Assignment 7

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Download all python codes from

```
https://github.com/pranav−159/
    ai1103_Probability_and_Random_variables/
    blob/main/Assignment_7/codes/
    experimental_verification_Assignment7.py
```

1 Problem

gov/stats/2015/statistics-I(1), Q.3(C)

Three points are chosen on the line of unit length.Find the probability that each the 3 line segments have length greater than $\frac{1}{4}$.

2 Solution

Let $X, Y \in \{0, 1\}$ be the random variables which represent the position of two points on the line of unit length.
Conditions which should be satisfied to have three line segments with length greater than $\frac{1}{4}$ are given in the below table.

Event	Condition
A	$\frac{1}{4} < X < \frac{3}{4}$
B	$\frac{1}{4} < Y < \frac{3}{4}$
C	$\frac{1}{4} < \lvert X - Y \rvert$

TABLE 0: Events and their conditions

Then the required event which solves the problem is ABC.
As A and B are independent events.

$$\Pr(ABC) = \Pr(A)\Pr(B)\Pr(C|AB) \tag{2.0.1}$$

As X and Y are uniformly distributed between 0 and 1.

$$\Pr(A) = \Pr\left(\frac{1}{4} < X < \frac{3}{4}\right) = \frac{1}{2} \tag{2.0.2}$$

$$\Pr(B) = \Pr\left(\frac{1}{4} < Y < \frac{3}{4}\right) = \frac{1}{2} \tag{2.0.3}$$

When event AB is known to occur,X and Y are uniformly distributed between $\frac{1}{4}$ and $\frac{3}{4}$

Conditional probability density functions of X and Y with condition AB are

$$f_{X|AB}(x) = \begin{cases} 2 & x \in \left(\frac{1}{4}, \frac{3}{4}\right) \\ 0 & \text{otherwise} \end{cases} \tag{2.0.4}$$

$$f_{Y|AB}(y) = \begin{cases} 2 & y \in \left(\frac{1}{4}, \frac{3}{4}\right) \\ 0 & \text{otherwise} \end{cases} \tag{2.0.5}$$

Let $Z = X - Y$.Then pdf of Z with condition AB can be given as

$$f_{Z|AB}(z) = \int_{-\infty}^{\infty} f_{X|AB}(y+z) f_{Y|AB}(y)\ dy \tag{2.0.6}$$

$$f_{Z|AB}(z) = \int_{\frac{1}{4}}^{\frac{3}{4}} f_{X|AB}(y+z) f_{Y|AB}(y)\ dy \tag{2.0.7}$$

if $z \in \left[0, \frac{1}{2}\right]$,

$$f_{Z|AB}(z) = \int_{\frac{1}{4}}^{\frac{3}{4}-z} f_{X|AB}(y+z) f_{Y|AB}(y)\ dy \tag{2.0.8}$$

$$f_{Z|AB}(z) = 4\left(\frac{1}{2} - z\right) \tag{2.0.9}$$

$$f_{Z|AB}(z) = 2 - 4z \tag{2.0.10}$$

if $z \in \left[-\frac{1}{2}, 0\right]$,

$$f_{Z|AB}(z) = \int_{\frac{1}{4}-z}^{\frac{3}{4}} f_{X|AB}(y+z) f_{Y|AB}(y)\, dy \tag{2.0.11}$$

$$f_{Z|AB}(z) = 4\left(\frac{1}{2} + z\right) \tag{2.0.12}$$

$$f_{Z|AB}(z) = 2 + 4z \tag{2.0.13}$$

if $z \in \left(-\infty, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$,

$$f_{Z|AB}(z) = 0 \tag{2.0.14}$$

Interval	$f_{X\|AB}$
$z \in \left[0, \frac{1}{2}\right]$	$2 - 4z$
$z \in \left[-\frac{1}{2}, 0\right]$	$2 + 4z$
$z \in \left(-\infty, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$	0

