Solutions to Plane Coordinate Geometry by S L Loney

G V V Sharma*

Contents

Abstract—**This book provides a vector approach to analytical geometry. The content and exercises are based on S L Loney's book on Plane Coordinate Geometry.**

1 Coordinates

1.1 1

1.1.1. Find the distance between the pair of points (2,3) and (5,7).
Solution: Let

$$\mathbf{A} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 5 \\ 7 \end{pmatrix} \tag{1.1.1.1}$$

$$\because \mathbf{B} - \mathbf{A} = \begin{pmatrix} 5-2 \\ 7-3 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \tag{1.1.1.2}$$

$$\|\mathbf{B} - \mathbf{A}\| = \left\|\begin{pmatrix} 3 \\ 4 \end{pmatrix}\right\| \tag{1.1.1.3}$$

$$= \sqrt{3^2 + 4^2} = \sqrt{25} = 5 \tag{1.1.1.4}$$

1.1.2.

1.1.3.

1.1.4. Find the distance between the following pair of points

$$\mathbf{A} = \begin{pmatrix} a \\ 0 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 0 \\ b \end{pmatrix} \tag{1.1.4.1}$$

Solution: The distance between two vectors is given by

$$\|\mathbf{A} - \mathbf{B}\| = \sqrt{\left(\mathbf{A} - \mathbf{B}\right)^\top \left(\mathbf{A} - \mathbf{B}\right)} \tag{1.1.4.2}$$

$$\tag{1.1.4.3}$$

$\because$

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} a \\ -b \end{pmatrix} \tag{1.1.4.4}$$

$$\|\mathbf{A} - \mathbf{B}\| = \sqrt{\begin{pmatrix} a & -b \end{pmatrix} \begin{pmatrix} a \\ -b \end{pmatrix}} \tag{1.1.4.5}$$

$$= \sqrt{\left(a^2 + b^2\right)} \tag{1.1.4.6}$$

1.1.5. Find the distance between the following pair of points

$$\mathbf{A} = \begin{pmatrix} b+c \\ c+a \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} c+a \\ a+b \end{pmatrix} \tag{1.1.5.1}$$

Solution: The distance between two vectors is given by

$$\|\mathbf{A} - \mathbf{B}\| = \sqrt{\left(\mathbf{A} - \mathbf{B}\right)^\top \left(\mathbf{A} - \mathbf{B}\right)} \tag{1.1.5.2}$$

$$\tag{1.1.5.3}$$

$$\because \mathbf{A} - \mathbf{B} = \begin{pmatrix} b+c \\ c+a \end{pmatrix} - \begin{pmatrix} c+a \\ a+b \end{pmatrix} \tag{1.1.5.4}$$

$$= \begin{pmatrix} b-a \\ c-b \end{pmatrix}, \tag{1.1.5.5}$$

$$(\mathbf{A} - \mathbf{B})^\top (\mathbf{A} - \mathbf{B}) = \sqrt{\begin{pmatrix} b-a & c-b \end{pmatrix} \begin{pmatrix} b-a \\ c-b \end{pmatrix}} \tag{1.1.5.6}$$

$$= \sqrt{\left((b-a)^2 + (c-b)^2\right)} \tag{1.1.5.7}$$

1.1.6.

1.1.7. Find the distance between the following pairs of points

$$\begin{pmatrix} am_1^2 \\ 2am_1 \end{pmatrix}, \begin{pmatrix} am_2^2 \\ 2am_2 \end{pmatrix} \tag{1.1.7.1}$$

Solution: The distance between two vectors is

*The author is with the Department of Electrical Engineering, Indian Institute of Technology, Hyderabad 502285 India e-mail: gadepall@iith.ac.in.

given by

$$\|\mathbf{A}-\mathbf{B}\| = \sqrt{\left(\mathbf{A}-\mathbf{B}\right)^{\top}\left(\mathbf{A}-\mathbf{B}\right)} \tag{1.1.7.2}$$

