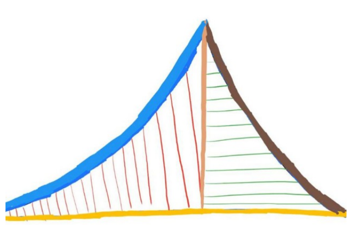
SIGNAL PROCESSING

Through GATE

G. V. V. Sharma



Contents

Introduction

This book provides solutions to signal processing problems in GATE.

Chapter 1

Harmonics

1.1 Let y(t)=x($4t$),where x(t) is a continous-time periodic signal of 100s.the fundamental period of y(t) is (**rounded off to the nearest integer**) (GATE IN 2023)

Solution:

Symbol	Value	Description
T	100	fundamental period of $x(t)$
T_1		fundamental period of y(t)
ω_0	$\frac{8\pi}{100}$	fundamental frequency of y(t)

Table 1: input parameters

From Table 1

Applying Fourier series:

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{j\,2\pi n\,t}{100}} \tag{1.1}$$

$$y(t) = x(4t) \tag{1.2}$$

$$y(t) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{j\,2\pi n\,(4t)}{100}} \tag{1.3}$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{\frac{j\,2\pi n\,t}{25}} \tag{1.4}$$

$$T_1 = 25\text{sec} \tag{1.5}$$

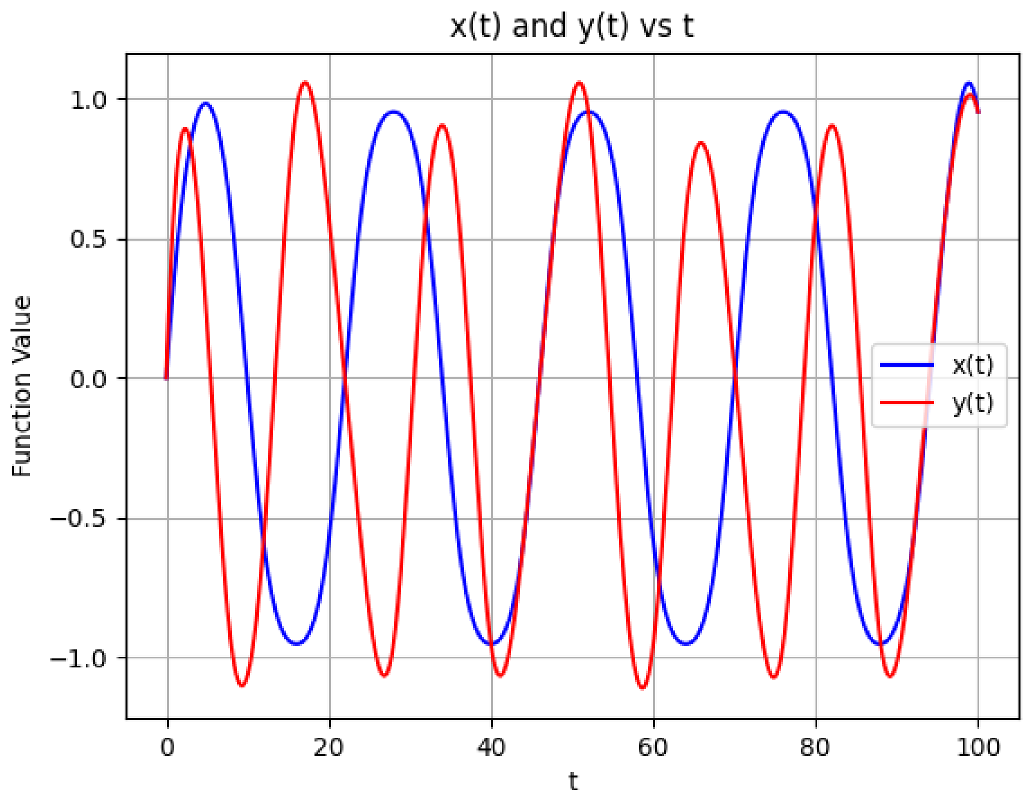
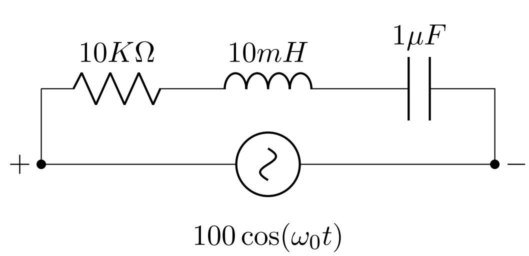


Figure 1: plot y(t) v/s t

1.2 In the circuit shown below, it is observed that the amplitude of voltage across the resistor is the same as the amplitude of the sorce voltage. What is the angular frequency ω_0(in rad/s)?



(A) 10^4

(B) 10^3

(C) $10^3\pi$

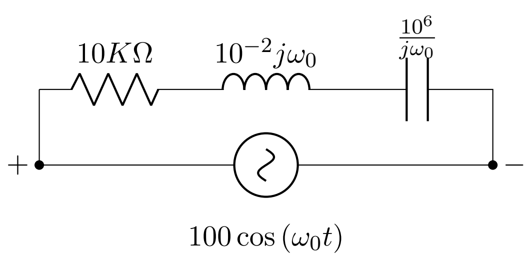
(D) $10^4\pi$

(GATE BM 2023)

Solution:

Symbols	Parameters	Value
R	Resistance	$10K\Omega$
L	Inductance	$10mH$
C	Capacitance	$1\mu F$
ω_0	Angular Frequency	
V_s	Source Voltage	

Table 1.2: Parameter Table



We have:

$$V_R = V_s \tag{1.6}$$

Using KVL:

$$V_s = V_R + V_C + V_L \tag{1.7}$$

By using (1.6) in (1.7):

$$V_C = -V_L \tag{1.8}$$

$$X_C = -X_L \tag{1.9}$$

$$\frac{1}{j\omega_0 C} = -j\omega_0 L \tag{1.10}$$

$$\frac{1}{LC} = -j^2\omega_0^2 \tag{1.11}$$

$$\omega_0^2 = \frac{1}{LC} \tag{1.12}$$

$$\omega_0 = \frac{1}{\sqrt{LC}} \tag{1.13}$$

$$= \frac{1}{\sqrt{10^{-2} \times 10^{-6}}} \tag{1.14}$$

$$= \frac{1}{10^{-4}} \tag{1.15}$$

$$= 10^4 rad/s \tag{1.16}$$

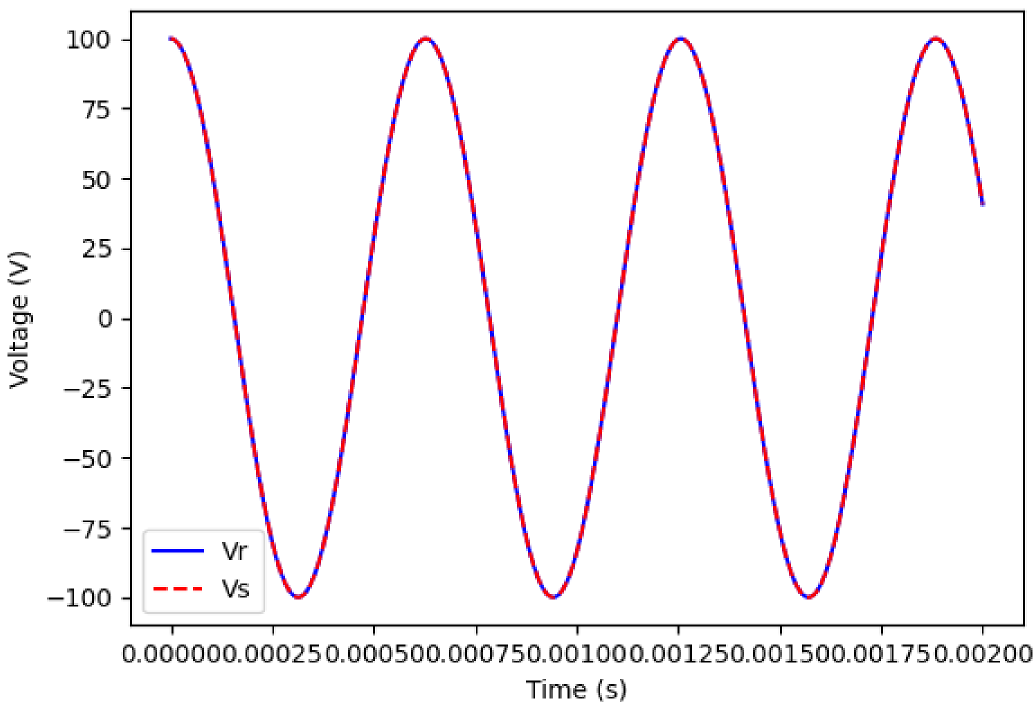


Figure 1.2: Voltage across Resistor and Source voltage

1.3 For a regular sinusoidal wave propagating in deep water having wave height of 3.5 m and wave period of 9 s, the wave steepness is _____ (round off to three decimal places). Gate 2023 NM 33

Solution:

(a) Deriving the formula for wavelength of deep water wave:

$$S = \frac{H}{\lambda} \tag{1.17}$$

Let's start with the linearized shallow water wave equation.

Symbol	Value	Description
H	$3.5m$	wave height
T	$9s$	wave period
S	?	wave steepness
λ		wave length
η		surface elevation of water

Table 1.3: Input Parameters

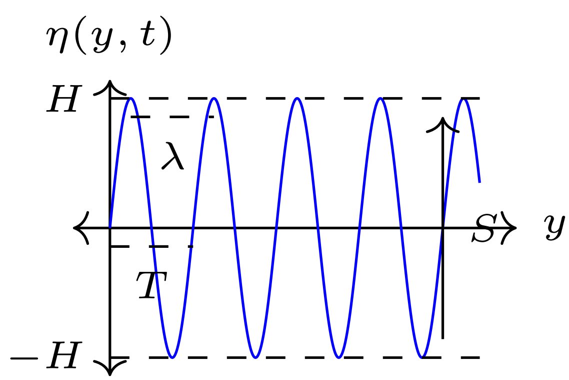


Figure 1.3: Sinusoidal wave

$$\frac{\partial^2 \eta}{\partial t^2} = g\frac{\partial \eta}{\partial y} \tag{1.18}$$

$$\eta = A\sin(ky - \omega t) \tag{1.19}$$

$$\frac{\partial^2 \eta}{\partial t^2} = -(\omega)^2 A\sin(ky - \omega t) \tag{1.20}$$

For deep water waves:

$$\frac{\partial \eta}{\partial y} \approx -k\eta \tag{1.21}$$

Using the equation (1.18).

$$\frac{\partial^2 \eta}{\partial t^2} = -gk\eta \tag{1.22}$$

$$= -gkA\sin(ky - \omega t) \tag{1.23}$$

where, k is wave number.

Comparing equations (1.20) and (1.23),

$$\omega^2 = gk \tag{1.24}$$

$$\omega = \frac{2\pi}{T} \tag{1.25}$$

$$k = \frac{2\pi}{\lambda} \tag{1.26}$$

$$\lambda = \frac{g \cdot T^2}{2\pi} \tag{1.27}$$

(b) Numerical computation:

$$\lambda = \frac{g \cdot T^2}{2\pi} \tag{1.28}$$

$$= \frac{9.81\,(9)^2}{2\pi} \tag{1.29}$$

$$= 126.53m \tag{1.30}$$

Using the equation (1.17).

$$S = \frac{3.5}{126.53} \tag{1.31}$$

$$= 0.028 \tag{1.32}$$

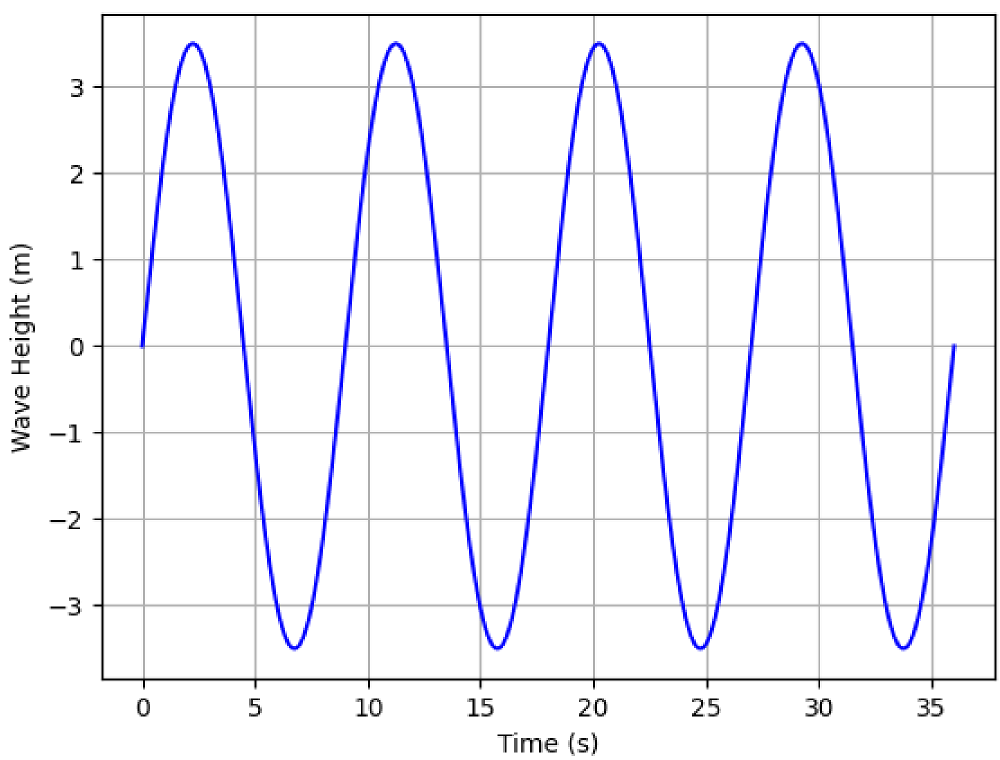
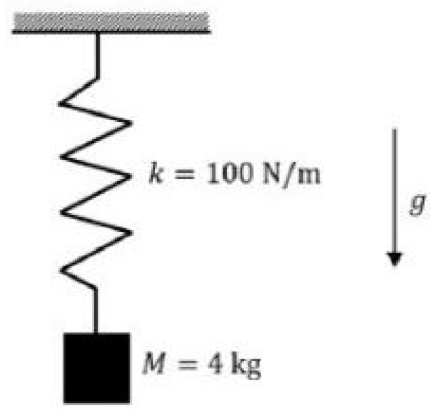


Figure 1.4: Sinusoidal wave

1.4 A spring mass system is shown in the figure . Take the value of acceleration due to gravity as $g = 9.81m/s^2$.The static deflection due to weight and the time period of the oscillations,respectively,are

(GATE 2023 XE)



Solution:

(a) Static deflection due to weight(sdw)

let x be sdw.

0 At mean position in equilibrium

1.4

$$Mg = kx \tag{1.33}$$

$$x = 39.24cm \tag{1.34}$$

(b) Time period of oscillation

$$F = -kx \tag{1.35}$$

$$m\left(\frac{d^2x}{dt^2}\right) = -kx \tag{1.36}$$

Initial Conditions be at extreme point of SHM

$$x(0) = 0.3924 \tag{1.37}$$

$$\frac{dx}{dt} = 0 \text{ at } t = 0 \text{ (released from rest)} \tag{1.38}$$

Taking Laplace transform:

$$m(s^2X(s) - sx(0) - mx'(0)) + kX(s) = 0 \tag{1.39}$$

$$X(s) = \frac{x(0)ms + mx'(0)}{ms^2 + k} \tag{1.40}$$

$$X(s) = x(0)\frac{s}{s^2 + \frac{k}{m}} + \left(x'(0)\sqrt{\frac{k}{m}}\right)\frac{\sqrt{\frac{k}{m}}}{s^2 + \left(\sqrt{\frac{k}{m}}\right)^2} \tag{1.41}$$

Taking Inverse Laplace Transform:

$$x(t) = x(0)\cos\left(\sqrt{\frac{k}{m}}t\right) + \left(x'(0)\sqrt{\frac{k}{m}}\right)\sin\left(\sqrt{\frac{k}{m}}t\right) \tag{1.42}$$

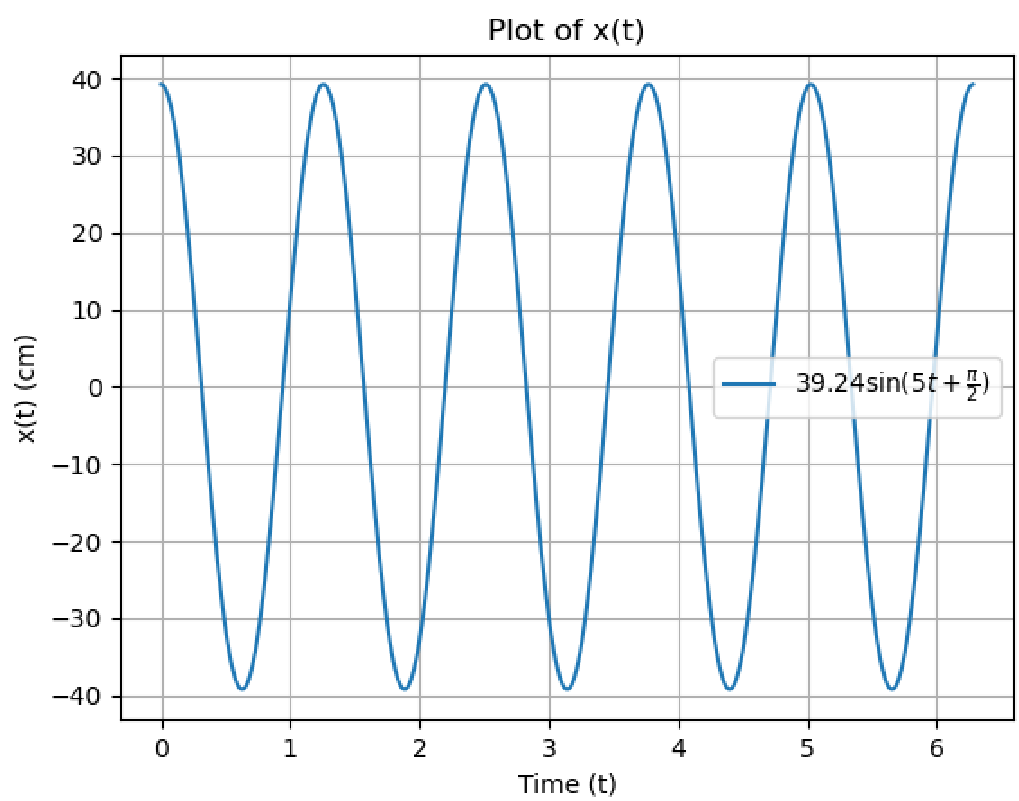
$$\tag{1.43}$$

Using 1.37 and 1.38

$$x(t) = 0.3924 \cos\left(\sqrt{\frac{k}{m}}t\right) \tag{1.44}$$

$$x(t) = 39.24 \sin\left(5t + \frac{\pi}{2}\right) \text{ cm} \tag{1.45}$$

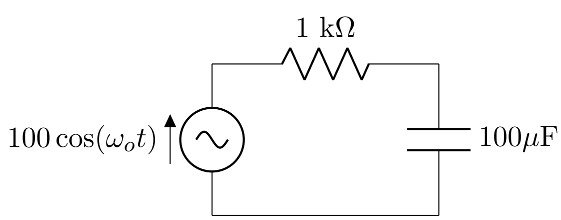
The static deflection due to weight and the time period of the oscillations,respectively are 39.24 cm and $\frac{2\pi}{5}$ s



Variable	**Description**	**Value**
M	weight of block	4 kg
K	spring constant	$100\frac{N}{m}$
x	Static deflection due to weight	39.24 cm
$x(t)$	Displacement of particle from mean position at time t	none
$x(0)$	Initial Displacement of particle from mean position	39.24cm
$x'(t)$	velocity of particle	none
$x'(0)$	initial velocity of particle	0

Table 1.4: input parameters

1.5 In the circuit shown below, the amplitudes of the voltage across the resistor and the capacitor are equal. What is the value of the angular frequency ω_o (in rad/s)? (Round off the answer to one decimal place.) (GATE BM 32 2023)



Solution:

Parameter	**Value**	**Description**
$v\,(t)$	$100cos\,(\omega_0 t)$	Input Voltage
R	1 kΩ	Resistance
C	100μF	Capacitance
ω_0	?	Angular Frequency
$Z_R = R$	10^3	Impedance for resistor
$Z_C = \frac{1}{j\omega C}$	$\frac{10^4}{j\omega_0}$	Impedance for capacitor
$Z = R + \frac{1}{j\omega C}$	$10^3 + \frac{10^4}{j\omega_0}$	Total Impedance

Table 1.5: Parameter Table

$$R \overset{\mathcal{F}}{\longleftrightarrow} R \tag{1.46}$$

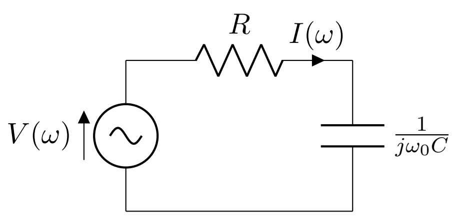
$$C \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{j\omega_0 C} \tag{1.47}$$

$$|V_R(\omega)| = |V_C(\omega)| \tag{1.48}$$

$$\implies |Z_R| = |Z_C| \tag{1.49}$$

$$10^3 = \frac{10^4}{\omega_0} \tag{1.50}$$

$$\therefore \omega_0 = 10.0 \tag{1.51}$$



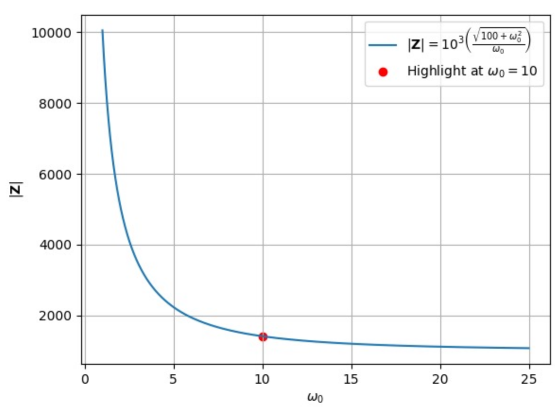


Figure 1.5: Plot of $|Z| = 10^3(\frac{\sqrt{100+\omega_0^2}}{\omega_0})$

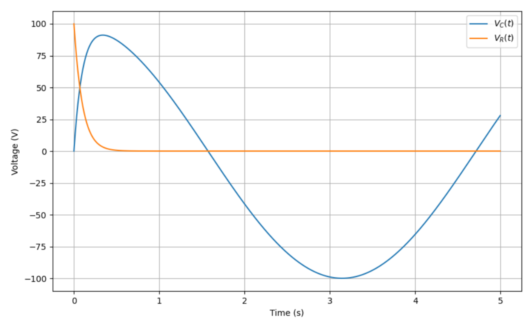


Figure 1.6: Plot of Voltage Across Capacitor and Resistor

1.6 Let $w^4 = 16j$. Which of the fo llowing can not be the value of w?

52 (A) $2e^{\frac{j2\pi}{8}}$

53 (B) $2e^{\frac{j\pi}{8}}$

54 (C) $2e^{\frac{j5\pi}{8}}$

55 (D) $2e^{\frac{j9\pi}{8}}$

(GATE 2023 EC)

Solution:

$$\left(w^4\right)^{\frac{1}{4}} = (16j)^{\frac{1}{4}} \tag{1.52}$$

Using De-Moivre's theorem for n^{th} root of w ,

$$w = 2j^{\frac{1}{4}} \tag{1.53}$$

$$e^{j\theta} = \cos\theta + j\sin\theta \tag{1.54}$$

Using equation (1.54) and put $\theta = (2n+1)\,\frac{\pi}{2}$

$$w = 2e^{\left[j(2n+1)\frac{\pi}{2}\right]\frac{1}{4}} \tag{1.55}$$

For different values of n ,

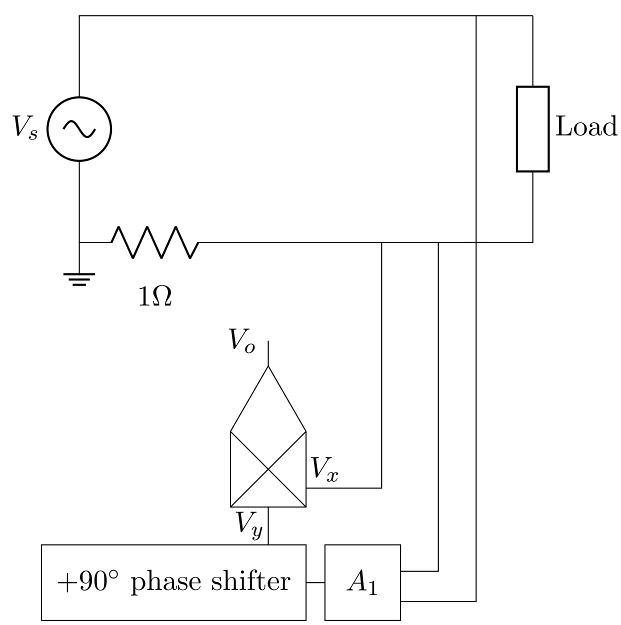
$$n = 0 \implies w = 2e^{\frac{j\pi}{8}} \tag{1.56}$$

$$n = 2 \implies w = 2e^{\frac{j5\pi}{8}} \tag{1.57}$$

$$n = 4 \implies w = 2e^{\frac{j9\pi}{8}} \tag{1.58}$$

Ans . (A) $2e^{\frac{j2\pi}{8}}$

1.7 In the diagram shown, the frequency of the sinusoidal source voltage V_s is 50 Hz.The load voltage is 230 V (RMS), and the load impedance is $\frac{230}{\sqrt{2}}+j\frac{230}{\sqrt{2}}$ Ω. The value of attenuator $A_1=\frac{1}{50\sqrt{2}}$.The multiplier output voltage $V_o = \frac{V_x V_y}{1V}$, where V_x and V_y are the inputs. The magnitude of the average value of the multiplier output V_0 is ____________________ V



GATE 2023 IN **Solution:**

(a) Let the curret in load be I

Parameter	Description	Value
V_s	sinusoidal Source voltage	230 V(RMS)
V_1	voltage across attenuator	
$V_x and V_y$	inputs voltages	
A_1	attenuator	$\frac{1}{50\sqrt{2}}$
Z	Load Impedance	$\frac{230}{\sqrt{2}}+j\frac{230}{\sqrt{2}}\ \Omega$
V_0	output voltage	$V_0=\frac{V_xV_y}{1V}$

Table 1.6: variables

$$I = \frac{V_s(peak)}{Z} \tag{1.59}$$

$$= \frac{230\sqrt{2}}{\frac{230}{\sqrt{2}}+j\frac{230}{\sqrt{2}}} \tag{1.60}$$

$$= \sqrt{2}(1-j) \tag{1.61}$$

(b) voltage at attenuator

$$V_1 = V_sA_1 \tag{1.62}$$

$$= 230\frac{1}{50\sqrt{2}}V \tag{1.63}$$

$$= \frac{4.6}{\sqrt{2}}V \tag{1.64}$$

$$V_y = 4.6\sin(\omega t + 90^\circ) \tag{1.65}$$

$$V_x = I \times 1\Omega \tag{1.66}$$

$$= 2\sqrt{2}\sin(\omega t - 45^\circ) \tag{1.67}$$

$$V_0 = 9.2\sqrt{2}(\frac{\cos(135) - \cos(2\omega t)}{2}) \tag{1.68}$$

$$= 4.6 - 4.6\sqrt{2}\cos(2\omega t) \tag{1.69}$$

$$\tag{1.70}$$

(c) Let f(t)$= 4.6 - 4.6\sqrt{2}\cos(2\omega t)$

$$V_o < avg > = \frac{1}{T}\int_0^T (4.6 - 4.6\sqrt{2}\cos(2\omega t))\,dt \tag{1.71}$$

$$= \frac{\omega}{\pi}\left[\int_0^{\frac{\pi}{\omega}} 4.6\,dt - 4.6\sqrt{2}\int_0^{\frac{\pi}{\omega}} \cos(2\omega t)\,dt\right] \tag{1.72}$$

$$= \frac{\omega}{\pi}\left[4.6\frac{\pi}{\omega} - 4.6\sqrt{2}\left[\frac{\sin(2\pi)}{2\omega}\right]\right] \tag{1.73}$$

$$= 4.6 \tag{1.74}$$

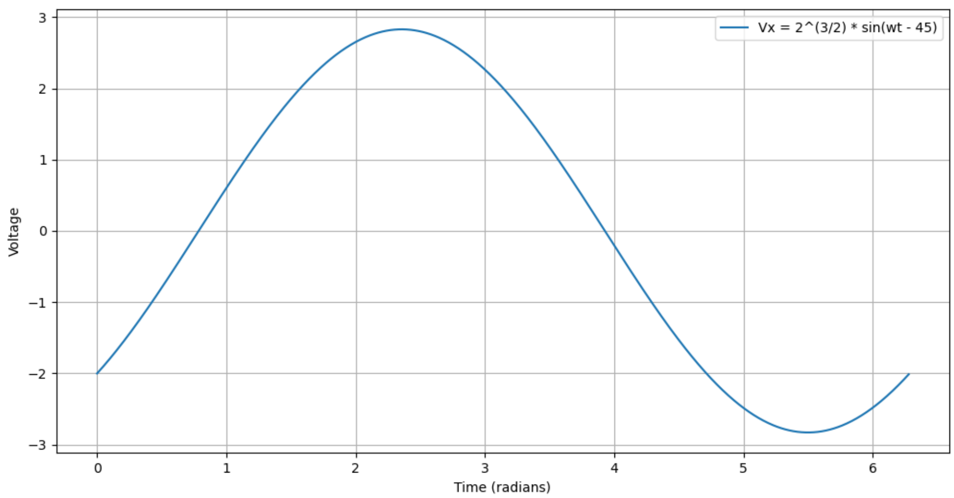


Figure 1.7: $plotofV_x$

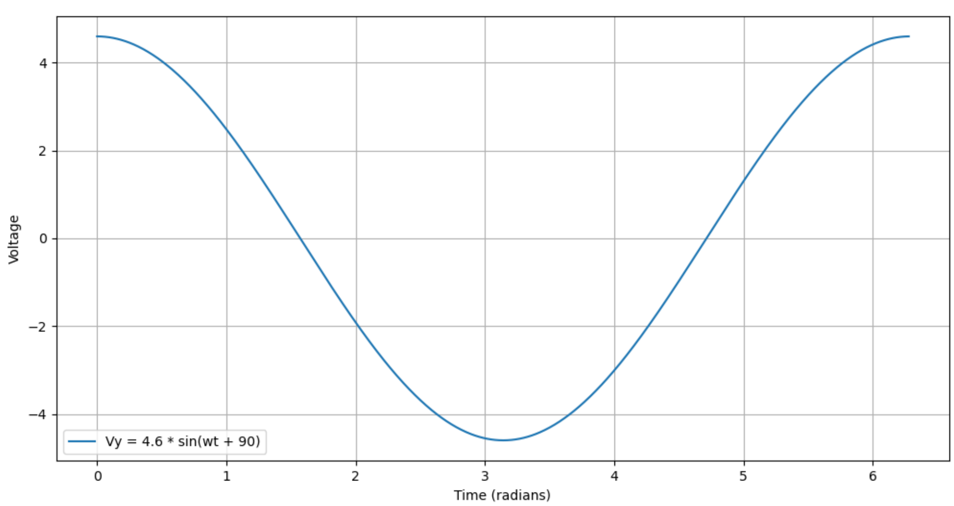


Figure 1.8: $plotofV_y$

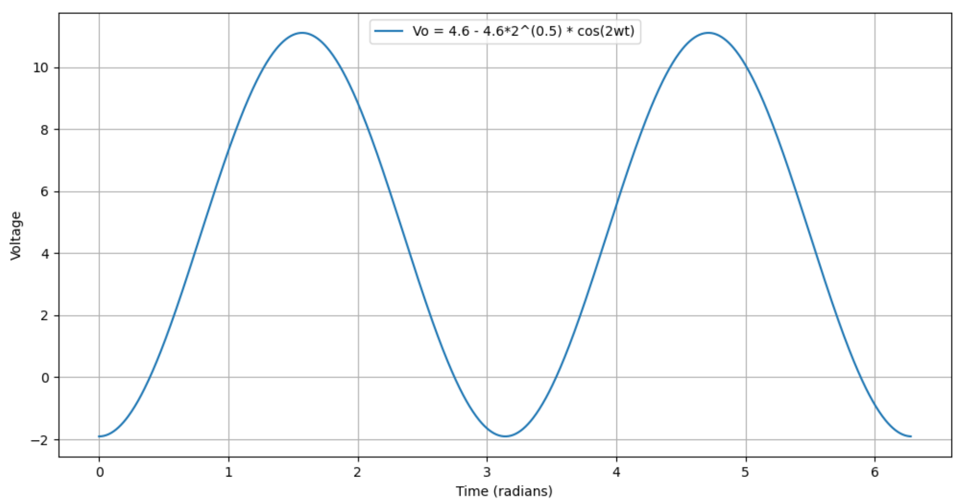


Figure 1.9: $plot of V_o$

Chapter 2

Filters

2.1 For the circuit given below, choose the angular frequency ω_0 at which voltage across capacitor has maximum amplitude?

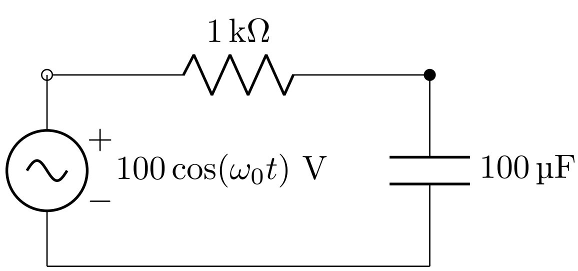


Figure 2.1: circuit

(A) 1000

(B) 100

(C) 1

(D) 0

(GATE BM 2023 Question 16)

Solution:

Parameter	Description	Value
$V_i(j\omega)$	Input voltage	100
$v_c(t)$	Potential difference across Capacitor	?
$V_c(s)$	Potential difference across Capacitor	$V_c(s)$
$H(s)$	Transfer function	$\frac{V_c(s)}{V_i(s)}$
V_o	Amplitude of input voltage	$100\,\mathrm{V}$
R	Resistance in circuit	$1\,\mathrm{k\Omega}$
C	Capacitance in circuit	$100\,\mu\mathrm{F}$
ω_o	Angular frequency of input voltage	ω_o

Table 2.1: input values

$$V_c\left(s\right) = \frac{V_1\left(s\right)\frac{1}{sC}}{R+\frac{1}{sC}} \tag{2.1}$$

$$\implies H\left(s\right) = \frac{1}{1+sRC} \tag{2.2}$$

$$\implies H\left(j\omega\right) = \frac{1}{1+j\omega RC} \tag{2.3}$$

$$\left|H\left(j\omega\right)\right| = \frac{1}{\sqrt{1+\left(\omega RC\right)^2}} \tag{2.4}$$

$$v_c\left(t\right) = \frac{100}{\sqrt{1+\left(\omega_o RC\right)^2}}\left(\cos\omega_o t + \arctan\left(\frac{1}{\omega_o RC}\right)\right) \tag{2.5}$$

Maximum amplitude of $v_c\left(t\right)$ occurs at $\omega_o = 0$

$$\therefore \omega_o = 0 \tag{2.6}$$

$\therefore$ maximum value of $v_c\left(t\right)$ at steady state is 100 Volts.

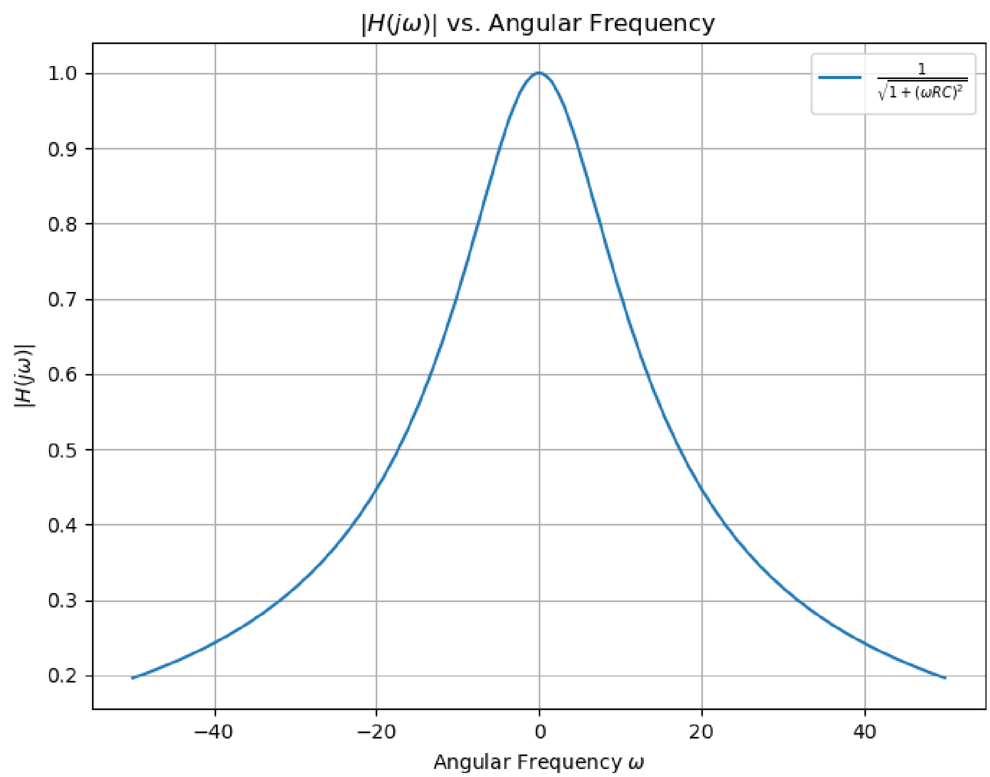


Figure 2.2: $|H(j\omega)|$

2.2 In the following circuit, the switch S is open for $t < 0$ and closed for $t \geq 0$. What is the steady state voltage (in Volts) across the capacitor when the switch is closed?

(GATE BM 2023 Question 30)

Solution:

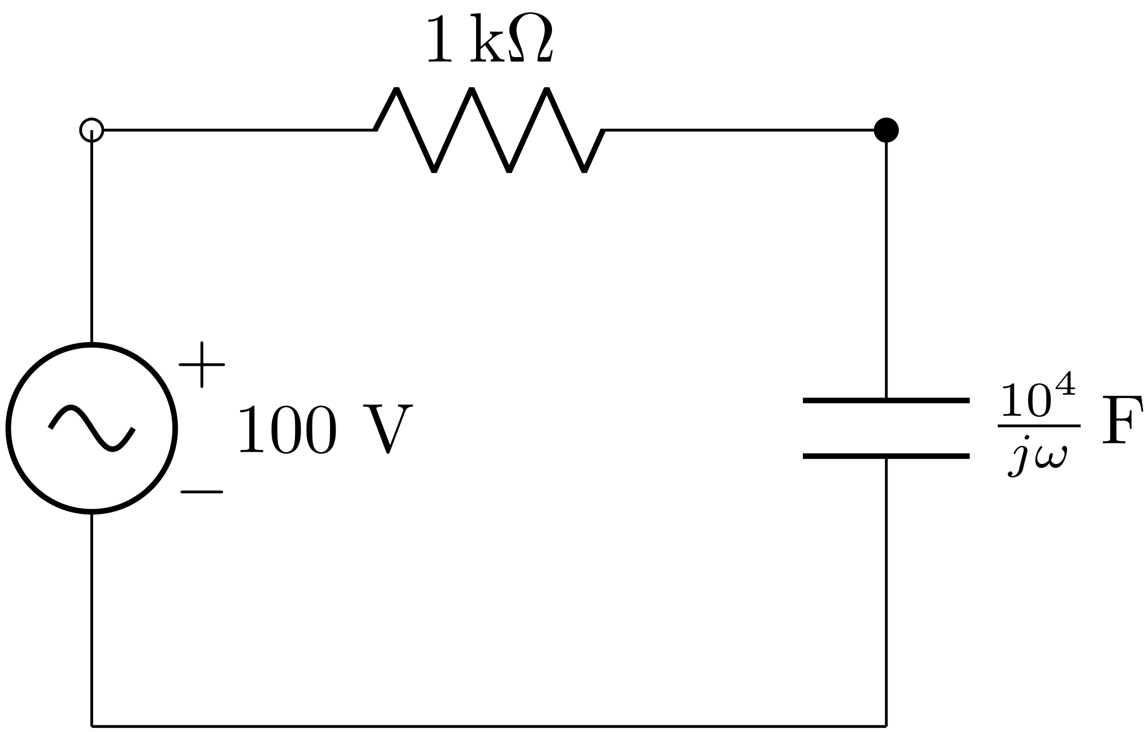


Figure 2.3: circuit in ω-domain

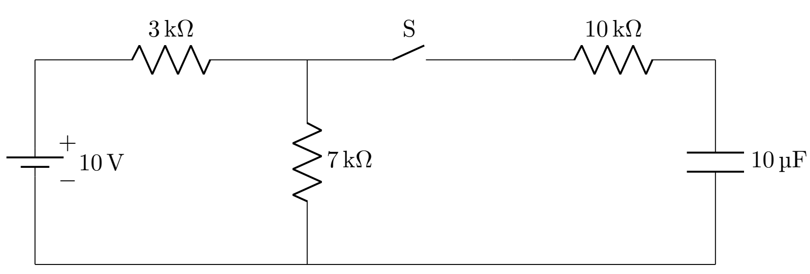
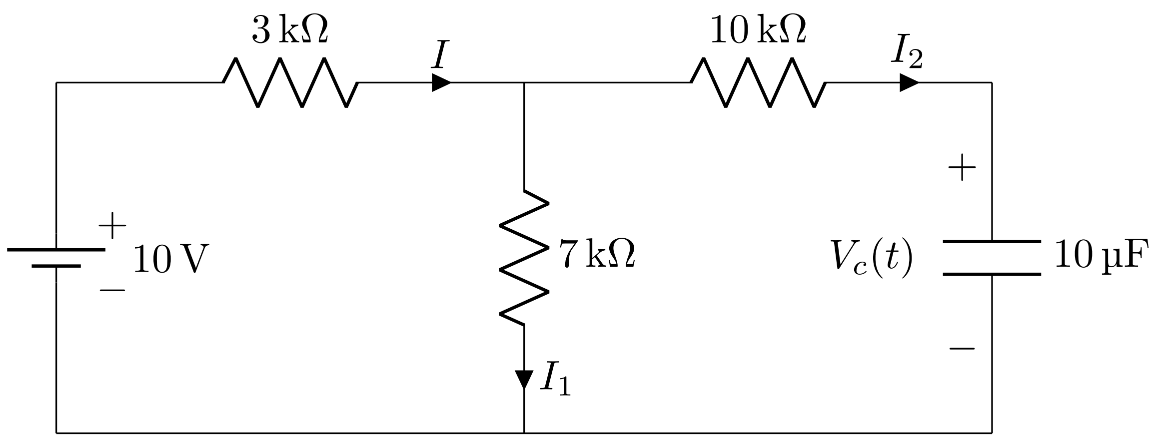


Figure 2.4: circuit

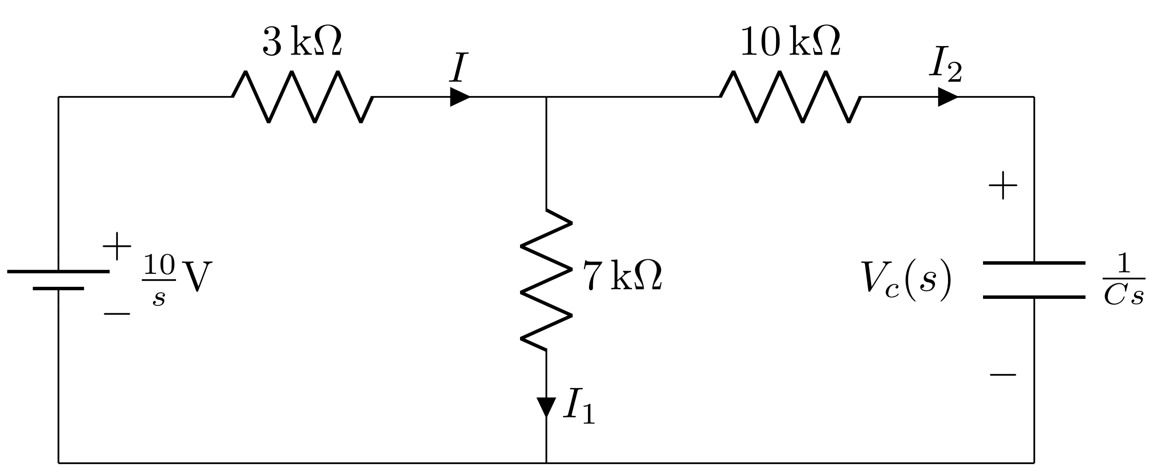
In steady state, no current flows through the capacitor.

$$I_2 = 0 \tag{2.7}$$

$$\begin{aligned} V_c &= (7\text{k}\Omega)\, I_1 && (2.8)\\ &= (7\text{k}\Omega)\, I && (2.9)\\ &= (7\text{k}\Omega)\, \frac{10\text{V}}{10\text{k}\Omega} && (2.10) \end{aligned}$$


3 kΩ
I
10 kΩ
I_2
+
10 V
−
7 kΩ
I_1
$V_c(t)$
10 µF
+
−

In s-domain:


3 kΩ
I
10 kΩ
I_2
+
$\frac{10}{s}$V
−
7 kΩ
I_1
$V_c(s)$
$\frac{1}{Cs}$
+
−

$$\implies I(s) = \frac{\frac{10}{s}\mathrm{V}}{3\mathrm{k}\Omega + \frac{(7\mathrm{k}\Omega)\left(10\mathrm{k}\Omega + \frac{1}{sC}\right)}{17\mathrm{k}\Omega + \frac{1}{sC}}} \tag{2.12}$$

$$I = I_1 + I_2 \tag{2.13}$$

$$I_1(7\mathrm{k}\Omega) = I_2\left(10\mathrm{k}\Omega + \frac{1}{sC}\right) \tag{2.14}$$

$$I_2(s) = \frac{7\mathrm{k}\Omega}{17\mathrm{k}\Omega + \frac{1}{sC}} I(s) \tag{2.15}$$

$$\implies I_2(s) = \frac{7\left(10^{-5}\right)}{0.121s + 1} \tag{2.16}$$

$$V_c(s) = I_2(s)\frac{1}{sC} \tag{2.17}$$

$$= \frac{7}{s(0.121s + 1)} \tag{2.18}$$

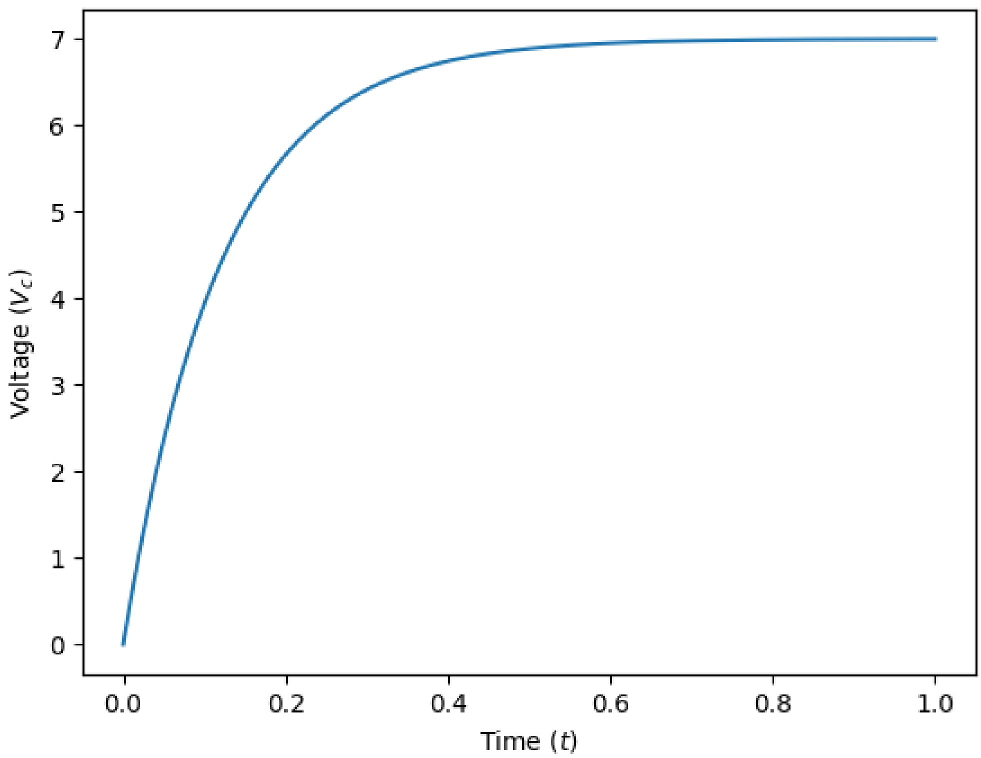
$$= 7\left(\frac{1}{s} - \frac{1}{s + \frac{1}{0.121}}\right) \tag{2.19}$$

Taking inverse Laplace transform:

$$V_c(t) = 7u(t)\left(1 - e^{-\frac{t}{-0.121}}\right) \tag{2.20}$$

In steady state $t \to \infty$. From (2.20):

$$\lim_{t\to\infty} V_c(t) = 7\mathrm{V} \tag{2.21}$$



2.3 A finite impulse response (FIR) filter has only two non-zero samples in its impulse response $h[n]$, namely $h[0] = h[1] = 1$. The Discrete Time Fourier Transform (DTFT) of $h[n]$ equals $H(e^{j\omega})$, as a function of the normalized angular frequency ω. For the range $|\omega| \leq \pi$, $\left|H(e^{j\omega})\right|$ is equal to

(A) $2\,|\cos(\omega)|$

(B) $2\,|\sin(\omega)|$

(C) $2\,\left|\cos(\frac{\omega}{2})\right|$

(D) $2\,\left|\sin(\frac{\omega}{2})\right|$

(GATE BM 2023 Question 17)

Solution:

Parameter	**Value**	**Description**
$h[n]$	-	impulse response
$h[0]$	1	impulse response at $n = 0$
$h[1]$	1	impulse response at $n = 1$
ω	$-\pi \leq \omega \leq \pi$	normalized frequency
$H(e^{j\omega})$	$\sum_{n=0}^{M} h[n]e^{-jn\omega}$	frequency response

Table 2.2: Input Parameters Table

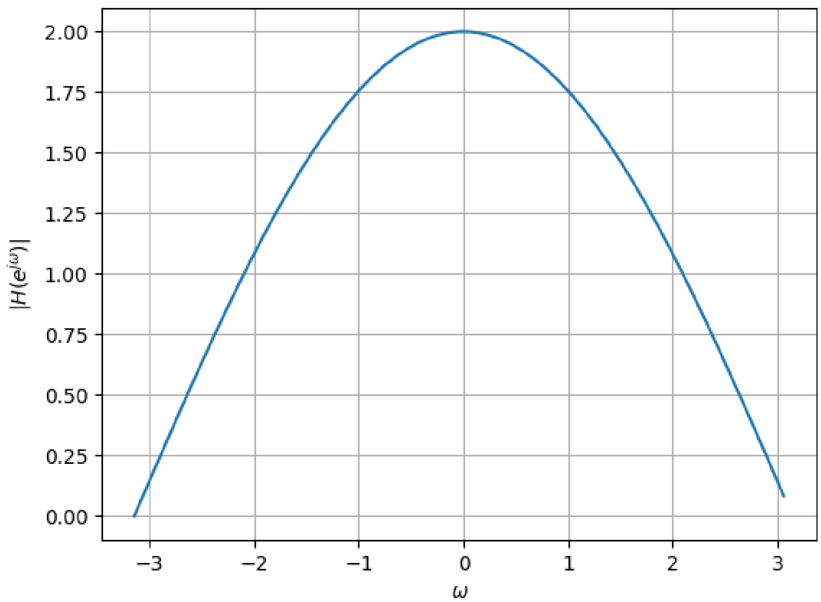
From Table 2.2,

$$H(e^{j\omega}) = 1 + e^{-j\omega} \tag{2.22}$$

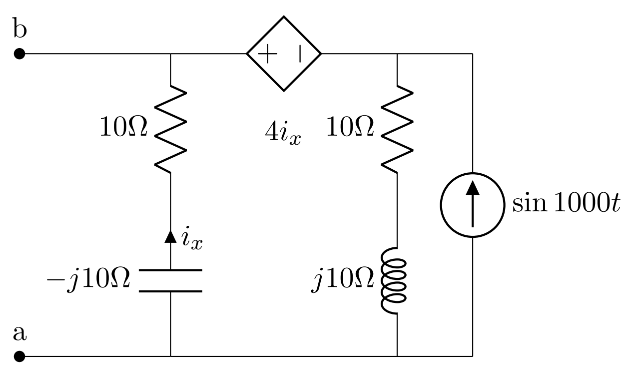
$$= e^{\frac{-j\omega}{2}}(e^{\frac{j\omega}{2}} + e^{\frac{-j\omega}{2}}) \tag{2.23}$$

$$= e^{\frac{-j\omega}{2}}(2\cos\left(\frac{\omega}{2}\right)) \tag{2.24}$$

$$\left|H(e^{j\omega})\right| = 2\left|\cos\left(\frac{\omega}{2}\right)\right| \tag{2.25}$$

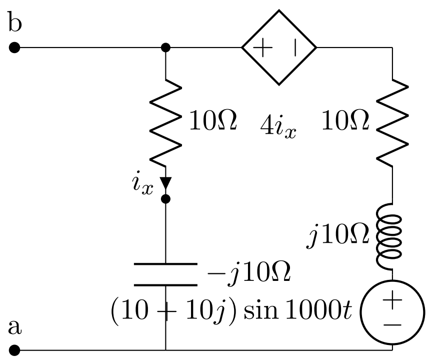


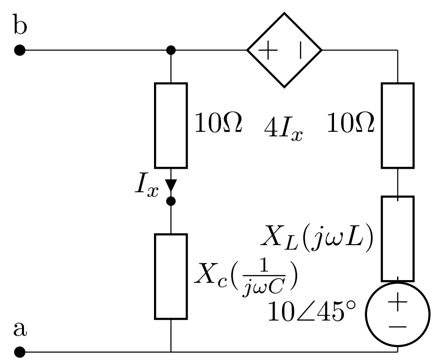
2.4 For the circuit shown,if $i = \sin 1000t$, the instantaneous value of the Thevenin's voltage(in volts) across the terminals a anb b at time t=5ms is



(GATE EE 2023)

Solution: By source transforming the given circuit we get





Solving using sinusoidal steady state analysis, Applying KVL we get,

$$10\angle 45° - (j\omega L)I_x - 10I_x + 4I_x - 10I_x - (\frac{1}{j\omega C})I_x = 0 \tag{2.26}$$

$$I_x = \frac{10\angle 45°}{16 + j\omega L + \frac{1}{j\omega C}} \tag{2.27}$$

$$V_{ab} = I_x(10 + \frac{1}{j\omega C}) \tag{2.28}$$

$$V_{ab} = \frac{10\angle 45°(10j\omega C + 1)}{16j\omega C - \omega^2 LC + 1} \tag{2.29}$$

From the question we can observe that,

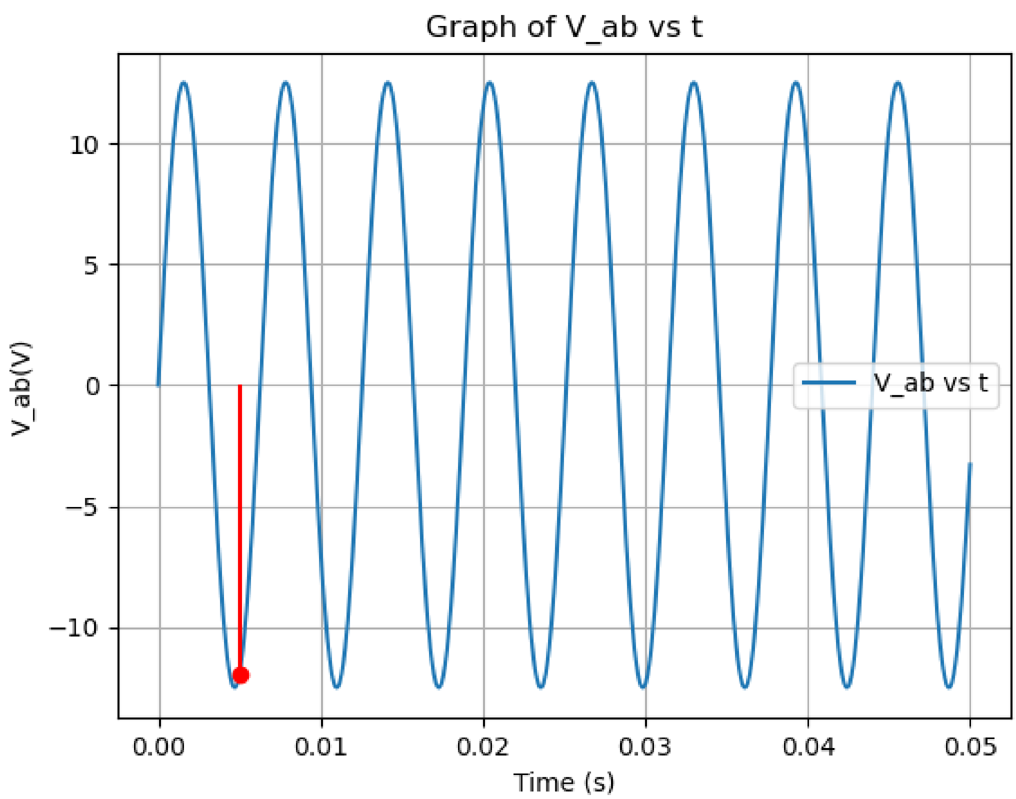
$$\frac{1}{\omega C} = \omega L \tag{2.30}$$

$$\omega^2 LC = 1 \tag{2.31}$$

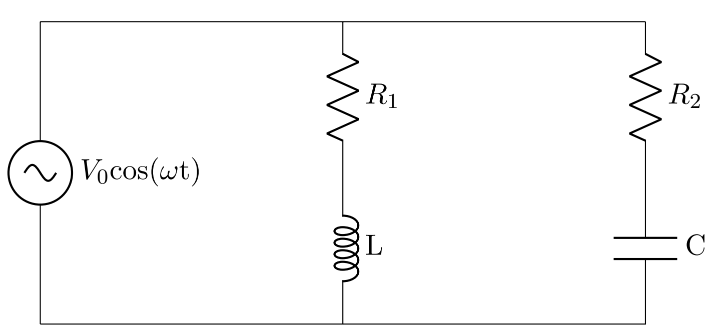
Substituting 2.31 and values of ωC and ωL

$$V_{ab} = 12.5\angle 0^\circ \tag{2.32}$$

$$V_{ab} = 12.5 \sin 1000t \tag{2.33}$$



2.5 In the circuit shown ,$\omega = 100\pi$rads/s, R1=R2=2.2Ω and L=7mH. the capacitance C for which Y_{in} is purely real is mF



(GATE IN 2023 Q46)

Solution:

variable	value	description	formulae
Y_{in}	-	Admittance of circuit	$\frac{R_1-Ls}{R_1^2-(Ls)^2}+\frac{R_2-\frac{1}{\text{sC}}}{R_2^2-\left(\frac{1}{\text{sC}}\right)^2}$
X_L	$7s\Omega$	Inductive reactance	sL
X_C	$\frac{1}{s\text{C}}\Omega$	Capacitive reactance	$\frac{1}{s\text{C}}$
s	100πj	Laplace complex frequency	$j\omega$
ω	100πrads/s	Angular frequency	-
V	$V_0\cos(\omega\text{t})$	voltage of source	-
R_1, R_2	2.2Ω	resistance of resistors	-

Table 2.3: Table: Input Parameters

From $Table$ 2.3

$$Y_{in}=\frac{R_1-Ls}{R_1^2-(Ls)^2}+\frac{R_2-\frac{1}{\text{sC}}}{R_2^2-\left(\frac{1}{\text{sC}}\right)^2} \tag{2.34}$$

$$Im\left(Y_{in}\right)=\frac{-Ls}{R_1^2-(Ls)^2}+\frac{-\frac{1}{\text{sC}}}{R_2^2-\left(\frac{1}{\text{sC}}\right)^2} \tag{2.35}$$

According to the question given, Y_{in} is purely real , so imaginary part should be equal to zero

Take the values from $Table$ 2.3

$$\frac{-1}{4.4} + \frac{\frac{1}{(100\pi)\mathrm{C}}}{(2.2)^2 + \left(\frac{1}{(100\pi)\mathrm{C}}\right)^2} = 0 \tag{2.36}$$

$$\frac{\frac{1}{(100\pi)\mathrm{C}}}{(2.2)^2 + \left(\frac{1}{(100\pi)\mathrm{C}}\right)^2} = \frac{1}{4.4} \tag{2.37}$$

$$(2.2)^2 - \frac{4.4}{(100\pi)\mathrm{C}} + \left(\frac{1}{(100\pi)\mathrm{C}}\right)^2 = 0 \tag{2.38}$$

$$\left(2.2 - \frac{1}{(100\pi)\mathrm{C}}\right)^2 = 0 \tag{2.39}$$

$$\frac{1}{(100\pi)\mathrm{C}} = 2.2 \tag{2.40}$$

$$\mathrm{C} = \frac{700}{484}\mathrm{mF} \tag{2.41}$$

$$\mathrm{C} = 1.446281\mathrm{mF} \tag{2.42}$$

The capacitance of capacitor C is 1.45mF

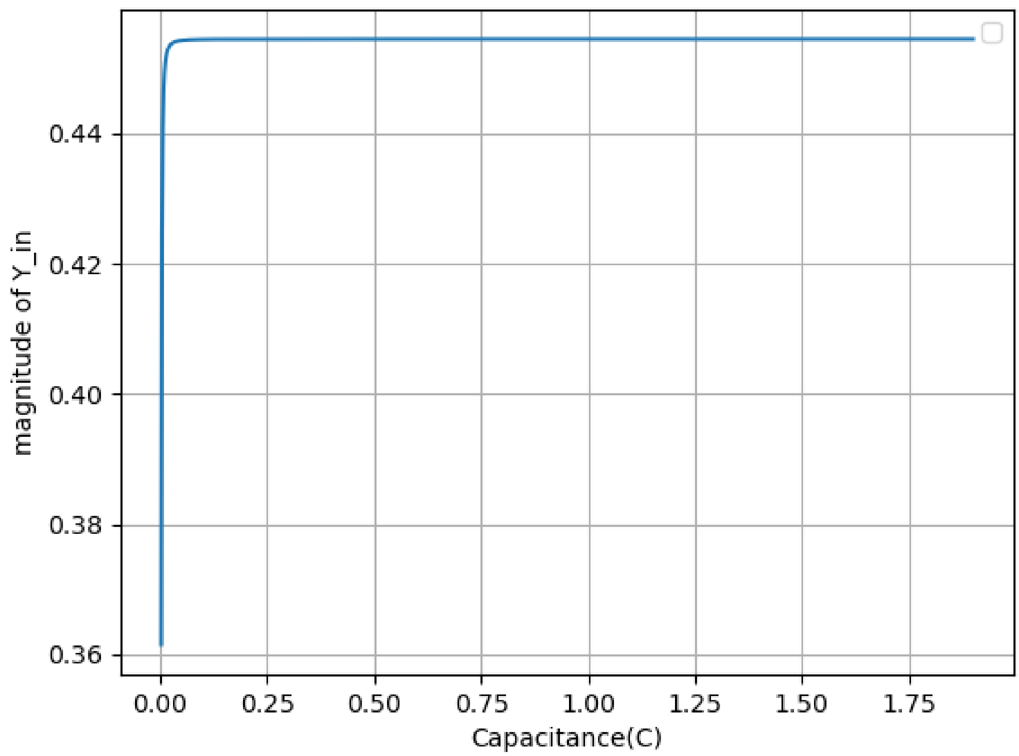


Figure 2.5: the plot of capacitance vs magnitude of Y_{in}

2.6 An input voltage in the form of a square wave of frequency $1\,kHz$ is given to a circuit, which results in the output shown schematically below. Which one of the following options is the CORRECT representation of the circuit? (GATE PH 2023 Q37)

(a)

(b)

(c)

(d)

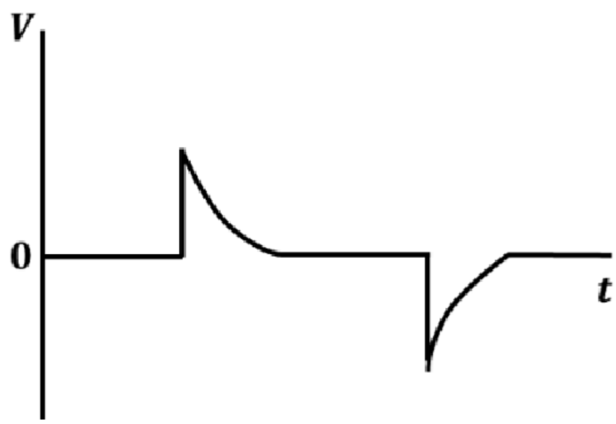
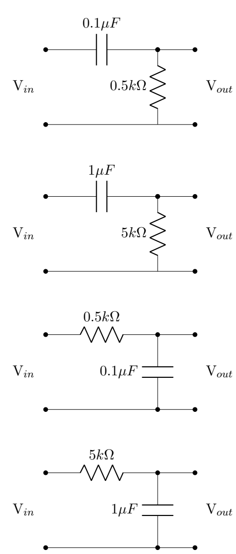


Figure 2.6:



(GATE 2023 PH 37)

Solution:

Input waveform is a square wave (Fig. 2.7), so we take its Fourier Transform

$$V_{in}(t) = 2\left(2\left[\frac{\left(t-\frac{T}{4}\right)}{T}\right] - \left[\frac{2\left(t-\frac{T}{4}\right)}{T}\right]\right) + 1 \tag{2.43}$$

Symbol	**Value**	**Description**
$V_{in}(t)$		Input Voltage
$\mathcal{V}_{in}(j\omega)$		Fourier Transform of $V_{in}(t)$
$V_{out}(t)$		Output Voltage
$\mathcal{V}_{out}(j\omega)$		Fourier Transform of $V_{out}(t)$
f	$\frac{\omega}{2\pi} = 1000Hz$	Input Wave Frequency
T	$\frac{2\pi}{\omega} = 10^{-3}s$	Input Wave Time Period
R	(a) $0.5k\Omega$	Resistance
	(b) $5k\Omega$	
C	(a) $0.1\mu F$	Capacitance
	(b) $1\mu F$	
τ	RC	Time Constant
Z	$R + \frac{1}{j\omega C}$	Impedance
$H(j\omega)$	$\frac{V_{out}}{V_{in}}$	General Transfer Function
$H_R(j\omega)$	$\frac{V_{R,out}}{V_{in}}$	Transfer Function for Resistor
$H_C(j\omega)$	$\frac{V_{C,out}}{V_{in}}$	Transfer Function for Capacitor

Table 2.4: Given Parameters

Fourier Series Coefficient:

$$c_k = \frac{1}{T}\int_T V_{in}(t)e^{-jk\omega t}dt \tag{2.44}$$

As square wave is even, $\sin(k\omega t)$ terms become zero. Cosine coefficients are:

$$a_n = \frac{2}{T}\int_T V_{in}(t)\cos\left(\frac{2\pi nt}{T}\right) \tag{2.45}$$

$$= \frac{4}{n\pi}\sin\left(\frac{n\pi}{2}\right)\cos(n\pi) \tag{2.46}$$

Fourier Series of $V_{in}(t)$:

$$V_{in}(t) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nt}{T}\right) \tag{2.47}$$

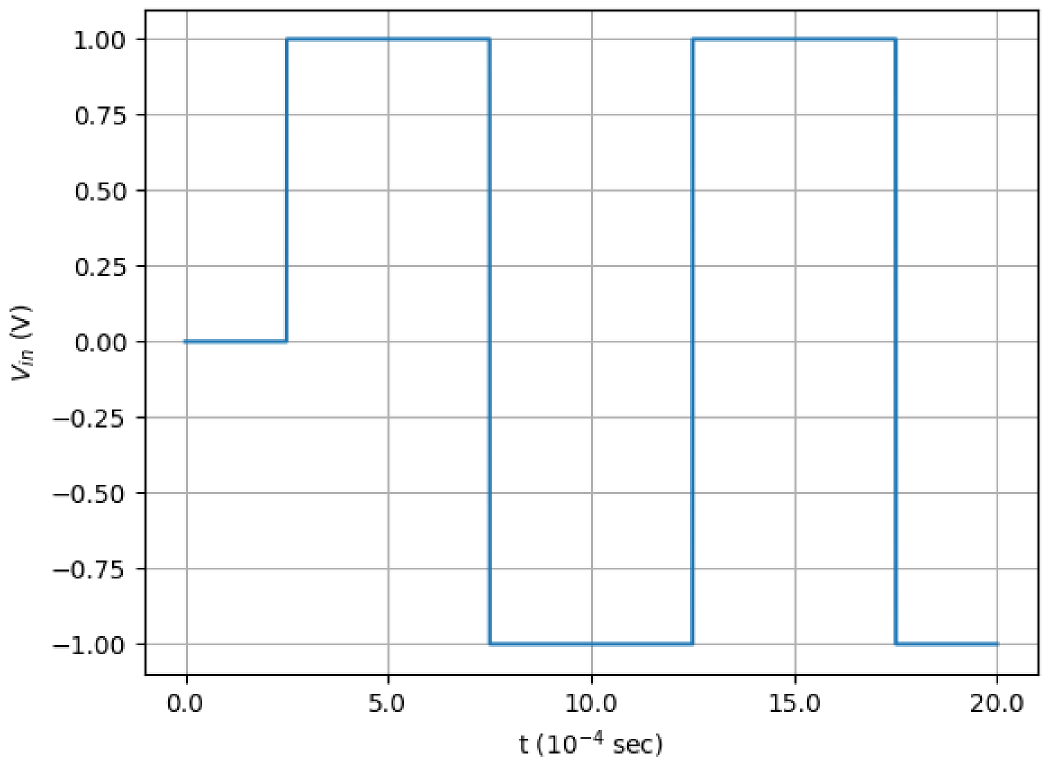


Figure 2.7: Input Square Waveform ($V_{in}(t)$)

Taking Fourier Transform of $V_{in}(t)$:

$$V_{in}(t) \stackrel{\mathcal{F}}{\longleftrightarrow} \mathcal{V}_{in}(j\omega) \tag{2.48}$$

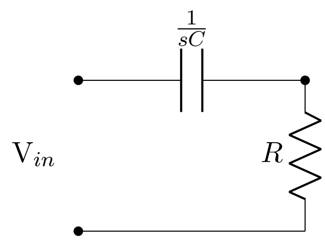


Figure 2.8: Series RC Circuit in s-domain

$$s = j\omega \tag{2.49}$$
$$\implies Z = R + \frac{1}{sC} \tag{2.50}$$
$$= R + \frac{1}{j\omega C} \tag{2.51}$$

$\mathcal{V}_{in}(j\omega)$ was input into all four circuits and $V_{out}(t)$ was calculated using the Transfer Functions of the RC Filters.

