Abstract Algebra

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1 Things Familiar and Less Familiar

1.1 Introduction

1. let S be a set having an operation * which assigns an element a*b of S for any $a, b \in S$. Let us assume that the following two rules hold:
   a) If a, b are any objects in S, then $a * b = a$.
   b) If a, b are any objects in S, then $a * b = b * a$.

   Show that S can have at most one object.
   Solution: From condition 1.1.1a, interchanging a, b,

   $$b * a = b \quad (1.1.1)$$

   and from condition 1.1.1b,

   $$b * a = a * b \quad (1.1.1)$$

   But from condition 1.1.1a,

   $$a * b = a \implies a = b \quad (1.1.1)$$

   Thus, S can have at most one object.
2. Let S be the set of all integers $0, \pm 1, \pm 2, \ldots, \pm n, \ldots$. For $a, b \in S$, define * by

   $$a * b = a - b \quad (1.1.2)$$

   Verify the following
   a) $a * b \neq b * a$ unless $a = b$
   b) $(a * b) * c \neq a * (b * c)$ in general. Under what conditions on a, b, c is

   $$(a * b) * c \neq a * (b * c) \quad ? \quad (1.1.2)$$

   c) The integer a has the property that $a * 0 = a$ for every $a \in S$.
   d) For $a \in S, a * a = 0$.

   Solution:
   a)

   $$a * b = b * a \quad (1.1.2)$$
   $$\implies a - b = b - a \quad (1.1.2)$$
   $$\text{or, } a = b \quad (1.1.2)$$

   b) Let $a = 1, b = 2, c = 4$. Then,

   $$a * b = -1, (a * b) * c = -1 - 4 = -5 \quad (1.1.2)$$
   $$b * c = -2, a * (b * c) = 1 + 2 = 3 \neq -5 \quad (1.1.2)$$

   Thus, for the given condition to be satisfied,

   $$(a - b) - c = a - (b - c) \quad (1.1.2)$$
   $$\implies c = 0 \quad (1.1.2)$$

   c)

   $$a * 0 = a - 0 = a \quad (1.1.2)$$

   d)

   $$a * a = a - a = 0 \quad (1.1.2)$$
3. Let S consist of the two objects $\square$ and $\triangle$. We define the operatin * on S by subjecting $\square$ and $\triangle$ to the following conditions.
   a) $\square * \triangle = \triangle = \triangle * \square$
   b) $\square * \square = \square$
   c) $\triangle * \triangle = \square$

   Verify by explicit calculation that if a, b, c are any elements of S, (i.e. a, b, c can be any of $\square$ or $\triangle$), then
   a) $a * b$ is in S
   b) $(a * b) * c = a * (b * c)$
   c) $a * b = b * a$
   d) There is a particular a in S such that $a * b = b * a = b$ for all $b \in S$
   e) Given $b \in S, b * b = a$, where a is the particular element in Part 1.1.3d.

   Solution: Let $\square = 1, \triangle = -1$. These satisfy all the given conditions.

a) $a * b \in [1, -1] \in S$.
b) Writing the truth table, $(a * b) * c = a * (b * c)$.
c) $a * b = b * a$ can be verified by writing the truth table.
d) For $a = 1, a * b = b * a = b$, for all $b \in S$.
e) For $a = 1$, if $b = -1, b * b = 1 = a$. This can be shown to be true for $b = 1$ as well.

1.2 Set Theory

1. Describe the following sets verbally
   a) S = {Mercury, Venus, Earth, . . . , Pluto}
   b) S = {Andhra Pradesh, Uttar Pradesh, . . . , Assam}

   Solution:
   a) Planets
   b) Indian states
2. Describe the following sets verbally
   a) $S = \{2, 4, 6, 8, \dots\}$
   b) $S = \{2, 4, 8, 16, \dots\}$
   c) $S = \{1, 4, 9, 16, 25, 36 \dots\}$

