# Fractal in Action

G. V. V. Sharma
Dept. of Electrical Engg.,
IIT Hyderabad.

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## Question

The centre of a circle is at (2,0). If one end of a diameter is at (6,0), then the other end is at :

1. $(0, 0)$
2. $(4, 0)$
3. $(-2, 0)$
4. $(-6, 0)$

## Solution

Let

$$\mathbf{O} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \ \mathbf{A} = \begin{pmatrix} 6 \\ 0 \end{pmatrix} \tag{1}$$

If $\mathbf{O}$ divides $AB$ in the ratio $k : 1$,

$$\mathbf{O} = \frac{(\mathbf{A} + k\mathbf{B})}{1 + k} \tag{2}$$

In this case, $\because k = 1$,

$$\mathbf{O} = \frac{(\mathbf{A} + \mathbf{B})}{2} \tag{3}$$

$$\implies \mathbf{B} = 2\mathbf{O} - \mathbf{A} \tag{4}$$

$$= 2 \begin{pmatrix} 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 6 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \tag{5}$$

## Question

ABCD is a rectangle formed by the points $A(-1, -1), B(-1, 6), C(3, 6)$ and $D(3, -1)$. $P, Q, R$ and $S$ are mid-points of sides $AB, BC, CD$ and $DA$ respectively. Show that the diagonals of the quadrilateral $PQRS$ bisect each other.

## Solution

From (2),

$$\mathbf{P} = \frac{\mathbf{A} + \mathbf{B}}{2}, \ \mathbf{Q} = \frac{\mathbf{B} + \mathbf{C}}{2} \tag{6}$$

$$\mathbf{R} = \frac{\mathbf{C} + \mathbf{D}}{2}, \ \mathbf{S} = \frac{\mathbf{D} + \mathbf{A}}{2} \tag{7}$$

Let $\mathbf{O}_1$ and $\mathbf{O}_2$ be the midpoints of $PR$ and $QS$ respectively

$$\mathbf{O}_1 = \frac{\mathbf{P} + \mathbf{R}}{2} = \frac{\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D}}{4} \tag{8}$$

$$\mathbf{O}_2 = \frac{\mathbf{Q} + \mathbf{S}}{2} = \frac{\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D}}{4} \tag{9}$$

Since

$$\mathbf{O}_1 = \mathbf{O}_2, \tag{10}$$

the diagonals bisect each other.

## Question

$AD$ is a median of $\Delta ABC$ with vertices $A\left(5, -6\right)$, $B\left(6, 4\right)$ and $C\left(0, 0\right)$. Length $AD$ is equal to:

1. $\sqrt{68}$
2. $2\sqrt{15}$
3. $\sqrt{101}$
4. $10$

## Solution

The midpoint of **BC** is

$$\mathbf{D} = \frac{\mathbf{B} + \mathbf{C}}{2} \tag{11}$$

$$= \frac{1}{2}\begin{pmatrix} 6 \\ 4 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \tag{12}$$

Since

$$\mathbf{A} - \mathbf{D} = \begin{pmatrix} 5 \\ -6 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -8 \end{pmatrix} \tag{13}$$

$$\implies \|\mathbf{A} - \mathbf{D}\| \triangleq \sqrt{(\mathbf{A} - \mathbf{D})^\top (\mathbf{A} - \mathbf{D})} \tag{14}$$

$$= \sqrt{\begin{pmatrix} 2 & -8 \end{pmatrix}\begin{pmatrix} 2 \\ -8 \end{pmatrix}} = \sqrt{2^2 + 8^2} = \sqrt{68} \tag{15}$$

## Question

If the distance between the points $(3, -5)$ and $(x, -5)$ is 15 units, then the values of $x$ are

