

# Abstract Algebra

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## 1 THINGS FAMILIAR AND LESS FAMILIAR

### 1.1 Introduction

1. let  $S$  be a set having an operation  $*$  which assigns an element  $a*b$  of  $S$  for any  $a, b \in S$ . Let us assume that the following two rules hold:

- a) If  $a, b$  are any objects in  $S$ , then  $a * b = a$ .  
 b) If  $a, b$  are any objects in  $S$ , then  $a * b = b * a$ .

Show that  $S$  can have at most one object.

**Solution:** From condition 1.1.1a, interchanging  $a, b$ ,

$$b * a = b \quad (1.1.1)$$

and from condition 1.1.1b,

$$b * a = a * b \quad (1.1.1)$$

But from condition 1.1.1a,

$$a * b = a \implies a = b \quad (1.1.1)$$

Thus,  $S$  can have at most one object.

2. Let  $S$  be the set of all integers  $0, \pm 1, \pm 2, \dots, \pm n, \dots$ . For  $a, b \in S$ , define  $*$  by

$$a * b = a - b \quad (1.1.2)$$

Verify the following

- a)  $a * b \neq b * a$  unless  $a = b$   
 b)  $(a * b) * c \neq a * (b * c)$  in general. Under what conditions on  $a, b, c$  is

$$(a * b) * c \neq a * (b * c) \quad ? \quad (1.1.2)$$

- c) The integer  $a$  has the property that  $a * 0 = a$  for every  $a \in S$ .

- d) For  $a \in S, a * a = 0$ .

**Solution:**

- a)

$$a * b = b * a \quad (1.1.2)$$

$$\implies a - b = b - a \quad (1.1.2)$$

$$\text{or, } a = b \quad (1.1.2)$$

- b) Let  $a = 1, b = 2, c = 4$ . Then,

$$a * b = -1, (a * b) * c = -1 - 4 = -5 \quad (1.1.2)$$

$$b * c = -2, a * (b * c) = 1 + 2 = 3 \neq -5 \quad (1.1.2)$$

Thus, for the given condition to be satisfied,

$$(a - b) - c = a - (b - c) \quad (1.1.2)$$

$$\implies c = 0 \quad (1.1.2)$$

- c)

$$a * 0 = a - 0 = a \quad (1.1.2)$$

- d)

$$a * a = a - a = 0 \quad (1.1.2)$$

3. Let  $S$  consist of the two objects  $\square$  and  $\Delta$ . We define the operation  $*$  on  $S$  by subjecting  $\square$  and  $\Delta$  to the following conditions.

a)  $\square * \Delta = \Delta = \Delta * \square$

b)  $\square * \square = \square$

c)  $\Delta * \Delta = \square$

Verify by explicit calculation that if  $a, b, c$  are any elements of  $S$ , (i.e.  $a, b, c$  can be any of  $\square$  or  $\Delta$ ), then

a)  $a * b$  is in  $S$

b)  $(a * b) * c = a * (b * c)$

c)  $a * b = b * a$

d) There is a particular  $a$  in  $S$  such that  $a * b = b * a = b$  for all  $b \in S$

e) Given  $b \in S, b * b = a$ , where  $a$  is the particular element in Part 1.1.3d.

**Solution:** Let  $\square = 1, \Delta = -1$ . These satisfy all the given conditions.

- $a * b \in [1, -1] \in S$ .
- Writing the truth table,  $(a * b) * c = a * (b * c)$ .
- $a * b = b * a$  can be verified by writing the truth table.
- For  $a = 1, a * b = b * a = b$ , for all  $b \in S$ .
- For  $a = 1$ , if  $b = -1, b * b = 1 = a$ . This can be shown to be true for  $b = 1$  as well.

## 1.2 Set Theory

- Describe the following sets verbally

- $S = \{\text{Mercury, Venus, Earth, ... , Pluto}\}$
- $S = \{\text{Andhra Pradesh, Uttar Pradesh, ... , Assam}\}$

**Solution:**

- Planets
- Indian states

- Describe the following sets verbally

- $S = \{2, 4, 6, 8, \dots\}$
- $S = \{2, 4, 8, 16, \dots\}$
- $S = \{1, 4, 9, 16, 25, 36, \dots\}$

**Solution:**

- Even numbers
- Powers of 2
- Squares of positive integers

- If  $A$  is the set of all residents of India,  $B$  the set of all Sri Lankan citizens, and  $C$  the set of all women in the world, describe the sets  $ABC, A - B, A - C, C - A$  verbally.

**Solution:**

- $ABC$  is the set of all women residents of India who are citizens of Sri Lanka.
  - $A - B = AB'$  is the set of all residents of India who are not Sri Lankan citizens.
  - $A - C = AC'$  is the set of all male residents of India.
  - $C - A = CA'$  is the set of all women who are not residing in India.
- If  $A = \{1, 4, 7, a\}$  and  $B = \{3, 4, 9, 11\}$  and you have been told that  $AB = \{4, 9\}$ , then what must  $a$  be?

