Solutions to Plane Coordinate Geometry by S L Loney

G V V Sharma*

Contents

***Abstract*—This book provides a vector approach to analytical geometry. The content and exercises are based on S L Loney's book on Plane Coordinate Geometry.**

1 Coordinates

1.1 1

1.1.1. Find the distance between the following pair of points (2,3) and (5,7).
Solution:

1.1.2.

1.1.3.

1.1.4.

1.1.5. Find the distance between the following pair of points

$$\mathbf{A} = \begin{pmatrix} b+c \\ c+a \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} c+a \\ a+b \end{pmatrix} \tag{1.1.5.1}$$

*The author is with the Department of Electrical Engineering, Indian Institute of Technology, Hyderabad 502285 India e-mail: gadepall@iith.ac.in.

The distance between two vectors is given by

$$\|\mathbf{A}-\mathbf{B}\| \tag{1.1.5.2}$$

$$\|\mathbf{A}-\mathbf{B}\| = \sqrt{\left(\mathbf{A}-\mathbf{B}\right)^\top\left(\mathbf{A}-\mathbf{B}\right)} \tag{1.1.5.3}$$

$$\|\mathbf{A}-\mathbf{B}\| = \sqrt{\left(\begin{pmatrix} b+c \\ c+a \end{pmatrix} - \begin{pmatrix} c+a \\ a+b \end{pmatrix}\right)^\top \left(\begin{pmatrix} b+c \\ c+a \end{pmatrix} - \begin{pmatrix} c+a \\ a+b \end{pmatrix}\right)} \tag{1.1.5.4}$$

$$\|\mathbf{A}-\mathbf{B}\| = \sqrt{\begin{pmatrix} b-a \\ c-b \end{pmatrix}^\top \begin{pmatrix} b-a \\ c-b \end{pmatrix}} \tag{1.1.5.5}$$

$$\|\mathbf{A}-\mathbf{B}\| = \sqrt{\begin{pmatrix} b-a & c-b \end{pmatrix} \begin{pmatrix} b-a \\ c-b \end{pmatrix}} \tag{1.1.5.6}$$

$$\|\mathbf{A}-\mathbf{B}\| = \sqrt{\left((b-a)^2 + (c-b)^2\right)} \tag{1.1.5.7}$$

$$\|\mathbf{A}-\mathbf{B}\| = \sqrt{a^2 + 2b^2 + c^2 - 2ab - 2cb} \tag{1.1.5.8}$$

1.1.6.

1.1.7. Find the distance between the following pairs of points

$$\begin{pmatrix} am_1^2 \\ 2am_1 \end{pmatrix}, \begin{pmatrix} am_2^2 \\ 2am_2 \end{pmatrix} \tag{1.1.7.1}$$

The distance between two vectors is given by

$$\|\mathbf{A}-\mathbf{B}\| = \sqrt{\left(\mathbf{A}-\mathbf{B}\right)^\top\left(\mathbf{A}-\mathbf{B}\right)} \tag{1.1.7.2}$$

Let $\mathbf{A} = \begin{pmatrix} am_1^2 \\ 2am_1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} am_2^2 \\ 2am_2 \end{pmatrix}$

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} am_1^2 \\ 2am_1 \end{pmatrix} - \begin{pmatrix} am_2^2 \\ 2am_2 \end{pmatrix} \tag{1.1.7.3}$$

$$= \begin{pmatrix} am_1^2 - am_2^2 \\ 2am_1 - 2am_2 \end{pmatrix} \tag{1.1.7.4}$$

$$= a\begin{pmatrix} m_1^2 - m_2^2 \\ 2\left(m_1 - m_2\right) \end{pmatrix} \tag{1.1.7.5}$$

$$= a\left(m_1 - m_2\right)\begin{pmatrix} m_1 + m_2 \\ 2 \end{pmatrix} \tag{1.1.7.6}$$

by using the property of $\|k\mathbf{A}\| = |k|\ \|\mathbf{A}\|$

$$\|\mathbf{A} - \mathbf{B}\| = \left\|a\left(m_1 - m_2\right)\begin{pmatrix} m_1 + m_2 \\ 2 \end{pmatrix}\right\| \tag{1.1.7.7}$$

$$= \left|a\left(m_1 - m_2\right)\right| \left\|\begin{pmatrix} m_1 + m_2 \\ 2 \end{pmatrix}\right\| \tag{1.1.7.8}$$

$$= \left|a\left(m_1 - m_2\right)\right| \sqrt{\begin{pmatrix} m_1 + m_2 \\ 2 \end{pmatrix}^\top \begin{pmatrix} m_1 + m_2 \\ 2 \end{pmatrix}} \tag{1.1.7.9}$$

$$= \left|a\left(m_1 - m_2\right)\right| \sqrt{\begin{pmatrix} m_1 + m_2 & 2 \end{pmatrix} \begin{pmatrix} m_1 + m_2 \\ 2 \end{pmatrix}} \tag{1.1.7.10}$$

$$= \left|a\left(m_1 - m_2\right)\right| \sqrt{\left(m_1 + m_2\right)^2 + \left(2\right)^2} \tag{1.1.7.11}$$

$$= \left|a\left(m_1 - m_2\right)\right| \sqrt{\left(m_1 + m_2\right)^2 + 4} \tag{1.1.7.12}$$

Distance between $(am_1^2, 2am_1)$ and $(am_2^2, 2am_2)$ is

$$= \left|a\left(m_1 - m_2\right)\right| \sqrt{\left(m_1 + m_2\right)^2 + 4} \tag{1.1.7.13}$$

1.1.8.
1.1.9.
1.1.10.
1.1.11.
1.1.12.
1.1.13.
1.1.14.
1.1.15.
1.1.16.
1.1.17.
1.1.18. Find the coordinates of the point which divides, internally and externally, the line joining (-3,-4) to (-8,7) in the ratio 7:5

Solution:
Let

$$\mathbf{A} = \begin{pmatrix} -3 \\ -4 \end{pmatrix} \tag{1.1.18.1}$$

$$\mathbf{B} = \begin{pmatrix} -8 \\ 7 \end{pmatrix} \tag{1.1.18.2}$$

a) Using section formula for internal division,

$$\mathbf{S} = \frac{7\begin{pmatrix} -8 \\ 7 \end{pmatrix} + 5\begin{pmatrix} -3 \\ -4 \end{pmatrix}}{(7+5)} \tag{1.1.18.3}$$

$$= \frac{1}{12}\begin{pmatrix} -71 \\ 29 \end{pmatrix} \tag{1.1.18.4}$$

b) Similarly, for external division,

$$\mathbf{S} = \frac{7\begin{pmatrix} -8 \\ 7 \end{pmatrix} - 5\begin{pmatrix} -3 \\ -4 \end{pmatrix}}{(7-5)} \quad = \frac{1}{2}\begin{pmatrix} -41 \\ 69 \end{pmatrix} \tag{1.1.18.5}$$

Fig. 1.1.21.1 plots the desired points.

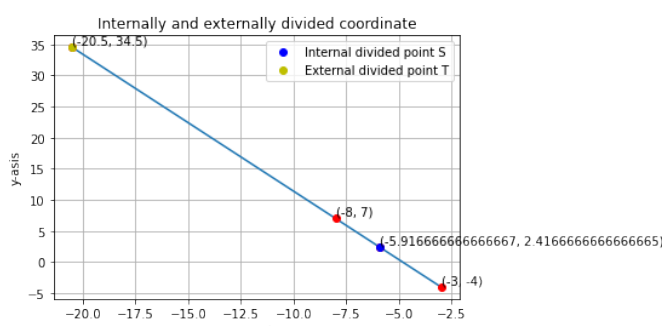

Fig. 1.1.18.1: Plot of coordinate of the point which divides internally and externally

1.1.19. The line joining the points (1, -2) and (-3,4) is trisected;Find the coordinates of the points of the trisection.

Solution: Let

$$\mathbf{A} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -3 \\ 4 \end{pmatrix} \tag{1.1.19.1}$$

Then,

$$\mathbf{Q} = \frac{2\begin{pmatrix}-3\\4\end{pmatrix} + 1\begin{pmatrix}1\\-2\end{pmatrix}}{(1+2)} \tag{1.1.19.2}$$

$$= \begin{pmatrix}\frac{-5}{3}\\2\end{pmatrix} \tag{1.1.19.3}$$

$$\mathbf{P} = \frac{1\begin{pmatrix}-5/3\\2\end{pmatrix} + 1\begin{pmatrix}1\\-2\end{pmatrix}}{(1+1)} \tag{1.1.19.4}$$

$$= \begin{pmatrix}\frac{-1}{3}\\0\end{pmatrix} \tag{1.1.19.5}$$

Fig. 1.1.22.1 verifies the result.

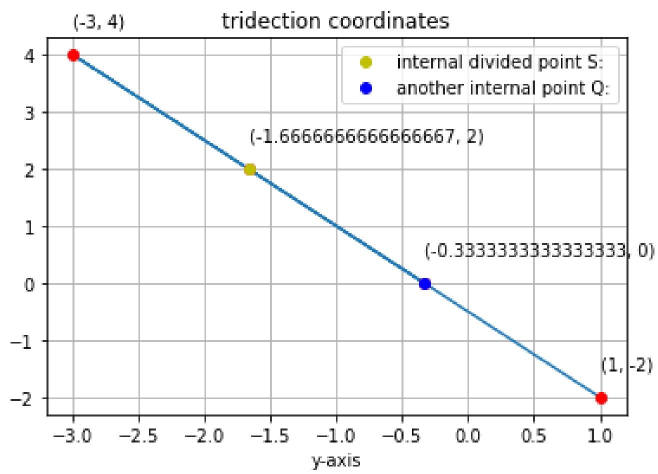

Fig. 1.1.19.1: Plot of coordinates

1.1.20.
1.1.21.
1.1.22.
1.1.23.
1.1.24.
1.1.25.
1.1.26.
1.1.27.
1.1.28.
1.1.29.
1.1.30.
1.1.31.
1.1.32.

1.2 2

1.2.1. Find the areas of the triangles the coordinates of whose angular points are respectively:

$$\mathbf{P} = \begin{pmatrix}1\\3\end{pmatrix}, \mathbf{Q} = \begin{pmatrix}-7\\6\end{pmatrix} \text{ and } \mathbf{R} = \begin{pmatrix}5\\-1\end{pmatrix} \tag{1.2.1.1}$$

Solution:

$$\mathbf{Q} - \mathbf{P} = \begin{pmatrix}-7\\6\end{pmatrix} - \begin{pmatrix}1\\3\end{pmatrix} \tag{1.2.1.2}$$

$$= \begin{pmatrix}-8\\3\end{pmatrix} \mathbf{R} - \mathbf{P} \quad = \begin{pmatrix}5\\-1\end{pmatrix} - \begin{pmatrix}1\\3\end{pmatrix} \tag{1.2.1.3}$$

$$= \begin{pmatrix}4\\-4\end{pmatrix} \tag{1.2.1.4}$$

$$\because \text{Area of the Triangle} = \frac{1}{2}\left\|(\mathbf{Q} - \mathbf{P}) \times (\mathbf{R} - \mathbf{P})\right\| \tag{1.2.1.5}$$

As the vector cross product of two vectors can also be expressed as the product of a skew-symmetric matrix and a vector.

$$\mathbf{A} \times \mathbf{B} = \begin{pmatrix}0 & -a_3 & a_2\\ a_3 & 0 & -a_1\\ -a_2 & a_1 & 0\end{pmatrix} \times \begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix} \tag{1.2.1.6}$$

Substituting values from equation 1.2.1.2 and 1.2.1.3 in above equation 1.2.1.6, we'll get

$$(\mathbf{Q} - \mathbf{P}) \times (\mathbf{R} - \mathbf{P}) = \begin{pmatrix}0 & 0 & 3\\ 0 & 0 & 8\\ -3 & -8 & 0\end{pmatrix} \times \begin{pmatrix}4\\-4\\0\end{pmatrix}$$
$$= \begin{pmatrix}0\\0\\20\end{pmatrix} \tag{1.2.1.7}$$

$$\therefore \frac{1}{2}\left\|(\mathbf{Q} - \mathbf{P}) \times (\mathbf{R} - \mathbf{P})\right\| = 10 \tag{1.2.1.8}$$

which is the desired area of the triangle

1.2.2. Find the distance between the following pair of points (2,3) and (5,7).

Solution:

1.2.3. Find the areas of the triangles the coordinates of whose angular points are respectively:

$$\mathbf{P} = \begin{pmatrix}5\\2\end{pmatrix}, \mathbf{Q} = \begin{pmatrix}-9\\-3\end{pmatrix} \text{ and } \mathbf{R} = \begin{pmatrix}-3\\-5\end{pmatrix} \tag{1.2.3.1}$$

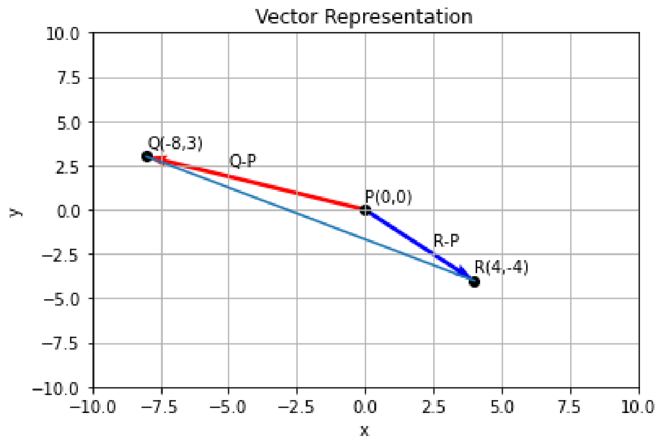

Fig. 1.2.1.1: Plot obtained from Python code

Solution:

$$\mathbf{Q}-\mathbf{P} = \begin{pmatrix}-9\\-3\end{pmatrix} - \begin{pmatrix}5\\2\end{pmatrix} \tag{1.2.3.2}$$

$$= \begin{pmatrix}-14\\-5\end{pmatrix} \mathbf{R}-\mathbf{P} \quad = \begin{pmatrix}-3\\-5\end{pmatrix} - \begin{pmatrix}5\\2\end{pmatrix} \tag{1.2.3.3}$$

$$= \begin{pmatrix}-8\\-7\end{pmatrix} \tag{1.2.3.4}$$

$$\because (\mathbf{Q}-\mathbf{P}) \times (\mathbf{R}-\mathbf{P}) = \begin{pmatrix}0 & 0 & -5\\0 & 0 & 14\\5 & -14 & 0\end{pmatrix} \times \begin{pmatrix}-8\\-7\\0\end{pmatrix} \tag{1.2.3.5}$$

$$= \begin{pmatrix}0\\0\\58\end{pmatrix} \implies \|(\mathbf{Q}-\mathbf{P}) \times (\mathbf{R}-\mathbf{P})\| = \sqrt{0^2+0^2+58^2} = 58, \tag{1.2.3.6}$$

area of the triangle is given by

$$\frac{1}{2}\|(\mathbf{Q}-\mathbf{P}) \times (\mathbf{R}-\mathbf{P})\| = \frac{1}{2}(58) = 29 \tag{1.2.3.7}$$

1.2.4. Find the area of the triangle formed by the points $\mathbf{P}\begin{pmatrix}a\\c+a\end{pmatrix}$, $\mathbf{Q}\begin{pmatrix}a\\c\end{pmatrix}$ and $\mathbf{R}\begin{pmatrix}-a\\c-a\end{pmatrix}$
Solution:

1.2.5. Find the area of the triangle the coordinates of whose angular points are respectively

$$\mathbf{A} = \begin{pmatrix}-1\\2\end{pmatrix}, \mathbf{B} = \begin{pmatrix}2\\3\end{pmatrix} \text{ and } \mathbf{C} = \begin{pmatrix}4\\-3\end{pmatrix} \tag{1.2.5.1}$$

Solution: We will be using vectors for calculating the area of the triangle formed by above three points.

$$\mathbf{B}-\mathbf{A} = \begin{pmatrix}2\\3\end{pmatrix} - \begin{pmatrix}-1\\2\end{pmatrix} \tag{1.2.5.2}$$

$$\mathbf{B}-\mathbf{A} = \begin{pmatrix}3\\1\end{pmatrix} \tag{1.2.5.3}$$

$$\mathbf{C}-\mathbf{A} = \begin{pmatrix}4\\-3\end{pmatrix} - \begin{pmatrix}-1\\2\end{pmatrix} \tag{1.2.5.4}$$

$$\mathbf{C}-\mathbf{A} = \begin{pmatrix}5\\-5\end{pmatrix} \tag{1.2.5.5}$$

$$\because \text{Area of the Triangle} = \frac{1}{2}\|(\mathbf{B}-\mathbf{A}) \times (\mathbf{C}-\mathbf{A})\| \tag{1.2.5.6}$$

As the vector cross product of two vectors can also be expressed as the product of a skew-symmetric matrix and a vector.

$$\mathbf{P} \times \mathbf{Q} = \begin{pmatrix}0 & -a_3 & a_2\\a_3 & 0 & -a_1\\-a_2 & a_1 & 0\end{pmatrix} \times \begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix} \tag{1.2.5.7}$$

Substituting values from equation (1.2.5.3) and (1.2.5.5) in above equation (1.2.5.7), we'll get:

$$(\mathbf{B}-\mathbf{A}) \times (\mathbf{C}-\mathbf{A}) = \begin{pmatrix}0 & 0 & 1\\0 & 0 & -3\\-1 & 3 & 0\end{pmatrix} \times \begin{pmatrix}5\\-5\\0\end{pmatrix} \tag{1.2.5.8}$$

$$(\mathbf{B}-\mathbf{A}) \times (\mathbf{C}-\mathbf{A}) = \begin{pmatrix}0\\0\\-20\end{pmatrix} \tag{1.2.5.9}$$

$$\frac{1}{2} \therefore \|(\mathbf{B}-\mathbf{A}) \times (\mathbf{C}-\mathbf{A})\| = 10 \tag{1.2.5.10}$$

Substituting value from equation (1.2.5.10) in equation (1.2.5.6), we'll get area of triangle:
$\implies \frac{1}{2}(20) = 10units^2$

1.2.6. The coordinates of the vertices of a triangle are (x_1, y_1), (x_2, y_2) and (x_3, y_3). The line joining the first two is divided in the ratio $l : k$, and the line joining his point of division to the opposite

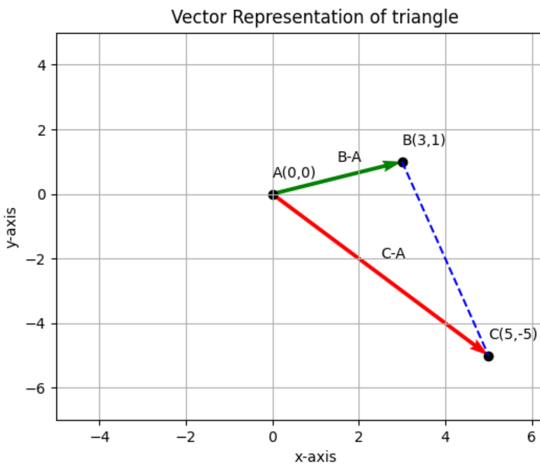

Fig. 1.2.5.1: Plot obtained from Python code

angular point is then divided in the ratio $m : k+l$. Find the coordinates of the latter point of section.

Solution: In Fig. 1.2.6.1

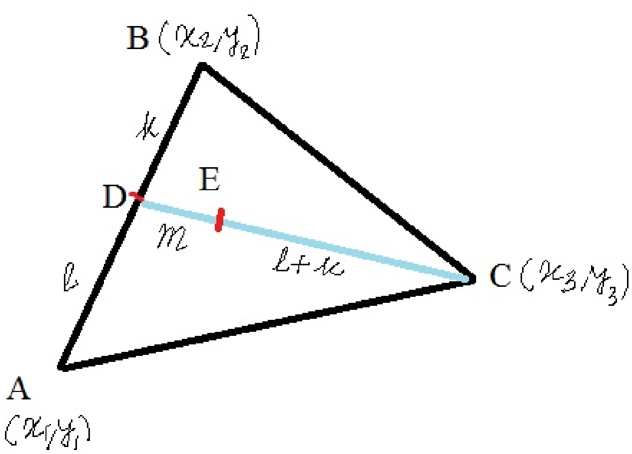

Fig. 1.2.6.1: Triangle ABC with vertices $\mathbf{A}\begin{pmatrix}2\\4\end{pmatrix}$, $\mathbf{B}\begin{pmatrix}0\\0\end{pmatrix}$, $\mathbf{C}\begin{pmatrix}4\\0\end{pmatrix}$, and $\begin{pmatrix}l\\m\\k\end{pmatrix} = \begin{pmatrix}0.5\\0.5\\0.5\end{pmatrix}$ are used for python plot.The point of section $\mathbf{P} = \begin{pmatrix}2\\0\end{pmatrix}$, $\mathbf{Q} = \begin{pmatrix}2\\1.33\end{pmatrix}$.

$$\mathbf{B} = \begin{pmatrix}x_1\\y_1\end{pmatrix}, \mathbf{C} = \begin{pmatrix}x_2\\y_2\end{pmatrix}, \mathbf{A} = \begin{pmatrix}x_3\\y_3\end{pmatrix} \tag{1.2.6.1}$$

Using section formula,

$$\mathbf{P} = \frac{l\mathbf{C} + k\mathbf{B}}{l+k} \tag{1.2.6.2}$$

Now,the line joining PA divided into the ratio m:k+l at point of division $\mathbf{Q}$ can be written by using section formula

$$\mathbf{Q} = \frac{m\mathbf{A} + (k+l)\,\mathbf{P}}{l+k+m} \tag{1.2.6.3}$$

From Eq.(1.2.6.2) substitute $\mathbf{P}$ in Eq.(1.2.6.3)

$$\mathbf{Q} = \frac{l\mathbf{C} + k\mathbf{B} + m\mathbf{A}}{l+k+m} \tag{1.2.6.4}$$

$$= \frac{1}{l+k+m}\begin{pmatrix}\mathbf{C} & \mathbf{B} & \mathbf{A}\end{pmatrix}\begin{pmatrix}l\\k\\m\end{pmatrix} \tag{1.2.6.5}$$

2 The Straight Line

2.1 6

2.1.1. Find the equation to the straight line passing through $(2,3)$ and perpendicular to the straight line: $4x - 3y = 10$.

Solution: The vector which is normal to $4x - 3y = 10$ from simple inspection is :$\begin{pmatrix}4\\-3\end{pmatrix}$. Clearly, the direction vector $\mathbf{m}$ of a line which is perpendicular to the given line is :

$$\mathbf{m} = \begin{pmatrix}3\\-4\end{pmatrix} \tag{2.1.1.1}$$

The equation of this line which is perpendicular to the given line and passing through $\mathbf{A} = \begin{pmatrix}x_A\\y_A\end{pmatrix}$ is then obtained as:

$$\mathbf{m}^{\mathbf{T}}\mathbf{x} = \mathbf{m}^{\mathbf{T}}\mathbf{A} \tag{2.1.1.2}$$

(2.1.1.2) simplifies to read:

$$\begin{pmatrix}3 & 4\end{pmatrix}\mathbf{x} = \mathbf{18} \tag{2.1.1.3}$$

Which in scalar form reads: $3x + 4y = 18$

Both the straight lines are plotted in Fig. 2.1.1.1 along with the point $A(2,3)$ using Python script.

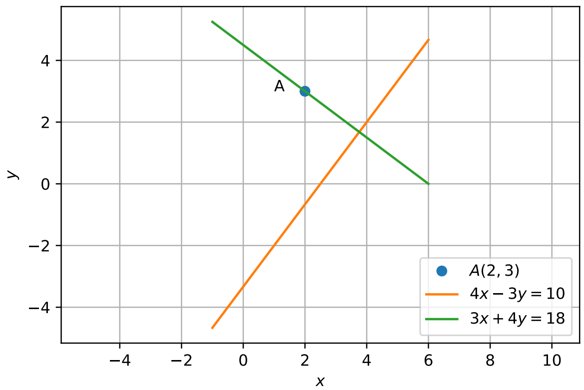

Fig. 2.1.1.1: Solution

3 The Circle

3.1 17

3.1.1. Find the equation to the circle which passes through the points $\begin{pmatrix}1\\-2\end{pmatrix}$ and $\begin{pmatrix}4\\-3\end{pmatrix}$ and which has its centre on the straight line $\begin{pmatrix}3 & 4\end{pmatrix}x = 7$.
Solution: The equation of circle can be expressed as

$$\mathbf{x}^T\mathbf{x} - 2\mathbf{C}^T\mathbf{x} + f = 0 \tag{3.1.1.1}$$

C is the centre and substituting the points in the equation of circle we get

$$2\begin{pmatrix}1 & -2\end{pmatrix}\mathbf{C} - f = 5 \tag{3.1.1.2}$$
$$2\begin{pmatrix}4 & -3\end{pmatrix}\mathbf{C} - f = 25 \tag{3.1.1.3}$$
$$\begin{pmatrix}3 & 4\end{pmatrix}\mathbf{C} = 7 \tag{3.1.1.4}$$

can be expressed in matrix form

$$\begin{pmatrix}3 & 4 & 0\\2 & -4 & -1\\8 & -6 & -1\end{pmatrix}\begin{pmatrix}\mathbf{C}\\f\end{pmatrix} = \begin{pmatrix}7\\5\\25\end{pmatrix} \tag{3.1.1.5}$$

Row reducing the augmented matrix

$$\begin{pmatrix}3 & 4 & 0 & 7\\2 & -4 & -1 & 5\\8 & -6 & -1 & 25\end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1/3} \begin{pmatrix}1 & \frac{4}{3} & 0 & \frac{7}{3}\\2 & -4 & -1 & 5\\8 & -6 & -1 & 25\end{pmatrix} \tag{3.1.1.6}$$

$$\xleftrightarrow[R_3 \leftarrow R_3 - 8R_1]{R_2 \leftarrow R_2 - 2R_1} \begin{pmatrix}1 & \frac{4}{3} & 0 & \frac{7}{3}\\0 & \frac{-20}{3} & -1 & \frac{1}{3}\\0 & \frac{-50}{3} & -1 & \frac{19}{3}\end{pmatrix} \tag{3.1.1.7}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{-3}{20}R_2} \begin{pmatrix}1 & \frac{4}{3} & 0 & \frac{7}{3}\\0 & 1 & \frac{3}{20} & \frac{-1}{20}\\0 & \frac{-50}{3} & -1 & \frac{19}{3}\end{pmatrix} \tag{3.1.1.8}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 + \frac{50}{3}R_2} \begin{pmatrix}1 & \frac{4}{3} & 0 & \frac{7}{3}\\0 & 1 & \frac{3}{20} & \frac{-1}{20}\\0 & 0 & \frac{3}{2} & \frac{11}{2}\end{pmatrix} \tag{3.1.1.9}$$

$$\mathbf{C} = \begin{pmatrix}\frac{47}{15}\\\frac{-3}{5}\end{pmatrix} f = \frac{11}{3} \tag{3.1.1.10}$$

$$f = \frac{11}{3} \tag{3.1.1.11}$$

The required circle equation,

$$\mathbf{x}^T\mathbf{x} - 2\begin{pmatrix}\frac{47}{15} & \frac{-3}{5}\end{pmatrix}\mathbf{x} + \frac{11}{3} = 0 \tag{3.1.1.12}$$

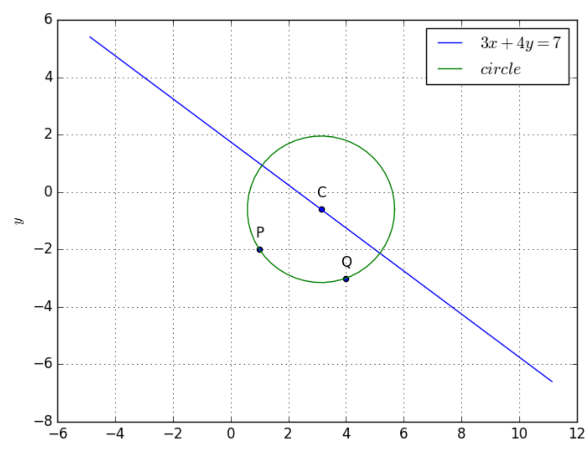

Fig. 3.1.1.1: Circle passing through point P and Q also centre lie on the line 3x+4y=7

Find the equation to the circle that passes through the points:

$$\mathbf{x_1} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \mathbf{x_2} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}, \mathbf{x_3} = \begin{pmatrix} 5 \\ -6 \end{pmatrix} \quad (3.1.1.13)$$

Solution: The equation of circle in vector form is given by:

$$\mathbf{x^T x} + 2\mathbf{x^T u} + f = 0 \quad (3.1.1.14)$$

Using $\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}$ in (3.1.1.14),

$$\mathbf{x_1}^T\mathbf{x_1} + 2\mathbf{x_1}^T\mathbf{u} + f = 0 \quad (3.1.1.15)$$

$$\mathbf{x_2}^T\mathbf{x_2} + 2\mathbf{x_2}^T\mathbf{u} + f = 0 \quad (3.1.1.16)$$

$$\mathbf{x_3}^T\mathbf{x_3} + 2\mathbf{x_3}^T\mathbf{u} + f = 0 \quad (3.1.1.17)$$

The above system can be written in matrix form as:

$$\begin{pmatrix} 2\mathbf{x_1}^T & 1 \\ 2\mathbf{x_2}^T & 1 \\ 2\mathbf{x_3}^T & 1 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ f \end{pmatrix} = \begin{pmatrix} -\mathbf{x_1}^T\mathbf{x_1} \\ -\mathbf{x_1}^T\mathbf{x_1} \\ -\mathbf{x_3}^T\mathbf{x_3} \end{pmatrix} \quad (3.1.1.18)$$

Substituting the values for $\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}$ in (3.1.1.18),

$$\begin{pmatrix} 2 & 4 & 1 \\ 6 & -8 & 1 \\ 10 & -12 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ f \end{pmatrix} = \begin{pmatrix} -5 \\ -25 \\ -61 \end{pmatrix} \quad (3.1.1.19)$$

Using row echelon form to reduce (3.1.1.19), we get:

$$\xleftrightarrow[R_3 \to R_3 - 5R_1]{R_2 \to R_2 - 3R_1} \begin{pmatrix} 2 & 4 & 1 & -5 \\ 0 & -20 & -2 & -10 \\ 0 & -32 & -4 & -36 \end{pmatrix} \quad (3.1.1.20)$$

$$\xleftrightarrow[R_3 \to -\frac{1}{4}R_3]{R_2 \to -\frac{1}{2}R_2} \begin{pmatrix} 2 & 4 & 1 & -5 \\ 0 & 10 & 1 & 5 \\ 0 & 8 & 1 & 9 \end{pmatrix} \quad (3.1.1.21)$$

$$\xleftrightarrow{R_3 \to 5R_3 - 4R_2} \begin{pmatrix} 2 & 4 & 1 & -5 \\ 0 & 10 & 1 & 5 \\ 0 & 0 & 1 & 25 \end{pmatrix} \quad (3.1.1.22)$$

$$\xleftrightarrow[R_2 \to \frac{1}{10}R_2]{R_1 \to \frac{1}{2}R_1} \begin{pmatrix} 1 & 2 & \frac{1}{2} & -\frac{5}{2} \\ 0 & 1 & \frac{1}{10} & \frac{2}{5} \\ 0 & 0 & 1 & 25 \end{pmatrix} \quad (3.1.1.23)$$

Back solving the system using (3.1.1.18) and (3.1.1.23), we get:

$$\begin{pmatrix} \mathbf{u} \\ f \end{pmatrix} = \begin{pmatrix} -11 \\ -2 \\ 25 \end{pmatrix} \quad (3.1.1.24)$$

$$\implies \mathbf{u} = \begin{pmatrix} -11 \\ -2 \end{pmatrix}, f = 25 \quad (3.1.1.25)$$

The equation of the circle that passes through the points $\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}$ is given by:

$$\mathbf{x^T x} + 2\begin{pmatrix} -11 \\ -2 \end{pmatrix}^T \mathbf{x} + f = 0 \quad (3.1.1.26)$$

$$\implies x^2 + y^2 - 22x - 4y + 25 = 0 \quad (3.1.1.27)$$

The plot of the circle is given below:

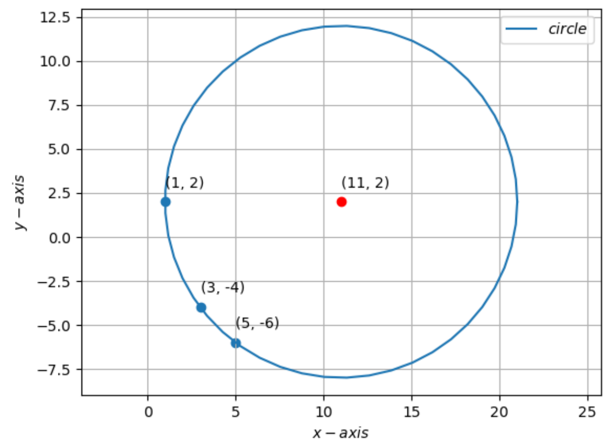

Fig. 3.1.1.2: Circle centered at $(11, 2)$ with radius 10.

3.1.2. Find the equation of circle passing through the points

$$\mathbf{x_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \mathbf{x_2} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \mathbf{x_3} = \begin{pmatrix} 8 \\ 2 \end{pmatrix} \quad (3.1.2.1)$$

Solution: Vector form of the equation of circle is :

$$\mathbf{x}^T\mathbf{x} + 2\mathbf{x}^T\mathbf{u} + f = 0 \quad (3.1.2.2)$$

For $\mathbf{x_1}$, $\mathbf{x_2}$ and $\mathbf{x_3}$ equation (3.1.2.2) can be written as:

$$\mathbf{x_1}^T\mathbf{x_1} + 2\mathbf{x_1}^T\mathbf{u} + f = 0 \quad (3.1.2.3)$$

$$\mathbf{x_2}^T\mathbf{x_2} + 2\mathbf{x_2}^T\mathbf{u} + f = 0 \quad (3.1.2.4)$$

$$\mathbf{x_3}^T\mathbf{x_3} + 2\mathbf{x_3}^T\mathbf{u} + f = 0 \quad (3.1.2.5)$$

In matrix form this can be written as :

$$\begin{pmatrix} 2\mathbf{x_1}^T & 1 \\ 2\mathbf{x_2}^T & 1 \\ 2\mathbf{x_3}^T & 1 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ f \end{pmatrix} = \begin{pmatrix} -\mathbf{x_1}^T\mathbf{x_1} \\ -\mathbf{x_1}^T\mathbf{x_1} \\ -\mathbf{x_3}^T\mathbf{x_3} \end{pmatrix} \tag{3.1.2.6}$$

By putting the values of $\mathbf{x_1}, \mathbf{x_2}$ and $\mathbf{x_3}$ from (3.1.2.1) in (3.1.2.6) we get :

$$\begin{pmatrix} 2 & 2 & 1 \\ 4 & -2 & 1 \\ 16 & 4 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ f \end{pmatrix} = \begin{pmatrix} -2 \\ -5 \\ -68 \end{pmatrix} \tag{3.1.2.7}$$

Using Gaussian Elimination method :

$$\xleftrightarrow[R_2 \leftarrow R_2 - 4R_1]{R_1 \leftarrow \frac{1}{2}R_1} \begin{pmatrix} 1 & 1 & \frac{1}{2} & -1 \\ 0 & -6 & -1 & -1 \\ 16 & 4 & 1 & -68 \end{pmatrix} \tag{3.1.2.8}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - 16R_1} \begin{pmatrix} 1 & 1 & \frac{1}{2} & -1 \\ 0 & -6 & -1 & -1 \\ 0 & -12 & -7 & -52 \end{pmatrix} \tag{3.1.2.9}$$

$$\xleftrightarrow[R_3 \leftarrow R_3 + 124R_2]{R_2 \leftarrow -\frac{1}{6}R_2} \begin{pmatrix} 1 & 1 & \frac{1}{2} & -1 \\ 0 & 1 & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & -5 & -50 \end{pmatrix} \tag{3.1.2.10}$$

Using (3.1.2.7) and (3.1.2.10) we get :

$$\begin{pmatrix} \mathbf{u} \\ f \end{pmatrix} = \begin{pmatrix} -\frac{9}{2} \\ -\frac{3}{2} \\ 10 \end{pmatrix} \tag{3.1.2.11}$$

$$\mathbf{u} = \begin{pmatrix} -\frac{9}{2} \\ -\frac{3}{2} \end{pmatrix} \tag{3.1.2.12}$$

$$f = 10 \tag{3.1.2.13}$$

By putting the values of $\mathbf{u}$ and f in (3.1.2.2) we get :

$$\mathbf{x}^T\mathbf{x} + 2\begin{pmatrix} -\frac{9}{2} \\ -\frac{3}{2} \end{pmatrix}^T \mathbf{x} + 10 = 0 \tag{3.1.2.14}$$

$$x^2 + y^2 - 9x - 6y + 10 = 0 \tag{3.1.2.15}$$

Plot of the circle given by equation (3.1.2.15) is as follows :

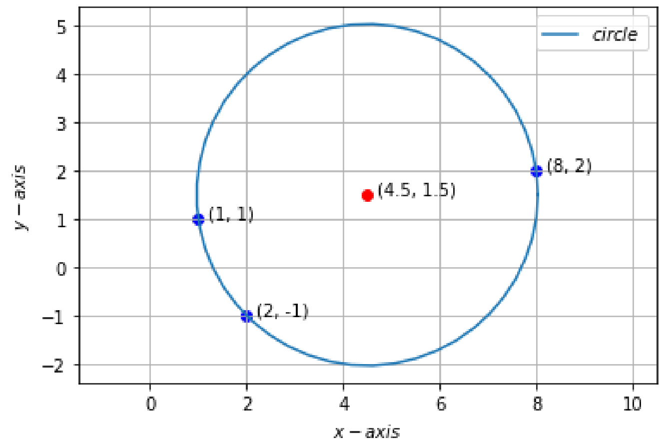

Fig. 3.1.2.1: A circle centered at (4.5, 1.5) with radius 3.53.

3.2 18

3.2.1. Write down the equation of the tangent to a circle passing through the point $\mathbf{p}$.
Equation of the circle and positional vector $\mathbf{p}$ is given as :

$$x^2 + y^2 - 3x + 10y = 15 \tag{3.2.1.1}$$

$$\mathbf{p} = \begin{pmatrix} 4 \\ -11 \end{pmatrix} \tag{3.2.1.2}$$

Solution:
General equation of the circle in vector form is :

$$\mathbf{x}^T\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{3.2.1.3}$$

In the vector form (3.2.1.1) can be written as :

$$\mathbf{x}^T\mathbf{x} + 2\begin{pmatrix} \frac{-3}{2} \\ 5 \end{pmatrix}^T \mathbf{x} - 15 = 0 \tag{3.2.1.4}$$

By comparing (3.2.1.3) and (3.2.1.4) we get :

$$\mathbf{u} = \begin{pmatrix} -\frac{3}{2} \\ 5 \end{pmatrix}, f = -15 \tag{3.2.1.5}$$

We know that the equation of tangent in the form of normal vector $(\mathbf{p} + \mathbf{u})$ and point $\mathbf{p}$ can be written as:

$$(\mathbf{p} + \mathbf{u})^T(\mathbf{x} - \mathbf{p}) = 0 \tag{3.2.1.6}$$

$$(\mathbf{p} + \mathbf{u})^T\mathbf{x} - \mathbf{p}^T\mathbf{p} - \mathbf{u}^T\mathbf{q} = 0 \tag{3.2.1.7}$$

Using (3.2.1.3), (3.2.1.7) will become :

$$(\mathbf{p} + \mathbf{u})^T\mathbf{x} + \mathbf{u}^T\mathbf{p} + f = 0 \tag{3.2.1.8}$$

By putting the values of $\mathbf{p}$, $\mathbf{u}$ and f from

(3.2.1.5) in (3.2.1.8) we get :

$$\begin{pmatrix} \frac{5}{2} & -6 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 4 & -11 \end{pmatrix} \begin{pmatrix} \frac{-3}{2} \\ 5 \end{pmatrix} - 15 = 0 \quad (3.2.1.9)$$

$$\begin{pmatrix} \frac{5}{2} & -6 \end{pmatrix} \mathbf{x} - 76 = 0 \quad (3.2.1.10)$$

Hence the equation of the tangent to the circle passing through the point $\mathbf{p}$ is:

$$\begin{pmatrix} \frac{5}{2} & -6 \end{pmatrix} \mathbf{x} = 76 \quad (3.2.1.11)$$

Plot of the tangent to a circle given by equation (3.2.1.11) is as follows :

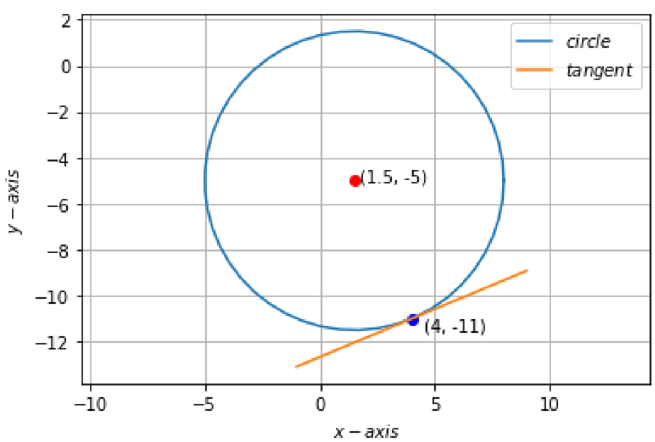

Fig. 3.2.1.1: Tangent to a circle centered at $(1.5, -5)$ with radius 6.5 passing through the point $(4, -11)$.

3.2.2. Find equation of the tangent to the circle

$$x^2 + y^2 = 4 \quad (3.2.2.1)$$

which is parallel to the line

$$x + 2y - 6 = 0 \quad (3.2.2.2)$$

Solution: The equations for the circle and line in (3.2.2.1) and (3.2.2.2) can be rewritten in vector form as:

$$\|\mathbf{x}\|^2 = 4 \quad (3.2.2.3)$$

$$\begin{pmatrix} 1 & 2 \end{pmatrix} \mathbf{x} = 6 \quad (3.2.2.4)$$

The center of the circle happens to be $(0, 0)$ The equation of a line is of the form:

$$\mathbf{n}^T \mathbf{x} = c \quad (3.2.2.5)$$

Where n is the normal to the line.

Comparing (3.2.2.5) to (3.2.2.4),

$$\mathbf{n} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (3.2.2.6)$$

Since the tangent is parallel to the line in (3.2.2.4), it will also have the same normal. The point of contact for a conic is given by:

$$\mathbf{v} = \mathbf{V}^{-1}(\kappa \mathbf{n} - \mathbf{u}) \quad (3.2.2.7)$$

where,

$$\kappa = \pm \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\mathbf{n}^T \mathbf{V}^{-1} \mathbf{n}}} \quad (3.2.2.8)$$

For a circle,

$$\mathbf{V} = \mathbf{I} \quad (3.2.2.9)$$

Using properties of identity matrix, we get:

$$\mathbf{I}^{-1} = \mathbf{I} \quad (3.2.2.10)$$

$$\mathbf{IX} = \mathbf{X} \quad (3.2.2.11)$$

Therefore (3.2.2.7) and (3.2.2.8) simplify to:

$$\kappa = \pm \sqrt{\frac{\mathbf{u}^T \mathbf{u} - f}{\mathbf{n}^T \mathbf{n}}} \quad (3.2.2.12)$$

$$\implies \mathbf{v} = \kappa \mathbf{n} - \mathbf{u} \quad (3.2.2.13)$$

Substituting the values, we get:

$$\kappa = \pm \sqrt{\frac{4}{\begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix}}} \quad (3.2.2.14)$$

$$\implies \kappa = \pm \sqrt{\frac{4}{5}} \quad (3.2.2.15)$$

$$\mathbf{q} = \pm \sqrt{\frac{4}{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (3.2.2.16)$$

$$\implies \mathbf{q_1} = \begin{pmatrix} \sqrt{\frac{4}{5}} \\ \sqrt{\frac{16}{5}} \end{pmatrix}, \mathbf{q_2} = -\begin{pmatrix} \sqrt{\frac{4}{5}} \\ \sqrt{\frac{16}{5}} \end{pmatrix} \quad (3.2.2.17)$$

Since there are two points of contact, there are two tangents parallel to (3.2.2.4) that have the same normal vector.

$$\implies \mathbf{n}^T \mathbf{q_1} = c_1 \quad (3.2.2.18)$$

$$\mathbf{n}^T \mathbf{q_2} = c_2 \quad (3.2.2.19)$$

Substituting the values, we get:

$$c_1 = 2\sqrt{5}, c_2 = -2\sqrt{5} \quad (3.2.2.20)$$

Therefore, the equation of the tangents are:

$$\begin{pmatrix}1 & 2\end{pmatrix}\mathbf{x} = 2\sqrt{5} \quad (3.2.2.21)$$

$$\begin{pmatrix}1 & 2\end{pmatrix}\mathbf{x} = -2\sqrt{5} \quad (3.2.2.22)$$

The plot of the circle with the tangents is given below:

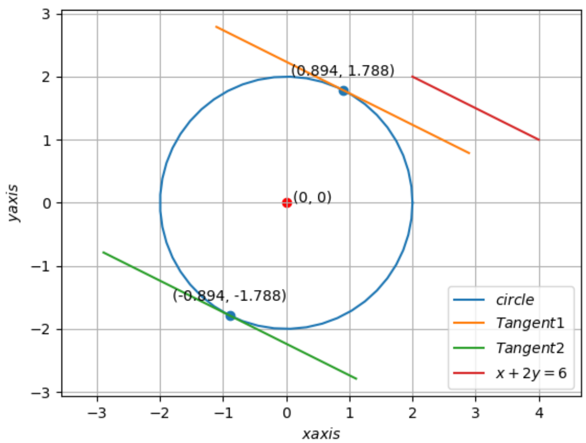

Fig. 3.2.2.1: Circle centered at $(0,0)$ with tangents parallel to line $x+2y=6$.