1. $12, -18$
2. $-12, 18$
3. $18, 5$
4. $-9, -12$

## Solution

$$\mathbf{A} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}, \ \mathbf{B} = \begin{pmatrix} x \\ -5 \end{pmatrix} \tag{16}$$

$$\implies \mathbf{A} - \mathbf{B} = \begin{pmatrix} 3 - x \\ -5 - (-5) \end{pmatrix} = \begin{pmatrix} 3 - x \\ 0 \end{pmatrix} \tag{17}$$

$$\implies \|\mathbf{A} - \mathbf{B}\| = \sqrt{\begin{pmatrix} 3 - x & 0 \end{pmatrix} \begin{pmatrix} 3 - x \\ 0 \end{pmatrix}} = \sqrt{(3 - x)^2} \tag{18}$$

$$\implies 15 = \pm (3 - x) \tag{19}$$

$$\implies x = -12, 18 \tag{20}$$

## Question

Solve the following system of linear equations algebraically

$$2x + 5y = -4$$
$$4x - 3y = 5 \tag{21}$$

## Solution

(21) can be expressed as

$$\begin{aligned} \mathbf{n}_1^\top \mathbf{x} &= c_1 \\ \mathbf{n}_2^\top \mathbf{x} &= c_2 \end{aligned} \tag{22}$$

where

$$\mathbf{n}_1 = \begin{pmatrix} 4 \\ -3 \end{pmatrix}, \ c_1 = 5 \tag{23}$$

$$\mathbf{n}_2 = \begin{pmatrix} 2 \\ 5 \end{pmatrix}, \ c_2 = -4 \tag{24}$$

(22) gives the normal forms of the equations given in (21) where

$$\mathbf{n}_1, \mathbf{n}_2 \tag{25}$$

are defined to be the normal vectors of the respective lines.

## Solution

(22) can be expressed as

$$\begin{pmatrix} \mathbf{n}_1 & \mathbf{n}_2 \end{pmatrix}^\top \mathbf{x} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \tag{26}$$

yielding the matrix equation

$$\begin{pmatrix} 2 & 5 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4 \\ 5 \end{pmatrix} \tag{27}$$

Writing the augmented matrix for using Gauss elimination

$$\left(\begin{array}{cc|c} 2 & 5 & -4 \\ 4 & -3 & 5 \end{array}\right) \xleftarrow{R_2 \to R_2 - 2R_1} \begin{pmatrix} 2 & 5 & -4 \\ 0 & -13 & 13 \end{pmatrix} \tag{28}$$

$$\begin{pmatrix} 2 & 5 & -4 \\ 0 & -13 & 13 \end{pmatrix} \xleftarrow{R_1 \to \frac{13}{5} R_1 + R_2} \begin{pmatrix} \frac{26}{5} & 0 & \frac{13}{5} \\ 0 & -13 & 13 \end{pmatrix} \tag{29}$$

$$\implies \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -1 \end{pmatrix} \tag{30}$$

## Question

Find the ratio in which the point $C\left(\frac{8}{5}, y\right)$ divides the line segment joining the points $A(1, 2)$ and $B(2, 3)$. Also, find the value of $y$.

## Solution

For collinearity,

$$\text{rank} \begin{pmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} = 2 \tag{31}$$

Performing row reduction,

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 8/5 \\ 2 & 3 & y \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & (2-1) & (\frac{8}{5}-1) \\ 0 & (3-2) & (y-3) \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & \frac{3}{5} \\ 0 & 1 & y-3 \end{pmatrix} \tag{32}$$

$$\xleftrightarrow{R_3 \to R_3 - R_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & \frac{3}{5} \\ 0 & 0 & y - \frac{18}{5} \end{pmatrix} \implies y = \frac{18}{5} \tag{33}$$

in order to get a 0 row.

## Question

The sum of the digits of a 2-digit number is 14. The number obtained by interchanging its digits exceeds the given number by 18. Find the number.

## Solution

Let the digits of the number be $x_1$(tens) and $x_2$(units).Given

$$x_1 + x_2 = 14 \tag{34}$$

$$10x_2 + x_1 = 18 + 10x_1 + x_2 \tag{35}$$

$$\implies x_1 - x_2 = -2 \tag{36}$$

The above equations can be expressed in matrix form as

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 14 \\ -2 \end{pmatrix} \tag{37}$$

$$\tag{38}$$

## Solution

If

$$\mathbf{A} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \tag{39}$$

$$\mathbf{A}^\top \mathbf{A} = \mathbf{I} \tag{40}$$

**A** is then defined to be an orthogonal matrix.

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 14 \\ -2 \end{pmatrix} \tag{41}$$

$$\implies 2\mathbf{I}\mathbf{x} = \begin{pmatrix} 12 \\ 16 \end{pmatrix} \tag{42}$$

$$\implies \mathbf{x} = \begin{pmatrix} 6 \\ 8 \end{pmatrix} \tag{43}$$

# Topics covered so far

1. Vectors
2. Section Formula
3. Norm
4. Gauss Elimination
5. Rank
6. Orthogonal matrix

## Question

If $\overrightarrow{a} = 2\hat{i} - \hat{j} + \hat{k}$ and $\overrightarrow{b} = \hat{i} + \hat{j} - \hat{k}$, then $\overrightarrow{a}$ and $\overrightarrow{b}$ are:

1. Collinear vectors which are not parallel
2. Parallel vectors
3. Perpendicular vectors
4. Unit vectors

## Solution

Let

$$\mathbf{a} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \tag{44}$$

Applying concept of rank from (31)

$$\text{rank} \begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 1 \end{pmatrix} = 2 \neq 1, \text{Not parallel} \tag{45}$$

Applying condition for perpendicularity:

$$\mathbf{a}^\top \mathbf{b} = \begin{pmatrix} 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 0 \implies \mathbf{a} \perp \mathbf{b} \tag{46}$$

## Question

If $\alpha, \beta$ and $\gamma$ are the angles which a line makes with positive directions of $x, y$ and $z$ axes respectively, then which of the following are not true?

1. $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$
2. $\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma = 2$
3. $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = -1$
4. $\cos \alpha + \cos \beta + \cos \gamma = 1$

## Solution

Let **m** represent the unit direction vector of the line. Then,

$$\mathbf{m} = \begin{pmatrix} \cos\alpha \\ \cos\beta \\ \cos\gamma \end{pmatrix} \tag{47}$$

with

$$\|\mathbf{m}\| = 1 \tag{48}$$

## Parametric Form

Also,

$$2x + 5y = -4 \tag{49}$$

$$\implies 2x = -4 - 5y \tag{50}$$

$$\implies \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} + y \begin{pmatrix} -\frac{5}{2} \\ 1 \end{pmatrix} \tag{51}$$

$$\mathbf{x} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} - \frac{5y}{2} \begin{pmatrix} 1 \\ -\frac{2}{5} \end{pmatrix} \tag{52}$$

$$= \mathbf{A} + k\mathbf{m} \tag{53}$$

$\mathbf{m}$ is defined to be the direction vector of the line.

## Question

$\overrightarrow{a}$, $\overrightarrow{b}$ and $\overrightarrow{c}$ are three mutually perpendicular unit vectors. If $\theta$ is the angle between $\overrightarrow{a}$ and $\left(\overrightarrow{2a} + \overrightarrow{3b} + \overrightarrow{6c}\right)$, find the value of $\cos\theta$.

## Solution

Given:

$$\mathbf{a}^\top \mathbf{b} = \mathbf{b}^\top \mathbf{c} = \mathbf{c}^\top \mathbf{a} = 0 \tag{54}$$

$$||\mathbf{a}|| = ||\mathbf{b}|| = ||\mathbf{c}|| = 1 \tag{55}$$

$$\cos\theta = \frac{\mathbf{a}^\top (2\mathbf{a} + 3\mathbf{b} + 6\mathbf{c})}{||\mathbf{a}||\,||2\mathbf{a} + 3\mathbf{b} + 6\mathbf{c}||} \tag{56}$$

Now,

$$\mathbf{a}^\top (2\mathbf{a} + 3\mathbf{b} + 6\mathbf{c}) = 2\mathbf{a}^\top \mathbf{a} + 3\mathbf{a}^\top \mathbf{b} + 6\mathbf{a}^\top \mathbf{c} = 2 + 0 + 0 = 2 \tag{57}$$

$$||\mathbf{a}||\,||2\mathbf{a} + 3\mathbf{b} + 6\mathbf{c}|| = ||2\mathbf{a} + 3\mathbf{b} + 6\mathbf{c}|| \tag{58}$$

## Solution

From (14) norm definition:

$$\left( ||2\mathbf{a} + 3\mathbf{b} + 6\mathbf{c}|| \right)^2 = ||4\mathbf{a}^2|| + ||9\mathbf{b}^2|| + ||36\mathbf{c}^2|| = 49 \tag{59}$$
$$\implies ||2\mathbf{a} + 3\mathbf{b} + 6\mathbf{c}|| = +7 \tag{60}$$
$$\implies \cos\theta = \frac{2}{7} \tag{61}$$

## Question

Find the position vector of point **C** which divides the line segment joining points **A** and **B** having position vectors $\hat{i} + 2\hat{j} - \hat{k}$ and $-\hat{i} + \hat{j} + \hat{k}$ respectively in the ratio $4 : 1$ externally. Further, find $|\overrightarrow{AB}| : |\overrightarrow{BC}|$.

