

Signal Processing

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CONTENTS

Abstract—This manual provides solved problems in signal processing from GATE exam papers.

- Two discrete time systems with impulse responses $h_1[n] = \delta[n - 1]$ and $h_2[n] = \delta[n - 2]$ are connected in cascade. The overall impulse response of the cascaded system is
 - $\delta[n - 1] + \delta[n - 2]$
 - $\delta[n - 4]$
 - $\delta[n - 3]$
 - $\delta[n - 1]\delta[n - 2]$

Solution:

$$h[n] = h_1[n] * h_2[n] = \delta[n - 1] * \delta[n - 2] \quad (1.1)$$

$$= \delta[n - 3] \quad (1.2)$$

- A continuous time LTI system is described by

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = 2\frac{dx(t)}{dt} + 4x(t) \quad (2.1)$$

Assuming zero initial conditions, the response $y(t)$ of the above system for the input $x(t) = e^{-2t}u(t)$ is given by

- $(e^t - e^{3t})u(t)$
- $(e^{-t} - e^{-3t})u(t)$
- $(e^{-t} + e^{-3t})u(t)$
- $(e^t + e^{3t})u(t)$

Solution:

Lemma 2.1 (Table of Laplace Transforms).

Time Function	Laplace transform of $f(t)$
$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
$u(t)$	$\frac{1}{s}, s > 0$
$g'(t)$	$sG(s) - g(0)$
$g''(t)$	$s^2G(s) - sg(0) - g'(0)$
$e^{-at}u(t)$	$\frac{1}{s+a}, s+a > 0$

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Lemma 2.2. Linearity of Laplace Transform

$$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\} \quad (2.2)$$

From Lemma-2.1 Laplace transform of $x(t) = e^{-2t}u(t)$ is given by

$$X(s) = \frac{1}{s+2} \quad (2.3)$$

Since initial conditions are zero. Laplace Transform of (2.1) gives

$$s^2Y(s) + 4sY(s) + 3Y(s) = 2sX(s) + 4X(s) \quad (2.4)$$

$$\begin{aligned} Y(s) &= \frac{2(s+2)}{s^2+4s+3}X(s) \\ &= \frac{1}{s+1} - \frac{1}{s+3} \end{aligned} \quad (2.5) \quad (2.6)$$

From Lemma-2.1. Inverse Laplace transform of $Y(s)$ is given by

$$y(t) = e^{-t}u(t) - e^{-3t}u(t) \quad (2.7)$$

$$= (e^{-t} - e^{-3t})u(t) \quad (2.8)$$

\therefore The required option is B. See Fig. 2.1.

- The transfer function for a discrete time LTI system is given by:

$$H(z) = \frac{2 - \frac{3}{4}z^{-1}}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}} \quad (3.1)$$

Consider the following statements:

S1: The system is stable and causal for ROC: $|z| > \frac{1}{2}$

S2: The system is stable but not causal for ROC: $|z| < \frac{1}{4}$

S3: The system is neither stable nor causal for ROC: $\frac{1}{4} < |z| < \frac{1}{2}$

Which one of the following statement are valid?

- Both S1 and S2 are true

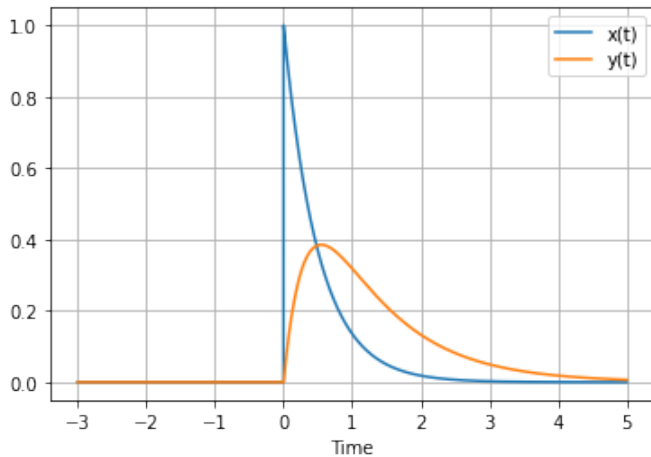


Fig. 2.1: Plot of input and output responses in time domain.

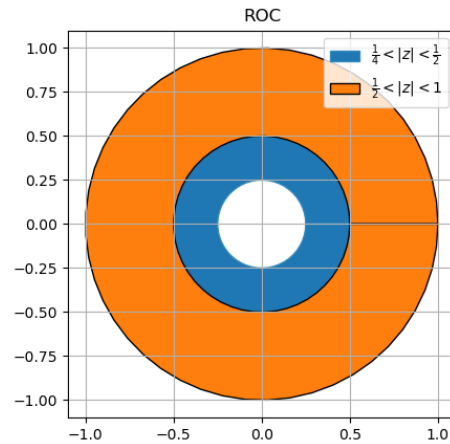


Fig. 3.1: ROC

- b) Both S2 and S3 are true
- c) Both S1 and S3 are true
- d) S1, S2 and S3 are all true

Solution: The given transfer function can be expressed as

$$H(z) = \frac{16 - 6z^{-1}}{8 - 6z^{-1} + z^{-2}} \quad (3.2)$$

$$= \frac{16 - 6z^{-1}}{(4 - z^{-1})(2 - z^{-1})} \quad (3.3)$$

$$= \frac{4}{4 - z^{-1}} + \frac{2}{2 - z^{-1}} \quad (3.4)$$

with poles at

$$z = \frac{1}{2}, z = \frac{1}{4} \quad (3.5)$$

- a) Since the ROC includes the unit circle, the system is stable. Also, the ROC extends outwards to infinity, so the system is causal as well. Hence S_1 is true.
 - b) When $\text{ROC} = \frac{1}{4} < |z| < \frac{1}{2}$, the unit circle is not included in the ROC. Hence, the system cannot be stable. Also, the ROC is an annulus, so the system is non-causal. So S_3 is true.
4. The impulse response of a system is $h(t) = tu(t)$. For an input $u(t-1)$, the output is
- a) $\frac{t^2}{2}u(t)$
 - b) $\frac{t(t-1)}{2}u(t-1)$
 - c) $\frac{(t-1)^2}{2}u(t-1)$

d) $\frac{t^2 - 1}{2}u(t-1)$

Solution:

