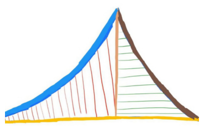
Matrices In Geometry



G. V. V. Sharma

About this Book

This book introduces matrices through high school coordinate geometry. This approach makes it easier for beginners to learn Python for scientific computing. All problems in the book are from NCERT mathematics textbooks from Class 9-12. Exercises are from CBSE board exam papers.

The content is sufficient for industry jobs and covers nearly all matrix prerequisites for machine learning. There is no copyright, so readers are free to print and share.

This book is dedicated to my high school teachers, Dr. G.N. Chandwani and Dr. Anand K. Tripathi.

August 12, 2025

Github: https://github.com/gadepall/matgeo

First manual appeared in January 2018
First edition published on July 10, 2024
In this edition, some incorrect solutions were removed. All figures redrawn. More problems added.

CONTENTS

1.1 Formulae

1.1.1. The *direction vector* of AB is defined as

$$\mathbf{m} = \mathbf{B} - \mathbf{A} = \kappa \begin{pmatrix} 1 \\ m \end{pmatrix} \tag{1.1.1.1}$$

where m is the slope of AB. We also say that

$$\mathbf{m} \equiv \begin{pmatrix} 1 \\ m \end{pmatrix} \tag{1.1.1.2}$$

1.1.2. The lines with direction vectors $\mathbf{m}_1$ and $\mathbf{m}_2$ respectively, are parallel if

$$\mathbf{m}_1 \equiv \mathbf{m}_2 \tag{1.1.2.1}$$

1.1.3. If $ABCD$ be a parallelogram with $AB \parallel CD$,

$$\mathbf{B} - \mathbf{A} = \mathbf{C} - \mathbf{D} \tag{1.1.3.1}$$

1.1.4. If $\mathbf{D}$ divides BC in the ratio $k : 1$,

$$\mathbf{D} = \frac{k\mathbf{C} + \mathbf{B}}{k + 1} \tag{1.1.4.1}$$

1.1.5. If $PQRS$ is formed by joining the mid points of $ABCD$,

$$\mathbf{P} = \frac{1}{2}(\mathbf{A} + \mathbf{B}), \ \mathbf{Q} = \frac{1}{2}(\mathbf{B} + \mathbf{C}) \tag{1.1.5.1}$$

$$\mathbf{R} = \frac{1}{2}(\mathbf{C} + \mathbf{D}), \ \mathbf{S} = \frac{1}{2}(\mathbf{D} + \mathbf{A}) \tag{1.1.5.2}$$

$$\implies \mathbf{P} - \mathbf{Q} = \mathbf{S} - \mathbf{R}. \tag{1.1.5.3}$$

Hence, $PQRS$ is a parallelogram from (1.1.3.1).

1.1.6. In 2D space, the basis vectors are defined as

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{1.1.6.1}$$

1.1.7. The length of a vector is defined as

$$\|\mathbf{x}\| \triangleq \sqrt{\mathbf{x}^\top \mathbf{x}} \tag{1.1.7.1}$$

For example, if

$$\mathbf{x} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \tag{1.1.7.2}$$

$$\mathbf{x}^\top \mathbf{x} = \begin{pmatrix} 3 & 4 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} \tag{1.1.7.3}$$

$$= 3 \times 3 + 4 \times 4 = 25 \tag{1.1.7.4}$$

yielding

$$\|\mathbf{x}\| = 5. \tag{1.1.7.5}$$

(1.1.7.3) is known as the scalar product.

1.1.8. The unit vector in the direction of $\mathbf{x}$ is

$$\frac{\mathbf{x}}{\|\mathbf{x}\|} \tag{1.1.8.1}$$

1.1.9. Points $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are defined to be collinear if

$$\text{rank}\begin{pmatrix}\mathbf{B}-\mathbf{A} & \mathbf{C}-\mathbf{A}\end{pmatrix} = 1 \tag{1.1.9.1}$$

1.1.10.

$$\text{rank}\mathbf{A} = \text{rank}\mathbf{A}^{\top} \tag{1.1.10.1}$$

1.1.11. In the 2D space, the unit direction vector is defined as

$$\mathbf{m} = \begin{pmatrix}\cos\alpha \\ \cos\beta\end{pmatrix} \tag{1.1.11.1}$$

where α, β are the angles made by the vector with the axes.

1.1.12. Code for plotting points and vector arithmetic

```
codes/book/points.py
```

1.1.13. Code for section formula

```
codes/book/section.py
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1.1.14. Code for matrix rank

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codes/book/rank.py
```

1.1.15. Code for vector length

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codes/book/dist.py
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1.2 Point Vectors

1.2.1 Find the values of x and y so that the vectors $2\hat{i} + 3\hat{j}$ and $x\hat{i} + y\hat{j}$ are equal.
Solution: From the given informatin,

$$\begin{pmatrix}2 \\ 3\end{pmatrix} = \begin{pmatrix}x \\ y\end{pmatrix} \tag{1.2.1.1}$$

$$\implies x = 2, y = 3 \tag{1.2.1.2}$$

1.2.2 Find the values of x, y, z so that the vectors $x\hat{i} + 2\hat{j} + z\hat{k}$ and $2\hat{i} + y\hat{j} + \hat{k}$ are equal.

1.2.3 Find the sum of the vectors $\mathbf{a} = \hat{i} - 2\hat{j} + \hat{k}$, $\mathbf{b} = -2\hat{i} + 4\hat{j} + 5\hat{k}$ and $\mathbf{c} = \hat{i} - 6\hat{j} - 7\hat{k}$.

1.2.4 Find the slope of a line, which passes through the origin and the mid point of the line segment joining the points **P**(0, -4) and **B**(8, 0).
Solution: The mid point of PB is

$$\mathbf{M} = \frac{1}{2}(\mathbf{P} + \mathbf{B}) = \begin{pmatrix} 4 \\ -2 \end{pmatrix} \tag{1.2.4.1}$$

which, from (1.1.1.1), is equal to the direction vector of OM, where **O** is the origin.

$$\because \mathbf{M} \equiv \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix}, m = -\frac{1}{2} \tag{1.2.4.2}$$

which, from (1.1.1.1), is the desired slope. See Fig. 1.2.4.1.

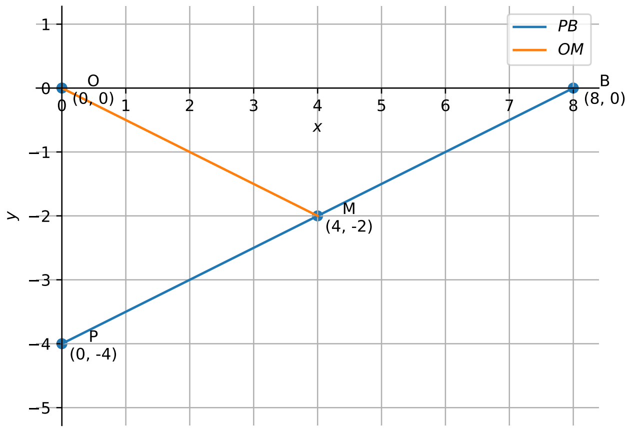


Fig. 1.2.4.1

1.2.5 Find the angle between x-axis and the line joining points (3, -1) and (4, -2).
Solution: The direction vector of the given line is

$$\begin{pmatrix} 4 \\ -2 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \implies m = -1 \tag{1.2.5.1}$$

Hence, the desired angle is 135°.

1.2.6 Plot the points (x, y) given in Table 1.2.6.

x	-2	-1	0	1	3
y	8	7	-1.25	3	-1

TABLE 1.2.6

1.2.7 A line passes through $\mathbf{A}(x_1, y_1)$ and $\mathbf{B}(h, k)$. If slope of the line is m, show that

$(k - y_1) = m(h - x_1)$.
Solution: The direction vector

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} h - x_1 \\ k - y_1 \end{pmatrix} \equiv \begin{pmatrix} 1 \\ \frac{k-y_1}{h-x_1} \end{pmatrix} \tag{1.2.7.1}$$

$$\implies m = \frac{k - y_1}{h - x_1}, \tag{1.2.7.2}$$

yielding the desired result.

1.2.8 Show that the line through the points $(4, 7, 8)$, $(2, 3, 4)$ is parallel to the line through the points $(-1, -2, 1)$, $(1, 2, 5)$.
Solution:

$$\begin{pmatrix} 4 \\ 7 \\ 8 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} \equiv \begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix} \tag{1.2.8.1}$$

which means that the given lines have the same direction vector and are hence parallel.

1.2.9 The vector having intial and terminal points as (-2, 5, 0) and (3, 7, 4), respectively is **Solution:** The desired vector is

$$\begin{pmatrix} 3 \\ 7 \\ 4 \end{pmatrix} - \begin{pmatrix} -2 \\ 5 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 4 \end{pmatrix} \tag{1.2.9.1}$$

1.2.10 Find the vector joining the points $\mathbf{P}(2, 3, 0)$ and $\mathbf{Q}(-1, -2, -4)$ directed from $\mathbf{P}$ to $\mathbf{Q}$.

1.2.11 Find the slope of lines
a) Passing through the points $(3, -2)$ and $(-1, 4)$
b) Passing through the points $(3, -2)$ and $(7, -2)$
c) passing through the points $(3, -2)$ and $(3, 4)$
d) Making inclination of $60°$ with the positive direction of x-axis.

1.2.12 If the points $\mathbf{A}(6, 1), \mathbf{B}(8, 2), \mathbf{C}(9, 4)$ and $\mathbf{D}(p, 3)$ are the vertices of a parallelogram, taken in order, find the value of p.

1.2.13 If $(1, 2), (4, y), (x, 6)$ and $(3, 5)$ are the vertices of a parallelogram taken in order, find x and y.

1.2.14 The fourth vertex $\mathbf{D}$ of a parallelogram $ABCD$ whose three vertices are $\mathbf{A}(-2, 3), \mathbf{B}(6, 7)$ and $\mathbf{C}(8, 3)$ is

1.2.15 Verify if the points $\mathbf{A}(4, 3), \mathbf{B}(6, 4), \mathbf{C}(5, -6)$ and $\mathbf{D}(-3, 5)$ are the vertices of a parallelogram.

1.2.16 $(-1, 2, 1), (1, -2, 5), (4, -7, 8)$ and $(2, -3, 4)$ are the vertices of a parallelogram.

1.2.17 Three vertices of a parallelogram $ABCD$ are $\mathbf{A}(3, -1, 2), \mathbf{B}(1, -2, 4)$ and $\mathbf{C}(-1, 1, 2)$. Find the coordinates of the fourth vertex.

1.2.18 If the origin is the centroid of the triangle PQR with vertices $\mathbf{P}(2a, 2, 6), \mathbf{Q}(-4, 3b, -10)$ and $R(8, 14, 2c)$, then find the values of a, b and c.

1.2.19 In which quadrant or on which axis do each of the points (-2, 4), (3, -1), (-1, 0), (1,

2) and (-3, -5) lie? Verify your answer by locating them on the Cartesian plane.

1.2.20 Without using distance formula, show that points $\mathbf{A}(-2,-1), \mathbf{B}(4,0), \mathbf{C}(3,3)$ and $\mathbf{D}(-3,2)$ are the vertices of a parallelogram.

Solution: From (1.1.3.1),

$$\mathbf{A}-\mathbf{B}=\mathbf{D}-\mathbf{C}=\begin{pmatrix}-6\\-1\end{pmatrix} \tag{1.2.20.1}$$

Hence, $ABCD$ is a parallelogram. See Fig. 1.2.20.1.

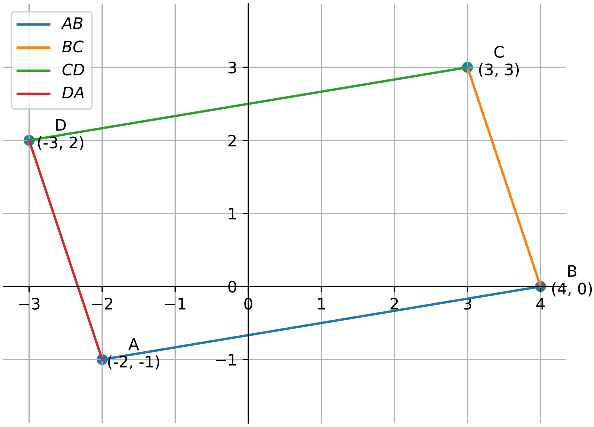


Fig. 1.2.20.1

1.2.21 The centroid of a triangle ABC is at the point $(1,1,1)$. If the coordinates of $\mathbf{A}$ and $\mathbf{B}$ are $(3,-5,7)$ and $(-1,7,-6)$, respectively find the coordinates of the point $\mathbf{C}$.

1.2.22 A girl walks 4 km towards west, then she walks 3 km in a direction 30° east of north and stops. Determine the girl's displacement from her initial point of departure.

Solution: See Fig. 1.2.22.1. Let the initial position be

$$\mathbf{A}=\begin{pmatrix}0\\0\end{pmatrix} \tag{1.2.22.1}$$

After going west, the position becomes

$$\mathbf{B}=\begin{pmatrix}-4\\0\end{pmatrix} \tag{1.2.22.2}$$

If the final position be $\mathbf{C}$, from the given information,

$$\mathbf{C}-\mathbf{B}=3\begin{pmatrix}\cos 60^\circ\\ \sin 60^\circ\end{pmatrix} \implies \mathbf{C}=\begin{pmatrix}-\frac{5}{2}\\ \frac{3\sqrt{3}}{2}\end{pmatrix} \tag{1.2.22.3}$$

which is the desired displacement.

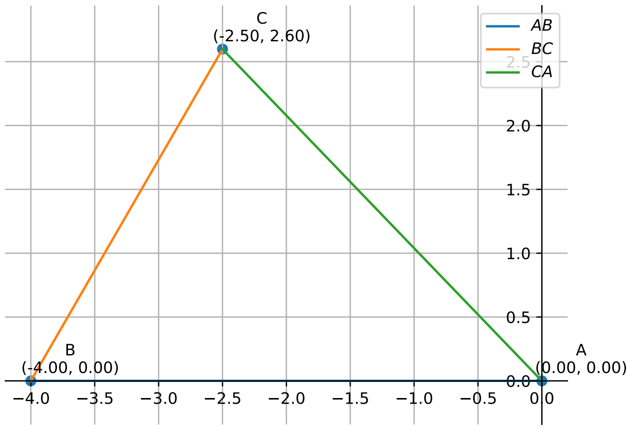


Fig. 1.2.22.1

1.2.23 Represent graphically a displacement of 40 km, 30° west of south.

1.2.24 Rain is falling vertically with a speed of 35 ms^{-1} . Winds starts blowing after sometime with a speed of 12 ms^{-1} in east to west direction. In which direction should a boy waiting at a bus stop hold his umbrella ?

1.2.25 A motorboat is racing towards north at 25 km/h and the water current in that region is 10 km/h in the direction of 60° east of south. Find the resultant velocity of the boat.

1.2.26 Rain is falling vertically with a speed of 35 ms^{-1} . A woman rides a bicycle with a speed of 12 ms^{-1} in east to west direction. What is the direction in which she should hold her umbrella ?

1.2.27 Rain is falling vertically with a speed of 30 ms^{-1}. A woman rides a bicycle with a speed of 10 ms^{-1} in the north to south direction. What is the direction in which she should hold her umbrella?

1.2.28 A man can swim with a speed of 4.0 km/h in still water. How long does he take to cross a river 1.0 km wide if the river flows steadily at 3.0 km/h and he makes his strokes normal to the river current? How far down the river does he go when he reaches the other bank ?

1.2.29 In a harbour, wind is blowing at the speed of 72 km/h and the flag on the mast of a boat anchored in the harbour flutters along the N-E direction. If the boat starts moving at a speed of 51 km/h to the north, what is the direction of the flag on the mast of the boat ?

1.3 CBSE

1.3.1 If in $\triangle ABC$, $\overrightarrow{BA} = 2\mathbf{a}$ and $\overrightarrow{BC} = 3\mathbf{b}$, then $\overrightarrow{AC}$ is ______. (12, 2023)

1.3.2 The coordinates of the three consecutive vertices of a parallelogram $ABCD$ are $\mathbf{A}(1,3)$, $\mathbf{B}(-1,2)$, and $\mathbf{C}(2,5)$. Find the coordinates of the fourth vertex $\mathbf{D}$. (10, 2021)

1.3.3 Points $\mathbf{A}(3,1)$, $\mathbf{B}(5,1)$, $\mathbf{C}(a,b)$, and $\mathbf{D}(4,3)$ are vertices of a parallelogram $ABCD$. Find the values of a and b. (10, 2019)

1.3.4 If $\mathbf{A}(1,3)$, $\mathbf{B}(-1,2)$, $\mathbf{C}(2,5)$ and $\mathbf{D}(x,4)$ are the vertices of a parallelogram $ABCD$, then the value of x is ______. (10, 2012)

1.3.5 If $(3,3),(6,y),(x,7)$ and $(5,6)$ are the vertices of a parallelogram taken in order, find the values of x and y. (10, 2011)

1.3.6 Show that the points $\mathbf{A}(6,2), \mathbf{B}(2,1), \mathbf{C}(1,5)$ and $\mathbf{D}(5,6)$ are the vertices of a square. (10, 2006)

1.3.7 Find the coordinates of the vertex $\mathbf{A}$ of a parallelogram $ABCD$ whose three vertices are given as $\mathbf{B}(0,0)$, $\mathbf{C}(3,0)$, and $\mathbf{D}(0,4)$. (10, 2024)

1.3.8 $ABCD$ is a rectangle formed by the points $\mathbf{A}(-1,-1), \mathbf{B}(-1,6), \mathbf{C}(3,6)$ and $\mathbf{D}(3,1), \mathbf{P}, \mathbf{Q}, \mathbf{R}$ and $\mathbf{S}$ are mid-points of sides AB, BC, CD and DA respectively. Show that diagonals of the quadrilateral $PQRS$ bisect each other. (10, 2024)

1.3.9 The center of a circle is at $(2,0)$. If one end of a diameter is at $(6,0)$, then find the other end. (10, 2024)

1.3.10 Find the ratio in which the point $(8,y)$ divides the line segment joining the points $(1,2)$ and $(2,3)$. Also, find the value of y. (10, 2024)

1.4 Section Formula

1.4.1 Find the coordinates of the point which divides the join of $(-1,7)$ and $(4,-3)$ in the ratio 2:3.

Solution: Using section formula (1.1.4.1), the desired point is

$$\frac{1}{1+\frac{3}{2}}\left(\begin{pmatrix}4\\-3\end{pmatrix}+\frac{3}{2}\begin{pmatrix}-1\\7\end{pmatrix}\right)=\begin{pmatrix}1\\3\end{pmatrix} \tag{1.4.1.1}$$

See Fig. 1.4.1.1

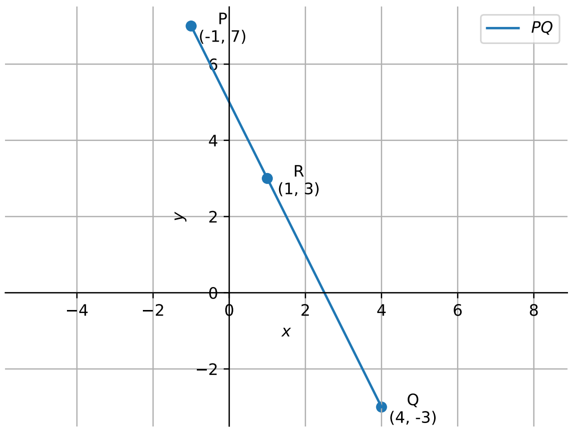


Fig. 1.4.1.1

1.4.2 Find the coordinates of the point $\mathbf{R}$ on the line segment joining the points $\mathbf{P}(-1,3)$ and $\mathbf{Q}(2,5)$ such that $PR = \frac{3}{5}PQ$.

1.4.3 Find the ratio in which the point $\mathbf{P}\left(\frac{3}{4}, \frac{5}{12}\right)$ divides the line segment joining the points $\mathbf{A}\left(\frac{1}{2}, \frac{3}{2}\right)$ and $\mathbf{B}(2,-5)$.

1.4.4 Find the coordinates of the point which divides the line segment joining the points $(4,-3)$ and $(8,5)$ in the ratio $3:1$ internally.

1.4.5 Find the coordinates of the point $\mathbf{P}$ on AD such that $AP : PD = 2 : 1$.

1.4.6 If the point $\mathbf{P}(2,1)$ lies on the line segment joining points $\mathbf{A}(4,2)$ and $\mathbf{B}(8,4)$, then

a) $AP = \frac{1}{3}AB$
b) $AP = PE$
c) $PB = \frac{1}{3}AB$
d) $AP = \frac{1}{2}AB$

1.4.7 Find the ratio in which the line segment joining the points $(-3,10)$ and $(6,-8)$ is divided by $(-1,6)$.

Solution: Using section formula,

$$\begin{pmatrix}-1\\6\end{pmatrix} = \frac{\begin{pmatrix}-3\\10\end{pmatrix} + k\begin{pmatrix}6\\-8\end{pmatrix}}{1+k} \tag{1.4.7.1}$$

$$\implies 7k\begin{pmatrix}1\\-2\end{pmatrix} = 2\begin{pmatrix}1\\-2\end{pmatrix} \tag{1.4.7.2}$$

$$\text{or, } k = \frac{2}{7}. \tag{1.4.7.3}$$

1.4.8 Find the position vector of the mid point of the vector joining the points $\mathbf{P}(2,\ 3,\ 4)$

and $\mathbf{Q}(4, 1, -2)$.
Solution: The desired vector is

$$\frac{1}{2}\begin{pmatrix}2\\3\\4\end{pmatrix}+\frac{1}{2}\begin{pmatrix}4\\1\\-2\end{pmatrix}=\begin{pmatrix}3\\2\\1\end{pmatrix} \tag{1.4.8.1}$$

1.4.9 If $\mathbf{A}$ and $\mathbf{B}$ are $(-2,-2)$ and $(2,-4)$, respectively, find the coordinates of $\mathbf{P}$ such that $AP = \frac{3}{7}AB$ and $\mathbf{P}$ lies on the line segment AB.
Solution: Using section formula,

$$\mathbf{P} = \frac{1}{1+\frac{3}{4}}\left(\begin{pmatrix}-2\\-2\end{pmatrix}+\frac{3}{4}\begin{pmatrix}2\\-4\end{pmatrix}\right)=\begin{pmatrix}\frac{-2}{7}\\\frac{-20}{7}\end{pmatrix} \tag{1.4.9.1}$$

1.4.10 Find the coordinates of the points of trisection of the line segment joining $(4,-1)$ and $(-2,3)$.
Solution: Using section formula,

$$\mathbf{R} = \frac{1}{1+\frac{1}{2}}\left(\begin{pmatrix}4\\-1\end{pmatrix}+\frac{1}{2}\begin{pmatrix}-2\\-3\end{pmatrix}\right)=\begin{pmatrix}2\\\frac{-5}{3}\end{pmatrix} \tag{1.4.10.1}$$

$$\mathbf{S} = \frac{1}{1+\frac{2}{1}}\left(\begin{pmatrix}4\\-1\end{pmatrix}+\frac{2}{1}\begin{pmatrix}-2\\-3\end{pmatrix}\right)=\begin{pmatrix}0\\\frac{-7}{3}\end{pmatrix} \tag{1.4.10.2}$$

which are the desired points of trisection.

1.4.11 Find the coordinates of the points which divide the line segment joining $A(-2,2)$ and $\mathbf{B}(2,8)$ into four equal parts.
Solution: Using section formula,

$$\mathbf{R}_k = \frac{\mathbf{B}+k\mathbf{A}}{1+k}, k = \frac{i}{n-i}, 0 < i < n \tag{1.4.11.1}$$

for $n = 4$. See Fig. 1.4.11.1.

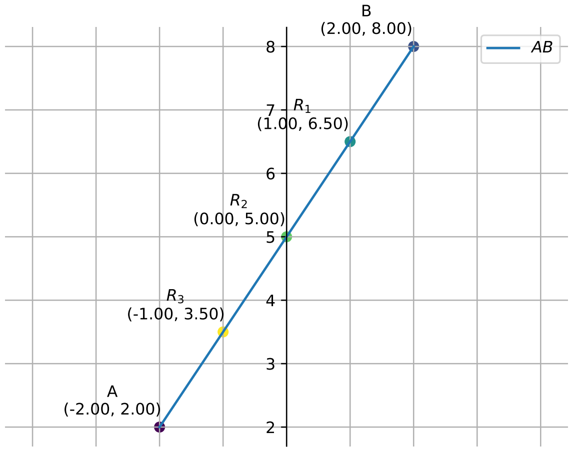


Fig. 1.4.11.1

1.4.12 In what ratio does the point $(-4, 6)$ divide the line segment joining the points $\mathbf{A}(-6, 0)$ and $\mathbf{B}(3, -8)$?

1.4.13 Given that $\mathbf{P}(3, 2, -4), \mathbf{Q}(5, 4, -6)$ and $\mathbf{R}(9, 8, -10)$ are collinear. Find the ratio in which $\mathbf{Q}$ divides PR.

1.4.14 Points $\mathbf{A}(-6, 10), \mathbf{B}(-4, 6)$ and $\mathbf{C}(3, -8)$ are collinear such that $AB = \frac{2}{9}AC$.

1.4.15 The point which divides the line segment joining the points $\mathbf{P}(7, –6)$ and $\mathbf{Q}(3, 4)$ in the ratio $1 : 2$ internally lies in which quadrant?

1.4.16 Find the coordinates of the points which trisect the line segment joining the points $\mathbf{P}(4, 2, -6)$ and $\mathbf{Q}(10, -16, 6)$.

1.4.17 Find the coordinates of the points of trisection (i.e. points dividing to three equal parts) of the line segment joining the points $\mathbf{A}(2, -2)$ and $\mathbf{B}(-7, 4)$.

1.4.18 Point $\mathbf{P}(5, -3)$ is one of the two points of trisection of line segment joining the points $\mathbf{A}(7, -2)$ and $\mathbf{B}(1, -5)$

1.4.19 Find the position vector of a point $\mathbf{R}$ which divides the line joining two points $\mathbf{P}$ and $\mathbf{Q}$ whose position vectors are $\hat{i} + 2\hat{j} - \hat{k}$ and $-\hat{i} + \hat{j} + \hat{k}$ respectively, in the ratio $2 : 1$

a) internally
b) externally

1.4.20 Find the coordinates of the point which divides the line segment joining the points which divides the line segment joining the points $(-2, 3, 5)$ and $(1, -4, 6)$ in the ratio

a) $2 : 3$ internally,
b) $2 : 3$ externally

1.4.21 Find the coordinates of the point which divides the line segment joining the points $(1, -2, 3)$ and $(3, 4, -5)$ in the ratio $2 : 3$

a) internally, and

b) externally

1.4.22 Let $\mathbf{A}(4,2), \mathbf{B}(6,5)$ and $\mathbf{C}(1,4)$ be the vertices of $\triangle ABC$.

a) The median from $\mathbf{A}$ meets BC at $\mathbf{D}$. Find the coordinates of the point $\mathbf{D}$.

b) Find the coordinates of points $\mathbf{Q}$ and $\mathbf{R}$ on medians BE and CF respectively such that $BQ : QE = 2 : 1$ and $CR : RF = 2 : 1$.

c) What do you observe?

d) If $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ are the vertices of $\triangle ABC$, find the coordinates of the centroid of the triangle.

Solution:

$$\mathbf{D} = \frac{\mathbf{B}+\mathbf{C}}{2} = \begin{pmatrix} \frac{7}{2} \\ \frac{9}{2} \end{pmatrix}, \ \mathbf{E} = \frac{\mathbf{A}+\mathbf{C}}{2} = \begin{pmatrix} \frac{5}{2} \\ 3 \end{pmatrix} \tag{1.4.22.1}$$

$$\mathbf{F} = \frac{\mathbf{A}+\mathbf{B}}{2} = \begin{pmatrix} 5 \\ \frac{7}{2} \end{pmatrix}, \ \mathbf{G} = \mathbf{Q} = \mathbf{R} = \frac{1}{3}\begin{pmatrix} 11 \\ 11 \end{pmatrix} \tag{1.4.22.2}$$

is the centroid. See Fig. 1.4.22.1.

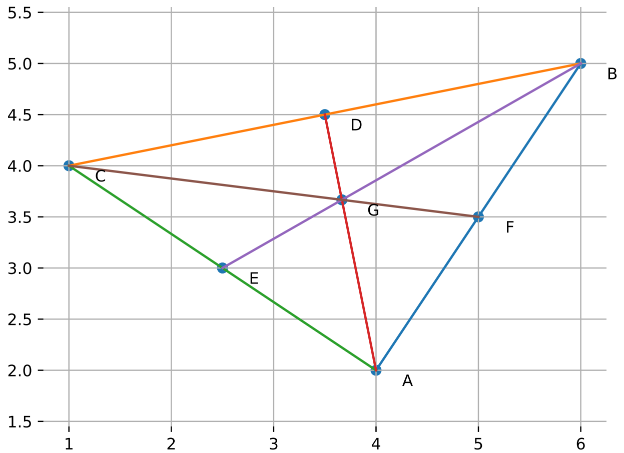


Fig. 1.4.22.1

1.4.23 Consider two points $\mathbf{P}$ and $\mathbf{Q}$ with position vectors $\overrightarrow{OP} = 3\vec{a} - 2\vec{b}$ and $\overrightarrow{OQ} = \vec{a} + \vec{b}$. Find the position vector of a point $\mathbf{R}$ which divides the line joining $\mathbf{P}$ and $\mathbf{Q}$ in the ratio $2 : 1$,

a) internally, and

b) externally.

1.4.24 If $\mathbf{P}(9a-2, -b)$ divides line segment joining $\mathbf{A}(3a+1, -3)$ and $\mathbf{B}(8a, 5)$ in the ratio 3:1, find the values of a and b.

1.4.25 Find the position vector of a point $\mathbf{R}$ which divides the line joining two points $\mathbf{P}$

and **Q** whose position vectors are $2\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - 3\mathbf{b}$ externally in the ratio 1 : 2.

1.4.26 The position vector of the point which divides the join of points $2\mathbf{a}$-$3\mathbf{b}$ and $\mathbf{a} + \mathbf{b}$ in the ratio 3:1 is ______.

1.4.27 If **a** and **b** are the postion vectors of **A** and **B**, respectively, find the position vector of a point **C** in BA produced such that $BC = 1.5BA$.

1.4.28 Find the position vector of a point **R** which divides the line joining two points **P** and **Q** whose position vectors are $(2\mathbf{a} + \mathbf{b})$ and $(\mathbf{a} - 3\mathbf{b})$ externally in the ratio 1 : 2. Also, show that **P** is the mid point of the line segment RQ.

1.5 CBSE

1.5.1 The centre of a circle whose end points of a diameter are $(-6, 3)$ and $(6, 4)$ is ______. (10, 2020)

1.5.2 Find the ratio in which the Y axis divides the line segment joining the points $(6, -4)$ and $(-2, -7)$. Also find the point of intersection. (10, 2020)

1.5.3 In what ratio does the X axis divide the line segment joining the points $\mathbf{A}(3, 6)$ and $\mathbf{B}(-12, -3)$? (10, 2023)

1.5.4 A circle has its center at $(4, 4)$. If one end of a diameter is $(4, 0)$, then find the coordinates of the other end. (10, 2022)

1.5.5 Find the coordinates of the point which divides the line segment joining the points $\mathbf{A}(7, -1)$ and $\mathbf{B}(-3, -4)$ in the ratio 2 : 3. (10, 2021)

1.5.6 The point which divides the line segment joining the points $(7, -6)$ and $(3, 4)$ in the ratio 1 : 2 is ______. (10, 2021)

1.5.7 If $\left(\frac{a}{3}, 4\right)$ is the midpoint of the line segment joining the points $(-6, 5)$ and $(-2, 3)$, then the value of a is (10, 2021)

1.5.8 Find the ratio in which $\mathbf{P}(4, 5)$ divides the line segment joining $\mathbf{A}(2, 3)$ and $\mathbf{B}(7, 8)$. (10, 2021)

1.5.9 Find the ratio in which the Y axis divides the line segment joining the points $\mathbf{A}(5, -6)$ and $\mathbf{B}(-1, -4)$. Also, find the coordinates of the point of intersection. (10, 2021)

1.5.10 Find the ratio in which the line segment joining the points $\mathbf{A}(1, -5)$ and $\mathbf{B}(-4, 5)$ is divided by the X axis. Also, find the coordinates of the point of division. (10, 2021)

1.5.11 The point **R** divides the line segment AB, where $\mathbf{A}(-4, 0)$ and $\mathbf{B}(0, 6)$ such that $AR = \frac{3}{4}AB$. Find the coordinates of **R**. (10, 2019)

1.5.12 In what ratio does the point $\mathbf{P}(-4, y)$ divide the line segment joining the points $\mathbf{A}(-6, 10)$ and $\mathbf{B}(3, -8)$? Hence, find the value of y. (10, 2019)

1.5.13 Find the ratio in which the Y axis divides the line segment joining the points $(-1, -4)$ and $(5, -6)$. Also find the coordinates of the point of intersection. (10, 2019)

1.5.14 Points **P** and **Q** trisect the line segment joining the points $\mathbf{A}(-2, 0)$ and $\mathbf{B}(0, 8)$ such that **P** is nearer to **A**. Find the coordinates of points **P** and **Q**. (10, 2019)

1.5.15 The midpoint of the line segment joining $\mathbf{A}(2a, 4)$ and $\mathbf{B}(-2, 3b)$ is $(1, 2a + 1)$. Find the values of a and b. (10, 2019)

1.5.16 Find the coordinates of a point **A** where AB is a diameter of the circle with center $(3, -1)$ and the point **B** is $(2, 6)$. (10, 2019)

1.5.17 The midpoint of the line segment joining $\mathbf{A}(2a, 4)$ and $\mathbf{B}(-2, 3b)$ is $(1, 2a + 1)$. Find the values of a and b. (10, 2019)

1.5.18 Find the coordinates of a point **A** where AB is the diameter of a circle whose center is $(2, -3)$ and **B** is the point $(1, 4)$. (10, 2019)

1.5.19 Find the ratio in which the segment joining the points $(1, 3)$ and $(4, 5)$ is divided by the X axis. Also find the coordinates of this point on the X axis. (10, 2019)

1.5.20 Find the coordinates of a point **A** where AB is a diameter of the circle with center $(-2, 2)$ and **B** is the point $(3, 4)$. (10, 2019)

1.5.21 Find the ratio in which $\mathbf{P}(4, m)$ divides the line segment joining the points $\mathbf{A}(2, 3)$ and $\mathbf{B}(6, -3)$. Hence, find m. (10, 2018)

1.5.22 **X** and **Y** are two points with position vectors $3\overrightarrow{a} + \overrightarrow{b}$ and $\overrightarrow{a} - 3\overrightarrow{b}$ respectively. Write the position vector of a point **V** which divides the line segment XY in the ratio 2 : 1 externally. (12, 2018)

1.5.23 Show that the points $\mathbf{A}\left(-2\hat{i} + 3\hat{j} + 5\hat{k}\right)$, $\mathbf{B}\left(\hat{i} + 2\hat{j} + 3\hat{k}\right)$ and $\mathbf{C}\left(7\hat{i} - \hat{k}\right)$ are collinear. (12, 2018)

1.5.24 A line intersects the Y axis and X axis at the points $\mathbf{P} = (0, b)$ and $\mathbf{Q} = (c, 0)$ respectively. If $(2, -5)$ is the midpoint of PQ, then find the coordinates of **P** and **Q**. (10, 2017)

1.5.25 In what ratio does the point $\left(\frac{24}{11}, y\right)$ divide the line segment joining the points $\mathbf{P} = (2, -2)$ and $\mathbf{Q} = (3, 7)$? Also find the value of y. (10, 2017)

1.5.26 Let **P** and **Q** be the points of trisection of the line segment joining the points $\mathbf{A}\left(2, -2\right)$ and $\mathbf{B}\left(-7, 4\right)$ such that **P** is nearer to **A**. Find the coordinates of **P** and **Q**. (10, 2016)

1.5.27 If the coordinates of points **A** and **B** are $(-2, -2)$ and $(2, -4)$ respectively, find the coordinates of **P** such that $AP = \frac{3}{7}AB$, and **P** lies on the line segment AB. (10, 2015)

1.5.28 $\mathbf{P}(5, -3)$ and $\mathbf{Q}(3, y)$ are the points of trisection of the line segment joining $\mathbf{A}(7, -2)$ and $\mathbf{B}(1, -5)$. Then y equals (10, 2012)

1.5.29 The coordinates of the point **P** dividing the line segment joining the points $\mathbf{A}(1, 3)$ and $\mathbf{B}(4, 6)$, in the ratio 2 : 1 are (10, 2012)

1.5.30 If the coordinates of one end of a diameter of a circle are $(2, 3)$ and the coordinates of its centre are $(-2, 5)$, then the coordinates of the other end of the diameter are ______. (10, 2012)

1.5.31 Find the coordinates of a point **P**, which lies on the line segment joining the points $\mathbf{A}(-2, 2)$ and $\mathbf{B}(2, -4)$ such that $AP = \frac{3}{7}AB$. (10, 2012)

1.5.32 Find the ratio in which the line segment joining the points $(1, -3)$ and $(4, 5)$ is divided by X axis. (10, 2012)

1.5.33 Find the ratio in which the Y axis divides the line segment joining the points $(5, -6)$ and $(-1, -4)$. Also find the coordinates of the point of intersection. (10, 2012)

1.5.34 The point **P** which divides the line segment joining the points $\mathbf{A}(2, -5)$ and $\mathbf{B}(5, 2)$ in the ratio 2:3 lies in which quadrant? (10, 2011)

1.5.35 The mid-point of segment AB is the point $\mathbf{P}(0, 4)$. If the coordinates of **B** are (-2, 3) then the coordinates of **A** are ______. (10, 2011)

1.5.36 Point $\mathbf{P}(x, 4)$ lies on the line segment joining the points $\mathbf{A}(-5, 8)$ and $\mathbf{B}(4, -10)$. Find the ratio in which point **P** divides the line segment AB. Also, find the value of x. (10, 2011)

1.5.37 The centre of a circle whose end points of a diameter are $(-6, 3)$ and $(6, 4)$ is ______. (10, 2020)

1.5.38 Find the ratio in which the Y axis divides the line segment joining the points $(6, -4)$ and $(-2, -7)$. Also find the point of intersection. (10, 2020)

1.5.39 The point $\mathbf{R}$ divides the line segment PQ in the ratio $3 : 1$ and $\mathbf{S}$ is the midpoint of the line segment PR. Find the position vector of $\mathbf{S}$ in terms of $\mathbf{P}$ and $\mathbf{Q}$. (12, 2024)

1.6 Rank

1.6.1 Prove that the three points (3, 0), (– 2, – 2) and (8, 2) are collinear.
Solution: From (1.1.9.1), the collinearity matrix can be expressed as

$$\begin{pmatrix} -5 & -2 \\ 5 & 2 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_1 + R_2} \begin{pmatrix} -5 & -2 \\ 0 & 0 \end{pmatrix} \tag{1.6.1.1}$$

which is a rank 1 matrix. The above process is known as row reduction, where we try to obtain zero rows in the matrix using arithmetic operations. The number of nonzero rows in the row reduced matrix (also known as *echelon form*) is defined as the rank. Fig. 1.6.1.1.

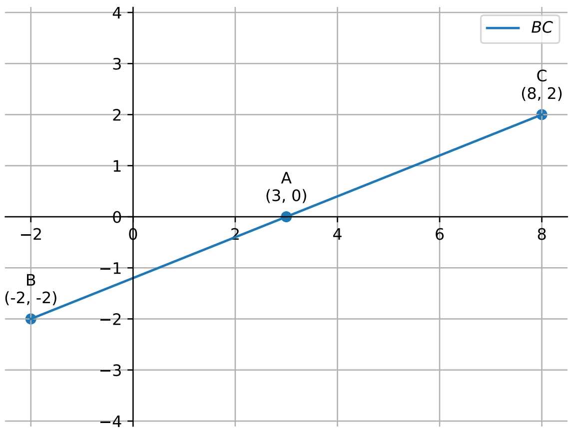


Fig. 1.6.1.1

1.6.2 Show that the points $\mathbf{A}(1, 2, 7), \mathbf{B}(2, 6, 3)$ and $\mathbf{C}(3, 10, -1)$ are collinear.
Solution: The matrix

$$\begin{pmatrix} \mathbf{B} - \mathbf{A} & \mathbf{C} - \mathbf{A} \end{pmatrix}^\top = \begin{pmatrix} 1 & 4 & -4 \\ 2 & 8 & -8 \end{pmatrix} \tag{1.6.2.1}$$

$$\xleftrightarrow{R_2 = R_2 - 2R_1} \begin{pmatrix} 1 & 4 & -4 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.6.2.2}$$

which has rank 1. Using (1.1.10.1), we conclude that the given points are collinear.

1.6.3 Determine if the points $(1,5),(2,3)$ and $(-2,-11)$ are collinear.

1.6.4 Show that the vectors $2\hat{i}-3\hat{j}+4\hat{k}$ and $-4\hat{i}+6\hat{j}-8\hat{k}$ are collinear.

1.6.5 Show that the points (2, 3, 4), (–1, –2, 1), (5, 8, 7) are collinear.

1.6.6 In each of the following, find the value of k, for which the points are collinear.

a) $(7,-2),(5,1),(3,k)$

b) $(8,1),(k,-4),(2,-5)$

1.6.7 Find a relation between x and y if the points $(x,y),(1,2)$ and $(7,0)$ are collinear.

1.6.8 If three points $(x,-1),(2,1)$ and $(4,5)$ are collinear, find the value of x.

1.6.9 If three points $(h,0),(a,b)$ and $(0,k)$ lie on a line, show that

$$\frac{a}{h}+\frac{b}{k}=1 \tag{1.6.9.1}$$

1.6.10 Show that the points $\mathbf{A}(1,-2,-8),\mathbf{B}(5,0,-2)$ and $\mathbf{C}(11,3,7)$ are collinear, and find the ratio in which $\mathbf{B}$ divides AC.

1.6.11 lf the points $\mathbf{A}(1,2),\mathbf{0}(0,0)$ and $\mathbf{C}(a,b)$ are collinear, then find the relation between a and b.

1.6.12 Point $(-4,2)$ lies on the line segment joining the points $\mathbf{A}(-4,6)$ and $\mathbf{B}(-4,-6)$.

1.6.13 The points $(0,5),(0,-9)$ and $(3,6)$ are collinear.

1.6.14 Points $\mathbf{A}(3,1),\mathbf{B}(12,-2)$ and $\mathbf{C}(0,2)$ cannot be the vertices of a triangle.

1.6.15 Find the value of m if the points $(5,1),(-2,-3)$ and $(8,2m)$ are collinear.

1.6.16 Find the values of k if the points $\mathbf{A}(k+1,2k),\mathbf{B}(3k,2k+3)$ and $\mathbf{C}(5k-1,5k)$ are collinear.

1.6.17 Using vectors, find the value of k such that the points $(k,-10,3)$, $(1,-1,3)$ and $(3,5,3)$ are collinear.

1.6.18 The points $\mathbf{A}(2,1)$, $\mathbf{B}(0,5)$, $\mathbf{C}(-1,2)$ are collinear.

1.6.19 The vectors $\lambda\hat{i}+\lambda\hat{j}+2\hat{k}$, $\hat{i}+\lambda\hat{j}-\hat{k}$ and $2\hat{i}-\hat{j}+\lambda\hat{k}$ are coplanar if $\lambda =$

1.6.20 Show that the points $(-2,3,5),(1,2,3)$ and $(7,0,-1)$ are collinear.

1.6.21 Show that points $\mathbf{A}(a,b+c),\mathbf{B}(b,c+a),\mathbf{C}(c,a+b)$ are collinear.

1.6.22 Show that the points $\mathbf{A}(2,-3,4),\mathbf{B}(-1,2,1)$ and $\mathbf{C}(0,\frac{1}{3},2)$ are collinear.

1.6.23 Are $\mathbf{A}(3,1),\mathbf{B}(6,4)$ and $\mathbf{C}(8,6)$ collinear?

1.6.24 Find the values of k if the points $\mathbf{A}(2,3),\mathbf{B}(4,k)$ and $\mathbf{C}(6,-3)$ are collinear.

1.6.25 Three points $\mathbf{P}(h,k),\mathbf{Q}(x_1,y_1)$ and $\mathbf{R}(x_2,y_2)$ lie on a line. Show that $(h-x_1)(y_2-y_1)=(k-y_1)(x_2-x_1)$.

1.6.26 Show that the points $\mathbf{P}(-2,3,5),\mathbf{Q}(1,2,3)$ and $\mathbf{R}(7,0,-1)$ are collinear.

1.6.27 Prove that the three points $(-4,6,10),(2,4,6)$ and $(14,0,-2)$ are collinear.

1.6.28 Show that the points $\mathbf{A}(-2\hat{i}+3\hat{j}+5\hat{k}),\mathbf{B}(\hat{i}+2\hat{j}+3\hat{k}$ and $\mathbf{C}(7\hat{i}-\hat{k})$ are collinear.

1.6.29 Show that the points $\mathbf{A}(2,3,-4),\mathbf{B}(1,-2,3)$ and $\mathbf{C}(3,8,-11)$ are collinear.

1.7 CBSE

1.7.1 The value of m which makes the points $(0,0)$, $(2m,-4)$, and $(3,6)$ collinear, is ______.
(10, 2022)

1.7.2 If $\mathbf{A}(1,2)$, $\mathbf{O}(0,0)$, and $\mathbf{C}(a,6)$ are collinear, then the value of a is ______.
(10, 2021)

1.7.3 Show that the points $\mathbf{A}(-2\hat{i} + 3\hat{j} + 5\hat{k})$, $\mathbf{B}(\hat{i} + 2\hat{j} + 3\hat{k})$, and $\mathbf{C}(7\hat{i} - \hat{k})$ are collinear. (12, 2019)

1.7.4 Using vectors, prove that the points $(2, -1, 3)$, $(3, -5, 1)$, and $(-1, 11, 9)$ are collinear. (12, 2019)

1.7.5 Find the value of p for which the points $(-5, 1)$, $(1, p)$, and $(4, -2)$ are collinear. (10, 2019)

1.7.6 Find a relation between x and y if the points $\mathbf{A}(x, y)$, $\mathbf{B}(-4, 6)$, and $\mathbf{C}(-2, 3)$ are collinear. (10, 2019)

1.7.7 For what value of p are the points $(2, 1)$, $(p, -1)$, and $(-1, 3)$ collinear? (10, 2019)

1.7.8 Using vectors, prove that the points $(2, -1, 3)$, $(3, -5, 1)$ and $(-1, 11, 9)$ are collinear. (12, 2018)

1.7.9 If the points $\mathbf{A} = (k + 1, 2k)$, $\mathbf{B} = (3k, 2k + 3)$, and $\mathbf{C} = (5k - 1, 5k)$ are collinear, then find the value of k. (10, 2017)

1.7.10 Find the relation between x and y if the points $\mathbf{A}(x, y)$, $\mathbf{B}(-5, 7)$ and $\mathbf{C}(-4, 5)$ are collinear. (10, 2015)

1.7.11 If the pair of equations $3x - y + 8 = 0$ and $6x - ry + 16 = 0$ represent coincident lines, then the value of r is ______. (10, 2023)

1.7.12 Find the value of k, if the points $\mathbf{P}(5, 4)$, $\mathbf{Q}(7, k)$ and $\mathbf{R}(9, -2)$ are collinear. (10, 2011)

1.7.13 Find the value of p for which the points (- 5, 1), $(1, p)$ and (4, - 2) are collinear. (10, 2006)

1.8 Length

1.8.1 Compute the magnitude of the following vectors:

$$\mathbf{a} = \hat{i} + \hat{j} + \hat{k} \tag{1.8.1.1}$$

$$\mathbf{b} = 2\hat{i} - 7\hat{j} - 3\hat{k} \tag{1.8.1.2}$$

$$\mathbf{c} = \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} - \frac{1}{3}\hat{k} \tag{1.8.1.3}$$

Solution: Let

$$\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 2 \\ -7 \\ 3 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{pmatrix} \tag{1.8.1.4}$$

Then

$$\mathbf{a}^\top \mathbf{a} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 3 \tag{1.8.1.5}$$

$$\implies \|\mathbf{a}\| = \sqrt{3}, \tag{1.8.1.6}$$

from (1.1.7.1). Similarly,

$$\|\mathbf{b}\| = \sqrt{\mathbf{b}^\top \mathbf{b}} = \sqrt{62}, \tag{1.8.1.7}$$

$$\|\mathbf{c}\| = \sqrt{\mathbf{c}^\top \mathbf{c}} = 1 \tag{1.8.1.8}$$

1.8.2 Find the distance between the following pairs of points

(i) $(2, 3, 5)$ and $(4, 3, 1)$

(ii) $(-3, 7, 2)$ and $(2, 4, -1)$

(iii) $(-1, 3, -4)$ and $(1, -3, 4)$

(iv) $(2, -1, 3)$ and $(-2, 1, 3)$

1.8.3 Find the lengths of the medians of the triangle with vertices $\mathbf{A}(0, 0, 6), \mathbf{B}(0, 4, 0)$ and $\mathbf{C}(6, 0, 0)$.

1.8.4 Find the coordinates of a point on Y axis which is at a distance of $5\sqrt{2}$ from the point $\mathbf{P}(3, -2, 5)$.

1.8.5 If $\mathbf{A}$ and $\mathbf{B}$ be the points $(3, 4, 5)$ and $(-1, 3, -7)$ respectively, find the equation of the set of the points $\mathbf{P}$ such that $PA^2 + PB^2 = K^2$ where K is a constant.

1.8.6 Find the distances between the following pairs of points

a) $(2, 3), (4, 1)$

b) $(-5, 7), (-1, 3)$

c) $(a, b), (-a, -b)$

Solution:

a)

$$\because \mathbf{A} - \mathbf{B} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}, \tag{1.8.6.1}$$

$$(\mathbf{A} - \mathbf{B})^\top (\mathbf{A} - \mathbf{B}) = 8 \tag{1.8.6.2}$$

Thus, the desired distance is

$$d = \|\mathbf{A} - \mathbf{B}\| = \sqrt{8} \tag{1.8.6.3}$$

b)

$$\mathbf{C} - \mathbf{D} = \begin{pmatrix} -5 \\ 7 \end{pmatrix} - \begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} -4 \\ 4 \end{pmatrix} \tag{1.8.6.4}$$

$$\implies (\mathbf{C} - \mathbf{D})^\top (\mathbf{C} - \mathbf{D}) = 32 \tag{1.8.6.5}$$

Thus,

$$d = \|\mathbf{C} - \mathbf{D}\| = 4\sqrt{2} \tag{1.8.6.6}$$

c)

$$\mathbf{E} - \mathbf{F} = \begin{pmatrix} a \\ b \end{pmatrix} - \begin{pmatrix} -a \\ -b \end{pmatrix} = \begin{pmatrix} 2a \\ 2b \end{pmatrix} \tag{1.8.6.7}$$

$$\implies (\mathbf{E} - \mathbf{F})^\top (\mathbf{E} - \mathbf{F}) = 4a^2 + 4b^2 \tag{1.8.6.8}$$

Thus,

$$d = \|\mathbf{E} - \mathbf{F}\| = 2\sqrt{a^2 + b^2} \tag{1.8.6.9}$$

1.8.7 Find the distance between the points $(0, 0)$ and $(36, 15)$.
Solution:

$$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ \mathbf{B} = \begin{pmatrix} 36 \\ 15 \end{pmatrix} \tag{1.8.7.1}$$

$$\implies \mathbf{d} = \|\mathbf{A} - \mathbf{B}\| = 39 \tag{1.8.7.2}$$

1.8.8 The distance between the points $\mathbf{A}(0, 6)$ and $\mathbf{B}(0, -2)$ is ______.

1.8.9 The distance of the point $\mathbf{P}(-6, 8)$ from the origin is ______.

1.8.10 The distance between the points $(0, 5)$ and $(-5, 0)$ is ______.

1.8.11 $AOBC$ is a rectangle whose three vertices are vertices $\mathbf{A}(0, 3), \mathbf{O}(0, 0)$ and $\mathbf{B}(5, 0)$. The length of its diagonal is ______.

1.8.12 The perimeter of a triangle with vertices $(0, 4), (0, 0)$ and $(3, 0)$ is ______.

1.8.13 If the distance between the points $(4, P)$ and $(1, 0)$ is 5, then the value of P is ______.

1.8.14 Find the points on the X axis which are at a distance on $2\sqrt{5}$ from the point $(7, -4)$. How many such points are there?

1.8.15 Find the value of a, if the if the distance between the points $\mathbf{A}(-3, -14)$ and $\mathbf{B}(a, -5)$ is 9 units.

1.8.16 Find a vector in the direction of vector $\overrightarrow{a} = \hat{i} - 2\hat{j}$ that has magnitude 7 units.

1.8.17 Find the point on the X axis which is equidistant from $(2, -5)$ and $(-2, 9)$.
Solution: The input parameters for this problem are available in Table 1.8.17

Symbol	Value	Description
$\mathbf{A}$	$\begin{pmatrix} 2 \\ -5 \end{pmatrix}$	First point
$\mathbf{B}$	$\begin{pmatrix} -2 \\ 9 \end{pmatrix}$	Second point
$\mathbf{O}$	?	Desired point

TABLE 1.8.17

If $\mathbf{O}$ lies on the x-axis and is equidistant from the points $\mathbf{A}$ and $\mathbf{B}$,

$$\|\mathbf{O} - \mathbf{A}\| = \|\mathbf{O} - \mathbf{B}\| \tag{1.8.17.1}$$

$$\implies \|\mathbf{O} - \mathbf{A}\|^2 = \|\mathbf{O} - \mathbf{B}\|^2 \tag{1.8.17.2}$$

$$\implies \|\mathbf{O}\|^2 - 2\mathbf{O}^\top\mathbf{A} + \|\mathbf{A}\|^2 = \|\mathbf{O}\|^2 - 2\mathbf{O}^\top\mathbf{B} + \|\mathbf{B}\|^2, \tag{1.8.17.3}$$

which can be simplified to obtain

$$(\mathbf{A} - \mathbf{B})^\top \mathbf{O} = \frac{\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2}{2}. \tag{1.8.17.4}$$

$$\because \mathbf{O} = x\mathbf{e}_1, \tag{1.8.17.5}$$

$$x = \frac{\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2}{2\,(\mathbf{A} - \mathbf{B})^\top \mathbf{e}_1.} \tag{1.8.17.6}$$

Substituting from Table 1.8.17 in (1.8.17.6), $x = -7$. Thus,

$$\mathbf{O} = \begin{pmatrix} -7 \\ 0 \end{pmatrix}. \tag{1.8.17.7}$$

See Fig. 1.8.17.1.

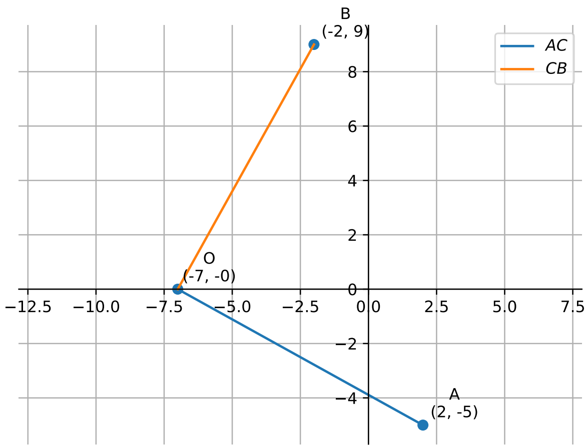


Fig. 1.8.17.1

1.8.18 Find the values of y for which the distance between the points $\mathbf{P}(2, -3)$ and $\mathbf{Q}(10, y)$ is 10 units.

1.8.19 If $\mathbf{Q}(0, 1)$ is equidistant from $\mathbf{P}(5, -3)$ and $\mathbf{R}(x, 6)$, find the values of x. Also find the distances QR and PR.

1.8.20 Find a relation between x and y such that the point (x, y) is equidistant from the point $(3, 6)$ and $(–3, 4)$.

1.8.21 Find a point on the X axis, which is equidistant from the points $\begin{pmatrix} 7 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

1.8.22 Find a point which is equidistant from the points $\mathbf{A}(-5, 4)$ and $\mathbf{B}(-1, 6)$. How many such points are there ?

1.8.23 If the point $\mathbf{A}(2, -4)$ is equidistant from $\mathbf{P}(3, 8)$ and $\mathbf{Q}(-10, y)$, find the values of y. Also find distance PQ.

1.8.24 If (a, b) is the mid-point of the line segment joining the point $\mathbf{A}(10, -6)$ and $\mathbf{B}(k, 4)$ and $a - 2b = 18$, find the value of a, b and the distance AB.

1.8.25 Find a relation between x and y such that the point (x, y) is equidistant from the points $(7, 1)$ and $(3, 5)$.

1.8.26 Find a point on the Y axis which is equidistant from the points $\mathbf{A}(6, 5)$ and $\mathbf{B}(-4, 3)$.

1.8.27 Find the equation of set of points **P** such that $PA^2 + PB^2 = 2k^2$, where **A** and **B** are the points $(3, 4, 5)$ and $(-1, 3, -7)$, respectively.

1.8.28 Find the equation of the set of the points **P** such that its distances from the points $\mathbf{A}(3, 4, -5)$ and $\mathbf{B}(-2, 1, 4)$ are equal.

1.9 CBSE

1.9.1 The distance between the points $(m, -n)$ and $(-m, n)$ is ______. (10, 2020)

1.9.2 The point on the X axis which is equidistant from $(-4, 0)$ and $(10, 0)$ is ______. (10, 2020)

1.9.3 $AOBC$ is a rectangle whose three vertices are $(0, -3)$, $(0, 0)$ and $(4, 0)$. The length of its diagonal is ______. (10, 2020)

1.9.4 If $\left|\overrightarrow{a}\right| = 4$ and $-3 \le \lambda \le 2$, then $\left|\lambda\overrightarrow{a}\right|$ lies in

a) $[0, 12]$
b) $[2, 3]$
c) $[8, 12]$
d) $[-12, 8]$

(12, 2020)

1.9.5 The distance between the point $(0, 2\sqrt{5})$ and $(-2\sqrt{5}, 0)$ is ______. (10, 2023)

1.9.6 If $\mathbf{Q} = (0, 1)$ is equidistant from $\mathbf{P} = (5, -3)$ and $\mathbf{R} = (x, 6)$, find the value of x. (10, 2023)

1.9.7 The distance of the point $(-6, 8)$ from the origin is ______. (10, 2023)

1.9.8 The distance between the points $(0, 0)$ and $(a - b, a + b)$ is ______. (10, 2022)

1.9.9 Find the distance between the points $\mathbf{A}\left(-\frac{7}{3}, 5\right)$ and $\mathbf{B}\left(\frac{2}{3}, 5\right)$. (10, 2021)

1.9.10 The distance between the points $\mathbf{A}(0, 6)$ and $\mathbf{B}(0, -2)$ is ______. (10, 2021)

1.9.11 If the distance between the points $(k, -2)$ and $(3, -6)$ is 10 units, find the positive value of k. (10, 2021)

1.9.12 Find the length of the segment joining $\mathbf{A}(-6, 7)$ and $\mathbf{B}(-1, -5)$. Also, find the midpoint of AB. (10, 2021)

1.9.13 A man goes 5 meters due west and then 12 meters due north. How far is he from the starting point? (10, 2021)

1.9.14 If $\mathbf{P}(2, 2)$, $\mathbf{Q}(-4, -4)$, and $\mathbf{R}(5, -8)$ are the vertices of a triangle $\triangle PQR$, then find the length of the median through **R**. (10, 2021)

1.9.15 If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are position vectors of the points $\mathbf{A}(2, 3, -4)$, $\mathbf{B}(3, -4, -5)$, and $\mathbf{C}(3, 2, -3)$ respectively, then $|\mathbf{a} + \mathbf{b} + \mathbf{c}|$ is equal to ______. (12, 2021)

1.9.16 Find the distance between the points (a, b) and $(-a, -b)$. (10, 2019)

1.9.17 Write the coordinates of a point **P** on the x-axis which is equidistant from the points $\mathbf{A}(-2, 0)$ and $\mathbf{B}(6, 0)$. (10, 2019)

1.9.18 Find the value of x if the distance between the points $\mathbf{A}(0, 0)$ and $\mathbf{B}(x, -4)$ is 5 units. (10, 2019)

1.9.19 Find the values of x for which the distance between the points $\mathbf{A}(x, 2)$ and $\mathbf{B}(9, 8)$ is 10 units. (10, 2019)

1.9.20 Find the point on the Y axis which is equidistant from the points $(5, -2)$ and $(-3, 2)$. (10, 2019)

1.9.21 Given vertices of a parallelogram $\mathbf{A}(-2,1)$, $\mathbf{B}(a,0)$, $\mathbf{C}(4,b)$, and $\mathbf{D}(1,2)$. Find the values of a and b. Hence, find the lengths of its sides. (10, 2018)

1.9.22 Find the value of y for which the distance between the points $\mathbf{P}(2,-3)$ and $\mathbf{Q}(10,y)$ is 10 units. (10, 2018)

1.9.23 If the point $\mathbf{P}(0,2)$ is equidistant from the points $\mathbf{Q}(3,k)$ and $\mathbf{R}(k,5)$, find the value of k. (10, 2018)

1.9.24 The x-coordinate of a point $\mathbf{P}$ is twice its y-coordinate. If $\mathbf{P}$ is equidistant from the points $\mathbf{Q}(2,-5)$ and $\mathbf{R}(-3,6)$, find the coordinates of $\mathbf{P}$. (10, 2018)

1.9.25 If the point $\mathbf{P}\left(x,y\right)$ is equidistant from the points $\mathbf{A}\left(a+b,b-a\right)$ and $\mathbf{B}\left(a-b,a+b\right)$, prove that $bx = ay$. (10, 2016)

1.9.26 Find the value of k, if the point $\mathbf{P}(2,4)$ is equidistant from the points $\mathbf{A}(5,k)$ and $\mathbf{B}(k,7)$. (10, 2012)

1.9.27 If a point $\mathbf{A}(0,2)$ is equidistant from the points $\mathbf{B}(3,p)$ and $\mathbf{C}(p,5)$, then find the value of p. (10, 2012)

1.9.28 If $\mathbf{A}$ and $\mathbf{B}$ are the points $(-6,7)$ and $(-1,-5)$ respectively, then the distance $2AB$ is equal to ______. (10, 2011)

1.9.29 Find the value of y for which the distance between the points $\mathbf{A}(3,-1)$ and $\mathbf{B}(11,y)$ is 10 units. (10, 2011)

1.9.30 If the distances of $\mathbf{P} = (x,y)$ from $\mathbf{A} = (5,1)$ and $\mathbf{B} = (-1,5)$ are equal, then prove that $3x = 2y$. (10, 2017)

1.9.31 AD is a median of $\triangle ABC$ with vertices $\mathbf{A}(5,-6)$, $\mathbf{B}(6,4)$, and $\mathbf{C}(0,0)$. Find the length of AD. (10, 2024)

1.9.32 If the distance between the points $(3,-5)$ and $(x,-5)$ is 15 units, then find the values of x. (10, 2024)

1.9.33 If $\mathbf{Q}(0,1)$ is equidistant from $\mathbf{P}(5,-3)$ and $\mathbf{R}(x,6)$, find the values of x.

1.9.34 Find the distance of the point $(-6,8)$ from origin. (10, 2024)

1.10 Unit Vector

1.10.1 For given vectors, $\mathbf{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\mathbf{b} = -\hat{i} + \hat{j} - \hat{k}$, find the unit vector in the direction of the vector $\mathbf{a} + \mathbf{b}$.

Solution:

$$\because \mathbf{a}+\mathbf{b} = \begin{pmatrix}2\\-1\\2\end{pmatrix} + \begin{pmatrix}-1\\1\\-1\end{pmatrix}, = \begin{pmatrix}1\\0\\1\end{pmatrix}, \tag{1.10.1.1}$$

$$\|\mathbf{a}+\mathbf{b}\| = \sqrt{2} \tag{1.10.1.2}$$

$$\implies \frac{\mathbf{a}+\mathbf{b}}{\|\mathbf{a}+\mathbf{b}\|} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix} \tag{1.10.1.3}$$

which, from (1.1.8.1) is the desired the unit vector.

1.10.2 Find the unit vector in the direction of sum of vectors $\mathbf{a}= 2\hat{i} - \hat{j} + \hat{k}$ and $\mathbf{b} = 2\hat{j} + \hat{k}$.

1.10.3 If $\mathbf{a}=\hat{i} + \hat{j} + 2\hat{k}$ and $\mathbf{b}=2\hat{i} + \hat{j} - 2\hat{k}$, find the unit vector in the direction of

a) 6**a**

b) 2**a**-**b**

1.10.4 Find the value of x for which $x(\hat{i}+\hat{j}+\hat{k})$ is a unit vector.
Solution:

$$\because \mathbf{x} = x\begin{pmatrix}1\\1\\1\end{pmatrix}, \|\mathbf{x}\| = 1 \implies x\sqrt{3} = 1 \tag{1.10.4.1}$$

$$\text{or, } x = \frac{1}{\sqrt{3}} \tag{1.10.4.2}$$

1.10.5 Find a vector in the direction of vector $5\hat{i}-\hat{j}+2\hat{k}$ which has magnitude 8 units.
Solution: Let the required vector be

$$c\begin{pmatrix}5\\-1\\2\end{pmatrix}. \tag{1.10.5.1}$$

From the given information,

$$\left\|c\begin{pmatrix}5\\-1\\2\end{pmatrix}\right\| = 8 \tag{1.10.5.2}$$

$$\implies |c| = \frac{4\sqrt{30}}{15} \tag{1.10.5.3}$$

1.10.6 Find a vector of magnitude 5 units, and parallel to the resultant of the vectors $\mathbf{a} = 2\hat{i}+3\hat{j}-\hat{k}$ and $\mathbf{b} = \hat{i}-2\hat{j}+\hat{k}$.
Solution:

$$\because \mathbf{a} = \begin{pmatrix}2\\3\\-1\end{pmatrix}, \mathbf{b} = \begin{pmatrix}1\\-2\\1\end{pmatrix} \tag{1.10.6.1}$$

$$\mathbf{a}+\mathbf{b} = \begin{pmatrix}3\\1\\0\end{pmatrix} \implies \|\mathbf{a}+\mathbf{b}\| = \sqrt{10} \tag{1.10.6.2}$$

From problem 1.10.1, the unit vector in the direction of $\mathbf{a}+\mathbf{b}$ is

$$\frac{\mathbf{a}+\mathbf{b}}{\|\mathbf{a}+\mathbf{b}\|} = \frac{1}{\sqrt{10}}\begin{pmatrix}3\\1\\0\end{pmatrix} \tag{1.10.6.3}$$

The desired vector can then be expressed as

$$\pm\frac{5}{\sqrt{10}}\begin{pmatrix}3\\1\\0\end{pmatrix} \tag{1.10.6.4}$$

1.10.7 Find a unit vector in the direction of $\overline{PQ}$, where **P** and **Q** have co-ordinates (5, 0, 8) and (3, 3, 2), respectively.

1.10.8 Find the unit vector in the direction of the vector $\mathbf{a} = \hat{i}+\hat{j}+2\hat{k}$.

1.10.9 Find the unit vector in the direction of vector $\overrightarrow{PQ}$, where **P** and **Q** are the points

(1, 2, 3) and (4, 5, 6), respectively.

1.10.10 The vector in the direction of the vector $\hat{i} - 2\hat{j} + 2\hat{k}$ that has magnitude 9 is

a) $\hat{i} - 2\hat{j} + 2\hat{k}$
b) $\hat{i} - 2\hat{j}$
c) $3(\hat{i} - 2\hat{j} + 2\hat{k})$
d) $9(\hat{i} - 2\hat{j} + 2\hat{k})$

1.10.11 Find a vector of magnitude 5 units, and parallel to the resultant of the vectors $\mathbf{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\mathbf{b} = \hat{i} - 2\hat{j} + \hat{k}$.

1.10.12 If $\mathbf{a} = \hat{i} + \hat{j} + \hat{k}, \mathbf{b} = 2\hat{i} - \hat{j} + 3\hat{k}$ and $\mathbf{c} = \hat{i} - 2\hat{j} + \hat{k}$, find a unit vector parallel to the vector $2\mathbf{a} - \mathbf{b} + 3\mathbf{c}$.

Solution:

$$2\mathbf{a} - \mathbf{b} + 3\mathbf{c} = \begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix} \implies \frac{2\mathbf{a} - \mathbf{b} + 3\mathbf{c}}{\|2\mathbf{a} - \mathbf{b} + 3\mathbf{c}\|} = \frac{1}{\sqrt{22}} \begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix} \tag{1.10.12.1}$$

1.10.13 If a line makes angles $90°, 135°, 45°$ with X, Y and Z axis respectivly. Find its direction cosines.

Solution: From (1.1.11.1), the direction vector is

$$\mathbf{A} = \begin{pmatrix} \cos 90° \\ \cos 135° \\ \cos 45° \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \tag{1.10.13.1}$$

1.10.14 Find the direction cosines of the vector joining the points **A** (1, 2, –3) and **B**(–1, –2, 1), directed from **A** to **B**.

Solution: The unit vector in the direction of AB is

$$\frac{\mathbf{B} - \mathbf{A}}{\|\mathbf{B} - \mathbf{A}\|} = \frac{1}{3} \begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix} \tag{1.10.14.1}$$

and the direction cosines are the elements of the above vector.

1.10.15 Show that the vector $\hat{i} + \hat{j} + \hat{k}$ is equally inclined to the axes OX, OY and OZ.

Solution: Since all entries of the given vector

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{1.10.15.1}$$

are equal, it is equally inclined to the axes.

1.10.16 Find the unit vector in the direction of vector $\vec{a} = 2\hat{i} + 3\hat{j} + \hat{k}$.

1.10.17 Find the unit vector in the direction of the sum of the vectors, $\vec{a} = 2\hat{i} + 2\hat{j} - 5\hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} + 3\hat{k}$.

1.10.18 Write the direction ratios of the vector $\vec{a} = \hat{i} + \hat{j} - \hat{k}$ and hence calculate its direction cosines.

1.10.19 If a line has direction ratios $2, -1, -2$, determine its direction cosines.

1.10.20 Find the direction cosines of the line passing through the two points $(-2, 4, -5)$ and

$(1, 2, 3)$.

1.10.21 If a line has the direction ratios –18, 12, –4, then what are its direction cosines?

Solution: Let

$$\mathbf{A} = \begin{pmatrix} -18 \\ 12 \\ -4 \end{pmatrix} \tag{1.10.21.1}$$

Then the unit direction vector of the line is

$$\frac{\mathbf{A}}{\|\mathbf{A}\|} = \begin{pmatrix} \frac{-9}{11} \\ \frac{6}{11} \\ \frac{-2}{11} \end{pmatrix} \tag{1.10.21.2}$$

1.10.22 Find the direction cosines of the sides of a triangle whose vertices are $\begin{pmatrix} 3 \\ 5 \\ -4 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -5 \\ -5 \\ -2 \end{pmatrix}$.

Solution: Let the vertices be

$$\mathbf{A} = \begin{pmatrix} 3 \\ 5 \\ -4 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -5 \\ -5 \\ -2 \end{pmatrix} \tag{1.10.22.1}$$

The direction vectors of the sides are,

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} 4 \\ 4 \\ -6 \end{pmatrix} = \mathbf{m_1}, \mathbf{B} - \mathbf{C} = \begin{pmatrix} 4 \\ 6 \\ 4 \end{pmatrix} = \mathbf{m_2}, \tag{1.10.22.2}$$

$$\mathbf{C} - \mathbf{A} = \begin{pmatrix} -8 \\ -10 \\ 2 \end{pmatrix} = \mathbf{m_3}, \tag{1.10.22.3}$$

The corresponding unit vectors are then obtained as

$$\begin{pmatrix} \frac{2}{\sqrt{17}} \\ \frac{2}{\sqrt{17}} \\ \frac{-3}{\sqrt{17}} \end{pmatrix}, \begin{pmatrix} \frac{2}{\sqrt{17}} \\ \frac{3}{\sqrt{17}} \\ \frac{2}{\sqrt{17}} \end{pmatrix}, \begin{pmatrix} \frac{-4}{\sqrt{42}} \\ \frac{-5}{\sqrt{42}} \\ \frac{1}{\sqrt{42}} \end{pmatrix} \tag{1.10.22.4}$$

1.10.23 Find the direction cosines of the vector $\hat{i} + 2\hat{j} + 3\hat{k}$.

Solution: The unit vector in the direction of the given vector is

$$\mathbf{A} = \frac{1}{\sqrt{14}} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \tag{1.10.23.1}$$

1.10.24 Find the direction cosines of the unit vector perpendicular to the plane $\overrightarrow{r} \cdot (6\hat{i} - 3\hat{j} - 2\hat{k}) + 1 = 0$ passing through the origin.

1.10.25 If a line makes angle $90°, 60°$ and $30°$ with the positive direction of X, Y and Z axes respectively, find its direction cosines.

1.10.26 Find the direction cosines of a line which makes equal angles with the coordinate axes.

Solution: Let α be the angle made by the line with the axes. The unit direction vector can be expressed as

$$\mathbf{x} = \begin{pmatrix} \cos\alpha \\ \cos\alpha \\ \cos\alpha \end{pmatrix} \implies \|\mathbf{x}\| = 1 \tag{1.10.26.1}$$

$$\text{or,} \quad \cos\alpha = \frac{1}{\sqrt{3}} \tag{1.10.26.2}$$

Thus the unit direction vector of the given line is

$$\mathbf{x} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{1.10.26.3}$$

1.10.27 If a unit vector $\overrightarrow{a}$ makes angles $\frac{\pi}{3}$ with $\hat{i}$, $\frac{\pi}{4}$ with $\hat{j}$ and an acute angle θ with $\hat{k}$, then find θ and hence, the components of $\overrightarrow{a}$.

Solution: From the given information,

$$\mathbf{a} = \begin{pmatrix} \cos\frac{\pi}{3} \\ \cos\frac{\pi}{4} \\ \cos\theta \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \cos\theta \end{pmatrix} \tag{1.10.27.1}$$

$$\because \|\mathbf{a}\| = 1, \tag{1.10.27.2}$$

$$\frac{1}{4} + \frac{1}{2} + \cos^2\theta = 1 \tag{1.10.27.3}$$

$$\implies \cos\theta = \frac{1}{2} \tag{1.10.27.4}$$

$\because$ θ is an acute angle. Hence

$$\mathbf{a} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix} \tag{1.10.27.5}$$

1.10.28 Write down a unit vector in XY-plane, making an angle of $30°$ with the positive direction of X axis.

1.10.29 A vector $\mathbf{r}$ is inclined at equal angles to the three axis. If the magnitude of $\mathbf{r}$ is $2\sqrt{3}$ units, find $\mathbf{r}$.

1.10.30 The direction cosines of the vector $(2\hat{i} + 2\hat{j} - \hat{k})$ are ____________.

1.10.31 A vector $\mathbf{r}$ has a magnitude 14 and direction ratios 2, 3, -6. Find the direction cosines and components of $\mathbf{r}$, given that $\mathbf{r}$ makes an acute angle with X axis.

1.11 CBSE

1.11.1 Find a vector $\overrightarrow{r}$ equally inclined to the three axes and whose magnitude is $3\sqrt{3}$ units. (12, 2020)

1.11.2 Unit vector along PQ, where coordinates of **P** and **Q** respectively are (2, 1, -1) and (4, 4, -7), is (12, 2023)

1.11.3 If a line makes 60° and 45° angles with the positive directions of the X axis and Z axis respectively, then find the angle that it makes with the positive direction of the Y-axis. Hence, write the direction cosines of the line. (12, 2023)

1.11.4 A vector of magnitude 9 units in the direction of the vector $-2\hat{i} - \hat{j} + 2\hat{k}$ is ______.

1.11.5 The scalar product of the vector $\overrightarrow{a} = \hat{i} + \hat{j} + \hat{k}$ with a unit vector along the sum of the vectors $\overrightarrow{b} = 2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\overrightarrow{c} = \hat{\lambda} + 2\hat{j} + 3\hat{k}$ is equal to 1. Find the value of λ and hence find the unit vector along $\overrightarrow{b} + \overrightarrow{c}$. (12, 2019)

1.11.6 Find the direction cosines of a line which makes equal angles with the coordinate axes. (12, 2019)

1.11.7 If a line has the direction ratios $-18, 12, -4$, then what are its direction cosines? (12, 2019)

1.11.8 Find the direction cosines of the line joining the points $\mathbf{P}(4, 3, -5)$ and $\mathbf{Q}(-2, 1, -8)$. (12, 2019)

1.11.9 Find a unit vector perpendicular to both the vectors $\overrightarrow{a} = \hat{i}-7\hat{j}+7\hat{k}$ and $\overrightarrow{b} = 3\hat{i}-2\hat{j}+2\hat{k}$. (12, 2018)

1.11.10 Find the direction cosines of the line joining points **P** $(4, 3, -5)$ and **Q** $(-2, 1, 8)$. (12, 2018)

1.11.11 The scalar product of the vector $\overrightarrow{a} = \hat{i} + \hat{j} + \hat{k}$ with a unit vector along the sum of the vector $\overrightarrow{b} = 2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\overrightarrow{c} = \lambda\hat{i} + 2\hat{j} + 3\hat{k}$ is equal to 1. Find the value of λ and hence find the unit vector along $\overrightarrow{b} + \overrightarrow{c}$. (12, 2018)

1.11.12 If the sum of two unit vectors is a unit vector, prove that the magnitude of their difference is $\sqrt{3}$. (12, 2018)

1.11.13 If a line makes angles $90°, 135°, 45°$ with the x,y and z axes respectively,find its direction cosines. (12, 2018)

1.11.14 If $\overrightarrow{a} = 4\hat{i} - \hat{j} + \hat{k}$ and $\overrightarrow{b} = 2\hat{i} - 2\hat{j} + \hat{k}$, then find a unit vector parallel to the vector $\overrightarrow{a} + \overrightarrow{b}$. (12, 2016)

1.11.15 Write the direction ratios of the vector $3\mathbf{a} + 2\mathbf{b}$ where $\mathbf{a} = \overrightarrow{i} + \overrightarrow{j} - 2\overrightarrow{k}$ and $\mathbf{b} = 2\overrightarrow{i} - 4\overrightarrow{j} + 5\overrightarrow{k}$. (12, 2015)

1.11.16 The Cartesian equation of a line AB is $\frac{2x-1}{12} = \frac{y+2}{2} = \frac{z-3}{3}$. Find the direction cosines of a line parallel to line AB. (12, 2023)

1.11.17 Find the direction cosines of a line whose Cartesian equation is given as $3x + 1 = 6y - 2 = 1 - z$. (12, 2022)

2.1 Formulae

2.1.1. The angle θ between $\mathbf{a}, \mathbf{b}$, is given by

$$\cos\theta = \frac{\mathbf{a}^\top \mathbf{b}}{\|\mathbf{a}\| \, \|\mathbf{b}\|} \tag{2.1.1.1}$$

2.1.2. The equation of a line is given by

$$\mathbf{x} = \mathbf{h} + \kappa \mathbf{m} \tag{2.1.2.1}$$

2.1.3. For

$$\mathbf{m}^\top \mathbf{n} = 0, \tag{2.1.3.1}$$

which means that $\mathbf{m} \perp \mathbf{n}$, (2.1.2.1) can be expressed as

$$\mathbf{n}^\top \mathbf{x} = \mathbf{n}^\top \mathbf{h} + \kappa \mathbf{n}^\top \mathbf{m} \tag{2.1.3.2}$$

$$\implies \mathbf{n}^\top \mathbf{x} = c \tag{2.1.3.3}$$

for

$$c = \mathbf{n}^\top \mathbf{h}. \tag{2.1.3.4}$$

$\mathbf{n}$ is defined to be the *normal vector* of the line. In 3D, (2.1.3.3) represents a plane.

2.1.4. Mathematically, the projection of $\mathbf{A}$ on $\mathbf{B}$ is defined as

$$\mathbf{C} = k\mathbf{B}, \text{ such that } (\mathbf{A} - \mathbf{C})^\top \mathbf{C} = 0 \tag{2.1.4.1}$$

yielding

$$(\mathbf{A} - k\mathbf{B})^\top \mathbf{B} = 0 \tag{2.1.4.2}$$

$$\text{or, } k = \frac{\mathbf{A}^\top \mathbf{B}}{\|\mathbf{B}\|^2} \implies \mathbf{C} = \frac{\mathbf{A}^\top \mathbf{B}}{\|\mathbf{B}\|^2}\mathbf{B} \tag{2.1.4.3}$$

2.1.5. If $\mathbf{A}, \mathbf{B}$ are unit vectors,

$$(\mathbf{A} - \mathbf{B})^\top (\mathbf{A} + \mathbf{B}) = \|\mathbf{A}\|^2 - \|\mathbf{B}\|^2 = 0 \tag{2.1.5.1}$$

2.1.6. If

$$\mathbf{A}^\top \mathbf{A} = \mathbf{I}, \tag{2.1.6.1}$$

then $\mathbf{A}$ is an *orthogonal* matrix. This also means that its rows and columns are unit vectors and mutually perpendicular.

2.1.7. The determinant

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1. \tag{2.1.7.1}$$

2.1.8. Let

$$\mathbf{A} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \equiv a_1 \overrightarrow{i} + a_2 \overrightarrow{j} + a_3 \overrightarrow{j}, \tag{2.1.8.1}$$

$$\mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \tag{2.1.8.2}$$

and

$$\begin{aligned} \mathbf{A}_{ij} &= \begin{pmatrix} a_i \\ a_j \end{pmatrix}, \\ \mathbf{B}_{ij} &= \begin{pmatrix} b_i \\ b_j \end{pmatrix}. \end{aligned} \tag{2.1.8.3}$$

2.1.9. The *cross product* or *vector product* of $\mathbf{A}, \mathbf{B}$ is defined as

$$\mathbf{A} \times \mathbf{B} = \begin{pmatrix} \begin{vmatrix} \mathbf{A}_{23} & \mathbf{B}_{23} \end{vmatrix} \\ \begin{vmatrix} \mathbf{A}_{31} & \mathbf{B}_{31} \end{vmatrix} \\ \begin{vmatrix} \mathbf{A}_{12} & \mathbf{B}_{12} \end{vmatrix} \end{pmatrix} \tag{2.1.9.1}$$

2.1.10. Verify that

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \tag{2.1.10.1}$$

$$\mathbf{A} \times \mathbf{A} = \mathbf{0} \tag{2.1.10.2}$$

2.1.11. If

$$\mathbf{A} \times \mathbf{B} = \mathbf{0}, \tag{2.1.11.1}$$

$\mathbf{A}$ and $\mathbf{B}$ are linearly independent, i.e., they are points on the same line.

2.1.12.

$$\|\mathbf{A} \times \mathbf{B}\| = \|\mathbf{A}\| \times \|\mathbf{B}\| \sin\theta \tag{2.1.12.1}$$

where θ is the angle between the vectors.

2.1.13.

$$ar\,(ABCD) = \frac{1}{2}\,((\mathbf{C} - \mathbf{A}) \times (\mathbf{D} - \mathbf{B})) \tag{2.1.13.1}$$

$$\tag{2.1.13.2}$$

2.1.14. The area of $\triangle ABC$ is

$$\frac{1}{2}\,\|(\mathbf{A} - \mathbf{B}) \times (\mathbf{A} - \mathbf{C})\| \tag{2.1.14.1}$$

2.1.15. The affine transformation is given by

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c} \tag{2.1.15.1}$$

where $\mathbf{c}$ is the translation vector.

2.1.16. The matrix

$$\mathbf{P} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{2.1.16.1}$$

is defined to be the rotation matrix.

2.1.17.

$$\mathbf{P}^\top \mathbf{P} = \mathbf{I} \tag{2.1.17.1}$$

$\mathbf{P}$ is known as as *orthogonal* matrix.

2.1.18. Given vertices $\mathbf{A}, \mathbf{C}$ of a square, the other two vertices are given by

$$\begin{aligned} \mathbf{B} &= \|\mathbf{C} - \mathbf{A}\| \cos\frac{\pi}{4} \mathbf{P}\mathbf{e}_1 + \mathbf{A} \\ \mathbf{D} &= \|\mathbf{C} - \mathbf{A}\| \cos\frac{\pi}{4} \mathbf{P}\mathbf{e}_2 + \mathbf{A} \end{aligned} \tag{2.1.18.1}$$

2.1.19. Code for orthogonality

```
codes/book/orth.py
```

2.1.20. Code for cross product

```
codes/book/cross.py
```

2.2 *Scalar Product*

2.2.1 Find the angle between two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ with magnitudes $\sqrt{3}$ and 2 respectively having $\overrightarrow{a} \cdot \overrightarrow{b} = \sqrt{6}$.
Solution: From the given information,

$$\|\mathbf{a}\| = \sqrt{3}, \|\mathbf{b}\| = 2, \mathbf{a}^\top \mathbf{b} = \sqrt{6} \tag{2.2.1.1}$$

Substituting in (2.1.1.1),

$$\cos\theta = \frac{1}{\sqrt{2}} \tag{2.2.1.2}$$

$$\text{or, } \theta = 45^\circ \tag{2.2.1.3}$$

2.2.2 Find the angle between the the vectors $\hat{i} - 2\hat{j} + 3\hat{k}$ and $3\hat{i} - 2\hat{j} + \hat{k}$.
Solution: Let

$$\mathbf{a} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}, \tag{2.2.2.1}$$

From problem 2.2.1,

$$\cos\theta = \frac{\mathbf{a}^\top \mathbf{b}}{\|\mathbf{a}\| \|\mathbf{b}\|} = \frac{10}{\sqrt{14} \times \sqrt{14}} = \frac{5}{7} \tag{2.2.2.2}$$

2.2.3 Evaluate the product $(3\overrightarrow{a} - 5\overrightarrow{b}) \cdot (2\overrightarrow{a} + 7\overrightarrow{b})$.
Solution:

$$(3\mathbf{a} - 5\mathbf{b})^\top (2\mathbf{a} + 7\mathbf{b}) = 3\mathbf{a}^\top (2\mathbf{a} + 7\mathbf{b}) - 5\mathbf{b}^\top (2\mathbf{a} + 7\mathbf{b})$$
$$= 6\|\mathbf{a}\|^2 - 35\|\mathbf{b}\|^2 + 11\mathbf{a}^\top \mathbf{b} \quad (2.2.3.1)$$

2.2.4 If the vertices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ of a triangle ABC are (1, 2, 3), (-1, 0, 0), (0, 1, 2), respectively, then find $\angle ABC$.
Solution: From the given information,

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}, \mathbf{C} - \mathbf{B} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \quad (2.2.4.1)$$

$$\implies \angle ABC = \cos^{-1} \frac{(\mathbf{A} - \mathbf{B})^\top (\mathbf{C} - \mathbf{B})}{\|\mathbf{A} - \mathbf{B}\| \|\mathbf{C} - \mathbf{B}\|} \quad (2.2.4.2)$$

$$= \cos^{-1} \frac{10}{\sqrt{102}} \quad (2.2.4.3)$$

$$(2.2.4.4)$$

2.2.5 The slope of a line is double of the slope of another line. If tangent of the angle between them is 1/3, find the slopes of the lines.
Solution: The direction vectors of the lines can be expressed as

$$\mathbf{m}_1 = \begin{pmatrix} 1 \\ m \end{pmatrix}, \mathbf{m}_2 = \begin{pmatrix} 1 \\ 2m \end{pmatrix} \quad (2.2.5.1)$$

If the angle between the lines be θ,

$$\tan\theta = \frac{1}{3} \implies \cos\theta = \frac{3}{\sqrt{10}} \quad (2.2.5.2)$$

Thus,

$$\frac{3}{\sqrt{10}} = \frac{\mathbf{m}_1^\top \mathbf{m}_2}{\|\mathbf{m}_1\| \|\mathbf{m}_2\|} \quad (2.2.5.3)$$

$$= \frac{2m^2 + 1}{\sqrt{m^2 + 1}\sqrt{4m^2 + 1}} \quad (2.2.5.4)$$

$$\implies \frac{9}{10} = \frac{4m^4 + 4m^2 + 1}{4m^4 + 5m^2 + 1} \quad (2.2.5.5)$$

$$\text{or, } 4m^4 - 5m^2 + 1 = 0 \quad (2.2.5.6)$$

yielding

$$m = \pm\frac{1}{2}, \pm 1 \quad (2.2.5.7)$$

2.2.6 Find the angle between the vectors $2\hat{i} - \hat{j} + \hat{k}$ and $3\hat{i} + 4\hat{j} - \hat{k}$.
2.2.7 The angles between two vectors $\mathbf{a}, \mathbf{b}$ with magnitude $\sqrt{3}, 4$ respectively, and $\mathbf{a} \cdot \mathbf{b} = 2\sqrt{3}$ is ______

2.2.8 Find angle between the lines, $\sqrt{3}x + y = 1$ and $x + \sqrt{3}y=1$.

Solution: From (2.1.3.3), the normal vectors of the given lines can be expressed as

$$\mathbf{n}_1 = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}, \mathbf{n}_2 = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} \tag{2.2.8.1}$$

The angle between the lines can then be obtained as

$$\cos\theta = \frac{\mathbf{n}_1^\top \mathbf{n}_2}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} = \frac{\sqrt{3}}{2} \tag{2.2.8.2}$$

$$\text{or, } \theta = 30^\circ \tag{2.2.8.3}$$

2.2.9 Find the angle between the lines

$$\overrightarrow{r} = 3\hat{i} - 2\hat{j} + 6\hat{k} + \lambda(2\hat{i} + \hat{j} + 2\hat{k}) \text{ and} \tag{2.2.9.1}$$

$$\overrightarrow{r} = (2\hat{j} - 5\hat{k}) + \mu(6\hat{i} + 3\hat{j} + 2\hat{k}) \tag{2.2.9.2}$$

Solution: The given lines can be expressed in the form of (2.1.2.1) as

$$\mathbf{x} = \begin{pmatrix} 3 \\ -2 \\ 6 \end{pmatrix} + \kappa_1 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \tag{2.2.9.3}$$

$$\mathbf{x} = \begin{pmatrix} 0 \\ 2 \\ -5 \end{pmatrix} + \kappa_2 \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix} \tag{2.2.9.4}$$

From the above, it is obvious that the direction vectors of the two lines are

$$\mathbf{m}_1 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \ \mathbf{m}_2 = \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix} \tag{2.2.9.5}$$

From (2.1.1.1), the angle between the two lines is obtained as

$$\cos\theta = \frac{19}{21} \tag{2.2.9.6}$$

2.2.10 The vectors $\mathbf{a} = 3\hat{i}-2\hat{j}+2\hat{k}$ and $\mathbf{b} = \hat{i}-2\hat{k}$ are the adjancent sides of a parallelogram. The acute angle between its diagonals is ______.

2.2.11 The plane $2x - 3y + 6z - 11 = 0$ makes an angle $\sin^{-1}(\alpha)$ with x-axis. The value of α is equal to

2.2.12 Find the angle between the vectors $2\hat{i} - \hat{j} + \hat{k}$ and $3\hat{i} + 4\hat{j} - \hat{k}$.

2.2.13 The angles between two vectors **a** and **b** with magnitude $\sqrt{3}$ and 4, respectively, and **a**, **b**$= 2\sqrt{3}$ is

2.2.14 The angle between the line

$$\overrightarrow{r} = (5\hat{i} - \hat{j} - 4\hat{k}) + \lambda(2\hat{i} - \hat{j} + \hat{k}) \tag{2.2.14.1}$$

and the plane

$$\overrightarrow{r} \cdot (3\hat{i} - 4\hat{j} - \hat{k}) + 5 = 0 \tag{2.2.14.2}$$

is $\sin^{-1}\left(\frac{5}{2\sqrt{91}}\right)$.

2.2.15 The sine of the angle between the straight line

$$\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5} \tag{2.2.15.1}$$

and the plane

$$2x - 2y + z = 5 \tag{2.2.15.2}$$

is ______.

Solution: The given line can be expressed in the form (2.1.2.1) as

$$\mathbf{x} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} + \kappa_1 \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} \tag{2.2.15.3}$$

Hence the direction vector of this line is

$$\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} \tag{2.2.15.4}$$

From (2.1.3.3), the normal vector of the given plane is

$$\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \tag{2.2.15.5}$$

Thus, the cosine of the angle between the two is obtained from (2.1.1.1) as

$$\frac{\sqrt{2}}{10}, \tag{2.2.15.6}$$

which is sine of the angle between the plane and the line.

2.2.16 The angle between the planes

$$\vec{r} \cdot (2\hat{i} - 3\hat{j} + \hat{k}) = 1 \text{ and} \tag{2.2.16.1}$$

$$\vec{r} \cdot (\hat{i} - \hat{j}) = 4 \tag{2.2.16.2}$$

is $\cos^{-1}\left(\frac{-5}{\sqrt{58}}\right)$.

2.2.17 Find the angle between the lines

$$y = (2 - \sqrt{3})(x + 5) \text{ and} \tag{2.2.17.1}$$

$$y = (2 + \sqrt{3})(x - 7). \tag{2.2.17.2}$$

2.2.18 The unit vector normal to the plane $x + 2y + 3z - 6 = 0$ is $\frac{1}{\sqrt{14}}\hat{i} + \frac{2}{\sqrt{14}}\hat{j} + \frac{3}{\sqrt{14}}\hat{k}$.

2.2.19 The scalar product of the vector $\hat{i} + \hat{j} + \hat{k}$ with a unit vector along the sum of vectors $2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\lambda\hat{i} + 2\hat{j} + 3\hat{k}$ is equal to one. Find the value of λ.

2.2.20 If the co-ordinates of the points $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ be $(1, 2, 3)$, $(4, 5, 7)$, $(-4, 3, -6)$ and $(2, 9, 2)$ respectively, then find the angle between the lines AB and CD.

2.2.21 If the angle between two lines is $\frac{\pi}{4}$ and slope of one of the lines is $\frac{1}{2}$, find the slope of the other line.

2.2.22 Find the angle between two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ with magnitudes 1 and 2 respectively and when $\overrightarrow{a} \cdot \overrightarrow{b} = 1$.

2.2.23 Find angle θ between the vectors $\overrightarrow{a} = \hat{i} + \hat{j} - \hat{k}$ and $\overrightarrow{b} = \hat{i} - \hat{j} + \hat{k}$.

2.2.24 If $\hat{i}+\hat{j}+\hat{k}, 2\hat{i}+5\hat{j}, 3\hat{i}+2\hat{j}-3\hat{k}$ and $\hat{i}-6\hat{j}-\hat{k}$ are the position vectors of points **A**, **B**, **C** and **D** respectively, then find the angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$. Deduce that $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are collinear.

2.2.25 Find the angle between the following pairs of lines.

a)

$$\overrightarrow{r} = 2\hat{i} - 5\hat{j} + \hat{k} + \lambda(3\hat{i} + 2\hat{j} + 6\hat{k}) \text{ and} \tag{2.2.25.1}$$
$$\overrightarrow{r} = 7\hat{i} - 6\hat{k} + \mu(\hat{i} + 2\hat{j} + 2\hat{k}) \tag{2.2.25.2}$$

b)

$$\overrightarrow{r} = 3\hat{i} + \hat{j} - 2\hat{k} + \lambda(\hat{i} - \hat{j} - 2\hat{k}) \text{ and} \tag{2.2.25.3}$$
$$\overrightarrow{r} = 2\hat{i} - \hat{j} - 56\hat{k} + \mu(3\hat{i} - 5\hat{j} - 4\hat{k}) \tag{2.2.25.4}$$

c)

$$\frac{x-2}{2} = \frac{y-1}{5} = \frac{z+3}{-3} \text{ and } \frac{x+2}{-1} = \frac{y-4}{8} = \frac{z-5}{4}. \tag{2.2.25.5}$$

d)

$$\frac{x}{2} = \frac{y}{2} = \frac{z}{1} \text{ and } \frac{x-5}{4} = \frac{y-2}{1} = \frac{z-3}{8}. \tag{2.2.25.6}$$

2.2.26 Find the angle between the pair of lines given by

$$\overrightarrow{r} = 3\hat{i} + 2\hat{j} - 4\hat{k} + \lambda(\hat{i} + 2\hat{j} + 2\hat{k}) \tag{2.2.26.1}$$
$$\text{and } \overrightarrow{r} = 5\hat{i} + 2\hat{j} + \mu(3\hat{i} + 2\hat{j} + 6\hat{k}) \tag{2.2.26.2}$$

2.2.27 Find the angle between the pair of lines:

$$\frac{x+3}{3} = \frac{y-1}{5} = \frac{z+3}{4} \tag{2.2.27.1}$$
$$\text{and } \frac{x+1}{1} = \frac{y-4}{1} = \frac{z+5}{2} \tag{2.2.27.2}$$

2.2.28 Find the angle between the two planes $2x + y - 2z = 5$ and $3x - 6y - 2z = 7$ using vector method.

2.2.29 Find the angle between the two planes $3x - 6y + 2z = 7$ and $2x + 2y - 2z = 5$.

2.2.30 Find the angle between the line $\frac{x+1}{2} = \frac{y}{3} = \frac{z-3}{6}$ and the plane $10x + 2y - 11z = 3$.

2.3 CBSE

2.3.1 The angle between the vectors $\hat{i} - \hat{j}$ and $\hat{j} - \hat{k}$ is ______. (12, 2020)

2.3.2 Find the angle between unit vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ so that $\sqrt{3}\overrightarrow{a}$ - $\overrightarrow{b}$ is also a unit vector. (12, 2020)

2.3.3 If $\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}$ are three non-zero unequal vectors such that $\overrightarrow{a} \cdot \overrightarrow{b} = \overrightarrow{a} \cdot \overrightarrow{c}$, then find the angle between $\overrightarrow{a}$ and $\overrightarrow{b} - \overrightarrow{c}$. (12, 2023)

2.3.4 $\overrightarrow{a}$ and $\overrightarrow{b}$ are two unit vectors such that

$$\left|2\overrightarrow{a} + 3\overrightarrow{b}\right| = \left|3\overrightarrow{a} - 2\overrightarrow{b}\right|. \tag{2.3.4.1}$$

Find the angle between $\overrightarrow{a}$ and $\overrightarrow{b}$. (12, 2023)

2.3.5 Find the angle between the line $\overrightarrow{r} = \hat{i}-\hat{j}+\hat{k}+\lambda(3\hat{i}-\hat{j}+2\hat{k})$ and the plane $\overrightarrow{r}\cdot(\hat{i}+\hat{j}+\hat{k}) = 3$. (12, 2019)

2.3.6 Find the magnitude of each of the vectors $\overrightarrow{\mathbf{a}}$ and $\overrightarrow{\mathbf{b}}$, having the same magnitude such that the angle between them is 60° and their scalar product is $\frac{9}{2}$. (12, 2018)

2.3.7 Find the acute angle between the planes $\mathbf{r}\cdot\left(\hat{i} - 2\hat{j} - 2\hat{k}\right) = 1$ and $\mathbf{r}\cdot\left(3\hat{i} - 6\hat{j} + 2\hat{k}\right) = 0$ (12, 2018)

2.3.8 If $\hat{i} + \hat{j} + k, 2\hat{i} + 5\hat{j}, 3\hat{i} + 2\hat{j} - 3k, \hat{i} - 6\hat{j} - k$ respectively are the position vectors of points A, B, C and D, then find the angle between the straight lines AB and CD. Find whether $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are collinear or not. (12, 2018)

2.3.9 If vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ are such that $\left|\overrightarrow{a}\right| = \frac{1}{2}$, $\left|\overrightarrow{b}\right| = \frac{4}{\sqrt{3}}$ and $\left|\overrightarrow{a} \times \overrightarrow{b}\right| = \frac{1}{\sqrt{3}}$, then find $\left|\overrightarrow{a} \cdot \overrightarrow{b}\right|$. (12, 2016)

2.3.10 If $\overrightarrow{a}$ and $\overrightarrow{b}$ are unit vectors, then what is the angle between $\overrightarrow{a}$ and $\overrightarrow{b}$ for $\overrightarrow{a} - \sqrt{2}\overrightarrow{b}$ to be a unit vector? (12, 2016)

2.3.11 Find the acute angle between the planes $\overrightarrow{r}\cdot\left(\hat{i} - 2\hat{j} - 2\hat{k}\right) = 1$ and $\overrightarrow{r}\cdot\left(3\hat{i} - 6\hat{j} + 2\hat{k}\right) = 0$. (12, 2019)

2.3.12 Find the angle made by the line $\frac{x}{1} = \frac{y}{-1} = \frac{z}{0}$ with the positive direction of Y axis. (12, 2024)

2.3.13 Find the angle which the line $\frac{x}{1} = \frac{y}{-1} = \frac{z}{2}$ makes with the positive direction of the Y axis. (12, 2024)

2.3.14 If a line makes an angle of 30° with the positive direction of X axis, 120° with the positive direction of Y axis, then find the angle which it makes with the positive direction of Z axis. (12, 2024)

2.3.15 The scalar product of the vector $\hat{i}+\hat{j}+\hat{k}$ with the unit vector along the sum of vectors $2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\lambda\hat{i} + 2\hat{j} + 3\hat{k}$ is equal to one. Find the value of λ. (10, 2009)

2.3.16 If $\mathbf{p}$ is a unit vector and $(\mathbf{x} - \mathbf{p}) \cdot (\mathbf{x} + \mathbf{p}) = 80$, then find $\|\mathbf{x}\|$. (10, 2009)

2.3.17 Find the properties of the triangle with vertices $(-4, 0), (4, 0)$, and $(0, 3)$. (10, 2024)

2.4 *Orthogonality*

2.4.1 Find the projection of the vector $\hat{i} + 3\hat{j} + 7\hat{k}$ on the vector $7\hat{i} - \hat{j} + 8\hat{k}$.
Solution: Let

$$\mathbf{A} = \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 7 \\ -1 \\ 8 \end{pmatrix} \tag{2.4.1.1}$$

The projection of **A** on **B** is defined as the foot of the perpendicular from **A** to **B** and obtained in (2.1.4.3). Substituting numerical values,

$$\mathbf{C} = \frac{10}{19}\begin{pmatrix} 7 \\ -1 \\ 8 \end{pmatrix} \tag{2.4.1.2}$$

2.4.2 Show that the points $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ with position vectors, $3\hat{i} - 4\hat{j} - 4\hat{k}, 2\hat{i} - \hat{j} + \hat{k}$ and $\hat{i} - 3\hat{j} - 5\hat{k}$, respectively, form the vertices of a right angled triangle.
Solution:

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix}, \mathbf{C} - \mathbf{B} = \begin{pmatrix} -1 \\ -2 \\ -6 \end{pmatrix}, \mathbf{C} - \mathbf{A} = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}, \tag{2.4.2.1}$$

$$\implies (\mathbf{B} - \mathbf{A})^{\top} (\mathbf{C} - \mathbf{A}) = 0 \tag{2.4.2.2}$$

Hence, $\triangle ABC$ is right angled at **A**.

2.4.3 Show that each of the given three vectors is a unit vector: $\frac{1}{7}(2\hat{i}+3\hat{j}+6\hat{k}), \frac{1}{7}(3\hat{i}-6\hat{j}+2\hat{k}), \frac{1}{7}(6\hat{i}+2\hat{j}-3\hat{k})$. Also, show that they are mutually perpendicular to each other.
Solution:

$$\mathbf{A} = \begin{pmatrix} \frac{2}{7} & \frac{3}{7} & \frac{6}{7} \\ \frac{3}{7} & -\frac{6}{7} & \frac{2}{7} \\ \frac{6}{7} & \frac{2}{7} & -\frac{3}{7} \end{pmatrix} \tag{2.4.3.1}$$

is an orthogonal matrix satisfying (2.1.6.1), which verifies the given conditions.

2.4.4 Let $\mathbf{a} = \hat{i} + 4\hat{j} + 2\hat{k}, \mathbf{b} = 3\hat{i} - 2\hat{j} + 7\hat{k}$ and $\mathbf{c} = 2\hat{i} - \hat{j} + 4\hat{k}$. Find a vector **d** which is perpendicular to both **a** and **b**, and $\mathbf{c} \cdot \mathbf{d}$=15.
Solution: From the given information,

$$\mathbf{a}^{\top}\mathbf{d} = 0 \tag{2.4.4.1}$$

$$\mathbf{b}^{\top}\mathbf{d} = 0 \tag{2.4.4.2}$$

$$\mathbf{c}^{\top}\mathbf{d} = 15 \tag{2.4.4.3}$$

yielding

$$\begin{pmatrix} \mathbf{a}^{\top} \\ \mathbf{b}^{\top} \\ \mathbf{c}^{\top} \end{pmatrix} \mathbf{d} = \begin{pmatrix} 0 \\ 0 \\ 15 \end{pmatrix} \tag{2.4.4.4}$$

$$\implies \begin{pmatrix} 1 & 4 & 2 \\ 3 & -2 & 7 \\ 2 & -1 & 4 \end{pmatrix} \mathbf{d} = \begin{pmatrix} 0 \\ 0 \\ 15 \end{pmatrix} \tag{2.4.4.5}$$

Forming the augmented matrix,

$$\left(\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 3 & -2 & 7 & 0 \\ 2 & -1 & 4 & 15 \end{array}\right) \xleftrightarrow[R_3 \leftarrow R_3 - 2R_1]{R_2 \leftarrow R_2 - 3R_1} \left(\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 0 & -14 & 1 & 0 \\ 0 & -9 & 0 & 15 \end{array}\right)$$
$$\xleftrightarrow{R_3 \leftarrow R_3 - \frac{9}{14}R_2} \left(\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 0 & -14 & 1 & 0 \\ 0 & 0 & -\frac{9}{14} & 15 \end{array}\right) \tag{2.4.4.6}$$

yielding

$$\mathbf{d} = \begin{pmatrix} \frac{160}{3} \\ -\frac{5}{3} \\ -\frac{70}{3} \end{pmatrix} \tag{2.4.4.7}$$

upon back substitution.

2.4.5 Show that the points $\mathbf{A}(2\hat{i} - \hat{j} + \hat{k}), \mathbf{B}(\hat{i} - 3\hat{j} - 5\hat{k}), \mathbf{C}(3\hat{i} - 4\hat{j} - 4\hat{k})$ are the vertices of a right angled triangle.

2.4.6 If $\overrightarrow{a} = 5\hat{i} - \hat{j} - 3k$ and $\overrightarrow{b} = \hat{i} + 3\hat{j} - 5\hat{k}$, then show that the vectors $\overrightarrow{a} + \overrightarrow{b}$ and $\overrightarrow{a} - \overrightarrow{b}$ are perpendicular.

2.4.7 Find the projection of the vector $\overrightarrow{a} = 2\hat{i} + 3\hat{j} + 2\hat{k}$ on the vector $\overrightarrow{b} = \hat{i} + 2\hat{j} + \hat{k}$.

2.4.8 Find a unit vector perpendicular to each of the vectors $(\overrightarrow{a} + \overrightarrow{b})$ and $(\overrightarrow{a} - \overrightarrow{b})$, where $\overrightarrow{a} = \hat{i} + \hat{j} + \hat{k}, \overrightarrow{b} = \hat{i} + 2\hat{j} + 3\hat{k}$.

2.4.9 Show that the vectors $2\hat{i} - \hat{j} + \hat{k}, \hat{i} - 3\hat{j} - 5\hat{k}$ and $3\hat{i} - 4\hat{j} - 4\hat{k}$ from the vertices of a right angled triangle.

Solution:

$$\mathbf{A} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 3 \\ -4 \\ -4 \end{pmatrix}, \tag{2.4.9.1}$$

$$\implies \mathbf{B} - \mathbf{C} = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}, \mathbf{C} - \mathbf{A} = \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix}, \tag{2.4.9.2}$$

$$\text{or, } (\mathbf{B} - \mathbf{C})^\top (\mathbf{C} - \mathbf{A}) = 0 \tag{2.4.9.3}$$

2.4.10 Find the projection of the vector $\hat{i} - \hat{j}$ on the vector $\hat{i} + \hat{j}$.

Solution: The given points are

$$\mathbf{A} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.4.10.1}$$

Since

$$\mathbf{A}^\top \mathbf{B} = 0, \tag{2.4.10.2}$$

from (2.1.4.3), the projection vector is the origin. See Fig. 2.4.10.1.

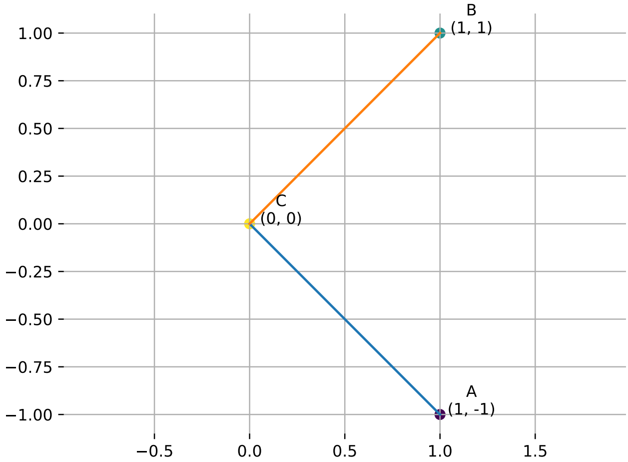


Fig. 2.4.10.1

2.4.11 Show that the line joining the origin to the point $\mathbf{P}(2, 1, 1)$ is perpendicular to the line determined by the points $\mathbf{A}(3, 5, -1), \mathbf{B}(4, 3, -1)$.

Solution:

$$(\mathbf{A} - \mathbf{B})^\top \mathbf{P} = \begin{pmatrix} -1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = 0 \quad \square \tag{2.4.11.1}$$

2.4.12 $ABCD$ is a rectangle formed by the points $\mathbf{A}(-1, -1), \mathbf{B}(-1, 4), \mathbf{C}(5, 4)$ and $\mathbf{D}(5, -1)$. $\mathbf{P}, \mathbf{Q}, \mathbf{R}$ and $\mathbf{S}$ are the mid-points of AB, BC, CD and DA respectively. Is the quadrilateral $PQRS$ a square? a rectangle? or a rhombus? Justify your answer.

Solution: See Fig. 2.4.12.1. From (1.1.5.3), $PQRS$ is a parallelogram.

$$\mathbf{P} = \frac{3}{2}, \ \mathbf{Q} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \ \mathbf{R} = \begin{pmatrix} 5 \\ \frac{3}{2} \end{pmatrix}, \ \mathbf{S} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \tag{2.4.12.1}$$

$$\implies (\mathbf{Q} - \mathbf{P})^\top (\mathbf{R} - \mathbf{Q}) \neq 0 \tag{2.4.12.2}$$

$$(\mathbf{R} - \mathbf{P})^\top (\mathbf{S} - \mathbf{Q}) = 0 \tag{2.4.12.3}$$

Therefore $PQRS$ is a rhombus.

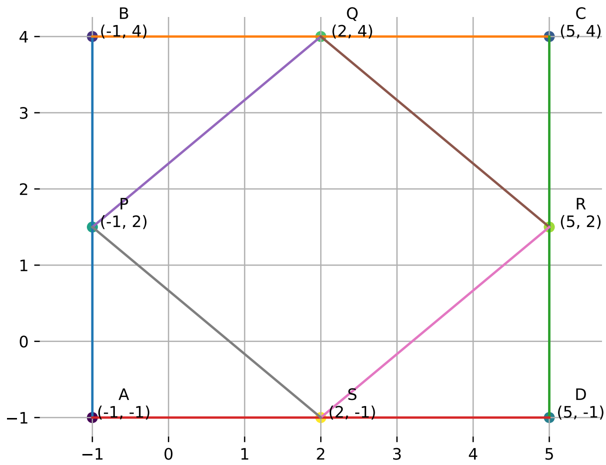


Fig. 2.4.12.1

2.4.13 $\overrightarrow{AB} = 3\hat{i} - \hat{j} + \hat{k}$ and $\overrightarrow{CD} = -3\hat{i} + 2\hat{j} + 4\hat{k}$ are two vectors. The position vectors of the points **A** and **C** are $6\hat{i} + 7\hat{j} + 4\hat{k}$ and $-9\hat{j} + 2\hat{k}$, respectively. Find the position vector of a point **P** on the line AB and a point **Q** on the line CD such that $\overrightarrow{PQ}$ is perpendicular to $\overrightarrow{AB}$ and $\overrightarrow{CD}$ both.

2.4.14 Find a unit vector perpendicular to each of the vectors $\overrightarrow{a} + \overrightarrow{b}$ and $\overrightarrow{a} - \overrightarrow{b}$, where $\overrightarrow{a} = 3\hat{i} + 2\hat{j} + 2\hat{k}$ and $\overrightarrow{b} = \hat{i} + 2\hat{j} - 2\hat{k}$.
Solution: Let the desired vector be **x**. Then,

$$\begin{pmatrix}\mathbf{a}+\mathbf{b} & \mathbf{a}-\mathbf{b}\end{pmatrix}^\top \mathbf{x} = 0 \tag{2.4.14.1}$$

$$\because \mathbf{a}+\mathbf{b} = \begin{pmatrix}\mathbf{a} & \mathbf{b}\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} \tag{2.4.14.2}$$

$$\mathbf{a}-\mathbf{b} = \begin{pmatrix}\mathbf{a} & \mathbf{b}\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix}, \tag{2.4.14.3}$$

(2.4.14.1) can be expressed as

$$\left\{\begin{pmatrix}\mathbf{a} & \mathbf{b}\end{pmatrix}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\right\}^\top \mathbf{x} = 0 \tag{2.4.14.4}$$

$$\implies \begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}^\top \begin{pmatrix}\mathbf{a} & \mathbf{b}\end{pmatrix}^\top \mathbf{x} = 0 \tag{2.4.14.5}$$

$$\implies \begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}^\top \begin{pmatrix}\mathbf{a} & \mathbf{b}\end{pmatrix}^\top \mathbf{x} = 0 \tag{2.4.14.6}$$

$$\text{or, } \begin{pmatrix}\mathbf{a} & \mathbf{b}\end{pmatrix}^\top \mathbf{x} = 0 \tag{2.4.14.7}$$

which can be expressed as

$$\begin{pmatrix} 3 & 2 & 2 \\ 1 & 2 & -2 \end{pmatrix} \xleftrightarrow[R_2=\frac{R_2}{4}]{R_2=3R_2-R_1} \begin{pmatrix} 3 & 2 & 2 \\ 0 & 1 & -2 \end{pmatrix} \tag{2.4.14.8}$$

$$\xleftrightarrow[R_1=\frac{R_1}{3}]{R_1=R_1-2R_2} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \end{pmatrix} \tag{2.4.14.9}$$

yielding

$$\begin{aligned} x_1 + 2x_3 &= 0 \\ x_2 - 2x_3 &= 0 \end{aligned} \implies \mathbf{x} = x_3 \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} \tag{2.4.14.10}$$

Thus, the desired unit vector is

$$\mathbf{x} = \frac{1}{3} \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} \tag{2.4.14.11}$$

2.4.15 Line joining the points (3, -4) and (-2, 6) is perpendicular to the line joining the points (-3, 6) and (9, -18).

2.4.16 Verify the following:

a) $(0, 7, -10), (1, 6, -6)$ and $(4, 9, -6)$ are the vertices of an isoceles triangle.
b) $(0, 7, 10), (-1, 6, 6)$ and $(-4, 9, 6)$ are the vertices of a right angled triangle.

2.4.17 Show that the line through the points $(1, -1, 2), (3, 4, -2)$ is perpendicular to the line through the points $(0, 3, 2)$ and $(3, 5, 6)$.

2.4.18 Find the values of p so that the lines $\frac{1-x}{3} = \frac{7y-14}{2p} = \frac{z-3}{2}$ and $\frac{7-7x}{3p} = \frac{y-5}{1} = \frac{6-z}{5}$ are at right angles.

2.4.19 If $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ are the points with position vectors $\hat{i} + \hat{j} - \hat{k}$, $2\hat{i} - \hat{j} + 3\hat{k}$, $2\hat{i} - 3\hat{k}$, $3\hat{i}$-$2\hat{j}$+$\hat{k}$, respectively, find the projection of $\overline{AB}$ along $\overline{CD}$.

2.4.20 Find the value of λ such that the vectors $\mathbf{a} = 2\hat{i} + \lambda\hat{j} + \hat{k}$ and $\mathbf{b} = \hat{i} + 2\hat{j} + 3\hat{k}$ are orthogonal.

2.4.21 The number of vectors of unit length perpendicular to the vectors $\mathbf{a} = 2\hat{i} + \hat{j} + 2\hat{k}$ and $\mathbf{b} = \hat{j} + \hat{k}$ is

2.4.22 Find the equation of a plane which bisects perpendicularly the line joining the points $\mathbf{A}(2, 3, 4)$ and $\mathbf{B}(4, 5, 8)$ at right angles.

2.4.23 Do the points $(3, 2), (-2, -3)$ and $(2, 3)$ form a triangle? If so, name the type of triangle formed.

2.4.24 Show that the points $(1, 7), (4, 2), (-1, -1)$ and $(-4, 4)$ are the vertices of a square.

2.4.25 Name the type of quadrilateral formed, if any, by the following points, and give reasons for your answer

a) $\mathbf{A}(-1, -2), \mathbf{B}(1, 0), \mathbf{C}(-1, 2), \mathbf{D}(-3, 0)$
b) $\mathbf{A}(-3, 5), \mathbf{B}(3, 1), \mathbf{C}(0, 3), \mathbf{D}(-1, -4)$
c) $\mathbf{A}(4, 5), \mathbf{B}(7, 6), \mathbf{C}(4, 3), \mathbf{D}(1, 2)$

Solution: See Table 2.4.25, Fig. 2.4.25.1, Fig. 2.4.25.2. and Fig. 2.4.25.3. In b), forming the collinearity matrix

$$\begin{pmatrix}\mathbf{B}-\mathbf{A} & \mathbf{C}-\mathbf{B}\end{pmatrix} = \begin{pmatrix} 6 & -3 \\ -4 & 2 \end{pmatrix} \xleftrightarrow{R_2 \to R_2 + \frac{2}{3}R_1} = \begin{pmatrix} 6 & -3 \\ 0 & 0 \end{pmatrix} \tag{2.4.25.1}$$

which is a rank 1 matrix. Hence, $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are collinear.

