Signal Processing

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Contents

***Abstract*—This manual provides solved problems in signal processing from GATE exam papers.**

1. The impulse response of a system is $h(t) = tu(t)$. For an input $u(t-1)$, the output is
   a) $\frac{t^2}{2}u(t)$
   b) $\frac{t(t-1)}{2}u(t-1)$
   c) $\frac{(t-1)^2}{2}u(t-1)$
   d) $\frac{t^2-1}{2}u(t-1)$

   Solution:

   Definition 1 (Laplace Transform)**.** *It is an integral transform that converts a function of a real variable t to a function of a complex variable s. The Laplace transform of $f(t)$ is denoted by $\mathcal{L}\{f(t)\}$ or $F(s)$.*

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt \tag{1.1}$$

   Remark. Laplace transform of $f(t) = t^n, n \geq 1$ is

$$F(s) = \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, s > 0 \tag{1.2}$$

   Proof. Basis Step: $n = 1$

$$\mathcal{L}\{t\} = \int_0^\infty e^{-st} t dt \tag{1.3}$$
$$= \left[\frac{te^{-st}}{-s}\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}dt \tag{1.4}$$
$$= 0 + \left[\frac{-1}{s^2}e^{-st}\right]_0^\infty, s > 0 \tag{1.5}$$
$$= \frac{1}{s^2}, s > 0 \tag{1.6}$$

Inductive Step:

$$\mathcal{L}\{t^n\} = \int_0^\infty e^{-st} t^n dt \tag{1.7}$$
$$= \left[\frac{t^n e^{-st}}{-s}\right]_0^\infty + \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}dt \tag{1.8}$$
$$= 0 + \frac{n}{s}\mathcal{L}\left\{t^{n-1}\right\}, s > 0 \tag{1.9}$$
$$= \frac{n}{s}\mathcal{L}\left\{t^{n-1}\right\}, s > 0 \tag{1.10}$$

To prove that if
eqrefec/2003/8eq:t holds for $n = k$, it holds for $n = k + 1$. From
eqrefec/2003/8eq:e

$$\mathcal{L}\left\{t^{k+1}\right\} = \frac{k+1}{s}\mathcal{L}\left\{t^k\right\} \tag{1.11}$$
$$= \frac{(k+1)k!}{s(s^{k+1})} = \frac{(k+1)!}{s^{k+2}}, s > 0 \tag{1.12}$$

By mathematical induction,
eqrefec/2003/8eq:t is true $\forall n \geq 1$ □

Lemma 1.1. *For any real number c,*

$$\mathcal{L}\{u(t-c)\} = \frac{e^{-cs}}{s}, s > 0 \tag{1.13}$$

Proof.

$$\mathcal{L}\{u(t-c)\} = \int_0^\infty e^{-st}u(t-c)dt = \int_c^\infty e^{-st}dt \tag{1.14}$$
$$= \left[-\frac{e^{-st}}{s}\right]_c^\infty = \frac{e^{-cs}}{s}, s > 0 \tag{1.15}$$

□

Definition 2 (Inverse Laplace Transform)**.** *It is the transformation of a Laplace transform into a function of time. If $F(s) = \mathcal{L}\{f(t)\}$, then the Inverse laplace transform of $F(s)$ is $\mathcal{L}^{-1}\{F(s)\} = f(t)$.*

Lemma 1.2 (t-shift rule)**.** *For any real number c,*

$$\mathcal{L}\{u(t-c)f(t-c)\} = e^{-cs}F(s) \tag{1.16}$$

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Proof.

$$\mathcal{L}\left\{u(t-c)f(t-c)\right\} = \int_0^\infty e^{-st}u(t-c)f(t-c)dt \tag{1.17}$$
$$= \int_c^\infty e^{-st}f(t-c)dt \tag{1.18}$$
$$= \int_0^\infty e^{-s(\tau+c)}f(\tau)d\tau \, (t=\tau+c) \tag{1.19}$$
$$= e^{-cs}\int_0^\infty e^{-s\tau}f(\tau)d\tau \tag{1.20}$$
$$= e^{-cs}F(s) \tag{1.21}$$

□

Corollary 0.1.

