Solutions to Plane Coordinate Geometry by S L Loney

G V V Sharma*

Contents

***Abstract*—This book provides a vector approach to analytical geometry. The content and exercises are based on S L Loney's book on Plane Coordinate Geometry.**

1 Coordinates

1.1 1

1.1.1. Find the distance between the pair of points (2,3) and (5,7).
Solution: Let

$$\mathbf{A} = \begin{pmatrix}2\\3\end{pmatrix}, \mathbf{B} = \begin{pmatrix}5\\7\end{pmatrix} \tag{1.1.1.1}$$

$$\because \mathbf{B}-\mathbf{A} = \begin{pmatrix}5-2\\7-3\end{pmatrix} = \begin{pmatrix}3\\4\end{pmatrix}, \tag{1.1.1.2}$$

$$\|\mathbf{B}-\mathbf{A}\| = \left\|\begin{pmatrix}3\\4\end{pmatrix}\right\| \tag{1.1.1.3}$$

$$= \sqrt{3^2+4^2} = \sqrt{25} = 5 \tag{1.1.1.4}$$

1.1.2.

1.1.3.

1.1.4. Find the distance between the following pair of points

$$\mathbf{A} = \begin{pmatrix}a\\0\end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix}0\\b\end{pmatrix} \tag{1.1.4.1}$$

Solution: The distance between two vectors is given by

$$\|\mathbf{A}-\mathbf{B}\| = \sqrt{\left(\mathbf{A}-\mathbf{B}\right)^\top\left(\mathbf{A}-\mathbf{B}\right)} \tag{1.1.4.2}$$

$$\tag{1.1.4.3}$$

$\because$

$$\mathbf{A}-\mathbf{B} = \begin{pmatrix}a\\-b\end{pmatrix} \tag{1.1.4.4}$$

$$\|\mathbf{A}-\mathbf{B}\| = \sqrt{\begin{pmatrix}a & -b\end{pmatrix}\begin{pmatrix}a\\-b\end{pmatrix}} \tag{1.1.4.5}$$

$$= \sqrt{\left(a^2+b^2\right)} \tag{1.1.4.6}$$

1.1.5. Find the distance between the following pair of points

$$\mathbf{A} = \begin{pmatrix}b+c\\c+a\end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix}c+a\\a+b\end{pmatrix} \tag{1.1.5.1}$$

Solution: The distance between two vectors is given by

$$\|\mathbf{A}-\mathbf{B}\| = \sqrt{\left(\mathbf{A}-\mathbf{B}\right)^\top\left(\mathbf{A}-\mathbf{B}\right)} \tag{1.1.5.2}$$

$$\tag{1.1.5.3}$$

$$\because \mathbf{A}-\mathbf{B} = \begin{pmatrix}b+c\\c+a\end{pmatrix} - \begin{pmatrix}c+a\\a+b\end{pmatrix} \tag{1.1.5.4}$$

$$= \begin{pmatrix}b-a\\c-b\end{pmatrix}, \tag{1.1.5.5}$$

$$(\mathbf{A}-\mathbf{B})^\top(\mathbf{A}-\mathbf{B}) = \sqrt{\begin{pmatrix}b-a & c-b\end{pmatrix}\begin{pmatrix}b-a\\c-b\end{pmatrix}} \tag{1.1.5.6}$$

$$= \sqrt{\left((b-a)^2+(c-b)^2\right)} \tag{1.1.5.7}$$

1.1.6.

1.1.7. Find the distance between the following pairs of points

$$\begin{pmatrix}am_1^2\\2am_1\end{pmatrix}, \begin{pmatrix}am_2^2\\2am_2\end{pmatrix} \tag{1.1.7.1}$$

Solution: The distance between two vectors is

*The author is with the Department of Electrical Engineering, Indian Institute of Technology, Hyderabad 502285 India e-mail: gadepall@iith.ac.in.

given by

$$\|\mathbf{A}-\mathbf{B}\| = \sqrt{\left(\mathbf{A}-\mathbf{B}\right)^{\top}\left(\mathbf{A}-\mathbf{B}\right)} \tag{1.1.7.2}$$

