Signal Processing

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Contents

***Abstract*—This manual provides solved problems in signal processing from GATE exam papers.**

1. A continuous time LTI system is described by

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = 2\frac{dx(t)}{dt} + 4x(t) \tag{1.1}$$

Assuming zero initial conditions, the response $y(t)$ of the above system for the input $x(t) = e^{-2t}u(t)$ is given by

a) $\left(e^{t} - e^{3t}\right)u(t)$
b) $\left(e^{-t} - e^{-3t}\right)u(t)$
c) $\left(e^{-t} + e^{-3t}\right)u(t)$
d) $\left(e^{t} + e^{3t}\right)u(t)$

Solution:

Lemma 1.1 (Table of Laplace Transforms)**.**

Time Function $f(t) = \mathcal{L}^{-1}\{F(s)\}$	***Laplace transform*** *of f(t)* $F(s) = \mathcal{L}\{f(t)\}$
$u(t)$	$\frac{1}{s},\ s > 0$
$g'(t)$	$sG(s) - g(0)$
$g''(t)$	$s^2G(s) - sg(0) - g'(0)$
$e^{-at}u(t)$	$\frac{1}{s+a},\ s + a > 0$

Lemma 1.2. *Linearity of Laplace Transform*

$$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\} \tag{1.2}$$

From Lemma-1.1 Laplace transform of $x(t) = e^{-2t}u(t)$ is given by

$$X(s) = \frac{1}{s+2} \tag{1.3}$$

Since initial conditions are zero. Laplace Transform of (1.1) gives

$$s^2Y(s) + 4sY(s) + 3Y(s) = 2sX(s) + 4X(s) \tag{1.4}$$

$$Y(s) = \frac{2(s+2)}{s^2+4s+3}X(s) \tag{1.5}$$

$$= \frac{1}{s+1} - \frac{1}{s+3} \tag{1.6}$$

From Lemma-1.1. Inverse Laplace transform of $Y(s)$ is given by

$$y(t) = e^{-t}u(t) - e^{-3t}u(t) \tag{1.7}$$

$$= \left(e^{-t} - e^{-3t}\right)u(t) \tag{1.8}$$

$\therefore$ The required option is B. See Fig. 1.1.

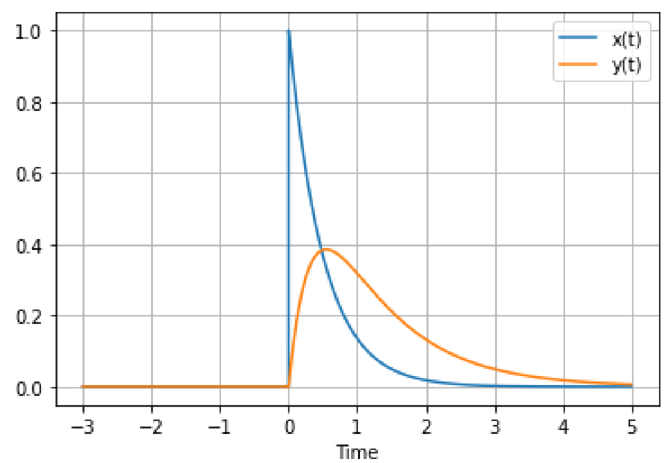

Fig. 1.1: Plot of input and output responses in time domain.

2. The transfer function for a discrete time LTI system is given by:

$$H(z) = \frac{2 - \frac{3}{4}z^{-1}}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}} \tag{2.1}$$

Consider the following statements:
S1: The system is stable and causal for ROC: $|z| > \frac{1}{2}$

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S2: The system is stable but not causal for ROC: $|z| < \frac{1}{4}$
S3: The system is neither stable nor causal for ROC: $\frac{1}{4} < |z| < \frac{1}{2}$
Which one of the following statement are valid?

a) Both S1 and S2 are true
b) Both S2 and S3 are true
c) Both S1 and S3 are true
d) S1, S2 and S3 are all true

Solution: The given transfer function can be expressed as

$$H(z) = \frac{16 - 6z^{-1}}{8 - 6z^{-1} + z^{-2}} \tag{2.2}$$
$$= \frac{16 - 6z^{-1}}{(4 - z^{-1})(2 - z^{-1})} \tag{2.3}$$
$$= \frac{4}{4 - z^{-1}} + \frac{2}{2 - z^{-1}} \tag{2.4}$$

with poles at

$$z = \frac{1}{2}, z = \frac{1}{4} \tag{2.5}$$

a) Since the ROC includes the unit circle, the system is stable. Also, the ROC extends outwards to infinity, so the system is causal as well. Hence S_1 is true.
b) When ROC $= \frac{1}{4} < |z| < \frac{1}{2}$, the unit circle is not included in the ROC. Hence, the system cannot be stable. Also, the ROC is an annulus, so the system is non-causal. So S_3 is true.

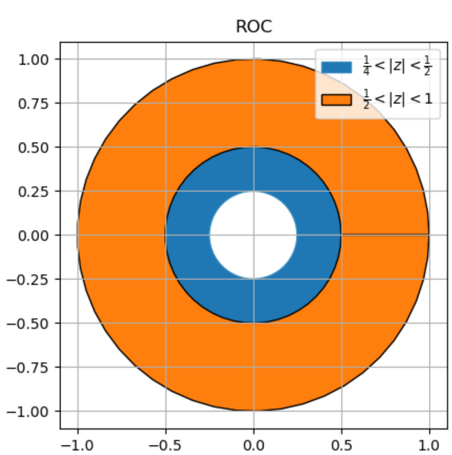

Fig. 2.1: ROC

3. The impulse response of a system is $h(t) = tu(t)$. For an input $u(t-1)$, the output is
   a) $\frac{t^2}{2}u(t)$
   b) $\frac{t(t-1)}{2}u(t-1)$
   c) $\frac{(t-1)^2}{2}u(t-1)$
   d) $\frac{t^2-1}{2}u(t-1)$

   Solution:

Definition 1 (Laplace Transform)**.** *It is an integral transform that converts a function of a real variable t to a function of a complex variable s. The Laplace transform of $f(t)$ is denoted by $\mathcal{L}\{f(t)\}$ or $F(s)$.*

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt \tag{3.1}$$

Remark. Laplace transform of $f(t) = t^n, n \geq 1$ is

$$F(s) = \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, s > 0 \tag{3.2}$$

Proof. Basis Step: $n = 1$

$$\mathcal{L}\{t\} = \int_0^\infty e^{-st} t dt \tag{3.3}$$
$$= \left[\frac{te^{-st}}{-s}\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st} dt \tag{3.4}$$
$$= 0 + \left[\frac{-1}{s^2}e^{-st}\right]_0^\infty, s > 0 \tag{3.5}$$
$$= \frac{1}{s^2}, s > 0 \tag{3.6}$$