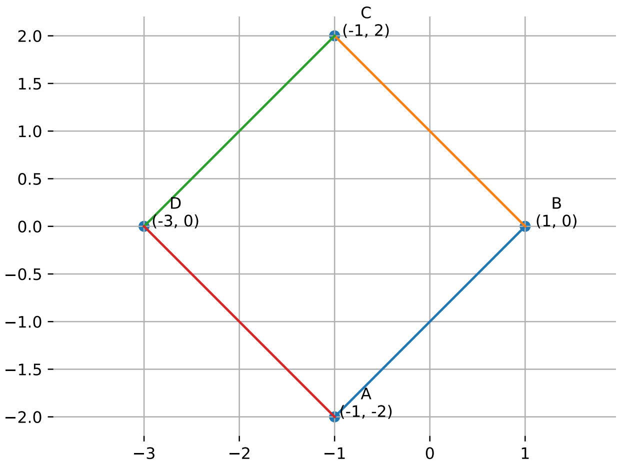


Fig. 2.4.25.1

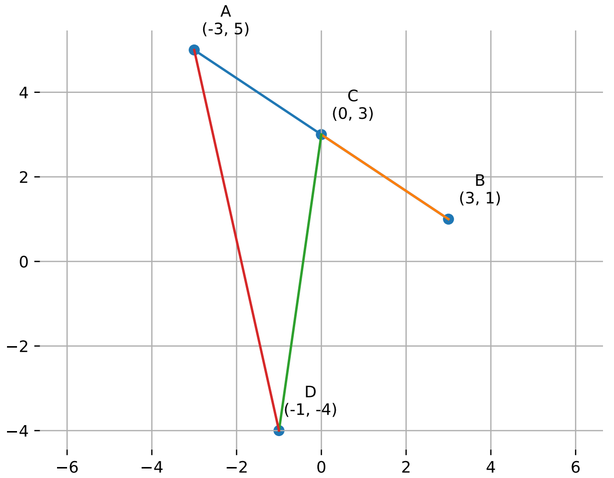


Fig. 2.4.25.2

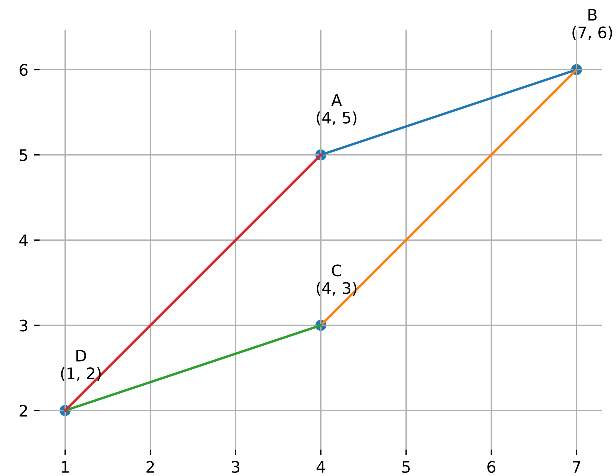


Fig. 2.4.25.3

	$\mathbf{B}-\mathbf{A}=\mathbf{C}-\mathbf{D}$?	$(\mathbf{B}-\mathbf{A})^{\top}(\mathbf{C}-\mathbf{B})=0$?	$(\mathbf{C}-\mathbf{A})^{\top}(\mathbf{D}-\mathbf{B})=0$	**Geometry**
a)	Yes	Yes	Yes	Square
b)	No	-	-	Triangle
c)	Yes	No	No	Parallelogram

TABLE 2.4.25

2.4.26 Check whether $(5,-2),(6,4)$ and $(7,-2)$ are the vertices of an isosceles triangle.

2.4.27 The perpendicular bisector of the line segment joining the points $\mathbf{A}(1,5)$ and $\mathbf{B}(4,6)$ cuts the y-axis at ______.

2.4.28 Find the coordinates of the point $\mathbf{Q}$ on the x-axis which lies on the perpendicular bisector of the line segment joining the points $\mathbf{A}(-5,-2)$ and $\mathbf{B}(4,-2)$. Name the type of triangle formed by points $\mathbf{Q},\mathbf{A}$ and $\mathbf{B}$.

2.4.29 The points $\mathbf{A}(2,9),\mathbf{B}(a,5)$ and $\mathbf{C}(5,5)$ are the vertices of a triangle $\mathbf{ABC}$ right angled at $\mathbf{B}$. Find the values of a and hence the area of $\triangle\mathbf{ABC}$.

2.4.30 If $\overrightarrow{a}=2\hat{i}+2\hat{j}3\hat{k},\overrightarrow{b}=-\hat{i}+2\hat{j}+\hat{k}$ and $\overrightarrow{c}=3\hat{i}+\hat{j}$ are such that $\overrightarrow{a}+\lambda\overrightarrow{b}$ is perpendicular to $\overrightarrow{c}$, then find the value of λ.

Solution:

$$\because (\mathbf{a}+\lambda\mathbf{b})^{\top}\mathbf{c}=0, \tag{2.4.30.1}$$

$$\lambda=-\frac{\mathbf{a}^{\top}\mathbf{c}}{\mathbf{b}^{\top}\mathbf{c}}=8, \tag{2.4.30.2}$$

upon substituting numerical values.

2.4.31 The point $\mathbf{A}(2,7)$ lies on the perpendicular bisector of line segment joining the points $\mathbf{P}(6,5)$ and $\mathbf{Q}(0,-4)$.

2.4.32 The points $\mathbf{A}(-1,-2),\mathbf{B}(4,3),\mathbf{C}(2,5)$ and $\mathbf{D}(-3,0)$ in that order form a rectangle.

2.4.33 Name the type of triangle formed by the points $\mathbf{A}(-5,6),\mathbf{B}(-4,-2)$, and $\mathbf{C}(7,5)$.

2.4.34 What type of a quadrilateral do the points $\mathbf{A}(2,-2),\mathbf{B}(7,3),\mathbf{C}(11,-1)$, and $\mathbf{D}(6,-6)$ taken in that order, form?

2.4.35 The point which lies on the perpendicular bisector of the line segment joining the points $\mathbf{A}(-2,-5)$ and $\mathbf{B}(2,5)$ is

a) $(0,0)$
b) $(0,2)$
c) $(2,0)$
d) $(-2,0)$

2.4.36 The points $(-4,0),(4,0),(0,3)$ are the vertices of

a) right triangle
b) isosceles triangle
c) equilateral triangle
d) scalene triangle

2.4.37 Find a vector of magnitude 6, which is perpendicular to both the vectors $2\hat{i}-\hat{j}+2\hat{k}$ and $4\hat{i}-\hat{j}+3\hat{k}$.

2.4.38 Without using the Baudhayana theorem, show that the points $\mathbf{A}(4,4),\mathbf{B}(3,5)$ and $\mathbf{C}(-1,-1)$ are the vertices of a right angled triangle.

Solution: See Fig. 2.4.38.1.

$$\mathbf{C}-\mathbf{A}=\begin{pmatrix}-5\\-5\end{pmatrix},\ \mathbf{A}-\mathbf{B}=\begin{pmatrix}1\\-1\end{pmatrix} \tag{2.4.38.1}$$

$$\implies (\mathbf{C}-\mathbf{A})^\top(\mathbf{A}-\mathbf{B})=0 \tag{2.4.38.2}$$

Thus, $AB \perp AC$.

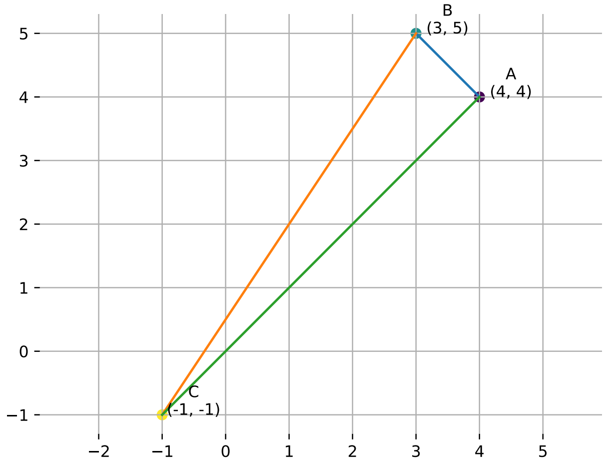


Fig. 2.4.38.1

2.4.39 Line through the points $(-2,6)$ and $(4,8)$ is perpendicular to the line through the points $(8,12)$ and $(x,24)$. Find the value of x.

2.4.40 Find the angle between the lines $y-\sqrt{3}x-5=0$ and $\sqrt{3}y-x+6=0$.

2.4.41 Are the points $\mathbf{A}(3,6,9),\mathbf{B}(10,20,30)$ and $\mathbf{C}(24,-41,5)$ the vertices of a right angled triangle?

2.4.42 Show that the points $\mathbf{A}(1,2,3),\mathbf{B}(-1,-2,-1),\mathbf{C}(2,3,2)$ and $\mathbf{D}(4,7,6)$ are the vertices of a parallelogram $ABCD$, but it is not a rectangle.

2.4.43 Show that the lines $\frac{x-5}{7}=\frac{y+2}{-5}=\frac{z}{1}$ and $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ are perpendicular to each other.

2.4.44 In the following cases, determine whether the given planes are parallel or perpendicular, and in case they are neither, find the angles between them.

a) $7x+5y+6z+30=0$ and $3x–y–10z+4=0$
b) $2x+y+3z–2=0$ and $x–2y+5=0$
c) $2x–2y+4z+5=0$ and $3x–3y+6z–1=0$
d) $2x–y+3z–1=0$ and $2x–y+3z+3=0$
e) $4x+8y+z–8=0$ and $y+z–4=0$

Solution: See Table 2.4.44.

TABLE 2.4.44

$\mathbf{n}_1$	$\mathbf{n}_1$	$\mathbf{n}_1^\top \mathbf{n}_2$	$\|\mathbf{n}_1\|$	$\|\mathbf{n}_2\|$	Angle
$\begin{pmatrix}7\\5\\6\end{pmatrix}$	$\begin{pmatrix}3\\-1\\-10\end{pmatrix}$	-44	$\sqrt{110}$	$\sqrt{110}$	$\cos^{-1} -\frac{2}{5}$
$\begin{pmatrix}2\\1\\3\end{pmatrix}$	$\begin{pmatrix}1\\-2\\0\end{pmatrix}$	0			perpendicular
$\begin{pmatrix}2\\-2\\4\end{pmatrix}$	$\begin{pmatrix}3\\-3\\6\end{pmatrix}$	36	$\sqrt{24}$	$\sqrt{54}$	parallel
$\begin{pmatrix}2\\-1\\3\end{pmatrix}$	$\begin{pmatrix}2\\-1\\3\end{pmatrix}$	14	$\sqrt{14}$	$\sqrt{14}$	parallel
$\begin{pmatrix}4\\8\\1\end{pmatrix}$	$\begin{pmatrix}0\\1\\1\end{pmatrix}$	9	9	$\sqrt{2}$	45°

2.5 CBSE

2.5.1 Show that the points $(7, 10)$, $(-2, 5)$ and $(3, 4)$ are vertices of an isosceles right triangle. (10, 2020)

2.5.2 The points $(-4, 0)$, $(4, 0)$, and $(0, 3)$ are the vertices of a:

a) right triangle
b) isosceles triangle
c) equilateral triangle
d) scalene triangle

(10, 2023)

2.5.3 Show that the points $(-2, 3)$, $(8, 3)$, and $(6, 7)$ are the vertices of a right-angled triangle. (10, 2023)

2.5.4 If $\overrightarrow{a} = 2\hat{i} + y\hat{j} + \hat{k}$ and $\overrightarrow{b} = \hat{i} + 2\hat{j} + 3\hat{k}$ are two vectors for which the vector $(\overrightarrow{a} + \overrightarrow{b})$ is perpendicular to the vector $(\overrightarrow{a} - \overrightarrow{b})$, then find all the possible values of y. (12, 2023)

2.5.5 Write the projection of the vector $(\overrightarrow{b} + \overrightarrow{c})$ on the vector $\overrightarrow{a}$, where $\overrightarrow{a} = 2\hat{i} - 2\hat{j} + \hat{k}$, $\overrightarrow{b} = \hat{i} + 2\hat{j} - 2\hat{k}$, and $\overrightarrow{c} = 2\hat{i} - \hat{j} + 4\hat{k}$. (12, 2023)

2.5.6 If $\overrightarrow{a} = 2\hat{i} - \hat{j} + \hat{k}$, $\overrightarrow{b} = \hat{i} + \hat{j} - 2\hat{k}$, and $\overrightarrow{c} = \hat{i} + 3\hat{j} - \hat{k}$, and the projection of vector $\overrightarrow{c} + \lambda\overrightarrow{b}$ on vector $\overrightarrow{a}$ is $2\sqrt{6}$, find the value of λ. (12, 2023)

2.5.7 If $\overrightarrow{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\overrightarrow{b} = 5\hat{i} - 3\hat{j} - 4\hat{k}$, then find the ratio of the projection of vector $\overrightarrow{a}$ on vector $\overrightarrow{b}$ to the projection of vector $\overrightarrow{b}$ on vector $\overrightarrow{a}$. (12, 2023)

2.5.8 Show that the three vectors $2\hat{i} - \hat{j} + \hat{k}$, $\hat{i} - 3\hat{j} - 5\hat{k}$, and $3\hat{i} - 4\hat{j} - 4\hat{k}$ form the vertices of a right-angled triangle. (12, 2023)

2.5.9 If $\overrightarrow{a} = 2\hat{i} + 2\hat{j} + 3\hat{k}$, $\overrightarrow{b} = -\hat{i} + 2\hat{j} + \hat{k}$ and $\overrightarrow{c} = 3\hat{i} + \hat{j}$, are such that the vector $(\overrightarrow{a} + \lambda\overrightarrow{b})$ is perpendicular to vector $\overrightarrow{c}$, then find the value of λ. (12, 2023)

2.5.10 What kind of triangle is formed with vertices $\mathbf{A}(0,2)$, $\mathbf{B}(-3,0)$, and $\mathbf{C}(3,0)$?

a) A right triangle
b) An equilateral triangle
c) An isosceles triangle
d) A scalene triangle

(10, 2021)

2.5.11 Check whether the points $\mathbf{P}(5,-2)$, $\mathbf{Q}(6,4)$, and $\mathbf{R}(7,-2)$ are the vertices of an isosceles triangle $\triangle PQR$. (10, 2021)

2.5.12 The points $\mathbf{A}(0,3)$, $\mathbf{B}(-2,a)$, and $\mathbf{C}(-1,4)$ are the vertices of a right triangle, right-angled at $\mathbf{A}$. Find the value of a.

(10, 2021)

2.5.13 If $\mathbf{a} = 2\hat{\imath} - \hat{\jmath} + 2\hat{k}$ and $\mathbf{b} = 5\hat{\imath} - 3\hat{\jmath} - 4\hat{k}$, then find the ratio $\frac{\text{projection of vector } \mathbf{a} \text{ on } \mathbf{b}}{\text{projection of vector } \mathbf{b} \text{ on vector } \mathbf{a}}$.

(12, 2021)

2.5.14 Let $\hat{a}$ and $\hat{b}$ be two unit vectors. If the vectors $\mathbf{c} = \hat{a} + 2\hat{b}$ and $\mathbf{d} = 5\hat{a} - 4\hat{b}$ are perpendicular to each other, then find the angle between the vectors $\hat{a}$ and $\hat{b}$.

(12, 2021)

2.5.15 Show that $|\mathbf{a}|\,\mathbf{b} + |\mathbf{b}|\,\mathbf{a}$ is perpendicular to $|\mathbf{a}|\,\mathbf{b} - |\mathbf{b}|\,\mathbf{a}$, for any two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$. (12, 2021)

2.5.16 Find the value of p for which the lines $\frac{1-x}{3} = \frac{2y-14}{2p} = \frac{z-3}{2}$ and $\frac{1-x}{3p} = \frac{y-5}{1} = \frac{6-z}{5}$ are perpendicular. (12, 2019)

2.5.17 Find a unit vector perpendicular to both the vectors $\overrightarrow{a}$ and $\overrightarrow{b}$, where $\overrightarrow{a} = \hat{\imath} - 7\hat{\jmath} + 7\hat{k}$ and $\overrightarrow{b} = 3\hat{\imath} - 2\hat{\jmath} + 2\hat{k}$. (12, 2019)

2.5.18 Let $\overrightarrow{a} = \hat{\imath} + 2\hat{\jmath} - 3\hat{k}$ and $\overrightarrow{b} = 3\hat{\imath} - \hat{\jmath} + 2\hat{k}$. Show that the vectors $\overrightarrow{a} + \overrightarrow{b}$ and $\overrightarrow{a} - \overrightarrow{b}$ are perpendicular to each other. (12, 2019)

2.5.19 Find the value of P for which the lines $\frac{1-x}{3} = \frac{2y-14}{2P} = \frac{z-3}{2}$ and $\frac{1-x}{3P} = \frac{y-5}{1} = \frac{6-z}{5}$ are perpendicular (12, 2018)

2.5.20 Let $\overrightarrow{a}$, $\overrightarrow{b}$ and $\overrightarrow{c}$ be three vectors such that $\left|\overrightarrow{a}\right| = 1$, $\left|\overrightarrow{b}\right| = 2$ and $\left|\overrightarrow{c}\right| = 3$. If the projection of $\overrightarrow{b}$ along $\overrightarrow{a}$ is equal to the projection of $\overrightarrow{c}$ along $\overrightarrow{a}$; and $\overrightarrow{b}$, $\overrightarrow{c}$ are perpendicular to each other, then find $\left|3\overrightarrow{a} - 2\overrightarrow{b} + 2\overrightarrow{c}\right|$. (12, 2018)

2.5.21 Find the value of P for which the lines $\frac{1-x}{3} = \frac{2y-14}{2p} = \frac{z-3}{2}$ and $\frac{1-x}{3p} = \frac{y-5}{1} = \frac{6-z}{5}$ are perpendicular. (12, 2018)

2.5.22 Let $\mathbf{a} = \hat{i} + 2\hat{j} - 3\hat{k}$ and $\mathbf{b} = 3\hat{i} - \hat{j} + 2\hat{k}$ be two vectors. Show that the vectors $(\mathbf{a} + \mathbf{b})$ and $(\mathbf{a} - \mathbf{b})$ are perpendicular to each other. (12, 2018)

2.5.23 Find the angle between the line $\overrightarrow{r} = \left(2\hat{\imath} - \hat{\jmath} + 3\hat{k}\right) + \lambda\left(3\hat{\imath} - \hat{\jmath} + 2\hat{k}\right)$ and the plane $\overrightarrow{r}.\left(\hat{\imath} + \hat{\jmath} + \hat{k}\right) = 3$. (12, 2018)

2.5.24 If $\left|\overrightarrow{a}\right| = 2$, $\left|\overrightarrow{b}\right| = 7$ and $\overrightarrow{a}\ X\ \overrightarrow{b} = 3\hat{i} + 2\hat{j} + 6\hat{k}$, find the angle between $\overrightarrow{a}$ and $\overrightarrow{b}$.

(12, 2018)

2.5.25 If $\mathbf{a} = 2\hat{\imath} - \hat{\jmath} - 2\hat{k}$ and $\mathbf{b} = 7\hat{\imath} + 2\hat{\jmath} - 3\hat{k}$, then express $\mathbf{b}$ in the form $\mathbf{b} = \mathbf{b_1} + \mathbf{b_2}$, where $\mathbf{b_1}$ is parallel to $\mathbf{a}$ and $\mathbf{b_2}$ is perpendicular to $\mathbf{a}$. (12, 2017)

2.5.26 Prove that the points $\left(3,0\right)$, $\left(6,4\right)$, and $\left(-1,3\right)$ are the vertices of a right-angled isosceles triangle. (10, 2016)

2.5.27 Write the number of vectors of unit length perpendicular to both the vectors $\mathbf{a} =$

$2\hat{i}+\hat{j}+2\hat{k}$ and $\mathbf{b}=\hat{j}+\hat{k}$. (12, 2016)

2.5.28 Find the projection of the vector $\mathbf{a}=2\overrightarrow{i}+3\overrightarrow{j}+2\overrightarrow{k}$ on the vector $\mathbf{b}=2\overrightarrow{i}+2\overrightarrow{j}+\overrightarrow{k}$. (12, 2015)

2.5.29 The points $\mathbf{A}(4,7)$, $\mathbf{B}(p,3)$ and $\mathbf{C}(7,3)$ are the vertices of a right triangle, right-angled at $\mathbf{B}$. Find the value of p. (10, 2015)

2.5.30 If the two lines $L_1: x=5, \frac{y}{3-\alpha}=\frac{z}{-2}$ and $L_2: x=2, \frac{y}{-1}=\frac{z}{z-\alpha}$ are perpendicular, then the value of α is ______. (12, 2021)

2.5.31 If two vertices of an equilateral triangle are $(3,0)$ and $(6,0)$, find the third vertex. (10, 2011)

2.5.32 Show that the points $(7,10)$, $(-2,5)$ and $(3,4)$ are vertices of an isosceles right triangle. (10, 2020)

2.5.33 Verify if the vectors $\overrightarrow{a}=6\hat{i}+2\hat{j}-8\hat{k}$, $\overrightarrow{b}=10\hat{i}-2\hat{j}-6\hat{k}$, $\overrightarrow{c}=4\hat{i}-4\hat{j}+2\hat{k}$ represent the sides of a right angled triangle. (12, 2024)

2.5.34 Show that the points $(-2,3)$, $(8,3)$ and $(6,7)$ are the vertices of a right-angled triangle. (10, 2024)

2.6 Vector Product

2.6.1 Find $\left|\overrightarrow{a}\times\overrightarrow{b}\right|$, if $\overrightarrow{a}=\hat{i}-7\hat{j}+7\hat{k}$ and $\overrightarrow{b}=3\hat{i}-2\hat{j}+2\hat{k}$.
Solution: From (2.1.8.3),

$$\begin{vmatrix}\mathbf{A}_{23} & \mathbf{B}_{23}\end{vmatrix}=\begin{vmatrix}-7 & -2\\ 7 & 2\end{vmatrix}=0 \tag{2.6.1.1}$$

$$\begin{vmatrix}\mathbf{A}_{31} & \mathbf{B}_{31}\end{vmatrix}=\begin{vmatrix}1 & 3\\ 7 & 2\end{vmatrix}=-19 \tag{2.6.1.2}$$

$$\begin{vmatrix}\mathbf{A}_{12} & \mathbf{B}_{12}\end{vmatrix}=\begin{vmatrix}1 & 3\\ -7 & -2\end{vmatrix}=19, \tag{2.6.1.3}$$

$$\|\mathbf{a}\times\mathbf{b}\|=\left\|\begin{pmatrix}\begin{vmatrix}\mathbf{A}_{23} & \mathbf{B}_{23}\end{vmatrix}\\ \begin{vmatrix}\mathbf{A}_{31} & \mathbf{B}_{31}\end{vmatrix}\\ \begin{vmatrix}\mathbf{A}_{12} & \mathbf{B}_{12}\end{vmatrix}\end{pmatrix}\right\|=19\sqrt{2} \tag{2.6.1.4}$$

from (2.1.9.1).

2.6.2 Find λ and μ if $(2\hat{i}+6\hat{j}+27\hat{k})\times(\hat{i}+\lambda\hat{j}+\mu\hat{k})=\overrightarrow{0}$.
Solution: From Formula 2.1.11, performing row reduction,

$$\begin{pmatrix}2 & 6 & 27\\ 1 & \lambda & \mu\end{pmatrix}\xleftrightarrow{R_2\leftarrow 2R_2-R_1}\begin{pmatrix}2 & 6 & 27\\ 0 & 2\lambda-6 & 2\mu-27\end{pmatrix} \tag{2.6.2.1}$$

For the above matrix to have rank 1,

$$\mu=\frac{27}{2},\lambda=3. \tag{2.6.2.2}$$

2.6.3 Find the area of the triangle with vertices $A(1,1,2)$, $B(2,3,5)$ and $C(1,5,5)$.

Solution:

$$\because \mathbf{B}-\mathbf{A} = \begin{pmatrix}1\\2\\3\end{pmatrix}, \mathbf{C}-\mathbf{A} = \begin{pmatrix}0\\4\\3\end{pmatrix}, \tag{2.6.3.1}$$

$$\frac{1}{2}\left\|\begin{pmatrix}1\\2\\3\end{pmatrix}\times\begin{pmatrix}0\\4\\3\end{pmatrix}\right\| = \frac{1}{2}\left\|\begin{pmatrix}-6\\3\\4\end{pmatrix}\right\| = \frac{\sqrt{61}}{2} \tag{2.6.3.2}$$

using (2.1.13.2), which is the the desired area.

2.6.4 Find the area of the parallelogram whose adjacent sides are determined by the vectors $\overrightarrow{a} = \hat{i} - \hat{j} + 3\hat{k}$ and $\overrightarrow{b} = 2\hat{i} - 7\hat{j} + \hat{k}$.

Solution: From (2.1.14.1), the desired area is obtained as

$$\left\|\begin{pmatrix}1\\-1\\3\end{pmatrix}\times\begin{pmatrix}2\\-7\\1\end{pmatrix}\right\| = \left\|\begin{pmatrix}20\\5\\-5\end{pmatrix}\right\| = 15\sqrt{2} \tag{2.6.4.1}$$

2.6.5 Find the area of a rhombus if its vertices are $A(3,0), B(4,5), C(-1,4)$ and $D(-2,-1)$ taken in order.

Solution: The area of the rhombus is

$$\left\|\left(\mathbf{A}-\mathbf{D}\right)\times\left(\mathbf{B}-\mathbf{A}\right)\right\| = \begin{vmatrix}5 & 1\\1 & 5\end{vmatrix} = 24 \tag{2.6.5.1}$$

2.6.6 Let the vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ be such that $|\overrightarrow{a}| = 3$ and $|\overrightarrow{b}| = \frac{\sqrt{2}}{3}$, then $\overrightarrow{a}\times\overrightarrow{b}$ is a unit vector, if the angle between $\overrightarrow{a}$ and $\overrightarrow{b}$ is ______.

Solution: From the given information and (2.1.12.1)

$$\|\mathbf{a}\times\mathbf{b}\| = \|\mathbf{a}\|\,\|\mathbf{b}\|\sin\theta = 1 \tag{2.6.6.1}$$

$$\implies \sin\theta = \frac{1}{\|\mathbf{a}\|\,\|\mathbf{b}\|} = \frac{1}{\sqrt{2}} \tag{2.6.6.2}$$

$$\implies \theta = \frac{\pi}{4} \tag{2.6.6.3}$$

2.6.7 Area of a rectangle having vertices A, B, C and D with position vectors $-\hat{i} + \frac{1}{2}\hat{j} + 4\hat{k}, \hat{i} + \frac{1}{2}\hat{j} + 4\hat{k}, \hat{i} - \frac{1}{2}\hat{j} + 4\hat{k}$ and $-\hat{i} - \frac{1}{2}\hat{j} + 4\hat{k}$, respectively is

Solution: Since

$$\mathbf{A}-\mathbf{B} = \begin{pmatrix}-2\\0\\0\end{pmatrix} \tag{2.6.7.1}$$

$$\mathbf{C}-\mathbf{B} = \begin{pmatrix}0\\-1\\0\end{pmatrix} \tag{2.6.7.2}$$

area of the rectangle is

$$\|(\mathbf{A}-\mathbf{B})\times(\mathbf{C}-\mathbf{D})\| = 2 \tag{2.6.7.3}$$

2.6.8 Find the area of the triangle whose vertices are

a) $(2,3), (-1,0), (2,-4)$

b) $(-5,-1), (3,-5), (5,2)$

Solution: See Table 2.6.8.

TABLE 2.6.8

	$\mathbf{A}-\mathbf{B}$	$\mathbf{A}-\mathbf{C}$	$\frac{1}{2}\|(\mathbf{A}-\mathbf{B})\times(\mathbf{A}-\mathbf{C})\|$
a)	$\begin{pmatrix}3\\3\end{pmatrix}$	$\begin{pmatrix}0\\7\end{pmatrix}$	$\frac{21}{2}$
b)	$\begin{pmatrix}-8\\4\end{pmatrix}$	$\begin{pmatrix}-10\\-3\end{pmatrix}$	32

2.6.9 The area of a triangle with vertices $\mathbf{A}(3,0), \mathbf{B}(7,0)$ and $\mathbf{C}(8,4)$ is ______.

2.6.10 Find the area of the triangle whose vertices are $(-8,4), (-6,6)$ and $(-3,9)$.

2.6.11 If $\mathbf{D}\left(\frac{-1}{2},\frac{5}{2}\right), \mathbf{E}(7,3)$ and $\mathbf{F}\left(\frac{7}{2},\frac{7}{2}\right)$ are the midpoints of sides of $\triangle ABC$, find the area of the $\triangle ABC$.

2.6.12 Find the sine of the angle between the vectors $\mathbf{a} = 3\hat{i}+\hat{j}+2\hat{k}$ and $\mathbf{b} = 2\hat{i}-2\hat{j}+4\hat{k}$.

2.6.13 Using vectors, find the area of $\triangle ABC$ with vertices A(1,2,3), B(2,-1,4) and C(4,5,-1).

2.6.14 Find the area of the parallelogram whose diagonals are $2\hat{i}-\hat{j}+\hat{k}$ and $\hat{i}+3\hat{j}-\hat{k}$.

2.6.15 Find the area of the triangle formed by joining the mid-points of the sides of the triangle whose vertices are $A(0,-1), B(2,1)$ and $C(0,3)$. Find the ratio of this area to the area of the given triangle.

Solution: Using (1.1.4.1), the mid point coordinates are given by

$$\mathbf{P} = \frac{1}{2}(\mathbf{A}+\mathbf{B}) = \begin{pmatrix}1\\0\end{pmatrix} \tag{2.6.15.1}$$

$$\mathbf{Q} = \frac{1}{2}(\mathbf{B}+\mathbf{C}) = \begin{pmatrix}1\\2\end{pmatrix} \tag{2.6.15.2}$$

$$\mathbf{R} = \frac{1}{2}(\mathbf{A}+\mathbf{C}) = \begin{pmatrix}0\\1\end{pmatrix} \tag{2.6.15.3}$$

$$\because \mathbf{P}-\mathbf{Q} = \begin{pmatrix}0\\-2\end{pmatrix}, \mathbf{Q}-\mathbf{R} = \begin{pmatrix}1\\1\end{pmatrix} \tag{2.6.15.4}$$

$$ar(PQR) = \frac{1}{2}\|(\mathbf{P}-\mathbf{Q})\times(\mathbf{Q}-\mathbf{R})\| = 1 \tag{2.6.15.5}$$

Similarly,

$$\mathbf{A}-\mathbf{B} = \begin{pmatrix}-2\\-2\end{pmatrix}, \mathbf{A}-\mathbf{C} = \begin{pmatrix}0\\-4\end{pmatrix} \tag{2.6.15.6}$$

$$\implies ar(ABC) = \frac{1}{2}\|(\mathbf{A}-\mathbf{B})\times(\mathbf{A}-\mathbf{C})\| = 4 \tag{2.6.15.7}$$

$$\implies \frac{ar\,(PQR)}{ar\,(ABC)} = \frac{1}{4} \tag{2.6.15.8}$$

See Fig. 2.6.15.1

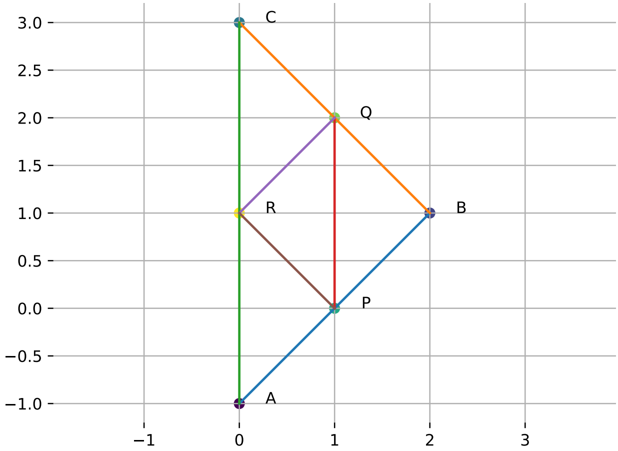


Fig. 2.6.15.1

2.6.16 Find the area of the quadrilateral whose vertices, taken in order, are $A(-4,-2), B(-3,-5), C(3,-2)$ and $D(2,3)$.

Solution: See Fig. 2.6.16.1

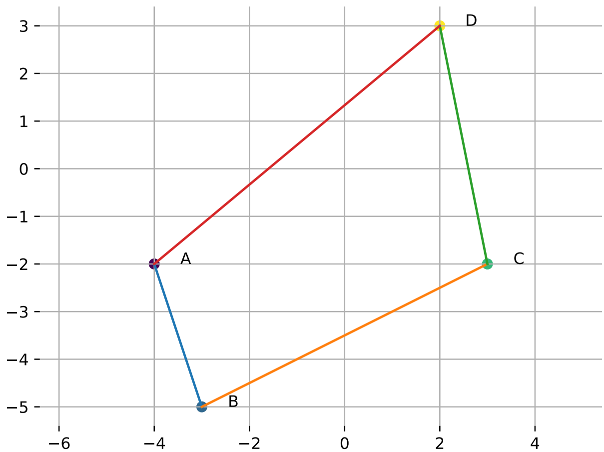


Fig. 2.6.16.1

$$\because \mathbf{A}-\mathbf{B} = \begin{pmatrix}-1\\3\end{pmatrix}, \mathbf{A}-\mathbf{D} = \begin{pmatrix}-6\\-5\end{pmatrix}, \tag{2.6.16.1}$$

$$\mathbf{B}-\mathbf{C} = \begin{pmatrix}-6\\-5\end{pmatrix}, \mathbf{B}-\mathbf{D} = \begin{pmatrix}-3\\-8\end{pmatrix}, \tag{2.6.16.2}$$

$$ar(ABD) = \frac{1}{2}\left\|(\mathbf{A}-\mathbf{B})\times(\mathbf{A}-\mathbf{D})\right\| = \frac{23}{2} \tag{2.6.16.3}$$

$$ar(BCD) = \frac{1}{2}\left\|(\mathbf{B}-\mathbf{C})\times(\mathbf{B}-\mathbf{D})\right\| = \frac{33}{2} \tag{2.6.16.4}$$

$$\implies ar(ABCD) = ar(ABD) + ar(BCD) = 28 \tag{2.6.16.5}$$

2.6.17 Find the area of region bounded by the triangle whose vertices are $(1,0),(2,2)$ and $(3,1)$.

2.6.18 Find the area of region bounded by the triangle whose vertices are $(-1,0),(1,3)$ and $(3,2)$.

2.6.19 Find the area of the $\triangle ABC$, coordinates of whose vertices are $\mathbf{A}(2,0),\mathbf{B}(4,5)$ and $\mathbf{C}(6,3)$.

2.6.20 The two adjacent sides of a parallelogram are $\mathbf{a} = 2\hat{i}-4\hat{j}+5\hat{k}$ and $\mathbf{b} = \hat{i}-2\hat{j}-3\hat{k}$. Find the unit vector parallel to its diagonal. Also, find its area.
Solution: The diagonals of the parallelogram are given by

$$\mathbf{a}+\mathbf{b} = \begin{pmatrix}3\\-6\\2\end{pmatrix}, \mathbf{a}-\mathbf{b} = \begin{pmatrix}1\\-2\\8\end{pmatrix} \tag{2.6.20.1}$$

and the corresponding unit vectors are

$$\frac{\mathbf{a}+\mathbf{b}}{\|\mathbf{a}+\mathbf{b}\|} = \begin{pmatrix}\frac{3}{\sqrt{45}}\\-\frac{6}{\sqrt{45}}\\\frac{2}{\sqrt{45}}\end{pmatrix}, \frac{\mathbf{a}-\mathbf{b}}{\|\mathbf{a}-\mathbf{b}\|} = \begin{pmatrix}\frac{1}{\sqrt{69}}\\-\frac{2}{\sqrt{69}}\\\frac{8}{\sqrt{69}}\end{pmatrix} \tag{2.6.20.2}$$

The area of the parallelogram is given by

$$\|\mathbf{a}\times\mathbf{b}\| = \left\|\begin{pmatrix}22\\-11\\0\end{pmatrix}\right\| = \sqrt{605} \tag{2.6.20.3}$$

2.6.21 Find the area of the triangle whose vertices are $(3,8),(-4,2)$ and $(5,1)$.

2.6.22 Find $\left|\vec{a}\times\vec{b}\right|$, if $\vec{a} = 2\hat{i}+\hat{j}+3\hat{k}$, and $\vec{b} = 3\hat{i}+5\hat{j}-2\hat{k}$.

2.6.23 Find the area of a triangle having the points $A(1,1,1),B(1,2,3)$ and $C(2,3,1)$ as its vertices.

2.6.24 Find the area of a parallelogram whose adjacent sides are given by the vectors $\vec{a} = 3\hat{i}+\hat{j}+4\hat{k}$ and $\vec{b} = \hat{i}-\hat{j}+\hat{k}$.

2.6.25 Find the area of a triangle formed by the points $A(5,2),B(4,7)$ and $(7,-4)$.

2.6.26 Find the area of the triangle formed by the points $P(-1.5,3),Q(6,-2)$ and $R(-3,4)$.

2.6.27 If $A(-5,7),B(-4,-5),C(-1,-6)$ and $D(4,5)$ are the vertices of a quadrilateral, find the area of quadrilateral ABCD.

2.6.28 Verify that a median of a triangle divides it into two triangles of equal areas for $\triangle ABC$ whose vertices are $\mathbf{A}(4,-6), \mathbf{B}(3,2)$, and $\mathbf{C}(5,2)$.

Solution:

$$\mathbf{D} = \frac{\mathbf{B}+\mathbf{C}}{2} = \begin{pmatrix}4\\0\end{pmatrix}, \tag{2.6.28.1}$$

$$\mathbf{A}-\mathbf{B} = \begin{pmatrix}1\\-4\end{pmatrix}, \mathbf{A}-\mathbf{D} = \begin{pmatrix}0\\-6\end{pmatrix} \tag{2.6.28.2}$$

$$\implies ar(ABD) = \frac{1}{2}\left\|(\mathbf{A}-\mathbf{B})\times(\mathbf{A}-\mathbf{D})\right\| = 3 \tag{2.6.28.3}$$

$$\mathbf{A}-\mathbf{C} = \begin{pmatrix}-1\\-8\end{pmatrix}, \mathbf{A}-\mathbf{D} = \begin{pmatrix}0\\-6\end{pmatrix} \tag{2.6.28.4}$$

$$\implies ar(ACD) = \frac{1}{2}\left\|(\mathbf{A}-\mathbf{C})\times(\mathbf{A}-\mathbf{D})\right\| \tag{2.6.28.5}$$

$$= 3 = ar(ABD) \tag{2.6.28.6}$$

See Fig. 2.6.28.1.

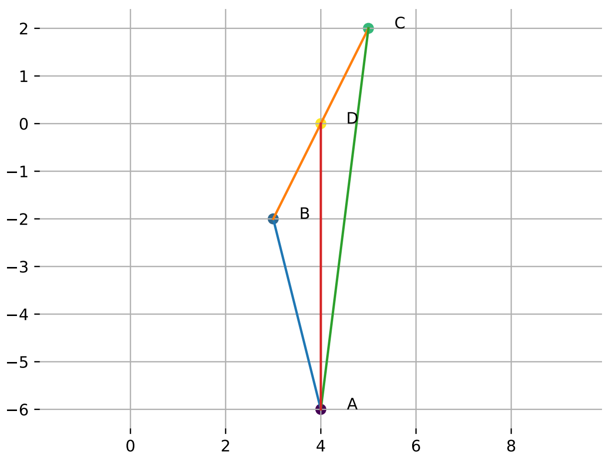


Fig. 2.6.28.1

2.6.29 The vertices of a $\triangle ABC$ are $\mathbf{A}(4,6), \mathbf{B}(1,5)$ and $\mathbf{C}(7,2)$. A line is drawn to intersect sides AB and AC at $\mathbf{D}$ and $\mathbf{E}$ respectively, such that $\frac{AD}{AB} = \frac{AE}{AC} = \frac{1}{4}$. Calculate the area of $\triangle ADE$ and compare it with the area of the $\triangle ABC$.

Solution: See Fig. 2.6.29.1.

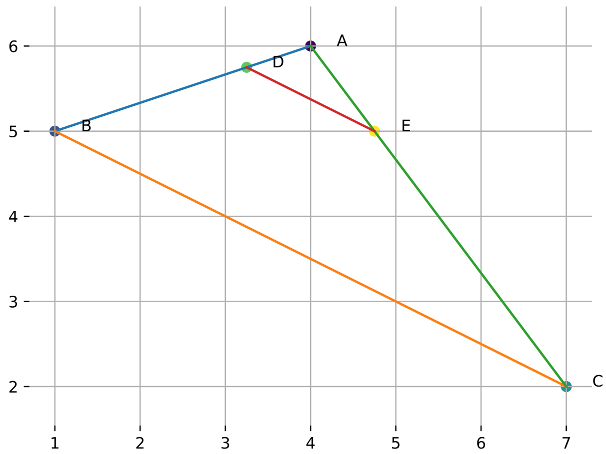


Fig. 2.6.29.1

Using section formula (1.1.4.1),

$$\mathbf{D} = \frac{3\mathbf{A}+\mathbf{B}}{4} = \frac{1}{4}\begin{pmatrix}13\\23\end{pmatrix} \tag{2.6.29.1}$$

$$\mathbf{E} = \frac{3\mathbf{A}+\mathbf{C}}{4} = \frac{1}{4}\begin{pmatrix}19\\20\end{pmatrix} \tag{2.6.29.2}$$

$$\mathbf{A}-\mathbf{D} = \frac{1}{4}\begin{pmatrix}3\\1\end{pmatrix}, \ \mathbf{A}-\mathbf{E} = \frac{1}{4}\begin{pmatrix}-3\\1\end{pmatrix} \tag{2.6.29.3}$$

$$\mathbf{A}-\mathbf{B} = \begin{pmatrix}3\\1\end{pmatrix}, \mathbf{B}-\mathbf{C} = \begin{pmatrix}-6\\3\end{pmatrix} \tag{2.6.29.4}$$

yielding

$$ar(ABD) = \frac{1}{2}\left\|(\mathbf{A}-\mathbf{D})\times(\mathbf{A}-\mathbf{E})\right\| = \frac{15}{32} \tag{2.6.29.5}$$

$$ar(ABC) = \frac{1}{2}\left\|(\mathbf{A}-\mathbf{B})\times(\mathbf{B}-\mathbf{C})\right\| = \frac{15}{2} \tag{2.6.29.6}$$

$$\implies \frac{ar\,(ADE)}{ar\,(ABC)} = \frac{1}{16} \tag{2.6.29.7}$$

2.6.30 If the area of triangle is 35 square units with vertices $(2,6), (5,-4)$ and $(k,4)$. Then k is ______.

2.6.31 Find values of k if area of triangle is 4 square units and vertices are

a) $(k,0), (4,0), (0,2)$

b) $(-2,0), (0,4), (0,k)$

2.6.32 Find the area of the triangle whose vertices are $(1,-1), (-4,6)$ and $(-3,5)$.

2.6.33 Find area of the triangle with vertices at the point given in each of the following:

a) $(1,0),(6,0),(4,3)$
b) $(2.7),(1,1),(10,8)$
c) $(-2,-3),(3,2),(-1,8)$

2.6.34 Find the area of region bounded by the triangle whose vertices are $(-1,1),(0,5)$ and $(3,2)$.

2.6.35 The value of $\hat{i}\cdot(\hat{j}\times\hat{k})+\hat{j}\cdot(\hat{i}\times\hat{k})+\hat{k}\cdot(\hat{i}\times\hat{j})$ is ______.

2.6.36 The value of $\hat{i}\cdot(\hat{j}\times\hat{k})+\hat{j}\cdot(\hat{i}\times\hat{k})+\hat{k}\cdot(\hat{i}\times\hat{j})$ is ______.

2.6.37 The vector from origin to the points **A** and **B** are $\mathbf{a}=2\hat{i}-3\hat{j}+2\hat{k}$ and $\mathbf{b}=2\hat{i}+3\hat{j}+\hat{k}$, respectively, then the area of $\triangle OAB$ is ______.

2.6.38 If $\mathbf{a}=\hat{i}+\hat{j}+\hat{k}$ and $\mathbf{b}=\hat{j}-\hat{k}$, find a vector $\mathbf{c}$ such that $\mathbf{a}\times\mathbf{c}=\mathbf{b}$ and $\mathbf{a}\cdot\mathbf{c}=3$.

2.6.39 The area of the quadrilateral ABCD, where A$(0,4,1)$, B$(2,3,-1)$, C$(4,5,0)$ and D$(2,6,2)$, is equal to ______.

2.6.40 Draw a quadrilateral in the Cartesian plane, whose vertices are

$$\mathbf{A}=\begin{pmatrix}-4\\5\end{pmatrix},\ \mathbf{B}=\begin{pmatrix}0\\7\end{pmatrix},\ \mathbf{C}=\begin{pmatrix}5\\-5\end{pmatrix},\ \mathbf{D}=\begin{pmatrix}-4\\-2\end{pmatrix}. \tag{2.6.40.1}$$

Also, find its area.

Solution: See Fig. 2.6.40.1. From (2.1.13.2),

$$ar\,(ABCD)=\frac{121}{2} \tag{2.6.40.2}$$

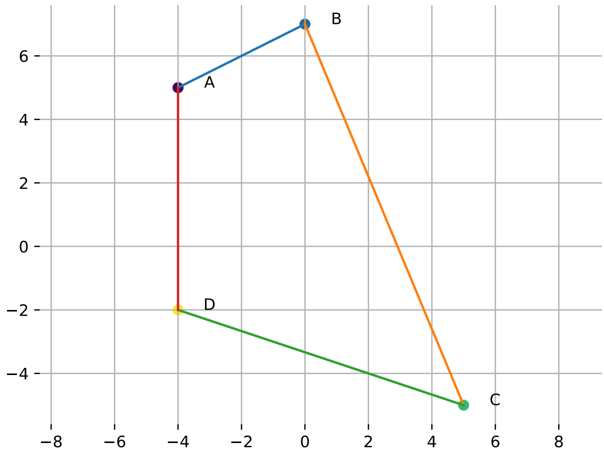


Fig. 2.6.40.1: Plot of quadrilateral $ABCD$

2.7 CBSE

2.7.1 The area of a triangle formed by vertices **O**, **A** and **B**, where $\overrightarrow{OA} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\overrightarrow{OB} = -3\hat{i} - 2\hat{j} + \hat{k}$ is (12, 2020)

2.7.2 Find the area of the triangle whose vertices are $(-1, 1)$, $(0, 5)$, and $(3, 2)$. (12, 2023)

2.7.3 If $\overrightarrow{a}$ and $\overrightarrow{b}$ are two vectors such that

$$\overrightarrow{a} = \hat{i} - \hat{j} + \hat{k} \tag{2.7.3.1}$$

and

$$\overrightarrow{b} = 2\hat{i} - \hat{j} - 3\hat{k}, \tag{2.7.3.2}$$

then find the vector $\overrightarrow{c}$, given that

$$\overrightarrow{a} \times \overrightarrow{c} = \overrightarrow{b} \tag{2.7.3.3}$$

and

$$\overrightarrow{a} \cdot \overrightarrow{c} = 4. \tag{2.7.3.4}$$

(12, 2023)

2.7.4 If

$$\overrightarrow{a} = 2\hat{i} + \hat{j} + 3\hat{k}, \tag{2.7.4.1}$$

$$\overrightarrow{b} = -\hat{i} + 2\hat{j} + \hat{k}, \tag{2.7.4.2}$$

and

$$\overrightarrow{c} = 3\hat{i} + \hat{j} + 2\hat{k}, \tag{2.7.4.3}$$

then find $\overrightarrow{a} \cdot (\overrightarrow{b} \times \overrightarrow{c})$. (12, 2023)

2.7.5 Find the area of the triangle with vertices $\mathbf{A}(-1, 0, -2)$, $\mathbf{B}(0, 2, 1)$, and $\mathbf{C}(-1, 4, 1)$. (12, 2023)

2.7.6 Find the area of the triangle with vertices $(2, 0)$, $(4, 5)$, and $(1, 4)$. (12, 2023)

2.7.7 Find the area of the quadrilateral $ABCD$ whose vertices are $\mathbf{A}(-4, -3)$, $\mathbf{B}(3, -1)$, $\mathbf{C}(0, 5)$, and $\mathbf{D}(-4, 2)$. (10, 2022)

2.7.8 Find $|\overrightarrow{a} \times \overrightarrow{b}|$, if $\overrightarrow{a} = 2\hat{i} + \hat{j} + 3\hat{k}$ and $\overrightarrow{b} = 3\hat{i} + 5\hat{j} - 2\hat{k}$. (12, 2019)

2.7.9 Find the area of the triangle whose vertices are $(1, 0)$, $(2, 2)$, and $(3, 1)$. (12, 2019)

2.7.10 Find the area of the triangle whose vertices are $(-1, 1)$, $(0, 5)$, and $(3, 2)$. (12, 2019)

2.7.11 Find the area of a triangle whose vertices are $(1, -1)$, $(-4, 6)$, and $(-3, -5)$. (10, 2019)

2.7.12 Find the area of the triangle formed by joining the midpoints of the sides of the triangle ABC, whose vertices are $\mathbf{A}(0, -1)$, $\mathbf{B}(2, 1)$, and $\mathbf{C}(0, 3)$. (10, 2019)

2.7.13 Given vertices $\mathbf{A}(-5, 7)$, $\mathbf{B}(-4, -5)$, $\mathbf{C}(-1, -6)$, and $\mathbf{D}(4, 5)$ of a quadrilateral. Find the area of quadrilateral $ABCD$. (10, 2018)

2.7.14 If θ is the angle between the two vectors $\hat{i} - 2\hat{j} + 3\hat{k}$ and $3\hat{i} - 2\hat{j} + \hat{k}$, find $\sin\theta$. (12, 2018)

2.7.15 Find the volume of a cuboid whose edges are given by $-3\hat{i}+7\hat{j}+5\hat{k}$, $-5\hat{i}+7\hat{j}-3\hat{k}$ and $7\hat{i}-5\hat{j}-3\hat{k}$.

(12, 2018)

2.7.16 Find $\left|\vec{a}\times\vec{b}\right|$, if $\vec{a}=2\hat{i}+\hat{j}+3\hat{k}$ and $\vec{b}=3\hat{i}+5\hat{j}-2\hat{k}$. (12, 2018)

2.7.17 Show that the points $\mathbf{A}=2\hat{i}-\hat{j}+\hat{k}$, $\mathbf{B}=\hat{i}-3\hat{j}-5\hat{k}$, and $\mathbf{C}=3\hat{i}-4\hat{j}-4\hat{k}$ respectively, are the vertices of a right-angled triangle. Hence, find the area of the triangle.

(12, 2017)

2.7.18 The vertices of $\triangle ABC$ are $\mathbf{A}\left(4,6\right)$, $\mathbf{B}\left(1,5\right)$, and $\mathbf{C}\left(7,2\right)$. A line-segment DE is drawn to intersect the sides AB and AC at $\mathbf{D}$ and $\mathbf{E}$ respectively such that $\frac{AD}{AB}=\frac{AE}{AC}=\frac{1}{3}$. Calculate the area of $\triangle ADE$ and compare it with the area of $\triangle ABC$. (10, 2016)

2.7.19 Find λ and μ if

$$\left(\hat{i}+3\hat{j}+9\hat{k}\right)\times\left(3\hat{i}-\lambda\hat{j}+\mu\hat{k}\right)=\vec{0}.$$

(12, 2016)

2.7.20 The two adjacent sides of a parallelogram are $2\hat{i}-4\hat{j}-5\hat{k}$ and $2\hat{i}+2\hat{j}+3\hat{k}$. Find the two unit vectors parallel to its diagonals. Using the diagonal vectorsFind the area of the parallelogram. (12, 2016)

2.7.21 Find the values of k so that the area of the triangle with vertices $(1,-1)$, $(-4,2k)$, and $(-k,-5)$ is 24 sq. units. (10, 2015)

2.7.22 The area of a triangle whose vertices are $(5,0)$, $(8,0)$ and $(8,4)$ (in sq. units) is

(10, 2012)

2.7.23 For what value of k, $(k>0)$, is the area of the triangle with vertices $(-2,5)$, $(k,-4)$, and $(2k+1,10)$ equal to 52 sq. units? (10, 2012)

2.7.24 If the vertices of a triangle are $(1,-3)$, $(4,p)$ and $(-9,7)$ and its area is 15 sq. units. Find the value(s) of p. (10, 2012)

2.7.25 Find the area of quadrilateral $ABCD$ whose vertices are $\mathbf{A}(-3,-1)$, $\mathbf{B}(-2,-4)$, $\mathbf{C}(4,-1)$ and $\mathbf{D}(3,4)$. (10, 2012)

2.7.26 Find the area of the quadrilateral $ABCD$, whose vertices are $\mathbf{A}(-3,-1)$, $\mathbf{B}(-2,-4)$, $\mathbf{C}(4,-1)$, and $\mathbf{D}(3,4)$. (10, 2011)

2.7.27 Find the area of triangle ABC, whose vertices are $\mathbf{A}(2,5)$, $\mathbf{B}(4,7)$ and $\mathbf{C}(6,2)$. (12, 2018)

2.7.28 Find the area of the triangle whose vertices are $(2,3),(3,5)$ and $(4,4)$ (12, 2018)

2.7.29 The area of a triangle formed by vertices $\mathbf{O}$, $\mathbf{A}$ and $\mathbf{B}$, where $\overrightarrow{OA}=\hat{i}+2\hat{j}+3\hat{k}$ and $\overrightarrow{OB}=-3\hat{i}-2\hat{j}+\hat{k}$ is (12, 2020)

2.7.30 Find the value of k so that the area of triangle ABC with $\mathbf{A}(k+1,1)$, $\mathbf{B}(4,-3)$ and $\mathbf{C}(7,-k)$ is 6 square units. (10, 2019)

2.7.31 Find the area of triangle ABC, whose vertices are $\mathbf{A}(2,5)$, $\mathbf{B}(4,7)$ and $\mathbf{C}(6,2)$. (12, 2018)

2.7.32 Find the area of the triangle whose vertices are $(2,3),(3,5)$ and $(4,4)$ (12, 2018)

2.7.33 Find the value of p if

$$(2\hat{i}+6\hat{j}+27\hat{k})\times(\hat{i}+3\hat{j}+p\hat{k})=\mathbf{0}.$$

(10, 2009)

2.8.1 If $(-4, 3)$ and $(4, 3)$ are two vertices of an equilateral triangle. Find the coordinates of the third vertex, given that the origin lies in the interior of the triangle.

2.8.2 $\mathbf{A}(6, 1), \mathbf{B}(8, 2)$ and $\mathbf{C}(9, 4)$ are three vertices of a parallelogram ABCD. If $\mathbf{C}$ is the midpoint of DC find the area of $\triangle ADE$.

2.8.3 Find $\left|\overrightarrow{x}\right|$, if for a unit vector $\overrightarrow{a}, (\overrightarrow{x} - \overrightarrow{a}) \cdot (\overrightarrow{x} + \overrightarrow{a})$=12.

Solution: From the given information,

$$(\mathbf{x} - \mathbf{a})^\top (\mathbf{x} + \mathbf{a}) = 12 \tag{2.8.3.1}$$

$$\implies \|\mathbf{x}\|^2 - \|\mathbf{a}\|^2 = 12 \tag{2.8.3.2}$$

$$\implies \|\mathbf{x}\| = \sqrt{13} \tag{2.8.3.3}$$

2.8.4 Let $\mathbf{a}$ and $\mathbf{b}$ be two unit vectors and θ the angle between them. Then $\mathbf{a} + \mathbf{b}$ is a unit vector if

a) $\theta = \frac{\pi}{4}$
b) $\theta = \frac{\pi}{3}$
c) $\theta = \frac{\pi}{2}$
d) $\theta = \frac{2\pi}{3}$

Solution:

$$\because \|\mathbf{a}\| = \|\mathbf{b}\| = 3 \|\mathbf{a} + \mathbf{b}\| = 1, \tag{2.8.4.1}$$

$$\|\mathbf{a} + \mathbf{b}\|^2 = 1^2 \tag{2.8.4.2}$$

$$\implies \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 + 2\mathbf{a}^\top\mathbf{b} = 1 \tag{2.8.4.3}$$

$$\implies (\|\mathbf{a}\| \|\mathbf{b}\| \cos\theta) = \frac{-1}{2} \tag{2.8.4.4}$$

$$\implies \cos\theta = \frac{-1}{2}, \text{or, } \theta = \frac{2\pi}{3} \tag{2.8.4.5}$$

2.8.5 A line makes angles α, β, γ and δ with the diagonals of a cube, prove that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}. \tag{2.8.5.1}$$

2.8.6 Assuming that straight lines work as the plane mirror for a point, find the image of the point $(1, 2)$ in the line $x - 3y + 4 = 0$.

2.8.7 Find $\left|\overrightarrow{a} - \overrightarrow{b}\right|$, if two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ are such that $\left|\overrightarrow{a}\right| = 2, \left|\overrightarrow{b}\right| = 3$ and $\overrightarrow{a} \cdot \overrightarrow{b} = 4$.

2.8.8 If $\overrightarrow{a}$ is a unit vector and $(\overrightarrow{x} - \overrightarrow{a}) \cdot (\overrightarrow{x} + \overrightarrow{a}) = 8$, then find $\left|\overrightarrow{x}\right|$.

2.8.9 Let $\overrightarrow{a}, \overrightarrow{b}$, and $\overrightarrow{c}$ are three vectors such that $\left|\overrightarrow{a}\right| = 3, \left|\overrightarrow{b}\right| = 4 \left|\overrightarrow{c}\right| = 5$ and each one of them being perpndicular to the sum of the other to, find $\left|\overrightarrow{a} + \overrightarrow{b} + \overrightarrow{c}\right|$.

2.8.10 If with reference to the right handed system of mutually perpendicular unit vectors $\hat{i}, \hat{j}$ and $\hat{k}, \overrightarrow{\alpha} = 3\hat{i} - \hat{j}, \overrightarrow{\beta} = 2\hat{i} + \hat{j} - 3\hat{k}$, then express $\overrightarrow{\beta}$ in the form $\overrightarrow{\beta} = \overrightarrow{\beta_1} + \overrightarrow{\beta_2}$ where $\overrightarrow{\beta_1}$ is parallel to $\overrightarrow{\alpha}$ and $\overrightarrow{\beta_2}$ is perpendicular to $\overrightarrow{\alpha}$.

2.8.11 The two opposite vertices of a square are $\mathbf{A}(-1, 2)$ and $\mathbf{C}(3, 2)$. Find the coordinates of the other two vertices.

Solution:

$$\mathbf{C} - \mathbf{A} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \implies \phi = 0^\circ \tag{2.8.11.1}$$

where ϕ is the angle made by AC with the x-axis. Also, the diagonal

$$d = \|\mathbf{C} - \mathbf{A}\| = 4 \tag{2.8.11.2}$$

a) We start with the square in Fig. 2.8.11.1, with vertices as columns of the matrix

$$\mathbf{y} = \frac{d}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \tag{2.8.11.3}$$

in (2.1.15.1).

b) The next square, obtained as

$$\mathbf{P}\mathbf{y}, \tag{2.8.11.4}$$

which is a rotated version of Fig. 2.8.11.1, is available in Fig. 2.8.11.2. The angle of rotation

$$\theta = \phi - \frac{\pi}{4} \tag{2.8.11.5}$$

c) The desired square is obtained using (2.1.15.1) as

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \begin{pmatrix} \mathbf{A} & \mathbf{A} & \mathbf{A} & \mathbf{A} \end{pmatrix} = \begin{pmatrix} -1 & 1 & 3 & 1 \\ 2 & 0 & 2 & 4 \end{pmatrix} \tag{2.8.11.6}$$

and available in Fig. 2.8.11.3. The 2nd and 4th columns in the above matrix are **B** and **C** respectively.

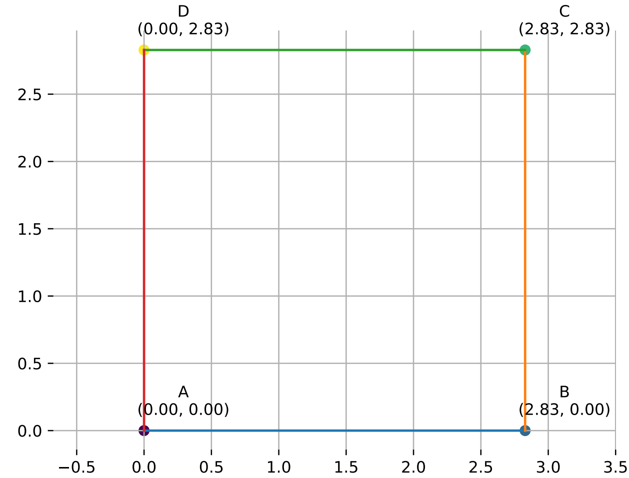


Fig. 2.8.11.1

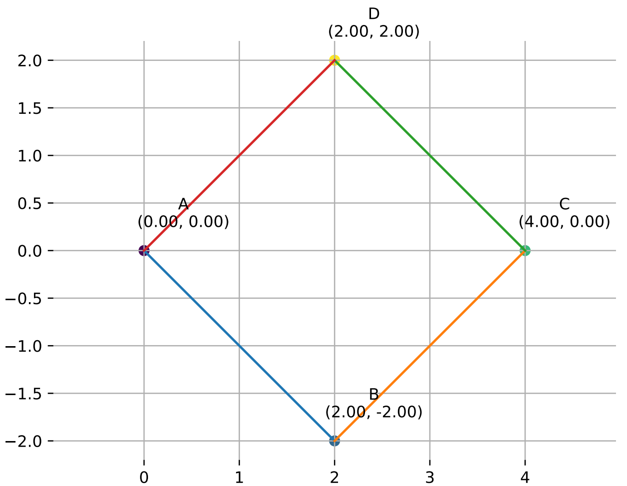


Fig. 2.8.11.2

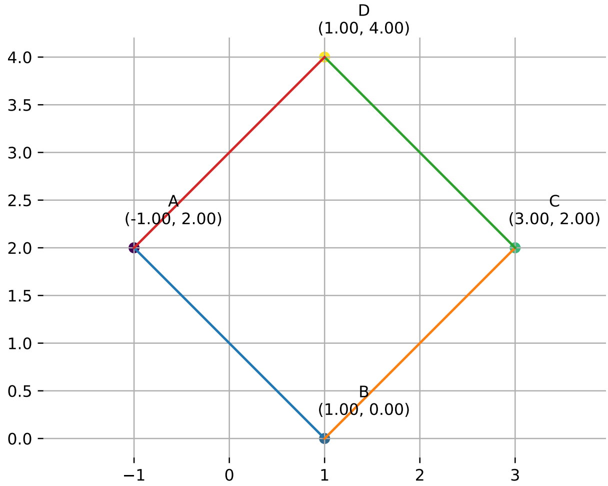


Fig. 2.8.11.3

2.8.12 Show that the tangent of an angle between the lines

$$\frac{x}{a} + \frac{y}{b} = 1 \text{ and} \tag{2.8.12.1}$$

$$\frac{x}{a} - \frac{y}{b} = 1 \tag{2.8.12.2}$$

is $\frac{2ab}{a^2-b^2}$.

2.8.13 Find $\left|\overrightarrow{a}\right|$ and $\left|\overrightarrow{b}\right|$, if $(\overrightarrow{a} + \overrightarrow{b}) \cdot (\overrightarrow{a} - \overrightarrow{b}) = 8$ and $\left|\overrightarrow{a}\right| = 8\left|\overrightarrow{b}\right|$.

Solution:

$$\because (\mathbf{a} + \mathbf{b})^\top (\mathbf{a} - \mathbf{b}) = 8, \|\mathbf{a}\| = 8\|\mathbf{b}\|, \tag{2.8.13.1}$$

$$\|\mathbf{a}\|^2 - \|\mathbf{b}\|^2 = 8 \tag{2.8.13.2}$$

$$\implies \|8\mathbf{b}\|^2 - \|\mathbf{b}\|^2 = 8 \tag{2.8.13.3}$$

$$\implies \|\mathbf{b}\| = \frac{2\sqrt{2}}{3\sqrt{7}} \tag{2.8.13.4}$$

Thus,

$$\|\mathbf{a}\| = 8\|\mathbf{b}\| = \frac{16\sqrt{2}}{3\sqrt{7}} \tag{2.8.13.5}$$

2.8.14 Three vectors $\overrightarrow{a}, \overrightarrow{b}$ and $\overrightarrow{c}$ satisfy the condition $\overrightarrow{a} + \overrightarrow{b} + \overrightarrow{c} = 0$. Evaluate the quantity $\mu = \overrightarrow{a} \cdot \overrightarrow{b} + \overrightarrow{b} \cdot \overrightarrow{c} + \overrightarrow{c} \cdot \overrightarrow{a}$, if $\left|\overrightarrow{a}\right| = 3, \left|\overrightarrow{b}\right| = 4$ and $\left|\overrightarrow{c}\right| = 2$.

2.8.15 Find the position vector of a point A in space such that $\overrightarrow{OA}$ is inclined at 60° to OX and at 45° to OY and $\left|\overrightarrow{OA}\right| = 10$ units.

2.8.16 Prove that the lines $x = py+q, z = ry+s$ and $x = p'y+q', z = r'y+s'$ are perpendicular if $pp' + rr' + 1 = 0$.

2.8.17 Show that the straight lines whose direction cosines are given by $2l + 2m - n = 0$ and $mn + nl + lm = 0$ are at right angles.

2.8.18 If $l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3$ are the direction cosines of the three mutually perpendcular lines, prove that the line whose direction cosines are propotional to $l_1 + l_2 + l_3, m_1 + m_2, m_3, n_1 + n_2 + n_3$ make angles with them.

2.8.19 If $\mathbf{r} \cdot \mathbf{a} = 0, \mathbf{r} \cdot \mathbf{b} = 0$ and $\mathbf{r} \cdot \mathbf{c} = 0$ for some non-zero vector $\mathbf{r}$, then the value of $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ is ______.

2.8.20 If $|\mathbf{a} + \mathbf{b}| = |\mathbf{a} - \mathbf{b}|$, then the vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal.

2.8.21 If $\mathbf{a}$ is any non-zero vector, then $(\mathbf{a} \cdot \hat{i})\hat{i}+(\mathbf{a} \cdot \hat{j})\hat{j}+(\mathbf{a} \cdot \hat{k})\, \hat{k}$ equals ______.

2.8.22 If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are the three vectors such that $\mathbf{a} + \mathbf{b} + \mathbf{c} = 0$ and $|\mathbf{a}| = 2$, $|\mathbf{b}|=3$, $|\mathbf{c}|=5$, the value of $\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{a}$ is

2.8.23 If a variable line in two adjacent positions has directions cosines l, m, n and $l+\delta l, m+\delta m, n + \delta n$, show that the small angle $\delta\theta$ between the two positions is given by

$$\delta\theta^2 = \delta l^2 + \delta m^2 + \delta n^2 \tag{2.8.23.1}$$

2.8.24 The vector $\mathbf{a} + \mathbf{b}$ bisects the angle between the non-collinear vectors $\mathbf{a}$ and $\mathbf{b}$ if ______.

2.8.25 If $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are mutually perpendicular vectors of equal magnitudes,show that the $\mathbf{A} + \mathbf{B} + \mathbf{C}$ is equally inclined to $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$.

2.8.26 The base of an equilateral triangle with side $2a$ lies along the y-axis such that the mid-point of the base is at the origin. Find the vertices of the triangle.

Solution:

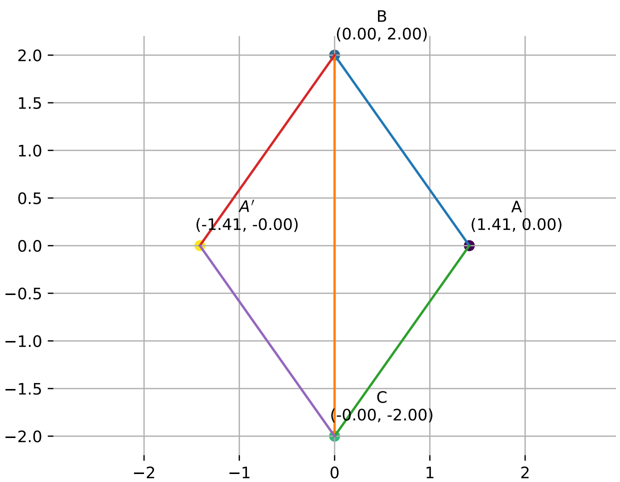


Fig. 2.8.26.1: $a = 2$.

See Fig. 2.8.26.1. Let the base be BC. From the given information,

$$\mathbf{B} = a\mathbf{e}_2, \mathbf{C} = -a\mathbf{e}_2 \tag{2.8.26.1}$$

Since $\mathbf{A}$ lies on the x-axis,

$$\mathbf{A} = k\mathbf{e}_1 \tag{2.8.26.2}$$

and

$$\|\mathbf{A} - \mathbf{C}\|^2 = (2a)^2 \tag{2.8.26.3}$$

$$\implies \|\mathbf{A}\|^2 + \|\mathbf{C}\|^2 - 2\mathbf{A}^\top\mathbf{C} = 4a^2 \tag{2.8.26.4}$$

$$\implies k^2 + a^2 = 4a^2 \tag{2.8.26.5}$$

yielding

$$k = \pm a\sqrt{3} \tag{2.8.26.6}$$

Thus,

$$\mathbf{A} = \pm\sqrt{3}a\mathbf{e}_1 \tag{2.8.26.7}$$

2.8.27 Find the magnitude of two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$, having the same magnitude and such that the angle between them is 60° and their scalar product is $\frac{1}{2}$.
Solution: Given

$$\|\mathbf{a}\| = \|\mathbf{b}\|, \cos\theta = \frac{1}{2}, \mathbf{a}^\top\mathbf{b} = \frac{1}{2}, \tag{2.8.27.1}$$

$$\implies \frac{1}{2} = \frac{\frac{1}{2}}{\|\mathbf{a}\|^2} \implies \|\mathbf{a}\| = \|\mathbf{b}\| = 1 \tag{2.8.27.2}$$

by using the definition of the scalar product in (2.1.1.1).

2.8.28 Show that $\left|\overrightarrow{a}\right|\overrightarrow{b} + \left|\overrightarrow{b}\right|\overrightarrow{a}$ is perpendicular to $\left|\overrightarrow{a}\right|\overrightarrow{b} - \left|\overrightarrow{b}\right|\overrightarrow{a}$, for any two nonzero vectors $\overrightarrow{a}$ and $\overrightarrow{b}$.
Solution:

$$\|\mathbf{a}\|\mathbf{b} + \|\mathbf{b}\|\mathbf{a} = \|\mathbf{a}\|\|\mathbf{b}\|\left(\frac{\mathbf{b}}{\|\mathbf{b}\|} + \frac{\mathbf{a}}{\|\mathbf{a}\|}\right) \tag{2.8.28.1}$$

$$\|\mathbf{a}\|\mathbf{b} - \|\mathbf{b}\|\mathbf{a} = \|\mathbf{a}\|\|\mathbf{b}\|\left(\frac{\mathbf{b}}{\|\mathbf{b}\|} - \frac{\mathbf{a}}{\|\mathbf{a}\|}\right) \tag{2.8.28.2}$$

$$\implies (\|\mathbf{a}\|\mathbf{b} + \|\mathbf{b}\|\mathbf{a})^\top(\|\mathbf{a}\|\mathbf{b} - \|\mathbf{b}\|\mathbf{a}) = 0 \tag{2.8.28.3}$$

from (2.1.5.1).

2.8.29 If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are unit vectors such that $\mathbf{a}+\mathbf{b}+\mathbf{c}=0$, then the value of $\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a}$ is

a) 1
b) 3
c) $\frac{-3}{2}$
d) None of these

Solution:

$$
\begin{aligned}
\|\mathbf{a}+\mathbf{b}+\mathbf{c}\|^2 &= 0 \\
\implies \|\mathbf{a}\|^2+\|\mathbf{b}\|^2+\|\mathbf{c}\|^2+2(\mathbf{a}^\top\mathbf{b}+\mathbf{b}^\top\mathbf{c}+\mathbf{c}^\top\mathbf{a}) &= 0 \\
\implies 3+2(\mathbf{a}^\top\mathbf{b}+\mathbf{b}^\top\mathbf{c}+\mathbf{c}^\top\mathbf{a}) &= 0 \\
\implies \mathbf{a}^\top\mathbf{b}+\mathbf{b}^\top\mathbf{c}+\mathbf{c}^\top\mathbf{a} &= -\frac{3}{2} \qquad (2.8.29.1)
\end{aligned}
$$

2.8.30 If either vector $\vec{a} = 0$ or $\vec{b} = 0$, then $\vec{a}.\vec{b}$=0. But the converse need not be true. Justify your answer with an example.
Solution:

$$
\mathbf{a} = \begin{pmatrix}1\\1\end{pmatrix}, \mathbf{b} = \begin{pmatrix}1\\-1\end{pmatrix} \qquad (2.8.30.1)
$$
$$
\implies \mathbf{a}^\top\mathbf{b} = 0 \qquad (2.8.30.2)
$$

2.8.31 If the points $\mathbf{A}(1,-2), \mathbf{B}(2,3), \mathbf{C}(a,2)$ and $\mathbf{D}(-4-3)$ form parallelogram, find the value of a and height of the parallelogram taking AB as base.

2.8.32 Ayush starts walking from his house to office. Instead of going to the office directly, he goes to a bank first, from there to his daughter school and then reaches the office what is the extra distance travelled by Ayush in reaching his office? If the house is situated at $(2,4)$, bank at $(5,8)$, school at $(13,14)$ and office at $(13,26)$ and coordinates are in km.

2.8.33 Prove that $(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) = |\mathbf{a}|^2+|\mathbf{b}|^2$, if and only if $\mathbf{a}, \mathbf{b}$ are perpendicular, given $\mathbf{a} \neq \mathbf{0}, \mathbf{b} \neq \mathbf{0}$.
Solution:

$$
\begin{aligned}
\because (\mathbf{a}+\mathbf{b})^\top(\mathbf{a}+\mathbf{b}) &= \|\mathbf{a}\|^2+\|\mathbf{b}\|^2, \qquad (2.8.33.1)\\
\|\mathbf{a}\|^2+\|\mathbf{b}\|^2+2\mathbf{a}^\top\mathbf{b} &= \|\mathbf{a}\|^2+\|\mathbf{b}\|^2 \qquad (2.8.33.2)\\
\implies \mathbf{a}^\top\mathbf{b} &= 0 \qquad (2.8.33.3)
\end{aligned}
$$

2.8.34 If l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of two mutually perpendicular lines, show that the direction cosines of the line perpendicular to both these are $m_1n_2 - m_2n_1, n_1l_2 - n_2l_1, l_1m_2 - l_2m_1$.
Solution:

$$
\mathbf{P} = \begin{pmatrix} l_1 & l_2 & m_1n_2-m_2n_1 \\ m_1 & m_2 & n_1l_2-n_2l_1 \\ n_1 & n_2 & l_1m_2-l_2m_1 \end{pmatrix} \qquad (2.8.34.1)
$$

satisfies (2.1.6.1). Hence, the three vectors are mutually perpendicular.

2.8.35 Find the angle between the lines whose direction ratios are a, b, c and $b-c, c-a, a-b$.
Solution:

$$
\because \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} b-c \\ c-a \\ a-b \end{pmatrix} = 0, \theta = \frac{\pi}{2} \qquad (2.8.35.1)
$$

2.8.36 The value of the expression $|\mathbf{a}\times\mathbf{b}|+(\mathbf{a}\cdot\mathbf{b})$ is ______

2.8.37 If $|\mathbf{a}\times\mathbf{b}|^2 + |\mathbf{a}\cdot\mathbf{b}|^2=144$ and $|\mathbf{a}|=4$, then $|\mathbf{b}|$ is equal to ______.

2.8.38 If the directions cosines of a line are (k,k,k) then

a) $k>0$
b) $0<k<1$
c) $k=1$
d) $k=\frac{1}{\sqrt{3}}$ or $-\frac{1}{\sqrt{3}}$

2.8.39 Find the angle between the lines whose direction cosines are given by the equations $l+m+n=0$, $l^2+m^2-n^2=0$.

2.9 CBSE

2.9.1 Jagdish has a field which is in the shape of a right-angled triangle AQC. He wants to leave a space in the form of a square $PQRS$ inside the field for growing wheat and the remaining space for growing vegetables. In the field, there is a pole marked as **O**. Based on the above information, answer the following equations

a) Taking **O** as the origin, $\mathbf{P}=(-200,0)$ and $\mathbf{Q}=(200,0)$. PQRS being a square, what are the coordinates of **R** and **S**?
b) i) What is the area of square $PQRS$?
   ii) What is the length of diagonal PR in $PQRS$?
c) If **S** divides CA in the ratio $K:1$, what is the value of K, where $\mathbf{A}=(200,800)$?

(10, 2023)

2.9.2 If $(-5,3)$ and $(5,3)$ are two vertices of an equilateral triangle, then the coordinates of the third vertex, given that the origin lies inside the triangle (take $\sqrt{3}=1.7$), are (10, 2023)

2.9.3 If

$$\left|\vec{a}\times\vec{b}\right|^2+\left|\vec{a}\cdot\vec{b}\right|^2=400 \text{ and } \left|\vec{b}\right|=5,$$

find the value of $\left|\vec{a}\right|$. (12, 2022)

2.9.4 If

$$\vec{a}=\hat{i}+\hat{j}+\hat{k}\vec{a}\cdot\vec{b}=1, \text{ and } \vec{a}\times\vec{b}=\hat{j}-\hat{k},$$

then find $\left|\vec{b}\right|$. (12, 2022)

2.9.5 If

$$\left|\vec{a}\right|=3,\ \left|\vec{b}\right|=2\sqrt{3} \text{ and } \vec{a}\cdot\vec{b}=6,$$

then find the value of $\left|\vec{a}\times\vec{b}\right|$. (12, 2022)

2.9.6 $\left|\vec{a}\right|=8$, $\left|\vec{b}\right|=3$, and $\vec{a}\cdot\vec{b}=12\sqrt{3}$, then the value of $\left|\vec{a}\times\vec{b}\right|$ is (12, 2022)

2.9.7 If

$$\vec{a}=2\hat{i}+\hat{j}+3\hat{k},\ \vec{b}=-\hat{i}+2\hat{j}+\hat{k}, \text{ and } \vec{c}=3\hat{i}+\hat{j}+2\hat{k},$$

then find $\vec{a}\cdot(\vec{b}\times\vec{c})$. (12, 2022)

2.9.8 $\vec{a}, \vec{b}, \vec{c}$, and $\vec{d}$ are four non-zero vectors such that

$$\vec{a} \times \vec{b} = \vec{c} \times \vec{d}$$

and

$$\vec{a} \times \vec{c} = 4\vec{b} \times \vec{d},$$

then show that $(\vec{a} - 2\vec{d})$ is parallel to $(2\vec{b} - \vec{c})$ where

$$\vec{a} \neq 2\vec{d}, \quad \vec{c} \neq 2\vec{b}.$$

(12, 2022)

2.9.9 If

$$\vec{a} = \hat{i} + \hat{j} + \hat{k},$$

$$\vec{a} \cdot \vec{b} = 1,$$

and

$$\vec{a} \times \vec{b} = \hat{j} - \hat{k},$$

then find $\left|\vec{b}\right|$. (12, 2022)

2.9.10 If $\vec{a}$ and $\vec{b}$ are two vectors such that

$$\left|\vec{a} + \vec{b}\right| = \left|\vec{b}\right|,$$

then prove that $(\vec{a} + 2\vec{b})$ is perpendicular to $\vec{a}$. (12, 2022)

2.9.11 If $\vec{a}$ and $\vec{b}$ are unit vectors and θ is the angle between them, then prove that

$$\sin\frac{\theta}{2} = \frac{1}{2}\left|\vec{a} - \vec{b}\right|.$$

(12, 2022)

2.9.12 If $\vec{a}$ and $\vec{b}$ are two unit vectors and θ is the angle between them, then prove that

$$\sin\frac{\theta}{2} = \frac{1}{2}\left|\vec{a} - \vec{b}\right|.$$

(12, 2022)

2.9.13 If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are the position vectors of the points **A**$(2, 3, -4)$, **B**$(3, -4, -5)$, and **C**$(3, 2, -3)$ respectively, then $\left|\vec{a} + \vec{b} + \vec{c}\right|$ is equal to: (12, 2022)

2.9.14 The two adjacent sides of a parallelogram are represented by $2\hat{i}+4\hat{j}-5\hat{k}$ and $\hat{i}+2\hat{j}+3\hat{k}$. Find the unit vectors parallel to its diagonals. Using the diagonal vectors, find the area of the parallelogram also. (12, 2022)

2.9.15 If the points **A**$(2, 0)$, **B**$(6, 1)$, and **C**(p, q) form a triangle of area 12 square units (positive only) and

$$2p + q = 10,$$

then find the values of p and q. (10, 2022)

2.9.16 Prove that three points **A**, **B**, and **C** with position vectors **a**, **b**, and **c** respectively are collinear if and only if $(\mathbf{b}\times\mathbf{c})+(\mathbf{c}\times\mathbf{a})+(\mathbf{a}\times\mathbf{b})=\mathbf{0}$. (12, 2021)

2.9.17 If $|\overrightarrow{a}|=2$, $|\overrightarrow{b}|=7$, and $\overrightarrow{a}\times\overrightarrow{b}=3\hat{i}+2\hat{j}+6\hat{k}$, find the angle between $\overrightarrow{a}$ and $\overrightarrow{b}$. (12, 2019)

2.9.18 Find the volume of a cuboid whose edges are given by $-3\hat{i}+7\hat{j}+5\hat{k}$, $-5\hat{i}+7\hat{j}-3\hat{k}$, and $7\hat{i}-5\hat{j}-3\hat{k}$. (12, 2019)

2.9.19 Let $\overrightarrow{a}$, $\overrightarrow{b}$, and $\overrightarrow{c}$ be three vectors such that $|\overrightarrow{a}|=1$, $|\overrightarrow{b}|=2$, and $|\overrightarrow{c}|=3$. If the projection of $\overrightarrow{b}$ along $\overrightarrow{a}$ is equal to the projection of $\overrightarrow{c}$ along $\overrightarrow{a}$, and $\overrightarrow{b}$ and $\overrightarrow{c}$ are perpendicular to each other, then find $|3\overrightarrow{a}-2\overrightarrow{b}+2\overrightarrow{c}|$. (12, 2019)

2.9.20 X and Y are two points with position vectors $3\overrightarrow{a}+\overrightarrow{b}$ and $\overrightarrow{a}-3\overrightarrow{b}$, respectively. Write the position vector of a point Z which divides the line segment XY in the ratio 2 : 1 externally. (12, 2019)

2.9.21 Let $\overrightarrow{\mathbf{a}}=4\hat{i}+5\hat{j}-\hat{k}$, $\overrightarrow{\mathbf{b}}=\hat{i}-4\hat{j}+5\hat{k}$, and $\overrightarrow{\mathbf{c}}=3\hat{i}+\hat{j}-\hat{k}$. Find a vector $\overrightarrow{\mathbf{d}}$ which is perpendicular to both $\overrightarrow{\mathbf{c}}$ and $\overrightarrow{\mathbf{b}}$ and satisfies $\overrightarrow{\mathbf{d}}\cdot\overrightarrow{\mathbf{a}}=21$. (12, 2018)

2.9.22 Let $\overrightarrow{a}$, $\overrightarrow{b}$, and $\overrightarrow{c}$ be three vectors such that $|\overrightarrow{a}|=1$, $|\overrightarrow{b}|=2$, and $|\overrightarrow{c}|=3$. If the projection of $\overrightarrow{b}$ along $\overrightarrow{a}$ is equal to the projection of $\overrightarrow{c}$ along $\overrightarrow{a}$, and $\overrightarrow{b}$ and $\overrightarrow{c}$ are perpendicular to each other, then find $|3\overrightarrow{a}-2\overrightarrow{b}+2\overrightarrow{c}|$. (12, 2018)

2.9.23 Given that vectors $\overrightarrow{a}$, $\overrightarrow{b}$, $\overrightarrow{c}$ form a triangle such that $\overrightarrow{a}=\overrightarrow{b}+\overrightarrow{c}$. Find p, q, r, s such that area of triangle is $5\sqrt{6}$ where $\overrightarrow{a}=p\hat{i}+q\hat{j}+r\hat{k}$, $\overrightarrow{b}=s\hat{i}+3\hat{j}+4\hat{k}$ and $\overrightarrow{c}=3\hat{i}+\hat{j}-2\hat{k}$. (12, 2016)

2.9.24 Find the co-ordinates of the point where the line $\overrightarrow{r}=(-\hat{i}-2\hat{j}-3\hat{k})+\lambda(3\hat{i}+4\hat{j}+3\hat{k})$ meets the plane which is perpendicular to the vector $\overrightarrow{n}=\hat{i}+\hat{j}+3\hat{k}$ and at a distance of $\frac{4}{\sqrt{11}}$ from origin. (12, 2016)

2.9.25 Find a unit vector perpendicular to each of the vectors $(\mathbf{a}+\mathbf{b})$ and $(\mathbf{a}-\mathbf{b})$ where $\mathbf{a}=\hat{i}+\hat{j}+\hat{k}$, $\mathbf{b}=\hat{i}+2\hat{j}+3\hat{k}$. (12, 2022)

2.9.26 If $f(\alpha)=\begin{pmatrix}\cos\alpha & -\sin\alpha & 0\\ \sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{pmatrix}$, then prove that

$$f(\alpha)\,f(-\beta)=f(\alpha-\beta)\,.$$

(12, 2023)

2.10 JEE

2.10.1. Consider 3 points

$$\mathbf{P}=(-\sin(\beta-\alpha),-\cos\beta),\mathbf{Q}=(\cos(\beta-\alpha),\sin\beta)$$

and

$$\mathbf{R}=(\cos(\beta-\alpha+\theta),\sin(\beta-\theta))$$

where $0<\alpha,\beta,\theta<\frac{\pi}{4}$. Then,

a) **P** lies on the line segment RQ
b) **Q** lies on the line segment PR
c) **R** lies on the line segment QP
d) **P**,**Q**,**R** are non-collinear

(2008)

2.10.2. Let **A**, **B**, **C** be vectors of length 3,4,5 respectively. Let $\mathbf{A} \perp \mathbf{B}+\mathbf{C}$, $\mathbf{B} \perp \mathbf{C}+\mathbf{A}$ and $\mathbf{C} \perp \mathbf{A}+\mathbf{B}$. Find the length of $\mathbf{A}+\mathbf{B}+\mathbf{C}$. (1981)

2.10.3. Find the unit vector perpendicular to the plane determined by $\mathbf{P}(1,-1,2)$, $\mathbf{Q}(2,0,-1)$ and $\mathbf{R}(0,2,1)$. (1983)

2.10.4. Find the area of the triangle whose vertices are $\mathbf{A}(1,-1,2)$, $\mathbf{B}(2,0,-1)$, $\mathbf{C}(3,-1,2)$. (1983)

2.10.5. $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and **D**, are four points in a plane respectively such that $(\mathbf{A}-\mathbf{D})\cdot(\mathbf{B}-\mathbf{C}) = (\mathbf{B}-\mathbf{D})\cdot(\mathbf{C}-\mathbf{A}) = 0$. The point **D**, then, is the ______ of $\triangle ABC$. (1984)

2.10.6. If $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are three non-coplanar vectors, then $\frac{\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}}{\mathbf{C}\times\mathbf{A}\cdot\mathbf{B}} + \frac{\mathbf{B}\cdot\mathbf{A}\times\mathbf{C}}{\mathbf{C}\cdot\mathbf{A}\times\mathbf{B}} =$. (1985)

2.10.7. $\mathbf{A} = (1,1,1)$, $\mathbf{C} = (0,1,-1)$ are given vectors, then find **B** that satisfies $\mathbf{A}\times\mathbf{B} = \mathbf{C}$ and $\mathbf{A}\cdot\mathbf{B} = 3$. (1985)

2.10.8. Let $\mathbf{b} = 4\hat{i} + 3\hat{j}$ and **c** be two vectors perpendicular to each other in the XY plane. All vectors in the same plane having projections 1 and 2 along **b** and **c**, respectively, are given by ______. (1987)

2.10.9. The components of a vector **a** along and perpendicular to a non-zero vector **b** are ______ and ______ respectively. (1988)

2.10.10. Given that $\mathbf{a} = (1,1,1)$, $\mathbf{c} = (0,1,-1)$, $\mathbf{a}\cdot\mathbf{b} = 3$ and $\mathbf{a}\times\mathbf{b} = \mathbf{c}$, $\mathbf{b} =$ ______. (1991)

2.10.11. A unit vector coplanar with $\hat{i}+\hat{j}+2\hat{k}$ and $\hat{i}+2\hat{j}+\hat{k}$ and perpendicular to $\hat{i}+\hat{j}+\hat{k}$ is ______. (1992)

2.10.12. A unit vector perpendicular to the plane determined by the points $\mathbf{P}(1,-1,2)$, $\mathbf{Q}(2,0,-1)$ and $\mathbf{R}(0,2,1)$ is ______. (1994)

2.10.13. If **b** and **c** are any two non-collinear unit vectors and **a** is any vector, then $(\mathbf{a}\cdot\mathbf{b})\mathbf{b} + (\mathbf{a}\cdot\mathbf{c})\mathbf{c} + \frac{\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})}{|\mathbf{b}\times\mathbf{c}|}(\mathbf{b}\times\mathbf{c}) =$ ______.

2.10.14. Let $\mathbf{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\mathbf{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\mathbf{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ be three non-zero vectors such that **c** is a unit vector perpendicular to both the vectors **a** and **b**. If the angle between **a** and **b** is $\frac{\pi}{6}$, then

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^2 = ?$$

(1986)

2.10.15. The number of vectors of unit length perpendicular to vectors $\mathbf{a} = (1,1,0)$ and $\mathbf{b} = (0,1,1)$ is (1987)

a) one
b) two
c) three
d) infinite
e) None of these

2.10.16. Let $\mathbf{a} = 2\hat{i} - \hat{j} + \hat{k}$, $\mathbf{b} = \hat{i} + 2\hat{j} - \hat{k}$ and $\mathbf{c} = \hat{i} + 2\hat{j} - 2\hat{k}$ be three vectors. A vector in the plane of **b** and **c**, whose projection on **a** is of magnitude $\sqrt{2/3}$, is ______. (1993)

a) $2\hat{i}+3\hat{j}-3\hat{k}$ b) $2\hat{i}+3\hat{j}+3\hat{k}$ c) $-2\hat{i}-\hat{j}+5\hat{k}$ d) $2\hat{i}+\hat{j}+5\hat{k}$

2.10.17. The vector $\frac{1}{3}\left(2\hat{i}-2\hat{j}+\hat{k}\right)$ is (1994)

a) a unit vector
b) makes an angle with the vector
c) parallel to the vector $-\hat{i}+\hat{j}-\frac{1}{2}\hat{k}$
d) perpendicular to the vector $3\hat{i}+2\hat{j}-2\hat{k}$

2.10.18. If $\mathbf{a}=\hat{i}+\hat{j}+\hat{k}, \mathbf{b}=4\hat{i}+3\hat{j}+4\hat{k}$ and $\mathbf{c}=\hat{i}+\alpha\hat{j}+\beta\hat{k}$ are linearly dependent vectors and $|c|=\sqrt{3}$, then (1998)

a) $\alpha=1,\beta=-1$
b) $\alpha=1,\beta=\pm1$
c) $\alpha=-1,\beta=\pm1$
d) $\alpha=\pm1,\beta=1$

2.10.19. For three vectors $\mathbf{u},\mathbf{v},\mathbf{w}$ which of the following expression is not equal to any of the remaining three? (1998)

a) $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$
b) $(\mathbf{v}\times\mathbf{w})\cdot\mathbf{u}$
c) $\mathbf{v}\cdot(\mathbf{u}\times\mathbf{w})$
d) $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}$

2.10.20. Which of the following expressions are meaningful? (1998)

a) $\mathbf{u}(\mathbf{v}\times\mathbf{w})$
b) $(\mathbf{u}\cdot\mathbf{v})\cdot\mathbf{w}$
c) $(\mathbf{u}\cdot\mathbf{v})\,\mathbf{w}$
d) $\mathbf{u}\times(\mathbf{v}\cdot\mathbf{w})$

2.10.21. Let $\mathbf{a}$ and $\mathbf{b}$ be two non-collinear unit vectors. If $\mathbf{u}=\mathbf{a}-(\mathbf{a}\cdot\mathbf{b})\,\mathbf{b}$ and $\mathbf{v}=\mathbf{a}\times\mathbf{b}$, then $|\mathbf{v}|$ is (1999)

a) $|\mathbf{u}|$
b) $|\mathbf{u}|+|\mathbf{u}\cdot\mathbf{a}|$
c) $|\mathbf{u}|+|\mathbf{u}\cdot\mathbf{b}|$
d) $|\mathbf{u}|+\mathbf{u}\cdot(\mathbf{a}+\mathbf{b})$

2.10.22. Let $\mathbf{A}$ be vector parallel to line of intersection of planes P_1 and P_2. Plane P_1 is parallel to the vectors $2\hat{j}+3\hat{k}$ and $4\hat{j}-3\hat{k}$ and P_2 is parallel to $\hat{j}-\hat{k}$ and $3\hat{i}+3\hat{j}$, then the angle between vector $\mathbf{A}$ and a given vector $2\hat{i}+\hat{j}-2\hat{k}$ is (2006)

a) $\frac{\pi}{2}$
b) $\frac{\pi}{4}$
c) $\frac{\pi}{6}$
d) $\frac{3\pi}{4}$

2.10.23. The vector(s) which is/are coplanar with vectors $\hat{i}+\hat{j}+2\hat{k}$ and $\hat{i}+2\hat{j}+\hat{k}$, and perpendicular to the vector $\hat{i}+\hat{j}+\hat{k}$ is/are (2011)

a) $\hat{j}-\hat{k}$
b) $\hat{i}+\hat{j}$
c) $\hat{i}-\hat{j}$
d) $\hat{j}+\hat{k}$

2.10.24. Let $\mathbf{x},\mathbf{y}$ and $\mathbf{z}$ be three vectors each of magnitude $\sqrt{2}$ and the angle between each pair of them is $\frac{\pi}{3}$. If $\mathbf{a}$ is a non-zero vector perpendicular to $\mathbf{x}$ and $\mathbf{y}\times\mathbf{z}$ and $\mathbf{b}$ is a non-zero vector perpendicular to $\mathbf{y}$ and $\mathbf{z}\times\mathbf{x}$, then (2014)

a) $\mathbf{b} = (\mathbf{b} \cdot \mathbf{z})(\mathbf{z} - \mathbf{x})$
b) $\mathbf{a} = (\mathbf{a} \cdot \mathbf{y})(\mathbf{y} - \mathbf{z})$
c) $\mathbf{a} \cdot \mathbf{b} = -(\mathbf{a} \cdot \mathbf{y})(\mathbf{b} \cdot \mathbf{z})$
d) $\mathbf{a} = -(\mathbf{a} \cdot \mathbf{y})(\mathbf{z} - \mathbf{y})$

2.10.25. In $\triangle PQR$, let $\mathbf{a} = \overrightarrow{QR}$, $\mathbf{b} = \overrightarrow{RP}$ and $\mathbf{c} = \overrightarrow{PQ}$. If $|\mathbf{a}| = 12$, $|\mathbf{b}| = 4\sqrt{3}$, $\mathbf{b} \cdot \mathbf{c} = 24$, then which of the following is (are)true? (2015)

a) $\frac{|\mathbf{c}|^2}{2} - |\mathbf{a}| = 12$
b) $\frac{|\mathbf{c}|^2}{2} + |\mathbf{a}| = 30$
c) $|\mathbf{a} \times \mathbf{b} + \mathbf{c} \times \mathbf{a}| = 48\sqrt{3}$
d) $\mathbf{a} \cdot \mathbf{b} = -72$

2.10.26. Let $\hat{\mathbf{u}} = u_1\hat{i} + u_2\hat{j} + u_3\hat{k}$ be a unit vector in $\mathbf{R}^3$ and $\hat{\mathbf{w}} = \frac{1}{\sqrt{6}}\left(\hat{i} + \hat{j} + 2\hat{k}\right)$. Given that there exists a vector $\mathbf{v}$ in $\mathbf{R}^3$ such that $|\hat{\mathbf{u}} \times \mathbf{v}| = 1$ and $\hat{\mathbf{w}} \cdot (\hat{\mathbf{u}} \times \mathbf{v}) = 1$. Which of the following statement(s) is(are) correct? (2016)

a) there is exactly one choice for such $\mathbf{v}$
b) There are infinitely many choices for such $\mathbf{v}$
c) If $\hat{u}$ lies in the xy-plane then $|u_1| = |u_2|$
d) If $\hat{u}$ lies in the xz-plane then $2|u_1| = |u_3|$

2.10.27. The scalar $\mathbf{A} \cdot ((\mathbf{B} + \mathbf{C}) \times (\mathbf{A} + \mathbf{B} + \mathbf{C}))$ equals (1981)

a) 0
b) $[\mathbf{A}\ \mathbf{B}\ \mathbf{C}] + [\mathbf{B}\ \mathbf{C}\ \mathbf{A}]$
c) $[\mathbf{A}\ \mathbf{B}\ \mathbf{C}]$
d) None of these

2.10.28. For non-zero vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$, $|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}| = |\mathbf{a}|\,|\mathbf{b}|\,|\mathbf{c}|$ holds if and only if

a) $\mathbf{a} \cdot \mathbf{b} = 0,\ \mathbf{b} \cdot \mathbf{c} = 0$
b) $\mathbf{b} \cdot \mathbf{c} = 0,\ \mathbf{c} \cdot \mathbf{a} = 0$
c) $\mathbf{c} \cdot \mathbf{a} = 0,\ \mathbf{a} \cdot \mathbf{b} = 0$
d) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{c} = \mathbf{c} \cdot \mathbf{a} = 0$

(1982 - 2 Marks)

2.10.29. The volume of the parallelopiped whose sides are given by $OA = 2\mathbf{i} - 2\mathbf{j}$, $OB = \mathbf{i} + \mathbf{j} - \mathbf{k}$, $OC = 3\mathbf{i} - \mathbf{k}$, is (1983)

a) $\frac{4}{13}$
b) 4
c) $\frac{2}{7}$
d) None of these

2.10.30. The points with position vectors $60\mathbf{i} + 3\mathbf{j}$, $40\mathbf{i} - 8\mathbf{j}$, $a\mathbf{i} - 52\mathbf{j}$ are collinear if

a) $a = -40$
b) $a = 40$
c) $a = 20$
d) None of these

(1983)

2.10.31. Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be three non coplanar vectors and $\mathbf{p}, \mathbf{q}, \mathbf{r}$ are vectors defined by the relations $\mathbf{p} = \frac{\mathbf{b} \times \mathbf{c}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}, \mathbf{q} = \frac{\mathbf{c} \times \mathbf{a}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}, \mathbf{r} = \frac{\mathbf{a} \times \mathbf{b}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$ then the value of the expression $(\mathbf{a} + \mathbf{b}) \cdot \mathbf{p} + (\mathbf{b} + \mathbf{c}) \cdot \mathbf{q} + (\mathbf{c} + \mathbf{a}) \cdot \mathbf{r}$ is equal to

a) 0
b) 1
c) 2
d) 3

(1988)

2.10.32. Let **p** and **q** be the position vectors of **P** and **Q** respectively, with respect to **O** and $|\mathbf{p}| = p$, $|\mathbf{q}| = q$. The points **R** and **S** divide PQ internally and externally in the ratio 2: 3 respectively. If OR and OS are perpendicular then

a) $9p^2 = 4q^2$
b) $4p^2 = 9q^2$
c) $9p = 4q$
d) $4p = 9q$

(1994)

2.10.33. Let α, β, γ be distinct real numbers. The points with position vectors $\alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k}$, $\beta\mathbf{i} + \gamma\mathbf{j} + \alpha\mathbf{k}$, $\gamma\mathbf{i} + \alpha\mathbf{j} + \beta\mathbf{k}$ (1994)

a) are collinear
b) form an equilateral triangle
c) form a scalene triangle
d) form a right angles triangle

2.10.34. Let $\mathbf{a} = \mathbf{i} - \mathbf{j}$, $\mathbf{b} = \mathbf{j} - \mathbf{k}$, $\mathbf{c} = \mathbf{k} - \mathbf{i}$. If **d** is a unit vector such that $\mathbf{a} \cdot \mathbf{d} = 0 = [\mathbf{b}\ \mathbf{c}\ \mathbf{d}]$, then **d** equals (1995)

a) $\pm\frac{\mathbf{i}+\mathbf{j}-2\mathbf{k}}{\sqrt{6}}$
b) $\pm\frac{\mathbf{i}+\mathbf{k}-\mathbf{k}}{\sqrt{3}}$
c) $\pm\frac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{3}}$
d) $\pm\mathbf{k}$

2.10.35. If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are non coplanar unit vectors such that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \frac{(\mathbf{b}+\mathbf{c})}{\sqrt{2}}$, then the angle between **a** and **b** is (1995)

a) $\frac{3\pi}{4}$
b) $\frac{\pi}{4}$
c) $\frac{\pi}{2}$
d) π

2.10.36. Let $\mathbf{u}, \mathbf{v}$ and **w** be vectors such that $\mathbf{u} + \mathbf{v} + \mathbf{w} = 0$. If $|\mathbf{u}| = 3$, $|\mathbf{v}| = 4$ and $|\mathbf{w}| = 5$, then $\mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{w} + \mathbf{u} \cdot \mathbf{w}$ is (1995)

a) 47
b) -25
c) 0
d) 25

2.10.37. If $\mathbf{a}, \mathbf{b}$ and **c** are three non-coplanar vectors then $(\mathbf{a} + \mathbf{b} + \mathbf{c}) \cdot [(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} + \mathbf{c})]$ equals (1995)

a) 0
b) $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$
c) $2\,[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$
d) $-[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

2.10.38. Let $\mathbf{a} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = \mathbf{i} + \mathbf{j}$. If **c** is a vector such that $\mathbf{a} \cdot \mathbf{c} = |\mathbf{c}|, |\mathbf{c} - \mathbf{a}| = 2\sqrt{2}$ and the angle between $(\mathbf{a} \times \mathbf{b})$ and **c** is 30°, then $|(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}| =$ (1999)

a) $\frac{2}{3}$
b) $\frac{3}{2}$
c) 2
d) 3

2.10.39. Let $\mathbf{a} = 2\mathbf{i}+\mathbf{j}+\mathbf{k}, \mathbf{b} = \mathbf{i}+2\mathbf{j}-\mathbf{k}$ and a unit vector $\mathbf{c}$ be coplanar. If $\mathbf{c}$ is perpendicular to $\mathbf{a}$, then $\mathbf{c} =$ (1999)

a) $\frac{1}{\sqrt{2}}(-\mathbf{j}+\mathbf{k})$
b) $\frac{1}{\sqrt{3}}(-\mathbf{i}-\mathbf{j}-\mathbf{k})$
c) $\frac{1}{\sqrt{5}}(\mathbf{i}-2\mathbf{j})$
d) $\frac{1}{\sqrt{3}}(\mathbf{i}-\mathbf{j}-\mathbf{k})$

2.10.40. If the vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ from the sides BC, CA and AB respectively of a triangle ABC, then (2000)

a) $\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a} = 0$
b) $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{c} = \mathbf{c}\cdot\mathbf{a}$
c) $\mathbf{a}\times\mathbf{b} = \mathbf{b}\times\mathbf{c} = \mathbf{c}\times\mathbf{a}$
d) $\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a} = 0$

2.10.41. Let the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{d}$ be such that $(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) = 0$. Let A and B be planes determined by the pairs of vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}, \mathbf{d}$ respectively. Then the angle between A and B is (2000)

a) 0
b) $\frac{\pi}{4}$
c) $\frac{\pi}{3}$
d) $\frac{\pi}{2}$

2.10.42. If $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are unit coplanar vectors, then the scalar triple product $[2\mathbf{a}-\mathbf{b}, 2\mathbf{b}-\mathbf{c}, 2\mathbf{c}-\mathbf{a}] =$ (2000)

a) 0
b) 1
c) $-\sqrt{3}$
d) $\sqrt{3}$

2.10.43. Let $\mathbf{a} = \mathbf{i}-\mathbf{k}, \mathbf{b} = x\mathbf{i}+\mathbf{j}+(1-x)\mathbf{k}$ and $\mathbf{c} = y\mathbf{i}+x\mathbf{j}+(1+x-y)\mathbf{k}$. Then $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$ depends on (2001)

a) only x
b) only y
c) Neither x nor y
d) both x and y

2.10.44. If $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are unit vectors, then
$|\mathbf{a}-\mathbf{b}|^2+|\mathbf{b}-\mathbf{c}|^2+|\mathbf{a}-\mathbf{b}|^2$ does not exceed (2001)

a) 4
b) 9
c) 8
d) 6

2.10.45. If $\mathbf{a}$ and $\mathbf{b}$ are two unit vectors such that $\mathbf{a}+2\mathbf{b}$ and $5\mathbf{a}-4\mathbf{b}$ are perpendicular to each other then the angle between $\mathbf{a}$ and $\mathbf{b}$ is (2002)

a) 45°
b) 60°
c) $\arccos\frac{1}{3}$
d) $\arccos\frac{2}{7}$

2.10.46. Let $\mathbf{V} = 2\mathbf{i}+\mathbf{j}-\mathbf{k}$ and $\mathbf{W} = \mathbf{i}+3\mathbf{k}$. If $\mathbf{U}$ is a unit vector, then the maximum value of the scalar triple product $(\mathbf{U}\ \mathbf{V}\ \mathbf{W})$ is (2002)

a) -1 b) $\sqrt{10}+\sqrt{6}$ c) $\sqrt{59}$ d) $\sqrt{60}$

2.10.47. The value of a so that the volume of parallelopiped formed by $\mathbf{i}+a\mathbf{j}+\mathbf{k}, \mathbf{j}+a\mathbf{k}$ and $a\mathbf{i}+\mathbf{k}$ becomes minimum is (2003)

a) -3 b) 3 c) $\frac{1}{\sqrt{3}}$ d) $\sqrt{3}$

2.10.48. If $\mathbf{a}=\mathbf{i}+\mathbf{j}+\mathbf{k}$, $\mathbf{a}\cdot\mathbf{b}=1$ and $\mathbf{a}\times\mathbf{b}=\mathbf{j}-\mathbf{k}$. Then $\mathbf{b}$ is (2004)

a) $\mathbf{i}-\mathbf{j}+\mathbf{k}$
c) $\mathbf{i}$
b) $2\mathbf{j}-\mathbf{k}$
d) $2\mathbf{i}$

2.10.49. The unit vector which is orthogonal to the vector $3\hat{i}+2\hat{j}+6\hat{k}$ and is coplanar with vectors $2\hat{i}+\hat{j}+\hat{k}$ and $\hat{i}-\hat{j}+\hat{k}$ is (2004)

a) $\frac{2\hat{i}-6\hat{j}+\hat{k}}{\sqrt{41}}$ b) $\frac{2\hat{i}-3\hat{j}}{\sqrt{13}}$ c) $\frac{3\hat{i}-\hat{k}}{\sqrt{10}}$ d) $\frac{4\hat{i}+3\hat{j}-3\hat{k}}{\sqrt{34}}$

2.10.50. A variable plane at a distance of one unit from the origin cuts the coordinate axes at $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$. If the centroid $\mathbf{D}(x, y, z)$ of triangle ABC satisfies the relation $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=k$, then the value of k is (2005)

a) 3 b) 1 c) $\frac{1}{3}$ d) 9

2.10.51. If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are three non-zero, non-coplanar vectors and

$$\mathbf{b_1}=\mathbf{b}-\frac{\mathbf{b}\cdot\mathbf{a}}{|\mathbf{a}^2|}\mathbf{a}, \mathbf{b_2}=\mathbf{b}+\frac{\mathbf{b}\cdot\mathbf{a}}{|\mathbf{a}^2|}\mathbf{a},$$

$$\mathbf{c_1}=\mathbf{c}-\frac{\mathbf{c}\cdot\mathbf{a}}{|\mathbf{a}^2|}\mathbf{a}+\frac{\mathbf{b}\cdot\mathbf{c}}{|\mathbf{c}^2|}\mathbf{b_1}, \mathbf{c_2}=\mathbf{c}-\frac{\mathbf{c}\cdot\mathbf{a}}{|\mathbf{a}^2|}\mathbf{a}+\frac{\mathbf{b_1}\cdot\mathbf{c}}{|\mathbf{b_1}^2|}\mathbf{b_1},$$

$$\mathbf{c_3}=\mathbf{c}-\frac{\mathbf{c}\cdot\mathbf{a}}{|\mathbf{c}^2|}\mathbf{a}+\frac{\mathbf{b}\cdot\mathbf{c}}{|\mathbf{c}^2|}\mathbf{b_1}, \mathbf{c_4}=\mathbf{c}-\frac{\mathbf{c}\cdot\mathbf{a}}{|\mathbf{c}^2|}\mathbf{a}=\frac{\mathbf{b}\cdot\mathbf{c}}{|\mathbf{b}^2|}\mathbf{b_1},$$

then the set of orthogonal vectors is (2005)

a) $(\mathbf{a}, \mathbf{b_1}, \mathbf{c_3})$ b) $(\mathbf{a}, \mathbf{b_1}, \mathbf{c_2})$ c) $(\mathbf{a}, \mathbf{b_1}, \mathbf{c_1})$ d) $(\mathbf{a}, \mathbf{b_2}, \mathbf{c_2})$

2.10.52. Let $\mathbf{a}=\hat{i}+2\hat{j}+\hat{k}, \mathbf{b}=\hat{i}-\hat{j}+\hat{k}$ and $\mathbf{c}=\hat{i}+\hat{j}-\hat{k}$. A vector in the plane of $\mathbf{a}$ and $\mathbf{b}$ whose projection on $\mathbf{c}$ is $\frac{1}{\sqrt{3}}$, is (2006)

a) $4\hat{i}-\hat{j}+4\hat{k}$ b) $3\hat{i}+\hat{j}-3\hat{k}$ c) $2\hat{i}+\hat{j}-2\hat{k}$ d) $4\hat{i}+\hat{j}-4\hat{k}$

2.10.53. The number of distinct real values of λ, for which the vectors $-\lambda^2\hat{i}+\hat{j}+\hat{k}, \hat{i}-\lambda^2\hat{j}+\hat{k}$ and $\hat{i}+\hat{j}-\lambda^2\hat{k}$ are coplanar, is (2007)

a) 1 b) 2 c) 3 d) 4

2.10.54. let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be unit vectors such that $\mathbf{a}+\mathbf{b}+\mathbf{c} = \mathbf{0}$. Which of the following are correct? (2007)

a) $\mathbf{a}\times\mathbf{b} = \mathbf{b}\times\mathbf{c} = \mathbf{c}\times\mathbf{a} = \mathbf{0}$
b) $\mathbf{a}\times\mathbf{b} = \mathbf{b}\times\mathbf{c} = \mathbf{c}\times\mathbf{a} \neq \mathbf{0}$
c) $\mathbf{a}\times\mathbf{b} = \mathbf{b}\times\mathbf{c} = \mathbf{a}\times\mathbf{c} \neq \mathbf{0}$
d) $\mathbf{a}\times\mathbf{b}, \mathbf{b}\times\mathbf{c}, \mathbf{c}\times\mathbf{a}$ are mutually perpendicular

2.10.55. The edges of a parallelopiped are of unit length and are parallel to non-coplanar unit vectors $\hat{a}, \hat{b}, \hat{c}$ such that $\hat{a}\cdot\hat{b} = \hat{b}\cdot\hat{c} = \hat{c}\cdot\hat{a} = \frac{1}{2}$. Then, the volume of the parallelopiped is (2008)

a) $\frac{1}{\sqrt{2}}$ b) $\frac{1}{2\sqrt{2}}$ c) $\frac{\sqrt{3}}{2}$ d) $\frac{1}{\sqrt{3}}$

2.10.56. Let two non-collinear unit vectors $\hat{a}$ and $\hat{b}$ form an acute angle. A point $\mathbf{P}$ moves so that at any time t the position vector $\overrightarrow{OP}$ (where $\mathbf{O}$ is the origin) is given by $\hat{\mathbf{a}}\cos t + \hat{\mathbf{b}}\sin t$. When $\mathbf{P}$ is farthest from origin $\mathbf{O}$, let M be the length of $\overrightarrow{OP}$ and $\hat{\mathbf{u}}$ be the unit vector along $\overrightarrow{OP}$. Then, (2008)

a) $\hat{\mathbf{u}} = \frac{\hat{\mathbf{a}}+\hat{\mathbf{b}}}{|\hat{\mathbf{a}}+\hat{\mathbf{b}}|}$ and $M = (1+\hat{\mathbf{a}}\cdot\hat{\mathbf{b}})^{\frac{1}{2}}$
b) $\hat{\mathbf{u}} = \frac{\hat{\mathbf{a}}-\hat{\mathbf{b}}}{|\hat{\mathbf{a}}-\hat{\mathbf{b}}|}$ and $M = (1+\hat{\mathbf{a}}\cdot\hat{\mathbf{b}})^{\frac{1}{2}}$
c) $\hat{\mathbf{u}} = \frac{\hat{\mathbf{a}}+\hat{\mathbf{b}}}{|\hat{\mathbf{a}}+\hat{\mathbf{b}}|}$ and $M = (1+2\hat{\mathbf{a}}\cdot\hat{\mathbf{b}})^{\frac{1}{2}}$
d) $\hat{\mathbf{u}} = \frac{\hat{\mathbf{a}}-\hat{\mathbf{b}}}{|\hat{\mathbf{a}}-\hat{\mathbf{b}}|}$ and $M = (1+2\hat{\mathbf{a}}\cdot\hat{\mathbf{b}})^{\frac{1}{2}}$

2.10.57. If $\mathbf{a}, \mathbf{b}, \mathbf{c},$ and $\mathbf{d}$ are unit vectors such that $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = 1$ and $\mathbf{a}\cdot\mathbf{c} = \frac{1}{2}$, then

a) $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are non-coplanar
b) $\mathbf{b}, \mathbf{c}, \mathbf{d}$ are non-coplanar
c) $\mathbf{b}, \mathbf{d}$ are non-parallel
d) $\mathbf{a}, \mathbf{d}$ are parallel and $\mathbf{b}, \mathbf{c}$ are parallel

(2009)

2.10.58. Let $\mathbf{P}, \mathbf{Q}, \mathbf{R}$ and $\mathbf{S}$ be the points on the plane with position vectors $-2\hat{\imath}-\hat{\jmath}, 4\hat{\imath}, 3\hat{\imath}+3\hat{\jmath}$ and $-3\hat{\imath}+2\hat{\jmath}$ respectively. The quadrilateral $PQRS$ must be a (2010)

a) parallelogram, which is neither a rhombus nor a rectangle
b) square
c) rectangle, but not a square
d) rhombus, but not a square

2.10.59. Two adjacent sides of a parallelogram $ABCD$ are given by $AB = 2\hat{\imath}+10\hat{\jmath}+11\hat{k}$ and $AD = \hat{\imath}+2\hat{\jmath}+2\hat{k}$. The side AD is rotated by an acute angle α in the plane of the parallelogram so that AD becomes AD'. If AD' makes a right angle with the side AB, then the cosine of the angle α is given by (2010)

a) $\frac{8}{9}$ b) $\frac{\sqrt{17}}{9}$ c) $\frac{1}{9}$ d) $\frac{4\sqrt{5}}{9}$

2.10.60. Let $\mathbf{a} = \hat{\imath}+\hat{\jmath}+\hat{k}$, $\mathbf{b} = \hat{\imath}-\hat{\jmath}+\hat{k}$ and $\mathbf{c} = \hat{\imath}-\hat{\jmath}-\hat{k}$ be three vectors. A vector $\mathbf{v}$ in the plane of $\mathbf{a}$ and $\mathbf{b}$, whose projection on $\mathbf{c}$ is $\frac{1}{\sqrt{3}}$, is given by (2011)

a) $\hat{i} - 3\hat{j} + 3\hat{k}$ b) $-3\hat{i} - 3\hat{j} - \hat{k}$ c) $3\hat{i} - \hat{j} + 3\hat{k}$ d) $\hat{i} + 3\hat{j} - 3\hat{k}$

2.10.61. If $\mathbf{a}$ and $\mathbf{b}$ are vectors such that $|\mathbf{a} + \mathbf{b}| = \sqrt{29}$ and

$$\mathbf{a} \times \left(2\hat{i} + 3\hat{j} + 4\hat{k}\right) = \left(2\hat{i} + 3\hat{j} + 4\hat{k}\right) \times \mathbf{b},$$

then a possible value of $(\mathbf{a} + \mathbf{b}) \cdot \left(-7\hat{i} + 2\hat{j} + 3\hat{k}\right)$ is (2012)

a) 0 b) 3 c) 4 d) 8

2.10.62. Find all values of λ such that $x, y, z \neq (0, 0, 0)$ and

$$\left(\hat{i} + \hat{j} + 3\hat{k}\right) x + \left(3\hat{i} - 3\hat{j} + \hat{k}\right) y + \left(-4\hat{i} + 5\hat{j}\right) z = \lambda \left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

where $\hat{i}, \hat{j}, \hat{k}$ are unit vectors along the coordinate axes. (1982)

2.10.63. A vector $\mathbf{A}$ has components A_1, A_2, A_3 in a right-handed rectangular Cartesian coordinate system oxyz. The coordinate system is rotated about the x-axis through an angle $\frac{\pi}{2}$. Find the components of $\mathbf{A}$ in the new coordinate system in terms of A_1, A_2, A_3. (1983)

2.10.64. The position vectors of the points $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{D}$ are $\left(3\hat{i} - 2\hat{j} - \hat{k}\right)$, $\left(2\hat{i} + 3\hat{j} - 4\hat{k}\right)$, $\left(-\hat{i} + \hat{j} + 2\hat{k}\right)$ and $\left(4\hat{i} + 5\hat{j} + \lambda\hat{k}\right)$ respectively. If the points $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{D}$ lie on a plane, find the value of λ. (1986)

2.10.65. Let OACB be a parallelogram with $\mathbf{O}$ at the origin and OC a diagonal. Let $\mathbf{D}$ be the midpoint of OA. Using vector methods prove that BD and CO intersect in the same ratio. Determine this ratio. (1988)

2.10.66. If vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are coplanar, show that

$$\begin{vmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{a} \cdot \mathbf{a} & \mathbf{a} \cdot \mathbf{b} & \mathbf{a} \cdot \mathbf{c} \\ \mathbf{b} \cdot \mathbf{a} & \mathbf{b} \cdot \mathbf{b} & \mathbf{b} \cdot \mathbf{c} \end{vmatrix} = \mathbf{0}$$

(1989)

2.10.67. In a triangle OAB, $\mathbf{E}$ is the midpoint of BO and $\mathbf{D}$ is a point on AB such that AD:DB=2:1. If OD and AE intersect at $\mathbf{P}$, determine the ratio OP:PD using vector methods. (1989)

2.10.68. Let $\mathbf{A} = 2\hat{i} + \hat{k}, \mathbf{B} = \hat{i} + \hat{j} + \hat{k}$, and $\mathbf{C} = 4\hat{i} - 3\hat{j} + 7\hat{k}$. Determine a vector $\mathbf{R}$ satisfying $\mathbf{R} \times \mathbf{B} = \mathbf{C} \times \mathbf{B}$ and $\mathbf{R} \cdot \mathbf{A} = 0$. (1990)

2.10.69. Determine the value of c so that for all real x, the vector $cx\hat{i} - 6\hat{j} - 3\hat{k}$ and $x\hat{i} + 2\hat{j} + 2cx\hat{k}$ make an obtuse angle with each other. (1991)

2.10.70. In a $\triangle ABC$, $\mathbf{D}$ and $\mathbf{E}$ are points on BC and AC respectively, such that $BD = 2DC$ and $AE = 3EC$. Let $\mathbf{P}$ be the point of intersection of AD and BE. Find $\frac{BP}{PE}$ using vector methods. (1993)

2.10.71. If the vectors $\mathbf{b}, \mathbf{c}, \mathbf{d}$ are not coplanar, then prove that the vector

$$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) + (\mathbf{a} \times \mathbf{c}) \times (\mathbf{d} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{d}) \times (\mathbf{b} \times \mathbf{c})$$

is parallel to $\mathbf{a}$. (1994)

2.10.72. The position vectors of the vertices $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ of a tetrahedron ABCD are $\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}, \hat{\mathbf{i}}$ and $3\hat{\mathbf{i}}$ respectively. The altitude from vertex $\mathbf{D}$ to the opposite face ABC meets the median line through $\mathbf{A}$ of the triangle ABC at a point $\mathbf{E}$. If the length of the side AD is 4 and the volume of the tetrahedron is $\frac{2\sqrt{2}}{3}$, find the position vector of the point $\mathbf{E}$ for all its possible positions. (1996)

2.10.73. Let $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ be unit vectors. Suppose that $\mathbf{A} \cdot \mathbf{B} = \mathbf{A} \cdot \mathbf{C} = 0$, and that the angle between $\mathbf{B}$ and $\mathbf{C}$ is $\frac{\pi}{6}$. Then $\mathbf{A} = \pm 2(\mathbf{B} \times \mathbf{C})$. (1981)

2.10.74. If $\mathbf{X} \cdot \mathbf{A} = 0$, $\mathbf{X} \cdot \mathbf{B} = 0$, $\mathbf{X} \cdot \mathbf{C} = 0$ for some non-zero vector $\mathbf{X}$, then $[\mathbf{A}\ \mathbf{B}\ \mathbf{C}] = 0$ (1983)

2.10.75. The points with position vectors $\mathbf{a} + \mathbf{b}$, $\mathbf{a} - \mathbf{b}$ and $\mathbf{a} + k\mathbf{b}$ are collinear for all real values of k. (1984)

2.10.76. For any three vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$, $(\mathbf{a} - \mathbf{b}) \cdot ((\mathbf{b} - \mathbf{c}) \times (\mathbf{c} - \mathbf{a})) = 2\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$. (1989)

2.10.77. The points $(-a, -b)$, $(0, 0)$, (a, b) and $\left(a^2, ab\right)$ are (1979)

a) Collinear
b) Vertices of a parallelogram
c) Vertices of a rectangle
d) None of these

2.10.78. The point $(4, 1)$ undergoes the following three transformations successively. (1980)

a) Reflection about the line $y = x$.
b) Translation through a distance 2 units along the positive direction of x-axis.
c) Rotation through an angle $\frac{\pi}{4}$ about the origin in the counter clockwise direction.