## Solution

We know that

$$\mathbf{C} = \frac{4\mathbf{B} - \mathbf{A}}{4 - 1} \tag{62}$$

Simplify the above for $\mathbf{C}$.

Two vertices of the parallelogram **ABCD** are given as **A**$(-1, 2, 1)$ and
**B**$(1, -2, 5)$. If the equation if the line passing through **C** and **D** is
$\frac{x-4}{1} = \frac{y+7}{-2} = \frac{z-8}{2}$, then find the distance between the sides $AB$ and $CD$.
Hence, find the area of parallelogram $ABCD$.

## Solution

Let the two parallel lines be

$$\mathbf{x} = \mathbf{A} + k_1 \mathbf{m} \tag{63}$$

$$\mathbf{x} = \mathbf{C} + k_2 \mathbf{m} \tag{64}$$

If $\mathbf{P}$ be a point on the second line,

$$\mathbf{P} = \mathbf{C} + k_2 \mathbf{m} \tag{65}$$

$$(\mathbf{A} - \mathbf{P})^\top \mathbf{m} = 0 \tag{66}$$

From the above,

$$(\mathbf{A} - \mathbf{C})^\top \mathbf{m} - k_2 \|\mathbf{m}\|^2 = 0 \tag{67}$$

$$\implies k_2 = \frac{(\mathbf{A} - \mathbf{C})^\top \mathbf{m}}{\|\mathbf{m}\|^2} \tag{68}$$

## Question

Find the equation of the line passing through the point of intersection of the lines $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$ and $\frac{x-1}{0} = \frac{y}{-3} = \frac{z-7}{2}$ and perpendicular to these given lines.

## Solution

Let the given lines be denoted by $\mathbf{x}_1$ and $\mathbf{x}_2$ respectively. From (53):

$$\mathbf{x}_1 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + k_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \mathbf{A} + k_1 \mathbf{m}_1 \tag{69}$$

$$\mathbf{x}_2 = \begin{pmatrix} 1 \\ 0 \\ 7 \end{pmatrix} + k_2 \begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix} = \mathbf{B} + k_2 \mathbf{m}_2 \tag{70}$$

## Solution

Let the unknown line in its parametric form be denoted as follows from (53).

$$\mathbf{x}_3 = \mathbf{C} + k_3\mathbf{m} \tag{71}$$

The two equations required to solve for the direction of line are

$$\mathbf{m}^\top \mathbf{m}_1 = 0 \tag{72}$$

$$\mathbf{m}^\top \mathbf{m}_2 = 0 \tag{73}$$

$$\implies \begin{pmatrix} \mathbf{m}_1 & \mathbf{m}_2 \end{pmatrix}^\top \mathbf{m} = 0 \tag{74}$$

yielding

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & 2 \end{pmatrix} \xleftarrow{R_1 \to 2R_2 + 3R_1} \begin{pmatrix} 3 & 0 & 13 \\ 0 & -3 & 2 \end{pmatrix} = 0 \tag{75}$$

$$\implies \begin{pmatrix} 3 & 0 & 13 \\ 0 & -3 & 2 \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = 0 \implies \mathbf{m} = \begin{pmatrix} -\frac{13}{3} \\ \frac{2}{3} \\ 1 \end{pmatrix} \tag{76}$$

## Solution

Equating (69) and (70),

$$\mathbf{A} + k_1\mathbf{m}_1 = \mathbf{B} + k_2\mathbf{m}_2 \tag{77}$$

$$\begin{pmatrix} \mathbf{m}_1 & \mathbf{m}_2 \end{pmatrix} \begin{pmatrix} k_1 \\ -k_2 \end{pmatrix} = \mathbf{B} - \mathbf{A} \tag{78}$$

From the above, $k_1$ and $k_2$ can be found by gauss elimination given in (29) and thus $\mathbf{C}$.