Inductive Step:

$$\mathcal{L}\{t^n\} = \int_0^\infty e^{-st} t^n dt \tag{3.7}$$
$$= \left[\frac{t^n e^{-st}}{-s}\right]_0^\infty + \frac{n}{s}\int_0^\infty t^{n-1} e^{-st} dt \tag{3.8}$$
$$= 0 + \frac{n}{s}\mathcal{L}\left\{t^{n-1}\right\}, s > 0 \tag{3.9}$$
$$= \frac{n}{s}\mathcal{L}\left\{t^{n-1}\right\}, s > 0 \tag{3.10}$$

To prove that if eqrefec/2003/8eq:t holds for $n = k$, it holds for $n = k + 1$. From eqrefec/2003/8eq:e

$$\mathcal{L}\left\{t^{k+1}\right\} = \frac{k+1}{s}\mathcal{L}\left\{t^{k}\right\} \tag{3.11}$$

$$= \frac{(k+1)k!}{s(s^{k+1})} = \frac{(k+1)!}{s^{k+2}}, s > 0 \tag{3.12}$$

By mathematical induction,
eqrefec/2003/8eq:t is true $\forall n \geq 1$ □

Lemma 3.1. *For any real number c,*

$$\mathcal{L}\left\{u(t-c)\right\} = \frac{e^{-cs}}{s}, s > 0 \tag{3.13}$$

Proof.

$$\mathcal{L}\left\{u(t-c)\right\} = \int_0^\infty e^{-st}u(t-c)dt = \int_c^\infty e^{-st}dt \tag{3.14}$$

$$= \left[-\frac{e^{-st}}{s}\right]_c^\infty = \frac{e^{-cs}}{s}, s > 0 \tag{3.15}$$

□

Definition 2 (Inverse Laplace Transform)**.** *It is the transformation of a Laplace transform into a function of time. If $F(s) = \mathcal{L}\{f(t)\}$, then the Inverse laplace transform of $F(s)$ is $\mathcal{L}^{-1}\{F(s)\} = f(t)$.*

Lemma 3.2 (t-shift rule)**.** *For any real number c,*

$$\mathcal{L}\left\{u(t-c)f(t-c)\right\} = e^{-cs}F(s) \tag{3.16}$$

Proof.

$$\mathcal{L}\left\{u(t-c)f(t-c)\right\} = \int_0^\infty e^{-st}u(t-c)f(t-c)dt \tag{3.17}$$

$$= \int_c^\infty e^{-st}f(t-c)dt \tag{3.18}$$

$$= \int_0^\infty e^{-s(\tau+c)}f(\tau)d\tau \; (t = \tau + c) \tag{3.19}$$

$$= e^{-cs}\int_0^\infty e^{-s\tau}f(\tau)d\tau \tag{3.20}$$

$$= e^{-cs}F(s) \tag{3.21}$$

□

Corollary 0.1.

$$\mathcal{L}^{-1}\left\{e^{-cs}F(s)\right\} = u(t-c)f(t-c) \tag{3.22}$$

Theorem 0.2 (Convolution theorem)**.** *Suppose $F(s) = \mathcal{L}\{f(t)\}, G(s) = \mathcal{L}\{g(t)\}$ exist, then,*

$$\mathcal{L}^{-1}\left\{F(s)G(s)\right\} = f(t) * g(t) \tag{3.23}$$

Given,

$$h(t) = tu(t) \tag{3.24}$$

$$x(t) = u(t-1) \tag{3.25}$$

To find: $y(t)$. We know,

$$y(t) = h(t) * x(t) \tag{3.26}$$

$$= \mathcal{L}^{-1}\left\{H(s)X(s)\right\} \tag{3.27}$$

From
eqrefec/2003/8eq:uf and
eqrefec/2003/8eq:t,

$$H(s) = e^0\mathcal{L}\left\{t\right\} = \frac{1}{s^2} \tag{3.28}$$

From
eqrefec/2003/8eq:u,

$$X(s) = \frac{e^{-s}}{s} \tag{3.29}$$

Substituting in
eqrefec/2003/8eq:def,

$$y(t) = \mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^3}\right\} \tag{3.30}$$

Consider

$$p(t) = \frac{t^2}{2} \tag{3.31}$$

From
eqrefec/2003/8eq:t

$$P(s) = \frac{2!}{2s^3} = \frac{1}{s^3} \tag{3.32}$$

Further, from
eqrefec/2003/8eq:cuf, for $c = 1$

$$\mathcal{L}^{-1}\left\{e^{-s}P(s)\right\} = u(t-1)p(t-1) \tag{3.33}$$

$$= u(t-1)\frac{(t-1)^2}{2} \tag{3.34}$$

$$\therefore y(t) = \frac{(t-1)^2}{2}u(t-1) \tag{3.35}$$

Option 3 is the correct answer.

$$h(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases} \tag{3.36}$$

$$x(t) = \begin{cases} 1, & t \geq 1 \\ 0, & t < 1 \end{cases} \tag{3.37}$$

$$y(t) = \begin{cases} \frac{(t-1)^2}{2}, & t \geq 1 \\ 0, & t < 1 \end{cases} \tag{3.38}$$

See Figs. 3.1, 3.2 and 3.3.