**Solution:**  $a = 9$

- If  $A \subset B, B \subset C$ , prove that  $A \subset C$

**Solution:** From the given information,

$$A + P = B, AP = 0, B + Q = C, BQ = 0 \quad (1.2.5.1)$$

$$\implies B + Q = A + P + Q = C, \quad (1.2.5.2)$$

$$\therefore BQ = 0,$$

$$AQ + PQ = 0 \implies AQ = 0, PQ = 0 \quad (1.2.5.3)$$

Hence,

$$A(P + Q) = 0 \implies A \subset C \quad (1.2.5.4)$$

- If  $A \subset B$  prove that  $A \cup C \subset B \cup C$  for any set  $C$ .

**Solution:** From the given information, there exists  $P$  such that

$$A + P = B, AP = 0 \quad (1.2.6.1)$$

Also,

$$B + C = A + P + C \quad (1.2.6.2)$$

$$\implies A + C \subset B + C \quad (1.2.6.3)$$

- Show that

$$A \cup B = B \cup A \quad (1.2.7.1)$$

$$A \cap B = B \cap A \quad (1.2.7.2)$$

- Prove that

$$(A - B) \cup (B - A) = (A \cup B) - (A \cap B) \quad (1.2.8.1)$$

**Solution:** Since

$$A - B = AB', \quad (1.2.8.2)$$

$$(A - B) \cup (B - A) = AB' + BA' \quad (1.2.8.3)$$

Also,

$$(A \cup B) - (A \cap B) = (A + B)(AB')' \quad (1.2.8.4)$$

$$= (A + B)(A' + B') \quad (1.2.8.5)$$

$$= AB' + BA' \quad (1.2.8.6)$$

- Prove that

$$(A) \cap (B \cup C) = (A \cap B) \cup (A \cap C) \quad (1.2.9.1)$$

**Solution:**

$$LHS = A(B + C) = AB + AC = RHS \quad (1.2.9.2)$$

- Prove that

$$(A) \cup (B \cap C) = (A \cup B) \cap (A \cup C) \quad (1.2.10.1)$$

**Solution:**

$$LHS = A + BC \quad (1.2.10.2)$$

$$RHS = (A + B)(A + C) \quad (1.2.10.3)$$

$$= A + A(B + C) + BC \quad (1.2.10.4)$$

$$= A(1 + B + C) + BC \quad (1.2.10.5)$$

$$= LHS \quad (1.2.10.6)$$

11. Write down all the subsets of  $S = \{1, 2, 3, 4\}$ .

**Solution:** Write a program for this.

12. If  $C$  is a subset of  $S$ , let  $C'$  denote the complement of  $C$  in  $S$ . Prove the *De Morgan Rules* for subsets  $A, B$  of  $S$ , namely,

a)  $(A \cup B)' = A' \cap B'$

b)  $(A \cap B)' = A' \cup B'$

**Solution:**

a)

$$(A + B)A'B' = AA'B' + BA'B' \quad (1.2.12.1)$$

$$= 0 \quad (1.2.12.2)$$

- b) Substituting  $A = A', B = B'$  in the above, the second result is obtained.

13. Let  $S$  be a set. For any two subsets of  $S$ , we define

$$A \oplus B = (A - B) \cup (B \cup A) \quad (1.2.13.1)$$

Prove that

a)  $A \oplus B = B \oplus A$ .

b)  $A \oplus \Phi = A$ .

c)  $A \cdot A = A$ .

d)  $A \oplus A = \Phi$ .

e)  $A \oplus (B \oplus C) = (A \oplus B) \oplus C$ .

f) If  $A \oplus B = A \oplus C$ , then  $B = C$ .

g)  $A \cdot (B + C) = A \cdot B + A \cdot C$ .

**Solution:** All can be proved using boolean logic.

14. If  $C$  is a finite set, let  $m(C)$  denote the number of elements in  $C$ . If  $A, B$  are finite sets, prove that

$$m(A \cup B) = m(A) + m(B) - m(A \cap B) \quad (1.2.14.1)$$

**Solution:**

$$A'B' = (A + B)' \quad (1.2.14.2)$$

$$\implies m(A'B') = m((A + B)') \quad (1.2.14.3)$$

$$= 1 - m(A + B) \quad (1.2.14.4)$$

$$\because A + B = A(B + B') + B \quad (1.2.14.5)$$

$$= B(A + 1) + AB' \quad (1.2.14.6)$$

$$= B + AB' \quad (1.2.14.7)$$

$$\implies m(A + B) = m(B + AB') \quad (1.2.14.8)$$

$$= m(B) + m(AB') \quad (1.2.14.9)$$

$$\because B(AB') = 0 \quad (1.2.14.10)$$

$$A = A(B + B') = AB + AB' \quad (1.2.14.11)$$

and

$$(AB)(AB') = 0, \because BB' = 0 \quad (1.2.14.12)$$

Hence,  $AB$  and  $AB'$  are mutually exclusive and

$$m(A) = m(AB) + m(AB') \quad (1.2.14.13)$$

$$\implies m(AB') = m(A) - m(AB) \quad (1.2.14.14)$$

Substituting (1.2.14.14) in (1.2.14.10),

$$m(A + B) = m(A) + m(B) - m(AB) \quad (1.2.14.15)$$

15. For three finite sets  $A, B, C$ , find a formula for  $m(A \cup B \cup C)$ . **Solution:** Extend the above.

16. Take a shot at finding  $m(\cup_{i=1}^n A_i)$ .

17. Show that if 80% of all Indians have gone to high school and 70% of all Indians read a daily newspaper, then *at least* 50% of all Indians have both gone to high school and read a daily newspaper.