$$\mathcal{L}^{-1}\left\{e^{-cs}F(s)\right\} = u(t-c)f(t-c) \tag{1.22}$$

Theorem 0.2 (Convolution theorem)**.** *Suppose* $F(s) = \mathcal{L}\left\{f(t)\right\}, G(s) = \mathcal{L}\left\{g(t)\right\}$ *exist, then,*

$$\mathcal{L}^{-1}\left\{F(s)G(s)\right\} = f(t) * g(t) \tag{1.23}$$

Given,

$$h(t) = tu(t) \tag{1.24}$$
$$x(t) = u(t-1) \tag{1.25}$$

To find: $y(t)$. We know,

$$y(t) = h(t) * x(t) \tag{1.26}$$
$$= \mathcal{L}^{-1}\left\{H(s)X(s)\right\} \tag{1.27}$$

From
eqrefec/2003/8eq:uf and
eqrefec/2003/8eq:t,

$$H(s) = e^0\mathcal{L}\left\{t\right\} = \frac{1}{s^2} \tag{1.28}$$

From
eqrefec/2003/8eq:u,

$$X(s) = \frac{e^{-s}}{s} \tag{1.29}$$

Substituting in
eqrefec/2003/8eq:def,

$$y(t) = \mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^3}\right\} \tag{1.30}$$

Consider

$$p(t) = \frac{t^2}{2} \tag{1.31}$$

From
eqrefec/2003/8eq:t

$$P(s) = \frac{2!}{2s^3} = \frac{1}{s^3} \tag{1.32}$$

Further, from
eqrefec/2003/8eq:cuf, for $c = 1$

$$\mathcal{L}^{-1}\left\{e^{-s}P(s)\right\} = u(t-1)p(t-1) \tag{1.33}$$
$$= u(t-1)\frac{(t-1)^2}{2} \tag{1.34}$$
$$\therefore y(t) = \frac{(t-1)^2}{2}u(t-1) \tag{1.35}$$

Option 3 is the correct answer.

$$h(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases} \tag{1.36}$$

$$x(t) = \begin{cases} 1, & t \geq 1 \\ 0, & t < 1 \end{cases} \tag{1.37}$$

$$y(t) = \begin{cases} \frac{(t-1)^2}{2}, & t \geq 1 \\ 0, & t < 1 \end{cases} \tag{1.38}$$

See Figs. 1.1, 1.2 and 1.3.

