Assignment 1

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Download all python codes from

https://github.com/V−Gopireddy/EE3900/blob/main/Assignment3/codes/Assignment−3.py

and latex-tikz codes from

https://github.com/V−gopireddy/EE3900/blob/main/Assignment3/Assignment−3.tex

1 Ramsey/4.4 Systems of circles/Q.2

Find the equation of a circle which cuts orthogonally the three circles

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix} 4 & -5 \end{pmatrix}\mathbf{x} + 6 = 0 \tag{1.0.1}$$

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix} 5 & -6 \end{pmatrix}\mathbf{x} + 7 = 0 \tag{1.0.2}$$

$$\mathbf{x}^T\mathbf{x} - \begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} - 1 = 0 \tag{1.0.3}$$

2 Solution

Lemma 2.1. *Tangent to a circle : Consider a circle*

$$\mathbf{x}^\top\mathbf{x} + 2\mathbf{c}^\top\mathbf{x} + f = 0 \tag{2.0.1}$$

Given a point $\mathbf{q}$ *on the circle, the tangent at that point is given as,*

$$(\mathbf{q} + \mathbf{c})^\top \mathbf{x} + \mathbf{c}^\top\mathbf{q} + f = 0 \tag{2.0.2}$$

Lemma 2.2. *Orthogonality of circles : Two circles are said to be orthogonal if the tangents at their points of intersection are perpendicular to each other.*
That implies, tangents to one circle at the points of contact are normals to the other circle. Given two circles,

$$\mathbf{x}^\top\mathbf{x} + 2\mathbf{c_1}^\top\mathbf{x} + f_1 = 0 \tag{2.0.3}$$

$$\mathbf{x}^\top\mathbf{x} + 2\mathbf{c_2}^\top\mathbf{x} + f_2 = 0 \tag{2.0.4}$$

They are orthogonal if

$$2\mathbf{c_1}^\top\mathbf{c_2} = f_1 + f_2 \tag{2.0.5}$$

Proof. Let the two circles (2.0.3) and (2.0.4) meet at a point $\mathbf{q}$ i.e $\mathbf{q}$ satisfies the equation of the circles

$$\mathbf{q}^\top\mathbf{q} + 2\mathbf{c_1}^\top\mathbf{q} + f_1 = 0 \tag{2.0.6}$$

$$\mathbf{q}^\top\mathbf{q} + 2\mathbf{c_2}^\top\mathbf{q} + f_2 = 0 \tag{2.0.7}$$

Eliminating quadratic term,

$$2\left(\mathbf{c_1}^T - \mathbf{c_2}^T\right)\mathbf{q} + f_1 - f_2 = 0 \tag{2.0.8}$$

Given the point of contact $\mathbf{q}$, the equation of tangent to circle (2.0.3) is

$$(\mathbf{q} + \mathbf{c_1})^\top \mathbf{x} + \mathbf{c_1}^\top\mathbf{q} + f_1 = 0 \tag{2.0.9}$$

As it is a normal to the second circle (2.0.4), it passes through the center of it

$$(\mathbf{q} + \mathbf{c_1})^\top (-\mathbf{c_2}) + \mathbf{c_1}^\top\mathbf{q} + f_1 = 0 \tag{2.0.10}$$

$$\implies \left(\mathbf{c_1}^T - \mathbf{c_2}^T\right)\mathbf{q} + f_1 - \mathbf{c_1}^T\mathbf{c_2} = 0 \tag{2.0.11}$$

$$\implies \left(\mathbf{c_1}^T - \mathbf{c_2}^T\right)\mathbf{q} = \mathbf{c_1}^T\mathbf{c_2} - f_1 \tag{2.0.12}$$

Substituting (2.0.12) in (2.0.8),

$$2\left(\mathbf{c_1}^T\mathbf{c_2} - f_1\right) + f_1 - f2 = 0 \tag{2.0.13}$$

$$\implies 2\mathbf{c_1}^\top\mathbf{c_2} = f_1 + f_2 \tag{2.0.14}$$

□

Lemma 2.3. *For given three circles,*

$$\mathbf{x}^\top\mathbf{x} + 2\mathbf{c_1}^\top\mathbf{x} + f_1 = 0 \tag{2.0.15}$$

$$\mathbf{x}^\top\mathbf{x} + 2\mathbf{c_2}^\top\mathbf{x} + f_2 = 0 \tag{2.0.16}$$

$$\mathbf{x}^\top\mathbf{x} + 2\mathbf{c_3}^\top\mathbf{x} + f_3 = 0 \tag{2.0.17}$$

