Abstract Algebra

Contents

1 Things Familiar and Less Familiar

1.1 Introduction

1. let S be a set having an operation * which assigns an element a*b of S for any $a, b \in S$. Let us assume that the following two rules hold:
   a) If a, b are any objects in S, then $a * b = a$.
   b) If a, b are any objects in S, then $a*b = b*a$.

   Show that S can have at most one object.
   Solution: From condition 1.1.1a, interchanging a, b,
   $$b * a = b \tag{1.1.1}$$
   and from condition 1.1.1b,
   $$b * a = a * b \tag{1.1.1}$$
   But from condition 1.1.1a,
   $$a * b = a \implies a = b \tag{1.1.1}$$
   Thus, S can have at most one object.
2. Let S be the set of all integers $0, \pm1, \pm2, \dots, \pm n, \dots$. For $a, b \in S$, define * by
   $$a * b = a - b \tag{1.1.2}$$
   Verify the following
   a) $a * b \neq b * a$ unless $a = b$
   b) $(a*b)*c \neq a*(b*c)$ in general. Under what conditions on a, b, c is
   $$(a * b) * c \neq a * (b * c) \quad ? \tag{1.1.2}$$
   c) The integer a has the property that $a*0 = a$ for every $a \in S$.
   d) For $a \in S, a * a = 0$.

   Solution:
   a)
   $$a * b = b * a \tag{1.1.2}$$
   $$\implies a - b = b - a \tag{1.1.2}$$
   $$\text{or, } a = b \tag{1.1.2}$$
   b) Let $a = 1, b = 2, c = 4$. Then,
   $$a * b = -1, (a * b) * c = -1 - 4 = -5 \tag{1.1.2}$$
   $$b * c = -2, a * (b * c) = 1 + 2 = 3 \neq -5 \tag{1.1.2}$$
   Thus, for the given condition to be satisfied,
   $$(a - b) - c = a - (b - c) \tag{1.1.2}$$
   $$\implies c = 0 \tag{1.1.2}$$
   c)
   $$a * 0 = a - 0 = a \tag{1.1.2}$$
   d)
   $$a * a = a - a = 0 \tag{1.1.2}$$
3. Let S consist of the two objects □ and △. We define the operatin * on S by subjecting □ and △ to the following conditions.
   a) $\square * \triangle = \triangle = \triangle * \square$
   b) $\square * \square = \square$
   c) $\triangle * \triangle = \square$

   Verify by explicit calculation that if a, b, c are any elements of S, (i.e. a, b, c can be any of □ or △), then
   a) $a * b$ is in S
   b) $(a * b) * c = a * (b * c)$
   c) $a * b = b * a$
   d) There is a particular a in S such that $a*b = b*a = b$ for all $b \in S$
   e) Given $b \in S, b * b = a$, where a is the particular element in Part 1.1.3d.

   Solution: Let $\square = 1, \triangle = -1$. These satisfy all the given conditions.
   a) $a * b \in [1, -1] \in S$.

b) Writing the truth table, $(a * b) * c = a * (b * c)$.
c) $a * b = b * a$ can be verified by writing the truth table.
d) For $a = 1, a * b = b * a = b$, for all $b \in S$.
e) For $a = 1$, if $b = -1, b * b = 1 = a$. This can be shown to be true for $b = 1$ as well.

1.2 Set Theory

1. Describe the following sets verbally
   a) $S = \{\text{Mercury, Venus, Earth, } \dots, \text{ Pluto}\}$
   b) $S = \{\text{Andhra Pradesh, Uttar Pradesh, } \dots, \text{ Assam}\}$

   Solution:
   a) Planets
   b) Indian states
2. Describe the following sets verbally
   a) $S = \{2, 4, 6, 8, \dots\}$
   b) $S = \{2, 4, 8, 16, \dots\}$
   c) $S = \{1, 4, 9, 16, 25, 36 \dots\}$

   Solution:
   a) Even numbers
   b) Powers of 2
   c) Squares of positive integers
3. If A is the set of all residents of India, B the set of all Sri Lankan citizens, and C the set of all women in the world, describe the sets $ABC, A - B, A - C, C - A$ verbally.
   Solution:
   a) ABC is the set of all women residents of India who are citizens of Sri Lanka.
   b) $A - B = AB'$ is the set of all residents of India who are not Sri Lankan citizens.
   c) $A - C = AC'$ is the set of all male residents of India.
   d) $C - A = CA'$ is the set of all women who are not residing in India.
4. If $A = \{1, 4, 7, a\}$ and $B = \{3, 4, 9, 11\}$ and you have been told that $AB = \{4, 9\}$, then what must a be?
   Solution: $a = 9$
5. If $A \subset B, B \subset C$, prove that $A \subset C$
   Solution: From the given information,

$$A + P = B, AP = 0, B + Q = C, BQ = 0 \tag{1.2.5.1}$$

$$\implies B + Q = A + P + Q = C, \tag{1.2.5.2}$$

$\because BQ = 0$,

$$AQ + PQ = 0 \implies AQ = 0, PQ = 0 \tag{1.2.5.3}$$

Hence,

$$A(P + Q) = 0 \implies A \subset C \tag{1.2.5.4}$$

6. If $A \subset B$ prove that $A \cup C \subset B \cup C$ for any set C.
   Solution: From the given information, there exists P such that

$$A + P = B, AP = 0 \tag{1.2.6.1}$$

Also,

$$B + C = A + P + C \tag{1.2.6.2}$$

$$\implies A + C \subset B + C \tag{1.2.6.3}$$

7. Show that

$$A \cup B = B \cup A \tag{1.2.7.1}$$

$$A \cap B = B \cap A \tag{1.2.7.2}$$

8. Prove that

$$(A - B) \cup (B - A) = (A \cup B) - (A \cap B) \tag{1.2.8.1}$$

Solution: Since

$$A - B = AB', \tag{1.2.8.2}$$

$$(A - B) \cup (B - A) = AB' + BA' \tag{1.2.8.3}$$

Also,

$$(A \cup B) - (A \cap B) = (A + B)(AB)' \tag{1.2.8.4}$$

$$= (A + B)(A' + B') \tag{1.2.8.5}$$

$$= AB' + BA' \tag{1.2.8.6}$$

9. Prove that

$$(A) \cap (B \cup C) = (A \cap B) \cup (A \cap C) \tag{1.2.9.1}$$

Solution:

$$LHS = A(B + C) = AB + AC = RHS \tag{1.2.9.2}$$

10. Prove that

$$(A) \cup (B \cap C) = (A \cup B) \cap (A \cup C) \tag{1.2.10.1}$$

Solution:

$$
\begin{aligned}
LHS &= A + BC && (1.2.10.2)\\
RHS &= (A+B)(A+C) && (1.2.10.3)\\
&= A + A(B+C) + BC && (1.2.10.4)\\
&= A(1+B+C) + BC && (1.2.10.5)\\
&= LHS && (1.2.10.6)
\end{aligned}
$$

11. Write down all the subsets of $S = \{1,2,3,4\}$.
Solution: Write a program for this.
12. If C is a subset of S, let C' denote the complement of C in S. Prove the *De Morgan Rules* for subsets A, B of S, namely,
a) $(A \cup B)' = A' \cap B'$
b) $(A \cap B)' = A' \cup B'$
Solution:
a)

$$
\begin{aligned}
(A+B)A'B' &= AA'B' + BA'B' && (1.2.12.1)\\
&= 0 && (1.2.12.2)
\end{aligned}
$$

b) Substituting $A = A', B = B'$ in the above, the second result is obtained.
13. Let S be a set. For any to subsets of S, we define

$$
A \oplus B = (A - B) \cup (B \cup A) \qquad (1.2.13.1)
$$

Prove that
a) $A \oplus B = B \oplus A$.
b) $A \oplus \Phi = A$.
c) $A \cdot A = A$.
d) $A \oplus A = \Phi$.
e) $A \oplus (B \oplus C) = (A \oplus B) \oplus C$.
f) If $A \oplus B = A \oplus C$, then $B = C$.
g) $A \cdot (B + C) = A \cdot B + A \cdot C$.
Solution: All can be proved using boolean logic.
14. If C is a finite set, let $m(C)$ denote the number of elements in C. If A, B are finite sets, prove that

$$
m(A \cup B) = m(A) + m(B) - m(A \cap B) \qquad (1.2.14.1)
$$

Solution:

$$
\begin{aligned}
A'B' &= (A+B)' && (1.2.14.2)\\
\implies m(A'B') &= m((A+B)') && (1.2.14.3)\\
&= 1 - m(A+B) && (1.2.14.4)\\
\because A + B &= A(B+B') + B && (1.2.14.5)\\
&= B(A+1) + AB' && (1.2.14.6)\\
&= B + AB' && (1.2.14.7)\\
\implies m(A+B) &= m(B + AB') && (1.2.14.8)\\
&= m(B) + m(AB') && (1.2.14.9)\\
&\because B(AB') = 0 && (1.2.14.10)
\end{aligned}
$$

$$
A = A(B+B') = AB + AB' \qquad (1.2.14.11)
$$

and

$$
(AB)(AB') = 0, \because BB' = 0 \qquad (1.2.14.12)
$$

Hence, AB and AB' are mutually exclusive and

$$
\begin{aligned}
m(A) &= m(AB) + m(AB') && (1.2.14.13)\\
\implies m(AB') &= m(A) - m(AB) && (1.2.14.14)
\end{aligned}
$$