**Solution:** Let  $A$  represent high school and  $B$  represent newspaper. Then,

$$\Pr(AB) = \Pr(A) + \Pr(B) - \Pr(A + B) \quad (1.2.17.1)$$

Since

$$\Pr(A + B) \leq 1, \quad (1.2.17.2)$$

$$\Pr(A) + \Pr(B) - \Pr(A + B) \geq \Pr(A) + \Pr(B) - 1 \quad (1.2.17.3)$$

$$\implies \Pr(AB) \geq 0.8 + 0.7 - 1 \quad (1.2.17.4)$$

$$= 0.5 \quad (1.2.17.5)$$

18. A public opinion poll shows that 90% of the population agreed with the government on the first decision, 84% on the second, and 74% on the third, for three decisions made by the government. At least what percentage of the population agreed with the government on all

three decisions.

**Solution:** Let the decisions be  $A, B, C$ . Then,

$$\Pr(AB) \geq \Pr(ABC), \quad (1.2.18.1)$$

$$\Pr(BC) \geq \Pr(ABC), \quad (1.2.18.2)$$

$$\Pr(CA) \geq \Pr(ABC) \quad (1.2.18.3)$$

Since

$$\begin{aligned} \Pr(A + B + C) &= \sum \Pr(A) \\ &\quad - \sum \Pr(AB) + \Pr(ABC), \\ \Rightarrow \Pr(A + B + C) + \sum \Pr(AB) \\ &= \sum \Pr(A) + \Pr(ABC), \quad (1.2.18.4) \end{aligned}$$

from (1.2.18.1),

$$\begin{aligned} \Pr(A + B + C) + 3\Pr(ABC) \\ \geq \sum \Pr(A) + \Pr(ABC), \\ \Rightarrow 2\Pr(ABC) \geq \sum \Pr(A) - \Pr(A + B + C) \quad (1.2.18.5) \end{aligned}$$

Since

$$\Pr(A + B + C) \leq 1, \quad (1.2.18.6)$$

$$-\Pr(A + B + C) \geq -1 \quad (1.2.18.7)$$

$$\Rightarrow 2\Pr(ABC) \geq \sum \Pr(A) - 1 \quad (1.2.18.8)$$

$$\begin{aligned} \text{or } \Pr(ABC) &\geq \frac{\sum \Pr(A) - 1}{2} \quad (1.2.18.9) \\ &= 0.74 \quad (1.2.18.10) \end{aligned}$$

19. In his book *A Tangled Tale*, Lewis Carroll proposed the following riddle about a group of disabled veterans. "Say that 70% have lost an eye, 75% an ear, 80% an arm, 85% a leg. What percentage, at least, must have lost all four?" Solve Lewis Carroll's problem.

**Solution:** Let  $A_i$  represent the events. Then,

$$\begin{aligned} \Pr\left(\sum_{i=1}^4 A_i\right) &= \sum_{i=1}^4 \Pr(A_i) - \sum_{i,j} \Pr(A_i A_j) \\ &\quad + \sum_{i,j,k} \Pr(A_i A_j A_k) - \Pr\left(\prod_{i=1}^4 A_i\right) \quad (1.2.19.1) \end{aligned}$$

Now,

$$\Pr(A_1 A_2) \geq \Pr(A_1 A_2 A_3) \geq \Pr(A_1 A_2 A_3 A_4) \quad (1.2.19.2)$$

which, upon substitution in (1.2.19.1) yields

$$\begin{aligned} \Pr\left(\sum_{i=1}^4 A_i\right) &\geq \frac{\sum_{i=1}^4 \Pr(A_i) - 1}{1 + {}^4C_2 - {}^4C_3} \quad (1.2.19.3) \\ &= 70\% \quad (1.2.19.4) \end{aligned}$$

20. Show, for finite sets  $A, B$ , that  $m(A \times B) = m(A) \times m(B)$ .

**Solution:** Basic principle of counting.

21. If  $S$  is a set having five elements,  
a) How many subsets does  $S$  have?  
b) How many subsets having four elements does  $S$  have?  
c) How many subsets having two elements does  $S$  have?

**Solution:**

- a)  $2^5 = 32$ .  
b)  ${}^5C_4 = 5$ .  
c)  ${}^5C_2 = 10$ .  
22. a) Show that a set having  $n$  elements has  $2^n$  subsets.  
b) If  $0 < m < n$ , how many subsets are there that have exactly  $m$  elements?