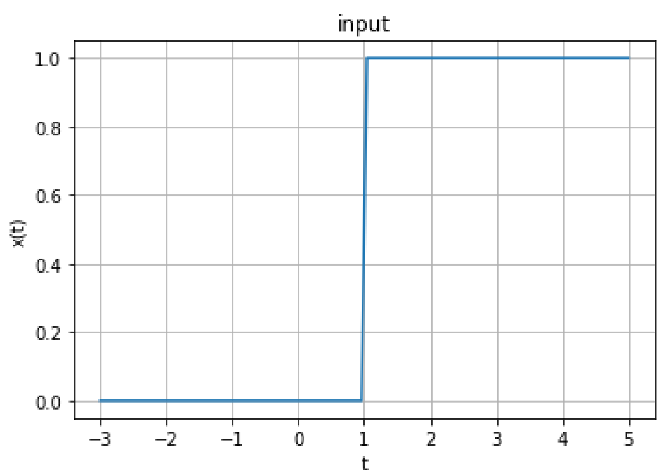

Fig. 1.1: Plot of $x(t)$

2. The input $x(t)$ and output $y(t)$ of a continous time signal are related as

$$y(t) = \int_{t-T}^{t} x(u)\,du \tag{2.1}$$

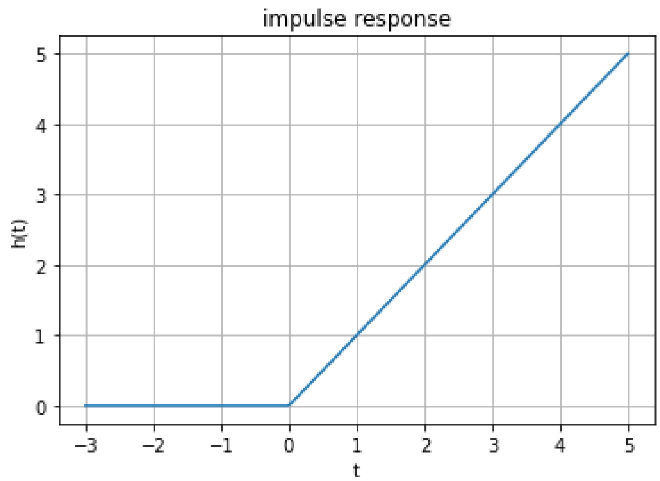

Fig. 1.2: Plot of $h(t)$

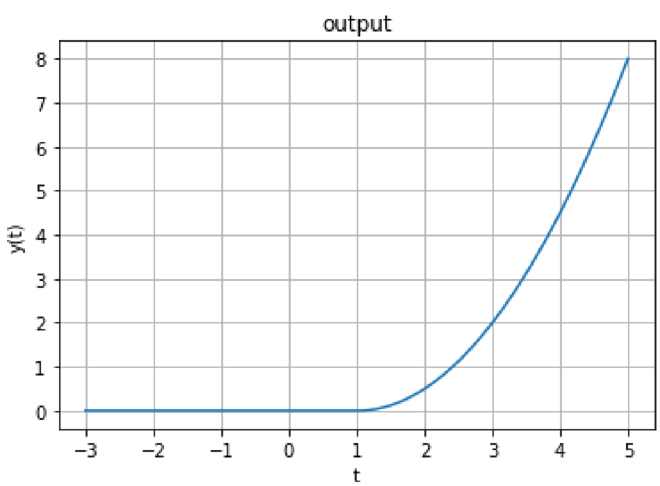

Fig. 1.3: Plot of $y(t)$

The system is:

a) Linear and Time-variant
b) Linear and Time-invariant
c) Non-Linear and Time-variant
d) Non-Linear and Time-invariant

Solution:

Definition 3. *We say that a system is* ***linear*** *if and only if it follows the Principle of Superposition, i.e Law of Additivity and Law of Homogeneity.*

Definition 4. *A system is said to be* ***time invariant*** *if the output signal does not depend on the absolute time, i.e a time delay on the input signal directly equates to the delay in the output signal.*

Lemma 2.1. *The system relating the input signal $x(t)$ and output signal $y(t)$, given by*

$$y(t) = \int_{t-T}^{t} x(u)\,du \tag{2.2}$$

is linear and time invariant in nature.

Proof. a) **Linearity and Time invariance**

From (3), we can say the system is linear if it follows both the laws of Additivity and Homogeneity.

Law of Additivity:

Let the two input signals be $x_1(t)$ and $x_2(t)$, and their corresponding output signals be $y_1(t)$ and $y_2(t)$, then:

$$y_1(t) = \int_{t-T}^{t} x_1(u)\,du \tag{2.3}$$

$$y_2(t) = \int_{t-T}^{t} x_2(u)\,du \tag{2.4}$$

$$y_1(t) + y_2(t) = \int_{t-T}^{t} [x_1(u) + x_2(u)]\,du \tag{2.5}$$

Now, consider the input signal of $x_1(t)+x_2(t)$, then the corresponding output signal is given by $y'(t)$:

$$y'(t) = \int_{t-T}^{t} [x_1(u) + x_2(u)]\,du \tag{2.6}$$

Clearly, from (2.5) and (2.6):

$$y'(t) = y_1(t) + y_2(t) \tag{2.7}$$

Thus, the Law of Additivity holds.

Law of Homogeneity:

Consider an input signal $kx(t)$, where k is any constant. Let the corresponding output be given by $y'(t)$, then:

$$y'(t) = \int_{t-T}^{t} kx(u)\,du \tag{2.8}$$

$$= k\int_{t-T}^{t} x(u)\,du \tag{2.9}$$

$$= ky(t) \tag{2.10}$$

Clearly, from (2.10),

$$y'(t) = ky(t) \tag{2.11}$$

Thus, the Law of Homogeneity holds.

Since both the Laws hold, the system satisfies the Principle of Superposition, and is thus, a **linear system**.

From (4) , to check for time-invariance, we would introduce a delay of t_0 in the output and input signals.
Delay in output signal:

$$y(t - t_0) = \int_{t-t_0-T}^{t-t_0} x(u)\, du \tag{2.12}$$

Now, we consider an input signal with a delay of t_0, given by $x(t - t_0)$, and let the corresponding output signal be given by $y'(t)$, then:

$$y'(t) = \int_{t-T}^{t} x(u - t_0)\, du \tag{2.13}$$

Substituting $a = u - t_0$:

$$y'(t) = \int_{t-t_0-T}^{t-t_0} x(a)\, da \tag{2.14}$$

Clearly, from (2.12) and (2.14):

$$y'(t) = y(t - t_0) \tag{2.15}$$

Thus, the system is **time-invariant**.
The correct option is **2) Linear and Time-invariant**

