Signal Processing

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Contents

***Abstract*—This manual provides solved problems in signal processing from GATE exam papers.**

1. A continuous time LTI system is described by

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = 2\frac{dx(t)}{dt} + 4x(t) \tag{1.1}$$

Assuming zero initial conditions, the response $y(t)$ of the above system for the input $x(t) = e^{-2t}u(t)$ is given by

a) $\left(e^{t} - e^{3t}\right)u(t)$
b) $\left(e^{-t} - e^{-3t}\right)u(t)$
c) $\left(e^{-t} + e^{-3t}\right)u(t)$
d) $\left(e^{t} + e^{3t}\right)u(t)$

Solution:

Lemma 1.1 (Table of Laplace Transforms)**.**

Time Function $f(t) = \mathcal{L}^{-1}\{F(s)\}$	***Laplace transform*** *of f(t)* $F(s) = \mathcal{L}\{f(t)\}$
$u(t)$	$\frac{1}{s},\ s > 0$
$g'(t)$	$sG(s) - g(0)$
$g''(t)$	$s^2G(s) - sg(0) - g'(0)$
$e^{-at}u(t)$	$\frac{1}{s+a},\ s + a > 0$

Lemma 1.2. *Linearity of Laplace Transform*

$$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\} \tag{1.2}$$

From Lemma-1.1 Laplace transform of $x(t) = e^{-2t}u(t)$ is given by

$$X(s) = \frac{1}{s+2} \tag{1.3}$$

Since initial conditions are zero. Laplace Transform of (1.1) gives

$$s^2Y(s) + 4sY(s) + 3Y(s) = 2sX(s) + 4X(s) \tag{1.4}$$

$$Y(s) = \frac{2(s+2)}{s^2+4s+3}X(s) \tag{1.5}$$

$$= \frac{1}{s+1} - \frac{1}{s+3} \tag{1.6}$$

From Lemma-1.1. Inverse Laplace transform of $Y(s)$ is given by

$$y(t) = e^{-t}u(t) - e^{-3t}u(t) \tag{1.7}$$

$$= \left(e^{-t} - e^{-3t}\right)u(t) \tag{1.8}$$

$\therefore$ The required option is B. See Fig. 1.1.

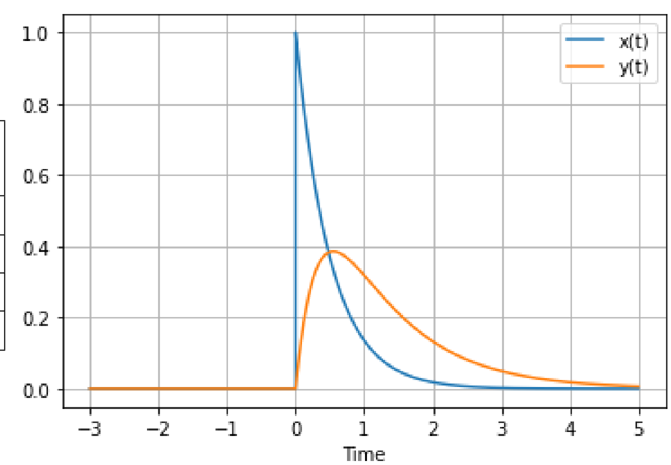

Fig. 1.1: Plot of input and output responses in time domain.

2. The impulse response of a system is $h(t) = tu(t)$. For an input $u(t-1)$, the output is

a) $\frac{t^2}{2}u(t)$
b) $\frac{t(t-1)}{2}u(t-1)$
c) $\frac{(t-1)^2}{2}u(t-1)$

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d) $\dfrac{t^2-1}{2}u(t-1)$

Solution:

Definition 1 (Laplace Transform)**.** *It is an integral transform that converts a function of a real variable t to a function of a complex variable s. The Laplace transform of $f(t)$ is denoted by $\mathcal{L}\{f(t)\}$ or $F(s)$.*

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt \tag{2.1}$$

Remark. Laplace transform of $f(t) = t^n, n \geq 1$ is

$$F(s) = \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, s > 0 \tag{2.2}$$

Proof. Basis Step: $n = 1$

$$\mathcal{L}\{t\} = \int_0^\infty e^{-st} t dt \tag{2.3}$$

$$= \left[\frac{te^{-st}}{-s}\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st} dt \tag{2.4}$$

$$= 0 + \left[\frac{-1}{s^2}e^{-st}\right]_0^\infty, s > 0 \tag{2.5}$$

$$= \frac{1}{s^2}, s > 0 \tag{2.6}$$

Inductive Step:

$$\mathcal{L}\{t^n\} = \int_0^\infty e^{-st} t^n dt \tag{2.7}$$

$$= \left[\frac{t^n e^{-st}}{-s}\right]_0^\infty + \frac{n}{s}\int_0^\infty t^{n-1} e^{-st} dt \tag{2.8}$$

$$= 0 + \frac{n}{s}\mathcal{L}\left\{t^{n-1}\right\}, s > 0 \tag{2.9}$$

$$= \frac{n}{s}\mathcal{L}\left\{t^{n-1}\right\}, s > 0 \tag{2.10}$$

To prove that if
eqrefec/2003/8eq:t holds for $n = k$, it holds for $n = k + 1$. From
eqrefec/2003/8eq:e

$$\mathcal{L}\left\{t^{k+1}\right\} = \frac{k+1}{s}\mathcal{L}\left\{t^k\right\} \tag{2.11}$$

$$= \frac{(k+1)k!}{s(s^{k+1})} = \frac{(k+1)!}{s^{k+2}}, s > 0 \tag{2.12}$$

By mathematical induction,
eqrefec/2003/8eq:t is true $\forall n \geq 1$ □

Lemma 2.1. *For any real number c,*

$$\mathcal{L}\{u(t-c)\} = \frac{e^{-cs}}{s}, s > 0 \tag{2.13}$$

Proof.

$$\mathcal{L}\{u(t-c)\} = \int_0^\infty e^{-st}u(t-c)dt = \int_c^\infty e^{-st}dt \tag{2.14}$$

$$= \left[-\frac{e^{-st}}{s}\right]_c^\infty = \frac{e^{-cs}}{s}, s > 0 \tag{2.15}$$

□

Definition 2 (Inverse Laplace Transform)**.** *It is the transformation of a Laplace transform into a function of time. If $F(s) = \mathcal{L}\{f(t)\}$, then the Inverse laplace transform of $F(s)$ is $\mathcal{L}^{-1}\{F(s)\} = f(t)$.*

Lemma 2.2 (t-shift rule)**.** *For any real number c,*

$$\mathcal{L}\{u(t-c)f(t-c)\} = e^{-cs}F(s) \tag{2.16}$$

Proof.