Then the final position of the point is given by the coordinates.

a) $\left(\frac{1}{\sqrt{2}}, \frac{7}{\sqrt{2}}\right)$ b) $\left(-\sqrt{2}, 7\sqrt{2}\right)$ c) $\left(\sqrt{2}, 7\sqrt{2}\right)$ d) $\left(-\frac{1}{\sqrt{2}}, \frac{7}{\sqrt{2}}\right)$

2.10.79. In a triangle PQR, let

$$\mathbf{a} = \overrightarrow{QR}, \quad \mathbf{b} = \overrightarrow{RP}, \quad \mathbf{c} = \overrightarrow{PQ}. \tag{2.10.79.1}$$

$\det \mathbf{a} = 3$, $\det \mathbf{b} = 4$, and

$$\frac{\mathbf{a} \cdot (\mathbf{c} - \mathbf{b})}{\mathbf{c} \cdot (\mathbf{a} - \mathbf{b})} = \frac{|\mathbf{a}|}{|\mathbf{a}| + |\mathbf{b}|}$$

then the value of $|\mathbf{a} \times \mathbf{b}|^2$ is ______. (2020)

2.10.80. Let $\alpha, \beta, \gamma, \delta$ be real numbers such that $\alpha^2 + \beta^2 + \gamma^2 \neq 0$ and $\alpha + \gamma = 1$. Suppose the point (3,2,-1) is the mirror image of the point (1,0,-1) with respect to the plane $\alpha x + \beta y + \gamma z = \delta$. Then which of the following statements is/are TRUE? (2020)

a) $\alpha + \beta = 2$
b) $\delta - \gamma = 3$
c) $\delta + \beta = 4$
d) $\alpha + \beta + \gamma = \delta$

2.10.81. Let a and b be positive real numbers. Suppose $\overrightarrow{PQ} = a\hat{i} + b\hat{j}$ and $\overrightarrow{PQ} = a\hat{i} - b\hat{j}$ are adjacent sides of a parallelogram $PQRS$. Let $\vec{u}$ and $\vec{v}$ be the projection vectors of $\vec{w} = \hat{i} + \hat{j}$ along $\overrightarrow{PQ}$ and $\overrightarrow{PS}$ respectively. If $|\vec{u}| + |\vec{v}| = |\vec{w}|$ and if the area ofthe parallelogram $PQRS$ is 8, then which of the following statements is/are TRUE? (2020)

a) $a + b = 4$
b) $a - b = 2$
c) The length of the diagonal PR of the parallelogram $PQRS$ is 4
d) $\overrightarrow{w}$ is an angle bisector of the vectors $\overrightarrow{PS}$ and $\overrightarrow{PS}$

2.10.82. Let **u**, **v**, and **w** be vectors in three-dimensional space, where **u** and **v** are unit vectors which are not perpendicular to each other and

$$\mathbf{u} \cdot \mathbf{w} = 1, \quad \mathbf{v} \cdot \mathbf{w} = 1, \quad \mathbf{u} \cdot \mathbf{v} = 4.$$

If the volume of the parallelepiped, whose adjacent sides are represented by the vectors **u**, **v**, and **w**, is $\sqrt{2}$, then the value of $|3\mathbf{u} + 5\mathbf{v}|$ is ______. (2021)

2.10.83. Let $\hat{i}, \hat{j}, \hat{k}$ be the unit vectors along the three positive coordinate axes. Let

$$\begin{aligned}\overrightarrow{a} &= 3\hat{i} + \hat{j} - \hat{k},\\ \overrightarrow{b} &= \hat{i} + b_2\hat{j} + b_3\hat{k}, \quad b_2, b_3 \in \mathbb{R},\\ \overrightarrow{c} &= c_1\hat{i} + c_2\hat{j} + c_3\hat{k}, \quad c_1, c_2, c_3 \in \mathbb{R}\end{aligned}$$

be three vectors such that $b_2b_3 > 0$, $\overrightarrow{a} \cdot \overrightarrow{b} = 0$, and

$$\begin{pmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 3 - c_1 \\ 1 - c_2 \\ -1 - c_3 \end{pmatrix}$$

Then, which of the following statements is/are TRUE? (2022)

a) $\overrightarrow{a} \cdot \overrightarrow{c} = 0$ b) $\overrightarrow{b} \cdot \overrightarrow{c} = 0$ c) $|\overrightarrow{b}| > \sqrt{10}$ d) $|\overrightarrow{c}| \leq \sqrt{11}$

2.10.84. Let Q be the cube with the set of vertices $\{(x_1, x_2, x_3) \mid x_1, x_2, x_3 \in \{0, 1\}\} \subset \mathbb{R}^3$. Let F be the set of all twelve lines containing the diagonals of the six faces of the cube Q. Let S be the set of all four lines containing the main diagonals of the cube Q; for instance, the line passing through the vertices $(0, 0, 0)$ and $(1, 1, 1)$ is in S. For lines λ_1 and λ_2, let $d(\lambda_1, \lambda_2)$ denote the shortest distance between them. Then the maximum value of $d(\lambda_1, \lambda_2)$, as λ_1 varies over F and λ_2 varies over S, is (2023)

a) $\frac{1}{6}$ b) $\frac{1}{8}$ c) $\frac{1}{3}$ d) $\frac{1}{12}$

2.10.85. Let **P** be the plane $3x + 2y + 3z = 16$ and let

$$S : \alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}, \quad \text{where } \alpha + \beta + \gamma = 7$$

and the distance of (α, β, γ) from the plane is $\frac{2}{\sqrt{22}}$. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be three distinct vectors in S such that $|\mathbf{u} - \mathbf{v}| = |\mathbf{v} - \mathbf{w}| = |\mathbf{w} - \mathbf{u}|$. Let V be the volume of the parallelepiped determined by vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$. Then the value of $\frac{80}{3}V$ is ______. (2023)

2.10.86. Let the position vectors of the points **P**, **Q**, **R**, and **S** be

$$\vec{a} = \hat{i} + 2\hat{j} - 5\hat{k}$$
$$\vec{b} = 3\hat{i} + 6\hat{j} + 3\hat{k}$$
$$\vec{c} = \frac{17}{5}\hat{i} + \frac{16}{5}\hat{j} + \frac{7}{5}\hat{k}$$
$$\vec{d} = 2\hat{i} + \hat{j} + \hat{k}$$

respectively. Then which of the following statements is true? (2023)

a) The points **P**, **Q**, **R**, and **S** are NOT coplanar
b) $\frac{\vec{b}+2\vec{d}}{3}$ is the position vector of a point which divides PR internally in the ratio $5:4$
c) $\frac{\vec{b}+2\vec{d}}{3}$ is the position vector of a point which divides PR externally in the ratio $5:4$
d) The square of the magnitude of the vector $\vec{b} \times \vec{d}$ is 95

2.10.87. Let $\mathbb{R}^3$ denote the three-dimensional space. Take two points $\mathbf{P} = (1, 2, 3)$ and $\mathbf{Q} = (4, 2, 7)$. Let $dist(\mathbf{X}, \mathbf{Y})$ denote the distance between two points **X** and **Y** in $\mathbb{R}^3$. Let

$$S = \{\mathbf{X} \in \mathbb{R}^3 \ : \ (dist(\mathbf{X}, \mathbf{P}))^2 - (dist(\mathbf{X}, \mathbf{Q}))^2 = 50\}$$

$$T = \{\mathbf{Y} \in \mathbb{R}^3 \ : \ (dist(\mathbf{Y}, \mathbf{Q}))^2 - (dist(\mathbf{Y}, \mathbf{P}))^2 = 50\}$$

Then which of the following statements is are TRUE? (2024)

a) There is a triangle whose area is 1 and all of whose vertices are from S.
b) There are two distinct points **L** and **M** in T such that each point on the line segment LM is also in T.
c) There are infinitely many rectangles of perimeter 48, two of whose vertices are from S and the other two vertices are from T.
d) There is a square of perimeter 48, two of whose vertices are from S and the other two vertices are from T.

3.1 Formulae

3.1.1. Construct a $\triangle ABC$ given $a, \angle B$ and $K = b + c$.
Solution: Using the cosine formula in $\triangle ABC$,

$$b^2 = a^2 + c^2 - 2ac\cos B \tag{3.1.1.1}$$

$$\implies (K-c)^2 = a^2 + c^2 - 2ac\cos B \tag{3.1.1.2}$$

$$\implies c = \frac{K^2 - a^2}{2(K - a\cos B)} \tag{3.1.1.3}$$

The coordinates of $\triangle ABC$ can then be expressed as

$$\mathbf{A} = c\begin{pmatrix}\cos B\\ \sin B\end{pmatrix}, \mathbf{B} = \mathbf{0}, \mathbf{C} = \begin{pmatrix}a\\ 0\end{pmatrix}. \tag{3.1.1.4}$$

3.1.2. Construct a $\triangle ABC$ given $\angle B, \angle C$ and $K = a + b + c$.
Solution:

$$a + b + c = K \tag{3.1.2.1}$$

$$b\cos C + c\cos B - a = 0 \tag{3.1.2.2}$$

$$b\sin C - c\sin B = 0 \tag{3.1.2.3}$$

resulting in the matrix equation

$$\begin{pmatrix}1 & 1 & 1\\ -1 & \cos C & \cos B\\ 0 & \sin C & -\sin B\end{pmatrix}\begin{pmatrix}a\\ b\\ c\end{pmatrix} = K\begin{pmatrix}1\\ 0\\ 0\end{pmatrix} \tag{3.1.2.4}$$

which can be solved to obtain all the sides. $\triangle ABC$ can then be plotted using

$$\mathbf{A} = \begin{pmatrix}a\\ b\end{pmatrix}, \mathbf{B} = \mathbf{0}, \mathbf{C} = \begin{pmatrix}a\\ 0\end{pmatrix} \tag{3.1.2.5}$$

3.2 Triangle

3.2.1 Draw a triangle ABC in which AB=4 cm, $BC = 6cm$ and $AC = 9$.
3.2.2 Draw a triangle ABC in which AB=5 cm. $BC = 6cm$ and $\angle ABC = 60°$.
3.2.3 Draw a parallelogram $ABCD$ in which $BC = 5$ cm, $AB = 3$ cm and $\angle ABC = 60°$, divide it into triangles ACB and ABD by the diagonal BD.
3.2.4 Construct the triangle $BD'C'$ similar to $\triangle BDC$ with scale factor $\frac{4}{3}$. Draw the line segment $D'A'$ parallel to DA where $\mathbf{A}^prime$ lies on extended side BA. Is $A'BC'D'$ a parallelogram?
3.2.5 Draw a triangle ABC in which $BC = 6$ cm, $CA = 5$ cm and $AB = 4$ cm.
3.2.6 Construct a triangle ABC in which $BC = 7cm, \angle B = 75°$ and $AB + AC = 13cm$.
Solution: From (3.1.1.3) and (3.1.1.4), we obtain Fig. 3.2.6.1.

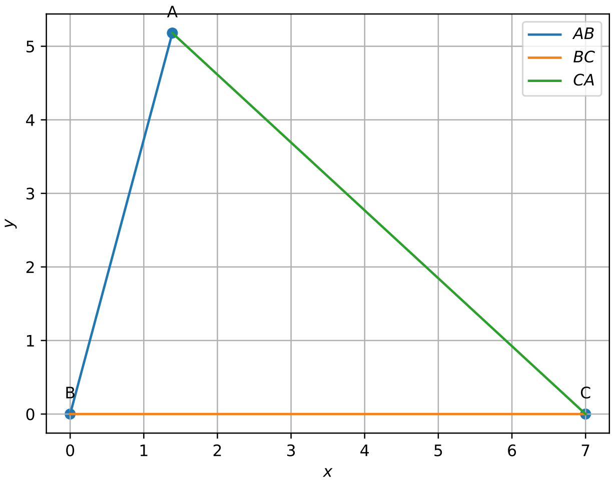


Fig. 3.2.6.1

3.2.7 Construct a triangle ABC in which $BC = 8cm$, $\angle B = 45^\circ$ and $AB - AC = 3.5cm$.
Solution: See Fig. 3.2.7.1.

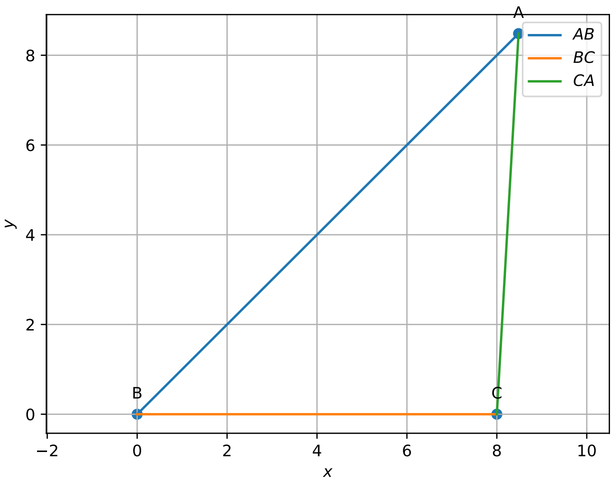


Fig. 3.2.7.1

3.2.8 Construct a triangle ABC in which $BC = 6cm$, $\angle B = 60^\circ$ and $AC - AB = 2cm$.
Solution: See Fig. 3.2.8.1 obtained by substituting $K = -2$.

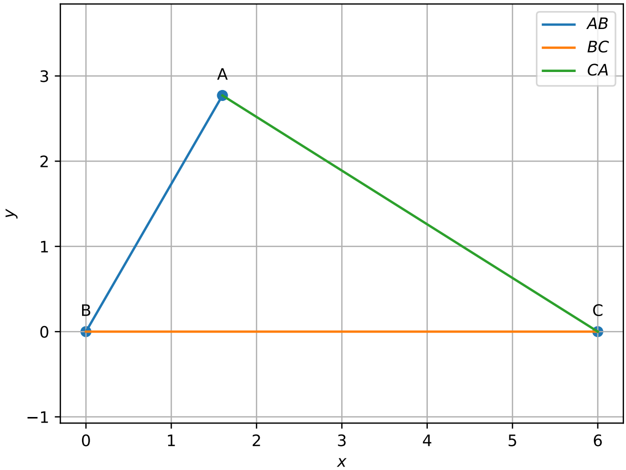


Fig. 3.2.8.1

3.2.9 Construct a right triangle whose base is $12cm$ and sum of its hypotenuse and other side is $18cm$.
Solution: For $a = 12, \angle B = 90°, b + c = 18$, we obtain Fig. 3.2.9.1.

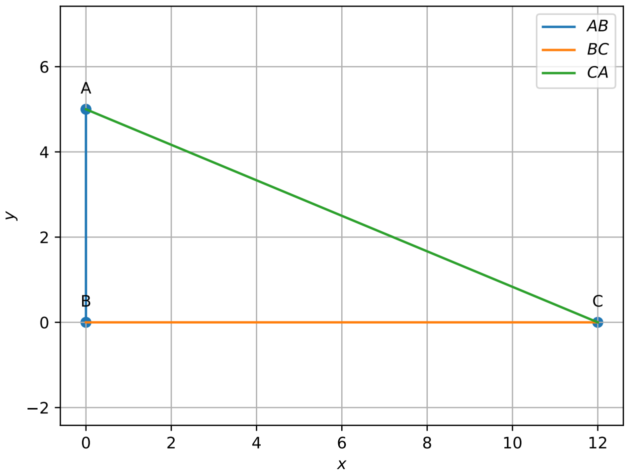


Fig. 3.2.9.1

3.2.10 Construct a triangle ABC in which $\angle B = 30°, \angle C = 90°$ and $AB + BC + CA = 11cm$.
Solution: From (3.1.2.4) and (3.1.2.5), Fig. 3.2.10.1 is generated.

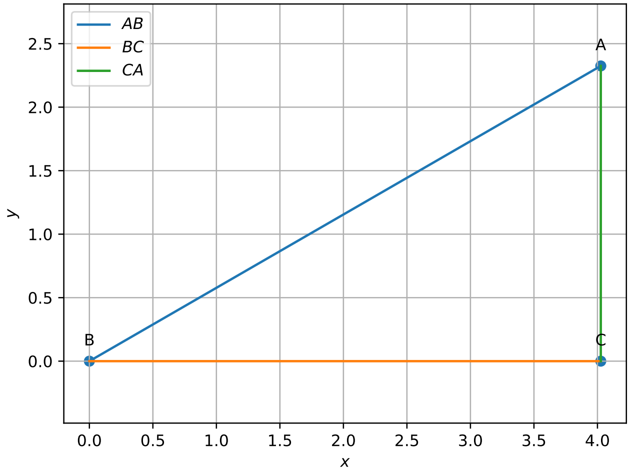


Fig. 3.2.10.1

3.2.11 Draw a right triangle ABC in which $BC = 12cm$, $AB = 5cm$ and $\angle B = 90°$.

3.2.12 Draw an isosceles triangle ABC in which $AB = AC = 6cm$ and $BC = 6cm$.

3.2.13 Draw a triangle ABC in which $AB = 5cm, BC = 6cm$ and $\angle ABC = 60°$.

3.2.14 Draw a triangle ABC in which $AB = 4cm, BC = 6cm$ and $AC = 9cm$.

3.2.15 Draw a triangle ABC in which $BC = 6cm, CA = 5cm$ and $AB = 4cm$.

3.2.16 Is it possible to construct a triangle with lengths of its sides as $4cm$, $3cm$ and $7cm$? Give reason for your answer.

3.2.17 Is it possible to construct a triangle with lengths of its sides as $9cm$, $7cm$ and $17cm$? Give reason for your answer.

3.2.18 Is it possible to construct a triangle with lengths of its sides as $8cm$, $7cm$ and $4cm$? Give reason for your answer.

3.2.19 Two sides of a triangle are of lengths $5cm$ and $1.5cm$. The length of the third side of the triangle cannot be

a) $3.6cm$
b) $4.1cm$
c) $3.8cm$
d) $3.4cm$

3.2.20 The construction of a triangle ABC, given that $BC = 6cm, \angle B = 45°$ is not possible when difference of AB and AC is equal to

a) $6.9cm$
b) $5.2cm$
c) $5.0cm$
d) $4.0cm$

3.2.21 The construction of a triangle ABC, given that $BC = 6cm, \angle C = 60°$ is possible when difference of AB and AC is equal to

a) $3.2cm$
b) $3.1cm$
c) $3cm$
d) $2.8cm$

3.2.22 Construct a triangle whose sides are $3.6cm$, $3.0cm$ and $4.8cm$. Bisect the smallest angle and measure each part.

3.2.23 Construct a triangle ABC in which $BC = 5cm$, $\angle B = 60^\circ$ and $AC + AB = 7.5cm$.

Construct each of the following and give justification :

3.2.24 A triangle if its perimeter is $10.4cm$ and two angles are $45^\circ$and 120°.

3.2.25 A triangle PQR given that $QR = 3cm$, $\angle PQR = 45^\circ$ and $QP - PR = 2cm$.

3.2.26 A right triangle when one side is $3.5cm$ and sum of other sides and the hypotenuse is $5.5cm$.

3.2.27 An equilateral triangle if its altitude is $3.2cm$.

Write true or false in each of the following. Give reasons for your answer:

3.2.28 A triangle ABC can be constructed in which $AB = 5cm$, $\angle A = 45^\circ$ and $BC+AC = 5cm$.

3.2.29 A triangle ABC can be constructed in which $BC = 6cm$, $\angle B = 30^\circ$ and $AC-AB = 4cm$.

3.2.30 A triangle ABC can be constructed in which $\angle B = 105^\circ, \angle C = 90^\circ$ and $AB+BC+AC = 10cm$.

3.2.31 A triangle ABC can be constructed in which $\angle B = 60^\circ, \angle C = 45^\circ$ and $AB+BC+AC = 12cm$.

3.2.32 Draw a right triangle ABC in which $BC = 12$ cm, $AB = 5$ cm and $\angle B = 90^\circ$.

3.3 CBSE

3.3.1 Draw a triangle $\triangle ABC$ with $BC = 6$ cm, $AB = 5$ cm, and $\angle ABC = 60^\circ$. (10, 2018)

3.3.2 Construct a triangle with sides $5cm$, $6cm$ and $7cm$. (10, 2019)

3.3.3 Construct an equilateral $\triangle ABC$ with each side $5cm$. (10, 2019)

3.3.4 Construct a right triangle in which sides (other than the hypotenuse) are $8cm$ and $6cm$. (10, 2019)

3.3.5 Construct a $\triangle ABC$ in which $CA = 6cm$, $AB = 5cm$ and $BAC = 45^\circ$. (10, 2019)

3.3.6 Construct a triangle ABC with side $BC = 6cm$, $\angle B = 45^\circ, \angle A = 105^\circ$. (10, 2019)

3.3.7 Write the steps of construction for drawing a $\triangle ABC$ in which $BC = 8$cm, $\angle B = 45^\circ$ and $\angle C = 30^\circ$. (10, 2018)

3.3.8 Construct a triangle ABC with side BC = 7 cm, $\angle B$=45°, $\angle A$=105°. (10, 2017)

3.3.9 Draw an isosceles $\triangle ABC$ in which $BC = 5.5cm$ and altitude $AL = 5.3cm$. (10, 2016)

3.3.10 Construct a right triangle ABC with AB= 6 cm, BC = 8 cm and $\angle$ B = 90°. Draw BD, the perpendicular from **B** on AC. Draw the circle through **B**, **C** and **D** and construct the tangents from **A** to this circle (10, 2015)

3.3.11 Construct a $\triangle$ ABC in which $AB = 6$ cm, $\angle$ A = 30° and $\angle$ B = 60°. (10, 2015)

3.3.12 Construct a triangle ABC in which $AB = 5$ cm, $BC = 6$ cm and $\angle ABC = 60^\circ$. (10, 2015)

3.3.13 Draw a triangle ABC with $BC = 7$ cm, $\angle B = 45^\circ$ and $\angle C = 60^\circ$. (10, 2012)

3.3.14 Construct a right triangle in which the sides, (other than the hypotenuse) are of length 6 cm and 8 cm. (10, 2012)

3.3.15 Construct a triangle ABC in which BC = 7 cm, and median AD = 5 cm, $\angle A = 60°$. Write the steps of construction also. (10, 2006)

3.4 Quadrilateral

3.4.1 Draw a quadrilateral in the Cartesian plane, whose vertices are (-4, 5), (0, 7), (5, -5) and (-4, -2).

3.4.2 Draw a parallelogram $ABCD$ in which $BC = 5cm, AB = 3cm$ and $\angle ABC = 60°$, divide it into triangles ACB and ABD by the diagonal BD.

3.4.3 Construct a square of side $3cm$.

3.4.4 Construct a rectangle whose adjacent sides are of lengths $5cm$ and $3.5cm$.

3.4.5 Construct a rhombus whose side is of length $3.4cm$ and one of its angles is 45°.

3.4.6 Construct a rhombus whose diagonals are 4 cm and 6 cm in lengths.

4.1 Formulae

4.1.1. The equation of a line is given by

$$y = mx + c \tag{4.1.1.1}$$

$$\implies \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ mx + c \end{pmatrix} = \begin{pmatrix} 0 \\ c \end{pmatrix} + x \begin{pmatrix} 1 \\ m \end{pmatrix} \tag{4.1.1.2}$$

yielding

$$\mathbf{x} = \mathbf{h} + \kappa \mathbf{m}. \tag{4.1.1.3}$$

where $\mathbf{h}$ is any point on the line and

$$\mathbf{m} = \begin{pmatrix} 1 \\ m \end{pmatrix} \tag{4.1.1.4}$$

is the direction vector.

4.1.2. For

$$\mathbf{m}^\top \mathbf{n} = 0, \tag{4.1.2.1}$$

(4.1.1.3) can be expressed as

$$\mathbf{n}^\top \mathbf{x} = \mathbf{n}^\top \mathbf{h} + \kappa \mathbf{n}^\top \mathbf{m} \tag{4.1.2.2}$$

$$\begin{aligned} \implies \mathbf{n}^\top (\mathbf{x} - \mathbf{h}) &= 0 \\ \text{or, } \mathbf{n}^\top \mathbf{x} &= c \end{aligned} \tag{4.1.2.3}$$

for

$$c = \mathbf{n}^\top \mathbf{h}. \tag{4.1.2.4}$$

where

$$\mathbf{n} = \begin{pmatrix} -m \\ 1 \end{pmatrix} \tag{4.1.2.5}$$

is defined to be the *normal vector* of the line. In 3D, (4.1.2.3) represents a plane.

4.1.3. If $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are collinear, from (4.1.2.3),

$$\mathbf{n}^\top \mathbf{A} = c \tag{4.1.3.1}$$

$$\mathbf{n}^\top \mathbf{B} = c \tag{4.1.3.2}$$

$$\mathbf{n}^\top \mathbf{C} = c \tag{4.1.3.3}$$

which can be expressed as

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix}^\top \mathbf{n} = c \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{4.1.3.4}$$

$$\equiv \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix}^\top \mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \tag{4.1.3.5}$$

$$\implies \begin{pmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix}^\top \begin{pmatrix} \mathbf{n} \\ -1 \end{pmatrix} = \mathbf{0} \tag{4.1.3.6}$$

4.1.4. The equation of a line that does not pass through the origin can be expressed as

$$\mathbf{n}^\top \mathbf{x} = 1 \tag{4.1.4.1}$$

4.1.5. Let the perpendicular distance from the origin to a line be p and the angle made by the perpendicular with the positive x-axis be θ. Then

$$p \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \tag{4.1.5.1}$$

is a point on the line as well as the normal vector. Hence, the equation of the line is

$$p \begin{pmatrix} \cos\theta & \sin\theta \end{pmatrix} \left\{ \mathbf{x} - p \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \right\} = 0 \tag{4.1.5.2}$$

$$\implies \begin{pmatrix} \cos\theta & \sin\theta \end{pmatrix} \mathbf{x} = p \tag{4.1.5.3}$$

4.1.6. Let $\mathbf{Q}$ be the foot of the perpendicular from $\mathbf{P}$ to the line

$$\mathbf{n}^\top \mathbf{x} = c \tag{4.1.6.1}$$

Then

$$PQ = \frac{\left|\mathbf{n}^\top \mathbf{P} - c\right|}{\|\mathbf{n}\|} \tag{4.1.6.2}$$

4.1.7. The foot of the perpendicular is given by

$$\begin{pmatrix} \mathbf{m} & \mathbf{n} \end{pmatrix}^\top \mathbf{Q} = \begin{pmatrix} \mathbf{m}^\top \mathbf{P} \\ c \end{pmatrix} \tag{4.1.7.1}$$

4.1.8. The distance between the parallel lines

$$\begin{aligned} \mathbf{n}^\top \mathbf{x} &= c_1 \\ \mathbf{n}^\top \mathbf{x} &= c_2 \end{aligned} \tag{4.1.8.1}$$

is given by

$$d = \frac{|c_1 - c_2|}{\|\mathbf{n}\|} \tag{4.1.8.2}$$

4.1.9. The reflection of point $\mathbf{Q}$ w.r.t a line is given by

$$\mathbf{R} = \mathbf{Q} - \frac{2\left(\mathbf{n}^\top\mathbf{Q} - c\right)}{\|\mathbf{n}\|}\mathbf{n} \tag{4.1.9.1}$$

4.1.10. The lines

$$\begin{aligned} L_1 : \quad & \mathbf{x} = \mathbf{A} + \kappa_1\mathbf{m_1} \\ L_2 : \quad & \mathbf{x} = \mathbf{B} + \kappa_2\mathbf{m_2} \end{aligned} \tag{4.1.10.1}$$

are coplanar if

$$\begin{pmatrix} \mathbf{m_1} & \mathbf{m_2} & \mathbf{B} - \mathbf{A} \end{pmatrix}\mathbf{n} = \mathbf{0} \tag{4.1.10.2}$$

where $\mathbf{n}$ is the normal vector of the plane. This is equivalent to

$$\text{nullity}\begin{pmatrix} \mathbf{M} & \mathbf{B} - \mathbf{A} \end{pmatrix} = 1 \tag{4.1.10.3}$$

where

$$\mathbf{M} = \begin{pmatrix} \mathbf{m_1} & \mathbf{m_2} \end{pmatrix} \tag{4.1.10.4}$$

4.1.11. Code for line in 2D using direction vector

codes/book/linear.py

4.1.12. Code for line in 2D using normal vector

codes/book/linorm.py

4.1.13. Code for line in 3D

codes/book/points3d.py

4.1.14. Code for plane

codes/book/plane.py

4.1.15. Code for foot of the perpendicular

codes/book/perp.py

4.1.16. Code for intersection of lines

codes/book/linsect.py

4.2 Parameters

Find the direction and normal vectors of each of the following lines

4.2.1 $2x + 3y = 9$
4.2.2 $x - \frac{y}{5} - 10 = 10$
4.2.3 $-2x + 3y = 6$
4.2.4 $x = 3y$

4.2.5 $2x = -5y$
4.2.6 $3x + 2 = 0$
4.2.7 $y - 2 = 0$
4.2.8 $5 = 2x$
4.2.9 $x + y = 4$
4.2.10 $x - y = 2$
4.2.11 $y = 3x$
4.2.12 $3 = 2x + y$
4.2.13 $y = x$
4.2.14 $x + y = 0$
4.2.15 $y = 2x$
4.2.16 $2 + 3y = 7x$
4.2.17 $y = x + 2$
4.2.18 $y = x - 2$
4.2.19 $y = -x + 2$
4.2.20 $x + 2y = 6$
4.2.21 $F = \frac{9}{5}C + 32$

Show that two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ where $b_1b_2 \neq 0$ are

4.2.22 parallel if $\frac{a_1}{b_1} = \frac{a_2}{b_2}$ and
4.2.23 perpendicular if $a_1a_2 - b_1b_2 = 0$.

4.3 Equation

Find the equation of line

4.3.1 passing through the point $\mathbf{P} = (-4, 3)$ with slope $\frac{1}{2}$.
Solution: From (4.1.2.5),

$$\mathbf{n} \equiv \begin{pmatrix} \frac{1}{2} \\ -1 \end{pmatrix} \implies \begin{pmatrix} \frac{1}{2} & -1 \end{pmatrix} \mathbf{x} = -5 \tag{4.3.1.1}$$

using (4.1.2.3). See Fig. 4.3.1.1.

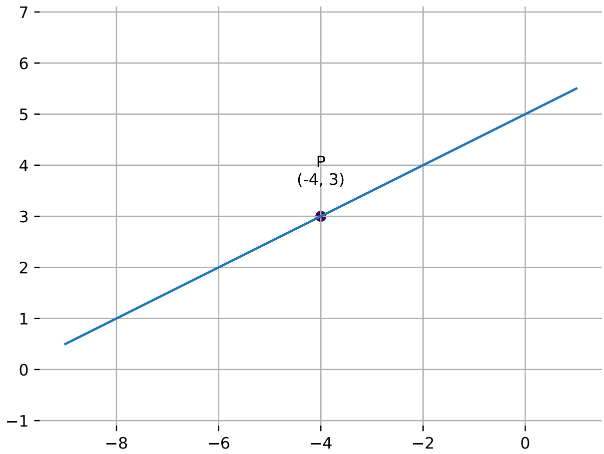


Fig. 4.3.1.1

4.3.2 passing through $\begin{pmatrix}0\\0\end{pmatrix}$ with slope m.

Solution:

$$\because \mathbf{n} = \begin{pmatrix}m\\-1\end{pmatrix}, \tag{4.3.2.1}$$

the desired equation is

$$\begin{pmatrix}m & -1\end{pmatrix}\left(\mathbf{x} - \begin{pmatrix}0\\0\end{pmatrix}\right) = 0 \tag{4.3.2.2}$$

$$\implies \begin{pmatrix}m & -1\end{pmatrix}\mathbf{x} = 0 \tag{4.3.2.3}$$

4.3.3 passing through $\mathbf{A} = \begin{pmatrix}2\\2\sqrt{3}\end{pmatrix}$ and inclined with the x-axis at an angle of 75°.

Solution:

$$\mathbf{m} = \begin{pmatrix}1\\\tan 75^\circ\end{pmatrix} \implies \mathbf{n} = \begin{pmatrix}2+\sqrt{3}\\-1\end{pmatrix} \tag{4.3.3.1}$$

$$\implies \begin{pmatrix}2+\sqrt{3} & -1\end{pmatrix}\mathbf{x} = \begin{pmatrix}2+\sqrt{3} & -1\end{pmatrix}\begin{pmatrix}2\\2\sqrt{3}\end{pmatrix} \tag{4.3.3.2}$$

$$= 4 \tag{4.3.3.3}$$

is the desired equation.

4.3.4 intersecting the y-axis at a distance of 2 units above the origin and making an angle of 30° with positive direction of the x-axis.

Solution:

$$\mathbf{n} = \begin{pmatrix} -\frac{1}{\sqrt{3}} \\ 1 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}. \tag{4.3.4.1}$$

Hence, the equation of the line is given by

$$\begin{pmatrix} -\frac{1}{\sqrt{3}} & 1 \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right) = 0 \tag{4.3.4.2}$$

$$\text{or, } \begin{pmatrix} -\frac{1}{\sqrt{3}} & 1 \end{pmatrix} \mathbf{x} = 2 \tag{4.3.4.3}$$

4.3.5 passing through (1, 2) and making angle 30° with y-axis.

4.3.6 passing through the points $(3, 4, -7)$ and $(1, -1, 6)$.

4.3.7 passing through the points $\mathbf{A}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\mathbf{B}\begin{pmatrix} 2 \\ -4 \end{pmatrix}$.

Solution: From (4.1.3.5),

$$\begin{pmatrix} -1 & 1 \\ 2 & -4 \end{pmatrix} \mathbf{n} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{4.3.7.1}$$

$$\implies \left(\begin{array}{cc|c} -1 & 1 & 1 \\ 2 & -4 & 1 \end{array}\right) \xleftrightarrow{R_2 \leftarrow R_2 + 2R_1} \left(\begin{array}{cc|c} -1 & 1 & 1 \\ 0 & -2 & 3 \end{array}\right) \tag{4.3.7.2}$$

$$\xleftrightarrow{R_1 \leftarrow 2R_1 + R_2} \left(\begin{array}{cc|c} -2 & 0 & 5 \\ 0 & -2 & 3 \end{array}\right) \implies \mathbf{n} = -\frac{1}{2}\begin{pmatrix} 5 \\ 3 \end{pmatrix} \tag{4.3.7.3}$$

Thus, from (4.1.4.1), the equation of the line is

$$\begin{pmatrix} 5 & 3 \end{pmatrix} \mathbf{x} = -2 \tag{4.3.7.4}$$

See Fig. 4.3.7.1.

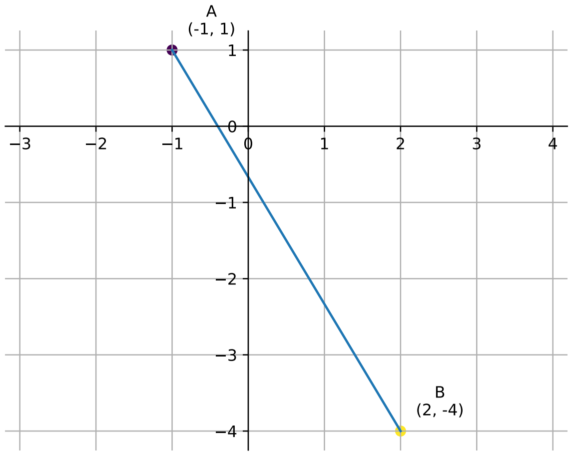


Fig. 4.3.7.1

4.3.8 The vector equation of the line

$$\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2}$$

is ______.

4.3.9 The vector equation of the line

$$\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2}$$

is ______.

4.3.10 The vertices of triangle PQR are $\mathbf{P}(2,1), \mathbf{Q}(-2,3), \mathbf{R}(4,5)$. Find the equation of the median through $\mathbf{R}$.

Solution: Using section formula, the mid point of PQ is

$$\mathbf{A} = \frac{\mathbf{P}+\mathbf{Q}}{2} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \tag{4.3.10.1}$$

Following the approach in Problem 4.3.7,

$$\left(\begin{array}{cc|c} 4 & 5 & 1 \\ 0 & 2 & 1 \end{array}\right) \xleftrightarrow[R_2 \leftarrow 4R_2]{R_1 \leftarrow 2R_1 - 5R_2} \left(\begin{array}{cc|c} 8 & 0 & -3 \\ 0 & 8 & 4 \end{array}\right) \implies \mathbf{n} = \frac{1}{8}\begin{pmatrix} -3 \\ 4 \end{pmatrix}$$

Thus, the equation of the line is

$$\begin{pmatrix} -3 & 4 \end{pmatrix} \mathbf{x} = 8 \tag{4.3.10.2}$$

4.3.11 Find the equation of the line intersecting the x-axis at a distance of 3 units to the left of origin with slope of -2.

Solution: From the given information,

$$\mathbf{A} = \begin{pmatrix} -3 \\ 0 \end{pmatrix}, \mathbf{n} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}. \tag{4.3.11.1}$$

The desired equation of the line is

$$\implies \begin{pmatrix} 2 & 1 \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} -3 \\ 0 \end{pmatrix} \right) = 0 \tag{4.3.11.2}$$

$$\text{or, } \begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = -6 \tag{4.3.11.3}$$

See Fig. 4.3.11.1.

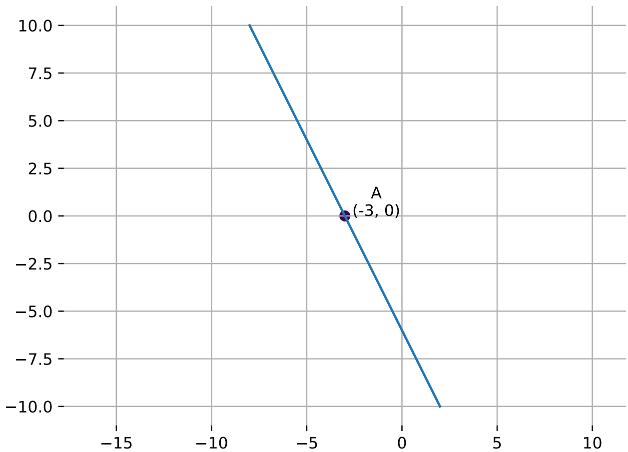


Fig. 4.3.11.1

4.3.12 Check which of the following are solutions of the equation $x - 2y = 4$ and which are not

a) $(0, 2)$
b) $(2, 0)$
c) $(4, 0)$
d) $(\sqrt{2}, 4\sqrt{2})$
e) $(1, 1)$

4.3.13 Equations of the diagonals of the square formed by the lines $x = 0$, $y = 0$, $x = 1$ and $y = 1$ are ______.

4.3.14 A line intersects the Y axis and X axis at the points **P** and **Q**, respectively. If $(2, 5)$ is the mid-point of PQ, then the coordinates of **P** and **Q** are ______.

4.3.15 Find the equations of the planes that pass through the points

a) $\mathbf{A} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 6 \\ 4 \\ -5 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -4 \\ -2 \\ 3 \end{pmatrix}$

b) $\mathbf{A} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix}$

Solution:

a) From (4.1.3.5),

$$\begin{pmatrix} 1 & 1 & -1 \\ 6 & 4 & -5 \\ -4 & -2 & 3 \end{pmatrix} \mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \implies \left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 6 & 4 & -5 & 1 \\ -4 & -2 & 3 & 1 \end{array}\right)$$

$$\xleftrightarrow[R_3 \leftarrow R_3 + 4R_1]{R_2 \leftarrow R_2 - 6R_1} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & -2 & 1 & -5 \\ 0 & 2 & -1 & 5 \end{array}\right) \xleftrightarrow[R_1 \leftarrow 2R_1 + R_2]{R_3 \leftarrow R_3 + R_2} \left(\begin{array}{ccc|c} 2 & 0 & -1 & -3 \\ 0 & 2 & -1 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Since we obtain a 0 row, the given points are collinear. The direction vector of the line is

$$\mathbf{m} = \mathbf{B} - \mathbf{C} \equiv \begin{pmatrix} 5 \\ 3 \\ -4 \end{pmatrix} \tag{4.3.15.1}$$

and the equation of a line is given by,

$$\mathbf{x} = \mathbf{A} + \kappa \mathbf{m} \tag{4.3.15.2}$$

$$= \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + \kappa \begin{pmatrix} 5 \\ 3 \\ -4 \end{pmatrix} \tag{4.3.15.3}$$

See Fig. 4.3.15.1.

b) In this case,

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ -2 & 2 & -1 \end{pmatrix} \mathbf{n} = \mathbf{1} \tag{4.3.15.4}$$

$$\implies \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 \\ -2 & 2 & -1 & 1 \end{array}\right) \xleftrightarrow[R_3 \leftarrow R_3 + 2R_1]{R_2 \leftarrow R_2 - R_1} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 4 & -1 & 3 \end{array}\right)$$

$$\xleftrightarrow[R_3 \leftarrow R_3 - 4R_2]{R_1 \leftarrow R_1 - R_2} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -5 & 3 \end{array}\right) \xleftrightarrow[R_2 \leftarrow 5R_2 + R_3]{R_1 \leftarrow 5R_1 - R_3} \left(\begin{array}{ccc|c} 5 & 0 & 0 & 2 \\ 0 & 5 & 0 & 3 \\ 0 & 0 & 5 & -3 \end{array}\right)$$

Hence, the equation of the plane is

$$\begin{pmatrix} 2 & 3 & -3 \end{pmatrix} \mathbf{x} = 5 \tag{4.3.15.5}$$

See Fig. 4.3.15.2.

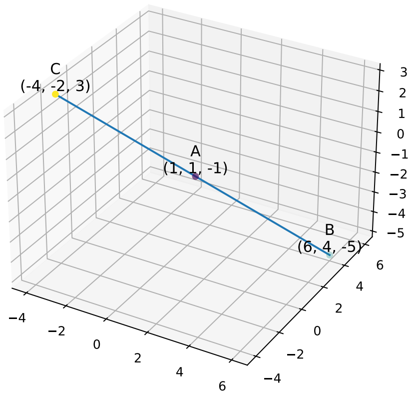


Fig. 4.3.15.1

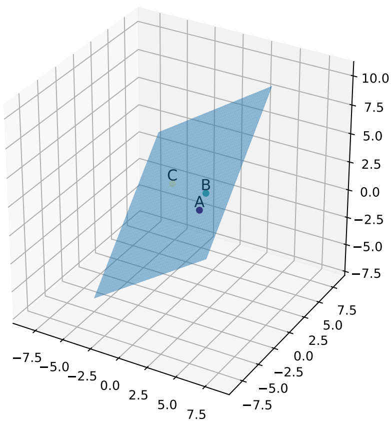


Fig. 4.3.15.2

4.3.16 Find the equation of the plane through the points $(2,1,0)$, $(3,-2,-2)$ and $(3,1,7)$.

4.3.17 A plane passes through the points $(2,0,0),(0,3,0)$ and $(0,0,4)$. The equation of the plane is ______.

4.3.18 If the intercept of a line between the coordinate axes is divided by the point (-5, 4) in the ratio 1:2 then find the equation of the line.

4.3.19 Find the equation of a line that cuts off equal intercepts on the coordinate axes and passes through the point $(2, 3)$.

Solution: Let $(a, 0)$ and $(0, a)$ be the intercept points.

$$\mathbf{m} = \begin{pmatrix} a \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ a \end{pmatrix} \equiv \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{4.3.19.1}$$

$$\implies \mathbf{n} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{4.3.19.2}$$

and the equation of the line is

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right) = 0 \tag{4.3.19.3}$$

$$\implies \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 5 \tag{4.3.19.4}$$

See Fig. 4.3.19.1.

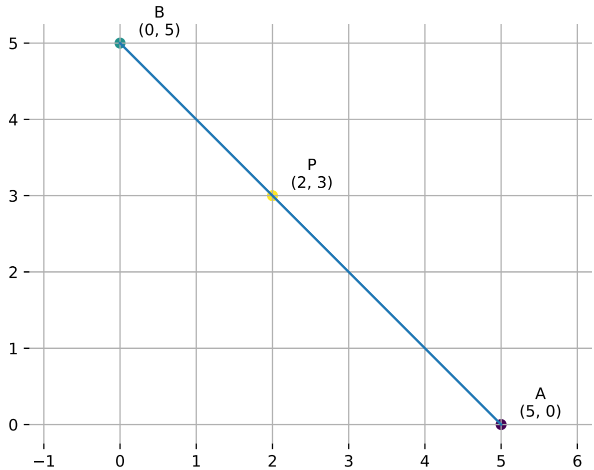


Fig. 4.3.19.1

4.3.20 Find the ratio in which the Y axis divides the line segment joining the points $(5, -6)$ and $(-1, -4)$. Also find the point of intersection.

4.3.21 In what ratio does the X axis divide the line segment joining the points $(-4, -6)$ and $(-1, 7)$? Find the coordinates of the point of division.

4.3.22 Find the ratio in which the line segment joining $\mathbf{A}(1, -5)$ and $\mathbf{B}(-4, 5)$ is divided by the X axis. Also find the coordinates of the point of division.

4.3.23 The line segment joining the points $\mathbf{A}(3, 2)$ and $\mathbf{B}(5, 1)$ is divided at the point $\mathbf{P}$ in the ratio 1:2 which lies on $3x - 18y + k = 0$. Find the value of k.

4.3.24 Find the ratio in which the line $2x + 3y - 5 = 0$ divides the line segment joining the points $(8, -9)$ and $(2, 1)$. Also find the coordinates of the point of division.

4.3.25 Find the ratio in which the YZ plane divides the line segment formed by joining the points $(-2, 4, 7)$ and $(3, -5, 8)$.

4.3.26 Find the ratio in which the line segment joining the points $(4, 8, 10)$ and $(6, 10, -8)$ is divided by the YZ plane.

4.3.27 Find the equation of the lines which pass through the point (3, 4) and cuts off intercepts from the coordinate axes such that their sum is 14.

4.3.28 Find the equation of the straight line which passes through the point (1, -2) and cuts off equal intercepts from axes.

4.3.29 Find the equation of a line passing through a point (2, 2) and cutting off intercepts on the axes whose sum is 9.

Solution: Let the intercept points be

$$\mathbf{P} = \begin{pmatrix} a \\ 0 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} 0 \\ b \end{pmatrix} \text{ and } \mathbf{R} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \tag{4.3.29.1}$$

be the given point. Forming the collinearity matrix from (4.1.3.6),

$$\begin{pmatrix} \mathbf{P} - \mathbf{Q} & \mathbf{P} - \mathbf{R} \end{pmatrix} = \begin{pmatrix} a & a-2 \\ -b & -2 \end{pmatrix} \tag{4.3.29.2}$$

which is singular if

$$ab - 2(a+b) = 0 \implies ab = 18 \tag{4.3.29.3}$$

$$\because a + b = 9. \tag{4.3.29.4}$$

$\therefore a, b$ are the roots of

$$x^2 - 9x + 18 = 0. \tag{4.3.29.5}$$

yielding

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \end{pmatrix} \tag{4.3.29.6}$$

Since

$$\mathbf{m} = \begin{pmatrix} a \\ -b \end{pmatrix}, \mathbf{n} = \begin{pmatrix} b \\ a \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{4.3.29.7}$$

Thus, the possible equations of the line are

$$\begin{pmatrix} 1 & 2 \end{pmatrix} \mathbf{x} = 6 \tag{4.3.29.8}$$

$$\begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = 6 \tag{4.3.29.9}$$

See Fig. 4.3.29.1.

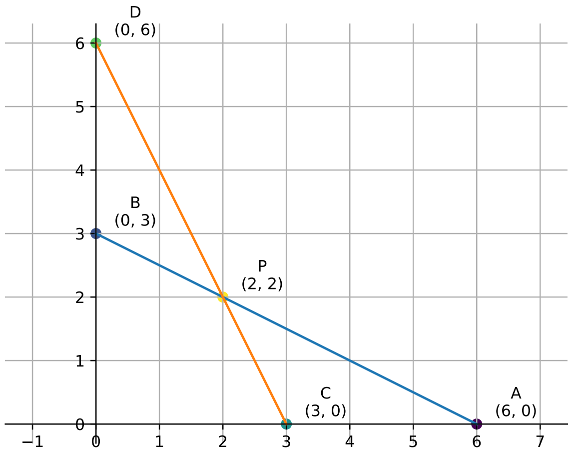


Fig. 4.3.29.1

4.3.30 Find the equation of the line which passes through the point (-4, 3) and the portion of the line intercepted between the axes is divided internally in ratio 5:3 by this point.

4.3.31 Consider the following population and year graph in Fig. 4.3.31.1. Find the slope of the line AB and using it, find what will be the population in the year 2010.

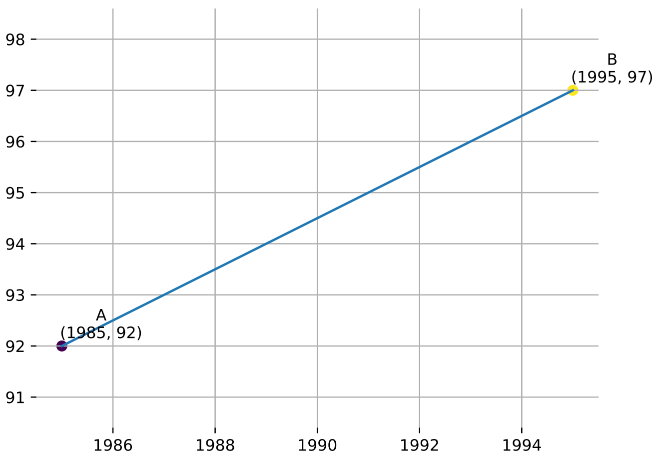


Fig. 4.3.31.1

Solution: The direction vector of the line in Fig. 4.3.31.1 is

$$\mathbf{m} = \mathbf{B} - \mathbf{A} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{4.3.31.1}$$

$$\implies \mathbf{n} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \tag{4.3.31.2}$$

The equation of the line is then given by

$$\mathbf{n}^{\top}(\mathbf{x} - \mathbf{A}) = 0 \tag{4.3.31.3}$$

$$\implies \begin{pmatrix} 1 & -2 \end{pmatrix} \mathbf{x} = 1801 \tag{4.3.31.4}$$

$$\implies \begin{pmatrix} 1 & -2 \end{pmatrix} \begin{pmatrix} 2010 \\ y \end{pmatrix} = 1801 \tag{4.3.31.5}$$

$$\implies y = \frac{209}{2} \tag{4.3.31.6}$$

4.3.32 Slope of a line which cuts off intercepts of equal length on the axes is ______.

4.3.33 If the coordinates of middle point of the portion of a line intercepted between the coordinate axes is (3, 2), then the equation of the line will be ______.

4.3.34 If the line $\frac{x}{a} + \frac{y}{b} = 1$ passes the points (2, -3) and (4, -5), then (a, b) is ______.

4.3.35 The intercepts made by the plane $2x - 3y + 5z + 4 = 0$ on the co-ordinate axis are $\left(-2, \frac{4}{3}, -\frac{4}{5}\right)$.

4.3.36 The line $\overrightarrow{r} = 2\hat{i} - 3\hat{j} - \hat{k} + \lambda(\hat{i} - \hat{j} + 2\hat{k})$ lies in the plane $\overrightarrow{r} \cdot (3\hat{i} + \hat{j} - \hat{k}) + 2 = 0$.

4.3.37 Find the equation of the line joining $(1, 2)$ and $(3, 6)$.

4.3.38 Find the equation of the line joining $(3, 1)$ and $(9, 3)$.

4.3.39 If the point (3, 4) lies on the line $3y = ax + 7$, find the value of a.

4.3.40 Find the equation of the line that passes through the point with position vector $2\hat{i} - \hat{j} + 4\hat{k}$ and is in direction $\hat{i} + 2\hat{j} - \hat{k}$.

4.3.41 The cartesian equation of a line is $\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2}$. Write its vector form.

4.3.42 Find the equation of the line that passes through the origin and $(5, -2, 3)$.

4.3.43 Find the equation of the line that passes through the points $(3, -2, -5), (3, -2, 6)$.

4.3.44 Find the coordinates of the point where the line through $(5, 1, 6)$ and $(3, 4, 1)$ crosses the YZ-plane.

4.3.45 Find the coordinates of the point where the line through $(5, 1, 6)$ and $(3, 4, 1)$ crosses the ZX-plane.

4.3.46 Find the coordinates of the point where the line through $(3, -4, -5)$ and $(2, -3, 1)$ crosses the plane $2x + y + z = 7$.

4.3.47 Find the equation of the line through $(-2, 3)$ with slope -4

4.3.48 Write the equation of the line through the points $(1, -1)$ and $(3, 5)$.

4.3.49 Write the equation of the lines for which $\tan\theta = \frac{1}{2}$, where θ is the inclination of the line and

a) y-intercept is $\frac{-3}{2}$
b) x-intercept is 4.

4.3.50 Find the equation of the lines which makes intercepts -3 and 2 on the x- and y-axes respectively.

4.3.51 Equation of a line is $3x - 4y + 10 = 0$, find its
  a) Slope
  b) x and y-intercepts.

4.3.52 Determine the ratio in which the line $2x + y - 4 = 0$ divides the line segment joining the points $\mathbf{A}(2, -2)$ and $\mathbf{B}(3, 7)$.
Solution: The given equation can be expressed as

$$\begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = 4 \tag{4.3.52.1}$$

Using section formula in (4.3.52.1),

$$\mathbf{n}^\top \left(\frac{k\mathbf{B}+\mathbf{A}}{k+1} \right) = c \tag{4.3.52.2}$$

$$\implies k = \frac{c - \mathbf{n}^\top \mathbf{A}}{\mathbf{n}^\top \mathbf{B} - c} \tag{4.3.52.3}$$

upon simplification. Substituting numerical values,

$$k = \frac{2}{9} \tag{4.3.52.4}$$

See Fig. 4.3.52.1.

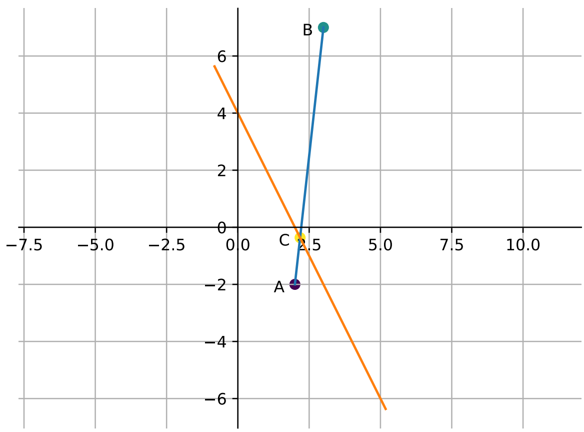


Fig. 4.3.52.1

4.3.53 The Fahrenheit temperature F and absolute temperature K satisfy a linear equation. Given that $K = 273$ when $F = 32$ and that $K = 373$ when $F = 212$. Express K in terms of F and find the value of F, when $K = 0$.

4.3.54 A line is such that its segment between the lines $5x - y + 4 = 0$ and $3x + 4y - 4 = 0$ is bisected at the point $(1, 5)$. Obtain its equation.

4.3.55 Find the vector equations of the plane passing through the points

$R(2,5,-3), S(-2,-3,5)$ and $T(5,3,-3)$.

4.3.56 Find the equation of the plane with intercepts 2, 3 and 4 on the x, y and z - axis respectively.

4.3.57 Show that the lines

$$\frac{x-a+d}{\alpha-\delta} = \frac{y-a}{\alpha} = \frac{z-a-d}{\alpha+\delta} \tag{4.3.57.1}$$

$$\text{and} \tag{4.3.57.2}$$

$$\frac{x-a+c}{\beta-\gamma} = \frac{y-b}{\beta} = \frac{z-b-c}{\beta+\gamma} \tag{4.3.57.3}$$

are coplanar.

4.3.58 Show that the lines

$$\frac{x+3}{-3} = \frac{y-1}{1} = \frac{z-5}{5} \text{ and} \tag{4.3.58.1}$$

$$\frac{x+1}{-1} = \frac{y-2}{2} = \frac{z-5}{5}. \tag{4.3.58.2}$$

are coplanar.

4.3.59 Find the equation for the line passing through the points $(-1,0,2)$ and $(3,4,6)$.

4.3.60 The Cartesian equation of a line is

$$\frac{x+3}{2} = \frac{y-5}{4} = \frac{z+6}{2} \tag{4.3.60.1}$$

Find the vector equation for the line.

4.4 CBSE

4.4.1 Show that the plane $x-5y-2z=1$ contains the line $\frac{x-5}{3}$ = y = $2-z$. (12, 2020)

4.4.2 If the equation of a line is

$$x = ay + b,$$
$$z = cy + d,$$

then find its parametric form. (12, 2023)

4.4.3 Equation of the line passing through the origin and making 30°, 60°, and 90° with the X, Y, Z axes respectively is ______. (12, 2023)

4.4.4 A line passes through the point with position vector $2\hat{i}-\hat{j}+4\hat{k}$ and is in the direction of the vector $\hat{i}+\hat{j}-2\hat{k}$. Find the equation of the line. (12, 2019)

4.4.5 Find the equation of the plane passing through the points having position vectors $\hat{i}+\hat{j}-2\hat{k}$, $2\hat{i}-\hat{j}+\hat{k}$, and $\hat{i}+2\hat{j}+\hat{k}$. (12, 2019)

4.4.6 Find the equation of the plane passing through the points $\mathbf{A}(2,5,-3)$, $\mathbf{B}(-2,-3,5)$ and $\mathbf{C}(5,3,-3)$. (12, 2019)

4.4.7 Find the value of x such that the four points with position vectors $\mathbf{A}(3\hat{i}+2\hat{j}+\hat{k})$, $\mathbf{B}(4\hat{i}+x\hat{j}+5\hat{k})$, $\mathbf{C}(4\hat{i}+2\hat{j}-2\hat{k})$, and $\mathbf{D}(6\hat{i}+5\hat{j}-\hat{k})$ are coplanar. (12, 2019)

4.4.8 Find the value of x such that the four points $\mathbf{A}(x,5,-1)$, $\mathbf{B}(3,2,1)$, $\mathbf{C}(4,5,5)$, and $\mathbf{D}(4,2,-2)$ are coplanar. (12, 2019)

4.4.9 Find the value of a so that the point $(3, a)$ lies on the line represented by $2x-3y = 5$. (10, 2019)

4.4.10 Point **P** divides the line segment joining the points **A**$(2, 1)$ and **B**$(5, -8)$ such that $\frac{AP}{AB} = \frac{1}{3}$. If **P** lies on the line $2x - y + k = 0$, find the value of k. (10, 2019)

4.4.11 The line segment joining the points **A**$(2, 1)$ and **B**$(5, -8)$ is trisected at the points **P** and **Q**, where **P** is nearer to **A**. If **P** lies on the line $2x - y + k = 0$, find the value of k. (10, 2018)

4.4.12 Find the equation of the plane passing through the points $(2, 5, -3), (-2, -3, 5)$ and $(5, 3, -3)$. Also find the point of intersection of this plane with the line passing through points $(3, 1, 5)$ and $(-1, -3, -1)$. (12, 2018)

4.4.13 A line passes through the point with position vector $2\hat{\imath} - \hat{\jmath} + 4\hat{k}$ and is in the direction of the vector $\hat{\imath} + \hat{\jmath} - 2\hat{k}$. Find the equation of the line. (12, 2018)

4.4.14 Find the value of λ, if four points with position vectors $\mathbf{P}_1 = 3\hat{\imath} + 6\hat{\jmath} + 9\hat{k}$, $\mathbf{P}_2 = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$, $\mathbf{P}_3 = 2\hat{\imath} + 3\hat{\jmath} + \hat{k}$, and $\mathbf{P}_4 = 4\hat{\imath} + 6\hat{\jmath} + \lambda\hat{k}$ are coplanar. (12, 2017)

4.4.15 Write the sum of intercepts cut off by the plane $\vec{r}.\left(2\hat{\imath} + \hat{\jmath} - \hat{k}\right) - 5 = 0$ on the three axes.

(12, 2016)

4.4.16 Find the equation of the plane with intercepts $3, -4$ and 2 on the three axes.

(12, 2016)

4.4.17 A point **P** divides the line segment joining the points **A**$(3, -5)$ and **B**$(-4, 8)$ such that $\frac{AP}{PB} = \frac{K}{1}$. If **P** lies on the line $x + y = 0$, then find the value of K. (10, 2012)

4.4.18 If the point **A**(x, y), **B**$(3, 6)$ and **C**$(-3, 4)$ are collinear, show that $x - 3y + 15 = 0$. (10, 2012)

4.4.19 **Assertion (A):** Point **P**(0,2) is the point of intersection of Y axis with the line $3x + 2y = 4$.

Reason (R): The distance of point **P**(0,2) from X axis is 2 units. (10, 2023)

4.4.20 The pair of linear equations $2x = 5y + 6$ and $15y = 6x - 18$ represents two lines which are:

a) intersecting
b) parallel
c) coincident
d) either intersecting or parallel

(10, 2023)

4.4.21 Write the equation of the line PQ passing through points **P**$(2, 2, 1)$ and **Q**$(5, 1, -2)$. Hence, find the y-coordinate of the point on the line PQ whose z-coordinate is -2.

(12, 2022)

4.4.22 Find the equation of a plane which passes through the point $(3, 2, 0)$ and contains the line $\frac{x-3}{1} = \frac{y-6}{5} = \frac{z-4}{4}$. (12, 2022)

4.4.23 If the graph of a pair of lines $x - 2y + 3 = 0$ and $2x - 4y = 5$ be drawn, that what type of lines are drawn ? (10, 2021)

4.4.24 If segment of the line intercepted between the co-ordinate-axes is bisected at the point **M**$(2, 3)$, then the equation of this line is ______. (12, 2021)

4.4.25 The equation of a line through $(2, -4)$ and parallel to X axis is ______. (12, 2021)

4.4.26 Find the equation of the median through vertex **A** of the triangle ABC, having vertices

$\mathbf{A}(2,5)$, $\mathbf{B}(-4,9)$ and $\mathbf{C}(-2,-1)$. (12, 2021)

4.4.27 Find the value of x such that the points $\mathbf{A}(3,2,1)$, $\mathbf{B}(4,x,5)$, $\mathbf{C}(4,2,-2)$ and $\mathbf{D}(6,5,-1)$ are coplanar. (12, 2017)

4.4.28 The x-coordinate of a point on the line joining the points $\mathbf{P}(2,2,1)$and $\mathbf{Q}(5,1,-2)$ is 4. Find its z-coordinate. (12, 2017)

4.4.29 Find the equation of plane passing through the points $\mathbf{A}(3,2,1)$, $\mathbf{B}(4,2,-2)$ and $\mathbf{C}(6,5,-1)$ and hence find the value of λ for which $\mathbf{A}(3,2,1)$, $\mathbf{B}(4,2,-2)$, $\mathbf{C}(6,5,-1)$ and $\mathbf{D}(\lambda,5,5)$ are coplanar. (12, 2016)

4.4.30 Show that the four points $\mathbf{A}(4,5,1)$, $\mathbf{B}(0,-1,-1)$, $\mathbf{C}(3,9,4)$ and $\mathbf{D}(-4,4,4)$ are coplanar. (12, 2016)

4.4.31 Find the equation of the line joining $\mathbf{A}(1,3)$ and $\mathbf{B}(0,0)$. Also, find k if $\mathbf{D}(k,0)$ is a point such that the area of $\triangle ABD$ is 3 square units. (12, 2021)

4.4.32 Show that the vectors $\hat{i}-2\hat{j}+3\hat{k}$, $2\hat{i}+3\hat{j}-4\hat{k}$, and $\hat{i}-3\hat{j}+5\hat{k}$ are coplanar. (12, 2018)

4.4.33 Find the value of x such that the four points with position vectors $\mathbf{A}(3\hat{i}+2\hat{j}+\hat{k})$, $\mathbf{B}(4\hat{i}+x\hat{j}+5\hat{k})$, $\mathbf{C}(4\hat{i}+2\hat{j}-2\hat{k})$, and $\mathbf{D}(6\hat{i}+5\hat{j}-\hat{k})$ are coplanar. (12, 2018)

4.4.34 Find the value of x, for which the four points $\mathbf{A}(x,1,-1)$, $\mathbf{B}(4,5,1)$, $\mathbf{C}(3,9,4)$ and $\mathbf{D}(-4,4,4)$ are coplanar. (12, 2018)

4.4.35 Find the value of x such that the four points $\mathbf{A}$ $(x,5,-1)$,$\mathbf{B}$ $(3,2,1)$,$\mathbf{C}$ $(4,5,5)$ and $\mathbf{D}$ $(4,2,-2)$ are coplanar. (12, 2018)

4.4.36 The area of the triangle formed by the line $\frac{x}{a}+\frac{y}{b}=1$ with the coordinate axes is ______. (10, 2023)

4.4.37 Find the vector equation of the line passing through the point (2, 3, -5) and making equal angles with the coordinate axes. (12, 2024)

4.4.38 Find the equation of the line which bisects the line segment joining points $A(2,3,4)$ and $B(4,5,8)$ and is perpendicular to the lines $\frac{x-8}{3}=\frac{y+19}{-16}=\frac{z-10}{7}$ and $\frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}$ (12, 2024)

4.5 Parallel

4.5.1 Find the vector equation of the line passing through $\begin{pmatrix}1 & 2 & 3\end{pmatrix}^\top$ and parallel to the planes $\begin{pmatrix}1 & -1 & 2\end{pmatrix}\mathbf{x}=5$ and $\begin{pmatrix}3 & 1 & 1\end{pmatrix}\mathbf{x}=6$.

Solution: The direction vector of the line is given by

$$\begin{pmatrix}1 & -1 & 2\\ 3 & 1 & 1\end{pmatrix}\mathbf{m}=0 \xleftrightarrow{R_2\to-\frac{3}{4}R_1+\frac{1}{4}R_2} \begin{pmatrix}1 & -1 & 2\\ 0 & 1 & -\frac{5}{4}\end{pmatrix}$$

$$\begin{pmatrix}1 & -1 & 2\\ 0 & 1 & -\frac{5}{4}\end{pmatrix} \xleftrightarrow{R_1\to R_1+R_2} \begin{pmatrix}1 & 0 & \frac{3}{4}\\ 0 & 1 & -\frac{5}{4}\end{pmatrix}$$

$$\implies \mathbf{m}=\begin{pmatrix}-3\\ 5\\ 4\end{pmatrix}$$

$\therefore$ the equation of the line is

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 5 \\ 4 \end{pmatrix} \tag{4.5.1.1}$$

4.5.2 Find the equation of the plane with an intercept 3 on the Y-axis and parallel to ZOX-Plane.

Solution: The normal vector to the ZOX plane is

$$\mathbf{n} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \tag{4.5.2.1}$$

Since, Y-axis has the intercept 3, the desired plane passes through the point

$$\mathbf{P} = \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}. \tag{4.5.2.2}$$

Thus, the equation of the plane is given by,

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{P}) = 0 \tag{4.5.2.3}$$

$$\implies \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \mathbf{x} = 3 \tag{4.5.2.4}$$

See Fig. 4.5.2.1.

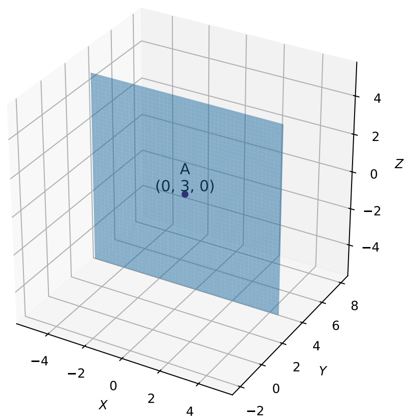


Fig. 4.5.2.1

4.5.3 Find the equation of the line parallel to the line $3x - 4y + 2 = 0$ and passing through the point (-2, 3).

Solution:

$$\begin{pmatrix}3 & -4\end{pmatrix}\mathbf{x} = \begin{pmatrix}3 & -4\end{pmatrix}\begin{pmatrix}-2\\3\end{pmatrix} = -18 \tag{4.5.3.1}$$

is the required equation of the line.

4.5.4 Find the equation of the line through the point (0, 2) making an angle $\frac{2\pi}{3}$ with the positive X-axis. Also find the equation of the line parallel to it and crossing the Y-axis at a distance of 2 units below the origin.

Solution: The equation of the first line is

$$\begin{pmatrix}\sqrt{3} & 1\end{pmatrix}\left(\mathbf{x} - \begin{pmatrix}0\\2\end{pmatrix}\right) = 0 \tag{4.5.4.1}$$

$$\implies \begin{pmatrix}\sqrt{3} & 1\end{pmatrix}\mathbf{x} = 2 \tag{4.5.4.2}$$

The equation of the second line is

$$\begin{pmatrix}\sqrt{3} & 1\end{pmatrix}\left(\mathbf{x} - \begin{pmatrix}0\\-2\end{pmatrix}\right) = 0 \tag{4.5.4.3}$$

$$\implies \begin{pmatrix}\sqrt{3} & 1\end{pmatrix}\mathbf{x} = -2 \tag{4.5.4.4}$$

See Fig. 4.5.4.1.

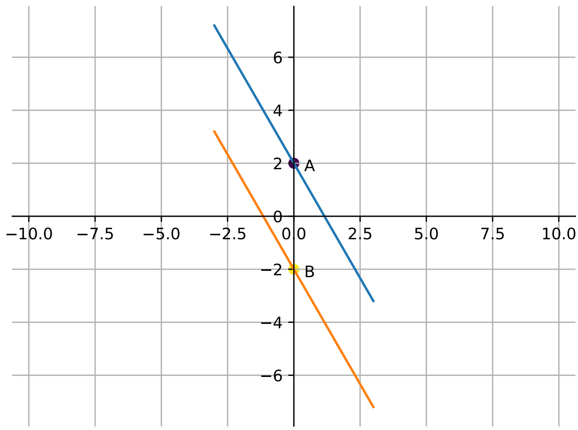


Fig. 4.5.4.1

4.5.5 Find the vector equation of the line which is parallel to the vector $3\hat{i} - 2\hat{j} + 6\hat{k}$ and passes through the point $(1, -2, 3)$.

4.5.6 Find the equations of the line passing through the point $(3, 0, 1)$ and parallel to the planes $x + 2y = 0$ and $3y - z = 0$.

4.5.7 The equation of a line, which is parallel to $2\hat{i}+\hat{j}+3\hat{k}$ and passes through the point $(5,-2,4)$ is $\frac{x-5}{2}=\frac{y+2}{-1}=\frac{z-4}{3}$.

4.5.8 The value of λ for which the vectors $3\hat{i}-6\hat{j}+\hat{k}$ and, $2\hat{i}-4\hat{j}+\lambda\hat{k}$ are parallel is

a) $\frac{2}{3}$

b) $\frac{3}{2}$

c) $\frac{5}{2}$

d) $\frac{2}{5}$

4.5.9 Equation of the line passing through (1, 2) and parallel to the line $y=3x-1$ is

a) $y+2=x+1$

b) $y+2=3(x+1)$

c) $y-2=3(x-1)$

d) $y-2=x-1$

4.5.10 Find the equation of the line which passes through the point $(1,2,3)$ and is parallel to the vector $3\hat{i}+2\hat{j}-2\hat{k}$

4.5.11 Find the cartesian equation of the line which passes through the point $(-2,4,-5)$ and parallel to the line given by $\frac{x+3}{3}=\frac{y-4}{5}=\frac{z+8}{6}$.

4.5.12 Find the equation of the plane passing through (a,b,c) and parallel to the plane $\overrightarrow{r}\cdot(\hat{i}+\hat{j}+\hat{k})=2$.

4.5.13 Find the equations of the lines parallel to axes and passing through $(2,3)$.

4.5.14 Find the equation of the line through the point $(5,2,-4)$ and which is parallel to the vector $3\hat{i}+2\hat{j}-8\hat{k}$.

4.6 CBSE

4.6.1 The distance between parallel planes $2x+y-2z-6=0$ and $4x+2y-4z=0$ is ____________ units. (12, 2020)

4.6.2 Find the equation of the line passing through $(2,1,-1)$ and parallel to the line $\overrightarrow{r}=(\hat{i}+\hat{j})+\lambda(2\hat{i}-\hat{j}+\hat{k})$. Also, find the distance between these two lines. (12, 2019)

4.6.3 Find the equation of the line which passes through the point $(-2,4,-5)$ and is parallel to the line $\frac{x+3}{3}=\frac{4-y}{5}=\frac{z+8}{6}$. (12, 2018)

4.6.4 Find the equation of a line passing through the point $(2,3,2)$ and parallel to the line $\overrightarrow{r}=\left(-2\hat{i}+3\hat{j}\right)+\lambda\left(2\hat{i}-3\hat{j}+6\hat{k}\right)$. Also, find the distance between these two lines. (12, 2018)

4.6.5 Find the equation of the line passing through $(2,1,-1)$ and parallel to the line $\overrightarrow{r}=\left(\hat{i}+\hat{j}\right)+\lambda\left(2\hat{i}-\hat{j}+\hat{k}\right)$. Also, find the distance between these two lines. (12, 2018)

4.6.6 Find the equation of the plane passing through the points having position vectors $\hat{i}+\hat{j}-2\hat{k}$, $2\hat{i}-\hat{j}+\hat{k}$ and $\hat{i}+2\hat{j}+\hat{k}$. Write the equation of the plane passing through a point $(2,3,7)$ and parllel to the plane obtained above. Hence, find the distance between the two parallel planes. (12, 2018)

4.6.7 Find the equation of the line which passes through the point $(3,4,5)$ and is parallel to the vector $2\hat{i}+2\hat{j}-3\hat{k}$. (12, 2018)

4.6.8 Find the equation of the plane passing through the points $(2, 2, -1), (3, 4, 2)$ and $(7, 0, 6)$. Also find the equation of the plane passing through $(4, 3, 1)$ and parallel to the plane obtained above. (12, 2018)

4.6.9 Find the equation of the plane containing two parallel lines $\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z}{3}$ and $\frac{x}{4} = \frac{y-2}{-2} = \frac{z+1}{6}$. Also, find if the plane thus obtained contains the line $\frac{x-2}{3} = \frac{y-1}{1} = \frac{z-2}{5}$ or not. (12, 2016)

4.6.10 Find the equation of the line passing through the point $(1, -3, 2)$ and parallel to the line

$$\mathbf{r} = (2 + \lambda)\hat{i} + \lambda\hat{j} + (-1 + \lambda)\hat{k}.$$

(12, 2024)

4.6.11 Find the equation of the line passing through the point $(1, -3, 2)$ and parallel to the line

$$\vec{r} = (2 + \lambda)\,\hat{i} + \lambda\hat{j} + (2\lambda - 1)\,\hat{k}.$$

(12, 2024)

4.7 Perpendicular

4.7.1 The line through the points $(h, 3)$ and $(4, 1)$ intersects the line $7x - 9y - 19 = 0$ at a right angle. Find the value of h.

Solution: The direction vectors of the given lines are

$$\begin{pmatrix} 4-h \\ -2 \end{pmatrix}, \begin{pmatrix} 9 \\ 7 \end{pmatrix} \tag{4.7.1.1}$$

$$\implies \begin{pmatrix} 9 & 7 \end{pmatrix} \begin{pmatrix} 4-h \\ -2 \end{pmatrix} = 0 \tag{4.7.1.2}$$

$$\implies h = \frac{22}{9} \tag{4.7.1.3}$$

See Fig. 4.7.1.1.

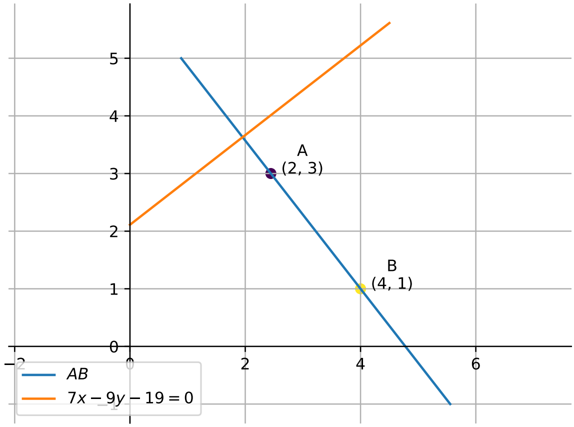


Fig. 4.7.1.1

4.7.2 If the lines $\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}$ and $\frac{x-1}{3k} = \frac{y-1}{1} = \frac{z-6}{-5}$ are perpendicular, find the value of k.

Solution: From the given information,

$$\mathbf{m}_1 = \begin{pmatrix} -3 \\ 2k \\ 2 \end{pmatrix}, \mathbf{m}_2 = \begin{pmatrix} 3k \\ 1 \\ -5 \end{pmatrix} \tag{4.7.2.1}$$

$$\implies \begin{pmatrix} -3 & 2k & 2 \end{pmatrix}^\top \begin{pmatrix} 3k \\ 1 \\ -5 \end{pmatrix} = 0 \tag{4.7.2.2}$$

$$\implies k = -\frac{10}{7} \tag{4.7.2.3}$$

4.7.3 Find the values of θ and p, if the equation $x\cos\theta + y\sin\theta = p$ is the normal form of the line $\sqrt{3}x + y + 2 = 0$.

Solution:

$$\mathbf{n} = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}, c = -2 \tag{4.7.3.1}$$

$$\implies \theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}, p = \frac{|c|}{\|\mathbf{n}\|} = 1 \tag{4.7.3.2}$$

4.7.4 Reduce the following equations into normal form. Find their perpendicular distances from the origin and angle between perpendicular and the positive X axis.

a) $x - \sqrt{3}y + 8 = 0$
b) $y - 2 = 0$
c) $x - y = 4$

Solution: See Table 4.7.4. (4.1.6.2) was used for computing the distance from the origin.

	$\mathbf{n}$	Angle	c	Distance
a)	$\begin{pmatrix}1\\-\sqrt{3}\end{pmatrix}$	$\tan^{-1}(-\sqrt{3}) = \frac{2\pi}{3}$	-8	4
b)	$\begin{pmatrix}0\\1\end{pmatrix}$	$\tan^{-1}\infty = \frac{\pi}{2}$	2	2
c)	$\begin{pmatrix}1\\-1\end{pmatrix}$	$\tan^{-1}(-1) = \frac{3\pi}{4}$	4	$2\sqrt{2}$

TABLE 4.7.4

4.7.5 Find the distance of the point $(-1, 1)$ from the line $12(x+6) = 5(y-2)$.
Solution:

$$\mathbf{n} = \begin{pmatrix}12\\-5\end{pmatrix}, c = -82 \tag{4.7.5.1}$$

$$\implies d = \frac{\left|\begin{pmatrix}12 & -5\end{pmatrix}\begin{pmatrix}-1\\1\end{pmatrix} - (-82)\right|}{\sqrt{12^2 + (-5)^2}} = 5 \tag{4.7.5.2}$$

4.7.6 In each of the following cases, determine the direction cosines of the normal to the plane and the distance from the origin.

a) $z = 2$
b) $x + y + z = 1$
c) $2x + 3y – z = 5$
d) $5y + 8 = 0$

Solution: See Table 4.7.6. (4.1.6.2) was used for computing the distance from the origin.

	$\mathbf{n}$	c	Distance
a)	$\begin{pmatrix}0\\0\\1\end{pmatrix}$	2	2
b)	$\begin{pmatrix}1\\1\\1\end{pmatrix}$	1	$\frac{1}{\sqrt{3}}$
c)	$\begin{pmatrix}2\\3\\-1\end{pmatrix}$	5	$\frac{5}{\sqrt{14}}$
d)	$\begin{pmatrix}0\\-5\\0\end{pmatrix}$	8	$\frac{8}{5}$

TABLE 4.7.6

4.7.7 Find the distance between parallel lines

a) $15x + 8y - 34 = 0$ and $15x + 8y + 31 = 0$

b) $l(x + y) + p = 0$ and $l(x + y) - r = 0$

Solution: From (4.1.8.1), the desired values are available in Table 4.7.7.

	$\mathbf{n}$	c_1	c_2	d
a)	$\begin{pmatrix}15\\8\end{pmatrix}$	34	-31	$\frac{65}{17}$
b)	$\begin{pmatrix}1\\1\end{pmatrix}$	$\frac{-p}{l}$	$\frac{r}{l}$	$\frac{\lvert p-r\rvert}{l\sqrt{2}}$

TABLE 4.7.7

4.7.8 Find the shortest distance between the lines l_1 and l_2 whose vector equations are

$$\overrightarrow{r} = \hat{i} + \hat{j} + \lambda(2\hat{i} - \hat{j} + \hat{k}) \tag{4.7.8.1}$$

$$\text{and } \overrightarrow{r} = 2\hat{i} + \hat{j} - \hat{k} + \mu(3\hat{i} - 5\hat{j} + 2\hat{k}) \tag{4.7.8.2}$$

4.7.9 Find the points on the x-axis, whose distances from the line $\frac{x}{3} + \frac{y}{4} = 1$ are 4 units.

Solution: Let the desired point be

$$\mathbf{P} = x\mathbf{e}_1 = \begin{pmatrix}x\\0\end{pmatrix} \tag{4.7.9.1}$$

From the distance formula,

$$d = \frac{\left|\mathbf{n}^\top\mathbf{P} - c\right|}{\|\mathbf{n}\|} = \frac{\left|x\mathbf{n}^\top\mathbf{e}_1 - c\right|}{\|\mathbf{n}\|} \tag{4.7.9.2}$$

$$\implies x = \frac{\pm d\,\|\mathbf{n}\| + c}{\mathbf{n}^\top\mathbf{e}_1} \tag{4.7.9.3}$$

Substituting

$$\mathbf{n} = \begin{pmatrix}4\\3\end{pmatrix}, c = 12, d = 4, \tag{4.7.9.4}$$

$$x = 8, -2 \tag{4.7.9.5}$$

See Fig. 4.7.9.1.

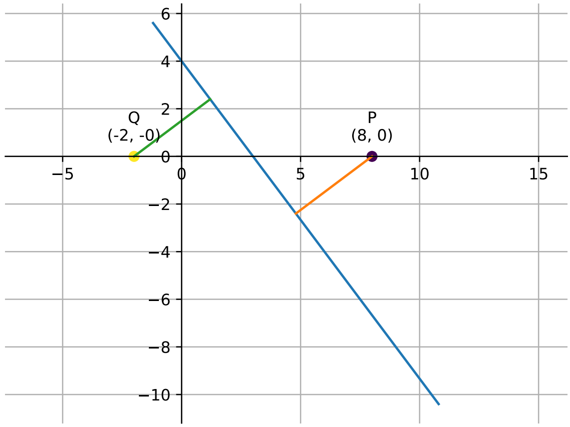


Fig. 4.7.9.1

4.7.10 What are the points on the y-axis whose distance from the line $\frac{x}{3} + \frac{y}{4} = 1$ is 4 units.
Solution: Following the approach in Problem 4.7.9,

$$y = \frac{\pm d \left\|\mathbf{n}\right\| + c}{\mathbf{n}^\top \mathbf{e}_2} = \frac{32}{3}, \frac{-8}{3}. \tag{4.7.10.1}$$

4.7.11 Show that the path of a moving point such that its distances from two lines $3x-2y = 5$ and $3x + 2y = 5$ are equal is a straight line.

4.7.12 Find the distance of the line $4x - y = 0$ from the point $\mathbf{P}(4, 1)$ measured along the line making an angle of 135° with the positive x-axis.

4.7.13 Find the distance between the lines l_1 and l_2 given by

$$\overrightarrow{r} = \hat{i} + 2\hat{j} - 4\hat{k} + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k}) \tag{4.7.13.1}$$

$$\text{and } \overrightarrow{r} = 3\hat{i} + 3\hat{j} - 5\hat{k} + \mu(2\hat{i} + 3\hat{j} + 6\hat{k}) \tag{4.7.13.2}$$

4.7.14 Find the distance of the plane $2x - 3y + 4z - 6 = 0$ from the origin.

4.7.15 Find the vector equation of the plane which is at a distance of $\frac{6}{\sqrt{29}}$ from the origin and its normal vector from the origin is $2\hat{i} - 3\hat{j} + 4\hat{k}$.

4.7.16 Find the coordinates of the foot of the perpendicular from $(-1, 3)$ to the line $3x - 4y - 16 = 0$.
Solution: Substituting

$$\mathbf{P} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}, \mathbf{n} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}, c = 16 \tag{4.7.16.1}$$

in (4.1.7.1), the desired foot of the perpendicular is then given by

$$\begin{pmatrix} 4 & 3 \\ 3 & -4 \end{pmatrix} \mathbf{Q} = \begin{pmatrix} \begin{pmatrix} 4 & 3 \end{pmatrix} \begin{pmatrix} -1 \\ 3 \end{pmatrix} \\ 16 \end{pmatrix} = \begin{pmatrix} 5 \\ 16 \end{pmatrix} \tag{4.7.16.2}$$

$$\implies \begin{pmatrix} 4 & 3 & 5 \\ 3 & -4 & 16 \end{pmatrix} \xleftrightarrow{R_2 = R_2 - \frac{3}{4} R_1} \begin{pmatrix} 4 & 3 & 5 \\ 0 & \frac{-25}{4} & \frac{49}{4} \end{pmatrix} \tag{4.7.16.3}$$

$$\xleftrightarrow{R_2 = \frac{-4}{25}} \begin{pmatrix} 4 & 3 & 5 \\ 0 & 1 & \frac{-49}{25} \end{pmatrix} \xleftrightarrow{R_1 = \frac{1}{4} R_1} \begin{pmatrix} 1 & \frac{3}{4} & \frac{5}{4} \\ 0 & 1 & \frac{-49}{25} \end{pmatrix} \tag{4.7.16.4}$$

$$\xleftrightarrow{R_1 = R_1 - \frac{3}{4} R_2} \begin{pmatrix} 1 & 0 & \frac{68}{25} \\ 0 & 1 & \frac{-49}{25} \end{pmatrix} \implies \mathbf{Q} = \begin{pmatrix} \frac{68}{25} \\ \frac{-49}{25} \end{pmatrix} \tag{4.7.16.5}$$

See Fig. 4.7.16.1.

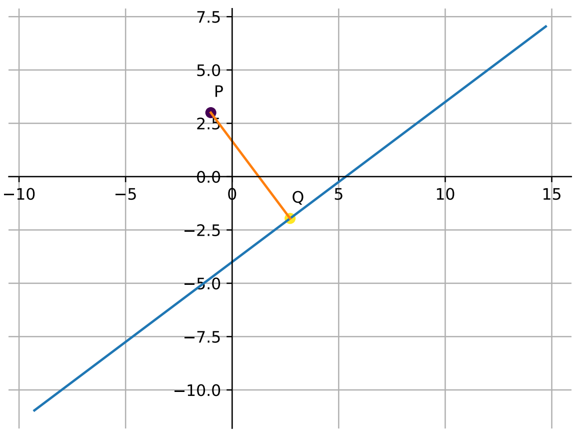


Fig. 4.7.16.1

4.7.17 Find the equation of a line whose perpendicular distance from the origin is 5 units and the angle made by the perpendicular with the positive X axis is 30°.
Solution: From (4.1.5.3), Thus, the equation of lines are

$$\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \mathbf{x} = \pm 5 \tag{4.7.17.1}$$

4.7.18 In the triangle ABC with vertices $\mathbf{A}(2,3)$, $\mathbf{B}(4,-1)$ and $\mathbf{C}(1,2)$, find the equation and length of altitude from the vertex $\mathbf{A}$.
Solution:

a) The normal vector of the altitude from $\mathbf{A}$ is,

$$\mathbf{m}_{BC} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \because \mathbf{n}_{BC} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{4.7.18.1}$$

The equation of the desired altitude is given by

$$\mathbf{m}_{BC}^{\top}\mathbf{x} = \mathbf{m}_{BC}^{\top}\mathbf{A} \tag{4.7.18.2}$$

$$\implies \begin{pmatrix} 1 & -1 \end{pmatrix}\mathbf{x} = -1 \tag{4.7.18.3}$$

b) The equation of line BC is given by,

$$\mathbf{n}_{BC}^{\top}\mathbf{x} = \mathbf{n}_{BC}^{\top}\mathbf{B} \tag{4.7.18.4}$$

$$\implies \begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} = 3 \tag{4.7.18.5}$$

From (4.1.6.2), the length of the desired altitude is

$$d = \sqrt{2} \tag{4.7.18.6}$$

4.7.19 Find the equation of a line drawn perpendicular to the line $\frac{x}{4} + \frac{y}{6} = 1$ through the point where it meets the y-axis.
Solution: The given line parameters are

$$\mathbf{n} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, c = 12, \mathbf{m} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}. \tag{4.7.19.1}$$

and the point on the y-axis is

$$\mathbf{A} = \begin{pmatrix} 0 \\ 6 \end{pmatrix}. \tag{4.7.19.2}$$

Thus, the equation of the desired line is

$$\mathbf{m}^{\top}(\mathbf{x} - \mathbf{A}) = 0 \tag{4.7.19.3}$$

$$\implies \begin{pmatrix} -2 & 3 \end{pmatrix}\mathbf{x} = -18 \tag{4.7.19.4}$$

4.7.20 Find the equation of line perpendicular to the line $x - 7y + 5 = 0$ and having x intercept 3.
Solution: The desired equation is

$$\begin{pmatrix} 7 & 1 \end{pmatrix}\left(\mathbf{x} - \begin{pmatrix} 3 \\ 0 \end{pmatrix}\right) = 0 \tag{4.7.20.1}$$

$$\implies \begin{pmatrix} 7 & 1 \end{pmatrix}\mathbf{x} = 21 \tag{4.7.20.2}$$

4.7.21 Find the equation of one of the sides of an isosceles right angled triangle whose hypotenuse is given by $3x + 4y = 4$ and the opposite vertex of the hypotenuse is (2, 2).

4.7.22 In what direction should a line be drawn through the point (1, 2) so that its point of intersection with line $x + y = 4$ is at a distance $\sqrt{6}3$.

4.7.23 Find the equation of a line which is equidistant from parallel lines $9x + 6y - 7 = 0$ and $3x + 2y + 6 = 0$.

Solution: Given

$$c_1 = \frac{7}{3}, c_2 = -6. \tag{4.7.23.1}$$

From (4.1.8.1), we need to find c such that,

$$|c - c_1| = |c - c_2| \implies c = \frac{c_1 + c_2}{2} = -\frac{11}{6}. \tag{4.7.23.2}$$

Hence, the desired equation is

$$\begin{pmatrix} 3 & 2 \end{pmatrix} \mathbf{x} = -\frac{11}{6} \tag{4.7.23.3}$$

See Fig. 4.7.23.1.

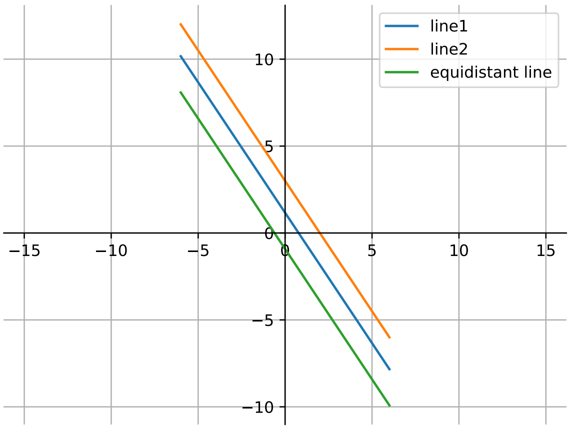


Fig. 4.7.23.1

4.7.24 Find the equation of the line passing through the point (5, 2) and perpendicular to the line joining the points (2, 3) and (3, -1).

4.7.25 Find the points on the line $x + y = 4$ which lie at a unit distance from the line $4x + 3y = 10$.

4.7.26 Find the equation of a straight line on which length of perpendicular from the origin is four units and the line makes on angle of 120° with the positive direction of X axis.

4.7.27 The equation of the straight line passing through the point (3, 2) and perpendicular to the line $y = x$ is ______.

4.7.28 Find the equation of the line passing through (-3, 5) and perpendicular to the line through the points (2, 5) and (-3, 6).
Solution: See Fig. 4.7.28.1.

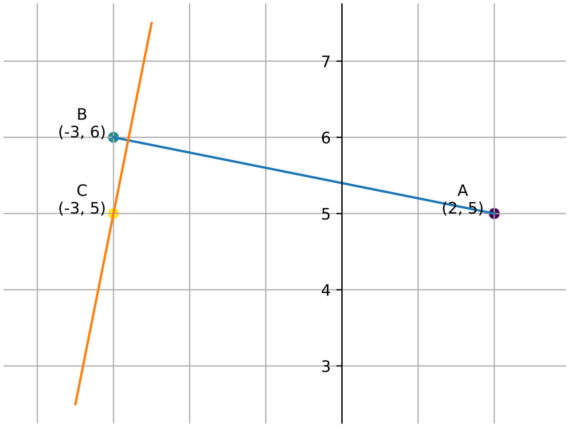


Fig. 4.7.28.1

The normal vector is

$$\mathbf{n} = \begin{pmatrix} 2 \\ 5 \end{pmatrix} - \begin{pmatrix} -3 \\ 6 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \end{pmatrix} \tag{4.7.28.1}$$

Thus, the equation of the line is

$$\begin{pmatrix} 5 & -1 \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} -3 \\ 5 \end{pmatrix} \right) = 0 \tag{4.7.28.2}$$

$$\implies \begin{pmatrix} 5 & -1 \end{pmatrix} \mathbf{x} = -20 \tag{4.7.28.3}$$

4.7.29 The perpendicular from the origin to a line meets it at the point $(-2, 9)$. Find the equation of the line.
Solution: It is obvious that the normal vector to the line is

$$\mathbf{n} = \begin{pmatrix} 2 \\ -9 \end{pmatrix} - \mathbf{0} = \begin{pmatrix} 2 \\ -9 \end{pmatrix} \tag{4.7.29.1}$$

Hence, the equation of the line is

$$\begin{pmatrix} 2 & -9 \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} 2 \\ -9 \end{pmatrix} \right) = 0 \tag{4.7.29.2}$$

$$\implies \begin{pmatrix} 2 & -9 \end{pmatrix} \mathbf{x} = 85 \tag{4.7.29.3}$$

See Fig. 4.7.29.1.

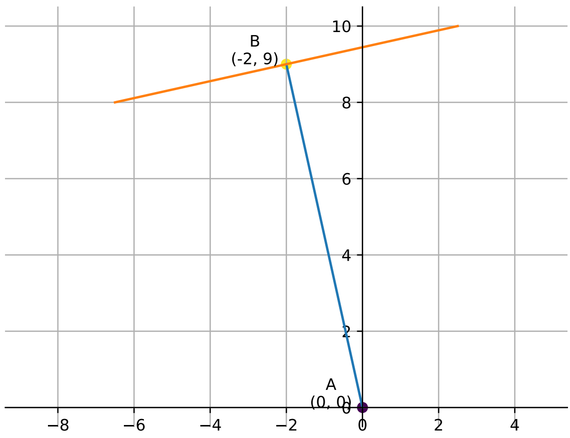


Fig. 4.7.29.1

4.7.30 Find the equation of the line passing through the point $(1, 2, -4)$ and perpendicular to the two lines

$$\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7} \text{ and} \tag{4.7.30.1}$$

$$\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5} \tag{4.7.30.2}$$

Solution: The direction vector of the desired line is given by

$$\begin{pmatrix} 3 & -16 & 7 \\ 3 & 8 & -5 \end{pmatrix} \mathbf{m} = 0 \xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 3 & -16 & 7 \\ 0 & 24 & -12 \end{pmatrix}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + \frac{2}{3}R_2} \begin{pmatrix} 3 & 0 & -1 \\ 0 & 24 & -12 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2/12} \begin{pmatrix} 3 & 0 & -1 \\ 0 & 2 & -1 \end{pmatrix}$$

yielding

$$\mathbf{m} = \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} \tag{4.7.30.3}$$

Hence the vector equation of the line passing through $(1, 2, -4)$ is,

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} + \kappa \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} \tag{4.7.30.4}$$

4.7.31 The perpendicular from the origin to the line $y = mx + c$ meets it at the point $(-1, 2)$. Find the values of m and c.

Solution: From Problem 4.7.29,

$$\mathbf{n} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \implies m = \frac{1}{2} \tag{4.7.31.1}$$

Also, from the given equation of the line and the given point,

$$c = \begin{pmatrix} -m & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \frac{5}{2} \tag{4.7.31.2}$$

4.7.32 A line perpendicular to the line segment joining the points $\mathbf{P}(1, 0)$ and $\mathbf{Q}(2, 3)$ divides it in the ratio $1 : n$. Find the equation of the line.
Solution: The direction vector of PQ is

$$\mathbf{Q} - \mathbf{P} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \tag{4.7.32.1}$$

Using section formula,

$$\mathbf{R} = \frac{\mathbf{Q} + n\mathbf{P}}{1 + n} \tag{4.7.32.2}$$

is the point of intersection. The equation of the desired line is

$$\mathbf{m}^{\top} (\mathbf{x} - \mathbf{R}) = 0 \tag{4.7.32.3}$$

$$\implies \begin{pmatrix} 1 & 3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & 3 \end{pmatrix} \begin{pmatrix} \frac{2+n}{1+n} \\ \frac{3}{1+n} \end{pmatrix} \tag{4.7.32.4}$$

$$= \frac{11 + n}{1 + n} \tag{4.7.32.5}$$

4.7.33 Find the vector equation of a plane which is at a distance of 7 units from the origin and normal to the vector $3\hat{i} + 5\hat{j} - 6\hat{k}$.
Solution: From the given information,

$$\mathbf{n} = \begin{pmatrix} 3 \\ 5 \\ -6 \end{pmatrix}, d = \frac{|c|}{\|\mathbf{n}\|} = 7 \tag{4.7.33.1}$$

$$\implies c = \pm 7\sqrt{70} \tag{4.7.33.2}$$

4.7.34 Find the equation of a plane which is at a distance $3\sqrt{3}$ units from origin and the normal to which is equally inclined to the coordinate axis.

4.7.35 If the line drawn from the point $(-2, -1, -3)$ meets a plane at right angle at the point $(1, -3, 3)$, find the equation of the plane.

4.7.36 Find the equation of the plane through the points $(2, 1, -1)$ and $(-1, 3, 4)$, and perpendicular to the plane $x - 2y + 4z = 10$.

4.7.37 If the foot of perpendicular drawn from the origin to a plane is $(5, -3, -2)$, then the equation of the plane is $\vec{r} \cdot (5\hat{i} - 3\hat{j} - 2\hat{k}) = 38$.

4.7.38 $\mathbf{P}(0, 2)$ is the point of intersection of Y axis and perpendicular bisector of line segment joining the points $\mathbf{A}(-1, 1)$ and $\mathbf{B}(3, 3)$.

4.7.39 The distance of the point $\mathbf{P}(2, 3)$ from the x-axis is ______.

4.7.40 Find the foot of perpendicular from the point $(2, 3, -8)$ to the line

$$\frac{4-x}{2} = \frac{y}{6} = \frac{1-z}{3}.$$

Also, find the perpendicular distance from the given point to the line.

4.7.41 Find the distance of a point $(2, 4, -1)$ from the line

$$\frac{x+5}{1} = \frac{y+3}{4} = \frac{z-6}{-9}.$$

4.7.42 Find the length and the foot of perpendicular from the point $\left(1, \frac{3}{2}, 2\right)$ to the plane $2x - 2y + 4z + 5 = 0$.

4.7.43 Show that the points $(\hat{i} - \hat{j} + 3\hat{k})$ and $3(\hat{i} + \hat{j} + \hat{k})$ are equidistant from the plane $\vec{r} \cdot (5\hat{i} + 2\hat{j} - 7\hat{k}) + 9 = 0$ and lie on opposite side of it.

4.7.44 The distance of the plane $\vec{r} \cdot \left(\frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} - \frac{6}{7}\hat{k}\right) = 1$ from the origin is ______.

4.7.45 The equation of the line passing through the point (1, 2) and perpendicular to the line $x + y + 1 = 0$ is ______.

4.7.46 The equations of the lines passing through the point (1, 0) and at a distance $\frac{\sqrt{3}}{2}$ from the origin, are ______.

4.7.47 The foot of perpendiculars from the point (2, 3) on the line $y = 3x + 4$ is given by ______.

4.7.48 A point equidistant from the lines $4x+3y+10 = 0$, $5x-12y+26 = 0$ and $7x+24y-50 = 0$ is ______.

4.7.49 The ratio in which the line $3x + 4y + 2 = 0$ divides the distance between the lines $3x + 4y + 5 = 0$ and $3x + 4y - 5 = 0$ is ______.

4.7.50 Find the vector equation of the line passing through $(1, 2, 3)$ and perpendicular to the plane $\vec{r} \cdot (\hat{i} + 2\hat{j} - 5\hat{k}) + 9 = 0$.

4.7.51 Find the equation of the plane passing through the point $(-1, 3, 2)$ and perpendicular to each of the planes $x + 2y + 3z = 5$ and $3x + 3y + z = 0$.

4.7.52 If the points $(1, 1, p)$ and $(-3, 0, 1)$ be equidistant from the plane $\vec{r} \cdot (3\hat{i}+4\hat{j}-12\hat{k})+13 = 0$, then find the value of p.

4.7.53 If **O** be the origin and the coordinates of **P** be $(1, 2, -3)$, then find the equation of the plane passing through **P** and perpendicular to OP.

4.7.54 Find the vector equations of the line passing through the point $(1, 2, -4)$ and perpendicular to the two lines

$$\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7} \text{ and } \frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}.$$

4.7.55 Distance between the two planes: $2x + 3y + 4z = 4$ and $4x + 6y + 8z = 12$ is ______.

4.7.56 Find the equation of the line whose perpendicular distance from the origin is 4 units and the angle which the normal makes with positive direction of x-axis is 15°.

4.7.57 Find the distance of the point $(3, -5)$ from the line $3x - 4y - 26 = 0$.

4.7.58 Find the distance between the parallel lines $3x - 4y + 7 = 0$ and $3x - 4y + 5 = 0$.

4.7.59 Find the equation of a line perpendicular to the line $x + 2y + 3 = 0$ and passing through the point $(1, -2)$.

4.7.60 Reduce the equation $\sqrt{3}x + y - 8 = 0$ into normal form. Find the values of p and ω.

4.7.61 Find the coordinates of the foot of the perpendicular drawn from the origin to the plane $2x - 3y + 4z - 6 = 0$.

4.7.62 Find the equation of the plane which passes through the point $(5, 2, -4)$ and perpendicular to the line with direction ratios $2, 3, -1$.

4.7.63 Find the equation of the plane that contains the point $(1, -1, 2)$ and is perpendicular to each of the planes $2x + 3y - 2z = 5$ and $x + 2y - 3z = 8$.

4.7.64 Find the distance between the point $\mathbf{P}(6, 5, 9)$ and the plane determined by the points $\mathbf{A}(3, -12), \mathbf{B}(5, 2, 4)$ and $\mathbf{C}(-1, -1, 6)$.

4.7.65 Find the distance of a point $(2, 5, -3)$ from the plane $\overrightarrow{r} \cdot (6\hat{i} - 3\hat{j} + 2\hat{k}) = 4$.

4.7.66 A line passes through (2, 2) and is perpendicular to the line $3x+y = 3$. Its y-intercept is ______.

4.8 CBSE

4.8.1 Find the equation of the plane passing through the points $(1, 0, -2)$, $(3, -1, 0)$ and perpendicular to the plane $2x - y + z = 8$. Also find the distance of the plane thus obtained from the origin. (12, 2020)

4.8.2 Find the values of λ for which the distance of the point $(2, 1, \lambda)$ from the plane $3x + 5y + 4z = 11$ is $2\sqrt{2}$ units. (12, 2023)

4.8.3 Find the distance of the point (a, b, c) from the X axis. (12, 2021)

4.8.4 If the lines $\frac{x-1}{-3} = \frac{y-2}{2\lambda} = \frac{z-3}{2}$ and $\frac{x-1}{3\lambda} = \frac{y-1}{2} = \frac{z-6}{-5}$ are perpendicular, find the value of λ. Hence, determine whether the lines intersect or not. (12, 2019)

4.8.5 Find the vector equation of the plane determined by the points $\mathbf{A}(3, -1, 2)$, $\mathbf{B}(5, 2, 4)$, and $\mathbf{C}(-1, -1, 6)$. Hence, find the distance of the plane, thus obtained, from the origin. (12, 2019)

4.8.6 Find the coordinates of the foot of the perpendicular $\mathbf{Q}$ drawn from $\mathbf{P}(3, 2, 1)$ to the plane $2x - y + z + 1 = 0$. Also, find the distance PQ and the image of the point $\mathbf{P}$ treating this plane as a mirror. (12, 2019)

4.8.7 Find the value of λ for which the lines $\frac{x-5}{5(\lambda+2)} = \frac{2-y}{5} = \frac{1-z}{-1}; \frac{x}{1} = \frac{y+\frac{1}{2}}{2\lambda} = \frac{z-1}{3}$. are perpendicular to each other. Hence, find whether the lines intersect or not. (12, 2019)

4.8.8 Find the equation of the plane passing through the point $(-1, 3, 2)$ and perpendicular to the planes $x + 2y + 3z = 5$ and $3x + 3y + z = 0$. (12, 2019)

4.8.9 Find the coordinates of the foot $\mathbf{Q}$ of the perpendicular drawn from the point $\mathbf{P}(1, 3, 4)$ to the plane $2x - y + z + 3 = 0$. Find the distance PQ and the image of $\mathbf{P}$ treating the plane as a mirror. (12, 2019)

4.8.10 Find the coordinates of the foot of the perpendicular $\mathbf{Q}$ drawn from $\mathbf{P}(3, 2, 1)$ to the plane $2x - y + z + 1 = 0$. Also, find distance PQ and the image of the point $\mathbf{P}$ treating this plane as a mirror. (12, 2018)

4.8.11 Find the vector equation of the plane determined by the points $\mathbf{A}\,(3, -1, 2)$, $\mathbf{B}\,(5, 2, 4)$, $\mathbf{C}\,(-1, -1, 6)$. Hence, find the distance of the plane, thus obtained, from the origin. (12, 2018)

4.8.12 Find the vector equation of the plane that contains the line $\mathbf{r} = \left(\hat{i} + \hat{j}\right) + \lambda\left(\hat{i} + 2\hat{j} - \hat{j}\right)$ and the point $(-1, 3, -4)$. Also,find the length of the perpendicular drawn from the point $(2,1,4)$ to the plane, thus obtained. (12, 2018)

4.8.13 Find the distance between the planes $\vec{r}.\left(2\hat{i}-3\hat{j}+6\hat{k}\right)-4=0$ and $\vec{r}.\left(6\hat{i}-9\hat{j}+18\hat{k}\right)+30=0$. (12, 2016)

4.8.14 Find the position vector of the foot of perpendicular and the perpendicular distance from the point **P** with position vector $2\hat{i}+3\hat{j}+\hat{k}$ to the plane $\mathbf{r}\cdot\left(2\hat{i}+\hat{j}+3\hat{k}\right)-26=0$. Also find image of **P** in the plane. (12, 2016)

4.8.15 A line l passes through point $(-1,3,-2)$ and is perpendicular to both the lines $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ and $\frac{x+2}{-3}=\frac{y-1}{2}=\frac{z+1}{5}$. Find the vector equation of the line l. Hence, obtain its distance from origin. (12, 2023)

4.8.16 Find the equation of the plane passing through the points $(2,1,0),(3,-2,-2)$ and $(1,1,7)$. Also, obtain its distance from the origin. (12, 2022)

4.8.17 The foot of a perpendicular drawn from the point $(-2,-1,-3)$ on a plane is $(1,-3,3)$. Find the equation of the plane. (12, 2022)

4.8.18 The distance between the planes $4x-4y+2z+5=0$ and $2x-2y+z+6=0$ is ______. (12, 2022)

4.8.19 If the distance of the point $(1,1,1)$ from the plane $x-y+z+\lambda=0$ is $\frac{5}{\sqrt{3}}$, find the value(s) of λ. (12, 2022)

4.8.20 Find the distance of the point $(2,3,4)$ measured along the line $\frac{x-4}{3}=\frac{y+5}{6}=\frac{z+1}{2}$ from the plane $3x+2y+2z+5=0$. (12, 2022)

4.8.21 Find the distance of the point $\mathbf{P}(4,3,2)$ from the plane determined by the points $\mathbf{A}(-1,6,-5),\mathbf{B}(-5,-2,3)$ and $\mathbf{C}(2,4,-5)$. (12, 2022)

4.8.22 The distance of the line $\mathbf{r}=(\hat{i}-\hat{j})+\lambda(\hat{i}+5\hat{j}+\hat{k})$ from the plane $\mathbf{r}\cdot(\hat{i}-\hat{j}+4\hat{k})=5$ is ______. (12, 2022)

4.8.23 Find the values of λ, for which the distance of point $(2,1,\lambda)$ from plane $3x+5y+4z=11$ is $2\sqrt{2}$ units. (12, 2022)

4.8.24 If the distance of the point $(1,1,1)$ from the plane $x-y+z+\lambda=0$ is $\frac{5}{\sqrt{3}}$, find the value(s) of λ. (12, 2022)

4.8.25 Find the coordinates of the foot of perpendicular drawn from the point $\mathbf{A}(-1,8,4)$ to the line joining the points $\mathbf{B}(0,-1,3)$ and $\mathbf{C}(2,-3,-1)$. Hence find the image of the point **A** in the line BC. (12, 2016)

4.8.26 The coordinates of the foot of the perpendicular drawn from the point $(2,-3,4)$ on the Y axis is ______. (12, 2020)

4.8.27 Find the equation of the plane passing through $(-1,3,2)$ and perpendicular to the planes $x+2y+3z=5$ and $3x+3y+z=0$. (12, 2018)

4.8.28 Find the equation of the plane passing through the intersection of the planes $\vec{r}\cdot\left(\hat{i}+\hat{j}+\hat{k}\right)=1$ and $\vec{r}\cdot\left(2\hat{i}+3\hat{j}-\hat{k}\right)+4=0$ and parallel to the X axis. Hence, find the distance of the plane from the X axis. (12, 2018)

4.8.29 Write the equation of the line passing through $(1,2,3)$ and perpendicular to the plane $\mathbf{r}\cdot(i+2j-5k)+9=0$. (12, 2015)

4.8.30 Find the equation of a line passing through the point $(2,3,2)$ and parallel to the line $\vec{r}=(-2\hat{i}+3\hat{j})+\lambda(2\hat{i}-3\hat{j}+6\hat{k})$. Also, find the distance between these two lines. (12, 2019)

4.8.31 Given $\mathbf{a}=2\hat{i}-\hat{j}+\hat{k}$, $\mathbf{b}=3\hat{i}-\hat{k}$ and $\mathbf{a}=2\hat{i}+\hat{j}-2\hat{k}$. Find a vector **d** which is perpendicular to both **a** and **b** and $\mathbf{c}\cdot\mathbf{d}=3$. (12, 2024)

4.8.32 Find the projection of vector $(\mathbf{b}+\mathbf{c})$ on vector $\mathbf{a}$, where $\mathbf{a} = 2\hat{i}+2\hat{j}+\hat{k}, \mathbf{b} = \hat{i}+3\hat{j}+\hat{k}$, and $\mathbf{c} = \hat{i}+\hat{k}$. (12, 2024)

4.8.33 Find the coordinates of the foot of the perpendicular drawn from the point $(2,3,-8)$ to the line $\frac{4-x}{2} = \frac{y}{6} = \frac{1-z}{3}$. Also, find the perpendicular distance of the given point from the line. (12, 2024)

4.8.34 Find the shortest distance between the lines L_1 & L_2, where

$$L_1 : \text{The line passing through } (2,-1,1) \text{ and parallel to } \frac{x}{1} = \frac{y}{1} = \frac{z}{3}$$

$$L_2 : \mathbf{r} = \hat{i} + (2\mu+1)\hat{j} - (\mu+2)\hat{k}.$$

(12, 2024)

4.8.35 Find the coordinates of the foot of the perpendicular drawn from the point $(0,1,2)$ on the X axis. (12, 2024)

4.8.36 Find the equation of the plane determined by the points $\mathbf{A}(3,-1,2), \mathbf{B}(5,2,4)$ and $\mathbf{C}(-1,-1,6)$. Also find the distance of the point $\mathbf{P}(6,5,9)$ from the plane. (12, 2009)

4.9 Angle

4.9.1 Two lines passing through the point (2, 3) intersect each other at an angle of 60°. If the slope of one line is 2, find the equation of the other line.

Solution: Using the scalar product

$$\cos 60^\circ = \frac{1}{2} = \frac{\begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}1\\ m\end{pmatrix}}{\sqrt{5}\sqrt{m^2+1}} \tag{4.9.1.1}$$

$$\implies 11m^2 + 16m - 1 = 0 \tag{4.9.1.2}$$

$$or, m = \frac{-8 \pm 5\sqrt{3}}{11} \tag{4.9.1.3}$$

So, the desired equation of the line is

$$\begin{pmatrix}\frac{-8\pm5\sqrt{3}}{11} & -1\end{pmatrix}\mathbf{x} = \begin{pmatrix}\frac{-8\pm5\sqrt{3}}{11} & -1\end{pmatrix}\begin{pmatrix}2\\ 3\end{pmatrix} \tag{4.9.1.4}$$

$$= \frac{-49 \pm 16\sqrt{3}}{11} \tag{4.9.1.5}$$

4.9.2 Find the equation of the lines through the point (3, 2) which make an angle of 45° with the line x–$2y = 3$.

Solution: Following the approach in Problem 4.9.1,

$$\cos 45^\circ = \frac{1}{\sqrt{2}} = \frac{\begin{pmatrix}2 & 1\end{pmatrix}\begin{pmatrix}1\\ m\end{pmatrix}}{\left\|\begin{pmatrix}2\\ 1\end{pmatrix}\right\| \left\|\begin{pmatrix}1\\ m\end{pmatrix}\right\|} \tag{4.9.2.1}$$

$$\implies 3m^2 - 8m - 3 = 0 \tag{4.9.2.2}$$

$$\text{or, } m = -\frac{1}{3}, 3 \tag{4.9.2.3}$$

Thus, the desired equations are

$$\begin{pmatrix} 1 & 3 \end{pmatrix} \left\{ \mathbf{x} - \begin{pmatrix} 3 \\ 2 \end{pmatrix} \right\} = 0 \tag{4.9.2.4}$$

$$\implies \begin{pmatrix} 1 & 3 \end{pmatrix} \mathbf{x} = 9 \tag{4.9.2.5}$$

and

$$\begin{pmatrix} 3 & -1 \end{pmatrix} \left\{ \mathbf{x} - \begin{pmatrix} 3 \\ 2 \end{pmatrix} \right\} = 0 \tag{4.9.2.6}$$

$$\implies \begin{pmatrix} 3 & -1 \end{pmatrix} \mathbf{x} = 7 \tag{4.9.2.7}$$

See Fig. 4.9.2.1.

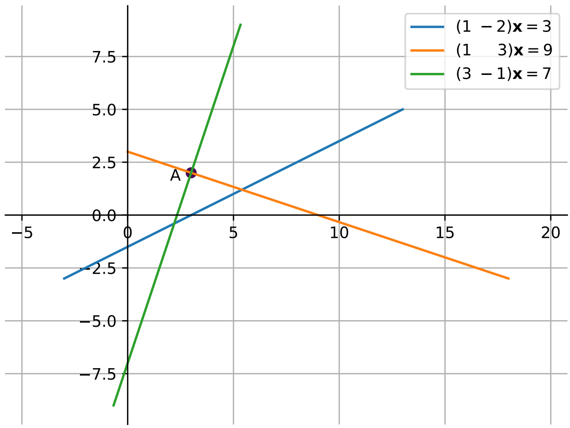


Fig. 4.9.2.1

4.9.3 Find the equations of the two lines through the origin which intersect the line $\frac{x-3}{2} = \frac{y-3}{1} = \frac{z}{1}$ at angles of $\frac{\pi}{3}$ each.

4.9.4 The equations of the lines which pass through the point (3, -2) and are inclined at 60° to the line $\sqrt{3}x + y = 1$ is

a) $y + 2 = 0, \ \sqrt{3}x - y - 2 - 3\sqrt{3} = 0$

b) $x - 2 = 0, \ \sqrt{3}x - y + 2 + 3\sqrt{3} = 0$

c) $\sqrt{3}x - y - 2 - 3\sqrt{3} = 0$

d) None of these

4.9.5 Equations of the lines through the point (3,2) and making an angle of 40° with the line $x - 2y = 3$ are ______.

4.10.1 Find the equation of the plane through the intersection of the planes $3x–y+2z–4 = 0$ and $x + y + z–2 = 0$ and the point $\begin{pmatrix}2\\2\\1\end{pmatrix}$.

Solution: The parameters of the given planes are

$$\mathbf{n}_1 = \begin{pmatrix}3\\-1\\2\end{pmatrix}, \mathbf{n}_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}, c_1 = 4, c_2 = 2. \tag{4.10.1.1}$$

The intersection of the planes is given as

$$\mathbf{n}_1^\top \mathbf{x} - c_1 + \lambda\left(\mathbf{n}_2^\top \mathbf{x} - c_2\right) = 0 \tag{4.10.1.2}$$

where

$$\lambda = \frac{c_1 - \mathbf{n}_1^\top \mathbf{P}}{\mathbf{n}_2^\top \mathbf{P} - c_2} = -\frac{2}{3} \tag{4.10.1.3}$$

upon substituting

$$\mathbf{P} = \begin{pmatrix}2\\2\\1\end{pmatrix}. \tag{4.10.1.4}$$

in (4.10.1.3) along with the numerical values in (4.10.1.1). Now, substituting (4.10.1.3) in (4.10.1.2), the equation of plane is

$$\begin{pmatrix}7 & -5 & 4\end{pmatrix}\mathbf{x} = 8 \tag{4.10.1.5}$$

4.10.2 The distance of the point of intersection of the lines $2x - 3y + 5 = 0$ and $3x + 4y = 0$ from the line $5x - 2y = 0$ is ______.

