Solutions to Analytical Plane Geometry by Hem Raj and Hukum Chand

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Contents

Abstract—**This book provides a vector approach to analytical geometry. The content and exercises are based on Hem Raj and hukum Chand's book on Analytical Plane Geometry.**

1 Coordinates

1.0.1. Find the distance between the following pair of points (11,16) and (23,21).
Let

$$\mathbf{A} = \begin{pmatrix} 11 \\ 16 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 23 \\ 21 \end{pmatrix} \tag{1.0.1.1}$$

Then,

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} 12 \\ 5 \end{pmatrix} \tag{1.0.1.2}$$

and

$$\|\mathbf{B} - \mathbf{A}\|^2 = (\mathbf{B} - \mathbf{A})^\top (\mathbf{B} - \mathbf{A}) = \begin{pmatrix} 12 & 5 \end{pmatrix} \begin{pmatrix} 12 \\ 5 \end{pmatrix} \tag{1.0.1.3}$$

$$\implies \|\mathbf{B} - \mathbf{A}\| = \sqrt{(12^2 + 5^2)} \tag{1.0.1.4}$$

$$= 13$$

1.0.2. Find the distance between the following pair of points (66,25) and (99,69).
Let

$$\mathbf{A} = \begin{pmatrix} 66 \\ 25 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 99 \\ 69 \end{pmatrix} \tag{1.0.2.1}$$

Then

$$\|\mathbf{B} - \mathbf{A}\| = \|\mathbf{C}\| \tag{1.0.2.2}$$

$$\|\mathbf{C}\|^2 = \mathbf{C}^\top \mathbf{C} \tag{1.0.2.3}$$

$$= \begin{pmatrix} 33 & 44 \end{pmatrix} \begin{pmatrix} 33 \\ 44 \end{pmatrix} \tag{1.0.2.4}$$

$$\implies \|\mathbf{C}\| = \sqrt{(33^2 + 44^2)} \tag{1.0.2.5}$$

$$= 55 \tag{1.0.2.6}$$

1.0.3. Find the distance between the following pairs of points with the axes at 60°

a) $\begin{pmatrix} 7 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 5 \end{pmatrix}$ **Solution:**
Let the given points in the angular axis be

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix} ; \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 6 \end{pmatrix} \tag{1.0.3.1}$$

From Fig. 1.0.3.1, the corresponding points in the rectangular axis are

$$x_3 = x_1 + y_1 \cos 60^\circ \tag{1.0.3.2}$$

$$y_3 = y_1 \cos 30^\circ \implies \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 & \cos 60^\circ \\ 0 & \cos 30^\circ \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \tag{1.0.3.3}$$

Similarly,

$$\begin{pmatrix} x_4 \\ y_4 \end{pmatrix} = \begin{pmatrix} 1 & \cos 60^\circ \\ 0 & \cos 30^\circ \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \tag{1.0.3.4}$$

In general, the rectangular coordinates can be express in terms of the angular coordinates through the linear transformation

$$\mathbf{x}_r = \mathbf{P}\mathbf{x}, \tag{1.0.3.5}$$

$$\text{where, } vecP = \begin{pmatrix} 1 & \cos\theta \\ 0 & \sin\theta \end{pmatrix} \tag{1.0.3.6}$$

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Thus, the distance between two points $\mathbf{x}_1, \mathbf{x}_2$ in the angular axis is given by

$$\|\mathbf{P}(\mathbf{x}_1 - \mathbf{x}_2)\| = \sqrt{(\mathbf{x}_1 - \mathbf{x}_2)^\top \mathbf{P}^\top \mathbf{P}(\mathbf{x}_1 - \mathbf{x}_2)} \tag{1.0.3.7}$$