$$\mathcal{L}\{u(t-c)f(t-c)\} = \int_0^\infty e^{-st}u(t-c)f(t-c)dt \tag{2.17}$$

$$= \int_c^\infty e^{-st}f(t-c)dt \tag{2.18}$$

$$= \int_0^\infty e^{-s(\tau+c)}f(\tau)d\tau \; (t = \tau + c) \tag{2.19}$$

$$= e^{-cs}\int_0^\infty e^{-s\tau}f(\tau)d\tau \tag{2.20}$$

$$= e^{-cs}F(s) \tag{2.21}$$

□

Corollary 0.1.

$$\mathcal{L}^{-1}\{e^{-cs}F(s)\} = u(t-c)f(t-c) \tag{2.22}$$

Theorem 0.2 (Convolution theorem)**.** *Suppose $F(s) = \mathcal{L}\{f(t)\}, G(s) = \mathcal{L}\{g(t)\}$ exist, then,*

$$\mathcal{L}^{-1}\{F(s)G(s)\} = f(t) * g(t) \tag{2.23}$$

Given,

$$h(t) = tu(t) \tag{2.24}$$

$$x(t) = u(t-1) \tag{2.25}$$

To find: $y(t)$. We know,

$$y(t) = h(t) * x(t) \tag{2.26}$$
$$= \mathcal{L}^{-1}\{H(s)X(s)\} \tag{2.27}$$

From
eqrefec/2003/8eq:uf and
eqrefec/2003/8eq:t,

$$H(s) = e^0\mathcal{L}\{t\} = \frac{1}{s^2} \tag{2.28}$$

From
eqrefec/2003/8eq:u,

$$X(s) = \frac{e^{-s}}{s} \tag{2.29}$$

Substituting in
eqrefec/2003/8eq:def,

$$y(t) = \mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^3}\right\} \tag{2.30}$$

Consider

$$p(t) = \frac{t^2}{2} \tag{2.31}$$

From
eqrefec/2003/8eq:t

$$P(s) = \frac{2!}{2s^3} = \frac{1}{s^3} \tag{2.32}$$

Further, from
eqrefec/2003/8eq:cuf, for $c = 1$

$$\mathcal{L}^{-1}\{e^{-s}P(s)\} = u(t-1)p(t-1) \tag{2.33}$$
$$= u(t-1)\frac{(t-1)^2}{2} \tag{2.34}$$
$$\therefore y(t) = \frac{(t-1)^2}{2}u(t-1) \tag{2.35}$$

Option 3 is the correct answer.

$$h(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases} \tag{2.36}$$

$$x(t) = \begin{cases} 1, & t \geq 1 \\ 0, & t < 1 \end{cases} \tag{2.37}$$

$$y(t) = \begin{cases} \frac{(t-1)^2}{2}, & t \geq 1 \\ 0, & t < 1 \end{cases} \tag{2.38}$$

See Figs. 2.1, 2.2 and 2.3.

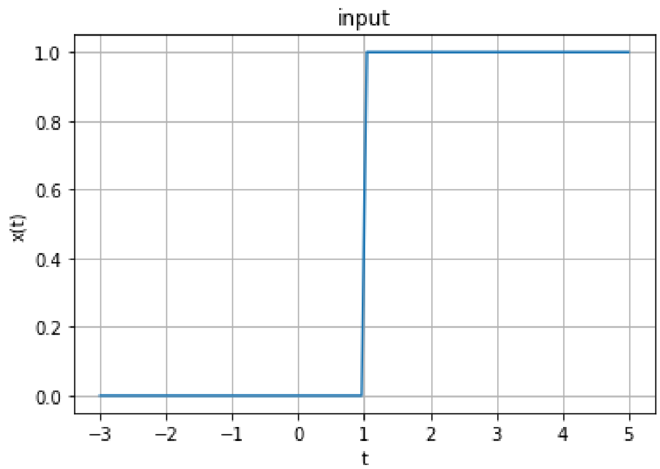

Fig. 2.1: Plot of $x(t)$

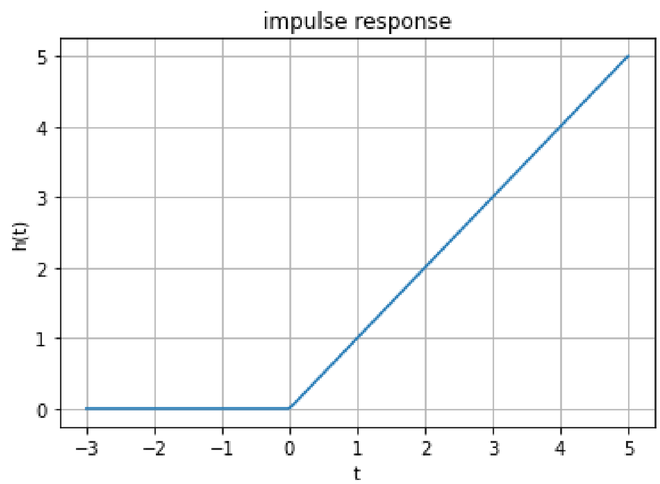

Fig. 2.2: Plot of $h(t)$

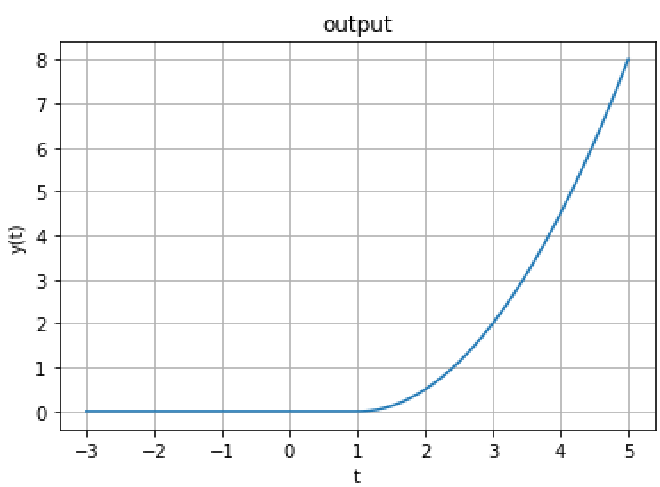

Fig. 2.3: Plot of $y(t)$

3. The DFT of a vector $\begin{pmatrix} a & b & c & d \end{pmatrix}$ is the vector $\begin{pmatrix} \alpha & \beta & \gamma & \delta \end{pmatrix}$. Consider the product

$$\begin{pmatrix} p & q & r & s \end{pmatrix} = \begin{pmatrix} a & b & c & d \end{pmatrix}\begin{pmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{pmatrix}$$