**Solution:**

- a) The number of subsets is

$$\sum_{k=0}^n {}^nC_k = 2^n \quad (1.2.22.1)$$

using the binomial theorem.

- b) The number of subsets having exactly  $m$  elements are  ${}^nC_m$ .

### 1.3 Mappings

1. For the given sets  $S, T$  determine if a mapping  $f : S \rightarrow T$  is clearly and unambiguously defined; if not, say why not.  
a)  $S$  = set of all women,  $T$  = set of all men,  $f(s)$  = husband of  $s$ .

## 1.4 Mathematical Induction

1. Prove that

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \quad (1.4.1.1)$$

by induction.

**Solution:**  $P(n+1)$  is

$$\begin{aligned} 1^2 + 2^2 + 3^2 + \cdots + n^2 + (n+1)^2 &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\ &= (n+1) \left( \frac{2n^2 + 7n + 7}{6} \right) \\ &= \frac{(n+1)(n+2)(2n+3)}{6} \quad (1.4.1.2) \end{aligned}$$

which is true. Hence, the given proposition is true for all  $n \geq 1$

2. Prove that

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[ \frac{n(n+1)}{2} \right]^2 \quad (1.4.2.1)$$

by induction.

**Solution:**  $P(n+1)$  is

$$\begin{aligned} 1^3 + 2^3 + 3^3 + \cdots + n^3 + (n+1)^3 &= \left[ \frac{n(n+1)}{2} \right]^2 + (n+1)^3 \\ &= (n+1)^2 \left( \frac{n^2 + 4n + 4}{4} \right) \\ &= \left[ \frac{(n+1)(n+2)}{2} \right]^2 \quad (1.4.2.2) \end{aligned}$$

which is true. Hence, the given proposition is true for all  $n \geq 1$ .

3. Prove that a set having  $n \geq 2$  elements has  $\frac{n(n-1)}{2}$  subsets having exactly 2 elements.
4. Prove that a set having  $n \geq 3$  elements has  $\frac{n(n-1)(n-2)}{3}$  subsets having exactly 3 elements.
5. If  $n \geq 4$  and  $S$  is a set having  $n$  elements, guess how many subsets having exactly 4 elements are there in  $S$ . Then verify your guess using mathematical induction.
6. If  $p$  is a prime and  $p \mid (a_1 a_2 a_3 \dots a_n)$ , then prove using induction that  $p \mid a_i$  for some  $i$  with  $1 \leq i \leq n$ .

7. If  $a \neq 1$ , prove that

$$1 + a + a^2 + \cdots + a^n = \frac{(a^{n+1} - 1)}{a - 1} \quad (1.4.7.1)$$

by induction.

**Solution:**  $P(n+1)$  can be expressed as

$$\begin{aligned} 1 + a + a^2 + \cdots + a^n + a^{n+1} &= \frac{(a^{n+1} - 1)}{a - 1} + a^{n+1} \\ &= \frac{(a^{n+2} - 1)}{a - 1} \quad (1.4.7.2) \end{aligned}$$

upon simplification. Hence, the given proposition is true for all  $n \geq 1$ .

8. By induction, show that

$$\begin{aligned} \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n \cdot (n+1)} &= \frac{n}{n+1} \quad (1.4.8.1) \end{aligned}$$

**Solution:**  $P(n+1)$  can be expressed as

$$\begin{aligned} \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n \cdot (n+1)} + \frac{1}{(n+1) \cdot (n+2)} &= \frac{n}{n+1} + \frac{1}{(n+1) \cdot (n+2)} \\ &= \frac{1}{n+1} \left[ n + \frac{1}{n+2} \right] \\ &= \frac{n+1}{n+2} \quad (1.4.8.2) \end{aligned}$$

upon simplification. Hence, the given proposition is true for all  $n \geq 1$ .

9. Suppose that  $P(n)$  is a proposition about positive integers  $n$  such that  $P(n_0)$  is valid, and if  $P(k)$  is true, so must be  $P(k+1)$ . What can you say about  $P(n)$ ? Prove your statement.
10. Let  $P(n)$  be a proposition about integers  $n$  such that  $P(1)$  is true and such that if  $P(j)$  is true for all positive integers  $j < k$ , then  $P(k)$  is true. Prove that  $P(n)$  is true for all positive integers  $n$ .
11. Given an example of a proposition that is *not* true for any positive integer, yet for which the induction step holds.
12. Prove by induction that a set having  $n$  elements has exactly  $2^n$  subsets.

**Solution:** Let  $S = \{1, 2\}$ . Then the subsets are

$$\{\phi\}, \{1\}, \{2\}, \{1, 2\} \quad (1.4.12.1)$$

For  $S = \{1, 2, 3\}$ , the subsets are

$$\{\phi\}, \{1\}, \{2\}, \{1, 2\} \quad (1.4.12.2)$$

$$\{3\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\} \quad (1.4.12.3)$$

Thus  $P(n+1)$  can be expressed as

$$2^n + 2^n = 2^{n+1} \quad (1.4.12.4)$$

Hence, the given proposition is true for all  $n \geq 1$ .