4.10.3 Find the vector equation of the plane passing through the intersection of the planes $\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 6$ and $\vec{r} \cdot (2\hat{i} + 3\hat{j} + 4\hat{k}) = -5$, and the point $(1, 1, 1)$.

4.10.4 Find the coordinates of the point where the line through the points $\mathbf{A}(3, 4, 1)$ and $\mathbf{B}(5, 1, 6)$ crosses the XY plane.

4.10.5 A person standing at the junction (crossing) of two straight paths represented by the equations

$$\begin{pmatrix}2 & -3\end{pmatrix}\mathbf{x} = -4 \tag{4.10.5.1}$$

and

$$\begin{pmatrix}3 & 4\end{pmatrix}\mathbf{x} = 5 \tag{4.10.5.2}$$

wants to reach the path whose equation is

$$\begin{pmatrix}6 & -7\end{pmatrix}\mathbf{x} = -8 \tag{4.10.5.3}$$

Find equation of the path that he should follow.

Solution: The junction of (4.10.5.1) and (4.10.5.2) is obtained as

$$\left(\begin{array}{cc|c} 2 & -3 & -4 \\ 3 & 4 & 5 \end{array}\right) \xleftrightarrow{R_2 \to 2R_2 - 3R_1} \left(\begin{array}{cc|c} 2 & -3 & -4 \\ 0 & 17 & 22 \end{array}\right)$$
$$\xleftrightarrow{R_1 \to 17R_1 + 3R_2} \left(\begin{array}{cc|c} 17 & 0 & -1 \\ 0 & 17 & 22 \end{array}\right) \implies \mathbf{A} = \frac{1}{17}\begin{pmatrix} -1 \\ 22 \end{pmatrix}$$

Clearly, the man should follow the path perpendicular to (4.10.5.3) from $\mathbf{A}$ to reach it in the shortest time. The normal vector of (4.10.5.3) is

$$\begin{pmatrix} 6 \\ -7 \end{pmatrix} \implies \mathbf{n} = \begin{pmatrix} 7 \\ 6 \end{pmatrix} \tag{4.10.5.4}$$

and the equation of the desired line is

$$\begin{pmatrix} 7 & 6 \end{pmatrix}\mathbf{x} = \frac{1}{17}\begin{pmatrix} 7 & 6 \end{pmatrix}\begin{pmatrix} -1 \\ 22 \end{pmatrix} = \frac{125}{17} \tag{4.10.5.5}$$

See Fig. 4.10.5.1.

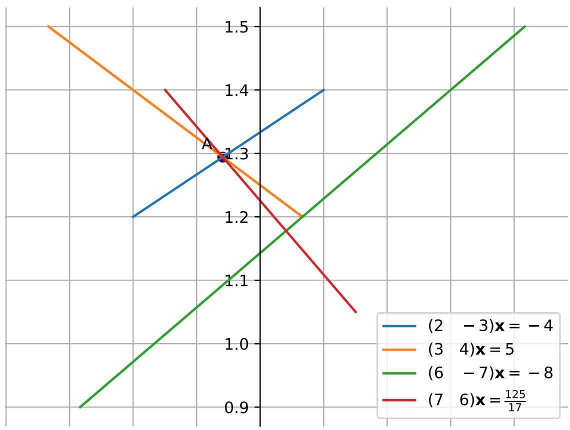


Fig. 4.10.5.1

4.10.6 If the lines $2x + y - 3 = 0, 5x + ky - 3 = 0$ and $3x - y - 2 = 0$ are concurrent, find the value of k.

4.10.7 Find the equation of the line parallel to y-axis and drawn through the point of intersection of the lines x − 7y + 5 = 0 and 3x + y = 0.

Solution: Following the approach in Problem 4.10.1, the desired equation is

$$\begin{pmatrix}1 & -7\end{pmatrix}\mathbf{x} - 5 + k\begin{pmatrix}3 & 1\end{pmatrix}\mathbf{x} = 0 \tag{4.10.7.1}$$

$$\implies \begin{pmatrix}1+3k & -7+k\end{pmatrix}\mathbf{x} = 5 \tag{4.10.7.2}$$

$$\implies \begin{pmatrix}1+3k \\ -7+k\end{pmatrix} = \alpha\begin{pmatrix}1 \\ 0\end{pmatrix} \text{or, } k = 7, \alpha = 22. \tag{4.10.7.3}$$

The desired equation is then given by

$$\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = \frac{5}{22} \tag{4.10.7.4}$$

The intersection of the lines is obtained using the augemented matrix as

$$\left(\begin{array}{cc|c}1 & -7 & -5 \\ 3 & 1 & 0\end{array}\right) \xleftrightarrow[R_1=22R_1+7R_2]{R_2=R_2-3R_1} \left(\begin{array}{cc|c}22 & 0 & -5 \\ 0 & 22 & 15\end{array}\right) \tag{4.10.7.5}$$

$$\implies \mathbf{x} = \frac{5}{22}\begin{pmatrix}-1 \\ 3\end{pmatrix} \tag{4.10.7.6}$$

See Fig. 4.10.7.1.

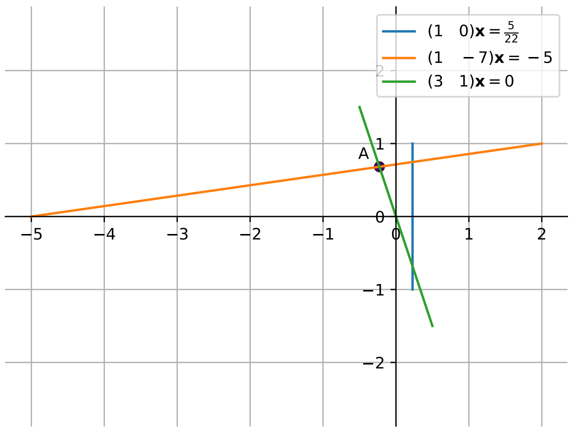


Fig. 4.10.7.1

4.10.8 Show that the area of the triangle formed by the lines $y = m_1x + c_1, y = m_2x + c_2$ and $x = 0$ is $\frac{(c_1-c_2)^2}{2|m_1-m_2|}$

4.10.9 Find the equation of the plane which contains the line of intersection of the planes $\overrightarrow{r} \cdot (\hat{i} + 2\hat{j} + 3\hat{k}) - 4 = 0, \overrightarrow{r} \cdot (2\hat{i} + \hat{j} - \hat{k}) + 5 = 0$ and which is perpendicular to the plane $\overrightarrow{r} \cdot (5\hat{i} + 3\hat{j} - 6\hat{k}) + 8 = 0$

4.10.10 Find the distance of the point $(-1, -5, -10)$ from the point of intersection of the line

$\overrightarrow{r} = 2\hat{i} - \hat{j} - 2\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})$ and the plane $\overrightarrow{r} \cdot (\hat{i} - \hat{j} + \hat{k}) = 5$.

4.10.11 Find the equation of the line passing through the point of intersection of the lines $4x + 7y - 3 = 0$ and $2x - 3y + 1 = 0$ that has equal intercepts on the axes.

Solution: From Problem 4.10.1, the intersection of the lines is given by

$$\begin{pmatrix} 4 + 2k & 7 - 3k \end{pmatrix} \mathbf{x} = 3 - k \tag{4.10.11.1}$$

$$\implies \begin{pmatrix} 4 + 2k \\ 7 - 3k \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{4.10.11.2}$$

from Problem 4.3.19, yielding,

$$\left(\begin{array}{cc|c} 1 & -2 & 4 \\ 1 & 3 & 7 \end{array}\right) \xleftrightarrow{R_2 = R_2 - R_1} \left(\begin{array}{cc|c} 1 & -2 & 4 \\ 0 & 5 & 3 \end{array}\right) \tag{4.10.11.3}$$

$$\text{or, } k = \frac{3}{5} \tag{4.10.11.4}$$

Substituting the above in (4.10.11.1), the desired equation is

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = \frac{6}{13} \tag{4.10.11.5}$$

See Fig. 4.10.11.1.

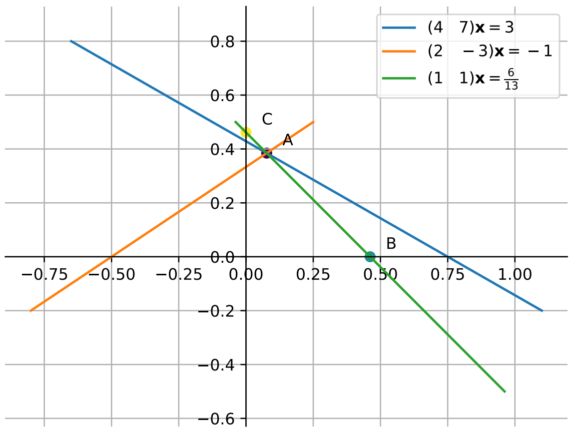


Fig. 4.10.11.1

4.10.12 The straight line $5x + 4y = 0$ passes through the point of intersection of the straight lines $x + 2y - 10 = 0$ and $2x + y + 5 = 0$.

4.10.13 Find the equation of the plane passing through the line of intersection of the planes $\overrightarrow{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 1$ and $\overrightarrow{r} \cdot (2\hat{i} + 3\hat{j} - \hat{k}) + 4 = 0$ and parallel to the X axis.

4.10.14 Find the value of p so that the three lines $3x + y - 2 = 0, px + 2y - 3 = 0$ and

$2x - y - 3 = 0$ may intersect at one point.
Solution: Performing row operations on the matrix

$$\begin{pmatrix} 3 & 1 & -2 \\ p & 2 & -3 \\ 2 & -1 & -3 \end{pmatrix} \xrightarrow[R_3=3R_3-2R_1]{R_2=3R_2-pR_1} \begin{pmatrix} 3 & 1 & -2 \\ 0 & 6-p & -9+2p \\ 0 & -5 & -5 \end{pmatrix}$$

$$\xrightarrow{R_3=R_3(6-p)+5R_2} \begin{pmatrix} 3 & 1 & -2 \\ 0 & 6-p & -9+2p \\ 0 & 0 & -75+15p \end{pmatrix}$$

$$\implies p = 5$$

Substituting this value in the above, we obtain

$$\begin{pmatrix} 3 & 1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.10.14.1}$$

yielding

$$\mathbf{x} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{4.10.14.2}$$

as the point of intersection. See Fig. 4.10.14.1.

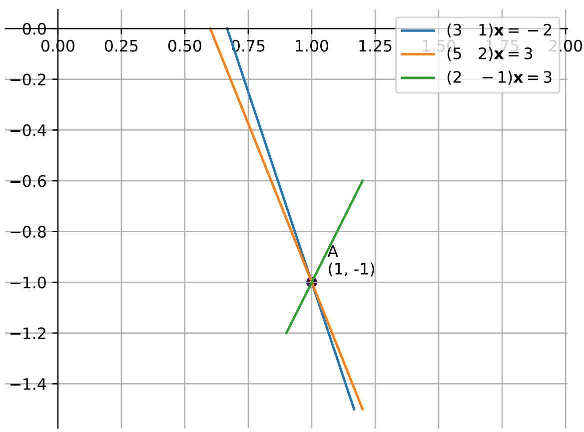


Fig. 4.10.14.1

4.10.15 Show that the lines

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$$

and

$$\frac{x-4}{5} = \frac{y-1}{2} = z$$

intersect. Also, find their point of intersection.

4.10.16 The area of the region bounded by the curve $y = x+1$ and the lines $x = 2$ and $x = 3$ is

a) $\frac{7}{2}$ sq units

b) $\frac{9}{2}$ sq units

c) $\frac{11}{2}$ sq units

d) $\frac{13}{2}$ sq units

4.10.17 Compute the area bounded by the line $x + 2y = 2$, $y - x = 1$ and $2x + y = 7$.

4.10.18 Find the area bounded by the lines $y = 4x + 5$, $y = 5 - x$ and $4y = x + 5$.

4.10.19 Find the equation of the plane which is perpendicular to the plane $5x+3y+6z+8 = 0$ and which contains the line of intersection of the planes $x + 2y + 3z - 4 = 0$ and $2x + y - z + 5 = 0$.

4.10.20 Point $\mathbf{P}(0, 2)$ is the point of intersection of the y-axis and the perpendicular bisector of line segment joining the points $\mathbf{A}(-1, 1)$ and $\mathbf{B}(3, 3)$.

4.10.21 Prove that the line through $\mathbf{A}(0, -1, -1)$ and $\mathbf{B}(4, 5, 1)$ intersects the line through $\mathbf{C}(3, 9, 4)$ and $\mathbf{D}(-4, 4, 4)$.

4.10.22 Find the equation of the plane through the intersection of the planes $\overrightarrow{r}\cdot(\hat{i}+3\hat{j})-6 = 0$ and $\overrightarrow{r} \cdot (3\hat{i} - \hat{j} - 4\hat{k}) = 0$, whose perpendicular distance from origin is unity.

4.10.23 Find the equation of the line passing through the point of intersection of $2x + y = 5$ and $x + 3y + 8 = 0$ and parallel to the line $3x + 4y = 7$.

4.10.24 Find the equations of the lines through the point of intersection of the line $x-y+1 = 0$ and $2x - 3y + 5 = 0$ and whose distance from the point (3, 2) is $\frac{7}{5}$.

4.11 CBSE

4.11.1 Find the coordinates of the point where the line $\frac{x-1}{3} = \frac{y+4}{7} = \frac{z+4}{2}$ cuts the XY plane. (12, 2020)

4.11.2 Find the coordinates of the point where the line through $(3, 4, 1)$ crosses the ZX plane. (12, 2023)

4.11.3 Find the equation of the line passing through $(2, -1, 2)$ and $(5, 3, 4)$ and the equation of the plane passing through $(2, 0, 3)$, $(1, 1, 5)$, and $(3, 2, 4)$. Also, find their point of intersection. (12, 2019)

4.11.4 Find the equation of the plane which contains the line of intersection of the planes $\overrightarrow{r} \cdot (\hat{i} + 2\hat{j} + 3\hat{k}) - 4 = 0$ and $\overrightarrow{r} \cdot (2\hat{i} + \hat{j} - \hat{k}) + 5 = 0$, and which is perpendicular to the plane $\overrightarrow{r} \cdot (5\hat{i} + 3\hat{j} - 6\hat{k}) + 8 = 0$. (12, 2019)

4.11.5 Find the equation of the plane passing through the points $(2, 5, -3)$, $(-2, -3, 5)$, and $(5, 3, -3)$. Also, find the point of intersection of this plane with the line passing through points $(3, 1, 5)$ and $(-1, -3, -1)$. (12, 2019)

4.11.6 Find the equation of the plane passing through the intersection of the planes $\overrightarrow{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 1$ and $\overrightarrow{r} \cdot (2\hat{i} + 3\hat{j} - \hat{k}) + 4 = 0$ and parallel to the X axis. Hence, find the distance of the plane from the X axis. (12, 2019)

4.11.7 Find the equation of the planes passing through the intersection of the planes $\overrightarrow{r} \cdot (3\hat{i} + 6\hat{j}) + 12 = 0$ and $\overrightarrow{r} \cdot (3\hat{i} - \hat{j} + 4\hat{k}) = 0$ and are at a unit distance from the origin. (12, 2019)

4.11.8 Find the coordinates of the point where the line $\frac{x+2}{1} = \frac{y-5}{3} = \frac{z+1}{5}$ cuts the YZ plane. (12, 2019)

4.11.9 Find the area of the triangle $\triangle ABC$ bounded by the lines $4x - y + 5 = 0, x + y - 5 = 0$ and $x - 4y + 5 = 0$. (12, 2019)

4.11.10 Point **A** lies on the line segment XY joining $\mathbf{X}(6, -6)$ and $\mathbf{Y}(-4, -1)$ in such a way that $\frac{XA}{XY} = \frac{2}{5}$. If point **A** also lies on the line $3x + k(y + 1) = 0$, find the value of k. (10, 2019)

4.11.11 Find the ratio in which the line $x - 3y = 0$ divides the line segment joining the points $(-2, -5)$ and $(6, 3)$. Find the coordinates of the point of intersection. (10, 2019)

4.11.12 Find the distance of the point $(-1, -5, -10)$ from the point of intersection of the line $\overrightarrow{r} = 2\hat{i} - \hat{j} + 2\hat{k} + \lambda\left(3\hat{i} + 4\hat{j} + 2\hat{k}\right)$ and the plane $\overrightarrow{r} \cdot \left(\hat{i} - \hat{j} + \hat{k}\right) = 5$. (12, 2018)

4.11.13 Find the equation of the plane passing through the points $(2, 5, -3)$, $(-2, -3, 5)$, and $(5, 3, -3)$. Also, find the point of intersection of this plane with the line passing through points $(3, 1, 5)$ and $(-1, -3, -1)$. (12, 2018)

4.11.14 Find the value of λ for which the lines $\frac{x-5}{5\lambda+2} = \frac{2-y}{5} = \frac{1-z}{-1}$ and $\frac{x}{1} = \frac{y+\frac{1}{2}}{2\lambda} = \frac{z-1}{3}$ are perpendicular to each other. Hence, find whether the lines intersect or not. (12, 2018)

4.11.15 Find the equation of the plane which contains the line of intersection of the planes $\overrightarrow{r}.\left(\hat{i} + 2\hat{j} + 3\hat{k}\right) - 4 = 0$, $\overrightarrow{r}.\left(2\hat{i} + \hat{j} - \hat{k}\right) + 5 = 0$ and which is perpendicular to the plane $\overrightarrow{r}.\left(5\hat{i} + 3\hat{j} - 6\hat{k}\right) + 8 = 0$. (12, 2018)

4.11.16 Find the equation of planes passing through the intersection of the planes $\overrightarrow{r} \cdot \left(2\hat{i} + 6\hat{j}\right) + 12 = 0$ and $\overrightarrow{r} \cdot \left(3\hat{i} - \hat{j} + 4\hat{k}\right) = 0$ and are at a unit distance from origin. (12, 2018)

4.11.17 Find the value of λ, so that the lines $\frac{1-x}{3} = \frac{7y-14}{\lambda} = \frac{z-3}{2}$ and $\frac{7-7x}{3\lambda} = \frac{y-5}{1} = \frac{6-z}{5}$ are at right angles. Also, find whether the lines are intersecting or not. (12, 2018)

4.11.18 Find the equation of the plane which contains the line of intersection of the planes $\overrightarrow{r}.\left(\hat{i} - 2\hat{j} + 3\hat{k}\right) - 4 = 0$ and $\overrightarrow{r}.\left(-2\hat{i} + \hat{j} + \hat{k}\right) + 5 = 0$ and whose intercept on X axis is equal to that of on Y axis. (12, 2016)

4.11.19 Find the coordinates of the point where the line through the points $\mathbf{P}(4, 3, 2)$ and $\mathbf{Q}(5, 1, 6)$ crosses the XZ plane. Also find the angle which this line makes with the XZ plane. (12, 2016)

4.11.20 Find the coordinates of the point where the line through the points $\mathbf{A}(3, 4, 1)$ and $\mathbf{B}(5, 1, 6)$ crosses the XZ plane. Also find the angle which this line makes with the XZ plane. (12, 2016)

4.11.21 Find the equation of the plane passing through the line of intersection of planes $\mathbf{r} \cdot \left(2\overrightarrow{i} + 2\overrightarrow{j} - 3\overrightarrow{k}\right) = 7$, $\mathbf{r} \cdot \left(2\overrightarrow{i} + 5\overrightarrow{j} + 3\overrightarrow{k}\right) = 9$ such that the intercepts made by the plane on X axis and Z axis are equal. (12, 2015)

4.11.22 Find the equations of the diagonals of the parallelogram $PQRS$ whose vertices are $\mathbf{P}(4, 2, -6)$, $\mathbf{Q}(5, -3, 1)$, $\mathbf{R}(12, 4, 5)$ and $\mathbf{S}(11, 9, -2)$. Use these equations to find the point of intersection of diagonals. (12, 2023)

4.11.23 Find the co-ordinates of the point where the line $\frac{x-3}{-1} = \frac{y+4}{1} = \frac{z+5}{6}$ crosses the plane passing through the points $\left(\frac{7}{2}, 0, 0\right), (0, 7, 0), (0, 0, 7)$. (12, 2022)

4.11.24 Find the equation of the plane through the line of intersection of the planes $\mathbf{r} \cdot (\hat{i} + 3\hat{j}) + 6 = 0$, $\mathbf{r} \cdot (3\hat{i} - \hat{j} - 4\hat{k}) = 0$ which is at a unit distance from the origin. (12, 2022)

4.11.25 Find the distance of the point $(1, -2, 9)$ from the point of intersection of the line $\mathbf{r} = 4\hat{i} + 2\hat{j} + 7\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})$ and the plane $\mathbf{r} \cdot (\hat{i} - \hat{j} + \hat{k}) = 10$. (12, 2022)

4.11.26 Find the area bounded by the curves $y = |x - 1|$ and $y = 1$. (12, 2022)

4.11.27 Find the coordinates of the point where the line through $(4, -3, -4)$ and $(3, -2, 2)$ crosses the plane $2x + y + z = 6$. (12, 2022)

4.11.28 Find the equation of the plane through the line of intersection of the planes $\mathbf{r} \cdot (i + 3j) + 6 = 0$ and $\mathbf{r} \cdot (3i - j - 4k) = 0$, which is at a unit distance from the origin. (12, 2021)

4.11.29 Find the area of the region bounded by the lines $3x - 2y + 1 = 0, 2x + 3y - 21 = 0$ and $x - 5y + 9 = 0$. (12, 2019)

4.11.30 Draw the graph of the equations $x - y + 1 = 0$ and $3x + 2y - 12 = 0$. Using this graph, find the values of x and y which satisfy both the equations. (10, 2021)

4.11.31 Find the area of the region bounded by the line $y = 3x + 2$, the X axis and the ordinates $x = -2$ and $x = 1$. (12, 2019)

4.11.32 Find the equation of the line passing through $(2, -1, 2)$ and $(5, 3, 4)$ and of the plane passing through $(2, 0, 3)$, $(1, 1, 5)$ and $(3, 2, 4)$. Also, find their point of intersection. (12, 2018)

4.11.33 Find the length of the intercept, cut off by the plane $2x + y - z = 5$ on the X axis. (12, 2018)

4.11.34 Find the area of the region bounded by the lines $3x - 2y + 1=0$, $2x + 3y - 21 = 0$, and $x - 5y + 9 = 0$. (12, 2018)

4.11.35 Find the area of the region bounded by the line $y = 3x + 2$, the X axis and the ordinates and the ordinates $x = -2$ and $x = 1$. (12, 2018)

4.11.36 Find the coordinates of the point where the line through the points $(3, -4, -5)$ and $(2, -3, 1)$, crosses the plane determined by the points $(1, 2, 3)$, $(4, 2, -3)$ and $(0, 4, 3)$. (12, 2017)

4.11.37 Show that the lines $\frac{x-1}{3} = \frac{y-1}{-1} = \frac{z+1}{0}$ and $\frac{x-4}{2} = \frac{y}{0} = \frac{z+1}{3}$ intersect. Find their point of intersection. (12, 2016)

4.11.38 Find the area of the triangular region whose sides have the equations $y = 2x + 1$, $y = 3x + 1$, and $x = 4$. (12, 2019)

4.11.39 Find the area of the region bounded by the lines $3x - 2y + 1 = 0, 2x + 3y - 21 = 0$ and $x - 5y + 9 = 0$. (12, 2019)

4.11.40 Find the area of the region bounded by the line $y = 3x + 2$, the X axis and the ordinates $x = -2$ and $x = 1$. (12, 2019)

4.12.1 For which values of a and b does the following pair of linear equations have an infinite number of solutions?

$$2x + 3y = 7 \tag{4.12.1.1}$$

$$(a-b)x + (a-b)y = 3a + b - 2 \tag{4.12.1.2}$$

4.12.2 For which value of k will the following pair of linear equations have no solution?

$$3x + y = 1 \tag{4.12.2.1}$$

$$(2k-1)x + (k-1)y = 2k + 1 \tag{4.12.2.2}$$

4.12.3 Find the values of k for which the line

$$(k-3)x - (4-k^2)y + k^2 - 7k + 6 = 0 \tag{4.12.3.1}$$

is

a) Parallel to the x-axis
b) Parallel to the Y axis
c) Passing through the origin

Solution:

$$\mathbf{n} = \begin{pmatrix} k-3 \\ -4+k^2 \end{pmatrix}, c = -k^2 + 7k - 6 \tag{4.12.3.2}$$

a)

$$\begin{pmatrix} k-3 \\ -4+k^2 \end{pmatrix} = \alpha \begin{pmatrix} 0 \\ 1 \end{pmatrix} \implies k = 3, \tag{4.12.3.3}$$

$$\implies \begin{pmatrix} 0 & 5 \end{pmatrix} \mathbf{x} = 6 \tag{4.12.3.4}$$

upon substituting from (4.12.3.2).

b) In this case,

$$\begin{pmatrix} k-3 \\ -4+k^2 \end{pmatrix} = \beta \begin{pmatrix} 1 \\ 0 \end{pmatrix} \implies k = \pm 2 \tag{4.12.3.5}$$

$$\implies \begin{pmatrix} -1 & 0 \end{pmatrix} \mathbf{x} = 4, \quad k = 2 \tag{4.12.3.6}$$

$$\begin{pmatrix} -5 & 0 \end{pmatrix} \mathbf{x} = -24, \quad k = -2 \tag{4.12.3.7}$$

c) In this case,

$$-k^2 + 7k - 6 = 0 \implies k = 1, k = 6 \tag{4.12.3.8}$$

$$\implies \begin{pmatrix} -2 & -3 \end{pmatrix} \mathbf{x} = 0, \quad k = 1 \tag{4.12.3.9}$$

$$\begin{pmatrix} 3 & 32 \end{pmatrix} \mathbf{x} = 0, \quad k = 6 \tag{4.12.3.10}$$

4.12.4 Find the equations of the lines, which cutoff intercepts on the axes whose sum and product are 1 and -6 respectively.

Solution: Let the intercepts be a and b. Then

$$a + b = 1, ab = -6 \tag{4.12.4.1}$$
$$\implies a = 3, b = -2 \tag{4.12.4.2}$$

Thus, the possible intercepts are

$$\begin{pmatrix} 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix} \tag{4.12.4.3}$$

From (4.1.3.5),

$$\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix} \mathbf{n} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{4.12.4.4}$$
$$\implies \mathbf{n} = \begin{pmatrix} \frac{1}{3} \\ -\frac{1}{2} \end{pmatrix} \tag{4.12.4.5}$$
$$\text{or, } \begin{pmatrix} 2 & -3 \end{pmatrix} \mathbf{x} = 6 \tag{4.12.4.6}$$

using (4.1.4.1). Similarly, the other line can be obtained as

$$\begin{pmatrix} 3 & -2 \end{pmatrix} \mathbf{x} = -6 \tag{4.12.4.7}$$

4.12.5 A ray of light passing through the point $\mathbf{P} = (1, 2)$ reflects on the x-axis at point $\mathbf{A}$ and the reflected ray passes through the point $\mathbf{Q} = (5, 3)$. Find the coordinates of $\mathbf{A}$.
Solution: From (4.1.9.1), the reflection of $\mathbf{Q}$ is

$$\mathbf{R} = \begin{pmatrix} 5 \\ -3 \end{pmatrix} \tag{4.12.5.1}$$

Letting

$$\mathbf{A} = \begin{pmatrix} x \\ 0 \end{pmatrix}, \tag{4.12.5.2}$$

since $\mathbf{P}, \mathbf{A}, \mathbf{R}$ are collinear, from (4.1.3.6),

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 5 & -3 \\ 1 & x & 0 \end{pmatrix} \xleftrightarrow[R_3=R_3-R_1]{R_2=R_2-R_1} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 4 & -5 \\ 0 & x-1 & -2 \end{pmatrix} \tag{4.12.5.3}$$
$$\xleftrightarrow{R_3=4R_3-(x-1)R_2} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 4 & -5 \\ 0 & 0 & 5x-13 \end{pmatrix} \implies x = \frac{13}{5} \tag{4.12.5.4}$$

See Fig. 4.12.5.1.

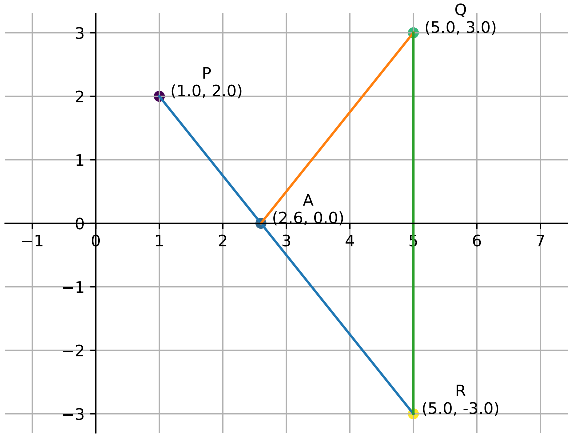


Fig. 4.12.5.1

4.12.6 The owner of a milk store finds that he can sell 980 litres of milk each week at ₹ 14/litre and 1220 litres of milk each week at ₹ 16/litre. Assuming a linear relationship between selling price and demand, how many litres could he sell weekly at ₹ 17/ litre?

4.12.7 Prove that in any $\triangle ABC$, cos A=$\frac{b^2+c^2-a^2}{2bc}$, where a, b, c are the magnitudes of the sides opposite to the vertices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ respectively.

4.12.8 Distance of the point (α, β, γ) from y-axis is

a) β
b) $|\beta|$
c) $|\beta + \gamma|$
d) $\sqrt{\alpha^2 + \gamma^2}$

4.12.9 The reflection of the point (α, β, γ) in the XY plane is

a) $\alpha, \beta, 0)$
b) $(0, 0, \gamma)$
c) $(-\alpha, -\beta, \gamma)$
d) $(\alpha, \beta, -\gamma)$

4.12.10 The plane $ax + by = 0$ is rotated about its line of intersection with the plane $z = 0$ through an angle α. Prove that the equation of the plane in its new position is

$$ax + by \pm (\sqrt{a^2 + b^2} \tan \alpha)z = 0.$$

4.12.11 The locus represented by $xy + yz = 0$ is

a) A pair of perpendicular lines
b) A pair of parallel lines
c) A pair of parallel planes

d) A pair of perpendicular planes

4.12.12 For what values of a and b the intercepts cut off on the coordinate axes by the line $ax + by + 8 = 0$ are equal in length but opposite in signs to those cut off by the line $2x - 3y = 0$ on the axes.

4.12.13 If the equation of the base of an equilateral triangle is $x + y = 2$ and the vertex is (2, -1), then find the length of the side of the triangle.

4.12.14 A variable line passes through a fixed point **P**. The algebraic sum of the perpendiculars drawn from the points (2, 0), (0, 2) and (1, 1) on the line is zero. Find the coordinates of the point **P**.

4.12.15 A straight line moves so that the sum of the reciprocals of its intercepts made on the axes is constant. Show that the line passes through a fixed point.

4.12.16 If the sum of the distances of a moving point in a plane from the axes is l, then finds the locus of the point.

4.12.17 $\mathbf{P}_1, \mathbf{P}_2$ are points on either of the two lines $y - \sqrt{3}\,|x| = 2$ at a distance of 5 units from their point of intersection. Find the coordinates of the foot of the perpendiculars drawn from $\mathbf{P}_1, \mathbf{P}_2$ on the bisector of the angle between the given lines.

4.12.18 If p is the length of perpendicular from the origin on the line $\frac{x}{a} + \frac{y}{b} = 1$ and a^2, p^2, b^2 are in A.P, then show that $a^4 + b^4 = 0$.

4.12.19 The point (4, 1) undergoes the following two successive transformations :

a) Reflection about the line $y = x$
b) Translation through a distance 2 units along the positive x-axis

Then the final coordinates of the point are

a) (4, 3)
b) (3, 4)
c) (1, 4)
d) $\frac{7}{2}, \frac{7}{2}$

4.12.20 One vertex of the equilateral with centroid at the origin and one side as $x + y - 2 = 0$ is

a) (-1, -1)
b) (2, 2)
c) (-2, -2)
d) (2, -2)

4.12.21 If a, b, c are is A.P., then the straight lines $ax + by + c = 0$ will always pass through ______.

4.12.22 The points (3, 4) and (2, -6) are situated on the ______ of the line $3x - 4y - 8 = 0$.

4.12.23 A point moves so that square of its distance from the point (3, -2) is numerically equal to its distance from the line $5x - 12y = 3$. The equation of its locus is

4.12.24 Locus of the mid-points of the portion of the line $x \sin\theta + y \cos\theta = p$ intercepted between the axes is ______.

State whether the following statements are true or false. Justify.

4.12.25 If the vertices of a triangle have integral coordinates, then the triangle can not be equilateral.

4.12.26 The line $\frac{x}{a} + \frac{y}{b} = 1$ moves in such a way that $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$, where c is a constant. The locus of the foot of the perpendicular from the origin on the given line is $x^2+y^2 = c^2$.

4.12.27 Match the following

1. The coordinates of the points **P** and **Q** on the line $x + 5y = 13$ which are at a distance of 2 units from the line $12x–5y+26 = 0$ are	a) (3, 1), (-7, 11)
2. The coordinates of the point on the line $x+y = 4$, which are at a unit distance from the line $4x + 3y–10 = 0$ are	b) $-\frac{1}{11}, \frac{11}{3}$, $\frac{4}{3}, \frac{7}{3}$
3. The coordinates of the point on the line joining $\mathbf{A}(–2, 5)$ and $\mathbf{B}(3, 1)$ such that $AP = PQ = QB$ are	c) $1, \frac{12}{5}$, $-3, \frac{16}{5}$

TABLE 4.12.27

4.12.28 The value of the λ, if the lines
$(2x + 3y + 4) + \lambda(6x - y + 12) = 0$ are

1. parallel to Y axis is	a) $\lambda = -\frac{3}{4}$
2. perpendicular to $7x + y - 4 = 0$ is	b) $\lambda = -\frac{1}{3}$
3. passes through (1, 2) is	c) $\lambda = -\frac{17}{41}$
4. parallel to X axis is	d) $\lambda = 3$

TABLE 4.12.28

4.12.29 The equation of the line through the intersection of the lines $2x - 3y = 0$ and $4x - 5y = 2$ and

1. through the point (2, 1) is	a) $2x - y = 4$
2. perpendicular to the line	b) $x + y - 5 = 0$
3. parallel to the line $3x - 4y + 5 = 0$ is	c) $x - y - 1 = 0$
4. equally inclined to the axes is	d) $3x - 4y - 1 = 0$

TABLE 4.12.29

4.12.30 Point $\mathbf{R}(h, k)$ divides a line segment between the axes in the ratio 1: 2. Find the equation of the line.

Solution: Choosing the intercept points in Problem 4.3.29,

$$\mathbf{R} = \frac{2\mathbf{A} + \mathbf{B}}{3} \implies \begin{pmatrix} h \\ k \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2a \\ b \end{pmatrix} \tag{4.12.30.1}$$

$$\text{or, } \begin{pmatrix} b \\ a \end{pmatrix} = \mathbf{n} \equiv \begin{pmatrix} 2k \\ h \end{pmatrix} \tag{4.12.30.2}$$

Thus, the equation of the line is given by,

$$\begin{pmatrix} 2k & h \end{pmatrix}\mathbf{x} = \begin{pmatrix} 2k & h \end{pmatrix}\begin{pmatrix} h \\ k \end{pmatrix} = 3hk \tag{4.12.30.3}$$

4.12.31 The tangent of the angle between the lines whose intercepts on the axes are $a, -b$ and $b, -a$, respectively, is

a) $\frac{a^2-b^2}{ab}$

b) $\frac{b^2-a^2}{2}$

c) $\frac{b^2-a^2}{2ab}$

d) none of these

4.12.32 Prove that the line through the point (x_1, y_1) and parallel to the line $Ax + By + C = 0$ is $A(x - x_1) + B(y - y_1) = 0$.

Solution: The equation of the desired line is

$$\begin{pmatrix} A & B \end{pmatrix}\left(\mathbf{x} - \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}\right) = 0 \tag{4.12.32.1}$$

$$\implies \begin{pmatrix} A & B \end{pmatrix}\mathbf{x} = Ax_1 + By_1 \tag{4.12.32.2}$$

4.12.33 If p and q are the lengths of perpendiculars from the origin to the lines $x\cos\theta - y\sin\theta = k\cos 2\theta$ and $x\sec\theta + y\operatorname{cosec}\theta = k$, respectively, prove that $p^2 + 4q^2 = k^2$

Solution: The line parameters are

$$\mathbf{n}_1 = \begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix}, c_1 = k\cos 2\theta \tag{4.12.33.1}$$

$$\mathbf{n}_2 = \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}, c_2 = \frac{1}{2}k\sin 2\theta \tag{4.12.33.2}$$

From (4.1.6.2),

$$p = \frac{\left|\mathbf{n}_1^\top\mathbf{x} - c_1\right|}{\|\mathbf{n}_1\|} = |k\cos 2\theta| \tag{4.12.33.3}$$

$$q = \frac{\left|\mathbf{n}_2^\top\mathbf{x} - c_2\right|}{\|\mathbf{n}_2\|} = \left|\frac{1}{2}k\sin 2\theta\right| \tag{4.12.33.4}$$

$$\implies p^2 + 4q^2 = k^2 \tag{4.12.33.5}$$

4.12.34 If p is the length of perpendicular from origin to the line whose intercepts on the axes are a and b, then show that

$$\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2} \tag{4.12.34.1}$$

Solution: Let the intercept points be

$$\begin{pmatrix} a \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ b \end{pmatrix}, \because \mathbf{n} = \begin{pmatrix} b \\ a \end{pmatrix}, \tag{4.12.34.2}$$

The line equation is,

$$\begin{pmatrix} b & a \end{pmatrix}\left(\mathbf{x} - \begin{pmatrix} a \\ 0 \end{pmatrix}\right) = 0 \tag{4.12.34.3}$$

$$\implies \begin{pmatrix} b & a \end{pmatrix}\mathbf{x} = ab \tag{4.12.34.4}$$

From (4.1.6.2), the perpendicular distance from the origin to the line is

$$p = \frac{ab}{\sqrt{a^2 + b^2}} \implies (4.12.34.1) \tag{4.12.34.5}$$

4.12.35 Find perpendicular distance from the origin to the line joining the points $(\cos\theta, \sin\theta)$ and $(\cos\phi, \sin\phi)$.

Solution: The equation of the line is

$$\begin{pmatrix} \sin\phi - \sin\theta & \cos\theta - \cos\phi \end{pmatrix}\mathbf{x} = \sin(\phi - \theta) \tag{4.12.35.1}$$

and from (4.1.6.2), the distance is

$$d = \frac{\sin(\phi - \theta)}{2\sin\left(\frac{\phi-\theta}{2}\right)} = \cos\left(\frac{\phi - \theta}{2}\right) \tag{4.12.35.2}$$

4.12.36 Prove that the products of the lengths of the perpendiculars drawn from the points $\begin{pmatrix} \sqrt{a^2 - b^2} & 0 \end{pmatrix}^\top$ and $\begin{pmatrix} -\sqrt{a^2 - b^2} & 0 \end{pmatrix}^\top$ to the line $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$ is b^2.

Solution: The input parameters for (4.1.6.2) are

$$\mathbf{n} = \begin{pmatrix} \frac{\cos\theta}{a} \\ \frac{\sin\theta}{b} \end{pmatrix}, \ c = 1, \ \mathbf{P} = \pm\begin{pmatrix} \sqrt{a^2 - b^2} \\ 0 \end{pmatrix} \tag{4.12.36.1}$$

The product of the distances is

$$d_1 d_2 = \frac{\left|(\mathbf{n}^\top\mathbf{P})^2 - c^2\right|}{\|\mathbf{n}\|} = \frac{\left|\frac{\cos^2\theta\left(a^2-b^2\right)}{a^2} - 1\right|}{\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}} \tag{4.12.36.2}$$

$$= \frac{\left(b^2\cos^2\theta + a^2\sin^2\theta\right)a^2b^2}{\left(b^2\cos^2\theta + a^2\sin^2\theta\right)a^2} = b^2 \tag{4.12.36.3}$$

4.12.37 **O** is the origin and **A** is (a, b, c). Find the direction cosines of the line OA and the equation of the plane through **A** at right angle at OA.

4.12.38 Two systems of rectangular axis have the same origin. If a plane cuts them at distances a, b, c and a', b', c', respectively, from the origin, prove that

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{a'^2} + \frac{1}{b'^2} + \frac{1}{c'^2}$$

.

4.12.39 Equation of the line passing through the point $(a\cos^3\theta, a\sin^3\theta)$ and perpendicular to the line $x\sec\theta + y\csc\theta = a$ is $x\cos\theta - y\sin\theta = \alpha\sin 2\theta$.

4.12.40 The distance between the lines $y = mx + c$, $andy = mx + c^2$ is

4.12.41 Find the area of the triangle formed by the lines $y - x = 0, x + y = 0$, and $x - k = 0$.

Solution: The vertices of the triangle can be expressed using the equations

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \mathbf{A} = \mathbf{0} \tag{4.12.41.1}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{B} = \begin{pmatrix} 0 \\ k \end{pmatrix} \tag{4.12.41.2}$$

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \mathbf{C} = \begin{pmatrix} k \\ 0 \end{pmatrix} \tag{4.12.41.3}$$

from which

$$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} k \\ -k \end{pmatrix}, \mathbf{C} = \begin{pmatrix} k \\ k \end{pmatrix} \tag{4.12.41.4}$$

are trivially obtained. Thus,

$$ar(ABC) = \frac{1}{2} \|(\mathbf{A} - \mathbf{B}) \times (\mathbf{A} - \mathbf{C})\| \tag{4.12.41.5}$$

$$= \frac{1}{2} \left\| \begin{pmatrix} -k \\ k \end{pmatrix} \times \begin{pmatrix} -k \\ -k \end{pmatrix} \right\| = k^2 \tag{4.12.41.6}$$

4.12.42 The lines $ax + 2y + 1 = 0$, $bx = 3y + 1 = 0$ and $cx + 4y + 1 = 0$ are concurrent if a, b, c are in G.P.

4.12.43 $\mathbf{P}(a, b)$ is the mid-point of the line segment between axes. Show that the equation of the line is $\frac{x}{a} + \frac{y}{b} = 2$

Solution: From Problem 4.3.29,

$$\mathbf{n} = \begin{pmatrix} b \\ a \end{pmatrix} \tag{4.12.43.1}$$

$$\implies \begin{pmatrix} b & a \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} a \\ b \end{pmatrix} \right) = 0 \tag{4.12.43.2}$$

$$\text{or, } \begin{pmatrix} b & a \end{pmatrix} \mathbf{x} = 2ab. \tag{4.12.43.3}$$

is the desired line equation.

4.12.44 Find the equation of the set of points which are equidistant from the points $(1, 2, 3)$ and $(3, 2, -1)$.

4.12.45 Find the equation of the set of points $\mathbf{P}$, the sum of whose distances from $\mathbf{A}(4, 0, 0)$ and $\mathbf{B}(-4, 0, 0)$ is equal to 10.

4.12.46 Find the values of θ and p, if the equation $x \cos\theta + y \sin\theta = p$ is the normal form of the line $\sqrt{3}x + y + 2 = 0$.

4.12.47 Find the image of the point $(3, 8)$ with respect to the line $x + 3y = 7$ assuming the line to be a plane mirror.

4.12.48 Prove that if a plane has the intercept a, b, c and is at a distance of p units from the origin, then $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{p^2}$.

4.12.49 The planes $2x - y + 4z = 5$ and $5x - 2.5y + 10z = 6$ are

a) Perpendicular
b) Parallel
c) Intersect y axis

d) Pass through $\left(0, 0, \frac{5}{4}\right)$

4.13 JEE

4.13.1. Consider the lines given by

$$L_1 : x + 3y - 5 = 0$$

$$L_2 : 3x - ky - 1 = 0$$

$$L_3 : 5x + 2y - 12 = 0.$$

Match the Statements/Expressions in Column I with the Statements/Expressions in Column II. (2008)

Column I	Column II
(A) L_1, L_2, L_3 are concurrent, if	(p) $k = 9$
(B) One of L_1, L_2, L_3 is parallel to at least one of the other two, if	(q) $k = \frac{-6}{5}$
(C) L_1, L_2, L_2 from a triangle, if	(r) $k = \frac{5}{6}$
(D) L_1, L_2, L_3 do not form a triangle, if	(s) $k = 5$

4.13.2. Lines

$$L_1 : y - x = 0$$

and

$$L_2 : 2x + y = 0$$

intersect the line

$$L_3 : y + 2 = 0$$

at **P** and **Q**, respectively. The bisector of the acute angle between L_1 and L_2 intersects L_3 at R.
STATEMENT-1 : The ratio $PR : RQ$ equals $2\sqrt{2} : \sqrt{5}$.
STATEMENT-2 : In any triangle, bisector of an angle divides the triangle into two similar triangles. (2007)

a) Statement-1 is True, Statement-2 is True; Statement-2 is not a correct explanation for Statement-1
b) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1
c) Statement-I is True, Statement-2 is False
d) Statement-1 is False, Statement-2 is True.

4.13.3. The area enclosed within the curve $|x| + |y| = 1$ is ______. (1981)

4.13.4. $y = 10^x$ is the reflection of $y = \log x$ in the line whose equation is ______. (1982)

4.13.5. The set of lines $ax + by + c = 0$, where $3a + 2b + 4c = 0$ are concurrent at the point ______. (1982)

4.13.6. Given the points $\mathbf{A}(0, 4)$ and $\mathbf{B}(0, -4)$,the equation of the locus of the point $\mathbf{p}(x, y)$,such that $|AP - BP| = 6$ is ______. (1983)

4.13.7. If a, b and c are in A.P, then the straight line $ax + by + c = 0$ will always pass through a fixed point whose coordinates are ______. (1984)

4.13.8. The orthocentre of the triangle formed by the lines $x + y = 1, 2x + 3y = 6$ and $4x - y + 4 = 0$ lies in the quadrant number ______. 1985

4.13.9. Let the algebric sum of the perpendicular distances from the points $(2, 0), (0, 2)$ and $(1, 1)$ to a variable straight line be zero; then the line passes through a fixed point whose coordinates are ______. (1991)

4.13.10. The vertices of a triangle are $\mathbf{A}(-1, -7)$, $\mathbf{B}(5, 1)$ and $\mathbf{C}(1, 10)$. The equation of the bisector of $\angle ABC$ is ______. (1993)

4.13.11. For a point $\mathbf{P}$ in the plane, let $d_1(\mathbf{P})$ and $d_2(\mathbf{P})$ be the distance of the point $\mathbf{P}$ from the lines $x - y = 0$ and $x + y = 0$ respectively. The area of the region R consisting of all points $\mathbf{P}$ lying in the first quadrant of the plane and satisfying $2 \leq d_1(\mathbf{P}) + d_2(\mathbf{P}) \leq 4$, is ______. (2014)

4.13.12. Let $\mathbf{P} = (-1, 0)$, $\mathbf{Q} = (0, 0)$ and $\mathbf{R} = \left(3, 3\sqrt{3}\right)$ be three points. The equation of the bisector of the angle PQR is (2007)

a) $\frac{\sqrt{3}}{2}x + y = 0$ b) $x + \sqrt{3}y = 0$ c) $\sqrt{3}x + y = 0$ d) $x + \frac{\sqrt{3}}{2}y = 0$

4.13.13. If one of the lines of $my^2 + (1 - m^2)xy - mx^2 = 80$ is a bisector of the angle between the lines $xy = 0$, then m is (2007)

a) 1 b) 2 c) $-\frac{1}{2}$ d) -2

4.13.14. The perpendicular bisector of the line segment joining $\mathbf{P}(1, 4)$ and $\mathbf{Q}(k, 3)$ has Y intercept -4. Then a possible value of k is (2008)

a) 1 b) 2 c) -2 d) -4

4.13.15. The shortest distance between the line $y - x = 1$ and the curve $x = y^2$ is (2009)

a) $\frac{2\sqrt{3}}{8}$ b) $\frac{3\sqrt{2}}{5}$ c) $\frac{\sqrt{3}}{4}$ d) $\frac{3\sqrt{2}}{8}$

4.13.16. The lines $p\left(p^2 + 1\right)x - y + q = 0$ and $\left(p^2 + 1\right)^2 x + \left(p^2 + 1\right)y + 2q = 0$ are perpendicular to a common line for (2009)

a) exactly one values of p
b) exactly two values of p
c) more than two values of p
d) no value of p

4.13.17. Three distinct points $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are given in the 2-dimensional coordinate plane such that the ratio of the distance of any one of them from the point $(1, 0)$ to the distance from the point $(-1, 0)$ is equal to $\frac{1}{3}$. Then the circumcentre of the triangle ABC is at the point: (2009)

a) $\left(\frac{5}{4}, 0\right)$ b) $\left(\frac{5}{2}, 0\right)$ c) $\left(\frac{5}{3}, 0\right)$ d) $(0, 0)$

4.13.18. The line L given by $\frac{x}{5} + \frac{y}{b} = 1$ passes through the point $(13, 32)$. The line K is parallel to the line L and has the equation $\frac{x}{c} + \frac{y}{3} = 1$. Then the distance between L and K is (2010)

a) $\sqrt{17}$ b) $\frac{17}{\sqrt{15}}$ c) $\frac{23}{\sqrt{17}}$ d) $\frac{23}{\sqrt{15}}$

4.13.19. If the line $2x+y=k$ passes through the point which divides the line segment joining the points $(1,1)$ and $(2,4)$ in the ratio 3:2, then k equals (2012)

a) $\frac{29}{5}$ b) 5 c) 6 d) $\frac{11}{5}$

4.13.20. A ray of light along $x+\sqrt{3}y=\sqrt{3}$ gets reflected upon reaching the X axis, the equation of the reflected ray is (2013)

a) $y=x+\sqrt{3}$
b) $\sqrt{3}y=x-\sqrt{3}$
c) $y=\sqrt{3}x-\sqrt{3}$
d) $\sqrt{3}y=x-1$

4.13.21. The X coordinate of the incentre of the triangle that has the coordinates of mid points of its sides as $(0,1)$, $(1,1)$ and $(1,0)$ is (2013)

a) $2+\sqrt{2}$ b) $2-\sqrt{2}$ c) $1+\sqrt{2}$ d) $1-\sqrt{2}$

4.13.22. Let PS be the median of the triangle with vertices $\mathbf{P}(2,2)$, $\mathbf{Q}(6,-1)$ and $\mathbf{R}(7,3)$. The equation of the line passing through $(1,-1)$ and parallel to PS is (2014)

a) $4x+7y+3=0$
b) $2x-9y-11=0$
c) $4x-7y-11=0$
d) $2x+9y+7=0$

4.13.23. Let a,b,c and d be non-zero numbers. If the point of intersection of the lines $4ax+2ay+c=0$ and $5bx+2by+d=0$ lies in the fourth quadrant and is equidistant from the two axes then (2014)

a) $3bc-2ad=0$
b) $3bc+2ad=0$
c) $2bc-3ad=0$
d) $2bc+3ad=0$

4.13.24. The number of points, having both co-ordinates as integers, that lie in the interior of the triangle with vertices $(0,0)$, $(0,41)$ and $(41,0)$ is (2015)

a) 820 b) 780 c) 901 d) 861

4.13.25. Two sides of a rhombus are along the lines, $x-y+1=0$ and $7x-y-5=0$. If its diagonals intersect at $(-1,-2)$, then which one of the following is a vertex of this rhombus? (2016)

a) $\left(\frac{1}{3},-\frac{8}{3}\right)$ b) $\left(-\frac{10}{3},-\frac{7}{3}\right)$ c) $(-3,-9)$ d) $(-3,-8)$

4.13.26. A straight the through a fixed point $(2,3)$ intersects the coordinate axes at distinct points $\mathbf{P}$ and $\mathbf{Q}$. If $\mathbf{O}$ is the origin and the rectangle $OPRQ$ is completed, then the locus of R is (2018)

a) $2x + 3y = xy$

b) $3x + 2y = xy$

c) $3x + 2y = 6xy$

d) $3x + 2y = 6$

4.13.27. Consider the set of all lines $px + qy + r = 0$ such that $3p + 2q + 4r = 0$. Which one of the following statements is true? (2019)

a) The lines are concurrent at the point $\left(\frac{3}{4}, \frac{1}{2}\right)$

b) Each line passes through the origin.

c) The lines are all parallel.

d) The lines are not concurrent.

4.13.28. Slope of a line passing through $\mathbf{P}(2, 3)$ and intersecting the line $x+y = 7$ at a distance of 4 units from $\mathbf{P}$, is (2019)

a) $\frac{1-\sqrt{5}}{1+\sqrt{5}}$

b) $\frac{1-\sqrt{7}}{1+\sqrt{7}}$

c) $\frac{\sqrt{7}-1}{\sqrt{7}+1}$

d) $\frac{\sqrt{5}-1}{\sqrt{5}+1}$

4.13.29. The straigth lines $x + y = 0$, $3x + y - 4 = 0$, $x + 3y - 4 = 0$ form a triangle which is (1983)

a) isosceles

b) equilateral

c) right angled

d) none of these

4.13.30. If $\mathbf{P} = (1, 0)$, $\mathbf{Q} = (-1, 0)$ and $\mathbf{R} = (2, 0)$ are three given points, then the locus of point $\mathbf{S}$ satisfing the relation $SQ^2 + SR^2 = 2SP^2$, is (1988)

a) a straight line parallel to X axis

b) a circle passing through the origin

c) a circle with the center at the origin

d) a straight line parallel to Y axis

4.13.31. Line L has intercepts a and b on the coordinate axes. When the axes are rotated through a given angle, keeping the origin fixed, line L has intercepts p and q. Then (1990)

a) $a^2 + b^2 = p^2 + q^2$

b) $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{p^2} + \frac{1}{q^2}$

c) $a^2 + p^2 = b^2 + q^2$

d) $\frac{1}{a^2} + \frac{1}{p^2} = \frac{1}{b^2} + \frac{1}{q^2}$

4.13.32. If the sum of distances of a point from two perpendicular lines in a plane is 1, then its locus is (1992)

a) square

b) straight line

c) circle

d) two intersecting lines

4.13.33. The locus of a variable point whose distance from $(-2, 0)$ is $\frac{2}{3}$ times its distance from the line $x = -\frac{9}{2}$ is (1994)

a) ellipse
b) hyperbola
c) parabola
d) none of these

4.13.34. The equations to a pair of opposite sides of parallogrqam are $x^2 - 5x + 6 = 0$ and $y^2 - 6y + 5 = 0$, the equations to its diagnaols are (1994)

a) $x + 4y = 13, y = 4x - 7$
b) $4x + y = 13, y = 4x - 7$
c) $4x + y = 13, 4y = x - 7$
d) $y - 4x = 13, y + 4x = 7$

4.13.35. The orthocenter of the triangle formed by the lines $xy = 0$ and $x + y = 1$ is (1995)

a) $\left(\frac{1}{2}, \frac{1}{2}\right)$
b) $\left(\frac{1}{3}, \frac{1}{3}\right)$
c) $(0, 0)$
d) $\left(\frac{1}{4}, \frac{1}{4}\right)$

4.13.36. Let PQR be a right angled triangle, right at $\mathbf{P}(2, 1)$. If the equation of the line QR is $2x + y = 3$, then the equation representing the pair of lines PQ and PR is (1990)

a) $3x^2 - 3y^2 + 8xy + 20x + 10y + 25 = 0$
b) $3x^2 - 3y^2 + 8xy - 20x - 10y + 25 = 0$
c) $3x^2 - 3y^2 + 8xy + 10x + 15y + 20 = 0$
d) $3x^2 - 3y^2 - 8xy - 10x - 15y - 20 = 0$

4.13.37. If x_1,x_2,x_3 as well as y_1,y_2,y_3, are in G.P with the same common ratio then the points (x_1,y_1),(x_2,y_2) and (x_3,y_3). (1999)

a) lie on a straight line
b) lie on ellipse
c) lie on circle
d) are vertices of a triangle

4.13.38. Let PS be the median of the triangle with vertices $\mathbf{P}(2, 2)$, $\mathbf{Q}(6, -1)$ and $\mathbf{R}(7, 3)$. The equation of the line passing through $(1, -1)$ and parallel to PS is (2000)

a) $2x - 9y - 7 = 0$
b) $2x - 9y - 11 = 0$
c) $2x + 9y - 11 = 0$
d) $2x + 9y + 7 = 0$

4.13.39. The incentre of the triangle with vertices $\left(1, \sqrt{3}\right)$, $(0, 0)$ and $(2, 0)$ is (2000)

a) $\left(1, \frac{\sqrt{3}}{2}\right)$
b) $\left(\frac{2}{3}, \frac{1}{\sqrt{3}}\right)$
c) $\left(\frac{2}{3}, \frac{\sqrt{3}}{2}\right)$
d) $\left(1, \frac{1}{\sqrt{3}}\right)$

4.13.40. the number of integer values of m, for which the x-coordinate of the point of intersection of the lines $3x + 4y = 9$ and $y = mx + 1$ is also an integer, is (2001)

a) 2
b) 0
c) 4
d) 1

4.13.41. Area of the parallelogram formed by the lines $y = mx$, $y = mx + 1$, $y = nx$ and $y = nx + 1$ equals (2001)

a) $\frac{|m+n|}{(m-n)^2}$ b) $\frac{2}{|m+n|}$ c) $\frac{1}{|m+n|}$ d) $\frac{1}{|m-n|}$

4.13.42. Let $0 < \alpha < \frac{\pi}{2}$ be a fixed angle. If $\mathbf{P} = (\cos\theta, \sin\theta)$ and $\mathbf{Q} = (\cos(\alpha-\theta), \sin(\alpha-\theta))$, then $\mathbf{Q}$ is obtained from $\mathbf{P}$ by (2002)

a) clockwise rotation around origin through an angle α
b) anticlockwise rotation around the orgin through an angle α
c) reflection in the line through origin with the slope $\tan\alpha$
d) reflection in the line through origin with slope $\tan\left(\frac{\alpha}{2}\right)$

4.13.43. Let $\mathbf{P} = (-1, 0)$,$\mathbf{Q} = (0, 0)$ and $\mathbf{R} = \left(3, \sqrt{3}\right)$ be three points. Then the equation of the bisector of the angle PQR is (2002)

a) $\frac{\sqrt{3}}{2}x + y = 0$ c) $\sqrt{3} + y = 0$
b) $x + \sqrt{3}y = 0$ d) $x + \frac{\sqrt{3}}{2}y = 0$

4.13.44. A straigth line through the origin $\mathbf{O}$ meets the parallel lines $4x+2y = 9$ and $2x+y+6 = 0$ at points $\mathbf{P}$ and $\mathbf{Q}$ respectively. Then the point $\mathbf{O}$ divides the segment PQ in the ratio (2002)

a) 12 b) 34 c) 21 d) 43

4.13.45. a) Two vertices of a triangle are $(5, -1)$ and $(2, -3)$. If the orthocentre of the triangle is the origin, find the coordinates of the third point.
b) Find the equation of the line which bisects the obtuse angle between the lines $x - 2y + 4 = 0$ and $4x - 3y + 2 = 0$.

(1979)

4.13.46. A stright line L is perpendicular to the line $5x-y+1$. The area of the triangle formed by L and the coordinate axes is 5. Find the equation of the line L. (1980)

4.13.47. The ends $\mathbf{A}$, $\mathbf{B}$ of a straight line segment of constant lenght c slide upon the fixed rectangular axes OX, OY respectively. If the rectangle $OAPB$ be completed, then show that the locus of the foot of perpendicular drawn from $\mathbf{P}$ to AB is

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = c^{\frac{2}{3}}$$

4.13.48. The vertices of a triangle are $[at_1t_2, a(t_1+t_2)]$, $[at_2t_3, a(t_2+t_3)]$, $[at_3t_1, a(t_3+t_1)]$. Find the orthocentre of the triangle. (1983)

4.13.49. The coordinates of $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are $(6, 3), (3, 5), (4, 2)$ respectively, and $\mathbf{P}$ is any point (x, y). Show that the ratio of the area of $\triangle$s PBC and ABC is $\left|\frac{x+y-2}{7}\right|$ (1983)

4.13.50. Two equal sides of an isosceles triangle are given by the equations $7x - y + 3 = 0$ and $x + y - 3 = 0$ and its third side passes through the point $(1, -10)$. Determine the equation of the third side. (1985)

4.13.51. One of the diameters of the circle circumscribing the rectangle $ABCD$ is $4y = x + 7$. If $\mathbf{A}$ and $\mathbf{B}$ are the points $(-3, 4)$ and $(5, 4)$ respectively, the find the area of rectangle. (1985)

4.13.52. Two sides of a rhombus $ABCD$ are parallel to the lines $y = x + 2$ and $y = 7x + 3$. If the diagonals of the rhombus intersect at the point $(1, 2)$ and the vertex $\mathbf{A}$ is on the Y axis, find the possible co-ordinates of $\mathbf{A}$. (1985)

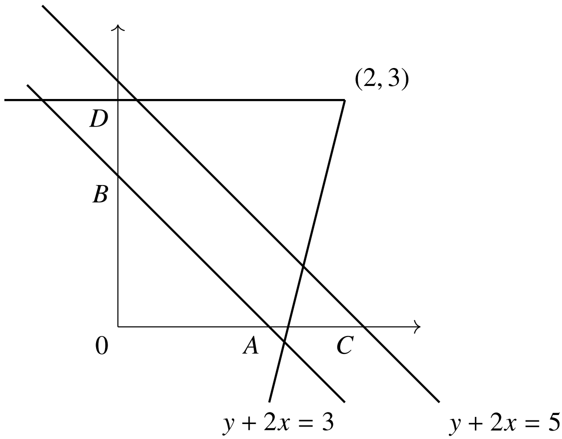
4.13.53. Lines $L_1 \equiv ax+by+c=0$ and $L_2 \equiv lx+my+n=0$ intersect at the point **P** and make an angle θ with each other. Find the equation of a line L different from L_2 which passes through **P** and makes same angle θ with L_1. (1988)

4.13.54. Let ABC be the triangle $AB = AC$. If **D** is the midpoint of BC, **E** is the foot of the perpendicular drawn from **D** to AC and **F** the mid-point of DE, prove that AF is perpendicular to BE. (1989)

4.13.55. Straight lines $3x+4y=5$ and $4x-3y=15$ intersect at the point **A**. Points **B** and **C** are chosen on these two lines such that $AB = AC$. Determine the possible equations of the line BC passing through the point $(1, 2)$. (1990)

4.13.56. A line cuts the X axis at **A**$(7, 0)$ and the Y axis at **B**$(0, -5)$. A variable line PQ is drawn perpendicular to AB cutting the X axis in **P** and the Y axis in **Q**. If AQ and BP intersect at **R**, find the locus of **R**. (1990)

4.13.57. Find the equation of the line passing through the point $(2, 3)$ and making intercept of length 2 units between the lines $y+2x=3$ and $y+2x=5$. (1991)



4.13.58. Show that all chords of the curve $3x^2-y^2-2x+4y=0$, which subtend a right angle at the origin, pass through a fixed point. Find the coordinates of the point. (1991)

4.13.59. Determine all values of α for which the point (α, α^2) lies inside the triangle formed by the lines. (1992)

$$2x+3y-1=0$$
$$x+2y-3=0$$
$$5x-6y-1=0$$

4.13.60. A line through **A** $(5, 4)$ meets the lines $x+3y+2=0$, $2x+y+4=0$ and $x-y-5=0$ at the points **B**, **C** and **D** respectively. If $\left(\frac{15}{AB}\right)^2+\left(\frac{10}{AC}\right)^2-\left(\frac{6}{AD}\right)^2$, find the equation of the line. (1993)

4.13.61. A rectangle $PQRS$ has its side PQ parallel to the line $y=mx$ and vertices **P**, **Q** and **S** on the lines $y=a$, $x=b$ and $x=-b$, respectively. Find the locus of the vertex **R**. (1996)

4.13.62. For points $\mathbf{P} = (x_1, y_1)$ and $\mathbf{Q} = (x_2, y_2)$ of the co-ordinate plane, a new distance $d(\mathbf{P}, \mathbf{Q})$ is defined by $d(\mathbf{P}, \mathbf{Q}) = |x_1-x_2| + |y_1-y_2|$. Let $\mathbf{O} = (0, 0)$ and $\mathbf{A} = (3, 2)$.

Prove that the set of points in the first quadrant which are equidistant (with respect to the new distance) from **O** and **A** consists of the union of a line segment of finite length and an infinite ray. Sketch this net in a labelled diagram. (2000)

4.13.63. Let ABC and PQR be any two triangles in the same plane. Assume that the perpendiculars from the points $\mathbf{A}, \mathbf{B}, \mathbf{C}$ to the sides QR, RP, PQ respectively are concurrent. Using vector methods or otherwise, prove that the perpendiculars from $\mathbf{P}, \mathbf{Q}, \mathbf{R}$ to BC, CA, AB respectively are also concurrent. (2000)

4.13.64. Let a, b, c be real numbers with $a^2 + b^2 + c^2 = 1$. Show that the equation

$$\begin{vmatrix} ax - by - c & bx + ay & cx + a \\ bx + ay & -ax + by - c & cy + b \\ cx + a & cy + b & -ax - by + c \end{vmatrix} = 0$$

represents a straight line. (2001)

4.13.65. A straight line L through the origin meets the lines $x + y = 1$ and $x + y = 3$ at **P** and **Q** respectively. Through **P** and **Q** two straight lines L_1, and L_2 are drawn, parallel to $2x - y = 5$ and $3x + y = 5$ respectively. Lines L_1 and L_2 intersect at **R**. Show that the locus of **R** as L varies, is a straight line. (2002)

4.13.66. A straight line L with negative slope passes through the point $(8, 2)$ and cuts the positive coordinate axes at points **P** and **Q**. Find the absolute minimum value of $OP + OQ$, as L varies, where **O** is the origin. (2002)

4.13.67. The area of the triangle formed by the intersection of a line parallel to X axis and passing through $\mathbf{P}(h, k)$ with the lines $y = x$ and $x + y = 2$ is $4h^2$. Find the locus of the point **P**. (2005)

4.13.68. The straight line $5x + 4y = 0$ passes through the point of intersection of the straight lines $x + 2y - 10 = 0$ and $2x + y + 5 = 0$. (1983)

4.13.69. The lines $2x + 3y + 19 = 0$ and $9x + 6y - 17 = 0$ cut the coordinates axes in concylic points. (1988)

4.13.70. If

$$\begin{pmatrix} a & a^2 & 1 + a^3 \\ b & b^2 & 1 + b^3 \\ c & c^2 & 1 + c^3 \end{pmatrix} = 0$$

and the vectors $\mathbf{A}$=(1,a,a^2), $\mathbf{B}$=(1,b,b^2), $\mathbf{C}$=(1,c,c^2) are co-planar, then the product abc=______. (1985)

4.13.71. If the vectors $a\hat{\imath} + \hat{\jmath} + \hat{k}, \hat{\imath} + b\hat{\jmath} + \hat{k}$ and $\hat{\imath} + \hat{\jmath} + c\hat{k}$ $(a \neq b \neq c \neq 1)$ are co-planar, then the value of the $\frac{1}{(1-a)} + \frac{1}{(1-b)} + \frac{1}{(1-c)}$=______. (1987)

4.13.72. A non-zero vector **a** is parallel to the line of intersection of the plane determined by the vectors $\hat{\imath}$, $\hat{\imath} + \hat{\jmath}$ and the plane determined by the vectors $\hat{\imath} - \hat{\jmath}$, $\hat{\imath} + \hat{k}$. The angle between **a** and the vector $\hat{\imath} - 2\hat{\jmath} + 2\hat{k}$ is ______. (1996)

4.13.73. Three lines

$$L_1 : \mathbf{r} = \lambda\hat{\imath}, \lambda \in R$$

$$L_2 : \mathbf{r} = \hat{k} + \mu\hat{\jmath}, \mu \in R$$

and

$$L_3 : \mathbf{r} = \hat{\imath} + \hat{\jmath} + \nu\hat{k}, \nu \in R$$

are given. For which point(s) **Q** on L_2 can we find a point **P** on L_1 and a point **R** on L_3 so that **P**, **Q** and **R** are collinear? (2019)

a) $\hat{k} - \frac{1}{2}\hat{j}$ b) $\hat{k}$ c) $\hat{k} + \hat{j}$ d) $\hat{k} + \frac{1}{2}\hat{j}$

4.13.74. If the straight lines $\frac{x-1}{2} = \frac{y+1}{k} = \frac{z}{2}$ and $\frac{x+1}{5} = \frac{y+1}{2} = \frac{z}{k}$ are coplanar, then the plane(s) containing these two lines is(are) (2012)

a) $y + 2z = -1$ b) $y + z = -1$ c) $y - z = -1$ d) $y - 2z = -1$

4.13.75. A line L passing through the origin is perpendicular to the lines

$$l_1 : (3 + t)\hat{\imath} + (1 + 2t)\hat{\jmath} + (4 + 2t)\hat{k}, \quad -\infty < t < \infty$$
$$l_2 : (3 + 2s)\hat{\imath} + (3 + 2s)\hat{\jmath} + (2 + s)\hat{k}, \quad -\infty < s < \infty$$

Then, the coordinate(s) of the point(s) on l_2 at a distance of $\sqrt{17}$ from the point of intersection of L and l_1 is (are) (2013)

a) $\left(\frac{7}{3}, \frac{7}{3}, \frac{5}{3}\right)$ b) $(-1, -1, 0)$ c) $(1, 1, 1)$ d) $\left(\frac{7}{9}, \frac{7}{9}, \frac{8}{9}\right)$

4.13.76. Two lines

$$L_1 : x = 5, \frac{y}{3 - \alpha} = \frac{z}{-2}$$

and

$$L_2 : x = \alpha, \frac{y}{-1} = \frac{z}{2 - \alpha}$$

are coplanar. Then α can take value(s) (2013)

a) 1 b) 2 c) 3 d) 4

4.13.77. From a point $\mathbf{P}(\lambda, \lambda, \lambda)$, perpendiculars PQ and PR are drawn respectively on the lines $y = x, z = 1$ and $y = -x, z = -1$. If $\angle QPR$ is a right angle, then the possible value(s) of λ is/(are) (2014)

a) $\sqrt{2}$ b) 1 c) -1 d) - $\sqrt{2}$

4.13.78. In R^3, consider the planes $P_1 : y = 0$ and $P_2 : x + z - 1$. Let P_3 be the plane different from P_1 and P_2 which passes through the intersection of P_1 and P_2. If the distance of the point $(0, 1, 0)$ from P_3 is 1 and the distance of point (α, β, γ) from P_3 is 2, then which of the following relations is (are) true (2015)

a) $2\alpha + \beta + 2\gamma + 2 = 0$ c) $2\alpha + \beta + 2\gamma - 10 = 0$
b) $2\alpha - \beta + 2\gamma + 4 = 0$ d) $2\alpha - \beta + 2\gamma - 8 = 0$

4.13.79. In R^3, let L be a straight line passing through the origin. Suppose that all the points on L are at a constant distance from two planes $P_1 : x + 2y - z + 1 = 0$ and $P_2 : 2x - y + z - 1 = 0$. Let M be the locus of the feet of the perpendicular drawn from the points on L to the plane P_1. Which of the following points lie(s) on M? (2015)

a) $\left(0, -\frac{5}{6}, -\frac{2}{3}\right)$
b) $\left(-\frac{1}{6}, -\frac{1}{3}, \frac{1}{6}\right)$
c) $\left(-\frac{5}{6}, 0, \frac{2}{3}\right)$
d) $\left(-\frac{1}{3}, 0, \frac{2}{3}\right)$

4.13.80. Consider a pyramid $OPQRS$ located in the first octant $(x \geq 0, y \geq 0, z \geq 0)$ with **O** as origin, and OP and OR along the X axis and the Y axis respectively. The base $OPQR$ of the pyramid is a square with $OP = 3$. The point **S** is directly above the mid-point **T** of diagonal OQ such that TS=3. Then (2016)

a) the acute angle between OQ and OS is $\frac{\pi}{3}$
b) the equation of the plane containg the triangle OQS is $x - y = 0$
c) the length of the perpendicular from **P** to the plane containing the triangle OQS is $\frac{3}{\sqrt{2}}$
d) the perpendicular distance from **O** to the straight line containing RS is $\sqrt{\frac{15}{2}}$

4.13.81. Let $P_1 : 2x + y - z = 3$ and $P_2 : x + 2y + z = 2$ be two planes. Then, which of the following statement(s) is(are) TRUE? (2018)

a) The line of intersection of P_1 and P_2 has direction ratios 1,2,-1
b) The line $\frac{3x-4}{9} = \frac{1-3y}{9} = \frac{z}{3}$ is perpendicular to the line of intersection of P_1 and P_2
c) The acute angle between P_1 and P_2 is $60°$.
d) If P_3 is the plane passing through the point $(4, 2, -2)$ and perpendicular to the line of intersection of P_1 and P_2, then the distance of the point $(2, 1, 1)$ from the plane P_3 is $\frac{2}{\sqrt{3}}$

4.13.82. Let L_1 and L_2 denote the lines
$\mathbf{r} = \hat{i} + \lambda\left(-\hat{i} + 2\hat{j} + 2\hat{k}\right), \lambda \in R$ and
$\mathbf{r} = \mu\left(2\hat{i} - \hat{j} + 2\hat{k}\right), \mu \in R$
respectively. If L_3 is a line which is perpendicular to both L_1 and L_2 and cuts both of them, then which of the following option describe(s) L_3? (2019)

a) $\mathbf{r} = \frac{2}{9}\left(4\hat{i} + \hat{j} + \hat{k}\right) + t\left(2\hat{i} + 2\hat{j} - \hat{k}\right), t \in R$
b) $\mathbf{r} = \frac{2}{9}\left(2\hat{i} - \hat{j} + 2\hat{k}\right) + t\left(2\hat{i} + 2\hat{j} - \hat{k}\right), t \in R$
c) $\mathbf{r} = t\left(2\hat{i} + 2\hat{j} - \hat{k}\right), t \in R$
d) $\mathbf{r} = \frac{1}{3}\left(2\hat{i} + \hat{k}\right) + t\left(2\hat{i} + 2\hat{j} - \hat{k}\right), t \in R$

4.13.83. Let a, b, c be distinct non-negative numbers. If the vectors $a\mathbf{i} + a\mathbf{j} + c\mathbf{k}$, $\mathbf{i} + \mathbf{k}$ and $c\mathbf{i} + c\mathbf{j} + b\mathbf{k}$ lie in a plane, then **c** is

a) the Arithmetic Mean of a and b
b) the Geometric Mean of a and b
c) the Harmonic Mean of a and b
d) equal to zero

(1993)

4.13.84. The value of k such that $\frac{x-4}{1} = \frac{y-2}{1} = \frac{z-k}{2}$ lies in the plane $2x - 4y + z = 7$, is (2003)

a) 7
b) -7
c) no real value
d) 4

4.13.85. If the lines $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-1}{4}$ and $\frac{x-3}{1} = \frac{y-k}{2} = \frac{z}{1}$ intersect, then the value of k is (2005)

a) $\frac{3}{2}$ b) $\frac{9}{2}$ c) $\frac{2}{9}$ d) $\frac{-3}{2}$

4.13.86. A plane which is perpendicular to two planes $2x - 2y + z = 0$ and $x - y + 2z = 4$ passes through $(1, -2, 1)$. The distance of the plane from the point $(1, 2, 2)$ is (2006)

a) 0 b) 1 c) $\sqrt{2}$ d) $2\sqrt{2}$

4.13.87. Let **P**$(3, 2, 6)$ be a point in space and **Q** be a point on the line

$$\mathbf{r} = (\hat{\imath} - \hat{\jmath} + 2\hat{k}) + \mu(-3\hat{\imath} + \hat{\jmath} + 5\hat{k}.$$

Then the value of μ for which the vector $\overrightarrow{PQ}$ is parallel to the plane $x - 4y + 3z = 1$ is (2009)

a) $\frac{1}{4}$ b) $-\frac{1}{4}$ c) $\frac{1}{8}$ d) $-\frac{1}{8}$

4.13.88. A line with positive direction cosines passes through the point **P**$(2, -1, 2)$ and makes equal angles with the coordinate axes. The line meets the plane $2x + y + z = 9$ at point **Q**. The length of the line segment PQ equals (2009)

a) 1 b) $\sqrt{2}$ c) $\sqrt{3}$ d) 2

4.13.89. Equation of the plane containing the straight line $\frac{x}{2} = \frac{y}{3} = \frac{z}{4}$ and perpendicular to the plane containing the straight lines $\frac{x}{3} = \frac{y}{4} = \frac{z}{2}$ and $\frac{x}{4} = \frac{y}{2} = \frac{z}{3}$ is (2010)

a) $x + 2y - 2z = 0$
b) $3x + 2y - 2z = 0$
c) $x - 2y + z = 0$
d) $5x + 2y - 4z = 0$

4.13.90. If the distance of the point **P**$(1, -2, 1)$ from the plane $x + 2y - 2z = \alpha$, where $\alpha > 0$, is 5, then the foot of the perpendicular from **P** to the plane is (2010)

a) $\left(\frac{8}{3}, \frac{4}{3}, \frac{-7}{3}\right)$ b) $\left(\frac{4}{3}, \frac{-4}{3}, \frac{1}{3}\right)$ c) $\left(\frac{1}{3}, \frac{2}{3}, \frac{10}{3}\right)$ d) $\left(\frac{2}{3}, \frac{-1}{3}, \frac{5}{2}\right)$

4.13.91. The point **P** is the intersection of the straight line joining the points **Q**$(2, 3, 5)$ and **R**$(1, -1, 4)$ with the plane $5x - 4y - z = 1$. If **S** is the foot of the perpendicular drawn from the point **T**$(2, 1, 4)$ to QR, then the length of the line segment PS is (2010)

a) $\frac{1}{\sqrt{2}}$ b) $\sqrt{2}$ c) 2 d) $2\sqrt{2}$

4.13.92. The equation of a plane passing through the line of intersection of the planes $x + 2y + 3z = 2$ and $x - y + z = 3$ and at a distance $\frac{2}{\sqrt{3}}$ from the point $(3, 1, -1)$ is (2012)

a) $5x - 11y + z = 17$ c) $x + y + z = \sqrt{3}$
b) $\sqrt{2}x + y = 3\sqrt{2} - 1$ d) $x - \sqrt{2}y = 1 - \sqrt{2}$

4.13.93. Let **P** be the image of the point $(3, 1, 7)$ with respect to the plane $x - y + z = 3$. Then the equation of the plane passing through **P** and containing the straight line $\frac{x}{1} = \frac{y}{2} = \frac{z}{1}$ is (2016)

a) $x + y - 3z = 0$ c) $x - 4y + 7z = 0$
b) $3x + z = 0$ d) $2x - y = 0$

4.13.94. The equation of the plane passing through the point $(1, 1, 1)$ and perpendicular to the planes $2x + y - 2z = 5$ and $3x - 6y - 2z = 7$ is (2017)

a) $14x + 2y - 15z = 1$ c) $14x + 2y + 15z = 31$
b) $14x - 2y + 15z = 27$ d) $-14x + 2y + 15z = 3$

4.13.95. Let L_1 and L_2 be the following straight lines

$$L_1 : \frac{x-1}{1} = \frac{y}{-1} = \frac{z-1}{3}, \quad L_2 : \frac{x-1}{-3} = \frac{y}{-1} = \frac{z-1}{1} \tag{4.13.95.1}$$

Suppose the straight line

$$L : \frac{x-a}{l} = \frac{y-1}{m} = \frac{z-y}{-2} \tag{4.13.95.2}$$

lies in the plane containing L_1 and L_2 and passes through the point of intersection of L_1 and L_2. If the line L bisects the acute angle between the lines L_1 and L_2, then which of the following statements is/are TRUE? (2020)

a) $\alpha - \gamma = 3$ c) $\alpha - \gamma = 1$
b) $l + m = 2$ d) $l + m = 0$

4.13.96. The area of the region $\{(x, y) : 0 \le x \le 9, 0 \le y \le 1, x \ge 3y, x + y \ge 2\}$ is (2021)

a) $\frac{11}{32}$ b) $\frac{35}{96}$ c) $\frac{37}{96}$ d) $\frac{13}{32}$

4.13.97. Let α, β, and γ be real numbers such that the system of linear equations

$$x + 2y + 3z = \alpha, \quad 4x + 5y + 6z = \beta, \quad 7x + 8y + 9z = \gamma - 1$$

is consistent. Let $|\mathbf{M}|$ represent the determinant of the matrix

$$\mathbf{M} = \begin{pmatrix} \alpha & 2 & \gamma \\ \beta & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

Let P be the plane containing all those $(\alpha, \beta, \gamma)$for which the above system of linear equations is consistent, and Let D be the square of the distance of the point $(0, 1, 0)$ from the plane P. (2021)

a) the value of $|\mathbf{M}|$ is: ______
b) the value of D is ______

4.13.98. Consider the lines L_1 and L_2 defined by

$$L_1 : x\sqrt{2} + y - 1 = 0, \quad L_2 : x\sqrt{2} - y + 1 = 0$$

For a fixed constant λ, let C be the locus of a point P such that the product of the distance of P from L_1 and the distance of P from L_2 is λ^2. The line $y = 2x + 1$ meets C at two points R and S, where the distance between R and S is $\sqrt{270}$. Let the perpendicular bisector of RS meet C at two distinct points R' and S'. Let D be the square of the distance between R' and S'. (2021)

a) The value of λ^2 is ______
b) The value of D is ______

4.13.99. Let P_1 and P_2 be two planes given by

- $P_1 : 10x + 15y + 12z - 60 = 0,$
- $P_2 : -2x + 5y + 4z - 20 = 0.$

Which of the following straight lines can be an edge of some tetrahedron whose two faces lie on P_1 and P_2 (2022)

a) $\frac{x-1}{0} = \frac{y-1}{0} = \frac{z-1}{5}$
b) $\frac{x-6}{-5} = \frac{y}{2} = \frac{z}{3}$
c) $\frac{x}{-2} = \frac{y-4}{5} = \frac{z}{4}$
d) $\frac{x}{1} = \frac{y-4}{-2} = \frac{z}{3}$

4.13.100. Let $\mathbf{S}$ be the reflection of a point $\mathbf{Q}$ with respect to the plane given by

$$\vec{r} = -(t+p)\hat{i} + t\hat{j} + (1+p)\hat{k}$$

where t, p are real parameters and $\hat{i}, \hat{j}, \hat{k}$ are the unit vectors along the three positive coordinate axes. If the position vectors of $\mathbf{Q}$ and $\mathbf{S}$ are $10\hat{i}+15\hat{j}+20\hat{k}$ and $\alpha\hat{i}+\beta\hat{j}+\gamma\hat{k}$ respectively, then which of the following is/are TRUE ? (2022)

a) $3(\alpha+\beta) = -101$
b) $3(\beta+\gamma) = -71$
c) $3(\gamma+\alpha) = -86$
d) $3(\alpha+\beta+\gamma) = -121$

4.13.101. Let p, q amd r be nonzero real numbers that are the 10^{th}, 100^{th} and 1000^{th} terms of a harmonic progression, respectively. Consider the following system of linear equations

$$x + y + z = 1$$
$$10x + 100y + 1000z = 0$$
$$qrx + pry + pqz = 0$$

(I) If $\frac{q}{r} = 10$, then the system of linear equations has

(II) If $\frac{p}{r} \neq 100$, then the system of linear equations has

(III) If $\frac{p}{q} \neq 10$, then the system of linear equations has

(IV) If $\frac{p}{q} = 10$, then the system of linear equations has

(A) $x = 0, y = \frac{10}{9}, z = -\frac{1}{9}$ as a solution
(B) $x = \frac{10}{9}, y = -\frac{1}{9}, z = 0$ as a solution
(C) infinitely many solutions
(D) no solution
(E) at least one solution

The correct option is (2022)

a) $(I) \to (E); (II) \to (C); (III) \to (D); (IV) \to (E)$
b) $(I) \to (B); (II) \to (D); (III) \to (D); (IV) \to (C)$
c) $(I) \to (B); (II) \to (C); (III) \to (A); (IV) \to (C)$
d) $(I) \to (E); (II) \to (D); (III) \to (A); (IV) \to (E)$

4.13.102. Let α,β and γ be real numbers. Consider the following system of linear equations:

$$x + 2y + z = 7$$
$$x + \alpha z = 11$$
$$2x - 3y + \beta z = \gamma$$

Match each entry in the following (2023)

(A) If $\beta = \frac{1}{2}(7\alpha - 3)$ and $\gamma = 28$, then the system has
(B) If $\beta = \frac{1}{2}(7\alpha - 3)$ and $\gamma \neq 28$, then the system has
(C) If $\beta \neq \frac{1}{2}(7\alpha - 3)$ where $\alpha = 1$ and $\gamma \neq 28$, then the system has
(D) If $\beta \neq \frac{1}{2}(7\alpha - 3)$ where $\alpha = 1$ and $\gamma = 28$, then the system has

(1) a unique solution
(2) no solution
(3) infinitely many solutions
(4) $x = 11, \quad y = -2, \quad z = 0$ as a solution
(5) $x = -15, \quad y = 4, \quad z = 0$ as a solution

Correct options

a) (A) → (3), (B) → (2), (C) → (1), (D) → (4)
b) (A) → (3), (B) → (2), (C) → (5), (D) → (4)
c) (A) → (2), (B) → (1), (C) → (4), (D) → (5)
d) (A) → (2), (B) → (1), (C) → (3), (D) → (3)

4.13.103. a) Let ℓ_1 and ℓ_2 be the lines $\mathbf{r_1} = \lambda(\hat{i} + \hat{j} + \hat{k})$ and $\mathbf{r_2} = (\hat{j} - \hat{k}) + \mu(\hat{i} + \hat{k})$ respectively. Let X be the set of all the planes H that contain the line ℓ_1. For a plane H, let $d(H)$ denote the smallest possible distance between the points of ℓ_2 and H.

b) Let H_0 be a plane in X for which $d(H_0)$ is the maximum value of $d(H)$ as H varies over all planes in X.

Match each entry in the following

(A) The value of $d(H_0)$ is
(B) The distance of the point $(0, 1, 2)$ from H_0 is
(C) The distance of origin from H_0 is
(D) The distance of origin from the point of intersection of planes $y = z$, $x = 1$ and H_0 is

(1) $\sqrt{3}$
(2) $\frac{1}{\sqrt{3}}$
(3) 0
(4) $\sqrt{2}$
(5) $\frac{1}{\sqrt{2}}$

The correct option is (2023)

a) (A) → (2), (B) → (4), (C) → (5), (D) → (1)
b) (A) → (5), (B) → (4), (C) → (3), (D) → (1)
c) (A) → (2), (B) → (1), (C) → (3), (D) → (2)
d) (A) → (5), (B) → (1), (C) → (4), (D) → (2)

4.13.104. Let $\mathbb{R}^2$ denote the Euclidean space. Let $S = \{(a, b, c) : a, b, c \in \mathbb{R},\ ax^2 + 2bxy + cy^2 > 0 \forall (x, y) \in \mathbb{R}^2 - \{(0, 0)\}\}$. Then which of the following statements is (are) TRUE? (2024)

a) $\left(2, \frac{7}{2}, 6\right) \in S$
b) If $\left(3, b, \frac{1}{12}\right) \in S$, then $|2b| < 1$.
c) For any given $(a, b, c) \in S$, the system of linear equations

$$ax + by = 1$$
$$bx + cy = -1$$

has a unique solution.

d) For any given $(a, b, c) \in S$, the system of linear equations

$$(a + 1)\,x + by = 0$$
$$bx + (c + 1)\,y = 0$$

has a unique solution.

4.13.105. Let

$$\overrightarrow{OP} = \frac{\alpha - 1}{\alpha}\hat{i} + \hat{j} + \hat{k}, \quad \overrightarrow{OQ} = \hat{i} + \frac{\beta - 1}{\beta}\hat{j} + \hat{k}, \quad \overrightarrow{OR} = \hat{i} + \hat{j} + \frac{1}{2}\hat{k}$$

be three vectors, where $\alpha, \beta \in \mathbb{R} - \{0\}$ and O denotes the origin. If

$$\left(\overrightarrow{OP} \times \overrightarrow{OQ}\right) \cdot \overrightarrow{OR} = 0$$

and the point $(\alpha, \beta, 2)$ lies on the plane

$$3x + 3y - z + l = 0,$$

then the value of l is ______. (2024)

4.13.106. Let $\gamma \in \mathbb{R}$ be such that the lines

$$L_1 : \frac{x + 11}{1} = \frac{y + 21}{2} = \frac{z + 29}{3}$$

and

$$L_2 : \frac{x + 16}{3} = \frac{y + 11}{2} = \frac{z + 4}{\gamma}$$

intersect. Let R_1 be the point of intersection of L_1 and L_2. Let $\mathbf{O} = (0, 0, 0)$ and $\hat{n}$ denote a unit normal vector to the plane containing both the lines L_1 and L_2. Match the following

(A) γ equals	(1) $-\hat{i} - \hat{j} + \hat{k}$
(B) A possible choice for $\hat{n}$ is	(2) $\sqrt{\frac{3}{2}}$
(C) $\mathbf{OR}_1$ equals	(3) 1
(D) A possible value of $\mathbf{OR}_1 \cdot \hat{n}$ is	(4) $\frac{1}{\sqrt{6}}\hat{i} - \frac{2}{\sqrt{6}}\hat{j} + \frac{1}{\sqrt{6}}\hat{k}$
	(5) $\sqrt{\frac{2}{3}}$

The correct option is (2024)

a) $(A) \to (3)$ $(B) \to (4)$ $(C) \to (1)$ $(D) \to (2)$
b) $(A) \to (5)$ $(B) \to (4)$ $(C) \to (1)$ $(D) \to (2)$
c) $(A) \to (3)$ $(B) \to (4)$ $(C) \to (1)$ $(D) \to (5)$
d) $(A) \to (3)$ $(B) \to (1)$ $(C) \to (4)$ $(D) \to (5)$

5 Matrices

5.1 Formulae

5.1.1. The characteristic equation for a matrix $\mathbf{A}$ is

$$f(\lambda) = \left|\mathbf{A} - \lambda\mathbf{I}\right| = 0 \tag{5.1.1.1}$$

5.1.2. Cayley-Hamilton theorem

$$f(\lambda) = f(\mathbf{A}) = 0 \tag{5.1.2.1}$$

5.1.3. Code for Cayley-Hamilton Theorem

```
codes/book/cayley.py
```

5.1.4. Code for balancing chemical equations

```
codes/book/chembal.py
```

5.2 Equation

Solve the following system of linear equations.

5.2.1

$$\begin{aligned} 2x + 5y &= 1 \\ 3x + 2y &= 7 \end{aligned}$$

5.2.2

$$\begin{aligned} 5x - 4y + 8 &= 0 \\ 7x + 6y - 9 &= 0 \end{aligned}$$

5.2.3

$$\begin{aligned} 9x + 3y + 12 &= 0 \\ 18x + 6y + 24 &= 0 \end{aligned}$$

5.2.4

$$\begin{aligned} 6x - 3y + 10 &= 0 \\ 2x - y + 9 &= 0 \end{aligned}$$

5.2.5

$$\begin{aligned} 3x + 2y &= 5 \\ 2x - 3y &= 7 \end{aligned}$$

5.2.6

$$\begin{aligned} 2x - 3y &= 8 \\ 4x - 6y &= 9 \end{aligned}$$

5.2.7

$$\begin{aligned} \frac{3}{2}x + \frac{5}{3}y &= 7 \\ 9x - 10y &= 14 \end{aligned}$$

5.2.8

$$\begin{aligned} 5x - 3y &= 11 \\ -10x + 6y &= -22 \end{aligned}$$

5.2.9

$$\begin{aligned} \frac{4}{3}x + 2y &= 8 \\ 2x + 3y &= 12 \end{aligned}$$

5.2.10

$$\begin{aligned} x + y &= 5 \\ 2x + y &= 10 \end{aligned}$$

5.2.11

$$x - y = 8$$
$$3x - 3y = 16$$

5.2.12

$$2x + y - 6 = 0$$
$$4x - 2y + 4 = 0$$

5.2.13

$$2x - 2y - 2 = 0$$
$$4x - 4y - 5 = 0$$

5.2.14

$$x - 3y - 3 = 0$$
$$3x - 9y - 2 = 0$$

5.2.15

$$2x + y = 5$$
$$3x + 2y = 8$$

5.2.16

$$3x - 5y = 20$$
$$6x - 10y = 40$$

5.2.17

$$x - 3y - 7 = 0$$
$$3x - 3y - 15 = 0$$

5.2.18

$$8x + 5y = 9$$
$$3x + 2y = 4$$

5.2.19

$$x + y = 14$$
$$x - y = 4$$

5.2.20

$$3x - y = 3$$
$$9x - 3y = 9$$

5.2.21

$$02x + 03y = 13$$
$$04x + 05y = 23$$

5.2.22

$$\sqrt{2x} + \sqrt{3y} = 0$$
$$\sqrt{3x} - \sqrt{8y} = 0$$

5.2.23

$$\frac{3x}{2} - \frac{5y}{2} = -2$$
$$\frac{x}{3} + \frac{y}{2} = \frac{13}{6}$$

5.2.24

$$px + qy = p - q$$
$$qx - py = p + q$$

5.2.25

$$ax + by = c$$
$$bx + ay = 1 + c$$

5.2.26

$$\frac{x}{a} - \frac{y}{b} = 0$$
$$ax + by = a^2 + b^2$$

5.2.27

$$152x - 378y = -74$$
$$-378x + 152y = -604$$

5.2.28

$$5x - 8y + 1 = 0$$
$$3x - \frac{24}{5}y + \frac{3}{5} = 0$$

5.2.29

$$x + 3y = 6$$
$$2x - 3y = 12$$

5.2.30

$$\begin{aligned} 7x - 15y &= 2 \\ x + 2y &= 3 \end{aligned}$$

5.2.31

$$\begin{aligned} 2x + 3y &= 8 \\ 4x + 6y &= 7 \end{aligned}$$

5.2.32

$$\begin{aligned} x + 2y - 4 &= 0 \\ 2x + 4y - 12 &= 0 \end{aligned}$$

5.2.33

$$\begin{aligned} x + 2y - 4 &= 0 \\ 2x + 4y - 12 &= 0 \end{aligned}$$

5.2.34

$$\begin{aligned} x + y &= 5 \\ 2x - 3y &= 4 \end{aligned}$$

5.2.35

$$\begin{aligned} 3x + 4y &= 10 \\ 2x - 2y &= 2 \end{aligned}$$

5.2.36

$$\begin{aligned} 3x - 5y - 4 &= 0 \\ 9x &= 2y + 7 \end{aligned}$$

5.2.37

$$\begin{aligned} \frac{x}{2} + \frac{2y}{3} &= -1 \\ x - \frac{y}{3} &= 0 \end{aligned}$$

5.2.38

$$\begin{aligned} \frac{1}{2x} + \frac{1}{3y} &= 2 \\ \frac{1}{3x} + \frac{1}{2y} &= \frac{13}{6} \end{aligned}$$

5.2.39

$$\begin{aligned} \frac{2}{\sqrt{x}} + \frac{3}{\sqrt{y}} &= 2 \\ \frac{4}{\sqrt{x}} - \frac{9}{\sqrt{y}} &= -1 \end{aligned}$$

5.2.40

$$\begin{aligned} \frac{4}{x} + 3y &= 14 \\ \frac{3}{x} - 4y &= 23 \end{aligned}$$

5.2.41

$$\begin{aligned} \frac{5}{x-1} + \frac{1}{y-2} &= 2 \\ \frac{6}{x-1} - \frac{3}{y-2} &= 1 \end{aligned}$$

5.2.42

$$\begin{aligned} \frac{7x - 2y}{xy} &= 5 \\ \frac{8x + 7y}{xy} &= 15 \end{aligned}$$

5.2.43

$$\begin{aligned} 6x + 3y &= 6xy \\ 2x + 4y &= 5xy \end{aligned}$$

5.2.44

$$\begin{aligned} \frac{10}{x+y} + \frac{2}{x-y} &= 4 \\ \frac{15}{x+y} - \frac{5}{x-y} &= -2 \end{aligned}$$

5.2.45

$$\begin{aligned} x + 2y &= 2 \\ 2x + 3y &= 3 \end{aligned}$$

5.2.46

$$\begin{aligned} 2x - y &= 5 \\ x + y &= 4 \end{aligned}$$

5.2.47

$$\begin{aligned} x + 3y &= 5 \\ 2x + 6y &= 8 \end{aligned}$$

5.2.48

$$\begin{aligned} x + y + z &= 1 \\ 2x + 3y + 2z &= 2 \\ ax + ay + 2az &= 4 \end{aligned}$$

5.2.49

$$\begin{aligned} 3x - y - 2z &= 2 \\ 2y - z &= -1 \\ 3x - 5y &= 3 \end{aligned}$$

5.2.50

$$\begin{aligned} 5x - y + 4z &= 5 \\ 2x + 3y + 5z &= 2 \\ 5x - 2y + 6z &= -1 \end{aligned}$$

5.2.51

$$\begin{aligned} 5x + 2y &= 4 \\ 7x + 3y &= 5 \end{aligned}$$

5.2.52

$$\begin{aligned} 2x - y &= -2 \\ 3x + 4y &= 3 \end{aligned}$$

5.2.53

$$\begin{aligned} 4x - 3y &= 3 \\ 3x - 5y &= 7 \end{aligned}$$

5.2.54

$$\begin{aligned} 5x + 2y &= 3 \\ 3x + 2y &= 5 \end{aligned}$$

5.2.55

$$\begin{aligned} \frac{2}{x} + \frac{3}{y} &= 13 \\ \frac{5}{x} + \frac{4}{y} &= -2 \end{aligned}$$

5.2.56

$$\begin{aligned} \frac{5}{x-1} + \frac{1}{y-2} &= 2 \\ \frac{6}{x-1} - \frac{3}{y-2} &= 1 \end{aligned}$$

5.2.57

$$\begin{aligned} 2x + y + z &= 1 \\ x - 2y - z &= \frac{3}{2} \\ 3y - 5z &= 9 \end{aligned}$$

5.2.58

$$\begin{aligned} x - y + z &= 4 \\ 2x + y - 3z &= 0 \\ x + y + z &= 2 \end{aligned}$$

5.2.59

$$\begin{aligned} 2x + 3y + 3z &= 5 \\ x - 2y + z &= -4 \\ 3x - y - 2z &= 3 \end{aligned}$$

5.2.60

$$\begin{aligned} x - y + 2z &= 7 \\ 3x + 4y - 5z &= -5 \\ 2x - y + 3z &= 12 \end{aligned}$$

5.2.61

$$\begin{aligned} x - y + 2z &= 1 \\ 2z - 3z &= 1 \\ 3x - 2y + 4z &= 2 \end{aligned}$$

5.2.62

$$\begin{aligned} 3x - 2y + 3z &= 8 \\ 2x + y - z &= 1 \\ 4x - 3y + 2z &= 4 \end{aligned}$$

5.2.63 Solve

$$\begin{pmatrix} x + y + z \\ x + z \\ y + z \end{pmatrix} = \begin{pmatrix} 9 \\ 5 \\ 7 \end{pmatrix}$$

5.2.64

$$\mathbf{X}+\mathbf{Y}=\begin{pmatrix}7 & 0\\ 2 & 5\end{pmatrix}$$
$$\mathbf{X}-\mathbf{Y}=\begin{pmatrix}3 & 0\\ 0 & 4\end{pmatrix}.$$

5.2.65

$$\mathbf{X}+\mathbf{Y}=\begin{pmatrix}5 & 2\\ 0 & 9\end{pmatrix}$$
$$\mathbf{X}-\mathbf{Y}=\begin{pmatrix}3 & 6\\ 0 & -1\end{pmatrix}$$

5.2.66

$$2\mathbf{X}+3\mathbf{Y}=\begin{pmatrix}2 & 3\\ 4 & 0\end{pmatrix}$$
$$3\mathbf{X}+2\mathbf{Y}=\begin{pmatrix}2 & -2\\ -1 & 5\end{pmatrix}$$

5.2.5 If

$$x\begin{pmatrix}2\\ 3\end{pmatrix}+y\begin{pmatrix}-1\\ 1\end{pmatrix}=\begin{pmatrix}10\\ 5\end{pmatrix} \tag{5.2.5.1}$$

find the values of x and y.

5.2.6 Solve $2x+3y=11$ and $2x+4y=-24$ and hence find the value of m for which $y=mx+3$.

5.2.7 For which values of p does the pair of equations given below have a unique solution.

$$4x+py+8=0 \tag{5.2.7.1}$$
$$2x+2y+2=0 \tag{5.2.7.2}$$

5.2.8 For what values of k will the following pair of linear equations have infinitely many solutions.

$$kx+3y-(k-3)=0 \tag{5.2.8.1}$$
$$12x+ky-k=0 \tag{5.2.8.2}$$

5.2.9 Find the values of a,b,c and d from the following equation

$$\begin{pmatrix}2a+b & a-2b\\ 5c-d & 4c+d\end{pmatrix}=\begin{pmatrix}4 & -3\\ 11 & 24\end{pmatrix} \tag{5.2.9.1}$$

5.2.10 Solve

$$(a-b)x+(a+b)y=a^2-2ab-b^2$$
$$(a+b)(x+y)=a^2+b^2$$

5.2.11 Solve

$$\frac{1}{3x+y}+\frac{1}{3x-y}=\frac{3}{4}$$
$$\frac{1}{2(3x+y)}-\frac{1}{2(3x-y)}=\frac{-1}{8}$$

5.3 CBSE

5.3.1 For what value of k, the system of linear equations

$$x + y + z = 2$$
$$2x + y - z = 3$$
$$3x + 2y + kz = 4$$

has a unique solution? (12, 2016)

5.3.2 The pair of linear equations $\frac{3x}{2} + \frac{5y}{3} = 7$ and $9x + 10y = 14$ is

(a) consistent
(b) inconsistent
(c) consistent with one solution
(d) consistent with many solutions

(12, 2020)

5.3.3 Solve the equation $x + 2y = 6$ and $2x - 5y = 12$ graphically. (10, 2022)

5.3.4 Solve the following equations for x and y using cross-multiplication method

$$(ax - by) + (a + 4b) = 0$$
$$(bx + ay) + (b - 4a) = 0$$

(10, 2022)

5.3.5 The pair of linear equations $2x = 5y + 6$ and $15y = 6x - 18$ represents two lines which are

a) intersecting
b) parallel
c) coincident
d) either intersecting or parallel

(10, 2023)

5.3.6 If the pair of equations $3x - y + 8 = 0$ and $6x - ry + 16 = 0$ represents coincident lines, then the value of r is ______. (10, 2023)

5.3.7 If the system of linear equations $2x + 3y = 7$ and $2ax + (a + b)y = 28$ have infinite number of solutions, then find the values of a and b. (10, 2023)

5.3.8 If $217x + 131y = 913$ and $131x + 217y = 827$, then solve the equations for the values of x and y. (10, 2023)

5.3.9 If $\begin{pmatrix} 2 & 0 \\ 5 & 4 \end{pmatrix} = \mathbf{P} + \mathbf{Q}$ is a symmetric and $\mathbf{Q}$ is a skew symmetric matrix, then $\mathbf{Q}$ is equal to ______. (12, 2023)

5.3.10 If $\mathbf{A} = \begin{pmatrix} -3 & -2 & -4 \\ 2 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 2 & 0 \\ -2 & -1 & -2 \\ 0 & -1 & 1 \end{pmatrix}$, then find AB and use it to solve the following system of equations

$$x - 2y = 3$$
$$2x - y - z = 2$$
$$-2y + z = 3$$

(12, 2023)

5.3.11 Find whether the following pair of linear equations are consistent or inconsistent

$$5x - 3y = 11, -10x + 6y = 22.$$

(10, 2021)

5.3.12 Solve for x and y

$$x + y = 6, 2x - 3y = 4.$$

(10, 2021)

5.3.13 Find out whether the pair of equations $2x + 3y = 0$ and $2x - 3y = 26$ is consistent or inconsistent. (10, 2021)

5.3.14 For what values of k, does the pair of linear equations $kx - 2y = 3$ and $3x + y = 5$ have a unique solution? (10, 2021)

5.3.15 What type of lines will you get by drawing the graph of the pair of equations $x - 2y + 3 = 0$ and $2x - 4y = 5$? (10, 2021)

5.3.16 Find the value of k for which the system of equations $x+2y = 5$ and $3x+ky+15 = 0$ has no solution. (10, 2021)

5.3.17 Solve the system of linear equations using the matrix method

$$\begin{aligned} 7x + 2y &= 11 \\ 4x - 7y &= 2 \end{aligned}$$

(12, 2021)

5.3.18 Using matrix method, solve the following system of equations

$$\begin{aligned} 2x - 3y + 5z &= 13 \\ 3x + 2y - 4z &= -2 \\ x + y - 2z &= -2. \end{aligned}$$

(12, 2019)

5.3.19 Using matrices solve the following system of linear equations

$$\begin{aligned} 2x + 3y + 10z &= 4 \\ 4x + 6y + 5z &= 1 \\ 6x + 9y - 20z &= 2 \end{aligned}$$

(12, 2019)

5.3.20 Find the solution of the pair of equations

$$\frac{3}{x} + \frac{8}{y} = -1; \frac{1}{x} - \frac{2}{y} = 2, x, y \neq 0$$

(10, 2019)

5.3.21 Find the value(s) of k for which the pair of equations

$$\begin{aligned} kx + 2y &= 3 \\ 3x + 6y &= 10 \end{aligned}$$

has a unique solution. (10, 2019)

5.3.22 Find the value(s) of k so that the pair of equations $x + 2y = 5$ and $3x + ky + 15 = 0$ has a unique solution. (10, 2019)

5.3.23 For what value of k, will the following pair of equations have infinitely many solutions

$$2x + 3y = 7 \text{ and } (k + 2)\,x - 3\,(1 - k)\,y = 5k + 1$$

(10, 2019)

5.3.24 Solve the following pair of linear equations

$$\begin{aligned} 3x - 5y &= 4 \\ 2y + 7 &= 9x \end{aligned}$$

(10, 2019)

5.3.25 Solve the following pair of linear equations

$$\begin{aligned} 3x + 4y &= 10 \\ 2x - 2y &= 2 \end{aligned}$$

(10, 2019)

5.3.26 For what value of k, does the system of linear equations

$$\begin{aligned} 2x + 3y &= 7 \\ (k - 1)\,x + (k + 2)\,y &= 3k \end{aligned}$$

have an infinite number of solutions ? (10, 2019)

5.3.27 Using matrices, solve the following system of linear equations

$$\begin{aligned} x + 2y - 3z &= -4 \\ 2x + 3y + 2z &= 2 \\ 3x - 3y - 4z &= 11. \end{aligned}$$

(12, 2018)

5.3.28 Determine the product $\begin{pmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{pmatrix}$ and use it to solve the system of equations

$$\begin{aligned} x - y + z &= 4 \\ x - 2y - 2z &= 9 \\ 2x + y + 3z &= 1. \end{aligned}$$

(12, 2017)

5.3.29 Solve the equations $x + 2y = 6$ and $2x - 5y = 12$ graphically. (12, 2022)

5.3.30 Solve the system of linear equations, using matrix method

$$7x + 2y = 11$$
$$4x - y = 2$$

(12, 2021)

5.3.31 Find the value of k for which the following pair of linear equations have infinitely many solutions.

$$2x + 3y = 7$$
$$(k + 1)x + (2k - 1)y = 4k + 1$$

(10, 2019)

5.3.32 If the system of linear equations

$$2x + 3y = 7$$
$$2ax + (a + b)\,y = 28$$

have infinite number of solutions, then find the values of a and b. (10, 2023)

5.3.33 If

$$217x + 131y = 913$$
$$131x + 217y = 827,$$

then solve the equations for the values of x and y. (10, 2023)

5.3.34 The pair of linear equations

$$2x = 5y + 6$$
$$15y = 6x - 18$$

represents two lines which are

a) intersecting
b) parallel
c) coincident
d) either intersecting or parallel

(10, 2023)

5.3.35 If the pair of equations

$$3x - y + 8 = 0$$
$$6x - ry + 16 = 0$$

represent coincident lines, then find the value of r. (10, 2023)

5.3.36 Solve the system of equations

$$\frac{bx}{a} - \frac{ay}{b} + a + b = 0$$
$$bx - ay + 2ab = 0.$$

(10, 2006)

5.3.37 Draw the graphs of the following equations

$$3x - 4y + 6 = 0$$
$$3x + y - 9 = 0$$

Also, determine the co-ordinates of the vertices of the triangle formed by these lines and the X axis. (10, 2006)

5.3.38 Find the value of x, if

$$3x + y = 1$$
$$2y - x = -5$$

(10, 2009)

5.3.39 Using matrices, solve the following system of equations

$$x + y + z = 6$$
$$x + 2z = 7$$
$$3x + y + z = 12$$

(12, 2009)

5.4 Inverse

Using elementary transformations, find the inverse of each of the following matrices

5.4.1 $\begin{pmatrix} 2 & 3 \\ -4 & -6 \end{pmatrix}$

5.4.2 $\begin{pmatrix} 1 & -2 \\ 4 & 3 \end{pmatrix}$

5.4.3 $\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$

5.4.4 $\begin{pmatrix} 2 & 4 \\ -1 & 2 \end{pmatrix}$

5.4.5 $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

5.4.6 $\begin{pmatrix} 10 & -2 \\ -5 & 1 \end{pmatrix}$

5.4.7 $\begin{pmatrix} 2 & 2 \\ 4 & 3 \end{pmatrix}$

5.4.8 $\begin{pmatrix} -1 & 5 \\ -3 & 2 \end{pmatrix}$

5.4.9 $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

5.4.10 $\begin{pmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{pmatrix}$

5.4.11 $\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$

5.4.12 $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

5.4.13 $\begin{pmatrix} 1 & 3 \\ 2 & 7 \end{pmatrix}$

5.4.14 $\begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix}$

5.4.15 $\begin{pmatrix} 2 & 1 \\ 7 & 4 \end{pmatrix}$

5.4.16 $\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$

5.4.17 $\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$

5.4.18 $\begin{pmatrix} 4 & 5 \\ 3 & 4 \end{pmatrix}$

5.4.19 $\begin{pmatrix} 3 & 10 \\ 2 & 7 \end{pmatrix}$

5.4.20 $\begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix}$

5.4.21 $\begin{pmatrix} 2 & -6 \\ 1 & -2 \end{pmatrix}$

5.4.22 $\begin{pmatrix} 6 & -3 \\ -2 & 1 \end{pmatrix}$

5.4.23 $\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$

5.4.24 $\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}$

5.4.25 $\begin{pmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{pmatrix}$

5.4.26 $\begin{pmatrix} 1 & 3 & -2 \\ -3 & 0 & -5 \\ 2 & 5 & 0 \end{pmatrix}$

5.4.27 $\begin{pmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix}$

5.4.28 $\begin{pmatrix} 2 & 4 \\ -5 & -1 \end{pmatrix}$

5.4.29 $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

5.4.30 $\begin{pmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{pmatrix}$

5.4.31 $\begin{pmatrix} 1 & 2 \\ 4 & 2 \end{pmatrix}$

5.4.32 $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{pmatrix}$

5.4.33 $\begin{pmatrix} 3 & -1 & -2 \\ 0 & 0 & 1 \\ 3 & -5 & 0 \end{pmatrix}$

5.4.34 $\begin{pmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{pmatrix}$

5.4.35 $\begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{pmatrix}$

5.4.36 $\begin{pmatrix} 2 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{pmatrix}$

5.4.37 $\begin{pmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & -9 \end{pmatrix}$

5.4.38 $\begin{pmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{pmatrix}$

5.4.39 $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{pmatrix}$

5.4.40 $\begin{pmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & 1 \end{pmatrix}$

5.4.41 $\begin{pmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{pmatrix}$

5.4.42 $\begin{pmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{pmatrix}$

5.4.43 $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{pmatrix}$

5.4.44 $\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix}$

5.4.45 $\begin{pmatrix} 1 & 2 & 4 \\ -1 & 3 & 0 \\ 4 & 1 & 0 \end{pmatrix}$

5.4.46 $\begin{pmatrix} 0 & \sin\alpha & -\cos\alpha \\ -\sin\alpha & 0 & \sin\beta \\ \cos\alpha & -\sin\beta & 0 \end{pmatrix}$

5.4.47 $\begin{pmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{pmatrix}$

5.4.48 $\begin{pmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{pmatrix}$

5.4.49 $\begin{pmatrix} 3 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 2 & 3 \end{pmatrix}$

5.4.50 $\begin{pmatrix} 102 & 18 & 36 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{pmatrix}$

5.4.51 $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

5.4.52 $\begin{pmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{pmatrix}$

5.4.53 $\begin{pmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{pmatrix}$

5.4.40 Verify that $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ for

a) $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 1 & -4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}$.

b) $\mathbf{A} = \begin{pmatrix} 3 & 7 \\ 2 & 5 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 6 & 8 \\ 7 & 9 \end{pmatrix}$.

5.4.41 If $\mathbf{A} = \begin{pmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{pmatrix}$, find $\mathbf{A}^{-1}$. Using $\mathbf{A}^{-1}$, solve the system of equations

$$2x - 3y + 5z = 11 \tag{5.4.41.1}$$
$$3x + 2y - 4z = -5 \tag{5.4.41.2}$$
$$x + y - 2z = -3 \tag{5.4.41.3}$$

5.4.42 Let $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 5 & 2 \\ 7 & 4 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 2 & 5 \\ 3 & 8 \end{pmatrix}$. Find a matrix $\mathbf{D}$ such that $\mathbf{CD} - \mathbf{AB} = 0$.

5.5 CBSE

5.5.1 If $\mathbf{A} = \begin{pmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ 5 & -2 & 6 \end{pmatrix}$, find $\mathbf{A}^{-1}$ and use it to solve the following system of equation

$$5x - y + 4z = 5$$
$$2x + 3y + 5z = 2$$
$$5x - 2y + 6z = -1$$

(12, 2020)

5.5.2 If $\mathbf{A} = \begin{pmatrix} 4x & 0 \\ 2x & 2x \end{pmatrix}$ and $\mathbf{A}^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}$, then $x =$ ______. (12, 2022)

5.5.3 If $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 3 & a & 1 \end{pmatrix}$ is non-singular matrix and $a \in A$, then the set A is ______. (12, 2021)

5.5.4 Value of k, for which $\mathbf{A} = \begin{pmatrix} k & 8 \\ 1 & 2k \end{pmatrix}$ is a singular matrix is ______. (12, 2021)

5.5.5 If $\mathbf{A} = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{pmatrix}$, then

a) $\mathbf{A}^{-1} = \mathbf{B}$
b) $\mathbf{A}^{-1} = 6\mathbf{B}$
c) $\mathbf{B}^{-1} = \mathbf{B}$
d) $\mathbf{B}^{-1} = \frac{1}{6}\mathbf{A}$

(12, 2021)

5.5.6 For $\mathbf{A} = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$, then $14\mathbf{A}^{-1}$ is given by (12, 2021)

5.5.7 Find the inverse of the following matrix, using elementary transformations

$$\mathbf{A} = \begin{pmatrix} 2 & 3 & 1 \\ 2 & 4 & 1 \\ 3 & 7 & 2 \end{pmatrix}.$$

(12, 2019)

5.5.8 If $\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & -2 & 1 \end{pmatrix}$, find $\mathbf{A}^{-1}$. Hence, solve the system of equations

$$\begin{aligned} x + y + z &= 6, \\ y + 3z &= 11, \\ \text{and} \quad x - 2y + z &= 0 \end{aligned}$$

(12, 2019)

5.5.9 Using elementary row transformations find the inverse of the matrix

$$\begin{pmatrix} 3 & 0 & -1 \\ 2 & 3 & 0 \\ 0 & 4 & 1 \end{pmatrix}$$

(12, 2019)

5.5.10 If $\mathbf{A} = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 1 & 2 \\ 5 & 1 & 1 \end{pmatrix}$, find $\mathbf{A}^{-1}$. Hence solve the system of equations

$$\begin{aligned} x + 3y + 4z &= 8 \\ 2x + y + 2z &= 5 \\ 5x + y + z &= 7 \end{aligned}$$

(12, 2018)

5.5.11 Find the inverse of the following matrix, using elementary transformations

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix}$$

(12, 2018)

5.5.12 If $\mathbf{A} = \begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 3 & 1 & 1 \end{pmatrix}$, find $\mathbf{A}^{-1}$. Hence solve the following system of equations

$$\begin{aligned} x + y + z &= 6 \\ x + 2z &= 7 \\ 3x + y + z &= 12. \end{aligned}$$

(12, 2018)

5.5.13 In the interval $\pi/2 < x < \pi$, find the value of x for which the matrix

$$\begin{pmatrix} 2\sin x & 3 \\ 1 & 2\sin x \end{pmatrix}$$

is singular. (12, 2015)

5.5.14 If $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 1 & -4 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}$, verify that $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$. (12, 2015)

5.5.15 Using elementary row transformations, find the inverse of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{pmatrix}.$$

(12, 2018)

5.5.16 If $\mathbf{A} = \begin{pmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{pmatrix}$, Find the $\mathbf{A}^{-1}$. Use it to solve the system of equations

$$\begin{aligned} 2x - 3y + 5z &= 11 \\ 3x - 2y - 4z &= -5 \\ x + y - 2z &= -3 \end{aligned}$$

(12, 2018)

5.5.17 If $\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & -2 & 1 \end{pmatrix}$, find $\mathbf{A}^{-1}$. Hence, solve the system of equations

$$\begin{aligned} x + y + z &= 6, \\ y + 3z &= 11 \\ x - 2y + z &= 0. \end{aligned}$$

5.5.18 Find the inverse of the following matrix, using elementary transformations

$$\mathbf{A} = \begin{pmatrix} 2 & 3 & 1 \\ 2 & 4 & 1 \\ 3 & 7 & 2 \end{pmatrix}$$

(12, 2018)

5.5.19 Using elementary row transformations, find the inverse of the matrix

$$\begin{pmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{pmatrix}$$

(12, 2018)

5.5.20 Using elementary row transformations, find the inverse of the matrix

$$\begin{pmatrix} 3 & 0 & -1 \\ 2 & 3 & 0 \\ 0 & 4 & 1 \end{pmatrix}.$$

(12, 2018)

5.5.21 Find matrix $\mathbf{A}$ such that

$$\begin{pmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{pmatrix} \mathbf{A} = \begin{pmatrix} -1 & -8 \\ 1 & -2 \\ 9 & 22 \end{pmatrix}$$

(12, 2017)

5.5.22 If $\mathbf{A} = \begin{pmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{pmatrix}$, find $\mathbf{A}^{-1}$. Hence using $\mathbf{A}^{-1}$ solve the system of equations

$$\begin{aligned} 2x - 3y + 5z &= 11 \\ 3x + 2y - 4z &= -5 \\ x + y - 2z &= -3 \end{aligned}$$

(12, 2017)

5.5.23 Using elementary row operations, find the inverse of the following matrix

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 3 \\ -5 & 3 & 1 \\ -3 & 2 & 3 \end{pmatrix}$$

(12, 2016)

5.5.24 Using elementary row operations find the inverse of matrix

$$\mathbf{A} = \begin{pmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{pmatrix}$$

and hence solve the following system of equations

$$\begin{aligned} 3x - 3y + 4z &= 21 \\ 2x - 3y + 4z &= 20 \\ -y + z &= 5. \end{aligned}$$

(12, 2016)

5.5.25 If $\mathbf{A} = \begin{pmatrix} 1 & 2 & -3 \\ 2 & 0 & -3 \\ 1 & 2 & 0 \end{pmatrix}$ then find $\mathbf{A}^{-1}$ and use it to solve the following system of equations

$$\begin{aligned} x + 2y - 3z &= 1 \\ 2x - 3z &= 2 \\ x + 2y &= 3 \end{aligned}$$

(12, 2024)

5.5.26 Find the product of the matrices $\begin{pmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & 4 \end{pmatrix} \begin{pmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{pmatrix}$, then find $\mathbf{AB}$ and use it to solve the system of linear equations

$$\begin{aligned} x - 2y &= 3 \\ 2x - y - z &= 2 \\ -2y + z &= 3 \end{aligned}$$

(12, 2024)

5.5.27 Find the product of the matrices $\begin{pmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{pmatrix}\begin{pmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{pmatrix}$ and hence solve the system of linear equations

$$\begin{aligned} x + 2y - 3z &= -4 \\ 2x + 3y + 2z &= 2 \\ 3x - 3y - 4z &= 11 \end{aligned}$$

(12, 2024)

5.5.28 Find λ for the matrix $\mathbf{A} = \begin{pmatrix} 2 & -1 & 1 \\ \lambda & 2 & 0 \\ 1 & -2 & 3 \end{pmatrix}$ to be invertible. (12, 2024)

5.5.29 If the inverse of the matrix $\begin{pmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$ is the matrix $\begin{pmatrix} 1 & 3 & 3 \\ 1 & \lambda & 3 \\ 1 & 3 & 4 \end{pmatrix}$, then find the value of λ.

