Signal Processing

G V V Sharma*

CONTENTS

Abstract—This manual provides solved problems in signal processing from GATE exam papers.

1. A casual system having the transfer function $H(s) = \frac{1}{s+2}$ is exicted with 10u(t). The time at which the output reaches 99% its steady state value is ?

Solution: Given the transfer function,

$$H(s) = \frac{1}{s+2} \tag{1.1}$$

$$\implies \frac{Y(s)}{X(s)} = \frac{1}{s+2} \tag{1.2}$$

and input signal,

$$x(t) = 10u(t) \tag{1.3}$$

By applying Laplace transform

$$\mathcal{L}\{x\}(s) = 10\mathcal{L}\{u\}(s) \tag{1.4}$$

We know that Laplace transform of unit step function (u(t)) is $\frac{1}{s}$ (discussed in class)

$$X(s) = \frac{10}{s} \tag{1.5}$$

$$\implies Y(s) = \frac{s}{s(s+2)} \tag{1.6}$$

$$\implies Y(s) = 5\left(\frac{1}{s} - \frac{1}{s+2}\right) \tag{1.7}$$

Applying inverse Laplace transform on Y(s) for output signal y(t)

$$y(t) = \mathcal{L}^{-1}{Y(s)}(t)$$
 (1.8)

$$\implies y(t) = 5(1 - e^{-2t})$$
 (1.9)

So the steady state value of output signal is 5. Time taken to reach 99% of its steady state

value will be

$$5(1 - e^{-2t}) = (0.99)5 \tag{1.10}$$

$$\implies e^{-2t} = 0.01$$
 (1.11)

$$\implies t = 2.3sec$$
 (1.12)

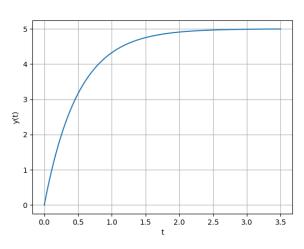


Fig. 1.1: Simulated plot of output signal y(t)

- 2. The region of convergence of Z-transform of the sequence $\left(\frac{5}{6}\right)^n u(n) \left(\frac{6}{5}\right)^n u(-n-1)$ must
- (A) $|Z| < \frac{5}{6}$
- (B) $|Z| > \frac{6}{5}$
- (C) $\frac{5}{6} < |Z| < \frac{6}{5}$
- (D) $\frac{6}{5} < |Z| < \infty$

^{*}The author is with the Department of Electrical Engineering, Indian Institute of Technology, Hyderabad 502285 India e-mail: gadepall@iith.ac.in. All content in this manual is released under GNU GPL. Free and open source.

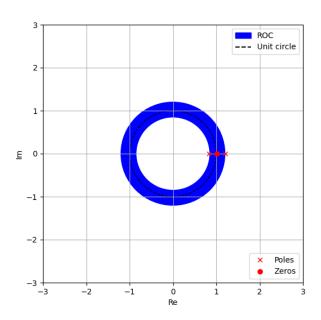


Fig. 2.1: Pole-zero plot of the system

Solution:

$$\left(\frac{5}{6}\right)^n u(n) \stackrel{Z}{\rightleftharpoons} \frac{1}{1 - \frac{5}{6}z^{-1}} \quad |z| > \frac{5}{6} \quad (2.1)$$

$$\left(\frac{6}{5}\right)^n u(-n-1) \stackrel{\mathcal{Z}}{\rightleftharpoons} = \frac{1}{\frac{6}{5}z^{-1}-1} \quad |z| < \frac{6}{5} \quad (2.2)$$

Therefore, the Z transform of the given sequence is

$$\frac{1}{1 - \frac{5}{6}z^{-1}} + \frac{1}{1 - \frac{6}{5}z^{-1}} = dfrac2 - \frac{61}{30}z^{-1} \left(1 - \frac{5}{6}z^{-1}\right) \left(1 - \frac{6}{5}z^{-1}\right) \frac{5}{6} < |z| < x_{5}^{6} t) = rect\left(\frac{t}{2}\right)$$
From the lemma 4.1, fourier transform of x(t)

See Fig. 2.1.

3. The dirac-delta function $\delta(t)$ is defines as

a)
$$\delta(t) = \begin{cases} 1, & t = 0 \\ 0, & otherwise \end{cases}$$
b) $\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & otherwise \end{cases}$
c) $\delta(t) = \begin{cases} 1, & t = 0 \\ 0, & otherwise \end{cases}$ and $\int_{-\infty}^{\infty} \delta(t)dt = 1$
d) $\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & otherwise \end{cases}$ and $\int_{-\infty}^{\infty} \delta(t)dt = 1$

Solution

Answer is 4.

4. The signal x(t) is described by

$$x(t) = \begin{cases} 1 & , \text{for } -1 \le t \le +1 \\ 0 & , otherwise \end{cases}$$
 (4.1)

Two of the angular frequencies at which its fourier transform becomes zero are

- a) $\pi.2\pi$
- b) $0.5\pi, 1.5\pi$
- c) $0.\pi$
- d) $2\pi, 2.5\pi$

Solution:

Lemma 4.1. The fourier transform of a rect function is sinc function

$$rect\left(\frac{t}{\tau}\right) \stackrel{\mathcal{F}}{\rightleftharpoons} \tau sinc\left(f\tau\right)$$
 (4.2)

Proof.

$$\int_{-\infty}^{\infty} rect\left(\frac{x}{\tau}\right) e^{-i2\pi xt} dx = \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} e^{-i2\pi xt} dx \quad (4.3)$$

$$= \left[\frac{e^{-i2\pi xt}}{-i2\pi t}\right]_{\frac{-\tau}{2}}^{\frac{\tau}{2}} \tag{4.4}$$

$$= \frac{e^{-i\pi t\tau} - e^{i\pi t\tau}}{-i2\pi t}$$
 (4.5)
$$= \tau \frac{\sin \pi t\tau}{\pi t\tau}$$
 (4.6)

$$= \tau \frac{\sin \pi t \tau}{\pi t \tau} \tag{4.6}$$

$$= \tau sinc(t\tau) \tag{4.7}$$

We can observe that

$$\int \left(1 - \frac{6}{5}z^{-1}\right) \frac{5}{6} < |z| < x \frac{6}{5}t = rect\left(\frac{t}{2}\right)$$

$$\tag{4.8}$$

(2.3)

$$x(f) = 2sinc(2f) \tag{4.9}$$

$$=2\frac{\sin(2\pi f)}{(2\pi f)}$$
 (4.10)

$$=2\frac{\sin(\omega)}{\omega}\tag{4.11}$$

x(f) is zero when $\omega = n\pi$, where $n \in I - \{0\}$. Hence, option ?? is true.