   Solution:
   a) Even numbers
   b) Powers of 2
   c) Squares of positive integers
3. If A is the set of all residents of India, B the set of all Sri Lankan citizens, and C the set of all women in the world, describe the sets $ABC, A - B, A - C, C - A$ verbally.
   Solution:
   a) ABC is the set of all women residents of India who are citizens of Sri Lanka.
   b) $A - B = AB'$ is the set of all residents of India who are not Sri Lankan citizens.
   c) $A - C = AC'$ is the set of all male residents of India.
   d) $C - A = CA'$ is the set of all women who are not residing in India.
4. If $A = \{1, 4, 7, a\}$ and $B = \{3, 4, 9, 11\}$ and you have been told that $AB = \{4, 9\}$, then what must a be?
   Solution: $a = 9$
5. If $A \subset B, B \subset C$, prove that $A \subset C$
   Solution: From the given information,

$$A + P = B, AP = 0, B + Q = C, BQ = 0 \tag{1.2.5.1}$$

$$\implies B + Q = A + P + Q = C, \tag{1.2.5.2}$$

$\because BQ = 0$,

$$AQ + PQ = 0 \implies AQ = 0, PQ = 0 \tag{1.2.5.3}$$

Hence,

$$A(P + Q) = 0 \implies A \subset C \tag{1.2.5.4}$$

6. If $A \subset B$ prove that $A \cup C \subset B \cup C$ for any set C.
   Solution: From the given information, there exists P such that

$$A + P = B, AP = 0 \tag{1.2.6.1}$$

Also,

$$B + C = A + P + C \tag{1.2.6.2}$$

$$\implies A + C \subset B + C \tag{1.2.6.3}$$

7. Show that

$$A \cup B = B \cup A \tag{1.2.7.1}$$

$$A \cap B = B \cap A \tag{1.2.7.2}$$

8. Prove that

$$(A - B) \cup (B - A) = (A \cup B) - (A \cap B) \tag{1.2.8.1}$$

Solution: Since

$$A - B = AB', \tag{1.2.8.2}$$

$$(A - B) \cup (B - A) = AB' + BA' \tag{1.2.8.3}$$

Also,

$$(A \cup B) - (A \cap B) = (A + B)(AB)' \tag{1.2.8.4}$$

$$= (A + B)(A' + B') \tag{1.2.8.5}$$

$$= AB' + BA' \tag{1.2.8.6}$$

9. Prove that

$$(A) \cap (B \cup C) = (A \cap B) \cup (A \cap C) \tag{1.2.9.1}$$

Solution:

$$LHS = A(B + C) = AB + AC = RHS \tag{1.2.9.2}$$

10. Prove that

$$(A) \cup (B \cap C) = (A \cup B) \cap (A \cup C) \tag{1.2.10.1}$$

Solution:

$$\begin{aligned} LHS &= A + BC && (1.2.10.2)\\ RHS &= (A+B)(A+C) && (1.2.10.3)\\ &= A + A(B+C) + BC && (1.2.10.4)\\ &= A(1+B+C) + BC && (1.2.10.5)\\ &= LHS && (1.2.10.6)\end{aligned}$$

11. Write down all the subsets of $S = \{1, 2, 3, 4\}$.
Solution: Write a program for this.
12. If C is a subset of S, let C' denote the complement of C in S. Prove the *De Morgan Rules* for subsets A, B of S, namely,
   a) $(A \cup B)' = A' \cap B'$
   b) $(A \cap B)' = A' \cup B'$

   Solution:

   a)

$$\begin{aligned} (A+B)A'B' &= AA'B' + BA'B' && (1.2.12.1)\\ &= 0 && (1.2.12.2)\end{aligned}$$

   b) Substituting $A = A', B = B'$ in the above, the second result is obtained.
13. Let S be a set. For any to subsets of S, we define

$$A \oplus B = (A - B) \cup (B \cup A) \qquad (1.2.13.1)$$

   Prove that
   a) $A \oplus B = B \oplus A$.
   b) $A \oplus \Phi = A$.
   c) $A \cdot A = A$.
   d) $A \oplus A = \Phi$.
   e) $A \oplus (B \oplus C) = (A \oplus B) \oplus C$.
   f) If $A \oplus B = A \oplus C$, then $B = C$.
   g) $A \cdot (B + C) = A \cdot B + A \cdot C$.

   Solution: All can be proved using boolean logic.
14. If C is a finite set, let $m(C)$ denote the number of elements in C. If A, B are finite sets, prove that

$$m(A \cup B) = m(A) + m(B) - m(A \cap B) \qquad (1.2.14.1)$$

   Solution:

$$\begin{aligned} A'B' &= (A+B)' && (1.2.14.2)\\ \implies m(A'B') &= m((A+B)') && (1.2.14.3)\\ &= 1 - m(A+B) && (1.2.14.4)\\ \because A + B &= A(B+B') + B && (1.2.14.5)\\ &= B(A+1) + AB' && (1.2.14.6)\\ &= B + AB' && (1.2.14.7)\\ \implies m(A+B) &= m(B + AB') && (1.2.14.8)\\ &= m(B) + m(AB') && (1.2.14.9)\\ \because B(AB') &= 0 && (1.2.14.10)\end{aligned}$$

$$A = A(B+B') = AB + AB' \qquad (1.2.14.11)$$

   and

$$(AB)(AB') = 0, \because BB' = 0 \qquad (1.2.14.12)$$

   Hence, AB and AB' are mutually exclusive and

$$\begin{aligned} m(A) &= m(AB) + m(AB') && (1.2.14.13)\\ \implies m(AB') &= m(A) - m(AB) && (1.2.14.14)\end{aligned}$$

