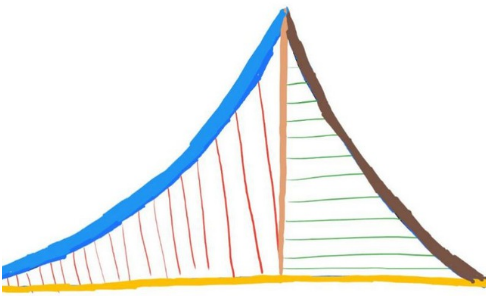
MATRICES
In Geometry



G. V. V. Sharma

About this Book

This book attempts to introduce matrices through high school coordinate geometry. This approach makes it easier for beginners to learn Python for scientific computing. All problems in the book are from NCERT mathematics textbooks from Class 9-12. The content is sufficient for industry jobs. There is no copyright, so readers are free to print and share.

July 18, 2024

Github: https://github.com/gadepall/matgeo

Contents

1.1 Formulae

1.1.1. The *direction vector* of AB is defined as

$$\mathbf{m} = \mathbf{B} - \mathbf{A} = \kappa \begin{pmatrix} 1 \\ m \end{pmatrix} \tag{1.1.1.1}$$

where m is the slope of AB. We also say that

$$\mathbf{m} \equiv \begin{pmatrix} 1 \\ m \end{pmatrix} \tag{1.1.1.2}$$

1.1.2. The lines with direction vectors $\mathbf{m}_1$ and $\mathbf{m}_2$ respectively, are parallel if

$$\mathbf{m}_1 \equiv \mathbf{m}_2 \tag{1.1.2.1}$$

1.1.3. If $ABCD$ be a parallelogram with $AB \parallel CD$,

$$\mathbf{B} - \mathbf{A} = \mathbf{C} - \mathbf{D} \tag{1.1.3.1}$$

1.1.4. If $\mathbf{D}$ divides BC in the ratio $k : 1$,

$$\mathbf{D} = \frac{k\mathbf{C} + \mathbf{B}}{k+1} \tag{1.1.4.1}$$

1.1.5. If $PQRS$ is formed by joining the mid points of $ABCD$,

$$\mathbf{P} = \frac{1}{2}(\mathbf{A} + \mathbf{B}), \ \mathbf{Q} = \frac{1}{2}(\mathbf{B} + \mathbf{C}) \tag{1.1.5.1}$$

$$\mathbf{R} = \frac{1}{2}(\mathbf{C} + \mathbf{D}), \ \mathbf{S} = \frac{1}{2}(\mathbf{D} + \mathbf{A}) \tag{1.1.5.2}$$

$$\implies \mathbf{P} - \mathbf{Q} = \mathbf{S} - \mathbf{R}. \tag{1.1.5.3}$$

Hence, $PQRS$ is a parallelogram from (1.1.3.1).

1.1.6. In 2D space, the basis vectors are defined as

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{1.1.6.1}$$

1.1.7. The length of a vector is defined as

$$\|\mathbf{x}\| \triangleq \sqrt{\mathbf{x}^\top \mathbf{x}} \tag{1.1.7.1}$$

For example, if

$$\mathbf{x} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \tag{1.1.7.2}$$

$$\mathbf{x}^\top \mathbf{x} = \begin{pmatrix} 3 & 4 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} \tag{1.1.7.3}$$

$$= 3 \times 3 + 4 \times 4 = 25 \tag{1.1.7.4}$$

yielding

$$\|\mathbf{x}\| = 5. \tag{1.1.7.5}$$

(1.1.7.3) is known as the scalar product.

1.1.8. The unit vector in the direction of $\mathbf{x}$ is

$$\frac{\mathbf{x}}{\|\mathbf{x}\|} \tag{1.1.8.1}$$

1.1.9. Points $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are defined to be collinear if

$$\text{rank}\begin{pmatrix}\mathbf{B} - \mathbf{A} & \mathbf{C} - \mathbf{A}\end{pmatrix} = 1 \tag{1.1.9.1}$$

1.1.10.

$$\text{rank}\mathbf{A} = \text{rank}\mathbf{A}^\top \tag{1.1.10.1}$$

1.1.11. In the 2D space, the unit direction vector is defined as

$$\mathbf{m} = \begin{pmatrix}\cos\alpha \\ \cos\beta\end{pmatrix} \tag{1.1.11.1}$$

where α, β are the angles made by the vector with the axes.

1.2 Direction

1.2.1 Find the values of x and y so that the vectors $2\hat{i} + 3\hat{j}$ and $x\hat{i} + y\hat{j}$ are equal.
Solution: From the given informatin,

$$\begin{pmatrix}2 \\ 3\end{pmatrix} = \begin{pmatrix}x \\ y\end{pmatrix} \tag{1.2.1.1}$$

$$\implies x = 2, y = 3 \tag{1.2.1.2}$$

1.2.2 Find the values of x, y, z so that the vectors $x\hat{i} + 2\hat{j} + z\hat{k}$ and $2\hat{i} + y\hat{j} + \hat{k}$ are equal.

1.2.3 Find the sum of the vectors $\mathbf{a} = \hat{i} - 2\hat{j} + \hat{k}$, $\mathbf{b} = -2\hat{i} + 4\hat{j} + 5\hat{k}$ and $\mathbf{c} = \hat{i} - 6\hat{j} - 7\hat{k}$.

1.2.4 Find the slope of a line, which passes through the origin and the mid point of the line segment joining the points $\mathbf{P}$(0,-4) and $\mathbf{B}$(8,0).
Solution: The mid point of PB is

$$\mathbf{M} = \frac{1}{2}(\mathbf{P} + \mathbf{B}) = \begin{pmatrix}4 \\ -2\end{pmatrix} \tag{1.2.4.1}$$

which, from (1.1.1.1), is equal to the direction vector of OM, where $\mathbf{O}$ is the origin.

$$\because \mathbf{M} \equiv \begin{pmatrix}1 \\ -\frac{1}{2}\end{pmatrix}, m = -\frac{1}{2} \tag{1.2.4.2}$$

which, from (1.1.1.1), is the desired slope. See Fig. 1.2.4.1.

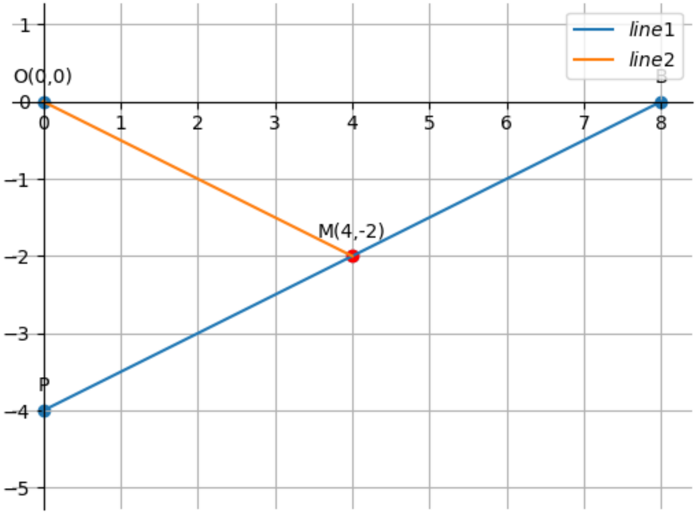


Fig. 1.2.4.1

1.2.5 Find the angle between x-axis and the line joining points (3,-1) and (4,-2).
Solution: The direction vector of the given line is

$$\begin{pmatrix} 4 \\ -2 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \implies m = -1 \tag{1.2.5.1}$$

Hence, the desired angle is 135°.

1.2.6 A line passes through $A(x_1, y_1)$ and $B(h, k)$. If slope of the line is m, show that $(k - y_1) = m(h - x_1)$.
Solution: The direction vector

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} h - x_1 \\ k - y_1 \end{pmatrix} \equiv \begin{pmatrix} 1 \\ \frac{k-y_1}{h-x_1} \end{pmatrix} \tag{1.2.6.1}$$

$$\implies m = \frac{k - y_1}{h - x_1}, \tag{1.2.6.2}$$

yielding the desired result.

1.2.7 Show that the line through the points $(4, 7, 8)$,$(2, 3, 4)$ is parallel to the line through the points $(-1, -2, 1)$,$(1, 2, 5)$.
Solution:

$$\begin{pmatrix} 4 \\ 7 \\ 8 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} \equiv \begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix} \tag{1.2.7.1}$$

which means that the given lines have the same direction vector and are hence parallel.

1.2.8 The vector having intial and terminal points as (-2,5,0) and (3,7,4),respectively is

a) $-\hat{i}+12\hat{j}+4\hat{k}$
b) $5\hat{i}+2\hat{j}-4\hat{k}$
c) $5\hat{i}+2\hat{j}+4\hat{k}$
d) $\hat{i}+\hat{j}+\hat{k}$

Solution: The desired vector is

$$\begin{pmatrix}3\\7\\4\end{pmatrix}-\begin{pmatrix}-2\\5\\0\end{pmatrix}=\begin{pmatrix}5\\2\\4\end{pmatrix} \tag{1.2.8.1}$$

1.2.9 Find the vector joining the points $P(2,3,0)$ and $Q(-1,-2,-4)$ directed from P to Q.

1.2.10 Without using distance formula, show that points $A(-2,-1), B(4,0), C(3,3)$ and $D(-3,2)$ are the vertices of a parallelogram.

Solution: From (1.1.3.1),

$$\mathbf{A}-\mathbf{B}=\mathbf{D}-\mathbf{C}=\begin{pmatrix}-6\\-1\end{pmatrix} \tag{1.2.10.1}$$

Hence, $ABCD$ is a parallelogram. See Fig. 1.2.10.1.

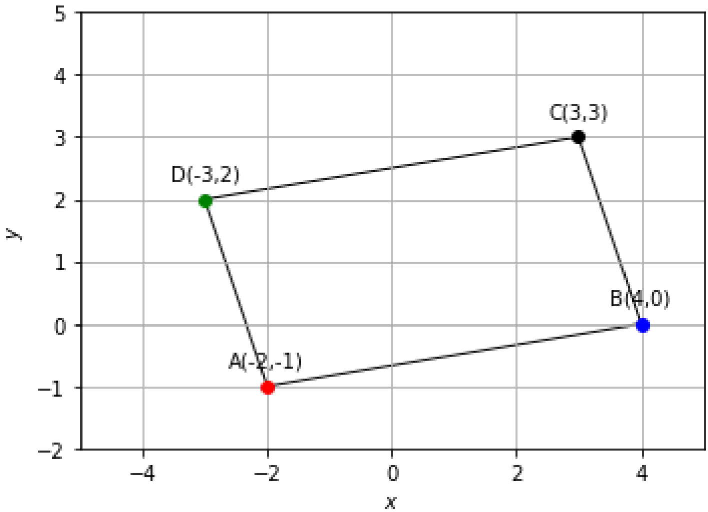


Fig. 1.2.10.1

1.2.11 If the points $A(6,1), B(8,2), C(9,4)$ and $D(p,3)$ are the vertices of a parallelogram, taken in order, find the value of p.

1.2.12 If $(1,2),(4,y),(x,6)$ and $(3,5)$ are the vertices of a parallelogram taken in order, find x and y.

Solution: Since $ABCD$ is a parallelogram,

$$\begin{pmatrix}4\\y\end{pmatrix}-\begin{pmatrix}1\\2\end{pmatrix}=\begin{pmatrix}x\\6\end{pmatrix}-\begin{pmatrix}3\\5\end{pmatrix} \tag{1.2.12.1}$$

$$\implies \begin{pmatrix}3\\y-2\end{pmatrix}=\begin{pmatrix}x-3\\1\end{pmatrix} \tag{1.2.12.2}$$

$$\text{or, } x=6, y=3. \tag{1.2.12.3}$$

See Fig. 1.2.12.1.

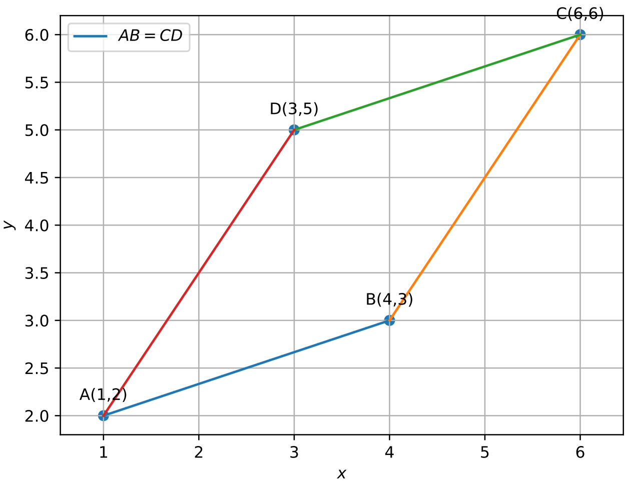


Fig. 1.2.12.1

1.2.13 The fourth vertex **D** of a parallelogram **ABCD** whose three vertices are $\mathbf{A}(-2,3), \mathbf{B}(6,7)$ and $\mathbf{C}(8,3)$ is

a) $(0,1)$
b) $(0,-1)$
c) $(-1,0)$
d) $(1,0)$

1.2.14 Verify if the points $\mathbf{A}(4,3), \mathbf{B}(6,4), \mathbf{C}(5,-6)$ and $\mathbf{D}(-3,5)$ are the vertices of a parallelogram.

1.2.15 A girl walks 4 km towards west, then she walks 3 km in a direction 30° east of north and stops. Determine the girl's displacement from her initial point of departure.

Solution: See Fig. 1.2.15.1. Let the initial position be

$$\mathbf{A}=\begin{pmatrix}0\\0\end{pmatrix} \tag{1.2.15.1}$$

After going west, the position becomes

$$\mathbf{B} = \begin{pmatrix} -4 \\ 0 \end{pmatrix} \tag{1.2.15.2}$$

If the final position be $\mathbf{C}$, from the given information,

$$\mathbf{C} - \mathbf{B} = 3\begin{pmatrix} \cos 60^\circ \\ \sin 60^\circ \end{pmatrix} \implies \mathbf{C} = \begin{pmatrix} \frac{-5}{2} \\ \frac{3\sqrt{3}}{2} \end{pmatrix} \tag{1.2.15.3}$$

which is the desired displacement.

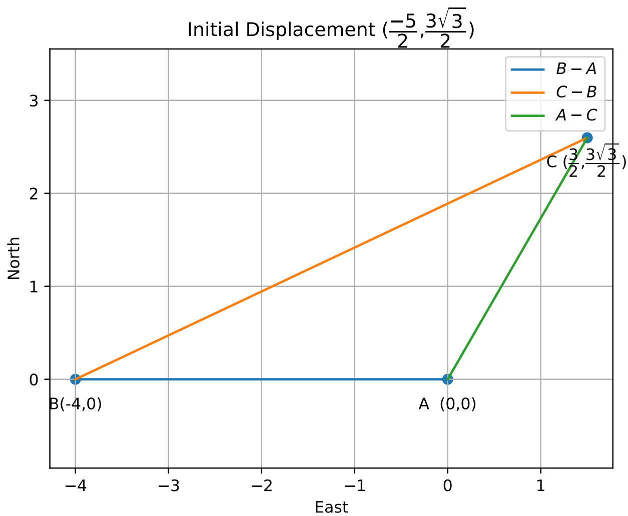


Fig. 1.2.15.1

1.3 Section Formula

1.3.1 Find the coordinates of the point which divides the join of $(-1, 7)$ and $(4, -3)$ in the ratio 2:3.
Solution: Using section formula (1.1.4.1), the desired point is

$$\frac{1}{1+\frac{3}{2}}\left(\begin{pmatrix} 4 \\ -3 \end{pmatrix} + \frac{3}{2}\begin{pmatrix} -1 \\ 7 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \tag{1.3.1.1}$$

See Fig. 1.3.1.1

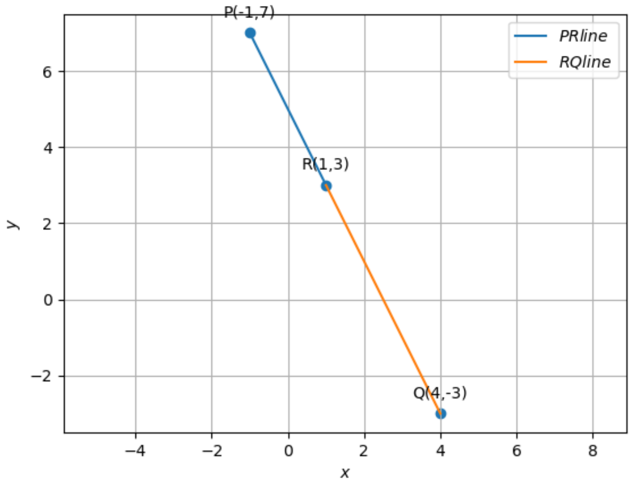


Fig. 1.3.1.1

1.3.2 Find the coordinates of the points of trisection of the line segment joining $(4, -1)$ and $(-2, 3)$.

Solution: Using section formula,

$$\mathbf{R} = \frac{1}{1+\frac{1}{2}}\left(\begin{pmatrix}4\\-1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}-2\\-3\end{pmatrix}\right) = \begin{pmatrix}2\\\frac{-5}{3}\end{pmatrix} \tag{1.3.2.1}$$

$$\mathbf{S} = \frac{1}{1+\frac{2}{1}}\left(\begin{pmatrix}4\\-1\end{pmatrix} + \frac{2}{1}\begin{pmatrix}-2\\-3\end{pmatrix}\right) = \begin{pmatrix}0\\\frac{-7}{3}\end{pmatrix} \tag{1.3.2.2}$$

which are the desired points of trisection. See Fig. 1.3.2.1

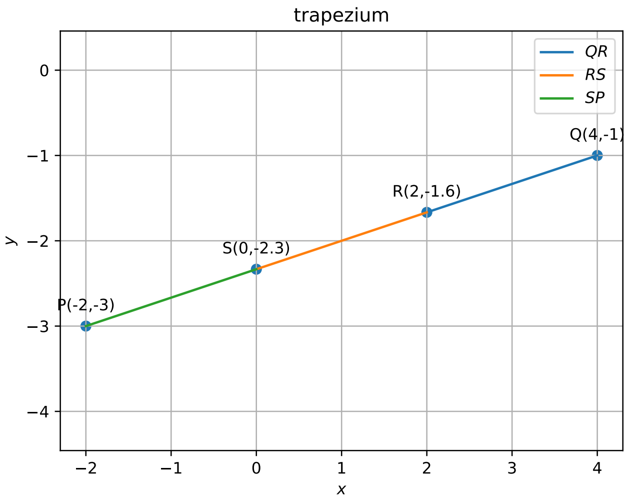


Fig. 1.3.2.1

1.3.3 Find the ratio in which the line segment joining the points $(-3, 10)$ and $(6, -8)$ is divided by $(-1, 6)$.

Solution: Using section formula,

$$\begin{pmatrix} -1 \\ 6 \end{pmatrix} = \frac{\begin{pmatrix} -3 \\ 10 \end{pmatrix} + k \begin{pmatrix} 6 \\ -8 \end{pmatrix}}{1 + n} \tag{1.3.3.1}$$

$$\implies 7k \begin{pmatrix} 1 \\ -2 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} \tag{1.3.3.2}$$

$$\text{or, } k = \frac{2}{7} \tag{1.3.3.3}$$

See Fig. 1.3.3.1.

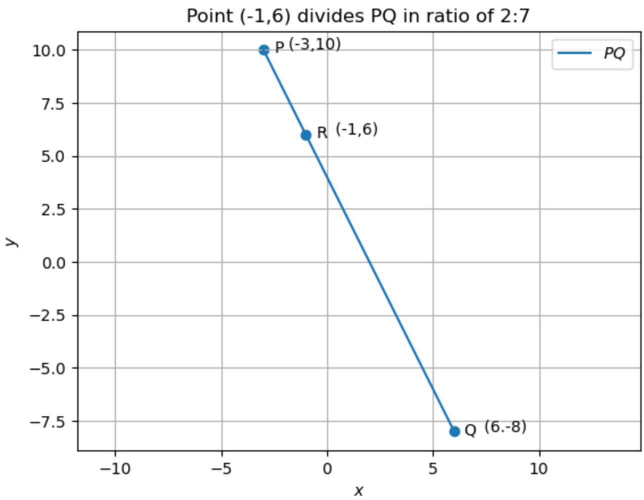


Fig. 1.3.3.1

1.3.4 Find the coordinates of a point A, where AB is the diameter of a circle whose centre is $C(2,-3)$ and B is $(1,4)$.
Solution:

$$\mathbf{C} = \frac{\mathbf{A}+\mathbf{B}}{2} \implies \mathbf{A} = 2\mathbf{C} - \mathbf{B} = \begin{pmatrix} 3 \\ -10 \end{pmatrix} \tag{1.3.4.1}$$

See Fig. 1.3.4.1.

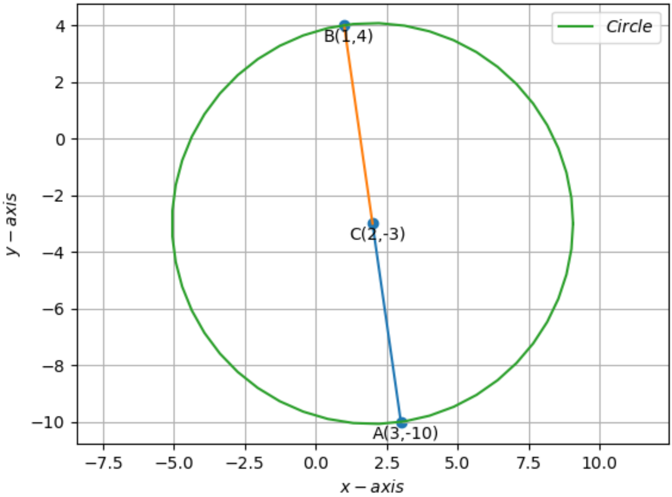


Fig. 1.3.4.1

1.3.5 If A and B are $(-2,-2)$ and $(2,-4)$, respectively, find the coordinates of P such that $AP = \frac{3}{7}AB$ and P lies on the line segment AB.
Solution: Using section formula,

$$\mathbf{P} = \frac{1}{1+\frac{3}{4}}\left(\begin{pmatrix}-2\\-2\end{pmatrix} + \frac{3}{4}\begin{pmatrix}2\\-4\end{pmatrix}\right) = \begin{pmatrix}\frac{-2}{7}\\\frac{-20}{7}\end{pmatrix} \tag{1.3.5.1}$$

See Fig. 1.3.5.1.

1.3.6 Find the coordinates of the points which divide the line segment joining $A(-2,2)$ and $B(2,8)$ into four equal parts.
Solution: Using section formula,

$$\mathbf{R}_k = \frac{\mathbf{B}+k\mathbf{A}}{1+k} \tag{1.3.6.1}$$

See Table 1.3.6 and Fig. 1.3.6.1

TABLE 1.3.6

k	$\mathbf{R}_k$
3	$\begin{pmatrix}-1\\\frac{7}{2}\end{pmatrix}$
1	$\begin{pmatrix}0\\5\end{pmatrix}$
$\frac{1}{3}$	$\begin{pmatrix}1\\\frac{13}{2}\end{pmatrix}$

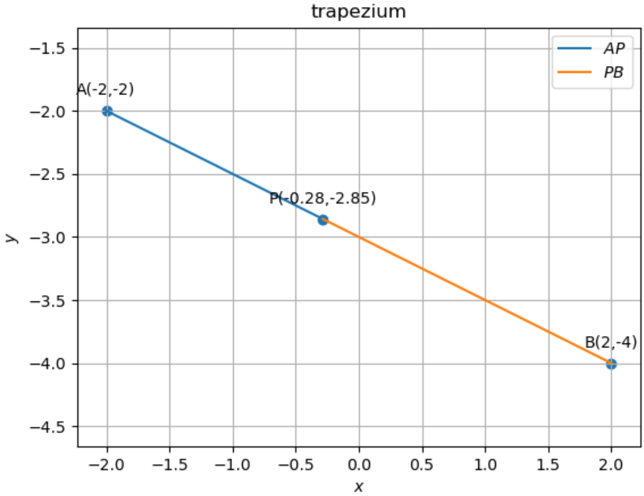


Fig. 1.3.5.1

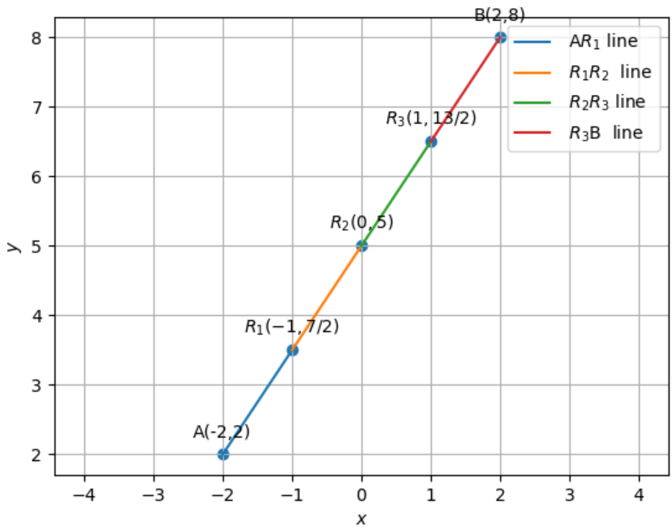


Fig. 1.3.6.1

1.3.7 Find the position vector of a point **R** which divides the line joining two points **P** and **Q** whose position vectors are $\hat{i}+2\hat{j}-\hat{k}$ and $-\hat{i}+\hat{j}+\hat{k}$ respectively, in the ratio 2 : 1

a) internally

b) externally

Solution: See Table 1.3.7.

TABLE 1.3.7

k	R_k
2	$\frac{1}{3}\begin{pmatrix}-1\\4\\1\end{pmatrix}$
-2	$\begin{pmatrix}-3\\0\\3\end{pmatrix}$

1.3.8 Find the position vector of the mid point of the vector joining the points **P**(2, 3, 4) and **Q**(4, 1, –2).

Solution: The desired vector is

$$\frac{1}{2}\begin{pmatrix}2\\3\\4\end{pmatrix}+\frac{1}{2}\begin{pmatrix}4\\1\\-2\end{pmatrix}=\begin{pmatrix}3\\2\\1\end{pmatrix} \tag{1.3.8.1}$$

1.3.9 Determine the ratio in which the line $2x+y-4=0$ divides the line segment joining the points $\mathbf{A}(2,-2)$ and $\mathbf{B}(3,7)$.

Solution: The given equation can be expressed as

$$\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} = 4 \tag{1.3.9.1}$$

Using section formula in (1.3.9.1),

$$\mathbf{n}^\top\left(\frac{k\mathbf{B}+\mathbf{A}}{k+1}\right) = c \tag{1.3.9.2}$$

$$\implies k = \frac{c-\mathbf{n}^\top\mathbf{A}}{\mathbf{n}^\top\mathbf{B}-c} \tag{1.3.9.3}$$

upon simplification. Substituting numerical values,

$$k = \frac{2}{9} \tag{1.3.9.4}$$

See Fig. 1.3.9.1.

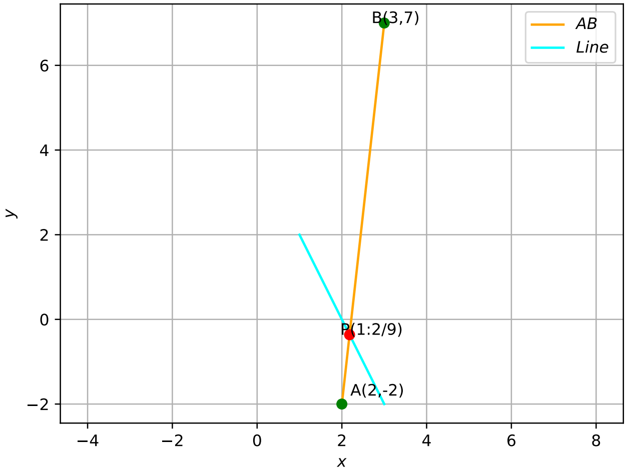


Fig. 1.3.9.1

1.3.10 Let $\mathbf{A}(4,2), \mathbf{B}(6,5)$ and $\mathbf{C}(1,4)$ be the vertices of $\triangle ABC$.

a) The median from $\mathbf{A}$ meets BC at $\mathbf{D}$. Find the coordinates of the point $\mathbf{D}$.
b) Find the coordinates of the point $\mathbf{P}$ on AD such that $AP : PD = 2 : 1$.
c) Find the coordinates of points $\mathbf{Q}$ and $\mathbf{R}$ on medians BE and CF respectively such that $BQ : QE = 2 : 1$ and $CR : RF = 2 : 1$.
d) What do you observe?
e) If $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ are the vertices of $\triangle ABC$, find the coordinates of the centroid of the triangle.

Solution:

$$\mathbf{D} = \frac{\mathbf{B}+\mathbf{C}}{2} = \begin{pmatrix} \frac{7}{2} \\ \frac{9}{2} \end{pmatrix} \tag{1.3.10.1}$$

$$\mathbf{E} = \frac{\mathbf{A}+\mathbf{C}}{2} = \begin{pmatrix} \frac{5}{2} \\ 3 \end{pmatrix} \tag{1.3.10.2}$$

$$\mathbf{F} = \frac{\mathbf{A}+\mathbf{B}}{2} = \begin{pmatrix} 5 \\ \frac{7}{2} \end{pmatrix} \tag{1.3.10.3}$$

$$\mathbf{P} = \mathbf{Q} = \mathbf{R} = \frac{1}{3}\begin{pmatrix} 11 \\ 11 \end{pmatrix} \tag{1.3.10.4}$$

$$\mathbf{G} = \frac{\mathbf{A}+\mathbf{B}+\mathbf{C}}{3} = \frac{1}{3}\begin{pmatrix} 11 \\ 11 \end{pmatrix} \tag{1.3.10.5}$$

is the centroid. See Fig. 1.3.10.1.

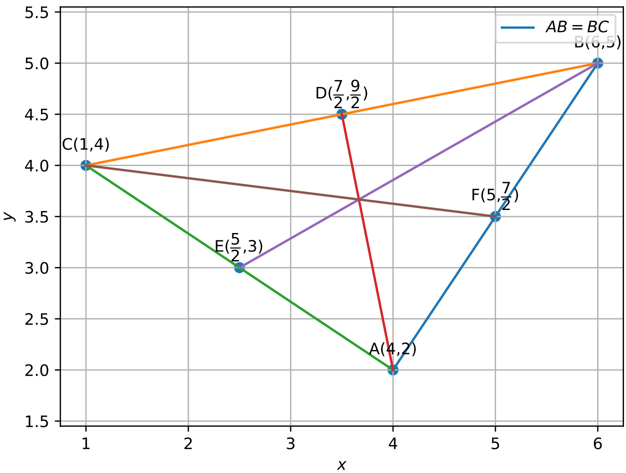


Fig. 1.3.10.1

1.3.11 Find the position vector of a point R which divides the line joining two points P and Q whose position vectors are $(2\mathbf{a}+\mathbf{b})$ and $(\mathbf{a}-3\mathbf{b})$ externally in the ratio 1 : 2. Also, show that P is the mid point of the line segment RQ.

Solution:

$$\mathbf{R} = \frac{\mathbf{Q}-2\mathbf{P}}{-1} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}, \tag{1.3.11.1}$$

$$\frac{(\mathbf{R}+\mathbf{Q})}{2} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \mathbf{P}. \tag{1.3.11.2}$$

See Fig. 1.3.11.1.

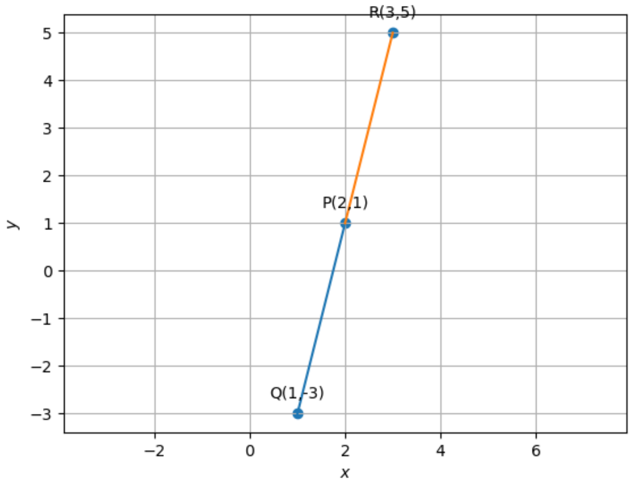


Fig. 1.3.11.1

1.3.12 The point which divides the line segment joining the points $\mathbf{P}(7,–6)$ and $\mathbf{Q}(3,4)$ in the ratio 1 : 2 internally lies in the

a) I quadrant
b) II quadrant
c) III quadrant
d) IV quadrant

1.3.13 If the point $\mathbf{P}(2,1)$ lies on the line segment joining points $\mathbf{A}(4,2)$ and $\mathbf{B}(8,4)$, then

a) $AP = \frac{1}{3}AB$
b) $AP = PE$
c) $PB = \frac{1}{3}AB$
d) $AP = \frac{1}{2}AB$

1.3.14 A line intersects the y-axis and x-axis at the points $\mathbf{P}$ and $\mathbf{Q}$, respectiveiy. lf $(2,5)$ is the mid-point of $\mathbf{PQ}$, then the coordinates of $\mathbf{P}$ and $\mathbf{Q}$ are, respectively

a) $(0,-5)$ and $(2,0)$
b) $(0,-10)$ and $(-4,0)$
c) $(0,4)$ and $(-10,0)$
d) $(0,-10)$ and $(4,0)$

1.3.15 Point $\mathbf{P}(5,-3)$ is one of the two points of trisection of line segment joining the points $\mathbf{A}(7,-2)$ and $\mathbf{B}(1,-5)$

1.3.16 Points $\mathbf{A}(-6,10), \mathbf{B}(-4,6)$ and $\mathbf{C}(3,-8)$ are collinear such that $\mathbf{AB} = \frac{2}{9}\mathbf{AC}$

1.3.17 In what ratio does the x-axis divide the line segment joining the points $(-4,-6)$ and $(-1,7)$? Find the coordinates of the point of division.

1.3.18 Find the ratio in which the point $\mathbf{P}\left(\frac{3}{4}, \frac{5}{12}\right)$ divides the line segment joining the points

$\mathbf{A}\left(\frac{1}{2}, \frac{3}{2}\right)$ and $\mathbf{B}(2, -5)$.

1.3.19 If $\mathbf{P}(9a-2, -b)$ divides line segment joining $\mathbf{A}(3a+1, -3)$ and $\mathbf{B}(8a, 5)$ in the ratio 3:1, find the values of a and b.

1.3.20 The line segment joining the points $\mathbf{A}(3, 2)$ and $\mathbf{B}(5, 1)$ is divided at the point $\mathbf{P}$ in the ratio 1:2 which lies on $3x - 18y + k = 0$. Find the value of k.

1.3.21 Find the coordinates of the point $\mathbf{R}$ on the line segment joining the points $\mathbf{P}(-1, 3)$ and $\mathbf{Q}(2, 5)$ such that $PR = \frac{3}{5}PQ$.

1.3.22 Find the ratio in which the line $2x + 3y - 5 = 0$ divides the line segment joining the points $(8, -9)$ and $(2, 1)$. Also find the coordinates of the point of division.

1.3.23 If $\mathbf{a}$ and $\mathbf{b}$ are the postion vectors of A and B, respectively, find the position vector of a point C in BA produced such that $BC = 1.5BA$.

1.3.24 The position vector of the point which divides the join of points $2\mathbf{a}$-$3\mathbf{b}$ and $\mathbf{a} + \mathbf{b}$ in the ratio 3:1 is

a) $\frac{3\mathbf{a}-2\mathbf{b}}{2}$
b) $\frac{7\mathbf{a}-8\mathbf{b}}{4}$
c) $\frac{3\mathbf{a}}{4}$
d) $\frac{5\mathbf{a}}{4}$

1.3.25 Find the ratio in which the line segment joining $A(1, -5)$ and $B(-4, 5)$ is divided by the x-axis. Also find the coordinates of the point of division.

1.3.26 Find the position vector of a point $\mathbf{R}$ which divides the line joining two points $\mathbf{P}$ and $\mathbf{Q}$ whose position vectors are $2\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - 3\mathbf{b}$ externally in the ratio 1 : 2.

1.4 Rank

1.4.1 By using the concept of equation of a line, prove that the three points (3, 0), (– 2, – 2) and (8, 2) are collinear.

Solution: From (1.1.9.1), the collinearity matrix can be expressed as

$$\begin{pmatrix} -5 & -2 \\ 5 & 2 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_1 + R_2} \begin{pmatrix} -5 & -2 \\ 0 & 0 \end{pmatrix} \tag{1.4.1.1}$$

which is a rank 1 matrix. The above process is known as row reduction, where we try to obtain zero rows in the matrix using arithmetic operations. The number of nonzero rows in the row reduced matrix (also known as *echelon form*) is defined as the rank. Fig. 1.4.1.1.

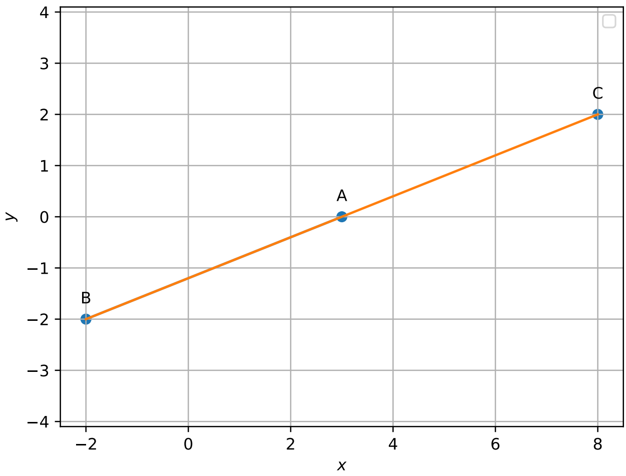


Fig. 1.4.1.1

1.4.2 Show that the points $\mathbf{A}(1,2,7), \mathbf{B}(2,6,3)$ and $\mathbf{C}(3,10,-1)$ are collinear.
Solution: The matrix

$$\begin{pmatrix} \mathbf{B}-\mathbf{A} & \mathbf{C}-\mathbf{A} \end{pmatrix}^\top = \begin{pmatrix} 1 & 4 & -4 \\ 2 & 8 & -8 \end{pmatrix} \tag{1.4.2.1}$$

$$\xleftrightarrow{R_2=R_2-2R_1} \begin{pmatrix} 1 & 4 & -4 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.4.2.2}$$

which has rank 1. Using (1.1.10.1), we conclude that the given points are collinear.

1.4.3 Determine if the points $(1,5), (2,3)$ and $(-2,-11)$ are collinear.

1.4.4 Show that the vectors $2\hat{i}-3\hat{j}+4\hat{k}$ and $-4\hat{i}+6\hat{j}-8\hat{k}$ are collinear.

1.4.5 Show that the points (2, 3, 4), (–1, –2, 1), (5, 8, 7) are collinear.

1.4.6 In each of the following, find the value of k, for which the points are collinear.

a) $(7,-2), (5,1), (3,k)$
b) $(8,1), (k,-4), (2,-5)$

1.4.7 Find a relation between x and y if the points $(x,y), (1,2)$ and $(7,0)$ are collinear.

1.4.8 If three points $(x,-1), (2,1)$ and $(4,5)$ are collinear, find the value of x.

1.4.9 If three points $(h,0), (a,b)$ and $(0,k)$ lie on a line, show that

$$\frac{a}{h}+\frac{b}{k}=1 \tag{1.4.9.1}$$

1.4.10 Show that the points $A(1,-2,-8), B(5,0,-2)$ and $C(11,3,7)$ are collinear, and find the ratio in which B divides AC.

1.4.11 If the points $\mathbf{A}(1,2), \mathbf{0}(0,0)$ and $\mathbf{C}(a,b)$ are collinear,then find the relation between a and b.

1.4.12 Point $(-4, 2)$ lies on the line segment joining the points $\mathbf{A}(-4, 6)$ and $\mathbf{B}(-4, -6)$.
1.4.13 The points $(0, 5), (0, -9)$ and $(3, 6)$ are collinear.
1.4.14 Points $\mathbf{A}(3, 1), \mathbf{B}(12, -2)$ and $\mathbf{C}(0, 2)$ cannot be the vertices of a triangle.
1.4.15 Find the value of m if the points $(5, 1), (-2, -3)$ and $(8, 2m)$ are collinear.
1.4.16 Find the values of k if the points $\mathbf{A}(k + 1, 2k), \mathbf{B}(3k, 2k + 3)$ and $\mathbf{C}(5k - 1, 5k)$ are collinear.
1.4.17 Using vectors, find the value of k such that the points $(k, -10, 3)$, $(1, -1, 3)$ and $(3, 5, 3)$ are collinear.
1.4.18 The points $\mathbf{A}(2, 1)$, $\mathbf{B}(0, 5)$, $\mathbf{C}(-1, 2)$ are collinear.
1.4.19 The vectors $\lambda\hat{i} + \lambda\hat{j} + 2\hat{k}$, $\hat{i} + \lambda\hat{j} - \hat{k}$ and $2\hat{i} - \hat{j} + \lambda\hat{k}$ are coplanar if

a) $\lambda = -2$
b) $\lambda = 0$
c) $\lambda = 1$
d) $\lambda = -1$

1.5 Length

1.5.1 Compute the magnitude of the following vectors:

$$\mathbf{a} = \hat{i} + \hat{j} + \hat{k} \tag{1.5.1.1}$$

$$\mathbf{b} = 2\hat{i} - 7\hat{j} - 3\hat{k} \tag{1.5.1.2}$$

$$\mathbf{c} = \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} - \frac{1}{3}\hat{k} \tag{1.5.1.3}$$

Solution: Let

$$\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 2 \\ -7 \\ 3 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{pmatrix} \tag{1.5.1.4}$$

Then

$$\mathbf{a}^\top\mathbf{a} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 3 \tag{1.5.1.5}$$

$$\implies \|\mathbf{a}\| = \sqrt{3}, \tag{1.5.1.6}$$

from (1.1.7.1). Similarly,

$$\|\mathbf{b}\| = \sqrt{\mathbf{b}^\top\mathbf{b}} = \sqrt{62}, \tag{1.5.1.7}$$

$$\|\mathbf{c}\| = \sqrt{\mathbf{c}^\top\mathbf{c}} = 1 \tag{1.5.1.8}$$

1.5.2 Find the value of x for which $x(\hat{i} + \hat{j} + \hat{k})$ is a unit vector.

Solution:

$$\because \mathbf{x} = x\begin{pmatrix}1\\1\\1\end{pmatrix}, \|\mathbf{x}\| = 1 \implies x\sqrt{3} = 1 \tag{1.5.2.1}$$

$$\text{or, } x = \frac{1}{\sqrt{3}} \tag{1.5.2.2}$$

1.5.3 For given vectors, $\mathbf{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\mathbf{b} = -\hat{i} + \hat{j} - \hat{k}$, find the unit vector in the direction of the vector $\mathbf{a} + \mathbf{b}$.
Solution:

$$\because \mathbf{a} + \mathbf{b} = \begin{pmatrix}2\\-1\\2\end{pmatrix} + \begin{pmatrix}-1\\1\\-1\end{pmatrix}, = \begin{pmatrix}1\\0\\1\end{pmatrix}, \tag{1.5.3.1}$$

$$\|\mathbf{a} + \mathbf{b}\| = \sqrt{2} \tag{1.5.3.2}$$

$$\implies \frac{\mathbf{a} + \mathbf{b}}{\|\mathbf{a} + \mathbf{b}\|} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix} \tag{1.5.3.3}$$

which, from (1.1.8.1) is the desired the unit vector.

1.5.4 Find a vector in the direction of vector $5\hat{i} - \hat{j} + 2\hat{k}$ which has magnitude 8 units.
Solution: Let the required vector be

$$c\begin{pmatrix}5\\-1\\2\end{pmatrix}. \tag{1.5.4.1}$$

From the given information,

$$\left\| c\begin{pmatrix}5\\-1\\2\end{pmatrix} \right\| = 8 \tag{1.5.4.2}$$

$$\implies |c| = \frac{4\sqrt{30}}{15} \tag{1.5.4.3}$$

1.5.5 Find the unit vector in the direction of sum of vectors $\mathbf{a}= 2\hat{i} - \hat{j} + \hat{k}$ and $\mathbf{b} = 2\hat{j} + \hat{k}$.

1.5.6 If $\mathbf{a}=\hat{i} + \hat{j} + 2\hat{k}$ and $\mathbf{b}=2\hat{i} + \hat{j} - 2\hat{k}$, find the unit vector in the direction of

a) 6**a**
b) 2**a-b**

1.5.7 Find a unit vector in the direction of $\overline{PQ}$, where P and Q have co-ordinates (5,0,8) and (3,3,2),respectively.

1.5.8 The vector in the direction of the vector $\hat{i} - 2\hat{j} + 2\hat{k}$ that has magnitude 9 is

a) $\hat{i} - 2\hat{j} + 2\hat{k}$
b) $\hat{i} - 2\hat{j}$
c) $3(\hat{i} - 2\hat{j} + 2\hat{k})$
d) $9(\hat{i} - 2\hat{j} + 2\hat{k})$

1.5.9 Find the unit vector in the direction of the vector $\mathbf{a} = \hat{i} + \hat{j} + 2\hat{k}$.

1.5.10 Find the unit vector in the direction of vector $\overrightarrow{PQ}$, where **P** and **Q** are the points (1, 2, 3) and (4, 5, 6), respectively.

1.5.11 Find a vector of magnitude 5 units, and parallel to the resultant of the vectors $\mathbf{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\mathbf{b} = \hat{i} - 2\hat{j} + \hat{k}$.

1.5.12 If $\mathbf{a} = \hat{i} + \hat{j} + \hat{k}, \mathbf{b} = 2\hat{i} - \hat{j} + 3\hat{k}$ and $\mathbf{c} = \hat{i} - 2\hat{j} + \hat{k}$, find a unit vector parallel to the vector $2\mathbf{a} - \mathbf{b} + 3\mathbf{c}$.

Solution:

$$2\mathbf{a} - \mathbf{b} + 3\mathbf{c} = \begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix} \implies \frac{2\mathbf{a} - \mathbf{b} + 3\mathbf{c}}{\|2\mathbf{a} - \mathbf{b} + 3\mathbf{c}\|} = \frac{1}{\sqrt{22}} \begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix} \tag{1.5.12.1}$$

1.5.13 Find a vector of magnitude 5 units, and parallel to the resultant of the vectors $\mathbf{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\mathbf{b} = \hat{i} - 2\hat{j} + \hat{k}$.

Solution:

$$\because \mathbf{a} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \tag{1.5.13.1}$$

$$\mathbf{a} + \mathbf{b} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \implies \|\mathbf{a} + \mathbf{b}\| = \sqrt{10} \tag{1.5.13.2}$$

From problem 1.5.3, the unit vector in the direction of $\mathbf{a} + \mathbf{b}$ is

$$\frac{\mathbf{a} + \mathbf{b}}{\|\mathbf{a} + \mathbf{b}\|} = \frac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \tag{1.5.13.3}$$

The desired vector can then be expressed as

$$\pm \frac{5}{\sqrt{10}} \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \tag{1.5.13.4}$$

1.5.14 If a line makes angles $90°, 135°, 45°$ with x,y and z-axis respectivly. Find its direction cosines.

Solution: From (1.1.11.1), the direction vector is

$$\mathbf{A} = \begin{pmatrix} \cos 90° \\ \cos 135° \\ \cos 45° \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \tag{1.5.14.1}$$

1.5.15 Find the direction cosines of the vector joining the points **A** (1, 2, –3) and **B**(–1, –2, 1), directed from **A** to **B**.

Solution: The unit vector in the direction of AB is

$$\frac{\mathbf{B}-\mathbf{A}}{\|\mathbf{B}-\mathbf{A}\|} = \frac{1}{3}\begin{pmatrix}-1\\-2\\2\end{pmatrix} \tag{1.5.15.1}$$

and the direction cosines are the elements of the above vector.

1.5.16 Show that the vector $\hat{i}+\hat{j}+\hat{k}$ is equally inclined to the axes OX, OY and OZ.

Solution: Since all entries of the given vector

$$\begin{pmatrix}1\\1\\1\end{pmatrix} \tag{1.5.16.1}$$

are equal, it is equally inclined to the axes.

1.5.17 If a line has the direction ratios –18, 12, –4, then what are its direction cosines?

Solution: Let

$$\mathbf{A} = \begin{pmatrix}-18\\12\\-4\end{pmatrix} \tag{1.5.17.1}$$

Then the unit direction vector of the line is

$$\frac{\mathbf{A}}{\|\mathbf{A}\|} = \begin{pmatrix}\frac{-9}{11}\\\frac{6}{11}\\\frac{-2}{11}\end{pmatrix} \tag{1.5.17.2}$$

1.5.18 Find the direction cosines of the sides of a triangle whose vertices are $\begin{pmatrix}3\\5\\-4\end{pmatrix}, \begin{pmatrix}-1\\1\\2\end{pmatrix}$ and $\begin{pmatrix}-5\\-5\\-2\end{pmatrix}$.

Solution: Let the vertices be

$$\mathbf{A} = \begin{pmatrix}3\\5\\-4\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-1\\1\\2\end{pmatrix}, \mathbf{C} = \begin{pmatrix}-5\\-5\\-2\end{pmatrix} \tag{1.5.18.1}$$

The direction vectors of the sides are,

$$\mathbf{A}-\mathbf{B} = \begin{pmatrix}4\\4\\-6\end{pmatrix} = \mathbf{m_1}, \mathbf{B}-\mathbf{C} = \begin{pmatrix}4\\6\\4\end{pmatrix} = \mathbf{m_2}, \tag{1.5.18.2}$$

$$\mathbf{C}-\mathbf{A} = \begin{pmatrix}-8\\-10\\2\end{pmatrix} = \mathbf{m_3}, \tag{1.5.18.3}$$

The corresponding unit vectors are then obtained as

$$\begin{pmatrix} \frac{2}{\sqrt{17}} \\ \frac{2}{\sqrt{17}} \\ \frac{-3}{\sqrt{17}} \end{pmatrix}, \begin{pmatrix} \frac{2}{\sqrt{17}} \\ \frac{3}{\sqrt{17}} \\ \frac{2}{\sqrt{17}} \end{pmatrix}, \begin{pmatrix} \frac{-4}{\sqrt{42}} \\ \frac{-5}{\sqrt{42}} \\ \frac{1}{\sqrt{42}} \end{pmatrix} \tag{1.5.18.4}$$

1.5.19 Find the direction cosines of the vector $\hat{i} + 2\hat{j} + 3\hat{k}$.
Solution: The unit vector in the direction of the given vector is

$$\mathbf{A} = \frac{1}{\sqrt{14}} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \tag{1.5.19.1}$$

1.5.20 Find the direction cosines of a line which makes equal angles with the coordinate axes.
Solution: Let α be the angle made by the line with the axes. The unit direction vector can be expressed as

$$\mathbf{x} = \begin{pmatrix} \cos\alpha \\ \cos\alpha \\ \cos\alpha \end{pmatrix} \implies \|\mathbf{x}\| = 1 \tag{1.5.20.1}$$

$$\text{or, } \cos\alpha = \frac{1}{\sqrt{3}} \tag{1.5.20.2}$$

Thus the unit direction vector of the given line is

$$\mathbf{x} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{1.5.20.3}$$

1.5.21 If a unit vector $\overrightarrow{a}$ makes angles $\frac{\pi}{3}$ with $\hat{i}$, $\frac{\pi}{4}$ with $\hat{j}$ and an acute angle θ with $\hat{k}$, then find θ and hence, the components of $\overrightarrow{a}$.
Solution: From the given information,

$$\mathbf{a} = \begin{pmatrix} \cos\frac{\pi}{3} \\ \cos\frac{\pi}{4} \\ \cos\theta \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \cos\theta \end{pmatrix} \tag{1.5.21.1}$$

$$\because \|\mathbf{a}\| = 1, \tag{1.5.21.2}$$

$$\frac{1}{4} + \frac{1}{2} + \cos^2\theta = 1 \tag{1.5.21.3}$$

$$\implies \cos\theta = \frac{1}{2} \tag{1.5.21.4}$$

$\because \theta$ is an acute angle. Hence

$$\mathbf{a} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix} \tag{1.5.21.5}$$

1.5.22 Write down a unit vector in XY-plane, making an angle of 30° with the positive direction of x-axis.

1.5.23 A vector **r** is inclined at equal angles to the three axis. If the magnitude of **r** is $2\sqrt{3}$ units, find **r**.

1.5.24 The direction cosines of the vector $(2\hat{i} + 2\hat{j} - \hat{k})$ are __________.

1.5.25 A vector **r** has a magnitude 14 and direction ratios 2, 3, -6. Find the direction cosines and components of **r**, given that **r** makes an acute angle with x-axis.

2.1 Formulae

2.1.1. The angle θ between $\mathbf{a}, \mathbf{b}$, is given by

$$\cos\theta = \frac{\mathbf{a}^\top \mathbf{b}}{\|\mathbf{a}\| \, \|\mathbf{b}\|} \tag{2.1.1.1}$$

2.1.2. The equation of a line is given by

$$\mathbf{x} = \mathbf{h} + \kappa\mathbf{m} \tag{2.1.2.1}$$

2.1.3. For

$$\mathbf{m}^\top \mathbf{n} = 0, \tag{2.1.3.1}$$

which means that $\mathbf{m} \perp \mathbf{n}$, (2.1.2.1) can be expressed as

$$\mathbf{n}^\top \mathbf{x} = \mathbf{n}^\top \mathbf{h} + \kappa \mathbf{n}^\top \mathbf{m} \tag{2.1.3.2}$$

$$\implies \mathbf{n}^\top \mathbf{x} = c \tag{2.1.3.3}$$

for

$$c = \mathbf{n}^\top \mathbf{h}. \tag{2.1.3.4}$$

$\mathbf{n}$ is defined to be the *normal vector* of the line. In 3D, (2.1.3.3) represents a plane.

2.1.4. Mathematically, the projection of $\mathbf{A}$ on $\mathbf{B}$ is defined as

$$\mathbf{C} = k\mathbf{B}, \text{ such that } (\mathbf{A} - \mathbf{C})^\top \mathbf{C} = 0 \tag{2.1.4.1}$$

yielding

$$(\mathbf{A} - k\mathbf{B})^\top \mathbf{B} = 0 \tag{2.1.4.2}$$

$$\text{or, } k = \frac{\mathbf{A}^\top \mathbf{B}}{\|\mathbf{B}\|^2} \implies \mathbf{C} = \frac{\mathbf{A}^\top \mathbf{B}}{\|\mathbf{B}\|^2}\mathbf{B} \tag{2.1.4.3}$$

2.1.5. If $\mathbf{A}, \mathbf{B}$ are unit vectors,

$$(\mathbf{A} - \mathbf{B})^\top (\mathbf{A} + \mathbf{B}) = \|\mathbf{A}\|^2 - \|\mathbf{B}\|^2 = 0 \tag{2.1.5.1}$$

2.1.6. If

$$\mathbf{A}^\top \mathbf{A} = \mathbf{I}, \tag{2.1.6.1}$$

then $\mathbf{A}$ is an *orthogonal* matrix. This also means that its rows and columns are unit vectors and mutually perpendicular.

2.1.7. The determinant

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1. \tag{2.1.7.1}$$

2.1.8. Let

$$\mathbf{A} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \equiv a_1 \overrightarrow{i} + a_2 \overrightarrow{j} + a_3 \overrightarrow{j}, \tag{2.1.8.1}$$

$$\mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \tag{2.1.8.2}$$

and

$$\begin{aligned} \mathbf{A}_{ij} &= \begin{pmatrix} a_i \\ a_j \end{pmatrix}, \\ \mathbf{B}_{ij} &= \begin{pmatrix} b_i \\ b_j \end{pmatrix}. \end{aligned} \tag{2.1.8.3}$$

2.1.9. The *cross product* or *vector product* of $\mathbf{A}, \mathbf{B}$ is defined as

$$\mathbf{A} \times \mathbf{B} = \begin{pmatrix} \begin{vmatrix} \mathbf{A}_{23} & \mathbf{B}_{23} \end{vmatrix} \\ \begin{vmatrix} \mathbf{A}_{31} & \mathbf{B}_{31} \end{vmatrix} \\ \begin{vmatrix} \mathbf{A}_{12} & \mathbf{B}_{12} \end{vmatrix} \end{pmatrix} \tag{2.1.9.1}$$

2.1.10. Verify that

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \tag{2.1.10.1}$$

$$\mathbf{A} \times \mathbf{A} = \mathbf{0} \tag{2.1.10.2}$$

2.1.11. If

$$\mathbf{A} \times \mathbf{B} = \mathbf{0}, \tag{2.1.11.1}$$

$\mathbf{A}$ and $\mathbf{B}$ are linearly independent, i.e., they are points on the same line.

2.1.12.

$$\|\mathbf{A} \times \mathbf{B}\| = \|\mathbf{A}\| \times \|\mathbf{B}\| \sin\theta \tag{2.1.12.1}$$

where θ is the angle between the vectors.

2.1.13.

$$ar\,(ABCD) = \frac{1}{2}\left((\mathbf{C} - \mathbf{A}) \times (\mathbf{D} - \mathbf{B})\right) \tag{2.1.13.1}$$

$$\tag{2.1.13.2}$$

2.1.14. The area of $\triangle ABC$ is

$$\frac{1}{2}\,\|(\mathbf{A} - \mathbf{B}) \times (\mathbf{A} - \mathbf{C})\| \tag{2.1.14.1}$$

2.1.15. The affine transformation is given by

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c} \tag{2.1.15.1}$$

where $\mathbf{c}$ is the translation vector.

2.1.16. The matrix

$$\mathbf{P} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{2.1.16.1}$$

is defined to be the rotation matrix.

2.1.17.

$$\mathbf{P}^\top \mathbf{P} = \mathbf{I} \tag{2.1.17.1}$$

$\mathbf{P}$ is known as as *orthogonal* matrix.

2.1.18. Given vertices $\mathbf{A}, \mathbf{C}$ of a square, the other two vertices are given by

$$\begin{aligned} \mathbf{B} &= \|\mathbf{C} - \mathbf{A}\| \cos\frac{\pi}{4} \mathbf{P}\mathbf{e}_1 + \mathbf{A} \\ \mathbf{D} &= \|\mathbf{C} - \mathbf{A}\| \cos\frac{\pi}{4} \mathbf{P}\mathbf{e}_2 + \mathbf{A} \end{aligned} \tag{2.1.18.1}$$

2.2 *Scalar Product*

2.2.1 Find the angle between two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ with magnitudes $\sqrt{3}$ and 2 respectively having $\overrightarrow{a} \cdot \overrightarrow{b} = \sqrt{6}$.
Solution: From the given information,

$$\|\mathbf{a}\| = \sqrt{3}, \|\mathbf{b}\| = 2, \mathbf{a}^\top \mathbf{b} = \sqrt{6} \tag{2.2.1.1}$$

Substituting in (2.1.1.1),

$$\cos\theta = \frac{1}{\sqrt{2}} \tag{2.2.1.2}$$

$$\text{or, } \theta = 45° \tag{2.2.1.3}$$

2.2.2 Find the angle between the the vectors $\hat{i} - 2\hat{j} + 3\hat{k}$ and $3\hat{i} - 2\hat{j} + \hat{k}$.
Solution: Let

$$\mathbf{a} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}, \tag{2.2.2.1}$$

From problem 2.2.1,

$$\cos\theta = \frac{\mathbf{a}^\top \mathbf{b}}{\|\mathbf{a}\| \|\mathbf{b}\|} = \frac{10}{\sqrt{14} \times \sqrt{14}} = \frac{5}{7} \tag{2.2.2.2}$$

2.2.3 Evaluate the product $(3\overrightarrow{a} - 5\overrightarrow{b}) \cdot (2\overrightarrow{a} + 7\overrightarrow{b})$.
Solution:

$$\begin{aligned} (3\mathbf{a} - 5\mathbf{b})^\top (2\mathbf{a} + 7\mathbf{b}) = 3\mathbf{a}^\top (2\mathbf{a} + 7\mathbf{b}) - 5\mathbf{b}^\top (2\mathbf{a} + 7\mathbf{b}) \\ = 6\|\mathbf{a}\|^2 - 35\|\mathbf{b}\|^2 + 11\mathbf{a}^\top \mathbf{b} \end{aligned} \tag{2.2.3.1}$$

2.2.4 If the vertices A, B, C of a triangle ABC are (1,2,3), (-1,0,0), (0,1,2), respectively, then find $\angle ABC$.

Solution: From the given information,

$$\mathbf{A}-\mathbf{B}=\begin{pmatrix}2\\2\\3\end{pmatrix}, \mathbf{C}-\mathbf{B}=\begin{pmatrix}1\\1\\2\end{pmatrix} \tag{2.2.4.1}$$

$$\implies \angle ABC = \cos^{-1}\frac{(\mathbf{A}-\mathbf{B})^\top(\mathbf{C}-\mathbf{B})}{\|\mathbf{A}-\mathbf{B}\|\,\|\mathbf{C}-\mathbf{B}\|} \tag{2.2.4.2}$$

$$= \cos^{-1}\frac{10}{\sqrt{102}} \tag{2.2.4.3}$$

$$\tag{2.2.4.4}$$

2.2.5 The slope of a line is double of the slope of another line. If tangent of the angle between them is 1/3, find the slopes of the lines.
Solution: The direction vectors of the lines can be expressed as

$$\mathbf{m}_1=\begin{pmatrix}1\\m\end{pmatrix}, \mathbf{m}_2=\begin{pmatrix}1\\2m\end{pmatrix} \tag{2.2.5.1}$$

If the angle between the lines be θ,

$$\tan\theta = \frac{1}{3} \implies \cos\theta = \frac{3}{\sqrt{10}} \tag{2.2.5.2}$$

Thus,

$$\frac{3}{\sqrt{10}} = \frac{\mathbf{m}_1^\top\mathbf{m}_2}{\|\mathbf{m}_1\|\,\|\mathbf{m}_2\|} \tag{2.2.5.3}$$

$$= \frac{2m^2+1}{\sqrt{m^2+1}\,\sqrt{4m^2+1}} \tag{2.2.5.4}$$

$$\implies \frac{9}{10} = \frac{4m^4+4m^2+1}{4m^4+5m^2+1} \tag{2.2.5.5}$$

$$\text{or, } 4m^4-5m^2+1=0 \tag{2.2.5.6}$$

yielding

$$m = \pm\frac{1}{2}, \pm 1 \tag{2.2.5.7}$$

2.2.6 Find angle between the lines, $\sqrt{3}x+y=1$ and $x+\sqrt{3}y$=1.
Solution: From (2.1.3.3), the normal vectors of the given lines can be expressed as

$$\mathbf{n}_1=\begin{pmatrix}\sqrt{3}\\1\end{pmatrix}, \mathbf{n}_2=\begin{pmatrix}1\\\sqrt{3}\end{pmatrix} \tag{2.2.6.1}$$

The angle between the lines can then be obtained as

$$\cos\theta = \frac{\mathbf{n}_1^\top\mathbf{n}_2}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|} = \frac{\sqrt{3}}{2} \tag{2.2.6.2}$$

$$\text{or, } \theta = 30° \tag{2.2.6.3}$$

2.2.7 Find the angle between the vectors $2\hat{i} - \hat{j} + \hat{k}$ and $3\hat{i} + 4\hat{j} - \hat{k}$.

2.2.8 The angles between two vectors $\mathbf{a}, \mathbf{b}$ with magnitude $\sqrt{3}, 4$ respectively, and $\mathbf{a} \cdot \mathbf{b} = 2\sqrt{3}$ is

a) $\frac{\pi}{6}$
b) $\frac{\pi}{3}$
c) $\frac{\pi}{2}$
d) $\frac{5\pi}{2}$

2.2.9 Find the angle between the lines

$$\vec{r} = 3\hat{i} - 2\hat{j} + 6\hat{k} + \lambda(2\hat{i} + \hat{j} + 2\hat{k}) \text{ and} \tag{2.2.9.1}$$

$$\vec{r} = (2\hat{j} - 5\hat{k}) + \mu(6\hat{i} + 3\hat{j} + 2\hat{k}) \tag{2.2.9.2}$$

Solution: The given lines can be expressed in the form of (2.1.2.1) as

$$\mathbf{x} = \begin{pmatrix} 3 \\ -2 \\ 6 \end{pmatrix} + \kappa_1 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \tag{2.2.9.3}$$

$$\mathbf{x} = \begin{pmatrix} 0 \\ 2 \\ -5 \end{pmatrix} + \kappa_2 \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix} \tag{2.2.9.4}$$

From the above, it is obvious that the direction vectors of the two lines are

$$\mathbf{m}_1 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \ \mathbf{m}_2 = \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix} \tag{2.2.9.5}$$

From (2.1.1.1), the angle between the two lines is obtained as

$$\cos\theta = \frac{19}{21} \tag{2.2.9.6}$$

2.2.10 The vectors $\mathbf{a} = 3\hat{i} - 2\hat{j} + 2\hat{k}$ and $\mathbf{b} = \hat{i} - 2\hat{k}$ are the adjancent sides of a parallelogram. The acute angle between its diagonals is ______.