5. Two discrete time systems with impulse responses $h_1[n] = \delta[n-1]$ and $h_2[n] = \delta[n-2]$ are connected in cascade. The overall impulse response of the cascaded system is

a)
$$\delta[n-1] + \delta[n-2]$$

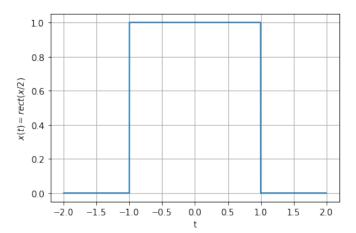


Fig. 4.1: Plot of x(t)

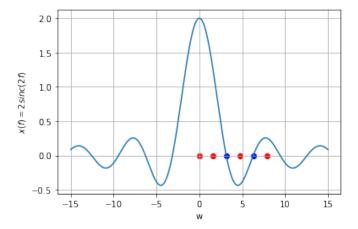


Fig. 4.2: Plot of the fourier transform

- b) $\delta[n-4]$
- c) $\delta[n-3]$
- d) $\delta[n-1]\delta[n-2]$

Solution:

$$h[n] = h_1[n] * h_2[n] = \delta[n-1] * \delta[n-2]$$

$$= \delta[n-3]$$
(5.2)

6. A continuous time LTI system is described by

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = 2\frac{dx(t)}{dt} + 4x(t)$$
(6.1)

Assuming zero initial conditions, the response y(t) of the above system for the input $x(t) = e^{-2t}u(t)$ is given by

a)
$$\left(e^{t}-e^{3t}\right)u\left(t\right)$$

b)
$$(e^{-t} - e^{-3t})u(t)$$

c)
$$\left(e^{-t} + e^{-3t}\right)u(t)$$

d) $\left(e^t + e^{3t}\right)u(t)$

Solution:

Lemma 6.1 (Table of Laplace Transforms).

Time Function	Laplace transform of f(t)
$f(t) = \mathcal{L}^{-1} \left\{ F(s) \right\}$	$F(s) = \mathcal{L}\{f(t)\}\$
u(t)	$\frac{1}{s}$, $s > 0$
g'(t)	sG(s) - g(0)
$g^{\prime\prime}\left(t\right)$	$s^2G(s) - sg(0) - g'(0)$
$e^{-at}u(t)$	$\frac{1}{s+a}$, $s+a>0$

Lemma 6.2. Linearity of Laplace Transform

$$\mathcal{L}\left\{af\left(t\right) + bg\left(t\right)\right\} = a\mathcal{L}\left\{f\left(t\right)\right\} + b\mathcal{L}\left\{g\left(t\right)\right\}$$
(6.2)

From Lemma-6.1 Laplace transform of $x(t) = e^{-2t}u(t)$ is given by

$$X(s) = \frac{1}{s+2} \tag{6.3}$$

Since initial conditions are zero. Laplace Transform of (6.1) gives

$$s^{2}Y(s) + 4sY(s) + 3Y(s) = 2sX(s) + 4X(s)$$

$$(6.4)$$

$$Y(s) = \frac{2(s+2)}{s^{2} + 4s + 3}X(s)$$

$$(6.5)$$

$$= \frac{1}{s+1} - \frac{1}{s+3}$$

$$(6.6)$$

From Lemma-6.1. Inverse Laplace transform of Y(s) is given by

$$y(t) = e^{-t}u(t) - e^{-3t}u(t)$$
 (6.7)

$$= (e^{-t} - e^{-3t}) u(t)$$
 (6.8)

- ... The required option is B. See Fig. 6.1.
- 7. The transfer function for a discrete time LTI system is given by:

$$H(z) = \frac{2 - \frac{3}{4}z^{-1}}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}$$
(7.1)

Consider the following statements:

S1: The system is stable and causal for ROC: $|z| > \frac{1}{2}$

S2: The system is stable but not causal for ROC: $|z| < \frac{1}{4}$

S3: The system is neither stable nor causal for ROC: $\frac{1}{4} < |z| < \frac{1}{2}$

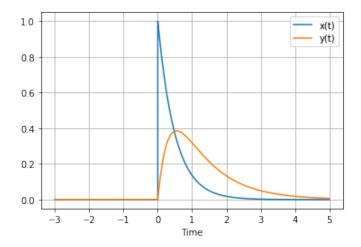


Fig. 6.1: Plot of input and output responses in time domain.

Which one of the following statement are valid?