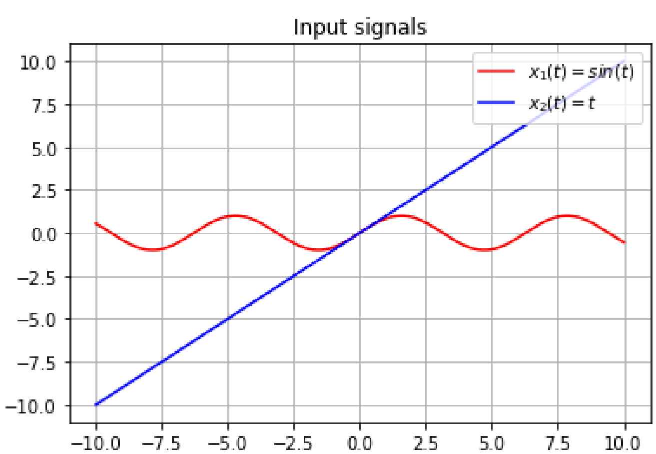

Fig. 2.1: $x_1(t) = \sin t$ and $x_2(t) = t$

b) **Calculating impulse response of LTI system**
Since the given system is an LTI system, it would possess an impulse response $h(t)$, which is the output of the system when the input signal is the Impulse function, given