Let $\mathbf{A} = \begin{pmatrix} am_1^2 \\ 2am_1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} am_2^2 \\ 2am_2 \end{pmatrix}$

$$\mathbf{A}-\mathbf{B} = \begin{pmatrix} am_1^2 \\ 2am_1 \end{pmatrix} - \begin{pmatrix} am_2^2 \\ 2am_2 \end{pmatrix} \tag{1.1.7.3}$$

$$= \begin{pmatrix} am_1^2 - am_2^2 \\ 2am_1 - 2am_2 \end{pmatrix} \tag{1.1.7.4}$$

$$= a\begin{pmatrix} m_1^2 - m_2^2 \\ 2\left(m_1 - m_2\right) \end{pmatrix} \tag{1.1.7.5}$$

$$= a\left(m_1 - m_2\right)\begin{pmatrix} m_1 + m_2 \\ 2 \end{pmatrix} \tag{1.1.7.6}$$

by using the property of $\|k\mathbf{A}\| = |k| \ \|\mathbf{A}\|$

$$\|\mathbf{A}-\mathbf{B}\| = \left\|a\left(m_1 - m_2\right)\begin{pmatrix} m_1 + m_2 \\ 2 \end{pmatrix}\right\| \tag{1.1.7.7}$$

$$= \left|a\left(m_1 - m_2\right)\right| \left\|\begin{pmatrix} m_1 + m_2 \\ 2 \end{pmatrix}\right\| \tag{1.1.7.8}$$

$$= \left|a\left(m_1 - m_2\right)\right| \sqrt{\begin{pmatrix} m_1 + m_2 \\ 2 \end{pmatrix}^{\top} \begin{pmatrix} m_1 + m_2 \\ 2 \end{pmatrix}} \tag{1.1.7.9}$$

$$= \left|a\left(m_1 - m_2\right)\right| \sqrt{\begin{pmatrix} m_1 + m_2 & 2 \end{pmatrix} \begin{pmatrix} m_1 + m_2 \\ 2 \end{pmatrix}} \tag{1.1.7.10}$$

$$= \left|a\left(m_1 - m_2\right)\right| \sqrt{\left(m_1 + m_2\right)^2 + \left(2\right)^2} \tag{1.1.7.11}$$

$$= \left|a\left(m_1 - m_2\right)\right| \sqrt{\left(m_1 + m_2\right)^2 + 4} \tag{1.1.7.12}$$

Distance between $(am_1^2, 2am_1)$ and $(am_2^2, 2am_2)$ is

$$= \left|a\left(m_1 - m_2\right)\right| \sqrt{\left(m_1 + m_2\right)^2 + 4} \tag{1.1.7.13}$$

1.1.8. Find the distance between the following pairs of points

$$\begin{pmatrix} 1 \\ -3 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \end{pmatrix} \tag{1.1.8.1}$$

Solution:

$$\because \mathbf{A}-\mathbf{B} = \begin{pmatrix} 1 \\ -3 \end{pmatrix} - \begin{pmatrix} -2 \\ 1 \end{pmatrix} \tag{1.1.8.2}$$

$$= \begin{pmatrix} 3 \\ -4 \end{pmatrix}, \tag{1.1.8.3}$$

Hence,

$$\|\mathbf{A}-\mathbf{B}\| = \sqrt{\left(\mathbf{A}-\mathbf{B}\right)^{\top}\left(\mathbf{A}-\mathbf{B}\right)} \tag{1.1.8.4}$$

$$= \sqrt{\begin{pmatrix} 3 & -4 \end{pmatrix}\begin{pmatrix} 3 \\ -4 \end{pmatrix}} \tag{1.1.8.5}$$

$$= \sqrt{3^2 + \left(-4\right)^2} \tag{1.1.8.6}$$

$$= 5 \tag{1.1.8.7}$$

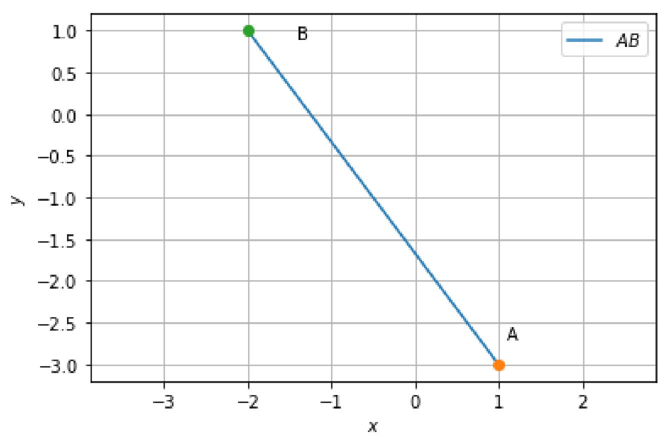

Fig. 1.1.8.1

1.1.9.

1.1.10. A line of length 10 and one end is at point (2, -3); if the abscissa of the other end be 10, Prove that its ordinate must be 3 or -9.

$$\begin{pmatrix} 2 \\ -3 \end{pmatrix}, \begin{pmatrix} 10 \\ y_1 \end{pmatrix} \tag{1.1.10.1}$$

Solution:

1.1.11.

1.1.12.

1.1.13.

1.1.14.

1.1.15.