- a) Both S1 and S2 are true
- b) Both S2 and S3 are true
- c) Both S1 and S3 are true
- d) S1, S2 and S3 are all true

Solution: The given transfer function can be expressed as

$$H(z) = \frac{16 - 6z^{-1}}{8 - 6z^{-1} + z^{-2}}$$
 (7.2)

$$=\frac{16-6z^{-1}}{(4-z^{-1})(2-z^{-1})}$$
 (7.3)

$$= \frac{4}{4 - z^{-1}} + \frac{2}{2 - z^{-1}} \tag{7.4}$$

with poles at

$$z = \frac{1}{2}, z = \frac{1}{4} \tag{7.5}$$

- a) Since the ROC includes the unit circle, the system is stable. Also, the ROC extends outwards to infinity, so the system is causal as well. Hence S_1 is true.
- b) When ROC = $\frac{1}{4} < |z| < \frac{1}{2}$, the unit circle is not included in the ROC. Hence, the system cannot be stable. Also, the ROC is an annulus, so the system is non-causal. So S_3 is true.
- 8. The impulse response of a system is h(t) = tu(t). For an input u(t-1), the output is

a)
$$\frac{t^2}{2}u(t)$$

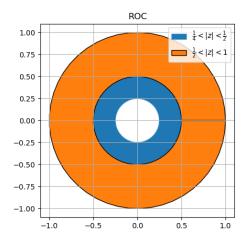


Fig. 7.1: ROC

b)
$$\frac{t(t-1)}{2}u(t-1)$$

c)
$$\frac{(t-1)^2}{2}u(t-1)$$

d)
$$\frac{t^2-1}{2}u(t-1)$$

Solution:

Definition 1 (Laplace Transform). It is an integral transform that converts a function of a real variable t to a function of a complex variable s. The Laplace transform of f(t) is denoted by $\mathcal{L}\{f(t)\}$ or F(s).

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt \qquad (8.1)$$

Remark. Laplace transform of $f(t) = t^n, n \ge 1$ is

$$F(s) = \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, s > 0$$
 (8.2)

Proof. Basis Step: n = 1

$$\mathcal{L}\{t\} = \int_0^\infty e^{-st} t dt \tag{8.3}$$

$$= \left[\frac{te^{-st}}{-s}\right]_0^\infty + \frac{1}{s} \int_0^\infty e^{-st} dt \qquad (8.4)$$

$$= 0 + \left[\frac{-1}{s^2} e^{-st} \right]_0^{\infty}, s > 0$$
 (8.5)

$$=\frac{1}{s^2}, s > 0 \tag{8.6}$$

Inductive Step:

$$\mathcal{L}\{t^{n}\} = \int_{0}^{\infty} e^{-st} t^{n} dt$$

$$= \left[\frac{t^{n} e^{-st}}{-s} \right]_{0}^{\infty} + \frac{n}{s} \int_{0}^{\infty} t^{n-1} e^{-st} dt$$

$$= 0 + \frac{n}{s} \mathcal{L}\{t^{n-1}\}, s > 0$$

$$= \frac{n}{s} \mathcal{L}\{t^{n-1}\}, s > 0$$
(8.9)
$$= \frac{n}{s} \mathcal{L}\{t^{n-1}\}, s > 0$$
(8.10)

To prove that if eqrefec/2003/8eq:t holds for n = k, it holds for n = k + 1. From eqrefec/2003/8eq:e

$$\mathcal{L}\left\{t^{k+1}\right\} = \frac{k+1}{s} \mathcal{L}\left\{t^{k}\right\}$$

$$= \frac{(k+1)k!}{s(s^{k+1})} = \frac{(k+1)!}{s^{k+2}}, s > 0$$
 (8.12)

By mathematical induction, eqrefec/2003/8eq:t is true $\forall n \ge 1$

Lemma 8.1. For any real number c,

$$\mathcal{L}\{u(t-c)\} = \frac{e^{-cs}}{s}, s > 0$$
 (8.13)

Proof.

$$\mathcal{L}\{u(t-c)\} = \int_0^\infty e^{-st} u(t-c) dt = \int_c^\infty e^{-st} dt$$

$$= \left[-\frac{e^{-st}}{s} \right]^\infty = \frac{e^{-cs}}{s}, s > 0 \quad (8.15)$$

Definition 2 (Inverse Laplace Transform). It is the transformation of a Laplace transform into a function of time. If $F(s) = \mathcal{L}\{f(t)\}$, then the Inverse laplace transform of F(s) is $\mathcal{L}^{-1}\{F(s)\} = f(t)$.

Lemma 8.2 (t-shift rule). *For any real number c*,

$$\mathcal{L}\left\{u(t-c)f(t-c)\right\} = e^{-cs}F(s) \tag{8.16}$$

Proof.

(8.7)
$$\mathcal{L}\{u(t-c)f(t-c)\} = \int_{0}^{\infty} e^{-st}u(t-c)f(t-c)dt$$
(8.8)
$$= \int_{c}^{\infty} e^{-st}f(t-c)dt$$
(8.18)
$$= \int_{0}^{\infty} e^{-s(\tau+c)}f(\tau)d\tau (t=\tau+c)$$
(8.19)
$$= e^{-cs} \int_{0}^{\infty} e^{-s\tau}f(\tau)d\tau$$
(8.20)
$$= e^{-cs}F(s)$$
(8.21)

Corollary 0.1.

$$\mathcal{L}^{-1}\left\{e^{-cs}F(s)\right\} = u(t-c)f(t-c) \tag{8.22}$$

Theorem 0.2 (Convolution theorem). Suppose $F(s) = \mathcal{L}\{f(t)\}, G(s) = \mathcal{L}\{g(t)\}\ exist,\ then,$

$$\mathcal{L}^{-1}\{F(s)G(s)\} = f(t) * g(t)$$
 (8.23)

Given,

$$h(t) = tu(t) \tag{8.24}$$

$$x(t) = u(t - 1) (8.25)$$

To find: y(t). We know,

$$y(t) = h(t) * x(t)$$
 (8.26)

$$= \mathcal{L}^{-1} \{ H(s)X(s) \}$$
 (8.27)

From eqrefec/2003/8eq:uf and eqrefec/2003/8eq:t,

$$H(s) = e^0 \mathcal{L}\{t\} = \frac{1}{s^2}$$
 (8.28)

From

egrefec/2003/8eq:u,

$$X(s) = \frac{e^{-s}}{s} {(8.29)}$$

Substituting in eqrefec/2003/8eq:def,

$$y(t) = \mathcal{L}^{-1} \left\{ \frac{e^{-s}}{s^3} \right\}$$
 (8.30)