## Question

Find the shortest distance between the lines whose vector equations are

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \kappa_1 \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$$

$$\mathbf{x} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + \kappa_2 \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \tag{79}$$

## Solution

From (78) the lines will intersect if

$$\text{rank} \begin{pmatrix} \mathbf{M} & \mathbf{B} - \mathbf{A} \end{pmatrix} = 2 \tag{80}$$

where

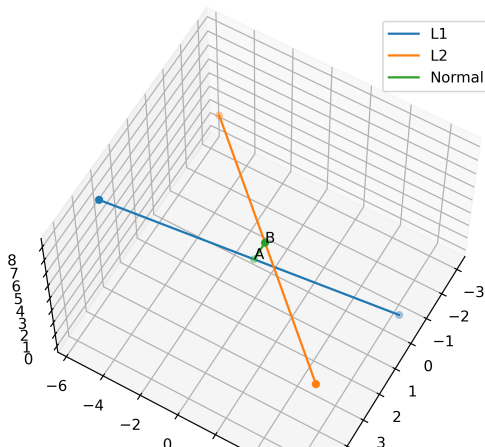$$\mathbf{M} = \begin{pmatrix} \mathbf{m_1} & \mathbf{m_2} \end{pmatrix} \tag{81}$$

## Solution

If $L_1, L_2$, do not intersect, let

$$\mathbf{x}_1 = \mathbf{A} + \kappa_1 \mathbf{m_1}$$
$$\mathbf{x}_2 = \mathbf{B} + \kappa_2 \mathbf{m_2}$$

(82)

be points on $L_1, L_2$ respectively, that are closest to each other.

## Solution

Then, from (82)

$$\mathbf{x_1} - \mathbf{x_2} = \mathbf{A} - \mathbf{B} + \begin{pmatrix} \mathbf{m_1} & \mathbf{m_2} \end{pmatrix} \begin{pmatrix} \kappa_1 \\ -\kappa_2 \end{pmatrix} \tag{83}$$

Also,

$$(\mathbf{x}_1 - \mathbf{x}_2)^\top \mathbf{m}_1 = (\mathbf{x}_1 - \mathbf{x}_2)^\top \mathbf{m}_2 = 0 \tag{84}$$

$$\implies (\mathbf{x}_1 - \mathbf{x}_2)^\top \begin{pmatrix} \mathbf{m_1} & \mathbf{m_2} \end{pmatrix} = \mathbf{0} \tag{85}$$

$$\text{or,} \ \mathbf{M}^\top (\mathbf{x}_1 - \mathbf{x}_2) = \mathbf{0} \tag{86}$$

$$\implies \mathbf{M}^\top (\mathbf{A} - \mathbf{B}) + \mathbf{M}^\top \mathbf{M} \begin{pmatrix} \kappa_1 \\ -\kappa_2 \end{pmatrix} = \mathbf{0} \tag{87}$$

from (83), yielding

$$\mathbf{M}^\top \mathbf{M} \begin{pmatrix} \kappa_1 \\ -\kappa_2 \end{pmatrix} = \mathbf{M}^\top (\mathbf{B} - \mathbf{A}) \tag{88}$$

This is known as the *least squares solution*.

## Question

Draw a circle of radius 6 cm. From a point 10 cm away from its centre, construct the pair of tangents to the circle and measure their lengths.

## Solution

The equation of the circle is given by

$$\|\mathbf{x} - \mathbf{O}\|^2 = r^2 \tag{89}$$

which can be expressed as

$$g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{90}$$

for

$$\mathbf{V} = \mathbf{I}, \mathbf{u} = -\mathbf{O}, f = \|\mathbf{O}\| - r^2, \tag{91}$$

Let (53) be the equation of the tangent from the point $\mathbf{h}$. Then, the intersection of (53) and (90) can be expressed as

$$(\mathbf{h} + \mu\mathbf{m})^\top \mathbf{V} (\mathbf{h} + \mu\mathbf{m}) + 2\mathbf{u}^\top (\mathbf{h} + \mu\mathbf{m}) + f = 0 \tag{92}$$

$$\implies \mu^2 \mathbf{m}^\top \mathbf{V} \mathbf{m} + 2\mu \mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u}) + g(\mathbf{h}) = 0 \tag{93}$$