Let $\mathbf{A} = \begin{pmatrix} am_1^2 \\ 2am_1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} am_2^2 \\ 2am_2 \end{pmatrix}$

$$\mathbf{A}-\mathbf{B} = \begin{pmatrix} am_1^2 \\ 2am_1 \end{pmatrix} - \begin{pmatrix} am_2^2 \\ 2am_2 \end{pmatrix} \tag{1.1.7.3}$$

$$= \begin{pmatrix} am_1^2 - am_2^2 \\ 2am_1 - 2am_2 \end{pmatrix} \tag{1.1.7.4}$$

$$= a\begin{pmatrix} m_1^2 - m_2^2 \\ 2\left(m_1 - m_2\right) \end{pmatrix} \tag{1.1.7.5}$$

$$= a\left(m_1 - m_2\right)\begin{pmatrix} m_1 + m_2 \\ 2 \end{pmatrix} \tag{1.1.7.6}$$

by using the property of $\|k\mathbf{A}\| = |k|\ \|\mathbf{A}\|$

$$\|\mathbf{A}-\mathbf{B}\| = \left\|a\left(m_1 - m_2\right)\begin{pmatrix} m_1 + m_2 \\ 2 \end{pmatrix}\right\| \tag{1.1.7.7}$$

$$= \left|a\left(m_1 - m_2\right)\right| \left\|\begin{pmatrix} m_1 + m_2 \\ 2 \end{pmatrix}\right\| \tag{1.1.7.8}$$

$$= \left|a\left(m_1 - m_2\right)\right| \sqrt{\begin{pmatrix} m_1 + m_2 \\ 2 \end{pmatrix}^{\top} \begin{pmatrix} m_1 + m_2 \\ 2 \end{pmatrix}} \tag{1.1.7.9}$$

$$= \left|a\left(m_1 - m_2\right)\right| \sqrt{\begin{pmatrix} m_1 + m_2 & 2 \end{pmatrix} \begin{pmatrix} m_1 + m_2 \\ 2 \end{pmatrix}} \tag{1.1.7.10}$$

$$= \left|a\left(m_1 - m_2\right)\right| \sqrt{\left(m_1 + m_2\right)^2 + \left(2\right)^2} \tag{1.1.7.11}$$

$$= \left|a\left(m_1 - m_2\right)\right| \sqrt{\left(m_1 + m_2\right)^2 + 4} \tag{1.1.7.12}$$

Distance between $(am_1^2, 2am_1)$ and $(am_2^2, 2am_2)$ is

$$= \left|a\left(m_1 - m_2\right)\right| \sqrt{\left(m_1 + m_2\right)^2 + 4} \tag{1.1.7.13}$$

1.1.8. Find the distance between the following pairs of points

$$\begin{pmatrix} 1 \\ -3 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \end{pmatrix} \tag{1.1.8.1}$$

Solution:

$$\because \mathbf{A}-\mathbf{B} = \begin{pmatrix} 1 \\ -3 \end{pmatrix} - \begin{pmatrix} -2 \\ 1 \end{pmatrix} \tag{1.1.8.2}$$

$$= \begin{pmatrix} 3 \\ -4 \end{pmatrix}, \tag{1.1.8.3}$$

Hence,

$$\|\mathbf{A}-\mathbf{B}\| = \sqrt{\left(\mathbf{A}-\mathbf{B}\right)^{\top}\left(\mathbf{A}-\mathbf{B}\right)} \tag{1.1.8.4}$$

$$= \sqrt{\begin{pmatrix} 3 & -4 \end{pmatrix}\begin{pmatrix} 3 \\ -4 \end{pmatrix}} \tag{1.1.8.5}$$

$$= \sqrt{3^2 + \left(-4\right)^2} \tag{1.1.8.6}$$

$$= 5 \tag{1.1.8.7}$$

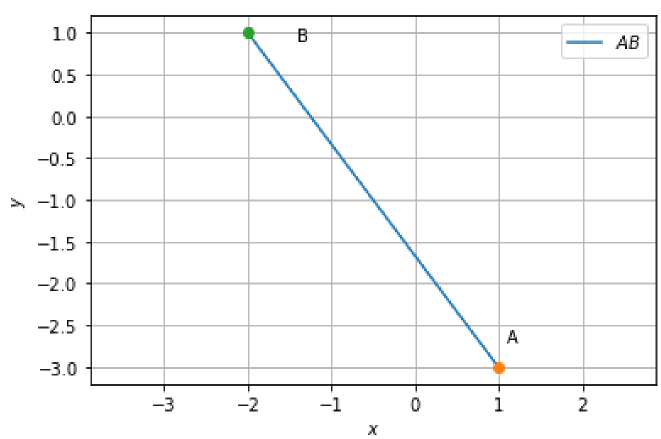

Fig. 1.1.8.1

1.1.9.

1.1.10.

1.1.11.

1.1.12.

1.1.13.

1.1.14.

