and sold together. Given that a bulb purchased at random is working after 12 months, What is the probability that it was manufactured in plant A?
Solution:
This problem involves Bayes theorem and Exponential distribution

- Probability that bulb is from Plant A = $\Pr(A) = \frac{1}{5}$
- Probability that bulb is from Plant B = $\Pr(B) = \frac{4}{5}$

One can use exponential distribution to find out the probability that the bulbs work after 12 months
Let X be a variable representing the lifetime of a bulb in months.
So X has a Cumulative distribution Function:

$$F_X(x, \lambda) = \begin{cases} 1 - e^{-\lambda x} & if \quad x \geq 0 \\ 0 & if \quad x < 0 \end{cases} \tag{206}$$

Let us denote that the bulbs works after 12 months with the variable W.

$\frac{1}{\lambda}$	Mean of distribution
x	Time to failure (in months)
λ_A	$\frac{1}{6}$
λ_B	$\frac{1}{2}$
$\Pr(X \leq k)$	$F_X(X, \lambda)$

$$\begin{aligned} \Pr(W \mid A) =& 1 - \Pr(\text{Fails within 12 months} \mid \text{A}) \\ =& 1 - F_X(12, \lambda_A) \qquad (207) \\ =& e^{-\lambda_A \times 12} \qquad (208) \end{aligned}$$

$$\begin{aligned} \Pr(W \mid B) =& 1 - \Pr(\text{Fails within 12 months} \mid \text{B}) \\ =& 1 - F_X(12, \lambda_B) \qquad (209) \\ =& e^{-\lambda_B \times 12} \qquad (210) \end{aligned}$$

From Bayes theorem,

$$\Pr(A \mid W) = \frac{Pr(A) \times Pr(W \mid A)}{Pr(A) \times Pr(W \mid A) + Pr(B) \times Pr(W \mid B)} \tag{211}$$

$$= \frac{Pr(A) \times e^{-\lambda_A \times 12}}{Pr(A) \times e^{-\lambda_A \times 12} + Pr(B) \times e^{-\lambda_B \times 12}} \tag{212}$$

Substituting the known values, we get

$$\Pr(A \mid W) = \frac{\frac{1}{5} \times e^{-2}}{\frac{1}{5} \times e^{-2} + \frac{4}{5} \times e^{-6}} \tag{213}$$

$$= 0.93173845935 \tag{214}$$

So the probability that the Bulb is manufactured in Plant A given that it works after a year is 0.93173845935.

20) The lifetime of two brands of bulbs X and Y are exponentially distributed with the mean life of 100 hours. Bulb X is switched on 15 hours after bulb Y has been switched on. The probability that bulb X fails before bulb Y is

(A) $\frac{15}{100}$

(B) $\frac{1}{2}$

(C) $\frac{85}{100}$

(D) 0

Solution: Let X and Y be exponential random variables which represent the lifetime of bulbs X and Y respectively, both with mean = 100. Using memorylessness property for exponential distribution, which states that :
An exponentially distributed random variable T obeys the relation

$$\Pr(T > t + s | T > s) = \Pr(T > t) \tag{215}$$

where $s, t \geq 0$
Proof : Using Complementary cumulative dis-

tributive function, we get

$$\Pr(T > t+s | T > s) = \frac{\Pr(T > t+s, T > s)}{\Pr(T > s)} \tag{216}$$

$$= \frac{\Pr(T > t+s)}{\Pr(T > s)} \tag{217}$$

$$= \frac{e^{-\lambda(t+s)}}{e^{-\lambda s}} \tag{218}$$

$$= e^{-\lambda t} \tag{219}$$

$$= \Pr(T > t) \tag{220}$$

Probability that bulb X fails before bulb Y given that bulb Y was functioning when bulb X was switched on

$$\Pr(Y > X + 15 | Y \geq 15) = \Pr(Y > X) \tag{221}$$

For both X and Y,

$$\lambda = \frac{1}{100} = 0.01 \tag{222}$$

Probability distribution function of exponential random variables is given by : For x,y ≥ 0

$$f_X(x) = \lambda e^{-\lambda x} \tag{223}$$

$$f_Y(y) = \lambda e^{-\lambda y} \tag{224}$$

Cumulative distribution function of exponential random variables is given by : For x ≥ 0

$$F_X(x) = 1 - e^{-\lambda x} \tag{225}$$

$$F_Y(x) = 1 - e^{-\lambda x} \tag{226}$$

$$\Pr(Y > X) = \int_{-\infty}^{\infty} F_Y(x) f_X(x) dx \tag{227}$$

$$= \int_{0}^{\infty} (1 - e^{-\lambda x}) \lambda e^{-\lambda x} \tag{228}$$

$$= \lambda \left(\frac{1}{2\lambda} e^{-2\lambda x} - \frac{1}{\lambda} e^{-\lambda x} \right) \Big|_0^\infty \tag{229}$$

$$= \left(\frac{1}{2} e^{-2\lambda x} - e^{-\lambda x} \right) \Big|_0^\infty \tag{230}$$

$$= \left(\frac{1}{2} e^{-0.02x} - e^{-0.01x} \right) \Big|_0^\infty \tag{231}$$

$$= \frac{1}{2} = 0.5 \tag{232}$$

$\therefore$The answer is option (b) $\frac{1}{2}$.

21) Let $X_1, X_2, \ldots,$ be a sequence of independent and identically distributed random variables with $P(X_1 = 1) = \frac{1}{4}$ and $P(X_1 = 2) = \frac{3}{4}$. If $\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$, for $n = 1, 2, \ldots,$ then $\lim_{n \to \infty} P(\bar{X}_n \leq 1.8)$ is equal to

Solution:

Given,

$$Pr(X_1 = 1) = \frac{1}{4}, Pr(X_2 = 2) = \frac{3}{4} \tag{32.1}$$

As $X_1, X_2, \ldots,$ are identically distributed random variables, $\forall i \in \{1, 2, \ldots, n\}$

$$Pr(X_i = 1) = \frac{1}{4}, Pr(X_i = 2) = \frac{3}{4} \tag{32.2}$$

Also,

$$\because P(X_i = 1) + P(X_i = 2) = 1 \tag{32.3}$$

$$\therefore X_i \in \{1, 2\} \tag{32.4}$$

Therefore, each X_i is a bernoulli distribution with

$$p = \frac{3}{4}, q = \frac{1}{4} \tag{32.5}$$

Let

$$X = \sum_{i=1}^{n} X_i \tag{32.6}$$

be a binomial distribution. Its CDF is

$$Pr(X \leq n + r) = \sum_{k=0}^{r} {}^nC_k p^k q^{n-k} \tag{32.7}$$

To find : $\lim_{n \to \infty} Pr(\bar{X}_n \leq a)$

$$\bar{X}_n \leq a \Rightarrow X \leq na \tag{32.8}$$

Substituting $a(= 1.8), p, q$, we get

$$\lim_{n\to\infty} Pr(\bar{X}_n \le 1.8) = \lim_{n\to\infty} P(X \le 1.8n) \tag{32.9}$$

$$= \sum_{k=0}^{0.8n} \frac{{}^nC_k 3^k}{4^n} \tag{32.10}$$

On solving (32.10), we get

$$\lim_{n\to\infty} P(\bar{X}_n \le 1.8) = 1 \tag{32.11}$$

22) Let X be the number of heads in 4 tosses of a fair coin by Person 1 and let Y be the number of heads in 4 tosses of a fair coin by Person 2. Assume that all the tosses are independent. Then the value of $\Pr(X = Y)$ correct up to three decimal places is ______ . **Solution:** Let $X \in \{0, 1, 2, 3, 4\}$ be the random variable representing the number of heads obtained by Person 1 in 4 tosses. Similarly, Let $Y \in \{0, 1, 2, 3, 4\}$ be the random variable representing the number of heads obtained by Person 2 in 4 tosses. Then X and Y are binomial distributions with parameter:

$$p = \frac{1}{2} \tag{233}$$

Then,

$$\Pr(X = i) = \begin{cases} {}^4C_k(p)^k(1-p)^{4-k} & i \in \{0, 1, 2, 3, 4\} \\ 0 & \text{otherwise} \end{cases} \tag{234}$$

$$\Pr(X = i) = \begin{cases} {}^4C_k(\frac{1}{2})^k(1-\frac{1}{2})^{4-k} & i \in \{0, 1, 2, 3, 4\} \\ 0 & \text{otherwise} \end{cases} \tag{235}$$

$$\Pr(X = i) = \begin{cases} {}^4C_k \times (\frac{1}{2})^4 & i \in \{0, 1, 2, 3, 4\} \\ 0 & \text{otherwise} \end{cases} \tag{236}$$

