Fractal in Action

G. V. V. Sharma
Dept. of Electrical Engg.,
IIT Hyderabad.

June 18, 2024

1. Class 10
2. Class 12
3. NCERT

Question

The centre of a circle is at (2,0). If one end of a diameter is at (6,0), then the other end is at :

1. $(0, 0)$
2. $(4, 0)$
3. $(-2, 0)$
4. $(-6, 0)$

$$\left(\frac{x_1+x_2+x_3}{2}, \frac{y_1+y_2+y_3}{2} \right)$$

P - Python

O - Optimization

M - Matrices & designs

P - Probability

C - Circuits

Solution

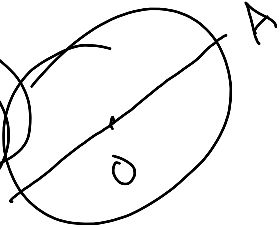
Let

$$\mathbf{O} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 6 \\ 0 \end{pmatrix} \tag{1}$$

→ on the circle

If $\mathbf{O}$ divides AB in the ratio $k : 1$,

$$\mathbf{O} = \frac{(\mathbf{A} + k\mathbf{B})}{1 + k} \tag{2}$$



In this case, $\because k = 1$,

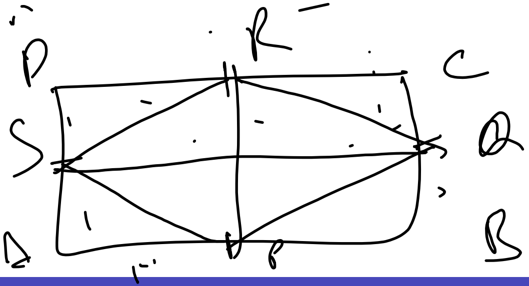
$$\mathbf{O} = \frac{(\mathbf{A} + \mathbf{B})}{2} \tag{3}$$

$$\implies \mathbf{B} = 2\mathbf{O} - \mathbf{A} \tag{4}$$

$$= 2\begin{pmatrix} 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 6 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \tag{5}$$

Question

$ABCD$ is a rectangle formed by the points $A(-1,-1)$, $B(-1,6)$, $C(3,6)$ and $D(3,-1)$. P, Q, R and S are mid-points of sides AB, BC, CD and DA respectively. Show that the diagonals of the quadrilateral $PQRS$ bisect each other.



Solution

From (2),

$$\mathbf{P} = \frac{\mathbf{A}+\mathbf{B}}{2}, \ \mathbf{Q} = \frac{\mathbf{B}+\mathbf{C}}{2} \tag{6}$$

$$\mathbf{R} = \frac{\mathbf{C}+\mathbf{D}}{2}, \ \mathbf{S} = \frac{\mathbf{D}+\mathbf{A}}{2} \tag{7}$$

Let $\mathbf{O}_1$ and $\mathbf{O}_2$ be the midpoints of PR and QS respectively

$$\mathbf{O}_1 = \frac{\mathbf{P}+\mathbf{R}}{2} = \frac{\mathbf{A}+\mathbf{B}+\mathbf{C}+\mathbf{D}}{4} \tag{8}$$

$$\mathbf{O}_2 = \frac{\mathbf{Q}+\mathbf{S}}{2} = \frac{\mathbf{A}+\mathbf{B}+\mathbf{C}+\mathbf{D}}{4} \tag{9}$$

Since

$$\mathbf{O}_1 = \mathbf{O}_2, \tag{10}$$

the diagonals bisect each other.

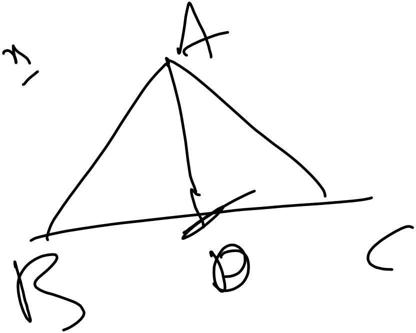
Question

AD is a median of ΔABC with vertices $A\,(5,-6)$, $B\,(6,4)$ and $C\,(0,0)$. Length AD is equal to:

1. $\sqrt{68}$
2. $2\sqrt{15}$
3. $\sqrt{101}$
4. 10

$\|x\| = \sqrt{x^t x}$

$\|A - D\|$



Solution

The midpoint of **BC** is

$$\mathbf{D} = \frac{\mathbf{B}+\mathbf{C}}{2} \tag{11}$$

$$= \frac{1}{2}\begin{pmatrix}6\\4\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}3\\2\end{pmatrix}, \tag{12}$$