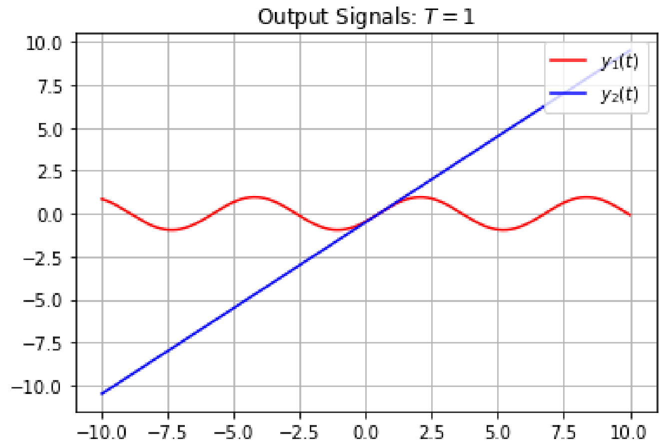

Fig. 2.2: $y_1(t)$ and $y_2(t)$

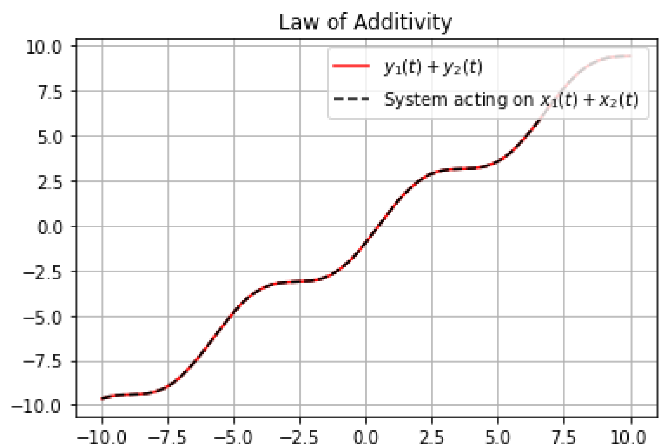

Fig. 2.3: Law of Additivity

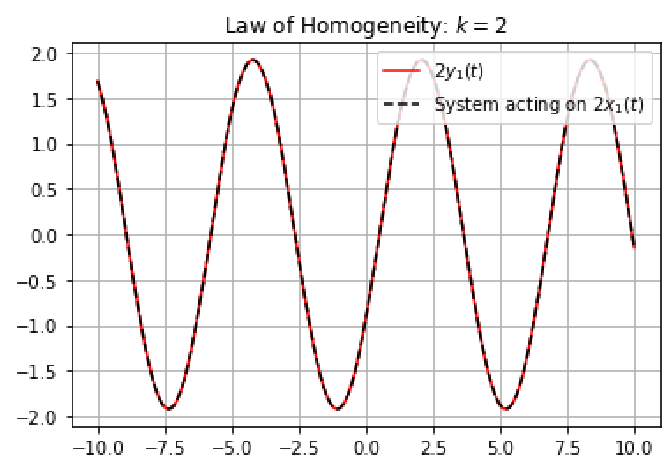

Fig. 2.4: Law of Homogeneity

by $\delta(t)$. Thus,

$$h(t) = \int_{t-T}^{t} \delta(u) du \tag{2.16}$$

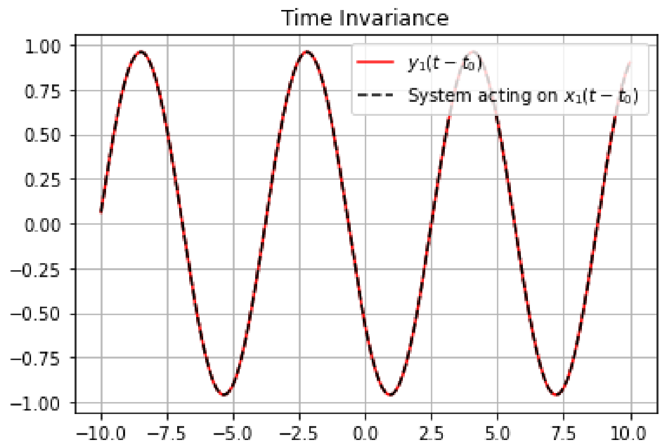

Fig. 2.5: Time invariance

The Impulse function can be loosely defined as:

$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & otherwise \end{cases} \; and \int_{-\infty}^{\infty} \delta(t)dt = 1 \tag{2.17}$$

Since the Impulse function is zero everywhere aside from $t = 0$, the non-zero value of integration is a result of $\delta(0)$. Thus, we can say $h(t)$ will be non-zero only if the limits of integration would include $t = 0$, i.e:

$$h(t) = \begin{cases} \int_{t-T}^{t} \delta(u)du & t - T < 0; t > 0 \\ 0 & otherwise \end{cases} \tag{2.18}$$

$$h(t) = \begin{cases} 1 & 0 < t < T \\ 0 & otherwise \end{cases} \tag{2.19}$$

c) **Expressing the impulse function in terms of $u(t)$**

The unit step signal, $u(t)$, is given by:

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & otherwise \end{cases} \tag{2.20}$$

On time-shifting $u(t)$ by T, we get:

$$u(t-T) = \begin{cases} 1 & t - T \geq 0 \\ 0 & otherwise \end{cases} = \begin{cases} 1 & t \geq T \\ 0 & otherwise \end{cases} \tag{2.21}$$

On subtracting (2.20) and (2.21), we get our impulse response $h(t)$ in terms of the unit step signal:

$$h(t) = u(t) - u(t-T) \tag{2.22}$$

d) **Expressing the impulse function in terms of $rect(t)$**

The unit rectangular signal, $rect(t)$ is given by:

$$rect(t) = \begin{cases} 1 & \frac{-1}{2} \leq t \leq \frac{1}{2} \\ 0 & otherwise \end{cases} \tag{2.23}$$

We can obtain the impulse response $h(t)$ in terms of $rect(t)$ using time scaling and shifting as follows:

$$rect\left(\frac{t}{\tau}\right) = \begin{cases} 1 & \frac{-1}{2} \leq \frac{t}{\tau} \leq \frac{1}{2} \\ 0 & otherwise \end{cases} = \begin{cases} 1 & \frac{-\tau}{2} \leq t \leq \frac{\tau}{2} \\ 0 & otherwise \end{cases} \tag{2.24}$$

Subsituting $\tau = T$:

$$rect\left(\frac{t}{T}\right) = \begin{cases} 1 & \frac{-T}{2} \leq t \leq \frac{T}{2} \\ 0 & otherwise \end{cases} \tag{2.25}$$

Now, we want to right-shift the signal by $\frac{T}{2}$:

$$rect\left(\frac{1}{T}\left(t - \frac{T}{2}\right)\right) = \begin{cases} 1 & 0 \leq t \leq T \\ 0 & otherwise \end{cases} = h(t) \tag{2.26}$$

Since the time shifting is to be performed on the variable t and not $\frac{t}{T}$

e) **Calculating the Fourier Transform of $h(t)$**

Let the Fourier Transform of $h(t)$ be given by $H(f)$ and of the rectangular signal, $rect(t)$ be given by $Y(f)$.

