Probability

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***Abstract*—This book provides solved examples on Probability from IES stats question papers.**

1 2015

1.1. For random variables X and Y, show that: $Var[Y] = E[Var(Y|X)] + Var[E(Y|X)]$ **Solution:**
Let the abbreviations LE and LIE denote linearity of expectations and law of iterated expectations respectively.

$$Var[Y] = E[Y^2] - [E(Y)]^2 \text{ (definition)} \tag{1.1.1}$$

$$= E[E(Y^2|X)] - (E[E(Y|X)])^2 \text{ (LIE)} \tag{1.1.2}$$

$$= E[E(Y^2|X)] - (E[E(Y|X)])^2 - E([E(Y|X)]^2) + E([E(Y|X)]^2) \tag{1.1.3}$$

$$= E[E(Y^2|X)] - E([E(Y|X)]^2) + E([E(Y|X)]^2) - (E[E(Y|X)])^2 \text{ (LE \& LIE)} \tag{1.1.4}$$

$$= Var[E(Y|X)] + E[Var(Y|X)] \text{ (definition)} \tag{1.1.5}$$

Hence, proved.

1.2. Let X be a Random Variable with $E[X] = 3$, $E[X^2] = 13$. Use Chebyshev's Inequality to obtain $\Pr(-2 < X < 8)$
Solution: Let X be a random variable with finite expected value $E[X]$ and finite non-zero variance σ^2. Then for any real number k > 0,

$$\Pr(|X - E[X]| \geq k\sigma) \leq \frac{1}{k^2} \tag{1}$$

computing the variance,

$$\sigma^2 = E[X^2] - E[X]^2$$

$$\implies \sigma^2 = 13 - 9 = 4 \tag{2}$$

$$\sigma = 2 \tag{3}$$

using (3),

$$\Pr(-2 < X < 8) = 1 - \Pr(|X - 3| > 5) \quad (4)$$

$$\Pr(|X - 3| > 5) = \Pr(|X - E[X]| > k\sigma) \quad (5)$$

$$k\sigma = 5$$

$$\implies 2k = 5$$

$$\therefore k = \frac{5}{2} \quad (6)$$

Using (1) , (5) and (6) in (4),

$$\Pr(-2 < X < 8) \geq 1 - \left(\frac{2}{5}\right)^2$$

$$\implies \Pr(-2 < X < 8) \geq \frac{21}{25} \quad (7)$$

1.3. Three points are chosen on the line of unit length.Find the probability that each the 3 line segments have length greater than $\frac{1}{4}$.

Solution:

Let $X, Y \in \{0, 1\}$ be the random variables which represent the position of two points on the line of unit length.

Conditions which should be satisfied to have three line segments with length greater than $\frac{1}{4}$ are given in the below table.

Event	**Condition**
A	$\frac{1}{4} < X < \frac{3}{4}$
B	$\frac{1}{4} < Y < \frac{3}{4}$
C	$\frac{1}{4} < X - Y$
D	$\frac{1}{4} < Y - X$

TABLE 1.3.1: Events and their conditions

Then the required event which solves the problem is $ABC+ABD$.

$$\Pr(ABC) = \Pr\left(\frac{1}{4} + Y < X, \frac{1}{4} < X, Y < \frac{3}{4}\right) \quad (1.3.1)$$

$$= \sum \Pr\left(Y = y | \frac{1}{4} < X, Y < \frac{3}{4}\right) \times \Pr\left(\frac{1}{4} + y < X, \frac{1}{4} < X < \frac{3}{4}\right) \quad (1.3.2)$$

$$= \int_{\frac{1}{4}}^{\frac{3}{4}} dy f_Y(y) \times \Pr\left(\frac{1}{4} + y < X, \frac{1}{4} < X < \frac{3}{4}\right) \quad (1.3.3)$$

$$= \int_{\frac{1}{4}}^{\frac{3}{4}} dy f_Y(y) \Pr\left(\frac{1}{4} + y < X < \frac{3}{4}\right) \quad (1.3.4)$$

As X is distributed uniformly between 0 and 1.

$$\Pr\left(\frac{1}{4} + y < X < \frac{3}{4}\right) = \begin{cases} \frac{1}{2} - y & y \in \left(0, \frac{1}{2}\right) \\ 0 & \text{otherwise} \end{cases} \quad (1.3.5)$$

Using (1.3.5),(1.3.4) can be written as

$$\Pr(ABC) = \int_{\frac{1}{4}}^{\frac{1}{2}} dy f_Y(y) \left(\frac{1}{2} - y\right) \quad (1.3.6)$$

As y is distributed uniformly between 0 and 1.

$$\Pr(ABC) = \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{1}{2} - y \, dy \quad (1.3.7)$$

$$= \frac{1}{32} \quad (1.3.8)$$

Similarly,we can find,

$$\Pr(ABD) = \frac{1}{32} \quad (1.3.9)$$

As C and D are mutually exclusive events.

$$\Pr(ABC + ABD) = \Pr(ABC) + \Pr(ABD) \quad (1.3.10)$$

$$= \frac{1}{16} \quad (1.3.11)$$

$\therefore$ probability that each of the three line segments have length greater than $\frac{1}{4}$ is $\frac{1}{16}$.

