

Abstract Algebra

G. V. V. Sharma

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1 THINGS FAMILIAR AND LESS FAMILIAR

1.1 Introduction

- let S be a set having an operation $*$ which assigns an element $a*b$ of S for any $a, b \in S$. Let us assume that the following two rules hold:
 - If a, b are any objects in S , then $a*b = a$.
 - If a, b are any objects in S , then $a*b = b*a$.

Show that S can have at most one object.

Solution: From condition 1.1.1a, interchanging a, b ,

$$b*a = b \quad (1.1.1)$$

and from condition 1.1.1b,

$$b*a = a*b \quad (1.1.1)$$

But from condition 1.1.1a,

$$a*b = a \implies a = b \quad (1.1.1)$$

Thus, S can have at most one object.

- Let S be the set of all integers $0, \pm 1, \pm 2, \dots, \pm n, \dots$. For $a, b \in S$, define $*$ by

$$a*b = a - b \quad (1.1.2)$$

Verify the following

- $a*b \neq b*a$ unless $a = b$
- $(a*b)*c \neq a*(b*c)$ in general. Under what conditions on a, b, c is

$$(a*b)*c \neq a*(b*c) \quad ? \quad (1.1.2)$$

- The integer a has the property that $a*0 = a$ for every $a \in S$.
- For $a \in S, a*a = 0$.

Solution:

a)

$$a*b = b*a \quad (1.1.2)$$

$$\implies a - b = b - a \quad (1.1.2)$$

$$\text{or, } a = b \quad (1.1.2)$$

b) Let $a = 1, b = 2, c = 4$. Then,

$$a*b = -1, (a*b)*c = -1 - 4 = -5 \quad (1.1.2)$$

$$b*c = -2, a*(b*c) = 1 + 2 = 3 \neq -5 \quad (1.1.2)$$

Thus, for the given condition to be satisfied,

$$(a - b) - c = a - (b - c) \quad (1.1.2)$$

$$\implies c = 0 \quad (1.1.2)$$

c)

$$a*0 = a - 0 = a \quad (1.1.2)$$

d)

$$a*a = a - a = 0 \quad (1.1.2)$$

- Let S consist of the two objects \square and \triangle . We define the operation $*$ on S by subjecting \square and \triangle to the following conditions.

$$\text{a) } \square * \triangle = \triangle = \triangle * \square$$

$$\text{b) } \square * \square = \square$$

$$\text{c) } \triangle * \triangle = \square$$

Verify by explicit calculation that if a, b, c are any elements of S , (i.e. a, b, c can be any of \square or \triangle), then

- a) $a * b$ is in S
- b) $(a * b) * c = a * (b * c)$
- c) $a * b = b * a$
- d) There is a particular a in S such that $a * b = b * a = b$ for all $b \in S$
- e) Given $b \in S, b * b = a$, where a is the particular element in Part 1.1.3d.

Solution: Let $\square = 1, \Delta = -1$. These satisfy all the given conditions.

- a) $a * b \in [1, -1] \in S$.
- b) Writing the truth table, $(a * b) * c = a * (b * c)$.
- c) $a * b = b * a$ can be verified by writing the truth table.
- d) For $a = 1, a * b = b * a = b$, for all $b \in S$.
- e) For $a = 1$, if $b = -1, b * b = 1 = a$. This can be shown to be true for $b = 1$ as well.

1.2 Set Theory

1. Describe the following sets verbally

- a) $S = \{\text{Mercury, Venus, Earth, } \dots, \text{Pluto}\}$
- b) $S = \{\text{Andhra Pradesh, Uttar Pradesh, } \dots, \text{Assam}\}$

Solution:

- a) Planets
- b) Indian states

2. Describe the following sets verbally

- a) $S = \{2, 4, 6, 8, \dots\}$
- b) $S = \{2, 4, 8, 16, \dots\}$
- c) $S = \{1, 4, 9, 16, 25, 36, \dots\}$

Solution:

- a) Even numbers
- b) Powers of 2
- c) Squares of positive integers

3. If A is the set of all residents of India, B the set of all Sri Lankan citizens, and C the set of all women in the world, describe the sets $ABC, A - B, A - C, C - A$ verbally.

Solution:

- a) ABC is the set of all women residents of India who are citizens of Sri Lanka.
- b) $A - B = AB'$ is the set of all residents of India who are not Sri Lankan citizens.
- c) $A - C = AC'$ is the set of all male residents of India.
- d) $C - A = CA'$ is the set of all women who are not residing in India.

4. If $A = \{1, 4, 7, a\}$ and $B = \{3, 4, 9, 11\}$ and you have been told that $AB = \{4, 9\}$, then what must

a be?

Solution: $a = 9$

5. If $A \subset B, B \subset C$, prove that $A \subset C$

Solution: From the given information,

$$A + P = B, AP = 0, B + Q = C, BQ = 0 \quad (1.2.5.1)$$

$$\implies B + Q = A + P + Q = C, \quad (1.2.5.2)$$

$$\therefore BQ = 0,$$

$$AQ + PQ = 0 \implies AQ = 0, PQ = 0 \quad (1.2.5.3)$$

Hence,

$$A(P + Q) = 0 \implies A \subset C \quad (1.2.5.4)$$

6. If $A \subset B$ prove that $A \cup C \subset B \cup C$ for any set C .

Solution: From the given information, there exists P such that

$$A + P = B, AP = 0 \quad (1.2.6.1)$$

Also,

$$B + C = A + P + C \quad (1.2.6.2)$$

$$\implies A + C \subset B + C \quad (1.2.6.3)$$

7. Show that

$$A \cup B = B \cup A \quad (1.2.7.1)$$

$$A \cap B = B \cap A \quad (1.2.7.2)$$

8. Prove that

$$(A - B) \cup (B - A) = (A \cup B) - (A \cap B) \quad (1.2.8.1)$$

Solution: Since

$$A - B = AB', \quad (1.2.8.2)$$

$$(A - B) \cup (B - A) = AB' + BA' \quad (1.2.8.3)$$

Also,

$$(A \cup B) - (A \cap B) = (A + B)(AB')' \quad (1.2.8.4)$$

$$= (A + B)(A' + B') \quad (1.2.8.5)$$

$$= AB' + BA' \quad (1.2.8.6)$$

9. Prove that

$$(A) \cap (B \cup C) = (A \cap B) \cup (A \cap C) \quad (1.2.9.1)$$

Solution:

$$LHS = A(B + C) = AB + AC = RHS \quad (1.2.9.2)$$

10. Prove that

$$(A) \cup (B \cap C) = (A \cup B) \cap (A \cup C) \quad (1.2.10.1)$$

Solution:

$$LHS = A + BC \quad (1.2.10.2)$$

$$RHS = (A + B)(A + C) \quad (1.2.10.3)$$

$$= A + A(B + C) + BC \quad (1.2.10.4)$$

$$= A(1 + B + C) + BC \quad (1.2.10.5)$$

$$= LHS \quad (1.2.10.6)$$

11. Write down all the subsets of $S = \{1, 2, 3, 4\}$.

Solution: Write a program for this.

12. If C is a subset of S , let C' denote the complement of C in S . Prove the *De Morgan Rules* for subsets A, B of S , namely,

a) $(A \cup B)' = A' \cap B'$

b) $(A \cap B)' = A' \cup B'$

Solution:

a)

$$(A + B)A'B' = AA'B' + BA'B' \quad (1.2.12.1)$$

$$= 0 \quad (1.2.12.2)$$

b) Substituting $A = A', B = B'$ in the above, the second result is obtained.

13. Let S be a set. For any two subsets of S , we define

$$A \oplus B = (A - B) \cup (B \cup A) \quad (1.2.13.1)$$

Prove that

a) $A \oplus B = B \oplus A$.

b) $A \oplus \Phi = A$.

c) $A \cdot A = A$.

d) $A \oplus A = \Phi$.

e) $A \oplus (B \oplus C) = (A \oplus B) \oplus C$.

f) If $A \oplus B = A \oplus C$, then $B = C$.

g) $A \cdot (B + C) = A \cdot B + A \cdot C$.

Solution: All can be proved using boolean logic.