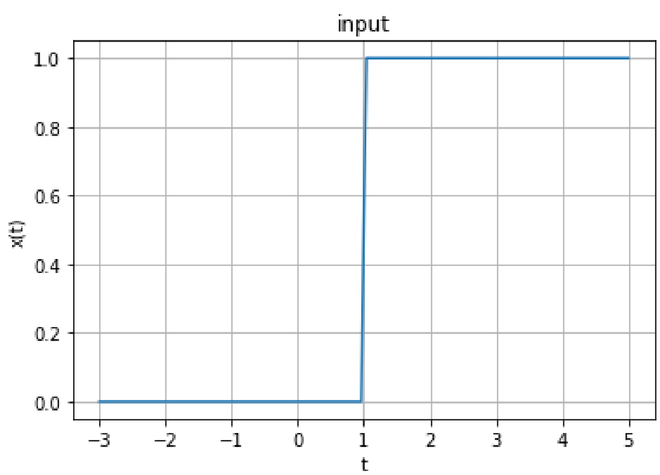

Fig. 3.1: Plot of $x(t)$

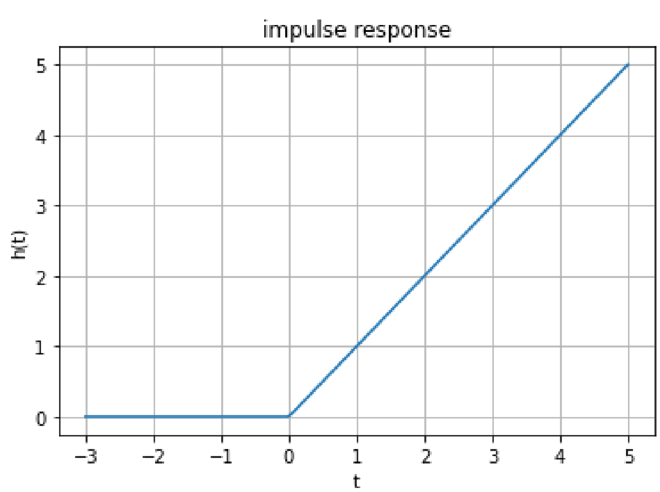

Fig. 3.2: Plot of $h(t)$

4. The DFT of a vector $\begin{pmatrix} a & b & c & d \end{pmatrix}$ is the vector

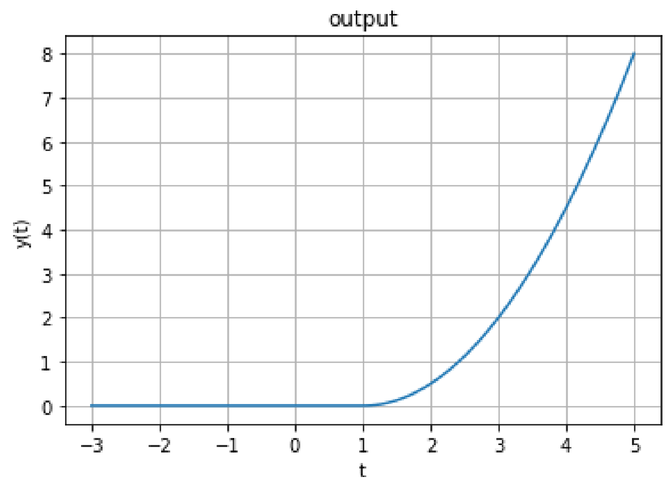

Fig. 3.3: Plot of $y(t)$

$\begin{pmatrix} \alpha & \beta & \gamma & \delta \end{pmatrix}$. Consider the product

$$\begin{pmatrix} p & q & r & s \end{pmatrix} = \begin{pmatrix} a & b & c & d \end{pmatrix} \begin{pmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{pmatrix} \tag{4.1}$$

The DFT of the vector $\begin{pmatrix} p & q & r & s \end{pmatrix}$ is a scaled version of

(A) $\begin{pmatrix} \alpha^2 & \beta^2 & \gamma^2 & \delta^2 \end{pmatrix}$

(B) $\begin{pmatrix} \sqrt{\alpha} & \sqrt{\beta} & \sqrt{\gamma} & \sqrt{\delta} \end{pmatrix}$

(C) $\begin{pmatrix} \alpha+\beta & \beta+\delta & \delta+\gamma & \gamma+\alpha \end{pmatrix}$

(D) $\begin{pmatrix} \alpha & \beta & \gamma & \delta \end{pmatrix}$

Solution:

Lemma 4.1. *Let*

$$\mathbf{T} = \begin{pmatrix} a & d & c & b \\ b & a & d & c \\ c & b & a & d \\ d & c & b & a \end{pmatrix} \tag{4.2}$$

Then, for

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = \mathbf{W} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}, \tag{4.3}$$

where $\mathbf{W}$ *is the DFT matrix,*

$$\mathbf{T} = \mathbf{W} \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & \delta \end{pmatrix} \mathbf{W}^{-1} \tag{4.4}$$

Let

$$\mathbf{x} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix};\ \mathbf{X} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = \mathbf{W}\mathbf{x};\ \mathbf{y} = \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} \tag{4.5}$$

Then

$$\mathbf{Y} = \mathbf{W}\mathbf{y} = \mathbf{W}\mathbf{T}\mathbf{x} \tag{4.6}$$
$$= \mathbf{W}\mathbf{T}\mathbf{W}^{-1}\mathbf{X} \tag{4.7}$$
$$= \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & \delta \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} \tag{4.8}$$
$$= \begin{pmatrix} \alpha^2 \\ \beta^2 \\ \gamma^2 \\ \delta^2 \end{pmatrix} \tag{4.9}$$

upon substituting from (4.4) and (4.5). Therefore option (A) is the correct option.

5. The input $x(t)$ and output $y(t)$ of a continous time signal are related as

$$y(t) = \int_{t-T}^{t} x(u)\,du \tag{5.1}$$

The system is:

a) Linear and Time-variant
b) Linear and Time-invariant
c) Non-Linear and Time-variant
d) Non-Linear and Time-invariant

Solution:

Definition 3. *We say that a system is* ***linear*** *if and only if it follows the Principle of Superposition, i.e Law of Additivity and Law of Homogeneity.*

Definition 4. *A system is said to be* ***time invariant*** *if the output signal does not depend on the absolute time, i.e a time delay on the input signal directly equates to the delay in the output signal.*

Lemma 5.1. *The system relating the input signal $x(t)$ and output signal $y(t)$, given by*

$$y(t) = \int_{t-T}^{t} x(u)\,du \tag{5.2}$$

is linear and time invariant in nature.

Proof. a) **Linearity and Time invariance**

From (3), we can say the system is linear if it follows both the laws of Additivity and Homogeneity.

Law of Additivity:

Let the two input signals be $x_1(t)$ and $x_2(t)$, and their corresponding output signals be $y_1(t)$ and $y_2(t)$, then:

$$y_1(t) = \int_{t-T}^{t} x_1(u)\,du \tag{5.3}$$
$$y_2(t) = \int_{t-T}^{t} x_2(u)\,du \tag{5.4}$$
$$y_1(t) + y_2(t) = \int_{t-T}^{t} [x_1(u) + x_2(u)]\,du \tag{5.5}$$

Now, consider the input signal of $x_1(t)+x_2(t)$, then the corresponding output signal is given by $y'(t)$:

$$y'(t) = \int_{t-T}^{t} [x_1(u) + x_2(u)]\,du \tag{5.6}$$

Clearly, from (5.5) and (5.6):

$$y'(t) = y_1(t) + y_2(t) \tag{5.7}$$

Thus, the Law of Additivity holds.

Law of Homogeneity:

Consider an input signal $kx(t)$, where k is any constant. Let the corresponding output be given by $y'(t)$, then:

$$y'(t) = \int_{t-T}^{t} kx(u)\,du \tag{5.8}$$
$$= k\int_{t-T}^{t} x(u)\,du \tag{5.9}$$
$$= ky(t) \tag{5.10}$$

Clearly, from (5.10),

$$y'(t) = ky(t) \tag{5.11}$$

Thus, the Law of Homogeneity holds.

Since both the Laws hold, the system satisfies the Principle of Superposition, and is thus, a **linear system**.

From (4) , to check for time-invariance, we would introduce a delay of t_0 in the output and input signals.