   Substituting (1.2.14.14) in (1.2.14.10),

$$m(A+B) = m(A) + m(B) - m(AB) \qquad (1.2.14.15)$$

15. For three finite sets A, B, C, find a formula for $m(A \cup b \cup C)$. **Solution:** Extend the above.
16. Take a shot at finding $m\left(\cup_{i=1}^{n} A_i\right)$.
17. Show that if 80% of all Indians have gone to high school and 70% of all Indians read a daily newspaper, then *at least* 50% of all Indians have both gone to high school and read a daily newspaper.
Solution: Let A represent high school and B represent newspaper. Then,

$$\Pr(AB) = \Pr(A) + \Pr(B) - \Pr(A+B) \qquad (1.2.17.1)$$

   Since

$$\Pr(A+B) \le 1, \qquad (1.2.17.2)$$

$$\Pr(A) + \Pr(B) - \Pr(A+B) \ge \Pr(A) + \Pr(B) - 1 \qquad (1.2.17.3)$$

$$\begin{aligned} \implies \Pr(AB) &\ge 0.8 + 0.7 - 1 && (1.2.17.4)\\ &= 0.5 && (1.2.17.5)\end{aligned}$$

18. A public opinion poll shows that 90% of the population agreed with the government on the first decision, 84% on the second, and 74% on the third, for three decisions made by the government. At least what percentage of the population agreed with the government on all

three decisions.
Solution: Let the decisions be A, B, C. Then,

$$\Pr(AB) \geq \Pr(ABC), \quad (1.2.18.1)$$
$$\Pr(BC) \geq \Pr(ABC), \quad (1.2.18.2)$$
$$\Pr(CA) \geq \Pr(ABC) \quad (1.2.18.3)$$

Since

$$\begin{aligned}\Pr(A+B+C) &= \sum \Pr(A) \\ &\quad - \sum \Pr(AB) + \Pr(ABC), \\ \implies \Pr(A+B+C) &+ \sum \Pr(AB) \\ &= \sum \Pr(A) + \Pr(ABC), \quad (1.2.18.4)\end{aligned}$$

from (1.2.18.1),

$$\begin{aligned}\Pr(A+B+C) &+ 3\Pr(ABC) \\ &\geq \sum \Pr(A) + \Pr(ABC), \\ \implies 2\Pr(ABC) &\geq \sum \Pr(A) - \Pr(A+B+C) \quad (1.2.18.5)\end{aligned}$$

Since

$$\Pr(A+B+C) \leq 1, \quad (1.2.18.6)$$
$$-\Pr(A+B+C) \geq -1 \quad (1.2.18.7)$$
$$\implies 2\Pr(ABC) \geq \sum \Pr(A) - 1 \quad (1.2.18.8)$$
$$\text{or } \Pr(ABC) \geq \frac{\sum \Pr(A) - 1}{2} \quad (1.2.18.9)$$
$$= 0.74 \quad (1.2.18.10)$$

19. In his book *A Tangled Tale*, Lewis Caroll proposed the following riddle about a group of disabled veterans. “Say that 70% have lost an eye, 75% an ear, 80% an arm, 85% a leg. What percentage, at least, must have lost all four?” Solve Lewis Caroll’s problem.
Solution: Let A_i represent the events. Then,

$$\Pr\left(\sum_{i=1}^{4} A_i\right) = \sum_{i=1}^{4} \Pr(A_i) - \sum_{i,j} \Pr\left(A_i A_j\right) + \sum_{i,j,k} \Pr\left(A_i A_j A_k\right) - \Pr\left(\prod_{i=1}^{4} A_i\right) \quad (1.2.19.1)$$

Now,

$$\Pr(A_1 A_2) \geq \Pr(A_1 A_2 A_3) \geq \Pr(A_1 A_2 A_3 A_4) \quad (1.2.19.2)$$

which, upon substitution in (1.2.19.1) yields

$$\Pr\left(\sum_{i=1}^{4} A_i\right) \geq \frac{\sum_{i=1}^{4} \Pr(A_i) - 1}{1 + {}^4C_2 - {}^4C_3} \quad (1.2.19.3)$$
$$= 70\% \quad (1.2.19.4)$$

20. Show, for finite sets A, B, that $m(A \times B) = m(A) \times m(B)$.
Solution: Basic principle of counting.
21. If S is a set having five elements,
    a) How many subsets does S have?
    b) How many subsets having four elements does S have?
    c) How many subsets having two elements does S have?