Definition 1 (Laplace Transform). *It is an integral transform that converts a function of a real variable t to a function of a complex variable s . The Laplace transform of $f(t)$ is denoted by $\mathcal{L}\{f(t)\}$ or $F(s)$.*

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt \quad (4.1)$$

Remark. Laplace transform of $f(t) = t^n, n \geq 1$ is

$$F(s) = \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, s > 0 \quad (4.2)$$

Proof. Basis Step: $n = 1$

$$\mathcal{L}\{t\} = \int_0^{\infty} e^{-st} t dt \quad (4.3)$$

$$= \left[\frac{te^{-st}}{-s} \right]_0^{\infty} + \frac{1}{s} \int_0^{\infty} e^{-st} dt \quad (4.4)$$

$$= 0 + \left[\frac{-1}{s^2} e^{-st} \right]_0^{\infty}, s > 0 \quad (4.5)$$

$$= \frac{1}{s^2}, s > 0 \quad (4.6)$$

Inductive Step:

$$\mathcal{L}\{t^n\} = \int_0^\infty e^{-st} t^n dt \quad (4.7)$$

$$= \left[\frac{t^n e^{-st}}{-s} \right]_0^\infty + \frac{n}{s} \int_0^\infty t^{n-1} e^{-st} dt \quad (4.8)$$

$$= 0 + \frac{n}{s} \mathcal{L}\{t^{n-1}\}, s > 0 \quad (4.9)$$

$$= \frac{n}{s} \mathcal{L}\{t^{n-1}\}, s > 0 \quad (4.10)$$

To prove that if
eqrefec/2003/8eq:t holds for $n = k$, it holds for
 $n = k + 1$. From
eqrefec/2003/8eq:e

$$\mathcal{L}\{t^{k+1}\} = \frac{k+1}{s} \mathcal{L}\{t^k\} \quad (4.11)$$

$$= \frac{(k+1)k!}{s(s^{k+1})} = \frac{(k+1)!}{s^{k+2}}, s > 0 \quad (4.12)$$

By mathematical induction,
eqrefec/2003/8eq:t is true $\forall n \geq 1$ \square

Lemma 4.1. For any real number c ,

$$\mathcal{L}\{u(t-c)\} = \frac{e^{-cs}}{s}, s > 0 \quad (4.13)$$

Proof.

$$\mathcal{L}\{u(t-c)\} = \int_0^\infty e^{-st} u(t-c) dt = \int_c^\infty e^{-st} dt \quad (4.14)$$

$$= \left[-\frac{e^{-st}}{s} \right]_c^\infty = \frac{e^{-cs}}{s}, s > 0 \quad (4.15)$$

\square

Definition 2 (Inverse Laplace Transform). It is the transformation of a Laplace transform into a function of time. If $F(s) = \mathcal{L}\{f(t)\}$, then the Inverse laplace transform of $F(s)$ is $\mathcal{L}^{-1}\{F(s)\} = f(t)$.

Lemma 4.2 (t-shift rule). For any real number c ,

$$\mathcal{L}\{u(t-c)f(t-c)\} = e^{-cs}F(s) \quad (4.16)$$

Proof.

$$\mathcal{L}\{u(t-c)f(t-c)\} = \int_0^\infty e^{-st} u(t-c)f(t-c) dt \quad (4.17)$$

$$= \int_c^\infty e^{-st} f(t-c) dt \quad (4.18)$$

$$= \int_0^\infty e^{-s(\tau+c)} f(\tau) d\tau \quad (t = \tau + c) \quad (4.19)$$

$$= e^{-cs} \int_0^\infty e^{-s\tau} f(\tau) d\tau \quad (4.20)$$

$$= e^{-cs} F(s) \quad (4.21)$$

\square

Corollary 0.1.

$$\mathcal{L}^{-1}\{e^{-cs}F(s)\} = u(t-c)f(t-c) \quad (4.22)$$

Theorem 0.2 (Convolution theorem). Suppose $F(s) = \mathcal{L}\{f(t)\}$, $G(s) = \mathcal{L}\{g(t)\}$ exist, then,

$$\mathcal{L}^{-1}\{F(s)G(s)\} = f(t) * g(t) \quad (4.23)$$

Given,

$$h(t) = tu(t) \quad (4.24)$$

$$x(t) = u(t-1) \quad (4.25)$$

To find: $y(t)$. We know,

$$y(t) = h(t) * x(t) \quad (4.26)$$

$$= \mathcal{L}^{-1}\{H(s)X(s)\} \quad (4.27)$$

From

eqrefec/2003/8eq:uf and

eqrefec/2003/8eq:t,

$$H(s) = e^0 \mathcal{L}\{t\} = \frac{1}{s^2} \quad (4.28)$$

From

eqrefec/2003/8eq:u,

$$X(s) = \frac{e^{-s}}{s} \quad (4.29)$$

Substituting in

eqrefec/2003/8eq:def,

$$y(t) = \mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^3}\right\} \quad (4.30)$$

Consider

$$p(t) = \frac{t^2}{2} \quad (4.31)$$

From

eqrefec/2003/8eq:t

$$P(s) = \frac{2!}{2s^3} = \frac{1}{s^3} \quad (4.32)$$

Further, from

eqrefec/2003/8eq:cuf, for $c = 1$

$$\mathcal{L}^{-1}\{e^{-s}P(s)\} = u(t-1)p(t-1) \quad (4.33)$$

$$= u(t-1)\frac{(t-1)^2}{2} \quad (4.34)$$

$$\therefore y(t) = \frac{(t-1)^2}{2}u(t-1) \quad (4.35)$$

Option 3 is the correct answer.

$$h(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases} \quad (4.36)$$

$$x(t) = \begin{cases} 1, & t \geq 1 \\ 0, & t < 1 \end{cases} \quad (4.37)$$

$$y(t) = \begin{cases} \frac{(t-1)^2}{2}, & t \geq 1 \\ 0, & t < 1 \end{cases} \quad (4.38)$$

See Figs. 4.1, 4.2 and 4.3.