The DFT of the vector $\begin{pmatrix} p & q & r & s \end{pmatrix}$ is a scaled version of

(A) $\begin{pmatrix} \alpha^2 & \beta^2 & \gamma^2 & \delta^2 \end{pmatrix}$
(B) $\begin{pmatrix} \sqrt{\alpha} & \sqrt{\beta} & \sqrt{\gamma} & \sqrt{\delta} \end{pmatrix}$
(C) $\begin{pmatrix} \alpha+\beta & \beta+\delta & \delta+\gamma & \gamma+\alpha \end{pmatrix}$
(D) $\begin{pmatrix} \alpha & \beta & \gamma & \delta \end{pmatrix}$

Solution:

Lemma 3.1. *Let*

$$\mathbf{T} = \begin{pmatrix} a & d & c & b \\ b & a & d & c \\ c & b & a & d \\ d & c & b & a \end{pmatrix} \tag{3.2}$$

Then, for

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = \mathbf{W} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}, \tag{3.3}$$

where $\mathbf{W}$ *is the DFT matrix,*

$$\mathbf{T} = \mathbf{W} \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & \delta \end{pmatrix} \mathbf{W}^{-1} \tag{3.4}$$

Let

$$\mathbf{x} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix};\ \mathbf{X} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = \mathbf{Wx};\ \mathbf{y} = \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} \tag{3.5}$$

Then

$$\mathbf{Y} = \mathbf{Wy} = \mathbf{WTx} \tag{3.6}$$
$$= \mathbf{WTW}^{-1}\mathbf{X} \tag{3.7}$$
$$= \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & \delta \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} \tag{3.8}$$
$$= \begin{pmatrix} \alpha^2 \\ \beta^2 \\ \gamma^2 \\ \delta^2 \end{pmatrix} \tag{3.9}$$

upon substituting from (3.4) and (3.5). Therefore option (A) is the correct option.

4. The input $x(t)$ and output $y(t)$ of a continous time signal are related as

$$y(t) = \int_{t-T}^{t} x(u)\,du \tag{4.1}$$

The system is:

a) Linear and Time-variant
b) Linear and Time-invariant
c) Non-Linear and Time-variant
d) Non-Linear and Time-invariant

Solution:

Definition 3. *We say that a system is* ***linear*** *if and only if it follows the Principle of Superposition, i.e Law of Additivity and Law of Homogeneity.*

Definition 4. *A system is said to be* ***time invariant*** *if the output signal does not depend on the absolute time, i.e a time delay on the input signal directly equates to the delay in the output signal.*

Lemma 4.1. *The system relating the input signal $x(t)$ and output signal $y(t)$, given by*

$$y(t) = \int_{t-T}^{t} x(u)\,du \tag{4.2}$$

is linear and time invariant in nature.

Proof. a) **Linearity and Time invariance**

From (3), we can say the system is linear if it follows both the laws of Additivity and Homogeneity.

Law of Additivity:

Let the two input signals be $x_1(t)$ and $x_2(t)$, and their corresponding output signals be $y_1(t)$ and $y_2(t)$, then:

$$y_1(t) = \int_{t-T}^{t} x_1(u)\,du \tag{4.3}$$
$$y_2(t) = \int_{t-T}^{t} x_2(u)\,du \tag{4.4}$$
$$y_1(t) + y_2(t) = \int_{t-T}^{t} [x_1(u) + x_2(u)]\,du \tag{4.5}$$

Now, consider the input signal of $x_1(t)+x_2(t)$, then the corresponding output signal is given by $y'(t)$:

$$y'(t) = \int_{t-T}^{t} [x_1(u) + x_2(u)]\,du \tag{4.6}$$

Clearly, from (4.5) and (4.6):

$$y'(t) = y_1(t) + y_2(t) \tag{4.7}$$

Thus, the Law of Additivity holds.

Law of Homogeneity:
Consider an input signal $kx(t)$, where k is any constant. Let the corresponding output be given by $y'(t)$, then:

$$y'(t) = \int_{t-T}^{t} kx(u)\,du \tag{4.8}$$
$$= k\int_{t-T}^{t} x(u)\,du \tag{4.9}$$
$$= ky(t) \tag{4.10}$$

Clearly, from (4.10),

$$y'(t) = ky(t) \tag{4.11}$$

Thus, the Law of Homogeneity holds.

Since both the Laws hold, the system satisfies the Principle of Superposition, and is thus, a **linear system**.

From (4) , to check for time-invariance, we would introduce a delay of t_0 in the output and input signals.
Delay in output signal:

$$y(t-t_0) = \int_{t-t_0-T}^{t-t_0} x(u)\,du \tag{4.12}$$

Now, we consider an input signal with a delay of t_0, given by $x(t-t_0)$, and let the corresponding output signal be given by $y'(t)$, then:

$$y'(t) = \int_{t-T}^{t} x(u-t_0)\,du \tag{4.13}$$

Substituting $a = u - t_0$:

$$y'(t) = \int_{t-t_0-T}^{t-t_0} x(a)\,da \tag{4.14}$$

Clearly, from (4.12) and (4.14):

$$y'(t) = y(t-t_0) \tag{4.15}$$

Thus, the system is **time-invariant**.
The correct option is **2) Linear and Time-invariant**