5.5.30 If inverse of matrix $\begin{pmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$ is the matrix $\begin{pmatrix} 1 & 3 & 3 \\ 1 & \lambda & 3 \\ 1 & 3 & 4 \end{pmatrix}$, then find the value of λ.

5.5.31 Solve the following system of equations, using matrices

$$\begin{aligned} \frac{2}{x} + \frac{3}{y} + \frac{10}{z} &= 4, \\ \frac{4}{x} - \frac{6}{y} + \frac{5}{z} &= 1, \\ \frac{6}{x} + \frac{9}{y} - \frac{20}{z} &= 2 \end{aligned}$$

where $x, y, z \neq 0$. (12, 2024)

5.5.32 If $\mathbf{A} = \begin{pmatrix} 1 & \cot x \\ -\cot x & 1 \end{pmatrix}$, then show that $\mathbf{A}^{\top}\mathbf{A}^{-1} = \begin{pmatrix} -\cos 2x & -\sin 2x \\ \sin 2x & -\cos 2x \end{pmatrix}$ (12, 2024)

5.5.33 Obtain the inverse of the following matrix using elementary opertions

$$\mathbf{A} = \begin{pmatrix} 3 & 0 & -1 \\ 2 & 3 & 0 \\ 0 & 4 & 1 \end{pmatrix}$$

(12, 2009)

5.6 Cayley-Hamilton Theoerm

5.6.1 Let $\mathbf{A} = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$, show that

$$\mathbf{A}^2 - 5\mathbf{A} + 7\mathbf{I} = \mathbf{0}. \tag{5.6.1.1}$$

Hence find $\mathbf{A}^{-1}$.

Solution: From (5.1.1.1),

$$\begin{vmatrix} 3-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} = 0 \tag{5.6.1.2}$$

$$\implies (3-\lambda)(2-\lambda) + 1 = 0 \tag{5.6.1.3}$$

$$\text{or, } \lambda^2 - 5\lambda + 7 = 0 \tag{5.6.1.4}$$

Using (5.1.2.1) in the above, (5.6.1.1) is obtained. Multiplying both sides of (5.6.1.1) by $\mathbf{A}^{-1}$,

$$\mathbf{A} - 5\mathbf{I} + 7\mathbf{A}^{-1} = \mathbf{0} \tag{5.6.1.5}$$

$$\implies \mathbf{A}^{-1} = \frac{1}{7}(5\mathbf{I} - \mathbf{A}) = \frac{1}{7}\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix} \tag{5.6.1.6}$$

5.6.2 Find $\mathbf{A}^2 - 5\mathbf{A} + 6\mathbf{I}$, if $\mathbf{A} = \begin{pmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{pmatrix}$.

5.6.3 If $\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{pmatrix}$, prove that

$$\mathbf{A}^3 - 6\mathbf{A}^2 + 7\mathbf{A} + 2\mathbf{I} = \mathbf{0}. \tag{5.6.3.1}$$

Solution: From (5.1.1.1), the characteristic equation is

$$\begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & 2-\lambda & 1 \\ 2 & 0 & 3-\lambda \end{vmatrix} = 0 \tag{5.6.3.2}$$

which can be expanded to obtain

$$\lambda^3 - 6\lambda^2 + 7\lambda + 2 = 0 \tag{5.6.3.3}$$

upon simplification. Using the Cayley-Hamilton theorem in (5.1.2.1), (5.6.3.1) is obtained.

5.6.4 If $\mathbf{A} = \begin{pmatrix} 3 & -2 \\ 4 & -2 \end{pmatrix}$ and $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, find k so that $\mathbf{A}^2 = k\mathbf{A} - 2\mathbf{I}$.

5.6.5 For the matrix $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$, find the numbers a and b such that $\mathbf{A}^2 + a\mathbf{A} + b\mathbf{I} = \mathbf{0}$.

5.6.6 For the matrix $\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & -1 & 3 \end{pmatrix}$. Show that $\mathbf{A}^3 - 6\mathbf{A}^2 + 5\mathbf{A} + 11\mathbf{I} = \mathbf{0}$. Hence, find

$\mathbf{A}^{-1}$.

5.6.7 If $\mathbf{A} = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$. Verify that $\mathbf{A}^3 - 6\mathbf{A}^2 + 9\mathbf{A} - 4\mathbf{I} = \mathbf{0}$ and hence find $\mathbf{A}^{-1}$.

5.6.8 If $\mathbf{A} = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}$ is such that $\mathbf{A}^2 = \mathbf{I}$, then

a) $1 + \alpha^2 + \beta\gamma = 0$
b) $1 - \alpha^2 + \beta\gamma = 0$
c) $1 - \alpha^2 - \beta\gamma = 0$
d) $1 + \alpha^2 - \beta\gamma = 0$

5.6.9 Let $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, show that $(a\mathbf{I} + b\mathbf{A})^n = a^n\mathbf{I} + na^{n-1}b\mathbf{A}$.

5.6.10 If $\mathbf{A} = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$, show that $\mathbf{A}^2 - 5\mathbf{A} + 7\mathbf{I} = \mathbf{0}$.

5.6.11 If $\mathbf{A}$ is square matrix such that $\mathbf{A}^2 = \mathbf{A}$, then $(\mathbf{I} + \mathbf{A})^3 - 7\mathbf{A}$ is equal to

a) $\mathbf{A}$
b) $\mathbf{I} - \mathbf{A}$
c) $\mathbf{I}$
d) $3\mathbf{A}$

5.6.12 If $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{pmatrix}$, then show that $\mathbf{A}^3 - 23\mathbf{A} - 40\mathbf{I} = \mathbf{0}$.

5.6.13 Show that the matrix $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$ satisfies the equation $\mathbf{A}^2 - 4\mathbf{A} + \mathbf{I} = \mathbf{0}$. Using this equation, find $\mathbf{A}^{-1}$.

5.7 CBSE

5.7.1 If $\mathbf{A}$ is an non-singular square matrix of order 3 such that $\mathbf{A}^2 = 3\mathbf{A}$, then value of $\left|\mathbf{A}\right|$ is (12, 2020)

5.7.2 If $\mathbf{A} = \begin{pmatrix} -3 & 2 \\ 1 & -1 \end{pmatrix}$ and $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, find scalar k so that $\mathbf{A}^2 + \mathbf{I} = k\mathbf{A}$. (12, 2020)

5.7.3 For the matrix $\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -2 \\ 2 & -1 & 3 \end{pmatrix}$, show that $\mathbf{A}^3 - 6\mathbf{A}^2 + 5\mathbf{A} + 11\mathbf{I} = 0$. Hence, find $\mathbf{A}^{-1}$. (12, 2022)

5.7.4 If $\mathbf{A}$ is a square matrix such that $\mathbf{A}^2 = \mathbf{A}$, then find $(2 + \mathbf{A})^3 - 19\mathbf{A}$. (12, 2022)

5.7.5 If $\mathbf{A} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, then $\mathbf{A}^2$ equals (12, 2021)

5.7.6 If $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then $\mathbf{A}^4 =$ ____________. (12, 2021)

5.7.7 If $\mathbf{A}$ is square matrix such that $\mathbf{A}^2 = \mathbf{A}$, then $(\mathbf{I} + \mathbf{A})^3 - 7\mathbf{A}$ is equal to (12, 2021)

5.7.8 Given that $\mathbf{A} = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}$ and $\mathbf{A}^2 = 3\mathbf{I}$, then

a) $1 + \alpha^2 + \beta\gamma = 0$

b) $1-\alpha^2-\beta\gamma=0$
c) $3-\alpha^2-\beta\gamma=0$
d) $3+\alpha^2+\beta\gamma=0$

(12, 2021)

5.7.9 If $\mathbf{A}=\begin{pmatrix}-3 & 6\\ -2 & 4\end{pmatrix}$, then show that $\mathbf{A}^3=\mathbf{A}$. (12, 2019)

5.7.10 If $\mathbf{A}=\begin{pmatrix}1 & -1 & 1\\ 2 & -1 & 0\\ 1 & 0 & 0\end{pmatrix}$, find $\mathbf{A}^2$ and show that $\mathbf{A}^2=\mathbf{A}^{-1}$. (12, 2019)

5.7.11 Show that for the matrix $\mathbf{A}=\begin{pmatrix}1 & 1 & 1\\ 1 & 2 & -3\\ 2 & -1 & 3\end{pmatrix}$, $\mathbf{A}^3-6\mathbf{A}^2+5\mathbf{A}+11\mathbf{I}=0$.

Hence, find $\mathbf{A}^{-1}$. (12, 2018)

5.7.12 If $\mathbf{A}=\begin{pmatrix}4 & 2\\ -1 & 1\end{pmatrix}$, show that $(\mathbf{A}-2\mathbf{I})(\mathbf{A}-3\mathbf{I})=0$. (12, 2018)

5.7.13 Given $\mathbf{A}=\begin{pmatrix}2 & -3\\ -4 & 7\end{pmatrix}$, compute $\mathbf{A}^{-1}$ and show that $2\mathbf{A}^{-1}=9\mathbf{I}-\mathbf{A}$. (12, 2018)

5.7.14 If $\mathbf{A}=\begin{pmatrix}-3 & 6\\ -2 & 4\end{pmatrix}$, then show that $\mathbf{A}^3=\mathbf{A}$. (12, 2018)

5.7.15 If

$$\mathbf{A}=\begin{pmatrix}1 & 0 & 2\\ 0 & 2 & 1\\ 2 & 0 & 3\end{pmatrix}$$

and $\mathbf{A}^3-6\mathbf{A}^2+7\mathbf{A}+k\mathbf{I}=0$ find k. (12, 2016)

5.8 *Application*

5.8.1 A trust fund has ₹30000 that must be invested in two different types of bonds. The first bond pays 5% interest per year, and the second bond pays 7% interest per year. Determine how to divide ₹30000 among the 2 types of bonds. If the trust fund must obtain an annual total interest of

a) ₹1800
b) ₹2000

Solution: Let the desired division for both cases be

$$\begin{aligned} x_1+x_2&=30000\\ 0.05x_1+0.07x_2&=1800\end{aligned} \implies \begin{pmatrix}1 & 1\\ 0.05 & 0.07\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}30000\\ 1800\end{pmatrix} \tag{5.8.1.1}$$

$$\begin{aligned} y_1+y_2&=30000\\ 0.05y_1+0.07y_2&=2000\end{aligned} \implies \begin{pmatrix}1 & 1\\ 0.05 & 0.07\end{pmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix}=\begin{pmatrix}30000\\ 2000\end{pmatrix} \tag{5.8.1.2}$$

yielding

$$\begin{pmatrix}1 & 1\\ 0.05 & 0.07\end{pmatrix}\begin{pmatrix}x_1 & y_1\\ x_2 & y_2\end{pmatrix}=\begin{pmatrix}30000 & 30000\\ 1800 & 2000\end{pmatrix} \tag{5.8.1.3}$$

with the augmented matrix followed by row reduction

$$\xleftrightarrow{R_2=R_2-0.05R_1}\left(\begin{array}{cc|cc}1 & 1 & 30000 & 30000\\ 0 & 0.02 & 300 & 500\end{array}\right)\xleftrightarrow{R_2=50R_2}\left(\begin{array}{cc|cc}1 & 1 & 30000 & 30000\\ 0 & 1 & 15000 & 25000\end{array}\right)$$

$$\xleftrightarrow{R_1=R_1-R_2}\left(\begin{array}{cc|cc}1 & 0 & 115000 & 5000\\ 0 & 1 & 15000 & 25000\end{array}\right)$$

Thus,

$$\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}15000\\15000\end{pmatrix},\begin{pmatrix}y_1\\y_2\end{pmatrix}=\begin{pmatrix}5000\\25000\end{pmatrix} \tag{5.8.1.4}$$

5.8.2 10 students of Class X took part in a Mathematics quiz. If the number of girls is 4 more than the number of boys, find the number of boys and girls who took part in the quiz.

5.8.3 5 pencils and 7 pens together cost ₹50 whereas 7 pencils and 5 pens together cost ₹46. Find the cost of one pencil and that of one pen.

5.8.4 Half the perimeter of a rectangular garden, whose length is $4m$, more than its width, is $36m$. Find the dimensions of the garden.

5.8.5 A part of monthly hostel charges is fixed and the remaining depends on the number of days one has taken food in the mess. When a student A takes food for 20 days she has to pay ₹1000 as hostel charges whereas a student B who takes food for 26 days, pays ₹1180 as hostel charges. Find the fixed charges and the cost of food per day.

5.8.6 A fraction becomes $\frac{1}{3}$ when 1 is subtracted from the numerator and it becomes $\frac{1}{4}$ when 8 is added to the denominator. Find the fraction.

5.8.7 Yash scored 40 marks in a test, getting 3 marks for each right answer and losing 1 mark for each wrong answer. Had 4 marks been awarded for each correct answer and 2 marks been deducted for each incorrect answer, then Yash would have scored 50 marks. How many questions were there in the test?

5.8.8 Places A and B are $100km$ apart on a highway. One car starts from A and another from B at the same time. If the car travel in the same direction at different speeds, they meet in 5hrs. If they travel towards each other, they meet in 1hr. What are the speeds of the two cars?

5.8.9 The area of a rectangle gets reduced by 9 square units, if its length is reduced by 5 units and breadth is increased by 3 units. If we increase the length by 3 units and the breadth by 2 units, the area increases by 67 square units. Find the dimensions of the rectangle.

5.8.10 Narayan tells his daughter, ‘Seven years ago, I was seven times as old as you were then. Also, 3 years from now, I shall be 3 times as old as you will be.’ Find their ages.

5.8.11 The coach of a cricket team buys 3 bats and 6 balls for ₹3900. Later, she buys another bat and 3 more balls of the same kind for ₹3300. Find the cost of a bat and ball.

5.8.12 The cost of $2kg$ of apples and $1kg$ of grapes on a day was found to be ₹160. After a month, the cost of $4kg$ of apples and $2kg$ of grapes is ₹300. Assuming that the costs remain unchanged, find the cost of a kg of apples and grapes.

5.8.13 The difference between two numbers is 26 and one number is three times the other. Find them.

5.8.14 The larger of two supplementary angles exceeds the smaller by 18 degrees. Find them.

5.8.15 The coach of a cricket team buys 7 balls and 6 balls for ₹3800. Later, she buys 3 bats and 5 balls for ₹1750. Find the cost of each bat and each ball.

5.8.16 The taxi charges in a city consist of a fixed charge together with the charges for the distance covered. For a distance of 10 km, the charge paid is ₹105 and for a distance of 15 km, the charge paid is ₹155. What are the fixed charges and the charge per km? How much does a person have to pay for travelling a distance of 25 km ?

5.8.17 A fraction becomes $\frac{9}{11}$ if 2 is added to both the numerator and the denominator. If 3 is added to both the numerator and the denominator, it becomes $\frac{5}{6}$. Find the fraction.

5.8.18 Five years hence, the age of Rahul will be three times that of his son. Five years ago, Rahul's age was seven times that of his son. What are their present ages?

5.8.19 If we add 1 to the numerator and subtract 1 from the denominator, a fraction reduces to 1. It becomes $\frac{1}{2}$ if we only add 1 to the denominator. What is the fraction?

5.8.20 Five years ago, Minu was thrice as old as Sonu. Ten years later, Minu will be twice as old as Sonu. How old are Minu and Sonu?

5.8.21 The Sum of the digits of a two-digit number is 9. Also, nine times this number is twice the number obtained by reversing the order of the digits. Find the number.

5.8.22 Meena went to a bank to withdraw ₹2000. She asked the cashier to give her ₹50 and ₹100 notes only. Meena got 25 notes in all. Find how many notes of ₹50 and ₹100 she received.

5.8.23 A lending library has a fixed charge for the first three days and an additional charge for each day thereafter. Sarita paid ₹27 for seven days, while Susheela paid ₹21 for five days. Find the fixed charge and the charge for each extra day.

5.8.24 The ages of two friends Ani and Bijoya differ by 3 years. Ani's father Dharam is twice as old as Ani and Bijoya is twice as old as sister Kanta. The ages of Kanta and Dharam differ by 30 years. Find the ages of Ani and Bijoya.

5.8.25 One says, "Give me a hundred, Friend! I shall then become twice as rich as you ". The other "if you give me ten, i shall be six times as rich as you ". Tell me What is the amount of their (respective) capital? [From the bijaganita of Bhaskara II].

5.8.26 A train covered a certain distance at a uniform speed. If the train would have been 10 km/h faster, it would have taken 2 hours less than the scheduled time. And, if the train were slower by 10 km/h; it would have taken 3 hours more than the scheduled time. Find the distance covered by the train.

5.8.27 The students of a class are made to stand in rows. If 3 students are extra in a row, there would be 1 row less. If 3 students are less in a row, there would be 2 rows more. Find the number of students in the class.

5.8.28 Ritu can row downstream $20km$ in 2 hours, and upstream $4km$ in 2 hours. Find her speed of rowing in still water and the speed of the current.

5.8.29 2 women and 5 men can together finish an embroidery work in 4 days, while 3 women and 6 men can finish it in 3 days. Find the time taken by 1 women alone to finish the work, and also that taken by 1 man alone.

5.8.30 Rambha travels $300km$ to her home partly by train and partly by bus. She takes 4 hours if she travels 60*km* by train and the remaining by bus. If she travels 100*km* by train and the remaining by bus, she takes 10 minutes longer. Find the speed of the train and the bus seperately

5.8.31 In a $\triangle ABC, \angle C = 3\angle B = 2(\angle A + \angle B)$. Find the three angles.

5.8.32 $ABCD$ is a cyclic quadrilateral with angles

$$A = 4y + 20, B = -7x + 5, C = -4x, D = 3y - 5. \tag{5.8.32.1}$$

Find them.

5.8.33 Draw the graphs of the equations $5x - y = 5$ and $3x - y = 3$. Determine the Co-ordinates of the vertices of the triangle formed by these lines and the y axis.

5.8.34 Draw the graphs of the equations $x - y + 1 = 0$ and $3x + 2y - 12 = 0$. Determine the coordinates of the vertices of the triangle formed by these lines and the axis and shade the triangular region.

5.8.35 The cost of 4*kg* onion, 3*kg* wheat and 2*kg* rice is ₹60. The cost of 2*kg* onion, 4*kg* wheat and 6*kg* rice is ₹90. The cost of 6*kg* onion, 2*kg* wheat and 3*kg* rice is ₹70. Find cost of each item per kg.

5.8.36 Romila went to a stationary shop and purchased 2 pencils and 3 erasers for ₹9. Her friend Sonali saw the new variety of pencils and erasers with Romila, and she also bought 4 pencils and 6 erasers of the same kind for ₹18. Represent this situation algebraically and graphically.

5.8.37 Champa went to a “Sale” to purchase some pants and skirts. When her friends asked her how many of each she had boughtc she answered, “The number of skirts is two less than twice the number of pants purchased. Also, the number of skirts is four less than four times the number of pants purchased”. Help her friends to find how many pants and skirts Champa bought.

5.8.38 Alwar tells his daughter, “Seven years ago, I was seven times as old as you were then. Also, three years from now, I shall be three times as old as you will be.” Represent this situation algebraically and graphically.

5.8.39 The cost of 2 pencils and 3 erasers is ₹9 and the cost of 4 pencils and 6 erasers is ₹18. Find the cost of each pencil and each eraser.

5.8.40 The ratio of incomes of two persons is 9:7 and the ratio of their expenditures is 4 : 3. If each of them manages to save ₹2000 per month, find their monthly incomes.

5.8.41 The sum of a two digit number and the number obtained by reversing the digits is 66. If the digits of the number differ by 2, find the number. How many such numbers are there?

5.8.42 A boat goes 30 km upstream and 44 km downstream in 10 hours. In 13 hours, it can go 40 km upstream and 55 km down-stream. Determine the speed of the stream and that of the boat in still water.

5.8.43 The sum of three numbers is 6. If we multiply the third number by 3 and add the second number to it, we get 11. By adding the first and third numbers, we get double of the second number. Find the numbers.

5.9 CBSE

5.9.1 A fraction becomes $\frac{1}{3}$ when 1 is subtracted from the numerator and it becomes $\frac{1}{4}$ when 8 is added to its denominator. Find the fraction. (10, 2020)

5.9.2 The present age of a father is three years more than three times the age of his son. Three years hence the father's age will be 10 years more than twice the age of the son. Determine their present ages. (10, 2020)

5.9.3 Two schools P and Q decided to award prizes to their students for two games of Hockey ₹x per students and cricket ₹y per student. School P decided to award a total of ₹$9,500$ for the two games to 5 and 4 students respectively; while school Q decided to award ₹$7,370$ for the two games to 4 and 3 students respectively. Based on the given information, answer the following questions :

a) Represent the following information algebraically (in terms of x and y).
b) i) What is the prize amount for hockey ?
   ii) Prize amount on which game is more and by how much ?
c) What will be the total prize amount if there are 2 students each from two games ?

(10, 2020)

5.9.4 Half of the difference between two numbers is 2. The sum of the greater number and twice the smaller number is 3. Find the numbers. (10, 2023)

5.9.5 The sum of the numerator and the denominator of a fraction is 18. If the denominator is increased by 2, the fraction reduces to $\frac{1}{3}$. Find the fraction. (10, 2021)

5.9.6 If 2 tables and 2 chairs cost ₹700 and 4 tables and 3 chairs cost ₹$1,250$, then find the cost of one table. (10, 2021)

5.9.7 A part of monthly hostel charges in a college hostel are fixed and the remaining depends on the number of days one has taken food in the mess. When a student A takes food for 25 days, he has to pay ₹$4,500$, whereas a student B who takes food for 30 days, has to pay ₹$5,200$. Find the fixed charges per month and the cost of food per day. (10, 2019)

5.9.8 A father's age is three times the sum of the ages of his two children. After 5 years his age will be two times the sum of their ages. Find the present age of the father.

5.9.9 A boat goes 30 km upstream and 44 km downstream in 10 hours. In 13 hours, it can go 40 km upstream and 55 km downstream. Determine the speed of the stream and that of the boat in still water. (10, 2019)

5.9.10 A fraction becomes $\frac{1}{3}$ when 2 is subtracted from the numerator and it becomes $\frac{1}{2}$ when 1 is subtracted from the denominator. Find the fraction. (10, 2019)

5.9.11 Sumit is 3 times as old as his son. Five years later, he shall be two and a half times as old as his son. How old is Sumit at present ? (10, 2019)

5.9.12 Ishan wants to donate a rectangular plot of land for a school in his village. When he was asked to give dimensions of the plot, he told that if its length is decreased by $50m$ and breadth is increased by $50m$, then its area will remain same, but if length is decreased by $10m$ and breadth is decreased by $20m$, then its area will decrease by $5300m^2$. Using matrices, find the dimensions of the plot. Also give reason why he wants to donate the plot for a school. (10, 2016)

5.9.13 A shopkeeper has 3 varieties of pens A, B and C. Meenu purchased 1 pen of each variety for a total of ₹21. Jeevan purchased 4 pens of A variety, 3 pens of B variety and 2 pens of C variety for ₹60. While Shikha purchased 6 pens of A variety , 2 pens of B variety and 3 pens of C variety for ₹70. Using matrix method, find cost of each variety of pen. (10, 2016)

5.9.14 On her birthday Seema decided to donate some money to children of an orphanage home. If there were 8 children less, every one would have got ₹10 more. However, if there were 16 children more, every one would have got ₹10 less. Using matrix method, find the number of children and the amount distributed by Seema. What values are reflected by Seema's decision ? (10, 2016)

5.9.15 A trust invested some money in two type of bonds. The first bond pays 10% interest and second bond pays 12% interest. The trust received ₹2, 800 as interest. However, if trust had interchanged money in bonds, they would have got ₹100 less as interest. Using matrix method, find the amount invested by the trust. Interest received on this amount will be given to an *ashram* as donation. Which value is reflected in this question ? (10, 2016)

5.9.16 A trust fund has ₹35, 000 is to be invested in two different types of bonds. The first bond pays 8% interest per annum which will be given to a *matha* and second bond pays 10% interest per annum which will be given to a temple. Using matrix multiplication, determine how to divide ₹35, 000 among two types of bonds if the trust fund obtains an annual total interest of ₹3, 200. What are the values reflected in this question? (10, 2016)

5.9.17 If

$$\begin{pmatrix} 2a+b & a-2b \\ 5c-d & 4c+3d \end{pmatrix} = \begin{pmatrix} 4 & -3 \\ 11 & 24 \end{pmatrix},$$

then the value of $a+b-c+2d$ is ______. (12, 2021)

5.10 Chemistry

5.10.1. Balance the following chemical equation.

$$HNO_3 + Ca(OH)_2 \to Ca(NO_3)_2 + H_2O \tag{5.10.1.1}$$

Solution: Let the balanced version of (5.10.1.1) be

$$x_1HNO_3 + x_2Ca(OH)_2 \to x_3Ca(NO_3)_2 + x_4H_2O \tag{5.10.1.2}$$

which results in the following equations:

$$(x_1 + 2x_2 - 2x_4)H = 0 \tag{5.10.1.3}$$

$$(x_1 - 2x_3)N = 0 \tag{5.10.1.4}$$

$$(3x_1 + 2x_2 - 6x_3 - x_4)O = 0 \tag{5.10.1.5}$$

$$(x_2 - x_3)Ca = 0 \tag{5.10.1.6}$$

which can be expressed as

$$x_1 + 2x_2 + 0x_3 - 2x_4 = 0 \tag{5.10.1.7}$$

$$x_1 + 0x_2 - 2x_3 + 0.x_4 = 0 \tag{5.10.1.8}$$

$$3x_1 + 2x_2 - 6x_3 - x_4 = 0 \tag{5.10.1.9}$$

$$0x_1 + x_2 - x_3 + 0.x_4 = 0 \tag{5.10.1.10}$$

resulting in the matrix equation

$$\begin{pmatrix} 1 & 2 & 0 & -2 \\ 1 & 0 & -2 & 0 \\ 3 & 2 & -6 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix} \mathbf{x} = \mathbf{0}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \tag{5.10.1.11}$$

(5.10.1.11) can be reduced as follows

$$\begin{pmatrix} 1 & 2 & 0 & -2 \\ 1 & 0 & -2 & 0 \\ 3 & 2 & -6 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix} \xleftrightarrow[R_3 \leftarrow \frac{R_3}{3} - R_1]{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 2 & 0 & -2 \\ 0 & -2 & -2 & 2 \\ 0 & -\frac{4}{3} & -2 & \frac{5}{3} \\ 0 & 1 & -1 & 0 \end{pmatrix} \tag{5.10.1.12}$$

$$\xleftrightarrow{R_2 \leftarrow -\frac{R_2}{2}} \begin{pmatrix} 1 & 2 & 0 & -2 \\ 0 & 1 & 1 & -1 \\ 0 & -\frac{4}{3} & -2 & \frac{5}{3} \\ 0 & 1 & -1 & 0 \end{pmatrix} \xleftrightarrow[R_4 \leftarrow R_4 - R_2]{R_3 \leftarrow R_3 + \frac{4}{3} R_2} \begin{pmatrix} 1 & 2 & 0 & -2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -\frac{2}{3} & \frac{1}{3} \\ 0 & 0 & -2 & 1 \end{pmatrix} \tag{5.10.1.13}$$

$$\xleftrightarrow[R_3 \leftarrow -\frac{3}{2} R_3]{R_1 \leftarrow R_1 - 2R_2} \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & -2 & 1 \end{pmatrix} \xleftrightarrow{R_4 \leftarrow R_4 + 2R_3} \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.10.1.14}$$

$$\xleftrightarrow[R_2 \leftarrow R_2 - R_3]{R_1 \leftarrow R_1 + 2R_3} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.10.1.15}$$

Thus,

$$x_1 = x_4, x_2 = \frac{1}{2} x_4, x_3 = \frac{1}{2} x_4 \tag{5.10.1.16}$$

$$\implies \quad \mathbf{x} = x_4 \begin{pmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 2 \end{pmatrix} \tag{5.10.1.17}$$

by substituting $x_4 = 2$. Hence, (5.10.1.2) finally becomes

$$2HNO_3 + Ca(OH)_2 \rightarrow Ca(NO_3)_2 + 2H_2O \tag{5.10.1.18}$$

5.10.2. Balance the following chemical equation.

$$NaOH + H_2SO_4 \rightarrow Na_2SO_4 + H_2O \tag{5.10.2.1}$$

5.10.3. Balance the following chemical equation.

$$Fe + H_2O \rightarrow Fe_3O_4 + H_2 \tag{5.10.3.1}$$

5.10.4. Write the balanced chemical equations for the following reaction.

$$BaCl_2 + K_2SO_4 \rightarrow BaSO_4 + KCl \tag{5.10.4.1}$$

5.10.5. Balance the following chemical equation.

$$\text{Zinc} + \text{Silver nitrate} \rightarrow \text{Zinc nitrate} + \text{Silver} \tag{5.10.5.1}$$

Solution: (5.10.5.1) can be written as

$$Zn + AgNO_3 \rightarrow Ag + Zn(NO_3)_2 \tag{5.10.5.2}$$

Suppose the balanced form of the equation is

$$x_1Zn + x_2AgNO_3 \rightarrow x_3Ag + x_4Zn(NO_3)_2, \tag{5.10.5.3}$$

which results in the following equations:

$$(x_1 - 2x_4)Zn = 0 \tag{5.10.5.4}$$
$$(x_2 - x_3)Ag = 0 \tag{5.10.5.5}$$
$$(x_3 - 2x_4)N = 0 \tag{5.10.5.6}$$
$$(3x_3 - 6x_4)O = 0 \tag{5.10.5.7}$$

which can be expressed as

$$x_1 + 0x_2 + 0x_3 - x_4 = 0 \tag{5.10.5.8}$$
$$0x_1 + x_2 - x_3 + 0x_4 = 0 \tag{5.10.5.9}$$
$$0x_1 + 0x_2 + x_3 - 2x_4 = 0 \tag{5.10.5.10}$$
$$0x_1 + 0x_2 + 3x_3 - 6x_4 = 0 \tag{5.10.5.11}$$

resulting in the matrix equation

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 3 & -6 \end{pmatrix} \mathbf{x} = \mathbf{0}, \ \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \tag{5.10.5.12}$$

(5.10.5.12) can be reduced as

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 3 & -6 \end{pmatrix} \xleftrightarrow{R_4 \leftarrow R_4 - 3R_3} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.10.5.13}$$

Thus,

$$x_1 = x_4, x_2 = 2x_4, x_3 = 2x_4 \implies \mathbf{x} = \begin{pmatrix} x_4 \\ 2x_4 \\ 2x_4 \\ x_4 \end{pmatrix} = x_4 \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix}, \; x_4 = 1 \tag{5.10.5.14}$$

Hence, (5.10.5.3) finally becomes

$$Zn + 2AgNO_3 \rightarrow 2Ag + Zn(NO_3)_2 \tag{5.10.5.15}$$

5.11 Physics

5.11.1. Determine the loop currents in Fig. 5.11.1.1.

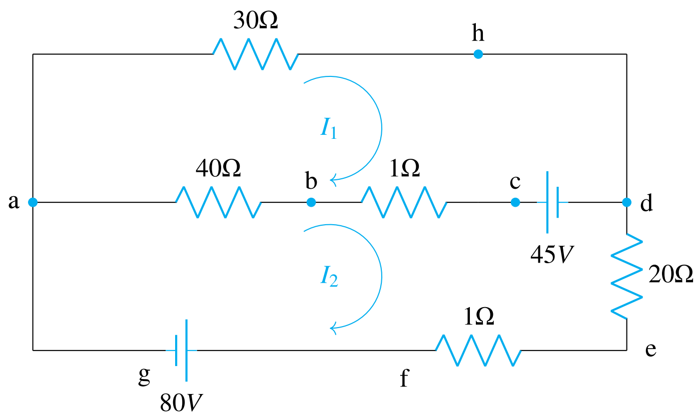


Fig. 5.11.1.1

5.11.2. Determine the current in each branch of the network shown in Fig. 5.11.2.1.

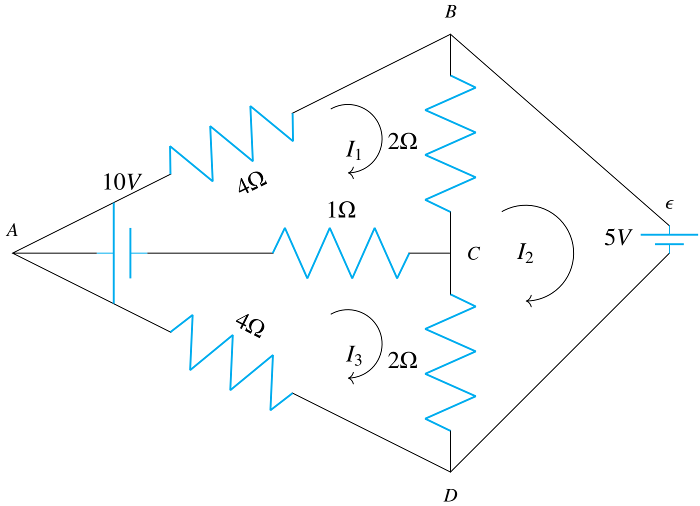


Fig. 5.11.2.1

5.11.3. In Fig. 5.11.3.1, a galvanometer of 15Ω resistance is connected across BD. Calculate the current through the galvanometer when a potential difference of $10V$ is maintained across AC.

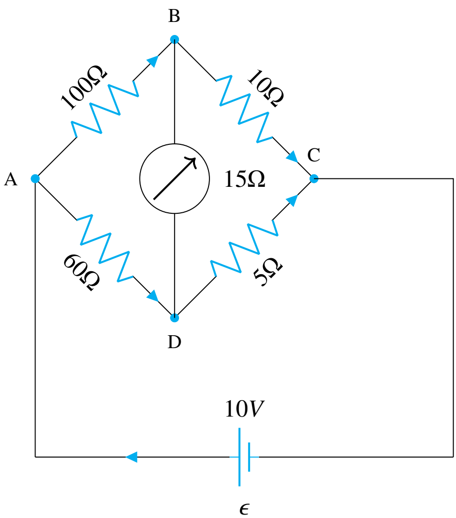


Fig. 5.11.3.1

5.11.4. Determine the current in each branch of the network shown in Fig. 5.11.4.1.

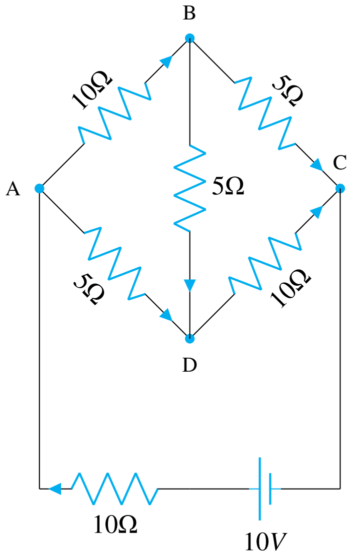


Fig. 5.11.4.1

5.12 CBSE

5.12.1 If $\mathbf{A} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$, find α satisfying $0 < \alpha < \frac{1}{2}$ when $\mathbf{A} + \mathbf{A}^{\mathrm{T}} = \sqrt{2}\mathbf{I}$.

5.12.2 Use elementary column operation $C_2 \to C_2 + 2C_1$ in the following matrix equation

$$\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

5.12.3 Write the number of all possible matrices of order 2×2 with each entry $1, 2$ or 3.

5.12.4 Solve for x

$$\begin{vmatrix} a+x & a-x & a-x \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix} = 0$$

5.12.5 If $x \in N$ and

$$\begin{vmatrix} x+3 & -2 \\ -3x & 2x \end{vmatrix} = 8$$

then find the value of x.

5.12.6 Show that $\triangle ABC$ is isosceles if

$$\begin{vmatrix} 1 & 1 & 1 \\ 1+\cos A & 1+\cos B & 1+\cos C \\ \cos^2 A + \cos A & \cos^2 B + \cos B & \cos^2 C + \cos C \end{vmatrix} = 0$$

5.12.7 Solve the following equations for x and y

$$(ax - by) + (a + 4b) = 0$$
$$(bx + ay) + (b - 4a) = 0$$

(12, 2022)

5.12.8 If $\mathbf{A} = \begin{pmatrix} 0 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 0 \end{pmatrix}$ and $\mathbf{I}$ is the identity matrix of order 2, show that $\mathbf{I} + \mathbf{A} = (\mathbf{I} - \mathbf{A})\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$.

5.13 JEE

5.13.1. If the system of linear equations

$$x + 2ay + az = 0$$
$$x + 3by + bz = 0$$
$$x + 4cy + cz = 0$$

has a non-zero solution, then a,b,c

(2003)

a) satisfy $a + 2b + 3c = 0$
b) are in A.P
c) are in G.P
d) are in H.P

5.13.2. If $\mathbf{A} = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$ and $\mathbf{A}^2 = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}$, then (2003)

a) $\alpha = 2ab, \beta = a^2 + b^2$
b) $\alpha = a^2 + b^2, \beta = ab$
c) $\alpha = a^2 + b^2, \beta = 2ab$
d) $\alpha = a^2 + b^2, \beta = a^2 - b^2$

5.13.3. The number of 3×3 non-singular matrices with four entries as 1 and all other entries as 0, is (2010)

a) 5
b) 6
c) atleast 7
d) less than 4

5.13.4. Let $\mathbf{A}$ be a 2×2 matrix with non-zero entries and let $\mathbf{A}^2 = \mathbf{I}$, where $\mathbf{I}$ is 2×2 identity matrix. Define
$Tr(\mathbf{A})$- sum of diagonal elements of $\mathbf{A}$ and
$|\mathbf{A}|$ - determinant of matrix $\mathbf{A}$.
Statement - 1: $Tr(\mathbf{A}) = 0$.
Statement - 2: $|\mathbf{A}| = 1$ (2010)

a) Statement - 1 is true, Statement - 2 is true; Statement - 2 is **not** a correct explanation for Statement-1.
b) Statement - 1 is true, Statement - 2 is false.
c) Statement - 1 is false, Statement - 2 is true.

d) Statement - 1 is true, Statement - 2 is true; Statement - 2 is a correct explanation for Statement-1.

5.13.5. The system of linear equations (2010)

$$x_1 + 2x_2 + x_3 = 3$$
$$2x_1 + 3x_2 + x_3 = 3$$
$$3x_1 + 5x_2 + 2x_3 = 1$$

a) exactly 3 solutions
b) a unique solution
c) no solution
d) infinite number of solutions

5.13.6. The number of values of k for which the linear equations $4x+ky+2z = 0$, $kx+4y+z = 0$ and $2x + 2y + z = 0$ possess a non zero solution is (2011)

a) 2
b) 1
c) zero
d) 3

5.13.7. Let **A** and **B** be two symmetrix matrices of order 3.
Statement - 1: $\mathbf{A}(\mathbf{BA})$ and $(\mathbf{AB})\mathbf{A}$ are symmetric matrices.
Statement - 2: **AB** is symmetric matrix if matrix multiplication of **A** with **B** is commutative.

a) Statement - 1 is true, Statement - 2 is true; Statement - 2 is **not** a correct explanation for Statement-1.
b) Statement - 1 is true, Statement - 2 is false.
c) Statement - 1 is false, Statement - 2 is true.
d) Statement - 1 is true, Statement - 2 is true; Statement - 2 is a correct explanation for Statement-1.

5.13.8. Let

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$$

If $\mathbf{u_1}$ and $\mathbf{u_2}$ are column matrices such that

$$\mathbf{Au_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ \mathbf{Au_2} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

then $\mathbf{u_1} + \mathbf{u_2}$ is equal to (2012)

a) $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$
b) $\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$
c) $\begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$
d) $\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$

5.13.9. Let **P** and **Q** be 3×3 matrices $\mathbf{P} \neq \mathbf{Q}$. If $\mathbf{P}^3 = \mathbf{Q}^3$ and $\mathbf{P}^2\mathbf{Q} = \mathbf{Q}^2\mathbf{P}$ then determinant of $\left(\mathbf{P}^2 + \mathbf{Q}^2\right)$ is equal to (2012)

a) -2 b) 1 c) 0 d) -1

5.13.10. If

$$\mathbf{P} = \begin{pmatrix} 1 & \alpha & 3 \\ 1 & 3 & 3 \\ 2 & 4 & 4 \end{pmatrix}$$

is the adjoint of a 3×3 matrix $\mathbf{A}$ and $|\mathbf{A}| = 4$, then α is equal to (2014)

a) 4 b) 11 c) 5 d) 0

5.13.11. If $\mathbf{A}$ is a 3×3 non-singular matrix such that $\mathbf{AA}' = \mathbf{A}'\mathbf{A}$ and $\mathbf{B} = \mathbf{A}^{-1}\mathbf{A}'$, then $\mathbf{BB}'$ equals (2014)

a) $\mathbf{B}^{-1}$ b) $\left(\mathbf{B}^{-1}\right)'$ c) $\mathbf{I} + \mathbf{B}$ d) $\mathbf{I}$

5.13.12. The set of all values of λ for which the system of linear equations

$$\begin{aligned} 2x_1 - 2x_2 + x_3 &= \lambda x_1 \\ 2x_1 - 3x_2 + 2x_3 &= \lambda x_2 \\ -x_1 + 2x_2 &= \lambda x_3 \end{aligned}$$

has a non-trivial solution (2015)

a) contains two elements
b) contains more than two elements
c) is an empty set
d) is a singleton

5.13.13. If

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{pmatrix}$$

is a matrix satisfying the equation $\mathbf{AA}^{\top} = 9\mathbf{I}$, where $\mathbf{I}$ is 3×3 identity matrix, then the ordered part (a, b) is equal to: (2015)

a) $(2, 1)$ b) $(-2, -1)$ c) $(2, -1)$ d) $(-2, 1)$

5.13.14. The system of linear equations

$$\begin{aligned} x + \lambda y - z &= 0 \\ \lambda x - y - z &= 0 \\ x + y - \lambda z &= 0 \end{aligned}$$

has a non-trivial solution for (2016)

a) exactly two values of λ
b) exactly three values of λ
c) inifinitely many values of λ
d) exactly one value of λ

5.13.15. If

$$\mathbf{A} = \begin{pmatrix} 5a & -b \\ 3 & 2 \end{pmatrix}$$

and $\mathbf{A}\,\mathrm{adj}\,(\mathbf{A}) = \mathbf{A}\mathbf{A}^\top$, then $5a + b$ is equal to (2016)

a) 4
b) 13
c) -1
d) 5

5.13.16. Let k be an integer such that triangle with vertices $(k, -3k)$, $(5, k)$, $(-k, 2)$ has area 28 sq. units. Then the orthocentre of this triangle is at the point (2017)

a) $\left(2, \frac{1}{2}\right)$
b) $\left(2, \frac{-1}{2}\right)$
c) $\left(1, \frac{3}{4}\right)$
d) $\left(1, \frac{-3}{4}\right)$

5.13.17. If $\mathbf{A} = \begin{pmatrix} 2 & -3 \\ -4 & 1 \end{pmatrix}$, then $\mathrm{adj}\left(3\mathbf{A}^2 + 12\mathbf{A}\right)$ is equal to (2017)

a) $\begin{pmatrix} 72 & -63 \\ -84 & 51 \end{pmatrix}$
b) $\begin{pmatrix} 72 & -84 \\ -63 & 51 \end{pmatrix}$
c) $\begin{pmatrix} 51 & 63 \\ 84 & 72 \end{pmatrix}$
d) $\begin{pmatrix} 51 & 84 \\ 63 & 72 \end{pmatrix}$

5.13.18. If the system of linear equations

$$x + ky + 3z = 0$$
$$3x + ky - 2z = 0$$
$$2x + 4y - 3z = 0$$

has a non-zero solution (x, y, z), then $\frac{xz}{y^2}$ is equal to (2018)

a) 10
b) -30
c) 30
d) -10

5.13.19. The system of linear equations (2019)

$$x + y + z = 2$$
$$2x + 3y + 2z = 5$$
$$2x + 3y + \left(a^2 - 1\right)z = a + 1$$

a) is consistent when $a = 4$
b) has a unique solution for $|a| = \sqrt{3}$
c) has infinitely many solutions for $a = 4$
d) is consistent when $|a| = \sqrt{3}$

5.13.20. If $\mathbf{A} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, then the matrix $\mathbf{A}^{-50}$ when $\theta = \frac{\pi}{12}$, is equal to: (2019)

a) $\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$

b) $\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$

c) $\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$

d) $\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$

5.13.21. If

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & n-1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 78 \\ 0 & 1 \end{pmatrix}.$$

then the inverse of $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ is (2019)

a) $\begin{pmatrix} 1 & 0 \\ 12 & 1 \end{pmatrix}$

b) $\begin{pmatrix} 1 & -13 \\ 0 & 1 \end{pmatrix}$

c) $\begin{pmatrix} 1 & -12 \\ 0 & 1 \end{pmatrix}$

d) $\begin{pmatrix} 1 & 0 \\ 13 & 1 \end{pmatrix}$

5.13.22. Let $\mathbf{A} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$. The only correct statement about the matrix $\mathbf{A}$ is (2004)

a) $\mathbf{A}^2 = \mathbf{I}$

c) $\mathbf{A}^{-1}$ does not exist

b) $\mathbf{A} = (-1)\mathbf{I}$, where $\mathbf{I}$ is a unit matrix

d) $\mathbf{A}$ is a zero matrix

5.13.23. Let $\mathbf{A} = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{pmatrix}$, and $10\mathbf{B} = \begin{pmatrix} 4 & 2 & 2 \\ -5 & 0 & \alpha \\ 1 & -2 & 3 \end{pmatrix}$. If $\mathbf{B}$ is the inversion of matrix $\mathbf{A}$, then α is (2004)

a) 5

b) -1

c) 2

d) -2

5.13.24. If $\mathbf{A}^2 - \mathbf{A} + \mathbf{I} = 0$, then the inverse of $\mathbf{A}$ is (2005)

a) $\mathbf{A} + \mathbf{I}$

b) $\mathbf{A}$

c) $\mathbf{A} - \mathbf{I}$

d) $\mathbf{I} - \mathbf{A}$

5.13.25. The system of equations

$$\alpha x + y + z = \alpha - 1$$
$$x + \alpha y + z = \alpha - 1$$
$$x + y + \alpha z = \alpha - 1$$

has infinite solutions, if α is (2005)

a) -2 b) either -2 or 1 c) not -2 d) 1

5.13.26. If $\mathbf{A}$ and $\mathbf{B}$ are square matrices of size $n \times n$ such that $\mathbf{A}^2 - \mathbf{B}^2 = (\mathbf{A} - \mathbf{B})(\mathbf{A} + \mathbf{B})$, then which of the following will be always true? (2006)

a) $\mathbf{A} = \mathbf{B}$
b) $\mathbf{AB} = \mathbf{BA}$
c) either of $\mathbf{A}$ or $\mathbf{B}$ is zero matrix
d) either of $\mathbf{A}$ or $\mathbf{B}$ is identity matrix

5.13.27. Let $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, a,b $\in$ N. Then (2006)

a) there cannot exist any $\mathbf{B}$ such that $\mathbf{AB} = \mathbf{BA}$
b) there exist more than one but finite number of $\mathbf{B}$'s such that $\mathbf{AB} = \mathbf{BA}$
c) there exists exactly one $\mathbf{B}$ such that $\mathbf{AB} = \mathbf{BA}$
d) there exist infinitely many $\mathbf{B}$'s such that $\mathbf{AB} = \mathbf{BA}$

5.13.28. Let $\mathbf{A}$ be a 2×2 matrix with real entries. Let $\mathbf{I}$ be the 2×2 identity matrix. Denote by tr($\mathbf{A}$),the sum of diagonal entries of $\mathbf{A}$. Assume that $\mathbf{A}^2 = \mathbf{I}$. (2008)
Statement-1 : If $\mathbf{A} \neq \mathbf{I}$ and $\mathbf{A} \neq -\mathbf{I}$,then det($\mathbf{A}$)=-1
Statement-2 : If $\mathbf{A} \neq \mathbf{I}$ and $\mathbf{A} \neq -\mathbf{I}$,then $tr(\mathbf{A}) \neq 0$.

a) Statement-1 is false, Statement-2 is true
b) Statement-1 is true, Statement-2 is true;Statement-2 is a correct explanation for Statement-1
c) Statement-1 is true, Statement-2 is true;Statement-2 is not a correct explanation for Statement-1
d) Statement-1 is true, Statement-2 is false

5.13.29. Let a,b,c be any real numbers. Suppose that there are real numbers x,y,z not all zero such that $x = cy + bz$, $y = az + cx$, and $z = bx + ay$. Then $a^2 + b^2 + c^2 + 2abc$ is equal to (2008)

a) 2 b) -1 c) 0 d) 1

5.13.30. Let $\mathbf{A}$ be a square matrix all of whose entries are integers. Then which of the following is true? (2008)

a) If $det(\mathbf{A}) \neq \pm 1$, then $\mathbf{A}^{-1}$ exists but all its entries are not necessarily integers
b) If $det(\mathbf{A}) \neq \pm 1$, then $\mathbf{A}^{-1}$ exists and all its entries are non integers
c) If $det(\mathbf{A}) = \pm 1$, then $\mathbf{A}^{-1}$ exists but all its entries are integers
d) If $det(\mathbf{A}) = \pm 1$, then $\mathbf{A}^{-1}$ need not exist

5.13.31. Let $\mathbf{A}$ and $\mathbf{B}$ be 3×3 matrices of real numbers, where $\mathbf{A}$ is symmetric, $\mathbf{B}$ is skew-symmetric and $(\mathbf{A}+\mathbf{B})(\mathbf{A}-\mathbf{B}) = (\mathbf{A}-\mathbf{B})(\mathbf{A}+\mathbf{B})$. If $(\mathbf{AB})^\top = (-1)^k \mathbf{AB}$, where $(\mathbf{AB})^\top$ is the transpose of the matrix $\mathbf{AB}$, then the possible values of k are ______. (2008)

5.13.32. Let $\mathbf{M}$ and $\mathbf{N}$ be two 3×3 non-singulr skew-symmetric matrices such that $\mathbf{MN} = \mathbf{NM}$. If $\mathbf{P}^\top$ denotes the transpose of $\mathbf{P}$, then $\mathbf{M}^2\mathbf{N}^2 \left(\mathbf{M}^\top\mathbf{N}^{-1}\right)^{-1} \left(\mathbf{MN}^{-1}\right)^\top$ is equal to

a) $\mathbf{M}^2$ b) $-\mathbf{N}^2$ c) $-\mathbf{M}^2$ d) $\mathbf{MN}$

(2011)

5.13.33. If the adjoint of a 3×3 matrix $\mathbf{P}$ is $\begin{pmatrix} 1 & 4 & 4 \\ 2 & 1 & 7 \\ 1 & 1 & 3 \end{pmatrix}$, then the possible value(s) of the determinant of $\mathbf{P}$ is (are) (2012)

a) -2 b) -1 c) 1 d) 2

5.13.34. For 3×3 matrices $\mathbf{M}$ and $\mathbf{N}$, which of the following statement(s) is (are) NOT correct?

a) $\mathbf{N}^\top\mathbf{MN}$ is symmetric or skew symmetric, according as $\mathbf{M}$ is symmetric or skew symmetric
b) $\mathbf{MN} - \mathbf{NM}$ is skew symmetric for all matrices $\mathbf{M}$ and $\mathbf{N}$.
c) $\mathbf{MN}$ is symmetric for all symmetric matrices $\mathbf{M}$ and $\mathbf{N}$.
d) $(\text{adj}\mathbf{M})(\text{adj}\mathbf{N}) = \text{adj}(\mathbf{MN})$ for all invertible matrices $\mathbf{M}$ and $\mathbf{N}$.

(2013)

5.13.35. Let ω be a complex cube root of unity with $\omega \neq 1$ and $\mathbf{P} = p_{ij}$ be a $n \times n$ matrix with $p_{ij} = \omega^{i+j}$. Then $p^2 \neq 0$, when $n =$ (2013)

a) 57 b) 55 c) 58 d) 56

5.13.36. Let $\mathbf{M}$ be a 2×2 symmetric matrix with integer entries. Then $\mathbf{M}$ is invertible if

a) The first column of $\mathbf{M}$ is the transpose of the second row of $\mathbf{M}$
b) The second row of $\mathbf{M}$ is the transpose of the first column of $\mathbf{M}$
c) $\mathbf{M}$ is a diagonal matrix with non-zero entries in the main diagonal
d) The product of entries in the main diagonal of $\mathbf{M}$ is not the square of an integer

(2014)

5.13.37. Let $\mathbf{M}$ and $\mathbf{N}$ be two 3×3 matrices such that $\mathbf{MN} = \mathbf{NM}$. Further, if $\mathbf{M} \neq \mathbf{N}^2$ and $\mathbf{M}^2 = \mathbf{N}^4$, then (2014)

a) determinant of $(\mathbf{M}^2 + \mathbf{N}^2)$ is 0
b) there is 3×3 non-zero matrix $\mathbf{U}$ such that $(\mathbf{M}^2 + \mathbf{MN}^2)\mathbf{U}$ is the zero matrix
c) determinant of $(\mathbf{M}^2 + \mathbf{MN}^2) \geq 1$
d) determinant of $(\mathbf{M}^2 + \mathbf{MN}^2)\mathbf{U}$ equals the zero matrix then $\mathbf{U}$ is the zero matrix

5.13.38. Let $\mathbf{X}$ and $\mathbf{Y}$ be two arbitrary, 3×3, non-zero, skew-symmetric matrices and Z be an arbitrary 3×3, non-zero, symmetric matrix. Then which of the following matrices is (are) skew symmetric? (2015)

a) $\mathbf{Y}^3Z^4 - Z^4\mathbf{Y}^3$ b) $\mathbf{X}^{44} + \mathbf{Y}^{44}$ c) $\mathbf{X}^4Z^3 - Z^3\mathbf{X}^4$ d) $\mathbf{X}^{23} + \mathbf{Y}^{23}$

5.13.39. Let $\mathbf{P} = \begin{pmatrix} 3 & -1 & -2 \\ 2 & 0 & \alpha \\ 3 & -5 & 0 \end{pmatrix}$, where $\alpha \in \mathbb{R}$. Suppose $\mathbf{Q} = \left[\mathrm{q_{ij}}\right]$ is a matrix such that $\mathbf{PQ} = k\mathbf{I}$, where $k \in \mathbb{R}, k \neq 0$ and $\mathbf{I}$ is the identity matrix of order 3. If $q_{23} = -\frac{k}{8}$ and $\det(\mathbf{Q}) = \frac{k^2}{2}$, then (2016)

a) $a = 0, k = 8$
b) $4a - k + 8 = 0$
c) $\det(\mathbf{P}\operatorname{adj}(\mathbf{Q})) = 2^9$
d) $\det(\mathbf{Q}\operatorname{adj}(\mathbf{P})) = 2^{13}$

5.13.40. Let $a, \lambda, \mu, \in \mathbb{R}$. Consider the system of linear equations

$$ax + 2y = \lambda$$

$$3x - 2y = \mu$$

Which of the following statement(s) is (are) correct?

a) If $a = -3$, then the system has infinitely many solutions for all value of λ and μ.
b) If $a \neq -3$, then the system has unique solution for all values of λ and μ.
c) If $\lambda + \mu = 0$, then the system has infinitely many solutions for $a = -3$.
d) If $\lambda + \mu \neq 0$, then the system has no solution for $a = -3$

(2016)

5.13.41. Which of the following is (are) not the square of a 3×3 matrix with real entries?

a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
c) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$
b) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$
d) $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

(2017)

5.13.42. Let S be set of all column matrix $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ such that $b_1, b_2, b_3 \in \mathbb{R}$ and the system of equations (in real variables)

$$-x + 2y + 5z = b_1$$

$$2x - 4y + 3z = b_2$$

$$x - 2y + 2z = b_3$$

has at least one solution. Then, which of the following system(s) (in real variables) has (have) at least one solution for each $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \in S$ (2018)

a) $x + 2y + 3z = b_1, 4y + 5z = b_2$ and $x + 2y + 6z = b_3$
b) $x + y + 3z = b_1, 5x + 2y + 6z = b_2$ and $-2x - y - 3z = b_3$
c) $-x + 2y - 5z = b_1, 2x - 4y + 10z = b_2$ and $x - 2y + 5z = b_3$
d) $sx + 2y + 5z = b_1, 2x + 3z = b_2, x + 4y - 5z = b_3$

5.13.43. Let $\mathbf{M} = \begin{pmatrix} 0 & 1 & a \\ 1 & 2 & 3 \\ 3 & b & 1 \end{pmatrix}$ and $\operatorname{adj}(\mathbf{M}) = \begin{pmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & 1 \end{pmatrix}$ where a and b are real numbers. Which of the following options is/are correct? (2019)

a) $a + b = 3$

b) $det(adj\,\mathbf{M}^2) = 81$

c) $(adj\mathbf{M})^{-1} + adj\mathbf{M}^{-1} = -\mathbf{M}$

d) If $\mathbf{M}\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix}$ then $\alpha - \beta + \gamma = 3$

5.13.44. Let

$$\mathbf{P}_1 = \mathbf{I} = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}, \mathbf{P}_2 = \begin{pmatrix}1&0&0\\0&0&1\\0&1&1\end{pmatrix}, \mathbf{P}_3 = \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}, \mathbf{P}_4 = \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix},$$

$$\mathbf{P}_5 = \begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}, \mathbf{P}_6 = \begin{pmatrix}0&0&1\\0&1&0\\1&0&1\end{pmatrix} \text{ and } \mathbf{X} = \sum_{k=1}^{6} \mathbf{P}_k \begin{pmatrix}2&1&3\\1&0&2\\3&2&1\end{pmatrix} \mathbf{P}_k^{\top}$$

Where $\mathbf{P}_k^{\top}$ denotes the transpose of matrix $\mathbf{P}_k$. Then which of the following options is/are correct? (2019)

a) $\mathbf{X}$ is a symmetric matrix

b) The sum of diagonal elements of $\mathbf{X}$ is 18

c) $\mathbf{X} - 30\mathbf{I}$ is an invertible matrix

d) If $\mathbf{X}\begin{pmatrix}1\\1\\1\end{pmatrix} = \alpha\begin{pmatrix}1\\1\\1\end{pmatrix}$, then α is 30

5.13.45. Let $x \in R$ and let

$$\mathbf{P} = \begin{pmatrix}1&1&1\\0&2&2\\0&0&3\end{pmatrix}, \mathbf{Q} = \begin{pmatrix}2&x&x\\0&4&0\\x&x&5\end{pmatrix} \text{ and } \mathbf{R} = \mathbf{PQP}^{-1}$$

Then which of the following options is/are correct? (2019)

a) $\det \mathbf{R} = \det\begin{pmatrix}2&x&x\\0&4&0\\x&x&5\end{pmatrix} + 8$, for all $x \in R$

b) For $x = 1$, there exists a unit vector $\alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}$ for which $\mathbf{R}\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

c) There exists a real number x such that $\mathbf{PQ} = \mathbf{QP}$

d) For $x = 0$, if $\mathbf{R} = \begin{pmatrix}1\\a\\b\end{pmatrix} = 6\begin{pmatrix}1\\a\\b\end{pmatrix}$, then a+b=5

5.13.46. Consider the set A of all determinants of order 3 with entries 0 or 1 only. Let B be the subset of A consisting of all determinants with value 1. Let C be the subset of A consisting of all determinants with value -1. Then (1981)

a) C is empty

b) B has as many elements as C

c) $\mathbf{A} = B \cup C$

d) B has twice as many elements as C

5.13.47. Let a, b, c be the real numbers. Then following system of equations in x, y and z

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1, \frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, -\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

has (1995)

a) no solution
c) infinitely many solutions
b) unique solution
d) finitely many solutions

5.13.48. If $\mathbf{A}$ and $\mathbf{B}$ are square matrices of equal degree, then which one is correct among the followings? (1995)

a) $\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}$
c) $\mathbf{A}-\mathbf{B}=\mathbf{B}-\mathbf{A}$
b) $\mathbf{A}+\mathbf{B}=\mathbf{A}-\mathbf{B}$
d) $\mathbf{AB}=\mathbf{BA}$

5.13.49. If the system of equations

$$x-ky-z=0,$$
$$kx-y-z=0,$$
$$x+y-z=0$$

has a non-zero solution, then the possible values of k are (2000)

a) -1,2 b) 1,2 c) 0,1 d) -1,1

5.13.50. The number of values of k for which the system of equations

$$(k+1)\,x+8y=4k;$$
$$kx+(k+3)\,y=3k-1$$

has infinitely many solutions is (2002)

a) 0 b) 1 c) 2 d) infinte

5.13.51. If $\mathbf{A}=\begin{pmatrix}\alpha & 0\\ 1 & 1\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}1 & 0\\ 5 & 1\end{pmatrix}$, then value of α for which $\mathbf{A}^2=\mathbf{B}$, is (2003)

a) 1
b) 4

c) 2
d) infinite

5.13.52. If the system of equations $x+ay=0, az+y=0$ and $ax+z=0$ has infinite solutions, then the value of a is (2003)

a) -1
b) 1
c) 0
d) no real values

5.13.53. Given

$$\begin{aligned} 2x - y + 2z &= 2, \\ x - 2y + z &= -4, \\ x + y + \lambda z &= 4 \end{aligned}$$

then the value of λ such that the given system of equation has NO solution, is (2004)

a) 3
b) 1
c) 0
d) -3

5.13.54. If $\mathbf{A} = \begin{pmatrix} \alpha & 2 \\ 2 & \alpha \end{pmatrix}$ and $\left|\mathbf{A}^3\right| = 125$ then the value α is (2004)

a) ± 1
b) ± 2
c) ± 3
d) ± 5

5.13.55. $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{pmatrix}$ and $\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\mathbf{A}^{-1} = \left(\frac{1}{6}\left(\mathbf{A}^2 + c\mathbf{A} + d\mathbf{I}\right)\right)$, then the value of c and d are (2005)

a) $(-6, -11)$
b) $(6, 11)$
c) $(-6, 11)$
d) $(6, -11)$

5.13.56. If $\mathbf{P} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$ and $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{Q} = \mathbf{PAP}^\top$ and $x = \mathbf{P}^\top\mathbf{Q}^{2005}\mathbf{P}$ then x is equal to

a) $\begin{pmatrix} 1 & 2005 \\ 0 & 1 \end{pmatrix}$
b) $\begin{pmatrix} 4 + 2005\sqrt{3} & 6015 \\ 2005 & 4 - 2005\sqrt{3} \end{pmatrix}$
c) $\frac{1}{4}\begin{pmatrix} 2 + \sqrt{3} & 1 \\ -1 & 2 - \sqrt{3} \end{pmatrix}$
d) $\frac{1}{4}\begin{pmatrix} 2005 & 2 - \sqrt{3} \\ 2 + \sqrt{3} & 2005 \end{pmatrix}$

5.13.57. The number of 3×3 matrices $\mathbf{A}$ whose entries are either 0 or 1 and for which the system $\mathbf{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ has exactly two distinct solutions is (2008)

a) 0 b) $2^9 - 1$ c) 168 d) 2

5.13.58. Let $\omega \neq 1$ be a cube root of unity and S be the set of all non-singular matrices of the form

$$\begin{pmatrix} 1 & a & b \\ \omega & 1 & c \\ \omega^2 & \omega & 1 \end{pmatrix}$$

where each of a, b and c is either ω or ω^2. Then the number of distinct matrices in the set S is (2008)

a) 2 b) 6 c) 4 d) 8

5.13.59. Let $\mathbf{P} = \left(a_{ij}\right)$ be a 3×3 matrix and let $\mathbf{Q} = \left(b_{ij}\right)$, where $b_{ij} = 2^{i+j} a_{ij}$ for $1 \leq i, j \leq 3$. If the determinant of $\mathbf{P}$ is 2, then the determinant of the matrix $\mathbf{Q}$ is (2012)

a) 2^{10} b) 2^{11} c) 2^{12} d) 2^{13}

5.13.60. If $\mathbf{P}$ is a 3×3 matrix such that $\mathbf{P}^\top = 2\mathbf{P} + \mathbf{I}$, where $\mathbf{P}^\top$ is the transpose of $\mathbf{P}$ and $\mathbf{I}$ is the 3×3 identity matrix, then there exists a column matrix $\mathbf{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ (2012)

a) $\mathbf{PX} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ b) $\mathbf{PX} = \mathbf{X}$

c) $\mathbf{PX} = 2\mathbf{X}$ d) $\mathbf{PX} = -\mathbf{X}$

5.13.61. Let $\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 16 & 4 & 1 \end{pmatrix}$ and $\mathbf{I}$ be the identity matrix of order 3. If $\mathbf{Q} = \left(q_{ij}\right)$ is a matrix such that $\mathbf{P}^{50} - \mathbf{Q} = \mathbf{I}$, then $\frac{q_{31}+q_{32}}{q_{21}}$ equals ($JEEAdv.$2016)

a) 52 b) 103 c) 201 d) 205

5.13.62. How many 3×3 matrices $\mathbf{M}$ with entries from $(0, 1, 2)$ are there, for which the sum of the diagonal entries of $\mathbf{M}^\top\mathbf{M}$ is 5? (2017)

a) 126 b) 198 c) 162 d) 135

5.13.63. Let $\mathbf{M} = \begin{pmatrix} \sin^4(\theta) & -1 - \sin^2(\theta) \\ 1 + \cos^2(\theta) & \cos^4(\theta) \end{pmatrix} = \alpha\mathbf{I} + \beta\mathbf{M}^{-1}$
Where $\alpha = \alpha(\theta)$ and $\beta = \beta(\theta)$ are real numbers, and $\mathbf{I}$ is the 2×2 identity matrix. If a* is the minimum of the set $(\alpha(\theta) : \theta \in [0, 2\pi))$ and b* is the minimum of the set $(\beta(\theta) : \theta \in [0, 2\pi))$. Then the value of a* + b* is (2019)

a) $-\frac{31}{16}$ b) $-\frac{17}{16}$ c) $-\frac{37}{16}$ d) $-\frac{29}{16}$

5.13.64. Let $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$ and $\mathbf{U_1}$, $\mathbf{U_2}$ and $\mathbf{U_3}$ are columns of a 3×3 matrix $\mathbf{U}$. If column matrices $\mathbf{U_1}$, $\mathbf{U_2}$ and $\mathbf{U_3}$ satisfy $\mathbf{AU_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{AU_2} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$, $\mathbf{AU_3} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ evaluate as directed in the following questions. (2006)

a) The value $|\mathbf{U}|$ is

i) 3 ii) -3 iii) $\frac{3}{2}$ iv) 2

b) The sum of the elements of the matrix $\mathbf{U}^{-1}$ is

i) -1 ii) 0 iii) 1 iv) 3

c) The value of $\begin{pmatrix} 3 & 2 & 0 \end{pmatrix} \mathbf{U} \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$ is

i) 5 ii) $\frac{5}{2}$ iii) 4 iv) $\frac{3}{2}$

5.13.65. Let $\mathcal{A}$ be the set of all 3×3 symmetric matrices all of whose entries are either 0 or 1. Five of these entries are 1 and four of them are 0. (2009)

a) The number of matrices in $\mathcal{A}$ is

i) less than 4
ii) at least 4 but less than 7
iii) at least 7 but less than 10
iv) at least 10

b) The number of matrices $\mathbf{A}$ in $\mathcal{A}$ for which the system of linear equations

$$\mathbf{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

has a unique solution, is

i) less than 4
ii) at least 4 but less than 7
iii) at least 7 but less than 10
iv) at least 10

c) The number of matrices $\mathbf{A}$ in $\mathcal{A}$ for which the system of linear equations

$$\mathbf{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

is inconsistent, is

i) 0 iii) 2
ii) more than 2 iv) 1

5.13.66. Let p be an odd prime number and $\mathbf{T_p}$ be the following set of 2×2 matrices

$$\mathbf{T}_p = \left\{ \mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c \in \{0, 1, 2, \ldots, p-1\} \right\}$$

(2010)

a) The number of $\mathbf{A}$ in $\mathbf{T}_p$ such that $\mathbf{A}$ is either symmetric or skew-symmetric or both, and det($\mathbf{A}$) divisibly by p is

i) $(p-1)^2$ ii) $2(p-1)$ iii) $(p-1)^2 + 1$ iv) $2p-1$

b) The number of $\mathbf{A}$ in $\mathbf{T}_p$ such that the trace of $\mathbf{A}$ is not divisible by p but det($\mathbf{A}$) is divisible by p is
[**Note:** The trace of a matrix is the sum of its diagonal entries.]

i) $(p-1)\left(p^2 - p + 1\right)$ ii) $p^3 - (p-1)^2$ iii) $(p-1)^2$ iv) $(p-1)\left(p^2 - 2\right)$

c) The number of $\mathbf{A}$ in $\mathbf{T}_p$ such that det($\mathbf{A}$) is not divisible by p is

i) $2p^2$ ii) $p^3 - 5p$ iii) $p^3 - 3p$ iv) $p^3 - p^2$

5.13.67. For what value of k do the following system of equations possess a non trivial (i.e., not all zero) solution over the set of rationals Q?

$$x + ky + 3z = 0$$
$$3x + ky - 2z = 0$$
$$2x + 3y - 4z = 0$$

For that value of k, find all the solutions of the system. (1979)

5.13.68. Consider the system of linear equations in x, y, z

$$(\sin 3\theta)x - y + z = 0$$
$$(\cos 2\theta)x + 4y + 3z = 0$$
$$2x + 7y + 7z = 0$$

Find the values of θ for which this system has non trivial solutions. (1986)

5.13.69. Let λ and α be real. Find the set of all values of λ for which the system of linear equations

$$\lambda x + (\sin\alpha)y + (\cos\alpha) = 0$$
$$x + (\cos\alpha)y + (\sin\alpha)z = 0$$
$$-x + (\sin\alpha)y - (\cos\alpha)z = 0$$

has a non trivial solution. For $\lambda = 1$, find all values of α. (1993)

5.13.70. If matrix $\mathbf{A} = \begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}$ where a, b, c are real positive numbers, $abc = 1$ and $\mathbf{A}^\top \mathbf{A} = \mathbf{I}$, then find the value of $a^3 + b^3 + c^3$. (2003)

5.13.71. If $\mathbf{M}$ is a 3×3 matrix, where $\det \mathbf{M} = 1$ and $\mathbf{M}\mathbf{M}^\top = \mathbf{I}$, where $\mathbf{I}$ is an identity matrix, prove that $\det(\mathbf{M} - \mathbf{I}) = 0$. (2004)

5.13.72. If $\mathbf{A} = \begin{pmatrix} a & 1 & 0 \\ 1 & b & d \\ 1 & b & c \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} a & 1 & 1 \\ 0 & d & c \\ f & g & h \end{pmatrix}$, $\mathbf{U} = \begin{pmatrix} f \\ g \\ h \end{pmatrix}$, $\mathbf{V} = \begin{pmatrix} a^2 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $\mathbf{AX} = \mathbf{U}$ has infinitely many solutions, prove that $\mathbf{BX} = \mathbf{V}$ has no unique solution. Also show that if $afd \neq 0$, then $\mathbf{BX} = \mathbf{V}$ has no solution. (2004)

5.13.73. Let M be a 3×3 invertible matrix with real entries and let I denote the 3×3 identity matrix. If $M^{-1} = \text{adj}(\text{adj } M)$ then which of the following statements is/are ALWAYS TRUE? (2020)

a) $M = I$

b) $\det M = 1$

c) $M^2 = I$

d) $(\text{adj } M)^2 = I$

5.13.74. The trace of a square matrix is defined to be the sum of its diagonal entries. If $\mathbf{A}$ is a 2×2 matrix, such that the trace of $\mathbf{A}$ is 3 and the trace of $\mathbf{A}^3$ is -18, then the value of the determinant of $\mathbf{A}$ is ______. (2020)

5.13.75. For any 3×3 matrix $\mathbf{M}$, let $|\mathbf{M}|$ denote the determinant of $\mathbf{M}$. Let

$$\mathbf{E} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 8 & 13 & 18 \end{pmatrix}, \quad \mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} 1 & 3 & 2 \\ 8 & 18 & 13 \\ 2 & 4 & 3 \end{pmatrix}$$

If $\mathbf{Q}$ is a nonsingular matrix of order 3×3, then which of the following statements is (are) TRUE? (2021)

a) $\mathbf{F} = \mathbf{PEP}$ and $\mathbf{P}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

b) $|\mathbf{EQ} + \mathbf{PFQ}^{-1}| = |\mathbf{EQ}| + |\mathbf{PFQ}^{-1}|$

c) $|(\mathbf{EF})^3| > |\mathbf{EF}|^2$

d) Sum of the diagonal entries of $\mathbf{P}^{-1}\mathbf{EP} + \mathbf{F}$ is equal to the sum of diagonal entries of $\mathbf{E} + \mathbf{P}^{-1}\mathbf{FP}$

5.13.76. For any 3×3 matrix $\mathbf{M}$, let $|\mathbf{M}|$ denote the determinant of $\mathbf{M}$. Let $\mathbf{I}$ be the 3×3 identity matrix. Let $\mathbf{E}$ and $\mathbf{F}$ be two 3×3 matrices such that $(\mathbf{I} - \mathbf{EF})$ is invertible. If $\mathbf{G} = (\mathbf{I} - \mathbf{EF})^{-1}$, then which of the following statements is (are) TRUE? (2021)

a) $|\mathbf{FE}| = |\mathbf{I} - \mathbf{FE}| \cdot |\mathbf{FGE}|$

b) $(\mathbf{I} - \mathbf{FE})(\mathbf{I} + \mathbf{FGE}) = \mathbf{I}$

c) $\mathbf{EFG} = \mathbf{GEF}$

d) $(\mathbf{I} - \mathbf{FE})(\mathbf{I} - \mathbf{FGE}) = \mathbf{I}$

5.13.77. If

$$M = \begin{pmatrix} \frac{5}{2} & \frac{3}{2} \\ [1ex] -\frac{3}{2} & \frac{1}{2} \end{pmatrix},$$

then which of the following matrices is equal to M^{2022}? (2022)

a) $\begin{pmatrix} 3034 & 3033 \\ -3033 & -3032 \end{pmatrix}$
c) $\begin{pmatrix} 3033 & 3032 \\ -3032 & -3031 \end{pmatrix}$
b) $\begin{pmatrix} 3034 & -3033 \\ 3033 & -3032 \end{pmatrix}$
d) $\begin{pmatrix} 3032 & 3031 \\ -3031 & -3030 \end{pmatrix}$

5.13.78. Let β be a real number. Consider the matrix

$$\mathbf{A} = \begin{pmatrix} \beta & 0 & 1 \\ 2 & 1 & -2 \\ 3 & 1 & -2 \end{pmatrix}$$

If $\mathbf{A}^7 - (\beta - 1)\mathbf{A}^6 - \beta\mathbf{A}^5$ is a singular matrix, then the value of 9β is ______. (2022)

5.13.79. Let $\mathbf{M} = \left(a_{ij}\right)$, $a_{ij} \in \{1, 2, 3\}$ be the 3×3 matrix such that $a_{ij} = 1$ if $j + 1$ is divisible by i, otherwise $a_{ij} = 0$. Then which of the following statements is(are) true? (2023)

a) $\mathbf{M}$ is invertible

b) There exists a nonzero column matrix $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ such that $\mathbf{M}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} -a_1 \\ -a_2 \\ -a_3 \end{pmatrix}$

c) The set $\{\mathbf{X} \in \mathbb{R}^3 : \mathbf{MX} = \mathbf{0}\} \neq \{\mathbf{0}\}$, where $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

d) The matrix $(\mathbf{M} - 2\mathbf{I})$ is invertible, where $\mathbf{I}$ is the 3×3 identity matrix

5.13.80. Let

$$\mathbf{R} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \{0, 3, 5, 7, 11, 13, 17, 19\}.$$

Then the number of invertible matrices in $\mathbf{R}$ is ______. (2023)

5.13.81. Let

$$S = \{\mathbf{A} = \begin{pmatrix} 0 & 1 & c \\ 1 & a & d \\ 1 & b & e \end{pmatrix} : a, b, c, d, e \in \{0, 1\} \text{ and } |\mathbf{A}| \in \{-1, 1\}\},$$

where $|\mathbf{A}|$ denotes the determinant of $\mathbf{A}$. Then the number of elements in S is ______. (2024)

5.13.82. Let α and β be the distinct roots of the equation $x^2 + x - 1 = 0$. Consider the set $T = \{1, \alpha, \beta\}$. For a 3×3 matrix $\mathbf{M} = \left(a_{ij}\right)_{3\times 3}$, define $R_i = a_{i1} + a_{i2} + a_{i3}$ and $C_j = a_{1j} + a_{2j} + a_{3j}$ for $i = 1, 2, 3$ and $j = 1, 2, 3$. Match the following

A The number of matrices $\mathbf{M} = \left(a_{ij}\right)_{3\times 3}$ with all entries in T such that $R_i = C_j = 0$ for all i, j, is

B The number of symmetric matrices $\mathbf{M} = \left(a_{ij}\right)_{3\times 3}$ with all entries in T such that $C_j = 0$ for all j, is

C Let $\mathbf{M} = \left(a_{ij}\right)_{3\times 3}$ be a skew-symmetric matrix such that $a_{ij} \in T$ for $i > j$. Then the number of elements in the set

$$\left\{\begin{pmatrix} x \\ y \\ z \end{pmatrix} : x, y, z \in \mathbb{R}, \mathbf{M}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & a_{12} & 0 \\ -a_{12} & 0 & a_{23} \\ 0 & -a_{23} & 0 \end{pmatrix}\right\}$$

is

D Let $\mathbf{M} = \left(a_{ij}\right)_{3\times 3}$ be a matrix with all entries in T such that $R_i = 0$ for all i. Then the absolute value of the determinant of $\mathbf{M}$ is

1 1
2 12
3 infinite

4 6
5 0

The correct option is (2024)

a) $A \to 4,\ B \to 2,\ C \to 5,\ D \to 1$
b) $A \to 2,\ B \to 4,\ C \to 1,\ D \to 5$
c) $A \to 2,\ B \to 4,\ C \to 3,\ D \to 5$
d) $A \to 1,\ B \to 5,\ C \to 3,\ D \to 4$

5.14 GATE

5.14.1. In the interconnection of ideal sources shown in Fig. 5.14.1.1, it is known that the $60V$ source is absorbing power.

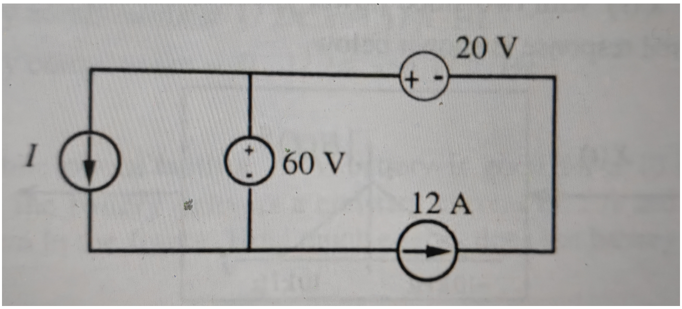


Fig. 5.14.1.1

Which of the following can be the value of current source I ?

a) 10 A b) 13 A c) 15 A d) 18 A

(EC 2009)

5.14.2. In the Fig. 5.14.2.1 shown below, what value of R_l maximizes the power delivered to R_l ?

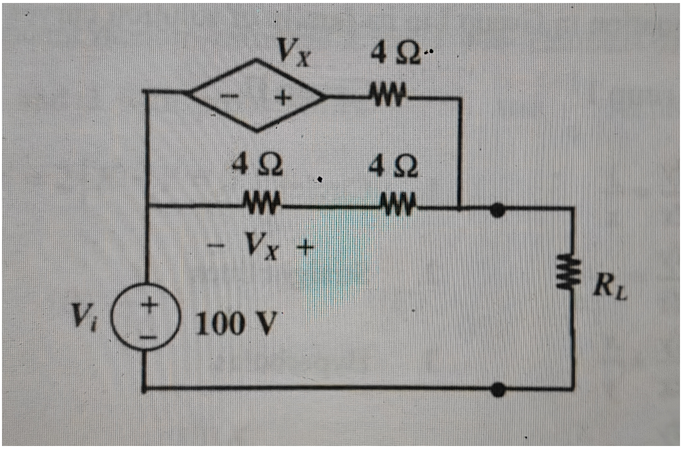


Fig. 5.14.2.1

a) 2.4Ω b) $\frac{8}{3}$Ω c) 4Ω d) 6Ω

(EC 2009)

5.14.3. The 4-point Discrete Fourier Transform (DFT) of a discrete time sequence $\{1, 0, 2, 3\}$ is

a) $[0, -2 + 2j, 2, -2, -2j]$
b) $[2, 2 + 2j, 6, 2 - 2j]$
c) $[6, 1 - 3j, 2, 1 + 3j]$
d) $[6, -1 + 3j, 0, -1 - 3j]$

(EC 2009)

6 Skew Lines

6.1 Formulae

6.1.1. The lines

$$
\begin{aligned}
L_1 : \quad & \mathbf{x} = \mathbf{A} + \kappa_1 \mathbf{m_1} \\
L_2 : \quad & \mathbf{x} = \mathbf{B} + \kappa_2 \mathbf{m_2}
\end{aligned}
\tag{6.1.1.1}
$$

will intersect if

$$
\mathbf{A} + \kappa_1 \mathbf{m_1} = \mathbf{B} + \kappa_2 \mathbf{m_2} \tag{6.1.1.2}
$$

$$
\Longrightarrow \begin{pmatrix} \mathbf{m_1} & \mathbf{m_2} \end{pmatrix} \begin{pmatrix} \kappa_1 \\ -\kappa_2 \end{pmatrix} = \mathbf{B} - \mathbf{A} \tag{6.1.1.3}
$$

$$
\Longrightarrow \operatorname{rank} \begin{pmatrix} \mathbf{M} & \mathbf{B} - \mathbf{A} \end{pmatrix} = 2 \tag{6.1.1.4}
$$

where

$$
\mathbf{M} = \begin{pmatrix} \mathbf{m_1} & \mathbf{m_2} \end{pmatrix} \tag{6.1.1.5}
$$

6.1.2. If L_1, L_2, do not intersect, let

$$
\begin{aligned}
\mathbf{x}_1 &= \mathbf{A} + \kappa_1 \mathbf{m_1} \\
\mathbf{x}_2 &= \mathbf{B} + \kappa_2 \mathbf{m_2}
\end{aligned}
\tag{6.1.2.1}
$$

be points on L_1, L_2 respectively, that are closest to each other. Then, from (6.1.2.1)

$$
\mathbf{x_1} - \mathbf{x_2} = \mathbf{A} - \mathbf{B} + \begin{pmatrix} \mathbf{m_1} & \mathbf{m_2} \end{pmatrix} \begin{pmatrix} \kappa_1 \\ -\kappa_2 \end{pmatrix} \tag{6.1.2.2}
$$

Also,

$$
(\mathbf{x}_1 - \mathbf{x}_2)^\top \mathbf{m}_1 = (\mathbf{x}_1 - \mathbf{x}_2)^\top \mathbf{m}_2 = 0 \tag{6.1.2.3}
$$

$$
\Longrightarrow (\mathbf{x}_1 - \mathbf{x}_2)^\top \begin{pmatrix} \mathbf{m_1} & \mathbf{m_2} \end{pmatrix} = \mathbf{0} \tag{6.1.2.4}
$$

$$
\text{or, } \mathbf{M}^\top (\mathbf{x}_1 - \mathbf{x}_2) = \mathbf{0} \tag{6.1.2.5}
$$

$$
\Longrightarrow \mathbf{M}^\top (\mathbf{A} - \mathbf{B}) + \mathbf{M}^\top \mathbf{M} \begin{pmatrix} \kappa_1 \\ -\kappa_2 \end{pmatrix} = \mathbf{0} \tag{6.1.2.6}
$$

from (6.1.2.2), yielding

$$
\mathbf{M}^\top \mathbf{M} \begin{pmatrix} \kappa_1 \\ -\kappa_2 \end{pmatrix} = \mathbf{M}^\top (\mathbf{B} - \mathbf{A}) \tag{6.1.2.7}
$$

This is known as the *least squares solution*.

6.1.3. Perform the eigendecompositions

$$
\mathbf{M}\mathbf{M}^\top = \mathbf{U}\mathbf{D}_1\mathbf{U}^\top \tag{6.1.3.1}
$$

$$
\mathbf{M}^\top\mathbf{M} = \mathbf{V}\mathbf{D}_2\mathbf{V}^\top \tag{6.1.3.2}
$$

6.1.4. The following expression is known as *singular value decomposition*

$$
\mathbf{M} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top \tag{6.1.4.1}
$$

where $\mathbf{\Sigma}$ is diagonal with entries obtained as in (6.3.1.21). Substituting in (6.1.2.7),

$$\mathbf{V\Sigma U}^\top \mathbf{U\Sigma V}^\top \boldsymbol{\kappa} = \mathbf{V\Sigma U}^\top (\mathbf{B} - \mathbf{A}) \tag{6.1.4.2}$$

$$\implies \mathbf{V\Sigma}^2\mathbf{V}^\top \boldsymbol{\kappa} = \mathbf{V\Sigma U}^\top (\mathbf{B} - \mathbf{A}) \tag{6.1.4.3}$$

$$\implies \boldsymbol{\kappa} = \left(\mathbf{V\Sigma}^2\mathbf{V}^\top\right)^{-1} \mathbf{V\Sigma U}^\top (\mathbf{B} - \mathbf{A}) \tag{6.1.4.4}$$

$$\implies \boldsymbol{\kappa} = \mathbf{V\Sigma}^{-2}\mathbf{V}^\top \mathbf{V\Sigma U}^\top (\mathbf{B} - \mathbf{A}) \tag{6.1.4.5}$$

$$\implies \boldsymbol{\kappa} = \mathbf{V\Sigma}^{-1}\mathbf{U}^\top (\mathbf{B} - \mathbf{A}) \tag{6.1.4.6}$$

where $\mathbf{\Sigma}^{-1}$ is obtained by inverting the nonzero elements of $\mathbf{\Sigma}$.

6.1.5. From (6.1.2.1),

$$\mathbf{x}_1 - \mathbf{x}_2 = \mathbf{A} + \kappa_1 \mathbf{m_1} - \mathbf{B} - \kappa_2 \mathbf{m_2} \tag{6.1.5.1}$$

$$= \mathbf{A} - \mathbf{B} + \mathbf{M}\boldsymbol{\kappa} \tag{6.1.5.2}$$

which, upon substitution from (6.1.4.1) yields

$$\mathbf{x}_1 - \mathbf{x}_2 = \mathbf{A} - \mathbf{B} + \mathbf{U\Sigma V}^\top \mathbf{V\Sigma}^{-1}\mathbf{U}^\top (\mathbf{B} - \mathbf{A}) \tag{6.1.5.3}$$

$$= (\mathbf{A} - \mathbf{B})\left(\mathbf{I} - \mathbf{U\Sigma\Sigma}^{-1}\mathbf{U}^\top\right) \tag{6.1.5.4}$$

Thus,

$$\|\mathbf{x}_1 - \mathbf{x}_2\| = \left\|(\mathbf{A} - \mathbf{B})\left(\mathbf{I} - \mathbf{U\Sigma\Sigma}^{-1}\mathbf{U}^\top\right)\right\| \tag{6.1.5.5}$$

6.1.6. Least squares solution

```
codes/book/skew_least.py
```

6.1.7. Least squares using builtin SVD

```
codes/book/skew_builtin.py
```

6.1.8. Code linking eigenvalues and singular values

```
codes/book/skew_svd.py
```

6.2 Least Squares

6.2.1 Find the shortest distance between the lines

$$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1} \text{ and} \tag{6.2.1.1}$$

$$\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1} \tag{6.2.1.2}$$

Solution: The given lines can be written as

$$\mathbf{x} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} + \kappa_1 \begin{pmatrix} 7 \\ -6 \\ 1 \end{pmatrix}$$
$$\mathbf{x} = \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix} + \kappa_2 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \tag{6.2.1.3}$$

with

$$\mathbf{A} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix}, \mathbf{M} = \begin{pmatrix} 7 & 1 \\ -6 & -2 \\ 1 & 1 \end{pmatrix} \tag{6.2.1.4}$$

Substituting the above in (6.1.1.4),

$$\left(\begin{array}{cc|c} 7 & 1 & 4 \\ -6 & -2 & 6 \\ 1 & 1 & 8 \end{array}\right) \xleftrightarrow[R_3 \leftarrow R_3 - \frac{1}{7}R_1]{R_2 \leftarrow R_2 + \frac{6}{7}R_1} \tag{6.2.1.5}$$

$$\left(\begin{array}{cc|c} 7 & 1 & 4 \\ 0 & -\frac{8}{7} & \frac{66}{7} \\ 0 & \frac{6}{7} & -\frac{52}{7} \end{array}\right) \xleftrightarrow{R_3 \leftarrow R_3 + \frac{3}{4}R_2} \tag{6.2.1.6}$$

$$\left(\begin{array}{cc|c} 2 & 3 & 1 \\ 0 & -\frac{7}{2} & \frac{1}{2} \\ 0 & 0 & -\frac{5}{14} \end{array}\right) \tag{6.2.1.7}$$

The rank of the matrix is 3. So the given lines are skew. From (6.1.2.7)

$$\begin{pmatrix} 7 & -6 & 1 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 7 & 1 \\ -6 & -2 \\ 1 & 1 \end{pmatrix} \boldsymbol{\kappa} = \begin{pmatrix} 7 & -6 & 1 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 6 \\ 8 \end{pmatrix} \tag{6.2.1.8}$$

$$\implies \begin{pmatrix} 86 & 20 \\ 20 & 6 \end{pmatrix} \boldsymbol{\kappa} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{6.2.1.9}$$

$$\implies \begin{pmatrix} \kappa_1 \\ -\kappa_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{6.2.1.10}$$

From (6.2.1.3), the closest points are **A** and **B** and the minimum distance between the lines is given by

$$\|\mathbf{B} - \mathbf{A}\| = \left\| \begin{pmatrix} 4 \\ 6 \\ 8 \end{pmatrix} \right\| = 2\sqrt{29} \tag{6.2.1.11}$$

See Fig. 6.2.1.1.

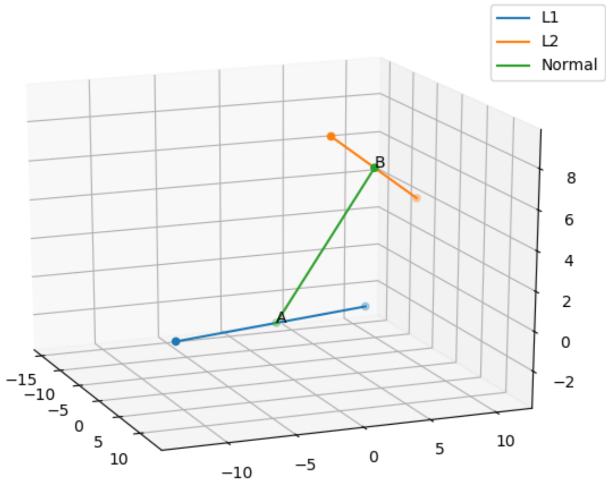


Fig. 6.2.1.1

6.2.2 Find the shortest distance between the lines whose vector equations are

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \kappa_1 \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \qquad \mathbf{x} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + \kappa_2 \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \tag{6.2.2.1}$$

Solution: In this case,

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} \tag{6.2.2.2}$$

$$\mathbf{M} = \begin{pmatrix} 1 & 2 \\ -3 & 3 \\ 2 & 1 \end{pmatrix}. \tag{6.2.2.3}$$

forming the matrix in (6.1.1.4),

$$\begin{pmatrix} 1 & 2 & 3 \\ -3 & 3 & 3 \\ 2 & 1 & 3 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 + 3R_1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 9 & 12 \\ 2 & 1 & 3 \end{pmatrix}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - 2R_1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 9 & 12 \\ 0 & -3 & -3 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow 3R_3 + R_2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 9 & 12 \\ 0 & 0 & 3 \end{pmatrix}$$

Clearly, the rank of this matrix is 3, and therefore, the lines are skew. From (6.1.2.7),

$$
\left(\begin{array}{cc|c} 14 & -5 & 0 \\ -5 & 14 & 18 \end{array}\right)
\xleftrightarrow{R_1 \leftarrow R_1 + R_2}
\left(\begin{array}{cc|c} 9 & 9 & 18 \\ -5 & 14 & 18 \end{array}\right)
$$

$$
\xleftrightarrow{R_1 \leftarrow \frac{R_1}{9}}
\left(\begin{array}{cc|c} 1 & 1 & 2 \\ -5 & 14 & 18 \end{array}\right)
\xleftrightarrow{R_2 \leftarrow R_2 + 5R_1}
\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 19 & 28 \end{array}\right)
$$

$$
\xleftrightarrow{R_1 \leftarrow 19R_1 - R_2}
\left(\begin{array}{cc|c} 19 & 0 & 10 \\ 0 & 19 & 28 \end{array}\right)
\xleftrightarrow[R_2 \leftarrow \frac{R_2}{9}]{R_1 \leftarrow \frac{R_1}{19}}
\left(\begin{array}{cc|c} 1 & 0 & \frac{10}{19} \\ 0 & 1 & \frac{28}{19} \end{array}\right)
$$

yielding

$$
\boldsymbol{\kappa} = \frac{1}{19}\begin{pmatrix} 10 \\ 28 \end{pmatrix} \tag{6.2.2.4}
$$

Substituting the above in (6.2.2.1),

$$
\mathbf{x}_1 = \frac{1}{19}\begin{pmatrix} 29 \\ 8 \\ 77 \end{pmatrix}, \mathbf{x}_2 = \frac{1}{19}\begin{pmatrix} 20 \\ 11 \\ 86 \end{pmatrix}. \tag{6.2.2.5}
$$

Thus, the required distance is

$$
\|\mathbf{x}_2 - \mathbf{x}_1\| = \frac{3}{\sqrt{19}} \tag{6.2.2.6}
$$

See Fig. 6.2.2.1.

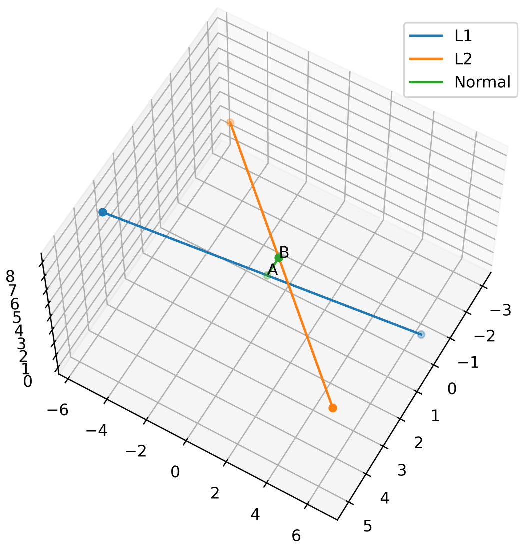


Fig. 6.2.2.1

6.2.3 Find the shortest distance between the lines l_1 and l_2 whose vector equations are

$$\overrightarrow{r} = \hat{i} + \hat{j} + \kappa(2\hat{i} - \hat{j} + \hat{k}) \text{ and} \tag{6.2.3.1}$$

$$\overrightarrow{r} = 2\hat{i} + \hat{j} - \hat{k} + \mu(3\hat{i} - 5\hat{j} + 2\hat{k}). \tag{6.2.3.2}$$

Solution: The given lines can be written in vector form as

$$\begin{aligned}\mathbf{x} &= \begin{pmatrix}1\\1\\0\end{pmatrix} + \kappa_1\begin{pmatrix}2\\-1\\1\end{pmatrix},\\ \mathbf{x} &= \begin{pmatrix}2\\1\\-1\end{pmatrix} + \kappa_2\begin{pmatrix}3\\-5\\2\end{pmatrix}\end{aligned} \tag{6.2.3.3}$$

$$\mathbf{M} = \begin{pmatrix}2 & 3\\-1 & -5\\1 & 2\end{pmatrix}, \mathbf{B} - \mathbf{A} = \begin{pmatrix}1\\0\\-1\end{pmatrix} \tag{6.2.3.4}$$

Substituting the above in (6.1.1.4),

$$\begin{pmatrix}2 & 3 & 1\\-1 & -5 & 0\\1 & 2 & -1\end{pmatrix} \xleftrightarrow[R_3 \leftarrow R_3 - \frac{1}{2}R_1]{R_2 \leftarrow R_2 + \frac{1}{2}R_1} \begin{pmatrix}2 & 3 & 1\\0 & -\frac{7}{2} & \frac{1}{2}\\0 & \frac{1}{2} & -\frac{3}{2}\end{pmatrix} \tag{6.2.3.5}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 + 7R_2} \begin{pmatrix}2 & 3 & 1\\0 & -\frac{7}{2} & \frac{1}{2}\\0 & 0 & -10\end{pmatrix} \tag{6.2.3.6}$$

The rank of the matrix is 3. So the given lines are skew. From (6.1.2.7),

$$\begin{pmatrix}2 & -1 & 1\\3 & -5 & 2\end{pmatrix}\begin{pmatrix}2 & 3\\-1 & -5\\1 & 2\end{pmatrix}\boldsymbol{\kappa} = \begin{pmatrix}2 & -1 & 1\\3 & -5 & 2\end{pmatrix}\begin{pmatrix}1\\0\\-1\end{pmatrix} \tag{6.2.3.7}$$

$$\implies \begin{pmatrix}6 & 13\\13 & 38\end{pmatrix}\boldsymbol{\kappa} = \begin{pmatrix}1\\1\end{pmatrix} \tag{6.2.3.8}$$

The augmented matrix of the above equation (6.2.3.8) is given by,

$$\left(\begin{array}{cc|c}6 & 13 & 1\\13 & 38 & 1\end{array}\right) \xleftrightarrow{R_2 \leftarrow R_2 - \frac{13}{6}R_1} \left(\begin{array}{cc|c}6 & 13 & 1\\0 & \frac{59}{6} & -\frac{7}{6}\end{array}\right) \tag{6.2.3.9}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - \frac{78}{59}R_2} \left(\begin{array}{cc|c}6 & 0 & \frac{150}{59}\\0 & \frac{59}{6} & -\frac{7}{6}\end{array}\right) \tag{6.2.3.10}$$

yielding

$$\begin{pmatrix}\kappa_1\\-\kappa_2\end{pmatrix} = \begin{pmatrix}\frac{25}{59}\\-\frac{7}{59}\end{pmatrix} \tag{6.2.3.11}$$

Substituting in (6.2.3.3),

$$\mathbf{x}_1 = \frac{1}{59}\begin{pmatrix}109\\34\\25\end{pmatrix}, \mathbf{x}_2 = \frac{1}{59}\begin{pmatrix}139\\24\\-45\end{pmatrix}. \tag{6.2.3.12}$$

The minimum distance between the lines is given by,

$$\|\mathbf{x}_2 - \mathbf{x}_1\| = \left\|\frac{1}{59}\begin{pmatrix}30\\-10\\-70\end{pmatrix}\right\| = \frac{10}{\sqrt{59}} \tag{6.2.3.13}$$

See Fig. 6.2.3.1.

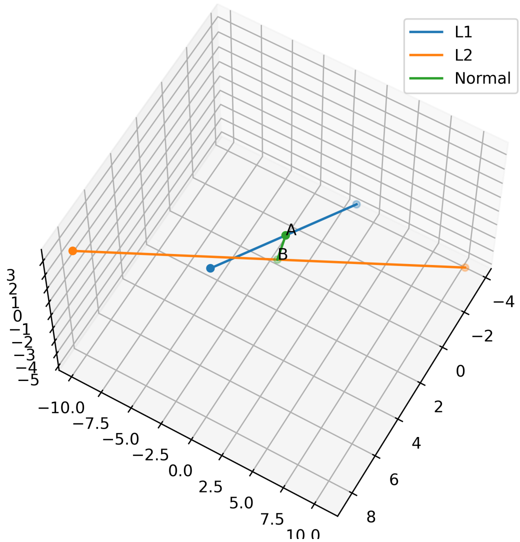


Fig. 6.2.3.1

6.2.4 Find the shortest distance between the lines given by

$$\overrightarrow{r} = (8 + 3\kappa\hat{i} - (9 + 16\kappa)\hat{j} + (10 + 7\kappa)\hat{k} \text{ and} \tag{6.2.4.1}$$
$$\overrightarrow{r} = 15\hat{i} + 29\hat{j} + 5\hat{k} + \mu(3\hat{i} + 8\hat{j} - 5\hat{k}). \tag{6.2.4.2}$$

6.2.5 Find the shortest distance between the lines

$$\overrightarrow{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \kappa(\hat{i} - \hat{j} + \hat{k}) \text{ and} \tag{6.2.5.1}$$
$$\overrightarrow{r} = 2\hat{i} - \hat{j} - \hat{k} + \mu(2\hat{i} + \hat{j} + 2\hat{k}) \tag{6.2.5.2}$$

6.2.6 Find the matrix $\mathbf{X}$ so that $\mathbf{X}\begin{pmatrix}1 & 2 & 3\\4 & 5 & 6\end{pmatrix} = \begin{pmatrix}-7 & -8 & -9\\2 & 4 & 6\end{pmatrix}$.

6.2.7 Find the shortest distance between the lines whose vector equations are

$$\vec{r} = (1-t)\hat{i} + (t-2)\hat{j} + (3-2t)\hat{k} \text{ and} \tag{6.2.7.1}$$

$$\vec{r} = (s+1)\hat{i} + (2s-1)\hat{j} - (2s+1)\hat{k} \tag{6.2.7.2}$$

6.2.8 Find the shortest distance between the lines $\vec{r} = 6\hat{i} + 2\hat{j} + 2\hat{k} + \lambda(\hat{i} - 2\hat{j} + 2\hat{k})$ and $\vec{r} = -4\hat{i} - \hat{k} + \mu(3\hat{i} - 2\hat{j} - 2\hat{k})$.

6.3 Singular Value Decomposition

6.3.1 Find the shortest distance between the lines whose vector equations are

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \tag{6.3.1.1}$$

and

$$\mathbf{x} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \tag{6.3.1.2}$$

Solution: From (6.2.2.3),

$$\mathbf{M}^\top\mathbf{M} = \begin{pmatrix} 1 & -3 & 2 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -3 & 3 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 14 & -5 \\ -5 & 14 \end{pmatrix} \tag{6.3.1.3}$$

$$\mathbf{M}\mathbf{M}^\top = \begin{pmatrix} 1 & 2 \\ -3 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -3 & 2 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 3 & 4 \\ 3 & 18 & -3 \\ 4 & -3 & 5 \end{pmatrix} \tag{6.3.1.4}$$

a) For $\mathbf{M}\mathbf{M}^\top$, the characteristic polynomial is

$$\text{char}\mathbf{M}\mathbf{M}^\top = \begin{vmatrix} \lambda - 5 & -3 & -4 \\ -3 & \lambda - 18 & 3 \\ -4 & 3 & \lambda - 5 \end{vmatrix} \tag{6.3.1.5}$$

$$= \lambda(\lambda - 9)(\lambda - 19) \tag{6.3.1.6}$$

Thus, the eigenvalues are given by

$$\lambda_1 = 19, \ \lambda_2 = 9, \ \lambda_3 = 0 \tag{6.3.1.7}$$

For λ_1, the augmented matrix formed from the eigenvalue-eigenvector equation is

$$\begin{pmatrix} -14 & 3 & 4 \\ 3 & -1 & -3 \\ 4 & -3 & -14 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{R_1+R_3}{-10}} \begin{pmatrix} 1 & 0 & 1 \\ 3 & -1 & -3 \\ 4 & -3 & -14 \end{pmatrix}$$

$$\xleftrightarrow[R_2 \leftarrow -R_2 + 3R_1]{R_3 \leftarrow R_3 - 4R_1} \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & -6 \\ 0 & -3 & -18 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 - 3R_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & -6 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence, the normalized eigenvector is

$$\mathbf{u}_1 = \frac{1}{\sqrt{38}}\begin{pmatrix}-1\\-6\\1\end{pmatrix} \tag{6.3.1.8}$$

For λ_2, the augmented matrix formed from the eigenvalue-eigenvector equation is

$$\begin{pmatrix}-4 & 3 & 4\\3 & 9 & -3\\4 & 3 & -4\end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_1 + R_3} \begin{pmatrix}-4 & 3 & 4\\3 & 9 & -3\\0 & 0 & 0\end{pmatrix}$$

$$\xleftrightarrow[R_2 \leftarrow \frac{4R_2+3R_1}{45}]{R_1 \leftarrow \frac{R_1-3R_2}{-4}} \begin{pmatrix}1 & 0 & -1\\0 & 1 & 0\\0 & 0 & 0\end{pmatrix}$$

Hence, the normalized eigenvector is

$$\mathbf{u}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix} \tag{6.3.1.9}$$

For λ_3, the augmented matrix formed from the eigenvalue-eigenvector equation is

$$\begin{pmatrix}5 & 3 & 4\\3 & 18 & -3\\4 & -3 & 5\end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{R_1+R_3}{9}} \begin{pmatrix}1 & 0 & 1\\3 & 18 & -3\\4 & -3 & 5\end{pmatrix}$$

$$\xleftrightarrow[R_2 \leftarrow R_2 - 3R_1]{R_3 \leftarrow R_3 - 4R_1} \begin{pmatrix}1 & 0 & 1\\0 & 18 & -6\\0 & -3 & 1\end{pmatrix} \xleftrightarrow[R_2 \leftarrow \frac{R_2}{6}]{R_3 \leftarrow R_3 + R_2} \begin{pmatrix}1 & 0 & 1\\0 & 3 & -1\\0 & 0 & 0\end{pmatrix}$$

yielding

$$\mathbf{u}_3 = \frac{1}{\sqrt{19}}\begin{pmatrix}-3\\1\\3\end{pmatrix} \tag{6.3.1.10}$$

Using (6.1.3.1), we see that

$$\mathbf{U} = \begin{pmatrix}-\frac{1}{\sqrt{38}} & \frac{1}{\sqrt{2}} & -\frac{3}{\sqrt{19}}\\-\frac{6}{\sqrt{38}} & 0 & \frac{1}{\sqrt{19}}\\\frac{1}{\sqrt{38}} & -\frac{1}{\sqrt{2}} & \frac{3}{\sqrt{19}}\end{pmatrix} \tag{6.3.1.11}$$

$$\mathbf{D_1} = \begin{pmatrix}19 & 0 & 0\\0 & 9 & 0\\0 & 0 & 0\end{pmatrix} \tag{6.3.1.12}$$

b) For $\mathbf{M}^\top\mathbf{M}$, the characteristic polynomial is

$$\text{char}\mathbf{M}^\top\mathbf{M} = \begin{vmatrix}\lambda - 14 & 5\\5 & \lambda - 14\end{vmatrix} \tag{6.3.1.13}$$

$$= (\lambda - 9)(\lambda - 19) \tag{6.3.1.14}$$

Thus, the eigenvalues are given by

$$\lambda_1 = 19, \ \lambda_2 = 9 \tag{6.3.1.15}$$

For λ_1, the augmented matrix formed from the eigenvalue-eigenvector equation is

$$\begin{pmatrix} -5 & -5 \\ -5 & -5 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 - R_2} \begin{pmatrix} 0 & 0 \\ -5 & -5 \end{pmatrix} \tag{6.3.1.16}$$

Hence, the normalized eigenvector is

$$\mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{6.3.1.17}$$

For λ_2, the augmented matrix formed from the eigenvalue-eigenvector equation is

$$\begin{pmatrix} 5 & -5 \\ -5 & 5 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 + R_2} \begin{pmatrix} 0 & 0 \\ 5 & -5 \end{pmatrix} \tag{6.3.1.18}$$

Hence, the normalized eigenvector is

$$\mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{6.3.1.19}$$

Thus, from (6.1.3.2),

$$\mathbf{V} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \ \mathbf{D_2} = \begin{pmatrix} 9 & 0 \\ 0 & 19 \end{pmatrix} \tag{6.3.1.20}$$

Using (6.3.1.15),

$$\mathbf{\Sigma} \triangleq \begin{pmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \sqrt{19} & 0 \\ 0 & 3 \\ 0 & 0 \end{pmatrix} \tag{6.3.1.21}$$

and substituting into (6.1.4.6),

$$\kappa = \frac{1}{19} \begin{pmatrix} 10 \\ 28 \end{pmatrix} \tag{6.3.1.22}$$

which agrees with (6.2.2.4).

6.3.2 Find the shortest distance between the lines l_1 and l_2 whose vector equations are

$$\overrightarrow{r} = \hat{i} + \hat{j} + \lambda(2\hat{i} - \hat{j} + \hat{k}) \text{ and} \tag{6.3.2.1}$$

$$\overrightarrow{r} = 2\hat{i} + \hat{j} - \hat{k} + \mu(3\hat{i} - 5\hat{j} + 2\hat{k}). \tag{6.3.2.2}$$

Solution:

a) To check whether the given lines are skew, from (6.2.3.4) and (6.1.1.4),

$$\begin{pmatrix} 2 & 3 & 1 \\ -1 & -5 & 0 \\ 1 & 2 & -1 \end{pmatrix} \xleftrightarrow[R_3 \leftarrow R_3 - \frac{1}{2}R_1]{R_2 \leftarrow R_2 + \frac{1}{2}R_1} \begin{pmatrix} 2 & 3 & 1 \\ 0 & -\frac{7}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{3}{2} \end{pmatrix}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 + 7R_2} \begin{pmatrix} 2 & 3 & 1 \\ 0 & -\frac{7}{2} & \frac{1}{2} \\ 0 & 0 & -10 \end{pmatrix}$$

The rank of the matrix is 3. So the given lines are skew.

b)

$$\mathbf{M}^\top \mathbf{M} = \begin{pmatrix} 2 & -1 & 1 \\ 3 & -5 & 2 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ -1 & -5 \\ 1 & 2 \end{pmatrix} \tag{6.3.2.3}$$

$$= \begin{pmatrix} 6 & 13 \\ 13 & 38 \end{pmatrix} \tag{6.3.2.4}$$

$$\mathbf{M}\mathbf{M}^\top = \begin{pmatrix} 2 & 3 \\ -1 & -5 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & -1 & 1 \\ 3 & -5 & 2 \end{pmatrix} \tag{6.3.2.5}$$

$$= \begin{pmatrix} 13 & -17 & 8 \\ -17 & 26 & -11 \\ 8 & -11 & 5 \end{pmatrix} \tag{6.3.2.6}$$

The characteristic polynomial of the matrix $\mathbf{M}\mathbf{M}^\top$ is given by,

$$\operatorname{char}\left(\mathbf{M}\mathbf{M}^\top\right) = \begin{vmatrix} 13-\lambda & -17 & 8 \\ -17 & 26-\lambda & -11 \\ 8 & -11 & 5-\lambda \end{vmatrix} \tag{6.3.2.7}$$

$$= -\lambda^3 + 44\lambda^2 - 59\lambda \tag{6.3.2.8}$$

resulting in

$$\mathbf{U} = \begin{pmatrix} \frac{12-\sqrt{17}}{\sqrt{5}\sqrt{68-6\sqrt{17}}} & \frac{12+\sqrt{17}}{\sqrt{5}\sqrt{68+6\sqrt{17}}} & -\frac{3}{\sqrt{59}} \\ \frac{1-3\sqrt{17}}{\sqrt{5}\sqrt{68-6\sqrt{17}}} & \frac{1+3\sqrt{17}}{\sqrt{5}\sqrt{68+6\sqrt{17}}} & \frac{1}{\sqrt{59}} \\ \frac{\sqrt{5}}{\sqrt{68-6\sqrt{17}}} & \frac{\sqrt{5}}{\sqrt{68+6\sqrt{17}}} & \frac{7}{\sqrt{59}} \end{pmatrix} \tag{6.3.2.9}$$

and

$$\mathbf{D}_1 = \begin{pmatrix} 22+5\sqrt{17} & 0 & 0 \\ 0 & 22-5\sqrt{17} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{6.3.2.10}$$

For $\mathbf{M}^\top\mathbf{M}$, the characteristic polynomial is

$$\operatorname{char}\left(\mathbf{M}^\top\mathbf{M}\right) = \begin{vmatrix} 6-\lambda & 13 \\ 13 & 38-\lambda \end{vmatrix} \tag{6.3.2.11}$$

$$= \lambda^2 - 44\lambda + 59 \tag{6.3.2.12}$$

Thus, the eigenvalues are given by

$$\lambda_1 = 22 + 5\sqrt{17},\ \lambda_2 = 22 - 5\sqrt{17} \tag{6.3.2.13}$$

resulting in

$$\mathbf{V} = \begin{pmatrix} \frac{-16-5\sqrt{17}}{\sqrt{850+160\sqrt{17}}} & \frac{13}{\sqrt{850-160\sqrt{17}}} \\ \frac{13}{\sqrt{850+160\sqrt{17}}} & \frac{-16+5\sqrt{17}}{\sqrt{850-160\sqrt{17}}} \end{pmatrix} \tag{6.3.2.14}$$

$$\mathbf{D}_2 = \begin{pmatrix} 22-5\sqrt{17} & 0 \\ 0 & 22+5\sqrt{17} \end{pmatrix} \tag{6.3.2.15}$$

Therefore,

$$\mathbf{\Sigma} = \begin{pmatrix} \sqrt{22+5\sqrt{17}} & 0 \\ 0 & \sqrt{22-5\sqrt{17}} \\ 0 & 0 \end{pmatrix} \tag{6.3.2.16}$$

and substituting into (6.1.5.5),

$$\boldsymbol{\lambda} = \begin{pmatrix} \frac{25}{59} \\ -\frac{7}{59} \end{pmatrix} \tag{6.3.2.17}$$

which agrees with (6.2.3.11).

6.3.3 Find the shortest distance between the lines given by

$$\vec{r} = (8 + 3\lambda\hat{i} - (9 + 16\lambda)\hat{j} + (10 + 7\lambda)\hat{k} \text{ and} \tag{6.3.3.1}$$

$$\vec{r} = 15\hat{i} + 29\hat{j} + 5\hat{k} + \mu(3\hat{i} + 8\hat{j} - 5\hat{k}). \tag{6.3.3.2}$$

6.3.4 Find the shortest distance between the lines

$$\vec{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k}) \text{ and} \tag{6.3.4.1}$$

$$\vec{r} = 2\hat{i} - \hat{j} - \hat{k} + \mu(2\hat{i} + \hat{j} + 2\hat{k}) \tag{6.3.4.2}$$

6.3.5 Find the shortest distance between the lines

$$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1} \text{ and} \tag{6.3.5.1}$$

$$\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1} \tag{6.3.5.2}$$

6.4 CBSE

6.4.1 Fit a straight line trend by the method of least squares and find the trend value for the year 2008 using the data from Table 6.4.1

TABLE 6.4.1: Table showing yearly trend of production of goods in lakh tonnes

Year	Production (in lakh tonnes)
2001	30
2002	35
2003	36
2004	32
2005	37
2006	40

(12, 2022)

6.4.2 Find the shortest distance between the lines $\vec{r} = 4\hat{i} - \hat{j} + \lambda\left(\hat{i} + 2\hat{j} - 3\hat{k}\right)$ and $\vec{r} = \hat{i} - \hat{j} + 2\hat{k} + \mu\left(2\hat{i} + 4\hat{j} - 5\hat{k}\right)$. (12, 2018)

6.4.3 Find the shortest distance between the lines $\mathbf{r} = 2\vec{i} - 5\vec{j} + \vec{k} + \lambda\left(3\vec{i} + 2\vec{j} + 6\vec{k}\right)$ and $\mathbf{r} = 7\vec{i} - 6\vec{k} + \mu\left(\vec{i} + 2\vec{j} + 2\vec{k}\right)$. (12, 2015)

6.4.4 Electrical transmission wires which are laid down in winters are stretched tightly to accommodate expansion in summers. Two such wires lie along the following lines

$$l_1 : \frac{x+1}{3} = \frac{y-3}{-2} = \frac{z+2}{-1}$$
$$l_2 : \frac{x}{-1} = \frac{y-7}{3} = \frac{z+7}{-2}$$

Based on the given information, answer the following questions

a) Are the l_1 and l_2 coplanar? Justify your answer.

b) Find the point of intersection of lines l_1 and l_2.

(12, 2022)

6.4.5 Find the distance between the lines $x = \frac{y-1}{2} = \frac{z-2}{3}$ and $x + 1 = \frac{y+2}{2} = \frac{z-1}{3}$. (12, 2022)

6.4.6 Find the shortest distance between the lines $\mathbf{r} = 3\hat{i} + 5\hat{j} + 7\hat{k} + \lambda(\hat{i} - 2\hat{j} + \hat{k}), \mathbf{r} = (-\hat{i} - \hat{j} - \hat{k}) + \mu(7\hat{i} - 6\hat{j} + \hat{k})$. (12, 2022)

6.4.7 Two motorcycles A and B are running at a speed more than the allowed speed on the road represented by the following lines

$$\mathbf{r} = \lambda(\hat{i} + 2\hat{j} - \hat{k})$$
$$\mathbf{r} = (3\hat{i} + 3\hat{j}) + \mu(2\hat{i} + \hat{j} + \hat{k})$$

Based on the following information, answer the following questions

a) Find the shortest distance between the given lines.

b) Find a point at which the motorcycles may collide.