Transfer Function:

$$H(j\omega) = \frac{V_{out}}{V_{in}} \tag{2.52}$$

(a) Option A

$$H_R(j\omega) = \frac{R}{R + \frac{1}{j\omega C}} \tag{2.53}$$
$$= \frac{j\omega RC}{1 + j\omega RC} \tag{2.54}$$
$$= \left(\frac{\omega RC}{\sqrt{1 + (\omega RC)^2}}\right) e^{j\tan^{-1}\left(\frac{1}{\omega RC}\right)} \tag{2.55}$$

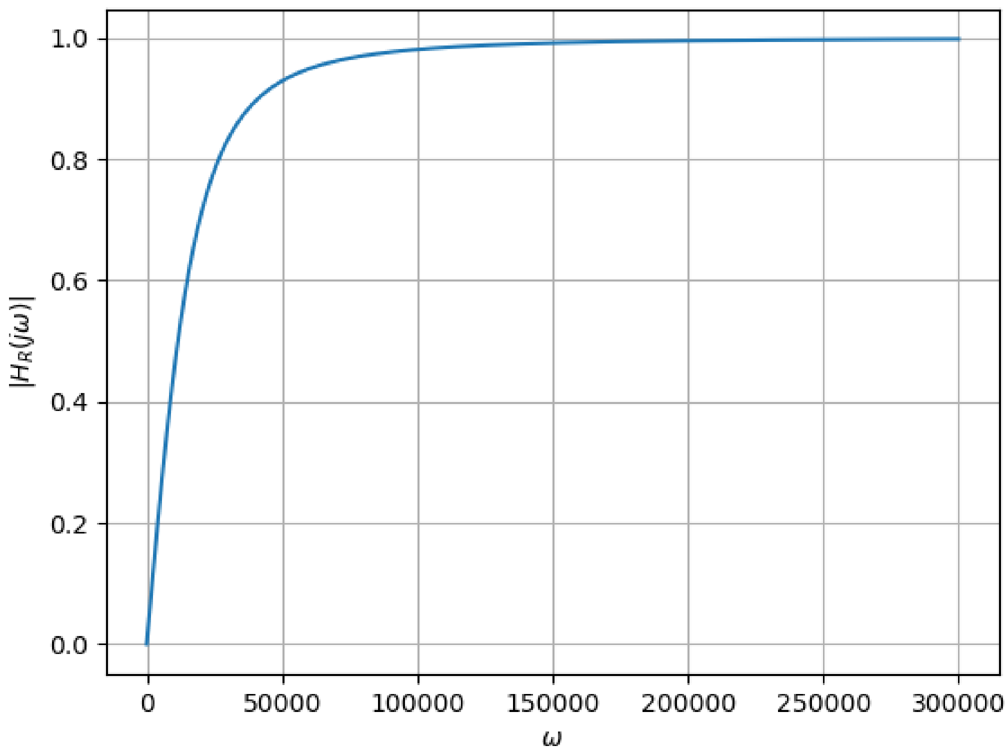


Figure 2.9: $|H_R(j\omega)|$ vs ω for $R = 0.5k\Omega$, $C = 0.1\mu F$

$$\implies \mathcal{V}_{out}(j\omega) = H_R(j\omega)\mathcal{V}_{in}(j\omega) \tag{2.56}$$

$$\tag{2.57}$$

Using (2.47) and (2.55),

$$V_{out}(t) = \sum_{n=1}^{\infty} \left(\frac{n\omega RC}{\sqrt{1 + (n\omega RC)^2}} \right) a_n \cos\left(n\omega t + \tan^{-1}\left(\frac{1}{n\omega RC} \right) \right) \tag{2.58}$$

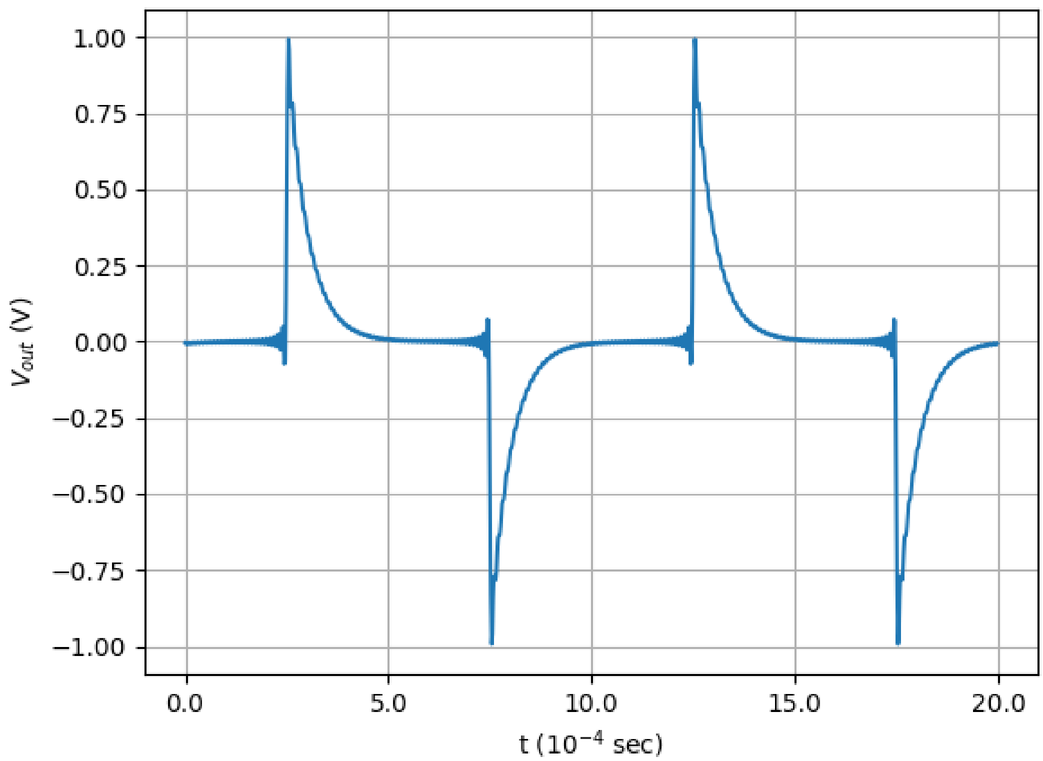


Figure 2.10: Opt A: $V_{out}(t)$ vs t

(b) Option B

$$H_R(j\omega) = \frac{R}{R + \frac{1}{j\omega C}} \tag{2.59}$$

$$= \frac{j\omega RC}{1 + j\omega RC} \tag{2.60}$$

$$= \left(\frac{\omega RC}{\sqrt{1 + (\omega RC)^2}} \right) e^{j \tan^{-1}\left(\frac{1}{\omega RC}\right)} \tag{2.61}$$

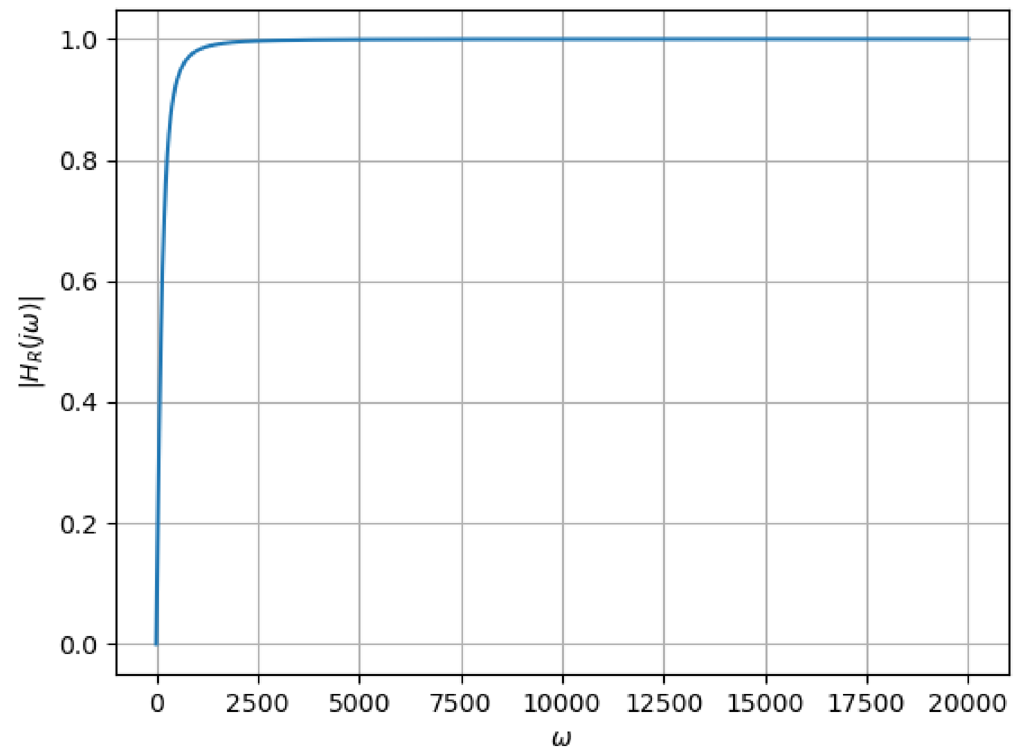


Figure 2.11: $|H_R(j\omega)|$ vs ω for $R = 5k\Omega$, $C = 1\mu F$

$$\implies \mathcal{V}_{out}(j\omega) = H_R(j\omega)\mathcal{V}_{in}(j\omega) \tag{2.62}$$

Using (2.47) and (2.61),

$$V_{out}(t) = \sum_{n=1}^{\infty} \left(\frac{n\omega RC}{\sqrt{1 + (n\omega RC)^2}} \right) a_n \cos\left(n\omega t + \tan^{-1}\left(\frac{1}{n\omega RC} \right) \right) \tag{2.63}$$

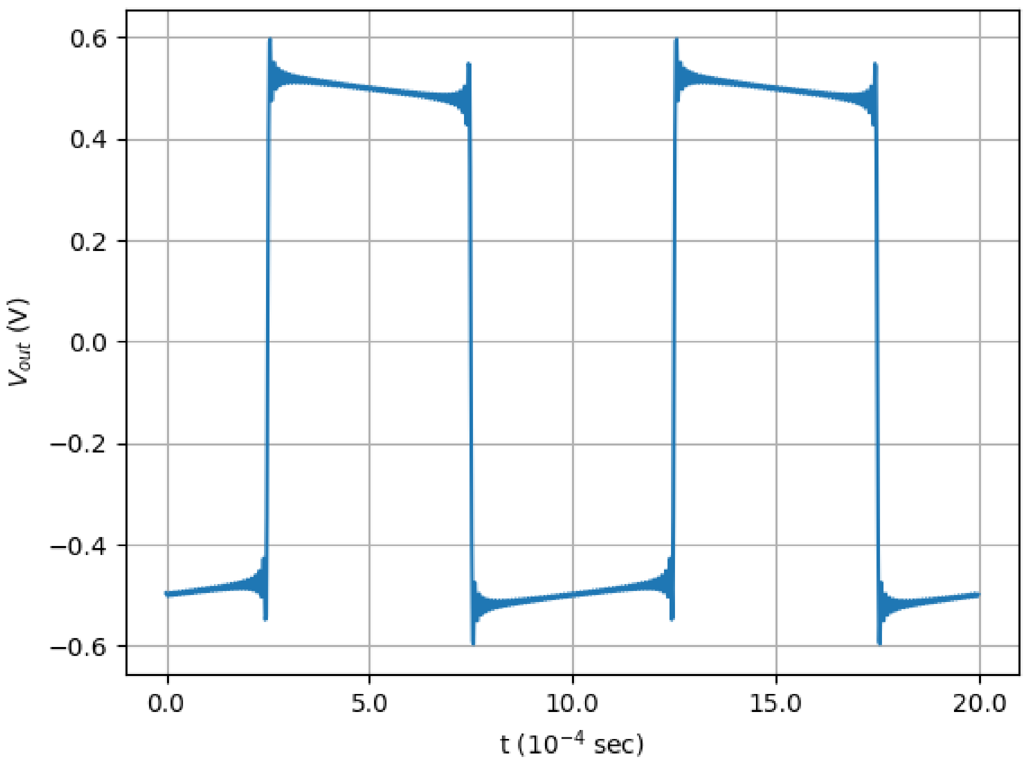


Figure 2.12: Opt B: $V_{out}(t)$ vs t

(c) Option C

$$H_C(j\omega) = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} \tag{2.64}$$

$$= \frac{1}{1 + j\omega RC} \tag{2.65}$$

$$= \left(\frac{1}{\sqrt{1 + (\omega RC)^2}}\right) e^{-j\tan^{-1}(\omega RC)} \tag{2.66}$$

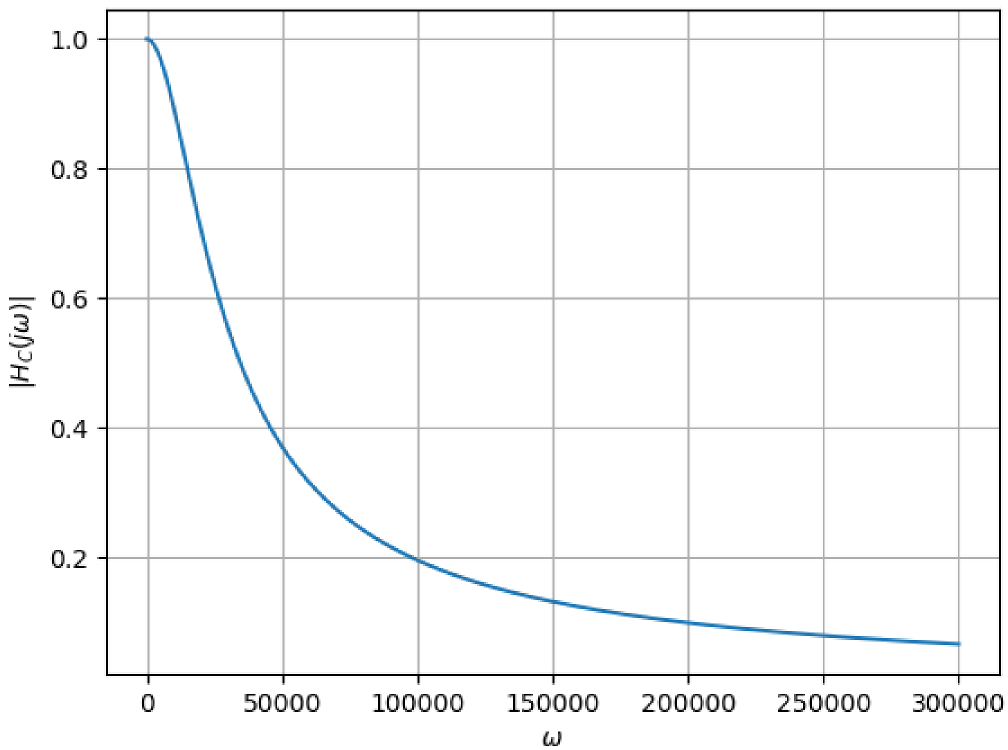


Figure 2.13: $|H_C(j\omega)|$ vs ω for $R = 0.5k\Omega$, $C = 0.1\mu F$

$$\implies \mathcal{V}_{out}(j\omega) = H_C(j\omega)\mathcal{V}_{in}(j\omega) \tag{2.67}$$

Using (2.47) and (2.66),

$$V_{out}(t) = \sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{1 + (n\omega RC)^2}} \right) a_n \cos\left(n\omega t - \tan^{-1}\left(n\omega RC\right)\right) \tag{2.68}$$

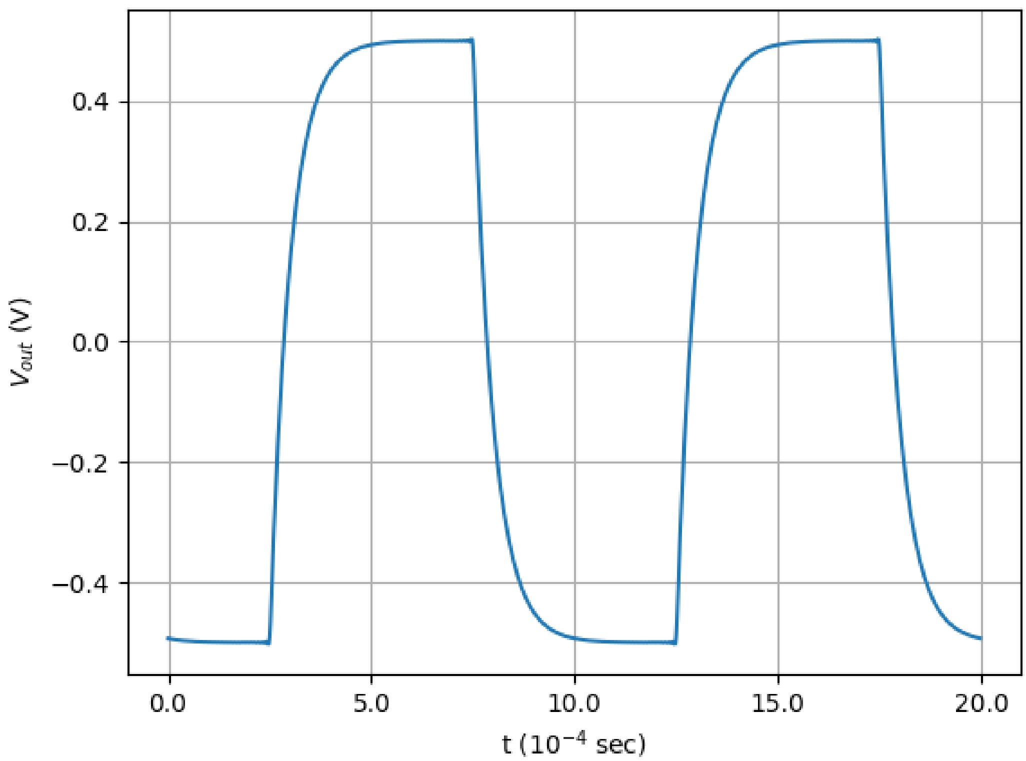


Figure 2.14: Opt C: $V_{out}(t)$ vs t

(d) Option D

$$H_C(j\omega) = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} \tag{2.69}$$

$$= \frac{1}{1 + j\omega RC} \tag{2.70}$$

$$= \left(\frac{1}{\sqrt{1 + (\omega RC)^2}}\right) e^{-j\tan^{-1}(\omega RC)} \tag{2.71}$$

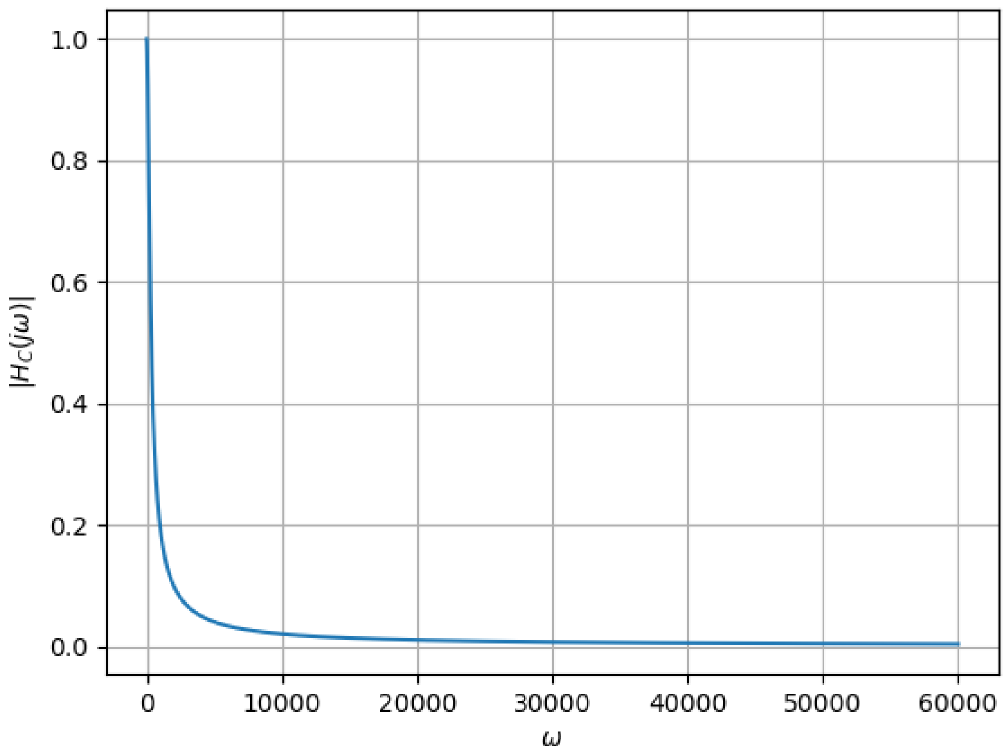


Figure 2.15: $|H_C(j\omega)|$ vs ω for $R = 5k\Omega$, $C = 1\mu F$

$$\implies \mathcal{V}_{out}(j\omega) = H_C(j\omega)\mathcal{V}_{in}(j\omega) \tag{2.72}$$

Using (2.47) and (2.71),

$$V_{out}(t) = \sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{1 + (n\omega RC)^2}} \right) a_n \cos\left(n\omega t - \tan^{-1}(n\omega RC)\right) \tag{2.73}$$

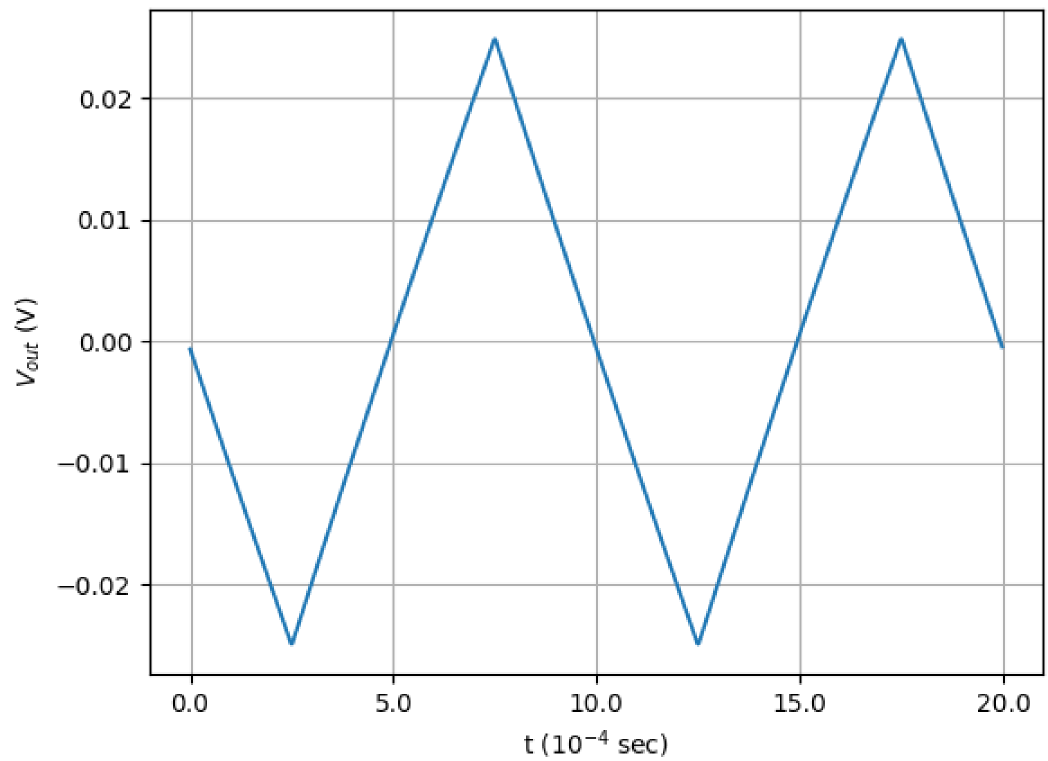


Figure 2.16: Opt D: $V_{out}(t)$ vs t

2.7 In the circuit shown below, switch S was closed for long time. If the switch is opened at $t = 0$, the maximum magnitude of the voltage V_R , in volts is (rounded off to the nearest integer) (GATE 2023 EC 35)

Solution:

$$At, t = 0^- \tag{2.74}$$

inductor acts as wire

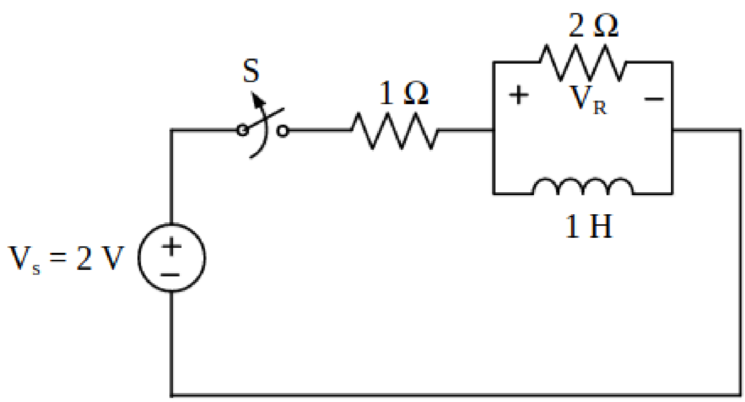


Figure 2.17:

parameter	description	value
$i\left(0^{-}\right)$	current at $t<0$	$2A$
$V_R\left(t\right)$	voltage across 2Ω	-2i(t)u(t)
L	inductance	$1H$
$i\left(t\right)$	current in small loop after $t=0$	$\frac{V_R(t)}{2}$
$I\left(s\right)$	$i\left(t\right)$ in laplace	$-$

Table 2.5: input parameters

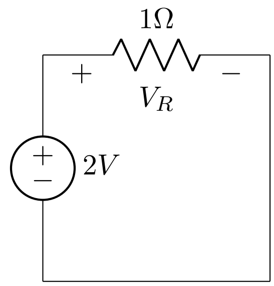


Figure 2.18: steady state circuit

apply KVL

$$-2+1i\left(0^{-}\right)=0 \tag{2.75}$$

$$i\left(0^{-}\right)=2A \tag{2.76}$$

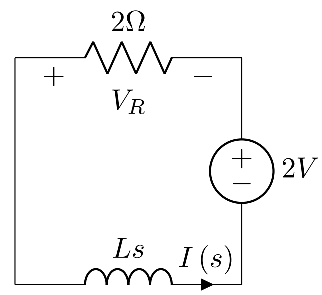


Figure 2.19: s domain circuit fot $t > 0$

$$2I(s) - 2V + LsI(s) = 0 \tag{2.77}$$

$$\implies I(s) = \frac{2}{s+2}A \tag{2.78}$$

applying inverse laplace transform

$$i(t) = 2e^{-2t}u(t)\,A \tag{2.79}$$

$$V_R(t) = -2i(t) \tag{2.80}$$

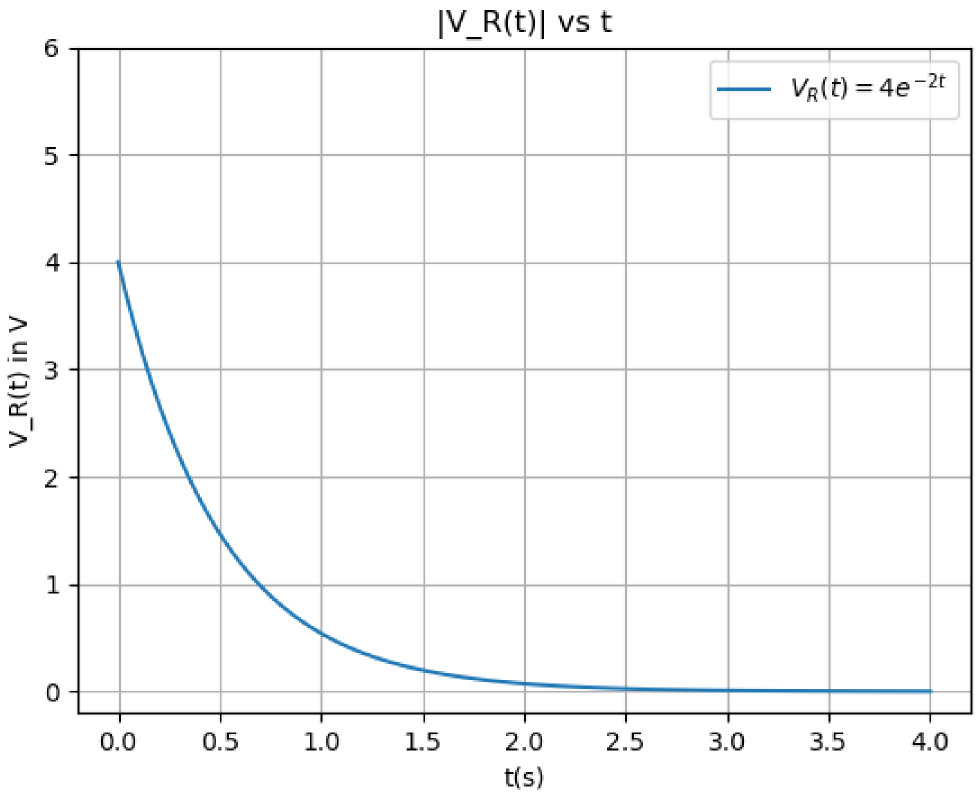
$$\implies V_R(t) = -4e^{-2t}u(t)\,V \tag{2.81}$$

As,

$$t \to 0 \tag{2.82}$$

$$\implies e^{-2t} \to 1 \tag{2.83}$$

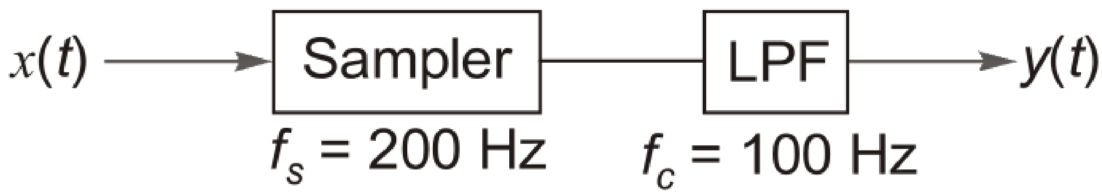
$$|V_R(max)| = 4V \tag{2.84}$$



2.8 A signal $x(t) = 2\cos(180\pi t)\cos(60\pi t)$ is sampled at 200 Hz and then passed through an ideal low pass filter having cut-off frequency of 100 Hz.

The maximum Frequency present in the filtered signal in Hz is ____ (Round off to the nearest integer.) (GATE 2023 EE) **Solution:** Given,

$$x(t) = \cos(240\pi t) + \cos(120\pi t) \tag{2.85}$$



symbol	value	description
$x(t)$	$2\cos(180\pi t)\cos(60\pi t)$	input signal
f_s	$200Hz$	sampling frequency
f_c	$100Hz$	cut-off frequency
$y(t)$		output signal
f_1	$120Hz$	first signal frequency
f_2	$60Hz$	second signal frequency

Table 2.6: Parameters

Aliased frequencies when f_1 frequency signal is sampled at $200Hz$

$$f_1, |f_s \pm f_1|, |2f_s \pm f_1| \dots \tag{2.86}$$

$$120, 80, 340, 280, 520 \dots \tag{2.87}$$

Aliased frequencies when f_2 frequency signal is sampled at $200Hz$

$$f_2, |f_s \pm f_2|, |2f_s \pm f_2| \dots \tag{2.88}$$

$$60, 140, 260, 340, 460 \dots \tag{2.89}$$

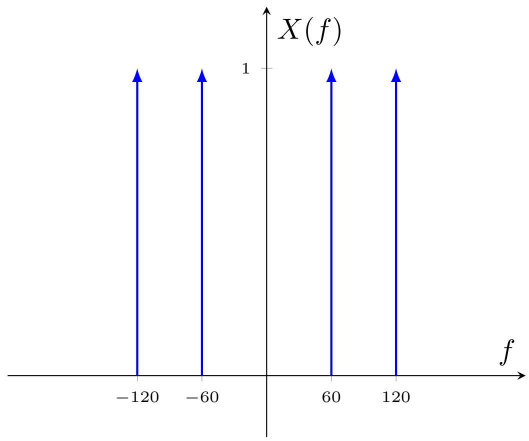


Figure 2.20: delta function of input signal

from table $f_c = 100Hz$

LPF output : $60Hz$, $80Hz$

Maximum Frequency present in the filtered signal is $80Hz$.

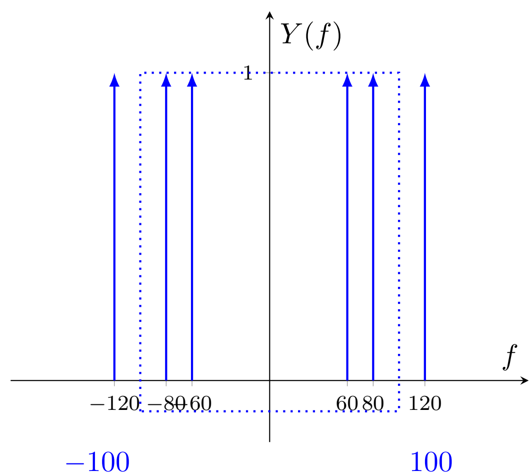
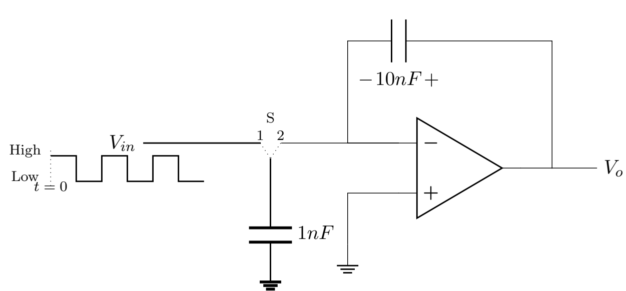


Figure 2.21: delta function of sampled and filtered signal

2.9 In the circuit shown, the input voltage $V_{in} = 100mV$. The switch and the opamp are ideal. At time $t = 0$, the intial charge stored in the $10nF$ capacitor is $1nC$, with the polarity as indicated in the figure. The switch S is controlled using a $1KHz$ square-wave voltage signal V_s as shown. Whenever V_s is 'High', S is in position '1' and when V_s is 'Low', S is in position '2'.

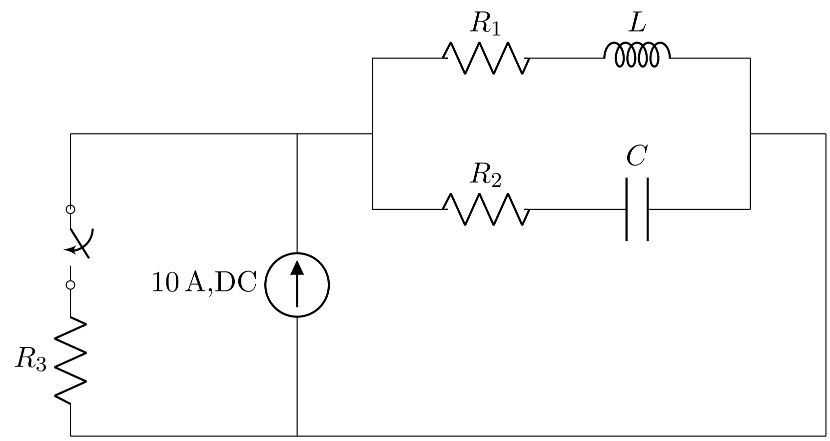
At $t = 20ms$, the magnitude of the voltage V_o will be

(GATE IN 2023)



Solution:

2.10 The value of parameters of the circuit shown in the figure are $R_1 = 2\Omega$,$R_2 = 2\Omega$,$R_3 = 3\Omega$,$L = 10mH$,$C = 100\mu F$. For time $t < 0$, the circuit is at steady state with the switch $'K'$ in closed condition. If the switch is opened at $t = 0$, the value of the voltage across the inductor (V_L) at $t = 0^+$ in Volts is ________ (Round off to 1 decimal

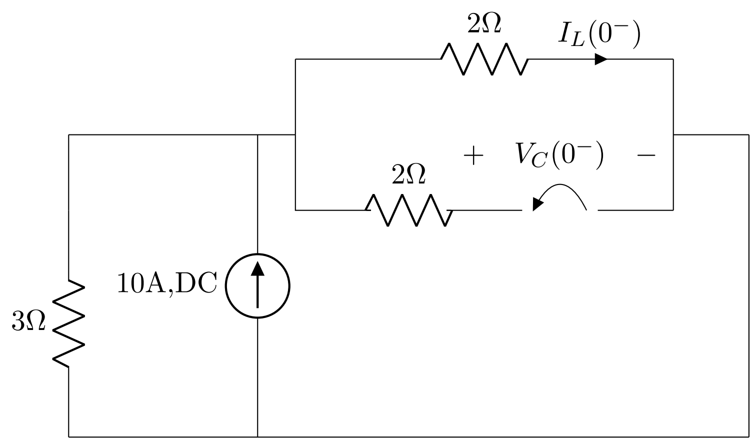


place). (GATE 2023 EE 29 Q) **Solution:**

Symbol	Value	Description
L	$10mH$	Inductance
C	$100muF$	Capacitance
R_1	2Ω	Resistance
R_2	2Ω	Resistance
R_3	3Ω	Resistance
V_L	??	Voltage across the inductor
V_C	??	Voltage across the capacitor
I_0	$10A$	DC current source
I_L	??	Current in inductor

Table 1: Input Parameter

At t=0^-, inductor behaves as wire and capacitor as open switch,

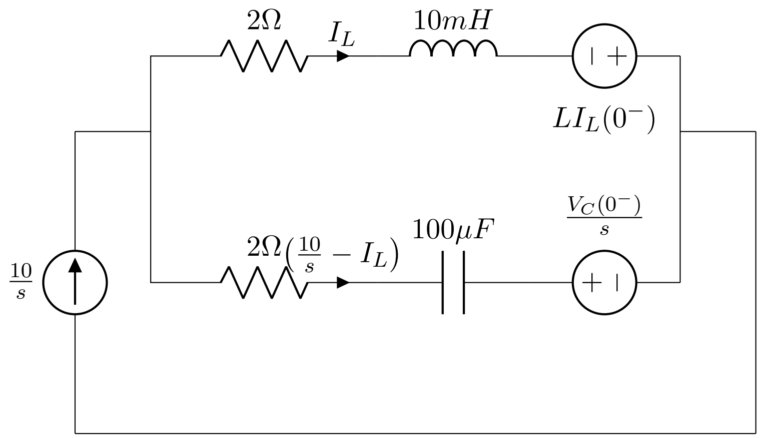


after current distribution

$$I_L(0^-) = 10A\left(\frac{3}{3+2}\right) = 6A \tag{2.90}$$

$$V_C(0^-) = 6 \times 2 = 12V \tag{2.91}$$

For $t > 0$, the switch is opened.



Using KVL,

$$2I_L + LsI_L - LI_L(0^-) - \frac{V_C(0^-)}{s} - \frac{1}{Cs}\left(\frac{10}{s} - I_L\right) - 2\left(\frac{10}{s} - I_L\right) = 0 \quad (2.92)$$

From (2.90), (2.91), (2.92)

$$I_L = \frac{6s^2 + 3200s + 10^7}{s\,(s^2 + 400s + 10^6)} \quad (2.93)$$

$$V_L(s) = I_L(sL) \quad (2.94)$$

Using (2.93)

$$V_L(s) = \frac{0.06s^2 + 32s + 10^5}{(s^2 + 400s + 10^6)} \quad (2.95)$$

Some Result:

$$\frac{1}{s^2 + 400s + 10^6} \overset{\mathcal{L}}{\longleftrightarrow} \left(e^{-200t}\right) \frac{\sin(400\sqrt{6}t)}{400\sqrt{6}} \quad (2.96)$$

$$\frac{s}{s^2 + 400s + 10^6} \overset{\mathcal{L}}{\longleftrightarrow} \left(e^{-200t}\right) \frac{\left(2\sqrt{6}\cos(400\sqrt{6}t) - \sin(400\sqrt{6}t)\right)}{2\sqrt{6}} \quad (2.97)$$

$$\frac{s^2}{s^2 + 400s + 10^6} \overset{\mathcal{L}}{\longleftrightarrow} \left(-e^{-200t}\right) \frac{\left(2300\sin(400\sqrt{6}t) + 400\sqrt{6}\cos(400\sqrt{6}t)\right)}{\sqrt{6}} \quad (2.98)$$

Inverse Laplace transform of (2.95) Using (2.96),(2.97), (2.98)

$$V_L(t) = e^{-200t}\left(-0.06\left(\frac{\left(2300\sin(400\sqrt{6}t) + 400\sqrt{6}\cos(400\sqrt{6}t)\right)}{\sqrt{6}}\right) + 32\left(\frac{\left(2\sqrt{6}\cos(400\sqrt{6}t) - \sin(400\sqrt{6}\right.}{2\sqrt{6}}\right.\right.$$

$$+ e^{-200t}\left(10^5\frac{\sin(400\sqrt{6}t)}{400\sqrt{6}}\right) \quad (2.99)$$

at t=0^+

$$V_L(0^+) = -24 + 32 = 8V \tag{2.100}$$

Hence at t=0^+ voltage across inductor is 8V

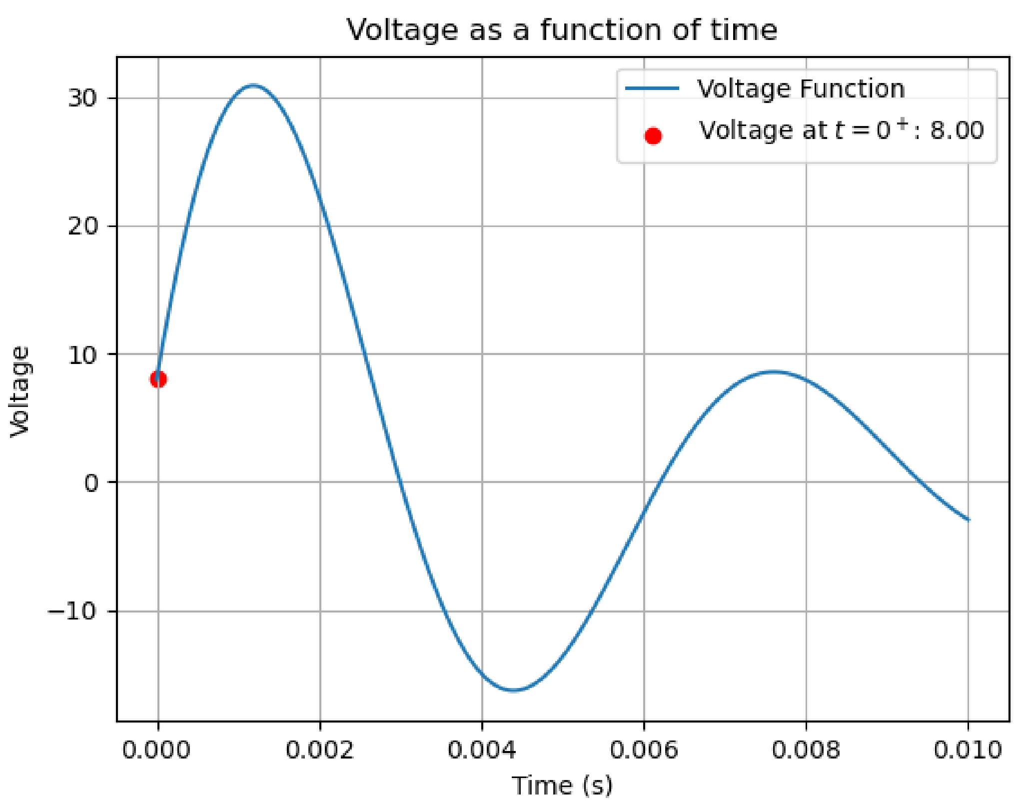
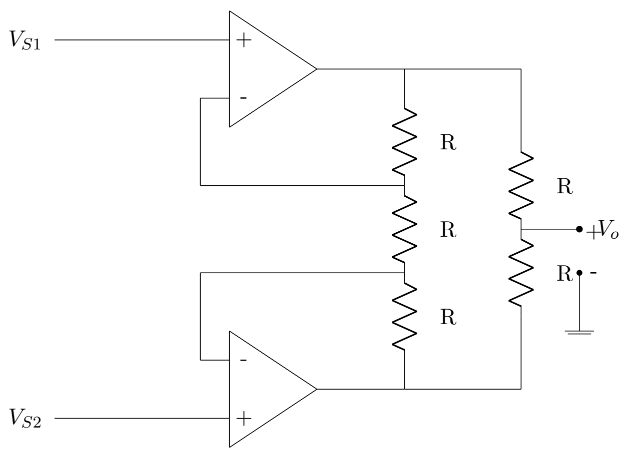


Figure 2.22: plot of voltage as function of t

2.11 The op amps in the circuit are ideal. The input signals are $V_{S1} = 3 + 0.10\sin(300t)$, V and $V_{S2} = -2 + 0.11\sin(300t)$ V. The average value of the voltage V_0 is _____ volts (rounded off to two decimal places). (GATE IN 2023) **Solution:**



Variable	Value	Description
V_{s1}	$3 + 0.10\sin(300t)$	Input voltages
V_{s2}	$-2 + 0.11\sin(300t)$	
R		Resistances of the resistors
V_o		Output voltage
V_1		Output voltage of V_{s1} opamp
V_2		Output voltage of V_{s2} opamp

Table 2.8: Input Parameters

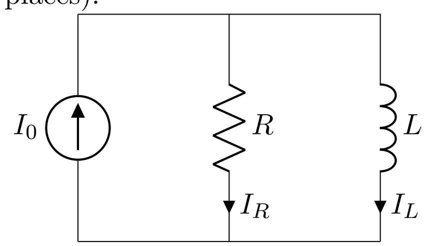
the current does not flow through op-amp. voltage drop by each R

$$= V_{s1} - V_{s2} \tag{2.101}$$

by KVL,

$$\begin{aligned}
V_{s2} - V_2 &= V_{s1} - V_{s2} && (2.102)\\
V_2 &= 2V_{s2} - V_{s1} && (2.103)\\
V_1 - V_{s1} &= V_{s1} - V_{s2} && (2.104)\\
V_1 &= 2V_{s1} - V_{s2} && (2.105)\\
V_o &= \frac{V_1 + V_2}{2} && (2.106)\\
&= \frac{V_{s1} + V_{s2}}{2} && (2.107)\\
&= \frac{3 + 0.10\sin(300t) + -2 + 0.11\sin(300t)}{2} && (2.108)\\
&= 0.5 + \frac{0.21\sin(300t)}{2} && (2.109)\\
V_{avg} &= \frac{1}{T}\int_0^T V(t)\,dt && (2.110)\\
&= \frac{300}{2\pi}\int_0^{\frac{2\pi}{300}} \left(0.5 + \frac{0.21\sin(300t)}{2}\right)\,dt && (2.111)\\
&= 0.5 && (2.112)
\end{aligned}$$

2.12 The R-L circuit with $R = 10k\Omega$ and $L = 1mH$ is excited by a step current $I_0 u(t)$. At $t = 0^-$, there is a current $I_L = I_0/5$ flowing through the inductor. The minimum time taken for the current through the inductor to reach 99% of its final value is $\ldots \mu s$ (rounded off to two decimal places).



(GATE IN 2023) **Solution:**

Transform	**Signal**
$\frac{1}{s(s+a)}$	$\frac{1}{a}(1 - e^{-at})$
$\frac{1}{s+a}$	e^{-at}

Table 2.9: Inverse Laplace transform pairs

$$I_0 u(t) = I_R + I_L \tag{2.113}$$

From KVL, we have:

$$(\frac{I_0}{s} - I_L(s))R - L(sI_L(s) - I_L(0^-)) = 0 \tag{2.114}$$

After Simplyfying we have:

$$I_L(s) = \frac{I_0 R + LsI_L(0^-)}{s(R + Ls)} \tag{2.115}$$

$$I_L(s) = \frac{I_0 R}{L} \frac{1}{s(s + \frac{R}{L})} + \frac{I_0}{5} \frac{1}{\frac{R}{L} + s} \tag{2.116}$$

From Table 2.9, we have:

$$I_L(t) = \frac{I_0 R}{L}\left[\frac{1}{\frac{R}{L}}(1 - e^{-\frac{R}{L}t})\right] + \frac{I_0}{5}e^{-\frac{R}{L}} \tag{2.117}$$

$$I_L(t) = I_0 - \frac{4}{5}I_0 e^{-\frac{R}{L}t} \tag{2.118}$$

$$I_L(t) = I_0 - \frac{4}{5}I_0 e^{-10^7 t} \tag{2.119}$$

$$\lim_{t\to\infty} I_L(t) = I_0 \tag{2.120}$$

Now time when current in inductor is 99% of its final value is given by:

$$0.99I_0 = I_0 - \frac{4}{5}I_0 e^{-\frac{R}{L}t} \tag{2.121}$$

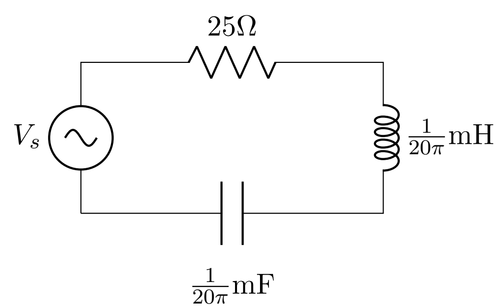
$$0.01I_0 = \frac{4}{5}I_0 e^{-\frac{R}{L}t} \tag{2.122}$$

$$t = \frac{L}{R}\ln(80) \tag{2.123}$$

$$t = 10^{-7}\ln(80)\mu s \tag{2.124}$$

$$t = 0.43\mu s \tag{2.125}$$

2.13 The voltage source $V_s = 10\sqrt{2}\sin(20000\pi t)$ V has an internal resistance of 50 ohms. The RMS value of the current through R is ___ (in mA) (rounded off to one decimal place).