$$= \sqrt{13} \tag{1.0.3.8}$$

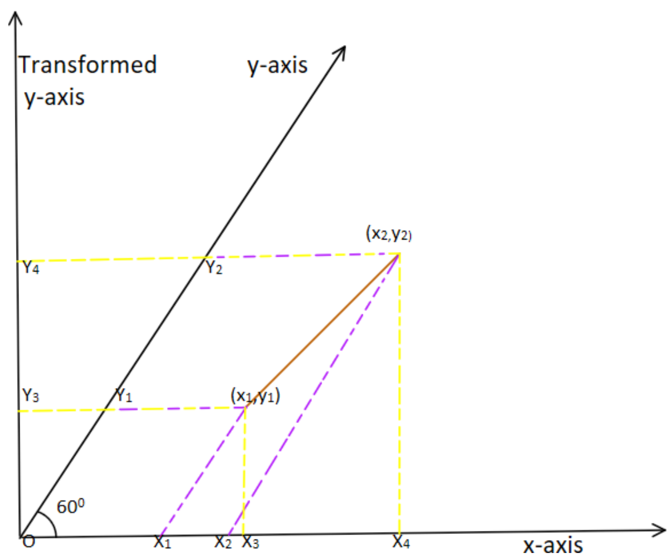

Fig. 1.0.3.1: Points defined on angular & rectangular axes

1.0.4. Find the area of the quadrilateral formed by the points

a)

$$\mathbf{A} = \begin{pmatrix}1\\1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}3\\5\end{pmatrix}, \mathbf{C} = \begin{pmatrix}-2\\4\end{pmatrix}, \mathbf{D} = \begin{pmatrix}-1\\-5\end{pmatrix}. \tag{1.0.4.1}$$

Solution: Let

$$\mathbf{A} = \begin{pmatrix}1\\1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}3\\5\end{pmatrix}, \mathbf{C} = \begin{pmatrix}-2\\4\end{pmatrix}, \mathbf{D} = \begin{pmatrix}-1\\-5\end{pmatrix}. \tag{1.0.4.2}$$

Then,

$$(\mathbf{B} - \mathbf{A}) = \begin{pmatrix}3\\5\end{pmatrix} - \begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}2\\4\end{pmatrix} \tag{1.0.4.3}$$

$$(\mathbf{D} - \mathbf{B}) = \begin{pmatrix}-1\\-5\end{pmatrix} - \begin{pmatrix}3\\5\end{pmatrix} = \begin{pmatrix}-4\\-10\end{pmatrix} \tag{1.0.4.4}$$

$$(\mathbf{A} - \mathbf{D}) = \begin{pmatrix}1\\1\end{pmatrix} - \begin{pmatrix}-1\\-5\end{pmatrix} = \begin{pmatrix}2\\6\end{pmatrix} \tag{1.0.4.5}$$

Row reducing the matrix formed by the vectors,

$$\tag{1.0.4.6}$$

$$\implies \begin{pmatrix}2 & 4 & 1\\-4 & -10 & 1\\2 & 6 & 1\end{pmatrix} \xleftrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix}2 & 4 & 1\\2 & 6 & 1\\-4 & -10 & 1\end{pmatrix} \xleftrightarrow{R_3 \leftrightarrow R_1 + R_2 + R_3} \tag{1.0.4.7}$$

The number of non-zero rows in the matrix = 3. Hence the matrix is full rank and AB, BD, DA are not collinear. See Fig. 1.0.4.1, which shows that the points given form a quadrilateral. Area of a $\triangle ABC$ is given by

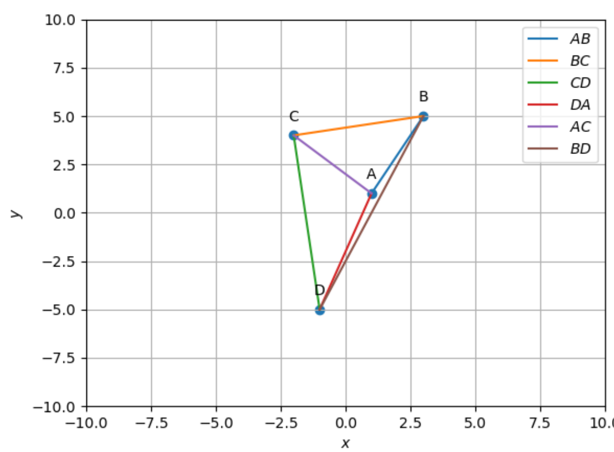

Fig. 1.0.4.1: Quadrilateral ABCD

$$\triangle ABC = \frac{1}{2}\begin{vmatrix}1 & 1 & 1\\1 & 3 & -2\\1 & 5 & 4\end{vmatrix} = 9 \tag{1.0.4.8}$$

