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# SIGNAL PROCESSING

## Through GATE

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# Contents

Introduction	iii
<b>1 Harmonics</b>	<b>1</b>
<b>2 Filters</b>	<b>15</b>
<b>3 Z-transform</b>	<b>49</b>
<b>4 Systems</b>	<b>51</b>
<b>5 Sequences</b>	<b>95</b>
<b>6 Sampling</b>	<b>105</b>
<b>7 Contour Integration</b>	<b>113</b>
<b>8 Laplace Transform</b>	<b>117</b>
<b>9 Fourier transform</b>	<b>183</b>
<b>10 FFT</b>	<b>213</b>
<b>11 Filter Design</b>	<b>219</b>



# Introduction

This book provides solutions to signal processing problems in GATE.



# Chapter 1

## Harmonics

- 1.1 Let  $y(t)=x(4t)$ , where  $x(t)$  is a continuous-time periodic signal of 100s. the fundamental period of  $y(t)$  is (**rounded off to the nearest integer**) (GATE IN 2023)

**Solution:**

Symbol	Value	Description
$T$	100	fundamental period of $x(t)$
$T_1$		fundamental period of $y(t)$
$\omega_0$	$\frac{8\pi}{100}$	fundamental frequency of $y(t)$

Table 1: input parameters

From Table 1

Applying Fourier series:

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{j 2\pi n t}{100}} \quad (1.1)$$

$$y(t) = x(4t) \quad (1.2)$$

$$y(t) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{j 2\pi n (4t)}{100}} \quad (1.3)$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{\frac{j 2\pi n t}{25}} \quad (1.4)$$

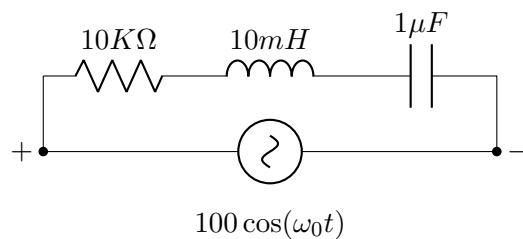
$$T_1 = 25\text{sec} \quad (1.5)$$





Figure 1: plot  $y(t)$  v/s  $t$

- 1.2 In the circuit shown below, it is observed that the amplitude of voltage across the resistor is the same as the amplitude of the source voltage. What is the angular frequency  $\omega_0$  (in rad/s)?



(GATE BM 2023) **Solution:**

- 1.3 For a regular sinusoidal wave propagating in deep water having wave height of 3.5 m and wave period of 9 s, the wave steepness is \_\_\_\_\_ (round off to three decimal places). Gate 2023 NM 33

**Solution:**

Symbol	Value	Description
$H$	3.5m	wave height
$T$	9s	wave period
$S$	?	wave steepness
$\lambda$		wave length
$\eta$		surface elevation of water

Table 1.2: Input Parameters

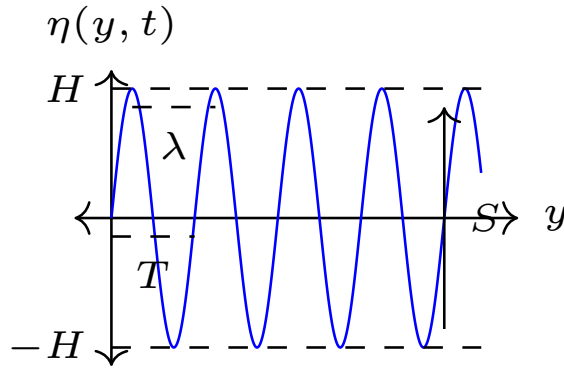


Figure 1.2: Sinusoidal wave

- (a) Deriving the formula for wavelength of deep water wave:

$$S = \frac{H}{\lambda} \quad (1.6)$$

Let's start with the linearized shallow water wave equation.

$$\frac{\partial^2 \eta}{\partial t^2} = g \frac{\partial \eta}{\partial y} \quad (1.7)$$

$$\eta = A \sin(ky - \omega t) \quad (1.8)$$

$$\frac{\partial^2 \eta}{\partial t^2} = -(\omega)^2 A \sin(ky - \omega t) \quad (1.9)$$

For deep water waves:

$$\frac{\partial \eta}{\partial y} \approx -k\eta \quad (1.10)$$

Using the equation (1.7).

$$\frac{\partial^2 \eta}{\partial t^2} = -gk\eta \quad (1.11)$$

$$= -gkA \sin(ky - \omega t) \quad (1.12)$$

where, k is wave number.

Comparing equations (1.9) and (1.12),

$$\omega^2 = gk \quad (1.13)$$

$$\omega = \frac{2\pi}{T} \quad (1.14)$$

$$k = \frac{2\pi}{\lambda} \quad (1.15)$$

$$\lambda = \frac{g \cdot T^2}{2\pi} \quad (1.16)$$

(b) Numerical computation:

$$\lambda = \frac{g \cdot T^2}{2\pi} \tag{1.17}$$

$$= \frac{9.81 \, (9)^2}{2\pi} \tag{1.18}$$

$$= 126.53m \tag{1.19}$$

Using the equation (1.6).

$$S = \frac{3.5}{126.53} \tag{1.20}$$

$$= 0.028 \tag{1.21}$$

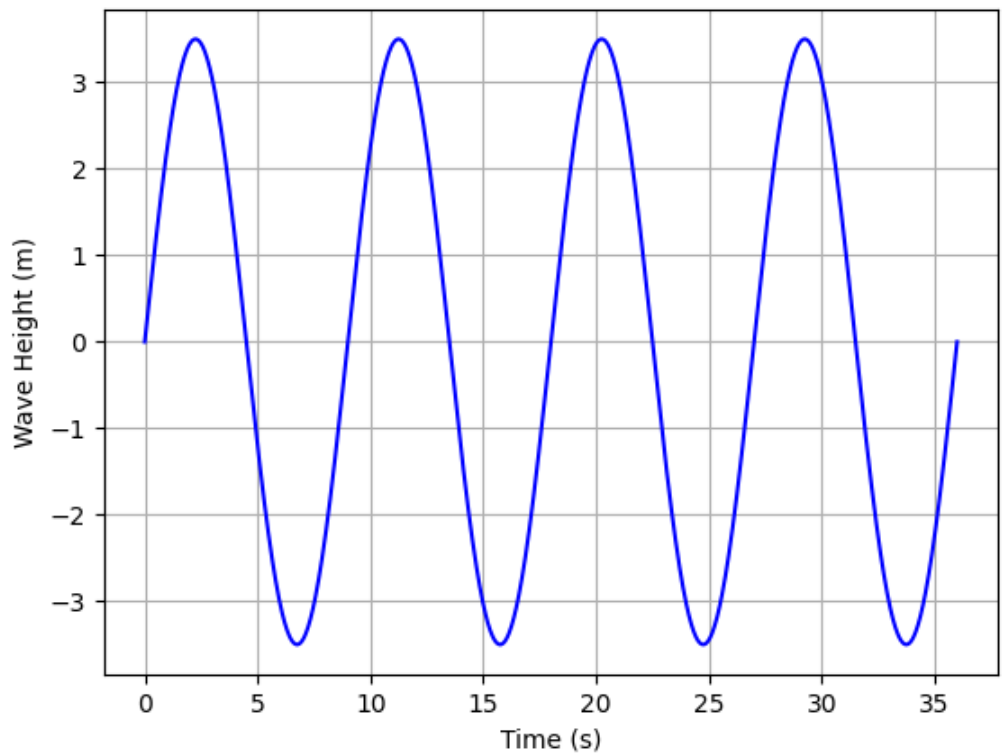


Figure 1.3: Sinusoidal wave

1.4 A spring mass system is shown in the figure . Take the value of acceleration due to gravity as  $g = 9.81m/s^2$ .The static deflection due to weight and the time period of the oscillations,respectively,are

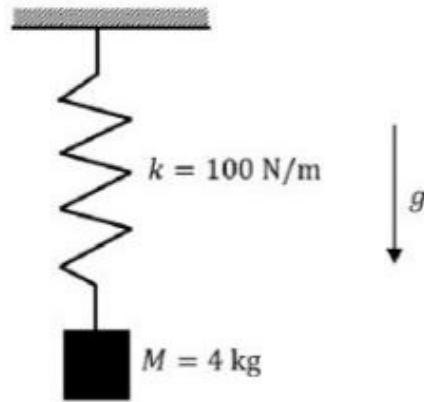
(GATE 2023 XE)

**Solution:**

(a) Static deflection due to weight(sdw)

let x be sdw.

0 At mean position in equilibrium



1.3

$$Mg = kx \quad (1.22)$$

$$x = 39.24 \text{ cm} \quad (1.23)$$

(b) Time period of oscillation

$$F = -kx \quad (1.24)$$

$$m \left( \frac{d^2x}{dt^2} \right) = -kx \quad (1.25)$$

Initial Conditions be at extreme point of SHM

$$x(0) = 0.3924 \quad (1.26)$$

$$\frac{dx}{dt} = 0 \text{ at } t = 0 \text{ (released from rest)} \quad (1.27)$$

Taking Laplace transform:

$$m(s^2 X(s) - sx(0) - mx'(0)) + kX(s) = 0 \quad (1.28)$$

$$X(s) = \frac{x(0)ms + mx'(0)}{ms^2 + k} \quad (1.29)$$

$$X(s) = x(0) \frac{s}{s^2 + \frac{k}{m}} + \left( x'(0) \sqrt{\frac{k}{m}} \right) \frac{\sqrt{\frac{k}{m}}}{s^2 + \left( \sqrt{\frac{k}{m}} \right)^2} \quad (1.30)$$

Taking Inverse Laplace Transform:

$$x(t) = x(0) \cos \left( \sqrt{\frac{k}{m}} t \right) + \left( x'(0) \sqrt{\frac{k}{m}} \right) \sin \left( \sqrt{\frac{k}{m}} t \right) \quad (1.31)$$

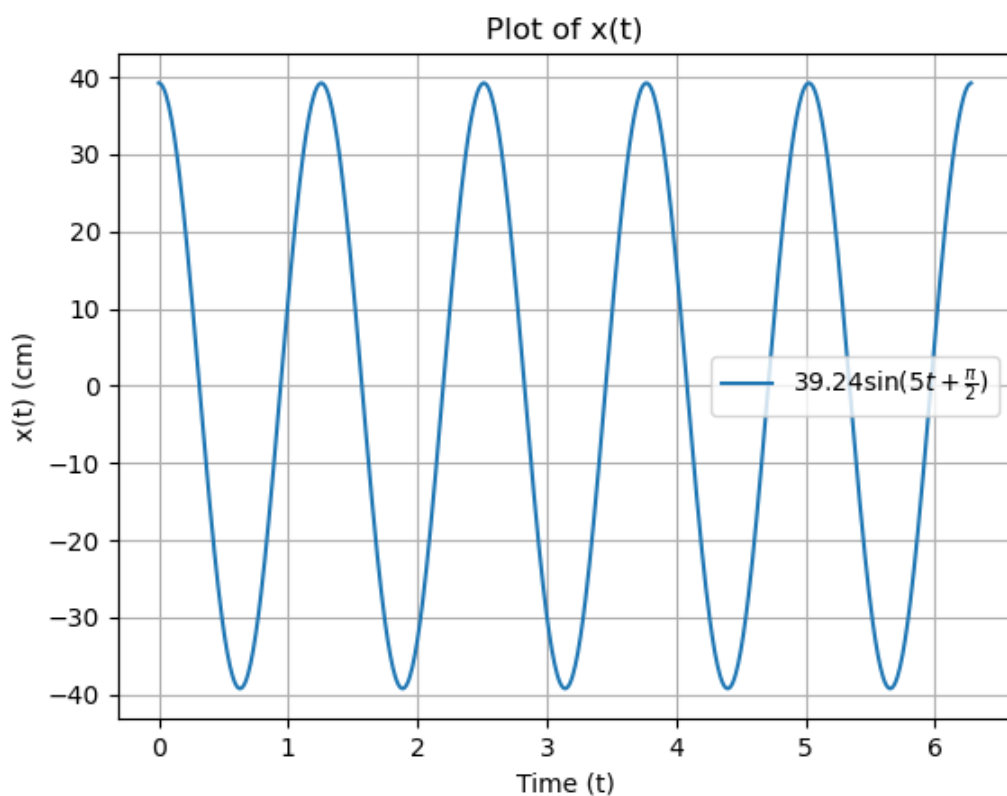
$$(1.32)$$

Using 1.26 and 1.27

$$x(t) = 0.3924 \cos \left( \sqrt{\frac{k}{m}} t \right) \quad (1.33)$$

$$x(t) = 39.24 \sin \left( 5t + \frac{\pi}{2} \right) \text{ cm} \quad (1.34)$$

The static deflection due to weight and the time period of the oscillations, respectively are 39.24 cm and  $\frac{2\pi}{5}$  s

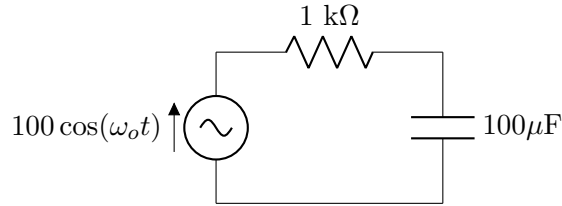


Variable	Description	Value
$M$	weight of block	4 kg
$K$	spring constant	$100\frac{N}{m}$
$x$	Static deflection due to weight	39.24 cm
$x(t)$	Displacement of particle from mean position at time $t$	none
$x(0)$	Initial Displacement of particle from mean position	39.24cm
$x'(t)$	velocity of particle	none
$x'(0)$	initial velocity of particle	0

Table 1.3: input parameters

- 1.5 In the circuit shown below, the amplitudes of the voltage across the resistor and the capacitor are equal. What is the value of the angular frequency  $\omega_o$  (in rad/s)? (Round off the answer to one decimal place.) (GATE BM 32 2023)





**Solution:**

Parameter	Value	Description
$v(t)$	$100 \cos(\omega_0 t)$	Input Voltage
$R$	$1 \text{ k}\Omega$	Resistance
$C$	$100 \mu\text{F}$	Capacitance
$\omega_0$	?	Angular Frequency
$Z_R = R$	$10^3$	Impedance for resistor
$Z_C = \frac{1}{j\omega C}$	$\frac{10^4}{j\omega_0}$	Impedance for capacitor
$Z = R + \frac{1}{j\omega C}$	$10^3 + \frac{10^4}{j\omega_0}$	Total Impedance

Table 1.4: Parameter Table

$$R \xleftrightarrow{\mathcal{F}} R \quad (1.35)$$

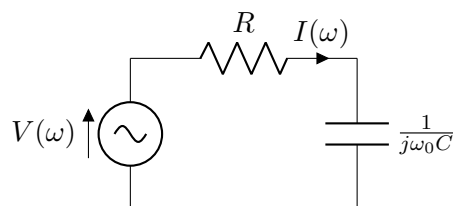
$$C \xleftrightarrow{\mathcal{F}} \frac{1}{j\omega_0 C} \quad (1.36)$$

$$|V_R(\omega)| = |V_C(\omega)| \quad (1.37)$$

$$\Rightarrow |Z_R| = |Z_C| \quad (1.38)$$

$$10^3 = \frac{10^4}{\omega_0} \quad (1.39)$$

$$\therefore \omega_0 = 10.0 \quad (1.40)$$



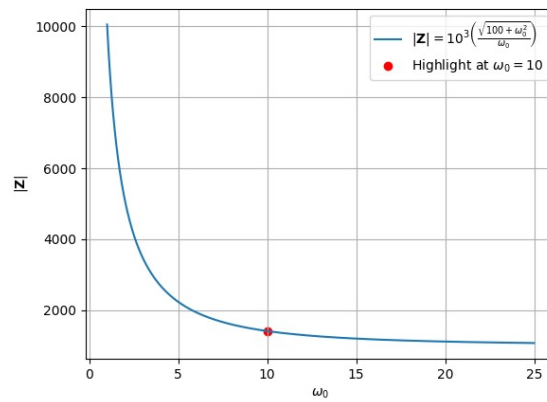


Figure 1.4: Plot of  $|Z| = 10^3 \left( \frac{\sqrt{100 + \omega_0^2}}{\omega_0} \right)$

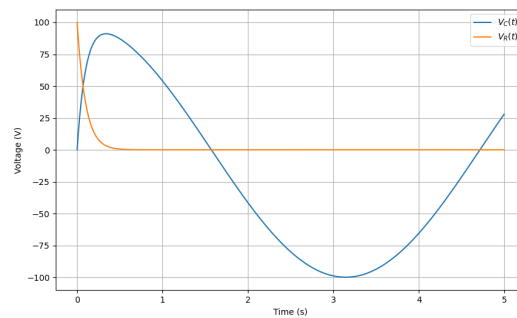


Figure 1.5: Plot of Voltage Across Capacitor and Resistor

1.6 Let  $w^4 = 16j$ . Which of the following can not be the value of  $w$ ?

52 (A)  $2e^{\frac{j2\pi}{8}}$

53 (B)  $2e^{\frac{j\pi}{8}}$

54 (C)  $2e^{\frac{j5\pi}{8}}$

55 (D)  $2e^{\frac{j9\pi}{8}}$

(GATE 2023 EC)

**Solution:**

$$(w^4)^{\frac{1}{4}} = (16j)^{\frac{1}{4}} \quad (1.41)$$

Using De-Moivre's theorem for  $n^{th}$  root of  $w$  ,

$$w = 2j^{\frac{1}{4}} \quad (1.42)$$

$$e^{j\theta} = \cos \theta + j \sin \theta \quad (1.43)$$

Using equation (1.43) and put  $\theta = (2n + 1) \frac{\pi}{2}$

$$w = 2e^{[j(2n+1)\frac{\pi}{2}]^{\frac{1}{4}}} \quad (1.44)$$

For different values of n ,

$$n = 0 \implies w = 2e^{\frac{j\pi}{8}} \quad (1.45)$$

$$n = 2 \implies w = 2e^{\frac{j5\pi}{8}} \quad (1.46)$$

$$n = 4 \implies w = 2e^{\frac{j9\pi}{8}} \quad (1.47)$$

Ans . (A)  $2e^{\frac{j2\pi}{8}}$



## Chapter 2

# Filters

2.1 For the circuit given below, choose the angular frequency  $\omega_0$  at which voltage across capacitor has maximum amplitude?



Figure 2.1: circuit

- (A) 1000
- (B) 100
- (C) 1
- (D) 0

(GATE BM 2023 Question 16)

**Solution:**

Parameter	Description	Value
$V_i(j\omega)$	Input voltage	100
$v_c(t)$	Potential difference across Capacitor	?
$V_c(s)$	Potential difference across Capacitor	$V_c(s)$
$H(s)$	Transfer function	$\frac{V_c(s)}{V_i(s)}$
$V_o$	Amplitude of input voltage	100 V
$R$	Resistance in circuit	1 k $\Omega$
$C$	Capacitance in circuit	100 $\mu$ F
$\omega_o$	Angular frequency of input voltage	$\omega_o$

Table 2.1: input values

$$V_c(s) = \frac{V_i(s) \frac{1}{sC}}{R + \frac{1}{sC}} \quad (2.1)$$

$$\Rightarrow H(s) = \frac{1}{1 + sRC} \quad (2.2)$$

$$\Rightarrow H(j\omega) = \frac{1}{1 + j\omega RC} \quad (2.3)$$

$$|H(j\omega)| = \frac{1}{\sqrt{1 + (\omega RC)^2}} \quad (2.4)$$

$$v_c(t) = \frac{100}{\sqrt{1 + (\omega_o RC)^2}} \left( \cos \omega_o t + \arctan \left( \frac{1}{\omega_o RC} \right) \right) \quad (2.5)$$

Maximum amplitude of  $v_c(t)$  occurs at  $\omega_o = 0$

$$\therefore \omega_o = 0 \quad (2.6)$$

$\therefore$  maximum value of  $v_c(t)$  at steady state is 100 Volts.

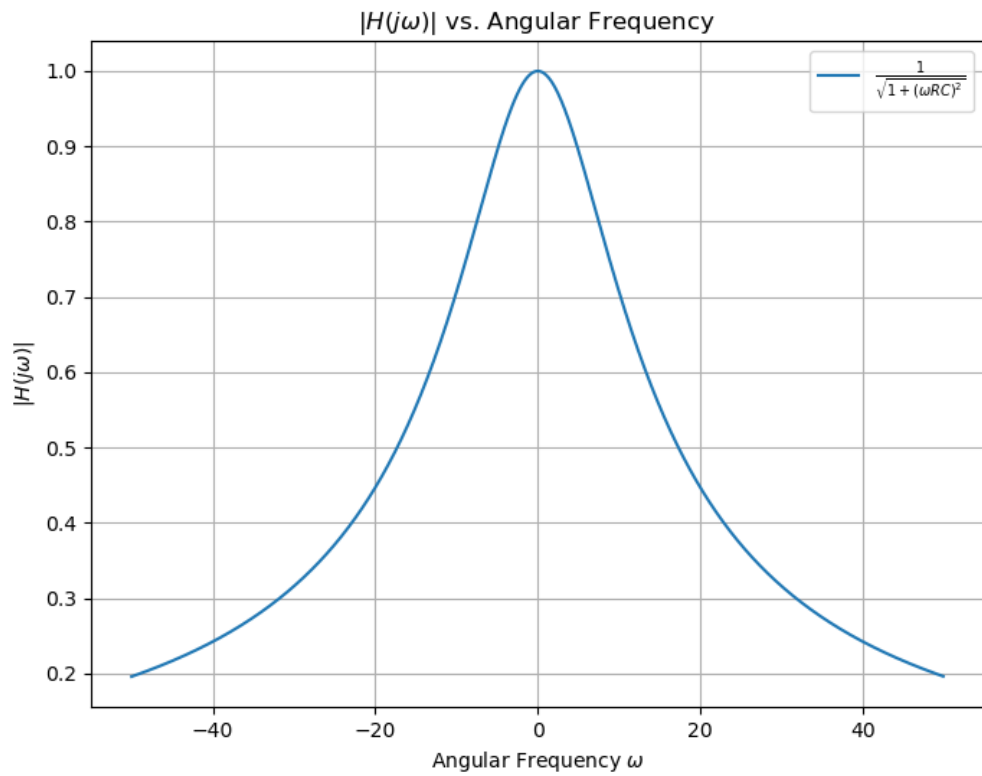


Figure 2.2:  $|H(j\omega)|$

2.2 In the following circuit, the switch S is open for  $t < 0$  and closed for  $t \geq 0$ . What is the steady state voltage (in Volts) across the capacitor when the switch is closed?

(GATE BM 2023 Question 30)

**Solution:**

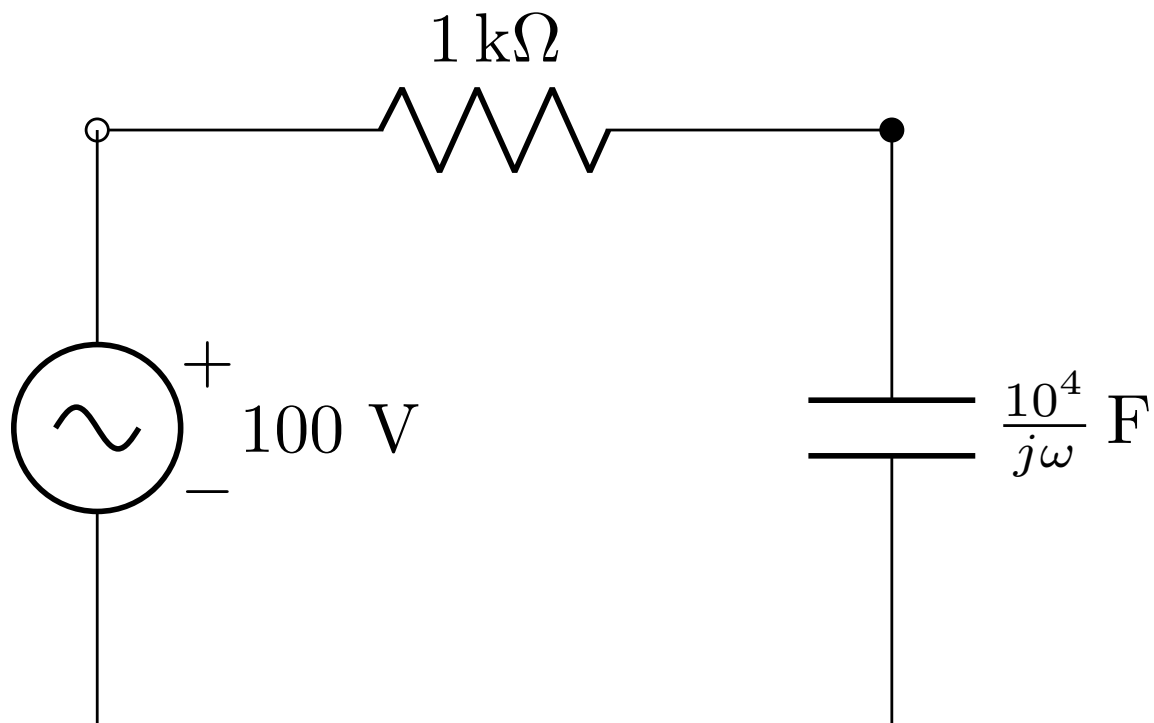


Figure 2.3: circuit in  $\omega$ -domain

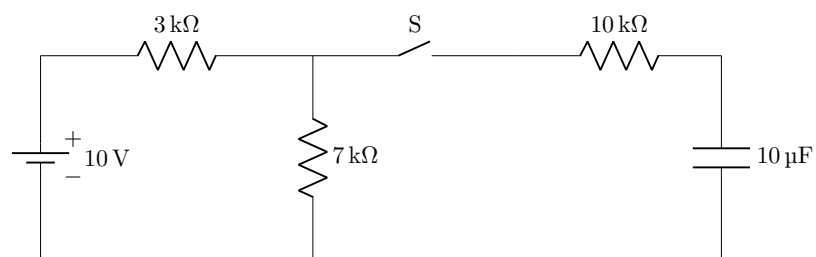


Figure 2.4: circuit

In steady state, no current flows through the capacitor.

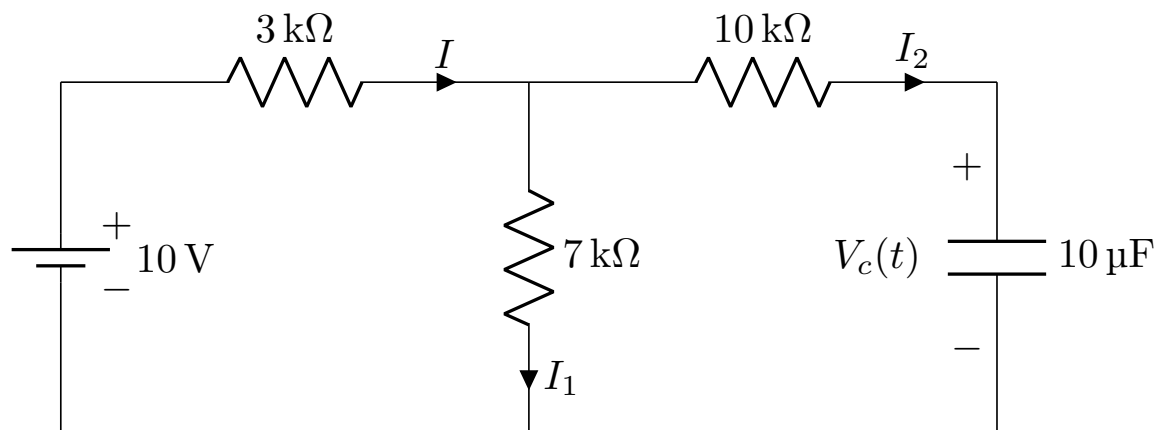
$$I_2 = 0 \quad (2.7)$$

$$V_c = (7\text{k}\Omega) I_1 \quad (2.8)$$

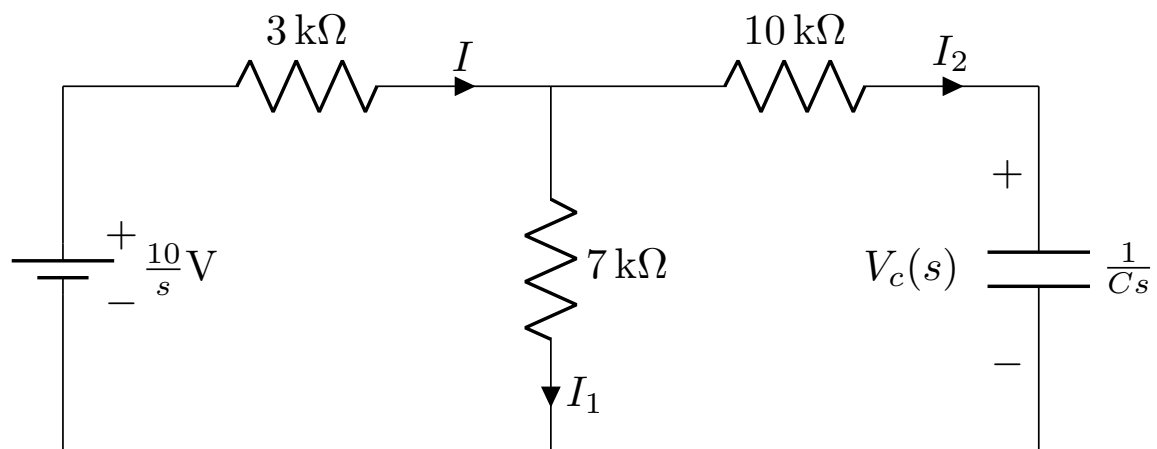
$$= (7\text{k}\Omega) I \quad (2.9)$$

$$\frac{10\text{V}}{18\text{k}\Omega} (7\text{k}\Omega) = \frac{10\text{V}}{10\text{k}\Omega} \quad (2.10)$$





In s-domain:



$$\Rightarrow I(s) = \frac{\frac{10}{s}V}{3k\Omega + \frac{(7k\Omega)(10k\Omega + \frac{1}{sC})}{17k\Omega + \frac{1}{sC}}} \quad (2.12)$$

$$I = I_1 + I_2 \quad (2.13)$$

$$I_1(7k\Omega) = I_2 \left( 10k\Omega + \frac{1}{sC} \right) \quad (2.14)$$

$$I_2(s) = \frac{7k\Omega}{17k\Omega + \frac{1}{sC}} I(s) \quad (2.15)$$

$$\Rightarrow I_2(s) = \frac{7(10^{-5})}{0.121s + 1} \quad (2.16)$$

$$V_c(s) = I_2(s) \frac{1}{sC} \quad (2.17)$$

$$= \frac{7}{s(0.121s + 1)} \quad (2.18)$$

$$= 7 \left( \frac{1}{s} - \frac{1}{s + \frac{1}{0.121}} \right) \quad (2.19)$$

Taking inverse Laplace transform:

$$V_c(t) = 7u(t) \left( 1 - e^{-\frac{t}{0.121}} \right) \quad (2.20)$$

In steady state  $t \rightarrow \infty$ . From (2.20):

$$\lim_{t \rightarrow \infty} V_c(t) = 7V \quad (2.21)$$

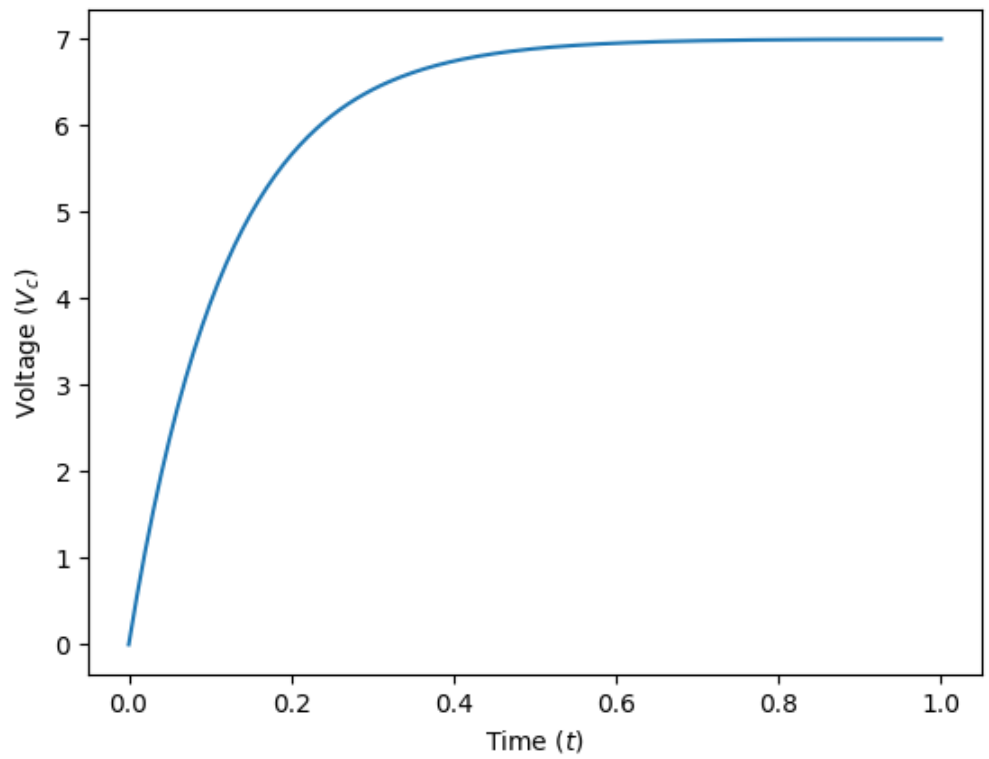


Figure 2.5:  $V_c(t)$

2.3 A finite impulse response (FIR) filter has only two non-zero samples in its impulse response  $h[n]$ , namely  $h[0] = h[1] = 1$ . The Discrete Time Fourier Transform (DTFT) of  $h[n]$  equals  $H(e^{j\omega})$ , as a function of the normalized angular frequency  $\omega$ . For the range  $|\omega| \leq \pi$ ,  $|H(e^{j\omega})|$  is equal to

(A)  $2|\cos(\omega)|$

(B)  $2|\sin(\omega)|$

(C)  $2|\cos(\frac{\omega}{2})|$

(D)  $2|\sin(\frac{\omega}{2})|$

(GATE BM 2023 Question 17)

**Solution:**

Parameter	Value	Description
$h[n]$	-	impulse response
$h[0]$	1	impulse response at $n = 0$
$h[1]$	1	impulse response at $n = 1$
$\omega$	$-\pi \leq \omega \leq \pi$	normalized frequency
$H(e^{j\omega})$	$\sum_{n=0}^M h[n]e^{-jn\omega}$	frequency response

Table 2.2: Input Parameters Table

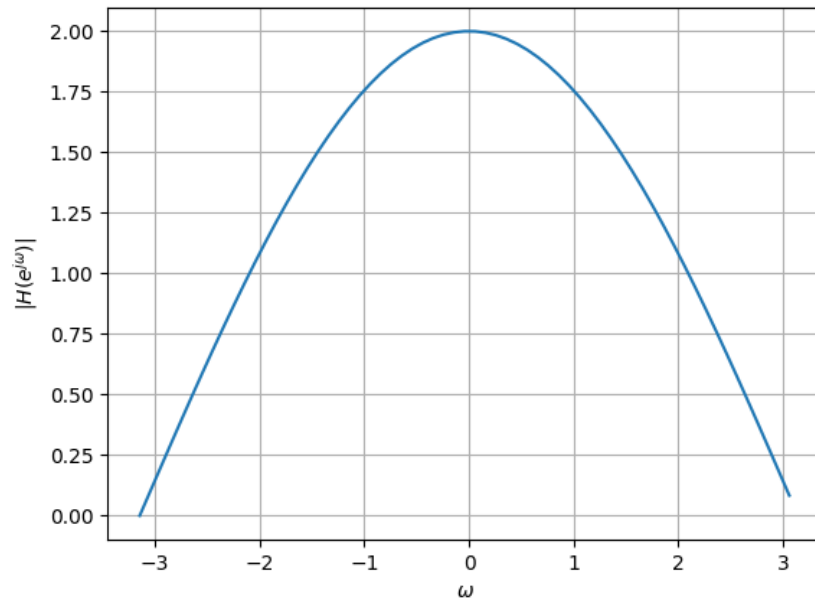
From Table 2.2,

$$H(e^{j\omega}) = 1 + e^{-j\omega} \quad (2.22)$$

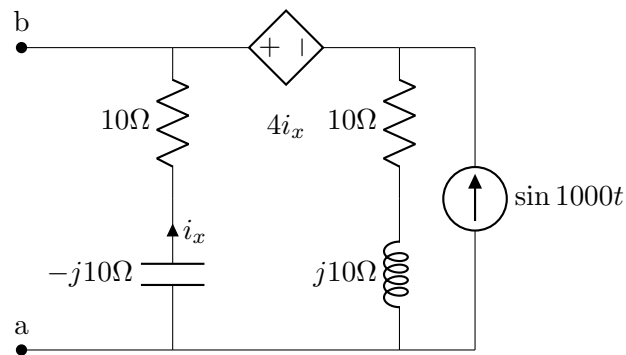
$$= e^{\frac{-j\omega}{2}} (e^{\frac{j\omega}{2}} + e^{\frac{-j\omega}{2}}) \quad (2.23)$$

$$= e^{\frac{-j\omega}{2}} (2 \cos\left(\frac{\omega}{2}\right)) \quad (2.24)$$

$$|H(e^{j\omega})| = 2 \left| \cos\left(\frac{\omega}{2}\right) \right| \quad (2.25)$$

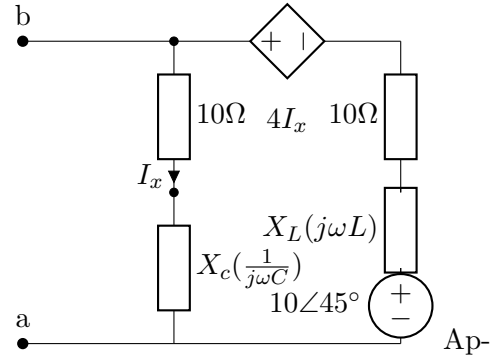
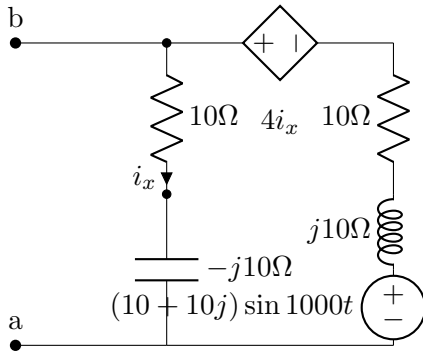


2.4 For the circuit shown, if  $i = \sin 1000t$ , the instantaneous value of the Thevenin's voltage (in volts) across the terminals a and b at time  $t=5\text{ms}$  is



(GATE EE 2023 )

**Solution:** By source transforming the given circuit we get



Solving using sinusoidal steady state analysis,  
 applying KVL we get,

$$10\angle 45^\circ - (j\omega L)I_x - 10I_x + 4I_x - 10I_x - \left(\frac{1}{j\omega C}\right)I_x = 0 \quad (2.26)$$

$$I_x = \frac{10\angle 45^\circ}{16 + j\omega L + \frac{1}{j\omega C}} \quad (2.27)$$

$$V_{ab} = I_x \left(10 + \frac{1}{j\omega C}\right) \quad (2.28)$$

$$V_{ab} = \frac{10\angle 45^\circ (10j\omega C + 1)}{16j\omega C - \omega^2 LC + 1} \quad (2.29)$$