Since

$$\mathbf{A}-\mathbf{D} = \begin{pmatrix}5\\-6\end{pmatrix} - \begin{pmatrix}3\\2\end{pmatrix} = \begin{pmatrix}2\\-8\end{pmatrix} \tag{13}$$

$$\implies \|\mathbf{A}-\mathbf{D}\| \triangleq \sqrt{(\mathbf{A}-\mathbf{D})^\top(\mathbf{A}-\mathbf{D})} \tag{14}$$

$$= \sqrt{\begin{pmatrix}2 & -8\end{pmatrix}\begin{pmatrix}2\\-8\end{pmatrix}} = \sqrt{2^2+8^2} = \sqrt{68} \tag{15}$$

Question

If the distance between the points $(3, -5)$ and $(x, -5)$ is 15 units, then the values of x are

1. $12, -18$
2. $-12, 18$
3. $18, 5$
4. $-9, -12$

Solution

$$\mathbf{A} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}, \ \mathbf{B} = \begin{pmatrix} x \\ -5 \end{pmatrix} \tag{16}$$

$$\implies \mathbf{A} - \mathbf{B} = \begin{pmatrix} 3 - x \\ -5 - (-5) \end{pmatrix} = \begin{pmatrix} 3 - x \\ 0 \end{pmatrix} \tag{17}$$

$$\implies ||\mathbf{A} - \mathbf{B}|| = \sqrt{\begin{pmatrix} 3 - x & 0 \end{pmatrix} \begin{pmatrix} 3 - x \\ 0 \end{pmatrix}} = \sqrt{(3 - x)^2} \tag{18}$$

$$\implies 15 = \pm(3 - x) \tag{19}$$

$$\implies x = -12, 18 \tag{20}$$

Question

Solve the following system of linear equations algebraically

$$\begin{aligned} 2x + 5y &= -4 \\ 4x - 3y &= 5 \end{aligned} \tag{21}$$

Solution

(21) can be expressed as

$$\begin{aligned} \mathbf{n}_1^\top \mathbf{x} &= c_1 \\ \mathbf{n}_2^\top \mathbf{x} &= c_2 \end{aligned} \tag{22}$$

where

$$\mathbf{n}_1 = \begin{pmatrix} 4 \\ -3 \end{pmatrix}, \ c_1 = 5 \tag{23}$$

$$\mathbf{n}_2 = \begin{pmatrix} 2 \\ 5 \end{pmatrix}, \ c_2 = -4 \tag{24}$$

(22) gives the normal forms of the equations given in (21) where

$$\mathbf{n}_1, \mathbf{n}_2 \tag{25}$$

are defined to be the normal vectors of the respective lines.

Solution

(22) can be expressed as

$$\begin{pmatrix} \mathbf{n}_1 & \mathbf{n}_2 \end{pmatrix}^\top \mathbf{x} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \tag{26}$$

yielding the matrix equation

$$\begin{pmatrix} 2 & 5 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4 \\ 5 \end{pmatrix} \tag{27}$$

Writing the augmented matrix for using Gauss elimination

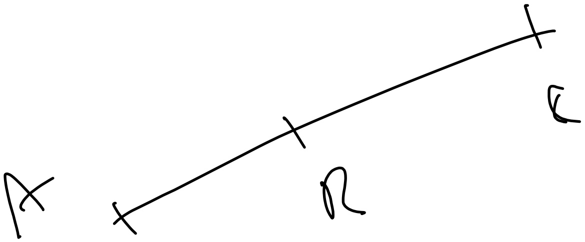
$$\left(\begin{array}{cc|c} 2 & 5 & -4 \\ 4 & -3 & 5 \end{array}\right) \xleftarrow{R_2 \to R_2 - 2R_1} \begin{pmatrix} 2 & 5 & -4 \\ 0 & -13 & 13 \end{pmatrix} \tag{28}$$

$$\begin{pmatrix} 2 & 5 & -4 \\ 0 & -13 & 13 \end{pmatrix} \xleftarrow{R_1 \to \frac{13}{5} R_1 + R_2} \begin{pmatrix} \frac{26}{5} & 0 & \frac{13}{5} \\ 0 & -13 & 13 \end{pmatrix} \tag{29}$$

$$\implies \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -1 \end{pmatrix} \tag{30}$$

Question

Find the ratio in which the point $C\left(\frac{8}{5}, y\right)$ divides the line segment joining the points $A\,(1,2)$ and $B\,(2,3)$. Also, find the value of y.



Solution

For collinearity,

$$\text{rank}\begin{pmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} = 2 \tag{31}$$

Performing row reduction,

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 8/5 \\ 2 & 3 & y \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & (2-1) & (\frac{8}{5}-1) \\ 0 & (3-2) & (y-3) \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & \frac{3}{5} \\ 0 & 1 & y-3 \end{pmatrix} \tag{32}$$

$$\xleftrightarrow{R_3 \to R_3 - R_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & \frac{3}{5} \\ 0 & 0 & y - \frac{18}{5} \end{pmatrix} \implies y = \frac{18}{5} \tag{33}$$

in order to get a 0 row.