(12, 2022)

6.4.8 Find the shortest distance between the lines $\mathbf{r} = (\lambda + 1)\hat{i} + (\lambda + 4)\hat{j} - (\lambda - 3)\hat{k}, \mathbf{r} = (3 - \mu)\hat{i} + (2\mu + 2)\hat{j} + (\mu + 6)\hat{k}$. (12, 2022)

6.4.9 Find the shortest distance between the lines $\frac{x-1}{2} = \frac{y+1}{3} = z, \frac{x+1}{5} = \frac{y-2}{1}, z = 2$ and hence write whether the lines are intersecting or not. (12, 2022)

6.4.10 Find the shortest distance between the lines

$$\frac{x-1}{2} = \frac{y+1}{3} = z; \frac{x+1}{5} = \frac{y-2}{1}, z = 2$$

and hence write whether the lines are intersecting or not. (12, 2021)

6.4.11 Find matrix $\mathbf{A}$ such that

$$\begin{pmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{pmatrix} \mathbf{A} = \begin{pmatrix} -1 & -8 \\ 1 & -2 \\ 9 & 22 \end{pmatrix}$$

(12, 2017)

6.4.12 Find the shortest distance between the lines

$$\mathbf{r} = (1 + \lambda)\hat{i} + (2 - \lambda)\hat{j} + (\lambda + 1)\hat{k};$$
$$\mathbf{r} = (2\hat{i} - \hat{j} - \hat{k}) + \mu(2\hat{i} + \hat{j} + 2\hat{k}).$$

(12, 2009)

7 Circle

7.1 Formulae

7.1.1. For a circle with centre $\mathbf{c}$ and radius r,

$$\mathbf{u} = -\mathbf{c}, f = \|\mathbf{u}\|^2 - r^2 \tag{7.1.1.1}$$

7.1.2. Given points , $\mathbf{x}_1,\ \mathbf{x}_2$ on the circle and the diameter

$$\mathbf{n}^\top \mathbf{x} = c, \tag{7.1.2.1}$$

the centre is given by

$$\begin{pmatrix} 2\mathbf{x}_1 & 2\mathbf{x}_2 & \mathbf{n} \\ 1 & 1 & 0 \end{pmatrix}^\top \begin{pmatrix} \mathbf{u} \\ f \end{pmatrix} = -\begin{pmatrix} \|\mathbf{x}_1\|^2 \\ \|\mathbf{x}_2\|^2 \\ c \end{pmatrix} \tag{7.1.2.2}$$

Solution: From (A.7.1.1),

$$\begin{aligned} \|\mathbf{x}_1\|^2 + 2\mathbf{u}^\top \mathbf{x}_1 + f = 0 \\ \|\mathbf{x}_2\|^2 + 2\mathbf{u}^\top \mathbf{x}_2 + f = 0 \end{aligned} \tag{7.1.2.3}$$

and (7.1.2.1) can be expressed as

$$\mathbf{u}^\top \mathbf{n} = -c \tag{7.1.2.4}$$

Clubbing (7.1.2.3) and (7.1.2.4), we obtain (7.1.2.2).

7.1.3. Given points $\mathbf{x}_1,\ \mathbf{x}_2,\ \mathbf{x}_3$ on the circle, the parameters are given by

$$\begin{pmatrix} 2\mathbf{x}_1 & 2\mathbf{x}_2 & 2\mathbf{x}_3 \\ 1 & 1 & 1 \end{pmatrix}^\top \begin{pmatrix} \mathbf{u} \\ f \end{pmatrix} = -\begin{pmatrix} \|\mathbf{x}_1\|^2 \\ \|\mathbf{x}_2\|^2 \\ \|\mathbf{x}_3\|^2 \end{pmatrix} \tag{7.1.3.1}$$

7.1.4. Code for circle

```
codes/book/circ.py
```

7.2 Equation

7.2.1 Find the coordinates of a point $\mathbf{A}$, where AB is the diameter of a circle whose centre is $\mathbf{C}(2, -3)$ and $\mathbf{B}$ is $(1, 4)$.

Solution:

$$\mathbf{C} = \frac{\mathbf{A} + \mathbf{B}}{2} \implies \mathbf{A} = 2\mathbf{C} - \mathbf{B} = \begin{pmatrix} 3 \\ -10 \end{pmatrix} \tag{7.2.1.1}$$

The radius is then obtained as

$$\|\mathbf{B} - \mathbf{C}\| = 5\sqrt{2} \tag{7.2.1.2}$$

See Fig. 7.2.1.1.

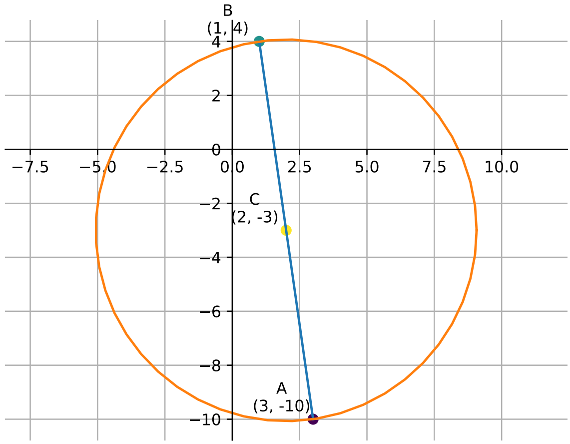


Fig. 7.2.1.1

7.2.2 Find the equation of the circle passing through the points $(4, 1)$ and $(6, 5)$ and whose centre is on the line $4x + y = 16$.

Solution: Following Appendix 7.1.2,

$$\mathbf{x}_1 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}, \ \mathbf{x}_2 = \begin{pmatrix} 6 \\ 5 \end{pmatrix}, \ \mathbf{n} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}, \ c = -16. \tag{7.2.2.1}$$

Substituting in (7.1.2.2),

$$\begin{pmatrix} 8 & 2 & 1 \\ 12 & 10 & 1 \\ -4 & -1 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ f \end{pmatrix} = \begin{pmatrix} 16 \\ -61 \\ -17 \end{pmatrix} \tag{7.2.2.2}$$

The augmented matrix is expressed as

$$\left(\begin{array}{ccc|c} 8 & 2 & 1 & -17 \\ 12 & 10 & 1 & -61 \\ -4 & -1 & 0 & 16 \end{array}\right) \tag{7.2.2.3}$$

Performing a sequence of row operations to transform into an Echelon form

$$\xleftrightarrow[R_2 \to R_2 + 3R_1]{R_3 \to R_3 + 2R_1} \left(\begin{array}{ccc|c} -4 & -1 & 0 & 16 \\ 0 & 7 & 1 & -13 \\ 0 & 0 & 1 & 15 \end{array}\right) \xleftrightarrow{R_2 \to R_2 - R_3} \left(\begin{array}{ccc|c} -4 & -1 & 0 & 16 \\ 0 & 7 & 0 & -28 \\ 0 & 0 & 1 & 15 \end{array}\right)$$

$$\xleftrightarrow{R_2 \to \frac{R_2}{7}, R_1 \to \frac{-R_1}{4}} \left(\begin{array}{ccc|c} 1 & \frac{1}{4} & 0 & -4 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 15 \end{array}\right) \xleftrightarrow{R_1 \to R_1 - \frac{1}{4} R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 15 \end{array}\right)$$

So, from (7.2.2.4)

$$\mathbf{u} = -\begin{pmatrix} 3 \\ 4 \end{pmatrix}, \ f = 15. \tag{7.2.2.4}$$

See Fig. 7.2.2.1.

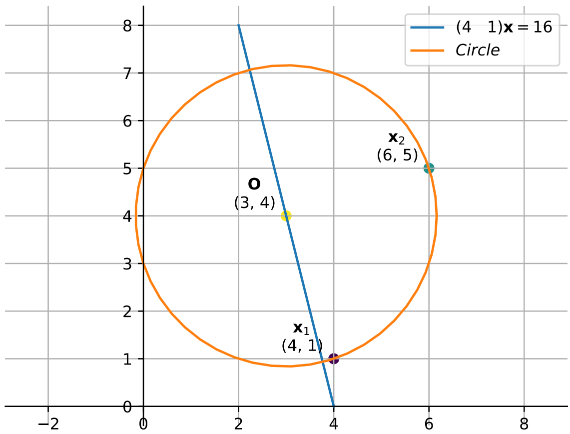


Fig. 7.2.2.1

7.2.3 Find the equation of the circle passing through the points $\mathbf{x}_1(2,3)$ and $\mathbf{x}_2(-1,1)$ and whose centre is on the line $x - 3y - 11 = 0$.
Solution: Substituting numerical values in (7.1.2.2),

$$\begin{pmatrix} 4 & 6 & 1 \\ -2 & 2 & 1 \\ -1 & 3 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ f \end{pmatrix} = \begin{pmatrix} -13 \\ -2 \\ 11 \end{pmatrix} \tag{7.2.3.1}$$

yielding

$$\mathbf{u} = \frac{1}{2}\begin{pmatrix} -7 \\ 5 \end{pmatrix}, \ f = -14. \tag{7.2.3.2}$$

7.2.4 A circle drawn with origin as the centre passes through $\left(\frac{13}{2}, 0\right)$. The point which does not lie in the interior of the circle is

a) $\left(\frac{-3}{4}, 1\right)$
b) $\left(2, \frac{7}{3}\right)$
c) $\left(5, \frac{-1}{2}\right)$
d) $\left(-6, \frac{-5}{2}\right)$

7.2.5 The point $\mathbf{P}(-2,4)$ lies on circle of radius 6 and center $\mathbf{C}(3,5)$.

7.2.6 Find the equation of the circle with radius 5 whose centre lies on the X axis and passes through the point $(2, 3)$.
Solution: See Fig. 7.2.6.1.

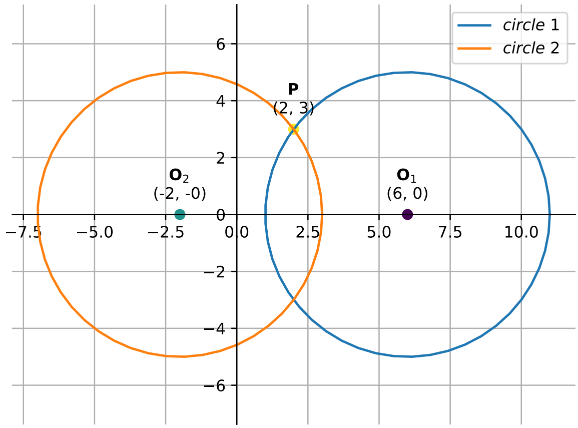


Fig. 7.2.6.1

From the given information, the following equations can be formulated using (A.7.1.1).

$$\|\mathbf{P}\|^2 + 2\mathbf{u}^\top\mathbf{P} + f = 0 \tag{7.2.6.1}$$

$$\mathbf{u} = k\mathbf{e}_1 \tag{7.2.6.2}$$

$$\|\mathbf{u}\|^2 - f = r^2 \tag{7.2.6.3}$$

where

$$\mathbf{P} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \text{ and } r = 5 \tag{7.2.6.4}$$

From (7.2.6.1) and (7.2.6.3),

$$\|\mathbf{P}\|^2 + 2\mathbf{u}^\top\mathbf{P} + \|\mathbf{u}\|^2 = r^2 \tag{7.2.6.5}$$

Substituting from (7.2.6.2) in the above,

$$k^2 + 2k\mathbf{e}_1^\top\mathbf{P} + \|\mathbf{P}\|^2 - r^2 = 0 \tag{7.2.6.6}$$

resulting in

$$k = -\mathbf{e}_1^\top\mathbf{P} \pm \sqrt{\left(\mathbf{e}_1^\top\mathbf{P}\right)^2 + r^2 - \|\mathbf{P}\|^2} \tag{7.2.6.7}$$

Substituting numerical values,

$$k = 2, -6 \tag{7.2.6.8}$$

resulting in circles with centre

$$-\mathbf{u} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 6 \\ 0 \end{pmatrix}. \tag{7.2.6.9}$$

This is verified in Fig. (7.2.6.1).

7.2.7 Find the equation of a circle with centre $(2, 2)$ and passing through the point $(4, 5)$.
Solution: From the given information

$$\mathbf{u} = -\begin{pmatrix} 2 \\ 2 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 4 \\ 5 \end{pmatrix} \tag{7.2.7.1}$$

$$\implies \|\mathbf{A}\|^2 + 2\mathbf{u}^\top \mathbf{A} + f = 0 \tag{7.2.7.2}$$

$$\implies f = -\|\mathbf{A}\|^2 - 2\mathbf{u}^\top \mathbf{A} = -5 \tag{7.2.7.3}$$

Hence the equation of circle is

$$\|\mathbf{x}\|^2 + 2\begin{pmatrix} -2 & -2 \end{pmatrix}\mathbf{x} - 5 = 0 \tag{7.2.7.4}$$

7.2.8 The centre of a circle is $(2a, a-7)$. Find the values of a if the circle passes through the point $(11, -9)$ and has diameter $10\sqrt{2}$ units.

7.2.9 A circle has its centre at the origin and a point $\mathbf{P}(5, 0)$ lies on it. The point $\mathbf{Q}(6, 8)$ lies outside the circle.

7.2.10 Does the point $(-2.5, 3.5)$ lie inside, outside or on the circle $x^2 + y^2 = 25$?
Solution: See Table 7.2.10.

Condition	Inference
$\|\mathbf{x} - \mathbf{O}\|^2 < r^2$	point lies inside the circle
$\|\mathbf{x} - \mathbf{O}\|^2 > r^2$	point lies outside the circle
$\|\mathbf{x} - \mathbf{O}\|^2 = r^2$	point lies on the circle

TABLE 7.2.10

The given circle equation can be expressed as

$$\|\mathbf{x}\|^2 = 25 \tag{7.2.10.1}$$

Let,

$$\mathbf{P} = \begin{pmatrix} -2.5 \\ 3.5 \end{pmatrix} \tag{7.2.10.2}$$

Since

$$\|\mathbf{P} - \mathbf{O}\|^2 = 18.5 < 25, \tag{7.2.10.3}$$

the point lies inside the given circle. See Fig. 7.2.10.1.

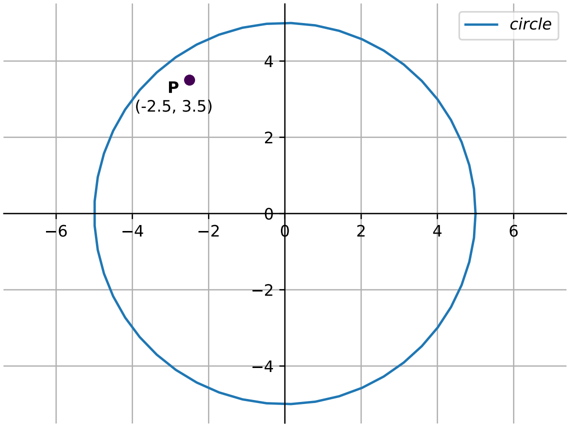


Fig. 7.2.10.1

7.2.11 Find the equation of the circle having (1, -2) as its centre and passing through the intersection of $3x + y = 14, 2x + 5y = 18$.

7.2.12 If the lines $2x - 3y = 5$ and $3x - 4y = 7$ are the diameters of a circle of area 154 square units, then obtain the equation of the circle.

7.2.13 Find the equation of the circle which passes through the points (2, 3) and (4, 5) and the centre lies on the straight line $y - 4x + 3 = 0$.

7.2.14 Find the equation of a circle passing through the point (7, 3) having radius 3 units and whose centre lies on the line $y = x - 1$.

7.2.15 Find the equation of a circle concentric with the circle $x^2 + y^2 - 6x + 12y + 15 = 0$ and has double of its area.

7.2.16 If one end of a diameter of the circle $x^2 + y^2 - 4x - 6y + 11 = 0$ is (3, 4), then find the coordinate of the other end of the diameter.

7.2.17 Find the centre of a circle passing though the points $(6, -6), (3, -7)$ and $(3, 3)$.

Solution: Substituting numerical values in (7.1.3.1),

$$\begin{pmatrix} 6 & -14 & 1 \\ 12 & -12 & 1 \\ 6 & 6 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ f \end{pmatrix} = \begin{pmatrix} -58 \\ -72 \\ -18 \end{pmatrix} \tag{7.2.17.1}$$

yielding

$$\mathbf{u} = \begin{pmatrix} -3 \\ 2 \end{pmatrix} \tag{7.2.17.2}$$

$$f = -12 \tag{7.2.17.3}$$

See Fig. 7.2.17.1.

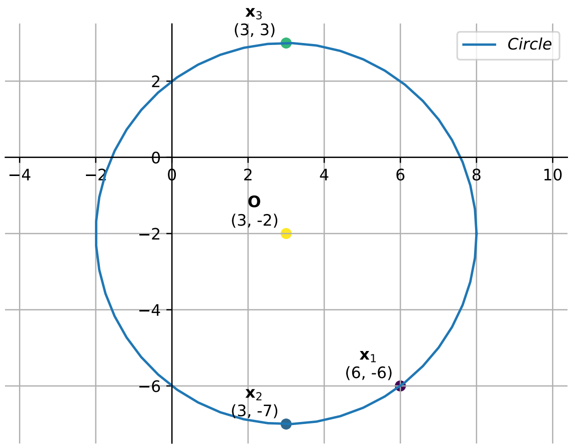


Fig. 7.2.17.1

7.2.18 Find the equation of the circle passing through $(0,0)$ and making intercepts a and b on the coordinate axes.

In each of the following exercises, find the equation of the circle with the following parameters

7.2.19 centre $(0,2)$ and radius 2

Solution: Substituting numerical values in (7.1.1.1),

$$\mathbf{u} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}, f = 0 \tag{7.2.19.1}$$

Thus, the equation of circle is obtained as

$$\|\mathbf{x}\|^2 - 2\begin{pmatrix} 0 & 2 \end{pmatrix}\mathbf{x} = 0 \tag{7.2.19.2}$$

7.2.20 centre $(-2,3)$ and radius 4

Solution: Given

$$\mathbf{u} = -\begin{pmatrix} -2 \\ 3 \end{pmatrix}, r = 4. \tag{7.2.20.1}$$

Substituting in (7.1.1.1),

$$f = -3 \tag{7.2.20.2}$$

The equation of the circle is then obtained as

$$\|\mathbf{x}\|^2 + 2\begin{pmatrix} 2 & -3 \end{pmatrix}\mathbf{x} - 3 = 0 \tag{7.2.20.3}$$

7.2.21 centre $\left(\frac{1}{2}, \frac{1}{4}\right)$ and radius $\frac{1}{12}$

Solution: Substituting numerical values in (7.1.1.1),

$$f = \frac{11}{36} \tag{7.2.21.1}$$

Thus, the equation of the circle is

$$\|\mathbf{x}\|^2 + \begin{pmatrix}-1 & -\frac{1}{2}\end{pmatrix}\mathbf{x} + \frac{11}{36} = 0 \tag{7.2.21.2}$$

7.2.22 centre $(1, 1)$ and radius $\sqrt{2}$

Solution: Substituting

$$r = \sqrt{2}, \ \mathbf{u} = \begin{pmatrix}-1\\-1\end{pmatrix} \tag{7.2.22.1}$$

in (7.1.1.1),

$$f = 0 \tag{7.2.22.2}$$

Thus, the equation of the circle is

$$\|\mathbf{x}\|^2 - 2\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 0 \tag{7.2.22.3}$$

7.2.23 The area of the circle centred at (1, 2) and passing through (4, 6) is

a) 5π
b) 10π
c) 25π
d) none of these

7.2.24 Equation of the circle with centre on the Y axis and passing through the orgin and the point (2, 3) is

a) $x^2 + y^2 + 6x + 6y + 3 = 0$
b) $x^2 + y^2 - 6x - 6y - 9 = 0$
c) $x^2 + y^2 - 6x - 6y + 9 = 0$
d) none of these

7.2.25 Equation of the circle with centre on the Y axis and passing through the origin and the point (2, 3) is

a) $x^2 + y^2 + 13y = 0$
b) $3x^2 + 3y^2 + 13x + 3 = 0$
c) $6x^2 + 6y^2 - 13x = 0$
d) $x^2 + y^2 + 13x + 3 = 0$

In each of the following exercises, find the centre and radius of the circles.

7.2.26 $x^2 + y^2 + 10x - 6y - 2 = 0$.

Solution: The circle parameters are

$$\mathbf{u} = \begin{pmatrix}5\\-3\end{pmatrix}, \ f = -2 \tag{7.2.26.1}$$

$$\implies \mathbf{c} = \begin{pmatrix}-5\\3\end{pmatrix}, \ r = \sqrt{\|\mathbf{u}\|^2 - f} = 6 \tag{7.2.26.2}$$

7.2.27 $x^2 + y^2 - 4x - 8y - 45 = 0$
Solution: The given circle can be expressed as

$$\|\mathbf{x}\|^2 + 2\begin{pmatrix}-2 & -4\end{pmatrix}\mathbf{x} - 45 = 0 \tag{7.2.27.1}$$

where

$$\mathbf{u} = \begin{pmatrix}-2\\-4\end{pmatrix}, \ f = -45 \tag{7.2.27.2}$$

$$\implies \mathbf{c} = \begin{pmatrix}2\\4\end{pmatrix}, \ r = \sqrt{65}. \tag{7.2.27.3}$$

7.2.28 $x^2 + y^2 - 8x + 10y - 12 = 0$
Solution: From the given informtion,

$$\mathbf{u} = \begin{pmatrix}-4\\5\end{pmatrix}, \ f = -12 \tag{7.2.28.1}$$

$$\implies \mathbf{c} = \begin{pmatrix}4\\-5\end{pmatrix}, \tag{7.2.28.2}$$

$$r = \sqrt{\|\mathbf{u}\|^2 - f} = \sqrt{53} \tag{7.2.28.3}$$

7.2.29 $2x^2 + 2y^2 - x = 0$
Solution: The given equation can be expressed as

$$\|\mathbf{x}\|^2 + 2\begin{pmatrix}\frac{-1}{4} & 0\end{pmatrix}\mathbf{x} = 0 \tag{7.2.29.1}$$

The centre of circle is then given by

$$\mathbf{u} = -\mathbf{c} = \begin{pmatrix}\frac{1}{4}\\0\end{pmatrix} \tag{7.2.29.2}$$

and the radius of circle is obtained as

$$r = \sqrt{\|\mathbf{u}\|^2 - f} = \frac{1}{4} \tag{7.2.29.3}$$

State whether the statements are True or False

7.2.30 The line $x + 3y = 0$ is a diameter of the circle $x^2 + y^2 + 6x + 2y = 0$.
7.2.31 The point (1, 2) lies inside the circle $x^2 + y^2 - 2x + 6y + 1 = 0$.

7.3 Miscellaneous

7.3.1 Find the equation of the circle passing through $(0, 0)$ and making intercepts a and b on the coordinate axes.
7.3.2 Find the equation of a circle with centre $(-a, -b)$ and radius $\sqrt{a^2 - b^2}$.
Solution: From (7.1.1.1),

$$\mathbf{u} = \begin{pmatrix}a\\b\end{pmatrix}, \ f = 2b^2 \tag{7.3.2.1}$$

Thus, the equation of circle is

$$\|\mathbf{x}\|^2 + 2\begin{pmatrix} a & b \end{pmatrix}\mathbf{x} + 2b^2 = 0 \tag{7.3.2.2}$$

7.3.3 The equation of a circle with origin as centre and passing through the vertices of an equilateral triangle whose median is of length $3a$ is

a) $x^2 + y^2 = 9a^2$
b) $x^2 + y^2 = 16a^2$
c) $x^2 + y^2 = 4a^2$
d) $x^2 + y^2 = a^2$

7.3.4 Show that the point (x, y) given by $x = \frac{2at}{1+t^2}$ and $y = \frac{a(1-t^2)}{1+t^2}$ lies on a circle for all real values of t such that -1≤ t ≤1 where a is any given real number.

7.3.5 If a circle passes through the point $(0, 0), (a, 0)$ and $(0, b)$ then find the coordinates of its centre.

7.3.6 The equation of the circle circumscribing the triangle whose sides are the lines $y = x + 2, 3y = 4x, 2y = 3x$ is ______

7.4 JEE

7.4.1. Consider

$$L_1 : 2x + 3y + p - 3 = 0$$
$$L_2 : 2x + 3y + p + 3 = 0$$

where p is a real number, and

$$C : x^2 + y^2 + 6x - 10y + 30 = 0$$

STATEMENT-1:If line L_1 is a chord of circle C, then line L_2 is not always a diameter of circle C

STATEMENT-2:If line L_1 is a diameter of circle C, then line L_2 is not a chord of circle C. (2008)

a) Statement-1 is True, statement-2 is True;Statement-2 is a correct explantion for Statement-1.
b) Statement-1 is True, statement-2 is True;Statement-2 is NOT a correct explantion for Statement-1.
c) Statement-1 is True, Statement-2 is False
d) Statement-1 is False, Statement-2 is True.

7.4.2. If **A** and **B** are points in the plane such that $\frac{PA}{PB} = K$(constant) for all **P** on a given circle, then the value of K cannot be equal to ______. (1982)

7.4.3. From the origin chords are drawn to the circle $(x - 1)^2 + y^2 = 1$. The equation of the locus of the mid-points of these chords is ______. (1985)

7.4.4. From the point **A** $(0, 3)$ on the circle

$$x^2 + 4x + (y - 3)^2 = 0,$$

a chord AB is drawn and extended to a point **M** such that $AM = 2AB$. The equation of the locus of **M** is (1986)

7.4.5. If a circle passes through the points of intersection of the coordinate axes with the lines $\lambda x - y + 1 = 0$ and $x - 2y + 3 = 0$, then the value of $\lambda =$ ______. (1991)

7.4.6. The equation of the locus of the mid-points of the circle $4x^2 + 4y^2 - 12x + 4y + 1 = 0$ that subtend an angle of $\frac{2\pi}{3}$ at its centre is ______. (1993)

7.4.7. The intercept of the line $y = x$ by the circle $x^2 + y^2 - 2x = 0$ is AB. Equation of the circle with AB as a diameter is ______. (1996)

7.4.8. For each natural number k, let C_k denote the circle with radius k centimetres and centre at the origin. On the circle C_k, a particle moves k centimetres in the counter-clockwise direction. After completing its motion on C_k, the particle moves to C_{k+1} in the radial direction. The motion of the particle continues in this manner. The particle starts at $(1, 0)$. If the particle crosses the positive direction of the X axis for the first time on the circle C_n, then $n =$______. (1997)

7.4.9. The straight line $2x - 3y = 1$ divides the circular region $x^2 + y^2 \leq 6$ into two parts. If S is $\{ (2, 3/4), (5/2, 3/4), (1/4, -1/4), (1/8, 1/4) \}$ then the number of point(s) in S lying inside the smaller part is ______. (2011)

7.4.10. For how many values of p, the circle $x^2 + y^2 + 2x + 4y - p = 0$ and the coordinate axes have exactly three common points? (2017)

7.4.11. If the chord $y = mx + 1$ of the circle $x^2 + y^2 = 1$ subtends an angle of measure $45°$ at the major segment of the circle then the value of m is (2002)

a) $2 \pm \sqrt{2}$ b) $-2 \pm \sqrt{2}$ c) $-1 \pm \sqrt{2}$ d) none of this

7.4.12. The centres of a set of circles, each of radius 3, lie on the circle $x^2 + y^2 = 25$.The locus of any point in the set is (2002)

a) $4 \leq x^2 + y^2 \leq 64$
b) $x^2 + y^2 \leq 25$
c) $x^2 + y^2 \geq 25$
d) $3 \leq x^2 + y^2 \leq 9$

7.4.13. The centre of the circle passing through $(0, 0)$ and $(1, 0)$ and touching the circle $x^2 + y^2 = 9$ is (2002)

a) $(1/2, 1/2)$ b) $\left(1/2, -\sqrt{2}\right)$ c) $(3/2, 1/2)$ d) $(1/2, 3/2)$

7.4.14. The equation of a circle with origin as a centre and passing through equilateral triangle whose median is of length $3a$ is (2002)

a) $x^2 + y^2 = 9a^2$
b) $x^2 + y^2 = 16a^2$
c) $x^2 + y^2 = 4a^2$
d) $x^2 + y^2 = a^2$

7.4.15. The lines $2x - 3y = 5$ and $3x - 4y = 7$ are diameters of a circle having area as 154 sq.units. Then the equation of the circle is (2003)

a) $x^2 + y^2 - 2x + 2y = 62$
b) $x^2 + y^2 + 2x - 2y = 62$
c) $x^2 + y^2 + 2x - 2y = 47$
d) $x^2 + y^2 - 2x + 2y = 47$

7.4.16. A variable circle passes through the fixed point $\mathbf{A}(p, q)$ and touches the X axis. The locus of the other end of the diameter through $\mathbf{A}$ is (2004)

a) $(y - q)^2 = 4px$
b) $(x - q)^2 = 4py$
c) $(y - p)^2 = 4qx$
d) $(x - p)^2 = 4qy$

7.4.17. If the lines $2x + 3y + 1 = 0$ and $3x - y - 4 = 0$ lie along diameter of a circle of circumference 10π, then the equation of the circle is (2004)

a) $x^2 + y^2 + 2x - 2y - 23 = 0$
b) $x^2 + y^2 - 2x - 2y - 23 = 0$
c) $x^2 + y^2 + 2x + 2y - 23 = 0$
d) $x^2 + y^2 - 2x + 2y - 23 = 0$

7.4.18. If the lines $3x - 4y - 7 = 0$ and $2x - 3y - 5 = 0$ are two diameters of a circle of area 49π square units, the equation of the circle is (2006)

a) $x^2 + y^2 + 2x - 2y - 47 = 0$
b) $x^2 + y^2 + 2x - 2y - 62 = 0$
c) $x^2 + y^2 - 2x + 2y - 62 = 0$
d) $x^2 + y^2 - 2x + 2y - 47 = 0$

7.4.19. Let $\mathbf{C}$ be the circle with centre $(0, 0)$ and radius 3 units. The equation of the locus of the mid points of the chords of the circle $\mathbf{C}$ that subtend an angle of $\frac{2\pi}{3}$ at its centre is (2006)

a) $x^2 + y^2 = \frac{3}{2}$
b) $x^2 + y^2 = 1$
c) $x^2 + y^2 = \frac{27}{4}$
d) $x^2 + y^2 = \frac{9}{4}$

7.4.20. The point diametrically opposite to the point $\mathbf{P}(1, 0)$ on the circle $x^2+y^2+2x+2y-3 = 0$ is (2008)

a) $(3, -4)$
b) $(-3, 4)$
c) $(-3, -4)$
d) $(3, 4)$

7.4.21. A square is inscribed in the circle $x^2 + y^2 - 2x + 4y + 3 = 0$. Its sides are parellel to the coordinate axes. Then one vertex of the square is (1980)

a) $\left(1 + \sqrt{2}, -2\right)$
b) $\left(1 - \sqrt{2}, -2\right)$
c) $\left(1, -2 + \sqrt{2}\right)$
d) none of these

7.4.22. The locus of the midpoint of a chord of the circle $x^2 + y^2 = 4$ which subtends a right angle at the origin is (1984)

a) $x + y = 2$
b) $x^2 + y^2 = 1$
c) $x^2 + y^2 = 2$
d) $x + y = 1$

7.4.23. The lines $2x - 3y = 5$ and $3x - 4y = 7$ are diameters of a circle of area 154 sq. units. The equation of this circle is (1989)

a) $x^2 + y^2 + 2x - 2y = 62$
b) $x^2 + y^2 + 2x - 2y = 47$
c) $x^2 + y^2 - 2x + 2y = 47$
d) $x^2 + y^2 - 2x + 2y = 62$

7.4.24. The centre of the circle passing through the points $(0,0)$, $(1,0)$ and touching the circle $x^2 + y^2 = 9$ is (1992)

a) $\left(\frac{3}{2}, \frac{1}{2}\right)$
b) $\left(\frac{1}{2}, \frac{3}{2}\right)$
c) $\left(\frac{1}{2}, -2\frac{1}{2}\right)$
d) none of these

7.4.25. The locus of the centre of a circle, which touches the circle is $x^2+y^2-6x-6y+14 = 0$ and also touches the y-axis, is given by the equation: (1993)

a) $x^2 - 6x - 10y + 14 = 0$
b) $x^2 - 10x - 6y + 14 = 0$
c) $y^2 - 6x - 10y + 14 = 0$
d) $y^2 - 10x - 6y + 14 = 0$

7.4.26. If two distinct chords, drawn from the point (p, q) on the circle $x^2 + y^2 = px + qy$ (where $pq \neq 0$) are bisected by the X axis, then which are true (1999)

a) $p^2 = q^2$
b) $p^2 = 8q^2$
c) $p^2 < 8q^2$
d) $p^2 > 8q^2$

7.4.27. The triangle PQR is inscribed in the circle $x^2+y^2 = 25$. If **Q** and **R** have co-ordinates $(3, 4)$ and $(-4, 3)$ respectively, then $\angle$QPR is equal to (2000)

a) $\frac{\pi}{2}$
b) $\frac{\pi}{3}$
c) $\frac{\pi}{4}$
d) $\frac{\pi}{6}$

7.4.28. Let AB be a chord of the circle $x^2 + y^2 = r^2$ subtending a right angle at the centre. Then the locus of the centroid of the triangle PAB as **P** moves on the circle is (2001)

a) a parabola
b) a circle
c) an ellipse
d) a pair of straight lines

7.4.29. If one of the diameters of the circle $x^2 + y^2 - 2x - 6y + 6 = 0$ is a chord to the circle with the centre $(2, 1)$, then the radius of the circle is (2004)

a) $\sqrt{3}$ b) $\sqrt{2}$ c) 3 d) 2

7.4.30. A circle is given by $x^2+(y-1)^2 = 1$, another circle C touches it externally and also the X axis, then the locus of its centre is (2005)

a) $\{(x, y):x^2 = 4y\} \bigcup \{(x, y):y\leq 0\}$
b) $\{(x, y):x^2 + (y-1)^2 = 4\} \bigcup \{(x, y):y\leq 0\}$
c) $\{(x, y):x^2 = y\} \bigcup \{(0, y):y\leq 0\}$
d) $\{(x, y):x^2 = 4y\} \bigcup \{(0, y):y\leq 0\}$

7.4.31. The circle passing through the point $(-1, 0)$ and touching the y-axis at $(0, 2)$ also passes through the point (2011)

a) $\left(-\frac{3}{2}, 0\right)$ b) $\left(-\frac{5}{2}, 2\right)$ c) $\left(-\frac{3}{2}, \frac{5}{2}\right)$ d) $(-4, 0)$

7.4.32. $ABCD$ is a square of side length 2 units. C_1 is the circle touching all the sides of the square $ABCD$ and C_2 is the *circumcircle* of square $ABCD$. L is a fixed line in same plane and **R** is a fixed point. (2006)

a) If **P** is any point of C_1 and **Q** is another point on C_2, then $\frac{PA^2+PB^2+PC^2+PD^2}{QA^2+QB^2+QC^2+QD^2}$

i) 0.75 ii) 1.25 iii) 1 iv) 0.5

b) If a circle is such that it touches the line L and the circle C_1 externally, such that both the circles are on the same side of the line, then locus of centre of the circle

i) ellipse ii) hyperbola iii) parabola iv) circle

c) A line L' through **A** is drawn parallel to BD. Point S moves such that its distances from the line BD and the vertex **A** are equal. If locus of S cuts L' at T_2 and T_3 and AC at T_1, then area of $\Delta T_1T_2T_3$ is

i) 1/2 sq.units ii) 2/3 sq.units iii) 1 sq.units iv) 2 sq.units

7.4.33. A circle C of radius 1 unit is inscribed in an equilateral triangle PQR. The points of contact of C with sides PQ, QR, RP are $\mathbf{D}, \mathbf{E}, \mathbf{F}$ respectively. The line PQ is given by the equation $\sqrt{3}x + y - 6 = 0$ and the point **D** is $\left(3\sqrt{3}/2, 3/2\right)$. Further, it is given that the origin and the centre of C are on same side of line PQ. The equation of circle C is (2008)

a) $(x - 2\sqrt{3})^2 + (y - 1)^2 = 1$
b) $(x - 2\sqrt{3})^2 + (y + 1/2)^2 = 1$
c) $(x - \sqrt{3})^2 + (y - 1)^2 = 1$
d) $(x - \sqrt{3})^2 + (y + 1)^2 = 1$

7.4.34. Find the equation of the circle whose radius is 5 and which touches the circle $x^2 + y^2 - 2x - 4y - 20 = 0$ at the point $(5, 5)$. (1978)

7.4.35. Through a fixed point (h, k) secants are drawn to the circle $x^2 + y^2 = r^2$. Show that the locus of the mid-points of the secants intercepted is $x^2 + y^2 = hx + ky$. (1983)

7.4.36. The abscissa of two points **A** and **B** are roots of the equation $x^2 + 2ax - b^2 = 0$ and their ordinates are roots of the equation $x^2 + 2px - q^2 = 0$. Find the equation and the radius of the circle with AB as diameter. (1984)

7.4.37. Let a given Line L_1 intersects the X and Y axes at **P** and **Q** respectively. Let another line L_2, perpendicular to L_1, cut the X and Y axes at **R** and **S**, respectively. Show that the locus of the point of intersection of PS and QR is a circle passing through origin. (1987)

7.4.38. The circle $x^2 + y^2 - 4x - y + 4 = 0$ is inscribed in a triangle which has two of its sides along the co-ordinate axes. The locus of circumcentre of the triangle is $x + y - xy + k(x^2 + y^2)^{1/2}$. Find k. (1987)

7.4.39. If $\left(m_i, \frac{1}{m_i}\right), m_i > 0, i = 1, 2, 3, 4$ are four distinct points on a circle, then show that $m_1m_2m_3m_4 = 1$ (1989)

7.4.40. Let a circle be given by $2x(x-a)+y(2y-b)=0$, $(a \neq 0, b \neq 0)$. Find the condition on a and b if two chords, each bisected by the X axis, can be drawn to the circle from $\left(a, \frac{b}{2}\right)$. (1992)

7.4.41. Consider a family of circles passing through two fixed points $\mathbf{A}(3,7)$ and $\mathbf{B}(6,5)$. Show that chords in which the circle $x^2+y^2-4x-6y-3=0$ cuts the members of the family are concurrent at a point. Find the coordinate of this point. (1993)

7.4.42. Find the intervals of values of a for which the line $y+x=0$ bisects two chords drawn from a point $\left(\frac{1+\sqrt{2}a}{2}, \frac{1-\sqrt{2}a}{2}\right)$ to the circle $2x^2+2y^2-\left(1+\sqrt{2}a\right)x-\left(1-\sqrt{2}a\right)y=0$. (1996)

7.4.43. A circle passes through three points **A**, **B** and C with the line segment AC as its diameter. A line passing through **A** intersects the chord BC at point **D** inside the circle. If angles DAB and CAB are α and β respectively and the distance between the point **A** and midpoint of the line segment DC is d, prove that the area of the circle is $\frac{\pi d^2 \cos^2\alpha}{\cos^2\alpha+\cos^2\beta+2\cos\alpha\cos\beta\cos(\beta-\alpha)}$. (1996)

7.4.44. Let C be any circle with centre $\left(0, \sqrt{2}\right)$. Prove that at the most two rational points can be there on C. (A rational point is a point both of whose coordinates are rational numbers) (1997)

7.4.45. Let **O** be the centre of the circle $x^2+y^2=r^2$, where $r > \frac{\sqrt{5}}{2}$. Suppose PQ is a chord of this circle and the equation of the line passing through **P** and **Q** is $2x+4y=5$. If the centre of the circumcircle of the triangle OPQ lies on the lies on the line $x+2y=4$, then the value of r is ______. (2020)

7.4.46. Consider a triangle Δ whose two sides lie on the X axis and the line $x+y+1=0$. If the orthocenter of Δ is $(1,1)$, then the equation of the circle passing through the vertices of the triangle Δ is (2021)

a) $x^2+y^2-3x+y=0$
b) $x^2+y^2+x+3y=0$
c) $x^2+y^2+2y-1=0$
d) $x^2+y^2+x+y=0$

8 Conics

8.1 Formulae

8.1.1. Let $\mathbf{q}$ be a point such that the ratio of its distance from a fixed point $\mathbf{F}$ and the distance (d) from a fixed line

$$L : \mathbf{n}^\top \mathbf{x} = c \tag{8.1.1.1}$$

is constant, given by

$$\frac{\|\mathbf{q} - \mathbf{F}\|}{d} = e \tag{8.1.1.2}$$

The locus of $\mathbf{q}$ is known as a conic section. The line L is known as the directrix and the point $\mathbf{F}$ is the focus. e is defined to be the eccentricity of the conic.

a) For $e = 1$, the conic is a parabola
b) For $e < 1$, the conic is an ellipse
c) For $e > 1$, the conic is a hyperbola

8.1.2. The equation of a conic with directrix $\mathbf{n}^\top \mathbf{x} = c$, eccentricity e and focus $\mathbf{F}$ is given by

$$g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{8.1.2.1}$$

where

$$\mathbf{V} = \|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{n}\mathbf{n}^\top, \tag{8.1.2.2}$$

$$\mathbf{u} = ce^2\mathbf{n} - \|\mathbf{n}\|^2 \mathbf{F}, \tag{8.1.2.3}$$

$$f = \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2e^2 \tag{8.1.2.4}$$

8.1.3. The eccentricity, directrices and foci of (8.1.2.1) are given by

$$e = \sqrt{1 - \frac{\lambda_1}{\lambda_2}} \tag{8.1.3.1}$$

$$\mathbf{n} = \sqrt{\lambda_2}\mathbf{p}_1,$$

$$c = \begin{cases} \frac{e\mathbf{u}^\top\mathbf{n} \pm \sqrt{e^2(\mathbf{u}^\top\mathbf{n})^2 - \lambda_2(e^2-1)(\|\mathbf{u}\|^2 - \lambda_2 f)}}{\lambda_2 e(e^2-1)} & e \neq 1 \\ \frac{\|\mathbf{u}\|^2 - \lambda_2 f}{2\mathbf{u}^\top\mathbf{n}} & e = 1 \end{cases} \tag{8.1.3.2}$$

$$\mathbf{F} = \frac{ce^2\mathbf{n} - \mathbf{u}}{\lambda_2} \tag{8.1.3.3}$$

8.1.4. For a symmetric matrix, from (A.7.7.1), we have the eigendecomposition

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^\top \tag{8.1.4.1}$$

where

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix}, \quad \mathbf{P}^\top\mathbf{P} = \mathbf{I} \tag{8.1.4.2}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{8.1.4.3}$$

8.1.5. Using the affine transformation in (2.1.15.1), the conic in (8.1.2.1) can be expressed in standard form as

$$\mathbf{y}^\top \left(\frac{\mathbf{D}}{f_0}\right)\mathbf{y} = 1 \qquad |\mathbf{V}| \neq 0 \tag{8.1.5.1}$$

$$\mathbf{y}^\top \mathbf{D}\mathbf{y} = -\eta \mathbf{e}_1^\top \mathbf{y} \qquad |\mathbf{V}| = 0 \tag{8.1.5.2}$$

where

$$f_0 = \mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f \neq 0 \tag{8.1.5.3}$$

$$\eta = 2\mathbf{u}^\top \mathbf{p}_1 \tag{8.1.5.4}$$

$$\mathbf{e}_1 = \begin{pmatrix}1\\0\end{pmatrix} \tag{8.1.5.5}$$

Solution: See Appendix B.1.5.

8.1.6. The equation of the minor and major axes for the ellipse/hyperbola are respectively given by

$$\mathbf{p}_i^\top (\mathbf{x} - \mathbf{c}) = 0, i = 1, 2 \tag{8.1.6.1}$$

8.1.7. The center/vertex of a conic section are given by

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \qquad \left|\mathbf{V}\right| \neq 0 \tag{8.1.7.1}$$

$$\begin{pmatrix}\mathbf{u}^\top + \frac{\eta}{2}\mathbf{p}_1^\top \\ \mathbf{V}\end{pmatrix}\mathbf{c} = \begin{pmatrix}-f \\ \frac{\eta}{2}\mathbf{p}_1 - \mathbf{u}\end{pmatrix} \qquad \left|\mathbf{V}\right| = 0 \tag{8.1.7.2}$$

8.1.8. The *focal length* of the standard parabola, defined to be the distance between the vertex and the focus, measured along the axis of symmetry, is $\left|\frac{\eta}{4\lambda_2}\right|$

8.1.9. For the standard hyperbola/ellipse, the length of the major axis is

$$2\sqrt{\left|\frac{f_0}{\lambda_1}\right|} \tag{8.1.9.1}$$

and the minor axis is

$$2\sqrt{\left|\frac{f_0}{\lambda_2}\right|} \tag{8.1.9.2}$$

Solution: See Appendix B.3.4.

8.1.10. The latus rectum of a conic section is the chord that passes through the focus and is perpendicular to the major axis. The length of the latus rectum for a conic is given by

$$l = \begin{cases} 2\frac{\sqrt{|f_0\lambda_1|}}{\lambda_2} & e \neq 1 \\ \frac{\eta}{\lambda_2} & e = 1 \end{cases} \tag{8.1.10.1}$$

Solution: See Appendix B.3.6.

8.1.11. Codes for parabola

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codes/book/parab.py
codes/book/parab_rot.py
```

8.1.12. Code for ellipse

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codes/book/ellipse.py
codes/book/ellipse_rot.py
```

