

# Solutions to Plane Coordinate Geometry by S L Loney

G V V Sharma\*

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**Abstract**—This book provides a vector approach to analytical geometry. The content and exercises are based on S L Loney's book on Plane Coordinate Geometry.

## 1 COORDINATES

### 1.1 1

- 1.1.1. Find the distance between the following pair of points (2,3) and (5,7).

**Solution:**

- 1.1.2. Find the coordinates of the point which divides, internally and externally, the line joining (-3,-4) to (-8,7) in the ratio 7:5

**Solution:**

Let

$$\mathbf{A} = \begin{pmatrix} -3 \\ -4 \end{pmatrix} \quad (1.1.2.1)$$

### 1.2 2

- 1.2.1. Find the areas of the triangles the coordinates of whose angular points are respectively:

$$\mathbf{P} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} -7 \\ 6 \end{pmatrix} \text{ and } \mathbf{R} = \begin{pmatrix} 5 \\ -1 \end{pmatrix} \quad (1.2.1.1)$$

$$\mathbf{B} = \begin{pmatrix} -8 \\ 7 \end{pmatrix} \quad (1.1.2.2)$$

- a) Using section formula for internal division,

$$\mathbf{S} = \frac{7 \begin{pmatrix} -8 \\ 7 \end{pmatrix} + 5 \begin{pmatrix} -3 \\ -4 \end{pmatrix}}{(7+5)} \quad (1.1.2.3)$$

$$= \frac{1}{12} \begin{pmatrix} -71 \\ 29 \end{pmatrix} \quad (1.1.2.4)$$

- b) Similarly, for external division,

$$\mathbf{S} = \frac{7 \begin{pmatrix} -8 \\ 7 \end{pmatrix} - 5 \begin{pmatrix} -3 \\ -4 \end{pmatrix}}{(7-5)} = \frac{1}{2} \begin{pmatrix} -41 \\ 69 \end{pmatrix} \quad (1.1.2.5)$$

Fig. 1.1.2.1 plots the desired points.

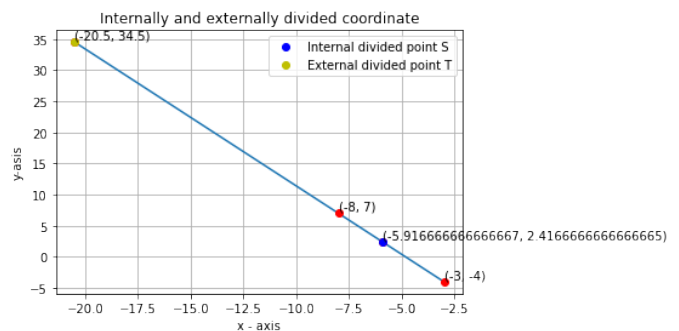


Fig. 1.1.2.1: Plot of coordinate of the point which divides internally and externally

\*The author is with the Department of Electrical Engineering, Indian Institute of Technology, Hyderabad 502285 India e-mail: gadepall@iith.ac.in. All content in this manual is released under GNU GPL. Free and open source.

**Solution:**

$$\mathbf{Q} - \mathbf{P} = \begin{pmatrix} -7 \\ 6 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad (1.2.1.2)$$

$$= \begin{pmatrix} -8 \\ 3 \end{pmatrix} \quad \mathbf{R} - \mathbf{P} = \begin{pmatrix} 5 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad (1.2.1.3)$$

$$= \begin{pmatrix} 4 \\ -4 \end{pmatrix} \quad (1.2.1.4)$$

$$\therefore \text{Area of the Triangle} = \frac{1}{2} \|(\mathbf{Q} - \mathbf{P}) \times (\mathbf{R} - \mathbf{P})\| \quad (1.2.1.5)$$

As the vector cross product of two vectors can also be expressed as the product of a skew-symmetric matrix and a vector.

$$\mathbf{A} \times \mathbf{B} = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \quad (1.2.1.6)$$

Substituting values from equation 1.2.1.2 and 1.2.1.3 in above equation 1.2.1.6, we'll get

$$\begin{aligned} (\mathbf{Q} - \mathbf{P}) \times (\mathbf{R} - \mathbf{P}) &= \begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & 8 \\ -3 & -8 & 0 \end{pmatrix} \times \begin{pmatrix} 4 \\ -4 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ 20 \end{pmatrix} \end{aligned} \quad (1.2.1.7)$$

$$\therefore \frac{1}{2} \|(\mathbf{Q} - \mathbf{P}) \times (\mathbf{R} - \mathbf{P})\| = 10 \quad (1.2.1.8)$$

which is the desired area of the triangle

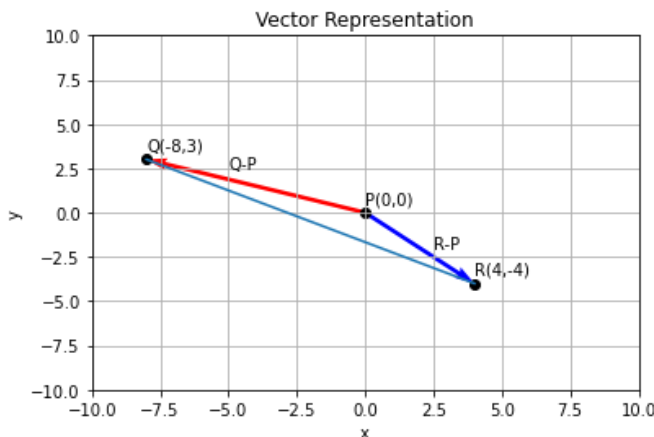


Fig. 1.2.1.1: Plot obtained from Python code

1.2.2. Find the distance between the following pair of points (2,3) and (5,7).

**Solution:**

1.2.3. Find the areas of the triangles the coordinates of whose angular points are respectively:

$$\mathbf{P} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} -9 \\ -3 \end{pmatrix} \text{ and } \mathbf{R} = \begin{pmatrix} -3 \\ -5 \end{pmatrix} \quad (1.2.3.1)$$

**Solution:**

$$\mathbf{Q} - \mathbf{P} = \begin{pmatrix} -9 \\ -3 \end{pmatrix} - \begin{pmatrix} 5 \\ 2 \end{pmatrix} \quad (1.2.3.2)$$

$$= \begin{pmatrix} -14 \\ -5 \end{pmatrix} \quad \mathbf{R} - \mathbf{P} = \begin{pmatrix} -3 \\ -5 \end{pmatrix} - \begin{pmatrix} 5 \\ 2 \end{pmatrix} \quad (1.2.3.3)$$

$$= \begin{pmatrix} -8 \\ -7 \end{pmatrix} \quad (1.2.3.4)$$

$$\therefore (\mathbf{Q} - \mathbf{P}) \times (\mathbf{R} - \mathbf{P}) = \begin{pmatrix} 0 & 0 & -5 \\ 0 & 0 & 14 \\ 5 & -14 & 0 \end{pmatrix} \times \begin{pmatrix} -8 \\ -7 \\ 0 \end{pmatrix} \quad (1.2.3.5)$$

$$= \begin{pmatrix} 0 \\ 0 \\ 58 \end{pmatrix} \Rightarrow \|(\mathbf{Q} - \mathbf{P}) \times (\mathbf{R} - \mathbf{P})\| = 58 \quad (1.2.3.6)$$

area of the triangle is given by

$$\frac{1}{2} \|(\mathbf{Q} - \mathbf{P}) \times (\mathbf{R} - \mathbf{P})\| = \frac{1}{2} (58) = 29 \quad (1.2.3.7)$$

1.2.4. Find the area of the triangle formed by the points  $\mathbf{P} \begin{pmatrix} a \\ c+a \end{pmatrix}$ ,  $\mathbf{Q} \begin{pmatrix} a \\ c \end{pmatrix}$  and  $\mathbf{R} \begin{pmatrix} -a \\ c-a \end{pmatrix}$

**Solution:**

1.2.5. Find the area of the triangle the coordinates of whose angular points are respectively

$$\mathbf{A} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \text{ and } \mathbf{C} = \begin{pmatrix} 4 \\ -3 \end{pmatrix} \quad (1.2.5.1)$$

**Solution:** We will be using vectors for calculating the area of the triangle formed by above three points.

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} - \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad (1.2.5.2)$$

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad (1.2.5.3)$$

$$\mathbf{C} - \mathbf{A} = \begin{pmatrix} 4 \\ -3 \end{pmatrix} - \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad (1.2.5.4)$$

$$\mathbf{C} - \mathbf{A} = \begin{pmatrix} 5 \\ -5 \end{pmatrix} \quad (1.2.5.5)$$

$$\therefore \text{Area of the Triangle} = \frac{1}{2} \|(\mathbf{B} - \mathbf{A}) \times (\mathbf{C} - \mathbf{A})\| \quad (1.2.5.6)$$

As the vector cross product of two vectors can also be expressed as the product of a skew-symmetric matrix and a vector.

$$\mathbf{P} \times \mathbf{Q} = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \quad (1.2.5.7)$$

Substituting values from equation (1.2.5.3) and (1.2.5.5) in above equation (1.2.5.7), we'll get:

$$(\mathbf{B} - \mathbf{A}) \times (\mathbf{C} - \mathbf{A}) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -3 \\ -1 & 3 & 0 \end{pmatrix} \times \begin{pmatrix} 5 \\ -5 \\ 0 \end{pmatrix} \quad (1.2.5.8)$$

$$(\mathbf{B} - \mathbf{A}) \times (\mathbf{C} - \mathbf{A}) = \begin{pmatrix} 0 \\ 0 \\ -20 \end{pmatrix} \quad (1.2.5.9)$$

$$\frac{1}{2} \therefore \|(\mathbf{B} - \mathbf{A}) \times (\mathbf{C} - \mathbf{A})\| = 10 \quad (1.2.5.10)$$

Substituting value from equation (1.2.5.10) in equation (1.2.5.6), we'll get area of triangle:

$$\Rightarrow \frac{1}{2}(20) = 10 \text{ units}^2$$

1.2.6. The coordinates of the vertices of a triangle are  $(x_1, y_1)$ ,  $(x_2, y_2)$  and  $(x_3, y_3)$ . The line joining the first two is divided in the ratio  $l : k$ , and the line joining his point of division to the opposite angular point is then divided in the ratio  $m : k + l$ . Find the coordinates of the latter point of section.

**Solution:** In Fig. 1.2.6.1

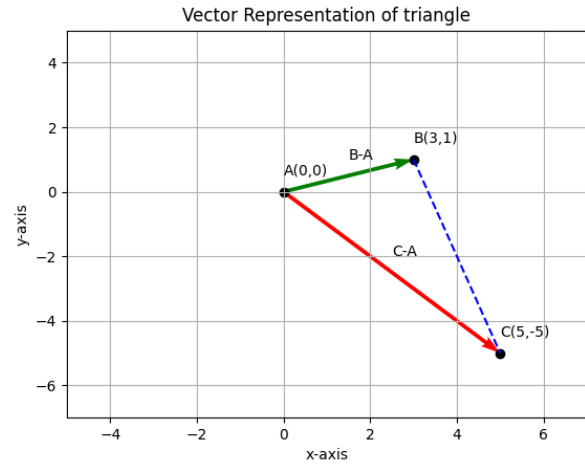


Fig. 1.2.5.1: Plot obtained from Python code

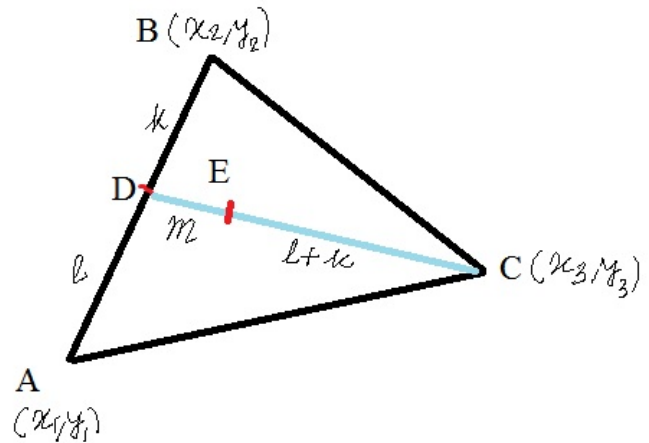


Fig. 1.2.6.1: Triangle ABC with vertices  $\mathbf{A} \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ ,  $\mathbf{B} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ ,  $\mathbf{C} \begin{pmatrix} 4 \\ 0 \end{pmatrix}$ , and  $\begin{pmatrix} l \\ m \\ k \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.5 \\ 0.5 \end{pmatrix}$  are used for python plot. The point of section  $\mathbf{P} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ ,  $\mathbf{Q} = \begin{pmatrix} 2 \\ 1.33 \end{pmatrix}$ .

$$\mathbf{B} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} \quad (1.2.6.1)$$

Using section formula,

$$\mathbf{P} = \frac{l\mathbf{C} + k\mathbf{B}}{l + k} \quad (1.2.6.2)$$

Now, the line joining PA divided into the ratio  $m:k+l$  at point of division  $\mathbf{Q}$  can be written by

using section formula

$$\mathbf{Q} = \frac{m\mathbf{A} + (k+l)\mathbf{P}}{l+k+m} \quad (1.2.6.3)$$

From Eq.(1.2.6.2) substitute  $\mathbf{P}$  in Eq.(1.2.6.3)

$$\mathbf{Q} = \frac{l\mathbf{C} + k\mathbf{B} + m\mathbf{A}}{l+k+m} \quad (1.2.6.4)$$

$$= \frac{1}{l+k+m} (\mathbf{C} \quad \mathbf{B} \quad \mathbf{A}) \begin{pmatrix} l \\ k \\ m \end{pmatrix} \quad (1.2.6.5)$$

## 2 THE STRAIGHT LINE

### 2.1 6

2.1.1. Find the equation to the straight line passing through (2, 3) and perpendicular to the straight line:  $4x - 3y = 10$ .

**Solution:** The vector which is normal to  $4x - 3y = 10$  from simple inspection is  $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$ . Clearly, the direction vector  $\mathbf{m}$  of a line which is perpendicular to the given line is :

$$\mathbf{m} = \begin{pmatrix} 3 \\ -4 \end{pmatrix} \quad (2.1.1.1)$$

The equation of this line which is perpendicular to the given line and passing through  $\mathbf{A} = \begin{pmatrix} x_A \\ y_A \end{pmatrix}$  is then obtained as:

$$\mathbf{m}^T \mathbf{x} = \mathbf{m}^T \mathbf{A} \quad (2.1.1.2)$$

(2.1.1.2) simplifies to read:

$$\begin{pmatrix} 3 & -4 \end{pmatrix} \mathbf{x} = 18 \quad (2.1.1.3)$$

Which in scalar form reads:  $3x + 4y = 18$

Both the straight lines are plotted in Fig. 2.1.1.1 along with the point A(2,3) using Python script.

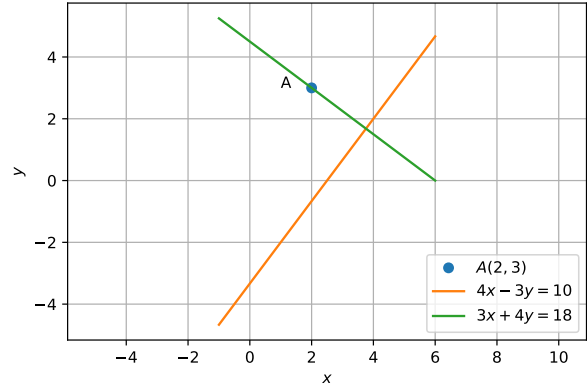


Fig. 2.1.1.1: Solution

$\mathbf{C}$  is the centre and substituting the points in the equation of circle we get

$$2 \begin{pmatrix} 1 & -2 \end{pmatrix} \mathbf{C} - f = 5 \quad (3.1.1.2)$$

$$2 \begin{pmatrix} 4 & -3 \end{pmatrix} \mathbf{C} - f = 25 \quad (3.1.1.3)$$

$$\begin{pmatrix} 3 & 4 \end{pmatrix} \mathbf{C} = 7 \quad (3.1.1.4)$$

can be expressed in matrix form

$$\begin{pmatrix} 3 & 4 & 0 \\ 2 & -4 & -1 \\ 8 & -6 & -1 \end{pmatrix} \begin{pmatrix} \mathbf{C} \\ f \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \\ 25 \end{pmatrix} \quad (3.1.1.5)$$

Row reducing the augmented matrix

$$\begin{pmatrix} 3 & 4 & 0 & 7 \\ 2 & -4 & -1 & 5 \\ 8 & -6 & -1 & 25 \end{pmatrix} \xrightarrow{R_1 \leftarrow R_1/3} \begin{pmatrix} 1 & \frac{4}{3} & 0 & \frac{7}{3} \\ 2 & -4 & -1 & 5 \\ 8 & -6 & -1 & 25 \end{pmatrix} \quad (3.1.1.6)$$

$$\xrightarrow{\begin{matrix} R_2 \leftarrow R_2 - 2R_1 \\ R_3 \leftarrow R_3 - 8R_1 \end{matrix}} \begin{pmatrix} 1 & \frac{4}{3} & 0 & \frac{7}{3} \\ 0 & -\frac{20}{3} & -1 & \frac{1}{3} \\ 0 & -\frac{50}{3} & -1 & \frac{19}{3} \end{pmatrix} \quad (3.1.1.7)$$

$$\xrightarrow{R_2 \leftarrow -\frac{3}{20}R_2} \begin{pmatrix} 1 & \frac{4}{3} & 0 & \frac{7}{3} \\ 0 & 1 & \frac{3}{20} & -\frac{1}{20} \\ 0 & -\frac{50}{3} & -1 & \frac{19}{3} \end{pmatrix} \quad (3.1.1.8)$$

## 3 THE CIRCLE

### 3.1 17

3.1.1. Find the equation to the circle which passes through the points  $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$  and  $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$  and which

has its centre on the straight line  $\begin{pmatrix} 3 & -4 \end{pmatrix} \mathbf{x} = 7$ .

**Solution:** The equation of circle can be expressed as

$$\mathbf{x}^T \mathbf{x} - 2\mathbf{C}^T \mathbf{x} + f = 0 \quad (3.1.1.1)$$

$$\xleftrightarrow{R_3 \leftarrow R_3 + \frac{50}{3}R_2} \begin{pmatrix} 1 & \frac{4}{3} & 0 & \frac{7}{3} \\ 0 & 1 & \frac{3}{20} & \frac{-1}{20} \\ 0 & 0 & \frac{3}{2} & \frac{11}{2} \end{pmatrix} \quad (3.1.1.9)$$

$$\mathbf{C} = \begin{pmatrix} \frac{47}{15} \\ \frac{-3}{5} \end{pmatrix} f = \frac{11}{3} \quad (3.1.1.10)$$

$$f = \frac{11}{3} \quad (3.1.1.11)$$

The required circle equation,

$$\mathbf{x}^T \mathbf{x} - 2 \begin{pmatrix} \frac{47}{15} & \frac{-3}{5} \end{pmatrix} \mathbf{x} + \frac{11}{3} = 0 \quad (3.1.1.12)$$

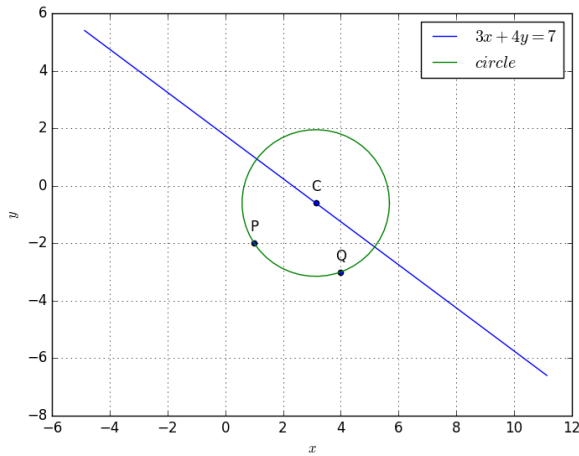


Fig. 3.1.1.1: Circle passing through point P and Q also centre lie on the line  $3x+4y=7$

Find the equation to the circle that passes through the points:

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \mathbf{x}_2 = \begin{pmatrix} 3 \\ -4 \end{pmatrix}, \mathbf{x}_3 = \begin{pmatrix} 5 \\ -6 \end{pmatrix} \quad (3.1.1.13)$$

**Solution:** The equation of circle in vector form is given by:

$$\mathbf{x}^T \mathbf{x} + 2\mathbf{x}^T \mathbf{u} + f = 0 \quad (3.1.1.14)$$

Using  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$  in (3.1.1.14),

$$\mathbf{x}_1^T \mathbf{x}_1 + 2\mathbf{x}_1^T \mathbf{u} + f = 0 \quad (3.1.1.15)$$

$$\mathbf{x}_2^T \mathbf{x}_2 + 2\mathbf{x}_2^T \mathbf{u} + f = 0 \quad (3.1.1.16)$$

$$\mathbf{x}_3^T \mathbf{x}_3 + 2\mathbf{x}_3^T \mathbf{u} + f = 0 \quad (3.1.1.17)$$

The above system can be written in matrix form as:

$$\begin{pmatrix} 2\mathbf{x}_1^T & 1 \\ 2\mathbf{x}_2^T & 1 \\ 2\mathbf{x}_3^T & 1 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ f \end{pmatrix} = \begin{pmatrix} -\mathbf{x}_1^T \mathbf{x}_1 \\ -\mathbf{x}_2^T \mathbf{x}_2 \\ -\mathbf{x}_3^T \mathbf{x}_3 \end{pmatrix} \quad (3.1.1.18)$$

Substituting the values for  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$  in (3.1.1.18),

$$\begin{pmatrix} 2 & 4 & 1 \\ 6 & -8 & 1 \\ 10 & -12 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ f \end{pmatrix} = \begin{pmatrix} -5 \\ -25 \\ -61 \end{pmatrix} \quad (3.1.1.19)$$

Using row echelon form to reduce (3.1.1.19), we get:

$$\xleftrightarrow{R_2 \rightarrow R_2 - 3R_1, R_3 \rightarrow R_3 - 5R_1} \begin{pmatrix} 2 & 4 & 1 & -5 \\ 0 & -20 & -2 & -10 \\ 0 & -32 & -4 & -36 \end{pmatrix} \quad (3.1.1.20)$$

$$\xleftrightarrow{R_2 \rightarrow -\frac{1}{2}R_2, R_3 \rightarrow -\frac{1}{4}R_3} \begin{pmatrix} 2 & 4 & 1 & -5 \\ 0 & 10 & 1 & 5 \\ 0 & 8 & 1 & 9 \end{pmatrix} \quad (3.1.1.21)$$

$$\xleftrightarrow{R_3 \rightarrow 5R_3 - 4R_2} \begin{pmatrix} 2 & 4 & 1 & -5 \\ 0 & 10 & 1 & 5 \\ 0 & 0 & 1 & 25 \end{pmatrix} \quad (3.1.1.22)$$

$$\xleftrightarrow{R_1 \rightarrow \frac{1}{2}R_1, R_2 \rightarrow \frac{1}{10}R_2} \begin{pmatrix} 1 & 2 & \frac{1}{2} & -\frac{5}{2} \\ 0 & 1 & \frac{1}{10} & \frac{2}{5} \\ 0 & 0 & 1 & 25 \end{pmatrix} \quad (3.1.1.23)$$

Back solving the system using (3.1.1.18) and (3.1.1.23), we get:

$$\begin{pmatrix} \mathbf{u} \\ f \end{pmatrix} = \begin{pmatrix} -11 \\ -2 \\ 25 \end{pmatrix} \quad (3.1.1.24)$$

$$\Rightarrow \mathbf{u} = \begin{pmatrix} -11 \\ -2 \end{pmatrix}, f = 25 \quad (3.1.1.25)$$

The equation of the circle that passes through the points  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$  is given by:

$$\mathbf{x}^T \mathbf{x} + 2 \begin{pmatrix} -11 \\ -2 \end{pmatrix}^T \mathbf{x} + f = 0 \quad (3.1.1.26)$$

$$\Rightarrow x^2 + y^2 - 22x - 4y + 25 = 0 \quad (3.1.1.27)$$

The plot of the circle is given below:

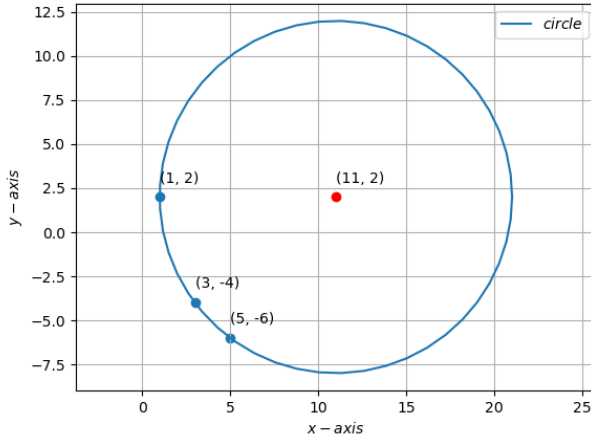


Fig. 3.1.1.2: Circle centered at (11, 2) with radius 10.

3.1.2. Find the equation of circle passing through the points

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \mathbf{x}_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \mathbf{x}_3 = \begin{pmatrix} 8 \\ 2 \end{pmatrix} \quad (3.1.2.1)$$

**Solution:** Vector form of the equation of circle is :

$$\mathbf{x}^T \mathbf{x} + 2\mathbf{x}^T \mathbf{u} + f = 0 \quad (3.1.2.2)$$

For  $\mathbf{x}_1$ ,  $\mathbf{x}_2$  and  $\mathbf{x}_3$  equation (3.1.2.2) can be written as:

$$\mathbf{x}_1^T \mathbf{x}_1 + 2\mathbf{x}_1^T \mathbf{u} + f = 0 \quad (3.1.2.3)$$

$$\mathbf{x}_2^T \mathbf{x}_2 + 2\mathbf{x}_2^T \mathbf{u} + f = 0 \quad (3.1.2.4)$$

$$\mathbf{x}_3^T \mathbf{x}_3 + 2\mathbf{x}_3^T \mathbf{u} + f = 0 \quad (3.1.2.5)$$

In matrix form this can be written as :

$$\begin{pmatrix} 2\mathbf{x}_1^T & 1 \\ 2\mathbf{x}_2^T & 1 \\ 2\mathbf{x}_3^T & 1 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ f \end{pmatrix} = \begin{pmatrix} -\mathbf{x}_1^T \mathbf{x}_1 \\ -\mathbf{x}_2^T \mathbf{x}_2 \\ -\mathbf{x}_3^T \mathbf{x}_3 \end{pmatrix} \quad (3.1.2.6)$$

By putting the values of  $\mathbf{x}_1$ ,  $\mathbf{x}_2$  and  $\mathbf{x}_3$  from (3.1.2.1) in (3.1.2.6) we get :

$$\begin{pmatrix} 2 & 2 & 1 \\ 4 & -2 & 1 \\ 16 & 4 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ f \end{pmatrix} = \begin{pmatrix} -2 \\ -5 \\ -68 \end{pmatrix} \quad (3.1.2.7)$$

Using Gaussian Elimination method :

$$\begin{matrix} R_1 \leftarrow \frac{1}{2} R_1 \\ R_2 \leftarrow R_2 - 4R_1 \end{matrix} \rightarrow \begin{pmatrix} 1 & 1 & \frac{1}{2} & -1 \\ 0 & -6 & -1 & -1 \\ 16 & 4 & 1 & -68 \end{pmatrix} \quad (3.1.2.8)$$

$$\begin{matrix} R_3 \leftarrow R_3 - 16R_1 \end{matrix} \rightarrow \begin{pmatrix} 1 & 1 & \frac{1}{2} & -1 \\ 0 & -6 & -1 & -1 \\ 0 & -12 & -7 & -52 \end{pmatrix} \quad (3.1.2.9)$$

$$\begin{matrix} R_2 \leftarrow -\frac{1}{6} R_2 \\ R_3 \leftarrow R_3 + 12R_2 \end{matrix} \rightarrow \begin{pmatrix} 1 & 1 & \frac{1}{2} & -1 \\ 0 & 1 & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & -5 & -50 \end{pmatrix} \quad (3.1.2.10)$$

Using (3.1.2.7) and (3.1.2.10) we get :

$$\begin{pmatrix} \mathbf{u} \\ f \end{pmatrix} = \begin{pmatrix} -\frac{9}{2} \\ -\frac{3}{2} \\ 10 \end{pmatrix} \quad (3.1.2.11)$$

$$\mathbf{u} = \begin{pmatrix} -\frac{9}{2} \\ -\frac{3}{2} \end{pmatrix} \quad (3.1.2.12)$$

$$f = 10 \quad (3.1.2.13)$$

By putting the values of  $\mathbf{u}$  and  $f$  in (3.1.2.2) we get :

$$\mathbf{x}^T \mathbf{x} + 2 \begin{pmatrix} -\frac{9}{2} \\ -\frac{3}{2} \end{pmatrix}^T \mathbf{x} + 10 = 0 \quad (3.1.2.14)$$

$$x^2 + y^2 - 9x - 6y + 10 = 0 \quad (3.1.2.15)$$

Plot of the circle given by equation (3.1.2.15) is as follows :

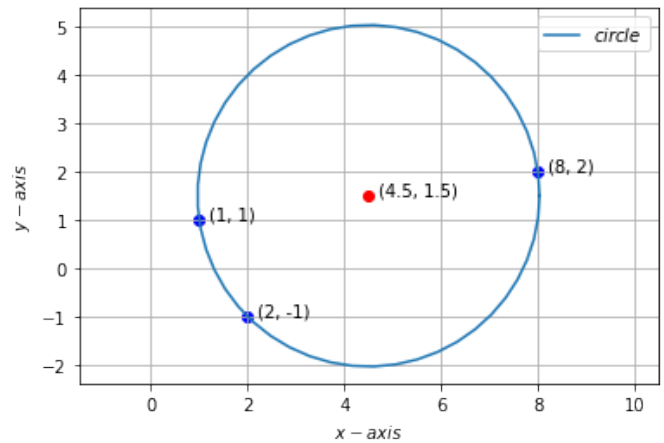


Fig. 3.1.2.1: A circle centered at (4.5, 1.5) with radius 3.53.

### 3.2 18

3.2.1. Write down the equation of the tangent to a circle passing through the point  $\mathbf{p}$ .

Equation of the circle and positional vector  $\mathbf{p}$  is given as :

$$x^2 + y^2 - 3x + 10y = 15 \quad (3.2.1.1)$$

$$\mathbf{p} = \begin{pmatrix} 4 \\ -11 \end{pmatrix} \quad (3.2.1.2)$$

**Solution:**

General equation of the circle in vector form is :

$$\mathbf{x}^T \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (3.2.1.3)$$

In the vector form (3.2.1.1) can be written as :

$$\mathbf{x}^T \mathbf{x} + 2 \begin{pmatrix} -\frac{3}{2} \\ 5 \end{pmatrix}^T \mathbf{x} - 15 = 0 \quad (3.2.1.4)$$

By comparing (3.2.1.3) and (3.2.1.4) we get :

$$\mathbf{u} = \begin{pmatrix} -\frac{3}{2} \\ 5 \end{pmatrix}, f = -15 \quad (3.2.1.5)$$

We know that the equation of tangent in the form of normal vector  $(\mathbf{p} + \mathbf{u})$  and point  $\mathbf{p}$  can be written as:

$$(\mathbf{p} + \mathbf{u})^T (\mathbf{x} - \mathbf{p}) = 0 \quad (3.2.1.6)$$

$$(\mathbf{p} + \mathbf{u})^T \mathbf{x} - \mathbf{p}^T \mathbf{p} - \mathbf{u}^T \mathbf{p} = 0 \quad (3.2.1.7)$$

Using (3.2.1.3), (3.2.1.7) will become :

$$(\mathbf{p} + \mathbf{u})^T \mathbf{x} + \mathbf{u}^T \mathbf{p} + f = 0 \quad (3.2.1.8)$$

By putting the values of  $\mathbf{p}$ ,  $\mathbf{u}$  and  $f$  from (3.2.1.5) in (3.2.1.8) we get :

$$\begin{pmatrix} \frac{5}{2} & -6 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 4 & -11 \end{pmatrix} \begin{pmatrix} -\frac{3}{2} \\ 5 \end{pmatrix} - 15 = 0 \quad (3.2.1.9)$$

$$\begin{pmatrix} \frac{5}{2} & -6 \end{pmatrix} \mathbf{x} - 76 = 0 \quad (3.2.1.10)$$

Hence the equation of the tangent to the circle passing through the point  $\mathbf{p}$  is:

$$\begin{pmatrix} \frac{5}{2} & -6 \end{pmatrix} \mathbf{x} = 76 \quad (3.2.1.11)$$

Plot of the tangent to a circle given by equation (3.2.1.11) is as follows :

3.2.2. Find equation of the tangent to the circle

$$x^2 + y^2 = 4 \quad (3.2.2.1)$$

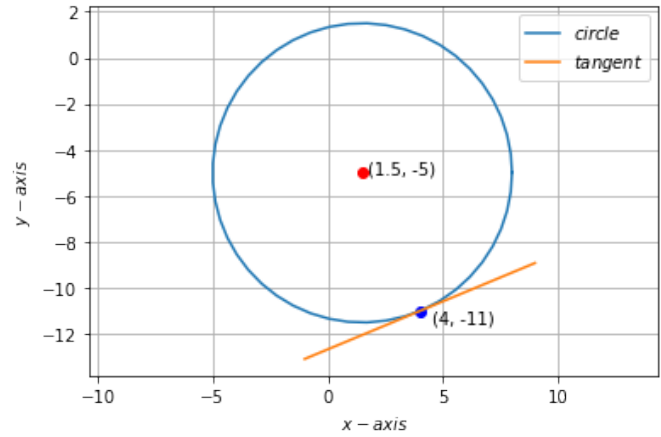


Fig. 3.2.1.1: Tangent to a circle centered at  $(1.5, -5)$  with radius 6.5 passing through the point  $(4, -11)$ .

which is parallel to the line

$$x + 2y - 6 = 0 \quad (3.2.2.2)$$

**Solution:** The equations for the circle and line in (3.2.2.1) and (3.2.2.2) can be rewritten in vector form as:

$$\|\mathbf{x}\|^2 = 4 \quad (3.2.2.3)$$

$$\begin{pmatrix} 1 & 2 \end{pmatrix} \mathbf{x} = 6 \quad (3.2.2.4)$$

The center of the circle happens to be  $(0, 0)$

The equation of a line is of the form:

$$\mathbf{n}^T \mathbf{x} = c \quad (3.2.2.5)$$

Where  $\mathbf{n}$  is the normal to the line.

Comparing (3.2.2.5) to (3.2.2.4),

$$\mathbf{n} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (3.2.2.6)$$

Since the tangent is parallel to the line in (3.2.2.4), it will also have the same normal.

The point of contact for a conic is given by:

$$\mathbf{v} = \mathbf{V}^{-1}(\kappa \mathbf{n} - \mathbf{u}) \quad (3.2.2.7)$$

where,

$$\kappa = \pm \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\mathbf{n}^T \mathbf{V}^{-1} \mathbf{n}}} \quad (3.2.2.8)$$

For a circle,

$$\mathbf{V} = \mathbf{I} \quad (3.2.2.9)$$

Using properties of identity matrix, we get:

$$\mathbf{I}^{-1} = \mathbf{I} \quad (3.2.2.10)$$

$$\mathbf{I}\mathbf{X} = \mathbf{X} \quad (3.2.2.11)$$

Therefore (3.2.2.7) and (3.2.2.8) simplify to:

$$\kappa = \pm \sqrt{\frac{\mathbf{u}^T \mathbf{u} - f}{\mathbf{n}^T \mathbf{n}}} \quad (3.2.2.12)$$

$$\Rightarrow \mathbf{v} = \kappa \mathbf{n} - \mathbf{u} \quad (3.2.2.13)$$

Substituting the values, we get:

$$\kappa = \pm \sqrt{\frac{4}{(1 \ 2) \begin{pmatrix} 1 \\ 2 \end{pmatrix}}} \quad (3.2.2.14)$$

$$\Rightarrow \kappa = \pm \sqrt{\frac{4}{5}} \quad (3.2.2.15)$$

$$\mathbf{q} = \pm \sqrt{\frac{4}{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (3.2.2.16)$$

$$\Rightarrow \mathbf{q}_1 = \begin{pmatrix} \sqrt{\frac{4}{5}} \\ \sqrt{\frac{16}{5}} \end{pmatrix}, \mathbf{q}_2 = -\begin{pmatrix} \sqrt{\frac{4}{5}} \\ \sqrt{\frac{16}{5}} \end{pmatrix} \quad (3.2.2.17)$$

Since there are two points of contact, there are two tangents parallel to (3.2.2.4) that have the same normal vector.

$$\Rightarrow \mathbf{n}^T \mathbf{q}_1 = c_1 \quad (3.2.2.18)$$

$$\mathbf{n}^T \mathbf{q}_2 = c_2 \quad (3.2.2.19)$$

Substituting the values, we get:

$$c_1 = 2\sqrt{5}, c_2 = -2\sqrt{5} \quad (3.2.2.20)$$

Therefore, the equation of the tangents are:

$$(1 \ 2)\mathbf{x} = 2\sqrt{5} \quad (3.2.2.21)$$

$$(1 \ 2)\mathbf{x} = -2\sqrt{5} \quad (3.2.2.22)$$

The plot of the circle with the tangents is given below:

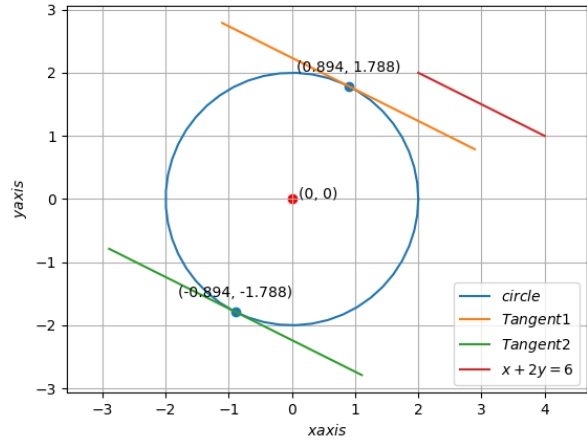


Fig. 3.2.2.1: Circle centered at (0,0) with tangents parallel to line  $x + 2y = 6$ .

The given equation can be written as:

$$-x^2 - xy + 6y^2 + 30y + 36 = 0 \quad (4.1.1.1)$$

$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix}$  of (4.1.1.1) becomes

$$\begin{vmatrix} -1 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 6 & 15 \\ 0 & 15 & 36 \end{vmatrix} = 0 \quad (4.1.1.2)$$

Expanding equation (4.1.1.2), we get zero.

Hence given equation represents a pair of straight lines.

The general equation second degree is given by

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \quad (4.1.1.3)$$

Let  $(\alpha, \beta)$  be their point of intersection, then

$$\begin{pmatrix} a & h \\ h & b \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -d \\ -e \end{pmatrix} \quad (4.1.1.4)$$

Given equation is

$$-x^2 - xy + 6y^2 + 30y + 36 = 0 \quad (4.1.1.5)$$

Substituting in (4.1.1.4)

$$\begin{pmatrix} -1 & -\frac{1}{2} \\ -\frac{1}{2} & 6 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ -15 \end{pmatrix} \quad (4.1.1.6)$$

$$\Rightarrow \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \frac{6}{5} \\ -\frac{12}{5} \end{pmatrix} \quad (4.1.1.7)$$

Hence, the intersection point is  $\begin{pmatrix} \frac{6}{5} \\ -\frac{12}{5} \end{pmatrix}$

#### 4 PAIR OF STRAIGHT LINES

##### 4.1 13

4.1.1. Prove that the following equations represent two straight lines, find also their point of intersection and the angle between them.

$$6y^2 - xy - x^2 + 30y + 36 = 0.$$

**Solution:**



Also, Verified using python code from

codes/Assignment\_5.py

From, Spectral decomposition,

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T \quad (4.1.1.8)$$

$$\mathbf{V} = \begin{pmatrix} -1 & \frac{-1}{2} \\ \frac{-1}{2} & 6 \end{pmatrix} \quad (4.1.1.9)$$

$$\mathbf{P} = \begin{pmatrix} 7-5\sqrt{2} & 7+5\sqrt{2} \\ 1 & 1 \end{pmatrix} \quad (4.1.1.10)$$

$$\mathbf{D} = \begin{pmatrix} \frac{5+5\sqrt{2}}{2} & 0 \\ 0 & \frac{5-5\sqrt{2}}{2} \end{pmatrix} \quad (4.1.1.11)$$

$\mathbf{P}$  and  $\mathbf{D}$  are also verified using python code from

codes/diagonalize1.py

Using, (4.1.1.7), (4.1.1.10) and (4.1.1.11) in,

$$u_1(x - \alpha) + u_2(y - \beta) = \pm \sqrt{-\frac{\lambda_2}{\lambda_1}}(v_1(x - \alpha) + v_2(y - \beta)) \quad (4.1.1.12)$$

$$\begin{aligned} &\Rightarrow (7 - 5\sqrt{2})\left(x - \frac{30}{23}\right) + \left(y + \frac{60}{23}\right) \\ &= \pm \sqrt{-\frac{\frac{5-5\sqrt{2}}{2}}{\frac{5+5\sqrt{2}}{2}}} \left( (7 - 5\sqrt{2})\left(x - \frac{6}{5}\right) + \left(y + \frac{12}{5}\right) \right) \end{aligned} \quad (4.1.1.13)$$

simplifying 4.1.1.13, we get:

$$-x + 2y + 6 = 0 \text{ and } x + 3y + 6 = 0 \quad (4.1.1.14)$$

$$\Rightarrow (-x + 2y + 6)(x + 3y + 6) = 0 \quad (4.1.1.15)$$

$$\therefore -x + 2y = -6, \quad x + 3y = -6 \quad (4.1.1.16)$$

Angle between two lines,  $\theta$  can be given by

$$n_1 = (-2, -1) \quad (4.1.1.17)$$

$$n_2 = (-3, 1) \quad (4.1.1.18)$$

$$\cos \theta = \frac{\mathbf{n}_1^T \mathbf{n}_2}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} \quad (4.1.1.19)$$

$$\cos \theta = \frac{(-2 \ -1) \begin{pmatrix} -3 \\ 1 \end{pmatrix}}{\sqrt{(-2)^2 + (-1)^2} \times \sqrt{+(-3)^2 + 1}} = \frac{1}{\sqrt{2}} \quad (4.1.1.20)$$

$$\Rightarrow \theta = 45^\circ \quad (4.1.1.21)$$

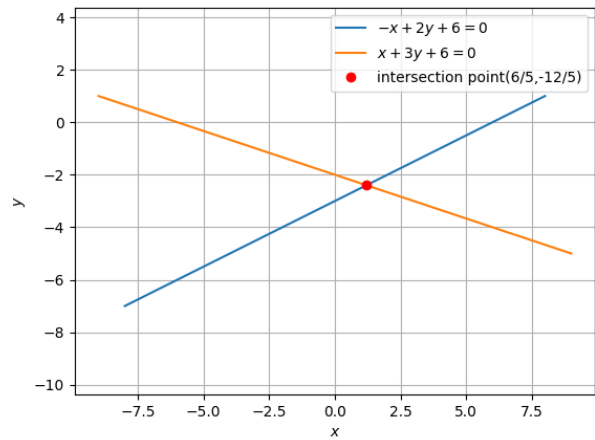


Fig. 4.1.1.1: plot showing intersection of lines.