2.2.11 The sine of the angle between the straight line

$$\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5} \tag{2.2.11.1}$$

and the plane

$$2x - 2y + z = 5 \tag{2.2.11.2}$$

is

a) $\frac{10}{6\sqrt{5}}$
b) $\frac{4}{5\sqrt{2}}$
c) $\frac{2\sqrt{3}}{5}$
d) $\frac{\sqrt{2}}{10}$

Solution: The given line can be expressed in the form (2.1.2.1) as

$$\mathbf{x} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} + \kappa_1 \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} \tag{2.2.11.3}$$

Hence the direction vector of this line is

$$\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} \tag{2.2.11.4}$$

From (2.1.3.3), the normal vector of the given plane is

$$\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \tag{2.2.11.5}$$

Thus, the cosine of the angle between the two is obtained from (2.1.1.1) as

$$\frac{\sqrt{2}}{10}, \tag{2.2.11.6}$$

which is sine of the angle between the plane and the line.

2.2.12 The plane $2x - 3y + 6z - 11 = 0$ makes an angle $\sin^{-1}(\alpha)$ with x-axis. The value of α is equal to

a) $\frac{\sqrt{3}}{2}$
b) $\frac{\sqrt{2}}{3}$
c) $\frac{2}{7}$
d) $\frac{3}{7}$

2.2.13 Find the angle between the vectors $2\hat{i} - \hat{j} + \hat{k}$ and $3\hat{i} + 4\hat{j} - \hat{k}$.

2.2.14 The angles between two vectors **a** and **b** with magnitude $\sqrt{3}$ and 4, respectively, and **a**, **b**= $2\sqrt{3}$ is

a) $\frac{\pi}{6}$
b) $\frac{\pi}{3}$
c) $\frac{\pi}{2}$
d) $\frac{5\pi}{2}$

2.2.15 The angle between the line

$$\overrightarrow{r} = (5\hat{i} - \hat{j} - 4\hat{k}) + \lambda(2\hat{i} - \hat{j} + \hat{k}) \tag{2.2.15.1}$$

and the plane

$$\overrightarrow{r} \cdot (3\hat{i} - 4\hat{j} - \hat{k}) + 5 = 0 \tag{2.2.15.2}$$

is $\sin^{-1}\left(\frac{5}{2\sqrt{91}}\right)$.

2.2.16 The angle between the planes

$$\vec{r} \cdot (2\hat{i} - 3\hat{j} + \hat{k}) = 1 \text{ and} \tag{2.2.16.1}$$

$$\vec{r} \cdot (\hat{i} - \hat{j}) = 4 \tag{2.2.16.2}$$

is $\cos^{-1}\left(\frac{-5}{\sqrt{58}}\right)$.

2.2.17 Find the angle between the lines

$$y = (2 - \sqrt{3})(x + 5) \text{ and} \tag{2.2.17.1}$$

$$y = (2 + \sqrt{3})(x - 7). \tag{2.2.17.2}$$

2.2.18 The unit vector normal to the plane $x + 2y + 3z - 6 = 0$ is $\frac{1}{\sqrt{14}}\hat{i} + \frac{2}{\sqrt{14}}\hat{j} + \frac{3}{\sqrt{14}}\hat{k}$.

2.2.19 The scalar product of the vector $\hat{i} + \hat{j} + \hat{k}$ with a unit vector along the sum of vectors $2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\lambda\hat{i} + 2\hat{j} + 3\hat{k}$ is equal to one. Find the value of λ.

2.3 Orthogonality

2.3.1 Name the type of quadrilateral formed, if any, by the following points,and give reasons for your answer

a) $A(-1, -2), B(1, 0), (C - 1, 2), D(-3, 0)$
b) $A(-3, 5), B(-3, 1), C(0, 3), D(-1, -4)$
c) $A(4, 5), B(7, 6), C(4, 3), D(1, 2)$

Solution: See Table 2.3.1, Fig. 2.3.1.1, Fig. 2.3.1.2. and Fig. 2.3.1.3. In b), forming the collinearity matrix

$$\begin{pmatrix}\mathbf{B} - \mathbf{A} & \mathbf{C} - \mathbf{B}\end{pmatrix} = \begin{pmatrix} 6 & -3 \\ -4 & 2 \end{pmatrix} \xleftrightarrow{R_2 \to R_2 + \frac{2}{3}R_1} = \begin{pmatrix} 6 & -3 \\ 0 & 0 \end{pmatrix} \tag{2.3.1.1}$$

which is a rank 1 matrix. Hence, $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are collinear.

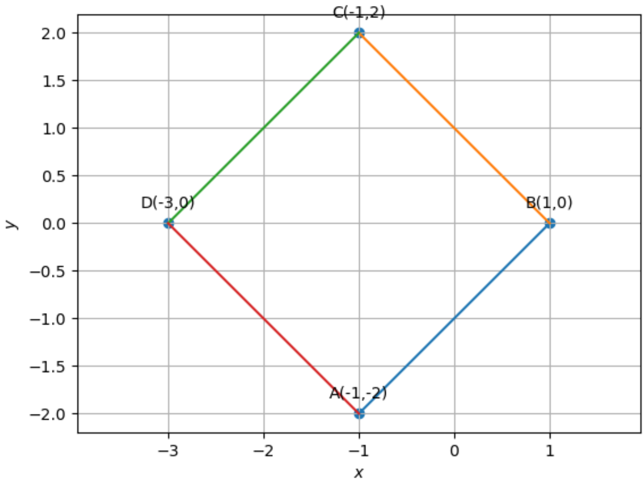


Fig. 2.3.1.1

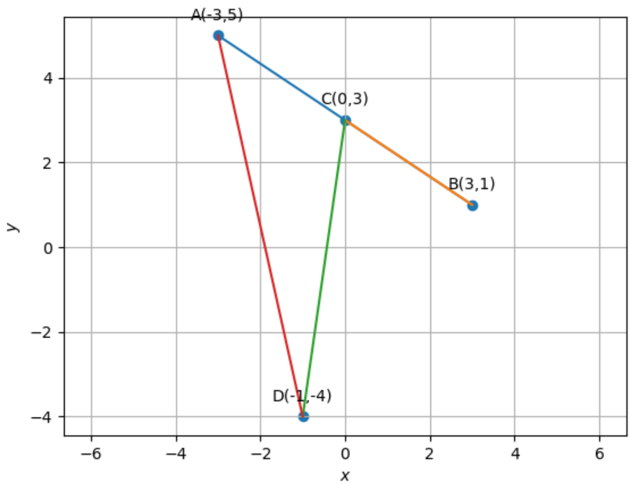


Fig. 2.3.1.2

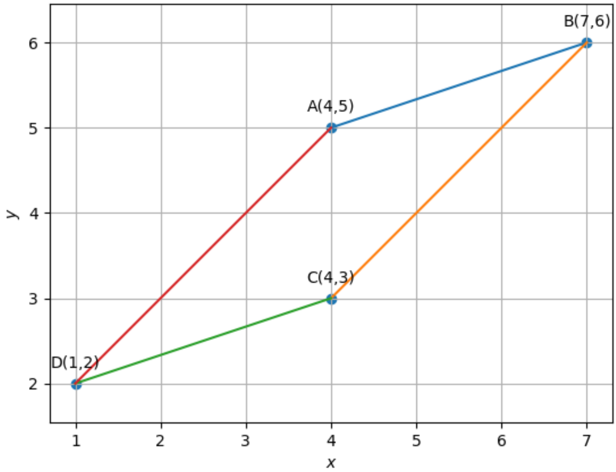


Fig. 2.3.1.3

	$\mathbf{B}-\mathbf{A}=\mathbf{C}-\mathbf{D}$?	$(\mathbf{B}-\mathbf{A})^\top(\mathbf{C}-\mathbf{B})=0$?	$(\mathbf{C}-\mathbf{A})^\top(\mathbf{D}-\mathbf{B})=0$	**Geometry**
a)	Yes	Yes	Yes	Square
b)	No	-	-	Triangle
c)	Yes	No	No	Parallelogram

TABLE 2.3.1

2.3.2 Find the projection of the vector $\hat{i}+3\hat{j}+7\hat{k}$ on the vector $7\hat{i}-\hat{j}+8\hat{k}$.
Solution: Let

$$\mathbf{A}=\begin{pmatrix}1\\3\\7\end{pmatrix},\mathbf{B}=\begin{pmatrix}7\\-1\\8\end{pmatrix} \tag{2.3.2.1}$$

The projection of **A** on **B** is defined as the foot of the perpendicular from **A** to **B** and obtained in (2.1.4.3). Substituting numerical values,

$$\mathbf{C}=\frac{10}{19}\begin{pmatrix}7\\-1\\8\end{pmatrix} \tag{2.3.2.2}$$

2.3.3 Find the projection of the vector $\hat{i}-\hat{j}$ on the vector $\hat{i}+\hat{j}$.
Solution: The given points are

$$\mathbf{A}=\begin{pmatrix}1\\-1\end{pmatrix},\mathbf{B}=\begin{pmatrix}1\\1\end{pmatrix} \tag{2.3.3.1}$$

Since

$$\mathbf{A}^{\top}\mathbf{B} = 0, \tag{2.3.3.2}$$

from (2.1.4.3), the projection vector is the origin. See Fig. 2.3.3.1.

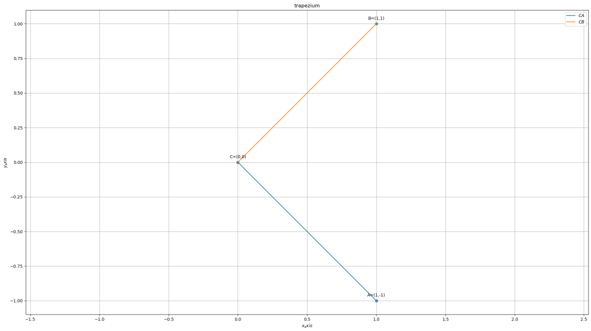


Fig. 2.3.3.1

2.3.4 Show that each of the given three vectors is a unit vector: $\frac{1}{7}(2\hat{i}+3\hat{j}+6\hat{k})$, $\frac{1}{7}(3\hat{i}-6\hat{j}+2\hat{k})$, $\frac{1}{7}(6\hat{i}+2\hat{j}-3\hat{k})$. Also,show that they are mutually perpendicular to each other.
Solution:

$$\mathbf{A} = \begin{pmatrix} \frac{2}{7} & \frac{3}{7} & \frac{6}{7} \\ \frac{3}{7} & -\frac{6}{7} & \frac{2}{7} \\ \frac{6}{7} & \frac{2}{7} & -\frac{3}{7} \end{pmatrix} \tag{2.3.4.1}$$

is an orthogonal matrix satisfying (2.1.6.1), which verifies the given conditions.

2.3.5 Show that the vectors $2\hat{i}-\hat{j}+\hat{k}, \hat{i}-3\hat{j}-5\hat{k}$ and $3\hat{i}-4\hat{j}-4\hat{k}$ from the vertices of a right angled triangle.
Solution:

$$\mathbf{A} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 3 \\ -4 \\ -4 \end{pmatrix}, \tag{2.3.5.1}$$

$$\implies \mathbf{B}-\mathbf{C} = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}, \mathbf{C}-\mathbf{A} = \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix}, \tag{2.3.5.2}$$

$$\text{or, } (\mathbf{B}-\mathbf{C})^{\top}(\mathbf{C}-\mathbf{A}) = 0 \tag{2.3.5.3}$$

2.3.6 Show that the points A, B and C with position vectors, $3\hat{i}-4\hat{j}-4\hat{k}, 2\hat{i}-\hat{j}+\hat{k}$ and $\hat{i}-3\hat{j}-5\hat{k}$, respectively, form the vertices of a right angled triangle.

Solution:

$$\mathbf{B}-\mathbf{A}=\begin{pmatrix}-1\\3\\5\end{pmatrix},\ \mathbf{C}-\mathbf{B}=\begin{pmatrix}-1\\-2\\-6\end{pmatrix},\ \mathbf{C}-\mathbf{A}=\begin{pmatrix}-2\\1\\-1\end{pmatrix}, \tag{2.3.6.1}$$

$$\implies (\mathbf{B}-\mathbf{A})^\top(\mathbf{C}-\mathbf{A})=0 \tag{2.3.6.2}$$

Hence, $\triangle ABC$ is right angled at $\mathbf{A}$.

2.3.7 Let $\mathbf{a}=\hat{i}+4\hat{j}+2\hat{k}, \mathbf{b}=3\hat{i}-2\hat{j}+7\hat{k}$ and $\mathbf{c}=2\hat{i}-\hat{j}+4\hat{k}$. Find a vector $\mathbf{d}$ which is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, and $\mathbf{c}\cdot\mathbf{d}$=15.

Solution: From the given information,

$$\mathbf{a}^\top\mathbf{d}=0 \tag{2.3.7.1}$$

$$\mathbf{b}^\top\mathbf{d}=0 \tag{2.3.7.2}$$

$$\mathbf{c}^\top\mathbf{d}=15 \tag{2.3.7.3}$$

yielding

$$\begin{pmatrix}\mathbf{a}^\top\\\mathbf{b}^\top\\\mathbf{c}^\top\end{pmatrix}\mathbf{d}=\begin{pmatrix}0\\0\\15\end{pmatrix} \tag{2.3.7.4}$$

$$\implies \begin{pmatrix}1&4&2\\3&-2&7\\2&-1&4\end{pmatrix}\mathbf{d}=\begin{pmatrix}0\\0\\15\end{pmatrix} \tag{2.3.7.5}$$

Forming the augmented matrix,

$$\left(\begin{array}{ccc|c}1&4&2&0\\3&-2&7&0\\2&-1&4&15\end{array}\right)\xleftrightarrow[R_3\leftarrow R_3-2R_1]{R_2\leftarrow R_2-3R_1}\left(\begin{array}{ccc|c}1&4&2&0\\0&-14&1&0\\0&-9&0&15\end{array}\right)$$
$$\xleftrightarrow{R_3\leftarrow R_3-\frac{9}{14}R_2}\left(\begin{array}{ccc|c}1&4&2&0\\0&-14&1&0\\0&0&-\frac{9}{14}&15\end{array}\right) \tag{2.3.7.6}$$

yielding

$$\mathbf{d}=\begin{pmatrix}\frac{160}{3}\\-\frac{5}{3}\\-\frac{70}{3}\end{pmatrix} \tag{2.3.7.7}$$

upon back substitution.

2.3.8 $ABCD$ is a rectangle formed by the points $\mathbf{A}(-1,-1), \mathbf{B}(-1,4), \mathbf{C}(5,4)$ and $\mathbf{D}(5,-1)$. $\mathbf{P},\mathbf{Q},\mathbf{R}$ and $\mathbf{S}$ are the mid-points of AB, BC, CD and DA respectively. Is the quadrilateral $PQRS$ a square? a rectangle? or a rhombus? Justify your answer.

Solution: See Fig. 2.3.8.1. From (1.1.5.3), $PQRS$ is a parallelogram.

$$\mathbf{P} = \frac{3}{2}, \mathbf{Q} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \mathbf{R} = \begin{pmatrix} 5 \\ \frac{3}{2} \end{pmatrix}, \mathbf{S} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \tag{2.3.8.1}$$

$$\implies (\mathbf{Q} - \mathbf{P})^\top (\mathbf{R} - \mathbf{Q}) \neq 0 \tag{2.3.8.2}$$

$$(\mathbf{R} - \mathbf{P})^\top (\mathbf{S} - \mathbf{Q}) = 0 \tag{2.3.8.3}$$

Therefore $PQRS$ is a rhombus.

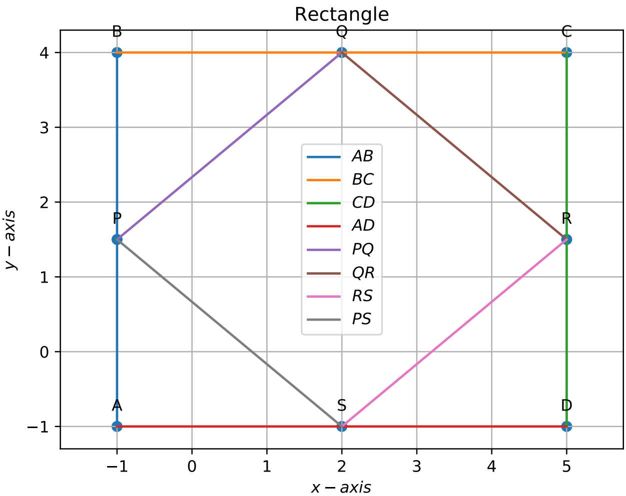


Fig. 2.3.8.1

2.3.9 Without using the Baudhayana theorem, show that the points $A(4,4), B(3,5)$ and $C(-1,-1)$ are the vertices of a right angled triangle. See Fig. 2.3.9.1.

$$\mathbf{C} - \mathbf{A} = \begin{pmatrix} -5 \\ -5 \end{pmatrix}, \mathbf{A} - \mathbf{B} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{2.3.9.1}$$

$$\implies (\mathbf{C} - \mathbf{A})^\top (\mathbf{A} - \mathbf{B}) = 0 \tag{2.3.9.2}$$

Thus, $AB \perp AC$.

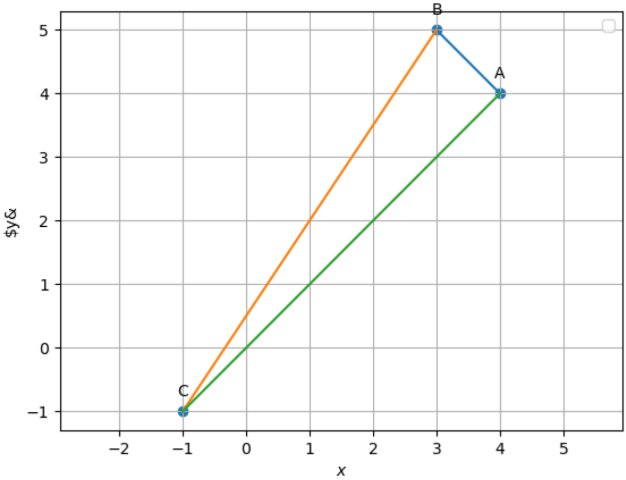


Fig. 2.3.9.1

2.3.10 The line through the points $(h, 3)$ and $(4, 1)$ intersects the line $7x - 9y - 19 = 0$ at a right angle. Find the value of h.

Solution: The direction vectors of the given lines are

$$\begin{pmatrix} 4-h \\ -2 \end{pmatrix}, \begin{pmatrix} 9 \\ 7 \end{pmatrix} \tag{2.3.10.1}$$

$$\implies \begin{pmatrix} 9 & 7 \end{pmatrix} \begin{pmatrix} 4-h \\ -2 \end{pmatrix} = 0 \tag{2.3.10.2}$$

$$\implies h = \frac{22}{9} \tag{2.3.10.3}$$

See Fig. 2.3.10.1.

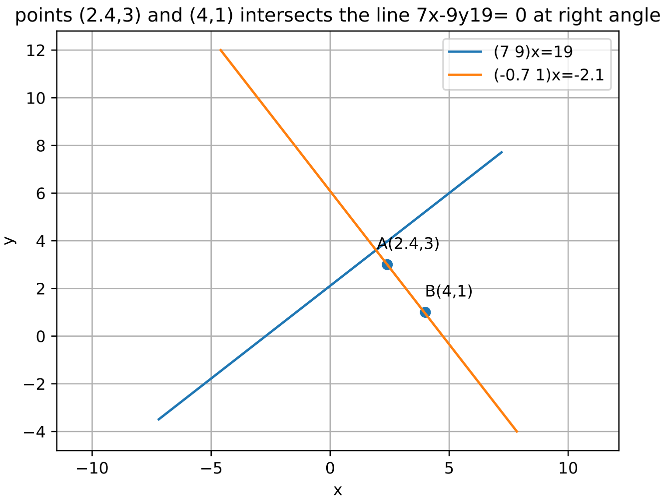


Fig. 2.3.10.1

2.3.11 In the following cases, determine whether the given planes are parallel or perpendicular, and in case they are neither, find the angles between them.

a) $7x + 5y + 6z + 30 = 0$ and $3x–y–10z + 4 = 0$

b) $2x + y + 3z–2 = 0$ and $x–2y + 5 = 0$

c) $2x–2y + 4z + 5 = 0$ and $3x–3y + 6z–1 = 0$

d) $2x–y + 3z–1 = 0$ and $2x–y + 3z + 3 = 0$

e) $4x + 8y + z–8 = 0$ and $y + z–4 = 0$

Solution: See Table 2.3.11.

TABLE 2.3.11

$\mathbf{n}_1$	$\mathbf{n}_1$	$\mathbf{n}_1^\top \mathbf{n}_2$	$\|\mathbf{n}_1\|$	$\|\mathbf{n}_2\|$	Angle
$\begin{pmatrix}7\\5\\6\end{pmatrix}$	$\begin{pmatrix}3\\-1\\-10\end{pmatrix}$	-44	$\sqrt{110}$	$\sqrt{110}$	$\cos^{-1} -\frac{2}{5}$
$\begin{pmatrix}2\\1\\3\end{pmatrix}$	$\begin{pmatrix}1\\-2\\0\end{pmatrix}$	0			perpendicular
$\begin{pmatrix}2\\-2\\4\end{pmatrix}$	$\begin{pmatrix}3\\-3\\6\end{pmatrix}$	36	$\sqrt{24}$	$\sqrt{54}$	parallel
$\begin{pmatrix}2\\-1\\3\end{pmatrix}$	$\begin{pmatrix}2\\-1\\3\end{pmatrix}$	14	$\sqrt{14}$	$\sqrt{14}$	parallel
$\begin{pmatrix}4\\8\\1\end{pmatrix}$	$\begin{pmatrix}0\\1\\1\end{pmatrix}$	9	9	$\sqrt{2}$	45°

2.3.12 Show that the line joining the origin to the point $P(2, 1, 1)$ is perpendicular to the line determined by the points $A(3, 5, -1), B(4, 3, -1)$.
Solution:

$$(\mathbf{A}-\mathbf{B})^\top \mathbf{P} = \begin{pmatrix}-1 & 2 & 0\end{pmatrix}\begin{pmatrix}2\\1\\1\end{pmatrix} = 0 \quad \square \tag{2.3.12.1}$$

2.3.13 If the lines $\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}$ and $\frac{x-1}{3k} = \frac{y-1}{1} = \frac{z-6}{-5}$ are perpendicular, find the value of k.
Solution: From the given information,

$$\mathbf{m}_1 = \begin{pmatrix}-3\\2k\\2\end{pmatrix}, \mathbf{m}_2 = \begin{pmatrix}3k\\1\\-5\end{pmatrix} \tag{2.3.13.1}$$

$$\implies \begin{pmatrix}-3 & 2k & 2\end{pmatrix}^\top \begin{pmatrix}3k\\1\\-5\end{pmatrix} = 0 \tag{2.3.13.2}$$

$$\implies k = -\frac{10}{7} \tag{2.3.13.3}$$

See Fig. 2.3.13.1

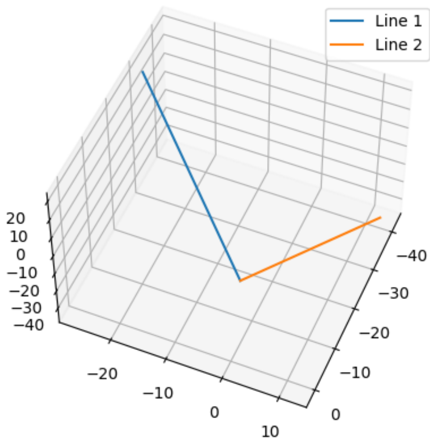


Fig. 2.3.13.1: lines represented for the given points and direction vector with k=$\frac{-10}{7}$

2.3.14 Find a unit vector perpendicular to each of the vector $\overrightarrow{a}+\overrightarrow{b}$ and $\overrightarrow{a}-\overrightarrow{b}$, where $\overrightarrow{a} = 3\hat{i}+2\hat{j}+2\hat{k}$ and $\overrightarrow{b} = \hat{i}+2\hat{j}-2\hat{k}$.

Solution: Let the desired vector be $\mathbf{x}$. Then,

$$\begin{pmatrix}\mathbf{a}+\mathbf{b} & \mathbf{a}-\mathbf{b}\end{pmatrix}^\top \mathbf{x} = 0 \tag{2.3.14.1}$$

$$\tag{2.3.14.2}$$

$$\because \mathbf{a}+\mathbf{b} = \begin{pmatrix}\mathbf{a} & \mathbf{b}\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} \tag{2.3.14.3}$$

$$\mathbf{a}-\mathbf{b} = \begin{pmatrix}\mathbf{a} & \mathbf{b}\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix}, \tag{2.3.14.4}$$

(2.3.14.2) can be expressed as

$$\left\{\begin{pmatrix}\mathbf{a} & \mathbf{b}\end{pmatrix}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\right\}^\top \mathbf{x} = 0 \tag{2.3.14.5}$$

$$\implies \begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}^\top \begin{pmatrix}\mathbf{a} & \mathbf{b}\end{pmatrix}^\top \mathbf{x} = 0 \tag{2.3.14.6}$$

$$\implies \begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}^\top \begin{pmatrix}\mathbf{a} & \mathbf{b}\end{pmatrix}^\top \mathbf{x} = 0 \tag{2.3.14.7}$$

$$\text{or, } \begin{pmatrix}\mathbf{a} & \mathbf{b}\end{pmatrix}^\top \mathbf{x} = 0 \tag{2.3.14.8}$$

which can be expressed as

$$\begin{pmatrix} 3 & 2 & 2 \\ 1 & 2 & -2 \end{pmatrix} \xleftrightarrow[R_2=\frac{R_2}{4}]{R_2=3R_2-R_1} \begin{pmatrix} 3 & 2 & 2 \\ 0 & 1 & -2 \end{pmatrix} \tag{2.3.14.9}$$

$$\xleftrightarrow[R_1=\frac{R_1}{3}]{R_1=R_1-2R_2} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \end{pmatrix} \tag{2.3.14.10}$$

yielding

$$\begin{aligned} x_1 + 2x_3 &= 0 \\ x_2 - 2x_3 &= 0 \end{aligned} \implies \mathbf{x} = x_3 \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} \tag{2.3.14.11}$$

Thus, the desired unit vector is

$$\mathbf{x} = \frac{1}{3} \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} \tag{2.3.14.12}$$

2.3.15 If $\overrightarrow{a} = 2\hat{i}+2\hat{j}3\hat{k}, \overrightarrow{b} = -\hat{i}+2\hat{j}+\hat{k}$ and $\overrightarrow{c} = 3\hat{i}+\hat{j}$ are such that $\overrightarrow{a}+\lambda\overrightarrow{b}$ is perpendicular to $\overrightarrow{c}$,then find the value of λ.

Solution:

$$\because (\mathbf{a} + \lambda\mathbf{b})^\top \mathbf{c} = 0, \tag{2.3.15.1}$$

$$\lambda = -\frac{\mathbf{a}^\top \mathbf{c}}{\mathbf{b}^\top \mathbf{c}} = 8, \tag{2.3.15.2}$$

upon substituting numerical values.

2.3.16 Check whether $(5,-2),(6,4)$ and $(7,-2)$ are the vertices of an isosceles triangle.

2.3.17 The perpendicular bisector of the line segment joining the points $\mathbf{A}(1,5)$ and $\mathbf{B}(4,6)$ cuts the y-axis at

a) $(0,13)$
b) $(0,-13)$
c) $(0,12)$
d) $(13,0)$

2.3.18 The point which lies on the perpendicular bisector of the line segment joining the points $\mathbf{A}(-2,-5)$ and $\mathbf{B}(2,5)$ is

a) $(0,0)$
b) $(0,2)$
c) $(2,0)$
d) $(-2,0)$

2.3.19 The points $(-4,0),(4,0),(0,3)$ are the vertices of

a) right triangle
b) isosceles triangle
c) equilateral triangle
d) scalene triangle

2.3.20 The point $\mathbf{A}(2,7)$ lies on the perpendicular bisector of line segment joining the points

$\mathbf{P}(6,5)$ and $\mathbf{Q}(0,-4)$.

2.3.21 The points $\mathbf{A}(-1,-2), \mathbf{B}(4,3), \mathbf{C}(2,5)$ and $\mathbf{D}(-3,0)$ in that order form a rectangle.

2.3.22 Name the type of triangle formed by the points $\mathbf{A}(-5,6), \mathbf{B}(-4,-2),$ and $\mathbf{C}(7,5)$.

2.3.23 What type of a quadrilateral do the points $\mathbf{A}(2,-2), \mathbf{B}(7,3), \mathbf{C}(11,-1),$ and $\mathbf{D}(6,-6)$ taken in that order, form?

2.3.24 Find the coordinates of the point $\mathbf{Q}$ on the x-axis which lies on the perpendicular bisector of the line segment joining the points $\mathbf{A}(-5,-2)$ and $\mathbf{B}(4,-2)$. Name the type of triangle formed by points $\mathbf{Q}, \mathbf{A}$ and $\mathbf{B}$.

2.3.25 The points $\mathbf{A}(2,9), \mathbf{B}(a,5)$ and $\mathbf{C}(5,5)$ are the vertices of a triangle **ABC** right angled at **B**. Find the values of a and hence the area of $\triangle\mathbf{ABC}$.

2.3.26 Find a vector of magnitude 6, which is perpendicular to both the vectors $2\hat{i} - \hat{j}+2\hat{k}$ and $4\hat{i} - \hat{j} + 3\hat{k}$.

2.3.27 If A,B,C,D are the points with position vectors $\hat{i}+\hat{j}-\hat{k}$, $2\hat{i}-\hat{j}+3\hat{k}$, $2\hat{i}-3\hat{k}$, $3\hat{i}$-2$\hat{j}$+$\hat{k}$, respectively, find the projection of $\overline{AB}$ along $\overline{CD}$.

2.3.28 Find the value of λ such that the vectors $\mathbf{a} = 2\hat{i} + \lambda\hat{j} + \hat{k}$ and $\mathbf{b} = \hat{i} + 2\hat{j} + 3\hat{k}$ are orthogonal.

a) 0
b) 1
c) $\frac{3}{2}$
d) $-\frac{5}{2}$

2.3.29 The number of vectors of unit length perpendicular to the vectors $\mathbf{a} = 2\hat{i} + \hat{j} + 2\hat{k}$ and $\mathbf{b} = \hat{j} + \hat{k}$ is

a) one
b) two
c) three
d) infinite

2.3.30 Find the equation of a plane which bisects perpendicularly the line joining the points A$(2,3,4)$ and B$(4,5,8)$ at right angles.

2.3.31 $\overrightarrow{AB} = 3\hat{i} - \hat{j} + \hat{k}$ and $\overrightarrow{CD} = -3\hat{i} + 2\hat{j} + 4\hat{k}$ are two vectors. The position vectors of the points A and C are $6\hat{i} + 7\hat{j} + 4\hat{k}$ and $-9\hat{j} + 2\hat{k}$, respectively. Find the position vector of a point P on the line AB and a point Q on the line CD such that $\overrightarrow{PQ}$ is perpendicular to $\overrightarrow{AB}$ and $\overrightarrow{CD}$ both.

2.3.32 Line joining the points (3,-4) and (-2,6) is perpendicular to the line joining the points (-3,6) and (9,-18).

2.4 Vector Product

2.4.1 Find $\left|\overrightarrow{a} \times \overrightarrow{b}\right|$, if $\overrightarrow{a} = \hat{i} - 7\hat{j} + 7\hat{k}$ and $\overrightarrow{b} = 3\hat{i} - 2\hat{j} + 2\hat{k}$.
Solution: From (2.1.8.3),

$$\left|\mathbf{A}_{23} \quad \mathbf{B}_{23}\right| = \begin{vmatrix} -7 & -2 \\ 7 & 2 \end{vmatrix} = 0 \tag{2.4.1.1}$$

$$\left|\mathbf{A}_{31} \quad \mathbf{B}_{31}\right| = \begin{vmatrix} 1 & 3 \\ 7 & 2 \end{vmatrix} = -19 \tag{2.4.1.2}$$

$$\left|\mathbf{A}_{12} \quad \mathbf{B}_{12}\right| = \begin{vmatrix} 1 & 3 \\ -7 & -2 \end{vmatrix} = 19, \tag{2.4.1.3}$$

$$\|\mathbf{a} \times \mathbf{b}\| = \left\|\begin{pmatrix} \left|\mathbf{A}_{23} \quad \mathbf{B}_{23}\right| \\ \left|\mathbf{A}_{31} \quad \mathbf{B}_{31}\right| \\ \left|\mathbf{A}_{12} \quad \mathbf{B}_{12}\right| \end{pmatrix}\right\| = 19\sqrt{2} \tag{2.4.1.4}$$

from (2.1.9.1).

2.4.2 Find λ and μ if $(2\hat{i} + 6\hat{j} + 27\hat{k}) \times (\hat{i} + \lambda\hat{j} + \mu\hat{k}) = \overrightarrow{0}$.
Solution: From Formula 2.1.11, performing row reduction,

$$\begin{pmatrix} 2 & 6 & 27 \\ 1 & \lambda & \mu \end{pmatrix} \xleftrightarrow{R_2 \leftarrow 2R_2 - R_1} \begin{pmatrix} 2 & 6 & 27 \\ 0 & 2\lambda - 6 & 2\mu - 27 \end{pmatrix} \tag{2.4.2.1}$$

For the above matrix to have rank 1,

$$\mu = \frac{27}{2}, \lambda = 3. \tag{2.4.2.2}$$

2.4.3 Find the area of the triangle with vertices $A(1, 1, 2), B(2, 3, 5)$ and $C(1, 5, 5)$.
Solution:

$$\because \mathbf{B} - \mathbf{A} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{C} - \mathbf{A} = \begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix}, \tag{2.4.3.1}$$

$$\frac{1}{2}\left\|\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \times \begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix}\right\| = \frac{1}{2}\left\|\begin{pmatrix} -6 \\ 3 \\ 4 \end{pmatrix}\right\| = \frac{\sqrt{61}}{2} \tag{2.4.3.2}$$

using (2.1.13.2), which is the the desired area.

2.4.4 Find the area of the parallelogram whose adjacent sides are determined by the vectors $\overrightarrow{a} = \hat{i} - \hat{j} + 3\hat{k}$ and $\overrightarrow{b} = 2\hat{i} - 7\hat{j} + \hat{k}$.
Solution: From (2.1.14.1), the desired area is obtained as

$$\left\|\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \times \begin{pmatrix} 2 \\ -7 \\ 1 \end{pmatrix}\right\| = \left\|\begin{pmatrix} 20 \\ 5 \\ -5 \end{pmatrix}\right\| = 15\sqrt{2} \tag{2.4.4.1}$$

2.4.5 Find the area of a rhombus if its vertices are $A(3, 0), B(4, 5), C(-1, 4)$ and $D(-2, -1)$ taken in order.

Solution: The area of the rhombus is

$$\left\|\left(\mathbf{A}-\mathbf{D}\right)\times\left(\mathbf{B}-\mathbf{A}\right)\right\| = \begin{vmatrix} 5 & 1 \\ 1 & 5 \end{vmatrix} = 24 \tag{2.4.5.1}$$

See Fig. 2.4.5.1.

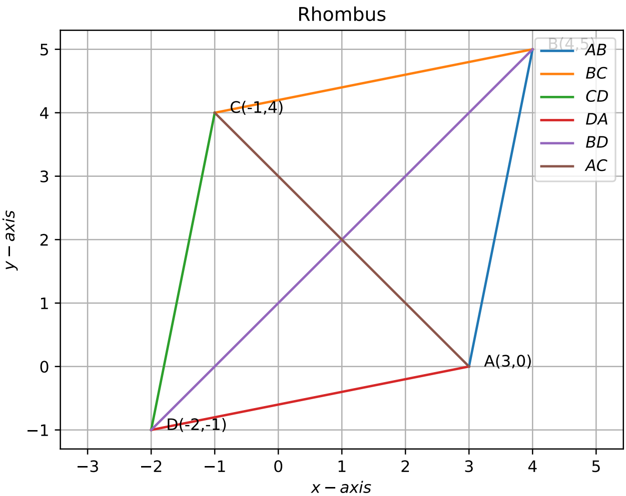


Fig. 2.4.5.1

2.4.6 Let the vectors $\vec{a}$ and $\vec{b}$ be such that $|\vec{a}| = 3$ and $|\vec{b}| = \frac{\sqrt{2}}{3}$, then $\vec{a}\times\vec{b}$ is a unit vector, if the angle between $\vec{a}$ and $\vec{b}$ is

a) $\frac{\pi}{6}$
b) $\frac{\pi}{4}$
c) $\frac{\pi}{3}$
d) $\frac{\pi}{2}$

Solution: From the given information and (2.1.12.1)

$$\|\mathbf{a}\times\mathbf{b}\| = \|\mathbf{a}\|\,\|\mathbf{b}\|\sin\theta = 1 \tag{2.4.6.1}$$

$$\implies \sin\theta = \frac{1}{\|\mathbf{a}\|\,\|\mathbf{b}\|} = \frac{1}{\sqrt{2}} \tag{2.4.6.2}$$

$$\implies \theta = \frac{\pi}{4} \tag{2.4.6.3}$$

2.4.7 Area of a rectangle having vertices A, B, C and D with position vectors $-\hat{\imath}+\frac{1}{2}\hat{\jmath}+4\hat{k}, \hat{\imath}+\frac{1}{2}\hat{\jmath}+4\hat{k}, \hat{\imath}-\frac{1}{2}\hat{\jmath}+4\hat{k}$ and $-\hat{\imath}-\frac{1}{2}\hat{\jmath}+4\hat{k}$, respectively is

a) $\frac{1}{2}$
b) 1

c) 2
d) 4

Solution: Since

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix} \tag{2.4.7.1}$$

$$\mathbf{C} - \mathbf{B} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \tag{2.4.7.2}$$

area of the rectangle is

$$\|(\mathbf{A} - \mathbf{B}) \times (\mathbf{C} - \mathbf{D})\| = 2 \tag{2.4.7.3}$$

See Fig. 2.4.7.1

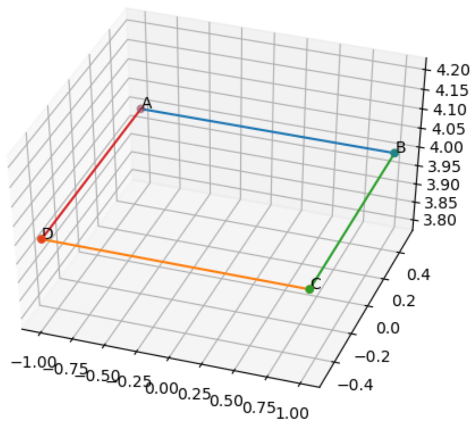


Fig. 2.4.7.1

2.4.8 Find the area of the triangle whose vertices are

a) $(2, 3), (-1, 0), (2, -4)$
b) $(-5, -1), (3, -5), (5, 2)$

Solution: See Table 2.4.8.

TABLE 2.4.8

	$\mathbf{A}-\mathbf{B}$	$\mathbf{A}-\mathbf{C}$	$\frac{1}{2}\|(\mathbf{A}-\mathbf{B})\times(\mathbf{A}-\mathbf{C})\|$
a)	$\begin{pmatrix}3\\3\end{pmatrix}$	$\begin{pmatrix}0\\7\end{pmatrix}$	$\frac{21}{2}$
b)	$\begin{pmatrix}-8\\4\end{pmatrix}$	$\begin{pmatrix}-10\\-3\end{pmatrix}$	32

2.4.9 Find the area of the triangle formed by joining the mid-points of the sides of the triangle whose vertices are $A(0,-1), B(2,1)$ and $C(0,3)$. Find the ratio of this area to the area of the given triangle.

Solution: Using (1.1.4.1), the mid point coordinates are given by

$$\mathbf{P} = \frac{1}{2}(\mathbf{A}+\mathbf{B}) = \begin{pmatrix}1\\0\end{pmatrix} \tag{2.4.9.1}$$

$$\mathbf{Q} = \frac{1}{2}(\mathbf{B}+\mathbf{C}) = \begin{pmatrix}1\\2\end{pmatrix} \tag{2.4.9.2}$$

$$\mathbf{R} = \frac{1}{2}(\mathbf{A}+\mathbf{C}) = \begin{pmatrix}0\\1\end{pmatrix} \tag{2.4.9.3}$$

$$\because \mathbf{P}-\mathbf{Q} = \begin{pmatrix}0\\-2\end{pmatrix}, \mathbf{Q}-\mathbf{R} = \begin{pmatrix}1\\1\end{pmatrix} \tag{2.4.9.4}$$

$$ar(PQR) = \frac{1}{2}\|(\mathbf{P}-\mathbf{Q})\times(\mathbf{Q}-\mathbf{R})\| = 1 \tag{2.4.9.5}$$

Similarly,

$$\mathbf{A}-\mathbf{B} = \begin{pmatrix}-2\\-2\end{pmatrix}, \mathbf{A}-\mathbf{C} = \begin{pmatrix}0\\-4\end{pmatrix} \tag{2.4.9.6}$$

$$\implies ar(ABC) = \frac{1}{2}\|(\mathbf{A}-\mathbf{B})\times(\mathbf{A}-\mathbf{C})\| = 4 \tag{2.4.9.7}$$

$$\implies \frac{ar\,(PQR)}{ar\,(ABC)} = \frac{1}{4} \tag{2.4.9.8}$$

See Fig. 2.4.9.1

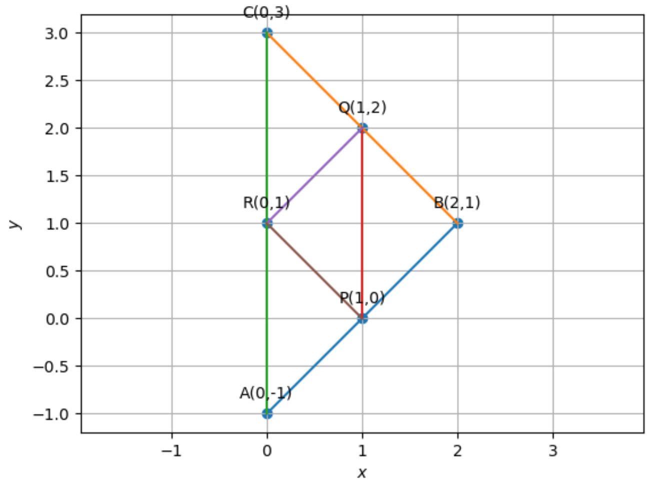


Fig. 2.4.9.1

2.4.10 Find the area of the quadrilateral whose vertices, taken in order, are $A(-4,-2), B(-3,-5), C(3,-2)$ and $D(2,3)$.

Solution: See Fig. 2.4.10.1

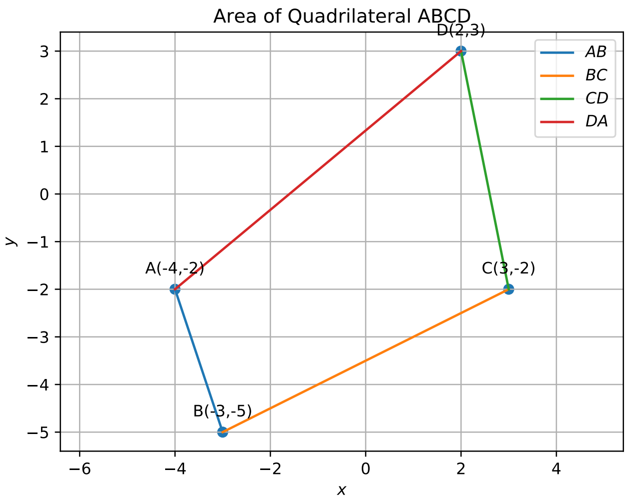


Fig. 2.4.10.1

$$\because \mathbf{A}-\mathbf{B}=\begin{pmatrix}-1\\3\end{pmatrix},\ \mathbf{A}-\mathbf{D}=\begin{pmatrix}-6\\-5\end{pmatrix}, \tag{2.4.10.1}$$

$$\mathbf{B}-\mathbf{C}=\begin{pmatrix}-6\\-5\end{pmatrix},\ \mathbf{B}-\mathbf{D}=\begin{pmatrix}-3\\-8\end{pmatrix}, \tag{2.4.10.2}$$

$$ar(ABD)=\frac{1}{2}\left\|(\mathbf{A}-\mathbf{B})\times(\mathbf{A}-\mathbf{D})\right\|=\frac{23}{2} \tag{2.4.10.3}$$

$$ar(BCD)=\frac{1}{2}\left\|(\mathbf{B}-\mathbf{C})\times(\mathbf{B}-\mathbf{D})\right\|=\frac{33}{2} \tag{2.4.10.4}$$

$$\implies ar(ABCD)=ar(ABD)+ar(BCD)=28 \tag{2.4.10.5}$$

2.4.11 Verify that a median of a triangle divides it into two triangles of equal areas for $\triangle ABC$ whose vertices are $\mathbf{A}(4,-6), \mathbf{B}(3,2),$ and $\mathbf{C}(5,2)$.
Solution:

$$\mathbf{D}=\frac{\mathbf{B}+\mathbf{C}}{2}=\begin{pmatrix}4\\0\end{pmatrix}, \tag{2.4.11.1}$$

$$\mathbf{A}-\mathbf{B}=\begin{pmatrix}1\\-4\end{pmatrix},\ \mathbf{A}-\mathbf{D}=\begin{pmatrix}0\\-6\end{pmatrix} \tag{2.4.11.2}$$

$$\implies ar(ABD)=\frac{1}{2}\left\|(\mathbf{A}-\mathbf{B})\times(\mathbf{A}-\mathbf{D})\right\|=3 \tag{2.4.11.3}$$

$$\mathbf{A}-\mathbf{C}=\begin{pmatrix}-1\\-8\end{pmatrix},\ \mathbf{A}-\mathbf{D}=\begin{pmatrix}0\\-6\end{pmatrix} \tag{2.4.11.4}$$

$$\implies ar(ACD)=\frac{1}{2}\left\|(\mathbf{A}-\mathbf{C})\times(\mathbf{A}-\mathbf{D})\right\| \tag{2.4.11.5}$$

$$=3=ar(ABD) \tag{2.4.11.6}$$

See Fig. 2.4.11.1.

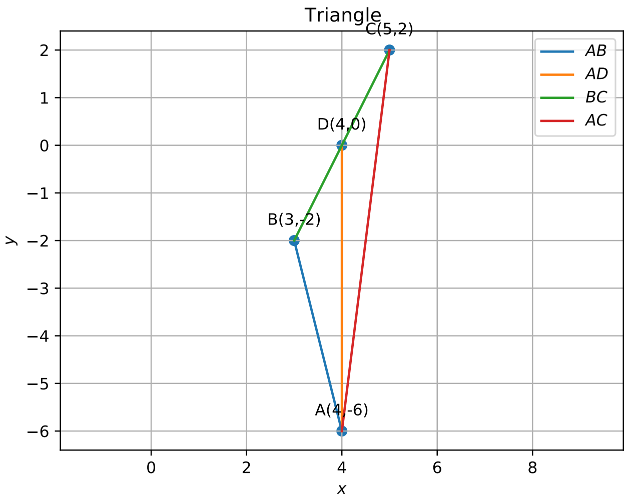


Fig. 2.4.11.1

2.4.12 The two adjacent sides of a parallelogram are $\mathbf{a} = 2\hat{i} - 4\hat{j} + 5\hat{k}$ and $\mathbf{b} = \hat{i} - 2\hat{j} - 3\hat{k}$. Find the unit vector parallel to its diagonal. Also, find its area.

Solution: The diagonals of the parallelogram are given by

$$\mathbf{a} + \mathbf{b} = \begin{pmatrix} 3 \\ -6 \\ 2 \end{pmatrix}, \mathbf{a} - \mathbf{b} = \begin{pmatrix} 1 \\ -2 \\ 8 \end{pmatrix} \tag{2.4.12.1}$$

and the corresponding unit vectors are

$$\frac{\mathbf{a} + \mathbf{b}}{\|\mathbf{a} + \mathbf{b}\|} = \begin{pmatrix} \frac{3}{\sqrt{45}} \\ -\frac{6}{\sqrt{45}} \\ \frac{2}{\sqrt{45}} \end{pmatrix}, \frac{\mathbf{a} - \mathbf{b}}{\|\mathbf{a} - \mathbf{b}\|} = \begin{pmatrix} \frac{1}{\sqrt{69}} \\ -\frac{2}{\sqrt{69}} \\ \frac{8}{\sqrt{69}} \end{pmatrix} \tag{2.4.12.2}$$

The area of the parallelogram is given by

$$\|\mathbf{a} \times \mathbf{b}\| = \left\| \begin{pmatrix} 22 \\ -11 \\ 0 \end{pmatrix} \right\| = \sqrt{605} \tag{2.4.12.3}$$

2.4.13 The vertices of a $\triangle ABC$ are $\mathbf{A}(4,6), \mathbf{B}(1,5)$ and $\mathbf{C}(7,2)$. A line is drawn to intersect sides AB and AC at $\mathbf{D}$ and $\mathbf{E}$ respectively, such that $\frac{AD}{AB} = \frac{AE}{AC} = \frac{1}{4}$. Calculate the area of $\triangle ADE$ and compare it with the area of the $\triangle ABC$.

Solution: See Fig. 2.4.13.1.

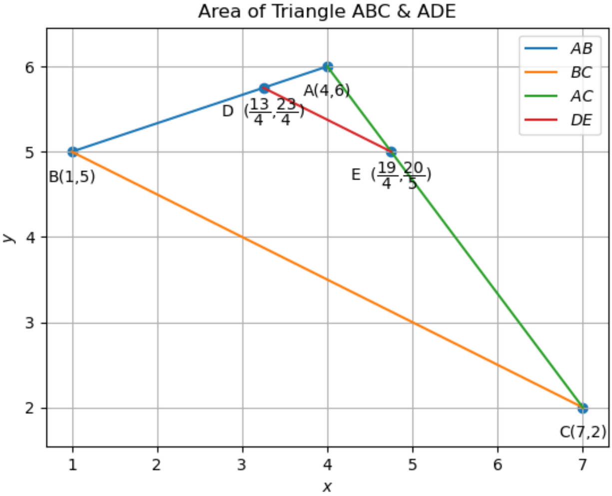


Fig. 2.4.13.1

Using section formula (1.1.4.1),

$$\mathbf{D} = \frac{3\mathbf{A}+\mathbf{B}}{4} = \frac{1}{4}\begin{pmatrix}13\\23\end{pmatrix} \tag{2.4.13.1}$$

$$\mathbf{E} = \frac{3\mathbf{A}+\mathbf{C}}{4} = \frac{1}{4}\begin{pmatrix}19\\20\end{pmatrix} \tag{2.4.13.2}$$

$$\mathbf{A}-\mathbf{D} = \frac{1}{4}\begin{pmatrix}3\\1\end{pmatrix}, \mathbf{A}-\mathbf{E} = \frac{1}{4}\begin{pmatrix}-3\\1\end{pmatrix} \tag{2.4.13.3}$$

$$\mathbf{A}-\mathbf{B} = \begin{pmatrix}3\\1\end{pmatrix}, \mathbf{B}-\mathbf{C} = \begin{pmatrix}-6\\3\end{pmatrix} \tag{2.4.13.4}$$

yielding

$$ar(ABD) = \frac{1}{2}\left\|(\mathbf{A}-\mathbf{D})\times(\mathbf{A}-\mathbf{E})\right\| = \frac{15}{32} \tag{2.4.13.5}$$

$$ar(ABC) = \frac{1}{2}\left\|(\mathbf{A}-\mathbf{B})\times(\mathbf{B}-\mathbf{C})\right\| = \frac{15}{2} \tag{2.4.13.6}$$

$$\implies \frac{ar\,(ADE)}{ar\,(ABC)} = \frac{1}{16} \tag{2.4.13.7}$$

2.4.14 Draw a quadrilateral in the Cartesian plane, whose vertices are

$$\mathbf{A} = \begin{pmatrix}-4\\5\end{pmatrix}, \mathbf{B} = \begin{pmatrix}0\\7\end{pmatrix}, \mathbf{C} = \begin{pmatrix}5\\-5\end{pmatrix}, \mathbf{D} = \begin{pmatrix}-4\\-2\end{pmatrix}. \tag{2.4.14.1}$$

Also, find its area.

Solution: See Fig. 2.4.14.1. From (2.1.13.2),

$$ar(ABCD) = \frac{121}{2} \tag{2.4.14.2}$$

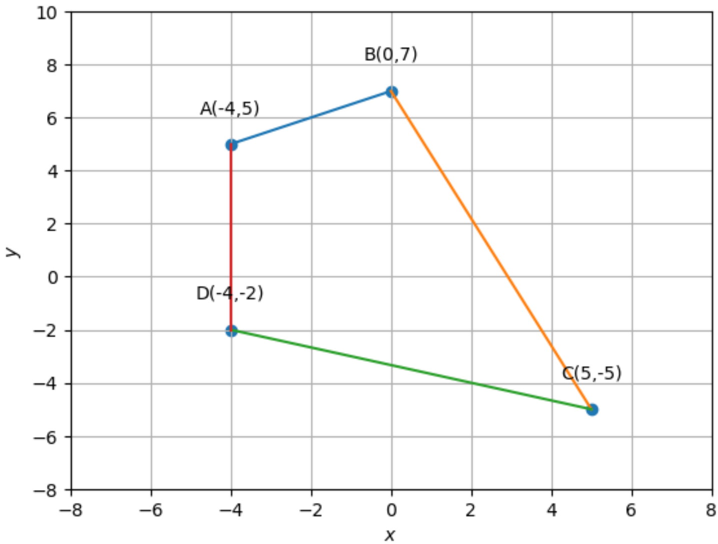


Fig. 2.4.14.1: Plot of quadrilateral $ABCD$

2.4.15 Find the area of region bounded by the triangle whose vertices are $(1,0),(2,2)$ and $(3,1)$.

2.4.16 Find the area of region bounded by the triangle whose vertices are $(-1,0),(1,3)$ and $(3,2)$.

2.4.17 Find the area of the $\triangle ABC$, coordinates of whose vertices are $\mathbf{A}(2,0), \mathbf{B}(4,5)$ and $\mathbf{C}(6,3)$.

2.4.18 The area of a triangle with vertices $\mathbf{A}(3,0), \mathbf{B}(7,0)$ and $\mathbf{C}(8,4)$ is

a) 14
b) 28
c) 8
d) 6

2.4.19 Find the area of the triangle whose vertices are $(-8,4),(-6,6)$ and $(-3,9)$.

2.4.20 If $\mathbf{D}\left(\frac{-1}{2},\frac{5}{2}\right), \mathbf{E}(7,3)$ and $\mathbf{F}\left(\frac{7}{2},\frac{7}{2}\right)$ are the midpoints of sides of $\triangle ABC$, find the area of the $\triangle ABC$.

2.4.21 Find the sine of the angle between the vectors $\mathbf{a} = 3\hat{i}+\hat{j}+2\hat{k}$ and $\mathbf{b} = 2\hat{i}-2\hat{j}+4\hat{k}$.

2.4.22 Using vectors, find the area of $\triangle ABC$ with vertices A(1,2,3), B(2,-1,4) and C(4,5,-1).

2.4.23 Find the area of the parallelogram whose diagonals are $2\hat{i}-\hat{j}+\hat{k}$ and $\hat{i}+3\hat{j}-\hat{k}$.

2.4.24 The vector from origin to the points A and B are $\mathbf{a} = 2\hat{i}-3\hat{j}+2\hat{k}$ and $\mathbf{b} = 2\hat{i}+3\hat{j}+\hat{k}$, respectively, then the area of $\triangle OAB$ is

a) 340
b) $\sqrt{25}$
c) $\sqrt{229}$
d) $\frac{1}{2}\sqrt{229}$

2.4.25 If $\mathbf{a} = \hat{i} + \hat{j} + \hat{k}$ and $\mathbf{b} = \hat{j} - \hat{k}$, find a vector $\mathbf{c}$ such that $\mathbf{a} \times \mathbf{c} = \mathbf{b}$ and $\mathbf{a} \cdot \mathbf{c} = 3$.