Consider

$$p(t) = \frac{t^2}{2} \tag{8.31}$$

From

eqrefec/2003/8eq:t

$$P(s) = \frac{2!}{2s^3} = \frac{1}{s^3}$$
 (8.32)

Further, from eqrefec/2003/8eq:cuf, for c = 1

$$\mathcal{L}^{-1}\left\{e^{-s}P(s)\right\} = u(t-1)p(t-1) \tag{8.33}$$

$$= u(t-1)\frac{(t-1)^2}{2}$$
 (8.34)

$$\therefore y(t) = \frac{(t-1)^2}{2}u(t-1)$$
 (8.35)

Option 3 is the correct answer.

$$h(t) = \begin{cases} t, & t \ge 0 \\ 0, & t < 0 \end{cases}$$
 (8.36)

$$x(t) = \begin{cases} 1, & t \ge 1 \\ 0, & t < 1 \end{cases}$$
 (8.37)

$$y(t) = \begin{cases} \frac{(t-1)^2}{2}, & t \ge 1\\ 0, & t < 1 \end{cases}$$
 (8.38)

See Figs. 8.1, 8.2 and 8.3.

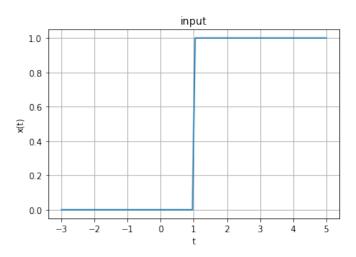


Fig. 8.1: Plot of x(t)

9. The DFT of a vector $\begin{pmatrix} a & b & c & d \end{pmatrix}$ is the vector

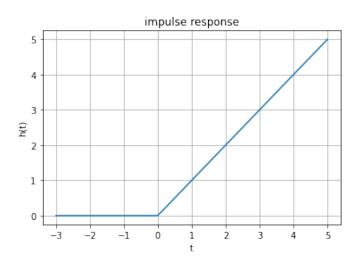


Fig. 8.2: Plot of h(t)

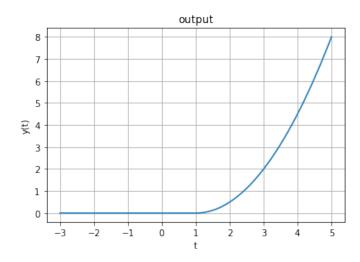


Fig. 8.3: Plot of y(t)

 $(\alpha \ \beta \ \gamma \ \delta)$. Consider the product

$$(p \quad q \quad r \quad s) = (a \quad b \quad c \quad d) \begin{pmatrix} a \quad b \quad c \quad d \\ d \quad a \quad b \quad c \\ c \quad d \quad a \quad b \\ b \quad c \quad d \quad a \end{pmatrix}$$

$$(9.1)$$

The DFT of the vector $(p \ q \ r \ s)$ is a scaled version of

(A)
$$\left(\alpha^2 \quad \beta^2 \quad \gamma^2 \quad \delta^2\right)$$

(B)
$$(\sqrt{\alpha} \quad \sqrt{\beta} \quad \sqrt{\gamma} \quad \sqrt{\delta})$$

(B)
$$\left(\sqrt{\alpha} \quad \sqrt{\beta} \quad \sqrt{\gamma'} \quad \sqrt{\delta}\right)$$

(C) $\left(\alpha + \beta \quad \beta + \delta \quad \delta + \gamma \quad \gamma + \alpha\right)$

(D)
$$(\alpha \beta \gamma \delta)$$

Solution:

Lemma 9.1. Let

$$\mathbf{T} = \begin{pmatrix} a & d & c & b \\ b & a & d & c \\ c & b & a & d \\ d & c & b & a \end{pmatrix} \tag{9.2}$$

Then, for

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = \mathbf{W} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}, \tag{9.3}$$

where **W** is the DFT matrix,

$$\mathbf{T} = \mathbf{W} \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & \delta \end{pmatrix} \mathbf{W}^{-1}$$
 (9.4)

Let

$$\mathbf{x} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}; \ \mathbf{X} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = \mathbf{W}\mathbf{x}; \ \mathbf{y} = \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} \tag{9.5}$$

Then

$$\mathbf{Y} = \mathbf{W}\mathbf{y} = \mathbf{W}\mathbf{T}\mathbf{x} \tag{9.6}$$

$$= \mathbf{W}\mathbf{T}\mathbf{W}^{-1}\mathbf{X} \tag{9.7}$$

$$= \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & \delta \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} \tag{9.8}$$

$$= \begin{pmatrix} \alpha^2 \\ \beta^2 \\ \gamma^2 \\ \delta^2 \end{pmatrix} \tag{9.9}$$

upon substituting from (9.4) and (9.5). Therefore option (A) is the correct option.

10. The input x(t) and output y(t) of a continuous time signal are related as

$$y(t) = \int_{t-T}^{t} x(u) \, du \tag{10.1}$$

The system is:

- a) Linear and Time-variant
- b) Linear and Time-invariant
- c) Non-Linear and Time-variant
- d) Non-Linear and Time-invariant

Solution:

Definition 3. We say that a system is **linear** if and only if it follows the Principle of Superposition, i.e Law of Additivity and Law of Homogeneity.

Definition 4. A system is said to be **time** invariant if the output signal does not depend on the absolute time, i.e a time delay on the input signal directly equates to the delay in the output signal.

Lemma 10.1. The system relating the input signal x(t) and output signal y(t), given by

$$y(t) = \int_{t-T}^{t} x(u) \, du \tag{10.2}$$

is linear and time invariant in nature.

Proof. a) Linearity and Time invariance

From $(\overline{3})$, we can say the system is linear if it follows both the laws of Additivity and Homogeneity.

