SIGNAL PROCESSING
Through GATE

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Contents

Introduction

This book provides solutions to signal processing problems in GATE.

Chapter 1

Harmonics

Chapter 2

Filters

2.1 For the circuit given below, choose the angular frequency ω_0 at which voltage across capacitor has maximum amplitude?
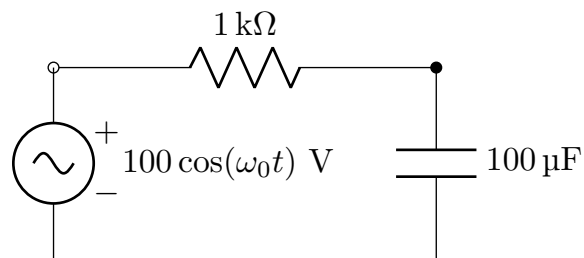

Figure 2.1: circuit

(A) 1000

(B) 100

(C) 1

(D) 0

Chapter 3

Z-transform

Chapter 4

Systems

4.1 Consider a unity-gain negative feedback system consisting of the plant $G(s)$ and a proportional-integral controller. Let the proportional gain and integral gain be 3 and 1, respectively. For a unit step reference input, the final values of the controller output and the plant output, respectively, are

$$G(s) = \frac{1}{(s-1)}$$

(GATE EE 2023)

Solution:

Parameter	Description	Value
K_p	Proportional Gain	3
K_i	Integral Gain	1
$r(t)$	Reference Input	$u(t)$
$w(t)$	Controller Output	?
$y(t)$	Plant Output	?
$e(t)$	Error Input	$r(t) - y(t)$

Table 1: Parameter Table

From theFig. 4.1:

$$E(s) = U(s) - Y(s) \tag{4.1}$$

$$W(s) = 3E(s) + \frac{1}{s}E(s) \tag{4.2}$$

$$Y(s) = G(s)W(s) \tag{4.3}$$

Some results:

$$tx(t) \xleftrightarrow{\mathcal{L}} -\frac{dX(s)}{ds} \tag{4.4}$$

$$e^{-at}x(t) \xleftrightarrow{\mathcal{L}} X(s+a) \tag{4.5}$$

By using (4.4) and (4.5):

$$e^{-t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+1}, Re(s) > -1 \tag{4.6}$$

$$te^{-t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{(s+1)^2}, Re(s) > -1 \tag{4.7}$$

Figure 4.1: Block Diagram of System

(a) **Plant Output:**

From (4.1) , (4.2) and (4.3):

$$Y(s) = \frac{3s+1}{s(s+1)^2}, Re(s) > -1 \tag{4.8}$$

Final Value Theorem:

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s) \tag{4.9}$$

Using (4.9) on Y(s):

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) \tag{4.10}$$

$$= 1 \tag{4.11}$$

Taking partial fraction of (4.8) :

$$Y(s) = \frac{1}{s} + \frac{2}{(s+1)^2} - \frac{1}{s+1} \tag{4.12}$$

Using (4.6) and (4.7):

$$\therefore y(t) = u(t) + 2te^{-t}u(t) - e^{-t}u(t) \tag{4.13}$$

(b) **Controller Output:**

From (4.2)

$$W(s) = \frac{3}{s} + \frac{1}{s^2} - Y(s)\left(3 + \frac{1}{s}\right) \tag{4.14}$$

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Substituting (4.8)

$$W\left(s\right) = \frac{\left(s-1\right)\left(3s+1\right)}{s\left(s+1\right)^2}, Re\left(s\right) > -1 \tag{4.15}$$

Using (4.9) on W(s)

$$\lim_{t\to\infty} w\left(t\right) = \lim_{s\to 0} sW\left(s\right) \tag{4.16}$$

$$= -1 \tag{4.17}$$

Taking partial fraction of equation(4.15) :

$$W\left(s\right) = -\frac{1}{s} - \frac{4}{\left(s+1\right)^2} + \frac{4}{s+1} \tag{4.18}$$

Using equations (4.6) and (4.7) and taking inverse lapalace transform:

$$w\left(t\right) = -u\left(t\right) - 4te^{-t}u\left(t\right) + 4e^{-t}u\left(t\right) \tag{4.19}$$

Figure 4.2: $w(t)$ converges at -1.