13. Prove by induction on  $n$  that  $n^3 - n$  is always divisible by 3.

**Solution:**  $P(n+1)$  can be expressed as

$$(n+1)^3 - (n+1) = n^3 + 3n^2 + 3n + 1 - n - 1 \quad (1.4.13.1)$$

$$= n^3 - n + 3(n^2 + n) \quad (1.4.13.2)$$

which is divisible by 3. Hence, the given proposition is true for all  $n \geq 1$ .

14. If  $p$  is a prime number, then prove that  $n^p - n$  is always divisible by  $p$ .

**Solution:**  $P(n+1)$  can be expressed as

$$(n+1)^p - (n+1) = n^p - n + p \sum_{k=1}^{n-1} n C_k p^{k-1} \quad (1.4.14.1)$$

$$\Rightarrow p \mid [(n+1)^p - (n+1)] \quad (1.4.14.2)$$

Hence, the given proposition is true for all  $n \geq 1$ .

15. Prove by induction that for a set having  $n$  elements the number of 1-1 mappings of this set onto itself is  $n!$ .

**Solution:** Let  $S = \{a, b, c\}$ . Then the possible 1-1 onto mappings are

$$\left\{ \begin{array}{l} a \mapsto b \\ b \mapsto c \\ c \mapsto a \end{array} \right\} \left\{ \begin{array}{l} a \mapsto b \\ b \mapsto a \\ c \mapsto c \end{array} \right\} \left\{ \begin{array}{l} a \mapsto c \\ b \mapsto b \\ c \mapsto a \end{array} \right\} \left\{ \begin{array}{l} a \mapsto a \\ b \mapsto c \\ c \mapsto b \end{array} \right\} \left\{ \begin{array}{l} a \mapsto a \\ b \mapsto b \\ c \mapsto c \end{array} \right\} \quad (1.4.15.1)$$

$$\text{b) } \left(\frac{2}{3} + \frac{3}{2}j\right)\left(\frac{2}{3} - \frac{3}{2}j\right)$$

$$\text{c) } (6+7j)(8-j)$$

**Solution:**

a)

$$(6-7j)(8+j) = \begin{pmatrix} 6 & 7 \\ -7 & 6 \end{pmatrix} \begin{pmatrix} 8 \\ 1 \end{pmatrix} \quad (1.5.1.1)$$

$$= \begin{pmatrix} 53 \\ -50 \end{pmatrix} = 53 - 50j \quad (1.5.1.2)$$

b)

$$\left(\frac{2}{3} + \frac{3}{2}j\right)\left(\frac{2}{3} - \frac{3}{2}j\right) = \begin{pmatrix} \frac{2}{3} & -\frac{3}{2} \\ \frac{3}{2} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} \frac{2}{3} \\ -\frac{3}{2} \end{pmatrix} \quad (1.5.1.3)$$

$$= \begin{pmatrix} \frac{97}{36} \\ 0 \end{pmatrix} = \frac{97}{36} \quad (1.5.1.4)$$

c)

$$(6+7j)8-j = [(6-7j)8+j]^* \quad (1.5.1.5)$$

$$= (53-50j)^* = 53+50j \quad (1.5.1.6)$$

2. Find  $z^{-1}$  for

$$\text{a) } z = 6 + 8j$$

$$\text{b) } z = 6 - 8j$$

$$\text{c) } z = \frac{1}{\sqrt{2}}(1+j)$$

**Solution:**

a)

$$z^{-1} = \frac{z^*}{|z|^2} = \frac{6-8j}{100} \quad (1.5.2.1)$$

b)

$$z^{-1} = \frac{6+8j}{100} \quad (1.5.2.2)$$

c)

$$z^{-1} = \frac{1-j}{\sqrt{2}} \quad (1.5.2.3)$$

3. Show that

$$(z^*)^{-1} = (z^{-1})^* \quad (1.5.3.1)$$

**Solution:** Since

$$zz^{-1} = 1, \quad (1.5.3.2)$$

$$(zz^{-1})^* = 1 \quad (1.5.3.3)$$

$$\Rightarrow (z)^*(z^{-1})^* = 1 \quad (1.5.3.4)$$

## 1.5 Complex Numbers

1. Multiply

$$\text{a) } (6-7j)(8+j)$$

yielding (1.5.3.1).