(GATE IN 2023)

Solution:

Parameter	**Value**
V_s	$10\sqrt{2}\sin(20000\pi t)$ V
R	??
R_{internal}	50 ohms
R_{net}	$R + R_{\text{internal}}$
ω	20000π

Table 2.10: Input Parameters

$$
\begin{aligned}
V(s) &= ZI(s) && (2.126)\\
\frac{V(s)}{I(s)} &= Z = R + R_{\text{internal}} + Ls + \frac{1}{Cs} && (2.127)\\
&= 50 + 25 + \frac{s}{20000\pi} + \frac{20000\pi}{s} && (2.128)\\
V_{\text{RMS}} &= \frac{10\sqrt{2}}{\sqrt{2}} = 10\,\text{V} && (2.129)\\
\text{Putting } s &= j\omega && (2.130)\\
\frac{V(j\omega)}{I(j\omega)} &= |Z| \cdot e^{\angle Z} && (2.131)\\
&= 75 + \frac{20000\pi j}{20000\pi} + \frac{20000\pi}{20000\pi j} && (2.132)\\
&= 75 + j - j && (2.133)\\
&= 75 && (2.134)\\
I_{\text{RMS}} &= \frac{V_{\text{RMS}}}{|Z|} && (2.135)\\
&= \frac{10}{75} \times 1000 = \frac{2000}{15} \approx 133.3\,\text{mA} && (2.136)
\end{aligned}
$$

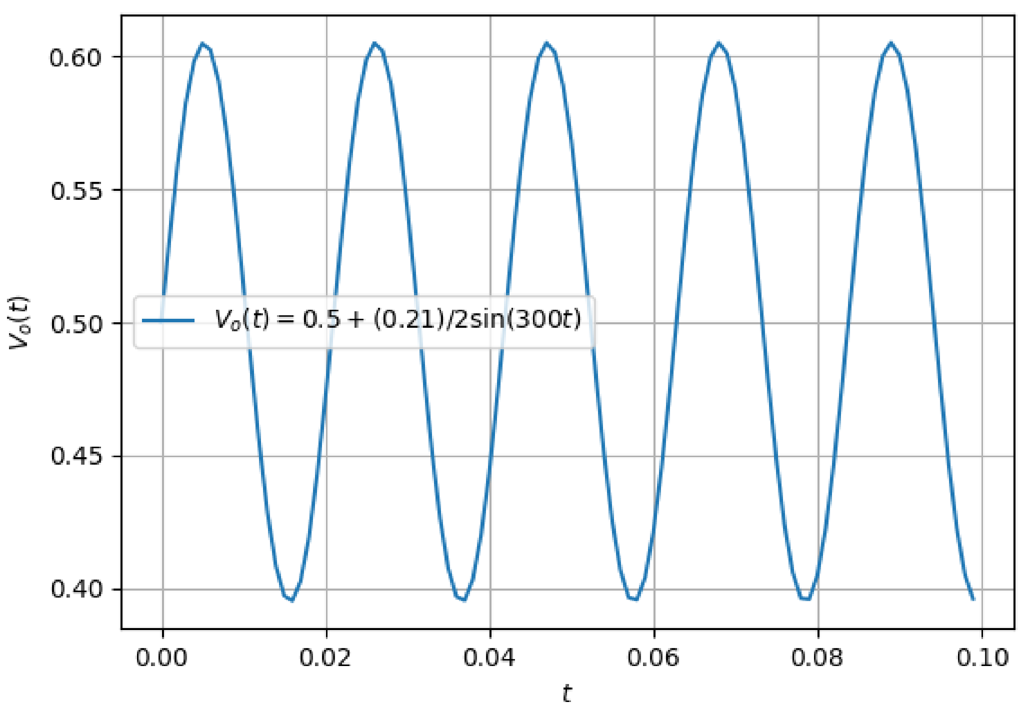
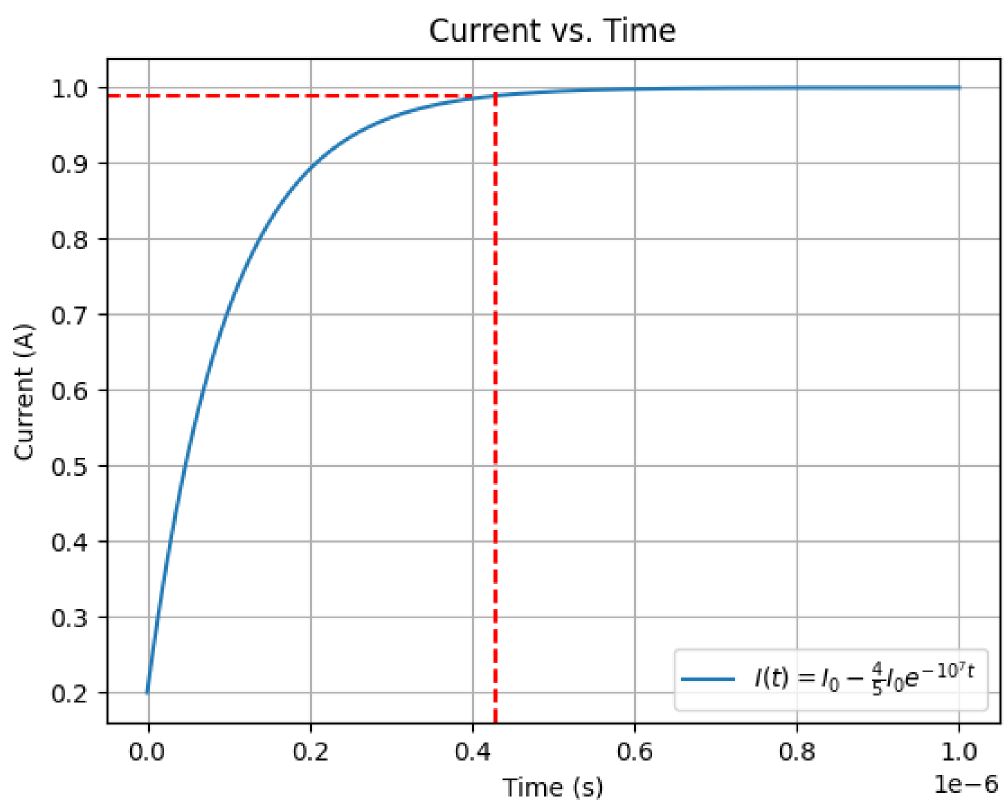
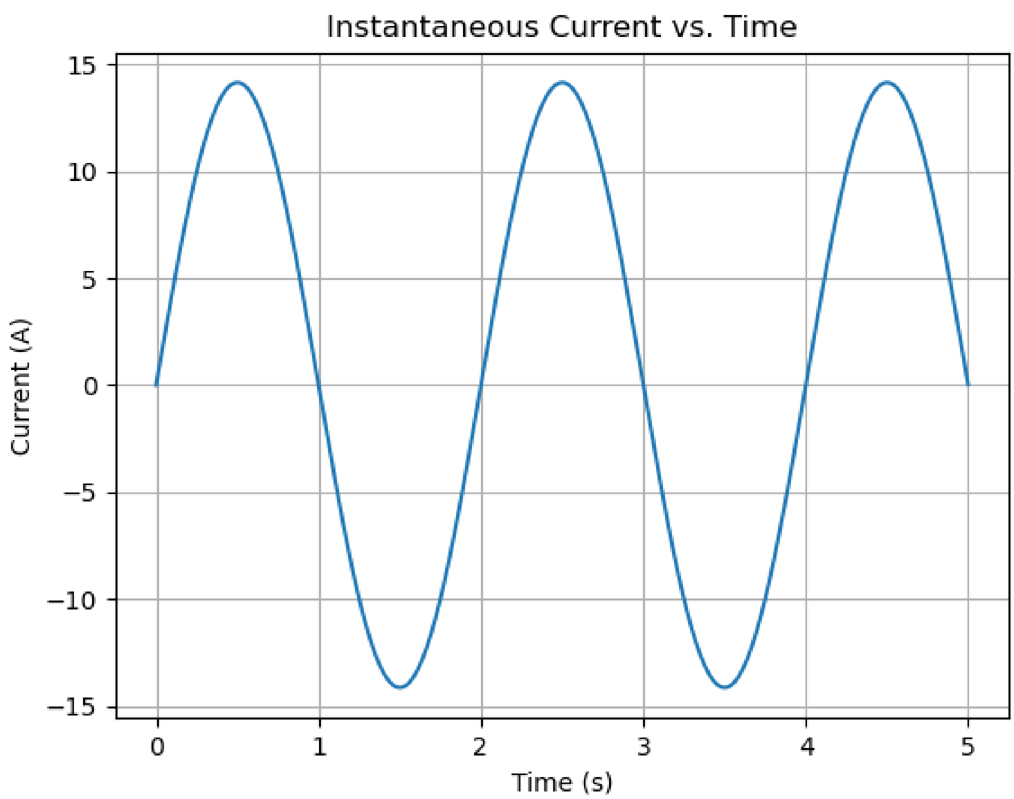


Figure 2.23: line plot



Current vs. Time
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
Current (A)
0.0
0.2
0.4
0.6
0.8
1.0
1e−6
Time (s)
$I(t) = I_0 - \frac{4}{5}I_0e^{-10^7t}$


Instantaneous Current vs. Time
Current (A)
Time (s)
15
10
5
0
−5
−10
−15
0
1
2
3
4
5

Chapter 3

Z-transform

Chapter 4

Systems

4.1 Consider a unity-gain negative feedback system consisting of the plant $G(s)$ and a proportional-integral controller. Let the proportional gain and integral gain be 3 and 1, respectively. For a unit step reference input, the final values of the controller output and the plant output, respectively, are

$$G(s) = \frac{1}{(s-1)}$$

(GATE EE 2023)

Solution:

Parameter	**Description**	**Value**
K_p	Proportional Gain	3
K_i	Integral Gain	1
$r(t)$	Reference Input	$u(t)$
$w(t)$	Controller Output	?
$y(t)$	Plant Output	?
$e(t)$	Error Input	$r(t) - y(t)$

Table 1: Parameter Table

From theFig. 4.1:

$$E(s) = U(s) - Y(s) \tag{4.1}$$

$$W(s) = 3E(s) + \frac{1}{s}E(s) \tag{4.2}$$

$$Y(s) = G(s)W(s) \tag{4.3}$$

Some results:

$$tx(t) \xleftrightarrow{\mathcal{L}} -\frac{dX(s)}{ds} \tag{4.4}$$

$$e^{-at}x(t) \xleftrightarrow{\mathcal{L}} X(s+a) \tag{4.5}$$

By using (4.4) and (4.5):

$$e^{-t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+1}, Re(s) > -1 \tag{4.6}$$

$$te^{-t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{(s+1)^2}, Re(s) > -1 \tag{4.7}$$

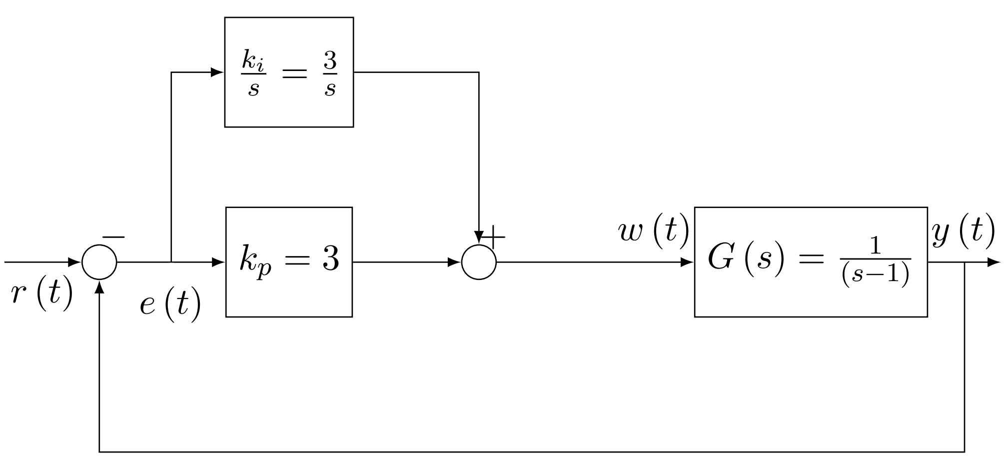


Figure 4.1: Block Diagram of System

(a) **Plant Output:**

From (4.1) , (4.2) and (4.3):

$$Y(s) = \frac{3s+1}{s(s+1)^2}, Re(s) > -1 \tag{4.8}$$

Final Value Theorem:

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s) \tag{4.9}$$

Using (4.9) on Y(s):

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) \tag{4.10}$$

$$= 1 \tag{4.11}$$

Taking partial fraction of (4.8) :

$$Y(s) = \frac{1}{s} + \frac{2}{(s+1)^2} - \frac{1}{s+1} \tag{4.12}$$

Using (4.6) and (4.7):

$$\therefore y(t) = u(t) + 2te^{-t}u(t) - e^{-t}u(t) \tag{4.13}$$

(b) **Controller Output:**

From (4.2)

$$W(s) = \frac{3}{s} + \frac{1}{s^2} - Y(s)\left(3 + \frac{1}{s}\right) \tag{4.14}$$

Substituting (4.8)

$$W\left(s\right)=\frac{\left(s-1\right)\left(3s+1\right)}{s\left(s+1\right)^{2}}, Re\left(s\right)>-1 \tag{4.15}$$

Using (4.9) on W(s)

$$\lim_{t\to\infty}w\left(t\right)=\lim_{s\to 0}sW\left(s\right) \tag{4.16}$$

$$=-1 \tag{4.17}$$

Taking partial fraction of equation(4.15) :

$$W\left(s\right)=-\frac{1}{s}-\frac{4}{\left(s+1\right)^{2}}+\frac{4}{s+1} \tag{4.18}$$

Using equations (4.6) and (4.7) and taking inverse lapalace transform:

$$w\left(t\right)=-u\left(t\right)-4te^{-t}u\left(t\right)+4e^{-t}u\left(t\right) \tag{4.19}$$

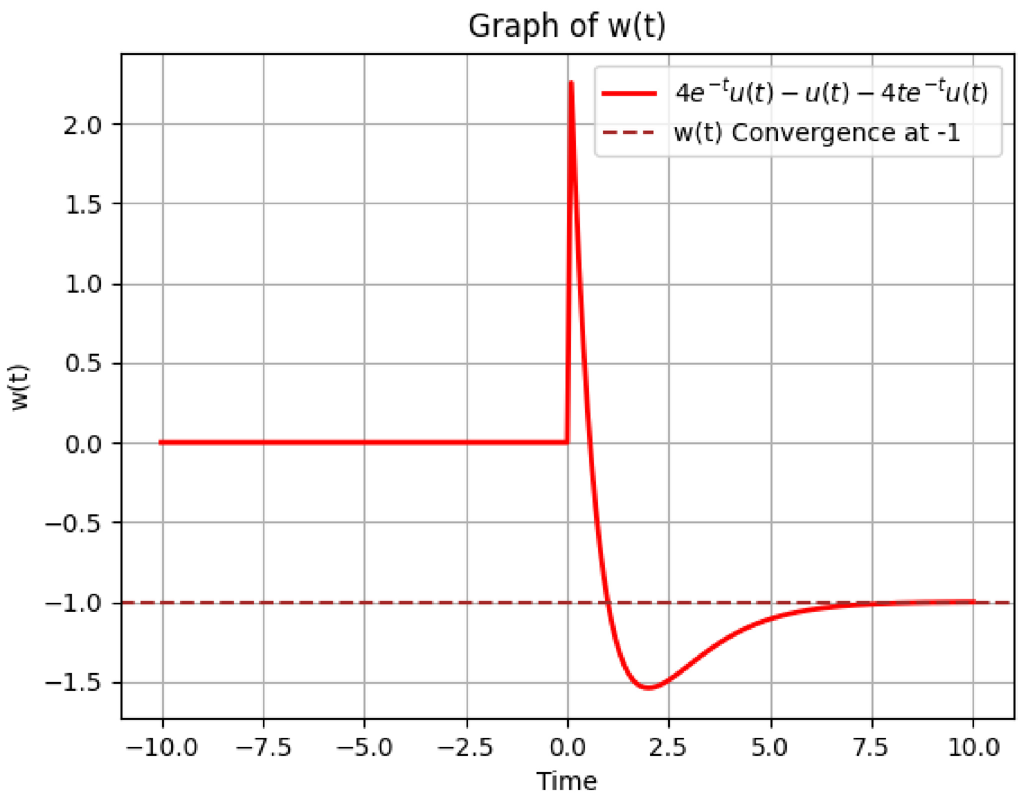


Figure 4.2: $w\left(t\right)$ converges at -1.

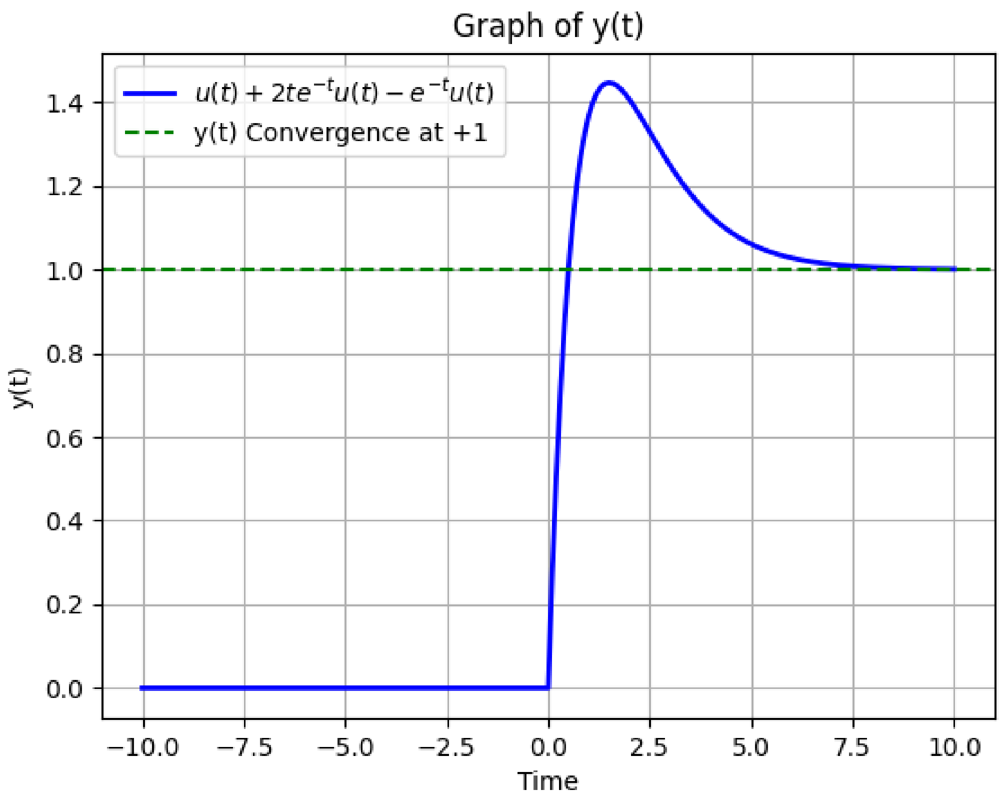


Figure 4.3: $y(t)$ converges at $+1$

4.2 Level (h) in a steam boiler is controlled by manipulating the flow rate (F) of the break-up(fresh) water using a proportional (P) controller. The transfer function between the output and the manipulated input is

$$\frac{h(s)}{F(s)} = \frac{0.25(1-s)}{s(2s+1)}$$

The measurement and the valve transfer functions are both equal to 1. A process engineer wants to tune the controller so that the closed loop response gives the decaying oscillations under the servo mode. Which one of the following is the CORRECT value of the controller gain to be used by the engineer?

(a) 0.25

(b) 2

(c) 4

(d) 6

GATE CH 2023

Solution:

Closed loop signal transfer function of the above block diagram can be given by,

$$T(s) = \frac{G(s)}{1+G(s)H(s)} \tag{4.20}$$

From Fig. 4.4 and Table 4.2 for a unit impulse, $X(s) = 1$

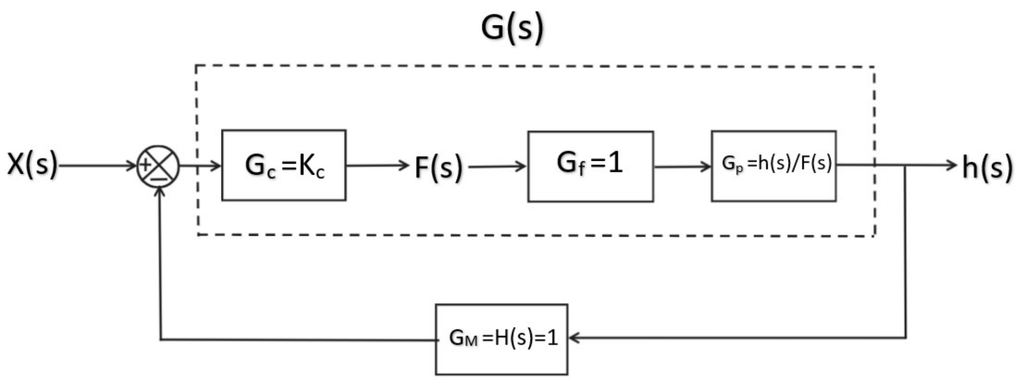


Figure 4.4: Closed loop Block diagram

$$h\left(s\right)=T\left(s\right)\times X\left(s\right) \tag{4.21}$$

$$h\left(s\right)=\frac{\left(1-s\right)K_c}{8s^2+\left(4-K_c\right)s+K_c} \tag{4.22}$$

$$\implies h\left(s\right)=\frac{\left(1-s\right)K_c}{8\left(s-s_1\right)\left(s-s_2\right)} \tag{4.23}$$

Where,

$$s_1=\frac{\left(K_c-4\right)}{16}+\sqrt{\left(\frac{K_c-4}{16}\right)^2-\frac{K_c}{8}} \tag{4.24}$$

$$s_2=\frac{\left(K_c-4\right)}{16}-\sqrt{\left(\frac{K_c-4}{16}\right)^2-\frac{K_c}{8}} \tag{4.25}$$

From (4.23) we get,

$$h(s) = \frac{K_c}{8(s_1 - s_2)} \left(\frac{1 - s_1}{s - s_1} - \frac{1 - s_2}{s - s_2} \right) \tag{4.26}$$

Now taking the inverse laplace transform we have,

$$h(t) = \frac{K_c}{8(s_1 - s_2)} \left[(1 - s_1) e^{s_1 t} - (1 - s_2) e^{s_2 t} \right] u(t) \tag{4.27}$$

$$\implies h(t) = e^{\frac{K_c - 4}{16}} \left(A_1 e^{\sqrt{\left(\frac{K_c-4}{16}\right)^2 - \frac{K_c}{8}}} - A_2 e^{-\sqrt{\left(\frac{K_c-4}{16}\right)^2 - \frac{K_c}{8}}} \right) u(t) \tag{4.28}$$

Where,

$$A_1 = \frac{K_c}{8} \left(\frac{1 - s_1}{s_1 - s_2} \right) \tag{4.29}$$

$$A_2 = \frac{K_c}{8} \left(\frac{1 - s_2}{s_1 - s_2} \right) \tag{4.30}$$

Now applying the condition for underdamped oscillations,

$$\left(\frac{K_c - 4}{16} \right)^2 - \frac{K_c}{8} < 0 \tag{4.31}$$

$$\implies K_c \in \left(20 - \sqrt{384}, 20 + \sqrt{384} \right) \tag{4.32}$$

For the system to be stable,

$$\frac{K_c - 4}{8} < 0 \tag{4.33}$$

$$\implies K_c < 4 \tag{4.34}$$

From (4.32) and (4.34)

$$K_c \in (0.4, 4) \tag{4.35}$$

(4.35) represents the ROC,($R_e\{s\} < 0$)

$$\implies K_c = 2 \tag{4.36}$$

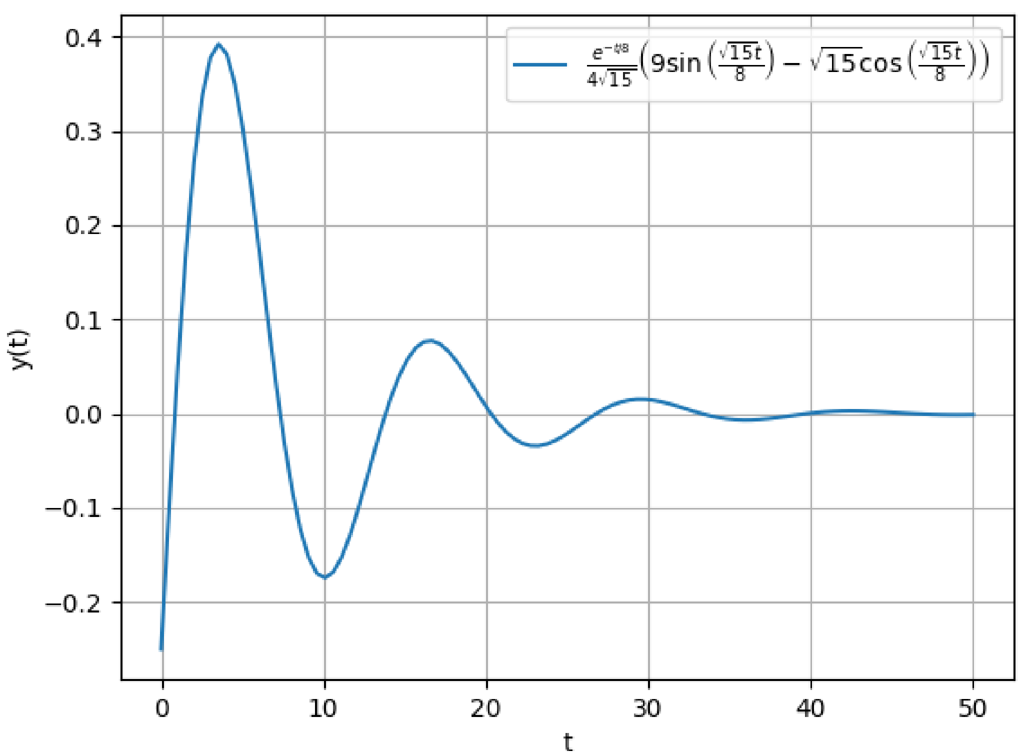


Figure 4.5: y(t) vs t graph

4.3 The figure shows a block of mass m = 20 kg attached to a pair of identical linear springs, each having a spring constant k = 1000 N/m. The block oscillates on a frictionless horizontal surface. Assuming free vibrations, the time taken by the block to complete ten oscillations is ____ seconds . (Rounded off to two decimal places) Take $\pi = 3.14$.

(GATE ME 2023)

Solution:

$$F = ma \tag{4.37}$$

$$F = -kx \tag{4.38}$$

$$\implies ma + kx = 0 \tag{4.39}$$

$$\therefore m\frac{d^2x}{dt^2} + kx = 0 \tag{4.40}$$

The Laplace transform of the terms is ,

$$x \overset{\mathcal{L}}{\longleftrightarrow} X(s) \tag{4.41}$$

$$\frac{d^2x}{dt^2} \overset{\mathcal{L}}{\longleftrightarrow} s^2X(s) - sx(0) - \dot{x}(0) \tag{4.42}$$

Using equation (4.41) and (4.42) in equation (4.40),

$$m\left(s^2X(s) - sx(0) - \dot{x}(0)\right) + kX(s) = 0 \tag{4.43}$$

$$ms^2X(s) - msA + kX(s) = 0 \tag{4.44}$$

$$X(s) = \frac{msA}{ms^2 + k} \tag{4.45}$$

$$= \frac{sA}{s^2 + \frac{k}{m}} \tag{4.46}$$

The inverse Laplace transform of such terms is given by,

$$\frac{s}{s^2 + a^2} \xleftrightarrow{\mathcal{L}^-} cos(at)\,u(t) \tag{4.47}$$

$\therefore$ the inverse Laplace of (4.46) is,

$$x(t) = Acos\left(\sqrt{\frac{k}{m}}t\right) \tag{4.48}$$

From equation (4.48) and Table 4.3 ,the time to complete one oscillation is,

$$T_n = \frac{2\pi}{\sqrt{\frac{k}{m}}} \tag{4.49}$$

$$= \frac{\pi}{5} \tag{4.50}$$

$\therefore$ the time required for 10 oscillations is ,

$$10T_n = 2\pi \tag{4.51}$$

$$= 6.28s \tag{4.52}$$

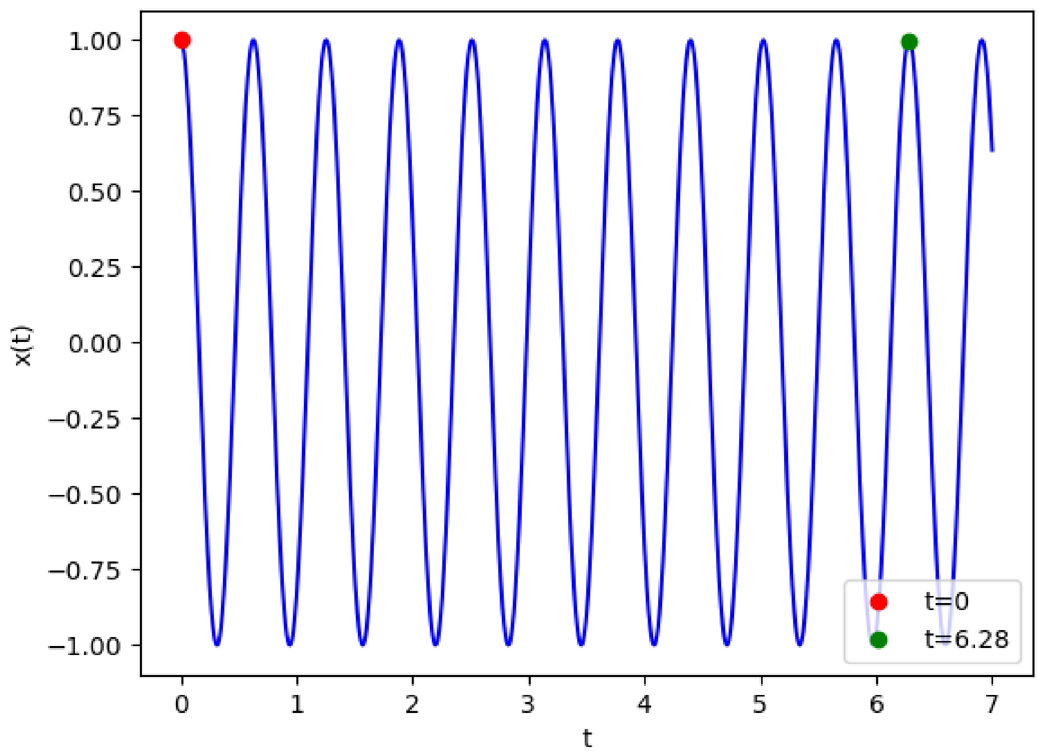


Figure 4.6: Plot of $x\left(t\right)$

4.4 A system has transfer function

$$\frac{Y(s)}{X(s)} = \frac{s-\pi}{s+\pi}$$

let $u(t)$ be the unit step function.The input $x(t)$ that results in a steady-state output $y(t) = sin(\pi t)$ is __. (GATE IN 2023)

Solution:

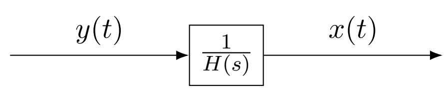


Figure 4.7: Block diagram of the inverse system

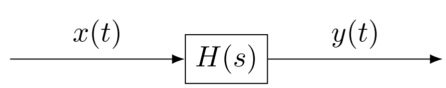


Figure 4.8: Block diagram of the system

$$H(s) = \frac{s-\pi}{s+\pi} \tag{4.53}$$

from Fig. 4.7

$$\frac{1}{H(s)} = \frac{s+\pi}{s-\pi} \tag{4.54}$$

Converting transfer function to frequency response, we get

$$\frac{1}{H(j\omega)} = \frac{j\omega+\pi}{j\omega-\pi} \tag{4.55}$$

from Table 4.4 $\omega = \pi$

$$\frac{1}{H(j\pi)} = \frac{j+1}{j-1} = -j = e^{-j\frac{\pi}{2}} \tag{4.56}$$

from (4.56)

$$\left|\frac{1}{H(j\pi)}\right| = 1 \quad \text{and} \quad \arg\left(\frac{1}{H(j\pi)}\right) = -90^\circ \tag{4.57}$$

$$y(t) = \sin(\pi t) \tag{4.58}$$

$$\sin(\pi t) \overset{\frac{\infty}{\mathcal{H}(|\omega)}}{\longleftrightarrow} \left|\frac{1}{H(j\omega)}\right| \sin\left(\pi t + \arg\left(\frac{1}{H(j\omega)}\right)\right) \tag{4.59}$$

Therefore, by (4.58) and (4.59), we get

$$x(t) = \sin\left(\pi t - \frac{\pi}{2}\right) \tag{4.60}$$

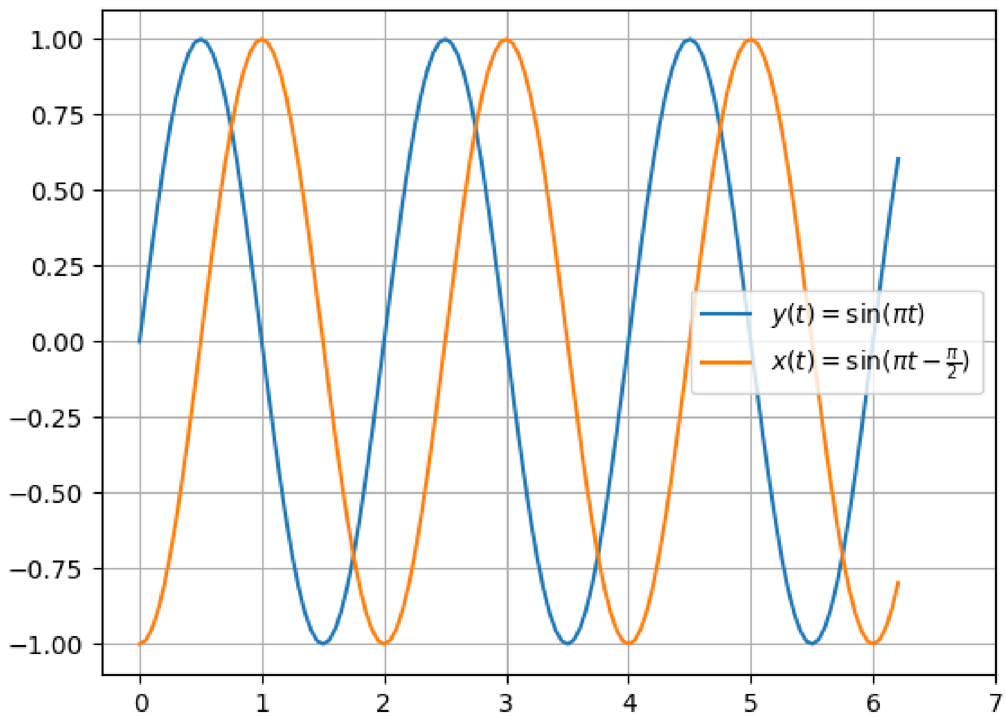


Figure 4.9: Plot of $x(t)$ and $y(t)$ taken from Python

4.5 Consider the complex function

$$f(z) = \frac{z^2 \sin z}{(z-\pi)^4}$$

At $z = \pi$, which of the following options is (are) correct?

(A) The order of the pole is 4

(B) The order of the pole is 3

(C) The residue at the pole is $\frac{\pi}{6}$

(D) The residue at the pole is $\frac{2\pi}{3}$

(GATE PH 2023) **Solution:**

(a) As the power of $(z-\pi)$ in denominator is 4 , so the order of the pole is 4.

(b)

$$\operatorname{Res}(f,\pi) = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[(z-\pi)^m f(z)\right]\Bigg|_{z=\pi} \tag{4.61}$$

$$\operatorname{Res}(f,\pi) = \frac{1}{3!}\frac{d^3}{dz^3}\left[(z-\pi)^4\frac{z^2\sin z}{(z-\pi)^4}\right]\Bigg|_{z=\pi} \tag{4.62}$$

$$\operatorname{Res}(f,\pi) = \frac{1}{3!}\frac{d^3}{dz^3}z^2\sin z\Bigg|_{z=\pi} \tag{4.63}$$

$$= \frac{1}{3!}(6\cos z - 6z\sin z - z^2\cos z)\Bigg|_{z=\pi} \tag{4.64}$$

$$\tag{4.65}$$

Since $\sin(\pi)=0$ and $\cos(\pi)=-1$, this simplifies to:

$$\operatorname{Res}(f,\pi) = \frac{\pi^2-6}{3!} = \frac{\pi^2-6}{6} \tag{4.66}$$

4.6 A buoy of virtual mass 30 kg oscillates in a fluid medium as a single degree of freedom system. If the total damping in the system is set as 188.5 N-s/m, such that the oscillation just ceases to occur, then the natural period of the system is ______ s (round off to one decimal place) (GATE MN 2023 question 63)

Solution:

The differential equation of the system is:

$$m\frac{d^2x\left(t\right)}{dt^2} + \lambda\frac{dx\left(t\right)}{dt} + m\omega_0^2 x\left(t\right) = 0 \tag{4.67}$$

Taking laplace transform:

$$ms^2X\left(s\right) + \lambda sX\left(s\right) + m\omega_o^2X\left(s\right) = 0 \tag{4.68}$$

$$\implies ms^2 + s\lambda + m\omega_o^2 = 0 \tag{4.69}$$

$$\therefore s = \frac{-\lambda \pm \sqrt{\lambda^2 - 4m^2\omega_o^2}}{2m} \tag{4.70}$$

where $\sqrt{\lambda^2 - 4m^2\omega_o^2}$ is ω_d.

From Table 4.6,

$$\omega_d = 0 \tag{4.71}$$

$$\implies \lambda = 2\omega_o m \tag{4.72}$$

$$\implies \omega_o \approx \pi \tag{4.73}$$

$$T_i = \frac{2\pi}{\omega_o} \tag{4.74}$$

$$\therefore t_i = 2 \text{ seconds} \tag{4.75}$$

To find x(t), we assume the initial amplitude of oscillations to be 1 meter and it is situated at extreme position.

$$s^2 X(s) + \lambda s X(s) + \omega_o^2 X(s) = sx(0) + x(0) \tag{4.76}$$

$$\implies X(s) = \frac{1+s}{s^2 + s\frac{\lambda}{m} + \omega_o^2} \tag{4.77}$$

substituting values from Table 4.6 and ω_o,

$$X(s) = \frac{1+s}{(s+\pi)^2} \tag{4.78}$$

Taking inverse laplace transform by method of partial fractions,

$$X(s) = \frac{1}{s+\pi} + \frac{1-\pi}{(s+\pi)^2} \tag{4.79}$$

$$\therefore x(t) = (1 + (1-\pi)t)\, e^{-\pi t} \tag{4.80}$$

4.7 Which of the following statement(s) is/are true?

(a) If an LTI system is causal, it is stable.

(b) A discrete time LTI system is causal if and only if its response to a step input $u[n]$ is 0 for $n < 0$.

(c) If a discrete time LTI system has an impulse response $h[n]$ of finite duration the system is stable.

(d) If the impulse response $0 < |h[n]| < 1$ for all n, then the LTI system is stable.

(GATE EE 2023 question 27)

Solution:

(a)

$$\text{Assume} \quad h(t) = e^{2t} \cdot u(t) \tag{4.81}$$

$$\mathcal{L}\{e^{2t}\} = \frac{1}{s-2} \quad Re(s) > 2 \tag{4.82}$$

This system is causal but not stable because ROC does not contian imaginary axis.

Therefore, this statement does not hold true.

(b) For a causal system, its response to the step input should indeed be zero for $n < 0$, as the system hasn't yet "seen" any input before time $n = 0$.

Mathematically, the output of an LTI system $y[n]$ can be represented as the convolution of the input $u[n]$ with the system's impulse response $h[n]$:

$$y[n] = \sum_{k=-\infty}^{\infty} h[n]\, u[n-k] \tag{4.83}$$

Now applying step input,

$$y[n] = \sum_{k=0}^{\infty} h[n]\, u[n-k] \tag{4.84}$$

This is because $u[n-k]$ is zero for $n < k$, hence, the summation only starts from $k = 0$.

For $n < 0$, $u[n-k] = 0$ for all k, because $n - k < 0$ when $n < 0$. Therefore, $y[n] = 0$ for $n < 0$.

Therefore, this statement is true.

(c)

$$\text{Assume} \quad h[n] = \begin{cases} n & \text{if } 0 \leq n \leq N \\ 0 & \text{otherwise} \end{cases} \tag{4.85}$$

$$y[n] = h[n] * u[n] \tag{4.86}$$

$$y[n] = \sum_{k=0}^{n} h[k]\, u[n-k] \tag{4.87}$$

$$= \sum_{k=0}^{n} k \tag{4.88}$$

The input response is finite but the output rsponse in not BIBO stable.

Therefore, this statement does not hold true.

(d)

$$\text{Assume} \quad h[n] = \frac{1}{2}u[n] \tag{4.89}$$

$$g[n] = \sum_{n=0}^{\infty} |h[n]| \tag{4.90}$$

$$= \sum_{n=0}^{\infty} \frac{1}{2} \tag{4.91}$$

$$\implies \sum_{n=0}^{\infty} |h[n]| \to \infty \tag{4.92}$$

Hence it is unstable.

Therefore, this statement does not hold true.

So, the answer is option (B).

4.8 The outlet concentration C_A of a plug flow reactor (PFR) is controlled by manipulating the inlet concentration C_{A0}.The following transfer function describes the dynamics of this PFR.

$$\frac{C_A(s)}{C_{A0}(s)} = e^{-\left(\frac{V}{F}\right)(k+s)}$$

In the above question, V=1m^3,F=0.1$m^3 min^{-1}$ and k=0.5min^{-1}.The measurement and valve transfer functions are both equal to 1.The ultimate gain, defined as the proportional controller gain that produces sustained oscillations, for this system is (GATE 2023 CH 61)

Solution:

4.9 For the block diagram shown in the figure, the transfer function $\frac{Y(s)}{R(s)}$ is

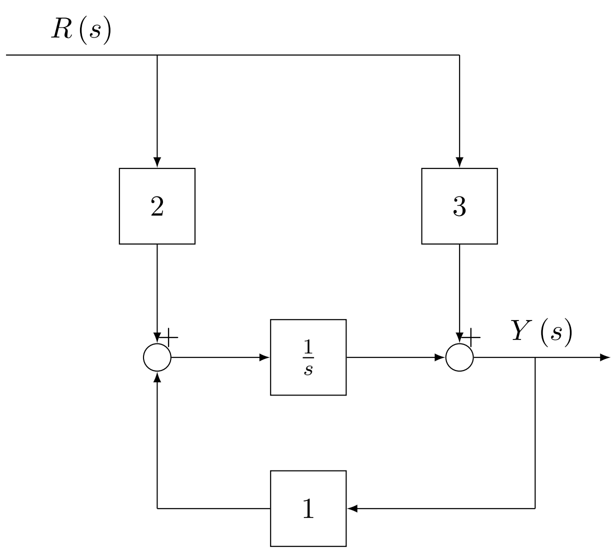


Figure 4.16: Block diagram

(GATE EE 2023)

Solution:

Parameter	**Description**	**Value**
$Y(s)$	Output node signal	
$R(s)$	Input node signal	
$\frac{Y(s)}{R(s)}$	Transfer function	?
P_1	Forward path gain a-b-c	$\frac{2}{s}$
P_2	Forward path gain a-c	3
Δ_1	Determinant of forward path a-b-c	1
Δ_2	Determinant of forward path a-c	1
Δ	Determinant of system	$1-\frac{1}{s}$
n	Number of forward path	2

Table 4.7: Parameters

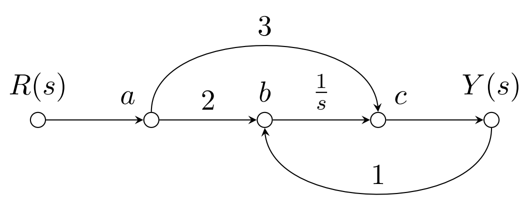


Figure 4.17: signal flow graph

$$P_1 = 2\left(\frac{1}{s}\right) = \frac{2}{s} \tag{4.93}$$

$$P_2 = 3 \tag{4.94}$$

$$\Delta_1 = 1-(0) = 1 \tag{4.95}$$

$$\Delta_2 = 1-(0) = 1 \tag{4.96}$$

$$L_1 = \frac{1}{s} \tag{4.97}$$

$$\Delta = 1 - L_1 = 1 - \frac{1}{s} \tag{4.98}$$

from Fig. 4.17 using Mason's Gain Formula,

$$\frac{Y(s)}{R(s)} = \frac{\sum_{i=1}^{n} P_i \Delta_i}{\Delta} \tag{4.99}$$

$$= \frac{P_1 \Delta_1 + P_2 \Delta_2}{\Delta} \tag{4.100}$$

$$= \frac{\frac{2}{s} + 3}{1 - \frac{1}{s}} \tag{4.101}$$

$$H(s) = \frac{Y(s)}{R(s)} = \frac{3s + 2}{s - 1} \tag{4.102}$$

$$H(s) = \frac{5}{s-1} + 3 \tag{4.103}$$

$$H(s) \overset{\mathcal{L}^{-\infty}}{\longleftrightarrow} h(t) \tag{4.104}$$

$$\frac{5}{s-1} \overset{\mathcal{L}^{-\infty}}{\longleftrightarrow} 5e^{t} \tag{4.105}$$

$$3 \overset{\mathcal{L}^{-\infty}}{\longleftrightarrow} 3\delta(t) \tag{4.106}$$

using (4.105) and (4.106),

$$h(t) = 5e^{t} + 3\delta(t) \tag{4.107}$$

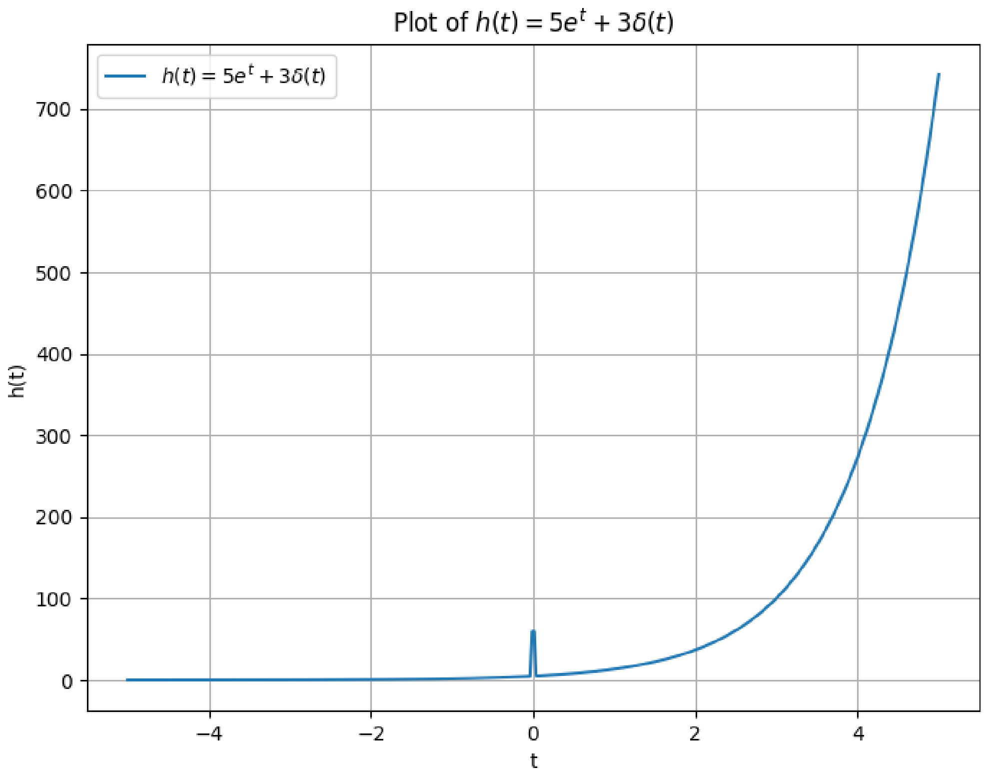


Figure 4.18: Plot of impulse response of the system

4.10 In the following block diagram, $R(s)$ and $D(s)$ are two inputs. The output Y(s) is expressed as $Y(s) = G_1(s)R(s) + G_2(s)D(s)$

$G_1(s)$ and $G_2(s)$ are given by

a) $G_1(s) = \frac{G(s)}{1+G(s)+G(s)H(s)}$ and $G_2(s) = \frac{G(s)}{1+G(s)+G(s)H(s)}$

b) $G_1(s) = \frac{G(s)}{1+G(s)+H(s)}$ and $G_2(s) = \frac{G(s)}{1+G(s)+H(s)}$

c) $G_1(s) = \frac{G(s)}{1+G(s)+H(s)}$ and $G_2(s) = \frac{G(s)}{1+G(s)+G(s)H(s)}$

d) $G_1(s) = \frac{G(s)}{1+G(s)+G(s)H(s)}$ and $G_2(s) = \frac{G(s)}{1+G(s)+H(s)}$

GATE 2023 EC Q.42

Solution:

By superposition principle, let

$$Y(s) = Y_1(s) + Y_2(s)$$

$$\text{where } Y_1(s) = \text{output considering only } R(s)$$

$$Y_2(s) = \text{Output considering only } D(s)$$

When only R(s) is present:

By Mason's gain formula,

$$\frac{Y_1\left(s\right)}{R(s)} = \sum_{k=1}^{n} \frac{P_k.\Delta_k}{\Delta} \tag{4.108}$$

Here,

$$n = 1\,(\text{i.e path } R - y_1 - y_2 - y_3 - Y_1) \tag{4.109}$$

$$P_1 = G\left(s\right) \tag{4.110}$$

$$\Delta_1 = 1\,(\text{As no isolated node is present}) \tag{4.111}$$

$$\Delta = 1 + G\left(s\right) + G\left(s\right)H\left(s\right) \tag{4.112}$$

Substituting in (4.108),

$$\frac{Y_1\left(s\right)}{R\left(s\right)} = \frac{G\left(s\right)}{1 + G\left(s\right) + G\left(s\right)H\left(s\right)} \tag{4.113}$$

$$Y_1\left(s\right) = \left[\frac{G\left(s\right)}{1 + G\left(s\right) + G\left(s\right)H\left(s\right)}\right]R\left(s\right) \tag{4.114}$$

Hence,

$$G_1\left(s\right) = \frac{G\left(s\right)}{1 + G\left(s\right) + G\left(s\right)H\left(s\right)} \tag{4.115}$$

When only $D\left(s\right)$ is present:

Here,

$$n = 1\,(\text{i.e path } D - y_1 - y_2 - Y_2) \tag{4.116}$$

$$P_1 = G\,(s) \tag{4.117}$$

$$\Delta_1 = 1 \tag{4.118}$$

$$\Delta = 1 + G\,(s) + G\,(s)\,H\,(s) \tag{4.119}$$

Substituting in (4.108),

$$\frac{Y_2\,(s)}{D\,(s)} = \frac{G\,(s)}{1 + G\,(s) + G\,(s)\,H\,(s)} \tag{4.120}$$

$$Y_2\,(s) = \left[\frac{G\,(s)}{1 + G\,(s) + G\,(s)\,H\,(s)}\right] D\,(s) \tag{4.121}$$

Hence,

$$G_2\,(s) = \frac{G\,(s)}{1 + G\,(s) + G\,(s)\,H\,(s)} \tag{4.122}$$

Option (a) is correct

4.11 In the table shown below, match the signal type with its spectral characteristics.GATE 2023 EC

(a) $(i) \rightarrow(a)$, $(ii) \rightarrow(b)$, $(iii) \rightarrow(c)$, $(iv) \rightarrow(d)$

(b) $(i) \rightarrow(a)$, $(ii) \rightarrow(c)$, $(iii) \rightarrow(b)$, $(iv) \rightarrow(d)$

(c) $(i) \rightarrow(d)$, $(ii) \rightarrow(b)$, $(iii) \rightarrow(c)$, $(iv) \rightarrow(a)$

(d) $(i) \rightarrow(a)$, $(ii) \rightarrow(c)$, $(iii) \rightarrow(d)$, $(iv) \rightarrow(b)$

Solution:

1. Continuous, aperiodic signal

$$X\left(f\right)=\int_{-\infty}^{\infty}x\left(t\right).e^{-j2\pi ft}\,dt \tag{4.123}$$

Let's consider the limit as f approaches a certain frequency f_0:

$$lim_{\epsilon\rightarrow 0}X\left(f_0+\epsilon\right)=lim_{\epsilon\rightarrow 0}\int_{-\infty}^{\infty}x\left(t\right).e^{-j2\pi ft}\,dt \tag{4.124}$$

By continuity of $x\left(t\right)$ we can interchange the limit and the integral:

$$=\int_{-\infty}^{\infty}x\left(t\right).lim_{\epsilon\rightarrow 0}e^{-j2\pi\left(f_0+\epsilon\right)t}\,dt \tag{4.125}$$

$$=\int_{-\infty}^{\infty}x\left(t\right).e^{-j2\pi f_0 t}\,dt \tag{4.126}$$

$$=X\left(f_0\right) \tag{4.127}$$

$$lim_{f_0-\epsilon}=X\left(f_0\right) \tag{4.128}$$

Therefore, X(f) is continuous for all frequencies f.

Let's assume $X\left(f\right)$ is periodic with period T

$$X\left(f+T\right)=X\left(f\right) \tag{4.129}$$

Now applying inverse Fourier transform

$$x(t) = \int_{-\infty}^{\infty} X(f).e^{j2\pi ft}\,df \tag{4.130}$$

$$= \int_{-\infty}^{\infty} X(f+T).e^{j2\pi(f+T)t}\,df \tag{4.131}$$

$$= \int_{-\infty}^{\infty} X(f).e^{j2\pi(f+T)t}\,df \tag{4.132}$$

$$= e^{j2\pi Tt}\int_{-\infty}^{\infty} .X(f).e^{2\pi ft}\,df \tag{4.133}$$

$e^{j2\pi Tt}$ is a periodic function of t, which contradicts that $x(t)$ is aperiodic,

Therefore, $X(f)$ cannot be periodic, hence it must be aperiodic.

For Example: Exponential Decay

From Table 4.10

$$x(t) = e^{-2t}.u(t) \tag{4.134}$$

$$X(f) = \int_{-\infty}^{\infty} e^{-2ft}u(t).e^{-j2\pi ft}\,dt \tag{4.135}$$

$$= \int_{0}^{\infty} e^{-(2+j2\pi f)t}\,dt \tag{4.136}$$

$$= \frac{1}{2(1+j\pi f)} \tag{4.137}$$

$\Longrightarrow$ $X(f)$ is continuous and aperiodic

2. Continuous, periodic signal

$$x(t) = x(t + kT) \tag{4.138}$$

$$x(t) \leftrightarrow X(f) \tag{4.139}$$

$$X(f) = \int_{-\infty}^{\infty} x(t).e^{-j2\pi ft}\,dt \tag{4.140}$$

$$X(f) = \int_{-\infty}^{\infty} x(t + kT).e^{-j2\pi ft}\,dt \tag{4.141}$$

$$\text{Let } \tau = t + kT \tag{4.142}$$

$$X(f) = \int_{-\infty}^{\infty} x(\tau).e^{-j2\pi f\tau}.e^{j2\pi fkT}\,d\tau \tag{4.143}$$

$e^{j2\pi fkT}$ is a Periodic function in frequency domain with period $\frac{1}{T}$ Hz

$$X(f) = e^{j2\pi fkT} \int_{-\infty}^{\infty} x(\tau).e^{-j2\pi f\tau}\,d\tau \tag{4.144}$$

$$\Longrightarrow X(f) \text{ is aperiodic} \tag{4.145}$$

$X(f)$ will be a sum of scaled copies of the Fourier transform of $x(t)$, each copy shifted by multiples of $\frac{1}{T}$Hz This results in a discrete frequency domain representation.

For example: Sine wave function

From Table 4.10

$$x(t) = sin(2\pi ft) \tag{4.146}$$

$$sin(f_0 t) = \frac{1}{2j}\left[e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}\right] \tag{4.147}$$

$$X(f) = \frac{1}{2j}\left[2\pi\delta(2\pi f - 2\pi f_0) - 2\pi\delta(2\pi f + 2\pi f_0)\right] \tag{4.148}$$

$$= j\pi\left[\delta(2\pi f + 2\pi f_0) - \delta(2\pi f - 2\pi f_0)\right] \tag{4.149}$$

$\Longrightarrow$ $X(f)$ is Discrete and Aperiodic Signal

3. Discrete, aperiodic signal

$$X\left(e^{j2\pi f}\right) = \sum_{n=-\infty}^{\infty} x[n] . e^{-j2\pi fn} \tag{4.150}$$

Let's consider the Discrete time Fourier Transform (DFT) of $x[n]$, denoted by $X[k]$ which is defined as:

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn} \tag{4.151}$$

The DFT is inherently periodic in frequency domain with period $\frac{1}{N}$.

As N approaches infinity, the spacing between frequency samples in the frequency domain $\frac{1}{N}$ approaches zero. Therefore, the DFT $X[k]$ approaches the continuous Fourier

Transform $X(f)$.

Since the DFT $X(f)$ is periodic in the frequency domain, its limiting form $X(f)$ is also periodic, continuous in frequency domain.

For example: Exponential Decay

From Table 4.10

$$x[n] = e^{n}.u[n] \tag{4.152}$$

$$X\left(e^{j2\pi f}\right) = \sum_{n=-\infty}^{\infty} e^{n}.u[n].e^{-j2\pi fn} \tag{4.153}$$

$$= \sum_{n=0}^{\infty} e^{(1-j2\pi f)^{n}} \tag{4.154}$$

$$= \frac{1}{1-e^{(1-j2\pi f)}} \tag{4.155}$$

$\Longrightarrow$ Continuous and Periodic signal

4. Discrete,periodic signal

$$X(f) = \sum_{n=-\infty}^{\infty} x[n].e^{-j2\pi fn} \tag{4.156}$$

The Discrete Fourier Transform(DFT) is periodic in frequency domain with period $\frac{1}{N}$

As N is finite , Therefore DFT has finite period $\frac{1}{N}$

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn} \tag{4.157}$$

The DFT computation involves summing over a finite number of samples . Therefore, the resulting spectrum $X[k]$ is inherently discrete.

For example: Sinusoidal Signal

From Table 4.10

$$x[n] = sin(2\pi f_0 n).u[n] \tag{4.158}$$

$$X\left(e^{j2\pi f}\right) = \sum_{n=-\infty}^{\infty} sin(2\pi f_0 n).u[n].e^{-j2\pi fn} \tag{4.159}$$

$$= \sum_{n=0}^{\infty} \frac{e^{j2\pi f_0 n} - e^{-j2\pi f_0 n}}{2j}.e^{-j2\pi fn} \tag{4.160}$$

$$= \frac{1}{2j}\sum_{n=0}^{\infty}\left(e^{j(2\pi f_0 n - 2\pi fn)} - e^{-j(2\pi f_0 n + 2\pi fn)}\right) \tag{4.161}$$

$$= \frac{1}{2j}\left(\frac{1}{1-e^{j(2\pi f_0 - 2\pi f)}} - \frac{1}{1-e^{-j(2\pi f_0 + 2\pi f)}}\right) \tag{4.162}$$

$\implies$ Discrete and Periodic

$(i) \rightarrow (a)$, $(ii) \rightarrow (c)$, $(iii) \rightarrow (b)$, $(iv) \rightarrow (d)$

4.12 The impulse response of an LTI system is $h(t) = \delta(t) + 0.5\delta(t-4)$, where $\delta(t)$ is continuous-time unit impulse signal.If the input signal $x(t) = \cos\left(\frac{7\pi t}{4}\right)$,the output is (GATE IN 2023)

Solution:

from Table 4.11

$$y(t) = x(t) * h(t) \tag{4.163}$$

$$= x(t) * (\delta(t) + 0.5\delta(t-4)) \tag{4.164}$$

$$= x(t) + 0.5x(t-4) \tag{4.165}$$

$$= \cos\left(\frac{7\pi t}{4}\right) + 0.5\cos\left(\frac{7\pi(t-4)}{4}\right) \tag{4.166}$$

$$= \cos\left(\frac{7\pi t}{4}\right) + 0.5\cos\left(\frac{7\pi t}{4} - 7\pi\right) \tag{4.167}$$

$$= \frac{1}{2}\cos\left(\frac{7\pi t}{4}\right) \tag{4.168}$$

$$= \frac{1}{2}x(t) \tag{4.169}$$

$$Y(f) = X(f)H(f) \tag{4.170}$$

$$= \frac{1}{2}\left[\delta\left(f - \frac{7}{8}\right) + \delta\left(f + \frac{7}{8}\right)\right]\left[1 + 0.5e^{-j8\pi f}\right] \tag{4.171}$$

$$
\begin{aligned}
Y(f) = &\frac{1}{2}\left(1+0.5e^{-j8\pi f}\right)\delta\left(f-\frac{7}{8}\right)+ \\
&\frac{1}{2}\left(1+0.5e^{-j8\pi f}\right)\delta\left(f+\frac{7}{8}\right) &(4.172)\\
&= \frac{1}{2}\left(1+0.5\left(-1\right)\right)\delta\left(f-\frac{7}{8}\right)+ \\
&\frac{1}{2}\left(1+0.5\left(-1\right)\right)\delta\left(f+\frac{7}{8}\right) &(4.173)\\
&= \frac{1}{4}\left[\delta\left(f-\frac{7}{8}\right)+\delta\left(f+\frac{7}{8}\right)\right] &(4.174)\\
&= \frac{1}{2}X\left(f\right) &(4.175)
\end{aligned}
$$

4.13 Consider the state-space description of an LTI system with matrices

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & -2 \end{bmatrix}, \quad D = 1$$

For the input, $\sin(\omega t)$, $\omega > 0$, the value of ω for which the steady-state output of the system will be zero, is __________ (Round off to the nearest integer). (GATE 2023 EE Q46)

Solution: The state-space representation of the system is given by:

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{4.176}$$

$$y(t) = Cx(t) + Du(t) \tag{4.177}$$

Transfer function given by:

$$T.F = C\left(sI - A\right)^{-1} B + D \tag{4.178}$$

$$\left(sI - A\right) = \begin{pmatrix} s & -1 \\ 1 & s+2 \end{pmatrix} \tag{4.179}$$

$$\left(sI - A\right)^{-1} = \frac{1}{s(s+2)+1} \begin{pmatrix} s+2 & 1 \\ -1 & s \end{pmatrix} \tag{4.180}$$

Referencing from equation (4.180), equation (4.178) becomes

$$T.F = \begin{pmatrix} \frac{3}{s^2+2s+1} & \frac{-2}{s^2+2s+1} \end{pmatrix} \begin{pmatrix} s+2 & 1 \\ -1 & s \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + 1 \tag{4.181}$$

$$= \begin{pmatrix} \frac{3}{s^2+2s+1} & \frac{-2}{s^2+2s+1} \end{pmatrix} \begin{pmatrix} 1 \\ s \end{pmatrix} + 1 \tag{4.182}$$

$$= \frac{s^2+4}{s^2+2s+1} \tag{4.183}$$

$$H(s) = T.F \tag{4.184}$$

$$H(s) = \frac{s^2+4}{s^2+2s+1} \tag{4.185}$$

Substituting $s = j\omega$ in equation (4.185),

$$H(j\omega) = \frac{4-(\omega)^2}{1+2j\omega-(\omega)^2} \tag{4.186}$$

Steady state output of system is zero:

$$4-(\omega)^2 = 0 \tag{4.187}$$

$$\omega = 2 \text{ rad/sec} \tag{4.188}$$

4.14 A causal, discrete time system is described by the difference equation $y[n] = 0.5y[n-1] + x[n]$, for all n, where $y[n]$ denotes the output sequence and $x[n]$ denotes the input sequence. Which of the following statements is/are TRUE?

(a) he system has an impulse response described by $0.5^n u[-n]$ where $u[n]$ is the unit step sequence.

(b) The system is stable in the bounded input, bounded output sense.

(c) The system has an infinite number of non-zero samples in its impulse response

(d) The system has a finite number of non-zero samples in its impulse response.

(GATE 2023 BM-26)

Solution:

$$y[n] = 0.5y[n-1] + x[n] \tag{4.189}$$

Taking Z-Transform

$$Y(z) = 0.5z^{-1}Y(z) + X(z) \tag{4.190}$$

$$\implies \frac{Y(z)}{X(z)} = \frac{1}{1-0.5z^{-1}} = H(z) \tag{4.191}$$

If $x[n]$ is impulse input

$$\implies Y(z) = H(z) = \frac{1}{1-0.5z^{-1}} \tag{4.192}$$

From (4.192) pole lies at $z = 0.5$

$$a^n u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-az^{-1}} \quad , |z| > a \tag{4.193}$$

From (4.192) , (4.193)

$$h[n] = 0.5^n u[n] \quad , |z| > 0.5 \tag{4.194}$$

Plotting $h[n]$ vs n

(a) From (4.194) , (a) is wrong

(b) As pole lies within unit circle (b) is true

(c) From (4.194) and Fig. 4.20 ,(c) is true and hence

(d) (d) is false

4.15 A closed loop system is shown in the figure where $k > 0$ and $\alpha > 0$. The steady state error due to a ramp input $\left(R(s) = \alpha s^{-2}\right)$ is given by (GATE 2023 EC 41)

(a) $\frac{2\alpha}{k}$

(b) $\frac{\alpha}{k}$

(c) $\frac{\alpha}{2k}$

(d) $\frac{\alpha}{4k}$

Solution: from table Open loop transfer function $G(s)$

$$G(s) = \frac{Y(s)}{E(s)} \tag{4.195}$$

$$= \frac{Y(s)}{R(s) - Y(s)} \tag{4.196}$$

$$Y(s) = \frac{R(s)\,G(s)}{1 + G(s)} \tag{4.197}$$

from eq 4.195 and eq (4.197)

$$G(s) = \frac{k}{s(s+2)} \tag{4.198}$$

$$Y(s) = \frac{\alpha k s^{-2}}{k + s(s+2)} \tag{4.199}$$

$$E(s) = R(s) - Y(s) \tag{4.200}$$

$$E(s) = \frac{\alpha(s+2)}{s(k + s(s+2))} \tag{4.201}$$

By Taking Inverse Laplace Transform of eq (4.198) and eq(4.199)

$$g(t) = \frac{k\left(1 - e^{-2t}\right)}{2} u(t) \tag{4.202}$$

$$\begin{aligned} y(t) &= \alpha t u(t) - \frac{2\alpha}{k} u(t) \\ &+ \frac{\alpha}{2k\sqrt{1-k}} \Bigg(2\sqrt{1-k} e^{\sqrt{1-k}t - 1} \\ &+ 2\sqrt{1-k} e^{-\sqrt{1-k}t - 1} \\ &+ (2-k) e^{\sqrt{1-k}t - 1} - (2-k) e^{-\sqrt{1-k}t - 1} \Bigg) u(t) \end{aligned} \tag{4.203}$$

$$e(t) = r(t) - y(t) \tag{4.204}$$

$$= \alpha t u(t) - y(t) \tag{4.205}$$

$$\begin{aligned} e(t) &= \frac{2\alpha}{k} u(t) \\ &- \frac{\alpha}{2k\sqrt{1-k}} \Bigg(2\sqrt{1-k} e^{\sqrt{1-k}t - 1} \\ &+ 2\sqrt{1-k} e^{-\sqrt{1-k}t - 1} \\ &+ (2-k) e^{\sqrt{1-k}t - 1} - (2-k) e^{-\sqrt{1-k}t - 1} \Bigg) u(t) \end{aligned} \tag{4.206}$$

$$e_s = \lim_{s \to 0} sE(s) \tag{4.207}$$

$$= \lim_{s \to 0} s \frac{R(s)}{1 + G(s)} \tag{4.208}$$

$$= \lim_{s \to 0} \frac{\alpha(s+2)}{s(s+2) + k} \tag{4.209}$$

$$e_s = \frac{2\alpha}{k} \tag{4.210}$$

PARAMETER	DESCRIPTION
G_c	Proportional controller's transfer function
G_f	Valve transfer function
G_p	Process transfer function
G_M	Measurement transfer function
$G(s)$	Open loop transfer function
$T(s)$	Transfer function of system

Table 4.2: PARAMETER TABLE 1

Parameter	**Description**	**Value**
k_i	spring constant	1000 N/m
m	mass of block	20Kg
k	Equivalent spring constant	$k_1 + k_2$ (parallel)
ω_n	Natural frequency	$\sqrt{\frac{k}{m}}$
T	Time period of an oscillation	$\frac{2\pi}{\omega_n}$
x	Displacement of block	
a	Acceleration of block	$\frac{d^2x}{dt^2}$
F	Force on block	
A	Amplitude of oscillation	x(0)

Table 4.3: Parameter Table

Variable	**Description**	**Value**
$x(t)$	input function	none
$y(t)$	output function	$\sin(\pi t)$
$H(s)$	Transfer-function	$\frac{s-\pi}{s+\pi}$

Table 4.4: Input parameters

Table 4.5: Input Parameters

Parameter	**Used to denote**	**Values**
m	order of pole at z = π	?
$Res(f, \pi)$	Residue of pole	?