4.1.2. Prove that the following equations represent two straight lines; and also find their point of intersection and the angle between them

$$x^2 - 5xy + 4y^2 + x + 2y - 2 = 0$$

**Solution:** Proving that given equation represents two straight lines The given equation is

$$x^2 - 5xy + 4y^2 + x + 2y - 2 = 0 \quad (4.1.2.1)$$

Comparing this to the standard equation,

$$\mathbf{V} = \begin{pmatrix} 1 & \frac{-5}{2} \\ \frac{-5}{2} & 4 \end{pmatrix} \quad (4.1.2.2)$$

$$\mathbf{u} = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} \quad (4.1.2.3)$$

$$f = -2 \quad (4.1.2.4)$$

$$\Rightarrow \mathbf{x}^T \begin{pmatrix} 1 & \frac{-5}{2} \\ \frac{-5}{2} & 4 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} \frac{1}{2} & 1 \end{pmatrix} \mathbf{x} - 2 = 0 \quad (4.1.2.5)$$

Equation (4.1.2.1) represents a pair of straight lines if

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \quad (4.1.2.6)$$

$$\delta = \begin{vmatrix} 1 & \frac{-5}{2} & \frac{1}{2} \\ \frac{-5}{2} & 4 & 1 \\ \frac{1}{2} & 1 & -2 \end{vmatrix} \quad (4.1.2.7)$$

$$= 0 \quad (4.1.2.8)$$

Hence, proved that given equation represents two straight lines. Finding point of intersection between the straight lines

$$\det V = \begin{vmatrix} 1 & \frac{-5}{2} \\ \frac{-5}{2} & 4 \end{vmatrix} \quad (4.1.2.9)$$

$$= \frac{-9}{4} < 0 \quad (4.1.2.10)$$

Thus, the two straight lines intersect. Let the equation of the straight lines be given as

$$\mathbf{n}_1^T \mathbf{x} = c_1 \quad (4.1.2.11)$$

$$\mathbf{n}_2^T \mathbf{x} = c_2 \quad (4.1.2.12)$$

with their slopes as  $\mathbf{m}_1$  and  $\mathbf{m}_2$  respectively. Then the equation of the pair of straight lines is

$$(\mathbf{n}_1^T \mathbf{x} - c_1)(\mathbf{n}_2^T \mathbf{x} - c_2) = 0 \quad (4.1.2.13)$$

Using (4.1.2.5) and (4.1.2.13),

$$(\mathbf{n}_1^T \mathbf{x} - c_1)(\mathbf{n}_2^T \mathbf{x} - c_2) = \mathbf{x}^T \begin{pmatrix} 1 & \frac{-5}{2} \\ \frac{-5}{2} & 4 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} \frac{1}{2} & 1 \end{pmatrix} \mathbf{x} - 2 \quad (4.1.2.14)$$

Comparing both sides,

$$c_2 \mathbf{n}_1 + c_1 \mathbf{n}_2 = -2 \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} \quad (4.1.2.15)$$

$$c_1 c_2 = -2 \quad (4.1.2.16)$$

Slopes of the lines are roots of the equation

$$cm^2 + 2bm + a = 0 \quad (4.1.2.17)$$

$$\Rightarrow m_i = \frac{-b \pm \sqrt{-|\mathbf{V}|}}{c} \quad (4.1.2.18)$$

$$\mathbf{n}_i = k_i \begin{pmatrix} -m_i \\ 1 \end{pmatrix} \quad (4.1.2.19)$$

Substituting (4.1.2.1) in (4.1.2.17),

$$4m^2 - 5m + 1 = 0 \quad (4.1.2.20)$$

$$\Rightarrow m_i = \frac{\frac{5}{2} \pm \frac{3}{2}}{4} \quad (4.1.2.21)$$

$$\Rightarrow m_1 = 1, m_2 = \frac{1}{4} \quad (4.1.2.22)$$

Therefore,

$$\mathbf{n}_1 = k_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad (4.1.2.23)$$

$$\mathbf{n}_2 = k_2 \begin{pmatrix} \frac{-1}{4} \\ 1 \end{pmatrix} \quad (4.1.2.24)$$

We know that

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (4.1.2.25)$$

$$k_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} * k_2 \begin{pmatrix} \frac{-1}{4} \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix} \quad (4.1.2.26)$$

$$\Rightarrow k_1 k_2 = 4 \quad (4.1.2.27)$$

Taking  $k_1 = 1, k_2 = 4$ , we get

$$\mathbf{n}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\mathbf{n}_2 = \begin{pmatrix} -1 \\ 4 \end{pmatrix} \quad (4.1.2.28)$$

For verifying values of  $\mathbf{n}_1$  and  $\mathbf{n}_2$ , we compute the convolution by representing  $\mathbf{n}_1$  as Toeplitz matrix,

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix} \quad (4.1.2.29)$$

Now, obtaining  $c_1$  and  $c_2$  using (4.1.2.28) and

(4.1.2.15)

$$(\mathbf{n}_1 \quad \mathbf{n}_2) \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2 \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} \quad (4.1.2.30)$$

$$\Rightarrow \begin{pmatrix} -1 & -1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix} \quad (4.1.2.31)$$

Row reducing the augmented matrix,

$$\begin{pmatrix} -1 & -1 & -1 \\ 1 & 4 & -2 \end{pmatrix} \xrightarrow{R_1 \leftarrow -R_1} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 4 & -2 \end{pmatrix} \quad (4.1.2.32)$$

$$\xrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & -3 \end{pmatrix} \quad (4.1.2.33)$$

$$\xrightarrow{R_1 \leftarrow R_1 - R_2} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{pmatrix} \quad (4.1.2.34)$$

$$\Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$c_1 = -1 \quad (4.1.2.35)$$

$$c_2 = 2 \quad (4.1.2.36)$$

Thus, equation of lines can be written as

$$(-1 \quad 1)\mathbf{x} = -1 \quad (4.1.2.37)$$

$$(-1 \quad 4)\mathbf{x} = 2 \quad (4.1.2.38)$$

Augmented matrix for these set of equations is

$$\begin{pmatrix} -1 & 1 & -1 \\ -1 & 4 & 2 \end{pmatrix} \xrightarrow{R_1 \leftarrow -R_1} \begin{pmatrix} 1 & -1 & 1 \\ -1 & 4 & 2 \end{pmatrix} \quad (4.1.2.39)$$

$$\xrightarrow{R_2 \leftarrow R_2 + R_1} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 3 & 3 \end{pmatrix} \xrightarrow{R_2 \leftarrow \frac{R_2}{3}} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad (4.1.2.40)$$

$$\xrightarrow{R_1 \leftarrow R_1 + R_2} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix} \quad (4.1.2.41)$$

Thus, the point of intersection is  $\mathbf{A} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ .

Using (4.1.2.28) and (4.1.2.36) in (4.1.2.13), equation of the pair of straight lines is

$$(x - y - 1)(x - 4y + 2) = 0 \quad (4.1.2.42)$$

Angle between lines Angle between pair of lines is,

$$\theta = \cos^{-1} \left( \frac{\mathbf{n}_1^T \mathbf{n}_2}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} \right) \quad (4.1.2.43)$$

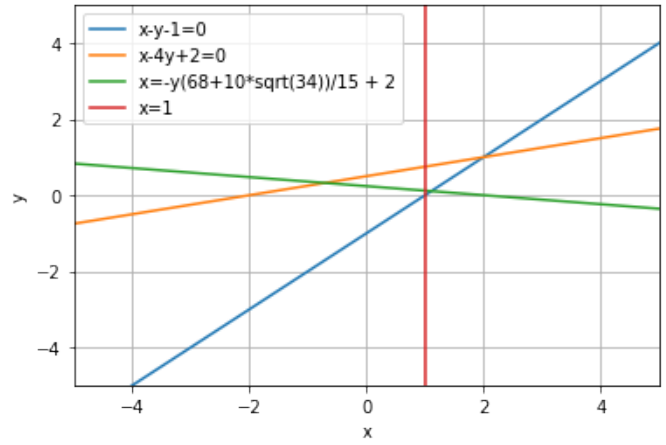


Fig. 4.1.2.1: Intersection of pair of original pair of straight lines and the pair of straight lines after affine transform

$$\mathbf{n}_1^T \mathbf{n}_2 = (-1 \quad 1) \begin{pmatrix} -1 \\ 4 \end{pmatrix} = 5 \quad (4.1.2.44)$$

$$\|\mathbf{n}_1\| = \sqrt{(-1)^2 + 1^2} = \sqrt{2} \quad (4.1.2.45)$$

$$\|\mathbf{n}_2\| = \sqrt{(-1)^2 + 4^2} = \sqrt{17} \quad (4.1.2.46)$$

Substituting these values (4.1.2.43)

$$\theta = 30.9^\circ \quad (4.1.2.47)$$

Hence, angle between the given pair of straight lines is  $30.9^\circ$ . Affine Transformation and Eigen Value decomposition First, verifying if  $\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f = 0$ . To do this, finding  $\mathbf{V}^{-1}$  by augmenting with identity matrix and row reducing as follows :

$$\begin{pmatrix} 1 & \frac{-5}{2} & 1 & 0 \\ \frac{-5}{2} & 4 & 0 & 1 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 + \frac{5}{2}R_1} \begin{pmatrix} 1 & \frac{-5}{2} & 1 & 0 \\ 0 & \frac{-9}{4} & \frac{5}{2} & 1 \end{pmatrix} \quad (4.1.2.48)$$

$$\xrightarrow{R_2 \leftarrow \frac{-4}{9}R_2} \begin{pmatrix} 1 & \frac{-5}{2} & 1 & 0 \\ 0 & 1 & \frac{-10}{9} & \frac{-4}{9} \end{pmatrix} \quad (4.1.2.49)$$

$$\xrightarrow{R_1 \leftarrow R_1 + \frac{5}{2}R_2} \begin{pmatrix} 1 & 0 & \frac{-16}{9} & \frac{-10}{9} \\ 0 & 1 & \frac{-10}{9} & \frac{-4}{9} \end{pmatrix} \quad (4.1.2.50)$$

$$\Rightarrow \mathbf{V}^{-1} = \begin{pmatrix} \frac{-16}{9} & \frac{-10}{9} \\ \frac{-10}{9} & \frac{-4}{9} \end{pmatrix} \quad (4.1.2.51)$$

$$u^T V^{-1} u - f = \begin{pmatrix} \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} \frac{-16}{9} & \frac{-10}{9} \\ \frac{-10}{9} & \frac{-4}{9} \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} - (-2)$$

$$= 0 \quad (4.1.2.53)$$

The characteristic equation of  $\mathbf{V}$  is given as :

$$|\lambda \mathbf{I} - \mathbf{V}| = \begin{vmatrix} \lambda - 1 & \frac{5}{2} \\ \frac{5}{2} & \lambda - 4 \end{vmatrix} = 0 \quad (4.1.2.54)$$

$$\Rightarrow (\lambda - 1)(\lambda - 4) - \frac{25}{4} = 0 \quad (4.1.2.55)$$

$$\Rightarrow 4\lambda^2 - 20\lambda - 9 = 0 \quad (4.1.2.56)$$

The roots of (4.1.2.56), i.e. the eigenvalues of  $\mathbf{V}$  are

$$\lambda_1 = \frac{5 + \sqrt{34}}{2}, \lambda_2 = \frac{5 - \sqrt{34}}{2} \quad (4.1.2.57)$$

The eigen vector  $\mathbf{p}$  is defined as,

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \quad (4.1.2.58)$$

$$\Rightarrow (\lambda \mathbf{I} - \mathbf{V})\mathbf{p} = 0 \quad (4.1.2.59)$$

$$\text{For } \lambda_1 = \frac{5 + \sqrt{34}}{2}$$

$$(\lambda_1 \mathbf{I} - \mathbf{V}) = \begin{pmatrix} \frac{3 + \sqrt{34}}{2} & \frac{5}{2} \\ \frac{5}{2} & \frac{-3 + \sqrt{34}}{2} \end{pmatrix} \quad (4.1.2.60)$$

To find  $\mathbf{p}_1$ , let's look at Augmented form of  $(\lambda_1 \mathbf{I} - \mathbf{V})$

$$\begin{pmatrix} \frac{3 + \sqrt{34}}{2} & \frac{5}{2} & 0 \\ \frac{5}{2} & \frac{-3 + \sqrt{34}}{2} & 0 \end{pmatrix} \quad (4.1.2.61)$$

$$\xrightarrow{R_1 \leftarrow \frac{2}{3 + \sqrt{34}} R_1} \begin{pmatrix} 1 & \frac{-3 + \sqrt{34}}{5} & 0 \\ \frac{5}{2} & \frac{-3 + \sqrt{34}}{2} & 0 \end{pmatrix} \quad (4.1.2.62)$$

$$\xrightarrow{R_2 \leftarrow \frac{2}{5} R_2 - R_1} \begin{pmatrix} 1 & \frac{-3 + \sqrt{34}}{5} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (4.1.2.63)$$

So we get

$$x_1 + \left( \frac{-3 + \sqrt{34}}{5} \right) x_2 = 0 \quad (4.1.2.64)$$

Thus, our eigenvector corresponding to  $\lambda_1$

$$\mathbf{p}_1 = \begin{pmatrix} \frac{3 - \sqrt{34}}{5} \\ 1 \end{pmatrix} \quad (4.1.2.65)$$

$$\text{For } \lambda_2 = \frac{5 - \sqrt{34}}{2}$$

$$(\lambda_2 \mathbf{I} - \mathbf{V}) = \begin{pmatrix} \frac{3 - \sqrt{34}}{2} & \frac{5}{2} \\ \frac{5}{2} & \frac{-3 - \sqrt{34}}{2} \end{pmatrix} \quad (4.1.2.66)$$

To find  $\mathbf{p}_2$ , let's look at Augmented form of  $(\lambda_2 \mathbf{I} - \mathbf{V})$

$$\begin{pmatrix} \frac{3 - \sqrt{34}}{2} & \frac{5}{2} & 0 \\ \frac{5}{2} & \frac{-3 - \sqrt{34}}{2} & 0 \end{pmatrix} \quad (4.1.2.67)$$

$$\xrightarrow{R_1 \leftarrow \frac{2}{3 - \sqrt{34}} R_1} \begin{pmatrix} 1 & \frac{-3 - \sqrt{34}}{5} & 0 \\ \frac{5}{2} & \frac{-3 - \sqrt{34}}{2} & 0 \end{pmatrix} \quad (4.1.2.68)$$

$$\xrightarrow{R_2 \leftarrow \frac{2}{5} R_2 - R_1} \begin{pmatrix} 1 & \frac{-3 - \sqrt{34}}{5} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (4.1.2.69)$$

So we get

$$x_1 + \left( \frac{-3 - \sqrt{34}}{5} \right) x_2 = 0 \quad (4.1.2.70)$$

Thus, our eigenvector corresponding to  $\lambda_2$

$$\mathbf{p}_2 = \begin{pmatrix} \frac{3 + \sqrt{34}}{5} \\ 1 \end{pmatrix} \quad (4.1.2.71)$$

We know  $\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T$ , where  $\mathbf{P}$  and the diagonal matrix  $\mathbf{D}$  are given as:

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad (4.1.2.72)$$

$$= \begin{pmatrix} \frac{5 + \sqrt{34}}{2} & 0 \\ 0 & \frac{5 - \sqrt{34}}{2} \end{pmatrix} \quad (4.1.2.73)$$

$$\mathbf{P} = (\mathbf{p}_1 \quad \mathbf{p}_2) \quad (4.1.2.74)$$

$$= \begin{pmatrix} \frac{3 - \sqrt{34}}{5} & \frac{3 + \sqrt{34}}{5} \\ 1 & 1 \end{pmatrix} \quad (4.1.2.75)$$

So, the equation of the pair of straight lines is

given by :

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f \quad |\mathbf{V}| \neq 0 \quad (4.1.2.76)$$

$$\mathbf{y}^T \begin{pmatrix} \frac{5 + \sqrt{34}}{2} & 0 \\ 0 & \frac{5 - \sqrt{34}}{2} \end{pmatrix} \mathbf{y} = 0 \quad (4.1.2.77)$$

$$\Rightarrow (y_1 \ y_2) \begin{pmatrix} \frac{5 + \sqrt{34}}{2} & 0 \\ 0 & \frac{5 - \sqrt{34}}{2} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0 \quad (4.1.2.78)$$

$$\Rightarrow (5 + \sqrt{34})y_1^2 + (5 - \sqrt{34})y_2^2 = 0 \quad (4.1.2.79)$$

So we get the equation of the pair of straight lines, as we can see this passes through the origin (0,0). The corresponding image is shown in Fig. 4.1.2.2

$$\mathbf{c} = -\mathbf{V}^{-1} \mathbf{u} \quad |\mathbf{V}| \neq 0 \quad (4.1.2.80)$$

$$\Rightarrow \mathbf{c} = -\begin{pmatrix} \frac{-16}{9} & \frac{-10}{9} \\ \frac{-10}{9} & \frac{-4}{9} \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad (4.1.2.81)$$

And,

$$\mathbf{P}^T = \begin{pmatrix} \frac{3 - \sqrt{34}}{5} & 1 \\ \frac{3 + \sqrt{34}}{5} & 1 \end{pmatrix} \quad (4.1.2.82)$$

Using affine transformation, we can express the equation as

$$\mathbf{x} = \mathbf{P} \mathbf{y} + \mathbf{c} \quad (4.1.2.83)$$

$$\Rightarrow \mathbf{x} = \begin{pmatrix} \frac{3 - \sqrt{34}}{5} & \frac{3 + \sqrt{34}}{5} \\ 1 & 1 \end{pmatrix} \mathbf{y} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad (4.1.2.84)$$

The corresponding image is shown in Fig. 4.1.2.1

4.1.3. Prove that the following equations represent two straight lines. Also find their point of intersection and the angle between them

$$3y^2 - 8xy - 3x^2 - 29x + 3y - 18 = 0 \quad (4.1.3.1)$$

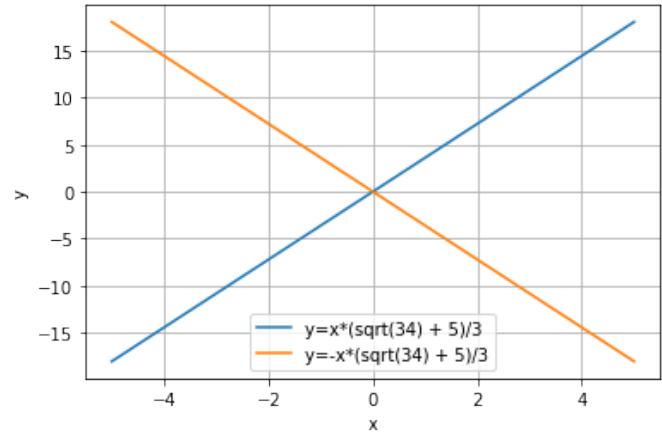


Fig. 4.1.2.2: Pair of straight lines passing through origin after eigenvalue decomposition

**Solution:**  $\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix}$  of (4.1.3.1) becomes

$$\begin{vmatrix} -3 & -4 & -\frac{29}{2} \\ -4 & 3 & \frac{3}{2} \\ -\frac{29}{2} & \frac{3}{2} & -18 \end{vmatrix} \quad (4.1.3.2)$$

Expanding equation (4.1.3.2), we get zero. Hence given equation represents a pair of straight lines. Slopes of the individual lines are roots of equation

$$cm^2 + 2bm + a = 0 \quad (4.1.3.3)$$

$$\Rightarrow 3m^2 - 8m - 3 = 0 \quad (4.1.3.4)$$

$$\text{Solving, } m = 3, -\frac{1}{3} \quad (4.1.3.5)$$

The normal vectors of the lines then become

$$\mathbf{n}_1 = \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} \quad (4.1.3.6)$$

$$\mathbf{n}_2 = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \quad (4.1.3.7)$$

Equations of the lines can therefore be written

as

$$\begin{pmatrix} \frac{1}{3} & 1 \end{pmatrix} \mathbf{x} = c \quad (4.1.3.8)$$

$$\Rightarrow \begin{pmatrix} 1 & 3 \end{pmatrix} \mathbf{x} = c_1, \quad (4.1.3.9)$$

$$\begin{pmatrix} -3 & 1 \end{pmatrix} \mathbf{x} = c_2 \quad (4.1.3.10)$$

$$\Rightarrow \left[ \begin{pmatrix} 1 & 3 \end{pmatrix} \mathbf{x} - c_1 \right] \left[ \begin{pmatrix} -3 & 1 \end{pmatrix} \mathbf{x} - c_2 \right] \quad (4.1.3.11)$$

represents the equation specified in (4.1.3.1)

Comparing the equations, we have

$$\begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 29 \\ -3 \end{pmatrix} \quad (4.1.3.12)$$

$$(4.1.3.13)$$

Row reducing the augmented matrix

$$\begin{pmatrix} 1 & -3 & 29 \\ 3 & 1 & -3 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 - 3 \times R_1} \begin{pmatrix} 1 & -3 & 29 \\ 0 & 10 & -90 \end{pmatrix} \quad (4.1.3.14)$$

$$\xrightarrow{R_2 \leftarrow R_2 \times \frac{1}{10}} \begin{pmatrix} 1 & -3 & 29 \\ 0 & 1 & -9 \end{pmatrix} \quad (4.1.3.15)$$

$$\xrightarrow{R_1 \leftarrow R_1 + 3 \times R_2} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -9 \end{pmatrix} \quad (4.1.3.16)$$

$$\Rightarrow c_2 = 2 \text{ and } c_1 = -9 \quad (4.1.3.17)$$

The individual line equations therefore become

$$\begin{pmatrix} 1 & 3 \end{pmatrix} \mathbf{x} = -9, \quad (4.1.3.18)$$

$$\begin{pmatrix} -3 & 1 \end{pmatrix} \mathbf{x} = 2 \quad (4.1.3.19)$$

Note that the convolution of the normal vectors, should satisfy the below condition

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} * \begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (4.1.3.20)$$

The LHS part of (4.1.3.20) can be rewritten using toeplitz matrix as

$$\begin{pmatrix} 1 & 0 \\ 3 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ -8 \\ 3 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (4.1.3.21)$$

The augmented matrix for the set of equations

represented in (4.1.3.18), (4.1.3.19) is

$$\begin{pmatrix} 1 & 3 & -9 \\ -3 & 1 & 2 \end{pmatrix} \quad (4.1.3.22)$$

Row reducing the matrix

$$\begin{pmatrix} 1 & 3 & -9 \\ -3 & 1 & 2 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 + 3 \times R_1} \begin{pmatrix} 1 & 3 & -9 \\ 0 & 10 & -25 \end{pmatrix} \quad (4.1.3.23)$$

$$\xrightarrow{R_1 \leftarrow R_1 - \frac{3}{10} \times R_2} \begin{pmatrix} 1 & 0 & -\frac{3}{2} \\ 0 & 10 & -25 \end{pmatrix} \quad (4.1.3.24)$$

$$\xrightarrow{R_2 \leftarrow \frac{R_2}{10}} \begin{pmatrix} 1 & 0 & -\frac{3}{2} \\ 0 & 1 & -\frac{5}{2} \end{pmatrix} \quad (4.1.3.25)$$

$$\text{Hence, the intersection point is } \begin{pmatrix} -\frac{3}{2} \\ -\frac{5}{2} \end{pmatrix} \quad (4.1.3.26)$$

Angle between two lines  $\theta$  can be given by

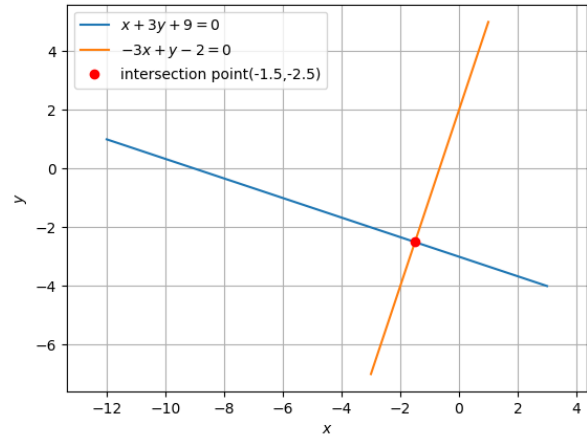


Fig. 4.1.3.1: plot showing intersection of lines

$$\cos \theta = \frac{\mathbf{n}_1^T \mathbf{n}_2}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} \quad (4.1.3.27)$$

$$\cos \theta = \frac{\begin{pmatrix} 1 & 3 \end{pmatrix} \begin{pmatrix} -3 \\ 1 \end{pmatrix}}{\sqrt{(3)^2 + 1} \times \sqrt{(-3)^2 + 1}} = 0 \quad (4.1.3.28)$$

$$\Rightarrow \theta = 90^\circ \quad (4.1.3.29)$$

4.1.4. Prove that the following equations represents two straight lines also find their point of inter-

section and angle between them.

$$y^2 + xy - 2x^2 - 5x - y - 2 = 0 \quad (4.1.4.1)$$

**Solution:**

$$\mathbf{V} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} -2 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} \quad (4.1.4.2)$$

$$\mathbf{u} = \begin{pmatrix} d \\ e \end{pmatrix} = \begin{pmatrix} \frac{-5}{2} \\ \frac{-1}{2} \end{pmatrix} \quad (4.1.4.3)$$

$$f = -2 \quad (4.1.4.4)$$

$$\begin{vmatrix} -2 & \frac{1}{2} & \frac{-5}{2} \\ \frac{1}{2} & 1 & \frac{-1}{2} \\ \frac{-5}{2} & \frac{-1}{2} & -2 \end{vmatrix} \xrightarrow[R_1 \rightarrow R_1 + R_3]{R_1 \rightarrow R_1 - R_2} \begin{vmatrix} 0 & 0 & 0 \\ \frac{1}{2} & 1 & \frac{-1}{2} \\ \frac{-5}{2} & \frac{-1}{2} & -2 \end{vmatrix} = 0 \quad (4.1.4.5)$$

Hence it represents the pair of straight lines. Now two intersecting lines are obtained when

$$|V| < 0 \implies \begin{vmatrix} -2 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{vmatrix} = \frac{-9}{4} < 0 \quad (4.1.4.6)$$

Let the pair of straight of lines be given by

$$\mathbf{n}_1^T \mathbf{x} = c_1 \quad (4.1.4.7)$$

$$\mathbf{n}_2^T \mathbf{x} = c_2 \quad (4.1.4.8)$$

The slopes of the lines are given by the roots of the polynomial

$$cm^2 + 2bm + a = 0 \quad (4.1.4.9)$$

$$m_1, m_2 = \frac{-\frac{1}{2} \pm \sqrt{\frac{9}{4}}}{1} \quad (4.1.4.10)$$

$$m_1 = 1, m_2 = -2 \quad (4.1.4.11)$$

$$\implies \mathbf{n}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \text{ and } \mathbf{n}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad (4.1.4.12)$$

$$(\mathbf{n}_1^T \mathbf{x} - c_1)(\mathbf{n}_2^T \mathbf{x} - c_2) = \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f \quad (4.1.4.13)$$

$$c_2 \mathbf{n}_1 + c_1 \mathbf{n}_2 = -2\mathbf{u} \quad (4.1.4.14)$$

$$c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = -2 \begin{pmatrix} \frac{-5}{2} \\ \frac{-1}{2} \end{pmatrix} \quad (4.1.4.15)$$

$$\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \quad (4.1.4.16)$$

Using row reduction we get

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 5 \end{pmatrix} \quad (4.1.4.17)$$

$$\xrightarrow[R_2 \leftarrow R_2 - 2R_1]{R_2 \leftarrow R_2 / -3} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \quad (4.1.4.18)$$

$$\xrightarrow{R_1 \leftarrow R_1 - R_2} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{pmatrix} \quad (4.1.4.19)$$

$$C = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad (4.1.4.20)$$

The convolution of the normal vectors, should satisfy the below condition

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix} * \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (4.1.4.21)$$

The LHS part of equation(2.0.20) can be rewritten using toeplitz matrix as

$$\begin{pmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (4.1.4.22)$$

Therefore the equation of lines is given by

$$\begin{pmatrix} -1 & 1 \end{pmatrix} \mathbf{x} = 2 \quad (4.1.4.23)$$

$$\begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = -1 \quad (4.1.4.24)$$

consider the augmented matrix

$$\begin{pmatrix} -1 & 1 & 2 \\ 2 & 1 & -1 \end{pmatrix} \quad (4.1.4.25)$$

$$\xrightarrow[R_2 \leftarrow R_2 - 2R_1]{R_1 \leftarrow -R_1} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad (4.1.4.26)$$

$$\xrightarrow[R_1 \leftarrow R_1 + R_2]{R_1 \leftarrow R_1 / 3} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} \quad (4.1.4.27)$$

Therefore point of intersection is  $\mathbf{A} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ .

Angle between two lines  $\theta$  can be given by

$$\cos \theta = \frac{\mathbf{n}_1^T \mathbf{n}_2}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} \quad (4.1.4.28)$$

$$\cos \theta = \frac{\begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}}{\sqrt{(1)^2 + 1} \times \sqrt{(2)^2 + 1}} \quad (4.1.4.29)$$

$$\theta = \cos^{-1}\left(\frac{-1}{\sqrt{10}}\right) \implies \theta = \tan^{-1}3 \quad (4.1.4.30)$$

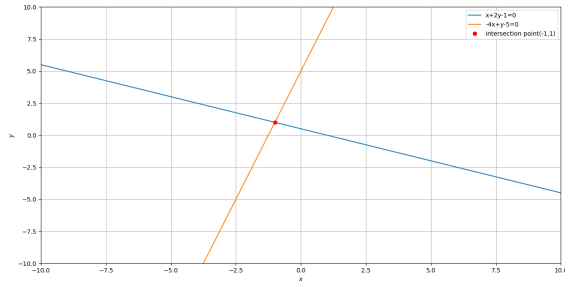


Fig. 4.1.4.1: plot showing intersection of lines

4.1.5. Prove that the equation

$$x^2 + 6xy + 9y^2 + 4x + 12y - 5 = 0 \quad (4.1.5.1)$$

represents two parallel lines.

**Solution:** The given equation (4.1.5.1) can be written as

$$\mathbf{x}^T \begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} 2 & 6 \end{pmatrix} \mathbf{x} - 5 = 0 \quad (4.1.5.2)$$

$$\mathbf{V} = \begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix} \quad \mathbf{u} = \begin{pmatrix} 2 \\ 6 \end{pmatrix} \quad f = -5 \quad (4.1.5.3)$$

Equation (4.1.5.1) represents pair of straight line as,

$$D = \begin{vmatrix} 1 & 3 & 2 \\ 3 & 9 & 6 \\ 2 & 6 & -5 \end{vmatrix} = 0 \quad (4.1.5.4)$$

Vector form of straight lines,

$$\mathbf{n}_1^T \mathbf{x} = \mathbf{c}_1 \quad (4.1.5.5)$$

$$\mathbf{n}_2^T \mathbf{x} = \mathbf{c}_2 \quad (4.1.5.6)$$

Equating their product with (4.1.5.2)

$$(\mathbf{n}_1^T \mathbf{x} - \mathbf{c}_1)(\mathbf{n}_2^T \mathbf{x} - \mathbf{c}_2) = \mathbf{x}^T \begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} 2 & 6 \end{pmatrix} \mathbf{x} - 5 \quad (4.1.5.7)$$

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} 1 \\ 6 \\ 9 \end{pmatrix} \quad (4.1.5.8)$$

$$c_2 \mathbf{n}_1 + c_1 \mathbf{n}_2 = -2 \begin{pmatrix} 2 \\ 6 \\ 6 \end{pmatrix} \quad (4.1.5.9)$$

$$c_1 c_2 = -5 \quad (4.1.5.10)$$

The slopes of the lines can be given by roots of the equation,

$$cm^2 + 2bm + a = 0 \quad (4.1.5.11)$$

$$m_i = \frac{-b \pm \sqrt{-|\mathbf{V}|}}{c} \quad (4.1.5.12)$$

$$\mathbf{n}_i = k_i \begin{pmatrix} -m_i \\ 1 \end{pmatrix} \quad (4.1.5.13)$$

From (4.1.5.2) equation (4.1.5.11) becomes

$$9m^2 + 6m + 1 = 0 \quad (4.1.5.14)$$

Using (4.1.5.3),

$$|\mathbf{V}| = \begin{vmatrix} 1 & 3 \\ 3 & 9 \end{vmatrix} = 0 \quad (4.1.5.15)$$

Substituting the values in (4.1.5.12),

$$m_i = \frac{-3 \pm 0}{9} \quad (4.1.5.16)$$

$$m_1 = m_2 = \frac{-1}{3} \quad (4.1.5.17)$$

Substituting values in (4.1.5.13)

$$\mathbf{n}_1 = k_1 \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} \quad (4.1.5.18)$$

$$\mathbf{n}_2 = k_2 \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} \quad (4.1.5.19)$$

Using the above values in (4.1.5.8),

$$k_1 k_2 = 9 \quad (4.1.5.20)$$

Taking  $k_1 = 3$  and  $k_2 = 3$  we get

$$\mathbf{n}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad (4.1.5.21)$$

$$\mathbf{n}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad (4.1.5.22)$$

Verifying  $\mathbf{n}_1$  and  $\mathbf{n}_2$  by computing the convolution by representing  $\mathbf{n}_1$  as Toeplitz matrix,

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} 1 & 0 \\ 3 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \\ 9 \end{pmatrix} \quad (4.1.5.23)$$



Finding the Angle between the lines,

$$\theta = \cos^{-1} \left( \frac{\mathbf{n}_1^T \mathbf{n}_2}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} \right) \quad (4.1.5.24)$$

$$\mathbf{n}_1^T \mathbf{n}_2 = \begin{pmatrix} 1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = 10 \quad (4.1.5.25)$$

$$\|\mathbf{n}_1\| = \sqrt{10} \quad \|\mathbf{n}_2\| = \sqrt{10} \quad (4.1.5.26)$$

Substituting (4.1.5.25) and (4.1.5.26) in (4.1.5.24) we get,

$$\theta = \cos^{-1}(1) \quad (4.1.5.27)$$

$$\theta = 0^\circ \quad (4.1.5.28)$$

From (4.1.5.17) and (4.1.5.28) shows the given equation (4.1.5.1) represents two parallel lines. Hence proved.

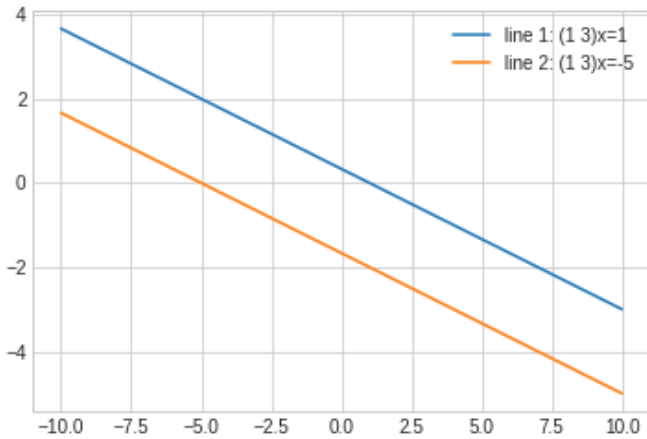


Fig. 4.1.5.1: Pair of straight lines plot generated using python

4.1.6. **Solution:** Find the value of k such that

$$6x^2 + 11xy - 10y^2 + x + 31y + k = 0 \quad (4.1.6.1)$$

represent pairs of straight lines.

From (4.1.6.1) we get,

$$\mathbf{V} = \begin{pmatrix} 6 & \frac{11}{2} \\ \frac{11}{2} & -10 \end{pmatrix} \quad (4.1.6.2)$$

$$\mathbf{u} = \begin{pmatrix} \frac{1}{2} \\ \frac{31}{2} \end{pmatrix} \quad (4.1.6.3)$$

$$f = k \quad (4.1.6.4)$$

Compute the slopes of lines given by the roots

of the polynomial  $-10m^2 + 11m + 6$

$$i.e., m_i = \frac{-b \pm \sqrt{-|V|}}{c} \quad (4.1.6.5)$$

$$\Rightarrow m = \frac{\frac{-11}{2} \pm \frac{19}{2}}{-10} \quad (4.1.6.6)$$

$$\Rightarrow m_1 = \frac{-2}{5}, m_2 = \frac{3}{2} \quad (4.1.6.7)$$

Let the pair of straight lines be given by

$$\mathbf{n}_1^T \mathbf{x} = c_1 \quad (4.1.6.8)$$

$$\mathbf{n}_2^T \mathbf{x} = c_2 \quad (4.1.6.9)$$

Here,

$$\mathbf{n}_1 = k_1 \begin{pmatrix} -m_1 \\ 1 \end{pmatrix} = k_1 \begin{pmatrix} \frac{2}{5} \\ 1 \end{pmatrix} \quad (4.1.6.10)$$

$$\mathbf{n}_2 = k_2 \begin{pmatrix} -m_2 \\ 1 \end{pmatrix} = k_2 \begin{pmatrix} \frac{-3}{2} \\ 1 \end{pmatrix} \quad (4.1.6.11)$$

We know that,

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (4.1.6.12)$$

Substituting (4.1.6.10) and (4.1.6.11) in the above equation, we get

$$k_1 \begin{pmatrix} \frac{2}{5} \\ 1 \end{pmatrix} * k_2 \begin{pmatrix} \frac{-3}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 11 \\ -10 \end{pmatrix} \quad (4.1.6.13)$$

$$\Rightarrow k_1 k_2 = -10 \quad (4.1.6.14)$$

By inspection, we get the values,  $k_1 = 5, k_2 = -2$ . Substituting the values of  $k_1$  and  $k_2$  in (4.1.6.10) and (4.1.6.11) respectively, we get

$$\mathbf{n}_1 = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \quad (4.1.6.15)$$

$$\mathbf{n}_2 = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \quad (4.1.6.16)$$

Using Teoplitz matrix representation, the convolution of  $\mathbf{n}_1$  with  $\mathbf{n}_2$ , is as follows:

$$\begin{pmatrix} 2 & 0 & 5 \\ 5 & 2 & 0 \\ 0 & 5 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 11 \\ -10 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (4.1.6.17)$$

Hence,  $\mathbf{n}_1$  and  $\mathbf{n}_2$  satisfies (4.1.6.12). We have,

$$c_2 \mathbf{n}_1 + c_1 \mathbf{n}_2 = -2\mathbf{u} \quad (4.1.6.18)$$

Substituting (4.1.6.15), (4.1.6.16) in (4.1.6.18), we get

$$\begin{pmatrix} 2 & 3 \\ 5 & -2 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2 \begin{pmatrix} \frac{1}{2} \\ \frac{31}{2} \end{pmatrix} \quad (4.1.6.19)$$

Solving for  $c_1$  and  $c_2$ , the augmented matrix is,

$$\begin{pmatrix} 2 & 3 & -1 \\ 5 & -2 & -31 \end{pmatrix} \xrightarrow[R_2 \leftarrow R_2 - 5R_1]{R_1 \leftarrow \frac{R_1}{2}} \begin{pmatrix} 1 & \frac{3}{2} & \frac{-1}{2} \\ 0 & \frac{-19}{2} & \frac{-37}{2} \end{pmatrix} \quad (4.1.6.20)$$

$$\xrightarrow[R_1 \leftarrow R_1 - \frac{3}{2}R_2]{R_2 \leftarrow \frac{R_2}{-19/2}} \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \end{pmatrix} \quad (4.1.6.21)$$

Hence we obtain,

$$c_1 = 3, c_2 = -5 \quad (4.1.6.22)$$

We know that,

$$f = k = c_1 c_2 \quad (4.1.6.23)$$

$$\Rightarrow \boxed{k = -15} \quad (4.1.6.24)$$

Hence the solution. Using (4.1.6.8) and (4.1.6.9), the equation of pair of straight lines is given by,

$$(2 \ 5)\mathbf{x} = 3 \quad (4.1.6.25)$$

$$(3 \ -2)\mathbf{x} = -5 \quad (4.1.6.26)$$

See Fig. 4.1.6.1

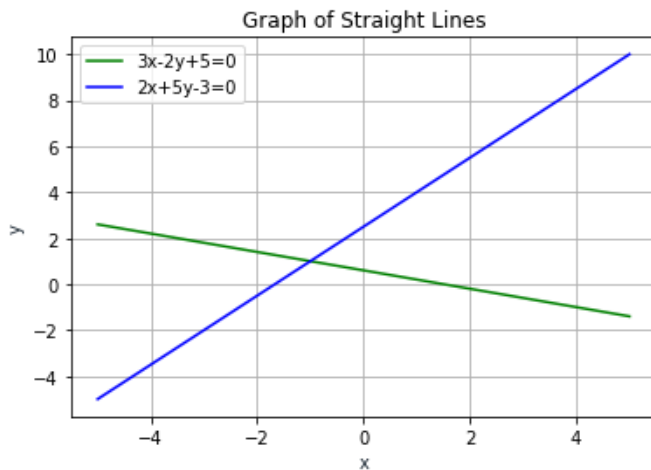


Fig. 4.1.6.1: Plot of two straight lines.

may represent pairs of straight lines,

$$12x^2 - 10xy + 2y^2 + 11x - 5y + k = 0 \quad (4.1.7.1)$$

**Solution:** The general equation of second degree is given by,

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \quad (4.1.7.2)$$

In vector form the equation (4.1.7.2) can be expressed as,

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (4.1.7.3)$$

where,

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad (4.1.7.4)$$

$$\mathbf{u} = \begin{pmatrix} d \\ e \end{pmatrix} \quad (4.1.7.5)$$

Now, comparing (4.1.7.2) to (4.1.7.1) we get,  $a = 12$ ,  $b = -5$ ,  $c = 2$ ,  $d = \frac{11}{2}$ ,  $e = -\frac{5}{2}$ ,  $f = k$ . Hence, substituting these values in (4.1.7.4) and (4.1.7.5) we get,

$$\mathbf{V} = \begin{pmatrix} 12 & -5 \\ -5 & 2 \end{pmatrix} \quad (4.1.7.6)$$

$$\mathbf{u} = \begin{pmatrix} \frac{11}{2} \\ -\frac{5}{2} \end{pmatrix} \quad (4.1.7.7)$$

(4.1.7.1) represents pair of straight lines if,

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \quad (4.1.7.8)$$

$$\begin{vmatrix} 12 & -5 & \frac{11}{2} \\ -5 & 2 & -\frac{5}{2} \\ \frac{11}{2} & -\frac{5}{2} & k \end{vmatrix} = 0 \quad (4.1.7.9)$$

$$\Rightarrow k = 2 \quad (4.1.7.10)$$

Lines intersect if

$$|\mathbf{V}| < 0 \quad (4.1.7.11)$$

$$|\mathbf{V}| = -1 < 0 \quad (4.1.7.12)$$

Hence Line intersect.