Serial number	Case	Probability of the case
1	$\Pr(X = 0)$	$\frac{{}^4C_0}{16} = \frac{1}{16}$
2	$\Pr(X = 1)$	$\frac{{}^4C_1}{16} = \frac{4}{16}$
3	$\Pr(X = 2)$	$\frac{{}^4C_2}{16} = \frac{6}{16}$
4	$\Pr(X = 3)$	$\frac{{}^4C_3}{16} = \frac{4}{16}$
5	$\Pr(X = 4)$	$\frac{{}^4C_4}{16} = \frac{1}{16}$

TABLE I: Probability distribution table of X

Similar is the distribution of Y. For finding $\Pr(X = Y)$, let $Y = y$,

$$\Pr(X = Y) = \frac{{}^4C_y}{16} \times \Pr(Y = y) \tag{237}$$

Generalizing this result,

$$\Pr(X = Y) = \sum_{y=0}^{4} \frac{{}^4C_y}{16} \times \Pr(Y = y) \tag{238}$$

$$= \sum_{y=0}^{4} \frac{{}^4C_y}{16} \times \frac{{}^4C_y}{16} \tag{239}$$

$$\Pr(X = Y) = \left(\frac{1}{16} \times \frac{1}{16}\right) + \left(\frac{4}{16} \times \frac{4}{16}\right) + \left(\frac{6}{16} \times \frac{6}{16}\right) + \left(\frac{4}{16} \times \frac{4}{16}\right) + \left(\frac{1}{16} \times \frac{1}{16}\right) \tag{240}$$

$$\Pr(X = Y) = \frac{1}{256} + \frac{16}{256} + \frac{36}{256} + \frac{16}{256} + \frac{1}{256} \tag{241}$$

$$= \frac{70}{256} \tag{242}$$

$$= \frac{35}{128} \tag{243}$$

$$= 0.2734375 \tag{244}$$

23) The probability density function of a random variable X is

$$f(x) = \begin{cases} \frac{1}{\lambda}e^{\left(-\frac{x}{\lambda}\right)}, & x > 0 \\ 0, & x \le 0 \end{cases} \tag{245}$$

where $\lambda > 0$. For testing the hypothesis $H_0 : \lambda = 3$ against $H_1 : \lambda = 5$, a test is given as "Reject H_0 if $X \geq 4.5$".The probability of type 1 error and power of the test are respectively:

a) 0.1353 and 0.4966
b) 0.1827 and 0.3791
c) 0.2021 and 0.4493
d) 0.2231 and 0.4066

Solution:

Definition 1. *A type 1 error occurs if the null hypothesis H_0 is rejected even if it is true.*

Definition 2. *The probability that the alternative hypothesis H_1 is true is defined to be Power of a given test.*

Given,

$$f_X(x) = \begin{cases} \frac{1}{\lambda} e^{\left(-\frac{x}{\lambda}\right)}, & x > 0 \\ 0, & x \leq 0 \end{cases} \tag{246}$$

Let cumulative distribution function be $F_X(x)$ for a given λ. Hence,

$$F_X(x) = \int_{-\infty}^{x} f_X(a) da \tag{247}$$

From the probability density function,

$$\implies F_X(4.5) = \int_{-\infty}^{x} f_X(a) da \tag{248}$$

$$= \int_{0}^{4.5} \frac{1}{\lambda} e^{\left(-\frac{a}{\lambda}\right)} da \tag{249}$$

$$= 1 - e^{-\frac{4.5}{\lambda}} \tag{250}$$

We need the probability for $X \geq 4.5$,hence required probability is,

$$1 - F_X(4.5) = e^{-\frac{4.5}{\lambda}} \tag{251}$$

From (251) we get probability that the given null hypothesis(H_0) is true is,

$$e^{-\frac{4.5}{3}} = 0.2231. \tag{252}$$

$\therefore$ The **probability of type 1 error is 0.2231**. From (251),we get the required probability that the given alternative hypothesis(H_1) is true is,

$$e^{-\frac{4.5}{5}} = 0.4066 \tag{253}$$

$\therefore$ The **power of the test is 0.4066**

24) Let E and F be any two events with $\Pr(E) = 0.4, \Pr(F) = 0.3$ and $\Pr(F|E) = 3\Pr(F|E')$. Then $\Pr(E|F)$ equals ......
Solution: Given
a) $\Pr(E) = 0.4$
b) $\Pr(F) = 0.3$
c) $\Pr(F|E) = 3\Pr(F|E')$

From given data

$$\Pr(F|E) = 3\Pr(F|E') \tag{254}$$

$$\frac{\Pr(FE)}{\Pr(E)} = 3 \times \frac{\Pr(FE')}{\Pr(E')} \tag{255}$$

$$\Pr(EF) = 2 \times \Pr(E'F) \tag{256}$$

We know that

$$\Pr(F) = \Pr(EF) + \Pr(E'F) \tag{257}$$

Using (256) and (257), we get

$$\Pr(F) = \frac{3}{2} \times \Pr(EF) \tag{258}$$

$$\frac{\Pr(EF)}{\Pr(F)} = \frac{2}{3} \tag{259}$$

$$\Pr(E|F) = \frac{2}{3} \approx 0.66 \tag{260}$$

25) Let a random variable X follow the exponential distribution with mean 2.

Define Y such that:

$$Y = [X - 2 | X > 2]$$

Then $E(Y)$ is equal to:

(A) $\frac{1}{4}$

(B) $\frac{1}{2}$

(C) 1

(D) 2

26) If A and B are two events and the probability $\Pr(B) \neq 1$,then

$$\frac{\Pr(A) - \Pr(A \cap B)}{1 - \Pr(B)} \tag{261}$$

equals

a) $\Pr(A|\bar{B})$

b) $\Pr(A|B)$

c) $\Pr(\bar{A}|B)$

d) $\Pr(\bar{A}|\bar{B})$

Solution:
Given A and B are two events,
We know that,

$$A = A(B + \bar{B}) \tag{262}$$
$$= AB + A\bar{B} \tag{263}$$

Since AB and $A\bar{B}$ are disjoint events,

$$\Pr(A) = \Pr(AB) + \Pr(A\bar{B}) \tag{264}$$

Hence,

$$\Pr(A\bar{B}) = \Pr(A) - \Pr(AB) \tag{265}$$

Since B and $\bar{B}$ are disjoint events,

$$\Pr(B) + \Pr(\bar{B}) = 1 \tag{266}$$
$$\Pr(\bar{B}) = 1 - \Pr(B) \tag{267}$$

We know that,

$$\Pr(A|\bar{B}) = \frac{\Pr(A\bar{B})}{\Pr(\bar{B})} \tag{268}$$

From (267) and (265)

$$\frac{\Pr(A) - \Pr(AB)}{1 - \Pr(B)} = \frac{\Pr(A\bar{B})}{\Pr(\bar{B})} \tag{269}$$

From (268)

$$\frac{\Pr(A) - \Pr(AB)}{1 - \Pr(B)} = \Pr(A|\bar{B}) \tag{270}$$

Hence **option A is correct**

27) If a random variable X assumes only positive integral values ,with the probability

$$P(X = x) = \frac{2}{3}\left(\frac{1}{3}\right)^{x-1}, x = 1, 2, 3, ..., \tag{271}$$

then $E(X)$ is

a) $\frac{2}{9}$

b) $\frac{2}{3}$

c) 1

d) $\frac{3}{2}$

Solution: Given that random variable X assumes only positive integral values and its probability is:

$$P(X = x) = \frac{2}{3}\left(\frac{1}{3}\right)^{x-1} \tag{272}$$

The expectation value E(X) is given by

$$E(X) = \sum_{i=1}^{\infty} i \times P(X = i) \tag{273}$$

Let $E(X) = S$
so,

$$S = \sum_{i=1}^{\infty} i \times P(X = i) \tag{274}$$

$$\implies S = \sum_{i=1}^{\infty} i \times \frac{2}{3}\left(\frac{1}{3}\right)^{i-1} \tag{275}$$

$$\implies S = \frac{2}{3} + \sum_{i=2}^{\infty} i \times \frac{2}{3}\left(\frac{1}{3}\right)^{i-1} \tag{276}$$

As

$$\sum_{i=2}^{\infty} i \times \frac{2}{3}\left(\frac{1}{3}\right)^{i-1} = \sum_{i=1}^{\infty}(i+1) \times \frac{2}{3}\left(\frac{1}{3}\right)^{i} \tag{277}$$

Now substituting (277) in (276)

$$\implies S = \frac{2}{3} + \sum_{i=1}^{\infty}(i+1) \times \frac{2}{3}\left(\frac{1}{3}\right)^{i} \tag{278}$$

$$\implies S = \frac{2}{3} + \sum_{i=1}^{\infty} i \times \frac{2}{3}\left(\frac{1}{3}\right)^{i} + \sum_{i=1}^{\infty} \frac{2}{3}\left(\frac{1}{3}\right)^{i} \tag{279}$$

