Abstract Algebra

Contents

1 Things Familiar and Less Familiar

1.1 Introduction

1. let S be a set having an operation * which assigns an element a*b of S for any $a, b \in S$. Let us assume that the following two rules hold:
   a) If a, b are any objects in S, then $a * b = a$.
   b) If a, b are any objects in S, then $a*b = b*a$.

   Show that S can have at most one object.
   Solution: From condition 1.1.1a, interchanging a, b,
   $$b * a = b \tag{1.1.1}$$
   and from condition 1.1.1b,
   $$b * a = a * b \tag{1.1.1}$$
   But from condition 1.1.1a,
   $$a * b = a \implies a = b \tag{1.1.1}$$
   Thus, S can have at most one object.
2. Let S be the set of all integers $0, \pm 1, \pm 2, \dots, \pm n, \dots$. For $a, b \in S$, define * by
   $$a * b = a - b \tag{1.1.2}$$
   Verify the following
   a) $a * b \neq b * a$ unless $a = b$
   b) $(a * b) * c \neq a * (b * c)$ in general. Under what conditions on a, b, c is
   $$(a * b) * c \neq a * (b * c) \quad ? \tag{1.1.2}$$
   c) The integer a has the property that $a * 0 = a$ for every $a \in S$.
   d) For $a \in S, a * a = 0$.

   Solution:
   a)
   $$a * b = b * a \tag{1.1.2}$$
   $$\implies a - b = b - a \tag{1.1.2}$$
   $$\text{or, } a = b \tag{1.1.2}$$
   b) Let $a = 1, b = 2, c = 4$. Then,
   $$a * b = -1, (a * b) * c = -1 - 4 = -5 \tag{1.1.2}$$
   $$b * c = -2, a * (b * c) = 1 + 2 = 3 \neq -5 \tag{1.1.2}$$
   Thus, for the given condition to be satisfied,
   $$(a - b) - c = a - (b - c) \tag{1.1.2}$$
   $$\implies c = 0 \tag{1.1.2}$$
   c)
   $$a * 0 = a - 0 = a \tag{1.1.2}$$
   d)
   $$a * a = a - a = 0 \tag{1.1.2}$$
3. Let S consist of the two objects □ and △. We define the operatin * on S by subjecting □ and △ to the following conditions.
   a) $\square * \triangle = \triangle = \triangle * \square$
   b) $\square * \square = \square$
   c) $\triangle * \triangle = \square$

   Verify by explicit calculation that if a, b, c are any elements of S, (i.e. a, b, c can be any of □ or △), then
   a) $a * b$ is in S
   b) $(a * b) * c = a * (b * c)$
   c) $a * b = b * a$
   d) There is a particular a in S such that $a * b = b * a = b$ for all $b \in S$
   e) Given $b \in S, b * b = a$, where a is the particular element in Part 1.1.3d.

Solution: Let $\square = 1, \triangle = -1$. These satisfy all the given conditions.

a) $a * b \in [1, -1] \in S$.
b) Writing the truth table, $(a * b) * c = a * (b * c)$.
c) $a * b = b * a$ can be verified by writing the truth table.
d) For $a = 1, a * b = b * a = b$, for all $b \in S$.
e) For $a = 1$, if $b = -1, b * b = 1 = a$. This can be shown to be true for $b = 1$ as well.

1.2 Set Theory

1. Describe the following sets verbally
   a) S = {Mercury, Venus, Earth, …, Pluto}
   b) S = {Andhra Pradesh, Uttar Pradesh, …, Assam}

   Solution:

   a) Planets
   b) Indian states
2. Describe the following sets verbally
   a) $S = \{2, 4, 6, 8, \ldots\}$
   b) $S = \{2, 4, 8, 16, \ldots\}$
   c) $S = \{1, 4, 9, 16, 25, 36 \ldots\}$

   Solution:

   a) Even numbers
   b) Powers of 2
   c) Squares of positive integers
3. If A is the set of all residents of India, B the set of all Sri Lankan citizens, and C the set of all women in the world, describe the sets $ABC, A - B, A - C, C - A$ verbally.

   Solution:

   a) ABC is the set of all women residents of India who are citizens of Sri Lanka.
   b) $A - B = AB'$ is the set of all residents of India who are not Sri Lankan citizens.
   c) $A - C = AC'$ is the set of all male residents of India.
   d) $C - A = CA'$ is the set of all women who are not residing in India.
4. If $A = \{1, 4, 7, a\}$ and $B = \{3, 4, 9, 11\}$ and you have been told that $AB = \{4, 9\}$, then what must a be?

   Solution: $a = 9$
5. If $A \subset B, B \subset C$, prove that $A \subset C$

   Solution: From the given information,
   $$A + P = B, AP = 0, B + Q = C, BQ = 0 \tag{1.2.5.1}$$
   $$\implies B + Q = A + P + Q = C, \tag{1.2.5.2}$$
   $\because BQ = 0$,
   $$AQ + PQ = 0 \implies AQ = 0, PQ = 0 \tag{1.2.5.3}$$
   Hence,
   $$A(P + Q) = 0 \implies A \subset C \tag{1.2.5.4}$$
6. If $A \subset B$ prove that $A \cup C \subset B \cup C$ for any set C.