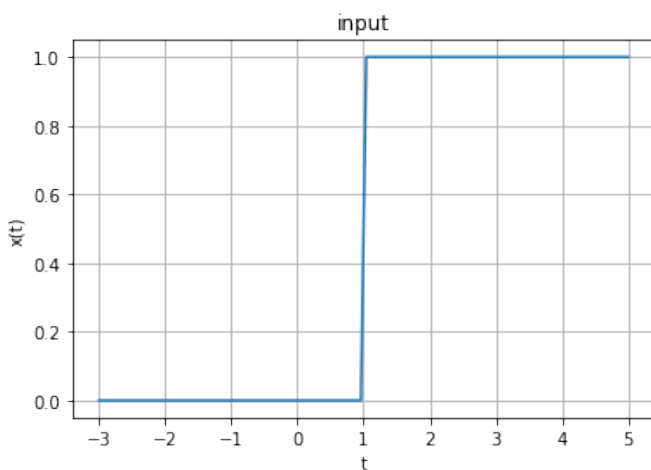


Fig. 4.1: Plot of $x(t)$

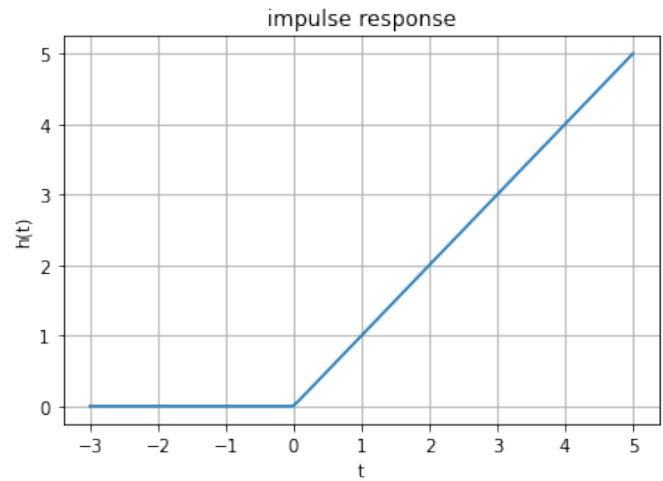


Fig. 4.2: Plot of $h(t)$

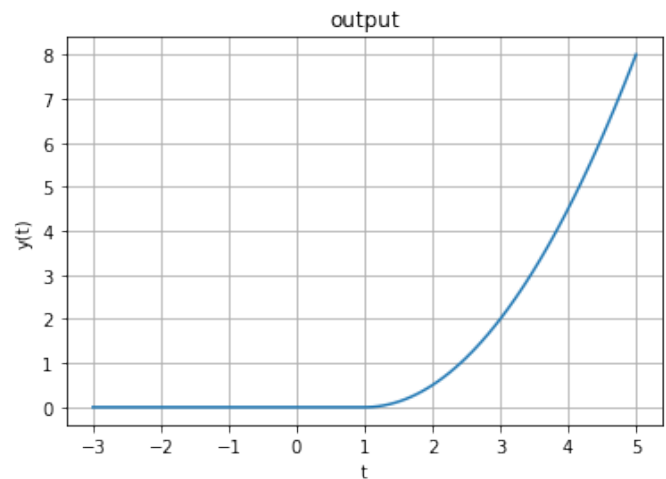


Fig. 4.3: Plot of $y(t)$

$(\alpha \ \beta \ \gamma \ \delta)$. Consider the product

$$(p \ q \ r \ s) = (a \ b \ c \ d) \begin{pmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{pmatrix} \quad (5.1)$$

The DFT of the vector $(p \ q \ r \ s)$ is a scaled version of

- (A) $(\alpha^2 \ \beta^2 \ \gamma^2 \ \delta^2)$
- (B) $(\sqrt{\alpha} \ \sqrt{\beta} \ \sqrt{\gamma} \ \sqrt{\delta})$
- (C) $(\alpha + \beta \ \beta + \delta \ \delta + \gamma \ \gamma + \alpha)$
- (D) $(\alpha \ \beta \ \gamma \ \delta)$

Solution:

5. The DFT of a vector $(a \ b \ c \ d)$ is the vector

Lemma 5.1. Let

$$\mathbf{T} = \begin{pmatrix} a & d & c & b \\ b & a & d & c \\ c & b & a & d \\ d & c & b & a \end{pmatrix} \quad (5.2)$$

Then, for

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = \mathbf{W} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}, \quad (5.3)$$

where \mathbf{W} is the DFT matrix,

$$\mathbf{T} = \mathbf{W} \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & \delta \end{pmatrix} \mathbf{W}^{-1} \quad (5.4)$$

Let

$$\mathbf{x} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}; \quad \mathbf{X} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = \mathbf{W}\mathbf{x}; \quad \mathbf{y} = \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} \quad (5.5)$$

Then

$$\mathbf{Y} = \mathbf{W}\mathbf{y} = \mathbf{W}\mathbf{T}\mathbf{x} \quad (5.6)$$

$$= \mathbf{W}\mathbf{T}\mathbf{W}^{-1}\mathbf{X} \quad (5.7)$$

$$= \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & \delta \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} \quad (5.8)$$

$$= \begin{pmatrix} \alpha^2 \\ \beta^2 \\ \gamma^2 \\ \delta^2 \end{pmatrix} \quad (5.9)$$

upon substituting from (5.4) and (5.5). Therefore option (A) is the correct option.

6. The input $x(t)$ and output $y(t)$ of a continuous time signal are related as

$$y(t) = \int_{t-T}^t x(u) du \quad (6.1)$$

The system is:

- Linear and Time-variant
- Linear and Time-invariant
- Non-Linear and Time-variant
- Non-Linear and Time-invariant

Solution:

Definition 3. We say that a system is **linear** if and only if it follows the Principle of Superposition, i.e. Law of Additivity and Law of Homogeneity.

Definition 4. A system is said to be **time invariant** if the output signal does not depend on the absolute time, i.e. a time delay on the input signal directly equates to the delay in the output signal.

Lemma 6.1. The system relating the input signal $x(t)$ and output signal $y(t)$, given by

$$y(t) = \int_{t-T}^t x(u) du \quad (6.2)$$

is linear and time invariant in nature.

Proof. a) Linearity and Time invariance

From (3), we can say the system is linear if it follows both the laws of Additivity and Homogeneity.