Figure 4.3: $y(t)$ converges at $+1$

4.2 Level (h) in a steam boiler is controlled by manipulating the flow rate (F) of the break-up(fresh) water using a proportional (P) controller. The transfer function between the output and the manipulated input is

$$\frac{h(s)}{F(s)} = \frac{0.25(1-s)}{s(2s+1)}$$

The measurement and the valve transfer functions are both equal to 1. A process engineer wants to tune the controller so that the closed loop response gives the decaying oscillations under the servo mode. Which one of the following is the CORRECT value of the controller gain to be used by the engineer?

(a) 0.25

(b) 2

(c) 4

(d) 6

GATE CH 2023

Solution:

4.3 The figure shows a block of mass m = 20 kg attached to a pair of identical linear springs, each having a spring constant k = 1000 N/m. The block oscillates on a frictionless horizontal surface. Assuming free vibrations, the time taken by the block to complete ten oscillations is _____ seconds . (Rounded off to two decimal places) Take π = 3.14. (GATE ME 2023)
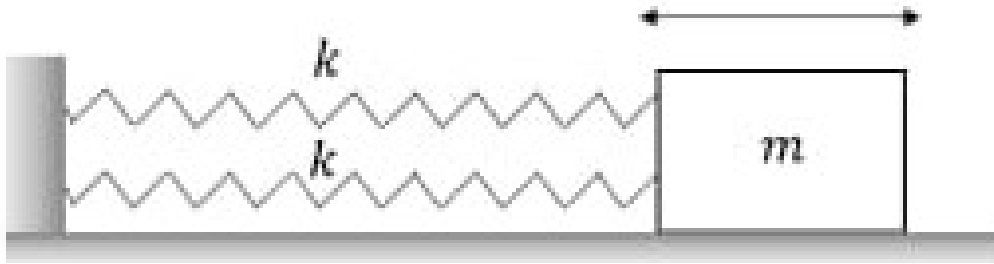
4.4 A system has transfer function

$$\frac{Y(s)}{X(s)} = \frac{s - \pi}{s + \pi}$$

let $u(t)$ be the unit step function.The input $x(t)$ that results in a steady-state output $y(t) = sin(\pi t)$ is __. (GATE IN 2023)

Solution:

Chapter 5

Sequences

5.1 Consider the discrete time signal $x[n] = u[-n+5] - u[n+3]$, where

$$u[n] = \begin{cases} 1; n \geq 0 \\ 0; n < 0 \end{cases}$$

The smallest n for which $x[n] = 0$ is?