Law of Additivity:

Let the two input signals be $x_1(t)$ and $x_2(t)$, and their corresponding output signals be $y_1(t)$ and $y_2(t)$, then:

$$y_1(t) = \int_{t-T}^{t} x_1(u) du$$
(10.3)

$$y_2(t) = \int_{t-T}^{t} x_2(u) du$$
 (10.4)

$$y_1(t) + y_2(t) = \int_{t-T}^t [x_1(u) + x_2(u)] du$$
(10.5)

Now, consider the input signal of $x_1(t)+x_2(t)$, then the corresponding output signal is given by y'(t):

$$y'(t) = \int_{t-T}^{t} [x_1(u) + x_2(u)] du \qquad (10.6)$$

Clearly, from (10.5) and (10.6):

$$y'(t) = y_1(t) + y_2(t)$$
 (10.7)

Thus, the Law of Additivity holds.

Law of Homogeneity:

Consider an input signal kx(t), where k is any constant. Let the corresponding output

be given by y'(t), then:

$$y'(t) = \int_{t-T}^{t} kx(u) \, du \tag{10.8}$$

$$= k \int_{t-T}^{t} x(u) \, du \tag{10.9}$$

$$= ky(t) \tag{10.10}$$

Clearly, from (10.10),

$$y'(t) = ky(t)$$
 (10.11)

Thus, the Law of Homogeneity holds.

Since both the Laws hold, the system satisfies the Principle of Superposition, and is thus, a **linear system**.

From (4), to check for time-invariance, we would introduce a delay of t_0 in the output and input signals.

Delay in output signal:

$$y(t - t_0) = \int_{t - t_0 - T}^{t - t_0} x(u) \, du \qquad (10.12)$$

Now, we consider an input signal with a delay of t_0 , given by $x(t - t_0)$, and let the corresponding output signal be given by y'(t), then:

$$y'(t) = \int_{t-T}^{t} x(u - t_0) du$$
 (10.13)

Substituting $a = u - t_0$:

$$y'(t) = \int_{t-t_0-T}^{t-t_0} x(a) \, da \tag{10.14}$$

Clearly, from (10.12) and (10.14):

$$y'(t) = y(t - t_0) \tag{10.15}$$

Thus, the system is **time-invariant**. The correct option is **2**) **Linear and Time-invariant**

b) Calculating impulse response of LTI system

Since the given system is an LTI system, it would possess an impulse response h(t), which is the output of the system when the input signal is the Impulse function, given

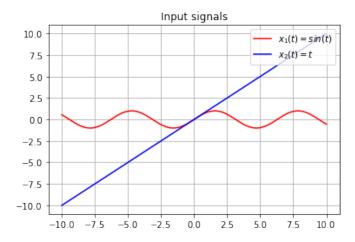


Fig. 10.1: $x_1(t) = \sin t$ and $x_2(t) = t$

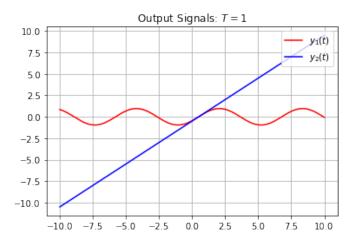


Fig. 10.2: $y_1(t)$ and $y_2(t)$

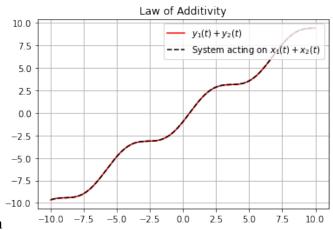


Fig. 10.3: Law of Additivity

by $\delta(t)$. Thus,

$$h(t) = \int_{t-T}^{t} \delta(u) du \qquad (10.16)$$

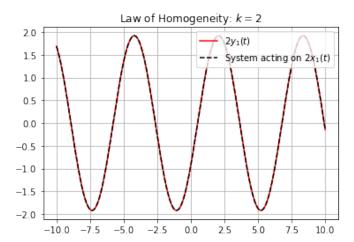


Fig. 10.4: Law of Homogeneity

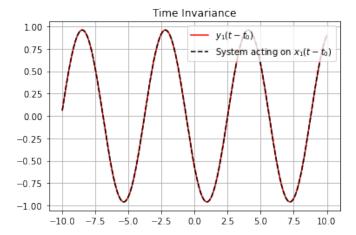


Fig. 10.5: Time invariance

The Impulse function can be loosely defined as:

$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & otherwise \end{cases} and \int_{-\infty}^{\infty} \delta(t)dt = 1$$
(10.17)

Since the Impulse function is zero everywhere aside from t = 0, the non-zero value of integration is a result of $\delta(0)$. Thus, we can say h(t) will be non-zero only if the limits of integration would include t = 0, i.e.

$$h(t) = \begin{cases} \int_{t-T}^{t} \delta(u) du & t - T < 0; t > 0\\ 0 & otherwise \end{cases}$$
(10.18)

$$h(t) = \begin{cases} 1 & 0 < t < T \\ 0 & otherwise \end{cases}$$
 (10.19)

c) Expressing the impulse function in terms of u(t)

The unit step signal, u(t), is given by:

$$u(t) = \begin{cases} 1 & t \ge 0 \\ 0 & otherwise \end{cases}$$
 (10.20)

On time-shifting u(t) by T, we get:

$$u(t-T) = \begin{cases} 1 & t-T \ge 0 \\ 0 & otherwise \end{cases} = \begin{cases} 1 & t \ge T \\ 0 & otherwise \end{cases}$$
(10.21)

On subtracting (10.20) and (10.21), we get our impulse response h(t) in terms of the unit step signal:

$$h(t) = u(t) - u(t - T)$$
 (10.22)

d) Expressing the impulse function in terms of rect(t). The unit rectangular signal, rect(t) is given by:

$$rect(t) = \begin{cases} 1 & \frac{-1}{2} \le t \le \frac{1}{2} \\ 0 & otherwise \end{cases}$$
 (10.23)