$$h(t) \overset{\mathcal{F}}{\rightleftharpoons} H(f) \tag{2.27}$$

$$rect(t) \overset{\mathcal{F}}{\rightleftharpoons} Y(f) \tag{2.28}$$

Then,

$$Y(f) = \int_{-\infty}^{\infty} rect(t)e^{-j2\pi ft}\,dt \tag{2.29}$$

From (2.23), we can write (2.29) as:

$$Y(f) = \int_{-\infty}^{\frac{-1}{2}} 0\,dt + \int_{\frac{-1}{2}}^{\frac{1}{2}} e^{-j2\pi f t}\,dt + \int_{\frac{1}{2}}^{\infty} 0\,dt \quad (2.30)$$

$$= \frac{e^{j\pi f} - e^{-j\pi f}}{j2\pi f} \quad (2.31)$$

$$= \frac{2j \sin \pi f}{j2\pi f} \quad (2.32)$$

$$= \frac{\sin(\pi f)}{\pi f} \quad (2.33)$$

$$= sinc(f) \quad (2.34)$$

where $sinc(t)$, the sampling function is defined as:

$$sinc(t) = \begin{cases} 1 & t = 0 \\ \frac{\sin(\pi t)}{\pi t} & otherwise \end{cases} \quad (2.35)$$

Let the Fourier Transform of a signal $x(t)$ be $X(f)$.

$$x(t) \overset{\mathcal{F}}{\rightleftharpoons} X(f) \quad (2.36)$$

When the signal $x(t)$ is time shifted by t_0, the resultant Fourier Transform is given by:

$$x(t \pm t_0) \overset{\mathcal{F}}{\rightleftharpoons} X(f)e^{\pm j2\pi f t_0} \quad (2.37)$$

And when the signal $x(t)$ is time scaled by α, the resulting Fourier Transform is given by:

$$x(\alpha t) \overset{\mathcal{F}}{\rightleftharpoons} \frac{1}{|\alpha|} X\left(\frac{f}{\alpha}\right) \quad (2.38)$$

Since we have already derived the Fourier Transform of $rect(t)$, we would use the properties mentioned above to find the Fourier Transform of $h(t)$:

$$rect(t) \overset{\mathcal{F}}{\rightleftharpoons} sinc(f) \quad (2.39)$$

Using (2.37):

$$rect\left(t - \frac{T}{2}\right) \overset{\mathcal{F}}{\rightleftharpoons} sinc(f)e^{-j(2\pi f)\frac{T}{2}} \quad (2.40)$$

$$rect\left(t - \frac{T}{2}\right) \overset{\mathcal{F}}{\rightleftharpoons} sinc(f)e^{-j\pi f T} \quad (2.41)$$

Using (2.38),

$$rect\left(\frac{1}{T}\left(t - \frac{T}{2}\right)\right) \overset{\mathcal{F}}{\rightleftharpoons} \frac{1}{\frac{1}{|T|}} sinc\left(\frac{f}{T}\right) e^{\frac{-j\pi f T}{T}} \quad (2.42)$$

$$h(t) \overset{\mathcal{F}}{\rightleftharpoons} T\,sinc\left(\frac{f}{T}\right) e^{-j\pi f} \quad (2.43)$$

$$\therefore H(f) = T\,sinc\left(\frac{f}{T}\right) e^{-j\pi f} \quad (2.44)$$

f) **An example**

Consider an input signal of $x(t) = \cos 2\pi f_0 t$. The Fourier Transform of $x(t)$ is given by:

$$x(t) = \cos 2\pi f_0 t \overset{\mathcal{F}}{\rightleftharpoons} \frac{1}{2}\left[\delta(f - f_0) + \delta(f + f_0)\right] \quad (2.45)$$

using the fact that

$$\cos 2\pi f_0 t = \frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2} \quad (2.46)$$

and the Fourier Transform of $e^{\pm j2\pi f_0 t}$ is given by:

$$e^{\pm j2\pi f_0 t} \overset{\mathcal{F}}{\rightleftharpoons} \delta(f \mp f_0) \quad (2.47)$$