Solution: From Fig. 1, the minimum value of n is given as

$$n = -3 \tag{5.1}$$
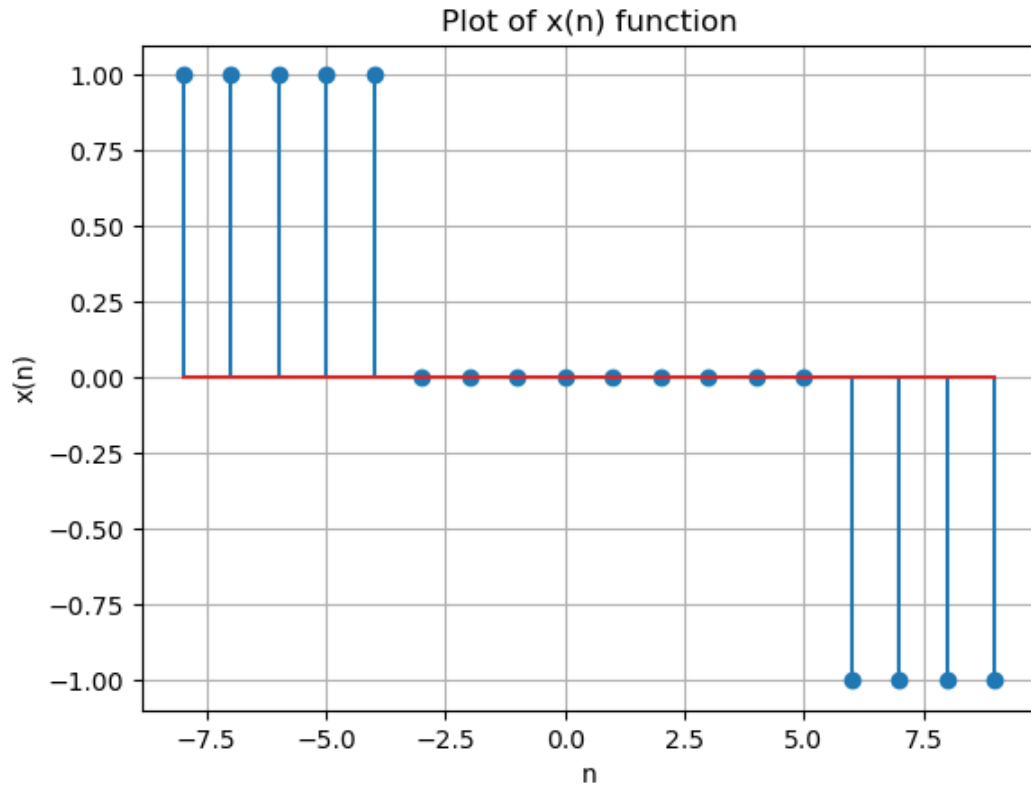
Figure 1: Plot of function $x(n)$ taken from python3

5.2 Two sequences $x_1[n]$ and $x_2[n]$ are described as follows:

$$x_1[0] = x_2[0] = 1 \tag{5.2}$$

$$x_1[1] = x_2[2] = 2 \tag{5.3}$$

$$x_1[2] = x_2[1] = 1 \tag{5.4}$$

$x_1[n] = x_2[n] = 0$ for all $n < 0$ and $n > 2$

If $x[n]$ is obtained by convoluting $x_1[n]$ with $x_2[n]$, which of the following equa-

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tions is/are TRUE?

(A) $x[2] = x[3]$

(B) $x[1] = 2$

(C) $x[4] = 3$

(D) $x[2] = 5$

Solution:

Chapter 6

Sampling

6.1 An 8 bit ADC converts analog voltage in the range of 0 to $+5\,V$ to the corresponding digital code as per the conversion characteristics shown in figure. For $V_{in} = 1.9922\,V$, which of the following digital output, given in hex, is true?

    (a) $64H$

    (b) $65H$

    (c) $66H$

    (d) $67H$

(GATE EE 40)

Solution:

Digital
Output in
Hex

03H

02H

01H
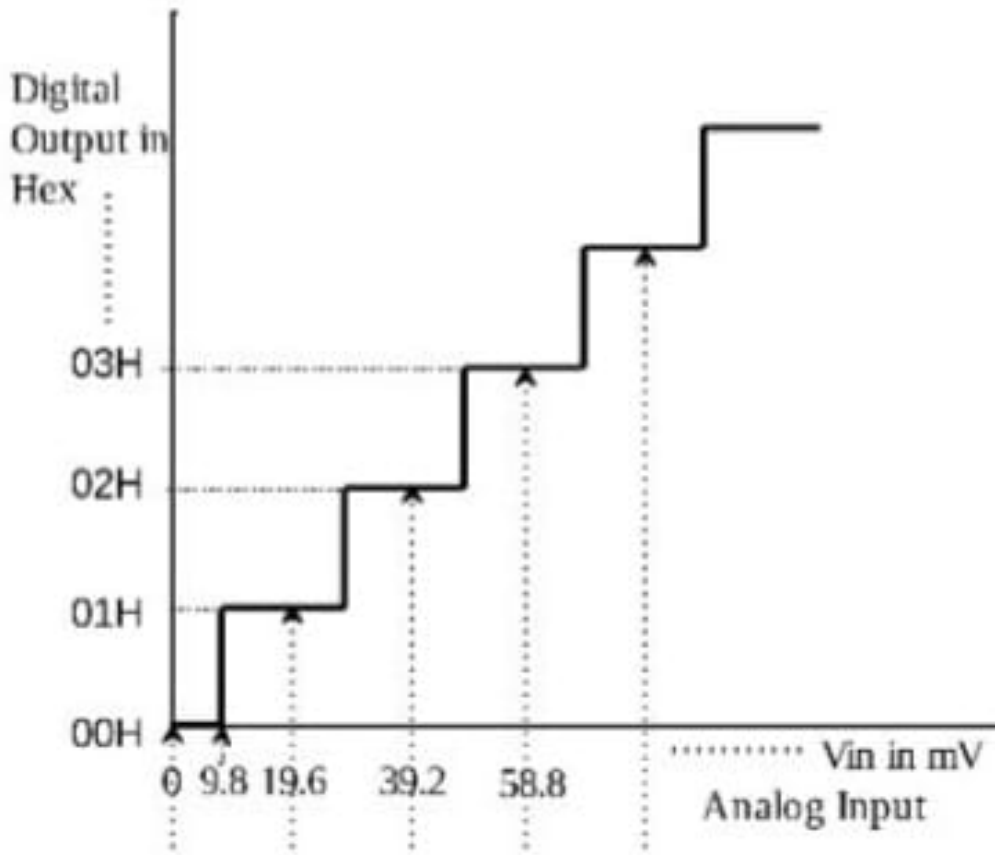
00H

0 9.8 19.6   39.2   58.8

Vin in mV
Analog Input

Figure 6.1:

Calculating the step-size:

$$\Delta V_{in} = \frac{V_{max} - V_{min}}{2^n - 1} \tag{6.1}$$

$$= \frac{5 - 0}{2^8 - 1} \tag{6.2}$$

$$= \frac{5}{255} \tag{6.3}$$

$$\implies V_{out} = \frac{V_{in}}{\Delta V_{in}} \tag{6.4}$$

$$= \frac{1.9922 \times 255}{5} \tag{6.5}$$

$$= 101.59 \tag{6.6}$$

$$\approx 102_{10} \tag{6.7}$$

Symbol	Value	Description
n	8	Number of bits of ADC
V_{min}	$0V$	Minimum Analog Voltage
V_{max}	$5V$	Maximum Analog Voltage
V_{in}	$1.9922V$	Input Voltage
V_{out}		Output Voltage

Table 6.1: Given Parameters
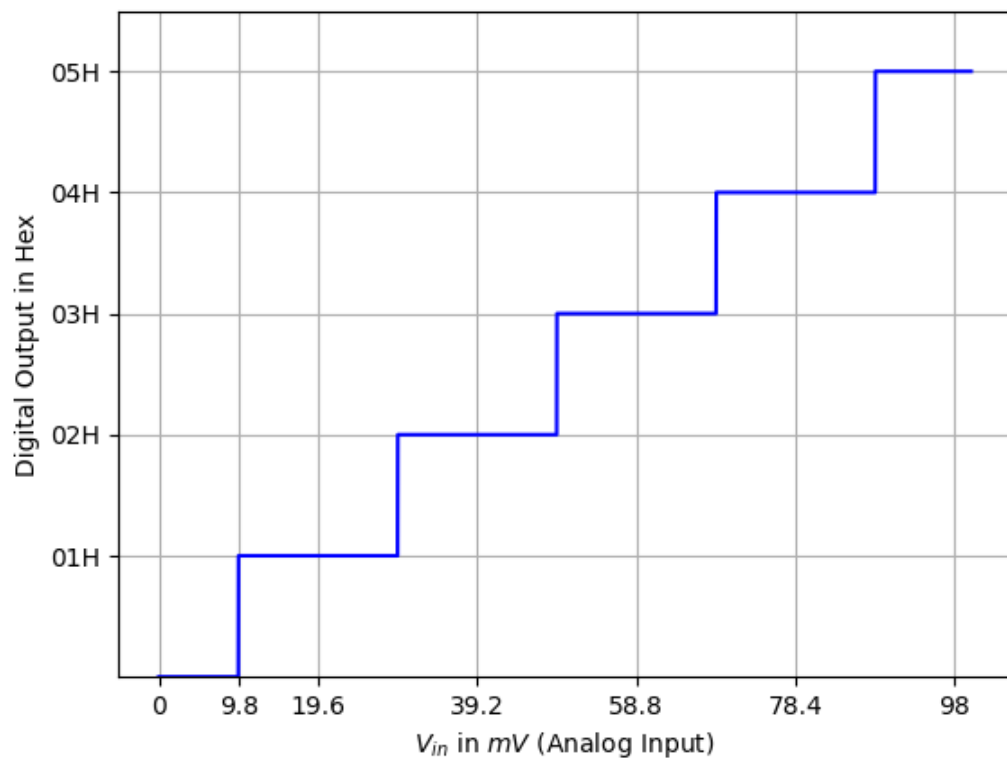
$\therefore$ correct answer is option (c).

