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EE22BTECH11059

Let (X, Y) have joint probability mass function

$$p(x, y) = \begin{cases} \frac{c}{2^{x+y+2}} & if x = 0, 1, 2, ... y = 0, 1, 2, ...; x \neq y \\ 0 & otherwise \end{cases} \tag{1}$$

Then which of the following is true?
(GATE ST 2023)

1) $c = \frac{1}{2}$
2) $c = \frac{1}{4}$
3) $c > 1$
4) X and Y are independent

Solution: For p(x, y) to be joint probability mass function

$$\sum_{y=-\infty}^{\infty} \sum_{x=-\infty}^{\infty} p(x, y) = 1 \, |x \neq y \tag{2}$$

$$\sum_{y=0}^{\infty} \sum_{x=0}^{\infty} \frac{c}{2^{x+y+2}} - \sum_{x=y} \frac{c}{2^{x+y+2}} = 1 \tag{3}$$

$$\sum_{y=0}^{\infty} \frac{c}{2^{y+2}} \sum_{x=0}^{\infty} 2^{-x} - \frac{c}{4} \sum_{x=0}^{\infty} \frac{1}{4^x} = 1 \tag{4}$$

$$\sum_{y=0}^{\infty} \frac{2c}{2^{y+2}} - \frac{c}{3} = 1 \tag{5}$$

$$\frac{2c}{4} \sum_{y=0}^{\infty} 2^{-y} - \frac{c}{3} = 1 \tag{6}$$

$$c - \frac{c}{3} = 1 \tag{7}$$

$$c = \frac{3}{2} \tag{8}$$

1) Marginal probability mass function of X

$$p_X(x) = \sum_{y=0}^{\infty} p(x, y) \, |x \neq y \tag{9}$$

$$= \sum_{y=0}^{\infty} \frac{3}{2^{x+y+3}} - p_{XY}(x, x) \tag{10}$$

$$= \frac{3}{2^{x+3}} \sum_{y=0}^{\infty} 2^{-y} - \frac{3}{2^{2x+3}} \tag{11}$$

$$= \frac{3}{2^{x+2}} - \frac{3}{2^{2x+3}} \tag{12}$$

2) Marginal cdf of X

$$F_X(x) = \sum_{i=0}^{x} p_X(X \leq x) \tag{13}$$

$$= \sum_{i=0}^{x} \left(\frac{3}{2^{x+2}} - \frac{3}{2^{2x+3}} \right) \tag{14}$$

$$= 1 + \frac{1}{2^{x+2}} - \frac{3}{2^{2x+3}} \tag{15}$$

3) Marginal probability mass function of Y

$$p_Y(y) = \sum_{x=0}^{\infty} p(x, y) \, |x \neq y \tag{16}$$

$$= \sum_{x=0}^{\infty} \frac{3}{2^{x+y+3}} - p_{XY}(y, y) \tag{17}$$

$$= \frac{3}{2^{y+3}} \sum_{x=0}^{\infty} 2^{-x} - \frac{3}{2^{2y+3}} \tag{18}$$

$$= \frac{3}{2^{y+2}} - \frac{3}{2^{2y+3}} \tag{19}$$

$$\tag{20}$$

4) Marginal cdf of Y

$$F_Y(y) = \sum_{i=0}^{y} p_Y(Y \leq y) \tag{21}$$

$$= \sum_{i=0}^{y} \left(\frac{3}{2^{y+2}} - \frac{3}{2^{2y+3}} \right) \tag{22}$$

$$= 1 + \frac{1}{2^{y+2}} - \frac{3}{2^{2y+3}} \tag{23}$$

5) For X and Y to be independent,

$$p(x, y) = p_X(x) p_Y(y) \tag{24}$$

$$p_X(x) p_Y(y) = \left(\frac{3}{2^{x+2}} - \frac{3}{2^{2x+3}}\right)\left(\frac{3}{2^{y+2}} - \frac{3}{2^{2y+3}}\right) \tag{25}$$

$$p(x, y) \neq p_X(x) p_Y(y) \tag{26}$$

Option (4) is incorrect.
∴ Only option (3) is correct.
Simulations:

1) Marginal pmf of X

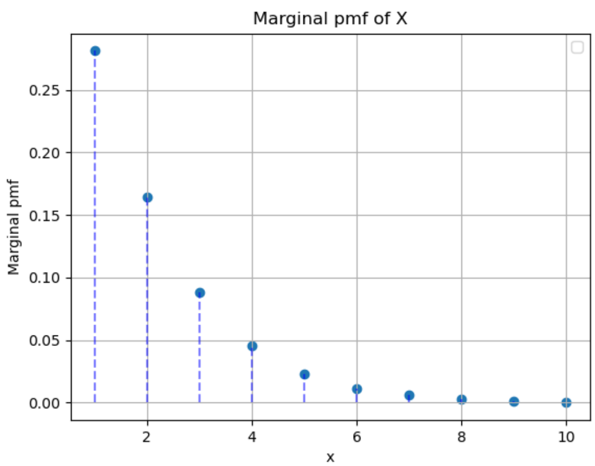

Fig. 1. Marginal pmf

Steps of simulation:

1) Define function p as $P(x, y) = \frac{3}{2^{x+y+3}}$ in C and run a loop to sum the function p for all values of y except $x \neq y$, for each value of x.
2) Store values of marginal pmf for each value of x in a dat file
3) Using plt.scatter function of python, plot the graph of Marginal pmf of X vs X. The same can be used to plot marginal pmf of Y