Question

The sum of the digits of a 2-digit number is 14. The number obtained by interchanging its digits exceeds the given number by 18. Find the number.

Solution

Let the digits of the number be x_1(tens) and x_2(units).Given

$$x_1 + x_2 = 14 \tag{34}$$

$$10x_2 + x_1 = 18 + 10x_1 + x_2 \tag{35}$$

$$\implies x_1 - x_2 = -2 \tag{36}$$

The above equations can be expressed in matrix form as

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 14 \\ -2 \end{pmatrix} \tag{37}$$

$$\tag{38}$$

Solution

If

$$\mathbf{A} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \tag{39}$$

$$\mathbf{A}^\top \mathbf{A} = \mathbf{I} \tag{40}$$

A is then defined to be an orthogonal matrix.

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 14 \\ -2 \end{pmatrix} \tag{41}$$

$$\implies 2\mathbf{I}\mathbf{x} = \begin{pmatrix} 12 \\ 16 \end{pmatrix} \tag{42}$$

$$\implies \mathbf{x} = \begin{pmatrix} 6 \\ 8 \end{pmatrix} \tag{43}$$

Topics covered so far

1. Vectors
2. Section Formula
3. Norm
4. Gauss Elimination
5. Rank
6. Orthogonal matrix

Question

If $\overrightarrow{a} = 2\hat{i} - \hat{j} + \hat{k}$ and $\overrightarrow{b} = \hat{i} + \hat{j} - \hat{k}$, then $\overrightarrow{a}$ and $\overrightarrow{b}$ are:

1. Collinear vectors which are not parallel
2. Parallel vectors
3. Perpendicular vectors
4. Unit vectors

Solution

Let

$$\mathbf{a} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \tag{44}$$

Applying concept of rank from (31)

$$\text{rank} \begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 1 \end{pmatrix} = 2 \neq 1, \text{Not parallel} \tag{45}$$

Applying condition for perpendicularity:

$$\mathbf{a}^\top \mathbf{b} = \begin{pmatrix} 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 0 \implies \mathbf{a} \perp \mathbf{b} \tag{46}$$

Question

If α,β and γ are the angles which a line makes with positive directions of x,y and z axes respectively, then which of the following are not true?

1. $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$
2. $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$
3. $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = -1$
4. $\cos\alpha + \cos\beta + \cos\gamma = 1$

Solution

Let $\mathbf{m}$ represent the unit direction vector of the line. Then,

$$\mathbf{m} = \begin{pmatrix} \cos\alpha \\ \cos\beta \\ \cos\gamma \end{pmatrix} \tag{47}$$

with

$$\|\mathbf{m}\| = 1 \tag{48}$$

Parametric Form

Also,

$$
\begin{aligned}
2x + 5y &= -4 && (49)\\
\implies 2x &= -4 - 5y && (50)\\
\implies \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} -2 \\ 0 \end{pmatrix} + y \begin{pmatrix} -\frac{5}{2} \\ 1 \end{pmatrix} && (51)\\
\mathbf{x} &= \begin{pmatrix} -2 \\ 0 \end{pmatrix} - \frac{5y}{2} \begin{pmatrix} 1 \\ -\frac{2}{5} \end{pmatrix} && (52)\\
&= \mathbf{A} + k\mathbf{m} && (53)
\end{aligned}
$$

m is defined to be the direction vector of the line.

Question

$\overrightarrow{a}, \overrightarrow{b}$ and $\overrightarrow{c}$ are three mutually perpendicular unit vectors. If θ is the angle between $\overrightarrow{a}$ and $\left(\overrightarrow{2a} + \overrightarrow{3b} + \overrightarrow{6c}\right)$, find the value of $\cos\theta$.

Solution

$$\frac{x_1^\top x_2}{\|x_1\|\|x_2\|}$$

$x_1^\top y = c_1$

$x_2^\top y = c_2$

Given:

$$\mathbf{a}^\top \mathbf{b} = \mathbf{b}^\top \mathbf{c} = \mathbf{c}^\top \mathbf{a} = 0 \tag{54}$$

$$||\mathbf{a}|| = ||\mathbf{b}|| = ||\mathbf{c}|| = 1 \tag{55}$$

$$\cos\theta = \frac{\mathbf{a}^\top (2\mathbf{a} + 3\mathbf{b} + 6\mathbf{c})}{||\mathbf{a}||||2\mathbf{a} + 3\mathbf{b} + 6\mathbf{c}||} \tag{56}$$