4 Pair of Straight Lines

4.1 13

4.1.1. Prove that the following equations represent two straight lines, find also their point of intersection and the angle between them.
$6y^2 - xy - x^2 + 30y + 36 = 0$.
Solution:
The given equation can be written as:

$$-x^2 - xy + 6y^2 + 30y + 36 = 0 \quad (4.1.1.1)$$

$\begin{vmatrix}\mathbf{V} & \mathbf{u}\\ \mathbf{u}^T & f\end{vmatrix}$ of (4.1.1.1) becomes

$$\begin{vmatrix}-1 & -\frac{1}{2} & 0\\ \frac{-1}{2} & 6 & 15\\ 0 & 15 & 36\end{vmatrix} = 0 \quad (4.1.1.2)$$

Expanding equation (4.1.1.2), we get zero.
Hence given equation represents a pair of straight lines.
The general equation second degree is given by

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \quad (4.1.1.3)$$

Let (α,β) be their point of intersection, then

$$\begin{pmatrix}a & h\\ h & b\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}-d\\ -e\end{pmatrix} \quad (4.1.1.4)$$

Given equation is

$$-x^2 - xy + 6y^2 + 30y + 36 = 0 \quad (4.1.1.5)$$

Substituting in (4.1.1.4)

$$\begin{pmatrix}-1 & \frac{-1}{2}\\ \frac{-1}{2} & 6\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}0\\ -15\end{pmatrix} \quad (4.1.1.6)$$

$$\implies \begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}\frac{6}{5}\\ \frac{-12}{5}\end{pmatrix} \quad (4.1.1.7)$$

Hence, the intersection point is $\begin{pmatrix}\frac{6}{5}\\ -\frac{12}{5}\end{pmatrix}$
Also, Verified using python code from

codes/Assignment_5.py

From, Spectral decomposition,

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T \quad (4.1.1.8)$$

$$\mathbf{V} = \begin{pmatrix}-1 & \frac{-1}{2}\\ \frac{-1}{2} & 6\end{pmatrix} \quad (4.1.1.9)$$

$$\mathbf{P} = \begin{pmatrix}7-5\sqrt{2} & 7+5\sqrt{2}\\ 1 & 1\end{pmatrix} \quad (4.1.1.10)$$

$$\mathbf{D} = \begin{pmatrix}\frac{5+5\sqrt{2}}{2} & 0\\ 0 & \frac{5-5\sqrt{2}}{2}\end{pmatrix} \quad (4.1.1.11)$$

P and D are also verified using python code from

codes/diagonalize1.py

Using, (4.1.1.7), (4.1.1.10) and (4.1.1.11) in,

$$u_1(x-\alpha) + u_2(y-\beta) = \pm\sqrt{-\frac{\lambda_2}{\lambda_1}}(v_1(x-\alpha) + v_2(y-\beta)) \quad (4.1.1.12)$$

$$\implies \left(7-5\sqrt{2}\right)\left(x-\frac{30}{23}\right) + \left(y+\frac{60}{23}\right)$$
$$= \pm\sqrt{-\frac{\frac{5-5\sqrt{2}}{2}}{\frac{5+5\sqrt{2}}{2}}}\left(\left(7-5\sqrt{2}\right)\left(x-\frac{6}{5}\right) + \left(y+\frac{12}{5}\right)\right) \quad (4.1.1.13)$$

simplifying 4.1.1.13, we get:

$$-x+2y+6=0 \text{ and } x+3y+6=0 \tag{4.1.1.14}$$

$$\implies (-x+2y+6)(x+3y+6)=0 \tag{4.1.1.15}$$

$$\therefore -x+2y=-6 \quad , \quad x+3y=-6 \tag{4.1.1.16}$$

Angle between two lines, θ can be given by

$$n_1 = (\mathbf{-2}, \mathbf{-1}) \tag{4.1.1.17}$$

$$n_2 = (\mathbf{-3}, \mathbf{1}) \tag{4.1.1.18}$$

$$\cos\theta = \frac{\mathbf{n_1}^T \mathbf{n_2}}{\|\mathbf{n_1}\| \, \|\mathbf{n_2}\|} \tag{4.1.1.19}$$

$$\cos\theta = \frac{\begin{pmatrix} -2 & -1 \end{pmatrix}\begin{pmatrix} -3 \\ 1 \end{pmatrix}}{\sqrt{(-2)^2+(-1)^2} \times \sqrt{+(-3)^2+1}} = \frac{1}{\sqrt{2}} \tag{4.1.1.20}$$

$$\implies \theta = 45^\circ \tag{4.1.1.21}$$

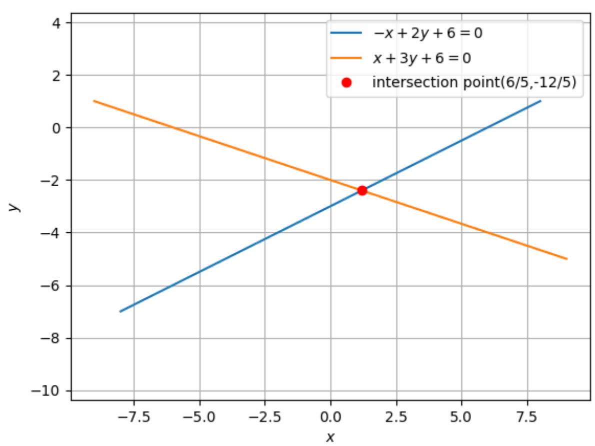

Fig. 4.1.1.1: plot showing intersection of lines.

4.1.2. Prove that the following equations represent two straight lines; and also find their point of intersection and the angle between them

$$x^2 - 5xy + 4y^2 + x + 2y - 2 = 0$$

Solution: Proving that given equation represents two straight lines The given equation is

$$x^2 - 5xy + 4y^2 + x + 2y - 2 = 0 \tag{4.1.2.1}$$

Comparing this to the standard equation,

$$\mathbf{V} = \begin{pmatrix} 1 & \frac{-5}{2} \\ \frac{-5}{2} & 4 \end{pmatrix} \tag{4.1.2.2}$$

$$\mathbf{u} = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} \tag{4.1.2.3}$$

$$f = -2 \tag{4.1.2.4}$$

$$\implies \mathbf{x}^T \begin{pmatrix} 1 & \frac{-5}{2} \\ \frac{-5}{2} & 4 \end{pmatrix}\mathbf{x} + 2\begin{pmatrix} \frac{1}{2} & 1 \end{pmatrix}\mathbf{x} - 2 = 0 \tag{4.1.2.5}$$

Equation (4.1.2.1) represents a pair of straight lines if

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \tag{4.1.2.6}$$

$$\delta = \begin{vmatrix} 1 & \frac{-5}{2} & \frac{1}{2} \\ \frac{-5}{2} & 4 & 1 \\ \frac{1}{2} & 1 & -2 \end{vmatrix} \tag{4.1.2.7}$$

$$= 0 \tag{4.1.2.8}$$

Hence, proved that given equation represents two straight lines. Finding point of intersection between the straight lines

$$\det V = \begin{vmatrix} 1 & \frac{-5}{2} \\ \frac{-5}{2} & 4 \end{vmatrix} \tag{4.1.2.9}$$

$$= \frac{-9}{4} < 0 \tag{4.1.2.10}$$

Thus, the two straight lines intersect. Let the equation of the straight lines be given as

$$\mathbf{n}_1^T \mathbf{x} = c_1 \tag{4.1.2.11}$$

$$\mathbf{n}_2^T \mathbf{x} = c_2 \tag{4.1.2.12}$$

with their slopes as $\mathbf{m}_1$ and $\mathbf{m}_2$ respectively. Then the equation of the pair of straight lines is

$$(\mathbf{n}_1^T \mathbf{x} - c_1)(\mathbf{n}_2^T \mathbf{x} - c_2) = 0 \tag{4.1.2.13}$$

Using (4.1.2.5) and (4.1.2.13),

$$(\mathbf{n}_1^T \mathbf{x} - c_1)(\mathbf{n}_2^T \mathbf{x} - c_2) = \mathbf{x}^T \begin{pmatrix} 1 & \frac{-5}{2} \\ \frac{-5}{2} & 4 \end{pmatrix}\mathbf{x} + 2\begin{pmatrix} \frac{1}{2} & 1 \end{pmatrix}\mathbf{x} - 2 \tag{4.1.2.14}$$

Comparing both sides,

$$c_2\mathbf{n}_1 + c_1\mathbf{n}_2 = -2\begin{pmatrix}\frac{1}{2}\\1\end{pmatrix} \tag{4.1.2.15}$$

$$c_1c_2 = -2 \tag{4.1.2.16}$$

Slopes of the lines are roots of the equation

$$cm^2 + 2bm + a = 0 \tag{4.1.2.17}$$

$$\implies m_i = \frac{-b \pm \sqrt{-\left|\mathbf{V}\right|}}{c} \tag{4.1.2.18}$$

$$\mathbf{n}_i = k_i\begin{pmatrix}-m_i\\1\end{pmatrix} \tag{4.1.2.19}$$

Substituting (4.1.2.1) in (4.1.2.17),

$$4m^2 - 5m + 1 = 0 \tag{4.1.2.20}$$

$$\implies m_i = \frac{\frac{5}{2} \pm \frac{3}{2}}{4} \tag{4.1.2.21}$$

$$\implies m_1 = 1, m_2 = \frac{1}{4} \tag{4.1.2.22}$$

Therefore,

$$\mathbf{n}_1 = k_1\begin{pmatrix}-1\\1\end{pmatrix} \tag{4.1.2.23}$$

$$\mathbf{n}_2 = k_2\begin{pmatrix}\frac{-1}{4}\\1\end{pmatrix} \tag{4.1.2.24}$$

We know that

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix}a\\2b\\c\end{pmatrix} \tag{4.1.2.25}$$

$$k_1\begin{pmatrix}-1\\1\end{pmatrix} * k_2\begin{pmatrix}\frac{-1}{4}\\1\end{pmatrix} = \begin{pmatrix}1\\-5\\4\end{pmatrix} \tag{4.1.2.26}$$

$$\implies k_1k_2 = 4 \tag{4.1.2.27}$$

Taking $k_1 = 1$, $k_2 = 4$, we get

$$\mathbf{n}_1 = \begin{pmatrix}-1\\1\end{pmatrix}$$

$$\mathbf{n}_2 = \begin{pmatrix}-1\\4\end{pmatrix} \tag{4.1.2.28}$$

For verifying values of $\mathbf{n}_1$ and $\mathbf{n}_2$, we compute the convolution by representing $\mathbf{n}_1$ as Toeplitz matrix,

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix}-1 & 0\\1 & -1\\0 & 1\end{pmatrix}\begin{pmatrix}-1\\4\end{pmatrix} = \begin{pmatrix}1\\-5\\4\end{pmatrix} \tag{4.1.2.29}$$

Now, obtaining c_1 and c_2 using (4.1.2.28) and (4.1.2.15)

$$\begin{pmatrix}\mathbf{n}_1 & \mathbf{n}_2\end{pmatrix}\begin{pmatrix}c_2\\c_1\end{pmatrix} = -2\begin{pmatrix}\frac{1}{2}\\1\end{pmatrix} \tag{4.1.2.30}$$

$$\implies \begin{pmatrix}-1 & -1\\1 & 4\end{pmatrix}\begin{pmatrix}c_2\\c_1\end{pmatrix} = \begin{pmatrix}-1\\-2\end{pmatrix} \tag{4.1.2.31}$$

Row reducing the augmented matrix,

$$\begin{pmatrix}-1 & -1 & -1\\1 & 4 & -2\end{pmatrix} \xleftrightarrow{R_1\leftarrow -R_1} \begin{pmatrix}1 & 1 & 1\\1 & 4 & -2\end{pmatrix} \tag{4.1.2.32}$$

$$\xleftrightarrow{R_2\leftarrow R_2-R_1} \begin{pmatrix}1 & 1 & 1\\0 & 3 & -3\end{pmatrix} \tag{4.1.2.33}$$

$$\xleftrightarrow{R_1\leftarrow R_1-R_2} \begin{pmatrix}1 & 0 & 2\\0 & 1 & -1\end{pmatrix} \tag{4.1.2.34}$$

$$\implies \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}c_2\\c_1\end{pmatrix} = \begin{pmatrix}2\\-1\end{pmatrix}$$

$$c_1 = -1 \tag{4.1.2.35}$$

$$c_2 = 2 \tag{4.1.2.36}$$

Thus, equation of lines can be written as

$$\begin{pmatrix}-1 & 1\end{pmatrix}\mathbf{x} = -1 \tag{4.1.2.37}$$

$$\begin{pmatrix}-1 & 4\end{pmatrix}\mathbf{x} = 2 \tag{4.1.2.38}$$

Augmented matrix for these set of equations is

$$\begin{pmatrix}-1 & 1 & -1\\-1 & 4 & 2\end{pmatrix} \xleftrightarrow{R_1\leftarrow -R_1} \begin{pmatrix}1 & -1 & 1\\-1 & 4 & 2\end{pmatrix} \tag{4.1.2.39}$$

$$\xleftrightarrow{R_2\leftarrow R_2+R_1} \begin{pmatrix}1 & -1 & 1\\0 & 3 & 3\end{pmatrix} \xleftrightarrow{R_2\leftarrow \frac{R_2}{3}} \begin{pmatrix}1 & -1 & 1\\0 & 1 & 1\end{pmatrix} \tag{4.1.2.40}$$

$$\xleftrightarrow{R_1\leftarrow R_1+R_2} \begin{pmatrix}1 & 0 & 2\\0 & 1 & 1\end{pmatrix} \tag{4.1.2.41}$$

Thus, the point of intersection is $\mathbf{A} = \begin{pmatrix}2\\1\end{pmatrix}$.

Using (4.1.2.28) and (4.1.2.36) in (4.1.2.13), equation of the pair of straight lines is

$$(x - y - 1)(x - 4y + 2) = 0 \tag{4.1.2.42}$$

Angle between lines Angle between pair of lines is,

$$\theta = \cos^{-1}\left(\frac{\mathbf{n}_1^T\mathbf{n}_2}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|}\right) \tag{4.1.2.43}$$

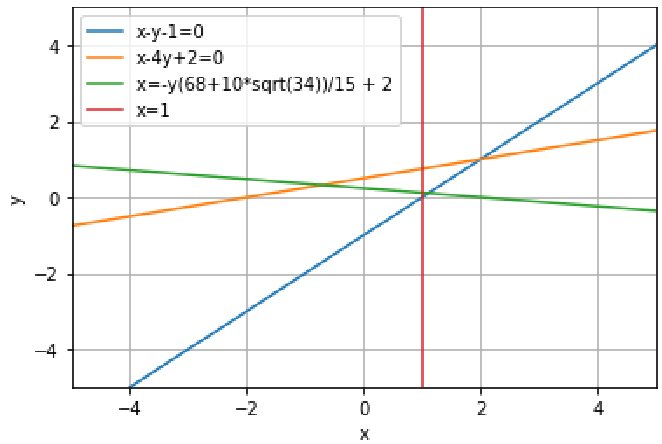

Fig. 4.1.2.1: Intersection of pair of original pair of straight lines and the pair of straight lines after affine transform

$$\mathbf{n}_1^T\mathbf{n}_2 = \begin{pmatrix}-1 & 1\end{pmatrix}\begin{pmatrix}-1\\4\end{pmatrix} = 5 \tag{4.1.2.44}$$

$$\|\mathbf{n}_1\| = \sqrt{(-1)^2+1^2} = \sqrt{2} \tag{4.1.2.45}$$

$$\|\mathbf{n}_2\| = \sqrt{(-1)^2+4^2} = \sqrt{17} \tag{4.1.2.46}$$

Substituting these values (4.1.2.43)

$$\theta = 30.9° \tag{4.1.2.47}$$

Hence, angle between the given pair of straight lines is 30.9° Affine Transformation and Eigen Value decomposition First, verifying if $\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f = 0$. To do this, finding V^{-1} by augmenting with identity matrix and row reducing as follows :

$$\begin{pmatrix}1 & \frac{-5}{2} & 1 & 0\\ \frac{-5}{2} & 4 & 0 & 1\end{pmatrix} \xleftrightarrow{R_2\leftarrow R_2+\frac{5}{2}R_1} \begin{pmatrix}1 & \frac{-5}{2} & 1 & 0\\ 0 & \frac{-9}{4} & \frac{5}{2} & 1\end{pmatrix} \tag{4.1.2.48}$$

$$\xleftrightarrow{R_2\leftarrow \frac{-4}{9}R_2} \begin{pmatrix}1 & \frac{-5}{2} & 1 & 0\\ 0 & 1 & \frac{-10}{9} & \frac{-4}{9}\end{pmatrix} \tag{4.1.2.49}$$

$$\xleftrightarrow{R_1\leftarrow R_1+\frac{5}{2}R_2} \begin{pmatrix}1 & 0 & \frac{-16}{9} & \frac{-10}{9}\\ 0 & 1 & \frac{-10}{9} & \frac{-4}{9}\end{pmatrix} \tag{4.1.2.50}$$

$$\implies \mathbf{V}^{-1} = \begin{pmatrix}\frac{-16}{9} & \frac{-10}{9}\\ \frac{-10}{9} & \frac{-4}{9}\end{pmatrix} \tag{4.1.2.51}$$

$$u^TV^{-1}u - f = \begin{pmatrix}\frac{1}{2} & 1\end{pmatrix}\begin{pmatrix}\frac{-16}{9} & \frac{-10}{9}\\ \frac{-10}{9} & \frac{-4}{9}\end{pmatrix}\begin{pmatrix}\frac{1}{2}\\1\end{pmatrix} - (-2) \tag{4.1.2.52}$$

$$= 0 \tag{4.1.2.53}$$

The characteristic equation of $\mathbf{V}$ is given as :

$$\left|\lambda\mathbf{I}-\mathbf{V}\right| = \begin{vmatrix}\lambda-1 & \frac{5}{2}\\ \frac{5}{2} & \lambda-4\end{vmatrix} = 0 \tag{4.1.2.54}$$

$$\implies (\lambda-1)(\lambda-4) - \frac{25}{4} = 0 \tag{4.1.2.55}$$

$$\implies 4\lambda^2 - 20\lambda - 9 = 0 \tag{4.1.2.56}$$

The roots of (4.1.2.56), i.e. the eigenvalues of $\mathbf{V}$ are

$$\lambda_1 = \frac{5+\sqrt{34}}{2}, \lambda_2 = \frac{5-\sqrt{34}}{2} \tag{4.1.2.57}$$

The eigen vector $\mathbf{p}$ is defined as,

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \tag{4.1.2.58}$$

$$\implies (\lambda\mathbf{I}-\mathbf{V})\mathbf{p} = 0 \tag{4.1.2.59}$$

For $\lambda_1 = \dfrac{5+\sqrt{34}}{2}$

$$(\lambda_1\mathbf{I}-\mathbf{V}) = \begin{pmatrix}\frac{3+\sqrt{34}}{2} & \frac{5}{2}\\ \frac{5}{2} & \frac{-3+\sqrt{34}}{2}\end{pmatrix} \tag{4.1.2.60}$$

To find $\mathbf{p}_1$, let's look at Augmented form of $(\lambda_1\mathbf{I}-\mathbf{V})$

$$\begin{pmatrix}\frac{3+\sqrt{34}}{2} & \frac{5}{2} & 0\\ \frac{5}{2} & \frac{-3+\sqrt{34}}{2} & 0\end{pmatrix} \tag{4.1.2.61}$$

$$\xleftrightarrow{R_1\leftarrow\frac{2}{3+\sqrt{34}}R_1} \begin{pmatrix}1 & \frac{-3+\sqrt{34}}{5} & 0\\ \frac{5}{2} & \frac{-3+\sqrt{34}}{2} & 0\end{pmatrix} \tag{4.1.2.62}$$

$$\xleftrightarrow{R_2\leftarrow\frac{2}{5}R_2-R_1} \begin{pmatrix}1 & \frac{-3+\sqrt{34}}{5} & 0\\ 0 & 0 & 0\end{pmatrix} \tag{4.1.2.63}$$

So we get

$$x_1 + \left(\frac{-3+\sqrt{34}}{5}\right)x_2 = 0 \tag{4.1.2.64}$$

Thus, our eigenvector corresponding to λ_1

$$\mathbf{p}_1 = \begin{pmatrix}\frac{3-\sqrt{34}}{5}\\ 1\end{pmatrix} \tag{4.1.2.65}$$

For $\lambda_2 = \dfrac{5-\sqrt{34}}{2}$

$$(\lambda_2\mathbf{I}-\mathbf{V}) = \begin{pmatrix} \frac{3-\sqrt{34}}{2} & \frac{5}{2} \\ \frac{5}{2} & \frac{-3-\sqrt{34}}{2} \end{pmatrix} \tag{4.1.2.66}$$

To find $\mathbf{p}_2$, let's look at Augmented form of $(\lambda_2\mathbf{I}-\mathbf{V})$

$$\begin{pmatrix} \frac{3-\sqrt{34}}{2} & \frac{5}{2} & 0 \\ \frac{5}{2} & \frac{-3-\sqrt{34}}{2} & 0 \end{pmatrix} \tag{4.1.2.67}$$

$$\xleftrightarrow{R_1 \leftarrow \frac{2}{3-\sqrt{34}}R_1} \begin{pmatrix} 1 & \frac{-3-\sqrt{34}}{5} & 0 \\ \frac{5}{2} & \frac{-3-\sqrt{34}}{2} & 0 \end{pmatrix} \tag{4.1.2.68}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{2}{5}R_2 - R_1} \begin{pmatrix} 1 & \frac{-3-\sqrt{34}}{5} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.1.2.69}$$

So we get

$$x_1 + \left(\frac{-3-\sqrt{34}}{5}\right)x_2 = 0 \tag{4.1.2.70}$$

Thus, our eigenvector corresponding to λ_2

$$\mathbf{p}_2 = \begin{pmatrix} \frac{3+\sqrt{34}}{5} \\ 1 \end{pmatrix} \tag{4.1.2.71}$$

We know $\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T$, where $\mathbf{P}$ and the diagonal matrix $\mathbf{D}$ are given as:

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{4.1.2.72}$$

$$= \begin{pmatrix} \frac{5+\sqrt{34}}{2} & 0 \\ 0 & \frac{5-\sqrt{34}}{2} \end{pmatrix} \tag{4.1.2.73}$$

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix} \tag{4.1.2.74}$$

$$= \begin{pmatrix} \frac{3-\sqrt{34}}{5} & \frac{3+\sqrt{34}}{5} \\ 1 & 1 \end{pmatrix} \tag{4.1.2.75}$$

So, the equation of the pair of straight lines is given by :

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f \qquad \left|\mathbf{V}\right| \neq 0 \tag{4.1.2.76}$$

$$\mathbf{y}^T \begin{pmatrix} \dfrac{5+\sqrt{34}}{2} & 0 \\ 0 & \dfrac{5-\sqrt{34}}{2} \end{pmatrix} \mathbf{y} = 0 \tag{4.1.2.77}$$

$$\implies \begin{pmatrix} y_1 & y_2 \end{pmatrix} \begin{pmatrix} \dfrac{5+\sqrt{34}}{2} & 0 \\ 0 & \dfrac{5-\sqrt{34}}{2} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0 \tag{4.1.2.78}$$

$$\implies (5+\sqrt{34})y_1^2 + (5-\sqrt{34})y_2^2 = 0 \tag{4.1.2.79}$$

So we get the equation of the pair of straight lines, as we can see this passes through the origin $(0,0)$. The corresponding image is shown in Fig. 4.1.2.2

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \qquad \left|\mathbf{V}\right| \neq 0 \tag{4.1.2.80}$$

$$\implies \mathbf{c} = -\begin{pmatrix} \frac{-16}{9} & \frac{-10}{9} \\ \frac{-10}{9} & \frac{-4}{9} \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{4.1.2.81}$$

And,

$$\mathbf{P}^T = \begin{pmatrix} \frac{3-\sqrt{34}}{5} & 1 \\ \frac{3+\sqrt{34}}{5} & 1 \end{pmatrix} \tag{4.1.2.82}$$

Using affine transformation, we can express the equation as

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c} \tag{4.1.2.83}$$

$$\implies \mathbf{x} = \begin{pmatrix} \frac{3-\sqrt{34}}{5} & \frac{3+\sqrt{34}}{5} \\ 1 & 1 \end{pmatrix} \mathbf{y} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{4.1.2.84}$$

The corresponding image is shown in Fig. 4.1.2.1

4.1.3. Prove that the following equations represent two straight lines.Also find their point of intersection and the angle between them

$$3y^2 - 8xy - 3x^2 - 29x + 3y - 18 = 0 \tag{4.1.3.1}$$

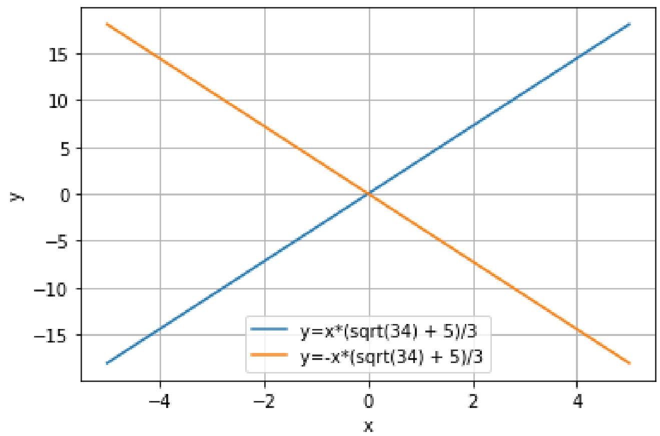

Fig. 4.1.2.2: Pair of straight lines passing through origin after eigenvalue decomposition

Solution: $\begin{vmatrix}\mathbf{V} & \mathbf{u}\\ \mathbf{u}^T & f\end{vmatrix}$ of (4.1.3.1) becomes

$$\begin{vmatrix}-3 & -4 & -\frac{29}{2}\\ -4 & 3 & \frac{3}{2}\\ -\frac{29}{2} & \frac{3}{2} & -18\end{vmatrix} \tag{4.1.3.2}$$

Expanding equation (4.1.3.2), we get zero. Hence given equation represents a pair of straight lines. Slopes of the individual lines are roots of equation

$$cm^2 + 2bm + a = 0 \tag{4.1.3.3}$$

$$\implies 3m^2 - 8m - 3 = 0 \tag{4.1.3.4}$$

$$\text{Solving, } m = 3, -\frac{1}{3} \tag{4.1.3.5}$$

The normal vectors of the lines then become

$$\mathbf{n_1} = \begin{pmatrix}\frac{1}{3}\\ 1\end{pmatrix} \tag{4.1.3.6}$$

$$\mathbf{n_2} = \begin{pmatrix}-3\\ 1\end{pmatrix} \tag{4.1.3.7}$$

Equations of the lines can therefore be written as

$$\begin{pmatrix}\frac{1}{3} & 1\end{pmatrix}\mathbf{x} = c \tag{4.1.3.8}$$

$$\implies \begin{pmatrix}1 & 3\end{pmatrix}\mathbf{x} = c_1, \tag{4.1.3.9}$$

$$\begin{pmatrix}-3 & 1\end{pmatrix}\mathbf{x} = c_2 \tag{4.1.3.10}$$

$$\implies \left[\begin{pmatrix}1 & 3\end{pmatrix}\mathbf{x} - c_1\right]\left[\begin{pmatrix}-3 & 1\end{pmatrix}\mathbf{x} - c_2\right] \tag{4.1.3.11}$$

represents the equation specified in (4.1.3.1) Comparing the equations, we have

$$\begin{pmatrix}1 & -3\\ 3 & 1\end{pmatrix}\begin{pmatrix}c_2\\ c_1\end{pmatrix} = \begin{pmatrix}29\\ -3\end{pmatrix} \tag{4.1.3.12}$$

$$\tag{4.1.3.13}$$

Row reducing the augmented matrix

$$\begin{pmatrix}1 & -3 & 29\\ 3 & 1 & -3\end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - 3\times R_1} \begin{pmatrix}1 & -3 & 29\\ 0 & 10 & -90\end{pmatrix} \tag{4.1.3.14}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 \times \frac{1}{10}} \begin{pmatrix}1 & -3 & 29\\ 0 & 1 & -9\end{pmatrix} \tag{4.1.3.15}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + 3\times R_2} \begin{pmatrix}1 & 0 & 2\\ 0 & 1 & -9\end{pmatrix} \tag{4.1.3.16}$$

$$\implies c_2 = 2 \text{ and } c_1 = -9 \tag{4.1.3.17}$$

The individual line equations therefore become

$$\begin{pmatrix}1 & 3\end{pmatrix}\mathbf{x} = -9, \tag{4.1.3.18}$$

$$\begin{pmatrix}-3 & 1\end{pmatrix}\mathbf{x} = 2 \tag{4.1.3.19}$$

Note that the convolution of the normal vectors, should satisfy the below condition

$$\begin{pmatrix}1\\ 3\end{pmatrix} * \begin{pmatrix}-3\\ 1\end{pmatrix} = \begin{pmatrix}a\\ 2b\\ c\end{pmatrix} \tag{4.1.3.20}$$

The LHS part of (4.1.3.20) can be rewritten using toeplitz matrix as

$$\begin{pmatrix}1 & 0\\ 3 & 1\\ 0 & 3\end{pmatrix}\begin{pmatrix}-3\\ 1\end{pmatrix} = \begin{pmatrix}-3\\ -8\\ 3\end{pmatrix} = \begin{pmatrix}a\\ 2b\\ c\end{pmatrix} \tag{4.1.3.21}$$

The augmented matrix for the set of equations represented in (4.1.3.18), (4.1.3.19) is

$$\begin{pmatrix} 1 & 3 & -9 \\ -3 & 1 & 2 \end{pmatrix} \tag{4.1.3.22}$$

Row reducing the matrix

$$\begin{pmatrix} 1 & 3 & -9 \\ -3 & 1 & 2 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 + 3 \times R_1} \begin{pmatrix} 1 & 3 & -9 \\ 0 & 10 & -25 \end{pmatrix} \tag{4.1.3.23}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - \frac{3}{10} \times R_2} \begin{pmatrix} 1 & 0 & -\frac{3}{2} \\ 0 & 10 & -25 \end{pmatrix} \tag{4.1.3.24}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{R_2}{10}} \begin{pmatrix} 1 & 0 & -\frac{3}{2} \\ 0 & 1 & -\frac{5}{2} \end{pmatrix} \tag{4.1.3.25}$$

$$\text{Hence, the intersection point is} \begin{pmatrix} -\frac{3}{2} \\ -\frac{5}{2} \end{pmatrix} \tag{4.1.3.26}$$

Angle between two lines θ can be given by

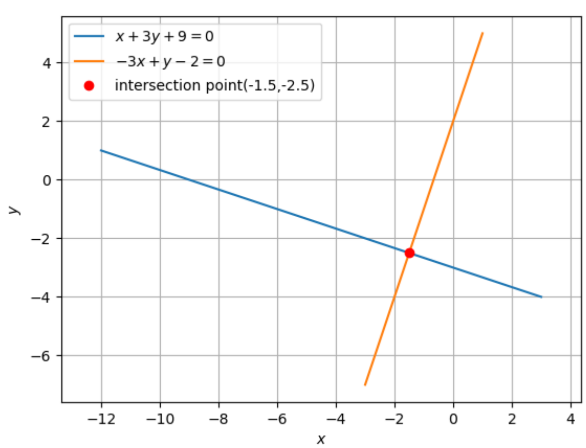

Fig. 4.1.3.1: plot showing intersection of lines

$$\cos\theta = \frac{\mathbf{n_1}^T \mathbf{n_2}}{\|\mathbf{n_1}\| \, \|\mathbf{n_2}\|} \tag{4.1.3.27}$$

$$\cos\theta = \frac{\begin{pmatrix} 1 & 3 \end{pmatrix} \begin{pmatrix} -3 \\ 1 \end{pmatrix}}{\sqrt{(3)^2 + 1} \times \sqrt{(-3)^2 + 1}} = 0 \tag{4.1.3.28}$$

$$\implies \theta = 90^\circ \tag{4.1.3.29}$$

4.1.4. Prove that the following equations represents two straight lines also find their point of intersection and angle between them.

$$y^2 + xy - 2x^2 - 5x - y - 2 = 0 \tag{4.1.4.1}$$

Solution:

$$\mathbf{V} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} -2 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} \tag{4.1.4.2}$$

$$\mathbf{u} = \begin{pmatrix} d \\ e \end{pmatrix} = \begin{pmatrix} \frac{-5}{2} \\ \frac{-1}{2} \end{pmatrix} \tag{4.1.4.3}$$

$$f = -2 \tag{4.1.4.4}$$

$$\begin{vmatrix} -2 & \frac{1}{2} & \frac{-5}{2} \\ \frac{1}{2} & 1 & \frac{-1}{2} \\ \frac{-5}{2} & \frac{-1}{2} & -2 \end{vmatrix} \xleftrightarrow[R_1 \to R_1 + R_3]{R_1 \to R_1 - R_2} \begin{vmatrix} 0 & 0 & 0 \\ \frac{1}{2} & 1 & \frac{-1}{2} \\ \frac{-5}{2} & \frac{-1}{2} & -2 \end{vmatrix} = 0 \tag{4.1.4.5}$$

Hence it represents the pair of straight lines. Now two intersecting lines are obtained when

$$\left|V\right| < 0 \implies \begin{vmatrix} -2 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{vmatrix} = \frac{-9}{4} < 0 \tag{4.1.4.6}$$

Let the pair of straight of lines be given by

$$\mathbf{n_1}^T \mathbf{x} = c_1 \tag{4.1.4.7}$$

$$\mathbf{n_2}^T \mathbf{x} = c_2 \tag{4.1.4.8}$$

The slopes of the lines are given by the roots of the polynomial

$$cm^2 + 2bm + a = 0 \tag{4.1.4.9}$$

$$m_1, m_2 = \frac{-\frac{1}{2} \pm \sqrt{\frac{9}{4}}}{1} \tag{4.1.4.10}$$

$$m_1 = 1, m_2 = -2 \tag{4.1.4.11}$$

$$\implies \mathbf{n_1} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} and \mathbf{n_2} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{4.1.4.12}$$

$$(\mathbf{n_1}^T \mathbf{x} - c_1)(\mathbf{n_2}^T \mathbf{x} - c_2) = \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f \tag{4.1.4.13}$$

$$c_2 \mathbf{n_1} + c_1 \mathbf{n_2} = -2\mathbf{u} \tag{4.1.4.14}$$

$$c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = -2 \left(\frac{-5}{2} \; \frac{-1}{2} \right) \tag{4.1.4.15}$$

$$\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \tag{4.1.4.16}$$

Using row reduction we get

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 5 \end{pmatrix} \tag{4.1.4.17}$$

$$\xleftrightarrow[R_2 \leftarrow R_2/-3]{R_2 \leftarrow R_2 - 2R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \tag{4.1.4.18}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - R_2} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{pmatrix} \tag{4.1.4.19}$$

$$C = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \tag{4.1.4.20}$$

The convolution of the normal vectors, should satisfy the below condition

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix} * \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \tag{4.1.4.21}$$

The LHS part of equation(2.0.20) can be rewritten using toeplitz matrix as

$$\begin{pmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \tag{4.1.4.22}$$

Therefore the equation of lines is given by

$$\begin{pmatrix} -1 & 1 \end{pmatrix} \mathbf{x} = 2 \tag{4.1.4.23}$$

$$\begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = -1 \tag{4.1.4.24}$$

consider the augmented matrix

$$\begin{pmatrix} -1 & 1 & 2 \\ 2 & 1 & -1 \end{pmatrix} \tag{4.1.4.25}$$

$$\xleftrightarrow[R_2 \leftarrow R_2 - 2R_1]{R_1 \leftarrow -R_1} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \tag{4.1.4.26}$$

$$\xleftrightarrow[R_1 \leftarrow R_1 + R_2]{R_1 \leftarrow R_1/3} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} \tag{4.1.4.27}$$

Therefore point of intersection is $\mathbf{A} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.
Angle between two lines θ can be given by

$$\cos\theta = \frac{\mathbf{n_1}^T \mathbf{n_2}}{\|\mathbf{n_1}\| \, \|\mathbf{n_2}\|} \tag{4.1.4.28}$$

$$\cos\theta = \frac{\begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}}{\sqrt{(1)^2 + 1} \times \sqrt{(2)^2 + 1}} \tag{4.1.4.29}$$

$$\theta = \cos^{-1}\left(\frac{-1}{\sqrt{10}}\right) \implies \theta = tan^{-1} 3 \tag{4.1.4.30}$$

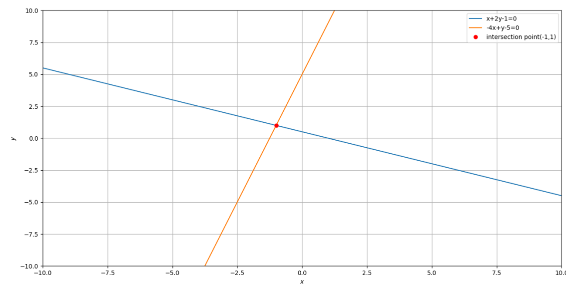

Fig. 4.1.4.1: plot showing intersection of lines

4.1.5. Prove that the equation

$$x^2 + 6xy + 9y^2 + 4x + 12y - 5 = 0 \tag{4.1.5.1}$$

represents two parallel lines.
Solution: The given equation (4.1.5.1) can be written as

$$\mathbf{x}^T \begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} 2 & 6 \end{pmatrix} \mathbf{x} - 5 = 0 \tag{4.1.5.2}$$

$$\mathbf{V} = \begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix} \quad \mathbf{u} = \begin{pmatrix} 2 \\ 6 \end{pmatrix} \quad f = -5 \tag{4.1.5.3}$$

Equation (4.1.5.1) represents pair of straight line as,

$$D = \begin{vmatrix} 1 & 3 & 2 \\ 3 & 9 & 6 \\ 2 & 6 & -5 \end{vmatrix} = 0 \tag{4.1.5.4}$$

Vector form of straight lines,

$$\mathbf{n_1}^T \mathbf{x} = \mathbf{c_1} \tag{4.1.5.5}$$

$$\mathbf{n_2}^T \mathbf{x} = \mathbf{c_2} \tag{4.1.5.6}$$

Equating their product with (4.1.5.2)

$$(\mathbf{n_1}^T \mathbf{x} - \mathbf{c_1})(\mathbf{n_2}^T \mathbf{x} - \mathbf{c_2}) = \mathbf{x}^T \begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} 2 & 6 \end{pmatrix} \mathbf{x} - 5 \tag{4.1.5.7}$$

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} 1 \\ 6 \\ 9 \end{pmatrix} \tag{4.1.5.8}$$

$$c_2 \mathbf{n_1} + c_1 \mathbf{n_2} = -2 \begin{pmatrix} 2 \\ 6 \end{pmatrix} \tag{4.1.5.9}$$

$$c_1 c_2 = -5 \tag{4.1.5.10}$$

The slopes of the lines can be given by roots of the equation,

$$cm^2 + 2bm + a = 0 \tag{4.1.5.11}$$

$$m_i = \frac{-b \pm \sqrt{-\left|\mathbf{V}\right|}}{c} \tag{4.1.5.12}$$

$$\mathbf{n_i} = k_i \begin{pmatrix} -m_i \\ 1 \end{pmatrix} \tag{4.1.5.13}$$

From (4.1.5.2) equation (4.1.5.11) becomes

$$9m^2 + 6m + 1 = 0 \tag{4.1.5.14}$$

Using (4.1.5.3),

$$\left|\mathbf{V}\right| = \begin{vmatrix} 1 & 3 \\ 3 & 9 \end{vmatrix} = 0 \tag{4.1.5.15}$$

Substituting the values in (4.1.5.12),

$$m_i = \frac{-3 \pm 0}{9} \tag{4.1.5.16}$$

$$m_1 = m_2 = \frac{-1}{3} \tag{4.1.5.17}$$

Substituting values in (4.1.5.13)

$$\mathbf{n_1} = k_1 \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} \tag{4.1.5.18}$$

$$\mathbf{n_2} = k_2 \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} \tag{4.1.5.19}$$

Using the above values in (4.1.5.8),

$$k_1 k_2 = 9 \tag{4.1.5.20}$$

Taking $k_1 = 3$ and $k_2 = 3$ we get

$$\mathbf{n_1} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \tag{4.1.5.21}$$

$$\mathbf{n_2} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \tag{4.1.5.22}$$

Verifying $\mathbf{n_1}$ and $\mathbf{n_2}$ by computing the convolution by representing $\mathbf{n_1}$ as Toeplitz matrix,

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \\ 9 \end{pmatrix} \tag{4.1.5.23}$$

Finding the Angle between the lines,

$$\theta = \cos^{-1}\left(\frac{\mathbf{n_1}^T \mathbf{n_2}}{\|\mathbf{n_1}\| \, \|\mathbf{n_2}\|}\right) \tag{4.1.5.24}$$

$$\mathbf{n_1}^T \mathbf{n_2} = \begin{pmatrix} 1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = 10 \tag{4.1.5.25}$$

$$\|\mathbf{n_1}\| = \sqrt{10} \quad \|\mathbf{n_2}\| = \sqrt{10} \tag{4.1.5.26}$$

Substituting (4.1.5.25) and (4.1.5.26) in (4.1.5.24) we get,

$$\theta = \cos^{-1}(1) \tag{4.1.5.27}$$

$$\theta = 0^\circ \tag{4.1.5.28}$$

From (4.1.5.17) and (4.1.5.28) shows the given equation (4.1.5.1) represents two parallel lines. Hence proved.