2.4.26 The area of the quadrilateral ABCD, where A$(0, 4, 1)$, B$(2, 3, -1)$, C$(4, 5, 0)$ and D$(2, 6, 2)$, is equal to

a) 9 sq. units
b) 18 sq. units
c) 27 sq. units
d) 81 sq. units

2.4.27 Find the area of region bounded by the triangle whose vertices are $(-1, 1), (0, 5)$ and $(3, 2)$.

2.4.28 The value of $\hat{i} \cdot (\hat{j} \times \hat{k}) + \hat{j} \cdot (\hat{i} \times \hat{k}) + \hat{k} \cdot (\hat{i} \times \hat{j})$ is

a) 0
b) -1
c) 1
d) 3

2.4.29 The value of $\hat{i} \cdot (\hat{j} \times \hat{k}) + \hat{j} \cdot (\hat{i} \times \hat{k}) + \hat{k} \cdot (\hat{i} \times \hat{j})$ is

a) 0
b) -1
c) 1
d) 3

2.5 Miscellaneous

2.5.1 The two opposite vertices of a square are $(-1, 2)$ and $(3, 2)$. Find the coordinates of the other two vertices.

Solution: Let

$$\mathbf{A} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \tag{2.5.1.1}$$

The given square is available in Fig. 2.5.1.1. Fig. 2.5.1.2 shows the translation of **A** to the origin. Fig. 2.5.1.3 shows the subsequent anticlockwise rotation of the square by 45°. From (2.1.15.1).

$$\mathbf{C} - \mathbf{A} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \phi = 0^\circ \tag{2.5.1.2}$$

where ϕ is the angle made by AC with the x-axis. Substituting numerical values in (2.1.18.1),

$$\mathbf{B} = 2\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{2.5.1.3}$$

Similarly,

$$\mathbf{D} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}. \tag{2.5.1.4}$$

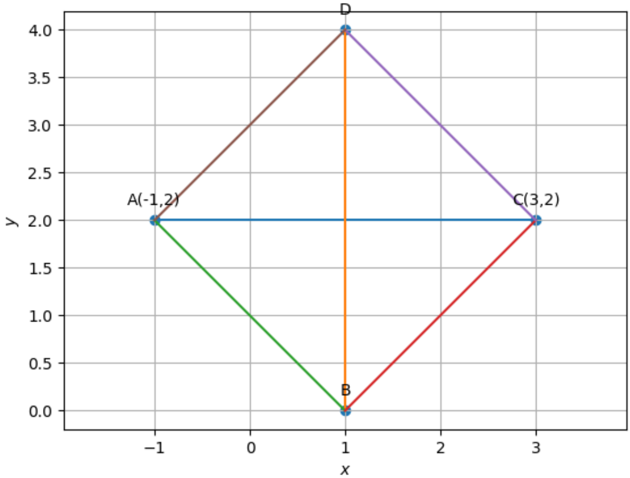


Fig. 2.5.1.1

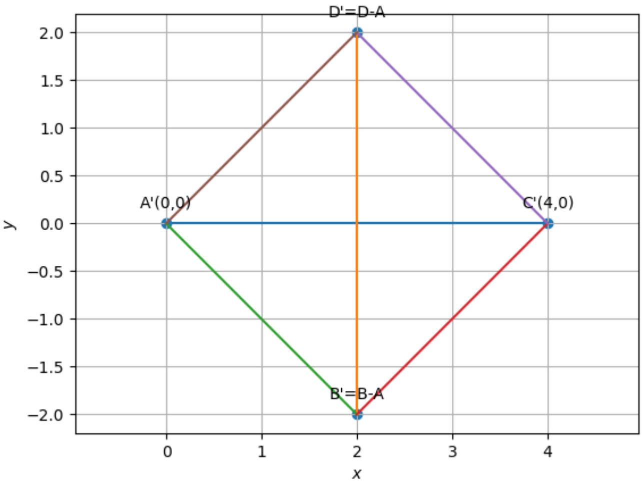


Fig. 2.5.1.2

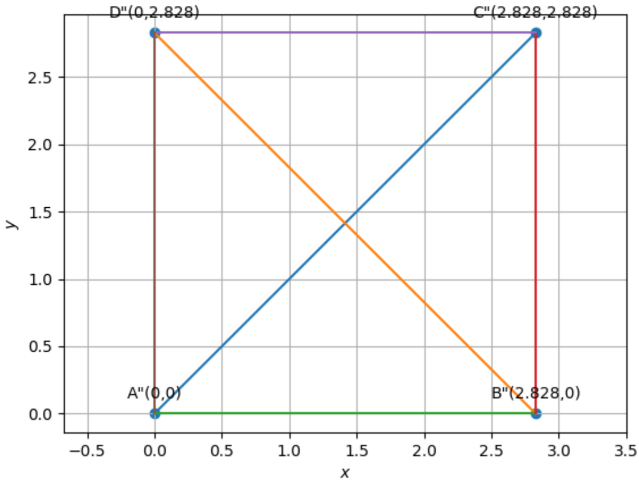


Fig. 2.5.1.3

2.5.2 The base of an equilateral triangle with side $2a$ lies along the y-axis such that the mid-point of the base is at the origin. Find vertices of the triangle.
Solution:

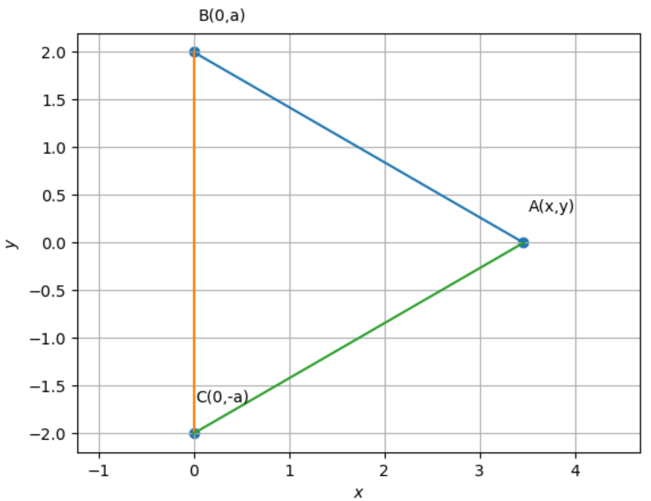


Fig. 2.5.2.1: $a = 2$.

Let the base be BC. From the given information,

$$\mathbf{B} = a\mathbf{e}_2, \mathbf{C} = -a\mathbf{e}_2 \tag{2.5.2.1}$$

Since $\mathbf{A}$ lies on the x-axis,

$$\mathbf{A} = k\mathbf{e}_1 \tag{2.5.2.2}$$

and

$$\|\mathbf{A} - \mathbf{C}\|^2 = (2a)^2 \tag{2.5.2.3}$$
$$\implies \|\mathbf{A}\|^2 + \|\mathbf{C}\|^2 - 2\mathbf{A}^\top\mathbf{C} = 4a^2 \tag{2.5.2.4}$$
$$\implies k^2 + a^2 = 4a^2 \tag{2.5.2.5}$$

yielding

$$k = \pm a\sqrt{3} \tag{2.5.2.6}$$

Thus,

$$\mathbf{A} = \pm\sqrt{3}a\mathbf{e}_1 \tag{2.5.2.7}$$

See Fig. 2.5.2.1.

2.5.3 Let $\mathbf{a}$ and $\mathbf{b}$ be two unit vectors and θ the angle between them. Then $\mathbf{a}+\mathbf{b}$ is a unit vector if

a) $\theta = \frac{\pi}{4}$
b) $\theta = \frac{\pi}{3}$
c) $\theta = \frac{\pi}{2}$
d) $\theta = \frac{2\pi}{3}$

Solution:

$$\because \|\mathbf{a}\| = \|\mathbf{b}\| = 3\,\|\mathbf{a} + \mathbf{b}\| = 1, \tag{2.5.3.1}$$
$$\|\mathbf{a} + \mathbf{b}\|^2 = 1^2 \tag{2.5.3.2}$$
$$\implies \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 + 2\mathbf{a}^\top\mathbf{b} = 1 \tag{2.5.3.3}$$
$$\implies (\|\mathbf{a}\|\,\|\mathbf{b}\|\cos\theta) = \frac{-1}{2} \tag{2.5.3.4}$$
$$\implies \cos\theta = \frac{-1}{2}, \text{or, } \theta = \frac{2\pi}{3} \tag{2.5.3.5}$$

2.5.4 Show that the tangent of an angle between the lines

$$\frac{x}{a} + \frac{y}{b} = 1 \text{ and} \tag{2.5.4.1}$$
$$\frac{x}{a} - \frac{y}{b} = 1 \tag{2.5.4.2}$$

is $\frac{2ab}{a^2-b^2}$.

2.5.5 Find $\left|\overrightarrow{x}\right|$, if for a unit vector $\overrightarrow{a}, (\overrightarrow{x} - \overrightarrow{a}) \cdot (\overrightarrow{x} + \overrightarrow{a})$=12.

Solution: From the given information,

$$(\mathbf{x}-\mathbf{a})^\top(\mathbf{x}+\mathbf{a}) = 12 \tag{2.5.5.1}$$
$$\implies \|\mathbf{x}\|^2 - \|\mathbf{a}\|^2 = 12 \tag{2.5.5.2}$$
$$\implies \|\mathbf{x}\| = \sqrt{13} \tag{2.5.5.3}$$

2.5.6 Find $\left|\overrightarrow{a}\right|$ and $\left|\overrightarrow{b}\right|$, if $(\overrightarrow{a}+\overrightarrow{b})\cdot(\overrightarrow{a}-\overrightarrow{b}) = 8$ and $\left|\overrightarrow{a}\right| = 8\left|\overrightarrow{b}\right|$.
Solution:

$$\because (\mathbf{a}+\mathbf{b})^\top(\mathbf{a}-\mathbf{b}) = 8, \|\mathbf{a}\| = 8\|\mathbf{b}\|, \tag{2.5.6.1}$$
$$\|\mathbf{a}\|^2 - \|\mathbf{b}\|^2 = 8 \tag{2.5.6.2}$$
$$\implies \|8\mathbf{b}\|^2 - \|\mathbf{b}\|^2 = 8 \tag{2.5.6.3}$$
$$\implies \|\mathbf{b}\| = \frac{2\sqrt{2}}{3\sqrt{7}} \tag{2.5.6.4}$$

Thus,

$$\|\mathbf{a}\| = 8\|\mathbf{b}\| = \frac{16\sqrt{2}}{3\sqrt{7}} \tag{2.5.6.5}$$

2.5.7 Find the magnitude of two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$, having the same magnitude and such that the angle between them is 60° and their scalar product is $\frac{1}{2}$.
Solution: Given

$$\|\mathbf{a}\| = \|\mathbf{b}\|, \cos\theta = \frac{1}{2}, \mathbf{a}^\top\mathbf{b} = \frac{1}{2}, \tag{2.5.7.1}$$
$$\implies \frac{1}{2} = \frac{\frac{1}{2}}{\|\mathbf{a}\|^2} \implies \|\mathbf{a}\| = \|\mathbf{b}\| = 1 \tag{2.5.7.2}$$

by using the definition of the scalar product in (2.1.1.1).

2.5.8 Show that $\left|\overrightarrow{a}\right|\overrightarrow{b} + \left|\overrightarrow{b}\right|\overrightarrow{a}$ is perpendicular to $\left|\overrightarrow{a}\right|\overrightarrow{b} - \left|\overrightarrow{b}\right|\overrightarrow{a}$, for any two nonzero vectors $\overrightarrow{a}$ and $\overrightarrow{b}$.
Solution:

$$\|\mathbf{a}\|\mathbf{b} + \|\mathbf{b}\|\mathbf{a} = \|\mathbf{a}\|\|\mathbf{b}\|\left(\frac{\mathbf{b}}{\|\mathbf{b}\|} + \frac{\mathbf{a}}{\|\mathbf{a}\|}\right) \tag{2.5.8.1}$$
$$\|\mathbf{a}\|\mathbf{b} - \|\mathbf{b}\|\mathbf{a} = \|\mathbf{a}\|\|\mathbf{b}\|\left(\frac{\mathbf{b}}{\|\mathbf{b}\|} - \frac{\mathbf{a}}{\|\mathbf{a}\|}\right) \tag{2.5.8.2}$$
$$\implies (\|\mathbf{a}\|\mathbf{b} + \|\mathbf{b}\|\mathbf{a})^\top(\|\mathbf{a}\|\mathbf{b} - \|\mathbf{b}\|\mathbf{a}) = 0 \tag{2.5.8.3}$$

from (2.1.5.1).

2.5.9 If **a**, **b**, **c** are unit vectors such that **a**+**b**+**c**=0, then the value of $\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a}$ is

a) 1
b) 3
c) $\frac{-3}{2}$
d) None of these

Solution:

$$\|\mathbf{a}+\mathbf{b}+\mathbf{c}\|^2 = 0$$
$$\implies \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 + \|\mathbf{c}\|^2 + 2(\mathbf{a}^\top\mathbf{b} + \mathbf{b}^\top\mathbf{c} + \mathbf{c}^\top\mathbf{a}) = 0$$
$$\implies 3 + 2(\mathbf{a}^\top\mathbf{b} + \mathbf{b}^\top\mathbf{c} + \mathbf{c}^\top\mathbf{a}) = 0$$
$$\implies \mathbf{a}^\top\mathbf{b} + \mathbf{b}^\top\mathbf{c} + \mathbf{c}^\top\mathbf{a} = -\frac{3}{2} \tag{2.5.9.1}$$

2.5.10 If either vector $\vec{a} = 0$ or $\vec{b} = 0$, then $\vec{a}.\vec{b}$=0. But the converse need not be true. Justify your answer with an example.
Solution:

$$\mathbf{a} = \begin{pmatrix}1\\1\end{pmatrix}, \mathbf{b} = \begin{pmatrix}1\\-1\end{pmatrix} \tag{2.5.10.1}$$
$$\implies \mathbf{a}^\top\mathbf{b} = 0 \tag{2.5.10.2}$$

2.5.11 Prove that $(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) = |\mathbf{a}|^2 + |\mathbf{b}|^2$, if and only if $\mathbf{a}, \mathbf{b}$ are perpendicular, given $\mathbf{a} \neq \mathbf{0}, \mathbf{b} \neq \mathbf{0}$.
Solution:

$$\because (\mathbf{a}+\mathbf{b})^\top(\mathbf{a}+\mathbf{b}) = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2, \tag{2.5.11.1}$$
$$\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 + 2\mathbf{a}^\top\mathbf{b} = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 \tag{2.5.11.2}$$
$$\implies \mathbf{a}^\top\mathbf{b} = 0 \tag{2.5.11.3}$$

2.5.12 If l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of two mutually perpendicular lines, show that the direction cosines of the line perpendicular to both these are $m_1n_2 - m_2n_1, n_1l_2 - n_2l_1, l_1m_2 - l_2m_1$.
Solution:

$$\mathbf{P} = \begin{pmatrix} l_1 & l_2 & m_1n_2 - m_2n_1 \\ m_1 & m_2 & n_1l_2 - n_2l_1 \\ n_1 & n_2 & l_1m_2 - l_2m_1 \end{pmatrix} \tag{2.5.12.1}$$

satisfies (2.1.6.1). Hence, the three vectors are mutually perpendicular.

2.5.13 Find the angle between the lines whose direction ratios are a, b, c and $b-c, c-a, a-b$.
Solution:

$$\because \begin{pmatrix} a & b & c \end{pmatrix}\begin{pmatrix} b-c \\ c-a \\ a-b \end{pmatrix} = 0, \theta = \frac{\pi}{2} \tag{2.5.13.1}$$

2.5.14 The value of the expression $|\mathbf{a}\times\mathbf{b}|$+ $(\mathbf{a}\cdot\mathbf{b})$ is ______

2.5.15 If $|\mathbf{a}\times\mathbf{b}|^2$ + $|\mathbf{a}\cdot\mathbf{b}|^2$=144 and $|\mathbf{a}|$=4, then $|\mathbf{b}|$ is equal to ______.

2.5.16 If the directions cosines of a line are (k, k, k) then

a) $k > 0$
b) $0 < k < 1$
c) $k = 1$
d) $k = \frac{1}{\sqrt{3}}$ or $-\frac{1}{\sqrt{3}}$

2.5.17 Find the position vector of a point A in space such that $\overrightarrow{OA}$ is inclined at 60° to OX and at 45° to OY and $\left|\overrightarrow{OA}\right| = 10$ units.

2.5.18 If $(-4, 3)$ and $(4, 3)$ are two vertices of an equilateral triangle. Find the coordinates of the third vertex, given that the origin lies in the interior of the triangle.

2.5.19 $\mathbf{A}(6, 1), \mathbf{B}(8, 2)$ and $\mathbf{C}(9, 4)$ are three vertices of a parallelogram ABCD. If $\mathbf{C}$ is the midpoint of DC find the area of $\triangle ADE$.

2.5.20 If the points $\mathbf{A}(1, -2), \mathbf{B}(2, 3), \mathbf{C}(a, 2)$ and $\mathbf{D}(-4 - 3)$ form parallelogram, find the value of a and height of the parallelogram taking AB as base.

2.5.21 Ayush starts walking from his house to office. Instead of going to the office directly, he goes to a bank first, from there to his daughter school and then reaches the office what is the extra distance travelled by Ayush in reaching his office? If the house is situated at $(2, 4)$, bank at $(5, 8)$, school at $(13, 14)$ and office at $(13, 26)$ and coordinates are in km.

2.5.22 Find the angle between the lines whose direction cosines are given by the equations $l + m + n = 0$, $l^2 + m^2 - n^2 = 0$.

2.5.23 If a variable line in two adjacent positions has directions cosines l, m, n and $l+\delta l, m+\delta m, n + \delta n$, show that the small angle $\delta\theta$ between the two positions is given by

$$\delta\theta^2 = \delta l^2 + \delta m^2 + \delta n^2 \tag{2.5.23.1}$$

2.5.24 The vector $\mathbf{a} + \mathbf{b}$ bisects the angle between the non-collinear vectors $\mathbf{a}$ and $\mathbf{b}$ if ______.

2.5.25 If $\mathbf{a}$ and $\mathbf{b}$ are adjacent sides of a rhombus, then $\mathbf{a} \cdot \mathbf{b}$=0.

2.5.26 If $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are mutually perpendicular vectors of equal magnitudes,show that the $\mathbf{A} + \mathbf{B} + \mathbf{C}$ is equally inclined to $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$.

2.5.27 Projection vector of $\mathbf{a}$ on $\mathbf{b}$ is

a) $\left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\right)$
b) $\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}$
c) $\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}$
d) $\left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\right)$

2.5.28 If $\mathbf{a}$ is any non-zero vector, then $(\mathbf{a} \cdot \hat{i})\hat{i}$+$(\mathbf{a} \cdot \hat{j})\hat{j}$+$(\mathbf{a} \cdot \hat{k})\ \hat{k}$ equals ______.

2.5.29 If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are the three vectors such that $\mathbf{a} + \mathbf{b} + \mathbf{c} = 0$ and $|\mathbf{a}| = 2$, $|\mathbf{b}|$=3, $|\mathbf{c}|$=5, the value of $\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{a}$ is

a) 0
b) 1
c) -19
d) 38

2.5.30 If $\mathbf{r} \cdot \mathbf{a} = 0, \mathbf{r} \cdot \mathbf{b} = 0$ and $\mathbf{r} \cdot \mathbf{c} = 0$ for some non-zero vector $\mathbf{r}$, then the value of $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ is ______.

2.5.31 If $|\mathbf{a} + \mathbf{b}| = |\mathbf{a} - \mathbf{b}|$, then the vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal.

2.5.32 Prove that the lines $x = py+q, z = ry+s$ and $x = p'y+q', z = r'y+s'$ are perpendicular if $pp' + rr' + 1 = 0$.

2.5.33 Show that the straight lines whose direction cosines are given by $2l + 2m - n = 0$

and $mn + nl + lm = 0$ are at right angles.

2.5.34 If $l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3$ are the direction cosines of the three mutually perpendcular lines, prove that the line whose direction cosines are propotional to $l_1 + l_2 + l_3, m_1 + m_2, m_3, n_1 + n_2 + n_3$ make angles with them.

3.1 Formulae

3.1.1. Construct a $\triangle ABC$ given $a, \angle B$ and $K = b + c$.
Solution: Using the cosine formula in $\triangle ABC$,

$$b^2 = a^2 + c^2 - 2ac\cos B \tag{3.1.1.1}$$

$$\implies (K - c)^2 = a^2 + c^2 - 2ac\cos B \tag{3.1.1.2}$$

$$\implies c = \frac{K^2 - a^2}{2(K - a\cos B)} \tag{3.1.1.3}$$

The coordinates of $\triangle ABC$ can then be expressed as

$$\mathbf{A} = c\begin{pmatrix}\cos B\\ \sin B\end{pmatrix}, \mathbf{B} = \mathbf{0}, \mathbf{C} = \begin{pmatrix}a\\ 0\end{pmatrix}. \tag{3.1.1.4}$$

3.1.2. Construct a $\triangle ABC$ given $\angle B, \angle C$ and $K = a + b + c$.
Solution:

$$a + b + c = K \tag{3.1.2.1}$$

$$b\cos C + c\cos B - a = 0 \tag{3.1.2.2}$$

$$b\sin C - c\sin B = 0 \tag{3.1.2.3}$$

resulting in the matrix equation

$$\begin{pmatrix}1 & 1 & 1\\ -1 & \cos C & \cos B\\ 0 & \sin C & -\sin B\end{pmatrix}\begin{pmatrix}a\\ b\\ c\end{pmatrix} = K\begin{pmatrix}1\\ 0\\ 0\end{pmatrix} \tag{3.1.2.4}$$

which can be solved to obtain all the sides. $\triangle ABC$ can then be plotted using

$$\mathbf{A} = \begin{pmatrix}a\\ b\end{pmatrix}, \mathbf{B} = \mathbf{0}, \mathbf{C} = \begin{pmatrix}a\\ 0\end{pmatrix} \tag{3.1.2.5}$$

3.2 Triangle

3.2.1 Construct a triangle ABC in which $BC = 7cm$, $\angle B = 75°$ and $AB + AC = 13cm$.
Solution: From (3.1.1.3) and (3.1.1.4), we obtain Fig. 3.2.1.1.

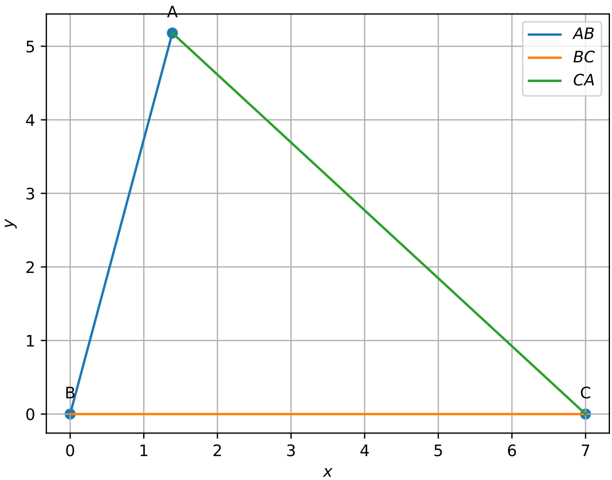


Fig. 3.2.1.1

3.2.2 Construct a triangle ABC in which $BC = 8cm$, $\angle B = 45°$ and $AB - AC = 3.5cm$.
Solution: See Fig. 3.2.2.1.

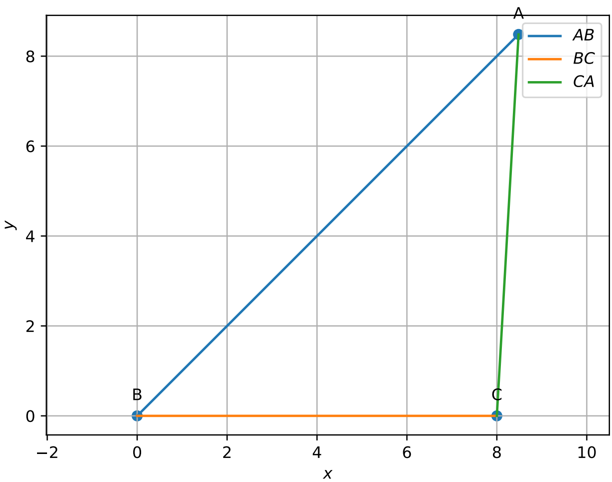


Fig. 3.2.2.1

3.2.3 Construct a triangle ABC in which $BC = 6cm$, $\angle B = 60°$ and $AC - AB = 2cm$.
Solution: See Fig. 3.2.3.1 obtained by substituting $K = -2$.

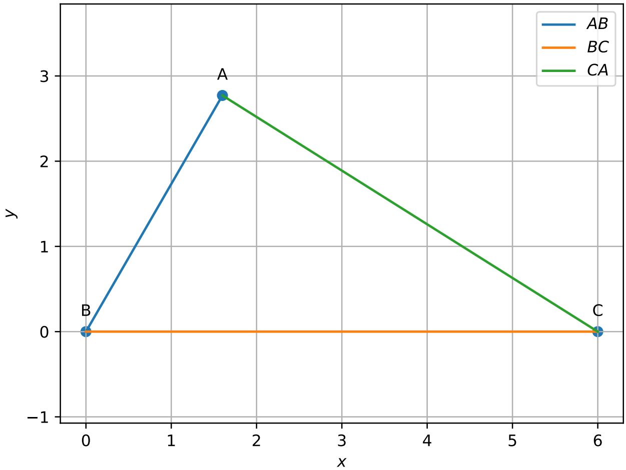


Fig. 3.2.3.1

3.2.4 Construct a right triangle whose base is $12cm$ and sum of its hypotenuse and other side is $18cm$.

Solution: For $a = 12, \angle B = 90°, b + c = 18$, we obtain Fig. 3.2.4.1.

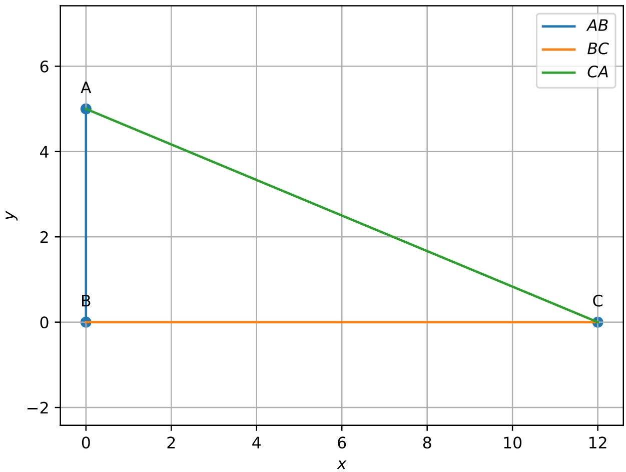


Fig. 3.2.4.1

3.2.5 Construct a triangle ABC in which $\angle B = 30°, \angle C = 90°$ and $AB + BC + CA = 11cm$.

Solution: From (3.1.2.4) and (3.1.2.5), Fig. 3.2.5.1 is generated.

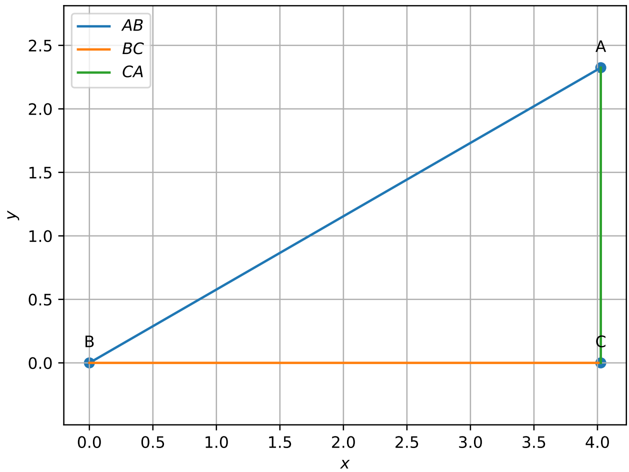


Fig. 3.2.5.1

3.2.6 Draw a right triangle ABC in which $BC = 12cm$, $AB = 5cm$ and $\angle B = 90°$.

3.2.7 Draw an isosceles triangle ABC in which $AB = AC = 6cm$ and $BC = 6cm$.

3.2.8 Draw a triangle ABC in which $AB = 5cm, BC = 6cm$ and $\angle ABC = 60°$.

3.2.9 Draw a triangle ABC in which $AB = 4cm, BC = 6cm$ and $AC = 9cm$.

3.2.10 Draw a triangle ABC in which $BC = 6cm, CA = 5cm$ and $AB = 4cm$.

3.2.11 Is it possible to construct a triangle with lengths of its sides as $4cm$, $3cm$ and $7cm$? Give reason for your answer.

3.2.12 Is it possible to construct a triangle with lengths of its sides as $9cm$, $7cm$ and $17cm$? Give reason for your answer.

3.2.13 Is it possible to construct a triangle with lengths of its sides as $8cm$, $7cm$ and $4cm$? Give reason for your answer.

3.2.14 Two sides of a triangle are of lengths $5cm$ and $1.5cm$. The length of the third side of the triangle cannot be

a) $3.6cm$
b) $4.1cm$
c) $3.8cm$
d) $3.4cm$

3.2.15 The construction of a triangle ABC, given that $BC = 6cm, \angle B = 45°$ is not possible when difference of AB and AC is equal to

a) $6.9cm$
b) $5.2cm$
c) $5.0cm$

d) $4.0cm$

3.2.16 The construction of a triangle ABC, given that $BC = 6cm$, $\angle C = 60^\circ$ is possible when difference of AB and AC is equal to

a) $3.2cm$
b) $3.1cm$
c) $3cm$
d) $2.8cm$

3.2.17 Construct a triangle whose sides are $3.6cm$, $3.0cm$ and $4.8cm$. Bisect the smallest angle and measure each part.

3.2.18 Construct a triangle ABC in which $BC = 5cm$, $\angle B = 60^\circ$ and $AC + AB = 7.5cm$.

Construct each of the following and give justification :

3.2.19 A triangle if its perimeter is $10.4cm$ and two angles are $45^\circ$and 120°.

3.2.20 A triangle PQR given that $QR = 3cm$, $\angle PQR = 45^\circ$ and $QP - PR = 2cm$.

3.2.21 A right triangle when one side is $3.5cm$ and sum of other sides and the hypotenuse is $5.5cm$.

3.2.22 An equilateral triangle if its altitude is $3.2cm$.

Write true or false in each of the following. Give reasons for your answer:

3.2.23 A triangle ABC can be constructed in which $AB = 5cm$, $\angle A = 45^\circ$ and $BC+AC = 5cm$.

3.2.24 A triangle ABC can be constructed in which $BC = 6cm$, $\angle B = 30^\circ$ and $AC-AB = 4cm$.

3.2.25 A triangle ABC can be constructed in which $\angle B = 105^\circ$, $\angle C = 90^\circ$ and $AB+BC+AC = 10cm$.

3.2.26 A triangle ABC can be constructed in which $\angle B = 60^\circ$, $\angle C = 45^\circ$ and $AB+BC+AC = 12cm$.

3.2.27 Draw a right triangle ABC in which $BC = 12$ cm, $AB = 5$ cm and $\angle B = 90^\circ$.

3.2.28 Draw a triangle ABC in which AB=4 cm, $BC = 6cm$ and $AC = 9$.

3.2.29 Draw a triangle ABC in which AB=5 cm. $BC = 6cm$ and $\angle ABC = 60^\circ$.

3.2.30 Draw a parallelogram $ABCD$ in which $BC = 5$ cm, $AB = 3$ cm and $\angle ABC = 60^\circ$, divide it into triangles ACB and ABD by the diagonal BD. Construct the triangle $BD'C'$ similar to $\triangle BDC$ with scale factor $\frac{4}{3}$. Draw the line segment $D'A'$ parallel to DA where **A**' lies on extended side BA. Is $A'BC'D'$ a parallelogram?

3.2.31 Draw a triangle ABC in which $BC = 6$ cm, $CA = 5$ cm and $AB = 4$ cm.

3.3 Quadrilateral

3.3.1 Draw a quadrilateral in the Cartesian plane, whose vertices are (-4, 5), (0, 7), (5, -5) and (-4, -2).

3.3.2 Draw a parallelogram $ABCD$ in which $BC = 5cm, AB = 3cm$ and $\angle ABC = 60^\circ$, divide it into triangles ACB and ABD by the diagonal BD.

3.3.3 Construct a square of side $3cm$.

3.3.4 Construct a rectangle whose adjacent sides are of lengths $5cm$ and $3.5cm$.

3.3.5 Construct a rhombus whose side is of length $3.4cm$ and one of its angles is 45°.

3.3.6 Construct a rhombus whose diagonals are 4 cm and 6 cm in lengths.

4.1 Formulae

4.1.1. The equation of a line is given by

$$y = mx + c \tag{4.1.1.1}$$

$$\implies \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ mx + c \end{pmatrix} = \begin{pmatrix} 0 \\ c \end{pmatrix} + x \begin{pmatrix} 1 \\ m \end{pmatrix} \tag{4.1.1.2}$$

yielding

$$\mathbf{x} = \mathbf{h} + \kappa \mathbf{m}. \tag{4.1.1.3}$$

where $\mathbf{h}$ is any point on the line and

$$\mathbf{m} = \begin{pmatrix} 1 \\ m \end{pmatrix} \tag{4.1.1.4}$$

is the direction vector.

4.1.2. For

$$\mathbf{m}^\top \mathbf{n} = 0, \tag{4.1.2.1}$$

(4.1.1.3) can be expressed as

$$\mathbf{n}^\top \mathbf{x} = \mathbf{n}^\top \mathbf{h} + \kappa \mathbf{n}^\top \mathbf{m} \tag{4.1.2.2}$$

$$\begin{aligned} \implies \mathbf{n}^\top (\mathbf{x} - \mathbf{h}) &= 0 \\ \text{or, } \mathbf{n}^\top \mathbf{x} &= c \end{aligned} \tag{4.1.2.3}$$

for

$$c = \mathbf{n}^\top \mathbf{h}. \tag{4.1.2.4}$$

where

$$\mathbf{n} = \begin{pmatrix} -m \\ 1 \end{pmatrix} \tag{4.1.2.5}$$

is defined to be the *normal vector* of the line. In 3D, (4.1.2.3) represents a plane.

4.1.3. If $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are collinear, from (4.1.2.3),

$$\mathbf{n}^\top \mathbf{A} = c \tag{4.1.3.1}$$

$$\mathbf{n}^\top \mathbf{B} = c \tag{4.1.3.2}$$

$$\mathbf{n}^\top \mathbf{C} = c \tag{4.1.3.3}$$

which can be expressed as

$$\begin{pmatrix}\mathbf{A} & \mathbf{B} & \mathbf{C}\end{pmatrix}^\top \mathbf{n} = c\begin{pmatrix}1\\1\\1\end{pmatrix} \tag{4.1.3.4}$$

$$\equiv \begin{pmatrix}\mathbf{A} & \mathbf{B} & \mathbf{C}\end{pmatrix}^\top \mathbf{n} = \begin{pmatrix}1\\1\\1\end{pmatrix}, \tag{4.1.3.5}$$

$$\implies \begin{pmatrix}1 & 1 & 1\\ \mathbf{A} & \mathbf{B} & \mathbf{C}\end{pmatrix}^\top \begin{pmatrix}\mathbf{n}\\-1\end{pmatrix} = \mathbf{0} \tag{4.1.3.6}$$

4.1.4. The equation of a line that does not pass through the origin can be expressed as

$$\mathbf{n}^\top \mathbf{x} = 1 \tag{4.1.4.1}$$

4.1.5. Let the perpendicular distance from the origin to a line be p and the angle made by the perpendicular with the positive x-axis be θ. Then

$$p\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} \tag{4.1.5.1}$$

is a point on the line as well as the normal vector. Hence, the equation of the line is

$$p\begin{pmatrix}\cos\theta & \sin\theta\end{pmatrix}\left\{\mathbf{x} - p\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}\right\} = 0 \tag{4.1.5.2}$$

$$\implies \begin{pmatrix}\cos\theta & \sin\theta\end{pmatrix}\mathbf{x} = p \tag{4.1.5.3}$$

4.1.6. Let $\mathbf{Q}$ be the foot of the perpendicular from $\mathbf{P}$ to the line

$$\mathbf{n}^\top \mathbf{x} = c \tag{4.1.6.1}$$

From (4.1.1.3)

$$\mathbf{Q} = \mathbf{P} + k\mathbf{n} \tag{4.1.6.2}$$

$$\implies PQ = \|\mathbf{Q} - \mathbf{P}\| = |k|\,\|\mathbf{n}\| \tag{4.1.6.3}$$

is the distance from $\mathbf{Q}$ to the line in (4.1.6.1). From (4.1.6.2),

$$\mathbf{n}^\top \mathbf{Q} = \mathbf{n}^\top \mathbf{P} + k\,\|\mathbf{n}\|^2 \tag{4.1.6.4}$$

$$\implies |k| = \frac{\left|\mathbf{n}^\top (\mathbf{Q} - \mathbf{P})\right|}{\|\mathbf{n}\|^2} \tag{4.1.6.5}$$

$$\implies PQ = |k|\,\|\mathbf{n}\| = \frac{\left|\mathbf{n}^\top \mathbf{P} - c\right|}{\|\mathbf{n}\|} \tag{4.1.6.6}$$

upon substituting from (4.1.6.3).

4.1.7. The foot of the perpendicular is given by

$$\begin{pmatrix}\mathbf{m} & \mathbf{n}\end{pmatrix}^\top \mathbf{Q} = \begin{pmatrix}\mathbf{m}^\top \mathbf{P}\\ c\end{pmatrix} \tag{4.1.7.1}$$

4.1.8. The distance between the parallel lines

$$\mathbf{n}^\top \mathbf{x} = c_1$$
$$\mathbf{n}^\top \mathbf{x} = c_2 \tag{4.1.8.1}$$

is given by

$$d = \frac{|c_1 - c_2|}{\|\mathbf{n}\|} \tag{4.1.8.2}$$

4.1.9. The reflection of point $\mathbf{Q}$ w.r.t a line is given by

$$\mathbf{R} = \mathbf{Q} - \frac{2\left(\mathbf{n}^\top \mathbf{Q} - c\right)}{\|\mathbf{n}\|}\mathbf{n} \tag{4.1.9.1}$$

4.2 Equation

Find the equation of line

4.2.1 passing through the point $\mathbf{P} = (-4, 3)$ with slope $\frac{1}{2}$.
Solution: From (4.1.2.5),

$$\mathbf{n} \equiv \begin{pmatrix} \frac{1}{2} \\ -1 \end{pmatrix} \implies \begin{pmatrix} \frac{1}{2} & -1 \end{pmatrix} \mathbf{x} = -5 \tag{4.2.1.1}$$

using (4.1.2.3). See Fig. 4.2.1.1.

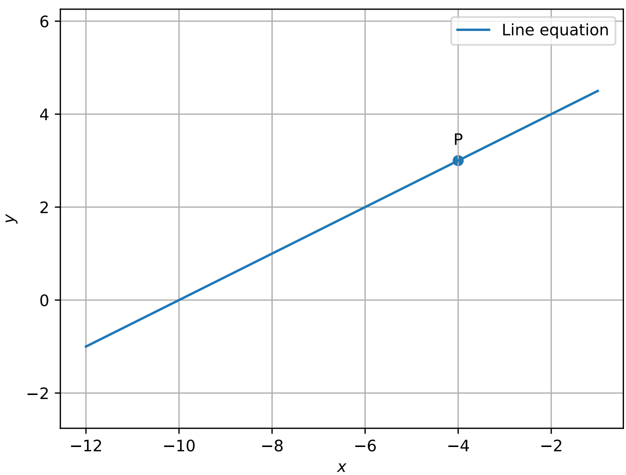


Fig. 4.2.1.1

4.2.2 passing through $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ with slope m.

Solution:

$$\because \mathbf{n} = \begin{pmatrix} m \\ -1 \end{pmatrix}, \tag{4.2.2.1}$$

the desired equation is

$$\begin{pmatrix} m & -1 \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) = 0 \tag{4.2.2.2}$$

$$\implies \begin{pmatrix} m & -1 \end{pmatrix} \mathbf{x} = 0 \tag{4.2.2.3}$$

4.2.3 passing through $\mathbf{A} = \begin{pmatrix} 2 \\ 2\sqrt{3} \end{pmatrix}$ and inclined with the x-axis at an angle of 75°.

Solution:

$$\mathbf{n} = \begin{pmatrix} -1 \\ 2+\sqrt{3} \end{pmatrix} \tag{4.2.3.1}$$

$$\implies \begin{pmatrix} -1 & 2+\sqrt{3} \end{pmatrix} \mathbf{x} = \begin{pmatrix} -1 & 2+\sqrt{3} \end{pmatrix} \begin{pmatrix} 2 \\ 2\sqrt{3} \end{pmatrix} \tag{4.2.3.2}$$

$$= 4\left(\sqrt{3}+1\right) \tag{4.2.3.3}$$

is the desired equation. See Fig. 4.2.3.1.

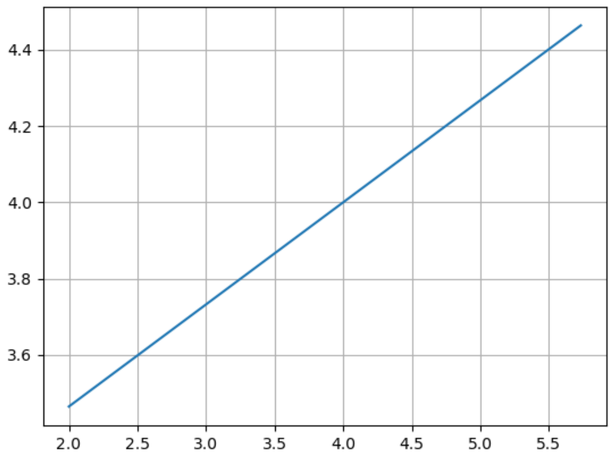


Fig. 4.2.3.1

4.2.4 intersecting the x-axis at a distance of 3 units to the left of origin with slope of -2.

Solution: From the given information,

$$\mathbf{A} = \begin{pmatrix} -3 \\ 0 \end{pmatrix}, \mathbf{n} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}. \tag{4.2.4.1}$$

The desired equation of the line is

$$\implies \begin{pmatrix} 2 & 1 \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} -3 \\ 0 \end{pmatrix} \right) = 0 \tag{4.2.4.2}$$

$$\text{or, } \begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = -6 \tag{4.2.4.3}$$

See Fig. 4.2.4.1.

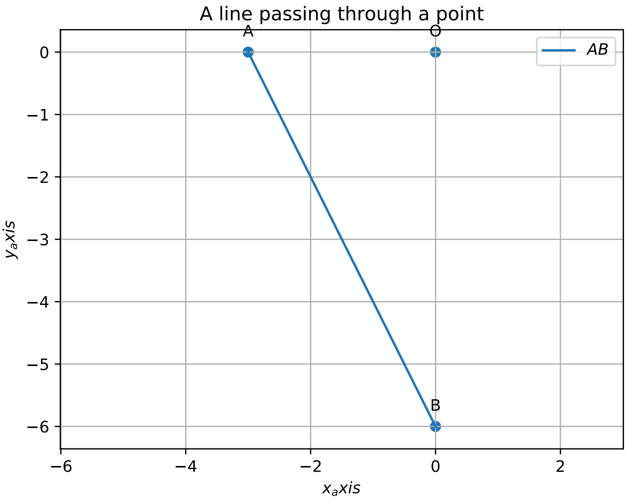


Fig. 4.2.4.1

4.2.5 intersecting the y-axis at a distance of 2 units above the origin and making an angle of 30° with positive direction of the x-axis.
Solution:

$$\mathbf{n} = \begin{pmatrix} -\frac{1}{\sqrt{3}} \\ 1 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}. \tag{4.2.5.1}$$

Hence, the equation of the line is given by

$$\begin{pmatrix} -\frac{1}{\sqrt{3}} & 1 \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right) = 0 \tag{4.2.5.2}$$

$$\text{or, } \begin{pmatrix} -\frac{1}{\sqrt{3}} & 1 \end{pmatrix} \mathbf{x} = 2 \tag{4.2.5.3}$$

See Fig. 4.2.5.1.

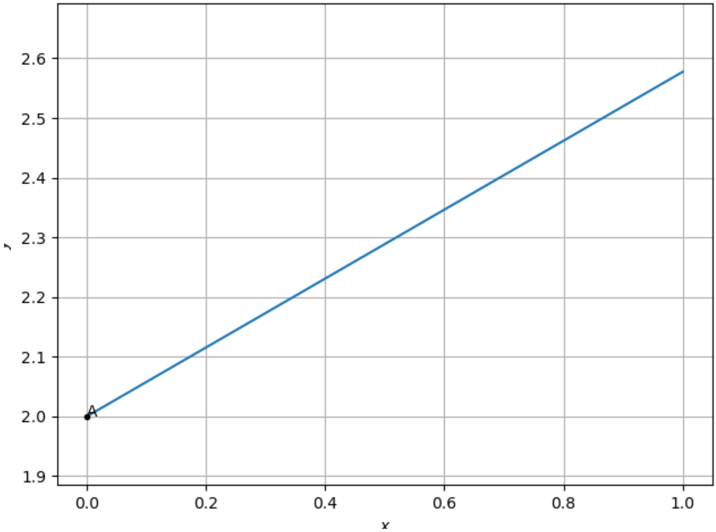


Fig. 4.2.5.1

4.2.6 passing through (1,2) and making angle 30° with y-axis.

4.2.7 passing through the points $\mathbf{A}\begin{pmatrix}-1\\1\end{pmatrix}$ and $\mathbf{B}\begin{pmatrix}2\\-4\end{pmatrix}$.

Solution: From (4.1.3.5),

$$\begin{pmatrix}-1 & 1\\2 & -4\end{pmatrix}\mathbf{n} = \begin{pmatrix}1\\1\end{pmatrix} \tag{4.2.7.1}$$

$$\implies \left(\begin{array}{cc|c}-1 & 1 & 1\\2 & -4 & 1\end{array}\right) \xleftrightarrow{R_2\leftarrow R_2+2R_1} \left(\begin{array}{cc|c}-1 & 1 & 1\\0 & -2 & 3\end{array}\right) \tag{4.2.7.2}$$

$$\xleftrightarrow{R_1\leftarrow 2R_1+R_2} \left(\begin{array}{cc|c}-2 & 0 & 5\\0 & -2 & 3\end{array}\right) \implies \mathbf{n} = -\frac{1}{2}\begin{pmatrix}5\\3\end{pmatrix} \tag{4.2.7.3}$$

Thus, from (4.1.4.1), the equation of the line is

$$\begin{pmatrix}5 & 3\end{pmatrix}\mathbf{x} = -2 \tag{4.2.7.4}$$

See Fig. 4.2.7.1.

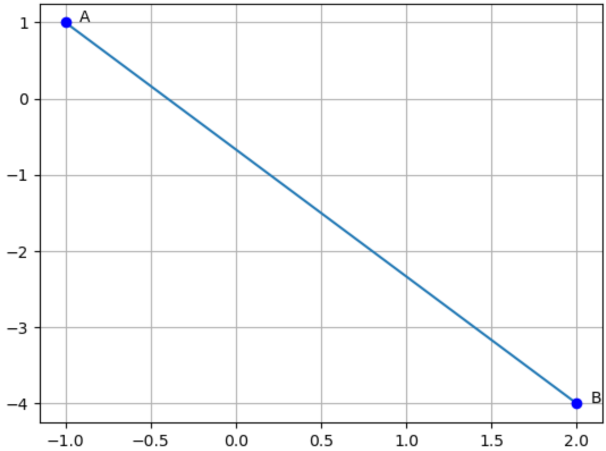


Fig. 4.2.7.1

4.2.8 passing through the points $(3, 4, -7)$ and $(1, -1, 6)$.

4.2.9 The vector equation of the line

$$\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2}$$

is ____________.

4.2.10 The vector equation of the line

$$\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2}$$

is ____________.

4.2.11 The vertices of triangle PQR are $\mathbf{P}(2, 1), \mathbf{Q}(-2, 3), \mathbf{R}(4, 5)$. Find the equation of the median through $\mathbf{R}$.

Solution:

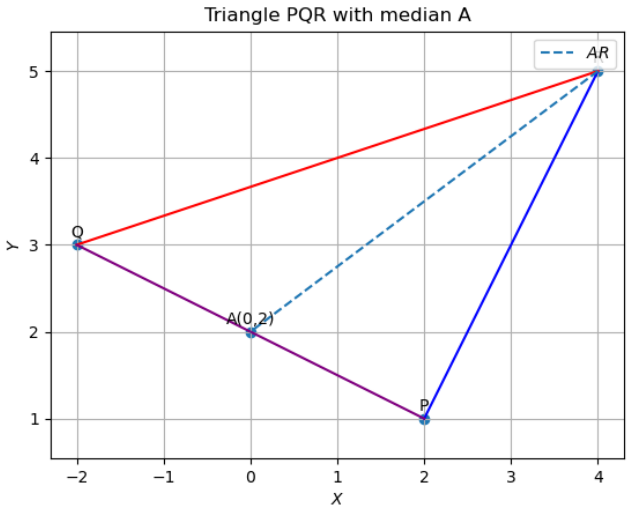


Fig. 4.2.11.1

See Fig. 4.2.11.1. Using section formula, the mid point of PQ is

$$\mathbf{A} = \frac{\mathbf{P}+\mathbf{Q}}{2} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \tag{4.2.11.1}$$

Following the approach in Problem 4.2.7,

$$\left(\begin{array}{cc|c} 4 & 5 & 1 \\ 0 & 2 & 1 \end{array}\right) \xleftrightarrow[R_2 \leftarrow 4R_2]{R_1 \leftarrow 2R_1 - 5R_2} \left(\begin{array}{cc|c} 8 & 0 & -3 \\ 0 & 8 & 4 \end{array}\right) \implies \mathbf{n} = \frac{1}{8}\begin{pmatrix} -3 \\ 4 \end{pmatrix}$$

Thus, the equation of the line is

$$\begin{pmatrix} -3 & 4 \end{pmatrix}\mathbf{x} = 8 \tag{4.2.11.2}$$

4.2.12 Find the equations of the planes that pass through the points

a) $\mathbf{A} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 6 \\ 4 \\ -5 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -4 \\ -2 \\ 3 \end{pmatrix}$

b) $\mathbf{A} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix}$

Solution:

a) From (4.1.3.5),

$$\begin{pmatrix} 1 & 1 & -1 \\ 6 & 4 & -5 \\ -4 & -2 & 3 \end{pmatrix}\mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{4.2.12.1}$$

$$\implies \left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 6 & 4 & -5 & 1 \\ -4 & -2 & 3 & 1 \end{array}\right)$$

$$\xleftrightarrow[R_3 \leftarrow R_3 + 4R_1]{R_2 \leftarrow R_2 - 6R_1} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & -2 & 1 & -5 \\ 0 & 2 & -1 & 5 \end{array}\right)$$

$$\xleftrightarrow[R_1 \leftarrow 2R_1 + R_2]{R_3 \leftarrow R_3 + R_2} \left(\begin{array}{ccc|c} 2 & 0 & -1 & -3 \\ 0 & 2 & -1 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Since we obtain a 0 row, the given points are collinear. The direction vector of the line is

$$\mathbf{m} = \mathbf{B} - \mathbf{C} \equiv \begin{pmatrix} 5 \\ 3 \\ -4 \end{pmatrix} \tag{4.2.12.2}$$

and the equation of a line is given by,

$$\mathbf{x} = \mathbf{A} + \kappa\mathbf{m} \tag{4.2.12.3}$$

$$= \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + \kappa \begin{pmatrix} 5 \\ 3 \\ -4 \end{pmatrix} \tag{4.2.12.4}$$

See Fig. 4.2.12.1.

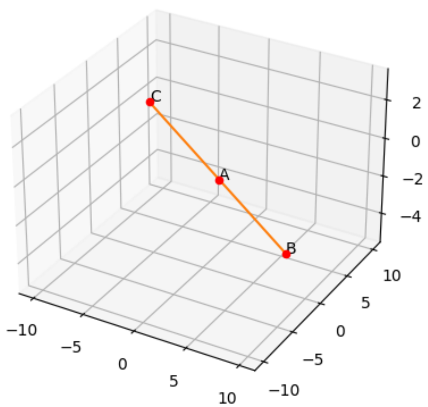


Fig. 4.2.12.1

b) In this case,

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ -2 & 2 & -1 \end{pmatrix} \mathbf{n} = \mathbf{1} \tag{4.2.12.5}$$

$$\implies \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 \\ -2 & 2 & -1 & 1 \end{array}\right)$$

$$\xleftrightarrow[R_3 \leftarrow R_3 + 2R_1]{R_2 \leftarrow R_2 - R_1} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 4 & -1 & 3 \end{array}\right)$$

$$\xleftrightarrow[R_3 \leftarrow R_3 - 4R_2]{R_1 \leftarrow R_1 - R_2} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -5 & 3 \end{array}\right)$$

$$\xleftrightarrow[R_2 \leftarrow 5R_2 + R_3]{R_1 \leftarrow 5R_1 - R_3} \left(\begin{array}{ccc|c} 5 & 0 & 0 & 2 \\ 0 & 5 & 0 & 3 \\ 0 & 0 & 5 & -3 \end{array}\right)$$

Hence, the equation of the plane is

$$\begin{pmatrix} 2 & 3 & -3 \end{pmatrix} \mathbf{x} = 5 \tag{4.2.12.6}$$

4.2.13 Find the equation of the plane through the points $(2, 1, 0)$, $(3, -2, -2)$ and $(3, 1, 7)$.

4.2.14 A plane passes through the points $(2, 0, 0)(0, 3, 0)$ and $(0, 0, 4)$. The equation of the plane is ____________.

4.2.15 If the intercept of a line between the coordinate axes is divided by the point (-5,4) in the ratio 1:2 then find the equation of the line.

4.2.16 Find the equation of a line that cuts off equal intercepts on the coordinate axes and passes through the point $(2, 3)$.

Solution: Let $(a, 0)$ and $(0, a)$ be the intercept points.

$$\mathbf{m} = \begin{pmatrix} a \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ a \end{pmatrix} \equiv \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{4.2.16.1}$$

$$\implies \mathbf{n} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{4.2.16.2}$$

and the equation of the line is

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right) = 0 \tag{4.2.16.3}$$

$$\implies \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 5 \tag{4.2.16.4}$$

See Fig. 4.2.16.1.

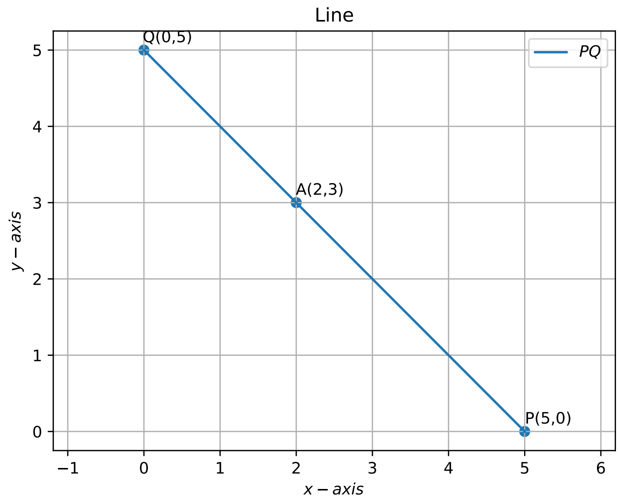


Fig. 4.2.16.1

4.2.17 Find the equation of a line passing through a point (2,2) and cutting off intercepts on the axes whose sum is 9.

Solution: Let the intercept points be

$$\mathbf{P} = \begin{pmatrix} a \\ 0 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} 0 \\ b \end{pmatrix} \text{ and } \mathbf{R} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \tag{4.2.17.1}$$

be the given point. Forming the collinearity matrix from (4.1.3.6),

$$\begin{pmatrix} \mathbf{P} - \mathbf{Q} & \mathbf{P} - \mathbf{R} \end{pmatrix} = \begin{pmatrix} a & a-2 \\ -b & -2 \end{pmatrix} \tag{4.2.17.2}$$

which is singular if

$$ab - 2(a+b) = 0 \implies ab = 18 \tag{4.2.17.3}$$

$$\because a + b = 9. \tag{4.2.17.4}$$

$\therefore a, b$ are the roots of

$$x^2 - 9x + 18 = 0. \tag{4.2.17.5}$$

yielding

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \end{pmatrix} \tag{4.2.17.6}$$

Since

$$\mathbf{m} = \begin{pmatrix} a \\ -b \end{pmatrix}, \mathbf{n} = \begin{pmatrix} b \\ a \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{4.2.17.7}$$

Thus, the possible equations of the line are

$$\begin{pmatrix}1 & 2\end{pmatrix}\mathbf{x} = 6 \tag{4.2.17.8}$$

$$\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} = 6 \tag{4.2.17.9}$$

See Fig. 4.2.17.1.

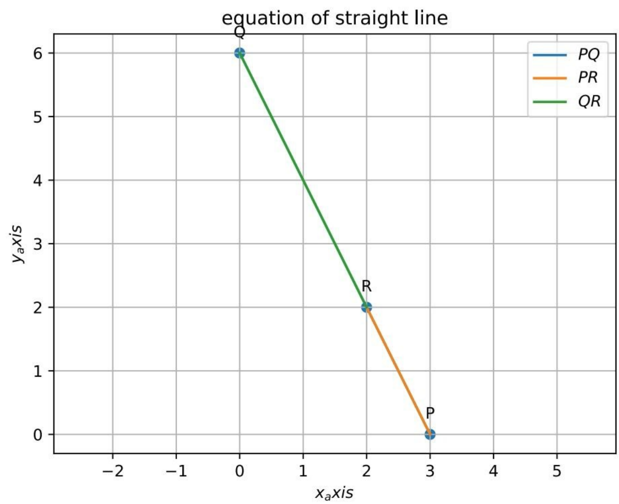


Fig. 4.2.17.1

4.2.18 Find the equation of the lines which passes the point (3,4) and cuts off intercepts from the coordinate axes such that their sum is 14.

4.2.19 Find the equation of the straight line which passes through the point (1, -2) and cuts off equal intercepts from axes.

4.2.20 Find the equation of the line which passes through the point (-4,3) and the portion of the line intercepted between the axes is divided internally in ratio 5:3 by this point.

4.2.21 Consider the following population and year graph. Find the slope of the line AB and using it, find what will be the population in the year 2010.

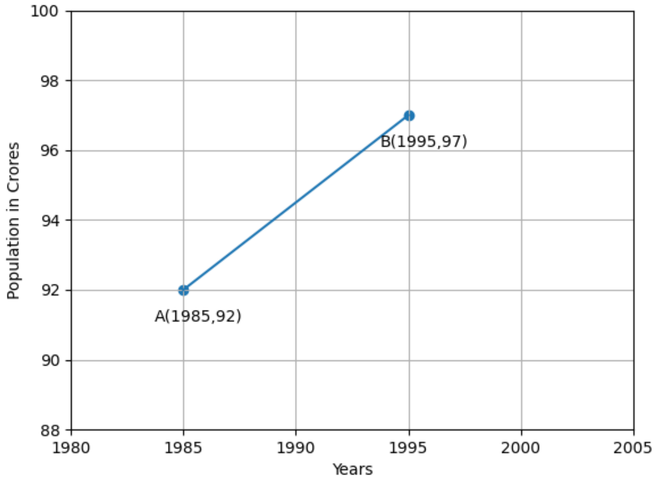


Fig. 4.2.21.1

Solution: The direction vector of the line in Fig. 4.2.21.1 is

$$\mathbf{m} = \mathbf{B} - \mathbf{A} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{4.2.21.1}$$

$$\implies \mathbf{n} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \tag{4.2.21.2}$$

The equation of the line is then given by

$$\mathbf{n}^\top(\mathbf{x} - \mathbf{A}) = 0 \tag{4.2.21.3}$$

$$\implies \begin{pmatrix} 1 & -2 \end{pmatrix} \mathbf{x} = 1801 \tag{4.2.21.4}$$

$$\implies \begin{pmatrix} 1 & -2 \end{pmatrix} \begin{pmatrix} 2010 \\ y \end{pmatrix} = 1801 \tag{4.2.21.5}$$

$$\implies y = \frac{209}{2} \tag{4.2.21.6}$$

4.2.22 Slope of a line which cuts off intercepts of equal length on the axes is

a) -1
b) -0
c) 2
d) $\sqrt{3}$

4.2.23 If the coordinates of middle point of the portion of a line intercepted between the coordinate axes is (3,2),then the equation of the line will be

a) $2x + 3y = 12$
b) $3x + 2y = 12$
c) $4x - 3y = 6$

d) $5x - 2y = 10$

4.2.24 If the line $\frac{x}{a} + \frac{y}{b} = 1$ passes the points (2,-3) and (4,-5), then (a, b) is

a) (1,1)
b) (-1,1)
c) (1,-1)
d) (-1,-1)

4.2.25 The intercepts made by the plane $2x - 3y + 5z + 4 = 0$ on the co-ordinate axis are $\left(-2, \frac{4}{3}, -\frac{4}{5}\right)$.

4.2.26 The line $\overrightarrow{r} = 2\hat{i} - 3\hat{j} - \hat{k} + \lambda(\hat{i} - \hat{j} + 2\hat{k})$ lies in the plane $\overrightarrow{r} \cdot (3\hat{i} + \hat{j} - \hat{k}) + 2 = 0$.

4.3 Parallel

4.3.1 Find the vector equation of the line passing through $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}^\top$ and parallel to the planes $\begin{pmatrix} 1 & -1 & 2 \end{pmatrix}\mathbf{x} = 5$ and $\begin{pmatrix} 3 & 1 & 1 \end{pmatrix}\mathbf{x} = 6$.

Solution: The direction vector of the line is given by

$$\begin{pmatrix} 1 & -1 & 2 \\ 3 & 1 & 1 \end{pmatrix}\mathbf{m} = 0 \xleftrightarrow{R_2 \to -\frac{3}{4}R_1 + \frac{1}{4}R_2} \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -\frac{5}{4} \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -\frac{5}{4} \end{pmatrix} \xleftrightarrow{R_1 \to R_1 + R_2} \begin{pmatrix} 1 & 0 & \frac{3}{4} \\ 0 & 1 & -\frac{5}{4} \end{pmatrix}$$

$$\implies \mathbf{m} = \begin{pmatrix} -3 \\ 5 \\ 4 \end{pmatrix}$$

$\therefore$ the equation of the line is

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 5 \\ 4 \end{pmatrix} \tag{4.3.1.1}$$

4.3.2 Find the equation of the plane with an intercept 3 on the Y-axis and parallel to ZOX-Plane.

Solution: The normal vector to the ZOX plane is

$$\mathbf{n} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \tag{4.3.2.1}$$

Since, Y-axis has the intercept 3, the desired plane passes through the point

$$\mathbf{P} = \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}. \tag{4.3.2.2}$$

Thus, the equation of the plane is given by,

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{P}) = 0 \tag{4.3.2.3}$$

$$\implies \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}\mathbf{x} = 3 \tag{4.3.2.4}$$

See Fig. 4.3.2.1.

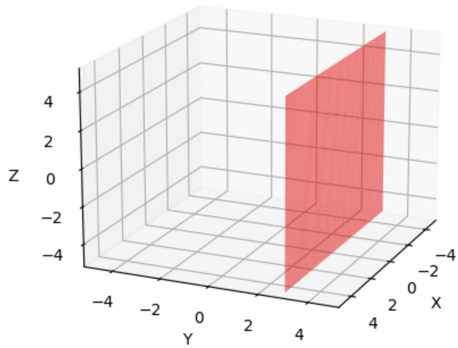


Fig. 4.3.2.1

4.3.3 Find the equation of the line parallel to the line $3x - 4y + 2 = 0$ and passing through the point (-2,3).

Solution:

$$\begin{pmatrix} 3 & -4 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 3 & -4 \end{pmatrix} \begin{pmatrix} -2 \\ 3 \end{pmatrix} = -18 \tag{4.3.3.1}$$

is the required equation of the line.