We can obtain the impulse response h(t) in terms of rect(t) using time scaling and shifting as follows:

$$rect\left(\frac{t}{\tau}\right) = \begin{cases} 1 & \frac{-1}{2} \le \frac{t}{\tau} \le \frac{1}{2} \\ 0 & otherwise \end{cases} = \begin{cases} 1 & \frac{-\tau}{2} \le t \le \frac{\tau}{2} \\ 0 & otherwise \end{cases}$$

$$(10.24)$$

Substituting $\tau = T$:

$$rect\left(\frac{t}{T}\right) = \begin{cases} 1 & \frac{-T}{2} \le t \le \frac{T}{2} \\ 0 & otherwise \end{cases}$$
 (10.25)

Now, we want to right-shift the signal by $\frac{T}{2}$:

$$rect\left(\frac{1}{T}\left(t - \frac{T}{2}\right)\right) = \begin{cases} 1 & 0 \le t \le T \\ 0 & otherwise \end{cases} = h(t)$$
(10.26)

Since the time shifting is to be performed on the variable t and not $\frac{t}{T}$

e) Calculating the Fourier Transform of h(t)Let the Fourier Transform of h(t) be given by H(f) and of the rectangular signal, rect(t) be given by Y(f).

$$h(t) \stackrel{\mathcal{F}}{\rightleftharpoons} H(f)$$
 (10.27)

$$rect(t) \stackrel{\mathcal{F}}{\rightleftharpoons} Y(f)$$
 (10.28)

Then,

$$Y(f) = \int_{-\infty}^{\infty} rect(t)e^{-j2\pi ft} dt \qquad (10.29)$$

From (10.23), we can write (10.29) as:

$$Y(f) = \int_{-\infty}^{\frac{-1}{2}} 0 \, dt + \int_{\frac{-1}{2}}^{\frac{1}{2}} e^{-j2\pi ft} \, dt + \int_{\frac{1}{2}}^{\infty} 0 \, dt \quad (10.30)$$

$$= \frac{e^{j\pi f} - e^{-j\pi f}}{j2\pi f} \quad (10.31)$$

$$= \frac{2j\sin\pi f}{j2\pi f} \quad (10.32)$$

$$= \frac{\sin(\pi f)}{\pi f} \quad (10.33)$$

$$= sinc(f) \quad (10.34)$$

where sinc(t), the sampling function is defined as:

$$sinc(t) = \begin{cases} 1 & t = 0\\ \frac{\sin(\pi t)}{\pi t} & otherwise \end{cases}$$
 (10.35)

Let the Fourier Transform of a signal x(t) be X(f).

$$x(t) \stackrel{\mathcal{F}}{\rightleftharpoons} X(f) \tag{10.36}$$

When the signal x(t) is time shifted by t_0 , the resultant Fourier Transform is given by:

$$x(t \pm t_0) \stackrel{\mathcal{F}}{\rightleftharpoons} X(f)e^{\pm j2\pi f t_0}$$
 (10.37)

And when the signal x(t) is time scaled by α , the resulting Fourier Transform is given by:

$$x(\alpha t) \stackrel{\mathcal{F}}{\rightleftharpoons} \frac{1}{|\alpha|} X \left(\frac{f}{\alpha} \right)$$
 (10.38)

Since we have already derived the Fourier Transform of rect(t), we would use the properties mentioned above to find the Fourier Transform of h(t):

$$rect(t) \stackrel{\mathcal{F}}{\rightleftharpoons} sinc(f)$$
 (10.39)

Using (10.37):

$$rect\left(t - \frac{T}{2}\right) \stackrel{\mathcal{F}}{\rightleftharpoons} sinc(f)e^{-j(2\pi f)\frac{T}{2}}$$
 (10.40)

$$rect\left(t - \frac{T}{2}\right) \stackrel{\mathcal{F}}{\rightleftharpoons} sinc(f)e^{-j\pi fT}$$
 (10.41)

Using (10.38),

$$rect\left(\frac{1}{T}\left(t - \frac{T}{2}\right)\right) \stackrel{\mathcal{F}}{\rightleftharpoons} \frac{1}{\frac{1}{|T|}} sinc\left(\frac{f}{T}\right) e^{\frac{-j\pi fT}{T}}$$
 (10.42)

$$h(t) \stackrel{\mathcal{F}}{\rightleftharpoons} T sinc\left(\frac{f}{T}\right) e^{-j\pi f}$$
 (10.43)

$$\therefore H(f) = T \operatorname{sinc}\left(\frac{f}{T}\right) e^{-j\pi f} \qquad (10.44)$$

f) An example

Consider an input signal of $x(t) = \cos 2\pi f_0 t$. The Fourier Transform of x(t) is given by:

$$x(t) = \cos 2\pi f_0 t \stackrel{\mathcal{F}}{\rightleftharpoons} \frac{1}{2} \left[\delta(f - f_0) + \delta(f + f_0) \right]$$
(10.45)

using the fact that

$$\cos 2\pi f_0 t = \frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2}$$
 (10.46)

and the Fourier Transform of $e^{\pm j2\pi f_0 t}$ is given by:

$$e^{\pm j2\pi f_0 t} \stackrel{\mathcal{F}}{\rightleftharpoons} \delta(f \mp f_0)$$
 (10.47)

The output signal will be given by:

$$y(t) = \int_{t-T}^{t} \cos 2\pi f_0 u \, du \qquad (10.48)$$

$$= \frac{1}{2\pi f_0} \left[\sin 2\pi f_0 t - \sin 2\pi f_0 (t - T) \right]$$
 (10.49)

$$= \frac{\sin \pi f_0 T}{\pi f_0} \left[\cos 2\pi f_0 \left(t - \frac{T}{2} \right) \right]$$
 (10.50)

$$= T sinc(f_0 T) \cos 2\pi f_0 \left(t - \frac{T}{2} \right)$$
 (10.51)