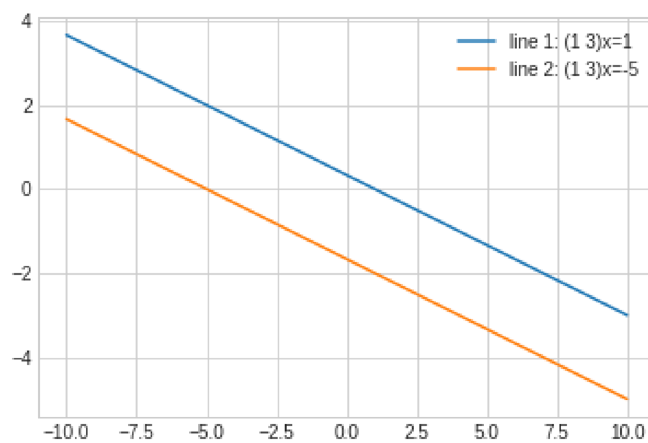

Fig. 4.1.5.1: Pair of straight lines plot generated using python

4.1.6. **Solution:** Find the value of k such that

$$6x^2 + 11xy - 10y^2 + x + 31y + k = 0 \tag{4.1.6.1}$$

represent pairs of straight lines.
From (4.1.6.1) we get,

$$\mathbf{V} = \begin{pmatrix} 6 & \frac{11}{2} \\ \frac{11}{2} & -10 \end{pmatrix} \tag{4.1.6.2}$$

$$\mathbf{u} = \begin{pmatrix} \frac{1}{2} \\ \frac{31}{2} \end{pmatrix} \tag{4.1.6.3}$$

$$f = k \tag{4.1.6.4}$$

Compute the slopes of lines given by the roots of the polynomial $-10m^2 + 11m + 6$

$$i.e., m_i = \frac{-b \pm \sqrt{-\left|\mathbf{V}\right|}}{c} \tag{4.1.6.5}$$

$$\implies m = \frac{\frac{-11}{2} \pm \frac{19}{2}}{-10} \tag{4.1.6.6}$$

$$\implies m_1 = \frac{-2}{5}, m_2 = \frac{3}{2} \tag{4.1.6.7}$$

Let the pair of straight lines be given by

$$\mathbf{n}_1^T\mathbf{x} = c_1 \tag{4.1.6.8}$$

$$\mathbf{n}_2^T\mathbf{x} = c_2 \tag{4.1.6.9}$$

Here,

$$\mathbf{n}_1 = k_1\begin{pmatrix}-m_1\\1\end{pmatrix} = k_1\begin{pmatrix}\frac{2}{5}\\1\end{pmatrix} \tag{4.1.6.10}$$

$$\mathbf{n}_2 = k_2\begin{pmatrix}-m_2\\1\end{pmatrix} = k_2\begin{pmatrix}\frac{-3}{2}\\1\end{pmatrix} \tag{4.1.6.11}$$

We know that,

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix}a\\2b\\c\end{pmatrix} \tag{4.1.6.12}$$

Substituting (4.1.6.10) and (4.1.6.11) in the above equation, we get

$$k_1\begin{pmatrix}\frac{2}{5}\\1\end{pmatrix} * k_2\begin{pmatrix}\frac{-3}{2}\\1\end{pmatrix} = \begin{pmatrix}6\\11\\-10\end{pmatrix} \tag{4.1.6.13}$$

$$\implies k_1k_2 = -10 \tag{4.1.6.14}$$

By inspection, we get the values, $k_1 = 5, k_2 = -2$. Substituting the values of k_1 and k_2 in (4.1.6.10) and (4.1.6.11) respectively, we get

$$\mathbf{n}_1 = \begin{pmatrix}2\\5\end{pmatrix} \tag{4.1.6.15}$$

$$\mathbf{n}_2 = \begin{pmatrix}3\\-2\end{pmatrix} \tag{4.1.6.16}$$

Using Teoplitz matrix representation, the convolution of $\mathbf{n}_1$ with $\mathbf{n}_2$, is as follows:

$$\begin{pmatrix}2&0&5\\5&2&0\\0&5&2\end{pmatrix}\begin{pmatrix}3\\-2\\0\end{pmatrix} = \begin{pmatrix}6\\11\\-10\end{pmatrix} = \begin{pmatrix}a\\2b\\c\end{pmatrix} \tag{4.1.6.17}$$

Hence, $\mathbf{n}_1$ and $\mathbf{n}_2$ satisfies (4.1.6.12). We have,

$$c_2\mathbf{n}_1 + c_1\mathbf{n}_2 = -2\mathbf{u} \tag{4.1.6.18}$$

Substituting (4.1.6.15), (4.1.6.16) in (4.1.6.18), we get

$$\begin{pmatrix}2&3\\5&-2\end{pmatrix}\begin{pmatrix}c_2\\c_1\end{pmatrix} = -2\begin{pmatrix}\frac{1}{2}\\\frac{31}{2}\end{pmatrix} \tag{4.1.6.19}$$

Solving for c_1 and c_2, the augmented matrix is,

$$\begin{pmatrix}2&3&-1\\5&-2&-31\end{pmatrix} \xleftrightarrow[R_2\leftarrow R_2-5R_1]{R_1\leftarrow\frac{R_1}{2}} \begin{pmatrix}1&\frac{3}{2}&\frac{-1}{2}\\0&\frac{-19}{2}&\frac{-57}{2}\end{pmatrix} \tag{4.1.6.20}$$

$$\xleftrightarrow[R_1\leftarrow R_1-\frac{3}{2}R_2]{R_2\leftarrow\frac{R_2}{-19/2}} \begin{pmatrix}1&0&-5\\0&1&3\end{pmatrix} \tag{4.1.6.21}$$

Hence we obtain,

$$c_1 = 3, c_2 = -5 \tag{4.1.6.22}$$

We know that,

$$f = k = c_1c_2 \tag{4.1.6.23}$$

$$\implies \boxed{k = -15} \tag{4.1.6.24}$$

Hence the solution. Using (4.1.6.8) and (4.1.6.9), the equation of pair of straight lines is given by,

$$\begin{pmatrix}2&5\end{pmatrix}\mathbf{x} = 3 \tag{4.1.6.25}$$

$$\begin{pmatrix}3&-2\end{pmatrix}\mathbf{x} = -5 \tag{4.1.6.26}$$

See Fig. 4.1.6.1

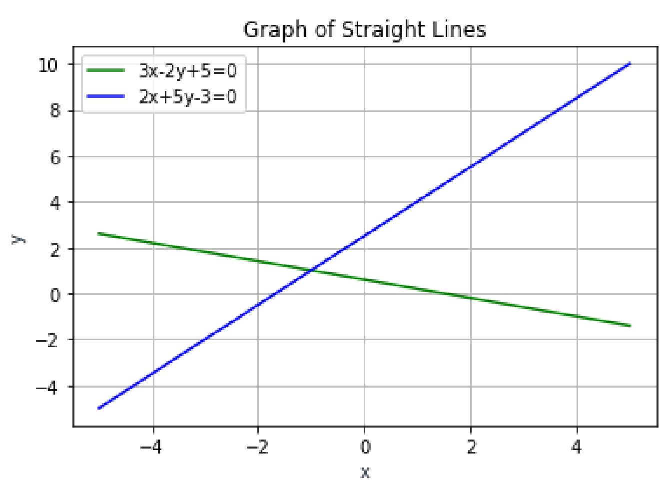

Fig. 4.1.6.1: Plot of two straight lines.

4.1.7. Find the value of k so that following equation may represent pairs of straight lines,

$$12x^2 - 10xy + 2y^2 + 11x - 5y + k = 0 \tag{4.1.7.1}$$

Solution: The general equation of second degree is given by,

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{4.1.7.2}$$

In vector from the equation (4.1.7.2) can be expressed as,

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{4.1.7.3}$$

where,

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \tag{4.1.7.4}$$

$$\mathbf{u} = \begin{pmatrix} d \\ e \end{pmatrix} \tag{4.1.7.5}$$

Now, comparing (4.1.7.2) to (4.1.7.1) we get, a =12, b=-5, c = 2, d = $\frac{11}{2}$,e = -$\frac{5}{2}$, f = k. Hence, substituting these values in (4.1.7.4) and (4.1.7.5) we get,

$$\mathbf{V} = \begin{pmatrix} 12 & -5 \\ -5 & 2 \end{pmatrix} \tag{4.1.7.6}$$

$$\mathbf{u} = \begin{pmatrix} \frac{11}{2} \\ -\frac{5}{2} \end{pmatrix} \tag{4.1.7.7}$$

(4.1.7.1) represents pair of straight lines if,

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \tag{4.1.7.8}$$

$$\begin{vmatrix} 12 & -5 & \frac{11}{2} \\ -5 & 2 & -\frac{5}{2} \\ \frac{11}{2} & -\frac{5}{2} & k \end{vmatrix} = 0 \tag{4.1.7.9}$$

$$\implies k = 2 \tag{4.1.7.10}$$

Lines Intercept if

$$|\mathbf{V}| < 0 \tag{4.1.7.11}$$

$$|\mathbf{V}| = -1 < 0 \tag{4.1.7.12}$$

Hence Line intercept.
Let (α,β) be their point of intersection, then

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -d \\ -e \end{pmatrix} \tag{4.1.7.13}$$

Substituting in (4.1.7.13)

$$\begin{pmatrix} 12 & -5 \\ -5 & 2 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -\frac{11}{2} \\ \frac{5}{2} \end{pmatrix} \tag{4.1.7.14}$$

$$\implies \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -\frac{3}{2} \\ -\frac{5}{2} \end{pmatrix} \tag{4.1.7.15}$$

Spectral Decomposition of $\mathbf{V}$ is given as

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T \tag{4.1.7.16}$$

$$\mathbf{V} = \begin{pmatrix} 12 & -5 \\ -5 & 2 \end{pmatrix} \tag{4.1.7.17}$$

$$\mathbf{P} = \begin{pmatrix} -1 - \sqrt{2} & -1 + \sqrt{2} \\ 1 & 1 \end{pmatrix} \tag{4.1.7.18}$$

$$\mathbf{D} = \begin{pmatrix} 7 + 5\sqrt{2} & 0 \\ 0 & 7 - 5\sqrt{2} \end{pmatrix} \tag{4.1.7.19}$$

Using Spectral decomposition concept and substution

$$u_1(x-\alpha) + u_2(y-\beta) = \pm\sqrt{-\frac{\lambda_2}{\lambda_1}}(v_1(x-\alpha) + v_2(y-\beta)) \tag{4.1.7.20}$$

Substituting (4.1.7.15), (4.1.7.18) and (4.1.7.19) in (4.1.7.20)

$$\left(-1 - \sqrt{2}\right)\left(x - \frac{-3}{2}\right) + \left(y - \frac{-5}{2}\right)$$
$$= \pm\sqrt{-\frac{7 + 5\sqrt{2}}{7 - 5\sqrt{2}}}\left(\left(-1 + \sqrt{2}\right)\left(x - \frac{-3}{2}\right) + \left(y - \frac{-5}{2}\right)\right) \tag{4.1.7.21}$$

Simplifying (4.1.7.21),

$$-6x + 2y - 4 = 0 \text{ and } -2x + y - \frac{1}{2} = 0 \tag{4.1.7.22}$$

$$\implies (-6x + 2y - 4)\left(-2x + y - \frac{1}{2}\right) = 0 \tag{4.1.7.23}$$

Thus the equation of lines are

$$\begin{pmatrix} -6 & 2 \end{pmatrix}\mathbf{x} = 4 \tag{4.1.7.24}$$

$$\begin{pmatrix} -2 & 1 \end{pmatrix}\mathbf{x} = \frac{1}{2} \tag{4.1.7.25}$$

Hence, Plot is shown below

4.1.8. Find the value of k so that the following

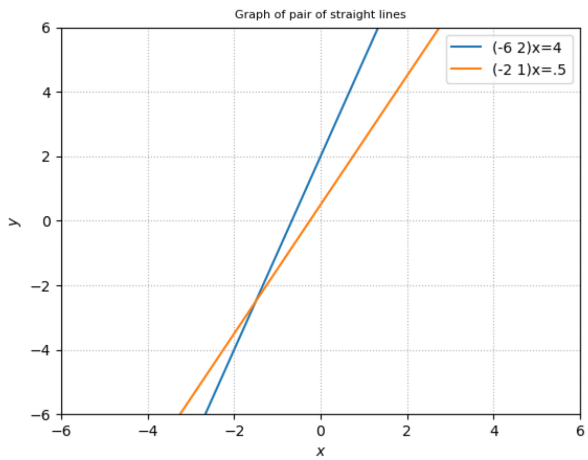

Fig. 4.1.7.1: Pair of lines

equation may represent pair of straight lines:

$$12x^2 + kxy + 2y^2 + 11x - 5y + 2 = 0 \tag{4.1.8.1}$$

Solution:

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} 12 & \frac{k}{2} \\ \frac{k}{2} & 2 \end{pmatrix} \tag{4.1.8.2}$$

$$\mathbf{u} = \begin{pmatrix} d \\ e \end{pmatrix} = \begin{pmatrix} \frac{11}{2} \\ -\frac{5}{2} \end{pmatrix} \tag{4.1.8.3}$$

The equation (4.1.8.1) represents pair of straight lines if

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \tag{4.1.8.4}$$

$$\implies \begin{vmatrix} 12 & \frac{k}{2} & \frac{11}{2} \\ \frac{k}{2} & 2 & -\frac{5}{2} \\ \frac{11}{2} & -\frac{5}{2} & 2 \end{vmatrix} = 0 \tag{4.1.8.5}$$

$$\implies \begin{vmatrix} 24 & k & 11 \\ k & 4 & -5 \\ 11 & -5 & 4 \end{vmatrix} = 0 \tag{4.1.8.6}$$

$$\implies 24\begin{vmatrix} 4 & -5 \\ -5 & 4 \end{vmatrix} - k\begin{vmatrix} k & -5 \\ 11 & 4 \end{vmatrix} + 11\begin{vmatrix} k & 4 \\ 11 & -5 \end{vmatrix} = 0 \tag{4.1.8.7}$$

$$\implies 2k^2 + 55k + 350 = 0 \tag{4.1.8.8}$$

$$\implies (10 + k)(2k + 35) = 0 \tag{4.1.8.9}$$

$$\implies k = -10$$

$$k = -\frac{35}{2} \tag{4.1.8.10}$$

Therefore, for $k = -10$ and $k = -\frac{35}{2}$ the given equation represents pair of straight lines.
Now Lets find equation of lines for $k = -10$. Substitute $k = -10$ in (4.1.8.1). We get equation of pair of straight lines as:

$$12x^2 - 10xy + 2y^2 + 11x - 5y + 2 = 0 \tag{4.1.8.11}$$

From (4.1.8.1), (4.1.8.2), (4.1.8.3) we get

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 12 & -5 \\ -5 & 2 \end{pmatrix} \tag{4.1.8.12}$$

$$\mathbf{u} = \begin{pmatrix} \frac{11}{2} \\ -\frac{5}{2} \end{pmatrix} \tag{4.1.8.13}$$

If $\left|\mathbf{V}\right| < 0$ then two lines will intersect.

$$\left|\mathbf{V}\right| = \begin{vmatrix} 12 & -5 \\ -5 & 2 \end{vmatrix} \tag{4.1.8.14}$$

$$\implies \left|\mathbf{V}\right| = -1 \tag{4.1.8.15}$$

$$\implies \left|\mathbf{V}\right| < 0 \tag{4.1.8.16}$$

Therefore the lines will intersect.
The equation of two lines is given by

$$\mathbf{n_1}^T\mathbf{x} = c_1 \tag{4.1.8.17}$$

$$\mathbf{n_2}^T\mathbf{x} = c_2 \tag{4.1.8.18}$$

Equating their product with (4.1.8.1)

$$(\mathbf{n_1}^T\mathbf{x} - c_1)(\mathbf{n_2}^T\mathbf{x} - c_2) = \mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{4.1.8.19}$$

$$\implies \mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} = \begin{pmatrix} 12 \\ -10 \\ 2 \end{pmatrix} \tag{4.1.8.20}$$

$$c_2\mathbf{n_1} + c_1\mathbf{n_2} = -2\mathbf{u} = -2\begin{pmatrix} \frac{11}{2} \\ -\frac{5}{2} \end{pmatrix} \tag{4.1.8.21}$$

$$c_1c_2 = f = 2 \tag{4.1.8.22}$$

The slopes of the lines are given by roots of equation

$$cm^2 + 2bm + a = 0 \tag{4.1.8.23}$$
$$\implies 2m^2 - 10m + 12 = 0 \tag{4.1.8.24}$$
$$m_i = \frac{-b \pm \sqrt{-\left|\mathbf{V}\right|}}{c} \tag{4.1.8.25}$$
$$\implies m_i = \frac{5 \pm \sqrt{1}}{2} \tag{4.1.8.26}$$
$$\implies m_1 = 3 \tag{4.1.8.27}$$
$$m_2 = 2 \tag{4.1.8.28}$$

The normal vector for two lines is given by

$$\mathbf{n_i} = k_i \begin{pmatrix} -m_i \\ 1 \end{pmatrix} \tag{4.1.8.29}$$
$$\implies \mathbf{n_1} = k_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix} \tag{4.1.8.30}$$
$$\mathbf{n_2} = k_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} \tag{4.1.8.31}$$

Substituting (4.1.8.30),(4.1.8.31) in (4.1.8.20). we get

$$k_1 k_2 = 2 \tag{4.1.8.32}$$

The possible combinations of (k_1,k_2) are (1,2), (2,1), (-1,-2) and (-2,-1).
lets assume $k_1 = 1$,$k_2 = 2$ we get

$$\implies \mathbf{n_1} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \tag{4.1.8.33}$$
$$\mathbf{n_2} = \begin{pmatrix} -4 \\ 2 \end{pmatrix} \tag{4.1.8.34}$$

We verify obtained $\mathbf{n_1}$,$\mathbf{n_2}$ using Toeplitz matrix

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} -3 & 0 \\ 1 & -3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -4 \\ 2 \end{pmatrix} = \begin{pmatrix} 12 \\ -10 \\ 2 \end{pmatrix} \tag{4.1.8.35}$$
$$\implies \mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} 12 \\ -10 \\ 2 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \tag{4.1.8.36}$$

Therefore the obtained $\mathbf{n_1}$,$\mathbf{n_2}$ are correct.
Substitute (4.1.8.33), (4.1.8.34) in (4.1.8.21) and calculate for c_1 and c_2

$$c_2 \begin{pmatrix} -3 \\ 1 \end{pmatrix} + c_1 \begin{pmatrix} -4 \\ 2 \end{pmatrix} = \begin{pmatrix} -11 \\ -5 \end{pmatrix} \tag{4.1.8.37}$$

Solve using row reduction technique.

$$\implies \begin{pmatrix} -4 & -3 & -11 \\ 2 & 1 & -5 \end{pmatrix} \tag{4.1.8.38}$$
$$\xleftrightarrow{R_2 \leftarrow 2R_2 + R_1} \begin{pmatrix} -4 & -3 & -11 \\ 0 & -1 & -21 \end{pmatrix} \tag{4.1.8.39}$$
$$\xleftrightarrow{R_1 \leftarrow R_1 - 3R_2} \begin{pmatrix} -4 & 0 & 52 \\ 0 & -1 & -21 \end{pmatrix} \tag{4.1.8.40}$$
$$\implies \begin{pmatrix} 1 & 0 & -13 \\ 0 & 1 & 21 \end{pmatrix} \tag{4.1.8.41}$$
$$\implies c_1 = -13 \tag{4.1.8.42}$$
$$c_2 = 21 \tag{4.1.8.43}$$

Substituting (4.1.8.33),(4.1.8.34),(4.1.8.42),(4.1.8.43) in (4.1.8.17) and (4.1.8.18). We get equation of two straight lines.

$$\begin{pmatrix} -3 & 1 \end{pmatrix} \mathbf{x} = -13 \tag{4.1.8.44}$$
$$\begin{pmatrix} -4 & 2 \end{pmatrix} \mathbf{x} = 21 \tag{4.1.8.45}$$

The plot of these two lines is shown in Fig. 4.1.8.1.

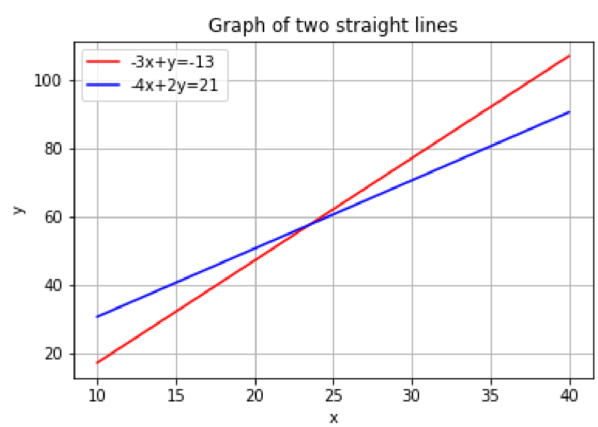

Fig. 4.1.8.1: Pair of straight lines for $k = -10$

Now Lets find equation of lines for $k = -\frac{35}{2}$. Substitute $k = -\frac{35}{2}$ in (4.1.8.1). We get equation of pair of straight lines as:

$$12x^2 - \frac{35}{2}xy + 2y^2 + 11x - 5y + 2 = 0 \tag{4.1.8.46}$$

From (4.1.8.1), (4.1.8.2), (4.1.8.3) we get

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 12 & -\frac{35}{4} \\ -\frac{35}{4} & 2 \end{pmatrix} \tag{4.1.8.47}$$

$$\mathbf{u} = \begin{pmatrix} \frac{11}{2} \\ -\frac{5}{2} \end{pmatrix} \tag{4.1.8.48}$$

If $\left|\mathbf{V}\right| < 0$ then two lines will intersect.

$$\left|\mathbf{V}\right| = \begin{vmatrix} 12 & -\frac{35}{4} \\ -\frac{35}{4} & 2 \end{vmatrix} \tag{4.1.8.49}$$

$$\implies \left|\mathbf{V}\right| = -\frac{841}{16} \tag{4.1.8.50}$$

$$\implies \left|\mathbf{V}\right| < 0 \tag{4.1.8.51}$$

Therefore the lines will intersect.
Now from (4.1.8.20),

$$\implies \mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} = \begin{pmatrix} 12 \\ -\frac{35}{2} \\ 2 \end{pmatrix} \tag{4.1.8.52}$$

The slopes of the lines are given by roots of equation (4.1.8.23)

$$\implies 2m^2 - \frac{35}{2}m + 12 = 0 \tag{4.1.8.53}$$

$$m_i = \frac{-b \pm \sqrt{-\left|\mathbf{V}\right|}}{c} \tag{4.1.8.54}$$

$$\implies m_i = \frac{\frac{35}{4} \pm \sqrt{\frac{841}{16}}}{2} \tag{4.1.8.55}$$

$$\implies m_1 = 8 \tag{4.1.8.56}$$

$$m_2 = \frac{3}{4} \tag{4.1.8.57}$$

The normal vector for two lines is given by (4.1.8.29)

$$\implies \mathbf{n_1} = k_1\begin{pmatrix} -8 \\ 1 \end{pmatrix} \tag{4.1.8.58}$$

$$\mathbf{n_2} = k_2\begin{pmatrix} -\frac{3}{4} \\ 1 \end{pmatrix} \tag{4.1.8.59}$$

Substituting (4.1.8.58),(4.1.8.59) in (4.1.8.52). we get

$$k_1k_2 = 2 \tag{4.1.8.60}$$

The possible combinations of (k_1,k_2) are (1,2), (2,1), (-1,-2) and (-2,-1).
lets assume $k_1 = 1$,$k_2 = 2$ we get

$$\implies \mathbf{n_1} = \begin{pmatrix} -8 \\ 1 \end{pmatrix} \tag{4.1.8.61}$$

$$\mathbf{n_2} = \begin{pmatrix} -\frac{3}{2} \\ 2 \end{pmatrix} \tag{4.1.8.62}$$

We verify obtained $\mathbf{n_1}$,$\mathbf{n_2}$ using Toeplitz matrix

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} -8 & 0 \\ 1 & -8 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -\frac{3}{2} \\ 2 \end{pmatrix} = \begin{pmatrix} 12 \\ -\frac{35}{2} \\ 2 \end{pmatrix} \tag{4.1.8.63}$$

$$\implies \mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} 12 \\ -\frac{35}{2} \\ 2 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \tag{4.1.8.64}$$

Therefore the obtained $\mathbf{n_1}$,$\mathbf{n_2}$ are correct.
Substitute (4.1.8.61), (4.1.8.62) in (4.1.8.21) we get

$$c_2\begin{pmatrix} -8 \\ 1 \end{pmatrix} + c_1\begin{pmatrix} -\frac{3}{2} \\ 2 \end{pmatrix} = \begin{pmatrix} -11 \\ -5 \end{pmatrix} \tag{4.1.8.65}$$

Solve using row reduction technique.

$$\implies \begin{pmatrix} -\frac{3}{2} & -8 & -11 \\ 2 & 1 & -5 \end{pmatrix} \tag{4.1.8.66}$$

$$\xleftrightarrow{R_1 \leftarrow 2R_1} \begin{pmatrix} -3 & -16 & -22 \\ 2 & 1 & -5 \end{pmatrix} \tag{4.1.8.67}$$

$$\xleftrightarrow{R_2 \leftarrow 3R_2 + 2R_1} \begin{pmatrix} -3 & -16 & -22 \\ 0 & -29 & -59 \end{pmatrix} \tag{4.1.8.68}$$

$$\xleftrightarrow{R_1 \leftarrow 29R_1 - 16R_2} \begin{pmatrix} -87 & 0 & 306 \\ 0 & -29 & -59 \end{pmatrix} \tag{4.1.8.69}$$

$$\implies \begin{pmatrix} 1 & 0 & -\frac{102}{29} \\ 0 & 1 & \frac{59}{29} \end{pmatrix} \tag{4.1.8.70}$$

$$\implies c_1 = -\frac{102}{29} \tag{4.1.8.71}$$

$$c_2 = \frac{59}{29} \tag{4.1.8.72}$$

Substituting (4.1.8.61),(4.1.8.62),(4.1.8.71),(4.1.8.72) in (4.1.8.17) and (4.1.8.18). we get equation of two straight lines.

$$\begin{pmatrix} -8 & 1 \end{pmatrix}\mathbf{x} = -\frac{102}{29} \tag{4.1.8.73}$$

$$\begin{pmatrix} -\frac{3}{2} & 2 \end{pmatrix}\mathbf{x} = \frac{59}{29} \tag{4.1.8.74}$$

4.1.9. Find the value of k so that the following equation may represent a pair of straight lines

-

$$6x^2 + xy + ky^2 - 11x + 43y - 35 = 0 \quad (4.1.9.1)$$

Solution: The given second degree equation is, Comparing coefficients of (4.1.9.1) we get,

$$\mathbf{V} = \begin{pmatrix} 6 & \frac{1}{2} \\ \frac{1}{2} & k \end{pmatrix} \quad (4.1.9.2)$$

$$\mathbf{u} = \begin{pmatrix} -\frac{11}{2} \\ \frac{43}{2} \end{pmatrix} \quad (4.1.9.3)$$

$$f = -35 \quad (4.1.9.4)$$

The given second degree equation (4.1.9.1) will represent a pair of straight line if,

$$\begin{vmatrix} 6 & \frac{1}{2} & -\frac{11}{2} \\ \frac{1}{2} & k & \frac{43}{2} \\ -\frac{11}{2} & \frac{43}{2} & -35 \end{vmatrix} = 0 \quad (4.1.9.5)$$

Expanding the determinant,

$$k + 12 = 0 \quad (4.1.9.6)$$

$$\implies k = -12 \quad (4.1.9.7)$$

Hence, from (4.1.9.7) we find that for $k = -12$, the given second degree equation (4.1.9.1) represents pair of straight lines. For the appropriate value of k, (4.1.9.1) becomes,

$$6x^2 + xy - 12y^2 - 11x + 43y - 35 = 0 \quad (4.1.9.8)$$

Let the pair of straight lines in vector form is given by

$$\mathbf{n_1}^T\mathbf{x} = c_1 \quad (4.1.9.9)$$

$$\mathbf{n_2}^T\mathbf{x} = c_2 \quad (4.1.9.10)$$

The pair of straight lines is given by,

$$(\mathbf{n_1}^T\mathbf{x} - c_1)(\mathbf{n_2}^T\mathbf{x} - c_2) = \mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^\mathbf{T}\mathbf{x} + f = 0 \quad (4.1.9.11)$$

Putting the values of $\mathbf{V}$ and $\mathbf{u}$ we get,

$$\mathbf{x}^T\begin{pmatrix} 6 & \frac{1}{2} \\ \frac{1}{2} & -12 \end{pmatrix}\mathbf{x} + 2\begin{pmatrix} -\frac{11}{2} & \frac{43}{2} \end{pmatrix}\mathbf{x} - 35 = 0 \quad (4.1.9.12)$$

Hence, from (4.1.9.12) we get,

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} 6 \\ 1 \\ -12 \end{pmatrix} \quad (4.1.9.13)$$

$$c_2\mathbf{n_1} + c_1\mathbf{n_2} = -2\begin{pmatrix} -\frac{11}{2} \\ \frac{43}{2} \end{pmatrix} \quad (4.1.9.14)$$

$$c_1c_2 = -35 \quad (4.1.9.15)$$

The slopes of the pair of straight lines are given by the roots of the polynomial,

$$cm^2 + 2bm + a = 0 \quad (4.1.9.16)$$

$$\implies m_i = \frac{-b \pm \sqrt{-\det(V)}}{c} \quad (4.1.9.17)$$

$$\mathbf{n_i} = k\begin{pmatrix} -m_i \\ 1 \end{pmatrix} \quad (4.1.9.18)$$

Substituting the values in above equations (4.1.9.16) we get,

$$-12m^2 + m + 6 = 0 \quad (4.1.9.19)$$

$$\implies m_i = \frac{-\frac{1}{2} \pm \sqrt{-(-\frac{289}{4})}}{-12} \quad (4.1.9.20)$$

Solving equation (4.1.9.20) we get ,

$$m_1 = -\frac{2}{3} \quad (4.1.9.21)$$

$$m_2 = \frac{3}{4} \quad (4.1.9.22)$$

Hence putting the values of m_1 and m_2 in (4.1.9.18) we get

$$\mathbf{n_1} = k_1\begin{pmatrix} \frac{2}{3} \\ 1 \end{pmatrix} \quad (4.1.9.23)$$

$$\mathbf{n_2} = k_2\begin{pmatrix} -\frac{3}{4} \\ 1 \end{pmatrix} \quad (4.1.9.24)$$

Putting values of $\mathbf{n_1}$ and $\mathbf{n_2}$ in (4.1.9.13) we get,

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} -\frac{3k_2}{4} & 0 \\ k_2 & -\frac{3k_2}{4} \\ 0 & k_2 \end{pmatrix}\begin{pmatrix} \frac{2k_1}{3} \\ k_1 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ -12 \end{pmatrix} \quad (4.1.9.25)$$

$$\implies \begin{pmatrix} -\frac{1}{2}k_1k_2 \\ -\frac{1}{12}k_1k_2 \\ k_1k_2 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ -12 \end{pmatrix} \quad (4.1.9.26)$$

Thus, from (4.1.9.26), $k_1k_2 = -12$. Possible combinations of (k_1, k_2) are (6,-2), (-6,2), (3,-4), (-3,4) Lets assume $k_1 = 3$, $k_2 = -4$, then we get,

$$\mathbf{n_1} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \tag{4.1.9.27}$$

$$\mathbf{n_2} = \begin{pmatrix} 3 \\ -4 \end{pmatrix} \tag{4.1.9.28}$$

From equation (4.1.9.14) we get

$$\begin{pmatrix} \mathbf{n_1} & \mathbf{n_2} \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2\mathbf{u} \tag{4.1.9.29}$$

$$\begin{pmatrix} 2 & 3 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2 \begin{pmatrix} -\frac{11}{2} \\ \frac{43}{2} \end{pmatrix} \tag{4.1.9.30}$$

Hence we get the following equations,

$$2c_2 + 3c_1 = 11 \tag{4.1.9.31}$$

$$3c_2 - 4c_1 = -43 \tag{4.1.9.32}$$

The augmented matrix of (4.1.9.31) ,(4.1.9.32) is,

$$\begin{pmatrix} 2 & 3 & 11 \\ 3 & -4 & -43 \end{pmatrix} \xleftrightarrow{R_1 = \frac{1}{2} R_1} \begin{pmatrix} 1 & \frac{3}{2} & \frac{11}{2} \\ 3 & -4 & -43 \end{pmatrix} \tag{4.1.9.33}$$

$$\xleftrightarrow{R_2 = R_2 - 3R_1} \begin{pmatrix} 1 & \frac{3}{2} & \frac{11}{2} \\ 0 & -\frac{17}{2} & -\frac{119}{2} \end{pmatrix} \tag{4.1.9.34}$$

$$\xleftrightarrow{R_2 = -\frac{2}{17}} \begin{pmatrix} 1 & \frac{3}{2} & \frac{11}{2} \\ 0 & 1 & 7 \end{pmatrix} \tag{4.1.9.35}$$

$$\xleftrightarrow{R_1 = R_1 - \frac{3}{2} R_2} \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 7 \end{pmatrix} \tag{4.1.9.36}$$

$$\tag{4.1.9.37}$$

Hence we get,

$$c_1 = -5 \tag{4.1.9.38}$$

$$c_2 = 7 \tag{4.1.9.39}$$

Hence (4.1.9.9), (4.1.9.10) can be modified as follows,

$$\begin{pmatrix} 2 & 3 \end{pmatrix} \mathbf{x} = -5 \tag{4.1.9.40}$$

$$\begin{pmatrix} 3 & -4 \end{pmatrix} \mathbf{x} = 7 \tag{4.1.9.41}$$

The figure below corresponds to the pair of straight lines represented by (4.1.9.40) and (4.1.9.41).

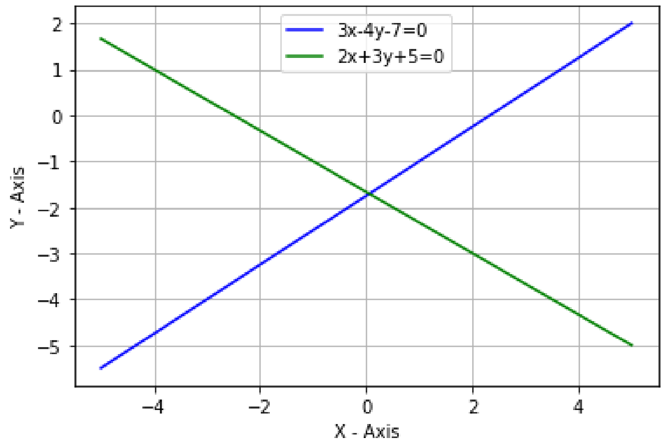

Fig. 4.1.9.1: Pair of Straight Lines

4.1.10. Find the value of k so that following equation may represent pairs of straight lines,

$$kxy - 8x + 9y - 12 = 0 \tag{4.1.10.1}$$

Solution: The general equation of second degree is given by,

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{4.1.10.2}$$

In vector from the equation (4.1.10.2) canb be expressed as,

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \tag{4.1.10.3}$$

where,

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \tag{4.1.10.4}$$

$$\mathbf{u} = \begin{pmatrix} d \\ e \end{pmatrix} \tag{4.1.10.5}$$

Now, comparing equation (4.1.10.2) to (4.1.10.1) we get, a = c = 0, b = $\left(\frac{k}{2}\right)$, d = -4, e = $\left(\frac{9}{2}\right)$, f = -12. Hence, substituting these values in equation (4.1.10.4) and (4.1.10.5) we get,

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 0 & \frac{k}{2} \\ \frac{k}{2} & 0 \end{pmatrix} \tag{4.1.10.6}$$

$$\mathbf{u} = \begin{pmatrix} -4 \\ \frac{9}{2} \end{pmatrix} \tag{4.1.10.7}$$

Now equation (4.1.10.1) represents pair of straight lines if,

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \tag{4.1.10.8}$$

$$\begin{vmatrix} 0 & \frac{k}{2} & -4 \\ \frac{k}{2} & 0 & \frac{9}{2} \\ -4 & \frac{9}{2} & -12 \end{vmatrix} = 0 \tag{4.1.10.9}$$

$$\implies k = 0, k = 6 \tag{4.1.10.10}$$

Substituting (4.1.10.10) in (4.1.10.1) we get,

$$6xy - 8x + 9y - 12 = 0 \tag{4.1.10.11}$$

$$-8x + 9y - 12 = 0 \tag{4.1.10.12}$$

Hence value of k = 6 represents pair of straight lines. Also it can be verified that the pair of lines intersect as,

$$\left|\mathbf{V}\right| = \begin{vmatrix} 0 & 3 \\ 3 & 0 \end{vmatrix} < 0 \tag{4.1.10.13}$$

Let the pair of straight lines is given by,

$$\mathbf{n_1}^T\mathbf{x} = c1 \tag{4.1.10.14}$$

$$\mathbf{n_2}^T\mathbf{x} = c2 \tag{4.1.10.15}$$

Now equating the product of equation (4.1.10.14) and (4.1.10.15) with (4.1.10.3) we get,

$$(\mathbf{n_1}^T\mathbf{x} - c1)(\mathbf{n_2}^T\mathbf{x} - c2) = \tag{4.1.10.16}$$

$$\mathbf{x}^T \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix} \mathbf{x} + 2\begin{pmatrix} -4 & \frac{9}{2} \end{pmatrix} \mathbf{x} - 12 \tag{4.1.10.17}$$

$$\implies n_1 * n_2 = \{0, 6, 0\} \tag{4.1.10.18}$$

$$c_1n_1 + c_2n_2 = \begin{pmatrix} 8 \\ -9 \end{pmatrix} \tag{4.1.10.19}$$

$$c_1c_2 = -12. \tag{4.1.10.20}$$

Now the slopes of line is given by roots of polynomial,

$$cm^2 + 2bm + a = 0 \tag{4.1.10.21}$$

$$\implies 2bm = 0 \tag{4.1.10.22}$$

$$\implies m = 0 \tag{4.1.10.23}$$

Also

$$m_i = \frac{-b \pm \sqrt{-|V|}}{c} \tag{4.1.10.24}$$

$$\implies m_i = \frac{-0 \pm \sqrt{9}}{0} \tag{4.1.10.25}$$

$$\therefore m_1 = 0 \tag{4.1.10.26}$$

$$m_2 = \infty \tag{4.1.10.27}$$

The normal vector to the two lines is given by,

$$n_i = k_i \begin{pmatrix} -m_i \\ 1 \end{pmatrix} \tag{4.1.10.28}$$

$$\implies n_1 = k_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{4.1.10.29}$$

$$n_2 = k_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{4.1.10.30}$$

Also,

$$k_1k_2 = 6 \tag{4.1.10.31}$$

Let $k_1 = 2$ and $k_2 = 3$

$$\implies n_1 = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \tag{4.1.10.32}$$

$$n_2 = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \tag{4.1.10.33}$$

We verify obtained n_1 and n_2 using Toeplitz matrix,

$$n_1 * n_2 = \begin{pmatrix} 0 & 0 \\ 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \\ 0 \end{pmatrix} \tag{4.1.10.34}$$

Hence (4.1.10.18) and (4.1.10.34) are same. Hence verified.
Now substituting it in (4.1.10.19) we get,

$$c_2 \begin{pmatrix} 0 \\ 2 \end{pmatrix} + c_1 \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 8 \\ -9 \end{pmatrix} \tag{4.1.10.35}$$

Solve using Row reduction Technique we get,

$$\implies \begin{pmatrix} 3 & 0 & 8 \\ 0 & 2 & -9 \end{pmatrix} \tag{4.1.10.36}$$

$$\xleftrightarrow{R_1 \leftarrow R_1/3} \begin{pmatrix} 1 & 0 & 8/3 \\ 0 & 2 & -9 \end{pmatrix} \tag{4.1.10.37}$$

$$\xleftrightarrow{R_2 \leftarrow R_2/2} \begin{pmatrix} 1 & 0 & 8/3 \\ 0 & 1 & -9/2 \end{pmatrix} \tag{4.1.10.38}$$

$$\implies c_1 = \frac{8}{3} \tag{4.1.10.39}$$

$$c_2 = \frac{-9}{2} \tag{4.1.10.40}$$

substituting the values of c_1, c_2 and equa-

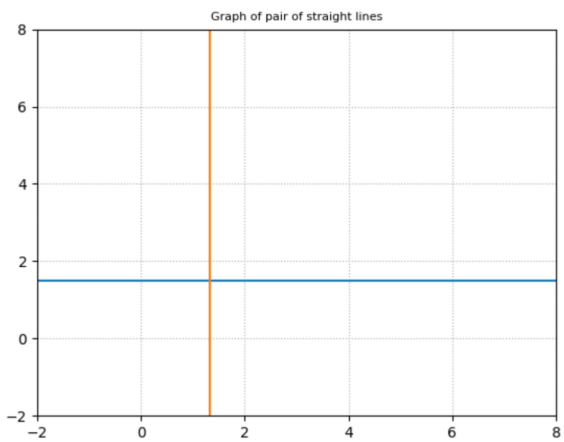

Fig. 4.1.10.1: Intersection of 2 lines

tion (4.1.10.32) and (4.1.10.33) to equation (4.1.10.14) and (4.1.10.15) we get equation of two straight lines.

$$\implies \begin{pmatrix} 0 & 2 \end{pmatrix} \mathbf{x} = \frac{8}{3} \tag{4.1.10.41}$$

$$\begin{pmatrix} 3 & 0 \end{pmatrix} \mathbf{x} = \frac{-9}{2} \tag{4.1.10.42}$$

Hence the equation of pair of straight lines are,

$$\left(\begin{pmatrix} 0 & 2 \end{pmatrix} \mathbf{x} - \frac{8}{3}\right)\left(\begin{pmatrix} 3 & 0 \end{pmatrix} \mathbf{x} - \frac{-9}{2}\right) = 0 \tag{4.1.10.43}$$

Hence, Plot of the equation (4.1.10.43) is shown in Figure.4.1.10.1 Now for value of k = 0 does not represent pair of straight lines.as,

$$\left|\mathbf{V}\right| = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} \not< 0 \tag{4.1.10.44}$$

Hence, Plot of the equation $\begin{pmatrix} -8 & 9 \end{pmatrix} \mathbf{x} = 12$ is shown in figure 4.1.10.2,

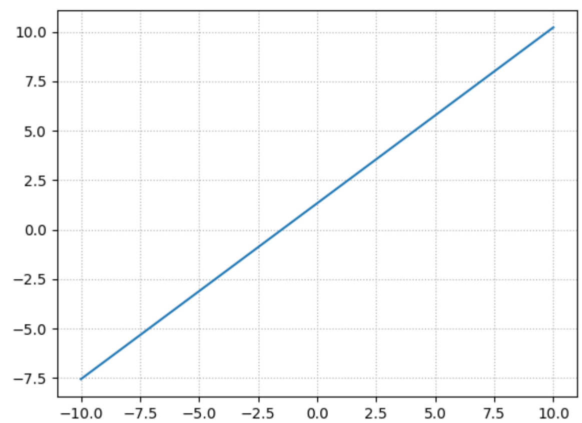

Fig. 4.1.10.2: Intersection of 2 lines

4.1.11. Find the value of k such that

$$x^2 + \frac{10}{3}(xy) + y^2 - 5x - 7y + k = 0 \tag{4.1.11.1}$$

represent pairs of straight lines.
Solution: From (4.1.11.1),

$$\mathbf{V} = \begin{pmatrix} 1 & \frac{5}{3} \\ \frac{5}{3} & 1 \end{pmatrix} \tag{4.1.11.2}$$

$$\mathbf{u}^T = \begin{pmatrix} \frac{-5}{2} & \frac{-7}{2} \end{pmatrix} \tag{4.1.11.3}$$

and

$$\begin{vmatrix} 1 & \frac{5}{3} & \frac{-5}{2} \\ \frac{5}{3} & 1 & \frac{-7}{2} \\ \frac{-5}{2} & \frac{-7}{2} & k \end{vmatrix} = 0 \tag{4.1.11.4}$$

$$\implies \left(k - \left(\frac{49}{4}\right)\right) - \frac{5}{3}\left(\frac{5}{3}k - \frac{35}{4}\right) - \frac{5}{2}\left(\frac{-35}{6} + \frac{5}{2}\right) = 0 \tag{4.1.11.5}$$

$$\implies \frac{64}{k}36 - \frac{128}{12} = 0 \tag{4.1.11.6}$$

$$\implies \boxed{k = 6} \tag{4.1.11.7}$$

Substituting (4.1.11.7) in (4.1.11.1), we get

$$x^2 + \frac{10}{3}(xy) + y^2 - 5x - 7y + 6 = 0 \tag{4.1.11.8}$$

Hence value of k=6 represents pair of straight lines. Substituting value of k =6 in (4.1.11.4)

$$\delta = \begin{vmatrix} 1 & \frac{5}{3} & \frac{-5}{2} \\ \frac{5}{3} & 1 & \frac{-7}{2} \\ \frac{-5}{2} & \frac{-7}{2} & 6 \end{vmatrix} \tag{4.1.11.9}$$