Delay in output signal:

$$y(t-t_0) = \int_{t-t_0-T}^{t-t_0} x(u)\, du \tag{5.12}$$

Now, we consider an input signal with a delay of t_0, given by $x(t-t_0)$, and let the corresponding output signal be given by $y'(t)$, then:

$$y'(t) = \int_{t-T}^{t} x(u-t_0)\, du \tag{5.13}$$

Substituting $a = u - t_0$:

$$y'(t) = \int_{t-t_0-T}^{t-t_0} x(a)\, da \tag{5.14}$$

Clearly, from (5.12) and (5.14):

$$y'(t) = y(t-t_0) \tag{5.15}$$

Thus, the system is **time-invariant**.
The correct option is **2) Linear and Time-invariant**

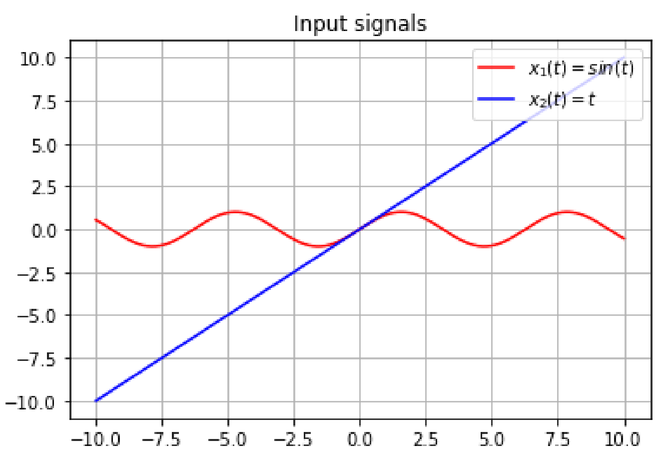

Fig. 5.1: $x_1(t) = \sin t$ and $x_2(t) = t$

b) **Calculating impulse response of LTI system**

Since the given system is an LTI system, it would possess an impulse response $h(t)$, which is the output of the system when the input signal is the Impulse function, given by $\delta(t)$. Thus,

$$h(t) = \int_{t-T}^{t} \delta(u) du \tag{5.16}$$

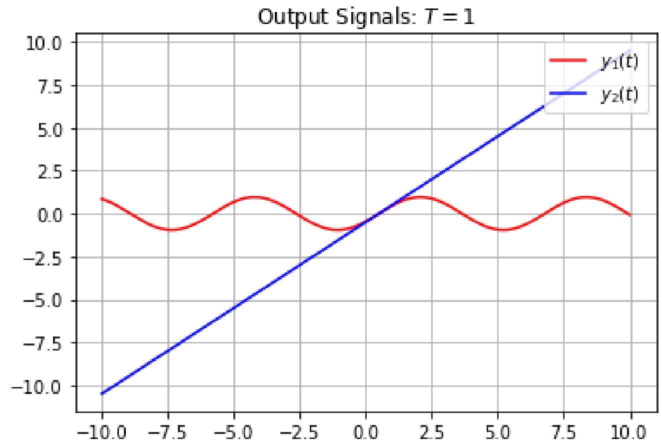

Fig. 5.2: $y_1(t)$ and $y_2(t)$

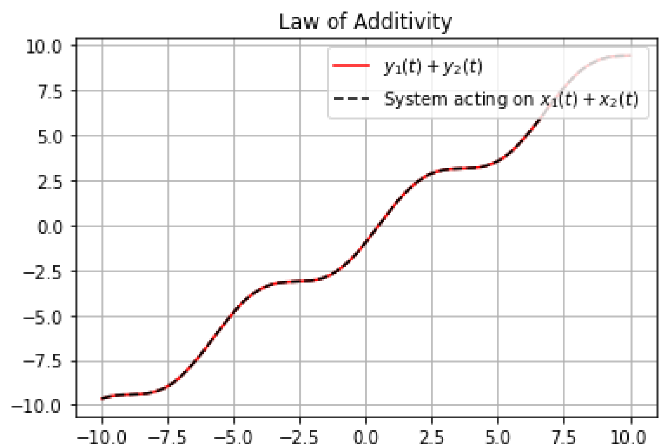

Fig. 5.3: Law of Additivity

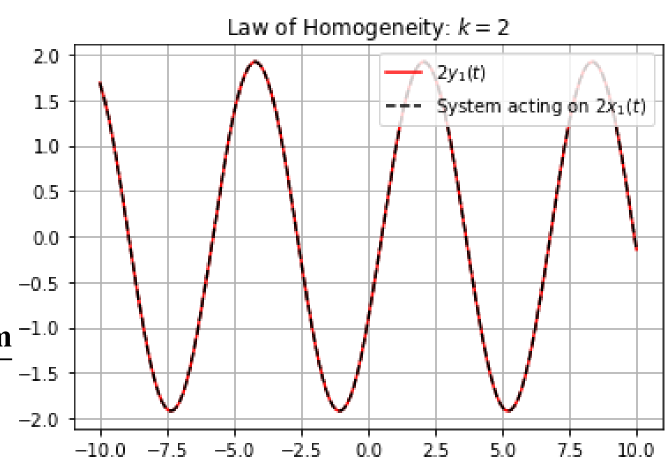

Fig. 5.4: Law of Homogeneity

The Impulse function can be loosely defined as:

$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & otherwise \end{cases} \quad and \int_{-\infty}^{\infty} \delta(t) dt = 1 \tag{5.17}$$

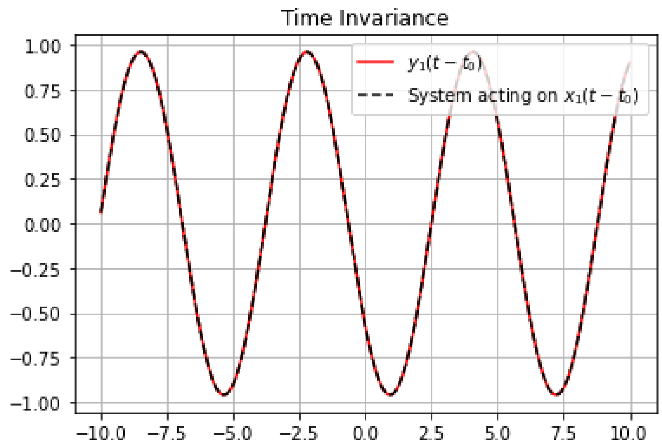

Fig. 5.5: Time invariance

Since the Impulse function is zero everywhere aside from $t = 0$, the non-zero value of integration is a result of $\delta(0)$. Thus, we can say $h(t)$ will be non-zero only if the limits of integration would include $t = 0$, i.e:

$$h(t) = \begin{cases} \int_{t-T}^{t} \delta(u)du & t-T < 0; t > 0 \\ 0 & otherwise \end{cases} \tag{5.18}$$

$$h(t) = \begin{cases} 1 & 0 < t < T \\ 0 & otherwise \end{cases} \tag{5.19}$$

c) **Expressing the impulse function in terms of** $u(t)$

The unit step signal, $u(t)$, is given by:

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & otherwise \end{cases} \tag{5.20}$$

On time-shifting $u(t)$ by T, we get:

$$u(t-T) = \begin{cases} 1 & t-T \geq 0 \\ 0 & otherwise \end{cases} = \begin{cases} 1 & t \geq T \\ 0 & otherwise \end{cases} \tag{5.21}$$

On subtracting (5.20) and (5.21), we get our impulse response $h(t)$ in terms of the unit step signal:

$$h(t) = u(t) - u(t-T) \tag{5.22}$$

d) **Expressing the impulse function in terms of** $rect(t)$

The unit rectangular signal, $rect(t)$ is given by:

$$rect(t) = \begin{cases} 1 & \frac{-1}{2} \leq t \leq \frac{1}{2} \\ 0 & otherwise \end{cases} \tag{5.23}$$