Now,

$$\mathbf{a}^\top (2\mathbf{a} + 3\mathbf{b} + 6\mathbf{c}) = 2\mathbf{a}^\top \mathbf{a} + 3\mathbf{a}^\top \mathbf{b} + 6\mathbf{a}^\top \mathbf{c} = 2 + 0 + 0 = 2 \tag{57}$$

$$||\mathbf{a}||||2\mathbf{a} + 3\mathbf{b} + 6\mathbf{c}|| = ||2\mathbf{a} + 3\mathbf{b} + 6\mathbf{c}|| \tag{58}$$

Solution

From (14) norm definition:

$$(||2\mathbf{a}+3\mathbf{b}+6\mathbf{c}||)^2 = ||4\mathbf{a}^2|| + ||9\mathbf{b}^2|| + ||36\mathbf{c}^2|| = 49 \tag{59}$$
$$\implies ||2\mathbf{a}+3\mathbf{b}+6\mathbf{c}|| = +7 \tag{60}$$
$$\implies \cos\theta = \frac{2}{7} \tag{61}$$

Question

Find the position vector of point **C** which divides the line segment joining points **A** and **B** having position vectors $\hat{i} + 2\hat{j} - \hat{k}$ and $-\hat{i} + \hat{j} + \hat{k}$ respectively in the ratio 4 : 1 externally. Further, find $|\overrightarrow{AB}| : |\overrightarrow{BC}|$.

Solution

We know that

$$\mathbf{C} = \frac{4\mathbf{B} - \mathbf{A}}{4 - 1} \tag{62}$$

Simplify the above for **C**.

Question

Two vertices of the parallelogram **ABCD** are given as **A** $(-1, 2, 1)$ and **B** $(1, -2, 5)$. If the equation if the line passing through **C** and **D** is $\frac{x-4}{1} = \frac{y+7}{-2} = \frac{z-8}{2}$, then find the distance between the sides AB and CD. Hence, find the area of parallelogram $ABCD$.

Solution

Let the two parallel lines be

$$\mathbf{x} = \mathbf{A} + k_1\mathbf{m} \tag{63}$$
$$\mathbf{x} = \mathbf{C} + k_2\mathbf{m} \tag{64}$$

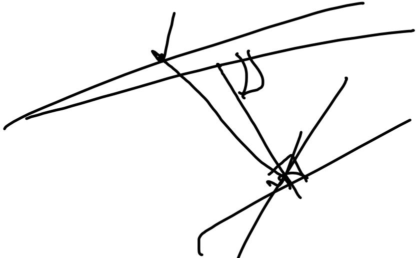
If $\mathbf{P}$ be a point on the second line,

$$\mathbf{P} = \mathbf{C} + k_2\mathbf{m} \tag{65}$$
$$\left(\mathbf{A} - \mathbf{P}\right)^{\top}\mathbf{m} = 0 \tag{66}$$

From the above,

$$\left(\mathbf{A} - \mathbf{C}\right)^{\top}\mathbf{m} - k_2\|\mathbf{m}\|^2 = 0 \tag{67}$$
$$\implies k_2 = \frac{\left(\mathbf{A} - \mathbf{C}\right)^{\top}\mathbf{m}}{\|\mathbf{m}\|^2} \tag{68}$$

Question



Find the equation of the line passing through the point of intersection of the lines $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$ and $\frac{x-1}{0} = \frac{y}{-3} = \frac{z-7}{2}$ and perpendicular to these given lines.

Solution

$m_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $m_2 = \begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix}$

Let the given lines be denoted by $\mathbf{x}_1$ and $\mathbf{x}_2$ respectively. From (53):

$$\mathbf{x}_1 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + k_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \mathbf{A} + k_1 \mathbf{m}_1 \tag{69}$$

$$\mathbf{x}_2 = \begin{pmatrix} 1 \\ 0 \\ 7 \end{pmatrix} + k_2 \begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix} = \mathbf{B} + k_2 \mathbf{m}_2 \tag{70}$$

Solution

Let the unknown line in its parametric form be denoted as follows from (53).

$$\mathbf{x}_3 = \mathbf{C} + k_3\mathbf{m} \tag{71}$$

The two equations required to solve for the direction of line are

$$\mathbf{m}^\top \mathbf{m}_1 = 0 \tag{72}$$

$$\mathbf{m}^\top \mathbf{m}_2 = 0 \tag{73}$$

$$\implies \begin{pmatrix} \mathbf{m}_1 & \mathbf{m}_2 \end{pmatrix}^\top \mathbf{m} = 0 \tag{74}$$

yielding

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & 2 \end{pmatrix} \xleftarrow{R_1 \to 2R_2 + 3R_1} \begin{pmatrix} 3 & 0 & 13 \\ 0 & -3 & 2 \end{pmatrix} = 0 \tag{75}$$

$$\implies \begin{pmatrix} 3 & 0 & 13 \\ 0 & -3 & 2 \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = 0 \implies \mathbf{m} = \begin{pmatrix} -\frac{13}{3} \\ \frac{2}{3} \\ 1 \end{pmatrix} \tag{76}$$