Simplyfying the above determinant , we get

$$\delta = 0 \tag{4.1.11.10}$$

(4.1.11.8) represents two straight lines

$$\det(V) = \begin{vmatrix} 1 & \frac{5}{3} \\ \frac{5}{3} & 1 \end{vmatrix} < 0 \tag{4.1.11.11}$$

Since $\det(V) < 0$ lines would intersect each other

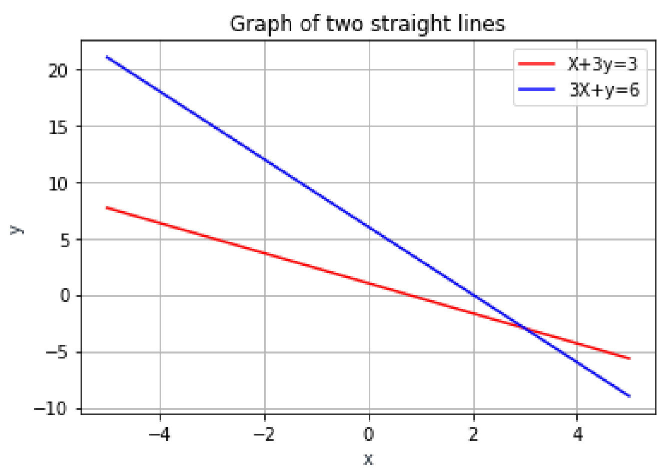

Fig. 4.1.11.1: Pair of straight lines

$$\mathbf{n_1} * \mathbf{n_2} = \{1, \frac{10}{3}, 1\} \tag{4.1.11.12}$$

$$c_2\mathbf{n_1} + c_1\mathbf{n_2} = -2\begin{pmatrix} \frac{-5}{2} \\ \frac{-7}{2} \end{pmatrix} \tag{4.1.11.13}$$

$$c_1c_2 = 6 \tag{4.1.11.14}$$

The slopes of the lines are given by the roots of the polynomial

$$cm^2 + 2bm + a = 0 \tag{4.1.11.15}$$

$$\implies m_i = \frac{-b \pm \sqrt{-\det(V)}}{c} \tag{4.1.11.16}$$

$$\mathbf{n_i} = k\begin{pmatrix} -m_i \\ 1 \end{pmatrix} \tag{4.1.11.17}$$

Substituting in above equations (4.1.11.15) we get,

$$m^2 + \frac{10}{3}m + 1 = 0 \tag{4.1.11.18}$$

$$\implies m_i = \frac{\frac{-10}{3} \pm \sqrt{-(\frac{-16}{9})}}{1} \tag{4.1.11.19}$$

Solving equation (4.1.11.19) we have ,

$$m_1 = \frac{-1}{3} \tag{4.1.11.20}$$

$$m_2 = -3 \tag{4.1.11.21}$$

$$\mathbf{n_1} = k_1\begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} \tag{4.1.11.22}$$

$$\mathbf{n_2} = k_2\begin{pmatrix} 3 \\ 1 \end{pmatrix} \tag{4.1.11.23}$$

Substituting equations (4.1.11.22), (4.1.11.23) in equation (4.1.11.12) we get

$$k_1k_2 = 1 \tag{4.1.11.24}$$

Possible combination of (k_1, k_2) is (1,1) Lets assume $k_1 = 1$, $k_2 = 1$, we get

$$\mathbf{n_1} = \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} \tag{4.1.11.25}$$

$$\mathbf{n_2} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \tag{4.1.11.26}$$

we have:

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \tag{4.1.11.27}$$

Convolution of $\mathbf{n_1}$ and $\mathbf{n_2}$ can be done by converting $\mathbf{n_1}$ into a teoplitz matrix and multiplying with $\mathbf{n_2}$
From equation (4.1.11.25) and (4.1.11.26)

$$\mathbf{n_1} = \begin{pmatrix} \frac{1}{3} & 0 \\ 1 & \frac{1}{3} \\ 0 & 1 \end{pmatrix} \mathbf{n_2} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \tag{4.1.11.28}$$

$$\implies \begin{pmatrix} \frac{1}{3} & 0 \\ 1 & \frac{1}{3} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{10}{3} \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \tag{4.1.11.29}$$

c_1 and c_2 can be obtained as,

$$\begin{pmatrix} \mathbf{n_1} & \mathbf{n_2} \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2\mathbf{u} \tag{4.1.11.30}$$

$$\begin{pmatrix} \mathbf{n_1} & \mathbf{n_2} \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2\begin{pmatrix} \frac{-5}{2} \\ \frac{-7}{2} \end{pmatrix} \tag{4.1.11.31}$$

Substituting (4.1.11.25) and (4.1.11.26) in (4.1.11.31), the augmented matrix is,

$$\begin{pmatrix} \frac{1}{3} & 3 & 5 \\ 1 & 1 & 7 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow 3\times R_1} \begin{pmatrix} 1 & 9 & 15 \\ 1 & 1 & 7 \end{pmatrix} \tag{4.1.11.32}$$

$$\begin{pmatrix} 1 & 9 & 15 \\ 1 & 1 & 7 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 9 & 15 \\ 0 & -8 & -8 \end{pmatrix} \tag{4.1.11.33}$$

$$\begin{pmatrix} 1 & 9 & 15 \\ 0 & -8 & -8 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 \div -8} \begin{pmatrix} 1 & 9 & 15 \\ 0 & 1 & 1 \end{pmatrix} \tag{4.1.11.34}$$

$$\begin{pmatrix} 1 & 9 & 15 \\ 0 & 1 & 1 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 - 9\times R_2} \begin{pmatrix} 1 & 0 & 6 \\ 0 & 1 & 1 \end{pmatrix} \tag{4.1.11.35}$$

From above we get

$$c_1 = 1 \tag{4.1.11.36}$$

$$c_2 = 6 \tag{4.1.11.37}$$

Hence pair of straight lines are

$$\begin{pmatrix} \frac{1}{3} & 1 \end{pmatrix} \mathbf{x} = 1 \tag{4.1.11.38}$$

$$\begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} = 6 \tag{4.1.11.39}$$

4.1.12. Prove that the equation

$$12x^2 + 7xy - 10y^2 + 13x + 45y - 35 = 0 \tag{4.1.12.1}$$

represents two straight lines and find the angle between the lines.
Solution: The above equation can be expressed as

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \tag{4.1.12.2}$$

where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 12 & \frac{7}{2} \\ \frac{7}{2} & -10 \end{pmatrix} \tag{4.1.12.3}$$

$$\mathbf{u} = \begin{pmatrix} \frac{13}{2} \\ \frac{45}{2} \end{pmatrix} \tag{4.1.12.4}$$

$$f = -35 \tag{4.1.12.5}$$

(4.1.12.2) represents a pair of straight lines if

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \tag{4.1.12.6}$$

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = \begin{vmatrix} 12 & \frac{7}{2} & \frac{13}{2} \\ \frac{7}{2} & -10 & \frac{45}{2} \\ \frac{13}{2} & \frac{45}{2} & -35 \end{vmatrix} \tag{4.1.12.7}$$

$$\implies 12 \begin{vmatrix} -10 & \frac{45}{2} \\ \frac{45}{2} & -35 \end{vmatrix} - \frac{7}{2} \begin{vmatrix} \frac{7}{2} & \frac{45}{2} \\ \frac{13}{2} & -35 \end{vmatrix} + \frac{13}{2} \begin{vmatrix} \frac{7}{2} & -10 \\ \frac{13}{2} & \frac{45}{2} \end{vmatrix} = 0 \tag{4.1.12.8}$$

$$\tag{4.1.12.9}$$

The lines intercept if

$$\left|\mathbf{V}\right| < 0 \tag{4.1.12.10}$$

$$\left|\mathbf{V}\right| = -\frac{529}{4} < 0 \tag{4.1.12.11}$$

From (4.1.12.8) and (4.1.12.11) it can be concluded that the given equation represents a pair of intersecting lines. Let the equations of lines be

$$\mathbf{n_1}^T \mathbf{x} = c_1 \tag{4.1.12.12}$$

$$\mathbf{n_2}^T \mathbf{x} = c_2 \tag{4.1.12.13}$$

Since (4.1.12.2) represents a pair of straight lines it must satisfy

$$(\mathbf{n_1}^T \mathbf{x} - c_1)(\mathbf{n_1}^T \mathbf{x} - c_1) = \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \tag{4.1.12.14}$$

where

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} = \begin{pmatrix} 12 \\ 7 \\ -10 \end{pmatrix} \tag{4.1.12.15}$$

$$c_2 \mathbf{n_1} + c_1 \mathbf{n_2} = -2\mathbf{u} \tag{4.1.12.16}$$

$$c_1 c_2 = f \tag{4.1.12.17}$$

Slopes of the lines can be obtained by solving

$$cm^2 + 2bm + a = 0 \tag{4.1.12.18}$$

$$-10m^2 + 7m + 12 = 0 \tag{4.1.12.19}$$

$$\implies m_1 = \frac{-4}{5}, m_2 = \frac{3}{2} \tag{4.1.12.20}$$

The normal vectors can be expressed in terms of corresponding slopes of lines as

$$\mathbf{n} = k\begin{pmatrix}-m\\1\end{pmatrix} \tag{4.1.12.21}$$

$$\implies \mathbf{n_1} = k_1\begin{pmatrix}\frac{4}{5}\\1\end{pmatrix} \tag{4.1.12.22}$$

$$\mathbf{n_2} = k_2\begin{pmatrix}-\frac{3}{2}\\1\end{pmatrix} \tag{4.1.12.23}$$

Substituing (4.1.12.22) and (4.1.12.23) in (4.1.12.15) we get

$$k_1k_2 = -10 \tag{4.1.12.24}$$

Assuming $k_1 = 5$ and $k_2 = -2$

$$\mathbf{n_1} = \begin{pmatrix}4\\5\end{pmatrix}, \mathbf{n_2} = \begin{pmatrix}3\\-2\end{pmatrix} \tag{4.1.12.25}$$

Verification using Toeplitz matrix

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix}4 & 0\\5 & 4\\0 & 5\end{pmatrix}\begin{pmatrix}3\\-2\end{pmatrix} = \begin{pmatrix}12\\7\\-10\end{pmatrix} \tag{4.1.12.26}$$

From (4.1.12.16) we have

$$c_2\begin{pmatrix}4\\5\end{pmatrix} + c_1\begin{pmatrix}3\\-2\end{pmatrix} = \begin{pmatrix}-13\\-45\end{pmatrix} \tag{4.1.12.27}$$

Solving the augmented matrix

$$\begin{pmatrix}4 & 3 & -13\\5 & -2 & -45\end{pmatrix} \xleftrightarrow{R_2\leftarrow 4R_2-5R_1} \begin{pmatrix}4 & 3 & -13\\0 & -23 & -115\end{pmatrix} \tag{4.1.12.28}$$

$$\xleftrightarrow{R_2\leftarrow -\frac{R_2}{23}} \begin{pmatrix}4 & 3 & -13\\0 & 1 & 5\end{pmatrix} \xleftrightarrow{R_1\leftarrow R_1-3R_2} \begin{pmatrix}4 & 0 & -28\\0 & 1 & 5\end{pmatrix} \tag{4.1.12.29}$$

$$\xleftrightarrow{R_1\leftarrow \frac{R_1}{4}} \begin{pmatrix}1 & 0 & -7\\0 & 1 & 5\end{pmatrix} \tag{4.1.12.30}$$

$$\implies c_1 = -7,\ c_2 = 5 \tag{4.1.12.31}$$

Thus the equation of lines are

$$\begin{pmatrix}4 & 5\end{pmatrix}\mathbf{x} = 5 \tag{4.1.12.32}$$

$$\begin{pmatrix}3 & -2\end{pmatrix}\mathbf{x} = -7 \tag{4.1.12.33}$$

The angle between the lines can be expressed interms of normal vectors

$$\mathbf{n_1} = \begin{pmatrix}4\\5\end{pmatrix}, \quad \mathbf{n_2} = \begin{pmatrix}3\\-2\end{pmatrix} \tag{4.1.12.34}$$

as

$$\cos\theta = \frac{\mathbf{n_1}^T\mathbf{n_2}}{\|\mathbf{n_1}\|\,\|\mathbf{n_2}\|} \tag{4.1.12.35}$$

$$\implies \theta = \cos^{-1}(\frac{2}{\sqrt{533}}) = \tan^{-1}(\frac{23}{2}) \tag{4.1.12.36}$$

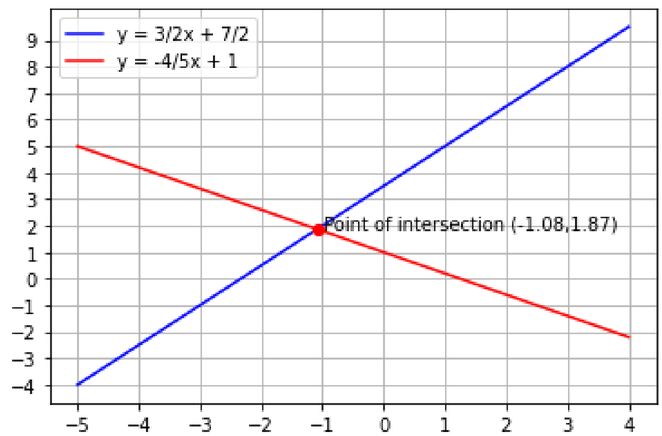

Fig. 4.1.12.1

4.1.13. Find the value of k so that the following equation may represent the pair of staright lines:

$$2x^2 + xy - y^2 + kx + 6y - 9 = 0 \tag{4.1.13.1}$$

Solution: We need to find the value of k for which (4.1.13.1) represents a pair of straight lines.
Converting (4.1.13.1) into vector form, we get

$$\mathbf{x}^T\begin{pmatrix}2 & 1/2\\1/2 & -1\end{pmatrix}\mathbf{x} + 2\begin{pmatrix}k/2\\3\end{pmatrix}\mathbf{x} - 9 = 0 \tag{4.1.13.2}$$

Here, we have

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix}2 & 1/2\\1/2 & -1\end{pmatrix} \tag{4.1.13.3}$$

$$\mathbf{u} = \begin{pmatrix}k/2\\3\end{pmatrix} \tag{4.1.13.4}$$

$$f = -9 \tag{4.1.13.5}$$

The above represents a pair of straight lines if

$$\begin{vmatrix}\mathbf{V} & \mathbf{u}\\\mathbf{u}^T & f\end{vmatrix} = 0 \tag{4.1.13.6}$$

Since (4.1.13.1) represents a pair of straight lines, then by (4.1.13.6), we have

$$\begin{vmatrix} 2 & 1/2 & k/2 \\ 1/2 & -1 & 3 \\ k/2 & 3 & -9 \end{vmatrix} = 0 \tag{4.1.13.7}$$

By solving, above determinant we get

$$2(9-9) + \frac{-1}{2}(\frac{-9}{2} + \frac{-3k}{2}) + \frac{k}{2}(\frac{3}{2} + \frac{k}{2}) = 0 \tag{4.1.13.8}$$

$$\frac{(9+3k)}{4} + \frac{k(3+k)}{4} = 0 \tag{4.1.13.9}$$

$$k^2 + 6k + 9 = 0 \tag{4.1.13.10}$$

$$(k+3)^2 = 0 \tag{4.1.13.11}$$

$$k = -3 \tag{4.1.13.12}$$

Hence by (4.1.13.12), we have

$$2x^2 + xy - y^2 - 3x + 6y - 9 = 0 \tag{4.1.13.13}$$

represents family of straight lines for $k = -3$. To find the staright lines, we write each of thrm in their vector form as

$$\mathbf{n_1}^T\mathbf{x} = c_1 \tag{4.1.13.14}$$

$$\mathbf{n_2}^T\mathbf{x} = c_2 \tag{4.1.13.15}$$

Equating the product of above with (4.1.13.2), we have

$$\left(\mathbf{n_1}^T\mathbf{x} - c_1\right)\left(\mathbf{n_2}^T\mathbf{x} - c_2\right) = \mathbf{x}^T\begin{pmatrix} 2 & 1/2 \\ 1/2 & -1 \end{pmatrix}\mathbf{x} + 2\begin{pmatrix} k/2 \\ 3 \end{pmatrix}\mathbf{x} - 9 \tag{4.1.13.16}$$

$$\implies \mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \tag{4.1.13.17}$$

$$c_2\mathbf{n_1} + c_1\mathbf{n_1} = -2\begin{pmatrix} -3/2 \\ 3 \end{pmatrix} \tag{4.1.13.18}$$

$$c_1c_2 = -9 \tag{4.1.13.19}$$

Here, the slope of these lines are given by the roots of the polynomial

$$-m^2 + m + 2 = 0 \tag{4.1.13.20}$$

$$m^2 - m - 2 = 0 \tag{4.1.13.21}$$

$$m = \frac{1 \pm \sqrt{1+8}}{2} \tag{4.1.13.22}$$

$$m_1 = \frac{1+3}{2} = 2 \tag{4.1.13.23}$$

$$m_2 = \frac{1-3}{2} = -1 \tag{4.1.13.24}$$

$$n_1 = k_1\begin{pmatrix} -2 \\ 1 \end{pmatrix} \tag{4.1.13.25}$$

$$n_2 = k_2\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{4.1.13.26}$$

Substituing (4.1.13.25) and (4.1.13.26) in (4.1.13.17), we get

$$k_1k_2 = -1 \tag{4.1.13.27}$$

Taking $k_1 = -1$ and $k_2 = 1$, we get

$$n_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \tag{4.1.13.28}$$

$$n_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{4.1.13.29}$$

Substituting in (4.1.13.18) for above values ofn_1 and n_2

$$\begin{pmatrix} n_1 n_2 \end{pmatrix}\begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \end{pmatrix} \tag{4.1.13.30}$$

$$\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \end{pmatrix} \tag{4.1.13.31}$$

Solving (4.1.13.31),

$$\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \end{pmatrix} \xleftrightarrow{r_2=r_2+2r_1} \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ -9 \end{pmatrix} \tag{4.1.13.32}$$

$$\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ -9 \end{pmatrix} \xleftrightarrow{r_2=r_2/3} \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \end{pmatrix} \tag{4.1.13.33}$$

$$\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \end{pmatrix} \xLeftrightarrow{r_1 = r_1 - r_2}$$

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \end{pmatrix} \quad (4.1.13.34)$$

Hence, we found out

$$c_1 = -3 \quad (4.1.13.35)$$

$$c_2 = 3 \quad (4.1.13.36)$$

Thus, pair of staright lines are

$$\begin{pmatrix} 2 & -1 \end{pmatrix}\mathbf{x} = -3 \quad (4.1.13.37)$$

$$\begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} = 3 \quad (4.1.13.38)$$

where

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \quad (4.1.13.39)$$

The plot of above is shown below

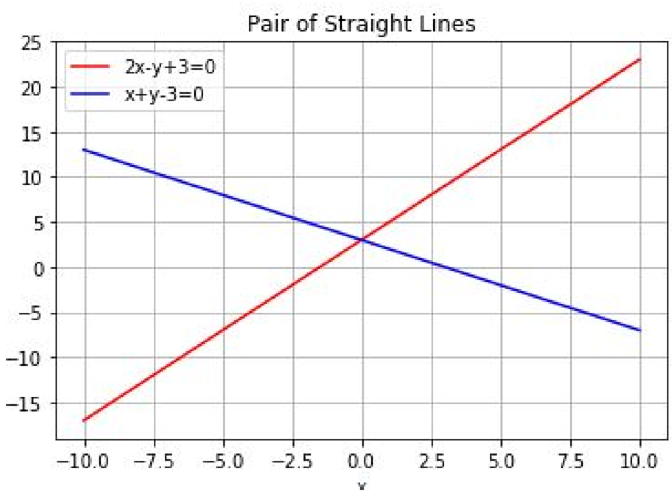

Fig. 4.1.13.1: Pair of Straight Lines

4.1.14. Prove that the equation $12x^2+7xy-10y^2+13x+45y-35=0$ represents two straight lines and find the angle between them.

Solution: The general second order equation is given by ,

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \quad (4.1.14.1)$$

Given,

$$12x^2 + 7xy - 10y^2 + 13x + 45y - 35 = 0 \quad (4.1.14.2)$$

The above equation can be expressed as

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \quad (4.1.14.3)$$

where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 12 & \frac{7}{2} \\ \frac{7}{2} & -10 \end{pmatrix} \quad (4.1.14.4)$$

$$\mathbf{u} = \begin{pmatrix} \frac{13}{2} \\ \frac{45}{2} \end{pmatrix} \quad (4.1.14.5)$$

$$f = -35 \quad (4.1.14.6)$$

(4.1.14.3) represents a pair of straight lines if

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \quad (4.1.14.7)$$

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = \begin{vmatrix} 12 & \frac{7}{2} & \frac{13}{2} \\ \frac{7}{2} & -10 & \frac{45}{2} \\ \frac{13}{2} & \frac{45}{2} & -35 \end{vmatrix} \quad (4.1.14.8)$$

$$\implies 12\begin{vmatrix} -10 & \frac{45}{2} \\ \frac{45}{2} & -35 \end{vmatrix} - \frac{7}{2}\begin{vmatrix} \frac{7}{2} & \frac{45}{2} \\ \frac{13}{2} & -35 \end{vmatrix} + \frac{13}{2}\begin{vmatrix} \frac{7}{2} & -10 \\ \frac{13}{2} & \frac{45}{2} \end{vmatrix} = 0 \quad (4.1.14.9)$$

The lines intercept if

$$\left|\mathbf{V}\right| < 0 \quad (4.1.14.10)$$

$$\left|\mathbf{V}\right| = -\frac{529}{4} < 0 \quad (4.1.14.11)$$

From (4.1.14.9) and (4.1.14.11) it can be concluded that the given equation represents a pair of intersecting lines.
Let (α, β) be their point of intersection, then

$$\begin{pmatrix} 12 & \frac{7}{2} \\ \frac{7}{2} & -10 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \frac{-13}{2} \\ -\frac{45}{2} \end{pmatrix} \quad (4.1.14.12)$$

$$\implies \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad (4.1.14.13)$$

From *Spectral theorem,* $\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T$ (4.1.14.14)

$$\mathbf{V} = \begin{pmatrix} 12 & \frac{7}{2} \\ \frac{7}{2} & -10 \end{pmatrix} \quad (4.1.14.15)$$

$$\mathbf{P} = \begin{pmatrix} \frac{-\sqrt{533}-22}{2} & \frac{-22+\sqrt{533}}{2} \\ 1 & 1 \end{pmatrix} \quad (4.1.14.16)$$

$$\mathbf{D} = \begin{pmatrix} 1 + \frac{\sqrt{533}}{2} & 0 \\ 0 & 1 - \frac{\sqrt{533}}{2} \end{pmatrix} \quad (4.1.14.17)$$

Using *Spectral decomposition* of matrix we can express equation as

$$u_1(x-\alpha)+u_2(y-\beta)=\pm\sqrt{-\frac{\lambda_2}{\lambda_1}}(v_1(x-\alpha)+v_2(y-\beta)) \tag{4.1.14.18}$$

Substituting values in above equation we get;

$$\frac{\sqrt{533}-22}{2}(x+1)+(y-2) = \pm\sqrt{-\frac{1-\frac{\sqrt{533}}{2}}{1+\frac{\sqrt{533}}{2}}}\left(\frac{-22-\sqrt{533}}{2}(x+1)+(y-2)\right) \tag{4.1.14.19}$$

Simplifying (4.1.14.19),

$$3x-2y+7=0 \text{ and } 4x+5y-5=0 \tag{4.1.14.20}$$

$$\implies (3x-2y+7)(4x+5y-5)=0 \tag{4.1.14.21}$$

Thus the equation of lines are

$$\begin{pmatrix}4 & 5\end{pmatrix}\mathbf{x}=5 \tag{4.1.14.22}$$

$$\begin{pmatrix}3 & -2\end{pmatrix}\mathbf{x}=-7 \tag{4.1.14.23}$$

Angle between the straight lines: The angle between the lines can be expressed in terms of normal vectors

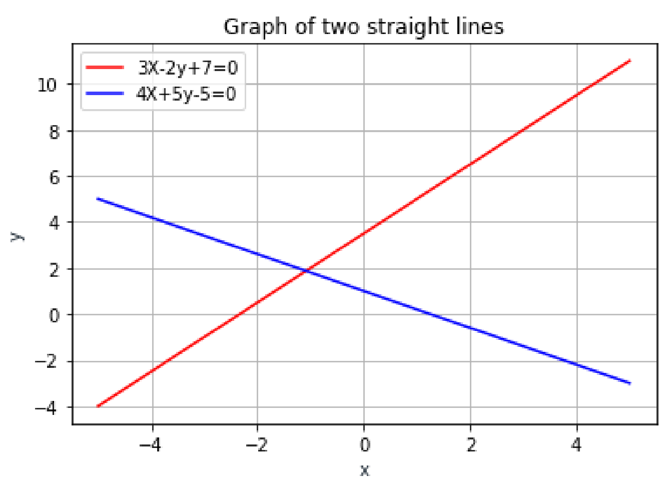

Fig. 1: Pair of straight lines

$$\mathbf{n_1}=\begin{pmatrix}4\\5\end{pmatrix}, \quad \mathbf{n_2}=\begin{pmatrix}3\\-2\end{pmatrix} \tag{4.1.14.24}$$

$$\cos\theta=\frac{\mathbf{n_1}^T\mathbf{n_2}}{\|\mathbf{n_1}\|\,\|\mathbf{n_2}\|} \tag{4.1.14.25}$$

$$\implies \theta=\cos^{-1}\left(\frac{2}{\sqrt{533}}\right)=\tan^{-1}\left(\frac{23}{2}\right) \tag{4.1.14.26}$$

4.1.15. Find the value of h so that the equation

$$6x^2+2hxy+12y^2+22x+31y+20=0 \tag{4.1.15.1}$$

may represent two straight lines.

Solution: The general equation second degree is given by

$$ax^2+2bxy+cy^2+2dx+2ey+f=0 \tag{4.1.15.2}$$

(4.1.15.2) represents pair of straight lines if

$$\begin{vmatrix}a & h & d\\ h & c & e\\ d & e & f\end{vmatrix}=0 \tag{4.1.15.3}$$

From (4.1.15.3), given equation represents pair of straight lines if

$$\begin{vmatrix}6 & h & 11\\ h & 12 & \frac{31}{2}\\ 11 & \frac{31}{2} & 20\end{vmatrix}=0 \tag{4.1.15.4}$$

$$\implies h=\frac{17}{2} \text{ or } h=\frac{171}{20} \tag{4.1.15.5}$$

Verify (4.1.15.5) using python code from

```
https://github.com/shreeprasadbhat/matrix−
    theory/tree/master/assignment5/codes/
    solve_determinant.py
```

The general equation second degree is given by

$$ax^2+2bxy+cy^2+2dx+2ey+f=0 \tag{4.1.15.6}$$

Let (α,β) be their point of intersection, then

$$\begin{pmatrix}a & h\\ h & b\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}=\begin{pmatrix}-d\\ -e\end{pmatrix} \tag{4.1.15.7}$$

Under *Affine transformation*,

$$\mathbf{x} = \mathbf{M}\mathbf{y} + c \tag{4.1.15.8}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{4.1.15.9}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X + \alpha \\ Y + \beta \end{pmatrix} \tag{4.1.15.10}$$

(4.1.15.6) under transformation (4.1.15.10) will become,

$$aX^2 + 2bXY + cY^2 = 0 \tag{4.1.15.11}$$

$$\begin{pmatrix} X & Y \end{pmatrix}\begin{pmatrix} a & h \\ h & b \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = 0 \tag{4.1.15.12}$$

$$\begin{pmatrix} X & Y \end{pmatrix}\begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = 0 \tag{4.1.15.13}$$

$$\begin{pmatrix} X' & Y' \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} X' \\ Y' \end{pmatrix} = 0 \tag{4.1.15.14}$$

where $X' = Xu_1 + Yu_2$ and $Y' = Xv_1 + Yv_2$

$$\implies \lambda_1(X')^2 + \lambda_2(Y')^2 = 0 \tag{4.1.15.15}$$

This is called *Spectral decomposition* of matrix

$$X' = \pm\sqrt{-\frac{\lambda_2}{\lambda_1}}Y' \tag{4.1.15.16}$$

$$u_1X + u_2Y = \pm\sqrt{-\frac{\lambda_2}{\lambda_1}}(v_1X + v_2Y) \tag{4.1.15.17}$$

$$u_1(x-\alpha) + u_2(y-\beta) = \pm\sqrt{-\frac{\lambda_2}{\lambda_1}}(v_1(x-\alpha) + v_2(y-\beta)) \tag{4.1.15.18}$$

Given equation is

$$6x^2 + 17xy + 12y^2 + 22x + 31y + 20 = 0 \tag{4.1.15.19}$$

Substituting in (4.1.15.7)

$$\begin{pmatrix} 6 & \frac{17}{2} \\ \frac{17}{2} & 12 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -11 \\ -\frac{31}{2} \end{pmatrix} \tag{4.1.15.20}$$

$$\implies \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \tag{4.1.15.21}$$

Verify (4.1.15.21) using python code from

```
https://github.com/shreeprasadbhat/matrix−
    theory/tree/master/assignment5/codes/
    find_intersection.py
```

Taking $h = \frac{17}{2}$

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T \tag{4.1.15.22}$$

$$\mathbf{V} = \begin{pmatrix} 6 & \frac{17}{2} \\ \frac{17}{2} & 12 \end{pmatrix} \tag{4.1.15.23}$$

$$\mathbf{P} = \begin{pmatrix} \frac{-5\sqrt{13}-6}{17} & \frac{-6+5\sqrt{13}}{17} \\ 1 & 1 \end{pmatrix} \tag{4.1.15.24}$$

$$\mathbf{D} = \begin{pmatrix} 9 - \frac{5\sqrt{13}}{2} & 0 \\ 0 & 9 + \frac{5\sqrt{13}}{2} \end{pmatrix} \tag{4.1.15.25}$$

Verify (4.1.15.24) and (4.1.15.25) using python code from

```
https://github.com/shreeprasadbhat/matrix−
    theory/tree/master/assignment5/codes/
    diagonalize1.py
```

Substituting (4.1.15.21), (4.1.15.24) and (4.1.15.25) in (4.1.15.18),

$$\frac{-5\sqrt{13}-6}{17}(x+1) + (y-2) = \pm\sqrt{-\frac{9+\frac{5\sqrt{13}}{2}}{9-\frac{5\sqrt{13}}{2}}}\left(\frac{-6+5\sqrt{13}}{17}(x+1) + (y+2)\right) \tag{4.1.15.26}$$

Simplifying (4.1.15.26),

$$2x + 3y + 4 = 0 \text{ and } 3x + 4y + 5 = 0 \tag{4.1.15.27}$$

$$\implies (2x+3y+4)(3x+4y+5) = 0 \tag{4.1.15.28}$$

Verify (4.1.15.27) using python code from

```
https://github.com/shreeprasadbhat/matrix−
    theory/tree/master/assignment5/codes/
    calculate1.py
```

Taking $h = \frac{171}{20}$

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T \tag{4.1.15.29}$$

$$\mathbf{V} = \begin{pmatrix} 6 & \frac{171}{2} \\ \frac{171}{2} & 12 \end{pmatrix} \tag{4.1.15.30}$$

$$\mathbf{P} = \begin{pmatrix} \frac{-\sqrt{3649}-20}{57} & \frac{-20+\sqrt{3649}}{57} \end{pmatrix} \tag{4.1.15.31}$$

$$\mathbf{D} = \begin{pmatrix} 9 - \frac{3\sqrt{3649}}{20} & 0 \\ 0 & 9 + \frac{3\sqrt{3649}}{20} \end{pmatrix} \tag{4.1.15.32}$$

Verify (4.1.15.31) and (4.1.15.32) using python code from

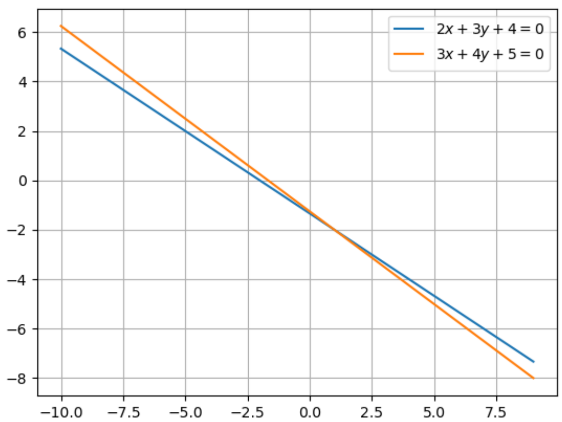

Fig. 1: Pair of straight lines $3x + 4y + 5 = 0$ and $2x + 3y + 4 = 0$

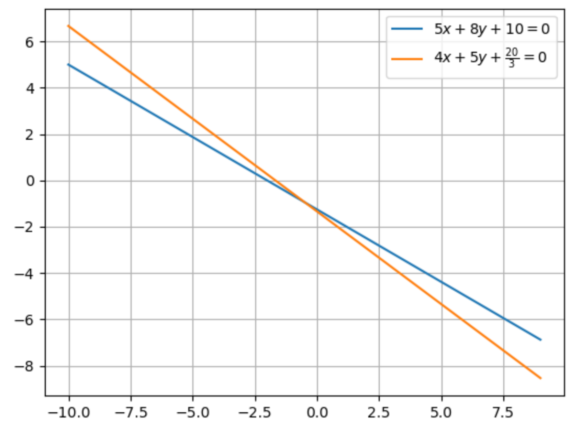

Fig. 1: Pair of straight lines $4x + 5y + \frac{20}{3} = 0$ and $5x + 8y + 10 = 0$

```
https://github.com/shreeprasadbhat/matrix−
    theory/tree/master/assignment5/codes/
    diagonalize2.py
```

Substituting (4.1.15.21),(4.1.15.31) and (4.1.15.32) in (4.1.15.18),

$$\frac{-\sqrt{3649}-20}{57}(x+1)+(y-2) = \pm\sqrt{-\frac{9+\frac{3\sqrt{3649}}{20}}{9-\frac{3\sqrt{3649}}{20}}}\left(\frac{-20+\sqrt{3649}}{57}(x+1)+(y+2)\right) \quad (4.1.15.33)$$

Simplifying (4.1.15.32),

$$2x + 3y + 4 = 0 \text{ and } 3x + 4y + 5 = 0 \quad (4.1.15.34)$$

$$\implies (2x + 3y + 4)(3x + 4y + 5) = 0 \quad (4.1.15.35)$$

Verify (4.1.15.33) using python code from

```
https://github.com/shreeprasadbhat/matrix−
    theory/tree/master/assignment5/codes/
    calculate2.py
```

5 General Equation. Tracing of Curves

5.1 40

5.1.1. What conics do the following equation represent? When possible, find the centres and also their equations referred to the centre

$$12x^2 - 23xy + 10y^2 - 25x + 26y = 14 \quad (5.1.1.1)$$

Solution: The given equation (5.1.1.1) can be expressed as

$$\mathbf{x}^T\begin{pmatrix}12 & \frac{-23}{2}\\ \frac{-23}{2} & 10\end{pmatrix}\mathbf{x} + 2\begin{pmatrix}\frac{-25}{2} & 13\end{pmatrix}\mathbf{x} - 14 = 0 \quad (5.1.1.2)$$

where

$$\mathbf{V} = \begin{pmatrix}12 & \frac{-23}{2}\\ \frac{-23}{2} & 10\end{pmatrix} \quad (5.1.1.3)$$

$$\mathbf{u} = \begin{pmatrix}\frac{-25}{2}\\ 13\end{pmatrix} \quad (5.1.1.4)$$

$$f = -14 \quad (5.1.1.5)$$

$$\det(\mathbf{V}) = \begin{vmatrix}12 & \frac{-23}{2}\\ \frac{-23}{2} & 10\end{vmatrix} \quad (5.1.1.6)$$

$$\implies \det(\mathbf{V}) = \frac{-49}{4} \quad (5.1.1.7)$$

$$\implies \det(\mathbf{V}) < 0 \quad (5.1.1.8)$$

Since $\det(\mathbf{V}) < 0$ the given equation (5.1.1.2) represents the hyperbola The characteristic equation of $\mathbf{V}$ is obtained by evaluating the determinant

$$\left| V - \lambda \mathbf{I} \right| = 0 \tag{5.1.1.9}$$

$$\begin{vmatrix} 12-\lambda & \frac{-23}{2} \\ \frac{-23}{2} & 10-\lambda \end{vmatrix} = 0 \tag{5.1.1.10}$$

$$\implies 4\lambda^2 - 88\lambda - 49 = 0 \tag{5.1.1.11}$$

The eigenvalues are the roots of equation 5.1.1.11 is given by

$$\lambda_1 = \frac{22 + \sqrt{533}}{2} \tag{5.1.1.12}$$

$$\lambda_2 = \frac{22 - \sqrt{533}}{2} \tag{5.1.1.13}$$

The eigenvector $\mathbf{p}$ is defined as

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \tag{5.1.1.14}$$

$$\implies (\mathbf{V} - \lambda\mathbf{I})\mathbf{p} = 0 \tag{5.1.1.15}$$

For $\lambda_1 = \frac{22-\sqrt{533}}{2}$,

$$(\mathbf{V} - \lambda_1\mathbf{I}) = \begin{pmatrix} \frac{\sqrt{553}+2}{2} & \frac{-23}{2} \\ \frac{-23}{2} & \frac{\sqrt{533}-2}{2} \end{pmatrix} \tag{5.1.1.16}$$

By row reduction ,

$$\begin{pmatrix} \frac{\sqrt{533}+2}{2} & \frac{-23}{2} \\ \frac{-23}{2} & \frac{\sqrt{533}-2}{2} \end{pmatrix} \tag{5.1.1.17}$$

$$\xleftrightarrow{R_1 = \frac{R_1}{\frac{\sqrt{533}+2}{2}}} \begin{pmatrix} 1 & \frac{2-\sqrt{533}}{23} \\ \frac{-23}{2} & \frac{\sqrt{533}-2}{2} \end{pmatrix} \tag{5.1.1.18}$$

$$\xleftrightarrow{R_2 = R_2 + \frac{23}{2}R_1} \begin{pmatrix} 1 & \frac{2-\sqrt{533}}{23} \\ 0 & 0 \end{pmatrix} \tag{5.1.1.19}$$

Subsituting equation 5.1.1.19 in equation 5.1.1.15 we get

$$\begin{pmatrix} 1 & \frac{2-\sqrt{533}}{23} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{5.1.1.20}$$

Where, $\mathbf{p} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$
Let $v_2 = t$

$$v_1 = \frac{-t(2 - \sqrt{533})}{23} \tag{5.1.1.21}$$

Eigen vector $\mathbf{p_1}$ is given by

$$\mathbf{p_1} = \begin{pmatrix} \frac{-t(2-\sqrt{533})}{23} \\ t \end{pmatrix} \tag{5.1.1.22}$$

Let $t = 1$, we get

$$\mathbf{p_1} = \begin{pmatrix} \frac{\sqrt{533}-2}{23} \\ 1 \end{pmatrix} \tag{5.1.1.23}$$

For $\lambda_2 = \frac{22+\sqrt{533}}{2}$,

$$(\mathbf{V} - \lambda_2\mathbf{I}) = \begin{pmatrix} \frac{2-\sqrt{553}}{2} & \frac{-23}{2} \\ \frac{-23}{2} & \frac{-2-\sqrt{533}}{2} \end{pmatrix} \tag{5.1.1.24}$$

By row reduction ,

$$\begin{pmatrix} \frac{2-\sqrt{533}}{2} & \frac{-23}{2} \\ \frac{-23}{2} & \frac{-2-\sqrt{533}}{2} \end{pmatrix} \tag{5.1.1.25}$$

$$\xleftrightarrow{R_1 = \frac{R_1}{\frac{2-\sqrt{533}}{2}}} \begin{pmatrix} 1 & \frac{2+\sqrt{533}}{23} \\ \frac{-23}{2} & \frac{-2-\sqrt{533}}{2} \end{pmatrix} \tag{5.1.1.26}$$

$$\xleftrightarrow{R_2 = R_2 + \frac{23}{2}R_1} \begin{pmatrix} 1 & \frac{2-\sqrt{533}}{23} \\ 0 & 0 \end{pmatrix} \tag{5.1.1.27}$$

Subsituting equation 5.1.1.27 in equation 5.1.1.15 we get

$$\begin{pmatrix} 1 & \frac{2+\sqrt{533}}{23} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{5.1.1.28}$$

Where, $\mathbf{p} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$
Let $v_2 = t$

$$v_1 = \frac{-t(2 + \sqrt{533})}{23} \tag{5.1.1.29}$$

Eigen vector $\mathbf{p_2}$ is given by

$$\mathbf{p_2} = \begin{pmatrix} \frac{-t(2+\sqrt{533})}{23} \\ t \end{pmatrix} \tag{5.1.1.30}$$

Let $t = 1$, we get

$$\mathbf{p_2} = \begin{pmatrix} \frac{-\sqrt{533}-2}{23} \\ 1 \end{pmatrix} \tag{5.1.1.31}$$

By eigen decompostion $\mathbf{V}$ can be represented by

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T \tag{5.1.1.32}$$

where

$$\mathbf{P} = \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} \end{pmatrix} \tag{5.1.1.33}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{5.1.1.34}$$

Substituting equations 5.1.1.23, 5.1.1.31 in equation 5.1.1.33 we get

$$\mathbf{P} = \begin{pmatrix} \frac{\sqrt{533}-2}{23} & \frac{-\sqrt{533}-2}{23} \\ 1 & 1 \end{pmatrix} \tag{5.1.1.35}$$

Substituting equations 5.1.1.12, 5.1.1.13 in 5.1.1.34 we get

$$\mathbf{D} = \begin{pmatrix} \frac{22-\sqrt{533}}{2} & 0 \\ 0 & \frac{22+\sqrt{533}}{2} \end{pmatrix} \tag{5.1.1.36}$$

Centre of the hyperbola is given by

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \tag{5.1.1.37}$$

$$\implies \mathbf{c} = -\begin{pmatrix} \frac{-40}{49} & \frac{-46}{49} \\ \frac{-46}{49} & \frac{-48}{49} \end{pmatrix} \begin{pmatrix} \frac{-25}{2} \\ 13 \end{pmatrix} \tag{5.1.1.38}$$

$$\implies \mathbf{c} = \begin{pmatrix} \frac{40}{49} & \frac{46}{49} \\ \frac{46}{49} & \frac{48}{49} \end{pmatrix} \begin{pmatrix} \frac{-25}{2} \\ 13 \end{pmatrix} \tag{5.1.1.39}$$

$$\implies \mathbf{c} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{5.1.1.40}$$

Since,

$$\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f = 26 > 0 \tag{5.1.1.41}$$

there isn't a need to swap axes
In hyperbola,

$$axes = \begin{cases} \sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}-f}{\lambda_1}} \\ \sqrt{\frac{f-\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}}{\lambda_2}} \end{cases} \tag{5.1.1.42}$$

From above equations we can say that,

$$\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}} = \frac{2\sqrt{13}}{\sqrt{22+\sqrt{533}}} \tag{5.1.1.43}$$

$$\sqrt{\frac{f - \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}}{\lambda_2}} = \frac{2\sqrt{13}}{\sqrt{\sqrt{533}-22}} \tag{5.1.1.44}$$

Now (5.1.1.2) can be written as,

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f \tag{5.1.1.45}$$

where ,

$$\mathbf{y} = \mathbf{P}^T(\mathbf{x} - \mathbf{c}) \tag{5.1.1.46}$$

To get $\mathbf{y}$,

$$\mathbf{y} = \mathbf{P}^T\mathbf{x} - \mathbf{P}^T\mathbf{c} \tag{5.1.1.47}$$

$$\mathbf{y} = \begin{pmatrix} \frac{\sqrt{533}-2}{23} & 1 \\ \frac{-\sqrt{533}-2}{23} & 1 \end{pmatrix}\mathbf{x} - \begin{pmatrix} \frac{\sqrt{533}-2}{23} & 1 \\ \frac{-\sqrt{533}-2}{23} & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{5.1.1.48}$$

$$\mathbf{y} = \begin{pmatrix} \frac{\sqrt{533}-2}{23} & 1 \\ \frac{-\sqrt{533}-2}{23} & 1 \end{pmatrix}\mathbf{x} - \begin{pmatrix} \frac{2(\sqrt{533}-2)}{23} + 1 \\ \frac{2(-\sqrt{533}-2)}{23} + 1 \end{pmatrix} \tag{5.1.1.49}$$

Subsituting the eqauations (5.1.1.41), (5.1.1.36) in equation (5.1.1.45)

$$\mathbf{y}^T \begin{pmatrix} \frac{22+\sqrt{533}}{2} & 0 \\ 0 & \frac{22-\sqrt{533}}{2} \end{pmatrix}\mathbf{y} - 26 = 0 \tag{5.1.1.50}$$

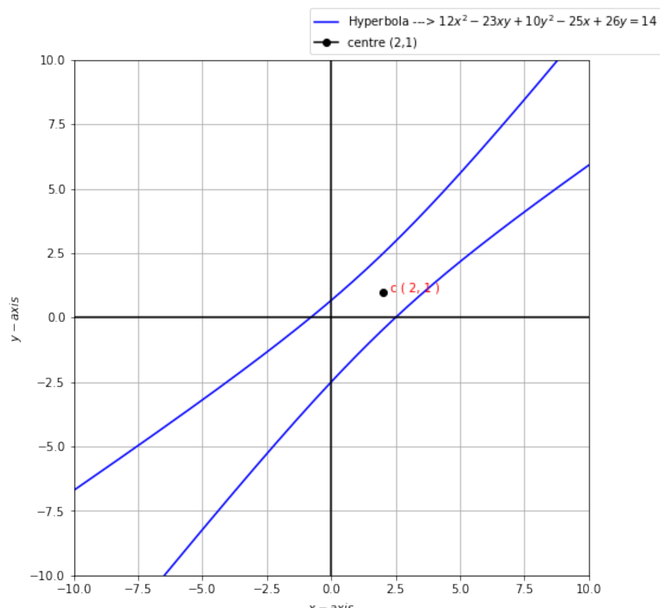

Fig. 5.1.1.1: Hyperbola when origin is shifted

The figure 5.1.1.1 verifies the given equation (5.1.1.2) as hyperbola with centre $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

5.1.2. What conic does the following equation represent.

$$13x^2 - 18xy + 37y^2 + 2x + 14y - 2 = 0 \tag{5.1.2.1}$$

Find the center.