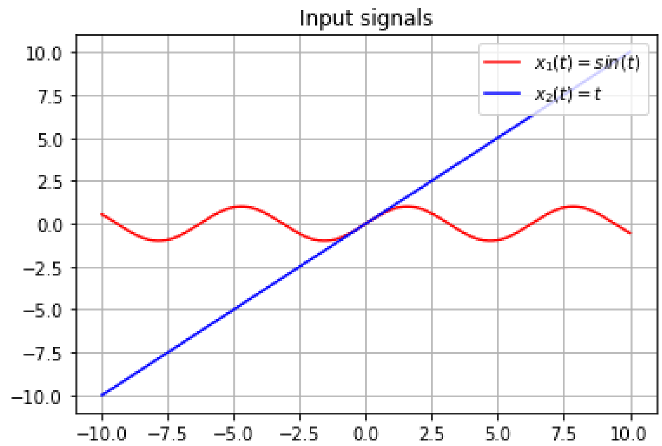

Fig. 4.1: $x_1(t) = \sin t$ and $x_2(t) = t$

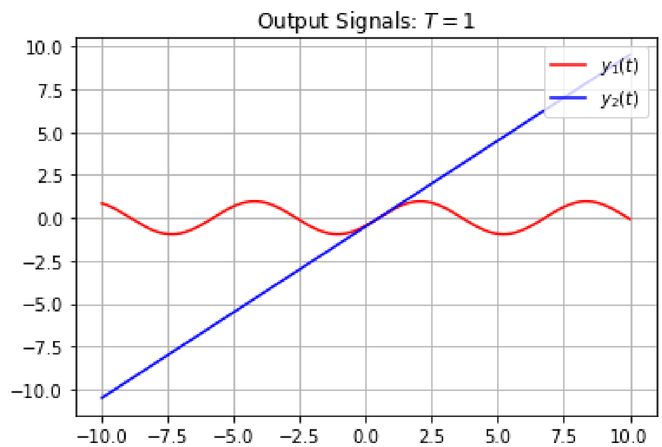

Fig. 4.2: $y_1(t)$ and $y_2(t)$

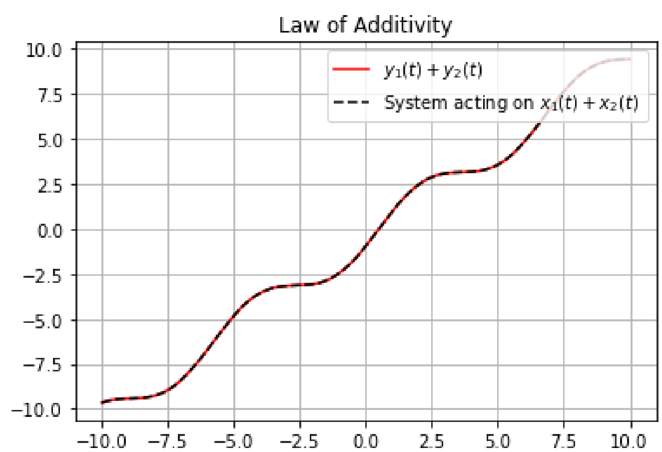

Fig. 4.3: Law of Additivity

b) **Calculating impulse response of LTI system**
Since the given system is an LTI system, it would possess an impulse response $h(t)$, which is the output of the system when the input signal is the Impulse function, given

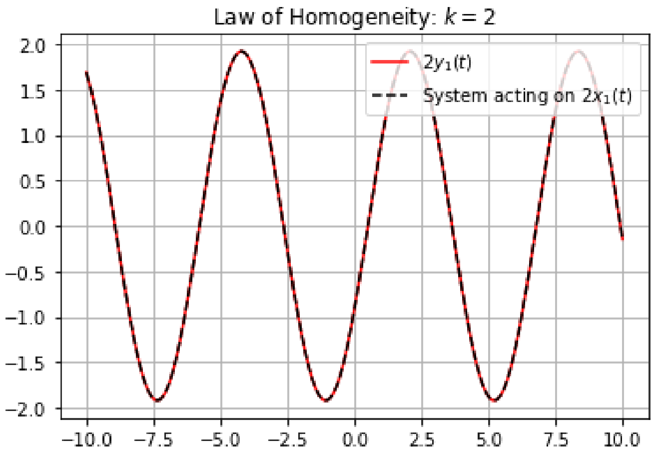

Fig. 4.4: Law of Homogeneity

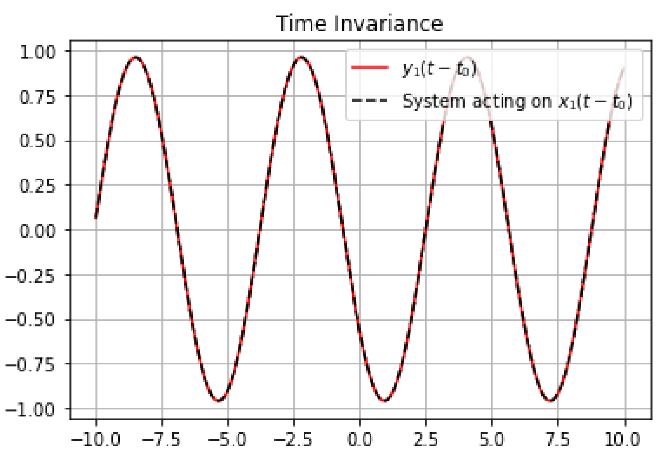

Fig. 4.5: Time invariance

by $\delta(t)$. Thus,

$$h(t) = \int_{t-T}^{t} \delta(u)du \tag{4.16}$$

The Impulse function can be loosely defined as:

$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & otherwise \end{cases} \; and \; \int_{-\infty}^{\infty} \delta(t)dt = 1 \tag{4.17}$$

Since the Impulse function is zero everywhere aside from $t = 0$, the non-zero value of integration is a result of $\delta(0)$. Thus, we can say $h(t)$ will be non-zero only if the limits of integration would include $t = 0$, i.e:

$$h(t) = \begin{cases} \int_{t-T}^{t} \delta(u)du & t - T < 0; t > 0 \\ 0 & otherwise \end{cases} \tag{4.18}$$

$$h(t) = \begin{cases} 1 & 0 < t < T \\ 0 & otherwise \end{cases} \tag{4.19}$$

c) **Expressing the impulse function in terms of $u(t)$**
The unit step signal, $u(t)$, is given by:

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & otherwise \end{cases} \tag{4.20}$$

On time-shifting $u(t)$ by T, we get:

$$u(t - T) = \begin{cases} 1 & t - T \geq 0 \\ 0 & otherwise \end{cases} = \begin{cases} 1 & t \geq T \\ 0 & otherwise \end{cases} \tag{4.21}$$

On subtracting (4.20) and (4.21), we get our impulse response $h(t)$ in terms of the unit step signal:

$$h(t) = u(t) - u(t - T) \tag{4.22}$$

d) **Expressing the impulse function in terms of $rect(t)$**
The unit rectangular signal, $rect(t)$ is given by:

$$rect(t) = \begin{cases} 1 & \frac{-1}{2} \leq t \leq \frac{1}{2} \\ 0 & otherwise \end{cases} \tag{4.23}$$

We can obtain the impulse response $h(t)$ in terms of $rect(t)$ using time scaling and shifting as follows:

$$rect\left(\frac{t}{\tau}\right) = \begin{cases} 1 & \frac{-1}{2} \leq \frac{t}{\tau} \leq \frac{1}{2} \\ 0 & otherwise \end{cases} = \begin{cases} 1 & \frac{-\tau}{2} \leq t \leq \frac{\tau}{2} \\ 0 & otherwise \end{cases} \tag{4.24}$$

Subsituting $\tau = T$:

$$rect\left(\frac{t}{T}\right) = \begin{cases} 1 & \frac{-T}{2} \leq t \leq \frac{T}{2} \\ 0 & otherwise \end{cases} \tag{4.25}$$