Figure 6.2:

Chapter 7

Contour Integration

Chapter 8

Laplace Transform

8.1 The number of zeroes of the polynomial $P(s) = s^3 + 2s^2 + 5s + 80$ in the right side of the plane? (GATE IN 2023)

Solution: The table below shows the Routh array of the nth- order characteristic polynomial :

$$a_0 s^n + a_1 s^{n-1} ...... + a_{n-1} s^1 + a_n s^0 \tag{8.1}$$

s^n	a_0	a_2	a_4	...
s^{n-1}	a_1	a_3	a_5	...
s^{n-2}	$b_1 = \dfrac{a_1 a_2 - a_3 a_0}{a_1}$	$b_2 = \dfrac{a_1 a_4 - a_5 a_0}{a_1}$	...	..
s^{n-3}	$c_1 = \dfrac{b_1 a_3 - b_2 a_1}{b_1}$	$\vdots$		
$\vdots$	$\vdots$	$\vdots$		
s^1	$\vdots$	$\vdots$		
s^0	a_n			

Table 8.1: Routh Array

Characteristic Equation:

$$s^3 + 2s^2 + 5s + 80 = 0 \tag{8.2}$$

From Table 8.1:

| s^3	1	5
s^2	2	80
s^1	$\dfrac{2 \times 5 - 80 \times 1}{2} = -35$	
s^0	$\dfrac{-35 \times 80}{-35} = 80$	

Table 8.2:

From Table 8.2:

Since there are 2 sign changes in the first column of the Routh tabulation. So, the number of zeros in the right half of the s-plane will be 2.

Figure 8.1:

8.2 The circuit shown in the figure is initially in the steady state with the switch K in open condition and $\overline{K}$ in closed condition. The switch K is closed and $\overline{K}$ is opened simultaneously at the instant $t = t_1$, where $t_1 > 0$. The minimum value of t_1 in milliseconds such that there is no transient in the voltage across the 100 μF capacitor, is _____ (Round off to 2 decimal places) (GATE EE 2023)
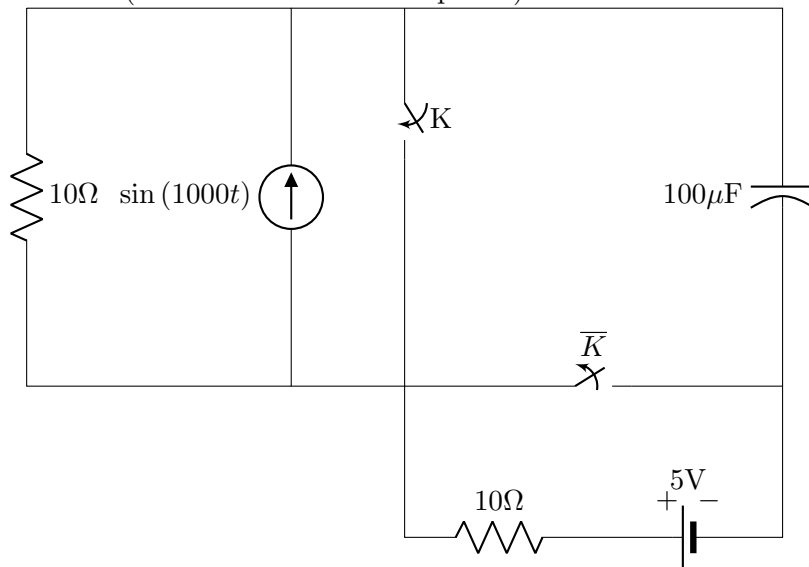
8.3 $y = e^{mx} + e^{-mx}$ is the solution of which differential equation?