Solution: The general second degree equation can be expressed as follows,

$$\mathbf{x}^{\mathbf{T}}\mathbf{V}\mathbf{x} + 2\mathbf{u}^{\mathbf{T}}\mathbf{x} + f = 0 \tag{5.1.2.2}$$

From the given second degree equation we get,

$$\mathbf{V} = \begin{pmatrix} 13 & -9 \\ -9 & 37 \end{pmatrix} \tag{5.1.2.3}$$

$$\mathbf{u} = \begin{pmatrix} 1 \\ 7 \end{pmatrix} \tag{5.1.2.4}$$

$$f = -2 \tag{5.1.2.5}$$

Expanding the determinant of $\mathbf{V}$ we observe,

$$\begin{vmatrix} 13 & -9 \\ -9 & 37 \end{vmatrix} = 400 > 0 \tag{5.1.2.6}$$

Hence from (5.1.2.6) we conclude that given equation is an ellipse. The characteristic equation of $\mathbf{V}$ is given as follows,

$$\left|\lambda\mathbf{I} - \mathbf{V}\right| = \begin{vmatrix} \lambda - 13 & 9 \\ 9 & \lambda - 37 \end{vmatrix} = 0 \tag{5.1.2.7}$$

$$\implies \lambda^2 - 50\lambda + 400 = 0 \tag{5.1.2.8}$$

Hence the characteristic equation of $\mathbf{V}$ is given by (5.1.2.8). The roots of (5.1.2.8) i.e the eigenvalues are given by

$$\lambda_1 = 10, \lambda_2 = 40 \tag{5.1.2.9}$$

The eigen vector $\mathbf{p}$ is defined as,

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \tag{5.1.2.10}$$

$$\implies (\lambda\mathbf{I} - \mathbf{V})\,\mathbf{p} = 0 \tag{5.1.2.11}$$

for $\lambda_1 = 10$,

$$(\lambda_1\mathbf{I} - \mathbf{V}) = \begin{pmatrix} -3 & 9 \\ 9 & -27 \end{pmatrix} \xleftrightarrow[R_1 = \frac{1}{3}R_1]{R_2 = R_2 + 3R_1} \begin{pmatrix} -1 & 3 \\ 0 & 0 \end{pmatrix} \tag{5.1.2.12}$$

$$\implies \mathbf{p_1} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \tag{5.1.2.13}$$

Again, for $\lambda_2 = 40$,

$$(\lambda_2\mathbf{I} - \mathbf{V}) = \begin{pmatrix} 27 & 9 \\ 9 & 3 \end{pmatrix} \xleftrightarrow[R_1 = \frac{1}{27}R_1]{R_2 = R_2 - R_1} \begin{pmatrix} 1 & \frac{1}{3} \\ 0 & 0 \end{pmatrix} \tag{5.1.2.14}$$

$$\implies \mathbf{p_2} = \begin{pmatrix} -1 \\ 3 \end{pmatrix} \tag{5.1.2.15}$$

Again, Hence from the equation

$$\mathbf{V} = \mathbf{PDP^{-1}P} \quad = \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix} \tag{5.1.2.16}$$

$$\mathbf{D} = \begin{pmatrix} 10 & 0 \\ 0 & 40 \end{pmatrix} \tag{5.1.2.17}$$

Now (5.1.2.2) can be written as,

$$\mathbf{y^T D y} = \mathbf{u^T V^{-1} u} - f \qquad \left|\mathbf{V}\right| \neq 0 \tag{5.1.2.18}$$

And,

$$\mathbf{c} = -\mathbf{V^{-1} u} \qquad \left|\mathbf{V}\right| \neq 0 \tag{5.1.2.19}$$

$$\mathbf{y} = \mathbf{P^T}\,(\mathbf{x} - \mathbf{c}) \tag{5.1.2.20}$$

The centre/vertex of the conic section in (5.1.2.2) is given by $\mathbf{c}$ in (5.1.2.19). We compute $\mathbf{V^{-1}}$ as follows,

$$\begin{pmatrix} 13 & -9 & 1 & 0 \\ -9 & 37 & 0 & 1 \end{pmatrix} \xleftrightarrow[R_2 = \frac{13}{400}R_2]{R_2 = R_2 + \frac{9}{13}R_1} \begin{pmatrix} 13 & -9 & 1 & 0 \\ 0 & 1 & \frac{9}{400} & \frac{13}{400} \end{pmatrix} \tag{5.1.2.21}$$

$$\xleftrightarrow[R_1 = R_1 + \frac{9}{13}R_2]{R_1 = \frac{1}{13}R_1} \begin{pmatrix} 1 & 0 & \frac{37}{400} & \frac{9}{400} \\ 0 & 1 & \frac{9}{400} & \frac{13}{400} \end{pmatrix} \tag{5.1.2.22}$$

Hence $\mathbf{V^{-1}}$ is given by,

$$\mathbf{V^{-1}} = \begin{pmatrix} \frac{37}{400} & \frac{9}{400} \\ \frac{9}{400} & \frac{13}{400} \end{pmatrix} \tag{5.1.2.23}$$

Now $\mathbf{u^T V^{-1} u}$ is given by,

$$\mathbf{u^T V^{-1} u} = \frac{1}{400}\begin{pmatrix} 1 & 7 \end{pmatrix}\begin{pmatrix} 37 & 9 \\ 9 & 13 \end{pmatrix}\begin{pmatrix} 1 \\ 7 \end{pmatrix} = 2 \tag{5.1.2.24}$$

And, $\mathbf{V^{-1} u}$ is given by,

$$\mathbf{V^{-1} u} = \frac{1}{400}\begin{pmatrix} 100 \\ 100 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{5.1.2.25}$$

By putting the value of (5.1.2.25), the center of the ellipse is given by (5.1.2.19) as follows,

$$\mathbf{c} = -\frac{1}{4}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{4} \\ -\frac{1}{4} \end{pmatrix} \tag{5.1.2.26}$$

Also the semi-major axis (a) and semi-minor

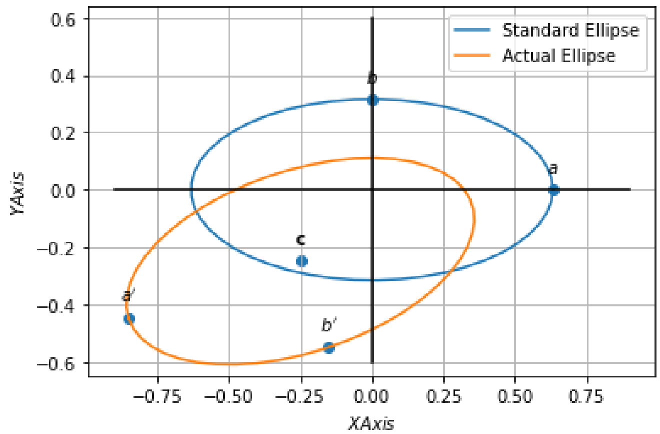

Fig. 5.1.2.1: Graphical representation of the ellipse

axis (b) of the ellipse are given by,

$$a = \sqrt{\frac{\mathbf{u}^{\mathbf{T}}\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}} = \frac{\sqrt{10}}{5} \tag{5.1.2.27}$$

$$b = \sqrt{\frac{\mathbf{u}^{\mathbf{T}}\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_2}} = \frac{\sqrt{10}}{10} \tag{5.1.2.28}$$

Finally from (5.1.2.18), the equation of ellipse is given by,

$$\mathbf{y}^{\mathbf{T}}\begin{pmatrix}10 & 0\\ 0 & 40\end{pmatrix}\mathbf{y} = 4 \tag{5.1.2.29}$$

The following figure 5.1.2.1 is the graphical representation of the ellipse in (5.1.2.29),

5.1.3. What conic does the following equation represent?

$$y^2 - 2\sqrt{3}xy + 3x^2 + 6x - 4y + 5 = 0 \tag{5.1.3.1}$$

Find the center.

Solution: The general second degree equation can be expressed as follows,

$$\mathbf{x}^{\mathbf{T}}\mathbf{V}\mathbf{x} + 2\mathbf{u}^{\mathbf{T}}\mathbf{x} + f = 0 \tag{5.1.3.2}$$

From the given second degree equation we get,

$$\mathbf{V} = \begin{pmatrix}3 & -\sqrt{3}\\ -\sqrt{3} & 1\end{pmatrix} \tag{5.1.3.3}$$

$$\mathbf{u} = \begin{pmatrix}3\\ -2\end{pmatrix} \tag{5.1.3.4}$$

$$f = 5 \tag{5.1.3.5}$$

Expanding the determinant of $\mathbf{V}$ we observe,

$$\begin{vmatrix}3 & -\sqrt{3}\\ -\sqrt{3} & 1\end{vmatrix} = 0 \tag{5.1.3.6}$$

Also

$$\begin{vmatrix}\mathbf{V} & \mathbf{u}\\ \mathbf{u}^T & f\end{vmatrix} = \begin{vmatrix}3 & -\sqrt{3} & 3\\ -\sqrt{3} & 1 & -2\\ 3 & -2 & 5\end{vmatrix} \neq 0 \tag{5.1.3.7}$$

Hence from (5.1.3.6) and (5.1.3.7) we conclude that given equation is a parabola. The characteristic equation of $\mathbf{V}$ is given as follows,

$$\left|\mathbf{V} - \lambda\mathbf{I}\right| = \begin{vmatrix}3-\lambda & -\sqrt{3}\\ -\sqrt{3} & 1-\lambda\end{vmatrix} = 0 \tag{5.1.3.8}$$

$$\implies \lambda^2 - 4\lambda = 0 \tag{5.1.3.9}$$

Hence the characteristic equation of $\mathbf{V}$ is given by (5.1.3.9). The roots of (5.1.3.9) i.e the eigenvalues are given by

$$\lambda_1 = 0, \lambda_2 = 4 \tag{5.1.3.10}$$

The eigen vector $\mathbf{p}$ is defined as,

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \tag{5.1.3.11}$$

$$\implies (\mathbf{V} - \lambda\mathbf{I})\,\mathbf{p} = 0 \tag{5.1.3.12}$$

for $\lambda_1 = 0$,

$$(\mathbf{V} - \lambda_1\mathbf{I}) = \begin{pmatrix}3 & -\sqrt{3}\\ -\sqrt{3} & 1\end{pmatrix} \xleftrightarrow[R_1=\frac{1}{\sqrt{3}}R_1]{R_2=R_1+R_2} \begin{pmatrix}\sqrt{3} & -1\\ 0 & 0\end{pmatrix} \tag{5.1.3.13}$$

Substiuting equation 5.1.3.13 in equation 5.1.3.12 and upon normalizing we get we get

$$\implies \mathbf{p_1} = \begin{pmatrix}1/2\\ \sqrt{3}/2\end{pmatrix} \tag{5.1.3.14}$$

Again, for $\lambda_2 = 4$,

$$(\mathbf{V} - \lambda_2\mathbf{I}) = \begin{pmatrix}-1 & -\sqrt{3}\\ -\sqrt{3} & -3\end{pmatrix} \xleftrightarrow[R_1=-\sqrt{3}R_1]{R_2=-\sqrt{3}R_1+R_2} \begin{pmatrix}1 & \sqrt{3}\\ 0 & 0\end{pmatrix} \tag{5.1.3.15}$$

Substiuting equation 5.1.3.15 in equation 5.1.3.12 and upon normalizing we get

$$\mathbf{p_2} = \begin{pmatrix}-\sqrt{3}/2\\ 1/2\end{pmatrix} \tag{5.1.3.16}$$

The matrix $\mathbf{P}$,

$$\mathbf{P} = \begin{pmatrix}\mathbf{p_1} & \mathbf{p_2}\end{pmatrix} = \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix} \tag{5.1.3.17}$$

$$\mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} \tag{5.1.3.18}$$

$$\eta = 2\mathbf{p_1}^T\mathbf{u} = 3 - 2\sqrt{3} \tag{5.1.3.19}$$

The focal length of the parabola is given by:

$$\left|\frac{\eta}{\lambda_2}\right| = \left|\frac{3-2\sqrt{3}}{4}\right| = 0.116 \tag{5.1.3.20}$$

When $\left|\mathbf{V}\right| = 0$, (5.1.3.2) can be written as

$$\mathbf{y}^\mathbf{T}\mathbf{D}\mathbf{y} = -\eta\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{y} \tag{5.1.3.21}$$

And the vertex $\mathbf{c}$ is given by

$$\begin{pmatrix} \mathbf{u}^\mathbf{T} + \frac{\eta}{2}\mathbf{p}_1^\mathbf{T} \\ \mathbf{V} \end{pmatrix}\mathbf{c} = \begin{pmatrix} -f \\ \frac{\eta}{2}\mathbf{p_1} - \mathbf{u} \end{pmatrix} \tag{5.1.3.22}$$

Substituting the found values

$$\mathbf{u}^T + \frac{\eta}{2}\mathbf{p_1}^T = \begin{pmatrix}3 & -2\end{pmatrix} + \frac{3-2\sqrt{3}}{2}\begin{pmatrix}\frac{1}{2} & \frac{\sqrt{3}}{2}\end{pmatrix} \tag{5.1.3.23}$$

$$\implies \mathbf{u}^T + \frac{\eta}{2}\mathbf{p_1}^T = \begin{pmatrix}\frac{15-2\sqrt{3}}{4} & \frac{-14+3\sqrt{3}}{4}\end{pmatrix} \tag{5.1.3.24}$$

$$\frac{\eta}{2}\mathbf{p_1} - \mathbf{u} = \begin{pmatrix}\frac{-9-2\sqrt{3}}{4} \\ \frac{2+3\sqrt{3}}{4}\end{pmatrix} \tag{5.1.3.25}$$

using equations (5.1.3.4),(5.1.3.5),(5.1.3.14),(5.1.3.24),(5.1.3.25) and (5.1.3.14) in (5.1.3.22)

$$\begin{pmatrix} \frac{15-2\sqrt{3}}{4} & \frac{-14+3\sqrt{3}}{4} \\ 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}\mathbf{c} = \begin{pmatrix} -5 \\ \frac{-9-2\sqrt{3}}{4} \\ \frac{2+3\sqrt{3}}{4} \end{pmatrix} \tag{5.1.3.26}$$

By performing row reductions on augmented matrix

$$\begin{pmatrix} \frac{15-2\sqrt{3}}{4} & \frac{-14+3\sqrt{3}}{4} & -5 \\ 3 & -\sqrt{3} & \frac{(-9-2\sqrt{3})}{4} \\ -\sqrt{3} & 1 & \frac{2+3\sqrt{3}}{4} \end{pmatrix} R_2 \leftrightarrow R_1$$

$$\begin{pmatrix} 3 & -\sqrt{3} & \frac{(-9-2\sqrt{3})}{4} \\ \frac{15-2\sqrt{3}}{4} & \frac{-14+3\sqrt{3}}{4} & -5 \\ -\sqrt{3} & 1 & \frac{2+3\sqrt{3}}{4} \end{pmatrix} \tag{5.1.3.27}$$

$$\begin{pmatrix} 3 & -\sqrt{3} & \frac{(-9-2\sqrt{3})}{4} \\ \frac{15-2\sqrt{3}}{4} & \frac{-14+3\sqrt{3}}{4} & -5 \\ -\sqrt{3} & 1 & \frac{2+3\sqrt{3}}{4} \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - \frac{15-2\sqrt{3}}{12}R_1}$$

$$\begin{pmatrix} 3 & -\sqrt{3} & \frac{(-9-2\sqrt{3})}{4} \\ 0 & 2(\sqrt{3}-2) & \frac{(4\sqrt{3}-39)}{16} \\ \sqrt{3} & 1 & \frac{2+3\sqrt{3}}{4} \end{pmatrix} \tag{5.1.3.28}$$

Therefore,

$$\begin{pmatrix} 3 & -\sqrt{3} & \frac{(-9-2\sqrt{3})}{4} \\ 0 & 2(\sqrt{3}-2) & \frac{(4\sqrt{3}-39)}{16} \\ -\sqrt{3} & 1 & \frac{(2+3\sqrt{3})}{4} \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 + \frac{1}{\sqrt{3}}R_1}$$

$$\begin{pmatrix} 3 & -\frac{433}{250} & -\frac{311}{100} \\ 0 & -\frac{107}{200} & -2 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.1.3.29}$$

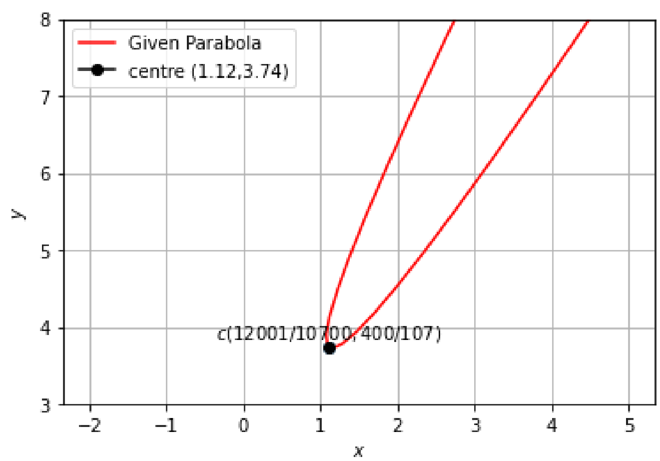

Fig. 5.1.3.1: Parabola with the center c

$$\begin{pmatrix}3 & -\frac{433}{250} & -\frac{311}{100}\\ 0 & -\frac{107}{200} & -2\\ 0 & 0 & 0\end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{R_1}{3}}$$

$$\begin{pmatrix}1 & -\frac{433}{750} & -\frac{311}{300}\\ 0 & -\frac{107}{200} & -2\\ 0 & 0 & 0\end{pmatrix} \tag{5.1.3.30}$$

$$\begin{pmatrix}1 & -\frac{433}{750} & -\frac{311}{300}\\ 0 & -\frac{107}{200} & -2\\ 0 & 0 & 0\end{pmatrix} \xleftrightarrow{R_2 \leftarrow \frac{-200}{107} R_2}$$

$$\begin{pmatrix}1 & -\frac{433}{750} & -\frac{311}{300}\\ 0 & 1 & \frac{400}{107}\\ 0 & 0 & 0\end{pmatrix} \tag{5.1.3.31}$$

$$\begin{pmatrix}1 & -\frac{433}{750} & -\frac{311}{300}\\ 0 & 1 & \frac{400}{107}\\ 0 & 0 & 0\end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 + \frac{433}{750} R_2}$$

$$\begin{pmatrix}1 & 0 & \frac{12001}{10700}\\ 0 & 1 & \frac{400}{107}\\ 0 & 0 & 0\end{pmatrix} \tag{5.1.3.32}$$

On solving for values of $\mathbf{c}$ from (5.1.3.32) The vertex of parabola is $\mathbf{c}=\begin{pmatrix}\frac{12001}{10700}\\ \frac{400}{107}\end{pmatrix}$.

5.1.4. What conics do the following equation represent? When possible, find the centres and also their equations referred to the centre.

$$2x^2 - 72xy + 23y^2 - 4x - 2y - 48 = 0 \tag{5.1.4.1}$$

Solution:

5.1.5. What conic does the given equations represent?

$$6x^2 - 5xy - 6y^2 + 14x + 5y + 4 = 0 \tag{5.1.5.1}$$

Solution: The above equation can be expressed in the form

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{5.1.5.2}$$

Comparing equation we get

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix}6 & \frac{-5}{2}\\ \frac{-5}{2} & -6\end{pmatrix} \tag{5.1.5.3}$$

$$\mathbf{u} = \begin{pmatrix}7\\ \frac{5}{2}\end{pmatrix} \tag{5.1.5.4}$$

$$f = 4 \tag{5.1.5.5}$$

The above equation (5.1.5.2) represents a pair of straight lines if

$$\begin{vmatrix}\mathbf{V} & \mathbf{u}\\ \mathbf{u}^T & f\end{vmatrix} = 0 \tag{5.1.5.6}$$

Verify the given equation as if it is pair of straight lines

$$\Delta = \begin{vmatrix}6 & \frac{-5}{2} & 7\\ \frac{-5}{2} & -6 & \frac{5}{2}\\ 7 & \frac{5}{2} & 4\end{vmatrix} \tag{5.1.5.7}$$

$$\implies 6\begin{vmatrix}-6 & \frac{5}{2}\\ \frac{5}{2} & 4\end{vmatrix} - \frac{-5}{2}\begin{vmatrix}-\frac{5}{2} & \frac{5}{2}\\ 7 & 4\end{vmatrix} + 7\begin{vmatrix}-\frac{5}{2} & -6\\ 7 & \frac{5}{2}\end{vmatrix} = 0 \tag{5.1.5.8}$$

$$\implies \Delta = 0 \tag{5.1.5.9}$$

Since equation (5.1.5.6) is satisfied, we could say that the given equation represents two straight lines

$$\Delta_V = \begin{vmatrix}6 & \frac{-5}{2}\\ \frac{-5}{2} & -6\end{vmatrix} < 0 \tag{5.1.5.10}$$

Let the equations of lines be,

$$\left(\mathbf{n_1}^T\mathbf{x} - c_1\right)\left(\mathbf{n_1}^T\mathbf{x} - c_1\right) = \mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{5.1.5.11}$$

$$\left(\mathbf{n_1}^T\mathbf{x} - c_1\right)\left(\mathbf{n_2}^T\mathbf{x} - c_2\right) = \mathbf{x}^T\begin{pmatrix}6 & \frac{-5}{2}\\ \frac{-5}{2} & -6\end{pmatrix}\mathbf{x} + 2\begin{pmatrix}7 & \frac{5}{2}\end{pmatrix}\mathbf{x} + 4 \tag{5.1.5.12}$$

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix}a\\ 2b\\ c\end{pmatrix} = \begin{pmatrix}6\\ -5\\ -6\end{pmatrix} \tag{5.1.5.13}$$

$$c_2\mathbf{n_1} + c_1\mathbf{n_2} = -2\begin{pmatrix}7\\ \frac{5}{2}\end{pmatrix} \tag{5.1.5.14}$$

$$c_1c_2 = 4 \tag{5.1.5.15}$$

The slopes of the lines are given by the roots of the polynomial

$$cm^2 + 2bm + a = 0 \tag{5.1.5.16}$$

$$\implies m_i = \frac{-b \pm \sqrt{-\Delta_V}}{c} \tag{5.1.5.17}$$

$$\mathbf{n_i} = k\begin{pmatrix}-m_i\\ 1\end{pmatrix} \tag{5.1.5.18}$$

Substituting the given data in above equations

(5.1.5.16) we get,

$$-6m^2 - 5m + 6 = 0 \quad (5.1.5.19)$$

$$\implies m_i = \frac{\frac{-5}{2} \pm \sqrt{-(\frac{-169}{4})}}{-6} \quad (5.1.5.20)$$

Solving equation (5.1.5.20) we get,

$$m_1 = -\frac{3}{2}, m_2 = \frac{2}{3} \quad (5.1.5.21)$$

$$= \mathbf{n_1} = \begin{pmatrix} -3 \\ -2 \end{pmatrix}, \mathbf{n_2} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} \quad (5.1.5.22)$$

We know that,

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (5.1.5.23)$$

Verification using Toeplitz matrix, From equation (5.1.5.22)

$$\mathbf{n_1} = \begin{pmatrix} -3 & 0 \\ -2 & -3 \\ 0 & -2 \end{pmatrix} \mathbf{n_2} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} \quad (5.1.5.24)$$

$$\implies \begin{pmatrix} -3 & 0 \\ -2 & -3 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ -5 \\ -6 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (5.1.5.25)$$

$\implies$ Equation (5.1.5.22) satisfies (5.1.5.23)
c_1 and c_2 can be obtained as,

$$\begin{pmatrix} \mathbf{n_1} & \mathbf{n_2} \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2\mathbf{u} \quad (5.1.5.26)$$

Substituting (5.1.5.22) in (5.1.5.26), the augmented matrix is,

$$\begin{pmatrix} -3 & -2 & 14 \\ -2 & 3 & 5 \end{pmatrix} \xleftrightarrow[R_2 \leftarrow R_2 + 2R_1]{R_1 \leftarrow -R_1/3} \begin{pmatrix} 1 & \frac{2}{3} & -\frac{14}{3} \\ 0 & \frac{13}{3} & -\frac{13}{3} \end{pmatrix} \quad (5.1.5.27)$$

$$\xleftrightarrow[R_1 \leftarrow R_1 - \frac{2}{3}R_2]{R_2 \leftarrow \frac{3}{13}R_2} \begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & -1 \end{pmatrix} \quad (5.1.5.28)$$

$$\implies c_1 = -4, c_2 = -1 \quad (5.1.5.29)$$

Equations (5.1.5.11), can be modified as,from (5.1.5.22) and (5.1.5.29) in we get,

$$\begin{pmatrix} -3 & -2 \end{pmatrix} \mathbf{x} = -4 \quad (5.1.5.30)$$

$$\begin{pmatrix} -2 & 3 \end{pmatrix} \mathbf{x} = -1 \quad (5.1.5.31)$$

$$\implies (-3x - 2y + 4)(-2x + 3y + 1) = 0$$

$$\implies \boxed{(3x + 2y - 4)(2x - 3y - 1) = 0} \quad (5.1.5.32)$$

The angle between the lines can be expressed as,

$$\mathbf{n_1} = \begin{pmatrix} -3 \\ -2 \end{pmatrix}, \quad \mathbf{n_2} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} \quad (5.1.5.33)$$

$$\cos\theta = \frac{\mathbf{n_1}^T \mathbf{n_2}}{\|\mathbf{n_1}\| \|\mathbf{n_2}\|} \quad (5.1.5.34)$$

$$\implies \quad \theta = \cos^{-1}\left(\frac{0}{\sqrt{169}}\right) = 90°. \quad (5.1.5.35)$$

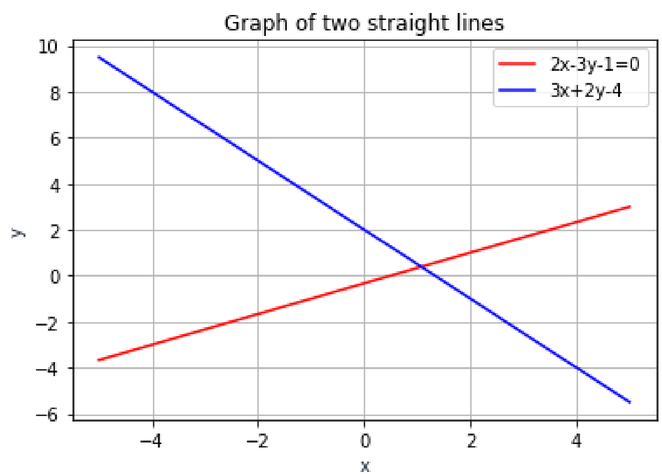

Fig. 5.1.5.1: Pair of straight lines

5.1.6. What conic does the following equation represent? Find its equation and centre.

$$3x^2 - 8xy - 3y^2 + 10x - 13y + 8 = 0$$

Solution: The general equation of second degree can be expressed as

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (5.1.6.1)$$

where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad (5.1.6.2)$$

$$\mathbf{u}^T = \begin{pmatrix} d & e \end{pmatrix} \quad (5.1.6.3)$$

From (5.1.6.2) and (5.1.6.3)

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 3 & -4 \\ -4 & -3 \end{pmatrix} \tag{5.1.6.4}$$

$$\mathbf{u} = \begin{pmatrix} 5 \\ -\frac{13}{2} \end{pmatrix} \tag{5.1.6.5}$$

$$\left|\mathbf{V}\right| = \begin{vmatrix} 3 & -4 \\ -4 & 3 \end{vmatrix} = -25 \tag{5.1.6.6}$$

$$\implies \left|\mathbf{V}\right| < 0 \tag{5.1.6.7}$$

Since $\mathbf{V} = \mathbf{V}^T$, there exists an orthogonal matrix $\mathbf{P}$ such that

$$\mathbf{PVP}^T = \mathbf{D} = diag\begin{pmatrix} \lambda_1 & \lambda_2 \end{pmatrix} \tag{5.1.6.8}$$

or equivalently

$$\mathbf{V} = \mathbf{PDP}^T \tag{5.1.6.9}$$

Eigen vectors of real symmetric matrix $\mathbf{V}$ are orthogonal. The characteristic equation of $\mathbf{V}$ is obtained by evaluating the determinant

$$\left|\lambda\mathbf{I} - \mathbf{V}\right| = \begin{vmatrix} \lambda - 3 & 4 \\ 4 & \lambda + 3 \end{vmatrix} = 0 \tag{5.1.6.10}$$

$$\implies \lambda^2 - 25 = 0 \tag{5.1.6.11}$$

$$\implies \lambda_1 = -5, \lambda_2 = 5 \tag{5.1.6.12}$$

From (5.1.6.7) and (5.1.6.12) the equation represents a hyperbola. The eigen vector $\mathbf{p}$ is defined as

$$\mathbf{Vp} = \lambda\mathbf{p} \tag{5.1.6.13}$$

$$\implies (\lambda\mathbf{I} - \mathbf{V})\mathbf{p} = 0 \tag{5.1.6.14}$$

For $\lambda_1 = -5$:

$$(\lambda_1\mathbf{I} - \mathbf{V}) = \begin{pmatrix} -8 & 4 \\ 4 & -2 \end{pmatrix} \xleftrightarrow[R_2 \leftarrow \frac{R_2}{2}]{R_1 \leftarrow -\frac{R_1}{4}} \begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix} \tag{5.1.6.15}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix} \tag{5.1.6.16}$$

$$\implies \mathbf{p_1} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{5.1.6.17}$$

Similarly, the eigenvector corresponding to λ_2 can be obtained as

$$\mathbf{p_2} = \frac{1}{\sqrt{5}}\begin{pmatrix} -1 \\ 2 \end{pmatrix} \tag{5.1.6.18}$$

The orthogonal eigen-vector matrix

$$\mathbf{P} = \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \tag{5.1.6.19}$$

$$\mathbf{D} = \begin{pmatrix} -5 & 0 \\ 0 & 5 \end{pmatrix} \tag{5.1.6.20}$$

Let $\mathbf{x} = \mathbf{Py} + \mathbf{c}$ with $\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u}$. Substituting in (5.1.6.1)

$$\mathbf{y}^T\mathbf{Dy} = \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f \tag{5.1.6.21}$$

with centre

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} = \begin{pmatrix} -\frac{41}{25} \\ \frac{1}{50} \end{pmatrix} \tag{5.1.6.22}$$

and minor and major axes parameters as

$$\sqrt{\frac{\lambda_1}{f - \mathbf{u^T V^{-1} u}}} = \sqrt{\frac{500}{33}}, \quad \sqrt{\frac{\lambda_2}{\mathbf{u^T V^{-1} u} - f}} = \sqrt{\frac{500}{33}} \tag{5.1.6.23}$$

The equation of hyperbola is

$$\frac{y_2^2}{\frac{33}{500}} - \frac{y_1^2}{\frac{33}{500}} = 1 \tag{5.1.6.24}$$

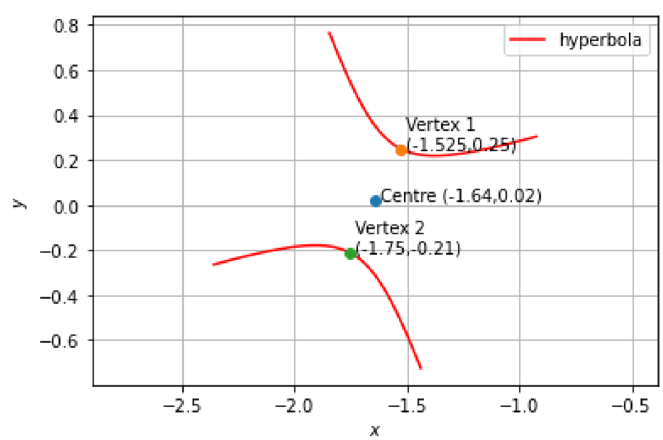

Fig. 5.1.6.1

5.1.7. Find the asymptotes of the hyperbola given below and also the equations to their conjugate hyperbolas.
$8x^2 + 10xy - 3y^2 - 2x + 4y - 2 = 0$ **Solution:** The

above equation can be expressed in the form

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{5.1.7.1}$$

Comparing equation we get

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 8 & 5 \\ 5 & -3 \end{pmatrix} \tag{5.1.7.2}$$

$$\mathbf{u} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \tag{5.1.7.3}$$

$$f = -2 \tag{5.1.7.4}$$

Expanding the Determinant of $\mathbf{V}$.

$$\Delta_V = \begin{vmatrix} 8 & 5 \\ 5 & -3 \end{vmatrix} < 0 \tag{5.1.7.5}$$

Hence from (5.1.7.5) given equation represents the hyperbola The characteristic equation of $\mathbf{V}$ is obtained by evaluating the determinant

$$\left| V - \lambda\mathbf{I} \right| = 0 \tag{5.1.7.6}$$

$$\begin{vmatrix} 8-\lambda & 5 \\ 5 & -3-\lambda \end{vmatrix} = 0 \tag{5.1.7.7}$$

$$(8-\lambda)(-3-\lambda) - 25 = 0 \tag{5.1.7.8}$$

$$\lambda_1 = \frac{5+\sqrt{221}}{2} \tag{5.1.7.9}$$

$$\lambda_2 = \frac{5-\sqrt{221}}{2} \tag{5.1.7.10}$$

The eigenvector $\mathbf{p}$ is defined as

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \tag{5.1.7.11}$$

$$\implies (\mathbf{V} - \lambda\mathbf{I})\mathbf{p} = 0 \tag{5.1.7.12}$$

For $\lambda_1 = \frac{5+\sqrt{221}}{2}$,

$$(\mathbf{V} - \lambda_1\mathbf{I}) = \begin{pmatrix} \frac{11-\sqrt{221}}{2} & 5 \\ 5 & \frac{-11-\sqrt{221}}{2} \end{pmatrix} \tag{5.1.7.13}$$

By row reduction ,

$$\begin{pmatrix} \frac{11-\sqrt{221}}{2} & 5 \\ 5 & \frac{-11-\sqrt{221}}{2} \end{pmatrix} \tag{5.1.7.14}$$

$$\xleftrightarrow{R_1 \leftarrow R_2} \begin{pmatrix} \frac{-11-\sqrt{221}}{2} & 5 \\ \frac{11-\sqrt{221}}{2} & 5 \end{pmatrix} \tag{5.1.7.15}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - \frac{11-\sqrt{221}}{10} R_1} \begin{pmatrix} 5 & \frac{-11-\sqrt{221}}{2} \\ 0 & 0 \end{pmatrix} \tag{5.1.7.16}$$

$$\xleftrightarrow{R_1 \leftarrow R_1/5} \begin{pmatrix} 1 & \frac{-11-\sqrt{221}}{10} \\ 0 & 0 \end{pmatrix} \tag{5.1.7.17}$$

Subsituting equation 5.1.7.17 in equation 5.1.7.12 we get

$$\begin{pmatrix} 1 & \frac{-11-\sqrt{221}}{10} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{5.1.7.18}$$

Where, $\mathbf{p} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ Let $v_2 = t$

$$v_1 = \frac{t(11+\sqrt{221})}{10} \tag{5.1.7.19}$$

Eigen vector $\mathbf{p_1}$ is given by

$$\mathbf{p_1} = \begin{pmatrix} \frac{t(11+\sqrt{221})}{10} \\ t \end{pmatrix} \tag{5.1.7.20}$$

Let $t = 1$, we get

$$\mathbf{p_1} = \begin{pmatrix} \frac{11+\sqrt{221}}{10} \\ 1 \end{pmatrix} \tag{5.1.7.21}$$

For $\lambda_2 = \frac{5-\sqrt{221}}{2}$,

$$(\mathbf{V} - \lambda_2\mathbf{I}) = \begin{pmatrix} \frac{11+\sqrt{221}}{2} & 5 \\ 5 & \frac{-11+\sqrt{221}}{2} \end{pmatrix} \tag{5.1.7.22}$$

By row reduction ,

$$\begin{pmatrix} \frac{11+\sqrt{221}}{2} & 5 \\ 5 & \frac{-11+\sqrt{221}}{2} \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_2 + \frac{11-\sqrt{221}}{10} R_1} \begin{pmatrix} \frac{11+\sqrt{221}}{2} & 5 \\ 0 & 0 \end{pmatrix} \tag{5.1.7.23}$$

$$\xleftrightarrow{R_1 \leftarrow \frac{R_1}{\frac{11+\sqrt{221}}{10}}} \begin{pmatrix} 1 & \frac{10}{11+\sqrt{221}} \\ 0 & 0 \end{pmatrix} \tag{5.1.7.24}$$

Substiuting equation 5.1.7.24 in equation 5.1.7.12 we get

$$\begin{pmatrix} 1 & \frac{10}{11+\sqrt{221}} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{5.1.7.25}$$

Where, $\mathbf{p} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ Let $v_2 = t$

$$v_1 = \frac{-t(10)}{11+\sqrt{221}} \tag{5.1.7.26}$$

Eigen vector $\mathbf{p_2}$ is given by

$$\mathbf{p_2} = \begin{pmatrix} \frac{-t(10)}{11+\sqrt{221}} \\ t \end{pmatrix} \tag{5.1.7.27}$$

Let $t = 1$, we get

$$\mathbf{p_2} = \begin{pmatrix} \frac{(-10)}{11+\sqrt{221}} \\ 1 \end{pmatrix} \quad (5.1.7.28)$$

By eigen decompostion $\mathbf{V}$ can be represented by

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T \quad (5.1.7.29)$$

where

$$\mathbf{P} = \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} \end{pmatrix} \quad (5.1.7.30)$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad (5.1.7.31)$$

Substituting equations 5.1.7.21, 5.1.7.28 in equation 5.1.7.30 we get

$$\mathbf{P} = \begin{pmatrix} \frac{11+\sqrt{221}}{10} & \frac{-10}{11+\sqrt{221}} \\ 1 & 1 \end{pmatrix} \quad (5.1.7.32)$$

Substituting equations 5.1.7.9, 5.1.7.10 in 5.1.7.31 we get

$$\mathbf{D} = \begin{pmatrix} \frac{5+\sqrt{221}}{2} & 0 \\ 0 & \frac{5-\sqrt{221}}{2} \end{pmatrix} \quad (5.1.7.33)$$

Centre of the hyperbola is given by

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \quad (5.1.7.34)$$

$$\implies \mathbf{c} = -\begin{pmatrix} \frac{3}{49} & \frac{5}{49} \\ \frac{5}{49} & \frac{-8}{49} \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad (5.1.7.35)$$

$$\implies \mathbf{c} = \begin{pmatrix} \frac{-3}{49} & \frac{-5}{49} \\ \frac{-5}{49} & \frac{8}{49} \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad (5.1.7.36)$$

$$\implies \mathbf{c} = \begin{pmatrix} \frac{-1}{7} \\ \frac{3}{7} \end{pmatrix} \quad (5.1.7.37)$$

Since,

$$\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f = 1 > 0 \quad (5.1.7.38)$$

there isn't a need to swap axes In hyperbola,

$$axes = \begin{cases} \sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}-f}{\lambda_1}} \\ \sqrt{\frac{f-\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}}{\lambda_2}} \end{cases} \quad (5.1.7.39)$$

From above equations we can say that,

$$\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}} = \sqrt{\frac{2}{5+\sqrt{221}}} \quad (5.1.7.40)$$

$$\sqrt{\frac{f - \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}}{\lambda_2}} = \sqrt{\frac{2}{5-\sqrt{221}}} \quad (5.1.7.41)$$

Now we have,

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f \quad (5.1.7.42)$$

where ,

$$\mathbf{y} = \mathbf{P}^T(\mathbf{x} - \mathbf{c}) \quad (5.1.7.43)$$

To get $\mathbf{y}$,

$$\mathbf{y} = \mathbf{P}^T\mathbf{x} - \mathbf{P}^T\mathbf{c} \quad (5.1.7.44)$$

$$\mathbf{y} = \begin{pmatrix} \frac{11+\sqrt{221}}{10} & 1 \\ \frac{-10}{11+\sqrt{221}} & 1 \end{pmatrix}\mathbf{x} - \begin{pmatrix} \frac{11+\sqrt{221}}{10} & 1 \\ \frac{-10}{11+\sqrt{221}} & 1 \end{pmatrix}\begin{pmatrix} \frac{-1}{7} \\ \frac{3}{7} \end{pmatrix} \quad (5.1.7.45)$$

$$\mathbf{y} = \begin{pmatrix} \frac{11+\sqrt{221}}{10} & 1 \\ \frac{-10}{11+\sqrt{221}} & 1 \end{pmatrix}\mathbf{x} - \begin{pmatrix} \frac{-11-\sqrt{221}}{70} + \frac{3}{7} \\ \frac{10}{(7)11+(7)\sqrt{221}} + \frac{3}{7} \end{pmatrix} \quad (5.1.7.46)$$

Subsituting the eqauations (5.1.7.38), (5.1.7.33) in equation (5.1.7.42)

$$\implies \mathbf{y}^T\begin{pmatrix} \frac{5+\sqrt{221}}{2} & 0 \\ 0 & \frac{5-\sqrt{221}}{2} \end{pmatrix}\mathbf{y} + 2 = 0 \quad (5.1.7.47)$$

Asymptotes of hyperbola Equation of a hyperbola and the combined equation of the Asymptotes differ only in the constant term.

$$8x^2 + 10xy - 3y^2 - 2x + 4y + K = 0 \quad (5.1.7.48)$$

The above equation can be expressed in the form

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \quad (5.1.7.49)$$

Comparing equation we get

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 8 & 5 \\ 5 & -3 \end{pmatrix} \quad (5.1.7.50)$$

$$\mathbf{u} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad (5.1.7.51)$$

$$f = K \quad (5.1.7.52)$$

$$\Delta = \begin{vmatrix} 8 & 5 & -1 \\ 5 & -3 & 2 \\ -1 & 2 & K \end{vmatrix} \tag{5.1.7.53}$$

$$\implies K = -1 \tag{5.1.7.54}$$

Similar way expanding the Determinant of $\mathbf{V}$.