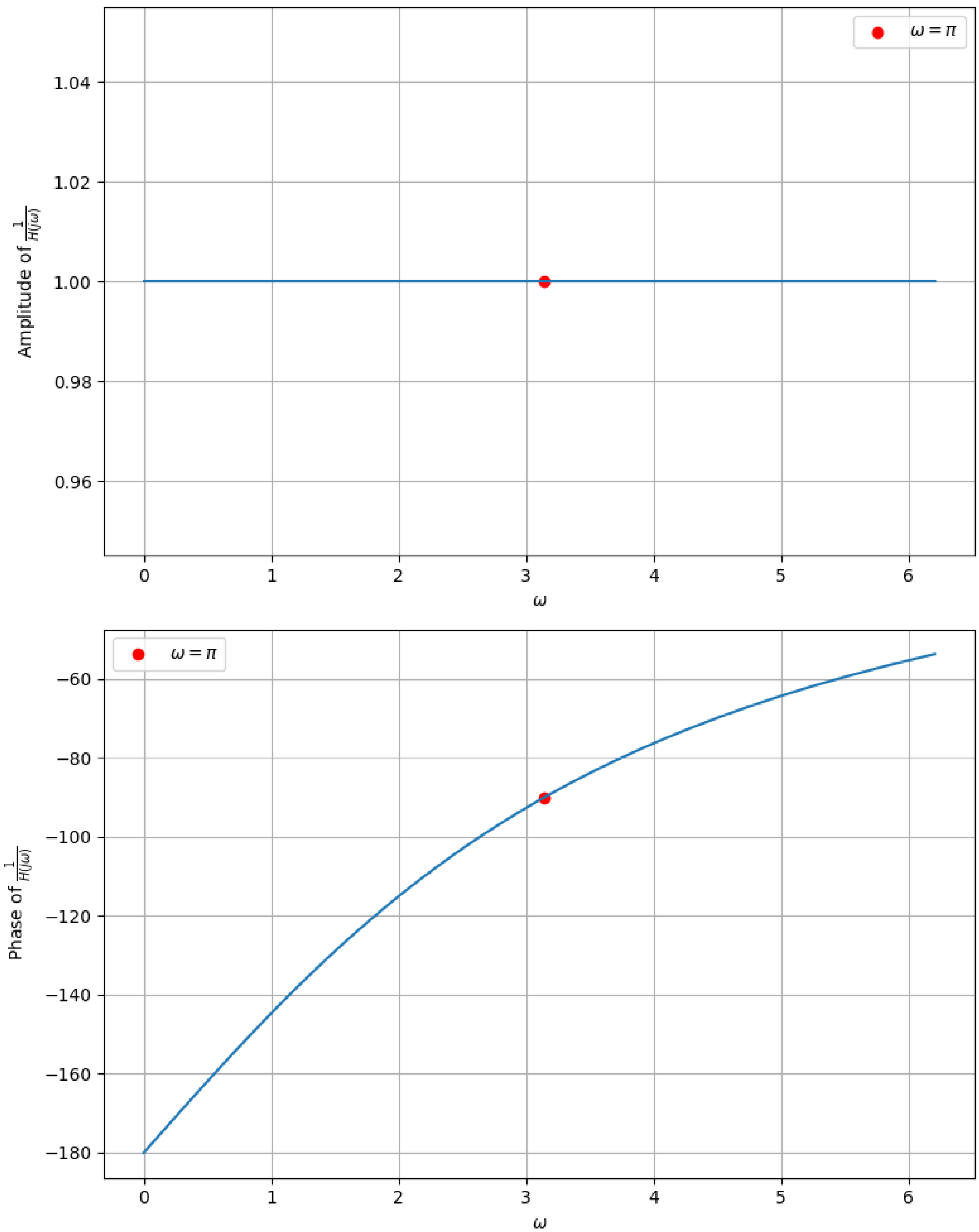


Figure 4.10: Amplitude and phase of $\frac{1}{H(j\omega)}$

Parameter	**Description**	**Value**
$X(s)$	position in laplace domain	$X(s)$
$x(t)$	position of buoy w.r.t time	$x(t)$
m	mass of buoy	30kg
λ	damping coeffecient of the system	$188.5 \approx 60\pi$
ω_o	natural angular frequency of the system	?
ω_d	damping frequency of the system	0 rad s^{-1}

Table 4.6: input values

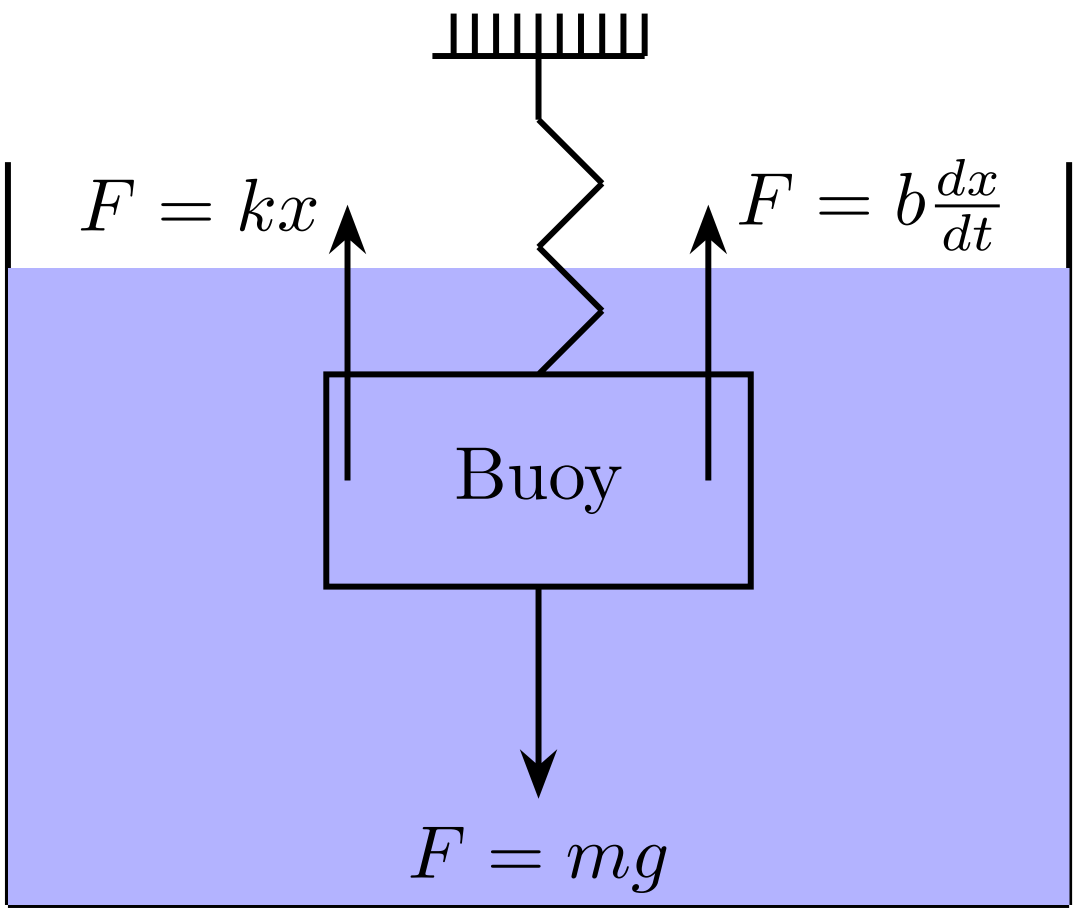


Figure 4.11: Figure given in question

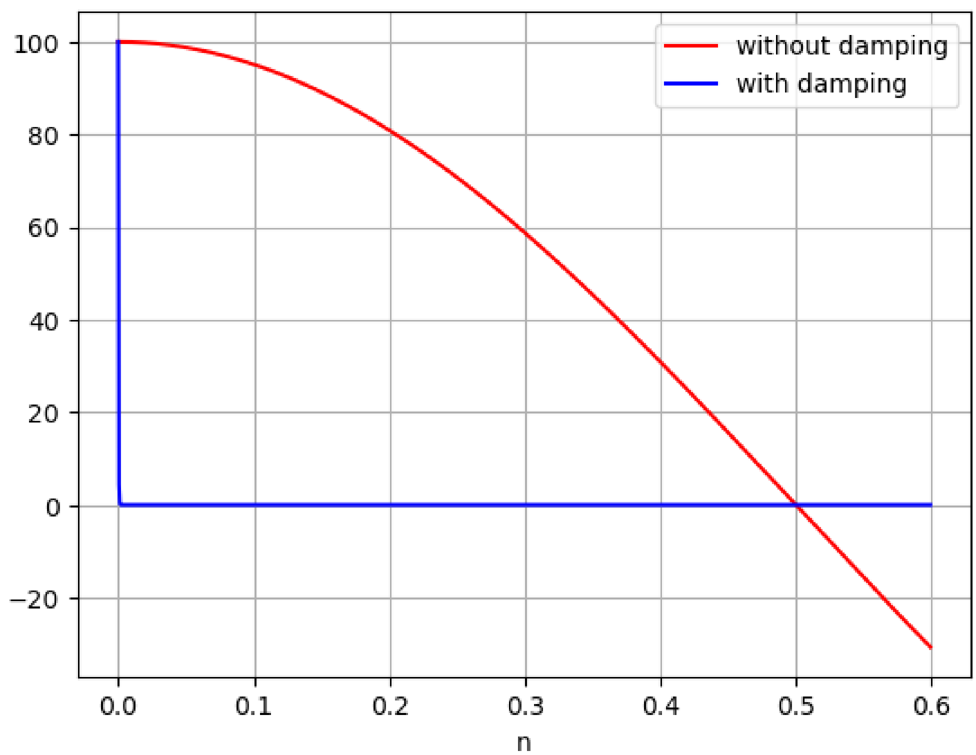


Figure 4.12: $x\,(t)$ with and with out damping

Term	**Description**
n	Number of foward paths
Δ_k	Associated path factor
P_k	Path gain of the k^{th} forward path
Δ	Determinant of Signal flow graph

Table 4.8: Mason's Gain formula parameters

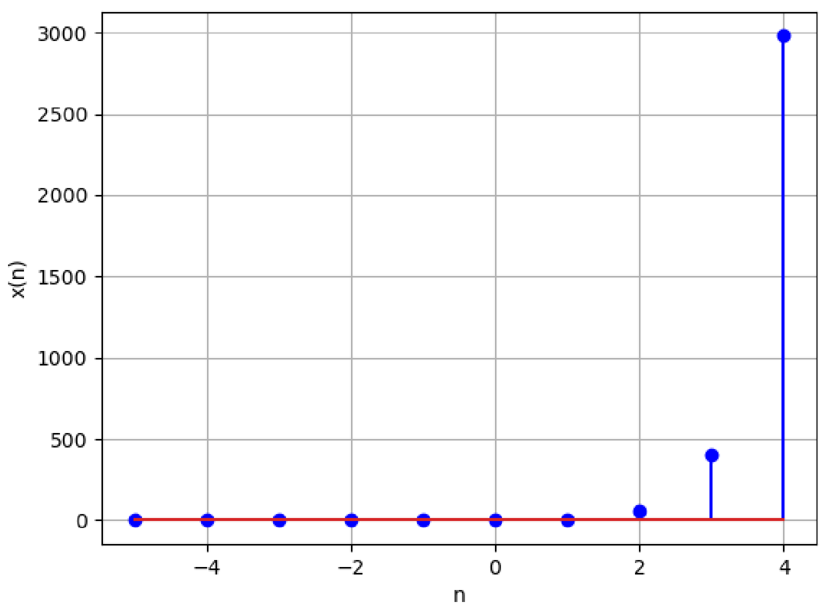


Figure 4.13: graph of $h(t) = e^{2t} \cdot u(t)$

Signal Type	Spectral Characterstics
(i) Continuous, aperiodic	(a) Continuous, aperiodic
(ii) Continuous, periodic	(b) Continuous, periodic
(iii) Discrete, aperiodic	(c) Discrete, aperiodic
(iii) Discrete, periodic	(d) Discrete, periodic

Table 4.9:

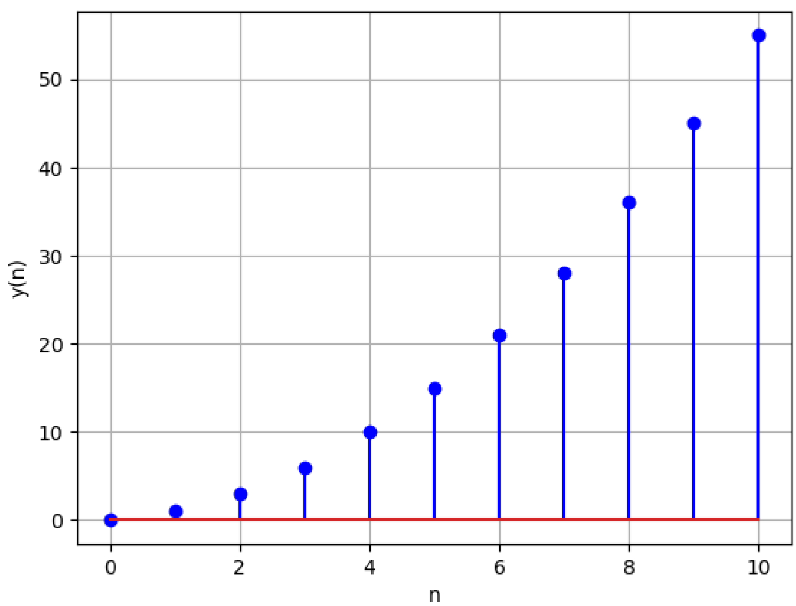


Figure 4.14: graph of $y\left[n\right]$

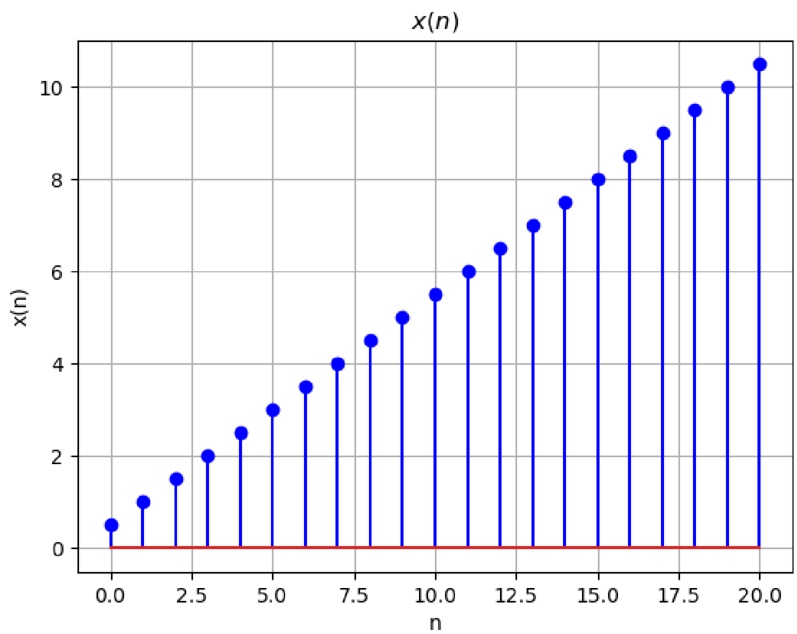
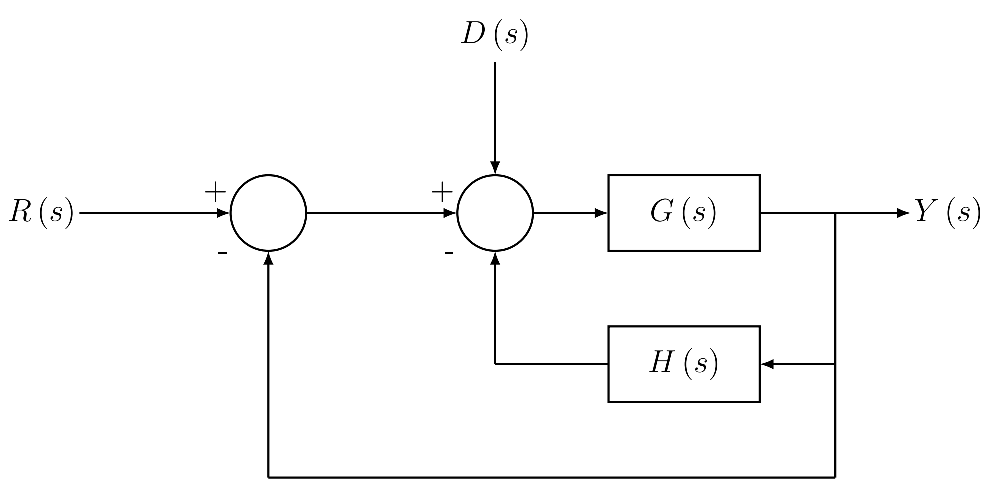
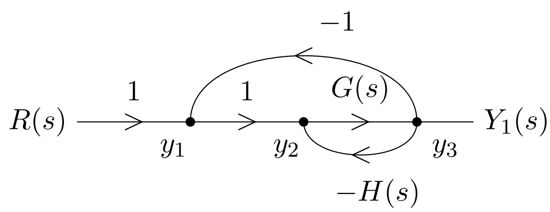
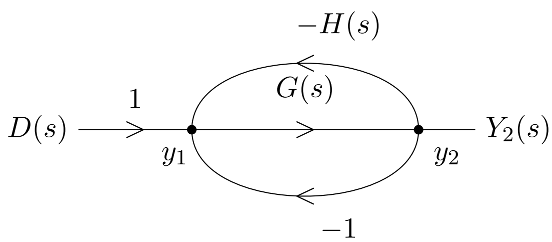


Figure 4.15: graph of $g[n]$







Parameter	Description
$x\left(t\right)$	Continuous Time Signal
$x\left(f\right)$	Fourier Transform of a Signal
$x\left[n\right]$	Discrete Time Signal
$X\left[k\right]$	Tthe amplitude and phase of the k^{th}frequency component of the input signal $x\left[n\right]$
$u\left(t\right)=\begin{cases}0 & t<0\\ 1 & t\geq 0\end{cases}$	Unit step signal in continuous time
$u\left[n\right]=\begin{cases}0 & n<0\\ 1 & n\geq 0\end{cases}$	Unit step signal Discrete time
$x\left(t\right)=e^{-2t}.u\left(t\right)$	Exponential Decay Continuous time Signal
$x\left(t\right)=sin\left(2\pi ft\right)$	Sine wave Continuous time Signal
$x\left[n\right]=e^{n}.u\left[n\right]$	Exponential Decay Discrete time Signal
$x\left[n\right]=sin\left(2\pi f_0 n\right).u\left[n\right]$	Sinusoidal Discrete time Signal

Table 4.10: Input Parameters

Variable	Description	value
$\delta\left(t\right)$	Dirac delta function	∞ if t=0; 0 in other cases $\int_{-\infty}^{\infty}\delta\left(t\right)=1$
$h\left(t\right)$	impulse response	$\delta\left(t\right)$+0.5$\delta\left(t-4\right)$
$x\left(t\right)$	input signal	$x(t)=cos\left(\frac{7\pi t}{4}\right)$
$y\left(t\right)$	output signal	x$(t)*$ h(t)
$\mathcal{F}\left(\cos at\right)$	Fourier transform of $\cos at$	$0.5\left[\delta\left(f-\frac{a}{2\pi}\right)+\delta\left(f+\frac{a}{2\pi}\right)\right]$
$X\left(f\right)$	Fourier transform of $x\left(t\right)$	$0.5\left[\delta\left(f-\frac{7}{8}\right)+\delta\left(f+\frac{7}{8}\right)\right]$
$H\left(f\right)$	Fourier transform of $h\left(t\right)$	1+0.5$e^{-j8\pi f}$
$Y\left(f\right)$	Fourier transform of $y\left(t\right)$	$X\left(f\right)H\left(f\right)$

Table 4.11: A Table with input parameters

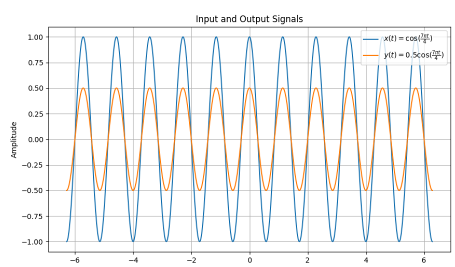


Figure 4.19: Graph showing x(t) and y(t)

Parameter	Value
System Matrix, A	$\begin{pmatrix}0 & 1\\-1 & -2\end{pmatrix}$
Input Matrix, B	$\begin{pmatrix}0\\1\end{pmatrix}$
Output Matrix, C	$\begin{pmatrix}3 & -2\end{pmatrix}$
Feedthrough Matrix, D	$\begin{pmatrix}1\end{pmatrix}$
Input Signal, $u(t)$	$\sin(\omega t)$, $\omega>0$

Table 4.12: Input Parameters

Parameter	Value	Description
$x[n]$	?	Input Sequence
$y[n]$	?	Output Sequence

Table 4.13: Input parameters table

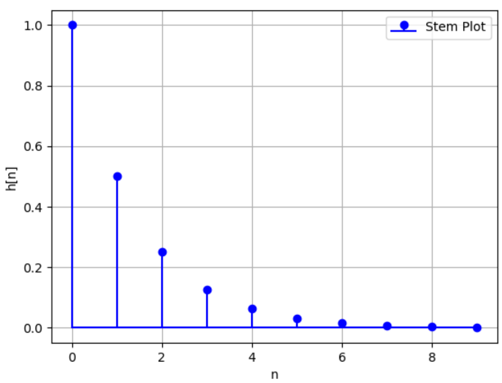
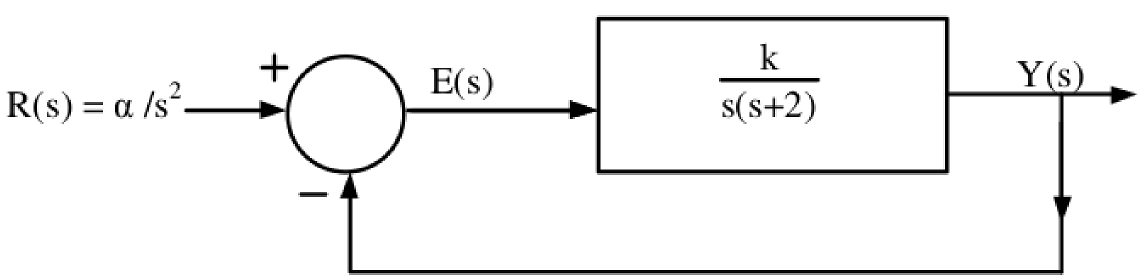


Figure 4.20: Plot of $h[n]$ vs n



Symbol	Parameters	Value
$R(s)$	Laplace transform Ramp input signal r(t)	αs^{-2}
$G(s)$	Open Loop transfer function	$\frac{Y(s)}{E(s)} = \frac{k}{s(s+2)}$
$Y(s)$	Laplace transform of the output signal y(t)	?
$E(s)$	Laplace transform of the error signal e(t)	R(s) - Y(s)
$E(s)$	Laplace transform of the error signal e(t)	R(s) - Y(s)
e_s	Steady State Error	?

Table 4.14: Parameters

Chapter 5

Sequences

5.1 Consider the discrete time signal $x[n] = u[-n+5] - u[n+3]$, where

$$u[n] = \begin{cases} 1; n \geq 0 \\ 0; n < 0 \end{cases}$$

The smallest n for which $x[n] = 0$ is? (GATE IN 2023)

Solution: From Fig. 1, the minimum value of n is given as

$$n = -3 \tag{5.1}$$

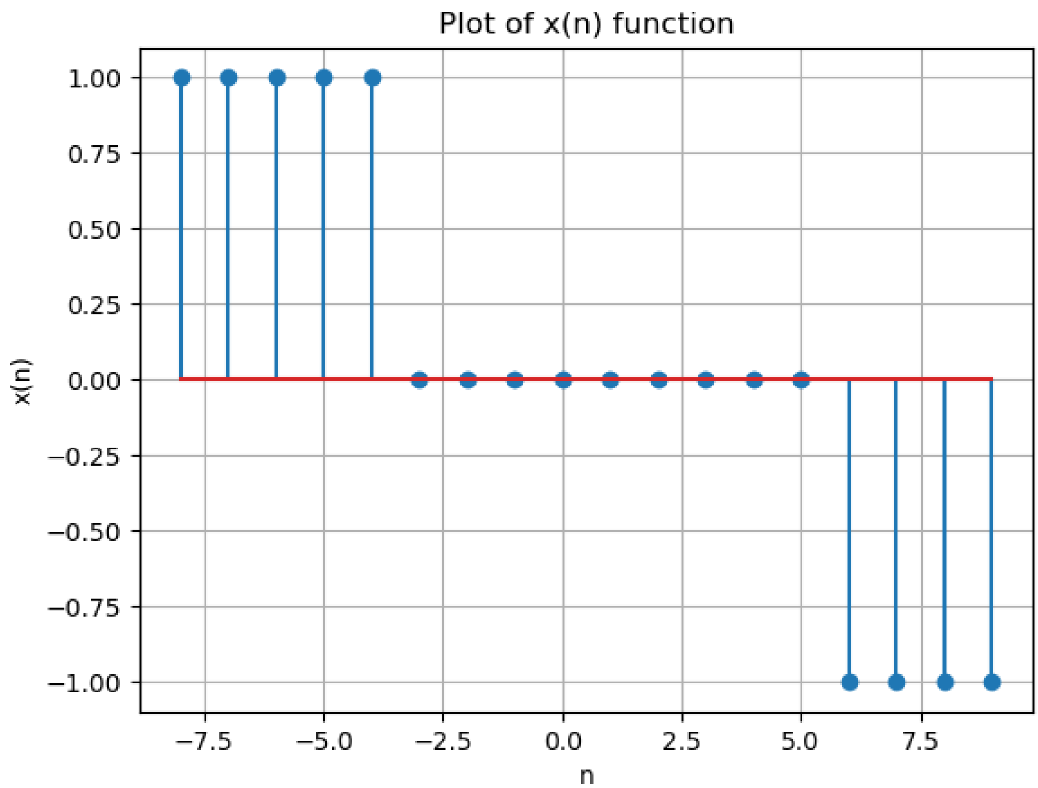


Figure 1: Plot of function $x(n)$ taken from python3

5.2 Two sequences $x_1[n]$ and $x_2[n]$ are described as follows:

$$x_1[0] = x_2[0] = 1 \tag{5.2}$$

$$x_1[1] = x_2[2] = 2 \tag{5.3}$$

$$x_1[2] = x_2[1] = 1 \tag{5.4}$$

$x_1[n] = x_2[n] = 0$ for all $n < 0$ and $n > 2$

If $x[n]$ is obtained by convoluting $x_1[n]$ with $x_2[n]$, which of the following equations is/are TRUE?

(A) $x[2] = x[3]$

(B) $x[1] = 2$

(C) $x[4] = 3$

(D) $x[2] = 5$

(GATE 2023 BM 47) **Solution:** From the data given:

$$x_1 = [1, 2, 1] \tag{5.5}$$

$$x_2 = [1, 1, 2] \tag{5.6}$$

Takin the convolution, we get

$$x = [1, 3, 5, 5, 2] \tag{5.7}$$

$$\tag{5.8}$$

Hence we get:

$$x[0] = 1 \tag{5.9}$$

$$x[1] = 3 \tag{5.10}$$

$$x[2] = 5 \tag{5.11}$$

$$x[3] = 5 \tag{5.12}$$

$$x[4] = 2 \tag{5.13}$$

$$\tag{5.14}$$

Comparing this with the options, we see that options (A) and (D) match

$\implies$ (A), (D)

5.3 A series (S) is given as S=1+3+5+7+9+..... The sum of the first 50 terms of S is _____________ (GATE 2023 BT 32)

Solution:

Variable	**Description**	**Value**
$x(0)$	First term of AP	1
$x(1)$	Second term of AP	3
d	Common difference of AP $(x(2) - x(1))$	2
$x(n)$	n^{th} term of sequence	$(2n+1)u(n)$

Table 5.1: input parameters

For an AP,

$$X(z) = \frac{x(0)}{1 - z^{-1}} + \frac{dz^{-1}}{(1 - z^{-1})^2} \tag{5.15}$$

$$\implies X(z) = \frac{1}{(1 - z^{-1})} + \frac{2z^{-1}}{(1 - z^{-1})^2}, |z| > 1 \tag{5.16}$$

$$y(n) = x(n) * u(n) \tag{5.17}$$

$$Y(z) = X(z) U(z) \tag{5.18}$$

$$Y(z) = \frac{1}{(1 - z^{-1})^2} + \frac{2z^{-1}}{(1 - z^{-1})^3} \tag{5.19}$$

$$\implies Y(z) = \frac{(z^{-1} + 1)}{(1 - z^{-1})^3}, |z| > 1 \tag{5.20}$$

Using Contour Integration to find the inverse Z-transform,

$$y(49) = \frac{1}{2\pi j} \oint_C Y(z)\, z^{48}\, dz \tag{5.21}$$

$$= \frac{1}{2\pi j} \oint_C \frac{(z^{-1} + 1)z^{48}}{(1 - z^{-1})^3}\, dz \tag{5.22}$$

We can observe that the pole is repeated 3 times and thus $m = 3$,

$$R = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} \left((z-a)^m f(z)\right) \tag{5.23}$$

$$\implies R = \frac{1}{(2)!} \lim_{z \to 1} \frac{d^2}{dz^2} \left((z-1)^3 \frac{(z^{-1}+1)z^{51}}{(z-1)^3} \right) \tag{5.24}$$

$$\implies R = \frac{1}{2} \lim_{z \to 1} \frac{d^2}{dz^2} (z^{50} + z^{51}) \tag{5.25}$$

$$\implies R = 2500 \tag{5.26}$$

$$\therefore y(50) = 2500 \tag{5.27}$$

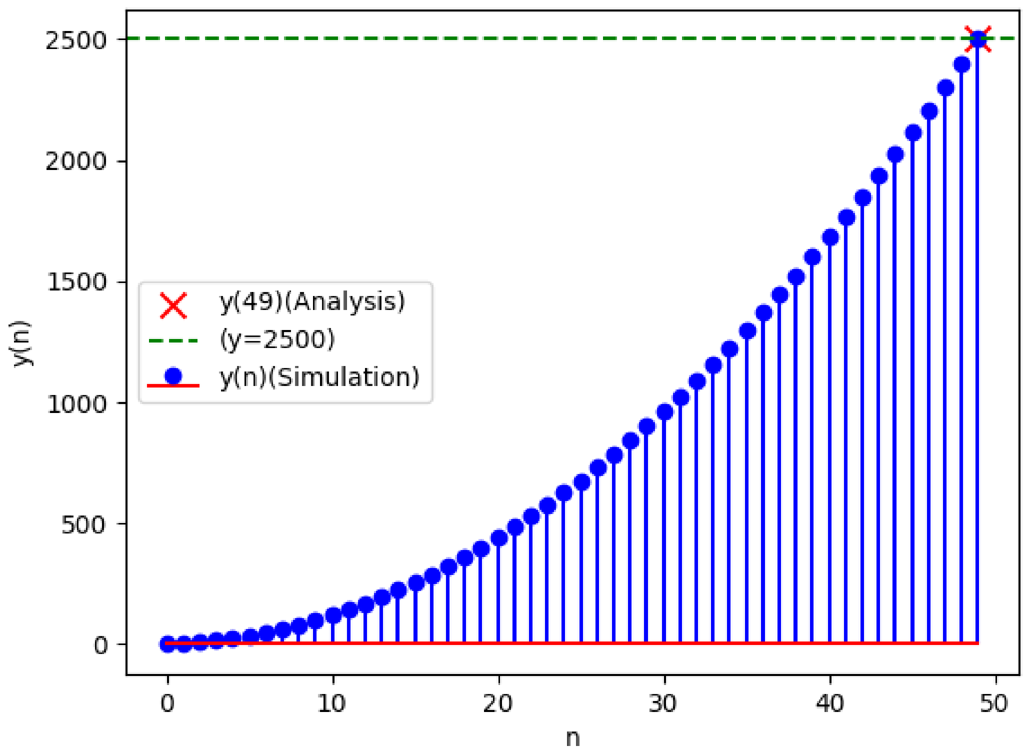


Figure 5.2: Analysis vs Simulation

5.4 For the signals x(t) and y(t) shown in the figure, $z(t) = x(t) * y(t)$ is maximum at $t = T_1$. Then T_1 in seconds is .... (Round off to the nearest integer) **Solution:**

5.5 A series of natural numbers F_1, F_2, F_3, F_4, F_5, F_6, F_7,....obeys $F_{n+1} = F_n + F_{n-1}$ for all integers n $\geq$ 2. If $F_6 = 37$, and $F_7 = 60$, then what is F_1?

[GATE CS 2023]

Solution:

Parameter	**Description**	**Value**
$x(6)$	Seventh term of the sequence	60
$x(5)$	Sixth term of the sequence	37
$x(1)$	Second term of the sequence	?
$x(0)$	First term of the sequence	?

Table 5.2: input values

Taking z-transform of $X(z)$:

$$X(z) = x(0) + z^{-1}x(1) + \sum_{n=2}^{\infty} x(n) z^{-n} \tag{5.28}$$

$$= x(0) + z^{-1}x(1) + z^{-1} \sum_{n=1}^{\infty} x(n+1) z^{-n} \tag{5.29}$$

$$= x(0) + z^{-1}x(1) + z^{-1} \sum_{n=1}^{\infty} (x(n) + x(n-1)) z^{-n} \tag{5.30}$$

$$= x(0) + z^{-1}x(1) + z^{-1} \left(X(z) - x(1) + z^{-1}X(z)\right) \tag{5.31}$$

$$\implies X(z) = \frac{x(0) + (x(1) - x(0)) z^{-1}}{1 - z^{-1} - z^{-2}} \tag{5.32}$$

Using Contour Integration to find the inverse Z-transform,

$$x(n) = \frac{1}{2\pi j} \oint_C X(z) \; z^{n-1} \, dz \tag{5.33}$$

$$= \frac{1}{2\pi j} \oint_C \frac{z^n (x(1) - x(0) + x(0) z)}{z^2 - z - 1} \, dz \tag{5.34}$$

By residue theorem:

$$x(n) = \frac{1}{(0)!} \lim_{z \to \frac{1+\sqrt{5}}{2}} \frac{d}{dz} \left((z + \frac{1+\sqrt{5}}{2}) X(z) \right) + \frac{1}{(0)!} \lim_{z \to \frac{1-\sqrt{5}}{2}} \frac{d}{dz} \left(\left(z + \frac{1-\sqrt{5}}{2} \right) X(z) \right) \tag{5.35}$$

On simplifying we get,

$$x(n) = (x(1) - x(0)) \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right) + (x(0)) \Bigg(\left(\frac{1+\sqrt{5}}{2} \right)^{n+1} - \Bigg(\frac{1-\sqrt{5}}{2} \tag{5.36}$$

From the values in Table 5.2:

$$5(x(1) - x(0)) + 8x(0) = 37 \tag{5.37}$$

$$8(x(1) - x(0)) + 13x(0) = 60 \tag{5.38}$$

$$\implies x(1) = 5, x(0) = 4 \tag{5.39}$$

$$\therefore x(0) = 4 \tag{5.40}$$

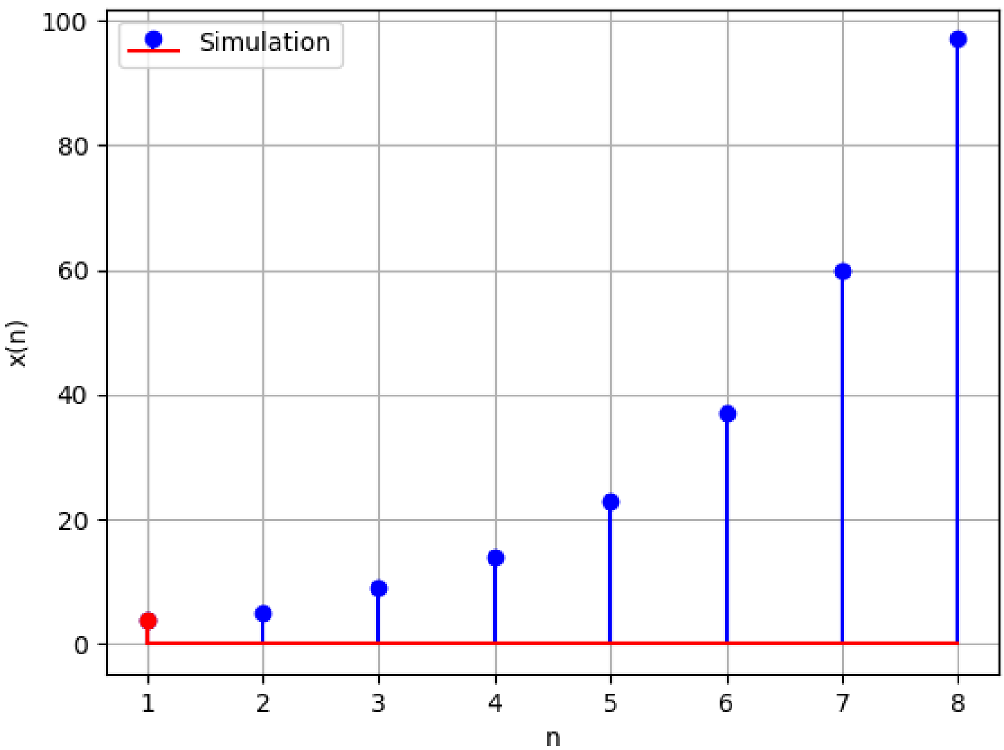


Figure 5.3: Terms of the given sequence

5.6 The Lucas sequence $L_n$is defined by the recurrence relation:

$$L_n = L_{n-1} + L_{n-2}, forn \geq 3$$

with L_1=1 and L_2=3

Which one of the option given is TRUE?

(a) $L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$

(b) $L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{3}\right)^n$

(c) $L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{3}\right)^n$

(d) $L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n$

(GATE 2023 CS 15)

Solution: Initial condition L_1=1 and L_2=3

$$L_n = L_{n-1} + L_{n-2} \tag{5.41}$$

Assume $L_{n+1} = x(n)$

$$x(n) =[x(n-1) + x(n-2) - 3]u(n-2) + u(n) + 2u(n-1) \tag{5.42}$$

$$X(z) =z^{-1}(X(z) - 1) + z^{-2}X(z) - 3\frac{z^{-2}}{1-z^{-1}} + \frac{1}{1-z^{-1}} + 2\frac{z^{-1}}{1-z^{-1}} \tag{5.43}$$

$$X(z)(1 - z^{-1} - z^{-2})(1 - z^{-1}) = 1 + z^{-1} - 2z^{-2} \tag{5.44}$$

$$X(z) = \frac{1 + z^{-1} - 2z^{-2}}{(1 - z^{-1} - z^{-2})(1 - z^{-1})} \tag{5.45}$$

$$X(z) = \frac{A}{1-z^{-1}} + \frac{B}{1-\alpha z^{-1}} + \frac{C}{1-\beta z^{-1}} \tag{5.46}$$

Where, $\alpha = \dfrac{1+\sqrt{5}}{2}$ and $\beta = \dfrac{1-\sqrt{5}}{2}$

using partial fractions,

$$X(z) = \frac{\alpha+2}{(\alpha-\beta)(1-\alpha z^{-1})} + \frac{\beta+2}{(\beta-\alpha)(1-\beta z^{-1})} \tag{5.47}$$

$$a^n u(n) \stackrel{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-az^{-1}} \quad |\, z\, | > |\, a\, |$$

Substituting this result,

$$x(n) = \frac{\alpha+2}{(\alpha-\beta)}(\alpha^n u(n)) - \frac{\beta+2}{(\alpha-\beta)}(\beta^n u(n)) \tag{5.48}$$

$$x(n) = \frac{(5+\sqrt{5})(1+\sqrt{5})^n - (5-\sqrt{5})(1-\sqrt{5})^n}{2^{n+1}\sqrt{5}} u(n) \tag{5.49}$$

$$x(n) = \frac{(1+\sqrt{5})^{n+1} + (1-\sqrt{5})^{n+1}}{2^{n+1}} u(n) \tag{5.50}$$

$\therefore L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$ option 1 is correct.

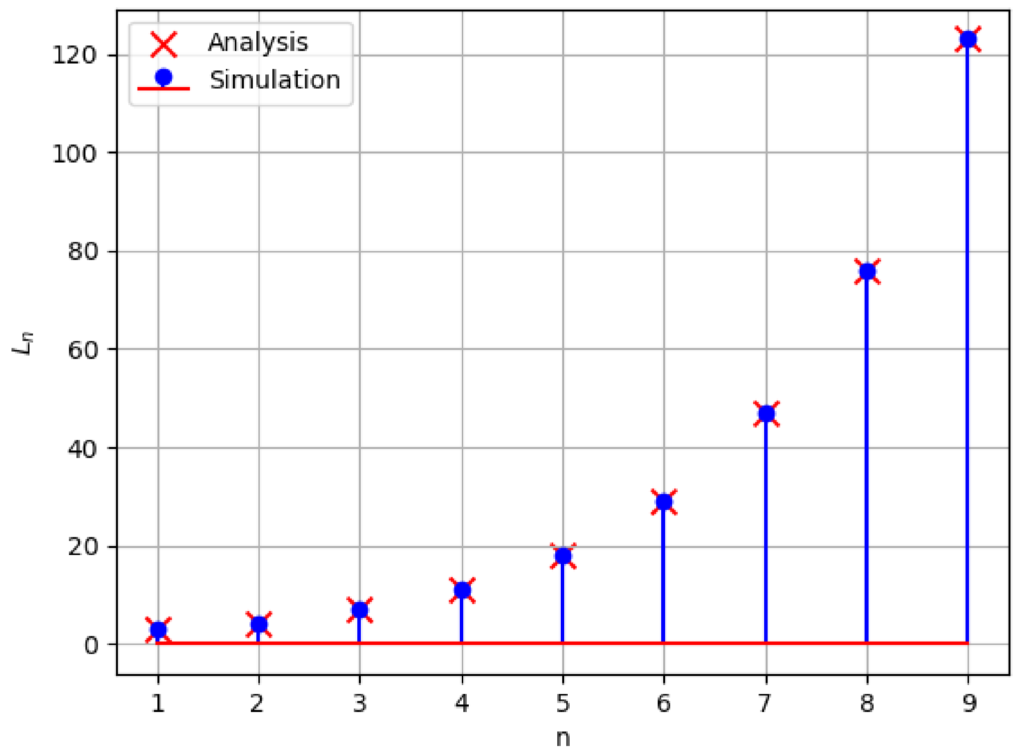


Figure 5.4: $L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$

Chapter 6

Sampling

6.1 An 8 bit ADC converts analog voltage in the range of 0 to $+5\,V$ to the corresponding digital code as per the conversion characteristics shown in figure. For $V_{in} = 1.9922\,V$, which of the following digital output, given in hex, is true?

(a) $64H$

(b) $65H$

(c) $66H$

(d) $67H$

(GATE EE 40)

Solution:

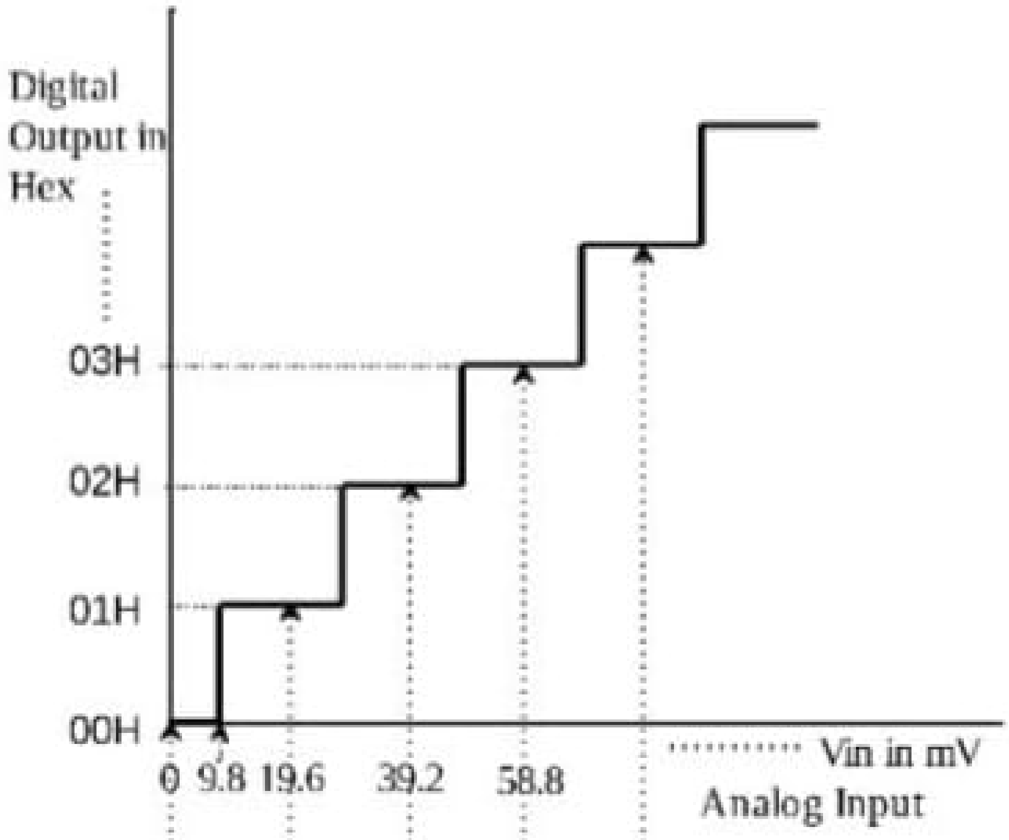


Figure 6.1:

Calculating the step-size:

$$
\begin{aligned}
\Delta V_{in} &= \frac{V_{max} - V_{min}}{2^n - 1} && (6.1)\\
&= \frac{5 - 0}{2^8 - 1} && (6.2)\\
&= \frac{5}{255} && (6.3)\\
\implies V_{out} &= \frac{V_{in}}{\Delta V_{in}} && (6.4)\\
&= \frac{1.9922 \times 255}{5} && (6.5)\\
&= 101.59 && (6.6)\\
&\approx 102_{10} && (6.7)
\end{aligned}
$$

Symbol	**Value**	**Description**
n	8	Number of bits of ADC
V_{min}	$0V$	Minimum Analog Voltage
V_{max}	$5V$	Maximum Analog Voltage
V_{in}	$1.9922V$	Input Voltage
V_{out}		Output Voltage

Table 6.1: Given Parameters

$\therefore$ correct answer is option (c).

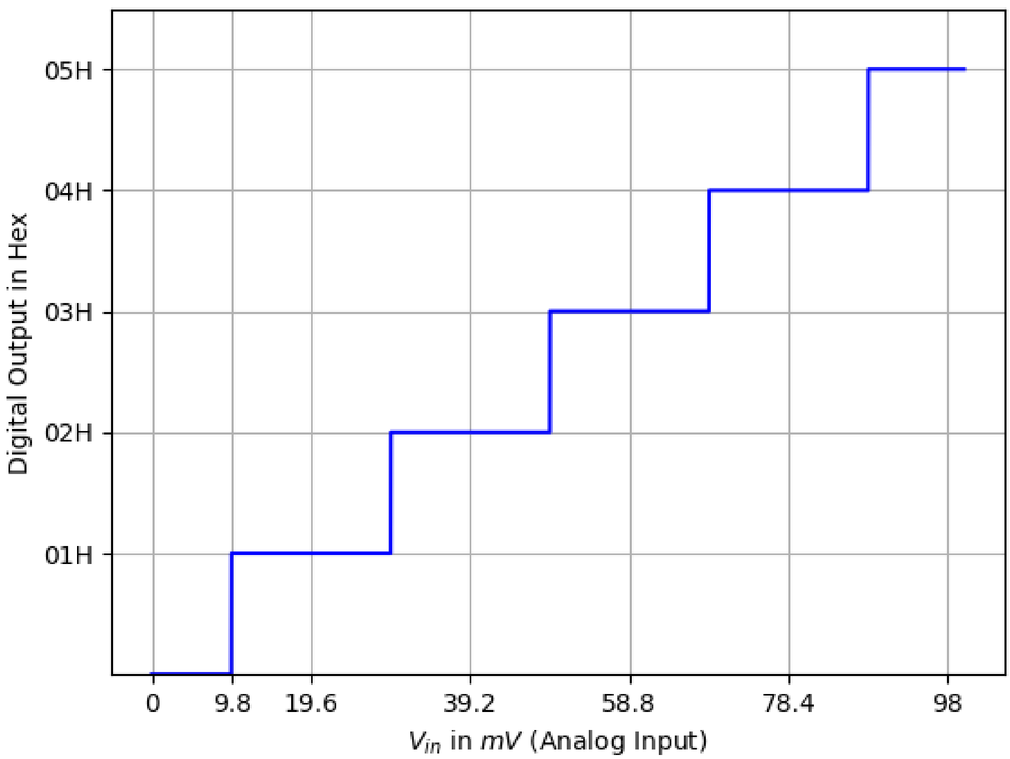
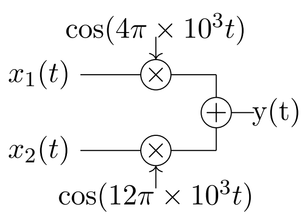


Figure 6.2:

6.2 Let x1(t) and x2(t) be two band-limited signals having bandwidth B $= 4\pi \times 10^3$ rad/s each. In the figure below, the Nyquist sampling frequency, in rad/s, required to sample y(t), is



(a) $20\pi \times 10^3$

(b) $40\pi \times 10^3$

(c) $8\pi \times 10^3$

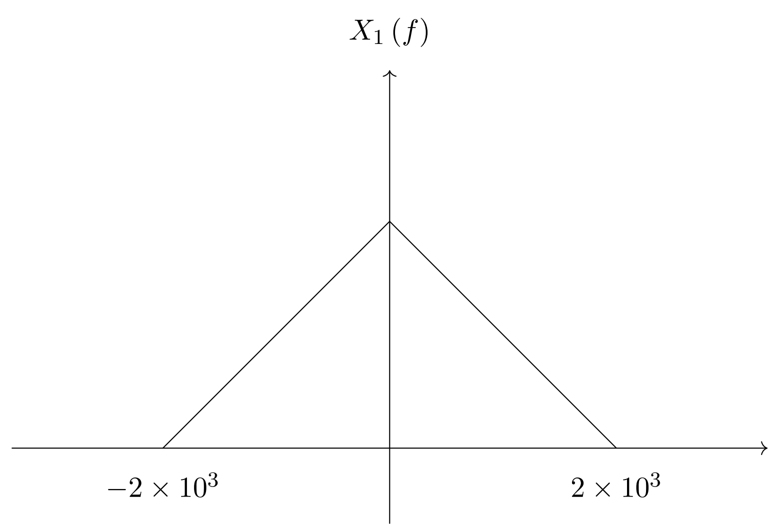
(d) $32\pi \times 10^3$

(GATE EC 50)

Solution:

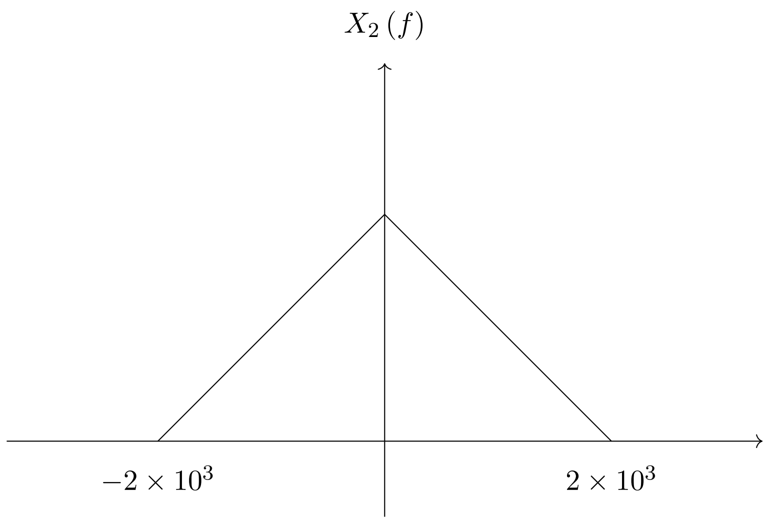
Symbol	Description	Value
f_1	Frequency of $\cos\left(4\pi \times 10^3\right)$	2×10^3
f_2	Frequency of $\cos\left(12\pi \times 10^3\right)$	6×10^3
f_m	Maximum frequency of the output signal	-
ω_m	-	$2\pi f_m$
ω_s	Nyquist sampling rate	$2\omega_m$

Table 6.2: Table of parameters



From question figure ,

$$y(t) = x_1(t) \times cos\left(4\pi \times 10^3 t\right) + x_2(t) \times cos\left(12\pi \times 10^3 t\right) \tag{6.9}$$

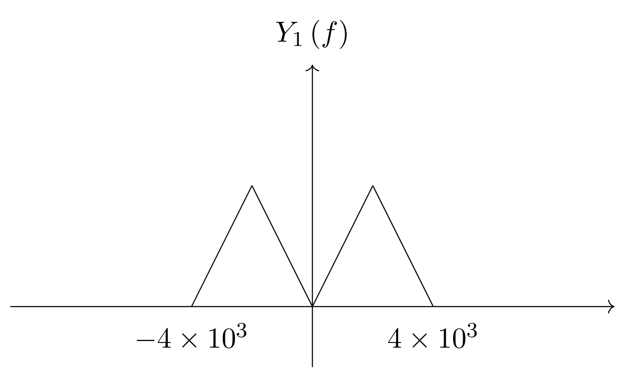


$$y(t) = y_1(t) + y_2(t) \tag{6.10}$$

$$Y(f) = Y_1(f) + Y_2(f) \tag{6.11}$$

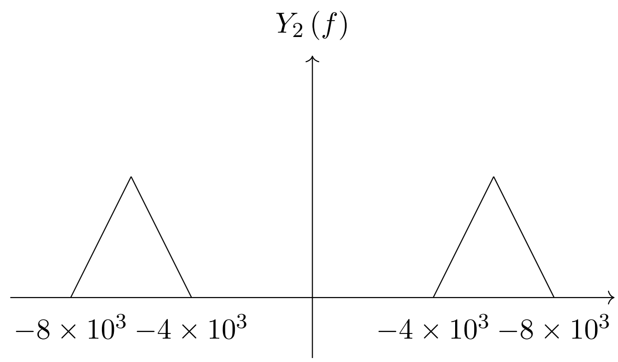
$$Y_1(f) = X_1(f) * \frac{1}{2}[\delta(f - f_1) + \delta(f + f_1)] \tag{6.12}$$

$$= \frac{1}{2}[X_1(f - f_1) + X_1(f + f_1)] \tag{6.13}$$

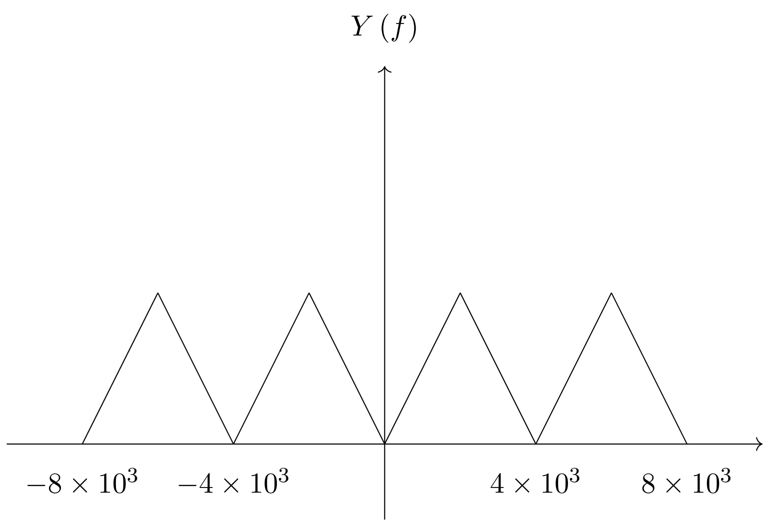


$$Y_2(f) = X_2(f) * \frac{1}{2}[\delta(f - f_2) + \delta(f + f_2)] \tag{6.14}$$

$$= \frac{1}{2}[X_2(f - f_2) + X_2(f + f_2)] \tag{6.15}$$



From (6.11):



From table,

$$\omega_m = 16\pi \times 10^3 rad/sec. \tag{6.16}$$

$$\omega_s = 2\omega_m = 32\pi \times 10^3 rad/sec. \tag{6.17}$$

6.3 An 8 bit successive approximation Analog to Digital Converter (ADC) has a clock frequency of 1 MHz. Assume that the start conversion and end conversion signals occupy one clock cycle each. Among the following options, what is the maximum frequency that this ADC can sample without aliasing?

a) 0.9 kHz

b) 9.9 kHz

c) 49.9 kHz

d) 99.9 kHz

(GATE BM 2023)

Solution:

6.4 The period of the discrete-time signal x[n] described by the equation below is N = (Round off to the nearest integer).

$$x[n] = 1 + 3\sin\left(\frac{15\pi}{8}n + \frac{3\pi}{4}\right) - 5\sin\left(\frac{\pi}{3}n - \frac{\pi}{4}\right)$$

(GATE 2023 EE) **Solution:**

Parameter	**Description**	**Value**
f_1	Sinusoid1 Frequency	15/16
f_2	Sinusoid2 Frequency	6

Table 6.3: Given parameters list

The time period must be an integer for a discrete-time signal.

$$T_1 = \frac{1}{f_1} = \frac{16}{15} \tag{6.18}$$

$$T_2 = \frac{1}{f_2} = 6 \tag{6.19}$$

$$N = \text{LCM}(T_1, T_2) = 48 \tag{6.20}$$

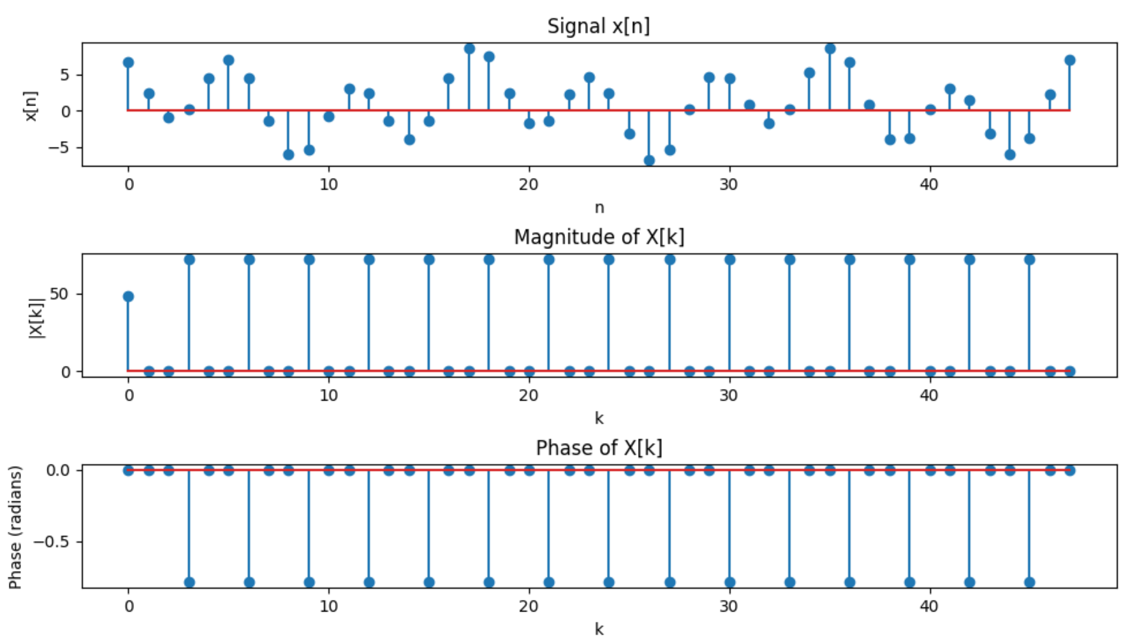
The Time Period of the signal is $N = 48$.

Let's find the Discrete Fourier Transform ($X[k]$):

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn} \tag{6.21}$$

$$X[k] = \sum_{n=0}^{47} \left(1 + 3\sin\left(\frac{15\pi}{8}n + \frac{3\pi}{4}\right) - 5\sin\left(\frac{\pi}{3}n - \frac{\pi}{4}\right)\right) \cdot e^{-j\frac{2\pi}{48}kn} \tag{6.22}$$

$$X[k] = \begin{cases} 48 & \text{if } k = 0 \\ 50.9117 - 50.9117j & \text{if } k = 3 \\ 0 & \text{otherwise} \end{cases} \tag{6.23}$$



Chapter 7

Contour Integration

7.1 The value of the contour integral, $\oint_C \frac{z+2}{z^2+2z+2}\, dz$, where the contour C is $\{z : |z + 1 - \frac{3}{2}i| = 1\}$, taken in the counter clockwise direction, is

(A) $-\pi(1 + j)$

(B) $\pi(1 + j)$

(C) $\pi(1 - j)$

(D) $-\pi(1 - j)$

(GATE EC 2023)

Solution:

$$I = \oint_C \frac{z+2}{z^2+2z+2}\, dz; \quad C = \left| z + 1 - \frac{3}{2}i \right| = 1 \tag{7.1}$$

The poles are given by $(z + 1)^2 + 1 = 0$

$$z + 1 = \pm\sqrt{-1} \tag{7.2}$$

$z = -1 + j$, $z = -1 - j$

where $-1 - i$ lies outside C and $z = (-1, 1)$ lies inside C, by the Residue Theorem:

$$\oint_C f(z)\,dz = 2\pi i\,\mathrm{Res}(f(z), z = -1 + j) \tag{7.3}$$

$$= 2\pi i \left(\frac{z+2}{2(z+1)}\right)\bigg|_{z=-1+i} \tag{7.4}$$

$$= 2\pi i \left(\frac{-1+j+2}{2(-1+j+1)}\right) \tag{7.5}$$

$$= \pi(1 + j). \tag{7.6}$$

Therefore, the correct answer is option (B).

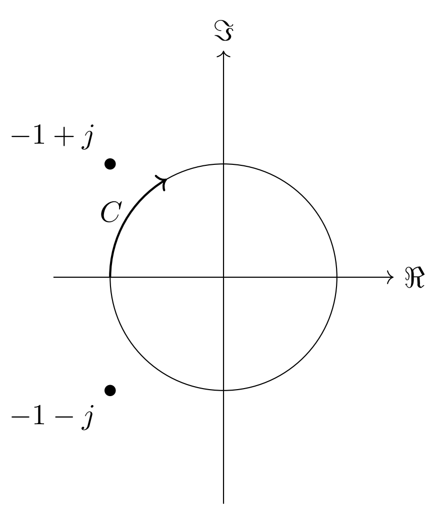


Figure 7.1: Contour C and poles

7.2 The function $f(z) = \frac{1}{z-1}$ of a complex variable z is integrated on a closed contour in an anti-clockwise direction. For which of the following contours, does this integral have a non-zero value?