Let  $(\alpha, \beta)$  be their point of intersection, then

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -d \\ -e \end{pmatrix} \quad (4.1.7.13)$$

4.1.7. Find the value of  $k$  so that following equation

Substituting in (4.1.7.13)

$$\begin{pmatrix} 12 & -5 \\ -5 & 2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -\frac{11}{2} \\ \frac{5}{2} \end{pmatrix} \quad (4.1.7.14)$$

$$\Rightarrow \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -\frac{3}{2} \\ -\frac{5}{2} \end{pmatrix} \quad (4.1.7.15)$$

Spectral Decomposition of  $\mathbf{V}$  is given as

$$\mathbf{V} = \mathbf{PDP}^T \quad (4.1.7.16)$$

$$\mathbf{V} = \begin{pmatrix} 12 & -5 \\ -5 & 2 \end{pmatrix} \quad (4.1.7.17)$$

$$\mathbf{P} = \begin{pmatrix} -1 - \sqrt{2} & -1 + \sqrt{2} \\ 1 & 1 \end{pmatrix} \quad (4.1.7.18)$$

$$\mathbf{D} = \begin{pmatrix} 7 + 5\sqrt{2} & 0 \\ 0 & 7 - 5\sqrt{2} \end{pmatrix} \quad (4.1.7.19)$$

Using Spectral decomposition concept and substitution

$$u_1(x - \alpha) + u_2(y - \beta) = \pm \sqrt{\frac{\lambda_2}{\lambda_1}}(v_1(x - \alpha) + v_2(y - \beta)) \quad (4.1.7.20)$$

Substituting (4.1.7.15), (4.1.7.18) and (4.1.7.19) in (4.1.7.20)

$$\begin{aligned} & (-1 - \sqrt{2})\left(x - \frac{-3}{2}\right) + \left(y - \frac{-5}{2}\right) \\ &= \pm \sqrt{\frac{7 + 5\sqrt{2}}{7 - 5\sqrt{2}}} \left( (-1 + \sqrt{2})\left(x - \frac{-3}{2}\right) + \left(y - \frac{-5}{2}\right) \right) \end{aligned} \quad (4.1.7.21)$$

Simplifying (4.1.7.21),

$$-6x + 2y - 4 = 0 \text{ and } -2x + y - \frac{1}{2} = 0 \quad (4.1.7.22)$$

$$\Rightarrow (-6x + 2y - 4)\left(-2x + y - \frac{1}{2}\right) = 0 \quad (4.1.7.23)$$

Thus the equation of lines are

$$(-6 \ 2)\mathbf{x} = 4 \quad (4.1.7.24)$$

$$(-2 \ 1)\mathbf{x} = \frac{1}{2} \quad (4.1.7.25)$$

Hence, Plot is shown below

4.1.8. Find the value of  $k$  so that the following

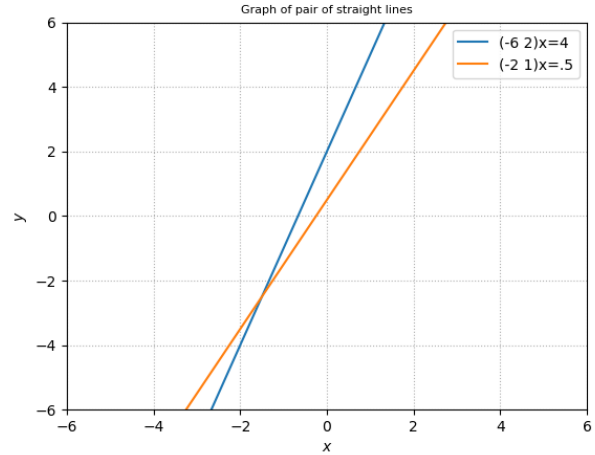


Fig. 4.1.7.1: Pair of lines

equation may represent pair of straight lines:

$$12x^2 + kxy + 2y^2 + 11x - 5y + 2 = 0 \quad (4.1.8.1)$$

**Solution:**

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} 12 & \frac{k}{2} \\ \frac{k}{2} & 2 \end{pmatrix} \quad (4.1.8.2)$$

$$\mathbf{u} = \begin{pmatrix} d \\ e \end{pmatrix} = \begin{pmatrix} \frac{11}{2} \\ -\frac{5}{2} \end{pmatrix} \quad (4.1.8.3)$$

The equation (4.1.8.1) represents pair of straight lines if

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \quad (4.1.8.4)$$

$$\Rightarrow \begin{vmatrix} 12 & \frac{k}{2} & \frac{11}{2} \\ \frac{k}{2} & 2 & -\frac{5}{2} \\ \frac{11}{2} & -\frac{5}{2} & 2 \end{vmatrix} = 0 \quad (4.1.8.5)$$

$$\Rightarrow \begin{vmatrix} 24 & k & 11 \\ k & 4 & -5 \\ 11 & -5 & 4 \end{vmatrix} = 0 \quad (4.1.8.6)$$

$$\Rightarrow 24 \begin{vmatrix} 4 & -5 \\ -5 & 4 \end{vmatrix} - k \begin{vmatrix} k & -5 \\ 11 & 4 \end{vmatrix} + 11 \begin{vmatrix} k & 4 \\ 11 & -5 \end{vmatrix} = 0 \quad (4.1.8.7)$$

$$\Rightarrow 2k^2 + 55k + 350 = 0 \quad (4.1.8.8)$$

$$\Rightarrow (10 + k)(2k + 35) = 0 \quad (4.1.8.9)$$

$$\Rightarrow k = -10$$

$$k = -\frac{35}{2} \quad (4.1.8.10)$$

Therefore, for  $k = -10$  and  $k = -\frac{35}{2}$  the given

equation represents pair of straight lines.

Now Lets find equation of lines for  $k = -10$ .  
Substitute  $k = -10$  in (4.1.8.1). We get equation of pair of straight lines as:

$$12x^2 - 10xy + 2y^2 + 11x - 5y + 2 = 0 \quad (4.1.8.11)$$

From (4.1.8.1), (4.1.8.2), (4.1.8.3) we get

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 12 & -5 \\ -5 & 2 \end{pmatrix} \quad (4.1.8.12)$$

$$\mathbf{u} = \begin{pmatrix} \frac{11}{2} \\ -\frac{5}{2} \end{pmatrix} \quad (4.1.8.13)$$

If  $|\mathbf{V}| < 0$  then two lines will intersect.

$$|\mathbf{V}| = \begin{vmatrix} 12 & -5 \\ -5 & 2 \end{vmatrix} \quad (4.1.8.14)$$

$$\Rightarrow |\mathbf{V}| = -1 \quad (4.1.8.15)$$

$$\Rightarrow |\mathbf{V}| < 0 \quad (4.1.8.16)$$

Therefore the lines will intersect.

The equation of two lines is given by

$$\mathbf{n}_1^T \mathbf{x} = c_1 \quad (4.1.8.17)$$

$$\mathbf{n}_2^T \mathbf{x} = c_2 \quad (4.1.8.18)$$

Equating their product with (4.1.8.1)

$$\begin{aligned} (\mathbf{n}_1^T \mathbf{x} - c_1)(\mathbf{n}_2^T \mathbf{x} - c_2) \\ = \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \end{aligned} \quad (4.1.8.19)$$

$$\Rightarrow \mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} = \begin{pmatrix} 12 \\ -10 \\ 2 \end{pmatrix} \quad (4.1.8.20)$$

$$c_2 \mathbf{n}_1 + c_1 \mathbf{n}_2 = -2\mathbf{u} = -2 \begin{pmatrix} \frac{11}{2} \\ -\frac{5}{2} \end{pmatrix} \quad (4.1.8.21)$$

$$c_1 c_2 = f = 2 \quad (4.1.8.22)$$

The slopes of the lines are given by roots of

equation

$$cm^2 + 2bm + a = 0 \quad (4.1.8.23)$$

$$\Rightarrow 2m^2 - 10m + 12 = 0 \quad (4.1.8.24)$$

$$m_i = \frac{-b \pm \sqrt{-|\mathbf{V}|}}{c} \quad (4.1.8.25)$$

$$\Rightarrow m_i = \frac{5 \pm \sqrt{1}}{2} \quad (4.1.8.26)$$

$$\Rightarrow m_1 = 3 \quad (4.1.8.27)$$

$$m_2 = 2 \quad (4.1.8.28)$$

The normal vector for two lines is given by

$$\mathbf{n}_i = k_i \begin{pmatrix} -m_i \\ 1 \end{pmatrix} \quad (4.1.8.29)$$

$$\Rightarrow \mathbf{n}_1 = k_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix} \quad (4.1.8.30)$$

$$\mathbf{n}_2 = k_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad (4.1.8.31)$$

Substituting (4.1.8.30), (4.1.8.31) in (4.1.8.20). we get

$$k_1 k_2 = 2 \quad (4.1.8.32)$$

The possible combinations of  $(k_1, k_2)$  are (1,2), (2,1), (-1,-2) and (-2,-1).

lets assume  $k_1 = 1, k_2 = 2$  we get

$$\Rightarrow \mathbf{n}_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \quad (4.1.8.33)$$

$$\mathbf{n}_2 = \begin{pmatrix} -4 \\ 2 \end{pmatrix} \quad (4.1.8.34)$$

We verify obtained  $\mathbf{n}_1, \mathbf{n}_2$  using Toeplitz matrix

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} -3 & 0 \\ 1 & -3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -4 \\ 2 \end{pmatrix} = \begin{pmatrix} 12 \\ -10 \\ 2 \end{pmatrix} \quad (4.1.8.35)$$

$$\Rightarrow \mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} 12 \\ -10 \\ 2 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (4.1.8.36)$$

Therefore the obtained  $\mathbf{n}_1, \mathbf{n}_2$  are correct.

Substitute (4.1.8.33), (4.1.8.34) in (4.1.8.21) and calculate for  $c_1$  and  $c_2$

$$c_2 \begin{pmatrix} -3 \\ 1 \end{pmatrix} + c_1 \begin{pmatrix} -4 \\ 2 \end{pmatrix} = \begin{pmatrix} -11 \\ -5 \end{pmatrix} \quad (4.1.8.37)$$

Solve using row reduction technique.

$$\Rightarrow \begin{pmatrix} -4 & -3 & -11 \\ 2 & 1 & -5 \end{pmatrix} \quad (4.1.8.38)$$

$$\xleftrightarrow{R_2 \leftarrow 2R_2 + R_1} \begin{pmatrix} -4 & -3 & -11 \\ 0 & -1 & -21 \end{pmatrix} \quad (4.1.8.39)$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - 3R_2} \begin{pmatrix} -4 & 0 & 52 \\ 0 & -1 & -21 \end{pmatrix} \quad (4.1.8.40)$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & -13 \\ 0 & 1 & 21 \end{pmatrix} \quad (4.1.8.41)$$

$$\Rightarrow c_1 = -13 \quad (4.1.8.42)$$

$$c_2 = 21 \quad (4.1.8.43)$$

Substituting (4.1.8.33),(4.1.8.34),(4.1.8.42),(4.1.8.43) in (4.1.8.17) and (4.1.8.18). We get equation of two straight lines.

$$(-3 \ 1)\mathbf{x} = -13 \quad (4.1.8.44)$$

$$(-4 \ 2)\mathbf{x} = 21 \quad (4.1.8.45)$$

The plot of these two lines is shown in Fig. 4.1.8.1.

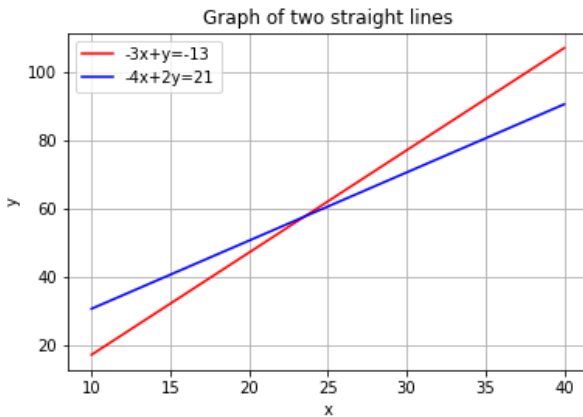


Fig. 4.1.8.1: Pair of straight lines for  $k = -10$

Now Lets find equation of lines for  $k = -\frac{35}{2}$ . Substitute  $k = -\frac{35}{2}$  in (4.1.8.1). We get equation of pair of straight lines as:

$$12x^2 - \frac{35}{2}xy + 2y^2 + 11x - 5y + 2 = 0 \quad (4.1.8.46)$$

From (4.1.8.1), (4.1.8.2), (4.1.8.3) we get

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 12 & -\frac{35}{4} \\ -\frac{35}{4} & 2 \end{pmatrix} \quad (4.1.8.47)$$

$$\mathbf{u} = \begin{pmatrix} \frac{11}{2} \\ \frac{5}{2} \end{pmatrix} \quad (4.1.8.48)$$

If  $|\mathbf{V}| < 0$  then two lines will intersect.

$$|\mathbf{V}| = \begin{vmatrix} 12 & -\frac{35}{4} \\ -\frac{35}{4} & 2 \end{vmatrix} \quad (4.1.8.49)$$

$$\Rightarrow |\mathbf{V}| = -\frac{841}{16} \quad (4.1.8.50)$$

$$\Rightarrow |\mathbf{V}| < 0 \quad (4.1.8.51)$$

Therefore the lines will intersect. Now from (4.1.8.20),

$$\Rightarrow \mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} = \begin{pmatrix} 12 \\ -\frac{35}{2} \\ 2 \end{pmatrix} \quad (4.1.8.52)$$

The slopes of the lines are given by roots of equation (4.1.8.23)

$$\Rightarrow 2m^2 - \frac{35}{2}m + 12 = 0 \quad (4.1.8.53)$$

$$m_i = \frac{-b \pm \sqrt{-|\mathbf{V}|}}{c} \quad (4.1.8.54)$$

$$\Rightarrow m_i = \frac{\frac{35}{4} \pm \sqrt{\frac{841}{16}}}{2} \quad (4.1.8.55)$$

$$\Rightarrow m_1 = 8 \quad (4.1.8.56)$$

$$m_2 = \frac{3}{4} \quad (4.1.8.57)$$

The normal vector for two lines is given by (4.1.8.29)

$$\Rightarrow \mathbf{n}_1 = k_1 \begin{pmatrix} -8 \\ 1 \end{pmatrix} \quad (4.1.8.58)$$

$$\mathbf{n}_2 = k_2 \begin{pmatrix} -\frac{3}{4} \\ 1 \end{pmatrix} \quad (4.1.8.59)$$

Substituting (4.1.8.58),(4.1.8.59) in (4.1.8.52). we get

$$k_1 k_2 = 2 \quad (4.1.8.60)$$

The possible combinations of  $(k_1, k_2)$  are (1,2), (2,1), (-1,-2) and (-2,-1).

lets assume  $k_1 = 1, k_2 = 2$  we get

$$\Rightarrow \mathbf{n}_1 = \begin{pmatrix} -8 \\ 1 \end{pmatrix} \quad (4.1.8.61)$$

$$\mathbf{n}_2 = \begin{pmatrix} -\frac{3}{2} \\ 2 \end{pmatrix} \quad (4.1.8.62)$$

We verify obtained  $\mathbf{n}_1, \mathbf{n}_2$  using Toeplitz matrix

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} -8 & 0 \\ 1 & -8 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{3}{2} \\ 2 \end{pmatrix} = \begin{pmatrix} 12 \\ -\frac{35}{2} \\ 2 \end{pmatrix} \quad (4.1.8.63)$$

$$\Rightarrow \mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} 12 \\ -\frac{35}{2} \\ 2 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (4.1.8.64)$$

Therefore the obtained  $\mathbf{n}_1, \mathbf{n}_2$  are correct.

Substitute (4.1.8.61), (4.1.8.62) in (4.1.8.21) we get

$$c_2 \begin{pmatrix} -8 \\ 1 \end{pmatrix} + c_1 \begin{pmatrix} -\frac{3}{2} \\ 2 \end{pmatrix} = \begin{pmatrix} -11 \\ -5 \end{pmatrix} \quad (4.1.8.65)$$

Solve using row reduction technique.

$$\Rightarrow \begin{pmatrix} -\frac{3}{2} & -8 & -11 \\ 2 & 1 & -5 \end{pmatrix} \quad (4.1.8.66)$$

$$\xleftrightarrow{R_1 \leftarrow 2R_1} \begin{pmatrix} -3 & -16 & -22 \\ 2 & 1 & -5 \end{pmatrix} \quad (4.1.8.67)$$

$$\xleftrightarrow{R_2 \leftarrow 3R_2 + 2R_1} \begin{pmatrix} -3 & -16 & -22 \\ 0 & -29 & -59 \end{pmatrix} \quad (4.1.8.68)$$

$$\xleftrightarrow{R_1 \leftarrow 29R_1 - 16R_2} \begin{pmatrix} -87 & 0 & 306 \\ 0 & -29 & -59 \end{pmatrix} \quad (4.1.8.69)$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & -\frac{102}{29} \\ 0 & 1 & \frac{59}{29} \end{pmatrix} \quad (4.1.8.70)$$

$$\Rightarrow c_1 = -\frac{102}{29} \quad (4.1.8.71)$$

$$c_2 = \frac{59}{29} \quad (4.1.8.72)$$

Substituting (4.1.8.61), (4.1.8.62), (4.1.8.71), (4.1.8.72) in (4.1.8.17) and (4.1.8.18). we get equation of two straight lines.

$$\begin{pmatrix} -8 & 1 \end{pmatrix} \mathbf{x} = -\frac{102}{29} \quad (4.1.8.73)$$

$$\begin{pmatrix} -\frac{3}{2} & 2 \end{pmatrix} \mathbf{x} = \frac{59}{29} \quad (4.1.8.74)$$

4.1.9. Find the value of  $k$  so that the following equation may represent a pair of straight lines

$$6x^2 + xy + ky^2 - 11x + 43y - 35 = 0 \quad (4.1.9.1)$$

**Solution:** The given second degree equation is, Comparing coefficients of (4.1.9.1) we get,

$$\mathbf{V} = \begin{pmatrix} 6 & \frac{1}{2} \\ \frac{1}{2} & k \end{pmatrix} \quad (4.1.9.2)$$

$$\mathbf{u} = \begin{pmatrix} -\frac{11}{2} \\ \frac{43}{2} \end{pmatrix} \quad (4.1.9.3)$$

$$f = -35 \quad (4.1.9.4)$$

The given second degree equation (4.1.9.1) will represent a pair of straight line if,

$$\begin{vmatrix} 6 & \frac{1}{2} & -\frac{11}{2} \\ \frac{1}{2} & k & \frac{43}{2} \\ -\frac{11}{2} & \frac{43}{2} & -35 \end{vmatrix} = 0 \quad (4.1.9.5)$$

Expanding the determinant,

$$k + 12 = 0 \quad (4.1.9.6)$$

$$\Rightarrow k = -12 \quad (4.1.9.7)$$

Hence, from (4.1.9.7) we find that for  $k = -12$ , the given second degree equation (4.1.9.1) represents pair of straight lines. For the appropriate value of  $k$ , (4.1.9.1) becomes,

$$6x^2 + xy - 12y^2 - 11x + 43y - 35 = 0 \quad (4.1.9.8)$$

Let the pair of straight lines in vector form is given by

$$\mathbf{n}_1^T \mathbf{x} = c_1 \quad (4.1.9.9)$$

$$\mathbf{n}_2^T \mathbf{x} = c_2 \quad (4.1.9.10)$$

The pair of straight lines is given by,

$$(\mathbf{n}_1^T \mathbf{x} - c_1)(\mathbf{n}_2^T \mathbf{x} - c_2) = \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (4.1.9.11)$$

Putting the values of  $\mathbf{V}$  and  $\mathbf{u}$  we get,

$$\mathbf{x}^T \begin{pmatrix} 6 & \frac{1}{2} \\ \frac{1}{2} & -12 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} -\frac{11}{2} & \frac{43}{2} \end{pmatrix} \mathbf{x} - 35 = 0 \quad (4.1.9.12)$$

Hence, from (4.1.9.12) we get,

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} 6 \\ 1 \\ -12 \end{pmatrix} \quad (4.1.9.13)$$

$$c_2 \mathbf{n}_1 + c_1 \mathbf{n}_2 = -2 \begin{pmatrix} -\frac{11}{2} \\ \frac{43}{2} \end{pmatrix} \quad (4.1.9.14)$$

$$c_1 c_2 = -35 \quad (4.1.9.15)$$

The slopes of the pair of straight lines are given by the roots of the polynomial,

$$cm^2 + 2bm + a = 0 \quad (4.1.9.16)$$

$$\Rightarrow m_i = \frac{-b \pm \sqrt{-\det(V)}}{c} \quad (4.1.9.17)$$

$$\mathbf{n}_i = k \begin{pmatrix} -m_i \\ 1 \end{pmatrix} \quad (4.1.9.18)$$

Substituting the values in above equations (4.1.9.16) we get,

$$-12m^2 + m + 6 = 0 \quad (4.1.9.19)$$

$$\Rightarrow m_i = \frac{-\frac{1}{2} \pm \sqrt{-\left(-\frac{289}{4}\right)}}{-12} \quad (4.1.9.20)$$

Solving equation (4.1.9.20) we get ,

$$m_1 = -\frac{2}{3} \quad (4.1.9.21)$$

$$m_2 = \frac{3}{4} \quad (4.1.9.22)$$

Hence putting the values of  $m_1$  and  $m_2$  in (4.1.9.18) we get

$$\mathbf{n}_1 = k_1 \begin{pmatrix} \frac{2}{3} \\ 1 \end{pmatrix} \quad (4.1.9.23)$$

$$\mathbf{n}_2 = k_2 \begin{pmatrix} -\frac{3}{4} \\ 1 \end{pmatrix} \quad (4.1.9.24)$$

Putting values of  $\mathbf{n}_1$  and  $\mathbf{n}_2$  in (4.1.9.13) we get,

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} -\frac{3k_2}{4} & 0 \\ k_2 & -\frac{3k_2}{4} \\ 0 & k_2 \end{pmatrix} \begin{pmatrix} \frac{2k_1}{3} \\ k_1 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ -12 \end{pmatrix} \quad (4.1.9.25)$$

$$\Rightarrow \begin{pmatrix} -\frac{1}{2}k_1k_2 \\ -\frac{1}{12}k_1k_2 \\ k_1k_2 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ -12 \end{pmatrix} \quad (4.1.9.26)$$

Thus, from (4.1.9.26),  $k_1k_2 = -12$ . Possible

combinations of  $(k_1, k_2)$  are  $(6, -2)$ ,  $(-6, 2)$ ,  $(3, -4)$ ,  $(-3, 4)$  Lets assume  $k_1 = 3$ ,  $k_2 = -4$ , then we get,

$$\mathbf{n}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad (4.1.9.27)$$

$$\mathbf{n}_2 = \begin{pmatrix} 3 \\ -4 \end{pmatrix} \quad (4.1.9.28)$$

From equation (4.1.9.14) we get

$$\begin{pmatrix} \mathbf{n}_1 & \mathbf{n}_2 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2\mathbf{u} \quad (4.1.9.29)$$

$$\begin{pmatrix} 2 & 3 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2 \begin{pmatrix} -\frac{11}{2} \\ \frac{43}{2} \end{pmatrix} \quad (4.1.9.30)$$

Hence we get the following equations,

$$2c_2 + 3c_1 = 11 \quad (4.1.9.31)$$

$$3c_2 - 4c_1 = -43 \quad (4.1.9.32)$$

The augmented matrix of (4.1.9.31), (4.1.9.32) is,

$$\begin{pmatrix} 2 & 3 & 11 \\ 3 & -4 & -43 \end{pmatrix} R_1 = \frac{1}{2} R_1 \begin{pmatrix} 1 & \frac{3}{2} & \frac{11}{2} \end{pmatrix} \quad (4.1.9.33)$$

$$\xrightarrow{R_2 = R_2 - 3R_1} \begin{pmatrix} 1 & \frac{3}{2} & \frac{11}{2} \\ 0 & -\frac{17}{2} & -\frac{119}{2} \end{pmatrix} \quad (4.1.9.34)$$

$$\xrightarrow{R_2 = -\frac{2}{17} R_2} \begin{pmatrix} 1 & \frac{3}{2} & \frac{11}{2} \\ 0 & 1 & 7 \end{pmatrix} \quad (4.1.9.35)$$

$$\xrightarrow{R_1 = R_1 - \frac{3}{2} R_2} \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 7 \end{pmatrix} \quad (4.1.9.36)$$

$$\quad (4.1.9.37)$$

Hence we get,

$$c_1 = -5 \quad (4.1.9.38)$$

$$c_2 = 7 \quad (4.1.9.39)$$

Hence (4.1.9.9), (4.1.9.10) can be modified as follows,

$$\begin{pmatrix} 2 & 3 \end{pmatrix} \mathbf{x} = -5 \quad (4.1.9.40)$$

$$\begin{pmatrix} 3 & -4 \end{pmatrix} \mathbf{x} = 7 \quad (4.1.9.41)$$

The figure below corresponds to the pair of straight lines represented by (4.1.9.40) and

(4.1.9.41).

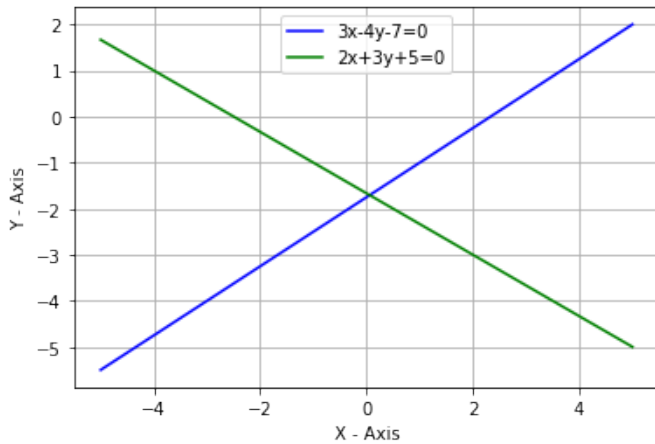


Fig. 4.1.9.1: Pair of Straight Lines

4.1.10. Find the value of  $k$  so that following equation may represent pairs of straight lines,

$$kxy - 8x + 9y - 12 = 0 \quad (4.1.10.1)$$

**Solution:** The general equation of second degree is given by,

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \quad (4.1.10.2)$$

In vector form from the equation (4.1.10.2) can be expressed as,

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (4.1.10.3)$$

where,

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad (4.1.10.4)$$

$$\mathbf{u} = \begin{pmatrix} d \\ e \end{pmatrix} \quad (4.1.10.5)$$

Now, comparing equation (4.1.10.2) to (4.1.10.1) we get,  $a = c = 0$ ,  $b = \left(\frac{k}{2}\right)$ ,  $d = -4$ ,  $e = \left(\frac{9}{2}\right)$ ,  $f = -12$ . Hence, substituting these values in equation (4.1.10.4) and (4.1.10.5) we get,

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 0 & \frac{k}{2} \\ \frac{k}{2} & 0 \end{pmatrix} \quad (4.1.10.6)$$

$$\mathbf{u} = \begin{pmatrix} -4 \\ \frac{9}{2} \end{pmatrix} \quad (4.1.10.7)$$

Now equation (4.1.10.1) represents pair of

straight lines if,

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \quad (4.1.10.8)$$

$$\begin{vmatrix} 0 & \frac{k}{2} & -4 \\ \frac{k}{2} & 0 & \frac{9}{2} \\ -4 & \frac{9}{2} & -12 \end{vmatrix} = 0 \quad (4.1.10.9)$$

$$\Rightarrow k = 0, k = 6 \quad (4.1.10.10)$$

Substituting (4.1.10.10) in (4.1.10.1) we get,

$$6xy - 8x + 9y - 12 = 0 \quad (4.1.10.11)$$

$$-8x + 9y - 12 = 0 \quad (4.1.10.12)$$

Hence value of  $k = 6$  represents pair of straight lines. Also it can be verified that the pair of lines intersect as,

$$|\mathbf{V}| = \begin{vmatrix} 0 & 3 \\ 3 & 0 \end{vmatrix} < 0 \quad (4.1.10.13)$$

Let the pair of straight lines is given by,

$$\mathbf{n}_1^T \mathbf{x} = c_1 \quad (4.1.10.14)$$

$$\mathbf{n}_2^T \mathbf{x} = c_2 \quad (4.1.10.15)$$

Now equating the product of equation (4.1.10.14) and (4.1.10.15) with (4.1.10.3) we get,

$$(\mathbf{n}_1^T \mathbf{x} - c_1)(\mathbf{n}_2^T \mathbf{x} - c_2) = \quad (4.1.10.16)$$

$$\mathbf{x}^T \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} -4 & \frac{9}{2} \end{pmatrix} \mathbf{x} - 12 \quad (4.1.10.17)$$

$$\Rightarrow n_1 * n_2 = \{0, 6, 0\} \quad (4.1.10.18)$$

$$c_1 n_1 + c_2 n_2 = \begin{pmatrix} 8 \\ -9 \end{pmatrix} \quad (4.1.10.19)$$

$$c_1 c_2 = -12. \quad (4.1.10.20)$$

Now the slopes of line is given by roots of polynomial,

$$cm^2 + 2bm + a = 0 \quad (4.1.10.21)$$

$$\Rightarrow 2bm = 0 \quad (4.1.10.22)$$



$$\Rightarrow m = 0 \quad (4.1.10.23)$$

Also

$$m_i = \frac{-b \pm \sqrt{-|V|}}{c} \quad (4.1.10.24)$$

$$\Rightarrow m_i = \frac{-0 \pm \sqrt{9}}{0} \quad (4.1.10.25)$$

$$\therefore m_1 = 0 \quad (4.1.10.26)$$

$$m_2 = \infty \quad (4.1.10.27)$$

The normal vector to the two lines is given by,

$$n_i = k_i \begin{pmatrix} -m_i \\ 1 \end{pmatrix} \quad (4.1.10.28)$$

$$\Rightarrow n_1 = k_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (4.1.10.29)$$

$$n_2 = k_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (4.1.10.30)$$

Also,

$$k_1 k_2 = 6 \quad (4.1.10.31)$$

Let  $k_1 = 2$  and  $k_2 = 3$

$$\Rightarrow n_1 = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \quad (4.1.10.32)$$

$$n_2 = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \quad (4.1.10.33)$$

We verify obtained  $n_1$  and  $n_2$  using Toeplitz matrix,

$$n_1 * n_2 = \begin{pmatrix} 0 & 0 \\ 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \\ 0 \end{pmatrix} \quad (4.1.10.34)$$

Hence (4.1.10.18) and (4.1.10.34) are same.  
Hence verified.

Now substituting it in (4.1.10.19) we get,

$$c_2 \begin{pmatrix} 0 \\ 2 \end{pmatrix} + c_1 \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 8 \\ -9 \end{pmatrix} \quad (4.1.10.35)$$

Solve using Row reduction Technique we get,

$$\Rightarrow \begin{pmatrix} 3 & 0 & 8 \\ 0 & 2 & -9 \end{pmatrix} \quad (4.1.10.36)$$

$$\xleftrightarrow{R_1 \leftarrow R_1/3} \begin{pmatrix} 1 & 0 & 8/3 \\ 0 & 2 & -9 \end{pmatrix} \quad (4.1.10.37)$$

$$\xleftrightarrow{R_2 \leftarrow R_2/2} \begin{pmatrix} 1 & 0 & 8/3 \\ 0 & 1 & -9/2 \end{pmatrix} \quad (4.1.10.38)$$

$$\Rightarrow c_1 = \frac{8}{3} \quad (4.1.10.39)$$

$$c_2 = \frac{-9}{2} \quad (4.1.10.40)$$

substituting the values of  $c_1$ ,  $c_2$  and equa-

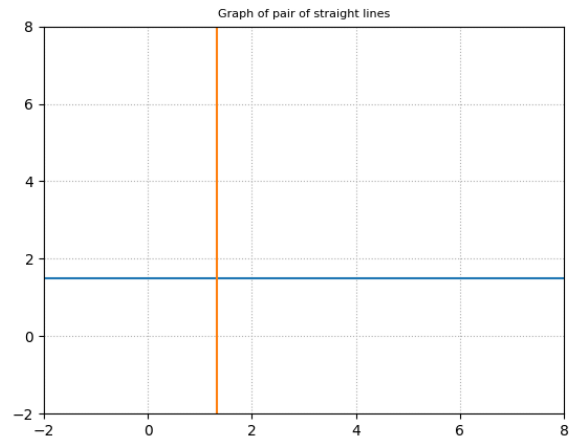


Fig. 4.1.10.1: Intersection of 2 lines

tion (4.1.10.32) and (4.1.10.33) to equation (4.1.10.14) and (4.1.10.15) we get equation of two straight lines.

$$\Rightarrow \begin{pmatrix} 0 & 2 \end{pmatrix} \mathbf{x} = \frac{8}{3} \quad (4.1.10.41)$$

$$\begin{pmatrix} 3 & 0 \end{pmatrix} \mathbf{x} = \frac{-9}{2} \quad (4.1.10.42)$$

Hence the equation of pair of straight lines are,

$$\left( \begin{pmatrix} 0 & 2 \end{pmatrix} \mathbf{x} - \frac{8}{3} \right) \left( \begin{pmatrix} 3 & 0 \end{pmatrix} \mathbf{x} - \frac{-9}{2} \right) = 0 \quad (4.1.10.43)$$

Hence, Plot of the equation (4.1.10.43) is shown in Figure.4.1.10.1 Now for value of  $k =$

0 does not represent pair of straight lines.as,

$$|\mathbf{V}| = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} \neq 0 \quad (4.1.10.44)$$

Hence, Plot of the equation  $(-8 \ 9)\mathbf{x} = 12$  is shown in figure 4.1.10.2,

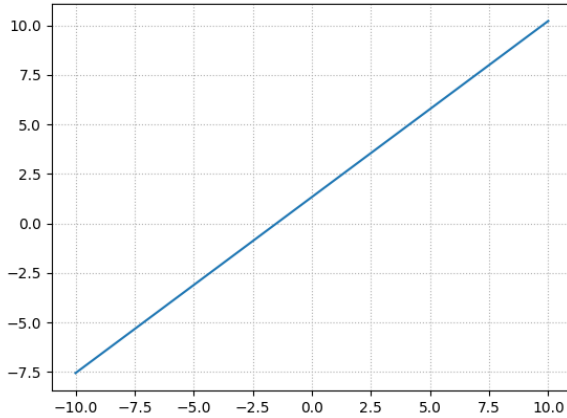


Fig. 4.1.10.2: Intersection of 2 lines

4.1.11. Find the value of k such that

$$x^2 + \frac{10}{3}(xy) + y^2 - 5x - 7y + k = 0 \quad (4.1.11.1)$$

represent pairs of straight lines.

**Solution:** From (4.1.11.1),

$$\mathbf{V} = \begin{pmatrix} 1 & \frac{5}{3} \\ \frac{5}{3} & 1 \end{pmatrix} \quad (4.1.11.2)$$

$$\mathbf{u}^T = \begin{pmatrix} -\frac{5}{2} & -\frac{7}{2} \end{pmatrix} \quad (4.1.11.3)$$

and

$$\begin{vmatrix} 1 & \frac{5}{3} & -\frac{5}{2} \\ \frac{5}{3} & 1 & -\frac{7}{2} \\ -\frac{5}{2} & -\frac{7}{2} & k \end{vmatrix} = 0 \quad (4.1.11.4)$$

$$\Rightarrow \left(k - \left(\frac{49}{4}\right)\right) - \frac{5}{3} \left(\frac{5}{3}k - \frac{35}{4}\right) - \frac{5}{2} \left(\frac{-35}{6} + \frac{5}{2}\right) = 0 \quad (4.1.11.5)$$

$$\Rightarrow \frac{64}{k} 36 - \frac{128}{12} = 0 \quad (4.1.11.6)$$

$$\Rightarrow \boxed{k = 6} \quad (4.1.11.7)$$

Substituting (4.1.11.7) in (4.1.11.1), we get

$$x^2 + \frac{10}{3}(xy) + y^2 - 5x - 7y + 6 = 0 \quad (4.1.11.8)$$

Hence value of k=6 represents pair of straight lines. Substituting value of k =6 in (4.1.11.4)

$$\delta = \begin{vmatrix} 1 & \frac{5}{3} & -\frac{5}{2} \\ \frac{5}{3} & 1 & -\frac{7}{2} \\ -\frac{5}{2} & -\frac{7}{2} & 6 \end{vmatrix} \quad (4.1.11.9)$$

Simplifying the above determinant, we get

$$\delta = 0 \quad (4.1.11.10)$$

(4.1.11.8) represents two straight lines

$$\det(\mathbf{V}) = \begin{vmatrix} 1 & \frac{5}{3} \\ \frac{5}{3} & 1 \end{vmatrix} < 0 \quad (4.1.11.11)$$

Since  $\det(\mathbf{V}) < 0$  lines would intersect each other

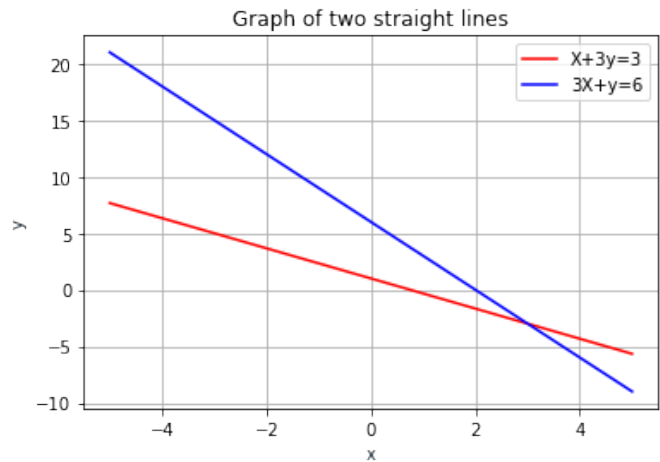


Fig. 4.1.11.1: Pair of straight lines

$$\mathbf{n}_1 * \mathbf{n}_2 = \{1, \frac{10}{3}, 1\} \quad (4.1.11.12)$$

$$c_2 \mathbf{n}_1 + c_1 \mathbf{n}_2 = -2 \begin{pmatrix} -\frac{5}{2} \\ -\frac{7}{2} \end{pmatrix} \quad (4.1.11.13)$$

$$c_1 c_2 = 6 \quad (4.1.11.14)$$

The slopes of the lines are given by the roots of the polynomial

$$cm^2 + 2bm + a = 0 \quad (4.1.11.15)$$

$$\Rightarrow m_i = \frac{-b \pm \sqrt{-\det(\mathbf{V})}}{c} \quad (4.1.11.16)$$

$$\mathbf{n}_i = k \begin{pmatrix} -m_i \\ 1 \end{pmatrix} \quad (4.1.11.17)$$

Substituting in above equations (4.1.11.15) we

get,

$$m^2 + \frac{10}{3}m + 1 = 0 \quad (4.1.11.18)$$

$$\Rightarrow m_i = \frac{\frac{-10}{3} \pm \sqrt{-\left(\frac{-16}{9}\right)}}{1} \quad (4.1.11.19)$$

Solving equation (4.1.11.19) we have ,

$$m_1 = \frac{-1}{3} \quad (4.1.11.20)$$

$$m_2 = -3 \quad (4.1.11.21)$$

$$\mathbf{n}_1 = k_1 \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} \quad (4.1.11.22)$$

$$\mathbf{n}_2 = k_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad (4.1.11.23)$$

Substituting equations (4.1.11.22), (4.1.11.23) in equation (4.1.11.12) we get

$$k_1 k_2 = 1 \quad (4.1.11.24)$$

Possible combination of  $(k_1, k_2)$  is (1,1) Lets assume  $k_1 = 1, k_2 = 1$ , we get

$$\mathbf{n}_1 = \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} \quad (4.1.11.25)$$

$$\mathbf{n}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad (4.1.11.26)$$

we have:

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (4.1.11.27)$$

Convolution of  $\mathbf{n}_1$  and  $\mathbf{n}_2$  can be done by converting  $\mathbf{n}_1$  into a teoplitz matrix and multiplying with  $\mathbf{n}_2$

From equation (4.1.11.25) and (4.1.11.26)

$$\mathbf{n}_1 = \begin{pmatrix} \frac{1}{3} & 0 \\ 1 & \frac{1}{3} \\ 0 & 1 \end{pmatrix} \mathbf{n}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad (4.1.11.28)$$

$$\Rightarrow \begin{pmatrix} \frac{1}{3} & 0 \\ 1 & \frac{1}{3} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{10}{3} \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (4.1.11.29)$$

$c_1$  and  $c_2$  can be obtained as,

$$\begin{pmatrix} \mathbf{n}_1 & \mathbf{n}_2 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2\mathbf{u} \quad (4.1.11.30)$$

$$\begin{pmatrix} \mathbf{n}_1 & \mathbf{n}_2 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2 \begin{pmatrix} -\frac{5}{2} \\ \frac{7}{2} \end{pmatrix} \quad (4.1.11.31)$$

Substituting (4.1.11.25) and (4.1.11.26) in (4.1.11.31), the augmented matrix is,

$$\begin{pmatrix} \frac{1}{3} & 3 & 5 \\ 1 & 1 & 7 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow 3 \times R_1} \begin{pmatrix} 1 & 9 & 15 \\ 1 & 1 & 7 \end{pmatrix} \quad (4.1.11.32)$$

$$\begin{pmatrix} 1 & 9 & 15 \\ 1 & 1 & 7 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 9 & 15 \\ 0 & -8 & -8 \end{pmatrix} \quad (4.1.11.33)$$

$$\begin{pmatrix} 1 & 9 & 15 \\ 0 & -8 & -8 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 \div -8} \begin{pmatrix} 1 & 9 & 15 \\ 0 & 1 & 1 \end{pmatrix} \quad (4.1.11.34)$$

$$\begin{pmatrix} 1 & 9 & 15 \\ 0 & 1 & 1 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 - 9 \times R_2} \begin{pmatrix} 1 & 0 & 6 \\ 0 & 1 & 1 \end{pmatrix} \quad (4.1.11.35)$$

From above we get

$$c_1 = 1 \quad (4.1.11.36)$$

$$c_2 = 6 \quad (4.1.11.37)$$

Hence pair of straight lines are

$$\begin{pmatrix} \frac{1}{3} & 1 \end{pmatrix} \mathbf{x} = 1 \quad (4.1.11.38)$$

$$\begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} = 6 \quad (4.1.11.39)$$

4.1.12. Prove that the equation

$$12x^2 + 7xy - 10y^2 + 13x + 45y - 35 = 0 \quad (4.1.12.1)$$

represents two straight lines and find the angle between the lines.