4.3.4 Find the equation of the line through the point (0,2) making an angle $\frac{2\pi}{3}$ with the positive X-axis. Also find the equation of the line parallel to it and crossing the Y-axis at a distance of 2 units below the origin.

Solution: The equation of the first line is

$$\begin{pmatrix} \sqrt{3} & 1 \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right) = 0 \tag{4.3.4.1}$$

$$\implies \begin{pmatrix} \sqrt{3} & 1 \end{pmatrix} \mathbf{x} = 2 \tag{4.3.4.2}$$

The equation of the second line is

$$\begin{pmatrix} \sqrt{3} & 1 \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} 0 \\ -2 \end{pmatrix} \right) = 0 \tag{4.3.4.3}$$

$$\implies \begin{pmatrix} \sqrt{3} & 1 \end{pmatrix} \mathbf{x} = -2 \tag{4.3.4.4}$$

See Fig. 4.3.4.1.

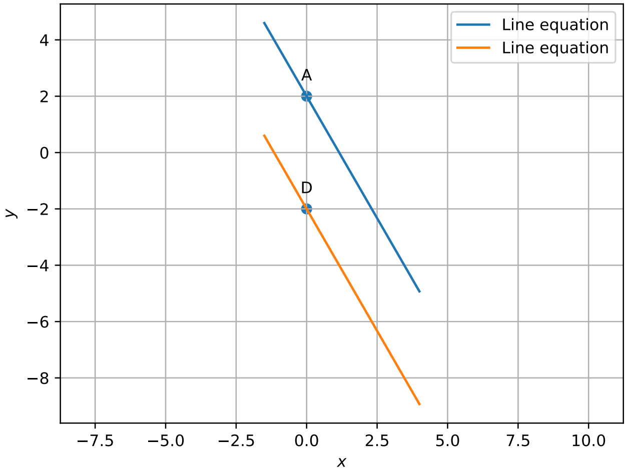


Fig. 4.3.4.1

4.3.5 Find the vector equation of the line which is parallel to the vector $3\hat{i} - 2\hat{j} + 6\hat{k}$ and passes through the point $(1, -2, 3)$.

4.3.6 Find the equations of the line passing through the point $(3, 0, 1)$ and parallel to the planes $x + 2y = 0$ and $3y - z = 0$.

4.3.7 The equation of a line, which is parallel to $2\hat{i} + \hat{j} + 3\hat{k}$ and passes through the point $(5, -2, 4)$ is $\frac{x-5}{2} = \frac{y+2}{-1} = \frac{z-4}{3}$.

4.3.8 The value of λ for which the vectors $3\hat{i} - 6\hat{j} + \hat{k}$ and, $2\hat{i} - 4\hat{j} + \lambda\hat{k}$ are parallel is

a) $\frac{2}{3}$
b) $\frac{3}{2}$
c) $\frac{5}{2}$
d) $\frac{2}{5}$

4.3.9 Equation of the line passing through (1,2) and parallel to the line $y = 3x - 1$ is

a) $y + 2 = x + 1$
b) $y + 2 = 3(x + 1)$
c) $y - 2 = 3(x - 1)$
d) $y - 2 = x - 1$

4.4 Perpendicular

4.4.1 Find the values of θ and p, if the equation $x\cos\theta + y\sin\theta = p$ is the normal form of the line $\sqrt{3}x + y + 2 = 0$.

Solution:

$$\mathbf{n} = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}, c = -2 \tag{4.4.1.1}$$

$$\implies \theta = \tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3}, p = \frac{|c|}{\|\mathbf{n}\|} = 1 \tag{4.4.1.2}$$

See Fig. 4.4.1.1.

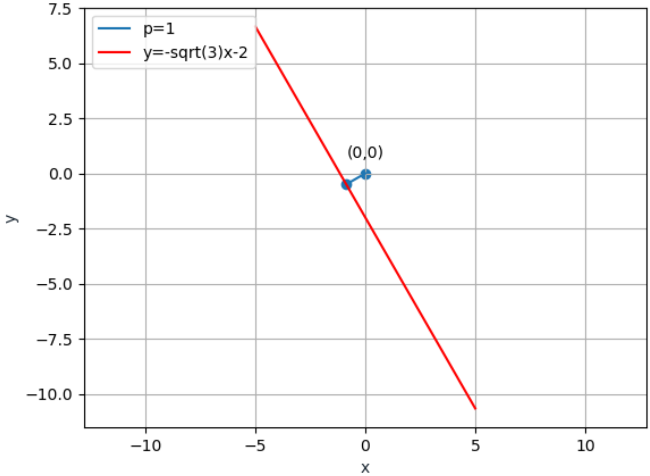


Fig. 4.4.1.1

4.4.2 Reduce the following equations into normal form. Find their perpendicular distances from the origin and angle between perpendicular and the positive x-axis.

a) $x - \sqrt{3}y + 8 = 0$

b) $y - 2 = 0$

c) $x - y = 4$

Solution: See Table 4.4.2. (4.1.6.6) was used for computing the distance from the origin.

	$\mathbf{n}$	Angle	c	Distance
a)	$\begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}$	$\tan^{-1}(-\sqrt{3}) = \frac{2\pi}{3}$	-8	4
b)	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$\tan^{-1}\infty = \frac{\pi}{2}$	2	2
c)	$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$	$\tan^{-1}(-1) = \frac{3\pi}{4}$	4	$2\sqrt{2}$

TABLE 4.4.2

4.4.3 In each of the following cases, determine the direction cosines of the normal to the plane and the distance from the origin.

a) $z = 2$
b) $x + y + z = 1$
c) $2x + 3y – z = 5$
d) $5y + 8 = 0$

Solution: See Table 4.4.3. (4.1.6.6) was used for computing the distance from the origin.

	$\mathbf{n}$	c	Distance
a)	$\begin{pmatrix}0\\0\\1\end{pmatrix}$	2	2
b)	$\begin{pmatrix}1\\1\\1\end{pmatrix}$	1	$\frac{1}{\sqrt{3}}$
c)	$\begin{pmatrix}2\\3\\-1\end{pmatrix}$	5	$\frac{5}{\sqrt{14}}$
d)	$\begin{pmatrix}0\\-5\\0\end{pmatrix}$	8	$\frac{8}{5}$

TABLE 4.4.3

4.4.4 Find the distance of the point $(-1, 1)$ from the line $12\,(x + 6) = 5\,(y - 2)$.
Solution:

$$\mathbf{n} = \begin{pmatrix}12\\-5\end{pmatrix}, c = -82 \tag{4.4.4.1}$$

$$\implies d = \frac{\left|\begin{pmatrix}12 & -5\end{pmatrix}\begin{pmatrix}-1\\1\end{pmatrix} - (-82)\right|}{\sqrt{12^2 + (-5)^2}} = 5 \tag{4.4.4.2}$$

See Fig. 4.4.4.1.

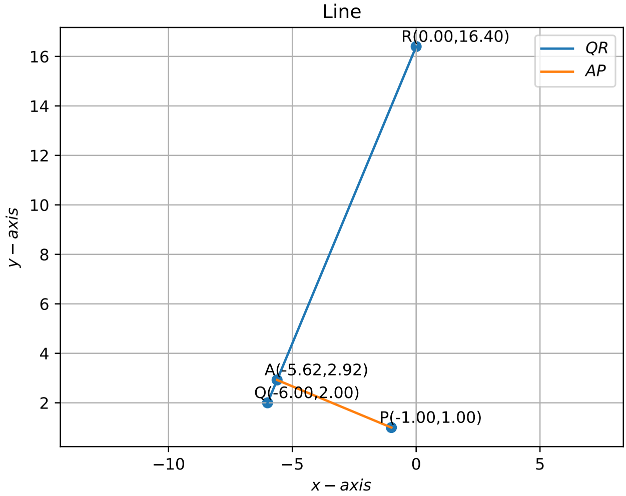


Fig. 4.4.4.1

4.4.5 Find the coordinates of the foot of the perpendicular from $(-1, 3)$ to the line $3x - 4y - 16 = 0$.

Solution: Substituting

$$\mathbf{P} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}, \mathbf{n} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}, c = 16 \tag{4.4.5.1}$$

in (4.1.7.1), the desired foot of the perpendicular is then given by

$$\begin{pmatrix} 4 & 3 \\ 3 & -4 \end{pmatrix} \mathbf{Q} = \begin{pmatrix} \begin{pmatrix} 4 & 3 \end{pmatrix} \begin{pmatrix} -1 \\ 3 \end{pmatrix} \\ 16 \end{pmatrix} = \begin{pmatrix} 5 \\ 16 \end{pmatrix} \tag{4.4.5.2}$$

$$\implies \begin{pmatrix} 4 & 3 & 5 \\ 3 & -4 & 16 \end{pmatrix} \xleftrightarrow{R_2 = R_2 - \frac{3}{4} R_1} \begin{pmatrix} 4 & 3 & 5 \\ 0 & \frac{-25}{4} & \frac{49}{4} \end{pmatrix} \tag{4.4.5.3}$$

$$\xleftrightarrow{R_2 = \frac{-4}{25}} \begin{pmatrix} 4 & 3 & 5 \\ 0 & 1 & \frac{-49}{25} \end{pmatrix} \xleftrightarrow{R_1 = \frac{1}{4} R_1} \begin{pmatrix} 1 & \frac{3}{4} & \frac{5}{4} \\ 0 & 1 & \frac{-49}{25} \end{pmatrix} \tag{4.4.5.4}$$

$$\xleftrightarrow{R_1 = R_1 - \frac{3}{4} R_2} \begin{pmatrix} 1 & 0 & \frac{68}{25} \\ 0 & 1 & \frac{-49}{25} \end{pmatrix} \implies \mathbf{Q} = \begin{pmatrix} \frac{68}{25} \\ \frac{-49}{25} \end{pmatrix} \tag{4.4.5.5}$$

See Fig. 4.4.5.1.

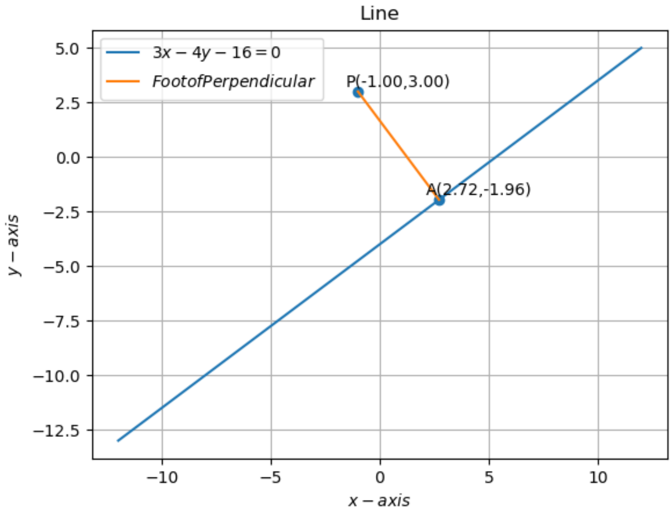


Fig. 4.4.5.1

4.4.6 In the triangle ABC with vertices $\mathbf{A}(2,3)$, $\mathbf{B}(4,-1)$ and $\mathbf{C}(1,2)$, find the equation and length of altitude from the vertex $\mathbf{A}$.

Solution:

a) The normal vector of the altitude from $\mathbf{A}$ is,

$$\mathbf{m}_{BC} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \because \mathbf{n}_{BC} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{4.4.6.1}$$

The equation of the desired altitude is given by

$$\mathbf{m}_{BC}^{\top}\mathbf{x} = \mathbf{m}_{BC}^{\top}\mathbf{A} \tag{4.4.6.2}$$

$$\implies \begin{pmatrix} 1 & -1 \end{pmatrix}\mathbf{x} = -1 \tag{4.4.6.3}$$

b) The equation of line BC is given by,

$$\mathbf{n}_{BC}^{\top}\mathbf{x} = \mathbf{n}_{BC}^{\top}\mathbf{B} \tag{4.4.6.4}$$

$$\implies \begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} = 3 \tag{4.4.6.5}$$

From (4.1.6.6), the length of the desired altitude is

$$d = \sqrt{2} \tag{4.4.6.6}$$

See Fig. 4.4.6.1.

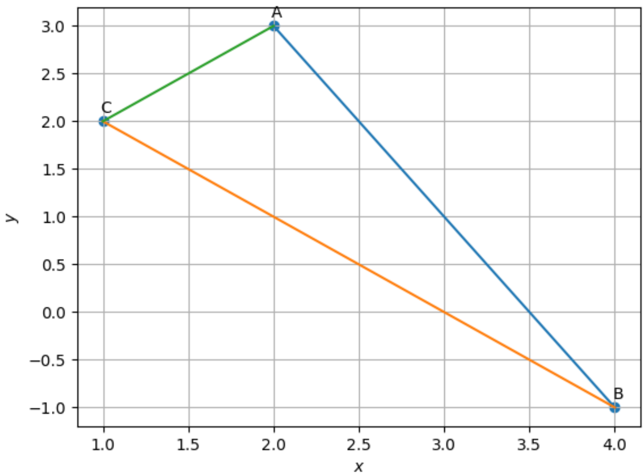


Fig. 4.4.6.1

4.4.7 Find the points on the x-axis, whose distances from the line $\frac{x}{3} + \frac{y}{4} = 1$ are 4 units.
Solution: Let the desired point be

$$\mathbf{P} = x\mathbf{e}_1 = \begin{pmatrix} x \\ 0 \end{pmatrix} \tag{4.4.7.1}$$

From the distance formula,

$$d = \frac{\left|\mathbf{n}^\top \mathbf{P} - c\right|}{\|\mathbf{n}\|} = \frac{\left|x\mathbf{n}^\top \mathbf{e}_1 - c\right|}{\|\mathbf{n}\|} \tag{4.4.7.2}$$

$$\implies x = \frac{\pm d \|\mathbf{n}\| + c}{\mathbf{n}^\top \mathbf{e}_1} \tag{4.4.7.3}$$

Substituting

$$\mathbf{n} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}, c = 12, d = 4, \tag{4.4.7.4}$$

$$x = 8, -2 \tag{4.4.7.5}$$

See Fig. 4.4.7.1.

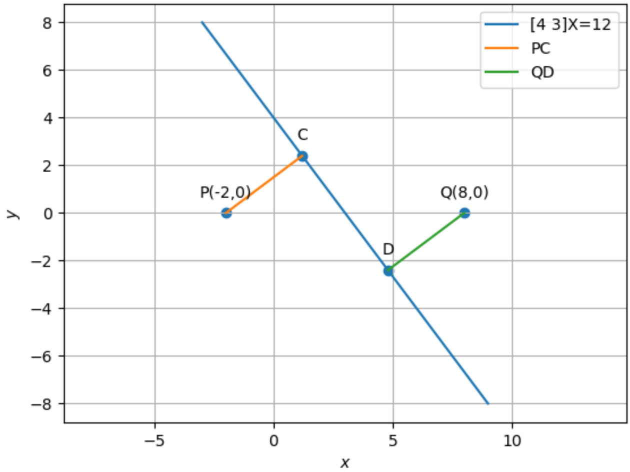


Fig. 4.4.7.1

4.4.8 What are the points on the y-axis whose distance from the line $\frac{x}{3} + \frac{y}{4} = 1$ is 4 units.
Solution: Following the approach in Problem 4.4.7,

$$y = \frac{\pm d \left\|\mathbf{n}\right\| + c}{\mathbf{n}^\top \mathbf{e}_2} = \frac{32}{3}, \frac{-8}{3}. \tag{4.4.8.1}$$

See Fig. 4.4.8.1.

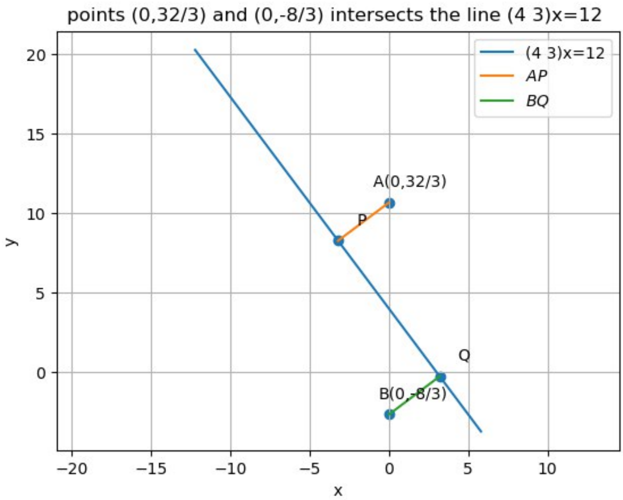


Fig. 4.4.8.1

4.4.9 Find the distance between parallel lines

a) $15x + 8y - 34 = 0$ and $15x + 8y + 31 = 0$

b) $l(x + y) + p = 0$ and $l(x + y) - r = 0$

Solution: From (4.1.8.1), the desired values are available in Table 4.4.9.

	$\mathbf{n}$	c_1	c_2	d
a)	$\begin{pmatrix} 15 \\ 8 \end{pmatrix}$	34	-31	$\frac{65}{17}$
b)	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$	$\frac{-p}{l}$	$\frac{r}{l}$	$\frac{\lvert p-r\rvert}{l\sqrt{2}}$

TABLE 4.4.9

4.4.10 Find the equation of line which is equidistant from parallel lines $9x+6y-7 = 0$ and $3x + 2y + 6 = 0$.

Solution: Given

$$c_1 = \frac{7}{3}, \; c_2 = -6. \tag{4.4.10.1}$$

From (4.1.8.1), we need to find c such that,

$$|c - c_1| = |c - c_2| \implies c = \frac{c_1 + c_2}{2} = -\frac{11}{6}. \tag{4.4.10.2}$$

Hence, the desired equation is

$$\begin{pmatrix} 3 & 2 \end{pmatrix} \mathbf{x} = -\frac{11}{6} \tag{4.4.10.3}$$

See Fig. 4.4.10.1.

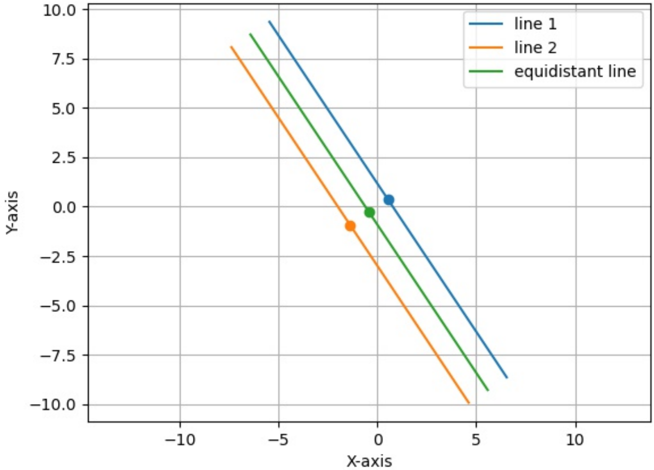


Fig. 4.4.10.1

4.4.11 Find the equation of line drawn perpendicular to the line $\frac{x}{4} + \frac{y}{6} = 1$ through the point where it meets the y-axis
Solution: The given line parameters are

$$\mathbf{n} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, c = 12, \mathbf{m} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}. \tag{4.4.11.1}$$

and the point on the y-axis is

$$\mathbf{A} = \begin{pmatrix} 0 \\ 6 \end{pmatrix}. \tag{4.4.11.2}$$

Thus, the equation of the desired line is

$$\mathbf{m}^\top (\mathbf{x} - \mathbf{A}) = 0 \tag{4.4.11.3}$$

$$\implies \begin{pmatrix} -2 & 3 \end{pmatrix} \mathbf{x} = -18 \tag{4.4.11.4}$$

See Fig. 4.4.11.1.

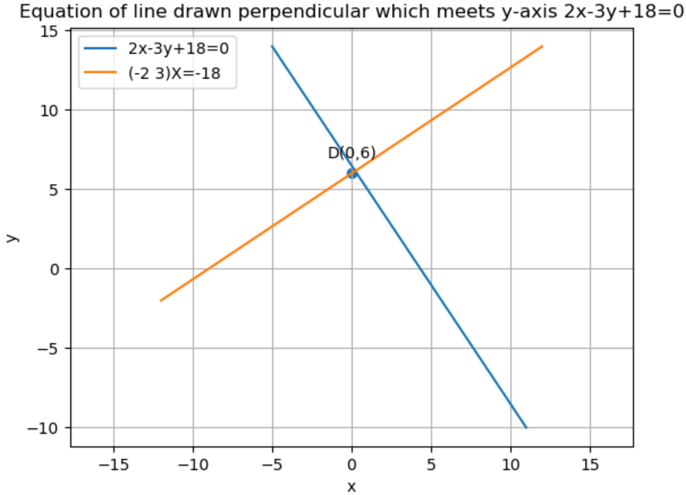


Fig. 4.4.11.1

4.4.12 Find the equation of line whose perpendicular distance from the origin is 5 units and the angle made by the perpendicular with the positive x-axis is 30°.
Solution: From (4.1.5.3), Thus, the equation of lines are

$$\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \mathbf{x} = \pm 5 \tag{4.4.12.1}$$

See Fig. 4.4.12.1.

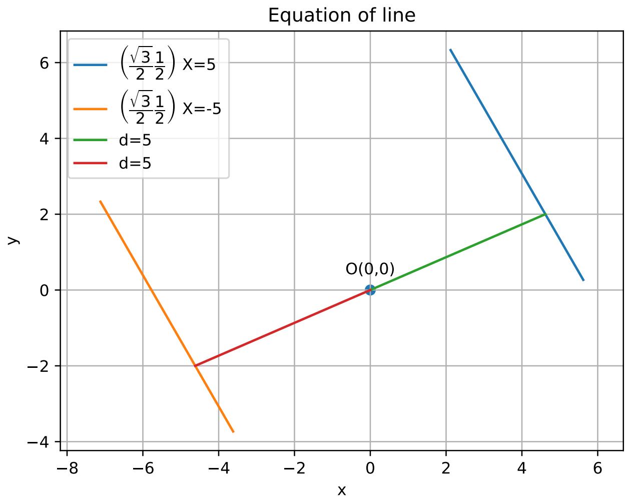


Fig. 4.4.12.1

4.4.13 Find the equation of the line passing through (-3,5) and perpendicular to the line through the points (2,5) and (-3,6).

Solution:

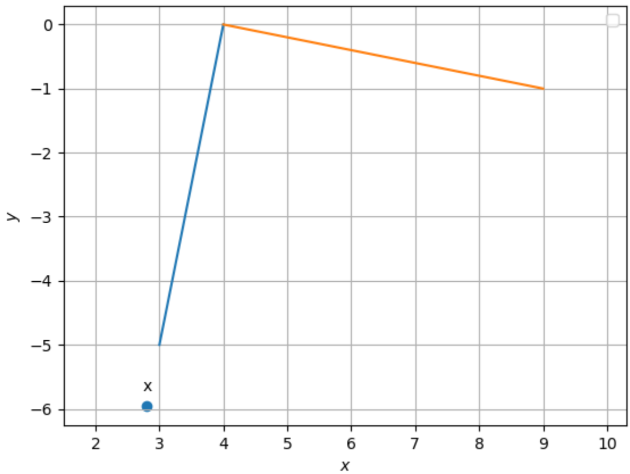


Fig. 4.4.13.1

The normal vector is

$$\mathbf{n} = \begin{pmatrix} 2 \\ 5 \end{pmatrix} - \begin{pmatrix} -3 \\ 6 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \end{pmatrix} \tag{4.4.13.1}$$

Thus, the equation of the line is

$$\begin{pmatrix} 5 & -1 \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} -3 \\ 5 \end{pmatrix} \right) = 0 \tag{4.4.13.2}$$

$$\implies \begin{pmatrix} 5 & -1 \end{pmatrix} \mathbf{x} = -20 \tag{4.4.13.3}$$

See Fig. 4.4.13.1.

4.4.14 The perpendicular from the origin to a line meets it at the point $(-2, 9)$. Find the equation of the line.

Solution: It is obvious that the normal vector to the line is

$$\mathbf{n} = \begin{pmatrix} 2 \\ -9 \end{pmatrix} - \mathbf{0} = \begin{pmatrix} 2 \\ -9 \end{pmatrix} \tag{4.4.14.1}$$

Hence, the equation of the line is

$$\begin{pmatrix} 2 & -9 \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} 2 \\ -9 \end{pmatrix} \right) = 0 \tag{4.4.14.2}$$

$$\implies \begin{pmatrix} 2 & -9 \end{pmatrix} \mathbf{x} = 85 \tag{4.4.14.3}$$

See Fig. 4.4.14.1.

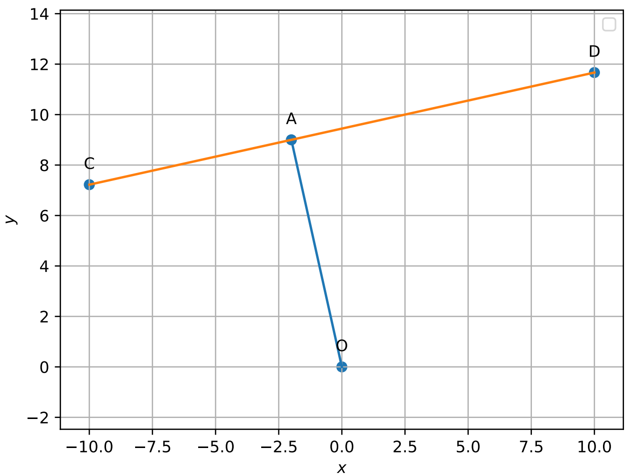


Fig. 4.4.14.1

4.4.15 Find the equation of line perpendicular to the line $x - 7y + 5 = 0$ and having x intercept 3

Solution: The desired equation is

$$\begin{pmatrix} 7 & 1 \end{pmatrix}\left(\mathbf{x} - \begin{pmatrix} 3 \\ 0 \end{pmatrix}\right) = 0 \tag{4.4.15.1}$$

$$\implies \begin{pmatrix} 7 & 1 \end{pmatrix}\mathbf{x} = 21 \tag{4.4.15.2}$$

See Fig. 4.4.15.1.

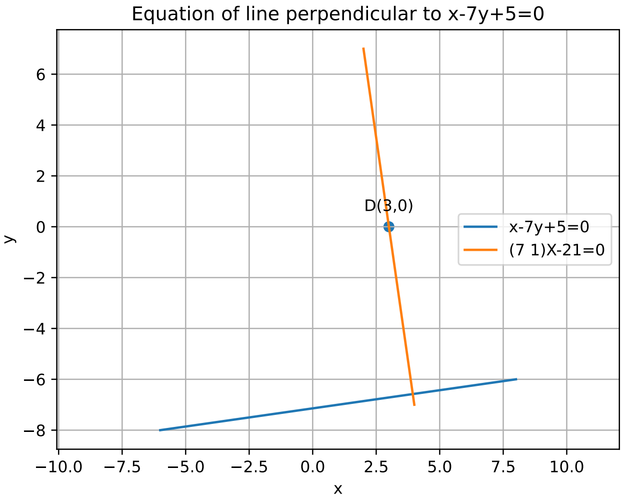


Fig. 4.4.15.1

4.4.16 Find the equation of the line passing through the point $(1, 2, -4)$ and perpendicular to the two lines

$$\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7} \text{ and} \tag{4.4.16.1}$$

$$\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5} \tag{4.4.16.2}$$

Solution: The direction vector of the desired line is given by

$$\begin{pmatrix} 3 & -16 & 7 \\ 3 & 8 & -5 \end{pmatrix}\mathbf{m} = 0 \xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 3 & -16 & 7 \\ 0 & 24 & -12 \end{pmatrix}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + \frac{2}{3}R_2} \begin{pmatrix} 3 & 0 & -1 \\ 0 & 24 & -12 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2/12} \begin{pmatrix} 3 & 0 & -1 \\ 0 & 2 & -1 \end{pmatrix}$$

yielding

$$\mathbf{m} = \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} \tag{4.4.16.3}$$

Hence the vector equation of the line passing through $(1, 2, -4)$ is,

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} + \kappa \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} \tag{4.4.16.4}$$

4.4.17 The perpendicular from the origin to the line $y = mx + c$ meets it at the point $(-1, 2)$. Find the values of m and c.

Solution: From Problem 4.4.14,

$$\mathbf{n} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \implies m = \frac{1}{2} \tag{4.4.17.1}$$

Also, from the given equation of the line and the given point,

$$c = \begin{pmatrix} -m & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \frac{5}{2} \tag{4.4.17.2}$$

See Fig. 4.4.17.1.

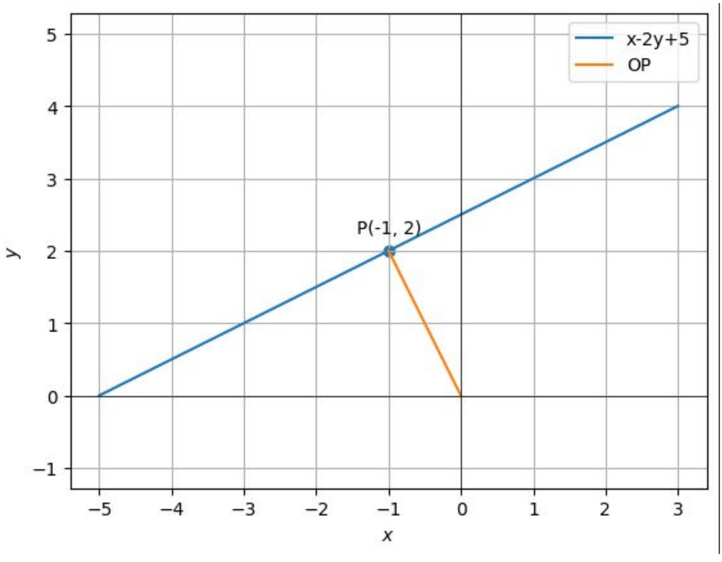


Fig. 4.4.17.1: Graph

4.4.18 A line perpendicular to the line segment joining the points $\mathbf{P}(1, 0)$ and $\mathbf{Q}(2, 3)$ divides it in the ratio $1 : n$. Find the equation of the line.

Solution: The direction vector of PQ is

$$\mathbf{Q} - \mathbf{P} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \tag{4.4.18.1}$$

Using section formula,

$$\mathbf{R} = \frac{\mathbf{Q} + n\mathbf{P}}{1+n} \tag{4.4.18.2}$$

is the point of intersection. The equation of the desired line is

$$\mathbf{m}^\top (\mathbf{x} - \mathbf{R}) = 0 \tag{4.4.18.3}$$

$$\implies \begin{pmatrix} 1 & 3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & 3 \end{pmatrix} \begin{pmatrix} \frac{2+n}{1+n} \\ \frac{3}{1+n} \end{pmatrix} \tag{4.4.18.4}$$

$$= \frac{11+n}{1+n} \tag{4.4.18.5}$$

See Fig. 4.4.18.1.

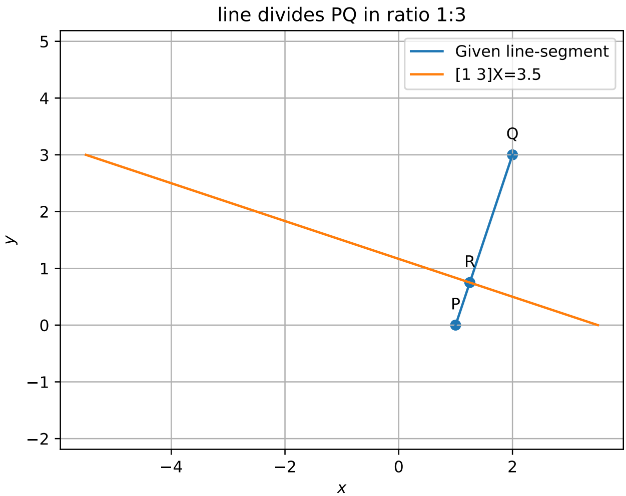


Fig. 4.4.18.1

4.4.19 Find the vector equation of a plane which is at a distance of 7 units from the origin and normal to the vector $3\hat{i} + 5\hat{j} - 6\hat{k}$.
Solution: From the given information,

$$\mathbf{n} = \begin{pmatrix} 3 \\ 5 \\ -6 \end{pmatrix}, d = \frac{|c|}{\|\mathbf{n}\|} = 7 \tag{4.4.19.1}$$

$$\implies c = \pm 7\sqrt{70} \tag{4.4.19.2}$$

4.4.20 Find the equation of a plane which is at a distance $3\sqrt{3}$ units from origin and the normal to which is equally inclined to the coordinate axis.

4.4.21 If the line drawn from the point $(-2, -1, -3)$ meets a plane at right angle at the point $(1, -3, 3)$, find the equation of the plane.

4.4.22 Find the equation of the plane through the points $(2, 1, -1)$ and $(-1, 3, 4)$, and perpendicular to the plane $x - 2y + 4z = 10$.

4.4.23 If the foot of perpendicular drawn from the origin to a plane is $(5, -3, -2)$, then the equation of the plane is $\vec{r} \cdot (5\hat{i} - 3\hat{j} - 2\hat{k}) = 38$.

4.4.24 $\mathbf{P}(0, 2)$ is the point of intersection of y-axis and perpendicular bisector of line segment joining the points $\mathbf{A}(-1, 1)$ and $\mathbf{B}(3, 3)$.

4.4.25 The distance of the point $\mathbf{P}(2, 3)$ from the x-axis is

a) 2
b) 3
c) 1
d) 5

4.4.26 Find the foot of perpendicular from the point $(2, 3, -8)$ to the line

$$\frac{4-x}{2} = \frac{y}{6} = \frac{1-z}{3}.$$

Also, find the perpendicular distance from the given point to the line.

4.4.27 Find the distance of a point $(2, 4, -1)$ from the line

$$\frac{x+5}{1} = \frac{y+3}{4} = \frac{z-6}{-9}$$

.

4.4.28 Find the length and the foot of perpendicular from the point $\left(1, \frac{3}{2}, 2\right)$ to the plane $2x - 2y + 4z + 5 = 0$.

4.4.29 Show that the points $(\hat{i} - \hat{j} + 3\hat{k})$ and $3(\hat{i} + \hat{j} + \hat{k})$ are equidistant from the plane $\vec{r} \cdot (5\hat{i} + 2\hat{j} - 7\hat{k}) + 9 = 0$ and lie on opposite side of it.

4.4.30 The distance of the plane $\vec{r} \cdot \left(\frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} - \frac{6}{7}\hat{k}\right) = 1$ from the origin is

a) 1
b) 7
c) $\frac{1}{7}$
d) None of these

4.4.31 Find the equation of the line passing through the point (5,2) and perpendicular to the line joining the points (2,3) and (3, -1).

4.4.32 Find the points on the line $x + y = 4$ which lie at a unit distance from the line $4x + 3y = 10$.

4.4.33 Find the equation of a straight line on which length of perpendicular from the origin is four units and the line makes on angle of $120°$ with the positive direction of x-axis.

4.4.34 Find the equation of one of the sides of an isosceles right angled triangle whose hypotenuse is given by $3x + 4y = 4$ and the opposite vertex of the hypotenuse is (2,2).

4.4.35 In what direction should a line be drawn through the point (1,2) so that its point of intersection with line $x + y = 4$ is at a distance $\sqrt{6}3$.

4.4.36 The equation of the straight line passing through the point (3,2) and perpendicular to the line $y = x$ is

a) $x - y = 5$
b) $x + y = 5$
c) $x + y = 1$
d) $x - y = 1$

4.4.37 The equation of the line passing through the point (1,2) and perpendicular to the line $x + y + 1 = 0$ is

a) $y - x + 1 = 0$
b) $y - x - 1 = 0$
c) $y - x + 2 = 0$
d) $y - x - 1 = 0$

4.4.38 The distance of the point of intersection of the lines $2x - 3y + 5 = 0$ and $3x + 4y = 0$ from the line $5x - 2y = 0$ is

a) $\frac{130}{17\sqrt{29}}$
b) $\frac{13}{7\sqrt{29}}$
c) $\frac{130}{7}$
d) none of these

4.4.39 The equations of the lines passing through the point (1,0) and at a distance $\frac{\sqrt{3}}{2}$ from the origin, are

a) $\sqrt{3}x + y - \sqrt{3} = 0,\ \sqrt{3}x - y - \sqrt{3} = 0$
b) $\sqrt{3}x + y + \sqrt{3} = 0,\ \sqrt{3}x - y + \sqrt{3} = 0$
c) $x + \sqrt{3}y - \sqrt{3} = 0,\ \sqrt{3}y - \sqrt{3} = 0$
d) None of these.

4.4.40 The coordinates of the foot of perpendiculars from the point (2,3) on the line $y = 3x+4$ is given by

a) $\frac{37}{10}, \frac{-1}{10}$
b) $\frac{-1}{10}, \frac{37}{10}$
c) $\frac{10}{37}$, -10
d) $\frac{2}{3}, \frac{-1}{3}$

4.4.41 A point equidistant from the lines $4x+3y+10 = 0$, $5x-12y+26 = 0$ and $7x+24y-50 = 0$ is

a) (1,-1)
b) (1,1)
c) (0,0)
d) (0,1)

4.4.42 A line passes through (2,2) and is perpendicular to the line $3x + y = 3$. Its y-intercept is

a) $\frac{1}{3}$
b) $\frac{2}{3}$
c) 1
d) $\frac{4}{3}$

4.4.43 The ratio in which the line $3x + 4y + 2 = 0$ divides the distance between the lines $3x + 4y + 5 = 0$ and $3x + 4y - 5 = 0$ is

a) 1:2
b) 3:7
c) 2:3
d) 2:5

4.5 Angle

4.5.1 Two lines passing through the point (2, 3) intersect each other at an angle of 60°. If slope of one line is 2, find the equation of the other line.
Solution: Using the scalar product

$$\cos 60^\circ = \frac{1}{2} = \frac{\begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ m \end{pmatrix}}{\sqrt{5}\sqrt{m^2+1}} \tag{4.5.1.1}$$

$$\implies 11m^2 + 16m - 1 = 0 \tag{4.5.1.2}$$

$$or, m = \frac{-8 \pm 5\sqrt{3}}{11} \tag{4.5.1.3}$$

So, the desired equation of the line is

$$\begin{pmatrix} \frac{-8\pm5\sqrt{3}}{11} & -1 \end{pmatrix}\mathbf{x} = \begin{pmatrix} \frac{-8\pm5\sqrt{3}}{11} & -1 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} \tag{4.5.1.4}$$

$$= \frac{-49 \pm 16\sqrt{3}}{11} \tag{4.5.1.5}$$

See Fig. 4.5.1.1.

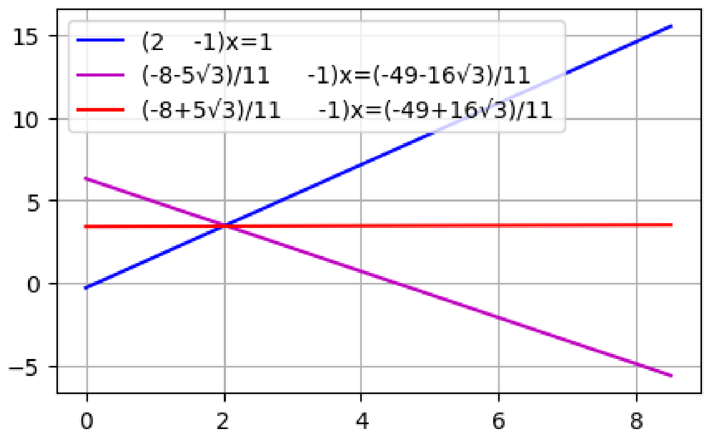


Fig. 4.5.1.1

4.5.2 Find the equation of the lines through the point (3, 2) which make an angle of 45° with the line x–$2y$ = 3.

Solution: Following the approach in Problem 4.5.1,

$$\cos 45^\circ \frac{1}{\sqrt{2}} = \frac{\begin{pmatrix} 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ m \end{pmatrix}}{\left\|\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right\| \left\|\begin{pmatrix} 1 \\ m \end{pmatrix}\right\|} \tag{4.5.2.1}$$

$$\implies 3m^2 - 8m - 3 = 0 \tag{4.5.2.2}$$

$$\text{or, } m = -\frac{1}{3}, 3 \tag{4.5.2.3}$$

Thus, the desired equations are

$$\begin{pmatrix} 3 & -1 \end{pmatrix}\left\{\mathbf{x} - \begin{pmatrix} 3 \\ 2 \end{pmatrix}\right\} = 0 \tag{4.5.2.4}$$

$$\implies \begin{pmatrix} 3 & -1 \end{pmatrix}\mathbf{x} = 7 \tag{4.5.2.5}$$

and

$$\begin{pmatrix} 1 & 3 \end{pmatrix}\left\{\mathbf{x} - \begin{pmatrix} 3 \\ 2 \end{pmatrix}\right\} = 0 \tag{4.5.2.6}$$

$$\implies \begin{pmatrix} 1 & 3 \end{pmatrix}\mathbf{x} = 9 \tag{4.5.2.7}$$

See Fig. 4.5.2.1.

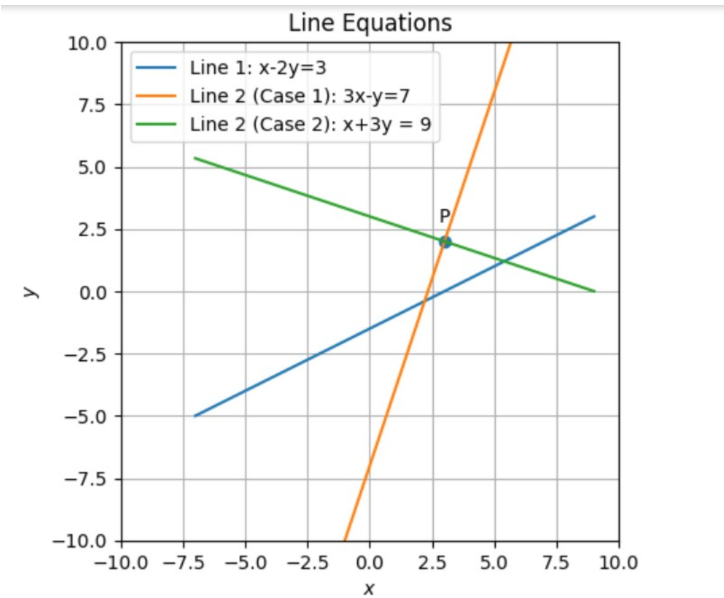


Fig. 4.5.2.1

4.5.3 Find the equations of the two lines through the origin which intersect the line $\frac{x-3}{2} =$

$\frac{y-3}{1} = \frac{z}{1}$ at angles of $\frac{\pi}{3}$ each.

4.5.4 The equations of the lines which pass through the point (3, -2) and are inclined at 60° to the line $\sqrt{3}x + y = 1$ is

a) $y + 2 = 0,\ \sqrt{3}x - y - 2 - 3\sqrt{3} = 0$
b) $x - 2 = 0,\ \sqrt{3}x - y + 2 + 3\sqrt{3} = 0$
c) $\sqrt{3}x - y - 2 - 3\sqrt{3} = 0$
d) None of these

4.5.5 Equations of the lines through the point (3,2) and making an angle of 40° with the line $x - 2y = 3$ are ______.

4.6 Intersection

4.6.1 Find the equation of the plane through the intersection of the planes $3x$–$y + 2z$–$4 = 0$ and $x + y + z$–$2 = 0$ and the point $\begin{pmatrix}2\\2\\1\end{pmatrix}$.

Solution: The parameters of the given planes are

$$\mathbf{n}_1 = \begin{pmatrix}3\\-1\\2\end{pmatrix}, \mathbf{n}_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}, c_1 = 4, c_2 = 2. \tag{4.6.1.1}$$

The intersection of the planes is given as

$$\mathbf{n}_1^\top \mathbf{x} - c_1 + \lambda\left(\mathbf{n}_2^\top \mathbf{x} - c_2\right) = 0 \tag{4.6.1.2}$$

where

$$\lambda = \frac{c_1 - \mathbf{n}_1^\top \mathbf{P}}{\mathbf{n}_2^\top \mathbf{P} - c_2} = -\frac{2}{3} \tag{4.6.1.3}$$

upon substituting

$$\mathbf{P} = \begin{pmatrix}2\\2\\1\end{pmatrix}. \tag{4.6.1.4}$$

in (4.6.1.3) along with the numerical values in (4.6.1.1). Now, substituting (4.6.1.3) in (4.6.1.2), the equation of plane is

$$\begin{pmatrix}7 & -5 & 4\end{pmatrix}\mathbf{x} = 8 \tag{4.6.1.5}$$

4.6.2 Find the equation of the line parallel to y-axis and drawn through the point of intersection of the lines x – 7y + 5 = 0 and 3x + y = 0.

Solution: Following the approach in Problem 4.6.1, the desired equation is

$$\begin{pmatrix}1 & -7\end{pmatrix}\mathbf{x} - 5 + k\begin{pmatrix}3 & 1\end{pmatrix}\mathbf{x} = 0 \tag{4.6.2.1}$$

$$\implies \begin{pmatrix}1 + 3k & -7 + k\end{pmatrix}\mathbf{x} = 5 \tag{4.6.2.2}$$

$$\implies \begin{pmatrix}1 + 3k\\-7 + k\end{pmatrix} = \alpha\begin{pmatrix}1\\0\end{pmatrix} \text{or, } k = 7, \alpha = 22. \tag{4.6.2.3}$$

The desired equation is then given by

$$\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = \frac{5}{22} \tag{4.6.2.4}$$

See Fig. 4.6.2.1.

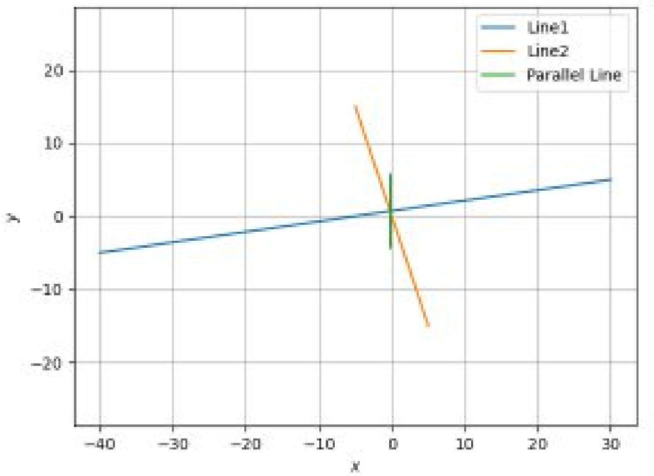


Fig. 4.6.2.1

4.6.3 A person standing at the junction (crossing) of two straight paths represented by the equations

$$\begin{pmatrix}2 & -3\end{pmatrix}\mathbf{x} = -4 \tag{4.6.3.1}$$

and

$$\begin{pmatrix}3 & 4\end{pmatrix}\mathbf{x} = 5 \tag{4.6.3.2}$$

wants to reach the path whose equation is

$$\begin{pmatrix}6 & -7\end{pmatrix}\mathbf{x} = -8 \tag{4.6.3.3}$$

Find equation of the path that he should follow.
Solution: The junction of (4.6.3.1) and (4.6.3.2) is obtained as

$$\left(\begin{array}{cc|c}2 & -3 & -4\\3 & 4 & 5\end{array}\right) \xleftrightarrow{R_2\to 2R_2-3R_1} \left(\begin{array}{cc|c}2 & -3 & -4\\0 & 17 & 22\end{array}\right)$$
$$\xleftrightarrow{R_1\to 17R_1+3R_2} \left(\begin{array}{cc|c}17 & 0 & -1\\0 & 17 & 22\end{array}\right) \implies \mathbf{A} = \frac{1}{17}\begin{pmatrix}-1\\22\end{pmatrix}$$

Clearly, the man should follow the path perpendicular to (4.6.3.3) from $\mathbf{A}$ to reach it in the shortest time. The normal vector of (4.6.3.3) is

$$\begin{pmatrix} 6 \\ -7 \end{pmatrix} \implies \mathbf{n} = \begin{pmatrix} 7 \\ 6 \end{pmatrix} \tag{4.6.3.4}$$

and the equation of the desired line is

$$\begin{pmatrix} 7 & 6 \end{pmatrix} \mathbf{x} = \frac{1}{17} \begin{pmatrix} 7 & 6 \end{pmatrix} \begin{pmatrix} -1 \\ 22 \end{pmatrix} = \frac{125}{17} \tag{4.6.3.5}$$

See Fig. 4.6.3.1.

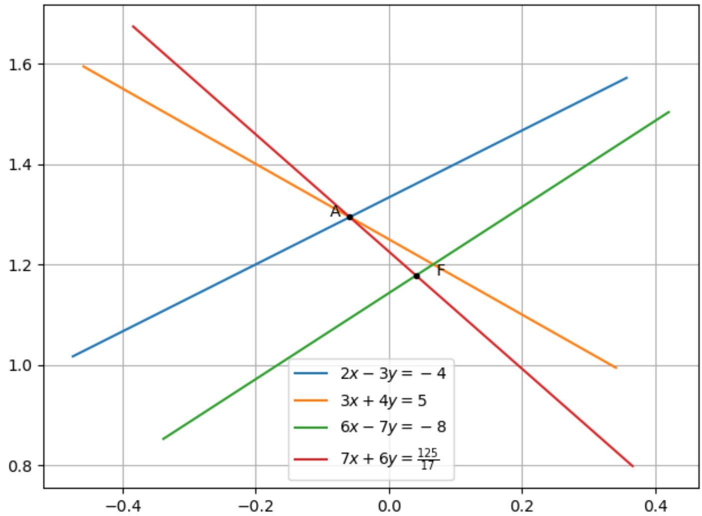


Fig. 4.6.3.1: AF is the required line.

4.6.4 Find the equation of the line passing through the point of intersection of the lines $4x + 7y - 3 = 0$ and $2x - 3y + 1 = 0$ that has equal intercepts on the axes.
Solution: From Problem 4.6.1, the intersection of the lines is given by

$$\begin{pmatrix} 4 + 2k & 7 - 3k \end{pmatrix} \mathbf{x} = 3 - k \tag{4.6.4.1}$$

$$\implies \begin{pmatrix} 4 + 2k \\ 7 - 3k \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{4.6.4.2}$$

from Problem 4.2.16, yielding,

$$\left(\begin{array}{cc|c} 1 & -2 & 4 \\ 1 & 3 & 7 \end{array}\right) \xleftrightarrow{R_2 = R_2 - R_1} \left(\begin{array}{cc|c} 1 & -2 & 4 \\ 0 & 5 & 3 \end{array}\right) \tag{4.6.4.3}$$

$$\text{or, } k = \frac{3}{5} \tag{4.6.4.4}$$

Substituting the above in (4.6.4.1), the desired equation is

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = \frac{6}{13} \tag{4.6.4.5}$$

See Fig. 4.6.4.1.

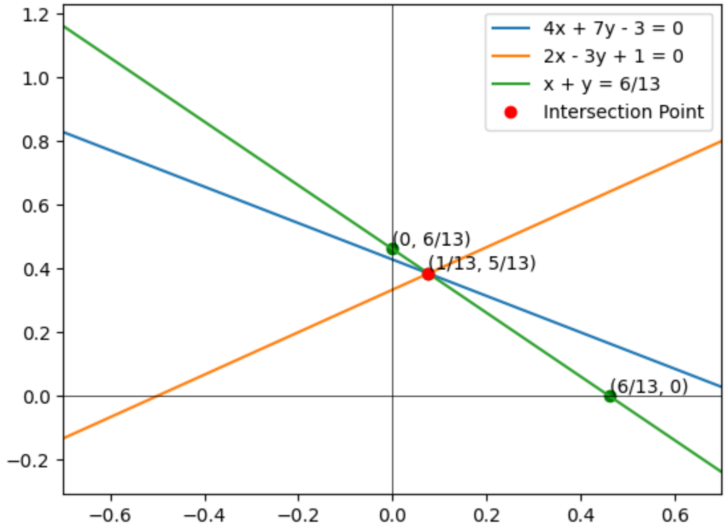


Fig. 4.6.4.1

4.6.5 Find the value of p so that the three lines $3x + y - 2 = 0, px + 2y - 3 = 0$ and $2x - y - 3 = 0$ may intersect at one point.

Solution: Performing row operations on the matrix

$$\begin{pmatrix} 3 & 1 & -2 \\ p & 2 & -3 \\ 2 & -1 & -3 \end{pmatrix} \xrightarrow[R_3=3R_3-2R_1]{R_2=3R_2-pR_1} \begin{pmatrix} 3 & 1 & -2 \\ 0 & 6-p & -9+2p \\ 0 & -5 & -5 \end{pmatrix}$$

$$\xrightarrow{R_3=R_3(6-p)+5R_2} \begin{pmatrix} 3 & 1 & -2 \\ 0 & 6-p & -9+2p \\ 0 & 0 & -75+15p \end{pmatrix}$$

$$\implies p = 5$$

See Fig. 4.6.5.1.

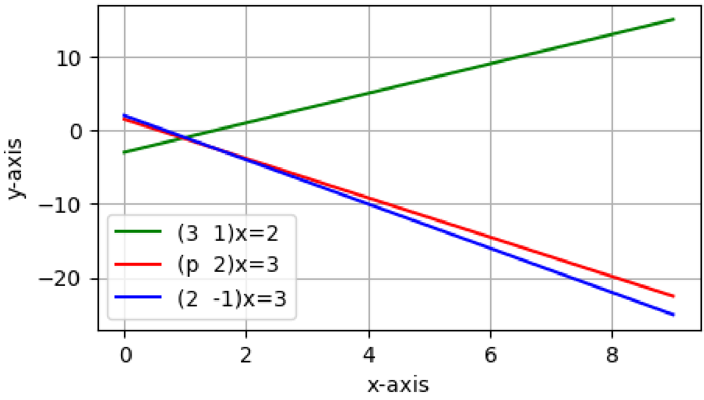


Fig. 4.6.5.1

4.6.6 Show the lines

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$$

$$\text{and } \frac{x-4}{5} = \frac{y-1}{2} = z \text{ intersect .}$$

Also, find their point of intersection.

4.6.7 The area of the region bounded by the curve $y = x + 1$ and the lines $x = 2$ and $x = 3$ is

a) $\frac{7}{2}$ sq units
b) $\frac{9}{2}$ sq units
c) $\frac{11}{2}$ sq units
d) $\frac{13}{2}$ sq units

4.6.8 The area of the region bounded by the curve $x = 2+3$ and the y lines $y = 1$ and $y = -1$ is

a) 4 sq units
b) $\frac{3}{2}$ sq units
c) 6 sq units
d) 8 sq units

4.6.9 Compute the area bounded by the line $x + 2y = 2$, $y - x = 1$ and $2x + y = 7$.

4.6.10 Find the area bounded by the lines $y = 4x + 5$, $y = 5 - x$ and $4y = x + 5$.

4.6.11 Find the equation of the plane which is perpendicular to the plane $5x+3y+6z+8 = 0$ and which contains the line of intersection of the planes $x + 2y + 3z - 4 = 0$ and $2x + y - z + 5 = 0$.

4.6.12 Point $\mathbf{P}(0, 2)$ is the point of intersection of y-axis and perpendicular bisector of line segment joining the points $\mathbf{A}(-1, 1)$ and $\mathbf{B}(3, 3)$.

4.6.13 Prove that the line through A$(0, -1, -1)$ and B$(4, 5, 1)$ intersects the line through C$(3, 9, 4)$ and D$(-4, 4, 4)$.

4.6.14 Find the equation of the plane through the intersection of the planes $\vec{r} \cdot (\hat{i} + 3\hat{j}) - 6 = 0$

and $\overrightarrow{r} \cdot (3\hat{i} - \hat{j} - 4\hat{k}) = 0$, whose perpendicular distance from origin is unity.

4.6.15 Find the equation of the line passing through the point of intersection of $2x + y = 5$ and $x + 3y + 8 = 0$ and parallel the line $3x + 4y = 7$.

4.6.16 Find the equations of the lines through the point of intersection of the line $x - y + 1 = 0$ and $2x - 3y + 5 = 0$ and whose distance from the point (3,2) is $\frac{7}{5}$.

4.6.17 Equations of diagonals of the square formed by the lines $x = 0$, $y = 0$, $x = 1$ and $y = 1$ are

a) $y = x$, $y + x = 1$
b) $y = x, x + y = 2$
c) $2y = x$, $y + x = \frac{1}{3}$
d) $y = 2x$, $y + 2x = 1$

4.6.18 The straight line $5x + 4y = 0$ passes through the point of intersection of the straight lines $x + 2y - 10 = 0$ and $2x + y + 5 = 0$.

4.7 Miscellaneous

4.7.1 Find the values of k for which the line

$$(k-3)x - (4-k^2)y + k^2 - 7k + 6 = 0 \tag{4.7.1.1}$$

is

a) Parallel to the x-axis
b) Parallel to the y-axis
c) Passing through the origin

Solution:

$$\mathbf{n} = \begin{pmatrix} k-3 \\ -4+k^2 \end{pmatrix}, c = -k^2 + 7k - 6 \tag{4.7.1.2}$$

a)

$$\begin{pmatrix} k-3 \\ -4+k^2 \end{pmatrix} = \alpha \begin{pmatrix} 0 \\ 1 \end{pmatrix} \implies k = 3, \tag{4.7.1.3}$$

$$\implies \begin{pmatrix} 0 & 5 \end{pmatrix} \mathbf{x} = 6 \tag{4.7.1.4}$$

upon substituting from (4.7.1.2).

b) In this case,

$$\begin{pmatrix} k-3 \\ -4+k^2 \end{pmatrix} = \beta \begin{pmatrix} 1 \\ 0 \end{pmatrix} \implies k = \pm 2 \tag{4.7.1.5}$$

$$\implies \begin{pmatrix} -1 & 0 \end{pmatrix} \mathbf{x} = 4, \quad k = 2 \tag{4.7.1.6}$$

$$\begin{pmatrix} -5 & 0 \end{pmatrix} \mathbf{x} = -24, \quad k = -2 \tag{4.7.1.7}$$

c) In this case,

$$-k^2 + 7k - 6 = 0 \implies k = 1, k = 6 \tag{4.7.1.8}$$

$$\implies \begin{pmatrix} -2 & -3 \end{pmatrix} \mathbf{x} = 0, \quad k = 1 \tag{4.7.1.9}$$

$$\begin{pmatrix} 3 & 32 \end{pmatrix} \mathbf{x} = 0, \quad k = 6 \tag{4.7.1.10}$$

4.7.2 Find the equations of the lines, which cutoff intercepts on the axes whose sum and product are 1 and -6 respectively.
Solution: Let the intercepts be a and b. Then

$$a + b = 1, ab = -6 \tag{4.7.2.1}$$

$$\implies a = 3, b = -2 \tag{4.7.2.2}$$

Thus, the possible intercepts are

$$\begin{pmatrix} 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix} \tag{4.7.2.3}$$

From (4.1.3.5),

$$\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix} \mathbf{n} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{4.7.2.4}$$

$$\implies \mathbf{n} = \begin{pmatrix} \frac{1}{3} \\ -\frac{1}{2} \end{pmatrix} \tag{4.7.2.5}$$

$$\text{or, } \begin{pmatrix} 2 & -3 \end{pmatrix} \mathbf{x} = 6 \tag{4.7.2.6}$$

using (4.1.4.1). Similarly, the other line can be obtained as

$$\begin{pmatrix} 3 & -2 \end{pmatrix} \mathbf{x} = -6 \tag{4.7.2.7}$$

See Fig. 4.7.2.1.

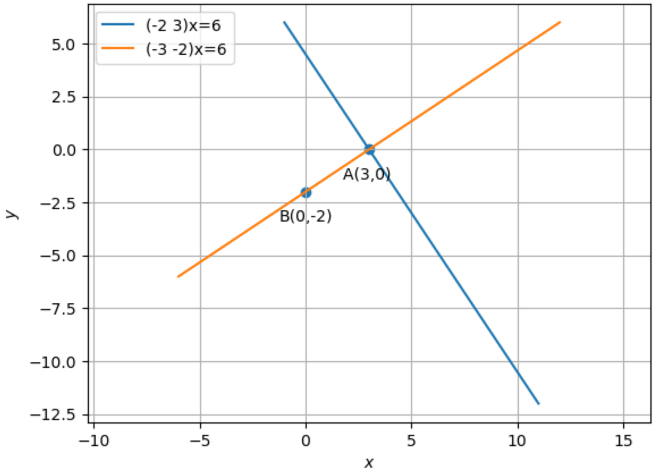


Fig. 4.7.2.1

4.7.3 A ray of light passing through the point $\mathbf{P} = (1, 2)$ reflects on the x-axis at point $\mathbf{A}$ and the reflected ray passes through the point $\mathbf{Q} = (5, 3)$. Find the coordinates of $\mathbf{A}$.
Solution: From (4.1.9.1), the reflection of $\mathbf{Q}$ is

$$\mathbf{R} = \begin{pmatrix} 5 \\ -3 \end{pmatrix} \tag{4.7.3.1}$$

Letting

$$\mathbf{A} = \begin{pmatrix} x \\ 0 \end{pmatrix}, \tag{4.7.3.2}$$

since $\mathbf{P}, \mathbf{A}, \mathbf{R}$ are collinear, from (4.1.3.6),

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 5 & -3 \\ 1 & x & 0 \end{pmatrix} \xleftrightarrow[R_3 = R_3 - R_1]{R_2 = R_2 - R_1} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 4 & -5 \\ 0 & x-1 & -2 \end{pmatrix} \tag{4.7.3.3}$$

$$\xleftrightarrow{R_3 = 4R_3 - (x-1)R_2} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 4 & -5 \\ 0 & 0 & 5x-13 \end{pmatrix} \implies x = \frac{13}{5} \tag{4.7.3.4}$$

See Fig. 4.7.3.1.