The Fourier transform of $\cos 2\pi f_0 \left(t - \frac{T}{2}\right)$ can be obtained using (10.38) and (10.37) as follows:

$$\cos t = \frac{1}{2} \left[e^{jt} + e^{-jt} \right]$$

$$(10.52)$$

$$\cos t \stackrel{\mathcal{F}}{\rightleftharpoons} \frac{1}{2} \left[\delta \left(f - \frac{1}{2\pi} \right) + \delta \left(f + \frac{1}{2\pi} \right) \right]$$

$$(10.53)$$

$$\cos \left(t - \frac{T}{2} \right) \stackrel{\mathcal{F}}{\rightleftharpoons} \frac{e^{j\pi fT}}{2} \left[\delta \left(f - \frac{1}{2\pi} \right) + \delta \left(f + \frac{1}{2\pi} \right) \right]$$

$$(10.54)$$

$$\cos 2\pi f_0 \left(t - \frac{T}{2} \right) \stackrel{\mathcal{F}}{\rightleftharpoons} \frac{e^{j\pi \frac{f}{2\pi f_0}T}}{2\pi f_0} \frac{\delta(\frac{f}{2\pi f_0} - \frac{1}{2\pi}) + \delta(\frac{f}{2\pi f_0} + \frac{1}{2\pi})}{2}$$

$$(10.55)$$

$$\cos 2\pi f_0 \left(t - \frac{T}{2} \right) \stackrel{\mathcal{F}}{\rightleftharpoons} \frac{e^{j\pi \frac{f}{2f_0}T}}{4\pi f_0} \left(\delta \left(\frac{f - f_0}{2\pi f_0} \right) + \delta \left(\frac{f + f_0}{2\pi f_0} \right) \right)$$

$$(10.56)$$

Therefore, the Fourier Transform of the output signal

y(t) from (10.51) is given by:

$$y(t) \stackrel{\mathcal{F}}{\rightleftharpoons} \frac{T \operatorname{sinc}(f_0 T)}{4\pi f_0} e^{j\pi \frac{f}{2f_0} T} \left(\delta \left(\frac{f - f_0}{2\pi f_0} \right) + \delta \left(\frac{f + f_0}{2\pi f_0} \right) \right)$$

$$(10.57)$$

$$y(t) \stackrel{\mathcal{F}}{\rightleftharpoons} k e^{j\pi \frac{f}{2f_0} T} \left(\delta \left(\frac{f - f_0}{2\pi f_0} \right) + \delta \left(\frac{f + f_0}{2\pi f_0} \right) \right)$$

$$(10.58)$$

where $k = \frac{T sinc(f_0 T)}{4\pi f_0}$. Substituting $2\pi f_0 = 1$ and T = 1:

$$y(t) \stackrel{\mathcal{F}}{\rightleftharpoons} ke^{j\pi^2 f} \left(\delta \left(f - \frac{1}{2\pi} \right) + \delta \left(f + \frac{1}{2\pi} \right) \right)$$
 (10.59)

$$y(t) \stackrel{\mathcal{F}}{\rightleftharpoons} ke^{j\frac{\pi}{2}} \delta\left(f - \frac{1}{2\pi}\right) + ke^{j-\frac{\pi}{2}} \delta\left(f + \frac{1}{2\pi}\right) \quad (10.60)$$

using the multiplication property of the Delta function:

$$x(t)\delta(t - t_1) = x(t_1)\delta(t - t_1)$$
 (10.61)

Since, $e^{j\frac{\pi}{2}} = j$ and $e^{-j\frac{\pi}{2}} = -j$, we finally get:

$$y(t) \stackrel{\mathcal{F}}{\rightleftharpoons} kj \left[\delta \left(f - \frac{1}{2\pi} \right) - \delta \left(f + \frac{1}{2\pi} \right) \right]$$
 (10.62)

Clearly, the Fourier transform of y(t) can be manipulated to represent a sinusoidal wave, which is given

$$sin(t) \stackrel{\mathcal{F}}{\rightleftharpoons} \frac{-j}{2} \left[\delta \left(f - \frac{1}{2\pi} \right) - \delta \left(f + \frac{1}{2\pi} \right) \right]$$
 (10.63)

The attenuation happens for the same values of f, as depicted in the graphs of the Fourier Transforms given below.

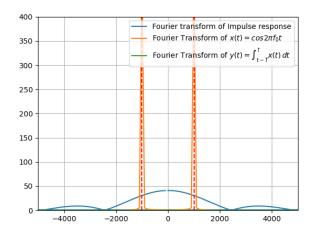


Fig. 10.6: Fourier Transform of Impulse response h(t)

11. Let the state-space representation on an LTI system be $\dot{x}(t) = Ax(t) + Bu(t)$, y(t) = Cx(t) +

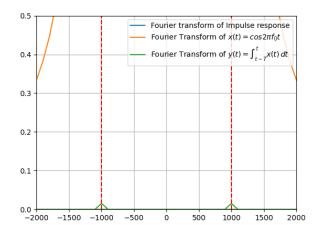


Fig. 10.7: Fourier Transform of Impulse response h(t) zoomed in

du(t) where A,B,C are matrices, d is a scalar, u(t) is the input to the system, and y(t) is its output. Let $B = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^{\top}$ and d = 0. Which one of the following options for A and C will ensure that the transfer function of this LTI system is

$$H(s) = \frac{1}{s^3 + 3s^2 + 2s + 1}$$
 (11.1)

(A)
$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix}$$
 and $C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$
(B) $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -2 & -1 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$
(C) $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$

(B)
$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -2 & -1 \end{pmatrix}$$
 and $C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$

(C)
$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix}$$
 and $C = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$

(D)
$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -2 & -1 \end{pmatrix}$$
 and $C = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$

Solution: From the given information,

$$\begin{pmatrix} \dot{x}(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} A & B \\ C & d \end{pmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}$$
 (11.2)