Law of Additivity:

Let the two input signals be $x_1(t)$ and $x_2(t)$, and their corresponding output signals be $y_1(t)$ and $y_2(t)$, then:

$$y_1(t) = \int_{t-T}^t x_1(u) du \quad (6.3)$$

$$y_2(t) = \int_{t-T}^t x_2(u) du \quad (6.4)$$

$$y_1(t) + y_2(t) = \int_{t-T}^t [x_1(u) + x_2(u)] du \quad (6.5)$$

Now, consider the input signal of $x_1(t) + x_2(t)$, then the corresponding output signal is given by $y'(t)$:

$$y'(t) = \int_{t-T}^t [x_1(u) + x_2(u)] du \quad (6.6)$$

Clearly, from (6.5) and (6.6):

$$y'(t) = y_1(t) + y_2(t) \quad (6.7)$$

Thus, the Law of Additivity holds.

Law of Homogeneity:

Consider an input signal $kx(t)$, where k is any constant. Let the corresponding output

be given by $y'(t)$, then:

$$y'(t) = \int_{t-T}^t kx(u) du \quad (6.8)$$

$$= k \int_{t-T}^t x(u) du \quad (6.9)$$

$$= ky(t) \quad (6.10)$$

Clearly, from (6.10),

$$y'(t) = ky(t) \quad (6.11)$$

Thus, the Law of Homogeneity holds.

Since both the Laws hold, the system satisfies the Principle of Superposition, and is thus, a **linear system**.

From (4), to check for time-invariance, we would introduce a delay of t_0 in the output and input signals.

Delay in output signal:

$$y(t - t_0) = \int_{t-t_0-T}^{t-t_0} x(u) du \quad (6.12)$$

Now, we consider an input signal with a delay of t_0 , given by $x(t - t_0)$, and let the corresponding output signal be given by $y'(t)$, then:

$$y'(t) = \int_{t-T}^t x(u - t_0) du \quad (6.13)$$

Substituting $a = u - t_0$:

$$y'(t) = \int_{t-t_0-T}^{t-t_0} x(a) da \quad (6.14)$$

Clearly, from (6.12) and (6.14):

$$y'(t) = y(t - t_0) \quad (6.15)$$

Thus, the system is **time-invariant**.

The correct option is **2) Linear and Time-invariant**

b) Calculating impulse response of LTI system

Since the given system is an LTI system, it would possess an impulse response $h(t)$, which is the output of the system when the input signal is the Impulse function, given