    Solution:
    a) $2^5 = 32$.
    b) ${}^5C_4 = 5$.
    c) ${}^5C_2 = 10$.
22. a) Show that a set having n elements has 2^n subsets.
    b) If $0 < m < n$, how many subsets are there that have exactly m elements?

    Solution:
    a) The number of subsets is

    $$\sum_{k=0}^{n} {}^nC_k = 2^n \quad (1.2.22.1)$$

    using the binomial theorem.
    b) The number of subsets having exactly m elements are ${}^nC_m$.

1.3 Mappings

1. For the given sets S, T determine if a mapping $f : S \to T$ is clearly and unambiguously defined; if not, say why not.
    a) S = set of all women, T = set of all men, $f(s)$ = husband of s.

1.4 Mathematical Induction

1. Prove that
$$1^2+2^2+3^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6} \tag{1.4.1.1}$$
by induction.
Solution: $P(n+1)$ is
$$\begin{aligned}1^2+2^2+3^2+\cdots+n^2+(n+1)^2\\ &=\frac{n(n+1)(2n+1)}{6}+(n+1)^2\\ &=(n+1)\left(\frac{2n^2+7n+7}{6}\right)\\ &=\frac{(n+1)(n+2)(2n+3)}{6}\end{aligned} \tag{1.4.1.2}$$
which is true. Hence, the given proposition is true for all $n \geq 1$
2. Prove that
$$1^3+2^3+3^3+\cdots+n^3=\left[\frac{n(n+1)}{2}\right]^2 \tag{1.4.2.1}$$
by induction.
Solution: $P(n+1)$ is
$$\begin{aligned}1^3+2^3+3^3+\cdots+n^3+(n+1)^3\\ &=\left[\frac{n(n+1)}{2}\right]^2+(n+1)^3\\ &=(n+1)^2\left(\frac{n^2+4n+4}{4}\right)\\ &=\left[\frac{(n+1)(n+2)}{2}\right]^2\end{aligned} \tag{1.4.2.2}$$
which is true. Hence, the given proposition is true for all $n \geq 1$.
3. Prove that a set having $n \geq 2$ elements has $\frac{n(n-1)}{2}$ subsets having exactly 2 elements.
4. Prove that a set having $n \geq 3$ elements has $\frac{n(n-1)(n-2)}{3}$ subsets having exactly 3 elements.
5. If $n \geq 4$ and S is a set having n elements, guess how many subsets having exactly 4 elements are there in S. Then verify your guess using mathematical induction.
6. If p is a prime and $p \mid (a_1a_2a_3 \ldots a_n)$, then prove using induction that $p \mid a_i$ for some i with $1 \leq i \leq n$.
7. If $a \neq 1$, prove that
$$1+a+a^2+\cdots+a^n=\frac{\left(a^{n+1}-1\right)}{a-1} \tag{1.4.7.1}$$
by induction.
Solution: $P(n+1)$ can be expressed as
$$\begin{aligned}1+a+a^2+\cdots+a^n+a^{n+1}\\ &=\frac{\left(a^{n+1}-1\right)}{a-1}+a^{n+1}\\ &=\frac{\left(a^{n+2}-1\right)}{a-1}\end{aligned} \tag{1.4.7.2}$$
upon simplification. Hence, the given proposition is true for all $n \geq 1$.
8. By induction, show that
$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n\cdot(n+1)} = \frac{n}{n+1} \tag{1.4.8.1}$$
Solution: $P(n+1)$ can be expressed as
$$\begin{aligned}\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n\cdot(n+1)}+\frac{1}{(n+1)\cdot(n+2)}\\ &=\frac{n}{n+1}+\frac{1}{(n+1)\cdot(n+2)}\\ &=\frac{1}{n+1}\left[n+\frac{1}{n+2}\right]\\ &=\frac{n+1}{n+2}\end{aligned} \tag{1.4.8.2}$$
upon simplification. Hence, the given proposition is true for all $n \geq 1$.