14. If C is a finite set, let $m(C)$ denote the number of elements in C . If A, B are finite sets, prove that

$$m(A \cup B) = m(A) + m(B) - m(A \cap B) \quad (1.2.14.1)$$

Solution:

$$A'B' = (A + B)' \quad (1.2.14.2)$$

$$\implies m(A'B') = m((A + B)') \quad (1.2.14.3)$$

$$= 1 - m(A + B) \quad (1.2.14.4)$$

$$\because A + B = A(B + B') + B \quad (1.2.14.5)$$

$$= B(A + 1) + AB' \quad (1.2.14.6)$$

$$= B + AB' \quad (1.2.14.7)$$

$$\implies m(A + B) = m(B + AB') \quad (1.2.14.8)$$

$$= m(B) + m(AB') \quad (1.2.14.9)$$

$$\because B(AB') = 0 \quad (1.2.14.10)$$

$$A = A(B + B') = AB + AB' \quad (1.2.14.11)$$

and

$$(AB)(AB') = 0, \because BB' = 0 \quad (1.2.14.12)$$

Hence, AB and AB' are mutually exclusive and

$$m(A) = m(AB) + m(AB') \quad (1.2.14.13)$$

$$\implies m(AB') = m(A) - m(AB) \quad (1.2.14.14)$$

Substituting (1.2.14.14) in (1.2.14.10),

$$m(A + B) = m(A) + m(B) - m(AB) \quad (1.2.14.15)$$

15. For three finite sets A, B, C , find a formula for $m(A \cup B \cup C)$. **Solution:** Extend the above.

16. Take a shot at finding $m(\bigcup_{i=1}^n A_i)$.

17. Show that if 80% of all Indians have gone to high school and 70% of all Indians read a daily newspaper, then *at least* 50% of all Indians have both gone to high school and read a daily newspaper.

Solution: Let A represent high school and B represent newspaper. Then,

$$\Pr(AB) = \Pr(A) + \Pr(B) - \Pr(A + B) \quad (1.2.17.1)$$

Since

$$\Pr(A + B) \leq 1, \quad (1.2.17.2)$$

$$\Pr(A) + \Pr(B) - \Pr(A + B) \geq \Pr(A) + \Pr(B) - 1 \quad (1.2.17.3)$$

$$\implies \Pr(AB) \geq 0.8 + 0.7 - 1 \quad (1.2.17.4)$$

$$= 0.5 \quad (1.2.17.5)$$

18. A public opinion poll shows that 90% of the population agreed with the government on the first decision, 84% on the second, and 74% on the third, for three decisions made by the government. At least what percentage of the population agreed with the government on all three decisions.

Solution: Let the decisions be A, B, C . Then,

$$\Pr(AB) \geq \Pr(ABC), \quad (1.2.18.1)$$

$$\Pr(BC) \geq \Pr(ABC), \quad (1.2.18.2)$$

$$\Pr(CA) \geq \Pr(ABC) \quad (1.2.18.3)$$

Since

$$\begin{aligned} \Pr(A + B + C) &= \sum \Pr(A) \\ &\quad - \sum \Pr(AB) + \Pr(ABC), \\ \Rightarrow \Pr(A + B + C) + \sum \Pr(AB) \\ &= \sum \Pr(A) + \Pr(ABC), \end{aligned} \quad (1.2.18.4)$$

from (1.2.18.1),

$$\begin{aligned} \Pr(A + B + C) + 3\Pr(ABC) \\ &\geq \sum \Pr(A) + \Pr(ABC), \\ \Rightarrow 2\Pr(ABC) &\geq \sum \Pr(A) - \Pr(A + B + C) \end{aligned} \quad (1.2.18.5)$$

Since

$$\Pr(A + B + C) \leq 1, \quad (1.2.18.6)$$

$$-\Pr(A + B + C) \geq -1 \quad (1.2.18.7)$$

$$\Rightarrow 2\Pr(ABC) \geq \sum \Pr(A) - 1 \quad (1.2.18.8)$$

$$\text{or } \Pr(ABC) \geq \frac{\sum \Pr(A) - 1}{2} \quad (1.2.18.9)$$

$$= 0.74 \quad (1.2.18.10)$$

19. In his book *A Tangled Tale*, Lewis Carroll proposed the following riddle about a group of disabled veterans. "Say that 70% have lost an eye, 75% an ear, 80% an arm, 85% a leg. What percentage, at least, must have lost all four?" Solve Lewis Carroll's problem.

Solution: Let A_i represent the events. Then,

$$\begin{aligned} \Pr\left(\sum_{i=1}^4 A_i\right) &= \sum_{i=1}^4 \Pr(A_i) - \sum_{i,j} \Pr(A_i A_j) \\ &\quad + \sum_{i,j,k} \Pr(A_i A_j A_k) - \Pr\left(\prod_{i=1}^4 A_i\right) \end{aligned} \quad (1.2.19.1)$$

Now,

$$\Pr(A_1 A_2) \geq \Pr(A_1 A_2 A_3) \geq \Pr(A_1 A_2 A_3 A_4) \quad (1.2.19.2)$$

which, upon substitution in (1.2.19.1) yields

$$\Pr\left(\sum_{i=1}^4 A_i\right) \geq \frac{\sum_{i=1}^4 \Pr(A_i) - 1}{1 + {}^4C_2 - {}^4C_3} \quad (1.2.19.3)$$

$$= 70\% \quad (1.2.19.4)$$

20. Show, for finite sets A, B , that $m(A \times B) = m(A) \times m(B)$.

Solution: Basic principle of counting.

21. If S is a set having five elements,
 a) How many subsets does S have?
 b) How many subsets having four elements does S have?
 c) How many subsets having two elements does S have?

Solution:

$$\text{a) } 2^5 = 32.$$

$$\text{b) } {}^5C_4 = 5.$$

$$\text{c) } {}^5C_2 = 10.$$

22. a) Show that a set having n elements has 2^n subsets.
 b) If $0 < m < n$, how many subsets are there that have exactly m elements?

Solution:

- a) The number of subsets is

$$\sum_{k=0}^n {}^nC_k = 2^n \quad (1.2.22.1)$$

using the binomial theorem.

- b) The number of subsets having exactly m elements are nC_m .

1.3 Mappings

1. For the given sets S, T determine if a mapping $f : S \rightarrow T$ is clearly and unambiguously defined; if not, say why not.
 a) S = set of all women, T = set of all men, $f(s)$ = husband of s .
 b) S = set of all positive integers, $T = S$, $f(s) = s - 1$.
 c) S = set of positive integers, T = set of nonnegative integers, $f(s) = s - 1$.
 d) S = set of nonnegative integers, $T = S$, $f(s) = s - 1$.

- e) S = set of all integers, $T = S$, $f(s) = s - 1$.
 f) S = set of all real numbers, $T = S$, $f(s) = \sqrt{s}$.
 g) S = set of all positive real numbers, $T = S$, $f(s) = \sqrt{s}$.

Solution:

- a) Not all women have husbands. So the mapping is not clearly defined.
 b) For every integer s , $s - 1$ is an integer. So the mapping is defined.
 c) $0 \notin S$, so the mapping is defined.
 d) $f(0) = -1 \notin S$. So the mapping is not defined.
 e) $f(-1) \notin S$, so the mapping is not defined.
 f) $f(s) \in S \forall s$. So the mapping is defined.
2. In those parts of Problem 1.3.1 where f does define a function, determine if it is 1-1, onto, or both. **Solution:**
- a) For $f(s) = s - 1$, $s \in \mathbb{Z}$, the mapping is a bijection.
 b) For $s \in \mathbb{N}$, $f(s) = s - 1 \in \mathbb{W}$, the mapping is a bijection.
 c) For $s \in S$, $f(s) \in S$ and vice-versa. So the mapping is a bijection.
3. If f is a 1-1 mapping of S onto T , prove that f^{-1} is a 1-1 mapping of T onto S .

Solution: By definition,

$$\begin{aligned} s_1 = s_2 \in S &\implies f(s_1) = f(s_2) \in T \\ t_1 = t_2 \in T &\implies \exists s_1 = s_2 \in S \ni f(s_1) = f(s_2). \end{aligned} \quad (1.3.3.1)$$

Let $g = f^{-1}$. Then,

$$f(s_i) = t_i \implies g(t_i) = s_i. \quad (1.3.3.2)$$

From (1.3.3.1),

$$\begin{aligned} g(t_1) = g(t_2) \in S &\implies t_1 = t_2 \in T \\ t_1 = t_2 \in T &\implies \exists g(t_1) = g(t_2) \in S \end{aligned} \quad (1.3.3.3)$$

(1.3.3.3) shows that $g = f^{-1}$ is also 1-1.