$(A)|z-2| = 0.01$

$(B)|z-1| = 0.1$

$(C)|z-3| = 5$

$(D)|z| = 2$

(GATE 2023 BM)

Solution: Cauchy's Integral Formula and Residue Theorem.

$$\oint_c f(z) = 2\pi j Res\left[f(z), z_0\right] \tag{7.7}$$

$$Res\left[f(z), z_0\right] = \lim_{z \to z_0} \left[(z - z_0) f(z)\right] \tag{7.8}$$

Here z_0 is pole of the f(z)

Using (7.7)

$$\oint_c \frac{1}{z-1} dz = 2\pi j Res\left[\frac{1}{z-1}, 1\right] \tag{7.9}$$

(a) For option A the pole is outside the contour, then Residue is zero.

$$\implies \oint_c \frac{1}{z-1} dz = 2\pi j (0) \tag{7.10}$$

$$\implies 0 \tag{7.11}$$

(b) For option B the pole is inside the contour.

Then, using (7.8)

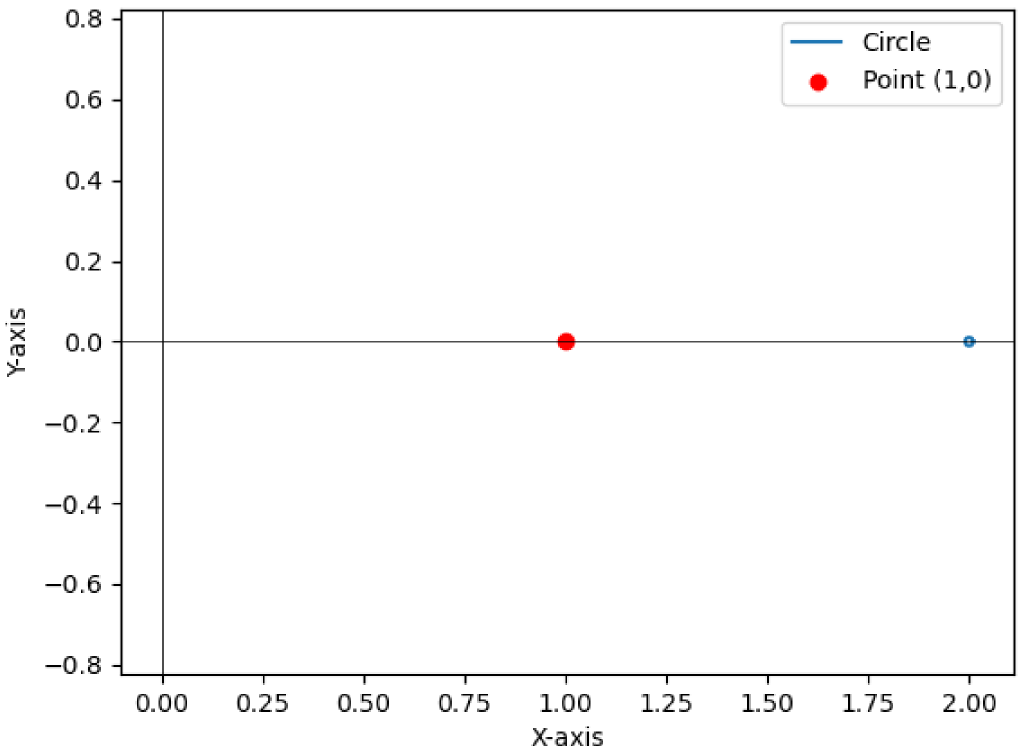


Figure 7.2: graph of option A

$$Res\left[\frac{1}{z-1},1\right] = \lim_{z\to 1}(z-1)\frac{1}{z-1} \tag{7.12}$$

$$= 1 \tag{7.13}$$

$$\implies \oint_c \frac{1}{z-1}dz = 2\pi j\,(1) \tag{7.14}$$

$$\implies 2\pi j \tag{7.15}$$

(c) For option C the pole is inside the contour.

Then, using (7.8)

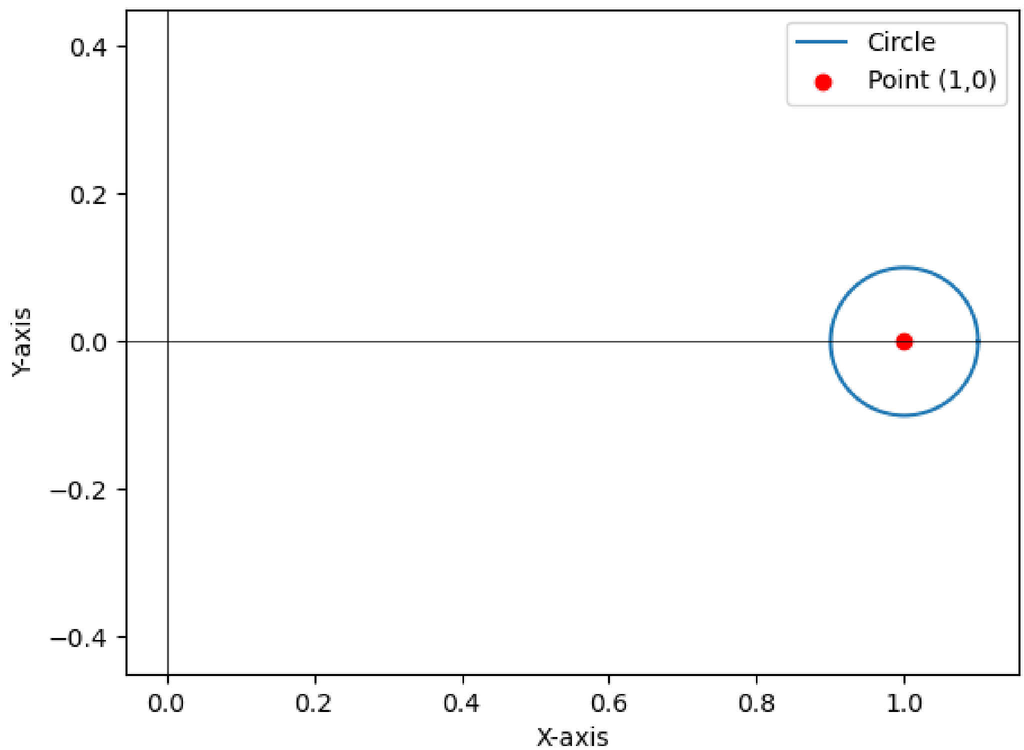


Figure 7.3: graph of option B

$$Res\left[\frac{1}{z-1},1\right] = \lim_{z\to 1}(z-1)\frac{1}{z-1} \tag{7.16}$$

$$= 1 \tag{7.17}$$

$$\implies \oint_c \frac{1}{z-1}dz = 2\pi j\,(1) \tag{7.18}$$

$$\implies 2\pi j \tag{7.19}$$

(d) For option D the pole is inside the contour.

Then, using (7.8)

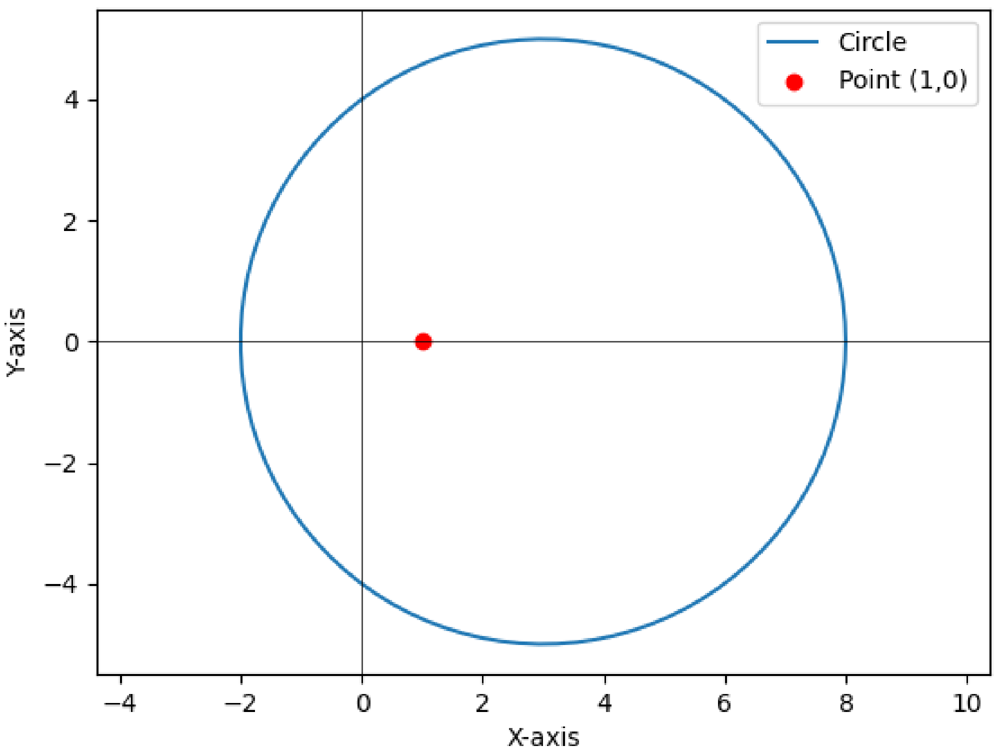


Figure 7.4: graph of option C

$$Res\left[\frac{1}{z-1},1\right] = \lim_{z\to 1}(z-1)\frac{1}{z-1} \tag{7.20}$$

$$= 1 \tag{7.21}$$

$$\implies \oint_c \frac{1}{z-1}dz = 2\pi j\,(1) \tag{7.22}$$

$$\implies 2\pi j \tag{7.23}$$

We can conclude that for options B,C,D contours have the non-zero value for this integral.

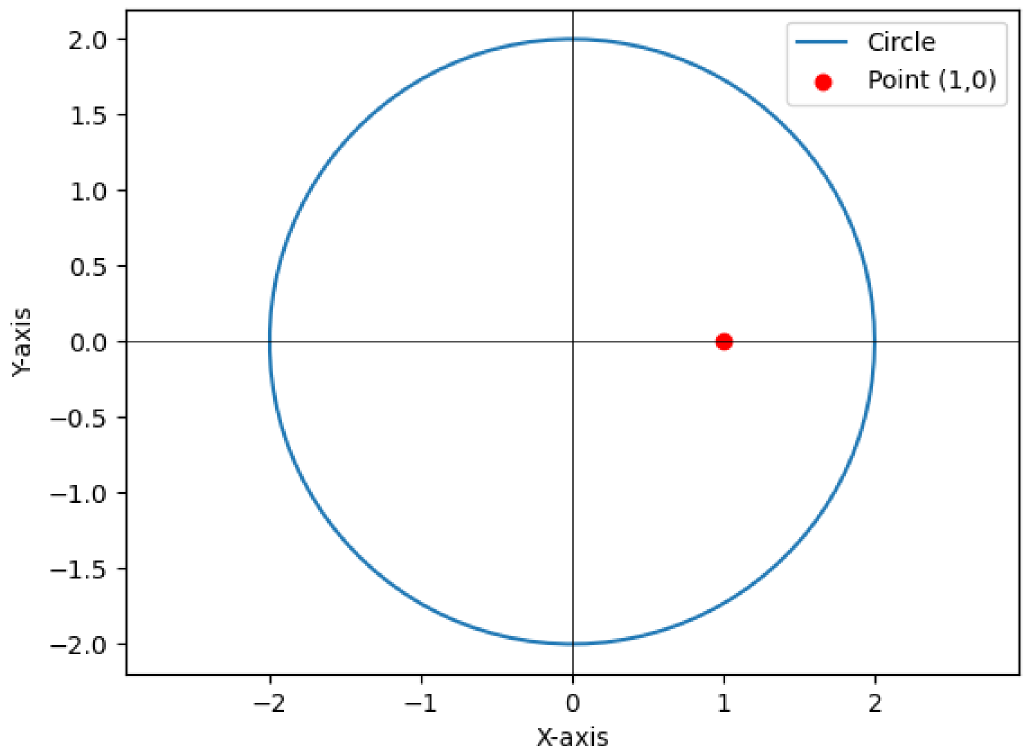


Figure 7.5: graph of option D

7.3 Consider the contour integral $\oint \frac{dz}{z^4+z^3-2z^2}$, along the curve $|z| = 3$ oriented in the counterclockwise direction. If $\text{Res}[f, z_0]$ denotes the residue of $f(z)$ at the point z_0, then which of the following are TRUE?

- (A) $\text{Res}[f, 0] = -\frac{1}{4}$
- (B) $\text{Res}[f, 1] = \frac{1}{3}$
- (C) $\text{Res}[f, -2] = -\frac{1}{12}$
- (D) $\text{Res}[f, 2] = -1$

(GATE NM 2023)

Solution:

$$\frac{dz}{z^4 + z^3 - 2z^2} = \frac{dz}{z^2(z-1)(z+2)} \tag{7.24}$$

Poles: $z = 0, 1, -2$

Curve: $|z| = 3$, all poles inside it

Given the function $f(z) = \frac{1}{z^2(z-1)(z+2)}$, with poles at $z = 0$, $z = 1$, and $z = -2$, and considering the curve $|z| = 3$, where all poles are inside it. We want to find the residues of $f(z)$ at these poles.

The general formula for finding the residue at a pole z_0 is:

$$\text{Res}(f, z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} \left((z - z_0)^n f(z)\right) \tag{7.25}$$

where n denotes how many times the pole is repeated.

(a) For $z_0 = 0$, where $n = 2$, we have:

$$\text{Res}(f, 0) = \frac{1}{1!} \lim_{z \to 0} \frac{d}{dz} \left(\frac{1}{(z-1)(z+2)} \right) \tag{7.26}$$

$$= -\frac{1}{4} \tag{7.27}$$

(b) For $z_0 = 1$, where $n = 1$, we have:

$$\text{Res}(f, 1) = \frac{1}{(1-1)!} \lim_{z \to 1} \frac{z-1}{z^2(z+2)} \tag{7.28}$$

$$= \frac{1}{3} \tag{7.29}$$

(c) For $z_0 = -2$, where $n = 1$, we have:

$$\operatorname{Res}(f, -2) = \frac{1}{(1-1)!} \lim_{z \to -2} \frac{z - (-2)}{z^2(z-1)} \tag{7.30}$$

$$= -\frac{1}{12} \tag{7.31}$$

Therefore, the correct answers are: options A, B, and C.

Chapter 8

Laplace Transform

8.1 The number of zeroes of the polynomial $P(s) = s^3 + 2s^2 + 5s + 80$ in the right side of the plane? (GATE IN 2023)

Solution: The table below shows the Routh array of the n^{th}- order characteristic polynomial :

$$a_0 s^n + a_1 s^{n-1} ...... + a_{n-1} s^1 + a_n s^0 \tag{8.1}$$

s^n	a_0	a_2	a_4	...
s^{n-1}	a_1	a_3	a_5	...
s^{n-2}	$b_1 = \frac{a_1 a_2 - a_3 a_0}{a_1}$	$b_2 = \frac{a_1 a_4 - a_5 a_0}{a_1}$	...	..
s^{n-3}	$c_1 = \frac{b_1 a_3 - b_2 a_1}{b_1}$	$\vdots$		
$\vdots$	$\vdots$	$\vdots$		
s^1	$\vdots$	$\vdots$		
s^0	a_n			

Table 8.1: Routh Array

Characteristic Equation:

$$s^3 + 2s^2 + 5s + 80 = 0 \tag{8.2}$$

From Table 8.1:

s^3	1	5
s^2	2	80
s^1	$\dfrac{2 \times 5 - 80 \times 1}{2} = -35$	
s^0	$\dfrac{-35 \times 80}{-35} = 80$	

Table 8.2:

From Table 8.2:

Since there are 2 sign changes in the first column of the Routh tabulation. So, the number of zeros in the right half of the s-plane will be 2.

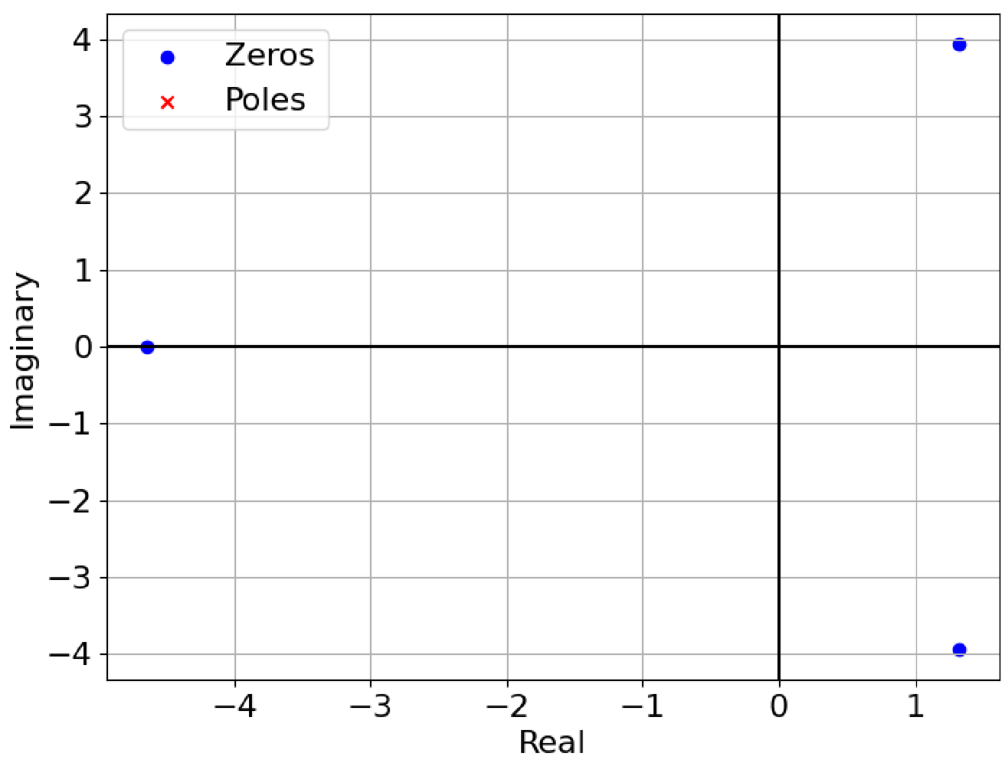
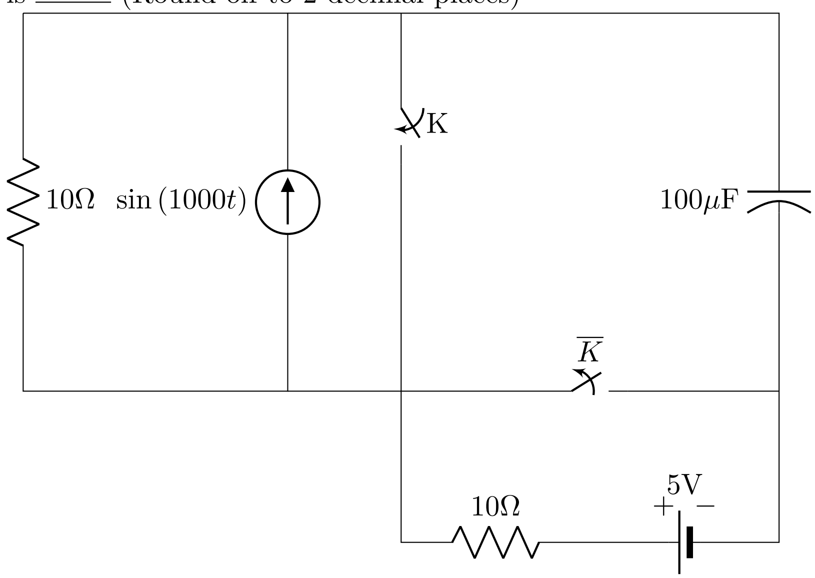


Figure 8.1:

8.2 The circuit shown in the figure is initially in the steady state with the switch K in open condition and $\overline{K}$ in closed condition. The switch K is closed and $\overline{K}$ is opened simultaneously at the instant $t = t_1$, where $t_1 > 0$. The minimum value of t_1 in milliseconds such that there is no transient in the voltage across the 100 μF capacitor, is ______ (Round off to 2 decimal places) (GATE EE 2023)



Solution:

(a) Switch K is open and $\overline{K}$ is closed.

Using Current divider rule,

$$I_1(j\omega) = \frac{10}{10 + \frac{1}{j\omega C}} \tag{8.3}$$

$$V_1(j\omega) = \frac{10}{1 + 10j\omega C} \tag{8.4}$$

$$|V_1(j\omega)| = 5\sqrt{2} \tag{8.5}$$

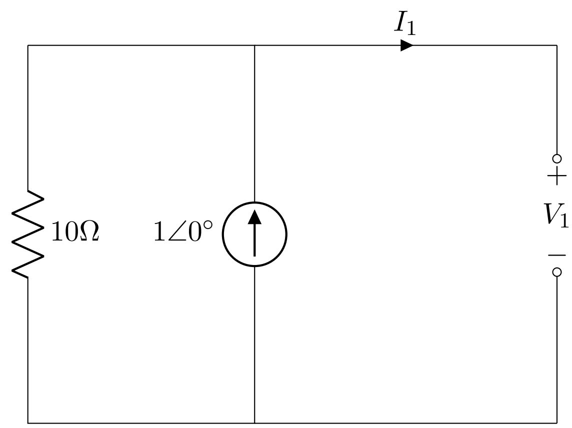


Figure 8.2: K is open and $\overline{K}$ is closed

From Table 8.3

$$V_1\left(t\right) = 5\sqrt{2}\sin\left(\omega t - \frac{\pi}{4}\right) \tag{8.6}$$

(b) Switch K is closed and $\overline{K}$ is open.

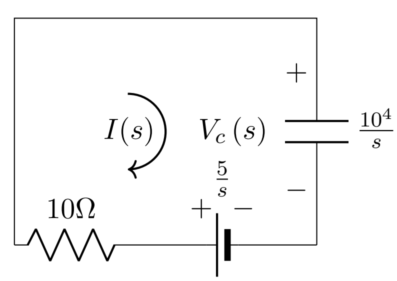


Figure 8.3: K is closed and $\overline{K}$ is open

The capacitor is charged. Thus, acts as a voltage source.

From eq(8.6) and Table 8.4

$$V_1(s) = \frac{5000 - 5s}{s^2 + 10^6} \tag{8.7}$$

$$I(s) = \frac{\frac{5}{s} - V_1(s)}{10 + \frac{10^4}{s}} \tag{8.8}$$

$$V_c(s) = \frac{5}{s} - 10\left(\frac{5 - V_1(s)}{1 + 10^{-3}s}\right) \tag{8.9}$$

For transient analysis,

$$\frac{5 - V_1(s)}{1 + 10^{-3}s} = 0 \tag{8.10}$$

$$\implies V_1(s) = 5 \tag{8.11}$$

$$\frac{10^7}{(s^2 + 10^6)(s + 10^3)} = \frac{5}{s} \tag{8.12}$$

$$\frac{5}{s + 10^3} + \frac{10^3 - s}{s^2 + 10^6} = \frac{5}{s} \tag{8.13}$$

$$\frac{-s}{s^2 + 10^6} + \frac{10^3}{s^2 + 10^6} + \frac{1}{s + 10^3} = \frac{1}{s} \tag{8.14}$$

From Table 8.4

$$-\cos(1000t_1) + \sin(1000t_1) + e^{-10^3 t_1} = 1 \tag{8.15}$$

$$\implies t_1 \approx 1.57\text{msec} \tag{8.16}$$

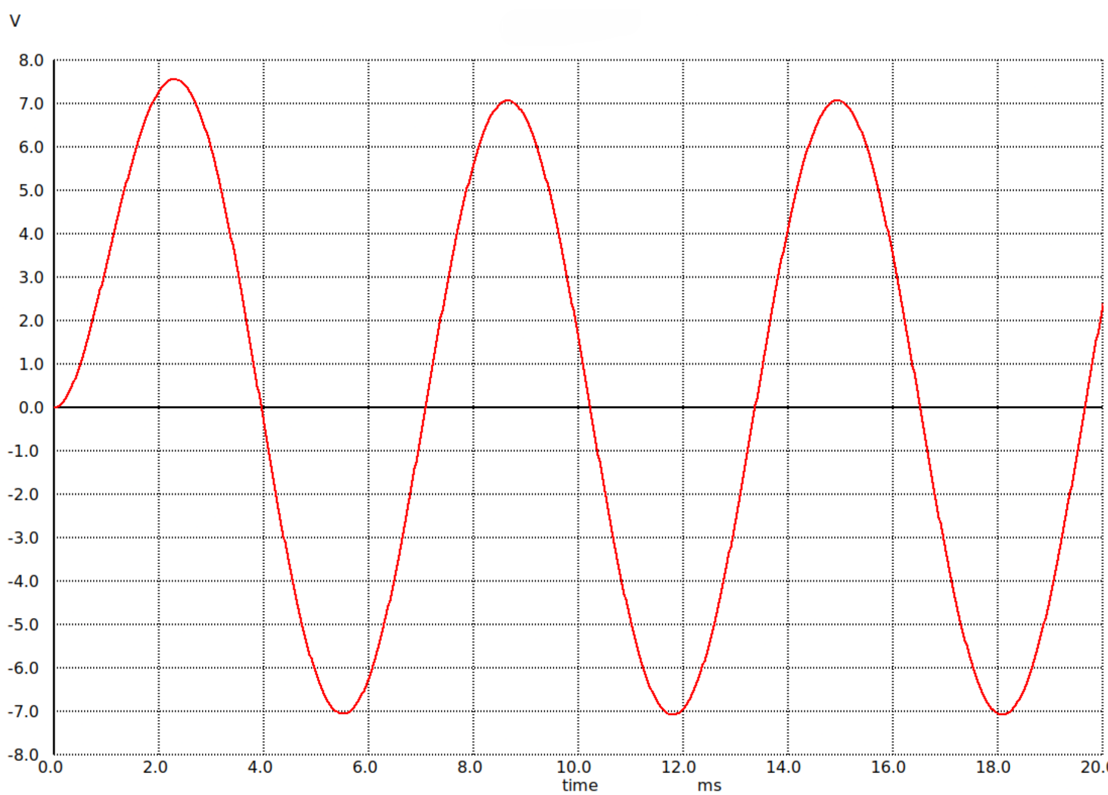


Figure 8.4: plot of V_1 vs time

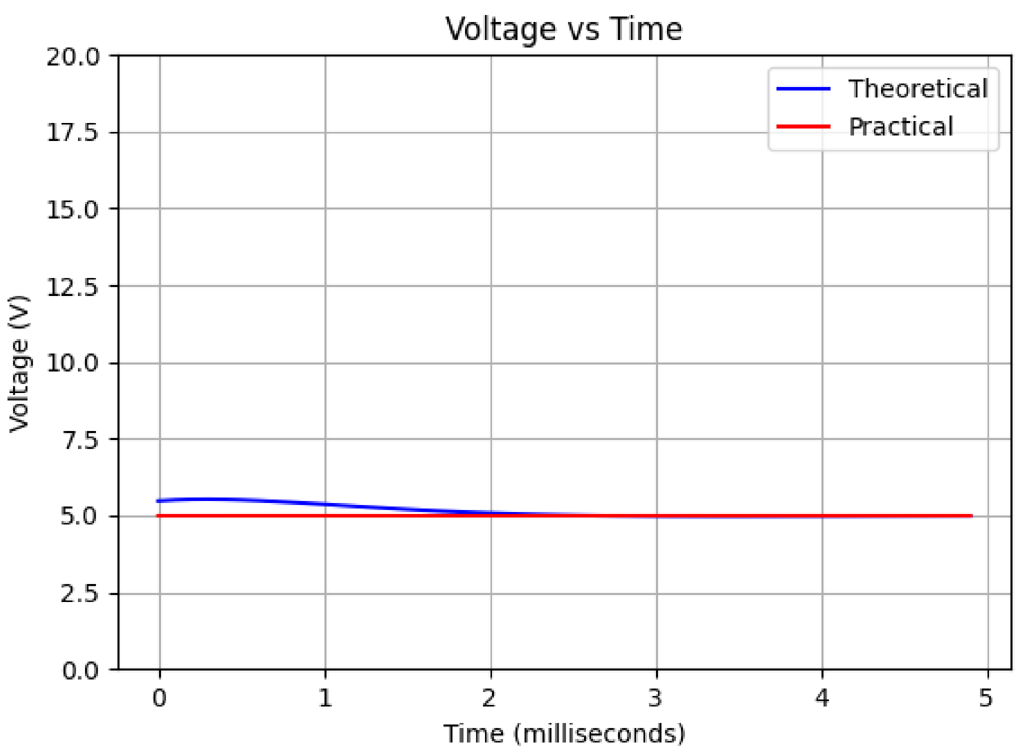


Figure 8.5: plot of V_c vs time

Parameter	Description	Remarks
ω	frequency of sine-wave	1000 rad s^{-1}
$V_1(t)$	Voltage across capacitor	$\lvert V_1(j\omega)\rvert \sin(\omega t - \angle V_1(j\omega))$
$\angle V_1(j\omega)$	phase of $V_1(j\omega)$	$\frac{-\pi}{4}$
C	Capacitance	100μF

Table 8.3: Parameters

S Domain	Time Domain
$\frac{1}{s}$	$u(t)$
$\frac{-s}{a^2+s^2}$	$-\cos(at)$
$\frac{a}{a^2+s^2}$	$\sin(at)$
$\frac{1}{s+a}$	e^{-at}

Table 8.4: Laplace transforms

8.3 $y = e^{mx} + e^{-mx}$ is the solution of which differential equation?

1. $\frac{dy}{dx} - my = 0$

2. $\frac{dy}{dx} + my = 0$

3. $\frac{d^2y}{dx^2} + m^2y = 0$

4. $\frac{d^2y}{dx^2} - m^2y = 0$

(GATE AG 2023) **Solution:**

8.4 A cascade control strategy is shown in the figure below. The transfer function between the output (y) and the secondary disturbance (d_2) is defined as

$$G_{d2}(s) = \frac{y(s)}{d_2(s)}$$

. Which one of the following is the CORRECT expression for the transfer function $G_{d2}(s)$?

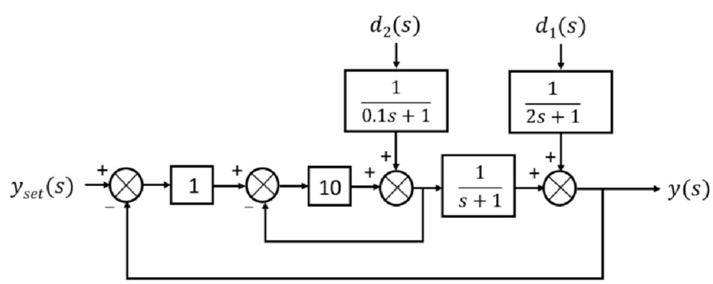


Figure 8.6:

A. $\frac{1}{(11s+21)(0.1s+1)}$

B. $\frac{1}{(s+1)(0.1s+1)}$

C. $\frac{(s+1)}{(s+2)(0.1s+1)}$

D. $\frac{(s+1)}{(s+1)(0.1s+1)}$

(GATE CH 2023) **Solution:**

Variable	Description
$d_1(s)$	Primary disturbance
$d_2(s)$	Secondary disturbance
$G_{d2}(s)$	Transfer function between $y(s)$ and $d_2(s)$
$y_{set}(s)$	Set point for desired output
$y(s)$	Output

Table 8.5: Input Parameters

Variable	Description	value
P_1	Forward path gain e-c-d	$\frac{1}{(0.1s+1)(s+1)}$
Δ_1	Determinant of forward path e-c-d	1
Δ	Determinant of system	$1+\frac{10}{s+1}+10$
n	Number of forward path	1

Table 8.6: Defined Parameters

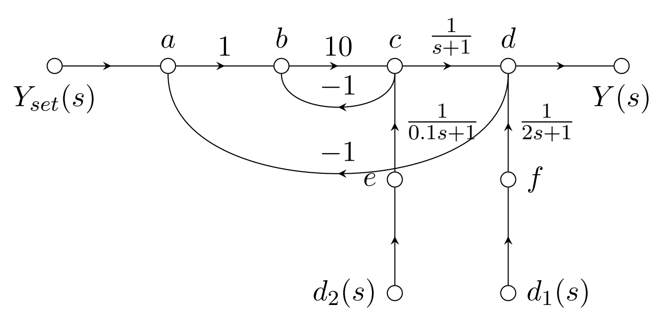


Figure 8.7: signal flow graph

Using Mason's Gain formula for the above Signal flow graph,

$$G_{d2}(s) = \frac{y(s)}{d_2(s)} = \frac{\sum_{i=1}^{n} P_i \Delta_i}{\Delta} \tag{8.17}$$

$$= \frac{P_1 \Delta_1}{\Delta} \tag{8.18}$$

$$= \frac{\frac{1}{(0.1s+1)(s+1)}}{1 + \frac{10}{s+1} + 10} \tag{8.19}$$

$$= \frac{\frac{1}{(0.1s+1)(s+1)}}{\frac{11s+21}{s+1}} \tag{8.20}$$

$$\implies G_{d2}(s) = \frac{1}{(11s+21)(0.1s+1)} \tag{8.21}$$

Now taking the inverse laplace transform we have,

$$G_{d2}(t) = \mathcal{L}^{-1}\left(\frac{10}{(s+10)\,(11s+21)}\right) \tag{8.22}$$

$$= \mathcal{L}^{-1}\left(\frac{-10}{89\,(x+10)} + \frac{110}{89\,(11x+21)}\right) \tag{8.23}$$

$$= \frac{-10e^{-10t}}{89} u(t) + \frac{10e^{\frac{-21t}{11}}}{89} u(t) \tag{8.24}$$

$$= \left(\frac{10\left(e^{\frac{-21t}{11}} - e^{-10t}\right)}{89}\right) u(t) \tag{8.25}$$

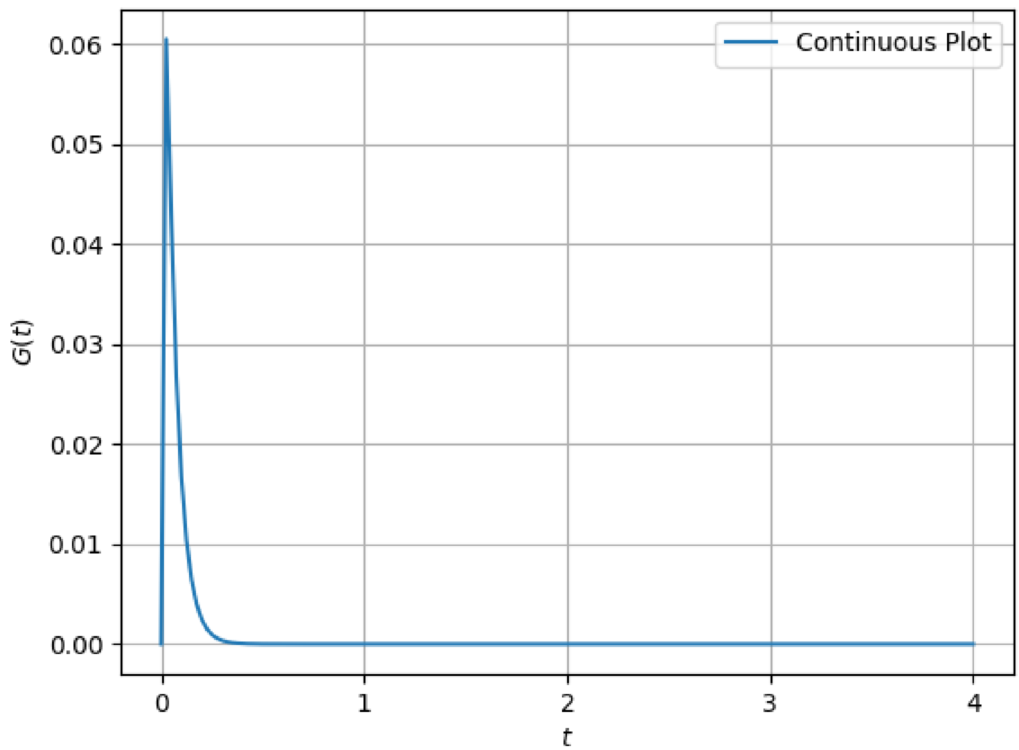
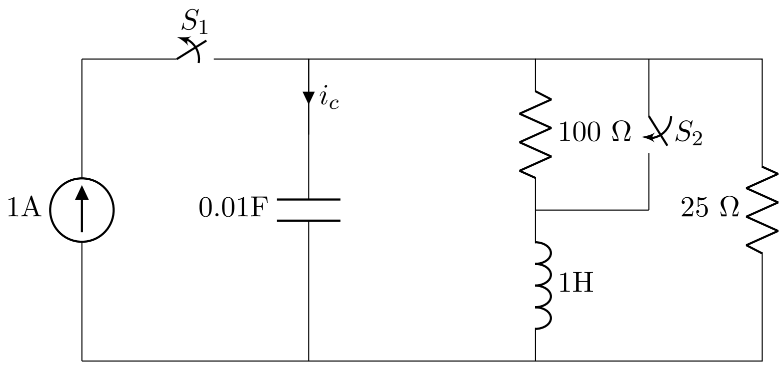


Figure 8.8:

8.5 In the differential equation $\frac{dy}{dx} + \alpha xy = 0, \alpha$ is a positive constant. If $y = 1.0$ at $x = 0.0$, and $y = 0.8$ at $x = 1.0$, the value of α is (rounded off to three decimal places). (GATE CE 2023) **Solution:**

8.6 The switch S_1 was closed and S_2 was open for a long time. At t=0,switch S_1 is opened and S_2 is closed,simultaneously. The value of $i_c(0^+)$, in amperes, is (GATE EC 44)



Solution:

1) Switch S_1 was closed and S_2 was open

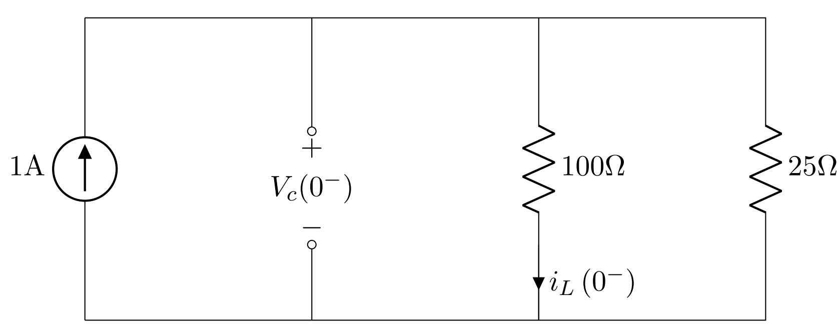


Figure 8.9: S_1 is closed and S_2 is open

$$R_{eff} = \frac{25\,(100)}{(25+100)}\Omega \tag{8.26}$$

$$R_{eff} = 20\Omega \tag{8.27}$$

$$V_c\left(0^-\right) = 1\left(R_{eff}\right) \tag{8.28}$$

$$V_c\left(0^-\right) = 20V \tag{8.29}$$

2) Switch S_1 is open and S_2 was closed

At $t = 0^+$ The capacitor is charged. Thus, it acts as a voltage source.The inductor acts as the current source.

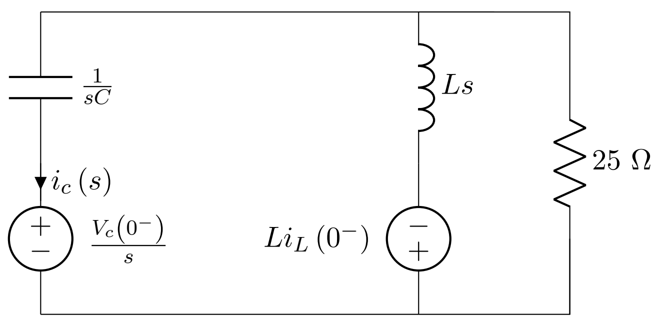


Figure 8.10: S_1 is open and S_2 is closed

By Superposition Theorem

Case (i):

$$\left(\frac{25\left(Ls\right)}{25+Ls} + \frac{1}{sC}\right) i_c^1\left(s\right) + \frac{V_c\left(0^-\right)}{s} = 0 \tag{8.30}$$

$$i_c^1\left(s\right) = -\frac{V_c\left(0^-\right)}{s}\left(\frac{25sC + LCs^2}{25LCs^2 + Ls + 25}\right) \tag{8.31}$$

$$i_c^1\left(s\right) = -\left(\frac{5 + 0.2s}{0.25s^2 + s + 25}\right) \tag{8.32}$$

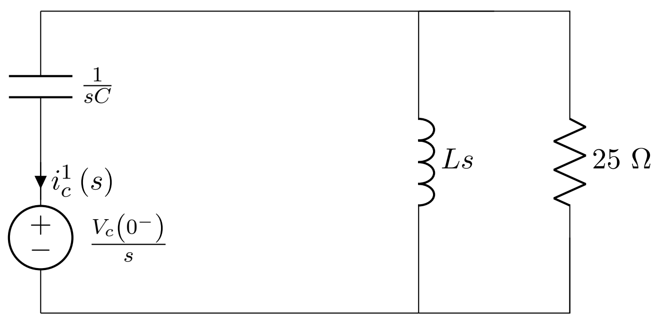


Figure 8.11: Circuit 4

Case (ii):

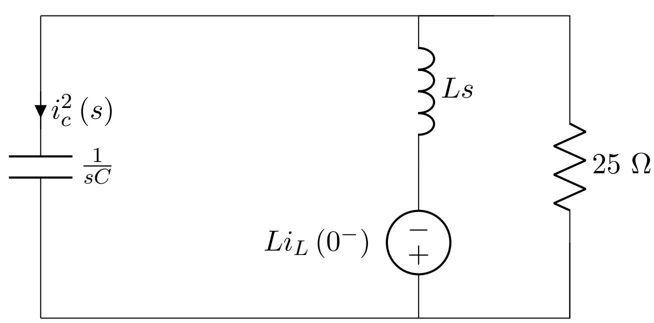


Figure 8.12: Circuit 5

$$Li_L\left(0^-\right) + \left(\frac{25}{25sC+1} + Ls\right) i_c^2(s) = 0 \tag{8.33}$$

$$0.2 = -\left(\frac{25LCs^2 + Ls + 25}{25sC+1}\right) i_c^2(s) \tag{8.34}$$

$$i_c^2(s) = -\left(\frac{0.05s + 0.2}{0.25s^2 + s + 25}\right) \tag{8.35}$$

From eq (8.32) and eq (8.35)

$$i_c(s) = i_c^1(s) + i_c^2(s) \tag{8.36}$$

$$i_c(s) = -\left(\frac{0.25s + 5.2}{0.25s^2 + s + 25}\right) \tag{8.37}$$

Using Inverse Laplace Transform

From eq (8.37)

$$i_c(t) = -e^{-2t}\left(\cos\left(4\sqrt{6}\,t\right) + \frac{18.8}{4\sqrt{6}}\sin\left(4\sqrt{6}\,t\right)\right) \tag{8.38}$$

$$i_c\left(0^+\right) = -1A \tag{8.39}$$

Parameter	Description	Remarks
$V_c(0^-)$	Voltage across capacitor when t<0	20V
$i_L(0^-)$	current across inductor when t<0	0.2
$i_L(0^+)$	current across inductor when t>0	0.2
C	Capacitance	0.01F
L	Inductance	1H

Table 8.7: Parameters

8.7 The continuous time signal $x(t)$ is described by:

$$x(t) = \begin{cases} 1, & \text{if } 0 \le t \le 1 \\ 0, & \text{elsewhere} \end{cases} \tag{8.40}$$

If $y(t)$ represents $x(t)$ convolved with itself, which of the following options is/are TRUE?

A $y(t) = 0$ for all $t < 0$

B $y(t) = 0$ for all $t > 1$

C $y(t) = 0$ for all $t > 3$

D $\int_{0.1}^{0.75} \frac{dy(t)}{dt}$ dt $\neq 0$

Solution:

8.8 The Z-transform of a discrete signal $x(n)$ is

$$X(z) = \frac{4z}{\left(z - \frac{1}{5}\right)\left(z - \frac{2}{3}\right)(z-3)} \text{ with ROC} = R \tag{8.41}$$

Which one of the following statements is TRUE?

(a) Discrete time Fourier transform of $x[n]$ converges if R is $|z| > 3$

(b) Discrete time Fourier transform of $x[n]$ converges if R is $\frac{2}{3} < |z| < 3$

(c) Discrete time Fourier transform of $x[n]$ converges if R is such that $x[n]$ is a left-sided sequence.

(d) Discrete time Fourier transform of $x[n]$ converges if R is such that $x[n]$is a right-sided sequence.

GATE EE 2023

Solution: Poles of $X(z)$ are located at $z = \frac{1}{5}$, $z = \frac{2}{3}$, and $z = 3$.
For DTFT to converge, the ROC of Z-transform of $x[n]$ should contain unit circle.

(a) If ROC is $|z| > 3$, it does not include unit circle

Option (a) is wrong.

(b) If ROC is $\frac{2}{3} < |z| < 3$, the ROC includes unit circle.

So, option (b) is correct.

(c) If $x(n)$ is a left-sided sequence, then ROC will be $|z| < \frac{1}{5}$, which does not include the unit circle.

Option (c) is wrong.

(d) If $x(n)$ is a right-sided sequence, then the ROC is $|z| > 3$, which does not include the unit circle.

Option (d) is wrong.

Hence, the correct option is (b).

8.9 The phase margin of the transfer function $G(s) = \frac{2(1-s)}{(1+s)^2}$ is ______ degrees. (rounded off to the nearest integer). (GATE IN 2023)

Solution:

Parameters	**Description**
ω_c	crossover frequency
$\angle G(j\omega)$	phase angle of the transfer function
PM	$\angle G(j\omega_c) + 180°$; Phase Margin

Table 8.8: Parameters

Considering $s = j\omega$,

$$G(j\omega) = \frac{2(1-j\omega)}{(1+j\omega)^2} \tag{8.42}$$

$$= \frac{2(1-j\omega)^3}{|1+j\omega|^4} \tag{8.43}$$

$$= \frac{2}{(1+\omega^2)^2}(1-j\omega)^3 \tag{8.44}$$

$$= \frac{2}{\sqrt{1+\omega^2}}(e^{-j\omega})^3 \tag{8.45}$$

$$\implies |G(j\omega)| = \frac{2}{\sqrt{1+\omega^2}} \tag{8.46}$$

$$\implies \angle G(j\omega) = 3\tan^{-1}(-\omega) \tag{8.47}$$

At $\omega = \omega_c$, $Gain = 0$

$$\implies |G(j\omega_c)| = 1 \tag{8.48}$$

$$\frac{2}{\sqrt{1+\omega_c^2}} = 1 \tag{8.49}$$

$$\implies \omega_c = \sqrt{3} \tag{8.50}$$

$$\angle G(j\omega_c) = 3\tan^{-1}(-\sqrt{3}) \tag{8.51}$$

$$= -180° \tag{8.52}$$

From Table 8.8,

$$PM = 0^\circ \tag{8.53}$$

8.10 Consider the second-order linear differential equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0,\ x \geq 1$$

with the initial conditions

$$y(x = 1) = 6, \quad \frac{dy}{dx}\Big|_{x=1} = 2.$$

Then the value of y at $x = 2$ is ________.

GATE ME 2023

Solution:

Symbol	Value	Description
y(x)	?	Function
y(1)	6	Initial Condition
y'(1)	2	Initial Condition

Table 8.9: Given Information

By Euler-Cauchy substitution for the given question,

$$x = e^t, \ \ t \geq 0 \tag{8.54}$$

$$\implies \frac{dt}{dx} = e^{-t} \tag{8.55}$$

$$\implies \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = e^{-t}\frac{dy}{dt} \tag{8.56}$$

$$\implies \frac{d^2y}{dx^2} = \frac{d^2y}{dt^2}\left(\frac{dt}{dx}\right)^2 + \frac{d^2t}{dx^2}\left(\frac{dy}{dt}\right) \tag{8.57}$$

$$= e^{-2t}\frac{d^2y}{dt^2} + e^{-2t}\frac{dy}{dt} \tag{8.58}$$

So the given equation becomes

$$e^{2t}\left(e^{-2t}\frac{d^2y}{dt^2}-e^{-2t}\frac{dy}{dt}\right)+e^{-t}\left(e^{-t}\frac{dy}{dt}\right)-y=0 \tag{8.59}$$

$$\frac{d^2y}{dt^2}-\frac{dy}{dt}+\frac{dy}{dt}-y=0 \tag{8.60}$$

$$\frac{d^2y}{dt^2}-y=0,\ \ t\geq 0 \tag{8.61}$$

Taking Laplace Transform,

$$s^2Y(s)-sy(0)-y'(0)-Y(s)=0 \tag{8.62}$$

$$Y(s)=\frac{sy(0)+y'(0)}{s^2-1} \tag{8.63}$$

$$=\frac{6s+2}{s^2-1} \tag{8.64}$$

By partial fractions,

$$Y(s)=\frac{4}{s-1}+\frac{2}{s+1} \tag{8.65}$$

Taking Inverse Laplace using

$$\frac{1}{s+a}\overset{\mathcal{L}^{-\prime}}{\longleftrightarrow}e^{-at} \tag{8.66}$$

We get,

$$y(t)=4e^{t}+2e^{-t} \tag{8.67}$$

From (8.54)

$$y\,(t) = 4x + \frac{2}{x} \tag{8.68}$$

$$\Longrightarrow y\,(2) = 9 \tag{8.69}$$

8.11 The transfer function of a measuring instrument is

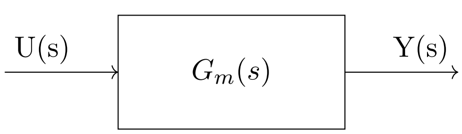
$$G_m(s) = \frac{1.05}{2s+1} exp(-s)$$

At time $t = 0$, a step change of +1 unit is introduced in the input of this instrument.The time taken by the instrument to show an increase of 1 unit in its output is(rounded off to two decimal places).

(GATE CH 2023) **Solution:**

Parameter	Description	Value
$G_m(s)$	Transfer function	$\frac{Y(s)}{U(s)}$
Y(s)	Laplace transform of the output	?
U(s)	Laplace transform of the input	$\frac{1}{s}$

Table 8.10: Given parameters



$$G_m(s) = \frac{1.05}{2s+1} e^{-s} \tag{8.70}$$

$$\because Y(s) = G_m(s).U(s) \tag{8.71}$$

$$\implies Y(s) = \frac{1}{s}.\frac{1.05}{2s+1} e^{-s} \tag{8.72}$$

By splitting into partial fractions,we get

$$Y(s) = \left[\frac{1.05}{s} - \frac{1.05}{s+0.5}\right] e^{-s} \tag{8.73}$$

As we know,

$$\mathcal{L}[e^{-at}] \longleftrightarrow \frac{1}{s+a} \tag{8.74}$$

$$\mathcal{L}[f(t-1)] \longleftrightarrow e^{-s}F(s) \tag{8.75}$$

By taking inverse laplce we get,

$$y(t) = 1.05\left[1 - e^{\frac{-(t-1)}{2}}\right]u(t-1) \tag{8.76}$$

$$\frac{1}{1.05} = 1 - e^{\frac{-(t-1)}{2}} \tag{8.77}$$

$$\frac{-(t-1)}{2} = \ln(\frac{0.05}{1.05}) \tag{8.78}$$

$$\implies t = 7.073 \tag{8.79}$$

8.12 The laplace transform of $x_1(t) = e^{-t}u(t)$ is $X_1(s)$, where $u(t)$ is the unit step function. The laplace transform of $x_2(t) = e^{t}u(-t)$ is $X_2(s)$. Which one of the following statements is TRUE?

(a) The region of convergence of $X_1(s)$ is $Re(s) \geq 0$

(b) The region of convergence of $X_2(s)$ is confined to the left half-plane of s.

(c) The region of convergence of $X_1(s)$ is confined to the right half-plane of s.

(d) the imaginary axis in the s-plane is included in both the region of convergence of $X_1(s)$ and the region of convergence of $X_2(s)$.

(GATE BM 2023)

Solution:

Symbols	Description
$X_1(s)$	Laplace transform of $x_1(t)$
$X_2(s)$	Laplace transform of $x_2(t)$
$u(t)$	Unit step function

Table 8.11: Parameters, Descriptions

(a) Laplace transform of $x_1(t)$ is given by :

$$X_1(s) = \int_{-\infty}^{\infty} e^{-t}e^{-st}u(t)\,dt \tag{8.80}$$

Let $s = \sigma + j\omega$:

$$X_1(s) = \int_{0}^{\infty} e^{-t(\sigma+1)}e^{-tj\beta}\,dt \tag{8.81}$$

$$= \left[\frac{-e^{-t(\sigma+1)}e^{-tj\beta}}{(\sigma+1)+j\beta}\right]_0^{\infty} \tag{8.82}$$

For $X_1(s)$ to be convergent, $\left|-e^{-t(\sigma+1)}e^{-tj\beta}\right|$ must converge $\forall t\epsilon\,(0,\infty)$, so:

$$\left|e^{-tj\beta}\right| = |1|\,, \forall\beta\epsilon\mathbb{R} \implies Im\,(s)\,\epsilon\mathbb{R} \tag{8.83}$$

$$\sigma + 1 > 0 \implies \Re\,(s) > -1 \tag{8.84}$$

Putting the limits :

$$X_1(s) = \frac{1}{s+1}, \Re\,(s) > -1 \tag{8.85}$$

(b) Laplace transform of $x_2(t)$ is given by :

$$X_2(s) = \int_{-\infty}^{\infty} e^t e^{-st} u(-t)\,dt \tag{8.86}$$

Let $s = \sigma + j\omega$:

$$= \int_{-\infty}^{0} e^{t(1-\sigma)}e^{-tj\beta}\,dt \tag{8.87}$$

$$= \left[\frac{e^{t(1-\sigma)}e^{-tj\beta}}{(1-\sigma)-j\beta}\right]_{-\infty}^{0} \tag{8.88}$$

For $X_2(s)$ to be convergent, $\left|e^{t(1-\sigma)}e^{-tj\beta}\right|$ must converge $\forall t\epsilon\,(-\infty,0)$, so:

$$\left|e^{-tj\beta}\right| = |1|\,, \forall\beta\epsilon\mathbb{R} \implies Im\,(s)\,\epsilon\mathbb{R} \tag{8.89}$$

$$1 - \sigma > 0 \implies \Re\,(s) < 1 \tag{8.90}$$

Putting the limits

$$X_2(s) = \frac{1}{1-s}, \Re\,(s) < 1 \tag{8.91}$$

Based on the overlap of regions of convergence of $X_1(s)$ and $X_2(s)$ from 8.19 , we can conclude that option 4) is correct .

8.13 Given that $\frac{dy}{dx} = 2x + y$ and $y = 1$,when $x = 0$ Using Runge-Kutta fourth order method,the value of y at $x = 0.2$ is (GATE 2023 AG 50)

Solution: By using runge kutta 4 th order method

Variable	Description	Value
$x(n-1)$	value of x before runge kutta iteration	0
$y(n-1)$	value of y before runge kutta iteration	1
$y(n)$	value of y after runge kutta iteration	??
$x(n)$	value of x after runge kutta iteration	?
h	step size	0.1

Table 8.12: Variables Used

$$y(n) = y(n-1) + \frac{h}{6}[(2x(n-1) + y(n-1)(6 + 3h + h^2 + \frac{h^3}{4}) + (6h + 2h^2 + \frac{h^3}{2})] \tag{8.92}$$

assume step size as 0.1 and initial conditions as $x = 0$ and $y = 1$

$$y(n) = 1 + (6 + 3(0.1) + 0.1^2 + \frac{0.1^3}{4}) + (6(0.1) + 2(0.1)^2 + \frac{0.1^3}{2}) \tag{8.93}$$

$$\implies y_n = 1.115 \tag{8.94}$$

cosidering outputs of last iteration as inputs of next iteration

$$y(n) = 1.155 + \frac{0.1}{6}[(2(0.1) + 1.155(6 + 3(0.1) + (0.1)^2 + \frac{(0.1)^3}{4}) + (6(0.1) + 2(0.1)^2 + \frac{(0.1)^3}{2}) \tag{8.95}$$

so at $x = 0.2$ value of y is 1.29

analysis

$$\frac{dy}{dx} = 2x + y \tag{8.96}$$

$$ye^{-x} = \int 2xe^{-x}dx \tag{8.97}$$

$$\implies ye^{-x} = -2(x+1)e^{-x} + c \tag{8.98}$$

by using intial conditions

$$c = 3 \tag{8.99}$$

$$\implies y = 3e^{x} - 2(x+1) \tag{8.100}$$

8.14 The magnitude and phase plots shown in the figure match with the transfer- function

(a) $\frac{10000}{s^2+2s+10000}$

(b) $\frac{10000}{s^2+2s+10000}e^{-0.05s}$

(c) $\frac{10000}{s^2+2s+10000}e^{-0.5\times10^{-12}s}$

(d) $\frac{100}{s^2+2s+100}$

(GATE IN 2023) **Solution:** Drawing bode plots for four options.

$$\implies H(s) = \frac{k}{s^2+2s+k}e^{as} \tag{8.101}$$

$$H(j\omega) = \frac{k}{k-\omega^2+2j\omega}e^{aj\omega} \tag{8.102}$$

$$|H(j\omega)| = \frac{k}{\sqrt{\left(k-\omega^2\right)^2+4\omega^2}} \tag{8.103}$$

$$\implies \phi(H(j\omega)) = \left(-\tan^{-1}\left(\frac{2\omega}{k-\omega^2}\right)-a\omega\right) \tag{8.104}$$

From the graphs , the answer is b

8.15 The Laplace transform of the continuous-time signal $x(t) = e^{-3t}u(t-5)$ is ____, where $u(t)$ denotes the continuous-time unit step signal.

A) $\frac{e^{-5s}}{s+3}$, Real$\{s\} > -3$

B) $\frac{e^{-5(s-3)}}{s-3}$, Real$\{s\} > 3$

C) $\frac{e^{-5(s+3)}}{s+3}$, Real$\{s\} > -3$

D) $\frac{e^{-5(s-3)}}{s+3}$, Real$\{s\} > -3$

Solution:

Parameter	**Description**	**Value**
$x(t)$	Given Function	$x(t) = e^{-3t}u(t)$
$X(s)$	Laplace Transform of $x(t)$	$\frac{e^{-5(s+3)}}{s+3}$

Table 1: Table of parameters

$$e^{-3t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+3} \quad \Re(s) > -3 \tag{8.105}$$

Using time shifting,

$$e^{-3(t-5)}u(t-5) \overset{\mathcal{L}}{\longleftrightarrow} \frac{e^{-5s}}{s+3} \tag{8.106}$$

$$e^{-15}e^{-3(t-5)}u(t-5) \overset{\mathcal{L}}{\longleftrightarrow} e^{-15}\frac{e^{-5s}}{s+3} \tag{8.107}$$

$$e^{-3t}u(t-5) \overset{\mathcal{L}}{\longleftrightarrow} \frac{e^{-5(s+3)}}{s+3} \tag{8.108}$$

$$\therefore x(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{e^{-5(s+3)}}{s+3} \quad \Re(s) > -3 \tag{8.109}$$

8.16 The solution $x(t)$, $t \geq 0$, to the differential equation $\ddot{x} = -k\dot{x}, k > 0$ with initial conditions $x(0) = 1$ and $\dot{x}(0) = 0$ is

(A) $x(t) = 2e^{-kt} + 2kt - 1$

(B) $x(t) = 2e^{-kt} - 1$

(C) $x(t) = 1$

(D) $x(t) = 2e^{-kt} - kt - 1$

GATE 2023 , IN

Solution:

Differential equation	$\ddot{x} = -k\dot{x}$
Initial conditions	$x(0) = 1$ and $\dot{x}(0) = 0$
$x(t)$	?

Table 8.14: Parameter Table

$$\implies \frac{d^2x(t)}{dt^2} = -k\frac{dx(t)}{dt} \tag{8.110}$$

Taking Laplace transform on both sides,

$$\frac{d^2x(t)}{dx^2} \xleftrightarrow{\mathcal{L}} s^2X(s) - sx(0) - \dot{x}(0) \tag{8.111}$$

$$\frac{dx(t)}{dx} \xleftrightarrow{\mathcal{L}} sX(s) - x(0) \tag{8.112}$$

From Table (8.111) , (8.112)

$$s^2X(s) - sx(0) - \dot{x}(0) = -k(sX(s) - x(0)) \tag{8.113}$$

$$s^2X(s) - s = -k(sX(s) - 1) \tag{8.114}$$

$$sX(s)(s+k) = (s+k) \tag{8.115}$$

$$X(s) = \frac{1}{s} \quad , s \neq -k \tag{8.116}$$

$$x(t) = u(t) \tag{8.117}$$

$$\implies x(t) = 1 \quad (t \geq 0) \tag{8.118}$$

Thus, the correct option is (C)

8.17 Consider the differential equation

$$x^2\frac{d^2y}{dx^2}+4x\frac{dy}{dx}+2y=0 \quad \text{for } x\geq 1$$

with initial conditions $y=0$ and $\frac{dy}{dx}=1$ at $x=1$. The value of y at $x=2$ is ?

(GATE AE 54 2023)

Solution: Given :

$$x^2\frac{d^2y}{dx^2}+4x\frac{dy}{dx}+2y=0 \quad \text{for } x\geq 1 \tag{8.119}$$

Using Euler Substitution :

$$x = e^t \tag{8.120}$$

$$\frac{dy}{dx} = e^{-t}\frac{dy}{dt} \tag{8.121}$$

$$\frac{d^2y}{dx^2} = e^{-2t}\frac{d^2y}{dt^2}-e^{-2t}\frac{dy}{dt} \tag{8.122}$$

Substituting equations (8.120),(8.121),(8.122) in equation (8.119):

$$\frac{d^2y}{dt^2}+3\frac{dy}{dt}+2y=0 \tag{8.123}$$

Taking Laplace on Both Sides:

$$s^2Y(s)-sy(1)-y'(1)+3sY(s)-y(1)+2Y(s)=0 \tag{8.124}$$

We know that:

$$y(1) = 0 \tag{8.125}$$

$$y'(1) = 1 \tag{8.126}$$

Substituting equations (8.125),(8.126) in (8.124)

$$s^2Y(s) + 3sY(s) + 2Y(s) = 1 \tag{8.127}$$

$$\implies Y(s) = \frac{1}{s^2 + 3s + 2} \tag{8.128}$$

$$\implies Y(s) = \frac{1}{s+1} - \frac{1}{s+2} \tag{8.129}$$

Taking Inverse Laplace:

$$y(t) = (e^{-t} - e^{-2t})u(t) \tag{8.130}$$

Substituting $x = e^t$:

$$y(x) = e^{-\ln x} - e^{-2\ln x}, \quad x \geq 1 \tag{8.131}$$

$$y(x) = \frac{1}{x} - \frac{1}{x^2}, \quad x \geq 1 \tag{8.132}$$

$$\therefore y(2) = 0.25 \tag{8.133}$$

8.18 Consider a lead compensator of the form

$K(s) = \frac{1+\frac{s}{a}}{1+\frac{s}{\beta a}}, \quad \beta > 1, \quad a > 0$

The frequency at which this compensator produces maximum phase lead is $4\,\text{rad/s}$. At this frequency, the gain amplification provided by the controller, assuming an asymptotic Bode-magnitude plot of $K(s)$, is $6\,\text{dB}$. The values of a and β, respectively, are

(a) $1, 16$

(b) $2, 4$

(c) $3, 5$

(d) $2.66, 2.25$

(GATE EE 2023) **Solution:**

Parameter	Value
Transfer Function	$K(s) = \frac{1+\frac{s}{a}}{1+\frac{s}{\beta a}}$
Maximum Phase Lead Frequency	$\omega_m = 4\,\text{rad/s}$
Gain Amplification at ω_m	$20\log_{10}\lvert K(j\omega_m)\rvert = 6\,\text{dB}$
Conditions	$\beta > 1$, $a > 0$

Table 8.15: Given Parameters

$$K(s) = \frac{1+\frac{s}{a}}{1+\frac{s}{a\beta}}$$

$$K(s) = \frac{s+a}{a} \cdot \frac{a\beta}{s+a\beta} \tag{8.134}$$
$$= \beta \frac{s+a}{s+a\beta} \tag{8.135}$$

1. The max phase lead is: $\omega_m = \sqrt{a \cdot a \cdot \beta}$

2. If $G(s) = \frac{k(s+a)}{s(s+a\beta)}$ has to act as a lead compensator, then $a\beta$ must be greater than a, i.e., $a\beta > a$.