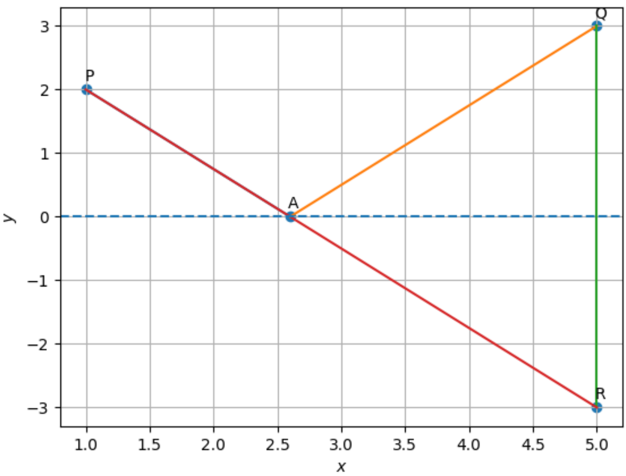


Fig. 4.7.3.1

4.7.4 Prove that in any $\triangle ABC$, cos A=$\frac{b^2+c^2-a^2}{2bc}$, where a,b,c are the magnitudes of the sides opposite to the vertices A,B,C respectively.

4.7.5 Distance of the point (α,β,γ) from y-axis is
   a) β
   b) $|\beta|$
   c) $|\beta+\gamma|$
   d) $\sqrt{\alpha^2+\gamma^2}$

4.7.6 The reflection of the point (α,β,γ) in the xy-plane is
   a) $\alpha,\beta,0)$
   b) $(0,0,\gamma)$
   c) $(-\alpha,-\beta,\gamma)$
   d) $(\alpha,\beta,-\gamma)$

4.7.7 The plane $ax+by=0$ is rotated about its line of intersection with the plane $z=0$ through an angle α. Prove that the equation of the plane in its new position is

$$ax+by\pm(\sqrt{a^2+b^2}\tan\alpha)z=0.$$

4.7.8 The locus represented by $xy+yz=0$ is
   a) A pair of perpendicular lines
   b) A pair of parallel lines
   c) A pair of parallel planes
   d) A pair of perpendicular planes

4.7.9 For what values of a and b the intercepts cut off on the coordinate axes by the line $ax+by+8=0$ are equal in length but opposite in signs to those cut off by the line $2x-3y=0$ on the axes.

4.7.10 If the equation of the base of an equilateral triangle is $x + y = 2$ and the vertex is (2,-1), then find the length of the side of the triangle.

4.7.11 A variable line passes through a fixed point **P**. The algebraic sum of the perpendiculars drawn from the points (2,0), (0,2) and (1,1) on the line is zero. Find the coordinates of the point **P**.

4.7.12 A straight line moves so that the sum of the reciprocals of its intercepts made on axes is constant. Show that the line passes through a fixed point.

4.7.13 If the sum of the distances of a moving point in a plane from the axes is l, then finds the locus of the point.

4.7.14 $\mathbf{P}_1, \mathbf{P}_2$ are points on either of the two lines $y - \sqrt{3}\,|x| = 2$ at a distance of 5 units from their point of intersection. Find the coordinates of the root of perpendiculars drawn from P_1, P_2 on the bisector of the angle between the given lines.

4.7.15 If p is the length of perpendicular from the origin on the lien $\frac{x}{a} + \frac{y}{b} = 1$ and a^2, p^2, b^2 are in A.P, then show that $a^4 + b^4 = 0$.

4.7.16 The point (4,1) undergoes the following two successive transformations :

a) Reflection about the line $y = x$
b) Translation through a distance 2 units along the positive x-axis

Then the final coordinates of the point are

a) (4,3)
b) (3,4)
c) (1,4)
d) $\frac{7}{2}, \frac{7}{2}$

4.7.17 One vertex of the equilateral with centroid at the origin and one side as $x + y - 2 = 0$ is

a) (-1,-1)
b) (2,2)
c) (-2-2)
d) (2,-2)

4.7.18 If a, b, c are is A.P., then the straight lines $ax + by + c = 0$ will always pass through ______.

4.7.19 The points (3,4) and (2,-6) are situated on the ______ of the line $3x - 4y - 8 = 0$.

4.7.20 A point moves so that square of its distance from the point (3,-2) is numerically equal to its distance from the line $5x - 12y = 3$. The equation of its locus is

4.7.21 Locus of the mid-points of the portion of the line $x \sin\theta + y \cos\theta = p$ intercepted between the axes is ______.

State whether the following statements are true or false. Justify.

4.7.22 If the vertices of a triangle have integral coordinates, then the triangle can not be equilateral.

4.7.23 The line $\frac{x}{a} + \frac{y}{b} = 1$ moves in such a way that $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$, where c is a constant. The locus of the foot of the perpendicular from the origin on the given line is $x^2 + y^2 = c^2$.

4.7.24 Match the following

1. The coordinates of the points P and Q on the line x + 5y = 13 which are at a distance of 2 units from the line 12x – 5y + 26 = 0 are	a) (3,1),(-7,11)
2. The coordinates of the point on the line x + y = 4, which are at a unit distance from the line 4x + 3y – 10 = 0 are	b) $-\frac{1}{11}, \frac{11}{3}$, $\frac{4}{3}, \frac{7}{3}$
3. The coordinates of the point on the line joining A (–2, 5) and B (3, 1) such that AP = PQ = QB are	c) $1, \frac{12}{5}$, $-3, \frac{16}{5}$

TABLE 4.7.24

4.7.25 The value of the λ, if the lines $(2x + 3y + 4) + \lambda(6x - y + 12) = 0$ are

1. parallel to y-axis is	a) $\lambda = -\frac{3}{4}$
2. perpendicular to $7x + y - 4 = 0$ is	b) $\lambda = -\frac{1}{3}$
3. passes through (1,2) is	c) $\lambda = -\frac{17}{41}$
4. parallel to x axis is	d) $\lambda = 3$

TABLE 4.7.25

4.7.26 The equation of the line through the intersection of the lines $2x - 3y = 0$ and $4x - 5y = 2$ and

1. through the point (2,1) is	a) $2x - y = 4$
2. perpendicular to the line	b) $x + y - 5 = 0$
3. parallel to the line $3x - 4y + 5 = 0$ is	c) $x - y - 1 = 0$
4. equally inclined to the axes is	d) $3x - 4y - 1 = 0$

TABLE 4.7.26

4.7.27 Point $\mathbf{R}(h, k)$ divides a line segment between the axes in the ratio 1: 2. Find the equation of the line.

Solution: Choosing the intercept points in Problem 4.2.17,

$$\mathbf{R} = \frac{2\mathbf{A} + \mathbf{B}}{3} \implies \begin{pmatrix} h \\ k \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2a \\ b \end{pmatrix} \tag{4.7.27.1}$$

$$\text{or, } \begin{pmatrix} b \\ a \end{pmatrix} = \mathbf{n} \equiv \begin{pmatrix} 2k \\ h \end{pmatrix} \tag{4.7.27.2}$$

Thus, the equation of the line is given by,

$$\begin{pmatrix} 2k & h \end{pmatrix}\mathbf{x} = \begin{pmatrix} 2k & h \end{pmatrix}\begin{pmatrix} h \\ k \end{pmatrix} = 3hk \tag{4.7.27.3}$$

4.7.28 The tangent of angle between the lines whose intercepts on the axes are $a, -b$ and $b, -a$, respectively, is

a) $\frac{a^2-b^2}{ab}$
b) $\frac{b^2-a^2}{2}$
c) $\frac{b^2-a^2}{2ab}$
d) none of these

4.7.29 Prove that the line through the point (x_1, y_1) and parallel to the line $Ax+By+C=0$ is $A(x-x_1)+B(y-y_1)=0$.
Solution: The equation of the desired line is

$$\begin{pmatrix} A & B \end{pmatrix}\left(\mathbf{x}-\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}\right)=0 \tag{4.7.29.1}$$

$$\implies \begin{pmatrix} A & B \end{pmatrix}\mathbf{x}=Ax_1+By_1 \tag{4.7.29.2}$$

4.7.30 If p and q are the lengths of perpendiculars from the origin to the lines $x\cos\theta - y\sin\theta = k\cos 2\theta$ and $x\sec\theta + y\operatorname{cosec}\theta = k$, respectively, prove that $p^2+4q^2=k^2$
Solution: The line parameters are

$$\mathbf{n}_1=\begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix}, c_1 = k\cos 2\theta \tag{4.7.30.1}$$

$$\mathbf{n}_2=\begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}, c_2 = \frac{1}{2}k\sin 2\theta \tag{4.7.30.2}$$

From (4.1.6.6),

$$p=\frac{\left|\mathbf{n}_1^\top\mathbf{x}-c_1\right|}{\|\mathbf{n}_1\|}=|k\cos 2\theta| \tag{4.7.30.3}$$

$$q=\frac{\left|\mathbf{n}_2^\top\mathbf{x}-c_2\right|}{\|\mathbf{n}_2\|}=\left|\frac{1}{2}k\sin 2\theta\right| \tag{4.7.30.4}$$

$$\implies p^2+4q^2=k^2 \tag{4.7.30.5}$$

4.7.31 If p is the length of perpendicular from origin to the line whose intercepts on the axes are a and b, then show that

$$\frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2} \tag{4.7.31.1}$$

Solution: Let the intercept points be

$$\begin{pmatrix} a \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ b \end{pmatrix}, \because \mathbf{n}=\begin{pmatrix} b \\ a \end{pmatrix}, \tag{4.7.31.2}$$

The line equation is,

$$\begin{pmatrix} b & a \end{pmatrix}\left(\mathbf{x}-\begin{pmatrix} a \\ 0 \end{pmatrix}\right)=0 \tag{4.7.31.3}$$

$$\implies \begin{pmatrix} b & a \end{pmatrix}\mathbf{x}=ab \tag{4.7.31.4}$$

From (4.1.6.6), the perpendicular distance from the origin to the line is

$$p = \frac{ab}{\sqrt{a^2 + b^2}} \implies (4.7.31.1) \tag{4.7.31.5}$$

4.7.32 Find perpendicular distance from the origin to the line joining the points $(\cos\theta, \sin\theta)$ and $(\cos\phi, \sin\phi)$.

Solution: The equation of the line is

$$\begin{pmatrix}\sin\phi - \sin\theta & \cos\theta - \cos\phi\end{pmatrix} \mathbf{x} = \sin(\phi - \theta) \tag{4.7.32.1}$$

and from (4.1.6.6), the distance is

$$d = \frac{\sin(\phi - \theta)}{2\sin\left(\frac{\phi-\theta}{2}\right)} = \cos\left(\frac{\phi - \theta}{2}\right) \tag{4.7.32.2}$$

4.7.33 Prove that the products of the lengths of the perpendiculars drawn from the points $\begin{pmatrix}\sqrt{a^2 - b^2} & 0\end{pmatrix}^\top$ and $\begin{pmatrix}-\sqrt{a^2 - b^2} & 0\end{pmatrix}^\top$ to the line $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$ is b^2.

Solution: The input parameters for (4.1.6.6) are

$$\mathbf{n} = \begin{pmatrix}\frac{\cos\theta}{a} \\ \frac{\sin\theta}{b}\end{pmatrix}, c = 1, \mathbf{P} = \pm\begin{pmatrix}\sqrt{a^2 - b^2} \\ 0\end{pmatrix} \tag{4.7.33.1}$$

The product of the distances is

$$d_1 d_2 = \frac{\left|\left(\mathbf{n}^\top \mathbf{P}\right)^2 - c^2\right|}{\|\mathbf{n}\|} = \frac{\left|\frac{\cos^2\theta\left(a^2 - b^2\right)}{a^2} - 1\right|}{\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}} \tag{4.7.33.2}$$

$$= \frac{\left(b^2\cos^2\theta + a^2\sin^2\theta\right)a^2 b^2}{\left(b^2\cos^2\theta + a^2\sin^2\theta\right)a^2} = b^2 \tag{4.7.33.3}$$

4.7.34 O is the origin and A is (a, b, c). Find the direction cosines of the line OA and the equation of the plane through A at right angle at OA.

4.7.35 Two systems of rectangular axis have the same origin. If a plane cuts them at distances a, b, c and a', b', c', respectively, from the origin, prove that

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{a'^2} + \frac{1}{b'^2} + \frac{1}{c'^2}$$

.

4.7.36 Equation of the line passing through the point $(a\cos^3\theta, a\sin^3\theta)$ and perpendicular to the line $x\sec\theta + y\csc\theta = a$ is $x\cos\theta - y\sin\theta = \alpha\sin 2\theta$.

4.7.37 The distance between the lines $y = mx + c$, and $y = mx + c^2$ is

a) $\frac{c_1 - c_2}{\sqrt{m+1}}$
b) $\frac{|c_1 - c_2|}{\sqrt{1+m^2}}$
c) $\frac{c^2 - c^1}{\sqrt{1+m^2}}$
d) 0

4.7.38 Find the area of triangle formed by the lines $y - x = 0, x + y = 0,$ and $x - k = 0$.

Solution: The vertices of the triangle can be expressed using the equations

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \mathbf{A} = \mathbf{0} \tag{4.7.38.1}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{B} = \begin{pmatrix} 0 \\ k \end{pmatrix} \tag{4.7.38.2}$$

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \mathbf{C} = \begin{pmatrix} k \\ 0 \end{pmatrix} \tag{4.7.38.3}$$

from which

$$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} k \\ -k \end{pmatrix}, \mathbf{C} = \begin{pmatrix} k \\ k \end{pmatrix} \tag{4.7.38.4}$$

are trivially obtained. Thus,

$$ar(ABC) = \frac{1}{2} \|(\mathbf{A} - \mathbf{B}) \times (\mathbf{A} - \mathbf{C})\| \tag{4.7.38.5}$$

$$= \frac{1}{2} \left\| \begin{pmatrix} -k \\ k \end{pmatrix} \times \begin{pmatrix} -k \\ -k \end{pmatrix} \right\| = k^2 \tag{4.7.38.6}$$

4.7.39 The lines $ax + 2y + 1 = 0$, $bx = 3y + 1 = 0$ and $cx + 4y + 1 = 0$ are concurrent if a, b, c are in G.P.

4.7.40 $P(a, b)$ is the mid-point of the line segment between axes. Show that the equation of the line is $\frac{x}{a} + \frac{y}{b} = 2$

Solution: From Problem 4.2.17,

$$\mathbf{n} = \begin{pmatrix} b \\ a \end{pmatrix} \tag{4.7.40.1}$$

$$\implies \begin{pmatrix} b & a \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} a \\ b \end{pmatrix} \right) = 0 \tag{4.7.40.2}$$

$$\text{or, } \begin{pmatrix} b & a \end{pmatrix} \mathbf{x} = 2ab. \tag{4.7.40.3}$$

is the desired line equation.

5 Skew Lines

5.1 Formulae

5.1.1. The lines

$$\begin{aligned} L_1 : \quad \mathbf{x} &= \mathbf{A} + \kappa_1 \mathbf{m_1} \\ L_2 : \quad \mathbf{x} &= \mathbf{B} + \kappa_2 \mathbf{m_2} \end{aligned} \tag{5.1.1.1}$$

will intersect if

$$\mathbf{A} + \kappa_1 \mathbf{m_1} = \mathbf{B} + \kappa_2 \mathbf{m_2} \tag{5.1.1.2}$$

$$\implies \begin{pmatrix} \mathbf{m_1} & \mathbf{m_2} \end{pmatrix} \begin{pmatrix} \kappa_1 \\ -\kappa_2 \end{pmatrix} = \mathbf{B} - \mathbf{A} \tag{5.1.1.3}$$

$$\implies \text{rank} \begin{pmatrix} \mathbf{M} & \mathbf{B} - \mathbf{A} \end{pmatrix} = 2 \tag{5.1.1.4}$$

where

$$\mathbf{M} = \begin{pmatrix} \mathbf{m_1} & \mathbf{m_2} \end{pmatrix} \tag{5.1.1.5}$$

5.1.2. If L_1, L_2, do not intersect, let

$$\begin{aligned} \mathbf{x}_1 &= \mathbf{A} + \kappa_1 \mathbf{m_1} \\ \mathbf{x}_2 &= \mathbf{B} + \kappa_2 \mathbf{m_2} \end{aligned} \tag{5.1.2.1}$$

be points on L_1, L_2 respectively, that are closest to each other. Then, from (5.1.2.1)

$$\mathbf{x_1} - \mathbf{x_2} = \mathbf{A} - \mathbf{B} + \begin{pmatrix} \mathbf{m_1} & \mathbf{m_2} \end{pmatrix} \begin{pmatrix} \kappa_1 \\ -\kappa_2 \end{pmatrix} \tag{5.1.2.2}$$

Also,

$$(\mathbf{x}_1 - \mathbf{x}_2)^\top \mathbf{m}_1 = (\mathbf{x}_1 - \mathbf{x}_2)^\top \mathbf{m}_2 = 0 \tag{5.1.2.3}$$

$$\implies (\mathbf{x}_1 - \mathbf{x}_2)^\top \begin{pmatrix} \mathbf{m_1} & \mathbf{m_2} \end{pmatrix} = \mathbf{0} \tag{5.1.2.4}$$

$$\text{or, } \mathbf{M}^\top (\mathbf{x}_1 - \mathbf{x}_2) = \mathbf{0} \tag{5.1.2.5}$$

$$\implies \mathbf{M}^\top (\mathbf{A} - \mathbf{B}) + \mathbf{M}^\top \mathbf{M} \begin{pmatrix} \kappa_1 \\ -\kappa_2 \end{pmatrix} = \mathbf{0} \tag{5.1.2.6}$$

from (5.1.2.2), yielding

$$\mathbf{M}^\top \mathbf{M} \begin{pmatrix} \kappa_1 \\ -\kappa_2 \end{pmatrix} = \mathbf{M}^\top (\mathbf{B} - \mathbf{A}) \tag{5.1.2.7}$$

This is known as the *least squares solution*.

5.1.3. Perform the eigendecompositions

$$\mathbf{M}\mathbf{M}^\top = \mathbf{U}\mathbf{D}_1\mathbf{U}^\top \tag{5.1.3.1}$$

$$\mathbf{M}^\top\mathbf{M} = \mathbf{V}\mathbf{D}_2\mathbf{V}^\top \tag{5.1.3.2}$$

5.1.4. The following expression is known as *singular value decomposition*

$$\mathbf{M} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top \tag{5.1.4.1}$$

where $\mathbf{\Sigma}$ is diagonal with entries obtained as in (5.3.1.21). Substituting in (5.1.2.7),

$$\mathbf{V\Sigma U}^{\top}\mathbf{U\Sigma V}^{\top}\boldsymbol{\kappa} = \mathbf{V\Sigma U}^{\top}(\mathbf{B}-\mathbf{A}) \tag{5.1.4.2}$$
$$\implies \mathbf{V\Sigma}^{2}\mathbf{V}^{\top}\boldsymbol{\kappa} = \mathbf{V\Sigma U}^{\top}(\mathbf{B}-\mathbf{A}) \tag{5.1.4.3}$$
$$\implies \boldsymbol{\kappa} = \left(\mathbf{V\Sigma}^{2}\mathbf{V}^{\top}\right)^{-1}\mathbf{V\Sigma U}^{\top}(\mathbf{B}-\mathbf{A}) \tag{5.1.4.4}$$
$$\implies \boldsymbol{\kappa} = \mathbf{V\Sigma}^{-2}\mathbf{V}^{\top}\mathbf{V\Sigma U}^{\top}(\mathbf{B}-\mathbf{A}) \tag{5.1.4.5}$$
$$\implies \boldsymbol{\kappa} = \mathbf{V\Sigma}^{-1}\mathbf{U}^{\top}(\mathbf{B}-\mathbf{A}) \tag{5.1.4.6}$$

where $\mathbf{\Sigma}^{-1}$ is obtained by inverting the nonzero elements of $\mathbf{\Sigma}$.

5.1.5. From (5.1.2.1),

$$\mathbf{x}_1 - \mathbf{x}_2 = \mathbf{A} + \kappa_1\mathbf{m_1} - \mathbf{B} - \kappa_2\mathbf{m_2} \tag{5.1.5.1}$$
$$= \mathbf{A} - \mathbf{B} + \mathbf{M}\boldsymbol{\kappa} \tag{5.1.5.2}$$

which, upon substitution from (5.1.4.1) yields

$$\mathbf{x}_1 - \mathbf{x}_2 = \mathbf{A} - \mathbf{B} + \mathbf{U\Sigma V}^{\top}\mathbf{V\Sigma}^{-1}\mathbf{U}^{\top}(\mathbf{B}-\mathbf{A}) \tag{5.1.5.3}$$
$$= (\mathbf{A}-\mathbf{B})\left(\mathbf{I} - \mathbf{U\Sigma\Sigma}^{-1}\mathbf{U}^{\top}\right) \tag{5.1.5.4}$$

Thus,

$$\|\mathbf{x}_1 - \mathbf{x}_2\| = \left\|(\mathbf{A}-\mathbf{B})\left(\mathbf{I} - \mathbf{U\Sigma\Sigma}^{-1}\mathbf{U}^{\top}\right)\right\| \tag{5.1.5.5}$$

5.2 *Least Squares*

5.2.1 Find the shortest distance between the lines

$$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1} \text{ and} \tag{5.2.1.1}$$
$$\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1} \tag{5.2.1.2}$$

Solution: The given lines can be written as

$$\mathbf{x} = \begin{pmatrix}-1\\-1\\-1\end{pmatrix} + \kappa_1\begin{pmatrix}7\\-6\\1\end{pmatrix}$$
$$\mathbf{x} = \begin{pmatrix}3\\5\\7\end{pmatrix} + \kappa_2\begin{pmatrix}1\\-2\\1\end{pmatrix} \tag{5.2.1.3}$$

with

$$\mathbf{A} = \begin{pmatrix}-1\\-1\\-1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}3\\5\\7\end{pmatrix}, \mathbf{M} = \begin{pmatrix}7 & 1\\-6 & -2\\1 & 1\end{pmatrix} \tag{5.2.1.4}$$

Substituting the above in (5.1.1.4),

$$\left(\begin{array}{cc|c} 7 & 1 & 4 \\ -6 & -2 & 6 \\ 1 & 1 & 8 \end{array}\right) \xleftrightarrow[R_3 \leftarrow R_3 - \frac{1}{7}R_1]{R_2 \leftarrow R_2 + \frac{6}{7}R_1} \tag{5.2.1.5}$$

$$\left(\begin{array}{cc|c} 7 & 1 & 4 \\ 0 & -\frac{8}{7} & \frac{66}{7} \\ 0 & \frac{6}{7} & -\frac{52}{7} \end{array}\right) \xleftrightarrow{R_3 \leftarrow R_3 + \frac{3}{4}R_2} \tag{5.2.1.6}$$

$$\left(\begin{array}{cc|c} 2 & 3 & 1 \\ 0 & -\frac{7}{2} & \frac{1}{2} \\ 0 & 0 & -\frac{5}{14} \end{array}\right) \tag{5.2.1.7}$$

The rank of the matrix is 3. So the given lines are skew. From (5.1.2.7)

$$\begin{pmatrix} 7 & -6 & 1 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 7 & 1 \\ -6 & -2 \\ 1 & 1 \end{pmatrix} \boldsymbol{\kappa} = \begin{pmatrix} 7 & -6 & 1 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 6 \\ 8 \end{pmatrix} \tag{5.2.1.8}$$

$$\implies \begin{pmatrix} 86 & 20 \\ 20 & 6 \end{pmatrix} \boldsymbol{\kappa} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{5.2.1.9}$$

$$\implies \begin{pmatrix} \kappa_1 \\ -\kappa_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{5.2.1.10}$$

From (5.2.1.3), the closest points are **A** and **B** and the minimum distance between the lines is given by

$$\|\mathbf{B} - \mathbf{A}\| = \left\| \begin{pmatrix} 4 \\ 6 \\ 8 \end{pmatrix} \right\| = 2\sqrt{29} \tag{5.2.1.11}$$

See Fig. 5.2.1.1.

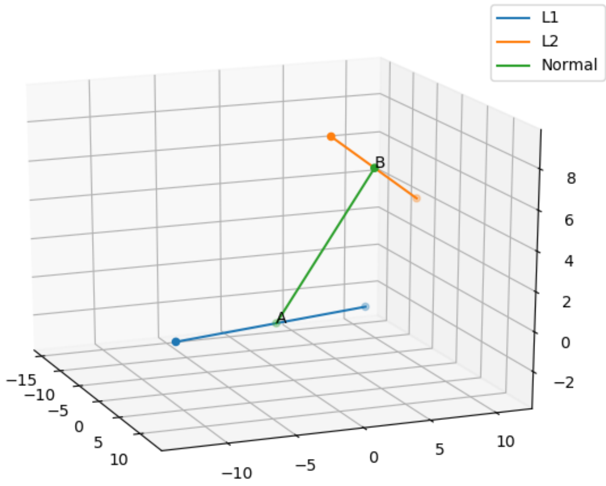


Fig. 5.2.1.1

5.2.2 Find the shortest distance between the lines whose vector equations are

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \kappa_1 \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$$
$$\mathbf{x} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + \kappa_2 \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \tag{5.2.2.1}$$

Solution: In this case,

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} \tag{5.2.2.2}$$

$$\mathbf{M} = \begin{pmatrix} 1 & 2 \\ -3 & 3 \\ 2 & 1 \end{pmatrix}. \tag{5.2.2.3}$$

forming the matrix in (5.1.1.4),

$$\begin{pmatrix} 1 & 2 & 3 \\ -3 & 3 & 3 \\ 2 & 1 & 3 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 + 3R_1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 9 & 12 \\ 2 & 1 & 3 \end{pmatrix}$$
$$\xleftrightarrow{R_3 \leftarrow R_3 - 2R_1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 9 & 12 \\ 0 & -3 & -3 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow 3R_3 + R_2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 9 & 12 \\ 0 & 0 & 3 \end{pmatrix}$$

Clearly, the rank of this matrix is 3, and therefore, the lines are skew. From (5.1.2.7),

$$\left(\begin{array}{cc|c} 14 & -5 & 0 \\ -5 & 14 & 18 \end{array}\right) \xleftrightarrow{R_1 \leftarrow R_1+R_2} \left(\begin{array}{cc|c} 9 & 9 & 18 \\ -5 & 14 & 18 \end{array}\right)$$

$$\xleftrightarrow{R_1 \leftarrow \frac{R_1}{9}} \left(\begin{array}{cc|c} 1 & 1 & 2 \\ -5 & 14 & 18 \end{array}\right) \xleftrightarrow{R_2 \leftarrow R_2+5R_1} \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 19 & 28 \end{array}\right)$$

$$\xleftrightarrow{R_1 \leftarrow 19R_1-R_2} \left(\begin{array}{cc|c} 19 & 0 & 10 \\ 0 & 19 & 28 \end{array}\right) \xleftrightarrow[R_2 \leftarrow \frac{R_2}{9}]{R_1 \leftarrow \frac{R_1}{19}} \left(\begin{array}{cc|c} 1 & 0 & \frac{10}{19} \\ 0 & 1 & \frac{28}{19} \end{array}\right)$$

yielding

$$\boldsymbol{\kappa} = \frac{1}{19}\begin{pmatrix} 10 \\ 28 \end{pmatrix} \tag{5.2.2.4}$$

Substituting the above in (5.2.2.1),

$$\mathbf{x}_1 = \frac{1}{19}\begin{pmatrix} 29 \\ 8 \\ 77 \end{pmatrix}, \mathbf{x}_2 = \frac{1}{19}\begin{pmatrix} 20 \\ 11 \\ 86 \end{pmatrix}. \tag{5.2.2.5}$$

Thus, the required distance is

$$\|\mathbf{x}_2 - \mathbf{x}_1\| = \frac{3}{\sqrt{19}} \tag{5.2.2.6}$$

See Fig. 5.2.2.1.

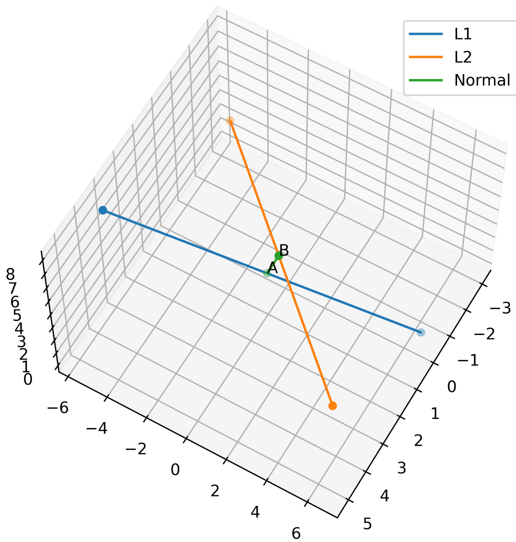


Fig. 5.2.2.1

5.2.3 Find the shortest distance between the lines l_1 and l_2 whose vector equations are

$$\vec{r} = \hat{i} + \hat{j} + \kappa(2\hat{i} - \hat{j} + \hat{k}) \text{ and} \tag{5.2.3.1}$$

$$\vec{r} = 2\hat{i} + \hat{j} - \hat{k} + \mu(3\hat{i} - 5\hat{j} + 2\hat{k}). \tag{5.2.3.2}$$

Solution: The given lines can be written in vector form as

$$\begin{aligned} \mathbf{x} &= \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \kappa_1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \\ \mathbf{x} &= \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \kappa_2 \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} \end{aligned} \tag{5.2.3.3}$$

$$\mathbf{M} = \begin{pmatrix} 2 & 3 \\ -1 & -5 \\ 1 & 2 \end{pmatrix}, \mathbf{B} - \mathbf{A} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \tag{5.2.3.4}$$

Substituting the above in (5.1.1.4),

$$\begin{pmatrix} 2 & 3 & 1 \\ -1 & -5 & 0 \\ 1 & 2 & -1 \end{pmatrix} \xleftrightarrow[R_3 \leftarrow R_3 - \frac{1}{2}R_1]{R_2 \leftarrow R_2 + \frac{1}{2}R_1} \begin{pmatrix} 2 & 3 & 1 \\ 0 & -\frac{7}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{3}{2} \end{pmatrix} \tag{5.2.3.5}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 + 7R_2} \begin{pmatrix} 2 & 3 & 1 \\ 0 & -\frac{7}{2} & \frac{1}{2} \\ 0 & 0 & -10 \end{pmatrix} \tag{5.2.3.6}$$

The rank of the matrix is 3. So the given lines are skew. From (5.1.2.7),

$$\begin{pmatrix} 2 & -1 & 1 \\ 3 & -5 & 2 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ -1 & -5 \\ 1 & 2 \end{pmatrix} \boldsymbol{\kappa} = \begin{pmatrix} 2 & -1 & 1 \\ 3 & -5 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \tag{5.2.3.7}$$

$$\implies \begin{pmatrix} 6 & 13 \\ 13 & 38 \end{pmatrix} \boldsymbol{\kappa} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{5.2.3.8}$$

The augmented matrix of the above equation (5.2.3.8) is given by,

$$\left(\begin{array}{cc|c} 6 & 13 & 1 \\ 13 & 38 & 1 \end{array}\right) \xleftrightarrow{R_2 \leftarrow R_2 - \frac{13}{6}R_1} \left(\begin{array}{cc|c} 6 & 13 & 1 \\ 0 & \frac{59}{6} & -\frac{7}{6} \end{array}\right) \tag{5.2.3.9}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - \frac{78}{59}R_2} \left(\begin{array}{cc|c} 6 & 0 & \frac{150}{59} \\ 0 & \frac{59}{6} & -\frac{7}{6} \end{array}\right) \tag{5.2.3.10}$$

yielding

$$\begin{pmatrix} \kappa_1 \\ -\kappa_2 \end{pmatrix} = \begin{pmatrix} \frac{25}{59} \\ -\frac{7}{59} \end{pmatrix} \tag{5.2.3.11}$$

Substituting in (5.2.3.3),

$$\mathbf{x}_1 = \frac{1}{59}\begin{pmatrix}109\\34\\25\end{pmatrix}, \mathbf{x}_2 = \frac{1}{59}\begin{pmatrix}139\\24\\-45\end{pmatrix}. \tag{5.2.3.12}$$

The minimum distance between the lines is given by,

$$\|\mathbf{x}_2 - \mathbf{x}_1\| = \left\|\frac{1}{59}\begin{pmatrix}30\\-10\\-70\end{pmatrix}\right\| = \frac{10}{\sqrt{59}} \tag{5.2.3.13}$$

See Fig. 5.2.3.1.

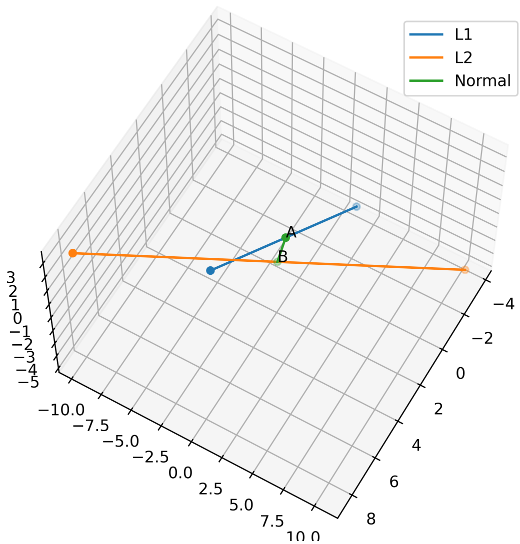


Fig. 5.2.3.1

5.2.4 Find the shortest distance between the lines given by

$$\overrightarrow{r} = (8 + 3\kappa\hat{i} - (9 + 16\kappa)\hat{j} + (10 + 7\kappa)\hat{k} \text{ and} \tag{5.2.4.1}$$

$$\overrightarrow{r} = 15\hat{i} + 29\hat{j} + 5\hat{k} + \mu(3\hat{i} + 8\hat{j} - 5\hat{k}). \tag{5.2.4.2}$$

5.2.5 Find the shortest distance between the lines

$$\overrightarrow{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \kappa(\hat{i} - \hat{j} + \hat{k}) \text{ and} \tag{5.2.5.1}$$

$$\overrightarrow{r} = 2\hat{i} - \hat{j} - \hat{k} + \mu(2\hat{i} + \hat{j} + 2\hat{k}) \tag{5.2.5.2}$$

5.3.1 Find the shortest distance between the lines whose vector equations are

$$\mathbf{x} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda_1 \begin{pmatrix}1\\-3\\2\end{pmatrix} \tag{5.3.1.1}$$

and

$$\mathbf{x} = \begin{pmatrix}4\\5\\6\end{pmatrix} + \lambda_2 \begin{pmatrix}2\\3\\1\end{pmatrix} \tag{5.3.1.2}$$

Solution: From (5.2.2.3),

$$\mathbf{M}^\top\mathbf{M} = \begin{pmatrix}1 & -3 & 2\\2 & 3 & 1\end{pmatrix}\begin{pmatrix}1 & 2\\-3 & 3\\2 & 1\end{pmatrix} = \begin{pmatrix}14 & -5\\-5 & 14\end{pmatrix} \tag{5.3.1.3}$$

$$\mathbf{M}\mathbf{M}^\top = \begin{pmatrix}1 & 2\\-3 & 3\\2 & 1\end{pmatrix}\begin{pmatrix}1 & -3 & 2\\2 & 3 & 1\end{pmatrix} = \begin{pmatrix}5 & 3 & 4\\3 & 18 & -3\\4 & -3 & 5\end{pmatrix} \tag{5.3.1.4}$$

a) For $\mathbf{M}\mathbf{M}^\top$, the characteristic polynomial is

$$\operatorname{char}\mathbf{M}\mathbf{M}^\top = \begin{vmatrix}\lambda-5 & -3 & -4\\-3 & \lambda-18 & 3\\-4 & 3 & \lambda-5\end{vmatrix} \tag{5.3.1.5}$$

$$= \lambda(\lambda-9)(\lambda-19) \tag{5.3.1.6}$$

Thus, the eigenvalues are given by

$$\lambda_1 = 19, \ \lambda_2 = 9, \ \lambda_3 = 0 \tag{5.3.1.7}$$

For λ_1, the augmented matrix formed from the eigenvalue-eigenvector equation is

$$\begin{pmatrix}-14 & 3 & 4\\3 & -1 & -3\\4 & -3 & -14\end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{R_1+R_3}{-10}} \begin{pmatrix}1 & 0 & 1\\3 & -1 & -3\\4 & -3 & -14\end{pmatrix}$$

$$\xleftrightarrow[R_2 \leftarrow -R_2+3R_1]{R_3 \leftarrow R_3-4R_1} \begin{pmatrix}1 & 0 & 1\\0 & -1 & -6\\0 & -3 & -18\end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3-3R_2} \begin{pmatrix}1 & 0 & 1\\0 & -1 & -6\\0 & 0 & 0\end{pmatrix}$$

Hence, the normalized eigenvector is

$$\mathbf{u}_1 = \frac{1}{\sqrt{38}}\begin{pmatrix}-1\\-6\\1\end{pmatrix} \tag{5.3.1.8}$$

For λ_2, the augmented matrix formed from the eigenvalue-eigenvector equation is

$$\begin{pmatrix} -4 & 3 & 4 \\ 3 & 9 & -3 \\ 4 & 3 & -4 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_1 + R_3} \begin{pmatrix} -4 & 3 & 4 \\ 3 & 9 & -3 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xleftrightarrow[R_2 \leftarrow \frac{4R_2 + 3R_1}{45}]{R_1 \leftarrow \frac{R_1 - 3R_2}{-4}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence, the normalized eigenvector is

$$\mathbf{u}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \tag{5.3.1.9}$$

For λ_3, the augmented matrix formed from the eigenvalue-eigenvector equation is

$$\begin{pmatrix} 5 & 3 & 4 \\ 3 & 18 & -3 \\ 4 & -3 & 5 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{R_1 + R_3}{9}} \begin{pmatrix} 1 & 0 & 1 \\ 3 & 18 & -3 \\ 4 & -3 & 5 \end{pmatrix}$$

$$\xleftrightarrow[R_2 \leftarrow R_2 - 3R_1]{R_3 \leftarrow R_3 - 4R_1} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 18 & -6 \\ 0 & -3 & 1 \end{pmatrix} \xleftrightarrow[R_2 \leftarrow \frac{R_2}{6}]{R_3 \leftarrow R_3 + R_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 3 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

yielding

$$\mathbf{u}_3 = \frac{1}{\sqrt{19}} \begin{pmatrix} -3 \\ 1 \\ 3 \end{pmatrix} \tag{5.3.1.10}$$

Using (5.1.3.1), we see that

$$\mathbf{U} = \begin{pmatrix} -\frac{1}{\sqrt{38}} & \frac{1}{\sqrt{2}} & -\frac{3}{\sqrt{19}} \\ -\frac{6}{\sqrt{38}} & 0 & \frac{1}{\sqrt{19}} \\ \frac{1}{\sqrt{38}} & -\frac{1}{\sqrt{2}} & \frac{3}{\sqrt{19}} \end{pmatrix} \tag{5.3.1.11}$$

$$\mathbf{D_1} = \begin{pmatrix} 19 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.3.1.12}$$

b) For $\mathbf{M}^\top\mathbf{M}$, the characteristic polynomial is

$$\text{char}\mathbf{M}^\top\mathbf{M} = \begin{vmatrix} \lambda - 14 & 5 \\ 5 & \lambda - 14 \end{vmatrix} \tag{5.3.1.13}$$

$$= (\lambda - 9)(\lambda - 19) \tag{5.3.1.14}$$

Thus, the eigenvalues are given by

$$\lambda_1 = 19, \ \lambda_2 = 9 \tag{5.3.1.15}$$

For λ_1, the augmented matrix formed from the eigenvalue-eigenvector equation is

$$\begin{pmatrix} -5 & -5 \\ -5 & -5 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 - R_2} \begin{pmatrix} 0 & 0 \\ -5 & -5 \end{pmatrix} \tag{5.3.1.16}$$

Hence, the normalized eigenvector is

$$\mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{5.3.1.17}$$

For λ_2, the augmented matrix formed from the eigenvalue-eigenvector equation is

$$\begin{pmatrix} 5 & -5 \\ -5 & 5 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 + R_2} \begin{pmatrix} 0 & 0 \\ 5 & -5 \end{pmatrix} \tag{5.3.1.18}$$

Hence, the normalized eigenvector is

$$\mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{5.3.1.19}$$

Thus, from (5.1.3.2),

$$\mathbf{V} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \ \mathbf{D_2} = \begin{pmatrix} 9 & 0 \\ 0 & 19 \end{pmatrix} \tag{5.3.1.20}$$

Using (5.3.1.15),

$$\mathbf{\Sigma} \triangleq \begin{pmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \sqrt{19} & 0 \\ 0 & 3 \\ 0 & 0 \end{pmatrix} \tag{5.3.1.21}$$

and substituting into (5.1.4.6),

$$\kappa = \frac{1}{19} \begin{pmatrix} 10 \\ 28 \end{pmatrix} \tag{5.3.1.22}$$

which agrees with (5.2.2.4).

5.3.2 Find the shortest distance between the lines l_1 and l_2 whose vector equations are

$$\overrightarrow{r} = \hat{i} + \hat{j} + \lambda(2\hat{i} - \hat{j} + \hat{k}) \text{ and} \tag{5.3.2.1}$$

$$\overrightarrow{r} = 2\hat{i} + \hat{j} - \hat{k} + \mu(3\hat{i} - 5\hat{j} + 2\hat{k}). \tag{5.3.2.2}$$

Solution:

a) To check whether the given lines are skew, from (5.2.3.4) and (5.1.1.4),

$$\begin{pmatrix} 2 & 3 & 1 \\ -1 & -5 & 0 \\ 1 & 2 & -1 \end{pmatrix} \xleftrightarrow[R_3 \leftarrow R_3 - \frac{1}{2}R_1]{R_2 \leftarrow R_2 + \frac{1}{2}R_1} \begin{pmatrix} 2 & 3 & 1 \\ 0 & -\frac{7}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{3}{2} \end{pmatrix}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 + 7R_2} \begin{pmatrix} 2 & 3 & 1 \\ 0 & -\frac{7}{2} & \frac{1}{2} \\ 0 & 0 & -10 \end{pmatrix}$$

The rank of the matrix is 3. So the given lines are skew.

b)

$$\mathbf{M}^\top \mathbf{M} = \begin{pmatrix} 2 & -1 & 1 \\ 3 & -5 & 2 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ -1 & -5 \\ 1 & 2 \end{pmatrix} \tag{5.3.2.3}$$

$$= \begin{pmatrix} 6 & 13 \\ 13 & 38 \end{pmatrix} \tag{5.3.2.4}$$

$$\mathbf{M}\mathbf{M}^\top = \begin{pmatrix} 2 & 3 \\ -1 & -5 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & -1 & 1 \\ 3 & -5 & 2 \end{pmatrix} \tag{5.3.2.5}$$

$$= \begin{pmatrix} 13 & -17 & 8 \\ -17 & 26 & -11 \\ 8 & -11 & 5 \end{pmatrix} \tag{5.3.2.6}$$

The characteristic polynomial of the matrix $\mathbf{M}\mathbf{M}^\top$ is given by,

$$\operatorname{char}\left(\mathbf{M}\mathbf{M}^\top\right) = \begin{vmatrix} 13-\lambda & -17 & 8 \\ -17 & 26-\lambda & -11 \\ 8 & -11 & 5-\lambda \end{vmatrix} \tag{5.3.2.7}$$

$$= -\lambda^3 + 44\lambda^2 - 59\lambda \tag{5.3.2.8}$$

resulting in

$$\mathbf{U} = \begin{pmatrix} \frac{12-\sqrt{17}}{\sqrt{5}\sqrt{68-6\sqrt{17}}} & \frac{12+\sqrt{17}}{\sqrt{5}\sqrt{68+6\sqrt{17}}} & -\frac{3}{\sqrt{59}} \\ \frac{1-3\sqrt{17}}{\sqrt{5}\sqrt{68-6\sqrt{17}}} & \frac{1+3\sqrt{17}}{\sqrt{5}\sqrt{68+6\sqrt{17}}} & \frac{1}{\sqrt{59}} \\ \frac{\sqrt{5}}{\sqrt{68-6\sqrt{17}}} & \frac{\sqrt{5}}{\sqrt{68+6\sqrt{17}}} & \frac{7}{\sqrt{59}} \end{pmatrix} \tag{5.3.2.9}$$

and

$$\mathbf{D}_1 = \begin{pmatrix} 22+5\sqrt{17} & 0 & 0 \\ 0 & 22-5\sqrt{17} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.3.2.10}$$

For $\mathbf{M}^\top\mathbf{M}$, the characteristic polynomial is

$$\operatorname{char}\left(\mathbf{M}^\top\mathbf{M}\right) = \begin{vmatrix} 6-\lambda & 13 \\ 13 & 38-\lambda \end{vmatrix} \tag{5.3.2.11}$$

$$= \lambda^2 - 44\lambda + 59 \tag{5.3.2.12}$$

Thus, the eigenvalues are given by

$$\lambda_1 = 22 + 5\sqrt{17}, \ \lambda_2 = 22 - 5\sqrt{17} \tag{5.3.2.13}$$

resulting in

$$\mathbf{V} = \begin{pmatrix} \frac{-16-5\sqrt{17}}{\sqrt{850+160\sqrt{17}}} & \frac{13}{\sqrt{850-160\sqrt{17}}} \\ \frac{13}{\sqrt{850+160\sqrt{17}}} & \frac{-16+5\sqrt{17}}{\sqrt{850-160\sqrt{17}}} \end{pmatrix} \tag{5.3.2.14}$$

$$\mathbf{D}_2 = \begin{pmatrix} 22-5\sqrt{17} & 0 \\ 0 & 22+5\sqrt{17} \end{pmatrix} \tag{5.3.2.15}$$

Therefore,

$$\boldsymbol{\Sigma} = \begin{pmatrix} \sqrt{22+5\sqrt{17}} & 0 \\ 0 & \sqrt{22-5\sqrt{17}} \\ 0 & 0 \end{pmatrix} \tag{5.3.2.16}$$

and substituting into (5.1.5.5),

$$\boldsymbol{\lambda} = \begin{pmatrix} \frac{25}{59} \\ -\frac{7}{59} \end{pmatrix} \tag{5.3.2.17}$$

which agrees with (5.2.3.11).

5.3.3 Find the shortest distance between the lines given by

$$\overrightarrow{r} = (8+3\lambda\hat{i} - (9+16\lambda)\hat{j} + (10+7\lambda)\hat{k} \text{ and} \tag{5.3.3.1}$$

$$\overrightarrow{r} = 15\hat{i} + 29\hat{j} + 5\hat{k} + \mu(3\hat{i} + 8\hat{j} - 5\hat{k}). \tag{5.3.3.2}$$

5.3.4 Find the shortest distance between the lines

$$\overrightarrow{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k}) \text{ and} \tag{5.3.4.1}$$

$$\overrightarrow{r} = 2\hat{i} - \hat{j} - \hat{k} + \mu(2\hat{i} + \hat{j} + 2\hat{k}) \tag{5.3.4.2}$$

5.3.5 Find the shortest distance between the lines

$$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1} \text{ and} \tag{5.3.5.1}$$

$$\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1} \tag{5.3.5.2}$$

6 Circle

6.1 Formulae

6.1.1. For a circle with centre $\mathbf{c}$ and radius r,

$$\mathbf{u} = -\mathbf{c}, f = \|\mathbf{u}\|^2 - r^2 \tag{6.1.1.1}$$

6.1.2. Given points , $\mathbf{x}_1,\ \mathbf{x}_2$ on the circle and the diameter

$$\mathbf{n}^\top \mathbf{x} = c, \tag{6.1.2.1}$$

the centre is given by

$$\begin{pmatrix} 2\mathbf{x}_1 & 2\mathbf{x}_2 & \mathbf{n} \\ 1 & 1 & 0 \end{pmatrix}^\top \begin{pmatrix} \mathbf{u} \\ f \end{pmatrix} = -\begin{pmatrix} \|\mathbf{x}_1\|^2 \\ \|\mathbf{x}_2\|^2 \\ c \end{pmatrix} \tag{6.1.2.2}$$

Solution: From (A.7.1.1),

$$\begin{aligned} \|\mathbf{x}_1\|^2 + 2\mathbf{u}^\top \mathbf{x}_1 + f = 0 \\ \|\mathbf{x}_2\|^2 + 2\mathbf{u}^\top \mathbf{x}_2 + f = 0 \end{aligned} \tag{6.1.2.3}$$

and (6.1.2.1) can be expressed as

$$\mathbf{u}^\top \mathbf{n} = -c \tag{6.1.2.4}$$

Clubbing (6.1.2.3) and (6.1.2.4), we obtain (6.1.2.2).

6.1.3. Given points $\mathbf{x}_1,\ \mathbf{x}_2,\ \mathbf{x}_3$ on the circle, the parameters are given by

$$\begin{pmatrix} 2\mathbf{x}_1 & 2\mathbf{x}_2 & 2\mathbf{x}_3 \\ 1 & 1 & 1 \end{pmatrix}^\top \begin{pmatrix} \mathbf{u} \\ f \end{pmatrix} = -\begin{pmatrix} \|\mathbf{x}_1\|^2 \\ \|\mathbf{x}_2\|^2 \\ \|\mathbf{x}_3\|^2 \end{pmatrix} \tag{6.1.3.1}$$

6.2 Equation

6.2.1 Find the equation of the circle passing through the points $(4, 1)$ and $(6, 5)$ and whose centre is on the line $4x + y = 16$.
Solution: Following Appendix 6.1.2,

$$\mathbf{x}_1 = \begin{pmatrix} 4 \\ 1 \end{pmatrix},\ \mathbf{x}_2 = \begin{pmatrix} 6 \\ 5 \end{pmatrix},\ \mathbf{n} = \begin{pmatrix} 4 \\ 1 \end{pmatrix},\ c = -16. \tag{6.2.1.1}$$

Substituting in (6.1.2.2),

$$\begin{pmatrix} -4 & -1 & 0 \\ 12 & 10 & 1 \\ 8 & 2 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ f \end{pmatrix} = \begin{pmatrix} 16 \\ -61 \\ -17 \end{pmatrix} \tag{6.2.1.2}$$

The augmented matrix is expressed as

$$\left(\begin{array}{ccc|c} -4 & -1 & 0 & 16 \\ 12 & 10 & 1 & -61 \\ 8 & 2 & 1 & -17 \end{array}\right) \tag{6.2.1.3}$$

Performing a sequence of row operations to transform into an Echelon form

$$\xleftrightarrow[R_2 \to R_2+3R_1]{R_3 \to R_3+2R_1} \left(\begin{array}{ccc|c} -4 & -1 & 0 & 16 \\ 0 & 7 & 1 & -13 \\ 0 & 0 & 1 & 15 \end{array}\right) \tag{6.2.1.4}$$

$$\xleftrightarrow{R_2 \to R_2-R_3} \left(\begin{array}{ccc|c} -4 & -1 & 0 & 16 \\ 0 & 7 & 0 & -28 \\ 0 & 0 & 1 & 15 \end{array}\right) \tag{6.2.1.5}$$

$$\xleftrightarrow{R_2 \to \frac{R_2}{7}, R_1 \to \frac{-R_1}{4}} \left(\begin{array}{ccc|c} 1 & \frac{1}{4} & 0 & -4 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 15 \end{array}\right) \tag{6.2.1.6}$$

$$\xleftrightarrow{R_1 \to R_1-\frac{1}{4}R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 15 \end{array}\right) \tag{6.2.1.7}$$

So, from (6.2.1.7)

$$\mathbf{u} = -\begin{pmatrix} 3 \\ 4 \end{pmatrix}, \quad f = 15. \tag{6.2.1.8}$$

See Fig. 6.2.1.1.

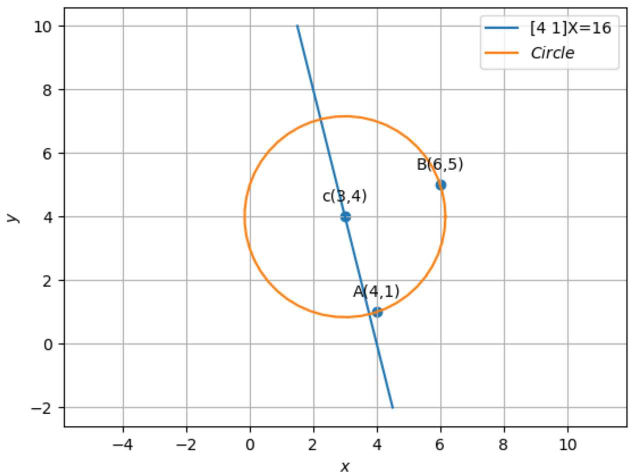


Fig. 6.2.1.1

6.2.2 Find the equation of the circle passing through the points $\mathbf{x}_1(2,3)$ and $\mathbf{x}_2(-1,1)$ and whose centre is on the line $x - 3y - 11 = 0$.

Solution: Substituting numerical values in (6.1.2.2),

$$\begin{pmatrix} 4 & 6 & 1 \\ -2 & 2 & 1 \\ -1 & 3 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ f \end{pmatrix} = \begin{pmatrix} -13 \\ -2 \\ 11 \end{pmatrix} \tag{6.2.2.1}$$

yielding

$$\mathbf{u} = \frac{1}{2}\begin{pmatrix} -7 \\ 5 \end{pmatrix}, \quad f = -14. \tag{6.2.2.2}$$

See Fig. 6.2.2.1.

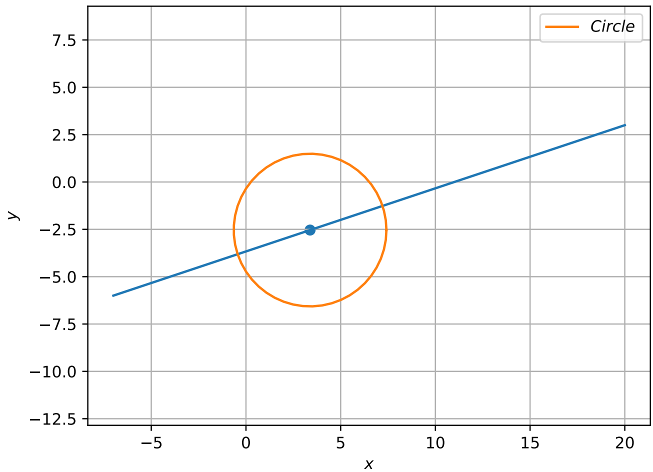


Fig. 6.2.2.1

6.2.3 Find the equation of the circle with radius 5 whose centre lies on x-axis and passes through the point $(2, 3)$.
Solution: See Fig. 6.2.3.1.

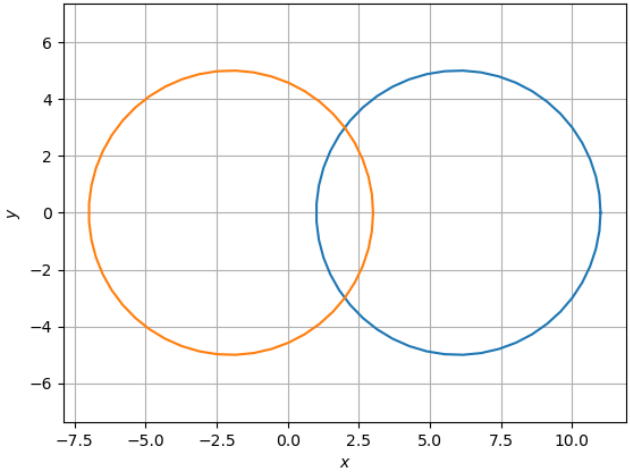


Fig. 6.2.3.1

From the given information, the following equations can be formulated using (A.7.1.1).

$$\|\mathbf{P}\|^2 + 2\mathbf{u}^\top\mathbf{P} + f = 0 \tag{6.2.3.1}$$

$$\mathbf{u} = k\mathbf{e}_1 \tag{6.2.3.2}$$

$$\|\mathbf{u}\|^2 - f = r^2 \tag{6.2.3.3}$$

where

$$\mathbf{P} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \text{ and } r = 5 \tag{6.2.3.4}$$

From (6.2.3.1) and (6.2.3.3),

$$\|\mathbf{P}\|^2 + 2\mathbf{u}^\top\mathbf{P} + \|\mathbf{u}\|^2 = r^2 \tag{6.2.3.5}$$

Substituting from (6.2.3.2) in the above,

$$k^2 + 2k\mathbf{e}_1^\top\mathbf{P} + \|\mathbf{P}\|^2 - r^2 = 0 \tag{6.2.3.6}$$

resulting in

$$k = -\mathbf{e}_1^\top\mathbf{P} \pm \sqrt{\left(\mathbf{e}_1^\top\mathbf{P}\right)^2 + r^2 - \|\mathbf{P}\|^2} \tag{6.2.3.7}$$

Substituting numerical values,

$$k = 2, -6 \tag{6.2.3.8}$$

resulting in circles with centre

$$-\mathbf{u} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 6 \\ 0 \end{pmatrix}. \tag{6.2.3.9}$$

This is verified in Fig. (6.2.3.1).

6.2.4 Find the equation of a circle with centre $(2, 2)$ and passing through the point $(4, 5)$.
Solution: From the given information

$$\mathbf{u} = -\begin{pmatrix} 2 \\ 2 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 4 \\ 5 \end{pmatrix} \tag{6.2.4.1}$$

$$\implies \|\mathbf{A}\|^2 + 2\mathbf{u}^\top \mathbf{A} + f = 0 \tag{6.2.4.2}$$

$$\implies f = -\|\mathbf{A}\|^2 - 2\mathbf{u}^\top \mathbf{A} = -5 \tag{6.2.4.3}$$

Hence the equation of circle is

$$\|\mathbf{x}\|^2 + 2\begin{pmatrix} -2 & -2 \end{pmatrix}\mathbf{x} - 5 = 0 \tag{6.2.4.4}$$

See Fig. 6.2.4.1.

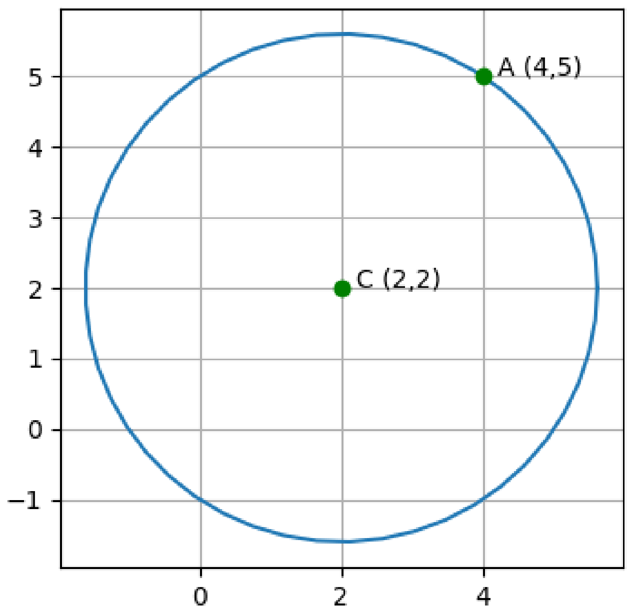


Fig. 6.2.4.1

6.2.5 Does the point $(-2.5, 3.5)$ lie inside, outside or on the circle $x^2 + y^2 = 25$?
Solution: See Table 6.2.5.

Condition	Inference
$\|\mathbf{x}-\mathbf{O}\|^2 < r^2$	point lies inside the circle
$\|\mathbf{x}-\mathbf{O}\|^2 > r^2$	point lies outside the circle
$\|\mathbf{x}-\mathbf{O}\|^2 = r^2$	point lies on the circle

TABLE 6.2.5

The given circle equation can be expressed as

$$\|\mathbf{x}\|^2 = 25 \tag{6.2.5.1}$$

Let,

$$\mathbf{P} = \begin{pmatrix} -2.5 \\ 3.5 \end{pmatrix} \tag{6.2.5.2}$$

Since

$$\|\mathbf{P}-\mathbf{O}\|^2 = 18.5 < 25, \tag{6.2.5.3}$$

the point lies inside the given circle. See Fig. 6.2.5.1.

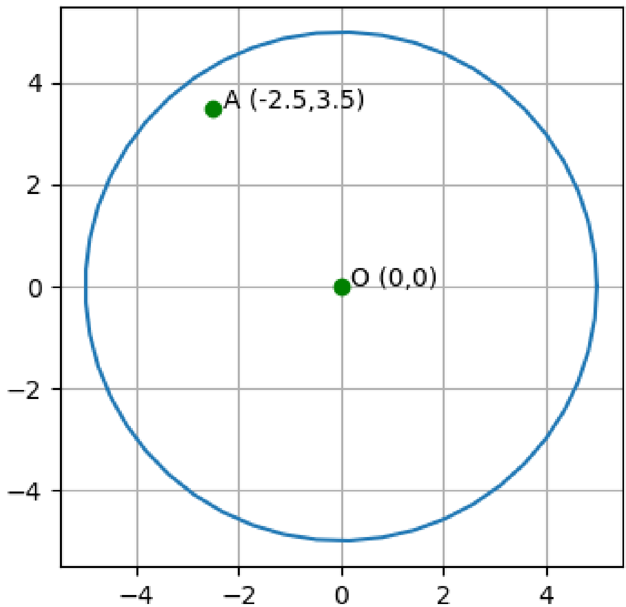


Fig. 6.2.5.1

6.2.6 Find the centre of a circle passing though the points $(6,-6),(3,-7)$ and $(3,3)$.
Solution: Substituting numerical values in (6.1.3.1),

$$\begin{pmatrix} 6 & -14 & 1 \\ 12 & -12 & 1 \\ 6 & 6 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ f \end{pmatrix} = \begin{pmatrix} -58 \\ -72 \\ -18 \end{pmatrix} \tag{6.2.6.1}$$

yielding

$$\mathbf{u} = \begin{pmatrix} -3 \\ 2 \end{pmatrix} \tag{6.2.6.2}$$

$$f = -12 \tag{6.2.6.3}$$

See Fig. 6.2.6.1.