1.5 Complex Numbers

1. Multiply
   a) $(6-7\jmath)(8+\jmath)$
   b) $\left(\frac{2}{3}+\frac{3}{2}\jmath\right)\left(\frac{2}{3}-\frac{3}{2}\jmath\right)$
   c) $(6+7\jmath)(8-\jmath)$

   Solution:

   a)
$$(6-7\jmath)(8+\jmath)=\begin{pmatrix}6 & 7\\ -7 & 6\end{pmatrix}\begin{pmatrix}8\\ 1\end{pmatrix} \tag{1.5.1.1}$$
$$=\begin{pmatrix}53\\ -50\end{pmatrix}=53-50\jmath \tag{1.5.1.2}$$

b)

$$\left(\frac{2}{3}+\frac{3}{2}\jmath\right)\left(\frac{2}{3}-\frac{3}{2}\jmath\right)=\begin{pmatrix}\frac{2}{3} & -\frac{3}{2}\\ \frac{3}{2} & \frac{2}{3}\end{pmatrix}\begin{pmatrix}\frac{2}{3}\\ -\frac{3}{2}\end{pmatrix} \tag{1.5.1.3}$$

$$=\begin{pmatrix}\frac{97}{36}\\ 0\end{pmatrix}=\frac{97}{36} \tag{1.5.1.4}$$

c)

$$(6+7\jmath)\,8-\jmath = \left[(6-7\jmath)\,8+\jmath\right]^* \tag{1.5.1.5}$$

$$=(53-50\jmath)^* = 53+50\jmath \tag{1.5.1.6}$$

2. Find z^{-1} for
   a) $z = 6+8\jmath$
   b) $z = 6-8\jmath$
   c) $z = \frac{1}{\sqrt{2}}(1+\jmath)$

   Solution:

   a)

$$z^{-1} = \frac{z^*}{|z|^2} = \frac{6-8\jmath}{100} \tag{1.5.2.1}$$

   b)

$$z^{-1} = \frac{6+8\jmath}{100} \tag{1.5.2.2}$$

   c)

$$z^{-1} = \frac{1-\jmath}{\sqrt{2}} \tag{1.5.2.3}$$

3. Show that

$$(z^*)^{-1} = \left(z^{-1}\right)^* \tag{1.5.3.1}$$

   Solution: Since

$$zz^{-1} = 1, \tag{1.5.3.2}$$

$$\left(zz^{-1}\right)^* = 1 \tag{1.5.3.3}$$

$$\implies (z)^*\left(z^{-1}\right)^* = 1 \tag{1.5.3.4}$$

   yielding (1.5.3.1).

4. Find

$$(\cos\theta + \jmath\sin\theta)^{-1} \tag{1.5.4.1}$$

   Solution:

$$(\cos\theta + \jmath\sin\theta)^{-1} = \cos\theta - \jmath\sin\theta \tag{1.5.4.2}$$

5. Verify the following
   a) $(z^*)^* = z$
   b) $(z+w)^* = z^* + w^*$
   c) $z + z^* = 2\mathrm{Re}\,(z)$
   d) $z - z^* = 2\jmath\mathrm{Im}\,(z)$

   Solution:

   a) For

$$z = a + \jmath b, \tag{1.5.5.1}$$

$$z^* = a - \jmath b, \tag{1.5.5.2}$$

$$\implies (z^*)^* = a + \jmath b = z \tag{1.5.5.3}$$

   b) For

$$\begin{aligned} z &= z_1 + \jmath z_2\\ w &= w_1 + \jmath w_2, \end{aligned} \tag{1.5.5.4}$$

$$(z+w)^* = (z_1 + \jmath z_2 + w_1 + \jmath w_2)^* \tag{1.5.5.5}$$

$$= (z_1 - \jmath z_2) + (w_1 - \jmath w_2) \tag{1.5.5.6}$$

$$= z^* + w^* \tag{1.5.5.7}$$

   c) For

$$z = a + \jmath b, \tag{1.5.5.8}$$

$$z^* = a - \jmath b, \tag{1.5.5.9}$$

$$\implies (z + z^*) = a + \jmath b + a - jb \tag{1.5.5.10}$$

$$= 2a = 2\mathrm{Re}\,(z) \tag{1.5.5.11}$$

   d) For

$$z = a + \jmath b, \tag{1.5.5.12}$$

$$z^* = a - \jmath b, \tag{1.5.5.13}$$

$$\implies (z - z^*) = a + \jmath b - a + jb \tag{1.5.5.14}$$

$$= 2\jmath b = 2\mathrm{Im}\,(z) \tag{1.5.5.15}$$

6. Show that z is real if and only if $z^* = z$ and is purely imaginary if and only if $z^* = -z$. **Solution:** Let

$$z = a + \jmath b. \tag{1.5.6.1}$$

   Then

$$z^* = a - \jmath b. \tag{1.5.6.2}$$

   If

$$z^* = z, \tag{1.5.6.3}$$

$$a + \jmath b = a - \jmath b \tag{1.5.6.4}$$

$$\implies b = 0 \tag{1.5.6.5}$$

and z is real. If z is real,

$$z = a \tag{1.5.6.6}$$
$$\implies z^* = a \tag{1.5.6.7}$$
$$\text{or, } z = z^* \tag{1.5.6.8}$$

Similarly, the other property can be proved.

