Solutions to Analytical Plane Geometry by Hem Raj and Hukum Chand

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Contents

Abstract—**This book provides a vector approach to analytical geometry. The content and exercises are based on Hem Raj and hukum Chand's book on Analytical Plane Geometry.**

1 Coordinates

1.0.1. Find the distance between the following pair of points (11,16) and (23,21).
Let

$$\mathbf{A} = \begin{pmatrix}11\\16\end{pmatrix}, \mathbf{B} = \begin{pmatrix}23\\21\end{pmatrix} \tag{1.0.1.1}$$

Then,

$$\mathbf{B}-\mathbf{A} = \begin{pmatrix}12\\5\end{pmatrix} \tag{1.0.1.2}$$

and

$$\|\mathbf{B}-\mathbf{A}\|^2 = (\mathbf{B}-\mathbf{A})^\top(\mathbf{B}-\mathbf{A}) = \begin{pmatrix}12 & 5\end{pmatrix}\begin{pmatrix}12\\5\end{pmatrix} \tag{1.0.1.3}$$

$$\implies \|\mathbf{B}-\mathbf{A}\| = \sqrt{(12^2+5^2)} \tag{1.0.1.4}$$

$$= 13$$

1.0.2. Find the distance between the following pair of points (66,25) and (99,69).
Let

$$\mathbf{A} = \begin{pmatrix}66\\25\end{pmatrix}, \mathbf{B} = \begin{pmatrix}99\\69\end{pmatrix} \tag{1.0.2.1}$$

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Then

$$\|\mathbf{B}-\mathbf{A}\| = \|\mathbf{C}\| \tag{1.0.2.2}$$

$$\|\mathbf{C}\|^2 = \mathbf{C}^\top\mathbf{C} \tag{1.0.2.3}$$

$$= \begin{pmatrix}33 & 44\end{pmatrix}\begin{pmatrix}33\\44\end{pmatrix} \tag{1.0.2.4}$$

$$\implies \|\mathbf{C}\| = \sqrt{(33^2+44^2)} \tag{1.0.2.5}$$

$$= 55 \tag{1.0.2.6}$$

1.0.3. Find the distance between the following pairs of points with the axes at 60°

a) $\begin{pmatrix}7\\6\end{pmatrix}$ and $\begin{pmatrix}4\\5\end{pmatrix}$ **Solution:** 2/solution/2/1/Assignment1.tex

1.0.4. Find the area of the quadrilateral formed by the points

a)

$$\mathbf{A} = \begin{pmatrix}1\\1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}3\\5\end{pmatrix}, \mathbf{C} = \begin{pmatrix}-2\\4\end{pmatrix}, \mathbf{D} = \begin{pmatrix}-1\\-5\end{pmatrix}. \tag{1.0.4.1}$$

Solution: Let

$$\mathbf{A} = \begin{pmatrix}1\\1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}3\\5\end{pmatrix}, \mathbf{C} = \begin{pmatrix}-2\\4\end{pmatrix}, \mathbf{D} = \begin{pmatrix}-1\\-5\end{pmatrix}. \tag{1.0.4.2}$$

Then,

$$(\mathbf{B}-\mathbf{A}) = \begin{pmatrix}3\\5\end{pmatrix} - \begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}2\\4\end{pmatrix} \tag{1.0.4.3}$$

$$(\mathbf{D}-\mathbf{B}) = \begin{pmatrix}-1\\-5\end{pmatrix} - \begin{pmatrix}3\\5\end{pmatrix} = \begin{pmatrix}-4\\-10\end{pmatrix} \tag{1.0.4.4}$$

$$(\mathbf{A}-\mathbf{D}) = \begin{pmatrix}1\\1\end{pmatrix} - \begin{pmatrix}-1\\-5\end{pmatrix} = \begin{pmatrix}2\\6\end{pmatrix} \tag{1.0.4.5}$$

Row reducing the matrix formed by the

vectors,

$$\begin{pmatrix} 2 & 4 & 1 \\ -4 & -10 & 1 \\ 2 & 6 & 1 \end{pmatrix} \tag{1.0.4.6}$$

$$\implies \begin{pmatrix} 2 & 4 & 1 \\ -4 & -10 & 1 \\ 2 & 6 & 1 \end{pmatrix} \xleftrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 2 & 4 & 1 \\ 2 & 6 & 1 \\ -4 & -10 & 1 \end{pmatrix} \xleftrightarrow{R_3 \leftrightarrow R_1+R_2+R_3} \begin{pmatrix} 2 & 4 & 1 \\ 2 & 6 & 1 \\ 0 & 0 & 3 \end{pmatrix} \xleftrightarrow{R_2 \leftrightarrow R_2 - R_1} \begin{pmatrix} 2 & 4 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \tag{1.0.4.7}$$