Dividing with 3 on both sides in (275) gives

$$\frac{S}{3} = \sum_{i=1}^{\infty} i \times \frac{2}{3}\left(\frac{1}{3}\right)^{i} \tag{280}$$

Now substituting (280) in (279) gives

$$\implies S = \frac{2}{3} + \frac{S}{3} + \sum_{i=1}^{\infty} \frac{2}{3}\left(\frac{1}{3}\right)^{i} \tag{281}$$

$$\implies \frac{2S}{3} = \frac{2}{3} + \frac{2}{3}\sum_{i=1}^{\infty}\left(\frac{1}{3}\right)^{i} \tag{282}$$

$$\implies \frac{2S}{3} = \frac{2}{3}\left(1 + \sum_{i=1}^{\infty}\left(\frac{1}{3}\right)^{i}\right) \tag{283}$$

$$\implies S = 1 + \sum_{i=1}^{\infty}\left(\frac{1}{3}\right)^{i} \tag{284}$$

$$\implies S = 1 + \frac{\frac{1}{3}}{1 - \frac{1}{3}} \tag{285}$$

$$\implies S = 1 + \frac{1}{2} = \frac{3}{2} \tag{286}$$

$$\implies E(X) = S = \frac{3}{2} \tag{287}$$

$\therefore$ **Option D is correct**

28) Let X, Y be continuous random variables with joint density function

$$f_{X,Y}(x,y) = \begin{cases} e^{-y}(1-e^{-x}) & \text{if } 0 < x < y < \infty \\ e^{-x}(1-e^{-y}) & \text{if } 0 < y \leq x < \infty \end{cases}$$

Then The value of $E[X+Y]$ is **Solution:** Let $g(X,Y) = X + Y$ We know that,

$$E[g(X,Y)] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x,y) f_{X,Y}(x,y) dxdy$$

Then,

$$E[X+Y] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (x+y) f_{X,Y}(x,y)\, dxdy$$
$$= \int_{0}^{+\infty}\int_{0}^{+\infty} (x+y) f_{X,Y}(x,y)\, dxdy$$
$$= \int_{0}^{+\infty}\left(\int_{0}^{+\infty} x f_{X,Y}(x,y)\, dx + \int_{0}^{+\infty} y f_{X,Y}(x,y)\, dx\right.$$

First we will calculate the $\int_0^{+\infty} y f_{X,Y}(x,y)\, dx$, $\int_0^{+\infty} x f_{X,Y}(x,y)\, dx$ seperately.

consider,

$$\int_0^{+\infty} y f_{X,Y}(x,y)\,dx$$
$$= \int_0^{y} ye^{-y}(1-e^{-x})\,dx + \int_y^{+\infty} ye^{-x}(1-e^{-y})\,dx$$
$$= (ye^{-y})(y+e^{-y}-1) + y(1-e^{-y})e^{-y}$$
$$= y^2e^{-y}$$

So,

$$\int_0^{+\infty} y f_{X,Y}(x,y)\,dx = y^2e^{-y} \tag{37.1}$$

Now consider,

$$\int_0^{+\infty} x f_{X,Y}(x,y)\,dx$$
$$= \int_0^{y} xe^{-y}(1-e^{-x})\,dx + \int_y^{+\infty} xe^{-x}(1-e^{-y})$$
$$= e^{-y}\left(\frac{y^2}{2} + e^{-y}(y+1) - 1\right) + (1-e^{-y})(e^{-y}(y+1))$$
$$= \frac{y^2e^{-y}}{2} + ye^{-y}$$

So,

$$\int_0^{+\infty} x f_{X,Y}(x,y)\,dx = \frac{y^2e^{-y}}{2} + ye^{-y} \tag{37.2}$$

From Eq 37.1 and 37.2

$$E[X+Y] = \int_0^{+\infty} \left(\frac{y^2e^{-y}}{2} + ye^{-y} + y^2e^{-y}\right) dy$$
$$= \int_0^{+\infty} \left(\frac{3}{2}y^2e^{-y} + ye^{-y}\right) dy$$
$$= \left(-\frac{3}{2}(y^2+2y+2)e^{-y} + (-e^{-y}(y+1))\right)\Big|_0^{+\infty}$$
$$= \frac{3}{2}\times 2 + 1$$
$$= 4$$

So,

$$E[X+Y] = 4$$

29) Suppose customers arrive at an ATM facility according to Poisson process with rate 5 customers per hour. The probability (rounded off to two decimal places) that no customer arrives at the ATM facility from 1:00pm to 1:18pm.

Solution: Given, Poisson rate

$$\lambda = 5 \tag{288}$$

The time interval is given as 1:00 pm to 1:18 pm Then, the length of the interval

$$\tau = \frac{18}{60} - \frac{0}{60} \tag{289}$$
$$= \frac{3}{10} \tag{290}$$

Thus, if X is the number of arrivals in that interval, we can write

$$X \sim Poisson\,(\lambda\tau) = Poisson\left(\frac{3}{2}\right) \tag{291}$$

We know that, if $X\,(n)$ has a Poisson distribution whose parameter is k then

$$\Pr(X=n) = \left(\frac{k^n e^{-k}}{n!}\right) \tag{292}$$

CDF is:

$$F(X=n) = \sum_{x=0}^{n} \left(\frac{k^n e^{-k}}{n!}\right) \tag{293}$$

And also,

$$\Pr(x < X \le y) = F(y) - F(x) \tag{294}$$

Given,

$$n = 0 \tag{295}$$

So from (294)

$$\Pr(X=0) = F(0) \tag{296}$$

Therefore, the probability that no customer arrives at the ATM facility from 1:00pm to 1:18pm is

$\Pr(X = 0)$

$$= \frac{e^{\frac{-3}{2}}\left(\frac{3}{2}\right)^0}{0!} \quad (297)$$
$$= e^{-3/2} \quad (298)$$
$$\sim 0.22 \quad (299)$$

30) Let the cumulative distribution function of the random variable X be given by

$$F_X(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x < 1/2 \\ (1+x)/2 & 1/2 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$

Then $\Pr(X = 1/2) = ?$ **Solution:**
Given,

$$F_X(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x < 1/2 \\ \frac{(1+x)}{2} & 1/2 \le x < 1 \\ 1 & x \ge 1 \end{cases} \quad (24.1)$$

$$\Pr(X = 1/2) = \Pr(X \le 1/2) - \Pr(X < 1/2) \quad (24.2)$$

$$\implies \Pr(X = 1/2) = F_X\left(\frac{1}{2}\right) - F_X\left(\frac{1}{2}^-\right) \quad (24.3)$$

Using (24.1) in (24.3),

$$\implies \Pr(X = 1/2) = \frac{(1 + 1/2)}{2} - (1/2) \quad (24.4)$$

$$\implies \Pr(X = 1/2) = (3/4) - (1/2) \quad (24.5)$$

$$\therefore \Pr(X = 1/2) = 1/4 \quad (24.5)$$

The cdf plot of random variable X is as shown in Fig. 6

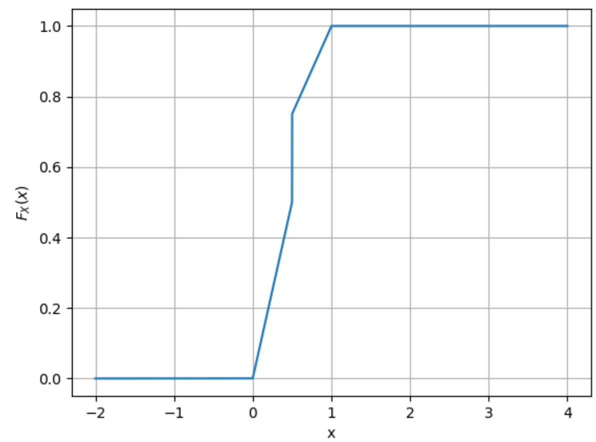

Fig. 6: cdf plot of random variable X

31) Let A and b be two events such that $\Pr(B) = \frac{3}{4}$ and $\Pr(A + B') = \frac{1}{2}$.If A and B are independent, then $\Pr(A)$ equals
Solution:
Given,

$$\Pr(B) = \frac{3}{4} \quad (300)$$

$$\Pr(A + B') = \frac{1}{2} \quad (301)$$

we know that,

$$\Pr(B') = 1 - \Pr(B) \quad (302)$$

using (300) in (302),

$$\Pr(B') = \frac{1}{4} \quad (303)$$

we know that,

$$\Pr(A + B') = \Pr(A) + \Pr(B') - \Pr(A, B') \quad (304)$$

A and B are independent $\iff$ A and B' are independent

$$\Pr(A + B') = \Pr(A) + \Pr(B') - \Pr(A)\Pr(B') \quad (305)$$

using (301) and (303) in (305),

$$\frac{1}{2} = \Pr(A) + \frac{1}{4} - \frac{\Pr(A)}{4} \tag{306}$$

$$\frac{1}{4} = \frac{3\Pr(A)}{4} \tag{307}$$

$$\therefore \Pr(A) = \frac{1}{3} \tag{308}$$

32) Let (X, Y) have a bivariate normal distribution with the joint probability density function

$$f_{X,Y}(x, y) = \frac{1}{\pi} e^{\left(\frac{3}{2}xy - \frac{25}{32}x^2 - 2y^2\right)} \tag{309}$$

$$-\infty < x, y < \infty \tag{310}$$

Then $E(XY)$ equals

Solution:

Given probability density function for (X, Y)

$$f_{X,Y}(x, y) = \frac{1}{\pi} e^{\left(\frac{3}{2}xy - \frac{25}{32}x^2 - 2y^2\right)} \tag{311}$$

$$-\infty < x, y < \infty \tag{312}$$

Joint pdf of bivariate normal distribution $N(\mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \rho)$ is

$$f_{X,Y}(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-p^2}} \times e^{\frac{-1}{2(1-p^2)}\left[\left[\frac{(x-\mu_x)}{\sigma_x}\right]^2 + \left[\frac{(y-\mu_y)}{\sigma_y}\right]^2 - 2\rho\left[\frac{(x-\mu_x)}{\sigma_x}\frac{(y-\mu_y)}{\sigma_y}\right]\right]} \tag{313}$$

Comparing (313) and (311) we get We

μ_x	μ_y	σ_x	σ_y	ρ
0	0	1	$\frac{5}{8}$	$\frac{3}{5}$

TABLE II: Table 1

need to find $E(XY)$

$$E(XY) = \rho\sigma_x\sigma_y + \mu_x\mu_y \tag{314}$$

Substituting values in table(II) in (314) we get

$$E(XY) = \frac{3}{8} \tag{315}$$

$$\therefore 8E(XY) = 3 \tag{316}$$

33) Let X_1 be an exponential random variable with mean 1 and X_2 a gamma random variable with mean 2 and variance 2. If X_1 and X_2 are independently distributed, then $\Pr(X_1 < X_2)$ is equal to ....

Solution:

We know that,

$$f_{X_1}(x) = \begin{cases} 0 & x < 0 \\ \lambda e^{\lambda x} & 0 \le x < \infty \end{cases} \tag{317}$$

Given,

$$E(X_1) = \frac{1}{\lambda} = 1 \tag{318}$$

$$\implies \lambda = 1 \tag{319}$$

Therefore,

$$f_{X_1}(x) = \begin{cases} 0 & x < 0 \\ e^{-x} & 0 \le x < \infty \end{cases} \tag{320}$$

We know that,

$$f_{X_2}(x) = \begin{cases} 0 & x < 0 \\ \dfrac{x^{\alpha-1}e^{\left(\frac{-x}{\beta}\right)}}{\beta^\alpha\Gamma(\alpha)} & 0 \le x < \infty \end{cases} \tag{321}$$

Given,

$$E(X_2) = \alpha\beta = 2 \tag{322}$$

$$V(X_2) = \alpha\beta^2 = 2 \tag{323}$$

Solving 322 and 323, we get, $\alpha = 2$, $\beta = 1$ and $\Gamma(2) = 1$

Therefore,

$$f_{X_2}(x) = \begin{cases} 0 & x < 0 \\ xe^{-x} & 0 \le x < \infty \end{cases} \tag{324}$$

Calculating the CDF of $f_{X_2}(x)$,

$$F_{X_2}(x) = \int_0^x f_{X_2}(x) \tag{325}$$

$$F_{X_2}(x) = \begin{cases} 0 & x < 0 \\ \dfrac{\gamma(\alpha, \frac{x}{\beta})}{\Gamma(\alpha)} & 0 \le x < \infty \end{cases} \tag{326}$$

For $\alpha = 2$ and $\beta = 1$
Alternately, we have CDF of X_1 and X_2 given by

$$F_{X_1}(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-x} & 0 \le x < \infty \end{cases} \tag{327}$$

$$F_{X_2}(x) = \begin{cases} 0 & x < 0 \\ \dfrac{\gamma(2, x)}{\Gamma(2)} & 0 \le x < \infty \end{cases} \tag{328}$$

Thus,

$$\Pr(X_1 \le X_2) = \int_{-\infty}^{\infty} F_{X_1}(x) f_{X_2}(x) dx \tag{329}$$

$$= \int_0^{\infty} (1 - e^{-x})(xe^{-x}) dx \tag{330}$$

$$= \frac{3}{4} \tag{331}$$

$$= 0.75 \tag{332}$$

34) Let $\Omega = (0, 1]$ be the sample space and let $P(.)$ be a probability distribution given by

$$P((0, x]) = \begin{cases} \frac{x}{2} & 0 \le x < \frac{1}{2} \\ x & \frac{1}{2} \le x \le 1 \end{cases} \tag{333}$$

Find $P\left(\frac{1}{2}\right)$
Solution:
CDF of X is defined as,

$$F_X(x) = \Pr(X \le x) \tag{334}$$

$\because x > 0$

$$F_X(x) = P((0, x]) \tag{335}$$

Thus, CDF of X is given by

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \frac{x}{2} & 0 \le x < \frac{1}{2} \\ x & \frac{1}{2} \le x \le 1 \\ 1 & x \ge 1 \end{cases} \tag{336}$$

$$\Pr\left(\frac{1}{2}\right) = F\left(\frac{1}{2}\right) - F\left(\frac{1}{2}^-\right) \tag{337}$$

$$= \frac{1}{2} - \frac{1/2}{2} \tag{338}$$

$$= \frac{1}{4} \tag{339}$$

The plot of CDF is given in the Figure 7

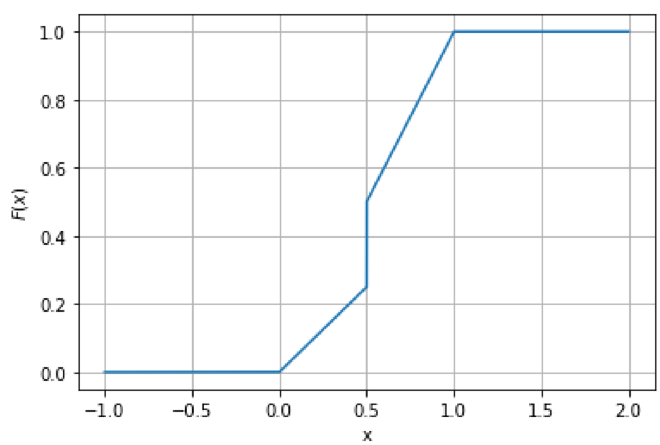

Fig. 7: CDF of X

35) Let (X, Y) be a two-dimensional random variable such that $E(X) = E(Y) = 1/2$, $Var(X) = Var(Y) = 1$ and $Cov(X, Y) = 1/2$. Then, $P(|X - Y| > 6)$ is

a) less than 1/6

b) equal to 1/2

c) equal to 1/3

d) greater than 1/2

Solution:
Given,

$$E(X) = E(Y) = 3 \tag{340}$$

$$Var(X) = Var(Y) = 1 \tag{341}$$

$$Cov(X, Y) = 1/2 \tag{342}$$

Now,

$$Var(X) = E(X^2) - (E(X))^2 \tag{343}$$

Substituting given values, we get,

$$1 = E(X^2) - 3^2 \tag{344}$$

So,

$$E(X^2) = 10 \tag{345}$$

Similarly for Y,

$$E(Y^2) = 10 \tag{346}$$

Also,

$$Cov(X, Y) = E(XY) - E(X)E(Y) \tag{347}$$

Substituting given values, we get,

$$1/2 = E(XY) - (3)(3) \tag{348}$$

So,

$$E(XY) = 19/2 \tag{349}$$

Let Z be a random variable defined as

$$Z = X - Y \tag{350}$$

Then using (340),

$$E(Z) = E(X - Y) = E(X) - E(Y) = 0 \tag{351}$$

Now, using (351)

$$Var(Z) = E(Z^2) - (E(Z))^2 = E(Z^2) \tag{352}$$

$$Var(Z) = E((X - Y)^2) \tag{353}$$

$$Var(Z) = E(X^2) + E(Y^2) - 2E(XY) \tag{354}$$

Using (345), (346) and (349),

$$Var(Z) = 10 + 10 - 2 \times 19/2 \tag{355}$$

$$Var(Z) = 1 \tag{356}$$

Theorem 0.1. *(Chebychev's Inequality) Let T be an arbitrary random variable, with finite mean E(T), then for all* $a > 0$,