   Solution: From the given information, there exists P such that
   $$A + P = B, AP = 0 \tag{1.2.6.1}$$
   Also,
   $$B + C = A + P + C \tag{1.2.6.2}$$
   $$\implies A + C \subset B + C \tag{1.2.6.3}$$
7. Show that
   $$A \cup B = B \cup A \tag{1.2.7.1}$$
   $$A \cap B = B \cap A \tag{1.2.7.2}$$
8. Prove that
   $$(A - B) \cup (B - A) = (A \cup B) - (A \cap B) \tag{1.2.8.1}$$
   Solution: Since
   $$A - B = AB', \tag{1.2.8.2}$$
   $$(A - B) \cup (B - A) = AB' + BA' \tag{1.2.8.3}$$
   Also,
   $$(A \cup B) - (A \cap B) = (A + B)(AB)' \tag{1.2.8.4}$$
   $$= (A + B)(A' + B') \tag{1.2.8.5}$$
   $$= AB' + BA' \tag{1.2.8.6}$$
9. Prove that
   $$(A) \cap (B \cup C) = (A \cap B) \cup (A \cap C) \tag{1.2.9.1}$$

Solution:

$$LHS = A(B+C) = AB + AC = RHS \tag{1.2.9.2}$$

10. Prove that

$$(A) \cup (B \cap C) = (A \cup B) \cap (A \cup C) \tag{1.2.10.1}$$

Solution:

$$\begin{aligned} LHS &= A + BC && (1.2.10.2)\\ RHS &= (A+B)(A+C) && (1.2.10.3)\\ &= A + A(B+C) + BC && (1.2.10.4)\\ &= A(1+B+C) + BC && (1.2.10.5)\\ &= LHS && (1.2.10.6) \end{aligned}$$

11. Write down all the subsets of $S = \{1, 2, 3, 4\}$.
Solution: Write a program for this.
12. If C is a subset of S, let C' denote the complement of C in S. Prove the *De Morgan Rules* for subsets A, B of S, namely,
a) $(A \cup B)' = A' \cap B'$
b) $(A \cap B)' = A' \cup B'$
Solution:
a)

$$\begin{aligned} (A+B)A'B' &= AA'B' + BA'B' && (1.2.12.1)\\ &= 0 && (1.2.12.2) \end{aligned}$$

b) Substituting $A = A', B = B'$ in the above, the second result is obtained.
13. Let S be a set. For any to subsets of S, we define

$$A \oplus B = (A - B) \cup (B \cup A) \tag{1.2.13.1}$$

Prove that
a) $A \oplus B = B \oplus A$.
b) $A \oplus \Phi = A$.
c) $A \cdot A = A$.
d) $A \oplus A = \Phi$.
e) $A \oplus (B \oplus C) = (A \oplus B) \oplus C$.
f) If $A \oplus B = A \oplus C$, then $B = C$.
g) $A \cdot (B + C) = A \cdot B + A \cdot C$.
Solution: All can be proved using boolean logic.
14. If C is a finite set, let $m(C)$ denote the number of elements in C. If A, B are finite sets, prove that

$$m(A \cup B) = m(A) + m(B) - m(A \cap B) \tag{1.2.14.1}$$

Solution:

$$\begin{aligned} A'B' &= (A+B)' && (1.2.14.2)\\ \implies m(A'B') &= m((A+B)') && (1.2.14.3)\\ &= 1 - m(A+B) && (1.2.14.4) \end{aligned}$$

$$\begin{aligned} \because A + B &= A(B+B') + B && (1.2.14.5)\\ &= B(A+1) + AB' && (1.2.14.6)\\ &= B + AB' && (1.2.14.7)\\ \implies m(A+B) &= m(B+AB') && (1.2.14.8)\\ &= m(B) + m(AB') && (1.2.14.9)\\ \because B(AB') &= 0 && (1.2.14.10) \end{aligned}$$

$$A = A(B+B') = AB + AB' \tag{1.2.14.11}$$

and

$$(AB)(AB') = 0, \because BB' = 0 \tag{1.2.14.12}$$

Hence, AB and AB' are mutually exclusive and

$$\begin{aligned} m(A) &= m(AB) + m(AB') && (1.2.14.13)\\ \implies m(AB') &= m(A) - m(AB) && (1.2.14.14) \end{aligned}$$

Substituting (1.2.14.14) in (1.2.14.10),

$$m(A+B) = m(A) + m(B) - m(AB) \tag{1.2.14.15}$$

15. For three finite sets A, B, C, find a formula for $m(A \cup b \cup C)$. **Solution:** Extend the above.
16. Take a shot at finding $m\left(\cup_{i=1}^{n} A_i\right)$.
17. Show that if 80% of all Indians have gone to high school and 70% of all Indians read a daily newspaper, then *at least* 50% of all Indians have both gone to high school and read a daily newspaper.
Solution: Let A represent high school and B represent newspaper. Then,

$$\Pr(AB) = \Pr(A) + \Pr(B) - \Pr(A+B) \tag{1.2.17.1}$$

Since

$$\Pr(A+B) \le 1, \quad (1.2.17.2)$$

$$\Pr(A)+\Pr(B)-\Pr(A+B) \ge \Pr(A)+\Pr(B)-1 \quad (1.2.17.3)$$

$$\implies \Pr(AB) \ge 0.8+0.7-1 \quad (1.2.17.4)$$

$$= 0.5 \quad (1.2.17.5)$$

18. A public opinion poll shows that 90% of the population agreed with the government on the first decision, 84% on the second, and 74% on the third, for three decisions made by the government. At least what percentage of the population agreed with the government on all three decisions.
Solution: Let the decisions be A, B, C. Then,

$$\Pr(AB) \ge \Pr(ABC), \quad (1.2.18.1)$$

$$\Pr(BC) \ge \Pr(ABC), \quad (1.2.18.2)$$

$$\Pr(CA) \ge \Pr(ABC) \quad (1.2.18.3)$$

Since

$$\Pr(A+B+C) = \sum \Pr(A) - \sum \Pr(AB) + \Pr(ABC),$$

$$\implies \Pr(A+B+C) + \sum \Pr(AB) = \sum \Pr(A) + \Pr(ABC), \quad (1.2.18.4)$$