The number of non-zero rows in the matrix = 3. Hence the matrix is full rank and AB, BD, DA are not collinear. See Fig. 1.0.4.1, which shows that the points given form a quadrilateral. Area of a $\triangle ABC$ is given by

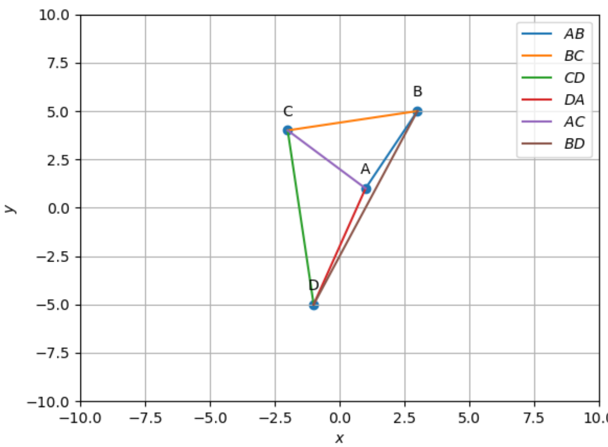

Fig. 1.0.4.1: Quadrilateral ABCD

$$\triangle ABC = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & -2 \\ 1 & 5 & 4 \end{vmatrix} = 9 \tag{1.0.4.8}$$

Area of $\triangle ACD$ is given by

$$\triangle ACD = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ 1 & -2 & -1 \\ 1 & 4 & -5 \end{vmatrix} = 11.5 \tag{1.0.4.9}$$

Thus, the area of quadrilateral ABCD = 9+11.5 = 20.5.

b)

$$\mathbf{P} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}, \mathbf{R} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}, \mathbf{S} = \begin{pmatrix} -2 \\ -2 \end{pmatrix} \tag{1.0.4.10}$$

Solution:

In Fig. 1.0.4.2, Area of a Quadrilateral PQRS=

$$Area(\triangle PQR) + Area(\triangle PRS) = \tag{1.0.4.11}$$

$$\frac{1}{2}\left\|(\mathbf{Q}-\mathbf{P}) \times (\mathbf{Q}-\mathbf{R})\right\| + \frac{1}{2}\left\|(\mathbf{S}-\mathbf{P}) \times (\mathbf{S}-\mathbf{R})\right\| \tag{1.0.4.12}$$

For two vectors

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \tag{1.0.4.13}$$

$$\|\mathbf{a} \times \mathbf{b}\| = |(a_1 b_2 - a_2 b_1)| \tag{1.0.4.14}$$

$$\mathbf{Q} - \mathbf{P} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \tag{1.0.4.15}$$

$$\mathbf{Q} - \mathbf{R} = \begin{pmatrix} 6 \\ 1 \end{pmatrix} \tag{1.0.4.16}$$

$$\mathbf{S} - \mathbf{P} = \begin{pmatrix} -4 \\ -3 \end{pmatrix} \tag{1.0.4.17}$$

$$\mathbf{S} - \mathbf{R} = \begin{pmatrix} 1 \\ -6 \end{pmatrix} \tag{1.0.4.18}$$

Using equation (1.0.4.14)

$$\frac{1}{2}\left\|(\mathbf{Q}-\mathbf{P}) \times (\mathbf{Q}-\mathbf{R})\right\| = \frac{1}{2}|(-23)| = 11.5 \tag{1.0.4.19}$$

$$\frac{1}{2}\left\|(\mathbf{S}-\mathbf{P}) \times (\mathbf{S}-\mathbf{R})\right\| = \frac{1}{2}|(27)| = 13.5 \tag{1.0.4.20}$$

Substituting values from equation (1.0.4.19) and (1.0.4.20) in equation (1.0.4.12),the desired area is

$$11.5 + 13.5 = 25 \tag{1.0.4.21}$$

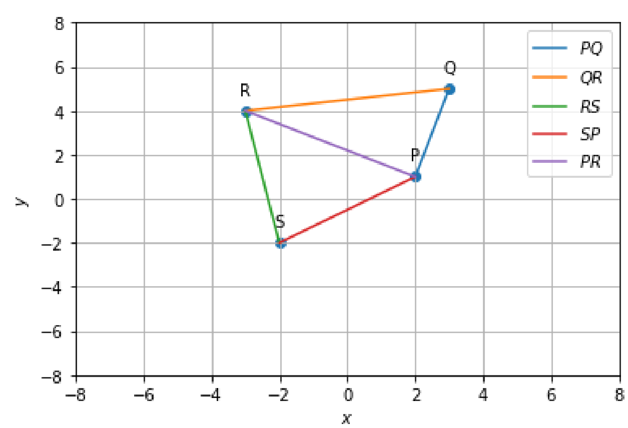

Fig. 1.0.4.2: Quadrilateral PQRS