**Solution:** The above equation can be expressed as

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (4.1.12.2)$$

where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 12 & \frac{7}{2} \\ \frac{7}{2} & -10 \end{pmatrix} \quad (4.1.12.3)$$

$$\mathbf{u} = \begin{pmatrix} \frac{13}{2} \\ \frac{45}{2} \end{pmatrix} \quad (4.1.12.4)$$

$$f = -35 \quad (4.1.12.5)$$

(4.1.12.2) represents a pair of straight lines if

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \quad (4.1.12.6)$$

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = \begin{vmatrix} 12 & \frac{7}{2} & \frac{13}{2} \\ \frac{7}{2} & -10 & \frac{45}{2} \\ \frac{13}{2} & \frac{45}{2} & -35 \end{vmatrix} \quad (4.1.12.7)$$

$$\Rightarrow 12 \begin{vmatrix} -10 & \frac{45}{2} \\ \frac{45}{2} & -35 \end{vmatrix} - \frac{7}{2} \begin{vmatrix} \frac{7}{2} & \frac{45}{2} \\ \frac{13}{2} & -35 \end{vmatrix} + \frac{13}{2} \begin{vmatrix} \frac{7}{2} & -10 \\ \frac{13}{2} & \frac{45}{2} \end{vmatrix} = 0 \quad (4.1.12.8)$$

$$(4.1.12.9)$$

The lines intersect if

$$|\mathbf{V}| < 0 \quad (4.1.12.10)$$

$$|\mathbf{V}| = -\frac{529}{4} < 0 \quad (4.1.12.11)$$

From (4.1.12.8) and (4.1.12.11) it can be concluded that the given equation represents a pair of intersecting lines. Let the equations of lines be

$$\mathbf{n}_1^T \mathbf{x} = c_1 \quad (4.1.12.12)$$

$$\mathbf{n}_2^T \mathbf{x} = c_2 \quad (4.1.12.13)$$

Since (4.1.12.2) represents a pair of straight lines it must satisfy

$$(\mathbf{n}_1^T \mathbf{x} - c_1)(\mathbf{n}_2^T \mathbf{x} - c_2) = \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (4.1.12.14)$$

where

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} = \begin{pmatrix} 12 \\ 7 \\ -10 \end{pmatrix} \quad (4.1.12.15)$$

$$c_2 \mathbf{n}_1 + c_1 \mathbf{n}_2 = -2\mathbf{u} \quad (4.1.12.16)$$

$$c_1 c_2 = f \quad (4.1.12.17)$$

Slopes of the lines can be obtained by solving

$$cm^2 + 2bm + a = 0 \quad (4.1.12.18)$$

$$-10m^2 + 7m + 12 = 0 \quad (4.1.12.19)$$

$$\Rightarrow m_1 = \frac{-4}{5}, m_2 = \frac{3}{2} \quad (4.1.12.20)$$

The normal vectors can be expressed in terms

of corresponding slopes of lines as

$$\mathbf{n} = k \begin{pmatrix} -m \\ 1 \end{pmatrix} \quad (4.1.12.21)$$

$$\Rightarrow \mathbf{n}_1 = k_1 \begin{pmatrix} \frac{4}{5} \\ 1 \end{pmatrix} \quad (4.1.12.22)$$

$$\mathbf{n}_2 = k_2 \begin{pmatrix} -\frac{3}{2} \\ 1 \end{pmatrix} \quad (4.1.12.23)$$

Substituting (4.1.12.22) and (4.1.12.23) in (4.1.12.15) we get

$$k_1 k_2 = -10 \quad (4.1.12.24)$$

Assuming  $k_1 = 5$  and  $k_2 = -2$

$$\mathbf{n}_1 = \begin{pmatrix} 4 \\ 5 \end{pmatrix}, \mathbf{n}_2 = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \quad (4.1.12.25)$$

Verification using Toeplitz matrix

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} 4 & 0 \\ 5 & 4 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 12 \\ 7 \\ -10 \end{pmatrix} \quad (4.1.12.26)$$

From (4.1.12.16) we have

$$c_2 \begin{pmatrix} 4 \\ 5 \end{pmatrix} + c_1 \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} -13 \\ -45 \end{pmatrix} \quad (4.1.12.27)$$

Solving the augmented matrix

$$\begin{pmatrix} 4 & 3 & -13 \\ 5 & -2 & -45 \end{pmatrix} \xrightarrow{R_2 \leftarrow 4R_2 - 5R_1} \begin{pmatrix} 4 & 3 & -13 \\ 0 & -23 & -115 \end{pmatrix} \quad (4.1.12.28)$$

$$\xrightarrow{R_2 \leftarrow -\frac{R_2}{23}} \begin{pmatrix} 4 & 3 & -13 \\ 0 & 1 & 5 \end{pmatrix} \xrightarrow{R_1 \leftarrow R_1 - 3R_2} \begin{pmatrix} 4 & 0 & -28 \\ 0 & 1 & 5 \end{pmatrix} \quad (4.1.12.29)$$

$$\xrightarrow{R_1 \leftarrow \frac{R_1}{4}} \begin{pmatrix} 1 & 0 & -7 \\ 0 & 1 & 5 \end{pmatrix} \quad (4.1.12.30)$$

$$\Rightarrow c_1 = -7, c_2 = 5 \quad (4.1.12.31)$$

Thus the equation of lines are

$$(4 \ 5) \mathbf{x} = 5 \quad (4.1.12.32)$$

$$(3 \ -2) \mathbf{x} = -7 \quad (4.1.12.33)$$

The angle between the lines can be expressed in terms of normal vectors

$$\mathbf{n}_1 = \begin{pmatrix} 4 \\ 5 \end{pmatrix}, \mathbf{n}_2 = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \quad (4.1.12.34)$$

as

$$\cos \theta = \frac{\mathbf{n}_1^T \mathbf{n}_2}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} \quad (4.1.12.35)$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{2}{\sqrt{533}}\right) = \tan^{-1}\left(\frac{23}{2}\right) \quad (4.1.12.36)$$

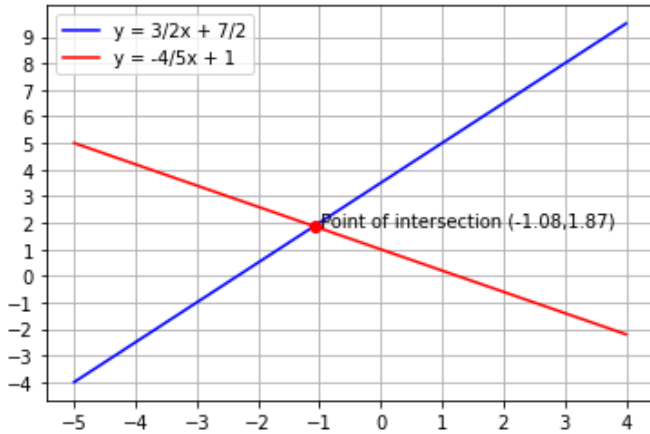


Fig. 4.1.12.1

4.1.13. Find the value of  $k$  so that the following equation may represent the pair of straight lines:

$$2x^2 + xy - y^2 + kx + 6y - 9 = 0 \quad (4.1.13.1)$$

**Solution:** We need to find the value of  $k$  for which (4.1.13.1) represents a pair of straight lines.

Converting (4.1.13.1) into vector form, we get

$$\mathbf{x}^T \begin{pmatrix} 2 & 1/2 \\ 1/2 & -1 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} k/2 \\ 3 \end{pmatrix} \mathbf{x} - 9 = 0 \quad (4.1.13.2)$$

Here, we have

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 2 & 1/2 \\ 1/2 & -1 \end{pmatrix} \quad (4.1.13.3)$$

$$\mathbf{u} = \begin{pmatrix} k/2 \\ 3 \end{pmatrix} \quad (4.1.13.4)$$

$$f = -9 \quad (4.1.13.5)$$

The above represents a pair of straight lines if

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \quad (4.1.13.6)$$

Since (4.1.13.1) represents a pair of straight lines, then by (4.1.13.6), we have

$$\begin{vmatrix} 2 & 1/2 & k/2 \\ 1/2 & -1 & 3 \\ k/2 & 3 & -9 \end{vmatrix} = 0 \quad (4.1.13.7)$$

By solving, above determinant we get

$$2(9 - 9) + \frac{-1}{2}\left(\frac{-9}{2} + \frac{-3k}{2}\right) + \frac{k}{2}\left(\frac{3}{2} + \frac{k}{2}\right) = 0 \quad (4.1.13.8)$$

$$\frac{(9 + 3k)}{4} + \frac{k(3 + k)}{4} = 0 \quad (4.1.13.9)$$

$$k^2 + 6k + 9 = 0 \quad (4.1.13.10)$$

$$(k + 3)^2 = 0 \quad (4.1.13.11)$$

$$k = -3 \quad (4.1.13.12)$$

Hence by (4.1.13.12), we have

$$2x^2 + xy - y^2 - 3x + 6y - 9 = 0 \quad (4.1.13.13)$$

represents family of straight lines for  $k = -3$ .

To find the straight lines, we write each of them in their vector form as

$$\mathbf{n}_1^T \mathbf{x} = c_1 \quad (4.1.13.14)$$

$$\mathbf{n}_2^T \mathbf{x} = c_2 \quad (4.1.13.15)$$

Equating the product of above with (4.1.13.2), we have

$$\begin{aligned} & (\mathbf{n}_1^T \mathbf{x} - c_1)(\mathbf{n}_2^T \mathbf{x} - c_2) = \\ & \mathbf{x}^T \begin{pmatrix} 2 & 1/2 \\ 1/2 & -1 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} k/2 \\ 3 \end{pmatrix} \mathbf{x} - 9 \end{aligned} \quad (4.1.13.16)$$

$$\Rightarrow \mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \quad (4.1.13.17)$$

$$c_2 \mathbf{n}_1 + c_1 \mathbf{n}_2 = -2 \begin{pmatrix} -3/2 \\ 3 \end{pmatrix} \quad (4.1.13.18)$$

$$c_1 c_2 = -9 \quad (4.1.13.19)$$

Here, the slope of these lines are given by the

roots of the polynomial

$$-m^2 + m + 2 = 0 \quad (4.1.13.20)$$

$$m^2 - m - 2 = 0 \quad (4.1.13.21)$$

$$m = \frac{1 \pm \sqrt{1+8}}{2} \quad (4.1.13.22)$$

$$m_1 = \frac{1+3}{2} = 2 \quad (4.1.13.23)$$

$$m_2 = \frac{1-3}{2} = -1 \quad (4.1.13.24)$$

$$n_1 = k_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad (4.1.13.25)$$

$$n_2 = k_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (4.1.13.26)$$

Substituting (4.1.13.25) and (4.1.13.26) in (4.1.13.17), we get

$$k_1 k_2 = -1 \quad (4.1.13.27)$$

Taking  $k_1 = -1$  and  $k_2 = 1$ , we get

$$n_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad (4.1.13.28)$$

$$n_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (4.1.13.29)$$

Substituting in (4.1.13.18) for above values of  $n_1$  and  $n_2$

$$(n_1 n_2) \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \end{pmatrix} \quad (4.1.13.30)$$

$$\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \end{pmatrix} \quad (4.1.13.31)$$

Solving (4.1.13.31),

$$\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \end{pmatrix} \xrightarrow{r_2=r_2+2r_1} \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ -9 \end{pmatrix} \quad (4.1.13.32)$$

$$\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ -9 \end{pmatrix} \xrightarrow{r_2=r_2/3} \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \end{pmatrix} \quad (4.1.13.33)$$

$$\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \end{pmatrix} \xrightarrow{r_1=r_1-r_2} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \end{pmatrix} \quad (4.1.13.34)$$

Hence, we found out

$$c_1 = -3 \quad (4.1.13.35)$$

$$c_2 = 3 \quad (4.1.13.36)$$

Thus, pair of straight lines are

$$(2 \ -1) \mathbf{x} = -3 \quad (4.1.13.37)$$

$$(1 \ 1) \mathbf{x} = 3 \quad (4.1.13.38)$$

where

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \quad (4.1.13.39)$$

The plot of above is shown below

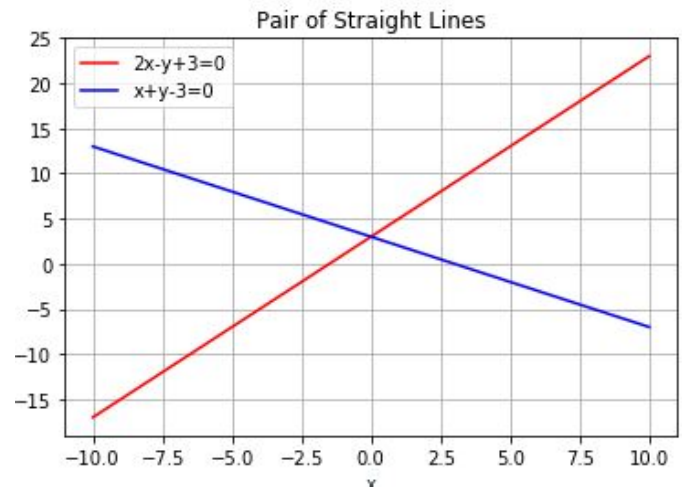


Fig. 4.1.13.1: Pair of Straight Lines

4.1.14. Prove that the equation  $12x^2 + 7xy - 10y^2 + 13x + 45y - 35 = 0$  represents two straight lines and find the angle between them.

**Solution:** The general second order equation is given by ,

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \quad (4.1.14.1)$$

Given,

$$12x^2 + 7xy - 10y^2 + 13x + 45y - 35 = 0 \quad (4.1.14.2)$$

The above equation can be expressed as

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (4.1.14.3)$$

where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 12 & \frac{7}{2} \\ \frac{7}{2} & -10 \end{pmatrix} \quad (4.1.14.4)$$

$$\mathbf{u} = \begin{pmatrix} \frac{13}{2} \\ \frac{45}{2} \end{pmatrix} \quad (4.1.14.5)$$

$$f = -35 \quad (4.1.14.6)$$

(4.1.14.3) represents a pair of straight lines if

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \quad (4.1.14.7)$$

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = \begin{vmatrix} 12 & \frac{7}{2} & \frac{13}{2} \\ \frac{7}{2} & -10 & \frac{45}{2} \\ \frac{13}{2} & \frac{45}{2} & -35 \end{vmatrix} \quad (4.1.14.8)$$

$$\Rightarrow 12 \begin{vmatrix} -10 & \frac{45}{2} \\ \frac{45}{2} & -35 \end{vmatrix} - \frac{7}{2} \begin{vmatrix} \frac{7}{2} & \frac{45}{2} \\ \frac{13}{2} & -35 \end{vmatrix} + \frac{13}{2} \begin{vmatrix} \frac{7}{2} & -10 \\ \frac{13}{2} & \frac{45}{2} \end{vmatrix} = 0 \quad (4.1.14.9)$$

The lines intersect if

$$|\mathbf{V}| < 0 \quad (4.1.14.10)$$

$$|\mathbf{V}| = -\frac{529}{4} < 0 \quad (4.1.14.11)$$

From (4.1.14.9) and (4.1.14.11) it can be concluded that the given equation represents a pair of intersecting lines.

Let  $(\alpha, \beta)$  be their point of intersection, then

$$\begin{pmatrix} 12 & \frac{7}{2} \\ \frac{7}{2} & -10 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -\frac{13}{2} \\ -\frac{45}{2} \end{pmatrix} \quad (4.1.14.12)$$

$$\Rightarrow \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad (4.1.14.13)$$

From *Spectral theorem*,  $\mathbf{V} = \mathbf{PDP}^T$  (4.1.14.14)

$$\mathbf{V} = \begin{pmatrix} 12 & \frac{7}{2} \\ \frac{7}{2} & -10 \end{pmatrix} \quad (4.1.14.15)$$

$$\mathbf{P} = \begin{pmatrix} \frac{-\sqrt{533}-22}{2} & \frac{-22+\sqrt{533}}{2} \\ 1 & 1 \end{pmatrix} \quad (4.1.14.16)$$

$$\mathbf{D} = \begin{pmatrix} 1 + \frac{\sqrt{533}}{2} & 0 \\ 0 & 1 - \frac{\sqrt{533}}{2} \end{pmatrix} \quad (4.1.14.17)$$

Using *Spectral decomposition* of matrix we can

express equation as

$$u_1(x - \alpha) + u_2(y - \beta) = \pm \sqrt{-\frac{\lambda_2}{\lambda_1}}(v_1(x - \alpha) + v_2(y - \beta)) \quad (4.1.14.18)$$

Substituting values in above equation we get;

$$\begin{aligned} & \frac{\sqrt{533}-22}{2}(x+1) + (y-2) \\ &= \pm \sqrt{-\frac{1-\frac{\sqrt{533}}{2}}{1+\frac{\sqrt{533}}{2}}} \left( \frac{-22-\sqrt{533}}{2}(x+1) + (y-2) \right) \end{aligned} \quad (4.1.14.19)$$

Simplifying (4.1.14.19),

$$3x - 2y + 7 = 0 \text{ and } 4x + 5y - 5 = 0$$

$$(4.1.14.20)$$

$$\Rightarrow (3x - 2y + 7)(4x + 5y - 5) = 0 \quad (4.1.14.21)$$

Thus the equation of lines are

$$(4 \ 5)\mathbf{x} = 5 \quad (4.1.14.22)$$

$$(3 \ -2)\mathbf{x} = -7 \quad (4.1.14.23)$$

Angle between the straight lines: The angle

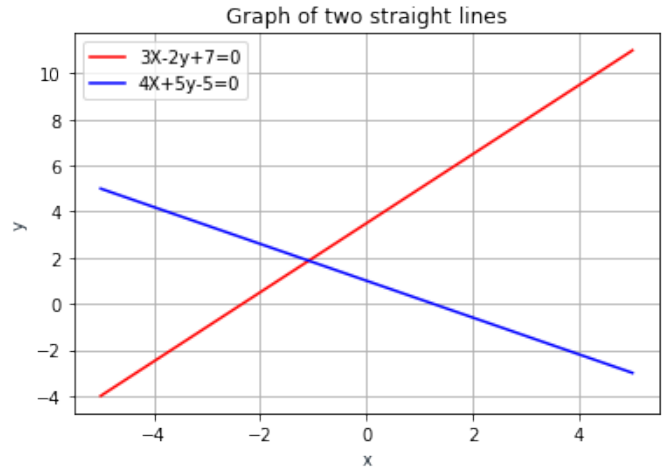


Fig. 1: Pair of straight lines

between the lines can be expressed in terms of normal vectors

$$\mathbf{n}_1 = \begin{pmatrix} 4 \\ 5 \end{pmatrix}, \quad \mathbf{n}_2 = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \quad (4.1.14.24)$$

$$\cos \theta = \frac{\mathbf{n}_1^T \mathbf{n}_2}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} \quad (4.1.14.25)$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{2}{\sqrt{533}}\right) = \tan^{-1}\left(\frac{23}{2}\right) \quad (4.1.14.26)$$

4.1.15. Find the value of  $h$  so that the equation

$$6x^2 + 2hxy + 12y^2 + 22x + 31y + 20 = 0 \quad (4.1.15.1)$$

may represent two straight lines.

**Solution:** The general equation second degree is given by

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \quad (4.1.15.2)$$

(4.1.15.2) represents pair of straight lines if

$$\begin{vmatrix} a & h & d \\ h & c & e \\ d & e & f \end{vmatrix} = 0 \quad (4.1.15.3)$$

From (4.1.15.3), given equation represents pair of straight lines if

$$\begin{vmatrix} 6 & h & 11 \\ h & 12 & \frac{31}{2} \\ 11 & \frac{31}{2} & 20 \end{vmatrix} = 0 \quad (4.1.15.4)$$

$$\Rightarrow h = \frac{17}{2} \text{ or } h = \frac{171}{20} \quad (4.1.15.5)$$

Verify (4.1.15.5) using python code from

[https://github.com/shreeprasadbhat/matrix-theory/tree/master/assignment5/codes/solve\\_determinant.py](https://github.com/shreeprasadbhat/matrix-theory/tree/master/assignment5/codes/solve_determinant.py)

The general equation second degree is given by

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \quad (4.1.15.6)$$

Let  $(\alpha, \beta)$  be their point of intersection, then

$$\begin{pmatrix} a & h \\ h & b \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -d \\ -e \end{pmatrix} \quad (4.1.15.7)$$

Under Affine transformation,

$$\mathbf{x} = \mathbf{M}\mathbf{y} + \mathbf{c} \quad (4.1.15.8)$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \quad (4.1.15.9)$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X + \alpha \\ Y + \beta \end{pmatrix} \quad (4.1.15.10)$$

(4.1.15.6) under transformation (4.1.15.10) will become,

$$aX^2 + 2bXY + cY^2 = 0 \quad (4.1.15.11)$$

$$\begin{pmatrix} X & Y \end{pmatrix} \begin{pmatrix} a & h \\ h & b \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = 0 \quad (4.1.15.12)$$

$$\begin{pmatrix} X & Y \end{pmatrix} \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = 0 \quad (4.1.15.13)$$

$$\begin{pmatrix} X' & Y' \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} X' \\ Y' \end{pmatrix} = 0 \quad (4.1.15.14)$$

where  $X' = Xu_1 + Yv_1$  and  $Y' = Xu_2 + Yv_2$

$$\Rightarrow \lambda_1(X')^2 + \lambda_2(Y')^2 = 0 \quad (4.1.15.15)$$

This is called *Spectral decomposition* of matrix

$$X' = \pm \sqrt{-\frac{\lambda_2}{\lambda_1}} Y' \quad (4.1.15.16)$$

$$u_1X + u_2Y = \pm \sqrt{-\frac{\lambda_2}{\lambda_1}} (v_1X + v_2Y) \quad (4.1.15.17)$$

$$u_1(x - \alpha) + u_2(y - \beta) = \pm \sqrt{-\frac{\lambda_2}{\lambda_1}} (v_1(x - \alpha) + v_2(y - \beta)) \quad (4.1.15.18)$$

Given equation is

$$6x^2 + 17xy + 12y^2 + 22x + 31y + 20 = 0 \quad (4.1.15.19)$$

Substituting in (4.1.15.7)

$$\begin{pmatrix} 6 & \frac{17}{2} \\ \frac{17}{2} & 12 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -11 \\ -\frac{31}{2} \end{pmatrix} \quad (4.1.15.20)$$

$$\Rightarrow \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad (4.1.15.21)$$

Verify (4.1.15.21) using python code from

[https://github.com/shreeprasadbhat/matrix-theory/tree/master/assignment5/codes/find\\_intersection.py](https://github.com/shreeprasadbhat/matrix-theory/tree/master/assignment5/codes/find_intersection.py)



Taking  $h = \frac{17}{2}$

$$\mathbf{V} = \mathbf{PDP}^T \quad (4.1.15.22)$$

$$\mathbf{V} = \begin{pmatrix} 6 & \frac{17}{2} \\ \frac{17}{2} & 12 \end{pmatrix} \quad (4.1.15.23)$$

$$\mathbf{P} = \begin{pmatrix} \frac{-5\sqrt{13}-6}{17} & \frac{-6+5\sqrt{13}}{17} \\ 1 & 1 \end{pmatrix} \quad (4.1.15.24)$$

$$\mathbf{D} = \begin{pmatrix} 9 - \frac{5\sqrt{13}}{2} & 0 \\ 0 & 9 + \frac{5\sqrt{13}}{2} \end{pmatrix} \quad (4.1.15.25)$$

Verify (4.1.15.24) and (4.1.15.25) using python code from

<https://github.com/shreeprasadbhat/matrix-theory/tree/master/assignment5/codes/diagonalize1.py>

Substituting (4.1.15.21), (4.1.15.24) and (4.1.15.25) in (4.1.15.18),

$$\begin{aligned} & \frac{-5\sqrt{13}-6}{17}(x+1) + (y-2) \\ &= \pm \sqrt{-\frac{9 + \frac{5\sqrt{13}}{2}}{9 - \frac{5\sqrt{13}}{2}}} \left( \frac{-6 + 5\sqrt{13}}{17}(x+1) + (y+2) \right) \end{aligned} \quad (4.1.15.26)$$

Simplifying (4.1.15.26),

$$2x + 3y + 4 = 0 \text{ and } 3x + 4y + 5 = 0 \quad (4.1.15.27)$$

$$\Rightarrow (2x + 3y + 4)(3x + 4y + 5) = 0 \quad (4.1.15.28)$$

Verify (4.1.15.27) using python code from

<https://github.com/shreeprasadbhat/matrix-theory/tree/master/assignment5/codes/calculate1.py>

Taking  $h = \frac{171}{20}$

$$\mathbf{V} = \mathbf{PDP}^T \quad (4.1.15.29)$$

$$\mathbf{V} = \begin{pmatrix} 6 & \frac{171}{2} \\ \frac{171}{2} & 12 \end{pmatrix} \quad (4.1.15.30)$$

$$\mathbf{P} = \begin{pmatrix} \frac{-\sqrt{3649}-20}{57} & \frac{-20+\sqrt{3649}}{57} \\ 1 & 1 \end{pmatrix} \quad (4.1.15.31)$$

$$\mathbf{D} = \begin{pmatrix} 9 - \frac{3\sqrt{3649}}{20} & 0 \\ 0 & 9 + \frac{3\sqrt{3649}}{20} \end{pmatrix} \quad (4.1.15.32)$$

Verify (4.1.15.31) and (4.1.15.32) using python 5.1.1. code from

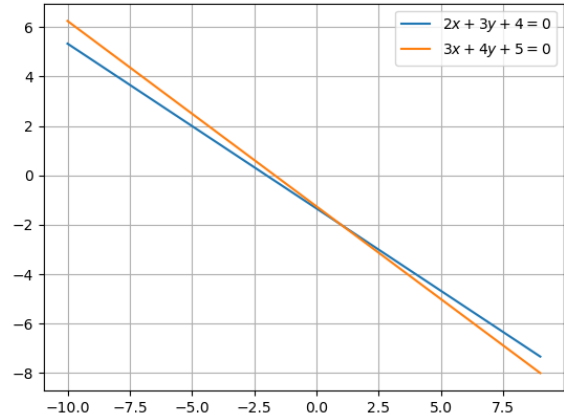


Fig. 1: Pair of straight lines  $3x + 4y + 5 = 0$  and  $2x + 3y + 4 = 0$

<https://github.com/shreeprasadbhat/matrix-theory/tree/master/assignment5/codes/diagonalize2.py>

Substituting (4.1.15.21), (4.1.15.31) and (4.1.15.32) in (4.1.15.18),

$$\begin{aligned} & \frac{-\sqrt{3649}-20}{57}(x+1) + (y-2) \\ &= \pm \sqrt{-\frac{9 + \frac{3\sqrt{3649}}{20}}{9 - \frac{3\sqrt{3649}}{20}}} \left( \frac{-20 + \sqrt{3649}}{57}(x+1) + (y+2) \right) \end{aligned} \quad (4.1.15.33)$$

Simplifying (4.1.15.32),

$$2x + 3y + 4 = 0 \text{ and } 3x + 4y + 5 = 0 \quad (4.1.15.34)$$

$$\Rightarrow (2x + 3y + 4)(3x + 4y + 5) = 0 \quad (4.1.15.35)$$

Verify (4.1.15.33) using python code from

<https://github.com/shreeprasadbhat/matrix-theory/tree/master/assignment5/codes/calculate2.py>

## 5 GENERAL EQUATION. TRACING OF CURVES

### 5.1 40

What conics do the following equation represent? When possible, find the centres and also

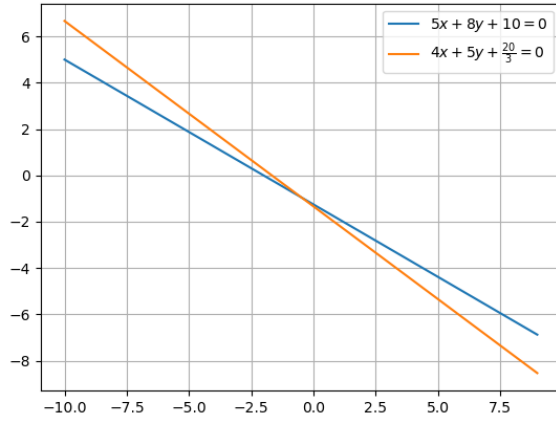


Fig. 1: Pair of straight lines  $4x + 5y + \frac{20}{3} = 0$  and  $5x + 8y + 10 = 0$

their equations referred to the centre

$$12x^2 - 23xy + 10y^2 - 25x + 26y = 14 \quad (5.1.1.1)$$

**Solution:** The given equation (5.1.1.1) can be expressed as

$$\mathbf{x}^T \begin{pmatrix} 12 & \frac{-23}{2} \\ \frac{-23}{2} & 10 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} \frac{-25}{2} & 13 \end{pmatrix} \mathbf{x} - 14 = 0 \quad (5.1.1.2)$$

where

$$\mathbf{V} = \begin{pmatrix} 12 & \frac{-23}{2} \\ \frac{-23}{2} & 10 \end{pmatrix} \quad (5.1.1.3)$$

$$\mathbf{u} = \begin{pmatrix} \frac{-25}{2} \\ 13 \end{pmatrix} \quad (5.1.1.4)$$

$$f = -14 \quad (5.1.1.5)$$

$$\det(\mathbf{V}) = \begin{vmatrix} 12 & \frac{-23}{2} \\ \frac{-23}{2} & 10 \end{vmatrix} \quad (5.1.1.6)$$

$$\Rightarrow \det(\mathbf{V}) = \frac{-49}{4} \quad (5.1.1.7)$$

$$\Rightarrow \det(\mathbf{V}) < 0 \quad (5.1.1.8)$$

Since  $\det(\mathbf{V}) < 0$  the given equation (5.1.1.2) represents the hyperbola. The characteristic equation of  $\mathbf{V}$  is obtained by evaluating the determinant

$$| \mathbf{V} - \lambda \mathbf{I} | = 0 \quad (5.1.1.9)$$

$$\begin{vmatrix} 12 - \lambda & \frac{-23}{2} \\ \frac{-23}{2} & 10 - \lambda \end{vmatrix} = 0 \quad (5.1.1.10)$$

$$\Rightarrow 4\lambda^2 - 88\lambda - 49 = 0 \quad (5.1.1.11)$$

The eigenvalues are the roots of equation 5.1.1.11 is given by

$$\lambda_1 = \frac{22 + \sqrt{533}}{2} \quad (5.1.1.12)$$

$$\lambda_2 = \frac{22 - \sqrt{533}}{2} \quad (5.1.1.13)$$

The eigenvector  $\mathbf{p}$  is defined as

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \quad (5.1.1.14)$$

$$\Rightarrow (\mathbf{V} - \lambda\mathbf{I})\mathbf{p} = 0 \quad (5.1.1.15)$$

For  $\lambda_1 = \frac{22 - \sqrt{533}}{2}$ ,

$$(\mathbf{V} - \lambda_1\mathbf{I}) = \begin{pmatrix} \frac{\sqrt{533}+2}{2} & \frac{-23}{2} \\ \frac{-23}{2} & \frac{\sqrt{533}-2}{2} \end{pmatrix} \quad (5.1.1.16)$$

By row reduction,

$$\begin{pmatrix} \frac{\sqrt{533}+2}{2} & \frac{-23}{2} \\ \frac{-23}{2} & \frac{\sqrt{533}-2}{2} \end{pmatrix} \quad (5.1.1.17)$$

$$\xrightarrow{R_1 = \frac{R_1}{\frac{\sqrt{533}+2}{2}}} \begin{pmatrix} 1 & \frac{2-\sqrt{533}}{23} \\ \frac{-23}{2} & \frac{\sqrt{533}-2}{2} \end{pmatrix} \quad (5.1.1.18)$$

$$\xrightarrow{R_2 = R_2 + \frac{23}{2}R_1} \begin{pmatrix} 1 & \frac{2-\sqrt{533}}{23} \\ 0 & 0 \end{pmatrix} \quad (5.1.1.19)$$

Substituting equation 5.1.1.19 in equation 5.1.1.15 we get

$$\begin{pmatrix} 1 & \frac{2-\sqrt{533}}{23} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (5.1.1.20)$$

Where,  $\mathbf{p} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

Let  $v_2 = t$

$$v_1 = \frac{-t(2 - \sqrt{533})}{23} \quad (5.1.1.21)$$

Eigen vector  $\mathbf{p}_1$  is given by

$$\mathbf{p}_1 = \begin{pmatrix} \frac{-t(2 - \sqrt{533})}{23} \\ t \end{pmatrix} \quad (5.1.1.22)$$

Let  $t = 1$ , we get

$$\mathbf{p}_1 = \begin{pmatrix} \frac{\sqrt{533}-2}{23} \\ 1 \end{pmatrix} \quad (5.1.1.23)$$

For  $\lambda_2 = \frac{22+\sqrt{533}}{2}$ ,

$$(\mathbf{V} - \lambda_2 \mathbf{I}) = \begin{pmatrix} \frac{2-\sqrt{533}}{2} & \frac{-23}{2} \\ \frac{-23}{2} & \frac{-2-\sqrt{533}}{2} \end{pmatrix} \quad (5.1.1.24)$$

By row reduction ,

$$\begin{pmatrix} \frac{2-\sqrt{533}}{2} & \frac{-23}{2} \\ \frac{-23}{2} & \frac{-2-\sqrt{533}}{2} \end{pmatrix} \quad (5.1.1.25)$$

$$\xleftrightarrow{R_1 = \frac{R_1}{\frac{2-\sqrt{533}}{2}}} \begin{pmatrix} 1 & \frac{2+\sqrt{533}}{23} \\ \frac{-23}{2} & \frac{-2-\sqrt{533}}{2} \end{pmatrix} \quad (5.1.1.26)$$

$$\xleftrightarrow{R_2 = R_2 + \frac{23}{2} R_1} \begin{pmatrix} 1 & \frac{2+\sqrt{533}}{23} \\ 0 & 0 \end{pmatrix} \quad (5.1.1.27)$$

Substituting equation 5.1.1.27 in equation 5.1.1.15 we get

$$\begin{pmatrix} 1 & \frac{2+\sqrt{533}}{23} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (5.1.1.28)$$

Where,  $\mathbf{p} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

Let  $v_2 = t$

$$v_1 = \frac{-t(2 + \sqrt{533})}{23} \quad (5.1.1.29)$$

Eigen vector  $\mathbf{p}_2$  is given by

$$\mathbf{p}_2 = \begin{pmatrix} \frac{-t(2+\sqrt{533})}{23} \\ t \end{pmatrix} \quad (5.1.1.30)$$

Let  $t = 1$ , we get

$$\mathbf{p}_2 = \begin{pmatrix} \frac{-\sqrt{533}-2}{23} \\ 1 \end{pmatrix} \quad (5.1.1.31)$$

By eigen decomposition  $\mathbf{V}$  can be represented by

$$\mathbf{V} = \mathbf{PDP}^T \quad (5.1.1.32)$$

where

$$\mathbf{P} = (\mathbf{p}_1 \quad \mathbf{p}_2) \quad (5.1.1.33)$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad (5.1.1.34)$$

Substituting equations 5.1.1.23, 5.1.1.31 in

equation 5.1.1.33 we get

$$\mathbf{P} = \begin{pmatrix} \frac{\sqrt{533}-2}{23} & \frac{-\sqrt{533}-2}{23} \\ 1 & 1 \end{pmatrix} \quad (5.1.1.35)$$

Substituting equations 5.1.1.12, 5.1.1.13 in 5.1.1.34 we get

$$\mathbf{D} = \begin{pmatrix} \frac{22-\sqrt{533}}{2} & 0 \\ 0 & \frac{22+\sqrt{533}}{2} \end{pmatrix} \quad (5.1.1.36)$$

Centre of the hyperbola is given by

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \quad (5.1.1.37)$$

$$\Rightarrow \mathbf{c} = -\begin{pmatrix} \frac{-40}{49} & \frac{-46}{49} \\ \frac{-46}{49} & \frac{-48}{49} \end{pmatrix} \begin{pmatrix} \frac{-25}{2} \\ 13 \end{pmatrix} \quad (5.1.1.38)$$

$$\Rightarrow \mathbf{c} = \begin{pmatrix} \frac{40}{49} & \frac{46}{49} \\ \frac{46}{49} & \frac{48}{49} \end{pmatrix} \begin{pmatrix} \frac{-25}{2} \\ 13 \end{pmatrix} \quad (5.1.1.39)$$

$$\Rightarrow \mathbf{c} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad (5.1.1.40)$$

Since,

$$\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f = 26 > 0 \quad (5.1.1.41)$$

there isn't a need to swap axes

In hyperbola,

$$axes = \begin{cases} \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}} \\ \sqrt{\frac{f - \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u}}{\lambda_2}} \end{cases} \quad (5.1.1.42)$$

From above equations we can say that,

$$\sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}} = \frac{2\sqrt{13}}{\sqrt{22 + \sqrt{533}}} \quad (5.1.1.43)$$

$$\sqrt{\frac{f - \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u}}{\lambda_2}} = \frac{2\sqrt{13}}{\sqrt{\sqrt{533} - 22}} \quad (5.1.1.44)$$

Now (5.1.1.2) can be written as,

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f \quad (5.1.1.45)$$

where ,

$$\mathbf{y} = \mathbf{P}^T(\mathbf{x} - \mathbf{c}) \quad (5.1.1.46)$$

To get  $\mathbf{y}$ ,

$$\mathbf{y} = \mathbf{P}^T \mathbf{x} - \mathbf{P}^T \mathbf{c} \quad (5.1.1.47)$$

$$\mathbf{y} = \begin{pmatrix} \frac{\sqrt{533}-2}{23} & 1 \\ -\frac{\sqrt{533}-2}{23} & 1 \end{pmatrix} \mathbf{x} - \begin{pmatrix} \frac{\sqrt{533}-2}{23} & 1 \\ -\frac{\sqrt{533}-2}{23} & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad (5.1.1.48)$$

$$\mathbf{y} = \begin{pmatrix} \frac{\sqrt{533}-2}{23} & 1 \\ -\frac{\sqrt{533}-2}{23} & 1 \end{pmatrix} \mathbf{x} - \begin{pmatrix} \frac{2(\sqrt{533}-2)}{23} + 1 \\ \frac{2(-\sqrt{533}-2)}{23} + 1 \end{pmatrix} \quad (5.1.1.49)$$

Substituting the equations (5.1.1.41), (5.1.1.36) in equation (5.1.1.45)

$$\mathbf{y}^T \begin{pmatrix} \frac{22+\sqrt{533}}{2} & 0 \\ 0 & \frac{22-\sqrt{533}}{2} \end{pmatrix} \mathbf{y} - 26 = 0 \quad (5.1.1.50)$$

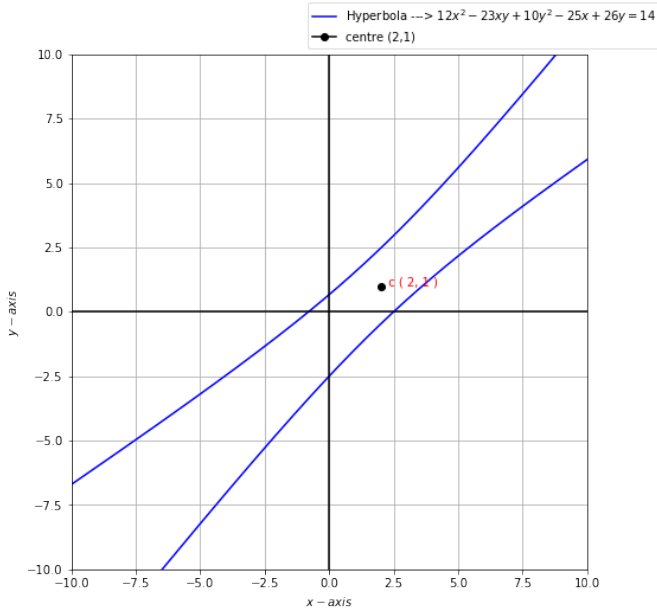


Fig. 5.1.1.1: Hyperbola when origin is shifted

The figure 5.1.1.1 verifies the given equation (5.1.1.2) as hyperbola with centre  $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

5.1.2. What conic does the following equation represent.

$$13x^2 - 18xy + 37y^2 + 2x + 14y - 2 = 0 \quad (5.1.2.1)$$

Find the center.