7. Verify the commutative law of multiplication $zw = wz$ in $\mathbb{C}$.
Solution: Let

$$z = a + \jmath b \tag{1.5.7.1}$$
$$w = x - \jmath y \tag{1.5.7.2}$$

Then

$$zw = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{1.5.7.3}$$
$$= \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \tag{1.5.7.4}$$
$$= wz \tag{1.5.7.5}$$

8. Show that for $z \neq 0$, $|z|^{-1} = \frac{1}{|z|}$.
Solution: Let

$$z = re^{\jmath\theta}. \tag{1.5.8.1}$$

Then

$$z^{-1} = \frac{1}{r} e^{-\jmath\theta} \tag{1.5.8.2}$$
$$\implies \left|z^{-1}\right| = \frac{1}{r} \tag{1.5.8.3}$$

9. Find
a) $\left|6 - 4\jmath\right|$.
b) $\left|\frac{1}{2} + \frac{2}{3}\jmath\right|$.
c) $\left|\frac{1}{\sqrt{2}}(1 + \jmath)\right|$
Solution:
a)

$$\left|6 - 4\jmath\right| = \sqrt{6^2 + 4^2} = 2\sqrt{13} \tag{1.5.9.1}$$

b)

$$\left|\frac{1}{2} + \frac{2}{3}\jmath\right| = \frac{5}{6} \tag{1.5.9.2}$$

c)

$$\left|\frac{1}{\sqrt{2}}(1 + \jmath)\right| = \frac{1}{\sqrt{2}}\left|(1 + \jmath)\right| = 1 \tag{1.5.9.3}$$

10. Show that $|z^*| = |z|$.
Solution: Let

$$z = re^{\jmath\theta} \tag{1.5.10.1}$$

Then

$$z^* = re^{-\jmath\theta} \tag{1.5.10.2}$$
$$\implies |z^*| = r = |z|. \tag{1.5.10.3}$$

11. Find the polar form for
a) $z = \frac{\sqrt{2}}{2} - \frac{1}{\sqrt{2}}\jmath$.
b) $z = 4\jmath$.
c) $z = \frac{6}{\sqrt{2}} + \frac{6}{\sqrt{2}}\jmath$.
d) $z = -\frac{13}{2} + \frac{39}{2\sqrt{3}}\jmath$.
Solution:
a)

$$|z| = \sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}. \tag{1.5.11.1}$$
$$= 1 \tag{1.5.11.2}$$

and

$$\angle z = -\tan^{-1}\frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} \tag{1.5.11.3}$$
$$= \frac{\pi}{4} \tag{1.5.11.4}$$

b)

$$|z| = 4, \angle z = \frac{\pi}{2}. \tag{1.5.11.5}$$

c)

$$|z| = \frac{6}{\sqrt{2}}, \angle z = \frac{\pi}{4}. \tag{1.5.11.6}$$

d)

$$|z| = \frac{13}{2}\sqrt{1 + 3} \tag{1.5.11.7}$$
$$= 13 \tag{1.5.11.8}$$

and

$$\angle z = \pi - \tan^{-1}\frac{\frac{39}{2\sqrt{3}}}{\frac{13}{2}} \tag{1.5.11.9}$$
$$= \pi - \tan^{-1}\sqrt{3} = \frac{2\pi}{3} \tag{1.5.11.10}$$

12. Prove that

$$\left(\cos\left(\frac{\theta}{2}\right) + \jmath\sin\left(\frac{\theta}{2}\right)\right)^2 = \cos(\theta) + \jmath\sin(\theta) \tag{1.5.12.1}$$

Solution: The L.H.S can be expressed as

$$\left(e^{\jmath\theta}\right)^2 = e^{\jmath\theta} \tag{1.5.12.2}$$

13. Show that

$$\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\jmath\right)^3 = -1 \tag{1.5.13.1}$$