1.1.16. Find coordinates of a point which divides the line joining the points (1, 3) and (2, 7) in the ratio 3 : 4

Solution: The point dividing the line AB in the ratio $m : n$ is given by

$$\mathbf{P} = \frac{m\mathbf{B} + n\mathbf{A}}{m+n} \tag{1.1.16.1}$$

Let $\mathbf{A} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 \\ 7 \end{pmatrix}, m = 3, n = 4$

From (1.1.16.1)

$$= \frac{3\begin{pmatrix} 2 \\ 7 \end{pmatrix} + 4\begin{pmatrix} 1 \\ 3 \end{pmatrix}}{3+4} \tag{1.1.16.2}$$

$$= \frac{\begin{pmatrix} 6 \\ 21 \end{pmatrix} + \begin{pmatrix} 4 \\ 12 \end{pmatrix}}{7} \tag{1.1.16.3}$$

$$= \frac{\begin{pmatrix} 10 \\ 33 \end{pmatrix}}{7} \tag{1.1.16.4}$$

$$= \begin{pmatrix} \frac{10}{7} \\ \frac{33}{7} \end{pmatrix} \tag{1.1.16.5}$$

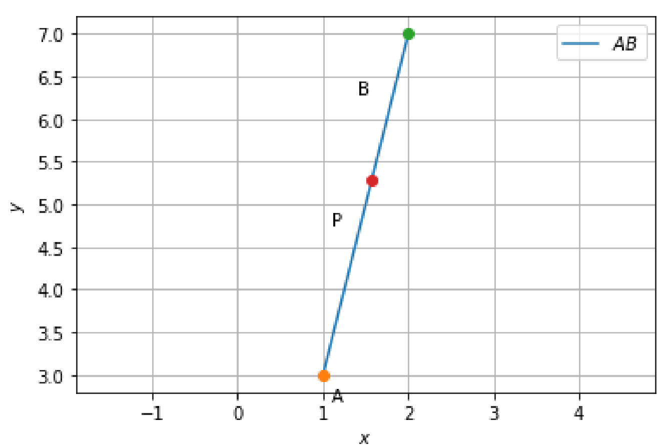

Fig. 1.1.16.1

1.1.17.

1.1.18. Find coordinates of the point which divides, in-ternally and externally, the line joining (-1, 2) to (4,-5) in the ratio 2 : 3

$$\mathbf{A} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 4 \\ -5 \end{pmatrix} \tag{1.1.18.1}$$

Solution: The coordinates of point $\mathbf{P}$, inter-nally dividing the line AB in the ratio $m : n$ is given by

$$\mathbf{P} = \frac{m\mathbf{B} + n\mathbf{A}}{m+n} \tag{1.1.18.2}$$

Let $\mathbf{A} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 4 \\ -5 \end{pmatrix}, m = 2, n = 3.$

From (1.1.18.2)

$$\mathbf{P} = \frac{2\begin{pmatrix} 4 \\ -5 \end{pmatrix} + 3\begin{pmatrix} -1 \\ 2 \end{pmatrix}}{2+3} \tag{1.1.18.3}$$

$$= \begin{pmatrix} 1 \\ \frac{-4}{5} \end{pmatrix} \tag{1.1.18.4}$$

The coordinates of point $\mathbf{Q}$, externally dividing the line AB in the ratio $m : n$ is given by

$$\mathbf{Q} = \frac{m\mathbf{B} - n\mathbf{A}}{m-n} \tag{1.1.18.5}$$

From (1.1.18.5)

$$\mathbf{Q} = \frac{2\begin{pmatrix} 4 \\ -5 \end{pmatrix} - 3\begin{pmatrix} -1 \\ 2 \end{pmatrix}}{2-3} \tag{1.1.18.6}$$

$$= \begin{pmatrix} -11 \\ 16 \end{pmatrix} \tag{1.1.18.7}$$

See Fig. (1.1.18.1)

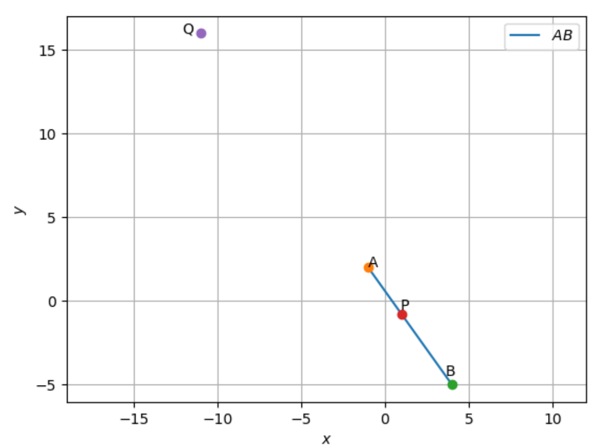

Fig. 1.1.18.1

1.1.19. Find the coordinates of the point which divides, internally and externally, the line joining (-3,-4) to (-8,7) in the ratio 7:5