Solution

$$\begin{pmatrix} 1 & 0 \\ 2 & -3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} k_1 \\ -k_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 7 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

Equating (69) and (70),

$$\mathbf{A} + k_1\mathbf{m}_1 = \mathbf{B} + k_2\mathbf{m}_2 \tag{77}$$

$$\begin{pmatrix} \mathbf{m}_1 & \mathbf{m}_2 \end{pmatrix} \begin{pmatrix} k_1 \\ -k_2 \end{pmatrix} = \mathbf{B} - \mathbf{A} \tag{78}$$

From the above, k_1 and k_2 can be found by gauss elimination given in (29) and thus $\mathbf{C}$.

$$Mx = c \qquad M^T M x = M^T c$$

$$x = (M^T M)^{-1} M^T c$$

Question

Find the shortest distance between the lines whose vector equations are

$$\begin{aligned} \mathbf{x} &= \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \kappa_1 \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \\ \mathbf{x} &= \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + \kappa_2 \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \end{aligned} \tag{79}$$

Solution

From (78) the lines will intersect if

$$\operatorname{rank}\begin{pmatrix}\mathbf{M} & \mathbf{B}-\mathbf{A}\end{pmatrix} = 2 \tag{80}$$

where

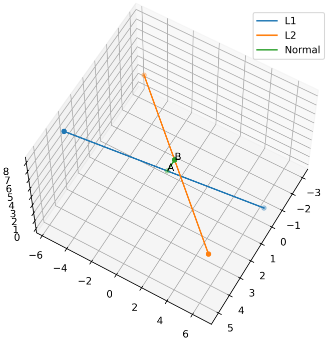
$$\mathbf{M} = \begin{pmatrix}\mathbf{m_1} & \mathbf{m_2}\end{pmatrix} \tag{81}$$

Solution

If L_1, L_2, do not intersect, let

$$\begin{aligned} \mathbf{x}_1 &= \mathbf{A} + \kappa_1 \mathbf{m_1} \\ \mathbf{x}_2 &= \mathbf{B} + \kappa_2 \mathbf{m_2} \end{aligned} \tag{82}$$

be points on L_1, L_2 respectively, that are closest to each other.



Figure

Solution

Then, from (82)

$$\mathbf{x_1} - \mathbf{x_2} = \mathbf{A} - \mathbf{B} + \begin{pmatrix} \mathbf{m_1} & \mathbf{m_2} \end{pmatrix} \begin{pmatrix} \kappa_1 \\ -\kappa_2 \end{pmatrix} \tag{83}$$

Also,

$$\begin{aligned} (\mathbf{x}_1 - \mathbf{x}_2)^\top \mathbf{m}_1 = (\mathbf{x}_1 - \mathbf{x}_2)^\top \mathbf{m}_2 = 0 \quad (84) \\ \implies (\mathbf{x}_1 - \mathbf{x}_2)^\top \begin{pmatrix} \mathbf{m_1} & \mathbf{m_2} \end{pmatrix} = \mathbf{0} \quad (85) \\ \text{or, } \mathbf{M}^\top (\mathbf{x}_1 - \mathbf{x}_2) = \mathbf{0} \quad (86) \\ \implies \mathbf{M}^\top (\mathbf{A} - \mathbf{B}) + \mathbf{M}^\top \mathbf{M} \begin{pmatrix} \kappa_1 \\ -\kappa_2 \end{pmatrix} = \mathbf{0} \quad (87) \end{aligned}$$

from (83), yielding

$$\mathbf{M}^\top \mathbf{M} \begin{pmatrix} \kappa_1 \\ -\kappa_2 \end{pmatrix} = \mathbf{M}^\top (\mathbf{B} - \mathbf{A}) \tag{88}$$

This is known as the *least squares solution*.

Question

Draw a circle of radius 6 cm. From a point 10 cm away from its centre, construct the pair of tangents to the circle and measure their lengths.