$$\Delta_V = \begin{vmatrix} 8 & 5 \\ 5 & -3 \end{vmatrix} < 0 \tag{5.1.7.55}$$

From (5.1.7.55) we could say that the given equation represents two straight lines Let the equations of lines be,

$$\left(\mathbf{n_1}^T\mathbf{x} - c_1\right)\left(\mathbf{n_1}^T\mathbf{x} - c_1\right) = \mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{5.1.7.56}$$

$$\left(\mathbf{n_1}^T\mathbf{x} - c_1\right)\left(\mathbf{n_2}^T\mathbf{x} - c_2\right) = \mathbf{x}^T\begin{pmatrix} 8 & 5 \\ 5 & -3 \end{pmatrix}\mathbf{x} + 2\begin{pmatrix} -1 & 2 \end{pmatrix}\mathbf{x} - 1 \tag{5.1.7.57}$$

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} = \begin{pmatrix} 8 \\ 10 \\ -3 \end{pmatrix} \tag{5.1.7.58}$$

$$c_2\mathbf{n_1} + c_1\mathbf{n_2} = -2\begin{pmatrix} -1 \\ 2 \end{pmatrix} \tag{5.1.7.59}$$

$$c_1c_2 = -1 \tag{5.1.7.60}$$

The slopes of the lines are given by the roots of the polynomial

$$cm^2 + 2bm + a = 0 \tag{5.1.7.61}$$

$$\implies m_i = \frac{-b \pm \sqrt{-\Delta_V}}{c} \tag{5.1.7.62}$$

$$\mathbf{n_i} = k\begin{pmatrix} -m_i \\ 1 \end{pmatrix} \tag{5.1.7.63}$$

Substituting the given data in above equations (5.1.7.61) we get,

$$-3m^2 + 10m + 8 = 0 \tag{5.1.7.64}$$

$$m_1 = 4, m_2 = \frac{-2}{3} \tag{5.1.7.65}$$

$$= \mathbf{n_1} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}, \mathbf{n_2} = \begin{pmatrix} -2 \\ -3 \end{pmatrix} \tag{5.1.7.66}$$

We know that,

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \tag{5.1.7.67}$$

Verification using Toeplitz matrix, From equation (5.1.7.66)

$$\mathbf{n_1} = \begin{pmatrix} -4 & 0 \\ 1 & -4 \\ 0 & -1 \end{pmatrix}\mathbf{n_2} = \begin{pmatrix} -2 \\ -3 \end{pmatrix} \tag{5.1.7.68}$$

$$\implies \begin{pmatrix} -4 & 0 \\ 1 & -4 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 10 \\ -3 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \tag{5.1.7.69}$$

$\implies$ Equation (5.1.7.66) satisfies (5.1.7.67)

c_1 and c_2 can be obtained as,

$$\begin{pmatrix} \mathbf{n_1} & \mathbf{n_2} \end{pmatrix}\begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2\mathbf{u} \tag{5.1.7.70}$$

Substituting (5.1.7.66) in (5.1.7.70), the augmented matrix is,

$$\begin{pmatrix} -4 & -2 & -2 \\ 1 & -3 & 4 \end{pmatrix} \xleftrightarrow[R_2 \leftarrow R_2 - R_1]{R_1 \leftarrow -R_1/4} \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{7}{2} & \frac{7}{2} \end{pmatrix} \tag{5.1.7.71}$$

$$\xleftrightarrow[R_1 \leftarrow R_1 - \frac{1}{2}R_2]{R_2 \leftarrow -\frac{2}{7}R_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} \tag{5.1.7.72}$$

$$\implies c_1 = 1, c_2 = -1 \tag{5.1.7.73}$$

Equations (5.1.7.56), can be modified as,from (5.1.7.66) and (5.1.7.73) in we get,

$$\begin{pmatrix} -4 & 1 \end{pmatrix}\mathbf{x} = 1 \tag{5.1.7.74}$$

$$\begin{pmatrix} -2 & -3 \end{pmatrix}\mathbf{x} = -1 \tag{5.1.7.75}$$

$$\implies (-4x + y - 1)(-2x - 3y + 1) = 0$$
$$\implies \boxed{(4x - y + 1)(2x + 3y - 1) = 0} \tag{5.1.7.76}$$

The angle between the lines can be expressed as,

$$\mathbf{n_1} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}, \quad \mathbf{n_2} = \begin{pmatrix} -2 \\ -3 \end{pmatrix} \tag{5.1.7.77}$$

$$\cos\theta = \frac{\mathbf{n_1}^T\mathbf{n_2}}{\|\mathbf{n_1}\|\,\|\mathbf{n_2}\|} \tag{5.1.7.78}$$

$$\implies \quad \theta = \cos^{-1}\left(\frac{0}{\sqrt{221}}\right) = 90°. \tag{5.1.7.79}$$

Equation of Asymptotes: The characteristic equation of $\mathbf{V}$ is obtained by evaluating the determinant (5.1.7.50)

$$\left| V - \lambda\mathbf{I} \right| = 0 \qquad (5.1.7.80)$$

$$\begin{vmatrix} 8-\lambda & 5 \\ 5 & -3-\lambda \end{vmatrix} = 0 \qquad (5.1.7.81)$$

$$(8-\lambda)(-3-\lambda) - 25 = 0 \qquad (5.1.7.82)$$

$$\lambda_1 = \frac{5+\sqrt{221}}{2} \qquad (5.1.7.83)$$

$$\lambda_2 = \frac{5-\sqrt{221}}{2} \qquad (5.1.7.84)$$

The eigenvector $\mathbf{p}$ is defined as

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \qquad (5.1.7.85)$$

$$\implies (\mathbf{V} - \lambda\mathbf{I})\mathbf{p} = 0 \qquad (5.1.7.86)$$

For $\lambda_1 = \frac{5+\sqrt{221}}{2}$,

$$(\mathbf{V} - \lambda_1\mathbf{I}) = \begin{pmatrix} \frac{11-\sqrt{221}}{2} & 5 \\ 5 & \frac{-11-\sqrt{221}}{2} \end{pmatrix} \qquad (5.1.7.87)$$

By row reduction ,

$$\begin{pmatrix} \frac{11-\sqrt{221}}{2} & 5 \\ 5 & \frac{-11-\sqrt{221}}{2} \end{pmatrix} \qquad (5.1.7.88)$$

$$\xleftrightarrow{R_1 \leftarrow R_2} \begin{pmatrix} \frac{-11-\sqrt{221}}{2} & 5 \\ \frac{11-\sqrt{221}}{2} & 5 \end{pmatrix} \qquad (5.1.7.89)$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - \frac{11-\sqrt{221}}{10}R_1} \begin{pmatrix} 5 & \frac{-11-\sqrt{221}}{2} \\ 0 & 0 \end{pmatrix} \qquad (5.1.7.90)$$

$$\xleftrightarrow{R_1 \leftarrow R_1/5} \begin{pmatrix} 1 & \frac{-11-\sqrt{221}}{10} \\ 0 & 0 \end{pmatrix} \qquad (5.1.7.91)$$

Subsituting equation 5.1.7.91 in equation 5.1.7.86 we get

$$\begin{pmatrix} 1 & \frac{-11-\sqrt{221}}{10} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (5.1.7.92)$$

Where, $\mathbf{p} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ Let $v_2 = t$

$$v_1 = \frac{t(11+\sqrt{221})}{10} \qquad (5.1.7.93)$$

Eigen vector $\mathbf{p_1}$ is given by

$$\mathbf{p_1} = \begin{pmatrix} \frac{t(11+\sqrt{221})}{10} \\ t \end{pmatrix} \qquad (5.1.7.94)$$

Let $t = 1$, we get

$$\mathbf{p_1} = \begin{pmatrix} \frac{11+\sqrt{221}}{10} \\ 1 \end{pmatrix} \qquad (5.1.7.95)$$

For $\lambda_2 = \frac{5-\sqrt{221}}{2}$,

$$(\mathbf{V} - \lambda_2\mathbf{I}) = \begin{pmatrix} \frac{11+\sqrt{221}}{2} & 5 \\ 5 & \frac{-11+\sqrt{221}}{2} \end{pmatrix} \qquad (5.1.7.96)$$

By row reduction ,

$$\begin{pmatrix} \frac{11+\sqrt{221}}{2} & 5 \\ 5 & \frac{-11+\sqrt{221}}{2} \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_2 + \frac{11-\sqrt{221}}{10}R_1} \begin{pmatrix} \frac{11+\sqrt{221}}{2} & 5 \\ 0 & 0 \end{pmatrix} \qquad (5.1.7.97)$$

$$\xleftrightarrow{R_1 \leftarrow \frac{R_1}{\frac{11+\sqrt{221}}{10}}} \begin{pmatrix} 1 & \frac{10}{11+\sqrt{221}} \\ 0 & 0 \end{pmatrix} \qquad (5.1.7.98)$$

Subsituting equation 5.1.7.98 in equation 5.1.7.86 we get

$$\begin{pmatrix} 1 & \frac{10}{11+\sqrt{221}} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (5.1.7.99)$$

Where, $\mathbf{p} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ Let $v_2 = t$

$$v_1 = \frac{-t(10)}{11+\sqrt{221}} \qquad (5.1.7.100)$$

Eigen vector $\mathbf{p_2}$ is given by

$$\mathbf{p_2} = \begin{pmatrix} \frac{-t(10)}{11+\sqrt{221}} \\ t \end{pmatrix} \qquad (5.1.7.101)$$

Let $t = 1$, we get

$$\mathbf{p_2} = \begin{pmatrix} \frac{(-10)}{11+\sqrt{221}} \\ 1 \end{pmatrix} \qquad (5.1.7.102)$$

By eigen decompostion $\mathbf{V}$ can be represented by

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T \qquad (5.1.7.103)$$

where

$$\mathbf{P} = \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} \end{pmatrix} \qquad (5.1.7.104)$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \qquad (5.1.7.105)$$

Substituting equations 5.1.7.95, 5.1.7.102 in equation 5.1.7.104 we get

$$\mathbf{P} = \begin{pmatrix} \frac{11+\sqrt{221}}{10} & \frac{-10}{11+\sqrt{221}} \\ 1 & 1 \end{pmatrix} \tag{5.1.7.106}$$

$$\mathbf{D} = \begin{pmatrix} \frac{5+\sqrt{221}}{2} & 0 \\ 0 & \frac{5-\sqrt{221}}{2} \end{pmatrix} \tag{5.1.7.107}$$

Centre of the hyperbola is given by

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \tag{5.1.7.108}$$

$$\implies \mathbf{c} = -\begin{pmatrix} \frac{3}{49} & \frac{5}{49} \\ \frac{5}{49} & \frac{-8}{49} \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix} \tag{5.1.7.109}$$

$$\implies \mathbf{c} = \begin{pmatrix} \frac{-3}{49} & \frac{-5}{49} \\ \frac{-5}{49} & \frac{8}{49} \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix} \tag{5.1.7.110}$$

$$\implies \mathbf{c} = \begin{pmatrix} \frac{-1}{7} \\ \frac{3}{7} \end{pmatrix} \tag{5.1.7.111}$$

Since,

$$\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f = 0 \tag{5.1.7.112}$$

Now,

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f \tag{5.1.7.113}$$

where ,

$$\mathbf{y} = \mathbf{P}^T(\mathbf{x} - \mathbf{c}) \tag{5.1.7.114}$$

To get $\mathbf{y}$,

$$\mathbf{y} = \mathbf{P}^T\mathbf{x} - \mathbf{P}^T\mathbf{c} \tag{5.1.7.115}$$

$$\mathbf{y} = \begin{pmatrix} \frac{11+\sqrt{221}}{10} & 1 \\ \frac{-10}{11+\sqrt{221}} & 1 \end{pmatrix}\mathbf{x} - \begin{pmatrix} \frac{11+\sqrt{221}}{10} & 1 \\ \frac{-10}{11+\sqrt{221}} & 1 \end{pmatrix}\begin{pmatrix} \frac{-1}{7} \\ \frac{3}{7} \end{pmatrix} \tag{5.1.7.116}$$

$$\mathbf{y} = \begin{pmatrix} \frac{11+\sqrt{221}}{10} & 1 \\ \frac{-10}{11+\sqrt{221}} & 1 \end{pmatrix}\mathbf{x} - \begin{pmatrix} \frac{-11-\sqrt{221}}{70} + \frac{3}{7} \\ \frac{10}{(7)11+(7)\sqrt{221}} + \frac{3}{7} \end{pmatrix} \tag{5.1.7.117}$$

Subsituting the eqauations (5.1.7.112), (5.1.7.107) in equation (5.1.7.113) Equation of asymptotes is

$$\implies \mathbf{y}^T\begin{pmatrix} \frac{5+\sqrt{221}}{2} & 0 \\ 0 & \frac{5-\sqrt{221}}{2} \end{pmatrix}\mathbf{y} + 1 = 0 \tag{5.1.7.118}$$

And the Equations of Conjugate hyperbola is 2(Equation of Asymptotes)- Equation of hyperbola.

$$\implies \mathbf{y}^T\begin{pmatrix} \frac{5+\sqrt{221}}{2} & 0 \\ 0 & \frac{5-\sqrt{221}}{2} \end{pmatrix}\mathbf{y} = 0 \tag{5.1.7.119}$$

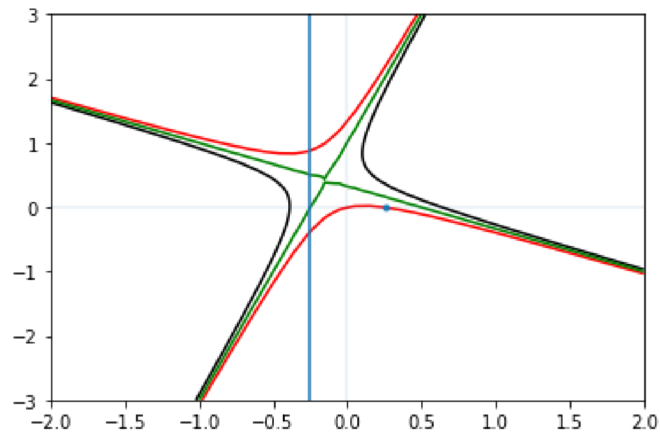

Fig. 5.1.7.1: Hyperbola with assymptotes and its conjugate

5.1.8. What conics do the following equation represents? When possible, find the center and the equation reffered to the center.

$$55x^2 - 120xy + 20y^2 + 64x - 48y = 0 \tag{5.1.8.1}$$

Solution: The general equation of second degree can be represented as:

$$\mathbf{X}^T\mathbf{V}\mathbf{X} + 2\mathbf{u}^T\mathbf{X} + f = 0 \tag{5.1.8.2}$$

The above 5.1.8.1 can also be written as:

$$\mathbf{X}^T\begin{pmatrix} 55 & -60 \\ -60 & 20 \end{pmatrix}\mathbf{X} + 2\begin{pmatrix} 32 & -24 \end{pmatrix}\mathbf{X} + 0 = 0 \tag{5.1.8.3}$$

So,

$$\mathbf{V} = \begin{pmatrix} 55 & -60 \\ -60 & 20 \end{pmatrix} \tag{5.1.8.4}$$

and

$$\mathbf{u} = \begin{pmatrix} 32 \\ -24 \end{pmatrix} \tag{5.1.8.5}$$

$$f = 0 \tag{5.1.8.6}$$

Now,

$$\det \mathbf{V} = \begin{vmatrix} 55 & -60 \\ -60 & 20 \end{vmatrix} \quad (5.1.8.7)$$

$$\implies \det \mathbf{V} = -2500 < 0 \quad (5.1.8.8)$$

As $\det \mathbf{V} < 0$, so we can say that the above conic section 5.1.8.1 is hyperbola.Now,

$$\mathbf{V}^{-1} = \frac{1}{-2500} \begin{pmatrix} 20 & 60 \\ 60 & 55 \end{pmatrix} \quad (5.1.8.9)$$

The center of this hyperbola will be:

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \quad (5.1.8.10)$$

$$\implies \mathbf{c} = \frac{1}{2500} \begin{pmatrix} 20 & 60 \\ 60 & 55 \end{pmatrix} \begin{pmatrix} 32 \\ -24 \end{pmatrix} \quad (5.1.8.11)$$

$$\implies \mathbf{c} = \begin{pmatrix} -\frac{8}{25} \\ \frac{6}{25} \end{pmatrix} \quad (5.1.8.12)$$

$$(5.1.8.13)$$

Now the characteristic equation of $\mathbf{V}$ is obtained as:

$$\left|\mathbf{V} - \lambda \mathbf{I}\right| = 0 \quad (5.1.8.14)$$

$$\implies \begin{vmatrix} 55 - \lambda & -60 \\ -60 & 20 - \lambda \end{vmatrix} = 0 \quad (5.1.8.15)$$

$$\implies \lambda^2 - 75\lambda - 2500 = 0 \quad (5.1.8.16)$$

The eigen values are given by:

$$\lambda_1 = 100 \quad (5.1.8.17)$$

$$\lambda_2 = -25 \quad (5.1.8.18)$$

The eigen vector $\mathbf{P}$ is defined as:

$$\mathbf{VP} = \lambda \mathbf{P} \quad (5.1.8.19)$$

$$\implies (\mathbf{V} - \lambda \mathbf{I})\mathbf{P} = \mathbf{0} \quad (5.1.8.20)$$

For λ_1=100,

$$(\mathbf{V} - \lambda_1 \mathbf{I}) = \begin{pmatrix} -45 & -60 \\ -60 & -80 \end{pmatrix} \quad (5.1.8.21)$$

By row reduction,

$$\begin{pmatrix} -45 & -60 \\ -60 & -80 \end{pmatrix} \xleftrightarrow[R_1 \leftarrow R_1/(-5)]{R_2 \leftarrow R_2/(-5)} \quad (5.1.8.22)$$

$$\begin{pmatrix} 9 & 12 \\ 12 & 16 \end{pmatrix} \xleftrightarrow[R_1 \leftarrow R_1/3]{R_2 \leftarrow R_2/4} \quad (5.1.8.23)$$

$$\begin{pmatrix} 3 & 4 \\ 3 & 4 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 3 & 4 \\ 0 & 0 \end{pmatrix} \quad (5.1.8.24)$$

So,

$$(\mathbf{V} - \lambda_1 \mathbf{I})\mathbf{P_1} = \mathbf{0} \quad (5.1.8.25)$$

$$\implies \begin{pmatrix} 3 & 4 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (5.1.8.26)$$

$$\implies \mathbf{P_1} = \begin{pmatrix} -\frac{4}{3} \\ 1 \end{pmatrix} \quad (5.1.8.27)$$

Similarly, For λ_2=100,

$$(\mathbf{V} - \lambda_2 \mathbf{I}) = \begin{pmatrix} 80 & -60 \\ -60 & 45 \end{pmatrix} \quad (5.1.8.28)$$

By row reduction,

$$\begin{pmatrix} 80 & -60 \\ -60 & 45 \end{pmatrix} \xleftrightarrow[R_1 \leftarrow R_1/5]{R_2 \leftarrow R_2/5} \quad (5.1.8.29)$$

$$\begin{pmatrix} 16 & -12 \\ -12 & 9 \end{pmatrix} \xleftrightarrow[R_1 \leftarrow R_1/4]{R_2 \leftarrow R_2/(-3)} \quad (5.1.8.30)$$

$$\begin{pmatrix} 4 & -3 \\ 4 & -3 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 4 & -3 \\ 0 & 0 \end{pmatrix} \quad (5.1.8.31)$$

So,

$$(\mathbf{V} - \lambda_2 \mathbf{I})\mathbf{P_2} = \mathbf{0} \quad (5.1.8.32)$$

$$\implies \begin{pmatrix} 4 & -3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (5.1.8.33)$$

$$\implies \mathbf{P_2} = \begin{pmatrix} 1 \\ \frac{4}{3} \end{pmatrix} \quad (5.1.8.34)$$

By eigen decomposition $\mathbf{V}$ can also be written as:

$$\mathbf{V} = \mathbf{PDP}^T \quad (5.1.8.35)$$

where

$$\mathbf{P} = \begin{pmatrix} \mathbf{P_1} & \mathbf{P_2} \end{pmatrix} \quad (5.1.8.36)$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad (5.1.8.37)$$

So,

$$\mathbf{P} = \begin{pmatrix} -\frac{4}{3} & 1 \\ 1 & \frac{4}{3} \end{pmatrix} \quad (5.1.8.38)$$

$$\mathbf{D} = \begin{pmatrix} 100 & 0 \\ 0 & -25 \end{pmatrix} \quad (5.1.8.39)$$

and

$$\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f = 16 > 0 \quad (5.1.8.40)$$

So, the axes are:

$$a = \sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}} = \frac{2}{5} \quad (5.1.8.41)$$

$$b = \sqrt{\frac{f - \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}}{\lambda_2}} = \frac{4}{5} \quad (5.1.8.42)$$

Now, the equation 5.1.8.1 can be written as:

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f \quad (5.1.8.43)$$

where,

$$\mathbf{y} = \mathbf{P}^T(\mathbf{x} - \mathbf{c}) \quad (5.1.8.44)$$

So,

$$\mathbf{y}^T\begin{pmatrix}100 & 0\\ 0 & -25\end{pmatrix}\mathbf{y} = 16 \quad (5.1.8.45)$$

$$\implies \mathbf{y}^T\begin{pmatrix}100 & 0\\ 0 & -25\end{pmatrix}\mathbf{y} - 16 = 0 \quad (5.1.8.46)$$

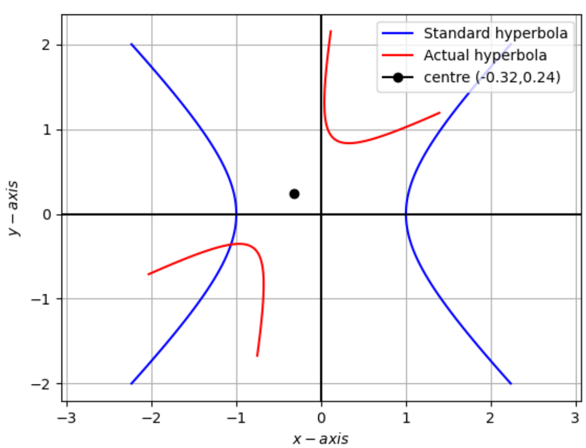

Fig. 5.1.8.1: Comparison of the Standerd and Actual Hyperbola

5.1.9. Find the asymptotes of the given hyperbola and also the equation to its conjugate hyperbola

$$19x^2 + 24xy + y^2 - 22x - 6y = 0 \quad (5.1.9.1)$$

Solution: The general equation of second degree is given by

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \quad (5.1.9.2)$$

and can be expressed as

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \quad (5.1.9.3)$$

where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix}a & b\\ b & c\end{pmatrix} \quad (5.1.9.4)$$

$$\mathbf{u} = \begin{pmatrix}d\\ e\end{pmatrix} \quad (5.1.9.5)$$

Comparing equations 5.1.9.1 and 5.1.9.3 we get

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix}19 & 12\\ 12 & 1\end{pmatrix} \quad (5.1.9.6)$$

$$\mathbf{u} = \begin{pmatrix}-11\\ -3\end{pmatrix} \quad (5.1.9.7)$$

$$f = 0 \quad (5.1.9.8)$$

Expanding the Determinant of $\mathbf{V}$.

$$\Delta_V = \begin{vmatrix}19 & 12\\ 12 & 1\end{vmatrix} < 0 \quad (5.1.9.9)$$

Hence from 5.1.9.9 given equation represents the hyperbola.
The characteristic equation of $\mathbf{V}$ is obtained by evaluating the determinant

$$\left| V - \lambda\mathbf{I} \right| = 0 \quad (5.1.9.10)$$

$$\begin{vmatrix}19-\lambda & 12\\ 12 & 1-\lambda\end{vmatrix} = 0 \quad (5.1.9.11)$$

$$(19-\lambda)(1-\lambda) - 144 = 0 \quad (5.1.9.12)$$

$$\lambda_1 = -5, \lambda_2 = 25 \quad (5.1.9.13)$$

The eigenvector $\mathbf{p}$ is defined as

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \quad (5.1.9.14)$$

$$\implies (\mathbf{V} - \lambda\mathbf{I})\mathbf{p} = 0 \quad (5.1.9.15)$$

For $\lambda_1 = -5$,

$$(\mathbf{V} - \lambda_1\mathbf{I}) = \begin{pmatrix}19+5 & 12\\ 12 & 1+5\end{pmatrix} \quad (5.1.9.16)$$

By row reduction ,

$$\begin{pmatrix}24 & 12\\ 12 & 6\end{pmatrix} \quad (5.1.9.17)$$

$$\xleftrightarrow{R_2\leftarrow 2R_2 - R_1}\begin{pmatrix}24 & 12\\ 0 & 0\end{pmatrix} \quad (5.1.9.18)$$

$$\xleftrightarrow{R_1\leftarrow \frac{R_1}{12}}\begin{pmatrix}2 & 1\\ 0 & 0\end{pmatrix} \quad (5.1.9.19)$$

Subsituting equation 5.1.9.19 in equation

5.1.9.15 we get

$$\begin{pmatrix}2 & 1\\0 & 0\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \tag{5.1.9.20}$$

Where, $\mathbf{p}=\begin{pmatrix}u_1\\u_2\end{pmatrix}$ Let $u_1 = t$

$$u_2 = -2t \tag{5.1.9.21}$$

Eigen vector $\mathbf{p_1}$ is given by

$$\mathbf{p_1} = \begin{pmatrix}t\\-2t\end{pmatrix} \tag{5.1.9.22}$$

Let $t = 1$, we get

$$\mathbf{p_1} = \begin{pmatrix}1\\-2\end{pmatrix} \tag{5.1.9.23}$$

For $\lambda_2 = 25$,

$$(\mathbf{V}-\lambda_2\mathbf{I}) = \begin{pmatrix}19-25 & 12\\12 & 1-25\end{pmatrix} \tag{5.1.9.24}$$

By row reduction ,

$$\begin{pmatrix}-6 & 12\\12 & -24\end{pmatrix} \tag{5.1.9.25}$$

$$\xleftrightarrow{R_2\leftarrow R_2+2R_1}\begin{pmatrix}-6 & 12\\0 & 0\end{pmatrix} \tag{5.1.9.26}$$

$$\xleftrightarrow{R_1\leftarrow \frac{R_1}{6}}\begin{pmatrix}-1 & 2\\0 & 0\end{pmatrix} \tag{5.1.9.27}$$

Subsituting equation 5.1.9.27 in equation 5.1.9.15 we get

$$\begin{pmatrix}-1 & 2\\0 & 0\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \tag{5.1.9.28}$$

Where, $\mathbf{p}=\begin{pmatrix}v_1\\v_2\end{pmatrix}$ Let $v_1 = t$

$$v_2 = \frac{t}{2} \tag{5.1.9.29}$$

Eigen vector $\mathbf{p_2}$ is given by

$$\mathbf{p_2} = \begin{pmatrix}t\\\frac{t}{2}\end{pmatrix} \tag{5.1.9.30}$$

Let $t = 1$, we get

$$\mathbf{p_2} = \begin{pmatrix}1\\\frac{1}{2}\end{pmatrix} \tag{5.1.9.31}$$

By eigen decompostion $\mathbf{V}$ can be represented by

$$\mathbf{V} = \mathbf{PDP}^T \tag{5.1.9.32}$$

where

$$\mathbf{P} = \begin{pmatrix}\mathbf{p_1} & \mathbf{p_2}\end{pmatrix} \tag{5.1.9.33}$$

$$\mathbf{D} = \begin{pmatrix}\lambda_1 & 0\\0 & \lambda_2\end{pmatrix} \tag{5.1.9.34}$$

Substituting equations 5.1.9.23, 5.1.9.31 in equation 5.1.9.33 we get

$$\mathbf{P} = \begin{pmatrix}1 & 1\\-2 & \frac{1}{2}\end{pmatrix} \tag{5.1.9.35}$$

Substituting equation 5.1.9.13 in 5.1.9.34 we get

$$\mathbf{D} = \begin{pmatrix}-5 & 0\\0 & 25\end{pmatrix} \tag{5.1.9.36}$$

Equation of a hyperbola and the combined equation of the Asymptotes differ only in the constant term.

$$19x^2 + 24xy + y^2 - 22x - 6y + K = 0 \tag{5.1.9.37}$$

The above equation can be expressed in the form

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{5.1.9.38}$$

Comparing equation we get

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix}19 & 12\\12 & 1\end{pmatrix} \tag{5.1.9.39}$$

$$\mathbf{u} = \begin{pmatrix}-11\\-3\end{pmatrix} \tag{5.1.9.40}$$

$$f = K \tag{5.1.9.41}$$

$$\Delta = \begin{vmatrix}19 & 12 & -11\\12 & 1 & -3\\-11 & -3 & K\end{vmatrix} \tag{5.1.9.42}$$

Since the equations represent pair of straight lines, equating the determinant to zero, we can get the value of K

$$\implies K = 4 \tag{5.1.9.43}$$

Let (α,β) be their point of intersection, then

$$\begin{pmatrix}a & b\\b & c\end{pmatrix}\begin{pmatrix}\alpha\\\beta\end{pmatrix}=\begin{pmatrix}-d\\-e\end{pmatrix} \tag{5.1.9.44}$$

Substituting the values, we obtain,

$$\begin{pmatrix}19 & 12\\12 & 1\end{pmatrix}\begin{pmatrix}\alpha\\\beta\end{pmatrix} = \begin{pmatrix}11\\3\end{pmatrix} \tag{5.1.9.45}$$

$$\text{We get, } \alpha = \frac{1}{5}, \beta = \frac{3}{5} \tag{5.1.9.46}$$

Using Affine transformation and Spectral decomposition, we get

$$X' = \pm\sqrt{-\frac{\lambda_2}{\lambda_1}}Y' \tag{5.1.9.47}$$

$$\text{where } X' = Xu_1 + Yu_2 \tag{5.1.9.48}$$

$$Y' = Xv_1 + Yv_2 \tag{5.1.9.49}$$

$$X = x - \alpha \text{ and } Y = y - \beta \tag{5.1.9.50}$$

Therefore,

$$u_1(x-\alpha) + u_2(y-\beta) = \pm\sqrt{-\frac{\lambda_2}{\lambda_1}}(v_1(x-\alpha) + v_2(y-\beta)) \tag{5.1.9.51}$$

Substituting values, we get

$$(x - \frac{1}{5}) - 2(y - \frac{3}{5}) = \pm\sqrt{\frac{25}{5}}(x - \frac{1}{5}) + \frac{1}{2}(y - \frac{3}{5}) \tag{5.1.9.52}$$

Simplifying above equation

$$8x + 9y - 7 = 0 \tag{5.1.9.53}$$

$$12x + y + 7 = 0 \tag{5.1.9.54}$$

$$\implies (8x + 9y - 7)(12x + y + 7) = 0 \tag{5.1.9.55}$$

Thus the equation of lines are

$$\begin{pmatrix}8 & 9\end{pmatrix}\mathbf{x} = 7 \tag{5.1.9.56}$$

$$\begin{pmatrix}12 & 1\end{pmatrix}\mathbf{x} = -7 \tag{5.1.9.57}$$

The Equation of Conjugate hyperbola is given by:

2(Equation of Asymptotes)- Equation of hyperbola.

From Eq 5.1.9.1 and 5.1.9.37, we obtain equation of Conjugate hyperbola as:-

$$19x^2 + 24xy + y^2 - 22x - 6y + 8 = 0 \tag{5.1.9.58}$$

The general equation of second degree is given by

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{5.1.9.59}$$

comparing equation 5.1.9.58 with the general equation of second degree given at 5.1.9.59, it can be expressed as

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{5.1.9.60}$$

where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix}a & b\\b & c\end{pmatrix} \tag{5.1.9.61}$$

$$\mathbf{u} = \begin{pmatrix}d\\e\end{pmatrix} \tag{5.1.9.62}$$

Comparing equations 5.1.9.58 and 5.1.9.60 we get

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix}19 & 12\\12 & 1\end{pmatrix} \tag{5.1.9.63}$$

$$\mathbf{u} = \begin{pmatrix}-11\\-3\end{pmatrix} \tag{5.1.9.64}$$

$$f = 8 \tag{5.1.9.65}$$

Therefore, the equation of the conjugate hyperbola is as given below:-

$$\mathbf{x}^T\begin{pmatrix}19 & 12\\12 & 1\end{pmatrix}\mathbf{x} + 2\begin{pmatrix}-11 & -3\end{pmatrix}\mathbf{x} + 8 = 0 \tag{5.1.9.66}$$

5.2 41

5.2.1. Trace the parabola:

$$(x - 4y)^2 = 51y \tag{5.2.1.1}$$

Solution: Expanding the given equation, we have,

$$x^2 - 8xy + 16y^2 - 51y = 0 \tag{5.2.1.2}$$

The general equation of second degree is given by

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{5.2.1.3}$$

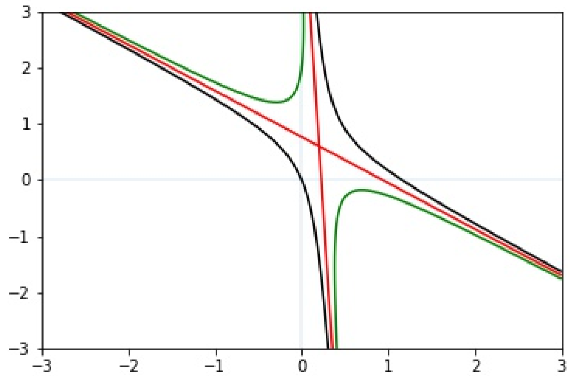

Fig. 5.1.9.1: Hyperbola, Conjugate Hyperbola and Asymptotes

and can be expressed as

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{5.2.1.4}$$

where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \tag{5.2.1.5}$$

$$\mathbf{u}^T = \begin{pmatrix} d & e \end{pmatrix} \tag{5.2.1.6}$$

From equation (5.2.1.2) , we get

$$\mathbf{V} = \begin{pmatrix} 1 & -4 \\ -4 & 16 \end{pmatrix} \tag{5.2.1.7}$$

$$\mathbf{u} = \begin{pmatrix} 0 \\ -\frac{51}{2} \end{pmatrix} \tag{5.2.1.8}$$

$$f = 0 \tag{5.2.1.9}$$

Expanding the determinant of $\mathbf{V}$ we observe,

$$\begin{vmatrix} 1 & -4 \\ -4 & 16 \end{vmatrix} = 0 \tag{5.2.1.10}$$

Therefore, (5.2.1.2) is a parabola.
The characteristic equation of $\mathbf{V}$ is given as follows,

$$\left|\lambda\mathbf{I} - \mathbf{V}\right| = \begin{vmatrix} \lambda - 1 & 4 \\ 4 & \lambda - 16 \end{vmatrix} = 0 \tag{5.2.1.11}$$

$$\implies \lambda^2 - 17\lambda = 0 \tag{5.2.1.12}$$

The eigenvalues are given by

$$\lambda_1 = 0, \lambda_2 = 17 \tag{5.2.1.13}$$

For $\lambda_1 = 0$, the eigen vector $\mathbf{p}$ is given by

$$\mathbf{V}\mathbf{p} = 0 \tag{5.2.1.14}$$

Row reducing $\mathbf{V}$

$$\begin{pmatrix} 1 & -4 \\ -4 & 16 \end{pmatrix} \xleftrightarrow[R_2=R_2+R_1]{R_2=R_2/4} \begin{pmatrix} 1 & -4 \\ 0 & 0 \end{pmatrix} \tag{5.2.1.15}$$

$$\implies \mathbf{p}_1 = \frac{1}{\sqrt{17}}\begin{pmatrix} -4 \\ -1 \end{pmatrix} \tag{5.2.1.16}$$

Similarly,

$$\mathbf{p}_2 = \frac{1}{\sqrt{17}}\begin{pmatrix} -1 \\ 4 \end{pmatrix} \tag{5.2.1.17}$$

Thus,

$$\mathbf{P} = \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} \end{pmatrix} = \frac{1}{\sqrt{17}}\begin{pmatrix} -4 & -1 \\ -1 & 4 \end{pmatrix} \tag{5.2.1.18}$$

The equation of the parabola is:

$$\mathbf{y^TDy} = -2\eta\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{y} \tag{5.2.1.19}$$

where

$$\eta = \mathbf{u}^T\mathbf{p_1} = \frac{51}{2\sqrt{17}} \tag{5.2.1.20}$$

and focal length of the parabola is given by

$$\frac{\left|2\mathbf{u}^T\mathbf{p_1}\right|}{\lambda_2} = \frac{3}{\sqrt{17}} \tag{5.2.1.21}$$

Now,

$$\begin{pmatrix} \mathbf{u^T} + \eta\mathbf{p_1^T} \\ \mathbf{V} \end{pmatrix}\mathbf{c} = \begin{pmatrix} -f \\ \eta\mathbf{p_1} - \mathbf{u} \end{pmatrix} \tag{5.2.1.22}$$

using equations (5.2.1.7), (5.2.1.8) and (5.2.1.22)

$$\begin{pmatrix} -6 & -27 \\ 1 & -4 \\ -4 & 16 \end{pmatrix}\mathbf{c} = \begin{pmatrix} 0 \\ -6 \\ 24 \end{pmatrix} \tag{5.2.1.23}$$

Forming the augmented matrix and row reducing it:

$$\begin{pmatrix} -6 & -27 & 0 \\ 1 & -4 & -6 \\ -4 & 16 & 24 \end{pmatrix} \tag{5.2.1.24}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 + 4R_2} \begin{pmatrix} -6 & -27 & 0 \\ 1 & -4 & -6 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.2.1.25}$$

$$\xleftrightarrow{R_1 \leftarrow R_1/(-6)} \begin{pmatrix} 1 & 9/2 & 0 \\ 1 & -4 & -6 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.2.1.26}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 9/2 & 0 \\ 0 & -17/2 & -6 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.2.1.27}$$

$$\xleftrightarrow{R_2 \leftarrow (-\frac{2}{17})R_2} \begin{pmatrix} 1 & 9/2 & 0 \\ 0 & 1 & 12/17 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.2.1.28}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - (\frac{9}{2})R_2} \begin{pmatrix} 1 & 0 & -54/17 \\ 0 & 1 & 12/17 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.2.1.29}$$

Thus the vertex is:

$$\mathbf{c} = \begin{pmatrix} -\frac{54}{17} \\ \frac{12}{17} \end{pmatrix} \tag{5.2.1.30}$$

$$\approx \begin{pmatrix} -3.18 \\ 0.71 \end{pmatrix} \tag{5.2.1.31}$$

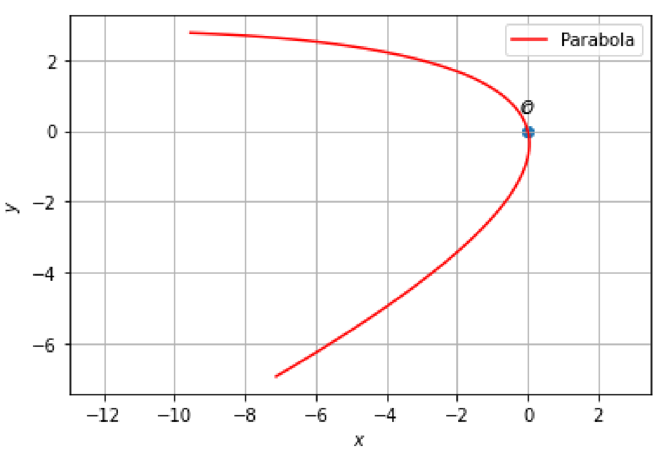

Fig. 5.2.1.1: Parabola

5.2.2. Trace the curve

$$(x - y)^2 = x + y + 1 \tag{5.2.2.1}$$

Solution:
We have given equation as :

$$(x - y)^2 = x + y + 1 \tag{5.2.2.2}$$

$$\implies x^2 - 2xy + y^2 - x - y - 1 = 0 \tag{5.2.2.3}$$

The general equation of second degree is given by

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{5.2.2.4}$$

and can be expressed as

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \tag{5.2.2.5}$$

where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \tag{5.2.2.6}$$

$$\mathbf{u}^T = \begin{pmatrix} d & e \end{pmatrix} \tag{5.2.2.7}$$

Comparing (5.2.2.3) with (5.2.2.4), we get

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \tag{5.2.2.8}$$

$$\mathbf{u}^T = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} \tag{5.2.2.9}$$

$$f = -1 \tag{5.2.2.10}$$

Expanding the determinant of V we observe,

$$|\mathbf{V}| = \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} = 0 \tag{5.2.2.11}$$

Also

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = \begin{vmatrix} 1 & -1 & -\frac{1}{2} \\ -1 & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -1 \end{vmatrix} \neq 0 \tag{5.2.2.12}$$

Hence from (5.2.2.11) and (5.2.2.12) we conclude that given equation is an parabola.The characteristic equation of $\mathbf{V}$ is given as follows,

$$\left|\lambda \mathbf{I} - \mathbf{V}\right| = \begin{vmatrix} \lambda - 1 & -1 \\ -1 & \lambda - 1 \end{vmatrix} = 0 \tag{5.2.2.13}$$

$$\implies (\lambda - 1)^2 - 1 = 0 \tag{5.2.2.14}$$

The eigenvalues are the roots of (5.2.2.14) given by

$$\lambda_1 = 0, \lambda_2 = 2 \tag{5.2.2.15}$$

The eigenvector $\mathbf{p}$ is defined as:

$$\mathbf{V}\mathbf{p} = \lambda \mathbf{p} \tag{5.2.2.16}$$

$$\implies (\lambda \mathbf{I} - \mathbf{V})\,\mathbf{p} = 0 \tag{5.2.2.17}$$

where λ is the eigenvalue. For $\lambda_1 = 0$,

$$\mathbf{V}\mathbf{p} = 0 \tag{5.2.2.18}$$

Row reducing $\mathbf{V}$ yields,

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 + R_1} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \tag{5.2.2.19}$$

Similarly, the eigenvector corresponding to λ_2 can be obtained as

$$(\lambda_2 \mathbf{I} - \mathbf{V}) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \tag{5.2.2.20}$$

It is easy to verify that

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1} = \mathbf{P}\mathbf{D}\mathbf{P}^{T} \quad \because \mathbf{P}^{-1} = \mathbf{P}^{T} \tag{5.2.2.21}$$

$$\text{or, } \mathbf{D} = \mathbf{P}^{T}\mathbf{V}\mathbf{P} \tag{5.2.2.22}$$

From equation (5.2.2.19) and (5.2.2.20), we have

$$\mathbf{p_1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{and, } \mathbf{p_2} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{5.2.2.23}$$

Thus, the eigenvector rotation matrix and the eigenvalue matrix are

$$\mathbf{P} = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{5.2.2.24}$$

$$\mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \tag{5.2.2.25}$$

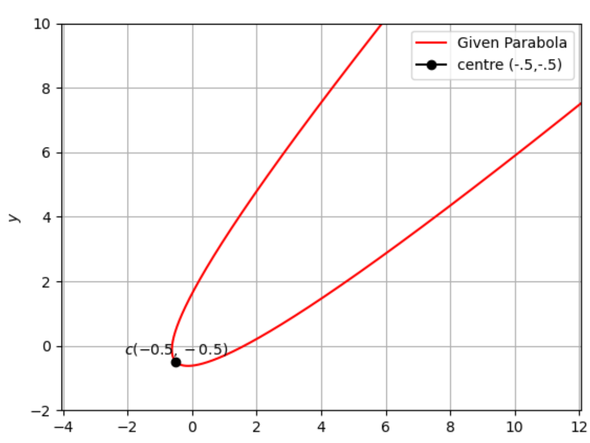

Fig. 5.2.2.1: Parabola with the center c

The focal length of the parabola is given by

$$\frac{\left|2\mathbf{u}^T\mathbf{p_1}\right|}{\lambda_2} = \frac{\sqrt{2}}{2} = \sqrt{2} \tag{5.2.2.26}$$

and its equation is

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = -2\eta\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{y} \tag{5.2.2.27}$$

where,

$$\eta = \mathbf{u}^T\mathbf{p_1} = -\frac{1}{\sqrt{2}} \tag{5.2.2.28}$$

$$\begin{pmatrix} \mathbf{u}^T + \eta\mathbf{p_1}^T \\ \mathbf{V} \end{pmatrix}\mathbf{c} = \begin{pmatrix} -f \\ \eta\mathbf{p_1} - \mathbf{u} \end{pmatrix} \tag{5.2.2.29}$$

$$\implies \begin{pmatrix} -1 & -1 \\ 1 & -1 \\ -1 & 1 \end{pmatrix}\mathbf{c} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{5.2.2.30}$$