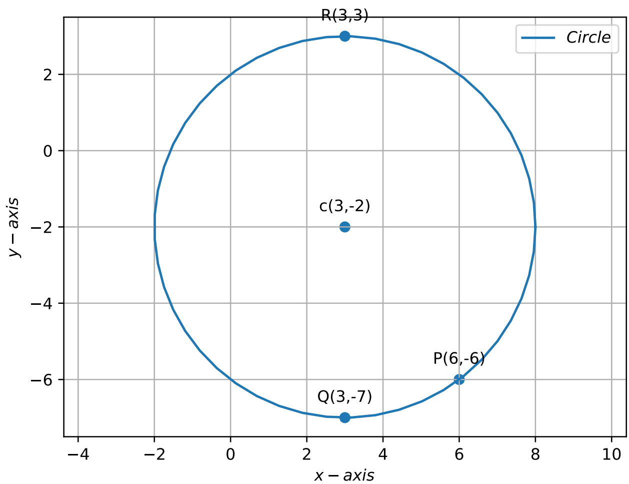


Fig. 6.2.6.1

6.2.7 Find the equation of the circle passing through $(0,0)$ and making intercepts a and b on the coordinate axes.

In each of the following exercises, find the equation of the circle with the following parameters

6.2.8 centre $(0,2)$ and radius 2
Solution: Substituting numerical values in (6.1.1.1),

$$\mathbf{u} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}, f = 0 \tag{6.2.8.1}$$

Thus, the equation of circle is obtained as

$$\|\mathbf{x}\|^2 - 2\begin{pmatrix} 0 & 2 \end{pmatrix}\mathbf{x} = 0 \tag{6.2.8.2}$$

See Fig. 6.2.8.1.

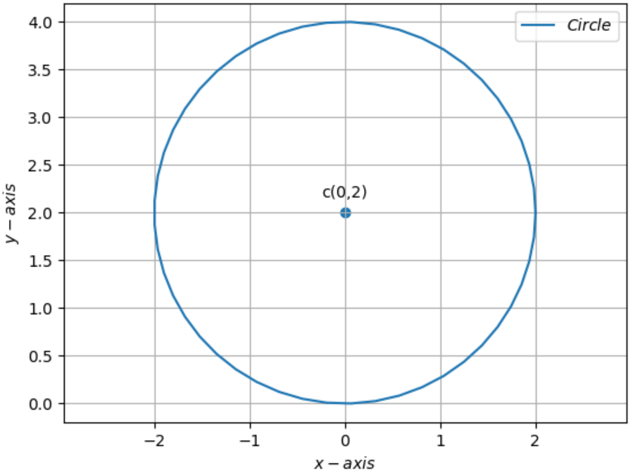


Fig. 6.2.8.1

6.2.9 centre $(-2, 3)$ and radius 4

Solution: Given

$$\mathbf{u} = -\begin{pmatrix} -2 \\ 3 \end{pmatrix}, r = 4. \tag{6.2.9.1}$$

Substituting in (6.1.1.1),

$$f = -3 \tag{6.2.9.2}$$

The equation of the circle is then obtained as

$$\|\mathbf{x}\|^2 + 2\begin{pmatrix} 2 & -3 \end{pmatrix}\mathbf{x} - 3 = 0 \tag{6.2.9.3}$$

See Fig. 6.2.9.1.

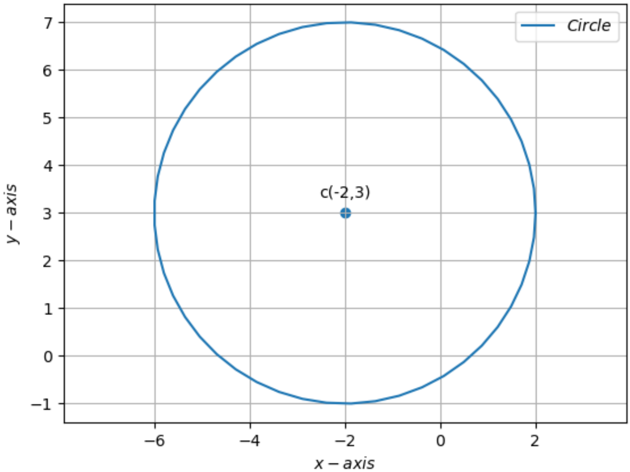


Fig. 6.2.9.1

6.2.10 centre $\left(\frac{1}{2}, \frac{1}{4}\right)$ and radius $\frac{1}{12}$

Solution: Substituting numerical values in (6.1.1.1),

$$f = \frac{11}{36} \tag{6.2.10.1}$$

Thus, the equation of the circle is

$$\|\mathbf{x}\|^2 + \begin{pmatrix} -1 & -\frac{1}{2} \end{pmatrix} \mathbf{x} + \frac{11}{36} = 0 \tag{6.2.10.2}$$

See Fig. 6.2.10.1.

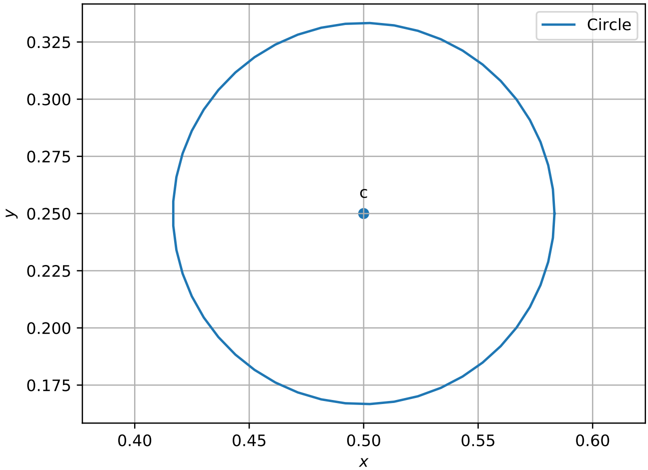


Fig. 6.2.10.1

6.2.11 centre $(1, 1)$ and radius $\sqrt{2}$

Solution: Substituting

$$r = \sqrt{2}, \ \mathbf{u} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \tag{6.2.11.1}$$

in (6.1.1.1),

$$f = 0 \tag{6.2.11.2}$$

Thus, the equation of the circle is

$$\|\mathbf{x}\|^2 - 2\begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} = 0 \tag{6.2.11.3}$$

See Fig. 6.2.11.1.

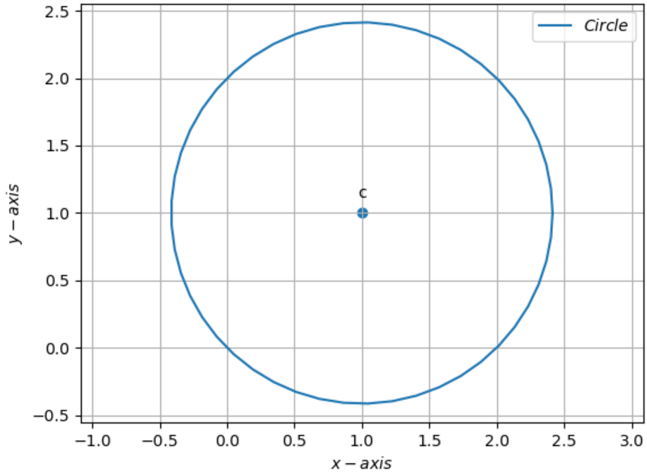


Fig. 6.2.11.1

In each of the following exercises, find the centre and radius of the circles.

6.2.12 $x^2 + y^2 + 10x - 6y - 2 = 0.$

Solution: The circle parameters are

$$\mathbf{u} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}, \ f = -2 \tag{6.2.12.1}$$

$$\implies \mathbf{c} = \begin{pmatrix} -5 \\ 3 \end{pmatrix}, \ r = \sqrt{\|\mathbf{u}\|^2 - f} = 6 \tag{6.2.12.2}$$

See Fig. 6.2.12.1.

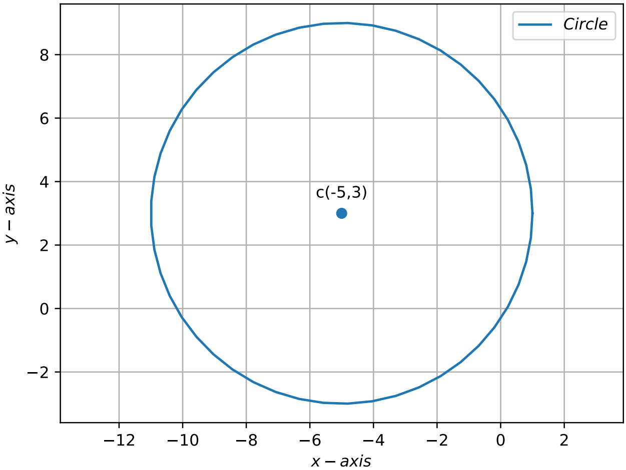


Fig. 6.2.12.1

6.2.13 $x^2 + y^2 - 4x - 8y - 45 = 0$

Solution: The given circle can be expressed as

$$\|\mathbf{x}\|^2 + 2\begin{pmatrix}-2 & -4\end{pmatrix}\mathbf{x} - 45 = 0 \tag{6.2.13.1}$$

where

$$\mathbf{u} = \begin{pmatrix}-2\\-4\end{pmatrix}, f = -45 \tag{6.2.13.2}$$

$$\implies \mathbf{c} = \begin{pmatrix}2\\4\end{pmatrix}, r = \sqrt{65}. \tag{6.2.13.3}$$

See Fig. 6.2.13.1.

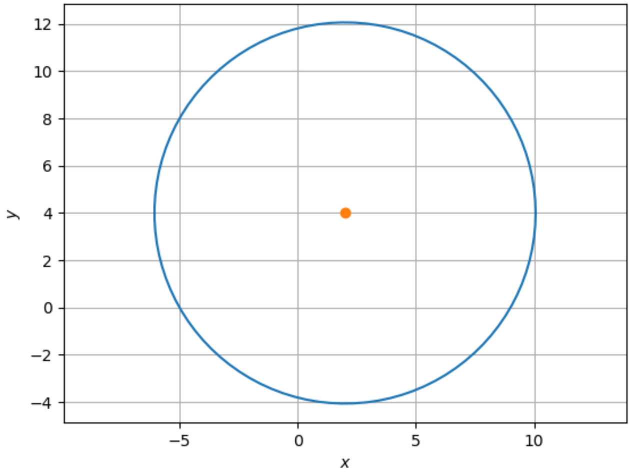


Fig. 6.2.13.1

6.2.14 $x^2 + y^2 - 8x + 10y - 12 = 0$

Solution: From the given informtion,

$$\mathbf{u} = \begin{pmatrix} -4 \\ 5 \end{pmatrix}, \ f = -12 \tag{6.2.14.1}$$

$$\implies \mathbf{c} = \begin{pmatrix} 4 \\ -5 \end{pmatrix}, \tag{6.2.14.2}$$

$$r = \sqrt{\|\mathbf{u}\|^2 - f} = \sqrt{53} \tag{6.2.14.3}$$

See Fig. 6.2.14.1.

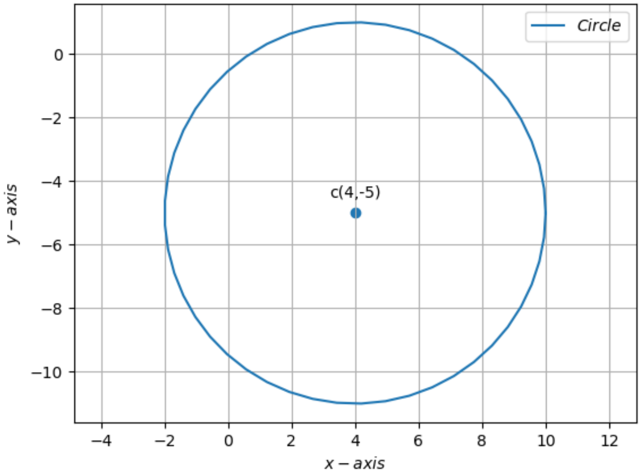


Fig. 6.2.14.1

6.2.15 $2x^2 + 2y^2 - x = 0$

Solution: The given equation can be expressed as

$$\|\mathbf{x}\|^2 + 2\begin{pmatrix} \frac{-1}{4} & 0 \end{pmatrix}\mathbf{x} = 0 \tag{6.2.15.1}$$

The centre of circle is then given by

$$\mathbf{u} = -\mathbf{c} = \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix} \tag{6.2.15.2}$$

and the radius of circle is obtained as

$$r = \sqrt{\|\mathbf{u}\|^2 - f} = \frac{1}{4} \tag{6.2.15.3}$$

See Fig. 6.2.15.1.

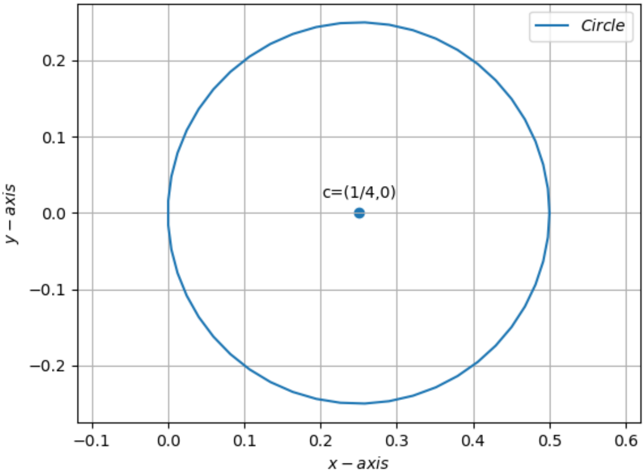


Fig. 6.2.15.1

6.2.16 The area of the circle centred at (1,2) and passing through (4,6) is
   a) 5π
   b) 10π
   c) 25π
   d) none of these

6.2.17 Equation of the circle with centre on the Y-axis and passing through the orgin and the point (2,3) is
   a) $x^2 + y^2 + 6x + 6y + 3 = 0$
   b) $x^2 + y^2 - 6x - 6y - 9 = 0$
   c) $x^2 + y^2 - 6x - 6y + 9 = 0$
   d) none of these

6.2.18 Equation of the circle with centre on the y-axis and passing through the origin and the point (2,3) is
   a) $x^2 + y^2 + 13y = 0$
   b) $3x^2 + 3y^2 + 13x + 3 = 0$
   c) $6x^2 + 6y^2 - 13x = 0$
   d) $x^2 + y^2 + 13x + 3 = 0$

6.2.19 Find the equation of a circle concentric with the circle $x^2 + y^2 - 6x + 12y + 15 = 0$ and has double of its area.

6.2.20 If one end of a diameter of the circle $x^2 + y^2 - 4x - 6y + 11 = 0$ is (3,4), then find the coordinate of the other end of the diameter.

6.2.21 Find the equation of the circle having (1,-2) as its centre and passing through $3x+y = 14, 2x + 5y = 18$.

6.2.22 If the lines 2x-3y=5 and 3x-4y=7 are the diameters of a circie of area 154 square units, then obtian the equation of the circle.

6.2.23 Find the equation of the circle which passes through the points (2,3) and (4,5) and the centre lies on the straight line y-4x+3=0.

6.2.24 Find the equation of a circle passing through the point (7,3) having radius 3 units and whose centre lies on the line $y = x - 1$.

State whether the statements are True or False

6.2.25 The line $x^2 + 3y = 0$ is a diameter of the circle $x^2 + y^2 + 6x + 2y = 0$.

6.2.26 The point (1,2) lies inside the circle $x^2 + y^2 - 2x + 6y + 1 = 0$.

6.3 Miscellaneous

6.3.1 Find the equation of the circle passing through $(0, 0)$ and making intercepts a and b on the coordinate axes.

6.3.2 Find the equation of a circle with centre $(-a, -b)$ and radius $\sqrt{a^2 - b^2}$.
Solution: From (6.1.1.1),

$$\mathbf{u} = \begin{pmatrix} a \\ b \end{pmatrix}, \ f = 2b^2 \tag{6.3.2.1}$$

Thus, the equation of circle is

$$\|\mathbf{x}\|^2 + 2\begin{pmatrix} a & b \end{pmatrix}\mathbf{x} + 2b^2 = 0 \tag{6.3.2.2}$$

6.3.3 The equation of a circle with origin as centre and passing through the vertices of an equilateral triangle whose median is of length $3a$ is

a) $x^2 + y^2 = 9a^2$
b) $x^2 + y^2 = 16a^2$
c) $x^2 + y^2 = 4a^2$
d) $x^2 + y^2 = a^2$

6.3.4 Show that the point (x, y) given by $x = \frac{2at}{1+t^2}$ and $y = \frac{a(1-t^2)}{1+t^2}$ lies on a circle for all real values of t such that -1≤ t ≤1 where a is any given real number.

6.3.5 If a circle passes through the point $(0, 0), (a, 0)$ and $(0, b)$ then find the coordinates of its centre.

6.3.6 The equation of the circle circumscribing the triangle whose sides are the lines $y = x + 2, 3y = 4x, 2y = 3x$ is ______

7 Conics

7.1 Formulae

7.1.1. The equation of a conic with directrix $\mathbf{n}^\top \mathbf{x} = c$, eccentricity e and focus $\mathbf{F}$ is given by

$$g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{7.1.1.1}$$

where

$$\mathbf{V} = \|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{n}\mathbf{n}^\top, \tag{7.1.1.2}$$

$$\mathbf{u} = ce^2\mathbf{n} - \|\mathbf{n}\|^2 \mathbf{F}, \tag{7.1.1.3}$$

$$f = \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2e^2 \tag{7.1.1.4}$$

7.1.2. The eccentricity, directrices and foci of (7.1.1.1) are given by

$$e = \sqrt{1 - \frac{\lambda_1}{\lambda_2}} \tag{7.1.2.1}$$

$$\begin{aligned} \mathbf{n} &= \sqrt{\lambda_2}\mathbf{p}_1, \\ c &= \begin{cases} \frac{e\mathbf{u}^\top\mathbf{n} \pm \sqrt{e^2(\mathbf{u}^\top\mathbf{n})^2 - \lambda_2(e^2-1)(\|\mathbf{u}\|^2 - \lambda_2 f)}}{\lambda_2 e(e^2-1)} & e \neq 1 \\ \frac{\|\mathbf{u}\|^2 - \lambda_2 f}{2\mathbf{u}^\top\mathbf{n}} & e = 1 \end{cases} \end{aligned} \tag{7.1.2.2}$$

$$\mathbf{F} = \frac{ce^2\mathbf{n} - \mathbf{u}}{\lambda_2} \tag{7.1.2.3}$$

7.1.3. For a symmetric matrix, from (A.7.7.1), we have the eigendecomposition

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^\top \tag{7.1.3.1}$$

where

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix}, \quad \mathbf{P}^\top\mathbf{P} = \mathbf{I} \tag{7.1.3.2}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{7.1.3.3}$$

7.1.4. Using the affine transformation in (2.1.15.1), the conic in (7.1.1.1) can be expressed in standard form as

$$\mathbf{y}^\top \left(\frac{\mathbf{D}}{f_0}\right) \mathbf{y} = 1 \qquad |\mathbf{V}| \neq 0 \tag{7.1.4.1}$$

$$\mathbf{y}^\top \mathbf{D} \mathbf{y} = -\eta \mathbf{e}_1^\top \mathbf{y} \qquad |\mathbf{V}| = 0 \tag{7.1.4.2}$$

where

$$f_0 = \mathbf{u}^\top \mathbf{V}^{-1} \mathbf{u} - f \neq 0 \tag{7.1.4.3}$$

$$\eta = 2\mathbf{u}^\top \mathbf{p}_1 \tag{7.1.4.4}$$

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{7.1.4.5}$$

Solution: See Appendix B.1.5.

7.1.5.

a) The directrices for the standard conic are given by

$$\mathbf{e}_1^\top \mathbf{y} = \pm\sqrt{\left|\frac{f_0\lambda_2}{\lambda_1\left(\lambda_2-\lambda_1\right)}\right|} \qquad e \neq 1 \tag{7.1.5.1}$$

$$\mathbf{e}_1^\top \mathbf{y} = \frac{\eta}{2\lambda_2} \qquad e = 1 \tag{7.1.5.2}$$

b) The foci of the standard conic are given by

$$\mathbf{F} = \begin{cases} \pm\sqrt{\left|f_0\left(\frac{1}{\lambda_2}-\frac{1}{\lambda_1}\right)\right|}\mathbf{e}_1 & e \neq 1 \\ -\frac{\eta}{4\lambda_2}\mathbf{e}_1 & e = 1 \end{cases} \tag{7.1.5.3}$$

7.1.6. The equation of the minor and major axes for the ellipse/hyperbola are respectively given by

$$\mathbf{p}_i^\top\left(\mathbf{x}-\mathbf{c}\right) = 0, i = 1, 2 \tag{7.1.6.1}$$

The axis of symmetry for the parabola is also given by (7.1.6.1).

7.1.7. The center of the standard ellipse/hyperbola, defined to be the mid point of the line joining the foci, is the origin.

7.1.8. The principal (major) axis of the standard ellipse/hyperbola, defined to be the line joining the two foci is the x-axis.

7.1.9. The minor axis of the standard ellipse/hyperbola, defined to be the line orthogonal to the x-axis is the y-axis.

7.1.10. The axis of symmetry of the standard parabola, defined to be the line perpendicular to the directrix and passing through the focus, is the x- axis.

7.1.11. The point where the parabola intersects its axis of symmetry is called the vertex. For the standard parabola, the vertex is the origin.

7.1.12. The *focal length* of the standard parabola, , defined to be the distance between the vertex and the focus, measured along the axis of symmetry, is $\left|\frac{\eta}{4\lambda_2}\right|$

7.1.13. For the standard hyperbola/ellipse, the length of the major axis is

$$2\sqrt{\left|\frac{f_0}{\lambda_1}\right|} \tag{7.1.13.1}$$

and the minor axis is

$$2\sqrt{\left|\frac{f_0}{\lambda_2}\right|} \tag{7.1.13.2}$$

Solution: See Appendix B.3.4.

7.1.14. The latus rectum of a conic section is the chord that passes through the focus and is perpendicular to the major axis. The length of the latus rectum for a conic is given by

$$l = \begin{cases} 2\frac{\sqrt{|f_0\lambda_1|}}{\lambda_2} & e \neq 1 \\ \frac{\eta}{\lambda_2} & e = 1 \end{cases} \tag{7.1.14.1}$$

Solution: See Appendix B.3.6.

7.2 Equation

In the each of the following exercises, find the coordinates of the focus, vertex, eccentricity, axis of the conic section, the equation of the directrix and the length of the latus rectum.

7.2.1 $y^2 = 12x$

Solution: See Table 7.2.5 and Fig. 7.2.1.1.

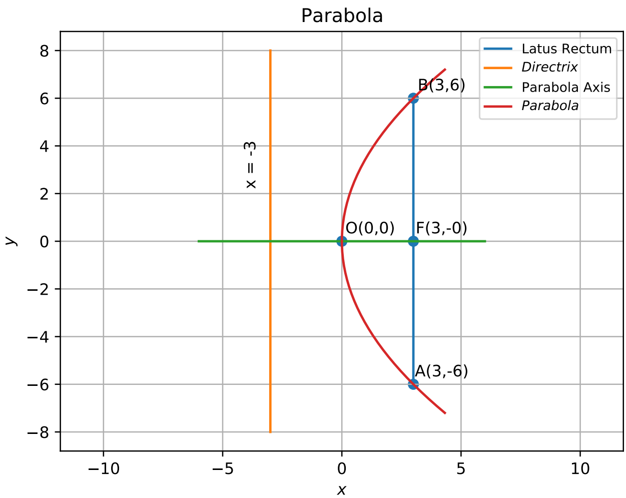


Fig. 7.2.1.1

7.2.2 $y^2 = -8x$

Solution: See Table 7.2.5 and Fig. 7.2.2.1.

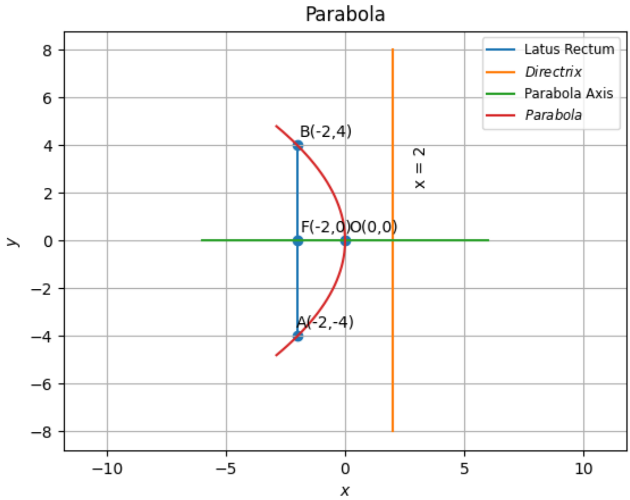


Fig. 7.2.2.1: Graph

7.2.3 $\frac{x^2}{36} + \frac{y^2}{16} = 1$
Solution: See Table 7.2.5 and Fig. 7.2.3.1.

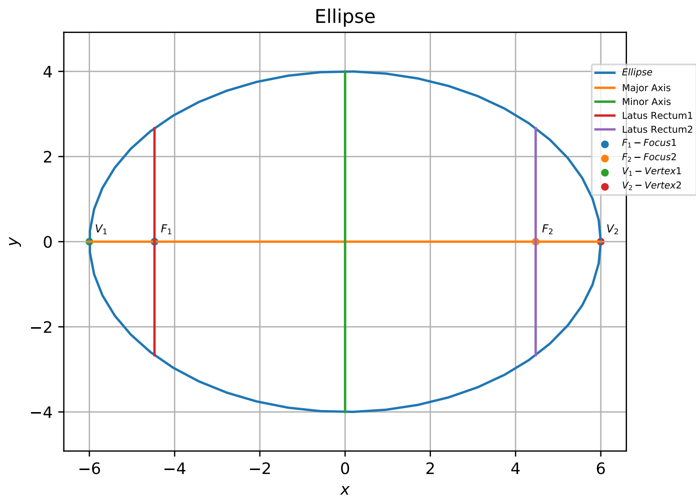


Fig. 7.2.3.1

7.2.4 $\frac{x^2}{16} + \frac{y^2}{9} = 1$
Solution: See Table 7.2.5 and Fig. 7.2.4.1.

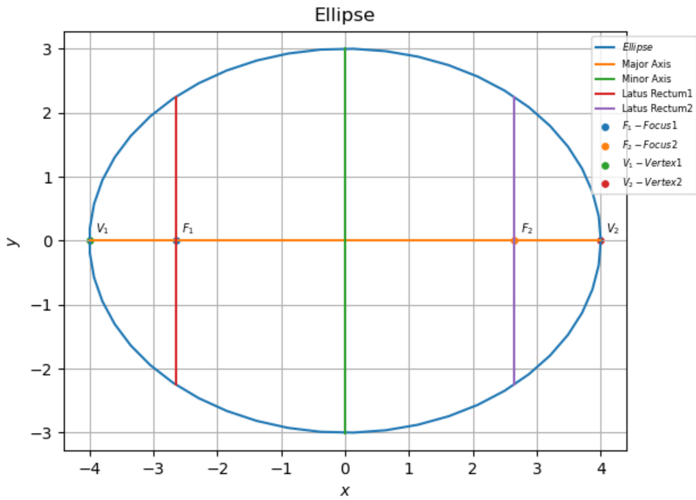


Fig. 7.2.4.1

7.2.5 $\frac{x^2}{16} - \frac{y^2}{9} = 1.$

Solution: See Table 7.2.5 and Fig. 7.2.5.1.

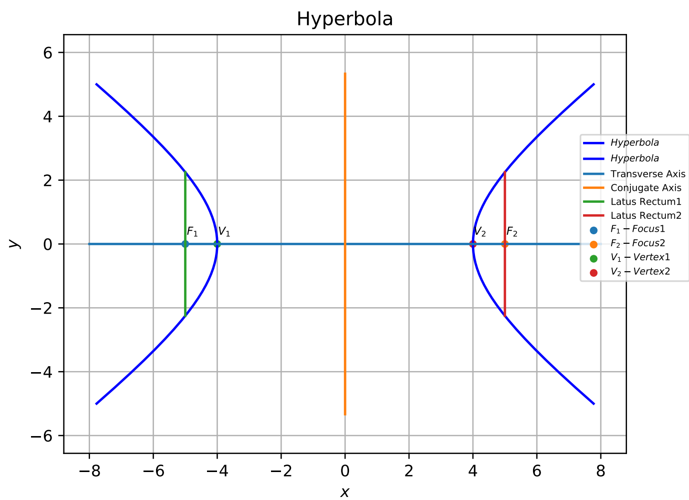


Fig. 7.2.5.1

TABLE 7.2.5

Input				Output		
Conic	$\mathbf{V}$	$\mathbf{u}$	f	$\mathbf{F}$	Directrix	Latus Rectum
$y^2 = 12x$	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$	$-6\mathbf{e}_1$	0	$3\mathbf{e}_1$	$\mathbf{e}_1^\top \mathbf{x} = -3$	12
$y^2 = -8x$	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$	$4\mathbf{e}_1$	0	$2\mathbf{e}_1$	$\mathbf{e}_1^\top \mathbf{x} = 2$	8
$\frac{x^2}{36} + \frac{y^2}{16} = 1$	$\begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix}$	$\mathbf{0}$	-144	$2\sqrt{5}\mathbf{e}_1$	$\mathbf{e}_1^\top \mathbf{x} = \frac{18}{\sqrt{5}}$	$\frac{16}{3}$
$\frac{x^2}{16} + \frac{y^2}{9} = 1$	$\begin{pmatrix} 9 & 0 \\ 0 & 16 \end{pmatrix}$	$\mathbf{0}$	-144	$\pm\sqrt{7}\mathbf{e}_1$	$\mathbf{e}_1^\top \mathbf{x} = \frac{16}{\sqrt{7}}$	$\frac{9}{2}$
$\frac{x^2}{16} - \frac{y^2}{9} = 1$	$\begin{pmatrix} 9 & 0 \\ 0 & -16 \end{pmatrix}$	$\mathbf{0}$	-144	$\pm 5\mathbf{e}_1$	$\mathbf{e}_1^\top \mathbf{x} = \frac{16}{5}$	$\frac{9}{2}$

7.2.6 $\frac{x^2}{4} + \frac{y^2}{25} = 1$

Solution: From Table 7.2.10, it can be seen that this is not a standard ellipse, since $\lambda_1 > \lambda_2$. Hence $\mathbf{P}$ plays a role and we need to use the affine transformation

$$\mathbf{x} = \mathbf{P}\mathbf{y} \tag{7.2.6.1}$$

So the value of λ_1 and λ_2 need to be interchanged for all calculations and in (7.1.5.1), $\mathbf{e}_2$ becomes the normal vector. See Fig. 7.2.6.1.

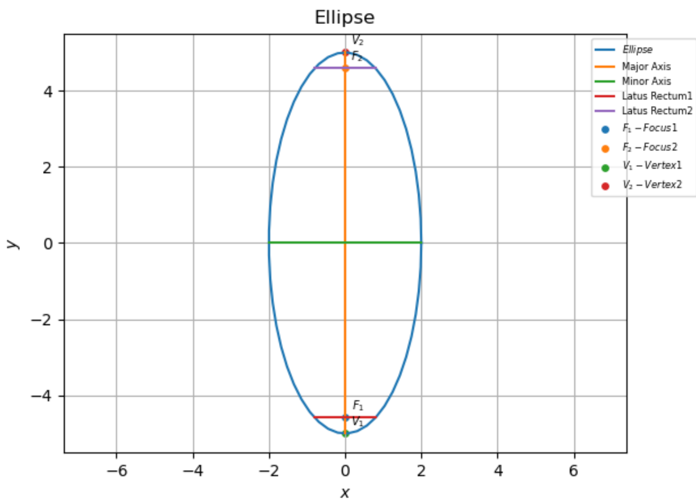


Fig. 7.2.6.1

7.2.7 $5y^2 - 9x^2 = 36.$

Solution:
See Table 7.2.10 and Fig. 7.2.7.1. In Table 7.2.10, **P** shifts the negative eigenvalue to get the hyperbola in standard form.

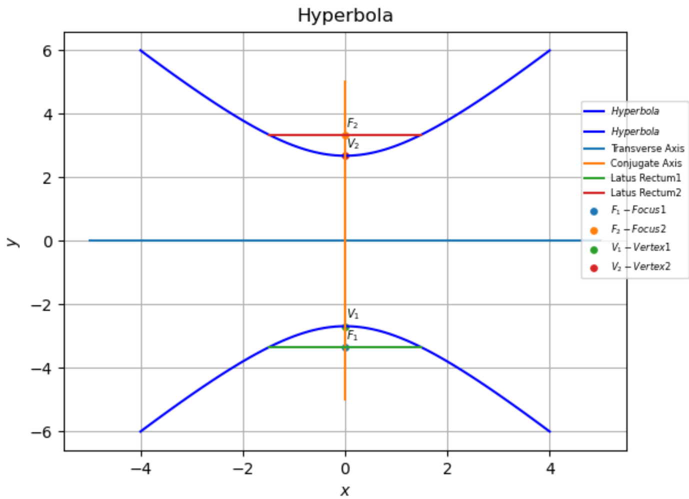


Fig. 7.2.7.1

7.2.8 $\frac{y^2}{9} - \frac{x^2}{27} = 1.$
Solution:
See Table 7.2.10 and Fig. 7.2.8.1.

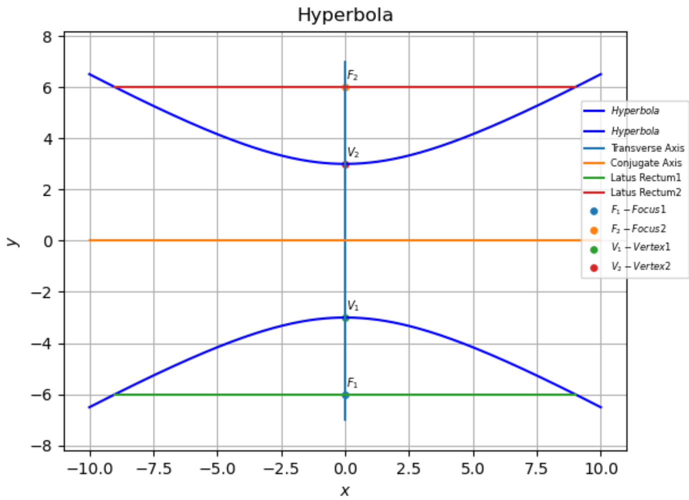


Fig. 7.2.8.1

7.2.9 $x^2 = -16y$

Solution: See Table 7.2.10 and Fig. 7.2.9.1.

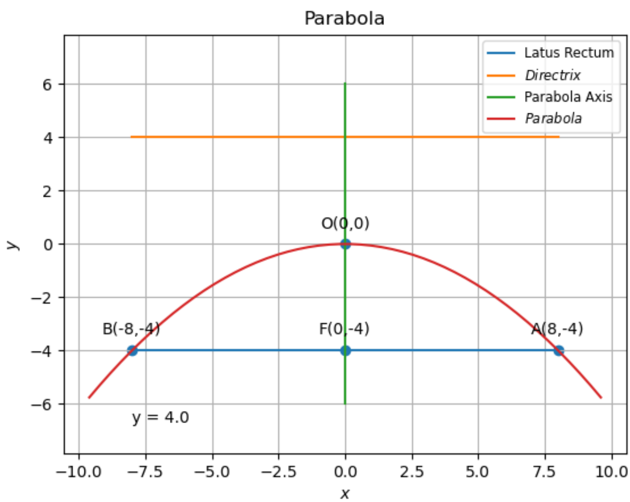


Fig. 7.2.9.1

7.2.10 x^2=6y

Solution: See Table 7.2.10 and Fig. 7.2.10.1.

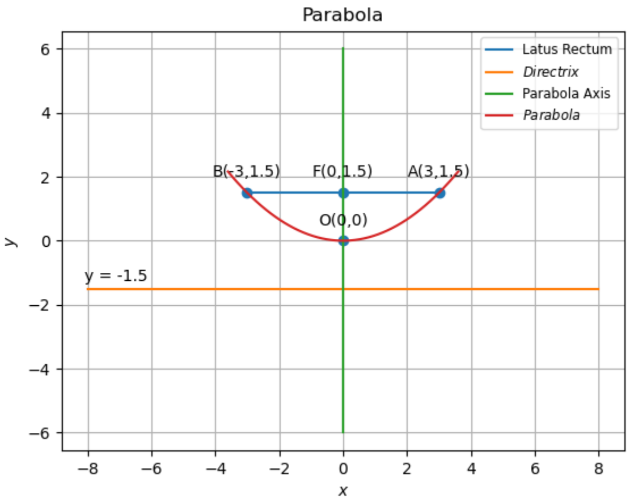


Fig. 7.2.10.1

TABLE 7.2.10

Input				Intermediate	Output		
Conic	$\mathbf{V}$	$\mathbf{u}$	f	$\mathbf{P}$	$\mathbf{F}$	Directrix	Latus Rectum
$\frac{x^2}{4}+\frac{y^2}{25}=1$	$\begin{pmatrix}25 & 0\\ 0 & 4\end{pmatrix}$	$\mathbf{0}$	-100	$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$	$\pm\sqrt{21}\mathbf{e}_2$	$\mathbf{e}_2^\top\mathbf{x}=\pm\frac{25}{\sqrt{21}}$	$\frac{8}{5}$
$5y^2-9x^2=36$	$\begin{pmatrix}-9 & 0\\ 0 & 5\end{pmatrix}$	$\mathbf{0}$	-36	$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$	$\pm3\sqrt{\frac{5}{14}}\mathbf{e}_2$	$\mathbf{e}_2^\top\mathbf{x}=\pm\frac{18}{\sqrt{70}}$	$4\frac{\sqrt{5}}{3}$
$\frac{y^2}{9}-\frac{x^2}{27}=1$	$\begin{pmatrix}-1 & 0\\ 0 & 3\end{pmatrix}$	$\mathbf{0}$	-27	$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$	$\pm6\mathbf{e}_2$	$\mathbf{e}_2^\top\mathbf{x}=\pm\frac{3}{2}$	18
$x^2=-16y$	$\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$	$\begin{pmatrix}0\\ 8\end{pmatrix}$	0	$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$	$-4\mathbf{e}_2$	$\mathbf{e}_2^\top\mathbf{x}=4$	16
$x^2=6y$	$\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$	$-\begin{pmatrix}0\\ 3\end{pmatrix}$	0	$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$	$\frac{3}{2}\mathbf{e}_2$	$\mathbf{e}_2^\top\mathbf{x}=-\frac{3}{2}$	6

7.2.11 x^2=-9y

7.2.12 $\frac{x^2}{25}+\frac{y^2}{100}=1$

7.2.13 $\frac{x^2}{49}+\frac{y^2}{36}=1$

7.2.14 $\frac{x^2}{100}+\frac{y^2}{400}=1$

7.2.15 $36x^2+4y^2=144$

7.2.16 $16x^2+y^2=16$

7.2.17 $4x^2+9y^2=36$

7.2.18 $y^2=10x$

In each of the following exercises, find the equation of the conic, that satisfies the given conditions.

7.2.19 foci $(\pm4,0)$, latus rectum of length 12.

Solution: The given information is available in Table 7.2.19. Since two foci are given, the conic cannot be a parabola.

a) The direction vector of F_1F_2 is the normal vector of the directrix. Hence,

$$\mathbf{n} = \mathbf{F_1} - \mathbf{F_2} \equiv \mathbf{e}_1 \tag{7.2.19.1}$$

Substituting in (7.1.1.2), (7.1.1.3) and (7.1.1.4),

$$\mathbf{V} = \begin{pmatrix} 1-e^2 & 0 \\ 0 & 1 \end{pmatrix} \tag{7.2.19.2}$$

$$\mathbf{u} = ce^2\mathbf{e}_1 - \mathbf{F} \tag{7.2.19.3}$$

$$f = 16 - c^2e^2 \tag{7.2.19.4}$$

b) From (7.2.19.2),

$$\lambda_1 = 1 - e^2, \ \lambda_2 = 1 \tag{7.2.19.5}$$

which upon substituting in (7.1.14.1), along with the value of the latus rectum from Table 7.2.19

$$6\left(1 - e^2\right) = \sqrt{|f|} \tag{7.2.19.6}$$

c) The centre of the conic is given by

$$\mathbf{c} = \frac{\mathbf{F_1} + \mathbf{F_2}}{2} = \mathbf{0} \tag{7.2.19.7}$$

From (7.2.19.2), it is obvious that $\mathbf{V}$ is invertible. Hence, from (7.2.19.7) and (B.1.5.9),

$$\mathbf{u} = \mathbf{0} \tag{7.2.19.8}$$

Substituting the above in (7.2.19.3),

$$\mathbf{F} = ce^2\mathbf{e}_1 \implies \|\mathbf{F}\| = 4 = ce^2 \tag{7.2.19.9}$$

d) From (7.1.4.3), (7.2.19.8) and (7.2.19.4),

$$36\left(1 - e^2\right)^2 = 16 - c^2e^2 \tag{7.2.19.10}$$

From (7.2.19.9) and (7.2.19.10)

$$\frac{4}{e\sqrt{e^2 - 1}} = 6 \tag{7.2.19.11}$$

$$\implies 9e^2\left(e^2 - 1\right) = 4 \tag{7.2.19.12}$$

$$\implies 9e^4 - 9e^2 - 4 = 0 \tag{7.2.19.13}$$

$$\text{or, } \left(3e^2 - 4\right)\left(12e^2 + 1\right) = 0 \tag{7.2.19.14}$$

yielding

$$e = \frac{4}{3} \tag{7.2.19.15}$$

as the only viable solution.

The equation of the conic is then obtained as

$$\mathbf{x}^\top \begin{pmatrix} -\frac{1}{3} & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + 4 = 0 \tag{7.2.19.16}$$

See Fig. 7.2.19.1.

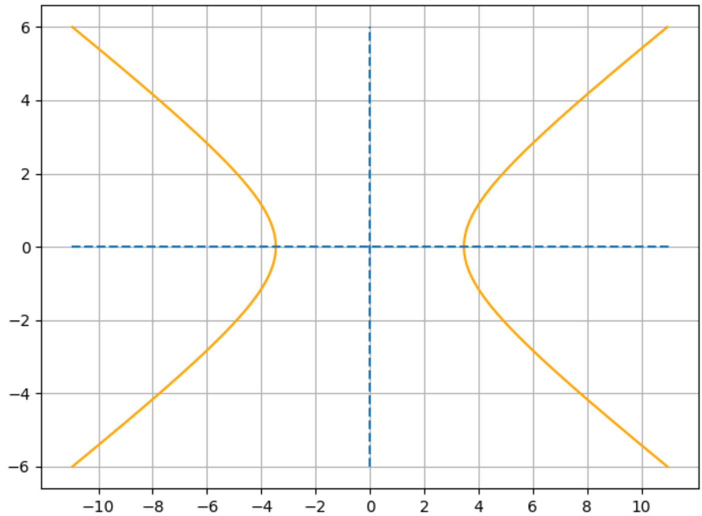


Fig. 7.2.19.1

Parameter	Description	Value
$\mathbf{F_1}$	Focus 1 of hyperbola	$\begin{pmatrix} 4 \\ 0 \end{pmatrix}$
$\mathbf{F_2}$	Focus 2 of hyperbola	$\begin{pmatrix} -4 \\ 0 \end{pmatrix}$
l	Length of latus rectum	12

TABLE 7.2.19

7.2.20 eccentricity $e = \frac{4}{3}$, vertices

$$\mathbf{P_1} = \begin{pmatrix} 7 \\ 0 \end{pmatrix}, \ \mathbf{P_2} = \begin{pmatrix} -7 \\ 0 \end{pmatrix} \tag{7.2.20.1}$$

Solution: The major axis of a conic is the chord which passes through the vertices of the conic. The direction vector of the major axis in this case is

$$\mathbf{P}_2 - \mathbf{P}_1 \equiv \mathbf{e}_1 = \mathbf{n} \tag{7.2.20.2}$$

which is the normal vector for the directrix. Since $e > 1$, the conic is a hyperbola. Substituting (7.2.20.2) in (7.1.1.2), (7.1.1.3) and (7.1.1.4),

$$\mathbf{V} = \begin{pmatrix} 1-e^2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{7}{9} & 0 \\ 0 & 1 \end{pmatrix} \tag{7.2.20.3}$$

$$\mathbf{u} = ce^2\mathbf{e}_1 - \mathbf{F} \tag{7.2.20.4}$$

$$f = 16 - c^2e^2 \tag{7.2.20.5}$$

Thus,

$$\mathbf{V} = \begin{pmatrix} 1-e^2 & 0 \\ 0 & 1 \end{pmatrix} \tag{7.2.20.6}$$

$$\mathbf{u} = ce^2\mathbf{e}_1 - \mathbf{F} \tag{7.2.20.7}$$

$$f = \|\mathbf{F}\|^2 - c^2e^2 \tag{7.2.20.8}$$

The centre of the hyperbola is

$$\mathbf{c} = \frac{\mathbf{P}_1 + \mathbf{P}_2}{2} = \mathbf{0} = \mathbf{u} \tag{7.2.20.9}$$

from (B.1.5.9). Substituting $\mathbf{P}_1$ and $\mathbf{P}_2$ in (7.1.1.1),

$$\mathbf{P}_1^\top \mathbf{V}\mathbf{P}_1 + 2\mathbf{u}^\top\mathbf{P}_1 + f = 0 \tag{7.2.20.10}$$

$$\mathbf{P}_2^\top \mathbf{V}\mathbf{P}_2 + 2\mathbf{u}^\top\mathbf{P}_2 + f = 0 \tag{7.2.20.11}$$

$$\implies f = \mathbf{P}_1^\top \mathbf{V}\mathbf{P}_1 = 49\left(e^2 - 1\right) = \frac{343}{9} \tag{7.2.20.12}$$

upon adding (7.2.20.11) and (7.2.20.10) and simplifying. Therefore, the equation of the conic is

$$\mathbf{x}^\top \begin{pmatrix} -\frac{7}{9} & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + \frac{343}{9} = 0 \tag{7.2.20.13}$$

See Fig. 7.2.20.1.

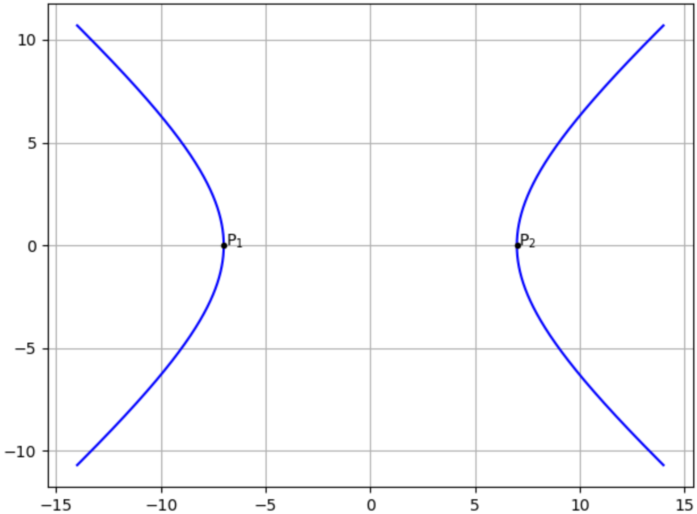


Fig. 7.2.20.1

7.2.21 centre at $\mathbf{c}(0,0)$, major axis on the y-axis and passes through the points $\mathbf{P}(3,2)$ and $\mathbf{Q}(1,6)$.

Solution: Since the major axis is along the y-axis,

$$\mathbf{n} = \mathbf{e}_2 \tag{7.2.21.1}$$

Thus,

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1-e^2 \end{pmatrix} \tag{7.2.21.2}$$

Since

$$\mathbf{c} = \mathbf{0}, \mathbf{u} = \mathbf{0}. \tag{7.2.21.3}$$

From (7.1.1.1),

$$\mathbf{P}^\top \mathbf{V}\mathbf{P} + 2\mathbf{u}^\top \mathbf{P} + f = 0 \tag{7.2.21.4}$$

$$\mathbf{Q}^\top \mathbf{V}\mathbf{Q} + 2\mathbf{u}^\top \mathbf{Q} + f = 0 \tag{7.2.21.5}$$

yielding

$$4e^2 - f = 13 \tag{7.2.21.6}$$

$$36e^2 - f = 37 \tag{7.2.21.7}$$

which can be formulated as the matrix equation

$$\begin{pmatrix} 4 & -1 \\ 36 & -1 \end{pmatrix} \begin{pmatrix} e^2 \\ f \end{pmatrix} = \begin{pmatrix} 13 \\ 37 \end{pmatrix} \tag{7.2.21.8}$$

The augmented matrix is given by,

$$\left(\begin{array}{cc|c} 4 & -1 & 13 \\ 36 & -1 & 37 \end{array}\right) \xleftrightarrow{R_1 \leftarrow -\frac{R_1}{8}} \left(\begin{array}{cc|c} 4 & 0 & 3 \\ 36 & -1 & 37 \end{array}\right)$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - 9R_1} \left(\begin{array}{cc|c} 4 & 0 & 3 \\ 0 & -1 & 10 \end{array}\right) \xleftrightarrow[R_2 \leftarrow -R_2]{R_1 \leftarrow \frac{R_1}{4}} \left(\begin{array}{cc|c} 1 & 0 & \frac{3}{4} \\ 0 & 1 & -10 \end{array}\right)$$

Thus,

$$e^2 = \frac{3}{4}, \ f = -10 \tag{7.2.21.9}$$

and the equation of the conic is given by

$$\mathbf{x}^\top \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{pmatrix} \mathbf{x} - 10 = 0 \tag{7.2.21.10}$$

See Fig. 7.2.21.1.

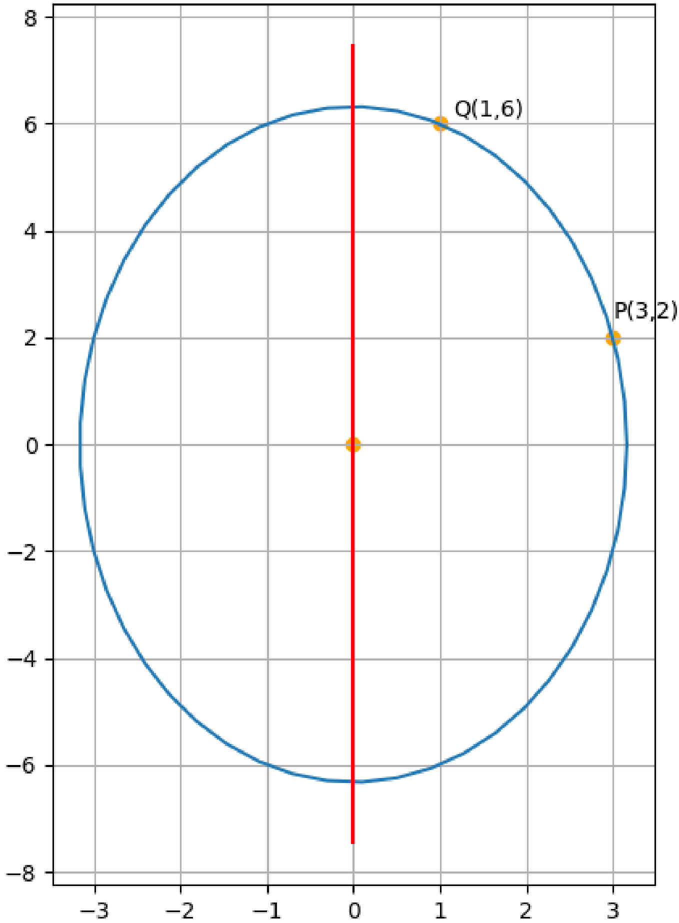


Fig. 7.2.21.1: Graph

7.2.22 major axis on the x-axis and passes through the points (4,3) and (6,2).
Solution: In this case,

$$\mathbf{n} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{7.2.22.1}$$

Thus,

$$\mathbf{V} = \begin{pmatrix} 1 - e^2 & 0 \\ 0 & 1 \end{pmatrix} \tag{7.2.22.2}$$

$$\tag{7.2.22.3}$$

Since

$$\mathbf{c} = \mathbf{0}, \mathbf{u} = \mathbf{0}. \tag{7.2.22.4}$$

From (7.1.1.1),

$$\mathbf{P}^\top \mathbf{V}\mathbf{P} + 2\mathbf{u}^\top \mathbf{P} + f = 0 \tag{7.2.22.5}$$

$$\mathbf{Q}^\top \mathbf{V}\mathbf{Q} + 2\mathbf{u}^\top \mathbf{Q} + f = 0 \tag{7.2.22.6}$$

yielding

$$16e^2 - f = 25 \tag{7.2.22.7}$$

$$36e^2 - f = 40 \tag{7.2.22.8}$$

which can be formulated as the matrix equation

$$\begin{pmatrix} 16 & -1 \\ 36 & -1 \end{pmatrix} \begin{pmatrix} e^2 \\ f \end{pmatrix} = \begin{pmatrix} 25 \\ 40 \end{pmatrix} \tag{7.2.22.9}$$

and can be solved using the augmented matrix.

$$\begin{pmatrix} 16 & -1 & 25 \\ 36 & -1 & 40 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 - R_2} \begin{pmatrix} -20 & 0 & -15 \\ 36 & -1 & 40 \end{pmatrix}$$

$$\xleftrightarrow[R_2 \leftarrow -R_2]{R_1 \leftarrow \frac{R_1}{-5}} \begin{pmatrix} 4 & 0 & 3 \\ -36 & 1 & -40 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 + 9R_1} \begin{pmatrix} 4 & 0 & 3 \\ 0 & 1 & -13 \end{pmatrix}$$

$$\xleftrightarrow{R_1 \leftarrow \frac{R_1}{4}} \begin{pmatrix} 1 & 0 & \frac{3}{4} \\ 0 & 1 & -13 \end{pmatrix}$$

Thus,

$$e^2 = \frac{3}{4}, \ f = -13 \tag{7.2.22.10}$$

and the equation of the conic is given by

$$\mathbf{x}^\top \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - 13 = 0 \tag{7.2.22.11}$$

See Fig. 7.2.22.1.

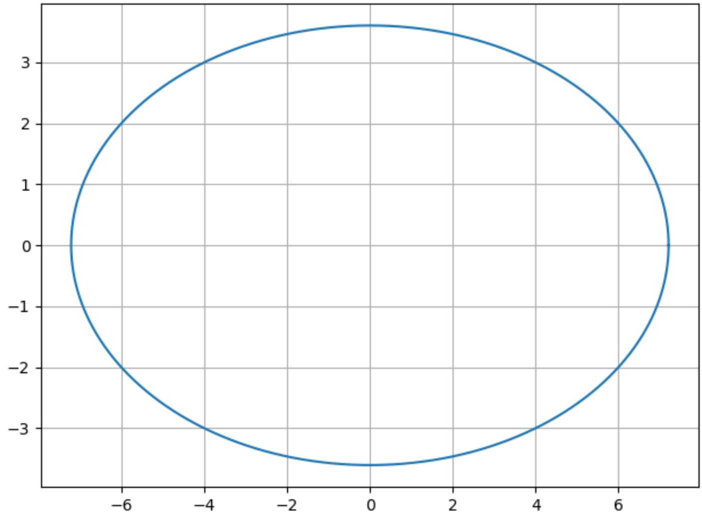


Fig. 7.2.22.1: Locus of the required ellipse.

7.2.23 vertices $\begin{pmatrix} 0 \\ \pm 3 \end{pmatrix}$ and foci $\begin{pmatrix} 0 \\ \pm 5 \end{pmatrix}$.

Solution: Following the approach in the earlier problems, it is obvious that

$$\mathbf{n} = \mathbf{e}_2, \mathbf{c} = \mathbf{u} = \mathbf{0}. \tag{7.2.23.1}$$

Consequently,

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 - e^2 \end{pmatrix} \tag{7.2.23.2}$$

$$\mathbf{F} = ce^2 \mathbf{e}_2 \implies \|\mathbf{F}\| = ce^2 = 5 \tag{7.2.23.3}$$

$$f = 25 - c^2 e^2 \tag{7.2.23.4}$$

Since the vertices are on the conic,

$$\mathbf{v_1}^\top \mathbf{V} \mathbf{v_1} + 2\mathbf{u}^\top \mathbf{v_1} + f = 0 \tag{7.2.23.5}$$

$$\implies 9\left(1 - e^2\right) + f = 0 \tag{7.2.23.6}$$

$$\tag{7.2.23.7}$$

Solving (7.2.23.7), (7.2.23.3) and (7.2.23.4),

$$c = \frac{9}{5}, \quad e = \frac{5}{3}, \tag{7.2.23.8}$$

yielding

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{16}{9} \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad f = 16. \tag{7.2.23.9}$$

Thus, the desired equation of the hyperbola is

$$\mathbf{x}^\top \begin{pmatrix} 1 & 0 \\ 0 & -\frac{16}{9} \end{pmatrix} \mathbf{x} + 16 = 0 \tag{7.2.23.10}$$

See Fig. 7.2.23.1.

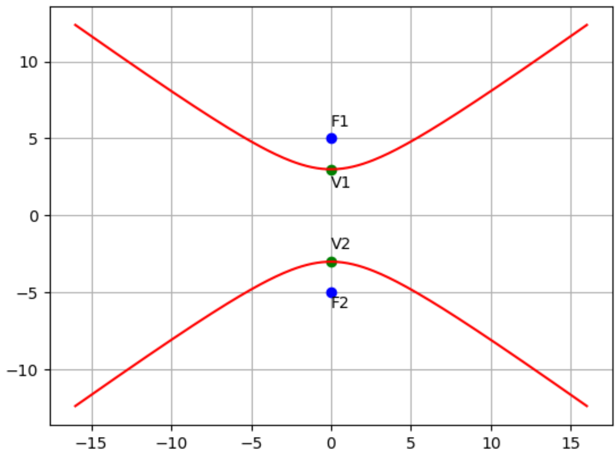


Fig. 7.2.23.1: Figure 1

7.2.24 vertices $(0, \pm 5)$, foci $(0, \pm 8)$.
7.2.25 focus (6,0); directrix x=-6
7.2.26 focus (0,-3); directrix y=3
7.2.27 vertex (0,0); focus (3,0)
7.2.28 vertex (0,0); focus (-2,0)
7.2.29 vertex (0,0) passing through (2,3) and axis is along x-axis
7.2.30 vertex (0,0) passing through (5,2) symmetric with respect to y-axis
7.2.31 vertices $(\pm 5, 0)$, foci$(\pm 4, 0)$.
7.2.32 vertices $(\pm 0, 13)$, foci$(0, \pm 5)$.
7.2.33 vertices $(\pm 6, 0)$, foci$(\pm 4, 0)$.
7.2.34 ends of major axis $(\pm 3, 0)$, ends of minor axis $(0, \pm 2)$.
7.2.35 ends of major axis $(0, \pm \sqrt{5})$, ends of minor axis $(\pm 1, 0)$.
7.2.36 length of major axis 26, foci $(\pm 5, 0)$.
7.2.37 length of minor axis 16, foci $(0, \pm 6)$.
7.2.38 foci $(\pm 3, 0), a = 4$.
7.2.39 vertex (0,4), focus (0,2).
7.2.40 vertex (-3,0), directrix $x + 5 = 0$.
7.2.41 focus (0,-3) and directrix $y = 3$.
7.2.42 directrix x=0, focus at (6,0).

7.2.43 vertex at (0,4), focus at (0,2).
7.2.44 focus at (-1,2), directrix $x - 2y + 3 = 0$.
7.2.45 vertices $(\pm 5, 0)$, foci $(\pm 7, 0)$.
7.2.46 vertices (0 ± 7) ,e $= \frac{4}{3}$.
7.2.47 foci $(0, \pm \sqrt{10})$, passing through (2,3).
7.2.48 vertices at $(0, \pm 6)$, eccentricity $\frac{5}{3}$.
7.2.49 focus (-1,-2), directrix $x - 2y + 3 = 0$.
7.2.50 eccentricity $\frac{3}{2}$, foci $(\pm 2, 0)$.
7.2.51 eccentricity $\frac{2}{3}$, latus rectum 5, centre (0,0).
7.2.52 If the parabola $y^2 = 4ax$ passes through the point (3,2), then the length of its latus rectum is
7.2.53 Find the eccentricity of the hyperbola $9y^2 - 4x^2 = 36$.
7.2.54 Equation of the hyperbola with eccentricty $\frac{3}{2}$ and foci at $(\pm 2, 0)$ is

a) $\frac{x^2}{4} - \frac{y^2}{5} = \frac{4}{9}$
b) $\frac{x^2}{9} - \frac{y^2}{9} = \frac{4}{9}$
c) $\frac{x^2}{4} - \frac{y^2}{9} = 1$
d) none of these.

7.2.55 Given the ellipse with equation $9x^2 + 25y^2 = 225$, find the eccentricity and foci.
7.2.56 Find the equation of the set of all points whose distance from (0,4) is $\frac{2}{3}$ of their distance from the line $y = 9$.
7.2.57 The equation of the ellipse whose focus is (1,-1), directrix $x-y-3 = 0$ and eccentricity $\frac{1}{2}$ is

a) $7x^2 + 2xy + 7y^2 - 10x + 10y + 7 = 0$
b) $7x^2 + 2xy + 7y^2 + 7 = 0$
c) $7x^2 + 2xy + 7y^2 + 10x - 10y - 7 = 0$
d) none

7.2.58 The length of the latus rectum of the ellipse $3x^2 + y^2 = 12$ is

a) 4
b) 3
c) 8
d) $4\sqrt{3}$

7.3 Miscellaneous

7.3.1 The cable of a uniformly loaded suspension bridge hangs in the form of a parabola. The roadway which is horizontal and 100 m long is supported by vertical wires attached to the cable, the longest wire being 30 m and the shortest being 6 m. Find the length of a supporting wire attached to the roadway 18 m from the middle.
Solution: The parameters are then listed in Table 7.3.1.