4. If f is a 1-1 mapping of S onto T , prove that $f^{-1} \circ f = i_S$.

Solution: For $s \in S$, $t \in T$,

$$f(s) = t \implies g(t) = s \quad (1.3.4.1)$$

$$\text{or, } g \circ f(s) = s \implies (g \circ f) = i_S \quad \square \quad (1.3.4.2)$$

5. If $g : S \rightarrow T$ and $f : T \rightarrow U$ are both onto, then $f \circ g : S \rightarrow U$ is also onto.

Solution: From the given information,

$$g(S) = T, f(T) = U \quad (1.3.5.1)$$

$$\implies (f \circ g)(S) = U \quad \square \quad (1.3.5.2)$$

6. If $f : S \rightarrow T$ is onto and $g : T \rightarrow U$ and $h : T \rightarrow U$ are such that $g \circ f = h \circ f$, prove that $g = h$.

Solution: From the given information,

$$g \circ f = h \circ f \quad (1.3.6.1)$$

$$\implies (g - h) \circ f = 0 \quad (1.3.6.2)$$

$$\text{or, } g = h \quad (1.3.6.3)$$

7. If $g : S \rightarrow T$, $h : S \rightarrow T$, and if $f : T \rightarrow U$ is 1-1, show that if $f \circ g = f \circ h$, then $g = h$.
 8. Let S be the set of all integers and $T = \{1, -1\}$; $f : S \rightarrow T$ is defined by

$$f(s) = \begin{cases} 1 & s \text{ even} \\ -1 & s \text{ odd} \end{cases}. \quad (1.3.8.1)$$

- a) Does this define a function from S to T ?
 b) Show that

$$f(s_1 + s_2) = f(s_1)f(s_2) \quad (1.3.8.2)$$

What does this say about the integers?

- c) Is $f(s_1 s_2) = f(s_1)f(s_2)$ also true?

Solution:

- a) Yes, f is a function.
 b) See Table 1.3.8.

$f(s_1)$	$f(s_2)$	$f(s_1) + f(s_2)$
1	1	1
1	-1	-1
-1	-1	1
-1	1	-1

TABLE 1.3.8

- c) No. If s_1, s_2 are odd,

$$s_1 s_2 \text{ odd} \quad (1.3.8.3)$$

$$f(s_1 s_2) = -1 \neq f(s_1)f(s_2) \quad (1.3.8.4)$$

9. Let S be the set of all real numbers. Define

$$f : S \rightarrow S | f(s) = s^2, \quad (1.3.9.1)$$

$$g : S \rightarrow S | g(s) = s + 1, \quad (1.3.9.2)$$

- a) Find $f \circ g$.

- b) Find $g \circ f$.
 c) Is $f \circ g = g \circ f$?

Solution:

a)

$$(f \circ g)(s) = (s + 1)^2 \quad (1.3.9.3)$$

b)

$$(g \circ f)(s) = s^2 + 1 \quad (1.3.9.4)$$

c) From (1.3.9.3) and (1.3.9.4) $f \circ g \neq g \circ f$.

10. Let S be the set of all real numbers and for $a, b \in S$, where $a \neq 0$; define $f_{a,b}(s) = as + b$.

a) Show that $f_{a,b} \circ f_{c,d} = f_{u,v}$ for some real u, v .
 Give explicit values for u, v in terms of a, b, c and d .

b) Is $f_{a,b} \circ f_{c,d} = f_{c,d} \circ f_{a,b}$ always?

c) Find all $f_{a,b}$ such that $f_{a,b} \circ f_{1,1} = f_{1,1} \circ f_{a,b}$.

d) Show that f^{-1} exists and find its form.

Solution:

a)

$$f_{a,b} \circ f_{c,d} = c(as + b) + d \quad (1.3.10.1)$$

$$= cas + cb + d \quad (1.3.10.2)$$

$$= us + v \quad (1.3.10.3)$$

$$\implies u = ca, v = bc + d \quad (1.3.10.4)$$

b) From (1.3.10.1),

$$f_{c,d} \circ f_{a,b} = cas + ad + b \quad (1.3.10.5)$$

Thus, from (1.3.10.1) and (1.3.10.5)

$$f_{a,b} \circ f_{c,d} = f_{c,d} \circ f_{a,b} \quad (1.3.10.6)$$

$$\implies bc + d = ad + b \quad (1.3.10.7)$$

c) From (1.3.10.7),

$$f_{a,b} \circ f_{1,1} = f_{1,1} \circ f_{a,b} \quad (1.3.10.8)$$

$$\implies as + b + 1 = as + a + b \quad (1.3.10.9)$$

$$\text{or, } a = 1. \quad (1.3.10.10)$$

Thus,

$$f_{a,b} = s + b \quad (1.3.10.11)$$

d) From the definition,

$$f_{a,b}(s) = as + b \quad (1.3.10.12)$$

$$\implies s = \frac{f_{a,b}(s) - b}{a} \quad (1.3.10.13)$$

$$\text{or, } f_{a,b}^{-1}(s) = \frac{s - b}{a} \quad (1.3.10.14)$$

11. Let S be the set of all positive integers. Define $f : S \rightarrow S$ by $f(1) = 2, f(2) = 3, f(3) = 1$ and $f(s) = s$ for any other $s \in S$. Show that $f \circ f \circ f = i_S$. What is f^{-1} in this case?

Solution: For $s \in \{1, 2, 3\}$, it is obvious. For $s \notin \{1, 2, 3\}$,

$$(f \circ f)(s) = f(s) = s \quad (1.3.11.1)$$

$$\implies (f \circ f \circ f)(s) = s \quad \square \quad (1.3.11.2)$$

It is easy to verify that

$$f^{-1}(s) = f(s) = ss \notin \{1, 2, 3\} \quad (1.3.11.3)$$

Also,

$$f^{-1}(2) = f(1), f^{-1}(3) = f(2), f^{-1}(1) = f(3), \quad (1.3.11.4)$$

1.4 $A(S)$ (The set of 1-1 mappings of S onto itself)

1. If $s_1 \neq s_2$ are in S , show that there is an $f \in A(S)$ such that $f(s_1) = s_2$.

Solution: By definition of a 1-1 mapping, it is obvious.

2. If $s_1 \in S$, let $H = \{f \in A(S) \mid f(s_1) = s_1\}$. Show that:

a) $i \in H$.

b) If $f, g \in H$, then $fg \in H$.

c) If $f \in H$, then $f^{-1} \in H$.

Solution:

a) $\because i(s_1) = s_1, i \in H$.

b) $fg(s_1) = f(s_1) = s_1$.

c) $f(s_1) = s_1 \implies f^{-1}(s_1) = s_1 \implies f^{-1} \in H$.

3. Suppose that $s_1 \neq s_2$ are in S and $f(s_1) = s_2$, where $f \in A(S)$. Then if H is as in Problem 1.4.2 and $K = \{g \in A(S) \mid g(s_2) = s_2\}$, show that:

a) If $g \in K$, then $f^{-1}gf \in H$.

b) If $h \in H$, then there is some $g \in K$ such that $h = f^{-1}gf$.

Solution:

a) From the given information,

$$f^{-1}gf(s_1) = f^{-1}g(s_2) = f^{-1}(s_2) = s_1 \quad (1.4.3.1)$$

Hence,

$$f^{-1}gf \in H \quad (1.4.3.2)$$

b) The h was found in the previous part.

4. If $f, g, h \in A(S)$, show that $(f^{-1}gf)(f^{-1}hf) = f^{-1}(gh)f$. What can you say about $(f^{-1}gf)^n$?

Solution: From the given information,

$$(f^{-1}gf)(f^{-1}hf) = f^{-1}g(ff^{-1})hf \quad (1.4.4.1)$$

$$= f^{-1}(gh)f \quad (1.4.4.2)$$

Similarly,

$$(f^{-1}gf)^n = f^{-1}g^n f \quad (1.4.4.3)$$

5. If $f, g \in A(S)$ and $fg = gf$, show that:

a) $(fg)^2 = f^2g^2$.

b) $(fg)^{-1} = f^{-1}g^{-1}$.