from (1.2.18.1),

$$\Pr(A+B+C) + 3\Pr(ABC) \ge \sum \Pr(A) + \Pr(ABC),$$

$$\implies 2\Pr(ABC) \ge \sum \Pr(A) - \Pr(A+B+C) \quad (1.2.18.5)$$

Since

$$\Pr(A+B+C) \le 1, \quad (1.2.18.6)$$

$$-\Pr(A+B+C) \ge -1 \quad (1.2.18.7)$$

$$\implies 2\Pr(ABC) \ge \sum \Pr(A) - 1 \quad (1.2.18.8)$$

$$\text{or} \Pr(ABC) \ge \frac{\sum \Pr(A) - 1}{2} \quad (1.2.18.9)$$

$$= 0.74 \quad (1.2.18.10)$$

19. In his book *A Tangled Tale*, Lewis Caroll proposed the following riddle about a group of disabled veterans. "Say that 70% have lost an eye, 75% an ear, 80% an arm, 85% a leg. What percentage, at least, must have lost all four?" Solve Lewis Caroll's problem.
Solution: Let A_i represent the events. Then,

$$\Pr\left(\sum_{i=1}^{4} A_i\right) = \sum_{i=1}^{4} \Pr(A_i) - \sum_{i,j} \Pr\left(A_i A_j\right) + \sum_{i,j,k} \Pr\left(A_i A_j A_k\right) - \Pr\left(\prod_{i=1}^{4} A_i\right) \quad (1.2.19.1)$$

Now,

$$\Pr(A_1 A_2) \ge \Pr(A_1 A_2 A_3) \ge \Pr(A_1 A_2 A_3 A_4) \quad (1.2.19.2)$$

which, upon substitution in (1.2.19.1) yields

$$\Pr\left(\sum_{i=1}^{4} A_i\right) \ge \frac{\sum_{i=1}^{4} \Pr(A_i) - 1}{1 + {}^4C_2 - {}^4C_3} \quad (1.2.19.3)$$

$$= 70\% \quad (1.2.19.4)$$

20. Show, for finite sets A, B, that $m(A \times B) = m(A) \times m(B)$.
Solution: Basic principle of counting.
21. If S is a set having five elements,
    a) How many subsets does S have?
    b) How many subsets having four elements does S have?
    c) How many subsets having two elements does S have?

Solution:
a) $2^5 = 32$.
b) ${}^5C_4 = 5$.
c) ${}^5C_2 = 10$.

22. a) Show that a set having n elements has 2^n subsets.
    b) If $0 < m < n$, how many subsets are there that have exactly m elements?

Solution:
a) The number of subsets is

$$\sum_{k=0}^{n} {}^nC_k = 2^n \quad (1.2.22.1)$$

using the binomial theorem.
b) The number of subsets having exactly m elements are ${}^nC_m$.

1.3 Mappings

1. For the given sets S, T determine if a mapping $f : S \to T$ is clearly and unambiguously defined; if not, say why not.
   a) S = set of all women, T = set of all men, $f(s)$ = husband of s.

1.4 The Integers

1. Find (a, b) and express (a, b) as $ma + nb$ for
   a) $(116, -84)$
   b) $(85, 65)$
   c) $(72, 26)$
   d) $(72, 25)$

   Solution:

   a) Using the extended Euclid algorithm,

$$\begin{pmatrix} 116 & 1 & 0 \\ -84 & 0 & 1 \end{pmatrix} \tag{1.4.1.1}$$
$$\xleftrightarrow{R_3 \leftarrow R_1 + R_2} \begin{pmatrix} 32 & 1 & 1 \end{pmatrix} \tag{1.4.1.2}$$
$$\xleftrightarrow{R_4 \leftarrow R_2 + 2R_3} \begin{pmatrix} -20 & 2 & 3 \end{pmatrix} \tag{1.4.1.3}$$
$$\xleftrightarrow{R_5 \leftarrow R_4 + R_3} \begin{pmatrix} 12 & 3 & 4 \end{pmatrix} \tag{1.4.1.4}$$
$$\xleftrightarrow{R_6 \leftarrow R_5 + R_4} \begin{pmatrix} -8 & 5 & 7 \end{pmatrix} \tag{1.4.1.5}$$
$$\xleftrightarrow{R_7 \leftarrow R_6 + R_5} \begin{pmatrix} 4 & 8 & 11 \end{pmatrix} \tag{1.4.1.6}$$
$$\xleftrightarrow{R_8 \leftarrow R_7 + 2R_6} \begin{pmatrix} 0 & 21 & 29 \end{pmatrix} \tag{1.4.1.7}$$

   Thus,

$$4 = (8)116 + 11(-84) \tag{1.4.1.8}$$

   b)

$$\begin{pmatrix} 85 & 1 & 0 \\ 65 & 0 & 1 \end{pmatrix} \tag{1.4.1.9}$$
$$\xleftrightarrow{R_3 \leftarrow R_1 - R_2} \begin{pmatrix} 20 & 1 & -1 \end{pmatrix} \tag{1.4.1.10}$$
$$\xleftrightarrow{R_4 \leftarrow R_2 - 3R_3} \begin{pmatrix} 5 & -3 & 4 \end{pmatrix} \tag{1.4.1.11}$$
$$\xleftrightarrow{R_5 \leftarrow R_3 - 4R_4} \begin{pmatrix} 0 & 13 & -17 \end{pmatrix} \tag{1.4.1.12}$$

   Thus,

$$5 = (-3)85 + 4(65) \tag{1.4.1.13}$$

   c)

$$\begin{pmatrix} 72 & 1 & 0 \\ 26 & 0 & 1 \end{pmatrix} \tag{1.4.1.14}$$
$$\xleftrightarrow{R_3 \leftarrow R_1 - 2R_2} \begin{pmatrix} 20 & 1 & -2 \end{pmatrix} \tag{1.4.1.15}$$
$$\xleftrightarrow{R_4 \leftarrow R_2 - R_3} \begin{pmatrix} 6 & -1 & 3 \end{pmatrix} \tag{1.4.1.16}$$
$$\xleftrightarrow{R_5 \leftarrow R_3 - 3R_4} \begin{pmatrix} 2 & 4 & -11 \end{pmatrix} \tag{1.4.1.17}$$
$$\xleftrightarrow{R_6 \leftarrow R_4 - 3R_5} \begin{pmatrix} 0 & -13 & 36 \end{pmatrix} \tag{1.4.1.18}$$