We can obtain the impulse response $h(t)$ in terms of $rect(t)$ using time scaling and shifting as follows:

$$rect\left(\frac{t}{\tau}\right) = \begin{cases} 1 & \frac{-1}{2} \leq \frac{t}{\tau} \leq \frac{1}{2} \\ 0 & otherwise \end{cases} = \begin{cases} 1 & \frac{-\tau}{2} \leq t \leq \frac{\tau}{2} \\ 0 & otherwise \end{cases} \tag{5.24}$$

Subsituting $\tau = T$:

$$rect\left(\frac{t}{T}\right) = \begin{cases} 1 & \frac{-T}{2} \leq t \leq \frac{T}{2} \\ 0 & otherwise \end{cases} \tag{5.25}$$

Now, we want to right-shift the signal by $\frac{T}{2}$:

$$rect\left(\frac{1}{T}\left(t - \frac{T}{2}\right)\right) = \begin{cases} 1 & 0 \leq t \leq T \\ 0 & otherwise \end{cases} = h(t) \tag{5.26}$$

Since the time shifting is to be performed on the variable t and not $\frac{t}{T}$

e) **Calculating the Fourier Transform of** $h(t)$

Let the Fourier Transform of $h(t)$ be given by $H(f)$ and of the rectangular signal, $rect(t)$ be given by $Y(f)$.

$$h(t) \overset{\mathcal{F}}{\rightleftharpoons} H(f) \tag{5.27}$$

$$rect(t) \overset{\mathcal{F}}{\rightleftharpoons} Y(f) \tag{5.28}$$

Then,

$$Y(f) = \int_{-\infty}^{\infty} rect(t) e^{-j2\pi ft}\, dt \tag{5.29}$$

From (5.23), we can write (5.29) as:

$$Y(f) = \int_{-\infty}^{\frac{-1}{2}} 0\, dt + \int_{\frac{-1}{2}}^{\frac{1}{2}} e^{-j2\pi ft}\, dt + \int_{\frac{1}{2}}^{\infty} 0\, dt \tag{5.30}$$

$$= \frac{e^{j\pi f} - e^{-j\pi f}}{j2\pi f} \tag{5.31}$$

$$= \frac{2j \sin \pi f}{j2\pi f} \tag{5.32}$$

$$= \frac{\sin(\pi f)}{\pi f} \tag{5.33}$$

$$= sinc(f) \tag{5.34}$$

where $sinc(t)$, the sampling function is defined as:

$$sinc(t) = \begin{cases} 1 & t = 0 \\ \frac{\sin(\pi t)}{\pi t} & otherwise \end{cases} \tag{5.35}$$

Let the Fourier Transform of a signal $x(t)$ be $X(f)$.

$$x(t) \overset{\mathcal{F}}{\rightleftharpoons} X(f) \tag{5.36}$$

When the signal $x(t)$ is time shifted by t_0, the resultant Fourier Transform is given by:

$$x(t \pm t_0) \overset{\mathcal{F}}{\rightleftharpoons} X(f)e^{\pm j2\pi f t_0} \tag{5.37}$$

And when the signal $x(t)$ is time scaled by α, the resulting Fourier Transform is given by:

$$x(\alpha t) \overset{\mathcal{F}}{\rightleftharpoons} \frac{1}{|\alpha|} X\left(\frac{f}{\alpha}\right) \tag{5.38}$$

Since we have already derived the Fourier Transform of $rect(t)$, we would use the properties mentioned above to find the Fourier Transform of $h(t)$:

$$rect(t) \overset{\mathcal{F}}{\rightleftharpoons} sinc(f) \tag{5.39}$$

Using (5.37):

$$rect\left(t - \frac{T}{2}\right) \overset{\mathcal{F}}{\rightleftharpoons} sinc(f)e^{-j(2\pi f)\frac{T}{2}} \tag{5.40}$$

$$rect\left(t - \frac{T}{2}\right) \overset{\mathcal{F}}{\rightleftharpoons} sinc(f)e^{-j\pi f T} \tag{5.41}$$

Using (5.38),

$$rect\left(\frac{1}{T}\left(t - \frac{T}{2}\right)\right) \overset{\mathcal{F}}{\rightleftharpoons} \frac{1}{\frac{1}{|T|}} sinc\left(\frac{f}{T}\right) e^{\frac{-j\pi f T}{T}} \tag{5.42}$$

$$h(t) \overset{\mathcal{F}}{\rightleftharpoons} T\,sinc\left(\frac{f}{T}\right) e^{-j\pi f} \tag{5.43}$$

$$\therefore H(f) = T\,sinc\left(\frac{f}{T}\right) e^{-j\pi f} \tag{5.44}$$

f) **An example**

Consider an input signal of $x(t) = \cos 2\pi f_0 t$. The Fourier Transform of $x(t)$ is given by:

$$x(t) = \cos 2\pi f_0 t \overset{\mathcal{F}}{\rightleftharpoons} \frac{1}{2}\left[\delta(f - f_0) + \delta(f + f_0)\right] \tag{5.45}$$

using the fact that

$$\cos 2\pi f_0 t = \frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2} \tag{5.46}$$

and the Fourier Transform of $e^{\pm j2\pi f_0 t}$ is given by:

$$e^{\pm j2\pi f_0 t} \overset{\mathcal{F}}{\rightleftharpoons} \delta(f \mp f_0) \tag{5.47}$$

The output signal will be given by:

$$y(t) = \int_{t-T}^{t} \cos 2\pi f_0 u \, du \tag{5.48}$$

$$= \frac{1}{2\pi f_0}\left[\sin 2\pi f_0 t - \sin 2\pi f_0 (t - T)\right] \tag{5.49}$$

$$= \frac{\sin \pi f_0 T}{\pi f_0}\left[\cos 2\pi f_0 \left(t - \frac{T}{2}\right)\right] \tag{5.50}$$

$$= T\,sinc(f_0 T)\cos 2\pi f_0 \left(t - \frac{T}{2}\right) \tag{5.51}$$

The Fourier transform of $\cos 2\pi f_0 \left(t - \frac{T}{2}\right)$ can be obtained using (5.38) and (5.37) as follows:

$$\cos t = \frac{1}{2}\left[e^{jt} + e^{-jt}\right] \tag{5.52}$$

$$\cos t \overset{\mathcal{F}}{\rightleftharpoons} \frac{1}{2}\left[\delta\left(f - \frac{1}{2\pi}\right) + \delta\left(f + \frac{1}{2\pi}\right)\right] \tag{5.53}$$

$$\cos\left(t - \frac{T}{2}\right) \overset{\mathcal{F}}{\rightleftharpoons} \frac{e^{j\pi f T}}{2}\left[\delta\left(f - \frac{1}{2\pi}\right) + \delta\left(f + \frac{1}{2\pi}\right)\right] \tag{5.54}$$

$$\cos 2\pi f_0 \left(t - \frac{T}{2}\right) \overset{\mathcal{F}}{\rightleftharpoons} \frac{e^{j\pi \frac{f}{2\pi f_0} T}}{2\pi f_0} \frac{\delta(\frac{f}{2\pi f_0} - \frac{1}{2\pi}) + \delta(\frac{f}{2\pi f_0} + \frac{1}{2\pi})}{2} \tag{5.55}$$

$$\cos 2\pi f_0 \left(t - \frac{T}{2}\right) \overset{\mathcal{F}}{\rightleftharpoons} \frac{e^{j\pi \frac{f}{2f_0} T}}{4\pi f_0}\left(\delta\left(\frac{f - f_0}{2\pi f_0}\right) + \delta\left(\frac{f + f_0}{2\pi f_0}\right)\right) \tag{5.56}$$