From the question we can observe that,

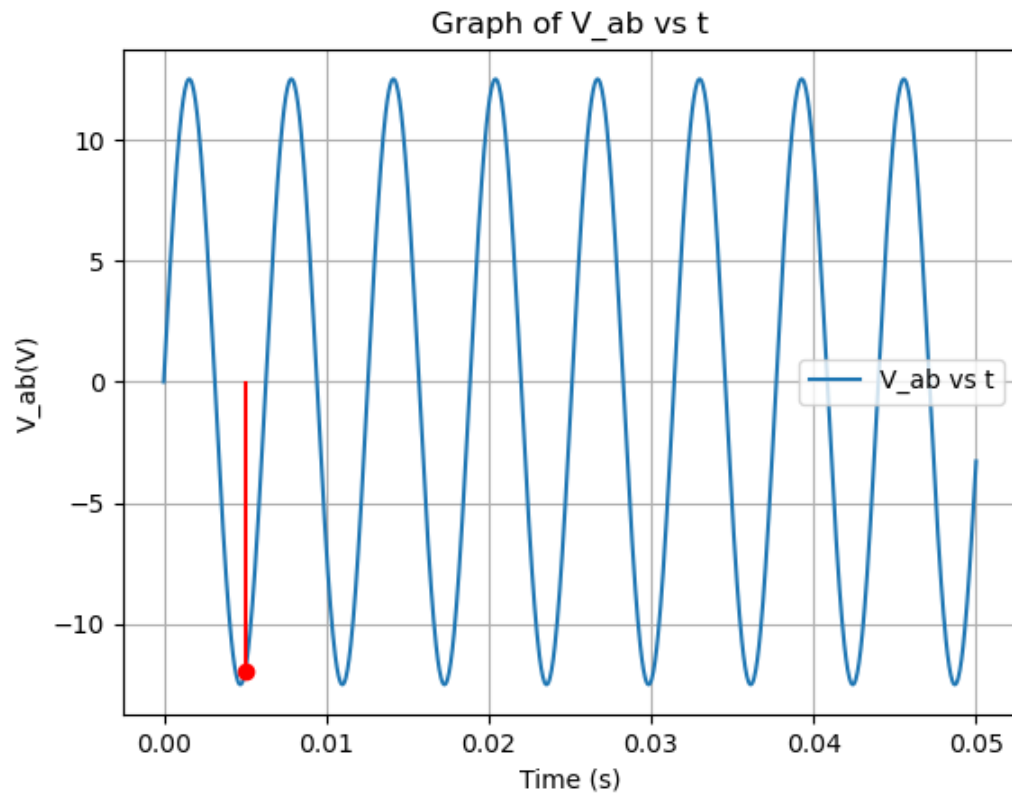
$$\frac{1}{\omega C} = \omega L \quad (2.30)$$

$$\omega^2 LC = 1 \quad (2.31)$$

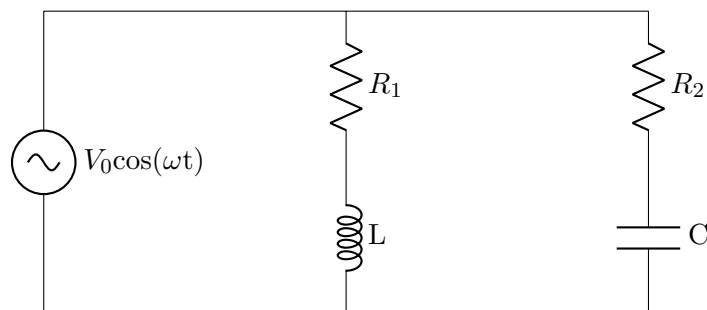
Substituting 2.31 and values of  $\omega C$  and  $\omega L$

$$V_{ab} = 12.5 \angle 0^\circ \quad (2.32)$$

$$V_{ab} = 12.5 \sin 1000t \quad (2.33)$$



2.5 In the circuit shown , $\omega = 100\pi\text{rad/s}$ ,  $R_1=R_2=2.2\Omega$  and  $L=7\text{mH}$ . the capacitance  $C$  for which  $Y_{in}$  is purely real is mF





(GATE IN 2023 Q46)

**Solution:**

variable	value	description	formulae
$Y_{in}$	-	Admittance of circuit	$\frac{R_1 - Ls}{R_1^2 - (Ls)^2} + \frac{R_2 - \frac{1}{sC}}{R_2^2 - (\frac{1}{sC})^2}$
$X_L$	$7s\Omega$	Inductive reactance	$sL$
$X_C$	$\frac{1}{sC}\Omega$	Capacitive reactance	$\frac{1}{sC}$
$s$	$100\pi j$	Laplace complex frequency	$j\omega$
$\omega$	$100\pi \text{rads/s}$	Angular frequency	-
$V$	$V_0 \cos(\omega t)$	voltage of source	-
$R_1, R_2$	$2.2\Omega$	resistance of resistors	-

Table 2.3: Table: Input Parameters

From *Table 2.3*

$$Y_{in} = \frac{R_1 - Ls}{R_1^2 - (Ls)^2} + \frac{R_2 - \frac{1}{sC}}{R_2^2 - (\frac{1}{sC})^2} \quad (2.34)$$

$$Im(Y_{in}) = \frac{-Ls}{R_1^2 - (Ls)^2} + \frac{-\frac{1}{sC}}{R_2^2 - (\frac{1}{sC})^2} \quad (2.35)$$

According to the question given,  $Y_{in}$  is purely real , so imaginary part should be equal to zero

Take the values from *Table 2.3*

$$\frac{-1}{4.4} + \frac{\frac{1}{(100\pi)C}}{(2.2)^2 + \left(\frac{1}{(100\pi)C}\right)^2} = 0 \quad (2.36)$$

$$\frac{\frac{1}{(100\pi)C}}{(2.2)^2 + \left(\frac{1}{(100\pi)C}\right)^2} = \frac{1}{4.4} \quad (2.37)$$

$$(2.2)^2 - \frac{4.4}{(100\pi)C} + \left(\frac{1}{(100\pi)C}\right)^2 = 0 \quad (2.38)$$

$$\left(2.2 - \frac{1}{(100\pi)C}\right)^2 = 0 \quad (2.39)$$

$$\frac{1}{(100\pi)C} = 2.2 \quad (2.40)$$

$$C = \frac{700}{484} \text{mF} \quad (2.41)$$

$$C = 1.446281 \text{mF} \quad (2.42)$$

The capacitance of capacitor C is 1.45mF

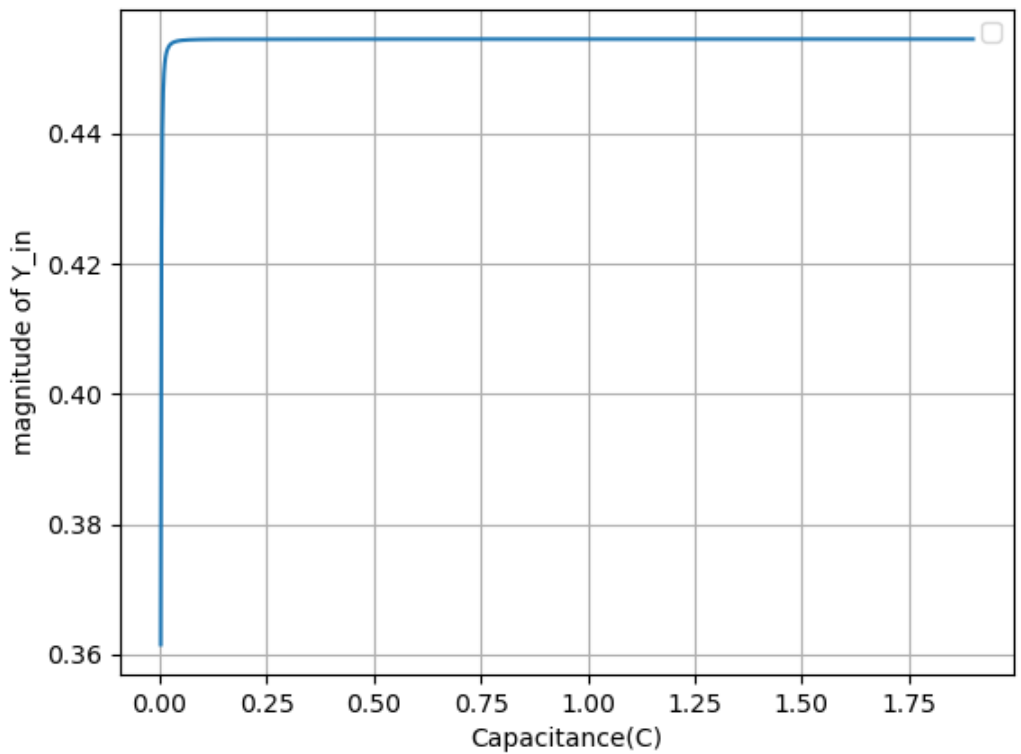


Figure 2.6: the plot of capacitance vs magnitude of  $Y_{in}$

2.6 An input voltage in the form of a square wave of frequency  $1\text{ kHz}$  is given to a circuit, which results in the output shown schematically below. Which one of the following options is the CORRECT representation of the circuit? (GATE PH 2023 Q37)

- (a)
- (b)
- (c)
- (d)

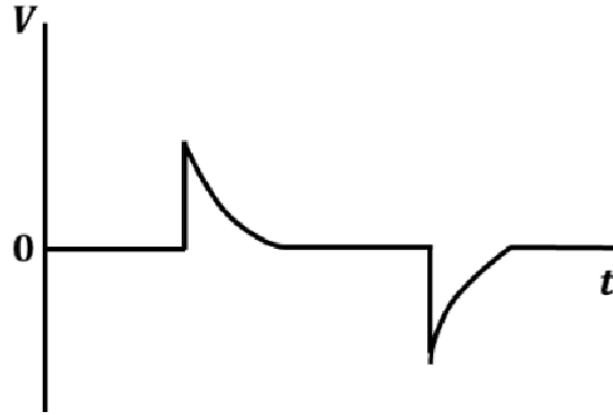
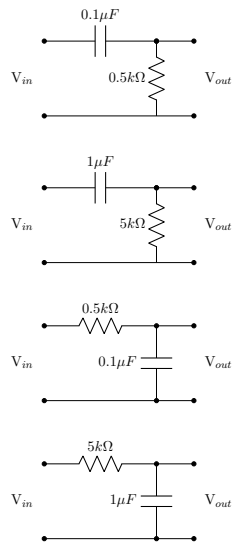


Figure 2.7:



(GATE 2023 PH 37)

**Solution:**

Input waveform is a square wave (Fig. 2.8), so we take its Fourier Transform

$$V_{in}(t) = 2 \left( 2 \left[ \frac{(t - \frac{T}{4})}{T} \right] - \left[ \frac{2(t - \frac{T}{4})}{T} \right] \right) + 1 \quad (2.43)$$

Symbol	Value	Description
$V_{in}(t)$		Input Voltage
$\mathcal{V}_{in}(j\omega)$		Fourier Transform of $V_{in}(t)$
$V_{out}(t)$		Output Voltage
$\mathcal{V}_{out}(j\omega)$		Fourier Transform of $V_{out}(t)$
$f$	$\frac{\omega}{2\pi} = 1000Hz$	Input Wave Frequency
$T$	$\frac{2\pi}{\omega} = 10^{-3}s$	Input Wave Time Period
$R$	(a) $0.5k\Omega$	Resistance
	(b) $5k\Omega$	
$C$	(a) $0.1\mu F$	Capacitance
	(b) $1\mu F$	
$\tau$	$RC$	Time Constant
$Z$	$R + \frac{1}{j\omega C}$	Impedance
$H(j\omega)$	$\frac{V_{out}}{V_{in}}$	General Transfer Function
$H_R(j\omega)$	$\frac{V_{R,out}}{V_{in}}$	Transfer Function for Resistor
$H_C(j\omega)$	$\frac{V_{C,out}}{V_{in}}$	Transfer Function for Capacitor

Table 2.4: Given Parameters

Fourier Series Coefficient:

$$c_k = \frac{1}{T} \int_T V_{in}(t) e^{-jk\omega t} dt \quad (2.44)$$

As square wave is even,  $\sin(k\omega t)$  terms become zero. Cosine coefficients are:

$$a_n = \frac{2}{T} \int_T V_{in}(t) \cos\left(\frac{2\pi nt}{T}\right) dt \quad (2.45)$$

$$= \frac{4}{n\pi} \sin\left(\frac{n\pi}{2}\right) \cos(n\pi) \quad (2.46)$$

Fourier Series of  $V_{in}(t)$ :

$$V_{in}(t) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nt}{T}\right) \quad (2.47)$$

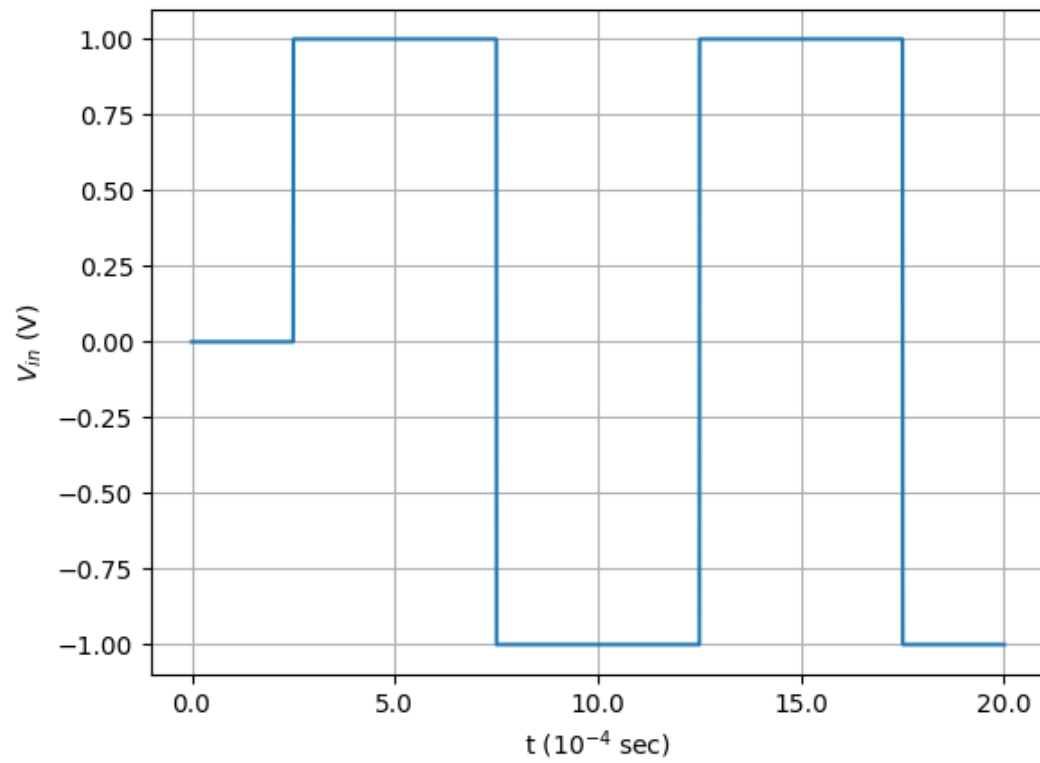


Figure 2.8: Input Square Waveform ( $V_{in}(t)$ )

Taking Fourier Transform of  $V_{in}(t)$ :

$$V_{in}(t) \xleftrightarrow{\mathcal{F}} V_{in}(j\omega) \quad (2.48)$$

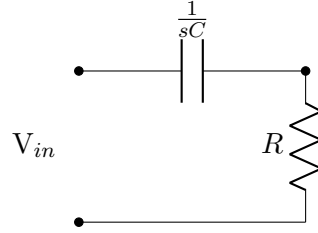


Figure 2.9: Series RC Circuit in s-domain

$$s = j\omega \quad (2.49)$$

$$\Rightarrow Z = R + \frac{1}{sC} \quad (2.50)$$

$$= R + \frac{1}{j\omega C} \quad (2.51)$$

$\mathcal{V}_{in}(j\omega)$  was input into all four circuits and  $V_{out}(t)$  was calculated using the Transfer Functions of the  $RC$  Filters.

Transfer Function:

$$H(j\omega) = \frac{V_{out}}{V_{in}} \quad (2.52)$$

(a) Option A

$$H_R(j\omega) = \frac{R}{R + \frac{1}{j\omega C}} \quad (2.53)$$

$$= \frac{j\omega RC}{1 + j\omega RC} \quad (2.54)$$

$$= \left( \frac{\omega RC}{\sqrt{1 + (\omega RC)^2}} \right) e^{j \tan^{-1}(\frac{1}{\omega RC})} \quad (2.55)$$

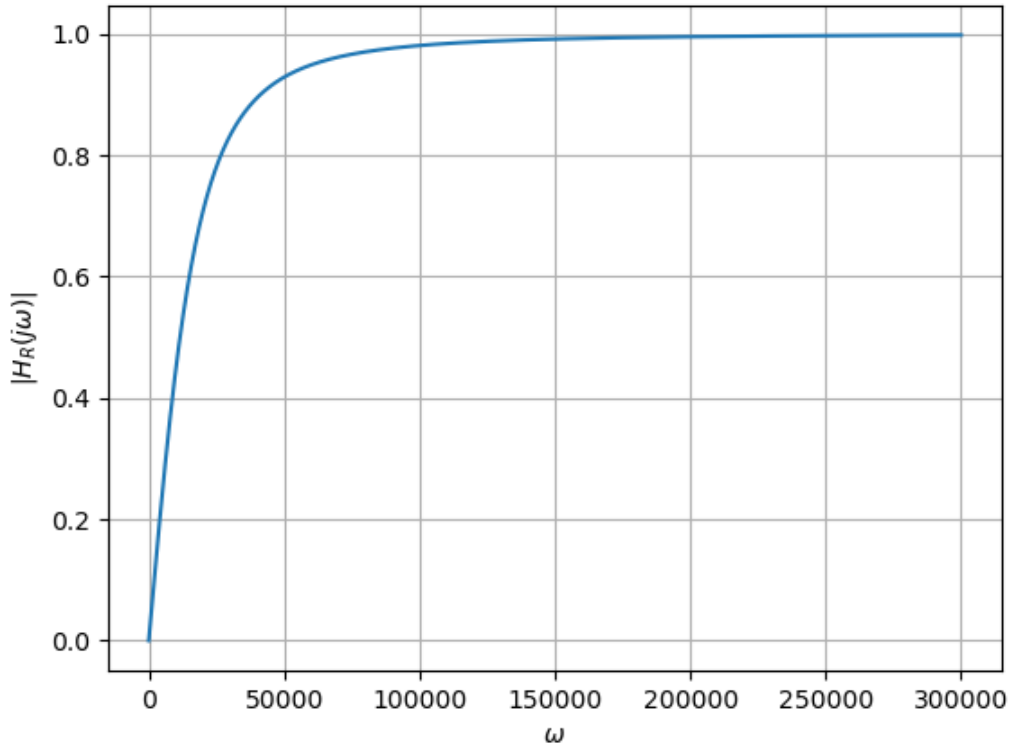


Figure 2.10:  $|H_R(j\omega)|$  vs  $\omega$  for  $R = 0.5k\Omega$ ,  $C = 0.1\mu F$

$$\implies \mathcal{V}_{out}(j\omega) = H_R(j\omega)\mathcal{V}_{in}(j\omega) \quad (2.56)$$

$$(2.57)$$

Using (2.47) and (2.55),

$$V_{out}(t) = \sum_{n=1}^{\infty} \left( \frac{n\omega RC}{\sqrt{1 + (n\omega RC)^2}} \right) a_n \cos \left( n\omega t + \tan^{-1} \left( \frac{1}{n\omega RC} \right) \right) \quad (2.58)$$



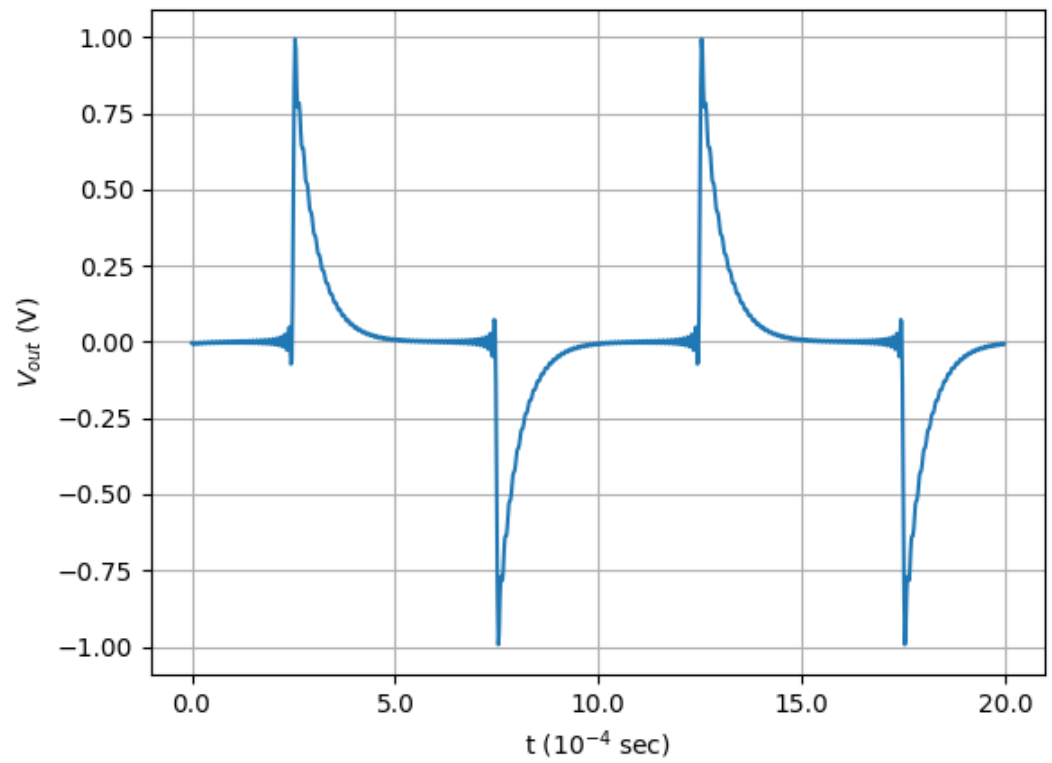


Figure 2.11: Opt A:  $V_{out}(t)$  vs  $t$

(b) Option B

$$H_R(j\omega) = \frac{R}{R + \frac{1}{j\omega C}} \quad (2.59)$$

$$= \frac{j\omega RC}{1 + j\omega RC} \quad (2.60)$$

$$= \left( \frac{\omega RC}{\sqrt{1 + (\omega RC)^2}} \right) e^{j \tan^{-1}(\frac{1}{\omega RC})} \quad (2.61)$$

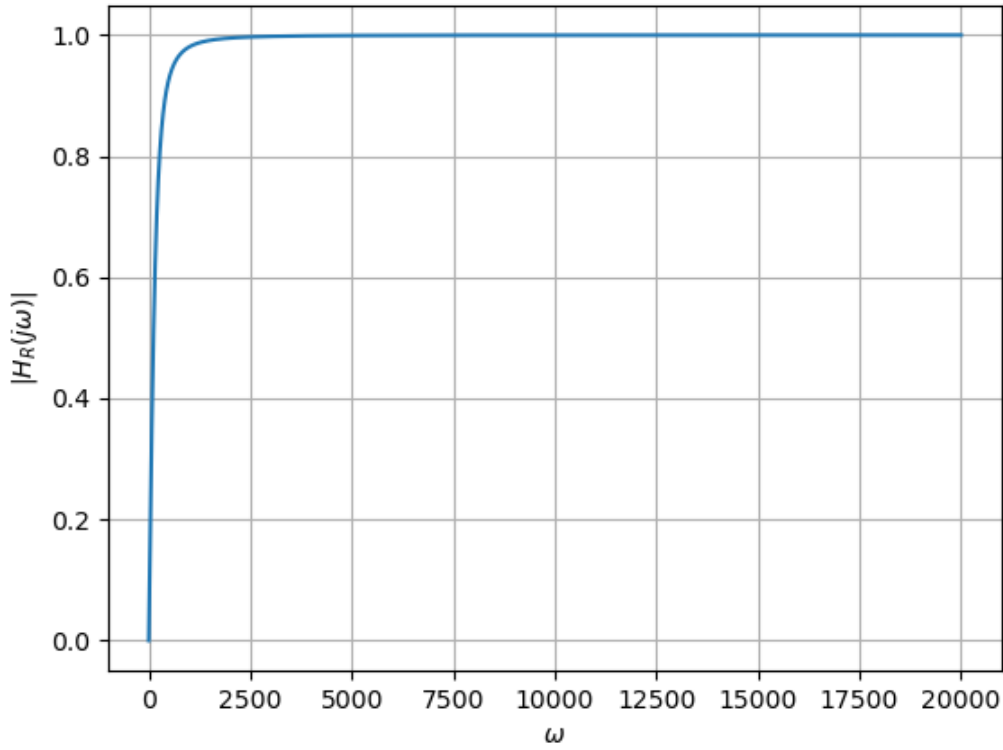


Figure 2.12:  $|H_R(j\omega)|$  vs  $\omega$  for  $R = 5k\Omega$ ,  $C = 1\mu F$

$$\Rightarrow \mathcal{V}_{out}(j\omega) = H_R(j\omega)\mathcal{V}_{in}(j\omega) \quad (2.62)$$

Using (2.47) and (2.61),

$$V_{out}(t) = \sum_{n=1}^{\infty} \left( \frac{n\omega RC}{\sqrt{1 + (n\omega RC)^2}} \right) a_n \cos \left( n\omega t + \tan^{-1} \left( \frac{1}{n\omega RC} \right) \right) \quad (2.63)$$

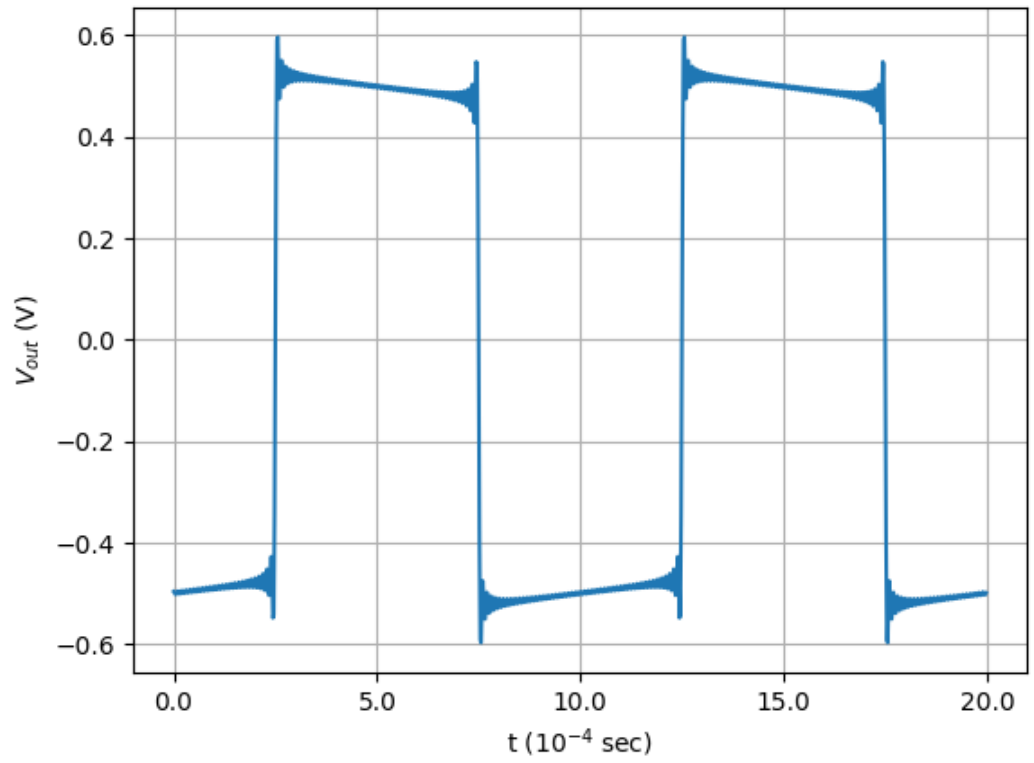


Figure 2.13: Opt B:  $V_{out}(t)$  vs  $t$

(c) Option C

$$H_C(j\omega) = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} \quad (2.64)$$

$$= \frac{1}{1 + j\omega RC} \quad (2.65)$$

$$= \left( \frac{1}{\sqrt{1 + (\omega RC)^2}} \right) e^{-j \tan^{-1}(\omega RC)} \quad (2.66)$$

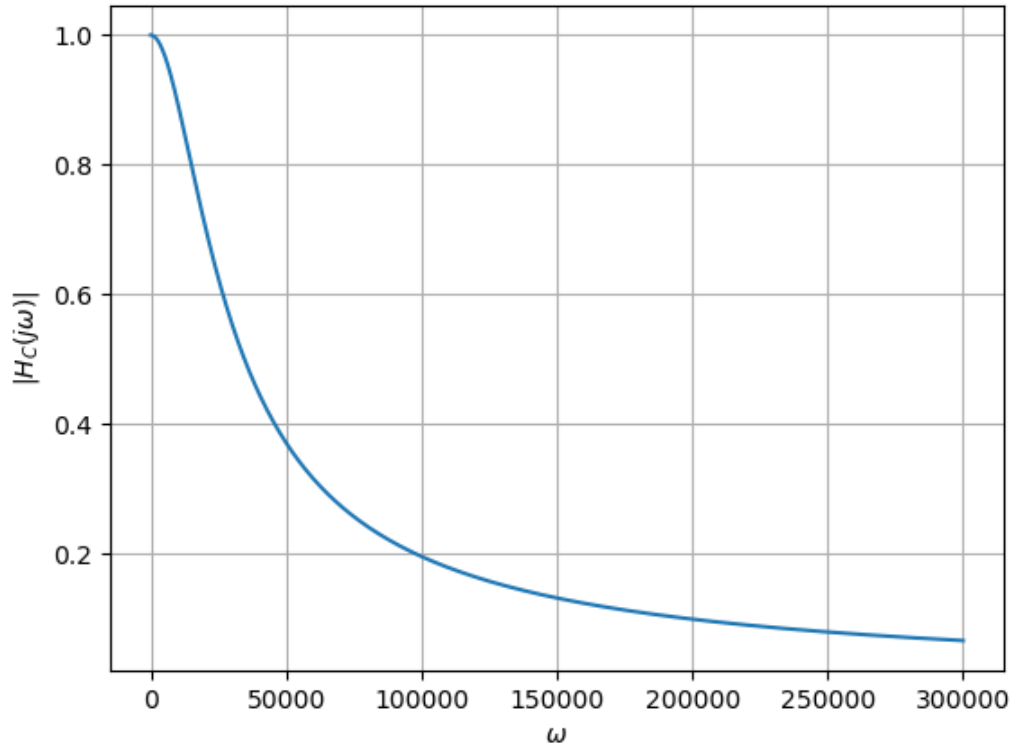


Figure 2.14:  $|H_C(j\omega)|$  vs  $\omega$  for  $R = 0.5k\Omega$ ,  $C = 0.1\mu F$

$$\Rightarrow \mathcal{V}_{out}(j\omega) = H_C(j\omega)\mathcal{V}_{in}(j\omega) \quad (2.67)$$

Using (2.47) and (2.66),

$$V_{out}(t) = \sum_{n=1}^{\infty} \left( \frac{1}{\sqrt{1 + (n\omega RC)^2}} \right) a_n \cos(n\omega t - \tan^{-1}(n\omega RC)) \quad (2.68)$$

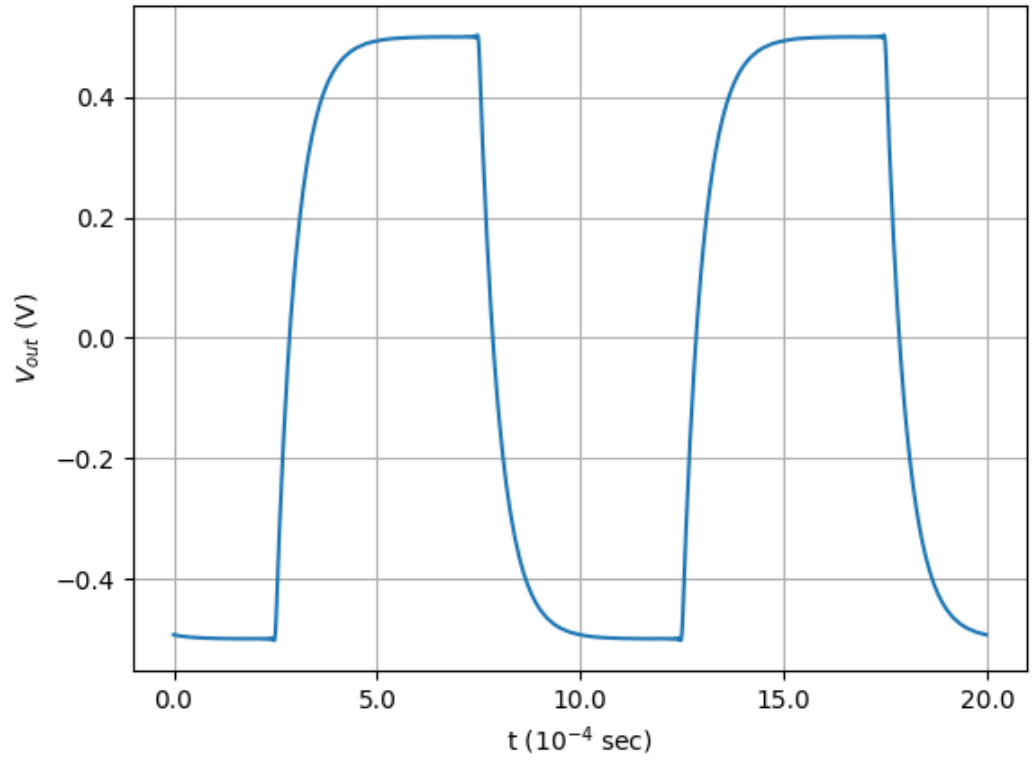


Figure 2.15: Opt C:  $V_{out}(t)$  vs  $t$

(d) Option D

$$H_C(j\omega) = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} \quad (2.69)$$

$$= \frac{1}{1 + j\omega RC} \quad (2.70)$$

$$= \left( \frac{1}{\sqrt{1 + (\omega RC)^2}} \right) e^{-j \tan^{-1}(\omega RC)} \quad (2.71)$$

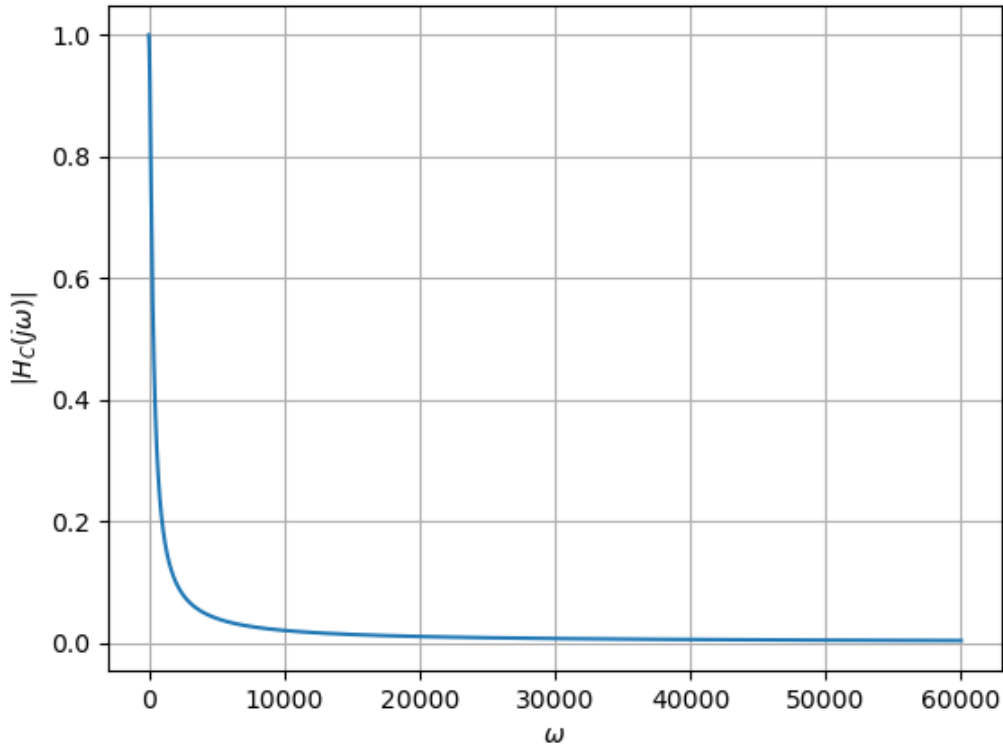


Figure 2.16:  $|H_C(j\omega)|$  vs  $\omega$  for  $R = 5k\Omega$ ,  $C = 1\mu F$

$$\implies \mathcal{V}_{out}(j\omega) = H_C(j\omega)\mathcal{V}_{in}(j\omega) \quad (2.72)$$

Using (2.47) and (2.71),

$$V_{out}(t) = \sum_{n=1}^{\infty} \left( \frac{1}{\sqrt{1 + (n\omega RC)^2}} \right) a_n \cos(n\omega t - \tan^{-1}(n\omega RC)) \quad (2.73)$$

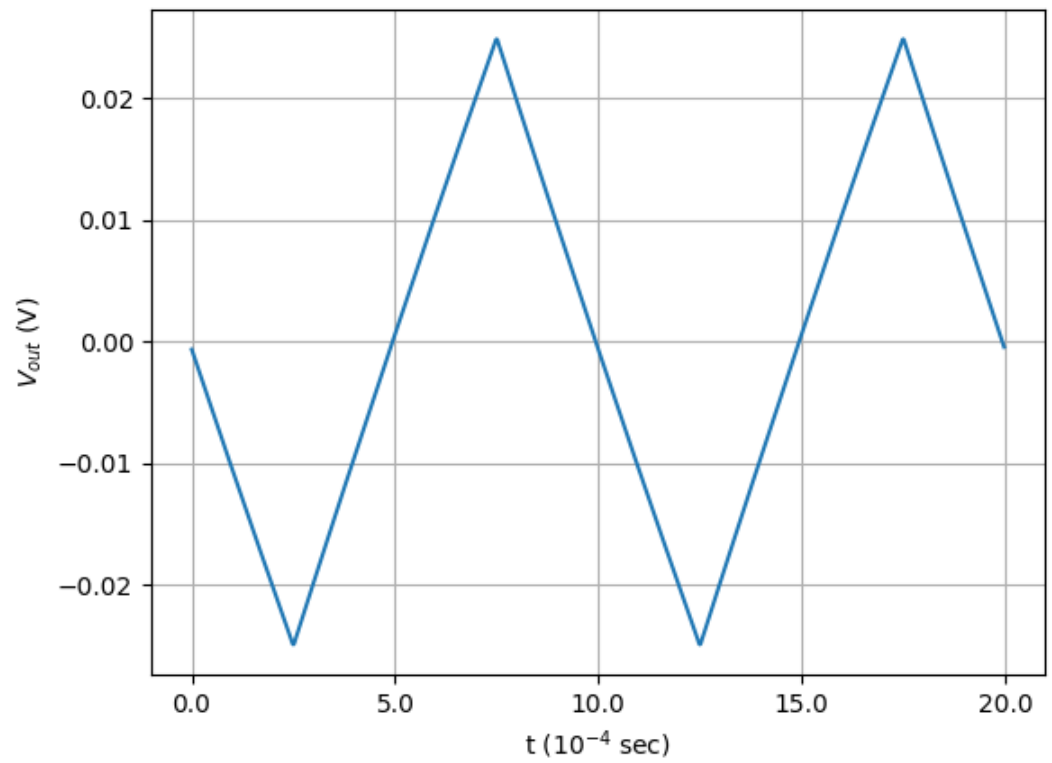


Figure 2.17: Opt D:  $V_{out}(t)$  vs  $t$

2.7 In the circuit shown below, switch S was closed for long time. If the switch is opened at  $t = 0$ , the maximum magnitude of the voltage  $V_R$ , in volts is (rounded off to the nearest integer) (GATE 2023 EC 35)

**Solution:**

$$\text{At, } t = 0^- \quad (2.74)$$

inductor acts as wire

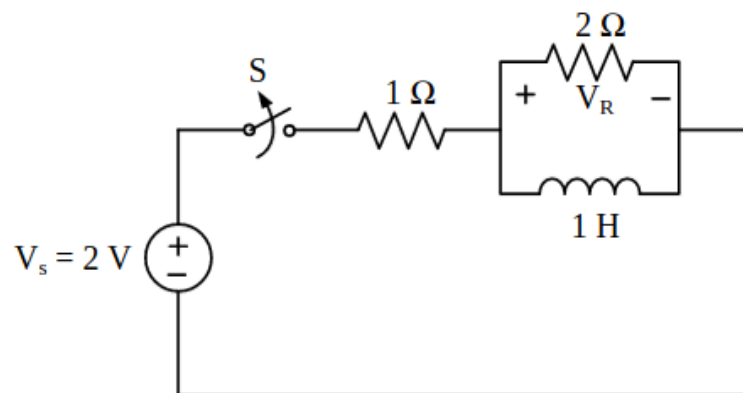


Figure 2.18:

parameter	description	value
$i(0^-)$	current at $t < 0$	$2A$
$V_R(t)$	voltage across $2\Omega$	$-2i(t)u(t)$
$L$	inductance	$1H$
$i(t)$	current in small loop after $t = 0$	$\frac{V_R(t)}{2}$
$I(s)$	$i(t)$ in laplace	$-$

Table 2.5: input parameters

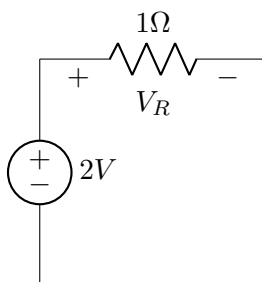


Figure 2.19: steady state circuit

apply KVL

$$-2 + 1i(0^-) = 0 \quad (2.75)$$

$$i(0^-) = 2A \quad (2.76)$$



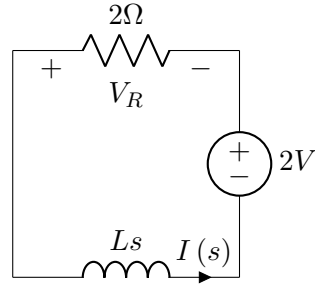


Figure 2.20: s domain circuit for  $t > 0$

$$2I(s) - 2V + LsI(s) = 0 \quad (2.77)$$

$$\implies I(s) = \frac{2}{s+2} A \quad (2.78)$$

applying inverse laplace transform

$$i(t) = 2e^{-2t}u(t) A \quad (2.79)$$

$$V_R(t) = -2i(t) \quad (2.80)$$

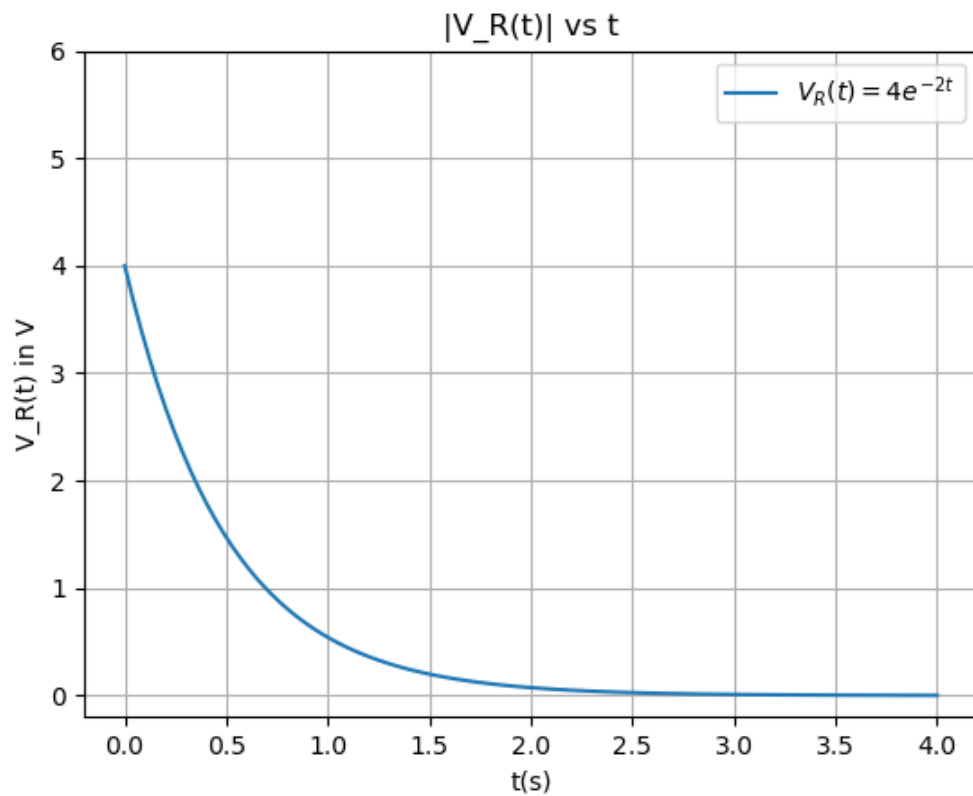
$$\implies V_R(t) = -4e^{-2t}u(t) V \quad (2.81)$$

As,

$$t \rightarrow 0 \quad (2.82)$$

$$\implies e^{-2t} \rightarrow 1 \quad (2.83)$$

$$|V_R(max)| = 4V \quad (2.84)$$



2.8 A signal  $x(t) = 2 \cos(180\pi t) \cos(60\pi t)$  is sampled at 200 Hz and then passed through an ideal low pass filter having cut-off frequency of 100 Hz.