   Thus,

$$2 = (4)72 + (-11)26 \tag{1.4.1.19}$$

   d)

$$\begin{pmatrix} 72 & 1 & 0 \\ 25 & 0 & 1 \end{pmatrix} \tag{1.4.1.20}$$
$$\xleftrightarrow{R_3 \leftarrow R_1 - 2R_2} \begin{pmatrix} 22 & 1 & -2 \end{pmatrix} \tag{1.4.1.21}$$
$$\xleftrightarrow{R_4 \leftarrow R_2 - R_3} \begin{pmatrix} 3 & -1 & 3 \end{pmatrix} \tag{1.4.1.22}$$
$$\xleftrightarrow{R_5 \leftarrow R_3 - 7R_4} \begin{pmatrix} 1 & 8 & -23 \end{pmatrix} \tag{1.4.1.23}$$

   Thus,

$$1 = (8)72 + (-23)25 \tag{1.4.1.24}$$

2. Show that the following are true
   a) $1 \mid n$ for all n .
   b) If $m \neq 0$, then $m \mid 0$.
   c) If $m \mid n$ and $n \mid q$, then $m \mid q$.
   d) If $m \mid n$ and $n \mid q$, then $m \mid (un + vq)$ for all v, u.
   e) If $m \mid 1$, then $m = 1$ or $m = -1$.
   f) If $m \mid n$, and $n \mid m$, then $m = \pm n$.

   Solution:

   a) $n = 1 \times n$.
   b) $0 = 0 \times m$.
   c) Let

$$n = cm, q = dn. \tag{1.4.2.1}$$

   Then

$$q = (cdn)m \implies m \mid q \tag{1.4.2.2}$$

   d) Let

$$n = cm, q = dn. \tag{1.4.2.3}$$

Then

$$un + vq = ucm + vdn \quad (1.4.2.4)$$
$$= (uc + vdc)m \quad (1.4.2.5)$$
$$\implies m \mid (un + vq) \quad (1.4.2.6)$$

e) If

$$1 = cm, \quad (1.4.2.7)$$
$$c = 1, m = 1 \quad (1.4.2.8)$$
$$c = -1, m = -1 \quad (1.4.2.9)$$

f)

$$n = cm, m = dn \quad (1.4.2.10)$$
$$\implies mn = cdmn \quad (1.4.2.11)$$
$$\text{or, } cd = 1 \quad (1.4.2.12)$$

Thus, either

$$c = d = 1, \implies n = m, \quad (1.4.2.13)$$
$$\text{or, } c = d = -1, \implies n = -m \quad (1.4.2.14)$$

3. Show that

$$(ma, mb) = m(a, b) \quad m > 0. \quad (1.4.3.1)$$

Solution: Let

$$(a, b) = xa + yb \quad (1.4.3.2)$$

Then,

$$(ma, mb) = xma + ymb = m(xa + yb) \quad (1.4.3.3)$$
$$= m(a, b) \quad (1.4.3.4)$$

4. Show that if $a \mid m$ and $b \mid m$, and $(a, b) = 1$, then $(ab) \mid m$.
Solution: From the given information,

$$m = ac, \quad m = bd, \quad (1.4.4.1)$$
$$ax + by = 1 \quad (1.4.4.2)$$

Multiplying both sides of (1.4.4.2) by m

$$max + mby = m \quad (1.4.4.3)$$
$$\implies ab(dx + cy) = m \quad (1.4.4.4)$$

upon substituting from (1.4.4.1). Hence, $(ab) \mid m$.

5. Factor the following into primes
a) 36
b) 120
c) 720
d) 5040

Solution:
a) $36 = 2^2 \times 3^2$.
b) $120 = 2^3 \times 3 \times 5$.
c) $720 = 2^4 \times 3^2 \times 5$.
d) $5040 = 2^2 \times 3^2 \times 5 \times 7$.

6. If $m = p_1^{a_1} \dots p_k^{a_k}$, and $n = p_1^{b_1} \dots p_k^{b_k}$, where p_i are distinct primes and a_i, b_i are nonnegative, express (m, n) as $p_1^{c_1} \dots p_k^{c_k}$ by describing the cs in terms of the as and bs.
Solution: Let

$$m = 36 = 2^2 \times 3^2 \quad (1.4.6.1)$$
$$n = 720 = 2^4 \times 3^2 \times 5 \quad (1.4.6.2)$$

Then,

$$k = 3 \quad (1.4.6.3)$$
$$p_1 = 2, p_2 = 3, p_3 = 5 \quad (1.4.6.4)$$
$$a_1 = 2, a_2 = 2, a_3 = 0 \quad (1.4.6.5)$$
$$b_1 = 4, b_2 = 2, b_3 = 1 \quad (1.4.6.6)$$

and

$$(36, 720) = 2^2 \times 3^2 \quad (1.4.6.7)$$
$$\implies c_i = \min(a_i, b_i) \quad (1.4.6.8)$$

7. Define the least common multiplie (LCM) of positive integers m and n to be the smallest positive integer v such that both $m \mid v$ and $n \mid v$.
a) Show that

$$v = \frac{mn}{(m, n)} \quad (1.4.7.1)$$

b) In terms of the factorization of m and n given in problem **??** what is v?

8. Find the least common multiples of the following pairs
a) $(116, -84)$
b) $(85, 65)$
c) $(72, 26)$
d) $(72, 25)$

Solution:
a) 2436.
b) 1105.
c) 936.
d) 1800.

9. If $m, n > 0$ are two integers, show that we can

find integers u, v with $-\frac{n}{2} \leq v \leq \frac{n}{2}$ such that $m = un + v$.

10. To check that a given integer $n > 1$ is a prime, prove that it is enough to show that n is not divisible by any prime p with $p \leq \sqrt{n}$.