$$\Pr\left(|T - E(T)| \geq a\right) \leq \frac{Var(T)}{a^2} \tag{357}$$

Proof. Let A be a non-negative random variable and $a > 0$ be any real number. Define a new random variable B by

$$B = \begin{cases} a & A \geq a \\ 0 & A < a \end{cases} \tag{358}$$

Then clearly $B \leq A$ and by monotonicity,

$$E(B) \leq E(A) \tag{359}$$

$$E(B) = a \Pr(B = a) + 0 \Pr(B = 0) \tag{360}$$

$$E(B) = a \Pr(A \geq a) \tag{361}$$

By (359) and (361),

$$a \Pr(A \geq a) \leq E(A) \tag{362}$$

$$\Pr(A \geq a) \leq \frac{E(A)}{a} \tag{363}$$

Set $A = (T - E(T))^2$. Then,

$$\Pr(|T - E(T)| \geq a) = \Pr\left(A \geq a^2\right) \tag{364}$$

Using (363),

$$\Pr(|T - E(T)| \geq a) \leq \frac{E(A)}{a^2} \tag{365}$$

$$\Pr(|T - E(T)| \geq a) \leq \frac{E(T - E(T))^2}{a^2} \tag{366}$$

$$\Pr(|T - E(T)| \geq a) \leq \frac{Var(T)}{a^2} \tag{367}$$

□

Applying Chebychev's Inequality for Z with $a = 6$, we get,

$$\Pr(|Z - E(Z)| \geq 6) \leq \frac{Var(Z)}{6^2} \tag{368}$$

Using (351) and (356),

$$\Pr(|Z - 0| \geq 6) \leq \frac{1}{36} \tag{369}$$

As $Z = X - Y$,

$$\Pr(|X - Y| \geq 6) \leq \frac{1}{36} \tag{370}$$

36) Let $X_1, X_2...$ be a sequence of independent and identically distributed random variable with

$$\Pr(X_1 = -1) = \Pr(X_1 = 1) = 1/2 \tag{371}$$

Suppose for the standard normal random variable Z,

$$\Pr(-0.1 \leq Z \leq 0.1) = 0.08. \tag{372}$$

If $S_n = \sum_{i=1}^{n^2} X_i$, then $\lim_{n\to\infty} \Pr\left(S_n > \frac{n}{10}\right) =$

a) 0.42

b) 0.46

c) 0.50

d) 0.54

Solution:

$$p_{X_i}(n) = \Pr(X_i = n) = \begin{cases} \frac{1}{2}, & \text{if n= 1 or n=-} \\ 0, & \text{otherwise} \end{cases}$$

$$\implies \mu = E(X_i) = 1/2(1 - 1) = 0 \tag{373}$$

$$\implies \sigma^2 = E(X_i{}^2) - \mu^2 = \frac{1}{2}(1 + 1) - 0 = 1 \tag{374}$$

Using Central Limit Theorem,we can say that for a series of random and identical variables X_i with the Mean =

μ and variance = σ^2 where i $\in$ 1,2...n

$$\text{Let } \overline{X_n} \equiv \frac{\sum_{i=1}^{n} X_i}{n} \quad (375)$$

$$\text{Then } \lim_{n\to\infty} \sqrt{n}(\overline{X_n} - \mu) = N(0, \sigma^2) \quad (376)$$

$$\implies \lim_{n\to\infty} \frac{S_n}{n} = N(0,1) \quad (377)$$

$$\implies S_n = nN(0,1) \quad (378)$$

$$\implies \lim_{n\to\infty} \Pr\left(nN(0,1) > \frac{n}{10}\right) \quad (379)$$

$$\implies \lim_{n\to\infty} \Pr\left(N(0,1) > \frac{1}{10}\right) = Q(0.1) \quad (380)$$

Now using (372)

$$\implies Q(0.1) + (1 - Q(-0.1)) + 0.08 = 1 \quad (381)$$

Now as $N(0,1)$symmetric about 0

$$\implies 2 \times Q(0.1) + 0.08 = 1 \quad (382)$$

$$\implies Q(0.1) = 0.46 \quad (383)$$

$$\implies \lim_{n\to\infty} \Pr\left(S_n > \frac{n}{10}\right) = 0.46 \quad (384)$$

Hence final solution is option 2) or 0.46

37) Consider an amusement park where visitors are arriving according to a Poisson process with rate 1. Upon arrival, a visitor spends a random amount of time in the park and then departs. The time spent by the visitors is independent of one another, as well as of the arrival process and have common probability density function

$$f(x) = \begin{cases} e^{-x}, & x > 0 \\ 0, & otherwise \end{cases} \quad (385)$$

If at a given point, there are 10 visitors in the park, and p is the probability that there will be exactly two more arrivals before the next departure, then $\frac{1}{p}$ equals..... **Solution:**

Symbol	Representation
X_1	Arrival time of P_1
$X_1 + X_2$	Arrival time of P_2
$X_1 + X_2 + X_3$	Arrival time of P_3
$Y_1, ..., Y_{10}$	Departure times of the 10 people in park currently
$X_1 + Y_{11}$	Departure time of P_1
$X_1 + X_2 + Y_{12}$	Departure time of P_2

TABLE III: Notations

According to the question, we want the following events to occur in order:

a) First visitor, P_1 arrives while no one leaves

b) Second visitor P_2 arrives while no one leaves

c) One or more person leaves before the third visitor P_3 arrives

Let the above events be E_1, E_2 and E_3 respectively. Thus the required probability

$$= \Pr(E_1 E_2 E_3) \quad (386)$$

$$= \Pr(E_1) \Pr(E_2|E_1) \Pr(E_3|E_1 E_2) \quad (387)$$

First we present the following result which shall be useful later. For $n > 0$,

$$\int_0^\infty x e^{-nx} dx = \frac{1}{n^2} \quad (388)$$

The above can be derived using integration by parts as follows

$$\int_0^\infty xe^{-nx}dx = -\frac{xe^{-nx}}{n}\Big|_0^\infty + \frac{1}{n}\int_0^\infty e^{-nx}dx \tag{389}$$

$$= -\frac{e^{-nx}}{n^2}\Big|_0^\infty \tag{390}$$

$$= \frac{1}{n^2} \tag{391}$$

Next we note that X_1, X_2 and X_3 are identical random variables having Poisson distribution with rate 1. Thus for $i \in \{1, 2, 3\}$,

$$\lambda = 1 * X_i = X_i \tag{392}$$

$$k = 1 \tag{393}$$

$$\implies f_{X_i}(x) = \begin{cases} \frac{x^1 e^{-x}}{1!} = xe^{-x} & x > 0 \\ 0 & otherwise \end{cases} \tag{394}$$

Also $Y_1, ..., Y_{12}$ are identical random variables. Thus for $i \in \{1, ..., 12\}$, as given in question,

$$f_{Y_i}(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & otherwise \end{cases} \tag{395}$$

$$\implies F_{Y_i}(x) = \begin{cases} 1 - e^{-x} & x > 0 \\ 0 & otherwise \end{cases} \tag{396}$$

Now we find $\Pr(E_1)$, $\Pr(E_2|E_1)$ and $\Pr(E_3|E_1E_2)$ in order to find the required probability from eq (387).

$$\Pr(E_1) = \Pr(Y_1, ..., Y_{10} > X_1) \tag{397}$$

$$= \int_{-\infty}^{\infty} \Pr(Y_1, ..., Y_{10} > x|X_1 = x) \tag{398}$$

$$= \int_{-\infty}^{\infty} (1 - F_{Y_1}(x))^{10} f_{X_1}(x)dx \tag{399}$$

$$= \int_0^\infty xe^{-11x}dx \tag{400}$$

$$= \frac{1}{121} \tag{401}$$

$$\Pr(E_2|E_1) = \Pr(Y_1, ..., Y_{10}, X_1 + Y_{11} > X_1 + X_2|Y_1, ..., Y_{10} > X_1) \tag{402}$$

Using memoryless property of exponential random variable,

$$\Pr(E_2|E_1) = \Pr(Y_1, ..., Y_{11} > X_2) \tag{403}$$

$$= \int_{-\infty}^{\infty} \Pr(Y_1, ...Y_{11} > x|X_2 = x) \tag{404}$$

$$= \int_{-\infty}^{\infty} (1 - F_{Y_1}(x))^{11} f_{X_2}(x)dx \tag{405}$$

$$= \int_0^\infty xe^{-12x}dx \tag{406}$$

$$= \frac{1}{144} \tag{407}$$

$$\Pr(E_3|E_1E_2) = \Pr(min(Y_1, ..., Y_{10}, X_1{+}Y_{11}, X_1{+}X_2{+}Y_{12}) < X_1{+}X_2{+}X_3|Y_1, ..., Y_{10}, X_1{+}Y_{11} > X_1{+}X_2) \tag{408}$$