Solution: From the given information,

a)

$$(fg)^2 = (fg)(fg) \quad (1.4.5.1)$$

$$= (fg)(gf) = fg^2f \quad (1.4.5.2)$$

$$= f(fg^2) = f^2g^2 \quad (1.4.5.3)$$

b) Since

$$(fg)^{-1}fg = i, \quad (1.4.5.4)$$

$$(fg)^{-1}fgg^{-1} = g^{-1} \quad (1.4.5.5)$$

$$\implies (fg)^{-1}f = g^{-1} \quad (1.4.5.6)$$

$$\implies (fg)^{-1}ff^{-1} = g^{-1}f^{-1} \quad (1.4.5.7)$$

$$\text{or, } (fg)^{-1} = g^{-1}f^{-1} \quad (1.4.5.8)$$

6. Push the result of Problem 1.4.5, for the same f and g , to show that

$$(fg)^m = f^m g^m \quad (1.4.6.1)$$

for all integers m .

Solution: Using induction,

$$(fg)^{m+1} = (fg)^m (fg) \quad (1.4.6.2)$$

$$= f^m g^m fg = f^m g^m gf \quad (1.4.6.3)$$

$$= f f^m g^m g \quad (1.4.6.4)$$

yielding (1.4.6.1).

7.

8. If $f, g \in A(S)$ and $(fg)^2 = f^2g^2$, prove that $fg = gf$.

Solution:

$$(fg)^2 = f^2g^2 \quad (1.4.8.1)$$

$$\implies fgfg = ffgg \quad (1.4.8.2)$$

$$\implies f^{-1}fgfg = f^{-1}ffgg \quad (1.4.8.3)$$

$$\implies gfg = fgg \quad (1.4.8.4)$$

$$\implies gfgg^{-1} = fggg^{-1} \quad (1.4.8.5)$$

yielding the desired result.

1.5 The Integers

1. Find (a, b) and express (a, b) as $ma + nb$ for

a) $(116, -84)$

b) $(85, 65)$

c) $(72, 26)$

d) $(72, 25)$

Solution:

a) Using the extended Euclid algorithm,

$$\begin{pmatrix} 116 & 1 & 0 \\ -84 & 0 & 1 \end{pmatrix} \quad (1.5.1.1)$$

$$\xleftrightarrow{R_3 \leftarrow R_1 + R_2} \begin{pmatrix} 32 & 1 & 1 \\ -20 & 2 & 3 \end{pmatrix} \quad (1.5.1.2)$$

$$\xleftrightarrow{R_4 \leftarrow R_2 + 2R_3} \begin{pmatrix} 12 & 3 & 4 \\ -8 & 5 & 7 \end{pmatrix} \quad (1.5.1.3)$$

$$\xleftrightarrow{R_5 \leftarrow R_4 + R_3} \begin{pmatrix} 4 & 8 & 11 \\ -8 & 5 & 7 \end{pmatrix} \quad (1.5.1.4)$$

$$\xleftrightarrow{R_6 \leftarrow R_5 + R_4} \begin{pmatrix} 4 & 8 & 11 \\ -8 & 5 & 7 \end{pmatrix} \quad (1.5.1.5)$$

$$\xleftrightarrow{R_7 \leftarrow R_6 + R_5} \begin{pmatrix} 4 & 8 & 11 \\ 0 & 21 & 29 \end{pmatrix} \quad (1.5.1.6)$$

$$\xleftrightarrow{R_8 \leftarrow R_7 + 2R_6} \begin{pmatrix} 0 & 21 & 29 \end{pmatrix} \quad (1.5.1.7)$$

Thus,

$$4 = (8)116 + 11(-84) \quad (1.5.1.8)$$

b)

$$\begin{pmatrix} 85 & 1 & 0 \\ 65 & 0 & 1 \end{pmatrix} \quad (1.5.1.9)$$

$$\xleftrightarrow{R_3 \leftarrow R_1 - R_2} \begin{pmatrix} 20 & 1 & -1 \\ 5 & -3 & 4 \end{pmatrix} \quad (1.5.1.10)$$

$$\xleftrightarrow{R_4 \leftarrow R_2 - 3R_3} \begin{pmatrix} 5 & -3 & 4 \\ 0 & 13 & -17 \end{pmatrix} \quad (1.5.1.11)$$

$$\xleftrightarrow{R_5 \leftarrow R_3 - 4R_4} \begin{pmatrix} 0 & 13 & -17 \end{pmatrix} \quad (1.5.1.12)$$

Thus,

$$5 = (-3)85 + 4(65) \quad (1.5.1.13)$$

c)

$$\begin{pmatrix} 72 & 1 & 0 \\ 26 & 0 & 1 \end{pmatrix} \quad (1.5.1.14)$$

$$\xleftrightarrow{R_3 \leftarrow R_1 - 2R_2} \begin{pmatrix} 20 & 1 & -2 \\ 26 & 0 & 1 \end{pmatrix} \quad (1.5.1.15)$$

$$\xleftrightarrow{R_4 \leftarrow R_2 - R_3} \begin{pmatrix} 6 & -1 & 3 \\ 26 & 0 & 1 \end{pmatrix} \quad (1.5.1.16)$$

$$\xleftrightarrow{R_5 \leftarrow R_3 - 3R_4} \begin{pmatrix} 2 & 4 & -11 \\ 26 & 0 & 1 \end{pmatrix} \quad (1.5.1.17)$$

$$\xleftrightarrow{R_6 \leftarrow R_4 - 3R_5} \begin{pmatrix} 0 & -13 & 36 \\ 26 & 0 & 1 \end{pmatrix} \quad (1.5.1.18)$$

Thus,

$$2 = (4)72 + (-11)26 \quad (1.5.1.19)$$

d)

$$\begin{pmatrix} 72 & 1 & 0 \\ 25 & 0 & 1 \end{pmatrix} \quad (1.5.1.20)$$

$$\xleftrightarrow{R_3 \leftarrow R_1 - 2R_2} \begin{pmatrix} 22 & 1 & -2 \\ 25 & 0 & 1 \end{pmatrix} \quad (1.5.1.21)$$

$$\xleftrightarrow{R_4 \leftarrow R_2 - R_3} \begin{pmatrix} 3 & -1 & 3 \\ 25 & 0 & 1 \end{pmatrix} \quad (1.5.1.22)$$

$$\xleftrightarrow{R_5 \leftarrow R_3 - 7R_4} \begin{pmatrix} 1 & 8 & -23 \\ 25 & 0 & 1 \end{pmatrix} \quad (1.5.1.23)$$

Thus,

$$1 = (8)72 + (-23)25 \quad (1.5.1.24)$$

2. Show that the following are true

- $1 \mid n$ for all n .
- If $m \neq 0$, then $m \mid 0$.
- If $m \mid n$ and $n \mid q$, then $m \mid q$.
- If $m \mid n$ and $n \mid q$, then $m \mid (un + vq)$ for all v, u .
- If $m \mid 1$, then $m = 1$ or $m = -1$.
- If $m \mid n$, and $n \mid m$, then $m = \pm n$.

Solution:

- $n = 1 \times n$.
- $0 = 0 \times m$.
- Let

$$n = cm, q = dn. \quad (1.5.2.1)$$

Then

$$q = (cdn)m \implies m \mid q \quad (1.5.2.2)$$

d) Let

$$n = cm, q = dn. \quad (1.5.2.3)$$

Then

$$un + vq = ucm + vdn \quad (1.5.2.4)$$

$$= (uc + vdc)m \quad (1.5.2.5)$$

$$\implies m \mid (un + vq) \quad (1.5.2.6)$$

e) If

$$1 = cm, \quad (1.5.2.7)$$

$$c = 1, m = 1 \quad (1.5.2.8)$$

$$c = -1, m = -1 \quad (1.5.2.9)$$

f)

$$n = cm, m = dn \quad (1.5.2.10)$$

$$\implies mn = cdmn \quad (1.5.2.11)$$

$$\text{or, } cd = 1 \quad (1.5.2.12)$$

Thus, either

$$c = d = 1, \implies n = m, \quad (1.5.2.13)$$

$$\text{or, } c = d = -1, \implies n = -m \quad (1.5.2.14)$$

3. Show that

$$(ma, mb) = m(a, b) \quad m > 0. \quad (1.5.3.1)$$

Solution: Let

$$(a, b) = xa + yb \quad (1.5.3.2)$$

Then,

$$(ma, mb) = xma + ymb = m(xa + yb) \quad (1.5.3.3)$$

$$= m(a, b) \quad (1.5.3.4)$$

4. Show that if $a \mid m$ and $b \mid m$, and $(a, b) = 1$, then $(ab) \mid m$.**Solution:** From the given information,

$$m = ac, \quad (1.5.4.1)$$

$$m = bd,$$

$$ax + by = 1 \quad (1.5.4.2)$$

Multiplying both sides of (1.5.4.2) by m

$$max + mby = m \quad (1.5.4.3)$$

$$\implies ab(dx + cy) = m \quad (1.5.4.4)$$

upon substituting from (1.5.4.1). Hence, $(ab) \mid m$.

