GATE 2023 BM

EE23BTECH11020 - Raghava Ganji*

GATE 2023 BM.48: The function $f(z) = \frac{1}{z-1}$ of a complex variable z on a closed contour in an anti-clockwise direction.For which of the following contours, does this integral have a non-zero value?
$(A)|z-2| = 0.01$
$(B)|z-1| = 0.1$
$(C)|z-3| = 5$
$(D)|z| = 2$
Solution:
Cauchy's Integral Formula and Residue Theorem.

$$\oint_c f(z) = 2\pi j Res\left[f(z), z_0\right] \tag{1}$$

$$Res\left[f(z), z_0\right] = \lim_{z \to z_0} \left[(z - z_0) f(z)\right] \tag{2}$$

Here z_0 is pole of the f(z)
Using (1)

$$\oint_c \frac{1}{z-1} dz = 2\pi j Res\left[\frac{1}{z-1}, 1\right] \tag{3}$$

1) For option A the pole is outside the contour, then Residue is zero.

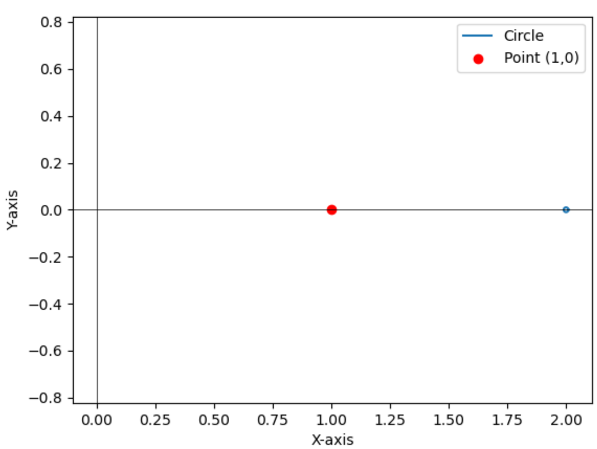

Fig. 1. graph of option A

$$\implies \oint_c \frac{1}{z-1} dz = 2\pi j (0) \tag{4}$$

$$\implies 0 \tag{5}$$

2) For option B the pole is inside the contour. Then, using (2)

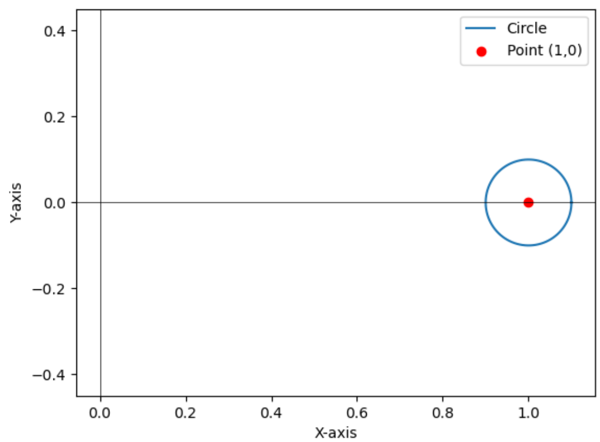

Fig. 2. graph of option B

$$Res\left[\frac{1}{z-1}, 1\right] = \lim_{z \to 1} (z-1) \frac{1}{z-1} \tag{6}$$

$$= 1 \tag{7}$$

$$\implies \oint_c \frac{1}{z-1} dz = 2\pi j (1) \tag{8}$$

$$\implies 2\pi j \tag{9}$$

3) For option C the pole is inside the contour. Then, using (2)

$$Res\left[\frac{1}{z-1}, 1\right] = \lim_{z \to 1} (z-1) \frac{1}{z-1} \tag{10}$$

$$= 1 \tag{11}$$

$$\implies \oint_c \frac{1}{z-1} dz = 2\pi j (1) \tag{12}$$

$$\implies 2\pi j \tag{13}$$

4) For option D the pole is inside the contour. Then, using (2)

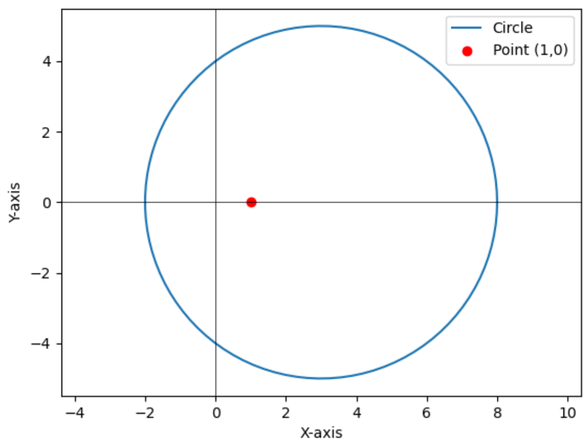

Fig. 3. graph of option C

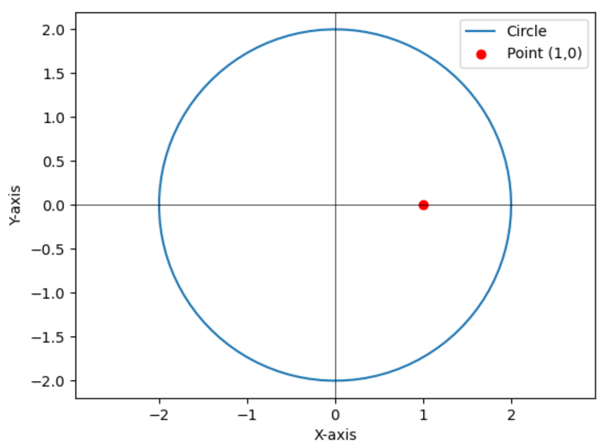

Fig. 4. graph of option D

$$Res\left[\frac{1}{z-1},1\right] = \lim_{z\to 1}(z-1)\frac{1}{z-1} \tag{14}$$

$$= 1 \tag{15}$$

$$\implies \oint_c \frac{1}{z-1}dz = 2\pi j(1) \tag{16}$$

$$\implies 2\pi j \tag{17}$$

We can conclude that for options B,C,D contours have the non-zero value for this integral.