Taking Laplace transform on both sides,

$$\begin{pmatrix} sX(s) \\ Y(s) \end{pmatrix} = \begin{pmatrix} A & B \\ C & d \end{pmatrix} \begin{pmatrix} X(s) \\ U(s) \end{pmatrix}$$
(11.3)

$$\implies sX(s) = AX(s) + BU(s)$$
 (11.4)

$$\implies X(s) = (sI - A)^{-1}BU(s) \tag{11.5}$$

$$\implies Y(s) = CX(s) + dU(s)$$

$$= C(sI - A)^{-1}BU(s) + dU(s)$$
(11.7)

By definition,

$$Y(s) = H(s)U(s) \tag{11.8}$$

$$\implies H(s) = C(sI - A)^{-1}B + d$$
 (11.9)

$$=\frac{1}{s^3+3s^2+2s+1} \qquad (11.10)$$

$$\implies C(sI - A)^{-1}B + d = \frac{1}{s^3 + 3s^2 + 2s + 1}$$
(11.11)

Now we cross verify the options with eq 11.11. By using a python script,

(A)

$$C(sI - A)^{-1}B + d = \frac{1}{s^3 + 3s^2 + 2s + 1}$$
(11.12)

(B)

$$C(sI - A)^{-1}B + d = \frac{1}{s^3 + 1s^2 + 2s + 3}$$
(11.13)

(C)

$$C(sI - A)^{-1}B + d = \frac{s^2}{s^3 + 3s^2 + 2s + 1}$$
(11.14)

(D)

$$C(sI - A)^{-1}B + d = \frac{s^2}{s^3 + 1s^2 + 2s + 3}$$
(11.15)

Hence A is the correct option.

- 12. Consider a real-valued base-band signal x(t), band limited to 10 kHz. The Nyquist rate for the signal $y(t) = x(t)x(1 + \frac{t}{2})$ is
 - a) 15 kHz
 - b) 30 kHz
 - c) 60 kHz
 - d) 20 kHz

Solution: Let

$$x(t) = \cos(20k\pi t) = \cos\{2\pi(10k)t\}$$
(12.1)

$$\implies B_x = 10kHz \tag{12.2}$$

where B_x is the bandwidth of x(t). Then

$$y(t) = \cos(30k\pi t) + \cos(10k\pi t)$$
 (12.3)

$$\implies B_y = \frac{30}{2} \qquad = 15kHz \tag{12.4}$$

Thus, the Nyquist rate is

$$B = 2B_{v} \tag{12.5}$$

$$= 30kHz \tag{12.6}$$

Figs. 12.1- 12.5 show the sampling theorem in action. Fig. 12.5 shows how the violation of the the Nyquist criterion results in distortion during reconstruction.

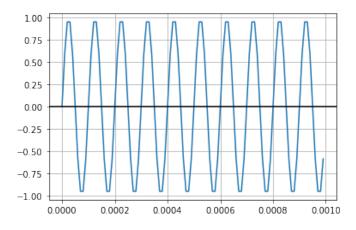


Fig. 12.1: x(t):Sinusoidal signal with freq=10kHz

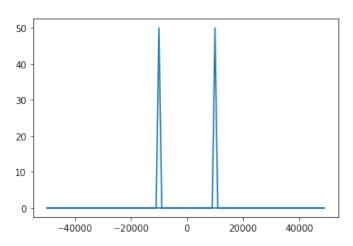


Fig. 12.2: DFT of x(t). Bandwidth = 10000

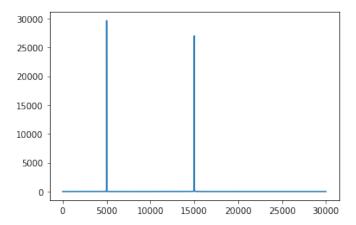


Fig. 12.3: DFT of y(t). Bandwidth = 15000

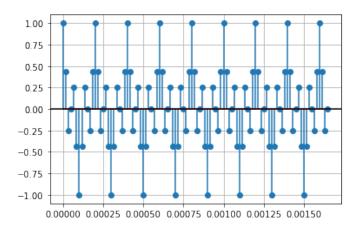


Fig. 12.4: stem plot of y(t) sampled at 60kHz

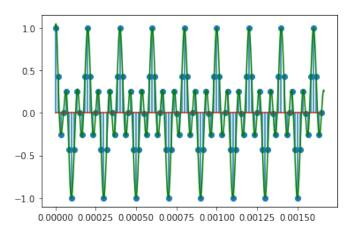


Fig. 12.5: Shannon interpolation of y(t)

13. Consider the signals $x[n] = 2^{n-1}u[-n + 2]$ and $y[n] = 2^{-n+2}u[n + 1]$, where u[n] is the unit step sequence. Let $X(e^{j\omega})$ and $Y(e^{j\omega})$ the discrete-time Fourier transform of x[n] and

y[n], respectively. The value of the integral

$$\frac{1}{2\pi} \int_0^{2\pi} X(e^{j\omega}) Y(e^{-j\omega}) d\omega \tag{13.1}$$

is Solution: Since

$$a^{n}u(-n-1) \stackrel{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1-az^{-1}} \quad |z| < |a|, \quad (13.2)$$

$$X(z) = \frac{4z^{-3}}{1 - 2z^{-1}} \quad |z| < 2, \tag{13.3}$$

$$Y(z) = \frac{8z}{1 - \frac{1}{2}z^{-1}} \quad |z| > \frac{1}{2}$$
 (13.4)

and

$$x(n) * y(-n) \stackrel{\mathcal{Z}}{\rightleftharpoons} \frac{32z^{-4}}{(1 - 2z^{-1})\left(1 - \frac{1}{2}z\right)} |z| < 2,$$

$$(13.5)$$

$$\implies z(n) = x(n) * y(-n) = (4 - n)2^{n+1}u[4 - n]$$

$$(13.6)$$

$$\therefore \frac{1}{2\pi} \int_0^{2\pi} X(e^{j\omega}) Y(e^{-j\omega}) d\omega = z[0] = 8$$
(13.7)