5. Factor the following into primes

- 36
- 120

- c) 720
d) 5040

Solution:

- a) $36 = 2^2 \times 3^2$.
b) $120 = 2^3 \times 3 \times 5$.
c) $720 = 2^4 \times 3^2 \times 5$.
d) $5040 = 2^2 \times 3^2 \times 5 \times 7$.

6. If $m = p_1^{a_1} \dots p_k^{a_k}$, and $n = p_1^{b_1} \dots p_k^{b_k}$, where p_i are distinct primes and a_i, b_i are nonnegative, express (m, n) as $p_1^{c_1} \dots p_k^{c_k}$ by describing the c_i in terms of the a_i and b_i .

Solution: Let

$$m = 36 = 2^2 \times 3^2 \quad (1.5.6.1)$$

$$n = 720 = 2^4 \times 3^2 \times 5 \quad (1.5.6.2)$$

Then,

$$k = 3 \quad (1.5.6.3)$$

$$p_1 = 2, p_2 = 3, p_3 = 5 \quad (1.5.6.4)$$

$$a_1 = 2, a_2 = 2, a_3 = 0 \quad (1.5.6.5)$$

$$b_1 = 4, b_2 = 2, b_3 = 1 \quad (1.5.6.6)$$

and

$$(36, 720) = 2^2 \times 3^2 \quad (1.5.6.7)$$

$$\implies c_i = \min(a_i, b_i) \quad (1.5.6.8)$$

7. Define the least common multiple (LCM) of positive integers m and n to be the smallest positive integer v such that both $m \mid v$ and $n \mid v$.

- a) Show that

$$v = \frac{mn}{(m, n)} \quad (1.5.7.1)$$

- b) In terms of the factorization of m and n given in problem 1.5.6 what is v ?

8. Find the least common multiples of the following pairs

- a) (116, -84)
b) (85, 65)
c) (72, 26)
d) (72, 25)

Solution:

- a) 2436.
b) 1105.
c) 936.
d) 1800.

9. If $m, n > 0$ are two integers, show that we can find integers u, v with $-\frac{n}{2} \leq v \leq \frac{n}{2}$ such that $m = un + v$.

10. To check that a given integer $n > 1$ is a prime, prove that it is enough to show that n is not divisible by any prime p with $p \leq \sqrt{n}$.

1.6 Mathematical Induction

1. Prove that

$$1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6} \quad (1.6.1.1)$$

by induction.

Solution: $P(n+1)$ is

$$\begin{aligned} &1^2 + 2^2 + 3^2 + \dots + n^2 + (n+1)^2 \\ &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\ &= (n+1) \left(\frac{2n^2 + 7n + 7}{6} \right) \\ &= \frac{(n+1)(n+2)(2n+3)}{6} \quad (1.6.1.2) \end{aligned}$$

which is true. Hence, the given proposition is true for all $n \geq 1$

2. Prove that

$$1^3 + 2^3 + 3^3 + \dots + n^3 = \left[\frac{n(n+1)}{2} \right]^2 \quad (1.6.2.1)$$

by induction.

Solution: $P(n+1)$ is

$$\begin{aligned} &1^3 + 2^3 + 3^3 + \dots + n^3 + (n+1)^3 \\ &= \left[\frac{n(n+1)}{2} \right]^2 + (n+1)^3 \\ &= (n+1)^2 \left(\frac{n^2 + 4n + 4}{4} \right) \\ &= \left[\frac{(n+1)(n+2)}{2} \right]^2 \quad (1.6.2.2) \end{aligned}$$

which is true. Hence, the given proposition is true for all $n \geq 1$.

3. Prove that a set having $n \geq 2$ elements has $\frac{n(n-1)}{2}$ subsets having exactly 2 elements.
4. Prove that a set having $n \geq 3$ elements has $\frac{n(n-1)(n-2)}{3}$ subsets having exactly 3 elements.
5. If $n \geq 4$ and S is a set having n elements, guess how many subsets having exactly 4 elements are there in S . Then verify your guess using mathematical induction.

6. If p is a prime and $p \mid (a_1 a_2 a_3 \dots a_n)$, then prove using induction that $p \mid a_i$ for some i with $1 \leq i \leq n$.
7. If $a \neq 1$, prove that

$$1 + a + a^2 + \cdots + a^n = \frac{(a^{n+1} - 1)}{a - 1} \quad (1.6.7.1)$$

by induction.

Solution: $P(n + 1)$ can be expressed as

$$\begin{aligned} 1 + a + a^2 + \cdots + a^n + a^{n+1} \\ &= \frac{(a^{n+1} - 1)}{a - 1} + a^{n+1} \\ &= \frac{(a^{n+2} - 1)}{a - 1} \quad (1.6.7.2) \end{aligned}$$

upon simplification. Hence, the given proposition is true for all $n \geq 1$.

8. By induction, show that

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n \cdot (n+1)} = \frac{n}{n+1} \quad (1.6.8.1)$$

Solution: $P(n + 1)$ can be expressed as

$$\begin{aligned} \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n \cdot (n+1)} + \frac{1}{(n+1) \cdot (n+2)} \\ = \frac{n}{n+1} + \frac{1}{(n+1) \cdot (n+2)} \\ = \frac{1}{n+1} \left[n + \frac{1}{n+2} \right] \\ = \frac{n+1}{n+2} \quad (1.6.8.2) \end{aligned}$$

upon simplification. Hence, the given proposition is true for all $n \geq 1$.

9. Suppose that $P(n)$ is a proposition about positive integers n such that $P(n_0)$ is valid, and if $P(k)$ is true, so must be $P(k+1)$. What can you say about $P(n)$? Prove your statement.
10. Let $P(n)$ be a proposition about integers n such that $P(1)$ is true and such that if $P(j)$ is true for all positive integers $j < k$, then $P(k)$ is true. Prove that $P(n)$ is true for all positive integers n .
11. Given an example of a proposition that is *not* true for any positive integer, yet for which the induction step holds.

12. Prove by induction that a set having n elements has exactly 2^n subsets.

Solution: Let $S = \{1, 2\}$. Then the subsets are

$$\{\phi\}, \{1\}, \{2\}, \{1, 2\} \quad (1.6.12.1)$$

For $S = \{1, 2, 3\}$, the subsets are

$$\{\phi\}, \{1\}, \{2\}, \{1, 2\} \quad (1.6.12.2)$$

$$\{3\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\} \quad (1.6.12.3)$$

Thus $P(n+1)$ can be expressed as

$$2^n + 2^n = 2^{n+1} \quad (1.6.12.4)$$

Hence, the given proposition is true for all $n \geq 1$.

13. Prove by induction on n that $n^3 - n$ is always divisible by 3.

Solution: $P(n + 1)$ can be expressed as

$$\begin{aligned}(n+1)^3 - (n+1) &= n^3 + 3n^2 + 3n + 1 - n - 1 \\ &= n^3 - n + 3(n^2 + n)\end{aligned}\tag{1.6.13.1}\tag{1.6.13.2}$$

which is divisible by 3. Hence, the given proposition is true for all $n \geq 1$.