O	Lowest point of cable	$\begin{pmatrix}0\\0\end{pmatrix}$
d	Length of the cable	100 m
d_1	Length of longest wire	30 m
d_2	Length of shortest wire	6 m
A	End point of cable	$\begin{pmatrix}\frac{d}{2}\\d_1-d_2\end{pmatrix}$
B	End point of cable	$\begin{pmatrix}-\frac{d}{2}\\d_1-d_2\end{pmatrix}$

TABLE 7.3.1: points

For the conic,

$$\mathbf{V} = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}. \tag{7.3.1.1}$$

Points $\mathbf{O}, \mathbf{A}$, and $\mathbf{B}$ are on conic, so we have

$$\mathbf{O}^\top\mathbf{V}\mathbf{O} + 2\mathbf{u}^\top\mathbf{O} + f = 0 \tag{7.3.1.2}$$
$$\mathbf{A}^\top\mathbf{V}\mathbf{A} + 2\mathbf{u}^\top\mathbf{A} + f = 0 \tag{7.3.1.3}$$
$$\mathbf{B}^\top\mathbf{V}\mathbf{B} + 2\mathbf{u}^\top\mathbf{B} + f = 0 \tag{7.3.1.4}$$

which can be expressed as

$$2\mathbf{O}^\top\mathbf{u} + f = -\mathbf{O}^\top\mathbf{V}\mathbf{O} \tag{7.3.1.5}$$
$$2\mathbf{A}^\top\mathbf{u} + f = -\mathbf{A}^\top\mathbf{V}\mathbf{A} \tag{7.3.1.6}$$
$$2\mathbf{B}^\top\mathbf{u} + f = -\mathbf{B}^\top\mathbf{V}\mathbf{B} \tag{7.3.1.7}$$

leading to the matrix equation

$$\begin{pmatrix}2\mathbf{O}^\top & 1\\2\mathbf{A}^\top & 1\\2\mathbf{B}^\top & 1\end{pmatrix}\begin{pmatrix}\mathbf{u}\\f\end{pmatrix} = -\begin{pmatrix}\mathbf{O}^\top\mathbf{V}\mathbf{O}\\\mathbf{A}^\top\mathbf{V}\mathbf{A}\\\mathbf{B}^\top\mathbf{V}\mathbf{B}\end{pmatrix} \tag{7.3.1.8}$$

Substituting numerical values in the above equation,

$$\begin{pmatrix}0 & 0 & 1\\100 & 48 & 1\\-100 & 48 & 1\end{pmatrix}\begin{pmatrix}\mathbf{u}\\f\end{pmatrix} = -\begin{pmatrix}0\\-2500\\-2500\end{pmatrix} \tag{7.3.1.9}$$

$$\implies f = 0 \text{ and } \mathbf{u} = \begin{pmatrix}0\\-\frac{625}{12}\end{pmatrix} \tag{7.3.1.10}$$

So, the equation of the parabola is

$$\mathbf{x}^\top\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\mathbf{x} + 2\begin{pmatrix}0 & -\frac{625}{12}\end{pmatrix}\mathbf{x} = 0 \tag{7.3.1.11}$$

The desired point can be expressed as

$$\mathbf{D} = \begin{pmatrix} 18 \\ x_2 \end{pmatrix} \tag{7.3.1.12}$$

Substituting this in the parabola equation,

$$18^2 - \frac{6}{625}\lambda_2 = 0 \tag{7.3.1.13}$$

$$\implies \lambda_2 = \frac{1944}{625} \tag{7.3.1.14}$$

Thus, the length of a supporting wire attached to the roadway 18m from the middle is

$$\lambda_2 + d_2 = \frac{5694}{625}m \tag{7.3.1.15}$$

See Fig. 7.3.1.1.

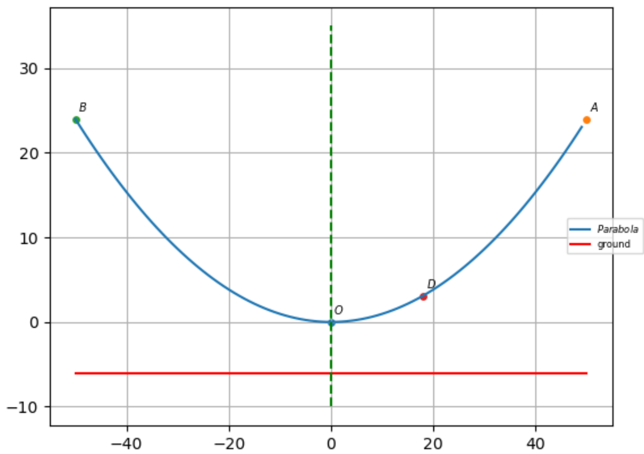


Fig. 7.3.1.1

7.3.2 Find the area of the triangle formed by the lines joining the vertex of the parabola

$$x^2 = 12y \tag{7.3.2.1}$$

to the ends of its latus rectum.

Solution: Rewriting (7.3.2.1) in matrix form,

$$\mathbf{x}^\top \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + 2\begin{pmatrix} 0 & -6 \end{pmatrix} \mathbf{x} = 0 \tag{7.3.2.2}$$

The above parabola can be expressed in standard form using

$$\mathbf{x} = \mathbf{P}\mathbf{y} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{y} \tag{7.3.2.3}$$

yielding

$$\mathbf{y}^\top \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x} + 2\begin{pmatrix} -6 & 0 \end{pmatrix}\mathbf{x} = 0 \tag{7.3.2.4}$$

Hence, from (7.1.2.2),

$$\mathbf{n} = \mathbf{e}_1 \tag{7.3.2.5}$$

$$c = -\frac{36}{2 \times 6} = -3 \tag{7.3.2.6}$$

Substituting in (7.1.2.3) yields

$$\mathbf{F} = 3\mathbf{e}_1 \tag{7.3.2.7}$$

Thus, the equation of the latus rectum is

$$\mathbf{x} = \mathbf{F} + \kappa\mathbf{e}_2 \tag{7.3.2.8}$$

Substituting in (7.3.2.4) and simplifying,

$$\kappa = \pm 6 \tag{7.3.2.9}$$

Thus, the ends of the latus rectum are

$$\mathbf{y} = \begin{pmatrix} 3 \\ \pm 6 \end{pmatrix} \tag{7.3.2.10}$$

The relevant parameters with respect to (7.3.2.2) can now be obtained using (7.3.2.3). See Fig. 7.3.2.1. The area of the required triangle is

$$\text{ar}\,(\triangle OAB) = \frac{1}{2}\begin{vmatrix} 6 & 3 \\ -6 & 3 \end{vmatrix} = 18 \tag{7.3.2.11}$$

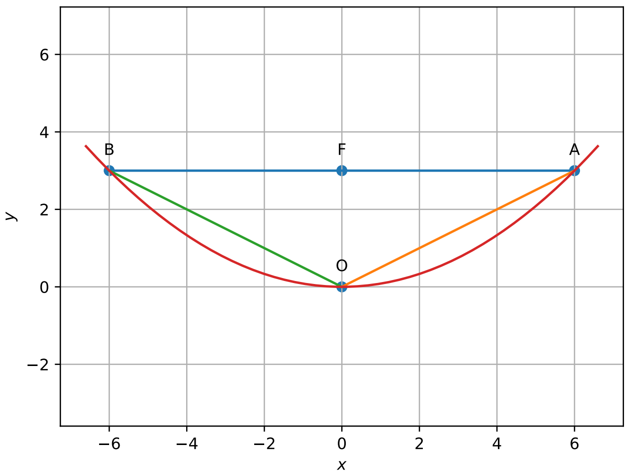


Fig. 7.3.2.1

7.3.3 A man running a racecourse notes that the sum of the distances from the two flag posts from him is always 10 m and the distance between the flag posts is 8 m. Find the equation of the posts traced by the man.

7.3.4 Find the coordinates of a point on the parabola $y^2 = 8x$ whose focal distance is 4.

7.3.5 Show that the set of all points such that the difference of their distances from (4,0) and (-4,0) is always equal to 2 represent a hyperbola.

7.3.6 If the distance between the foci of a hyperbola is 16 and its eccentricity is $\sqrt{2}$, then obtain the equation of the hyperbola.

7.3.7 The eccentricity of the hyperbola whose latus rectum is 8 and conjugate axis is equal to half of the distance between the foci is

a) $\frac{4}{3}$
b) $\frac{4}{\sqrt{3}}$
c) $\frac{2}{\sqrt{3}}$
d) none of these

7.3.8 The distance between the foci of a hyperbola is 16 and its eccentricity is ≤ 2. Its equation is

a) $x^2 - y^2 = 3^2$
b) $\frac{x^2}{4-} \frac{y^2}{9} = 1$
c) $2x - 3y^2 = 7$
d) none of these

7.3.9 If the latus rectum of an ellipse is equal to half of minor axis, then find its eccentricity.

7.3.10 If the eccentricity of an ellipse is $\frac{5}{8}$ and the distance between its foci is 10 then find latus rectum of the ellipse.

7.3.11 Find the distance between the directrices of the ellipse $\frac{x^2}{36} + \frac{y^2}{20}$

7.3.12 Find the equation of the set of all points the sum of whose distances from the points (3,0) and (9,0) is 12.

7.3.13 If P is a point on the ellipse $\frac{x^2}{16} + \frac{y^2}{25} = 1$ whose foci are s and s' then $Ps + Ps' = 8$.

7.3.14 An arch is in the form of a parabola with its axis vertical. The arch is 10m high and 5m wide at the base. How wide is it 2m from the vertex of the parabola?

7.3.15 An equilateral triangle is inscribed in the parabola $y^2 = 4ax$,where one vertex is at the vertex of the parabola. Find the length of the side of the triangle.

7.3.16 An arch is in the form of a semi-ellipse. It is 8 m wide and 2 m high at the centre. Find the height of the arch at a point 1.5 m from one end.

7.3.17 A rod of length 12cm moves with its ends always touching the coordinate axes. Determine the equation of locus of a point P on the rod, which is 3cm from the end in contact with $x - axis$.

8.1 Formulae

8.1.1 The points of intersection of the line

$$L: \quad \mathbf{x} = \mathbf{h} + \kappa \mathbf{m} \quad \kappa \in \mathbb{R} \tag{8.1.1.1}$$

with the conic section in (7.1.1.1) are given by

$$\mathbf{x}_i = \mathbf{h} + \kappa_i \mathbf{m} \tag{8.1.1.2}$$

where

$$\kappa_i = \frac{1}{\mathbf{m}^\top \mathbf{V} \mathbf{m}} \left(-\mathbf{m}^\top \left(\mathbf{V}\mathbf{h} + \mathbf{u} \right) \pm \sqrt{\left[\mathbf{m}^\top \left(\mathbf{V}\mathbf{h} + \mathbf{u} \right) \right]^2 - g\left(\mathbf{h} \right) \left(\mathbf{m}^\top \mathbf{V} \mathbf{m} \right)} \right) \tag{8.1.1.3}$$

See B.3.1 for proof.

8.1.2 (7.1.1.1) represents a pair of straight lines if the matrix

$$\begin{pmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^\top & f \end{pmatrix} \tag{8.1.2.1}$$

is singular.

8.1.3 The intersection of two conics with parameters $\mathbf{V}_i, \mathbf{u}_i, f_i,\ i = 1, 2$ is defined as

$$\mathbf{x}^\top \left(\mathbf{V}_1 + \mu \mathbf{V}_2 \right) \mathbf{x} + 2 \left(\mathbf{u}_1 + \mu \mathbf{u}_2 \right)^\top \mathbf{x} + \left(f_1 + \mu f_2 \right) = 0 \tag{8.1.3.1}$$

8.1.4 From (8.1.2.1), (8.1.3.1) represents a pair of straight lines if

$$\begin{vmatrix} \mathbf{V}_1 + \mu \mathbf{V}_2 & \mathbf{u}_1 + \mu \mathbf{u}_2 \\ \left(\mathbf{u}_1 + \mu \mathbf{u}_2 \right)^\top & f_1 + \mu f_2 \end{vmatrix} = 0 \tag{8.1.4.1}$$

8.2 Chords

8.2.1 Find the area between the curves $y = x$ and $y = x^2$.
Solution:

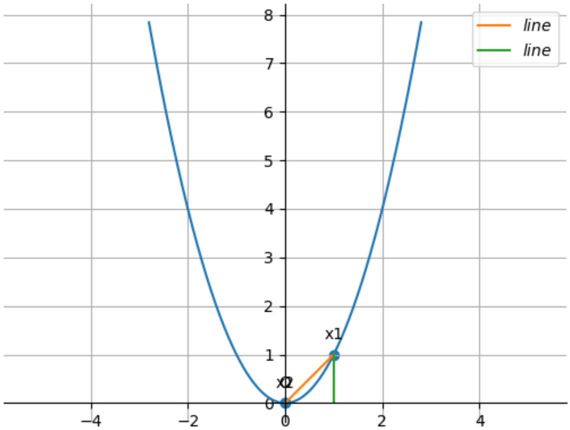


Fig. 8.2.1.1

The given curve can be expressed as a conic with parameters

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix}, f = 0 \tag{8.2.1.1}$$

The given line parameters are

$$\mathbf{h} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{8.2.1.2}$$

Substituting the given parameters in (8.1.1.3),

$$\mathbf{x}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{8.2.1.3}$$

From Fig. 8.2.1.1, the area bounded by the curve $y = x^2$ and line $y = x$ is given by

$$\int_0^1 \left(x - \frac{x^2}{2} \right) dx = \frac{1}{6} \tag{8.2.1.4}$$

8.2.2 Find the area of the region bounded by the curve $y^2 = x$ and the lines $x = 1$ and $x = 4$ and the axis in the first quadrant.
Solution:

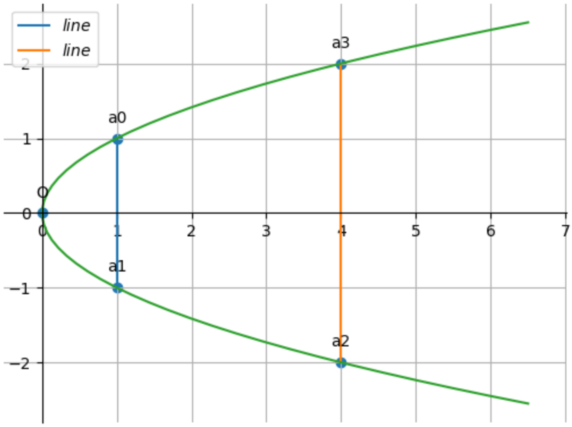


Fig. 8.2.2.1

The parameters of the conic are

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u} = -\frac{1}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, f = 0 \tag{8.2.2.1}$$

For the line $x - 1 = 0$, the parameters are

$$\mathbf{q}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{m}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{8.2.2.2}$$

Substituting from the above in (8.1.1.3),

$$\kappa_i = 1, -1 \tag{8.2.2.3}$$

yilelding the points of intersection

$$\mathbf{a}_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \mathbf{a}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{8.2.2.4}$$

Similarly, for the line $x - 4 = 0$

$$\mathbf{q_1} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \mathbf{m_1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{8.2.2.5}$$

yielding

$$\kappa_i = 2, -2 \tag{8.2.2.6}$$

from which, the points of intersection are

$$\mathbf{a_3} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \mathbf{a_2} = \begin{pmatrix} 4 \\ -2 \end{pmatrix} \tag{8.2.2.7}$$

See Fig. 8.2.2.1. Thus, the area of the parabola in between the lines $x = 1$ and $x = 4$ is given by

$$\int_0^4 \sqrt{x}\,dx - \int_0^1 \sqrt{x}\,dx = 14/3 \tag{8.2.2.8}$$

8.2.3 Find the area of the region bounded by the curve $y^2 = 9x$ and the lines $x = 2$ and $x = 4$ and the axis in the first quadrant.

Solution:

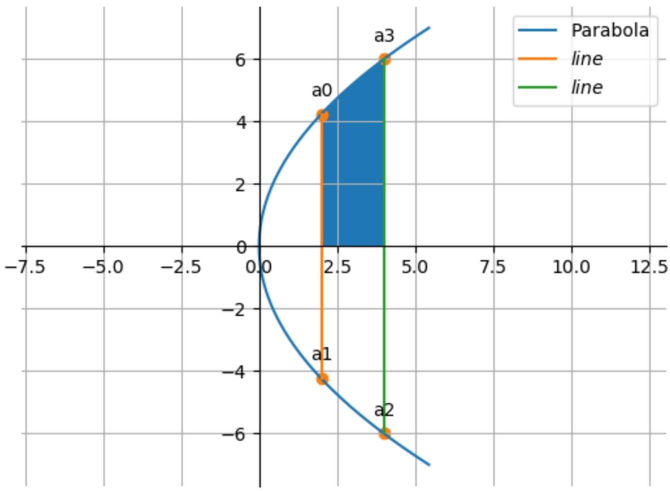


Fig. 8.2.3.1

The parameters of the conic are

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u} = \frac{9}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, f = 0. \tag{8.2.3.1}$$

The parameters of the line $x - 2 = 0$ are

$$\mathbf{q_2} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \mathbf{m_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{8.2.3.2}$$

Substituting in (8.1.1.3),

$$\kappa_i = \pm 3\sqrt{2} \tag{8.2.3.3}$$

yielding

$$\mathbf{a_0} = \begin{pmatrix} 2 \\ 3\sqrt{2} \end{pmatrix}, \mathbf{a_1} = \begin{pmatrix} 2 \\ -3\sqrt{2} \end{pmatrix}. \tag{8.2.3.4}$$

Similarly, for the line $x - 4 = 0$,

$$\mathbf{q_1} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \mathbf{m_1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{8.2.3.5}$$

yielding

$$\kappa_i = \pm 6. \tag{8.2.3.6}$$

Thus,

$$\mathbf{a_3} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}, \mathbf{a_2} = \begin{pmatrix} 4 \\ -6 \end{pmatrix} \tag{8.2.3.7}$$

and from Fig. 8.2.3.1, the desired area of the parabola is

$$\int_0^4 3\sqrt{x}\,dx - \int_0^2 3\sqrt{x}\,dx = 16 - 4\sqrt{2} \tag{8.2.3.8}$$

8.2.4 Find the area of the region bounded by $x^2 = 4y$, $y = 2$, $y = 4$ and the y-axis in the first quadrant.

8.2.5 Find the area of the region bounded by the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$

8.2.6 Find the area of the region bounded by the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$

8.2.7 Find the area of the region in the first quadrant enclosed by the x-axis, line $x = \sqrt{3}y$ and circle $x^2 + y^2 = 4$.

Solution:

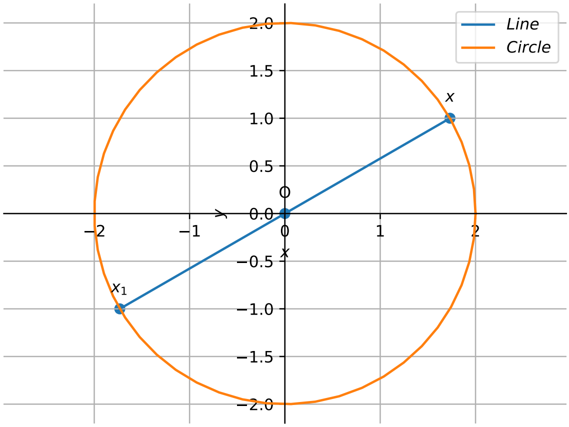


Fig. 8.2.7.1

From the given information, the parameters of the circle and line are

$$f = -4, \mathbf{u} = \mathbf{0}, \mathbf{V} = \mathbf{I}, \mathbf{m} = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}, \mathbf{h} = \mathbf{0} \tag{8.2.7.1}$$

Substituting the above parameters in (8.1.1.3),

$$\mu = \sqrt{3} \tag{8.2.7.2}$$

yielding the desired point of intersection as

$$\mathbf{x} = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} \tag{8.2.7.3}$$

Note that we have chosen only the point of intersection in the first quadrant as shown in Fig. 8.2.7.1. From (8.2.7.1), the angle between the given line and the x axis is

$$\theta = 30^\circ \tag{8.2.7.4}$$

and the area of the sector is

$$\frac{\theta}{360}\pi r^2 = \frac{\pi}{3} \tag{8.2.7.5}$$

8.2.8 Find the area of the smaller part of the circle $x^2 + y^2 = a^2$ cut off by the line $x = \frac{a}{\sqrt{2}}$.
Solution:

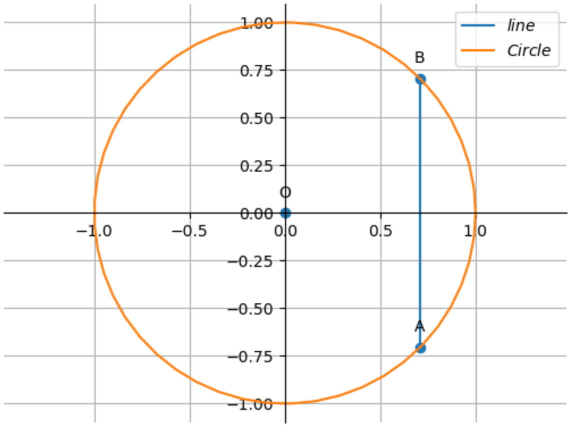


Fig. 8.2.8.1

The given circle can be expressed as a conic with parameters

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u} = 0, f = -a^2 \tag{8.2.8.1}$$

The given line parameters are

$$\mathbf{h} = \begin{pmatrix} \frac{a}{\sqrt{2}} \\ 0 \end{pmatrix}, \mathbf{m} = \mathbf{e}_2. \tag{8.2.8.2}$$

Substituting the above in (8.1.1.3),

$$\kappa = \pm \frac{a}{\sqrt{2}} \tag{8.2.8.3}$$

yielding the points of intersection of the line with circle as

$$\mathbf{A} = \begin{pmatrix} \frac{a}{\sqrt{2}} \\ -\frac{a}{\sqrt{2}} \end{pmatrix}, \mathbf{B} = \begin{pmatrix} \frac{a}{\sqrt{2}} \\ \frac{a}{\sqrt{2}} \end{pmatrix} \tag{8.2.8.4}$$

From Fig. 8.2.8.1, the total area of the portion is given by

$$ar(APQ) = 2ar(APR) \tag{8.2.8.5}$$

$$= 2\int_0^{\frac{a}{\sqrt{2}}} \sqrt{a^2 - x^2}\, dx \tag{8.2.8.6}$$

$$= \frac{a^2}{2}\left(1 + \frac{\pi}{2}\right) \tag{8.2.8.7}$$

8.2.9 The area between $x = y^2$ and $x = 4$ is divided into two equal parts by the line $x = a$, find the value of a.

Solution:

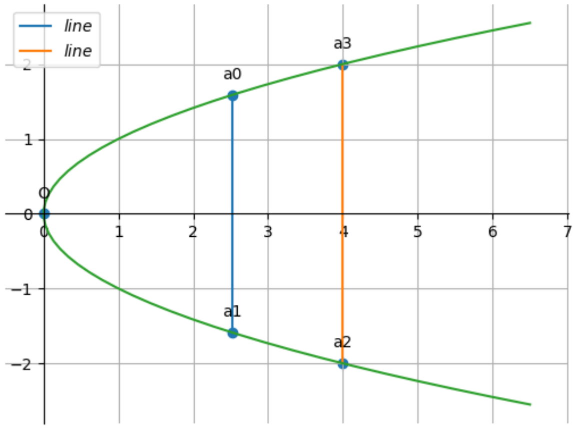


Fig. 8.2.9.1

The given conic parameters are

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u} = -\frac{1}{2}\mathbf{e}_1 f = 0 \tag{8.2.9.1}$$

The parameters of the lines are

$$\mathbf{q}_2 = \begin{pmatrix} a \\ 0 \end{pmatrix}, \mathbf{m}_2 = \mathbf{e}_2 \tag{8.2.9.2}$$

Substituting the above values in (8.1.1.3),

$$\mu_i = a, -a \tag{8.2.9.3}$$

yielding the points of intersection as

$$\mathbf{a_0} = \begin{pmatrix} a \\ a \end{pmatrix}, \mathbf{a_1} = \begin{pmatrix} a \\ -a \end{pmatrix} \tag{8.2.9.4}$$

Similarly, for the line $x - 4 = 0$,

$$\mathbf{q_1} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \mathbf{m_1} = \mathbf{e}_2 \tag{8.2.9.5}$$

yielding

$$\mu_i = 2, -2 \tag{8.2.9.6}$$

and

$$\mathbf{a}_3 = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \mathbf{a}_2 = \begin{pmatrix} 4 \\ -2 \end{pmatrix}. \tag{8.2.9.7}$$

Area between parabola and the line $x = 4$ is divided equally by the line $x = a$. Thus, from Fig. 8.2.9.1,

$$A_1 = \int_0^a \sqrt{x}\,dx \tag{8.2.9.8}$$

$$A_2 = \int_a^4 \sqrt{x}\,dx \tag{8.2.9.9}$$

$$\text{and } A_1 = A_2 \tag{8.2.9.10}$$

$$\implies a = 4^{\frac{2}{3}} \tag{8.2.9.11}$$

8.2.10 Find the area of the region bounded by the parabola $y = x^2$ and $y = |x|$.

8.2.11 Find the area bounded by the curve $x^2 = 4y$ and the line $x = 4y - 2$.

Solution:

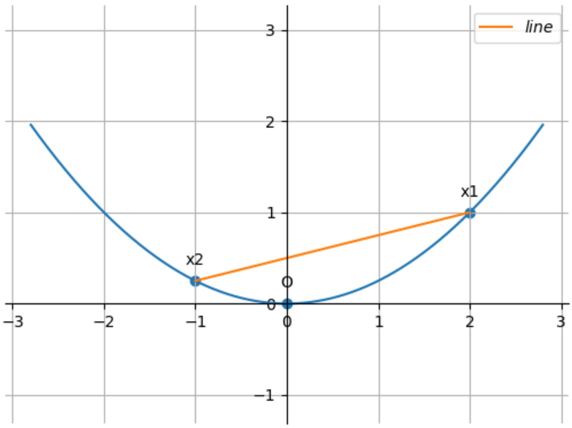


Fig. 8.2.11.1

The given curve can be expressed as a conic with parameters

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}, f = 0 \tag{8.2.11.1}$$

The parameters of the given line are

$$\mathbf{q} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \tag{8.2.11.2}$$

The points of intersection can then be obtained from (8.1.1.3)

$$\therefore \mathbf{x}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \mathbf{x}_2 = \begin{pmatrix} -1 \\ \frac{1}{4} \end{pmatrix} \tag{8.2.11.3}$$

The desired area is then obtained from Fig. 8.2.11.1 as

$$A = \int_{x_2}^{x_1} [f(x) - g(x)]\, dx \tag{8.2.11.4}$$

$$= \int_{-1}^{2} \left(\frac{x+2}{4} - \frac{x^2}{4} \right) dx \tag{8.2.11.5}$$

$$= \frac{9}{8} \tag{8.2.11.6}$$

8.2.12 Find the area of the region bounded by the curve $y^2 = 4x$ and the line $x = 3$.

8.2.13 Area lying in the first quadrant and bounded by the circle $x^2 + y^2 = 4$ and the lines $x = 0$ and $x = 2$ is

a) π

b) $\frac{\pi}{2}$

c) $\frac{\pi}{3}$

d) $\frac{\pi}{4}$

8.2.14 Find the area of the region bounded by the curve $y^2 = 4x$, y-axis and the line $y = 3$.

Solution: In this case,

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{8.2.14.1}$$

$$\mathbf{u} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \tag{8.2.14.2}$$

$$f = 0 \tag{8.2.14.3}$$

For the given line $y = 3$, the parameters are

$$\mathbf{h} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{8.2.14.4}$$

The intersection of the line with the conic is obtained from (8.1.1.3) as

$$\kappa = \frac{9}{4} \tag{8.2.14.5}$$

The point of contact is given as

$$\mathbf{a}_0 = \begin{pmatrix} \frac{9}{4} \\ 3 \end{pmatrix} \tag{8.2.14.6}$$

From Fig. 8.2.14.1, the desired area of the region is obtaioned as

$$\int_0^3 \frac{y^2}{4}\,dy = \frac{1}{12}\left[y^3\right]_0^3 \tag{8.2.14.7}$$

$$= \frac{1}{12}(27-0) \tag{8.2.14.8}$$

$$= \frac{9}{4}\text{ sq.units} \tag{8.2.14.9}$$

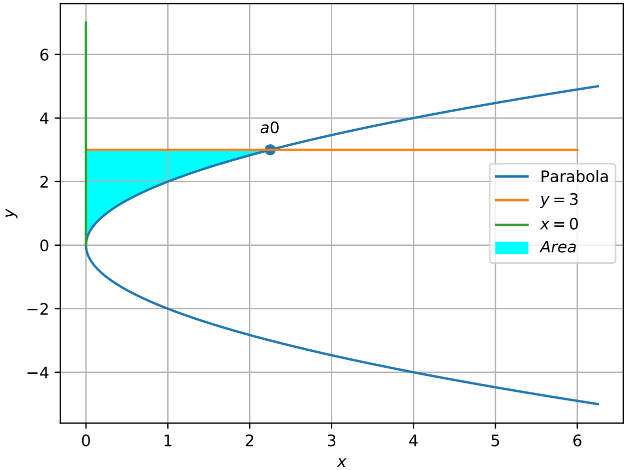


Fig. 8.2.14.1

8.2.15 Find the area of the region bounded by the curve $x^2 = 4y$ and the lines $y = 2$ and $y = 4$ and the y-axis in the first quadrant.

Solution:

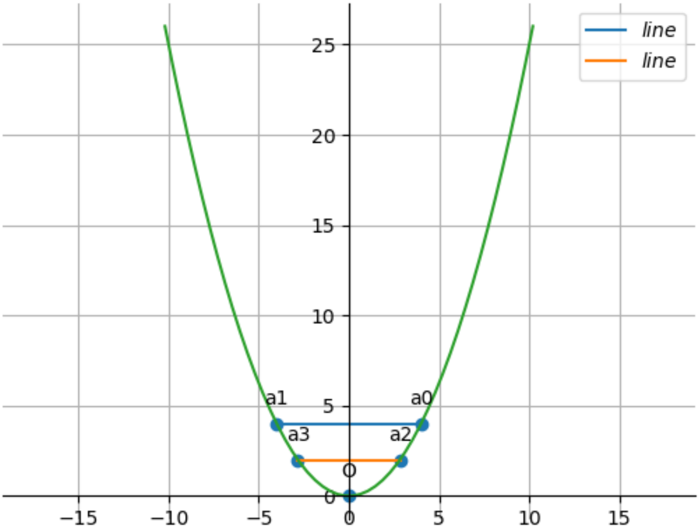


Fig. 8.2.15.1

The conic parameters are

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}, f = 0 \tag{8.2.15.1}$$

The vector parameters of $y - 4 = 0$ are

$$\mathbf{h}_1 = \begin{pmatrix} 0 \\ 4 \end{pmatrix}, \mathbf{m}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{8.2.15.2}$$

Substituting the above in (8.1.1.3),

$$\kappa_i = 4, -4 \tag{8.2.15.3}$$

yielding the points of intersection with the parabola as

$$\mathbf{a}_0 = \begin{pmatrix} 4 \\ 4 \end{pmatrix}, \mathbf{a}_1 = \begin{pmatrix} -4 \\ 4 \end{pmatrix} \tag{8.2.15.4}$$

Similarly, for the line $y - 2 = 0$, the vector parameters are

$$\mathbf{h}_2 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \mathbf{m}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{8.2.15.5}$$

yielding

$$\kappa_i = 2.8, -2.8 \tag{8.2.15.6}$$

and the points of intersection

$$\mathbf{a}_2 = \begin{pmatrix} 2.8 \\ 2 \end{pmatrix}, \mathbf{a}_3 = \begin{pmatrix} -2.8 \\ 2 \end{pmatrix} \tag{8.2.15.7}$$

From Fig. 8.2.15.1, the area of the parabola between the lines $y = 2$ and $y = 4$ is given by

$$\int_0^4 2\sqrt{y}\,dy - \int_0^2 2\sqrt{y}\,dy = 6.895 \tag{8.2.15.8}$$

8.2.16 Find the area enclosed by the parabola $4y = 3x^2$ and the line $2y = 3x + 12$.
Solution:

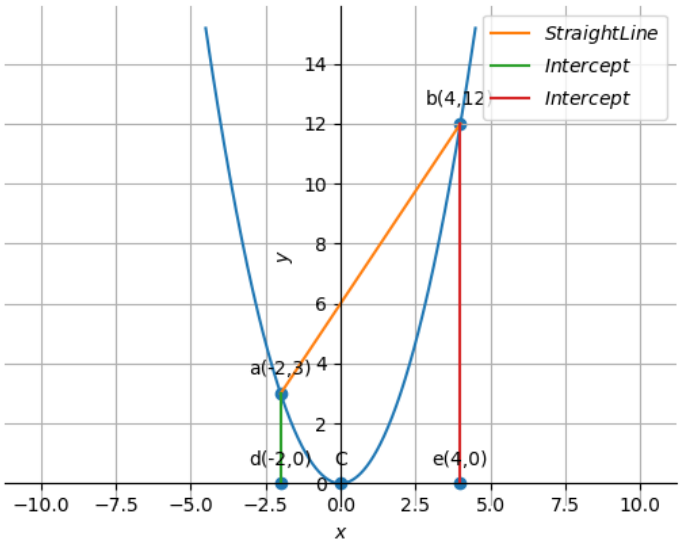


Fig. 8.2.16.1

The parameters of the given conic are

$$\mathbf{V} = \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}, f = 0. \tag{8.2.16.1}$$

For the line, the parameters are

$$\mathbf{h} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \tag{8.2.16.2}$$

yielding

$$\kappa = -2.5, 2.7 \tag{8.2.16.3}$$

upon substitution in (8.1.1.3) resulting in the points of intersection

$$\mathbf{A} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 4 \\ 12 \end{pmatrix}. \tag{8.2.16.4}$$

From Fig. 8.2.16.1, the desired area is

$$\int_{-2}^{4} \frac{3x+12}{2}\,dx - \int_{-2}^{4} \frac{3x^2}{4}\,dx = 27 \tag{8.2.16.5}$$

8.2.17 Find the area of the smaller region bounded by the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and the line $\frac{x}{3} + \frac{y}{2} = 1$.

Solution:

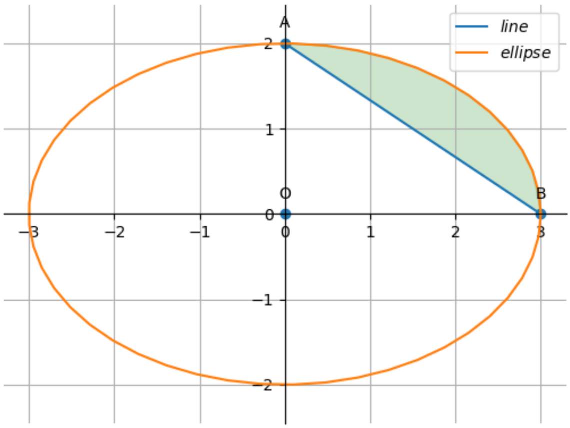


Fig. 8.2.17.1

The given ellipse can be expressed as conics with parameters

$$\mathbf{V} = \begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix}, \mathbf{u} = 0, f = -(a^2b^2). \tag{8.2.17.1}$$

The line parameters are

$$\mathbf{h} = \begin{pmatrix} a \\ 0 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} \frac{1}{b} \\ -\frac{1}{a} \end{pmatrix}. \tag{8.2.17.2}$$

Substituting the given parameters in (8.1.1.3),

$$\mu = 0, -6 \tag{8.2.17.3}$$

yielding the points of intersection

$$\mathbf{A} = \begin{pmatrix} a \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ b \end{pmatrix}. \tag{8.2.17.4}$$

From Fig. 8.2.17.1, the desired area is

$$\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\, dx - \int_0^a \frac{b}{a}(a-x)\, dx$$

$$= \frac{ab}{2}\left(\frac{\pi}{2} - 1\right) = 3\left(\frac{\pi}{2} - 1\right) \tag{8.2.17.5}$$

upon substituting $a = 3, b = 2$.

8.2.18 Find the area of the region bounded by the curve $x^2 = y$ and the lines $y = x + 2$ and the x axis.

8.2.19 Find the area bounded by the curve $y = x|x|$, x-axis and the ordinates x=-1 and x=1.

8.2.20 Find the area of the region bounded by the curves $y = x^2 + 2$, $y = x$, $x = 0$ and $x = 3$.

8.2.21 Find the smaller area enclosed by the circle $x^2 + y^2 = 4$ and the line $x + y = 2$.

Solution:

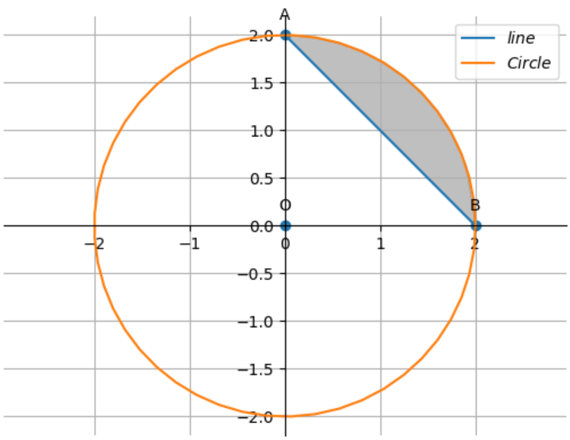


Fig. 8.2.21.1

The given circle can be expressed as conics with parameters,

$$\mathbf{V} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}, \mathbf{u} = 0, f = -16 \tag{8.2.21.1}$$

The line parameters are

$$\mathbf{h} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} \tag{8.2.21.2}$$

Substituting the parameters in (8.1.1.3),

$$\kappa = 0, -4 \tag{8.2.21.3}$$

yielding the points of intersection as

$$\mathbf{A} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \tag{8.2.21.4}$$

From Fig. 8.2.21.1, the desired area is

$$\int_0^2 \sqrt{4 - x^2}\, dx - \int_0^2 (2 - x)\, dx = \pi - 2 \tag{8.2.21.5}$$

8.2.22 Find the area of the region bounded by the curves $y^2 = 9x$, $y = 3x$.
8.2.23 Find the area of the region bounded by the parabola $y^2 = 2px$, $x^2 = 2py$.
8.2.24 Find the area of the region bounded by the curve $y = x^2$ and $y = x + 6$ and $x = 0$.
8.2.25 Find the area of the region bounded by the curve $y^2 = 4x$, $x^2 = 4y$.
8.2.26 Find the area of the region included between $y^2 = 9x$ and $y = x$
8.2.27 Find the area of the region enclosed by the parabola $x^2 = y$ and the line $y = x + 2$
8.2.28 Find the area of region bounded by the line $x = 2$ and the parabola $y^2 = 8x$
8.2.29 Sketch the region $(x, 0) : y = \sqrt{4 - x^2}$ and x-axis. Find the area of the region using integration.
8.2.30 Calculate the area under the curve $y = 2\sqrt{x}$ included between the lines $x = 0$ and $x = 1$.
8.2.31 Using integration, find the area of the region bounded by the line $2y = 5x + 7$, x-axis and the lines $x = 2$ and $x = 8$.
8.2.32 Draw a rough sketch of the curve $y = \sqrt{x - 1}$ in the interval $[1, 5]$. Find the area under the curve and between the lines $x = 1$ and $x = 5$.
8.2.33 Determine the area under the curve $y = \sqrt{a^2 - x^2}$ included between the lines $x = 0$ and $x = a$
8.2.34 Find the area of the region bounded by $y = \sqrt{x}$ and $y = x$.
8.2.35 Find the area enclosed by the curve $y = -x^2$ and the straight line $x + y + 2 = 0$.
8.2.36 Find the area bounded by the curve $y = \sqrt{x}$, $x = 2y + 3$ in the first quadrant and x-axis.
8.2.37 Draw a rough sketch of the region $(x, y) : y^2 \leqslant 6ax$ and $x^2 + y^2 \leqslant 16a^2$.
8.2.38 Draw a rough sketch of the given curve $y = 1 + |x + 1|$, $x = -3$, $x = 3$, $y = 0$, and find the area of the region bounded by them, using integration.
8.2.39 The area of the region bounded by the curve $x^2 = 4y$ and the straight line $x = 4y - 2$ is

a) $\frac{3}{8}$ sq units
b) $\frac{5}{8}$ sq units
c) $\frac{7}{8}$ sq units

d) $\frac{9}{8}$ sq units

8.2.40 The area of the region bounded by the curve $y = \sqrt{16 - x^2}$ and x-axis is

a) 8 sq units
b) 20π sq units
c) 16π sq units
d) 256π sq units

8.2.41 Area of the region in the first quadrant enclosed by the x-axis, the line $y = x$ and the circle $x^2 + y^2 = 32$ is

a) 16π sq units
b) 4π sq units
c) 32π sq units
d) 24π sq units

8.2.42 The area of the region bounded by parabola $y^2 = x$ and the straight line $2y = x$ is

a) $\frac{4}{3}$ sq units
b) 1 sq units
c) $\frac{2}{3}$ sq units
d) $\frac{1}{3}$ sq units

8.2.43 Find the equation of a circle whose centre is (3,1) and which cuts off a chord of length 6 units on the line $2x - 5y + 18 = 0$.

8.3 Curves

8.3.1 Find the area bounded by the curves $(x - 1)^2 + y^2 = 1$ and $x^2 + y^2 = 1$.
Solution: The conic parameters for the two circles can be expressed as

$$\mathbf{V}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \mathbf{u}_1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \ f_1 = 0, \quad \mathbf{V}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \mathbf{u}_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ f_2 = -1. \tag{8.3.1.1}$$

On substituting from (8.3.1.1) in (8.1.4.1), we obtain

$$\begin{vmatrix} 1 + \mu & 0 & -1 \\ 0 & 1 + \mu & 0 \\ -1 & 0 & -\mu \end{vmatrix} = 0 \tag{8.3.1.2}$$

yileding

$$\implies \mu = -1. \tag{8.3.1.3}$$

Substituting (8.3.1.1) in (8.1.3.1), we obtain

$$\mathbf{x}^\top \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + 2\begin{pmatrix} -1 & 0 \end{pmatrix} \mathbf{x} + 1 = 0 \tag{8.3.1.4}$$

$$\implies \begin{pmatrix} -2 & 0 \end{pmatrix} \mathbf{x} = -1 \tag{8.3.1.5}$$

Therefore the intersection of the two circles is a line with parameters

$$\mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \mathbf{h} = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}. \tag{8.3.1.6}$$

The intersection parameters of the chord in (8.3.1.5) with the first circle in (8.3.1.1) is obtained from (8.1.1.3) as

$$\kappa_i = \pm \frac{\sqrt{3}}{2} \tag{8.3.1.7}$$

Hence the point of intersection are obtained from (8.1.1.2) as

$$\mathbf{a}_0 = \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}, \mathbf{a}_2 = \begin{pmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix}. \tag{8.3.1.8}$$

The desired area of region is given as

$$\begin{aligned} 2\left(\int_0^{\frac{1}{2}} \sqrt{1-(x-1)^2}dx + \int_{\frac{1}{2}}^1 \sqrt{1-x^2}dx\right) \\ = 2\left[\frac{1}{2}(x-1)\sqrt{1-(x-1)^2} + \frac{1}{2}\sin^{-1}(x-1)\right]_0^{\frac{1}{2}} \\ + 2\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right]_{\frac{1}{2}}^1 \\ = \frac{2\pi}{3} - \frac{\sqrt{3}}{2} \end{aligned} \tag{8.3.1.9}$$

See Fig. 8.3.1.1.

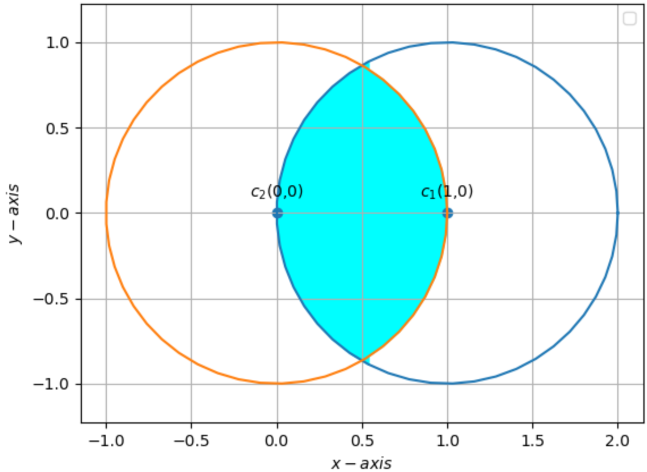


Fig. 8.3.1.1

8.3.2 Find the area of the circle $4x^2 + 4y^2 = 9$ which is interior to the parabola $x^2 = 4y$.
8.3.3 Find the area of the circle $x^2 + y^2 = 16$ exterior to the parabola $y^2 = 6x$.
8.3.4 Find the area of the region bounded by the curve $y^2 = 2x$ and $x^2 + y^2 = 4x$.

9 Tangent And Normal

9.1 Formulae

9.1.1 If L in (8.1.1.1) touches (7.1.1.1) at exactly one point $\mathbf{q}$,

$$\mathbf{m}^\top (\mathbf{V}\mathbf{q} + \mathbf{u}) = 0 \tag{9.1.1.1}$$

9.1.2 Given the point of contact $\mathbf{q}$, the equation of a tangent to (7.1.1.1) is

$$(\mathbf{V}\mathbf{q} + \mathbf{u})^\top \mathbf{x} + \mathbf{u}^\top \mathbf{q} + f = 0 \tag{9.1.2.1}$$

9.1.3 Given the point of contact $\mathbf{q}$, the equation of the normal to (7.1.1.1) is

$$(\mathbf{V}\mathbf{q} + \mathbf{u})^\top \mathbf{R} (\mathbf{x} - \mathbf{q}) = 0 \tag{9.1.3.1}$$

9.1.4 If $\mathbf{V}^{-1}$ exists, given the normal vector $\mathbf{n}$, the tangent points of contact to (7.1.1.1) are given by

$$\begin{aligned} \mathbf{q}_i &= \mathbf{V}^{-1} (\kappa_i \mathbf{n} - \mathbf{u}), i = 1, 2 \\ \text{where } \kappa_i &= \pm \sqrt{\frac{f_0}{\mathbf{n}^\top \mathbf{V}^{-1} \mathbf{n}}} \end{aligned} \tag{9.1.4.1}$$

9.1.5 If $\mathbf{V}$ is not invertible, given the normal vector $\mathbf{n}$, the point of contact to (7.1.1.1) is given by the matrix equation

$$\begin{pmatrix} (\mathbf{u} + \kappa \mathbf{n})^\top \\ \mathbf{V} \end{pmatrix} \mathbf{q} = \begin{pmatrix} -f \\ \kappa \mathbf{n} - \mathbf{u} \end{pmatrix} \tag{9.1.5.1}$$

$$\text{where } \kappa = \frac{\mathbf{p}_1^\top \mathbf{u}}{\mathbf{p}_1^\top \mathbf{n}}, \quad \mathbf{V}\mathbf{p}_1 = 0 \tag{9.1.5.2}$$

9.1.6 For a conic/hyperbola, a line with normal vector $\mathbf{n}$ cannot be a tangent if

$$\frac{\mathbf{u}^\top \mathbf{V}^{-1} \mathbf{u} - f}{\mathbf{n}^\top \mathbf{V}^{-1} \mathbf{n}} < 0 \tag{9.1.6.1}$$

9.1.7 For a circle, the points of contact are

$$\mathbf{q}_{ij} = \left(\pm r \frac{\mathbf{n}_j}{\left\| \mathbf{n}_j \right\|} - \mathbf{u} \right), \quad i, j = 1, 2 \tag{9.1.7.1}$$

9.1.8 A point $\mathbf{h}$ lies on a normal to the conic in (7.1.1.1) if

$$\begin{aligned} &\left(\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u}) \right)^2 \left(\mathbf{n}^\top \mathbf{V} \mathbf{n} \right) \\ &\qquad - 2 \left(\mathbf{m}^\top \mathbf{V} \mathbf{n} \right) \left(\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u}) \mathbf{n}^\top (\mathbf{V}\mathbf{h} + \mathbf{u}) \right) \\ &\qquad\qquad + g(\mathbf{h}) \left(\mathbf{m}^\top \mathbf{V} \mathbf{n} \right)^2 = 0 \end{aligned} \tag{9.1.8.1}$$

9.1.9 A point $\mathbf{h}$ lies on a tangent to the conic in (7.1.1.1) if

$$\mathbf{m}^\top \left[(\mathbf{V}\mathbf{h} + \mathbf{u}) (\mathbf{V}\mathbf{h} + \mathbf{u})^\top - \mathbf{V} g(\mathbf{h}) \right] \mathbf{m} = 0 \tag{9.1.9.1}$$

9.1 Find the points on the curve $x^2 + y^2 - 2x - 3 = 0$ at which the tangents are parallel to the x-axis.
Solution: Given that

$$\mathbf{u} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, f = -3 \tag{9.1.1}$$

Hence, the centre and radius are given as

$$\mathbf{c} = -\mathbf{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, r = \sqrt{\|\mathbf{u}\|^2 - f} = 2 \tag{9.1.2}$$

From (9.1.7.1), the points of contact for the tangent are given by

$$\mathbf{q}_{ij} = \left(\pm r \frac{\mathbf{n}_j}{\|\mathbf{n}_j\|} - \mathbf{u} \right) \text{ i,j} = 1, 2 \tag{9.1.3}$$

Since, tangents are parallel to the x-axis, the normal is given as

$$\mathbf{n} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{9.1.4}$$

Substituting in (9.1.3) we get

$$\mathbf{q}_{11} = \left(\pm 2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right) \tag{9.1.5}$$

$$= \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{9.1.6}$$

Hence, the two points of contact are

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ -2 \end{pmatrix} \tag{9.1.7}$$

See Fig. 9.1.1.

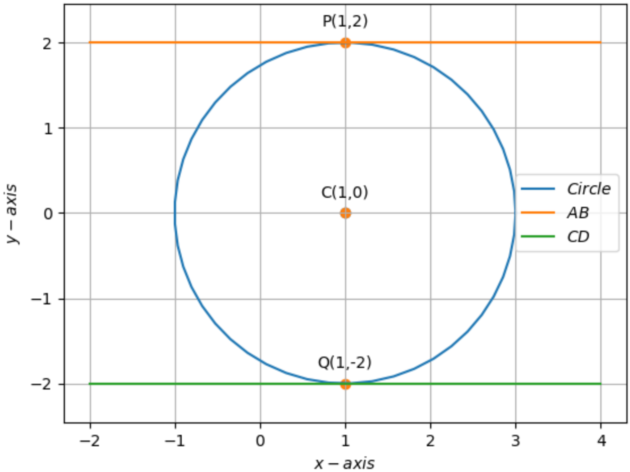


Fig. 9.1.1

9.2 Find the equation of a circle of redius 5 which is touching another circle $x^2 + y^2 - 2x - 4y - 20 = 0$ at (5,5).

9.3 The equation of the circle having centre at (3,-4) and touching the line $5x+12y-12 = 0$ is ______

9.4 Find the equation of the circle which touches both the axes in first quadrant and whose radius is a.

9.5 Find the equation of the circle which touches x-axis and whose centre is $(1, 2)$

9.6 If the lines $3x - 4y + 4 = 0$ and $6x - 8y - 7 = 0$ are tangents to a circle, then find the radius of the circle.

9.7 Find the equation of a circle which touches both the axes and the line $3x-4y+8 = 0$ and lies in the third quadrant.

9.8 At what points on the curve $x^2 + y^2 - 2x - 4y + 1 = 0$, the tangents are parallel to the y-axis?

9.9 The shortest distance from the point (2,7) to the circle $x^2 + y^2 - 14x - 10y - 151 = 0$ is equal to 5.

9.10 If the line $lx + my = 1$ is a tangent to the circle $x^2 + y^2 = a^2$, then the point $(1, m)$ lies an a circle.

9.3 Conic

9.3.1 Find the slope of the tangent to the curve $y = \frac{x-1}{x-2}$, $x \neq 2$ at $x = 10$.
Solution: The given equation of the curve can be rearranged as

$$xy - x - 2y + 1 = 0 \tag{9.3.1.1}$$

$$\implies \mathbf{x}^\top \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -1 & -2 \end{pmatrix} \mathbf{x} + 1 = 0 \tag{9.3.1.2}$$

Thus,

$$\mathbf{V} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \tag{9.3.1.3}$$

$$\mathbf{u} = -\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} \tag{9.3.1.4}$$

$$f = 1 \tag{9.3.1.5}$$

$\because q_1 = 10$, the point of contact can be obtained as

$$\mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 10 \\ \frac{9}{8} \end{pmatrix} \tag{9.3.1.6}$$

From (9.1.1.1), the normal vector of the tangent to (9.3.1.2) is

$$\mathbf{n} = \begin{pmatrix} 1 \\ 64 \end{pmatrix} \implies \mathbf{m} = \begin{pmatrix} 1 \\ \frac{-1}{64} \end{pmatrix} \tag{9.3.1.7}$$

The eigenvector matrix

$$\begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{9.3.1.8}$$

which implies that the conic is a 45° rotated hyperbola. See Fig. 9.3.1.1.

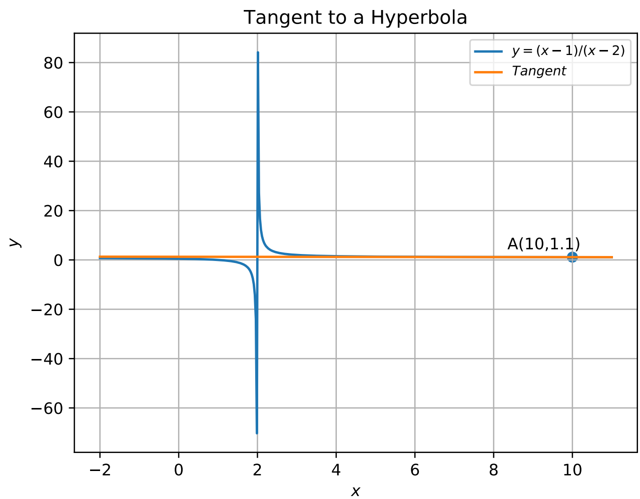


Fig. 9.3.1.1

9.3.2 Find a point on the curve

$$y = (x-2)^2 \tag{9.3.2.1}$$

at which a tangent is parallel to the chord joining the points (2,0) and (4,4).
Solution:

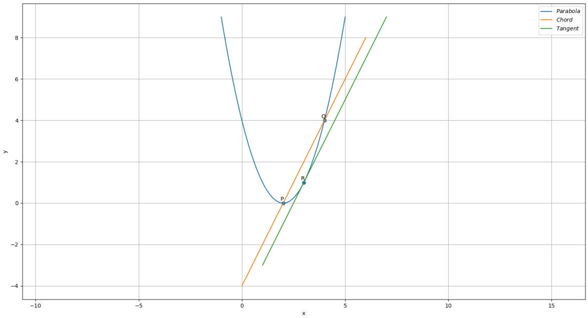


Fig. 9.3.2.1

The equation of the conic can be represented as

$$\mathbf{x}^\top \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + 2\begin{pmatrix} -2 & \frac{-1}{2} \end{pmatrix} \mathbf{x} + 4 = 0 \tag{9.3.2.2}$$

So,

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u}^\top = \begin{pmatrix} -2 & \frac{-1}{2} \end{pmatrix}, f = 4 \tag{9.3.2.3}$$

The direction vector of the line passing through (2,0) and (4,4) is

$$\mathbf{m} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \implies \mathbf{n} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}. \tag{9.3.2.4}$$

The eigenvector corresponding to the zero eigenvalue is

$$\mathbf{p}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{9.3.2.5}$$

In (9.1.5.1),

$$\kappa = \frac{\begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} -2 \\ \frac{-1}{2} \end{pmatrix}}{\begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix}} = \frac{1}{2} \tag{9.3.2.6}$$

Substituting κ, from (9.1.5.1),

$$\begin{pmatrix} \left[\begin{pmatrix} -2 \\ \frac{-1}{2} \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 2 \\ -1 \end{pmatrix}\right]^\top \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \mathbf{q} = \begin{pmatrix} -4 \\ \frac{1}{2}\begin{pmatrix} 2 \\ -1 \end{pmatrix} - \begin{pmatrix} -2 \\ \frac{-1}{2} \end{pmatrix} \end{pmatrix} \tag{9.3.2.7}$$

$$\implies \begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{q} = \begin{pmatrix} -4 \\ 3 \\ 0 \end{pmatrix} \tag{9.3.2.8}$$

yielding

$$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{q} = \begin{pmatrix} -4 \\ 3 \end{pmatrix} \tag{9.3.2.9}$$

The augmented matrix is

$$\left(\begin{array}{cc|c} -1 & -1 & -4 \\ 1 & 0 & 3 \end{array}\right) \xleftrightarrow{R_1 \leftarrow R_1 + 2R_2} \left(\begin{array}{cc|c} 1 & -1 & 2 \\ 1 & 0 & 3 \end{array}\right)$$
$$\xleftrightarrow{R_2 \leftarrow R_2 - R_1} \left(\begin{array}{cc|c} 1 & -1 & 2 \\ 0 & 1 & 1 \end{array}\right) \xleftrightarrow{R_1 \leftarrow R_1 + R_2} \left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 1 \end{array}\right)$$
$$\implies \mathbf{q} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

which is the desired point of contact. See Fig. 9.3.2.1.

9.3.3 Find the equation of all lines having slope -1 that are tangents to the curve

$$y = \frac{1}{x-1}, x \neq 1 \tag{9.3.3.1}$$

Solution:

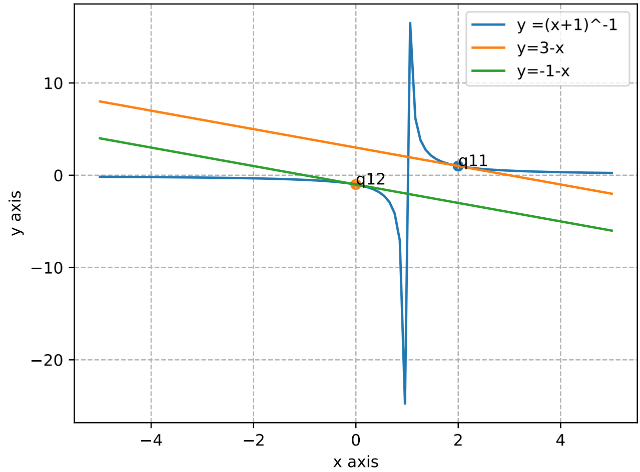


Fig. 9.3.3.1

From the given information,

$$\mathbf{V} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix}, f = -1, m = -1 \tag{9.3.3.2}$$

From the above, the normal vector is

$$\mathbf{n} = \begin{pmatrix} -m \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{9.3.3.3}$$

From (9.1.4.1), the point(s) of contact are given by

$$\mathbf{q} = \mathbf{V}^{-1}(k_i\mathbf{n} - \mathbf{u}) \text{ where,} \tag{9.3.3.4}$$

$$k_i = \pm\sqrt{\frac{f_0}{\mathbf{n}^\top\mathbf{V}^{-1}\mathbf{n}}} \tag{9.3.3.5}$$

$$f_0 = f + \mathbf{u}^\top\mathbf{V}^{-1}\mathbf{u} \tag{9.3.3.6}$$

Substituting from (9.3.3.3) and (9.3.3.2) in the above,

$$\mathbf{q} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}. \tag{9.3.3.7}$$

From (9.1.2.1), the equations of tangents are given by

$$(\mathbf{V}\mathbf{q} + \mathbf{u})^\top\mathbf{x} + \mathbf{u}^\top\mathbf{q} + f = 0 \tag{9.3.3.8}$$

yielding

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} + 1 = 0 \tag{9.3.3.9}$$

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} - 3 = 0 \tag{9.3.3.10}$$

$$\tag{9.3.3.11}$$

See Fig. 9.3.3.1.

9.3.4 Find the equation of all lines having slope 2 which are tangents to the curve

$$y = \frac{1}{x-3}, x \neq 3 \tag{9.3.4.1}$$

Solution:

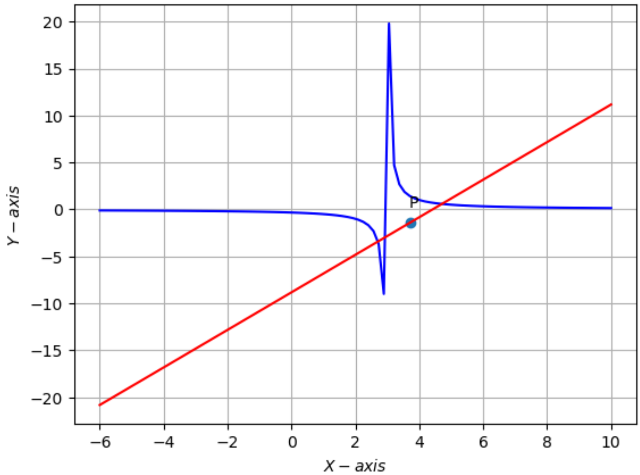


Fig. 9.3.4.1

From the given information

$$\mathbf{V} = \begin{pmatrix}0 & \frac{1}{2}\\ \frac{1}{2} & 0\end{pmatrix}, \mathbf{u} = \begin{pmatrix}0\\ -\frac{3}{2}\end{pmatrix}, f = -1, m = 2 \tag{9.3.4.2}$$

$$\implies \mathbf{n} = \begin{pmatrix}-m\\ 1\end{pmatrix} = \begin{pmatrix}-2\\ 1\end{pmatrix} \tag{9.3.4.3}$$

$$\tag{9.3.4.4}$$

Hence, the given curve is a hyperbola. Substituting numerical values, we obtain the condition in (9.1.6), which implies that the line with slope 2 is not a tangent. This can be verified from Fig. 9.3.4.1.