1.5 Mathematical Induction

1. Prove that
$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \quad (1.5.1.1)$$
by induction.
Solution: $P(n+1)$ is
$$\begin{aligned} &1^2 + 2^2 + 3^2 + \cdots + n^2 + (n+1)^2 \\ &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\ &= (n+1)\left(\frac{2n^2+7n+7}{6}\right) \\ &= \frac{(n+1)(n+2)(2n+3)}{6} \end{aligned} \quad (1.5.1.2)$$
which is true. Hence, the given proposition is true for all $n \geq 1$

2. Prove that
$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2 \quad (1.5.2.1)$$
by induction.
Solution: $P(n+1)$ is
$$\begin{aligned} &1^3 + 2^3 + 3^3 + \cdots + n^3 + (n+1)^3 \\ &= \left[\frac{n(n+1)}{2}\right]^2 + (n+1)^3 \\ &= (n+1)^2\left(\frac{n^2+4n+4}{4}\right) \\ &= \left[\frac{(n+1)(n+2)}{2}\right]^2 \end{aligned} \quad (1.5.2.2)$$
which is true. Hence, the given proposition is true for all $n \geq 1$.

3. Prove that a set having $n \geq 2$ elements has $\frac{n(n-1)}{2}$ subsets having exactly 2 elements.
4. Prove that a set having $n \geq 3$ elements has $\frac{n(n-1)(n-2)}{3}$ subsets having exactly 3 elements.
5. If $n \geq 4$ and S is a set having n elements, guess how many subsets having exactly 4 elements are there in S. Then verify your guess using mathematical induction.
6. If p is a prime and $p \mid (a_1 a_2 a_3 \ldots a_n)$, then prove using induction that $p \mid a_i$ for some i with $1 \leq i \leq n$.
7. If $a \neq 1$, prove that
$$1 + a + a^2 + \cdots + a^n = \frac{\left(a^{n+1} - 1\right)}{a-1} \quad (1.5.7.1)$$
by induction.
Solution: $P(n+1)$ can be expressed as
$$\begin{aligned} &1 + a + a^2 + \cdots + a^n + a^{n+1} \\ &= \frac{\left(a^{n+1}-1\right)}{a-1} + a^{n+1} \\ &= \frac{\left(a^{n+2}-1\right)}{a-1} \end{aligned} \quad (1.5.7.2)$$
upon simplification. Hence, the given proposition is true for all $n \geq 1$.

8. By induction, show that
$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n\cdot(n+1)} = \frac{n}{n+1} \quad (1.5.8.1)$$
Solution: $P(n+1)$ can be expressed as
$$\begin{aligned} &\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n\cdot(n+1)} + \frac{1}{(n+1)\cdot(n+2)} \\ &= \frac{n}{n+1} + \frac{1}{(n+1)\cdot(n+2)} \\ &= \frac{1}{n+1}\left[n + \frac{1}{n+2}\right] \\ &= \frac{n+1}{n+2} \end{aligned} \quad (1.5.8.2)$$
upon simplification. Hence, the given proposition is true for all $n \geq 1$.

9. Suppose that $P(n)$ is a proposition about positive integers n such that $P(n_0)$ is valid, and if $P(k)$ is true, so must be $P(k+1)$. What can you say about $P(n)$? Prove your statement.
10. Let $P(n)$ be a proposition about integers n such that $P(1)$ is true and such that if $P(j)$ is true for all positive integers $j < k$, then $P(k)$ is true. Prove that $P(n)$ is true for all positive integers n.
11. Given an example of a proposition that is *not* true for any positive integer, yet for which the

induction step holds.

12. Prove by induction that a set having n elements has exactly 2^n subsets.
Solution: Let $S = \{1, 2\}$. Then the subsets are

$$\{\phi\}, \{1\}, \{2\}, \{1, 2\} \tag{1.5.12.1}$$

For $S = \{1, 2, 3\}$, the subsets are

$$\{\phi\}, \{1\}, \{2\}, \{1, 2\} \tag{1.5.12.2}$$
$$\{3\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\} \tag{1.5.12.3}$$

Thus $P(n+1)$ can be expressed as

$$2^n + 2^n = 2^{n+1} \tag{1.5.12.4}$$

Hence, the given proposition is true for all $n \geq 1$.

13. Prove by induction on n that $n^3 - n$ is always divisible by 3.
Solution: $P(n+1)$ can be expressed as

$$(n+1)^3 - (n+1) = n^3 + 3n^2 + 3n + 1 - n - 1 \tag{1.5.13.1}$$
$$= n^3 - n + 3\left(n^2 + n\right) \tag{1.5.13.2}$$

which is divisible by 3. Hence, the given proposition is true for all $n \geq 1$.

14. If p is a prime number, then prove that $n^p - n$ is always divisible by p.
Solution: $P(n+1)$ can be expressed as

$$(n+1)^p - (n+1) = n^p - n + p \sum_{k=1}^{n-1} {}^nC_k p^{k-1} \tag{1.5.14.1}$$
$$\implies p \mid \left[(n+1)^p - (n+1)\right] \tag{1.5.14.2}$$

Hence, the given proposition is true for all $n \geq 1$

15. Prove by induction that for a set having n elements the number of $1-1$ mappings of this set onto itself is $n!$.
Solution: Let $S = \{a, b, c\}$. Then the possible 1-1 onto mappings are

$$\begin{cases} a \mapsto b \\ b \mapsto c \\ c \mapsto a \end{cases} \begin{cases} a \mapsto b \\ b \mapsto a \\ c \mapsto c \end{cases} \begin{cases} a \mapsto c \\ b \mapsto b \\ c \mapsto a \end{cases} \begin{cases} a \mapsto a \\ b \mapsto c \\ c \mapsto b \end{cases} \begin{cases} a \mapsto a \\ b \mapsto b \\ c \mapsto c \end{cases} \begin{cases} a \mapsto a \\ b \mapsto c \\ c \mapsto b \end{cases} \tag{1.5.15.1}$$

1.6 Complex Numbers

1. Multiply
   a) $(6 - 7j)(8 + j)$
   b) $\left(\frac{2}{3} + \frac{3}{2}j\right)\left(\frac{2}{3} - \frac{3}{2}j\right)$
   c) $(6 + 7j)(8 - j)$