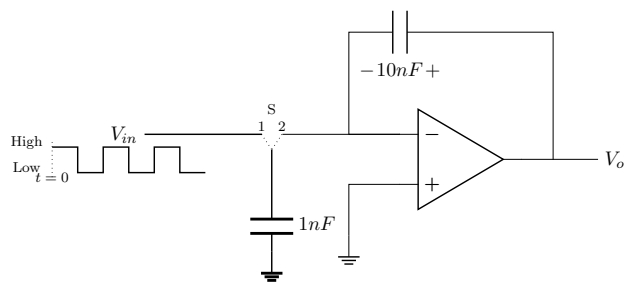
The maximum Frequency present in the filtered signal in Hz is \_\_\_\_\_ (Round off to the nearest integer.)

(GATE 2023 EE) **Solution:**

2.9 In the circuit shown, the input voltage  $V_{in} = 100mV$ . The switch and the opamp are ideal. At time  $t = 0$ , the initial charge stored in the  $10nF$  capacitor is  $1nC$ , with the polarity as indicated in the figure. The switch  $S$  is controlled using a  $1KHz$  square-wave voltage signal  $V_s$  as shown. Whenever  $V_s$  is 'High',  $S$  is in position '1' and when  $V_s$  is 'Low',  $S$  is in position '2'.

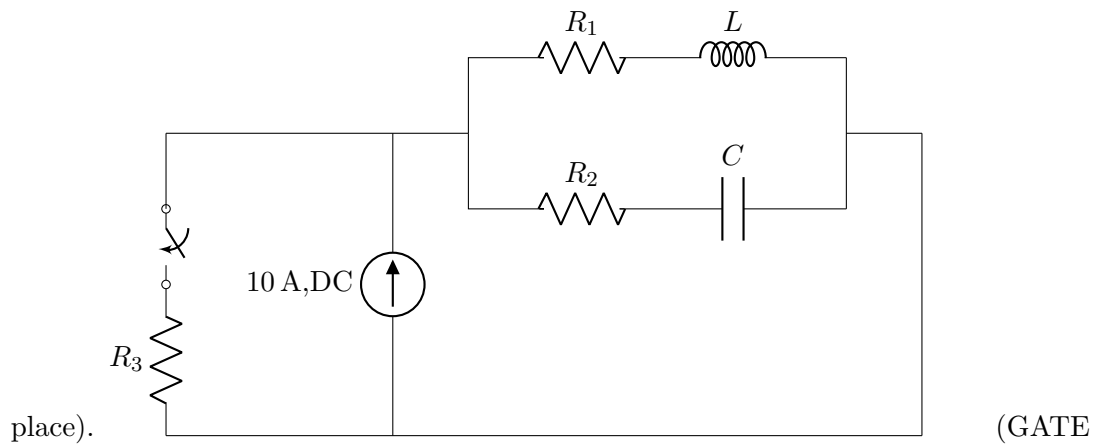
At  $t = 20ms$ , the magnitude of the voltage  $V_o$  will be

(GATE IN 2023)



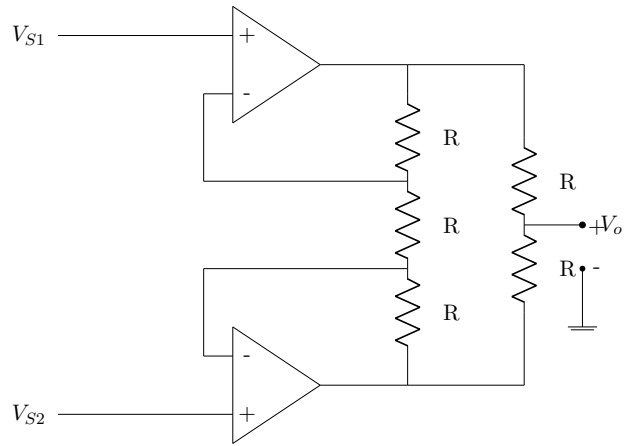
**Solution:**

2.10 The value of parameters of the circuit shown in the figure are  $R_1 = 2\Omega, R_2 = 2\Omega, R_3 = 3\Omega, L = 10\text{mH}, C = 100\mu\text{F}$ . For time  $t < 0$ , the circuit is at steady state with the switch ' $K$ ' in closed condition. If the switch is opened at  $t = 0$ , the value of the voltage across the inductor ( $V_L$ ) at  $t = 0^+$  in Volts is \_\_\_\_\_ (Round off to 1 decimal



2023 EE 29Q) **Solution:**

- 2.11 The op amps in the circuit are ideal. The input signals are  $V_{S1} = 3 + 0.10 \sin(300t)$ , V and  $V_{S2} = -2 + 0.11 \sin(300t)$  V. The average value of the voltage  $V_0$  is \_\_\_\_\_ volts (rounded off to two decimal places). (GATE IN 2023)





## Chapter 3

# Z-transform





## Chapter 4

# Systems

- 4.1 Consider a unity-gain negative feedback system consisting of the plant  $G(s)$  and a proportional-integral controller. Let the proportional gain and integral gain be 3 and 1, respectively. For a unit step reference input, the final values of the controller output and the plant output, respectively, are

$$G(s) = \frac{1}{(s-1)}$$

(GATE EE 2023)

**Solution:**

Parameter	Description	Value
$K_p$	Proportional Gain	3
$K_i$	Integral Gain	1
$r(t)$	Reference Input	$u(t)$
$w(t)$	Controller Output	?
$y(t)$	Plant Output	?
$e(t)$	Error Input	$r(t) - y(t)$

Table 1: Parameter Table

From the Fig. 4.1:

$$E(s) = U(s) - Y(s) \quad (4.1)$$

$$W(s) = 3E(s) + \frac{1}{s}E(s) \quad (4.2)$$

$$Y(s) = G(s)W(s) \quad (4.3)$$

Some results:

$$tx(t) \xleftrightarrow{\mathcal{L}} -\frac{dX(s)}{ds} \quad (4.4)$$

$$e^{-at}x(t) \xleftrightarrow{\mathcal{L}} X(s+a) \quad (4.5)$$

By using (4.4) and (4.5):

$$e^{-t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+1}, \text{Re}(s) > -1 \quad (4.6)$$

$$te^{-t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{(s+1)^2}, \text{Re}(s) > -1 \quad (4.7)$$

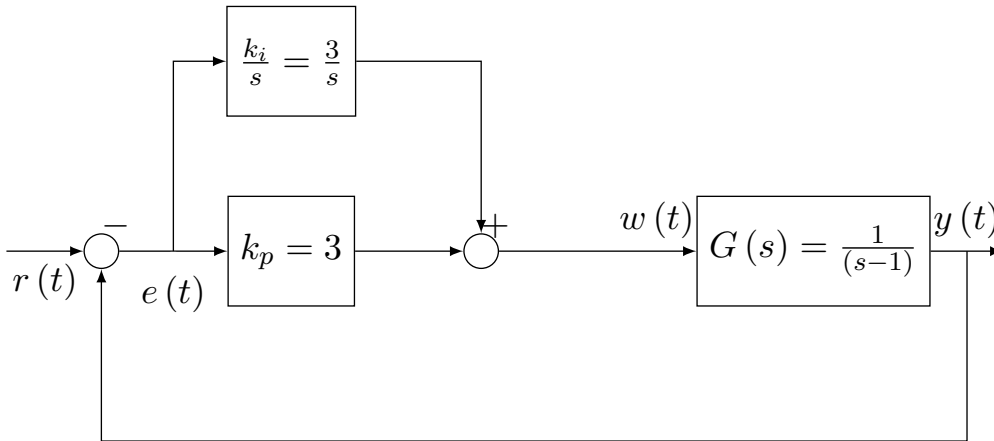


Figure 4.1: Block Diagram of System

(a) **Plant Output:**

From (4.1) , (4.2) and (4.3):

$$Y(s) = \frac{3s+1}{s(s+1)^2}, \text{Re}(s) > -1 \quad (4.8)$$

Final Value Theorem:

$$\lim_{t \rightarrow \infty} x(t) = \lim_{s \rightarrow 0} sX(s) \quad (4.9)$$

Using (4.9) on Y(s):

$$\lim_{t \rightarrow \infty} y(t) = \lim_{s \rightarrow 0} sY(s) \quad (4.10)$$

$$= 1 \quad (4.11)$$

Taking partial fraction of (4.8) :

$$Y(s) = \frac{1}{s} + \frac{2}{(s+1)^2} - \frac{1}{s+1} \quad (4.12)$$

Using (4.6) and (4.7):

$$\therefore y(t) = u(t) + 2te^{-t}u(t) - e^{-t}u(t) \quad (4.13)$$

(b) **Controller Output:**

From (4.2)

$$W(s) = \frac{3}{s} + \frac{1}{s^2} - Y(s) \left( 3 + \frac{1}{s} \right) \quad (4.14)$$

Substituting (4.8)

$$W(s) = \frac{(s-1)(3s+1)}{s(s+1)^2}, \operatorname{Re}(s) > -1 \quad (4.15)$$

Using (4.9) on  $W(s)$

$$\lim_{t \rightarrow \infty} w(t) = \lim_{s \rightarrow 0} sW(s) \quad (4.16)$$

$$= -1 \quad (4.17)$$

Taking partial fraction of equation(4.15) :

$$W(s) = -\frac{1}{s} - \frac{4}{(s+1)^2} + \frac{4}{s+1} \quad (4.18)$$

Using equations (4.6) and (4.7) and taking inverse lapalace transform:

$$w(t) = -u(t) - 4te^{-t}u(t) + 4e^{-t}u(t) \quad (4.19)$$

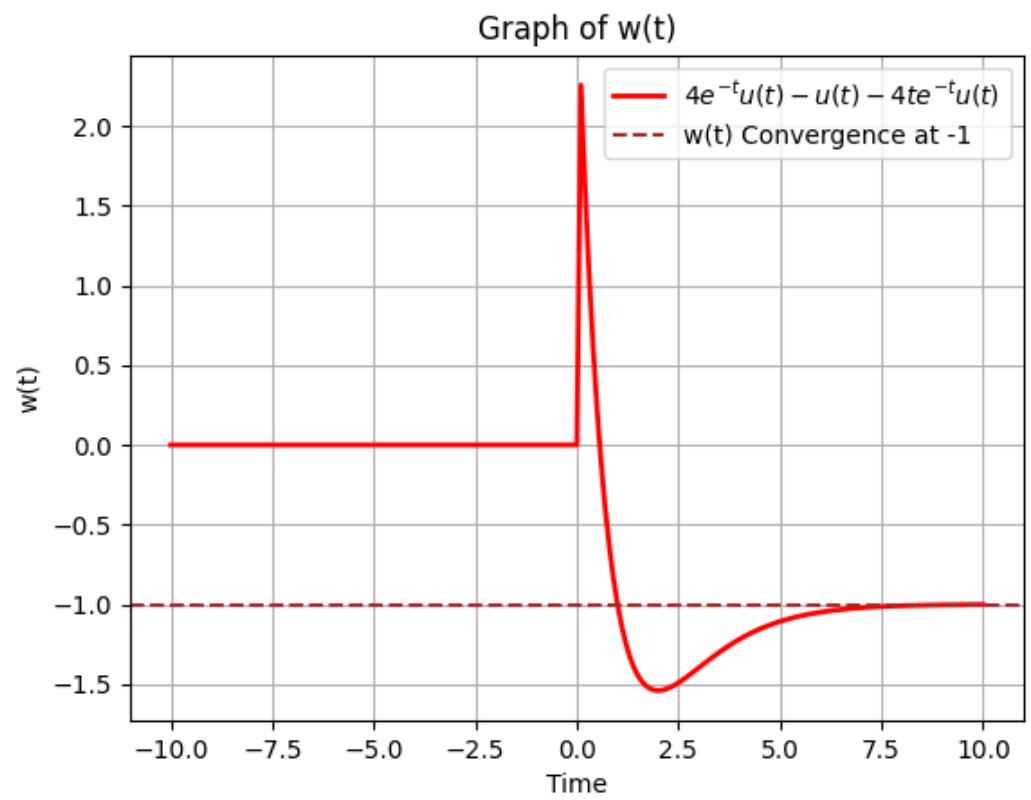


Figure 4.2:  $w(t)$  converges at -1.

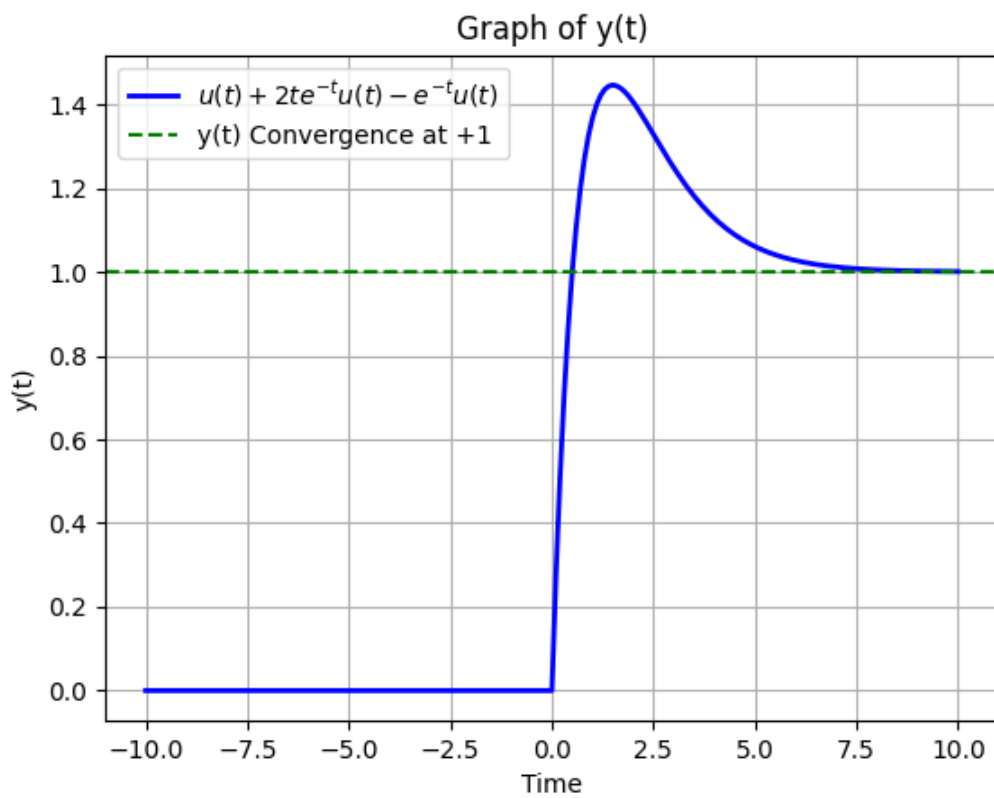


Figure 4.3:  $y(t)$  converges at +1

4.2 Level ( $h$ ) in a steam boiler is controlled by manipulating the flow rate ( $F$ ) of the break-up(fresh) water using a proportional ( $P$ ) controller. The transfer function between the output and the manipulated input is

$$\frac{h(s)}{F(s)} = \frac{0.25(1-s)}{s(2s+1)}$$

The measurement and the valve transfer functions are both equal to 1. A process engineer wants to tune the controller so that the closed loop response gives the decaying oscillations under the servo mode. Which one of the following is the CORRECT value of the controller gain to be used by the engineer?

- (a) 0.25
- (b) 2
- (c) 4
- (d) 6

GATE CH 2023

**Solution:**

Closed loop signal transfer function of the above block diagram can be given by,

$$T(s) = \frac{G(s)}{1 + G(s)H(s)} \quad (4.20)$$

From Fig. 4.4 and Table 4.2 for a unit impulse,  $X(s) = 1$

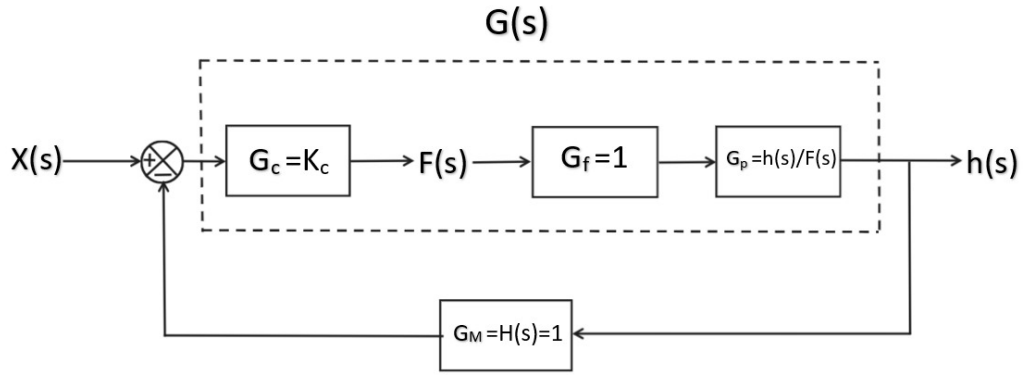


Figure 4.4: Closed loop Block diagram

$$h(s) = T(s) \times X(s) \quad (4.21)$$

$$h(s) = \frac{(1-s)K_c}{8s^2 + (4-K_c)s + K_c} \quad (4.22)$$

$$\Rightarrow h(s) = \frac{(1-s)K_c}{8(s-s_1)(s-s_2)} \quad (4.23)$$

Where,

$$s_1 = \frac{(K_c - 4)}{16} + \sqrt{\left(\frac{K_c - 4}{16}\right)^2 - \frac{K_c}{8}} \quad (4.24)$$

$$s_2 = \frac{(K_c - 4)}{16} - \sqrt{\left(\frac{K_c - 4}{16}\right)^2 - \frac{K_c}{8}} \quad (4.25)$$



From (4.23) we get,

$$h(s) = \frac{K_c}{8(s_1 - s_2)} \left( \frac{1 - s_1}{s - s_1} - \frac{1 - s_2}{s - s_2} \right) \quad (4.26)$$

Now taking the inverse laplace transform we have,

$$h(t) = \frac{K_c}{8(s_1 - s_2)} [(1 - s_1)e^{s_1 t} - (1 - s_2)e^{s_2 t}] u(t) \quad (4.27)$$

$$\Rightarrow h(t) = e^{\frac{K_c - 4}{16}} \left( A_1 e^{\sqrt{\left(\frac{K_c - 4}{16}\right)^2 - \frac{K_c}{8}}} - A_2 e^{-\sqrt{\left(\frac{K_c - 4}{16}\right)^2 - \frac{K_c}{8}}} \right) u(t) \quad (4.28)$$

Where,

$$A_1 = \frac{K_c}{8} \left( \frac{1 - s_1}{s_1 - s_2} \right) \quad (4.29)$$

$$A_2 = \frac{K_c}{8} \left( \frac{1 - s_2}{s_1 - s_2} \right) \quad (4.30)$$

Now applying the condition for underdamped oscillations,

$$\left( \frac{K_c - 4}{16} \right)^2 - \frac{K_c}{8} < 0 \quad (4.31)$$

$$\Rightarrow K_c \in (20 - \sqrt{384}, 20 + \sqrt{384}) \quad (4.32)$$

For the system to be stable,

$$\frac{K_c - 4}{8} < 0 \quad (4.33)$$

$$\Rightarrow K_c < 4 \quad (4.34)$$

From (4.32) and (4.34)

$$K_c \in (0.4, 4) \quad (4.35)$$

(4.35) represents the  $ROC, (Re\{s\} < 0)$

$$\implies K_c = 2 \quad (4.36)$$

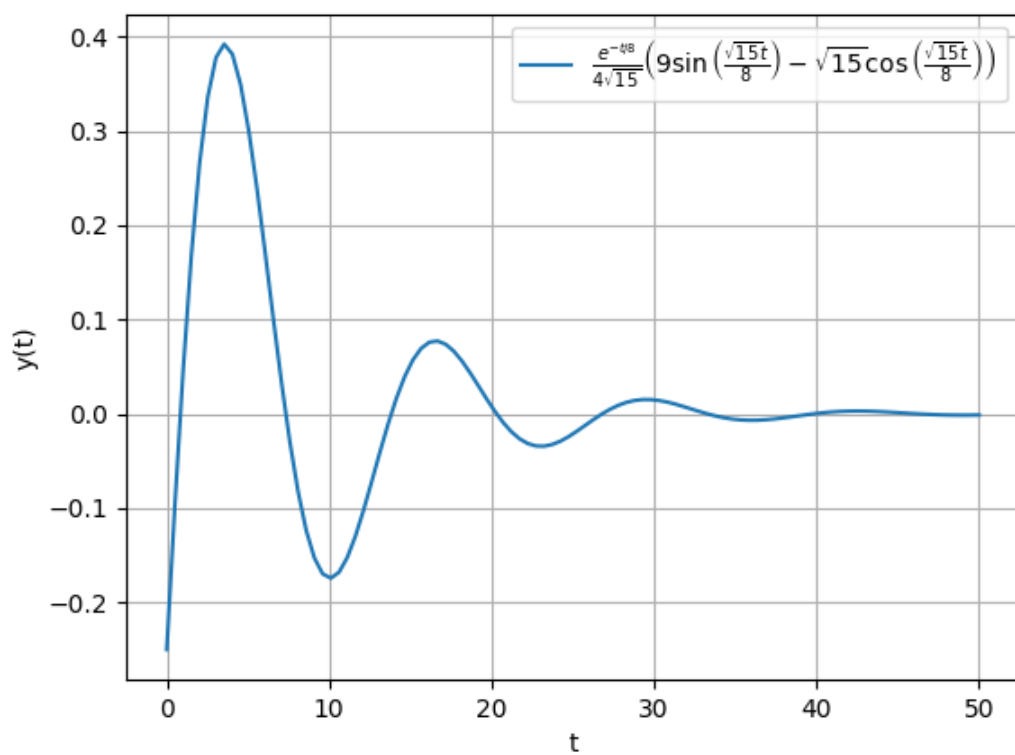
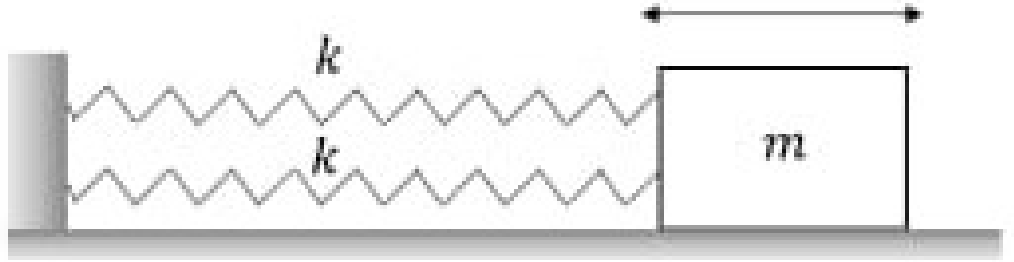


Figure 4.5:  $y(t)$  vs  $t$  graph

- 4.3 The figure shows a block of mass  $m = 20$  kg attached to a pair of identical linear springs, each having a spring constant  $k = 1000$  N/m. The block oscillates on a frictionless horizontal surface. Assuming free vibrations, the time taken by the block to complete ten oscillations is \_\_\_\_\_ seconds . (Rounded off to two decimal places)  
Take  $\pi = 3.14$ .  
(GATE ME 2023)



**Solution:**

$$F = ma \quad (4.37)$$

$$F = -kx \quad (4.38)$$

$$\implies ma + kx = 0 \quad (4.39)$$

$$\therefore m \frac{d^2x}{dt^2} + kx = 0 \quad (4.40)$$

The Laplace transform of the terms is ,

$$x \xrightarrow{\mathcal{L}} X(s) \quad (4.41)$$

$$\frac{d^2x}{dt^2} \xrightarrow{\mathcal{L}} s^2 X(s) - sx(0) - \dot{x}(0) \quad (4.42)$$

Using equation (4.41) and (4.42) in equation (4.40),

$$m (s^2 X (s) - sx (0) - \dot{x} (0)) + kX (s) = 0 \quad (4.43)$$

$$ms^2 X (s) - msA + kX (s) = 0 \quad (4.44)$$

$$X (s) = \frac{msA}{ms^2 + k} \quad (4.45)$$

$$= \frac{sA}{s^2 + \frac{k}{m}} \quad (4.46)$$

The inverse Laplace transform of such terms is given by,

$$\frac{s}{s^2 + a^2} \xleftrightarrow{\mathcal{L}^{-1}} \cos (at) u (t) \quad (4.47)$$

$\therefore$  the inverse Laplace of (4.46) is,

$$x (t) = A \cos \left( \sqrt{\frac{k}{m}} t \right) \quad (4.48)$$

From equation (4.48) and Table 4.3 ,the time to complete one oscillation is,

$$T_n = \frac{2\pi}{\sqrt{\frac{k}{m}}} \quad (4.49)$$

$$= \frac{\pi}{5} \quad (4.50)$$

$\therefore$  the time required for 10 oscillations is ,

$$10T_n = 2\pi \quad (4.51)$$

$$= 6.28s \quad (4.52)$$

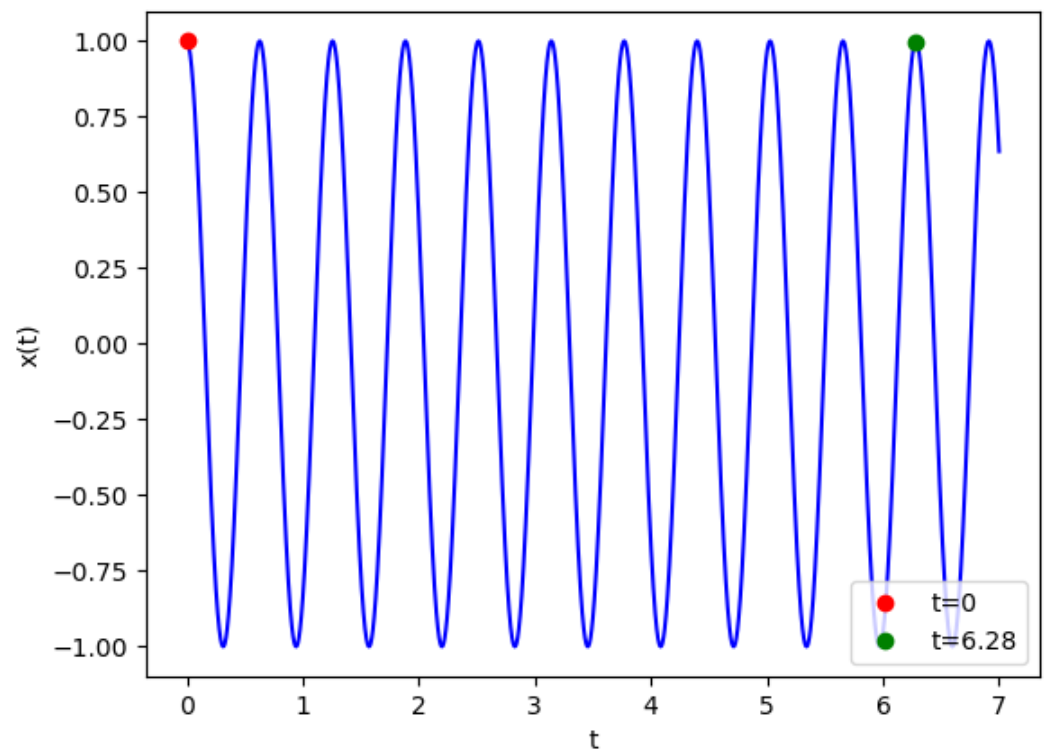


Figure 4.6: Plot of  $x(t)$

4.4 A system has transfer function

$$\frac{Y(s)}{X(s)} = \frac{s - \pi}{s + \pi}$$

let  $u(t)$  be the unit step function. The input  $x(t)$  that results in a steady-state output  $y(t) = \sin(\pi t)$  is \_\_\_\_\_. (GATE IN 2023)

**Solution:**

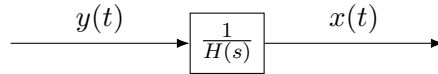


Figure 4.7: Block diagram of the inverse system

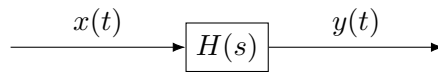


Figure 4.8: Block diagram of the system

$$H(s) = \frac{s - \pi}{s + \pi} \quad (4.53)$$

from Fig. 4.7

$$\frac{1}{H(s)} = \frac{s + \pi}{s - \pi} \quad (4.54)$$

Converting transfer function to frequency response, we get

$$\frac{1}{H(j\omega)} = \frac{j\omega + \pi}{j\omega - \pi} \quad (4.55)$$

from Table 4.4  $\omega = \pi$

$$\frac{1}{H(j\pi)} = \frac{j + 1}{j - 1} = -j = e^{-j\frac{\pi}{2}} \quad (4.56)$$

from (4.56)

$$\left| \frac{1}{H(j\pi)} \right| = 1 \quad \text{and} \quad \arg \left( \frac{1}{H(j\pi)} \right) = -90^\circ \quad (4.57)$$

$$y(t) = \sin(\pi t) \quad (4.58)$$

$$\sin(\pi t) \xleftrightarrow{\mathcal{H}(\omega)} \left| \frac{1}{H(j\omega)} \right| \sin \left( \pi t + \arg \left( \frac{1}{H(j\omega)} \right) \right) \quad (4.59)$$

Therefore, by (4.58) and (4.59), we get

$$x(t) = \sin \left( \pi t - \frac{\pi}{2} \right) \quad (4.60)$$

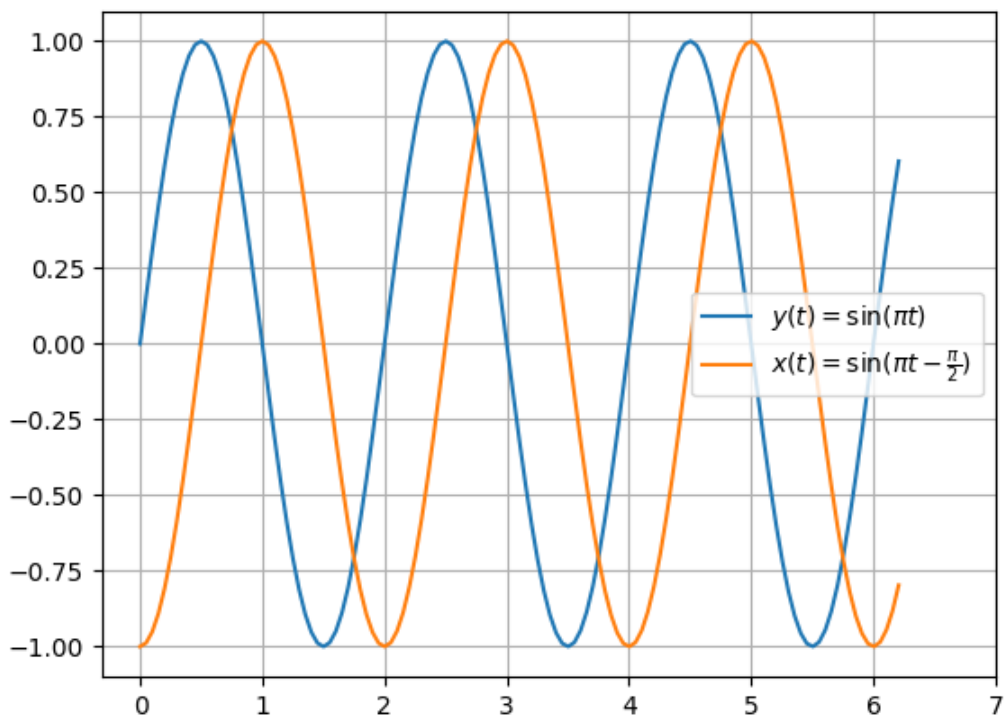


Figure 4.9: Plot of  $x(t)$  and  $y(t)$  taken from Python

4.5 Consider the complex function

$$f(z) = \frac{z^2 \sin z}{(z - \pi)^4}$$

At  $z = \pi$ , which of the following options is (are) correct?

(A) The order of the pole is 4

(B) The order of the pole is 3

(C) The residue at the pole is  $\frac{\pi}{6}$



(D) The residue at the pole is  $\frac{2\pi}{3}$

(GATE PH 2023)

- 4.6 A buoy of virtual mass 30 kg oscillates in a fluid medium as a single degree of freedom system. If the total damping in the system is set as 188.5 N-s/m, such that the oscillation just ceases to occur, then the natural period of the system is \_\_\_\_\_ s (round off to one decimal place) (GATE MN 2023 question 63)

**Solution:**

The differential equation of the system is:

$$m \frac{d^2 x(t)}{dt^2} + \lambda \frac{dx(t)}{dt} + m\omega_o^2 x(t) = 0 \quad (4.61)$$

Taking laplace transform:

$$ms^2 X(s) + \lambda s X(s) + m\omega_o^2 X(s) = 0 \quad (4.62)$$

$$\implies ms^2 + s\lambda + m\omega_o^2 = 0 \quad (4.63)$$

$$\therefore s = \frac{-\lambda \pm \sqrt{\lambda^2 - 4m^2\omega_o^2}}{2m} \quad (4.64)$$

where  $\sqrt{\lambda^2 - 4m^2\omega_o^2}$  is  $\omega_d$ .

From Table 4.5,

$$\omega_d = 0 \quad (4.65)$$

$$\implies \lambda = 2\omega_o m \quad (4.66)$$

$$\implies \omega_o \approx \pi \quad (4.67)$$

$$T_i = \frac{2\pi}{\omega_o} \quad (4.68)$$

$$\therefore t_i = 2 \text{ seconds} \quad (4.69)$$

To find  $x(t)$ , we assume the initial amplitude of oscillations to be 1 meter and it is

situated at extreme position.

$$s^2 X(s) + \lambda s X(s) + \omega_o^2 X(s) = s x(0) + x(0) \quad (4.70)$$

$$\Rightarrow X(s) = \frac{1+s}{s^2 + s \frac{\lambda}{m} + \omega_o^2} \quad (4.71)$$

substituting values from Table 4.5 and  $\omega_o$ ,

$$X(s) = \frac{1+s}{(s+\pi)^2} \quad (4.72)$$

Taking inverse laplace transform by method of partial fractions,

$$X(s) = \frac{1}{s+\pi} + \frac{1-\pi}{(s+\pi)^2} \quad (4.73)$$

$$\therefore x(t) = (1 + (1-\pi)t) e^{-\pi t} \quad (4.74)$$

4.7 Which of the following statement(s) is/are true?

- (a) If an LTI system is causal, it is stable.
- (b) A discrete time LTI system is causal if and only if its response to a step input  $u[n]$  is 0 for  $n < 0$ .
- (c) If a discrete time LTI system has an impulse response  $h[n]$  of finite duration the system is stable.
- (d) If the impulse response  $0 < |h[n]| < 1$  for all  $n$ , then the LTI system is stable.

(GATE EE 2023 question 27)

**Solution:**

(a)

$$\text{Assume } h(t) = e^{2t} \cdot u(t) \quad (4.75)$$

$$\mathcal{L}\{e^{2t}\} = \frac{1}{s-2} \quad \text{Re}(s) > 2 \quad (4.76)$$

This system is causal but not stable because ROC does not contain imaginary axis.

Therefore, this statement does not hold true.

- (b) For a causal system, its response to the step input should indeed be zero for  $n < 0$ , as the system hasn't yet "seen" any input before time  $n = 0$ .

Mathematically, the output of an LTI system  $y[n]$  can be represented as the convolution of the input  $u[n]$  with the system's impulse response  $h[n]$ :

$$y[n] = \sum_{k=-\infty}^{\infty} h[n] u[n-k] \quad (4.77)$$

Now applying step input,

$$y[n] = \sum_{k=0}^{\infty} h[n] u[n-k] \quad (4.78)$$

This is because  $u[n-k]$  is zero for  $n < k$ , hence, the summation only starts from  $k = 0$ .

For  $n < 0$ ,  $u[n-k] = 0$  for all  $k$ , because  $n-k < 0$  when  $n < 0$ . Therefore,  $y[n] = 0$  for  $n < 0$ .

Therefore, this statement is true.

(c)

$$\text{Assume } h[n] = \begin{cases} n & \text{if } 0 \leq n \leq N \\ 0 & \text{otherwise} \end{cases} \quad (4.79)$$

$$y[n] = h[n] * u[n] \quad (4.80)$$

$$y[n] = \sum_{k=0}^n h[k] u[n-k] \quad (4.81)$$

$$= \sum_{k=0}^n k \quad (4.82)$$

The input response is finite but the output response is not BIBO stable.

Therefore, this statement does not hold true.

(d)

$$\text{Assume } h[n] = \frac{1}{2}u[n] \quad (4.83)$$

$$g[n] = \sum_{n=0}^{\infty} |h[n]| \quad (4.84)$$

$$= \sum_{n=0}^{\infty} \frac{1}{2} \quad (4.85)$$

$$\Rightarrow \sum_{n=0}^{\infty} |h[n]| \rightarrow \infty \quad (4.86)$$

Hence it is unstable.

Therefore, this statement does not hold true.

So, the answer is option (B).