Solution

The equation of the circle is given by

$$\|\mathbf{x} - \mathbf{O}\|^2 = r^2 \tag{89}$$

which can be expressed as

$$\mathrm{g}\left(\mathbf{x}\right) = \mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{90}$$

for

$$\mathbf{V} = \mathbf{I}, \mathbf{u} = -\mathbf{O}, f = \|\mathbf{O}\| - r^2, \tag{91}$$

Let (53) be the equation of the tangent from the point $\mathbf{h}$. Then, the intersection of (53) and (90) can be expressed as

$$\left(\mathbf{h} + \mu\mathbf{m}\right)^\top \mathbf{V}\left(\mathbf{h} + \mu\mathbf{m}\right) + 2\mathbf{u}^\top \left(\mathbf{h} + \mu\mathbf{m}\right) + f = 0 \tag{92}$$

$$\implies \mu^2\mathbf{m}^\top\mathbf{V}\mathbf{m} + 2\mu\mathbf{m}^\top \left(\mathbf{V}\mathbf{h} + \mathbf{u}\right) + g\left(\mathbf{h}\right) = 0 \tag{93}$$

For (93) to have exactly one root, the discriminant

Solution

$$\left\{\mathbf{m}^\top (\mathbf{Vh}+\mathbf{u})\right\}^2 - g(\mathbf{h})\,\mathbf{m}^\top \mathbf{Vm} = 0 \tag{94}$$

and

$$\mu = -\frac{\mathbf{m}^\top (\mathbf{Vh}+\mathbf{u})}{\mathbf{m}^\top \mathbf{Vm}} \tag{95}$$

is obtained. (94) can be expressed as

$$\mathbf{m}^\top (\mathbf{Vh}+\mathbf{u})^\top (\mathbf{Vh}+\mathbf{u})\,\mathbf{m} - g(\mathbf{h})\,\mathbf{m}^\top \mathbf{Vm} = 0 \tag{96}$$

$$\implies \mathbf{m}^\top \mathbf{\Sigma} \mathbf{m} = 0 \tag{97}$$

where

$$\mathbf{\Sigma} = (\mathbf{Vh}+\mathbf{u})(\mathbf{Vh}+\mathbf{u})^\top - g(\mathbf{h})\,\mathbf{V} \tag{98}$$

Using the eigenvalue decomposition

$$\mathbf{P}^\top \mathbf{\Sigma} \mathbf{P} = \mathbf{D}, \tag{99}$$

in (97),

Solution

$$\mathbf{m}^\top \mathbf{P}\mathbf{D}\mathbf{P}^\top \mathbf{m} = 0 \tag{100}$$

$$\implies \mathbf{v}^\top \mathbf{D}\mathbf{v} = 0 \tag{101}$$

where

$$\mathbf{v} = \mathbf{P}^\top \mathbf{m} \tag{102}$$

(101) can be expressed as

$$\lambda_1 v_1^2 - \lambda_2 v_2^2 = 0 \tag{103}$$

$$\implies \mathbf{v} = \begin{pmatrix} \sqrt{|\lambda_2|} \\ \pm\sqrt{|\lambda_1|} \end{pmatrix} \tag{104}$$

after some algebra. From (104) and (102) we obtain

$$\mathbf{m} = \mathbf{P} \begin{pmatrix} \sqrt{|\lambda_2|} \\ \pm\sqrt{|\lambda_1|} \end{pmatrix} \tag{105}$$

SVD

Perform the eigendecompositions

$$\mathbf{M}\mathbf{M}^\top = \mathbf{U}\mathbf{D}_1\mathbf{U}^\top \tag{106}$$

$$\mathbf{M}^\top\mathbf{M} = \mathbf{V}\mathbf{D}_2\mathbf{V}^\top \tag{107}$$

The following expression is known as *singular value decomposition*

$$\mathbf{M} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top \tag{108}$$

where $\mathbf{\Sigma}$ is diagonal with

$$\mathbf{\Sigma} \triangleq \begin{pmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \\ 0 & 0 \end{pmatrix} \tag{109}$$

Substituting in (88),

SVD

$$\mathbf{V\Sigma U}^\top \mathbf{U\Sigma V}^\top \boldsymbol{\kappa} = \mathbf{V\Sigma U}^\top (\mathbf{B} - \mathbf{A}) \tag{110}$$

$$\implies \mathbf{V\Sigma}^2\mathbf{V}^\top \boldsymbol{\kappa} = \mathbf{V\Sigma U}^\top (\mathbf{B} - \mathbf{A}) \tag{111}$$

$$\implies \boldsymbol{\kappa} = \left(\mathbf{V\Sigma}^2\mathbf{V}^\top\right)^{-1} \mathbf{V\Sigma U}^\top (\mathbf{B} - \mathbf{A}) \tag{112}$$

$$\implies \boldsymbol{\kappa} = \mathbf{V\Sigma}^{-2}\mathbf{V}^\top \mathbf{V\Sigma U}^\top (\mathbf{B} - \mathbf{A}) \tag{113}$$

$$\implies \boldsymbol{\kappa} = \mathbf{V\Sigma}^{-1}\mathbf{U}^\top (\mathbf{B} - \mathbf{A}) \tag{114}$$