**Solution:** The general second degree equation can be expressed as follows,

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (5.1.2.2)$$

From the given second degree equation we get,

$$\mathbf{V} = \begin{pmatrix} 13 & -9 \\ -9 & 37 \end{pmatrix} \quad (5.1.2.3)$$

$$\mathbf{u} = \begin{pmatrix} 1 \\ 7 \end{pmatrix} \quad (5.1.2.4)$$

$$f = -2 \quad (5.1.2.5)$$

Expanding the determinant of  $\mathbf{V}$  we observe,

$$\begin{vmatrix} 13 & -9 \\ -9 & 37 \end{vmatrix} = 400 > 0 \quad (5.1.2.6)$$

Hence from (5.1.2.6) we conclude that given equation is an ellipse. The characteristic equation of  $\mathbf{V}$  is given as follows,

$$|\lambda \mathbf{I} - \mathbf{V}| = \begin{vmatrix} \lambda - 13 & 9 \\ 9 & \lambda - 37 \end{vmatrix} = 0 \quad (5.1.2.7)$$

$$\Rightarrow \lambda^2 - 50\lambda + 400 = 0 \quad (5.1.2.8)$$

Hence the characteristic equation of  $\mathbf{V}$  is given by (5.1.2.8). The roots of (5.1.2.8) i.e the eigenvalues are given by

$$\lambda_1 = 10, \lambda_2 = 40 \quad (5.1.2.9)$$

The eigen vector  $\mathbf{p}$  is defined as,

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \quad (5.1.2.10)$$

$$\Rightarrow (\lambda \mathbf{I} - \mathbf{V})\mathbf{p} = 0 \quad (5.1.2.11)$$

for  $\lambda_1 = 10$ ,

$$(\lambda_1 \mathbf{I} - \mathbf{V}) = \begin{pmatrix} -3 & 9 \\ 9 & -27 \end{pmatrix} \xrightarrow[R_1 = \frac{1}{3}R_1]{R_2 = R_2 + 3R_1} \begin{pmatrix} -1 & 3 \\ 0 & 0 \end{pmatrix} \quad (5.1.2.12)$$

$$\Rightarrow \mathbf{p}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad (5.1.2.13)$$

Again, for  $\lambda_2 = 40$ ,

$$(\lambda_2 \mathbf{I} - \mathbf{V}) = \begin{pmatrix} 27 & 9 \\ 9 & 3 \end{pmatrix} \xrightarrow[R_1 = \frac{1}{27}R_1]{R_2 = R_2 - R_1} \begin{pmatrix} 1 & \frac{1}{3} \\ 0 & 0 \end{pmatrix} \quad (5.1.2.14)$$

$$\Rightarrow \mathbf{p}_2 = \begin{pmatrix} -1 \\ 3 \end{pmatrix} \quad (5.1.2.15)$$

Again, Hence from the equation

$$\mathbf{V} = \mathbf{PDP}^{-1}\mathbf{P} = (\mathbf{p}_1 \ \mathbf{p}_2) = \begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix} \quad (5.1.2.16)$$

$$\mathbf{D} = \begin{pmatrix} 10 & 0 \\ 0 & 40 \end{pmatrix} \quad (5.1.2.17)$$

Now (5.1.2.2) can be written as,

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f \quad |\mathbf{V}| \neq 0 \quad (5.1.2.18)$$

And,

$$\mathbf{c} = -\mathbf{V}^{-1} \mathbf{u} \quad |\mathbf{V}| \neq 0 \quad (5.1.2.19)$$

$$\mathbf{y} = \mathbf{P}^T (\mathbf{x} - \mathbf{c}) \quad (5.1.2.20)$$

The centre/vertex of the conic section in (5.1.2.2) is given by  $\mathbf{c}$  in (5.1.2.19). We compute  $\mathbf{V}^{-1}$  as follows,

$$\begin{pmatrix} 13 & -9 & 1 & 0 \\ -9 & 37 & 0 & 1 \end{pmatrix} \xrightarrow[R_2 = \frac{13}{400}R_2]{R_2 = R_2 + \frac{9}{13}R_1} \begin{pmatrix} 13 & -9 & 1 & 0 \\ 0 & 1 & \frac{9}{400} & \frac{13}{400} \end{pmatrix} \quad (5.1.2.21)$$

$$\xrightarrow[R_1 = R_1 + \frac{9}{13}R_2]{R_1 = \frac{1}{13}R_1} \begin{pmatrix} 1 & 0 & \frac{37}{400} & \frac{9}{400} \\ 0 & 1 & \frac{9}{400} & \frac{13}{400} \end{pmatrix} \quad (5.1.2.22)$$

Hence  $\mathbf{V}^{-1}$  is given by,

$$\mathbf{V}^{-1} = \begin{pmatrix} \frac{37}{400} & \frac{9}{400} \\ \frac{9}{400} & \frac{13}{400} \end{pmatrix} \quad (5.1.2.23)$$

Now  $\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u}$  is given by,

$$\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} = \frac{1}{400} \begin{pmatrix} 1 & 7 \end{pmatrix} \begin{pmatrix} 37 & 9 \\ 9 & 13 \end{pmatrix} \begin{pmatrix} 1 \\ 7 \end{pmatrix} = 2 \quad (5.1.2.24) \quad 5.1.3.$$

And,  $\mathbf{V}^{-1} \mathbf{u}$  is given by,

$$\mathbf{V}^{-1} \mathbf{u} = \frac{1}{400} \begin{pmatrix} 100 \\ 100 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (5.1.2.25)$$

By putting the value of (5.1.2.25), the center of the ellipse is given by (5.1.2.19) as follows,

$$\mathbf{c} = -\frac{1}{4} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{4} \\ -\frac{1}{4} \end{pmatrix} \quad (5.1.2.26)$$

Also the semi-major axis ( $a$ ) and semi-minor

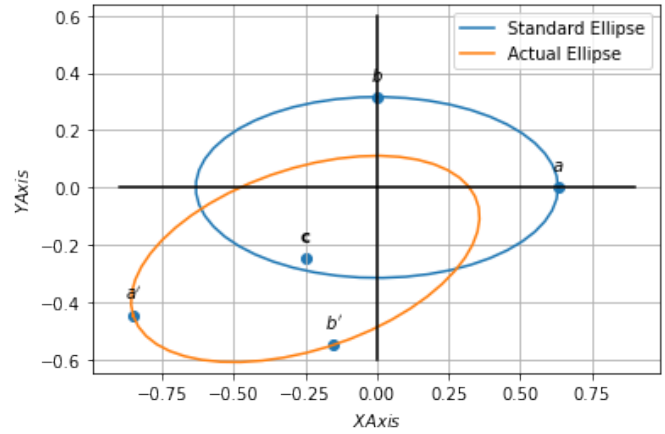


Fig. 5.1.2.1: Graphical representation of the ellipse

axis ( $b$ ) of the ellipse are given by,

$$a = \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}} = \frac{\sqrt{10}}{5} \quad (5.1.2.27)$$

$$b = \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_2}} = \frac{\sqrt{10}}{10} \quad (5.1.2.28)$$

Finally from (5.1.2.18), the equation of ellipse is given by,

$$\mathbf{y}^T \begin{pmatrix} 10 & 0 \\ 0 & 40 \end{pmatrix} \mathbf{y} = 4 \quad (5.1.2.29)$$

The following figure 5.1.2.1 is the graphical representation of the ellipse in (5.1.2.29),

5.1.3. What conic does the following equation represent?

$$y^2 - 2\sqrt{3}xy + 3x^2 + 6x - 4y + 5 = 0 \quad (5.1.3.1)$$

Find the center.

**Solution:** The general second degree equation can be expressed as follows,

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (5.1.3.2)$$

From the given second degree equation we get,

$$\mathbf{V} = \begin{pmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix} \quad (5.1.3.3)$$

$$\mathbf{u} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \quad (5.1.3.4)$$

$$f = 5 \quad (5.1.3.5)$$

Expanding the determinant of  $\mathbf{V}$  we observe,

$$\begin{vmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{vmatrix} = 0 \quad (5.1.3.6)$$

Also

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = \begin{vmatrix} 3 & -\sqrt{3} & 3 \\ -\sqrt{3} & 1 & -2 \\ 3 & -2 & 5 \end{vmatrix} \neq 0 \quad (5.1.3.7)$$

Hence from (5.1.3.6) and (5.1.3.7) we conclude that given equation is a parabola. The characteristic equation of  $\mathbf{V}$  is given as follows,

$$|\mathbf{V} - \lambda \mathbf{I}| = \begin{vmatrix} 3 - \lambda & -\sqrt{3} \\ -\sqrt{3} & 1 - \lambda \end{vmatrix} = 0 \quad (5.1.3.8)$$

$$\implies \lambda^2 - 4\lambda = 0 \quad (5.1.3.9)$$

Hence the characteristic equation of  $\mathbf{V}$  is given by (5.1.3.9). The roots of (5.1.3.9) i.e the eigenvalues are given by

$$\lambda_1 = 0, \lambda_2 = 4 \quad (5.1.3.10)$$

The eigen vector  $\mathbf{p}$  is defined as,

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \quad (5.1.3.11)$$

$$\implies (\mathbf{V} - \lambda\mathbf{I})\mathbf{p} = 0 \quad (5.1.3.12)$$

for  $\lambda_1 = 0$ ,

$$(\mathbf{V} - \lambda_1\mathbf{I}) = \begin{pmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix} \xrightarrow[R_1 = \frac{1}{\sqrt{3}}R_1]{R_2 = R_1 + R_2} \begin{pmatrix} \sqrt{3} & -1 \\ 0 & 0 \end{pmatrix} \quad (5.1.3.13)$$

Substituting equation 5.1.3.13 in equation 5.1.3.12 and upon normalizing we get we get

$$\implies \mathbf{p}_1 = \begin{pmatrix} 1/2 \\ \sqrt{3}/2 \end{pmatrix} \quad (5.1.3.14)$$

Again, for  $\lambda_2 = 4$ ,

$$(\mathbf{V} - \lambda_2\mathbf{I}) = \begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & -3 \end{pmatrix} \xrightarrow[R_1 = -\sqrt{3}R_1]{R_2 = -\sqrt{3}R_1 + R_2} \begin{pmatrix} 1 & \sqrt{3} \\ 0 & 0 \end{pmatrix} \quad (5.1.3.15)$$

Substituting equation 5.1.3.15 in equation 5.1.3.12 and upon normalizing we get

$$\mathbf{p}_2 = \begin{pmatrix} -\sqrt{3}/2 \\ 1/2 \end{pmatrix} \quad (5.1.3.16)$$

The matrix  $\mathbf{P}$ ,

$$\mathbf{P} = (\mathbf{p}_1 \ \mathbf{p}_2) = \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix} \quad (5.1.3.17)$$

$$\mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} \quad (5.1.3.18)$$

$$\eta = 2\mathbf{p}_1^T \mathbf{u} = 3 - 2\sqrt{3} \quad (5.1.3.19)$$

The focal length of the parabola is given by:

$$\left| \frac{\eta}{\lambda_2} \right| = \left| \frac{3 - 2\sqrt{3}}{4} \right| = 0.116 \quad (5.1.3.20)$$

When  $|\mathbf{V}| = 0$ , (5.1.3.2) can be written as

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = -\eta \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{y} \quad (5.1.3.21)$$

And the vertex  $\mathbf{c}$  is given by

$$\begin{pmatrix} \mathbf{u}^T + \frac{\eta}{2}\mathbf{p}_1^T \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \frac{\eta}{2}\mathbf{p}_1 - \mathbf{u} \end{pmatrix} \quad (5.1.3.22)$$

Substituting the found values

$$\mathbf{u}^T + \frac{\eta}{2}\mathbf{p}_1^T = \begin{pmatrix} 3 & -2 \end{pmatrix} + \frac{3 - 2\sqrt{3}}{2} \begin{pmatrix} 1/2 & \sqrt{3}/2 \end{pmatrix} \quad (5.1.3.23)$$

$$\implies \mathbf{u}^T + \frac{\eta}{2}\mathbf{p}_1^T = \begin{pmatrix} \frac{15-2\sqrt{3}}{4} & \frac{-14+3\sqrt{3}}{4} \end{pmatrix} \quad (5.1.3.24)$$

$$\frac{\eta}{2}\mathbf{p}_1 - \mathbf{u} = \begin{pmatrix} \frac{-9-2\sqrt{3}}{4} \\ \frac{2+3\sqrt{3}}{4} \end{pmatrix} \quad (5.1.3.25)$$

using equations (5.1.3.4), (5.1.3.5), (5.1.3.14), (5.1.3.24), (5.1.3.25) and (5.1.3.14) in (5.1.3.22)

$$\begin{pmatrix} \frac{15-2\sqrt{3}}{4} & \frac{-14+3\sqrt{3}}{4} \\ 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -5 \\ \frac{-9-2\sqrt{3}}{4} \\ \frac{2+3\sqrt{3}}{4} \end{pmatrix} \quad (5.1.3.26)$$

By performing row reductions on augmented

matrix

$$\begin{pmatrix} \frac{15-2\sqrt{3}}{4} & \frac{-14+3\sqrt{3}}{4} & -5 \\ 3 & -\sqrt{3} & \frac{(-9-2\sqrt{3})}{4} \\ -\sqrt{3} & 1 & \frac{2+3\sqrt{3}}{4} \end{pmatrix} R_2 \leftrightarrow R_1$$

$$\begin{pmatrix} 3 & -\sqrt{3} & \frac{(-9-2\sqrt{3})}{4} \\ \frac{15-2\sqrt{3}}{4} & \frac{-14+3\sqrt{3}}{4} & -5 \\ -\sqrt{3} & 1 & \frac{2+3\sqrt{3}}{4} \end{pmatrix} \quad (5.1.3.27)$$

$$\begin{pmatrix} 3 & -\sqrt{3} & \frac{(-9-2\sqrt{3})}{4} \\ \frac{15-2\sqrt{3}}{4} & \frac{-14+3\sqrt{3}}{4} & -5 \\ -\sqrt{3} & 1 & \frac{2+3\sqrt{3}}{4} \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - \frac{15-2\sqrt{3}}{12} R_1}$$

$$\begin{pmatrix} 3 & -\sqrt{3} & \frac{(-9-2\sqrt{3})}{4} \\ 0 & 2(\sqrt{3}-2) & \frac{(4\sqrt{3}-39)}{16} \\ \sqrt{3} & 1 & \frac{2+3\sqrt{3}}{4} \end{pmatrix} \quad (5.1.3.28)$$

Therefore,

$$\begin{pmatrix} 3 & -\sqrt{3} & \frac{(-9-2\sqrt{3})}{4} \\ 0 & 2(\sqrt{3}-2) & \frac{(4\sqrt{3}-39)}{16} \\ -\sqrt{3} & 1 & \frac{2+3\sqrt{3}}{4} \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 + \frac{1}{\sqrt{3}} R_1}$$

$$\begin{pmatrix} 3 & -\sqrt{3} & \frac{(-9-2\sqrt{3})}{4} \\ 0 & 2(\sqrt{3}-2) & \frac{(4\sqrt{3}-39)}{16} \\ 0 & 0 & 0 \end{pmatrix} \quad (5.1.3.29) \quad 5.1.4.$$

$$\begin{pmatrix} 3 & -\frac{433}{250} & -\frac{311}{100} \\ 0 & -\frac{107}{200} & -2 \\ 0 & 0 & 0 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{R_1}{3}}$$

$$\begin{pmatrix} 1 & -\frac{433}{750} & -\frac{311}{300} \\ 0 & -\frac{107}{200} & -2 \\ 0 & 0 & 0 \end{pmatrix} \quad (5.1.3.30)$$

$$\begin{pmatrix} 1 & -\frac{433}{750} & -\frac{311}{300} \\ 0 & -\frac{107}{200} & -2 \\ 0 & 0 & 0 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow -\frac{200}{107} R_2}$$

$$\begin{pmatrix} 1 & -\frac{433}{750} & -\frac{311}{300} \\ 0 & 1 & \frac{400}{107} \\ 0 & 0 & 0 \end{pmatrix} \quad (5.1.3.31)$$

$$\begin{pmatrix} 1 & -\frac{433}{750} & -\frac{311}{300} \\ 0 & 1 & \frac{400}{107} \\ 0 & 0 & 0 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 + \frac{433}{750} R_2}$$

$$\begin{pmatrix} 1 & 0 & \frac{12001}{10700} \\ 0 & 1 & \frac{400}{107} \\ 0 & 0 & 0 \end{pmatrix} \quad (5.1.3.32)$$

On solving for values of  $\mathbf{c}$  from (5.1.3.32) The vertex of parabola is  $\mathbf{c} = \begin{pmatrix} \frac{12001}{10700} \\ \frac{400}{107} \end{pmatrix}$ .

What conics do the following equation represent? When possible, find the centres and also their equations referred to the centre.

$$2x^2 - 72xy + 23y^2 - 4x - 2y - 48 = 0$$

(5.1.4.1)

**Solution:**

5.1.5. What conic does the given equations represent?

$$6x^2 - 5xy - 6y^2 + 14x + 5y + 4 = 0 \quad (5.1.5.1)$$

**Solution:** The above equation can be expressed in the form

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (5.1.5.2)$$

Comparing equation we get

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 6 & -\frac{5}{2} \\ -\frac{5}{2} & -6 \end{pmatrix} \quad (5.1.5.3)$$

$$\mathbf{u} = \begin{pmatrix} 7 \\ \frac{5}{2} \end{pmatrix} \quad (5.1.5.4)$$

$$f = 4 \quad (5.1.5.5)$$

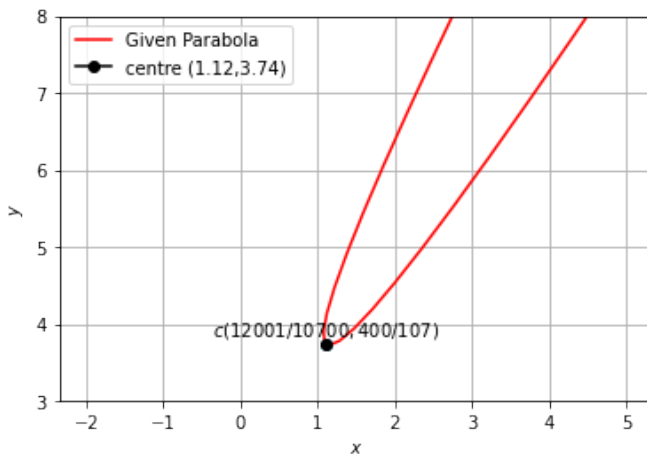


Fig. 5.1.3.1: Parabola with the center  $\mathbf{c}$

The above equation (5.1.5.2) represents a pair of straight lines if

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \quad (5.1.5.6)$$

Verify the given equation as if it is pair of straight lines

$$\Delta = \begin{vmatrix} 6 & \frac{-5}{2} & 7 \\ \frac{-5}{2} & -6 & \frac{5}{2} \\ 7 & \frac{5}{2} & 4 \end{vmatrix} \quad (5.1.5.7)$$

$$\Rightarrow 6 \begin{vmatrix} -6 & \frac{5}{2} \\ \frac{5}{2} & 4 \end{vmatrix} - \frac{-5}{2} \begin{vmatrix} -\frac{5}{2} & \frac{5}{2} \\ 7 & 4 \end{vmatrix} + 7 \begin{vmatrix} -\frac{5}{2} & -6 \\ 7 & \frac{5}{2} \end{vmatrix} = 0 \quad (5.1.5.8)$$

$$\Rightarrow \Delta = 0 \quad (5.1.5.9)$$

Since equation (5.1.5.6) is satisfied, we could say that the given equation represents two straight lines

$$\Delta_V = \begin{vmatrix} 6 & \frac{-5}{2} \\ \frac{-5}{2} & -6 \end{vmatrix} < 0 \quad (5.1.5.10)$$

Let the equations of lines be,

$$(\mathbf{n}_1^T \mathbf{x} - c_1)(\mathbf{n}_2^T \mathbf{x} - c_2) = \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (5.1.5.11)$$

$$(\mathbf{n}_1^T \mathbf{x} - c_1)(\mathbf{n}_2^T \mathbf{x} - c_2) = \mathbf{x}^T \begin{pmatrix} 6 & \frac{-5}{2} \\ \frac{-5}{2} & -6 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} 7 & \frac{5}{2} \end{pmatrix} \mathbf{x} + 4 \quad (5.1.5.12)$$

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} = \begin{pmatrix} 6 \\ -5 \\ -6 \end{pmatrix} \quad (5.1.5.13)$$

$$c_2 \mathbf{n}_1 + c_1 \mathbf{n}_2 = -2 \begin{pmatrix} 7 \\ \frac{5}{2} \end{pmatrix} \quad (5.1.5.14)$$

$$c_1 c_2 = 4 \quad (5.1.5.15)$$

The slopes of the lines are given by the roots of the polynomial

$$cm^2 + 2bm + a = 0 \quad (5.1.5.16)$$

$$\Rightarrow m_i = \frac{-b \pm \sqrt{-\Delta_V}}{c} \quad (5.1.5.17)$$

$$\mathbf{n}_i = k \begin{pmatrix} -m_i \\ 1 \end{pmatrix} \quad (5.1.5.18)$$

Substituting the given data in above equations

(5.1.5.16) we get,

$$-6m^2 - 5m + 6 = 0 \quad (5.1.5.19)$$

$$\Rightarrow m_i = \frac{\frac{-5}{2} \pm \sqrt{-\left(\frac{-169}{4}\right)}}{-6} \quad (5.1.5.20)$$

Solving equation (5.1.5.20) we get,

$$m_1 = -\frac{3}{2}, m_2 = \frac{2}{3} \quad (5.1.5.21)$$

$$= \mathbf{n}_1 = \begin{pmatrix} -3 \\ -2 \end{pmatrix}, \mathbf{n}_2 = \begin{pmatrix} -2 \\ 3 \end{pmatrix} \quad (5.1.5.22)$$

We know that,

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (5.1.5.23)$$

Verification using Toeplitz matrix, From equation (5.1.5.22)

$$\mathbf{n}_1 = \begin{pmatrix} -3 & 0 \\ -2 & -3 \\ 0 & -2 \end{pmatrix} \mathbf{n}_2 = \begin{pmatrix} -2 \\ 3 \end{pmatrix} \quad (5.1.5.24)$$

$$\Rightarrow \begin{pmatrix} -3 & 0 \\ -2 & -3 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ -5 \\ -6 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (5.1.5.25)$$

$\Rightarrow$  Equation (5.1.5.22) satisfies (5.1.5.23)  
 $c_1$  and  $c_2$  can be obtained as,

$$(\mathbf{n}_1 \quad \mathbf{n}_2) \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2\mathbf{u} \quad (5.1.5.26)$$

Substituting (5.1.5.22) in (5.1.5.26), the augmented matrix is,

$$\begin{pmatrix} -3 & -2 & 14 \\ -2 & 3 & 5 \end{pmatrix} \xrightarrow[R_2 \leftarrow R_2 + 2R_1]{R_1 \leftarrow -R_1/3} \begin{pmatrix} 1 & \frac{2}{3} & -\frac{14}{3} \\ 0 & \frac{13}{3} & -\frac{13}{3} \end{pmatrix} \quad (5.1.5.27)$$

$$\xrightarrow[R_1 \leftarrow R_1 - \frac{2}{3}R_2]{R_2 \leftarrow \frac{3}{13}R_2} \begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & -1 \end{pmatrix} \quad (5.1.5.28)$$

$$\Rightarrow c_1 = -4, c_2 = -1 \quad (5.1.5.29)$$

Equations (5.1.5.11), can be modified as,from



(5.1.5.22) and (5.1.5.29) in we get,

$$\begin{pmatrix} -3 & -2 \end{pmatrix} \mathbf{x} = -4 \quad (5.1.5.30)$$

$$\begin{pmatrix} -2 & 3 \end{pmatrix} \mathbf{x} = -1 \quad (5.1.5.31)$$

$$\Rightarrow (-3x - 2y + 4)(-2x + 3y + 1) = 0$$

$$\Rightarrow \boxed{(3x + 2y - 4)(2x - 3y - 1) = 0} \quad (5.1.5.32)$$

The angle between the lines can be expressed as,

$$\mathbf{n}_1 = \begin{pmatrix} -3 \\ -2 \end{pmatrix}, \quad \mathbf{n}_2 = \begin{pmatrix} -2 \\ 3 \end{pmatrix} \quad (5.1.5.33)$$

$$\cos \theta = \frac{\mathbf{n}_1^T \mathbf{n}_2}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} \quad (5.1.5.34)$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{0}{\sqrt{169}}\right) = 90^\circ. \quad (5.1.5.35)$$

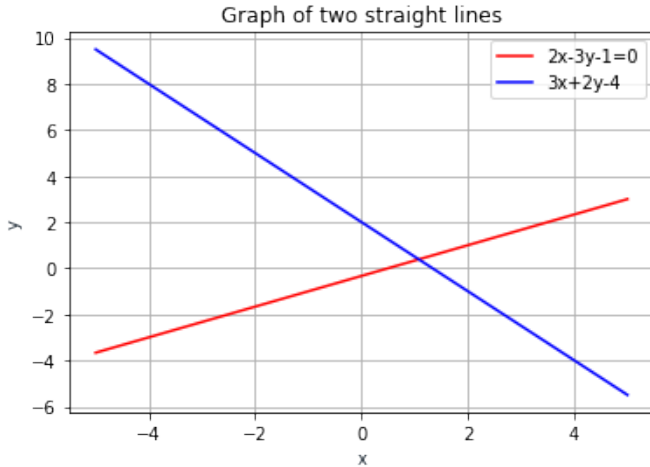


Fig. 5.1.5.1: Pair of straight lines

5.1.6. What conic does the following equation represent? Find its equation and centre.

$$3x^2 - 8xy - 3y^2 + 10x - 13y + 8 = 0$$

**Solution:** The general equation of second degree can be expressed as

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (5.1.6.1)$$

where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad (5.1.6.2)$$

$$\mathbf{u}^T = \begin{pmatrix} d & e \end{pmatrix} \quad (5.1.6.3)$$

From (5.1.6.2) and (5.1.6.3)

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 3 & -4 \\ -4 & -3 \end{pmatrix} \quad (5.1.6.4)$$

$$\mathbf{u} = \begin{pmatrix} 5 \\ -\frac{13}{2} \end{pmatrix} \quad (5.1.6.5)$$

$$|\mathbf{V}| = \begin{vmatrix} 3 & -4 \\ -4 & -3 \end{vmatrix} = -25 \quad (5.1.6.6)$$

$$\Rightarrow |\mathbf{V}| < 0 \quad (5.1.6.7)$$

Since  $\mathbf{V} = \mathbf{V}^T$ , there exists an orthogonal matrix  $\mathbf{P}$  such that

$$\mathbf{PVP}^T = \mathbf{D} = \text{diag}(\lambda_1 \quad \lambda_2) \quad (5.1.6.8)$$

or equivalently

$$\mathbf{V} = \mathbf{PDP}^T \quad (5.1.6.9)$$

Eigen vectors of real symmetric matrix  $\mathbf{V}$  are orthogonal. The characteristic equation of  $\mathbf{V}$  is obtained by evaluating the determinant

$$|\lambda \mathbf{I} - \mathbf{V}| = \begin{vmatrix} \lambda - 3 & 4 \\ 4 & \lambda + 3 \end{vmatrix} = 0 \quad (5.1.6.10)$$

$$\Rightarrow \lambda^2 - 25 = 0 \quad (5.1.6.11)$$

$$\Rightarrow \lambda_1 = -5, \lambda_2 = 5 \quad (5.1.6.12)$$

From (5.1.6.7) and (5.1.6.12) the equation represents a hyperbola. The eigen vector  $\mathbf{p}$  is defined as

$$\mathbf{Vp} = \lambda \mathbf{p} \quad (5.1.6.13)$$

$$\Rightarrow (\lambda \mathbf{I} - \mathbf{V})\mathbf{p} = 0 \quad (5.1.6.14)$$

For  $\lambda_1 = -5$  :

$$(\lambda_1 \mathbf{I} - \mathbf{V}) = \begin{pmatrix} -8 & 4 \\ 4 & -2 \end{pmatrix} \xrightarrow[R_2 \leftarrow \frac{R_2}{2}]{R_1 \leftarrow \frac{R_1}{4}} \begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix} \quad (5.1.6.15)$$

$$\xrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix} \quad (5.1.6.16)$$

$$\Rightarrow \mathbf{p}_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad (5.1.6.17)$$

Similarly, the eigenvector corresponding to  $\lambda_2$

can be obtained as

$$\mathbf{p}_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad (5.1.6.18)$$

The orthogonal eigen-vector matrix

$$\mathbf{P} = (\mathbf{p}_1 \ \mathbf{p}_2) = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \quad (5.1.6.19)$$

$$\mathbf{D} = \begin{pmatrix} -5 & 0 \\ 0 & 5 \end{pmatrix} \quad (5.1.6.20)$$

Let  $\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c}$  with  $\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u}$ . Substituting in (5.1.6.1)

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f \quad (5.1.6.21)$$

with centre

$$\mathbf{c} = -\mathbf{V}^{-1} \mathbf{u} = \begin{pmatrix} -\frac{41}{25} \\ \frac{1}{50} \end{pmatrix} \quad (5.1.6.22)$$

and minor and major axes parameters as

$$\sqrt{\frac{\lambda_1}{f - \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u}}} = \sqrt{\frac{500}{33}}, \quad \sqrt{\frac{\lambda_2}{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}} = \sqrt{\frac{500}{33}} \quad (5.1.6.23)$$

The equation of hyperbola is

$$\frac{y_2^2}{\frac{33}{500}} - \frac{y_1^2}{\frac{33}{500}} = 1 \quad (5.1.6.24)$$

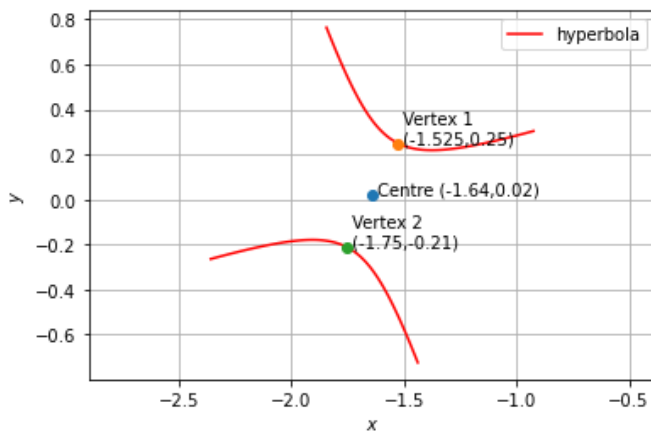


Fig. 5.1.6.1

5.1.7. Find the asymptotes of the hyperbola given below and also the equations to their conjugate hyperbolas.

$8x^2 + 10xy - 3y^2 - 2x + 4y - 2 = 0$  **Solution:** The

above equation can be expressed in the form

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (5.1.7.1)$$

Comparing equation we get

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 8 & 5 \\ 5 & -3 \end{pmatrix} \quad (5.1.7.2)$$

$$\mathbf{u} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad (5.1.7.3)$$

$$f = -2 \quad (5.1.7.4)$$

Expanding the Determinant of  $\mathbf{V}$ .

$$\Delta_V = \begin{vmatrix} 8 & 5 \\ 5 & -3 \end{vmatrix} < 0 \quad (5.1.7.5)$$

Hence from (5.1.7.5) given equation represents the hyperbola. The characteristic equation of  $\mathbf{V}$  is obtained by evaluating the determinant

$$|\mathbf{V} - \lambda \mathbf{I}| = 0 \quad (5.1.7.6)$$

$$\begin{vmatrix} 8 - \lambda & 5 \\ 5 & -3 - \lambda \end{vmatrix} = 0 \quad (5.1.7.7)$$

$$(8 - \lambda)(-3 - \lambda) - 25 = 0 \quad (5.1.7.8)$$

$$\lambda_1 = \frac{5 + \sqrt{221}}{2} \quad (5.1.7.9)$$

$$\lambda_2 = \frac{5 - \sqrt{221}}{2} \quad (5.1.7.10)$$

The eigenvector  $\mathbf{p}$  is defined as

$$\mathbf{V}\mathbf{p} = \lambda \mathbf{p} \quad (5.1.7.11)$$

$$\Rightarrow (\mathbf{V} - \lambda \mathbf{I})\mathbf{p} = 0 \quad (5.1.7.12)$$

For  $\lambda_1 = \frac{5 + \sqrt{221}}{2}$ ,

$$(\mathbf{V} - \lambda_1 \mathbf{I}) = \begin{pmatrix} \frac{11 - \sqrt{221}}{2} & 5 \\ 5 & \frac{-11 - \sqrt{221}}{2} \end{pmatrix} \quad (5.1.7.13)$$

By row reduction,

$$\begin{pmatrix} \frac{11 - \sqrt{221}}{2} & 5 \\ 5 & \frac{-11 - \sqrt{221}}{2} \end{pmatrix} \quad (5.1.7.14)$$

$$\xleftrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} \frac{-11 - \sqrt{221}}{2} & 5 \\ \frac{11 - \sqrt{221}}{2} & 5 \end{pmatrix} \quad (5.1.7.15)$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - \frac{11 - \sqrt{221}}{10} R_1} \begin{pmatrix} 5 & \frac{-11 - \sqrt{221}}{2} \\ 0 & 0 \end{pmatrix} \quad (5.1.7.16)$$

$$\xleftrightarrow{R_1 \leftarrow R_1/5} \begin{pmatrix} 1 & \frac{-11 - \sqrt{221}}{10} \\ 0 & 0 \end{pmatrix} \quad (5.1.7.17)$$

Substituting equation 5.1.7.17 in equation

5.1.7.12 we get

$$\begin{pmatrix} 1 & \frac{-11-\sqrt{221}}{10} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (5.1.7.18)$$

Where,  $\mathbf{p} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$  Let  $v_2 = t$

$$v_1 = \frac{t(11 + \sqrt{221})}{10} \quad (5.1.7.19)$$

Eigen vector  $\mathbf{p}_1$  is given by

$$\mathbf{p}_1 = \begin{pmatrix} \frac{t(11 + \sqrt{221})}{10} \\ t \end{pmatrix} \quad (5.1.7.20)$$

Let  $t = 1$ , we get

$$\mathbf{p}_1 = \begin{pmatrix} \frac{11 + \sqrt{221}}{10} \\ 1 \end{pmatrix} \quad (5.1.7.21)$$

For  $\lambda_2 = \frac{5 - \sqrt{221}}{2}$ ,

$$(\mathbf{V} - \lambda_2 \mathbf{I}) = \begin{pmatrix} \frac{11 + \sqrt{221}}{2} & 5 \\ 5 & \frac{-11 + \sqrt{221}}{2} \end{pmatrix} \quad (5.1.7.22)$$

By row reduction ,

$$\begin{pmatrix} \frac{11 + \sqrt{221}}{2} & 5 \\ 5 & \frac{-11 + \sqrt{221}}{2} \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_2 + \frac{11 - \sqrt{221}}{10} R_1} \begin{pmatrix} \frac{11 + \sqrt{221}}{2} & 5 \\ 0 & 0 \end{pmatrix} \quad (5.1.7.23)$$

$$\xleftrightarrow{R_1 \leftarrow \frac{R_1}{\frac{11 + \sqrt{221}}{10}}} \begin{pmatrix} 1 & \frac{10}{11 + \sqrt{221}} \\ 0 & 0 \end{pmatrix} \quad (5.1.7.24)$$

Substituting equation 5.1.7.24 in equation 5.1.7.12 we get

$$\begin{pmatrix} 1 & \frac{10}{11 + \sqrt{221}} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (5.1.7.25)$$

Where,  $\mathbf{p} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$  Let  $v_2 = t$

$$v_1 = \frac{-t(10)}{11 + \sqrt{221}} \quad (5.1.7.26)$$

Eigen vector  $\mathbf{p}_2$  is given by

$$\mathbf{p}_2 = \begin{pmatrix} \frac{-t(10)}{11 + \sqrt{221}} \\ t \end{pmatrix} \quad (5.1.7.27)$$

Let  $t = 1$ , we get

$$\mathbf{p}_2 = \begin{pmatrix} \frac{(-10)}{11 + \sqrt{221}} \\ 1 \end{pmatrix} \quad (5.1.7.28)$$

By eigen decomposition  $\mathbf{V}$  can be represented by

$$\mathbf{V} = \mathbf{P} \mathbf{D} \mathbf{P}^T \quad (5.1.7.29)$$

where

$$\mathbf{P} = (\mathbf{p}_1 \quad \mathbf{p}_2) \quad (5.1.7.30)$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad (5.1.7.31)$$

Substituting equations 5.1.7.21, 5.1.7.28 in equation 5.1.7.30 we get

$$\mathbf{P} = \begin{pmatrix} \frac{11 + \sqrt{221}}{10} & \frac{-10}{11 + \sqrt{221}} \\ 1 & 1 \end{pmatrix} \quad (5.1.7.32)$$

Substituting equations 5.1.7.9, 5.1.7.10 in 5.1.7.31 we get

$$\mathbf{D} = \begin{pmatrix} \frac{5 + \sqrt{221}}{2} & 0 \\ 0 & \frac{5 - \sqrt{221}}{2} \end{pmatrix} \quad (5.1.7.33)$$

Centre of the hyperbola is given by

$$\mathbf{c} = -\mathbf{V}^{-1} \mathbf{u} \quad (5.1.7.34)$$

$$\Rightarrow \mathbf{c} = -\begin{pmatrix} \frac{3}{49} & \frac{5}{49} \\ \frac{5}{49} & \frac{-8}{49} \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad (5.1.7.35)$$

$$\Rightarrow \mathbf{c} = \begin{pmatrix} \frac{-3}{49} & \frac{-5}{49} \\ \frac{-5}{49} & \frac{8}{49} \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad (5.1.7.36)$$

$$\Rightarrow \mathbf{c} = \begin{pmatrix} \frac{-1}{7} \\ \frac{3}{7} \end{pmatrix} \quad (5.1.7.37)$$

Since,

$$\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f = 1 > 0 \quad (5.1.7.38)$$

there isn't a need to swap axes In hyperbola,

$$axes = \begin{cases} \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}} \\ \sqrt{\frac{f - \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u}}{\lambda_2}} \end{cases} \quad (5.1.7.39)$$

From above equations we can say that,

$$\sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}} = \sqrt{\frac{2}{5 + \sqrt{221}}} \quad (5.1.7.40)$$

$$\sqrt{\frac{f - \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u}}{\lambda_2}} = \sqrt{\frac{2}{5 - \sqrt{221}}} \quad (5.1.7.41)$$

Now we have,

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f \quad (5.1.7.42)$$

where ,

$$\mathbf{y} = \mathbf{P}^T (\mathbf{x} - \mathbf{c}) \quad (5.1.7.43)$$

To get  $\mathbf{y}$ ,

$$\mathbf{y} = \mathbf{P}^T \mathbf{x} - \mathbf{P}^T \mathbf{c} \quad (5.1.7.44)$$

$$\mathbf{y} = \begin{pmatrix} \frac{11+\sqrt{221}}{11+\sqrt{221}} & 1 \\ \frac{-10}{11+\sqrt{221}} & 1 \end{pmatrix} \mathbf{x} - \begin{pmatrix} \frac{11+\sqrt{221}}{11+\sqrt{221}} & 1 \\ \frac{-10}{11+\sqrt{221}} & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{7} \\ \frac{3}{7} \end{pmatrix} \quad (5.1.7.45)$$

$$\mathbf{y} = \begin{pmatrix} \frac{11+\sqrt{221}}{11+\sqrt{221}} & 1 \\ \frac{-10}{11+\sqrt{221}} & 1 \end{pmatrix} \mathbf{x} - \begin{pmatrix} \frac{-11-\sqrt{221}}{(7)11+(7)\sqrt{221}} + \frac{3}{7} \\ \frac{70}{(7)11+(7)\sqrt{221}} + \frac{3}{7} \end{pmatrix} \quad (5.1.7.46)$$

Substituting the equations (5.1.7.38), (5.1.7.33) in equation (5.1.7.42)

$$\Rightarrow \mathbf{y}^T \begin{pmatrix} \frac{5+\sqrt{221}}{2} & 0 \\ 0 & \frac{5-\sqrt{221}}{2} \end{pmatrix} \mathbf{y} + 2 = 0 \quad (5.1.7.47)$$

Asymptotes of hyperbola Equation of a hyperbola and the combined equation of the Asymptotes differ only in the constant term.

$$8x^2 + 10xy - 3y^2 - 2x + 4y + K = 0 \quad (5.1.7.48)$$

The above equation can be expressed in the form

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (5.1.7.49)$$

Comparing equation we get

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 8 & 5 \\ 5 & -3 \end{pmatrix} \quad (5.1.7.50)$$

$$\mathbf{u} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad (5.1.7.51)$$

$$f = K \quad (5.1.7.52)$$

$$\Delta = \begin{vmatrix} 8 & 5 & -1 \\ 5 & -3 & 2 \\ -1 & 2 & K \end{vmatrix} \quad (5.1.7.53)$$

$$\Rightarrow K = -1 \quad (5.1.7.54)$$

Similar way expanding the Determinant of  $\mathbf{V}$ .

$$\Delta_V = \begin{vmatrix} 8 & 5 \\ 5 & -3 \end{vmatrix} < 0 \quad (5.1.7.55)$$

From (5.1.7.55) we could say that the given equation represents two straight lines Let the equations of lines be,

$$(\mathbf{n}_1^T \mathbf{x} - c_1)(\mathbf{n}_1^T \mathbf{x} - c_1) = \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (5.1.7.56)$$

$$(\mathbf{n}_1^T \mathbf{x} - c_1)(\mathbf{n}_2^T \mathbf{x} - c_2) = \mathbf{x}^T \begin{pmatrix} 8 & 5 \\ 5 & -3 \end{pmatrix} \mathbf{x} + 2(-1 \ 2) \mathbf{x} - 1 \quad (5.1.7.57)$$

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} = \begin{pmatrix} 8 \\ 10 \\ -3 \end{pmatrix} \quad (5.1.7.58)$$

$$c_2 \mathbf{n}_1 + c_1 \mathbf{n}_2 = -2 \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad (5.1.7.59)$$

$$c_1 c_2 = -1 \quad (5.1.7.60)$$

The slopes of the lines are given by the roots of the polynomial

$$cm^2 + 2bm + a = 0 \quad (5.1.7.61)$$

$$\Rightarrow m_i = \frac{-b \pm \sqrt{-\Delta_V}}{c} \quad (5.1.7.62)$$

$$\mathbf{n}_i = k \begin{pmatrix} -m_i \\ 1 \end{pmatrix} \quad (5.1.7.63)$$

Substituting the given data in above equations (5.1.7.61) we get,

$$-3m^2 + 10m + 8 = 0 \quad (5.1.7.64)$$

$$m_1 = 4, m_2 = \frac{-2}{3} \quad (5.1.7.65)$$

$$= \mathbf{n}_1 = \begin{pmatrix} -4 \\ 1 \end{pmatrix}, \mathbf{n}_2 = \begin{pmatrix} -2 \\ -3 \end{pmatrix} \quad (5.1.7.66)$$

We know that,

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (5.1.7.67)$$

Verification using Toeplitz matrix, From equa-

tion (5.1.7.66)

$$\mathbf{n}_1 = \begin{pmatrix} -4 & 0 \\ 1 & -4 \\ 0 & -1 \end{pmatrix} \mathbf{n}_2 = \begin{pmatrix} -2 \\ -3 \end{pmatrix} \quad (5.1.7.68)$$

$$\Rightarrow \begin{pmatrix} -4 & 0 \\ 1 & -4 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 10 \\ -3 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (5.1.7.69)$$

$\Rightarrow$  Equation (5.1.7.66) satisfies (5.1.7.67)  
 $c_1$  and  $c_2$  can be obtained as,

$$(\mathbf{n}_1 \quad \mathbf{n}_2) \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2\mathbf{u} \quad (5.1.7.70)$$

Substituting (5.1.7.66) in (5.1.7.70), the augmented matrix is,

$$\begin{pmatrix} -4 & -2 & -2 \\ 1 & -3 & 4 \end{pmatrix} \xrightarrow[R_2 \leftarrow R_2 - R_1]{R_1 \leftarrow -R_1/4} \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{7}{2} & \frac{7}{2} \end{pmatrix} \quad (5.1.7.71)$$

$$\xrightarrow[R_1 \leftarrow R_1 - \frac{1}{2}R_2]{R_2 \leftarrow -\frac{2}{7}R_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} \quad (5.1.7.72)$$

$$\Rightarrow c_1 = 1, c_2 = -1 \quad (5.1.7.73)$$

Equations (5.1.7.56), can be modified as, from (5.1.7.66) and (5.1.7.73) in we get,

$$\begin{pmatrix} -4 & 1 \end{pmatrix} \mathbf{x} = 1 \quad (5.1.7.74)$$

$$\begin{pmatrix} -2 & -3 \end{pmatrix} \mathbf{x} = -1 \quad (5.1.7.75)$$

$$\begin{aligned} \Rightarrow (-4x + y - 1)(-2x - 3y + 1) &= 0 \\ \Rightarrow \boxed{(4x - y + 1)(2x + 3y - 1) = 0} &\quad (5.1.7.76) \end{aligned}$$

The angle between the lines can be expressed as,

$$\mathbf{n}_1 = \begin{pmatrix} -4 \\ 1 \end{pmatrix}, \quad \mathbf{n}_2 = \begin{pmatrix} -2 \\ -3 \end{pmatrix} \quad (5.1.7.77)$$

$$\cos \theta = \frac{\mathbf{n}_1^T \mathbf{n}_2}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} \quad (5.1.7.78)$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{0}{\sqrt{221}}\right) = 90^\circ. \quad (5.1.7.79)$$