TABLE 0: Pdf of Z in different intervals

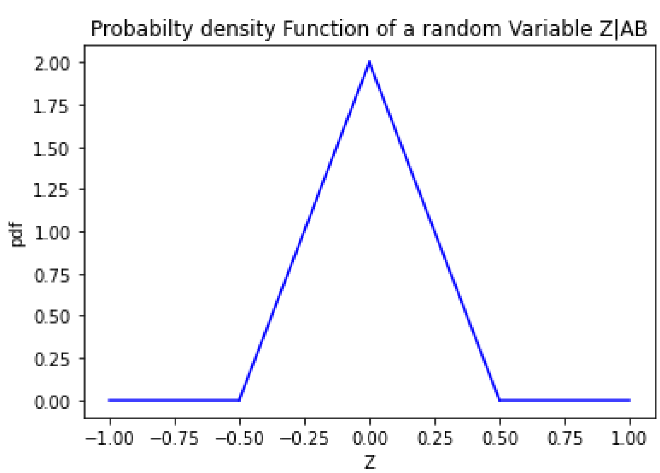

Fig. 0: pdf Z with AB already occurred

$$\Pr(C|AB) = \int_{|z|>\frac{1}{4}} f_{z|AB}\, dz \tag{2.0.15}$$

$$= \int_{-\frac{1}{2}}^{-\frac{1}{4}} f_{z|AB}\, dz + \int_{\frac{1}{4}}^{\frac{1}{2}} f_{z|AB}\, dz \tag{2.0.16}$$

$$= \frac{1}{8} + \frac{1}{8} \tag{2.0.17}$$

$$= \frac{1}{4} \tag{2.0.18}$$

$$\Pr(ABC) = \Pr(A)\Pr(B)\Pr(C|AB) \tag{2.0.19}$$

$$\Pr(ABC) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{4} \tag{2.0.20}$$

$$\Pr(ABC) = \frac{1}{16} \tag{2.0.21}$$

$\therefore$ probability that each of the three line segments have length greater than $\frac{1}{4}$ is $\frac{1}{16}$.

3 Alternate Method

Let $X, Y \in \{0, 1\}$ be the random variables which represent the position of two points on the line of unit length.
Conditions which should be satisfied to have three line segments with length greater than $\frac{1}{4}$ are given

Event	Condition
A	$\frac{1}{4} < X < \frac{3}{4}$
B	$\frac{1}{4} < Y < \frac{3}{4}$
C	$\frac{1}{4} < X - Y$
D	$\frac{1}{4} < Y - X$

TABLE 0: Events and their conditions

in the below table.
Then the required event which solves the problem

is $ABC+ABD$.

$$\Pr(ABC) = \Pr\left(\frac{1}{4} + Y < X, \frac{1}{4} < X, Y < \frac{3}{4}\right) \quad (3.0.1)$$

$$= \sum \Pr\left(Y = y | \frac{1}{4} < X, Y < \frac{3}{4}\right) \times \Pr\left(\frac{1}{4} + y < X, \frac{1}{4} < X < \frac{3}{4}\right) \quad (3.0.2)$$

$$= \int_{\frac{1}{4}}^{\frac{3}{4}} dy f_Y(y) \times \Pr\left(\frac{1}{4} + y < X, \frac{1}{4} < X < \frac{3}{4}\right) \quad (3.0.3)$$

$$= \int_{\frac{1}{4}}^{\frac{3}{4}} dy f_Y(y) \Pr\left(\frac{1}{4} + y < X < \frac{3}{4}\right) \quad (3.0.4)$$

As X is distributed uniformly between 0 and 1.

$$\Pr\left(\frac{1}{4} + y < X < \frac{3}{4}\right) = \begin{cases} \frac{1}{2} - y & y \in \left(0, \frac{1}{2}\right) \\ 0 & \text{otherwise} \end{cases} \quad (3.0.5)$$

Using (3.0.5),(3.0.4) can be written as

$$\Pr(ABC) = \int_{\frac{1}{4}}^{\frac{1}{2}} dy f_Y(y) \left(\frac{1}{2} - y\right) \quad (3.0.6)$$

As y is distributed uniformly between 0 and 1.

$$\Pr(ABC) = \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{1}{2} - y \; dy \quad (3.0.7)$$

$$= \frac{1}{32} \quad (3.0.8)$$

Similarly,we can find,

$$\Pr(ABD) = \frac{1}{32} \quad (3.0.9)$$

As C and D are mutually exclusive events.

$$\Pr(ABC + ABD) = \Pr(ABC) + \Pr(ABD) \quad (3.0.10)$$

$$= \frac{1}{16} \quad (3.0.11)$$

$\therefore$ probability that each of the three line segments have length greater than $\frac{1}{4}$ is $\frac{1}{16}$.