Now, we want to right-shift the signal by $\frac{T}{2}$:

$$rect\left(\frac{1}{T}\left(t - \frac{T}{2}\right)\right) = \begin{cases} 1 & 0 \leq t \leq T \\ 0 & otherwise \end{cases} = h(t) \tag{4.26}$$

Since the time shifting is to be performed on the variable t and not $\frac{t}{T}$

e) **Calculating the Fourier Transform of $h(t)$**
Let the Fourier Transform of $h(t)$ be given by $H(f)$ and of the rectangular signal, $rect(t)$ be given by $Y(f)$.

$$h(t) \overset{\mathcal{F}}{\rightleftharpoons} H(f) \tag{4.27}$$

$$rect(t) \overset{\mathcal{F}}{\rightleftharpoons} Y(f) \tag{4.28}$$

Then,

$$Y(f) = \int_{-\infty}^{\infty} rect(t)e^{-j2\pi ft}\,dt \tag{4.29}$$

From (4.23), we can write (4.29) as:

$$\begin{aligned} Y(f) &= \int_{-\infty}^{\frac{-1}{2}} 0\,dt + \int_{\frac{-1}{2}}^{\frac{1}{2}} e^{-j2\pi ft}\,dt + \int_{\frac{1}{2}}^{\infty} 0\,dt && (4.30)\\ &= \frac{e^{j\pi f} - e^{-j\pi f}}{j2\pi f} && (4.31)\\ &= \frac{2j\sin\pi f}{j2\pi f} && (4.32)\\ &= \frac{\sin(\pi f)}{\pi f} && (4.33)\\ &= sinc(f) && (4.34)\end{aligned}$$

where $sinc(t)$, the sampling function is defined as:

$$sinc(t) = \begin{cases} 1 & t = 0 \\ \frac{\sin(\pi t)}{\pi t} & otherwise \end{cases} \tag{4.35}$$

Let the Fourier Transform of a signal $x(t)$ be $X(f)$.

$$x(t) \overset{\mathcal{F}}{\rightleftharpoons} X(f) \tag{4.36}$$

When the signal $x(t)$ is time shifted by t_0, the resultant Fourier Transform is given by:

$$x(t \pm t_0) \overset{\mathcal{F}}{\rightleftharpoons} X(f)e^{\pm j2\pi f t_0} \tag{4.37}$$

And when the signal $x(t)$ is time scaled by α, the resulting Fourier Transform is given by:

$$x(\alpha t) \overset{\mathcal{F}}{\rightleftharpoons} \frac{1}{|\alpha|}X\left(\frac{f}{\alpha}\right) \tag{4.38}$$

Since we have already derived the Fourier Transform of $rect(t)$, we would use the properties mentioned above to find the Fourier Transform of $h(t)$:

$$rect(t) \overset{\mathcal{F}}{\rightleftharpoons} sinc(f) \tag{4.39}$$

Using (4.37):

$$rect\left(t - \frac{T}{2}\right) \overset{\mathcal{F}}{\rightleftharpoons} sinc(f)e^{-j(2\pi f)\frac{T}{2}} \tag{4.40}$$

$$rect\left(t - \frac{T}{2}\right) \overset{\mathcal{F}}{\rightleftharpoons} sinc(f)e^{-j\pi fT} \tag{4.41}$$

Using (4.38),

$$rect\left(\frac{1}{T}\left(t - \frac{T}{2}\right)\right) \overset{\mathcal{F}}{\rightleftharpoons} \frac{1}{\frac{1}{|T|}} sinc\left(\frac{f}{T}\right)e^{\frac{-j\pi fT}{T}} \tag{4.42}$$

$$h(t) \overset{\mathcal{F}}{\rightleftharpoons} T\,sinc\left(\frac{f}{T}\right)e^{-j\pi f} \tag{4.43}$$

$$\therefore H(f) = T\,sinc\left(\frac{f}{T}\right)e^{-j\pi f} \tag{4.44}$$

f) **An example**
Consider an input signal of $x(t) = \cos 2\pi f_0 t$. The Fourier Transform of $x(t)$ is given by:

$$x(t) = \cos 2\pi f_0 t \overset{\mathcal{F}}{\rightleftharpoons} \frac{1}{2}\left[\delta(f - f_0) + \delta(f + f_0)\right] \tag{4.45}$$

using the fact that

$$\cos 2\pi f_0 t = \frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2} \tag{4.46}$$

and the Fourier Transform of $e^{\pm j2\pi f_0 t}$ is given by:

$$e^{\pm j2\pi f_0 t} \overset{\mathcal{F}}{\rightleftharpoons} \delta(f \mp f_0) \tag{4.47}$$

The output signal will be given by:

$$\begin{aligned} y(t) &= \int_{t-T}^{t} \cos 2\pi f_0 u\,du && (4.48)\\ &= \frac{1}{2\pi f_0}\left[\sin 2\pi f_0 t - \sin 2\pi f_0 (t - T)\right] && (4.49)\\ &= \frac{\sin\pi f_0 T}{\pi f_0}\left[\cos 2\pi f_0\left(t - \frac{T}{2}\right)\right] && (4.50)\\ &= T\,sinc(f_0 T)\cos 2\pi f_0\left(t - \frac{T}{2}\right) && (4.51)\end{aligned}$$

The Fourier transform of $\cos 2\pi f_0\left(t - \frac{T}{2}\right)$ can be obtained using (4.38) and (4.37) as follows:

$$\cos t = \frac{1}{2}\left[e^{jt} + e^{-jt}\right] \tag{4.52}$$

$$\cos t \overset{\mathcal{F}}{\rightleftharpoons} \frac{1}{2}\left[\delta\left(f - \frac{1}{2\pi}\right) + \delta\left(f + \frac{1}{2\pi}\right)\right] \tag{4.53}$$

$$\cos\left(t - \frac{T}{2}\right) \overset{\mathcal{F}}{\rightleftharpoons} \frac{e^{j\pi f T}}{2}\left[\delta\left(f - \frac{1}{2\pi}\right) + \delta\left(f + \frac{1}{2\pi}\right)\right] \tag{4.54}$$

$$\cos 2\pi f_0\left(t - \frac{T}{2}\right) \overset{\mathcal{F}}{\rightleftharpoons} \frac{e^{j\pi\frac{f}{2\pi f_0}T}}{2\pi f_0}\,\frac{\delta(\frac{f}{2\pi f_0} - \frac{1}{2\pi}) + \delta(\frac{f}{2\pi f_0} + \frac{1}{2\pi})}{2} \tag{4.55}$$

$$\cos 2\pi f_0\left(t - \frac{T}{2}\right) \overset{\mathcal{F}}{\rightleftharpoons} \frac{e^{j\pi\frac{f}{2f_0}T}}{4\pi f_0}\left(\delta\left(\frac{f - f_0}{2\pi f_0}\right) + \delta\left(\frac{f + f_0}{2\pi f_0}\right)\right) \tag{4.56}$$