14. If p is a prime number, then prove that $n^p - n$ is always divisible by p .

Solution: $P(n + 1)$ can be expressed as

$$(n+1)^p - (n+1) = n^p - n + p \sum_{k=1}^{n-1} {}^nC_k p^{k-1} \quad (1.6.14.1)$$

$$\implies p \mid [(n+1)^p - (n+1)] \quad (1.6.14.2)$$

Hence, the given proposition is true for all $n \geq 1$

15. Prove by induction that for a set having n elements the number of 1-1 mappings of this set onto itself is $n!$.

Solution: Let $S = \{a, b, c\}$. Then the possible 1-1 onto mappings are

$$\begin{array}{cccccc} \left\{ \begin{array}{l} a \mapsto b \\ b \mapsto c \\ c \mapsto a \end{array} \right. & \left\{ \begin{array}{l} a \mapsto b \\ b \mapsto a \\ c \mapsto c \end{array} \right. & \left\{ \begin{array}{l} a \mapsto c \\ b \mapsto b \\ c \mapsto a \end{array} \right. & \left\{ \begin{array}{l} a \mapsto a \\ b \mapsto c \\ c \mapsto b \end{array} \right. & \left\{ \begin{array}{l} a \mapsto a \\ b \mapsto b \\ c \mapsto c \end{array} \right. & \left\{ \begin{array}{l} a \mapsto b \\ b \mapsto c \\ c \mapsto a \end{array} \right. \\ & & & & & \\ & & & & & (1.6.15.1) \end{array}$$

1.7 Complex Numbers

1. Multiply

- a) $(6 - 7j)(8 + j)$
 b) $\left(\frac{2}{3} + \frac{3}{2}j\right)\left(\frac{2}{3} - \frac{3}{2}j\right)$
 c) $(6 + 7j)(8 - j)$

Solution:

a)

$$(6 - 7j)(8 + j) = \begin{pmatrix} 6 & 7 \\ -7 & 6 \end{pmatrix} \begin{pmatrix} 8 \\ 1 \end{pmatrix} \quad (1.7.1.1)$$

$$= \begin{pmatrix} 53 \\ -50 \end{pmatrix} = 53 - 50j \quad (1.7.1.2)$$

b)

$$\left(\frac{2}{3} + \frac{3}{2}j\right)\left(\frac{2}{3} - \frac{3}{2}j\right) = \begin{pmatrix} \frac{2}{3} & -\frac{3}{2} \\ \frac{3}{2} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} \frac{2}{3} \\ -\frac{3}{2} \end{pmatrix} \quad (1.7.1.3)$$

$$= \begin{pmatrix} \frac{97}{36} \\ 0 \end{pmatrix} = \frac{97}{36} \quad (1.7.1.4)$$

c)

$$\begin{aligned} (6 + 7j)8 - j &= [(6 - 7j)8 + j]^* \quad (1.7.1.5) \\ &= (53 - 50j)^* = 53 + 50j \quad (1.7.1.6) \end{aligned}$$

2. Find z^{-1} for

- a) $z = 6 + 8j$
 b) $z = 6 - 8j$
 c) $z = \frac{1}{\sqrt{2}}(1 + j)$

Solution:

a)

$$z^{-1} = \frac{z^*}{|z|^2} = \frac{6 - 8j}{100} \quad (1.7.2.1)$$

b)

$$z^{-1} = \frac{6 + 8j}{100} \quad (1.7.2.2)$$

c)

$$z^{-1} = \frac{1 - j}{\sqrt{2}} \quad (1.7.2.3)$$

3. Show that

$$(z^*)^{-1} = (z^{-1})^* \quad (1.7.3.1)$$

Solution: Since

$$zz^{-1} = 1, \quad (1.7.3.2)$$

$$(zz^{-1})^* = 1 \quad (1.7.3.3)$$

$$\Rightarrow (z)^*(z^{-1})^* = 1 \quad (1.7.3.4)$$

yielding (1.7.3.1).

4. Find

$$(\cos \theta + j \sin \theta)^{-1} \quad (1.7.4.1)$$

Solution:

$$(\cos \theta + j \sin \theta)^{-1} = \cos \theta - j \sin \theta \quad (1.7.4.2)$$

5. Verify the following

a) $(z^*)^* = z$

b) $(z + w)^* = z^* + w^*$

c) $z + z^* = 2\text{Re}(z)$

d) $z - z^* = 2j\text{Im}(z)$

Solution:

a) For

$$z = a + jb, \quad (1.7.5.1)$$

$$z^* = a - jb, \quad (1.7.5.2)$$

$$\Rightarrow (z^*)^* = a + jb = z \quad (1.7.5.3)$$

b) For

$$z = z_1 + jz_2 \quad (1.7.5.4)$$

$$w = w_1 + jw_2,$$

$$(z + w)^* = (z_1 + jz_2 + w_1 + jw_2)^* \quad (1.7.5.5)$$

$$= (z_1 - jz_2) + (w_1 - jw_2) \quad (1.7.5.6)$$

$$= z^* + w^* \quad (1.7.5.7)$$

c) For

$$z = a + jb, \quad (1.7.5.8)$$

$$z^* = a - jb, \quad (1.7.5.9)$$

$$\Rightarrow (z + z^*) = a + jb + a - jb \quad (1.7.5.10)$$

$$= 2a = 2\text{Re}(z) \quad (1.7.5.11)$$

d) For

$$z = a + jb, \quad (1.7.5.12)$$

$$z^* = a - jb, \quad (1.7.5.13)$$

$$\Rightarrow (z - z^*) = a + jb - a - jb \quad (1.7.5.14)$$

$$= 2jb = 2j\text{Im}(z) \quad (1.7.5.15)$$

6. Show that z is real if and only if $z^* = z$ and is purely imaginary if and only if $z^* = -z$.

Solution: Let

$$z = a + jb. \quad (1.7.6.1)$$

Then

$$z^* = a - jb. \quad (1.7.6.2)$$

If

$$z^* = z, \quad (1.7.6.3)$$

$$a + jb = a - jb \quad (1.7.6.4)$$

$$\implies b = 0 \quad (1.7.6.5)$$

and z is real. If z is real,

$$z = a \quad (1.7.6.6)$$

$$\implies z^* = a \quad (1.7.6.7)$$

$$\text{or, } z = z^* \quad (1.7.6.8)$$

Similarly, the other property can be proved.

7. Verify the commutative law of multiplication $zw = wz$ in \mathbb{C} .

Solution: Let

$$z = a + jb \quad (1.7.7.1)$$

$$w = x - jy \quad (1.7.7.2)$$

Then

$$zw = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad (1.7.7.3)$$

$$= \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \quad (1.7.7.4)$$

$$= wz \quad (1.7.7.5)$$

8. Show that for $z \neq 0$, $|z|^{-1} = \frac{1}{|z|}$.

Solution: Let

$$z = re^{j\theta}. \quad (1.7.8.1)$$

Then

$$z^{-1} = \frac{1}{r} e^{-j\theta} \quad (1.7.8.2)$$

$$\implies |z^{-1}| = \frac{1}{r} \quad (1.7.8.3)$$

9. Find

a) $|6 - 4j|$.

b) $\left| \frac{1}{2} + \frac{2}{3}j \right|$.

c) $\left| \frac{1}{\sqrt{2}} (1 + j) \right|$

Solution:

a)

$$|6 - 4j| = \sqrt{6^2 + 4^2} = 2\sqrt{13} \quad (1.7.9.1)$$

b)

$$\left| \frac{1}{2} + \frac{2}{3}j \right| = \frac{5}{6} \quad (1.7.9.2)$$

c)

$$\left| \frac{1}{\sqrt{2}} (1 + j) \right| = \frac{1}{\sqrt{2}} |(1 + j)| = 1 \quad (1.7.9.3)$$

10. Show that $|z^*| = |z|$.

Solution: Let

$$z = re^{j\theta} \quad (1.7.10.1)$$

Then

$$z^* = re^{-j\theta} \quad (1.7.10.2)$$

$$\implies |z^*| = r = |z|. \quad (1.7.10.3)$$

11. Find the polar form for

a) $z = \frac{\sqrt{2}}{2} - \frac{1}{\sqrt{2}}j$.

b) $z = 4j$.

c) $z = \frac{6}{\sqrt{2}} + \frac{6}{\sqrt{2}}j$.

d) $z = -\frac{13}{2} + \frac{39}{2\sqrt{3}}j$.