The equation of a circle S *which cuts these circles orthogonally is given by,*

$$\mathbf{x}^\top\mathbf{x} + 2\mathbf{c}^\top\mathbf{x} + f = 0 \tag{2.0.18}$$

Where

$$\begin{pmatrix} \mathbf{c} \\ f \end{pmatrix} = \begin{pmatrix} 2\mathbf{c_1}^\top & -1 \\ 2\mathbf{c_2}^\top & -1 \\ 2\mathbf{c_3}^\top & -1 \end{pmatrix}^{-1} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \tag{2.0.19}$$

Proof. Since S is orthogonal to (2.0.15), (2.0.16) and (2.0.17) we have,

$$2\mathbf{c_1}^\top \mathbf{c} - f = f_1 \tag{2.0.20}$$
$$2\mathbf{c_2}^\top \mathbf{c} - f = f_2 \tag{2.0.21}$$
$$2\mathbf{c_3}^\top \mathbf{c} - f = f_3 \tag{2.0.22}$$
$$\implies \begin{pmatrix} 2\mathbf{c_1}^\top & -1 \\ 2\mathbf{c_2}^\top & -1 \\ 2\mathbf{c_3}^\top & -1 \end{pmatrix} \begin{pmatrix} \mathbf{c} \\ f \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \tag{2.0.23}$$
$$\tag{2.0.24}$$

□

Let the required equation of the circle be

$$\mathbf{x}^\top \mathbf{x} + 2\mathbf{c}^\top \mathbf{x} + f = 0 \tag{2.0.25}$$

It is orthogonal to the circles (1.0.1), (1.0.2) and (1.0.3)

$$\implies \begin{pmatrix} 4 & -5 & -1 \\ 5 & -6 & -1 \\ -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} \mathbf{c} \\ f \end{pmatrix} = \begin{pmatrix} 6 \\ 7 \\ -1 \end{pmatrix} \tag{2.0.26}$$

Therefore the augmented matrix can be transformed as,

$$\begin{pmatrix} 4 & -5 & -1 & 6 \\ 5 & -6 & -1 & 7 \\ -1 & -1 & -1 & -1 \end{pmatrix} \tag{2.0.27}$$
$$\xleftrightarrow{R_1 \leftrightarrow -R_3} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 5 & -6 & -1 & 7 \\ 4 & -5 & -1 & 6 \end{pmatrix} \tag{2.0.28}$$
$$\xleftrightarrow[R_2 \leftarrow R_2 - 5R_1, R_2 \leftarrow -\frac{R_2}{11}]{R_3 \leftarrow R_3 - 4R_1} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & \frac{6}{11} & \frac{-2}{11} \\ 0 & -9 & -5 & 2 \end{pmatrix} \tag{2.0.29}$$
$$\xleftrightarrow[R_1 \leftarrow R_1 - R_2]{R_3 \leftarrow R_3 + 9R_2} \begin{pmatrix} 1 & 0 & \frac{5}{11} & \frac{13}{11} \\ 0 & 1 & \frac{6}{11} & \frac{-2}{11} \\ 0 & 0 & \frac{-1}{11} & \frac{4}{11} \end{pmatrix} \tag{2.0.30}$$
$$\xleftrightarrow[R_1 \leftarrow R_1 + 5R_3, R_3 \leftarrow -11R_3]{R_2 \leftarrow R_2 + 6R_3} \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -4 \end{pmatrix} \tag{2.0.31}$$
$$\implies \mathbf{c} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, f = -4 \tag{2.0.32}$$

The required equation of circle,

$$S = \mathbf{x}^\top \mathbf{x} + \begin{pmatrix} 6 & 4 \end{pmatrix} \mathbf{x} - 4 = 0 \tag{2.0.33}$$

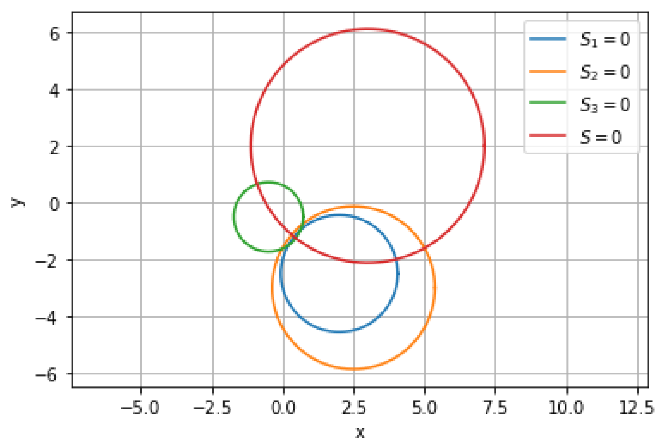

Fig. 0: Plot of circles