Therefore, the Fourier Transform of the output signal $y(t)$ from (4.51) is given by:

$$y(t) \overset{\mathcal{F}}{\rightleftharpoons} \frac{T\,sinc(f_0 T)}{4\pi f_0} e^{j\pi\frac{f}{2f_0}T}\left(\delta\left(\frac{f - f_0}{2\pi f_0}\right) + \delta\left(\frac{f + f_0}{2\pi f_0}\right)\right) \tag{4.57}$$

$$y(t) \overset{\mathcal{F}}{\rightleftharpoons} k e^{j\pi\frac{f}{2f_0}T}\left(\delta\left(\frac{f - f_0}{2\pi f_0}\right) + \delta\left(\frac{f + f_0}{2\pi f_0}\right)\right) \tag{4.58}$$

where $k = \frac{T\,sinc(f_0 T)}{4\pi f_0}$. Substituting $2\pi f_0 = 1$ and $T = 1$:

$$y(t) \overset{\mathcal{F}}{\rightleftharpoons} k e^{j\pi^2 f}\left(\delta\left(f - \frac{1}{2\pi}\right) + \delta\left(f + \frac{1}{2\pi}\right)\right) \tag{4.59}$$

$$y(t) \overset{\mathcal{F}}{\rightleftharpoons} k e^{j\frac{\pi}{2}}\delta\left(f - \frac{1}{2\pi}\right) + k e^{j\frac{-\pi}{2}}\delta\left(f + \frac{1}{2\pi}\right) \tag{4.60}$$

using the multiplication property of the Delta function:

$$x(t)\delta(t - t_1) = x(t_1)\delta(t - t_1) \tag{4.61}$$

Since , $e^{j\frac{\pi}{2}} = j$ and $e^{-j\frac{\pi}{2}} = -j$, we finally get:

$$y(t) \overset{\mathcal{F}}{\rightleftharpoons} kj\left[\delta\left(f - \frac{1}{2\pi}\right) - \delta\left(f + \frac{1}{2\pi}\right)\right] \tag{4.62}$$

Clearly, the Fourier transform of $y(t)$ can be manipulated to represent a sinusoidal wave, which is given by:

$$sin(t) \overset{\mathcal{F}}{\rightleftharpoons} \frac{-j}{2}\left[\delta\left(f - \frac{1}{2\pi}\right) - \delta\left(f + \frac{1}{2\pi}\right)\right] \tag{4.63}$$

The attenuation happens for the same values of f, as depicted in the graphs of the Fourier Transforms given below.

□

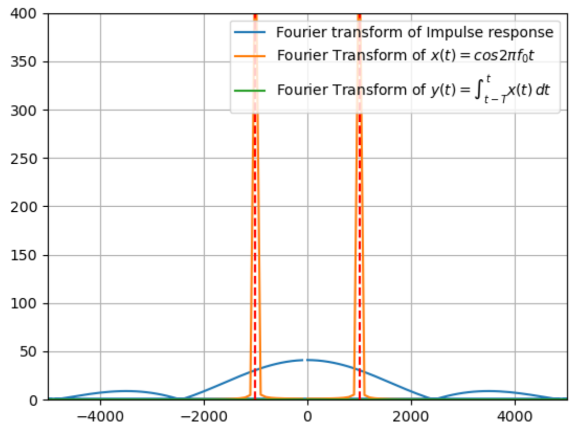

Fig. 4.6: Fourier Transform of Impulse response $h(t)$

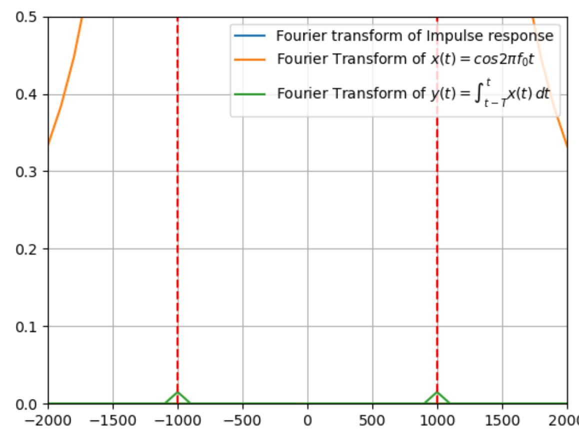

Fig. 4.7: Fourier Transform of Impulse response $h(t)$ zoomed in

5. Let the state-space representation on an LTI system be $\dot{x}(t) = Ax(t) + Bu(t)$, $y(t) = Cx(t) + du(t)$ where A,B,C are matrices, d is a scalar, u(t) is the input to the system, and y(t) is its output. Let $B = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^\top$ and $d = 0$. Which one of the following options for A and C will ensure that the transfer function of this LTI system is

$$H(s) = \frac{1}{s^3 + 3s^2 + 2s + 1} \tag{5.1}$$

(A) $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$

(B) $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -2 & -1 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$

(C) $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$

(D) $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -2 & -1 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$

Solution: From the given information,

$$\begin{pmatrix} \dot{x}(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} A & B \\ C & d \end{pmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \tag{5.2}$$

Taking Laplace transform on both sides,

$$\begin{pmatrix} sX(s) \\ Y(s) \end{pmatrix} = \begin{pmatrix} A & B \\ C & d \end{pmatrix} \begin{pmatrix} X(s) \\ U(s) \end{pmatrix} \tag{5.3}$$

$$\implies sX(s) = AX(s) + BU(s) \tag{5.4}$$

$$\implies X(s) = (sI - A)^{-1} BU(s) \tag{5.5}$$

$$\implies Y(s) = CX(s) + dU(s) \tag{5.6}$$

$$= C(sI - A)^{-1} BU(s) + dU(s) \tag{5.7}$$

By definition,

$$Y(s) = H(s)U(s) \tag{5.8}$$

$$\implies H(s) = C(sI - A)^{-1} B + d \tag{5.9}$$

$$= \frac{1}{s^3 + 3s^2 + 2s + 1} \tag{5.10}$$

$$\implies C(sI - A)^{-1} B + d = \frac{1}{s^3 + 3s^2 + 2s + 1} \tag{5.11}$$

Now we cross verify the options with eq 5.11. By using a python script,

(A)

$$C(sI-A)^{-1}B+d = \frac{1}{s^3 + 3s^2 + 2s + 1} \tag{5.12}$$

(B)

$$C(sI-A)^{-1}B+d = \frac{1}{s^3 + 1s^2 + 2s + 3} \tag{5.13}$$

(C)

$$C(sI-A)^{-1}B+d = \frac{s^2}{s^3 + 3s^2 + 2s + 1} \tag{5.14}$$

(D)

$$C(sI-A)^{-1}B+d = \frac{s^2}{s^3 + 1s^2 + 2s + 3} \tag{5.15}$$

Hence A is the correct option.