4. Find

$$(\cos \theta + j \sin \theta)^{-1} \quad (1.5.4.1)$$

**Solution:**

$$(\cos \theta + j \sin \theta)^{-1} = \cos \theta - j \sin \theta \quad (1.5.4.2)$$

5. Verify the following

a)  $(z^*)^* = z$

b)  $(z + w)^* = z^* + w^*$

c)  $z + z^* = 2\text{Re}(z)$

d)  $z - z^* = 2j\text{Im}(z)$

**Solution:**

a) For

$$z = a + jb, \quad (1.5.5.1)$$

$$z^* = a - jb, \quad (1.5.5.2)$$

$$\Rightarrow (z^*)^* = a + jb = z \quad (1.5.5.3)$$

b) For

$$z = z_1 + jz_2 \quad (1.5.5.4)$$

$$w = w_1 + jw_2,$$

$$(z + w)^* = (z_1 + jz_2 + w_1 + jw_2)^* \quad (1.5.5.5)$$

$$= (z_1 - jz_2) + (w_1 - jw_2) \quad (1.5.5.6)$$

$$= z^* + w^* \quad (1.5.5.7)$$

c) For

$$z = a + jb, \quad (1.5.5.8)$$

$$z^* = a - jb, \quad (1.5.5.9)$$

$$\Rightarrow (z + z^*) = a + jb + a - jb \quad (1.5.5.10)$$

$$= 2a = 2\text{Re}(z) \quad (1.5.5.11)$$

d) For

$$z = a + jb, \quad (1.5.5.12)$$

$$z^* = a - jb, \quad (1.5.5.13)$$

$$\Rightarrow (z - z^*) = a + jb - a + jb \quad (1.5.5.14)$$

$$= 2jb = 2\text{Im}(z) \quad (1.5.5.15)$$

6. Show that  $z$  is real if and only if  $z^* = z$  and is purely imaginary if and only if  $z^* = -z$ .

**Solution:** Let

$$z = a + jb. \quad (1.5.6.1)$$

Then

$$z^* = a - jb. \quad (1.5.6.2)$$

If

$$z^* = z, \quad (1.5.6.3)$$

$$a + jb = a - jb \quad (1.5.6.4)$$

$$\Rightarrow b = 0 \quad (1.5.6.5)$$

and  $z$  is real. If  $z$  is real,

$$z = a \quad (1.5.6.6)$$

$$\Rightarrow z^* = a \quad (1.5.6.7)$$

$$\text{or, } z = z^* \quad (1.5.6.8)$$

Similarly, the other property can be proved.

7. Verify the commutative law of multiplication

$zw = wz$  in  $\mathbb{C}$ .

**Solution:** Let

$$z = a + jb \quad (1.5.7.1)$$

$$w = x - jy \quad (1.5.7.2)$$

Then

$$zw = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad (1.5.7.3)$$

$$= \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \quad (1.5.7.4)$$

$$= wz \quad (1.5.7.5)$$

8. Show that for  $z \neq 0$ ,  $|z|^{-1} = \frac{1}{|z|}$ .

**Solution:** Let

$$z = re^{j\theta}. \quad (1.5.8.1)$$

Then

$$z^{-1} = \frac{1}{r}e^{-j\theta} \quad (1.5.8.2)$$

$$\Rightarrow |z^{-1}| = \frac{1}{r} \quad (1.5.8.3)$$

9. Find

a)  $|6 - 4j|$ .

b)  $\left|\frac{1}{2} + \frac{2}{3}j\right|$ .

c)  $\left|\frac{1}{\sqrt{2}}(1 + j)\right|$

**Solution:**

a)

$$|6 - 4j| = \sqrt{6^2 + 4^2} = 2\sqrt{13} \quad (1.5.9.1)$$

b)

$$\left|\frac{1}{2} + \frac{2}{3}j\right| = \frac{5}{6} \quad (1.5.9.2)$$

c)

$$\left| \frac{1}{\sqrt{2}} (1 + j) \right| = \frac{1}{\sqrt{2}} |(1 + j)| = 1 \quad (1.5.9.3)$$

10. Show that  $|z^*| = |z|$ .**Solution:** Let

$$z = re^{j\theta} \quad (1.5.10.1)$$

Then

$$z^* = re^{-j\theta} \quad (1.5.10.2)$$

$$\Rightarrow |z^*| = r = |z|. \quad (1.5.10.3)$$

11. Find the polar form for

a)  $z = \frac{\sqrt{2}}{2} - \frac{1}{\sqrt{2}}j$ .

b)  $z = 4j$ .

c)  $z = \frac{6}{\sqrt{2}} + \frac{6}{\sqrt{2}}j$ .

d)  $z = -\frac{13}{2} + \frac{39}{2\sqrt{3}}j$ .

**Solution:**

a)

$$|z| = \sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}. \quad (1.5.11.1)$$

$$= 1 \quad (1.5.11.2)$$

and

$$\angle z = -\tan^{-1} \frac{\frac{1}{\sqrt{2}}}{\frac{\sqrt{2}}{2}} \quad (1.5.11.3)$$

$$= \frac{\pi}{4} \quad (1.5.11.4)$$

b)

$$|z| = 4, \angle z = \frac{\pi}{2}. \quad (1.5.11.5)$$

c)

$$|z| = \frac{6}{\sqrt{2}}, \angle z = \frac{\pi}{4}. \quad (1.5.11.6)$$

d)

$$|z| = \frac{13}{2} \sqrt{1+3} \quad (1.5.11.7)$$

$$= 13 \quad (1.5.11.8)$$

and

$$\angle z = \pi - \tan^{-1} \frac{\frac{39}{2\sqrt{3}}}{\frac{13}{2}} \quad (1.5.11.9)$$

$$= \pi - \tan^{-1} \sqrt{3} = \frac{2\pi}{3} \quad (1.5.11.10)$$

12. Prove that

$$\left( \cos\left(\frac{\theta}{2}\right) + j \sin\left(\frac{\theta}{2}\right) \right)^2 = \cos(\theta) + j \sin(\theta) \quad (1.5.12.1)$$