Therefore, the Fourier Transform of the output signal $y(t)$ from (5.51) is given by:

$$y(t) \overset{\mathcal{F}}{\rightleftharpoons} \frac{T\,sinc(f_0 T)}{4\pi f_0} e^{j\pi \frac{f}{2f_0} T}\left(\delta\left(\frac{f - f_0}{2\pi f_0}\right) + \delta\left(\frac{f + f_0}{2\pi f_0}\right)\right) \tag{5.57}$$

$$y(t) \overset{\mathcal{F}}{\rightleftharpoons} k e^{j\pi \frac{f}{2f_0} T}\left(\delta\left(\frac{f - f_0}{2\pi f_0}\right) + \delta\left(\frac{f + f_0}{2\pi f_0}\right)\right) \tag{5.58}$$

where $k = \frac{T\,sinc(f_0 T)}{4\pi f_0}$. Substituting $2\pi f_0 = 1$ and $T = 1$:

$$y(t) \overset{\mathcal{F}}{\rightleftharpoons} k e^{j\pi^2 f}\left(\delta\left(f - \frac{1}{2\pi}\right) + \delta\left(f + \frac{1}{2\pi}\right)\right) \tag{5.59}$$

$$y(t) \overset{\mathcal{F}}{\rightleftharpoons} k e^{j\frac{\pi}{2}}\delta\left(f - \frac{1}{2\pi}\right) + k e^{j\frac{-\pi}{2}}\delta\left(f + \frac{1}{2\pi}\right) \tag{5.60}$$

using the multiplication property of the Delta function:

$$x(t)\delta(t - t_1) = x(t_1)\delta(t - t_1) \tag{5.61}$$

Since , $e^{j\frac{\pi}{2}} = j$ and $e^{-j\frac{\pi}{2}} = -j$, we finally get:

$$y(t) \overset{\mathcal{F}}{\rightleftharpoons} kj\left[\delta\left(f - \frac{1}{2\pi}\right) - \delta\left(f + \frac{1}{2\pi}\right)\right] \tag{5.62}$$

Clearly, the Fourier transform of $y(t)$ can be manipulated to represent a sinusoidal wave, which is given by:

$$sin(t) \overset{\mathcal{F}}{\rightleftharpoons} \frac{-j}{2}\left[\delta\left(f-\frac{1}{2\pi}\right)-\delta\left(f+\frac{1}{2\pi}\right)\right] \tag{5.63}$$

The attenuation happens for the same values of f, as depicted in the graphs of the Fourier Transforms given below.

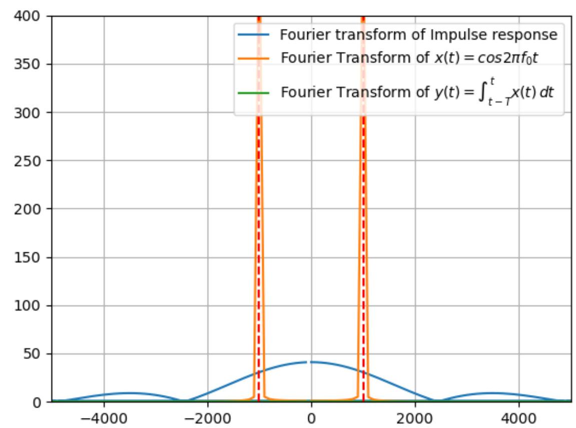

Fig. 5.6: Fourier Transform of Impulse response $h(t)$

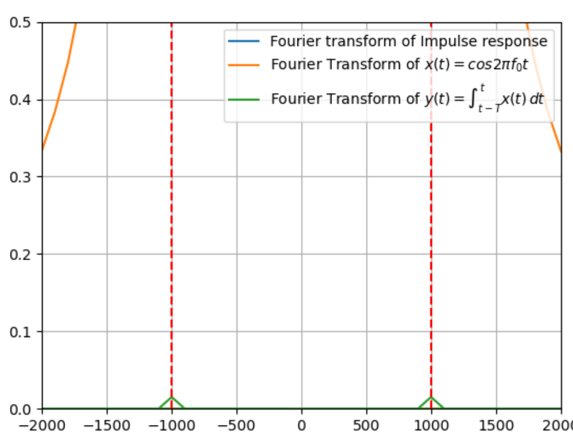

Fig. 5.7: Fourier Transform of Impulse response $h(t)$ zoomed in

□

6. Let the state-space representation on an LTI system be $\dot{x}(t) = Ax(t) + Bu(t)$, $y(t) = Cx(t) + du(t)$ where A,B,C are matrices, d is a scalar, u(t) is the input to the system, and y(t) is its output. Let $B = \begin{pmatrix}0 & 0 & 1\end{pmatrix}^\top$ and $d = 0$. Which one of the following options for A and C will ensure that the transfer function of this LTI system is

$$H(s) = \frac{1}{s^3+3s^2+2s+1} \tag{6.1}$$

(A) $A = \begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -1 & -2 & -3\end{pmatrix}$ and $C = \begin{pmatrix}1 & 0 & 0\end{pmatrix}$

(B) $A = \begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -3 & -2 & -1\end{pmatrix}$ and $C = \begin{pmatrix}1 & 0 & 0\end{pmatrix}$

(C) $A = \begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -1 & -2 & -3\end{pmatrix}$ and $C = \begin{pmatrix}0 & 0 & 1\end{pmatrix}$

(D) $A = \begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -3 & -2 & -1\end{pmatrix}$ and $C = \begin{pmatrix}0 & 0 & 1\end{pmatrix}$

Solution: From the given information,

$$\begin{pmatrix}\dot{x}(t)\\ y(t)\end{pmatrix} = \begin{pmatrix}A & B\\ C & d\end{pmatrix}\begin{pmatrix}x(t)\\ u(t)\end{pmatrix} \tag{6.2}$$