Forming the augmented matrix and row reducing it:

$$\begin{pmatrix} -1 & -1 & 1 \\ 1 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 + R_1} \begin{pmatrix} -1 & -1 & 1 \\ 0 & -2 & 1 \\ -1 & 1 & 0 \end{pmatrix} \xleftrightarrow[R_1 \leftarrow -1R_1]{R_3 \leftarrow R_3 - R_1}$$
$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & -2 & 1 \\ 0 & 2 & -1 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 + R_2} \begin{pmatrix} 1 & 1 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\xleftrightarrow[R_1 \leftarrow R_1 - R_2]{R_1 \leftarrow \frac{R_1}{-2}} \begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix} \tag{5.2.2.31}$$

So,

$$\mathbf{c} = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} \tag{5.2.2.32}$$

5.2.3. Trace the parabola

$$(4x + 3y + 15)^2 = 5(3x - 4y) \tag{5.2.3.1}$$

Solution: The given equation can be rewritten as

$$16x^2 + 24xy + 9y^2 + 105x + 110y + 225 = 0 \tag{5.2.3.2}$$

Comparing this to the standard equation,

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 16 & 12 \\ 12 & 9 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} \frac{105}{2} \\ 55 \end{pmatrix}, \quad f = 225 \tag{5.2.3.3}$$

The characteristic equation of $\mathbf{V}$ is given as

$$\left|\lambda \mathbf{I} - \mathbf{V}\right| = 0 \tag{5.2.3.4}$$

$$\implies \begin{vmatrix} \lambda - 16 & -12 \\ -12 & \lambda - 9 \end{vmatrix} = 0 \tag{5.2.3.5}$$

$$\implies \lambda^2 - 25\lambda = 0 \tag{5.2.3.6}$$

The eigenvalues are the roots of the equation (5.2.3.6), which are as follows :

$$\lambda_1 = 0, \quad \lambda_2 = 25 \tag{5.2.3.7}$$

The eigen vector $\mathbf{p}$ is defined as,

$$\mathbf{V}\mathbf{p} = \lambda \mathbf{p} \tag{5.2.3.8}$$

$$\implies (\lambda \mathbf{I} - \mathbf{V})\mathbf{p} = 0 \tag{5.2.3.9}$$

For $\lambda_1 = 0$

$$(\lambda_1 \mathbf{I} - \mathbf{V}) = \begin{pmatrix} -16 & -12 \\ -12 & -9 \end{pmatrix} \xleftrightarrow[R_2 \leftarrow R_2 - 3R_1]{R_1 \leftarrow \frac{1}{4}R_1} \begin{pmatrix} -4 & -3 \\ 0 & 0 \end{pmatrix} \tag{5.2.3.10}$$

$$\implies \mathbf{p_1} = \frac{1}{5}\begin{pmatrix} -3 \\ 4 \end{pmatrix} \tag{5.2.3.11}$$

For $\lambda_2 = 25$

$$(\lambda_2 \mathbf{I} - \mathbf{V}) = \begin{pmatrix} 9 & -12 \\ -12 & 16 \end{pmatrix} \xleftrightarrow[R_2 \leftarrow R_2 + 4R_1]{R_1 \leftarrow \frac{1}{3}R_1} \begin{pmatrix} 3 & -4 \\ 0 & 0 \end{pmatrix} \tag{5.2.3.12}$$

$$\implies \mathbf{p_2} = \frac{1}{5}\begin{pmatrix} 4 \\ 3 \end{pmatrix} \tag{5.2.3.13}$$

So, using Eigenvalue decomposition, $\mathbf{P}^T\mathbf{V}\mathbf{P} = \mathbf{D}$, where

$$\mathbf{P} = \frac{1}{5}\begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix} \tag{5.2.3.14}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 25 \end{pmatrix} \tag{5.2.3.15}$$

Then, for the parabola

$$\text{focal length} = \left|\frac{2\eta}{\lambda_2}\right| \tag{5.2.3.16}$$

$$\eta = \mathbf{p}_1^T \mathbf{u} = \frac{25}{2} \tag{5.2.3.17}$$

Substituting values from (5.2.3.17) and (5.2.3.7) in (5.2.3.16), we get

$$\text{focal length} = 1 \tag{5.2.3.18}$$

The standard equation of the parabola is given by

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = -2\eta\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{y} \tag{5.2.3.19}$$

And the vertex $\mathbf{c}$ is given by

$$\begin{pmatrix} \mathbf{u}^T + \eta \mathbf{p}_1^T \\ \mathbf{V} \end{pmatrix}\mathbf{c} = \begin{pmatrix} -f \\ \eta \mathbf{p}_1 - \mathbf{u} \end{pmatrix} \tag{5.2.3.20}$$

Substituting values from (5.2.3.3),(5.2.3.17),(5.2.3.11) in (5.2.3.20),

$$\begin{pmatrix} 45 & 65 \\ 16 & 12 \\ 12 & 9 \end{pmatrix}\mathbf{c} = \begin{pmatrix} -225 \\ -60 \\ -45 \end{pmatrix} \tag{5.2.3.21}$$

To find $\mathbf{c}$, performing row reduction on the augmented matrix as follows:

$$\begin{pmatrix} 45 & 65 & -225 \\ 16 & 12 & -60 \\ 12 & 9 & -45 \end{pmatrix} \xleftrightarrow[R_1 \leftarrow \frac{1}{45}R_1]{R_3 \leftarrow R_3 - \frac{3}{4}R_2} \begin{pmatrix} 1 & \frac{13}{9} & -5 \\ 16 & 12 & -60 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.2.3.22}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - 16R_1} \begin{pmatrix} 1 & \frac{13}{9} & -5 \\ 0 & \frac{-100}{9} & 20 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.2.3.23}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{-9}{100}R_2} \begin{pmatrix} 1 & \frac{13}{9} & -5 \\ 0 & 1 & \frac{-9}{5} \\ 0 & 0 & 0 \end{pmatrix} \tag{5.2.3.24}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - \frac{13}{9}R_2} \begin{pmatrix} 1 & 0 & \frac{-12}{5} \\ 0 & 1 & \frac{-9}{5} \\ 0 & 0 & 0 \end{pmatrix} \tag{5.2.3.25}$$

Thus,

$$\mathbf{c} = \begin{pmatrix} \frac{-12}{5} \\ \frac{-9}{5} \end{pmatrix} = \begin{pmatrix} -2.4 \\ -1.8 \end{pmatrix} \tag{5.2.3.26}$$

5.2.4. Trace the parabola

$$16x^2 + 24xy + 9y^2 - 5x - 10y + 1 = 0$$

Solution: Compare the given equation with the

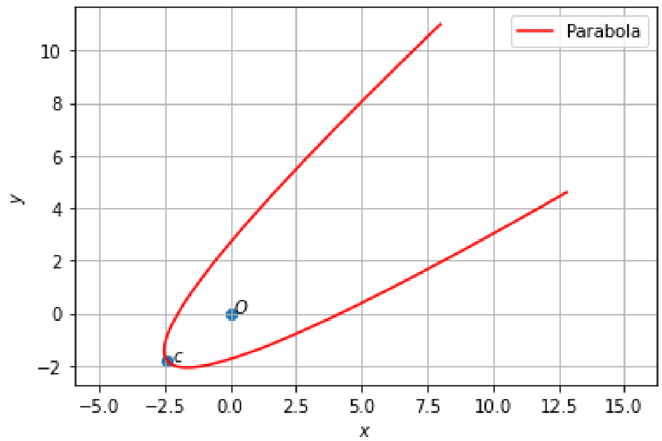

Fig. 5.2.3.1: Parabola with vertex c

standard form

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{5.2.4.1}$$

Write the values Of V and u as follows

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 16 & 12 \\ 12 & 9 \end{pmatrix} \quad \mathbf{u} = \begin{pmatrix} -\frac{5}{2} \\ -5 \end{pmatrix} \quad f = 1 \tag{5.2.4.2}$$

The characteristic equation of $\mathbf{V}$ is given as

$$\left|\lambda\mathbf{I} - \mathbf{V}\right| = 0 \tag{5.2.4.3}$$

$$\implies \begin{vmatrix} \lambda - 16 & -12 \\ -12 & \lambda - 9 \end{vmatrix} = 0 \tag{5.2.4.4}$$

$$\implies \lambda^2 - 25\lambda = 0 \tag{5.2.4.5}$$

The eigenvalues are the roots of the equation (5.2.4.5) are

$$\lambda_1 = 0, \quad \lambda_2 = 25 \tag{5.2.4.6}$$

The eigen vector $\mathbf{p}$ is defined as,

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \tag{5.2.4.7}$$

$$\implies (\lambda\mathbf{I} - \mathbf{V})\mathbf{p} = 0 \tag{5.2.4.8}$$

For $\lambda_1 = 0$

$$(\lambda_1\mathbf{I} - \mathbf{V}) = \begin{pmatrix} -16 & -12 \\ -12 & -9 \end{pmatrix} \xleftrightarrow[R_2 \leftarrow R_2 - 3R_1]{R_1 \leftarrow \frac{1}{4}R_1} \begin{pmatrix} -4 & -3 \\ 0 & 0 \end{pmatrix} \tag{5.2.4.9}$$

$$\implies \mathbf{p_1} = \frac{1}{5}\begin{pmatrix} -3 \\ 4 \end{pmatrix} \tag{5.2.4.10}$$

For $\lambda_2 = 25$

$$(\lambda_2\mathbf{I} - \mathbf{V}) = \begin{pmatrix} 9 & -12 \\ -12 & 16 \end{pmatrix} \xleftrightarrow[R_2 \leftarrow R_2 + 4R_1]{R_1 \leftarrow \frac{1}{3}R_1} \begin{pmatrix} 3 & -4 \\ 0 & 0 \end{pmatrix} \tag{5.2.4.11}$$

$$\implies \mathbf{p_2} = \frac{1}{5}\begin{pmatrix} 4 \\ 3 \end{pmatrix} \tag{5.2.4.12}$$

Use Eigenvalue decomposition, $\mathbf{P}^T\mathbf{V}\mathbf{P} = \mathbf{D}$, where

$$\mathbf{P} = \frac{1}{5}\begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix} \tag{5.2.4.13}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 25 \end{pmatrix} \tag{5.2.4.14}$$

Focal length of the parabola is given as

$$\text{focal length} = \left|\frac{2\eta}{\lambda_2}\right| \tag{5.2.4.15}$$

$$\eta = \mathbf{p}_1^T\mathbf{u} = -\frac{5}{2} \tag{5.2.4.16}$$

Substituting values from (5.2.4.16) and (5.2.4.6) in (5.2.4.15), we get

$$\text{focal length} = \frac{1}{5} \tag{5.2.4.17}$$

The standard equation of the parabola is given by

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = -2\eta\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{y} \tag{5.2.4.18}$$

And the vertex $\mathbf{c}$ is given by

$$\begin{pmatrix} \mathbf{u}^T + \eta\mathbf{p}_1^T \\ \mathbf{V} \end{pmatrix}\mathbf{c} = \begin{pmatrix} -f \\ \eta\mathbf{p}_1 - \mathbf{u} \end{pmatrix} \tag{5.2.4.19}$$

Substituting values from (5.2.4.2),(5.2.4.16),(5.2.4.10) in (5.2.4.19),

$$\begin{pmatrix} -1 & -7 \\ 16 & 12 \\ 12 & 9 \end{pmatrix}\mathbf{c} = \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} \tag{5.2.4.20}$$

To find $\mathbf{c}$, performing row reduction on the augmented matrix as follows:

$$\begin{pmatrix} -1 & -7 & -1 \\ 16 & 12 & 4 \\ 12 & 9 & 3 \end{pmatrix} \xleftrightarrow[R_1\leftarrow -R_1]{R_3\leftarrow R_3-\frac{3}{4}R_2} \begin{pmatrix} 1 & 7 & 1 \\ 16 & 12 & 4 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.2.4.21}$$

$$\xleftrightarrow{R_2\leftarrow R_2-16R_1} \begin{pmatrix} 1 & 7 & 1 \\ 0 & -100 & -12 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.2.4.22}$$

$$\xleftrightarrow{R_2\leftarrow \frac{-1}{100}R_2} \begin{pmatrix} 1 & 7 & 1 \\ 0 & 1 & \frac{3}{25} \\ 0 & 0 & 0 \end{pmatrix} \tag{5.2.4.23}$$

$$\xleftrightarrow{R_1\leftarrow R_1-7R_2} \begin{pmatrix} 1 & 0 & \frac{4}{25} \\ 0 & 1 & \frac{3}{25} \\ 0 & 0 & 0 \end{pmatrix} \tag{5.2.4.24}$$

Thus,

$$\mathbf{c} = \begin{pmatrix} \frac{4}{25} \\ \frac{3}{25} \end{pmatrix} \tag{5.2.4.25}$$

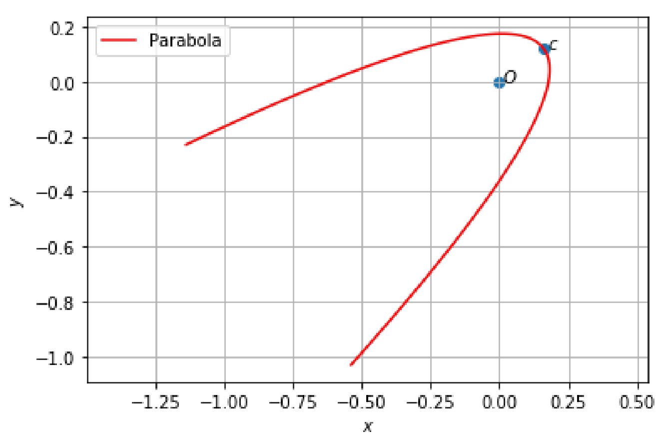

Fig. 5.2.4.1: Parabola with vertex c

5.2.5. Trace the parabola

$$9x^2 + 24xy + 16y^2 - 4y - x + 7 = 0 \tag{5.2.5.1}$$

Solution: The general second degree equation can be expressed as

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{5.2.5.2}$$

Comparing (5.2.5.1) and (5.2.5.2) we get

$$\mathbf{V} = \begin{pmatrix} 9 & 12 \\ 12 & 16 \end{pmatrix} \tag{5.2.5.3}$$

$$\mathbf{u} = \begin{pmatrix} \frac{-1}{2} \\ -2 \end{pmatrix} \tag{5.2.5.4}$$

$$f = 7 \tag{5.2.5.5}$$

The characteristic equation of $\mathbf{V}$ is given as

$$\left|\mathbf{V} - \lambda\mathbf{I}\right| = 0 \tag{5.2.5.6}$$

$$\implies \begin{vmatrix} 9-\lambda & 12 \\ 12 & 16-\lambda \end{vmatrix} = 0 \tag{5.2.5.7}$$

$$\implies \lambda^2 - 25\lambda = 0 \tag{5.2.5.8}$$

The roots of (5.2.5.8) are eigenvalue of $\mathbf{V}$ and are given by

$$\lambda_1 = 0, \lambda_2 = 25$$

The eigenvector $\mathbf{p}$ is defined as

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \tag{5.2.5.9}$$

$$\implies (\mathbf{V} - \lambda\mathbf{I})\mathbf{p} = 0 \tag{5.2.5.10}$$

For $\lambda_1 = 0$

$$(\mathbf{V} - \lambda\mathbf{I}) = \begin{pmatrix} 9 & 12 \\ 12 & 16 \end{pmatrix} \xleftrightarrow{R_2=R_2-\frac{4}{3}R_1} \begin{pmatrix} 9 & 12 \\ 0 & 0 \end{pmatrix} \tag{5.2.5.11}$$

Substituting equation (5.2.5.11) in equation (5.2.5.10) and upon normalization we get

$$\mathbf{p_1} = \frac{1}{5}\begin{pmatrix} -4 \\ 3 \end{pmatrix} \tag{5.2.5.12}$$

For $\lambda_2 = 25$

$$(\mathbf{V} - \lambda\mathbf{I}) = \begin{pmatrix} -16 & 12 \\ 12 & -9 \end{pmatrix} \xleftrightarrow{R_2=R_2+\frac{3}{4}R_1} \begin{pmatrix} -16 & 12 \\ 0 & 0 \end{pmatrix} \tag{5.2.5.13}$$

Substituting equation (5.2.5.13) in equation (5.2.5.10) and upon normalization we get

$$\mathbf{p_2} = \frac{1}{5}\begin{pmatrix} 3 \\ 4 \end{pmatrix} \tag{5.2.5.14}$$

The matrix $\mathbf{P}$ and $\mathbf{D}$ are

$$\mathbf{P} = \begin{pmatrix} \mathbf{p1} & \mathbf{p2} \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -4 & 3 \\ 3 & 4 \end{pmatrix} \tag{5.2.5.15}$$

and

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 25 \end{pmatrix} \tag{5.2.5.16}$$

Then for the parabola

$$\eta = 2\mathbf{p_1}^T\mathbf{u} = -\frac{8}{5} \tag{5.2.5.17}$$

$$focal\ length = \left|\frac{\eta}{\lambda_2}\right| = \frac{8}{125} \tag{5.2.5.18}$$

For parabola $\left|\mathbf{V}\right| = 0$,so equation (5.2.5.2) can be written as

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = -\eta\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{y} \tag{5.2.5.19}$$

And the vertex $\mathbf{c}$ is given by

$$\begin{pmatrix} \mathbf{u}^T + \frac{\eta}{2}\mathbf{p_1}^T \\ \mathbf{V} \end{pmatrix}\mathbf{c} = \begin{pmatrix} -f \\ \frac{\eta}{2}\mathbf{p_1} - \mathbf{u} \end{pmatrix} \tag{5.2.5.20}$$

Substituting values from (5.2.5.3), (5.2.5.4), (5.2.5.5), (5.2.5.12), (5.2.5.17) in (5.2.5.20)

$$\begin{pmatrix} \frac{7}{50} & -\frac{124}{50} \\ 9 & 12 \\ 12 & 16 \end{pmatrix}\mathbf{c} = \begin{pmatrix} -7 \\ \frac{57}{50} \\ \frac{76}{50} \end{pmatrix} \tag{5.2.5.21}$$

To find $\mathbf{c}$,performing row reduction in augmented matrix as follows

$$\begin{pmatrix} \frac{7}{50} & -\frac{124}{50} & -7 \\ 9 & 12 & \frac{57}{50} \\ 12 & 16 & \frac{76}{50} \end{pmatrix} \xleftrightarrow[R_1 \leftarrow \frac{50}{7}R_1]{R_3 \leftarrow R_3 - \frac{4}{3}R_2} \begin{pmatrix} 1 & -\frac{124}{7} & -50 \\ 9 & 12 & \frac{57}{50} \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - 9R_1} \begin{pmatrix} 1 & -\frac{124}{7} & -50 \\ 0 & \frac{1200}{7} & \frac{22557}{50} \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{7}{1200}R_2} \begin{pmatrix} 1 & -\frac{124}{7} & -50 \\ 0 & 1 & \frac{52633}{20000} \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + \frac{124}{7}R_2} \begin{pmatrix} 1 & 0 & -\frac{16911}{5000} \\ 0 & 1 & \frac{52633}{20000} \\ 0 & 0 & 0 \end{pmatrix}$$

Thus

$$\mathbf{c} = \begin{pmatrix} -\frac{16911}{5000} \\ \frac{52633}{20000} \end{pmatrix} \tag{5.2.5.22}$$

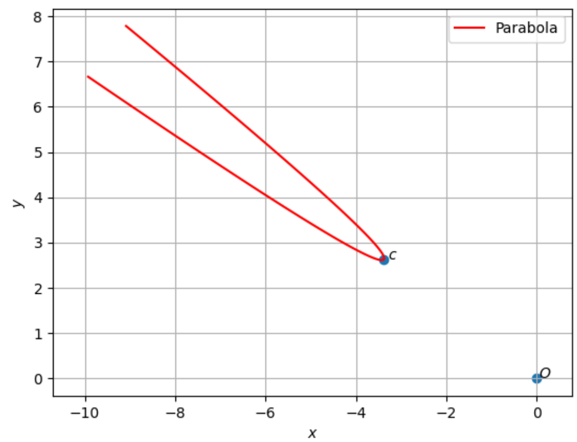

Fig. 5.2.5.1: Graph of $9x^2{+}24xy{+}16y^2{-}4y{-}x{+}7 = 0$

5.2.6. Trace the parabola and find its focus.

$$144y^2 - 120xy + 25x^2 + 619x - 272y + 663 = 0 \tag{5.2.6.1}$$

Solution: The general second degree equation can be expressed as follows,

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{5.2.6.2}$$

where,

$$\mathbf{V} = \begin{pmatrix} 144 & -60 \\ -60 & 25 \end{pmatrix} \tag{5.2.6.3}$$

$$\mathbf{u} = \begin{pmatrix} \frac{619}{2} \\ -136 \end{pmatrix} \tag{5.2.6.4}$$

$$f = 663 \tag{5.2.6.5}$$

a) Expanding the determinant of $\mathbf{V}$ we observe,

$$\begin{vmatrix} 144 & -60 \\ -60 & 25 \end{vmatrix} = 0 \tag{5.2.6.6}$$

Also

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = \begin{vmatrix} 144 & -60 & \frac{619}{2} \\ -60 & 25 & -136 \\ \frac{619}{2} & -136 & 663 \end{vmatrix} \tag{5.2.6.7}$$

$$\neq 0 \tag{5.2.6.8}$$

Hence from (5.2.6.6) and (5.2.6.8) we conclude that given equation is an parabola. The characteristic equation of $\mathbf{V}$ is given as follows,

$$\left|\lambda\mathbf{I} - \mathbf{V}\right| = \begin{vmatrix}\lambda - 144 & 60 \\ 60 & \lambda - 25\end{vmatrix} = 0 \quad (5.2.6.9)$$

$$\implies \lambda^2 - 169\lambda = 0 \quad (5.2.6.10)$$

Hence the characteristic equation of $\mathbf{V}$ is given by (5.2.6.10). The roots of (5.2.6.10) i.e the eigenvalues are given by

$$\lambda_1 = 0, \lambda_2 = 169 \quad (5.2.6.11)$$

b) For $\lambda_1 = 0$, the eigen vector $\mathbf{p}$ is given by

$$\mathbf{Vp} = 0 \quad (5.2.6.12)$$

Row reducing $\mathbf{V}$ yields

$$\implies \begin{pmatrix}-144 & 60 \\ 60 & -25\end{pmatrix} \xleftrightarrow[R_2=R_2+5R_1]{R_1=\frac{R_1}{12}} \begin{pmatrix}-12 & 5 \\ 0 & 0\end{pmatrix} \quad (5.2.6.13)$$

$$\implies \mathbf{p}_1 = \frac{1}{13}\begin{pmatrix}5 \\ 12\end{pmatrix} \quad (5.2.6.14)$$

Similarly,

$$\mathbf{p}_2 = \frac{1}{13}\begin{pmatrix}12 \\ -5\end{pmatrix} \quad (5.2.6.15)$$

Thus, the eigenvector rotation matrix and the eigenvalue matrix are

$$\mathbf{P} = \begin{pmatrix}\mathbf{p_1} & \mathbf{p_2}\end{pmatrix} = \frac{1}{13}\begin{pmatrix}5 & 12 \\ 12 & -5\end{pmatrix} \quad (5.2.6.16)$$

$$\mathbf{D} = \begin{pmatrix}0 & 0 \\ 0 & 169\end{pmatrix} \quad (5.2.6.17)$$

The focal length of the parabola is given by

$$\frac{\left|2\mathbf{u}^T\mathbf{p_1}\right|}{\lambda_2} = \frac{13}{169} = \frac{1}{13} \quad (5.2.6.18)$$

and its equation is

$$\mathbf{y^T D y} = -\eta\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{y} \quad (5.2.6.19)$$

where

$$\eta = 2\mathbf{u}^T\mathbf{p_1} = -13 \quad (5.2.6.20)$$

and the vertex $\mathbf{c}$ is given by

$$\begin{pmatrix}\mathbf{u^T} + \frac{\eta}{2}\mathbf{p_1^T} \\ \mathbf{V}\end{pmatrix}\mathbf{c} = \begin{pmatrix}-f \\ \frac{\eta}{2}\mathbf{p_1} - \mathbf{u}\end{pmatrix} \quad (5.2.6.21)$$

using equations (5.2.6.4),(5.2.6.5) and (5.2.6.14)

$$\begin{pmatrix}307 & -142 \\ 144 & -60 \\ -60 & 25\end{pmatrix}\mathbf{c} = \begin{pmatrix}-663 \\ -312 \\ 130\end{pmatrix} \quad (5.2.6.22)$$

Forming the augmented matrix and row reducing it:

$$\begin{pmatrix}307 & -142 & -663 \\ 144 & -60 & -312 \\ -60 & 25 & 130\end{pmatrix} \quad (5.2.6.23)$$

$$R_2 \leftrightarrow \frac{R_2}{12}$$

$$\begin{pmatrix}307 & -142 & -663 \\ 12 & -5 & -26 \\ -60 & 25 & 130\end{pmatrix} \quad (5.2.6.24)$$

$$R_3 \leftrightarrow R_3 + 5R_2$$

$$\begin{pmatrix}307 & -142 & -663 \\ 12 & -5 & -26 \\ 0 & 0 & 0\end{pmatrix} \quad (5.2.6.25)$$

$$R_1 \leftrightarrow \frac{R_1}{307}$$

$$\begin{pmatrix}1 & \frac{-142}{307} & \frac{-663}{307} \\ 0 & \frac{169}{307} & \frac{-26}{307} \\ 0 & 0 & 0\end{pmatrix} \quad (5.2.6.26)$$

$$R_2 \leftrightarrow R_2 - 12R_1$$

$$\begin{pmatrix}1 & \frac{-142}{307} & \frac{-663}{307} \\ 0 & 1 & \frac{-26}{307} \\ 0 & 0 & 0\end{pmatrix} \quad (5.2.6.27)$$

$$R_1 \leftrightarrow R_1 + (142/307)R_2$$

$$\begin{pmatrix}1 & 0 & -29/13 \\ 0 & 1 & -2/13 \\ 0 & 0 & 0\end{pmatrix} \quad (5.2.6.28)$$

Thus the vertex $\mathbf{c}$ is:

$$\mathbf{c} = \begin{pmatrix}-29/13 \\ -2/13\end{pmatrix} \quad (5.2.6.29)$$

The direction vector of axis of symmetry is given by :

$$\mathbf{m} = \mathbf{V}\mathbf{c} + \mathbf{u} \tag{5.2.6.30}$$

$$= \begin{pmatrix} 144 & -60 \\ -60 & 25 \end{pmatrix} \begin{pmatrix} -\frac{29}{13} \\ -\frac{2}{13} \end{pmatrix} + \begin{pmatrix} \frac{619}{2} \\ -\frac{272}{2} \end{pmatrix} \tag{5.2.6.31}$$

$$= \begin{pmatrix} -\frac{5}{2} \\ -6 \end{pmatrix} \tag{5.2.6.32}$$

$$\mathbf{m} = \frac{13}{2} \tag{5.2.6.33}$$

$$\implies \frac{\mathbf{m}}{||\mathbf{m}||} = \begin{pmatrix} -\frac{5}{13} \\ -\frac{12}{13} \end{pmatrix} \tag{5.2.6.34}$$

The focus is given by:

$$\mathbf{F} = \mathbf{c} - \left(\frac{\mathbf{m}}{||\mathbf{m}|| \times a} \right) \tag{5.2.6.35}$$

$$= \begin{pmatrix} -\frac{29}{13} \\ -\frac{2}{13} \end{pmatrix} - \left(\begin{pmatrix} -\frac{5}{13} \\ -\frac{12}{13} \end{pmatrix} \right) \times \frac{1}{52} \tag{5.2.6.36}$$

$$= \begin{pmatrix} -\frac{1503}{676} \\ -\frac{23}{169} \end{pmatrix} \tag{5.2.6.37}$$

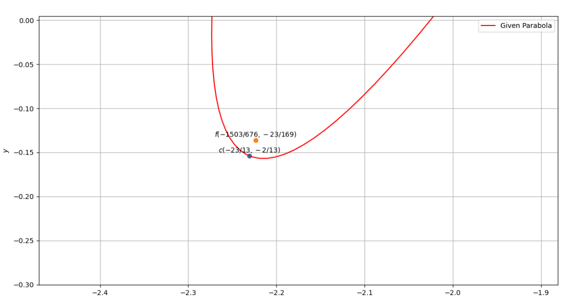

Fig. 5.2.6.1: Traced parabola

5.2.7. Trace the parabola

$$16x^2 - 24xy + 9y^2 + 32x + 86y - 39 = 0 \tag{5.2.7.1}$$

Solution: The general equation of a second degree can be expressed as:

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{5.2.7.2}$$

Comparing (5.2.7.1) and (5.2.7.2)

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 16 & -12 \\ -12 & 9 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} 16 \\ 43 \end{pmatrix}, \quad f = -39 \tag{5.2.7.3}$$

Eigen Values: The characteristic equation of $\mathbf{V}$ is given as

$$\left| \lambda \mathbf{I} - \mathbf{V} \right| = 0 \tag{5.2.7.4}$$

$$\implies \begin{vmatrix} \lambda - 16 & 12 \\ 12 & \lambda - 9 \end{vmatrix} = 0 \tag{5.2.7.5}$$

$$\implies \lambda^2 - 25\lambda = 0 \tag{5.2.7.6}$$

The eigenvalues are the roots of the equation (5.2.7.6), which are as follows:

$$\lambda_1 = 0, \quad \lambda_2 = 25 \tag{5.2.7.7}$$

Eigen Vectors: The eigen vector $\mathbf{p}$ is defined as

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \tag{5.2.7.8}$$

$$\implies (\lambda \mathbf{I} - \mathbf{V})\mathbf{p} = 0 \tag{5.2.7.9}$$

For $\lambda_1 = 0$

$$(\lambda_1 \mathbf{I} - \mathbf{V}) = \begin{pmatrix} -16 & 12 \\ 12 & -9 \end{pmatrix} \xleftrightarrow[R_2 \leftarrow R_2 + 3R_1]{R_1 \leftarrow \frac{1}{4}R_1} \begin{pmatrix} -4 & 3 \\ 0 & 0 \end{pmatrix} \tag{5.2.7.10}$$

$$\implies \mathbf{p_1} = \frac{1}{5} \begin{pmatrix} 3 \\ 4 \end{pmatrix} \tag{5.2.7.11}$$

For $\lambda_2 = 25$

$$(\lambda_2 \mathbf{I} - \mathbf{V}) = \begin{pmatrix} 9 & 12 \\ 12 & 1 \end{pmatrix} \xleftrightarrow[R_2 \leftarrow R_2 - 4R_1]{R_1 \leftarrow \frac{1}{3}R_1} \begin{pmatrix} 3 & 4 \\ 0 & 0 \end{pmatrix} \tag{5.2.7.12}$$

$$\implies \mathbf{p_2} = \frac{1}{5} \begin{pmatrix} -4 \\ 3 \end{pmatrix} \tag{5.2.7.13}$$

Eigen Value Decomposition: Using EVD, we can write

$$\mathbf{D} = \mathbf{P}\mathbf{V}\mathbf{P}^T \tag{5.2.7.14}$$

From (5.2.7.11) and (5.2.7.13)

$$\mathbf{P} = \frac{1}{5}\begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix} \tag{5.2.7.15}$$

From (5.2.7.7)

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 25 \end{pmatrix} \tag{5.2.7.16}$$

Parabola

$$\text{Focal Length} = \left|\frac{2\eta}{\lambda_2}\right| \tag{5.2.7.17}$$

From (5.2.7.11) and (5.2.7.3)

$$\eta = \mathbf{p}_1^T\mathbf{u} = 44 \tag{5.2.7.18}$$

Substituting values of (5.2.7.18) and (5.2.7.7) in (5.2.7.17), we get

$$\text{Focal Length} = \left|\frac{88}{25}\right| = 3.52 \tag{5.2.7.19}$$

The standard equation of parabola is given by:

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = -2\eta\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{y} \tag{5.2.7.20}$$

And the vertex $\mathbf{c}$ is:

$$\begin{pmatrix} \mathbf{u}^T + \eta\mathbf{p_1}^T \\ \mathbf{V} \end{pmatrix}\mathbf{c} = \begin{pmatrix} -f \\ \eta\mathbf{p_1} - \mathbf{u} \end{pmatrix} \tag{5.2.7.21}$$

From (5.2.7.3) (5.2.7.18) and (5.2.7.11),

$$\begin{pmatrix} \frac{212}{5} & \frac{391}{5} \\ 16 & -12 \\ -12 & 9 \end{pmatrix}\mathbf{c} = \begin{pmatrix} 39 \\ \frac{52}{5} \\ \frac{-39}{5} \end{pmatrix} \tag{5.2.7.22}$$

To find $\mathbf{c}$, perform row reduction on the augmented matrix as follows:

$$\begin{pmatrix} \frac{212}{5} & \frac{391}{5} & 39 \\ 16 & -12 & \frac{52}{5} \\ -12 & 9 & \frac{-39}{5} \end{pmatrix} \xleftrightarrow[R_1 \leftarrow \frac{5}{212}R_1]{R_3 \leftarrow R_3 + \frac{3}{4}R_2} \begin{pmatrix} 1 & \frac{391}{212} & \frac{195}{212} \\ 16 & -12 & \frac{52}{5} \\ 0 & 0 & 0 \end{pmatrix} \tag{5.2.7.23}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - 16R_1} \begin{pmatrix} 1 & \frac{391}{212} & \frac{195}{212} \\ 0 & \frac{-2200}{53} & \frac{-1144}{265} \\ 0 & 0 & 0 \end{pmatrix} \tag{5.2.7.24}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{-53}{2200}R_2} \begin{pmatrix} 1 & \frac{391}{212} & \frac{195}{212} \\ 0 & 1 & \frac{13}{125} \\ 0 & 0 & 0 \end{pmatrix} \tag{5.2.7.25}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - \frac{391}{212}R_2} \begin{pmatrix} 1 & 0 & \frac{4823}{6625} \\ 0 & 1 & \frac{13}{125} \\ 0 & 0 & 0 \end{pmatrix} \tag{5.2.7.26}$$

Hence,

$$\mathbf{c} = \begin{pmatrix} \frac{4823}{6625} \\ \frac{13}{125} \end{pmatrix} = \begin{pmatrix} 0.728 \\ 0.104 \end{pmatrix} \tag{5.2.7.27}$$

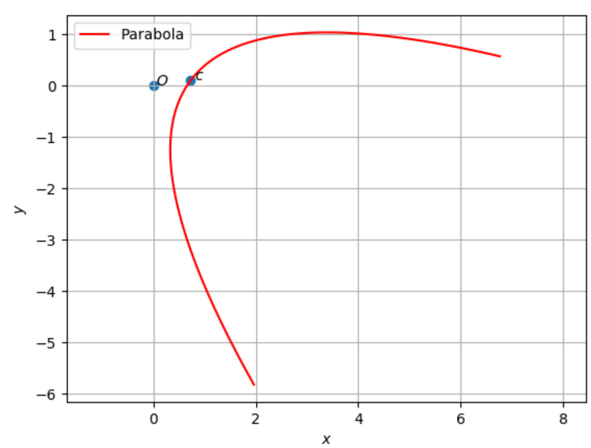

Fig. 5.2.7.1: Parabola with vertex c

5.2.8. Trace the following parabola

$$4x^2 - 4xy + y^2 - 12x + 6y + 9 = 0 \tag{5.2.8.1}$$

Solution: The given quadratic equation can be written in the matrix form as

$$\mathbf{x}^T\begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}\mathbf{x} + 2\begin{pmatrix} -6 & 3 \end{pmatrix}\mathbf{x} + 9 = 0 \tag{5.2.8.2}$$

Calculating the parameters,we get

$$\left|\mathbf{V}\right| = \begin{vmatrix} 4 & -2 \\ -2 & 1 \end{vmatrix} = 0 \tag{5.2.8.3}$$

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = \begin{vmatrix} 4 & -2 & -6 \\ -2 & 1 & 3 \\ -6 & 3 & 9 \end{vmatrix} = 0 \tag{5.2.8.4}$$

Therefore the given parabola equation is a degenerate.The quadratic equation corresponds to a pair of coincident straight lines.
The characteristic equation of $\mathbf{V}$ will be

$$\left|\mathbf{V} - \lambda\mathbf{I}\right| = \begin{vmatrix} 4-\lambda & -2 \\ -2 & 1-\lambda \end{vmatrix} \tag{5.2.8.5}$$

$$= \lambda^2 - 5\lambda \tag{5.2.8.6}$$

$$\lambda_1 = 0, \lambda_2 = 5 \tag{5.2.8.7}$$

The eigen vectors are the nullspace of the matrix $\mathbf{V} - \lambda\mathbf{I}$.For $\lambda_1 = 0$

$$\begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} \xleftrightarrow{R_2 = 2R_2 + R_1} \begin{pmatrix} 4 & -2 \\ 0 & 0 \end{pmatrix} \tag{5.2.8.8}$$

$$p_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{5.2.8.9}$$

Therefore the normalized eigen vector will be

$$p_1 = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix} \tag{5.2.8.10}$$

For $\lambda_2 = 5$

$$\begin{pmatrix} -1 & -2 \\ -2 & -4 \end{pmatrix} \xleftrightarrow{R_2 = R_2 - 2R_1} \begin{pmatrix} -1 & -2 \\ 0 & 0 \end{pmatrix} \tag{5.2.8.11}$$

$$p_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \tag{5.2.8.12}$$

Therefore the normalized eigen vector will be

$$p_2 = \begin{pmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} \tag{5.2.8.13}$$

Therefore the transformation matrix will be

$$\mathbf{P} = \begin{pmatrix} p_1 & p_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \tag{5.2.8.14}$$

The value of η will be

$$\eta = 2p_1^T\mathbf{u} \tag{5.2.8.15}$$

$$= 2\begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}\begin{pmatrix} -6 \\ 3 \end{pmatrix} \tag{5.2.8.16}$$

$$= 0 \tag{5.2.8.17}$$

A point on the line can be found by using to following formula

$$\begin{pmatrix} \mathbf{u}^T + \frac{\eta}{2}p_1^T \\ \mathbf{V} \end{pmatrix} c = \begin{pmatrix} -f \\ \frac{\eta}{2}p_1 - \mathbf{u} \end{pmatrix} \tag{5.2.8.18}$$

$$\begin{pmatrix} \mathbf{u}^T \\ \mathbf{V} \end{pmatrix} c = \begin{pmatrix} -f \\ -\mathbf{u} \end{pmatrix} \tag{5.2.8.19}$$

$$\begin{pmatrix} -6 & 3 \\ 4 & -2 \\ -2 & 1 \end{pmatrix} c = \begin{pmatrix} -9 \\ 6 \\ -3 \end{pmatrix} \tag{5.2.8.20}$$

Writing it in augmented form, we get

$$\begin{pmatrix} -6 & 3 & -9 \\ 4 & -2 & 6 \\ -2 & 1 & -3 \end{pmatrix} \xleftrightarrow{R_3 = R_3 - \frac{R_1}{3}} \begin{pmatrix} -6 & 3 & -9 \\ 4 & -2 & 6 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.2.8.21}$$

$$\xleftrightarrow{R_2 = \frac{3}{2}R_2 + R_1} \begin{pmatrix} -6 & 3 & -9 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.2.8.22}$$

Therefore we can see that the point $c = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ lies on the line. Equation of the straight line Applying affine transformation we get

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = -\eta\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{y} \tag{5.2.8.23}$$

$$\mathbf{y}^T\begin{pmatrix} 0 & 0 \\ 0 & 5 \end{pmatrix}\mathbf{y} = 0 \tag{5.2.8.24}$$

$$5y^2 = 0 \tag{5.2.8.25}$$

Therefore the transformed line is $y = 0$,which in vector form will be $\begin{pmatrix} 0 & 1 \end{pmatrix}\mathbf{y} = 0$.

Taking the Inverse affine transformation we get

$$\begin{pmatrix}0 & 1\end{pmatrix}\left(P^T(\mathbf{x}-c)\right) = 0 \tag{5.2.8.26}$$

$$\begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\end{pmatrix}(\mathbf{x}-c) = 0 \tag{5.2.8.27}$$

$$\begin{pmatrix}-\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\end{pmatrix}(\mathbf{x}-c) = 0 \tag{5.2.8.28}$$

$$\begin{pmatrix}-\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\end{pmatrix}\mathbf{x} - \begin{pmatrix}-\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\end{pmatrix}\begin{pmatrix}1 \\ -1\end{pmatrix} = 0 \tag{5.2.8.29}$$

$$\begin{pmatrix}-\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\end{pmatrix}\mathbf{x} + \frac{3}{\sqrt{5}} = 0 \tag{5.2.8.30}$$

$$\begin{pmatrix}2 & -1\end{pmatrix}\mathbf{x} = 3 \tag{5.2.8.31}$$

Therefore the equation of coincident lines is $(2x - y - 3) = 0$.

5.2.9. Trace the central conic,

$$2x^2 - 2xy + y^2 + 2x - 2y = 0 \tag{5.2.9.1}$$

Solution: The general equation of a second degree (In algebraic form) can be expressed as,

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{5.2.9.2}$$

The general equation of a second degree (In vector form) can be expressed as,

$$\mathbf{x}^{\mathbf{T}}\mathbf{V}\mathbf{x} + 2\mathbf{u}^{\mathbf{T}}\mathbf{x} + f = 0 \tag{5.2.9.3}$$

Comparing (5.2.9.1) with (5.2.9.2) , we get,

$$a = 2,\ b = -1,\ c = 1,\ d = 1,\ e = -1\ and\ f = 0 \tag{5.2.9.4}$$

where,

$$\mathbf{V} = \begin{pmatrix}a & b \\ b & c\end{pmatrix} = \begin{pmatrix}2 & -1 \\ -1 & 1\end{pmatrix} = \mathbf{V}^{\mathbf{T}} \tag{5.2.9.5}$$

$$\implies \mathbf{V} = \begin{pmatrix}2 & -1 \\ -1 & 1\end{pmatrix} \tag{5.2.9.6}$$

and

$$\mathbf{u} = \begin{pmatrix}1 \\ -1\end{pmatrix} \tag{5.2.9.7}$$

Finding the determinant of V we obtain,

$$|\mathbf{V}| = 1 > 0 \tag{5.2.9.8}$$

which means the given central conic is an ellipse which can be proven more effectively using,

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^{\mathbf{T}} \tag{5.2.9.9}$$

where **P** is a matrix of Eigen vectors and **D** is a diagonal matrix of Eigen values which will be computed subsequently.
Computing Eigen values for **V** using the characteristic equation of the matrix, we get the following quadratic equation in terms of λ

$$\lambda^2 - 3\lambda + 1 = 0 \tag{5.2.9.10}$$

$$\implies \lambda_1 = \frac{3+\sqrt{5}}{2}\ and\ \lambda_2 = \frac{3-\sqrt{5}}{2} \tag{5.2.9.11}$$

Eigen vectors can be computed using the following equation,

$$(\lambda\mathbf{I} - \mathbf{V})\mathbf{p} = 0 \tag{5.2.9.12}$$

Solving this for λ_1 and λ_2 respectively and normalizing them we obtain,

$$\mathbf{p_1} = \sqrt{\frac{2}{5-\sqrt{5}}}\begin{pmatrix}1 \\ \frac{1-\sqrt{5}}{2}\end{pmatrix} \tag{5.2.9.13}$$

$$\mathbf{p_2} = \sqrt{\frac{2}{5+\sqrt{5}}}\begin{pmatrix}1 \\ \frac{\sqrt{5}+1}{2}\end{pmatrix} \tag{5.2.9.14}$$

Simplifying,

$$\implies \mathbf{P} = \begin{pmatrix}\sqrt{\frac{2}{5-\sqrt{5}}} & \sqrt{\frac{2}{5+\sqrt{5}}} \\ \frac{1-\sqrt{5}}{\sqrt{5\sqrt{2}-\sqrt{10}}} & \frac{1+\sqrt{5}}{\sqrt{5\sqrt{2}+\sqrt{10}}}\end{pmatrix} \tag{5.2.9.15}$$

$$\mathbf{D} = \begin{pmatrix}\frac{3+\sqrt{5}}{2} & 0 \\ 0 & \frac{3-\sqrt{5}}{2}\end{pmatrix} \tag{5.2.9.16}$$

Using (5.2.9.9) can verify that it holds which means that the given central conic is an ellipse.