For (93) to have exactly one root, the discriminant

## Solution

$$\left\{ \mathbf{m}^\top \left( \mathbf{Vh} + \mathbf{u} \right) \right\}^2 - g\left( \mathbf{h} \right) \mathbf{m}^\top \mathbf{Vm} = 0 \tag{94}$$

and

$$\mu = -\frac{\mathbf{m}^\top \left( \mathbf{Vh} + \mathbf{u} \right)}{\mathbf{m}^\top \mathbf{Vm}} \tag{95}$$

is obtained. (94) can be expressed as

$$\mathbf{m}^\top \left( \mathbf{Vh} + \mathbf{u} \right)^\top \left( \mathbf{Vh} + \mathbf{u} \right) \mathbf{m} - g\left( \mathbf{h} \right) \mathbf{m}^\top \mathbf{Vm} = 0 \tag{96}$$

$$\implies \mathbf{m}^\top \mathbf{\Sigma m} = 0 \tag{97}$$

where

$$\mathbf{\Sigma} = \left( \mathbf{Vh} + \mathbf{u} \right) \left( \mathbf{Vh} + \mathbf{u} \right)^\top - g\left( \mathbf{h} \right) \mathbf{V} \tag{98}$$

Using the eigenvalue decomposition

$$\mathbf{P}^\top \mathbf{\Sigma P} = \mathbf{D}, \tag{99}$$

in (97),

## Solution

$$\mathbf{m}^\top \mathbf{P} \mathbf{D} \mathbf{P}^\top \mathbf{m} = 0 \tag{100}$$

$$\implies \mathbf{v}^\top \mathbf{D} \mathbf{v} = 0 \tag{101}$$

where

$$\mathbf{v} = \mathbf{P}^\top \mathbf{m} \tag{102}$$

(101) can be expressed as

$$\lambda_1 v_1^2 - \lambda_2 v_2^2 = 0 \tag{103}$$

$$\implies \mathbf{v} = \begin{pmatrix} \sqrt{|\lambda_2|} \\ \pm\sqrt{|\lambda_1|} \end{pmatrix} \tag{104}$$

after some algebra. From (104) and (102) we obtain

$$\mathbf{m} = \mathbf{P} \begin{pmatrix} \sqrt{|\lambda_2|} \\ \pm\sqrt{|\lambda_1|} \end{pmatrix} \tag{105}$$

## SVD

Perform the eigendecompositions

$$\mathbf{M}\mathbf{M}^\top = \mathbf{U}\mathbf{D}_1\mathbf{U}^\top \tag{106}$$

$$\mathbf{M}^\top\mathbf{M} = \mathbf{V}\mathbf{D}_2\mathbf{V}^\top \tag{107}$$

The following expression is known as *singular value decomposition*

$$\mathbf{M} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^\top \tag{108}$$

where $\boldsymbol{\Sigma}$ is diagonal with

$$\boldsymbol{\Sigma} \triangleq \begin{pmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \\ 0 & 0 \end{pmatrix} \tag{109}$$

Substituting in (88),

# SVD

$$\mathbf{V\Sigma U^\top U\Sigma V^\top}\kappa = \mathbf{V\Sigma U^\top}\left(\mathbf{B} - \mathbf{A}\right) \tag{110}$$

$$\implies \mathbf{V\Sigma^2 V^\top}\kappa = \mathbf{V\Sigma U^\top}\left(\mathbf{B} - \mathbf{A}\right) \tag{111}$$

$$\implies \kappa = \left(\mathbf{V\Sigma^2 V^\top}\right)^{-1}\mathbf{V\Sigma U^\top}\left(\mathbf{B} - \mathbf{A}\right) \tag{112}$$

$$\implies \kappa = \mathbf{V\Sigma^{-2}V^\top V\Sigma U^\top}\left(\mathbf{B} - \mathbf{A}\right) \tag{113}$$

$$\implies \kappa = \mathbf{V\Sigma^{-1}U^\top}\left(\mathbf{B} - \mathbf{A}\right) \tag{114}$$

where $\mathbf{\Sigma}^{-1}$ is obtained by inverting the nonzero elements of $\mathbf{\Sigma}$. From (82),

$$\mathbf{x}_1 - \mathbf{x}_2 = \mathbf{A} + \kappa_1\mathbf{m_1} - \mathbf{B} - \kappa_2\mathbf{m_2} \tag{115}$$

$$= \mathbf{A} - \mathbf{B} + \mathbf{M}\kappa \tag{116}$$

which, upon substitution from (108) yields

$$\mathbf{x}_1 - \mathbf{x}_2 = \mathbf{A} - \mathbf{B} + \mathbf{U\Sigma V^\top V\Sigma^{-1}U^\top}\left(\mathbf{B} - \mathbf{A}\right) \tag{117}$$

$$= \left(\mathbf{A} - \mathbf{B}\right)\left(\mathbf{I} - \mathbf{U\Sigma\Sigma^{-1}U^\top}\right) \tag{118}$$