Equation of Asymptotes: The characteristic equation of  $\mathbf{V}$  is obtained by evaluating the

determinant (5.1.7.50)

$$|V - \lambda \mathbf{I}| = 0 \quad (5.1.7.80)$$

$$\begin{vmatrix} 8 - \lambda & 5 \\ 5 & -3 - \lambda \end{vmatrix} = 0 \quad (5.1.7.81)$$

$$(8 - \lambda)(-3 - \lambda) - 25 = 0 \quad (5.1.7.82)$$

$$\lambda_1 = \frac{5 + \sqrt{221}}{2} \quad (5.1.7.83)$$

$$\lambda_2 = \frac{5 - \sqrt{221}}{2} \quad (5.1.7.84)$$

The eigenvector  $\mathbf{p}$  is defined as

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \quad (5.1.7.85)$$

$$\Rightarrow (\mathbf{V} - \lambda\mathbf{I})\mathbf{p} = 0 \quad (5.1.7.86)$$

For  $\lambda_1 = \frac{5 + \sqrt{221}}{2}$ ,

$$(\mathbf{V} - \lambda_1 \mathbf{I}) = \begin{pmatrix} \frac{11 - \sqrt{221}}{2} & 5 \\ 5 & \frac{-11 - \sqrt{221}}{2} \end{pmatrix} \quad (5.1.7.87)$$

By row reduction ,

$$\begin{pmatrix} \frac{11 - \sqrt{221}}{2} & 5 \\ 5 & \frac{-11 - \sqrt{221}}{2} \end{pmatrix} \quad (5.1.7.88)$$

$$\xrightarrow{R_1 \leftarrow R_2} \begin{pmatrix} \frac{-11 - \sqrt{221}}{2} & 5 \\ \frac{11 - \sqrt{221}}{2} & 5 \end{pmatrix} \quad (5.1.7.89)$$

$$\xrightarrow{R_2 \leftarrow R_2 - \frac{11 - \sqrt{221}}{10} R_1} \begin{pmatrix} 5 & \frac{-11 - \sqrt{221}}{2} \\ 0 & \frac{2}{0} \end{pmatrix} \quad (5.1.7.90)$$

$$\xrightarrow{R_1 \leftarrow R_1/5} \begin{pmatrix} 1 & \frac{-11 - \sqrt{221}}{10} \\ 0 & 0 \end{pmatrix} \quad (5.1.7.91)$$

Substituting equation 5.1.7.91 in equation 5.1.7.86 we get

$$\begin{pmatrix} 1 & \frac{-11 - \sqrt{221}}{10} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (5.1.7.92)$$

Where,  $\mathbf{p} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$  Let  $v_2 = t$

$$v_1 = \frac{t(11 + \sqrt{221})}{10} \quad (5.1.7.93)$$

Eigen vector  $\mathbf{p}_1$  is given by

$$\mathbf{p}_1 = \begin{pmatrix} \frac{t(11 + \sqrt{221})}{10} \\ t \end{pmatrix} \quad (5.1.7.94)$$

Let  $t = 1$ , we get

$$\mathbf{p}_1 = \begin{pmatrix} \frac{11+\sqrt{221}}{10} \\ 1 \end{pmatrix} \quad (5.1.7.95)$$

For  $\lambda_2 = \frac{5-\sqrt{221}}{2}$ ,

$$(\mathbf{V} - \lambda_2 \mathbf{I}) = \begin{pmatrix} \frac{11+\sqrt{221}}{2} & 5 \\ 5 & \frac{-11+\sqrt{221}}{2} \end{pmatrix} \quad (5.1.7.96)$$

By row reduction ,

$$\begin{pmatrix} \frac{11+\sqrt{221}}{2} & 5 \\ 5 & \frac{-11+\sqrt{221}}{2} \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_2 + \frac{11-\sqrt{221}}{10} R_1} \begin{pmatrix} \frac{11+\sqrt{221}}{2} & 5 \\ 0 & 0 \end{pmatrix} \quad (5.1.7.97)$$

$$\xleftrightarrow{R_1 \leftarrow \frac{R_1}{\frac{11+\sqrt{221}}{10}}} \begin{pmatrix} 1 & \frac{10}{11+\sqrt{221}} \\ 0 & 0 \end{pmatrix} \quad (5.1.7.98)$$

Substituting equation 5.1.7.98 in equation 5.1.7.86 we get

$$\begin{pmatrix} 1 & \frac{10}{11+\sqrt{221}} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (5.1.7.99)$$

Where,  $\mathbf{p} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$  Let  $v_2 = t$

$$v_1 = \frac{-t(10)}{11 + \sqrt{221}} \quad (5.1.7.100)$$

Eigen vector  $\mathbf{p}_2$  is given by

$$\mathbf{p}_2 = \begin{pmatrix} \frac{-t(10)}{11+\sqrt{221}} \\ t \end{pmatrix} \quad (5.1.7.101)$$

Let  $t = 1$ , we get

$$\mathbf{p}_2 = \begin{pmatrix} \frac{(-10)}{11+\sqrt{221}} \\ 1 \end{pmatrix} \quad (5.1.7.102)$$

By eigen decomposition  $\mathbf{V}$  can be represented by

$$\mathbf{V} = \mathbf{PDP}^T \quad (5.1.7.103)$$

where

$$\mathbf{P} = (\mathbf{p}_1 \quad \mathbf{p}_2) \quad (5.1.7.104)$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad (5.1.7.105)$$

Substituting equations 5.1.7.95, 5.1.7.102 in

equation 5.1.7.104 we get

$$\mathbf{P} = \begin{pmatrix} \frac{11+\sqrt{221}}{10} & \frac{-10}{11+\sqrt{221}} \\ 1 & 1 \end{pmatrix} \quad (5.1.7.106)$$

$$\mathbf{D} = \begin{pmatrix} \frac{5+\sqrt{221}}{2} & 0 \\ 0 & \frac{5-\sqrt{221}}{2} \end{pmatrix} \quad (5.1.7.107)$$

Centre of the hyperbola is given by

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \quad (5.1.7.108)$$

$$\Rightarrow \mathbf{c} = -\begin{pmatrix} \frac{3}{49} & \frac{5}{49} \\ \frac{5}{49} & \frac{-8}{49} \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad (5.1.7.109)$$

$$\Rightarrow \mathbf{c} = \begin{pmatrix} \frac{-3}{49} & \frac{-5}{49} \\ \frac{-5}{49} & \frac{8}{49} \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad (5.1.7.110)$$

$$\Rightarrow \mathbf{c} = \begin{pmatrix} \frac{-1}{7} \\ \frac{3}{7} \end{pmatrix} \quad (5.1.7.111)$$

Since,

$$\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f = 0 \quad (5.1.7.112)$$

Now,

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f \quad (5.1.7.113)$$

where ,

$$\mathbf{y} = \mathbf{P}^T (\mathbf{x} - \mathbf{c}) \quad (5.1.7.114)$$

To get  $\mathbf{y}$ ,

$$\mathbf{y} = \mathbf{P}^T \mathbf{x} - \mathbf{P}^T \mathbf{c} \quad (5.1.7.115)$$

$$\mathbf{y} = \begin{pmatrix} \frac{11+\sqrt{221}}{10} & 1 \\ \frac{-10}{11+\sqrt{221}} & 1 \end{pmatrix} \mathbf{x} - \begin{pmatrix} \frac{11+\sqrt{221}}{10} & 1 \\ \frac{-10}{11+\sqrt{221}} & 1 \end{pmatrix} \begin{pmatrix} \frac{-1}{7} \\ \frac{3}{7} \end{pmatrix} \quad (5.1.7.116)$$

$$\mathbf{y} = \begin{pmatrix} \frac{11+\sqrt{221}}{10} & 1 \\ \frac{-10}{11+\sqrt{221}} & 1 \end{pmatrix} \mathbf{x} - \begin{pmatrix} \frac{-11-\sqrt{221}}{70} + \frac{3}{7} \\ \frac{70}{(7)11+(7)\sqrt{221}} + \frac{3}{7} \end{pmatrix} \quad (5.1.7.117)$$

Substituting the equations (5.1.7.112), (5.1.7.107) in equation (5.1.7.113) Equation of asymptotes is

$$\Rightarrow \mathbf{y}^T \begin{pmatrix} \frac{5+\sqrt{221}}{2} & 0 \\ 0 & \frac{5-\sqrt{221}}{2} \end{pmatrix} \mathbf{y} + 1 = 0 \quad (5.1.7.118)$$

And the Equations of Conjugate hyperbola is 2(Equation of Asymptotes)- Equation of hyper-

bola.

$$\Rightarrow \mathbf{y}^T \begin{pmatrix} \frac{5+\sqrt{221}}{2} & 0 \\ 0 & \frac{5-\sqrt{221}}{2} \end{pmatrix} \mathbf{y} = 0 \quad (5.1.7.119)$$

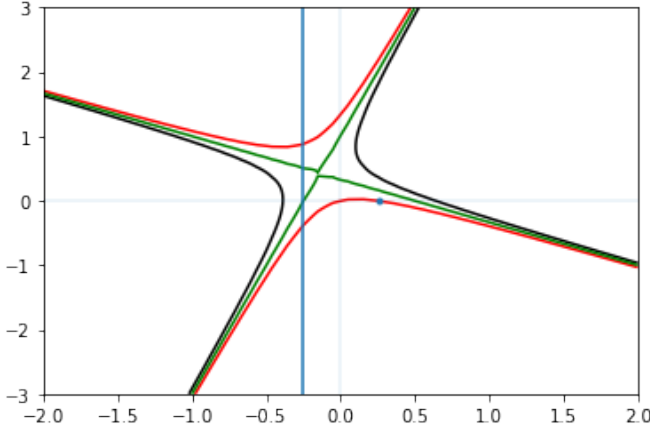


Fig. 5.1.7.1: Hyperbola with assymptotes and its conjugate

5.1.8. What conics do the following equation represents? When possible, find the center and the equation referred to the center.

$$55x^2 - 120xy + 20y^2 + 64x - 48y = 0 \quad (5.1.8.1)$$

**Solution:** The general equation of second degree can be represented as:

$$\mathbf{X}^T \mathbf{V} \mathbf{X} + 2\mathbf{u}^T \mathbf{X} + f = 0 \quad (5.1.8.2)$$

The above 5.1.8.1 can also be written as:

$$\mathbf{X}^T \begin{pmatrix} 55 & -60 \\ -60 & 20 \end{pmatrix} \mathbf{X} + 2 \begin{pmatrix} 32 & -24 \end{pmatrix} \mathbf{X} + 0 = 0 \quad (5.1.8.3)$$

So,

$$\mathbf{V} = \begin{pmatrix} 55 & -60 \\ -60 & 20 \end{pmatrix} \quad (5.1.8.4)$$

and

$$\mathbf{u} = \begin{pmatrix} 32 \\ -24 \end{pmatrix} \quad (5.1.8.5)$$

$$f = 0 \quad (5.1.8.6)$$

Now,

$$\det \mathbf{V} = \begin{vmatrix} 55 & -60 \\ -60 & 20 \end{vmatrix} \quad (5.1.8.7)$$

$$\Rightarrow \det \mathbf{V} = -2500 < 0 \quad (5.1.8.8)$$

As  $\det \mathbf{V} < 0$ , so we can say that the above conic section 5.1.8.1 is hyperbola. Now,

$$\mathbf{V}^{-1} = \frac{1}{-2500} \begin{pmatrix} 20 & 60 \\ 60 & 55 \end{pmatrix} \quad (5.1.8.9)$$

The center of this hyperbola will be:

$$\mathbf{c} = -\mathbf{V}^{-1} \mathbf{u} \quad (5.1.8.10)$$

$$\Rightarrow \mathbf{c} = \frac{1}{2500} \begin{pmatrix} 20 & 60 \\ 60 & 55 \end{pmatrix} \begin{pmatrix} 32 \\ -24 \end{pmatrix} \quad (5.1.8.11)$$

$$\Rightarrow \mathbf{c} = \begin{pmatrix} -\frac{8}{25} \\ \frac{6}{25} \end{pmatrix} \quad (5.1.8.12)$$

$$(5.1.8.13)$$

Now the characteristic equation of  $\mathbf{V}$  is obtained as:

$$|\mathbf{V} - \lambda \mathbf{I}| = 0 \quad (5.1.8.14)$$

$$\Rightarrow \begin{vmatrix} 55 - \lambda & -60 \\ -60 & 20 - \lambda \end{vmatrix} = 0 \quad (5.1.8.15)$$

$$\Rightarrow \lambda^2 - 75\lambda - 2500 = 0 \quad (5.1.8.16)$$

The eigen values are given by:

$$\lambda_1 = 100 \quad (5.1.8.17)$$

$$\lambda_2 = -25 \quad (5.1.8.18)$$

The eigen vector  $\mathbf{P}$  is defined as:

$$\mathbf{V} \mathbf{P} = \lambda \mathbf{P} \quad (5.1.8.19)$$

$$\Rightarrow (\mathbf{V} - \lambda \mathbf{I}) \mathbf{P} = \mathbf{0} \quad (5.1.8.20)$$

For  $\lambda_1 = 100$ ,

$$(\mathbf{V} - \lambda_1 \mathbf{I}) = \begin{pmatrix} -45 & -60 \\ -60 & -80 \end{pmatrix} \quad (5.1.8.21)$$

By row reduction,

$$\begin{pmatrix} -45 & -60 \\ -60 & -80 \end{pmatrix} \xrightarrow[R_1 \leftarrow R_1 / (-5)]{R_2 \leftarrow R_2 / (-5)} \quad (5.1.8.22)$$

$$\begin{pmatrix} 9 & 12 \\ 12 & 16 \end{pmatrix} \xrightarrow[R_1 \leftarrow R_1 / 3]{R_2 \leftarrow R_2 / 4} \quad (5.1.8.23)$$

$$\begin{pmatrix} 3 & 4 \\ 3 & 4 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 3 & 4 \\ 0 & 0 \end{pmatrix} \quad (5.1.8.24)$$

So,

$$(\mathbf{V} - \lambda_1 \mathbf{I})\mathbf{P}_1 = \mathbf{0} \quad (5.1.8.25)$$

$$\Rightarrow \begin{pmatrix} 3 & 4 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (5.1.8.26)$$

$$\Rightarrow \mathbf{P}_1 = \begin{pmatrix} -\frac{4}{3} \\ 1 \end{pmatrix} \quad (5.1.8.27)$$

Similarly, For  $\lambda_2=100$ ,

$$(\mathbf{V} - \lambda_2 \mathbf{I}) = \begin{pmatrix} 80 & -60 \\ -60 & 45 \end{pmatrix} \quad (5.1.8.28)$$

By row reduction,

$$\begin{pmatrix} 80 & -60 \\ -60 & 45 \end{pmatrix} \xrightarrow[R_1 \leftarrow R_1/5]{R_2 \leftarrow R_2/5} \quad (5.1.8.29)$$

$$\begin{pmatrix} 16 & -12 \\ -12 & 9 \end{pmatrix} \xrightarrow[R_1 \leftarrow R_1/4]{R_2 \leftarrow R_2/(-3)} \quad (5.1.8.30)$$

$$\begin{pmatrix} 4 & -3 \\ 4 & -3 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 4 & -3 \\ 0 & 0 \end{pmatrix} \quad (5.1.8.31)$$

So,

$$(\mathbf{V} - \lambda_2 \mathbf{I})\mathbf{P}_2 = \mathbf{0} \quad (5.1.8.32)$$

$$\Rightarrow \begin{pmatrix} 4 & -3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (5.1.8.33)$$

$$\Rightarrow \mathbf{P}_2 = \begin{pmatrix} 1 \\ \frac{4}{3} \end{pmatrix} \quad (5.1.8.34)$$

By eigen decomposition  $\mathbf{V}$  can also be written as:

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T \quad (5.1.8.35)$$

where

$$\mathbf{P} = (\mathbf{P}_1 \quad \mathbf{P}_2) \quad (5.1.8.36)$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad (5.1.8.37)$$

So,

$$\mathbf{P} = \begin{pmatrix} -\frac{4}{3} & 1 \\ 1 & \frac{4}{3} \end{pmatrix} \quad (5.1.8.38)$$

$$\mathbf{D} = \begin{pmatrix} 100 & 0 \\ 0 & -25 \end{pmatrix} \quad (5.1.8.39)$$

and

$$\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f = 16 > 0 \quad (5.1.8.40)$$

So, the axes are:

$$a = \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}} = \frac{2}{5} \quad (5.1.8.41)$$

$$b = \sqrt{\frac{f - \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u}}{\lambda_2}} = \frac{4}{5} \quad (5.1.8.42)$$

Now, the equation 5.1.8.1 can be written as:

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f \quad (5.1.8.43)$$

where,

$$\mathbf{y} = \mathbf{P}^T (\mathbf{x} - \mathbf{c}) \quad (5.1.8.44)$$

So,

$$\mathbf{y}^T \begin{pmatrix} 100 & 0 \\ 0 & -25 \end{pmatrix} \mathbf{y} = 16 \quad (5.1.8.45)$$

$$\Rightarrow \mathbf{y}^T \begin{pmatrix} 100 & 0 \\ 0 & -25 \end{pmatrix} \mathbf{y} - 16 = 0 \quad (5.1.8.46)$$

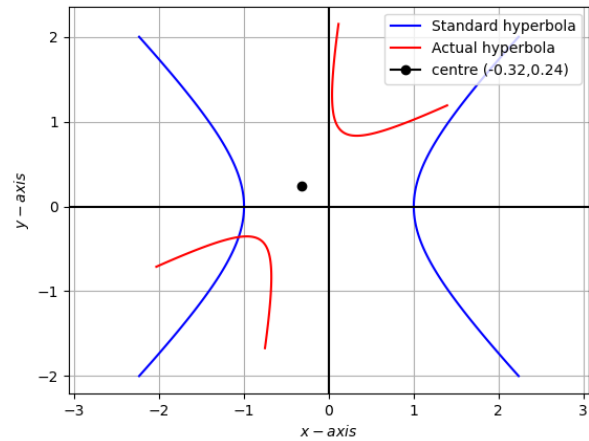


Fig. 5.1.8.1: Comparison of the Standard and Actual Hyperbola

5.1.9. Find the asymptotes of the given hyperbola and also the equation to its conjugate hyperbola

$$19x^2 + 24xy + y^2 - 22x - 6y = 0 \quad (5.1.9.1)$$

**Solution:** The general equation of second degree is given by

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \quad (5.1.9.2)$$

and can be expressed as

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (5.1.9.3)$$



where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad (5.1.9.4)$$

$$\mathbf{u} = \begin{pmatrix} d \\ e \end{pmatrix} \quad (5.1.9.5)$$

Comparing equations 5.1.9.1 and 5.1.9.3 we get

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 19 & 12 \\ 12 & 1 \end{pmatrix} \quad (5.1.9.6)$$

$$\mathbf{u} = \begin{pmatrix} -11 \\ -3 \end{pmatrix} \quad (5.1.9.7)$$

$$f = 0 \quad (5.1.9.8)$$

Expanding the Determinant of  $\mathbf{V}$ .

$$\Delta_V = \begin{vmatrix} 19 & 12 \\ 12 & 1 \end{vmatrix} < 0 \quad (5.1.9.9)$$

Hence from 5.1.9.9 given equation represents the hyperbola.

The characteristic equation of  $\mathbf{V}$  is obtained by evaluating the determinant

$$| \mathbf{V} - \lambda \mathbf{I} | = 0 \quad (5.1.9.10)$$

$$\begin{vmatrix} 19 - \lambda & 12 \\ 12 & 1 - \lambda \end{vmatrix} = 0 \quad (5.1.9.11)$$

$$(19 - \lambda)(1 - \lambda) - 144 = 0 \quad (5.1.9.12)$$

$$\lambda_1 = -5, \lambda_2 = 25 \quad (5.1.9.13)$$

The eigenvector  $\mathbf{p}$  is defined as

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \quad (5.1.9.14)$$

$$\implies (\mathbf{V} - \lambda\mathbf{I})\mathbf{p} = 0 \quad (5.1.9.15)$$

For  $\lambda_1 = -5$ ,

$$(\mathbf{V} - \lambda_1\mathbf{I}) = \begin{pmatrix} 19 + 5 & 12 \\ 12 & 1 + 5 \end{pmatrix} \quad (5.1.9.16)$$

By row reduction ,

$$\begin{pmatrix} 24 & 12 \\ 12 & 6 \end{pmatrix} \quad (5.1.9.17)$$

$$\xleftrightarrow{R_2 \leftarrow 2R_2 - R_1} \begin{pmatrix} 24 & 12 \\ 0 & 0 \end{pmatrix} \quad (5.1.9.18)$$

$$\xleftrightarrow{R_1 \leftarrow \frac{R_1}{12}} \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \quad (5.1.9.19)$$

Substituting equation 5.1.9.19 in equation

5.1.9.15 we get

$$\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (5.1.9.20)$$

Where,  $\mathbf{p} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$  Let  $u_1 = t$

$$u_2 = -2t \quad (5.1.9.21)$$

Eigen vector  $\mathbf{p}_1$  is given by

$$\mathbf{p}_1 = \begin{pmatrix} t \\ -2t \end{pmatrix} \quad (5.1.9.22)$$

Let  $t = 1$ , we get

$$\mathbf{p}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad (5.1.9.23)$$

For  $\lambda_2 = 25$ ,

$$(\mathbf{V} - \lambda_2\mathbf{I}) = \begin{pmatrix} 19 - 25 & 12 \\ 12 & 1 - 25 \end{pmatrix} \quad (5.1.9.24)$$

By row reduction ,

$$\begin{pmatrix} -6 & 12 \\ 12 & -24 \end{pmatrix} \quad (5.1.9.25)$$

$$\xleftrightarrow{R_2 \leftarrow R_2 + 2R_1} \begin{pmatrix} -6 & 12 \\ 0 & 0 \end{pmatrix} \quad (5.1.9.26)$$

$$\xleftrightarrow{R_1 \leftarrow \frac{R_1}{-6}} \begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix} \quad (5.1.9.27)$$

Substituting equation 5.1.9.27 in equation 5.1.9.15 we get

$$\begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (5.1.9.28)$$

Where,  $\mathbf{p} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$  Let  $v_1 = t$

$$v_2 = \frac{t}{2} \quad (5.1.9.29)$$

Eigen vector  $\mathbf{p}_2$  is given by

$$\mathbf{p}_2 = \begin{pmatrix} t \\ \frac{t}{2} \end{pmatrix} \quad (5.1.9.30)$$

Let  $t = 1$ , we get

$$\mathbf{p}_2 = \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} \quad (5.1.9.31)$$

By eigen decomposition  $\mathbf{V}$  can be represented

by

$$\mathbf{V} = \mathbf{PDP}^T \quad (5.1.9.32)$$

where

$$\mathbf{P} = (\mathbf{p}_1 \quad \mathbf{p}_2) \quad (5.1.9.33)$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad (5.1.9.34)$$

Substituting equations 5.1.9.23, 5.1.9.31 in equation 5.1.9.33 we get

$$\mathbf{P} = \begin{pmatrix} 1 & 1 \\ -2 & \frac{1}{2} \end{pmatrix} \quad (5.1.9.35)$$

Substituting equation 5.1.9.13 in 5.1.9.34 we get

$$\mathbf{D} = \begin{pmatrix} -5 & 0 \\ 0 & 25 \end{pmatrix} \quad (5.1.9.36)$$

Equation of a hyperbola and the combined equation of the Asymptotes differ only in the constant term.

$$19x^2 + 24xy + y^2 - 22x - 6y + K = 0 \quad (5.1.9.37)$$

The above equation can be expressed in the form

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (5.1.9.38)$$

Comparing equation we get

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 19 & 12 \\ 12 & 1 \end{pmatrix} \quad (5.1.9.39)$$

$$\mathbf{u} = \begin{pmatrix} -11 \\ -3 \end{pmatrix} \quad (5.1.9.40)$$

$$f = K \quad (5.1.9.41)$$

$$\Delta = \begin{vmatrix} 19 & 12 & -11 \\ 12 & 1 & -3 \\ -11 & -3 & K \end{vmatrix} \quad (5.1.9.42)$$

Since the equations represent pair of straight lines, equating the determinant to zero, we can get the value of K

$$\implies K = 4 \quad (5.1.9.43)$$

Let  $(\alpha, \beta)$  be their point of intersection, then

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -d \\ -e \end{pmatrix} \quad (5.1.9.44)$$

Substituting the values, we obtain,

$$\begin{pmatrix} 19 & 12 \\ 12 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 11 \\ 3 \end{pmatrix} \quad (5.1.9.45)$$

$$\text{We get, } \alpha = \frac{1}{5}, \beta = \frac{3}{5} \quad (5.1.9.46)$$

Using Affine transformation and Spectral decomposition, we get

$$X' = \pm \sqrt{-\frac{\lambda_2}{\lambda_1}} Y' \quad (5.1.9.47)$$

$$\text{where } X' = Xu_1 + Yu_2 \quad (5.1.9.48)$$

$$Y' = Xv_1 + Yv_2 \quad (5.1.9.49)$$

$$X = x - \alpha \text{ and } Y = y - \beta \quad (5.1.9.50)$$

Therefore,

$$\begin{aligned} u_1(x - \alpha) + u_2(y - \beta) = \\ \pm \sqrt{-\frac{\lambda_2}{\lambda_1}} (v_1(x - \alpha) + v_2(y - \beta)) \end{aligned} \quad (5.1.9.51)$$

Substituting values, we get

$$\begin{aligned} (x - \frac{1}{5}) - 2(y - \frac{3}{5}) = \\ \pm \sqrt{\frac{25}{5}} (x - \frac{1}{5}) + \frac{1}{2} (y - \frac{3}{5}) \end{aligned} \quad (5.1.9.52)$$

Simplifying above equation

$$8x + 9y - 7 = 0 \quad (5.1.9.53)$$

$$12x + y + 7 = 0 \quad (5.1.9.54)$$

$$\implies (8x + 9y - 7)(12x + y + 7) = 0 \quad (5.1.9.55)$$

Thus the equation of lines are

$$(8 \quad 9)\mathbf{x} = 7 \quad (5.1.9.56)$$

$$(12 \quad 1)\mathbf{x} = -7 \quad (5.1.9.57)$$

The Equation of Conjugate hyperbola is given by:

2(Equation of Asymptotes)- Equation of hyperbola.

From Eq 5.1.9.1 and 5.1.9.37, we obtain

equation of Conjugate hyperbola as:-

$$19x^2 + 24xy + y^2 - 22x - 6y + 8 = 0 \quad (5.1.9.58)$$

The general equation of second degree is given by

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \quad (5.1.9.59)$$

comparing equation 5.1.9.58 with the general equation of second degree given at 5.1.9.59, it can be expressed as

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (5.1.9.60)$$

where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad (5.1.9.61)$$

$$\mathbf{u} = \begin{pmatrix} d \\ e \end{pmatrix} \quad (5.1.9.62)$$

Comparing equations 5.1.9.58 and 5.1.9.60 we get

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 19 & 12 \\ 12 & 1 \end{pmatrix} \quad (5.1.9.63)$$

$$\mathbf{u} = \begin{pmatrix} -11 \\ -3 \end{pmatrix} \quad (5.1.9.64)$$

$$f = 8 \quad (5.1.9.65)$$

Therefore, the equation of the conjugate hyperbola is as given below:-

$$\mathbf{x}^T \begin{pmatrix} 19 & 12 \\ 12 & 1 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} -11 & -3 \end{pmatrix} \mathbf{x} + 8 = 0 \quad (5.1.9.66)$$

## 5.2 41

5.2.1. Trace the parabola:

$$(x - 4y)^2 = 51y \quad (5.2.1.1)$$

**Solution:** Expanding the given equation, we have,

$$x^2 - 8xy + 16y^2 - 51y = 0 \quad (5.2.1.2)$$

The general equation of second degree is given by

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \quad (5.2.1.3)$$

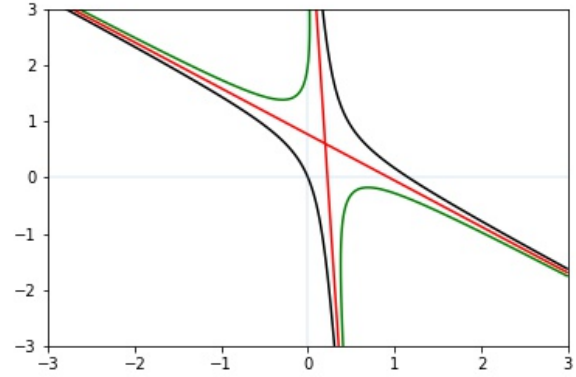


Fig. 5.1.9.1: Hyperbola, Conjugate Hyperbola and Asymptotes

and can be expressed as

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (5.2.1.4)$$

where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad (5.2.1.5)$$

$$\mathbf{u}^T = \begin{pmatrix} d & e \end{pmatrix} \quad (5.2.1.6)$$

From equation (5.2.1.2), we get

$$\mathbf{V} = \begin{pmatrix} 1 & -4 \\ -4 & 16 \end{pmatrix} \quad (5.2.1.7)$$

$$\mathbf{u} = \begin{pmatrix} 0 \\ -\frac{51}{2} \end{pmatrix} \quad (5.2.1.8)$$

$$f = 0 \quad (5.2.1.9)$$

Expanding the determinant of  $\mathbf{V}$  we observe,

$$\begin{vmatrix} 1 & -4 \\ -4 & 16 \end{vmatrix} = 0 \quad (5.2.1.10)$$

Therefore, (5.2.1.2) is a parabola.

The characteristic equation of  $\mathbf{V}$  is given as follows,

$$|\lambda \mathbf{I} - \mathbf{V}| = \begin{vmatrix} \lambda - 1 & 4 \\ 4 & \lambda - 16 \end{vmatrix} = 0 \quad (5.2.1.11)$$

$$\Rightarrow \lambda^2 - 17\lambda = 0 \quad (5.2.1.12)$$

The eigenvalues are given by

$$\lambda_1 = 0, \lambda_2 = 17 \quad (5.2.1.13)$$

For  $\lambda_1 = 0$ , the eigen vector  $\mathbf{p}$  is given by

$$\mathbf{V} \mathbf{p} = 0 \quad (5.2.1.14)$$

Row reducing  $\mathbf{V}$

$$\begin{pmatrix} 1 & -4 \\ -4 & 16 \end{pmatrix} \xrightarrow[R_2=R_2+R_1]{R_2=R_2/4} \begin{pmatrix} 1 & -4 \\ 0 & 0 \end{pmatrix} \quad (5.2.1.15)$$

$$\Rightarrow \mathbf{p}_1 = \frac{1}{\sqrt{17}} \begin{pmatrix} -4 \\ -1 \end{pmatrix} \quad (5.2.1.16)$$

Similarly,

$$\mathbf{p}_2 = \frac{1}{\sqrt{17}} \begin{pmatrix} -1 \\ 4 \end{pmatrix} \quad (5.2.1.17)$$

Thus,

$$\mathbf{P} = (\mathbf{p}_1 \quad \mathbf{p}_2) = \frac{1}{\sqrt{17}} \begin{pmatrix} -4 & -1 \\ -1 & 4 \end{pmatrix} \quad (5.2.1.18)$$

The equation of the parabola is:

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = -2\eta(1 \ 0) \mathbf{y} \quad (5.2.1.19)$$

where

$$\eta = \mathbf{u}^T \mathbf{p}_1 = \frac{51}{2\sqrt{17}} \quad (5.2.1.20)$$

and focal length of the parabola is given by

$$\frac{|2\mathbf{u}^T \mathbf{p}_1|}{\lambda_2} = \frac{3}{\sqrt{17}} \quad (5.2.1.21)$$

Now,

$$\begin{pmatrix} \mathbf{u}^T + \eta \mathbf{p}_1^T \\ \mathbf{v} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \eta \mathbf{p}_1 - \mathbf{u} \end{pmatrix} \quad (5.2.1.22)$$

using equations (5.2.1.7), (5.2.1.8) and (5.2.1.22)

$$\begin{pmatrix} -6 & -27 \\ 1 & -4 \\ -4 & 16 \end{pmatrix} \mathbf{c} = \begin{pmatrix} 0 \\ -6 \\ 24 \end{pmatrix} \quad (5.2.1.23)$$

Forming the augmented matrix and row reduc-

ing it:

$$\begin{pmatrix} -6 & -27 & 0 \\ 1 & -4 & -6 \\ -4 & 16 & 24 \end{pmatrix} \quad (5.2.1.24)$$

$$\xleftrightarrow{R_3 \leftarrow R_3 + 4R_2} \begin{pmatrix} -6 & -27 & 0 \\ 1 & -4 & -6 \\ 0 & 0 & 0 \end{pmatrix} \quad (5.2.1.25)$$

$$\xleftrightarrow{R_1 \leftarrow R_1 / (-6)} \begin{pmatrix} 1 & 9/2 & 0 \\ 1 & -4 & -6 \\ 0 & 0 & 0 \end{pmatrix} \quad (5.2.1.26)$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 9/2 & 0 \\ 0 & -17/2 & -6 \\ 0 & 0 & 0 \end{pmatrix} \quad (5.2.1.27)$$

$$\xleftrightarrow{R_2 \leftarrow (-\frac{2}{17})R_2} \begin{pmatrix} 1 & 9/2 & 0 \\ 0 & 1 & 12/17 \\ 0 & 0 & 0 \end{pmatrix} \quad (5.2.1.28)$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - (\frac{9}{2})R_2} \begin{pmatrix} 1 & 0 & -54/17 \\ 0 & 1 & 12/17 \\ 0 & 0 & 0 \end{pmatrix} \quad (5.2.1.29)$$

Thus the vertex is:

$$\mathbf{c} = \begin{pmatrix} -\frac{54}{17} \\ \frac{12}{17} \end{pmatrix} \quad (5.2.1.30)$$

$$\approx \begin{pmatrix} -3.18 \\ 0.71 \end{pmatrix} \quad (5.2.1.31)$$

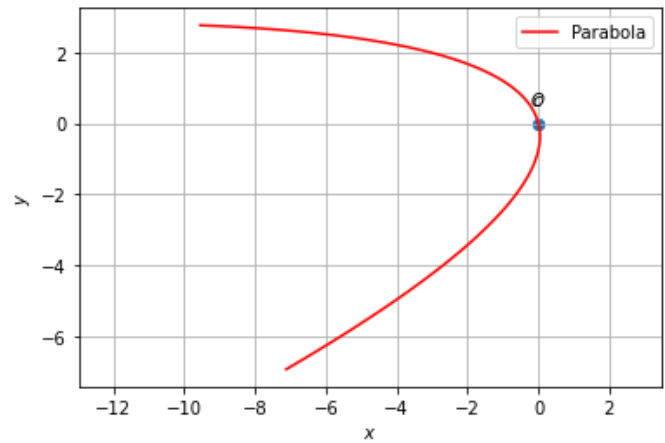


Fig. 5.2.1.1: Parabola

### 5.2.2. Trace the curve

$$(x - y)^2 = x + y + 1 \quad (5.2.2.1)$$

**Solution:**

We have given equation as :

$$(x - y)^2 = x + y + 1 \quad (5.2.2.2)$$

$$\Rightarrow x^2 - 2xy + y^2 - x - y - 1 = 0 \quad (5.2.2.3)$$

The general equation of second degree is given by

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \quad (5.2.2.4)$$

and can be expressed as

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (5.2.2.5)$$

where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad (5.2.2.6)$$

$$\mathbf{u}^T = \begin{pmatrix} d & e \end{pmatrix} \quad (5.2.2.7)$$

Comparing (5.2.2.3) with (5.2.2.4), we get

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad (5.2.2.8)$$

$$\mathbf{u}^T = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} \quad (5.2.2.9)$$

$$f = -1 \quad (5.2.2.10)$$

Expanding the determinant of  $\mathbf{V}$  we observe,

$$|\mathbf{V}| = \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} = 0 \quad (5.2.2.11)$$

Also

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = \begin{vmatrix} 1 & -1 & -\frac{1}{2} \\ -1 & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -1 \end{vmatrix} \neq 0 \quad (5.2.2.12)$$

Hence from (5.2.2.11) and (5.2.2.12) we conclude that given equation is an parabola. The characteristic equation of  $\mathbf{V}$  is given as follows,

$$|\lambda \mathbf{I} - \mathbf{V}| = \begin{vmatrix} \lambda - 1 & -1 \\ -1 & \lambda - 1 \end{vmatrix} = 0 \quad (5.2.2.13)$$

$$\Rightarrow (\lambda - 1)^2 - 1 = 0 \quad (5.2.2.14)$$

The eigenvalues are the roots of (5.2.2.14) given by

$$\lambda_1 = 0, \lambda_2 = 2 \quad (5.2.2.15)$$

The eigenvector  $\mathbf{p}$  is defined as:

$$\mathbf{V} \mathbf{p} = \lambda \mathbf{p} \quad (5.2.2.16)$$

$$\Rightarrow (\lambda \mathbf{I} - \mathbf{V}) \mathbf{p} = 0 \quad (5.2.2.17)$$

where  $\lambda$  is the eigenvalue. For  $\lambda_1 = 0$ ,

$$\mathbf{V} \mathbf{p} = 0 \quad (5.2.2.18)$$

Row reducing  $\mathbf{V}$  yields,

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 + R_1} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \quad (5.2.2.19)$$

Similarly, the eigenvector corresponding to  $\lambda_2$  can be obtained as

$$(\lambda_2 \mathbf{I} - \mathbf{V}) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad (5.2.2.20)$$

It is easy to verify that

$$\mathbf{V} = \mathbf{P} \mathbf{D} \mathbf{P}^{-1} = \mathbf{P} \mathbf{D} \mathbf{P}^T \quad \because \mathbf{P}^{-1} = \mathbf{P}^T \quad (5.2.2.21)$$

$$\text{or, } \mathbf{D} = \mathbf{P}^T \mathbf{V} \mathbf{P} \quad (5.2.2.22)$$

From equation (5.2.2.19) and (5.2.2.20), we have

$$\mathbf{p}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } \mathbf{p}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (5.2.2.23)$$

Thus, the eigenvector rotation matrix and the eigenvalue matrix are

$$\mathbf{P} = \frac{1}{\sqrt{2}} (\mathbf{p}_1 \quad \mathbf{p}_2) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad (5.2.2.24)$$

$$\mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \quad (5.2.2.25)$$

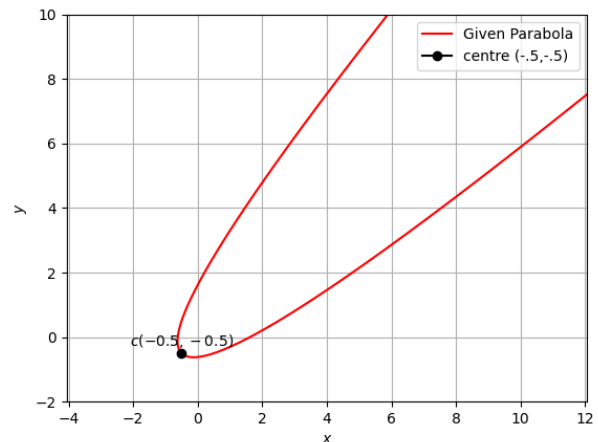


Fig. 5.2.2.1: Parabola with the center  $c$

The focal length of the parabola is given by

$$\frac{|2\mathbf{u}^T \mathbf{p}_1|}{\lambda_2} = \frac{\sqrt{2}}{2} = \sqrt{2} \quad (5.2.2.26)$$

and its equation is

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = -2\eta(1 \ 0) \mathbf{y} \quad (5.2.2.27)$$

where,

$$\eta = \mathbf{u}^T \mathbf{p}_1 = -\frac{1}{\sqrt{2}} \quad (5.2.2.28)$$

$$\begin{pmatrix} \mathbf{u}^T + \eta \mathbf{p}_1^T \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \eta \mathbf{p}_1 - \mathbf{u} \end{pmatrix} \quad (5.2.2.29)$$

$$\Rightarrow \begin{pmatrix} -1 & -1 \\ 1 & -1 \\ -1 & 1 \end{pmatrix} \mathbf{c} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad (5.2.2.30)$$

Forming the augmented matrix and row reducing it:

$$\begin{aligned} \begin{pmatrix} -1 & -1 & 1 \\ 1 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix} &\xrightarrow{R_2 \leftarrow R_2 + R_1} \begin{pmatrix} -1 & -1 & 1 \\ 0 & -2 & 1 \\ -1 & 1 & 0 \end{pmatrix} \xrightarrow{\begin{matrix} R_3 \leftarrow R_3 - R_1 \\ R_1 \leftarrow -1R_1 \end{matrix}} \begin{pmatrix} 1 & 1 & -1 \\ 0 & -2 & 1 \\ 0 & 2 & -1 \end{pmatrix} \\ &\xrightarrow{R_3 \leftarrow R_3 + R_2} \begin{pmatrix} 1 & 1 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{\begin{matrix} R_1 \leftarrow \frac{R_1}{-2} \\ R_1 \leftarrow R_1 - R_2 \end{matrix}} \begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix} \end{aligned} \quad (5.2.2.31)$$

So,

$$\mathbf{c} = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} \quad (5.2.2.32)$$

### 5.2.3. Trace the parabola

$$(4x + 3y + 15)^2 = 5(3x - 4y) \quad (5.2.3.1)$$

**Solution:** The given equation can be rewritten as

$$16x^2 + 24xy + 9y^2 + 105x + 110y + 225 = 0 \quad (5.2.3.2)$$

Comparing this to the standard equation,

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 16 & 12 \\ 12 & 9 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} \frac{105}{2} \\ 55 \end{pmatrix}, \quad f = 225 \quad (5.2.3.3)$$

The characteristic equation of  $\mathbf{V}$  is given as

$$|\lambda \mathbf{I} - \mathbf{V}| = 0 \quad (5.2.3.4)$$

$$\Rightarrow \begin{vmatrix} \lambda - 16 & -12 \\ -12 & \lambda - 9 \end{vmatrix} = 0 \quad (5.2.3.5)$$

$$\Rightarrow \lambda^2 - 25\lambda = 0 \quad (5.2.3.6)$$

The eigenvalues are the roots of the equation (5.2.3.6), which are as follows :

$$\lambda_1 = 0, \quad \lambda_2 = 25 \quad (5.2.3.7)$$

The eigen vector  $\mathbf{p}$  is defined as,

$$\mathbf{V} \mathbf{p} = \lambda \mathbf{p} \quad (5.2.3.8)$$

$$\Rightarrow (\lambda \mathbf{I} - \mathbf{V}) \mathbf{p} = 0 \quad (5.2.3.9)$$

For  $\lambda_1 = 0$

$$(\lambda_1 \mathbf{I} - \mathbf{V}) = \begin{pmatrix} -16 & -12 \\ -12 & -9 \end{pmatrix} \xrightarrow{\begin{matrix} R_1 \leftarrow \frac{1}{4}R_1 \\ R_2 \leftarrow R_2 - 3R_1 \end{matrix}} \begin{pmatrix} -4 & -3 \\ 0 & 0 \end{pmatrix} \quad (5.2.3.10)$$

$$\Rightarrow \mathbf{p}_1 = \frac{1}{5} \begin{pmatrix} -3 \\ 4 \end{pmatrix} \quad (5.2.3.11)$$