Taking Laplace transform on both sides,

$$\begin{pmatrix}sX(s)\\ Y(s)\end{pmatrix} = \begin{pmatrix}A & B\\ C & d\end{pmatrix}\begin{pmatrix}X(s)\\ U(s)\end{pmatrix} \tag{6.3}$$

$$\implies sX(s) = AX(s) + BU(s) \tag{6.4}$$

$$\implies X(s) = (sI-A)^{-1}BU(s) \tag{6.5}$$

$$\implies Y(s) = CX(s) + dU(s) \tag{6.6}$$

$$= C(sI-A)^{-1}BU(s) + dU(s) \tag{6.7}$$

By definition,

$$Y(s) = H(s)U(s) \tag{6.8}$$

$$\implies H(s) = C(sI-A)^{-1}B + d \tag{6.9}$$

$$= \frac{1}{s^3+3s^2+2s+1} \tag{6.10}$$

$$\implies C(sI-A)^{-1}B + d = \frac{1}{s^3+3s^2+2s+1} \tag{6.11}$$

Now we cross verify the options with eq 6.11. By using a python script,

(A)

$$C(sI-A)^{-1}B+d = \frac{1}{s^3+3s^2+2s+1} \tag{6.12}$$

(B)

$$C(sI-A)^{-1}B+d = \frac{1}{s^3+1s^2+2s+3} \tag{6.13}$$

(C)

$$C(sI-A)^{-1}B+d = \frac{s^2}{s^3+3s^2+2s+1} \quad (6.14)$$

(D)

$$C(sI-A)^{-1}B+d = \frac{s^2}{s^3+1s^2+2s+3} \quad (6.15)$$

Hence A is the correct option.

7. Consider a real-valued base-band signal $x(t)$, band limited to 10 kHz. The Nyquist rate for the signal $y(t) = x(t)\, x\left(1+\frac{t}{2}\right)$ is
   a) 15 kHz
   b) 30 kHz
   c) 60 kHz
   d) 20 kHz

   Solution:

Definition 5 (Dirac-delta impulse)**.**

$$\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & otherwise \end{cases} \quad (7.1)$$

Lemma 7.1 (Shifting property of $\delta(t)$)**.** *If $g(t)$ is a continuous and finite function at $t = a$ then*

$$\int_{-\infty}^{\infty} \delta(t-a)\, g(t)\, dt = g(a) \quad (7.2)$$

We also have

$$\int_{-\infty}^{\infty} \delta(t-a)\, \delta(t-b)\, dt = \delta(a-b) \quad (7.3)$$

Theorem 0.3. *Fourier transform of shifted impulse is the complex exponential.*

$$G(f) = \mathcal{F}\{\delta(t-a)\} = e^{-i2\pi fa} \quad (7.4)$$

Proof.

$$G(f) = \int_{-\infty}^{\infty} \delta(t-a)\, e^{-i2\pi ft} dt \quad (7.5)$$

$$= e^{-i2\pi fa} \quad (7.6)$$

□

Corollary 0.4. *Inverse Fourier Transform of the complex exponential must be the shifted impulse. So*

$$\mathcal{F}^{-1}\left\{e^{-2\pi fa}\right\} = \int_{-\infty}^{\infty} e^{-2\pi fa} e^{i2\pi ft} df \quad (7.7)$$

$$= \int_{-\infty}^{\infty} e^{i2\pi f(t-a)} df \quad (7.8)$$

$$= \int_{-\infty}^{\infty} e^{-i2\pi f(t-a)} df \quad (7.9)$$

$$= \delta(t-a) \quad (7.10)$$

Theorem 0.5. *The Fourier transform of $g(t) = e^{i2\pi at}$ is given by*

$$G(f) = \mathcal{F}\left\{e^{i2\pi at}\right\} = \delta(f-a) \quad (7.11)$$

Proof.

$$G(f) = \int_{-\infty}^{\infty} e^{i2\pi at} e^{-i2\pi ft} dt \quad (7.12)$$

$$= \int_{-\infty}^{\infty} e^{i2\pi t(a-f)} dt \quad (7.13)$$

$$= \delta(f-a) \quad (7.14)$$

□

Lemma 7.2 (Linearity of Fourier Transform)**.**

$$\mathcal{F}\{c_1 g(t) + c_2 h(t)\} = c_1\mathcal{F}\{g(t)\} + c_2\mathcal{F}\{h(t)\} \quad (7.15)$$

Lemma 7.3. *Let $x(t)$ be a signal, its Fourier Transform be of the form*

$$G_x(f) = c_1\delta(f-a_1A) + c_2\delta(f-a_2A) + ... \quad (7.16)$$

where $c_i \in \mathbb{C}$ and $a_i \in \mathbb{R}$. Then the frequencies present in the signal are a_jA where $a_j \in \mathbb{R}^+$

Let $x(t) = \cos(2\pi At)$, where $A = 10kHz$.

$$\cos(2\pi At) = \frac{e^{i2\pi At} + e^{-i2\pi At}}{2} \quad (7.17)$$

The Fourier transform of $x(t)$

$$G_x(f) = \int_{-\infty}^{\infty} \frac{e^{i2\pi At} + e^{-i2\pi At}}{2} e^{-i2\pi ft} dt \quad (7.18)$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} e^{-i2\pi t(f-A)} dt + \int_{-\infty}^{\infty} e^{-i2\pi t(A+F)}\right] \quad (7.19)$$

$$= \frac{1}{2}\left[\delta(f-A) + \delta(f+A)\right] \quad (7.20)$$

∴ All the energy of the sinusoidal wave is entirely localized at the frequencies given by

$|f| = A$.

$$y(t) = \cos(2\pi At)\cos\left(2\pi A\left(1+\frac{t}{2}\right)\right) \tag{7.21}$$
$$= \frac{1}{2}\left(\cos(2\pi A + 3\pi At) + \cos(2\pi A - \pi At)\right) \tag{7.22}$$
$$= \frac{1}{2}\left(\cos(3\pi At) + \cos(\pi At)\right) \tag{7.23}$$

Fourier Transform of $y(t)$ is given by

$$G_y(f) = \frac{1}{4}\left[\delta\left(f-\frac{3A}{2}\right)+\delta\left(f+\frac{3A}{2}\right)\right] + \frac{1}{4}\left[\delta\left(f-\frac{A}{2}\right)+\delta\left(f+\frac{A}{2}\right)\right] \tag{7.24}$$

From lemma 7.3 we can conclude that the frequencies present in signal $y(t)$ are $\frac{A}{2}$, $\frac{3A}{2}$

Lemma 7.4. *Multiplication property of Fourier Transform*

$$\textit{If } x(t) \overset{\mathcal{F}}{\rightleftharpoons} X(f) \tag{7.25}$$
$$y(t) \overset{\mathcal{F}}{\rightleftharpoons} Y(f) \tag{7.26}$$

Then

$$x(t)\,y(t) \overset{\mathcal{F}}{\rightleftharpoons} X(f) * Y(f) \tag{7.27}$$

where $*$ represents convolution

Lemma 7.5.

$$\delta(t-t_0) * g(t) = g(t-t_0) \tag{7.28}$$

Lemma 7.6. *Computing $G_y(f)$ using convolution*

$$x(t) = \cos(2\pi At) \tag{7.29}$$
$$x(t) \overset{\mathcal{F}}{\rightleftharpoons} X_1(f) = \frac{1}{2}\left[\delta(f-A)+\delta(f+A)\right] \tag{7.30}$$
$$x\left(1+\frac{t}{2}\right) = \cos(\pi At) \tag{7.31}$$
$$x\left(1+\frac{t}{2}\right) \overset{\mathcal{F}}{\rightleftharpoons} X_2(f) = \frac{1}{2}\left[\delta\left(f-\frac{A}{2}\right)+\delta\left(f+\frac{A}{2}\right)\right] \tag{7.32}$$