   Solution:

   a)
$$(6 - 7j)(8 + j) = \begin{pmatrix} 6 & 7 \\ -7 & 6 \end{pmatrix}\begin{pmatrix} 8 \\ 1 \end{pmatrix} \tag{1.6.1.1}$$
$$= \begin{pmatrix} 53 \\ -50 \end{pmatrix} = 53 - 50j \tag{1.6.1.2}$$

   b)
$$\left(\frac{2}{3} + \frac{3}{2}j\right)\left(\frac{2}{3} - \frac{3}{2}j\right) = \begin{pmatrix} \frac{2}{3} & -\frac{3}{2} \\ \frac{3}{2} & \frac{2}{3} \end{pmatrix}\begin{pmatrix} \frac{2}{3} \\ -\frac{3}{2} \end{pmatrix} \tag{1.6.1.3}$$
$$= \begin{pmatrix} \frac{97}{36} \\ 0 \end{pmatrix} = \frac{97}{36} \tag{1.6.1.4}$$

   c)
$$(6 + 7j)\, 8 - j = \left[(6 - 7j)\, 8 + j\right]^* \tag{1.6.1.5}$$
$$= (53 - 50j)^* = 53 + 50j \tag{1.6.1.6}$$

2. Find z^{-1} for
   a) $z = 6 + 8j$
   b) $z = 6 - 8j$
   c) $z = \frac{1}{\sqrt{2}}(1 + j)$

   Solution:

   a)
$$z^{-1} = \frac{z^*}{|z|^2} = \frac{6 - 8j}{100} \tag{1.6.2.1}$$

   b)
$$z^{-1} = \frac{6 + 8j}{100} \tag{1.6.2.2}$$

   c)
$$z^{-1} = \frac{1 - j}{\sqrt{2}} \tag{1.6.2.3}$$

3. Show that

$$(z^*)^{-1} = \left(z^{-1}\right)^* \tag{1.6.3.1}$$

Solution: Since

$$zz^{-1} = 1, \tag{1.6.3.2}$$
$$\left(zz^{-1}\right)^* = 1 \tag{1.6.3.3}$$
$$\implies (z)^* \left(z^{-1}\right)^* = 1 \tag{1.6.3.4}$$

yielding (1.6.3.1).

4. Find

$$(\cos\theta + \jmath \sin\theta)^{-1} \tag{1.6.4.1}$$

Solution:

$$(\cos\theta + \jmath \sin\theta)^{-1} = \cos\theta - \jmath \sin\theta \tag{1.6.4.2}$$

5. Verify the following
   a) $(z^*)^* = z$
   b) $(z + w)^* = z^* + w^*$
   c) $z + z^* = 2\mathrm{Re}\,(z)$
   d) $z - z^* = 2\jmath\mathrm{Im}\,(z)$

Solution:

a) For

$$z = a + \jmath b, \tag{1.6.5.1}$$
$$z^* = a - \jmath b, \tag{1.6.5.2}$$
$$\implies (z^*)^* = a + \jmath b = z \tag{1.6.5.3}$$

b) For

$$\begin{aligned} z &= z_1 + \jmath z_2 \\ w &= w_1 + \jmath w_2, \end{aligned} \tag{1.6.5.4}$$
$$(z + w)^* = (z_1 + \jmath z_2 + w_1 + \jmath w_2)^* \tag{1.6.5.5}$$
$$= (z_1 - \jmath z_2) + (w_1 - \jmath w_2) \tag{1.6.5.6}$$
$$= z^* + w^* \tag{1.6.5.7}$$

c) For

$$z = a + \jmath b, \tag{1.6.5.8}$$
$$z^* = a - \jmath b, \tag{1.6.5.9}$$
$$\implies (z + z^*) = a + \jmath b + a - jb \tag{1.6.5.10}$$
$$= 2a = 2\mathrm{Re}\,(z) \tag{1.6.5.11}$$

d) For

$$z = a + \jmath b, \tag{1.6.5.12}$$
$$z^* = a - \jmath b, \tag{1.6.5.13}$$
$$\implies (z - z^*) = a + \jmath b - a + jb \tag{1.6.5.14}$$
$$= 2\jmath b = 2\mathrm{Im}\,(z) \tag{1.6.5.15}$$

6. Show that z is real if and only if $z^* = z$ and is purely imaginary if and only if $z^* = -z$.

Solution: Let

$$z = a + \jmath b. \tag{1.6.6.1}$$

Then

$$z^* = a - \jmath b. \tag{1.6.6.2}$$

If

$$z^* = z, \tag{1.6.6.3}$$
$$a + \jmath b = a - \jmath b \tag{1.6.6.4}$$
$$\implies b = 0 \tag{1.6.6.5}$$

and z is real. If z is real,

$$z = a \tag{1.6.6.6}$$
$$\implies z^* = a \tag{1.6.6.7}$$
$$\text{or, } z = z^* \tag{1.6.6.8}$$

Similarly, the other property can be proved.

7. Verify the commutative law of multiplication $zw = wz$ in $\mathbb{C}$.

Solution: Let

$$z = a + \jmath b \tag{1.6.7.1}$$
$$w = x - \jmath y \tag{1.6.7.2}$$

Then

$$zw = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{1.6.7.3}$$
$$= \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \tag{1.6.7.4}$$
$$= wz \tag{1.6.7.5}$$

8. Show that for $z \neq 0$, $|z|^{-1} = \frac{1}{|z|}$.

Solution: Let

$$z = re^{\jmath\theta}. \tag{1.6.8.1}$$

Then

$$z^{-1} = \frac{1}{r} e^{-\jmath\theta} \tag{1.6.8.2}$$
$$\implies \left|z^{-1}\right| = \frac{1}{r} \tag{1.6.8.3}$$

9. Find
   a) $\left|6 - 4\jmath\right|$.
   b) $\left|\frac{1}{2} + \frac{2}{3}\jmath\right|$.
   c) $\left|\frac{1}{\sqrt{2}}\left(1 + \jmath\right)\right|$