4.8 The outlet concentration  $C_A$  of a plug flow reactor (PFR) is controlled by manipulating the inlet concentration  $C_{A0}$ . The following transfer function describes the dynamics of this PFR.

$$\frac{C_A(s)}{C_{A0}(s)} = e^{-(\frac{V}{F})(k+s)}$$

In the above question,  $V=1m^3$ ,  $F=0.1m^3min^{-1}$  and  $k=0.5min^{-1}$ . The measurement and valve transfer functions are both equal to 1. The ultimate gain, defined as the proportional controller gain that produces sustained oscillations, for this system is (GATE 2023 CH 61)

**Solution:**

4.9 For the block diagram shown in the figure, the transfer function  $\frac{Y(s)}{R(s)}$  is

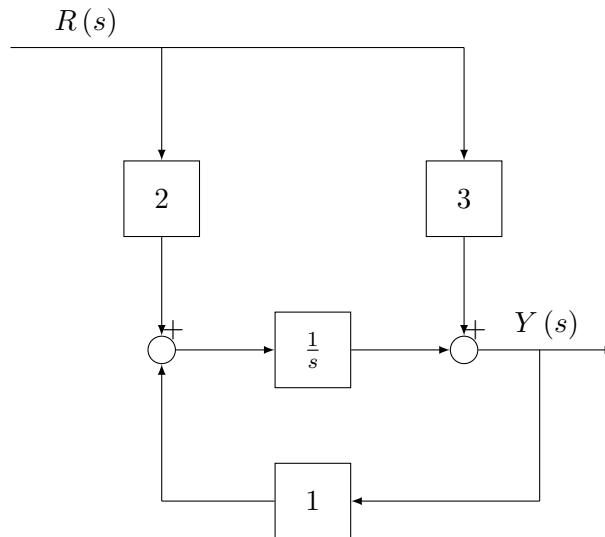


Figure 4.16: Block diagram

(GATE EE 2023)

**Solution:**

Parameter	Description	Value
$Y(s)$	Output node signal	
$R(s)$	Input node signal	
$\frac{Y(s)}{R(s)}$	Transfer function	?
$P_1$	Forward path gain a-b-c	$\frac{2}{s}$
$P_2$	Forward path gain a-c	3
$\Delta_1$	Determinant of forward path a-b-c	1
$\Delta_2$	Determinant of forward path a-c	1
$\Delta$	Determinant of system	$1 - \frac{1}{s}$
$n$	Number of forward path	2

Table 4.6: Parameters

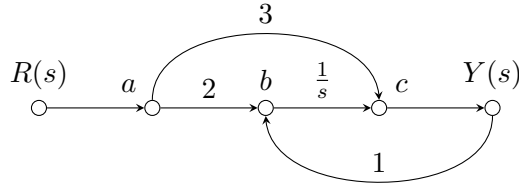


Figure 4.17: signal flow graph

$$P_1 = 2 \left( \frac{1}{s} \right) = \frac{2}{s} \quad (4.87)$$

$$P_2 = 3 \quad (4.88)$$

$$\Delta_1 = 1 - (0) = 1 \quad (4.89)$$

$$\Delta_2 = 1 - (0) = 1 \quad (4.90)$$

$$L_1 = \frac{1}{s} \quad (4.91)$$

$$\Delta = 1 - L_1 = 1 - \frac{1}{s} \quad (4.92)$$



from Fig. 4.17 using Mason's Gain Formula,

$$\frac{Y(s)}{R(s)} = \frac{\sum_{i=1}^n P_i \Delta_i}{\Delta} \quad (4.93)$$

$$= \frac{P_1 \Delta_1 + P_2 \Delta_2}{\Delta} \quad (4.94)$$

$$= \frac{\frac{2}{s} + 3}{1 - \frac{1}{s}} \quad (4.95)$$

$$H(s) = \frac{Y(s)}{R(s)} = \frac{3s + 2}{s - 1} \quad (4.96)$$

$$H(s) = \frac{5}{s - 1} + 3 \quad (4.97)$$

$$H(s) \xLeftrightarrow[-\infty]{} h(t) \quad (4.98)$$

$$\frac{5}{s - 1} \xLeftrightarrow[-\infty]{} 5e^t \quad (4.99)$$

$$3 \xLeftrightarrow[-\infty]{} 3\delta(t) \quad (4.100)$$

using (4.99) and (4.100),

$$h(t) = 5e^t + 3\delta(t) \quad (4.101)$$

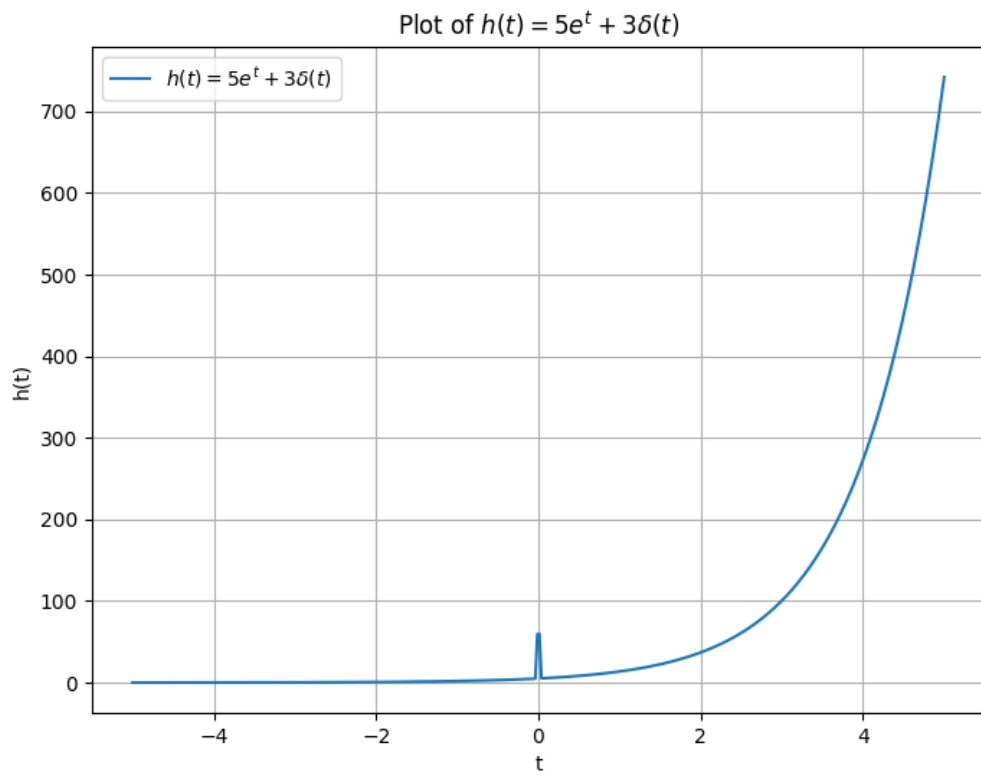


Figure 4.18: Plot of impulse response of the system

4.10 In the following block diagram,  $R(s)$  and  $D(s)$  are two inputs. The output  $Y(s)$  is expressed as  $Y(s) = G_1(s)R(s) + G_2(s)D(s)$ .

$G_1(s)$  and  $G_2(s)$  are given by

GATE 2023 EC Question 42

**Solution:**

4.11 In the table shown below, match the signal type with its spectral characteristics. GATE

2023 EC

(a)  $(i) \rightarrow (a)$  ,  $(ii) \rightarrow (b)$  ,  $(iii) \rightarrow (c)$  ,  $(iv) \rightarrow (d)$

(b)  $(i) \rightarrow (a)$  ,  $(ii) \rightarrow (c)$  ,  $(iii) \rightarrow (b)$  ,  $(iv) \rightarrow (d)$

(c)  $(i) \rightarrow (d)$  ,  $(ii) \rightarrow (b)$  ,  $(iii) \rightarrow (c)$  ,  $(iv) \rightarrow (a)$

(d)  $(i) \rightarrow (a)$  ,  $(ii) \rightarrow (c)$  ,  $(iii) \rightarrow (d)$  ,  $(iv) \rightarrow (b)$

**Solution:**

4.12 The impulse response of an LTI system is  $h(t) = \delta(t) + 0.5\delta(t - 4)$ , where  $\delta(t)$  is continuous-time unit impulse signal. If the input signal  $x(t) = \cos\left(\frac{7\pi t}{4}\right)$ , the output is (GATE IN 2023)

**Solution:**

from Table 4.8

$$y(t) = x(t) * h(t) \quad (4.102)$$

$$= x(t) * (\delta(t) + 0.5\delta(t - 4)) \quad (4.103)$$

$$= x(t) + 0.5x(t - 4) \quad (4.104)$$

$$= \cos\left(\frac{7\pi t}{4}\right) + 0.5 \cos\left(\frac{7\pi(t - 4)}{4}\right) \quad (4.105)$$

$$= \cos\left(\frac{7\pi t}{4}\right) + 0.5 \cos\left(\frac{7\pi t}{4} - 7\pi\right) \quad (4.106)$$

$$= \frac{1}{2} \cos\left(\frac{7\pi t}{4}\right) \quad (4.107)$$

$$= \frac{1}{2} x(t) \quad (4.108)$$

$$Y(f) = X(f) H(f) \quad (4.109)$$

$$= \frac{1}{2} \left[ \delta\left(f - \frac{7}{8}\right) + \delta\left(f + \frac{7}{8}\right) \right] [1 + 0.5e^{-j8\pi f}] \quad (4.110)$$

$$Y(f) = \frac{1}{2} \left( 1 + 0.5e^{-j8\pi f} \right) \delta \left( f - \frac{7}{8} \right) + \frac{1}{2} \left( 1 + 0.5e^{-j8\pi f} \right) \delta \left( f + \frac{7}{8} \right) \quad (4.111)$$

$$= \frac{1}{2} (1 + 0.5(-1)) \delta \left( f - \frac{7}{8} \right) + \frac{1}{2} (1 + 0.5(-1)) \delta \left( f + \frac{7}{8} \right) \quad (4.112)$$

$$= \frac{1}{4} \left[ \delta \left( f - \frac{7}{8} \right) + \delta \left( f + \frac{7}{8} \right) \right] \quad (4.113)$$

$$= \frac{1}{2} X(f) \quad (4.114)$$

4.13 Consider the state-space description of an LTI system with matrices

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & -2 \end{bmatrix}, \quad D = 1$$

For the input,  $\sin(\omega t)$ ,  $\omega > 0$ , the value of  $\omega$  for which the steady-state output of the system will be zero, is \_\_\_\_\_ (Round off to the nearest integer). (GATE 2023 EE Q46)

**Solution:** The state-space representation of the system is given by:

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (4.115)$$

$$y(t) = Cx(t) + Du(t) \quad (4.116)$$

Transfer function given by:

$$T.F = C \left( sI - A \right)^{-1} B + D \quad (4.117)$$

$$\left( sI - A \right) = \begin{pmatrix} s & -1 \\ 1 & s+2 \end{pmatrix} \quad (4.118)$$

$$\left( sI - A \right)^{-1} = \frac{1}{s(s+2)+1} \begin{pmatrix} s+2 & 1 \\ -1 & s \end{pmatrix} \quad (4.119)$$

Referencing from equation (4.119), equation (4.117) becomes

$$T.F = \begin{pmatrix} \frac{3}{s^2+2s+1} & \frac{-2}{s^2+2s+1} \end{pmatrix} \begin{pmatrix} s+2 & 1 \\ -1 & s \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + 1 \quad (4.120)$$

$$= \begin{pmatrix} \frac{3}{s^2+2s+1} & \frac{-2}{s^2+2s+1} \end{pmatrix} \begin{pmatrix} 1 \\ s \end{pmatrix} + 1 \quad (4.121)$$

$$= \frac{s^2 + 4}{s^2 + 2s + 1} \quad (4.122)$$

$$H(s) = T.F \quad (4.123)$$

$$H(s) = \frac{s^2 + 4}{s^2 + 2s + 1} \quad (4.124)$$

Substituting  $s = j\omega$  in equation (4.124),

$$H(j\omega) = \frac{4 - (\omega)^2}{1 + 2j\omega - (\omega)^2} \quad (4.125)$$

Steady state output of system is zero:

$$4 - (\omega)^2 = 0 \quad (4.126)$$

$$\omega = 2 \text{ rad/sec} \quad (4.127)$$



4.14 A causal, discrete time system is described by the difference equation  $y[n] = 0.5y[n - 1] + x[n]$ , for all  $n$ , where  $y[n]$  denotes the output sequence and  $x[n]$  denotes the input sequence. Which of the following statements is/are TRUE?

- (a) The system has an impulse response described by  $0.5^n u[-n]$  where  $u[n]$  is the unit step sequence.
- (b) The system is stable in the bounded input, bounded output sense.
- (c) The system has an infinite number of non-zero samples in its impulse response.
- (d) The system has a finite number of non-zero samples in its impulse response.

(GATE 2023 BM-26)

**Solution:**

4.15 A closed loop system is shown in the figure where  $k > 0$  and  $\alpha > 0$ . The steady state error due to a ramp input ( $R(s) = \alpha s^{-2}$ ) is given by (GATE 2023 EC 41)

(a)  $\frac{2\alpha}{k}$

(b)  $\frac{\alpha}{k}$

(c)  $\frac{\alpha}{2k}$

(d)  $\frac{\alpha}{4k}$

**Solution:**

PARAMETER	DESCRIPTION
$G_c$	Proportional controller's transfer function
$G_f$	Valve transfer function
$G_p$	Process transfer function
$G_M$	Measurement transfer function
$G(s)$	Open loop transfer function
$T(s)$	Transfer function of system

Table 4.2: PARAMETER TABLE 1

Parameter	Description	Value
$k_i$	spring constant	1000 N/m
m	mass of block	20Kg
k	Equivalent spring constant	$k_1 + k_2$ (parallel)
$\omega_n$	Natural frequency	$\sqrt{\frac{k}{m}}$
T	Time period of an oscillation	$\frac{2\pi}{\omega_n}$
x	Displacement of block	
a	Acceleration of block	$\frac{d^2x}{dt^2}$
F	Force on block	
A	Amplitude of oscillation	$x(0)$

Table 4.3: Parameter Table

Variable	Description	Value
$x(t)$	input function	none
$y(t)$	output function	$\sin(\pi t)$
$H(s)$	Transfer-function	$\frac{s-\pi}{s+\pi}$

Table 4.4: Input parameters

Parameter	Description	Value
$X(s)$	position in laplace domain	$X(s)$
$x(t)$	position of buoy w.r.t time	$x(t)$
$m$	mass of buoy	30kg
$\lambda$	damping coefficient of the system	$188.5 \approx 60\pi$
$\omega_o$	natural angular frequency of the system	?
$\omega_d$	damping frequency of the system	$0 \text{ rad s}^{-1}$

Table 4.5: input values

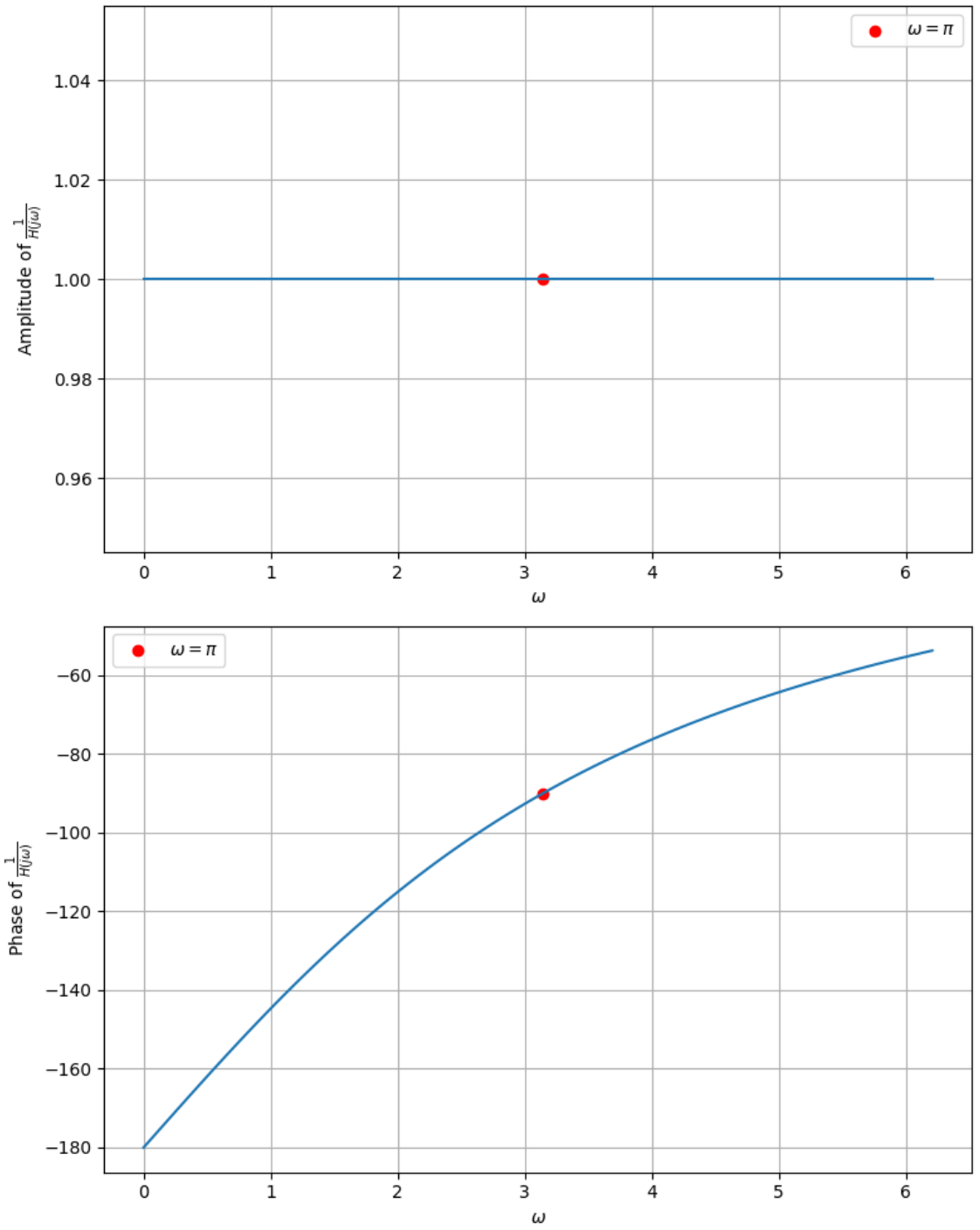


Figure 4.10: Amplitude and phase of  $\frac{1}{H(j\omega)}$

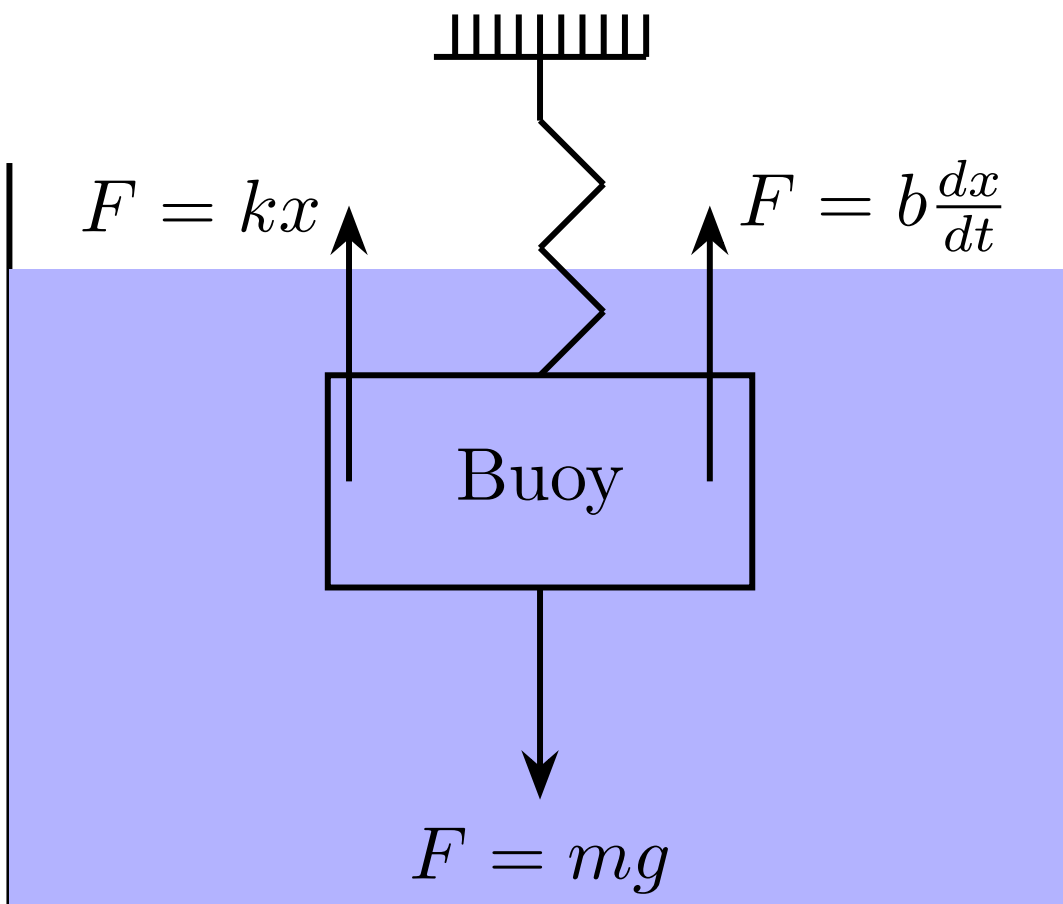


Figure 4.11: Figure given in question

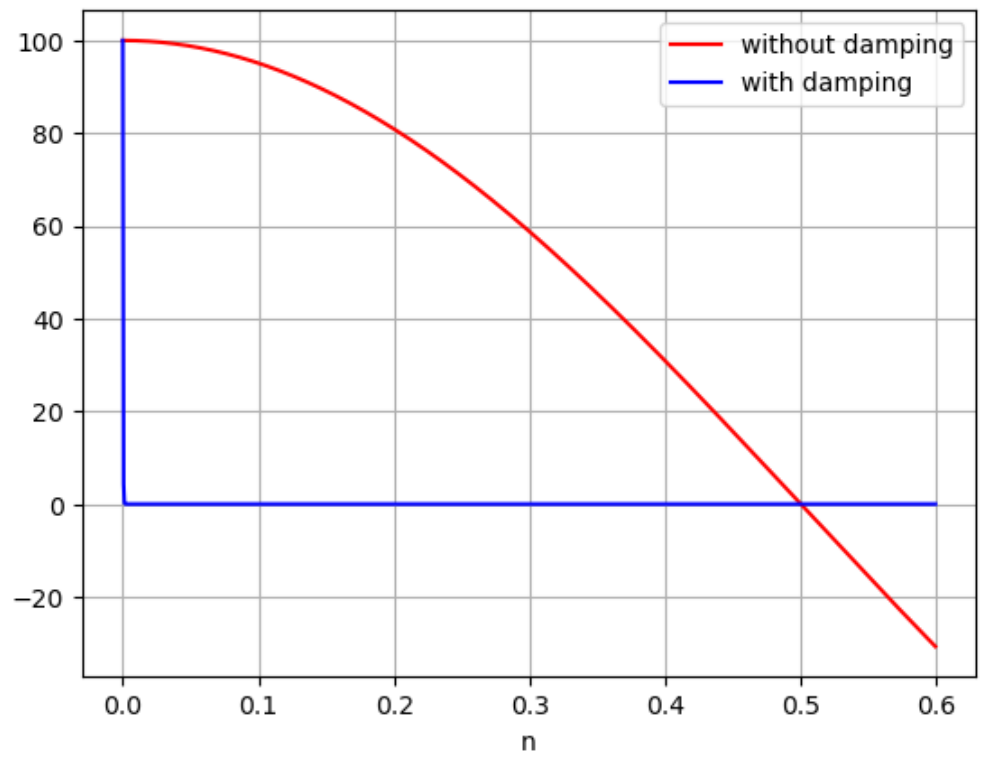


Figure 4.12:  $x(t)$  with and with out damping

Signal Type	Spectral Characteristics
(i) Continuous, aperiodic	(a) Continuous, aperiodic
(ii) Continuous, periodic	(b) Continuous, periodic
(iii) Discrete, aperiodic	(c) Discrete, aperiodic
(iii) Discrete, periodic	(c) Discrete, periodic

Table 4.7:

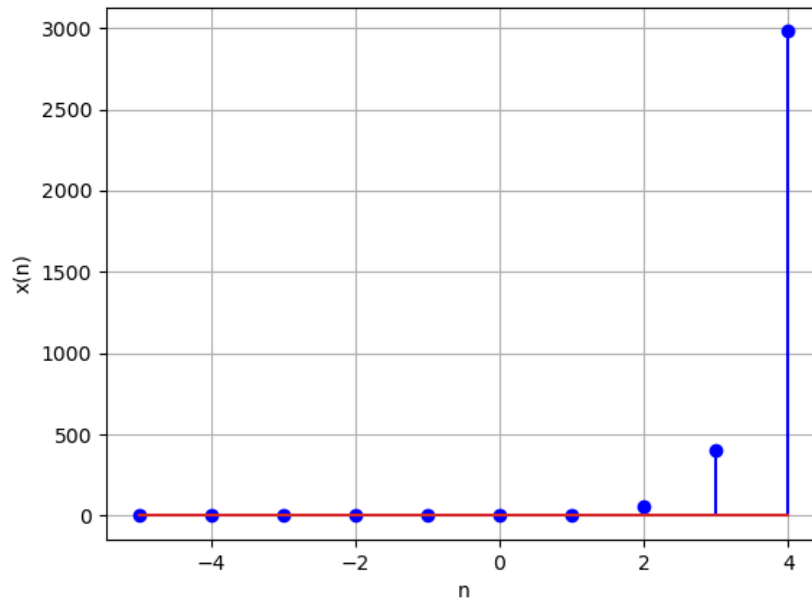


Figure 4.13: graph of  $h(t) = e^{2t} \cdot u(t)$

Variable	Description	value
$\delta(t)$	Dirac delta function	$\infty$ if $t=0$ ; 0 in other cases $\int_{-\infty}^{\infty} \delta(t) dt = 1$
$h(t)$	impulse response	$\delta(t) + 0.5\delta(t - 4)$
$x(t)$	input signal	$x(t) = \cos\left(\frac{7\pi t}{4}\right)$
$y(t)$	output signal	$x(t) * h(t)$
$\mathcal{F}(\cos at)$	Fourier transform of $\cos at$	$0.5 \left[ \delta\left(f - \frac{a}{2\pi}\right) + \delta\left(f + \frac{a}{2\pi}\right) \right]$
$X(f)$	Fourier transform of $x(t)$	$0.5 \left[ \delta\left(f - \frac{7}{8}\right) + \delta\left(f + \frac{7}{8}\right) \right]$
$H(f)$	Fourier transform of $h(t)$	$1 + 0.5e^{-j8\pi f}$
$Y(f)$	Fourier transform of $y(t)$	$X(f)H(f)$

Table 4.8: A Table with input parameters



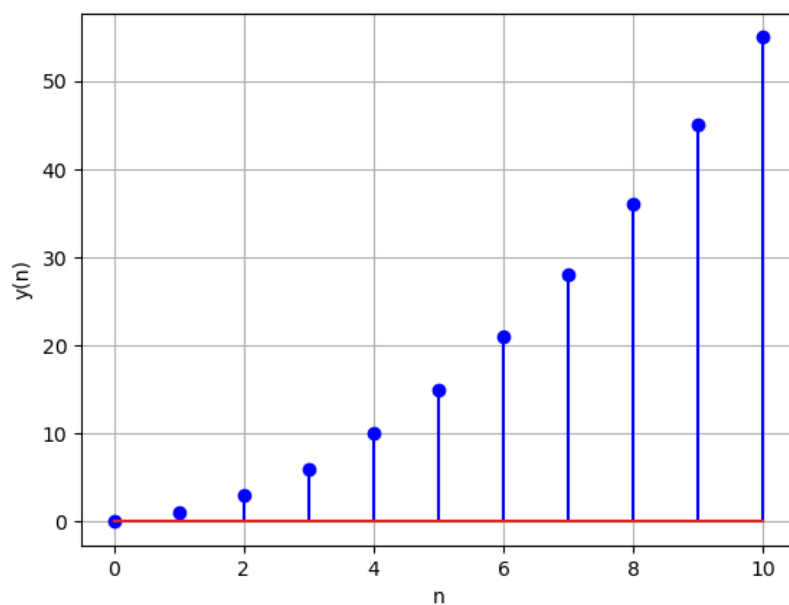


Figure 4.14: graph of  $y[n]$

Parameter	Value
System Matrix, $A$	$\begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}$
Input Matrix, $B$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
Output Matrix, $C$	$\begin{pmatrix} 3 & -2 \end{pmatrix}$
Feedthrough Matrix, $D$	$\begin{pmatrix} 1 \end{pmatrix}$
Input Signal, $u(t)$	$\sin(\omega t), \omega > 0$

Table 4.9: Input Parameters

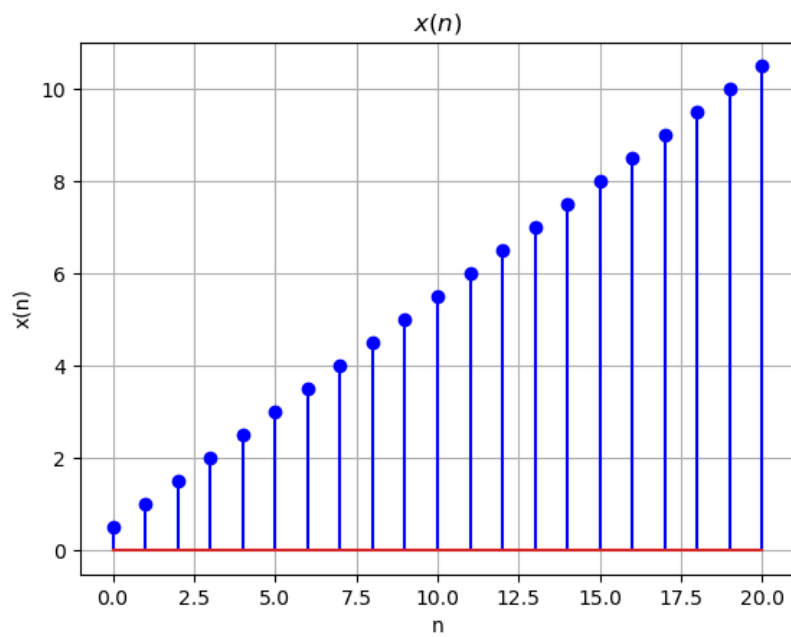
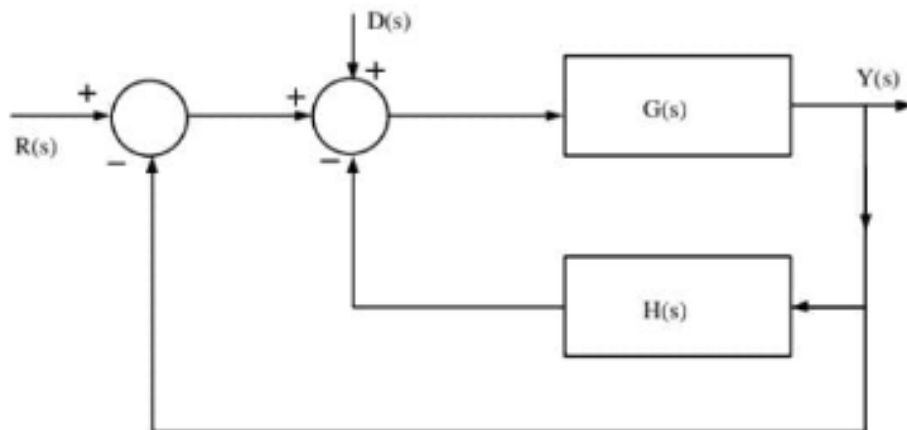


Figure 4.15: graph of  $g[n]$



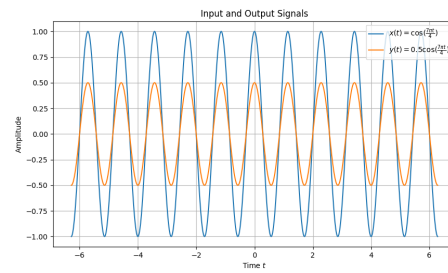
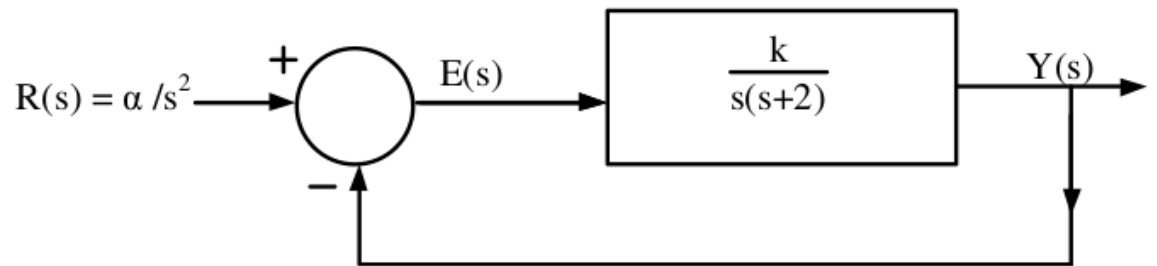


Figure 4.19: Graph showing  $x(t)$  and  $y(t)$





## Chapter 5

# Sequences

5.1 Consider the discrete time signal  $x[n] = u[-n + 5] - u[n + 3]$ , where

$$u[n] = \begin{cases} 1; n \geq 0 \\ 0; n < 0 \end{cases}$$

The smallest  $n$  for which  $x[n] = 0$  is? (GATE IN 2023)

**Solution:** From Fig. 1, the minimum value of  $n$  is given as

$$n = -3 \tag{5.1}$$

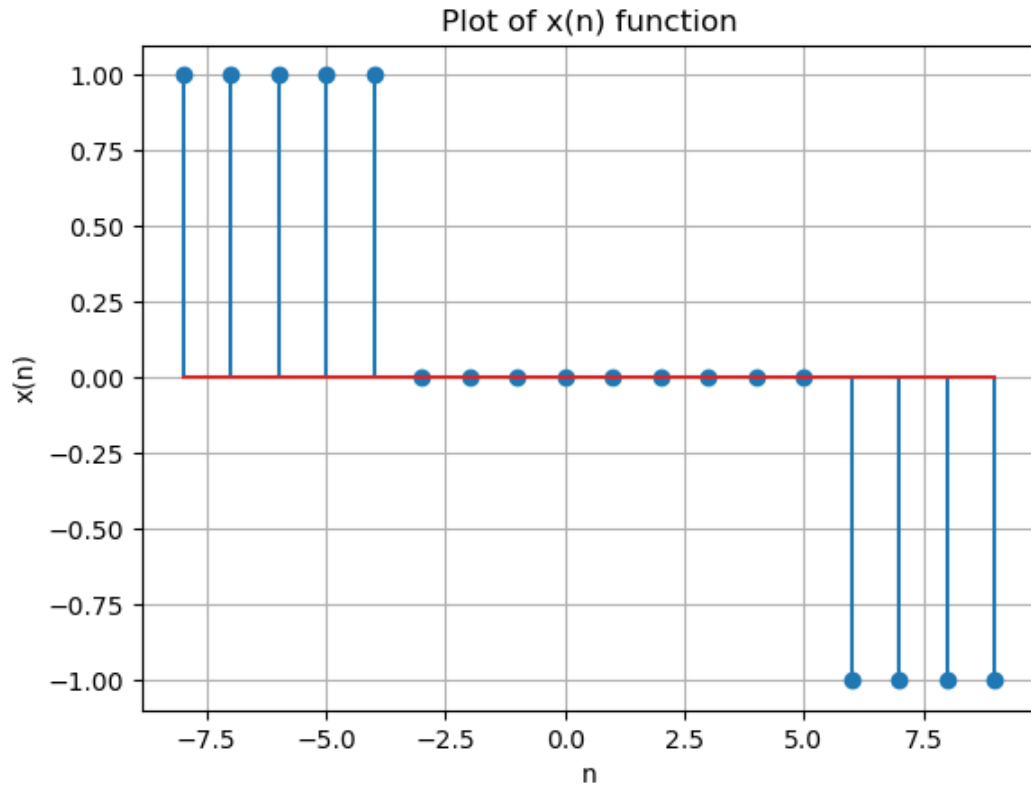


Figure 1: Plot of function  $x(n)$  taken from python3

5.2 Two sequences  $x_1[n]$  and  $x_2[n]$  are described as follows:

$$x_1[0] = x_2[0] = 1 \quad (5.2)$$

$$x_1[1] = x_2[2] = 2 \quad (5.3)$$

$$x_1[2] = x_2[1] = 1 \quad (5.4)$$

$$x_1[n] = x_2[n] = 0 \text{ for all } n < 0 \text{ and } n > 2$$

If  $x[n]$  is obtained by convoluting  $x_1[n]$  with  $x_2[n]$ , which of the following equa-

tions is/are TRUE?

(A)  $x[2] = x[3]$

(B)  $x[1] = 2$

(C)  $x[4] = 3$

(D)  $x[2] = 5$

(GATE 2023 BM 47) **Solution:** From the data given:

$$x_1 = [1, 2, 1] \tag{5.5}$$

$$x_2 = [1, 1, 2] \tag{5.6}$$

Takin the convolution, we get

$$x = [1, 3, 5, 5, 2] \tag{5.7}$$

$$\tag{5.8}$$

Hence we get:

$$x[0] = 1 \tag{5.9}$$

$$x[1] = 3 \tag{5.10}$$

$$x[2] = 5 \tag{5.11}$$

$$x[3] = 5 \tag{5.12}$$

$$x[4] = 2 \tag{5.13}$$

$$\tag{5.14}$$

Comparing this with the options, we see that options (A) and (D) match

$\implies$  (A), (D)



5.3 A series ( $S$ ) is given as  $S=1+3+5+7+9+\dots$ . The sum of the first 50 terms of  $S$  is

(GATE 2023 BT 32)

**Solution:**

Variable	Description	Value
$x(0)$	First term of $AP$	1
$x(1)$	Second term of $AP$	3
$d$	Common difference of $AP$ ( $x(2) - x(1)$ )	2
$x(n)$	$n^{th}$ term of sequence	$(2n + 1)u(n)$

Table 5.1: input parameters

For an  $AP$ ,

$$X(z) = \frac{x(0)}{1 - z^{-1}} + \frac{dz^{-1}}{(1 - z^{-1})^2} \quad (5.15)$$

$$\Rightarrow X(z) = \frac{1}{(1 - z^{-1})} + \frac{2z^{-1}}{(1 - z^{-1})^2}, |z| > 1 \quad (5.16)$$

$$y(n) = x(n) * u(n) \quad (5.17)$$

$$Y(z) = X(z)U(z) \quad (5.18)$$

$$Y(z) = \frac{1}{(1 - z^{-1})^2} + \frac{2z^{-1}}{(1 - z^{-1})^3} \quad (5.19)$$

$$\Rightarrow Y(z) = \frac{(z^{-1} + 1)}{(1 - z^{-1})^3}, |z| > 1 \quad (5.20)$$

Using Contour Integration to find the inverse  $Z$ -transform,

$$y(49) = \frac{1}{2\pi j} \oint_C Y(z) z^{48} dz \quad (5.21)$$

$$= \frac{1}{2\pi j} \oint_C \frac{(z^{-1} + 1)z^{48}}{(1 - z^{-1})^3} dz \quad (5.22)$$

We can observe that the pole is repeated 3 times and thus  $m = 3$ ,

$$R = \frac{1}{(m-1)!} \lim_{z \rightarrow a} \frac{d^{m-1}}{dz^{m-1}} ((z-a)^m f(z)) \quad (5.23)$$

$$\implies R = \frac{1}{(2)!} \lim_{z \rightarrow 1} \frac{d^2}{dz^2} \left( (z-1)^3 \frac{(z^{-1}+1)z^{51}}{(z-1)^3} \right) \quad (5.24)$$

$$\implies R = \frac{1}{2} \lim_{z \rightarrow 1} \frac{d^2}{dz^2} (z^{50} + z^{51}) \quad (5.25)$$

$$\implies R = 2500 \quad (5.26)$$

$$\therefore y(50) = 2500 \quad (5.27)$$

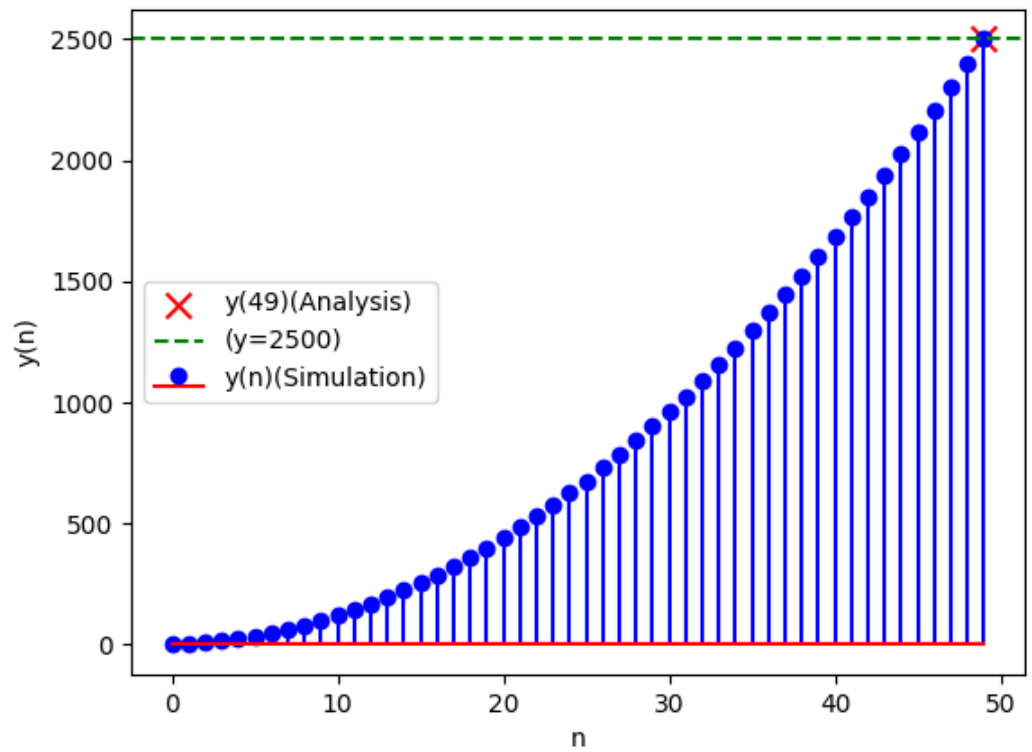


Figure 5.2: Analysis vs Simulation

5.4 For the signals  $x(t)$  and  $y(t)$  shown in the figure,  $z(t) = x(t) * y(t)$  is maximum at  $t = T_1$ . Then  $T_1$  in seconds is ..... (Round off to the nearest integer) **Solution:**

5.5 A series of natural numbers  $F_1, F_2, F_3, F_4, F_5, F_6, F_7, \dots$  obeys  $F_{n+1} = F_n + F_{n-1}$  for all integers  $n \geq 2$ . If  $F_6 = 37$ , and  $F_7 = 60$ , then what is  $F_1$ ?  
[GATE CS 2023]

**Solution:**

5.6 The Lucas sequence  $L_n$  is defined by the recurrence relation:

$$L_n = L_{n-1} + L_{n-2}, \text{ for } n \geq 3$$

with  $L_1=1$  and  $L_2=3$

Which one of the options given is TRUE?