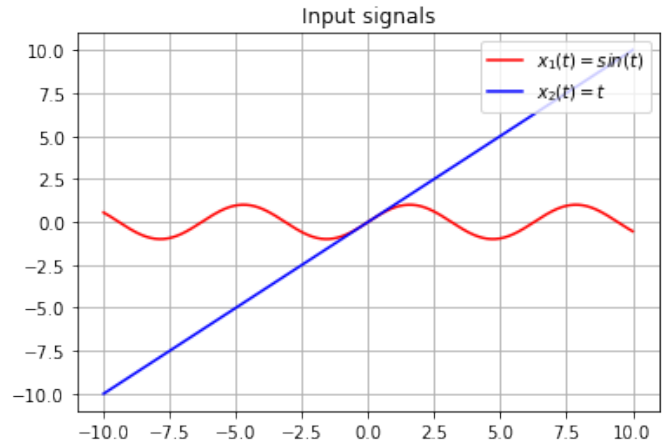


Fig. 6.1: $x_1(t) = \sin t$ and $x_2(t) = t$

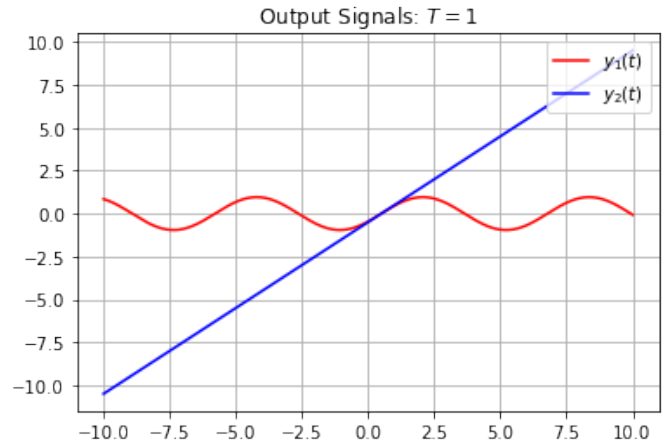


Fig. 6.2: $y_1(t)$ and $y_2(t)$

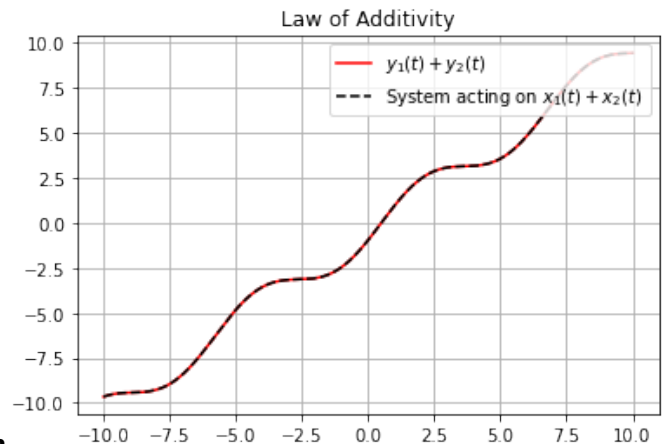


Fig. 6.3: Law of Additivity

by $\delta(t)$. Thus,

$$h(t) = \int_{t-T}^t \delta(u) du \quad (6.16)$$

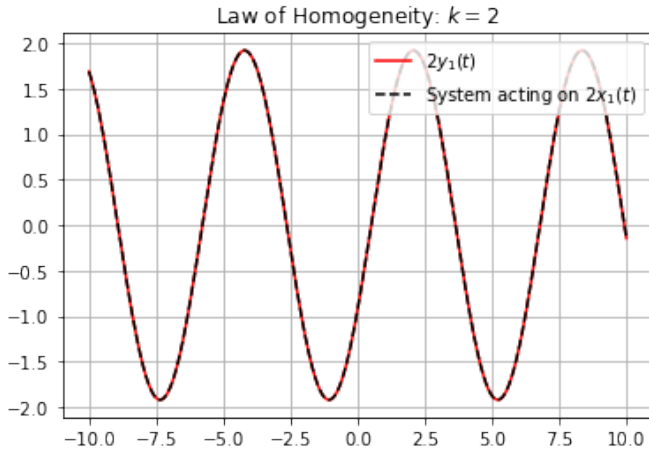


Fig. 6.4: Law of Homogeneity

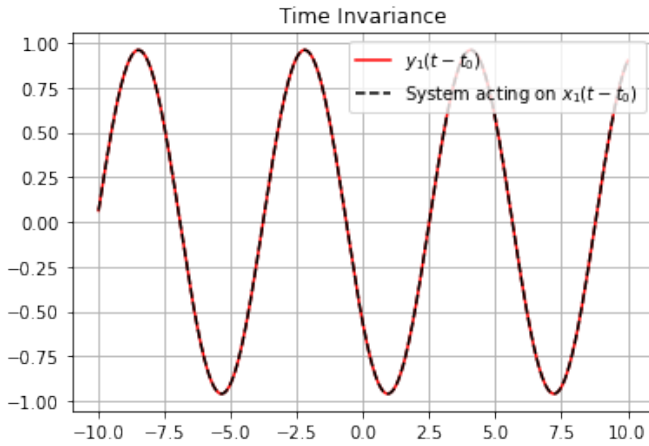


Fig. 6.5: Time invariance

The Impulse function can be loosely defined as:

$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & \text{otherwise} \end{cases} \text{ and } \int_{-\infty}^{\infty} \delta(t) dt = 1 \quad (6.17)$$

Since the Impulse function is zero everywhere aside from $t = 0$, the non-zero value of integration is a result of $\delta(0)$. Thus, we can say $h(t)$ will be non-zero only if the limits of integration would include $t = 0$, i.e:

$$h(t) = \begin{cases} \int_{t-T}^t \delta(u) du & t - T < 0; t > 0 \\ 0 & \text{otherwise} \end{cases} \quad (6.18)$$

$$h(t) = \begin{cases} 1 & 0 < t < T \\ 0 & \text{otherwise} \end{cases} \quad (6.19)$$

c) **Expressing the impulse function in terms of $u(t)$**

The unit step signal, $u(t)$, is given by:

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (6.20)$$

On time-shifting $u(t)$ by T , we get:

$$u(t - T) = \begin{cases} 1 & t - T \geq 0 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & t \geq T \\ 0 & \text{otherwise} \end{cases} \quad (6.21)$$

On subtracting (6.20) and (6.21), we get our impulse response $h(t)$ in terms of the unit step signal:

$$h(t) = u(t) - u(t - T) \quad (6.22)$$

d) **Expressing the impulse function in terms of $rect(t)$**

The unit rectangular signal, $rect(t)$ is given by:

$$rect(t) = \begin{cases} 1 & -\frac{1}{2} \leq t \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \quad (6.23)$$

We can obtain the impulse response $h(t)$ in terms of $rect(t)$ using time scaling and shifting as follows:

$$rect\left(\frac{t}{\tau}\right) = \begin{cases} 1 & -\frac{1}{2} \leq \frac{t}{\tau} \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & -\frac{\tau}{2} \leq t \leq \frac{\tau}{2} \\ 0 & \text{otherwise} \end{cases} \quad (6.24)$$

Substituting $\tau = T$:

$$rect\left(\frac{t}{T}\right) = \begin{cases} 1 & -\frac{T}{2} \leq t \leq \frac{T}{2} \\ 0 & \text{otherwise} \end{cases} \quad (6.25)$$

Now, we want to right-shift the signal by $\frac{T}{2}$:

$$rect\left(\frac{1}{T}\left(t - \frac{T}{2}\right)\right) = \begin{cases} 1 & 0 \leq t \leq T \\ 0 & \text{otherwise} \end{cases} = h(t) \quad (6.26)$$

Since the time shifting is to be performed on the variable t and not $\frac{t}{T}$

e) **Calculating the Fourier Transform of $h(t)$**

Let the Fourier Transform of $h(t)$ be given by $H(f)$ and of the rectangular signal,

$rect(t)$ be given by $Y(f)$.

$$h(t) \stackrel{\mathcal{F}}{\rightleftharpoons} H(f) \quad (6.27)$$

$$rect(t) \stackrel{\mathcal{F}}{\rightleftharpoons} Y(f) \quad (6.28)$$

Then,

$$Y(f) = \int_{-\infty}^{\infty} rect(t) e^{-j2\pi ft} dt \quad (6.29)$$

From (6.23), we can write (6.29) as:

$$Y(f) = \int_{-\infty}^{-\frac{1}{2}} 0 dt + \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-j2\pi ft} dt + \int_{\frac{1}{2}}^{\infty} 0 dt \quad (6.30)$$

$$= \frac{e^{j\pi f} - e^{-j\pi f}}{j2\pi f} \quad (6.31)$$

$$= \frac{2j \sin \pi f}{j2\pi f} \quad (6.32)$$

$$= \frac{\sin(\pi f)}{\pi f} \quad (6.33)$$

$$= sinc(f) \quad (6.34)$$

where $sinc(t)$, the sampling function is defined as:

$$sinc(t) = \begin{cases} 1 & t = 0 \\ \frac{\sin(\pi t)}{\pi} & \text{otherwise} \end{cases} \quad (6.35)$$

Let the Fourier Transform of a signal $x(t)$ be $X(f)$.

$$x(t) \stackrel{\mathcal{F}}{\rightleftharpoons} X(f) \quad (6.36)$$

When the signal $x(t)$ is time shifted by t_0 , the resultant Fourier Transform is given by:

$$x(t \pm t_0) \stackrel{\mathcal{F}}{\rightleftharpoons} X(f) e^{\pm j2\pi f t_0} \quad (6.37)$$

And when the signal $x(t)$ is time scaled by α , the resulting Fourier Transform is given by:

$$x(\alpha t) \stackrel{\mathcal{F}}{\rightleftharpoons} \frac{1}{|\alpha|} X\left(\frac{f}{\alpha}\right) \quad (6.38)$$