Solution:

a)

$$\left|6-4\jmath\right| = \sqrt{6^2+4^2} = 2\sqrt{13} \quad (1.6.9.1)$$

b)

$$\left|\frac{1}{2}+\frac{2}{3}\jmath\right| = \frac{5}{6} \quad (1.6.9.2)$$

c)

$$\left|\frac{1}{\sqrt{2}}(1+\jmath)\right| = \frac{1}{\sqrt{2}}\left|(1+\jmath)\right| = 1 \quad (1.6.9.3)$$

10. Show that $|z^*| = |z|$.
Solution: Let

$$z = re^{\jmath\theta} \quad (1.6.10.1)$$

Then

$$z^* = re^{-\jmath\theta} \quad (1.6.10.2)$$
$$\implies |z^*| = r = |z|. \quad (1.6.10.3)$$

11. Find the polar form for
a) $z = \frac{\sqrt{2}}{2} - \frac{1}{\sqrt{2}}\jmath$.
b) $z = 4\jmath$.
c) $z = \frac{6}{\sqrt{2}} + \frac{6}{\sqrt{2}}\jmath$.
d) $z = -\frac{13}{2} + \frac{39}{2\sqrt{3}}\jmath$.
Solution:
a)

$$|z| = \sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}. \quad (1.6.11.1)$$
$$= 1 \quad (1.6.11.2)$$

and

$$\angle z = -\tan^{-1}\frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} \quad (1.6.11.3)$$
$$= \frac{\pi}{4} \quad (1.6.11.4)$$

b)

$$|z| = 4, \angle z = \frac{\pi}{2}. \quad (1.6.11.5)$$

c)

$$|z| = \frac{6}{\sqrt{2}}, \angle z = \frac{\pi}{4}. \quad (1.6.11.6)$$

d)

$$|z| = \frac{13}{2}\sqrt{1+3} \quad (1.6.11.7)$$
$$= 13 \quad (1.6.11.8)$$

and

$$\angle z = \pi - \tan^{-1}\frac{\frac{39}{2\sqrt{3}}}{\frac{13}{2}} \quad (1.6.11.9)$$
$$= \pi - \tan^{-1}\sqrt{3} = \frac{2\pi}{3} \quad (1.6.11.10)$$

12. Prove that

$$\left(\cos\left(\frac{\theta}{2}\right) + \jmath\sin\left(\frac{\theta}{2}\right)\right)^2 = \cos(\theta) + \jmath\sin(\theta) \quad (1.6.12.1)$$

Solution: The L.H.S can be expressed as

$$\left(e^{\jmath\theta}\right)^2 = e^{\jmath\theta} \quad (1.6.12.2)$$

13. Show that

$$\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\jmath\right)^3 = -1 \quad (1.6.13.1)$$

Solution:

$$\frac{1}{2} + \frac{\sqrt{3}}{2}\jmath = e^{\frac{\jmath\pi}{3}} \quad (1.6.13.2)$$
$$\implies \left(e^{\frac{\jmath\pi}{3}}\right)^3 = e^{\jmath\pi} = -1 \quad (1.6.13.3)$$

14. Show that

$$(\cos(\theta) + \jmath\sin(\theta))^m = \cos(m\theta) + \jmath\sin(m\theta) \quad (1.6.14.1)$$

for all integers m. **Solution:** It is easy to verify that

$$(\cos(\theta) + \jmath\sin(\theta))^2 = \cos(2\theta) + \jmath\sin(2\theta) \quad (1.6.14.2)$$

Then

$$(\cos(\theta) + \jmath\sin(\theta))^{k+1} = (\cos(\theta) + \jmath\sin(\theta))^k (\cos(m\theta) + \jmath\sin(m\theta))$$
$$= \cos\left[(k+1)\theta + \jmath\sin((k+1)\theta)\right] \quad (1.6.14.3)$$

By induction, (1.6.14.1) is proved.

15. Show that

$$(\cos(\theta) + \jmath\sin(\theta))^r = \cos(r\theta) + \jmath\sin(r\theta) \quad (1.6.15.1)$$

for all rational numbers r. **Solution:** Let

$$r = \frac{m}{n}, (\cos(\theta) + \jmath \sin(\theta))^{\frac{1}{n}} = \cos(\alpha) + \jmath \sin(\alpha) \tag{1.6.15.2}$$

Then

$$(\cos(\alpha) + \jmath \sin(\alpha))^n = (\cos(\theta) + \jmath \sin(\theta)) \tag{1.6.15.3}$$

$$\implies \cos(n\alpha) + \jmath \sin(n\alpha) = (\cos(\theta) + \jmath \sin(\theta)) \tag{1.6.15.4}$$

$$\text{or, } \alpha = \frac{\theta}{n} \tag{1.6.15.5}$$

yielding

$$(\cos(\theta) + \jmath \sin(\theta))^{\frac{1}{n}} = \cos\left(\frac{\theta}{n}\right) + \jmath \sin\left(\frac{\theta}{n}\right) \tag{1.6.15.6}$$

Using (1.6.14.1) and (1.6.15.6),

$$(\cos(\theta) + \jmath \sin(\theta))^{\frac{m}{n}} = \cos\left(\frac{m\theta}{n}\right) + \jmath \sin\left(\frac{m\theta}{n}\right) \tag{1.6.15.7}$$

16. If $z \in \mathbb{C}$ and $n \geq 1$ is any positive integer, show that there are n distinct complex numbers such that $z = w^n$. **Solution:** Let

$$z = \cos(\theta) + \jmath \sin(\theta) \tag{1.6.16.1}$$

then using (1.6.15.6),

$$w = \cos\left(\frac{2\pi k + \theta}{n}\right) + \jmath \sin\left(\frac{2\pi k + \theta}{n}\right), k = 0, \dots, n-1 \tag{1.6.16.2}$$

which are the distinct roots.