8.1.13. Code for hyperbola

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codes/book/hyper.py
codes/book/hyper_rot.py
```

8.2 Equation

8.2.1 The equation of the conic whose focus is $(1, -1)$, directrix $x-y-3 = 0$ and eccentricity $\frac{1}{2}$ is ______.
Solution: In (8.1.1.2), substituting

$$\mathbf{q} = \begin{pmatrix} x \\ y \end{pmatrix} \mathbf{F} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e = \frac{1}{2}, \tag{8.2.1.1}$$

$$4\left[(x+1)^2 + (y-1)^2\right] = \left(\frac{x-y+3}{\sqrt{1^2 + (-1)^2}}\right)^2 \tag{8.2.1.2}$$

$$= \left(\frac{x-y+3}{\sqrt{2}}\right)^2 \tag{8.2.1.3}$$

yielding

$$7\left(x^2 + y^2\right) + 2xy + 10\left(-x + y\right) + 7 = 0 \tag{8.2.1.4}$$

Comparing the above with (8.1.2.1),

$$\mathbf{V} = \begin{pmatrix} 7 & 1 \\ 1 & 7 \end{pmatrix}, \mathbf{u} = 5\begin{pmatrix} -1 \\ 1 \end{pmatrix}, f = 7. \tag{8.2.1.5}$$

The above can also be obtained from (8.1.2.2) - (8.1.2.4) using the parameters in Table 8.2.1.

Variable	**Description**	**Value**
$\mathbf{n}$	Normal of Directrix	$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$
c	c of Directrix	3
e	Eccentricity of conic	$\frac{1}{2}$
$\mathbf{F}$	Focus of conic	$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

TABLE 8.2.1

The eigenvalues of $\mathbf{V}$ are obtained using

$$\left|\mathbf{V} - \lambda\mathbf{I}\right| = 0, \tag{8.2.1.6}$$

$$\implies \begin{vmatrix} 7-\lambda & 1 \\ 1 & 7-\lambda \end{vmatrix} = 0 \tag{8.2.1.7}$$

$$\text{or, } \lambda^2 - 14\lambda + 48 = 0 \tag{8.2.1.8}$$

$$\implies \lambda_1, \lambda_2 = 6, 8 \tag{8.2.1.9}$$

$$\tag{8.2.1.10}$$

The eigenvector of $\mathbf{V}$ corresponding to $\lambda = 6$ is obtained as

$$\begin{pmatrix} 7-\lambda & 1 \\ 1 & 7-\lambda \end{pmatrix}\mathbf{x} = 0 \tag{8.2.1.11}$$

$$\implies \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\mathbf{x} = 0 \tag{8.2.1.12}$$

$$\implies \mathbf{x} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{8.2.1.13}$$

Similarly, the eigenvector corresponding to $\lambda = 8$ can be obtained as

$$\mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{8.2.1.14}$$

resulting in the spectral decomposition

$$\begin{pmatrix} 7 & 1 \\ 1 & 7 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 6 & 0 \\ 0 & 8 \end{pmatrix}\begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \tag{8.2.1.15}$$

$$\text{or, } \mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^{\top}, \quad \mathbf{P}^{\top}\mathbf{P} = \mathbf{I} \tag{8.2.1.16}$$

Also,

$$\mathbf{P} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{8.2.1.17}$$

$$\implies \theta = \frac{3\pi}{4} \tag{8.2.1.18}$$

From (8.1.5.3) and (8.1.7.1), the corresponding standard ellipse parameters are

$$f_0 = \mathbf{u}^{\mathrm{T}}\mathbf{V}^{-1}\mathbf{u} - f = \frac{25}{48}\begin{pmatrix} -1 & 1 \end{pmatrix}\begin{pmatrix} 7 & -1 \\ -1 & 7 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} - 7 = \frac{4}{3} \tag{8.2.1.19}$$

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} = -\frac{5}{48}\begin{pmatrix} 7 & -1 \\ -1 & 7 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \frac{5}{6}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{8.2.1.20}$$

yielding the standard form of the ellipse from (8.1.5.1) as

$$\mathbf{y}^{\top}\left(\frac{\mathbf{D}}{f_0}\right)\mathbf{y} = 1 \tag{8.2.1.21}$$

$$\implies \mathbf{y}^{T}\begin{pmatrix} \frac{9}{2} & 0 \\ 0 & 6 \end{pmatrix}\mathbf{y} = 1 \tag{8.2.1.22}$$

From (8.1.9.1) and (8.1.9.2), the lengthes of the axes of are obtained as

$$a = \sqrt{\frac{f_0}{\lambda_1}} = \sqrt{\frac{2}{9}} \tag{8.2.1.23}$$

$$b = \sqrt{\frac{f_0}{\lambda_2}} = \sqrt{\frac{1}{6}} \tag{8.2.1.24}$$

and from (8.1.10.1), the length of the latus rectum of the ellipse is

$$l = 2\frac{\sqrt{|f_0\lambda_1|}}{\lambda_2} = \frac{1}{\sqrt{2}}. \tag{8.2.1.25}$$

From (8.1.6.1), the equations of the axes are

$$\text{Minor Axis}: \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} x\mathbf{x} = 0 \tag{8.2.1.26}$$

$$\text{Major Axis}: \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \mathbf{x} = 0 \tag{8.2.1.27}$$

See Fig. 8.2.1.1 for verification.

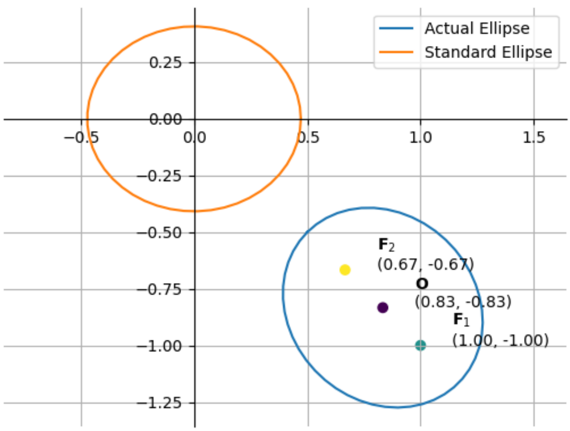

Fig. 8.2.1.1

In the each of the following exercises, find the coordinates of the focus, vertex, eccentricity, axis of the conic section, the equation of the directrix and the length of the latus rectum.

8.2.2 $y^2 = 12x$

Solution: See Table 8.2.6 and Fig. 8.2.2.1. Problem 8.1.3 was used to obtain the conic and directrix parameters. The latus rectum is obtained using (8.1.10.1)

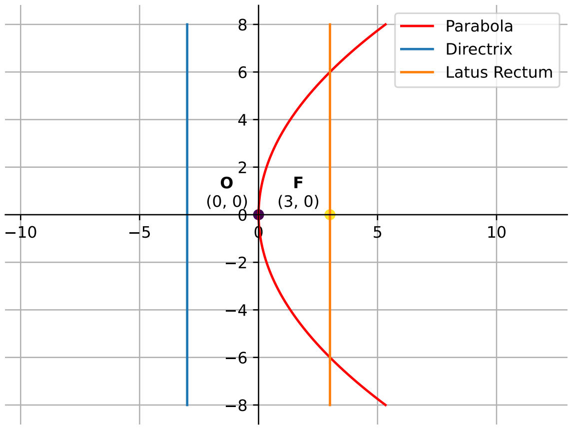

Fig. 8.2.2.1

8.2.3 $y^2 = -8x$

8.2.4 $\frac{x^2}{36} + \frac{y^2}{16} = 1$

Solution: See Table 8.2.6 and Fig. 8.2.4.1.

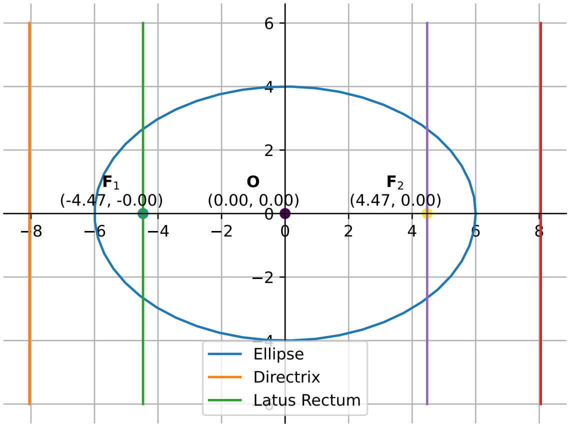

Fig. 8.2.4.1

8.2.5 $\frac{x^2}{16} + \frac{y^2}{9} = 1$

8.2.6 $\frac{x^2}{16} - \frac{y^2}{9} = 1$.

Solution: See Table 8.2.6 and Fig. 8.2.6.1.

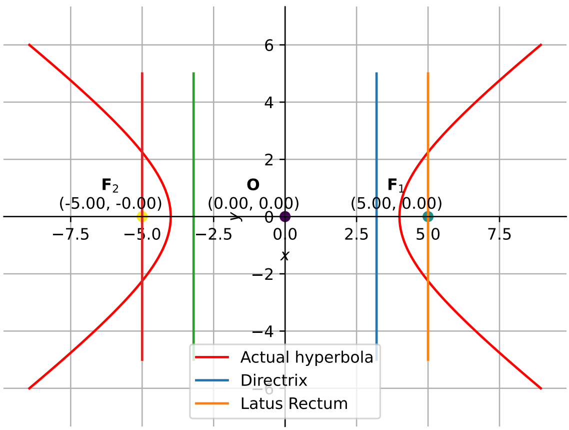

Fig. 8.2.6.1

TABLE 8.2.6

Input				Output		
Conic	$\mathbf{V}$	$\mathbf{u}$	f	$\mathbf{F}$	Directrix	Latus Rectum
$y^2 = 12x$	$\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}$	$-6\mathbf{e}_1$	0	$3\mathbf{e}_1$	$\mathbf{e}_1^\top\mathbf{x} = -3$	12
$y^2 = -8x$	$\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}$	$4\mathbf{e}_1$	0	$2\mathbf{e}_1$	$\mathbf{e}_1^\top\mathbf{x} = 2$	8
$\frac{x^2}{36} + \frac{y^2}{16} = 1$	$\begin{pmatrix}4 & 0\\0 & 9\end{pmatrix}$	$\mathbf{0}$	-144	$2\sqrt{5}\mathbf{e}_1$	$\mathbf{e}_1^\top\mathbf{x} = \frac{18}{\sqrt{5}}$	$\frac{16}{3}$
$\frac{x^2}{16} + \frac{y^2}{9} = 1$	$\begin{pmatrix}9 & 0\\0 & 16\end{pmatrix}$	$\mathbf{0}$	-144	$\pm\sqrt{7}\mathbf{e}_1$	$\mathbf{e}_1^\top\mathbf{x} = \frac{16}{\sqrt{7}}$	$\frac{9}{2}$
$\frac{x^2}{16} - \frac{y^2}{9} = 1$	$\begin{pmatrix}9 & 0\\0 & -16\end{pmatrix}$	$\mathbf{0}$	-144	$\pm 5\mathbf{e}_1$	$\mathbf{e}_1^\top\mathbf{x} = \frac{16}{5}$	$\frac{9}{2}$

8.2.7 $16x^2 + y^2 = 16$

8.2.8 $4x^2 + 9y^2 = 36$

8.2.9 $y^2 = 10x$

8.2.10 $\frac{x^2}{4} + \frac{y^2}{25} = 1$

Solution: From Table 8.2.10, it can be seen that this is not a standard ellipse, since $\lambda_1 > \lambda_2$. Hence $\mathbf{P}$ plays a role and we need to use the affine transformation

$$\mathbf{x} = \mathbf{P}\mathbf{y} \tag{8.2.10.1}$$

So the value of λ_1 and λ_2 need to be interchanged for all calculations and $\mathbf{e}_2$ becomes the normal vector of the directrix. See Fig. 8.2.10.1.

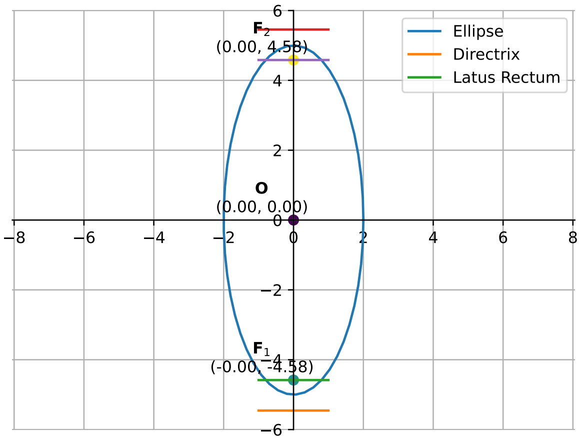

Fig. 8.2.10.1

TABLE 8.2.10

Input				Intermediate	Output		
Conic	$\mathbf{V}$	$\mathbf{u}$	f	$\mathbf{P}$	$\mathbf{F}$	Directrix	Latus Rectum
$\frac{x^2}{4}+\frac{y^2}{25}=1$	$\begin{pmatrix}25 & 0\\0 & 4\end{pmatrix}$	$\mathbf{0}$	-100	$\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$	$\pm\sqrt{21}\mathbf{e}_2$	$\mathbf{e}_2^\top\mathbf{x}=\pm\frac{25}{\sqrt{21}}$	$\frac{8}{5}$
$5y^2-9x^2=36$	$\begin{pmatrix}-9 & 0\\0 & 5\end{pmatrix}$	$\mathbf{0}$	-36	$\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$	$\pm 3\sqrt{\frac{5}{14}}\mathbf{e}_2$	$\mathbf{e}_2^\top\mathbf{x}=\pm\frac{18}{\sqrt{70}}$	$4\frac{\sqrt{5}}{3}$
$\frac{y^2}{9}-\frac{x^2}{27}=1$	$\begin{pmatrix}-1 & 0\\0 & 3\end{pmatrix}$	$\mathbf{0}$	-27	$\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$	$\pm 6\mathbf{e}_2$	$\mathbf{e}_2^\top\mathbf{x}=\pm\frac{3}{2}$	18
$x^2=-16y$	$\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$	$\begin{pmatrix}0\\8\end{pmatrix}$	0	$\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$	$-4\mathbf{e}_2$	$\mathbf{e}_2^\top\mathbf{x}=4$	16
$x^2=6y$	$\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$	$-\begin{pmatrix}0\\3\end{pmatrix}$	0	$\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$	$\frac{3}{2}\mathbf{e}_2$	$\mathbf{e}_2^\top\mathbf{x}=-\frac{3}{2}$	6

8.2.11 $\frac{x^2}{100}+\frac{y^2}{400}=1$

8.2.12 $36x^2+4y^2=144$

8.2.13 $5y^2-9x^2=36$.

Solution:

See Table 8.2.10 and Fig. 8.2.13.1. In Table 8.2.10, $\mathbf{P}$ shifts the negative eigenvalue to get the hyperbola in standard form.

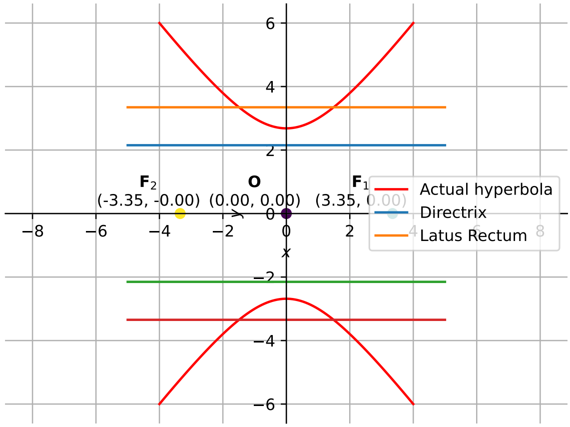

Fig. 8.2.13.1

8.2.14 $\frac{y^2}{9} - \frac{x^2}{27} = 1.$

8.2.15 $x^2 = -16y$

Solution: See Table 8.2.10 and Fig. 8.2.15.1.

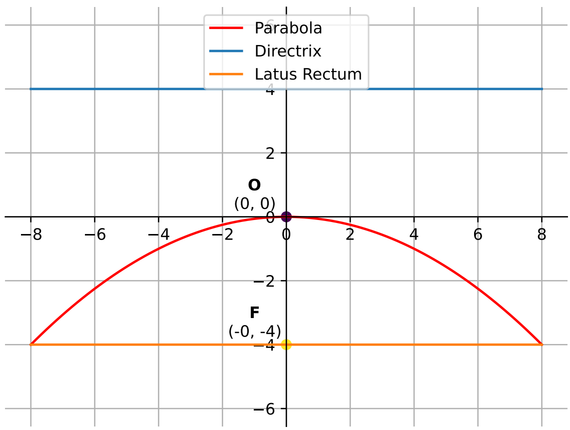

Fig. 8.2.15.1

8.2.16 $x^2 = 6y$

8.2.17 $x^2 = -9y$
8.2.18 $\frac{x^2}{25} + \frac{y^2}{100} = 1$
8.2.19 $\frac{x^2}{49} + \frac{y^2}{36} = 1$

In each of the following exercises, find the equation of the conic, that satisfies the given conditions.

8.2.20 vertex (0,0) passing through (2,3) and axis is along X axis.
8.2.21 vertex (0,0) passing through (5,2) symmetric with respect to Y axis.
8.2.22 vertices $(\pm 5, 0)$, foci$(\pm 4, 0)$.
8.2.23 vertices $(\pm 0, 13)$, foci$(0, \pm 5)$.
8.2.24 vertices $(\pm 6, 0)$, foci$(\pm 4, 0)$.
8.2.25 vertices $(0, \pm 5)$, foci $(0, \pm 8)$.
8.2.26 focus (6,0); directrix $x = -6$
8.2.27 focus $(0, -3)$; directrix $y = 3$
8.2.28 vertex (0,0); focus (3,0)
8.2.29 vertex (0,0); focus $(-2, 0)$
8.2.30 foci $(\pm 4, 0)$, latus rectum of length 12.

Solution: The given information is available in Table 8.2.30. Since two foci are given, the conic cannot be a parabola.

a) The direction vector of F_1F_2 is the normal vector of the directrix. Hence,

$$\mathbf{n} = \mathbf{F_1} - \mathbf{F_2} \equiv \mathbf{e}_1 \tag{8.2.30.1}$$

Substituting in (8.1.2.2), (8.1.2.3) and (8.1.2.4),

$$\mathbf{V} = \begin{pmatrix} 1 - e^2 & 0 \\ 0 & 1 \end{pmatrix} \tag{8.2.30.2}$$

$$\mathbf{u} = ce^2\mathbf{e}_1 - \mathbf{F} \tag{8.2.30.3}$$

$$f = 16 - c^2e^2 \tag{8.2.30.4}$$

b) From (8.2.30.2),

$$\lambda_1 = 1 - e^2, \ \lambda_2 = 1 \tag{8.2.30.5}$$

which upon substituting in (8.1.10.1), along with the value of the latus rectum from Table 8.2.30

$$6\left(1 - e^2\right) = \sqrt{|f|} \tag{8.2.30.6}$$

c) The centre of the conic is given by

$$\mathbf{c} = \frac{\mathbf{F_1} + \mathbf{F_2}}{2} = \mathbf{0} \tag{8.2.30.7}$$

From (8.2.30.2), it is obvious that $\mathbf{V}$ is invertible. Hence, from (8.2.30.7) and (B.1.5.9),

$$\mathbf{u} = \mathbf{0} \tag{8.2.30.8}$$

Substituting the above in (8.2.30.3),

$$\mathbf{F} = ce^2\mathbf{e}_1 \implies \|\mathbf{F}\| = 4 = ce^2 \tag{8.2.30.9}$$

d) From (8.1.5.3), (8.2.30.8) and (8.2.30.4),

$$36\left(1-e^2\right)^2 = 16 - c^2e^2 \tag{8.2.30.10}$$

From (8.2.30.9) and (8.2.30.10)

$$\frac{4}{e\sqrt{e^2-1}} = 6 \tag{8.2.30.11}$$

$$\implies 9e^2\left(e^2-1\right) = 4 \tag{8.2.30.12}$$

$$\implies 9e^4 - 9e^2 - 4 = 0 \tag{8.2.30.13}$$

$$\text{or, } \left(3e^2-4\right)\left(12e^2+1\right) = 0 \tag{8.2.30.14}$$

yielding

$$e = \frac{2}{\sqrt{3}} \tag{8.2.30.15}$$

as the only viable solution.

The equation of the conic is then obtained as

$$\mathbf{x}^\top \begin{pmatrix} -\frac{1}{3} & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + 4 = 0 \tag{8.2.30.16}$$

See Fig. 8.2.30.1.

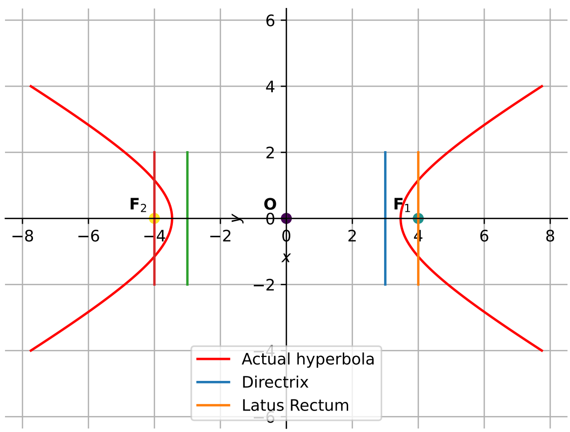

Fig. 8.2.30.1

Parameter	Description	Value
$\mathbf{F_1}$	Focus 1 of hyperbola	$\begin{pmatrix}4\\0\end{pmatrix}$
$\mathbf{F_2}$	Focus 2 of hyperbola	$\begin{pmatrix}-4\\0\end{pmatrix}$
l	Length of latus rectum	12

TABLE 8.2.30

8.2.31 ends of major axis $(\pm 3, 0)$, ends of minor axis $(0, \pm 2)$.
8.2.32 ends of major axis $(0, \pm\sqrt{5})$, ends of minor axis $(\pm 1, 0)$.
8.2.33 length of major axis 26, foci $(\pm 5, 0)$.
8.2.34 length of minor axis 16, foci $(0, \pm 6)$.
Solution: $\because$ 2 foci exist for this conic, it must be an ellipse or a hyperbola.

$$\therefore \mathbf{F}_1 = \begin{pmatrix}0\\6\end{pmatrix}, \mathbf{F}_2 = \begin{pmatrix}0\\-6\end{pmatrix}, \tag{8.2.34.1}$$

$$\mathbf{u} = \frac{\mathbf{F_1} + \mathbf{F_2}}{2} = \begin{pmatrix}0\\0\end{pmatrix} \tag{8.2.34.2}$$

$$\mathbf{n} \equiv \mathbf{F_1} - \mathbf{F_2} \equiv \begin{pmatrix}0\\1\end{pmatrix} \tag{8.2.34.3}$$

Using (8.2.34.3) in (8.1.2.2),

$$\mathbf{V} = \begin{pmatrix}1 & 0\\0 & 1-e^2\end{pmatrix} \tag{8.2.34.4}$$

From (8.1.9.2), the length of the minor axis

$$16 = 2\sqrt{\frac{|f|}{|1-e^2|}} \tag{8.2.34.5}$$

$$\implies 8\sqrt{|1-e^2|} = \sqrt{|f|} \tag{8.2.34.6}$$

From (8.1.3.3), substituting from (8.2.34.1) and (8.2.34.3),

$$\pm ce^2 = 6 \tag{8.2.34.7}$$

Substituting all known values in (8.1.3.2),

$$c = \pm\frac{1}{e}\sqrt{\frac{|f|}{|e^2-1|}} \tag{8.2.34.8}$$

In (8.2.34.6)-(8.2.34.8), there are 3 unkowns, (c, f, e). Upon solving, we get

$$e = \frac{3}{4}, c = \pm\frac{32}{3}, |f| = 28. \tag{8.2.34.9}$$

Let $\mathbf{x} = \begin{pmatrix} 0 \\ \alpha \end{pmatrix}$ be a vertex of the conic on the minor axis. Substituting in (8.1.2.1),

$$\frac{7\alpha^2}{16} + f = 0 \implies f < 0 \tag{8.2.34.10}$$

$$\text{or, } f = -28. \tag{8.2.34.11}$$

Thus, the desired equation of the conic is

$$\mathbf{x}^\top \begin{pmatrix} 1 & 0 \\ 0 & \frac{7}{16} \end{pmatrix} \mathbf{x} - 28 = 0 \tag{8.2.34.12}$$

See Fig. 8.2.34.1.

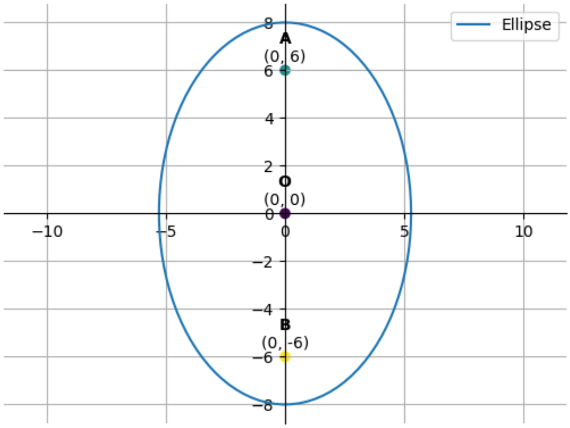

Fig. 8.2.34.1

8.2.35 foci $(\pm 3, 0), a = 4$.
8.2.36 vertex (0,4), focus (0,2).
8.2.37 vertex $(-3, 0)$, directrix $x + 5 = 0$.
8.2.38 eccentricity $e = \frac{4}{3}$, vertices

$$\mathbf{P_1} = \begin{pmatrix} 7 \\ 0 \end{pmatrix}, \ \mathbf{P_2} = \begin{pmatrix} -7 \\ 0 \end{pmatrix} \tag{8.2.38.1}$$

Solution: The major axis of a conic is the chord which passes through the vertices of the conic. The direction vector of the major axis in this case is

$$\mathbf{P}_2 - \mathbf{P}_1 \equiv \mathbf{e}_1 = \mathbf{n} \tag{8.2.38.2}$$

which is the normal vector for the directrix. Since $e > 1$, the conic is a hyperbola.

Substituting (8.2.38.2) in (8.1.2.2), (8.1.2.3) and (8.1.2.4),

$$\mathbf{V} = \begin{pmatrix} 1-e^2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{7}{9} & 0 \\ 0 & 1 \end{pmatrix} \tag{8.2.38.3}$$

The centre of the hyperbola is

$$\mathbf{c} = \frac{\mathbf{P}_1 + \mathbf{P}_2}{2} = \mathbf{0} = \mathbf{u} \tag{8.2.38.4}$$

from (B.1.5.9). Substituting $\mathbf{P}_1$ and $\mathbf{P}_2$ in (8.1.2.1),

$$\mathbf{P}_1^\top \mathbf{V}\mathbf{P}_1 + 2\mathbf{u}^\top \mathbf{P}_1 + f = 0 \tag{8.2.38.5}$$

$$\mathbf{P}_2^\top \mathbf{V}\mathbf{P}_2 + 2\mathbf{u}^\top \mathbf{P}_2 + f = 0 \tag{8.2.38.6}$$

$$\implies f = \mathbf{P}_1^\top \mathbf{V}\mathbf{P}_1 = 49\left(e^2 - 1\right) = \frac{343}{9} \tag{8.2.38.7}$$

upon adding (8.2.38.6) and (8.2.38.5) and simplifying. Therefore, the equation of the conic is

$$\mathbf{x}^\top \begin{pmatrix} -\frac{7}{9} & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + \frac{343}{9} = 0 \tag{8.2.38.8}$$

See Fig. 8.2.38.1.

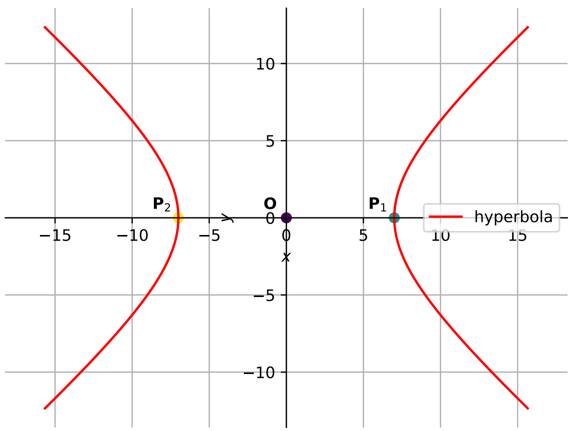

Fig. 8.2.38.1

8.2.39 centre at $\mathbf{c}(0,0)$, major axis on the Y axis and passes through the points $\mathbf{P}(3,2)$ and $\mathbf{Q}(1,6)$.

Solution: Since the major axis is along the y-axis,

$$\mathbf{n} = \mathbf{e}_2 \tag{8.2.39.1}$$

Thus,

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 - e^2 \end{pmatrix} \tag{8.2.39.2}$$

Since

$$\mathbf{c} = \mathbf{0}, \mathbf{u} = \mathbf{0}. \tag{8.2.39.3}$$

From (8.1.2.1),

$$\mathbf{P}^\top \mathbf{V} \mathbf{P} + 2\mathbf{u}^\top \mathbf{P} + f = 0 \tag{8.2.39.4}$$

$$\mathbf{Q}^\top \mathbf{V} \mathbf{Q} + 2\mathbf{u}^\top \mathbf{Q} + f = 0 \tag{8.2.39.5}$$

yielding

$$4e^2 - f = 13 \tag{8.2.39.6}$$

$$36e^2 - f = 37 \tag{8.2.39.7}$$

which can be formulated as the matrix equation

$$\begin{pmatrix} 4 & -1 \\ 36 & -1 \end{pmatrix} \begin{pmatrix} e^2 \\ f \end{pmatrix} = \begin{pmatrix} 13 \\ 37 \end{pmatrix} \tag{8.2.39.8}$$

The augmented matrix is given by,

$$\left(\begin{array}{cc|c} 4 & -1 & 13 \\ 36 & -1 & 37 \end{array}\right) \xleftrightarrow{R_1 \leftarrow -\frac{R_1}{8}} \left(\begin{array}{cc|c} 4 & 0 & 3 \\ 36 & -1 & 37 \end{array}\right)$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - 9R_1} \left(\begin{array}{cc|c} 4 & 0 & 3 \\ 0 & -1 & 10 \end{array}\right) \xleftrightarrow[R_2 \leftarrow -R_2]{R_1 \leftarrow \frac{R_1}{4}} \left(\begin{array}{cc|c} 1 & 0 & \frac{3}{4} \\ 0 & 1 & -10 \end{array}\right)$$

Thus,

$$e^2 = \frac{3}{4}, \ f = -10 \tag{8.2.39.9}$$

and the equation of the conic is given by

$$\mathbf{x}^\top \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{pmatrix} \mathbf{x} - 10 = 0 \tag{8.2.39.10}$$

See Fig. 8.2.39.1.

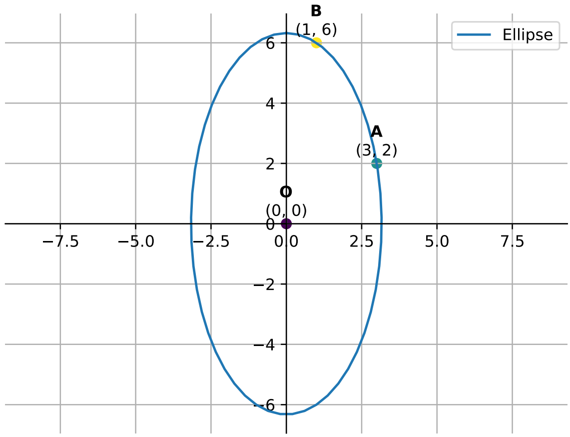

Fig. 8.2.39.1

8.2.40 focus $(0, -3)$ and directrix $y = 3$.
8.2.41 directrix $x = 0$, focus at (6,0).
8.2.42 vertex at (0,4), focus at (0,2).
8.2.43 focus at $(-1, 2)$, directrix $x - 2y + 3 = 0$.
8.2.44 vertices $(\pm 5, 0)$, foci $(\pm 7, 0)$.
8.2.45 vertices $(0 \pm 7), e = \frac{4}{3}$.
8.2.46 foci $(0,\pm\sqrt{10})$, passing through (2,3).
8.2.47 vertices at $(0, \pm 6)$, eccentricity $\frac{5}{3}$.
8.2.48 focus $(-1, -2)$, directrix $x - 2y + 3 = 0$.
8.2.49 eccentricity $\frac{3}{2}$, foci $(\pm 2, 0)$.
8.2.50 major axis on the X axis and passes through the points (4,3) and (6,2).
Solution: In this case,

$$\mathbf{n} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{8.2.50.1}$$

Thus,

$$\mathbf{V} = \begin{pmatrix} 1 - e^2 & 0 \\ 0 & 1 \end{pmatrix} \tag{8.2.50.2}$$

$$\tag{8.2.50.3}$$

Since

$$\mathbf{c} = \mathbf{0}, \mathbf{u} = \mathbf{0}. \tag{8.2.50.4}$$

From (8.1.2.1),

$$\mathbf{P}^\top \mathbf{V}\mathbf{P} + 2\mathbf{u}^\top \mathbf{P} + f = 0 \tag{8.2.50.5}$$

$$\mathbf{Q}^\top \mathbf{V}\mathbf{Q} + 2\mathbf{u}^\top \mathbf{Q} + f = 0 \tag{8.2.50.6}$$

yielding

$$16e^2 - f = 25 \tag{8.2.50.7}$$

$$36e^2 - f = 40 \tag{8.2.50.8}$$

which can be formulated as the matrix equation

$$\begin{pmatrix} 16 & -1 \\ 36 & -1 \end{pmatrix}\begin{pmatrix} e^2 \\ f \end{pmatrix} = \begin{pmatrix} 25 \\ 40 \end{pmatrix} \tag{8.2.50.9}$$

and can be solved using the augmented matrix.

$$\begin{aligned} \begin{pmatrix} 16 & -1 & 25 \\ 36 & -1 & 40 \end{pmatrix} &\xleftrightarrow{R_1 \leftarrow R_1 - R_2} \begin{pmatrix} -20 & 0 & -15 \\ 36 & -1 & 40 \end{pmatrix} \\ \xleftrightarrow[R_2 \leftarrow -R_2]{R_1 \leftarrow \frac{R_1}{-5}} \begin{pmatrix} 4 & 0 & 3 \\ -36 & 1 & -40 \end{pmatrix} &\xleftrightarrow{R_2 \leftarrow R_2 + 9R_1} \begin{pmatrix} 4 & 0 & 3 \\ 0 & 1 & -13 \end{pmatrix} \\ &\xleftrightarrow{R_1 \leftarrow \frac{R_1}{4}} \begin{pmatrix} 1 & 0 & \frac{3}{4} \\ 0 & 1 & -13 \end{pmatrix} \end{aligned}$$

Thus,

$$e^2 = \frac{3}{4}, \; f = -13 \tag{8.2.50.10}$$

and the equation of the conic is given by

$$\mathbf{x}^\top \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x} - 13 = 0 \tag{8.2.50.11}$$

See Fig. 8.2.50.1.

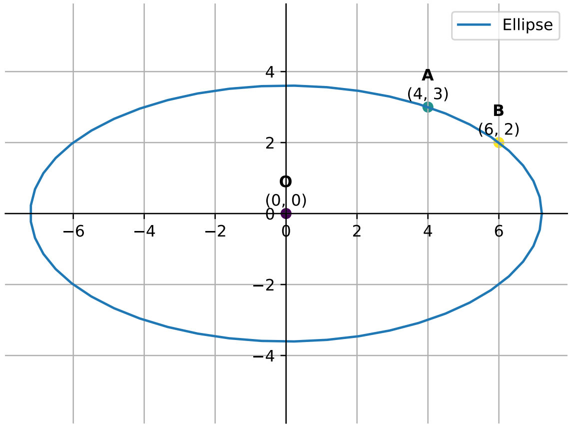

Fig. 8.2.50.1

8.2.51 vertices $\begin{pmatrix} 0 \\ \pm 3 \end{pmatrix}$ and foci $\begin{pmatrix} 0 \\ \pm 5 \end{pmatrix}$.

Solution: Following the approach in the earlier problems, it is obvious that

$$\mathbf{n} = \mathbf{e}_2, \mathbf{c} = \mathbf{u} = \mathbf{0}. \tag{8.2.51.1}$$

Consequently,

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 - e^2 \end{pmatrix} \tag{8.2.51.2}$$

$$\mathbf{F} = ce^2\mathbf{e}_2 \implies \|\mathbf{F}\| = ce^2 = 5 \tag{8.2.51.3}$$

$$f = 25 - c^2e^2 \tag{8.2.51.4}$$

Since the vertices are on the conic,

$$\mathbf{v_1}^\top \mathbf{V}\mathbf{v_1} + 2\mathbf{u}^\top \mathbf{v_1} + f = 0 \tag{8.2.51.5}$$

$$\implies 9\left(1 - e^2\right) + f = 0 \tag{8.2.51.6}$$

$$\tag{8.2.51.7}$$

Solving (8.2.51.7), (8.2.51.3) and (8.2.51.4),

$$c = \frac{9}{5}, \ e = \frac{5}{3}, \tag{8.2.51.8}$$

yielding

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{16}{9} \end{pmatrix}, \ \mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ f = 16. \tag{8.2.51.9}$$

Thus, the desired equation of the hyperbola is

$$\mathbf{x}^\top \begin{pmatrix} 1 & 0 \\ 0 & -\frac{16}{9} \end{pmatrix} \mathbf{x} + 16 = 0 \tag{8.2.51.10}$$

See Fig. 8.2.51.1.

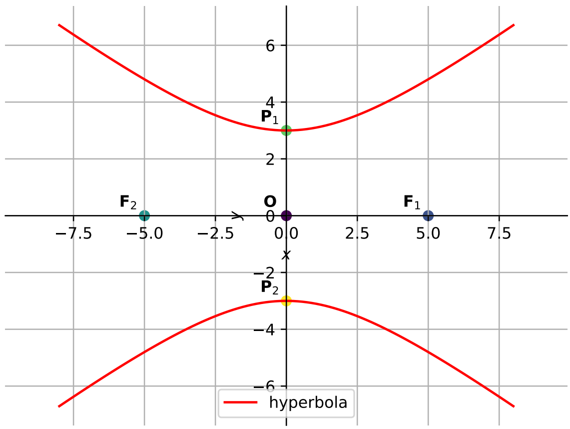

Fig. 8.2.51.1: Figure 1

8.2.52 eccentricity $\frac{2}{3}$, latus rectum 5, centre (0,0).

8.2.53 If the parabola $y^2 = 4ax$ passes through the point (3,2), then the length of its latus rectum is ______.

8.2.54 Find the eccentricity of the hyperbola $9y^2 - 4x^2 = 36$.

8.2.55 Equation of the hyperbola with eccentricty $\frac{3}{2}$ and foci at $(\pm 2, 0)$ is ______.

8.2.56 Given the ellipse with equation $9x^2 + 25y^2 = 225$, find the eccentricity and foci.

8.2.57 Find the equation of the set of all points whose distance from (0,4) is $\frac{2}{3}$ of their distance from the line $y = 9$.

8.2.58 The length of the latus rectum of the ellipse $3x^2 + y^2 = 12$ is ______.

8.3 Miscellaneous

8.3.1 The cable of a uniformly loaded suspension bridge hangs in the form of a parabola. The roadway which is horizontal and 100m long is supported by vertical wires attached to the cable, the longest wire being 30m and the shortest being 6m. Find the length of a supporting wire attached to the roadway 18m from the middle.

8.3.2 Find the area of the triangle formed by the lines joining the vertex of the parabola $x^2 = 12y$ to the ends of its latus rectum.

8.3.3 A man running a racecourse notes that the sum of the distances from the two flag posts from him is always 10m and the distance between the flag posts is 8m. Find the equation of the posts traced by the man.

8.3.4 Find the coordinates of a point on the parabola $y^2 = 8x$ whose focal distance is 4.

8.3.5 Show that the set of all points such that the difference of their distances from (4,0) and $(-4, 0)$ is always equal to 2 represent a hyperbola.

8.3.6 If the distance between the foci of a hyperbola is 16 and its eccentricity is $\sqrt{2}$, then obtain the equation of the hyperbola.

8.3.7 The eccentricity of the hyperbola whose latus rectum is 8 and conjugate axis is equal to half of the distance between the foci is ______.

8.3.8 The distance between the foci of a hyperbola is 16 and its eccentricity is ≤ 2. lts equation is ______.

8.3.9 If the latus rectum of an ellipse is equal to half of minor axis, then find its eccentricity.

8.3.10 If the eccentricity of an ellipse is $\frac{5}{8}$ and the distance between its foci is 10 then find latus rectum of the ellipse.

8.3.11 Find the distance between the directrices of the ellipse $\frac{x^2}{36} + \frac{y^2}{20}$.

8.3.12 Find the equation of the set of all points the sum of whose distances from the points (3,0) and (9,0) is 12.

8.3.13 If P is a point on the ellipse $\frac{x^2}{16} + \frac{y^2}{25} = 1$ whose foci are s and s' then $Ps + Ps' = 8$.

8.3.14 An arch is in the form of a parabola with its axis vertical. The arch is 10m high and 5m wide at the base. How wide is it 2m from the vertex of the parabola?

8.3.15 An equilateral triangle is inscribed in the parabola $y^2 = 4ax$, where one vertex is at the vertex of the parabola. Find the length of the side of the triangle.

8.3.16 An arch is in the form of a semi-ellipse. It is 8m wide and 2m high at the centre. Find the height of the arch at a point 1.5m from one end.

8.3.17 A rod of length 12cm moves with its ends always touching the coordinate axes. Determine the equation of locus of a point P on the rod, which is 3cm from the end in contact with X axis.

8.4 JEE

8.4.1. STATEMENT-1: The curve $y = \frac{-x^2}{2} + x + 1$ is symmetric with respect to the line $x = 1$.
STATEMENT-2: A Parabola is symmetric about its axis. (2007)

a) Statement-1 is True,Statement-2 is True;Statement-2 is a correct explanation for Statement-1
b) Statement-1 is True,Statement-2 is True;Statement-2 is NOT a correct explanation for Statement-1
c) Statement-1 is True,Statement-2 is False
d) Statement-1 is False,Statement-2 is True

8.4.2. A parabola has the origin as its focus and $x = 2$ as the directrix. Then the vertex of the parabola is at (2009)

a) $(0, 2)$ b) $(1, 0)$ c) $(0, 1)$ d) $(2, 0)$

8.4.3. The ellipse $x^2 + y^2 = 4$ is inscribed in a rectangle aligned with the coordinate axes, which in turn is inscribed in another ellipse that passes through the point$(4, 0)$. Then the equation of the ellipse is: (2009)

a) $x^2 + 12y^2 + 16$
b) $4x^2 + 48y^2 = 48$
c) $4x^2 + 64y^2 = 48$
d) $x^2 + 16y^2 = 16$

8.4.4. Equation of the ellipse whose axes are the coordinates and which passes through the point $(-3, 1)$ and has eccentricity $\sqrt{\frac{2}{5}}$ is (2011)

a) $5x^2 + 3y^2 - 48 = 0$
b) $3x^2 + 5y^2 - 15 = 0$
c) $5x^2 + 3y^2 - 32 = 0$
d) $3x^2 + 5y^2 - 32 = 0$

8.4.5. An ellipse is drawn by taking a diameter of the circle $(x - 1)^2 + y^2 = 1$ as its semi minor axis and a diameter of the circle $x^2 + (y - 2)^2 = 4$ as semi-major axis. If the centre of the ellipse is at the origin and its axes are the coordinate axes, then the equation of the ellipse is

(2012)

a) $4x^2 + y^2 = 4$
b) $x^2 + 4y^2 = 8$
c) $4x^2 + y^2 = 8$
d) $x^2 + 4y^2 = 1$

8.4.6. The equation of the circle passing through the foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$, and having centre at $(0, 3)$ is

(2013)

a) $x^2 + y^2 - 6y - 7 = 0$
b) $x^2 + y^2 - 6y + 7 = 0$
c) $x^2 + y^2 - 6y - 5 = 0$
d) $x^2 + y^2 - 6y + 5 = 0$

8.4.7. Let **O** be the vertex and **Q** be any point on the parabola, $x^2 = 8y$. If the point **P** divides the line segment OQ internally in the ratio $(1 : 3)$, then the locus of **P** is

(2015)

a) $y^2 = 2x$ b) $x^2 = 2y$ c) $x^2 = y$ d) $y^2 = x$

8.4.8. Let **P** be the point on the parabola, $y^2 = 8x$ which is a minimum distance from the centre **C** of the circle, passing through **C** and having its centre at **P** is

(2016)

a) $x^2 + y^2 - \frac{x}{4} + 2y - 24 = 0$
b) $x^2 + y^2 - 4x + 9y + 18 = 0$
c) $x^2 + y^2 - 4x + 8y + 12 = 0$
d) $x^2 + y^2 - x + 4y - 12 = 0$

8.4.9. The eccentricity of the hyperbola whose length of the latus rectum is equal to 8 and the length of its conjugate axis is equal to half of the distance between its foci, is

(2016)

a) $\frac{2}{\sqrt{3}}$
b) $\sqrt{3}$
c) $\frac{4}{3}$
d) $\frac{4}{\sqrt{3}}$

8.4.10. Match the conics in column 1 with the statements/expressions in column 2 (2009)

Column I

a) Circle
b) Parabola
c) Ellipse
d) Hyperbola

Column II

a) The locus of the point (h, k) for which the line $hx + ky = 1$ touches the circle $x^2 + y^2 = 4$
b) Points z in the complex plane satisfying $|z + 2| - |z - 2| = \pm 3$
c) Points of the conic have parametric representations $x = \sqrt{3}\left(\frac{1-t^2}{1+t^2}\right)$, $y = \frac{2t}{1+t^2}$
d) The eccentricity of the conic lies in the interval $1 \leq x < \infty$

8.4.11. Let $H : \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where $a > b > 0$, be a hyperbola in XY-plane whose conjugate axis LM subtends an angle of 60° at one of its vertices **N**. Let the area of the triangle LMN be $4\sqrt{3}$. Match the following.

List 1

a) The length of the conjugate axis of H is
b) The eccentricity of H is
c) The distance between the foci of H is
d) The length of the latus rectum of H is

List 2

a) 8
b) $\frac{4}{\sqrt{3}}$
c) $\frac{2}{\sqrt{3}}$
d) 4

8.4.12. If $\mathbf{P} = (x, y)$, $\mathbf{F}_1 = (3, 0)$, $\mathbf{F}_2 = (-3, 0)$ and $16x^2 + 25y^2 = 400$, then $PF_1 + PF_2$ equals (1998)

a) 8
b) 6
c) 10
d) 12

8.4.13. Let a hyperbola passes through the focus of the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$. The transverse and conjugate axes of this hyperbola coincide with the major and minor axes of the given ellipse, also the product of eccentricities of given ellipse and hyperbola is 1, then (2006)

a) the equation of hyperbola is $\frac{x^2}{9} - \frac{y^2}{16} = 1$
b) the equation of hyperbola is $\frac{x^2}{9} - \frac{y^2}{25} = 1$
c) focus of hyperbola is $(5, 0)$
d) vertex of hyperbola is $\left(5\sqrt{3}, 0\right)$

8.4.14. Let $\mathbf{P}(x_1, y_1)$ and $\mathbf{Q}(x_2, y_2)$, $y_1 < 0, y_2 < 0$, be the end points of the latus rectum of the ellipse $x^2 + 4y^2 = 4$. The equations of parabolas with latus rectum PQ are (2008)

a) $x^2 + 2\sqrt{3}y = 3 + \sqrt{3}$
b) $x^2 - 2\sqrt{3}y = 3 + \sqrt{3}$
c) $x^2 + 2\sqrt{3}y = 3 - \sqrt{3}$
d) $x^2 - 2\sqrt{3}y = 3 - \sqrt{3}$

8.4.15. In a triangle ABC with fixed base BC, the vertex **A** moves such that

$$\cos B + \cos C = 4\sin^2\frac{A}{2}.$$

If a, b and c denote the lengths of the sides of the triangle opposite to the angles A, B and C, respectively, then (2009)

a) $b + c = 4a$
b) $b + c = 2a$
c) locus of the point **A** is an ellipse
d) locus of the point **A** is a pair of straight lines

8.4.16. Let the eccentricity of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be reciprocal to that of the elipse $x^2 + 4y^2 = 4$. If the hyperbola passes through a focus of the elipse, then (2011)

a) the equation of the hyperbola is $\frac{x^2}{3} - \frac{y^2}{2} = 1$
b) a focus of the hypebola is $(2, 0)$
c) the eccentricity of the hyperbola is $\sqrt{\frac{5}{3}}$
d) the equation of the hyperbola is $x^2 - 3y^2 = 3$

8.4.17. Let **P** and **Q** be distinct points on the parabola $y^2 = 2x$ such that a circle with PQ as diameter passes through the vertex **O** of the parabola. If **P** lies in the first quadrant and the area of the triangle ΔOPQ is $3\sqrt{2}$, then which of the following is (are) the coordinates of **P**? (2015)

a) $\left(4, 2\sqrt{2}\right)$
b) $\left(9, 3\sqrt{2}\right)$
c) $\left(\frac{1}{4}, \frac{1}{\sqrt{2}}\right)$
d) $\left(1, \sqrt{2}\right)$

8.4.18. The equation $\frac{x^2}{1-r} - \frac{y^2}{1+r} = 1, r > 1$ represents (1981)

a) an ellipse
b) a hyperbola
c) a circle
d) none of there

8.4.19. Each of the four inequalities give below defines a region in xy plane.One of these four regions does not have the following property.For any two points $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$ is also in region.The inequality defining this region is: (1981)

a) $x^2 + 2y^2 \leq 1$
b) Max $|x|, |y| \leq 1$
c) $x^2 - y^2 \leq 1$
d) $y^2 - x \leq 0$

8.4.20. The equation $2x^2 + 3y^2 - 8x - 18y + 35 = k$ represents (1994)

a) no locus if $k<0$
b) an ellipse if $k<0$
c) a point if $k = 0$
d) a hyperbola if $k>0$

8.4.21. Let E be the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and C be the circle $x^2 + y^2 = 9$. Let **P** and **Q** be the points $(1, 2)$ and $(2, 1)$ respectively.Then (1994)

a) **Q** lies inside C but outside E
b) **Q** lies outside both C and E
c) **P** lies inside both C and E
d) **P** lies inside C but outside E

8.4.22. The radius of the circle passing through the foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ and having its centre at $(0, 3)$ is (1995)

a) 4
b) 3
c) $\sqrt{\frac{1}{2}}$
d) $\frac{7}{2}$

8.4.23. The curve described parametrically by $x = t^2 + t + 1$, $y = t^2 - t + 1$ represents (1999)

a) a pair of straight lines
b) an ellipse
c) a parabola
d) a hyperbola

8.4.24. If the line $x - 1 = 0$ is the directrix of parabola $y^2 - kx + 8 = 0$, then one of the values of K is (2000)

a) $\frac{1}{8}$
b) 8
c) 4
d) $\frac{1}{4}$

8.4.25. The equation of the directrix of the parabola $y^2 + 4y + 4x + 2 = 0$ is (2001)

a) $x = -1$
b) $x = 1$
c) $x = -\frac{3}{2}$
d) $x = \frac{3}{2}$

8.4.26. The locus of the mid-point of the line segment joining the focus to a moving point on the parabola $y^2 = 4ax$ is another parabola with directrix (2002)

a) x=-a
b) x=-a/2
c) x=a
d) x=a/2

8.4.27. For hyperbola $\frac{x^2}{\cos^2\alpha} - \frac{y^2}{\sin^2\alpha} = 1$ which of the following remains constant with change in α (2003)

a) abscissae of vertices
b) abscissae of foci
c) eccentricity
d) directrix

8.4.28. The axis of the parabola is along the line $y = x$ and the distance of its vertex and focus from origin are $\sqrt{2}$ and $2\sqrt{2}$ respectively. If the vertex and focus both lie in the first quadrant,then the equation of the parabola is (2006)

a) $(x+y)^2 = (x-y-2)$ c) $(x-y)^2 = 4(x+y-2)$
b) $(x-y)^2 = (x+y-2)$ d) $(x-y)^2 = 8(x+y-2)$

8.4.29. A hyperbola, having the transverse axis of length $2\sin\theta$, is confocal with the ellipse $3x^2 + 4y^2 = 12$. Then its equation is (2007)

a) $x^2 \operatorname{cosec}^2\theta - y^2 \sec^2\theta = 1$ c) $x^2 \sin^2\theta - y^2 \cos^2\theta = 1$
b) $x^2 \sec^2\theta - y^2 \operatorname{cosec}^2\theta = 1$ d) $x^2 \cos^2\theta - y^2 \sin^2\theta = 1$

8.4.30. Let a and b be non-zero real numbers. Then, the equation $(ax^2+by^2+c)(x^2-5xy+6y^2 = 0)$ represents (2008)

a) four straight lines, when $c = 0$ and a, b are of the same sign.
b) two straight lines and a circle, when $a = b$,and c is of sign opposite to that of a.
c) two straight lines and a hyperbola, when a and b are of the same sign and c is of opposite to that of a.
d) a circle and an ellipse, when a and b are of the same sign and c is of sign opposite to that of a.

8.4.31. Consider a branch of the hyperbola $x^2 - 2y^2 - 2\sqrt{2}x - 4\sqrt{2}y - 6 = 0$ with vertex at a point **A**. Let **B** be one of the end points of its latus rectum. If **C** is the focus of the hyperbola nearest to the point **A**, then the area of the triangle ABC is (2008)

a) $1 - \sqrt{\frac{2}{3}}$ b) $\sqrt{\frac{3}{2}} - 1$ c) $1 + \sqrt{\frac{2}{3}}$ d) $\sqrt{\frac{3}{2}} + 1$

8.4.32. The line passing through the extremity **A** of the major axis and extremity **B** of the minor axis of the ellipse $x^2 + 9y^2 = 9$ meets its auxillary circle at the point **M**. Then the area of the triangle with the vertices at **A**, **M** and the origin **O** is (2009)

a) $\frac{31}{10}$ b) $\frac{29}{10}$ c) $\frac{21}{10}$ d) $\frac{27}{10}$

8.4.33. The locus of the orthocentre of the triangle formed by the lines

$$(1+p)\,x - py + p\,(1+p) = 0,$$

$$(1+q)\,x - qy + q\,(1+q) = 0,$$

and $y = 0$, where $p \neq q$, is (2009)

a) a hyperbola b) a parabola c) an ellipse d) a straight line

8.4.34. Let (x, y) be any point on the parabola $y^2 = 4x$. Let **P** be the point that divides the line segment from $(0, 0)$ to (x, y) in the ratio $1 : 3$. Then the locus of **P** is (2011)

a) $x^2 = y$ b) $y^2 = 2x$ c) $y^2 = x$ d) $x^2 = 2y$

8.4.35. The ellipse $E_1 : \frac{x^2}{9} + \frac{y^2}{4} = 1$ is inscribed in a rectangle **R** whose sides are parallel to the coordinate axes. Another ellipse E_2 passing through the point $(0, 4)$ circumscribes the rectangle **R**. The eccentricity of the ellipse E_2 is (2012)

a) $\frac{\sqrt{2}}{2}$ b) $\frac{\sqrt{3}}{2}$ c) $\frac{1}{2}$ d) $\frac{3}{4}$

8.4.36. Two sets A and B are as under: $A = \{(a, b) \in \mathbb{R} \times \mathbb{R} : |a-5| < 1 \text{ and } |b-5| < 1\}$ $B = \left\{(a, b) \in \mathbb{R} \times \mathbb{R} : 4(a-6)^2 + 9(b-5)^2 \le 36\right\}$ (2018)

a) $A \subset B$
b) $A \cap B$
c) neither $A \subset B$ nor $B \subset A$
d) $B \subset A$

8.4.37. Axis of a parabola lies along X axis. If its vertex and focus are at a distance 2 and 4 respectively from origin, on the positive X axis then which of the following points does not lie on it? (2018)

a) $\left(5, 2\sqrt{6}\right)$ b) $(8, 6)$ c) $\left(6, 4\sqrt{2}\right)$ d) $(4, -4)$

8.4.38. Let $0 < \theta < \pi/2$. If the eccentricty of the hyperbola $\frac{x^2}{\cos^2\theta} - \frac{y^2}{\sin^2\theta} = 1$ is greater than 2, then the length of its latus rectum lies in the interval (2019)

a) $(5, \infty)$ b) $\left(\frac{3}{2}, 3\right]$ c) $(2, 3]$ d) $\left(1, \frac{3}{2}\right]$

8.4.39. The equation of the locus of the point whose distances from the point **P** and the line AB are equal, is

a) $9x^2 + y^2 - 6xy - 54x - 62y + 241 = 0$
b) $x^2 + 9y^2 + 6xy - 54x - 62y - 241 = 0$
c) $9x^2 + 9y^2 - 6xy - 54x - 62y - 241 = 0$
d) $x^2 + y^2 - 2xy + 27x + 31y - 120 = 0$

8.4.40. Let **P** be a point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $0 < b < a$. Let the line parallel to the X axis passing through **P** meet the circle $x^2 + y^2 = a^2$ at the point **Q** such that **P** and **Q** are on the same side of the X axis. For two positive real numbers r and s, find the locus of the point **R** on PQ such that $PR = r$ as **P** varies over the ellipse. (2001)

8.4.41. Let a and b be positive real numbers such that $a > 1$ and $b < a$. Let **P** be a point in the first quadrant that lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2}$. Suppose the tangent to the hyperbola at **P** passes through the point (1, 0), and suppose the normal to the hyperbola at **P** cuts off equal intercepts on the coordinate axes. Let $\triangle$ denote the area of the triangle formed by the tangent at **P**, the normal at **P** and the X axis. If e denotes the eccentricity of the hyperbola, then which of the following statements is/are TRUE? (2020)

a) $1 < e < \sqrt{2}$
b) $\sqrt{2} < e < 2$
c) $\triangle = a^4$
d) $\triangle = b^4$

8.4.42. Consider the ellipse $\frac{x^2}{4} + \frac{y^2}{3} = 1$. Let $\mathbf{H}(\alpha, 0), 0 < \alpha < 2$, be a point. A straight line drawn through **H** parallel to the Y axis crosses the ellipse and its auxiliary circle at points **E** and **F** respectively, in the first quadrant. The tangent to the ellipse at the point **E** intersects the positive X axis at a point **G**. Suppose the straight line joining **F** and the origin makes an angle ϕ with the positive X axis

(I) if $\phi = \frac{3}{4}$, then the area of the triangle FGH is

(II) if $\phi = \frac{\pi}{3}$, then the area of the triangle FGH is

(III) if $\phi = \frac{\pi}{6}$, then the area of the triangle FGH is

(IV) if $\phi = \frac{\pi}{12}$, then the area of the triangle FGH is

(A) $\frac{(\sqrt{3}-1)^4}{8}$
(B) 1
(C) $\frac{3}{4}$
(D) $\frac{1}{2\sqrt{3}}$
(E) $\frac{3\sqrt{3}}{2}$

The correct option is (2022)

a) $(I) \rightarrow (R); (II) \rightarrow (S); (III) \rightarrow (Q); (IV) \rightarrow (P)$
b) $(I) \rightarrow (R); (II) \rightarrow (T); (III) \rightarrow (S); (IV) \rightarrow (P)$
c) $(I) \rightarrow (Q); (II) \rightarrow (T); (III) \rightarrow (S); (IV) \rightarrow (P)$
d) $(I) \rightarrow (Q); (II) \rightarrow (S); (III) \rightarrow (Q); (IV) \rightarrow (P)$

9 Intersection of Conics

9.1 Formulae

9.1.1 The points of intersection of the line

$$L : \quad \mathbf{x} = \mathbf{h} + \kappa\mathbf{m} \quad \kappa \in \mathbb{R} \tag{9.1.1.1}$$

with the conic section in (8.1.2.1) are given by

$$\mathbf{x}_i = \mathbf{h} + \kappa_i\mathbf{m} \tag{9.1.1.2}$$

where

$$\kappa_i = \frac{1}{\mathbf{m}^\top\mathbf{V}\mathbf{m}}\left(-\mathbf{m}^\top\left(\mathbf{V}\mathbf{h}+\mathbf{u}\right) \pm \sqrt{\left[\mathbf{m}^\top\left(\mathbf{V}\mathbf{h}+\mathbf{u}\right)\right]^2 - g\left(\mathbf{h}\right)\left(\mathbf{m}^\top\mathbf{V}\mathbf{m}\right)}\right) \tag{9.1.1.3}$$

See B.3.1 for proof.

9.1.2 (9.1.1.2) can be expressed as the solution matrix

$$\mathbf{X} = \begin{pmatrix}\mathbf{h} & \mathbf{m}\end{pmatrix}\begin{pmatrix}\mathbf{1} & \boldsymbol{\kappa}\end{pmatrix}^\top \tag{9.1.2.1}$$

where

$$\boldsymbol{\kappa} = \begin{pmatrix}\kappa_1 \\ \kappa_2\end{pmatrix}. \tag{9.1.2.2}$$

9.1.3 (8.1.2.1) represents a pair of straight lines if the matrix

$$\begin{pmatrix}\mathbf{V} & \mathbf{u} \\ \mathbf{u}^\top & f\end{pmatrix} \tag{9.1.3.1}$$

is singular.

9.1.4 The intersection of two conics with parameters $\mathbf{V}_i, \mathbf{u}_i, f_i,\ i = 1, 2$ is defined as

$$\mathbf{x}^\top\left(\mathbf{V}_1 + \mu\mathbf{V}_2\right)\mathbf{x} + 2\left(\mathbf{u}_1 + \mu\mathbf{u}_2\right)^\top\mathbf{x} + \left(f_1 + \mu f_2\right) = 0 \tag{9.1.4.1}$$

9.1.5 From (9.1.3.1), (9.1.4.1) represents a pair of straight lines if

$$\begin{vmatrix}\mathbf{V}_1 + \mu\mathbf{V}_2 & \mathbf{u}_1 + \mu\mathbf{u}_2 \\ \left(\mathbf{u}_1 + \mu\mathbf{u}_2\right)^\top & f_1 + \mu f_2\end{vmatrix} = 0 \tag{9.1.5.1}$$

9.2 Chords

9.2.1 Find the area bounded by the curve $y = \sqrt{x}$, $x = 2y + 3$ in the first quadrant and x-axis.

9.2.2 Draw a rough sketch of the region $(x, y) : y^2 \leqslant 6ax$ and $x^2 + y^2 \leqslant 16a^2$.

9.2.3 Draw a rough sketch of the given curve $y = 1 + |x + 1|$, $x = -3$, $x = 3$, $y = 0$, and find the area of the region bounded by them, using integration.

9.2.4 The area of the region bounded by the curve $x^2 = 4y$ and the straight line $x = 4y - 2$ is ______.

9.2.5 The area of the region bounded by the curve $y = \sqrt{16 - x^2}$ and x-axis is ______.

9.2.6 Area of the region in the first quadrant enclosed by the x-axis, the line $y = x$ and the circle $x^2 + y^2 = 32$ is ______.

9.2.7 The area of the region bounded by parabola $y^2 = x$ and the straight line $2y = x$ is ______.

9.2.8 Find the equation of a circle whose centre is (3,1) and which cuts off a chord of length 6 units on the line $2x - 5y + 18 = 0$.

9.2.9 Find the area between the curves $y = x$ and $y = x^2$.

Solution:

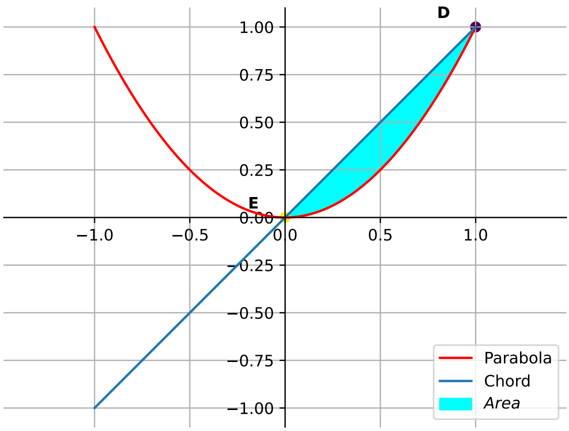

Fig. 9.2.9.1

The given curve can be expressed as a conic with parameters

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix}, f = 0 \tag{9.2.9.1}$$

The given line parameters are

$$\mathbf{h} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{9.2.9.2}$$

Substituting the given parameters in (9.1.1.3),

$$\mathbf{x}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{9.2.9.3}$$

From Fig. 9.2.9.1, the area bounded by the curve $y = x^2$ and line $y = x$ is given by

$$\int_0^1 \left(x - \frac{x^2}{2} \right) dx = \frac{1}{6} \tag{9.2.9.4}$$

9.2.10 Find the area of the region bounded by the curve $y^2 = x$ and the lines $x = 1$ and $x = 4$ and the axis in the first quadrant.

Solution:

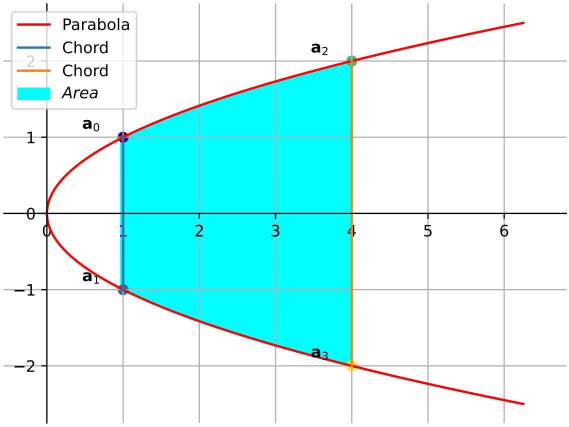

Fig. 9.2.10.1

The parameters of the conic are

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u} = -\frac{1}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, f = 0 \tag{9.2.10.1}$$

For the line $x - 1 = 0$, the parameters are

$$\mathbf{h}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{m}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{9.2.10.2}$$

Substituting from the above in (9.1.1.3),

$$\kappa_i = 1, -1 \tag{9.2.10.3}$$

yilelding the points of intersection

$$\mathbf{a}_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \mathbf{a}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{9.2.10.4}$$

Similarly, for the line $x - 4 = 0$

$$\mathbf{q_1} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \mathbf{m_1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{9.2.10.5}$$

yielding

$$\kappa_i = 2, -2 \tag{9.2.10.6}$$

from which, the points of intersection are

$$\mathbf{a_3} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \mathbf{a_2} = \begin{pmatrix} 4 \\ -2 \end{pmatrix} \tag{9.2.10.7}$$

See Fig. 9.2.10.1. Thus, the area of the parabola in between the lines $x = 1$ and $x = 4$ is given by

$$\int_0^4 \sqrt{x}\,dx - \int_0^1 \sqrt{x}\,dx = 14/3 \tag{9.2.10.8}$$

9.2.11 Find the area of the region bounded by the curve $y^2 = 9x$ and the lines $x = 2$ and $x = 4$ and the axis in the first quadrant.

9.2.12 Find the area of the region bounded by $x^2 = 4y$, $y = 2$, $y = 4$ and the y-axis in the first quadrant.

9.2.13 Find the area of the region bounded by the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$

9.2.14 Find the area of the region bounded by the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$

9.2.15 Determine the area under the curve $y = \sqrt{a^2 - x^2}$ included between the lines $x = 0$ and $x = a$

9.2.16 Find the area of the region bounded by $y = \sqrt{x}$ and $y = x$.

9.2.17 Find the area enclosed by the curve $y = -x^2$ and the straight line $x + y + 2 = 0$.

9.2.18 Find the area of the region in the first quadrant enclosed by the x-axis, line $x = \sqrt{3}y$ and circle $x^2 + y^2 = 4$.

Solution:

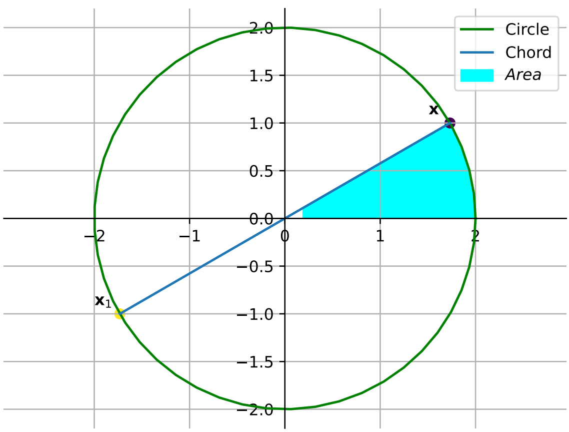

Fig. 9.2.18.1

From the given information, the parameters of the circle and line are

$$f = -4, \mathbf{u} = \mathbf{0}, \mathbf{V} = \mathbf{I}, \mathbf{m} = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}, \mathbf{h} = \mathbf{0} \tag{9.2.18.1}$$

Substituting the above parameters in (9.1.1.3),

$$\kappa = \sqrt{3} \tag{9.2.18.2}$$

yielding the desired point of intersection as

$$\mathbf{x} = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} \tag{9.2.18.3}$$

Note that we have chosen only the point of intersection in the first quadrant as shown in Fig. 9.2.18.1. From (9.2.18.1), the angle between the given line and the X axis is

$$\theta = 30° \tag{9.2.18.4}$$

and the area of the sector is

$$\frac{\theta}{360}\pi r^2 = \frac{\pi}{3} \tag{9.2.18.5}$$

9.2.19 Find the area of the smaller part of the circle $x^2 + y^2 = a^2$ cut off by the line $x = \frac{a}{\sqrt{2}}$.
9.2.20 The area between $x = y^2$ and $x = 4$ is divided into two equal parts by the line $x = a$, find the value of a.
9.2.21 Find the area of the region bounded by the parabola $y = x^2$ and $y = |x|$.
9.2.22 Find the area bounded by the curve $x^2 = 4y$ and the line $x = 4y - 2$.
9.2.23 Find the area of the region bounded by the curve $y^2 = 4x$ and the line $x = 3$.
9.2.24 Area lying in the first quadrant and bounded by the circle $x^2 + y^2 = 4$ and the lines $x = 0$ and $x = 2$ is ______
9.2.25 Find the area of the region bounded by the curve $y^2 = 4x$, y-axis and the line $y = 3$.
9.2.26 Find the area of the region bounded by the curve $x^2 = 4y$ and the lines $y = 2$ and $y = 4$ and the y-axis in the first quadrant.
9.2.27 Find the area enclosed by the parabola $4y = 3x^2$ and the line $2y = 3x + 12$.
9.2.28 Calculate the area under the curve $y = 2\sqrt{x}$ included between the lines $x = 0$ and $x = 1$.
9.2.29 Using integration, find the area of the region bounded by the line $2y = 5x + 7$, x-axis and the lines $x = 2$ and $x = 8$.
9.2.30 Draw a rough sketch of the curve $y = \sqrt{x-1}$ in the interval $[1, 5]$. Find the area under the curve and between the lines $x = 1$ and $x = 5$.
9.2.31 Find the area of the region bounded by the curve $y^2 = 4x$, $x^2 = 4y$.
9.2.32 Find the area of the region included between $y^2 = 9x$ and $y = x$
9.2.33 Find the area of the region enclosed by the parabola $x^2 = y$ and the line $y = x + 2$
9.2.34 Find the area of region bounded by the line $x = 2$ and the parabola $y^2 = 8x$
9.2.35 Sketch the region $(x, 0) : y = \sqrt{4 - x^2}$ and x-axis. Find the area of the region using integration.
9.2.36 Find the area of the smaller region bounded by the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and the line $\frac{x}{3} + \frac{y}{2} = 1$.

Solution:

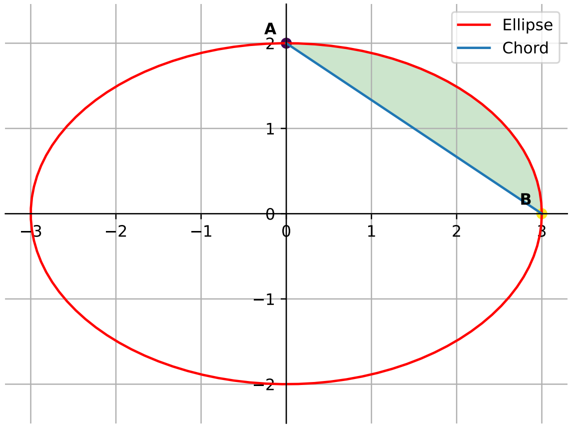

Fig. 9.2.36.1

The given ellipse can be expressed as conics with parameters

$$\mathbf{V} = \begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix}, \mathbf{u} = 0, f = -(a^2b^2). \tag{9.2.36.1}$$

The line parameters are

$$\mathbf{h} = \begin{pmatrix} a \\ 0 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} \frac{1}{b} \\ -\frac{1}{a} \end{pmatrix}. \tag{9.2.36.2}$$

Substituting the given parameters in (9.1.1.3),

$$\kappa = 0, -6 \tag{9.2.36.3}$$

yielding the points of intersection

$$\mathbf{A} = \begin{pmatrix} a \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ b \end{pmatrix}. \tag{9.2.36.4}$$

From Fig. 9.2.36.1, the desired area is

$$\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\, dx - \int_0^a \frac{b}{a}(a - x)\, dx = \frac{ab}{2}\left(\frac{\pi}{2} - 1\right) = 3\left(\frac{\pi}{2} - 1\right) \tag{9.2.36.5}$$

upon substituting $a = 3, b = 2$.

9.2.37 Find the area of the region bounded by the curve $x^2 = y$ and the lines $y = x + 2$ and the x axis.

9.2.38 Find the area bounded by the curve $y = x|x|$, x-axis and the ordinates x=-1 and x=1.

9.2.39 Find the area of the region bounded by the curves $y = x^2 + 2$, $y = x$, $x = 0$ and $x = 3$.

9.2.40 Find the smaller area enclosed by the circle $x^2 + y^2 = 4$ and the line $x + y = 2$.
9.2.41 Find the area of the region bounded by the curves $y^2 = 9x$, $y = 3x$.
9.2.42 Find the area of the region bounded by the parabola $y^2 = 2px$, $x^2 = 2py$.
9.2.43 Find the area of the region bounded by the curve $y = x^2$ and $y = x + 6$ and $x = 0$.

9.3 CBSE

9.3.1 Find the area of the region in the first quadrant enclosed by the X axis, the line $y = x$ and the circle $x^2 + y^2 = 32$. (12, 2018)
9.3.2 Find the area of the smaller region bounded by the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and the line $\frac{x}{3} + \frac{y}{2} = 1$. (12, 2018)
9.3.3 Find the area enclosed by the parabola $4y = 3x^2$ and the line $2y = 3x + 12$. (12, 2015)
9.3.4 Find the area of the region enclosed by the curve $y = x^2$, the X axis and the ordinates $x = -2$ and $x = 1$. (10, 2022)
9.3.5 Find the area of the region enclosed by line $y = \sqrt{3}x$, semi-circle $y = \sqrt{4 - x^2}$ and X axis in first quadrant. (10, 2022)
9.3.6 Find the area of the smaller region enclosed by the curve $4x^2 + 4y^2 = 9$ and the line $2x + 2y = 3$. (12, 2022)
9.3.7 Find the area of the region enclosed by the curves $y^2 = x$, $x = \frac{1}{4}$, $y = 0$ and $x = 1$. (12, 2022)
9.3.8 Find the area of the region $\{(x, y) : x^2 \le y \le x\}$. (12, 2021)
9.3.9 Find the area of the region

$$\left\{(x, y) : 0 \le y \le x^2, 0 \le y \le x + 2, -1 \le x \le 3\right\}.$$

(12, 2019)
9.3.10 If the area of the region bounded by the curve $y^2 = 4ax$ and the line $x = 4a$ is $\frac{256}{3}$ sq. units, then find the value of a, where $a > 0$. (12, 2022)
9.3.11 If the area of the region bounded by the line $y = mx$ and the curve $x^2 = y$ is $\frac{32}{3}$ sq. units, then find the positive value of m. (12, 2022)
9.3.12 If the area between the curves $x = y^2$ and $x = 4$ is divided into two equal parts by the line $x = a$, then find the value of a. (12, 2022)
9.3.13 Find the area bounded by the ellipse $x^2 + 4y^2 = 16$ and the ordinates $x = 0$ and $x = 2$. (12, 2022)
9.3.14 Find the area of the smaller region bounded by the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and the line $\frac{x}{3} + \frac{y}{2} = 1$. (12, 2019)
9.3.15 Find the equation of tangent to the curve $y = \sqrt{3x - 2}$ which is parallel to the line $4x - 2y + 5 = 0$. Also write the equation of the normal to the curve at the point of contact. (12, 2018)
9.3.16 Find the area bounded by the curve $y = \sqrt{x}$, the Y axis, and the lines $y = 0$ and $y = 3$. (12, 2024)
9.3.17 Find the area of the region bounded by the curve $y^2 = 4x$ and $x = 1$. (12, 2024)

9.3.18 Find the aea of the region bounded by curve $y^2 = 4x$ and the X axis between $x = 0$ and $x = 1$. (12, 2024)

9.3.19 If A_1 denotes the area of region bounded by $y^2 = 4x$ and X axis in the first quadrant and A_2 denotes the area of region bounded by $y^2 = 4x$, $x = 4$, find $A_1 : A_2$. (12, 2024)

9.3.20 Find the area of the region included between the parabola $y^2 = x$ and the line $x+y = 2$. (12, 2009)

9.4 Quadratic Equations

Find the roots of the following quadratic equations grapically.

9.4.1 $(x-2)^2 + 1 = 2x - 3$

9.4.2 $x(2x+3) = x^2 + 1$

9.4.3 $(x+2)^3 = x^3 - 4$

9.4.4 $x^2 - 3x - 10 = 0$

9.4.5 $2x^2 + x - 6 = 0$

9.4.6 $\sqrt{2}x^2 + 7x + 5\sqrt{2} = 0$

9.4.7 $2x^2 - x + \frac{1}{8} = 0$

9.4.8 $100x^2 - 20x + 1 = 0$

9.4.9 $2x^2 - 3x + 5 = 0$

9.4.10 $3x^2 - 4\sqrt{3}x + 4 = 0$

9.4.11 $2x^2 - 6x + 3 = 0$

9.4.12 $(x+1)^2 = 2(x-3)$

9.4.13 $x^2 - 2x = (-2)(3-x)$

9.4.14 $(x-3)(2x-1) = x(x+5)$

9.4.15 $(2x-1)(x-3) = (x+5)(x-1)$

9.4.16 $2x^2 - 7x + 3 = 0$

9.4.17 $2x^2 + x - 4 = 0$

9.4.18 $4x^2 + 4\sqrt{3}x + 3 = 0$

9.4.19 $2x^2 + x + 4 = 0$

9.4.20 $x - \frac{1}{x} = 3, x \neq 0$

9.4.21 $\frac{1}{x+4} - \frac{1}{x-7} = \frac{11}{30}, x \neq -4, 7$

9.4.22 $2x^2 - 5x + 3 = 0.$

9.4.23 $6x^2 - x - 2 = 0.$

9.4.24 $3x^2 - 2\sqrt{6}x + 2 = 0.$

9.4.25 $5x^2 - 6x - 2 = 0.$

9.4.26 $4x^2 + 3x + 5 = 0.$

9.4.27 $3x^2 - 5x + 2 = 0$

9.4.28 $x^2 + 4x + 5 = 0$

9.4.29 $2x^2 - 2\sqrt{2}x + 1 = 0$

9.4.30 $x + \frac{1}{x} = 3, x \neq 0$

9.4.31 $\frac{1}{x} - \frac{1}{x-2} = 3, x \neq 0, 2$

9.4.32 $3x^2 - 2x + \frac{1}{3} = 0.$

9.4.33 $x^2 - 4x + 3 = 0.$

9.4.34 $2x^2 - 4x + 3 = 0.$

Find the values of k for each of the following quadratic equations, so that they have equal roots. Verify your solution graphically.

9.4.21 $2x^2 = kx - 3 = 0.$

9.4.22 $kx(x-2) + 6 = 0.$

Represent the following situations graphically.

9.4.23 Janaki and Jivanti together have 45 marbles. Both of them lost 5 marbles each, and the product of the number of marbles they have is 124. We would like to find out how many marbles they had to start with.

9.4.24 A cottage industry produces a certain number of toys in a day. The cost of production of each toy (in rupees) was found to be 55 minus the number of toys produced in a day. On a particular day, the total cost of production was ₹750. We would like to find out the number of toys produced on that day.

9.4.25 Find two numbers whose sum is 27 and product is 182.

9.4.26 Find two consecutive positive integers, sum of whose squares is 365.

9.4.27 The altitude of a right triangle is 7 cm less than its base. If the hypotenuse is 13 cm, find the other two sides.

Solution: The input parameters are available in Table 9.4.27. Using Baudhayana's

Variable	Description	Value
BC	Hypotenuse of the triangle	13 cm
AB	Base of the triangle	x cm
AC	Altitude of the triangle	$x - 7$ cm

TABLE 9.4.27

theorem,

$$x^2 + (x - 7)^2 = 13^2 \tag{9.4.27.1}$$

$$\implies y = x^2 - 7x - 60 = 0 \tag{9.4.27.2}$$

which can be expressed as the conic

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{9.4.27.3}$$

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} -\frac{7}{2} \\ -\frac{1}{2} \end{pmatrix}, f = -60. \tag{9.4.27.4}$$

To find the roots of (9.4.27.1), we find the points of intersection of the conic with the x-axis

$$\mathbf{x} = \mathbf{h} + k\mathbf{m} \tag{9.4.27.5}$$

$$\mathbf{h} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{9.4.27.6}$$

using (9.1.1.3). The values of k are given by

$$k_i = \frac{1}{1}\left(\frac{7}{2} \pm \sqrt{\left(\frac{7}{2}\right)^2 + 60}\right) \tag{9.4.27.7}$$

$$\implies k_1 = -5, k_1 = 12 \tag{9.4.27.8}$$

Hence the points of intersection are

$$\mathbf{h} + k\mathbf{m} = \begin{pmatrix} -5 \\ 0 \end{pmatrix}, \begin{pmatrix} 12 \\ 0 \end{pmatrix} \tag{9.4.27.9}$$

See Fig. 9.4.27.1 Hence the solutions of (9.4.27.1) are $x = -5$ and $x = 12$. We reject $x = -5$ as the length of the side cannot be negative. Hence, the lengths of the sides are

$$AB = 12\ cm\, AC = 7\ cm\, BC = 13\ cm \tag{9.4.27.10}$$

See Fig. 9.4.27.1.

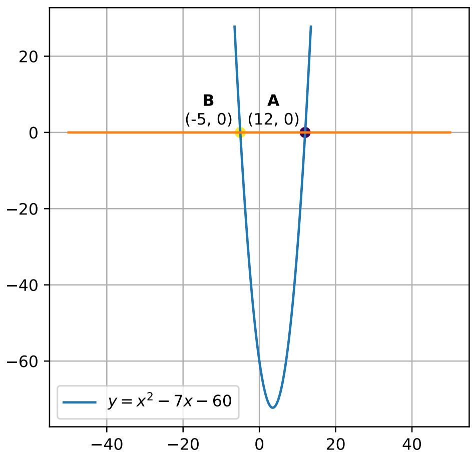

Fig. 9.4.27.1: Intersection of $y = x^2 - 7x - 60$ with the x-axis

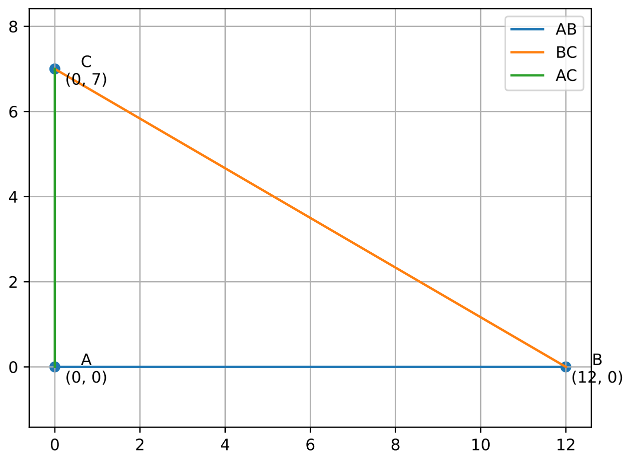

Fig. 9.4.27.2

9.4.28 A cottage industry produces a certain number of pottery articles in a day. It was observed on a particular day that the cost of production of each article (in rupees) was 3 more than twice the number of articles produced on that day. If the total cost of production on that day was ₹90, find the number of articles produced and the cost of each article.

9.4.29 Is it possible to design a rectangular mango grove whose length is twice its breadth, and the area is $800m^2$? If so, find its length and breadth.

9.4.30 Is the following situation possible? If so, determine their present ages.
The sum of the ages of the two friends is 20 years. Four years ago, the product of their ages in years was 48.

9.4.31 Is it possible to design a rectangular park of perimeter 80m and area of $400m^2$. If so, find its length and breadth.

9.4.32 The area of rectangular plot is $528m^2$. The length of the plot (in metres) is one more than twice its breadth. We need to find the length and breadth of the plot.

9.4.33 The product of two consecutive positive integers is 306. We need to find the integers.

9.4.34 Rohan's mother is 26 years older than him. The product of their ages (in years) 3 years from now will be 360. We would like to find Rohan's present age.

9.4.35 A train travels a distance of $480km$ at a unifom speed. If the speed had been $8km/h$ less, then it would have taken 3 hours more to cover the same distance. We need to find the speed of the train.

9.4.36 The sum of the reciprocals of Ram's ages, (in years) 3 years ago and 5 years from now is $\frac{1}{3}$. Find his present age.

9.4.37 In a class test, the sum of Shefali's marks in Mathematics and English is 30. Had she got 2 marks more in Mathematics and 3 marks less in English, the product of their marks would have been 210. Find her marks in the two subjects.

9.4.38 The diagonal of a rectangular field is 60 metres more than the shorter side. If the longer side is 30 metres more than the shorter side, find the sides of the field.

9.4.39 The difference of squares of two numbers is 180. The square of the smaller number is 8 times the larger number. Find the two numbers.

9.4.40 A train travels 360 km at a uniform speed. If the speed had been 5 km/hr more, it would have taken 1 hour less for the same journey. Find the speed of the train.

9.4.41 Two water taps together can fill a tank in $9\frac{3}{8}$ hours. The tap of larger diameter takes 10 hours. The tap of larger diameter takes 10 hours less than the smaller one to fill the tank seperately. Find the time in which each tap can seperately fill the tank.

9.4.42 An express train takes 1 hour less than a passenger train to travel 132 km between Mysuru and Bengaluru (without taking into consideration the time they stop at intermediate statioons). If the average speed of the express train is 11 km/h more than that of the passenger train, find the average speed of the two trains.

9.4.43 Sum of the areas of two squares is $468m^2$. If the difference of their perimeter is 24m, find the sides of the two squares.

9.4.44 A charity trust decides to build a prayer hall having a carpet area of 300 square metres with its length one metre more than twice its breath. What should be the length and breadth of the hall?

9.4.45 A motor boat whose speed is 18 km/h in still water takes 1 hour more to go 24 km upstream than to return downstream to the same spot. Find the speed of the stream.

9.4.46 A pole has to be erected at a point on the boundary of a circular park of diameter 1.3 metres in such a way that the difference of its distances from two diametrically opposite fixed gates A and B on the boundary is 7 metres. Is it possible to do so? If yes, at what distances from the two gatees should the pole be erected?

9.4.47 The area of a rectangle plot is $528m^2$. The length of the plot (in metres) is one more than twice its breadth. We need to find the length and breadth of the plot.

9.4.48 Find two consecutive odd positive integers, sum of whose squares is 290.

9.4.49 A rectangular park is to be designed whose breadth is 3 m less than its length. Its area is to be 4 square metres than the area of a park that has already been made in the shape of a isoceles triangle with its base as the breadth of the rectangular park and of altitude 12 m. Find its length and breadth.

9.5 CBSE

9.5.1 Find the roots of the equation $x^2 + 3x - 10 = 0$ (10, 2023)

9.5.2 If α, β are zeroes of the polynomial $x^2 - 1$, then find the value of $(\alpha + \beta)$. (10, 2023)

9.5.3 If α, β are the zeroes of the polynomial $p(x) = 4x^2 - 3x - 7$, then find $\frac{1}{\alpha} + \frac{1}{\beta}$. (10, 2023)

9.5.4 If one zero of the polynomial $6x^2 + 37x - (k - 2)$ is the reciprocal of the other, then what is the value of k? (10, 2023)

9.5.5 Find the zeroes of the polynomial $p(x) = x^2 + 4x + 3$. (10, 2023)

9.5.6 Find the sum and product of the roots of the quadratic equation $2x^2 - 9x + 4 = 0$. (10, 2023)

9.5.7 Find the discriminant of the quadratic equation $4x^2 - 5 = 0$ and hence comment on the nature of roots of the equation. (10, 2023)

9.5.8 If one zero of polynomial $p(x) = 6x^2 + 37x - (k - 2)$ is reciprocal of the other, then find the value of k. (10, 2023)

9.5.9 Find the value of p for which one root of the quadratic equation $px^2 - 14x + 8 = 0$ is 6 times the other. (10, 2023)

9.5.10 If the sum of zeroes of the polynomial $p(x) = 2x^2 - k\sqrt{2}x + 1$ is $\sqrt{2}$, then find the value of k. (10, 2024)

9.5.11 Two pipes running together can fill a tank in $11\frac{1}{9}$ minutes. If one pipe takes 5 minutes more than the other to fill the tank separately, find the time in which each pipe would fill the tank separately. (10, 2016)

9.5.12 Which of the following quadratic equations has a sum of its roots as 4 ?

a) $2x^2 - 4x + 8 = 0$
b) $-x^2 - 4x + 4 = 0$
c) $\sqrt{2}x^2 - \frac{4}{\sqrt{2}}x + 1 = 0$
d) $4x^2 - 4x + 4 = 0$

(10, 2023)

9.5.13 A motor boat whose speed is 24 km/hr in still water takes 1 hour more to go $32km$ upstream than to return downstream to the same spot. Find the speed of the stream. (10, 2016)

9.5.14 The speed of a boat in still water is 11 km/hr. It can go 12 km upstream and return downstream to the original point in 2 hours 45 minutes. Find the speed of the stream. (10, 2006)

9.6 Curves

9.6.1 Find the area bounded by the curves $(x-1)^2 + y^2 = 1$ and $x^2 + y^2 = 1$.
Solution: The conic parameters for the two circles can be expressed as

$$\begin{aligned} \mathbf{V}_1 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \mathbf{u}_1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \ f_1 = 0, \\ \mathbf{V}_2 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \mathbf{u}_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ f_2 = -1. \end{aligned} \tag{9.6.1.1}$$

On substituting from (9.6.1.1) in (9.1.5.1), we obtain

$$\begin{vmatrix} 1+\mu & 0 & -1 \\ 0 & 1+\mu & 0 \\ -1 & 0 & -\mu \end{vmatrix} = 0 \tag{9.6.1.2}$$

yileding

$$\mu = -1. \tag{9.6.1.3}$$

Substituting (9.6.1.1) in (9.1.4.1), we obtain

$$\mathbf{x}^\top \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + 2\begin{pmatrix} -1 & 0 \end{pmatrix} \mathbf{x} + 1 = 0 \tag{9.6.1.4}$$

$$\implies \begin{pmatrix} -2 & 0 \end{pmatrix} \mathbf{x} = -1 \tag{9.6.1.5}$$

Therefore the intersection of the two circles is a line with parameters

$$\mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \mathbf{h} = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}. \tag{9.6.1.6}$$

The intersection parameters of the chord in (9.6.1.5) with the first circle in (9.6.1.1) is obtained from (9.1.1.3) as

$$\mu_i = \pm\frac{\sqrt{3}}{2} \tag{9.6.1.7}$$

Hence the point of intersection are obtained from (9.1.1.2) as

$$\mathbf{a}_0 = \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}, \mathbf{a}_2 = \begin{pmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix}. \tag{9.6.1.8}$$

The desired area of region is given as

$$2\left(\int_0^{\frac{1}{2}} \sqrt{1-(x-1)^2}dx + \int_{\frac{1}{2}}^1 \sqrt{1-x^2}dx\right)$$
$$= 2\left[\frac{1}{2}(x-1)\sqrt{1-(x-1)^2} + \frac{1}{2}\sin^{-1}(x-1)\right]_0^{\frac{1}{2}}$$
$$+ 2\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right]_{\frac{1}{2}}^1 = \frac{2\pi}{3} - \frac{\sqrt{3}}{2} \quad (9.6.1.9)$$

See Fig. 9.6.1.1.

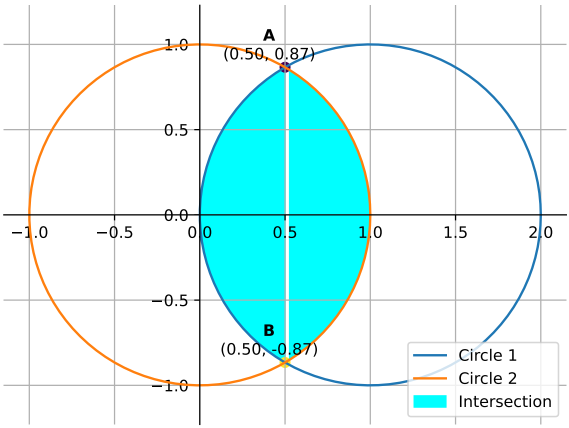

Fig. 9.6.1.1

9.6.2 Find the area of the circle $4x^2 + 4y^2 = 9$ which is interior to the parabola $x^2 = 4y$.
9.6.3 Find the area of the circle $x^2 + y^2 = 16$ exterior to the parabola $y^2 = 6x$.
9.6.4 Find the area of the region bounded by the curve $y^2 = 2x$ and $x^2 + y^2 = 4x$.
9.6.5 Solve

$$\frac{1}{3x+y} + \frac{1}{3x-y} = \frac{3}{4}$$
$$\frac{1}{2(3x+y)} - \frac{1}{2(3x-y)} = \frac{-1}{8}$$

9.7 CBSE

9.7.1 Find the area of the region

$$\left\{(x,y) : x^2 + y^2 \leq 16a^2 \text{ and } y^2 \leq 6ax\right\}$$

(12, 2018)

9.7.2 Find the area of the region $\left\{(x, y) : y^2 \leq 4x, 4x^2 + 4y^2 \leq 9\right\}$. (12, 2015)

9.7.3 Find the area of the region bounded by the curves $(x-1)^2 + y^2 = 1$ and $x^2 + y^2 = 1$. (12, 2019)

9.7.4 Prove that the curves $y^2 = 4x$ and $x^2 = 4y$ divide the area of the square bounded by sides $x = 0, x = 4, y = 4$, and $y = 0$ into three equal parts. (12, 2018)

9.7.5 Using integration, find the area of the region bounded by the parabola $y^2 = 4x$ and the circle $4x^2 + 4y^2 = 9$. (12, 2018)

9.7.6 Find the area of the region lying above X axis and included between the circle $x^2 + y^2 = 8x$ and inside of the parabola $y^2 = 4x$. (12, 2018)

9.7.7 Find the solution of the pair of equations

$$\frac{3}{x} + \frac{8}{y} = -1; \frac{1}{x} - \frac{2}{y} = 2, x, y \neq 0$$

(10, 2019)

9.8 JEE

9.8.1. The line $x + 3y = 0$ is a diameter of the circle $x^2 + y^2 - 6x + 2y = 0$. (1989)

9.8.2. The centre of the circle inscribed in square formed by the lines $x^2 - 8x + 12 = 0$ and $y^2 - 14y + 45 = 0$, is (2003)

a) $(4, 7)$ b) $(7, 4)$ c) $(9, 4)$ d) $(4, 9)$

9.8.3. If a circle is passing through the point (a, b) and it is cutting the circle $x^2 + y^2 = k^2$ orthogonally, then the equation of the locus of its centre is (1988)

a) $2ax + 2by - (a^2 + b^2 + k^2) = 0$
b) $2ax + 2by - (a^2 - b^2 + k^2) = 0$
c) $x^2 + y^2 - 3ax - 4by + (a^2 + b^2 - k^2) = 0$
d) $x^2 + y^2 - 2ax - 3by + (a^2 - b^2 - k^2) = 0$

9.8.4. A circle S passes through the point $(0, 1)$ and is orthogonal to the circle $(x-1)^2 + y^2 = 16$ and $x^2 + y^2 = 1$. Then (2014)

a) Radius of S is 8
b) Radius of S is 7
c) Centre of S is $(-7, 1)$
d) Centre of S is $(-8, 1)$

9.8.5. Let **S** be the focus of the parabola $y^2 = 8x$ and let PQ be the common chord of the circle $x^2 + y^2 - 2x - 4y = 0$ and the given parabola. The area of the triangle PQS is (2012)

9.8.6. Let the curve C be the mirror image of the parabola $y^2 = 4x$ with respect to the line $x + y + 4 = 0$. If **A** and **B** are the points of the intersection of C with the line $y = -5$, then the distance between **A** and **B** is ______. (2015)

9.8.7. If a chord, which is not tangent, of the parabola $y^2 = 16x$ has equation $2x + y = p$, and midpoint (h, k), then which of the following is (are) possible value(s) of p, h and k? (2017)

a) $p = -2, h = 2, k = -4$
b) $p = -1, h = 1, k = 3$
c) $p = 2, h = 3, k = -4$
d) $p = 5, h = 4, k = -3$

9.8.8. If one end of a focal chord of the parabola, $y^2 = 16x$ is at $(1, 4)$, then the length of this focal chord is (2019)

a) 25 b) 22 c) 24 d) 20

9.8.9. If chord PQ subtends an angle θ at the vertex of $y^2 = 4ax$ (2013)

a) $\frac{2}{3}\sqrt{7}$ b) $\frac{-2}{3}\sqrt{7}$ c) $\frac{2}{3}\sqrt{5}$ d) $\frac{-2}{3}\sqrt{5}$

9.8.10. Let a,r,s,t be nonzero real numbers. Let $\mathbf{P}(at^2, 2as)$,$\mathbf{Q}$,$\mathbf{R}(as^2, 2as)$ be distinct points on the parabola $y^2 = 4ax$. Suppose that PQ is the focal chord and lines QR and PK are parallel,where K is the point $(2a, 0)$. The value of r is (2014)

a) $\frac{-1}{t}$ b) $\frac{t^2+1}{t}$ c) $\frac{1}{t}$ d) $\frac{t^2-1}{t}$

9.8.11. Let $\mathbf{F_1}(x_1,0)$ and $\mathbf{F_2}(x_2,0)$ for $x_1 < 0$ and $x_2 > 0$, be the focii of the ellipse $\frac{x^2}{9} + \frac{y^2}{8} = 1$. Suppose a parabola having vertex at the origin and focus at $\mathbf{F_2}$ intersects the ellipse at point $\mathbf{M}$ in the first quadrant and the point $\mathbf{N}$ in the first quadrant. The orthocentre of the triangle F_1MN is (2016)

a) $\left(\frac{-9}{10}, 0\right)$ b) $\left(\frac{2}{3}, 0\right)$ c) $\left(\frac{9}{10}, 0\right)$ d) $\left(\frac{2}{3}, \sqrt{6}\right)$

9.8.12. Through the vertex $\mathbf{O}$ of parabola $y^2 = 4x$, chords OP and OQ are drawn at right angles to one another. Show that for all positions of $\mathbf{P}$, PQ cuts the axis of the parabola at a fixed point. Also find the locus of the middle point of PQ. (1994)

9.8.13. Show that the locus of a point that divides a chord of slope 2 of the parabola $y^2 = 4x$ internally in the ratio 1 : 2 is a parabola. Find the vertex of this parabola. (1995)

9.8.14. If the circle $C_1 : x^2 + y^2 = 16$ intersects another circle C_2 of radius 5 in such a manner that the common chord is of maximum length and has a slope equal to $\frac{3}{4}$, then the coordinates of the centre of C_2 are ______. (1988)

9.8.15. If the two circles $(x - 1)^2 + (y - 3)^2 = r^2$ and $x^2 + y^2 - 8x + 2y + 8 = 0$ intersect in two distinct points, then (2003)

a) $r > 2$ b) $2 < r < 8$ c) $r < 2$ d) $r = 2$

9.8.16. If the circles $x^2 + y^2 + 2ax + cy + a = 0$ and $x^2 + y^2 - 3ax + dy - 1 = 0$ intersect in two distinct points $\mathbf{P}$ and $\mathbf{Q}$, then the line $5x + by - a = 0$ passes through $\mathbf{P}$ and $\mathbf{Q}$ for (2005)

a) exactly one value of a
b) no value of a
c) infinitely many values of a
d) exactly two values of a

9.8.17. If a circle passes through the point (a, b) and cuts the circle $x^2+y^2 = p^2$ orthogonally, then the equation of the locus of its centre is (2005)

a) $x^2+y^2-3ax-4by+\left(a^2+b^2-p^2\right)=0$
b) $2ax+2by-\left(a^2-b^2+p^2\right)=0$
c) $x^2+y^2-2ax-3by+\left(a^2-b^2-p^2\right)=0$
d) $2ax+2by-\left(a^2+b^2+p^2\right)=0$

9.8.18. If **P** and **Q** are the points of intersection of the circles $x^2+y^2+3x+7y+2p-5=0$ and $x^2+y^2+2x+2y-p^2=0$ then there is a circle passing through **P**, **Q** and $(1, 1)$ for: (2009)

a) all except one value of p
b) all except two values of p
c) exactly one value of p
d) all value of p

9.8.19. Two circles $x^2+y^2=6$ and $x^2+y^2-6x+8=0$ are given. Then the equation of the circle through their points of intersection and the point $(1, 1)$ is (1980)

a) $x^2+y^2-6x+4=0$
b) $x^2+y^2-3x+1=0$
c) $x^2+y^2-4y+2=0$
d) none of these

9.8.20. The equation of circle passing through $(1, 1)$ and points of intersection of the circles $x^2+y^2+13x-3y=0$ and $2x^2+2y^2+4x-7y-25=0$ is (1983)

a) $4x^2+4y^2-30x-10y-25=0$
b) $4x^2+4y^2+30x-13y-25=0$
c) $4x^2+4y^2-17x-10y+25=0$
d) none of these

9.8.21. If the two circles $(x-1)^2+(y-3)^2=r^2$ and $x^2+y^2-8x+2y+8=0$ intersect in two distinct points, then (1989)

a) $2<r<8$
b) $r<2$
c) $r=2$
d) $r>2$

9.8.22. The circles $x^2-10x+16=0$ and $x^2+y^2=r^2$ intersect each other in the two distinct points if (1994)

a) $r<2$
b) $r>8$
c) $2<r<8$
d) $2\le r\le 8$

9.8.23. If the circles $x^2+y^2+2x+2ky+k=0$ intersect orthogonally, then k is (2000)

a) 2 or $-\frac{3}{2}$
b) -2 or $-\frac{3}{2}$
c) 2 or $\frac{3}{2}$
d) -2 or $\frac{3}{2}$

9.8.24. A line $y=mx+1$ intersects the circle $(x-3)^2+(y+2)^2=25$ at the points **P** and **Q**. If the mid point of the line segment PQ has X coordinate $-\frac{3}{5}$, then which one of the following options is correct? (2019)

a) $2 \leq m < 4$ b) $-3 \leq m < -1$ c) $4 \leq m < 6$ d) $6 \leq m < 8$

9.8.25. If a circle passes through the point (a, b) and cuts the circle $x^2 + y^2 = 4$ orthogonally, then the locus of its centre is (2004)

a) $2ax - 2by - \left(a^2 + b^2 + 4\right) = 0$
b) $2ax + 2by - \left(a^2 + b^2 + 4\right) = 0$
c) $2ax - 2by + \left(a^2 + b^2 + 4\right) = 0$
d) $2ax + 2by + \left(a^2 + b^2 + 4\right) = 4$

9.8.26. Intercept on the line $y = x$ by the circle $x^2 + y^2 - 2x = 0$ is AB. Equation of the circle on AB as a diameter is (2004)

a) $x^2 + y^2 + x - y = 0$
b) $x^2 + y^2 - x + y = 0$
c) $x^2 + y^2 + x + y = 0$
d) $x^2 + y^2 - x - y = 0$

9.8.27. If the pair of lines $ax^2 + 2(a + b)xy + by^2 = 0$ lie along diameters of a circle and divide the circle into four sectors such that the area of one of the sectors is thrice the area of another sector then (2005)

a) $3a^2 - 10ab + 3b^2 = 0$
b) $3a^2 - 2ab + 3b^2 = 0$
c) $3a^2 + 10ab + 3b^2 = 0$
d) $3a^2 + 2ab + 3b^2 = 0$

9.8.28. The differential equation of the family of circles with fixed radius 5 units and centre on the line $y = 2$ is

a) $(x - 2)\, y'^2 = 25 - (y - 2)^2$
b) $(y - 2)\, y'^2 = 25 - (y - 2)^2$
c) $(y - 2)^2\, y'^2 = 25 - (y - 2)^2$
d) $(x - 2)^2\, y'^2 = 25 - (y - 2)^2$

9.8.29. Let **A** and **B** be two distinct points on the parabola $y^2 = 4x$. If the axis of the parabola touches a circle of radius r having AB as diameter, then the slope of the line joining **A** and **B** can be (2010)

a) $\frac{-1}{r}$ b) $\frac{1}{r}$ c) $\frac{2}{r}$ d) $\frac{2}{r}$

9.8.30. Let E_1 and E_2 be two elipses whose centres are at the orgin. The major axes of E_1 and E_2 lie along the x-axis and the y-axis,respectively. Let S be the circle $x^2 + (y - 1)^2 = 2$. The straight line $x + y = 3$ touches the curves S, E_1 and E_2 at **P**, **Q** and **R** respectively. Suppose that $PQ = PR = \frac{2\sqrt{2}}{3}$. If e_1 and e_2 are the eccentricities of E_1 and E_2, respectively, then the correct expression(s) is (are) (2015)

a) $e_1^2 + e_2^2 = \frac{43}{40}$ b) $e_1 e_2 = \frac{\sqrt{7}}{2\sqrt{10}}$ c) $\left|e_1^2 - e_2^2\right| = \frac{5}{8}$ d) $e_1 e_2 = \frac{\sqrt{3}}{4}$

9.8.31. Consider a circle with its centre lying on focus of the parabola $y^2 = 2px$ such that it touches the directrix of the parabola. Then a point of intersection of the circle and the parabola is (1995S)

a) $\left(\frac{p}{2}, p\right)$ or $\left(\frac{p}{2}, -p\right)$
b) $\left(\frac{p}{2}, -\frac{p}{2}\right)$
c) $\left(-\frac{p}{2}, p\right)$
d) $\left(-\frac{p}{2}, -\frac{p}{2}\right)$

9.8.32. The centre of the circle passing through the point $(0, 1)$ and touching the curve $y = x^2$ at $(2, 4)$ is (1983)

a) $\left(\frac{-16}{5}, \frac{27}{10}\right)$ b) $\left(\frac{-16}{7}, \frac{53}{10}\right)$ c) $\left(\frac{-16}{5}, \frac{53}{10}\right)$ d) none of these

9.8.33. The points of intersection of the line $4x - 3y - 10 = 0$ and the circle $x^2 + y^2 - 2x + 4y - 20 = 0$ are ______. (1983)

9.8.34. The equation of the line passing through the points of intersection of the circles $3x^2 + 3y^2 - 2x + 12y - 9 = 0$ and $x^2 + y^2 + 6x + 2y - 15 = 0$ is ______. (1986)

9.8.35. An ellipse intersects the hyperbola $2x^2 - 2y^2 = 1$ orthogonally. The eccentricity of the ellipse is reciprocal of that of the hyperbola. If the axes of the ellipse are along the coordinate axes, then (2009)

a) equation of the ellipse is $x^2 + 2y^2 = 2$
b) the foci of ellipse are $(\pm 1, 0)$
c) equation of the ellipse is $x^2 + 2y^2 = 4$
d) the foci of ellipse are $\left(\pm \sqrt{2}, 0\right)$

9.8.36. If the circle $x^2 + y^2 = a^2$ intersects the hyperbola $xy = c^2$ in four points $\mathbf{P}(x_1, y_1)$, $\mathbf{Q}(x_2, y_2)$, $\mathbf{R}(x_3, y_3)$, $\mathbf{S}(x_4, y_4)$, then (1998)

a) $x_1 + x_2 + x_3 + x_4 = 0$
b) $y_1 + y_2 + y_3 + y_4 = 0$
c) $x_1x_2x_3x_4 = c^4$
d) $y_1y_2y_3y_4 = c^4$

10.1 Formulae

10.1.1 If L in (9.1.1.1) touches (8.1.2.1) at exactly one point $\mathbf{q}$,

$$\mathbf{m}^\top (\mathbf{V}\mathbf{q} + \mathbf{u}) = 0 \tag{10.1.1.1}$$

10.1.2 Given the point of contact $\mathbf{q}$, the equation of a tangent to (8.1.2.1) is

$$(\mathbf{V}\mathbf{q} + \mathbf{u})^\top \mathbf{x} + \mathbf{u}^\top \mathbf{q} + f = 0 \tag{10.1.2.1}$$

10.1.3 Given the point of contact $\mathbf{q}$, the equation of the normal to (8.1.2.1) is

$$(\mathbf{V}\mathbf{q} + \mathbf{u})^\top \mathbf{R}(\mathbf{x} - \mathbf{q}) = 0 \tag{10.1.3.1}$$

10.1.4 If $\mathbf{V}^{-1}$ exists, given the normal vector $\mathbf{n}$, the tangent points of contact to (8.1.2.1) are given by

$$\begin{aligned} \mathbf{q}_i &= \mathbf{V}^{-1}(\kappa_i \mathbf{n} - \mathbf{u}), i = 1, 2 \\ \text{where } \kappa_i &= \pm\sqrt{\frac{f_0}{\mathbf{n}^\top \mathbf{V}^{-1}\mathbf{n}}} \end{aligned} \tag{10.1.4.1}$$

10.1.5 (10.1.4.1) can be expressed as the solution matrix

$$\mathbf{Q} = \mathbf{V}^{-1}\begin{pmatrix}\mathbf{u} & \mathbf{n}\end{pmatrix}\begin{pmatrix}-\mathbf{1} & \boldsymbol{\kappa}\end{pmatrix}^\top \tag{10.1.5.1}$$

10.1.6 If $\mathbf{V}$ is not invertible, given the normal vector $\mathbf{n}$, the point of contact to (8.1.2.1) is given by the matrix equation

$$\begin{pmatrix}(\mathbf{u} + \kappa\mathbf{n})^\top \\ \mathbf{V}\end{pmatrix}\mathbf{q} = \begin{pmatrix}-f \\ \kappa\mathbf{n} - \mathbf{u}\end{pmatrix} \quad \text{where } \kappa = \frac{\mathbf{p}_1^\top \mathbf{u}}{\mathbf{p}_1^\top \mathbf{n}}, \quad \mathbf{V}\mathbf{p}_1 = 0 \tag{10.1.6.1}$$

10.1.7 For a conic/hyperbola, a line with normal vector $\mathbf{n}$ cannot be a tangent if

$$\frac{\mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f}{\mathbf{n}^\top \mathbf{V}^{-1}\mathbf{n}} < 0 \tag{10.1.7.1}$$

10.1.8 For a circle, the points of contact are

$$\mathbf{q}_{ij} = \left(\pm r\frac{\mathbf{n}_j}{\left\|\mathbf{n}_j\right\|} - \mathbf{u}\right), \quad i, j = 1, 2 \tag{10.1.8.1}$$

10.1.9 A point $\mathbf{h}$ lies on a normal to the conic in (8.1.2.1) if

$$\begin{aligned} &\left(\mathbf{m}^\top(\mathbf{V}\mathbf{h} + \mathbf{u})\right)^2 \left(\mathbf{n}^\top \mathbf{V}\mathbf{n}\right) \\ &\quad - 2\left(\mathbf{m}^\top \mathbf{V}\mathbf{n}\right)\left(\mathbf{m}^\top(\mathbf{V}\mathbf{h} + \mathbf{u})\mathbf{n}^\top(\mathbf{V}\mathbf{h} + \mathbf{u})\right) \\ &\quad\quad + g(\mathbf{h})\left(\mathbf{m}^\top \mathbf{V}\mathbf{n}\right)^2 = 0 \end{aligned} \tag{10.1.9.1}$$

10.1.10 A point $\mathbf{h}$ lies on a tangent to the conic in (8.1.2.1) if

$$\mathbf{m}^\top \left[(\mathbf{V}\mathbf{h} + \mathbf{u})(\mathbf{V}\mathbf{h} + \mathbf{u})^\top - \mathbf{V}g(\mathbf{h})\right]\mathbf{m} = 0 \tag{10.1.10.1}$$

10.1 Find the equation of a circle of radius 5 which is touching another circle $x^2 + y^2 - 2x - 4y - 20 = 0$ at (5,5).

10.2 The equation of the circle having centre at (3,-4) and touching the line $5x+12y-12 = 0$ is ______

10.3 Find the equation of the circle which touches both the axes in first quadrant and whose radius is a.

10.4 Find the equation of the circle which touches x-axis and whose centre is $(1, 2)$

10.5 lf the lines $3x-4y+4 = 0$ and $6x-8y-7 = 0$ are tangents to a circle, then find the radius of the circle.

10.6 Find the equation of a circle which touches both the axes and the line $3x-4y+8 = 0$ and lies in the third quadrant.

10.7 At what points on the curve $x^2 + y^2 - 2x - 4y + 1 = 0$, the tangents are parallel to the y-axis?

10.8 The shortest distance from the point (2,7) to the circle $x^2 + y^2 - 14x - 10y - 151 = 0$ is equal to 5.

10.9 lf the line $lx + my = 1$ is a tangent to the circle $x^2 + y^2 = a^2$, then the point $(1, m)$ lies an a circle.

10.10 Find the points on the curve $x^2 + y^2 - 2x - 3 = 0$ at which the tangents are parallel to the x-axis.

Solution: Given that

$$\mathbf{u} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \ f = -3 \tag{10.10.1}$$

Hence, the centre and radius are given as

$$\mathbf{c} = -\mathbf{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ r = \sqrt{||\mathbf{u}||^2 - f} = 2 \tag{10.10.2}$$

From (10.1.8.1), the points of contact for the tangent are given by

$$\mathbf{q}_{ij} = \left(\pm r \frac{\mathbf{n}_j}{\left\|\mathbf{n}_j\right\|} - \mathbf{u} \right) \ \text{i,j} = 1, 2 \tag{10.10.3}$$

Since, tangents are parallel to the x-axis, the normal is given as

$$\mathbf{n} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{10.10.4}$$

Substituting in (10.10.3) we get

$$\mathbf{q}_{11} = \left(\pm 2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right) \tag{10.10.5}$$

$$= \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{10.10.6}$$

Hence, the two points of contact are

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ -2 \end{pmatrix} \tag{10.10.7}$$

See Fig. 10.10.1.

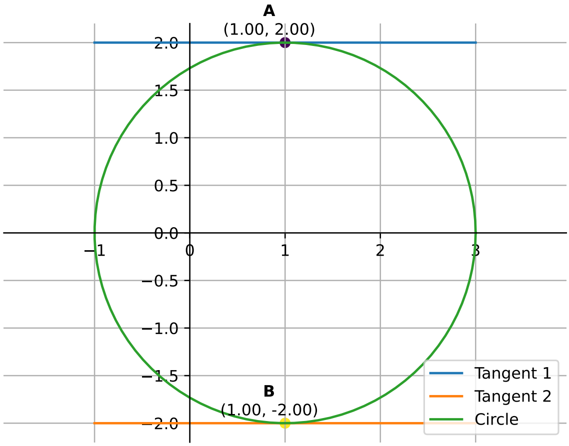

Fig. 10.10.1

10.3 Conic

10.3.1 Find the slope of the tangent to the curve $y = \frac{x-1}{x-2}$, $x \neq 2$ at $x = 10$.
Solution: The given equation of the curve can be rearranged as

$$xy - x - 2y + 1 = 0 \tag{10.3.1.1}$$

$$\implies \mathbf{x}^\top \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -1 & -2 \end{pmatrix} \mathbf{x} + 1 = 0 \tag{10.3.1.2}$$

Thus,

$$\mathbf{V} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \tag{10.3.1.3}$$

$$\mathbf{u} = -\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} \tag{10.3.1.4}$$

$$f = 1 \tag{10.3.1.5}$$

$\because q_1 = 10$, the point of contact can be obtained as

$$\mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 10 \\ \frac{9}{8} \end{pmatrix} \tag{10.3.1.6}$$

From (10.1.1.1), the normal vector of the tangent to (10.3.1.2) is

$$\mathbf{n} = \begin{pmatrix} 1 \\ 64 \end{pmatrix} \implies \mathbf{m} = \begin{pmatrix} 1 \\ \frac{-1}{64} \end{pmatrix} \tag{10.3.1.7}$$

The eigenvector matrix

$$\begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{10.3.1.8}$$

which implies that the conic is a 45° rotated hyperbola. See Fig. 10.3.1.1.

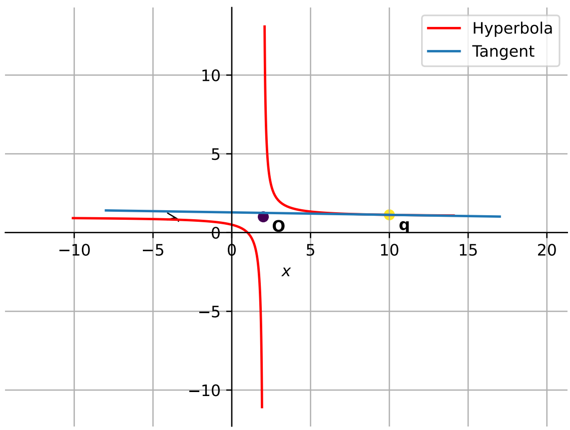

Fig. 10.3.1.1

10.3.2 Prove that the curves $y^2 = 4x$ and $x^2 + y^2 - 6x + 1 = 0$ touch each each other at the point (1,2).

10.3.3 Find the equation of the normal lines to the curve $3x^2 - y^2 = 8$ which are parallel to the line $x + 3y = 4$.

10.3.4 The equation of the normal to the curve $3x^2 - y^2 = 8$ which is parallel to the line $x + 3y = 8$ is ______.

10.3.5 The equation of the tangent to the curve $(1 + y^2) = 2 - x$, where it crosses the x-axis is ______.

10.3.6 Find the condition that the curves $2x = y^2$ and $2xy = k$ intersect orthogonally.

10.3.7 Prove that the curves $xy = 4$ and $x^2 + y^2 = 8$ touch each other.

10.3.8 Find the angle of intersection of the curves $y = 4 - x^2$ and $y = x^2$.

10.3.9 Find a point on the curve

$$y = (x - 2)^2 \tag{10.3.9.1}$$

at which a tangent is parallel to the chord joining the points (2,0) and (4,4).

Solution:

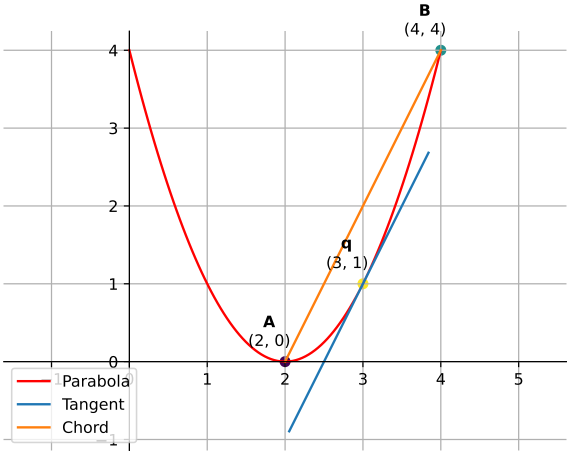

Fig. 10.3.9.1

The equation of the conic can be represented as

$$\mathbf{x}^\top \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} -2 & \frac{-1}{2} \end{pmatrix} \mathbf{x} + 4 = 0 \tag{10.3.9.2}$$

So,

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u}^\top = \begin{pmatrix} -2 & \frac{-1}{2} \end{pmatrix}, f = 4 \tag{10.3.9.3}$$

The direction vector of the line passing through (2,0) and (4,4) is

$$\mathbf{m} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \implies \mathbf{n} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}. \tag{10.3.9.4}$$

The eigenvector corresponding to the zero eigenvalue is

$$\mathbf{p}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{10.3.9.5}$$

In (10.1.6.1),

$$\kappa = \frac{\begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} -2 \\ \frac{-1}{2} \end{pmatrix}}{\begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix}} = \frac{1}{2} \tag{10.3.9.6}$$

Substituting κ, from (10.1.6.1),

$$\begin{pmatrix} \left[\begin{pmatrix} -2 \\ \frac{-1}{2} \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 2 \\ -1 \end{pmatrix}\right]^{\top} \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \mathbf{q} = \begin{pmatrix} -4 \\ \frac{1}{2}\begin{pmatrix} 2 \\ -1 \end{pmatrix} - \begin{pmatrix} -2 \\ \frac{-1}{2} \end{pmatrix} \end{pmatrix} \tag{10.3.9.7}$$

$$\implies \begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{q} = \begin{pmatrix} -4 \\ 3 \\ 0 \end{pmatrix} \tag{10.3.9.8}$$

yielding

$$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{q} = \begin{pmatrix} -4 \\ 3 \end{pmatrix} \tag{10.3.9.9}$$

The augmented matrix is

$$\left(\begin{array}{cc|c} -1 & -1 & -4 \\ 1 & 0 & 3 \end{array}\right) \xleftrightarrow{R_1 \leftarrow R_1 + 2R_2} \left(\begin{array}{cc|c} 1 & -1 & 2 \\ 1 & 0 & 3 \end{array}\right)$$
$$\xleftrightarrow{R_2 \leftarrow R_2 - R_1} \left(\begin{array}{cc|c} 1 & -1 & 2 \\ 0 & 1 & 1 \end{array}\right) \xleftrightarrow{R_1 \leftarrow R_1 + R_2} \left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 1 \end{array}\right)$$
$$\implies \mathbf{q} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

which is the desired point of contact. See Fig. 10.3.9.1.

10.3.10 The line $y = mx + 1$ is a tangent to the curve $y^2 = 4x$, find the value of m.

10.3.11 Find the normal at the point (1,1) on the curve

$$2y + x^2 = 3 \tag{10.3.11.1}$$

10.3.12 If the line $y = \sqrt{3}x + K$ touches the parabola $x^2 = 16y$, then find the value of K.

10.3.13 If the line $y = mx + 1$ is tangent to the parabola $y^2 = 4x$ then find the value of m.

State whether the statements are True or False

10.3.14 The line $lx + my + n = 0$ will touch the parabola $y^2 = 4ax$ if $ln = am^2$,

10.3.15 The line $2x + 3y = 12$ touches the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 2$ at the point (3,2).

10.3.16 Find the equation of all lines having slope -1 that are tangents to the curve

$$y = \frac{1}{x-1}, x \neq 1 \tag{10.3.16.1}$$

Solution: From the given information,

$$\mathbf{V} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix}, f = -1, m = -1 \tag{10.3.16.2}$$

From the above, the normal vector is

$$\mathbf{n} = \begin{pmatrix} -m \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{10.3.16.3}$$

Substituting from (10.3.16.3) and (10.3.16.2) in (10.1.4.1), the point(s) of contact are given by

$$\mathbf{q} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}. \tag{10.3.16.4}$$

From (10.1.2.1), the equations of tangents are

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} + 1 = 0 \tag{10.3.16.5}$$
$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} - 3 = 0 \tag{10.3.16.6}$$
$$\tag{10.3.16.7}$$

See Fig. 10.3.16.1.

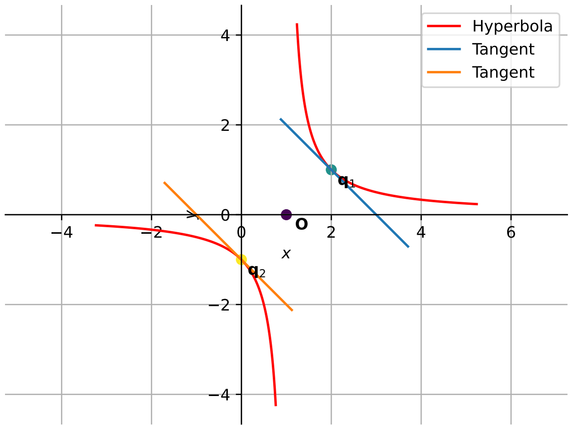

Fig. 10.3.16.1

10.3.17 Find the equation of all lines having slope 2 which are tangents to the curve

$$y = \frac{1}{x-3}, x \neq 3 \tag{10.3.17.1}$$

10.3.18 Find points on the curve $\frac{x^2}{9} + \frac{y^2}{16} = 1$ at which the tangents are

a) parallel to x-axis

b) parallel to y-axis

Solution: The parameters of the given conic are

$$\mathbf{V} = \begin{pmatrix} 16 & 0 \\ 0 & 9 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, f = -144 \tag{10.3.18.1}$$

a) The normal vector in this case is

$$\mathbf{n_1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{10.3.18.2}$$

which can be used along with the parameters in (10.3.18.1) to obtain

$$\mathbf{q_1} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}, \mathbf{q_2} = \begin{pmatrix} 0 \\ -4 \end{pmatrix} \tag{10.3.18.3}$$

using (10.1.4.1).

b) Simlarly, choosing

$$\mathbf{n_2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{10.3.18.4}$$

$$\mathbf{q_3} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \mathbf{q_4} = \begin{pmatrix} -3 \\ 0 \end{pmatrix} \tag{10.3.18.5}$$

See Fig. 10.3.18.1.

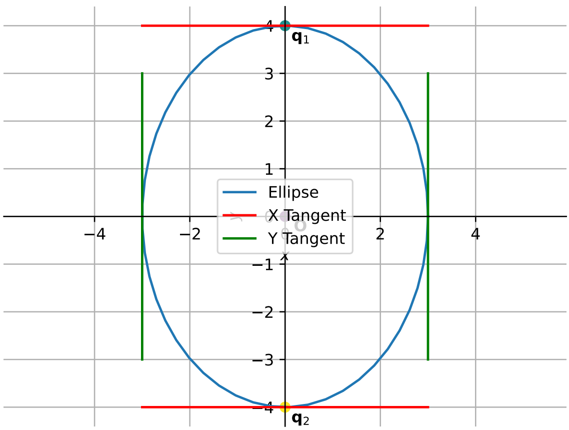

Fig. 10.3.18.1

10.3.19 Find the equation of the tangent line to the curve

$$y = x^2 - 2x + 7 \tag{10.3.19.1}$$

a) parallel to the line $2x - y + 9 = 0$.

b) perpendicular to the line $5y - 15x = 13$.

10.3.20 Find the equation of the tangent to the curve

$$y = \sqrt{3x - 2} \tag{10.3.20.1}$$

which is parallel to the line

$$4x - 2y + 5 = 0 \tag{10.3.20.2}$$

10.3.21 Find the point at which the line $y = x + 1$ is a tangent to the curve $y^2 = 4x$.

10.3.22 The point on the curve

$$x^2 = 2y \tag{10.3.22.1}$$

which is nearest to the point $\mathbf{P} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$ is ______.

Solution: We rewrite the conic (10.3.22.1) in matrix form.

$$\mathbf{x}^\top \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + 2\begin{pmatrix} 0 & -1 \end{pmatrix} \mathbf{x} = 0 \tag{10.3.22.2}$$

Comparing with the general equation of the conic,

$$\mathbf{V}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ \mathbf{u}_0 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \ f_0 = 0 \tag{10.3.22.3}$$

Therefore, the equation of the normal where $\mathbf{q}$ is the point of contact and

$$\mathbf{R} \triangleq \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{10.3.22.4}$$

is

$$(\mathbf{V}_0\mathbf{q} + \mathbf{u}_0)^\top \mathbf{R}\left(\begin{pmatrix} 0 \\ 5 \end{pmatrix} - \mathbf{q}\right) = 0 \tag{10.3.22.5}$$

Substituting appropriate values and simplifying, we get

$$\mathbf{q}^\top \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mathbf{q} + 2\begin{pmatrix} -2 & 0 \end{pmatrix} \mathbf{q} = 0 \tag{10.3.22.6}$$

which can be expressed as

$$\frac{1}{2}\left\{\mathbf{q}^\top \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mathbf{q} + 2\begin{pmatrix} -2 & 0 \end{pmatrix} \mathbf{q} \ + \ \mathbf{q}^\top \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^\top \mathbf{q} + 2\begin{pmatrix} -2 & 0 \end{pmatrix} \mathbf{q}\right\} \ = \ 0 \tag{10.3.22.7}$$

yielding

$$\mathbf{q}^\top \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \mathbf{q} + 2\begin{pmatrix} -2 & 0 \end{pmatrix} \mathbf{q} = 0 \tag{10.3.22.8}$$

(10.3.22.8) also looks like a conic with parameters

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ \mathbf{u} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \ f = 0 \tag{10.3.22.9}$$

The eigenparameters of $\mathbf{V}$ are

$$\mathbf{P} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \ \mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{10.3.22.10}$$

Applying the affine transformation

$$\mathbf{q} = \mathbf{P}\mathbf{y} + \mathbf{c} \tag{10.3.22.11}$$

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} \tag{10.3.22.12}$$

$$f_0 = \mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f = 0 \tag{10.3.22.13}$$

$\because$ $\det V = -\frac{1}{4} \neq 0$, using (B.4.9.1), (10.3.22.8) represents a pair of straight lines. From (B.4.8.2), (10.3.22.10), (A.8.3.4) and (A.8.3.7),

$$\mathbf{y} = \kappa \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}. \tag{10.3.22.14}$$

Hence, using (10.3.22.11),

$$\mathbf{q} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} + \kappa \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}, \tag{10.3.22.15}$$

which, upon substituting in (10.3.22.2) and solving for κ yields

$$\kappa = \pm\sqrt{2}, -2. \tag{10.3.22.16}$$

Thus, the points of contact are

$$\mathbf{q} = \left\{ \begin{pmatrix} \pm 2\sqrt{2} \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} \tag{10.3.22.17}$$

The nearest point out of these three candidates for $\mathbf{q}$ is $\begin{pmatrix} \pm 2\sqrt{2} \\ 4 \end{pmatrix}$. See Fig. 10.3.22.1.

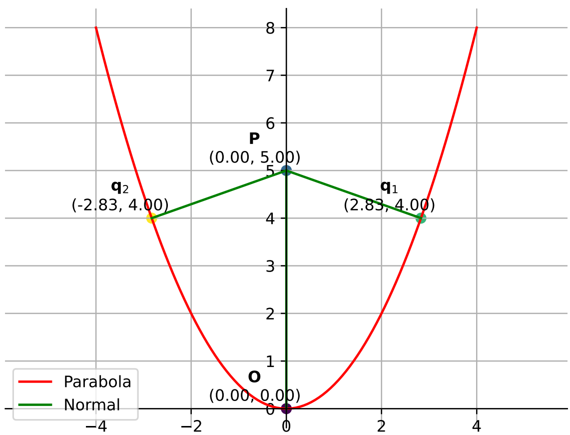

Fig. 10.3.22.1

10.3.23 Find the equation of the normal to curve $x^2 = 4y$ which passes through the point (1, 2).

Solution: The conic parameters are

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}, f = 0 \tag{10.3.23.1}$$

Choosing the direction and normal vectors as

$$\mathbf{m} = \begin{pmatrix} 1 \\ m \end{pmatrix}, \quad \mathbf{n} = \begin{pmatrix} -m \\ 1 \end{pmatrix}, \tag{10.3.23.2}$$

and substituting these values in (10.1.9.1), we obtain

$$m = 1 \tag{10.3.23.3}$$

as the only real solution. Thus,

$$\mathbf{m} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{10.3.23.4}$$

and the equation of the normal is then obtained as

$$\mathbf{m}^\top (\mathbf{x} - \mathbf{h}) = 0 \tag{10.3.23.5}$$

$$\implies \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{10.3.23.6}$$

$$= 3 \tag{10.3.23.7}$$

See Fig. 10.3.23.1.

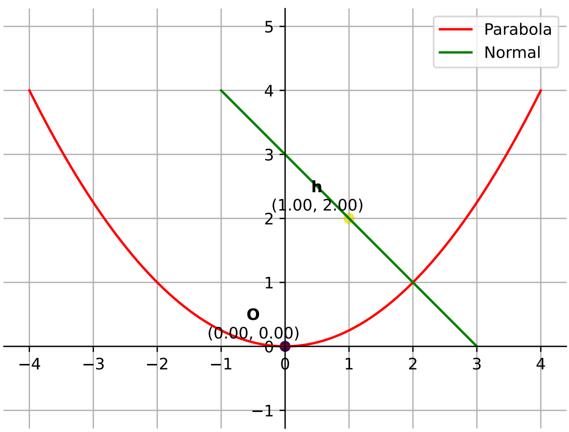

Fig. 10.3.23.1

10.3.24 Find points on the curve $\frac{x^2}{4} + \frac{y^2}{25} = 1$ at which the tangents are

a) parallel to x-axis

b) parallel to y-axis

10.3.25 Find the equation of all lines having slope 2 which are tangents to the curve

$$y + \frac{2}{x-3} = 0 \tag{10.3.25.1}$$

10.3.26 Find the point at which the tangent to the curve $y = \sqrt{4x-3} - 1$ has its slope $\frac{2}{3}$.

10.3.27 Prove that the curves $x = y^2$ and $xy = k$ cut at right angles if $8k^2 = 1$.

10.4 CBSE

10.4.1 Find the equations of the tangent and the normal, to the curve $16x^2 + 9y^2 = 145$ at the point (x_1, y_1), where $x_1 = 2$ and $y_1 > 0$. (12, 2018)

10.4.2 Find the equations of the tangent and normal to the curve $y = \frac{x-7}{(x-2)(x-3)}$ at the point where it cuts the X axis. (12, 2018)

10.4.3 The point at which the normal to the curve

$$y = x + \frac{1}{x}, x > 0$$

is perpendicular to the line

$$3x - 4y - 7 = 0$$

is ______. (12, 2021)

10.4.4 The points on the curve

$$\frac{x^2}{9} + \frac{y^2}{16} = 1$$

at which the tangents are parallel to the Y axis are ______. (12, 2021)

10.4.5 Find the point on the curve y^2=$4x$, which is nearest to the point $(2, -8)$. (12, 2018)

10.4.6 Find the equation of the normal to the curve $x^2 = 4y$ which passes through the point $(-1, 4)$. (12, 2018)

10.4.7 Find the equation of tangent to the curve $y = \sqrt{3x-2}$ which is parallel to the line $4x - 2y + 5 = 0$. Also write the equation of the normal to the curve at the point of contact. (12, 2018)

10.4.8 For which value of m is the line

$$y = mx + 1$$

a tangent to the curve

$$y^2 = 4x$$

(12, 2022)

10.5 Construction

10.5.1 Let ABC be a right triangle in which $AB = 6cm, BC = 8cm$ and $\angle B = 90°$. BD is the perpendicular from **B** on AC. The circle through **B**, **C**, **D** is drawn. Construct the tangents from **A** to this circle.

10.5.2 Draw a line segment AB of length 8cm. Taking **A** as centre, draw a circle of radius 4cm and taking **B** as centre, draw another circle of radius 3cm. Construct tangents to each circle from the centre of the circle.

10.5.3 Draw a pair of tangents to a circle of radius 5 cm which are inclined to each other at an angle of 60°.

10.5.4 Draw a circle of radius 3 cm. Take two points **P** and **Q** on one of its extended diameter each at a distance of 7 cm from its centre. Draw tangents to the circle from these two points **P** and **Q**.

10.5.5 Construct a tangent to a circle of radius 4cm from a point on the concentric circle of radius 6cm and measure its length. Also verify the measurement by actual calculation.

10.5.6 From a point **Q**, the length of the tangent to a circle is $24cm$ and the distance of **Q** from the centre is $25cm$. Find the radius of the circle. Draw the circle and the tangents.

10.5.7 To draw a pair of tangents to a circle which are inclined to each other at an angle of 60°, it is required to draw tangents at end points of those two radii of the circle, the angle between them should be

a) 135°
b) 90°
c) 60°
d) 120°

10.5.8 Draw two concentric circles of radii 3 cm and 5 cm. Taking a point on outer circle construct the pair of tangents to the other. Measure the length of a tangent and verify it by actual calculation.

10.5.9 Draw a circle of radius 4 cm .Construct a pair of tangents to it, the angle detween which is 60°. Also justify the construction. Measure the distance between the centre of the circle and the point of intersection of tangents.

10.5.10 Construct a tangent to a circle of radius 4 cm from a point which is at a distance of 6 cm from its centre.

Write True or False and give reasons for your answer in each of the following

10.5.11 A pair of tangents can be constructed from a point **h** to a circle of radius 3.5 cm situated at a distance of 3 cm from the centre.

10.5.12 A pair of tangents can be constructed to a circle inclined at an angle of 170°.

10.6 CBSE

10.6.1 Draw a circle of radius $2.5cm$. Take a point **P** outside the circle at a distance of $7cm$ from the center. Then construct a pair of tangents to the circle from point **P**.
(10, 2022)

10.6.2 Write the steps of construction for constructing a pair of tangents to a circle of radius $4cm$ from a point **P**, at a distance of $7cm$ from its center **O**. (10, 2022)

10.6.3 Draw a pair of tangents to a circle of radius $4cm$ which are inclined to each other at an angle of $45°$. (10, 2021)

10.6.4 Draw a circle of radius $5cm$. From a point $8cm$ away from its centre, construct a pair of tangents to the circle. (10, 2021)

10.6.5 Write the steps of construction of a circle of diameter $6cm$ and drawing of a pair of tangents to the circle from a point $5cm$ away from the centre. (10, 2021)

10.6.6 Draw two concentric circles of radii $2cm$ and $5cm$. Take a point **P** on the outer circle and construct a pair of tangents PA and PB to the smaller circle. Measure PA. (10, 2019)

10.6.7 Draw a pair of tangents to a circle of radius 3 cm, which are inclined to each other at an angle of $60°$. (10, 2011)

10.6.8 Construct a pair of tangents to a circle of radius $4cm$ from a point **P** lying outside the circle at a distance of $6cm$ from the centre. (10, 2023)

10.6.9 Draw a circle of radius $3cm$. From a point **P** lying outside the circle at a distance of $6cm$ from its centre, construct two tangents PA and PB to the circle. (10, 2023)

10.6.10 Draw a circle of radius $3.5cm$. Take a point **P** outside the circle at a distance of $7cm$ from the centre of the circle and construct a pair of tangents to the circle from that point. (10, 2020)

10.6.11 Draw a circle of radius 4 cm. Draw two tangents to the circle inclined at an angle of $60°$ to each other. (10, 2016)

10.7 JEE

10.7.1. The line $2x + y = 1$ is the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If this line passes through the point of intersection of the nearest directrix and the X axis, then the eccentricity of the hyperbola is ______. (2010)

10.7.2. If the normal of the parabola $y^2 = 4x$ drawn at the end points of its latusrectum are the tangents of th circle $(x-3)^2 + (y+2)^2 = r^2$, then the value of r^2 is ______. (2015)

10.7.3. If two tangents drawn from a point **P** to the parabola $y^2 = 4x$ are at right angles, then the locus of **P** is (2010)

a) $2x + 1 = 0$ b) $x = -1$ c) $2x - 1 = 0$ d) $x = 1$

10.7.4. Statement -1 : An equation of a common tangent to the parabola $y^2 = 16\sqrt{3}x$ and the ellipse $2x^2 + y^2 = 4$ is $y = 2x + 2\sqrt{3}$
Statement -2 : If the line $y = mx + \frac{4\sqrt{3}}{m}$ $(m \neq 0)$is a common tangent to the parabola $y^2 = 16\sqrt{3}x$ and the ellipse $2x^2 + y^2 = 4$, then m satisfies $m^4 + 2m^2 = 24$ (2012)

a) Statement-1 is false, statement-2 is true.
b) Statement-1 is true, statement-2 is true; statement-2 is correct explanation for statement-1.
c) Statement-1 is true, statement-2 is true; statement-2 is not a correct explanation for statement-1.
d) Statement-1 is true, statement-2 is false.

10.7.5. Given: A circle, $2x^2 + 2y^2 = 5$ and a parabola, $y^2 = 4\sqrt{5}x$.
Statement-1: An equation of a common tangent to these curves is $y = x + \sqrt{5}$.

Statement -2: If the line, $y = mx + \frac{\sqrt{5}}{m}$ $(m \neq 0)$ is their common tangent, then m satisfies $m^4 - 3m^2 + 2 = 0$. (2013)

a) Statement-1 is true; Statement-2 is true; Statement-2 is a correct explanation for Statement-1.
b) Statement-1 is true; Statement-2 is true; Statement-2 is not a correct explanation for Statement-1.
c) Statement-1 is true; Statement-2 is false.
d) Statement-1 is false; Statement-2 is true.

10.7.6. The locus of the foot of perpendicular drawn from the centre of the ellipse $x^2+3y^2 = 6$ on an tangent to it is (2014)

a) $(x^2 + y^2)^2 = 6x^2 + 2y^2$
b) $(x^2 + y^2)^2 = 6x^2 - 2y^2$
c) $(x^2 - y^2)^2 = 6x^2 + 2y^2$
d) $(x^2 - y^2)^2 = 6x^2 - 2y^2$

10.7.7. The slope of the line touching both the parabolas $y^2 = 4x$ and $x^2 = -32y$ is (2014)

a) $\frac{1}{8}$
b) $\frac{2}{3}$
c) $\frac{1}{2}$
d) $\frac{3}{2}$

10.7.8. The area (in sq. units) of the quadrilateral formed by the tangents at the end points of the latera recta to the ellipse $\frac{x^2}{9} + \frac{y^2}{5} = 1$ is (2015)

a) $\frac{27}{2}$
b) 27
c) $\frac{27}{4}$
d) 18

10.7.9. Match the statements in Column I with the properties in Column II. (2007)

Column I

a) Two intersecting circles
b) Twomutually external circles
c) Two circles, one strictly inside the other
d) Two branches of a hyperbola

Column II

a) have a common tangent
b) havea common normal
c) do not have a common tangent
d) do not have a common normal

10.7.10. line L : $y = mx + 3$ meets Y axis at $\mathbf{E}(0, 3)$ and the arc of the arc of the parabola $y^2 = 16x$, $0 \leq y \leq 6$ at the point $\mathbf{F}(x_o, y_o)$. The tangent to the parabola at $\mathbf{F}(x_o, y_o)$ intersects the Y axis at $\mathbf{G}(0, y_1)$. The slope m of the line L is chosen such that the area of the triangle EFG has a local maximum. (2013)
Match List 1 with List 2 and select the correct answer using the code given below the list

List 1

a) m=
b) Maximum area of ΔEFG
c) $y_o =$
d) $y_1 =$

List 2

a) $\frac{1}{2}$
b) 4
c) 2
d) 1

10.7.11. Columns 1, 2 and 3 contain conics and points of contact, respectively.

Sno	1	2	3
1	$x^2 + y^2 = a^2$	$my = m^2x + a$	$\left(\frac{a}{m^2}\right), \frac{2a}{m}$
2	$x^2 + a^2y^2 = a^2$	$y = mx + a\sqrt{m^2+1}$	$\left(\frac{-ma}{\sqrt{m^2+1}}, \frac{a}{\sqrt{m^2+1}}\right)$
3	$y^2 = 4ax$	$y = mx + \sqrt{a^2m^2-1}$	$\left(\frac{-a^2m}{\sqrt{a^2m^2+1}}, \frac{1}{\sqrt{a^2m^2+1}}\right)$
4	$x^2 - a^2y^2 = a^2$	$y = mx + \sqrt{a^2m^2-1}$	$\left(\frac{-a^2m}{\sqrt{a^2m^2-1}}, \frac{-1}{\sqrt{a^2m^2-1}}\right)$

a) For $a = \sqrt{2}$, if a tangent is drawn to a suitable conic (Column 1) at the point of contact $(-1, 1)$, then which of the following options is the only correct combination for obtaining its equation? (2009)

i) $(1, 1, 1)$ ii) $(1, 2, 2)$ iii) $(2, 2, 2)$ iv) $(3, 1, 1)$

b) If a tangent to a suitable conic (Column 1) is found to be $y = x + 8$ and its point of contact is $(8, 16)$, then which of the following options is the only correct combination? (2018)

i) $(1, 2, 2)$ ii) $(2, 4, 3)$ iii) $(3, 1, 1)$ iv) $(3, 2, 2)$

c) The tangent to a suitable conic (Column 1) at $(\sqrt{3}, \frac{1}{2})$ is found to be $\sqrt{3}x + 2y = 4$, then which of the following options is the only correct option ?

i) $(4, 3, 4)$ ii) $(4, 4, 4)$ iii) $(2, 3, 3)$ iv) $(2, 4, 3)$

10.7.12. On the ellipse $4x^2 + 9y^2 = 1$, the points at which the tangents are parallel to the line $8x = 9y$ are (1999)

a) $\left(\frac{2}{5}, \frac{1}{5}\right)$ b) $\left(-\frac{2}{5}, \frac{1}{5}\right)$ c) $\left(-\frac{2}{5}, -\frac{1}{5}\right)$ d) $\left(\frac{2}{5}, -\frac{1}{5}\right)$

10.7.13. The equations of the common tangents to the parabola $y = x^2$ and $y = -(x-2)^2$ is/are (2006)

a) $y = 4(x-1)$ b) $y = 0$ c) $y = -4(x-1)$ d) $y = -30x - 50$

10.7.14. The tangent PT and the normal PN to the parabola $y^2 = 4ax$ at a point **T** and **N**, respectively. The locus of the centroid of the traingle PTN is a parabola whose (2009)

a) vertex is $\left(\frac{2a}{3}, 0\right)$
b) directrix is $x = 0$
c) latus rectum is $\frac{2a}{3}$
d) focus is $(a, 0)$

10.7.15. Tangents are drawn to the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$, parallel to the straight line $2x - y = 1$. The points of contact of the tangents to the hyperbola are (2012)

a) $\left(\frac{9}{2\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ b) $\left(\frac{-9}{2\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$ c) $\left(3\sqrt{3}, -2\sqrt{2}\right)$ d) $\left(-3\sqrt{3}, 2\sqrt{2}\right)$

10.7.16. Consider the hyperbola H:$x^2 - y^2 = 1$ and a circle S with centre $\mathbf{N}(x_2, 0)$. Suppose that H and S touch each other at a point $\mathbf{P}(x_1, y_1)$ with $x_1 > 0$ and $y_1 > 0$. The common tangent to H and S at $\mathbf{P}$ intersects the X axis at point $\mathbf{M}$. If (l, m) is the centroid of the triangle PMN, then correct expressions(s) is(are)

(2015)

a) $\frac{dl}{dx_1} = 1 - \frac{1}{3x_1^2}$ for $x_1 > 1$
b) $\frac{dm}{dx_1} = \frac{x_1}{3\sqrt{x_1^2 - 1}}$ for $x_1 > 1$
c) $\frac{dl}{dx_1} = 1 + \frac{1}{3x_1^2}$ for $x_1 > 1$
d) $\frac{dm}{dy_1} = \frac{1}{3}$ for $y_1 > 0$

10.7.17. The circle C_1:$x^2 + y^2 = 3$, with centre at $\mathbf{O}$, intersects the parabola $x^2 = 2y$ at the point $\mathbf{P}$ in the first quadrant. Let the tangent to the circle C_1, at $\mathbf{P}$ touch the other two circles C_2 and C_3 at $\mathbf{R_2}$ and $\mathbf{R}_3$, respectively. Suppose C_2 and C_3 have equal radii $2\sqrt{3}$ and the centres $\mathbf{Q}_2$ and $\mathbf{Q}_3$, respectively. If $\mathbf{Q}_2$ and $\mathbf{Q}_3$ lie on the Y axis, then

(2016)

a) $Q_2Q_3 = 12$
b) $R_2R_3 = 4\sqrt{6}$
c) area of the triangle OR_2R_3 is $6\sqrt{2}$
d) area of the triangle PQ_2Q_3 is $4\sqrt{2}$

10.7.18. Let $\mathbf{P}$ be the point on the parabola $y^2 = 4x$ which is at the shortest distance from the center S of the circle $x^2 + y^2 - 4x - 16y + 64 = 0$. Let $\mathbf{Q}$ be the point on the circle dividing the line segment SP internally. Then (2016)

a) $SP = 2\sqrt{5}$
b) $SQ : QP = (\sqrt{5} + 1) : 2$
c) the x-intercept of the normal to the parabola at $\mathbf{P}$ is 6
d) the slope of the tangent to the circle at $\mathbf{Q}$ is $\frac{1}{2}$

10.7.19. If $2x - y + 1 = 0$ is a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{16} = 1$ then which of the following cannot be sides of a right angled triangle? (2017)

a) $a, 4, 1$ b) $a, 4, 2$ c) $2a, 8, 1$ d) $2a, 4, 1$

10.7.20. Consider two straight lines, each of which is tangent to both the circle $x^2 + y^2 = \frac{1}{2}$ and the parabola $y^2 = 4x$. Let these lines intersect at the point $\mathbf{Q}$. Consider the elipse whose center is at orgin $\mathbf{O}(0, 0)$ and whose semi-major axis is OQ. If the length of the minor axis of the elipse is $\sqrt{2}$, then which of the following statement(s) is(are) TRUE? (2018)

a) For the elipse, the eccentricity is $\frac{1}{\sqrt{2}}$ and the length of the latus rectum is 1
b) For the elipse, the eccentricity is $\frac{1}{2}$ and the length of the latus rectum is $\frac{1}{2}$
c) The area of the region bounded by the elipse between the lines $x = \frac{1}{\sqrt{2}}$ and $x = 1$ is $\frac{1}{4\sqrt{2}}(\pi - 2)$
d) The area of the region bounded by the elipse between the line $x = \frac{1}{\sqrt{2}}$ and $x = 1$ is $\frac{1}{16}(\pi - 2)$

10.7.21. If $x = 9$ is the chord of contact of the hyperbola $x^2 - y^2 = 9$, then the equation of the corresponding pair of tangents is (1999)

a) $9x^2 - 8y^2 + 18x - 9 = 0$
b) $9x^2 - 8y^2 - 18x + 9 = 0$
c) $9x^2 - 8y^2 - 18x - 9 = 0$
d) $9x^2 - 8y^2 + 18x + 9 = 0$

10.7.22. The equation of the common tangent touching the circle $(x - 3)^2 - kx + 8 = 0$ and the parabola $y^2 = 4x$ above the X axis is (2000)

a) $\sqrt{3}y = 3x + 1$
b) $\sqrt{3}y = -(x + 3)$
c) $\sqrt{3}y = x + 3$
d) $\sqrt{3}y = -(3x + 1)$

10.7.23. If a > 2b > 0 then the positive value of m for which $y = mx - b\sqrt{1 + m^2}$ is a common tangent to $x^2 + y^2 = b^2$ and $(x - a)^2 + y^2 = b^2$ is (2002)

a) $\frac{2b}{\sqrt{a^2-4b^2}}$
b) $\frac{2b}{a-2b}$
c) $\frac{\sqrt{a^2-4b^2}}{2b}$
d) $\frac{b}{a-2b}$

10.7.24. The equation of the common tangent to the curves $y^2 = 8x$ and $xy = -1$ is (2002)

a) $3y = 9x + 2$
b) $y = 2x + 1$
c) $2y = x + 8$
d) $y = x + 2$

10.7.25. The area of the quadrilateral formed by the tangents at the end points of the latus rectum to the ellipse $\frac{x^2}{9} + \frac{y^2}{5} = 1$, in sq. units is (2003)

a) $\frac{27}{4}$
b) 9
c) $\frac{27}{2}$
d) 27

10.7.26. The focal chord to $y^2 = 16x$ is tangent to $(x - 6)^2 + y^2 = 2$, then the possible values of the slope of this chord, are (2003)

a) $-1, 1$
b) $-2, 2$
c) $-2, -1/2$
d) $2, -1/2$

10.7.27. If tangents are drawn to ellipse $x^2 + 2y^2 = 2$,then the locus of the mid-point of the intercept made by the tangents between the coordinate axes is (2004)

a) $\frac{1}{2x^2} + \frac{1}{4y^2}$
b) $\frac{1}{4x^2} + \frac{1}{2x^2}$
c) $\frac{x^2}{2} + \frac{y^2}{4} = 1$
d) $\frac{x^2}{4} + \frac{y^2}{2} = 1$

10.7.28. The angle between the tangents drawn from the point $(1, 4)$ to the parabola $y^2 = 4x$ is (2004)

a) $\pi/6$ b) $\pi/4$ c) $\pi/3$ d) $\pi/2$

10.7.29. If the line $2x+\sqrt{6}y=2$ touches the hyperbola $x^2-2y^2=4$, then the point of contact is (2004)

a) $\left(-2, \sqrt{6}\right)$ b) $\left(-5, 2\sqrt{6}\right)$ c) $\left(\frac{1}{2}, \frac{1}{\sqrt{6}}\right)$ d) $\left(4, -\sqrt{6}\right)$

10.7.30. The minimum area of the triangle formed by the tangent to the $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and coordinate axes in sq. units is (2005)

a) ab b) $\frac{a^2+b^2}{2}$ c) $\frac{(a+b)^2}{2}$ d) $\frac{a^2+ab+b^2}{3}$

10.7.31. Tangent to the curve $y=x^2+6$ at a point $(1,7)$ touches the circle $x^2+y^2+16x+12y+c=0$ at a point **Q**. Then the coordinates of **Q** are (2005)

a) $(-6,-11)$ b) $(-9,-13)$ c) $(-10,-15)$ d) $(-6,-7)$

10.7.32. The common tangents to the circle $x^2+y^2=2$ and the parabola $y^2=8x$ touch the circle at the points **P**, **Q** and the parabola at the points **R**, **S**. Then the area of the quadrilateral **PQRS** is (2014)

a) 3 b) 6 c) 9 d) 15

10.7.33. A hyperbola passes through point $\mathbf{P}\left(\sqrt{2}, \sqrt{2}\right)$ and has foci at $(\pm 2, 0)$. Then the tangent to this hyperbola at **P** also passes through the point (2017)

a) $\left(-\sqrt{2}, -\sqrt{3}\right)$ b) $\left(3\sqrt{2}, 2\sqrt{3}\right)$ c) $\left(2\sqrt{2}, 3\sqrt{3}\right)$ d) $\left(\sqrt{3}, \sqrt{2}\right)$

10.7.34. The radius of a circle, having minimum area, which touches the curve $y=4-x^2$ and the lines, $y=|x|$ is (2018)

a) $4\left(\sqrt{2}+1\right)$ c) $2\left(\sqrt{2}-1\right)$
b) $2\left(\sqrt{2}+1\right)$ d) $4\left(\sqrt{2}-1\right)$

10.7.35. Tangents are drawn to the hyperbola $4x^2-y^2=36$ at the points **P** and **Q**. If these tangents intersect at the point $\mathbf{T}(0,3)$ then the area (in sq.units) of $\triangle PTQ$ is (2018)

a) $54\sqrt{3}$ b) $60\sqrt{3}$ c) $36\sqrt{3}$ d) $45\sqrt{5}$

10.7.36. Tangent and normal are drawn at $\mathbf{P}(16,16)$ on the parabola $y^2=16x$, which intersect the axis of the parabola at **A** and **B**, respectively. If **C** is the centre of the circle through the points **P**, **A** and **B** and $\angle CPB=\theta$, then the value of $\tan\theta$ is (2018)

a) 2 b) 3 c) $\frac{4}{3}$ d) $\frac{1}{2}$

10.7.37. If the tangent at $(1, 7)$ to the curve $x^2 = y - 6$ touches the circle $x^2 + y^2 + 16x + 12y + c = 0$ then the value of c is (2018)

a) 185 b) 85 c) 95 d) 195

10.7.38. Equation of a common tangent to the circle $x^2 + y^2 - 6x = 0$ and the parabola $y^2 = 4x$, is (2019)

a) $2\sqrt{3}y = 12x + 1$ c) $2\sqrt{3}y = -x - 12$
b) $\sqrt{3}y = x + 3$ d) $\sqrt{3}y = 3x + 1$

10.7.39. If the line $y = mx + 7\sqrt{3}$ is normal to the hyperbola $\frac{x^2}{24} - \frac{y^2}{18}$ then a value of m is (2019)

a) $\frac{\sqrt{5}}{2}$ b) $\frac{\sqrt{15}}{2}$ c) $\frac{2}{\sqrt{5}}$ d) $\frac{3}{\sqrt{5}}$

10.7.40. Let PQ be a focal chord of the parabola $y^2 = 4ax$. The tangents to the parabola at **P** and **Q** meet at a point lying on the line $y = 2x + a$, $a > 0$.

a) Length of the chord PQ is (2013)

i) $7a$ ii) $5a$ iii) $2a$ iv) $3a$

b) If $st = 1$, then the tangent at **P** and the normal at **M** to the parabola meet at a point whose ordinate is

i) $\frac{a(t^2+1)^2}{t^3}$ ii) $\frac{a(t^2+1)}{2t^3}$ iii) $\frac{1}{t}$ iv) $\frac{t^2-1}{t}$

10.7.41. If the tangents to the ellipse at **M** and **N** meet at **R** and the normal to the parabola at **M** meets the X axis at **Q**, then the ratio of area of the triangle MQR to the area of the quadrilateral MF_1NF_2 is (2016)

a) 3 : 4 b) 4 : 5 c) 5 : 8 d) 2 : 3

10.7.42. Suppose that the normals drawn at three different points on the parabola $y^2 = 4x$ pass through the point (h, k). Show that $h > 2$. (1981)

10.7.43. **A** is a point on the parabola $y^2 = 4ax$. The normal at **A** cuts the parabola again at point **B**. If AB subtends a right angle at the vertex of the parabola. Find the slope of AB. (1982)

10.7.44. Three normals are drawn from the point $(c, 0)$ to the curve $y^2 = x$. Show that c must be greater than $\frac{1}{2}$. One normal is always the X axis. Find c for which the other two normals are perpendicular to each other. (1991)

10.7.45. Let d be the perpendicular distance from the centre of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to the tangent drawn at a point **P** on the ellipse. If $\mathbf{F_1}$ and $\mathbf{F_2}$ are the two *foci* of the ellipse, then show that $(PF_1 - PF_2)^2 = 4a^2\left(1 - \frac{b^2}{d^2}\right)$. (1995)

10.7.46. Points **A**, **B** and **C** lie on a parabola $y^2 = 4ax$. The tangents to the parabola at **A**, **B** and **C** taken in pairs, intersect at points **P**, **Q** and **R**. Determine the ratios of the areas of triangles ABC and PQR. (1996)

10.7.47. From a point **A** common tangents are drawn to the circle $x^2+y^2 = \frac{a^2}{2}$ and the parabola $y^2 = 4ax$. Find the area of the quadrilateral formed by the common tangents, the chord of contact of the circle, and the chord of contact of the parabola. (1996)

10.7.48. A tangent to the ellipse $x^2+4y^2 = 4$ meets the ellipse $x^2+2y^2 = 6$ at **P** and **Q**. Prove that the tangents at **P** and **Q** of the ellipse $x^2 + 2y^2 = 6$ are at right angles. (1997)

10.7.49. The angle between a pair of tangents drawn from a point **P** to the parabola $y^2 = 4ax$ is 45°. Show that the locus of the point **P** is a hyperbola. (1998)

10.7.50. Consider the family of circles $x^2 + y^2 = r^2$, $2 < r < 5$. If in the first quadrant, the common tangent to a circle of this family and the ellipse $4x^2 + 25y^2 = 100$ meets the coordinate axes at **A** and **B**, then find the equation of the locus of the midpoint of AB. (1999)

10.7.51. Find the coordinates of all the points **P** on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2}=1$, for which the area of the triangle PON is maximum, where **O** denotes the origin and **N**, the foot of the perpendicular from **O** to the tangent at **P**. (1999)

10.7.52. Let ABC be an equilateral triangle inscribed in the circle $x^2 + y^2 = a^2$. Suppose perpendiculars from **A**, **B**, **C** to the major axis of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2}=1$, $(a > b)$ meet the ellipse respectively at **P**, **Q**, **R** such that **P**, **Q**, **R** lie on the same side of the major axis as **A**, **B**, **C** respectively. Prove that the normals to the ellipse drawn at the points **P**, **Q**, and **R** are concurrent. (2000)

10.7.53. Let C_1 and C_2 be respectively, the parabolas $x^2 = y-1$ and $y^2 = x-1$. Let **P** be any point on C_1 and **Q** be any point on C_2. Let $\mathbf{P}_1$ and $\mathbf{Q}_1$ be the reflections of **P** and **Q** respectively with respect to the line $y = x$. Prove that $\mathbf{P}_1$ lies on C_2, $\mathbf{Q}_1$ lies on C_1, and $PQ \geq \min(PP_1, QQ_1)$. Hence or otherwise determine points $\mathbf{P}_0$ and $\mathbf{Q}_0$ on the parabolas C_1 and C_2 respectively such that $P_0Q_0 \leq PQ$ for all pairs of points $(\mathbf{P}, \mathbf{Q})$ with **P** on C_1 and **Q** on C_2. (2000)

10.7.54. Prove that, in an ellipse, the perpendicular from a focus upon any tangent and the line joining the center of the ellipse to the point of contact meet on the corresponding directrix. (2002)

10.7.55. Normals are drawn from the point **P** with slopes m_1, m_2, m_3 to the parabola $y^2 = 4x$. If the locus of **P** with $m_1m_2 = \alpha$ is a part of the parabola itself, then find α. (2003)

10.7.56. A tangent is drawn to the parabola $y^2 - 2y - 4x + 5 = 0$ at a point **P** which cuts the directrix at the point **Q**. A point **R** is such that it divides QP externally in the ratio 1:2. Find the locus of the point **R**. (2004)

10.7.57. Tangents are drawn from any point on the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ to the circle $x^2 + y^2 = 9$. Find the locus of the midpoint of the chord of contact. (2005)

10.7.58. Find the equation of the common tangent in the 1st quadrant to the circle $x^2+y^2 = 16$ and the ellipse $\frac{x^2}{25}+\frac{y^2}{4}=1$. Also, find the length of the intercept of the tangent between the coordinate axes. (2005)

10.7.59. Let the point **B** be the reflection of the point **A** $(2, 3)$ with respect to the line $8x-6y-23 = 0$. Let T_A and T_B be circles of radii 2 and 1 with centres **A** and **B** respectively. Let T be a common tangent to the circles T_A and T_B such that both the circles are on the same side of T. If **C** is the point of intersection of T and the line passing through **A** and **B**, then the length of the line segment AC is ______. (2019)

10.7.60. Let the circles $C_1 : x^2 + y^2 = 9$ and $C_2 : (x-3)^2 + (y-4)^2 = 16$, intersect at the points **X** and **Y**. Suppose that another circle $C_3 : (x-h)^2 + (y-k)^2 = r^2$ satisfies the following conditions

a) Centre of C_3 is collinear with the centres of C_1 and C_2
b) C_1 and C_2 both lie inside C_3, and
c) C_3 touches C_1 at **M** and C_2 at **N**

Let the line through **X** and **Y** intersect C_3 at **Z** and **W**, and let a common tangent of C_1 and C_3 be a tangent to the parabola $x^2 = 8\alpha y$. Match the expressions given in the List-1 with those in List-2 below

Column 1	**Column 2**
(A) $2h + k$	(p) 6
(B) $\frac{\text{Length of } ZW}{\text{length of } XY}$	(q) $\sqrt{6}$
(C) $\frac{\text{Area of triangle } MZN}{\text{Area of triangle } ZMW}$	(r) $\frac{5}{4}$
	(s) $\frac{21}{5}$
	(t) $2\sqrt{6}$
	(u) $\frac{10}{3}$

10.7.61. Let RS be the diameter of the Circle $x^2 + y^2 = 1$, where **S** is the point $(1, 0)$. Let **P** be a variable point (other than **R** and **S**) on the circle and tangents to the circle at **S** and **P** meet at the point **Q**. The normal to the circle at **P** intersects a line drawn through **Q** parallel to RS at point **E**. Then the locus of **E** passes through the point(s) (2016)

a) $\left(\frac{1}{3}, \frac{1}{\sqrt{3}}\right)$ b) $\left(\frac{1}{4}, \frac{1}{2}\right)$ c) $\left(\frac{1}{3}, -\frac{1}{\sqrt{3}}\right)$ d) $\left(\frac{1}{4}, -\frac{1}{2}\right)$

10.7.62. Let T be a line passing through the points $\mathbf{P}(-2, 7)$ and $\mathbf{Q}(2, -5)$. Let F_1 be the set of all pairs of circles (S_1, S_2) such that T is tangent to S_1 at **P** and tangent to S_2 at **Q**, and also such that S_1 and S_2 touch each other at a point, say **M**. Let E_1 be the set representing the locus of **M** as the pair (S_1, S_2) varies in F_1. Let the set of all straight line segments joining a pair of distinct points of E_1 and passing through the point $\mathbf{R}(1, 1)$ be F_2. Then which of the following statements is (are) TRUE? (2018)

a) The point $(-2, 7)$ lies on E_1
b) The point $\left(\frac{4}{5}, \frac{7}{5}\right)$ does **NOT** lie on E_1
c) The point $\left(\frac{1}{3}, 1\right)$ lies on E_1
d) The point $\left(0, \frac{3}{2}\right)$ does not lie on E_1

10.7.63. Let PQ and RS be tangents at the extremities of the diameter PR of a circle of radius r. If PS and RQ intersect at a point **X** on the circumference of the circle, then $2r$ equals (2001S)

a) $\sqrt{PQ.RS}$ b) $(PQ + RS)$ c) $\frac{2PQ.RS}{PQ+RS}$ d) $\frac{\sqrt{(PQ^2+RS^2)}}{2}$

10.7.64. If the tangent at the point **P** on the circle $x^2 + y^2 + 6x + 6y = 2$ meets a straight line $5x - 2y + 6 = 0$ at a point on the Y axis, then the length of PQ is (2002S)

a) 4 b) $2\sqrt{5}$ c) 5 d) $3\sqrt{5}$

10.7.65. Circles with radii 3, 4 and 5 touch each other externally. If **P** is the point of intersection of tangents to these circles at their points of contact, find the distance of **P** from the points of contact. (2005)

10.7.66. No tangent can be drawn from point (5/2, 1) to circumcircle of triangle with vertices $\left(1, \sqrt{3}\right)$, $\left(1, -\sqrt{3}\right)$, and $\left(3, -\sqrt{3}\right)$. (1985)

10.7.67. Tangents are drawn from point (17, 7) to the circle $x^2 + y^2 = 169$.
STATEMENT-1:The tangents are mutually perpendicular.
STATEMENT-2:The locus of all points from which mutually perpendicular tangents can be drawn to a given circle is $x^2 + y^2 = 338$. (2007)

a) Statement-1 is True, statement-2 is True; Statement-2 is a correct explantion for Statement-1.
b) Statement-1 is True, statement-2 is True; Statement-2 is NOT a correct explantion for Statement-1.
c) Statement-1 is True, Statement-2 is False
d) Statement-1 is False, Statement-2 is True.

10.7.68. The lines $3x - 4y + 4 = 0$ and $6x - 8y - 7 = 0$ are tangents to the same circle. The radius of the circle is ______. (1984)

10.7.69. Let $x^2 + y^2 - 4x - 2y - 11 = 0$ be a circle. A pair of tangents from the point (4, 5) with a pair of radii form a quadrilateral of area ______. (1985)

10.7.70. The area of the triangle formed by the tangents from the point (4, 3) to the circle $x^2 + y^2 = 9$ and the line joining their point of contact is ______. (1987)

10.7.71. The area formed by the positive X axis and the normal and the tangent to the circle $x^2 + y^2 = 4$ at $\left(1, \sqrt{3}\right)$ is ______. (1989)

10.7.72. The chords of contact of the pair of tangents drawn from each point on the line $2x + y = 4$ to $x^2 + y^2 = 1$ pass through the point ______. (1997)

10.7.73. The centres of two circles C_1 and C_2 each of unit radius are at a distance of 6 units from each other. Let **P** be the midpoint of the line segment joining the centres of C_1 and C_2 and C be a circle touching circles C_1 and C_2 externally. If a common tangent to C_1 and C passing through **P** is also a common tangent to C_2 and C, then the radius of circle C is ______. (2009)

10.7.74. Consider a family of circles which are passing through the point (−1, 1), and are tangent to X axis. If (h, k) are the coordinate of the centre of the circles, then the set of values of k is given by the interval (2007)

a) $\frac{-1}{2} \le k \le \frac{1}{2}$ b) $k \le \frac{1}{2}$ c) $o \le k \le \frac{1}{2}$ d) $k \ge \frac{1}{2}$

10.7.75. The equations of the tangents drawn from the origin to the circle $x^2 + y^2 - 2rx - 2hy + h^2 = 0$, are (1988)

a) $x = 0$
b) $y = 0$
c) $(h^2 - r^2)x - 2rhy = 0$
d) $(h^2 - r^2)x + 2rhy = 0$

10.7.76. The number of common tangents to the circles $x^2 + y^2 = 4$ and $x^2 + y^2 - 6x - 8y = 24$ is (1998)

a) 0 b) 1 c) 3 d) 4

10.7.77. The angle between the pair of tangents drawn from the point **P** to the circle $x^2 + y^2 + 4x - 6y + 9\sin^2\alpha + 13\cos^2\alpha = 0$ is 2α. The equation of the locus of the point **P** is (1996)

a) $x^2 + y^2 + 4x - 6y + 4 = 0$
b) $x^2 + y^2 + 4x - 6y - 9 = 0$
c) $x^2 + y^2 + 4x - 6y - 4 = 0$
d) $x^2 + y^2 + 4x - 6y + 9 = 0$

10.7.78. Tangents drawn from the point $\mathbf{P}(1, 8)$ to the circle $x^2 + y^2 - 6x - 4y - 11 = 0$ touch the circle at the points **A** and **B**. The equation of the circumcircle of the triangle PAB is (2009)

a) $x^2 + y^2 + 4x - 6y + 19 = 0$
b) $x^2 + y^2 - 4x - 10y + 19 = 0$
c) $x^2 + y^2 - 4x + 6y - 29 = 0$
d) $x^2 + y^2 - 4x - 6y + 19 = 0$

10.7.79. The locus of the mid-point of the chord of contact of tangents drawn from points lying on the straight line $4x - 5y = 20$ to the circle $x^2 + y^2 = 9$ is (2012)

a) $20\left(x^2 + y^2\right) - 36x + 45y = 0$
b) $20\left(x^2 + y^2\right) + 36x - 45y = 0$
c) $36\left(x^2 + y^2\right) - 20x + 45y = 0$
d) $36\left(x^2 + y^2\right) + 20x - 45y = 0$

10.7.80. Let **A** be the centre of circle $x^2 + y^2 - 2x - 4y - 20 = 0$. Suppose that the tangents at the points $\mathbf{B}(1, 7)$ and $\mathbf{D}(4, -2)$ on the circle meet at point **C**. Find the area of the quadrilateral $ABCD$. (1981)

10.7.81. Two circles, each of radius 5 units, touch each other at $(1, 2)$. If the equation of common tangent is $4x + 3y = 10$, find the equations of circles. (1991)

10.7.82. Find the coordinates of the point at which the circles $x^2 + y^2 - 4x - 2y = -4$ and $x^2 + y^2 - 12x - 8y = -36$ touch each other. Also find equations of the common tangents touching the circles in the distinct points. (1993)

10.7.83. Let T_1, T_2 be two tangents drawn from $(2, 0)$ onto the circle $C : x^2 + y^2 = 1$. Determine the circles touching C and having T_1, T_2 as their pair of tangents. Further, find the equations of all possible common tangents to these circles, when taken two at a time. (1999)

10.7.84. Let $2x^2 + y^2 - 3xy = 0$ be the equation of pair of tangents drawn from the origin **O** to a circle of radius 3 with the centre in the first quadrant. If **A** is one of the points of contact, find the length of OA. (2001)

10.7.85. For the circle $x^2 + y^2 = r^2$, find the value of r for which the area enclosed by the tangents drawn from the point $\mathbf{P}(6, 8)$ to the circle and the chord of contact is maximum. (2003)

10.7.86. Let C_1 and C_2 be two circles with C_2 lying inside C_1. A circle C lying inside C_1 touches C_1 internally and C_2 externally. Identify the locus of centre of C. (2001)

10.7.87. Find the equation of circle touching the line $2x + 3y + 1 = 0$ at $(1, -1)$ and cutting orthogonally the circle having line segment joining $(0, 3)$ and $(-2, -1)$ as diameter. (2004)

10.7.88. Find the equations of the circle passing through $(-4, 3)$ and touching the lines $x+y = 2$ and $x - y = 2$ (1981)

10.7.89. Lines $5x + 12y - 10 = 0$ and $5x - 12y - 40 = 0$ touch a Circle C_1 of diameter 6. If the centre of C_1 lies in the first quadrant, find the equation of circle C_2 which is concentric with C_1 and cuts intercepts of length 8 on these lines. (1986)

10.7.90. A circle touches the line $y = x$ at a point **P** such that $OP = 4\sqrt{2}$, where **O** is the origin. The circle contains the point $(-10, 2)$ in its interior and the length of its chord on the line $x + y = 0$ is $6\sqrt{2}$. Determine the equation of circle. (1990)

10.7.91. The point of intersection of the tangents at the ends of the latus rectum of the parabola $y^2 = 4x$ is ______. (1994)

10.7.92. An ellipse has eccentricity $\frac{1}{2}$ and one focus at the point $\mathbf{P}\left(\frac{1}{2}, 1\right)$.Its one directrix is common tangent,nearer to the point **P**, to the circle $x^2 + y^2 = 1$ and the hyperbola $x^2 - y^2 = 1$.The equation of the ellipse, in the standard form, is ______. (1992)

10.7.93. Consider the parabola $y^2 = 8x$. Let Δ_1 be the area of the triangle formed by the end points of its latus rectum and the point $\mathbf{P}(\frac{1}{2}, 2)$ on the parabola and Δ_2 be the area of the triangle formed by drawing tangents at **P** and at the end points of the latus rectum. Then $\frac{\Delta_1}{\Delta_2}$ is (2011)

10.7.94. A circle touches the X axis and also touches the circle with centre at $(0, 3)$ and radius 2. The locus of the centre of the circle is (2005)

a) an ellipse   b) a circle   c) a hyperbola   d) a parabola

10.7.95. Circle(s) touching the X axis at a distance 3 from the origin and having an intercept of length $2\sqrt{7}$ on the Y axis is (are) (2013)

a) $x^2 + y^2 - 6x + 8y + 9 = 0$
b) $x^2 + y^2 - 6x + 7y + 9 = 0$
c) $x^2 + y^2 - 6x - 8y + 9 = 0$
d) $x^2 + y^2 - 6x - 7y + 9 = 0$

10.7.96. The number of the values of c such that the straight line $y = 4x + c$ touches the curves $(x^2/4) + y^2 = 1$ is (1998)

a) 0   b) 1   c) 2   d) infinite

10.7.97. Let $\mathbf{P}(a \sec\theta, b\tan\theta)$ and $\mathbf{Q}(a\sec\phi, b\tan\phi)$,where $\theta+\phi = \pi/2$, be two points on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If (h, k) is the point of intersection of the normals at **P** and **Q**, then k equal to (1999)

a) $\frac{a^2+b^2}{a}$ b) $-\left(\frac{a^2+b^2}{a}\right)$ c) $\frac{a^2+b^2}{b}$ d) $-\left(\frac{a^2+b^2}{b}\right)$

10.7.98. The normal at a point **P** on the ellipse $x^2 + 4y^2 = 16$ meets the x-axis at **Q**. If **M** is the mid point of the line segment PQ, then the locus of **M** interests the latus rectums of the given ellipse at the points (2009)

a) $\left(\pm\frac{3\sqrt{5}}{2}, \pm\frac{2}{7}\right)$ b) $\left(\pm\frac{3\sqrt{5}}{2}, \pm\sqrt{\frac{19}{4}}\right)$ c) $\left(\pm 2\sqrt{3}, \pm\frac{1}{7}\right)$ d) $\left(\pm 2\sqrt{3}, \pm\frac{4\sqrt{3}}{7}\right)$

10.7.99. Let $\mathbf{P}(6,3)$ be a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If the normal at the point **P** intersects the x-axis at $(9,0)$, then the eccentricity of the hyperbola is (2011)

a) $\sqrt{\frac{5}{2}}$ b) $\sqrt{\frac{3}{2}}$ c) $\sqrt{2}$ d) $\sqrt{3}$

10.7.100. The normal to the curve $x^2 + 2xy - 3y^2 = 0$ at $(0,1)$ (2015)

a) meets the curve again in the third quadrant.
b) meets the curve again in the fourth quadrant.
c) doesn't meet the curve again.
d) meets the curve again in the second quadrant.

10.7.101. $(3,0)$ is the point from which three normals are drawn to the parabola $y^2 = 4x$ which meet the parabola in the points **P**, **Q** and **R**. Match the following (2006)

Column I	**Column II**
a) Area of ΔPOR	a) 2
b) Radius of circumcircle of ΔPQR	b) $\frac{5}{2}$
c) Centroid of ΔPOR	c) $(\frac{5}{2}, 0)$
d) Circumcentre of ΔPQR	d) $(\frac{2}{3}, 0)$

10.7.102. Let L be a normal to the parabola $y^2 = 4x$. If L passes through the point $(9,6)$, then L is given by (2010)

a) $y - x + 3 = 0$ b) $y + 3x - 33 = 0$ c) $y + x - 15 = 0$ d) $y - 2x + 12 = 0$

10.7.103. If $x + y = k$ is normal $y^2 = 12x$,then k is (2000)

a) 3 b) 9 c) −9 d) −3

10.7.104. Let a, b and λ be positive real numbers. Suppose P is end point of the latus rectum of the parabola $y^2 = 4\lambda x$, and suppose the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ passes through the point P, If the tangents to the parabola and the ellipse at the point P are perpendicular to each other, then the eccentricity of the ellipse is: (2020)

a) $\frac{1}{\sqrt{2}}$ b) $\frac{1}{2}$ c) $\frac{1}{3}$ d) $\frac{2}{5}$

10.7.105. Consider the parabola $y^2 = 4x$. Let **S** be the focus of the parabola. A pair of tangents drawn to the parabola from the point $\mathbf{P} = (-2, 1)$ meet the parabola at $\mathbf{P}_1$ and $\mathbf{P}_2$. Let $\mathbf{Q}_1$ and $\mathbf{Q}_2$ be points on the lines SP_1 and SP_2 respectively such that PQ_1 is perpendicular to SP_1 and PQ_2 is perpendicular to SP_2. Then, which of the following is/are TRUE ? (2022)

a) $SQ_1 = 2$ b) $Q_1Q_2 = \frac{3\sqrt{10}}{5}$ c) $PQ_1 = 3$ d) $SQ_2 = 1$

10.7.106. Consider the hyperbola

$$\frac{x^2}{100} - \frac{y^2}{64} = 1$$

with foci at **S** and $\mathbf{S}_1$, where **S** lies on the positive x-axis. Let **P** be a point on the hyperbola in the first quadrant. Let $\angle SPS_1 = \alpha$, with $\alpha < \frac{\pi}{2}$. The straight line passing through the point **S** and having the same slope as that of the tangent at **P** to the hyperbola intersects the straight line S_1P at $\mathbf{P}_1$. Let δ be the distance of **P** from the straight line SP_1, and let $\beta = S_1P$. Then the greatest integer less than or equal to

$$\frac{\beta\delta}{9\sin\frac{\alpha}{2}}$$

is ______. (2022)

10.7.107. Let T_1 and T_2 be two distinct common tangents to the ellipse

$$E : \frac{x^2}{6} + \frac{y^2}{3} = 1$$

and the parabola

$$P : y = \frac{x^2}{12}.$$

Suppose that the tangent T_1 touches P and E at the points $\mathbf{A}_1$ and $\mathbf{A}_2$, respectively, and the tangent T_2 touches P and E at the points $\mathbf{A}_4$ and $\mathbf{A}_3$, respectively. Then which of the following statements is(are) true? (2023)

a) The area of the quadrilateral $A_1A_2A_3A_4$ is 35 square units.
b) The area of the quadrilateral $A_1A_2A_3A_4$ is 36 square units.
c) The tangents T_1 and T_2 meet the X axis at the point $(-3, 0)$.
d) The tangents T_1 and T_2 meet the X axis at the point $(6, 0)$.

10.7.108. Let **P** be a point on the parabola $y = ax^2$, where $a > 0$. The normal to the parabola at **P** meets the X axis at a point **Q**. The area of the triangle PFQ, where **F** is the focus of the parabola, is 120. If the slope m of the normal and a are both positive integers, then the pair (a, m) is

a) $(2,3)$ b) $(1,3)$ c) $(2,4)$ d) $(3,4)$

(2023)

10.7.109. Let C_1 be the circle of radius 1 with center at the origin. Let C_2 be the circle of radius r with center at the point $\mathbf{A} = (4,1)$, where $1 < r < 3$. Two distinct common tangents PQ and ST of C_1 and C_2 are drawn. The tangent PQ touches C_1 at $\mathbf{P}$ and C_2 at $\mathbf{Q}$. The tangent ST touches C_1 at $\mathbf{S}$ and C_2 at $\mathbf{T}$. Midpoints of the line segments PQ and ST are joined to form a line which meets the X axis at a point $\mathbf{B}$. If $AB = \sqrt{5}$, then the value of r^2 is ______. (2023)

10.7.110. Consider the ellipse

$$\frac{x^2}{9} + \frac{y^2}{4} = 1.$$

Let $\mathbf{S}(p,q)$ be a point in the first quadrant such that

$$\frac{p^2}{9} + \frac{q^2}{4} > 1.$$

Two tangents are drawn from $\mathbf{S}$ to the ellipse, of which one meets the ellipse at one end point of the minor axis and the other meets the ellipse at a point $\mathbf{T}$ in the fourth quadrant. Let $\mathbf{R}$ be the vertex of the ellipse with positive x-coordinate and $\mathbf{O}$ be the center of the ellipse. If the area of the triangle $\triangle ORT$ is $\frac{3}{2}$, then which of the following options is correct? (2024)

a) $q = 2, p = 3\sqrt{3}$ b) $q = 2, p = 4\sqrt{3}$ c) $q = 1, p = 5\sqrt{3}$ d) $q = 1, p = 6\sqrt{3}$

10.7.111. Let the straight line $y = 2x$ touch a circle with center $(0,a)$, $a > 0$, and radius r at a point $\mathbf{A}_1$. Let $\mathbf{B}_1$ be the point on the circle such that the line segment $\mathbf{A}_1\mathbf{B}_1$ is a diameter of the circle. Let $a + r = 5 + \sqrt{5}$. Match the following

(A) a equals	(1) $(-2,4)$
(B) r equals	(2) $\sqrt{5}$
(C) $\mathbf{A}_1$ equals	(3) $(-2,6)$
(D) $\mathbf{B}_1$ equals	(4) 5
	(5) $(2,4)$

The correct option is (2024)

a) $(A) \to (4)$ $(B) \to (2)$ $(C) \to (1)$ $(D) \to (3)$
b) $(A) \to (2)$ $(B) \to (4)$ $(C) \to (1)$ $(D) \to (3)$
c) $(A) \to (4)$ $(B) \to (2)$ $(C) \to (5)$ $(D) \to (3)$
d) $(A) \to (2)$ $(B) \to (4)$ $(C) \to (3)$ $(D) \to (5)$

11 Vector Algebra

11.1 Formulae

11.1.1. If $\mathbf{D}$ divides BC in the ratio $k : 1$,

$$\mathbf{D} = \frac{k\mathbf{C} + \mathbf{B}}{k+1} \tag{11.1.1.1}$$

11.1.2. Points $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ form a triangle if

$$p(\mathbf{A} - \mathbf{B}) + q(\mathbf{C} - \mathbf{B}) \tag{11.1.2.1}$$

$$= (p+q)\mathbf{B} - p\mathbf{A} - q\mathbf{C} = 0 \tag{11.1.2.2}$$

$$\implies p = 0, q = 0 \tag{11.1.2.3}$$

11.2 Examples

11.2.1. In Fig. 11.2.1.1, $AD \perp BC$ and $BE \perp AC$ are defined to be the altitudes of $\triangle ABC$. Let $\mathbf{H}$ be the intersection of the altitudes AD and BE as shown in Fig. 11.2.1.1. CH is extended to meet AB at $\mathbf{F}$. Show that $CF \perp AB$.

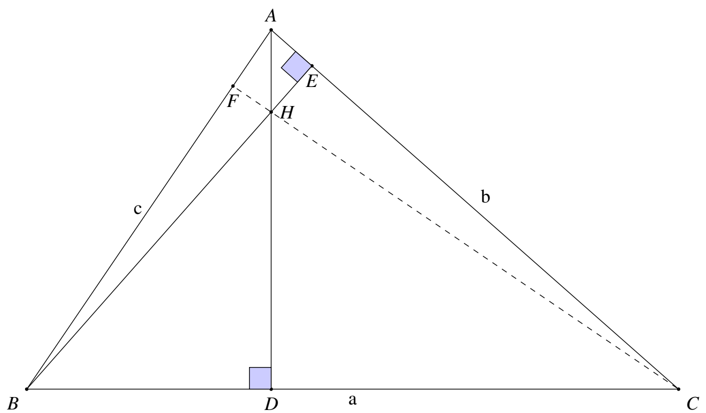

Fig. 11.2.1.1: Altitudes of a triangle meet at the orthocentre H

Solution: $\because AD \perp BC, BE \perp AC$,

$$(\mathbf{B} - \mathbf{C})^\top (\mathbf{H} - \mathbf{A}) = 0 \tag{11.2.1.1}$$

$$(\mathbf{C} - \mathbf{A})^\top (\mathbf{H} - \mathbf{B}) = 0 \tag{11.2.1.2}$$

Adding both the above and simplifying,

$$(\mathbf{B} - \mathbf{A})^\top (\mathbf{H} - \mathbf{C}) = 0 \tag{11.2.1.3}$$

$\implies CH \perp AB$, or $CF \perp AB$.

11.2.2. In Fig. 11.2.2.1 If $\mathbf{G}$ divides BE and CF in the ratios k_1 and k_2 respectively, show that

$$k_1 = k_2 = 2 \tag{11.2.2.1}$$

Solution: Using (11.1.1.1)

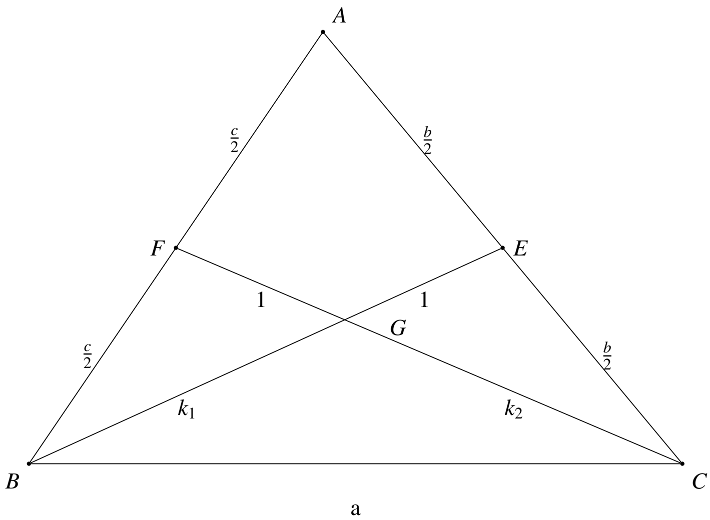

Fig. 11.2.2.1: $k_1 = k_2$.

$$\mathbf{E} = \left(\frac{\mathbf{A} + \mathbf{C}}{2}\right), \mathbf{F} = \left(\frac{\mathbf{A} + \mathbf{B}}{2}\right) \tag{11.2.2.2}$$

and

$$\mathbf{G} = \frac{k_1\mathbf{E} + \mathbf{B}}{k_1 + 1} = \frac{k_2\mathbf{F} + \mathbf{C}}{k_2 + 1} \tag{11.2.2.3}$$

$$\implies (k_2 + 1)\left[k_1\left(\frac{\mathbf{A} + \mathbf{C}}{2}\right) + \mathbf{B}\right] = (k_1 + 1)\left[k_2\left(\frac{\mathbf{A} + \mathbf{B}}{2}\right) + \mathbf{C}\right] \tag{11.2.2.4}$$

yielding

$$\left[k_2 + 1 - \frac{k_2(k_1 + 1)}{2}\right]\mathbf{B} - \left[\frac{k_2(k_1 + 1)}{2} - \frac{k_1(k_2 + 1)}{2}\right]\mathbf{A} - \left[k_1 + 1 - \frac{k_1(k_2 + 1)}{2}\right]\mathbf{C} = 0 \tag{11.2.2.5}$$

Comparing the above with (11.1.2.3), we obtain the equations

$$k_2 + 1 - \frac{k_2(k_1 + 1)}{2} = 0 \tag{11.2.2.6}$$

$$\frac{k_2(k_1 + 1)}{2} - \frac{k_1(k_2 + 1)}{2} = 0 \tag{11.2.2.7}$$

$$k_1 + 1 - \frac{k_1(k_2 + 1)}{2} = 0 \tag{11.2.2.8}$$

yielding

$$k_1 = k_2,\ k_1^2 - k_1 - 2 = 0 \tag{11.2.2.9}$$

$$\text{or,}\ (k_1 - 2)(k_1 + 1) = 0 \tag{11.2.2.10}$$

resulting in (11.2.2.1).

11.2.3. In Fig. 11.2.3.1, AG is extended to join BC at $\mathbf{D}$. Show that AD is also a median.

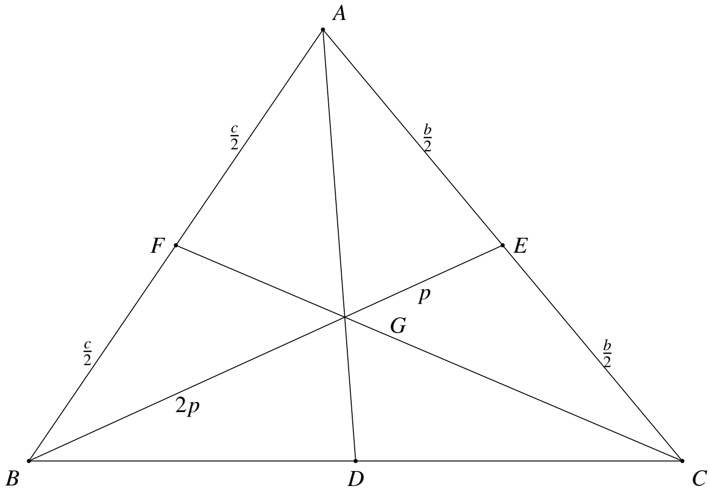

Fig. 11.2.3.1: $k_3 = 2, k_4 = 1$

Solution: Substituting $k_1 = k_2 = 2$ in (11.2.2.4),

$$\mathbf{G} = \frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} \tag{11.2.3.1}$$

Considering the ratios in Fig. 11.2.3.1,

$$\mathbf{G} = \frac{k_3\mathbf{D} + \mathbf{A}}{k_3 + 1},\ \mathbf{D} = \frac{k_4\mathbf{C} + \mathbf{B}}{k_4 + 1} \tag{11.2.3.2}$$

Substituting from (11.2.3.1) in the above,

$$(k_3 + 1)\left(\frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3}\right) = k_3\left(\frac{k_4\mathbf{C} + \mathbf{B}}{k_4 + 1}\right) + \mathbf{A} \tag{11.2.3.3}$$

which can be expressed as

$$(k_3 + 1)(k_4 + 1)(\mathbf{A} + \mathbf{B} + \mathbf{C}) = 3\left\{k_3(k_4\mathbf{C} + \mathbf{B}) + (k_4 + 1)\mathbf{A}\right\} \tag{11.2.3.4}$$

yielding

$$(k_3k_4 + k_3 - 2k_4 - 2)\mathbf{A} - (-k_3k_4 - k_4 + 2k_3 - 1)\mathbf{B} - (-k_3 - k_4 - 1 + 2k_3k_4)\mathbf{C} = \mathbf{0} \tag{11.2.3.5}$$

Comparing the above with (11.1.2.3),

$$p = -k_3k_4 - k_4 + 2k_3 - 1, \; q = -k_3 - k_4 - 1 + 2k_3k_4 \tag{11.2.3.6}$$

yielding

$$-k_3k_4 - k_4 + 2k_3 - 1 = 0 \tag{11.2.3.7}$$
$$-k_3 - k_4 - 1 + 2k_3k_4 = 0 \tag{11.2.3.8}$$

Subtracting (11.2.3.7) from (11.2.3.8),

$$3k_3 (k_4 - 1) = 0 \implies k_4 = 1 \tag{11.2.3.9}$$

which upon substituting in (11.2.3.7) yields

$$k_3 = 2 \tag{11.2.3.10}$$

11.2.4. In Fig. 11.2.4.1, OC is the radius and PC touches the circle at $\mathbf{C}$. Show that

$$OC \perp PC. \tag{11.2.4.1}$$

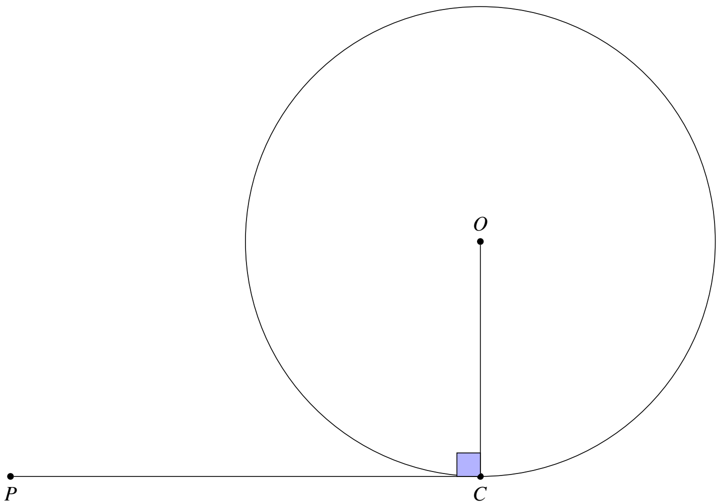

Fig. 11.2.4.1

Solution: The equation of PC can be expressed as

$$\mathbf{x} = \mathbf{C} + \mu\mathbf{m} \tag{11.2.4.2}$$

and the equation of the circle is

$$\|\mathbf{x} - \mathbf{O}\| = R \tag{11.2.4.3}$$

Substituting (11.2.4.2) in (11.2.4.3),

$$\|\mathbf{C} + \mu\mathbf{m} - \mathbf{O}\|^2 = R^2 \tag{11.2.4.4}$$

$$\implies \mu^2 \|\mathbf{m}\|^2 + 2\mu\mathbf{m}^\top (\mathbf{C} - \mathbf{O}) + \|\mathbf{C} - \mathbf{O}\|^2 - R^2 = 0 \tag{11.2.4.5}$$

The above equation has only one root. Hence the discriminant of the above quadratic should be zero. So,

$$\left\{\mathbf{m}^\top (\mathbf{C} - \mathbf{O})\right\}^2 - \|\mathbf{m}\|^2 \left\{\|\mathbf{C} - \mathbf{O}\|^2 - R^2\right\} = 0 \tag{11.2.4.6}$$

Since $\mathbf{C}$ is a point on the circle,

$$\|\mathbf{C} - \mathbf{O}\|^2 - R^2 = 0 \tag{11.2.4.7}$$

$$\implies \mathbf{m}^\top (\mathbf{C} - \mathbf{O}) = 0 \tag{11.2.4.8}$$

upon substituting in (11.2.4.6). Using the definition of the direction vector from (1.1.1.1)

$$\mathbf{m} = \mathbf{P} - \mathbf{C} \tag{11.2.4.9}$$

$$\implies (\mathbf{P} - \mathbf{C})^\top (\mathbf{C} - \mathbf{O}) = 0 \tag{11.2.4.10}$$

which is equivalent to (11.2.4.1).

11.2.5. In $\triangle ABC, \mathbf{D}, \mathbf{E}$ and $\mathbf{F}$ are respectively the mid-points of sides AB, BC and CA. Show that $\triangle ABC$ is divided into four congruent triangles by joining $\mathbf{D}, \mathbf{E}$ and $\mathbf{F}$.

11.2.6. The line-segment joining the mid-points of any two sides of a triangle is parallel to the third side and is half of it.

11.2.7. A line through the mid-point of a side of a triangle parallel to another side bisects the third side.

11.2.8. ABC is a triangle right angled at $\mathbf{C}$. A line through the mid-point M of hypotenuse AB and parallel to BC intersects AC at $\mathbf{D}$. Show that (i) $\mathbf{D}$ is the mid-point of AC (ii) $MD \perp AC$ (iii) $CM = MA = \frac{1}{2}AB$

11.2.9. AB is a line-segment. $\mathbf{P}$ and $\mathbf{Q}$ are points on opposite sides of AB such that each of them is equidistant from the points $\mathbf{A}$ and $\mathbf{B}$. Show that the line PQ is the perpendicular bisector of AB.

11.2.10. AB is a line segment and line l is its perpendicular bisector. If a point $\mathbf{P}$ lies on l, show that $\mathbf{P}$ is equidistant from $\mathbf{A}$ and $\mathbf{B}$.

11.2.11. $ABCD$ is a trapezium with $AB \parallel DC$. $\mathbf{E}$ and $\mathbf{F}$ are points on non-parallel sides AD and BC respectively such that EF is parallel to AB . Show that $\frac{AE}{ED} = \frac{BF}{FC}$.

11.2.12. $\mathbf{O}$ is a point in the interior of $\triangle ABC$. $\mathbf{D}$ is a point on OA. If $DE \parallel OB$ and $DF \parallel OC$. Show that $EF \parallel BC$.

11.2.13. $\mathbf{O}$ is a point in the interior of $\triangle PQR$. $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ are points on OP, OQ and OR respectively such that $AB \parallel PQ$ and $AC \parallel PR$. Show that $BC \parallel QR$.

11.2.14. The diagonals of a quadrilateral $ABCD$ intersect each other at the point $\mathbf{O}$ such that $\frac{AO}{BO} = \frac{CO}{DO}$. Show that $ABCD$ is a trapezium.

11.2.15. In parallelogram $ABCD$, two points $\mathbf{P}$ and $\mathbf{Q}$ are taken on diagonal BD such that $DP = BQ$. show that

a) $\triangle APD \cong \triangle CQB$

b) $AP = CQ$
c) $\triangle AQB \cong \triangle CPD$
d) $AQ = CP$
e) $APCQ$ is a parallelogram

11.2.16. $ABCD$ is a parallelogram and AP and CQ are perpendiculars from vertices **A** and **C** on diagonal BD. Show that
a) $\triangle APB \cong \triangle CQD$
b) $AP = CQ$

11.2.17. $ABCD$ is a trapezium in which $AB \parallel DC, BD$ is a diagonal and **E** is the mid-point of AD. A line is drawn through **E** $\parallel AB$ intersecting BC at **F**. Show that **F** is the mid-point of BC.

11.2.18. In a parallelogram $ABCD$, **E** and **F** are the mid-points of sides AB and CD respectively . Show that the line segments AF and EC trisect the diagonal BD.

11.2.19. Show that the line segments joining the mid-points of the opposite sides of a quadrilateral bisect each other.

11.2.20. $ABCD$ is a parallelogram in which **P** and **Q** are mid-points of opposite sides AB and CD. If AQ intersects DP at **S** and BQ intersects CP at **R**, show that:
a) $APCQ$ is a parallelogram.
b) $DPBQ$ is a parallelogram.
c) $PSQR$ is a parallelogram.

11.2.21. Two circles intersect at two points **A** and **B**. AD and AC are diameters to the two circles. Prove that **B** lies on the line segment DC.

11.2.22. There is one and only one circle passing through three non-collinear points.

11.2.23. If a line intersects two concentric circles (circles with the same centre) with centre **O** at **A**, **B**, **C** and **D**, prove that $AB = CD$.

11.2.24. If diagonals of a cyclic quadrilateral are diameters of the circle through the vertices of the quadrilateral, prove that it is a rectangle.

11.2.25. If circles are drawn taking two sides of a triangle as diameters, prove that the point of intersection of these circles lie on the third side.

11.3 JEE

11.3.1. Let **O** be the origin and let PQR be an arbitrary triangle. The point **S** is such that $OP \cdot OQ + OR \cdot OS = OR \cdot OP + OQ \cdot OS = OQ \cdot OR + OP \cdot OS$. Then the triangle PQR has **S** as its (2017)
a) Centroid
b) Circumcentre
c) Incentre
d) Orthocenter

11.3.2. From a point **O** inside the $\triangle ABC$, perpendiculars OD, OE, OF are drawn to the sides BC, CA, AB respectively. Prove that the perpendiculars from **A**, **B**, **C** to the sides EF, FD, DE are concurrent. (1978)

11.3.3. $A_1, A_2, \ldots A_n$ are the vertices of a regular plane polygon with n sides and **O** is its centre. Show that

$$\sum_{i=1}^{n-1} (OA_i \times OA_{i+1}) = (1-n)(OA_2 \times OA_1)$$

(1982)

11.3.4. If $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ are any four points in space, prove that

$$|AB \times CD + BC \times AD + CA \times BD| = 4\text{ar}(\triangle ABC)$$

(1987)

11.3.5. Tangent at a point $\mathbf{P}_1$ (other than $(0,0)$) on the curve $y = x^3$ meets the curve again at $\mathbf{P}_2$. The tangent at $\mathbf{P}_2$ meets the curve again at $\mathbf{P}_1$, and so on. Show that the abscissae of $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3 \ldots \mathbf{P}_n$, form a G.P. Also find the ratio

$$\frac{ar(\triangle P_1 P_2 P_3)}{ar(\triangle P_2 P_3 P_4)}$$

(1993)

11.3.6. Using co-ordinate geometry, prove that the three altitudes of any triangle are concurrent. (1998)

11.3.7. C_1 and C_2 are two concentric circles, the radius of C_2 being twice that of C_1. From a point **P** on C_2, tangents PA and PB are drawn to C_1. Prove that the centroid of the triangle PAB lies on C_1. (1998)

Appendix A
Triangle

Consider a triangle with vertices

$$\mathbf{A} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -4 \\ 6 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -3 \\ -5 \end{pmatrix} \tag{7.1}$$

A.1 Sides

A.1.1. The direction vector of AB is defined as

$$\mathbf{B} - \mathbf{A} \tag{A.1.1.1}$$

Find the direction vectors of AB, BC and CA.
Solution:

a) The Direction vector of AB is

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} -4 \\ 6 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -4 - 1 \\ 6 - (-1) \end{pmatrix} = \begin{pmatrix} -5 \\ 7 \end{pmatrix} \tag{A.1.1.2}$$

b) The Direction vector of BC is

$$\mathbf{C} - \mathbf{B} = \begin{pmatrix} -3 \\ -5 \end{pmatrix} - \begin{pmatrix} -4 \\ 6 \end{pmatrix} = \begin{pmatrix} -3 - (-4) \\ -5 - 6 \end{pmatrix} = \begin{pmatrix} 1 \\ -11 \end{pmatrix} \tag{A.1.1.3}$$

c) The Direction vector of CA is

$$\mathbf{A} - \mathbf{C} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \begin{pmatrix} -3 \\ -5 \end{pmatrix} = \begin{pmatrix} 1 - (-3) \\ -1 - (-5) \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \tag{A.1.1.4}$$

A.1.2. The length of side BC is

$$c = \|\mathbf{B} - \mathbf{A}\| \triangleq \sqrt{(\mathbf{B} - \mathbf{A})^\top (\mathbf{B} - \mathbf{A})} \tag{A.1.2.1}$$

where

$$\mathbf{A}^\top \triangleq \begin{pmatrix} 1 & -1 \end{pmatrix} \tag{A.1.2.2}$$

Similarly,

$$b = \|\mathbf{C} - \mathbf{B}\|, a = \|\mathbf{A} - \mathbf{C}\| \tag{A.1.2.3}$$

Find a, b, c.

a) From (A.1.1.2),

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} 5 \\ -7 \end{pmatrix}, \tag{A.1.2.4}$$

$$\implies c = \|\mathbf{B} - \mathbf{A}\| = \|\mathbf{A} - \mathbf{B}\| \tag{A.1.2.5}$$

$$= \sqrt{\begin{pmatrix} 5 & -7 \end{pmatrix} \begin{pmatrix} 5 \\ -7 \end{pmatrix}} = \sqrt{(5)^2 + (7)^2} \tag{A.1.2.6}$$

$$= \sqrt{74} \tag{A.1.2.7}$$

b) Similarly, from (A.1.1.3),

$$a = \|\mathbf{B} - \mathbf{C}\| = \sqrt{\begin{pmatrix} -1 & 11 \end{pmatrix}\begin{pmatrix} -1 \\ 11 \end{pmatrix}} \tag{A.1.2.8}$$

$$= \sqrt{(1)^2 + (11)^2} = \sqrt{122} \tag{A.1.2.9}$$

and from (A.1.1.4),

c)

$$b = \|\mathbf{A} - \mathbf{C}\| = \sqrt{\begin{pmatrix} 4 & 4 \end{pmatrix}\begin{pmatrix} 4 \\ 4 \end{pmatrix}} \tag{A.1.2.10}$$

$$= \sqrt{(4)^2 + (4)^2} = \sqrt{32} \tag{A.1.2.11}$$

A.1.3. Points $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are defined to be collinear if

$$\text{rank}\begin{pmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} = 2 \tag{A.1.3.1}$$

Are the given points in (7.1) collinear?
Solution: From (7.1),

$$\begin{pmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -4 & -3 \\ -1 & 6 & -5 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 + R_2} \begin{pmatrix} 1 & 1 & 1 \\ 1 & -4 & -3 \\ 0 & 2 & -8 \end{pmatrix} \tag{A.1.3.2}$$

$$\xleftrightarrow{R_2 \leftarrow R_1 - R_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 5 & 4 \\ 0 & 2 & -8 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 - \frac{2}{5} R_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 5 & 4 \\ 0 & 0 & \frac{-48}{5} \end{pmatrix} \tag{A.1.3.3}$$

There are no zero rows. So,

$$\text{rank}\begin{pmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} = 3 \tag{A.1.3.4}$$

Hence, the points $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are not collinear. This is visible in Fig. A.1.3.1.

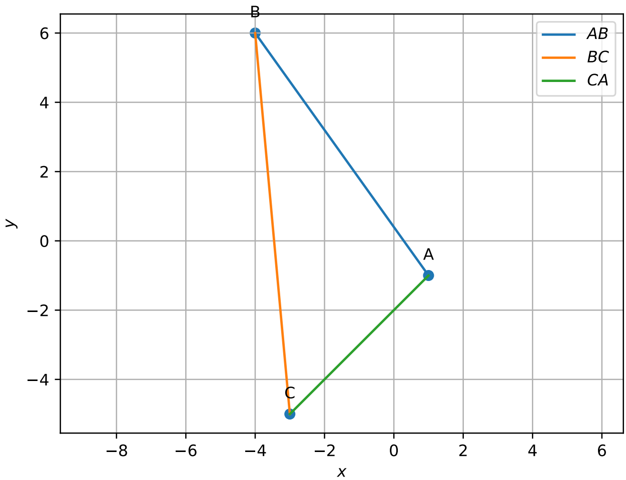

Fig. A.1.3.1: $\triangle ABC$

A.1.4. The parameteric form of the equation of AB is

$$\mathbf{x} = \mathbf{A} + k\mathbf{m} \quad k \neq 0, \tag{A.1.4.1}$$

where

$$\mathbf{m} = \mathbf{B} - \mathbf{A} \tag{A.1.4.2}$$

is the direction vector of AB. Find the parameteric equations of AB, BC and CA.
Solution: From (A.1.4.1) and (A.1.1.2), the parametric equation for AB is given by

$$AB : \mathbf{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + k \begin{pmatrix} -5 \\ 7 \end{pmatrix} \tag{A.1.4.3}$$

Similarly, from (A.1.1.3) and (A.1.1.4),

$$BC : \mathbf{x} = \begin{pmatrix} -4 \\ 6 \end{pmatrix} + k \begin{pmatrix} 1 \\ -11 \end{pmatrix} \tag{A.1.4.4}$$

$$CA : \mathbf{x} = \begin{pmatrix} -3 \\ -5 \end{pmatrix} + k \begin{pmatrix} 4 \\ 4 \end{pmatrix} \tag{A.1.4.5}$$

A.1.5. The normal form of the equation of AB is

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{A}) = 0 \tag{A.1.5.1}$$

where

$$\mathbf{n}^\top \mathbf{m} = \mathbf{n}^\top (\mathbf{B} - \mathbf{A}) = 0 \tag{A.1.5.2}$$

$$\text{or, } \mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{m} \tag{A.1.5.3}$$

Find the normal form of the equations of AB, BC and CA.
Solution:

a) From (A.1.1.3), the direction vector of side **BC** is

$$\mathbf{m} = \begin{pmatrix} 1 \\ -11 \end{pmatrix} \tag{A.1.5.4}$$

$$\implies \mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -11 \end{pmatrix} = \begin{pmatrix} -11 \\ -1 \end{pmatrix} \tag{A.1.5.5}$$

from (A.1.5.3). Hence, from (A.1.5.1), the normal equation of side BC is

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{B}) = 0 \tag{A.1.5.6}$$

$$\implies \begin{pmatrix} -11 & -1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -11 & -1 \end{pmatrix} \begin{pmatrix} -4 \\ 6 \end{pmatrix} \tag{A.1.5.7}$$

$$\implies BC : \quad \begin{pmatrix} 11 & 1 \end{pmatrix} \mathbf{x} = -38 \tag{A.1.5.8}$$

b) Similarly, for AB, from (A.1.1.2),

$$\mathbf{m} = \begin{pmatrix} -5 \\ 7 \end{pmatrix} \tag{A.1.5.9}$$

$$\implies \mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -5 \\ 7 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix} \tag{A.1.5.10}$$

and

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{A}) = 0 \tag{A.1.5.11}$$

$$\implies AB : \quad \mathbf{n}^\top \mathbf{x} = \begin{pmatrix} 7 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{A.1.5.12}$$

$$\implies \begin{pmatrix} 7 & 5 \end{pmatrix} \mathbf{x} = 2 \tag{A.1.5.13}$$

c) For CA, from (A.1.1.4),

$$\mathbf{m} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{A.1.5.14}$$

$$\implies \mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{A.1.5.15}$$

$$\tag{A.1.5.16}$$

$$\implies \mathbf{n}^\top (\mathbf{x} - \mathbf{C}) = 0 \tag{A.1.5.17}$$

$$\implies \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} -3 \\ -5 \end{pmatrix} = 2 \tag{A.1.5.18}$$

A.1.6. The area of $\triangle ABC$ is defined as

$$\frac{1}{2}\left\|(\mathbf{A}-\mathbf{B})\times(\mathbf{A}-\mathbf{C})\right\| \tag{A.1.6.1}$$

where

$$\mathbf{A}\times\mathbf{B} \triangleq \begin{vmatrix} 1 & -4 \\ -1 & 6 \end{vmatrix} \tag{A.1.6.2}$$

Find the area of $\triangle ABC$.

Solution: From (A.1.1.2) and (A.1.1.4),

$$\mathbf{A}-\mathbf{B} = \begin{pmatrix} 5 \\ -7 \end{pmatrix}, \mathbf{A}-\mathbf{C} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \tag{A.1.6.3}$$

$$\implies (\mathbf{A}-\mathbf{B})\times(\mathbf{A}-\mathbf{C}) = \begin{vmatrix} 5 & 4 \\ -7 & 4 \end{vmatrix} \tag{A.1.6.4}$$

$$= 5\times 4 - 4\times(-7) \tag{A.1.6.5}$$

$$= 48 \tag{A.1.6.6}$$

$$\implies \frac{1}{2}\left\|(\mathbf{A}-\mathbf{B})\times(\mathbf{A}-\mathbf{C})\right\| = \frac{48}{2} = 24 \tag{A.1.6.7}$$

which is the desired area.

A.1.7. Find the angles A, B, C if

$$\cos A \triangleq \frac{(\mathbf{B}-\mathbf{A})^\top \mathbf{C}-\mathbf{A}}{\left\|\mathbf{B}-\mathbf{A}\right\|\left\|\mathbf{C}-\mathbf{A}\right\|} \tag{A.1.7.1}$$

Solution:

a) From (A.1.1.2), (A.1.1.4), (A.1.2.7) and (A.1.2.11)

$$(\mathbf{B}-\mathbf{A})^\top(\mathbf{C}-\mathbf{A}) = \begin{pmatrix} -5 & 7 \end{pmatrix}\begin{pmatrix} -4 \\ -4 \end{pmatrix} \tag{A.1.7.2}$$

$$= -8 \tag{A.1.7.3}$$

$$\implies \cos A = \frac{-8}{\sqrt{74}\sqrt{32}} = \frac{-1}{\sqrt{37}} \tag{A.1.7.4}$$

$$\implies A = \cos^{-1}\frac{-1}{\sqrt{37}} \tag{A.1.7.5}$$

b) From (A.1.1.2), (A.1.1.3), (A.1.2.7) and (A.1.2.9)

$$(\mathbf{C}-\mathbf{B})^\top(\mathbf{A}-\mathbf{B}) = \begin{pmatrix} 1 & -11 \end{pmatrix}\begin{pmatrix} 5 \\ -7 \end{pmatrix} \tag{A.1.7.6}$$

$$= 82 \tag{A.1.7.7}$$

$$\implies \cos B = \frac{82}{\sqrt{74}\sqrt{122}} = \frac{41}{\sqrt{2257}} \tag{A.1.7.8}$$

$$\implies B = \cos^{-1}\frac{41}{\sqrt{2257}} \tag{A.1.7.9}$$

c) From (A.1.1.3), (A.1.1.4), (A.1.2.9) and (A.1.2.11)

$$(\mathbf{A}-\mathbf{C})^\top(\mathbf{B}-\mathbf{C}) = \begin{pmatrix}4 & 4\end{pmatrix}\begin{pmatrix}-1\\11\end{pmatrix} \tag{A.1.7.10}$$
$$= 40 \tag{A.1.7.11}$$
$$\implies \cos C = \frac{40}{\sqrt{32}\sqrt{122}} = \frac{5}{\sqrt{61}} \tag{A.1.7.12}$$
$$\implies C = \cos^{-1}\frac{5}{\sqrt{61}} \tag{A.1.7.13}$$

All codes for this section are available at

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codes/triangle/sides.py
```