(a)  $L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$

(b)  $L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{3}\right)^n$

(c)  $L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{3}\right)^n$

(d)  $L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n$

(GATE 2023 CS 15)

**Solution:**



## Chapter 6

# Sampling

6.1 An 8 bit ADC converts analog voltage in the range of 0 to  $+5\text{ V}$  to the corresponding digital code as per the conversion characteristics shown in figure. For  $V_{in} = 1.9922\text{ V}$ , which of the following digital output, given in hex, is true?

(a)  $64H$

(b)  $65H$

(c)  $66H$

(d)  $67H$

(GATE EE 40)

**Solution:**



Figure 6.1:

Calculating the step-size:

$$\Delta V_{in} = \frac{V_{max} - V_{min}}{2^n - 1} \quad (6.1)$$

$$= \frac{5 - 0}{2^8 - 1} \quad (6.2)$$

$$= \frac{5}{255} \quad (6.3)$$

$$\Rightarrow V_{out} = \frac{V_{in}}{\Delta V_{in}} \quad (6.4)$$

$$= \frac{1.9922 \times 255}{5} \quad (6.5)$$

$$= 101.59 \quad (6.6)$$

$$\approx 102_{10} \quad (6.7)$$



Symbol	Value	Description
$n$	8	Number of bits of ADC
$V_{min}$	0V	Minimum Analog Voltage
$V_{max}$	5V	Maximum Analog Voltage
$V_{in}$	1.9922V	Input Voltage
$V_{out}$		Output Voltage

Table 6.1: Given Parameters

$\therefore$  correct answer is option (c).

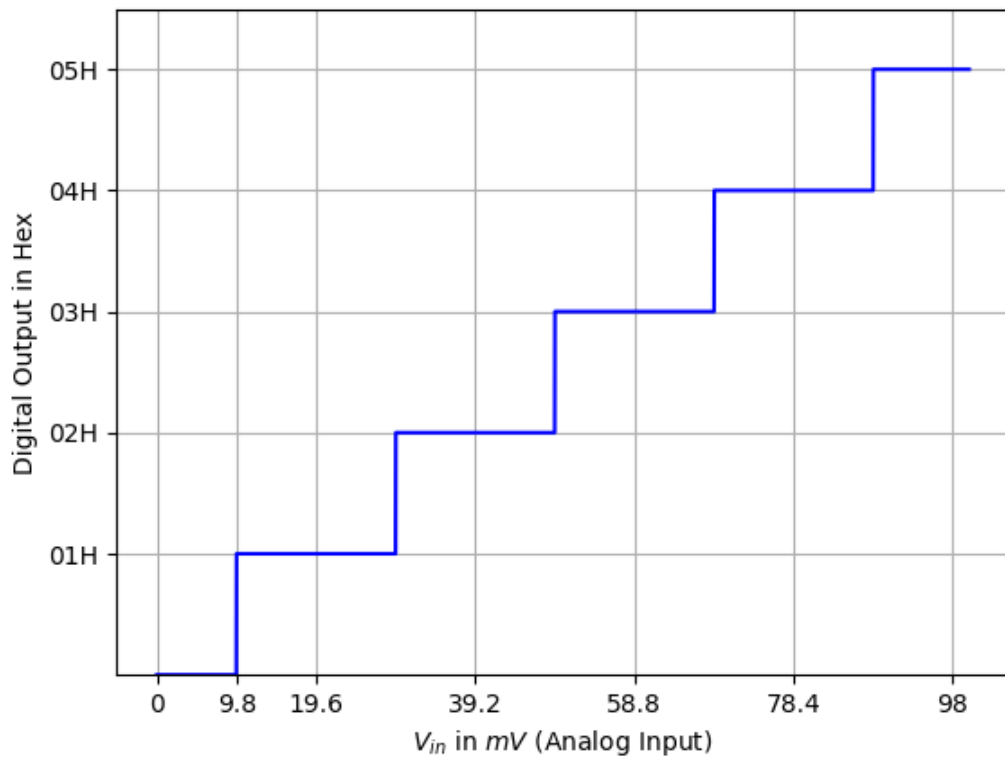
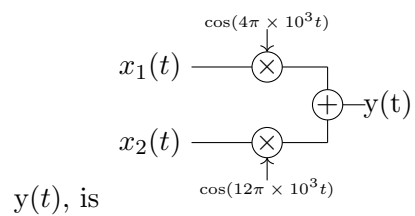


Figure 6.2:

6.2 Let  $x_1(t)$  and  $x_2(t)$  be two band-limited signals having bandwidth  $B = 4\pi \times 10^3$  rad/s each. In the figure below, the Nyquist sampling frequency, in rad/s, required to sample



(a)  $20\pi \times 10^3$

(b)  $40\pi \times 10^3$

(c)  $8\pi \times 10^3$

(d)  $32\pi \times 10^3$

(GATE EC 50)

**Solution:**

6.3 An 8 bit successive approximation Analog to Digital Converter (ADC) has a clock frequency of 1 MHz. Assume that the start conversion and end conversion signals occupy one clock cycle each. Among the following options, what is the maximum frequency that this ADC can sample without aliasing?

- a) 0.9 kHz
- b) 9.9 kHz
- c) 49.9 kHz
- d) 99.9 kHz

(GATE BM 2023)

**Solution:**

6.4 The period of the discrete-time signal  $x[n]$  described by the equation below is  $N =$   
(Round off to the nearest integer).

$$x[n] = 1 + 3 \sin \left( \frac{15\pi}{8}n + \frac{3\pi}{4} \right) - 5 \sin \left( \frac{\pi}{3}n - \frac{\pi}{4} \right)$$

(GATE 2023 EE) **Solution:**



## Chapter 7

# Contour Integration

7.1 The value of the contour integral,  $\oint_C \frac{z+2}{z^2+2z+2} dz$ , where the contour  $C$  is  $\{z : |z+1-\frac{3}{2}i| = 1\}$ , taken in the counter clockwise direction, is

(A)  $-\pi(1+j)$

(B)  $\pi(1+j)$

(C)  $\pi(1-j)$

(D)  $-\pi(1-j)$

(GATE EC 2023)

**Solution:**

$$I = \oint_C \frac{z+2}{z^2+2z+2} dz; \quad C = \left| z+1-\frac{3}{2}i \right| = 1 \quad (7.1)$$

The poles are given by  $(z+1)^2 + 1 = 0$

$$z+1 = \pm\sqrt{-1} \quad (7.2)$$

$$z = -1 + j, z = -1 - j$$

where  $-1 - i$  lies outside  $C$  and  $z = (-1, 1)$  lies inside  $C$ , by the Residue Theorem:

$$\oint_C f(z) dz = 2\pi i \operatorname{Res}(f(z), z = -1 + j) \quad (7.3)$$

$$= 2\pi i \left( \frac{z+2}{2(z+1)} \right) \Big|_{z=-1+i} \quad (7.4)$$

$$= 2\pi i \left( \frac{-1+j+2}{2(-1+j+1)} \right) \quad (7.5)$$

$$= \pi(1+j). \quad (7.6)$$

Therefore, the correct answer is option (B).

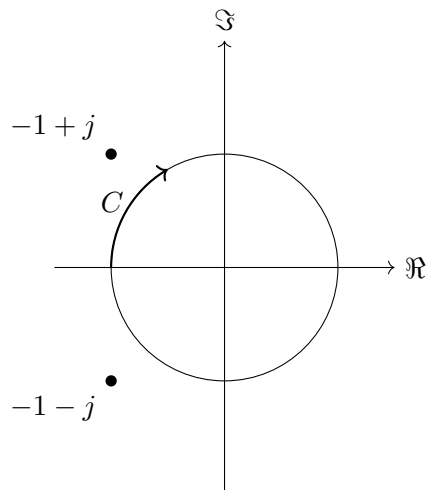


Figure 7.1: Contour  $C$  and poles



7.2 The function  $f(Z) = \frac{1}{Z-1}$  of a complex variable  $Z$  is integrated on a closed contour in an anti-clockwise direction. For which of the following contours, does this integral have a non-zero value?

(A)  $|z - 2| = 0.01$

(B)  $|z - 1| = 0.1$

(C)  $|z - 3| = 5$

(D)  $|z| = 2$

(GATE 2023 BM)

**Solution:**

7.3 Consider the contour integral  $\oint \frac{dz}{z^4+z^3-2z^2}$ , along the curve  $|z| = 3$  oriented in the counterclockwise direction. If  $\text{Res}[f, z_0]$  denotes the residue of  $f(z)$  at the point  $z_0$ , then which of the following are TRUE?

- (A)  $\text{Res}[f, 0] = -\frac{1}{4}$
- (B)  $\text{Res}[f, 1] = \frac{1}{3}$
- (C)  $\text{Res}[f, -2] = -\frac{1}{12}$
- (D)  $\text{Res}[f, 2] = -1$

(GATE NM 2023)

**Solution:**

## Chapter 8

# Laplace Transform

8.1 The number of zeroes of the polynomial  $P(s) = s^3 + 2s^2 + 5s + 80$  in the right side of the plane? (GATE IN 2023)

**Solution:** The table below shows the Routh array of the  $n^{th}$ - order characteristic polynomial :

$$a_0s^n + a_1s^{n-1} + \dots + a_{n-1}s^1 + a_ns^0 \quad (8.1)$$

$s^n$	$a_0$	$a_2$	$a_4$	...
$s^{n-1}$	$a_1$	$a_3$	$a_5$	...
$s^{n-2}$	$b_1 = \frac{a_1a_2 - a_3a_0}{a_1}$	$b_2 = \frac{a_1a_4 - a_5a_0}{a_1}$	...	..
$s^{n-3}$	$c_1 = \frac{b_1a_3 - b_2a_1}{b_1}$	$\vdots$		
$\vdots$	$\vdots$	$\vdots$		
$s^1$	$\vdots$	$\vdots$		
$s^0$	$a_n$			

Table 8.1: Routh Array

Characteristic Equation:

$$s^3 + 2s^2 + 5s + 80 = 0 \quad (8.2)$$

From Table 8.1:

$s^3$	1	5
$s^2$	2	80
$s^1$	$\frac{2 \times 5 - 80 \times 1}{2} = -35$	
$s^0$	$\frac{-35 \times 80}{-35} = 80$	

Table 8.2:

From Table 8.2:

Since there are 2 sign changes in the first column of the Routh tabulation. So, the number of zeros in the right half of the s-plane will be 2.

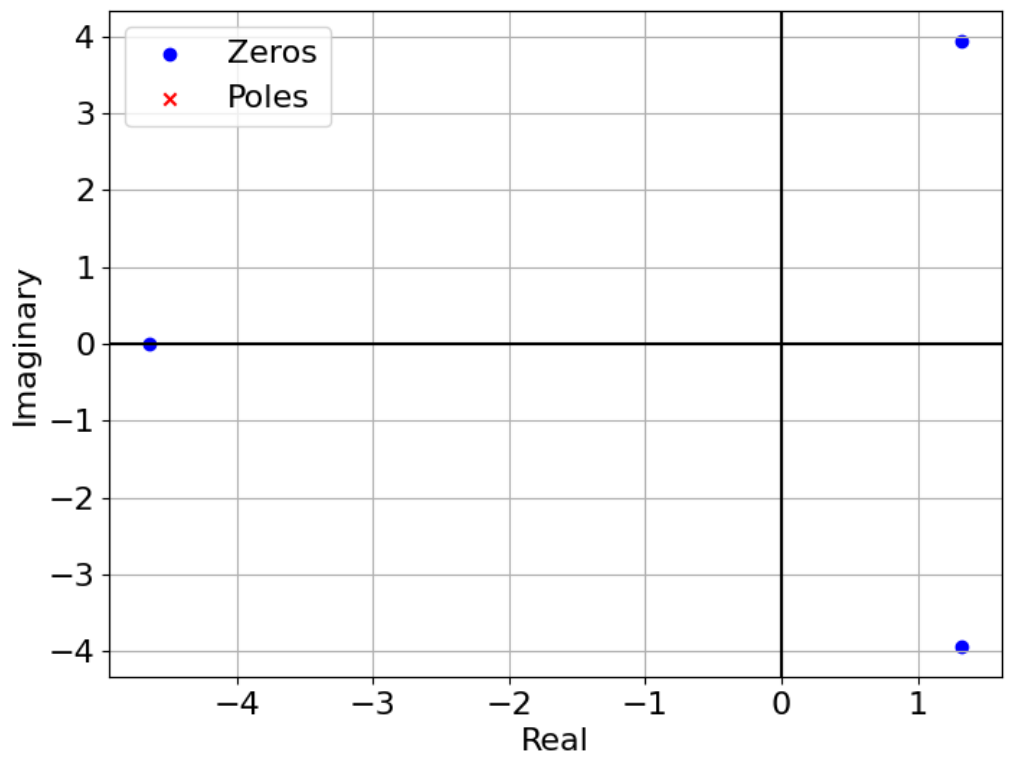
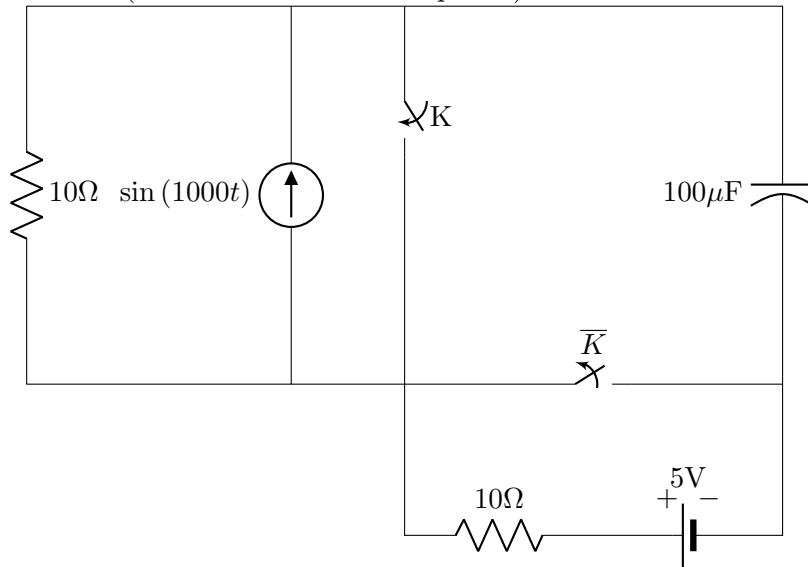


Figure 8.1:

8.2 The circuit shown in the figure is initially in the steady state with the switch K in open condition and  $\overline{K}$  in closed condition. The switch K is closed and  $\overline{K}$  is opened simultaneously at the instant  $t = t_1$ , where  $t_1 > 0$ . The minimum value of  $t_1$  in milliseconds such that there is no transient in the voltage across the  $100\ \mu F$  capacitor, is \_\_\_\_ (Round off to 2 decimal places) (GATE EE 2023)



**Solution:**

(a) Switch K is open and  $\overline{K}$  is closed.

Using Current divider rule,

$$I_1(j\omega) = \frac{10}{10 + \frac{1}{j\omega C}} \quad (8.3)$$

$$V_1(j\omega) = \frac{10}{1 + 10j\omega C} \quad (8.4)$$

$$|V_1(j\omega)| = 5\sqrt{2} \quad (8.5)$$

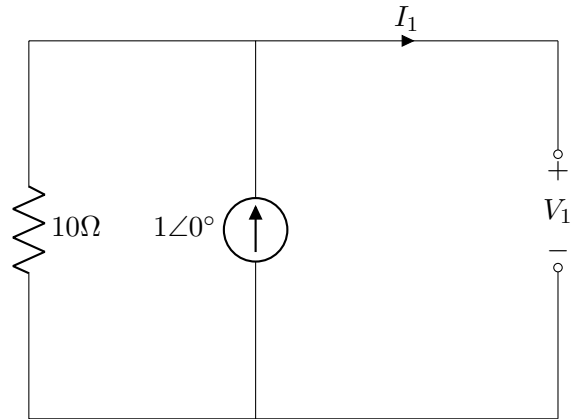


Figure 8.2: K is open and  $\overline{K}$  is closed

From Table 8.3

$$V_1(t) = 5\sqrt{2} \sin\left(\omega t - \frac{\pi}{4}\right) \quad (8.6)$$

(b) Switch K is closed and  $\overline{K}$  is open.

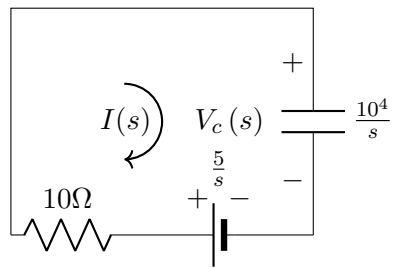


Figure 8.3: K is closed and  $\overline{K}$  is open

The capacitor is charged. Thus, acts as a voltage source.

From eq(8.6) and Table 8.4

$$V_1(s) = \frac{5000 - 5s}{s^2 + 10^6} \quad (8.7)$$

$$I(s) = \frac{\frac{5}{s} - V_1(s)}{10 + \frac{10^4}{s}} \quad (8.8)$$

$$V_c(s) = \frac{5}{s} - 10 \left( \frac{5 - V_1(s)}{1 + 10^{-3}s} \right) \quad (8.9)$$

For transient analysis,

$$\frac{5 - V_1(s)}{1 + 10^{-3}s} = 0 \quad (8.10)$$

$$\implies V_1(s) = 5 \quad (8.11)$$

$$\frac{10^7}{(s^2 + 10^6)(s + 10^3)} = \frac{5}{s} \quad (8.12)$$

$$\frac{5}{s + 10^3} + \frac{10^3 - s}{s^2 + 10^6} = \frac{5}{s} \quad (8.13)$$

$$\frac{-s}{s^2 + 10^6} + \frac{10^3}{s^2 + 10^6} + \frac{1}{s + 10^3} = \frac{1}{s} \quad (8.14)$$

From Table 8.4

$$-\cos(1000t_1) + \sin(1000t_1) + e^{-10^3 t_1} = 1 \quad (8.15)$$

$$\implies t_1 \approx 1.57 \text{msec} \quad (8.16)$$



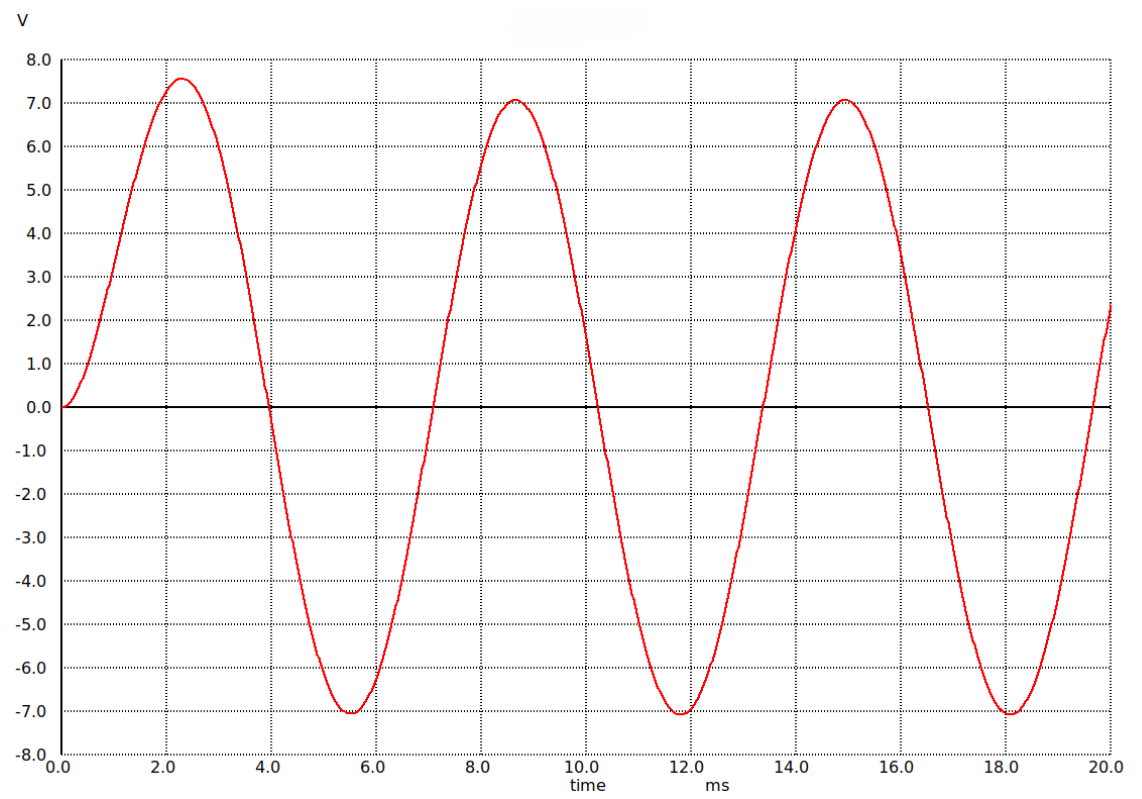


Figure 8.4: plot of  $V_1$  vs time

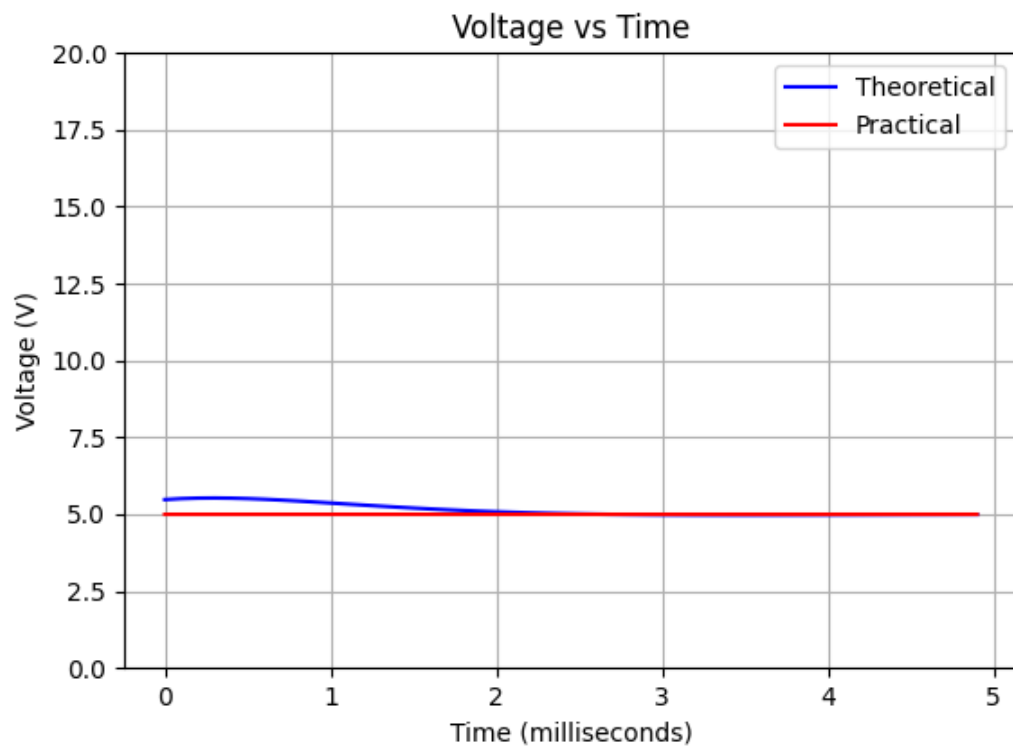


Figure 8.5: plot of  $V_c$  vs time

Parameter	Description	Remarks
$\omega$	frequency of sine-wave	1000 rad s <sup>-1</sup>
$V_1(t)$	Voltage across capacitor	$ V_1(j\omega)  \sin(\omega t - \angle V_1(j\omega))$
$\angle V_1(j\omega)$	phase of $V_1(j\omega)$	$\frac{-\pi}{4}$
$C$	Capacitance	100 $\mu$ F

Table 8.3: Parameters

S Domain	Time Domain
$\frac{1}{s}$	$u(t)$
$\frac{-s}{a^2+s^2}$	$-\cos(at)$
$\frac{a}{a^2+s^2}$	$\sin(at)$
$\frac{1}{s+a}$	$e^{-at}$

Table 8.4: Laplace transforms

8.3  $y = e^{mx} + e^{-mx}$  is the solution of which differential equation?

1.  $\frac{dy}{dx} - my = 0$
2.  $\frac{dy}{dx} + my = 0$
3.  $\frac{d^2y}{dx^2} + m^2y = 0$
4.  $\frac{d^2y}{dx^2} - m^2y = 0$

(GATE AG 2023) **Solution:**

8.4 A cascade control strategy is shown in the figure below. The transfer function between the output ( $y$ ) and the secondary disturbance ( $d_2$ ) is defined as

$$G_{d2}(s) = \frac{y(s)}{d_2(s)}$$

. Which one of the following is the CORRECT expression for the transfer function  $G_{d2}(s)$ ?



Figure 8.6:

- A.  $\frac{1}{(11s+21)(0.1s+1)}$
- B.  $\frac{1}{(s+1)(0.1s+1)}$
- C.  $\frac{(s+1)}{(s+2)(0.1s+1)}$
- D.  $\frac{(s+1)}{(s+1)(0.1s+1)}$

(GATE CH 2023) **Solution:**

Variable	Description
$d_1(s)$	Primary disturbance
$d_2(s)$	Secondary disturbance
$G_{d2}(s)$	Transfer function between $y(s)$ and $d_2(s)$
$y_{set}(s)$	Set point for desired output
$y(s)$	Output

Table 8.5: Input Parameters

Variable	Description	value
$P_1$	Forward path gain e-c-d	$\frac{1}{(0.1s+1)(s+1)}$
$\Delta_1$	Determinant of forward path e-c-d	1
$\Delta$	Determinant of system	$1 + \frac{10}{s+1} + 10$
$n$	Number of forward path	1

Table 8.6: Defined Parameters

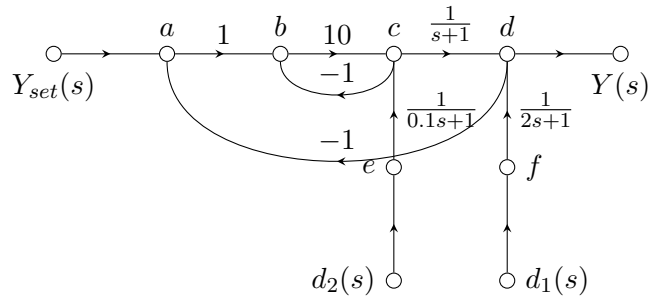


Figure 8.7: signal flow graph

Using Mason's Gain formula for the above Signal flow graph,

$$G_{d2}(s) = \frac{y(s)}{d_2(s)} = \frac{\sum_{i=1}^n P_i \Delta_i}{\Delta} \quad (8.17)$$

$$= \frac{P_1 \Delta_1}{\Delta} \quad (8.18)$$

$$= \frac{1}{\frac{(0.1s+1)(s+1)}{1 + \frac{10}{s+1} + 10}} \quad (8.19)$$

$$= \frac{1}{\frac{(0.1s+1)(s+1)}{\frac{11s+21}{s+1}}} \quad (8.20)$$

$$\Rightarrow G_{d2}(s) = \frac{1}{(11s + 21)(0.1s + 1)} \quad (8.21)$$

Now taking the inverse laplace transform we have,

$$G_{d2}(t) = \mathcal{L}^{-1} \left( \frac{10}{(s + 10)(11s + 21)} \right) \quad (8.22)$$

$$= \mathcal{L}^{-1} \left( \frac{-10}{89(x + 10)} + \frac{110}{89(11x + 21)} \right) \quad (8.23)$$

$$= \frac{-10e^{-10t}}{89} u(t) + \frac{10e^{-\frac{21t}{11}}}{89} u(t) \quad (8.24)$$

$$= \left( \frac{10 \left( e^{-\frac{21t}{11}} - e^{-10t} \right)}{89} \right) u(t) \quad (8.25)$$

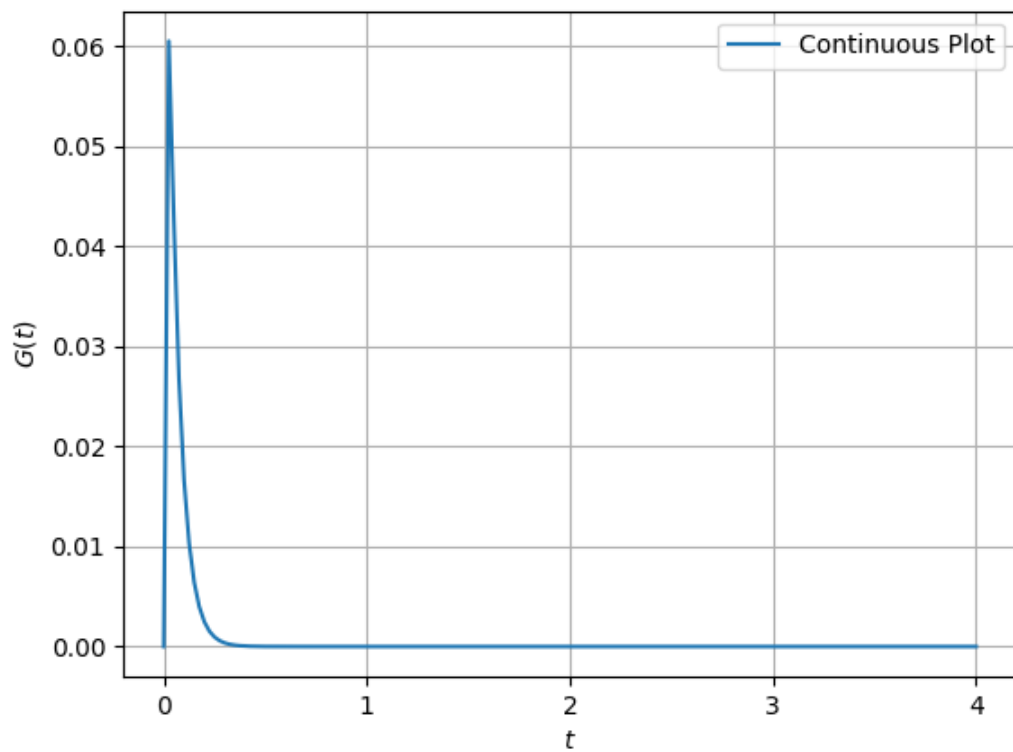
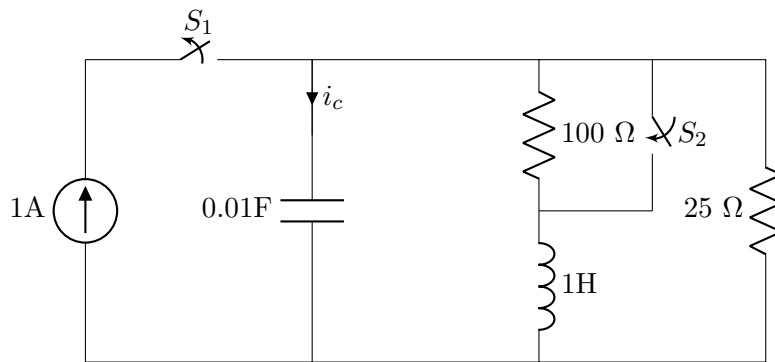


Figure 8.8:

8.5 In the differential equation  $\frac{dy}{dx} + \alpha xy = 0$ ,  $\alpha$  is a positive constant. If  $y = 1.0$  at  $x = 0.0$ , and  $y = 0.8$  at  $x = 1.0$ , the value of  $\alpha$  is (rounded off to three decimal places). (GATE CE 2023) **Solution:**

8.6 The switch  $S_1$  was closed and  $S_2$  was open for a long time. At  $t=0$ , switch  $S_1$  is opened and  $S_2$  is closed, simultaneously. The value of  $i_c(0^+)$ , in amperes, is (GATE EC 44)



**Solution:**

1) Switch  $S_1$  was closed and  $S_2$  was open

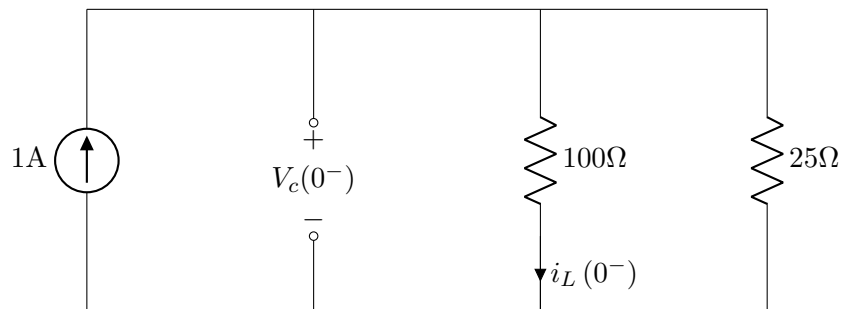


Figure 8.9:  $S_1$  is closed and  $S_2$  is open

$$R_{eff} = \frac{25(100)}{(25 + 100)} \Omega \quad (8.26)$$

$$R_{eff} = 20 \Omega \quad (8.27)$$



$$V_c(0^-) = 1(R_{eff}) \quad (8.28)$$

$$V_c(0^-) = 20V \quad (8.29)$$

2) Switch  $S_1$  is open and  $S_2$  was closed

At  $t = 0^+$  The capacitor is charged. Thus, it acts as a voltage source. The inductor acts as the current source.

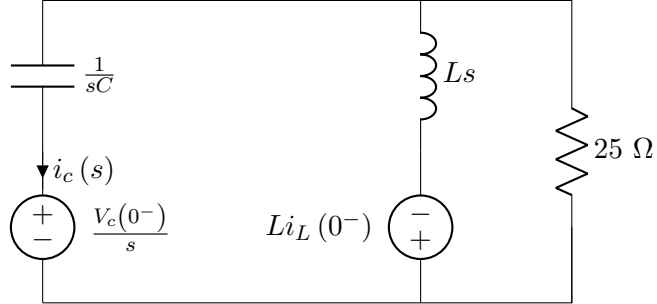


Figure 8.10:  $S_1$  is open and  $S_2$  is closed

By Superposition Theorem

Case (i):

$$\left( \frac{25(Ls)}{25 + Ls} + \frac{1}{sC} \right) i_c^1(s) + \frac{V_c(0^-)}{s} = 0 \quad (8.30)$$

$$i_c^1(s) = -\frac{V_c(0^-)}{s} \left( \frac{25sC + LCs^2}{25LCs^2 + Ls + 25} \right) \quad (8.31)$$

$$i_c^1(s) = -\left( \frac{5 + 0.2s}{0.25s^2 + s + 25} \right) \quad (8.32)$$

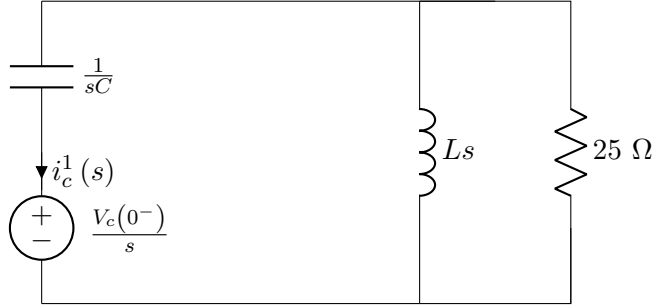


Figure 8.11: Circuit 4

Case (ii):

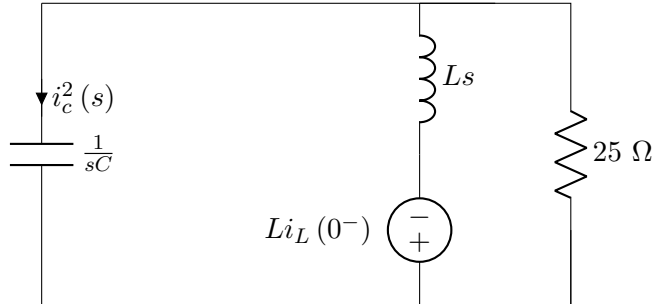


Figure 8.12: Circuit 5

$$Li_L(0^-) + \left( \frac{25}{25sC + 1} + Ls \right) i_c^2(s) = 0 \quad (8.33)$$

$$0.2 = - \left( \frac{25LCs^2 + Ls + 25}{25sC + 1} \right) i_c^2(s) \quad (8.34)$$

$$i_c^2(s) = - \left( \frac{0.05s + 0.2}{0.25s^2 + s + 25} \right) \quad (8.35)$$

From eq (8.32) and eq (8.35)

$$i_c(s) = i_c^1(s) + i_c^2(s) \quad (8.36)$$

$$i_c(s) = - \left( \frac{0.25s + 5.2}{0.25s^2 + s + 25} \right) \quad (8.37)$$

Using Inverse Laplace Transform

From eq (8.37)

$$i_c(t) = -e^{-2t} \left( \cos(4\sqrt{6}t) + \frac{18.8}{4\sqrt{6}} \sin(4\sqrt{6}t) \right) \quad (8.38)$$

$$i_c(0^+) = -1A \quad (8.39)$$

Parameter	Description	Remarks
$V_c(0^-)$	Voltage across capacitor when $t < 0$	20V
$i_L(0^-)$	current across inductor when $t < 0$	0.2
$i_L(0^+)$	current across inductor when $t > 0$	0.2
$C$	Capacitance	0.01F
$L$	Inductance	1H

Table 8.7: Parameters

8.7 The continuous time signal  $x(t)$  is described by:

$$x(t) = \begin{cases} 1, & \text{if } 0 \leq t \leq 1 \\ 0, & \text{elsewhere} \end{cases} \quad (8.40)$$

If  $y(t)$  represents  $x(t)$  convolved with itself, which of the following options is/are TRUE?

A  $y(t) = 0$  for all  $t < 0$

B  $y(t) = 0$  for all  $t > 1$

C  $y(t) = 0$  for all  $t > 3$

D  $\int_{0.1}^{0.75} \frac{dy(t)}{dt} dt \neq 0$

**Solution:**

8.8 The Z-transform of a discrete signal  $x(n)$  is

$$X(z) = \frac{4z}{\left(z - \frac{1}{5}\right)\left(z - \frac{2}{3}\right)(z - 3)} \text{ with ROC} = R \quad (8.41)$$

Which one of the following statements is TRUE?

- (a) Discrete time Fourier transform of  $x[n]$  converges if  $R$  is  $|z| > 3$
- (b) Discrete time Fourier transform of  $x[n]$  converges if  $R$  is  $\frac{2}{3} < |z| < 3$
- (c) Discrete time Fourier transform of  $x[n]$  converges if  $R$  is such that  $x[n]$  is a left-sided sequence.
- (d) Discrete time Fourier transform of  $x[n]$  converges if  $R$  is such that  $x[n]$  is a right-sided sequence.

GATE EE 2023

**Solution:** Poles of  $X(z)$  are located at  $z = \frac{1}{5}$ ,  $z = \frac{2}{3}$ , and  $z = 3$ .

For DTFT to converge, the ROC of Z-transform of  $x[n]$  should contain unit circle.