2 2016

2.1. Let the random variable X have the distribution $P(X = 0) = P(X = 3) = p$, $P(X = 1) = 1 - 3p$ for $0 \le p \le \frac{1}{2}$. What is the maximum value of V(X)?

A) 3
B) 4
C) 5
D) 6
E) none

Solution: Given, for $0 \le p \le \frac{1}{2}$,

$$P(X = 0) = p \tag{2.1.1}$$
$$P(X = 1) = 1 - 3p \tag{2.1.2}$$
$$P(X = 3) = p \tag{2.1.3}$$

Now consider $P(X = 1) = 1 - 3p$ for $p = \frac{1}{2}$. We get,

$$P(X = 1) = 1 - 3p \tag{2.1.4}$$
$$= 1 - (3)\left(\frac{1}{2}\right) \tag{2.1.5}$$
$$= 1 - \frac{3}{2} \tag{2.1.6}$$
$$= -\frac{1}{2} < 0 \tag{2.1.7}$$

Probability cannot be negative. But in equation (0.0.7) probability is negative, which is not possible.
Therefore, the question is not a proper one.

Answer : Option E

2.2. X_1 and X_2 are independent Poisson variables such that $\Pr(X_1 = 2) = \Pr(X_1 = 1)$ and $\Pr(X_2 = 2) = \Pr(X_2 = 3)$. What is the variance of $(X_1 - 2X_2)$?

a) 14
b) 4
c) 3
d) 2

Solution: For a Poisson variable X,

$$\Pr(X = k) = \frac{\lambda^k e^{-\lambda}}{k!} \tag{2.2.1}$$

Since $\Pr(X_1 = 2) = \Pr(X_1 = 1)$,

$$\frac{{\lambda_1}^2 e^{-\lambda_1}}{2!} = \frac{{\lambda_1}^1 e^{-\lambda_1}}{1!} \tag{2.2.2}$$
$$\lambda_1 = 2!/1! = 2 \tag{2.2.3}$$

Similarly, as $\Pr(X_2 = 2) = \Pr(X_2 = 3)$,

$$\frac{{\lambda_2}^2 e^{-\lambda_2}}{2!} = \frac{{\lambda_2}^3 e^{-\lambda_2}}{3!} \tag{2.2.4}$$
$$\lambda_2 = 3!/2! = 3 \tag{2.2.5}$$

Also we know for a Poisson variable X, the following holds true:

$$\mathrm{E}[X] = \lambda \tag{2.2.6}$$
$$\mathrm{Var}[X] = \lambda \tag{2.2.7}$$
$$\mathrm{Var}[X] = \mathrm{E}[X^2] - (\mathrm{E}[X])^2 \tag{2.2.8}$$

Now, for the variance of $(X_1 - 2X_2)$

$$\begin{aligned}\mathrm{Var}[X_1 - 2X_2] &= \mathrm{E}[(X_1 - 2X_2)^2] - (\mathrm{E}[X_1 - 2X_2])^2 \\ &= \mathrm{E}[X_1^2 + 4X_2^2 - 4X_1X_2] \\ &\quad - (\mathrm{E}[X_1] - 2\mathrm{E}[X_2])^2 \\ &= \mathrm{E}[X_1^2] - (\mathrm{E}[X_1])^2 + 4\mathrm{E}[X_2^2] \\ &\quad - 4(\mathrm{E}[X_2])^2) + 4\mathrm{E}[X_1X_2] \\ &\quad + 4\mathrm{E}[X_1]\mathrm{E}[X_2] \end{aligned} \tag{2.2.9}$$

Since the variables are independent:

$$\mathrm{E}[X_1X_2] = \mathrm{E}[X_1]\mathrm{E}[X_2] \tag{2.2.10}$$

Substituting equations (2.2.7) and (2.2.8), we get:

$$\begin{aligned}\mathrm{Var}[X_1 - 2X_2] &= \mathrm{Var}[X_1] + 4(\mathrm{Var}[X_2]) \\ &\quad - 4\mathrm{E}[X_1][X_2] + 4\mathrm{E}[X_1][X_2] \\ &= \lambda_1 + 4\lambda_2 = 2 + 4(3) = 14 \end{aligned} \tag{2.2.11}$$

Hence option (a) 14 is correct.