**Solution:** The L.H.S can be expressed as

$$\left( e^{j\theta/2} \right)^2 = e^{j\theta} \quad (1.5.12.2)$$

13. Show that

$$\left( \frac{1}{2} + \frac{\sqrt{3}}{2}j \right)^3 = -1 \quad (1.5.13.1)$$

**Solution:**

$$\frac{1}{2} + \frac{\sqrt{3}}{2}j = e^{j\pi/3} \quad (1.5.13.2)$$

$$\Rightarrow \left( e^{j\pi/3} \right)^3 = e^{j\pi} = -1 \quad (1.5.13.3)$$

14. Show that

$$(\cos(\theta) + j \sin(\theta))^m = \cos(m\theta) + j \sin(m\theta) \quad (1.5.14.1)$$

for all integers  $m$ . **Solution:** It is easy to verify that

$$(\cos(\theta) + j \sin(\theta))^2 = \cos(2\theta) + j \sin(2\theta) \quad (1.5.14.2)$$

Then

$$\begin{aligned} (\cos(\theta) + j \sin(\theta))^{k+1} &= (\cos(\theta) + j \sin(\theta))^k \\ &\quad (\cos(m\theta) + j \sin(m\theta)) \\ &= \cos[(k+1)\theta] + j \sin[(k+1)\theta] \end{aligned} \quad (1.5.14.3)$$

By induction, (1.5.14.1) is proved.

15. Show that

$$(\cos(\theta) + j \sin(\theta))^r = \cos(r\theta) + j \sin(r\theta) \quad (1.5.15.1)$$

for all rational numbers  $r$ . **Solution:** Let

$$r = \frac{m}{n}, (\cos(\theta) + j \sin(\theta))^{\frac{1}{n}} = \cos(\alpha) + j \sin(\alpha) \quad (1.5.15.2)$$



Then

$$(\cos(\alpha) + j \sin(\alpha))^n = (\cos(\theta) + j \sin(\theta)) \quad (1.5.15.3)$$

$$\implies \cos(n\alpha) + j \sin(n\alpha) = (\cos(\theta) + j \sin(\theta)) \quad (1.5.15.4)$$

$$\text{or, } \alpha = \frac{\theta}{n} \quad (1.5.15.5)$$

yielding

$$(\cos(\theta) + j \sin(\theta))^{\frac{1}{n}} = \cos\left(\frac{\theta}{n}\right) + j \sin\left(\frac{\theta}{n}\right) \quad (1.5.15.6)$$

Using (1.5.14.1) and (1.5.15.6),

$$(\cos(\theta) + j \sin(\theta))^{\frac{m}{n}} = \cos\left(\frac{m\theta}{n}\right) + j \sin\left(\frac{m\theta}{n}\right) \quad (1.5.15.7)$$

16. If  $z \in \mathbb{C}$  and  $n \geq 1$  is any positive integer, show that there are  $n$  distinct complex numbers such that  $z = w^n$ . **Solution:** Let

$$z = \cos(\theta) + j \sin(\theta) \quad (1.5.16.1)$$

then using (1.5.15.6),

$$w = \cos\left(\frac{2\pi k + \theta}{n}\right) + j \sin\left(\frac{2\pi k + \theta}{n}\right), k = 0, \dots, n-1 \quad (1.5.16.2)$$

which are the distinct roots.

17. Find the necessary and sufficient condition on  $k$  such that

$$\left(\cos\left(\frac{2\pi k}{n}\right) + j \sin\left(\frac{2\pi k}{n}\right)\right)^n = 1 \quad \text{and} \quad (1.5.17.1)$$

$$\left(\cos\left(\frac{2\pi k}{n}\right) + j \sin\left(\frac{2\pi k}{n}\right)\right)^m \neq 1 \quad 0 < m < n \quad (1.5.17.2)$$

**Solution:** From the above equations, using (1.5.14.1),

$$\frac{mk}{n} \notin \mathbb{Z} \quad (1.5.17.3)$$

18. Viewing the  $x$ - $y$  plane as the set of all complex numbers  $x + jy$ , show that multiplication by  $j$  induces a  $90^\circ$  rotation of the  $x$ - $y$  plane in counterclockwise direction.

**Solution:** The given multiplication can be ex-

pressed using matrices as

$$\begin{pmatrix} \cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad (1.5.18.1)$$

which is the multiplication of  $\begin{pmatrix} x \\ y \end{pmatrix}$  with a  $90^\circ$  rotation matrix.

19. In problem (1.5.18), interpret geometrically what multiplication by the complex number  $a + jb$  does to the  $x$ - $y$  plane.