6. Consider a real-valued base-band signal $x(t)$, band limited to 10 kHz. The Nyquist rate for the signal $y(t) = x(t)\, x\left(1 + \frac{t}{2}\right)$ is

a) 15 kHz
b) 30 kHz
c) 60 kHz
d) 20 kHz

Solution:

Definition 5 (Dirac-delta impulse)**.**

$$\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & otherwise \end{cases} \tag{6.1}$$

Lemma 6.1 (Shifting property of $\delta(t)$)**.** *If $g(t)$ is a continuous and finite function at $t = a$ then*

$$\int_{-\infty}^{\infty} \delta(t-a)\, g(t)\, dt = g(a) \tag{6.2}$$

We also have

$$\int_{-\infty}^{\infty} \delta(t-a)\, \delta(t-b)\, dt = \delta(a-b) \tag{6.3}$$

Theorem 0.3. *Fourier transform of shifted impulse is the complex exponential.*

$$G(f) = \mathcal{F}\{\delta(t-a)\} = e^{-i2\pi fa} \tag{6.4}$$

Proof.

$$G(f) = \int_{-\infty}^{\infty} \delta(t-a)\, e^{-i2\pi ft} dt \tag{6.5}$$

$$= e^{-i2\pi fa} \tag{6.6}$$

□

Corollary 0.4. *Inverse Fourier Transform of the complex exponential must be the shifted impulse. So*

$$\mathcal{F}^{-1}\left\{e^{-2\pi fa}\right\} = \int_{-\infty}^{\infty} e^{-2\pi fa} e^{i2\pi ft} df \tag{6.7}$$

$$= \int_{-\infty}^{\infty} e^{i2\pi f(t-a)} df \tag{6.8}$$

$$= \int_{-\infty}^{\infty} e^{-i2\pi f(t-a)} df \tag{6.9}$$

$$= \delta(t-a) \tag{6.10}$$

Theorem 0.5. *The Fourier transform of $g(t) = e^{i2\pi at}$ is given by*

$$G(f) = \mathcal{F}\left\{e^{i2\pi at}\right\} = \delta(f-a) \tag{6.11}$$

Proof.

$$G(f) = \int_{-\infty}^{\infty} e^{i2\pi at} e^{-i2\pi ft} dt \tag{6.12}$$
$$= \int_{-\infty}^{\infty} e^{i2\pi t(a-f)} dt \tag{6.13}$$
$$= \delta(f-a) \tag{6.14}$$

□

Lemma 6.2 (Linearity of Fourier Transform)**.**

$$\mathcal{F}\{c_1 g(t) + c_2 h(t)\} = c_1 \mathcal{F}\{g(t)\} + c_2 \mathcal{F}\{h(t)\} \tag{6.15}$$

Lemma 6.3. *Let $x(t)$ be a signal, its Fourier Transform be of the form*

$$G_x(f) = c_1 \delta(f - a_1 A) + c_2 \delta(f - a_2 A) + ... \tag{6.16}$$

where $c_i \in \mathbb{C}$ and $a_i \in \mathbb{R}$. Then the frequencies present in the signal are $a_j A$ where $a_j \in \mathbb{R}^+$

Let $x(t) = \cos(2\pi At)$, where $A = 10kHz$.

$$\cos(2\pi At) = \frac{e^{i2\pi At} + e^{-i2\pi At}}{2} \tag{6.17}$$

The Fourier transform of $x(t)$

$$G_x(f) = \int_{-\infty}^{\infty} \frac{e^{i2\pi At} + e^{-i2\pi At}}{2} e^{-i2\pi ft} dt \tag{6.18}$$
$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} e^{-i2\pi t(f-A)} dt + \int_{-\infty}^{\infty} e^{-i2\pi t(A+F)}\right] \tag{6.19}$$
$$= \frac{1}{2}\left[\delta(f-A) + \delta(f+A)\right] \tag{6.20}$$

∴ All the energy of the sinusoidal wave is entirely localized at the frequencies given by $|f| = A$.

$$y(t) = \cos(2\pi At)\cos\left(2\pi A\left(1 + \frac{t}{2}\right)\right) \tag{6.21}$$
$$= \frac{1}{2}\left(\cos(2\pi A + 3\pi At) + \cos(2\pi A - \pi At)\right) \tag{6.22}$$
$$= \frac{1}{2}\left(\cos(3\pi At) + \cos(\pi At)\right) \tag{6.23}$$

Fourier Transform of $y(t)$ is given by

$$G_y(f) = \frac{1}{4}\left[\delta\left(f - \frac{3A}{2}\right) + \delta\left(f + \frac{3A}{2}\right)\right] + \frac{1}{4}\left[\delta\left(f - \frac{A}{2}\right) + \delta\left(f + \frac{A}{2}\right)\right] \tag{6.24}$$

From lemma 6.3 we can conclude that the frequencies present in signal $y(t)$ are $\frac{A}{2}$, $\frac{3A}{2}$

Lemma 6.4. *Multiplication property of Fourier Transform*

$$\text{If } x(t) \overset{\mathcal{F}}{\rightleftharpoons} X(f) \tag{6.25}$$
$$y(t) \overset{\mathcal{F}}{\rightleftharpoons} Y(f) \tag{6.26}$$

Then

$$x(t)\,y(t) \overset{\mathcal{F}}{\rightleftharpoons} X(f) * Y(f) \tag{6.27}$$

where $*$ represents convolution

Lemma 6.5.

$$\delta(t - t_0) * g(t) = g(t - t_0) \tag{6.28}$$

Lemma 6.6. *Computing $G_y(f)$ using convolution*

$$x(t) = \cos(2\pi At) \tag{6.29}$$
$$x(t) \overset{\mathcal{F}}{\rightleftharpoons} X_1(f) = \frac{1}{2}\left[\delta(f-A) + \delta(f+A)\right] \tag{6.30}$$
$$x\left(1 + \frac{t}{2}\right) = \cos(\pi At) \tag{6.31}$$
$$x\left(1 + \frac{t}{2}\right) \overset{\mathcal{F}}{\rightleftharpoons} X_2(f) = \frac{1}{2}\left[\delta\left(f - \frac{A}{2}\right) + \delta\left(f + \frac{A}{2}\right)\right] \tag{6.32}$$