Substituting (1.2.14.14) in (1.2.14.10),

$$
m(A+B) = m(A) + m(B) - m(AB) \qquad (1.2.14.15)
$$

15. For three finite sets A, B, C, find a formula for $m(A \cup b \cup C)$. **Solution:** Extend the above.
16. Take a shot at finding $m\left(\cup_{i=1}^{n} A_i\right)$.
17. Show that if 80% of all Indians have gone to high school and 70% of all Indians read a daily newspaper, then *at least* 50% of all Indians have both gone to high school and read a daily newspaper.
Solution: Let A represent high school and B represent newspaper. Then,

$$
\Pr(AB) = \Pr(A) + \Pr(B) - \Pr(A+B) \qquad (1.2.17.1)
$$

Since

$$
\Pr(A+B) \le 1, \qquad (1.2.17.2)
$$

$$
\Pr(A) + \Pr(B) - \Pr(A+B) \ge \Pr(A) + \Pr(B) - 1 \qquad (1.2.17.3)
$$

$$
\begin{aligned}
\implies \Pr(AB) &\ge 0.8 + 0.7 - 1 && (1.2.17.4)\\
&= 0.5 && (1.2.17.5)
\end{aligned}
$$

18. A public opinion poll shows that 90% of the population agreed with the government on the first decision, 84% on the second, and 74% on the third, for three decisions made by the government. At least what percentage of the population agreed with the government on all

three decisions.

Solution: Let the decisions be A, B, C. Then,

$$\Pr(AB) \geq \Pr(ABC), \tag{1.2.18.1}$$
$$\Pr(BC) \geq \Pr(ABC), \tag{1.2.18.2}$$
$$\Pr(CA) \geq \Pr(ABC) \tag{1.2.18.3}$$

Since

$$\begin{aligned}\Pr(A+B+C) &= \sum \Pr(A) \\ &\quad - \sum \Pr(AB) + \Pr(ABC), \\ \implies \Pr(A+B+C) &+ \sum \Pr(AB) \\ &= \sum \Pr(A) + \Pr(ABC), \end{aligned}\tag{1.2.18.4}$$

from (1.2.18.1),

$$\begin{aligned}\Pr(A+B+C) &+ 3\Pr(ABC) \\ &\geq \sum \Pr(A) + \Pr(ABC), \\ \implies 2\Pr(ABC) &\geq \sum \Pr(A) - \Pr(A+B+C)\end{aligned}\tag{1.2.18.5}$$

Since

$$\Pr(A+B+C) \leq 1, \tag{1.2.18.6}$$
$$-\Pr(A+B+C) \geq -1 \tag{1.2.18.7}$$
$$\implies 2\Pr(ABC) \geq \sum \Pr(A) - 1 \tag{1.2.18.8}$$
$$\text{or} \Pr(ABC) \geq \frac{\sum \Pr(A) - 1}{2} \tag{1.2.18.9}$$
$$= 0.74 \tag{1.2.18.10}$$

19. In his book *A Tangled Tale*, Lewis Caroll proposed the following riddle about a group of disabled veterans. "Say that 70% have lost an eye, 75% an ear, 80% an arm, 85% a leg. What percentage, at least, must have lost all four?" Solve Lewis Caroll's problem.

Solution: Let A_i represent the events. Then,

$$\begin{aligned}\Pr\left(\sum_{i=1}^{4} A_i\right) &= \sum_{i=1}^{4} \Pr(A_i) - \sum_{i,j} \Pr\left(A_i A_j\right) \\ &+ \sum_{i,j,k} \Pr\left(A_i A_j A_k\right) - \Pr\left(\prod_{i=1}^{4} A_i\right)\end{aligned}\tag{1.2.19.1}$$

Now,

$$\Pr(A_1A_2) \geq \Pr(A_1A_2A_3) \geq \Pr(A_1A_2A_3A_4) \tag{1.2.19.2}$$

which, upon substitution in (1.2.19.1) yields

$$\Pr\left(\sum_{i=1}^{4} A_i\right) \geq \frac{\sum_{i=1}^{4} \Pr(A_i) - 1}{1 + {}^4C_2 - {}^4C_3} \tag{1.2.19.3}$$
$$= 70\% \tag{1.2.19.4}$$

20. Show, for finite sets A, B, that $m(A \times B) = m(A) \times m(B)$.

Solution: Basic principle of counting.

21. If S is a set having five elements,
    a) How many subsets does S have?
    b) How many subsets having four elements does S have?
    c) How many subsets having two elements does S have?

Solution:

a) $2^5 = 32$.
b) ${}^5C_4 = 5$.
c) ${}^5C_2 = 10$.

22. a) Show that a set having n elements has 2^n subsets.
    b) If $0 < m < n$, how many subsets are there that have exactly m elements?

Solution:

a) The number of subsets is

$$\sum_{k=0}^{n} {}^nC_k = 2^n \tag{1.2.22.1}$$

using the binomial theorem.

b) The number of subsets having exactly m elements are ${}^nC_m$.

1.3 Mappings

1. For the given sets S, T determine if a mapping $f : S \to T$ is clearly and unambiguously defined; if not, say why not.
    a) S = set of all women, T = set of all men, $f(s)$ = husband of s.