## Conic Section

Let $\mathbf{q}$ be a point such that the ratio of its distance from a fixed point $\mathbf{F}$ and the distance $(d)$ from a fixed line

$$L : \mathbf{n}^\top \mathbf{x} = c \qquad (119)$$

is constant, given by

$$\frac{\|\mathbf{q} - \mathbf{F}\|}{d} = e \qquad (120)$$

The locus of $\mathbf{q}$ is known as a conic section. The line $L$ is known as the directrix and the point $\mathbf{F}$ is the focus. $e$ is defined to be the eccentricity of the conic.

1. For $e = 1$, the conic is a parabola
2. For $e < 1$, the conic is an ellipse
3. For $e > 1$, the conic is a hyperbola

## Conic

The equation of a conic with directrix $\mathbf{n}^\top \mathbf{x} = c$, eccentricity $e$ and focus $\mathbf{F}$ is given by

$$g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{121}$$

where

$$\mathbf{V} = \|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{n}\mathbf{n}^\top, \tag{122}$$

$$\mathbf{u} = ce^2 \mathbf{n} - \|\mathbf{n}\|^2 \mathbf{F}, \tag{123}$$

$$f = \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2 e^2 \tag{124}$$

## Conic

Using Definition 46, for any point $\mathbf{x}$ on the conic,

$$\|\mathbf{x} - \mathbf{F}\|^2 = e^2 \frac{\left(\mathbf{n}^\top \mathbf{x} - c\right)^2}{\|\mathbf{n}\|^2}$$

$$\implies \|\mathbf{n}\|^2 \left(\mathbf{x} - \mathbf{F}\right)^\top \left(\mathbf{x} - \mathbf{F}\right) = e^2 \left(\mathbf{n}^\top \mathbf{x} - c\right)^2$$

$$\implies \|\mathbf{n}\|^2 \left(\mathbf{x}^\top \mathbf{x} - 2\mathbf{F}^\top \mathbf{x} + \|\mathbf{F}\|^2\right)$$

$$= e^2 \left(c^2 + \left(\mathbf{n}^\top \mathbf{x}\right)^2 - 2c\mathbf{n}^\top \mathbf{x}\right)$$

$$= e^2 \left(c^2 + \left(\mathbf{x}^\top \mathbf{n}\mathbf{n}^\top \mathbf{x}\right) - 2c\mathbf{n}^\top \mathbf{x}\right) \quad (125)$$

which can be expressed as (90) after simplification.

## Conic

The eccentricity, directrices and foci of (90) are given by

$$e = \sqrt{1 - \frac{\lambda_1}{\lambda_2}} \tag{126}$$

$$\mathbf{n} = \sqrt{\lambda_2}\mathbf{p}_1,$$

$$c = \begin{cases} \dfrac{e\mathbf{u}^\top\mathbf{n} \pm \sqrt{e^2\left(\mathbf{u}^\top\mathbf{n}\right)^2 - \lambda_2(e^2-1)(\|\mathbf{u}\|^2 - \lambda_2 f)}}{\lambda_2 e(e^2-1)} & e \neq 1 \\ \dfrac{\|\mathbf{u}\|^2 - \lambda_2 f}{2\mathbf{u}^\top\mathbf{n}} & e = 1 \end{cases} \tag{127}$$

$$\mathbf{F} = \frac{ce^2\mathbf{n} - \mathbf{u}}{\lambda_2} \tag{128}$$

## Conic

From (122), using the fact that $\mathbf{V}$ is symmetric with $\mathbf{V} = \mathbf{V}^\top$,

$$\mathbf{V}^\top \mathbf{V} = \left( \|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{n}\mathbf{n}^\top \right)^\top \left( \|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{n}\mathbf{n}^\top \right) \tag{129}$$

$$\implies \mathbf{V}^2 = \|\mathbf{n}\|^4 \mathbf{I} + e^4 \mathbf{n}\mathbf{n}^\top \mathbf{n}\mathbf{n}^\top - 2e^2 \|\mathbf{n}\|^2 \mathbf{n}\mathbf{n}^\top \tag{130}$$