We can simplify and write

$\Pr(E_3|E_1E_2) =$

$$1 - \Pr(Y_1, ..., Y_{10}, X_1 + Y_{11}, X_1 + X_2 + Y_{12} > X_1+X_2+X_3|Y_1, ..., Y_{10}, X_1+Y_{11} > X_1+X_2) \tag{409}$$

Using memoryless property of exponential random variable,

$$\Pr(E_3|E_1E_2) = 1 - \Pr(Y_1, ..., Y_{12} > X_3) \tag{410}$$

$$= 1 - \int_{-\infty}^{\infty} \Pr(Y_1, ...Y_{12} > x|X_3 = x) \tag{411}$$

$$= 1 - \int_{-\infty}^{\infty} (1 - F_{Y_1}(x))^{12} f_{X_3}(x) dx \tag{412}$$

$$= 1 - \int_{0}^{\infty} xe^{-13x} dx \tag{413}$$

$$= 1 - \frac{1}{169} \tag{414}$$

$$= \frac{168}{169} \tag{415}$$

Thus on substituting values in (387),

$$\Pr(E_1E_2E_3) = \frac{1}{121} \times \frac{1}{144} \times \frac{168}{169} \tag{416}$$

$$= 5.7 \times 10^{-5} \tag{417}$$

38) Let $\{X_n\}_{n\geq 1}$ be a sequence of independent and identically distributed random variables each having uniform distribution on [0,3]. Let Y be a random variable, independent of $\{X_n\}_{n\geq 1}$, having probability mass function

$$\Pr(Y = k) = \begin{cases} \frac{1}{(e-1)k!} & k = 1, 2, 3 \cdots \\ 0 & otherwise \end{cases} \tag{418}$$

Then $\Pr(max\{X_1, X_2, \cdots X_Y\} \leq 1)$ equals …………

Solution:

Given that $\{X_n\}_{n\geq 1}$ is having a uniform distribution on [0,3], so probability can be written as

$$\Pr(X_n)_{n\geq 1} = \begin{cases} \frac{1}{3} & 0 \leq X_n \leq 3 \\ 0 & otherwise \end{cases} \tag{419}$$

So,

$$\Pr(X_n \leq 1)_{n\geq 1} = \frac{1}{3} \tag{420}$$

Required probability

$$= \Pr(max\{X_1, X_2, \cdots X_Y\} \leq 1) \tag{421}$$

Since, $\{X_n\}_{n\geq 1}$ is a sequence of independent variables and Y is also independent of $\{X_n\}_{n\geq 1}$.

And also in (421), the index of $X_i's$ depends on Y, so number of terms depends on Y, like if $Y = 1$, then there is only X_1, if $Y = 2$, then there's X_1, X_2, so required probability

$$= \sum_{p=1}^{\infty} \Pr\left(max\{X_1, X_2, \cdots X_p\} \leq 1|Y = p\right) \cdot \Pr(Y = \tag{422}$$

$$= \sum_{p=1}^{\infty} \Pr\left(max\{X_1, X_2, \cdots X_p\} \leq 1\right) \cdot \Pr(Y = p) \tag{423}$$

$$= \sum_{p=1}^{\infty} \Pr\left(X_1, X_2, \cdots X_p \leq 1\right) \cdot \Pr(Y = p) \tag{424}$$

$$= \sum_{p=1}^{\infty} \Pr(X_1 \leq 1) \cdot \Pr(X_2 \leq 1) \cdots \Pr\left(X_{p-1} \leq 1\right) \cdot \Pr\left(X_p \leq 1\right) \cdot \Pr(Y = p) \tag{425}$$

$$= \sum_{p=1}^{\infty} \left(\frac{1}{3}\right)^p \left(\frac{1}{e-1}\right)\left(\frac{1}{p!}\right) \tag{426}$$

$$= \left(\frac{1}{e-1}\right)\left[\sum_{p=0}^{\infty} \left(\frac{1}{3}\right)^p \left(\frac{1}{p!}\right) - 1\right] \tag{427}$$

Using Taylor's Series of e^x in (427), Required probability

$$= \frac{e^{1/3}}{e-1} - \frac{1}{e-1} \tag{428}$$

$$= 0.23 \tag{429}$$

39) The characteristic function of a random variable X is given by

$$\phi_X(t) = \begin{cases} \frac{\sin t \cos t}{t} & t \neq 0 \\ 1 & t = 0 \end{cases} \tag{430}$$

Then $P\left(|X| \leq \frac{3}{2}\right) =$ **Solution:**
The pdf is given by

$$f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi_X(t)\, e^{-jxt} dt \tag{431}$$

If

$$g(x) \overset{\mathcal{H}}{\longleftrightarrow} FG(t) \tag{432}$$

$$\implies G(t) \overset{\mathcal{H}}{\longleftrightarrow} Fg(-x) \tag{433}$$

where $\left(\overset{\mathcal{H}}{\longleftrightarrow} F\right)$ represents Fourier transform and

$$G(t) = \int_{-\infty}^{\infty} g(x)\, e^{-j2\pi xt} dx \tag{434}$$

we know that the Fourier transform of rectangular function is sinc function

$$rect\left(\frac{x}{\tau}\right) \overset{\mathcal{H}}{\longleftrightarrow} F\tau sinc(t\tau) \tag{435}$$

from (433) we get

$$\tau sinc(t\tau) \overset{\mathcal{H}}{\longleftrightarrow} Frect\left(-\frac{x}{\tau}\right) \tag{436}$$

$$\implies rect\left(-\frac{x}{\tau}\right) = \int_{-\infty}^{\infty} \tau \frac{\sin \pi t\tau}{\pi t\tau} e^{-j2\pi xt} dt \tag{437}$$

substituting $\tau = \frac{2}{\pi}$ and changing $2\pi x \rightarrow x$ we get

$$\frac{1}{4} rect\left(\frac{-x}{4}\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(\frac{\sin 2t}{2t}\right) e^{-jxt} dt \tag{438}$$

So

$$f_X(x) = \frac{1}{4} rect\left(\frac{-x}{4}\right) \tag{439}$$

$$P\left(|X| \leq \frac{3}{2}\right) = \int_{\frac{-3}{2}}^{\frac{3}{2}} \frac{1}{4} dx \tag{440}$$

$$= \frac{3}{4} \tag{441}$$

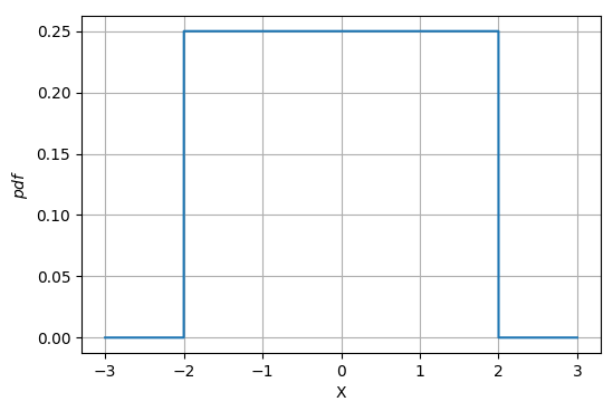

Fig. 8: $f_X(x)$

40) Let $\{X_j\}$ be a sequence of independent Bernoulli random variables with $\mathbb{P}(X_j = 1) = \frac{1}{4}$ and let $Y_n = \frac{1}{n}\sum_{j=1}^{n} X_j^2$. Then Y_n converges, in probability, to ________ .
Solution:
A sequence of random variables $Y_1, Y_2, Y_3 \ldots$ converges, in probability,

to a random variable Y if

$$\lim_{n\to\infty} \Pr(|Y_n - Y| \geq \epsilon) = 0 \quad \forall \epsilon > 0 \tag{442}$$

Similarly, a sequence of random variables $Y_1, Y_2, Y_3 \ldots$ converges, in mean square, to a random variable Y if

$$\lim_{n\to\infty} E(|Y_n - Y|^2) = 0 \tag{443}$$

A random variable converges, in probability, to a value if it converges, in mean square, to the same particular value by Markov's Inequality. Proof for this is: For any $\epsilon > 0$

$$\Pr(|Y_n - Y| \geq \epsilon) = \Pr(|Y_n - Y|^2 \geq \epsilon^2) \tag{444}$$

$$\Pr(|Y_n - Y| \geq \epsilon) \leq \frac{E\,|Y_n - Y|^2}{\epsilon^2} \quad \text{(by Markov's Inequality)} \tag{445}$$

$$\lim_{n\to\infty} E(|Y_n - Y|^2) = 0 \tag{446}$$

$$0 \leq \lim_{n\to\infty} \Pr(|Y_n - Y| \geq \epsilon) \leq \frac{0}{\epsilon^2} \tag{447}$$

$$\lim_{n\to\infty} \Pr(|Y_n - Y| \geq \epsilon) = 0 \quad \forall \epsilon > 0 \tag{448}$$