Since we have already derived the Fourier Transform of $rect(t)$, we would use the properties mentioned above to find the Fourier Transform of $h(t)$:

$$rect(t) \stackrel{\mathcal{F}}{\rightleftharpoons} sinc(f) \quad (6.39)$$

Using (6.37):

$$rect\left(t - \frac{T}{2}\right) \stackrel{\mathcal{F}}{\rightleftharpoons} sinc(f) e^{-j(2\pi f) \frac{T}{2}} \quad (6.40)$$

$$rect\left(t - \frac{T}{2}\right) \stackrel{\mathcal{F}}{\rightleftharpoons} sinc(f) e^{-j\pi f T} \quad (6.41)$$

Using (6.38),

$$rect\left(\frac{1}{T}\left(t - \frac{T}{2}\right)\right) \stackrel{\mathcal{F}}{\rightleftharpoons} \frac{1}{\left|\frac{1}{T}\right|} sinc\left(\frac{f}{\frac{1}{T}}\right) e^{-\frac{j\pi f T}{T}} \quad (6.42)$$

$$h(t) \stackrel{\mathcal{F}}{\rightleftharpoons} T sinc\left(\frac{f}{T}\right) e^{-j\pi f} \quad (6.43)$$

$$\therefore H(f) = T sinc\left(\frac{f}{T}\right) e^{-j\pi f} \quad (6.44)$$

f) An example

Consider an input signal of $x(t) = \cos 2\pi f_0 t$. The Fourier Transform of $x(t)$ is given by:

$$x(t) = \cos 2\pi f_0 t \stackrel{\mathcal{F}}{\rightleftharpoons} \frac{1}{2} [\delta(f - f_0) + \delta(f + f_0)] \quad (6.45)$$

using the fact that

$$\cos 2\pi f_0 t = \frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2} \quad (6.46)$$

and the Fourier Transform of $e^{\pm j2\pi f_0 t}$ is given by:

$$e^{\pm j2\pi f_0 t} \stackrel{\mathcal{F}}{\rightleftharpoons} \delta(f \mp f_0) \quad (6.47)$$

The output signal will be given by:

$$y(t) = \int_{t-T}^t \cos 2\pi f_0 u du \quad (6.48)$$

$$= \frac{1}{2\pi f_0} [\sin 2\pi f_0 t - \sin 2\pi f_0 (t - T)] \quad (6.49)$$

$$= \frac{\sin \pi f_0 T}{\pi f_0} \left[\cos 2\pi f_0 \left(t - \frac{T}{2}\right) \right] \quad (6.50)$$

$$= T sinc(f_0 T) \cos 2\pi f_0 \left(t - \frac{T}{2}\right) \quad (6.51)$$

The Fourier transform of $\cos 2\pi f_0 \left(t - \frac{T}{2}\right)$ can be obtained using (6.38) and (6.37) as follows:

$$\cos t = \frac{1}{2} [e^{jt} + e^{-jt}] \quad (6.52)$$

$$\cos t \stackrel{\mathcal{F}}{\rightleftharpoons} \frac{1}{2} \left[\delta\left(f - \frac{1}{2\pi}\right) + \delta\left(f + \frac{1}{2\pi}\right) \right] \quad (6.53)$$

$$\cos\left(t - \frac{T}{2}\right) \stackrel{\mathcal{F}}{\rightleftharpoons} \frac{e^{j\pi f T}}{2} \left[\delta\left(f - \frac{1}{2\pi}\right) + \delta\left(f + \frac{1}{2\pi}\right) \right] \quad (6.54)$$

$$\cos 2\pi f_0 \left(t - \frac{T}{2}\right) \stackrel{\mathcal{F}}{\rightleftharpoons} \frac{e^{j\pi \frac{f}{2\pi f_0} T}}{2\pi f_0} \frac{\delta\left(\frac{f}{2\pi f_0} - \frac{1}{2\pi}\right) + \delta\left(\frac{f}{2\pi f_0} + \frac{1}{2\pi}\right)}{2} \quad (6.55)$$

$$\cos 2\pi f_0 \left(t - \frac{T}{2}\right) \stackrel{\mathcal{F}}{\rightleftharpoons} \frac{e^{j\pi \frac{f}{2f_0} T}}{4\pi f_0} \left[\delta\left(\frac{f - f_0}{2\pi f_0}\right) + \delta\left(\frac{f + f_0}{2\pi f_0}\right) \right] \quad (6.56)$$

Therefore, the Fourier Transform of the output signal

$y(t)$ from (6.51) is given by:

$$y(t) \stackrel{\mathcal{F}}{\Leftrightarrow} \frac{T \text{sinc}(f_0 T)}{4\pi f_0} e^{j\pi \frac{f}{2f_0} T} \left(\delta\left(\frac{f-f_0}{2\pi f_0}\right) + \delta\left(\frac{f+f_0}{2\pi f_0}\right) \right) \quad (6.57)$$

$$y(t) \stackrel{\mathcal{F}}{\Leftrightarrow} k e^{j\pi \frac{f}{2f_0} T} \left(\delta\left(\frac{f-f_0}{2\pi f_0}\right) + \delta\left(\frac{f+f_0}{2\pi f_0}\right) \right) \quad (6.58)$$

where $k = \frac{T \text{sinc}(f_0 T)}{4\pi f_0}$. Substituting $2\pi f_0 = 1$ and $T = 1$:

$$y(t) \stackrel{\mathcal{F}}{\Leftrightarrow} k e^{j\pi^2 f} \left(\delta\left(f - \frac{1}{2\pi}\right) + \delta\left(f + \frac{1}{2\pi}\right) \right) \quad (6.59)$$

$$y(t) \stackrel{\mathcal{F}}{\Leftrightarrow} k e^{j\frac{\pi}{2}} \delta\left(f - \frac{1}{2\pi}\right) + k e^{j\frac{\pi}{2}} \delta\left(f + \frac{1}{2\pi}\right) \quad (6.60)$$

using the multiplication property of the Delta function:

$$x(t)\delta(t-t_1) = x(t_1)\delta(t-t_1) \quad (6.61)$$

Since, $e^{j\frac{\pi}{2}} = j$ and $e^{-j\frac{\pi}{2}} = -j$, we finally get:

$$y(t) \stackrel{\mathcal{F}}{\Leftrightarrow} k j \left[\delta\left(f - \frac{1}{2\pi}\right) - \delta\left(f + \frac{1}{2\pi}\right) \right] \quad (6.62)$$

Clearly, the Fourier transform of $y(t)$ can be manipulated to represent a sinusoidal wave, which is given by:

$$\sin(t) \stackrel{\mathcal{F}}{\Leftrightarrow} \frac{-j}{2} \left[\delta\left(f - \frac{1}{2\pi}\right) - \delta\left(f + \frac{1}{2\pi}\right) \right] \quad (6.63)$$

The attenuation happens for the same values of f , as depicted in the graphs of the Fourier Transforms given below.