- (a) If ROC is  $|z| > 3$ , it does not include unit circle

Option (a) is wrong.

- (b) If ROC is  $\frac{2}{3} < |z| < 3$ , the ROC includes unit circle.

So, option (b) is correct.

(c) If  $x(n)$  is a left-sided sequence, then ROC will be  $|z| < \frac{1}{5}$ , which does not include the unit circle.

Option (c) is wrong.

(d) If  $x(n)$  is a right-sided sequence, then the ROC is  $|z| > 3$ , which does not include the unit circle.

Option (d) is wrong.

Hence, the correct option is (b).

- 8.9 The phase margin of the transfer function  $G(s) = \frac{2(1-s)}{(1+s)^2}$  is \_\_\_\_\_ degrees. (rounded off to the nearest integer). (GATE IN 2023)

**Solution:**

Parameters	Description
$\omega_c$	crossover frequency
$\angle G(j\omega)$	phase angle of the transfer function
$PM$	$\angle G(j\omega_c) + 180^\circ$ ; Phase Margin

Table 8.8: Parameters

Considering  $s = j\omega$ ,

$$G(j\omega) = \frac{2(1 - j\omega)}{(1 + j\omega)^2} \quad (8.42)$$

$$= \frac{2(1 - j\omega)^3}{|1 + j\omega|^4} \quad (8.43)$$

$$= \frac{2}{(1 + \omega^2)^2} (1 - j\omega)^3 \quad (8.44)$$

$$= \frac{2}{\sqrt{1 + \omega^2}} (e^{-j\omega})^3 \quad (8.45)$$

$$\Rightarrow |G(j\omega)| = \frac{2}{\sqrt{1 + \omega^2}} \quad (8.46)$$

$$\Rightarrow \angle G(j\omega) = 3 \tan^{-1}(-\omega) \quad (8.47)$$

At  $\omega = \omega_c$ ,  $Gain = 0$

$$\Rightarrow |G(j\omega_c)| = 1 \quad (8.48)$$

$$\frac{2}{\sqrt{1 + \omega_c^2}} = 1 \quad (8.49)$$

$$\Rightarrow \omega_c = \sqrt{3} \quad (8.50)$$

$$\angle G(j\omega_c) = 3 \tan^{-1}(-\sqrt{3}) \quad (8.51)$$

$$= -180^\circ \quad (8.52)$$

From Table 8.8,

$$PM = 0^\circ \tag{8.53}$$



8.10 Consider the second-order linear differential equation

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - y = 0, \quad x \geq 1$$

with the initial conditions

$$y(x=1) = 6, \quad \left. \frac{dy}{dx} \right|_{x=1} = 2.$$

Then the value of  $y$  at  $x = 2$  is \_\_\_\_\_.

GATE ME 2023

**Solution:**

Symbol	Value	Description
$y(x)$	?	Function
$y(1)$	6	Initial Condition
$y'(1)$	2	Initial Condition

Table 8.9: Given Information

By Euler-Cauchy substitution for the given question,

$$x = e^t, \quad t \geq 0 \tag{8.54}$$

$$\implies \frac{dt}{dx} = e^{-t} \tag{8.55}$$

$$\implies \frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx} = e^{-t} \frac{dy}{dt} \tag{8.56}$$

$$\implies \frac{d^2 y}{dx^2} = \frac{d^2 y}{dt^2} \left( \frac{dt}{dx} \right)^2 + \frac{d^2 t}{dx^2} \left( \frac{dy}{dt} \right) \tag{8.57}$$

$$= e^{-2t} \frac{d^2 y}{dt^2} + e^{-2t} \frac{dy}{dt} \tag{8.58}$$

So the given equation becomes

$$e^{2t} \left( e^{-2t} \frac{d^2 y}{dt^2} - e^{-2t} \frac{dy}{dt} \right) + e^{-t} \left( e^{-t} \frac{dy}{dt} \right) - y = 0 \quad (8.59)$$

$$\frac{d^2 y}{dt^2} - \frac{dy}{dt} + \frac{dy}{dt} - y = 0 \quad (8.60)$$

$$\frac{d^2 y}{dt^2} - y = 0, \quad t \geq 0 \quad (8.61)$$

Taking Laplace Transform,

$$s^2 Y(s) - sy(0) - y'(0) - Y(s) = 0 \quad (8.62)$$

$$Y(s) = \frac{sy(0) + y'(0)}{s^2 - 1} \quad (8.63)$$

$$= \frac{6s + 2}{s^2 - 1} \quad (8.64)$$

By partial fractions,

$$Y(s) = \frac{4}{s-1} + \frac{2}{s+1} \quad (8.65)$$

Taking Inverse Laplace using

$$\frac{1}{s+a} \xleftrightarrow{\mathcal{L}^{-1}} e^{-at} \quad (8.66)$$

We get,

$$y(t) = 4e^t + 2e^{-t} \quad (8.67)$$

From (8.54)

$$y(t) = 4x + \frac{2}{x} \quad (8.68)$$

$$\implies y(2) = 9 \quad (8.69)$$

8.11 The transfer function of a measuring instrument is

$$G_m(s) = \frac{1.05}{2s + 1} \exp(-s)$$

At time  $t = 0$ , a step change of +1 unit is introduced in the input of this instrument. The time taken by the instrument to show an increase of 1 unit in its output is (rounded off to two decimal places).

(GATE CH 2023) **Solution:**

8.12 The laplace transform of  $x_1(t) = e^{-t}u(t)$  is  $X_1(s)$ , where  $u(t)$  is the unit step function. The laplace transform of  $x_2(t) = e^t u(-t)$  is  $X_2(s)$ . Which one of the following statements is TRUE?

- (a) The region of convergence of  $X_1(s)$  is  $\text{Re}(s) \geq 0$
- (b) The region of convergence of  $X_2(s)$  is confined to the left half-plane of  $s$ .
- (c) The region of convergence of  $X_1(s)$  is confined to the right half-plane of  $s$ .
- (d) the imaginary axis in the  $s$ -plane is included in both the region of convergence of  $X_1(s)$  and the region of convergence of  $X_2(s)$ .

(GATE BM 2023)

**Solution:**

Symbols	Description
$X_1(s)$	Laplace transform of $x_1(t)$
$X_2(s)$	Laplace transform of $x_2(t)$
$u(t)$	Unit step function

Table 8.10: Parameters, Descriptions

(a) Laplace transform of  $x_1(t)$  is given by :

$$X_1(s) = \int_{-\infty}^{\infty} e^{-t} e^{-st} u(t) dt \quad (8.70)$$

Let  $s = \sigma + j\omega$  :

$$X_1(s) = \int_0^{\infty} e^{-t(\sigma+1)} e^{-tj\beta} dt \quad (8.71)$$

$$= \left[ \frac{-e^{-t(\sigma+1)} e^{-tj\beta}}{(\sigma+1) + j\beta} \right]_0^{\infty} \quad (8.72)$$

For  $X_1(s)$  to be convergent,  $|-e^{-t(\sigma+1)} e^{-tj\beta}|$  must converge  $\forall t \in (0, \infty)$ , so:

$$|e^{-tj\beta}| = |1|, \forall \beta \in \mathbb{R} \implies \text{Im}(s) \in \mathbb{R} \quad (8.73)$$

$$\sigma + 1 > 0 \implies \Re(s) > -1 \quad (8.74)$$

Putting the limits :

$$X_1(s) = \frac{1}{s+1}, \Re(s) > -1 \quad (8.75)$$

(b) Laplace transform of  $x_2(t)$  is given by :

$$X_2(s) = \int_{-\infty}^{\infty} e^t e^{-st} u(-t) dt \quad (8.76)$$

Let  $s = \sigma + j\omega$  :

$$= \int_{-\infty}^0 e^{t(1-\sigma)} e^{-tj\beta} dt \quad (8.77)$$

$$= \left[ \frac{e^{t(1-\sigma)} e^{-tj\beta}}{(1-\sigma) - j\beta} \right]_{-\infty}^0 \quad (8.78)$$

For  $X_2(s)$  to be convergent,  $|e^{t(1-\sigma)} e^{-tj\beta}|$  must converge  $\forall t \in (-\infty, 0)$ , so:

$$|e^{-tj\beta}| = |1|, \forall \beta \in \mathbb{R} \implies \text{Im}(s) \in \mathbb{R} \quad (8.79)$$

$$1 - \sigma > 0 \implies \Re(s) < 1 \quad (8.80)$$

Putting the limits

$$X_2(s) = \frac{1}{1-s}, \Re(s) < 1 \quad (8.81)$$

Based on the overlap of regions of convergence of  $X_1(s)$  and  $X_2(s)$  from 8.18 , we can conclude that option 4) is correct .

8.13 Given that  $\frac{dy}{dx} = 2x + y$  and  $y = 1$ , when  $x = 0$  Using Runge-Kutta fourth order method, the value of  $y$  at  $x = 0.2$  is (GATE 2023 AG 50)

**Solution:**

8.14 The magnitude and phase plots shown in the figure match with the transfer- function

(a)  $\frac{10000}{s^2 + 2s + 10000}$

(b)  $\frac{10000}{s^2 + 2s + 10000} e^{-0.05s}$

(c)  $\frac{10000}{s^2 + 2s + 10000} e^{-0.5 \times 10^{-12}s}$

(d)  $\frac{100}{s^2 + 2s + 100}$

(GATE IN 2023) **Solution:**

8.15 The Laplace transform of the continuous-time signal  $x(t) = e^{-3t}u(t-5)$  is \_\_\_\_\_, where  $u(t)$  denotes the continuous-time unit step signal.

A)  $\frac{e^{-5s}}{s+3}$ ,  $\text{Real}\{s\} > -3$

B)  $\frac{e^{-5(s-3)}}{s-3}$ ,  $\text{Real}\{s\} > 3$

C)  $\frac{e^{-5(s+3)}}{s+3}$ ,  $\text{Real}\{s\} > -3$

D)  $\frac{e^{-5(s-3)}}{s+3}$ ,  $\text{Real}\{s\} > -3$

**Solution:**

Parameter	Description	Value
$x(t)$	Given Function	$x(t) = e^{-3t}u(t)$
$X(s)$	Laplace Transform of $x(t)$	$\frac{e^{-5(s+3)}}{s+3}$

Table 1: Table of parameters

$$e^{-3t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+3} \quad \Re(s) > -3 \quad (8.82)$$



Using time shifting,

$$e^{-3(t-5)}u(t-5) \xleftrightarrow{\mathcal{L}} \frac{e^{-5s}}{s+3} \quad (8.83)$$

$$e^{-15}e^{-3(t-5)}u(t-5) \xleftrightarrow{\mathcal{L}} e^{-15}\frac{e^{-5s}}{s+3} \quad (8.84)$$

$$e^{-3t}u(t-5) \xleftrightarrow{\mathcal{L}} \frac{e^{-5(s+3)}}{s+3} \quad (8.85)$$

$$\therefore x(t) \xleftrightarrow{\mathcal{L}} \frac{e^{-5(s+3)}}{s+3} \quad \Re(s) > -3 \quad (8.86)$$

8.16 The solution  $x(t), t \geq 0$ , to the differential equation  $\ddot{x} = -k\dot{x}, k > 0$  with initial conditions  $x(0) = 1$  and  $\dot{x}(0) = 0$  is

**Solution:**

8.17 Consider the differential equation

$$x^2 \frac{d^2 y}{dx^2} + 4x \frac{dy}{dx} + 2y = 0 \quad \text{for } x \geq 1$$

with initial conditions  $y = 0$  and  $\frac{dy}{dx} = 1$  at  $x = 1$ . The value of  $y$  at  $x = 2$  is ?

(GATE AE 54 2023)

**Solution:** Given :

$$x^2 \frac{d^2 y}{dx^2} + 4x \frac{dy}{dx} + 2y = 0 \quad \text{for } x \geq 1 \quad (8.87)$$

Using Euler Substitution :

$$x = e^t \quad (8.88)$$

$$\frac{dy}{dx} = e^{-t} \frac{dy}{dt} \quad (8.89)$$

$$\frac{d^2 y}{dx^2} = e^{-2t} \frac{d^2 y}{dt^2} - e^{-2t} \frac{dy}{dt} \quad (8.90)$$

Substituting equations (8.88),(8.89),(8.90) in equation (8.87):

$$\frac{d^2 y}{dt^2} + 3 \frac{dy}{dt} + 2y = 0 \quad (8.91)$$

Taking Laplace on Both Sides:

$$s^2 Y(s) - sy(1) - y'(1) + 3sY(s) - y(1) + 2Y(s) = 0 \quad (8.92)$$

We know that:

$$y(1) = 0 \quad (8.93)$$

$$y'(1) = 1 \quad (8.94)$$

Substituting equations (8.93),(8.94) in (8.92)

$$s^2Y(s) + 3sY(s) + 2Y(s) = 1 \quad (8.95)$$

$$\Rightarrow Y(s) = \frac{1}{s^2 + 3s + 2} \quad (8.96)$$

$$\Rightarrow Y(s) = \frac{1}{s+1} - \frac{1}{s+2} \quad (8.97)$$

Taking Inverse Laplace:

$$y(t) = (e^{-t} - e^{-2t})u(t) \quad (8.98)$$

Substituting  $x = e^t$ :

$$y(x) = e^{-\ln x} - e^{-2\ln x}, \quad x \geq 1 \quad (8.99)$$

$$y(x) = \frac{1}{x} - \frac{1}{x^2}, \quad x \geq 1 \quad (8.100)$$

$$\therefore y(2) = 0.25 \quad (8.101)$$

8.18 Consider a lead compensator of the form

$$K(s) = \frac{1 + \frac{s}{a}}{1 + \frac{s}{\beta a}}, \quad \beta > 1, \quad a > 0$$

The frequency at which this compensator produces maximum phase lead is 4 rad/s.

At this frequency, the gain amplification provided by the controller, assuming an asymptotic Bode-magnitude plot of  $K(s)$ , is 6 dB. The values of  $a$  and  $\beta$ , respectively, are

(a) 1, 16

(b) 2, 4

(c) 3, 5

(d) 2.66, 2.25

(GATE EE 2023) **Solution:**

Parameter	Value
Transfer Function	$K(s) = \frac{1 + \frac{s}{a}}{1 + \frac{s}{\beta a}}$
Maximum Phase Lead Frequency	$\omega_m = 4 \text{ rad/s}$
Gain Amplification at $\omega_m$	$20 \log_{10}  K(j\omega_m)  = 6 \text{ dB}$
Conditions	$\beta > 1, a > 0$

Table 8.12: Given Parameters

$$K(s) = \frac{1 + \frac{s}{a}}{1 + \frac{s}{a\beta}}$$

$$K(s) = \frac{s+a}{a} \cdot \frac{a\beta}{s+a\beta} \quad (8.102)$$

$$= \beta \frac{s+a}{s+a\beta} \quad (8.103)$$

1. The max phase lead is:  $\omega_m = \sqrt{a \cdot a \cdot \beta}$

2. If  $G(s) = \frac{k(s+a)}{s(s+a\beta)}$  has to act as a lead compensator, then  $a\beta$  must be greater than  $a$ , i.e.,  $a\beta > a$ .

Using the above properties we have:

$$\omega_m = \sqrt{a \cdot a \cdot \beta} = 4 \quad (8.104)$$

$$\beta > 1 \quad (8.105)$$

$$K(j\omega) = \frac{1 + \frac{j\omega}{a}}{1 + \frac{j\omega}{a\beta}} = \beta \frac{j\omega + a}{j\omega + a\beta} \quad (8.106)$$

$$|K(j\omega)| = \frac{\beta \sqrt{\omega^2 + a^2}}{\sqrt{\omega^2 + (a\beta)^2}} \quad (8.107)$$

Using Gain Amplification:

$$20 \log_{10} |K(j\omega_m)| = 6 \quad (8.108)$$

$$|K(j\omega_m)| \approx 2 \quad (8.109)$$

$$|K(j\omega_m)| = \frac{\beta \sqrt{(\omega_m)^2 + a^2}}{\sqrt{(\omega_m)^2 + (a\beta)^2}} = 2 \quad (8.110)$$

$$\frac{\beta \sqrt{16 + a^2}}{\sqrt{16 + (a\beta)^2}} = 2 \quad (8.111)$$

Solving the above equation, we get  $a \approx 2$  and  $\beta = 4$ .

Therefore, the correct answer is **(B) 2, 4**.

The Bode plots for the same are as follows:

8.19 Second order ordinary differential equation  $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$  has values  $y = 2$  and  $\frac{dy}{dx} = 1$  at  $x = 0$ . The value of  $y$  at  $x = 1$  is? (round off to three decimal places)  
[GATE-ES 2023]

**Solution:** We convert given second order differential equation to s domain using Laplace transform and solve for  $Y(s)$  and take inversion to get  $y(x)$ .

Symbol	Values	Description
$Y(s)$	$\frac{2s-1}{s^2-s-2}$	$y$ in s domain
$y(x)$	$e^{2x} + e^{-x}$	$y$ in x domain
$y(0)$	2	$y$ at $x = 0$
$y'(0)$	1	$y'(x)$ at $x = 0$
$u(x)$	$= \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{o.w} \end{cases}$	unit step function

Table 8.13: Parameters

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y \xleftrightarrow{\mathcal{L}} s^2Y(s) - sy(0) - y'(0) - sY(s) + y(0) - 2Y(s) \quad (8.112)$$

$$Y(s) (s^2 - s - 2) = 2s - 1 \quad (8.113)$$

$$\Rightarrow Y(s) = \frac{2s - 1}{s^2 - s - 2} \quad (8.114)$$

$$\Rightarrow Y(s) = \frac{1}{s - 2} + \frac{1}{s + 1} \quad (8.115)$$

For inversion of  $Y(s)$  in partial fractions-

$$\frac{b}{s + a} \xleftrightarrow{\mathcal{L}^{-1}} be^{-ax}u(x) \quad (8.116)$$

Where b, a are real numbers, we invert  $Y(s)$  to get  $y(x)$ :-



From (8.116)

$$Y(s) \xleftrightarrow{\mathcal{L}^{-1}} y(x)u(x) \quad (8.117)$$

$$y(x) = (e^{2x} + e^{-x}) u(x) \quad (8.118)$$

$$\Rightarrow y(1) = 7.757 \quad (8.119)$$

8.20 A continuous-time system that is initially at rest is described by,

$$\frac{dy(t)}{dt} + 3y(t) = 2x(t)$$

where  $x(t)$  is the input voltage and  $y(t)$  is the output voltage.

The impulse response of the system is?

(GATE 2023 EE) **Solution:**

8.21 Consider the equation  $\frac{dy}{dx} + ay = \sin \omega x$ , where  $a$  and  $\omega$  are constants. Given  $y = 1$  at  $x = 0$ , correct all the correct statement(s) from the following as  $x \rightarrow \infty$ .

(A)  $y \rightarrow 0$  if  $a \neq 0$

(B)  $y \rightarrow 1$  if  $a = 0$

(C)  $y \rightarrow A \exp(|a|x)$  if  $a < 0$ ;  $A$  is constant

(D)  $y \rightarrow B \sin(\omega x + C)$  if  $a > 0$ ;  $B$  and  $C$  are constants

(GATE AE 2023) **Solution:**  $y(0) = 1$

$$\frac{dy}{dx} + ay = \sin \omega x \quad (8.120)$$

Taking laplace transform on both sides

Function	Laplace transform
$\frac{dy}{dx}$	$xY - y(0)$
$y$	$Y$
$\sin \omega x$	$\frac{\omega}{\omega^2 + x^2}$

Table 8.14: Laplace transform

$$sY - y(0) + aY = \frac{\omega}{\omega^2 + s^2} \quad (8.121)$$

$$sY - 1 + aY = \frac{\omega}{\omega^2 + s^2} \quad (8.122)$$

$$\implies Y(s) = \frac{1}{s+a} \left( \frac{\omega}{\omega^2 + s^2} + 1 \right) \quad (8.123)$$

$$Y(s) = \frac{A}{s+a} + \frac{Bs+C}{\omega^2 + s^2} \quad (8.124)$$

Taking inverse laplace transform on both sides

$$y(x) = \mathcal{L}^{-1}\{Y(s)\} = Ae^{-ax} + (B \cos(\omega x) + C \sin(\omega x)) \quad (8.125)$$

$$y(x) = \mathcal{L}^{-1}\{Y(s)\} = Ae^{-ax} + B \sin(\omega x + C) \quad (8.126)$$

now as  $x \rightarrow \infty$

- (a)  $y \rightarrow 0$  if  $a \neq 0$  is not true as  $y$  depend on  $a, \omega$
- (b)  $y \rightarrow 1$  if  $a = 0$  is not true as  $y$  depend on  $\omega$
- (c)  $y \rightarrow A \exp(|a|x)$  if  $a < 0$  is true as  $B \sin(\omega x + C)$  is neglected compared to  $Ae^{-ax}$
- (d)  $y \rightarrow B \sin(\omega x + C)$  if  $a > 0$ ; is true as  $Ae^{-ax} \rightarrow 0$

$\therefore$  C,D are correct options

8.22 **Question :** The position  $x(t)$  of a particle, at constant  $\omega$ , is described by the equation

$$\frac{d^2x}{dt^2} = -\omega^2 x. \quad (8.127)$$

The initial conditions are  $x(t=0) = 1$  and  $\left. \frac{dx}{dt} \right|_{t=0} = 0$ . The position of the particle at  $t = \frac{3\pi}{\omega}$  is \_\_\_\_\_ (in integer). (GATE CH 2023) **Solution:**

8.23 **Question:** The initial value problem  $\frac{dy}{dt} + 2y = 0, y(0) = 1$  is solved numerically using the forward Euler's method with a constant and positive time step of  $\delta$ .

Let  $y_n$  represent the numerical solution obtained after  $n$  steps. The condition  $|y_{n+1}| \leq |y_n|$  is satisfied if and only if  $\delta$  does not exceed

(GATE ME 2023) **Solution:**

8.24 In the context of signals and systems, determine the phase cross-over frequency of the open-loop transfer function

$$G(s) = \frac{k \cdot s \cdot (1 + sT_1) \cdot (1 + sT_2)}{s}$$

with positive constants  $k, T_1, T_2$  are positive constants. The phase crossover frequency, in rad/s, is

(a)  $\frac{1}{\sqrt{T_1 T_2}}$

(b)  $\frac{1}{T_1 T_2}$

(c)  $\frac{1}{T_1 \sqrt{T_2}}$

(d)  $\frac{1}{\sqrt{T_2} T_1}$

(GATE EC 2023) **Solution:**

8.25 Which one of the options given is the inverse Laplace transform of  $\frac{1}{s^3-s}$ ?

$u(t)$  denotes the unit-step function.

(A)  $\left(-1 + \frac{1}{2}e^{-t} + \frac{1}{2}e^t\right) u(t)$

(B)  $\left(\frac{1}{3}e^{-t} - e^t\right) u(t)$

(C)  $\left(-1 + \frac{1}{2}e^{-(t-1)} + \frac{1}{2}e^{(t-1)}\right) u(t-1)$

(D)  $\left(-1 - \frac{1}{2}e^{-(t-1)} - \frac{1}{2}e^{(t-1)}\right) u(t-1)$

(GATE ME 2023)

**Solution:**

Using partial fraction,

$$\frac{1}{s^3-s} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s-1} \quad (8.128)$$

On Solving,

$$\implies A = -1 \quad B = \frac{1}{2} \quad C = \frac{1}{2} \quad (8.129)$$

$$X(s) = \frac{-1}{s} + \frac{1}{2(s+1)} + \frac{1}{2(s-1)} \quad (8.130)$$

$$\text{As } e^{-at}u(t) \xrightarrow{\mathcal{L}} \frac{1}{s+a} \quad \Re(s) > -a \quad (8.131)$$

$$\text{And } -e^{-at}u(-t) \xrightarrow{\mathcal{L}} \frac{1}{s+a} \quad \Re(s) < -a \quad (8.132)$$



Now,

$$\mathcal{L}^{-1}(X(s)) = x(t) \quad (8.133)$$

There are 4 cases possible,

$$x(t) = \begin{cases} \left(1 - \frac{1}{2}e^{-t} - \frac{1}{2}e^t\right) u(-t) & \Re(s) < -1 \\ -1u(-t) + \frac{1}{2}e^{-t}u(t) - \frac{1}{2}e^t u(-t) & \Re(s) \in (-1, 0) \\ -u(t) + \frac{1}{2}e^{-t}u(t) - \frac{1}{2}e^t u(-t) & \Re(s) \in (0, 1) \\ \left(-1 + \frac{1}{2}e^{-t} + \frac{1}{2}e^t\right) u(t) & \Re(s) > 1 \end{cases} \quad (8.134)$$

Thus the correct option is (A)  $\left(-1 + \frac{1}{2}e^{-t} + \frac{1}{2}e^t\right) u(t)$  for  $\Re(s) > 1$

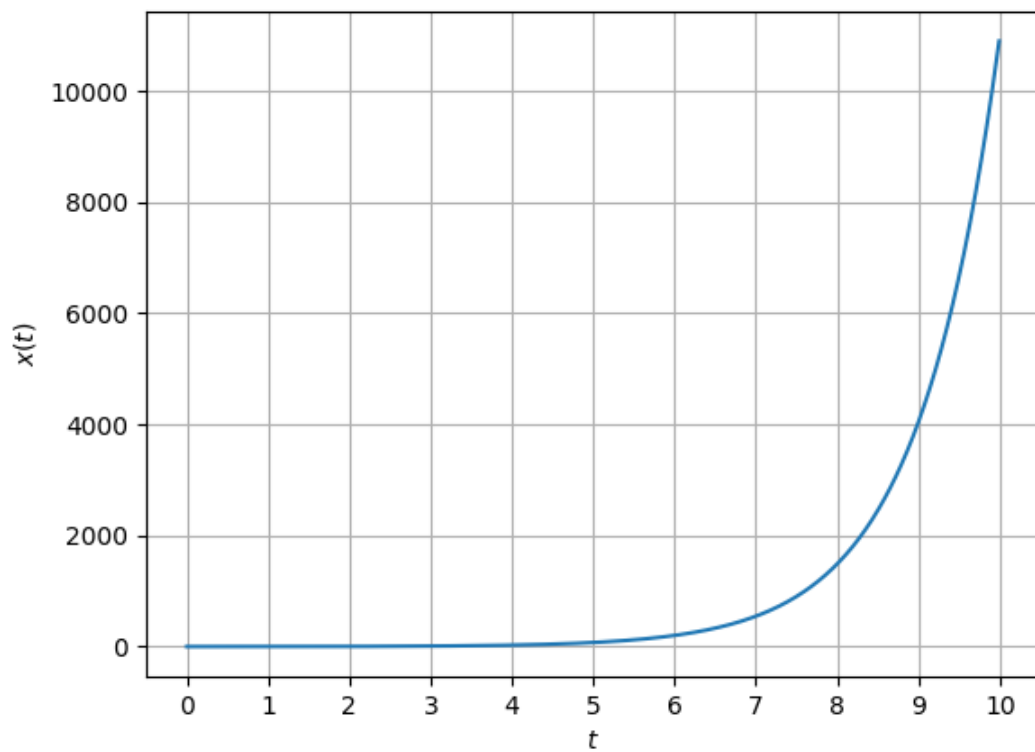


Figure 8.24: Plot for  $x(t) = (-1 + \frac{1}{2}e^{-t} + \frac{1}{2}e^t)u(t)$

8.26 The state equation of a second order system is

$$\dot{x}(t) = Ax(t), \quad x(0) \text{ is the initial condition.}$$

Suppose  $\lambda_1$  and  $\lambda_2$  are two distinct eigenvalues of  $A$ , and  $\nu_1$  and  $\nu_2$  are the corresponding eigenvectors. For constants  $\alpha_1$  and  $\alpha_2$ , the solution,  $x(t)$ , of the state equation is

(A)  $\sum_{i=1}^2 \alpha_i e^{\lambda_i t} \nu_i$

(B)  $\sum_{i=1}^2 \alpha_i e^{2\lambda_i t} \nu_i$

(C)  $\sum_{i=1}^2 \alpha_i e^{3\lambda_i t} \nu_i$

(D)  $\sum_{i=1}^2 \alpha_i e^{4\lambda_i t} \nu_i$

GATE EC 2023

**Solution:**

Variable	Description	Value
$x(t)$	state variable	-
$\dot{x}(t)$	derivative of $x(t)$ w.r.t $t$	$\frac{dx(t)}{dt}$
$A$	2x2 matrix	-
$\lambda_i$ for $i = 1, 2$	eigen values of $A$	-
$\nu_i$ for $i = 1, 2$	eigen vectors of $A$	-
$\alpha_i$ for $i = 1, 2$	component of $x(t)$ along $\nu_i$	-

Table 8.15: input parameters

**Theories and Proofs:**

$$A\mathbf{y} = \lambda\mathbf{y} \quad (8.135)$$

Eigen values of inverse of a matrix is reciprocal of eigen value of the given matrix

$$A^{-1}A\mathbf{y} = A^{-1}\lambda\mathbf{y} \quad (8.136)$$

$$\implies \mathbf{y} = \lambda A^{-1}\mathbf{y} \quad (8.137)$$

$$\implies A^{-1}\mathbf{y} = \frac{1}{\lambda}\mathbf{y} \quad (8.138)$$

Eigen value of a matrix shifts by the same amount as that of the matrix.

$$(A - \sigma I)\mathbf{y} = A\mathbf{y} - \sigma I\mathbf{y} \quad (8.139)$$

$$= \lambda\mathbf{y} - \sigma\mathbf{y} \quad (8.140)$$

$$= (\lambda - \sigma)\mathbf{y} \quad (8.141)$$

**Sol:**

Using Laplace transform:

Given Equation:

$$\dot{x}(t) = Ax(t) \quad (8.142)$$

$$\frac{dx(t)}{dt} = Ax(t) \quad (8.143)$$

Taking Laplace Transform:

$$\mathcal{L}\left(\frac{dx(t)}{dt}\right) = \mathcal{L}(Ax(t)) \quad (8.144)$$

$$sX(s) - x(0) = AX(s) \quad (8.145)$$

$$(sI - A)X(s) = x(0) \quad (8.146)$$

$$X(s) = (sI - A)^{-1}x(0) \quad (8.147)$$

From Table 8.15, we can write  $x(0)$  in terms of two linearly independent variables as

$$x(0) = \alpha_1 v_1 + \alpha_2 v_2 \quad (8.148)$$

$$= \sum_{i=1}^2 \alpha_i v_i \quad (8.149)$$

From (8.147), (8.149)

$$X(s) = (sI - A)^{-1} \left( \sum_{i=1}^2 \alpha_i v_i \right) \quad (8.150)$$

$$= \sum_{i=1}^2 (sI - A)^{-1} \alpha_i v_i \quad (8.151)$$

From (8.138), (8.141)

$$X(s) = \sum_{i=1}^2 \frac{1}{s - \lambda_i} (\alpha_i v_i) \quad (8.152)$$

Now take inverse Laplace Transform

$$\mathcal{L}^{-1}(X(s)) = \mathcal{L}^{-1}\left(\sum_{i=1}^2 \frac{1}{s - \lambda_i} (\alpha_i v_i)\right) \quad (8.153)$$

$$x(t) = \sum_{i=1}^2 e^{\lambda_i t} (\alpha_i v_i) \quad (8.154)$$

Hence the answer is option (A).

8.27 The continuous time signal  $x(t)$  is described by:

$$x(t) = \begin{cases} 1, & \text{if } 0 \leq t \leq 1 \\ 0, & \text{elsewhere} \end{cases} \quad (8.155)$$

If  $y(t)$  represents  $x(t)$  convolved with itself, which of the following options is/are TRUE?

(A)  $y(t) = 0$  for all  $t < 0$

(B)  $y(t) = 0$  for all  $t > 1$

(C)  $y(t) = 0$  for all  $t > 3$

(D)  $\int_{0.1}^{0.75} \frac{dy(t)}{dt} dt \neq 0$

GATE 2023 BM- Q 49

**Solution:**

Symbol	Description
$X(s)$	Laplace transform of $x(t)$
$Y(s)$	Laplace transform of $y(t)$
$u(t - t_0)$	Unit step function, $u(t - t_0) = 1, t \geq t_0$

Table 8.16: Parameters

$$y(t) = x(t) * x(t) \quad (8.156)$$

$$Y(s) = X(s) X(s) \quad (8.157)$$

$$= \left( \frac{1 - e^{-s}}{s} \right)^2 \quad (8.158)$$

$$= \frac{1 + e^{-2s} - 2e^{-s}}{s^2} \quad (8.159)$$

$$u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s} \quad (8.160)$$

$$tu(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s^2} \quad (8.161)$$

$$f(t) \xleftrightarrow{\mathcal{L}} F(s) \implies f(t+a) \xleftrightarrow{\mathcal{L}} e^{as} F(s) \quad (8.162)$$

Using (8.161) and (8.162),

$$y(t) = tu(t) + (t-2)u(t-2) - 2(t-1)u(t-1) \quad (8.163)$$

This can also be expressed as

$$y(t) = \begin{cases} 1 - |1-t|, & \text{if } 0 \leq t \leq 2 \\ 0, & \text{otherwise} \end{cases} \quad (8.164)$$

Checking (8.164) with every option,



(A) From Fig: 8.25 ,  $y(t) = 0, \forall t < 0$ , hence (A) is true

(B) From Fig: 8.25 ,  $y(t) \neq 0, \forall t \in [1, 2]$ , hence (B) is false

(C) From Fig: 8.25 ,  $y(t) = 0, \forall t > 3$ , hence (C) is true

(D) From Fig: 8.26 , area under graph between 0.75 and 0.1 is non-zero, hence (D) is true