Using lemma 7.4

$$G_y(f) = X_1(f) * X_2(f) \tag{7.33}$$
$$= \left(\frac{1}{2}\left[\delta(f-A)+\delta(f+A)\right]\right) * X_2(f) \tag{7.34}$$
$$= \frac{1}{2}\left\{\delta(f-A) * X_2(f) + \delta(f+A) * X_2(f)\right\} \tag{7.35}$$

Using (7.28)

$$G_y(f) = \frac{1}{2}\left(X_2(f-A) + X_2(f+A)\right) \tag{7.36}$$

$$G_y(f) = \frac{1}{4}\left[\delta\left(f-\frac{3A}{2}\right)+\delta\left(f+\frac{3A}{2}\right)\right] + \frac{1}{4}\left[\delta\left(f-\frac{A}{2}\right)+\delta\left(f+\frac{A}{2}\right)\right] \tag{7.37}$$

$$x(t) = \cos(20k\pi t) \tag{7.38}$$
$$\text{bandwidth of } x(t) = 10kHz \tag{7.39}$$
$$x\left(1+\frac{t}{2}\right) = \cos(20k\pi + 10k\pi t) \tag{7.40}$$
$$\text{bandwidth of } x\left(1+\frac{t}{2}\right) = 5kHz \tag{7.41}$$
$$\text{from (7.23) } y(t) = \cos(30k\pi t) + \cos(10k\pi t) \tag{7.42}$$
$$\text{bandwidth of } y(t) = \frac{30}{2}kHz \tag{7.43}$$
$$= 15kHz \tag{7.44}$$

$$\text{Nyquist rate} = 2 \times \text{maximum frequency} \tag{7.45}$$
$$= 30kHz \tag{7.46}$$

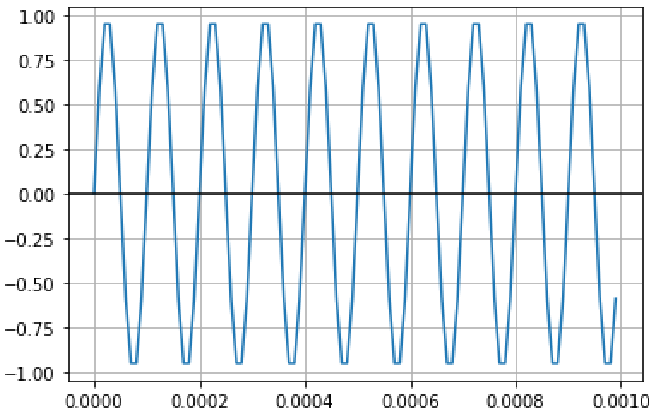

Fig. 7.1: $x(t)$:Sinusoidal signal with freq=10kHz

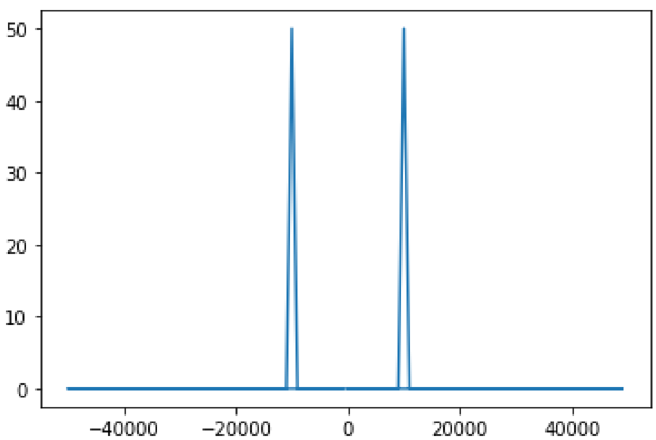

Fig. 7.2: DFT of $x(t)$. $Bandwidth = 10000$

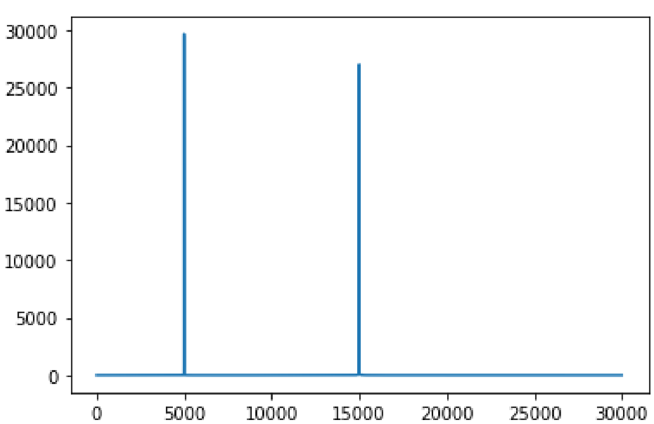

Fig. 7.3: DFT of $y(t)$.$Bandwidth = 15000$

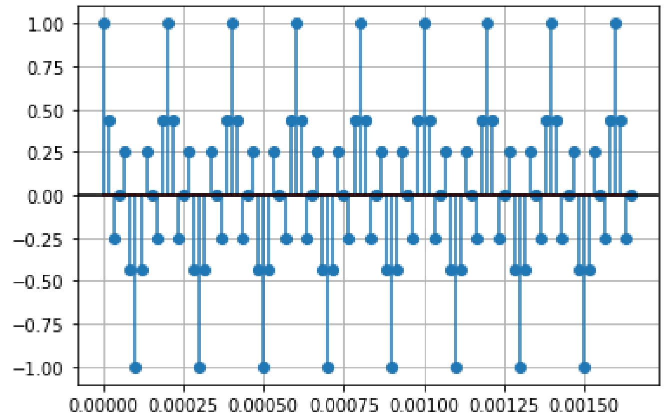

Fig. 7.4: stem plot of $y(t)$ sampled at 60kHz

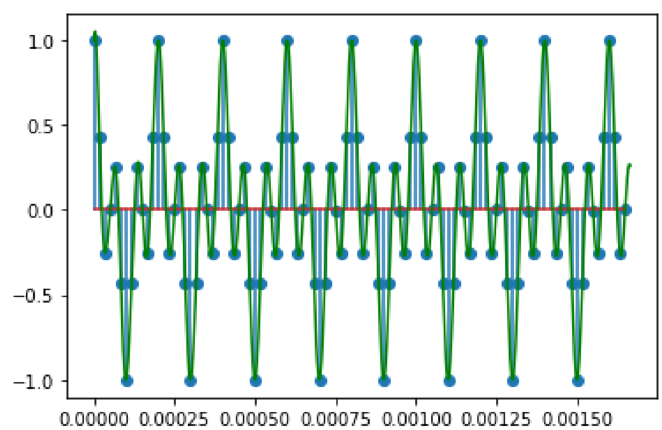

Fig. 7.5: Shannon interpolation of $y(t)$