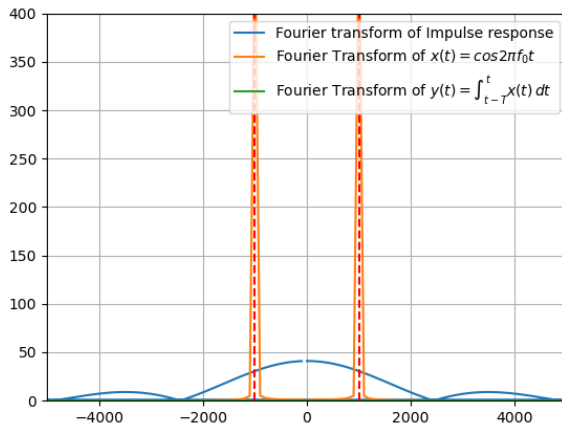


Fig. 6.6: Fourier Transform of Impulse response $h(t)$

□

7. Let the state-space representation on an LTI system be $\dot{x}(t) = Ax(t) + Bu(t)$, $y(t) = Cx(t) + du(t)$ where A,B,C are matrices, d is a scalar,

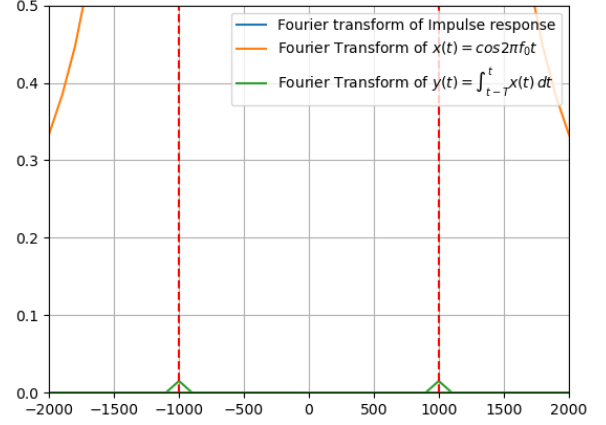


Fig. 6.7: Fourier Transform of Impulse response $h(t)$ zoomed in

$u(t)$ is the input to the system, and $y(t)$ is its output. Let $B = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T$ and $d = 0$. Which one of the following options for A and C will ensure that the transfer function of this LTI system is

$$H(s) = \frac{1}{s^3 + 3s^2 + 2s + 1} \quad (7.1)$$

- (A) $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$
- (B) $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -2 & -1 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$
- (C) $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$
- (D) $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -2 & -1 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$

Solution: From the given information,

$$\begin{pmatrix} \dot{x}(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} A & B \\ C & d \end{pmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \quad (7.2)$$

Taking Laplace transform on both sides,

$$\begin{pmatrix} sX(s) \\ Y(s) \end{pmatrix} = \begin{pmatrix} A & B \\ C & d \end{pmatrix} \begin{pmatrix} X(s) \\ U(s) \end{pmatrix} \quad (7.3)$$

$$\Rightarrow sX(s) = AX(s) + BU(s) \quad (7.4)$$

$$\Rightarrow X(s) = (sI - A)^{-1}BU(s) \quad (7.5)$$

$$\Rightarrow Y(s) = CX(s) + dU(s) \quad (7.6)$$

$$= C(sI - A)^{-1}BU(s) + dU(s) \quad (7.7)$$

By definition,

$$Y(s) = H(s)U(s) \quad (7.8)$$

$$\Rightarrow H(s) = C(sI - A)^{-1}B + d \quad (7.9)$$

$$= \frac{1}{s^3 + 3s^2 + 2s + 1} \quad (7.10)$$

$$\Rightarrow C(sI - A)^{-1}B + d = \frac{1}{s^3 + 3s^2 + 2s + 1} \quad (7.11)$$

Now we cross verify the options with eq 7.11.

By using a python script,

(A)

$$C(sI - A)^{-1}B + d = \frac{1}{s^3 + 3s^2 + 2s + 1} \quad (7.12)$$

(B)

$$C(sI - A)^{-1}B + d = \frac{1}{s^3 + 1s^2 + 2s + 3} \quad (7.13)$$

(C)

$$C(sI - A)^{-1}B + d = \frac{s^2}{s^3 + 3s^2 + 2s + 1} \quad (7.14)$$

(D)

$$C(sI - A)^{-1}B + d = \frac{s^2}{s^3 + 1s^2 + 2s + 3} \quad (7.15)$$

Hence A is the correct option.

8. Consider a real-valued base-band signal $x(t)$, band limited to 10 kHz. The Nyquist rate for the signal $y(t) = x(t)x\left(1 + \frac{t}{2}\right)$ is

- 15 kHz
- 30 kHz
- 60 kHz
- 20 kHz

Solution: Let

$$x(t) = \cos(20k\pi t) = \cos\{2\pi(10k)t\} \quad (8.1)$$

$$\Rightarrow B_x = 10kHz \quad (8.2)$$

where B_x is the bandwidth of $x(t)$. Then

$$y(t) = \cos(30k\pi t) + \cos(10k\pi t) \quad (8.3)$$

$$\Rightarrow B_y = \frac{30}{2} = 15kHz \quad (8.4)$$

Thus, the Nyquist rate is

$$B = 2B_y \quad (8.5)$$

$$= 30kHz \quad (8.6)$$

Figs. 8.1- 8.5 show the sampling theorem in action. Fig. 8.5 shows how the violation of the the Nyquist criterion results in distortion during reconstruction.