Using the above properties we have:

$$\omega_m = \sqrt{a \cdot a\beta} = 4 \tag{8.136}$$
$$\beta > 1 \tag{8.137}$$

$$K(j\omega) = \frac{1 + \frac{j\omega}{a}}{1 + \frac{j\omega}{a\beta}} \qquad = \beta \frac{j\omega + a}{j\omega + a\beta} \tag{8.138}$$

$$|K(j\omega)| = \frac{\beta\sqrt{\omega^2 + a^2}}{\sqrt{\omega^2 + (a\beta)^2}} \tag{8.139}$$

Using Gain Amplification:

$$20 \log_{10} |K(j\omega_m)| = 6 \tag{8.140}$$

$$|K(j\omega_m)| \approx 2 \tag{8.141}$$

$$|K(j\omega_m)| = \frac{\beta\sqrt{(\omega_m)^2 + a^2}}{\sqrt{(\omega_m)^2 + (a\beta)^2}} = 2 \tag{8.142}$$

$$\frac{\beta\sqrt{16 + a^2}}{\sqrt{16 + (a\beta)^2}} = 2 \tag{8.143}$$

Solving the above equation, we get $a \approx 2$ and $\beta = 4$.

Therefore, the correct answer is **(B) 2, 4**.

The Bode plots for the same are as follows:

8.19 Second order ordinary differential equation $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$ has values $y = 2$ and $\frac{dy}{dx} = 1$ at $x = 0$.The value of y at $x = 1$ is?($round\, off\;\; to\;\; three\;\; decimal\;\; places$) [GATE-ES 2023]

Solution: We convert given second order differential equation to s domain using Laplace transform and solve for $Y(s)$ and take inversion to get $y(x)$.

Symbol	**Values**	**Description**
$Y(s)$	$\frac{2s-1}{s^2-s-2}$	y in s domain
$y(x)$	$e^{2x} + e^{-x}$	y in x domain
$y(0)$	2	y at $x = 0$
$y'(0)$	1	$y'(x)$ at $x = 0$
$u(x)$	$= \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & o.w \end{cases}$	unit step function

Table 8.16: Parameters

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y \overset{\mathcal{L}}{\longleftrightarrow} s^2Y(s) - sy(0) - y'(0) - sY(s) + y(0) - 2Y(s) \tag{8.144}$$

$$Y(s)\left(s^2 - s - 2\right) = 2s - 1 \tag{8.145}$$

$$\Rightarrow Y(s) = \frac{2s-1}{s^2-s-2} \tag{8.146}$$

$$\Rightarrow Y(s) = \frac{1}{s-2} + \frac{1}{s+1} \tag{8.147}$$

For inversion of $Y(s)$ in partial fractions-

$$\frac{b}{s+a} \overset{\mathcal{L}^{-1}}{\longleftrightarrow} be^{-ax}u(x) \tag{8.148}$$

Where b, a are real numbers, we invert $Y(s)$ to get $y(x)$:-

From (8.148)

$$Y(s) \xleftrightarrow{\mathcal{L}^{-1}} y(x)u(x) \tag{8.149}$$

$$y(x) = \left(e^{2x} + e^{-x}\right) u(x) \tag{8.150}$$

$$\Rightarrow y(1) = 7.757 \tag{8.151}$$

8.20 A continuous-time system that is initially at rest is described by,

$$\frac{dy(t)}{dt} + 3y(t) = 2x(t)$$

where $x(t)$ is the input voltage and $y(t)$ is the output voltage.

The impulse response of the system is? (GATE EE 2023) **Solution:**

Parameter	**Value**	**Description**
$x(t)$	-	Input voltage
$y(t)$	-	Output voltage
$h(t)$	$\frac{y(t)}{x(t)}$	Impulse response
$X(s)$	-	Input voltage in s-domain
$Y(s)$	-	Output voltage in s-domain
$H(s)$	$\frac{Y(s)}{X(s)}$	Impulse response in s-domain

Table 8.17: Input Table

Given equation is,

$$\frac{dy(t)}{dt} + 3y(t) = 2x(t) \tag{8.152}$$

Applying Laplace transform,

$$x(t) \xleftrightarrow{\mathcal{L}} X(s) \tag{8.153}$$

$$y(t) \xleftrightarrow{\mathcal{L}} Y(s) \tag{8.154}$$

From the differentiation property,

$$\frac{dy(t)}{dt} \xleftrightarrow{\mathcal{L}} sY(s) \tag{8.155}$$

The equation becomes,

$$sY(s) + 3Y(s) = 2X(s) \tag{8.156}$$

$$Y(s)(s+3) = 2X(s) \tag{8.157}$$

$$H(s) = \frac{Y(s)}{X(s)} \tag{8.158}$$

$$H(s) = \frac{2}{s+3} \tag{8.159}$$

$$\frac{1}{s+a} \xleftrightarrow{\mathcal{L}} e^{-at}u(t) \tag{8.160}$$

Using these results,

$$h(t) = 2e^{-3t}u(t) \tag{8.161}$$

8.21 Consider the equation $\frac{dy}{dx} + ay = \sin \omega x$,where a and ω are constants.Given $y = 1$ at $x = 0$, correct all the correct statement(s) from the following as $x \to \infty$.

(A) $y \to 0$ if $a \neq 0$

(B) $y \to 1$ if $a = 0$

(C) $y \to Aexp(|a|x)$ if $a < 0$; A is constant

(D) $y \to B\sin(\omega x + C)$ if $a > 0$; B and C are constants

(GATE AE 2023) **Solution:** $y(0) = 1$

$$\frac{dy}{dx} + ay = \sin \omega x \tag{8.162}$$

Taking laplace transform on both sides

Function	**Laplace transform**
$\frac{dy}{dx}$	$xY - y(0)$
y	Y
$\sin \omega x$	$\frac{\omega}{\omega^2+x^2}$

Table 8.18: Laplace transform

$$sY - y(0) + aY = \frac{\omega}{\omega^2 + s^2} \tag{8.163}$$

$$sY - 1 + aY = \frac{\omega}{\omega^2 + s^2} \tag{8.164}$$

$$\implies Y(s) = \frac{1}{s+a}\left(\frac{\omega}{\omega^2 + s^2} + 1\right) \tag{8.165}$$

$$Y(s) = \frac{A}{s+a} + \frac{Bs + C}{\omega^2 + s^2} \tag{8.166}$$

Taking inverse laplace transform on both sides

$$y(x) = \mathcal{L}^{-1}\{Y(s)\} = Ae^{-ax} + (B\cos(\omega x) + C\sin(\omega x)) \tag{8.167}$$

$$y(x) = \mathcal{L}^{-1}\{Y(s)\} = Ae^{-ax} + B\sin(\omega x + C) \tag{8.168}$$

now as $x \to \infty$

(a) $y \to 0$ if $a \neq 0$ is not true as y depend on a, ω

(b) $y \to 1$ if $a = 0$ is not true as y depend on ω

(c) $y \to A\exp(|a|x)$ if $a < 0$ is true as $B\sin(\omega x + C)$ is neglected compared to Ae^{-ax}

(d) $y \to B\sin(\omega x + C)$ if $a > 0$; is true as $Ae^{-ax} \to 0$

$\therefore$ C,D are correct options

8.22 **Question** : The position $x(t)$ of a particle, at constant ω, is described by the equation

$$\frac{d^2x}{dt^2} = -\omega^2 x. \tag{8.169}$$

The initial conditions are $x(t=0) = 1$ and $\left.\frac{dx}{dt}\right|_{t=0} = 0$. The position of the particle at $t = \frac{3\pi}{\omega}$ is ___________ (in integer). (GATE CH 2023) **Solution:**

Table 8.19: Input Parameters

Parameter	**Description**
s	Complex frequency variable in Laplace domain
ω	Angular frequency
$X(s)$	Laplace transform of the function $x(t)$
$x(t)$	Time-domain function

$$s^2X(s) - sx(0) - \frac{dx}{dt}(0) + \omega^2 X(s) = 0$$

$$x(0) = 1 \quad \text{and} \quad \frac{dx}{dt}(0) = 0$$

$$s^2X(s) - s - \omega^2 X(s) = 0$$

$$(s^2 + \omega^2)X(s) = s$$

$$X(s) = \frac{s}{s^2 + \omega^2}$$

$$X(s) = \frac{s}{s^2 + \omega^2} = \frac{A}{s} + \frac{Bs + C}{s^2 + \omega^2}$$

Multiplying both sides by $s(s^2 + \omega^2)$, we get:

$$s = A(s^2 + \omega^2) + (Bs + C)s$$

This implies $A = 0$, $B = 1$, and $C = 0$. Therefore,

$$X(s) = \frac{1}{s^2 + \omega^2}$$

$$x(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2 + \omega^2}\right\} = \cos(\omega t)$$

Finally, evaluating $x(t)$ at $t = \frac{3\pi}{\omega}$, we have:

$$x\left(\frac{3\pi}{\omega}\right) = \cos\left(\omega \cdot \frac{3\pi}{\omega}\right) = \cos(3\pi) = -1$$

8.23 **Question**:The initial value problem $\frac{dy}{dt} + 2y = 0, y(0) = 1$ is solved numerically using the forward Euler's method with a constant and positive time step of δ.

Let y_n represent the numerical solution obtained after n steps. The condition $|y_{n+1}| \leq |y_n|$is satisfied if and only if δ does not exceed

(GATE ME 2023)

Solution: Numerical solution: -

By forward Euler's method formula

$$y(n+1) = y(n) + \delta f(x, y) \tag{8.170}$$

From question we get

$$\frac{dy}{dx} = -2y = f(x, y) \tag{8.171}$$

From (8.171) in (8.170)

$$y(n+1) - y(n) = -2\delta y(n) \tag{8.172}$$

$$y(n+1) = (1 - 2\delta)y(n) \tag{8.173}$$

$$y(n) \overset{Z}{\longleftrightarrow} Y(z) \tag{8.174}$$

$$y(n+1) \overset{Z}{\longleftrightarrow} zY(z) - y(0) \tag{8.175}$$

$$\implies zY(z) - y(0) = (1 - 2\delta)Y(z) \tag{8.176}$$

$$Y(z) = \frac{1}{z - 1 + 2\delta} \tag{8.177}$$

$$\frac{1}{z - (1 - 2\delta)} \overset{Z}{\longleftrightarrow} (1 - 2\delta)^n u(n) \tag{8.178}$$

For good approximation we choose $\delta = 0.4$

$$y(n) = (0.2)^n u(n) \tag{8.179}$$

Now using the condition given in question

$$|y(n+1)| \leq |y(n)| \tag{8.180}$$

$$|(1-2\delta)^2| \leq |1-2\delta| \tag{8.181}$$

$$|1-2\delta| \leq 1 \tag{8.182}$$

$$0 \leq \delta \leq 1 \tag{8.183}$$

From this we can say that the maximum value of δ is 1

Theoritical solution: -

By properties of Laplace transform: -

$$Y(s) = \mathcal{L}y(s) \tag{8.184}$$

$$\mathcal{L}y' = sY(s) - y(0) \tag{8.185}$$

Given equation: -

$$y' + 2y = 0 \tag{8.186}$$

$$\mathcal{L}(y' + 2y) = 0 \tag{8.187}$$

From (8.184) and (8.185)

$$sY(s) - 1 + 2Y(s) = 0 \tag{8.188}$$

$$\frac{1}{s+2} = Y(s) \tag{8.189}$$

$$y(t) = \mathcal{L}^{-1}Y(s) \tag{8.190}$$

$$\implies y(t) = \mathcal{L}^{-1}\left(\frac{1}{s+2}\right) \tag{8.191}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s+k}\right) = e^{-kt}u(t) \tag{8.192}$$

$$\implies y(t) = e^{-2t}u(t) \tag{8.193}$$

8.24 In the context of signals and systems, determine the phase cross-over frequency of the open-loop transfer function

$$G(s) = \frac{k \cdot s \cdot (1 + sT_1) \cdot (1 + sT_2)}{s}$$

with positive constants k, T_1, T_2 are positive constants.The phase crossover fequency,in rad/s,is

(a) $\frac{1}{\sqrt{T_1 T_2}}$

(b) $\frac{1}{T_1 T_2}$

(c) $\frac{1}{T_1 \sqrt{T_2}}$

(d) $\frac{1}{\sqrt{T_2} T_1}$

(GATE EC 2023)

Solution: The phase of $G(s)$

$$\angle G(s) = \angle(ks(1 + sT_1)(1 + sT_2)) - \angle s \tag{8.194}$$

$$= \angle ks + \angle(1 + sT_1) + \angle(1 + sT_2) - \angle s \tag{8.195}$$

The phase contribution of each term

$$\angle ks = \angle k + \angle s = 0 + \frac{\pi}{2} \tag{8.196}$$

$$= \frac{\pi}{2} \text{ radians} \tag{8.197}$$

$$\angle(1 + sT_1) = \tan^{-1}(0) + \tan^{-1}(sT_1) \tag{8.198}$$

$$= \tan^{-1}(sT_1) \tag{8.199}$$

$$\angle(1 + sT_2) = \tan^{-1}(0) + \tan^{-1}(sT_2) \tag{8.200}$$

$$= \tan^{-1}(sT_2) \tag{8.201}$$

$$\angle s = \frac{\pi}{2} \text{ radians} \tag{8.202}$$

So, the total phase of $G(s)$ becomes:

$$\angle G(s) = \frac{\pi}{2} + \tan^{-1}(sT_1) + \tan^{-1}(sT_2) - \frac{\pi}{2} \tag{8.203}$$

$$= \tan^{-1}(sT_1) + \tan^{-1}(sT_2) \tag{8.204}$$

the frequency at which the phase angle $\angle G(s)$ equals $-\pi$ radians.

$$\tan^{-1}(j\omega T_1) + \tan^{-1}(j\omega T_2) = -\pi \tag{8.205}$$

$$\tan^{-1}(j\omega T_1) + \tan^{-1}(j\omega T_2) = -\frac{\pi}{2} \tag{8.206}$$

$$\tan^{-1}(j\omega T_1) = -\frac{\pi}{2} - \tan^{-1}(j\omega T_2) \tag{8.207}$$

$$j\omega T_1 = \tan\left(-\frac{\pi}{2} - \tan^{-1}(j\omega T_2)\right) \tag{8.208}$$

$$j\omega T_1 = -\frac{1}{\tan(\tan^{-1}(j\omega T_2))} \tag{8.209}$$

$$j\omega T_1 = -\frac{1}{j\omega T_2} \tag{8.210}$$

$$\omega T_1 = \frac{1}{\omega T_2} \tag{8.211}$$

$$\omega^2 = \frac{1}{T_1 T_2} \tag{8.212}$$

$$\omega = \frac{1}{\sqrt{T_1 T_2}} \tag{8.213}$$

the phase cross-over frequency is

$$\frac{1}{\sqrt{T_1 T_2}}$$

8.25 Which one of the options given is the inverse Laplace transform of $\frac{1}{s^3-s}$?

$u(t)$ denotes the unit-step function.

(A) $\left(-1+\frac{1}{2}e^{-t}+\frac{1}{2}e^{t}\right)u(t)$

(B) $\left(\frac{1}{3}e^{-t}-e^{t}\right)u(t)$

(C) $\left(-1+\frac{1}{2}e^{-(t-1)}+\frac{1}{2}e^{(t-1)}\right)u(t-1)$

(D) $\left(-1-\frac{1}{2}e^{-(t-1)}-\frac{1}{2}e^{(t-1)}\right)u(t-1)$

(GATE ME 2023)

Solution:

Using partial fraction,

$$\frac{1}{s^3-s}=\frac{A}{s}+\frac{B}{s+1}+\frac{C}{s-1} \tag{8.214}$$

On Solving,

$$\implies A=-1 \quad B=\frac{1}{2} \quad C=\frac{1}{2} \tag{8.215}$$

$$X(s)=\frac{-1}{s}+\frac{1}{2(s+1)}+\frac{1}{2(s-1)} \tag{8.216}$$

$$\text{As} \quad e^{-at}u(t) \xrightarrow{\mathcal{L}} \frac{1}{s+a} \quad \Re(s)>-a \tag{8.217}$$

$$\text{And} \quad -e^{-at}u(-t) \xrightarrow{\mathcal{L}} \frac{1}{s+a} \quad \Re(s)<-a \tag{8.218}$$

Now,

$$\mathcal{L}^{-1}(X(s)) = x(t) \tag{8.219}$$

There are 4 cases possible,

$$x(t) = \begin{cases} \left(1 - \frac{1}{2}e^{-t} - \frac{1}{2}e^{t}\right)u(-t) & \Re(s) < -1 \\ \\ -1u(-t) + \frac{1}{2}e^{-t}u(t) - \frac{1}{2}e^{t}u(-t) & \Re(s) \in (-1, 0) \\ \\ -u(t) + \frac{1}{2}e^{-t}u(t) - \frac{1}{2}e^{t}u(-t) & \Re(s) \in (0, 1) \\ \\ \left(-1 + \frac{1}{2}e^{-t} + \frac{1}{2}e^{t}\right)u(t) & \Re(s) > 1 \end{cases} \tag{8.220}$$

Thus the correct option is (A) $\left(-1 + \frac{1}{2}e^{-t} + \frac{1}{2}e^{t}\right)u(t)$ for $\Re(s) > 1$

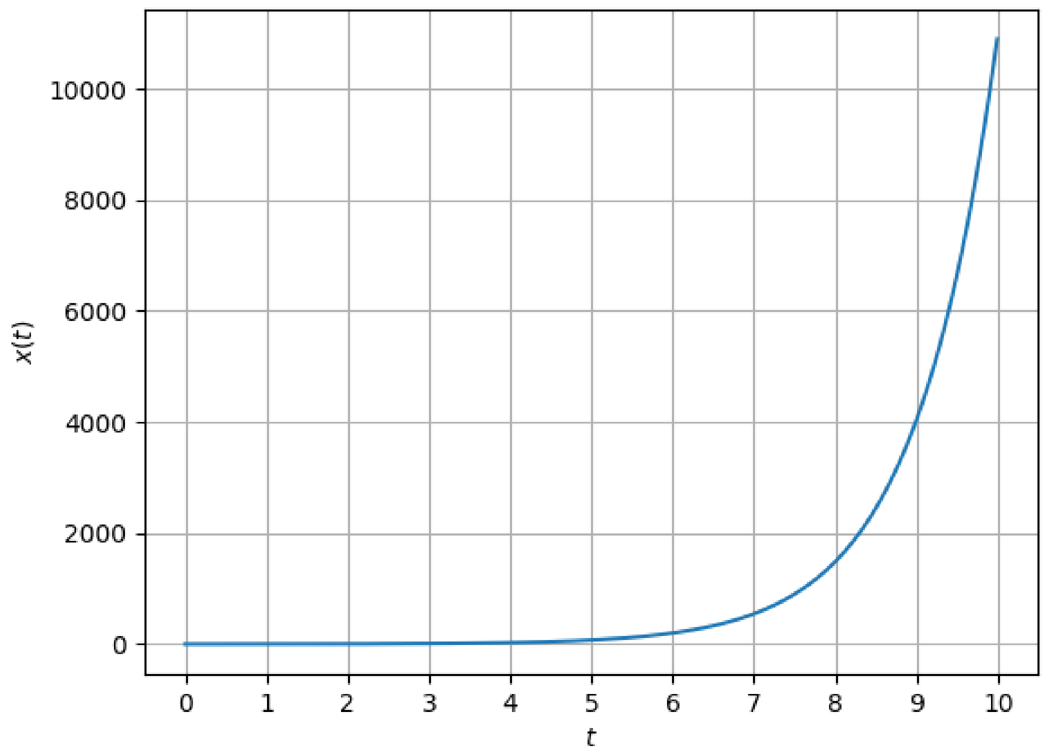


Figure 8.34: Plot for x(t)=$\left(-1+\frac{1}{2}e^{-t}+\frac{1}{2}e^{t}\right)u(t)$

8.26 The state equation of a second order system is

$\dot{x}(t) = Ax(t), \quad x(0)$ is the initial condition.

Suppose λ_1 and λ_2 are two distinct eigenvalues of A, and ν_1 and ν_2 are the corresponding eigenvectors. For constants α_1 and α_2, the solution, $x(t)$, of the state equation is

(A) $\sum_{i=1}^{2} \alpha_i e^{\lambda_i t} \nu_{\mathbf{i}}$

(B) $\sum_{i=1}^{2} \alpha_i e^{2\lambda_i t} \nu_{\mathbf{i}}$

(C) $\sum_{i=1}^{2} \alpha_i e^{3\lambda_i t} \nu_{\mathbf{i}}$

(D) $\sum_{i=1}^{2} \alpha_i e^{4\lambda_i t} \nu_{\mathbf{i}}$

GATE EC 2023

Solution:

Variable	**Description**	**Value**
$x(t)$	state variable	-
$\dot{x}(t)$	derivative of x(t) w.r.t t	$\frac{dx(t)}{dt}$
A	2x2 matrix	-
λ_i for $i = 1, 2$	eigen values of A	-
v_i for $i = 1, 2$	eigen vectors of A	-
α_i for $i = 1, 2$	component of $x(t)$ along v_i	-

Table 8.20: input parameters

Theories and Proofs:

$$A\mathbf{y} = \lambda\mathbf{y} \tag{8.221}$$

Eigen values of inverse of a matrix is reciprocal of eigen value of the given matrix

$$A^{-1}A\mathbf{y} = A^{-1}\lambda\mathbf{y} \tag{8.222}$$

$$\implies \mathbf{y} = \lambda A^{-1}\mathbf{y} \tag{8.223}$$

$$\implies A^{-1}\mathbf{y} = \frac{1}{\lambda}\mathbf{y} \tag{8.224}$$

Eigen value of a matrix shifts by the same amount as that of the matrix.

$$(A - \sigma I)\mathbf{y} = A\mathbf{y} - \sigma I\mathbf{y} \tag{8.225}$$

$$= \lambda\mathbf{y} - \sigma\mathbf{y} \tag{8.226}$$

$$= (\lambda - \sigma)\mathbf{y} \tag{8.227}$$

Sol:

Using Laplace transform:

Given Equation:

$$\dot{x}(t) = Ax(t) \tag{8.228}$$

$$\frac{dx(t)}{dt} = Ax(t) \tag{8.229}$$

Taking Laplace Transform:

$$\mathcal{L}\left(\frac{dx(t)}{dt}\right) = \mathcal{L}\left(Ax(t)\right) \tag{8.230}$$

$$sX(s) - x(0) = AX(s) \tag{8.231}$$

$$(sI - A)X(s) = x(0) \tag{8.232}$$

$$X(s) = (sI - A)^{-1}x(0) \tag{8.233}$$

From Table 8.20, we can write $x(0)$ in terms of two linearly independent variables as

$$x(0) = \alpha_1 v_1 + \alpha_2 v_2 \tag{8.234}$$

$$= \sum_{i=1}^{2} \alpha_i v_i \tag{8.235}$$

From (8.233), (8.235)

$$X(s) = (sI - A)^{-1}\left(\sum_{i=1}^{2} \alpha_i v_i\right) \tag{8.236}$$

$$= \sum_{i=1}^{2} (sI - A)^{-1} \alpha_i v_i \tag{8.237}$$

From (8.224), (8.227)

$$X(s) = \sum_{i=1}^{2} \frac{1}{s - \lambda_i} \left(\alpha_i v_i\right) \tag{8.238}$$

Now take inverse Laplace Transform

$$\mathcal{L}^{-1}\left(X(s)\right) = \mathcal{L}^{-1}\left(\sum_{i=1}^{2} \frac{1}{s-\lambda_i}\left(\alpha_i v_i\right)\right) \tag{8.239}$$

$$x(t) = \sum_{i=1}^{2} e^{\lambda_i t}\left(\alpha_i v_i\right) \tag{8.240}$$

Hence the answer is option (A).

8.27 The continuous time signal $x(t)$ is described by:

$$x(t) = \begin{cases} 1, & \text{if } 0 \le t \le 1 \\ 0, & \text{elsewhere} \end{cases} \tag{8.241}$$

If $y(t)$ represents $x(t)$ convolved with itself, which of the following options is/are TRUE?

(A) $y(t) = 0$ for all $t < 0$

(B) $y(t) = 0$ for all $t > 1$

(C) $y(t) = 0$ for all $t > 3$

(D) $\int_{0.1}^{0.75} \frac{dy(t)}{dt}$ dt $\neq 0$

GATE 2023 BM- Q 49

Solution:

Symbol	Description
$X(s)$	Laplace transform of $x(t)$
$Y(s)$	Laplace transform of $y(t)$
$u(t-t_0)$	Unit step function, $u(t-t_0) = 1, t \geq t_0$

Table 8.21: Parameters

$$y(t) = x(t) * x(t) \tag{8.242}$$

$$Y(s) = X(s) X(s) \tag{8.243}$$

$$= \left(\frac{1 - e^{-s}}{s}\right)^2 \tag{8.244}$$

$$= \frac{1 + e^{-2s} - 2e^{-s}}{s^2} \tag{8.245}$$

$$u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{s} \tag{8.246}$$

$$tu(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{s^2} \tag{8.247}$$

$$f(t) \stackrel{\mathcal{L}}{\longleftrightarrow} F(s) \implies f(t+a) \stackrel{\mathcal{L}}{\longleftrightarrow} e^{as} F(s) \tag{8.248}$$

Using (8.247) and (8.248),

$$y(t) = tu(t) + (t-2)u(t-2) - 2(t-1)u(t-1) \tag{8.249}$$

This can also be expressed as

$$y(t) = \begin{cases} 1 - |1 - t|, & \text{if } 0 \le t \le 2 \\ 0, & \text{otherwise} \end{cases} \tag{8.250}$$

Checking (8.250) with every option,

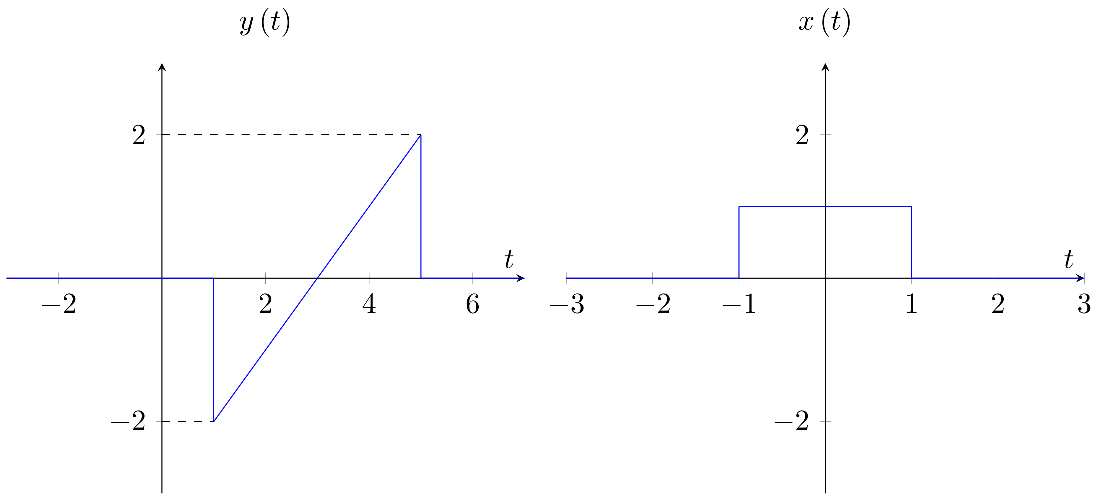
(A) From Fig: 8.35 , $y(t) = 0, \forall t < 0$, hence (A) is true

(B) From Fig: 8.35 ,$y(t) \neq 0, \forall t \in [1, 2]$, hence (B) is false

(C) From Fig: 8.35 , $y(t) = 0, \forall t > 3$, hence (C) is true

(D) From Fig: 8.36 , area under graph between 0.75 and 0.1 is non-zero, hence (D) is true

8.28 For the signals x(t) and y(t) shown in the figure, $z(t) = x(t) * y(t)$ is maximum at $t = T_1$. Then T_1 in seconds is .... (Round off to the nearest integer)



(GATE EE 2023 Q 31) **Solution:**

Variable	**values**	**Description**
$x(t)$	$u(t+1) - u(t-1)$	signal 1
$y(t)$	$y(t) = \begin{cases} t-3 & ; 1 \le n \le 5 \\ 0 & ; otherwise \end{cases}$	signal 2
$X(s)$	$\int_0^\infty x(t) e^{-st} dt$	Laplace transform of $x(t)$
$Y(s)$	$\int_0^\infty y(t) e^{-st} dt$	Laplace transform of $y(t)$
$\mathscr{L}^-\{Z(s)\}$	$f(t-c) u(t-c) = \mathscr{L}^-(e^{-cs} F(s))$	Inverse Laplace transform

Table 1: Input Parameters

Using laplace transform,

$$z(t) = x(t) * y(t) \tag{8.251}$$

$$Z(s) = X(s)\,Y(s) \tag{8.252}$$

$$X(s) = \frac{1}{s}\left(e^{s} - e^{-s}\right) \tag{8.253}$$

$$Y(s) = \frac{2s+1}{s^2}\left(e^{-s} - e^{-5s}\right) \tag{8.254}$$

$$Z(s) = \frac{2s+1}{s^3}\left(1 - e^{-4s} - e^{-2s} + e^{-6s}\right) \tag{8.255}$$

Now taking inverse laplace transform for each terms, $\mathscr{L}^{-}\{Z(s)\}$

$$\begin{aligned}
&= \left(2t + \frac{t^2}{2}\right) u(t) \\
&\quad - \left(2(t-4) + \frac{(t-4)^2}{2}\right) u(t-4) \\
&\quad - \left(2(t-2) + \frac{(t-2)^2}{2}\right) u(t-2) \\
&\quad + \left(2(t-6) + \frac{(t-6)^2}{2}\right) u(t-6)
\end{aligned}$$

From the plot 8.37, it is clear that $T_1 = 4$.

Now in time domain,

$$z(t) = x(t) * y(t) = y(t) * x(t) \tag{8.256}$$

$$z(t) = \int_{-\infty}^{\infty} y(\tau)\,x(t-\tau)\,d\tau \tag{8.257}$$

$x(\tau)$ is an even signal,

$$x(\tau) = x(-\tau) \tag{8.258}$$

$$x(-\tau) = \begin{cases} 1 & ; -1 \leq -\tau \leq 1 \\ 0 & ; \text{otherwise} \end{cases} \tag{8.259}$$

$$x(-\tau) \xleftrightarrow{\text{Time shifting}} x(t-\tau) \tag{8.260}$$

$$x(t-\tau) = \begin{cases} 1 & ; t-1 \leq t-\tau \leq t+1 \\ 0 & ; \text{otherwise} \end{cases} \tag{8.261}$$

For $z(t)$ to be maximum both $y(\tau)$ and $x(t-\tau)$ must be maximum,

$$\implies t-1 = 3 \quad \text{or} \quad t+1 = 5$$

$$t = T_1 = 4$$

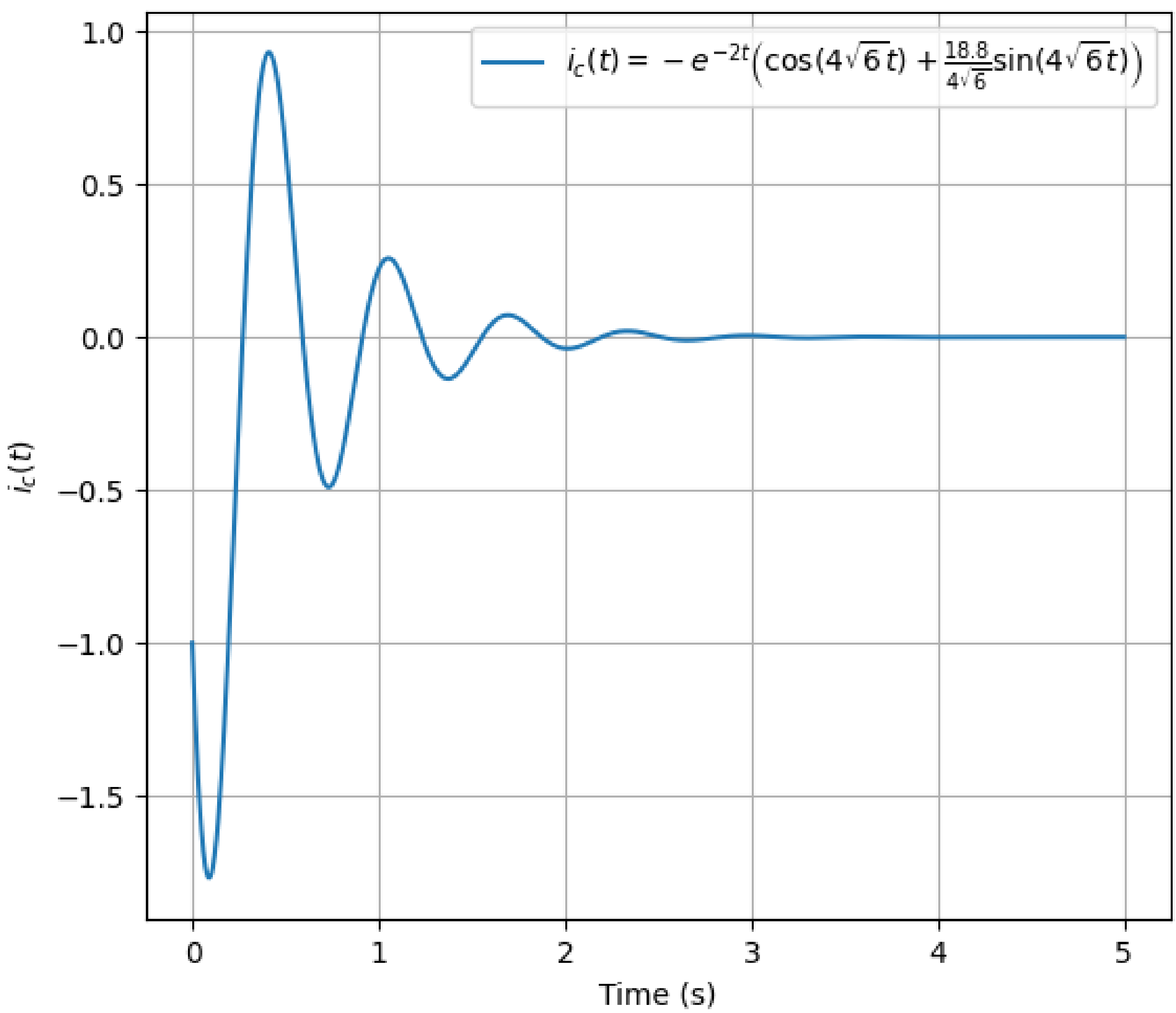


Figure 8.13: Plot of $i_c\,(t)$ vs time

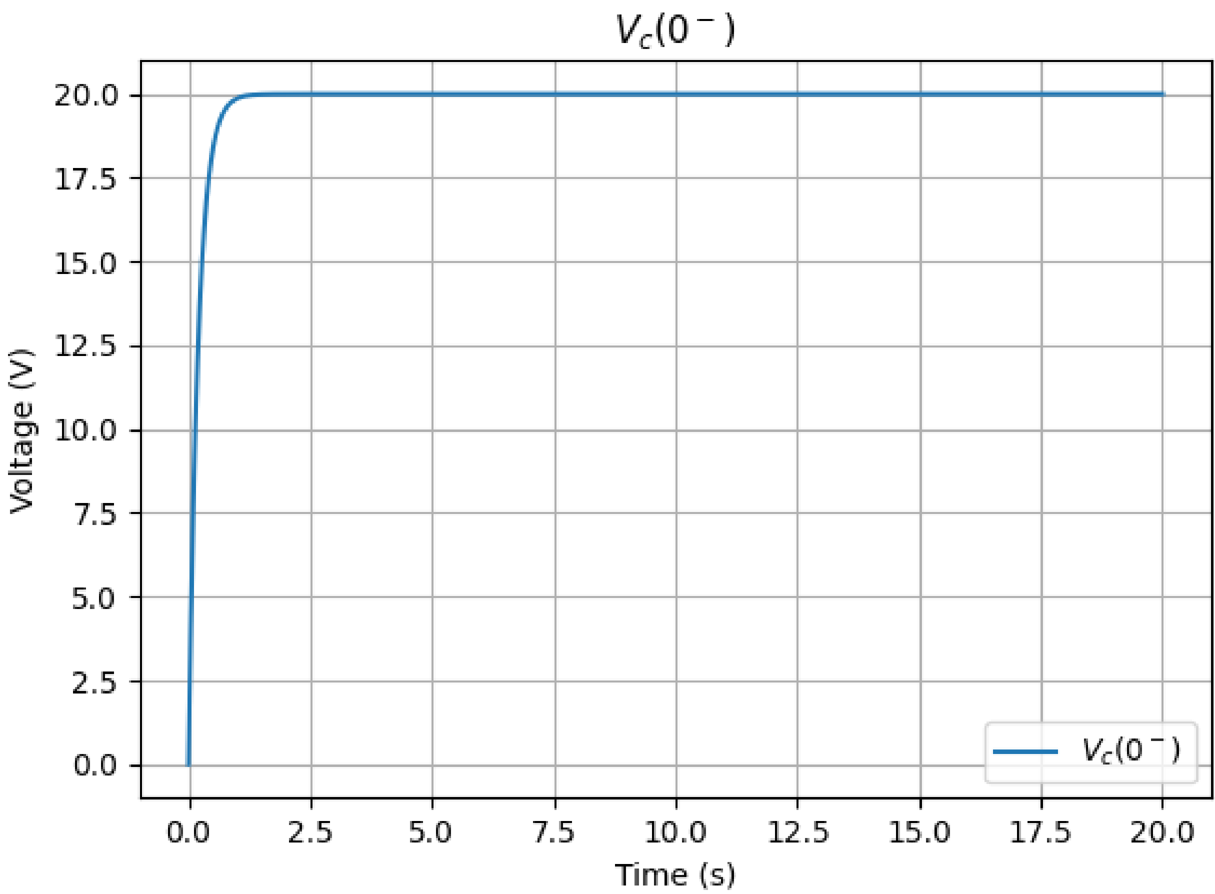


Figure 8.14: Plot of $V_c(0^-)$ vs time

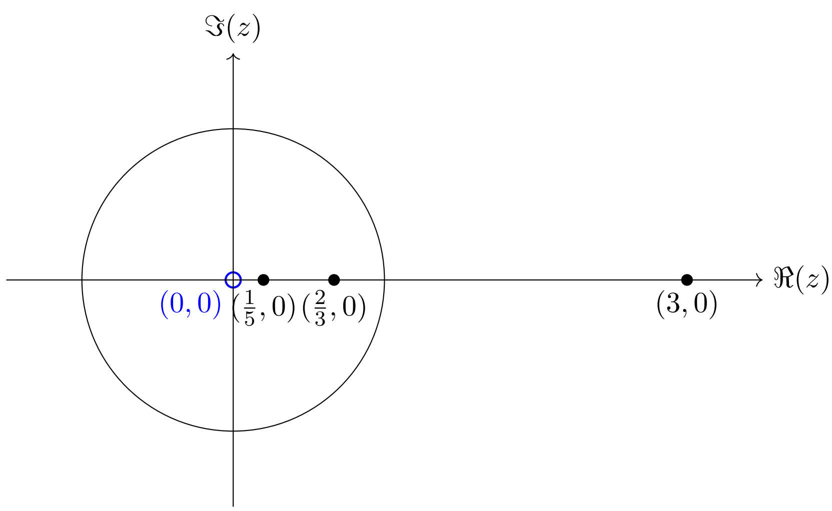


Figure 8.15: Representation of Poles

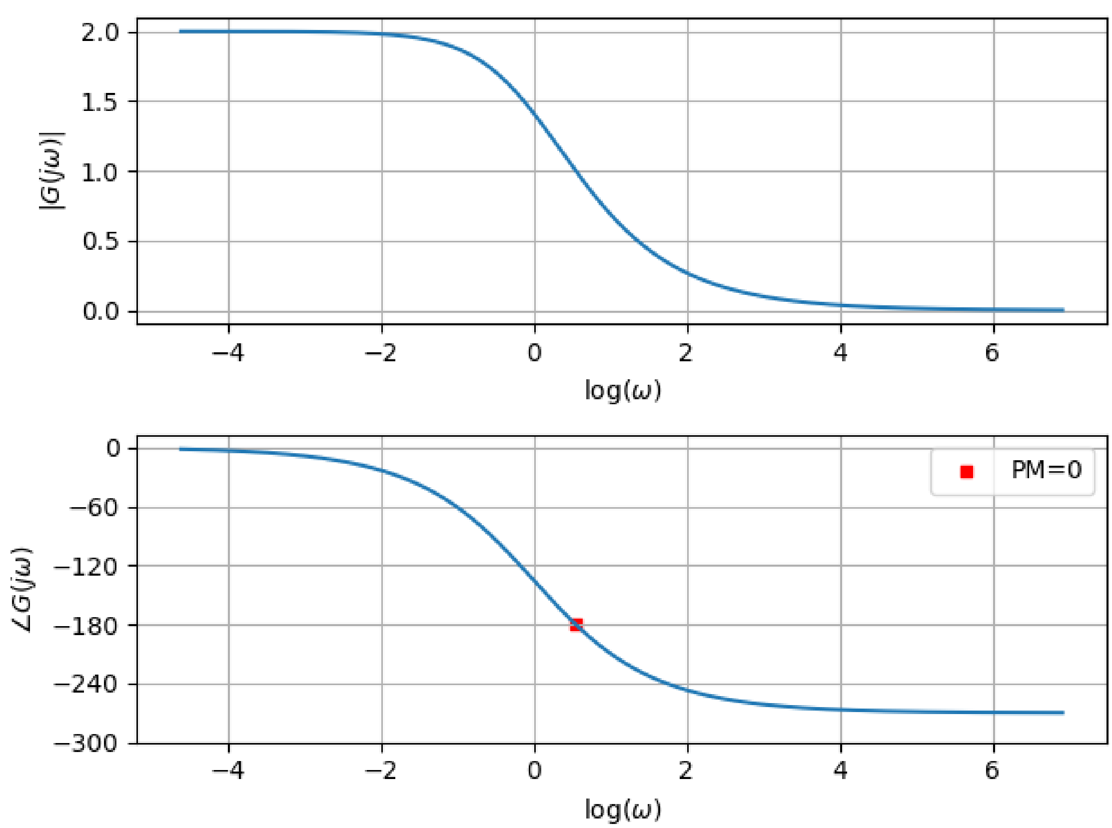


Figure 8.16: Bode Plot of Transfer Function $G(s)$

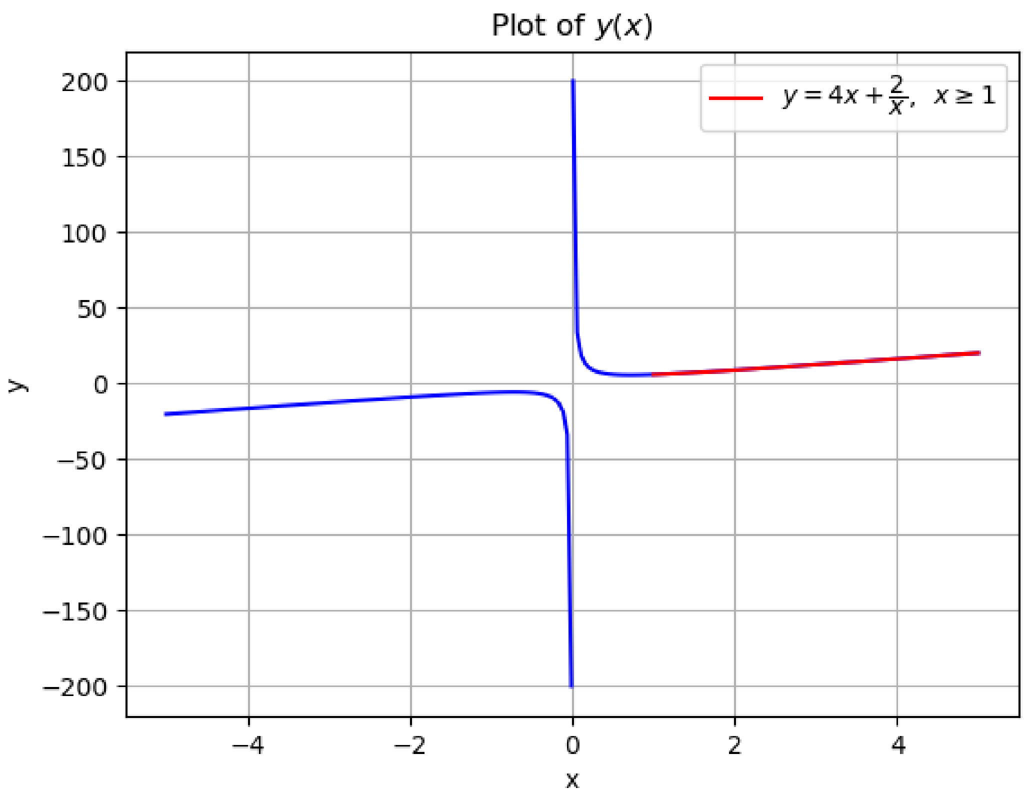


Figure 8.17: Plot of $y\,(x)$ v/s x

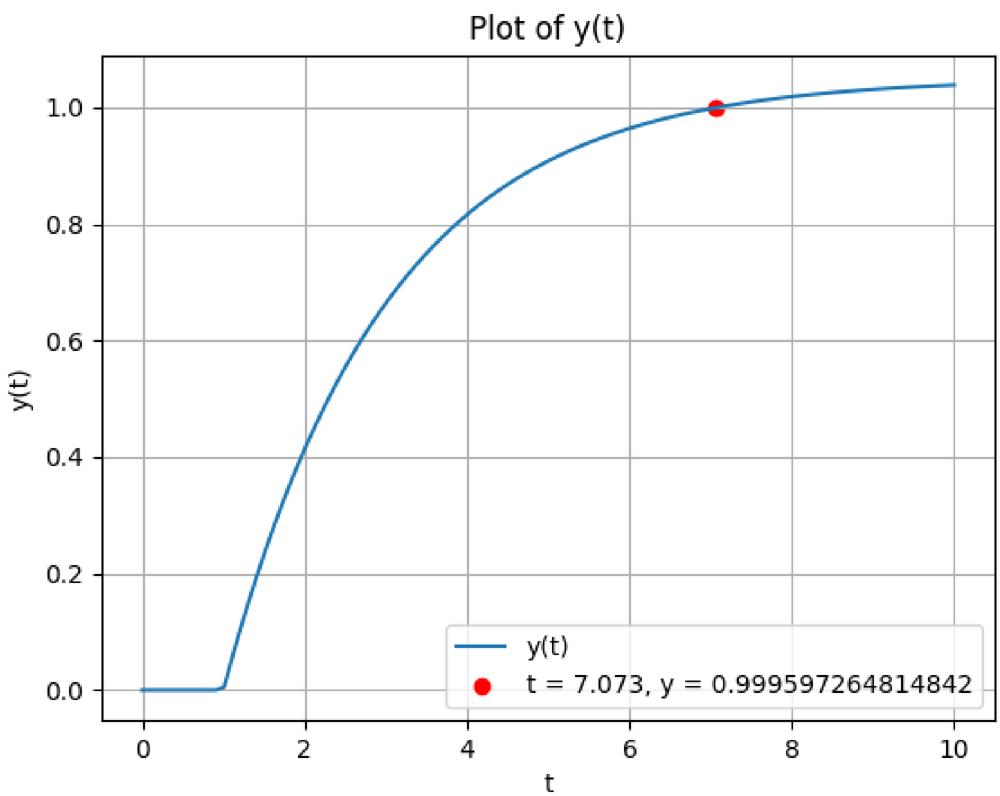


Figure 8.18: y(t) $= 1.05\left[1 - e^{\frac{t-1}{2}}\right]$

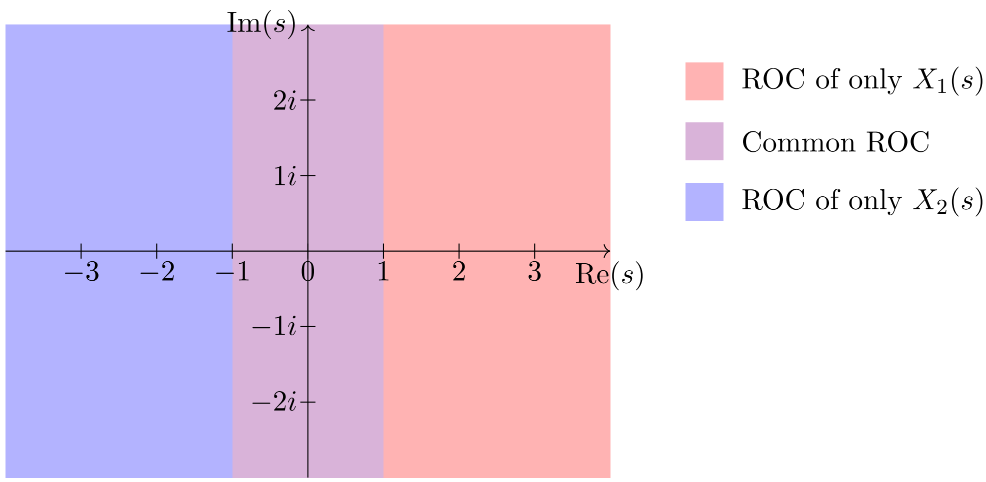


Figure 8.19: Representation of ROCs of $X_1(s)$ and $X_2(s)$

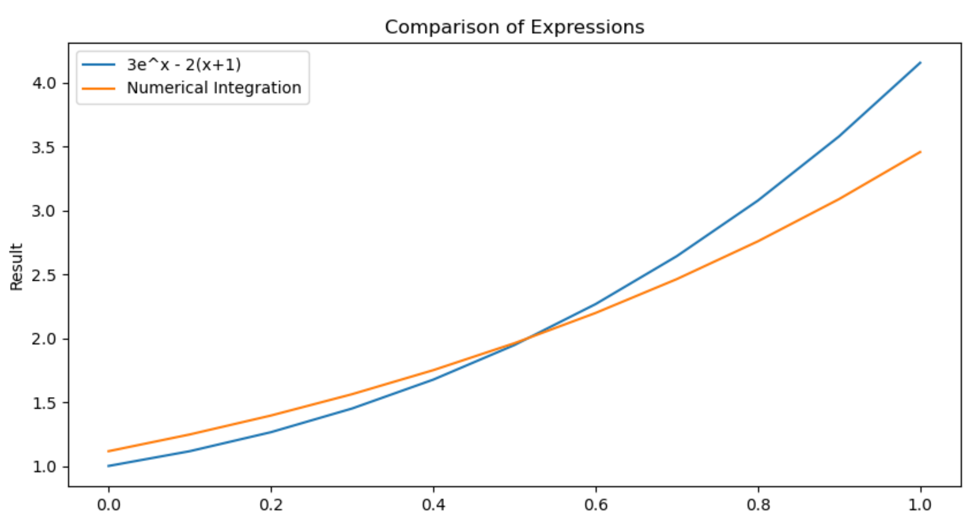
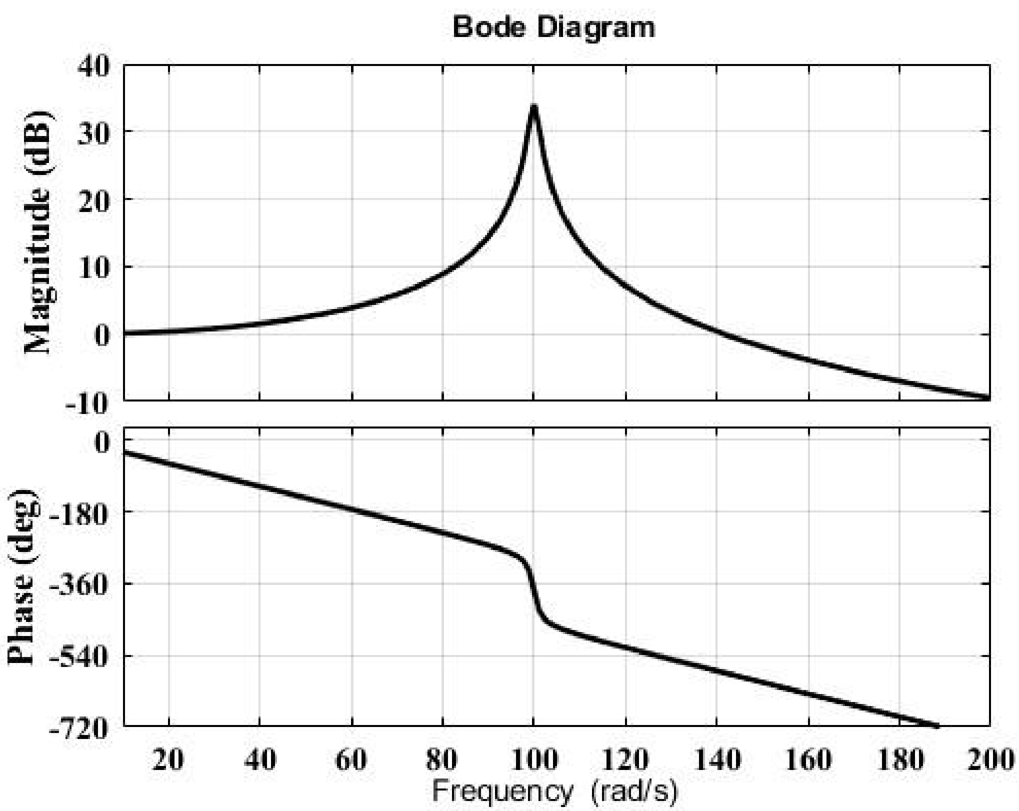


Figure 8.20: simulation vs analysis



Bode Diagram
Magnitude (dB)
40
30
20
10
0
-10
Phase (deg)
0
-180
-360
-540
-720
20
40
60
80
100
120
140
160
180
200
Frequency (rad/s)