Using lemma 6.4

$$G_y(f) = X_1(f) * X_2(f) \tag{6.33}$$
$$= \left(\frac{1}{2}\left[\delta(f-A) + \delta(f+A)\right]\right) * X_2(f) \tag{6.34}$$
$$= \frac{1}{2}\left\{\delta(f-A) * X_2(f) + \delta(f+A) * X_2(f)\right\} \tag{6.35}$$

Using (6.28)

$$G_y(f) = \frac{1}{2}\left(X_2(f-A) + X_2(f+A)\right) \tag{6.36}$$

$$G_y(f) = \frac{1}{4}\left[\delta\left(f - \frac{3A}{2}\right) + \delta\left(f + \frac{3A}{2}\right)\right] + \frac{1}{4}\left[\delta\left(f - \frac{A}{2}\right) + \delta\left(f + \frac{A}{2}\right)\right] \tag{6.37}$$

$$x(t) = \cos(20k\pi t) \tag{6.38}$$

$$\text{bandwidth of } x(t) = 10kHz \tag{6.39}$$

$$x\left(1 + \frac{t}{2}\right) = \cos(20k\pi + 10k\pi t) \tag{6.40}$$

$$\text{bandwidth of } x\left(1 + \frac{t}{2}\right) = 5kHz \tag{6.41}$$

$$\text{from (6.23) } y(t) = \cos(30k\pi t) + \cos(10k\pi t) \tag{6.42}$$

$$\text{bandwidth of } y(t) = \frac{30}{2}kHz \tag{6.43}$$

$$= 15kHz \tag{6.44}$$

$$\text{Nyquist rate} = 2 \times \text{maximum frequency} \tag{6.45}$$

$$= 30kHz \tag{6.46}$$

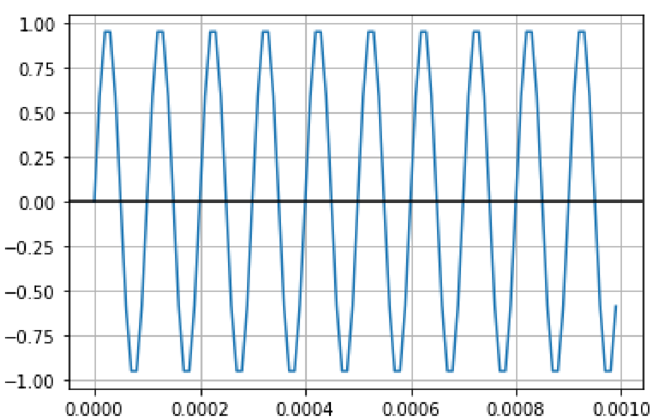

Fig. 6.1: $x(t)$:Sinusoidal signal with freq=10kHz

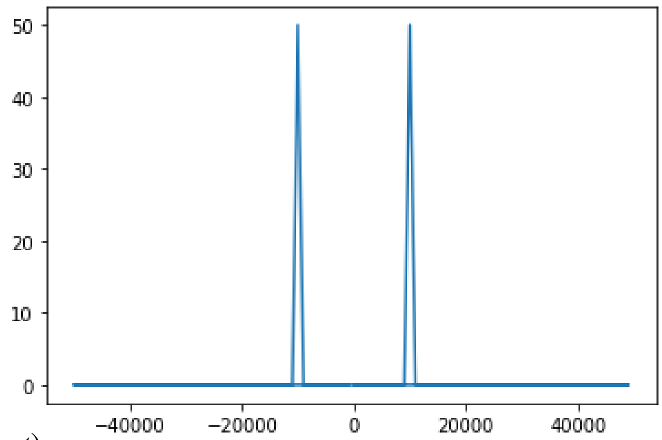

Fig. 6.2: DFT of $x(t)$. $Bandwidth = 10000$

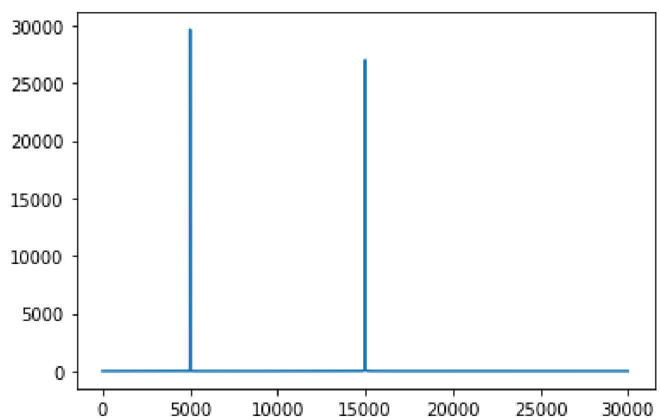

Fig. 6.3: DFT of $y(t)$.$Bandwidth = 15000$

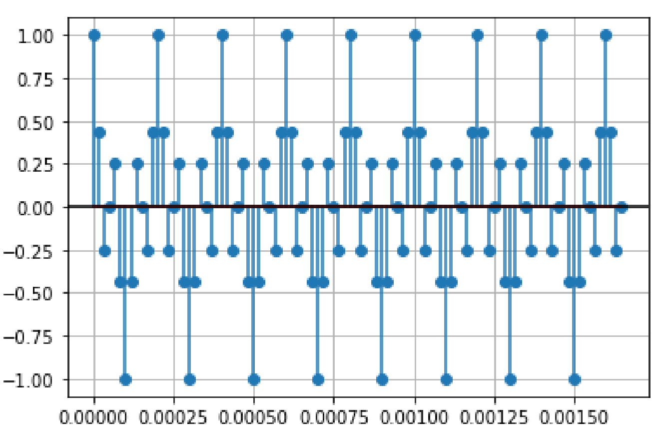

Fig. 6.4: stem plot of $y(t)$ sampled at 60kHz

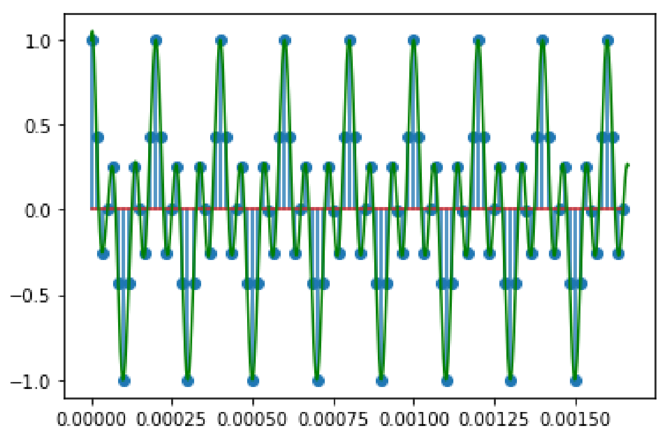

Fig. 6.5: Shannon interpolation of $y(t)$