Solution:

Let

$$\mathbf{A} = \begin{pmatrix} -3 \\ -4 \end{pmatrix} \tag{1.1.19.1}$$

$$\mathbf{B} = \begin{pmatrix} -8 \\ 7 \end{pmatrix} \tag{1.1.19.2}$$

a) Using section formula for internal division,

$$\mathbf{S} = \frac{7\begin{pmatrix} -8 \\ 7 \end{pmatrix} + 5\begin{pmatrix} -3 \\ -4 \end{pmatrix}}{(7+5)} \tag{1.1.19.3}$$

$$= \frac{1}{12}\begin{pmatrix} -71 \\ 29 \end{pmatrix} \tag{1.1.19.4}$$

b) Similarly, for external division,

$$\mathbf{S} = \frac{7\begin{pmatrix} -8 \\ 7 \end{pmatrix} - 5\begin{pmatrix} -3 \\ -4 \end{pmatrix}}{(7-5)} \qquad = \frac{1}{2}\begin{pmatrix} -41 \\ 69 \end{pmatrix} \tag{1.1.19.5}$$

Fig. 1.1.19.1 plots the desired points.

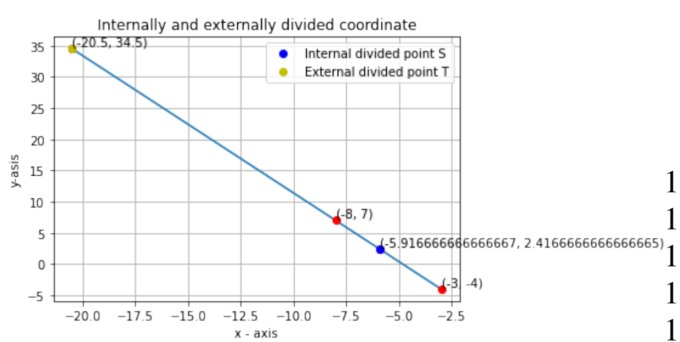

Fig. 1.1.19.1: Plot of coordinate of the point which divides internally and externally

1.1.20. The line joining the points (1, -2) and (-3,4) is trisected;Find the coordinates of the points of the trisection.
Solution: Let

$$\mathbf{A} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -3 \\ 4 \end{pmatrix} \tag{1.1.20.1}$$

Then,

$$\mathbf{Q} = \frac{2\begin{pmatrix} -3 \\ 4 \end{pmatrix} + 1\begin{pmatrix} 1 \\ -2 \end{pmatrix}}{(1+2)} \tag{1.1.20.2}$$

$$= \begin{pmatrix} \frac{-5}{3} \\ 2 \end{pmatrix} \tag{1.1.20.3}$$

$$\mathbf{P} = \frac{1\begin{pmatrix} -5/3 \\ 2 \end{pmatrix} + 1\begin{pmatrix} 1 \\ -2 \end{pmatrix}}{(1+1)} \tag{1.1.20.4}$$

$$= \begin{pmatrix} \frac{-1}{3} \\ 0 \end{pmatrix} \tag{1.1.20.5}$$

Fig. 1.1.20.1 verifies the result.

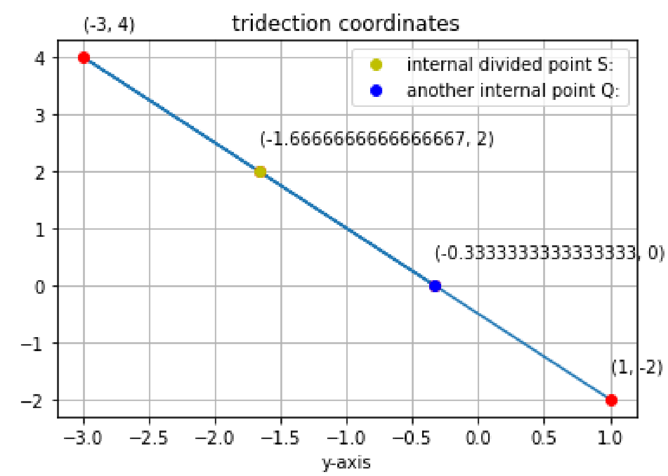

Fig. 1.1.20.1: Plot of coordinates

1.1.21.
1.1.22.
1.1.23.
1.1.24.
1.1.25.
1.1.26.
1.1.27.
1.1.28.
1.1.29.
1.1.30.
1.1.31.
1.1.32.
1.1.33.