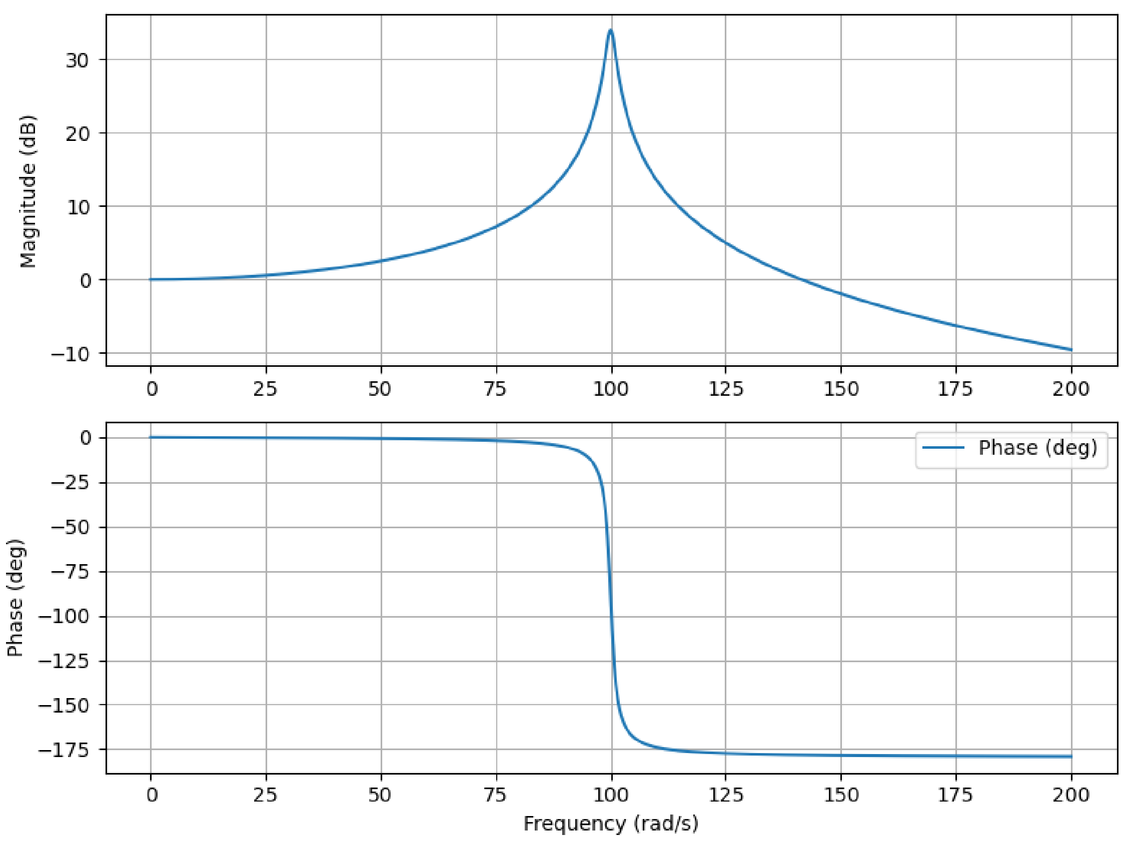


Figure 8.21: Bode plot of a $\frac{10000}{s^2+2s+10000}$

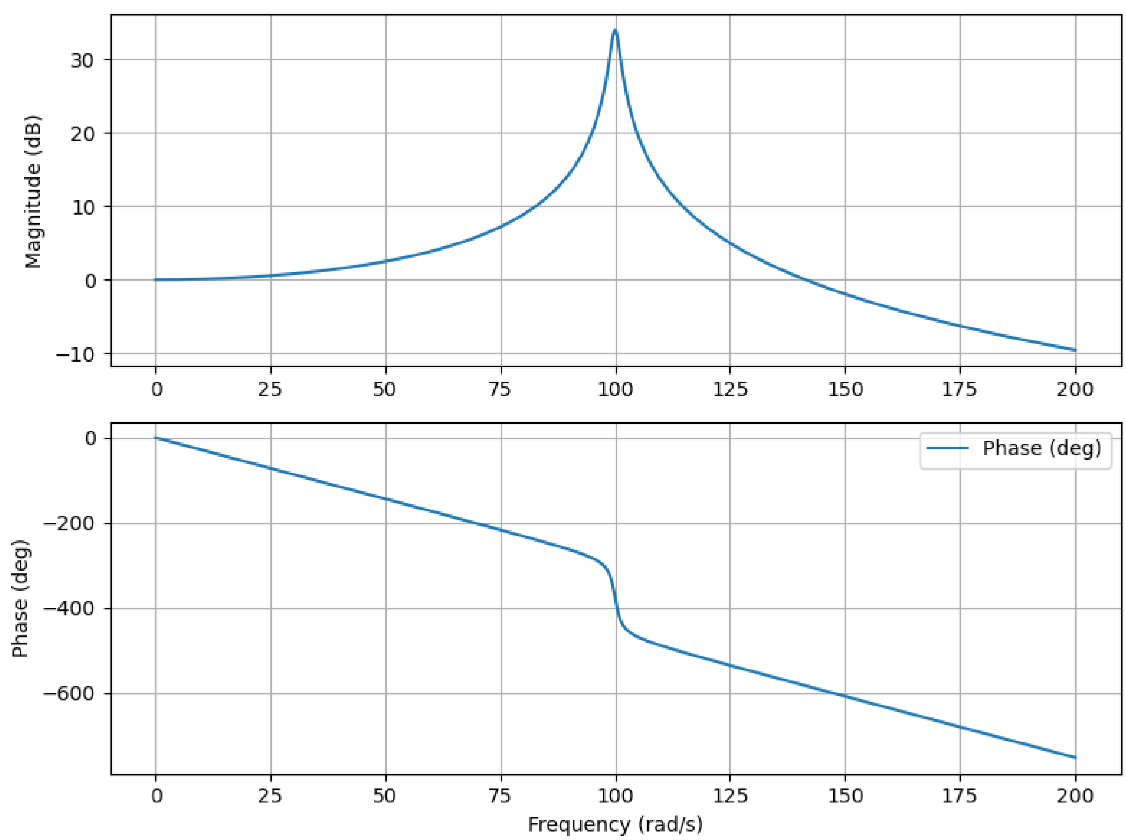


Figure 8.22: Bode plot of a $\frac{10000e^{-0.05s}}{s^2+2s+10000}$

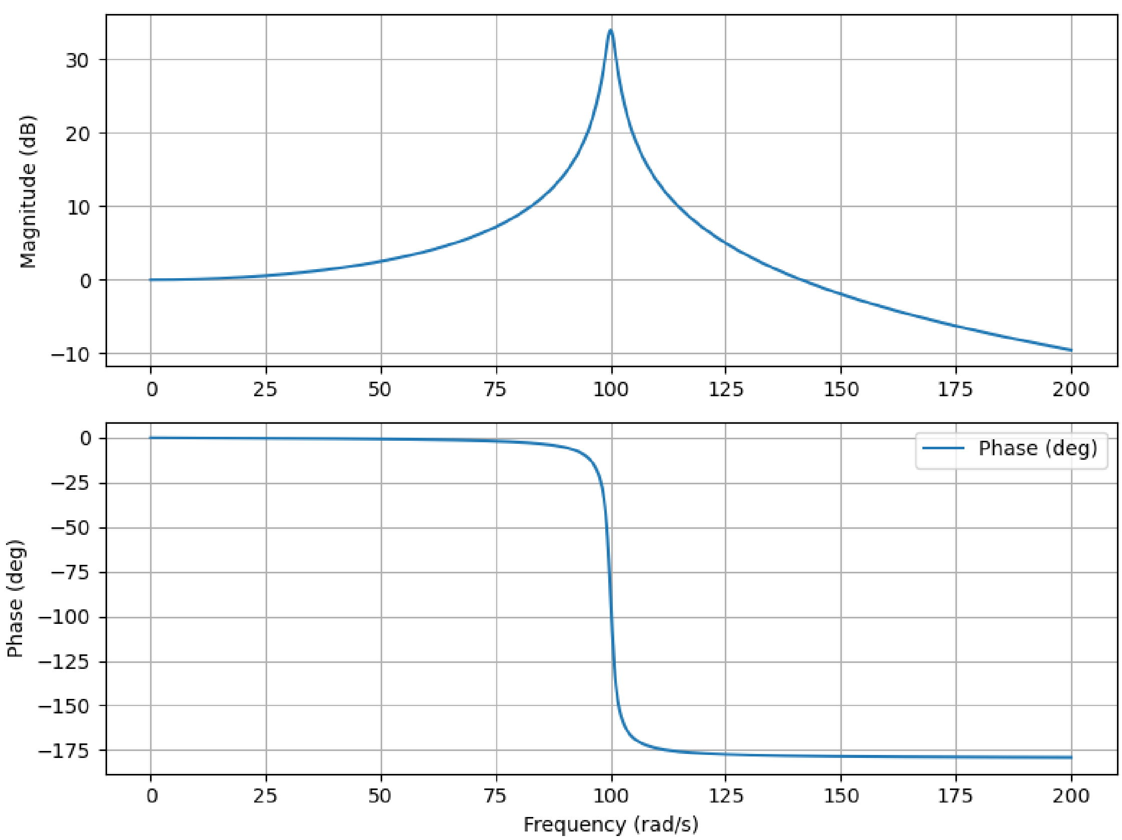


Figure 8.23: Bode plot of a $\frac{10000e^{0.5\times10^{-12}s}}{s^2+2s+10000}$

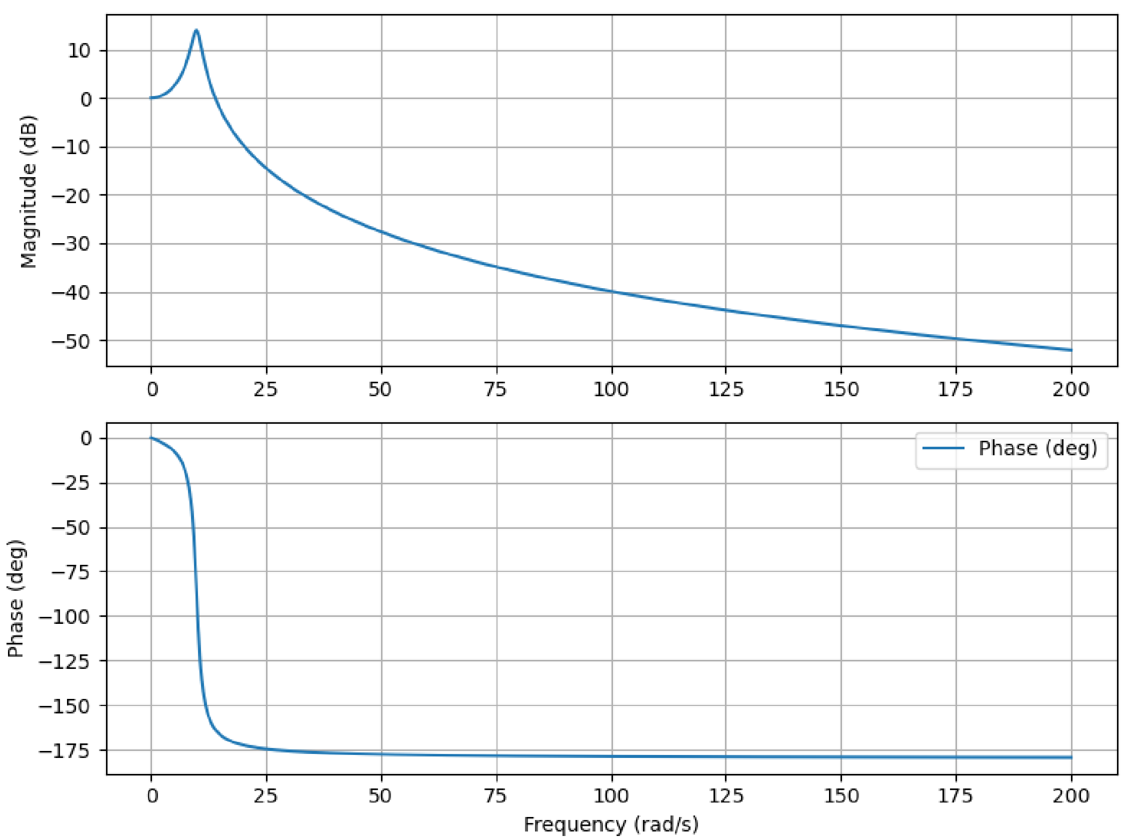


Figure 8.24: Bode plot of a $\frac{100}{s^2+2s+100}$

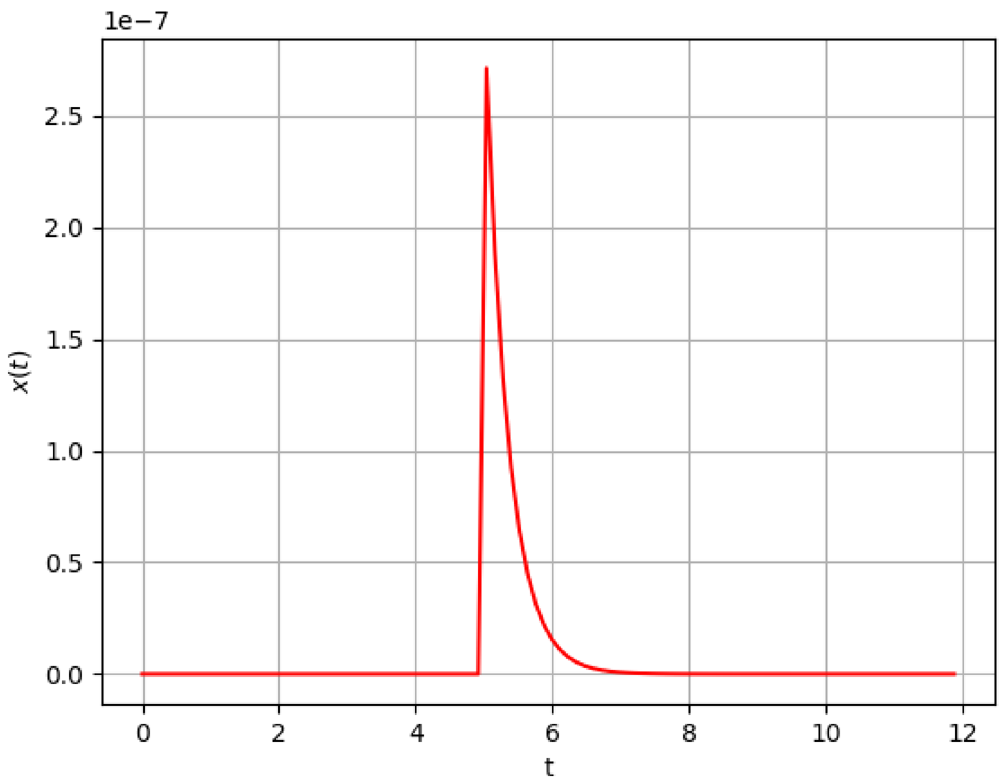


Figure 3: Plot of $x(t)$ vs t. See Table 1

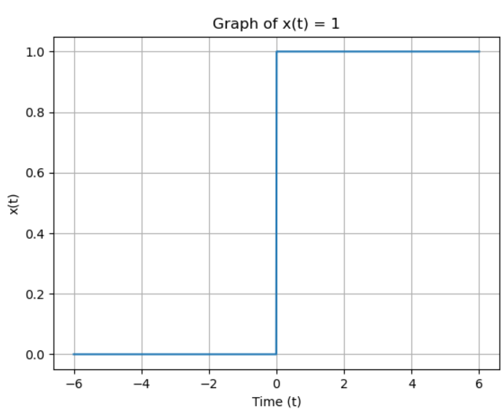


Figure 8.26: Plot of $x(t)$ v/s t

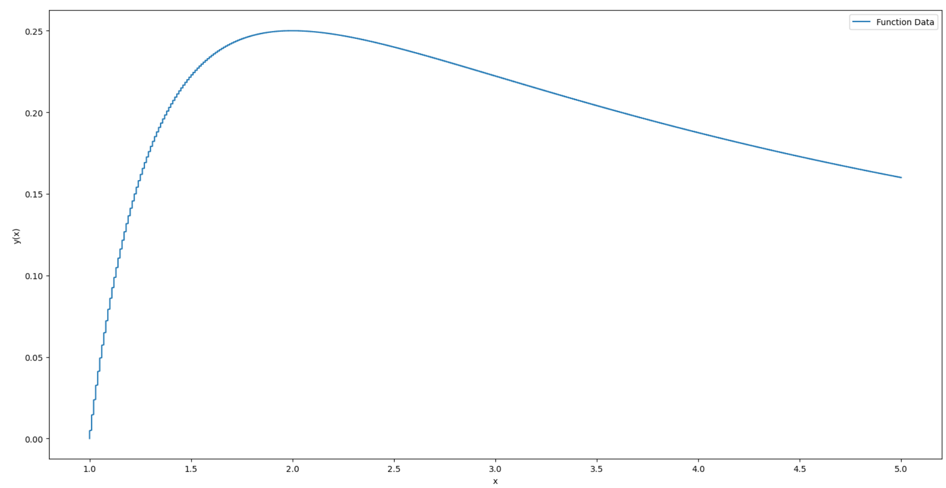


Figure 8.27: Plot of $y(x)$ vs x

Figure 8.28: Amplitude

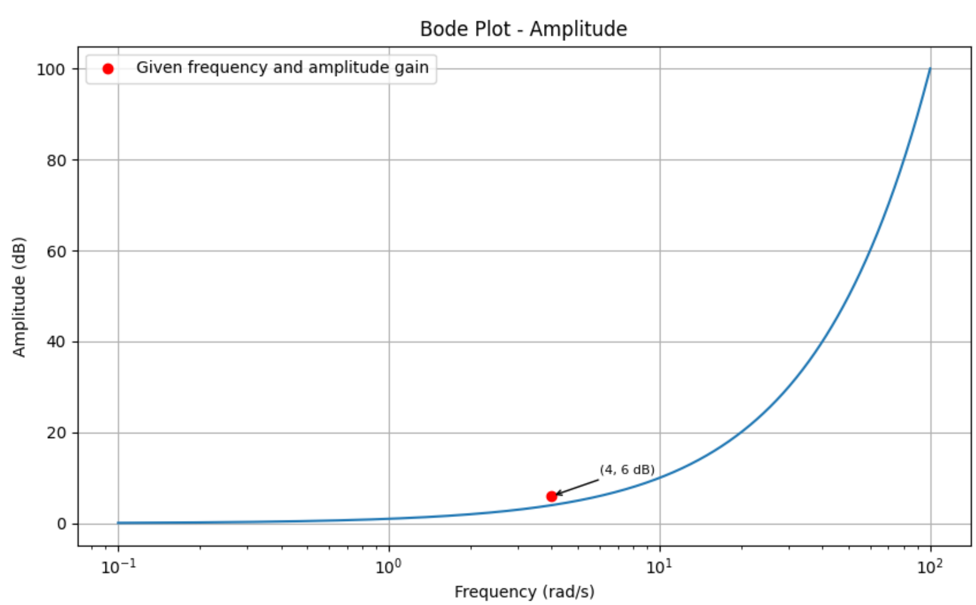
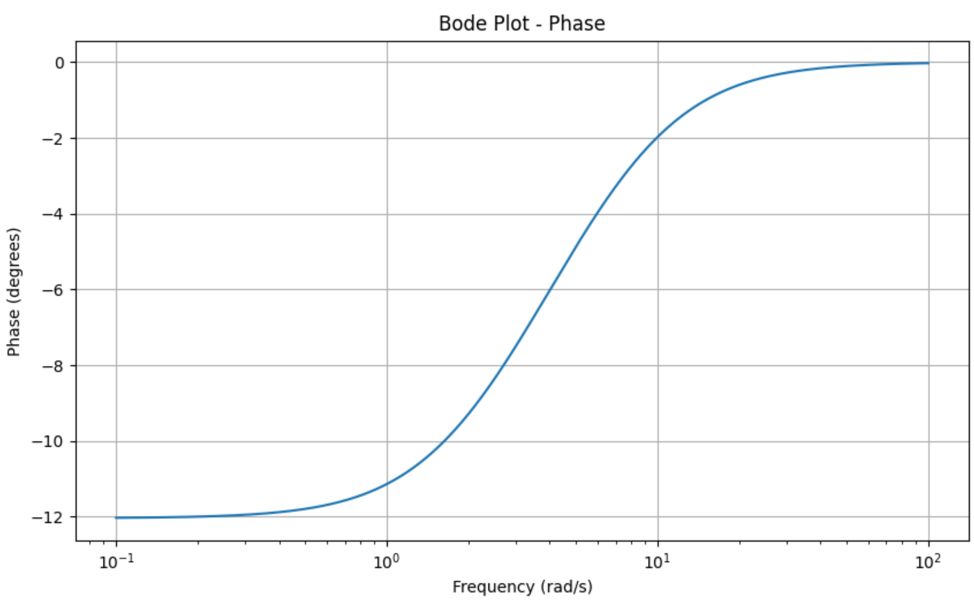


Figure 8.29: Phase Response



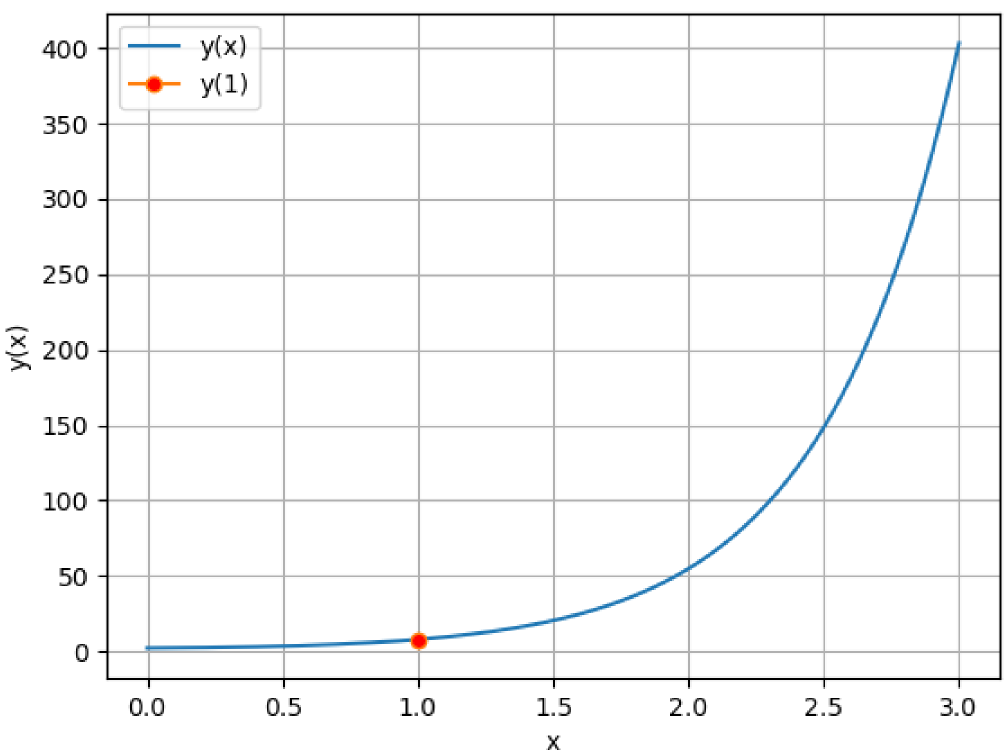


Figure 8.30: Plot of y(x)

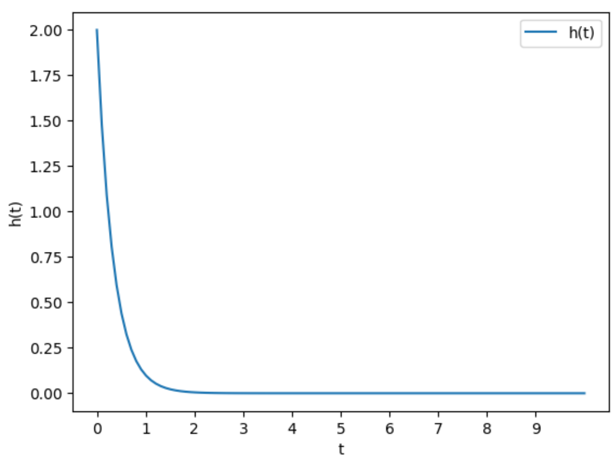


Figure 8.31: (a) Plot of $h\left(t\right)\ vs\ t$

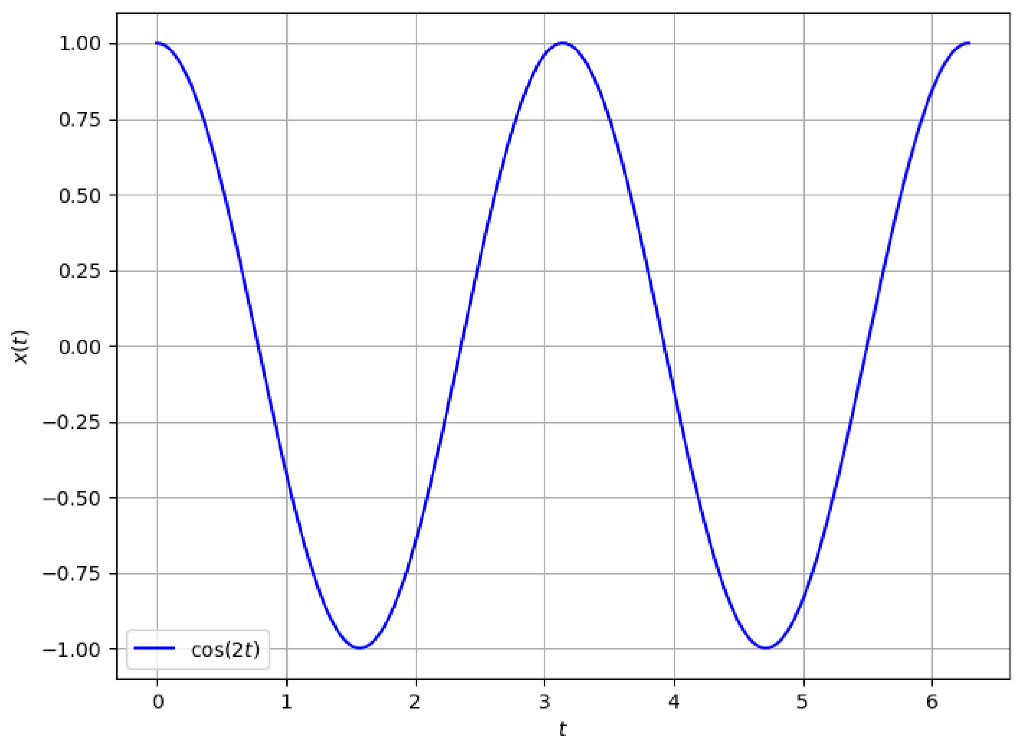


Figure 8.32: Graph of $x(t)$

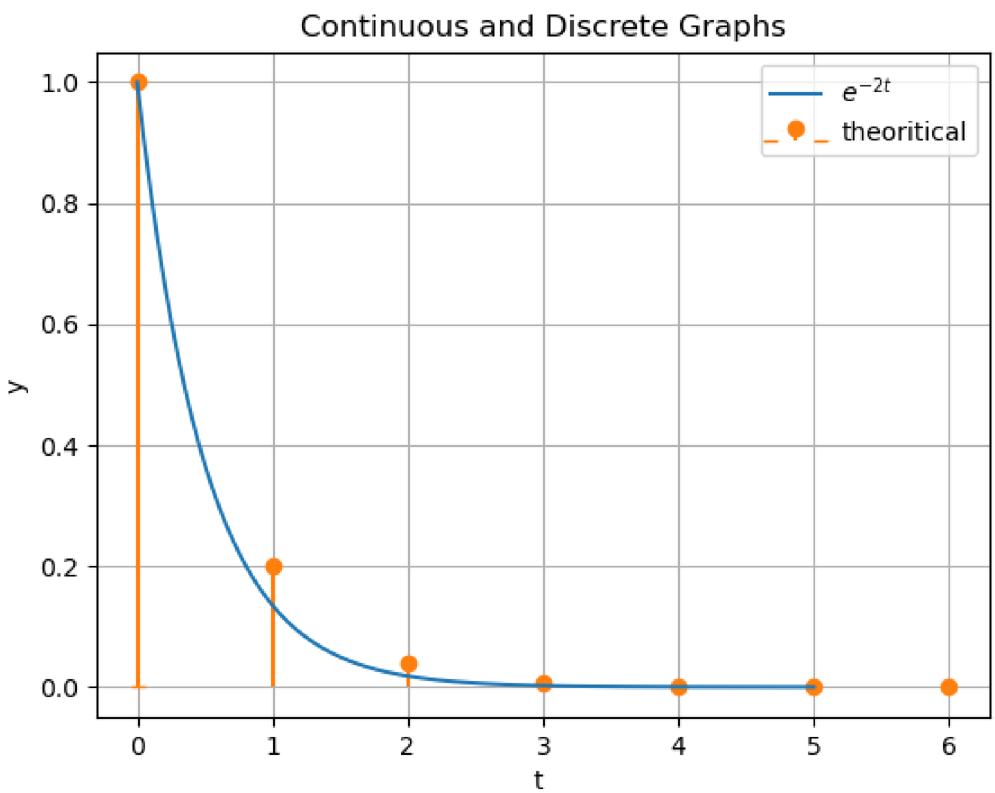
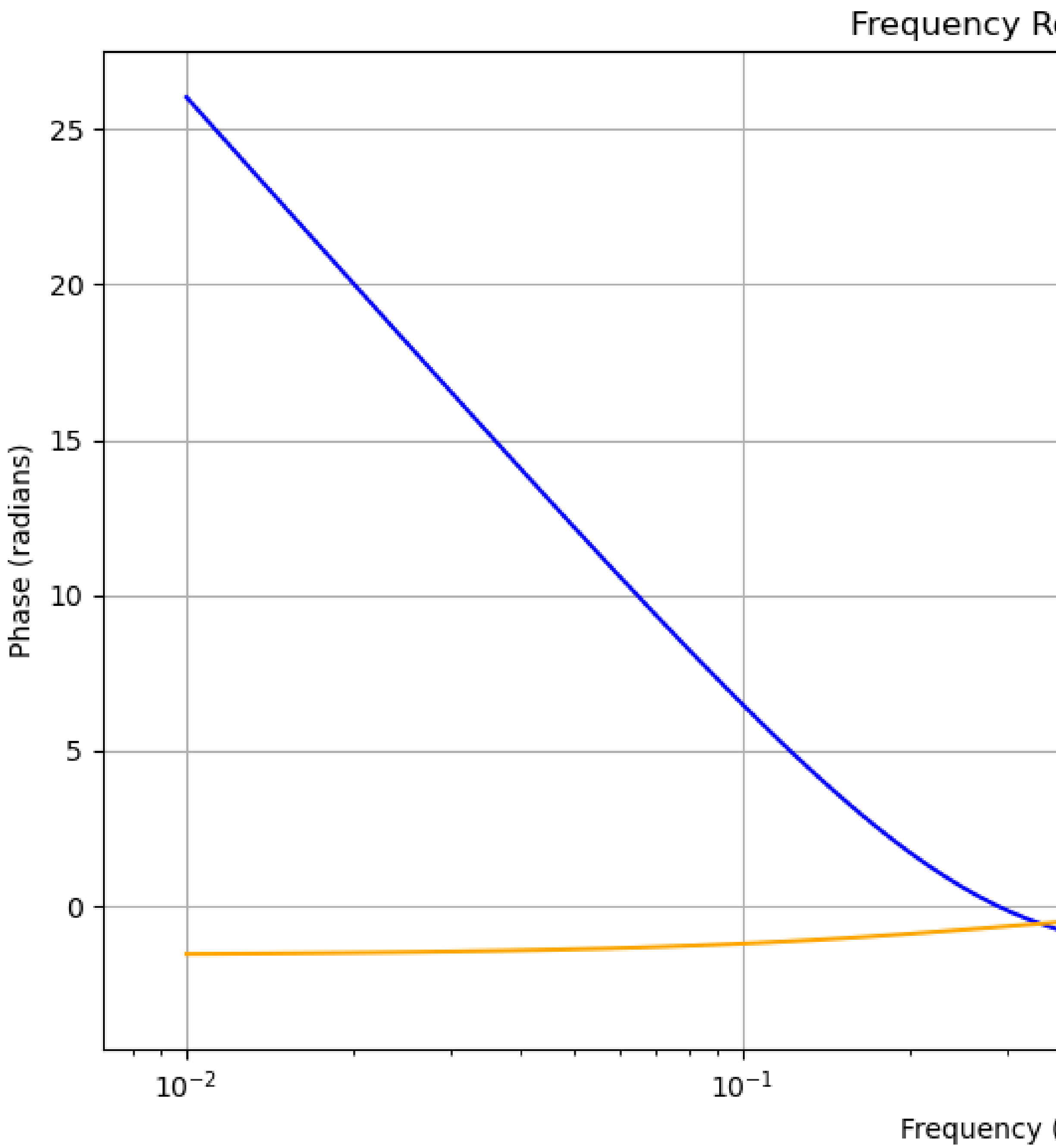


Figure 8.33: simulation vs analysis



Frequency R
Phase (radians)
25
20
15
10
5
0
10^{-2}
10^{-1}
Frequency (

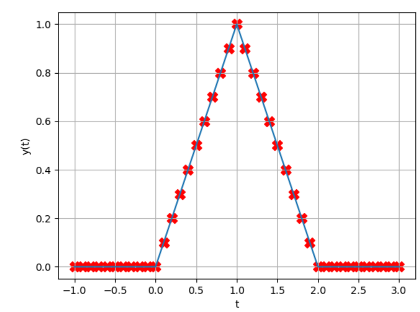


Figure 8.35: Stem Plot of $y(t)$ v/s t

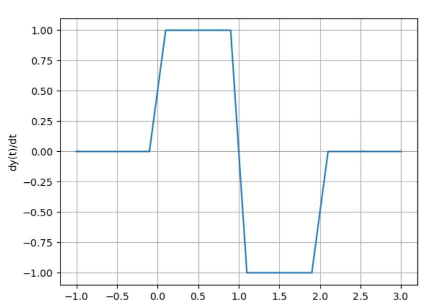


Figure 8.36: Stem Plot of $dy(t)/dt$ v/s t

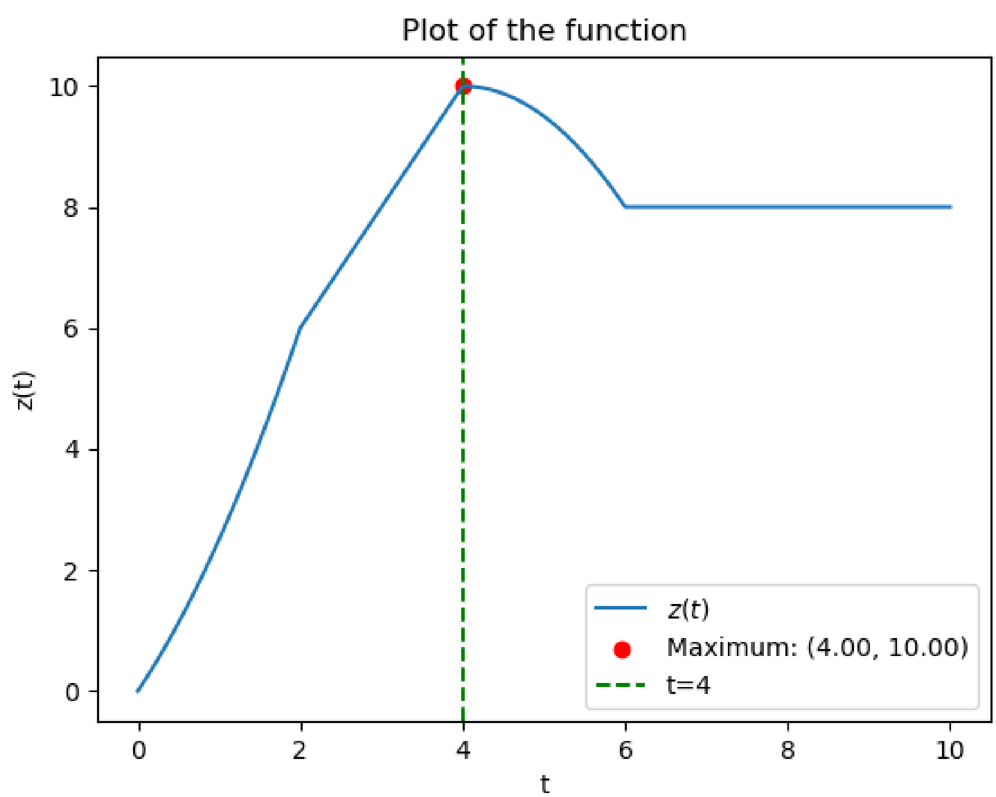


Figure 8.37: z(t) vs. t

Chapter 9

Fourier transform

9.1 Let a frequency modulated (FM) signal : $x(t) = A\cos(\omega_c t + k_f \int_{-\infty}^{t} m(\lambda)d\lambda)$, where $m(t)$is a message signal of bandwidth W. It is passed through a non-linear system with output $y(t) = 2x(t) + 5(x(t))^2$. Let $B_T$denote the FM bandwidth. The minimum value of ω_c required to recover $x(t)$ from $y(t)$ is:

(A) $B_T + W$

(B) $\frac{3}{2}B_T$

(C) $2B_T + W$

(D) $\frac{5}{2}B_T$

Solution:

9.2 Let an input $x[n]$ having discrete-time Fourier transform $X(e^{j\Omega}) = 1 - e^{-j\Omega} + 2e^{-3j\Omega}$ be passed through an LTI system. The frequency response of the LTI system is $H(e^{j\Omega}) = 1 - \frac{1}{2}e^{-2j\Omega}$ The output $y[n]$ of the system is

(GATE EC 2023) **Solution:**

Parameter	**Value**
$X(e^{j\omega})$	$1 - e^{-j\omega} + 2e^{-3j\omega}$
$H(e^{j\omega})$	$1 - \frac{1}{2}e^{-2j\omega}$
$Y(e^{j\omega})$	$X(e^{j\omega}) \cdot H(e^{j\omega})$
$y[n]$	?

Table 9.1: Parameters

$$y[n] = x[n] * h[n] \tag{9.1}$$

$$x(n) * h(n) \longleftrightarrow X(e^{j\omega}) \cdot H(e^{j\omega})$$

$$Y(e^{j\omega}) = X(e^{j\omega}) \cdot H(e^{j\omega}) \tag{9.2}$$

$$Y(e^{j\omega}) = (1 - e^{-j\omega} + 2e^{-3j\omega}) \cdot \left(1 - \frac{1}{2}e^{-2j\omega}\right) \tag{9.3}$$

$$= (1 - e^{-j\omega} + \frac{5}{2}e^{-3j\omega} - \frac{1}{2}e^{-2j\omega} - e^{-5j\omega}) \tag{9.4}$$

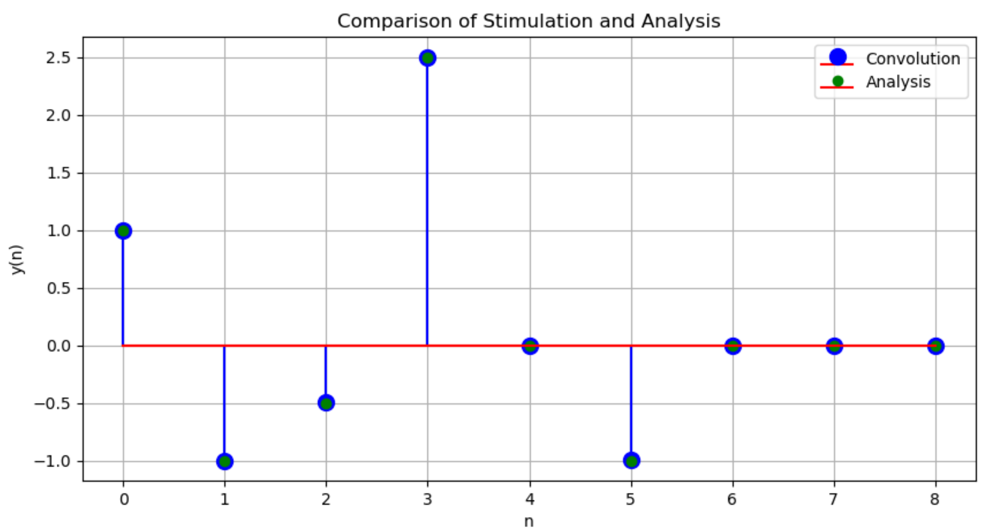


Figure 9.1: $y(n)$ vs n

9.3 The Fourier transform $X(\omega)$ of $x(t) = e^{-t^2}$ is

Note:$\int_{-\infty}^{\infty} e^{-y^2}\, dy = \sqrt{\pi}$

A) $\sqrt{\pi} e^{\frac{\omega^2}{2}}$

B) $\frac{e^{\frac{-\omega^2}{4}}}{2\sqrt{\pi}}$

C) $\sqrt{\pi} e^{\frac{-\omega^2}{4}}$

D) $\sqrt{\pi} e^{\frac{-\omega^2}{2}}$

Gate 2023 EC Question 28 **Solution:**

$$x'(t) = -2te^{-t^2} \tag{9.5}$$

$$x'(t) = -2tx(t) \tag{9.6}$$

doing fourier transform

$$j2\pi f X(f) = -2j\frac{dX(f)}{df} \tag{9.7}$$

$$\int_0^f \frac{dX(f)}{X(f)} = \int_0^f \frac{2\pi f df}{-2} \tag{9.8}$$

$$\frac{X(f)}{X(0)} = e^{\frac{-(2\pi f)^2}{4}} \tag{9.9}$$

$$X(0) = \int_{-\infty}^{\infty} x(t)\, dt = \sqrt{\pi} \tag{9.10}$$

$$X(f) = \sqrt{\pi} e^{\frac{-(2\pi f)^2}{4}} \tag{9.11}$$

$$X(f) = \sqrt{\pi} e^{-(\pi f)^2} \tag{9.12}$$

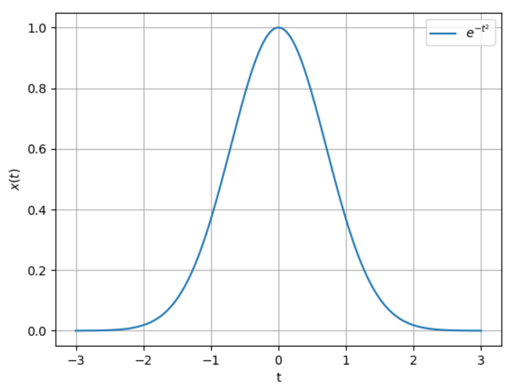


Figure 9.2: Graph of e^{-t^2}

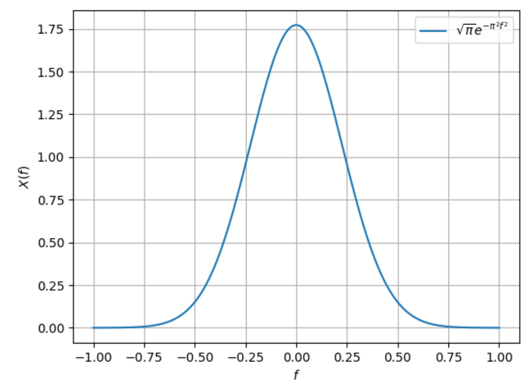


Figure 9.3: Graph of $X(f) = \sqrt{\pi}e^{-\pi^2 f^2}$

9.4 Let $x(t) = 10\cos(10.5\omega t)$ be passed through an LTI system with impulse response $h(t) = \pi\left(\frac{\sin(\omega t)}{\pi t}\right)^2 \cos(10\omega t)$. The output of the system is:

(GATE EC 2023)

A) $\frac{15}{4}\omega\cos(10.5\omega t)$

B) $\frac{15}{2}\omega\cos(10.5\omega t)$

C) $\frac{15}{8}\omega\cos(10.5\omega t)$

D) $15\omega\cos(10.5\omega t)$

Solution:

Symbol	Description	Value
$x(t)$	input	$10\cos(10.5\omega t)$
$h(t)$	impulse	$\pi\left(\frac{\sin(\omega t)}{\pi t}\right)^2\cos(10\omega t)$
$y(t)$	output	??

Table 9.2: Input Parameters

Given $h(t)$ is real and even. When a sinusoidal input is applied to an LTI system with an even impulse response, the output will also be sinusoidal. Hence, $y(t) =$

$A \cdot 10 \cos(10.5\omega t + \theta)$.

$$x(t) \rightarrow \boxed{\text{h(t)}} \rightarrow y(t)$$

$$\text{Let } f(t) = \pi \left(\frac{\sin(\omega t)}{\pi t}\right)^2 \tag{9.13}$$

$$h(t) = f(t)\cos(10\omega t) \tag{9.14}$$

Using

$$x_1(t) \cdot x_2(t) \overset{\mathcal{F}}{\longleftrightarrow} X_1(\omega) * X_2(\omega) \tag{9.15}$$

$$\left(\frac{\sin(\omega t)}{\pi t}\right) \cdot \left(\frac{\sin(\omega t)}{\pi t}\right) \overset{\mathcal{F}}{\longleftrightarrow} X_1(\omega) * X_2(\omega) \tag{9.16}$$

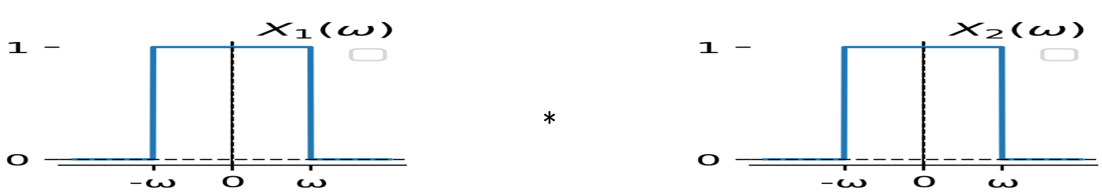


Figure 9.4:

$$\left(\frac{\sin(\omega t)}{\pi t}\right)^2 \overset{\mathcal{F}}{\longleftrightarrow} X_3(\omega) \tag{9.17}$$

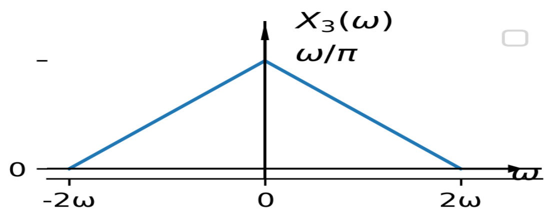


Figure 9.5:

$$\pi \left(\frac{\sin(\omega t)}{\pi t}\right)^2 \overset{\mathcal{F}}{\longleftrightarrow} X_4(\omega) \tag{9.18}$$

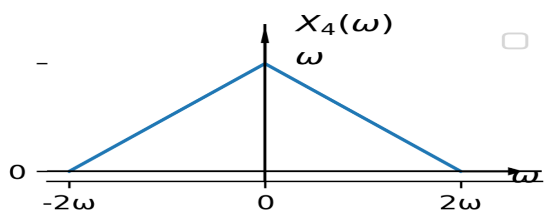


Figure 9.6:

From modulating property:

$$f(t)\cos(\omega_0 t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2}\left[F(\omega+\omega_0)+F(\omega-\omega_0)\right] \tag{9.19}$$

$$H(\omega) = \frac{1}{2}\left[F(\omega+10\omega)+F(\omega-10\omega)\right] \tag{9.20}$$

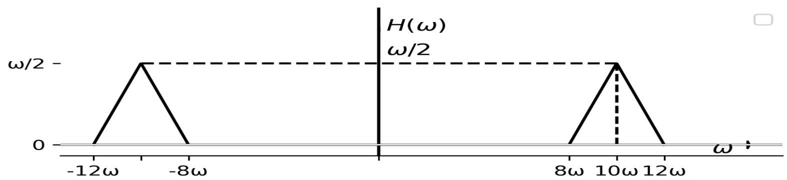


Figure 9.7:

$$\frac{\frac{\omega}{2} - 0}{10\omega - 12\omega} = \frac{|H(10.5\omega)| - 0}{10.5\omega - 12\omega} \tag{9.21}$$

$$A = |H(10.5\omega)| = \frac{3}{8}\omega \quad \text{and} \quad \theta = \angle H(10.5\omega) = 0^\circ \tag{9.22}$$

The output $y(t)$:

$$y(t) = \frac{3}{8}\omega \cdot 10\cos(10.5\omega t) \tag{9.23}$$

$$= \frac{15}{4}\omega\cos(10.5\omega t) \tag{9.24}$$

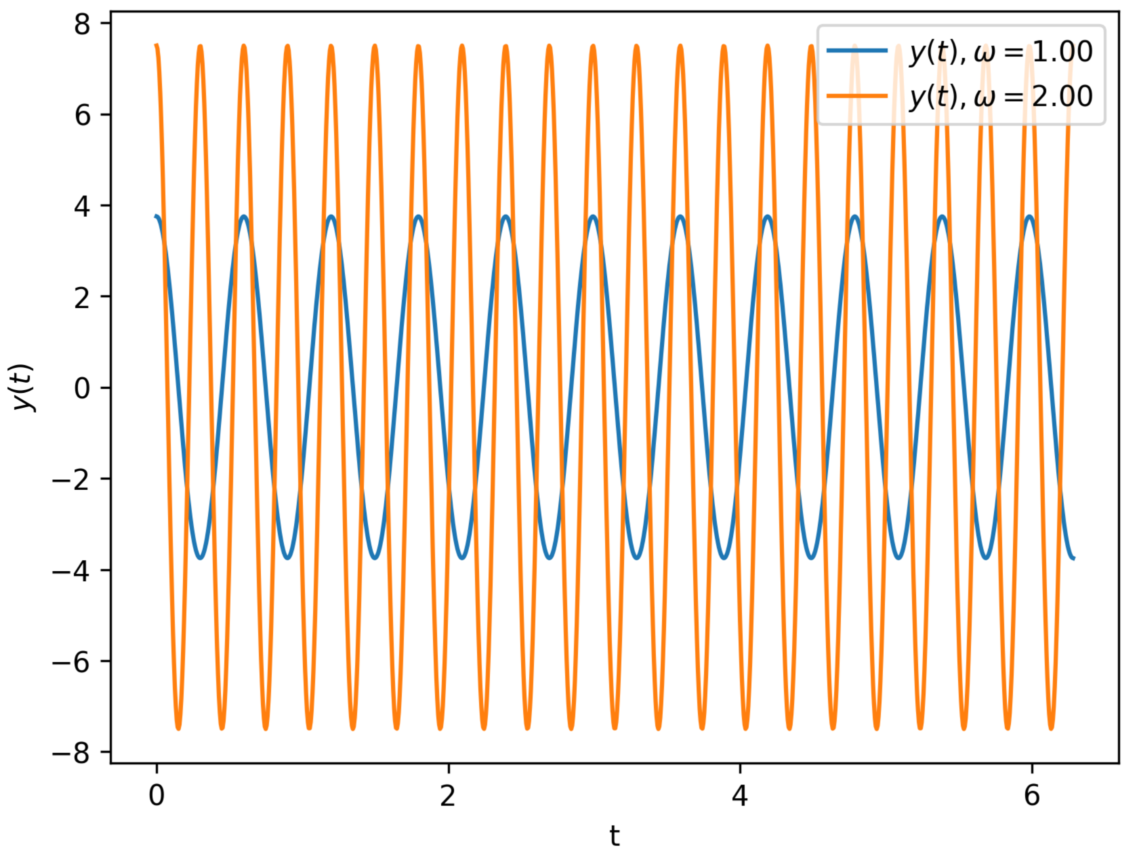


Figure 9.8:

9.5 Q27) Let m(t) be a strictly band-limited signal with bandwidth B and energy E. Assuming $\omega_0 = 10\mathrm{B}$, the energy in the signal $\mathrm{m}(\mathrm{t})\cos(\omega_0 \mathrm{t})$

(A) $\frac{\mathrm{E}}{4}$

(B) $\frac{\mathrm{E}}{2}$

(C) E

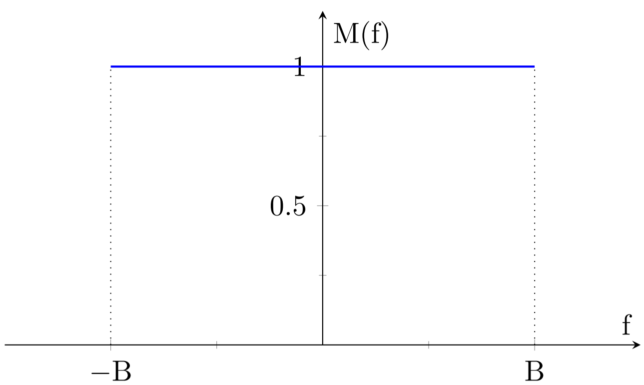
(D) 2E (GATE EC 2023)

Solution:

Variables	Conditions
M(f)	Fourier transform of m(t)
y(t)	$y(t)=m(t)\cos(2\pi f_0 t)$
Y(f)	Fourier transform of y(t)

Table of Parameters

Let us assume for a case of M(f),



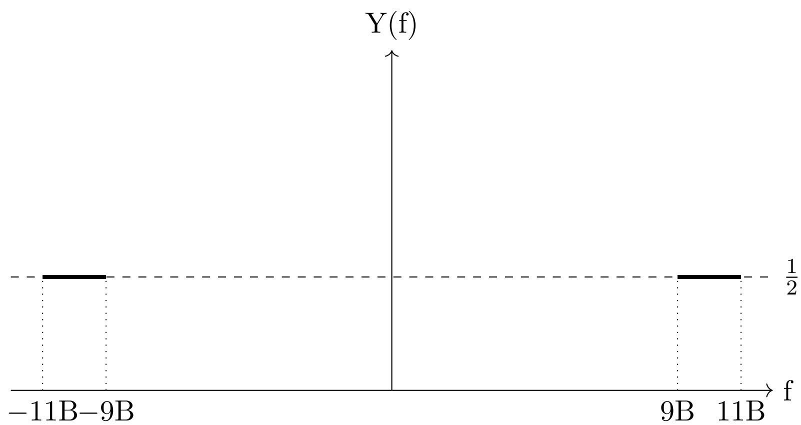
Energy (E) of the signal M(f) is given by,

$$E = \frac{1}{2\pi}\int_{-B}^{B} |M(f)|^2\, df = \frac{B}{\pi} \tag{9.25}$$

Fourier transform of y(t) is given by,

$$Y(f) = M(f) * \frac{1}{2}\left(\delta(f+f_0) + \delta(f-f_0)\right) \tag{9.26}$$

$$Y(f) = \frac{1}{2}\left(M(f+f_0) + M(f-f_0)\right) \tag{9.27}$$



Energy (E_1) of the signal Y(f) is given by,

$$E_1 = \frac{1}{2\pi}\left(\frac{2B}{4} + \frac{2B}{4}\right) = \frac{B}{2\pi} \tag{9.28}$$

So, from (9.25) and (9.28),

$$E_1 = \frac{E}{2} \tag{9.29}$$

Hence, option B is correct

9.6 The following function is defined over the interval $[-L, L]$:

$$f(x) = px^4 + qx^5$$

It is expressed as a Fourier series,

$$f(x) = a(0) + \sum_{n=1}^{\infty} \left\{ a(n) \sin\left(\frac{\pi x}{L}\right) + b(n) \cos\left(\frac{\pi x}{L}\right) \right\}$$

which options amongst the following are true?

(a) $a(n)$, $n = 1, 2, .., \infty$ depend on p

(b) $a(n)$, $n = 1, 2, .., \infty$ depend on q

(c) $b(n)$, $n = 1, 2, .., \infty$ depend on p

(d) $b(n)$, $n = 1, 2, .., \infty$ depend on q

(GATE 2023 CE Question 25)

Solution:

Parameter	**Description**
$f(x)$	Polynomial function
$2L$	Period of the Polynomial function
$c(n)$	Complex Fourier Coefficients
$a(0)$, $a(n)$, $b(n)$	Trigonometric Fourier Coefficients

Table 9.3: Parameters

The complex exponential Fourier Series of $f(x)$ is,

$$f(x) = \sum_{n=-\infty}^{\infty} c(n)\, e^{j\frac{\pi n x}{L}} \tag{9.30}$$

$$\implies c(n) = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{-j\frac{\pi n x}{L}}\, dx \tag{9.31}$$

$$c(n) = \frac{1}{2L}\int_{-L}^{L} \left(px^4 + qx^5\right) e^{-j\frac{\pi n x}{L}}\, dx \tag{9.32}$$

For $n = 0$,

$$\begin{aligned} c(0) &= \frac{1}{2L}\int_{-L}^{L} \left(px^4 + qx^5\right)\, dx && (9.33)\\ &= \frac{pL^4}{5} && (9.34) \end{aligned}$$

For $n \neq 0$,

$$\begin{aligned} c(n) &= \frac{1}{2L}\int_{-L}^{L} \left(px^4 + qx^5\right) e^{-j\frac{\pi n x}{L}}\, dx && (9.35)\\ &= \frac{pL^4}{2}\left(e^{j\pi n} - e^{-j\pi n}\right)\left(\frac{1}{j\pi n} + \frac{12}{(j\pi n)^3} + \frac{24}{(j\pi n)^5}\right)\\ &\quad - \frac{pL^4}{2}\left(e^{j\pi n} + e^{-j\pi n}\right)\left(\frac{4}{(j\pi n)^2} + \frac{24}{(j\pi n)^4}\right)\\ &\quad - \frac{qL^5}{2}\left(e^{j\pi n} + e^{-j\pi n}\right)\left(\frac{1}{j\pi n} + \frac{20}{(j\pi n)^3} + \frac{120}{(j\pi n)^5}\right)\\ &\quad + \frac{qL^5}{2}\left(e^{j\pi n} - e^{-j\pi n}\right)\left(\frac{5}{(j\pi n)^2} + \frac{60}{(j\pi n)^4} + \frac{120}{(j\pi n)^6}\right) && (9.36)\\ &= \left(pL^4\right)(-1)^n\left(\frac{4}{(\pi n)^2} - \frac{24}{(\pi n)^4}\right)\\ &\quad - \left(qL^5\right)(-1)^n\left(-\frac{j}{\pi n} + \frac{20j}{(\pi n)^3} - \frac{120j}{(\pi n)^5}\right) && (9.37) \end{aligned}$$

Given,

$$f(x) = a(0) + \sum_{n=1}^{\infty} \left\{ a(n) \sin\left(\frac{\pi n x}{L}\right) + b(n) \cos\left(\frac{\pi n x}{L}\right) \right\} \tag{9.38}$$

Finding the Fourier Coefficient $a(0)$,

$$a(0) = c(0) \tag{9.39}$$

$$\implies a(0) = \frac{pL^4}{5} \tag{9.40}$$

We know,

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j} \tag{9.41}$$

Finding the Fourier Coefficients $a(n)$,

$$a(n) = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{\pi n x}{L}\right) \, dx \tag{9.42}$$

$$a(n) = \frac{1}{L} \int_{-L}^{L} f(x) \left(\frac{e^{j\frac{\pi n x}{L}} - e^{-j\frac{\pi n x}{L}}}{2j} \right) \, dx \tag{9.43}$$

$$= \frac{1}{2Lj} \int_{-L}^{L} f(x) \, e^{j\frac{\pi n x}{L}} \, dx - \frac{1}{2Lj} \int_{-L}^{L} f(x) \, e^{-j\frac{\pi n x}{L}} \, dx \tag{9.44}$$

$$\implies a(n) = \frac{c(-n) - c(n)}{j} \tag{9.45}$$

$$a(n) = \left(-2qL^5\right)(-1)^n \left(\frac{1}{\pi n} - \frac{2}{(\pi n)^3} + \frac{120}{(\pi n)^5} \right) \tag{9.46}$$

We know,

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \tag{9.47}$$

Finding the Fourier Coefficients $b\left(n\right)$,

$$b\left(n\right) = \frac{1}{L}\int_{-L}^{L} f\left(x\right)\cos\left(\frac{\pi n x}{L}\right)\,dx \tag{9.48}$$

$$b\left(n\right) = \frac{1}{L}\int_{-L}^{L} f\left(x\right)\left(\frac{e^{j\frac{\pi n x}{L}} + e^{-j\frac{\pi n x}{L}}}{2}\right)\,dx \tag{9.49}$$

$$= \frac{1}{2L}\int_{-L}^{L} f\left(x\right)e^{j\frac{\pi n x}{L}}\,dx + \frac{1}{2L}\int_{-L}^{L} f\left(x\right)e^{-j\frac{\pi n x}{L}}\,dx \tag{9.50}$$

$$\implies b\left(n\right) = c\left(-n\right) + c\left(n\right) \tag{9.51}$$

$$b\left(n\right) = \left(2pL^4\right)\left(-1\right)^n\left(\frac{4}{\left(\pi n\right)^2} - \frac{24}{\left(\pi n\right)^4}\right) \tag{9.52}$$

Hence, options (b) and (c) are correct.

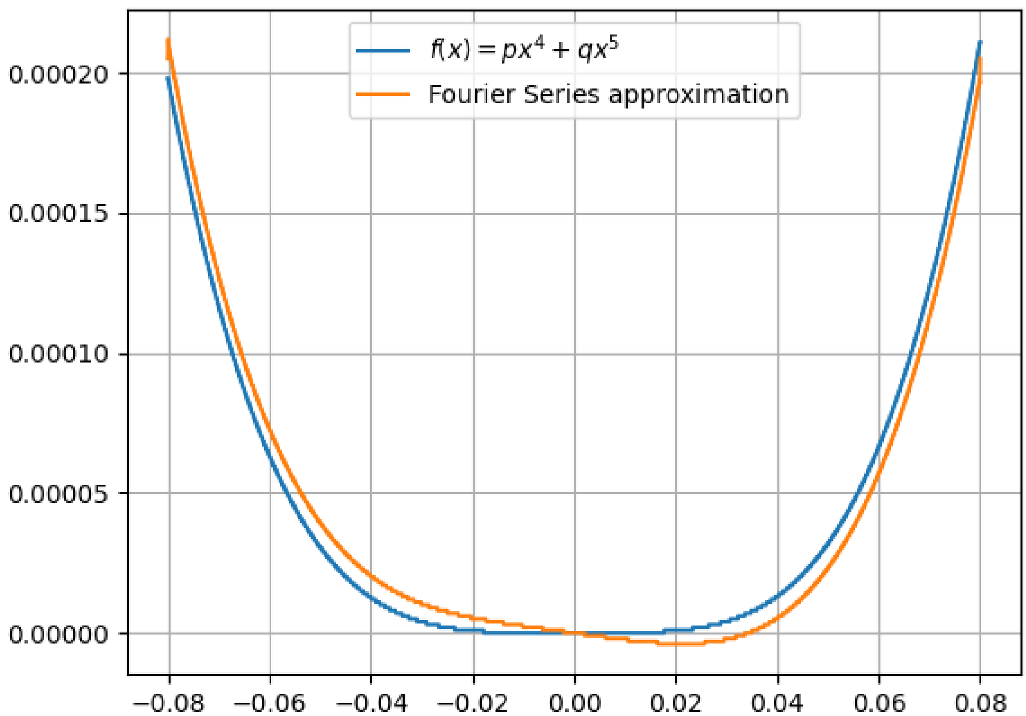


Figure 9.9: Fourier Series Approximation of $f(x)$ for $p = 5$, $q = 2$, $L = 0.08$

9.7 A continuous real-valued signal $x(t)$ has finite positive energy and $x(t) = 0, \forall\ t < 0$. From the list given below, select ALL the signals whose continuous-time Fourier transform is purely imaginary.

(a) $x(t) + x(-t)$

(b) $x(t) - x(-t)$

(c) $j(x(t) + x(-t))$

(d) $j(x(t) - x(-t))$

(GATE IN 2023)

Solution:

Parameter	Description
$x(t)$	Continuous real valued signal
t	time
f	frequency of the signal
$Y(f)$	Fourier Transfom of $y(t)$

Table 9.4: Variables and their descriptions

Fourier transform of a signal $y\left(t\right)$

$$\mathcal{F}\{y(t)\} = Y\left(f\right) \tag{9.53}$$

$$Y\left(f\right) = \int_{-\infty}^{\infty} y\left(t\right) e^{-j2\pi ft}\, dt \tag{9.54}$$

$$Y^*\left(f\right) = \int_{-\infty}^{\infty} y^*\left(t\right) e^{j2\pi ft}\, dt \tag{9.55}$$

Fourier transform is purely imaginary if $Y\left(f\right) + Y^*\left(f\right) = 0$

(a) $x\left(t\right) + x\left(-t\right)$

$$y(t) = x(t) + x(-t) \tag{9.56}$$

$$y^*(t) = y(t) \tag{9.57}$$

$$y(t) = y(-t) \tag{9.58}$$

$$Y(f) + Y^*(f) = \int_{-\infty}^{\infty} y(t)\, e^{-j2\pi ft}\ dt + \int_{-\infty}^{\infty} y^*(t)\, e^{j2\pi ft}\ dt \tag{9.59}$$

$$= 2\int_{-\infty}^{\infty} y(t)\, cos(2\pi ft)\ dt \tag{9.60}$$

$\therefore$ Fourier Transform is Purely real.

(b) $x(t) - x(-t)$

$$y(t) = x(t) - x(-t) \tag{9.61}$$

$$y^*(t) = y(t) = -y(-t) \tag{9.62}$$

$$Y(f) = \int_{-\infty}^{\infty} y(t)\, e^{-j2\pi ft}\ dt \tag{9.63}$$

$$Y^*(f) = -\int_{-\infty}^{\infty} y(-t)\, e^{j2\pi ft}\ dt \tag{9.64}$$

$$= -\int_{-\infty}^{\infty} y(t)\, e^{-j2\pi ft}\ dt \tag{9.65}$$

$$Y(f) + Y^*(f) = 0 \tag{9.66}$$

$\therefore$ Fourier Transform is purely imaginary.

(c) $j\left(x\left(t\right)+x\left(-t\right)\right)$

$$y\left(t\right)=j\left(x\left(t\right)+x\left(-t\right)\right) \tag{9.67}$$

$$y\left(-t\right)=y\left(t\right) \tag{9.68}$$

$$y^{*}\left(t\right)=-y(t) \tag{9.69}$$

$$Y\left(f\right)=\int_{-\infty}^{\infty}y\left(t\right)e^{-j2\pi ft}\ dt \tag{9.70}$$

$$Y^{*}\left(f\right)=-\int_{-\infty}^{\infty}y\left(t\right)e^{j2\pi ft}\ dt \tag{9.71}$$

$$=-\int_{-\infty}^{\infty}y\left(t\right)e^{-j2\pi ft}\ dt \tag{9.72}$$

$$Y\left(f\right)+Y^{*}\left(f\right)=0 \tag{9.73}$$

$\therefore$ Fourier Transform is Purely imaginary.

(d) $j\left(x\left(t\right)-x\left(-t\right)\right)$

$$y\left(t\right)=j\left(x\left(t\right)-x\left(-t\right)\right) \tag{9.74}$$

$$y\left(-t\right)=-y\left(t\right) \tag{9.75}$$

$$y^{*}\left(t\right)=-y\left(t\right) \tag{9.76}$$

$$Y\left(f\right)=\int_{-\infty}^{\infty}y\left(t\right)e^{-j2\pi ft}\ dt \tag{9.77}$$

$$Y^{*}\left(f\right)=-\int_{-\infty}^{\infty}y\left(t\right)e^{j2\pi ft}\ dt \tag{9.78}$$

$$=\int_{-\infty}^{\infty}y\left(t\right)e^{-j2\pi ft}\ dt \tag{9.79}$$

$$Y\left(f\right)+Y^{*}\left(f\right)=2\int_{-\infty}^{\infty}y\left(t\right)e^{-j2\pi ft}\ dt \tag{9.80}$$

$\therefore$ Fourier Transform is not Purely imaginary.

9.8 Let $x_1(t) = u(t + 1.5) - u(t - 1.5)$ and $x_2(t)$ is shown in the figure below. For $y(t) = x_1(t) * x_2(t)$, the $\int_{-\infty}^{\infty} y(t)\, dt$ is ___________.

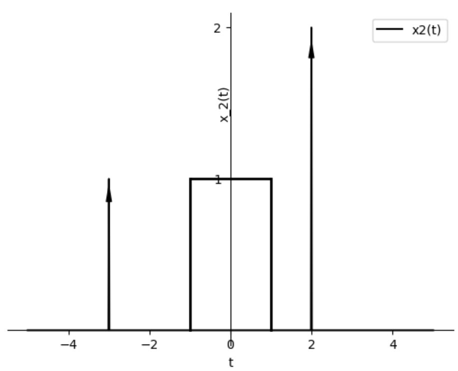


Figure 9.10: Figure

(GATE IN 2023)

Solution:

Input Parameters		
Function	**Expression**	**Description**
$x_1(t)$	$u(t + 1.5) - u(t - 1.5)$	Step function with delay and width parameters.
$X_1(f)$		Fourier Transform of $x_1(t)$.
$x_2(t)$	$\delta(t + 3) + \text{rect}\left(\frac{t}{2}\right) + 2\delta(t - 2)$	Impulse function followed by a rectangle and two impulses.
$X_2(f)$		Fourier Transform of $x_2(t)$.