1.1.15. Find coordinates of a point which divides the line joining the points (1, 3) and (2, 7) in the ratio 3 : 4

Solution: The point dividing the line AB in the ratio $m : n$ is given by

$$\mathbf{P} = \frac{m\mathbf{B} + n\mathbf{A}}{m + n} \tag{1.1.15.1}$$

Let $\mathbf{A} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 \\ 7 \end{pmatrix}, m = 3, n = 4$

From (1.1.15.1)

$$= \frac{3\begin{pmatrix} 2 \\ 7 \end{pmatrix} + 4\begin{pmatrix} 1 \\ 3 \end{pmatrix}}{3+4} \tag{1.1.15.2}$$

$$= \frac{\begin{pmatrix} 6 \\ 21 \end{pmatrix} + \begin{pmatrix} 4 \\ 12 \end{pmatrix}}{7} \tag{1.1.15.3}$$

$$= \frac{\begin{pmatrix} 10 \\ 33 \end{pmatrix}}{7} \tag{1.1.15.4}$$

$$= \begin{pmatrix} \frac{10}{7} \\ \frac{33}{7} \end{pmatrix} \tag{1.1.15.5}$$

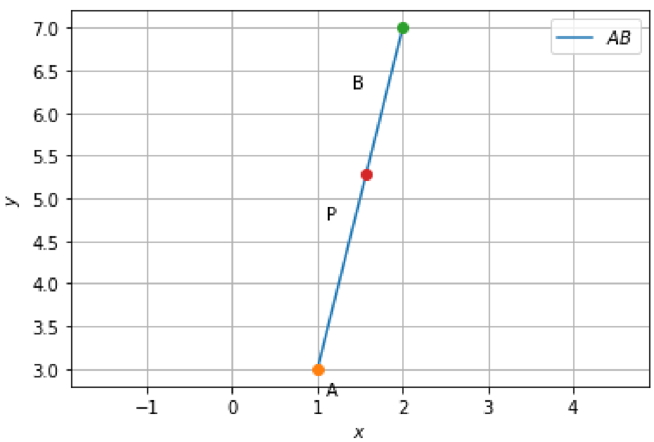

Fig. 1.1.15.1

1.1.16.

1.1.17. Find coordinates of the point which divides, in-ternally and externally, the line joining (-1, 2) to (4,-5) in the ratio 2 : 3

$$\mathbf{A} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 4 \\ -5 \end{pmatrix} \tag{1.1.17.1}$$

Solution: The coordinates of point **P**, internally dividing the line AB in the ratio $m : n$ is given by

$$\mathbf{P} = \frac{m\mathbf{B} + n\mathbf{A}}{m+n} \tag{1.1.17.2}$$

Let $\mathbf{A} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 4 \\ -5 \end{pmatrix}, m = 2, n = 3.$

From (**??**)

$$\mathbf{P} = \frac{2\begin{pmatrix} 4 \\ -5 \end{pmatrix} + 3\begin{pmatrix} -1 \\ 2 \end{pmatrix}}{2+3} \tag{1.1.17.3}$$

$$= \begin{pmatrix} 1 \\ \frac{-4}{5} \end{pmatrix} \tag{1.1.17.4}$$

The coordinates of point **Q**, externally dividing the line AB in the ratio $m : n$ is given by

$$\mathbf{Q} = \frac{m\mathbf{B} - n\mathbf{A}}{m-n} \tag{1.1.17.5}$$

From (**??**)

$$\mathbf{Q} = \frac{2\begin{pmatrix} 4 \\ -5 \end{pmatrix} - 3\begin{pmatrix} -1 \\ 2 \end{pmatrix}}{2-3} \tag{1.1.17.6}$$

$$= \begin{pmatrix} -11 \\ 16 \end{pmatrix} \tag{1.1.17.7}$$

See Fig. (**??**)

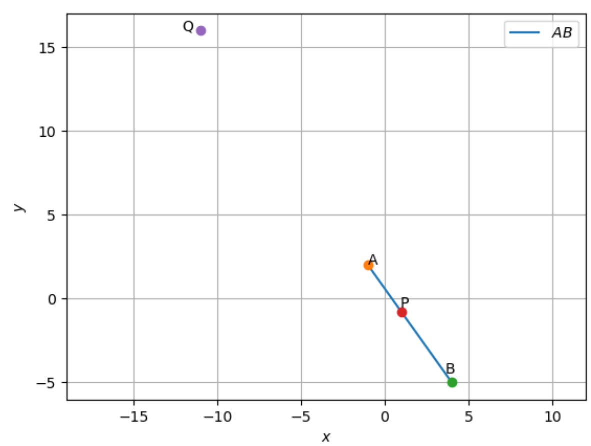

Fig. 1.1.17.1

1.1.18. Find the coordinates of the point which divides, internally and externally, the line joining (-3,-4) to (-8,7) in the ratio 7:5