Solution:

$$\frac{1}{2} + \frac{\sqrt{3}}{2}\jmath = e^{\frac{\jmath\pi}{3}} \tag{1.5.13.2}$$

$$\implies \left(e^{\frac{\jmath\pi}{3}}\right)^3 = e^{\jmath\pi} = -1 \tag{1.5.13.3}$$

14. Show that

$$(\cos(\theta) + \jmath\sin(\theta))^m = \cos(m\theta) + \jmath\sin(m\theta) \tag{1.5.14.1}$$

for all integers m. **Solution:** It is easy to verify that

$$(\cos(\theta) + \jmath\sin(\theta))^2 = \cos(2\theta) + \jmath\sin(2\theta) \tag{1.5.14.2}$$

Then

$$(\cos(\theta) + \jmath\sin(\theta))^{k+1} = (\cos(\theta) + \jmath\sin(\theta))^k$$
$$(\cos(m\theta) + \jmath\sin(m\theta))$$
$$= \cos\left[(k+1)\theta + \jmath\sin((k+1)\theta)\right] \tag{1.5.14.3}$$

By induction, (1.5.14.1) is proved.

15. Show that

$$(\cos(\theta) + \jmath\sin(\theta))^r = \cos(r\theta) + \jmath\sin(r\theta) \tag{1.5.15.1}$$

for all rational numbers r. **Solution:** Let

$$r = \frac{m}{n}, (\cos(\theta) + \jmath\sin(\theta))^{\frac{1}{n}} = \cos(\alpha) + \jmath\sin(\alpha) \tag{1.5.15.2}$$

Then

$$(\cos(\alpha) + \jmath\sin(\alpha))^n = (\cos(\theta) + \jmath\sin(\theta)) \tag{1.5.15.3}$$

$$\implies \cos(n\alpha) + \jmath\sin(n\alpha) = (\cos(\theta) + \jmath\sin(\theta)) \tag{1.5.15.4}$$

$$\text{or, } \alpha = \frac{\theta}{n} \tag{1.5.15.5}$$

yielding

$$(\cos(\theta) + \jmath\sin(\theta))^{\frac{1}{n}} = \cos\left(\frac{\theta}{n}\right) + \jmath\sin\left(\frac{\theta}{n}\right) \tag{1.5.15.6}$$

Using (1.5.14.1) and (1.5.15.6),

$$(\cos(\theta) + \jmath\sin(\theta))^{\frac{m}{n}} = \cos\left(\frac{m\theta}{n}\right) + \jmath\sin\left(\frac{m\theta}{n}\right) \tag{1.5.15.7}$$

16. If $z \in \mathbb{C}$ and $n \geq 1$ is any positive integer, show that there are n distinct complex numbers such that $z = w^n$. **Solution:** Let

$$z = \cos(\theta) + \jmath\sin(\theta) \tag{1.5.16.1}$$

then using (1.5.15.6),

$$w = \cos\left(\frac{2\pi k + \theta}{n}\right) + \jmath\sin\left(\frac{2\pi k + \theta}{n}\right), k = 0, \ldots, n-1 \tag{1.5.16.2}$$

which are the distinct roots.

17. Find the necessary and sufficient condition on k such that

$$\left(\cos\left(\frac{2\pi k}{n}\right) + \jmath\sin\left(\frac{2\pi k}{n}\right)\right)^n = 1 \quad \text{and} \tag{1.5.17.1}$$

$$\left(\cos\left(\frac{2\pi k}{n}\right) + \jmath\sin\left(\frac{2\pi k}{n}\right)\right)^m \neq 1 \quad 0 < m < n \tag{1.5.17.2}$$

Solution: From the above equations, using (1.5.14.1),

$$\frac{mk}{n} \notin \mathbb{Z} \tag{1.5.17.3}$$

18. Viewing the $x-y$ plane as the set of all complex numbers $x + \jmath y$, show that multiplication by j induces as $90°$ rotation of the $x - y$ plan in counterclockwise direction.
Solution: The given multiplication can be expressed using matrices as

$$\begin{pmatrix} \cos 90° & -\sin 90° \\ \sin 90° & \cos 90° \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{1.5.18.1}$$

which is the multiplication of $\begin{pmatrix} x \\ y \end{pmatrix}$ with a $90°$ rotation matrix.

19. In problem (1.5.18), interpret geometrically what multiplication by the complex number $a + \jmath b$ does to the $x - y$ plane.

Solution: The multiplication can be represented as

$$\sqrt{a^2+b^2}\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} \quad (1.5.19.1)$$

where

$$\cos\theta = \frac{a}{\sqrt{a^2+b^2}}\sin\theta \quad = \frac{b}{\sqrt{a^2+b^2}} \quad (1.5.19.2)$$

Geometrically, multiplication by $a + \jmath b$ results in rotation by θ and scaling by $\sqrt{a^2+b^2}$.