The output signal will be given by:

$$y(t) = \int_{t-T}^{t} \cos 2\pi f_0 u\,du \quad (2.48)$$

$$= \frac{1}{2\pi f_0}\left[\sin 2\pi f_0 t - \sin 2\pi f_0 (t - T)\right] \quad (2.49)$$

$$= \frac{\sin \pi f_0 T}{\pi f_0}\left[\cos 2\pi f_0 \left(t - \frac{T}{2}\right)\right] \quad (2.50)$$

$$= T\,sinc(f_0 T) \cos 2\pi f_0 \left(t - \frac{T}{2}\right) \quad (2.51)$$

The Fourier transform of $\cos 2\pi f_0 \left(t - \frac{T}{2}\right)$ can be obtained using (2.38) and (2.37) as follows:

$$\cos t = \frac{1}{2}\left[e^{jt} + e^{-jt}\right] \quad (2.52)$$

$$\cos t \overset{\mathcal{F}}{\rightleftharpoons} \frac{1}{2}\left[\delta\left(f - \frac{1}{2\pi}\right) + \delta\left(f + \frac{1}{2\pi}\right)\right] \quad (2.53)$$

$$\cos\left(t - \frac{T}{2}\right) \overset{\mathcal{F}}{\rightleftharpoons} \frac{e^{j\pi f T}}{2}\left[\delta\left(f - \frac{1}{2\pi}\right) + \delta\left(f + \frac{1}{2\pi}\right)\right] \quad (2.54)$$

$$\cos 2\pi f_0 \left(t - \frac{T}{2}\right) \overset{\mathcal{F}}{\rightleftharpoons} \frac{e^{j\pi \frac{f}{2\pi f_0} T}}{2\pi f_0} \frac{\delta(\frac{f}{2\pi f_0} - \frac{1}{2\pi}) + \delta(\frac{f}{2\pi f_0} + \frac{1}{2\pi})}{2} \quad (2.55)$$

$$\cos 2\pi f_0 \left(t - \frac{T}{2}\right) \overset{\mathcal{F}}{\rightleftharpoons} \frac{e^{j\pi \frac{f}{2f_0} T}}{4\pi f_0}\left(\delta\left(\frac{f - f_0}{2\pi f_0}\right) + \delta\left(\frac{f + f_0}{2\pi f_0}\right)\right) \quad (2.56)$$

Therefore, the Fourier Transform of the output signal $y(t)$ from (2.51) is given by:

$$y(t) \overset{\mathcal{F}}{\rightleftharpoons} \frac{T\,sinc(f_0 T)}{4\pi f_0} e^{j\pi \frac{f}{2f_0} T}\left(\delta\left(\frac{f - f_0}{2\pi f_0}\right) + \delta\left(\frac{f + f_0}{2\pi f_0}\right)\right) \quad (2.57)$$

$$y(t) \overset{\mathcal{F}}{\rightleftharpoons} k e^{j\pi \frac{f}{2f_0} T}\left(\delta\left(\frac{f - f_0}{2\pi f_0}\right) + \delta\left(\frac{f + f_0}{2\pi f_0}\right)\right) \quad (2.58)$$

where $k = \frac{T sinc(f_0 T)}{4\pi f_0}$. Substituting $2\pi f_0 = 1$ and $T = 1$:

$$y(t) \overset{\mathcal{F}}{\rightleftharpoons} ke^{j\pi^2 f}\left(\delta\left(f - \frac{1}{2\pi}\right) + \delta\left(f + \frac{1}{2\pi}\right)\right) \quad (2.59)$$

$$y(t) \overset{\mathcal{F}}{\rightleftharpoons} ke^{j\frac{\pi}{2}}\delta\left(f - \frac{1}{2\pi}\right) + ke^{j\frac{-\pi}{2}}\delta\left(f + \frac{1}{2\pi}\right) \quad (2.60)$$

using the multiplication property of the Delta function:

$$x(t)\delta(t - t_1) = x(t_1)\delta(t - t_1) \quad (2.61)$$

Since , $e^{j\frac{\pi}{2}} = j$ and $e^{-j\frac{\pi}{2}} = -j$, we finally get:

$$y(t) \overset{\mathcal{F}}{\rightleftharpoons} kj\left[\delta\left(f - \frac{1}{2\pi}\right) - \delta\left(f + \frac{1}{2\pi}\right)\right] \quad (2.62)$$

Clearly, the Fourier transform of $y(t)$ can be manipulated to represent a sinusoidal wave, which is given by:

$$sin(t) \overset{\mathcal{F}}{\rightleftharpoons} \frac{-j}{2}\left[\delta\left(f - \frac{1}{2\pi}\right) - \delta\left(f + \frac{1}{2\pi}\right)\right] \quad (2.63)$$

The attenuation happens for the same values of f, as depicted in the graphs of the Fourier Transforms given below.