Given in the question that $\{X_j\}$ is a sequence of random variables with

$$\Pr(X_j = 1) = \frac{1}{4} \tag{449}$$

$$\Pr(X_j = 0) + \Pr(X_j = 1) = 1 \tag{450}$$

$$\Pr(X_j = 0) = 1 - \frac{1}{4} = \frac{3}{4} \tag{451}$$

$$X_j \in \{0, 1\} \tag{452}$$

Since $0^2 = 0$ and $1^2 = 1$,

$$X_j^2 = X_j \quad \forall j \in \{1, 2, \ldots, n\} \tag{453}$$

Thus,

$$Y_n = \frac{1}{n}\sum_{j=1}^{n} X_j^2 \tag{454}$$

$$= \frac{1}{n}\sum_{j=1}^{n} X_j \tag{455}$$

$$\Pr(Y_n = y) = {}^{n}C_{ny}\left(\frac{1}{4}\right)^{ny}\left(\frac{3}{4}\right)^{n-ny} \tag{456}$$

Let us assume

$$k = ny \tag{457}$$

$$k \in \{0, 1, 2, \ldots, n-1, n\} \tag{458}$$

$$\Pr(Y_n = \frac{k}{n}) = {}^{n}C_{k}\left(\frac{1}{4}\right)^{k}\left(\frac{3}{4}\right)^{n-k} \tag{459}$$

$$E\left(\left|Y_n - \frac{1}{4}\right|^2\right) = E\left(Y_n^2 - \frac{1}{2}Y_n + \frac{1}{16}\right) \tag{460}$$

$$= E(Y_n^2) - \frac{1}{2}E(Y_n) + \frac{1}{16} \tag{461}$$

$$E(Y_n^2) = \sum_{k=0}^{n}\left(\frac{k}{n}\right)^2 \Pr\left(Y_n = \frac{k}{n}\right) \tag{462}$$

$$= \sum_{k=0}^{n}\left(\frac{k^2}{n^2}\right) {}^{n}C_{k}\left(\frac{1}{4}\right)^{k}\left(\frac{3}{4}\right)^{n-k} \tag{463}$$

$$E(Y_n^2) = 0 + \frac{1}{n^2} \times n\left(\frac{1}{4}\right)^{1}\left(\frac{3}{4}\right)^{n-1} + \sum_{k=2}^{n}\left(\frac{k}{n}\right)^2 \times \frac{n(n-1)}{k(k-1)} \times {}^{n-2}C_{k-2}\left(\frac{1}{4}\right)^{k}\left(\frac{3}{4}\right)^{n-k} \tag{464}$$

$$E(Y_n^2) = \frac{1}{4n}\left(\frac{3}{4}\right)^{n-1} + \frac{n-1}{n} \times \sum_{k=2}^{n}\left(\frac{k}{k-1}\right)^{n-2}C_{k-2}\left(\frac{1}{4}\right)^{k}\left(\frac{3}{4}\right)^{n-k} \quad (465)$$

$$E(Y_n^2) = \frac{1}{4n}\left(\frac{3}{4}\right)^{n-1} + \frac{n-1}{n}\left(\sum_{k=2}^{n} {}^{n-2}C_{k-2}\left(\frac{1}{4}\right)^{k}\left(\frac{3}{4}\right)^{n-k}\right) + \frac{n-1}{n}\left(\sum_{k=2}^{n}\frac{1}{k-1}{}^{n-2}C_{k-2}\left(\frac{1}{4}\right)^{k}\left(\frac{3}{4}\right)^{n-k}\right) \quad (466)$$

$$E(Y_n^2) = \frac{1}{4n}\left(\frac{3}{4}\right)^{n-1} + \frac{n-1}{n} \times \frac{1}{16}\left(\sum_{k=2}^{n} {}^{n-2}C_{k-2}\left(\frac{1}{4}\right)^{k-2}\left(\frac{3}{4}\right)^{(n-2)-(k-2)}\right) + \frac{1}{n}\left(\sum_{k=2}^{n}\frac{n-1}{k-1}{}^{n-2}C_{k-2}\left(\frac{1}{4}\right)^{k}\left(\frac{3}{4}\right)^{n-k}\right) \quad (467)$$

$$E(Y_n^2) = \frac{1}{4n}\left(\frac{3}{4}\right)^{n-1} + \frac{n-1}{16n}\left(\sum_{j=0}^{n-2} {}^{n-2}C_{j}\left(\frac{1}{4}\right)^{j}\left(\frac{3}{4}\right)^{(n-2)-j}\right) + \frac{1}{4n}\left(\sum_{k=2}^{n} {}^{n-1}C_{k-1}\left(\frac{1}{4}\right)^{k-1}\left(\frac{3}{4}\right)^{(n-1)-(k-1)}\right) \quad (468)$$

$$E(Y_n^2) = \frac{1}{4n}\left(\frac{3}{4}\right)^{n-1} + \frac{n-1}{16n}\left(\frac{1}{4}+\frac{3}{4}\right)^{n-2} + \frac{1}{4n}\left(\sum_{j=1}^{n-1} {}^{n-1}C_{j}\left(\frac{1}{4}\right)^{j}\left(\frac{3}{4}\right)^{(n-1)-j}\right) \quad (469)$$

$$E(Y_n^2) = \frac{1}{4n}\left(\frac{3}{4}\right)^{n-1} + \frac{n-1}{16n} + \frac{1}{4n}\left(\left(\frac{1}{4}+\frac{3}{4}\right)^{n-1} - \left(\frac{3}{4}\right)^{n-1}\right) \quad (470)$$

$$E(Y_n^2) = \frac{1}{4n}\left(\frac{3}{4}\right)^{n-1} + \frac{n-1}{16n} + \frac{1}{4n} - \frac{1}{4n}\left(\frac{3}{4}\right)^{n-1} \quad (471)$$

$$= \frac{1}{16} + \frac{3}{16n} \quad (472)$$

$$E(Y_n) = \sum_{k=0}^{n}\frac{k}{n}\Pr\left(Y_n = \frac{k}{n}\right) \quad (473)$$

$$= \sum_{k=0}^{n}\left(\frac{k}{n}\right){}^{n}C_k\left(\frac{1}{4}\right)^{k}\left(\frac{3}{4}\right)^{n-k} \quad (474)$$

$$= 0 + \sum_{k=1}^{n}\frac{k}{n}\times\frac{n}{k}\times{}^{n-1}C_{k-1}\left(\frac{1}{4}\right)^{k}\left(\frac{3}{4}\right)^{n-k} \quad (475)$$

$$= \frac{1}{4}\sum_{j=0}^{n-1}{}^{n-1}C_j\left(\frac{1}{4}\right)^{j}\left(\frac{3}{4}\right)^{(n-1)-j} \quad (476)$$

$$= \frac{1}{4}\left(\frac{1}{4}+\frac{3}{4}\right)^{n-1} \quad (477)$$

$$= \frac{1}{4} \quad (478)$$

Using equations (472) and (478) in (461),

$$E\left(\left|Y_n - \frac{1}{4}\right|^2\right) = \frac{1}{16} + \frac{3}{16n} - \frac{1}{2}\times\frac{1}{4} + \frac{1}{16} \quad (479)$$

$$= \frac{3}{16n} \quad (480)$$

$$\lim_{n\to\infty} E\left(\left|Y_n - \frac{1}{4}\right|^2\right) = \lim_{n\to\infty} \frac{3}{16n} \tag{481}$$
$$= \frac{3}{16} \lim_{n\to\infty} \frac{1}{n} \tag{482}$$
$$= 0 \tag{483}$$

Thus, Y_n converges, in mean square, to $\frac{1}{4}$ and hence Y_n converges, in probability, to $\frac{1}{4}$.