$$= \|\mathbf{n}\|^4 \mathbf{I} + e^4 \|\mathbf{n}\|^2 \mathbf{n}\mathbf{n}^\top - 2e^2 \|\mathbf{n}\|^2 \mathbf{n}\mathbf{n}^\top \tag{131}$$

$$= \|\mathbf{n}\|^4 \mathbf{I} + e^2 \left( e^2 - 2 \right) \|\mathbf{n}\|^2 \mathbf{n}\mathbf{n}^\top \tag{132}$$

$$= \|\mathbf{n}\|^4 \mathbf{I} + \left( e^2 - 2 \right) \|\mathbf{n}\|^2 \left( \|\mathbf{n}\|^2 \mathbf{I} - \mathbf{V} \right) \tag{133}$$

which can be expressed as

$$\mathbf{V}^2 + \left( e^2 - 2 \right) \|\mathbf{n}\|^2 \mathbf{V} - \left( e^2 - 1 \right) \|\mathbf{n}\|^4 \mathbf{I} = 0 \tag{134}$$

Using the Cayley-Hamilton theorem, (134) results in the characteristic equation,

$$\lambda^2 - \left( 2 - e^2 \right) \|\mathbf{n}\|^2 \lambda + \left( 1 - e^2 \right) \|\mathbf{n}\|^4 = 0 \tag{135}$$

which can be expressed as

## Conic

$$\left(\frac{\lambda}{\|\mathbf{n}\|^2}\right)^2 - (2 - e^2)\left(\frac{\lambda}{\|\mathbf{n}\|^2}\right) + (1 - e^2) = 0 \tag{136}$$

$$\implies \frac{\lambda}{\|\mathbf{n}\|^2} = 1 - e^2, 1 \tag{137}$$

$$\text{or, } \lambda_2 = \|\mathbf{n}\|^2, \lambda_1 = (1 - e^2)\lambda_2 \tag{138}$$

From (138), the eccentricity of (90) is given by (126). Multiplying both sides of (122) by $\mathbf{n}$,

$$\mathbf{V}\mathbf{n} = \|\mathbf{n}\|^2\mathbf{n} - e^2\mathbf{n}\mathbf{n}^\top\mathbf{n} \tag{139}$$

$$= \|\mathbf{n}\|^2\left(1 - e^2\right)\mathbf{n} \tag{140}$$

$$= \lambda_1\mathbf{n} \tag{141}$$

$$\tag{142}$$

from (138). Thus, $\lambda_1$ is the corresponding eigenvalue for $\mathbf{n}$.

## Conic

From (142), this implies that

$$\mathbf{p}_1 = \frac{\mathbf{n}}{\|\mathbf{n}\|} \tag{143}$$

$$\text{or, } \mathbf{n} = \|\mathbf{n}\|\mathbf{p}_1 = \sqrt{\lambda_2}\mathbf{p}_1 \tag{144}$$

from (138) . From (123) and (138),

$$\mathbf{F} = \frac{ce^2\mathbf{n} - \mathbf{u}}{\lambda_2} \tag{145}$$

$$\implies \|\mathbf{F}\|^2 = \frac{\left(ce^2\mathbf{n} - \mathbf{u}\right)^\top \left(ce^2\mathbf{n} - \mathbf{u}\right)}{\lambda_2^2} \tag{146}$$

$$\implies \lambda_2^2\|\mathbf{F}\|^2 = c^2e^4\lambda_2 - 2ce^2\mathbf{u}^\top\mathbf{n} + \|\mathbf{u}\|^2 \tag{147}$$

Also, (124) can be expressed as

$$\lambda_2\|\mathbf{F}\|^2 = f + c^2e^2 \tag{148}$$

## Conic

From (147) and (148),

$$c^2 e^4 \lambda_2 - 2ce^2 \mathbf{u}^\top \mathbf{n} + \|\mathbf{u}\|^2 = \lambda_2 \left( f + c^2 e^2 \right) \tag{149}$$

$$\implies \lambda_2 e^2 \left( e^2 - 1 \right) c^2 - 2ce^2 \mathbf{u}^\top \mathbf{n} + \|\mathbf{u}\|^2 - \lambda_2 f = 0 \tag{150}$$

yielding (127)