Table 9.5: Input Parameters

$$x_1(t) = u(t+1.5) - u(t-1.5) \tag{9.81}$$

$$x_1(t) = \mathrm{rect}\left(\frac{t}{3}\right) \tag{9.82}$$

$$\tag{9.83}$$

The Fourier Transform of $rect\left(\frac{t}{a}\right)$:

$$\text{rect}\left(\frac{t}{a}\right) \overset{\mathcal{F}}{\longleftrightarrow} a \times \sin(2\pi f \frac{a}{2}) \tag{9.84}$$

$$\text{where rect}\left(\frac{t}{a}\right) = \begin{cases} 1 & \text{if } |t| < \frac{a}{2} \\ 0 & \text{otherwise} \end{cases} \tag{9.85}$$

$$X_1(f) = 3\text{sinc}(1.5 \cdot 2\pi f) \tag{9.86}$$

$$x_2(t) = \delta(t+3) + \text{rect}\left(\frac{t}{2}\right) + 2\delta(t-2) \tag{9.87}$$

$$X_2(f) = e^{3j \cdot 2\pi f} + 2\text{sinc}(2\pi f) + 2e^{-2j \cdot 2\pi f} \tag{9.88}$$

$$Y(f) = X_1(f) \cdot X_2(f) \tag{9.89}$$

$$Y(f) = 3\text{sinc}(1.5 \cdot 2\pi f) \cdot (e^{3j \cdot 2\pi f} + 2\text{sinc}(2\pi f) + 2e^{-2j \cdot 2\pi f}) \tag{9.90}$$

$$y(t) = x_1(t) * x_2(t) \tag{9.91}$$

$$Y(f) \overset{\mathcal{F}}{\longleftrightarrow} y(t) \tag{9.92}$$

$$y(t) = \text{rect}\left(\frac{t+3}{3}\right) + 2\text{rect}\left(\frac{t-2}{3}\right) + (t+2.5)u(t+2.5) + (t-2.5)u(t-2.5) \tag{9.93}$$

$$-(t+0.5)u(t+0.5) - (t-0.5)u(t-0.5) \tag{9.94}$$

$$Y(f) = \int_{-\infty}^{\infty} y(t) e^{-j2\pi f t}\, dt \tag{9.95}$$

$$\int_{-\infty}^{\infty} y(t)\, dt = Y(0) \tag{9.96}$$

$$Y(0) = 3\text{sinc}(0) \cdot (e^0 + 2\text{sinc}(0) + 2e^0) \tag{9.97}$$

$$= 3 \cdot (1 + 2 + 2) \tag{9.98}$$

$$= 15 \tag{9.99}$$

Therefore, the value of $\int_{-\infty}^{\infty} y(t)\, dt$ is 15

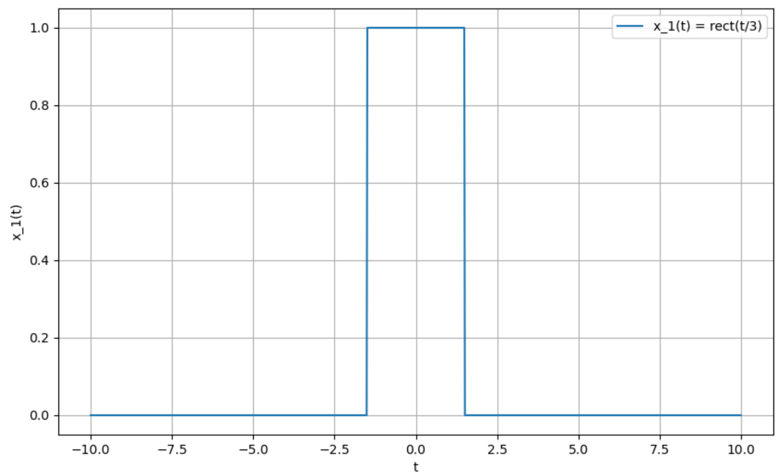


Figure 9.11: Graph of $x_1(t) = \text{rect}\left(\frac{t}{3}\right)$

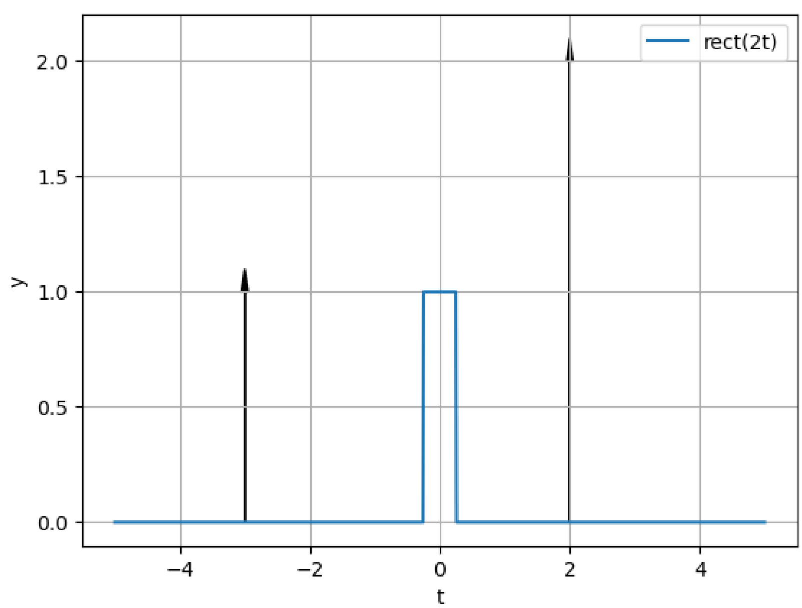


Figure 9.12: Graph of $x_2(t) = \delta(t+3) + \text{rect}(2t) + 2\delta(t-2)$

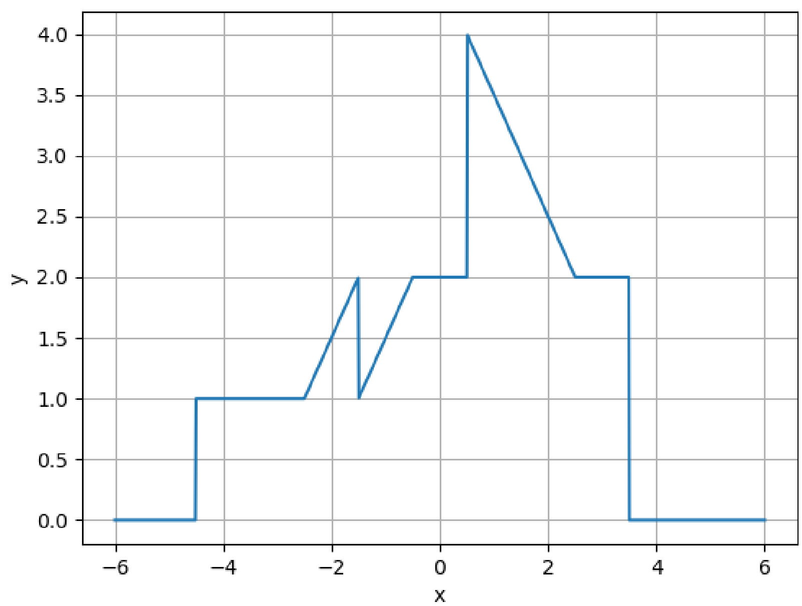


Figure 9.13: Graph of $y(t)$

9.9 Consider a discrete-time signal with period $N = 5$. Let the discrete-time Fourier series (DTFS) representation be $x[n] = \sum_{k=0}^{4} a_k e^{\frac{jk2\pi n}{5}}$ where $a_0 = 1$, $a_1 = 3j$, $a_2 = 2j$, $a_3 = -2j$, $a_4 = -3j$. The value of the sum $\sum_{n=0}^{4} x[n]\sin\left(\frac{4\pi n}{5}\right)$ is

(A) -10

(B) 10

(C) -2

(D) 2

Gate 2023 EC 47 **Solution:**

(a) Solving the question for N=5:

Parameter	**Value**	**Description**
N	5	Time period
$X(k)$	$\sum_{n=0}^{N-1} x(n)e^{\frac{-j2\pi kn}{N}}$	DFT formula
$X(0)$	5	DFT values
$X(1)$	$15j$	
$X(2)$	$10j$	
$X(3)$	$-10j$	
$X(4)$	$-15j$	

Table 9.6: Input Parameters

$$\sum_{n=0}^{4} x(n)\sin\left(\frac{4\pi n}{5}\right) = \sum_{n=0}^{4} x(n)\left[\frac{e^{\frac{j4\pi n}{5}} - e^{\frac{-j4\pi n}{5}}}{2j}\right] \tag{9.100}$$

$$= \frac{1}{2j}\left[\sum_{n=0}^{4} x(n)e^{\frac{j2\pi(2)n}{5}} - \sum_{n=0}^{4} x(n)e^{\frac{-j2\pi(2)n}{5}}\right] \tag{9.101}$$

Refering to the table 9.6.

$$X(k) = \sum_{n=0}^{4} x(n)e^{\frac{-j2\pi kn}{5}} \tag{9.102}$$

Referencing from equation (9.102), equation (9.101) can be written as:

$$\sum_{n=0}^{4} x(n)\sin\left(\frac{4\pi n}{5}\right) = \frac{1}{2j}\left[X(-2) - X(2)\right] \tag{9.103}$$

From the property of discrete Fourier series.

$$X(k) = X(k+N) \tag{9.104}$$

So, equation (9.103) becomes,

$$\sum_{n=0}^{4} x(n)\sin\left(\frac{4\pi n}{5}\right) = \frac{1}{2j}\left[X(3) - X(2)\right] \tag{9.105}$$

$$\sum_{n=0}^{4} x(n)\sin\left(\frac{4\pi n}{5}\right) = -10 \tag{9.106}$$

(b) Solving the question for N=8:

Parameter	**Value**	**Description**
N	8	Time period
$X(k)$	$\sum_{n=0}^{N-1} x(n)e^{\frac{-j2\pi kn}{N}}$	DFT formula
$X(0)$	8	DFT values
$X(1)$	$24j$	
$X(2)$	$16j$	
$X(3)$	$-16j$	
$X(4)$	$-24j$	
$X(5)$	0	
$X(6)$	0	
$X(7)$	0	

Table 9.7: Input Parameters

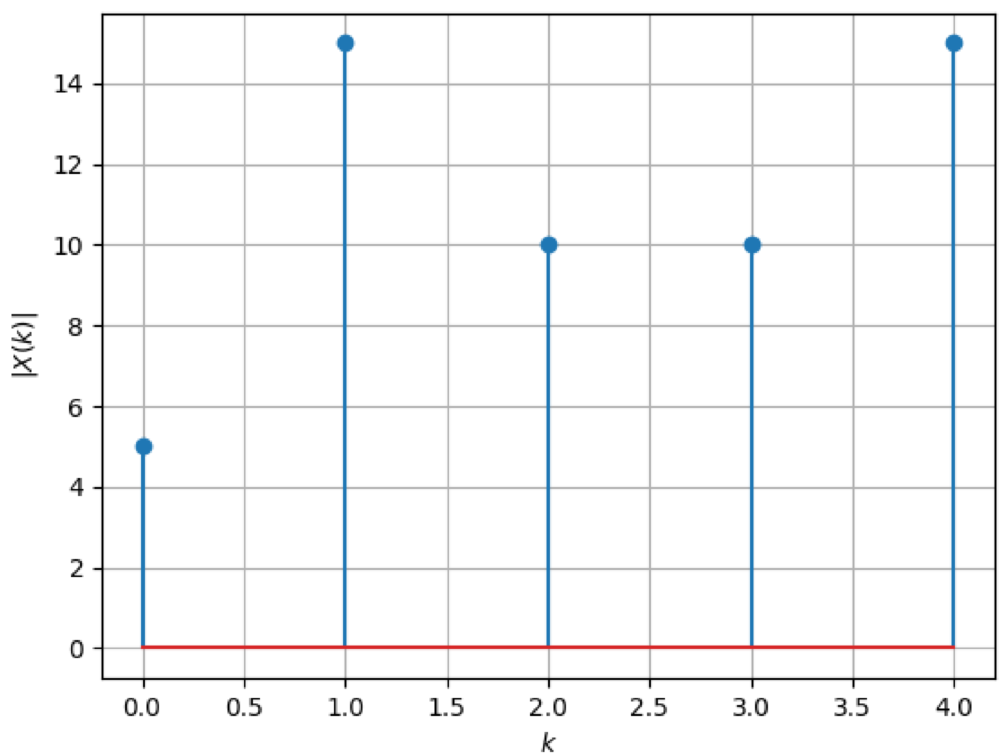


Figure 9.14: Amplitude of equation (9.102)

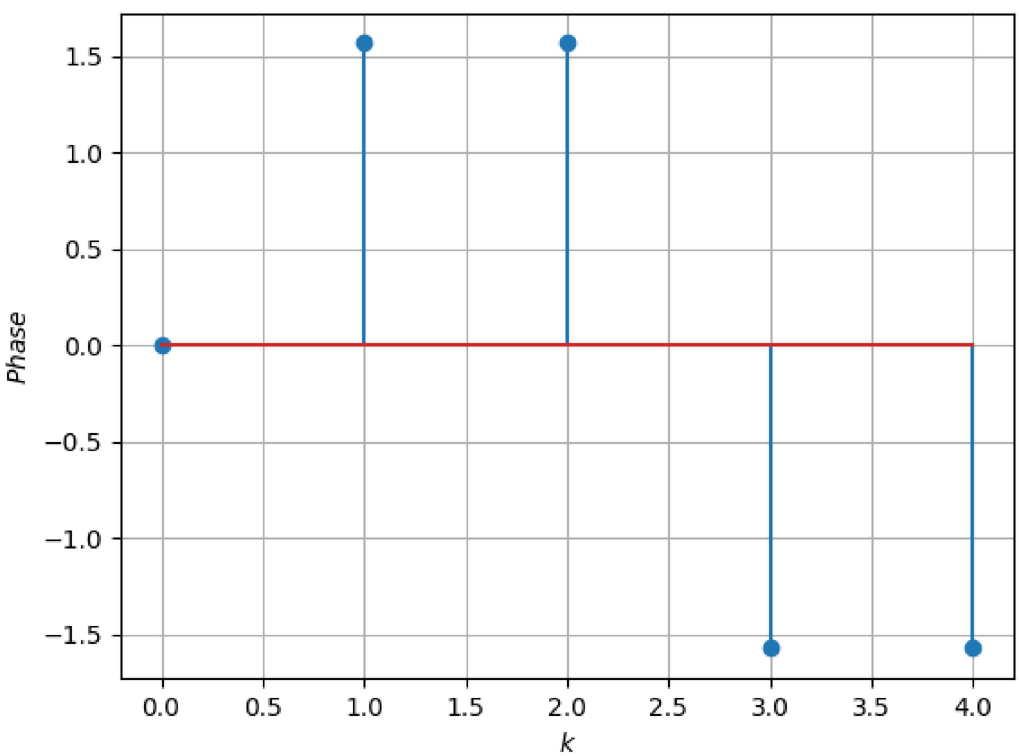


Figure 9.15: Phase of equation (9.102)

$$\sum_{n=0}^{7} x(n)\sin\left(\frac{4\pi n}{8}\right) = \sum_{n=0}^{7} x(n)\left[\frac{e^{\frac{j4\pi n}{8}} - e^{\frac{-j4\pi n}{8}}}{2j}\right] \tag{9.107}$$

$$= \frac{1}{2j}\left[\sum_{n=0}^{7} x(n)e^{\frac{j2\pi(2)n}{8}} - \sum_{n=0}^{7} x(n)e^{\frac{-j2\pi(2)n}{8}}\right] \tag{9.108}$$

Refering to the table 9.7.

$$X(k) = \sum_{n=0}^{7} x(n)e^{\frac{-j2\pi kn}{8}} \tag{9.109}$$

Referencing from equation(9.109), equation(9.108) can be written as:

$$\sum_{n=0}^{7} x(n)\sin\left(\frac{4\pi n}{8}\right) = \frac{1}{2j}\left[X(-2) - X(2)\right] \tag{9.110}$$

From the property of discrete Fourier series.

$$X(k) = X(k+N) \tag{9.111}$$

So, equation(9.110) becomes,

$$\sum_{n=0}^{7} x(n)\sin\left(\frac{4\pi n}{8}\right) = \frac{1}{2j}\left[X(6) - X(2)\right] \tag{9.112}$$

$$\sum_{n=0}^{7} x(n)\sin\left(\frac{4\pi n}{8}\right) = -8 \tag{9.113}$$

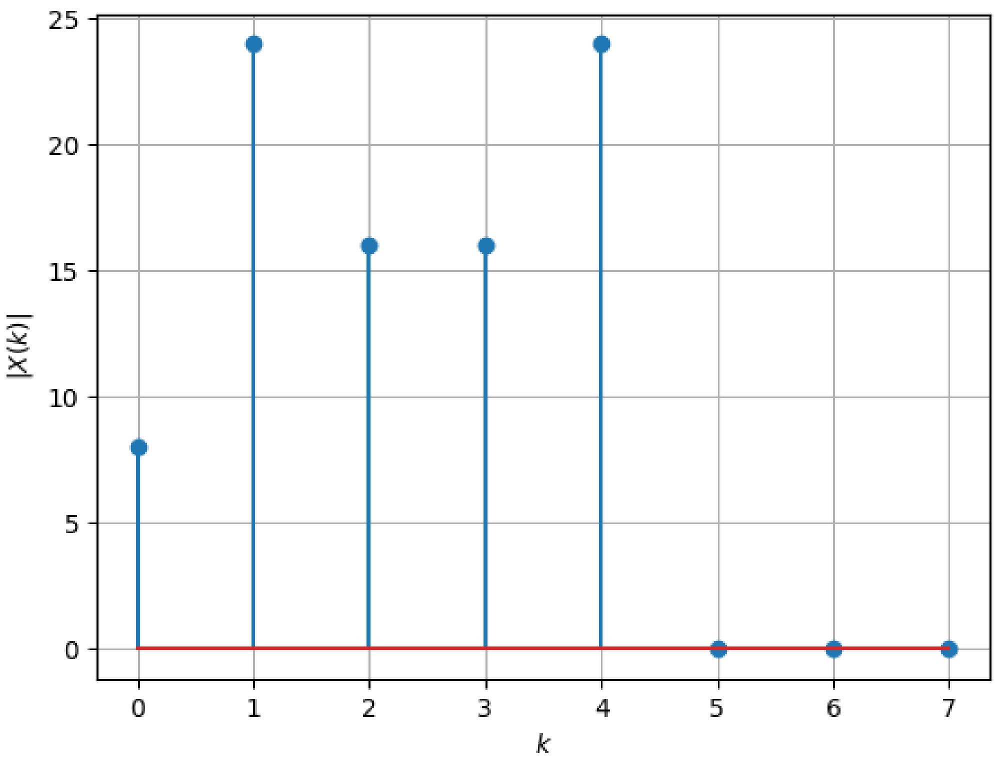


Figure 9.16: Amplitude of equation (9.109)

9.10 A continuous time, band-limited signal $x(t)$ has its Fourier transform described by:

$$X(f) = \begin{cases} 1 - \frac{|f|}{200} & \text{if } |f| \leq 200 \\ 0 & \text{if } |f| > 200 \end{cases}$$

The signal is uniformly sampled at a sampling rate of 600 Hz. The Fourier transform of the signal is $X_s(f)$. What is the value of $\frac{X_s(600)}{X_s(500)}$?

(GATE 2023 BM) **Solution:**

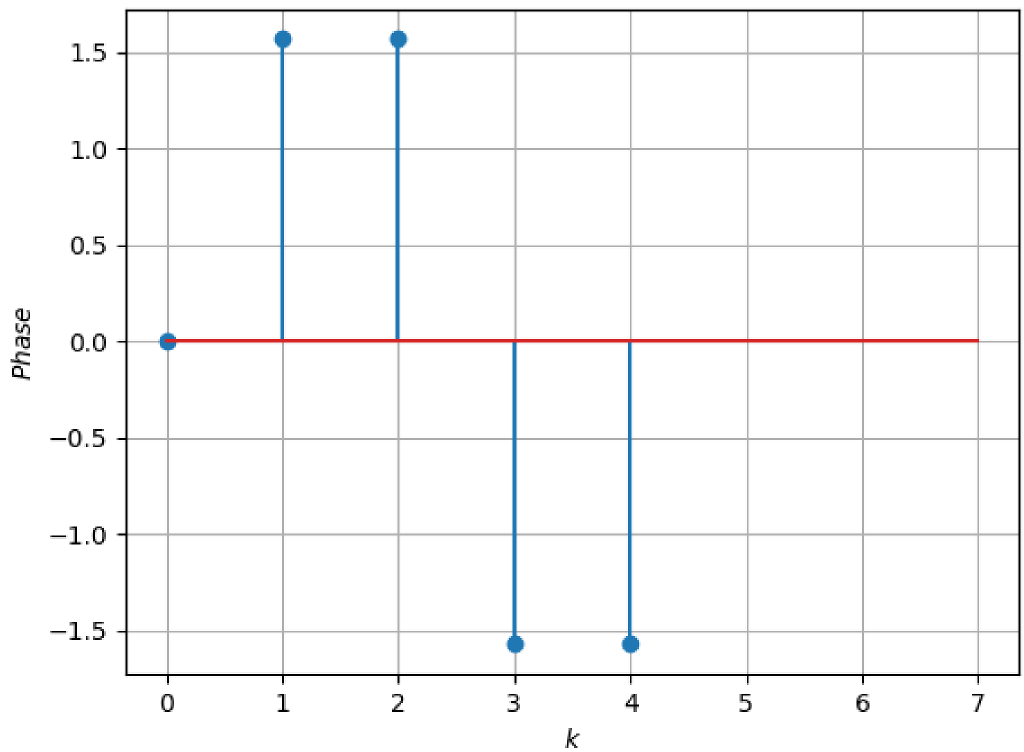


Figure 9.17: Phase of equation (9.109)

$$X_s(f) = \frac{1}{600}\sum_{k=-\infty}^{\infty} X(f - 600k) \tag{9.114}$$

$$\implies X_s(f + 600) = \frac{X(f)}{600} \tag{9.115}$$

Parameter	**Description**	**Value**
$X(f)$	Fourier transform of $x(t)$	$\begin{cases} 1 - \frac{\lvert f \rvert}{200} & \text{if } \lvert f \rvert \leq 200 \\ 0 & \text{if } \lvert f \rvert > 200 \end{cases}$
$X_s(f)$	Fourier transform of sampled signal	?

Table 9.8: Input Parameters

$$X_s(600) = \frac{X(0)}{600} \tag{9.116}$$

$$\implies X_s(600) = \frac{1}{600} \tag{9.117}$$

$$X_s(500) = \frac{X(-100)}{600} \tag{9.118}$$

$$\implies X_s(500) = \frac{1}{2 \cdot 600} \tag{9.119}$$

$$\frac{X_s(600)}{X_s(500)} = 2 \tag{9.120}$$

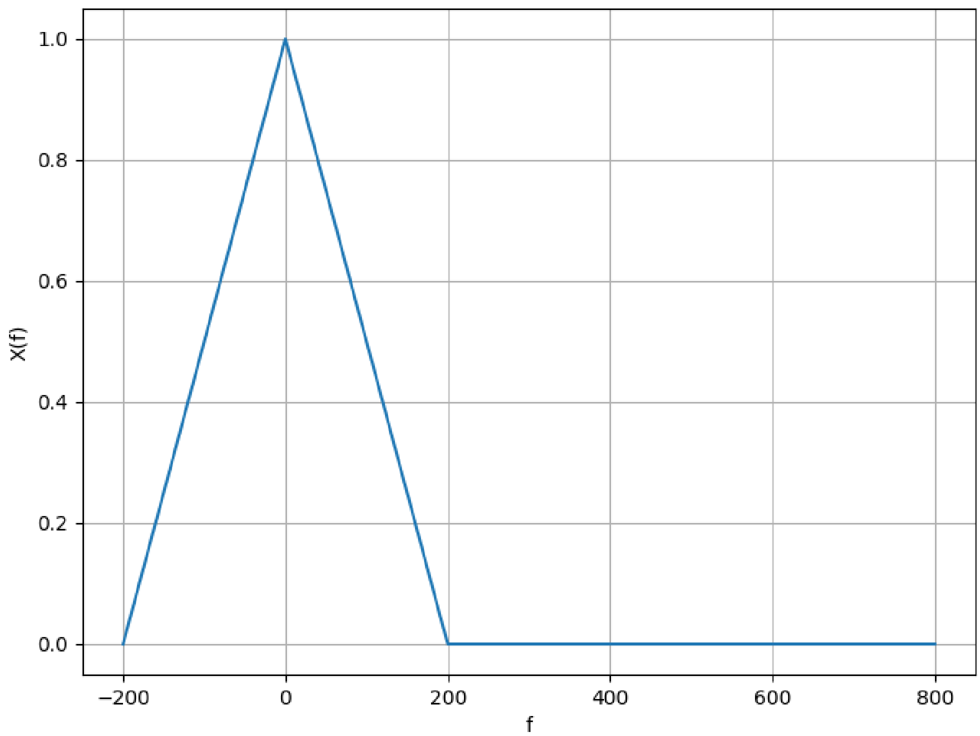


Figure 9.18: Plot of $X(f)$

9.11 The magnitude and phase plots of an LTI systems are shown in figure. Find the transfer function.

(a) $2.511e^{-0.0032s}$

(b) $\frac{e^{-2.514s}}{s+1}$

(c) $1.04e^{-2.514s}$

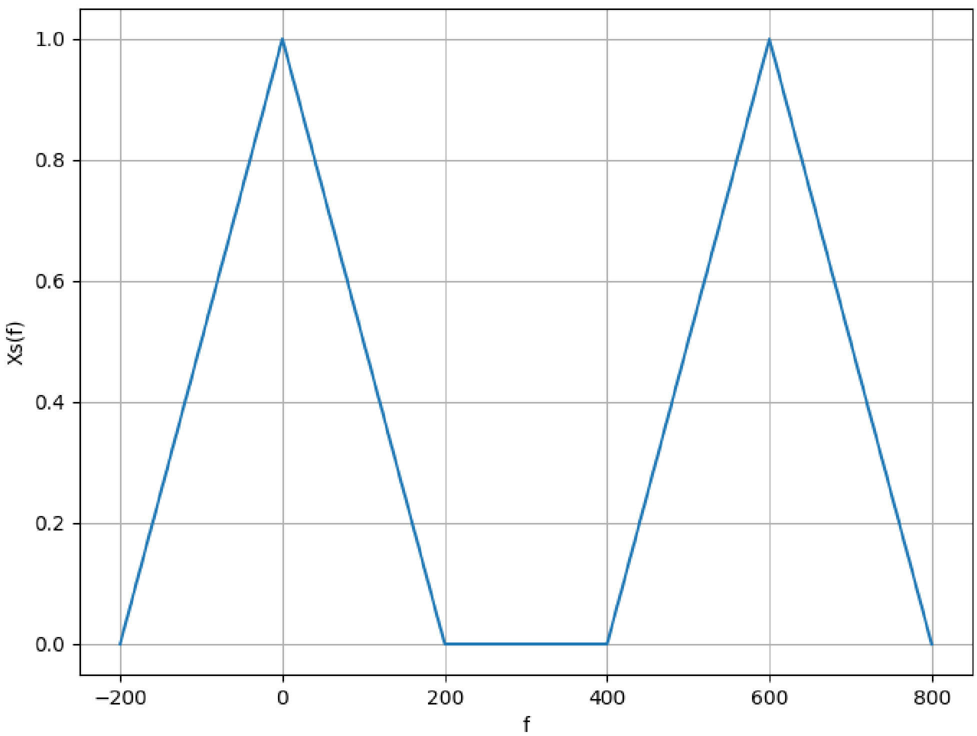


Figure 9.19: Plot of $X_s\left(f\right)$

(d) $2.511e^{-1.047s}$

(GATE EE 23)

Solution: From Fig. 9.20

$$\left|H\left(j\omega\right)\right| = 8 \tag{9.121}$$

$$\angle H\left(j\omega\right) = \frac{-\pi}{3}\omega \tag{9.122}$$

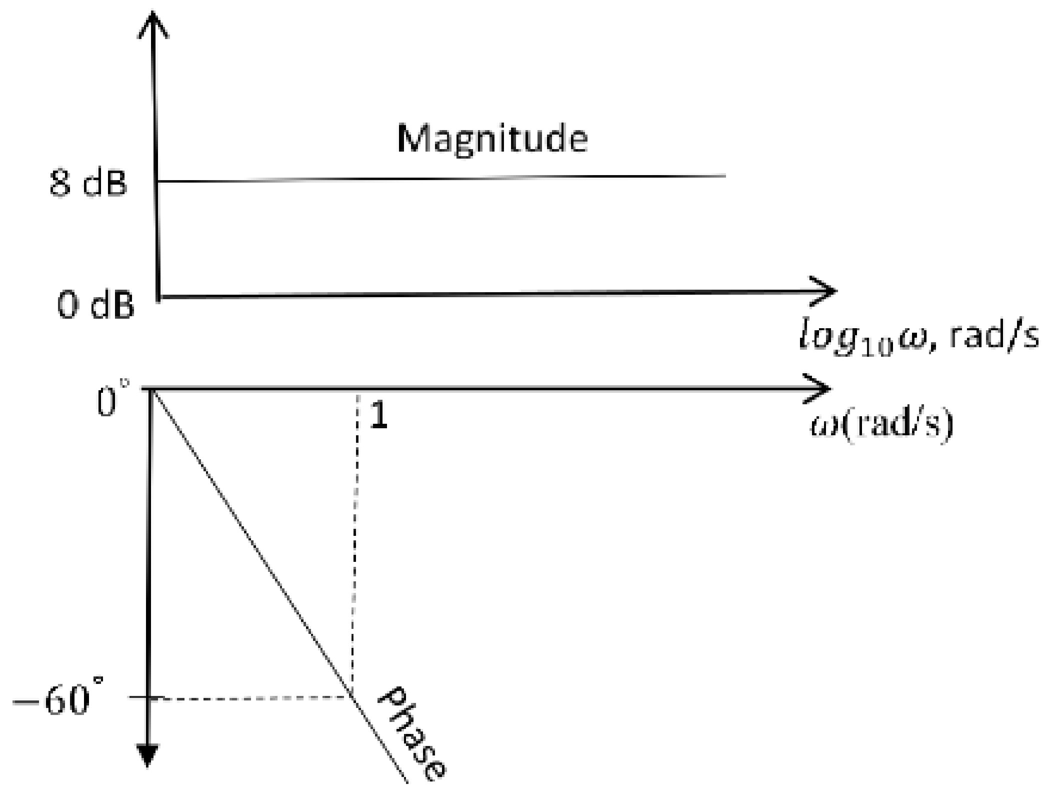


Figure 9.20:

Substituting the values from Fig. 9.20, magnitude of transfer function is:

$$8 = 20\log_{10}(|H(j\omega)|) \tag{9.123}$$

$$|H(j\omega)| = 10^{0.4} = 2.511 \tag{9.124}$$

Substituting the values from Fig. 9.20, The direction of the transfer function is:

$$\frac{H(j\omega)}{|H(j\omega)|} = e^{-j\frac{\pi}{3}\omega} \tag{9.125}$$

$$H(j\omega) = 2.511e^{-j\frac{\pi}{3}\omega} \tag{9.126}$$

$$= 2.511e^{-1.047s} \tag{9.127}$$

9.12 The value of the convolution of $f(x) = 3\cos(2x)$ and $g(x) = \frac{1}{3}\sin(2x)$ where $x \in [0, 2\pi)$, at $x = \frac{\pi}{3}$, is (Rounded off to 2 decimal places)

(GATE 2023 GE)

Solution:

$$f(w) = 3\cos(2w) \tag{9.128}$$
$$= \frac{3}{2}e^{j2w} + \frac{3}{2}e^{-j2w} \tag{9.129}$$
$$f(w) = \sum_{k=-\infty}^{\infty} c_k e^{-jwk} \tag{9.130}$$

by comparing (9.129) and (9.130)

$$c_2 = c_{-2} = \frac{3}{2} \tag{9.131}$$
$$c_k = 0 \quad o.w \tag{9.132}$$

$$g(w) = \frac{1}{3}\sin(2w) \tag{9.133}$$
$$= \frac{1}{6j}e^{j2w} - \frac{1}{6j}e^{-j2w} \tag{9.134}$$
$$g(w) = \sum_{k=-\infty}^{\infty} d_k e^{-jwk} \tag{9.135}$$

by comparing (9.134) and (9.135)

$$d_2 = \frac{-1}{6j} \tag{9.136}$$

$$d_{-2} = \frac{1}{6j} \tag{9.137}$$

$$d_k = 0 \quad o.w \tag{9.138}$$

The periodic convolution is multiply Fourier series coefficients is c(n)=c(k)*d(k)*p

$$\begin{aligned} c_2 &= c_2 * d_2 * p && (9.139)\\ &= \left(\frac{3}{2}\right)\left(\frac{-1}{6j}\right)(2\pi) && (9.140)\\ &= \frac{-\pi}{2j} && (9.141) \end{aligned}$$

and

$$\begin{aligned} c_{-2} &= c_{-2} * d_{-2} * p && (9.142)\\ &= \left(\frac{3}{2}\right)\left(\frac{1}{6j}\right)(2\pi) && (9.143)\\ &= \frac{\pi}{2j} && (9.144) \end{aligned}$$

$$f * g\,(x) = \sum_{n=-N}^{N} c_n e^{-j\frac{2\pi n x}{p}} \tag{9.145}$$

$$= c_{-2}e^{j2x} + c_2 e^{-j2x} \tag{9.146}$$

$$= \frac{\pi}{2j}e^{j2x} - \frac{\pi}{2j}e^{-j2x} \tag{9.147}$$

$$= \frac{\pi}{2j}\left(e^{j2x} - e^{-j2x}\right) \tag{9.148}$$

$$= \frac{\pi}{2j}\left(sin\,(2x)\,2j\right) \tag{9.149}$$

$$= \pi \sin\,(2x) \tag{9.150}$$

$$\tag{9.151}$$

at $x = \frac{\pi}{3}$

$$= \frac{\sqrt{3}\pi}{2} \tag{9.152}$$

$$\approx 3 \tag{9.153}$$

Therefore the convolution of f(x) and g(x) is 3

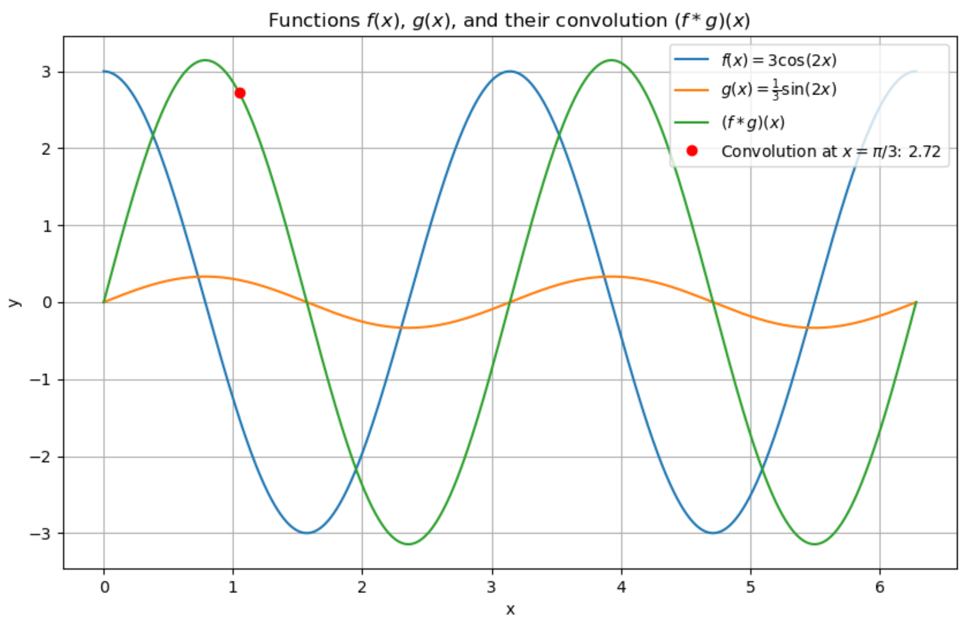


Figure 9.2: Plot of y vs x

9.13 A system is described by the following differential equation

$$0.01\frac{d^2y(t)}{dt^2} + 0.2\frac{dy(t)}{dt} + y(t) = 6x(t)$$

where time t is in seconds. If $x(t)$ is the unit step input applied at $t = 0$ s to this system, the magnitude of the output at $t = 1$ s is __________. (Round off the answer to two decimal places.) (GATE-2023.BM)

Solution: Given,

$$(0.01)\frac{d^2y(t)}{dt^2} + (0.2)\frac{dy(t)}{dt} + y(t) = 6x(t) \tag{9.154}$$

Parameter	**Description**	**Formulae/Value**
$x(t)$	The unit step input applied at $t = 0$ s to this system	$\begin{cases} 0, & \text{if } t < 0, \\ 1, & \text{if } t \geq 0. \end{cases}$
$y(t)$	A function of $x(t)$	-
$y(1)$	Value of y at $t = 1$	-

Table 9.9: Parameters

property:

$$\mathcal{L}\left\{\frac{d^n y(t)}{dt^n}\right\} = s^n Y(s) - s^{n-1} y(0) - \cdots - y^{(n-1)}(0) \tag{9.155}$$

Taking the Laplace transform of both sides (assuming zero initial conditions):

$$0.01s^2 Y(s) + 0.2sY(s) + Y(s) = \frac{6}{s} \tag{9.156}$$

$$\implies Y(s) = \frac{6}{s(0.01s^2 + 0.2s + 1)} \tag{9.157}$$

$$\implies Y(s) = \frac{6}{0.01(s)(s+10)^2} \tag{9.158}$$

Using partial fraction decomposition:

$$Y(s) = \frac{A}{s} + \frac{B}{s+10} + \frac{C}{(s+10)^2} \tag{9.159}$$

On solving, we get $A = 6, B = -6, C = -60$.

So,

$$Y(s) = \frac{6}{s} - \frac{6}{s+10} - \frac{60}{(s+10)^2} \tag{9.160}$$

From standard inverse laplace transforms:

$$\frac{1}{s+a} \longleftrightarrow e^{-at} \tag{9.161}$$

$$\frac{1}{(s+a)^2} \longleftrightarrow te^{-at} \tag{9.162}$$

Taking inverse Laplace transform of $Y(s)$,

$$y(t) = u(t)\left(6 - 6e^{-10t} - 60te^{-10t}\right) \tag{9.163}$$

At $t = 1s$

$$y(1) = u(1)\left(6 - 66e^{-10}\right) \tag{9.164}$$

$$y(1) = 6 - 66e^{-10} \tag{9.165}$$

approximately,

$$\implies y(1) = 5.99 \tag{9.166}$$

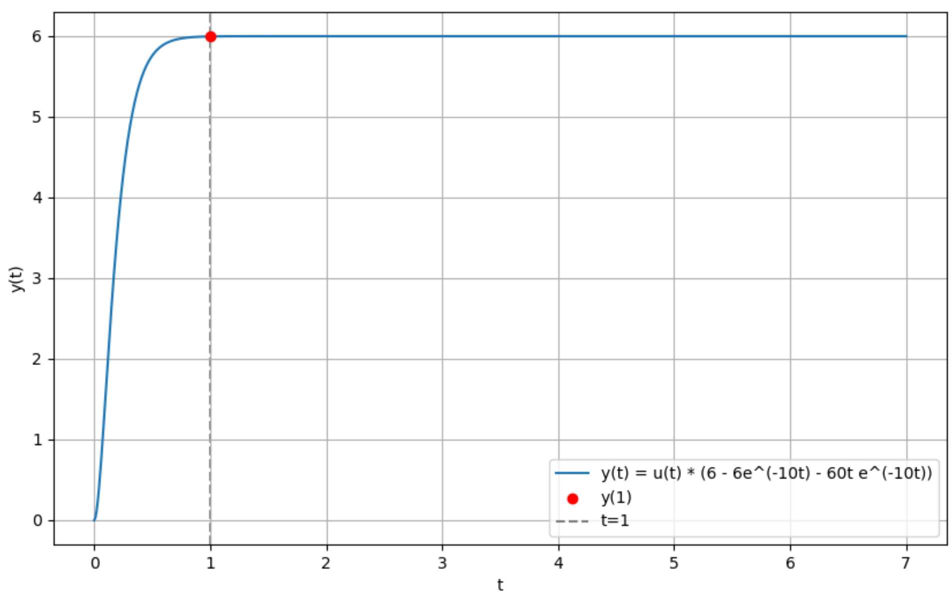


Figure 9.3: Plot of $y(t) = u(t)\left(6 - 6e^{-10t} - 60te^{-10t}\right)$

9.14 In the differential equation $\frac{dy}{dx} + \alpha xy = 0, \alpha$ is a positive constant. If $y = 1.0$ at $x = 0.0$, and $y = 0.8$ at $x = 1.0$, the value of α is (rounded off to three decimal places). (GATE CE 30 2023)

Solution:

Parameter	**Value**
x	0.0
	1.0
y	1.0
	0.8

Table 9.10: Given parameters

Let, $t = x$

$$\frac{dy}{dt} + \alpha t y = 0 \tag{9.167}$$

$$\int \frac{dy}{y} = -\int \alpha t dt \tag{9.168}$$

$$\ln(|y|) = -\frac{\alpha t^2}{2} + c \tag{9.169}$$

$$y(t) = e^c \cdot e^{-\frac{\alpha t^2}{2}} \tag{9.170}$$

Taking Fourier Transform:

where,

$$\frac{dy}{dt} \stackrel{\mathcal{F}}{\longleftrightarrow} j2\pi f Y(f) \tag{9.171}$$

$$aty(t) \stackrel{\mathcal{F}}{\longleftrightarrow} a\frac{j}{2\pi}\frac{d}{df}Y(f) \tag{9.172}$$

From equation (9.171) and (9.172):

$$\frac{4\pi^2 f}{\alpha}Y(f) + \frac{d}{df}Y(f) = 0 \tag{9.173}$$

$$Y(f) = Ke^{-\frac{4\pi^2 f^2}{2\alpha}} \tag{9.174}$$

Substituting x and y values:

$$c = \ln(1) = 0 \tag{9.175}$$

$$\alpha = -2\ln(0.8) = 0.446 \tag{9.176}$$

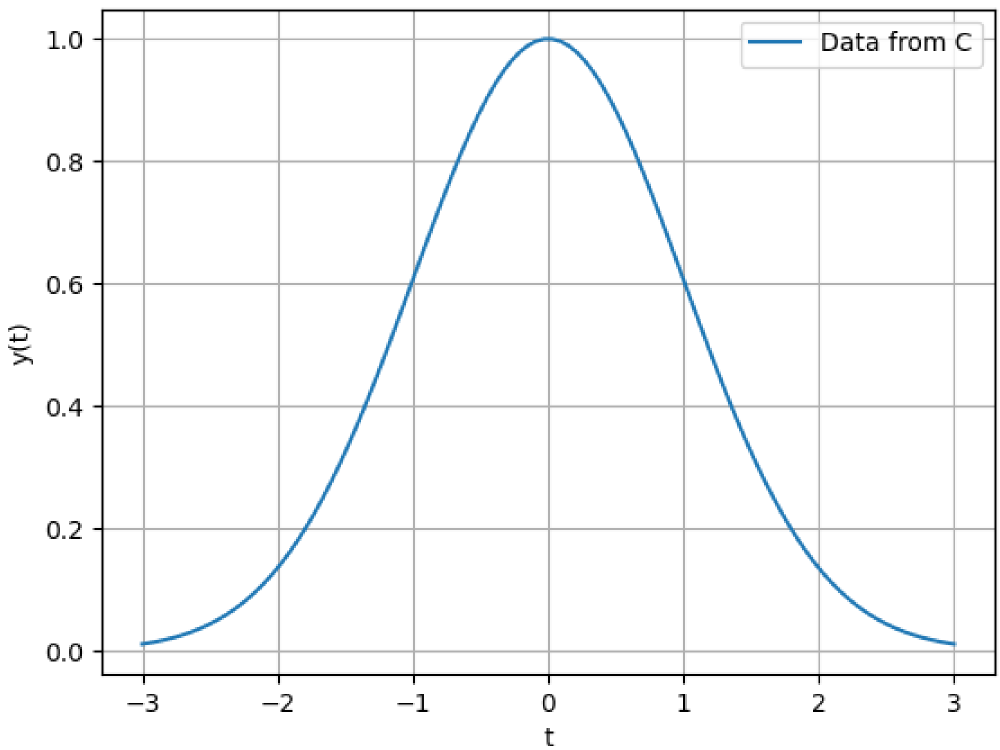


Figure 9.4: Graph of y(t)

9.15 The Fourier transform x(ω) of the signal x(t) is given by

$$X(\omega) = \begin{cases} 1, \text{for } |\omega| < \omega_0 \\ 0, \text{for } |\omega| > \omega_0 \end{cases}$$

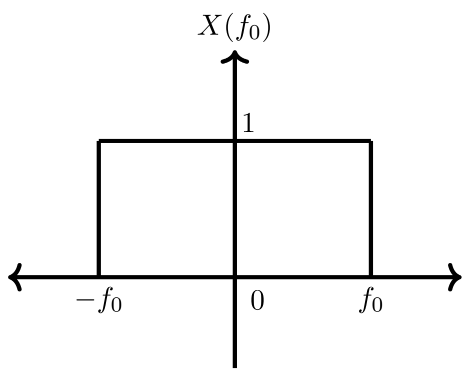
(A) x(t) tends to be an impulse as $W_0 \to \infty$.

(B) x(0) decreases as W_0 increases.

(C) At t $= \frac{\pi}{2W_0}$, x(t) = -$\frac{1}{\pi}$.

(D) At t $= \frac{\pi}{2W_0}$,   x$(t) = \frac{1}{\pi}$. (GATE EE 2023)

Solution:



By taking inverse Fourier transform,

$$x\left(t\right) = \frac{\sin\left(t\right)}{\pi t} \tag{9.177}$$

$$x\left(\frac{\pi}{2\left(2\pi f_0\right)}\right) = \frac{2\left(2\pi f_0\right)}{\pi^2} \tag{9.178}$$

So, option (C) and (D) are wrong.

$$x\left(0\right) = \lim_{t\to 0}\frac{\sin\left(2\pi f_0\right)t}{\pi t} = \frac{2\pi f_0}{\pi} \tag{9.179}$$

So, $x\left(0\right) \propto f_0 \Rightarrow$ Option (B) is wrong.

When $f_0 \rightarrow \infty$, $X\left(f_0\right)$ will be a D.C signal and inverse Fourier transform of a D.C signal will be impulse signal

So, option (A) is correct

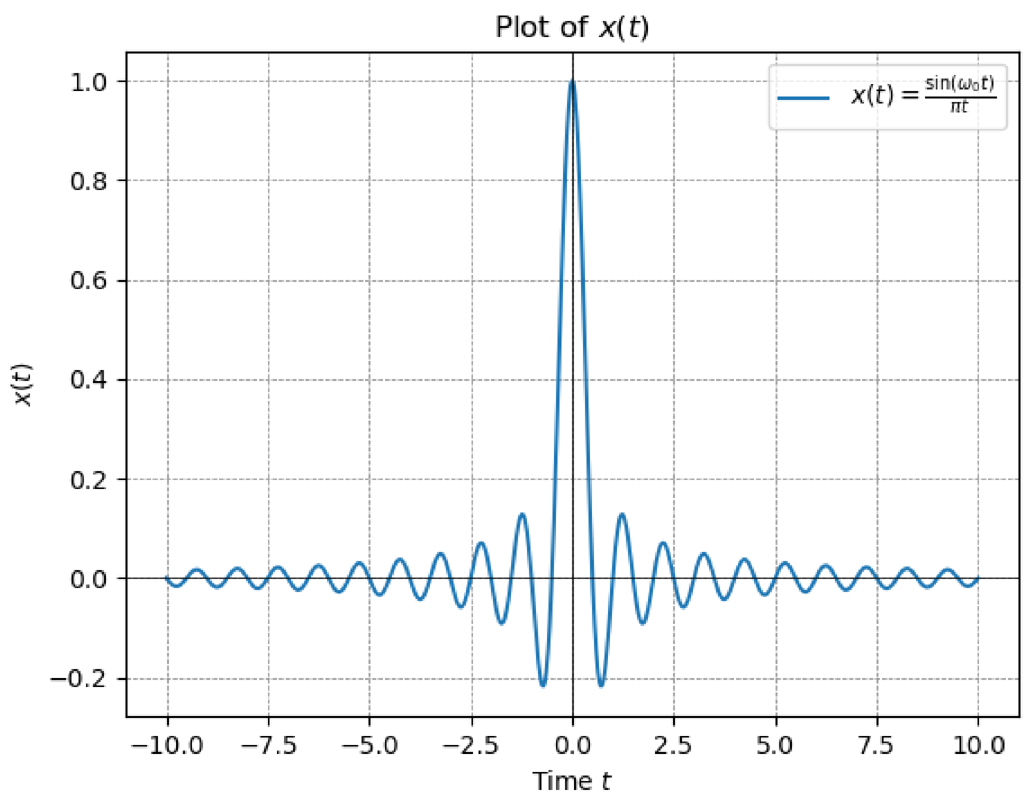


Figure 9.5: plot of X(t)

Chapter 10

FFT

10.1 The discrete-time Fourier transform of a signal x $[n]$ is $X(\Omega) = (1+\cos\Omega)\, e^{-j\Omega}$. Consider that $x_p[n]$ is a periodic signal of period $N = 5$ such that

$$x_p[n] = x[n], \text{for n= 0, 1, 2} \tag{10.1}$$

$$= 0, \text{for n= 3, 4} \tag{10.2}$$

Note that $x_p[n] = \sum_{k=0}^{N-1} a_k e^{j\frac{2\pi}{N}kn}$. The magnitude of the Fourier series coefficient a_3 is ________________ (Round off to 3 decimal places). (GATE EE 2023) **Solution:** Using Euler's form of representation of complex numbers,

$$e^{j\Omega} = \cos\Omega + j\sin\Omega \tag{10.3}$$

$X(\Omega)$ can be expressed as,

$$X(\Omega) = \frac{1}{2} + e^{-j\Omega} + \frac{e^{-j2\Omega}}{2} \tag{10.4}$$

Symbol	Value	Description
$X(\Omega)$	$(1+\cos\Omega)\, e^{-j\Omega}$	Frequency function
Ω	ωF_s	angular frequency
ω	$\omega \in (-\pi, \pi)$	radian frequency
F_s	$1Hz$	Sampling frequency
$X(\omega)$	$\sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$	D.T.F.T
$x(n)$	$x(n)$	Signal
$X(k)$	$\frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{-\frac{j2\pi}{N}kn}$	Fourier coefficient
N	5	Period of the signal

Table 10.1: variable description

As sampling frequency is $1Hz$ $(\omega = \Omega)$ from DTFT(discrete time fourier transform) we get,

$$X(\Omega) = \sum_{n=0}^{n=2} x(n) e^{-j\Omega n}, \Omega \in (-\pi, \pi) \tag{10.5}$$

$$\implies \sum_{n=0}^{n=2} x(n) e^{-j\Omega n} = \frac{1}{2} + e^{-j\Omega} + \frac{e^{-j2\Omega}}{2} \tag{10.6}$$

On comparing coefficients we get,

x(n)=$\left\{\frac{1}{2}, 1, \frac{1}{2}\right\}$

$$x_p(n) = \left\{\frac{1}{2}, 1, \frac{1}{2}, 0, 0\right\} \text{with period, N=5} \tag{10.7}$$

$$X(3) = \frac{1}{5} \sum_{n=0}^{4} x(n) e^{-\frac{j6\pi}{5} n} \tag{10.8}$$

$$|X(3)| = 0.038 \tag{10.9}$$

Fourier series coefficient	Real part	Imaginary part
X(0)	0.4000	0
X(1)	0.081	-0.249
X(2)	-0.031	-0.225
X(3)	-0.031	0.225
X(4)	0.081	0.249

Table 10.2: variable description

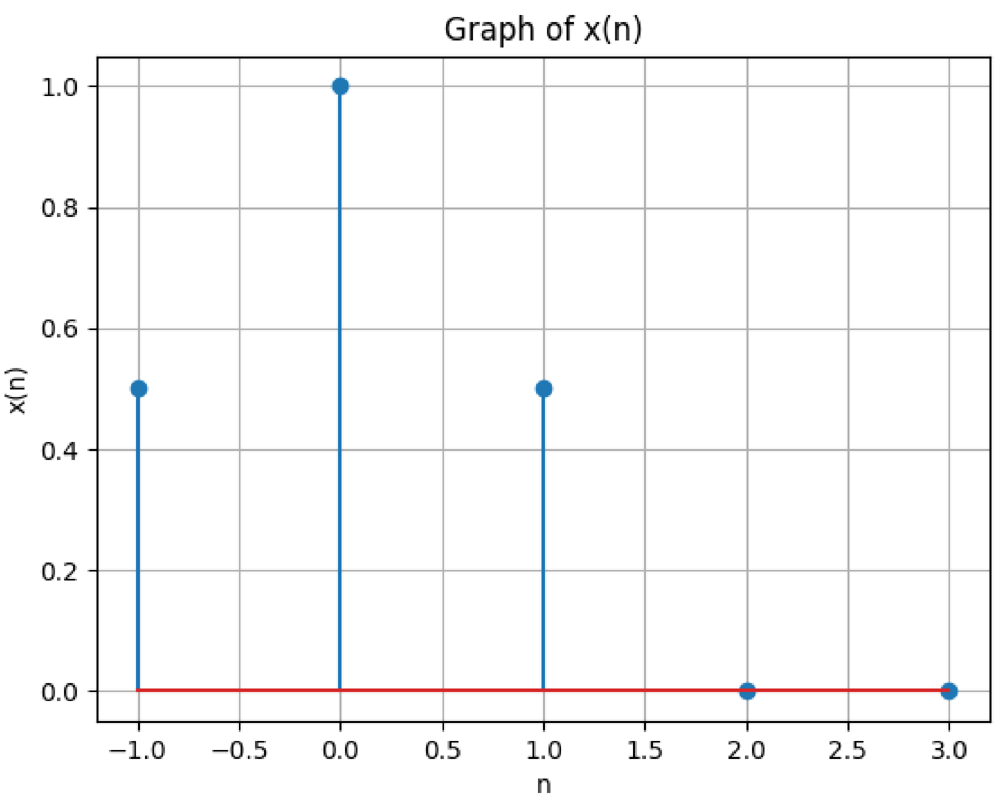


Figure 10.1: Plot of x(n) vs n

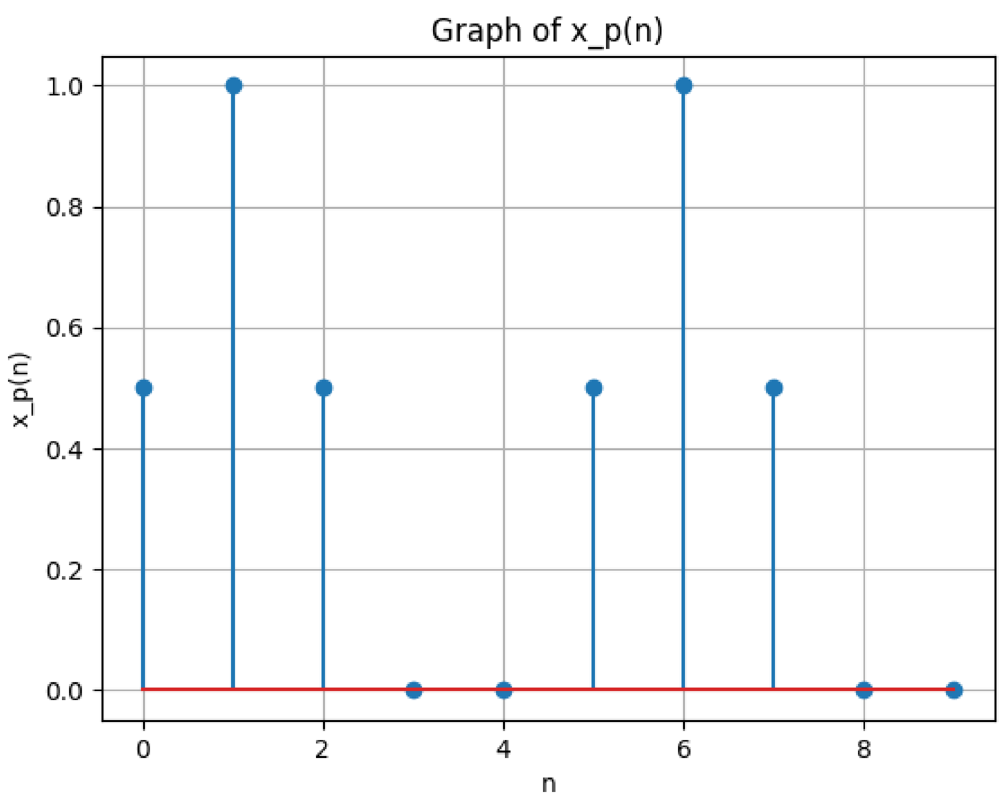


Figure 10.2: Plot of $x_p(n)$ vs n

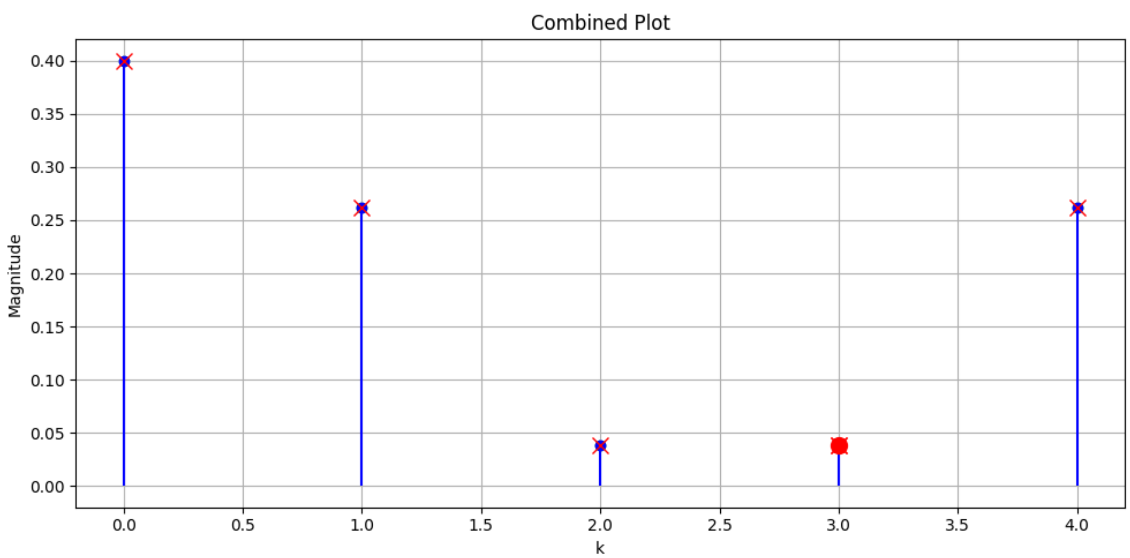


Figure 10.3: Plot of $x_p(n)$ vs n

Chapter 11

Filter Design

11.1 The time-dependent growth of a bacterial population is governed by the equation

$$\frac{dx}{dt} = x\left(1 - \frac{x}{200}\right) \tag{11.1}$$

where x is the population size at time t. The initial population size is $x_0 = 100$ at $x = 0$. As $t \to \infty$, the population size of bacteria asymptotically approaches .... (GATE BM 2023) **Solution:** The growth equation is given by,

Parameters	**Values**	**Description**
t		Time, Independent variable
$x(t)$		Population size at any time
$x(0)$	100	Initial population
h		Step size
$x(n)$		Discrete-Time approximation of $x(t)$

Table 11.1: Parameters

$$\frac{dx}{dt} = x\left(1 - \frac{x}{200}\right) \tag{11.2}$$

$$dx\,(t)dt = \frac{1}{200}x\,(200 - x) \tag{11.3}$$

$$\frac{1}{200}\left(\int_0^{x(t)} \frac{dx}{200 - x} + \int_0^{x(t)} \frac{dx}{x}\right) = \int_0^t \frac{dt}{200} \tag{11.4}$$

$$-\ln\,(200 - x)|_{100}^{x(t)} + \ln\,(x)|_{100}^{x(t)} = t \tag{11.5}$$

$$\frac{x\,(t)}{200 - x\,(t)} = e^t \tag{11.6}$$

$$\implies x\,(t) = \frac{200}{1 + e^{-t}} \tag{11.7}$$

We can also express the growth equation as,

$$\int_0^{x(t)} dx = \int_0^t xdt - \frac{1}{200}\int_0^t x^2 dt \tag{11.8}$$

Now approximating by trapezoidal rule of integration between t_{n-1} to t_n with the step size being h we have,

$$\begin{aligned} x\,(t_n) - x\,(t_{n-1}) =& \frac{h}{2}\left[x\,(t_n) + x\,(t_{n-1})\right] - \\ & \frac{h}{400}\left[x^2\,(t_n) + x^2\,(t_{n-1})\right] \end{aligned} \tag{11.9}$$

Next, replacing $t_n = hn$ we get the difference equation,

$$x\,(hn) = \frac{-\left(1 - \frac{h}{2}\right) + \sqrt{\left(1 - \frac{h}{2}\right)^2 + \frac{h}{100}p_1}}{h/200} \tag{11.10}$$

Where,

$$p_1 = \left[\left(1 - \frac{h}{2}\right)x\,(h\,(n-1)) - \frac{h}{400}x^2\,(h\,(n-1))\right] \tag{11.11}$$

And relabeling $hn \longleftrightarrow n$ we will be having the discrete time equation as,

$$x(n) = \frac{-\left(1-\frac{h}{2}\right)+\sqrt{\left(1-\frac{h}{2}\right)^2+\frac{h}{100}p_2}}{h/200} \tag{11.12}$$

Where,

$$p_2 = \left[\left(1-\frac{h}{2}\right)x(n-1)-\frac{h}{400}x^2(n-1)\right] \tag{11.13}$$

Now, plotting both the differential and the difference equations,

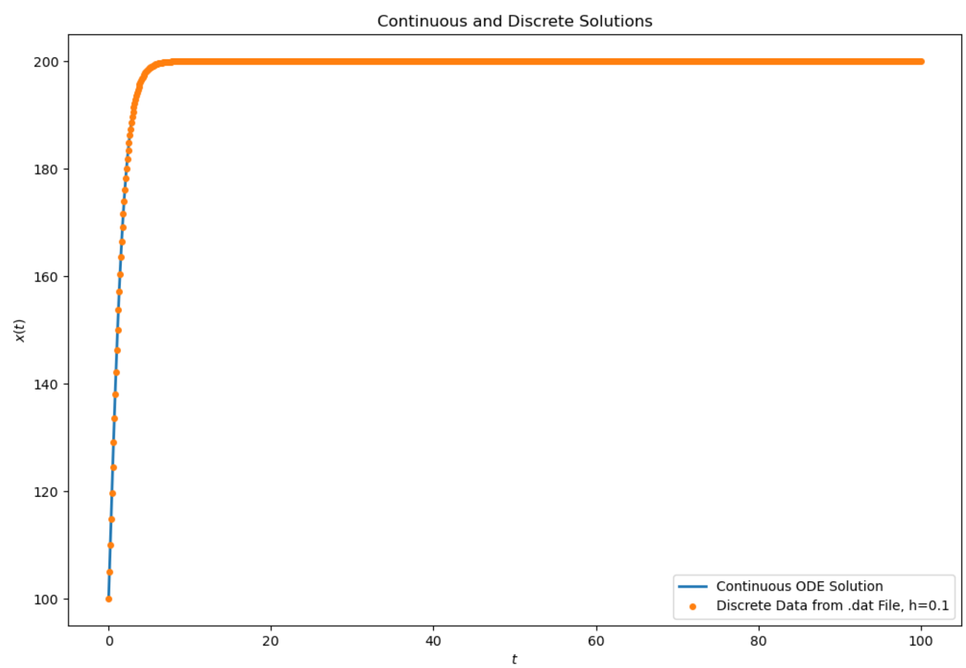


Figure 11.1: Plot in Continuous and Discrete Time

As we can see the discrete time plot is following the actual curve, which means (11.12) is indeed a good approximation of the original continuous-time equation.

And the population size approaches to 200 as $t \to \infty$.

Appendix A

Mason's Gain Formula

A.1 Mason's Gain Formula

Mason's Gain Formula, also known as Mason's Rule or the Signal Flow Graph Method, is a technique used in control systems. It provides a systematic way to analyze the transfer function of a LTI system, especially those with multiple feedback loops and complex interconnections.

parameters	**description**
$X(s)$	input signal transfer function
$Y(s)$	output signal transfer function
N	Total number of forward paths
P_i	Gain of the i^{th} forward path
$\triangle$	determinant of the graph
$\triangle_i$	path factor for the i^{th} path

Table A.1.1: Current Parameters

$$T(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{i=1}^{N} P_i \triangle_i}{\triangle} \tag{A.1.1}$$

From the signal-flow graph, we identify:

(a) Number of forward paths possible (N).

(b) Forward path gain for each path P_i.

(c) Number of individual loops in the system and there corresponding loop gain.

(d) Number of non-touching loops (i.e which do not share any common node) and their corresponding loop gain.

parameters	description
L_i	Loop gain of the i^{th} individual loop
$L_i^{'}$	Loop gain of the i^{th} non-touching loop

Table A.1.2: New Parameters

$$\triangle_i = 1 - \sum L_i \quad (L_i \text{ that doesn't touch the forward path}) \tag{A.1.2}$$

$$\triangle = 1 + \sum L_i - \sum_i \Pi_{j=0}^{i} \left(L_j^{'} \right). \tag{A.1.3}$$