where $\boldsymbol{\Sigma}^{-1}$ is obtained by inverting the nonzero elements of $\boldsymbol{\Sigma}$. From (82),

$$\mathbf{x}_1 - \mathbf{x}_2 = \mathbf{A} + \kappa_1 \mathbf{m_1} - \mathbf{B} - \kappa_2 \mathbf{m_2} \tag{115}$$

$$= \mathbf{A} - \mathbf{B} + \mathbf{M}\boldsymbol{\kappa} \tag{116}$$

which, upon substitution from (108) yields

$$\mathbf{x}_1 - \mathbf{x}_2 = \mathbf{A} - \mathbf{B} + \mathbf{U\Sigma V}^\top \mathbf{V\Sigma}^{-1}\mathbf{U}^\top (\mathbf{B} - \mathbf{A}) \tag{117}$$

$$= (\mathbf{A} - \mathbf{B}) \left(\mathbf{I} - \mathbf{U\Sigma\Sigma}^{-1}\mathbf{U}^\top\right) \tag{118}$$

Conic Section

Let $\mathbf{q}$ be a point such that the ratio of its distance from a fixed point $\mathbf{F}$ and the distance (d) from a fixed line

$$L : \mathbf{n}^\top \mathbf{x} = c \tag{119}$$

is constant, given by

$$\frac{\|\mathbf{q} - \mathbf{F}\|}{d} = e \tag{120}$$

The locus of $\mathbf{q}$ is known as a conic section. The line L is known as the directrix and the point $\mathbf{F}$ is the focus. e is defined to be the eccentricity of the conic.

1. For $e = 1$, the conic is a parabola
2. For $e < 1$, the conic is an ellipse
3. For $e > 1$, the conic is a hyperbola

Conic

The equation of a conic with directrix $\mathbf{n}^\top \mathbf{x} = c$, eccentricity e and focus $\mathbf{F}$ is given by

$$\mathrm{g}(\mathbf{x}) = \mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{121}$$

where

$$\mathbf{V} = \|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{n}\mathbf{n}^\top, \tag{122}$$

$$\mathbf{u} = ce^2\mathbf{n} - \|\mathbf{n}\|^2 \mathbf{F}, \tag{123}$$

$$f = \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2 e^2 \tag{124}$$

Conic

Using Definition 46, for any point $\mathbf{x}$ on the conic,

$$\begin{aligned}
\|\mathbf{x}-\mathbf{F}\|^2 = e^2\frac{\left(\mathbf{n}^\top\mathbf{x}-c\right)^2}{\|\mathbf{n}\|^2} \\
\implies \|\mathbf{n}\|^2(\mathbf{x}-\mathbf{F})^\top(\mathbf{x}-\mathbf{F}) = e^2\left(\mathbf{n}^\top\mathbf{x}-c\right)^2 \\
\implies \|\mathbf{n}\|^2\left(\mathbf{x}^\top\mathbf{x}-2\mathbf{F}^\top\mathbf{x}+\|\mathbf{F}\|^2\right) \\
= e^2\left(c^2+\left(\mathbf{n}^\top\mathbf{x}\right)^2-2c\mathbf{n}^\top\mathbf{x}\right) \\
= e^2\left(c^2+\left(\mathbf{x}^\top\mathbf{n}\mathbf{n}^\top\mathbf{x}\right)-2c\mathbf{n}^\top\mathbf{x}\right)
\end{aligned} \tag{125}$$

which can be expressed as (90) after simplification.

Conic

The eccentricity, directrices and foci of (90) are given by

$$
\begin{aligned}
e &= \sqrt{1-\frac{\lambda_1}{\lambda_2}} && (126)\\
\mathbf{n} &= \sqrt{\lambda_2}\mathbf{p}_1, \\
c &= \begin{cases} \frac{e\mathbf{u}^\top\mathbf{n} \pm \sqrt{e^2\left(\mathbf{u}^\top\mathbf{n}\right)^2 - \lambda_2(e^2-1)(\|\mathbf{u}\|^2-\lambda_2 f)}}{\lambda_2 e(e^2-1)} & e \neq 1 \\ \frac{\|\mathbf{u}\|^2-\lambda_2 f}{2\mathbf{u}^\top\mathbf{n}} & e = 1 \end{cases} && (127)\\
\mathbf{F} &= \frac{ce^2\mathbf{n}-\mathbf{u}}{\lambda_2} && (128)
\end{aligned}
$$