Solution:

Let

$$\mathbf{A} = \begin{pmatrix} -3 \\ -4 \end{pmatrix} \tag{1.1.18.1}$$

$$\mathbf{B} = \begin{pmatrix} -8 \\ 7 \end{pmatrix} \tag{1.1.18.2}$$

a) Using section formula for internal division,

$$\mathbf{S} = \frac{7\begin{pmatrix}-8\\7\end{pmatrix}+5\begin{pmatrix}-3\\-4\end{pmatrix}}{(7+5)} \tag{1.1.18.3}$$

$$= \frac{1}{12}\begin{pmatrix}-71\\29\end{pmatrix} \tag{1.1.18.4}$$

b) Similarly, for external division,

$$\mathbf{S} = \frac{7\begin{pmatrix}-8\\7\end{pmatrix}-5\begin{pmatrix}-3\\-4\end{pmatrix}}{(7-5)} \quad = \frac{1}{2}\begin{pmatrix}-41\\69\end{pmatrix} \tag{1.1.18.5}$$

Fig. **??** plots the desired points.

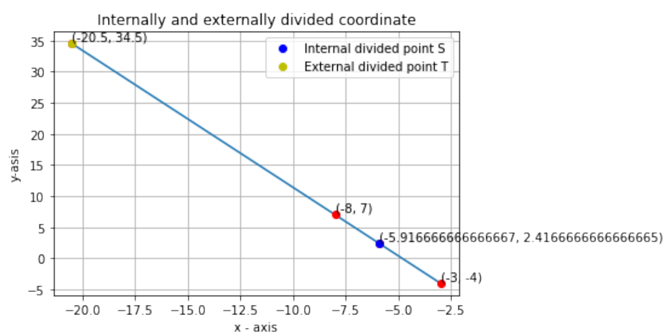

Fig. 1.1.18.1: Plot of coordinate of the point which divides internally and externally

1.1.19. The line joining the points (1, -2) and (-3,4) is trisected;Find the coordinates of the points of the trisection.

Solution: Let

$$\mathbf{A} = \begin{pmatrix}1\\-2\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-3\\4\end{pmatrix} \tag{1.1.19.1}$$

Then,

$$\mathbf{Q} = \frac{2\begin{pmatrix}-3\\4\end{pmatrix}+1\begin{pmatrix}1\\-2\end{pmatrix}}{(1+2)} \tag{1.1.19.2}$$

$$= \begin{pmatrix}\frac{-5}{3}\\2\end{pmatrix} \tag{1.1.19.3}$$

$$\mathbf{P} = \frac{1\begin{pmatrix}-5/3\\2\end{pmatrix}+1\begin{pmatrix}1\\-2\end{pmatrix}}{(1+1)} \tag{1.1.19.4}$$

$$= \begin{pmatrix}\frac{-1}{3}\\0\end{pmatrix} \tag{1.1.19.5}$$

Fig. **??** verifies the result.

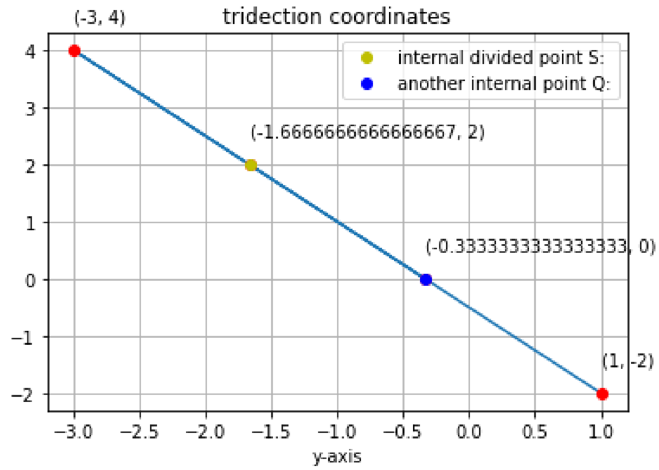

Fig. 1.1.19.1: Plot of coordinates

1.1.20.
1.1.21.
1.1.22.
1.1.23.
1.1.24.
1.1.25.
1.1.26.
1.1.27.
1.1.28.
1.1.29.
1.1.30.
1.1.31.
1.1.32.