Area of $\triangle ACD$ is given by

$$\triangle ACD = \frac{1}{2}\begin{vmatrix}1 & 1 & 1\\1 & -2 & -1\\1 & 4 & -5\end{vmatrix} = 11.5 \tag{1.0.4.9}$$

Thus, the area of quadrilateral ABCD = 9+11.5 = 20.5.

b)

$$\mathbf{P} = \begin{pmatrix}2\\1\end{pmatrix}, \mathbf{Q} = \begin{pmatrix}3\\5\end{pmatrix}, \mathbf{R} = \begin{pmatrix}-3\\4\end{pmatrix}, \mathbf{S} = \begin{pmatrix}-2\\-2\end{pmatrix} \tag{1.0.4.10}$$

Solution:

In Fig. 1.0.4.2, Area of a Quadrilateral PQRS=

$$Area(\triangle PQR) + Area(\triangle PRS) = \tag{1.0.4.11}$$

$$\frac{1}{2}\left\|(\mathbf{Q}-\mathbf{P})\times(\mathbf{Q}-\mathbf{R})\right\| + \frac{1}{2}\left\|(\mathbf{S}-\mathbf{P})\times(\mathbf{S}-\mathbf{R})\right\| \tag{1.0.4.12}$$

For two vectors

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \tag{1.0.4.13}$$

$$\|\mathbf{a}\times\mathbf{b}\| = |(a_1b_2 - a_2b_1)| \tag{1.0.4.14}$$

$$\mathbf{Q}-\mathbf{P} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \tag{1.0.4.15}$$

$$\mathbf{Q}-\mathbf{R} = \begin{pmatrix} 6 \\ 1 \end{pmatrix} \tag{1.0.4.16}$$

$$\mathbf{S}-\mathbf{P} = \begin{pmatrix} -4 \\ -3 \end{pmatrix} \tag{1.0.4.17}$$

$$\mathbf{S}-\mathbf{R} = \begin{pmatrix} 1 \\ -6 \end{pmatrix} \tag{1.0.4.18}$$

Using equation (1.0.4.14)

$$\frac{1}{2}\left\|(\mathbf{Q}-\mathbf{P})\times(\mathbf{Q}-\mathbf{R})\right\| = \frac{1}{2}|(-23)| = 11.5 \tag{1.0.4.19}$$

$$\frac{1}{2}\left\|(\mathbf{S}-\mathbf{P})\times(\mathbf{S}-\mathbf{R})\right\| = \frac{1}{2}|(27)| = 13.5 \tag{1.0.4.20}$$

Substituting values from equation (1.0.4.19) and (1.0.4.20) in equation (1.0.4.12),the desired area is

$$11.5 + 13.5 = 25 \tag{1.0.4.21}$$

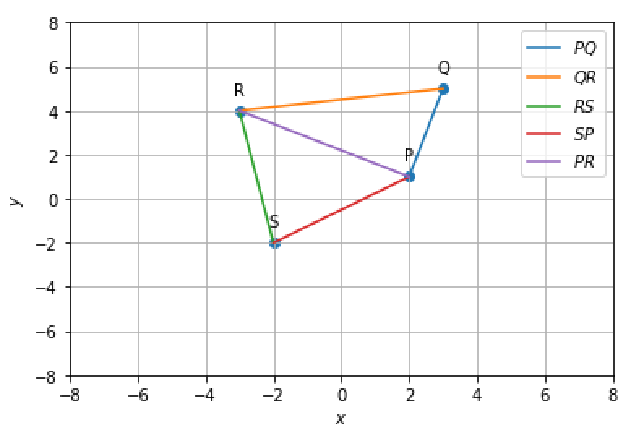

Fig. 1.0.4.2: Quadrilateral PQRS