1. $\frac{dy}{dx} - my = 0$

2. $\frac{dy}{dx} + my = 0$

3. $\frac{d^2y}{dx^2} + m^2y = 0$

4. $\frac{d^2y}{dx^2} - m^2y = 0$

(GATE AG 2023) **Solution:**

8.4 A cascade control strategy is shown in the figure below. The transfer function between the output (y) and the secondary disturbance (d_2) is defined as

$$G_{d2}(s) = \frac{y(s)}{d_2(s)}$$

. Which one of the following is the CORRECT expression for the transfer function $G_{d2}(s)$?

Figure 8.2:

A. $\frac{1}{(11s+21)(0.1s+1)}$

B. $\frac{1}{(s+1)(0.1s+1)}$

C. $\frac{(s+1)}{(s+2)(0.1s+1)}$

D. $\frac{(s+1)}{(s+1)(0.1s+1)}$

(GATE CH 2023) **Solution:**

8.5 In the differential equation $\frac{dy}{dx} + \alpha xy = 0$, α is a positive constant. If $y = 1.0$ at $x = 0.0$, and $y = 0.8$ at $x = 1.0$, the value of α is (rounded off to three decimal places). (GATE CE 2023) **Solution:**

8.6 The switch S_1 was closed and S_2 was open for a long time. At t=0,switch S_1 is opened and S_2 is closed,simultaneously. The value of $i_c(0^+)$, in amperes, is .   (GATE EC 2023)
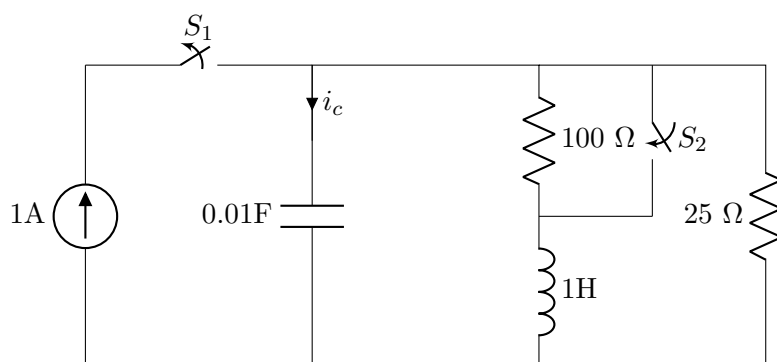
8.7 The continuous time signal $x(t)$ is described by:

$$x(t) = \begin{cases} 1, & \text{if } 0 \leq t \leq 1 \\ 0, & \text{elsewhere} \end{cases} \tag{8.3}$$

If $y(t)$ represents $x(t)$ convolved with itself, which of the following options is/are TRUE?

A $y(t) = 0$ for all $t < 0$

B $y(t) = 0$ for all $t > 1$

C $y(t) = 0$ for all $t > 3$

D $\int_{0.1}^{0.75} \frac{dy(t)}{dt} \, dt \neq 0$

Solution:

8.8 The Z-transform of a discrete signal $x(n)$ is

$$X(z) = \frac{4z}{\left(z - \frac{1}{5}\right)\left(z - \frac{2}{3}\right)(z - 3)} \quad \text{with ROC} = R \qquad (8.4)$$

Which one of the following statements is TRUE?