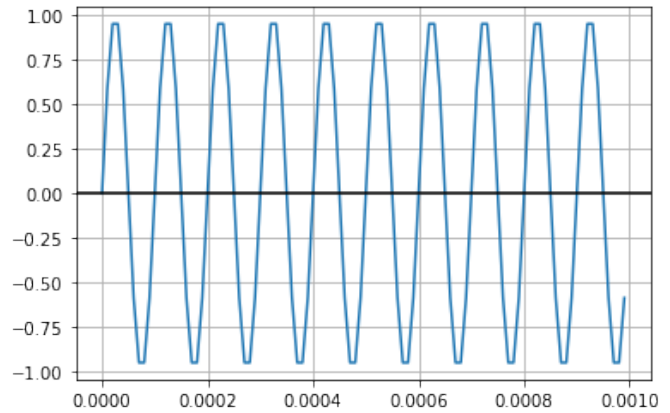


Fig. 8.1: $x(t)$: Sinusoidal signal with freq=10kHz

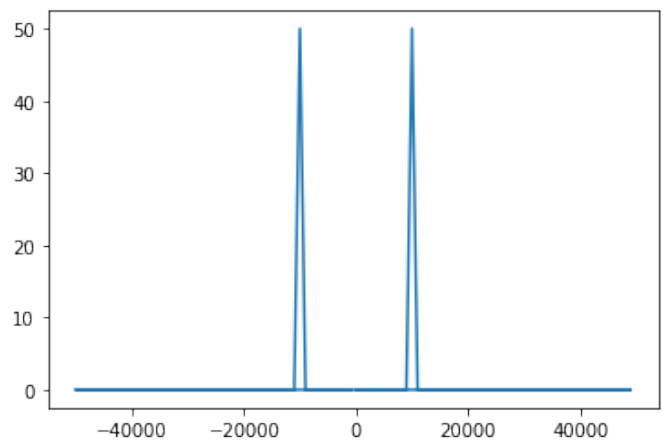
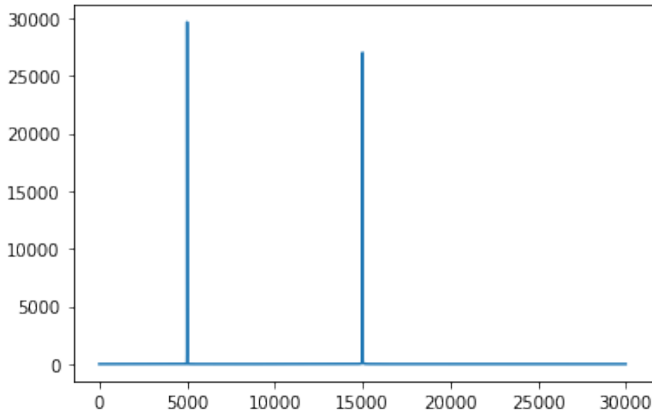
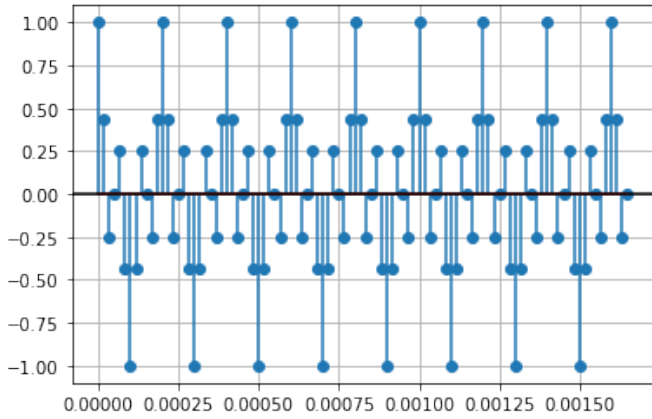
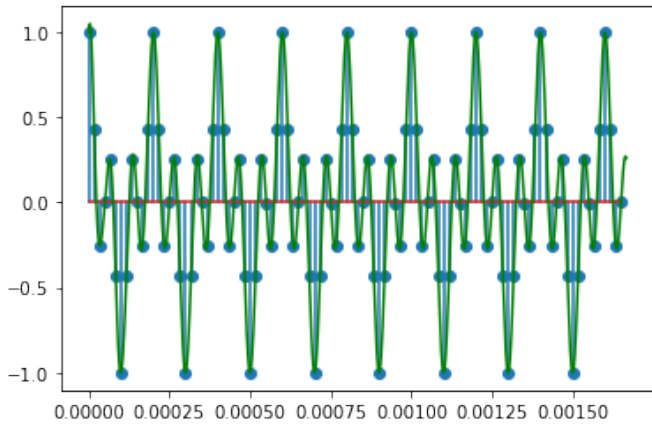


Fig. 8.2: DFT of $x(t)$. Bandwidth = 10000

9. Consider the signals $x[n] = 2^{n-1}u[-n + 2]$ and $y[n] = 2^{-n+2}u[n + 1]$, where $u[n]$ is the unit step sequence. Let $X(e^{j\omega})$ and $Y(e^{j\omega})$ the discrete-time Fourier transform of $x[n]$ and

Fig. 8.3: DFT of $y(t)$. Bandwidth = 15000Fig. 8.4: stem plot of $y(t)$ sampled at 60kHzFig. 8.5: Shannon interpolation of $y(t)$

is **Solution:** Since

$$a^n u(-n-1) \stackrel{Z}{\Longleftrightarrow} \frac{1}{1 - az^{-1}} \quad |z| < |a|, \quad (9.2)$$

$$X(z) = \frac{4z^{-3}}{1 - 2z^{-1}} \quad |z| < 2, \quad (9.3)$$

$$Y(z) = \frac{8z}{1 - \frac{1}{2}z^{-1}} \quad |z| > \frac{1}{2} \quad (9.4)$$

and

$$x(n) * y(-n) \stackrel{Z}{\Longleftrightarrow} \frac{32z^{-4}}{(1 - 2z^{-1})(1 - \frac{1}{2}z)} \quad |z| < 2, \quad (9.5)$$

$$\Rightarrow z(n) = x(n) * y(-n) = (4 - n)2^{n+1}u[4 - n] \quad (9.6)$$

$$\therefore \frac{1}{2\pi} \int_0^{2\pi} X(e^{j\omega})Y(e^{-j\omega})d\omega = z[0] = 8 \quad (9.7)$$

$y[n]$, respectively. The value of the integral

$$\frac{1}{2\pi} \int_0^{2\pi} X(e^{j\omega})Y(e^{-j\omega})d\omega \quad (9.1)$$