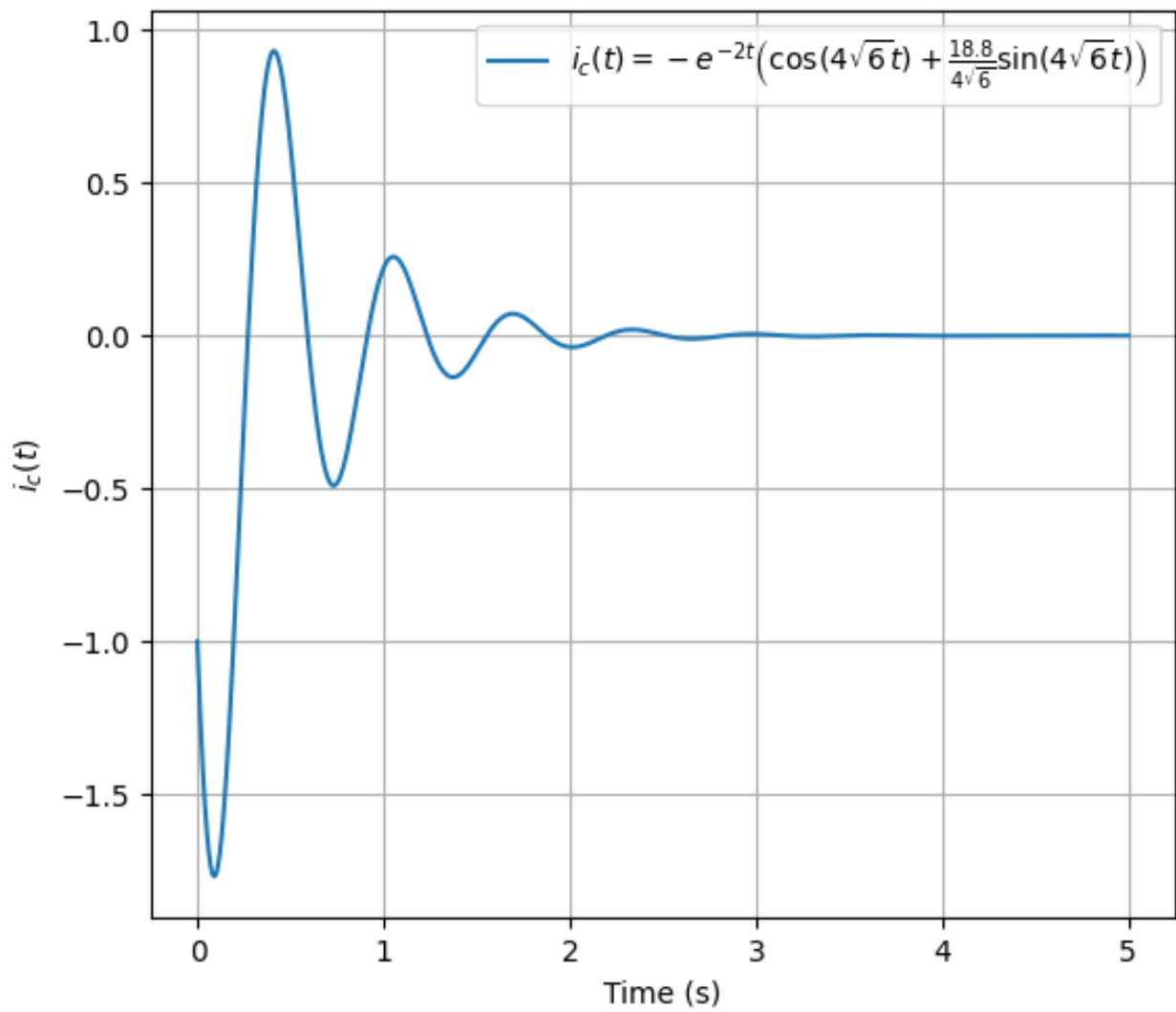


Figure 8.13: Plot of  $i_c(t)$  vs time

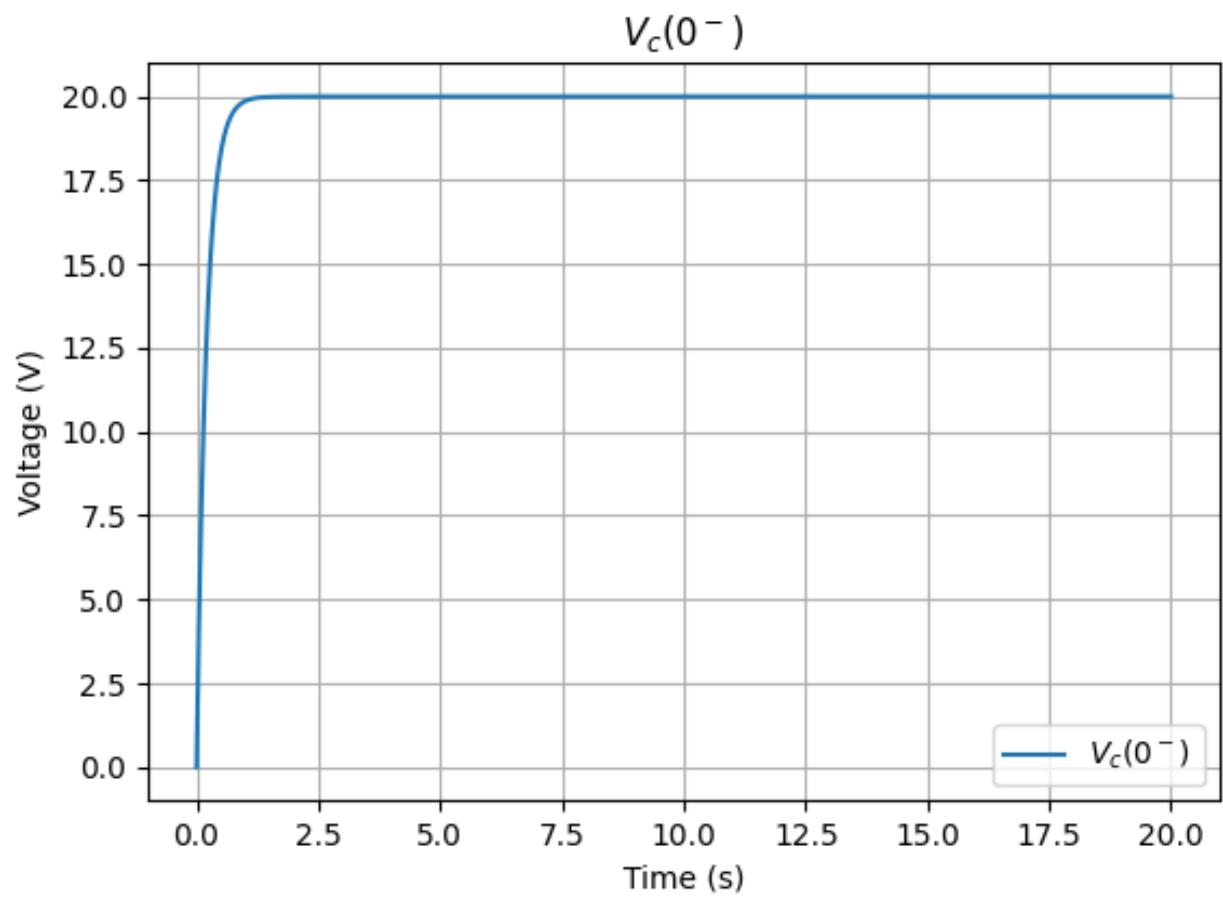


Figure 8.14: Plot of  $V_c(0^-)$  vs time

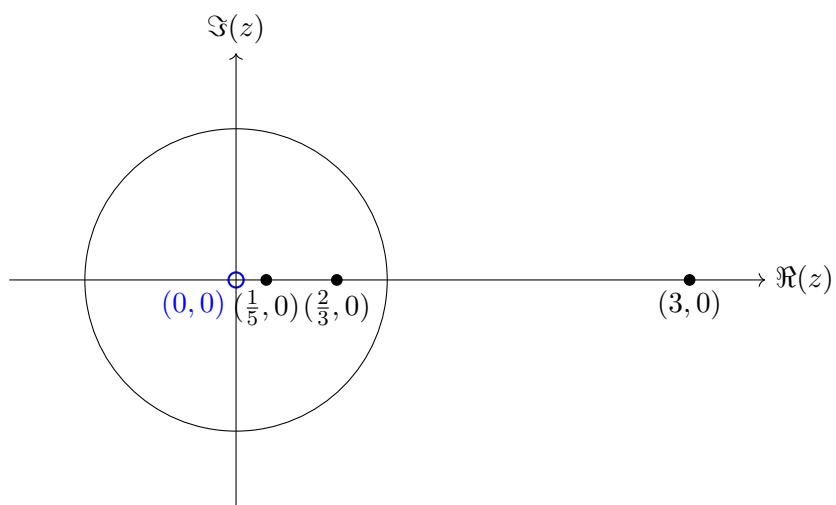


Figure 8.15: Representation of Poles

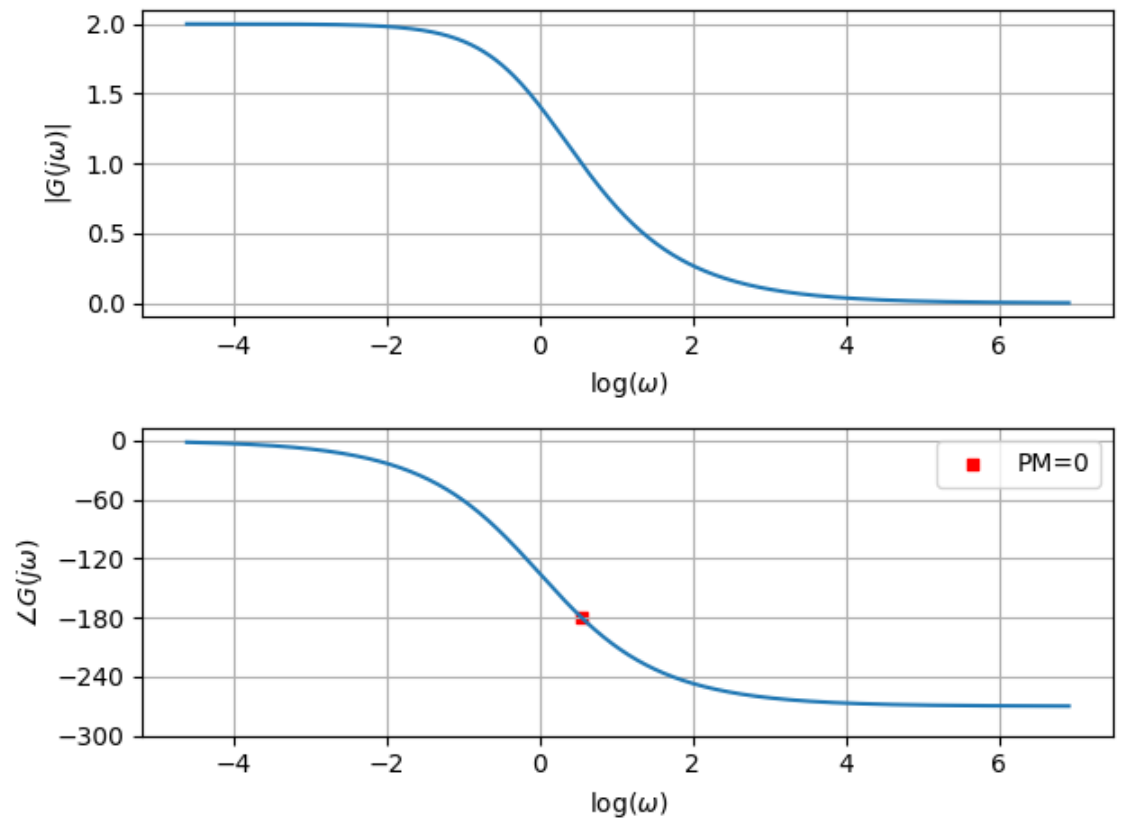


Figure 8.16: Bode Plot of Transfer Function  $G(s)$

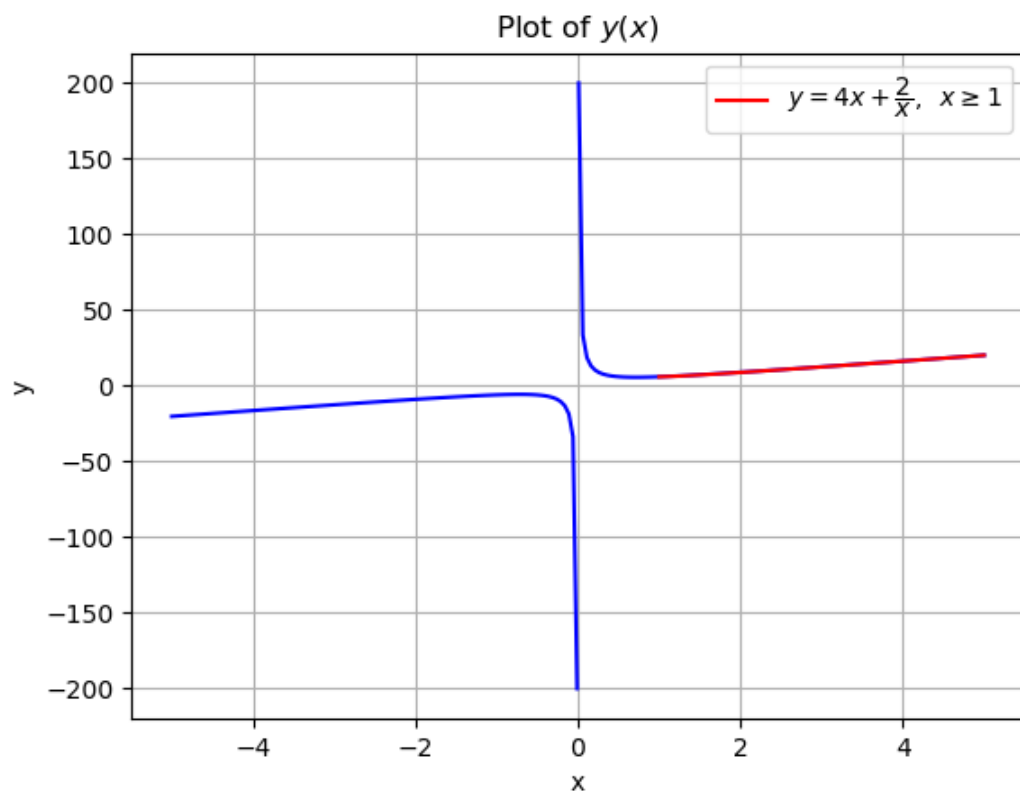


Figure 8.17: Plot of  $y(x)$  v/s  $x$

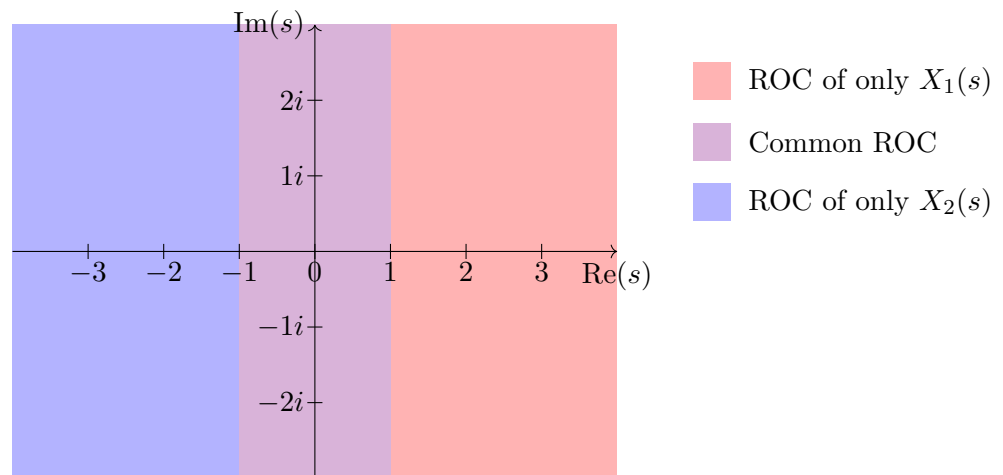
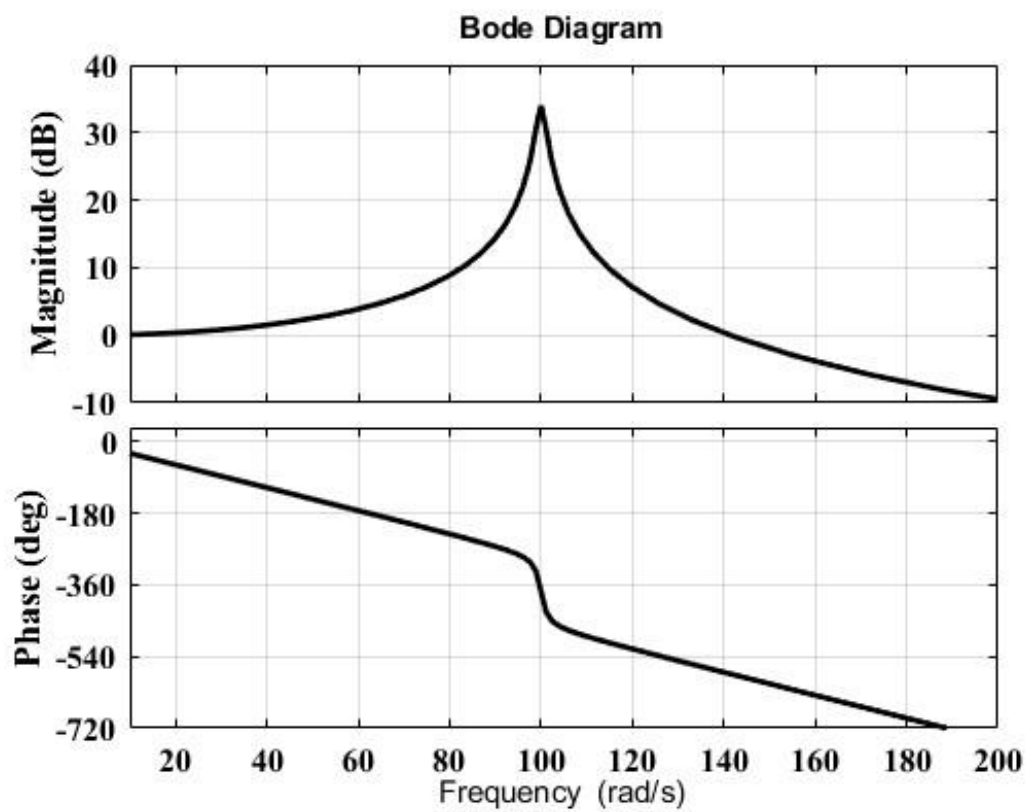


Figure 8.18: Representation of ROCs of  $X_1(s)$  and  $X_2(s)$



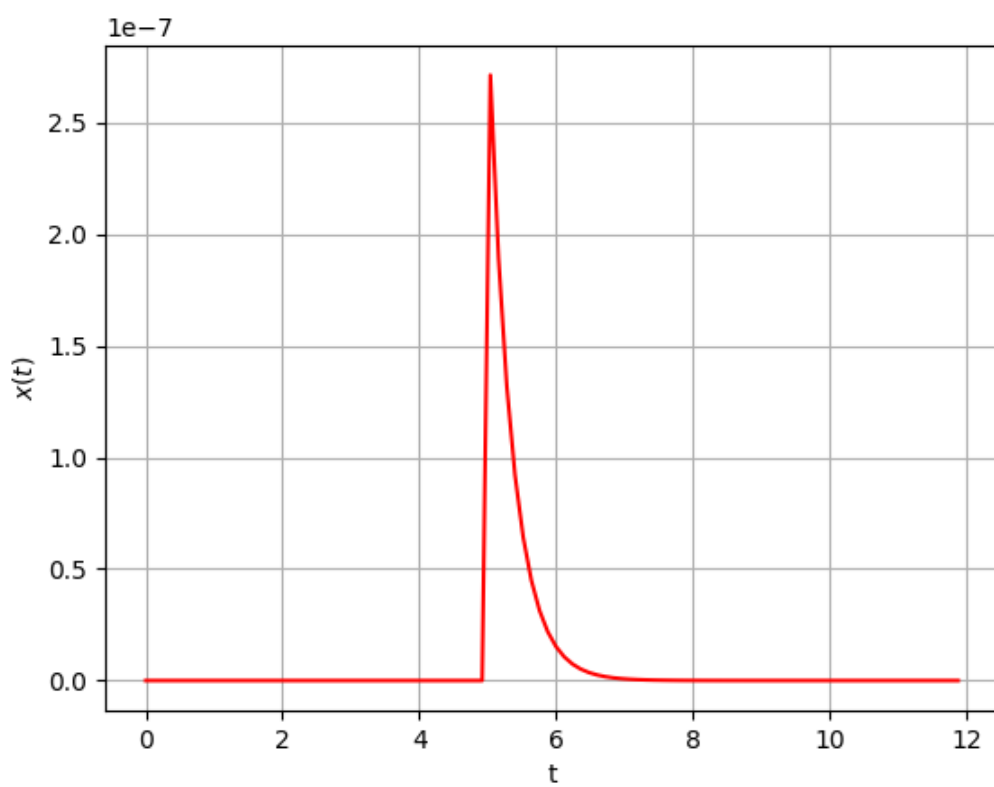


Figure 3: Plot of  $x(t)$  vs  $t$ . See Table 1



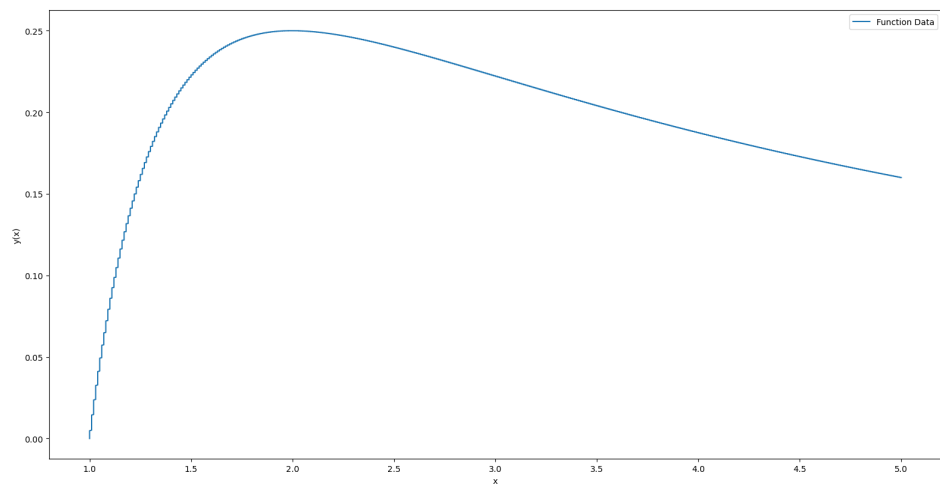


Figure 8.20: Plot of  $y(x)$  vs  $x$

Figure 8.21: Amplitude

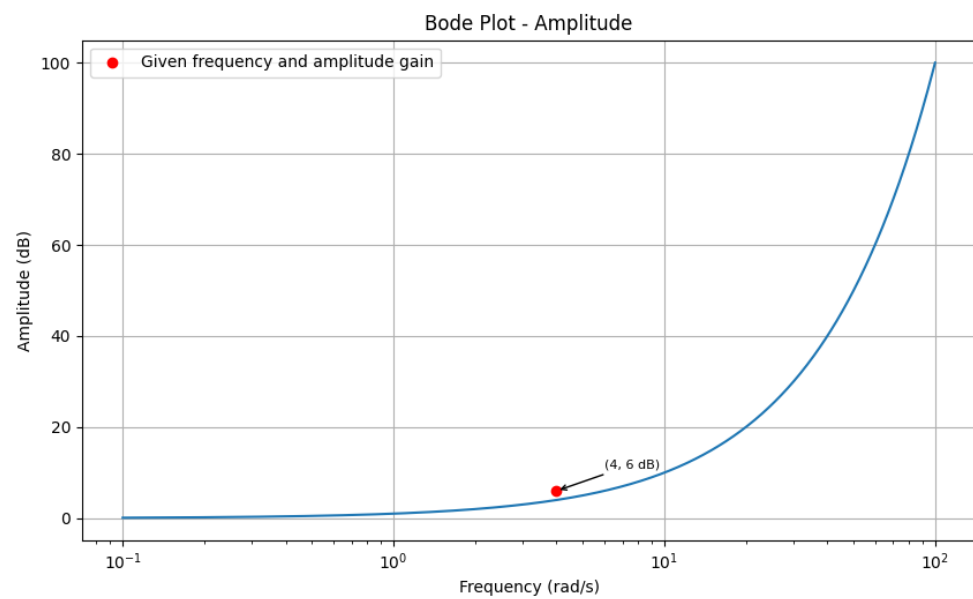
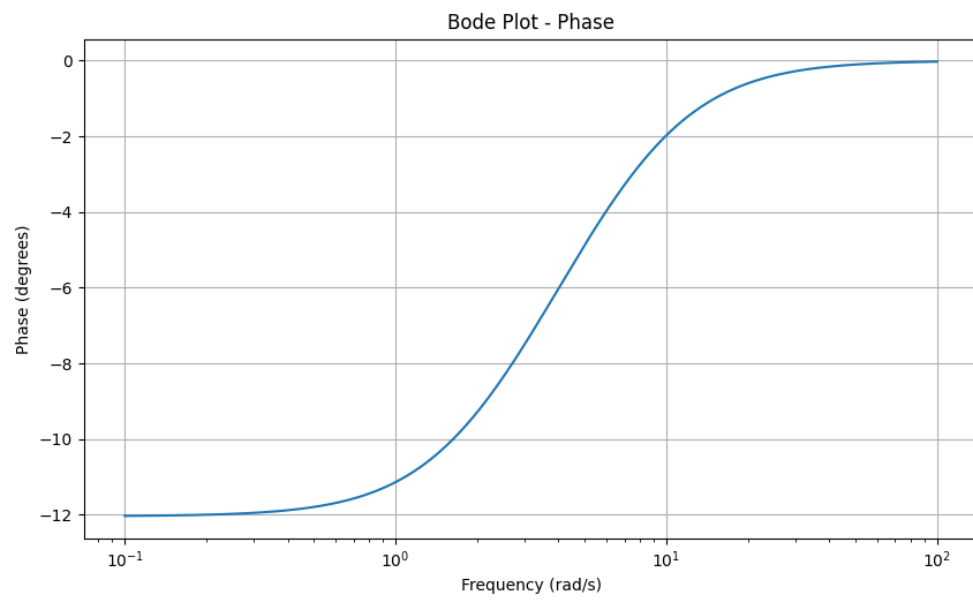


Figure 8.22: Phase Response



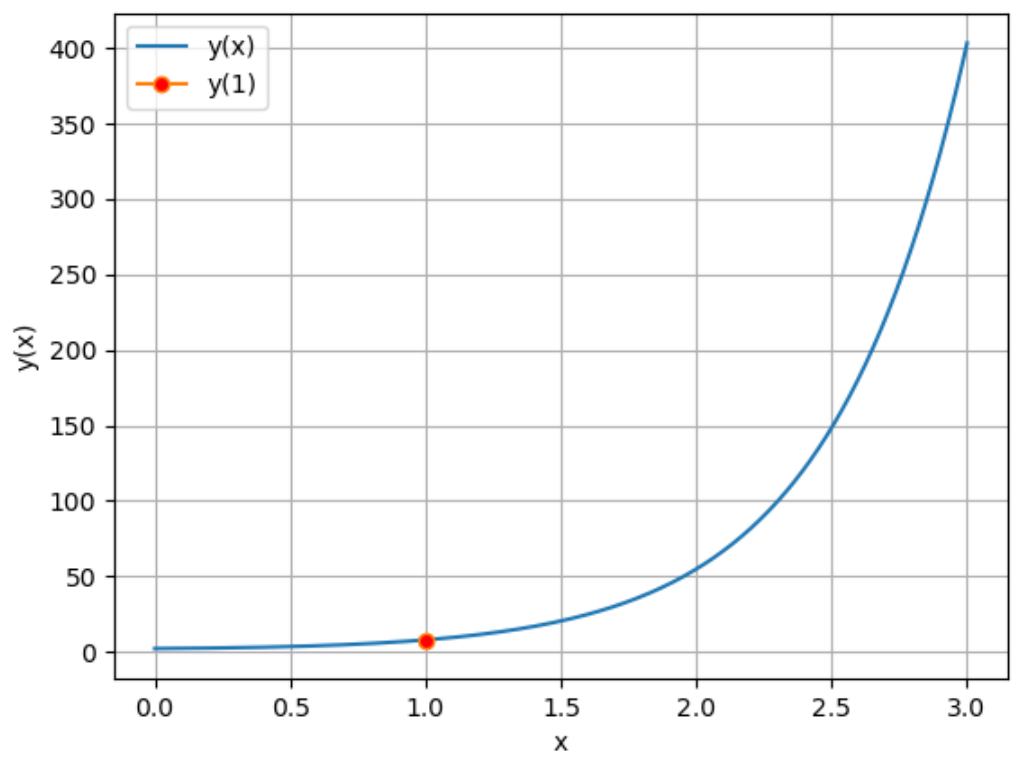


Figure 8.23: Plot of  $y(x)$

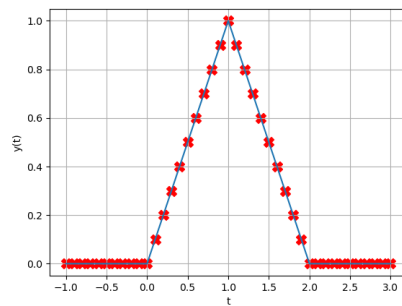


Figure 8.25: Stem Plot of  $y(t)$  v/s  $t$

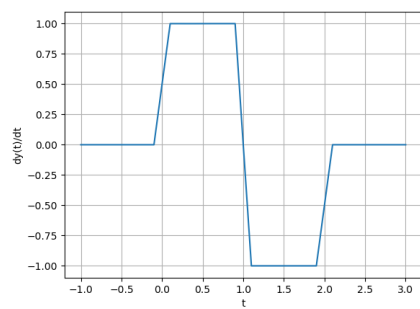


Figure 8.26: Stem Plot of  $dy(t)/dt$  v/s  $t$

## Chapter 9

# Fourier transform

9.1 Let a frequency modulated (FM) signal :  $x(t) = A \cos(\omega_c t + k_f \int_{-\infty}^t m(\lambda) d\lambda)$  , where  $m(t)$  is a message signal of bandwidth  $W$ . It is passed through a non-linear system with output  $y(t) = 2x(t) + 5(x(t))^2$ . Let  $B_T$  denote the FM bandwidth. The minimum value of  $\omega_c$  required to recover  $x(t)$  from  $y(t)$  is:

(A)  $B_T + W$

(B)  $\frac{3}{2}B_T$

(C)  $2B_T + W$

(D)  $\frac{5}{2}B_T$

**Solution:**

9.2 Let an input  $x[n]$  having discrete-time Fourier transform  $X(e^{j\Omega}) = 1 - e^{-j\Omega} + 2e^{-3j\Omega}$  be passed through an LTI system. The frequency response of the LTI system is  $H(e^{j\Omega}) = 1 - \frac{1}{2}e^{-2j\Omega}$ . The output  $y[n]$  of the system is

(GATE EC 2023) **Solution:**

Parameter	Value
$X(e^{j\omega})$	$1 - e^{-j\omega} + 2e^{-3j\omega}$
$H(e^{j\omega})$	$1 - \frac{1}{2}e^{-2j\omega}$
$Y(e^{j\omega})$	$X(e^{j\omega}) \cdot H(e^{j\omega})$
$y[n]$	?

Table 9.1: Parameters

$$y[n] = x[n] * h[n] \quad (9.1)$$

$$x(n) * h(n) \longleftrightarrow X(e^{j\omega}) \cdot H(e^{j\omega})$$

$$Y(e^{j\omega}) = X(e^{j\omega}) \cdot H(e^{j\omega}) \quad (9.2)$$

$$Y(e^{j\omega}) = (1 - e^{-j\omega} + 2e^{-3j\omega}) \cdot \left(1 - \frac{1}{2}e^{-2j\omega}\right) \quad (9.3)$$

$$= (1 - e^{-j\omega} + \frac{5}{2}e^{-3j\omega} - \frac{1}{2}e^{-2j\omega} - e^{-5j\omega}) \quad (9.4)$$

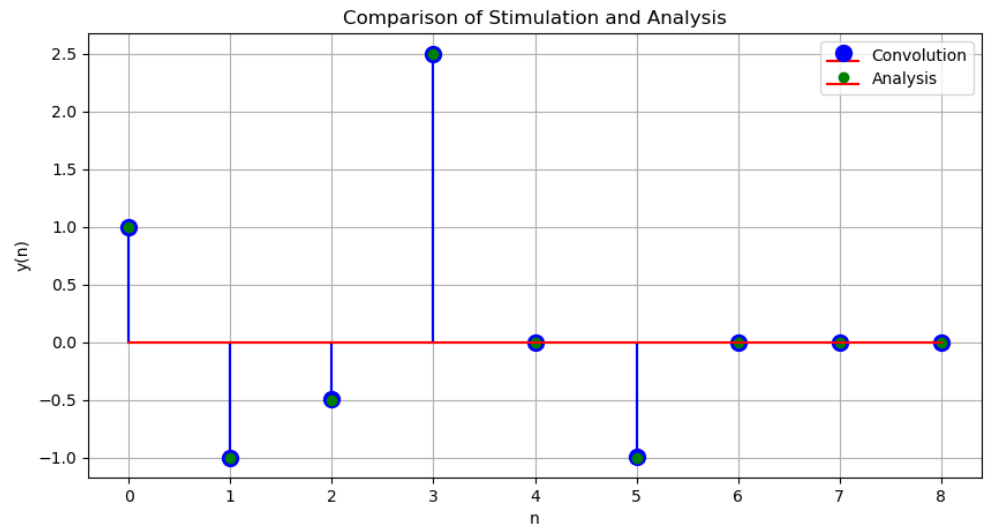


Figure 9.1:  $y(n)$  vs  $n$

9.3 The Fourier transform  $X(\omega)$  of  $x(t) = e^{-t^2}$  is

Note:  $\int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\pi}$

A)  $\sqrt{\pi} e^{\frac{\omega^2}{2}}$

B)  $\frac{e^{-\frac{\omega^2}{4}}}{2\sqrt{\pi}}$

C)  $\sqrt{\pi} e^{-\frac{\omega^2}{4}}$

D)  $\sqrt{\pi} e^{-\frac{\omega^2}{2}}$

Gate 2023 EC Question 28 **Solution:**

$$x'(t) = -2te^{-t^2} \quad (9.5)$$

$$x'(t) = -2tx(t) \quad (9.6)$$

doing fourier transform

$$j2\pi f X(f) = -2j \frac{dX(f)}{df} \quad (9.7)$$

$$\int_0^f \frac{dX(f)}{X(f)} = \int_0^f \frac{2\pi f df}{-2} \quad (9.8)$$

$$\frac{X(f)}{X(0)} = e^{\frac{-(2\pi f)^2}{4}} \quad (9.9)$$

$$X(0) = \int_{-\infty}^{\infty} x(t) dt = \sqrt{\pi} \quad (9.10)$$

$$X(f) = \sqrt{\pi} e^{\frac{-(2\pi f)^2}{4}} \quad (9.11)$$

$$X(f) = \sqrt{\pi} e^{-(\pi f)^2} \quad (9.12)$$

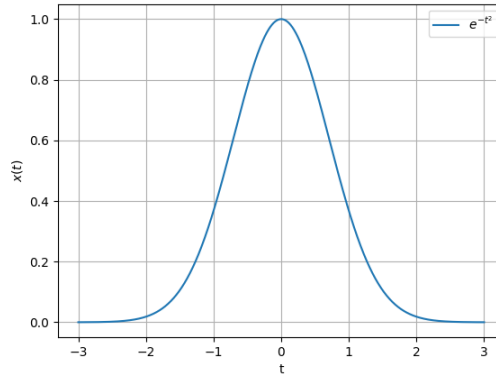


Figure 9.2: Graph of  $e^{-t^2}$



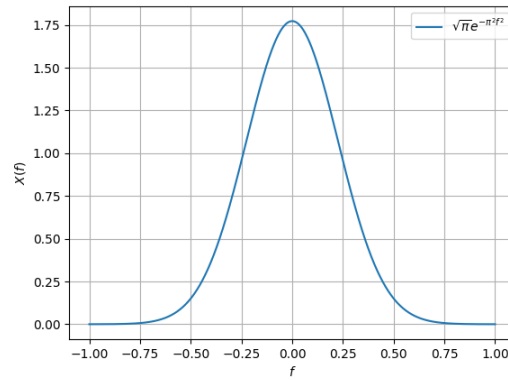


Figure 9.3: Graph of  $X(f) = \sqrt{\pi} e^{-\pi^2 f^2}$

9.4 Let  $x(t) = 10 \cos(10.5\omega t)$  be passed through an LTI system with impulse response

$$h(t) = \pi \left( \frac{\sin(\omega t)}{\pi t} \right)^2 \cos(10\omega t) . \text{ The output of the system is:}$$

(GATE EC 2023) **Solution:**

9.5 Q27) Let  $m(t)$  be a strictly band-limited signal with bandwidth  $B$  and energy  $E$ . Assuming  $\omega_0 = 10B$ , the energy in the signal  $m(t) \cos(\omega_0 t)$

(A)  $\frac{E}{4}$

(B)  $\frac{E}{2}$

(C)  $E$

(D)  $2E$

(GATE EC 2023)

**Solution:**

9.6 The following function is defined over the interval  $[-L, L]$  :

$$f(x) = px^4 + qx^5$$

It is expressed as a Fourier series,

$$f(x) = a(0) + \sum_{n=1}^{\infty} \left\{ a(n) \sin\left(\frac{\pi x}{L}\right) + b(n) \cos\left(\frac{\pi x}{L}\right) \right\}$$

which options amongst the following are true?

- (a)  $a(n)$ ,  $n = 1, 2, \dots, \infty$  depend on  $p$
- (b)  $a(n)$ ,  $n = 1, 2, \dots, \infty$  depend on  $q$
- (c)  $b(n)$ ,  $n = 1, 2, \dots, \infty$  depend on  $p$
- (d)  $b(n)$ ,  $n = 1, 2, \dots, \infty$  depend on  $q$

(GATE 2023 CE Question 25)

**Solution:**

Parameter	Description
$f(x)$	Polynomial function
$2L$	Period of the Polynomial function
$c(n)$	Complex Fourier Coefficients
$a(0), a(n), b(n)$	Trigonometric Fourier Coefficients

Table 9.2: Parameters

The complex exponential Fourier Series of  $f(x)$  is,

$$f(x) = \sum_{n=-\infty}^{\infty} c(n) e^{j \frac{\pi n x}{L}} \quad (9.13)$$

$$\Rightarrow c(n) = \frac{1}{2L} \int_{-L}^L f(x) e^{-j \frac{\pi n x}{L}} dx \quad (9.14)$$

$$c(n) = \frac{1}{2L} \int_{-L}^L (px^4 + qx^5) e^{-j \frac{\pi n x}{L}} dx \quad (9.15)$$

For  $n = 0$ ,

$$c(0) = \frac{1}{2L} \int_{-L}^L (px^4 + qx^5) dx \quad (9.16)$$

$$= \frac{pL^4}{5} \quad (9.17)$$

For  $n \neq 0$ ,

$$c(n) = \frac{1}{2L} \int_{-L}^L (px^4 + qx^5) e^{-j \frac{\pi n x}{L}} dx \quad (9.18)$$

$$\begin{aligned} &= \frac{pL^4}{2} (e^{j\pi n} - e^{-j\pi n}) \left( \frac{1}{j\pi n} + \frac{12}{(j\pi n)^3} + \frac{24}{(j\pi n)^5} \right) \\ &\quad - \frac{pL^4}{2} (e^{j\pi n} + e^{-j\pi n}) \left( \frac{4}{(j\pi n)^2} + \frac{24}{(j\pi n)^4} \right) \\ &\quad - \frac{qL^5}{2} (e^{j\pi n} + e^{-j\pi n}) \left( \frac{1}{j\pi n} + \frac{20}{(j\pi n)^3} + \frac{120}{(j\pi n)^5} \right) \\ &\quad + \frac{qL^5}{2} (e^{j\pi n} - e^{-j\pi n}) \left( \frac{5}{(j\pi n)^2} + \frac{60}{(j\pi n)^4} + \frac{120}{(j\pi n)^6} \right) \end{aligned} \quad (9.19)$$

$$\begin{aligned} &= (pL^4) (-1)^n \left( \frac{4}{(\pi n)^2} - \frac{24}{(\pi n)^4} \right) \\ &\quad - (qL^5) (-1)^n \left( -\frac{j}{\pi n} + \frac{20j}{(\pi n)^3} - \frac{120j}{(\pi n)^5} \right) \end{aligned} \quad (9.20)$$

Given,

$$f(x) = a(0) + \sum_{n=1}^{\infty} \left\{ a(n) \sin\left(\frac{\pi nx}{L}\right) + b(n) \cos\left(\frac{\pi nx}{L}\right) \right\} \quad (9.21)$$

Finding the Fourier Coefficient  $a(0)$ ,

$$a(0) = c(0) \quad (9.22)$$

$$\Rightarrow a(0) = \frac{pL^4}{5} \quad (9.23)$$

We know,

$$\sin \theta = \frac{e^{j\theta} - e^{-j\theta}}{2j} \quad (9.24)$$

Finding the Fourier Coefficients  $a(n)$ ,

$$a(n) = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{\pi nx}{L}\right) dx \quad (9.25)$$

$$a(n) = \frac{1}{L} \int_{-L}^L f(x) \left( \frac{e^{j\frac{\pi nx}{L}} - e^{-j\frac{\pi nx}{L}}}{2j} \right) dx \quad (9.26)$$

$$= \frac{1}{2Lj} \int_{-L}^L f(x) e^{j\frac{\pi nx}{L}} dx - \frac{1}{2Lj} \int_{-L}^L f(x) e^{-j\frac{\pi nx}{L}} dx \quad (9.27)$$

$$\Rightarrow a(n) = \frac{c(-n) - c(n)}{j} \quad (9.28)$$

$$a(n) = (-2qL^5) (-1)^n \left( \frac{1}{\pi n} - \frac{2}{(\pi n)^3} + \frac{120}{(\pi n)^5} \right) \quad (9.29)$$

We know,

$$\cos \theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \quad (9.30)$$

Finding the Fourier Coefficients  $b(n)$ ,

$$b(n) = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{\pi nx}{L}\right) dx \quad (9.31)$$

$$b(n) = \frac{1}{L} \int_{-L}^L f(x) \left( \frac{e^{j\frac{\pi nx}{L}} + e^{-j\frac{\pi nx}{L}}}{2} \right) dx \quad (9.32)$$

$$= \frac{1}{2L} \int_{-L}^L f(x) e^{j\frac{\pi nx}{L}} dx + \frac{1}{2L} \int_{-L}^L f(x) e^{-j\frac{\pi nx}{L}} dx \quad (9.33)$$

$$\Rightarrow b(n) = c(-n) + c(n) \quad (9.34)$$

$$b(n) = (2pL^4) (-1)^n \left( \frac{4}{(\pi n)^2} - \frac{24}{(\pi n)^4} \right) \quad (9.35)$$

Hence, options (b) and (c) are correct.

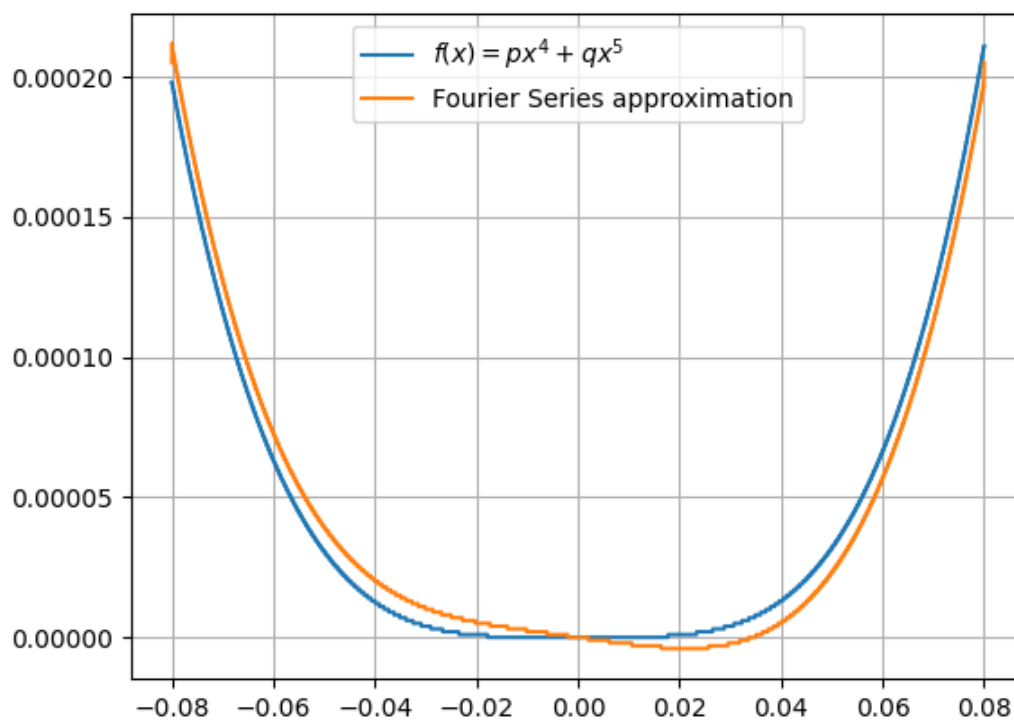


Figure 9.4: Fourier Series Approximation of  $f(x)$  for  $p = 5$ ,  $q = 2$ ,  $L = 0.08$

9.7 A continuous real-valued signal  $x(t)$  has finite positive energy and  $x(t) = 0, \forall t < 0$ .

From the list given below, select ALL the signals whose continuous-time Fourier transform is purely imaginary.

- (a)  $x(t) + x(-t)$
- (b)  $x(t) - x(-t)$
- (c)  $j(x(t) + x(-t))$
- (d)  $j(x(t) - x(-t))$

(GATE IN 2023)

**Solution:**

Parameter	Description
$x(t)$	Continuous real valued signal
$t$	time
$f$	frequency of the signal
$Y(f)$	Fourier Transform of $y(t)$

Table 9.3: Variables and their descriptions

Fourier transform of a signal  $y(t)$

$$\mathcal{F}\{y(t)\} = Y(f) \quad (9.36)$$

$$Y(f) = \int_{-\infty}^{\infty} y(t) e^{-j2\pi ft} dt \quad (9.37)$$

$$Y^*(f) = \int_{-\infty}^{\infty} y^*(t) e^{j2\pi ft} dt \quad (9.38)$$

Fourier transform is purely imaginary if  $Y(f) + Y^*(f) = 0$

(a)  $x(t) + x(-t)$



$$y(t) = x(t) + x(-t) \quad (9.39)$$

$$y^*(t) = y(t) \quad (9.40)$$

$$y(t) = y(-t) \quad (9.41)$$

$$Y(f) + Y^*(f) = \int_{-\infty}^{\infty} y(t) e^{-j2\pi ft} dt + \int_{-\infty}^{\infty} y^*(t) e^{j2\pi ft} dt \quad (9.42)$$

$$= 2 \int_{-\infty}^{\infty} y(t) \cos(2\pi ft) dt \quad (9.43)$$

$\therefore$  Fourier Transform is Purely real.

$$(b) \ x(t) - x(-t)$$

$$y(t) = x(t) - x(-t) \quad (9.44)$$

$$y^*(t) = y(t) = -y(-t) \quad (9.45)$$

$$Y(f) = \int_{-\infty}^{\infty} y(t) e^{-j2\pi ft} dt \quad (9.46)$$

$$Y^*(f) = - \int_{-\infty}^{\infty} y(-t) e^{j2\pi ft} dt \quad (9.47)$$

$$= - \int_{-\infty}^{\infty} y(t) e^{-j2\pi ft} dt \quad (9.48)$$

$$Y(f) + Y^*(f) = 0 \quad (9.49)$$

$\therefore$  Fourier Transform is purely imaginary.