**Solution:** The multiplication can be represented as

$$\sqrt{a^2 + b^2} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad (1.5.19.1)$$

where

$$\cos \theta = \frac{a}{\sqrt{a^2 + b^2}} \quad \sin \theta = \frac{b}{\sqrt{a^2 + b^2}} \quad (1.5.19.2)$$

Geometrically, multiplication by  $a + jb$  results in rotation by  $\theta$  and scaling by  $\sqrt{a^2 + b^2}$ .

20. Prove that

$$|z + w|^2 + |z - w|^2 = 2(|z|^2 + |w|^2) \quad (1.5.20.1)$$

**Solution:** Since

$$|z + w|^2 = (z + w)^*(z + w) \quad (1.5.20.2)$$

$$= |z|^2 + |w|^2 + 2z^*w \quad (1.5.20.3)$$

and

$$|z - w|^2 = (z - w)^*(z - w) \quad (1.5.20.4)$$

$$= |z|^2 + |w|^2 - 2z^*w, \quad (1.5.20.5)$$

$$|z + w|^2 + |z - w|^2 = 2(|z|^2 + |w|^2) \quad (1.5.20.6)$$

21. Consider the set  $A = a + bj$ ,  $a, b \in \mathbb{Z}$ . Prove that there is 1-1 correspondence of  $A$  onto  $\mathbb{N}$ .

22. If  $a$  is a (complex) root of the polynomial

$$x^n + \alpha_1 x^{n-1} + \dots + \alpha_{n-1} x + \alpha_n, \quad (1.5.22.1)$$

where the  $\alpha_i$  are real, show that  $\bar{a}$  must also be a root.

**Solution:** From the given information,

$$\bar{a}^n + \alpha_1 \bar{a}^{n-1} + \dots + \alpha_{n-1} \bar{a} + \alpha_n = 0 \quad (1.5.22.2)$$

Thus,  $\bar{a}$  is also a root of the given polynomial.

23. Find the necessary and sufficient conditions on

$z$  and  $w$  in order that

$$|z + w| = |z| + |w| \quad (1.5.23.1)$$

**Solution:**

$$|z + w|^2 = |z|^2 + |w|^2 + 2z^*w \quad (1.5.23.2)$$

$$(|z| + |w|)^2 = |z|^2 + |w|^2 + 2|z||w| \quad (1.5.23.3)$$

If the above expressions are equal,

$$z^*w = |z||w| \quad (1.5.23.4)$$

which is the desired condition.

24. Find the necessary and sufficient conditions on  $z_i$  in order that

$$\left| \sum_{i=1}^k z_i \right| = \sum_{i=1}^k |z_i| \quad (1.5.24.1)$$

**Solution:**

$$\left| \sum_{i=1}^k z_i \right|^2 = \sum_{i=1}^k |z_i|^2 + 2 \sum_{\substack{i=1, j=1 \\ i \neq j}}^k z_i^* z_j \quad (1.5.24.2)$$

$$\left( \sum_{i=1}^k |z_i| \right)^2 = \sum_{i=1}^k |z_i|^2 + 2 \sum_{\substack{i=1, j=1 \\ i \neq j}}^k |z_i| |z_j| \quad (1.5.24.3)$$

From (1.5.24.2) and (1.5.24.3),

$$\begin{aligned} \sum_{i=1}^k |z_i|^2 + 2 \sum_{\substack{i=1, j=1 \\ i \neq j}}^k z_i^* z_j \\ = \sum_{i=1}^k |z_i|^2 + 2 \sum_{\substack{i=1, j=1 \\ i \neq j}}^k |z_i| |z_j| \\ \Rightarrow \sum_{\substack{i=1, j=1 \\ i \neq j}}^k z_i^* z_j \\ = \sum_{\substack{i=1, j=1 \\ i \neq j}}^k |z_i| |z_j| \quad (1.5.24.4) \end{aligned}$$

which is the desired condition.

25. The complex number  $\theta$  is said to have *order*  $n \geq 1$  if  $\theta^n = 1$  and  $\theta^m \neq 1$  for  $0 < m < n$ . Show that if  $\theta$  has order  $n$  and  $\theta^k = 1$ , where  $k > 0$ , then  $n|k$ .

**Solution:** From the given information,

$$\theta^n = \theta^k = 1, k \geq n \quad (1.5.25.1)$$

If  $n \nmid k, k = mn + p, 0 < p < n$ , Then,

$$\theta^k = \theta^{mn+p} = \theta^p \neq 1, \quad (1.5.25.2)$$

which is a contradiction, Hence,  $n | k$ .

26. Find all complex numbers  $\theta$  having order  $n$ .

**Solution:** If

$$\theta^n = 1, \quad (1.5.26.1)$$

$$\theta^n = e^{i2\pi r}, 0 \leq r < n \quad (1.5.26.2)$$

yielding

$$\theta = \exp\left(j \frac{2\pi r}{n}\right) 0 \leq r < n \quad (1.5.26.3)$$