Solution:

a)

$$|z| = \sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}. \quad (1.7.11.1)$$

$$= 1 \quad (1.7.11.2)$$

and

$$\angle z = -\tan^{-1} \frac{\frac{1}{\sqrt{2}}}{\frac{\sqrt{2}}{2}} \quad (1.7.11.3)$$

$$= \frac{\pi}{4} \quad (1.7.11.4)$$

b)

$$|z| = 4, \angle z = \frac{\pi}{2}. \quad (1.7.11.5)$$

c)

$$|z| = \frac{6}{\sqrt{2}}, \angle z = \frac{\pi}{4}. \quad (1.7.11.6)$$

d)

$$|z| = \frac{13}{2} \sqrt{1+3} \quad (1.7.11.7)$$

$$= 13 \quad (1.7.11.8)$$

and

$$\angle z = \pi - \tan^{-1} \frac{\frac{39}{2\sqrt{3}}}{\frac{13}{2}} \quad (1.7.11.9)$$

$$= \pi - \tan^{-1} \sqrt{3} = \frac{2\pi}{3} \quad (1.7.11.10)$$

12. Prove that

$$\left(\cos\left(\frac{\theta}{2}\right) + j \sin\left(\frac{\theta}{2}\right) \right)^2 = \cos(\theta) + j \sin(\theta) \quad (1.7.12.1)$$

Solution: The L.H.S can be expressed as

$$(e^{j\theta})^2 = e^{j2\theta} \quad (1.7.12.2)$$

13. Show that

$$\left(\frac{1}{2} + \frac{\sqrt{3}}{2}j \right)^3 = -1 \quad (1.7.13.1)$$

Solution:

$$\frac{1}{2} + \frac{\sqrt{3}}{2}j = e^{j\frac{\pi}{3}} \quad (1.7.13.2)$$

$$\Rightarrow \left(e^{j\frac{\pi}{3}} \right)^3 = e^{j\pi} = -1 \quad (1.7.13.3)$$

14. Show that

$$(\cos(\theta) + j \sin(\theta))^m = \cos(m\theta) + j \sin(m\theta) \quad (1.7.14.1)$$

for all integers m . **Solution:** It is easy to verify that

$$(\cos(\theta) + j \sin(\theta))^2 = \cos(2\theta) + j \sin(2\theta) \quad (1.7.14.2)$$

Then

$$\begin{aligned} (\cos(\theta) + j \sin(\theta))^{k+1} &= (\cos(\theta) + j \sin(\theta))^k \\ &\quad (\cos(\theta) + j \sin(\theta)) \\ &= \cos[(k+1)\theta] + j \sin((k+1)\theta) \end{aligned} \quad (1.7.14.3)$$

By induction, (1.7.14.1) is proved.

15. Show that

$$(\cos(\theta) + j \sin(\theta))^r = \cos(r\theta) + j \sin(r\theta) \quad (1.7.15.1)$$

for all rational numbers r . **Solution:** Let

$$r = \frac{m}{n}, (\cos(\theta) + j \sin(\theta))^{\frac{1}{n}} = \cos(\alpha) + j \sin(\alpha) \quad (1.7.15.2)$$

Then

$$(\cos(\alpha) + j \sin(\alpha))^n = (\cos(\theta) + j \sin(\theta)) \quad (1.7.15.3)$$

$$\Rightarrow \cos(n\alpha) + j \sin(n\alpha) = (\cos(\theta) + j \sin(\theta)) \quad (1.7.15.4)$$

$$\text{or, } \alpha = \frac{\theta}{n} \quad (1.7.15.5)$$

yielding

$$(\cos(\theta) + j \sin(\theta))^{\frac{1}{n}} = \cos\left(\frac{\theta}{n}\right) + j \sin\left(\frac{\theta}{n}\right) \quad (1.7.15.6)$$

Using (1.7.14.1) and (1.7.15.6),

$$(\cos(\theta) + j \sin(\theta))^{\frac{m}{n}} = \cos\left(\frac{m\theta}{n}\right) + j \sin\left(\frac{m\theta}{n}\right) \quad (1.7.15.7)$$

16. If $z \in \mathbb{C}$ and $n \geq 1$ is any positive integer, show that there are n distinct complex numbers such that $z = w^n$. **Solution:** Let

$$z = \cos(\theta) + j \sin(\theta) \quad (1.7.16.1)$$

then using (1.7.15.6),

$$w = \cos\left(\frac{2\pi k + \theta}{n}\right) + j \sin\left(\frac{2\pi k + \theta}{n}\right), k = 0, \dots, n-1 \quad (1.7.16.2)$$

which are the distinct roots.

17. Find the necessary and sufficient condition on k such that

$$\left(\cos\left(\frac{2\pi k}{n}\right) + j \sin\left(\frac{2\pi k}{n}\right) \right)^n = 1 \quad \text{and} \quad (1.7.17.1)$$

$$\left(\cos\left(\frac{2\pi k}{n}\right) + j \sin\left(\frac{2\pi k}{n}\right) \right)^m \neq 1 \quad 0 < m < n \quad (1.7.17.2)$$

Solution: From the above equations, using (1.7.14.1),

$$\frac{mk}{n} \notin \mathbb{Z} \quad (1.7.17.3)$$

18. Viewing the x - y plane as the set of all complex numbers $x + jy$, show that multiplication by j induces as 90° rotation of the $x - y$ plan in counterclockwise direction.

Solution: The given multiplication can be expressed using matrices as

$$\begin{pmatrix} \cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad (1.7.18.1)$$

which is the multiplication of $\begin{pmatrix} x \\ y \end{pmatrix}$ with a 90° rotation matrix.

19. In problem (1.7.18), interpret geometrically what multiplication by the complex number $a + jb$ does to the $x - y$ plane.

Solution: The multiplication can be represented as

$$\sqrt{a^2 + b^2} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad (1.7.19.1)$$

where

$$\cos \theta = \frac{a}{\sqrt{a^2 + b^2}} \sin \theta = \frac{b}{\sqrt{a^2 + b^2}} \quad (1.7.19.2)$$

Geometrically, multiplication by $a + jb$ results in rotation by θ and scaling by $\sqrt{a^2 + b^2}$.

20. Prove that

$$|z + w|^2 + |z - w|^2 = 2(|z|^2 + |w|^2) \quad (1.7.20.1)$$

Solution: Since

$$|z + w|^2 = (z + w)^* (z + w) \quad (1.7.20.2)$$

$$= |z|^2 + |w|^2 + 2z^* w \quad (1.7.20.3)$$

and

$$|z - w|^2 = (z - w)^* (z - w) \quad (1.7.20.4)$$

$$= |z|^2 + |w|^2 - 2z^* w, \quad (1.7.20.5)$$

$$|z + w|^2 + |z - w|^2 = 2(|z|^2 + |w|^2) \quad (1.7.20.6)$$

21. Consider the set $A = a + bJ, a, b \in \mathbb{Z}$. Prove that there is 1-1 correspondence of A onto \mathbb{N} .

22. If a is a (complex) root of the polynomial

$$x^n + \alpha_1 x^{n-1} + \cdots + \alpha_{n-1} x + \alpha_n, \quad (1.7.22.1)$$

where the α_i are real, show that \bar{a} must also be a root.

Solution: From the given information,

$$\bar{a}^n + \alpha_1 \bar{a}^{n-1} + \cdots + \alpha_{n-1} \bar{a} + \alpha_n = 0 \quad (1.7.22.2)$$

Thus, \bar{a} is also a root of the given polynomial.

23. Find the necessary and sufficient conditions on z and w in order that

$$|z + w| = |z| + |w| \quad (1.7.23.1)$$

Solution:

$$|z + w|^2 = |z|^2 + |w|^2 + 2z^* w \quad (1.7.23.2)$$

$$(|z| + |w|)^2 = |z|^2 + |w|^2 + 2|z||w| \quad (1.7.23.3)$$

If the above expressions are equal,

$$z^* w = |z||w| \quad (1.7.23.4)$$

which is the desired condition.

24. Find the necessary and sufficient conditions on z_i in order that

$$\left| \sum_{i=1}^k z_i \right| = \sum_{i=1}^k |z_i| \quad (1.7.24.1)$$

Solution:

$$\left| \sum_{i=1}^k z_i \right|^2 = \sum_{i=1}^k |z_i|^2 + 2 \sum_{\substack{i=1, j=1 \\ i \neq j}}^k z_i^* z_j \quad (1.7.24.2)$$

$$\left(\sum_{i=1}^k |z_i| \right)^2 = \sum_{i=1}^k |z_i|^2 + 2 \sum_{\substack{i=1, j=1 \\ i \neq j}}^k |z_i| |z_j| \quad (1.7.24.3)$$

From (1.7.24.2) and (1.7.24.3),

$$\begin{aligned} \sum_{i=1}^k |z_i|^2 + 2 \sum_{\substack{i=1, j=1 \\ i \neq j}}^k z_i^* z_j \\ = \sum_{i=1}^k |z_i|^2 + 2 \sum_{\substack{i=1, j=1 \\ i \neq j}}^k |z_i| |z_j| \\ \Rightarrow \sum_{\substack{i=1, j=1 \\ i \neq j}}^k z_i^* z_j \\ = \sum_{\substack{i=1, j=1 \\ i \neq j}}^k |z_i| |z_j| \end{aligned} \quad (1.7.24.4)$$

which is the desired condition.

25. The complex number θ is said to have *order* $n \geq 1$ if $\theta^n = 1$ and $\theta^m \neq 1$ for $0 < m < n$. Show that if θ has order n and $\theta^k = 1$, where $k > 0$, then $n|k$.

Solution: From the given information,

$$\theta^n = \theta^k = 1, k \geq n \quad (1.7.25.1)$$

If $n \nmid k, k = mn + p, 0 < p < n$, Then,

$$\theta^k = \theta^{mn+p} = \theta^p \neq 1, \quad (1.7.25.2)$$

which is a contradiction, Hence, $n | k$.

26. Find all complex numbers θ having order n .

Solution: If

$$\theta^n = 1, \quad (1.7.26.1)$$

$$\theta^n = e^{j2\pi r}, 0 \leq r < n \quad (1.7.26.2)$$

yielding

$$\theta = \exp\left(j\frac{2\pi r}{n}\right) 0 \leq r < n \quad (1.7.26.3)$$

2 GROUPS

2.1 Definitions and Examples of Groups

1. Determine if the following sets G with the operation indicated form a group. If not, point out which of the group axioms fail.

- G = set of all integers, $a * b = a - b$.
- G = set of all integers, $a * b = a + b + ab$
- G = set of nonnegative integers, $a * b = a + b$.
- G = set of all rational numbers $\neq -1$, $a * b = a + b + ab$.
- G = set of all rational numbers with denominator divisible by 5 (written so that numerator and denominator are relatively prime), $a * b = a + b$.
- G a set having more than one element, $a * b = a \forall a, b \in G$.

Solution: The properties of a group are

- $a, b \in G \implies a * b \in G$.
 - $a, b, c \in G \implies a * (b * c) = (a * b) * c \in G$.
 - $\exists e \in G \ni a * i = i * a = a \forall a \in G$.
 - $a \in G \implies \exists b \in G \ni a * b = b * a = i$.
- a) From 2.1.1b,

$$a * (b * c) = a - (b - c) = a - b + c \quad (2.1.1.1)$$

$$(a * b) * c = (a - b) - c = a - b - c \quad (2.1.1.2)$$

$$\implies a * (b * c) \neq (a * b) * c \quad (2.1.1.3)$$

Thus, G is not a group.

b) i) From property 2.1.1b,

$$a * (b * c) = a * (b + c + bc) \quad (2.1.1.4)$$

$$= a + b + c + bc + a(b + c + bc) \quad (2.1.1.5)$$

$$= a + b + c + ab + bc + ca + abc \quad (2.1.1.6)$$

$$(a * b) * c = (a + b + ab) + c + c(a + b + ab) \quad (2.1.1.7)$$

$$= a + b + c + ab + bc + ca + abc \quad (2.1.1.8)$$

Thus, property 2.1.1b is satisfied.

ii) Since

$$a * i = a + i + ai \quad (2.1.1.9)$$

$$i * a = a + i + ai \quad (2.1.1.10)$$

property 2.1.1c is satisfied.

iii)

$$a * i = a + i + ai \quad (2.1.1.11)$$

$$i * a = a + i + ai \quad (2.1.1.12)$$

Thus, for property 2.1.1c to be satisfied,

$$i * a = a \quad (2.1.1.13)$$

$$\implies a + i + ai = a \quad (2.1.1.14)$$

$$\text{or, } i(1 + a) = 0 \quad (2.1.1.15)$$

$$\implies i = 0 \quad (2.1.1.16)$$

iv) If

$$a * b = b * a = i, \quad (2.1.1.17)$$

$$a + b + ab = 0 \quad (2.1.1.18)$$

$$\implies b = -\frac{a}{1 + a} \quad (2.1.1.19)$$

which is not finite for $a = -1$. Also, $b \notin G$ for $a = 1$. Thus, property 2.1.1d is violated and G is not a group.

c) In this case, for property 2.1.1c to be satisfied,

$$a * i = i * a = a, \quad (2.1.1.20)$$

$$\implies a + i = a \quad (2.1.1.21)$$

$$\text{or, } i = 0 \quad (2.1.1.22)$$

From property 2.1.1c,

$$a + b = 0 \implies b = -a \quad (2.1.1.23)$$

Thus, G is a group.

d) From problem 2.1.1b, it is easy to verify that G is a group, since we are now considering rational numbers.

e) From property 2.1.1c,

$$a * i = a, i * a = i \quad (2.1.1.24)$$

$$\implies a = i \quad (2.1.1.25)$$

From property 2.1.1d,

$$a * b = b * a = i \quad (2.1.1.26)$$

$$\implies i * b = i, b * i = b = i \quad (2.1.1.27)$$

Thus, G has only a single element i which is a contradiction. So G is not a group.

2. Let G be the set of all mappings

$$T_{a,b} \mid T_{a,b}(r) = ar + b, \quad a \neq 0, b, r \in \mathbb{R}, \quad (2.1.2.1)$$

show that the set $H = T_{a,b} \mid a = \pm 1, b \in \mathbb{R}$ forms a group under the $*$ of G .

Solution:

a)

$$(T_{a,b} * T_{a,c})(r) = a(ar + c) + b \quad (2.1.2.2)$$

$$= a^2r + ac + b = r + ac + b \quad (2.1.2.3)$$

$$= T_{1,ac+b} \in G \quad (2.1.2.4)$$

Similarly,

$$(T_{a,c} * T_{a,b})(r) = r + ab + c \quad (2.1.2.5)$$

b) If

$$(T_{a,b} * T_{a,c})(r) = T_{a,b}, \quad (2.1.2.6)$$

$$r + ab + c = ar + b \quad (2.1.2.7)$$

$$\implies a = 1, c = 0 \quad (2.1.2.8)$$

Thus,

$$i = T_{1,0} \quad (2.1.2.9)$$

c) If

$$(T_{a,b} * T_{a,c})(r) = (T_{a,c} * T_{a,b})(r) = T_{1,0}, \quad (2.1.2.10)$$

$$r + ab + c = r + ac + b = r \quad (2.1.2.11)$$

$$\implies b = \pm c \quad (2.1.2.12)$$

d) From (2.1.2.4),

$$T_{a,b} * (T_{a,c}(r) * T_{a,d})(r) = T_{a,b} * T_{1,ad+c} \quad (2.1.2.13)$$

$$= a(r + ad + b + c) + b \quad (2.1.2.14)$$

$$= ar + ab + ac + b + d \quad (2.1.2.15)$$

Similarly,

$$(T_{a,b} * T_{a,c})(r) * T_{a,d}(r) = T_{1,ac+b} * T_{a,d} \quad (2.1.2.16)$$

$$= (ar + d) + ac + b \quad (2.1.2.17)$$

$$= ar + ab + ac + b + d \quad (2.1.2.18)$$

which satisfies the associativity property.

Thus, $T_{a,b}^{-1} = T_{a,\pm b}$. and G is a group.

3. Let $H \subset G$, for G in problem 2.1.2d and $H = \{T_{a,b} \in G \mid a \text{ is rational, } b \text{ any real}\}$. Show that H is also a group.

4. Let $K \subset G$, for G in problem 2.1.2d and $K = \{T_{1,b} \in G \mid b \in \mathbb{R}\}$. Show that K is an Abelian group.

Solution: From (2.1.2.4),

$$T_{1,b} * T_{1,c} = T_{1,b+c} \quad (2.1.4.1)$$

$$= T_{1,c+b} \quad (2.1.4.2)$$

Thus, K is an Abelian group.

5. Let $S = \{(x, y) \mid x, y \in \mathbb{R}\}$ and consider $f, g \in A(S)$ defined by $f(x, y) = (-x, y)$ and $g(x, y) = (-y, x)$; f is the reflection about the y -axis and g is the rotation through 90° in a counterclockwise direction about the origin. We then define $G = \{f^i g^j \mid i = 0, 1; j = 0, 1, 2, 3\}$, and let $*$ in G be the product of elements in $A(S)$. Clearly, $f^2 = g^4 = \text{identity mapping}$; $(f * g)(x, y) = (fg)(x, y) = f(g(x, y)) = f(-y, x) = (y, x)$ and $(g * f)(x, y) = g(f(x, y)) = g(-x, y) = (-y, -x)$. Prove that $g * f = f * g^{-1}$, and that G is a group, is nonabelian, and is of order 8.

Solution: Let

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}. \quad (2.1.5.1)$$

Then

$$f(\mathbf{x}) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} \quad (2.1.5.2)$$

$$g(\mathbf{x}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{x} \quad (2.1.5.3)$$