For  $\lambda_2 = 25$

$$(\lambda_2 \mathbf{I} - \mathbf{V}) = \begin{pmatrix} 9 & -12 \\ -12 & 16 \end{pmatrix} \xrightarrow{\begin{matrix} R_1 \leftarrow \frac{1}{3}R_1 \\ R_2 \leftarrow R_2 + 4R_1 \end{matrix}} \begin{pmatrix} 3 & -4 \\ 0 & 0 \end{pmatrix} \quad (5.2.3.12)$$

$$\Rightarrow \mathbf{p}_2 = \frac{1}{5} \begin{pmatrix} 4 \\ 3 \end{pmatrix} \quad (5.2.3.13)$$

So, using Eigenvalue decomposition,  $\mathbf{P}^T \mathbf{V} \mathbf{P} = \mathbf{D}$ , where

$$\mathbf{P} = \frac{1}{5} \begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix} \quad (5.2.3.14)$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 25 \end{pmatrix} \quad (5.2.3.15)$$

Then, for the parabola

$$\text{focal length} = \left| \frac{2\eta}{\lambda_2} \right| \quad (5.2.3.16)$$

$$\eta = \mathbf{p}_1^T \mathbf{u} = \frac{25}{2} \quad (5.2.3.17)$$

Substituting values from (5.2.3.17) and (5.2.3.7) in (5.2.3.16), we get

$$\text{focal length} = 1 \quad (5.2.3.18)$$

The standard equation of the parabola is given by

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = -2\eta(1 \ 0) \mathbf{y} \quad (5.2.3.19)$$

And the vertex  $\mathbf{c}$  is given by

$$\begin{pmatrix} \mathbf{u}^T + \eta \mathbf{p}_1^T \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \eta \mathbf{p}_1 - \mathbf{u} \end{pmatrix} \quad (5.2.3.20)$$

Substituting values from (5.2.3.3), (5.2.3.17), (5.2.3.11) in (5.2.3.20),

$$\begin{pmatrix} 45 & 65 \\ 16 & 12 \\ 12 & 9 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -225 \\ -60 \\ -45 \end{pmatrix} \quad (5.2.3.21)$$

To find  $\mathbf{c}$ , performing row reduction on the augmented matrix as follows:

$$\begin{pmatrix} 45 & 65 & -225 \\ 16 & 12 & -60 \\ 12 & 9 & -45 \end{pmatrix} \xrightarrow[R_1 \leftarrow \frac{1}{45} R_1]{R_3 \leftarrow R_3 - \frac{3}{4} R_2} \begin{pmatrix} 1 & \frac{13}{9} & -5 \\ 16 & 12 & -60 \\ 0 & 0 & 0 \end{pmatrix} \quad (5.2.3.22)$$

$$\xrightarrow{R_2 \leftarrow R_2 - 16R_1} \begin{pmatrix} 1 & \frac{13}{9} & -5 \\ 0 & -\frac{100}{9} & 20 \\ 0 & 0 & 0 \end{pmatrix} \quad (5.2.3.23)$$

$$\xrightarrow{R_2 \leftarrow -\frac{9}{100} R_2} \begin{pmatrix} 1 & \frac{13}{9} & -5 \\ 0 & 1 & -\frac{9}{5} \\ 0 & 0 & 0 \end{pmatrix} \quad (5.2.3.24)$$

$$\xrightarrow{R_1 \leftarrow R_1 - \frac{13}{9} R_2} \begin{pmatrix} 1 & 0 & -\frac{12}{5} \\ 0 & 1 & -\frac{9}{5} \\ 0 & 0 & 0 \end{pmatrix} \quad (5.2.3.25)$$

Thus,

$$\mathbf{c} = \begin{pmatrix} -\frac{12}{5} \\ -\frac{9}{5} \end{pmatrix} = \begin{pmatrix} -2.4 \\ -1.8 \end{pmatrix} \quad (5.2.3.26)$$

#### 5.2.4. Trace the parabola

$$16x^2 + 24xy + 9y^2 - 5x - 10y + 1 = 0$$

**Solution:** Compare the given equation with the

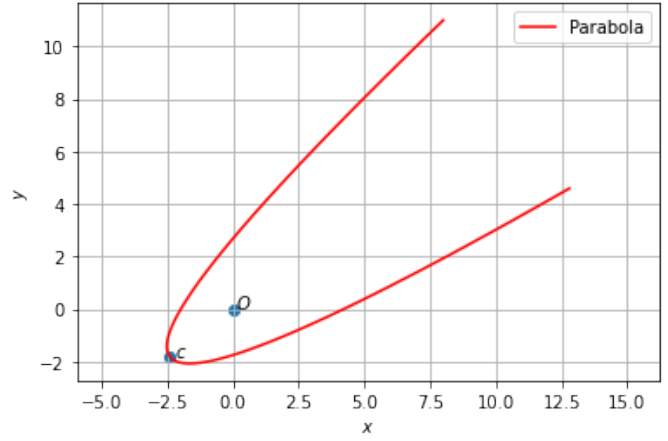


Fig. 5.2.3.1: Parabola with vertex  $\mathbf{c}$

standard form

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \quad (5.2.4.1)$$

Write the values Of  $\mathbf{V}$  and  $\mathbf{u}$  as follows

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 16 & 12 \\ 12 & 9 \end{pmatrix} \quad \mathbf{u} = \begin{pmatrix} -\frac{5}{2} \\ -5 \end{pmatrix} \quad f = 1 \quad (5.2.4.2)$$

The characteristic equation of  $\mathbf{V}$  is given as

$$|\lambda \mathbf{I} - \mathbf{V}| = 0 \quad (5.2.4.3)$$

$$\Rightarrow \begin{vmatrix} \lambda - 16 & -12 \\ -12 & \lambda - 9 \end{vmatrix} = 0 \quad (5.2.4.4)$$

$$\Rightarrow \lambda^2 - 25\lambda = 0 \quad (5.2.4.5)$$

The eigenvalues are the roots of the equation (5.2.4.5) are

$$\lambda_1 = 0, \quad \lambda_2 = 25 \quad (5.2.4.6)$$

The eigen vector  $\mathbf{p}$  is defined as,

$$\mathbf{V} \mathbf{p} = \lambda \mathbf{p} \quad (5.2.4.7)$$

$$\Rightarrow (\lambda \mathbf{I} - \mathbf{V}) \mathbf{p} = 0 \quad (5.2.4.8)$$

For  $\lambda_1 = 0$

$$(\lambda_1 \mathbf{I} - \mathbf{V}) = \begin{pmatrix} -16 & -12 \\ -12 & -9 \end{pmatrix} \xrightarrow[R_2 \leftarrow R_2 - 3R_1]{R_1 \leftarrow \frac{1}{4} R_1} \begin{pmatrix} -4 & -3 \\ 0 & 0 \end{pmatrix} \quad (5.2.4.9)$$

$$\Rightarrow \mathbf{p}_1 = \frac{1}{5} \begin{pmatrix} -3 \\ 4 \end{pmatrix} \quad (5.2.4.10)$$

For  $\lambda_2 = 25$

$$(\lambda_2 \mathbf{I} - \mathbf{V}) = \begin{pmatrix} 9 & -12 \\ -12 & 16 \end{pmatrix} \xrightarrow[R_2 \leftarrow R_2 + 4R_1]{R_1 \leftarrow \frac{1}{3}R_1} \begin{pmatrix} 3 & -4 \\ 0 & 0 \end{pmatrix} \quad (5.2.4.11)$$

$$\Rightarrow \mathbf{p}_2 = \frac{1}{5} \begin{pmatrix} 4 \\ 3 \end{pmatrix} \quad (5.2.4.12)$$

Use Eigenvalue decomposition,  $\mathbf{P}^T \mathbf{V} \mathbf{P} = \mathbf{D}$ , where

$$\mathbf{P} = \frac{1}{5} \begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix} \quad (5.2.4.13)$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 25 \end{pmatrix} \quad (5.2.4.14)$$

Focal length of the parabola is given as

$$\text{focal length} = \left| \frac{2\eta}{\lambda_2} \right| \quad (5.2.4.15)$$

$$\eta = \mathbf{p}_1^T \mathbf{u} = -\frac{5}{2} \quad (5.2.4.16)$$

Substituting values from (5.2.4.16) and (5.2.4.6) in (5.2.4.15), we get

$$\text{focal length} = \frac{1}{5} \quad (5.2.4.17)$$

The standard equation of the parabola is given by

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = -2\eta \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{y} \quad (5.2.4.18)$$

And the vertex  $\mathbf{c}$  is given by

$$\begin{pmatrix} \mathbf{u}^T + \eta \mathbf{p}_1^T \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \eta \mathbf{p}_1 - \mathbf{u} \end{pmatrix} \quad (5.2.4.19)$$

Substituting values from (5.2.4.2), (5.2.4.16), (5.2.4.10) in (5.2.4.19),

$$\begin{pmatrix} -1 & -7 \\ 16 & 12 \\ 12 & 9 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} \quad (5.2.4.20)$$

To find  $\mathbf{c}$ , performing row reduction on the

augmented matrix as follows:

$$\begin{pmatrix} -1 & -7 & -1 \\ 16 & 12 & 4 \\ 12 & 9 & 3 \end{pmatrix} \xrightarrow[R_1 \leftarrow -R_1]{R_3 \leftarrow R_3 - \frac{3}{4}R_2} \begin{pmatrix} 1 & 7 & 1 \\ 16 & 12 & 4 \\ 0 & 0 & 0 \end{pmatrix} \quad (5.2.4.21)$$

$$\xrightarrow{R_2 \leftarrow R_2 - 16R_1} \begin{pmatrix} 1 & 7 & 1 \\ 0 & -100 & -12 \\ 0 & 0 & 0 \end{pmatrix} \quad (5.2.4.22)$$

$$\xrightarrow{R_2 \leftarrow \frac{-1}{100}R_2} \begin{pmatrix} 1 & 7 & 1 \\ 0 & 1 & \frac{3}{25} \\ 0 & 0 & 0 \end{pmatrix} \quad (5.2.4.23)$$

$$\xrightarrow{R_1 \leftarrow R_1 - 7R_2} \begin{pmatrix} 1 & 0 & \frac{4}{25} \\ 0 & 1 & \frac{3}{25} \\ 0 & 0 & 0 \end{pmatrix} \quad (5.2.4.24)$$

Thus,

$$\mathbf{c} = \begin{pmatrix} \frac{4}{25} \\ \frac{3}{25} \end{pmatrix} \quad (5.2.4.25)$$

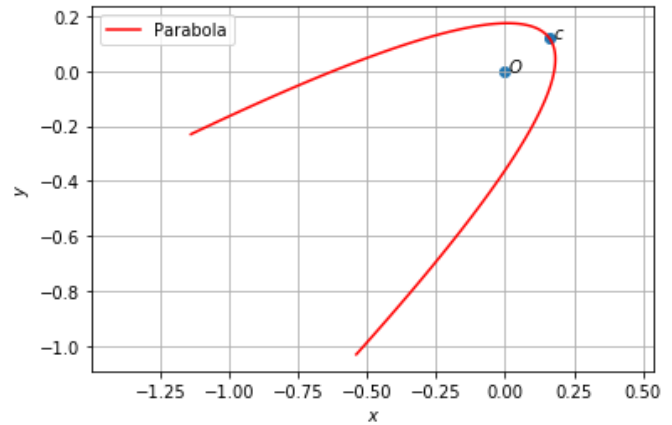


Fig. 5.2.4.1: Parabola with vertex  $\mathbf{c}$

### 5.2.5. Trace the parabola

$$9x^2 + 24xy + 16y^2 - 4y - x + 7 = 0 \quad (5.2.5.1)$$

**Solution:** The general second degree equation can be expressed as

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (5.2.5.2)$$



Comparing (5.2.5.1) and (5.2.5.2) we get

$$\mathbf{V} = \begin{pmatrix} 9 & 12 \\ 12 & 16 \end{pmatrix} \quad (5.2.5.3)$$

$$\mathbf{u} = \begin{pmatrix} \frac{-1}{2} \\ -2 \end{pmatrix} \quad (5.2.5.4)$$

$$f = 7 \quad (5.2.5.5)$$

The characteristic equation of  $\mathbf{V}$  is given as

$$|\mathbf{V} - \lambda \mathbf{I}| = 0 \quad (5.2.5.6)$$

$$\Rightarrow \begin{vmatrix} 9 - \lambda & 12 \\ 12 & 16 - \lambda \end{vmatrix} = 0 \quad (5.2.5.7)$$

$$\Rightarrow \lambda^2 - 25\lambda = 0 \quad (5.2.5.8)$$

The roots of (5.2.5.8) are eigenvalue of  $\mathbf{V}$  and are given by

$$\lambda_1 = 0, \lambda_2 = 25$$

The eigenvector  $\mathbf{p}$  is defined as

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \quad (5.2.5.9)$$

$$\Rightarrow (\mathbf{V} - \lambda\mathbf{I})\mathbf{p} = 0 \quad (5.2.5.10)$$

For  $\lambda_1 = 0$

$$(\mathbf{V} - \lambda\mathbf{I}) = \begin{pmatrix} 9 & 12 \\ 12 & 16 \end{pmatrix} \xrightarrow{R_2 = R_2 - \frac{4}{3}R_1} \begin{pmatrix} 9 & 12 \\ 0 & 0 \end{pmatrix} \quad (5.2.5.11)$$

Substituting equation (5.2.5.11) in equation (5.2.5.10) and upon normalization we get

$$\mathbf{p}_1 = \frac{1}{5} \begin{pmatrix} -4 \\ 3 \end{pmatrix} \quad (5.2.5.12)$$

For  $\lambda_2 = 25$

$$(\mathbf{V} - \lambda\mathbf{I}) = \begin{pmatrix} -16 & 12 \\ 12 & -9 \end{pmatrix} \xrightarrow{R_2 = R_2 + \frac{3}{4}R_1} \begin{pmatrix} -16 & 12 \\ 0 & 0 \end{pmatrix} \quad (5.2.5.13)$$

Substituting equation (5.2.5.13) in equation (5.2.5.10) and upon normalization we get

$$\mathbf{p}_2 = \frac{1}{5} \begin{pmatrix} 3 \\ 4 \end{pmatrix} \quad (5.2.5.14)$$

The matrix  $\mathbf{P}$  and  $\mathbf{D}$  are

$$\mathbf{P} = (\mathbf{p}_1 \ \mathbf{p}_2) = \frac{1}{5} \begin{pmatrix} -4 & 3 \\ 3 & 4 \end{pmatrix} \quad (5.2.5.15)$$

and

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 25 \end{pmatrix} \quad (5.2.5.16)$$

Then for the parabola

$$\eta = 2\mathbf{p}_1^T \mathbf{u} = -\frac{8}{5} \quad (5.2.5.17)$$

$$\text{focal length} = \left| \frac{\eta}{\lambda_2} \right| = \frac{8}{125} \quad (5.2.5.18)$$

For parabola  $|\mathbf{V}| = 0$ , so equation (5.2.5.2) can be written as

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = -\eta (1 \ 0) \mathbf{y} \quad (5.2.5.19)$$

And the vertex  $\mathbf{c}$  is given by

$$\left( \mathbf{u}^T + \frac{\eta}{2} \mathbf{p}_1^T \right) \mathbf{c} = \begin{pmatrix} -f \\ \frac{\eta}{2} \mathbf{p}_1 - \mathbf{u} \end{pmatrix} \quad (5.2.5.20)$$

Substituting values from (5.2.5.3), (5.2.5.4), (5.2.5.5), (5.2.5.12), (5.2.5.17) in (5.2.5.20)

$$\begin{pmatrix} \frac{7}{50} & -\frac{124}{50} \\ 9 & 12 \\ 12 & 16 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -7 \\ \frac{57}{50} \\ \frac{76}{50} \end{pmatrix} \quad (5.2.5.21)$$

To find  $\mathbf{c}$ , performing row reduction in augmented matrix as follows

$$\begin{pmatrix} \frac{7}{50} & -\frac{124}{50} & -7 \\ 9 & 12 & \frac{57}{50} \\ 12 & 16 & \frac{76}{50} \end{pmatrix} \xrightarrow[R_1 \leftarrow \frac{50}{7}R_1]{R_3 \leftarrow R_3 - \frac{4}{3}R_2} \begin{pmatrix} 1 & -\frac{124}{7} & -50 \\ 9 & 12 & \frac{57}{50} \\ 0 & 0 & 0 \end{pmatrix} \\ \xrightarrow{R_2 \leftarrow R_2 - 9R_1} \begin{pmatrix} 1 & -\frac{124}{7} & -50 \\ 0 & \frac{1200}{7} & \frac{22557}{50} \\ 0 & 0 & 0 \end{pmatrix} \\ \xrightarrow{R_2 \leftarrow \frac{7}{1200}R_2} \begin{pmatrix} 1 & -\frac{124}{7} & -50 \\ 0 & 1 & \frac{52633}{20000} \\ 0 & 0 & 0 \end{pmatrix} \\ \xrightarrow{R_1 \leftarrow R_1 + \frac{124}{7}R_2} \begin{pmatrix} 1 & 0 & -\frac{16911}{5000} \\ 0 & 1 & \frac{52633}{20000} \\ 0 & 0 & 0 \end{pmatrix}$$

Thus

$$\mathbf{c} = \begin{pmatrix} -\frac{16911}{5000} \\ \frac{52633}{20000} \\ 0 \end{pmatrix} \quad (5.2.5.22)$$

5.2.6. Trace the parabola and find its focus.

$$144y^2 - 120xy + 25x^2 + 619x - 272y + 663 = 0 \quad (5.2.6.1)$$

**Solution:** The general second degree equation

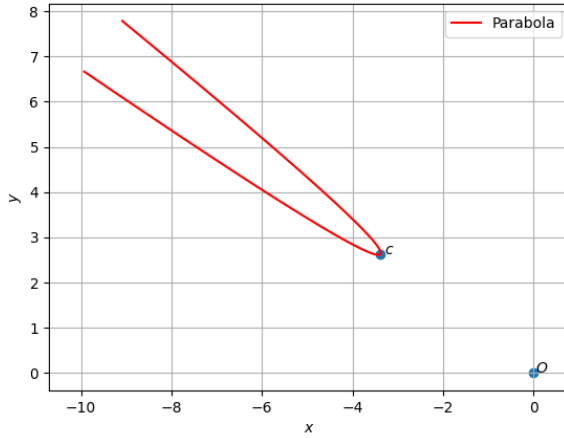


Fig. 5.2.5.1: Graph of  $9x^2 + 24xy + 16y^2 - 4y - x + 7 = 0$

can be expressed as follows,

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (5.2.6.2)$$

where,

$$\mathbf{V} = \begin{pmatrix} 144 & -60 \\ -60 & 25 \end{pmatrix} \quad (5.2.6.3)$$

$$\mathbf{u} = \begin{pmatrix} \frac{619}{2} \\ -136 \end{pmatrix} \quad (5.2.6.4)$$

$$f = 663 \quad (5.2.6.5)$$

a) Expanding the determinant of  $\mathbf{V}$  we observe,

$$\begin{vmatrix} 144 & -60 \\ -60 & 25 \end{vmatrix} = 0 \quad (5.2.6.6)$$

Also

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = \begin{vmatrix} 144 & -60 & \frac{619}{2} \\ -60 & 25 & -136 \\ \frac{619}{2} & -136 & 663 \end{vmatrix} \quad (5.2.6.7)$$

$$\neq 0 \quad (5.2.6.8)$$

Hence from (5.2.6.6) and (5.2.6.8) we conclude that given equation is an parabola. The characteristic equation of  $\mathbf{V}$  is given as

follows,

$$|\lambda \mathbf{I} - \mathbf{V}| = \begin{vmatrix} \lambda - 144 & 60 \\ 60 & \lambda - 25 \end{vmatrix} = 0 \quad (5.2.6.9)$$

$$\Rightarrow \lambda^2 - 169\lambda = 0 \quad (5.2.6.10)$$

Hence the characteristic equation of  $\mathbf{V}$  is given by (5.2.6.10). The roots of (5.2.6.10) i.e the eigenvalues are given by

$$\lambda_1 = 0, \lambda_2 = 169 \quad (5.2.6.11)$$

b) For  $\lambda_1 = 0$ , the eigen vector  $\mathbf{p}$  is given by

$$\mathbf{V} \mathbf{p} = 0 \quad (5.2.6.12)$$

Row reducing  $\mathbf{V}$  yields

$$\Rightarrow \begin{pmatrix} -144 & 60 \\ 60 & -25 \end{pmatrix} \xrightarrow[R_2=R_2+5R_1]{R_1=\frac{R_1}{12}} \begin{pmatrix} -12 & 5 \\ 0 & 0 \end{pmatrix} \quad (5.2.6.13)$$

$$\Rightarrow \mathbf{p}_1 = \frac{1}{13} \begin{pmatrix} 5 \\ 12 \end{pmatrix} \quad (5.2.6.14)$$

Similarly,

$$\mathbf{p}_2 = \frac{1}{13} \begin{pmatrix} 12 \\ -5 \end{pmatrix} \quad (5.2.6.15)$$

Thus, the eigenvector rotation matrix and the eigenvalue matrix are

$$\mathbf{P} = (\mathbf{p}_1 \quad \mathbf{p}_2) = \frac{1}{13} \begin{pmatrix} 5 & 12 \\ 12 & -5 \end{pmatrix} \quad (5.2.6.16)$$

$$\mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & 169 \end{pmatrix} \quad (5.2.6.17)$$

The focal length of the parabola is given by

$$\frac{|2\mathbf{u}^T \mathbf{p}_1|}{\lambda_2} = \frac{13}{169} = \frac{1}{13} \quad (5.2.6.18)$$

and its equation is

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = -\eta \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{y} \quad (5.2.6.19)$$

where

$$\eta = 2\mathbf{u}^T \mathbf{p}_1 = -13 \quad (5.2.6.20)$$

and the vertex  $\mathbf{c}$  is given by

$$\left( \mathbf{u}^T + \frac{\eta}{2} \mathbf{p}_1^T \right) \mathbf{c} = \left( \frac{-f}{\frac{\eta}{2} \mathbf{p}_1 - \mathbf{u}} \right) \quad (5.2.6.21)$$

using equations (5.2.6.4),(5.2.6.5) and (5.2.6.14)

$$\begin{pmatrix} 307 & -142 \\ 144 & -60 \\ -60 & 25 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -663 \\ -312 \\ 130 \end{pmatrix} \quad (5.2.6.22)$$

Forming the augmented matrix and row reducing it:

$$\begin{pmatrix} 307 & -142 & -663 \\ 144 & -60 & -312 \\ -60 & 25 & 130 \end{pmatrix} \quad (5.2.6.23)$$

$$R_2 \leftrightarrow \frac{R_2}{12}$$

$$\begin{pmatrix} 307 & -142 & -663 \\ 12 & -5 & -26 \\ -60 & 25 & 130 \end{pmatrix} \quad (5.2.6.24)$$

$$R_3 \leftrightarrow R_3 + 5R_2$$

$$\begin{pmatrix} 307 & -142 & -663 \\ 12 & -5 & -26 \\ 0 & 0 & 0 \end{pmatrix} \quad (5.2.6.25)$$

$$R_1 \leftrightarrow \frac{R_1}{307}$$

$$\begin{pmatrix} 1 & -\frac{142}{307} & -\frac{663}{307} \\ 0 & \frac{169}{307} & -\frac{26}{307} \\ 0 & 0 & 0 \end{pmatrix} \quad (5.2.6.26)$$

$$R_2 \leftrightarrow R_2 - 12R_1$$

$$\begin{pmatrix} 1 & -\frac{142}{307} & -\frac{663}{307} \\ 0 & 1 & \frac{307}{169} \\ 0 & 0 & \frac{307}{0} \end{pmatrix} \quad (5.2.6.27)$$

$$R_1 \leftrightarrow R_1 + (142/307)R_2$$

$$\begin{pmatrix} 1 & 0 & -29/13 \\ 0 & 1 & -2/13 \\ 0 & 0 & 0 \end{pmatrix} \quad (5.2.6.28)$$

Thus the vertex  $\mathbf{c}$  is:

$$\mathbf{c} = \begin{pmatrix} -29/13 \\ -2/13 \end{pmatrix} \quad (5.2.6.29)$$

The direction vector of axis of symmetry is

given by :

$$\mathbf{m} = \mathbf{Vc} + \mathbf{u} \quad (5.2.6.30)$$

$$= \begin{pmatrix} 144 & -60 \\ -60 & 25 \end{pmatrix} \begin{pmatrix} -\frac{29}{13} \\ -\frac{2}{13} \end{pmatrix} + \begin{pmatrix} \frac{619}{2} \\ \frac{272}{2} \end{pmatrix} \quad (5.2.6.31)$$

$$= \begin{pmatrix} 5 \\ -\frac{13}{2} \\ -6 \end{pmatrix} \quad (5.2.6.32)$$

$$\mathbf{m} = \frac{13}{2} \quad (5.2.6.33)$$

$$\Rightarrow \frac{\mathbf{m}}{\|\mathbf{m}\|} = \begin{pmatrix} 5 \\ -\frac{13}{12} \\ -\frac{13}{13} \end{pmatrix} \quad (5.2.6.34)$$

The focus is given by:

$$\mathbf{F} = \mathbf{c} - \left( \frac{\mathbf{m}}{\|\mathbf{m}\|} \times a \right) \quad (5.2.6.35)$$

$$= \begin{pmatrix} -\frac{29}{13} \\ -\frac{2}{13} \end{pmatrix} - \left( \begin{pmatrix} 5 \\ -\frac{13}{12} \\ -\frac{13}{13} \end{pmatrix} \times \frac{1}{52} \right) \quad (5.2.6.36)$$

$$= \begin{pmatrix} -\frac{1503}{676} \\ \frac{676}{23} \\ -\frac{169}{169} \end{pmatrix} \quad (5.2.6.37)$$

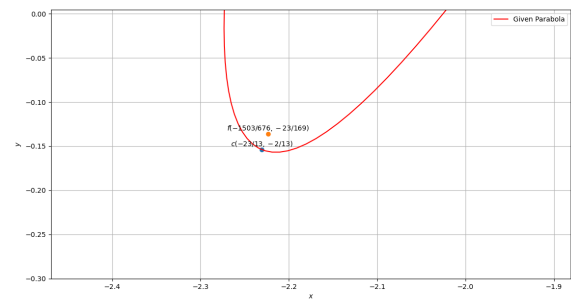


Fig. 5.2.6.1: Traced parabola

### 5.2.7. Trace the parabola

$$16x^2 - 24xy + 9y^2 + 32x + 86y - 39 = 0 \quad (5.2.7.1)$$

**Solution:** The general equation of a second

degree can be expressed as:

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (5.2.7.2)$$

Comparing (5.2.7.1) and (5.2.7.2)

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 16 & -12 \\ -12 & 9 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} 16 \\ 43 \end{pmatrix}, \quad f = -39 \quad (5.2.7.3)$$

Eigen Values: The characteristic equation of  $\mathbf{V}$  is given as

$$|\lambda \mathbf{I} - \mathbf{V}| = 0 \quad (5.2.7.4)$$

$$\Rightarrow \begin{vmatrix} \lambda - 16 & 12 \\ 12 & \lambda - 9 \end{vmatrix} = 0 \quad (5.2.7.5)$$

$$\Rightarrow \lambda^2 - 25\lambda = 0 \quad (5.2.7.6)$$

The eigenvalues are the roots of the equation (5.2.7.6), which are as follows:

$$\lambda_1 = 0, \quad \lambda_2 = 25 \quad (5.2.7.7)$$

Eigen Vectors: The eigen vector  $\mathbf{p}$  is defined as

$$\mathbf{V} \mathbf{p} = \lambda \mathbf{p} \quad (5.2.7.8)$$

$$\Rightarrow (\lambda \mathbf{I} - \mathbf{V}) \mathbf{p} = 0 \quad (5.2.7.9)$$

For  $\lambda_1 = 0$

$$(\lambda_1 \mathbf{I} - \mathbf{V}) = \begin{pmatrix} -16 & 12 \\ 12 & -9 \end{pmatrix} \xrightarrow[R_2 \leftarrow R_2 + 3R_1]{R_1 \leftarrow \frac{1}{4}R_1} \begin{pmatrix} -4 & 3 \\ 0 & 0 \end{pmatrix} \quad (5.2.7.10)$$

$$\Rightarrow \mathbf{p}_1 = \frac{1}{5} \begin{pmatrix} 3 \\ 4 \end{pmatrix} \quad (5.2.7.11)$$

For  $\lambda_2 = 25$

$$(\lambda_2 \mathbf{I} - \mathbf{V}) = \begin{pmatrix} 9 & 12 \\ 12 & 1 \end{pmatrix} \xrightarrow[R_2 \leftarrow R_2 - 4R_1]{R_1 \leftarrow \frac{1}{3}R_1} \begin{pmatrix} 3 & 4 \\ 0 & 0 \end{pmatrix} \quad (5.2.7.12)$$

$$\Rightarrow \mathbf{p}_2 = \frac{1}{5} \begin{pmatrix} -4 \\ 3 \end{pmatrix} \quad (5.2.7.13)$$

Eigen Value Decomposition: Using EVD, we can write

$$\mathbf{D} = \mathbf{P} \mathbf{V} \mathbf{P}^T \quad (5.2.7.14)$$

From (5.2.7.11) and (5.2.7.13)

$$\mathbf{P} = \frac{1}{5} \begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix} \quad (5.2.7.15)$$

From (5.2.7.7)

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 25 \end{pmatrix} \quad (5.2.7.16)$$

Parabola

$$\text{Focal Length} = \left| \frac{2\eta}{\lambda_2} \right| \quad (5.2.7.17)$$

From (5.2.7.11) and (5.2.7.3)

$$\eta = \mathbf{p}_1^T \mathbf{u} = 44 \quad (5.2.7.18)$$

Substituting values of (5.2.7.18) and (5.2.7.7) in (5.2.7.17), we get

$$\text{Focal Length} = \left| \frac{88}{25} \right| = 3.52 \quad (5.2.7.19)$$

The standard equation of parabola is given by:

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = -2\eta \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{y} \quad (5.2.7.20)$$

And the vertex  $\mathbf{c}$  is:

$$\begin{pmatrix} \mathbf{u}^T + \eta \mathbf{p}_1^T \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \eta \mathbf{p}_1 - \mathbf{u} \end{pmatrix} \quad (5.2.7.21)$$

From (5.2.7.3) (5.2.7.18) and (5.2.7.11),

$$\begin{pmatrix} \frac{212}{5} & \frac{391}{5} \\ 16 & -12 \\ -12 & 9 \end{pmatrix} \mathbf{c} = \begin{pmatrix} 39 \\ \frac{52}{5} \\ \frac{-39}{5} \end{pmatrix} \quad (5.2.7.22)$$

To find  $\mathbf{c}$ , perform row reduction on the aug-

mented matrix as follows:

$$\begin{pmatrix} \frac{212}{5} & \frac{391}{5} & 39 \\ 16 & -12 & \frac{52}{5} \\ -12 & 9 & -\frac{39}{5} \end{pmatrix} \xrightarrow[R_1 \leftarrow \frac{5}{212}R_1]{R_3 \leftarrow R_3 + \frac{3}{4}R_2} \begin{pmatrix} 1 & \frac{391}{212} & \frac{195}{212} \\ 16 & -12 & \frac{52}{5} \\ 0 & 0 & 0 \end{pmatrix} \quad (5.2.7.23)$$

$$\xrightarrow{R_2 \leftarrow R_2 - 16R_1} \begin{pmatrix} 1 & \frac{391}{212} & \frac{195}{212} \\ 0 & -\frac{2200}{53} & -\frac{1144}{265} \\ 0 & 0 & 0 \end{pmatrix} \quad (5.2.7.24)$$

$$\xrightarrow{R_2 \leftarrow \frac{-53}{2200}R_2} \begin{pmatrix} 1 & \frac{391}{212} & \frac{195}{212} \\ 0 & 1 & \frac{13}{125} \\ 0 & 0 & 0 \end{pmatrix} \quad (5.2.7.25)$$

$$\xrightarrow{R_1 \leftarrow R_1 - \frac{391}{212}R_2} \begin{pmatrix} 1 & 0 & \frac{4823}{6625} \\ 0 & 1 & \frac{13}{125} \\ 0 & 0 & 0 \end{pmatrix} \quad (5.2.7.26)$$

Hence,

$$\mathbf{c} = \begin{pmatrix} \frac{4823}{6625} \\ \frac{13}{125} \end{pmatrix} = \begin{pmatrix} 0.728 \\ 0.104 \end{pmatrix} \quad (5.2.7.27)$$

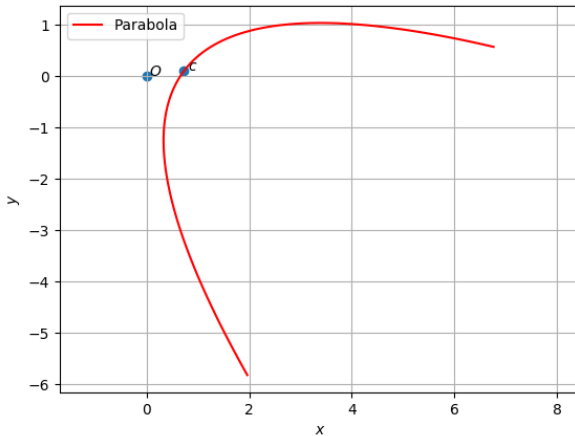


Fig. 5.2.7.1: Parabola with vertex c

5.2.8. Trace the following parabola

$$4x^2 - 4xy + y^2 - 12x + 6y + 9 = 0 \quad (5.2.8.1)$$

**Solution:** The given quadratic equation can be written in the matrix form as

$$\mathbf{x}^T \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} -6 & 3 \end{pmatrix} \mathbf{x} + 9 = 0 \quad (5.2.8.2)$$

Calculating the parameters, we get

$$|\mathbf{V}| = \begin{vmatrix} 4 & -2 \\ -2 & 1 \end{vmatrix} = 0 \quad (5.2.8.3)$$

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = \begin{vmatrix} 4 & -2 & -6 \\ -2 & 1 & 3 \\ -6 & 3 & 9 \end{vmatrix} = 0 \quad (5.2.8.4)$$

Therefore the given parabola equation is a degenerate. The quadratic equation corresponds to a pair of coincident straight lines.

The characteristic equation of  $\mathbf{V}$  will be

$$|\mathbf{V} - \lambda \mathbf{I}| = \begin{vmatrix} 4 - \lambda & -2 \\ -2 & 1 - \lambda \end{vmatrix} \quad (5.2.8.5)$$

$$= \lambda^2 - 5\lambda \quad (5.2.8.6)$$

$$\lambda_1 = 0, \lambda_2 = 5 \quad (5.2.8.7)$$

The eigen vectors are the nullspace of the matrix  $\mathbf{V} - \lambda \mathbf{I}$ . For  $\lambda_1 = 0$

$$\begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} \xrightarrow{R_2 = 2R_2 + R_1} \begin{pmatrix} 4 & -2 \\ 0 & 0 \end{pmatrix} \quad (5.2.8.8)$$

$$p_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (5.2.8.9)$$

Therefore the normalized eigen vector will be

$$p_1 = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix} \quad (5.2.8.10)$$

For  $\lambda_2 = 5$

$$\begin{pmatrix} -1 & -2 \\ -2 & -4 \end{pmatrix} \xrightarrow{R_2 = R_2 - 2R_1} \begin{pmatrix} -1 & -2 \\ 0 & 0 \end{pmatrix} \quad (5.2.8.11)$$

$$p_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad (5.2.8.12)$$

Therefore the normalized eigen vector will be

$$p_2 = \begin{pmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} \quad (5.2.8.13)$$

Therefore the transformation matrix will be

$$\mathbf{P} = (p_1 \ p_2) = \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \quad (5.2.8.14)$$

The value of  $\eta$  will be

$$\eta = 2p_1^T \mathbf{u} \quad (5.2.8.15)$$

$$= 2 \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} -6 \\ 3 \end{pmatrix} \quad (5.2.8.16)$$

$$= 0 \quad (5.2.8.17)$$

A point on the line can be found by using to following formula

$$\begin{pmatrix} \mathbf{u}^T + \frac{\eta}{2} p_1^T \\ \mathbf{V} \end{pmatrix} c = \begin{pmatrix} -f \\ \frac{\eta}{2} p_1 - \mathbf{u} \end{pmatrix} \quad (5.2.8.18)$$

$$\begin{pmatrix} \mathbf{u}^T \\ \mathbf{V} \end{pmatrix} c = \begin{pmatrix} -f \\ -\mathbf{u} \end{pmatrix} \quad (5.2.8.19)$$

$$\begin{pmatrix} -6 & 3 \\ 4 & -2 \\ -2 & 1 \end{pmatrix} c = \begin{pmatrix} -9 \\ 6 \\ -3 \end{pmatrix} \quad (5.2.8.20)$$

Writing it in augmented form, we get

$$\begin{pmatrix} -6 & 3 & -9 \\ 4 & -2 & 6 \\ -2 & 1 & -3 \end{pmatrix} \xrightarrow{R_3=R_3-\frac{R_1}{3}} \begin{pmatrix} -6 & 3 & -9 \\ 4 & -2 & 6 \\ 0 & 0 & 0 \end{pmatrix} \quad (5.2.8.21)$$

$$\xrightarrow{R_2=\frac{3}{2}R_2+R_1} \begin{pmatrix} -6 & 3 & -9 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (5.2.8.22)$$

Therefore we can see that the point  $c = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$  lies on the line. Equation of the straight line Applying affine transformation we get

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = -\eta \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{y} \quad (5.2.8.23)$$

$$\mathbf{y}^T \begin{pmatrix} 0 & 0 \\ 0 & 5 \end{pmatrix} \mathbf{y} = 0 \quad (5.2.8.24)$$

$$5y^2 = 0 \quad (5.2.8.25)$$

Therefore the transformed line is  $y = 0$ , which in vector form will be  $\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{y} = 0$ .

Taking the Inverse affine transformation we get

$$\begin{pmatrix} 0 & 1 \end{pmatrix} (P^T (\mathbf{x} - c)) = 0 \quad (5.2.8.26)$$

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} (\mathbf{x} - c) = 0 \quad (5.2.8.27)$$

$$\begin{pmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} (\mathbf{x} - c) = 0 \quad (5.2.8.28)$$

$$\begin{pmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \mathbf{x} - \begin{pmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0 \quad (5.2.8.29)$$

$$\begin{pmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \mathbf{x} + \frac{3}{\sqrt{5}} = 0 \quad (5.2.8.30)$$

$$\begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} = 3 \quad (5.2.8.31)$$

Therefore the equation of coincident lines is  $(2x - y - 3) = 0$ .

5.2.9. Trace the central conic,

$$2x^2 - 2xy + y^2 + 2x - 2y = 0 \quad (5.2.9.1)$$

**Solution:** The general equation of a second degree (In algebraic form) can be expressed as,

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \quad (5.2.9.2)$$

The general equation of a second degree (In vector form) can be expressed as,

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (5.2.9.3)$$

Comparing (5.2.9.1) with (5.2.9.2), we get,

$$a = 2, b = -1, c = 1, d = 1, e = -1 \text{ and } f = 0 \quad (5.2.9.4)$$

where,

$$\mathbf{V} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} = \mathbf{V}^T \quad (5.2.9.5)$$

$$\Rightarrow \mathbf{V} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \quad (5.2.9.6)$$

and

$$\mathbf{u} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (5.2.9.7)$$

Finding the determinant of  $\mathbf{V}$  we obtain,

$$|\mathbf{V}| = 1 > 0 \quad (5.2.9.8)$$

which means the given central conic is an ellipse which can be proven more effectively using,

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T \quad (5.2.9.9)$$

where  $\mathbf{P}$  is a matrix of Eigen vectors and  $\mathbf{D}$  is a diagonal matrix of Eigen values which will be computed subsequently.

Computing Eigen values for  $\mathbf{V}$  using the characteristic equation of the matrix, we get the following quadratic equation in terms of  $\lambda$

$$\lambda^2 - 3\lambda + 1 = 0 \quad (5.2.9.10)$$

$$\Rightarrow \lambda_1 = \frac{3 + \sqrt{5}}{2} \text{ and } \lambda_2 = \frac{3 - \sqrt{5}}{2} \quad (5.2.9.11)$$

Eigen vectors can be computed using the following equation,

$$(\lambda\mathbf{I} - \mathbf{V})\mathbf{p} = 0 \quad (5.2.9.12)$$

Solving this for  $\lambda_1$  and  $\lambda_2$  respectively and normalizing them we obtain,

$$\mathbf{p}_1 = \sqrt{\frac{2}{5 - \sqrt{5}}} \begin{pmatrix} 1 \\ \frac{1 - \sqrt{5}}{2} \end{pmatrix} \quad (5.2.9.13)$$

$$\mathbf{p}_2 = \sqrt{\frac{2}{5 + \sqrt{5}}} \begin{pmatrix} 1 \\ \frac{\sqrt{5} + 1}{2} \end{pmatrix} \quad (5.2.9.14)$$

Simplifying,

$$\Rightarrow \mathbf{P} = \begin{pmatrix} \sqrt{\frac{2}{5 - \sqrt{5}}} & \sqrt{\frac{2}{5 + \sqrt{5}}} \\ \frac{1 - \sqrt{5}}{\sqrt{5}\sqrt{2} - \sqrt{10}} & \frac{1 + \sqrt{5}}{\sqrt{5}\sqrt{2} + \sqrt{10}} \end{pmatrix} \quad (5.2.9.15)$$

$$\mathbf{D} = \begin{pmatrix} \frac{3 + \sqrt{5}}{2} & 0 \\ 0 & \frac{3 - \sqrt{5}}{2} \end{pmatrix} \quad (5.2.9.16)$$

Using (5.2.9.9) can verify that it holds which means that the given central conic is an ellipse.

The center of the ellipse can be computed using,

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \quad (5.2.9.17)$$

$$\Rightarrow \mathbf{c} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (5.2.9.18)$$

The parameters of the ellipse are computed as follows,

$$\sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}} = \sqrt{\frac{3 - \sqrt{5}}{2}} \quad (5.2.9.19)$$

$$\sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_2}} = \sqrt{\frac{3 + \sqrt{5}}{2}} \quad (5.2.9.20)$$

The angle of Rotation can be obtained by equating  $\mathbf{P}$  with the Rotation matrix which is,

$$\mathbf{P} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \quad (5.2.9.21)$$

Comparing (5.2.9.15) and (5.2.9.21) we get,

$$\theta = \frac{\pi}{5.66} \quad (5.2.9.22)$$

Using the Affine transformation we find out the actual ellipse,

$$\mathbf{y} = \mathbf{P}^T \mathbf{x} + \mathbf{c} \quad (5.2.9.23)$$

which means the actual ellipse is obtained by translating and rotating the standard ellipse w.r.t center,  $\mathbf{c}$  from (5.2.9.18) and angle of rotation,  $\theta$  from (5.2.9.22) respectively.

Using the above data along with  $\mathbf{o}$  (Origin), the center of the standard ellipse, the actual ellipse is plotted as follows.