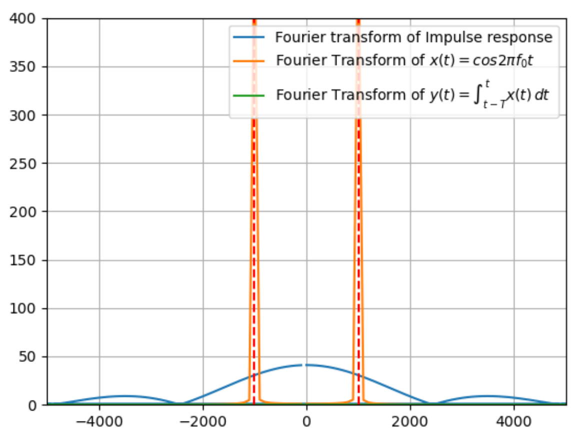

Fig. 2.6: Fourier Transform of Impulse response $h(t)$

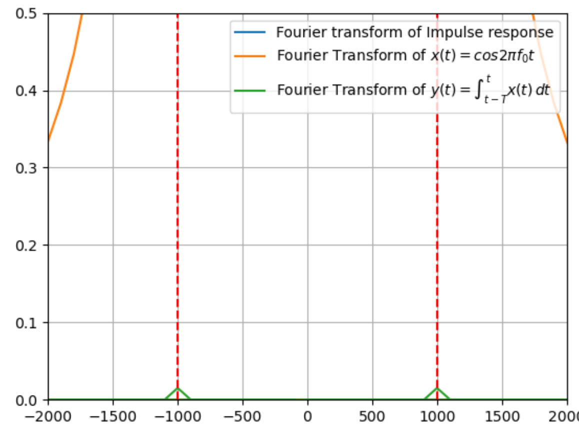

Fig. 2.7: Fourier Transform of Impulse response $h(t)$ zoomed in

□

3. Let the state-space representation on an LTI system be $\dot{x}(t) = Ax(t) + Bu(t)$, $y(t) = Cx(t) + du(t)$ where A,B,C are matrices, d is a scalar, u(t) is the input to the system, and y(t) is its output. Let $B = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^\top$ and $d = 0$. Which one of the following options for A and C will ensure that the transfer function of this LTI system is

$$H(s) = \frac{1}{s^3 + 3s^2 + 2s + 1} \quad (3.1)$$

(A) $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$

(B) $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -2 & -1 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$

(C) $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$

(D) $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -2 & -1 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$

Solution: From the given information,

$$\begin{pmatrix} \dot{x}(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} A & B \\ C & d \end{pmatrix}\begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \quad (3.2)$$

Taking Laplace transform on both sides,

$$\begin{pmatrix} sX(s) \\ Y(s) \end{pmatrix} = \begin{pmatrix} A & B \\ C & d \end{pmatrix}\begin{pmatrix} X(s) \\ U(s) \end{pmatrix} \quad (3.3)$$

$$\implies sX(s) = AX(s) + BU(s) \quad (3.4)$$

$$\implies X(s) = (sI - A)^{-1}BU(s) \quad (3.5)$$

$$\implies Y(s) = CX(s) + dU(s) \quad (3.6)$$

$$= C(sI - A)^{-1}BU(s) + dU(s) \quad (3.7)$$

By definition,

$$Y(s) = H(s)U(s) \quad (3.8)$$

$$\implies H(s) = C(sI - A)^{-1}B + d \quad (3.9)$$

$$= \frac{1}{s^3 + 3s^2 + 2s + 1} \quad (3.10)$$

$$\implies C(sI - A)^{-1}B + d = \frac{1}{s^3 + 3s^2 + 2s + 1} \tag{3.11}$$

Now we cross verify the options with eq 3.11. By using a python script,

(A)

$$C(sI-A)^{-1}B+d = \frac{1}{s^3 + 3s^2 + 2s + 1} \tag{3.12}$$

(B)

$$C(sI-A)^{-1}B+d = \frac{1}{s^3 + 1s^2 + 2s + 3} \tag{3.13}$$

(C)

$$C(sI-A)^{-1}B+d = \frac{s^2}{s^3 + 3s^2 + 2s + 1} \tag{3.14}$$

(D)

$$C(sI-A)^{-1}B+d = \frac{s^2}{s^3 + 1s^2 + 2s + 3} \tag{3.15}$$

Hence A is the correct option.