41) The variable x takes a value between 0 and 10 with uniform probability distribution. The variable y takes a value between 0 and 20 with uniform probability distribution.The probability that sum of variables $(x+y)$ being greater than 20 is

42) Robot Ltd. wishes to maintain enough safety stock during the lead time period between starting a new production run and its completion such that the probability of satisfying the customer demand during the lead time period is 95%. The lead time periods is 5 days and daily customer demand can be assumed to follow the Gaussian (normal) distribution with mean 50 units and a standard deviation of 10 units. Using $\phi^{-1}(0.95)$ =1.64 , where ϕ represents the cumulative distribution function of the standard normal random variable, the amount of safety stock that must be maintained by Robot Ltd. to achieve this demand fulfillment probability for the lead time period is ____ units (round off to two decimal places). **Solution:** Probability of satisfying customer demand is 0.95. Let Z be a standard normal

Symbol	definition	value
X	customer demand in lead time	-
X_1	normal R.V denotes daily customer demand	-
μ	Mean of X_1	50
σ	Standard deviation of X_1	10
ϕ	CDF of standard normal R.V	-

TABLE IV: Variables and their definitions

R.V such that,

$$Z = \frac{X_1 - \mu}{\sigma} \tag{484}$$

Referring table(IV) to use in (484),

$$Z = \frac{X_1 - 50}{10} \tag{485}$$

Given that,

$$\phi^{-1}(0.95) = 1.64 \tag{486}$$
$$\implies \phi(1.64) = 0.95 \tag{487}$$
$$\phi(1.64) = \Pr(Z \le 1.64) = 0.95 \tag{488}$$
$$\implies Z \le 1.64 \iff \frac{X_1 - 50}{10} \le 1.64 \tag{489}$$
$$\implies X_1 - 50 \le 1.64(10) \tag{490}$$
$$\therefore X_1 \le 66.4 \tag{491}$$

The demand in one day is independent of demand in the other day and the lead time is 5 days.

$$\implies X = 5(X_1) = 5(66.4) = 332 \tag{492}$$

Therefore the amount of safety stock that must be maintained by Robot Ltd. to achieve this demand fulfillment probability for the lead time period is 332 units.

43) Let X be a random variable having prob-

ability density function

$$f(x) = \begin{cases} \frac{3}{13}(1-x)(9-x) & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases} \tag{493}$$

Then $\frac{4}{3}E[X(X^2 - 15X + 27)]$ equals — (round of to two decimal places).

Solution:

Let X be the random variable. To find

$$\frac{4}{3}E[X(X^2 - 15X + 27)] \tag{494}$$

Let,

$$g(X) = X(X^2 - 15X + 27) \tag{495}$$
$$= X^3 - 15X^2 + 27X \tag{496}$$

Then for random variable X we have that,

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)\,dx \tag{497}$$

The probability distribution of X is,

$$f(x) = \begin{cases} \frac{3}{13}(1-x)(9-x) & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases} \tag{498}$$

Using 498 we have,

$$E[g(X)] = 0 + \int_0^1 g(x)f(x)\,dx + 0 \tag{499}$$

Where ,

$$f(x) = \frac{3}{13}(1-x)(9-x) \text{ and} \tag{500}$$
$$g(x) = x^3 - 15x^2 + 27x \tag{501}$$

Using Integration by substitution let,

$$t = x^3 - 15x^2 + 27x$$
$$dt = 3x^2 - 30x + 27$$
$$= 3(1-x)(9-x)$$

The corresponding limits are,

$$\text{For } x = 0 \implies t = 0^3 - 15 \times 0^2 + 27 \times 0 = 0 \tag{502}$$

$$\text{For } x = 1 \implies t = 1^3 - 15 \times 1^2 + 27 \times 1 = 13 \tag{503}$$

Therefore we have,

$$E[g(X)] = \frac{1}{13}\int_0^{13} t\,dt \tag{504}$$
$$= \frac{1}{13} \times \left(\frac{t^2}{2}\right)\Bigg|_0^{13} \tag{505}$$
$$= \frac{1}{13} \times \frac{13^2}{2} \tag{506}$$
$$= \frac{13}{2} \tag{507}$$

Thus,

$$\frac{4}{3}E[g(X)] = \frac{4}{3} \times \frac{13}{2} \tag{508}$$
$$= \frac{26}{3} \tag{509}$$
$$= 8.67 \text{ (rounded off)} \tag{510}$$

Therefore,

$$\frac{4}{3}E[X(X^2 - 15X + 27)] = 8.67 \tag{511}$$

44) Two independent events E and F are such that $P(E \cap F) = \frac{1}{6}, P(E^c \cap F^c) = \frac{1}{3}$ and $P(E) > P(F)$. Then $P(E)$ is

(A) $\frac{1}{2}$

(B) $\frac{2}{3}$

(C) $\frac{1}{3}$

(D) $\dfrac{1}{4}$

Solution:

If E and F are independent, E' and F' are also independent.

So,

$$\Pr(EF) = \Pr(E)\Pr(F) = \frac{1}{6} \tag{512}$$

$$\Pr(E'F') = \Pr(E')\Pr(F') = (1-\Pr(E))(1-\Pr(F)) = \frac{1}{3} \tag{513}$$

From (512) and (513)

$$\Pr(E) + \Pr(F) = \frac{5}{6} \tag{514}$$

From (512) and (514),

$$\Pr(E)\left(\frac{5}{6} - \Pr(E)\right) = \frac{1}{6}$$
$$\equiv \Pr(E) = \frac{1}{3}\text{or}\frac{1}{2}$$

$\Pr(E) = \frac{1}{2}$ satisfies $\Pr(E) > \Pr(F)$ while $\Pr(E) = \frac{1}{3}$ does not.

$\therefore \Pr(E) = \frac{1}{2}$

Solution: Option A

45) Let $Y_1, Y_2, ..., Y_{15}$ be a random sample of size 15 from the probability density function

$$f_y(y) = 3(1-y)^2, 0 < y < 1 \qquad \text{(Eq:1)}$$

Use the central limit theorem to approximate $P\left(\frac{1}{8} < \bar{Y} < \frac{3}{8}\right)$ **Solution:**

The **central limit theorem** states that whenever a random sample of size n is taken from any distribution with mean and variance, then the sample mean will be approximately normally distributed with mean and variance. The larger the value of the sample size, the better the approximation to the normal.

$$Z_n = \frac{\bar{Y} - \mu}{\frac{\sigma}{\sqrt{n}}} \tag{1.1}$$

From equation 1.1

$$\bar{Y} = Z_n\left(\frac{\sigma}{\sqrt{n}}\right) + \mu \tag{1.2}$$

$$\Pr\left(\frac{1}{8} < \bar{Y} < \frac{3}{8}\right) = \Pr\left(\frac{1}{8} < Z_n\left(\frac{\sigma}{\sqrt{n}}\right) + \mu < \frac{3}{8}\right) \tag{1.3}$$

$$= \Pr\left(\frac{\frac{1}{8} - \mu}{\frac{\sigma}{\sqrt{n}}} < Z_n < \frac{\frac{3}{8} - \mu}{\frac{\sigma}{\sqrt{n}}}\right) \tag{1.4}$$

$\bar{Y}$:Mean of the randomly selected 15 variables

$$\bar{Y} = \frac{Y_1 + Y_2 + ..Y_{15}}{15} \tag{1.5}$$

Mean of probability density function is

$$\mu = \int_{-\infty}^{\infty} yf(y)dy \tag{1.6}$$
$$= \int_0^1 y \times 3(1-y)^2 dy \tag{1.7}$$
$$= \frac{1}{4} \tag{1.8}$$

Variance of probability density function is

$$\sigma^2 = E[y^2] - (E[y])^2 \tag{1.9}$$
$$= \left(\int_0^1 y^2 f(y)dy\right) - \left(\frac{1}{4}\right)^2 \tag{1.10}$$

$$\int_0^1 y^2 f(y)dy = \int_0^1 y^2 \times 3(1-y)^2 dy \quad (1.11)$$

$$= 3\int_0^1 (y-y^2)^2 dy \quad (1.12)$$

$$= \frac{1}{10} \quad (1.13)$$

Substituting equation 1.13 in equation 1.10

$$\sigma^2 = \frac{1}{10} - \frac{1}{16} \quad (1.14)$$

$$= \frac{3}{80} \quad (1.15)$$

Using Q function in equation 1.4 we have,

$$\Pr\left(\frac{1}{8} < \bar{Y} < \frac{3}{8}\right) = \Pr\left(\frac{\frac{1}{8}-\mu}{\frac{\sigma}{\sqrt{n}}} < Z_n < \frac{\frac{3}{8}-\mu}{\frac{\sigma}{\sqrt{n}}}\right)$$

$$= \Pr\left(\frac{\frac{1}{8}-\mu(y)}{\frac{\sigma}{\sqrt{n}}} < Z_n < \frac{\frac{3}{8}-\mu(y)}{\frac{\sigma}{\sqrt{n}}}\right) \quad (1.16)$$

$$= Q\left(\frac{\frac{-1}{8}}{\sqrt{\frac{3}{80}}}\right) - Q\left(\frac{\frac{1}{8}}{\sqrt{\frac{3}{80}}}\right) \quad (1.17)$$

$$= 1 - 2Q\left(\frac{\frac{1}{8}}{\sqrt{\frac{3}{80}}}\right) \quad (1.18)$$

$$= 1 - 2Q\,(0.645) \quad (1.19)$$

$$= 0.9938 \quad (1.20)$$