20. Prove that

$$|z+w|^2 + |z-w|^2 = 2\left(|z|^2 + |w|^2\right) \quad (1.5.20.1)$$

Solution: Since

$$|z+w|^2 = (z+w)^* (z+w) \quad (1.5.20.2)$$
$$= |z|^2 + |w|^2 + 2z^* w \quad (1.5.20.3)$$

and

$$|z-w|^2 = (z+w)^* (z-w) \quad (1.5.20.4)$$
$$= |z|^2 + |w|^2 - 2z^* w, \quad (1.5.20.5)$$

$$|z+w|^2 + |z-w|^2 = 2\left(|z|^2 + |w|^2\right) \quad (1.5.20.6)$$

21. Consider the set $A = a + b\jmath, a, b \in \mathbb{Z}$. Prove that there is 1-1 correspondence of A onto $\mathbb{N}$.
22. If a is a (complex) root of the polynomial

$$x^n + \alpha_1 x^{n-1} + \cdots + \alpha_{n-1}x + \alpha_n, \quad (1.5.22.1)$$

where the α_i are real, show that $\bar{a}$ must also be a root.
Solution: From the given information,

$$\bar{a}^n + \alpha_1 \bar{a}^{n-1} + \cdots + \alpha_{n-1}\bar{a} + \alpha_n = 0 \quad (1.5.22.2)$$

Thus, $\bar{a}$ is also a root of the given polynomial.

23. Find the necessary and sufficient conditions on z and w in order that

$$|z+w| = |z| + |w| \quad (1.5.23.1)$$

Solution:

$$|z+w|^2 = |z|^2 + |w|^2 + 2z^* w \quad (1.5.23.2)$$
$$(|z| + |w|)^2 = |z|^2 + |w|^2 + 2|z||w| \quad (1.5.23.3)$$

If the above expressions are equal,

$$z^* w = |z||w| \quad (1.5.23.4)$$

which is the desired condition.

24. Find the necessary and sufficient conditions on z_i in order that

$$\left|\sum_{i=1}^{k} z_i\right| = \sum_{i=1}^{k} |z_i| \quad (1.5.24.1)$$

Solution:

$$\left|\sum_{i=1}^{k} z_i\right|^2 = \sum_{i=1}^{k} |z_i|^2 + 2\sum_{\substack{i=1,j=1\\ i\neq j}}^{k} {z_i}^* z_j \quad (1.5.24.2)$$

$$\left(\sum_{i=1}^{k} |z_i|\right)^2 = \sum_{i=1}^{k} |z_i|^2 + 2\sum_{\substack{i=1,j=1\\ i\neq j}}^{k} |z_i|\left|z_j\right| \quad (1.5.24.3)$$

From (1.5.24.2) and (1.5.24.3),

$$\sum_{i=1}^{k} |z_i|^2 + 2\sum_{\substack{i=1,j=1\\ i\neq j}}^{k} {z_i}^* z_j$$
$$= \sum_{i=1}^{k} |z_i|^2 + 2\sum_{\substack{i=1,j=1\\ i\neq j}}^{k} |z_i|\left|z_j\right|$$
$$\implies \sum_{\substack{i=1,j=1\\ i\neq j}}^{k} {z_i}^* z_j$$
$$= \sum_{\substack{i=1,j=1\\ i\neq j}}^{k} |z_i|\left|z_j\right| \quad (1.5.24.4)$$

which is the desired condition.

25. The complex number θ is said to have *order* $n \geq 1$ if $\theta^n = 1$ and $\theta^m \neq 1$ for $0 < m < n$. Show that if θ has order n and $\theta^k = 1$, where $k > 0$, then $n | k$.
Solution: From the given information,

$$\theta^n = \theta^k = 1, k \geq n \quad (1.5.25.1)$$

If $n \nmid k, k = mn + p, 0 < p < n$, Then,

$$\theta^k = \theta^{mn+p} = \theta^p \neq 1, \quad (1.5.25.2)$$

which is a contradiction, Hence, $n \mid k$.

26. Find all complex numbers θ having order n.
Solution: If

$$\theta^n = 1, \quad (1.5.26.1)$$
$$\theta^n = e^{\jmath 2\pi r}, 0 \leq r < n \quad (1.5.26.2)$$

yielding

$$\theta = \exp\left(\jmath\frac{2\pi r}{n}\right) 0 \leq r < n \qquad (1.5.26.3)$$