(a) Discrete time Fourier transform of $x[n]$ converges if R is $|z| > 3$

(b) Discrete time Fourier transform of $x[n]$ converges if R is $\frac{2}{3} < |z| < 3$

(c) Discrete time Fourier transform of $x[n]$ converges if R is such that $x[n]$ is a left-sided sequence.

(d) Discrete time Fourier transform of $x[n]$ converges if R is such that $x[n]$ is a right-sided sequence.

GATE EE 2023 **Solution:**

8.9 The phase margin of the transfer function $G(s) = \frac{2(1-s)}{(1+s)^2}$ is _____ degrees. (rounded off to the nearest integer). (GATE IN 2023)

Solution:

The gain crossover frequency ω_g is found from $|G(j\omega)| = 1$.

$$|G(j\omega)| = \frac{2|1-j\omega|}{|1+j\omega|^2} = \frac{2\sqrt{1+\omega^2}}{1+\omega^2} = \frac{2}{\sqrt{1+\omega^2}}$$

Setting this equal to 1:

$$\sqrt{1+\omega^2} = 2 \implies 1+\omega^2 = 4 \implies \omega^2 = 3 \implies \omega_g = \sqrt{3}$$

The phase at $\omega_g = \sqrt{3}$:

$$\angle G(j\omega) = \angle(1-j\omega) - 2\angle(1+j\omega) = -\tan^{-1}(\sqrt{3}) - 2\tan^{-1}(\sqrt{3})$$

$$= -60^\circ - 2(60^\circ) = -60^\circ - 120^\circ = -180^\circ$$

Phase margin:

$$PM = 180^\circ + \angle G(j\omega_g) = 180^\circ + (-180^\circ) = 0^\circ$$

Chapter 9

Fourier transform

9.1 The discrete-time Fourier transform of a signal $\mathrm{x}\,[n]$ is $X\,(\Omega)\;=\;(1+\cos\Omega)\,e^{-j\Omega}$. Consider that $x_p\,[n]$ is a periodic signal of period $N = 5$ such that

$$x_p\,[n] = x[n], \text{for n= 0, 1, 2} \tag{9.1}$$

$$= 0, \text{for n= 3, 4} \tag{9.2}$$

Note that $x_p\,[n] = \sum_{k=0}^{N-1} a_k e^{j\frac{2\pi}{N}kn}$. The magnitude of the Fourier series coefficient a_3 is _____ (Round off to 3 decimal places). (GATE EE 2023) **Solution:**

9.2 Let a frequency modulated (FM) signal : $x(t) = A\cos(\omega_c t + k_f \int_{-\infty}^{t} m(\lambda)d\lambda)$, where $m(t)$is a message signal of bandwidth W. It is passed through a non-linear system with output $y(t) = 2x(t) + 5(x(t))^2$. Let B_T denote the FM bandwidth. The minimum value of ω_c required to recover $x(t)$ from $y(t)$ is:

(A) $B_T + W$

(B) $\dfrac{3}{2}B_T$

(C) $2B_T + W$

(D) $\dfrac{5}{2}B_T$

Solution:

9.3 Let an input $x[n]$ having discrete-time Fourier transform $X(e^{j\Omega}) = 1 - e^{-j\Omega} + 2e^{-3j\Omega}$ be passed through an LTI system. The frequency response of the LTI system is $H(e^{j\Omega}) = 1 - \frac{1}{2}e^{-2j\Omega}$ The output $y[n]$ of the system is

(GATE EC 2023) **Solution:**

9.4 A continuous real-valued signal $x(t)$ has finite positive energy and $x(t) = 0$, $\forall\, t < 0$. From the list given below, select ALL the signals whose continuous-time Fourier transform is purely imaginary.

(a) $x(t) + x(-t)$

(b) $x(t) - x(-t)$

(c) $j(x(t) + x(-t))$

(d) $j(x(t) - x(-t))$

(GATE IN 2023)

Solution: