Problems in Linear Algebra

Contents

1 Introduction

1.1 Points

1. Find the distance between

$$\mathbf{P} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} 3 \\ -5 \end{pmatrix} \tag{1.1.1}$$

Solution: Two point are P(x_1, y_1) and Q(x_2, y_2). The distance between both points is d.

$$\mathbf{Z} = \mathbf{P} - \mathbf{Q} \tag{1.1.1}$$

Then the distance between P and Q is given by:

$$d = \|\mathbf{Z}\| \tag{1.1.1}$$

$$d = \|\mathbf{P} - \mathbf{Q}\| \tag{1.1.1}$$

So, the distance between given points P and Q is:

$$d = \sqrt{(-2-3)^2 + (4-(-5))^2} \tag{1.1.1}$$

$$d = \sqrt{25+81} \tag{1.1.1}$$

$$d = \sqrt{106} \tag{1.1.1}$$

So, the distance between P(-2,4) and Q(3,-5) is :

$$d = \sqrt{106} \tag{1.1.1}$$

2. Find the length of PQ for
   a) $\mathbf{P} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$;
   b) $\mathbf{P} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$;
   c) $\mathbf{P} = \begin{pmatrix} a \\ b \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} -b \\ a \end{pmatrix}$.

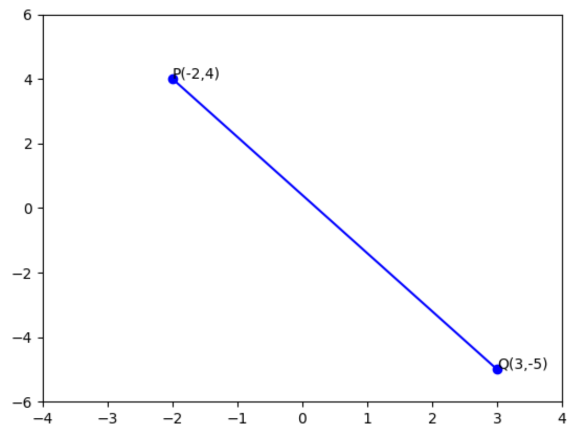

Fig. 1.1.1: Line between two points

Solution:

a) The distance between **P** and **Q** is given by:

$$d = \|\mathbf{P} - \mathbf{Q}\| \tag{1.1.2}$$
$$= \sqrt{(-1-2)^2 + (1+1)^2} \tag{1.1.2}$$
$$= \sqrt{9+4} = 3.6055 \tag{1.1.2}$$

b)

$$\mathbf{R} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} - \begin{pmatrix} -2 \\ 2 \end{pmatrix} \tag{1.1.2}$$
$$= \begin{pmatrix} 6 \\ 1 \end{pmatrix} \tag{1.1.2}$$

The desired distance between **P** and **Q** is

$$d = \|\mathbf{P} - \mathbf{Q}\| \tag{1.1.2}$$

From (1.1.2) and (1.1.2)

$$d = \|\mathbf{R}\| \tag{1.1.2}$$
$$= \sqrt{37} \tag{1.1.2}$$

3. Using direction vectors, show that $\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 7 \end{pmatrix}, \begin{pmatrix} 5 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ are the vertices of a parallelogram. **Solution:** Two lines are parallel if their respective

directional vectors are in the same ratio.
Let the points be denoted by:

$$\mathbf{A} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{1.1.3}$$

$$\mathbf{B} = \begin{pmatrix} 5 \\ 4 \end{pmatrix} \tag{1.1.3}$$

$$\mathbf{C} = \begin{pmatrix} 4 \\ 7 \end{pmatrix} \tag{1.1.3}$$

$$\mathbf{D} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \tag{1.1.3}$$

The directional vector of **AB** is

$$\begin{pmatrix} 2-5 \\ 1-4 \end{pmatrix} = \begin{pmatrix} -3 \\ -3 \end{pmatrix} \tag{1.1.3}$$

The directional vector of **BC** is

$$\begin{pmatrix} 5-4 \\ 4-7 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix} \tag{1.1.3}$$

The directional vector of **CD** is

$$\begin{pmatrix} 4-1 \\ 7-4 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} \tag{1.1.3}$$

The directional vector of **AD** is

$$\begin{pmatrix} 2-1 \\ 1-4 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix} \tag{1.1.3}$$

The directional vector of **AC** is

$$\begin{pmatrix} 2-4 \\ 1-7 \end{pmatrix} = \begin{pmatrix} -2 \\ -6 \end{pmatrix} \tag{1.1.3}$$

Since the directional vectors of **AB** and **CD** are in the same ratio, so **AB** and **CD** are parallel and also opposite to each other.
Similarly, the directional vectors of **BC** and **AD** are in the same ratio,hence **BC** and **AD** are parallel and opposite.
Since the two pairs of opposite sides are parallel, the given points are the vertices of the parallelogram.

Moreover the sum of the directional vectors of **AB** and **BC**

$$\begin{pmatrix} -3 \\ -3 \end{pmatrix} + \begin{pmatrix} 1 \\ -3 \end{pmatrix} = \begin{pmatrix} -3+1 \\ -3-3 \end{pmatrix} = \begin{pmatrix} -2 \\ -6 \end{pmatrix}$$

Thus **AB** + **BC**= **AC**, which satisfy parallelogram law of vector addition i.e vector sum of two adjacent side of a parallelogram is the diagonal vector of the parallelogram. See Fig. 1.1.3

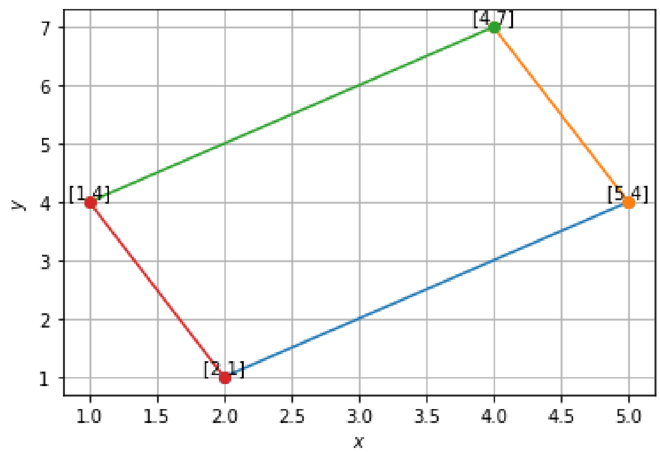

Fig. 1.1.3: This is the 2D diagram of the parallelogram with the given vertices

4. Using Baudhayana's theorem, show that the points $\begin{pmatrix} -3 \\ -4 \end{pmatrix}, \begin{pmatrix} 2 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} -6 \\ 10 \end{pmatrix}$ are the vertices of a right-angled traingle. Repeat using orthogonality. **Solution:** Say there exists two points $\mathbf{P}(x_1, y_1)$ and $\mathbf{Q}(x_2, y_2)$. The distance between the points is:

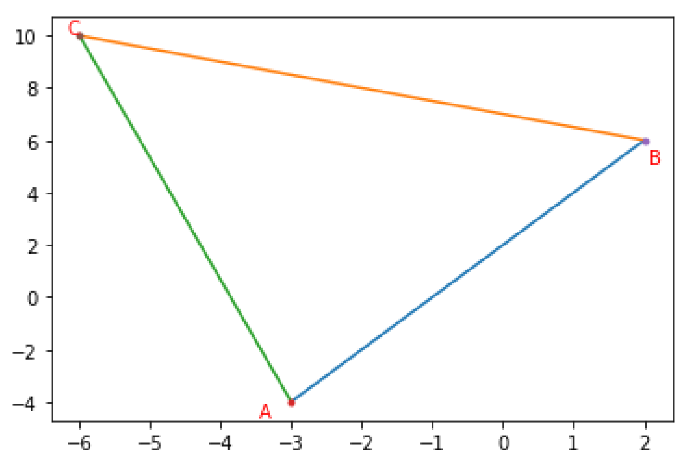

Fig. 1.1.4: Right Angled Triangle

$$\mathbf{Z = P - Q} \tag{1.1.4}$$

Distance between **P** and **Q** is given by

$$\|\mathbf{Z}\| = \|\mathbf{P\text{-}Q}\| \tag{1.1.4}$$

Let $\mathbf{P} = (-3, -4)$, $\mathbf{Q} = (2, 6)$ and $\mathbf{R} = (-6, 10)$.

a) Distance between **P** and **Q** is

$$||\mathbf{P\text{-}Q}|| = \sqrt{(-3-2)^2 + (-4-6)^2} = \sqrt{125} \tag{1.1.4}$$

b) Distance between **Q** and **R** is

$$||\mathbf{Q\text{-}R}|| = \sqrt{(2-(-6))^2 + (6-10)^2} = \sqrt{80} \tag{1.1.4}$$

c) Distance between **P** and **R** is

$$||\mathbf{P\text{-}R}|| = \sqrt{(-3-(-6))^2 + (-4-10)^2} = \sqrt{205} \tag{1.1.4}$$

Here, the largest distance is $\sqrt{205}$. To be vertices of a right angled triangle, we should have

$$||\mathbf{P\text{-}Q}||^2 + ||\mathbf{Q\text{-}R}||^2 = ||\mathbf{R\text{-}P}||^2 \tag{1.1.4}$$

$$(\sqrt{205})^2 = (\sqrt{125})^2 + (\sqrt{80})^2 \tag{1.1.4}$$

$$205 = 205 \tag{1.1.4}$$

So, the condition is satisfied. So, using Baudhayana's theorem, it is proved that 3 points given are vertices of a right angled triangle. Now, for orthogonality,

$$(\mathbf{P}-\mathbf{Q})^T(\mathbf{Q}-\mathbf{R}) = 0 \tag{1.1.4}$$

We have

a)

$$\mathbf{P}-\mathbf{Q} = (2-(-3), 6-(-4)) \tag{1.1.4}$$

$$\mathbf{P}-\mathbf{Q} = \begin{pmatrix} 5 \\ 10 \end{pmatrix} \tag{1.1.4}$$

b)

$$\mathbf{Q}-\mathbf{R} = (2-(-6), 6-10) \tag{1.1.4}$$

$$\mathbf{Q}-\mathbf{R} = \begin{pmatrix} 8 \\ -4 \end{pmatrix} \tag{1.1.4}$$

c)

$$\mathbf{P}-\mathbf{R} = (-3-(-6), -4-10) \tag{1.1.4}$$

$$\mathbf{P}-\mathbf{R} = \begin{pmatrix} 3 \\ -14 \end{pmatrix} \tag{1.1.4}$$

For orthogonality, product of transpose of one point and other must be 0. Here, checking for

$$\begin{pmatrix} 5 \\ 10 \end{pmatrix}^T \begin{pmatrix} 8 \\ -4 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \end{pmatrix}^T \begin{pmatrix} 8 \\ -4 \end{pmatrix} = 0 \tag{1.1.4}$$

Hence, using orthogonality, it is proved that the points form a right angled triangle.
Figure 1.1.4 Right anlged triangle where A=P and B=Q and C=R

5. Plot the points $\begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$ and prove that they are the vertices of a rectangle.
6. Show that $\mathbf{B} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ are the vertices of an isosceles triangle.
Solution:
Define a matrix **M** such that,

$$\mathbf{M} = \begin{pmatrix} \mathbf{B}-\mathbf{A} & \mathbf{C}-\mathbf{A} \end{pmatrix}^T \tag{1.1.6}$$

$$\mathbf{M} = \begin{pmatrix} -1 & -4 \\ 4 & -1 \end{pmatrix} \tag{1.1.6}$$

Using matrix transformation,

$$\mathbf{M} = \begin{pmatrix} -1 & -4 \\ 4 & -1 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow -R_1 - \frac{R_2}{4}} \begin{pmatrix} 0 & \frac{17}{4} \\ 4 & -1 \end{pmatrix} \tag{1.1.6}$$

$$\implies rank(\mathbf{M}) = 2 \tag{1.1.6}$$

Since the rank of matrix **M** is 2, the points form a triangle.

$$AB^2 = (\mathbf{A}-\mathbf{B})^T(\mathbf{A}-\mathbf{B}) \tag{1.1.6}$$
$$= 17 \tag{1.1.6}$$
$$BC^2 = (\mathbf{B}-\mathbf{C})^T(\mathbf{B}-\mathbf{C}) \tag{1.1.6}$$
$$= 34 \tag{1.1.6}$$
$$CA^2 = (\mathbf{C}-\mathbf{A})^T(\mathbf{C}-\mathbf{A}) \tag{1.1.6}$$
$$= 17 \tag{1.1.6}$$
$$\implies AB = AC \tag{1.1.6}$$

Hence, the triangle is isosceles. See Fig. 1.1.6

7. In the last question, find the distance of the vertex **A** of the triangle from the middle point of the base BC.
Solution:

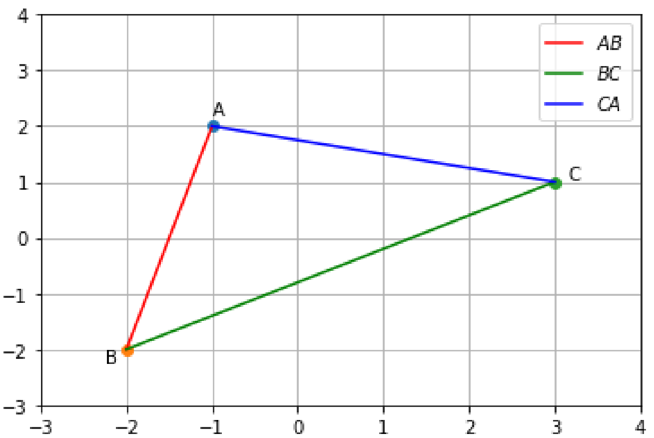

Fig. 1.1.6: Plot of the given points

From the given information,

$$a^2 = (\mathbf{B}-\mathbf{C})^T(\mathbf{B}-\mathbf{C}) \tag{1.1.7}$$
$$= 34 \tag{1.1.7}$$
$$c^2 = (\mathbf{A}-\mathbf{B})^T(\mathbf{A}-\mathbf{B}) \tag{1.1.7}$$
$$= 17 \tag{1.1.7}$$
$$b^2 = (\mathbf{C}-\mathbf{A})^T(\mathbf{C}-\mathbf{A}) \tag{1.1.7}$$
$$= 17 \tag{1.1.7}$$
$$\implies AB = AC \tag{1.1.7}$$

Thus, the required distance is AD where

$$\mathbf{D} = \frac{\mathbf{B}+\mathbf{C}}{2} = \begin{pmatrix} \frac{-1}{2} \\ \frac{-1}{2} \end{pmatrix} \tag{1.1.7}$$

and

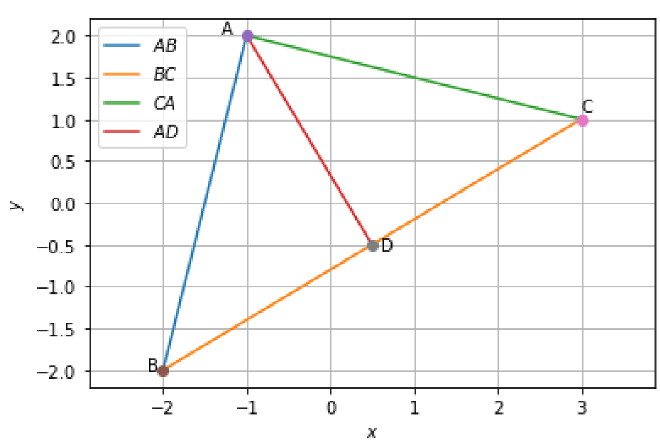

Fig. 1.1.7: plot

$$AD = \|\mathbf{A}-\mathbf{D}\| \tag{1.1.7}$$
$$= \frac{\sqrt{34}}{2} \tag{1.1.7}$$

8. Prove that the points $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ are the vertices of a square.
9. Prove that the points $\mathbf{A} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ are the vertices of a parallelogram. Find $\mathbf{E}, \mathbf{F}, \mathbf{G}, \mathbf{H}$, the mid points of AB, BC, CD, AD respectively. Show that EG and FH bisect each other.
Solution:

$$\mathbf{A}-\mathbf{B} = \begin{pmatrix} -4 \\ -1 \end{pmatrix} \tag{1.1.9}$$
$$= -(\mathbf{C}-\mathbf{D}) \tag{1.1.9}$$
$$\mathbf{B}-\mathbf{C} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \mathbf{A}-\mathbf{D} \tag{1.1.9}$$
$$\implies AB \parallel CD, BC \parallel AD \tag{1.1.9}$$

Hence, $ABCD$ is a parallelogram. Also,

$$\mathbf{E} = \frac{\mathbf{A}+\mathbf{B}}{2} = \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} \tag{1.1.9}$$
$$\mathbf{F} = \frac{\mathbf{B}+\mathbf{C}}{2} = \begin{pmatrix} \frac{5}{2} \\ \frac{3}{2} \end{pmatrix} \tag{1.1.9}$$
$$\mathbf{G} = \frac{\mathbf{C}+\mathbf{D}}{2} = \begin{pmatrix} 0 \\ \frac{3}{2} \end{pmatrix} \tag{1.1.9}$$
$$\mathbf{H} = \frac{\mathbf{A}+\mathbf{D}}{2} = \begin{pmatrix} \frac{-3}{2} \\ \frac{1}{2} \end{pmatrix} \tag{1.1.9}$$

and

$$\frac{\mathbf{E}+\mathbf{G}}{2} = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} \tag{1.1.9}$$
$$= \frac{\mathbf{F}+\mathbf{H}}{2} \tag{1.1.9}$$

See Fig. 1.1.9.

10. Prove that the points $\begin{pmatrix} 21 \\ -2 \end{pmatrix}$, $\begin{pmatrix} 15 \\ 10 \end{pmatrix}$, $\begin{pmatrix} -5 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -12 \end{pmatrix}$ are the vertices of a rectangle, and find the coordinates of its centre.

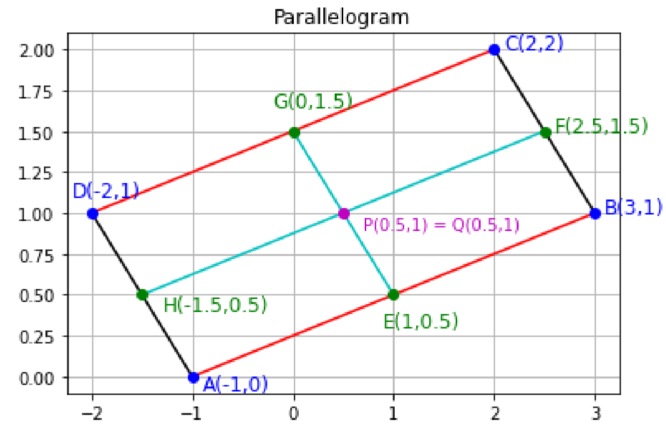

Fig. 1.1.9

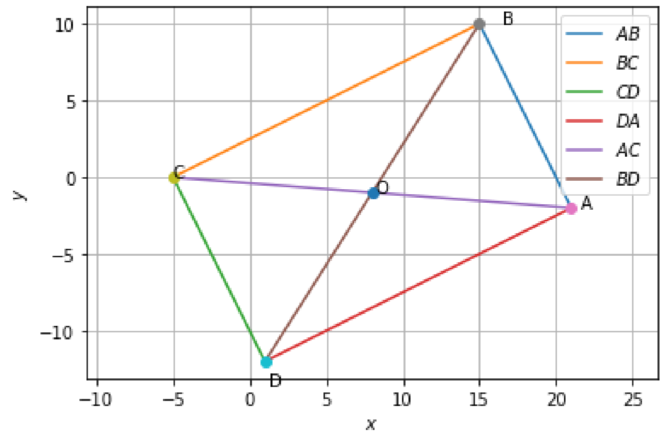

Fig. 1.1.10: plot

Solution: Let

$$\mathbf{A} = \begin{pmatrix} 21 \\ -2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 15 \\ 10 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -5 \\ 0 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 1 \\ -12 \end{pmatrix} \tag{1.1.10}$$