A.2 Formulae

A.1. The equation of a line is given by

$$y = mx + c \tag{A.1.1}$$
$$\implies \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\mx+c\end{pmatrix} = \begin{pmatrix}0\\c\end{pmatrix} + x\begin{pmatrix}1\\m\end{pmatrix} \tag{A.1.2}$$

yielding (4.1.1.3).

A.2. (A.1.1) can also be expressed as

$$y - mx = c \tag{A.2.1}$$
$$\implies \begin{pmatrix}-m & 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = c \tag{A.2.2}$$

yielding (4.1.2.3).

A.3. The direction vector is

$$\mathbf{m} = \begin{pmatrix}1\\m\end{pmatrix} \tag{A.3.1}$$

and the normal vector is

$$\mathbf{n} = \begin{pmatrix}-m\\1\end{pmatrix} \tag{A.3.2}$$

A.4. From (4.1.1.3), if $\mathbf{A}, \mathbf{D}$ and $\mathbf{C}$ are on the same line,

$$\mathbf{D} = \mathbf{A} + q\mathbf{m} \tag{A.4.1}$$
$$\mathbf{C} = \mathbf{D} + p\mathbf{m} \tag{A.4.2}$$
$$\implies p(\mathbf{D}-\mathbf{A}) + q(\mathbf{D}-\mathbf{C}) = 0, \quad p, q \neq 0 \tag{A.4.3}$$
$$\implies \mathbf{D} = \frac{p\mathbf{A}+q\mathbf{C}}{p+q} \tag{A.4.4}$$

yielding (1.1.4.1) upon substituting

$$k = \frac{p}{q}. \tag{A.4.5}$$

$(\mathbf{D} - \mathbf{A}), (\mathbf{D} - \mathbf{C})$ are then said to be *linearly dependent*.

A.5. If $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are collinear, from (4.1.2.3),

$$\mathbf{n}^\top \mathbf{A} = c \tag{A.5.1}$$

$$\mathbf{n}^\top \mathbf{B} = c \tag{A.5.2}$$

$$\mathbf{n}^\top \mathbf{C} = c \tag{A.5.3}$$

which can be expressed as

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix}^\top \mathbf{n} = c \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{A.5.4}$$

$$\equiv \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix}^\top \mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \tag{A.5.5}$$

$$\implies \begin{pmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix}^\top \begin{pmatrix} \mathbf{n} \\ -1 \end{pmatrix} = \mathbf{0} \tag{A.5.6}$$

yielding

$$\text{rank} \begin{pmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} = 2 \tag{A.5.7}$$

Rank is defined to be the number of linearly indpendent rows or columns of a matrix.

A.6. The equation of a line can also be expressed as

$$\mathbf{n}^\top \mathbf{x} = 1 \tag{A.6.1}$$

A.3 Median

A.3.1. If $\mathbf{D}$ divides BC in the ratio $k : 1$,

$$\mathbf{D} = \frac{k\mathbf{C} + \mathbf{B}}{k + 1} \tag{A.3.1.1}$$

Find the mid points $\mathbf{D}, \mathbf{E}, \mathbf{F}$ of the sides BC, CA and AB respectively.

Solution: Since $\mathbf{D}$ is the midpoint of BC,

$$k = 1, \tag{A.3.1.2}$$

$$\implies \mathbf{D} = \frac{\mathbf{C} + \mathbf{B}}{2} = \frac{1}{2} \begin{pmatrix} -7 \\ 1 \end{pmatrix} \tag{A.3.1.3}$$

Similarly,

$$\mathbf{E} = \frac{\mathbf{A}+\mathbf{C}}{2} = \begin{pmatrix} -1 \\ -3 \end{pmatrix} \tag{A.3.1.4}$$

$$\mathbf{F} = \frac{\mathbf{A}+\mathbf{B}}{2} = \frac{1}{2}\begin{pmatrix} -3 \\ 5 \end{pmatrix} \tag{A.3.1.5}$$

A.3.2. Find the equations of AD, BE and CF.

Solution: :

a) The direction vector of AD is

$$\mathbf{m} = \mathbf{D} - \mathbf{A} = \begin{pmatrix} \frac{-7}{2} \\ \frac{1}{2} \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -9 \\ 3 \end{pmatrix} \equiv \begin{pmatrix} -3 \\ 1 \end{pmatrix} \tag{A.3.2.1}$$

$$\implies \mathbf{n} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \tag{A.3.2.2}$$

Hence the normal equation of median AD is

$$\mathbf{n}^\top \left(\mathbf{x} - \mathbf{A}\right) = 0 \tag{A.3.2.3}$$

$$\implies \begin{pmatrix} 1 & 3 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = -2 \tag{A.3.2.4}$$

b) For BE,

$$\mathbf{m} = \mathbf{E} - \mathbf{B} = \begin{pmatrix} -1 \\ -3 \end{pmatrix} - \begin{pmatrix} -4 \\ 6 \end{pmatrix} = \begin{pmatrix} 3 \\ -9 \end{pmatrix} \equiv \begin{pmatrix} 1 \\ -3 \end{pmatrix} \tag{A.3.2.5}$$

$$\implies \mathbf{n} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \tag{A.3.2.6}$$

Hence the normal equation of median BE is

$$\mathbf{n}^\top \left(\mathbf{x} - \mathbf{B}\right) = 0 \tag{A.3.2.7}$$

$$\implies \begin{pmatrix} 3 & 1 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 3 & 1 \end{pmatrix}\begin{pmatrix} -4 \\ 6 \end{pmatrix} = -6 \tag{A.3.2.8}$$

c) For median CF,

$$\mathbf{m} = \mathbf{F} - \mathbf{C} = \begin{pmatrix} \frac{-3}{2} \\ \frac{5}{2} \end{pmatrix} - \begin{pmatrix} -3 \\ -5 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ \frac{15}{2} \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 5 \end{pmatrix} \tag{A.3.2.9}$$

$$\implies \mathbf{n} = \begin{pmatrix} 5 \\ -1 \end{pmatrix} \tag{A.3.2.10}$$

Hence the normal equation of median CF is

$$\mathbf{n}^\top \left(\mathbf{x} - \mathbf{C}\right) = 0 \tag{A.3.2.11}$$

$$\implies \begin{pmatrix} 5 & -1 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 5 & -1 \end{pmatrix}\begin{pmatrix} -3 \\ -5 \end{pmatrix} = -10 \tag{A.3.2.12}$$

A.3.3. Find the intersection $\mathbf{G}$ of BE and CF.

Solution: From (A.3.2.8) and (A.3.2.12), the equations of BE and CF are, respectively,

$$\begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -6 \end{pmatrix} \tag{A.3.3.1}$$

$$\begin{pmatrix} 5 & -1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -10 \end{pmatrix} \tag{A.3.3.2}$$

From (A.3.3.1) and (A.3.3.2) the augmented matrix is

$$\begin{pmatrix} 3 & 1 & -6 \\ 5 & -1 & -10 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 + R_2} \begin{pmatrix} 8 & 0 & -16 \\ 5 & -1 & -10 \end{pmatrix} \tag{A.3.3.3}$$

$$\xleftrightarrow{R_1 \leftarrow R_1/8} \begin{pmatrix} 1 & 0 & -2 \\ 5 & -1 & -10 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - 5R_1} \begin{pmatrix} 1 & 0 & -2 \\ 0 & -1 & 0 \end{pmatrix} \tag{A.3.3.4}$$

$$\xleftrightarrow{R_2 \leftarrow -R_2} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \end{pmatrix} \tag{A.3.3.5}$$

using Gauss elimination. Therefore,

$$\mathbf{G} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \tag{A.3.3.6}$$

A.3.4. Verify that

$$\frac{BG}{GE} = \frac{CG}{GF} = \frac{AG}{GD} = 2 \tag{A.3.4.1}$$

Solution:

a) From (A.3.1.4) and (A.3.3.6),

$$\mathbf{G} - \mathbf{B} = \begin{pmatrix} 2 \\ -6 \end{pmatrix}, \mathbf{E} - \mathbf{G} = \begin{pmatrix} 1 \\ -3 \end{pmatrix} \tag{A.3.4.2}$$

$$\implies \mathbf{G} - \mathbf{B} = 2\left(\mathbf{E} - \mathbf{G}\right) \tag{A.3.4.3}$$

$$\implies \|\mathbf{G} - \mathbf{B}\| = 2\,\|\mathbf{E} - \mathbf{G}\| \tag{A.3.4.4}$$

$$\text{or, } \frac{BG}{GE} = 2 \tag{A.3.4.5}$$

b) From (A.3.1.5) and (A.3.3.6),

$$\mathbf{F} - \mathbf{G} = \frac{1}{2}\begin{pmatrix} 1 \\ 5 \end{pmatrix}, \mathbf{G} - \mathbf{C} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \tag{A.3.4.6}$$

$$\implies \mathbf{G} - \mathbf{C} = 2\left(\mathbf{F} - \mathbf{G}\right) \tag{A.3.4.7}$$

$$\implies \|\mathbf{G} - \mathbf{C}\| = 2\,\|\mathbf{F} - \mathbf{G}\| \tag{A.3.4.8}$$

$$\text{or, } \frac{CG}{GF} = 2 \tag{A.3.4.9}$$

c) From (A.3.1.3) and (A.3.3.6),

$$\mathbf{G}-\mathbf{A}=\begin{pmatrix}-3\\1\end{pmatrix},\ \mathbf{D}-\mathbf{G}=\frac{1}{2}\begin{pmatrix}-3\\1\end{pmatrix} \tag{A.3.4.10}$$

$$\mathbf{G}-\mathbf{A}=2\left(\mathbf{D}-\mathbf{G}\right) \tag{A.3.4.11}$$

$$\implies \left\|\mathbf{G}-\mathbf{A}\right\|=2\left\|\mathbf{D}-\mathbf{G}\right\| \tag{A.3.4.12}$$

$$\text{or, } \frac{AG}{GD}=2 \tag{A.3.4.13}$$

From (A.3.4.5), (A.3.4.9), (A.3.4.13)

$$\frac{BG}{GE}=\frac{CG}{GF}=\frac{AG}{GD}=2 \tag{A.3.4.14}$$

A.3.5. Show that $\mathbf{A},\mathbf{G}$ and $\mathbf{D}$ are collinear.

Solution: Points $\mathbf{A},\mathbf{D},\mathbf{G}$ are defined to be collinear if

$$\text{rank}\begin{pmatrix}1 & 1 & 1\\ \mathbf{A} & \mathbf{D} & \mathbf{G}\end{pmatrix}=2 \tag{A.3.5.1}$$

$$\implies \begin{pmatrix}1 & 1 & 1\\ 1 & -\frac{7}{2} & -2\\ -1 & \frac{1}{2} & 0\end{pmatrix} \xleftrightarrow{R_3\leftarrow R_3+R_2} \begin{pmatrix}1 & 1 & 1\\ 1 & -\frac{7}{2} & -2\\ 0 & -3 & -2\end{pmatrix} \tag{A.3.5.2}$$

$$\xleftrightarrow{R_2\leftarrow R_2-R_1} \begin{pmatrix}1 & 1 & 1\\ 0 & -\frac{9}{2} & -3\\ 0 & -3 & -2\end{pmatrix} \xleftrightarrow{R_3\leftarrow R_3-\frac{2}{3}R_2} \begin{pmatrix}1 & 1 & 1\\ 0 & -\frac{9}{2} & -3\\ 0 & 0 & 0\end{pmatrix} \tag{A.3.5.3}$$

Thus, the matrix (A.3.5.1) has rank 2 and the points are collinear. Thus, the medians of a triangle meet at the point $\mathbf{G}$. See Fig. A.3.5.1.

A.3.6. Verify that

$$\mathbf{G}=\frac{\mathbf{A}+\mathbf{B}+\mathbf{C}}{3} \tag{A.3.6.1}$$

$\mathbf{G}$ is known as the *centroid* of $\triangle ABC$.

Solution:

$$\begin{aligned}\mathbf{G}&=\frac{\begin{pmatrix}1\\-1\end{pmatrix}+\begin{pmatrix}-4\\6\end{pmatrix}+\begin{pmatrix}-3\\-5\end{pmatrix}}{3}\\&=\begin{pmatrix}-2\\0\end{pmatrix}\end{aligned} \tag{A.3.6.2}$$

A.3.7. Verify that

$$\mathbf{A}-\mathbf{F}=\mathbf{E}-\mathbf{D} \tag{A.3.7.1}$$

The quadrilateral $AFDE$ is defined to be a parallelogram.

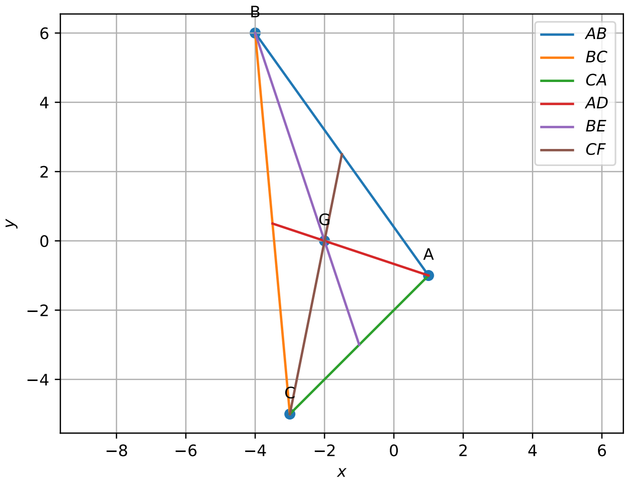

Fig. A.3.5.1: Medians of $\triangle ABC$ meet at $\mathbf{G}$.

Solution:

$$\mathbf{A}-\mathbf{F}=\begin{pmatrix}1\\-1\end{pmatrix}-\begin{pmatrix}\frac{-3}{2}\\\frac{5}{2}\end{pmatrix}=\begin{pmatrix}\frac{5}{2}\\\frac{-7}{2}\end{pmatrix} \tag{A.3.7.2}$$

$$\mathbf{E}-\mathbf{D}=\begin{pmatrix}-1\\-3\end{pmatrix}-\begin{pmatrix}\frac{-7}{2}\\\frac{1}{2}\end{pmatrix}=\begin{pmatrix}\frac{5}{2}\\\frac{-7}{2}\end{pmatrix} \tag{A.3.7.3}$$

$$\implies \mathbf{A}-\mathbf{F}=\mathbf{E}-\mathbf{D} \tag{A.3.7.4}$$

See Fig. A.3.7.1,
All codes for this section are available in

```
codes/triangle/medians.py
codes/triangle/pgm.py
```

A.4 Altitude

A.4.1. $\mathbf{D}_1$ is a point on BC such that

$$AD_1 \perp BC \tag{A.4.1.1}$$

and AD_1 is defined to be the altitude. Find the normal vector of AD_1.
Solution: The normal vector of AD_1 is the direction vector BC and is obtained from (A.1.1.3) as

$$\mathbf{n}=\begin{pmatrix}1\\-11\end{pmatrix} \tag{A.4.1.2}$$

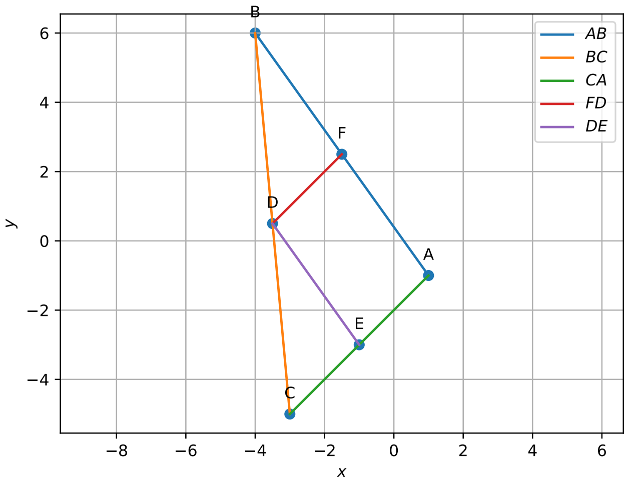

Fig. A.3.7.1: $AFDE$ forms a parallelogram in triangle ABC

A.4.2. Find the equation of AD_1.
Solution: The equation of AD_1 is

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{A}) = 0 \tag{A.4.2.1}$$

$$\implies \begin{pmatrix} -1 & 11 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -1 & 11 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = -12 \tag{A.4.2.2}$$

A.4.3. Find the equations of the altitudes BE_1 and CF_1 to the sides AC and AB respectively.

Solution:

a) From (A.1.1.4), the normal vector of CF_1 is

$$\mathbf{n} = \begin{pmatrix} -5 \\ 7 \end{pmatrix} \tag{A.4.3.1}$$

and the equation of CF_1 is

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{C}) = 0 \tag{A.4.3.2}$$

$$\implies \begin{pmatrix} -5 & 7 \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} -3 \\ -5 \end{pmatrix} \right) = 0 \tag{A.4.3.3}$$

$$\implies \begin{pmatrix} 5 & -7 \end{pmatrix} \mathbf{x} = 20, \tag{A.4.3.4}$$

b) Similarly, from (A.1.1.2), the normal vector of BE_1 is

$$\mathbf{n} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{A.4.3.5}$$

and the equation of BE_1 is

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{B}) = 0 \tag{A.4.3.6}$$

$$\implies \begin{pmatrix} 1 & 1 \end{pmatrix}\left(\mathbf{x} - \begin{pmatrix} -4 \\ 6 \end{pmatrix}\right) = 0 \tag{A.4.3.7}$$

$$\implies \begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} = 2, \tag{A.4.3.8}$$

A.4.4. Find the intersection $\mathbf{H}$ of BE_1 and CF_1.

Solution: The intersection of (A.4.3.8) and (A.4.3.4), is obtained from the matrix equation

$$\begin{pmatrix} 1 & 1 \\ 5 & -7 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 2 \\ 20 \end{pmatrix} \tag{A.4.4.1}$$

which can be solved as

$$\begin{pmatrix} 1 & 1 & 2 \\ 5 & -7 & 20 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - 5R_1} \begin{pmatrix} 1 & 1 & 2 \\ 0 & -12 & 10 \end{pmatrix} \tag{A.4.4.2}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{R_2}{-12}} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & \frac{-5}{6} \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 - R_2} \begin{pmatrix} 1 & 0 & \frac{17}{6} \\ 0 & 1 & \frac{-5}{6} \end{pmatrix} \tag{A.4.4.3}$$

yielding

$$\mathbf{H} = \frac{1}{6}\begin{pmatrix} 17 \\ -5 \end{pmatrix}, \tag{A.4.4.4}$$

See Fig. A.4.4.1

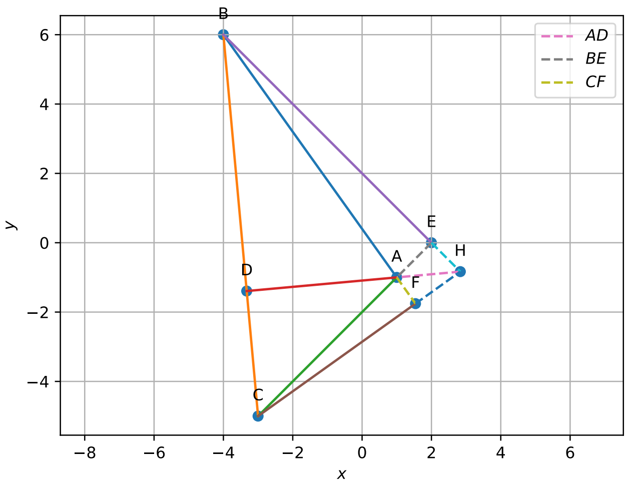

Fig. A.4.4.1: Altitudes BE_1 and CF_1 intersect at $\mathbf{H}$

A.4.5. Verify that

$$(\mathbf{A}-\mathbf{H})^\top(\mathbf{B}-\mathbf{C}) = 0 \tag{A.4.5.1}$$

Solution: From (A.4.4.4),

$$\mathbf{A}-\mathbf{H} = -\frac{1}{6}\begin{pmatrix}11\\1\end{pmatrix}, \mathbf{B}-\mathbf{C} = \begin{pmatrix}-1\\11\end{pmatrix} \tag{A.4.5.2}$$

$$\implies (\mathbf{A}-\mathbf{H})^\top(\mathbf{B}-\mathbf{C}) = \frac{1}{6}\begin{pmatrix}11 & 1\end{pmatrix}\begin{pmatrix}-1\\11\end{pmatrix} = 0 \tag{A.4.5.3}$$

A.4.6. Find the length of the altitude AD_1.
Solution: If the equation of BC be $\mathbf{n}^\top\mathbf{x} = c$, from (4.1.1.3),

$$\mathbf{D}_1 = \mathbf{A}+k\mathbf{n} \tag{A.4.6.1}$$

$$\implies AD_1 = \|\mathbf{D}_1-\mathbf{A}\| = |k|\,\|\mathbf{n}\| \tag{A.4.6.2}$$

From (A.4.6.1),

$$\mathbf{n}^\top\mathbf{D}_1 = \mathbf{n}^\top\mathbf{A}+k\|\mathbf{n}\|^2 \tag{A.4.6.3}$$

$$\implies |k| = \frac{\left|\mathbf{n}^\top(\mathbf{D}_1-\mathbf{A})\right|}{\|\mathbf{n}\|^2} \tag{A.4.6.4}$$

$$\implies AD_1 = |k|\,\|\mathbf{n}\| = \frac{\left|\mathbf{n}^\top\mathbf{A}-c\right|}{\|\mathbf{n}\|} \tag{A.4.6.5}$$

upon substituting from (A.4.6.2).

A.4.7. Find $\mathbf{D}_1$.
Solution: $\because \mathbf{D}_1$ lies on BC,

$$\mathbf{n}^\top\mathbf{D}_1 = c \tag{A.4.7.1}$$

and $\because AD_1 \perp BC$,

$$\mathbf{m}^\top(\mathbf{A}-\mathbf{D}_1) = 0 \tag{A.4.7.2}$$

$$\implies \mathbf{m}^\top\mathbf{D}_1 = \mathbf{m}^\top\mathbf{A} \tag{A.4.7.3}$$

Clubbing (A.4.7.1) and (A.4.7.3),

$$\begin{pmatrix}\mathbf{m} & \mathbf{n}\end{pmatrix}^\top\mathbf{D}_1 = \begin{pmatrix}\mathbf{m}^\top\mathbf{A}\\c\end{pmatrix} \tag{A.4.7.4}$$

A.4.8. All codes for this section are available at

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codes/triangle/altitude.py
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A.5 Perpendicular Bisector

A.5.1. The equation of the perpendicular bisector of BC is

$$\left(\mathbf{x}-\frac{\mathbf{B}+\mathbf{C}}{2}\right)^\top(\mathbf{B}-\mathbf{C}) = 0 \tag{A.5.1.1}$$

Substitute numerical values and find the equations of the perpendicular bisectors of AB, BC and CA.

Solution: From (A.1.1.2), (A.1.1.3), (A.1.1.4), (A.3.1.3), (A.3.1.4) and (A.3.1.5),

$$\frac{\mathbf{B}+\mathbf{C}}{\mathbf{2}} = \frac{1}{2}\begin{pmatrix}-7\\1\end{pmatrix}, \mathbf{B}-\mathbf{C} = \begin{pmatrix}-1\\11\end{pmatrix} \tag{A.5.1.2}$$

$$\frac{\mathbf{A}+\mathbf{B}}{\mathbf{2}} = \frac{1}{2}\begin{pmatrix}-3\\5\end{pmatrix}, \mathbf{A}-\mathbf{B} = \begin{pmatrix}5\\-7\end{pmatrix} \tag{A.5.1.3}$$

$$\frac{\mathbf{C}+\mathbf{A}}{\mathbf{2}} = \begin{pmatrix}-1\\-3\end{pmatrix}, \mathbf{C}-\mathbf{A} = \begin{pmatrix}-4\\-4\end{pmatrix} \tag{A.5.1.4}$$

yielding

$$(\mathbf{B}-\mathbf{C})^\top\left(\frac{\mathbf{B}+\mathbf{C}}{2}\right) = \begin{pmatrix}-1 & 11\end{pmatrix}\begin{pmatrix}-\frac{7}{2}\\\frac{1}{2}\end{pmatrix} = 9 \tag{A.5.1.5}$$

$$(\mathbf{A}-\mathbf{B})^\top\left(\frac{\mathbf{A}+\mathbf{B}}{2}\right) = \begin{pmatrix}5 & -7\end{pmatrix}\begin{pmatrix}-\frac{3}{2}\\\frac{5}{2}\end{pmatrix} = -25 \tag{A.5.1.6}$$

$$(\mathbf{C}-\mathbf{A})^\top\left(\frac{\mathbf{C}+\mathbf{A}}{2}\right) = \begin{pmatrix}-4 & -4\end{pmatrix}\begin{pmatrix}-1\\-3\end{pmatrix} = 16 \tag{A.5.1.7}$$

Thus, the perpendicular bisectors are obtained from (A.5.1.1) as

$$BC: \quad \begin{pmatrix}-1 & 11\end{pmatrix}\mathbf{x} = 9 \tag{A.5.1.8}$$

$$CA: \quad \begin{pmatrix}5 & -7\end{pmatrix}\mathbf{x} = -25 \tag{A.5.1.9}$$

$$AB: \quad \begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = -4 \tag{A.5.1.10}$$

A.5.2. Find the intersection $\mathbf{O}$ of the perpendicular bisectors of AB and AC.

Solution:

The intersection of (A.5.1.9) and (A.5.1.10), can be obtained as

$$\begin{pmatrix}5 & -7 & -25\\1 & 1 & -4\end{pmatrix} \xleftrightarrow{R_2 \leftarrow 5R_2 - R_1} \begin{pmatrix}5 & -7 & -25\\0 & 12 & 5\end{pmatrix} \tag{A.5.2.1}$$

$$\xleftrightarrow{R_1 \leftarrow \frac{12}{7}R_1 + R_2} \begin{pmatrix}\frac{60}{7} & 0 & \frac{-265}{7}\\0 & 12 & 5\end{pmatrix} \xleftrightarrow[R_1 \leftarrow \frac{7}{60}R_1]{R_2 \leftarrow \frac{1}{12}R_2} \begin{pmatrix}1 & 0 & \frac{-53}{12}\\0 & 1 & \frac{5}{12}\end{pmatrix} \tag{A.5.2.2}$$

$$\implies \mathbf{O} = \begin{pmatrix}\frac{-53}{12}\\\frac{5}{12}\end{pmatrix} \tag{A.5.2.3}$$

A.5.3. Verify that $\mathbf{O}$ satisfies (A.5.1.1). $\mathbf{O}$ is known as the circumcentre.

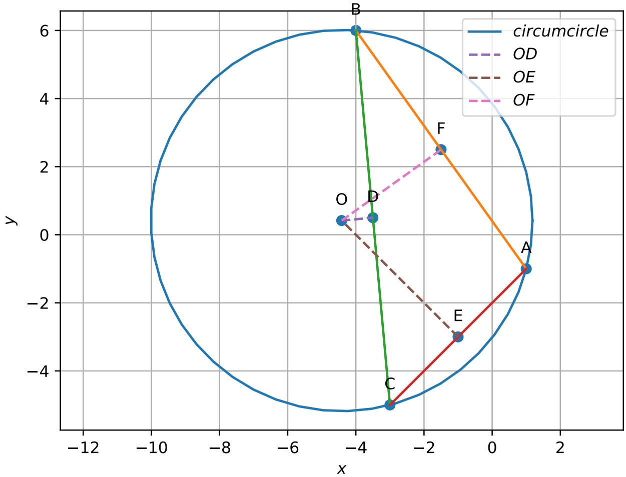

Fig. A.5.5.1: Circumcircle of $\triangle ABC$ with centre $\mathbf{O}$.

Solution: Substituing from (A.5.2.3) in (A.5.1.1),

$$\left(\mathbf{O} - \frac{\mathbf{B}+\mathbf{C}}{2}\right)^{\top}(\mathbf{B}-\mathbf{C})$$

$$= \left(\frac{1}{12}\begin{pmatrix}-53\\5\end{pmatrix} - \frac{1}{2}\begin{pmatrix}-7\\1\end{pmatrix}\right)^{\top}\begin{pmatrix}-1\\11\end{pmatrix}$$

$$= \frac{1}{12}\begin{pmatrix}-11 & -1\end{pmatrix}\begin{pmatrix}-1\\11\end{pmatrix} = 0 \quad \text{(A.5.3.1)}$$

A.5.4. Verify that

$$OA = OB = OC \quad \text{(A.5.4.1)}$$

A.5.5. Draw the circle with centre at $\mathbf{O}$ and radius

$$R = OA \quad \text{(A.5.5.1)}$$

This is known as the *circumradius*.
Solution: See Fig. A.5.5.1.

A.5.6. Verify that

$$\angle BOC = 2\angle BAC. \quad \text{(A.5.6.1)}$$

Solution:

a) To find the value of $\angle BOC$:

$$\mathbf{B}-\mathbf{O}=\begin{pmatrix}\frac{5}{12}\\\frac{67}{12}\end{pmatrix}, \mathbf{C}-\mathbf{O}=\begin{pmatrix}\frac{17}{12}\\\frac{-65}{12}\end{pmatrix} \tag{A.5.6.2}$$

$$\implies (\mathbf{B}-\mathbf{O})^\top(\mathbf{C}-\mathbf{O})=\frac{-4270}{144} \tag{A.5.6.3}$$

$$\implies \|\mathbf{B}-\mathbf{O}\|=\frac{\sqrt{4514}}{12}, \|\mathbf{C}-\mathbf{O}\|=\frac{\sqrt{4514}}{12} \tag{A.5.6.4}$$

Thus,

$$\cos BOC=\frac{(\mathbf{B}-\mathbf{O})^\top(\mathbf{C}-\mathbf{O})}{\|\mathbf{B}-\mathbf{O}\|\,\|\mathbf{C}-\mathbf{O}\|}=\frac{-4270}{4514} \tag{A.5.6.5}$$

$$\implies \angle BOC=\cos^{-1}\left(\frac{-4270}{4514}\right) \tag{A.5.6.6}$$

$$=161.07536^\circ \text{ or } 198.92464^\circ \tag{A.5.6.7}$$

b) To find the value of $\angle BAC$:

$$\mathbf{B}-\mathbf{A}=\begin{pmatrix}-5\\7\end{pmatrix}, \mathbf{C}-\mathbf{A}=\begin{pmatrix}-4\\-4\end{pmatrix} \tag{A.5.6.8}$$

$$\implies (\mathbf{B}-\mathbf{A})^\top(\mathbf{C}-\mathbf{A})=-8 \tag{A.5.6.9}$$

$$\|\mathbf{B}-\mathbf{A}\|=\sqrt{74}\,\|\mathbf{C}-\mathbf{A}\|=4\sqrt{2} \tag{A.5.6.10}$$

Thus,

$$\cos BAC=\frac{(\mathbf{B}-\mathbf{A})^\top(\mathbf{C}-\mathbf{A})}{\|\mathbf{B}-\mathbf{A}\|\,\|\mathbf{C}-\mathbf{A}\|}=\frac{-8}{4\sqrt{148}} \tag{A.5.6.11}$$

$$\implies \angle BAC=\cos^{-1}\left(\frac{-8}{4\sqrt{148}}\right) \tag{A.5.6.12}$$

$$=99.46232^\circ \tag{A.5.6.13}$$

From (A.5.6.13) and (A.5.6.7),

$$2\times\angle BAC=\angle BOC \tag{A.5.6.14}$$

A.5.7. Let

$$\mathbf{P}=\begin{pmatrix}\cos\theta & -\sin\theta\\\sin\theta & \cos\theta\end{pmatrix} \tag{A.5.7.1}$$

where

$$\theta=\angle BOC \tag{A.5.7.2}$$

Verify that

$$\mathbf{B}-\mathbf{O}=\mathbf{P}(\mathbf{C}-\mathbf{O}) \tag{A.5.7.3}$$

All codes for this section are available at

```
codes/triangle/perp−bisect.py
```

A.6 Angle Bisector

A.6.1. Let $\mathbf{D}_3, \mathbf{E}_3, \mathbf{F}_3$, be points on AB, BC and CA respectively such that

$$BD_3 = BF_3 = m, CD_3 = CE_3 = n, AE_3 = AF_3 = p. \tag{A.6.1.1}$$

Obtain m, n, p in terms of a, b, c obtained in Problem A.1.2.
Solution: From the given information,

$$a = m + n, \tag{A.6.1.2}$$
$$b = n + p, \tag{A.6.1.3}$$
$$c = m + p \tag{A.6.1.4}$$

which can be expressed as

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} m \\ n \\ p \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{A.6.1.5}$$

$$\implies \begin{pmatrix} m \\ n \\ p \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{A.6.1.6}$$

Using row reduction,

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \tag{A.6.1.7}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - R_1} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \end{array}\right) \tag{A.6.1.8}$$

$$\xleftrightarrow[R_1 \leftarrow R_1 - R_2]{R_3 \leftarrow R_3 + R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array}\right) \tag{A.6.1.9}$$

$$\xleftrightarrow[R_1 \leftarrow 2R_1 + R_3]{R_2 \leftarrow 2R_2 - R_3} \left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & -1 & 1 \\ 0 & 2 & 0 & 1 & 1 & -1 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array}\right) \tag{A.6.1.10}$$

yielding

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix} \tag{A.6.1.11}$$

Therefore,

$$\begin{aligned} p &= \frac{c+b-a}{2} = \frac{\sqrt{74}+\sqrt{32}-\sqrt{122}}{2} \\ m &= \frac{a+c-b}{2} = \frac{\sqrt{74}+\sqrt{122}-\sqrt{32}}{2} \\ n &= \frac{a+b-c}{2} = \frac{\sqrt{122}+\sqrt{32}-\sqrt{74}}{2} \end{aligned} \tag{A.6.1.12}$$

upon substituting from (A.1.2.7), (A.1.2.9) and (A.1.2.11).

A.6.2. Using section formula, find

$$\mathbf{D}_3 = \frac{m\mathbf{C}+n\mathbf{B}}{m+n}, \mathbf{E}_3 = \frac{n\mathbf{A}+p\mathbf{C}}{n+p}, \mathbf{F}_3 = \frac{p\mathbf{B}+m\mathbf{A}}{p+m} \tag{A.6.2.1}$$

A.6.3. Find the circumcentre and circumradius of $\triangle D_3E_3F_3$. These are the *incentre* and *inradius* of $\triangle ABC$.

A.6.4. Draw the circumcircle of $\triangle D_3E_3F_3$. This is known as the *incircle* of $\triangle ABC$.
Solution: See Fig. A.6.4.1

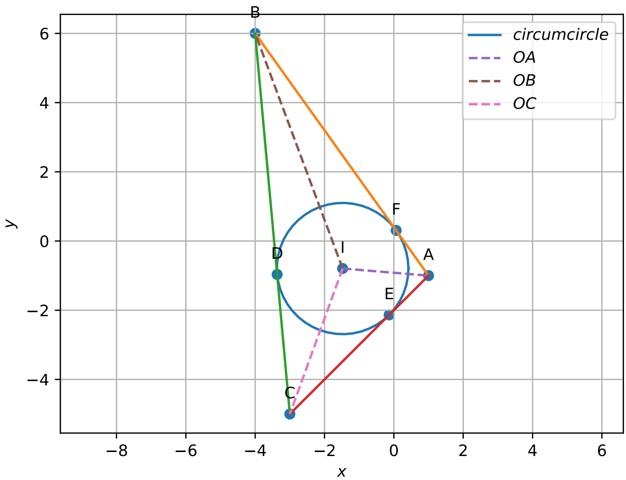

Fig. A.6.4.1: Incircle of $\triangle ABC$

A.6.5. Using (A.1.7.1) verify that

$$\angle BAI = \angle CAI. \tag{A.6.5.1}$$

AI is the bisector of $\angle A$.

A.6.6. Verify that BI, CI are also the angle bisectors of $\triangle ABC$. All codes for this section are available at

```
codes/triangle/ang-bisect.py
```

A.7 Eigenvalues and Eigenvectors

A.7.1. The equation of a circle is given by

$$\|\mathbf{x}\|^2 + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{A.7.1.1}$$

for

$$\mathbf{u} = -\mathbf{O}, f = \|\mathbf{O}\|^2 - r^2, \tag{A.7.1.2}$$

$\mathbf{O}$ being the incentre and r the inradius.

A.7.2. Compute

$$\mathbf{\Sigma} = (\mathbf{V}\mathbf{h} + \mathbf{u})(\mathbf{V}\mathbf{h} + \mathbf{u})^\top - g(\mathbf{h})\mathbf{V} \tag{A.7.2.1}$$

for $\mathbf{h} = \mathbf{A}$.

A.7.3. Find the roots of the equation

$$\left|\lambda\mathbf{I} - \mathbf{\Sigma}\right| = 0 \tag{A.7.3.1}$$

These are known as the eigenvalues of $\mathbf{\Sigma}$.

A.7.4. Find $\mathbf{p}$ such that

$$\mathbf{\Sigma}\mathbf{p} = \lambda\mathbf{p} \tag{A.7.4.1}$$

using row reduction. These are known as the eigenvectors of Σ.

A.7.5. Define

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \tag{A.7.5.1}$$

$$\mathbf{P} = \begin{pmatrix} \frac{\mathbf{p}_1}{\|\mathbf{p}_1\|} & \frac{\mathbf{p}_2}{\|\mathbf{p}_2\|} \end{pmatrix} \tag{A.7.5.2}$$

A.7.6. Verify that

$$\mathbf{P}^\top = \mathbf{P}^{-1}. \tag{A.7.6.1}$$

$\mathbf{P}$ is defined to be an orthogonal matrix.

A.7.7. Verify that

$$\mathbf{P}^\top \mathbf{\Sigma}\mathbf{P} = \mathbf{D}, \tag{A.7.7.1}$$

This is known as the spectral (eigenvalue) decomposition of a symmetric matrix

A.7.8. The direction vectors of the tangents from a point $\mathbf{h}$ to the circle in (8.1.2.1) are given by

$$\mathbf{m} = \mathbf{P}\begin{pmatrix} \sqrt{|\lambda_2|} \\ \pm\sqrt{|\lambda_1|} \end{pmatrix} \tag{A.7.8.1}$$

A.7.9. The points of contact of the pair of tangents to the circle in (8.1.2.1) from a point $\mathbf{h}$ are given by

$$\mathbf{x} = \mathbf{h} + \mu\mathbf{m} \tag{A.7.9.1}$$

where

$$\mu = -\frac{\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u})}{\mathbf{m}^\top\mathbf{V}\mathbf{m}} \tag{A.7.9.2}$$

for $\mathbf{m}$ in (A.7.8.1). Compute the points of contact. You should get the same points that you obtained in the previous section.
All codes for this section are available at

```
codes/triangle/tangpair.py
```

A.8 Formulae

A.8.1 The equation of the *incircle* is given by

$$\|\mathbf{x} - \mathbf{O}\|^2 = r^2 \tag{A.8.1.1}$$

which can be expressed as (8.1.2.1) using (A.7.1.2).

A.8.2 In Fig. A.6.4.1, let (A.7.9.1) be the equation of AB. Then, the intersection of (A.7.9.1) and (8.1.2.1) can be expressed as

$$(\mathbf{h} + \mu\mathbf{m})^\top \mathbf{V} (\mathbf{h} + \mu\mathbf{m}) + 2\mathbf{u}^\top (\mathbf{h} + \mu\mathbf{m}) + f = 0 \tag{A.8.2.1}$$

$$\implies \mu^2\mathbf{m}^\top\mathbf{V}\mathbf{m} + 2\mu\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u}) + g(\mathbf{h}) = 0 \tag{A.8.2.2}$$

For (A.8.2.2) to have exactly one root, the discriminant

$$\left\{\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u})\right\}^2 - g(\mathbf{h})\,\mathbf{m}^\top\mathbf{V}\mathbf{m} = 0 \tag{A.8.2.3}$$

and (A.7.9.2) is obtained.

A.8.3 (A.8.2.3) can be expressed as

$$\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u})^\top (\mathbf{V}\mathbf{h} + \mathbf{u})\,\mathbf{m} - g(\mathbf{h})\,\mathbf{m}^\top\mathbf{V}\mathbf{m} = 0 \tag{A.8.3.1}$$

$$\implies \mathbf{m}^\top\mathbf{\Sigma}\mathbf{m} = 0 \tag{A.8.3.2}$$

for $\mathbf{\Sigma}$ defined in (A.8.3.2). Substituting (A.7.7.1) in (A.8.3.2),

$$\mathbf{m}^\top\mathbf{P}\mathbf{D}\mathbf{P}^\top\mathbf{m} = 0 \tag{A.8.3.3}$$

$$\implies \mathbf{v}^\top\mathbf{D}\mathbf{v} = 0 \tag{A.8.3.4}$$

where

$$\mathbf{v} = \mathbf{P}^\top\mathbf{m} \tag{A.8.3.5}$$

(A.8.3.4) can be expressed as

$$\lambda_1 v_1^2 + \lambda_2 v_2^2 = 0 \tag{A.8.3.6}$$

$$\implies \mathbf{v} = \begin{pmatrix} \sqrt{|\lambda_2|} \\ \pm\sqrt{|\lambda_1|} \end{pmatrix} \tag{A.8.3.7}$$

after some algebra. From (A.8.3.7) and (A.8.3.5) we obtain (A.7.8.1).

A.9 Matrices

A.9.1. The matrix of the vertices of the triangle is defined as

$$\mathbf{P} = \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} \tag{A.9.1.1}$$

A.9.2. Obtain the direction matrix of the sides of $\triangle ABC$ defined as

$$\mathbf{M} = \begin{pmatrix} \mathbf{A} - \mathbf{B} & \mathbf{B} - \mathbf{C} & \mathbf{C} - \mathbf{A} \end{pmatrix} \tag{A.9.2.1}$$

Solution:

$$\mathbf{M} = \begin{pmatrix} \mathbf{A} - \mathbf{B} & \mathbf{B} - \mathbf{C} & \mathbf{C} - \mathbf{A} \end{pmatrix} \tag{A.9.2.2}$$

$$= \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \tag{A.9.2.3}$$

where the second matrix above is known as a *circulant* matrix. Note that the 2nd and 3rd row of the above matrix are circular shifts of the 1st row.

A.9.3. Obtain the normal matrix of the sides of $\triangle ABC$
Solution: Considering the roation matrix

$$\mathbf{R} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{A.9.3.1}$$

the normal matrix is obtained as

$$\mathbf{N} = \mathbf{R}\mathbf{M} \tag{A.9.3.2}$$

A.9.4. Obtain a, b, c.
Solution: The sides vector is obtained as

$$\mathbf{d} = \sqrt{\text{diag}(\mathbf{M}^\top \mathbf{M})} \tag{A.9.4.1}$$

A.9.5. Obtain the constant terms in the equations of the sides of the triangle.
Solution: The constants for the lines can be expressed in vector form as

$$\mathbf{c} = \text{diag}\left\{\left(\mathbf{N}^\top \mathbf{P}\right)\right\} \tag{A.9.5.1}$$

A.9.6. Obtain the mid point matrix for the sides of the triangle

Solution:

$$\begin{pmatrix}\mathbf{D} & \mathbf{E} & \mathbf{F}\end{pmatrix} = \frac{1}{2}\begin{pmatrix}\mathbf{A} & \mathbf{B} & \mathbf{C}\end{pmatrix}\begin{pmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{pmatrix} \tag{A.9.6.1}$$

A.9.7. Obtain the median direction matrix.
Solution: The median direction matrix is given by

$$\mathbf{M}_1 = \begin{pmatrix}\mathbf{A}-\mathbf{D} & \mathbf{B}-\mathbf{E} & \mathbf{C}-\mathbf{F}\end{pmatrix} \tag{A.9.7.1}$$

$$= \begin{pmatrix}\mathbf{A}-\frac{\mathbf{B}+\mathbf{C}}{2} & \mathbf{B}-\frac{\mathbf{C}+\mathbf{A}}{2} & \mathbf{C}-\frac{\mathbf{A}+\mathbf{B}}{2}\end{pmatrix} \tag{A.9.7.2}$$

$$= \begin{pmatrix}\mathbf{A} & \mathbf{B} & \mathbf{C}\end{pmatrix}\begin{pmatrix}1 & -\frac{1}{2} & -\frac{1}{2}\\ -\frac{1}{2} & 1 & -\frac{1}{2}\\ -\frac{1}{2} & -\frac{1}{2} & 1\end{pmatrix} \tag{A.9.7.3}$$

A.9.8. Obtain the median normal matrix.
A.9.9. Obtian the median equation constants.
A.9.10. Obtain the centroid by finding the intersection of the medians.
A.9.11. Find the normal matrix for the altitudes
Solution: The desired matrix is

$$\mathbf{M}_2 = \begin{pmatrix}\mathbf{B}-\mathbf{C} & \mathbf{C}-\mathbf{A} & \mathbf{A}-\mathbf{B}\end{pmatrix} \tag{A.9.11.1}$$

$$= \begin{pmatrix}\mathbf{A} & \mathbf{B} & \mathbf{C}\end{pmatrix}\begin{pmatrix}0 & -1 & 1\\ 1 & 0 & -1\\ -1 & 1 & 0\end{pmatrix} \tag{A.9.11.2}$$

A.9.12. Find the constants vector for the altitudes.
Solution: The desired vector is

$$\mathbf{c}_2 = \text{diag}\left\{\left(\mathbf{M}^\top\mathbf{P}\right)\right\} \tag{A.9.12.1}$$

A.9.13. Find the normal matrix for the perpendicular bisectors
Solution: The normal matrix is $\mathbf{M}_2$
A.9.14. Find the constants vector for the perpendicular bisectors.
Solution: The desired vector is

$$\mathbf{c}_3 = \text{diag}\left\{\mathbf{M}_2^\top\begin{pmatrix}\mathbf{D} & \mathbf{E} & \mathbf{F}\end{pmatrix}\right\} \tag{A.9.14.1}$$

A.9.15. Find the points of contact.
Solution: The points of contact are given by

$$\begin{pmatrix}\frac{m\mathbf{C}+n\mathbf{B}}{m+n} & \frac{n\mathbf{A}+p\mathbf{C}}{n+p} & \frac{p\mathbf{B}+m\mathbf{A}}{p+m}\end{pmatrix} = \begin{pmatrix}\mathbf{A} & \mathbf{B} & \mathbf{C}\end{pmatrix}\begin{pmatrix}0 & \frac{n}{b} & \frac{m}{c}\\ \frac{n}{a} & 0 & \frac{p}{c}\\ \frac{m}{a} & \frac{p}{b} & 0\end{pmatrix} \tag{A.9.15.1}$$

All codes for this section are available at

```
codes/triangle/mat−alg.py
```

Appendix B
Conic Section

B.1 Equation

B.1.1. Let $\mathbf{q}$ be a point such that the ratio of its distance from a fixed point $\mathbf{F}$ and the distance (d) from a fixed line

$$L : \mathbf{n}^\top \mathbf{x} = c \tag{B.1.1.1}$$

is constant, given by

$$\frac{\|\mathbf{q} - \mathbf{F}\|}{d} = e \tag{B.1.1.2}$$

The locus of $\mathbf{q}$ is known as a conic section. The line L is known as the directrix and the point $\mathbf{F}$ is the focus. e is defined to be the eccentricity of the conic.

a) For $e = 1$, the conic is a parabola
b) For $e < 1$, the conic is an ellipse
c) For $e > 1$, the conic is a hyperbola

B.1.2. The equation of a conic with directrix $\mathbf{n}^\top \mathbf{x} = c$, eccentricity e and focus $\mathbf{F}$ is given by

$$g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{B.1.2.1}$$

where

$$\mathbf{V} = \|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{n}\mathbf{n}^\top, \tag{B.1.2.2}$$

$$\mathbf{u} = ce^2 \mathbf{n} - \|\mathbf{n}\|^2 \mathbf{F}, \tag{B.1.2.3}$$

$$f = \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2 e^2 \tag{B.1.2.4}$$

Solution: Using Definition B.1.1 and (4.1.6.2), for any point $\mathbf{x}$ on the conic,

$$\|\mathbf{x} - \mathbf{F}\|^2 = e^2 \frac{(\mathbf{n}^\top \mathbf{x} - c)^2}{\|\mathbf{n}\|^2} \tag{B.1.2.5}$$

$$\implies \|\mathbf{n}\|^2 (\mathbf{x} - \mathbf{F})^\top (\mathbf{x} - \mathbf{F}) = e^2 \left(\mathbf{n}^\top \mathbf{x} - c\right)^2 \tag{B.1.2.6}$$

$$\implies \|\mathbf{n}\|^2 \left(\mathbf{x}^\top \mathbf{x} - 2\mathbf{F}^\top \mathbf{x} + \|\mathbf{F}\|^2\right) = e^2 \left(c^2 + \left(\mathbf{n}^\top \mathbf{x}\right)^2 - 2c\mathbf{n}^\top \mathbf{x}\right) \tag{B.1.2.7}$$

$$= e^2 \left(c^2 + \left(\mathbf{x}^\top \mathbf{n}\mathbf{n}^\top \mathbf{x}\right) - 2c\mathbf{n}^\top \mathbf{x}\right) \tag{B.1.2.8}$$

which can be expressed as (B.1.2.1) after simplification.

B.1.3. The eccentricity, directrices and foci of (B.1.2.1) are given by

$$e = \sqrt{1 - \frac{\lambda_1}{\lambda_2}} \tag{B.1.3.1}$$

$$\mathbf{n} = \sqrt{\lambda_2}\mathbf{p}_1,$$

$$c = \begin{cases} \frac{e\mathbf{u}^\top\mathbf{n} \pm \sqrt{e^2(\mathbf{u}^\top\mathbf{n})^2 - \lambda_2\left(e^2-1\right)\left(\|\mathbf{u}\|^2 - \lambda_2 f\right)}}{\lambda_2 e\left(e^2-1\right)} & e \neq 1 \\ \frac{\|\mathbf{u}\|^2 - \lambda_2 f}{2\mathbf{u}^\top\mathbf{n}} & e = 1 \end{cases} \tag{B.1.3.2}$$

$$\mathbf{F} = \frac{ce^2\mathbf{n} - \mathbf{u}}{\lambda_2} \tag{B.1.3.3}$$

Solution: From (B.1.2.2), using the fact that $\mathbf{V}$ is symmetric with $\mathbf{V} = \mathbf{V}^\top$,

$$\mathbf{V}^\top\mathbf{V} = \left(\|\mathbf{n}\|^2 \mathbf{I} - e^2\mathbf{n}\mathbf{n}^\top\right)^\top \left(\|\mathbf{n}\|^2 \mathbf{I} - e^2\mathbf{n}\mathbf{n}^\top\right) \tag{B.1.3.4}$$

$$\implies \mathbf{V}^2 = \|\mathbf{n}\|^4 \mathbf{I} + e^4\mathbf{n}\mathbf{n}^\top\mathbf{n}\mathbf{n}^\top - 2e^2 \|\mathbf{n}\|^2 \mathbf{n}\mathbf{n}^\top \tag{B.1.3.5}$$

$$= \|\mathbf{n}\|^4 \mathbf{I} + e^4 \|\mathbf{n}\|^2 \mathbf{n}\mathbf{n}^\top - 2e^2 \|\mathbf{n}\|^2 \mathbf{n}\mathbf{n}^\top \tag{B.1.3.6}$$

$$= \|\mathbf{n}\|^4 \mathbf{I} + e^2 \left(e^2 - 2\right) \|\mathbf{n}\|^2 \mathbf{n}\mathbf{n}^\top \tag{B.1.3.7}$$

$$= \|\mathbf{n}\|^4 \mathbf{I} + \left(e^2 - 2\right) \|\mathbf{n}\|^2 \left(\|\mathbf{n}\|^2 \mathbf{I} - \mathbf{V}\right) \tag{B.1.3.8}$$

which can be expressed as

$$\mathbf{V}^2 + \left(e^2 - 2\right) \|\mathbf{n}\|^2 \mathbf{V} - \left(e^2 - 1\right) \|\mathbf{n}\|^4 \mathbf{I} = 0 \tag{B.1.3.9}$$

Using the Cayley-Hamilton theorem, (B.1.3.9) results in the characteristic equation,

$$\lambda^2 - \left(2 - e^2\right) \|\mathbf{n}\|^2 \lambda + \left(1 - e^2\right) \|\mathbf{n}\|^4 = 0 \tag{B.1.3.10}$$

which can be expressed as

$$\left(\frac{\lambda}{\|\mathbf{n}\|^2}\right)^2 - \left(2 - e^2\right)\left(\frac{\lambda}{\|\mathbf{n}\|^2}\right) + \left(1 - e^2\right) = 0 \tag{B.1.3.11}$$

$$\implies \frac{\lambda}{\|\mathbf{n}\|^2} = 1 - e^2, 1 \tag{B.1.3.12}$$

$$\text{or, } \lambda_2 = \|\mathbf{n}\|^2, \lambda_1 = \left(1 - e^2\right)\lambda_2 \tag{B.1.3.13}$$

From (B.1.3.13), the eccentricity of (B.1.2.1) is given by (B.1.3.1). Multiplying both sides of (B.1.2.2) by $\mathbf{n}$,

$$\mathbf{V}\mathbf{n} = \|\mathbf{n}\|^2 \mathbf{n} - e^2\mathbf{n}\mathbf{n}^\top\mathbf{n} \tag{B.1.3.14}$$

$$= \|\mathbf{n}\|^2 \left(1 - e^2\right)\mathbf{n} \tag{B.1.3.15}$$

$$= \lambda_1\mathbf{n} \tag{B.1.3.16}$$

$$\tag{B.1.3.17}$$

from (B.1.3.13). Thus, λ_1 is the corresponding eigenvalue for $\mathbf{n}$. From (A.7.5.2) and

(B.1.3.17), this implies that

$$\mathbf{p}_1 = \frac{\mathbf{n}}{\|\mathbf{n}\|} \tag{B.1.3.18}$$

$$\text{or, } \mathbf{n} = \|\mathbf{n}\| \, \mathbf{p}_1 = \sqrt{\lambda_2}\mathbf{p}_1 \tag{B.1.3.19}$$

from (B.1.3.13) . From (B.1.2.3) and (B.1.3.13),

$$\mathbf{F} = \frac{ce^2\mathbf{n} - \mathbf{u}}{\lambda_2} \tag{B.1.3.20}$$

$$\implies \|\mathbf{F}\|^2 = \frac{\left(ce^2\mathbf{n} - \mathbf{u}\right)^\top \left(ce^2\mathbf{n} - \mathbf{u}\right)}{\lambda_2^2} \tag{B.1.3.21}$$

$$\implies \lambda_2^2 \, \|\mathbf{F}\|^2 = c^2e^4\lambda_2 - 2ce^2\mathbf{u}^\top\mathbf{n} + \|\mathbf{u}\|^2 \tag{B.1.3.22}$$

Also, (B.1.2.4) can be expressed as

$$\lambda_2 \, \|\mathbf{F}\|^2 = f + c^2e^2 \tag{B.1.3.23}$$

From (B.1.3.22) and (B.1.3.23),

$$c^2e^4\lambda_2 - 2ce^2\mathbf{u}^\top\mathbf{n} + \|\mathbf{u}\|^2 = \lambda_2\left(f + c^2e^2\right) \tag{B.1.3.24}$$

$$\implies \lambda_2e^2\left(e^2 - 1\right)c^2 - 2ce^2\mathbf{u}^\top\mathbf{n} + \|\mathbf{u}\|^2 - \lambda_2 f = 0 \tag{B.1.3.25}$$

yielding (B.1.3.3).

B.1.4. (B.1.2.1) represents

a) a parabola for $\left|\mathbf{V}\right| = 0$,
b) ellipse for $\left|\mathbf{V}\right| > 0$ and
c) hyperbola for $\left|\mathbf{V}\right| < 0$.

Solution: From (B.1.3.1),

$$\frac{\lambda_1}{\lambda_2} = 1 - e^2 \tag{B.1.4.1}$$

Also,

$$\left|\mathbf{V}\right| = \lambda_1\lambda_2 \tag{B.1.4.2}$$

yielding Table B.1.4.

Eccentricity	**Conic**	**Eigenvalue**	**Determinant**
$e = 1$	Parabola	$\lambda_1 = 0$	$\left|\mathbf{V}\right| = 0$
$e < 1$	Ellipse	$\lambda_1 > 0, \lambda_2 > 0$	$\left|\mathbf{V}\right| > 0$
$e > 1$	Hyperbola	$\lambda_1 < 0, \lambda_2 > 0$	$\left|\mathbf{V}\right| < 0$

TABLE B.1.4

B.1.5. Using the affine transformation in (2.1.15.1), the conic in (B.1.2.1) can be expressed in standard form as

$$\mathbf{y}^\top \left(\frac{\mathbf{D}}{f_0}\right)\mathbf{y} = 1 \qquad |\mathbf{V}| \neq 0 \tag{B.1.5.1}$$

$$\mathbf{y}^\top \mathbf{D}\mathbf{y} = -\eta \mathbf{e}_1^\top \mathbf{y} \qquad |\mathbf{V}| = 0 \tag{B.1.5.2}$$

where

$$f_0 = \mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f \neq 0 \tag{B.1.5.3}$$

$$\eta = 2\mathbf{u}^\top \mathbf{p}_1 \tag{B.1.5.4}$$

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{B.1.5.5}$$

Solution: Using (2.1.15.1), (B.1.2.1) can be expressed as

$$(\mathbf{P}\mathbf{y} + \mathbf{c})^\top \mathbf{V}(\mathbf{P}\mathbf{y} + \mathbf{c}) + 2\mathbf{u}^\top (\mathbf{P}\mathbf{y} + \mathbf{c}) + f = 0, \tag{B.1.5.6}$$

yielding

$$\mathbf{y}^\top \mathbf{P}^\top \mathbf{V}\mathbf{P}\mathbf{y} + 2(\mathbf{V}\mathbf{c} + \mathbf{u})^\top \mathbf{P}\mathbf{y} + \mathbf{c}^\top \mathbf{V}\mathbf{c} + 2\mathbf{u}^\top \mathbf{c} + f = 0 \tag{B.1.5.7}$$

From (B.1.5.7) and (A.7.7.1),

$$\mathbf{y}^\top \mathbf{D}\mathbf{y} + 2(\mathbf{V}\mathbf{c} + \mathbf{u})^\top \mathbf{P}\mathbf{y} + \mathbf{c}^\top (\mathbf{V}\mathbf{c} + \mathbf{u}) + \mathbf{u}^\top \mathbf{c} + f = 0 \tag{B.1.5.8}$$

When $\mathbf{V}^{-1}$ exists, choosing

$$\mathbf{V}\mathbf{c} + \mathbf{u} = \mathbf{0}, \quad \text{or}, \mathbf{c} = -\mathbf{V}^{-1}\mathbf{u}, \tag{B.1.5.9}$$

and substituting (B.1.5.9) in (B.1.5.8) yields (B.1.5.1). When $|\mathbf{V}| = 0, \lambda_1 = 0$ and

$$\mathbf{V}\mathbf{p}_1 = 0, \mathbf{V}\mathbf{p}_2 = \lambda_2 \mathbf{p}_2. \tag{B.1.5.10}$$

Substituting (8.1.4.2) in (B.1.5.8),

$$\begin{aligned}
&\mathbf{y}^\top \mathbf{D}\mathbf{y} + 2\left(\mathbf{c}^\top \mathbf{V} + \mathbf{u}^\top\right)\begin{pmatrix}\mathbf{p}_1 & \mathbf{p}_2\end{pmatrix}\mathbf{y} + \mathbf{c}^\top (\mathbf{V}\mathbf{c} + \mathbf{u}) + \mathbf{u}^\top \mathbf{c} + f = 0 \\
\implies &\mathbf{y}^\top \mathbf{D}\mathbf{y} + 2\begin{pmatrix}(\mathbf{c}^\top \mathbf{V} + \mathbf{u}^\top)\mathbf{p}_1 & (\mathbf{c}^\top \mathbf{V} + \mathbf{u}^\top)\mathbf{p}_2\end{pmatrix}\mathbf{y} + \mathbf{c}^\top (\mathbf{V}\mathbf{c} + \mathbf{u}) + \mathbf{u}^\top \mathbf{c} + f = 0 \\
\implies &\mathbf{y}^\top \mathbf{D}\mathbf{y} + 2\begin{pmatrix}\mathbf{u}^\top \mathbf{p}_1 & (\lambda_2 \mathbf{c}^\top + \mathbf{u}^\top)\mathbf{p}_2\end{pmatrix}\mathbf{y} + \mathbf{c}^\top (\mathbf{V}\mathbf{c} + \mathbf{u}) + \mathbf{u}^\top \mathbf{c} + f = 0
\end{aligned}$$

upon substituting from (B.1.5.10), yielding

$$\lambda_2 y_2^2 + 2\left(\mathbf{u}^\top \mathbf{p}_1\right) y_1 + 2y_2 (\lambda_2 \mathbf{c} + \mathbf{u})^\top \mathbf{p}_2 + \mathbf{c}^\top (\mathbf{V}\mathbf{c} + \mathbf{u}) + \mathbf{u}^\top \mathbf{c} + f = 0 \tag{B.1.5.11}$$

Thus, (B.1.5.11) can be expressed as (B.1.5.2) by choosing

$$\eta = 2\mathbf{u}^\top \mathbf{p}_1 \tag{B.1.5.12}$$

and $\mathbf{c}$ in (B.1.5.8) such that

$$2\mathbf{P}^\top (\mathbf{V}\mathbf{c} + \mathbf{u}) = \eta \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{B.1.5.13}$$

$$\mathbf{c}^\top (\mathbf{V}\mathbf{c} + \mathbf{u}) + \mathbf{u}^\top \mathbf{c} + f = 0 \tag{B.1.5.14}$$

B.1.6. The center/vertex of a conic section are given by

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \qquad |\mathbf{V}| \neq 0 \tag{B.1.6.1}$$

$$\begin{pmatrix} \mathbf{u}^\top + \frac{\eta}{2}\mathbf{p}_1^\top \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \frac{\eta}{2}\mathbf{p}_1 - \mathbf{u} \end{pmatrix} \qquad |\mathbf{V}| = 0 \tag{B.1.6.2}$$

Solution: $\because \mathbf{P}^\top\mathbf{P} = \mathbf{I}$, multiplying (B.1.5.13) by $\mathbf{P}$ yields

$$(\mathbf{V}\mathbf{c} + \mathbf{u}) = \frac{\eta}{2}\mathbf{p}_1, \tag{B.1.6.3}$$

which, upon substituting in (B.1.5.14) results in

$$\frac{\eta}{2}\mathbf{c}^\top\mathbf{p}_1 + \mathbf{u}^\top\mathbf{c} + f = 0 \tag{B.1.6.4}$$

(B.1.6.3) and (B.1.6.4) can be clubbed together to obtain (B.1.6.2).

B.1.7. In (2.1.15.1), substituting $\mathbf{y} = \mathbf{0}$, the center/vertex for the quadratic form is obtained as

$$\mathbf{x} = \mathbf{c}, \tag{B.1.7.1}$$

where $\mathbf{c}$ is derived as (B.1.6.1) and (B.1.6.2) in Appendix B.1.5.

B.2 Standard Conic

B.2.1. For the standard conic,

$$\mathbf{P} = \mathbf{I} \tag{B.2.1.1}$$

$$\mathbf{u} = \begin{cases} 0 & e \neq 1 \\ \frac{\eta}{2}\mathbf{e}_1 & e = 1 \end{cases} \tag{B.2.1.2}$$

$$\lambda_1 \begin{cases} = 0 & e = 1 \\ \neq 0 & e \neq 1 \end{cases} \tag{B.2.1.3}$$

where

$$\mathbf{I} = \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 \end{pmatrix} \tag{B.2.1.4}$$

is the identity matrix.

B.2.2. The center of the standard ellipse/hyperbola, defined to be the mid point of the line joining the foci, is the origin.

B.2.3. The principal (major) axis of the standard ellipse/hyperbola, defined to be the line joining the two foci is the x-axis.

Proof. From (B.2.7.3), it is obvious that the line joining the foci passes through the origin. Also, the direction vector of this line is $\mathbf{e}_1$. Thus, the principal axis is the x-axis. □

B.2.4. The minor axis of the standard ellipse/hyperbola, defined to be the line orthogonal to the x-axis is the y-axis.

B.2.5. The axis of symmetry of the standard parabola, defined to be the line perpendicular to the directrix and passing through the focus, is the x- axis.

Proof. From (B.2.7.7) and (B.2.7.3), the axis of the parabola can be expressed as

$$\mathbf{e}_2^\top \left(\mathbf{y} + \frac{\eta}{4\lambda_2}\mathbf{e}_1\right) = 0 \tag{B.2.5.1}$$

$$\implies \mathbf{e}_2^\top \mathbf{y} = 0, \tag{B.2.5.2}$$

which is the equation of the x-axis. □

B.2.6. The point where the parabola intersects its axis of symmetry is called the vertex. For the standard parabola, the vertex is the origin.

Proof. (B.2.5.2) can be expressed as

$$\mathbf{y} = \alpha \mathbf{e}_1. \tag{B.2.6.1}$$

Substituting (B.2.6.1) in (B.1.5.2),

$$\alpha^2 \mathbf{e}_1^\top \mathbf{D} \mathbf{e}_1 = -\eta\alpha \mathbf{e}_1^\top \mathbf{e}_1 \tag{B.2.6.2}$$

$$\implies \alpha = 0, \text{ or, } \mathbf{y} = \mathbf{0}. \tag{B.2.6.3}$$

□

B.2.7.

a) The directrices for the standard conic are given by

$$\mathbf{e}_1^\top \mathbf{y} = \pm\frac{1}{e}\sqrt{\frac{|f_0|}{\lambda_2\left(1-e^2\right)}} \qquad e \neq 1 \tag{B.2.7.1}$$

$$\mathbf{e}_1^\top \mathbf{y} = \frac{\eta}{2\lambda_2} \qquad e = 1 \tag{B.2.7.2}$$

b) The foci of the standard ellipse and hyperbola are given by

$$\mathbf{F} = \begin{cases} \pm e\sqrt{\frac{|f_0|}{\lambda_2\left(1-e^2\right)}}\mathbf{e}_1 & e \neq 1 \\ -\frac{\eta}{4\lambda_2}\mathbf{e}_1 & e = 1 \end{cases} \tag{B.2.7.3}$$

Proof. a) For the standard hyperbola/ellipse in (B.1.5.1), from (B.2.1.1), (B.1.3.2) and (B.2.1.2),

$$\mathbf{n} = \sqrt{\frac{\lambda_2}{f_0}}\mathbf{e}_1 \tag{B.2.7.4}$$

$$c = \pm\frac{\sqrt{-\frac{\lambda_2}{f_0}\left(e^2-1\right)\left(\frac{\lambda_2}{f_0}\right)}}{\frac{\lambda_2}{f_0}e\left(e^2-1\right)} \tag{B.2.7.5}$$

$$= \pm\frac{1}{e\sqrt{1-e^2}} \tag{B.2.7.6}$$

yielding (B.2.7.1) upon substituting from (B.1.3.1) and simplifying. For the standard parabola in (B.1.5.2), from (B.2.1.1), (B.1.3.2) and (B.2.1.2), noting that $f = 0$,

$$\mathbf{n} = \sqrt{\lambda_2}\mathbf{e}_1 \tag{B.2.7.7}$$

$$c = \frac{\left\|\frac{\eta}{2}\mathbf{e}_1\right\|^2}{2\left(\frac{\eta}{2}\right)(\mathbf{e_1})^\top \mathbf{n}} \tag{B.2.7.8}$$

$$\tag{B.2.7.9}$$

$$= \frac{\eta}{4\sqrt{\lambda_2}} \tag{B.2.7.10}$$

yielding (B.2.7.2).

b) For the standard ellipse/hyperbola, substituting from (B.2.7.6), (B.2.7.4), (B.2.1.2) and (B.1.3.1) in (B.1.3.3),

$$\mathbf{F} = \pm\frac{\left(\frac{1}{e\sqrt{1-e^2}}\right)\left(e^2\right)\sqrt{\frac{\lambda_2}{f_0}}\mathbf{e}_1}{\frac{\lambda_2}{f_0}} \tag{B.2.7.11}$$

yielding (B.2.7.3) after simplification. For the standard parabola, substituting from (B.2.7.10), (B.2.7.7), (B.2.1.2) and (B.1.3.1) in (B.1.3.3),

$$\mathbf{F} = \frac{\left(\frac{\eta}{4\sqrt{\lambda_2}}\right)\sqrt{\lambda_2}\mathbf{e}_1 - \frac{\eta}{2}\mathbf{e_1}}{\lambda_2} \tag{B.2.7.12}$$

$$\tag{B.2.7.13}$$

yielding (B.2.7.3) after simplification.

□

B.2.8. The *focal length* of the standard parabola, , defined to be the distance between the vertex and the focus, measured along the axis of symmetry, is $\left|\frac{\eta}{4\lambda_2}\right|$

B.3 Conic Lines

B.3.1. The points of intersection of the line

$$L: \quad \mathbf{x} = \mathbf{h} + \kappa\mathbf{m} \quad \kappa \in \mathbb{R} \tag{B.3.1.1}$$

with the conic section in (B.1.2.1) are given by

$$\mathbf{x}_i = \mathbf{h} + \kappa_i\mathbf{m} \tag{B.3.1.2}$$

where

$$\kappa_i = \frac{1}{\mathbf{m}^\top\mathbf{V}\mathbf{m}}\left(-\mathbf{m}^\top(\mathbf{V}\mathbf{h}+\mathbf{u}) \pm \sqrt{\left[\mathbf{m}^\top(\mathbf{V}\mathbf{h}+\mathbf{u})\right]^2 - \mathrm{g}(\mathbf{h})\left(\mathbf{m}^\top\mathbf{V}\mathbf{m}\right)}\right) \tag{B.3.1.3}$$

Solution: Substituting (B.3.1.1) in (B.1.2.1),

$$(\mathbf{h}+\kappa\mathbf{m})^\top\mathbf{V}(\mathbf{h}+\kappa\mathbf{m}) + 2\mathbf{u}^\top(\mathbf{h}+\kappa\mathbf{m}) + f = 0 \tag{B.3.1.4}$$

$$\implies \kappa^2\mathbf{m}^\top\mathbf{V}\mathbf{m} + 2\kappa\mathbf{m}^\top(\mathbf{V}\mathbf{h}+\mathbf{u}) + \mathbf{h}^\top\mathbf{V}\mathbf{h} + 2\mathbf{u}^\top\mathbf{h} + f = 0 \tag{B.3.1.5}$$

$$\text{or, } \kappa^2\mathbf{m}^\top\mathbf{V}\mathbf{m} + 2\kappa\mathbf{m}^\top(\mathbf{V}\mathbf{h}+\mathbf{u}) + \mathrm{g}(\mathbf{h}) = 0 \tag{B.3.1.6}$$

for g defined in (B.1.2.1). Solving the above quadratic in (B.3.1.6) yields (B.3.1.3).

B.3.2. The length of the chord in (B.3.1.1) is given by

$$\frac{2\sqrt{\left[\mathbf{m}^\top(\mathbf{V}\mathbf{h}+\mathbf{u})\right]^2-\left(\mathbf{h}^\top\mathbf{V}\mathbf{h}+2\mathbf{u}^\top\mathbf{h}+f\right)\left(\mathbf{m}^\top\mathbf{V}\mathbf{m}\right)}}{\mathbf{m}^\top\mathbf{V}\mathbf{m}}\left\|\mathbf{m}\right\| \tag{B.3.2.1}$$

Proof. The distance between the points in (B.3.1.2) is given by

$$\left\|\mathbf{x}_1-\mathbf{x}_2\right\|=\left|\kappa_1-\kappa_2\right|\left\|\mathbf{m}\right\| \tag{B.3.2.2}$$

Substituing κ_i from (B.3.1.3) in (B.3.2.2) yields (B.3.2.1). □

B.3.3. The affine transform for the conic section, preserves the norm. This implies that the length of any chord of a conic is invariant to translation and/or rotation.

Proof. Let

$$\mathbf{x}_i=\mathbf{P}\mathbf{y}_i+\mathbf{c} \tag{B.3.3.1}$$

be any two points on the conic. Then the distance between the points is given by

$$\left\|\mathbf{x}_1-\mathbf{x}_2\right\|=\left\|\mathbf{P}\left(\mathbf{y}_1-\mathbf{y}_2\right)\right\| \tag{B.3.3.2}$$

which can be expressed as

$$\left\|\mathbf{x}_1-\mathbf{x}_2\right\|^2=\left(\mathbf{y}_1-\mathbf{y}_2\right)^\top\mathbf{P}^\top\mathbf{P}\left(\mathbf{y}_1-\mathbf{y}_2\right) \tag{B.3.3.3}$$

$$=\left\|\mathbf{y}_1-\mathbf{y}_2\right\|^2 \tag{B.3.3.4}$$

since

$$\mathbf{P}^\top\mathbf{P}=\mathbf{I} \tag{B.3.3.5}$$

□

B.3.4. For the standard hyperbola/ellipse, the length of the major axis is

$$2\sqrt{\left|\frac{f_0}{\lambda_1}\right|} \tag{B.3.4.1}$$

and the minor axis is

$$2\sqrt{\left|\frac{f_0}{\lambda_2}\right|} \tag{B.3.4.2}$$

Solution: Since the major axis passes through the origin,

$$\mathbf{q}=\mathbf{0} \tag{B.3.4.3}$$

Further, from Corollary (B.2.3),

$$\mathbf{m}=\mathbf{e}_2, \tag{B.3.4.4}$$

and from (B.1.5.1),

$$\mathbf{V}=\frac{\mathbf{D}}{f_0},\mathbf{u}=0,f=-1 \tag{B.3.4.5}$$

Substituting the above in (B.3.2.1),

$$\frac{2\sqrt{\mathbf{e}_1^\top \frac{\mathbf{D}}{f_0}\mathbf{e}_1}}{\mathbf{e}_1^\top \frac{\mathbf{D}}{f_0}\mathbf{e}_1}\|\mathbf{e}_1\| \tag{B.3.4.6}$$

yielding (B.3.4.1). Similarly, for the minor axis, the only different parameter is

$$\mathbf{m} = \mathbf{e}_2, \tag{B.3.4.7}$$

Substituting the above in (B.3.2.1),

$$\frac{2\sqrt{\mathbf{e}_2^\top \frac{\mathbf{D}}{f_0}\mathbf{e}_2}}{\mathbf{e}_2^\top \frac{\mathbf{D}}{f_0}\mathbf{e}_2}\|\mathbf{e}_2\| \tag{B.3.4.8}$$

yielding (B.3.4.2).

B.3.5. The equation of the minor and major axes for the ellipse/hyperbola are respectively given by

$$\mathbf{p}_i^\top(\mathbf{x} - \mathbf{c}) = 0, i = 1, 2 \tag{B.3.5.1}$$

The axis of symmetry for the parabola is also given by (B.3.5.1).

Proof. From (B.2.3), the major/symmetry axis for the hyperbola/ellipse/parabola can be expressed using (2.1.15.1) as

$$\mathbf{e}_2^\top \mathbf{P}^\top(\mathbf{x} - \mathbf{c}) = 0 \tag{B.3.5.2}$$

$$\implies (\mathbf{P}\mathbf{e}_2)^\top(\mathbf{x} - \mathbf{c}) = 0 \tag{B.3.5.3}$$

yielding (B.3.5.1), and the proof for the minor axis is similar. □

B.3.6. The latus rectum of a conic section is the chord that passes through the focus and is perpendicular to the major axis. The length of the latus rectum for a conic is given by

$$l = \begin{cases} 2\frac{\sqrt{|f_0\lambda_1|}}{\lambda_2} & e \neq 1 \\ \frac{\eta}{\lambda_2} & e = 1 \end{cases} \tag{B.3.6.1}$$

Solution: The latus rectum is perpendicular to the major axis for the standard conic. Hence, from Corollary (B.2.3),

$$\mathbf{m} = \mathbf{e}_2, \tag{B.3.6.2}$$

Since it passes through the focus, from (B.2.7.3)

$$\mathbf{q} = \mathbf{F} = \pm e\sqrt{\frac{f_0}{\lambda_2(1 - e^2)}}\mathbf{e}_1 \tag{B.3.6.3}$$

for the standard hyperbola/ellipse. Also, from (B.1.5.1),

$$\mathbf{V} = \frac{\mathbf{D}}{f_0}, \mathbf{u} = 0, f = -1 \tag{B.3.6.4}$$

Substituting the above in (B.3.2.1), we obtain

$$\frac{2\sqrt{\left[\mathbf{e}_2^\top\left(\frac{\mathbf{D}}{f_0}e\sqrt{\frac{f_0}{\lambda_2(1-e^2)}}\mathbf{e}_1\right)\right]^2-\left(e\sqrt{\frac{f_0}{\lambda_2(1-e^2)}}\mathbf{e}_1^\top\frac{\mathbf{D}}{f_0}e\sqrt{\frac{f_0}{\lambda_2(1-e^2)}}\mathbf{e}_1-1\right)\left(\mathbf{e}_2^\top\frac{\mathbf{D}}{f_0}\mathbf{e}_2\right)}}{\mathbf{e}_2^\top\frac{\mathbf{D}}{f_0}\mathbf{e}_2}\|\mathbf{e}_2\| \tag{B.3.6.5}$$

Since

$$\mathbf{e}_2^\top\mathbf{D}\mathbf{e}_1=0,\mathbf{e}_1^\top\mathbf{D}\mathbf{e}_1=\lambda_1,\mathbf{e}_1^\top\mathbf{e}_1=1,\|\mathbf{e}_2\|=1,\mathbf{e}_2^\top\mathbf{D}\mathbf{e}_2=\lambda_2, \tag{B.3.6.6}$$

(B.3.6.5) can be expressed as

$$\frac{2\sqrt{\left(1-\frac{\lambda_1e^2}{\lambda_2(1-e^2)}\right)\left(\frac{\lambda_2}{f_0}\right)}}{\frac{\lambda_2}{f_0}} \tag{B.3.6.7}$$

$$=2\frac{\sqrt{f_0\lambda_1}}{\lambda_2} \qquad \left(\because e^2=1-\frac{\lambda_1}{\lambda_2}\right) \tag{B.3.6.8}$$

For the standard parabola, the parameters in (B.3.2.1) are

$$\mathbf{q}=\mathbf{F}=-\frac{\eta}{4\lambda_2}\mathbf{e}_1,\mathbf{m}=\mathbf{e}_1,\mathbf{V}=\mathbf{D},\mathbf{u}=\frac{\eta}{2}\mathbf{e}_1^\top,f=0 \tag{B.3.6.9}$$

Substituting the above in (B.3.2.1), the length of the latus rectum can be expressed as

$$\frac{2\sqrt{\left[\mathbf{e}_2^\top\left(\mathbf{D}\left(-\frac{\eta}{4\lambda_2}\mathbf{e}_1\right)+\frac{\eta}{2}\mathbf{e}_1\right)\right]^2-\left(\left(-\frac{\eta}{4\lambda_2}\mathbf{e}_1\right)^\top\mathbf{D}\left(-\frac{\eta}{4\lambda_2}\mathbf{e}_1\right)+2\frac{\eta}{2}\mathbf{e}_1^\top\left(-\frac{\eta}{4\lambda_2}\mathbf{e}_1\right)\right)\left(\mathbf{e}_2^\top\mathbf{D}\mathbf{e}_2\right)}}{\mathbf{e}_2^\top\mathbf{D}\mathbf{e}_2}\|\mathbf{e}_2\| \tag{B.3.6.10}$$

Since

$$\mathbf{e}_2^\top\mathbf{D}\mathbf{e}_1=0,\mathbf{e}_2^\top\mathbf{e}_2=0,\mathbf{e}_1^\top\mathbf{D}\mathbf{e}_1=0, \tag{B.3.6.11}$$

$$\mathbf{e}_1^\top\mathbf{e}_1=1,\|\mathbf{e}_1\|=1,\mathbf{e}_2^\top\mathbf{D}\mathbf{e}_2=\lambda_2, \tag{B.3.6.12}$$

(B.3.6.10) can be expressed as

$$2\frac{\sqrt{\frac{\eta^2}{4\lambda_2}\lambda_2}}{\lambda_2}=\frac{\eta}{\lambda_2} \tag{B.3.6.13}$$

B.4 Tangent and Normal

B.4.1. If L in (B.3.1.1) touches (B.1.2.1) at exactly one point $\mathbf{q}$,

$$\mathbf{m}^\top(\mathbf{V}\mathbf{q}+\mathbf{u})=0 \tag{B.4.1.1}$$

Proof. In this case, (B.3.1.6) has exactly one root. Hence, in (B.3.1.3)

$$\left[\mathbf{m}^\top(\mathbf{V}\mathbf{q}+\mathbf{u})\right]^2-\left(\mathbf{m}^\top\mathbf{V}\mathbf{m}\right)g(\mathbf{q})=0 \tag{B.4.1.2}$$

$\because$ $\mathbf{q}$ is the point of contact,

$$g(\mathbf{q}) = 0 \tag{B.4.1.3}$$

Substituting (B.4.1.3) in (B.4.1.2) and simplifying, we obtain (B.4.1.1). □

B.4.2. Given the point of contact $\mathbf{q}$, the equation of a tangent to (B.1.2.1) is

$$(\mathbf{V}\mathbf{q} + \mathbf{u})^\top \mathbf{x} + \mathbf{u}^\top \mathbf{q} + f = 0 \tag{B.4.2.1}$$

Proof. The normal vector is obtained from (B.4.1.1) as

$$\kappa\mathbf{n} = \mathbf{V}\mathbf{q} + \mathbf{u}, \kappa \in \mathbb{R} \tag{B.4.2.2}$$

From (B.4.2.2), the equation of the tangent is

$$(\mathbf{V}\mathbf{q} + \mathbf{u})^\top (\mathbf{x} - \mathbf{q}) = 0 \tag{B.4.2.3}$$

$$\implies (\mathbf{V}\mathbf{q} + \mathbf{u})^\top \mathbf{x} - \mathbf{q}^\top\mathbf{V}\mathbf{q} - \mathbf{u}^\top\mathbf{q} = 0 \tag{B.4.2.4}$$

which, upon substituting from (B.4.1.3) and simplifying yields (B.4.2.1) □

B.4.3. Given the point of contact $\mathbf{q}$, the equation of the normal to (B.1.2.1) is

$$(\mathbf{V}\mathbf{q} + \mathbf{u})^\top \mathbf{R}(\mathbf{x} - \mathbf{q}) = 0 \tag{B.4.3.1}$$

Proof. The direction vector of the tangent is obtained from (B.4.2.2) as as

$$\mathbf{m} = \mathbf{R}(\mathbf{V}\mathbf{q} + \mathbf{u}), \tag{B.4.3.2}$$

where $\mathbf{R}$ is the rotation matrix. From (B.4.3.2), the equation of the normal is given by (10.1.3.1) □

B.4.4. Given the tangent

$$\mathbf{n}^\top\mathbf{x} = c, \tag{B.4.4.1}$$

the point of contact to the conic in (B.1.2.1) is given by

$$\begin{pmatrix} \mathbf{n}^\top \\ \mathbf{m}^\top\mathbf{V} \end{pmatrix} \mathbf{q} = \begin{pmatrix} c \\ -\mathbf{m}^\top\mathbf{u} \end{pmatrix} \tag{B.4.4.2}$$

Proof. From (B.4.1.1),

$$\mathbf{m}^\top(\mathbf{V}\mathbf{q} + \mathbf{u}) = 0 \tag{B.4.4.3}$$

$$\implies \mathbf{m}^\top\mathbf{V}\mathbf{q} = -\mathbf{m}^\top\mathbf{u} \tag{B.4.4.4}$$

Combining (B.4.4.1) and (B.4.4.4), (B.4.4.2) is obtained.

□

B.4.5. If $\mathbf{V}^{-1}$ exists, given the normal vector $\mathbf{n}$, the tangent points of contact to (B.1.2.1) are given by

$$\mathbf{q}_i = \mathbf{V}^{-1}(\kappa_i\mathbf{n} - \mathbf{u}), i = 1, 2 \\ \text{where } \kappa_i = \pm\sqrt{\frac{f_0}{\mathbf{n}^\top\mathbf{V}^{-1}\mathbf{n}}} \tag{B.4.5.1}$$

Proof. From (B.4.2.2),

$$\mathbf{q} = \mathbf{V}^{-1}(\kappa\mathbf{n} - \mathbf{u}), \quad \kappa \in \mathbb{R} \tag{B.4.5.2}$$

Substituting (B.4.5.2) in (B.4.1.3),

$$(\kappa\mathbf{n} - \mathbf{u})^\top \mathbf{V}^{-1}(\kappa\mathbf{n} - \mathbf{u}) + 2\mathbf{u}^\top\mathbf{V}^{-1}(\kappa\mathbf{n} - \mathbf{u}) + f = 0 \tag{B.4.5.3}$$

$$\implies \kappa^2\mathbf{n}^\top\mathbf{V}^{-1}\mathbf{n} - \mathbf{u}^\top\mathbf{V}^{-1}\mathbf{u} + f = 0 \tag{B.4.5.4}$$

$$\text{or, } \kappa = \pm\sqrt{\frac{f_0}{\mathbf{n}^\top\mathbf{V}^{-1}\mathbf{n}}} \tag{B.4.5.5}$$

Substituting (B.4.5.5) in (B.4.5.2) yields (B.4.5.1). □

B.4.6. For a conic/hyperbola, a line with normal vector $\mathbf{n}$ cannot be a tangent if

$$\frac{\mathbf{u}^\top\mathbf{V}^{-1}\mathbf{u} - f}{\mathbf{n}^\top\mathbf{V}^{-1}\mathbf{n}} < 0 \tag{B.4.6.1}$$

B.4.7. If $\mathbf{V}$ is not invertible, given the normal vector $\mathbf{n}$, the point of contact to (B.1.2.1) is given by the matrix equation

$$\begin{pmatrix} (\mathbf{u} + \kappa\mathbf{n})^\top \\ \mathbf{V} \end{pmatrix} \mathbf{q} = \begin{pmatrix} -f \\ \kappa\mathbf{n} - \mathbf{u} \end{pmatrix} \tag{B.4.7.1}$$

$$\text{where } \kappa = \frac{\mathbf{p}_1^\top\mathbf{u}}{\mathbf{p}_1^\top\mathbf{n}}, \quad \mathbf{V}\mathbf{p}_1 = 0 \tag{B.4.7.2}$$

Proof. If $\mathbf{V}$ is non-invertible, it has a zero eigenvalue. If the corresponding eigenvector is $\mathbf{p}_1$, then,

$$\mathbf{V}\mathbf{p}_1 = 0 \tag{B.4.7.3}$$

From (B.4.2.2),

$$\kappa\mathbf{n} = \mathbf{V}\mathbf{q} + \mathbf{u}, \quad \kappa \in \mathbb{R} \tag{B.4.7.4}$$

$$\implies \kappa\mathbf{p}_1^\top\mathbf{n} = \mathbf{p}_1^\top\mathbf{V}\mathbf{q} + \mathbf{p}_1^\top\mathbf{u} \tag{B.4.7.5}$$

$$\text{or, } \kappa\mathbf{p}_1^\top\mathbf{n} = \mathbf{p}_1^\top\mathbf{u}, \quad \because \mathbf{p}_1^\top\mathbf{V} = 0, \quad (\text{ from (B.4.7.3)}) \tag{B.4.7.6}$$

yielding κ in (B.4.7.2). From (B.4.7.4),

$$\kappa\mathbf{q}^\top\mathbf{n} = \mathbf{q}^\top\mathbf{V}\mathbf{q} + \mathbf{q}^\top\mathbf{u} \tag{B.4.7.7}$$

$$\implies \kappa\mathbf{q}^\top\mathbf{n} = -f - \mathbf{q}^\top\mathbf{u} \quad \text{from (B.4.1.3)}, \tag{B.4.7.8}$$

$$\text{or, } (\kappa\mathbf{n} + \mathbf{u})^\top\mathbf{q} = -f \tag{B.4.7.9}$$

(B.4.7.4) can be expressed as

$$\mathbf{V}\mathbf{q} = \kappa\mathbf{n} - \mathbf{u}. \tag{B.4.7.10}$$

(B.4.7.9) and (B.4.7.10) clubbed together result in (B.4.7.1). □

B.4.8. The asymptotes of the hyperbola in (B.1.5.1), defined to be the lines that do not intersect the hyperbola, are given by

$$\left(\sqrt{|\lambda_1|} \quad \pm\sqrt{|\lambda_2|}\right)\mathbf{y} = 0 \tag{B.4.8.1}$$

Proof. From (B.1.5.1), it is obvious that the pair of lines represented by

$$\mathbf{y}^\top \mathbf{D}\mathbf{y} = 0 \tag{B.4.8.2}$$

do not intersect the conic

$$\mathbf{y}^\top \mathbf{D}\mathbf{y} = f_0 \tag{B.4.8.3}$$

Thus, (B.4.8.2) represents the asysmptotes of the hyperbola in (B.1.5.1) and can be expressed as

$$\lambda_1 y_1^2 + \lambda_2 y_1^2 = 0, \tag{B.4.8.4}$$

which can then be simplified using the steps in (A.8.3.4)- (A.8.3.7) to obtain (B.4.8.1). □

B.4.9. (B.1.2.1) represents a pair of straight lines if

$$\mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f = 0 \tag{B.4.9.1}$$

B.4.10. (B.1.2.1) represents a pair of straight lines if the matrix

$$\begin{pmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^\top & f \end{pmatrix} \tag{B.4.10.1}$$

is singular.

Proof. Let

$$\begin{pmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^\top & f \end{pmatrix}\mathbf{x} = \mathbf{0} \tag{B.4.10.2}$$

Expressing

$$\mathbf{x} = \begin{pmatrix} \mathbf{y} \\ y_3 \end{pmatrix}, \tag{B.4.10.3}$$

$$\begin{pmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^\top & f \end{pmatrix}\begin{pmatrix} \mathbf{y} \\ y_3 \end{pmatrix} = \mathbf{0} \tag{B.4.10.4}$$

$$\implies \mathbf{V}\mathbf{y} + y_3\mathbf{u} = \mathbf{0} \quad \text{and} \tag{B.4.10.5}$$

$$\mathbf{u}^\top \mathbf{y} + f y_3 = 0 \tag{B.4.10.6}$$

From (B.4.10.5) we obtain,

$$\mathbf{y}^\top \mathbf{V}\mathbf{y} + y_3\mathbf{y}^\top \mathbf{u} = \mathbf{0} \tag{B.4.10.7}$$

$$\implies \mathbf{y}^\top \mathbf{V}\mathbf{y} + y_3\mathbf{u}^\top \mathbf{y} = \mathbf{0} \tag{B.4.10.8}$$

yielding (B.4.9.1) upon substituting from (B.4.10.6). □

B.4.11. Using the affine transformation, (B.4.8.1) can be expressed as the lines

$$\left(\sqrt{|\lambda_1|} \quad \pm\sqrt{|\lambda_2|}\right)\mathbf{P}^\top(\mathbf{x}-\mathbf{c}) = 0 \tag{B.4.11.1}$$

B.4.12. The angle between the asymptotes can be expressed as

$$\cos\theta = \frac{|\lambda_1| - |\lambda_2|}{|\lambda_1| + |\lambda_2|} \tag{B.4.12.1}$$

Proof. The normal vectors of the lines in (B.4.11.1) are

$$\begin{aligned} \mathbf{n}_1 &= \mathbf{P}\begin{pmatrix}\sqrt{|\lambda_1|}\\ \sqrt{|\lambda_2|}\end{pmatrix} \\ \mathbf{n}_2 &= \mathbf{P}\begin{pmatrix}\sqrt{|\lambda_1|}\\ -\sqrt{|\lambda_2|}\end{pmatrix} \end{aligned} \tag{B.4.12.2}$$

The angle between the asymptotes is given by

$$\cos\theta = \frac{\mathbf{n_1}^\top\mathbf{n_2}}{\|\mathbf{n_1}\|\,\|\mathbf{n_2}\|} \tag{B.4.12.3}$$

The orthogonal matrix $\mathbf{P}$ preserves the norm, i.e.

$$\|\mathbf{n_1}\| = \left\|\mathbf{P}\begin{pmatrix}\sqrt{|\lambda_1|}\\ \sqrt{|\lambda_2|}\end{pmatrix}\right\| = \left\|\begin{pmatrix}\sqrt{|\lambda_1|}\\ \sqrt{|\lambda_2|}\end{pmatrix}\right\| \tag{B.4.12.4}$$

$$= \sqrt{|\lambda_1| + |\lambda_2|} = \|\mathbf{n_2}\| \tag{B.4.12.5}$$

It is easy to verify that

$$\mathbf{n_1}^\top\mathbf{n_2} = |\lambda_1| - |\lambda_2| \tag{B.4.12.6}$$

Thus, the angle between the asymptotes is obtained from (B.4.12.3) as (B.4.12.1). □

B.4.13. For a circle, the points of contact are

$$\mathbf{q}_{ij} = \left(\pm r\frac{\mathbf{n}_j}{\left\|\mathbf{n}_j\right\|} - \mathbf{u}\right), \quad i, j = 1, 2 \tag{B.4.13.1}$$

Proof. From (B.4.5.1), and (7.1.1.1),

$$\kappa_{ij} = \pm\frac{r}{\left\|\mathbf{n}_j\right\|} \tag{B.4.13.2}$$

□

B.4.14. A point $\mathbf{h}$ lies on a normal to the conic in (B.1.2.1) if

$$\left(\mathbf{m}^\top(\mathbf{V}\mathbf{h}+\mathbf{u})\right)^2\left(\mathbf{n}^\top\mathbf{V}\mathbf{n}\right) - 2\left(\mathbf{m}^\top\mathbf{V}\mathbf{n}\right)\left(\mathbf{m}^\top(\mathbf{V}\mathbf{h}+\mathbf{u})\mathbf{n}^\top(\mathbf{V}\mathbf{h}+\mathbf{u})\right) + g(\mathbf{h})\left(\mathbf{m}^\top\mathbf{V}\mathbf{n}\right)^2 = 0 \tag{B.4.14.1}$$

Proof. The point of contact for the normal passing through a point $\mathbf{h}$ is given by

$$\mathbf{q} = \mathbf{h} + \mu\mathbf{n} \tag{B.4.14.2}$$

From (B.4.1.1), the tangent at $\mathbf{q}$ satisfies

$$\mathbf{m}^\top(\mathbf{V}\mathbf{q}+\mathbf{u}) = 0 \tag{B.4.14.3}$$

Substituting (B.4.14.2) in (B.4.14.3),

$$\begin{aligned} &\mathbf{m}^\top(\mathbf{V}(\mathbf{h}+\mu\mathbf{n})+\mathbf{u}) = 0 && \text{(B.4.14.4)}\\ \implies &\mu\mathbf{m}^\top\mathbf{V}\mathbf{n} = -\mathbf{m}^\top(\mathbf{V}\mathbf{h}+\mathbf{u}) && \text{(B.4.14.5)} \end{aligned}$$

yielding

$$\mu = -\frac{\mathbf{m}^\top(\mathbf{V}\mathbf{h}+\mathbf{u})}{\mathbf{m}^\top\mathbf{V}\mathbf{n}}, \tag{B.4.14.6}$$

From (B.3.1.6),

$$\mu^2\mathbf{n}^\top\mathbf{V}\mathbf{n} + 2\mu\mathbf{n}^\top(\mathbf{V}\mathbf{h}+\mathbf{u}) + \mathrm{g}(\mathbf{h}) = 0 \tag{B.4.14.7}$$

From (B.4.14.6), (B.4.14.7) can be expressed as

$$\left(-\frac{\mathbf{m}^\top(\mathbf{V}\mathbf{h}+\mathbf{u})}{\mathbf{m}^\top\mathbf{V}\mathbf{n}}\right)^2\mathbf{n}^\top\mathbf{V}\mathbf{n} + 2\left(-\frac{\mathbf{m}^\top(\mathbf{V}\mathbf{h}+\mathbf{u})}{\mathbf{m}^\top\mathbf{V}\mathbf{n}}\right)\mathbf{n}^\top(\mathbf{V}\mathbf{h}+\mathbf{u}) + \mathrm{g}(\mathbf{h}) = 0 \tag{B.4.14.8}$$

yielding (B.4.14.1). □

B.4.15. A point $\mathbf{h}$ lies on a tangent to the conic in (B.1.2.1) if

$$\mathbf{m}^\top\left[(\mathbf{V}\mathbf{h}+\mathbf{u})(\mathbf{V}\mathbf{h}+\mathbf{u})^\top - \mathbf{V}\mathrm{g}(\mathbf{h})\right]\mathbf{m} = 0 \tag{B.4.15.1}$$

Proof. From (B.3.1.3) and (B.4.1.2)

$$\left[\mathbf{m}^\top(\mathbf{V}\mathbf{h}+\mathbf{u})\right]^2 - \left(\mathbf{m}^\top\mathbf{V}\mathbf{m}\right)\mathrm{g}(\mathbf{h}) = 0 \tag{B.4.15.2}$$

yielding (B.4.15.1). □

B.4.16. The normal vectors of the tangents to the conic in (B.1.2.1) from a point $\mathbf{h}$ are given by

$$\begin{aligned} \mathbf{n}_1 &= \mathbf{P}\begin{pmatrix}\sqrt{|\lambda_1|}\\ \sqrt{|\lambda_2|}\end{pmatrix}\\ \mathbf{n}_2 &= \mathbf{P}\begin{pmatrix}\sqrt{|\lambda_1|}\\ -\sqrt{|\lambda_2|}\end{pmatrix} \end{aligned} \tag{B.4.16.1}$$

where $\lambda_i, \mathbf{P}$ are the eigenparameters of

$$\mathbf{\Sigma} = (\mathbf{V}\mathbf{h}+\mathbf{u})(\mathbf{V}\mathbf{h}+\mathbf{u})^\top - (\mathrm{g}(\mathbf{h}))\mathbf{V}. \tag{B.4.16.2}$$

Proof. From (B.4.15.1) we obtain (B.4.16.2). Consequently, from (B.4.12.2), (B.4.16.1) can be obtained. □

B.4.17. (B.1.2.1) represents a pair of straight lines if the matrix

$$\begin{pmatrix}\mathbf{V} & \mathbf{u}\\ \mathbf{u}^\top & f\end{pmatrix} \tag{B.4.17.1}$$

is singular.

B.4.18. The intersection of two conics with parameters $\mathbf{V}_i, \mathbf{u}_i, f_i,\ i = 1, 2$ is defined as

$$\mathbf{x}^\top (\mathbf{V}_1 + \mu\mathbf{V}_2)\,\mathbf{x} + 2\,(\mathbf{u}_1 + \mu\mathbf{u}_2)^\top \mathbf{x} + (f_1 + \mu f_2) = 0 \tag{B.4.18.1}$$

B.4.19. From (B.4.17.1), (B.4.18.1) represents a pair of straight lines if

$$\begin{vmatrix} \mathbf{V}_1 + \mu\mathbf{V}_2 & \mathbf{u}_1 + \mu\mathbf{u}_2 \\ (\mathbf{u}_1 + \mu\mathbf{u}_2)^\top & f_1 + \mu f_2 \end{vmatrix} = 0 \tag{B.4.19.1}$$