Conic

From (122), using the fact that $\mathbf{V}$ is symmetric with $\mathbf{V} = \mathbf{V}^\top$,

$$\mathbf{V}^\top\mathbf{V} = \left(\|\mathbf{n}\|^2\mathbf{I} - e^2\mathbf{n}\mathbf{n}^\top\right)^\top \left(\|\mathbf{n}\|^2\mathbf{I} - e^2\mathbf{n}\mathbf{n}^\top\right) \tag{129}$$

$$\implies \mathbf{V}^2 = \|\mathbf{n}\|^4\mathbf{I} + e^4\mathbf{n}\mathbf{n}^\top\mathbf{n}\mathbf{n}^\top - 2e^2\|\mathbf{n}\|^2\mathbf{n}\mathbf{n}^\top \tag{130}$$

$$= \|\mathbf{n}\|^4\mathbf{I} + e^4\|\mathbf{n}\|^2\mathbf{n}\mathbf{n}^\top - 2e^2\|\mathbf{n}\|^2\mathbf{n}\mathbf{n}^\top \tag{131}$$

$$= \|\mathbf{n}\|^4\mathbf{I} + e^2\left(e^2 - 2\right)\|\mathbf{n}\|^2\mathbf{n}\mathbf{n}^\top \tag{132}$$

$$= \|\mathbf{n}\|^4\mathbf{I} + \left(e^2 - 2\right)\|\mathbf{n}\|^2\left(\|\mathbf{n}\|^2\mathbf{I} - \mathbf{V}\right) \tag{133}$$

which can be expressed as

$$\mathbf{V}^2 + \left(e^2 - 2\right)\|\mathbf{n}\|^2\mathbf{V} - \left(e^2 - 1\right)\|\mathbf{n}\|^4\mathbf{I} = 0 \tag{134}$$

Using the Cayley-Hamilton theorem, (134) results in the characteristic equation,

$$\lambda^2 - \left(2 - e^2\right)\|\mathbf{n}\|^2\lambda + \left(1 - e^2\right)\|\mathbf{n}\|^4 = 0 \tag{135}$$

which can be expressed as

Conic

$$
\begin{aligned}
\left(\frac{\lambda}{\|\mathbf{n}\|^2}\right)^2 - \left(2-e^2\right)\left(\frac{\lambda}{\|\mathbf{n}\|^2}\right) + \left(1-e^2\right) = 0 \qquad (136)\\
\implies \frac{\lambda}{\|\mathbf{n}\|^2} = 1-e^2, 1 \qquad (137)\\
\text{or, } \lambda_2 = \|\mathbf{n}\|^2, \lambda_1 = \left(1-e^2\right)\lambda_2 \qquad (138)
\end{aligned}
$$

From (138), the eccentricity of (90) is given by (126). Multiplying both sides of (122) by $\mathbf{n}$,

$$
\begin{aligned}
\mathbf{V}\mathbf{n} &= \|\mathbf{n}\|^2\mathbf{n} - e^2\mathbf{n}\mathbf{n}^\top\mathbf{n} && (139)\\
&= \|\mathbf{n}\|^2\left(1-e^2\right)\mathbf{n} && (140)\\
&= \lambda_1\mathbf{n} && (141)\\
& && (142)
\end{aligned}
$$

from (138). Thus, λ_1 is the corresponding eigenvalue for $\mathbf{n}$.

Conic

From (142), this implies that

$$\mathbf{p}_1 = \frac{\mathbf{n}}{\|\mathbf{n}\|} \tag{143}$$

$$\text{or, } \mathbf{n} = \|\mathbf{n}\|\mathbf{p}_1 = \sqrt{\lambda_2}\mathbf{p}_1 \tag{144}$$

from (138) . From (123) and (138),

$$\mathbf{F} = \frac{ce^2\mathbf{n} - \mathbf{u}}{\lambda_2} \tag{145}$$

$$\implies \|\mathbf{F}\|^2 = \frac{\left(ce^2\mathbf{n} - \mathbf{u}\right)^\top \left(ce^2\mathbf{n} - \mathbf{u}\right)}{\lambda_2^2} \tag{146}$$

$$\implies \lambda_2^2\|\mathbf{F}\|^2 = c^2e^4\lambda_2 - 2ce^2\mathbf{u}^\top\mathbf{n} + \|\mathbf{u}\|^2 \tag{147}$$

Also, (124) can be expressed as

$$\lambda_2\|\mathbf{F}\|^2 = f + c^2e^2 \tag{148}$$

Conic

From (147) and (148),

$$c^2 e^4 \lambda_2 - 2ce^2 \mathbf{u}^\top \mathbf{n} + \|\mathbf{u}\|^2 = \lambda_2 \left(f + c^2 e^2 \right) \tag{149}$$

$$\implies \lambda_2 e^2 \left(e^2 - 1 \right) c^2 - 2ce^2 \mathbf{u}^\top \mathbf{n} + \|\mathbf{u}\|^2 - \lambda_2 f = 0 \tag{150}$$

yielding (127)