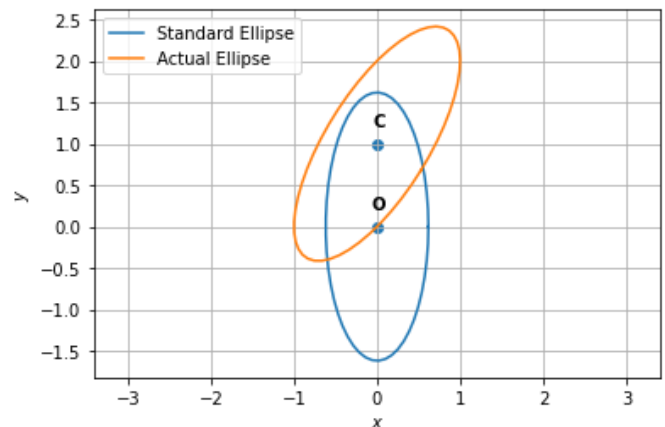


Fig. 5.2.9.1: Standard and Actual Ellipses

5.2.10. Trace the following central conic :

$$x^2 + y^2 + xy + x + y = 1 \quad (5.2.10.1)$$

**Solution:** General equation of second degree is given by :

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (5.2.10.2)$$

In the vector form (5.2.10.1) can be written as :

$$\mathbf{x}^T \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}^T \mathbf{x} - 1 = 0 \quad (5.2.10.3)$$

By comparing (5.2.10.2) and (5.2.10.3) we get :

$$\mathbf{V} = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, f = -1 \quad (5.2.10.4)$$

Eigen values for matrix  $\mathbf{V}$  can be calculated by solving :

$$\begin{vmatrix} 1 - \lambda & \frac{1}{2} \\ \frac{1}{2} & 1 - \lambda \end{vmatrix} = 0 \quad (5.2.10.5)$$

$$\lambda^2 - 2\lambda + \frac{3}{4} = 0 \quad (5.2.10.6)$$

$$\lambda_1 = \frac{3}{2}, \lambda_2 = \frac{1}{2} \quad (5.2.10.7)$$

By doing Eigenvalue Decomposition and Affine Transformation we get :

$$\mathbf{P}^{-1} \mathbf{V} \mathbf{P} = \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad (5.2.10.8)$$

$$\mathbf{x} = \mathbf{P} \mathbf{y} + \mathbf{c} \quad (5.2.10.9)$$

Where the matrix  $\mathbf{P}$  is normalised eigenbasis and  $\mathbf{c}$  is the center.

By putting the value of  $\mathbf{x}$  from (5.2.10.9) in (5.2.10.2) we get :

$$(\mathbf{P} \mathbf{y} + \mathbf{c})^T \mathbf{V} (\mathbf{P} \mathbf{y} + \mathbf{c}) + 2\mathbf{u}^T (\mathbf{P} \mathbf{y} + \mathbf{c}) + f = 0 \quad (5.2.10.10)$$

Further solving this we get :

$$\mathbf{V} \mathbf{c} + \mathbf{u} = 0 \implies \mathbf{c} = -\mathbf{V}^{-1} \mathbf{u} \quad (5.2.10.11)$$

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f \quad (5.2.10.12)$$

As

$$|\mathbf{V}| = \begin{vmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{vmatrix} = \frac{3}{4} > 0 \quad (5.2.10.13)$$

Equation (5.2.10.12) forms an ellipse centered at origin with major and minor axis given as :

$$a = \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}} \quad (5.2.10.14)$$

$$b = \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_2}} \quad (5.2.10.15)$$

Using Gauss Jordan Elimination on matrix  $\mathbf{V}$  :

$$\xleftrightarrow{R_2 \leftarrow \frac{1}{2} R_1 - R_2} \begin{pmatrix} 1 & \frac{1}{2} & : & 1 & 0 \\ 0 & \frac{3}{4} & : & \frac{1}{2} & -1 \end{pmatrix} \quad (5.2.10.16)$$

$$\xleftrightarrow{R_2 \leftarrow \frac{4}{3} R_2} \begin{pmatrix} 1 & \frac{1}{2} & : & 1 & 0 \\ 0 & 1 & : & \frac{2}{3} & \frac{4}{3} \end{pmatrix} \quad (5.2.10.17)$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - \frac{1}{2} R_2} \begin{pmatrix} 1 & 0 & : & \frac{4}{3} & \frac{2}{3} \\ 0 & 1 & : & \frac{2}{3} & \frac{4}{3} \end{pmatrix} \quad (5.2.10.18)$$

Therefore,

$$\mathbf{V}^{-1} = \begin{pmatrix} \frac{4}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{4}{3} \end{pmatrix} \quad (5.2.10.19)$$

Using (5.2.10.11) and (5.2.10.19) we get :

$$\mathbf{c} = -\mathbf{V}^{-1} \mathbf{u} = -\begin{pmatrix} \frac{4}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{4}{3} \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{-3}{10} \\ \frac{-3}{10} \end{pmatrix} \quad (5.2.10.20)$$

By putting the values of  $\mathbf{u}$ ,  $\mathbf{V}^{-1}$ ,  $f$ ,  $\lambda_1$  and  $\lambda_2$  in (5.2.10.14) and (5.2.10.15) respectively we get :

$$a = \sqrt{\frac{\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{4}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{4}{3} \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} - 1}{\frac{3}{2}}} = \frac{9}{10} \quad (5.2.10.21)$$

$$b = \sqrt{\frac{\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{4}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{4}{3} \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} - 1}{\frac{1}{2}}} = \frac{8}{5} \quad (5.2.10.22)$$

In the transformed space with Eigenbasis, an ellipse centered at origin with major and minor axis as  $a$  and  $b$  is traced as 'Standard Ellipse' in the plot.

And after doing Affine Transformation on  $\mathbf{y}$  as in (5.2.10.9) we get our 'Actual Ellipse'



centered at  $\mathbf{c}$  shown in the plot.

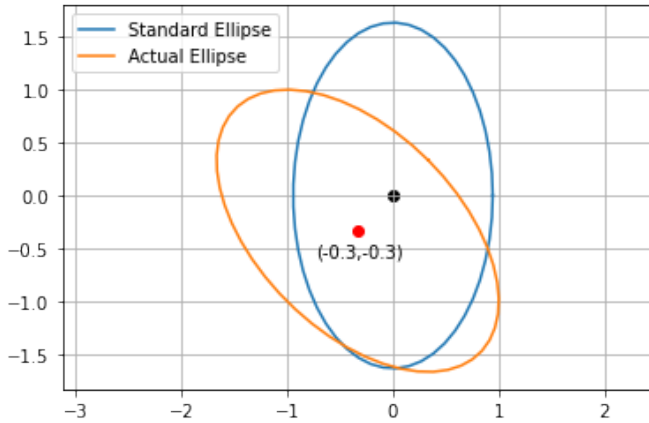


Fig. 5.2.10.1: Standard Ellipse centered at origin and Actual Ellipse centered at  $(-0.3, -0.3)$ .

5.2.11. Trace the following central conic:

$$2x^2 + 3xy - 2y^2 - 7x + y - 2 = 0 \quad (5.2.11.1)$$

**Solution:** Any second degree equation of the form:

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \quad (5.2.11.2)$$

Can be represented in matrix / vector form as:

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (5.2.11.3)$$

where,

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad (5.2.11.4)$$

$$\mathbf{u} = \begin{pmatrix} d & e \end{pmatrix} \quad (5.2.11.5)$$

Rewriting (5.2.11.1) in matrix form, we get:

$$\mathbf{x}^T \begin{pmatrix} 2 & \frac{3}{2} \\ \frac{3}{2} & -2 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} -\frac{7}{2} & \frac{1}{2} \end{pmatrix} \mathbf{x} - 2 = 0 \quad (5.2.11.6)$$

where,

$$\mathbf{V} = \begin{pmatrix} 2 & \frac{3}{2} \\ \frac{3}{2} & -2 \end{pmatrix} \quad (5.2.11.7)$$

$$\mathbf{u} = \begin{pmatrix} -\frac{7}{2} \\ \frac{1}{2} \end{pmatrix} \quad (5.2.11.8)$$

$$f = -2 \quad (5.2.11.9)$$

$$\det(\mathbf{V}) = \begin{vmatrix} 2 & \frac{3}{2} \\ \frac{3}{2} & -2 \end{vmatrix} = -\frac{25}{4} \quad (5.2.11.10)$$

As  $\det(\mathbf{V}) < 0$ , the given conic represents a hyperbola.

The characteristic equation of  $\mathbf{V}$  is given by the determinant:

$$|\mathbf{V} - \lambda \mathbf{I}| = 0 \quad (5.2.11.11)$$

$$\begin{vmatrix} 2 - \lambda & \frac{3}{2} \\ \frac{3}{2} & -2 - \lambda \end{vmatrix} = 0 \quad (5.2.11.12)$$

$$\Rightarrow \lambda^2 - \frac{25}{4} = 0 \quad (5.2.11.13)$$

The roots of (5.2.11.13) (the eigenvalues) are:

$$\lambda_1 = \frac{5}{2}, \lambda_2 = -\frac{5}{2} \quad (5.2.11.14)$$

The eigenvector  $\mathbf{p}$  is defined as:

$$\mathbf{V} \mathbf{p} = \lambda \mathbf{p} \quad (5.2.11.15)$$

$$\Rightarrow (\mathbf{V} - \lambda \mathbf{I}) \mathbf{p} = 0 \quad (5.2.11.16)$$

Evaluating (5.2.11.16) for  $\lambda_1 = \frac{5}{2}$ , we get:

$$(\mathbf{V} - \lambda_1 \mathbf{I}) = \begin{pmatrix} -\frac{1}{2} & \frac{3}{2} \\ \frac{3}{2} & -\frac{9}{2} \end{pmatrix} \quad (5.2.11.17)$$

Reducing the above equation to row-echelon form, we get:

$$\xleftrightarrow{R_2 \rightarrow R_2 + 3R_1} \begin{pmatrix} -\frac{1}{2} & \frac{3}{2} \\ 0 & 0 \end{pmatrix} \xleftrightarrow{R_1 \rightarrow -2R_1} \begin{pmatrix} 1 & -3 \\ 0 & 0 \end{pmatrix} \quad (5.2.11.18)$$

Substituting (5.2.11.18) in (5.2.11.16), we get:

$$\begin{pmatrix} 1 & -3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (5.2.11.19)$$

where,

$$\mathbf{p} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad (5.2.11.20)$$

Let  $v_2 = t$ . Then

$$v_1 = 3t \quad (5.2.11.21)$$

Let  $t = 1$ . The eigenvector  $\mathbf{p}_1$  is:

$$\mathbf{p}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad (5.2.11.22)$$

Similarly for  $\lambda_2 = -\frac{5}{2}$ , we get:

$$(\mathbf{V} - \lambda_2 \mathbf{I}) = \begin{pmatrix} \frac{9}{2} & \frac{3}{2} \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix} \xrightarrow[R_1 \rightarrow \frac{2}{9}R_1]{R_2 \rightarrow 3R_2 - R_1} \begin{pmatrix} 1 & \frac{1}{3} \\ 0 & 0 \end{pmatrix} \quad (5.2.11.23)$$

Substituting (5.2.11.23) in (5.2.11.16), we get:

$$\begin{pmatrix} 1 & \frac{1}{3} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (5.2.11.24)$$

where,

$$\mathbf{p} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad (5.2.11.25)$$

Let  $v_2 = t$ . Then

$$v_1 = \frac{-t}{3} \quad (5.2.11.26)$$

Let  $t = 1$ . The eigenvector  $\mathbf{p}_2$  is:

$$\mathbf{p}_2 = \begin{pmatrix} -\frac{1}{3} \\ 1 \end{pmatrix} \quad (5.2.11.27)$$

As  $\mathbf{V} = \mathbf{V}^T$ , there exists an orthogonal matrix  $\mathbf{P}$  such that:

$$\mathbf{PVP}^T = \mathbf{D} = \text{diag}(\lambda_1, \lambda_2) \quad (5.2.11.28)$$

$\mathbf{V}$  can be rewritten using the above equation as:

$$\mathbf{V} = \mathbf{PDP}^T \quad (5.2.11.29)$$

where,

$$\mathbf{P} = (\mathbf{p}_1 \quad \mathbf{p}_2) \quad (5.2.11.30)$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad (5.2.11.31)$$

Substituting the values -

$$\mathbf{P} = \begin{pmatrix} 3 & -\frac{1}{3} \\ 1 & 1 \end{pmatrix} \quad (5.2.11.32)$$

$$\mathbf{D} = \begin{pmatrix} \frac{5}{2} & 0 \\ 0 & -\frac{5}{2} \end{pmatrix} \quad (5.2.11.33)$$

The center of hyperbola is given by:

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \quad (5.2.11.34)$$

$$\Rightarrow \mathbf{c} = -\begin{pmatrix} \frac{8}{25} & \frac{6}{25} \\ \frac{6}{25} & \frac{-8}{25} \end{pmatrix} \begin{pmatrix} -\frac{7}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (5.2.11.35)$$

As

$$\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f = 5 > 0 \quad (5.2.11.36)$$

there is no requirement for swapping the axes (which will be evident from the equation below). The axes of the hyperbola are given by:

$$\text{axes} = \left\{ \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}}, \sqrt{\frac{f - \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u}}{\lambda_2}} \right\} \quad (5.2.11.37)$$

$$\Rightarrow \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}} = \sqrt{2} \quad (5.2.11.38)$$

$$\Rightarrow \sqrt{\frac{f - \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u}}{\lambda_2}} = \sqrt{2} \quad (5.2.11.39)$$

The standard form of conic is written as:

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f \quad (5.2.11.40)$$

where,

$$\mathbf{y} = \mathbf{P}^T (\mathbf{x} - \mathbf{c}) \quad (5.2.11.41)$$

$$\Rightarrow \mathbf{y}^T \begin{pmatrix} \frac{5}{2} & 0 \\ 0 & -\frac{5}{2} \end{pmatrix} \mathbf{y} - 5 = 0 \quad (5.2.11.42)$$

The plot of both the conics are given below:

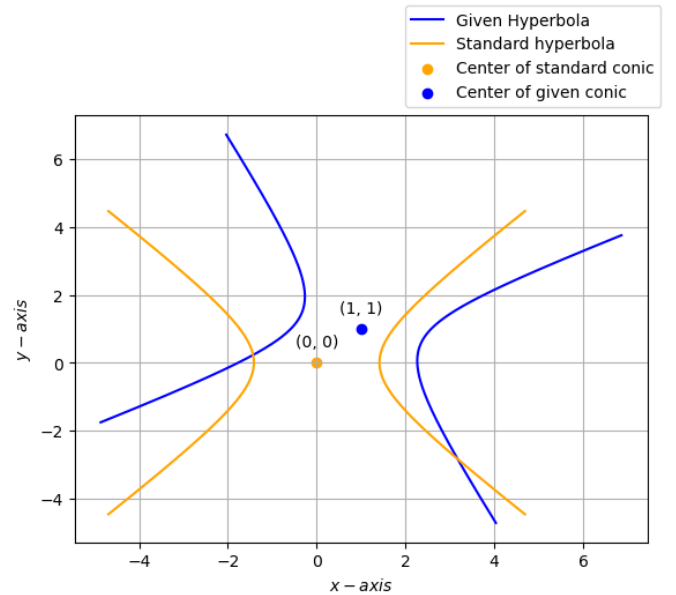


Fig. 5.2.11.1: Plot of given hyperbola and the standard hyperbola

5.2.12. Trace the following central conics:

$$40x^2 + 36xy + 25y^2 - 196x - 122y + 205 = 0 \quad (5.2.12.1)$$

**Solution:** The general equation of a second degree can be expressed as:

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \quad (5.2.12.2)$$

$$\Rightarrow \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (5.2.12.3)$$

where

$$\mathbf{V} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \mathbf{u} = \begin{pmatrix} d \\ e \end{pmatrix} \quad (5.2.12.4)$$

The given equation of the curve can be expressed as:

$$40x^2 + 2(18)xy + 25y^2 + 2(-98)x + 2(-61)y + 205 = 0 \quad (5.2.12.5)$$

Comparing (5.2.12.2), (5.2.12.4) and (5.2.12.5):

$$\mathbf{V} = \begin{pmatrix} 40 & \sqrt{18} \\ \sqrt{18} & 25 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} -98 \\ -61 \end{pmatrix} \text{ and } f = 205 \quad (5.2.12.6)$$

$$\Rightarrow |\mathbf{V}| = 982 \quad \text{and} \quad b^2 - ac = 18 - 40 \cdot 25 = -982 \quad (5.2.12.7)$$

Since  $|\mathbf{V}| > 0$  and  $b^2 < ac$ , (5.2.12.5) represent an ellipse.

The characteristic equation of  $\mathbf{V}$  is given as follows,

$$|\lambda \mathbf{I} - \mathbf{V}| = \begin{vmatrix} \lambda - 40 & \sqrt{18} \\ \sqrt{18} & \lambda - 25 \end{vmatrix} = 0 \quad (5.2.12.8)$$

$$\Rightarrow \lambda^2 - 65\lambda + 982 = 0 \quad (5.2.12.9)$$

Hence the characteristic equation of  $\mathbf{V}$  is given by (5.2.12.9). The roots of (5.2.12.9) i.e the eigenvalues are given by

$$\lambda_1 = \frac{65 + \sqrt{297}}{2}, \lambda_2 = \frac{65 - \sqrt{297}}{2} \quad (5.2.12.10)$$

The eigen vector  $\mathbf{p}$  is defined as,

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \quad (5.2.12.11)$$

$$\Rightarrow (\lambda \mathbf{I} - \mathbf{V})\mathbf{p} = 0 \quad (5.2.12.12)$$

$$\text{for } \lambda_1 = \frac{65 + \sqrt{297}}{2},$$

$$(\lambda_1 \mathbf{I} - \mathbf{V}) = \begin{pmatrix} \frac{\sqrt{297}-15}{2} & -\sqrt{18} \\ -\sqrt{18} & \frac{\sqrt{297}+15}{2} \end{pmatrix} \quad (5.2.12.13)$$

$$\xleftrightarrow{R_2=R_2+\frac{2\sqrt{18}}{\sqrt{297}-15}R_1} \begin{pmatrix} \frac{\sqrt{297}-15}{2} & -\sqrt{18} \\ 0 & 0 \end{pmatrix} \quad (5.2.12.14)$$

From (5.2.12.12) and (5.2.12.14)

$$\Rightarrow \mathbf{p}_1 = \begin{pmatrix} \sqrt{18} \\ \frac{\sqrt{297}-15}{2} \end{pmatrix} \quad (5.2.12.15)$$

$$\text{For } \lambda_2 = \frac{65 - \sqrt{297}}{2}$$

$$(\lambda_2 \mathbf{I} - \mathbf{V}) = \begin{pmatrix} \frac{-\sqrt{297}-15}{2} & -\sqrt{18} \\ -\sqrt{18} & \frac{15-\sqrt{297}}{2} \end{pmatrix} \quad (5.2.12.16)$$

$$\xleftrightarrow{R_2=R_2+\frac{2\sqrt{18}}{\sqrt{297}+15}R_1} \begin{pmatrix} \frac{\sqrt{297}+15}{2} & \sqrt{18} \\ 0 & 0 \end{pmatrix} \quad (5.2.12.17)$$

$$\Rightarrow \mathbf{p}_2 = \begin{pmatrix} -\sqrt{18} \\ \frac{\sqrt{297}+15}{2} \end{pmatrix} \quad (5.2.12.18)$$

using the affine transformation

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c}' \quad (5.2.12.19)$$

such that

$$\mathbf{P}^T \mathbf{V} \mathbf{P} = \mathbf{D} \quad \text{and} \quad \mathbf{P} = (\mathbf{p}_1 \quad \mathbf{p}_2), \quad \mathbf{P}^T = \mathbf{P}^{-1} \quad (5.2.12.20)$$

Where  $\mathbf{D}$  is a diagonal matrix, we get

$$\mathbf{D} = \begin{pmatrix} \frac{65 + \sqrt{297}}{2} & 0 \\ 0 & \frac{65 - \sqrt{297}}{2} \end{pmatrix} \quad (5.2.12.21)$$

Now (5.2.12.3) can be written as,

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f \quad |\mathbf{V}| \neq 0 \quad (5.2.12.22)$$

And,

$$\mathbf{c}' = -\mathbf{V}^{-1} \mathbf{u} \quad |\mathbf{V}| \neq 0 \quad (5.2.12.23)$$

$$\mathbf{y} = \mathbf{P}^T (\mathbf{x} - \mathbf{c}) \quad (5.2.12.24)$$

The centre of the conic section in (5.2.12.5) is given by  $\mathbf{c}'$  in (5.2.12.23). We compute  $\mathbf{V}^{-1}$  as

follows,

$$\begin{pmatrix} 40 & \sqrt{18} & 1 & 0 \\ \sqrt{18} & 25 & 0 & 1 \end{pmatrix} \xrightarrow[R_1 = \frac{1}{40}R_1]{R_2 = R_2 - \frac{\sqrt{18}}{40}R_1} \begin{pmatrix} 1 & \frac{\sqrt{18}}{40} & \frac{1}{40} & 0 \\ 0 & \frac{982}{40} & -\frac{\sqrt{18}}{40} & 1 \end{pmatrix} \quad (5.2.12.25)$$

$$\xrightarrow[R_1 = R_1 - \frac{\sqrt{18}}{40}R_2]{R_2 = \frac{40}{982}R_2} \begin{pmatrix} 1 & 0 & \frac{25}{982} & -\frac{\sqrt{18}}{982} \\ 0 & 1 & -\frac{\sqrt{18}}{982} & \frac{40}{982} \end{pmatrix} \quad (5.2.12.26)$$

Hence  $\mathbf{V}^{-1}$  is given by,

$$\mathbf{V}^{-1} = \begin{pmatrix} \frac{25}{982} & -\frac{\sqrt{18}}{982} \\ -\frac{\sqrt{18}}{982} & \frac{40}{982} \end{pmatrix} \quad (5.2.12.27)$$

Now  $\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u}$  is given by,

$$\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} = \frac{1}{982} \begin{pmatrix} -98 & -61 \end{pmatrix} \begin{pmatrix} 25 & -\sqrt{18} \\ -\sqrt{18} & 40 \end{pmatrix} \begin{pmatrix} -98 \\ -61 \end{pmatrix} \quad (5.2.12.28)$$

$$= 344.4203 \quad (5.2.12.29)$$

And,  $\mathbf{V}^{-1} \mathbf{u}$  is given by,

$$\mathbf{V}^{-1} \mathbf{u} = \frac{1}{982} \begin{pmatrix} 25 & -\sqrt{18} \\ -\sqrt{18} & 40 \end{pmatrix} \begin{pmatrix} -98 \\ -61 \end{pmatrix} \quad (5.2.12.30)$$

$$(5.2.12.31)$$

By putting the value of (5.2.12.30), the center of the ellipse is given by (5.2.12.23) as follows,

$$\mathbf{c}' = \begin{pmatrix} 2.231 \\ 2.061 \end{pmatrix} \quad (5.2.12.32)$$

Also the semi-major axis ( $a$ ) and semi-minor axis ( $b$ ) of the ellipse are given by,

$$a = \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}} = 1.8414 \quad (5.2.12.33)$$

$$b = \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_2}} = 2.416 \quad (5.2.12.34)$$

Finally from (5.2.12.22), the equation of ellipse is given by,

$$\mathbf{y}^T \begin{pmatrix} 41.116 & 0 \\ 0 & 23.883 \end{pmatrix} \mathbf{y} = 139.4203 \quad (5.2.12.35)$$

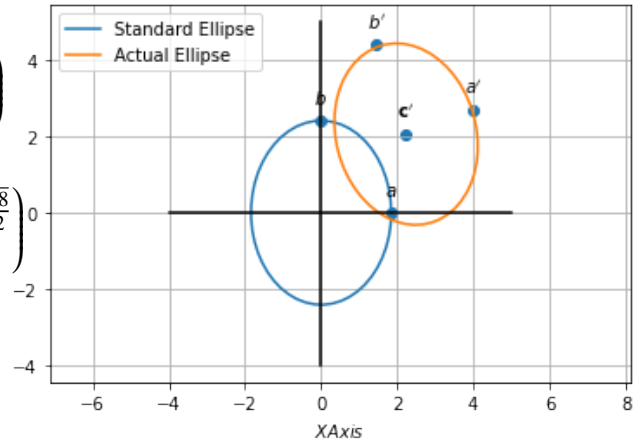


Fig. 5.2.12.1: Graphical representation of the actual curve  $40x^2 + 36xy + 25y^2 - 196x - 122y + 205 = 0$ , which represent an ellipse.

5.2.13. Trace the curve

$$35x^2 + 30y^2 + 32x - 108y - 12xy + 59 = 0 \quad (5.2.13.1)$$

**Solution:** The general equation of second degree is given by

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \quad (5.2.13.2)$$

and can be expressed as

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (5.2.13.3)$$

where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad (5.2.13.4)$$

$$\mathbf{u}^T = \begin{pmatrix} d & e \end{pmatrix} \quad (5.2.13.5)$$

Comparing (5.2.13.1) with (5.2.13.2), we get

$$\mathbf{V} = \begin{pmatrix} 35 & -6 \\ -6 & 30 \end{pmatrix} \quad (5.2.13.6)$$

$$\mathbf{u}^T = \begin{pmatrix} 16 & -54 \end{pmatrix} \quad (5.2.13.7)$$

If  $|\mathbf{V}| > 0$ , then (5.2.13.3) is an ellipse.

$$|\mathbf{V}| = \begin{vmatrix} 35 & -6 \\ -6 & 30 \end{vmatrix} = 1014 > 0 \quad (5.2.13.8)$$

(5.2.13.3) can be expressed as

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f \quad |\mathbf{V}| \neq 0 \quad (5.2.13.9)$$

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = -\eta \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{y} \quad |\mathbf{V}| = 0 \quad (5.2.13.10)$$

with center as

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \quad |\mathbf{V}| \neq 0 \quad (5.2.13.11)$$

Calculating the center for given curve we get,

$$\mathbf{c} = -\frac{1}{|35 \times 30 - 6 \times 6|} \begin{pmatrix} 30 & 6 \\ 6 & 35 \end{pmatrix} \begin{pmatrix} 16 \\ -54 \end{pmatrix} \quad (5.2.13.12)$$

$$= \frac{1}{1014} \begin{pmatrix} 156 \\ -1794 \end{pmatrix} \quad (5.2.13.13)$$

$$= \begin{pmatrix} \frac{2}{13} \\ -\frac{23}{13} \end{pmatrix} \quad (5.2.13.14)$$

For

$$|\mathbf{V}| > 0, \quad \text{or, } \lambda_1 > 0, \lambda_2 > 0 \quad (5.2.13.15)$$

(5.2.13.9) becomes

$$\lambda_1 y_1^2 + \lambda_2 y_2^2 = \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f \quad (5.2.13.16)$$

which is the equation of an ellipse with major and minor axes parameters

$$\sqrt{\frac{\lambda_1}{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}}, \sqrt{\frac{\lambda_2}{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}} \quad (5.2.13.17)$$

The characteristic equation of  $\mathbf{V}$  is obtained by evaluating the determinant

$$|\lambda \mathbf{I} - \mathbf{V}| = \begin{vmatrix} \lambda - 35 & 6 \\ 6 & \lambda - 30 \end{vmatrix} = 0 \quad (5.2.13.18)$$

$$\Rightarrow \lambda^2 - 65\lambda + 1014 = 0 \quad (5.2.13.19)$$

The eigenvalues are the roots of (5.2.13.19) given by

$$\lambda_1 = 39, \lambda_2 = 26 \quad (5.2.13.20)$$

Calculating the major and minor axes lengths

using (5.2.13.17), we get

$$\begin{aligned} \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} &= \\ &= (16 - 54) \frac{1}{1014} \begin{pmatrix} 30 & 6 \\ 6 & 35 \end{pmatrix} \begin{pmatrix} 16 \\ -54 \end{pmatrix} \\ &= \frac{1}{1014} (16 - 54) \begin{pmatrix} 156 \\ -1794 \end{pmatrix} \\ &= 98 \end{aligned}$$

$$\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f = 98 - 59 = 39 \quad (5.2.13.21)$$

$$\sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}} = \sqrt{\frac{39}{39}} = 1 \quad (5.2.13.22)$$

$$\sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_2}} = \sqrt{\frac{39}{26}} = \frac{\sqrt{6}}{2} \quad (5.2.13.23)$$

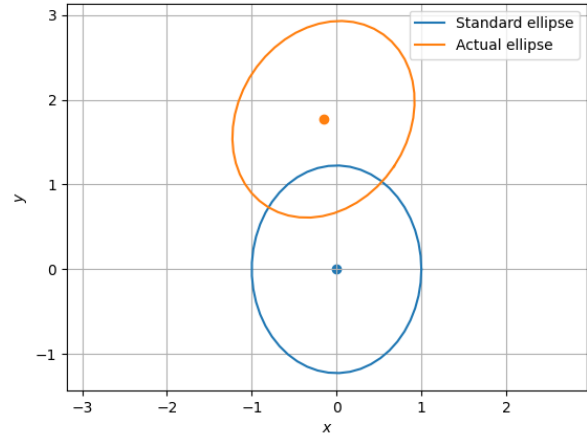


Fig. 5.2.13.1: Ellipse with center  $\begin{pmatrix} \frac{2}{13} & -\frac{23}{13} \end{pmatrix}$  and having the axes lengths as 1 and  $\frac{\sqrt{6}}{2}$

#### 5.2.14. Trace the curve

$$14x^2 - 4xy + 11y^2 - 44x - 58y + 71 = 0 \quad (5.2.14.1)$$

**Solution:** The general equation of second degree is given by

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \quad (5.2.14.2)$$

and can be expressed as

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (5.2.14.3)$$

where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad (5.2.14.4)$$

$$\mathbf{u}^T = (d \quad e) \quad (5.2.14.5)$$

Comparing (5.2.14.1) with (5.2.14.2), we get

$$\mathbf{V} = \begin{pmatrix} 14 & -2 \\ -2 & 11 \end{pmatrix} \quad (5.2.14.6)$$

$$\mathbf{u}^T = (-22 \quad -29) \quad (5.2.14.7)$$

If  $|\mathbf{V}| > 0$ , then (5.2.14.3) is an ellipse.

$$|\mathbf{V}| = \begin{vmatrix} 14 & -2 \\ -2 & 11 \end{vmatrix} = 150 > 0 \quad (5.2.14.8)$$

(5.2.14.3) can be expressed as

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f \quad |\mathbf{V}| \neq 0 \quad (5.2.14.9)$$

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = -\eta \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{y} \quad |\mathbf{V}| = 0 \quad (5.2.14.10)$$

with center as

$$\mathbf{c} = -\mathbf{V}^{-1} \mathbf{u} \quad |\mathbf{V}| \neq 0 \quad (5.2.14.11)$$

Calculating the center for given curve we get,

$$\mathbf{c} = -\frac{1}{|14 \times 11 - (-2 \times -2)|} \begin{pmatrix} 11 & 2 \\ 2 & 14 \end{pmatrix} \begin{pmatrix} -22 \\ -29 \end{pmatrix} \quad (5.2.14.12)$$

$$= \frac{1}{150} \begin{pmatrix} 300 \\ 450 \end{pmatrix} \quad (5.2.14.13)$$

$$= \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad (5.2.14.14)$$

For

$$|\mathbf{V}| > 0, \quad \text{or, } \lambda_1 > 0, \lambda_2 > 0 \quad (5.2.14.15)$$

(5.2.14.9) becomes

$$\lambda_1 y_1^2 + \lambda_2 y_2^2 = \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f \quad (5.2.14.16)$$

which is the equation of an ellipse with major and minor axes parameters

$$\sqrt{\frac{\lambda_1}{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}}, \sqrt{\frac{\lambda_2}{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}} \quad (5.2.14.17)$$

The characteristic equation of  $\mathbf{V}$  is obtained by

evaluating the determinant

$$|\lambda \mathbf{I} - \mathbf{V}| = \begin{vmatrix} \lambda - 14 & 2 \\ 2 & \lambda - 11 \end{vmatrix} = 0 \quad (5.2.14.18)$$

$$\implies \lambda^2 - 25\lambda + 150 = 0 \quad (5.2.14.19)$$

The eigenvalues are the roots of (5.2.14.19) given by

$$\lambda_1 = 15, \lambda_2 = 10 \quad (5.2.14.20)$$

The eigenvector  $\mathbf{p}$  is defined as

$$\mathbf{V} \mathbf{p} = \lambda \mathbf{p} \quad (5.2.14.21)$$

$$\implies (\lambda \mathbf{I} - \mathbf{V}) \mathbf{p} = 0 \quad (5.2.14.22)$$

where  $\lambda$  is the eigenvalue. For  $\lambda_1 = 15$ ,

$$(\lambda_1 \mathbf{I} - \mathbf{V}) = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \quad (5.2.14.23)$$

$$\implies \mathbf{p}_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad (5.2.14.24)$$

such that  $\|\mathbf{p}_1\| = 1$ . Similarly, the eigenvector corresponding to  $\lambda_2$  can be obtained as

$$\mathbf{p}_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (5.2.14.25)$$

It is easy to verify that

$$\mathbf{V} = \mathbf{P} \mathbf{D} \mathbf{P}^{-1} = \mathbf{P} \mathbf{D} \mathbf{P}^T \quad \because \mathbf{P}^{-1} = \mathbf{P}^T \quad (5.2.14.26)$$

$$\text{or, } \mathbf{D} = \mathbf{P}^T \mathbf{V} \mathbf{P} \quad (5.2.14.27)$$

where

$$\mathbf{P} = (\mathbf{p}_1 \quad \mathbf{p}_2) = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \quad (5.2.14.28)$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 15 & 0 \\ 0 & 10 \end{pmatrix} \quad (5.2.14.29)$$

Calculating the ellipse parameters using

(5.2.14.17), we get

$$\begin{aligned}\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} &= \\ &= (-22 - 29) \frac{1}{150} \begin{pmatrix} 11 & 2 \\ 2 & 14 \end{pmatrix} \begin{pmatrix} -22 \\ -29 \end{pmatrix} \\ &= \frac{1}{150} (300 \ 450) \begin{pmatrix} 22 \\ 29 \end{pmatrix} \\ &= 131\end{aligned}$$

$$\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f = 131 - 71 = 60 \quad (5.2.14.30)$$

$$\sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}} = \sqrt{\frac{60}{15}} = 2 \quad (5.2.14.31)$$

$$\sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_2}} = \sqrt{\frac{60}{10}} = \sqrt{6} \quad (5.2.14.32)$$

Thus, the given curve is found to be an ellipse from (5.2.14.8) with center at  $(2 \ 3)$  and the major and minor axes lengths are calculated as  $\sqrt{6}$ , 2. An ellipse with these parameters along with one having center as origin are plotted as shown.

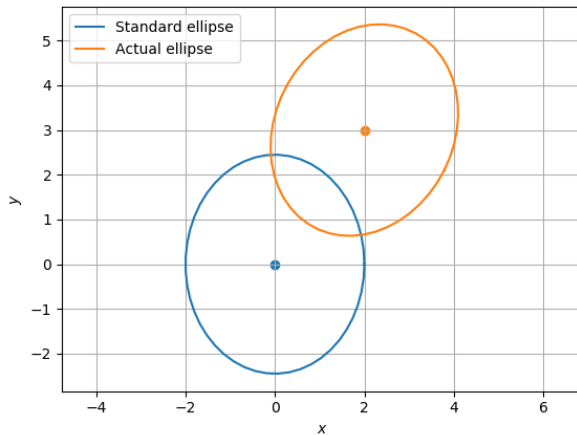


Fig. 5.2.14.1: Ellipse with center  $(2 \ 3)$  and having the axes lengths as  $\sqrt{6}$  and 2 along with an ellipse with center as origin

5.2.15. Trace the following

$$x^2 - 3xy + y^2 + 10x - 10y + 21 = 0 \quad (5.2.15.1)$$

**Solution:** The given quadratic equation can be

written in the matrix form as

$$\mathbf{x}^T \begin{pmatrix} 1 & -\frac{3}{2} \\ -\frac{3}{2} & 1 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} 5 & -5 \end{pmatrix} \mathbf{x} + 21 = 0 \quad (5.2.15.2)$$

Calculating the parameters, we get

$$|\mathbf{V}| = \begin{vmatrix} 1 & -\frac{3}{2} \\ -\frac{3}{2} & 1 \end{vmatrix} = -\frac{5}{4} \quad (5.2.15.3)$$

Since,  $|\mathbf{V}| < 0$ , therefore the given equation represents a hyperbola.

The characteristic equation of  $\mathbf{V}$  will be

$$|\mathbf{V} - \lambda \mathbf{I}| = \begin{vmatrix} 1 - \lambda & -\frac{3}{2} \\ -\frac{3}{2} & 1 - \lambda \end{vmatrix} = 0 \quad (5.2.15.4)$$

$$\Rightarrow 4\lambda^2 - 8\lambda - 5 = 0 \quad (5.2.15.5)$$

$$\Rightarrow \lambda_1 = \frac{5}{2}, \lambda_2 = -\frac{1}{2} \quad (5.2.15.6)$$

The eigen vector  $\mathbf{p}$  is given by

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \quad (5.2.15.7)$$

$$\Rightarrow \mathbf{V} - \lambda\mathbf{I}\mathbf{p} = 0 \quad (5.2.15.8)$$

For  $\lambda_1 = \frac{5}{2}$

$$\mathbf{V} - \lambda\mathbf{I} = \begin{pmatrix} 1 - \frac{5}{2} & -\frac{3}{2} \\ -\frac{3}{2} & 1 - \frac{5}{2} \end{pmatrix} \quad (5.2.15.9)$$

$$= \begin{pmatrix} -\frac{3}{2} & -\frac{3}{2} \\ -\frac{3}{2} & -\frac{3}{2} \end{pmatrix} \quad (5.2.15.10)$$

$$\begin{pmatrix} -\frac{3}{2} & -\frac{3}{2} \\ -\frac{3}{2} & -\frac{3}{2} \end{pmatrix} \xrightarrow{R_2=R_2-R_1} \begin{pmatrix} -\frac{3}{2} & -\frac{3}{2} \\ 0 & 0 \end{pmatrix} \quad (5.2.15.11)$$

$$\xrightarrow{R_1=R_1/-\frac{3}{2}} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad (5.2.15.12)$$

Substituting (5.2.15.12) in (5.2.15.8) we get

$$\mathbf{p}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad (5.2.15.13)$$

Therefore the normalized eigen vector will be

$$\mathbf{p}_1 = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \quad (5.2.15.14)$$

For  $\lambda_2 = -\frac{1}{2}$

$$\mathbf{V} - \lambda \mathbf{I} = \begin{pmatrix} 1 + \frac{1}{2} & -\frac{3}{2} \\ -\frac{3}{2} & 1 + \frac{1}{2} \end{pmatrix} \quad (5.2.15.15)$$

$$= \begin{pmatrix} \frac{3}{2} & -\frac{3}{2} \\ -\frac{3}{2} & -\frac{3}{2} \end{pmatrix} \quad (5.2.15.16)$$

$$\begin{pmatrix} -\frac{3}{2} & -\frac{3}{2} \\ -\frac{3}{2} & -\frac{3}{2} \end{pmatrix} \xrightarrow{R_2=R_2+R_1} \begin{pmatrix} -\frac{3}{2} & -\frac{3}{2} \\ 0 & 0 \end{pmatrix} \quad (5.2.15.17)$$

$$\xrightarrow{R_1=R_1/\frac{3}{2}} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \quad (5.2.15.18)$$

Substituting (5.2.15.18) in (5.2.15.8) we get

$$\mathbf{p}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (5.2.15.19)$$

Therefore the normalized eigen vector will be

$$\mathbf{p}_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \quad (5.2.15.20)$$

Eigen decomposition

Since  $\mathbf{V} = \mathbf{V}^T$  there exists an orthogonal matrix  $\mathbf{P}$  such that

$$\mathbf{P}\mathbf{P}^T = \mathbf{I} \quad (5.2.15.21)$$

$$\mathbf{P}\mathbf{V}\mathbf{P}^T = \mathbf{D} = \text{diag}(\lambda_1, \lambda_2) \quad (5.2.15.22)$$

or equivalently

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T \quad (5.2.15.23)$$

As

$$\mathbf{P} = \begin{pmatrix} p_1 & p_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \quad (5.2.15.24)$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad (5.2.15.25)$$

$$\Rightarrow \mathbf{D} = \begin{pmatrix} \frac{5}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \quad (5.2.15.26)$$

$$\mathbf{C} = -\mathbf{V}^{-1}\mathbf{u} \quad (5.2.15.27)$$

$$\Rightarrow \mathbf{C} = \begin{pmatrix} -\frac{4}{5} & -\frac{6}{5} \\ -\frac{6}{5} & -\frac{4}{5} \end{pmatrix} \begin{pmatrix} -5 \\ 5 \end{pmatrix} \quad (5.2.15.28)$$

$$= \begin{pmatrix} -2 \\ 2 \end{pmatrix} \quad (5.2.15.29)$$

$\therefore$  Centre  $\mathbf{C}$  is given by:

$$\begin{pmatrix} -2 \\ 2 \end{pmatrix} \quad (5.2.15.30)$$

Now Equation (5.2.15.1) can be written as

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - \mathbf{f} \quad (5.2.15.31)$$

$$(5.2.15.32)$$

where  $\mathbf{y}$  is given by:

$$\mathbf{y} = \mathbf{P}^T (\mathbf{x} - \mathbf{c}) \quad (5.2.15.33)$$

So

$$\mathbf{y}^T \begin{pmatrix} \frac{5}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \mathbf{y} = -1 \quad (5.2.15.34)$$

$$\Rightarrow \mathbf{y}^T \begin{pmatrix} \frac{5}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \mathbf{y} + 1 = 0 \quad (5.2.15.35)$$

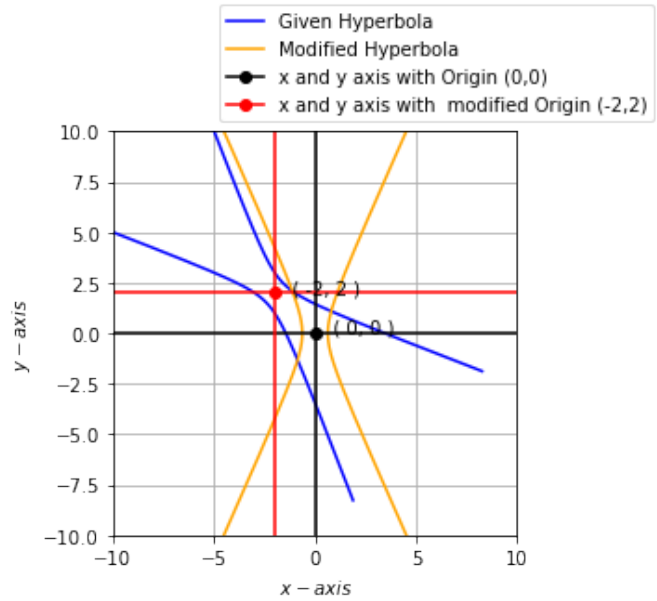


Fig. 5.2.15.1: Hyperbola plot when origin is shifted