Then,

$$\mathbf{A} - \mathbf{B}* = \begin{pmatrix} 6 \\ -12 \end{pmatrix} \tag{1.1.10}$$

$$\mathbf{B} - \mathbf{C} = \begin{pmatrix} 20 \\ 10 \end{pmatrix} \tag{1.1.10}$$

$$\mathbf{C} - \mathbf{D} = \begin{pmatrix} -6 \\ 12 \end{pmatrix} \tag{1.1.10}$$

$$\mathbf{D} - \mathbf{A} = \begin{pmatrix} -20 \\ -10 \end{pmatrix} \tag{1.1.10}$$

Since the directional vectors of **AB** and **CD** are in the same ratio, so **AB** and **CD** are parallel and also opposite to each other. Similarly, **BC** and **DA** are parallel and opposite. Hence ABCD is a parallelogram. Also,

$$(\mathbf{B} - \mathbf{A})^\top (\mathbf{C} - \mathbf{B}) = \begin{pmatrix} -6 & 12 \end{pmatrix} \begin{pmatrix} -20 \\ -10 \end{pmatrix} \tag{1.1.10}$$

$$= 0 \tag{1.1.10}$$

Therefore, one of the angle is right angle and ABCD is a rectangle. The center

$$\mathbf{O} = \frac{\mathbf{A} + \mathbf{C}}{2} \tag{1.1.10}$$

$$= \begin{pmatrix} 8 \\ -1 \end{pmatrix} \tag{1.1.10}$$

This is verified in Fig. 1.1.10.

11. Find the lengths of the medians of the triangle whose vertices are at the points $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -2 \end{pmatrix}$.
12. Find the coordinates of the points that divide the line joining the points $\begin{pmatrix} -35 \\ -20 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ -10 \end{pmatrix}$ into four equal parts.
13. Find the coordinates of the points of trisection of the line joining the points $\begin{pmatrix} -5 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 25 \\ 10 \end{pmatrix}$.
14. Prove that the middle point of the line joining the points $\begin{pmatrix} -5 \\ 12 \end{pmatrix}$ and $\begin{pmatrix} 9 \\ -2 \end{pmatrix}$ is a point of trisection of the line joining the points $\begin{pmatrix} -8 \\ -5 \end{pmatrix}$ and $\begin{pmatrix} 7 \\ 10 \end{pmatrix}$.
    Solution:
15. The points $\begin{pmatrix} 8 \\ 5 \end{pmatrix}$, $\begin{pmatrix} -7 \\ -5 \end{pmatrix}$ and $\begin{pmatrix} -5 \\ 5 \end{pmatrix}$ are three of the vertices of a parallelogram. Find the coordinates of the remaining vertex which is to be taken as opposite to $\begin{pmatrix} -7 \\ -5 \end{pmatrix}$.
16. The point $\begin{pmatrix} 2 \\ 6 \end{pmatrix}$ is the intesection of the diagonals of a parallelogram two of whose vertices are at the points $\begin{pmatrix} 7 \\ 16 \end{pmatrix}$ and $\begin{pmatrix} 10 \\ 2 \end{pmatrix}$. Find the coordinates of the remaining vertices.
17. Find the area of the triangle whose vertices are the points $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\begin{pmatrix} -4 \\ 7 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ -2 \end{pmatrix}$.
18. Find the coordinates of points which divide the join of $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\begin{pmatrix} -4 \\ 5 \end{pmatrix}$ externally in the ratio 2 : 3,

and also externally in the ratio 3 : 2.

19. Prove the centroid of $\triangle ABC$ is

$$\mathbf{O} = \frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} \tag{1.1.19}$$