17. Find the necessary and sufficient condition on k such that

$$\left(\cos\left(\frac{2\pi k}{n}\right) + \jmath \sin\left(\frac{2\pi k}{n}\right)\right)^n = 1 \quad \text{and} \tag{1.6.17.1}$$

$$\left(\cos\left(\frac{2\pi k}{n}\right) + \jmath \sin\left(\frac{2\pi k}{n}\right)\right)^m \neq 1 \quad 0 < m < n \tag{1.6.17.2}$$

Solution: From the above equations, using (1.6.14.1),

$$\frac{mk}{n} \notin \mathbb{Z} \tag{1.6.17.3}$$

18. Viewing the $x-y$ plane as the set of all complex numbers $x + \jmath y$, show that multiplication by j induces as $90°$ rotation of the $x - y$ plan in counterclockwise direction.
Solution: The given multiplication can be expressed using matrices as

$$\begin{pmatrix} \cos 90° & -\sin 90° \\ \sin 90° & \cos 90° \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{1.6.18.1}$$

which is the multiplication of $\begin{pmatrix} x \\ y \end{pmatrix}$ with a $90°$ rotation matrix.

19. In problem (1.6.18), interpret geometrically what multiplication by the complex number $a + \jmath b$ does to the $x - y$ plane.
Solution: The multiplication can be represented as

$$\sqrt{a^2 + b^2} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{1.6.19.1}$$

where

$$\cos\theta = \frac{a}{\sqrt{a^2 + b^2}} \sin\theta \qquad = \frac{b}{\sqrt{a^2 + b^2}} \tag{1.6.19.2}$$

Geometrically, multiplication by $a + \jmath b$ results in rotation by θ and scaling by $\sqrt{a^2 + b^2}$.

20. Prove that

$$|z + w|^2 + |z - w|^2 = 2\left(|z|^2 + |w|^2\right) \tag{1.6.20.1}$$

Solution: Since

$$|z + w|^2 = (z + w)^* (z + w) \tag{1.6.20.2}$$

$$= |z|^2 + |w|^2 + 2z^* w \tag{1.6.20.3}$$

and

$$|z - w|^2 = (z + w)^* (z - w) \tag{1.6.20.4}$$

$$= |z|^2 + |w|^2 - 2z^* w, \tag{1.6.20.5}$$

$$|z + w|^2 + |z - w|^2 = 2\left(|z|^2 + |w|^2\right) \tag{1.6.20.6}$$

21. Consider the set $A = a + b\jmath, a, b \in \mathbb{Z}$. Prove that there is 1-1 correspondence of A onto $\mathbb{N}$.

22. If a is a (complex) root of the polynomial

$$x^n + \alpha_1 x^{n-1} + \cdots + \alpha_{n-1} x + \alpha_n, \tag{1.6.22.1}$$

where the α_i are real, show that $\bar{a}$ must also be a root.
Solution: From the given information,

$$\bar{a}^n + \alpha_1 \bar{a}^{n-1} + \cdots + \alpha_{n-1} \bar{a} + \alpha_n = 0 \tag{1.6.22.2}$$

Thus, $\bar{a}$ is also a root of the given polynomial.

23. Find the necessary and sufficient conditions on z and w in order that

$$|z+w| = |z| + |w| \tag{1.6.23.1}$$

Solution:

$$|z+w|^2 = |z|^2 + |w|^2 + 2z^* w \tag{1.6.23.2}$$
$$(|z| + |w|)^2 = |z|^2 + |w|^2 + 2\,|z|\,|w| \tag{1.6.23.3}$$

If the above expressions are equal,

$$z^* w = |z|\,|w| \tag{1.6.23.4}$$

which is the desired condition.

24. Find the necessary and sufficient conditions on z_i in order that

$$\left|\sum_{i=1}^{k} z_i\right| = \sum_{i=1}^{k} |z_i| \tag{1.6.24.1}$$

Solution:

$$\left|\sum_{i=1}^{k} z_i\right|^2 = \sum_{i=1}^{k} |z_i|^2 + 2 \sum_{\substack{i=1,j=1\\ i\neq j}}^{k} {z_i}^* z_j \tag{1.6.24.2}$$
$$\left(\sum_{i=1}^{k} |z_i|\right)^2 = \sum_{i=1}^{k} |z_i|^2 + 2 \sum_{\substack{i=1,j=1\\ i\neq j}}^{k} |z_i|\left|z_j\right| \tag{1.6.24.3}$$

From (1.6.24.2) and (1.6.24.3),

$$\begin{aligned} &\sum_{i=1}^{k} |z_i|^2 + 2 \sum_{\substack{i=1,j=1\\ i\neq j}}^{k} {z_i}^* z_j \\ &\quad = \sum_{i=1}^{k} |z_i|^2 + 2 \sum_{\substack{i=1,j=1\\ i\neq j}}^{k} |z_i|\left|z_j\right| \\ &\qquad \implies \sum_{\substack{i=1,j=1\\ i\neq j}}^{k} {z_i}^* z_j \\ &\qquad\quad = \sum_{\substack{i=1,j=1\\ i\neq j}}^{k} |z_i|\left|z_j\right| \end{aligned} \tag{1.6.24.4}$$

which is the desired condition.

25. The complex number θ is said to have *order* $n \geq 1$ if $\theta^n = 1$ and $\theta^m \neq 1$ for $0 < m < n$. Show that if θ has order n and $\theta^k = 1$, where $k > 0$, then $n|k$.

Solution: From the given information,

$$\theta^n = \theta^k = 1, k \geq n \tag{1.6.25.1}$$

If $n \nmid k, k = mn + p, 0 < p < n$, Then,

$$\theta^k = \theta^{mn+p} = \theta^p \neq 1, \tag{1.6.25.2}$$

which is a contradiction, Hence, $n \mid k$.

26. Find all complex numbers θ having order n.

Solution: If

$$\theta^n = 1, \tag{1.6.26.1}$$
$$\theta^n = e^{\jmath 2\pi r}, 0 \leq r < n \tag{1.6.26.2}$$

yielding

$$\theta = \exp\left(\jmath \frac{2\pi r}{n}\right) 0 \leq r < n \tag{1.6.26.3}$$