The center of the ellipse can be computed using,

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \tag{5.2.9.17}$$

$$\implies \mathbf{c} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{5.2.9.18}$$

The parameters of the ellipse are computed as follows,

$$\sqrt{\frac{\mathbf{u^T V^{-1} u} - f}{\lambda_1}} = \sqrt{\frac{3 - \sqrt{5}}{2}} \tag{5.2.9.19}$$

$$\sqrt{\frac{\mathbf{u^T V^{-1} u} - f}{\lambda_2}} = \sqrt{\frac{3 + \sqrt{5}}{2}} \tag{5.2.9.20}$$

The angle of Rotation can be obtained by equating $\mathbf{P}$ with the Rotation matrix which is,

$$\mathbf{P} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & cos\theta \end{pmatrix} \tag{5.2.9.21}$$

Comparing (5.2.9.15) and (5.2.9.21) we get,

$$\theta = \frac{\pi}{5.66} \tag{5.2.9.22}$$

Using the Affine transformation we find out the actual ellipse,

$$\mathbf{y} = \mathbf{P^T x} + \mathbf{c} \tag{5.2.9.23}$$

which means the actual ellipse is obtained by translating and rotating the standard ellipse w.r.t center, $\mathbf{c}$ from (5.2.9.18) and angle of rotation, θ from (5.2.9.22) respectively.
Using the above data along with $\mathbf{o}$ (Origin), the center of the standard ellipse, the actual ellipse is plotted as follows.

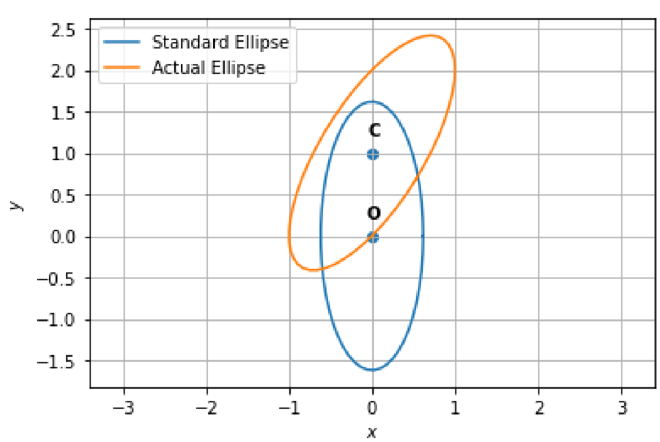

Fig. 5.2.9.1: Standard and Actual Ellipses

5.2.10. Trace the following central conic :

$$x^2 + y^2 + xy + x + y = 1 \tag{5.2.10.1}$$

Solution: General equation of second degree is given by :

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{5.2.10.2}$$

In the vector form (5.2.10.1) can be written as :

$$\mathbf{x}^T\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}\mathbf{x} + 2\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}^T \mathbf{x} - 1 = 0 \tag{5.2.10.3}$$

By comparing (5.2.10.2) and (5.2.10.3) we get :

$$\mathbf{V} = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, f = -1 \tag{5.2.10.4}$$

Eigen values for matrix $\mathbf{V}$ can be calculated by solving :

$$\begin{vmatrix} 1-\lambda & \frac{1}{2} \\ \frac{1}{2} & 1-\lambda \end{vmatrix} = 0 \tag{5.2.10.5}$$

$$\lambda^2 - 2\lambda + \frac{3}{4} = 0 \tag{5.2.10.6}$$

$$\lambda_1 = \frac{3}{2}, \lambda_2 = \frac{1}{2} \tag{5.2.10.7}$$

By doing Eigenvalue Decomposition and Affine Transformation we get :

$$\mathbf{P}^{-1}\mathbf{V}\mathbf{P} = \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{5.2.10.8}$$

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c} \tag{5.2.10.9}$$

Where the matrix $\mathbf{P}$ is normalised eigenbasis and $\mathbf{c}$ is the center.
By putting the value of $\mathbf{x}$ from (5.2.10.9) in (5.2.10.2) we get :

$$(\mathbf{P}\mathbf{y} + \mathbf{c})^T\mathbf{V}(\mathbf{P}\mathbf{y} + \mathbf{c}) + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{5.2.10.10}$$

Further solving this we get :

$$\mathbf{V}\mathbf{c} + \mathbf{u} = 0 \implies \mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \tag{5.2.10.11}$$

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f \tag{5.2.10.12}$$

As

$$\left|\mathbf{V}\right| = \begin{vmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{vmatrix} = \frac{3}{4} > 0 \tag{5.2.10.13}$$

Equation (5.2.10.12) forms an ellipse centered at origin with major and minor axis given as :

$$a = \sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}} \tag{5.2.10.14}$$

$$b = \sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_2}} \tag{5.2.10.15}$$

Using Gauss Jordan Elimination on matrix $\mathbf{V}$:

$$\xleftrightarrow{R_2 \leftarrow \frac{1}{2}R_1 - R_2} \begin{pmatrix} 1 & \frac{1}{2} & : & 1 & 0 \\ 0 & \frac{-3}{4} & : & \frac{1}{2} & -1 \end{pmatrix} \tag{5.2.10.16}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{-4}{3}R_2} \begin{pmatrix} 1 & \frac{1}{2} & : & 1 & 0 \\ 0 & 1 & : & \frac{-2}{3} & \frac{4}{3} \end{pmatrix} \tag{5.2.10.17}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - \frac{1}{2}R_2} \begin{pmatrix} 1 & 0 & : & \frac{4}{3} & \frac{-2}{3} \\ 0 & 1 & : & \frac{-2}{3} & \frac{4}{3} \end{pmatrix} \tag{5.2.10.18}$$

Therefore,

$$\mathbf{V}^{-1} = \begin{pmatrix} \frac{4}{3} & \frac{-2}{3} \\ \frac{4}{3} & \frac{-2}{3} \end{pmatrix} \tag{5.2.10.19}$$

Using (5.2.10.11) and (5.2.10.19) we get :

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} = -\begin{pmatrix} \frac{4}{3} & \frac{-2}{3} \\ \frac{4}{3} & \frac{-2}{3} \end{pmatrix}\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{-3}{10} \\ \frac{-3}{10} \end{pmatrix} \tag{5.2.10.20}$$

By putting the values of $\mathbf{u}$, $\mathbf{V}^{-1}$, f, λ_1 and λ_2 in (5.2.10.14) and (5.2.10.15) respectively we get :

$$a = \sqrt{\frac{\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} \frac{4}{3} & \frac{-2}{3} \\ \frac{4}{3} & \frac{-2}{3} \end{pmatrix}\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} - 1}{\frac{3}{2}}} = \frac{9}{10} \tag{5.2.10.21}$$

$$b = \sqrt{\frac{\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} \frac{4}{3} & \frac{-2}{3} \\ \frac{4}{3} & \frac{-2}{3} \end{pmatrix}\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} - 1}{\frac{1}{2}}} = \frac{8}{5} \tag{5.2.10.22}$$

In the transformed space with Eigenbasis, an ellipse centered at origin with major and minor axis as a and b is traced as 'Standard Ellipse' in the plot.

And after doing Affine Transformation on $\mathbf{y}$ as in (5.2.10.9) we get our 'Actual Ellipse' centered at $\mathbf{c}$ shown in the plot.

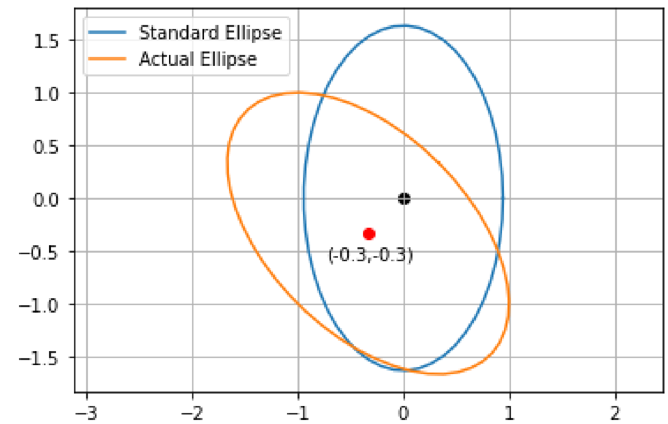

Fig. 5.2.10.1: Standard Ellipse centered at origin and Actual Ellipse centered at $(-0.3, -0.3)$.

5.2.11. Trace the following central conic:

$$2x^2 + 3xy - 2y^2 - 7x + y - 2 = 0 \tag{5.2.11.1}$$

Solution: Any second degree equation of the form:

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{5.2.11.2}$$

Can be represented in matrix / vector form as:

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{5.2.11.3}$$

where,

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \tag{5.2.11.4}$$

$$\mathbf{u} = \begin{pmatrix} d & e \end{pmatrix} \tag{5.2.11.5}$$

Rewriting (5.2.11.1) in matrix form, we get:

$$\mathbf{x}^T\begin{pmatrix} 2 & \frac{3}{2} \\ \frac{3}{2} & -2 \end{pmatrix}\mathbf{x} + 2\begin{pmatrix} -\frac{7}{2} & \frac{1}{2} \end{pmatrix} - 2 = 0 \tag{5.2.11.6}$$

where,

$$\mathbf{V} = \begin{pmatrix} 2 & \frac{3}{2} \\ \frac{3}{2} & -2 \end{pmatrix} \tag{5.2.11.7}$$

$$\mathbf{u} = \begin{pmatrix} -\frac{7}{2} \\ \frac{1}{2} \end{pmatrix} \tag{5.2.11.8}$$

$$f = -2 \tag{5.2.11.9}$$

$$det(\mathbf{V}) = \begin{vmatrix} 2 & \frac{3}{2} \\ \frac{3}{2} & -2 \end{vmatrix} = -\frac{25}{4} \tag{5.2.11.10}$$

As $det(\mathbf{V}) < 0$, the given conic represents a hyperbola.
The characteristic equation of $\mathbf{V}$ is given by the determinant:

$$\left|\mathbf{V} - \lambda\mathbf{I}\right| = 0 \tag{5.2.11.11}$$

$$\begin{vmatrix} 2-\lambda & \frac{3}{2} \\ \frac{3}{2} & -2-\lambda \end{vmatrix} = 0 \tag{5.2.11.12}$$

$$\implies \lambda^2 - \frac{25}{4} = 0 \tag{5.2.11.13}$$

The roots of (5.2.11.13) (the eigenvalues) are:

$$\lambda_1 = \frac{5}{2}, \lambda_2 = -\frac{5}{2} \tag{5.2.11.14}$$

The eigenvector $\mathbf{p}$ is defined as:

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \tag{5.2.11.15}$$

$$\implies (\mathbf{V} - \lambda\mathbf{I})\mathbf{p} = 0 \tag{5.2.11.16}$$

Evaluating (5.2.11.16) for $\lambda_1 = \frac{5}{2}$, we get:

$$(\mathbf{V} - \lambda_1\mathbf{I}) = \begin{pmatrix} -\frac{1}{2} & \frac{3}{2} \\ \frac{3}{2} & -\frac{9}{2} \end{pmatrix} \tag{5.2.11.17}$$

Reducing the above equation to row-echelon form, we get:

$$\xleftrightarrow{R_2 \to R_2 + 3R_1} \begin{pmatrix} -\frac{1}{2} & \frac{3}{2} \\ 0 & 0 \end{pmatrix} \xleftrightarrow{R_1 \to -2R_1} \begin{pmatrix} 1 & -3 \\ 0 & 0 \end{pmatrix} \tag{5.2.11.18}$$

Substituting (5.2.11.18) in (5.2.11.16), we get:

$$\begin{pmatrix} 1 & -3 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{5.2.11.19}$$

where,

$$\mathbf{p} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{5.2.11.20}$$

Let $v_2 = t$. Then

$$v_1 = 3t \tag{5.2.11.21}$$

Let $t = 1$. The eigenvector $\mathbf{p_1}$ is:

$$\mathbf{p_1} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \tag{5.2.11.22}$$

Similarly for $\lambda_2 = -\frac{5}{2}$, we get:

$$(\mathbf{V} - \lambda_2\mathbf{I}) = \begin{pmatrix} \frac{9}{2} & \frac{3}{2} \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix} \xleftrightarrow[R_1 \to \frac{2}{9}R_1]{R_2 \to 3R_2 - R_1} \begin{pmatrix} 1 & \frac{1}{3} \\ 0 & 0 \end{pmatrix} \tag{5.2.11.23}$$

Substituting (5.2.11.23) in (5.2.11.16), we get:

$$\begin{pmatrix} 1 & \frac{1}{3} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{5.2.11.24}$$

where,

$$\mathbf{p} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{5.2.11.25}$$

Let $v_2 = t$. Then

$$v_1 = \frac{-t}{3} \tag{5.2.11.26}$$

Let $t = 1$. The eigenvector $\mathbf{p_2}$ is:

$$\mathbf{p_2} = \begin{pmatrix} \frac{-1}{3} \\ 1 \end{pmatrix} \tag{5.2.11.27}$$

As $\mathbf{V} = \mathbf{V}^T$, there exists an orthogonal matrix P such that:

$$\mathbf{P}\mathbf{V}\mathbf{P}^T = \mathbf{D} = diag(\lambda_1, \lambda_2) \tag{5.2.11.28}$$

$\mathbf{V}$ can be rewritten using the above equation as:

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T \tag{5.2.11.29}$$

where,

$$\mathbf{P} = \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} \end{pmatrix} \tag{5.2.11.30}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{5.2.11.31}$$

Substituting the values -

$$\mathbf{P} = \begin{pmatrix} 3 & \frac{-1}{3} \\ 1 & 1 \end{pmatrix} \tag{5.2.11.32}$$

$$\mathbf{D} = \begin{pmatrix} \frac{5}{2} & 0 \\ 0 & \frac{-5}{2} \end{pmatrix} \tag{5.2.11.33}$$

The center of hyperbola is given by:

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \tag{5.2.11.34}$$

$$\implies \mathbf{c} = -\begin{pmatrix} \frac{8}{25} & \frac{6}{25} \\ \frac{6}{25} & \frac{-8}{25} \end{pmatrix}\begin{pmatrix} \frac{-7}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{5.2.11.35}$$

As

$$\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f = 5 > 0 \tag{5.2.11.36}$$

there is no requirement for swapping the axes (which will be evident from the equation below). The axes of the hyperbola are given by:

$$axes = \begin{cases} \sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}-f}{\lambda_1}} \\ \sqrt{\frac{f-\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}}{\lambda_2}} \end{cases} \tag{5.2.11.37}$$

$$\implies \sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}} = \sqrt{2} \tag{5.2.11.38}$$

$$\implies \sqrt{\frac{f - \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}}{\lambda_2}} = \sqrt{2} \tag{5.2.11.39}$$

The standard form of conic is written as:

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f \tag{5.2.11.40}$$

where,

$$\mathbf{y} = \mathbf{P}^T(\mathbf{x} - \mathbf{c}) \tag{5.2.11.41}$$

$$\implies \mathbf{y}^T\begin{pmatrix} \frac{5}{2} & 0 \\ 0 & \frac{-5}{2} \end{pmatrix}\mathbf{y} - 5 = 0 \tag{5.2.11.42}$$

The plot of both the conics are given below:

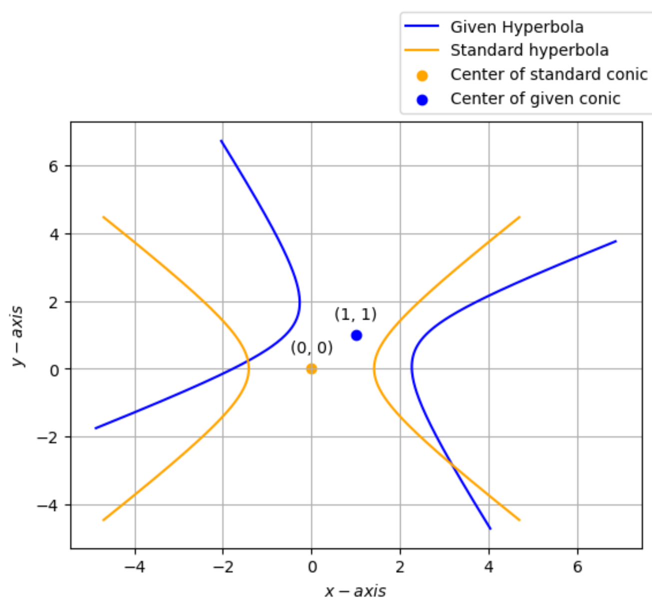

Fig. 5.2.11.1: Plot of given hyperbola and the standard hyperbola

5.2.12. Trace the following central conics:

$$40x^2 + 36xy + 25y^2 - 196x - 122y + 205 = 0 \tag{5.2.12.1}$$

Solution: The general equation of a second degree can be expressed as:

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{5.2.12.2}$$

$$\implies \mathbf{x^T}\mathbf{V}\mathbf{x} + 2\mathbf{u^T}\mathbf{x} + f = 0 \tag{5.2.12.3}$$

where

$$\mathbf{V} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \mathbf{u} = \begin{pmatrix} d \\ e \end{pmatrix} \tag{5.2.12.4}$$

The given equation of the curve can be expressed as:

$$40x^2 + 2(18)xy + 25y^2 + 2(-98)x + 2(-61)y + 205 = 0 \tag{5.2.12.5}$$

Comparing (5.2.12.2),(5.2.12.4) and (5.2.12.5):

$$\mathbf{V} = \begin{pmatrix} 40 & \sqrt{18} \\ \sqrt{18} & 25 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} -98 \\ -61 \end{pmatrix} and \quad f = 205 \tag{5.2.12.6}$$

$$\implies \left|\mathbf{V}\right| = 982 \quad and \quad b^2 - ac = 18 - 40.25 = -982 \tag{5.2.12.7}$$

Since $\left|\mathbf{V}\right| > 0$ and $b^2 < ac$, (5.2.12.5) represent an ellipse.
The characteristic equation of $\mathbf{V}$ is given as follows,

$$\left|\lambda\mathbf{I} - \mathbf{V}\right| = \begin{vmatrix} \lambda - 40 & \sqrt{18} \\ \sqrt{18} & \lambda - 25 \end{vmatrix} = 0 \tag{5.2.12.8}$$

$$\implies \lambda^2 - 65\lambda + 982 = 0 \tag{5.2.12.9}$$

Hence the characteristic equation of $\mathbf{V}$ is given by (5.2.12.9). The roots of (5.2.12.9) i.e the eigenvalues are given by

$$\lambda_1 = \frac{65 + \sqrt{297}}{2}, \lambda_2 = \frac{65 - \sqrt{297}}{2} \tag{5.2.12.10}$$

The eigen vector $\mathbf{p}$ is defined as,

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \tag{5.2.12.11}$$

$$\implies (\lambda\mathbf{I} - \mathbf{V})\,\mathbf{p} = 0 \tag{5.2.12.12}$$

for $\lambda_1 = \frac{65+\sqrt{297}}{2}$,

$$(\lambda_1\mathbf{I} - \mathbf{V}) = \begin{pmatrix} \frac{\sqrt{297}-15}{2} & -\sqrt{18} \\ -\sqrt{18} & \frac{\sqrt{297}+15}{2} \end{pmatrix} \quad (5.2.12.13)$$

$$\xleftrightarrow{R_2 = R_2 + \frac{2\sqrt{18}}{\sqrt{297}-15}R_1} \begin{pmatrix} \frac{\sqrt{297}-15}{2} & -\sqrt{18} \\ 0 & 0 \end{pmatrix} \quad (5.2.12.14)$$

From (5.2.12.12) and (5.2.12.14)

$$\implies \mathbf{p_1} = \begin{pmatrix} \sqrt{18} \\ \frac{\sqrt{297}-15}{2} \end{pmatrix} \quad (5.2.12.15)$$

For $\lambda_2 = \frac{65-\sqrt{297}}{2}$

$$(\lambda_2\mathbf{I} - \mathbf{V}) = \begin{pmatrix} \frac{-\sqrt{297}-15}{2} & -\sqrt{18} \\ -\sqrt{18} & \frac{15-\sqrt{297}}{2} \end{pmatrix} \quad (5.2.12.16)$$

$$\xleftrightarrow[R_1 = -R_1]{R_2 = R_2 + \frac{2\sqrt{18}}{\sqrt{297}+15}R_1} \begin{pmatrix} \frac{\sqrt{297}+15}{2} & \sqrt{18} \\ 0 & 0 \end{pmatrix} \quad (5.2.12.17)$$

$$\implies \mathbf{p_2} = \begin{pmatrix} -\sqrt{18} \\ \frac{\sqrt{297}+15}{2} \end{pmatrix} \quad (5.2.12.18)$$

using the affine transformation

$$\mathbf{x} = \mathbf{P}\mathbf{y} + c' \quad (5.2.12.19)$$

such that

$$\mathbf{P}^T\mathbf{V}\mathbf{P} = \mathbf{D} \quad and \quad \mathbf{P} = \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} \end{pmatrix}, \quad \mathbf{P}^T = \mathbf{P}^{-1} \quad (5.2.12.20)$$

Where $\mathbf{D}$ is a diagonal matrix, we get

$$\mathbf{D} = \begin{pmatrix} \frac{65+\sqrt{297}}{2} & 0 \\ 0 & \frac{65-\sqrt{297}}{2} \end{pmatrix} \quad (5.2.12.21)$$

Now (5.2.12.3) can be written as,

$$\mathbf{y^T D y} = \mathbf{u^T V^{-1} u} - f \quad \left|\mathbf{V}\right| \neq 0 \quad (5.2.12.22)$$

And,

$$\mathbf{c}' = -\mathbf{V^{-1}u} \quad \left|\mathbf{V}\right| \neq 0 \quad (5.2.12.23)$$

$$\mathbf{y} = \mathbf{P^T}(\mathbf{x} - \mathbf{c}) \quad (5.2.12.24)$$

The centre of the conic section in (5.2.12.5) is given by $\mathbf{c}'$ in (5.2.12.23). We compute $\mathbf{V^{-1}}$ as follows,

$$\left(\begin{array}{cc|cc} 40 & \sqrt{18} & 1 & 0 \\ \sqrt{18} & 25 & 0 & 1 \end{array}\right) \xleftrightarrow[R_1 = \frac{1}{40}R_1]{R_2 = R_2 - \frac{\sqrt{18}}{40}R_1} \left(\begin{array}{cc|cc} 1 & \frac{\sqrt{18}}{40} & \frac{1}{40} & 0 \\ 0 & \frac{982}{40} & -\frac{\sqrt{18}}{40} & 1 \end{array}\right) \quad (5.2.12.25)$$

$$\xleftrightarrow[R_1 = R_1 - \frac{\sqrt{18}}{40}R_2]{R_2 = \frac{40}{982}R_2} \left(\begin{array}{cc|cc} 1 & 0 & \frac{25}{982} & -\frac{\sqrt{18}}{982} \\ 0 & 1 & -\frac{\sqrt{18}}{982} & \frac{40}{982} \end{array}\right) \quad (5.2.12.26)$$

Hence $\mathbf{V^{-1}}$ is given by,

$$\mathbf{V^{-1}} = \begin{pmatrix} \frac{25}{982} & -\frac{\sqrt{18}}{982} \\ -\frac{\sqrt{18}}{982} & \frac{40}{982} \end{pmatrix} \quad (5.2.12.27)$$

Now $\mathbf{u^T V^{-1} u}$ is given by,

$$\mathbf{u^T V^{-1} u} = \frac{1}{982}\begin{pmatrix} -98 & -61 \end{pmatrix}\begin{pmatrix} 25 & -\sqrt{18} \\ -\sqrt{18} & 40 \end{pmatrix}\begin{pmatrix} -98 \\ -61 \end{pmatrix} \quad (5.2.12.28)$$

$$= 344.4203 \quad (5.2.12.29)$$

And, $\mathbf{V^{-1}u}$ is given by,

$$\mathbf{V^{-1}u} = \frac{1}{982}\begin{pmatrix} 25 & -\sqrt{18} \\ -\sqrt{18} & 40 \end{pmatrix}\begin{pmatrix} -98 \\ -61 \end{pmatrix} \quad (5.2.12.30)$$

$$(5.2.12.31)$$

By putting the value of (5.2.12.30), the center of the ellipse is given by (5.2.12.23) as follows,

$$\mathbf{c}' = \begin{pmatrix} 2.231 \\ 2.061 \end{pmatrix} \quad (5.2.12.32)$$

Also the semi-major axis (a) and semi-minor axis (b) of the ellipse are given by,

$$a = \sqrt{\frac{\mathbf{u^T V^{-1} u} - f}{\lambda_1}} = 1.8414 \quad (5.2.12.33)$$

$$b = \sqrt{\frac{\mathbf{u^T V^{-1} u} - f}{\lambda_2}} = 2.416 \quad (5.2.12.34)$$

Finally from (5.2.12.22), the equation of ellipse is given by,

$$\mathbf{y^T}\begin{pmatrix} 41.116 & 0 \\ 0 & 23.883 \end{pmatrix}\mathbf{y} = 139.4203 \quad (5.2.12.35)$$

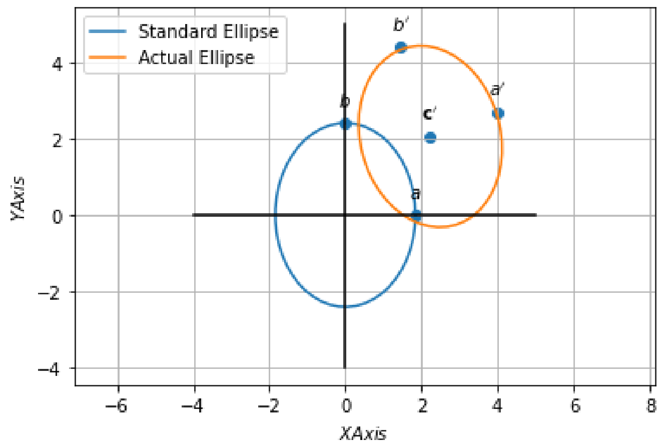

Fig. 5.2.12.1: Graphical representation of the actual curve $40x^2 + 36xy + 25y^2 - 196x - 122y + 205 = 0$, which represent an ellipse.

5.2.13. Trace the curve

$$35x^2 + 30y^2 + 32x - 108y - 12xy + 59 = 0 \tag{5.2.13.1}$$

Solution: The general equation of second degree is given by

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{5.2.13.2}$$

and can be expressed as

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{5.2.13.3}$$

where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \tag{5.2.13.4}$$

$$\mathbf{u}^T = \begin{pmatrix} d & e \end{pmatrix} \tag{5.2.13.5}$$

Comparing (5.2.13.1) with (5.2.13.2), we get

$$\mathbf{V} = \begin{pmatrix} 35 & -6 \\ -6 & 30 \end{pmatrix} \tag{5.2.13.6}$$

$$\mathbf{u}^T = \begin{pmatrix} 16 & -54 \end{pmatrix} \tag{5.2.13.7}$$

If $|\mathbf{V}| > 0$, then (5.2.13.3) is an ellipse.

$$|V| = \begin{vmatrix} 35 & -6 \\ -6 & 30 \end{vmatrix} = 1014 > 0 \tag{5.2.13.8}$$

(5.2.13.3) can be expressed as

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f \qquad |V| \neq 0 \tag{5.2.13.9}$$

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = -\eta\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{y} \qquad |V| = 0 \tag{5.2.13.10}$$

with center as

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \qquad |V| \neq 0 \tag{5.2.13.11}$$

Calculating the center for given curve we get,

$$\mathbf{c} = -\frac{1}{|35 * 30 - 6 * 6|}\begin{pmatrix} 30 & 6 \\ 6 & 35 \end{pmatrix}\begin{pmatrix} 16 \\ -54 \end{pmatrix} \tag{5.2.13.12}$$

$$= \frac{1}{1014}\begin{pmatrix} 156 \\ -1794 \end{pmatrix} \tag{5.2.13.13}$$

$$= \begin{pmatrix} \frac{2}{13} \\ \frac{-23}{13} \end{pmatrix} \tag{5.2.13.14}$$

For

$$|\mathbf{V}| > 0, \quad \text{or, } \lambda_1 > 0, \lambda_2 > 0 \tag{5.2.13.15}$$

(5.2.13.9) becomes

$$\lambda_1 y_1^2 + \lambda_2 y_1^2 = \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f \tag{5.2.13.16}$$

which is the equation of an ellipse with major and minor axes parameters

$$\sqrt{\frac{\lambda_1}{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}}, \sqrt{\frac{\lambda_2}{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}} \tag{5.2.13.17}$$

The characteristic equation of $\mathbf{V}$ is obtained by evaluating the determinant

$$\left|\lambda\mathbf{I} - \mathbf{V}\right| = \begin{vmatrix} \lambda - 35 & 6 \\ 6 & \lambda - 30 \end{vmatrix} = 0 \tag{5.2.13.18}$$

$$\implies \lambda^2 - 65\lambda + 1014 = 0 \tag{5.2.13.19}$$

The eigenvalues are the roots of (5.2.13.19) given by

$$\lambda_1 = 39, \lambda_2 = 26 \tag{5.2.13.20}$$

Calculating the major and minor axes lengths using (5.2.13.17), we get

$$\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} =$$
$$= \begin{pmatrix}16 - 54\end{pmatrix} \frac{1}{1014} \begin{pmatrix}30 & 6 \\ 6 & 35\end{pmatrix} \begin{pmatrix}16 \\ -54\end{pmatrix}$$
$$= \frac{1}{1014} \begin{pmatrix}16 & -54\end{pmatrix} \begin{pmatrix}156 \\ -1794\end{pmatrix}$$
$$= 98$$

$$\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f = 98 - 59 = 39 \quad (5.2.13.21)$$

$$\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}} = \sqrt{\frac{39}{39}} = 1 \quad (5.2.13.22)$$

$$\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_2}} = \sqrt{\frac{39}{26}} = \frac{\sqrt{6}}{2} \quad (5.2.13.23)$$

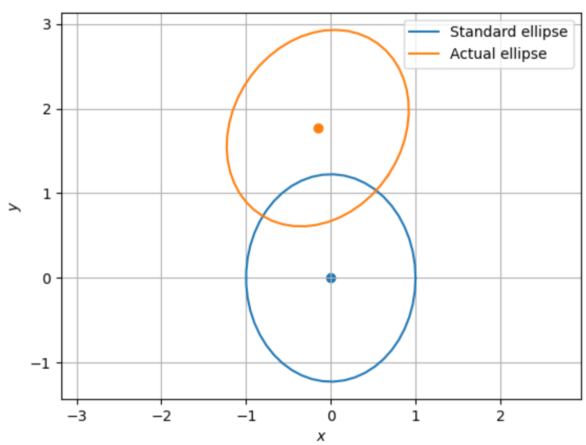

Fig. 5.2.13.1: Ellipse with center $\begin{pmatrix}\frac{2}{13} & \frac{-23}{13}\end{pmatrix}$ and having the axes lengths as 1 and $\frac{\sqrt{6}}{2}$

5.2.14. Trace the curve

$$14x^2 - 4xy + 11y^2 - 44x - 58y + 71 = 0 \quad (5.2.14.1)$$

Solution: The general equation of second degree is given by

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \quad (5.2.14.2)$$

and can be expressed as

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \quad (5.2.14.3)$$

where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix}a & b \\ b & c\end{pmatrix} \quad (5.2.14.4)$$
$$\mathbf{u}^T = \begin{pmatrix}d & e\end{pmatrix} \quad (5.2.14.5)$$

Comparing (5.2.14.1) with (5.2.14.2), we get

$$\mathbf{V} = \begin{pmatrix}14 & -2 \\ -2 & 11\end{pmatrix} \quad (5.2.14.6)$$
$$\mathbf{u}^T = \begin{pmatrix}-22 & -29\end{pmatrix} \quad (5.2.14.7)$$

If $|\mathbf{V}| > 0$, then (5.2.14.3) is an ellipse.

$$|V| = \begin{vmatrix}14 & -2 \\ -2 & 11\end{vmatrix} = 150 > 0 \quad (5.2.14.8)$$

(5.2.14.3) can be expressed as

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f \qquad |V| \neq 0 \quad (5.2.14.9)$$
$$\mathbf{y}^T\mathbf{D}\mathbf{y} = -\eta\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{y} \qquad |V| = 0 \quad (5.2.14.10)$$

with center as

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \qquad |V| \neq 0 \quad (5.2.14.11)$$

Calculating the center for given curve we get,

$$\mathbf{c} = -\frac{1}{|14 \times 11 - (-2 \times -2)|}\begin{pmatrix}11 & 2 \\ 2 & 14\end{pmatrix}\begin{pmatrix}-22 \\ -29\end{pmatrix} \quad (5.2.14.12)$$
$$= \frac{1}{150}\begin{pmatrix}300 \\ 450\end{pmatrix} \quad (5.2.14.13)$$
$$= \begin{pmatrix}2 \\ 3\end{pmatrix} \quad (5.2.14.14)$$

For

$$|\mathbf{V}| > 0, \quad \text{or, } \lambda_1 > 0, \lambda_2 > 0 \quad (5.2.14.15)$$

(5.2.14.9) becomes

$$\lambda_1 y_1^2 + \lambda_2 y_1^2 = \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f \quad (5.2.14.16)$$

which is the equation of an ellipse with major and minor axes parameters

$$\sqrt{\frac{\lambda_1}{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}}, \sqrt{\frac{\lambda_2}{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}} \quad (5.2.14.17)$$

The characteristic equation of $\mathbf{V}$ is obtained by evaluating the determinant

$$\left|\lambda\mathbf{I} - \mathbf{V}\right| = \begin{vmatrix} \lambda - 14 & 2 \\ 2 & \lambda - 11 \end{vmatrix} = 0 \quad (5.2.14.18)$$

$$\implies \lambda^2 - 25\lambda + 150 = 0 \quad (5.2.14.19)$$

The eigenvalues are the roots of (5.2.14.19) given by

$$\lambda_1 = 15, \lambda_2 = 10 \quad (5.2.14.20)$$

The eigenvector $\mathbf{p}$ is defined as

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \quad (5.2.14.21)$$

$$\implies (\lambda\mathbf{I} - \mathbf{V})\,\mathbf{p} = 0 \quad (5.2.14.22)$$

where λ is the eigenvalue. For $\lambda_1 = 15$,

$$(\lambda_1\mathbf{I} - \mathbf{V}) = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \quad (5.2.14.23)$$

$$\implies \mathbf{p}_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad (5.2.14.24)$$

such that $\|\mathbf{p}_1\| = 1$. Similarly, the eigenvector corresponding to λ_2 can be obtained as

$$\mathbf{p}_2 = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (5.2.14.25)$$

It is easy to verify that

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1} = \mathbf{P}\mathbf{D}\mathbf{P}^T \quad \because \mathbf{P}^{-1} = \mathbf{P}^T \quad (5.2.14.26)$$

$$\text{or, } \mathbf{D} = \mathbf{P}^T\mathbf{V}\mathbf{P} \quad (5.2.14.27)$$

where

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \quad (5.2.14.28)$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 15 & 0 \\ 0 & 10 \end{pmatrix} \quad (5.2.14.29)$$

Calculating the ellipse parameters using (5.2.14.17), we get

$$\begin{aligned} \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} &= \\ &= \begin{pmatrix} -22 & -29 \end{pmatrix} \frac{1}{150}\begin{pmatrix} 11 & 2 \\ 2 & 14 \end{pmatrix}\begin{pmatrix} -22 \\ -29 \end{pmatrix} \\ &= \frac{1}{150}\begin{pmatrix} 300 & 450 \end{pmatrix}\begin{pmatrix} 22 \\ 29 \end{pmatrix} \\ &= 131 \end{aligned}$$

$$\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f = 131 - 71 = 60 \quad (5.2.14.30)$$

$$\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}} = \sqrt{\frac{60}{15}} = 2 \quad (5.2.14.31)$$

$$\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_2}} = \sqrt{\frac{60}{10}} = \sqrt{6} \quad (5.2.14.32)$$

Thus, the given curve is found to be an ellipse from (5.2.14.8) with center at $\begin{pmatrix} 2 & 3 \end{pmatrix}$ and the major and minor axes lengths are calculated as $\sqrt{6}$, 2. An ellipse with these parameters along with one having center as origin are plotted as shown.

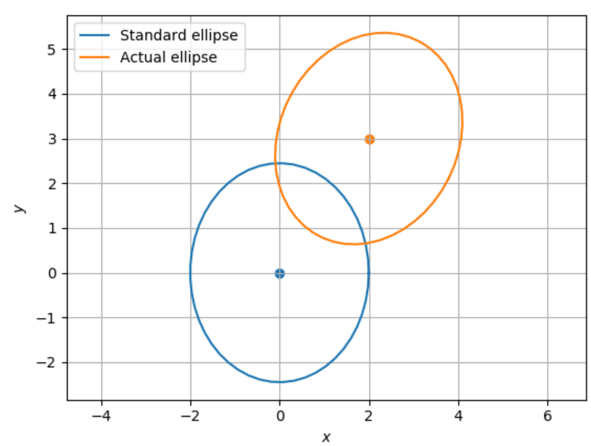

Fig. 5.2.14.1: Ellipse with center (2 3) and having the axes lengths as $\sqrt{6}$ and 2 along with an ellipse with center as origin

5.2.15. Trace the following

$$x^2 - 3xy + y^2 + 10x - 10y + 21 = 0 \quad (5.2.15.1)$$

Solution: The given quadratic equation can be written in the matrix form as

$$\mathbf{x}^T \begin{pmatrix} 1 & -\frac{3}{2} \\ -\frac{3}{2} & 1 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} 5 & -5 \end{pmatrix} \mathbf{x} + 21 = 0 \tag{5.2.15.2}$$

Calculating the parameters,we get

$$\left|\mathbf{V}\right| = \begin{vmatrix} 1 & -\frac{3}{2} \\ -\frac{3}{2} & 1 \end{vmatrix} = -\frac{5}{4} \tag{5.2.15.3}$$

Since, $\left|\mathbf{V}\right| < 0$, therefore the given equation represents a hyperbola.
The characteristic equation of $\mathbf{V}$ will be

$$\left|\mathbf{V} - \lambda\mathbf{I}\right| = \begin{vmatrix} 1-\lambda & -\frac{3}{2} \\ -\frac{3}{2} & 1-\lambda \end{vmatrix} = 0 \tag{5.2.15.4}$$

$$\implies 4\lambda^2 - 8\lambda - 5 = 0 \tag{5.2.15.5}$$

$$\implies \lambda_1 = \frac{5}{2}, \lambda_2 = -\frac{1}{2} \tag{5.2.15.6}$$

The eigen vector $\mathbf{p}$ is given by

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \tag{5.2.15.7}$$

$$\implies \mathbf{V} - \lambda\mathbf{I}\mathbf{p} = 0 \tag{5.2.15.8}$$

For $\lambda_1 = \frac{5}{2}$

$$\mathbf{V} - \lambda\mathbf{I} = \begin{pmatrix} 1-\frac{5}{2} & -\frac{3}{2} \\ -\frac{3}{2} & 1-\frac{5}{2} \end{pmatrix} \tag{5.2.15.9}$$

$$= \begin{pmatrix} -\frac{3}{2} & -\frac{3}{2} \\ -\frac{3}{2} & -\frac{3}{2} \end{pmatrix} \tag{5.2.15.10}$$

$$\begin{pmatrix} -\frac{3}{2} & -\frac{3}{2} \\ -\frac{3}{2} & -\frac{3}{2} \end{pmatrix} \xleftrightarrow{R_2=R_2-R_1} \begin{pmatrix} -\frac{3}{2} & -\frac{3}{2} \\ 0 & 0 \end{pmatrix} \tag{5.2.15.11}$$

$$\xleftrightarrow{R_1=R_1/-\frac{3}{2}} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \tag{5.2.15.12}$$

Substituting (5.2.15.12) in (5.2.15.8) we get

$$\mathbf{p_1} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{5.2.15.13}$$

Therefore the normalized eigen vector will be

$$\mathbf{p_1} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \tag{5.2.15.14}$$

For $\lambda_2 = -\frac{1}{2}$

$$\mathbf{V} - \lambda\mathbf{I} = \begin{pmatrix} 1+\frac{1}{2} & -\frac{3}{2} \\ -\frac{3}{2} & 1+\frac{1}{2} \end{pmatrix} \tag{5.2.15.15}$$

$$= \begin{pmatrix} \frac{3}{2} & -\frac{3}{2} \\ -\frac{3}{2} & -\frac{3}{2} \end{pmatrix} \tag{5.2.15.16}$$

$$\begin{pmatrix} -\frac{3}{2} & -\frac{3}{2} \\ -\frac{3}{2} & -\frac{3}{2} \end{pmatrix} \xleftrightarrow{R_2=R_2+R_1} \begin{pmatrix} -\frac{3}{2} & -\frac{3}{2} \\ 0 & 0 \end{pmatrix} \tag{5.2.15.17}$$

$$\xleftrightarrow{R_1=R_1/\frac{3}{2}} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \tag{5.2.15.18}$$

Substituting (5.2.15.18) in (5.2.15.8) we get

$$\mathbf{p_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{5.2.15.19}$$

Therefore the normalized eigen vector will be

$$\mathbf{p_2} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \tag{5.2.15.20}$$

Eigen decomposition
Since $\mathbf{V} = \mathbf{V}^T$ there exists an orthogonal matrix P such that

$$\mathbf{P}\mathbf{P}^T = \mathbf{I} \tag{5.2.15.21}$$

$$\mathbf{P}\mathbf{V}\mathbf{P}^T = \mathbf{D} = diag\left(\lambda_1\lambda_2\right) \tag{5.2.15.22}$$

or equivalently

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T \tag{5.2.15.23}$$

As

$$\mathbf{P} = \begin{pmatrix} p_1 & p_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \tag{5.2.15.24}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{5.2.15.25}$$

$$\implies \mathbf{D} = \begin{pmatrix} \frac{5}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \tag{5.2.15.26}$$

$$\mathbf{C} = -\mathbf{V}^{-1}\mathbf{u} \tag{5.2.15.27}$$

$$\implies \mathbf{C} = \begin{pmatrix} -\frac{4}{5} & -\frac{6}{5} \\ -\frac{6}{5} & -\frac{4}{5} \end{pmatrix} \begin{pmatrix} -5 \\ 5 \end{pmatrix} \tag{5.2.15.28}$$

$$= \begin{pmatrix} -2 \\ 2 \end{pmatrix} \tag{5.2.15.29}$$

$\therefore$ Centre C is given by:

$$\begin{pmatrix} -2 \\ 2 \end{pmatrix} \tag{5.2.15.30}$$

Now Equation (5.2.15.1) can be written as

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - \mathbf{f} \tag{5.2.15.31}$$
$$\tag{5.2.15.32}$$

where y is given by:

$$\mathbf{y} = \mathbf{P}^T(\mathbf{x} - \mathbf{c}) \tag{5.2.15.33}$$

So

$$\mathbf{y}^T\begin{pmatrix} \frac{5}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}\mathbf{y} = -1 \tag{5.2.15.34}$$

$$\implies \mathbf{y}^T\begin{pmatrix} \frac{5}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}\mathbf{y} + 1 = 0 \tag{5.2.15.35}$$

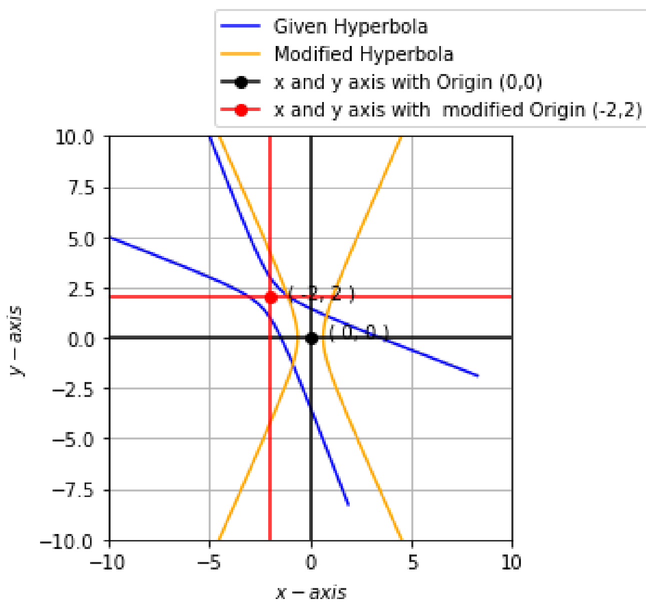

Fig. 5.2.15.1: Hyperbola plot when origin is shifted