(c)  $j(x(t) + x(-t))$

$$y(t) = j(x(t) + x(-t)) \quad (9.50)$$

$$y(-t) = y(t) \quad (9.51)$$

$$y^*(t) = -y(t) \quad (9.52)$$

$$Y(f) = \int_{-\infty}^{\infty} y(t) e^{-j2\pi ft} dt \quad (9.53)$$

$$Y^*(f) = - \int_{-\infty}^{\infty} y(t) e^{j2\pi ft} dt \quad (9.54)$$

$$= - \int_{-\infty}^{\infty} y(t) e^{-j2\pi ft} dt \quad (9.55)$$

$$Y(f) + Y^*(f) = 0 \quad (9.56)$$

$\therefore$  Fourier Transform is Purely imaginary.

(d)  $j(x(t) - x(-t))$

$$y(t) = j(x(t) - x(-t)) \quad (9.57)$$

$$y(-t) = -y(t) \quad (9.58)$$

$$y^*(t) = -y(t) \quad (9.59)$$

$$Y(f) = \int_{-\infty}^{\infty} y(t) e^{-j2\pi ft} dt \quad (9.60)$$

$$Y^*(f) = - \int_{-\infty}^{\infty} y(t) e^{j2\pi ft} dt \quad (9.61)$$

$$= \int_{-\infty}^{\infty} y(t) e^{-j2\pi ft} dt \quad (9.62)$$

$$Y(f) + Y^*(f) = 2 \int_{-\infty}^{\infty} y(t) e^{-j2\pi ft} dt \quad (9.63)$$

$\therefore$  Fourier Transform is not Purely imaginary.

9.8 Let  $x_1(t) = u(t + 1.5) - u(t - 1.5)$  and  $x_2(t)$  is shown in the figure below. For  $y(t) = x_1(t) * x_2(t)$ , the  $\int_{-\infty}^{\infty} y(t) dt$  is \_\_\_\_\_.

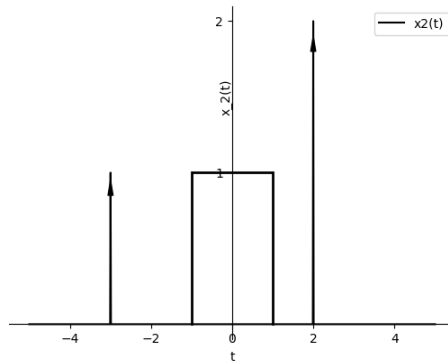


Figure 9.5: Figure

(GATE IN 2023)

**Solution:**

Input Parameters		
Function	Expression	Description
$x_1(t)$	$u(t + 1.5) - u(t - 1.5)$	Step function with delay and width parameters.
$X_1(f)$		Fourier Transform of $x_1(t)$ .
$x_2(t)$	$\delta(t + 3) + \text{rect}\left(\frac{t}{2}\right) + 2\delta(t - 2)$	Impulse function followed by a rectangle and two impulses.
$X_2(f)$		Fourier Transform of $x_2(t)$ .

Table 9.4: Input Parameters

$$x_1(t) = u(t + 1.5) - u(t - 1.5) \quad (9.64)$$

$$x_1(t) = \text{rect}\left(\frac{t}{3}\right) \quad (9.65)$$

$$(9.66)$$

The Fourier Transform of  $\text{rect}\left(\frac{t}{a}\right)$ :

$$\text{rect}\left(\frac{t}{a}\right) \xleftrightarrow{\mathcal{F}} a \times \sin(2\pi f \frac{a}{2}) \quad (9.67)$$

$$\text{where } \text{rect}\left(\frac{t}{a}\right) = \begin{cases} 1 & \text{if } |t| < \frac{a}{2} \\ 0 & \text{otherwise} \end{cases} \quad (9.68)$$

$$X_1(f) = 3\text{sinc}(1.5 \cdot 2\pi f) \quad (9.69)$$

$$x_2(t) = \delta(t+3) + \text{rect}\left(\frac{t}{2}\right) + 2\delta(t-2) \quad (9.70)$$

$$X_2(f) = e^{3j \cdot 2\pi f} + 2\text{sinc}(2\pi f) + 2e^{-2j \cdot 2\pi f} \quad (9.71)$$

$$Y(f) = X_1(f) \cdot X_2(f) \quad (9.72)$$

$$Y(f) = 3\text{sinc}(1.5 \cdot 2\pi f) \cdot (e^{3j \cdot 2\pi f} + 2\text{sinc}(2\pi f) + 2e^{-2j \cdot 2\pi f}) \quad (9.73)$$

$$y(t) = x_1(t) * x_2(t) \quad (9.74)$$

$$Y(f) \xleftrightarrow{\mathcal{F}} y(t) \quad (9.75)$$

$$y(t) = \text{rect}\left(\frac{t+3}{3}\right) + 2\text{rect}\left(\frac{t-2}{3}\right) + (t+2.5)u(t+2.5) + (t-2.5)u(t-2.5) \quad (9.76)$$

$$- (t+0.5)u(t+0.5) - (t-0.5)u(t-0.5) \quad (9.77)$$

$$Y(f) = \int_{-\infty}^{\infty} y(t)e^{-j2\pi ft} dt \quad (9.78)$$

$$\int_{-\infty}^{\infty} y(t) dt = Y(0) \quad (9.79)$$

$$Y(0) = 3\text{sinc}(0) \cdot (e^0 + 2\text{sinc}(0) + 2e^0) \quad (9.80)$$

$$= 3 \cdot (1 + 2 + 2) \quad (9.81)$$

$$= 15 \quad (9.82)$$

Therefore, the value of  $\int_{-\infty}^{\infty} y(t) dt$  is 15

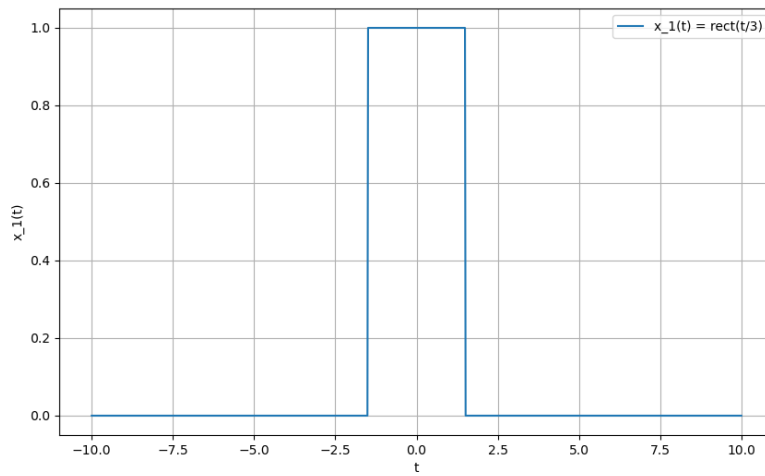


Figure 9.6: Graph of  $x_1(t) = \text{rect}\left(\frac{t}{3}\right)$

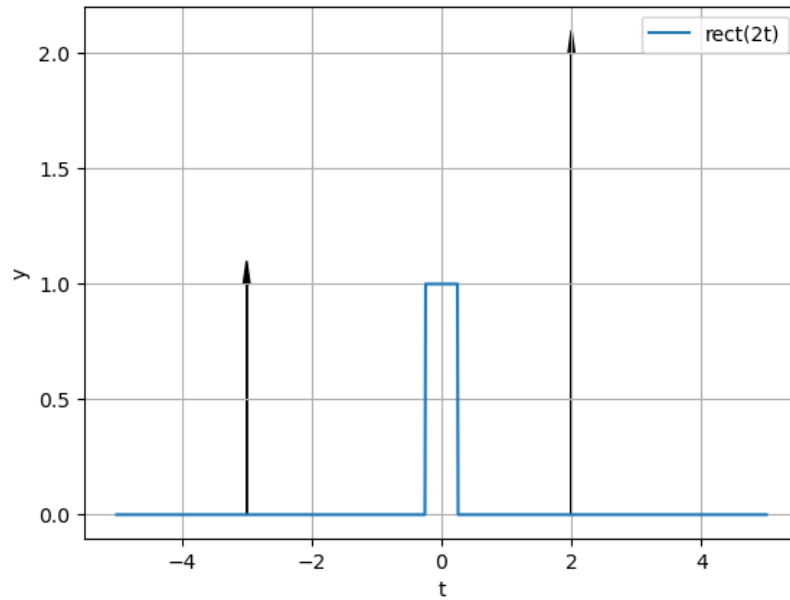


Figure 9.7: Graph of  $x_2(t) = \delta(t+3) + \text{rect}(2t) + 2\delta(t-2)$

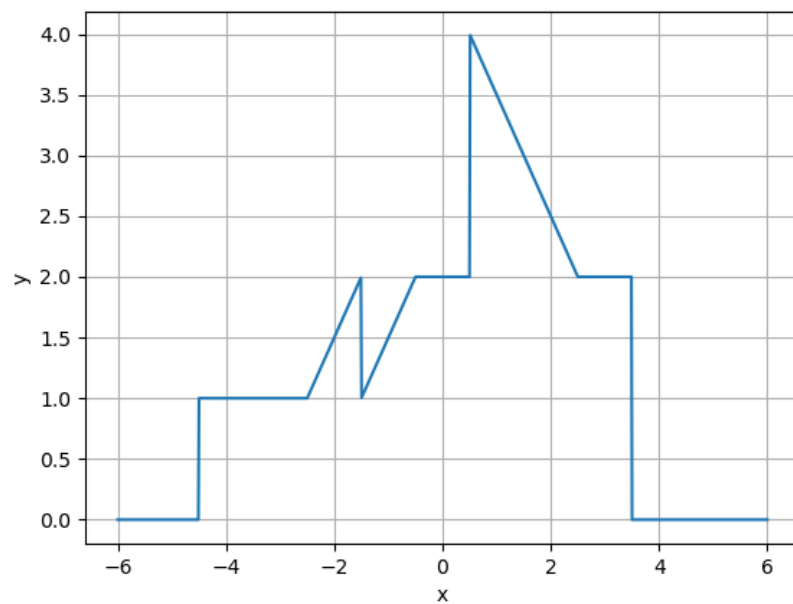


Figure 9.8: Graph of  $y(t)$

9.9 Consider a discrete-time signal with period  $N = 5$ . Let the discrete-time Fourier series (DTFS) representation be  $x[n] = \sum_{k=0}^4 a_k e^{j \frac{2\pi k n}{5}}$  where  $a_0 = 1$ ,  $a_1 = 3j$ ,  $a_2 = 2j$ ,  $a_3 = -2j$ ,  $a_4 = -3j$ . The value of the sum  $\sum_{n=0}^4 x[n] \sin\left(\frac{4\pi n}{5}\right)$  is

(A) -10

(B) 10

(C) -2

(D) 2

Gate 2023 EC 47 **Solution:**



9.10 A continuous time, band-limited signal  $x(t)$  has its Fourier transform described by:

$$X(f) = \begin{cases} 1 - \frac{|f|}{200} & \text{if } |f| \leq 200 \\ 0 & \text{if } |f| > 200 \end{cases}$$

The signal is uniformly sampled at a sampling rate of 600 Hz. The Fourier transform of the signal is  $X_s(f)$ . What is the value of  $\frac{X_s(600)}{X_s(500)}$ ?

(GATE 2023 BM) **Solution:**

Parameter	Description	Value
$X(f)$	Fourier transform of $x(t)$	$\begin{cases} 1 - \frac{ f }{200} & \text{if }  f  \leq 200 \\ 0 & \text{if }  f  > 200 \end{cases}$
$X_s(f)$	Fourier transform of sampled signal	?

Table 9.5: Input Parameters

$$X_s(f) = \frac{1}{600} \sum_{k=-\infty}^{\infty} X(f - 600k) \quad (9.83)$$

$$\Rightarrow X_s(f + 600) = \frac{X(f)}{600} \quad (9.84)$$

$$X_s(600) = \frac{X(0)}{600} \quad (9.85)$$

$$\Rightarrow X_s(600) = \frac{1}{600} \quad (9.86)$$

$$X_s(500) = \frac{X(-100)}{600} \quad (9.87)$$

$$\Rightarrow X_s(500) = \frac{1}{2 \cdot 600} \quad (9.88)$$

$$\frac{X_s(600)}{X_s(500)} = 2 \quad (9.89)$$

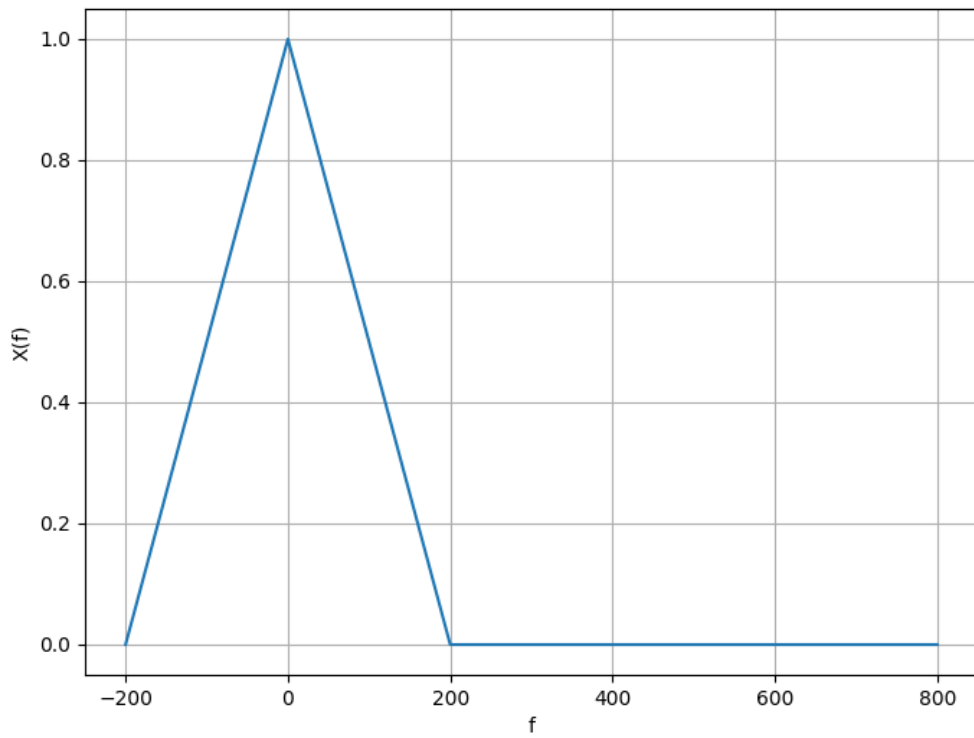


Figure 9.9: Plot of  $X(f)$

9.11 The magnitude and phase plots of an LTI systems are shown in figure. Find the transfer function.

(a)  $2.511e^{-0.0032s}$

(b)  $\frac{e^{-2.514s}}{s+1}$

(c)  $1.04e^{-2.514s}$

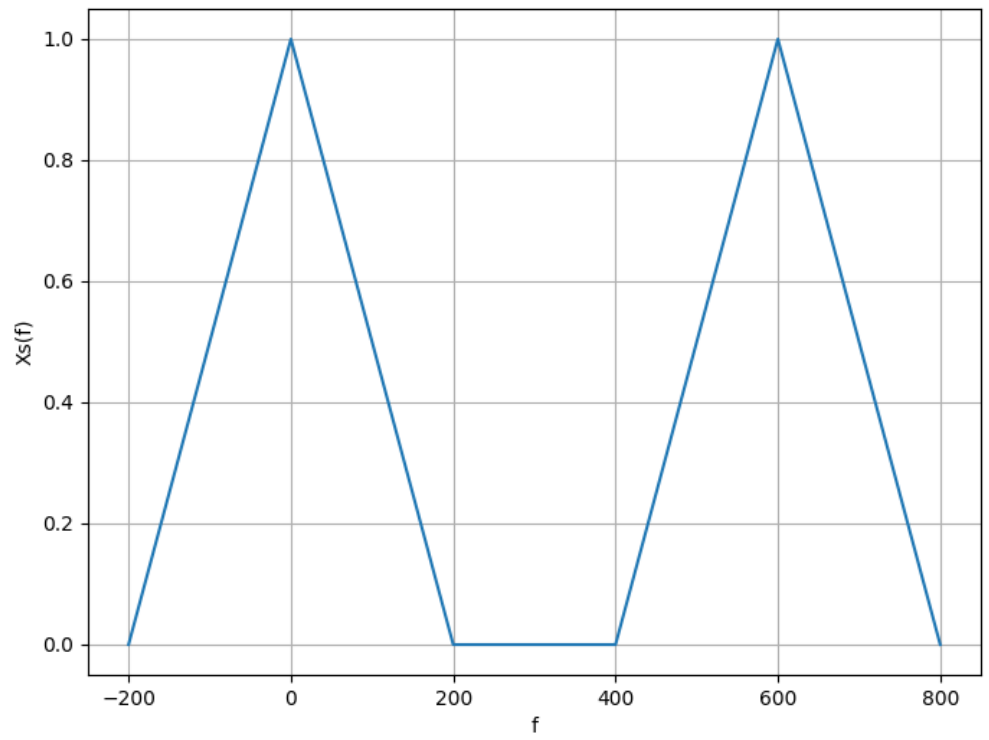


Figure 9.10: Plot of  $X_s(f)$

(d)  $2.511e^{-1.047s}$

(GATE EE 23)

**Solution:** From Fig. 9.11

$$|H(j\omega)| = 8 \quad (9.90)$$

$$\angle H(j\omega) = \frac{-\pi}{3}\omega \quad (9.91)$$

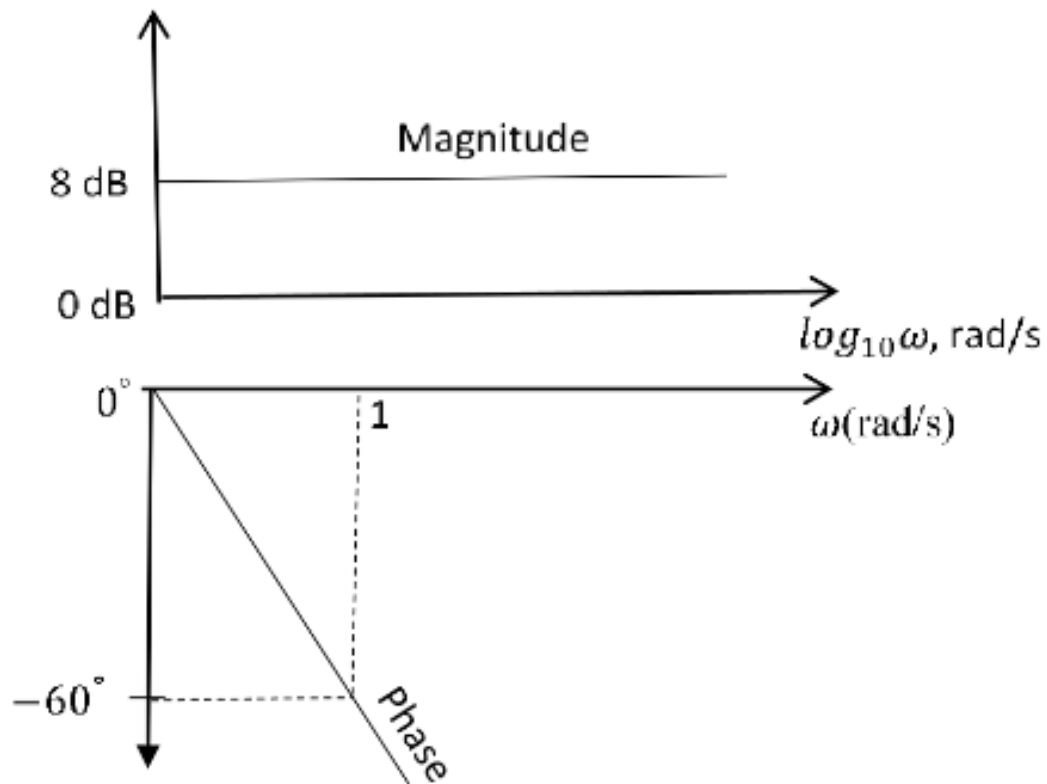


Figure 9.11:

Substituting the values from Fig. 9.11, magnitude of transfer function is:

$$8 = 20 \log_{10}(|H(j\omega)|) \quad (9.92)$$

$$|H(j\omega)| = 10^{0.4} = 2.511 \quad (9.93)$$

Substituting the values from Fig. 9.11, The direction of the transfer function is:

$$\frac{H(j\omega)}{|H(j\omega)|} = e^{-j\frac{\pi}{3}\omega} \quad (9.94)$$

$$H(j\omega) = 2.511e^{-j\frac{\pi}{3}\omega} \quad (9.95)$$

$$= 2.511e^{-1.047s} \quad (9.96)$$

9.12 The value of the convolution of  $f(x) = 3\cos(2x)$  and  $g(x) = \frac{1}{3}\sin(2x)$  where  $x \in [0, 2\pi)$ , at  $x = \frac{\pi}{3}$ , is (Rounded off to 2 decimal places)  
(GATE 2023 GE)

**Solution:**

9.13 A system is described by the following differential equation

$$0.01 \frac{d^2 y(t)}{dt^2} + 0.2 \frac{dy(t)}{dt} + y(t) = 6x(t)$$

where time  $t$  is in seconds. If  $x(t)$  is the unit step input applied at  $t = 0$  s to this system, the magnitude of the output at  $t = 1$  s is \_\_\_\_\_. (Round off the answer to two decimal places.) (GATE-2023.BM)

**Solution:**

9.14 In the differential equation  $\frac{dy}{dx} + \alpha xy = 0$ ,  $\alpha$  is a positive constant. If  $y = 1.0$  at  $x = 0.0$ , and  $y = 0.8$  at  $x = 1.0$ , the value of  $\alpha$  is (rounded off to three decimal places). (GATE CE 30 2023)

**Solution:**

Parameter	Value
$x$	0.0
	1.0
$y$	1.0
	0.8

Table 9.6: Given parameters

Let,  $t = x$

$$\frac{dy}{dt} + \alpha ty = 0 \quad (9.97)$$

$$\int \frac{dy}{y} = - \int \alpha t dt \quad (9.98)$$

$$\ln(|y|) = -\frac{\alpha t^2}{2} + c \quad (9.99)$$

$$y(t) = e^c \cdot e^{-\frac{\alpha t^2}{2}} \quad (9.100)$$

Taking Fourier Transform:

where,

$$\frac{dy}{dt} \xleftrightarrow{\mathcal{F}} j2\pi f Y(f) \quad (9.101)$$

$$\alpha ty(t) \xleftrightarrow{\mathcal{F}} \alpha \frac{j}{2\pi} \frac{d}{df} Y(f) \quad (9.102)$$



From equation (9.101) and (9.102):

$$\frac{4\pi^2 f}{\alpha} Y(f) + \frac{d}{df} Y(f) = 0 \quad (9.103)$$

$$Y(f) = K e^{-\frac{4\pi^2 f^2}{2\alpha}} \quad (9.104)$$

Substituting  $x$  and  $y$  values:

$$c = \ln(1) = 0 \quad (9.105)$$

$$\alpha = -2 \ln(0.8) = 0.446 \quad (9.106)$$

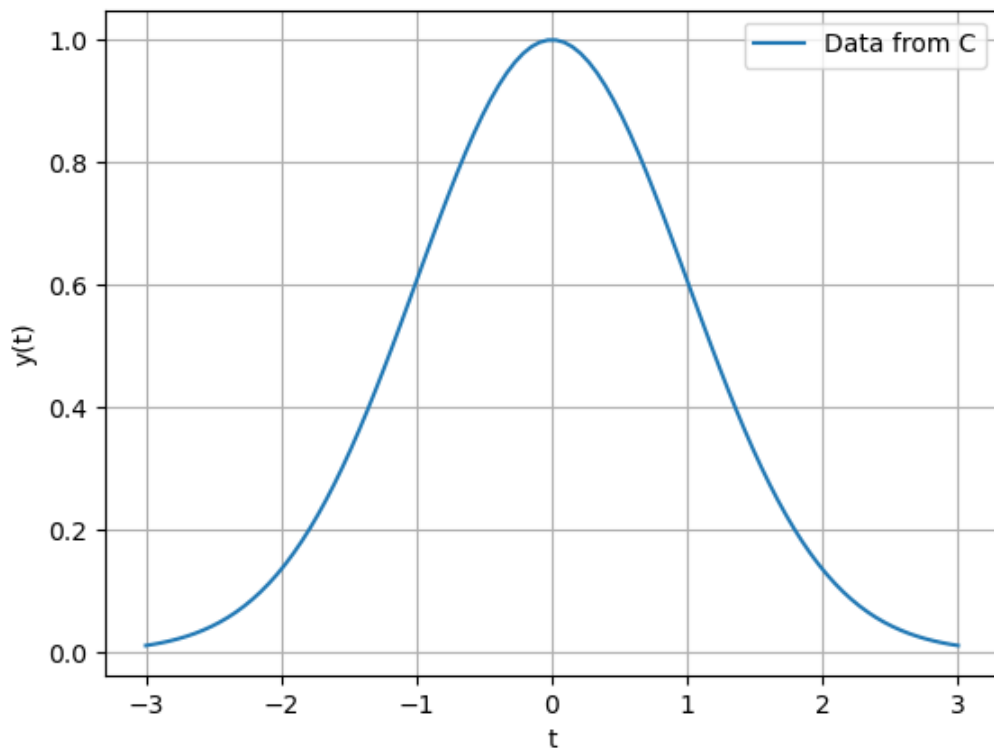


Figure 9.2: Graph of  $y(t)$

## Chapter 10

# FFT

10.1 The discrete-time Fourier transform of a signal  $x[n]$  is  $X(\Omega) = (1 + \cos \Omega) e^{-j\Omega}$ .

Consider that  $x_p[n]$  is a periodic signal of period  $N = 5$  such that

$$x_p[n] = x[n], \text{ for } n = 0, 1, 2 \quad (10.1)$$

$$= 0, \text{ for } n = 3, 4 \quad (10.2)$$

Note that  $x_p[n] = \sum_{k=0}^{N-1} a_k e^{j\frac{2\pi}{N}kn}$ . The magnitude of the Fourier series coefficient  $a_3$  is \_\_\_\_\_ (Round off to 3 decimal places). (GATE EE 2023) **Solution:**

Using Euler's form of representation of complex numbers,

$$e^{j\Omega} = \cos \Omega + j \sin \Omega \quad (10.3)$$

$X(\Omega)$  can be expressed as,

$$X(\Omega) = \frac{1}{2} + e^{-j\Omega} + \frac{e^{-j2\Omega}}{2} \quad (10.4)$$

Symbol	Value	Description
$X(\Omega)$	$(1 + \cos \Omega) e^{-j\Omega}$	Frequency function
$\Omega$	$\omega F_s$	angular frequency
$\omega$	$\omega \in (-\pi, \pi)$	radian frequency
$F_s$	$1Hz$	Sampling frequency
$X(\omega)$	$\sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$	D.T.F.T
$x(n)$	$x(n)$	Signal
$X(k)$	$\frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-\frac{j2\pi}{N} kn}$	Fourier coefficient
$N$	5	Period of the signal

Table 10.1: variable description

As sampling frequency is  $1Hz$  ( $\omega = \Omega$ ) from DTFT(discrete time fourier transform) we get,

$$X(\Omega) = \sum_{n=0}^{n=2} x(n) e^{-j\Omega n}, \Omega \in (-\pi, \pi) \quad (10.5)$$

$$\Rightarrow \sum_{n=0}^{n=2} x(n) e^{-j\Omega n} = \frac{1}{2} + e^{-j\Omega} + \frac{e^{-j2\Omega}}{2} \quad (10.6)$$

On comparing coefficients we get,

$$x(n) = \left\{ \frac{1}{2}, 1, \frac{1}{2} \right\}$$

$$x_p(n) = \left\{ \frac{1}{2}, 1, \frac{1}{2}, 0, 0 \right\} \text{ with period, } N=5 \quad (10.7)$$

$$X(3) = \frac{1}{5} \sum_{n=0}^4 x(n) e^{-j\frac{6\pi}{5}n} \quad (10.8)$$

$$|X(3)| = 0.038 \quad (10.9)$$

Fourier series coef- ficient	Real part	Imaginary part
X(0)	0.4000	0
X(1)	0.081	-0.249
X(2)	-0.031	-0.225
X(3)	-0.031	0.225
X(4)	0.081	0.249

Table 10.2: variable description

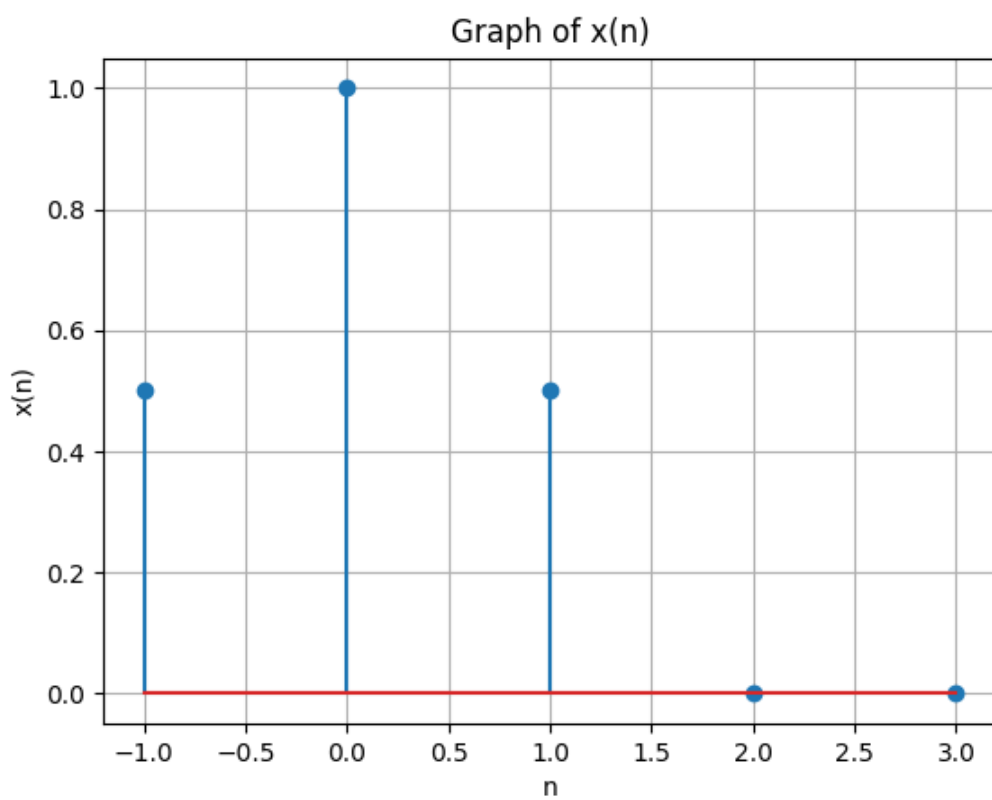


Figure 10.1: Plot of  $x(n)$  *vs*  $n$

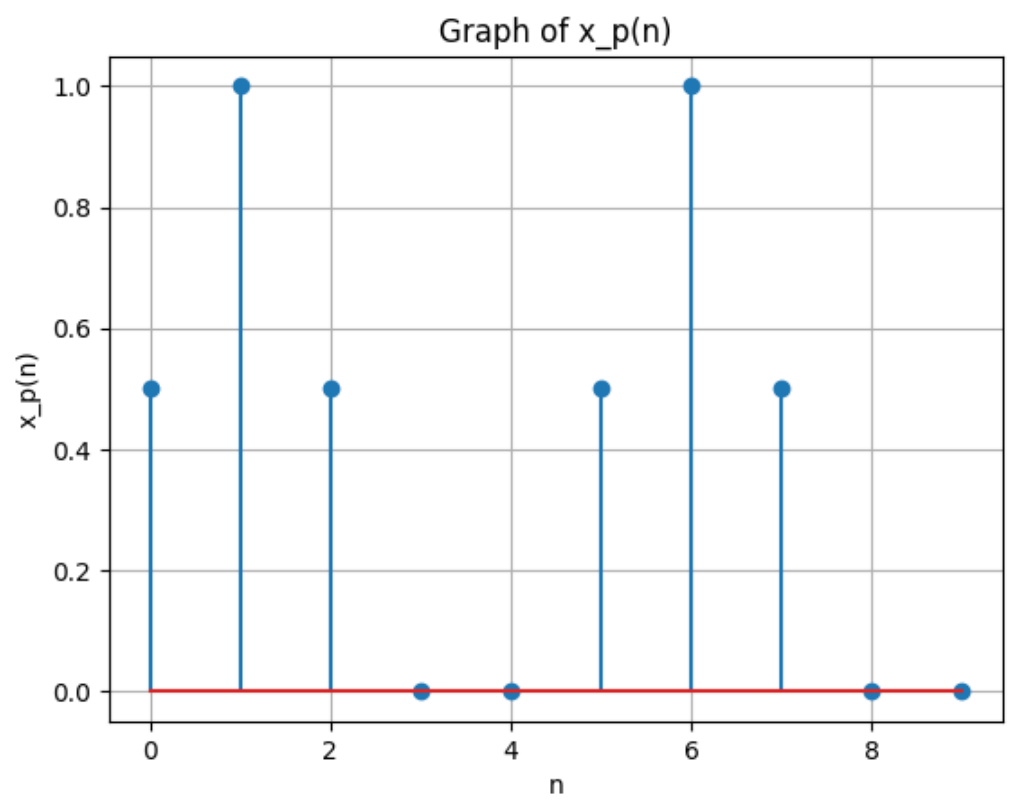


Figure 10.2: Plot of  $x_p(n)$  vs  $n$

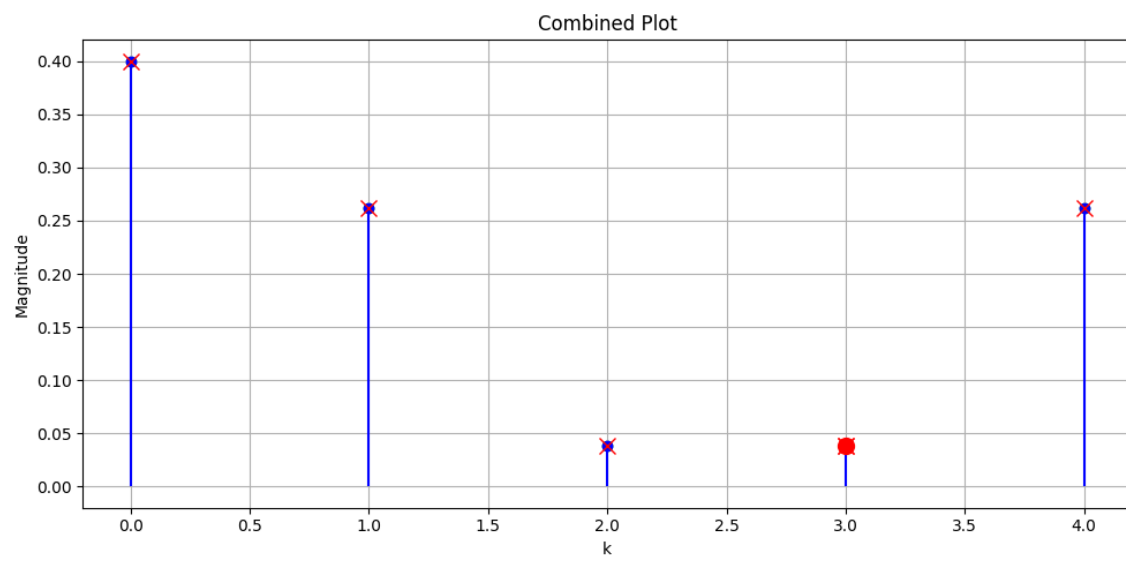


Figure 10.3: Plot of  $x_p(n)$  vs  $n$



# Chapter 11

## Filter Design

11.1 The time-dependent growth of a bacterial population is governed by the equation

$$\frac{dx}{dt} = x \left(1 - \frac{x}{200}\right) \tag{11.1}$$

where  $x$  is the population size at time  $t$ . The initial population size is  $x_0 = 100$  at  $x = 0$ . As  $t \rightarrow \infty$ , the population size of bacteria asymptotically approaches ...

(GATE BM 2023) **Solution:** The growth equation is given by,

Parameters	Values	Description
$t$		Time, Independent variable
$x(t)$		Population size at any time
$x(0)$	100	Initial population
$h$		Step size
$x(n)$		Discrete-Time approximation of $x(t)$

Table 11.1: Parameters

$$\frac{dx}{dt} = x \left(1 - \frac{x}{200}\right) \quad (11.2)$$

$$dx(t)dt = \frac{1}{200}x(200 - x) \quad (11.3)$$

$$\frac{1}{200} \left( \int_0^{x(t)} \frac{dx}{200 - x} + \int_0^{x(t)} \frac{dx}{x} \right) = \int_0^t \frac{dt}{200} \quad (11.4)$$

$$-\ln(200 - x)|_{100}^{x(t)} + \ln(x)|_{100}^{x(t)} = t \quad (11.5)$$

$$\frac{x(t)}{200 - x(t)} = e^t \quad (11.6)$$

$$\Rightarrow x(t) = \frac{200}{1 + e^{-t}} \quad (11.7)$$

We can also express the growth equation as,

$$\int_0^{x(t)} dx = \int_0^t x dt - \frac{1}{200} \int_0^t x^2 dt \quad (11.8)$$

Now approximating by trapezoidal rule of integration between  $t_{n-1}$  to  $t_n$  with the step size being  $h$  we have,

$$\begin{aligned} x(t_n) - x(t_{n-1}) &= \frac{h}{2} [x(t_n) + x(t_{n-1})] - \\ &\quad \frac{h}{400} [x^2(t_n) + x^2(t_{n-1})] \end{aligned} \quad (11.9)$$

Next, replacing  $t_n = hn$  we get the difference equation,

$$x(hn) = \frac{-(1 - \frac{h}{2}) + \sqrt{(1 - \frac{h}{2})^2 + \frac{h}{100}p_1}}{h/200} \quad (11.10)$$

Where,

$$p_1 = \left[ \left(1 - \frac{h}{2}\right) x(h(n-1)) - \frac{h}{400} x^2(h(n-1)) \right] \quad (11.11)$$

And relabeling  $hn \longleftrightarrow n$  we will be having the discrete time equation as,

$$x(n) = \frac{-(1 - \frac{h}{2}) + \sqrt{(1 - \frac{h}{2})^2 + \frac{h}{100}p_2}}{h/200} \quad (11.12)$$

Where,

$$p_2 = \left[ \left(1 - \frac{h}{2}\right) x(n-1) - \frac{h}{400} x^2(n-1) \right] \quad (11.13)$$

Now, plotting both the differential and the difference equations,

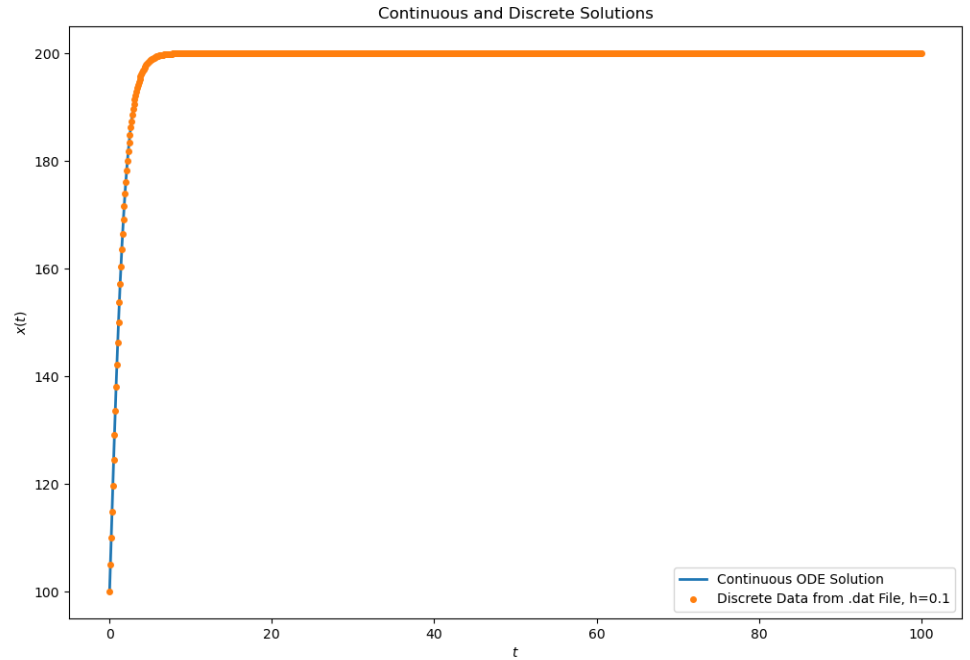


Figure 11.1: Plot in Continuous and Discrete Time

As we can see the discrete time plot is following the actual curve, which means (11.12) is indeed a good approximation of the original continuous-time equation.

And the population size approaches to 200 as  $t \rightarrow \infty$ .