9.3.5 Find points on the curve $\frac{x^2}{9} + \frac{y^2}{16} = 1$ at which the tangents are

a) parallel to x-axis

b) parallel to y-axis

Solution:

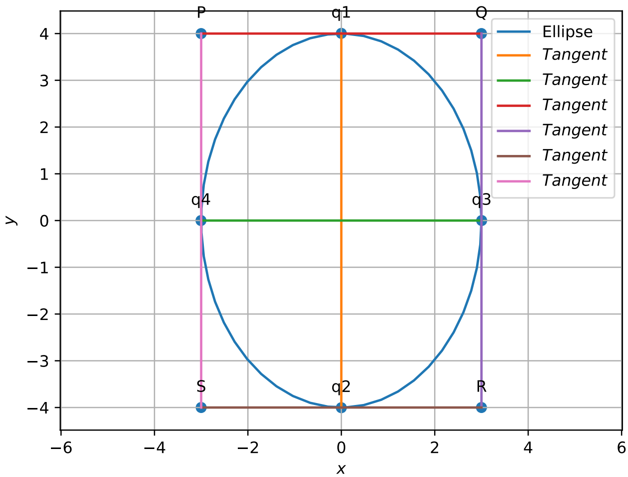


Fig. 9.3.5.1

The parameters of the given conic are

$$\lambda_1 = 16, \lambda_2 = 9 \tag{9.3.5.1}$$

$$\mathbf{V} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, f = -144 \tag{9.3.5.2}$$

a) The normal vector in this case is

$$\mathbf{n_1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{9.3.5.3}$$

which can be used along with the parameters in (9.3.5.2) to obtain

$$\mathbf{q_1} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}, \mathbf{q_2} = \begin{pmatrix} 0 \\ -4 \end{pmatrix} \tag{9.3.5.4}$$

using (9.1.4.1).

b) Simlarly, choosing

$$\mathbf{n_2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{9.3.5.5}$$

$$\mathbf{q_3} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \mathbf{q_4} = \begin{pmatrix} -3 \\ 0 \end{pmatrix} \tag{9.3.5.6}$$

See Fig. 9.3.5.1.

9.3.6 Find the equation of the tangent line to the curve

$$y = x^2 - 2x + 7 \tag{9.3.6.1}$$

a) parallel to the line $2x - y + 9 = 0$.
b) perpendicular to the line $5y - 15x = 13$.

Solution:

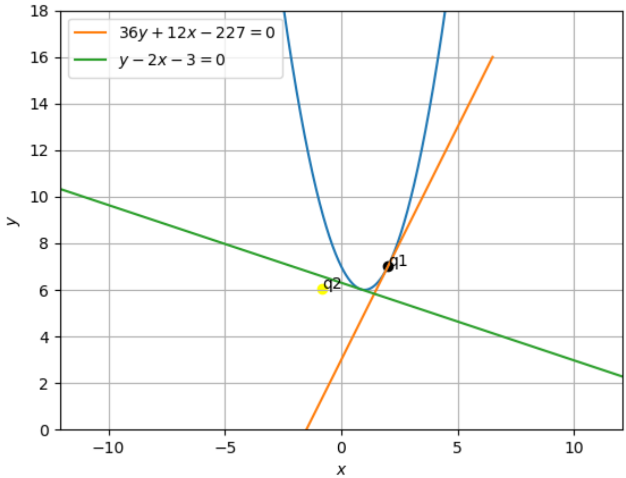


Fig. 9.3.6.1

The parameters of the given conic are

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = -\begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix}, f = 7 \tag{9.3.6.2}$$

a) In this case, the normal vector

$$\mathbf{n}_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \tag{9.3.6.3}$$

Since $\mathbf{V}$ is not invertible, the point of contact is given by (9.1.5.1) resulting in

$$\begin{pmatrix} \left(\begin{pmatrix} -1 \\ -\frac{1}{2} \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 2 \\ -1 \end{pmatrix}\right)^\top \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \mathbf{q}_1 = \begin{pmatrix} -7 \\ \frac{1}{2}\begin{pmatrix} 2 \\ -1 \end{pmatrix} - \begin{pmatrix} -1 \\ -\frac{1}{2} \end{pmatrix} \end{pmatrix} \tag{9.3.6.4}$$

By solving the above equation, we can get the point of contact as

$$\mathbf{q}_1 = \begin{pmatrix} 2 \\ 7 \end{pmatrix} \tag{9.3.6.5}$$

The tangent equation is then obtained as

$$\mathbf{n}_1^\top (\mathbf{x} - \mathbf{q}_1) = 0 \tag{9.3.6.6}$$

$$\implies \begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} + 3 = 0 \tag{9.3.6.7}$$

b) In this case,

$$\mathbf{n}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \tag{9.3.6.8}$$

resulting in

$$\begin{pmatrix} \left(\begin{pmatrix} -1 \\ -\frac{1}{2} \end{pmatrix} + -\frac{1}{6}\begin{pmatrix} 1 \\ 3 \end{pmatrix}\right)^\top \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \mathbf{q}_2 = \begin{pmatrix} -7 \\ -\frac{1}{6}\begin{pmatrix} 1 \\ 3 \end{pmatrix} - \begin{pmatrix} -1 \\ -\frac{1}{2} \end{pmatrix} \end{pmatrix} \tag{9.3.6.9}$$

$$\text{or, } \mathbf{q}_2 = \begin{pmatrix} \frac{5}{6} \\ \frac{217}{36} \end{pmatrix} \tag{9.3.6.10}$$

The tangent equation is

$$\mathbf{n}_2^\top (\mathbf{x} - \mathbf{q}_2) = 0 \tag{9.3.6.11}$$

$$\text{or, } \begin{pmatrix} 1 & 3 \end{pmatrix} \mathbf{x} = \frac{227}{12} \tag{9.3.6.12}$$

See Fig. 9.3.6.1.

9.3.7 Find the equation of the tangent to the curve

$$y = \sqrt{3x - 2} \tag{9.3.7.1}$$

which is parallel to the line

$$4x - 2y + 5 = 0 \tag{9.3.7.2}$$

Solution: The parameters for the given conic are

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{9.3.7.3}$$

$$\mathbf{u} = \begin{pmatrix} -3/2 \\ 0 \end{pmatrix}, \tag{9.3.7.4}$$

$$f = 2 \tag{9.3.7.5}$$

which represent a parabola. Following the approach in Problem 9.3.6,

$$\mathbf{p_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \mathbf{n} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \tag{9.3.7.6}$$

yielding the matrix equation

$$\begin{pmatrix} -3 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{q} = \begin{pmatrix} -41/16 \\ 0 \\ 3/4 \end{pmatrix} \tag{9.3.7.7}$$

$$\tag{9.3.7.8}$$

The augmented matrix for (9.3.7.7) can be expressed as

$$\xleftrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{cc|c} -3 & 0 & -41/16 \\ 0 & 1 & 0 \\ 0 & 0 & 3/4 \end{array}\right) \xleftrightarrow{-\frac{R_1}{-3} \leftarrow R_2} \left(\begin{array}{cc|c} 1 & 0 & 41/48 \\ 0 & 1 & 0 \\ 0 & 0 & 3/4 \end{array}\right)$$

$$\implies \mathbf{q} = \begin{pmatrix} \frac{41}{48} \\ \frac{3}{4} \end{pmatrix}$$

The equation of tangent is then obtained as

$$\begin{pmatrix} -2 & 1 \end{pmatrix} \mathbf{x} + \frac{23}{24} = 0 \tag{9.3.7.9}$$

See Fig. 9.3.7.1.

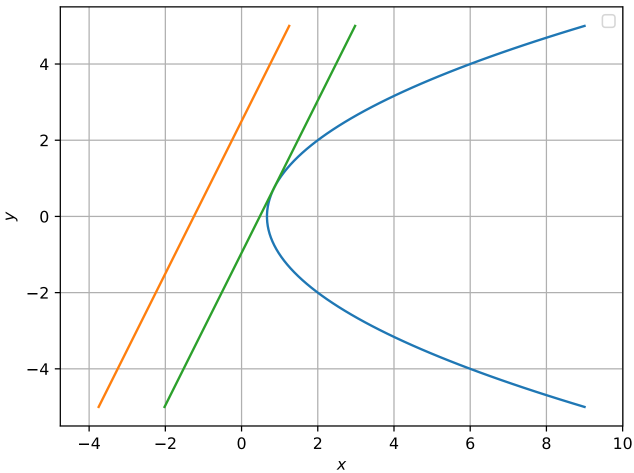


Fig. 9.3.7.1

9.3.8 Find the point at which the line $y = x + 1$ is a tangent to the curve $y^2 = 4x$.
Solution:

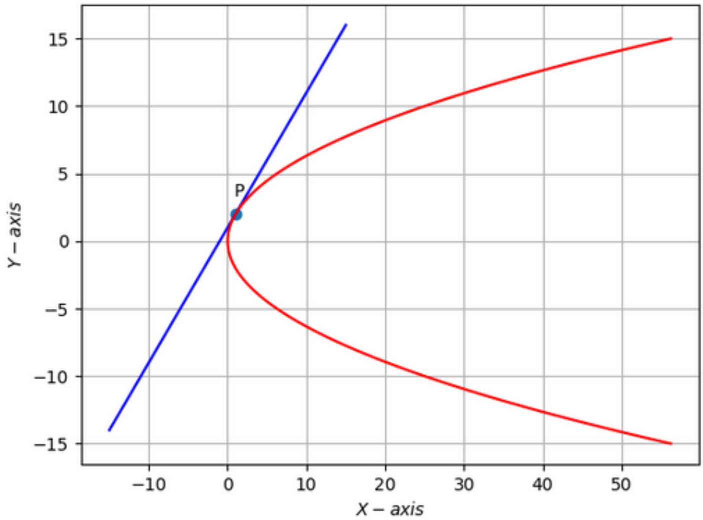


Fig. 9.3.8.1

The parameters of the conic are

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} -2 & 0 \end{pmatrix}, f = 0 \tag{9.3.8.1}$$

Following the approach in Problem 9.3.6, since

$$\mathbf{n} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{9.3.8.2}$$

we obtain

$$\mathbf{q} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{9.3.8.3}$$

See Fig. 9.3.8.1.

9.3.9 The point on the curve

$$x^2 = 2y \tag{9.3.9.1}$$

which is nearest to the point $\mathbf{P} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$ is

a) $\begin{pmatrix} 2\sqrt{2} \\ 4 \end{pmatrix}$

b) $\begin{pmatrix} 2\sqrt{2} \\ 0 \end{pmatrix}$

c) $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

d) $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$

Solution: We rewrite the conic (9.3.9.1) in matrix form.

$$\mathbf{x}^\top \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} 0 & -1 \end{pmatrix} \mathbf{x} = 0 \tag{9.3.9.2}$$

Comparing with the general equation of the conic,

$$\mathbf{V}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ \mathbf{u}_0 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \ f_0 = 0 \tag{9.3.9.3}$$

Therefore, the equation of the normal where $\mathbf{q}$ is the point of contact and

$$\mathbf{R} \triangleq \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{9.3.9.4}$$

is

$$(\mathbf{V}_0\mathbf{q} + \mathbf{u}_0)^\top \mathbf{R} \left(\begin{pmatrix} 0 \\ 5 \end{pmatrix} - \mathbf{q} \right) = 0 \tag{9.3.9.5}$$

Substituting appropriate values and simplifying, we get

$$\mathbf{q}^\top \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mathbf{q} + 2 \begin{pmatrix} -2 & 0 \end{pmatrix} \mathbf{q} = 0 \tag{9.3.9.6}$$

which can be expressed as

$$\frac{1}{2} \left\{ \mathbf{q}^\top \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mathbf{q} + 2 \begin{pmatrix} -2 & 0 \end{pmatrix} \mathbf{q} \right. \\ \left. + \mathbf{q}^\top \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^\top \mathbf{q} + 2 \begin{pmatrix} -2 & 0 \end{pmatrix} \mathbf{q} \right\} = 0 \tag{9.3.9.7}$$

yielding

$$\mathbf{q}^\top \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \mathbf{q} + 2 \begin{pmatrix} -2 & 0 \end{pmatrix} \mathbf{q} = 0 \tag{9.3.9.8}$$

(9.3.9.8) also looks like a conic with parameters

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ \mathbf{u} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \ f = 0 \tag{9.3.9.9}$$

The eigenparameters of $\mathbf{V}$ are

$$\mathbf{P} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \ \mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{9.3.9.10}$$

Applying the affine transformation

$$\mathbf{q} = \mathbf{P}\mathbf{y} + \mathbf{c} \tag{9.3.9.11}$$

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} \tag{9.3.9.12}$$

$$f_0 = \mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f = 0 \tag{9.3.9.13}$$

$\because \det V = -\frac{1}{4} \neq 0$, using (B.4.9.1), (9.3.9.8) represents a pair of straight lines. From (B.4.8.2), (9.3.9.10), (A.8.3.4) and (A.8.3.7),

$$\mathbf{y} = \kappa \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}. \tag{9.3.9.14}$$

Hence, using (9.3.9.11),

$$\mathbf{q} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} + \kappa \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}, \tag{9.3.9.15}$$

which, upon substituting in (9.3.9.2) and solving for κ yields

$$\kappa = \pm\sqrt{2}, -2. \tag{9.3.9.16}$$

Thus, the points of contact are

$$\mathbf{q} = \left\{ \begin{pmatrix} \pm 2\sqrt{2} \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} \tag{9.3.9.17}$$

The nearest point out of these three candidates for $\mathbf{q}$ is $\begin{pmatrix} \pm 2\sqrt{2} \\ 4 \end{pmatrix}$. See Fig. 9.3.9.1.

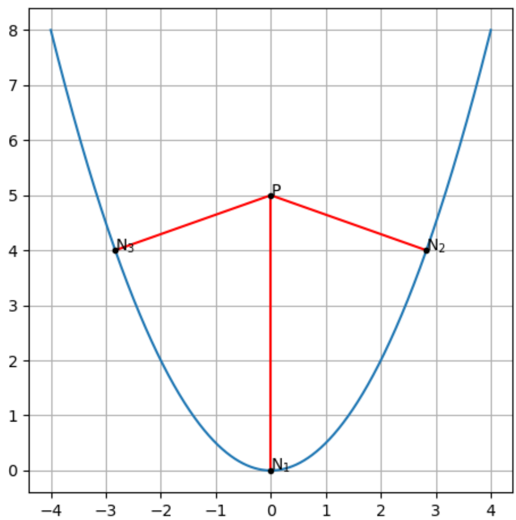


Fig. 9.3.9.1

9.3.10 Find the equation of the normal to curve $x^2 = 4y$ which passes through the point (1, 2).

Solution:

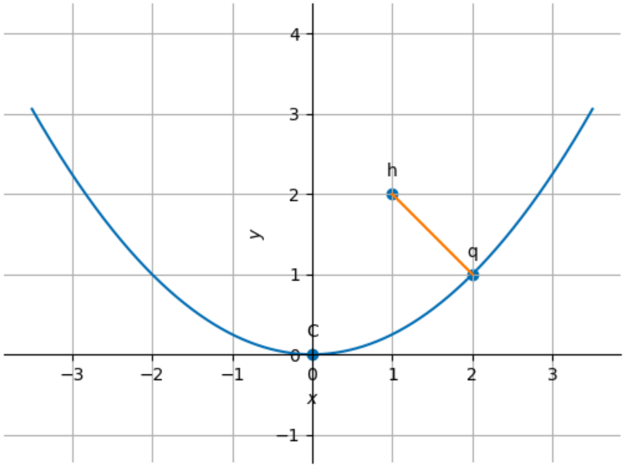


Fig. 9.3.10.1

The conic parameters are

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}, f = 0 \tag{9.3.10.1}$$

Choosing the direction and normal vectors as

$$\mathbf{m} = \begin{pmatrix} 1 \\ m \end{pmatrix}, \quad \mathbf{n} = \begin{pmatrix} -m \\ 1 \end{pmatrix}, \tag{9.3.10.2}$$

and substituting these values in (9.1.8.1), we obtain

$$m = 1 \tag{9.3.10.3}$$

as the only real solution. Thus,

$$\mathbf{m} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{9.3.10.4}$$

and the equation of the normal is then obtained as

$$\mathbf{m}^\top (\mathbf{x} - \mathbf{h}) = 0 \tag{9.3.10.5}$$

$$\implies \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{9.3.10.6}$$

$$= 3 \tag{9.3.10.7}$$

See Fig. 9.3.10.1.

9.3.11 The line $y = mx + 1$ is a tangent to the curve $y^2 = 4x$, find the value of m.

Solution:

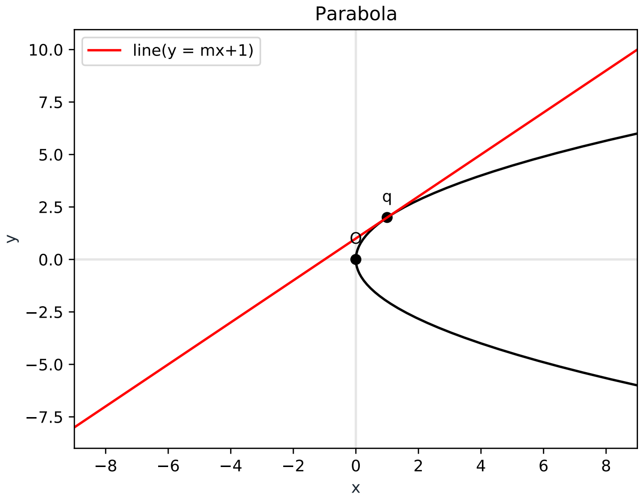


Fig. 9.3.11.1

The parameters for the given conic are

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}, f = 0 \tag{9.3.11.1}$$

The given tangent can be expressed in parametric form as

$$\mathbf{x} = \mathbf{e}_2 + \mu \mathbf{m} \tag{9.3.11.2}$$

Substituting from (9.3.11.2) and (9.3.11.1) in (9.1.9.1) and solving, we obtain

$$m = 1. \tag{9.3.11.3}$$

See Fig. 9.3.11.1.

9.3.12 Find the normal at the point (1,1) on the curve

$$2y + x^2 = 3 \tag{9.3.12.1}$$

Solution: Use (9.1.1.1) with

$$\mathbf{m} = \begin{pmatrix} 1 \\ m \end{pmatrix} \tag{9.3.12.2}$$

9.3.13 If the line $y = \sqrt{3}x + K$ touches the parabola $x^2 = 16y$, then find the value of K.

9.3.14 If the line $y = mx + 1$ is tangent to the parabola $y^2 = 4x$ then find the value of m.

9.3.15 Find the condition that the curves $2x = y^2$ and $2xy = k$ intersect orthogonally.

9.3.16 Prove that the curves $xy = 4$ and $x^2 + y^2 = 8$ touch each other.
9.3.17 Find the angle of intersection of the curves $y = 4 - x^2$ and $y = x^2$.
9.3.18 Prove that the curves $y^2 = 4x$ and $x^2 + y^2 - 6x + 1 = 0$ touch each each other at the point (1,2).
9.3.19 Find the equation of the normal lines to the curve $3x^2 - y^2 = 8$ which are parallel to the line $x + 3y = 4$.
9.3.20 The equation of the normal to the curve $3x^2 - y^2 = 8$ which is parallel to the line $x + 3y = 8$ is

a) $3x - y = 8$
b) $3x + y + 8 = 0$
c) $x + 3y + 8 = 0$
d) $x + 3y = 0$

9.3.21 The equation of the tangent to the curve $(1 + y^2) = 2 - x$, where it crosses the x-axis is

a) $x + 5y = 2$
b) $x - 5y = 2$
c) $5x - y = 2$
d) $5x + y = 2$

State whether the statements are True or False

9.22 The line $lx + my + n = 0$ will touch the parabola $y^2 = 4ax$ if $ln = am^2$,
9.23 The line $2x + 3y = 12$ touches the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 2$ at the point (3,2).

9.4 Construction

9.4.1 Let ABC be a right triangle in which $AB = 6cm, BC = 8cm$ and $\angle B = 90°$. BD is the perpendicular from **B** on AC. The circle through **B**, **C**, **D** is drawn. Construct the tangents from **A** to this circle.
9.4.2 Draw a line segment AB of length 8cm. Taking **A** as centre, draw a circle of radius 4cm and taking **B** as centre, draw another circle of radius 3cm. Construct tangents to each circle from the centre of the circle.
9.4.3 Draw a pair of tangents to a circle of radius 5 cm which are inclined to each other at an angle of 60°.
9.4.4 Draw a circle of radius 3 cm. Take two points **P** and **Q** on one of its extended diameter each at a distance of 7 cm from its centre. Draw tangents to the circle from these two points **P** and **Q**.
9.4.5 Construct a tangent to a circle of radius 4cm from a point on the concentric circle of radius 6cm and measure its length. Also verify the measurement by actual calculation.

9.4.6 From a point **Q**, the length of the tangent to a circle is $24cm$ and the distance of **Q** from the centre is $25cm$. Find the radius of the circle. Draw the circle and the tangents.
9.4.7 To draw a pair of tangents to a circle which are inclined to each other at an angle of 60°, it is required to draw tangents at end points of those two radii of the circle, the angle between them should be

a) 135°
b) 90°
c) 60°
d) 120°

9.4.8 Draw two concentric circles of radii 3 cm and 5 cm. Taking a point on outer circle construct the pair of tangents to the other. Measure the length of a tangent and verify it by actual calculation.

9.4.9 Draw a circle of radius 4 cm .Construct a pair of tangents to it, the angle between which is 60°. Also justify the construction. Measure the distance between the centre of the circle and the point of intersection of tangents.

9.4.10 Construct a tangent to a circle of radius 4 cm from a point which is at a distance of 6 cm from its centre.

Write True or False and give reasons for your answer in each of the following

9.4.11 A pair of tangents can be constructed from a point **h** to a circle of radius 3.5 cm situated at a distance of 3 cm from the centre.

9.4.12 A pair of tangents can be constructed to a circle inclined at an angle of 170°.

Appendix A
Triangle

Consider a triangle with vertices

$$\mathbf{A} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -4 \\ 6 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -3 \\ -5 \end{pmatrix} \tag{12.1}$$

A.1 Sides

A.1.1. The direction vector of AB is defined as

$$\mathbf{B} - \mathbf{A} \tag{A.1.1.1}$$

Find the direction vectors of AB, BC and CA.

Solution:

a) The Direction vector of AB is

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} -4 \\ 6 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -4 - 1 \\ 6 - (-1) \end{pmatrix} = \begin{pmatrix} -5 \\ 7 \end{pmatrix} \tag{A.1.1.2}$$

b) The Direction vector of BC is

$$\mathbf{C} - \mathbf{B} = \begin{pmatrix} -3 \\ -5 \end{pmatrix} - \begin{pmatrix} -4 \\ 6 \end{pmatrix} = \begin{pmatrix} -3 - (-4) \\ -5 - 6 \end{pmatrix} = \begin{pmatrix} 1 \\ -11 \end{pmatrix} \tag{A.1.1.3}$$

c) The Direction vector of CA is

$$\mathbf{A} - \mathbf{C} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \begin{pmatrix} -3 \\ -5 \end{pmatrix} = \begin{pmatrix} 1 - (-3) \\ -1 - (-5) \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \tag{A.1.1.4}$$

A.1.2. The length of side BC is

$$c = \|\mathbf{B} - \mathbf{A}\| \triangleq \sqrt{(\mathbf{B} - \mathbf{A})^\top (\mathbf{B} - \mathbf{A})} \tag{A.1.2.1}$$

where

$$\mathbf{A}^{\top} \triangleq \begin{pmatrix} 1 & -1 \end{pmatrix} \tag{A.1.2.2}$$

Similarly,

$$b = \|\mathbf{C} - \mathbf{B}\|, \; a = \|\mathbf{A} - \mathbf{C}\| \tag{A.1.2.3}$$

Find a, b, c.

a) From (A.1.1.2),

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} 5 \\ -7 \end{pmatrix}, \tag{A.1.2.4}$$

$$\implies c = \|\mathbf{B} - \mathbf{A}\| = \|\mathbf{A} - \mathbf{B}\| \tag{A.1.2.5}$$

$$= \sqrt{\begin{pmatrix} 5 & -7 \end{pmatrix} \begin{pmatrix} 5 \\ -7 \end{pmatrix}} = \sqrt{(5)^2 + (7)^2} \tag{A.1.2.6}$$

$$= \sqrt{74} \tag{A.1.2.7}$$

b) Similarly, from (A.1.1.3),

$$a = \|\mathbf{B} - \mathbf{C}\| = \sqrt{\begin{pmatrix} -1 & 11 \end{pmatrix} \begin{pmatrix} -1 \\ 11 \end{pmatrix}} \tag{A.1.2.8}$$

$$= \sqrt{(1)^2 + (11)^2} = \sqrt{122} \tag{A.1.2.9}$$

and from (A.1.1.4),

c)

$$b = \|\mathbf{A} - \mathbf{C}\| = \sqrt{\begin{pmatrix} 4 & 4 \end{pmatrix} \begin{pmatrix} 4 \\ 4 \end{pmatrix}} \tag{A.1.2.10}$$

$$= \sqrt{(4)^2 + (4)^2} = \sqrt{32} \tag{A.1.2.11}$$

A.1.3. Points $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are defined to be collinear if

$$\text{rank} \begin{pmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} = 2 \tag{A.1.3.1}$$

Are the given points in (12.1) collinear?
Solution: From (12.1),

$$\begin{pmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -4 & -3 \\ -1 & 6 & -5 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 + R_2} \begin{pmatrix} 1 & 1 & 1 \\ 1 & -4 & -3 \\ 0 & 2 & -8 \end{pmatrix} \tag{A.1.3.2}$$

$$\xleftrightarrow{R_2 \leftarrow R_1 - R_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 5 & 4 \\ 0 & 2 & -8 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 - \frac{2}{5} R_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 5 & 4 \\ 0 & 0 & \frac{-48}{5} \end{pmatrix} \tag{A.1.3.3}$$

There are no zero rows. So,

$$\operatorname{rank}\begin{pmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} = 3 \tag{A.1.3.4}$$

Hence, the points $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are not collinear. This is visible in Fig. A.1.3.1.

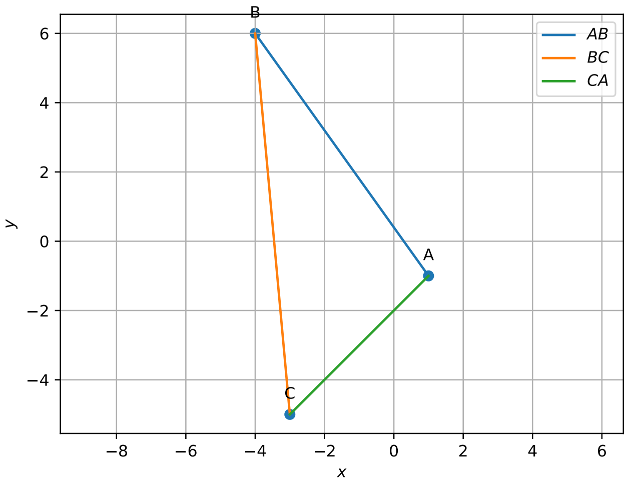


Fig. A.1.3.1: $\triangle ABC$

A.1.4. The parameteric form of the equation of AB is

$$\mathbf{x} = \mathbf{A} + k\mathbf{m} \quad k \neq 0, \tag{A.1.4.1}$$

where

$$\mathbf{m} = \mathbf{B} - \mathbf{A} \tag{A.1.4.2}$$

is the direction vector of AB. Find the parameteric equations of AB, BC and CA.
Solution: From (A.1.4.1) and (A.1.1.2), the parametric equation for AB is given by

$$AB : \mathbf{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + k\begin{pmatrix} -5 \\ 7 \end{pmatrix} \tag{A.1.4.3}$$

Similarly, from (A.1.1.3) and (A.1.1.4),

$$BC : \mathbf{x} = \begin{pmatrix} -4 \\ 6 \end{pmatrix} + k\begin{pmatrix} 1 \\ -11 \end{pmatrix} \tag{A.1.4.4}$$

$$CA : \mathbf{x} = \begin{pmatrix} -3 \\ -5 \end{pmatrix} + k\begin{pmatrix} 4 \\ 4 \end{pmatrix} \tag{A.1.4.5}$$

A.1.5. The normal form of the equation of AB is

$$\mathbf{n}^{\top}(\mathbf{x}-\mathbf{A})=0 \tag{A.1.5.1}$$

where

$$\mathbf{n}^{\top}\mathbf{m}=\mathbf{n}^{\top}(\mathbf{B}-\mathbf{A})=0 \tag{A.1.5.2}$$

$$\text{or, } \mathbf{n}=\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}\mathbf{m} \tag{A.1.5.3}$$

Find the normal form of the equations of AB, BC and CA.
Solution:

a) From (A.1.1.3), the direction vector of side **BC** is

$$\mathbf{m}=\begin{pmatrix}1\\-11\end{pmatrix} \tag{A.1.5.4}$$

$$\implies \mathbf{n}=\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}\begin{pmatrix}1\\-11\end{pmatrix}=\begin{pmatrix}-11\\-1\end{pmatrix} \tag{A.1.5.5}$$

from (A.1.5.3). Hence, from (A.1.5.1), the normal equation of side BC is

$$\mathbf{n}^{\top}(\mathbf{x}-\mathbf{B})=0 \tag{A.1.5.6}$$

$$\implies \begin{pmatrix}-11 & -1\end{pmatrix}\mathbf{x}=\begin{pmatrix}-11 & -1\end{pmatrix}\begin{pmatrix}-4\\6\end{pmatrix} \tag{A.1.5.7}$$

$$\implies BC: \quad \begin{pmatrix}11 & 1\end{pmatrix}\mathbf{x}=-38 \tag{A.1.5.8}$$

b) Similarly, for AB, from (A.1.1.2),

$$\mathbf{m}=\begin{pmatrix}-5\\7\end{pmatrix} \tag{A.1.5.9}$$

$$\implies \mathbf{n}=\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}\begin{pmatrix}-5\\7\end{pmatrix}=\begin{pmatrix}7\\5\end{pmatrix} \tag{A.1.5.10}$$

and

$$\mathbf{n}^{\top}(\mathbf{x}-\mathbf{A})=0 \tag{A.1.5.11}$$

$$\implies AB: \quad \mathbf{n}^{\top}\mathbf{x}=\begin{pmatrix}7 & 5\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix} \tag{A.1.5.12}$$

$$\implies \begin{pmatrix}7 & 5\end{pmatrix}\mathbf{x}=2 \tag{A.1.5.13}$$

c) For CA, from (A.1.1.4),

$$\mathbf{m} = \begin{pmatrix}1\\1\end{pmatrix} \tag{A.1.5.14}$$

$$\implies \mathbf{n} = \begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\-1\end{pmatrix} \tag{A.1.5.15}$$

$$\tag{A.1.5.16}$$

$$\implies \mathbf{n}^\top(\mathbf{x} - \mathbf{C}) = 0 \tag{A.1.5.17}$$

$$\implies \begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = \begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}-3\\-5\end{pmatrix} = 2 \tag{A.1.5.18}$$

A.1.6. The area of $\triangle ABC$ is defined as

$$\frac{1}{2}\left\|(\mathbf{A} - \mathbf{B}) \times (\mathbf{A} - \mathbf{C})\right\| \tag{A.1.6.1}$$

where

$$\mathbf{A} \times \mathbf{B} \triangleq \begin{vmatrix}1 & -4\\-1 & 6\end{vmatrix} \tag{A.1.6.2}$$

Find the area of $\triangle ABC$.

Solution: From (A.1.1.2) and (A.1.1.4),

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix}5\\-7\end{pmatrix}, \mathbf{A} - \mathbf{C} = \begin{pmatrix}4\\4\end{pmatrix} \tag{A.1.6.3}$$

$$\implies (\mathbf{A} - \mathbf{B}) \times (\mathbf{A} - \mathbf{C}) = \begin{vmatrix}5 & 4\\-7 & 4\end{vmatrix} \tag{A.1.6.4}$$

$$= 5 \times 4 - 4 \times (-7) \tag{A.1.6.5}$$

$$= 48 \tag{A.1.6.6}$$

$$\implies \frac{1}{2}\left\|(\mathbf{A} - \mathbf{B}) \times (\mathbf{A} - \mathbf{C})\right\| = \frac{48}{2} = 24 \tag{A.1.6.7}$$

which is the desired area.

A.1.7. Find the angles A, B, C if

$$\cos A \triangleq \frac{(\mathbf{B} - \mathbf{A})^\top \mathbf{C} - \mathbf{A}}{\left\|\mathbf{B} - \mathbf{A}\right\|\left\|\mathbf{C} - \mathbf{A}\right\|} \tag{A.1.7.1}$$

Solution:

a) From (A.1.1.2), (A.1.1.4), (A.1.2.7) and (A.1.2.11)

$$(\mathbf{B}-\mathbf{A})^\top(\mathbf{C}-\mathbf{A}) = \begin{pmatrix}-5 & 7\end{pmatrix}\begin{pmatrix}-4\\-4\end{pmatrix} \tag{A.1.7.2}$$

$$= -8 \tag{A.1.7.3}$$

$$\implies \cos A = \frac{-8}{\sqrt{74}\sqrt{32}} = \frac{-1}{\sqrt{37}} \tag{A.1.7.4}$$

$$\implies A = \cos^{-1}\frac{-1}{\sqrt{37}} \tag{A.1.7.5}$$

b) From (A.1.1.2), (A.1.1.3), (A.1.2.7) and (A.1.2.9)

$$(\mathbf{C}-\mathbf{B})^\top(\mathbf{A}-\mathbf{B}) = \begin{pmatrix}1 & -11\end{pmatrix}\begin{pmatrix}5\\-7\end{pmatrix} \tag{A.1.7.6}$$

$$= 82 \tag{A.1.7.7}$$

$$\implies \cos B = \frac{82}{\sqrt{74}\sqrt{122}} = \frac{41}{\sqrt{2257}} \tag{A.1.7.8}$$

$$\implies B = \cos^{-1}\frac{41}{\sqrt{2257}} \tag{A.1.7.9}$$

c) From (A.1.1.3), (A.1.1.4), (A.1.2.9) and (A.1.2.11)

$$(\mathbf{A}-\mathbf{C})^\top(\mathbf{B}-\mathbf{C}) = \begin{pmatrix}4 & 4\end{pmatrix}\begin{pmatrix}-1\\11\end{pmatrix} \tag{A.1.7.10}$$

$$= 40 \tag{A.1.7.11}$$

$$\implies \cos C = \frac{40}{\sqrt{32}\sqrt{122}} = \frac{5}{\sqrt{61}} \tag{A.1.7.12}$$

$$\implies C = \cos^{-1}\frac{5}{\sqrt{61}} \tag{A.1.7.13}$$

All codes for this section are available at

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codes/triangle/sides.py
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A.2 Formulae

A.1. The equation of a line is given by

$$y = mx + c \tag{A.1.1}$$

$$\implies \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\mx+c\end{pmatrix} = \begin{pmatrix}0\\c\end{pmatrix} + x\begin{pmatrix}1\\m\end{pmatrix} \tag{A.1.2}$$

yielding (4.1.1.3).

A.2. (A.1.1) can also be expressed as

$$y - mx = c \tag{A.2.1}$$

$$\implies \begin{pmatrix}-m & 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = c \tag{A.2.2}$$

yielding (4.1.2.3).

A.3. The direction vector is

$$\mathbf{m} = \begin{pmatrix} 1 \\ m \end{pmatrix} \tag{A.3.1}$$

and the normal vector is

$$\mathbf{n} = \begin{pmatrix} -m \\ 1 \end{pmatrix} \tag{A.3.2}$$

A.4. From (4.1.1.3), if $\mathbf{A}, \mathbf{D}$ and $\mathbf{C}$ are on the same line,

$$\mathbf{D} = \mathbf{A} + q\mathbf{m} \tag{A.4.1}$$

$$\mathbf{C} = \mathbf{D} + p\mathbf{m} \tag{A.4.2}$$

$$\implies p(\mathbf{D} - \mathbf{A}) + q(\mathbf{D} - \mathbf{C}) = 0, \quad p, q \neq 0 \tag{A.4.3}$$

$$\implies \mathbf{D} = \frac{p\mathbf{A} + q\mathbf{C}}{p+q} \tag{A.4.4}$$

yielding (1.1.4.1) upon substituting

$$k = \frac{p}{q}. \tag{A.4.5}$$

$(\mathbf{D} - \mathbf{A}), (\mathbf{D} - \mathbf{C})$ are then said to be *linearly dependent*.

A.5. If $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are collinear, from (4.1.2.3),

$$\mathbf{n}^\top \mathbf{A} = c \tag{A.5.1}$$

$$\mathbf{n}^\top \mathbf{B} = c \tag{A.5.2}$$

$$\mathbf{n}^\top \mathbf{C} = c \tag{A.5.3}$$

which can be expressed as

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix}^\top \mathbf{n} = c \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{A.5.4}$$

$$\equiv \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix}^\top \mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \tag{A.5.5}$$

$$\implies \begin{pmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix}^\top \begin{pmatrix} \mathbf{n} \\ -1 \end{pmatrix} = \mathbf{0} \tag{A.5.6}$$

yielding

$$\text{rank} \begin{pmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} = 2 \tag{A.5.7}$$

Rank is defined to be the number of linearly indpendent rows or columns of a matrix.

A.6. The equation of a line can also be expressed as

$$\mathbf{n}^\top \mathbf{x} = 1 \tag{A.6.1}$$

A.3.1. If $\mathbf{D}$ divides BC in the ratio $k:1$,

$$\mathbf{D} = \frac{k\mathbf{C} + \mathbf{B}}{k+1} \tag{A.3.1.1}$$

Find the mid points $\mathbf{D}, \mathbf{E}, \mathbf{F}$ of the sides BC, CA and AB respectively.
Solution: Since $\mathbf{D}$ is the midpoint of BC,

$$k = 1, \tag{A.3.1.2}$$

$$\implies \mathbf{D} = \frac{\mathbf{C} + \mathbf{B}}{2} = \frac{1}{2}\begin{pmatrix} -7 \\ 1 \end{pmatrix} \tag{A.3.1.3}$$

Similarly,

$$\mathbf{E} = \frac{\mathbf{A} + \mathbf{C}}{2} = \begin{pmatrix} -1 \\ -3 \end{pmatrix} \tag{A.3.1.4}$$

$$\mathbf{F} = \frac{\mathbf{A} + \mathbf{B}}{2} = \frac{1}{2}\begin{pmatrix} -3 \\ 5 \end{pmatrix} \tag{A.3.1.5}$$

A.3.2. Find the equations of AD, BE and CF.

Solution: :

a) The direction vector of AD is

$$\mathbf{m} = \mathbf{D} - \mathbf{A} = \begin{pmatrix} \frac{-7}{2} \\ \frac{1}{2} \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -9 \\ 3 \end{pmatrix} \equiv \begin{pmatrix} -3 \\ 1 \end{pmatrix} \tag{A.3.2.1}$$

$$\implies \mathbf{n} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \tag{A.3.2.2}$$

Hence the normal equation of median AD is

$$\mathbf{n}^\top \left(\mathbf{x} - \mathbf{A}\right) = 0 \tag{A.3.2.3}$$

$$\implies \begin{pmatrix} 1 & 3 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = -2 \tag{A.3.2.4}$$

b) For BE,

$$\mathbf{m} = \mathbf{E} - \mathbf{B} = \begin{pmatrix} -1 \\ -3 \end{pmatrix} - \begin{pmatrix} -4 \\ 6 \end{pmatrix} = \begin{pmatrix} 3 \\ -9 \end{pmatrix} \equiv \begin{pmatrix} 1 \\ -3 \end{pmatrix} \tag{A.3.2.5}$$

$$\implies \mathbf{n} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \tag{A.3.2.6}$$

Hence the normal equation of median BE is

$$\mathbf{n}^\top \left(\mathbf{x} - \mathbf{B}\right) = 0 \tag{A.3.2.7}$$

$$\implies \begin{pmatrix} 3 & 1 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 3 & 1 \end{pmatrix}\begin{pmatrix} -4 \\ 6 \end{pmatrix} = -6 \tag{A.3.2.8}$$

c) For median CF,

$$\mathbf{m} = \mathbf{F} - \mathbf{C} = \begin{pmatrix} \frac{-3}{2} \\ \frac{5}{2} \end{pmatrix} - \begin{pmatrix} -3 \\ -5 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ \frac{15}{2} \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 5 \end{pmatrix} \tag{A.3.2.9}$$

$$\implies \mathbf{n} = \begin{pmatrix} 5 \\ -1 \end{pmatrix} \tag{A.3.2.10}$$

Hence the normal equation of median CF is

$$\mathbf{n}^\top \left(\mathbf{x} - \mathbf{C}\right) = 0 \tag{A.3.2.11}$$

$$\implies \begin{pmatrix} 5 & -1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 5 & -1 \end{pmatrix} \begin{pmatrix} -3 \\ -5 \end{pmatrix} = -10 \tag{A.3.2.12}$$

A.3.3. Find the intersection $\mathbf{G}$ of BE and CF.

Solution: From (A.3.2.8) and (A.3.2.12), the equations of BE and CF are, respectively,

$$\begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -6 \end{pmatrix} \tag{A.3.3.1}$$

$$\begin{pmatrix} 5 & -1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -10 \end{pmatrix} \tag{A.3.3.2}$$

From (A.3.3.1) and (A.3.3.2) the augmented matrix is

$$\begin{pmatrix} 3 & 1 & -6 \\ 5 & -1 & -10 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 + R_2} \begin{pmatrix} 8 & 0 & -16 \\ 5 & -1 & -10 \end{pmatrix} \tag{A.3.3.3}$$

$$\xleftrightarrow{R_1 \leftarrow R_1/8} \begin{pmatrix} 1 & 0 & -2 \\ 5 & -1 & -10 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - 5R_1} \begin{pmatrix} 1 & 0 & -2 \\ 0 & -1 & 0 \end{pmatrix} \tag{A.3.3.4}$$

$$\xleftrightarrow{R_2 \leftarrow -R_2} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \end{pmatrix} \tag{A.3.3.5}$$

using Gauss elimination. Therefore,

$$\mathbf{G} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \tag{A.3.3.6}$$

A.3.4. Verify that

$$\frac{BG}{GE} = \frac{CG}{GF} = \frac{AG}{GD} = 2 \tag{A.3.4.1}$$

Solution:

a) From (A.3.1.4) and (A.3.3.6),

$$\mathbf{G} - \mathbf{B} = \begin{pmatrix} 2 \\ -6 \end{pmatrix}, \mathbf{E} - \mathbf{G} = \begin{pmatrix} 1 \\ -3 \end{pmatrix} \tag{A.3.4.2}$$

$$\implies \mathbf{G} - \mathbf{B} = 2\left(\mathbf{E} - \mathbf{G}\right) \tag{A.3.4.3}$$

$$\implies \|\mathbf{G} - \mathbf{B}\| = 2\left\|\mathbf{E} - \mathbf{G}\right\| \tag{A.3.4.4}$$

$$\text{or, } \frac{BG}{GE} = 2 \tag{A.3.4.5}$$

b) From (A.3.1.5) and (A.3.3.6),

$$\mathbf{F}-\mathbf{G}=\frac{1}{2}\begin{pmatrix}1\\5\end{pmatrix},\ \mathbf{G}-\mathbf{C} \quad =\begin{pmatrix}1\\5\end{pmatrix} \tag{A.3.4.6}$$

$$\implies \mathbf{G}-\mathbf{C}=2\left(\mathbf{F}-\mathbf{G}\right) \tag{A.3.4.7}$$

$$\implies \left\|\mathbf{G}-\mathbf{C}\right\|=2\left\|\mathbf{F}-\mathbf{G}\right\| \tag{A.3.4.8}$$

$$\text{or, } \frac{CG}{GF}=2 \tag{A.3.4.9}$$

c) From (A.3.1.3) and (A.3.3.6),

$$\mathbf{G}-\mathbf{A}=\begin{pmatrix}-3\\1\end{pmatrix},\ \mathbf{D}-\mathbf{G}=\frac{1}{2}\begin{pmatrix}-3\\1\end{pmatrix} \tag{A.3.4.10}$$

$$\mathbf{G}-\mathbf{A}=2\left(\mathbf{D}-\mathbf{G}\right) \tag{A.3.4.11}$$

$$\implies \left\|\mathbf{G}-\mathbf{A}\right\|=2\left\|\mathbf{D}-\mathbf{G}\right\| \tag{A.3.4.12}$$

$$\text{or, } \frac{AG}{GD}=2 \tag{A.3.4.13}$$

From (A.3.4.5), (A.3.4.9), (A.3.4.13)

$$\frac{BG}{GE}=\frac{CG}{GF}=\frac{AG}{GD}=2 \tag{A.3.4.14}$$

A.3.5. Show that $\mathbf{A},\mathbf{G}$ and $\mathbf{D}$ are collinear.

Solution: Points $\mathbf{A},\mathbf{D},\mathbf{G}$ are defined to be collinear if

$$\operatorname{rank}\begin{pmatrix}1 & 1 & 1\\ \mathbf{A} & \mathbf{D} & \mathbf{G}\end{pmatrix}=2 \tag{A.3.5.1}$$

$$\implies \begin{pmatrix}1 & 1 & 1\\ 1 & -\frac{7}{2} & -2\\ -1 & \frac{1}{2} & 0\end{pmatrix} \xleftrightarrow{R_3\leftarrow R_3+R_2} \begin{pmatrix}1 & 1 & 1\\ 1 & -\frac{7}{2} & -2\\ 0 & -3 & -2\end{pmatrix} \tag{A.3.5.2}$$

$$\xleftrightarrow{R_2\leftarrow R_2-R_1} \begin{pmatrix}1 & 1 & 1\\ 0 & -\frac{9}{2} & -3\\ 0 & -3 & -2\end{pmatrix} \xleftrightarrow{R_3\leftarrow R_3-\frac{2}{3}R_2} \begin{pmatrix}1 & 1 & 1\\ 0 & -\frac{9}{2} & -3\\ 0 & 0 & 0\end{pmatrix} \tag{A.3.5.3}$$

Thus, the matrix (A.3.5.1) has rank 2 and the points are collinear. Thus, the medians of a triangle meet at the point $\mathbf{G}$. See Fig. A.3.5.1.

A.3.6. Verify that

$$\mathbf{G}=\frac{\mathbf{A}+\mathbf{B}+\mathbf{C}}{3} \tag{A.3.6.1}$$

$\mathbf{G}$ is known as the *centroid* of $\triangle ABC$.

Solution:

$$\mathbf{G}=\frac{\begin{pmatrix}1\\-1\end{pmatrix}+\begin{pmatrix}-4\\6\end{pmatrix}+\begin{pmatrix}-3\\-5\end{pmatrix}}{3} =\begin{pmatrix}-2\\0\end{pmatrix} \tag{A.3.6.2}$$

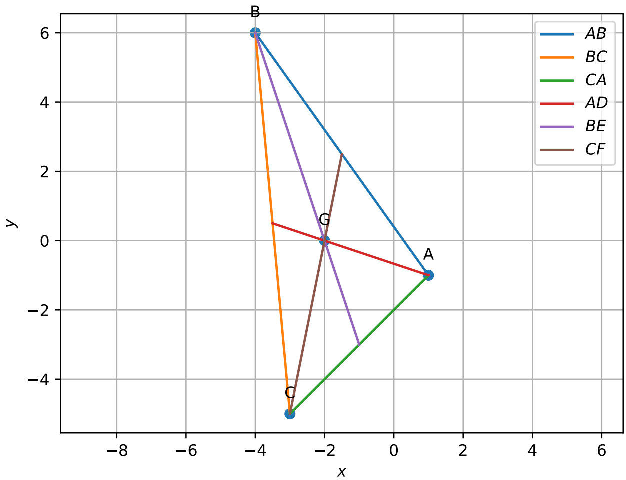

Fig. A.3.5.1: Medians of $\triangle ABC$ meet at $\mathbf{G}$.

A.3.7. Verify that

$$\mathbf{A} - \mathbf{F} = \mathbf{E} - \mathbf{D} \tag{A.3.7.1}$$

The quadrilateral $AFDE$ is defined to be a parallelogram.

Solution:

$$\mathbf{A} - \mathbf{F} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \begin{pmatrix} \frac{-3}{2} \\ \frac{5}{2} \end{pmatrix} = \begin{pmatrix} \frac{5}{2} \\ \frac{-7}{2} \end{pmatrix} \tag{A.3.7.2}$$

$$\mathbf{E} - \mathbf{D} = \begin{pmatrix} -1 \\ -3 \end{pmatrix} - \begin{pmatrix} \frac{-7}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{5}{2} \\ \frac{-7}{2} \end{pmatrix} \tag{A.3.7.3}$$

$$\implies \mathbf{A} - \mathbf{F} = \mathbf{E} - \mathbf{D} \tag{A.3.7.4}$$

See Fig. A.3.7.1,
All codes for this section are available in

```
codes/triangle/medians.py
codes/triangle/pgm.py
```

A.4 Altitude

A.4.1. $\mathbf{D}_1$ is a point on BC such that

$$AD_1 \perp BC \tag{A.4.1.1}$$

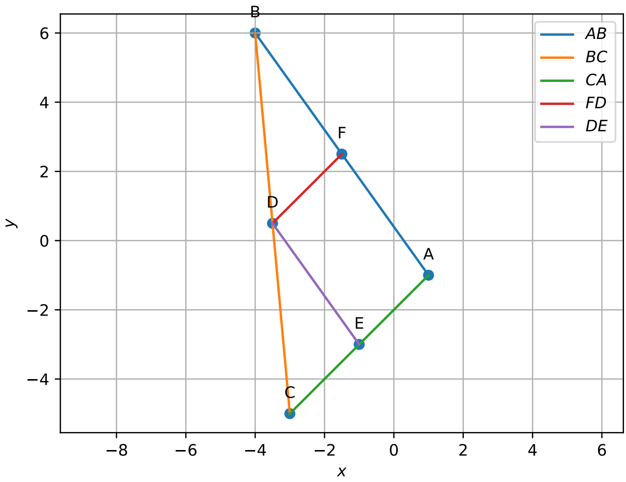

Fig. A.3.7.1: $AFDE$ forms a parallelogram in triangle ABC

and AD_1 is defined to be the altitude. Find the normal vector of AD_1.
Solution: The normal vector of AD_1 is the direction vector BC and is obtained from (A.1.1.3) as

$$\mathbf{n} = \begin{pmatrix} 1 \\ -11 \end{pmatrix} \tag{A.4.1.2}$$

A.4.2. Find the equation of AD_1.
Solution: The equation of AD_1 is

$$\mathbf{n}^\top(\mathbf{x} - \mathbf{A}) = 0 \tag{A.4.2.1}$$

$$\implies \begin{pmatrix} -1 & 11 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -1 & 11 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = -12 \tag{A.4.2.2}$$

A.4.3. Find the equations of the altitudes BE_1 and CF_1 to the sides AC and AB respectively.

Solution:

a) From (A.1.1.4), the normal vector of CF_1 is

$$\mathbf{n} = \begin{pmatrix} -5 \\ 7 \end{pmatrix} \tag{A.4.3.1}$$

and the equation of CF_1 is

$$\mathbf{n}^\top(\mathbf{x}-\mathbf{C}) = 0 \tag{A.4.3.2}$$

$$\implies \begin{pmatrix}-5 & 7\end{pmatrix}\left(\mathbf{x}-\begin{pmatrix}-3\\-5\end{pmatrix}\right) = 0 \tag{A.4.3.3}$$

$$\implies \begin{pmatrix}5 & -7\end{pmatrix}\mathbf{x} = 20, \tag{A.4.3.4}$$

b) Similarly, from (A.1.1.2), the normal vector of BE_1 is

$$\mathbf{n} = \begin{pmatrix}1\\1\end{pmatrix} \tag{A.4.3.5}$$

and the equation of BE_1 is

$$\mathbf{n}^\top(\mathbf{x}-\mathbf{B}) = 0 \tag{A.4.3.6}$$

$$\implies \begin{pmatrix}1 & 1\end{pmatrix}\left(\mathbf{x}-\begin{pmatrix}-4\\6\end{pmatrix}\right) = 0 \tag{A.4.3.7}$$

$$\implies \begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 2, \tag{A.4.3.8}$$

A.4.4. Find the intersection $\mathbf{H}$ of BE_1 and CF_1.

Solution: The intersection of (A.4.3.8) and (A.4.3.4), is obtained from the matrix equation

$$\begin{pmatrix}1 & 1\\5 & -7\end{pmatrix}\mathbf{x} = \begin{pmatrix}2\\20\end{pmatrix} \tag{A.4.4.1}$$

which can be solved as

$$\begin{pmatrix}1 & 1 & 2\\5 & -7 & 20\end{pmatrix} \xleftrightarrow{R_2\leftarrow R_2-5R_1} \begin{pmatrix}1 & 1 & 2\\0 & -12 & 10\end{pmatrix} \tag{A.4.4.2}$$

$$\xleftrightarrow{R_2\leftarrow \frac{R_2}{-12}} \begin{pmatrix}1 & 1 & 2\\0 & 1 & \frac{-5}{6}\end{pmatrix} \xleftrightarrow{R_1\leftarrow R_1-R_2} \begin{pmatrix}1 & 0 & \frac{17}{6}\\0 & 1 & \frac{-5}{6}\end{pmatrix} \tag{A.4.4.3}$$

yielding

$$\mathbf{H} = \frac{1}{6}\begin{pmatrix}17\\-5\end{pmatrix}, \tag{A.4.4.4}$$

See Fig. A.4.4.1

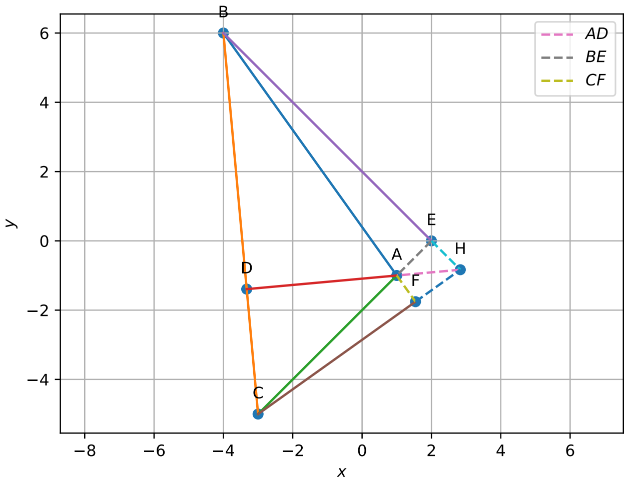

Fig. A.4.4.1: Altitudes BE_1 and CF_1 intersect at $\mathbf{H}$

A.4.5. Verify that

$$(\mathbf{A}-\mathbf{H})^\top(\mathbf{B}-\mathbf{C}) = 0 \tag{A.4.5.1}$$

Solution: From (A.4.4.4),

$$\mathbf{A}-\mathbf{H} = -\frac{1}{6}\begin{pmatrix}11\\1\end{pmatrix}, \mathbf{B}-\mathbf{C} = \begin{pmatrix}-1\\11\end{pmatrix} \tag{A.4.5.2}$$

$$\implies (\mathbf{A}-\mathbf{H})^\top(\mathbf{B}-\mathbf{C}) = \frac{1}{6}\begin{pmatrix}11 & 1\end{pmatrix}\begin{pmatrix}-1\\11\end{pmatrix} = 0 \tag{A.4.5.3}$$

All codes for this section are available at

codes/triangle/altitude.py

A.5 Perpendicular Bisector

A.5.1. The equation of the perpendicular bisector of BC is

$$\left(\mathbf{x}-\frac{\mathbf{B}+\mathbf{C}}{2}\right)(\mathbf{B}-\mathbf{C}) = 0 \tag{A.5.1.1}$$

Substitute numerical values and find the equations of the perpendicular bisectors of AB, BC and CA.

Solution: From (A.1.1.2), (A.1.1.3), (A.1.1.4), (A.3.1.3), (A.3.1.4) and (A.3.1.5),

$$\frac{\mathbf{B}+\mathbf{C}}{\mathbf{2}} = \frac{1}{2}\begin{pmatrix}-7\\1\end{pmatrix}, \mathbf{B}-\mathbf{C} = \begin{pmatrix}-1\\11\end{pmatrix} \tag{A.5.1.2}$$

$$\frac{\mathbf{A}+\mathbf{B}}{\mathbf{2}} = \frac{1}{2}\begin{pmatrix}-3\\5\end{pmatrix}, \mathbf{A}-\mathbf{B} = \begin{pmatrix}5\\-7\end{pmatrix} \tag{A.5.1.3}$$

$$\frac{\mathbf{C}+\mathbf{A}}{\mathbf{2}} = \begin{pmatrix}-1\\-3\end{pmatrix}, \mathbf{C}-\mathbf{A} = \begin{pmatrix}-4\\-4\end{pmatrix} \tag{A.5.1.4}$$

yielding

$$(\mathbf{B}-\mathbf{C})^\top\left(\frac{\mathbf{B}+\mathbf{C}}{2}\right) = \begin{pmatrix}-1 & 11\end{pmatrix}\begin{pmatrix}-\frac{7}{2}\\\frac{1}{2}\end{pmatrix} = 9 \tag{A.5.1.5}$$

$$(\mathbf{A}-\mathbf{B})^\top\left(\frac{\mathbf{A}+\mathbf{B}}{2}\right) = \begin{pmatrix}5 & -7\end{pmatrix}\begin{pmatrix}-\frac{3}{2}\\\frac{5}{2}\end{pmatrix} = -25 \tag{A.5.1.6}$$

$$(\mathbf{C}-\mathbf{A})^\top\left(\frac{\mathbf{C}+\mathbf{A}}{2}\right) = \begin{pmatrix}-4 & -4\end{pmatrix}\begin{pmatrix}-1\\-3\end{pmatrix} = 16 \tag{A.5.1.7}$$

Thus, the perpendicular bisectors are obtained from (A.5.1.1) as

$$BC: \quad \begin{pmatrix}-1 & 11\end{pmatrix}\mathbf{x} = 9 \tag{A.5.1.8}$$

$$CA: \quad \begin{pmatrix}5 & -7\end{pmatrix}\mathbf{x} = -25 \tag{A.5.1.9}$$

$$AB: \quad \begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = -4 \tag{A.5.1.10}$$

A.5.2. Find the intersection **O** of the perpendicular bisectors of AB and AC.
Solution:
The intersection of (A.5.1.9) and (A.5.1.10), can be obtained as

$$\begin{pmatrix}5 & -7 & -25\\1 & 1 & -4\end{pmatrix} \xleftrightarrow{R_2\leftarrow 5R_2-R_1} \begin{pmatrix}5 & -7 & -25\\0 & 12 & 5\end{pmatrix} \tag{A.5.2.1}$$

$$\xleftrightarrow{R_1\leftarrow \frac{12}{7}R_1+R_2} \begin{pmatrix}\frac{60}{7} & 0 & \frac{-265}{7}\\0 & 12 & 5\end{pmatrix} \xleftrightarrow[R_1\leftarrow\frac{7}{60}R_1]{R_2\leftarrow\frac{1}{12}R_2} \begin{pmatrix}1 & 0 & \frac{-53}{12}\\0 & 1 & \frac{5}{12}\end{pmatrix} \tag{A.5.2.2}$$

$$\implies \mathbf{O} = \begin{pmatrix}\frac{-53}{12}\\\frac{5}{12}\end{pmatrix} \tag{A.5.2.3}$$

A.5.3. Verify that **O** satisfies (A.5.1.1). **O** is known as the circumcentre.
Solution: Substituing from (A.5.2.3) in (A.5.1.1),

$$\begin{aligned}&\left(\mathbf{O}-\frac{\mathbf{B}+\mathbf{C}}{2}\right)^\top(\mathbf{B}-\mathbf{C})\\ &= \left(\frac{1}{12}\begin{pmatrix}-53\\5\end{pmatrix}-\frac{1}{2}\begin{pmatrix}-7\\1\end{pmatrix}\right)^\top\begin{pmatrix}-1\\11\end{pmatrix}\\ &= \frac{1}{12}\begin{pmatrix}-11 & -1\end{pmatrix}\begin{pmatrix}-1\\11\end{pmatrix} = 0\end{aligned} \tag{A.5.3.1}$$

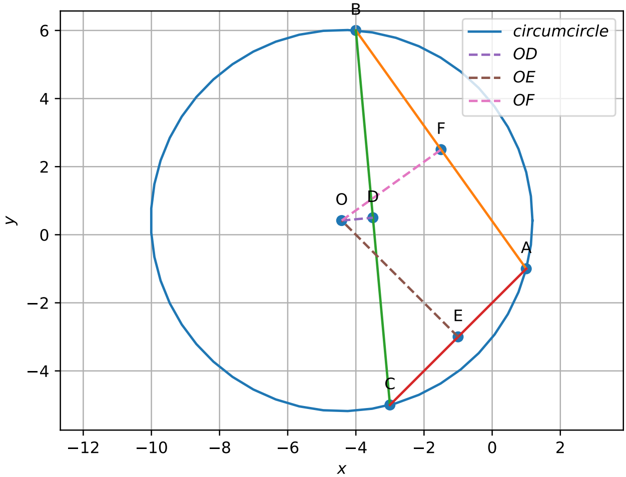

Fig. A.5.5.1: Circumcircle of $\triangle ABC$ with centre $\mathbf{O}$.

A.5.4. Verify that

$$OA = OB = OC \tag{A.5.4.1}$$

A.5.5. Draw the circle with centre at $\mathbf{O}$ and radius

$$R = OA \tag{A.5.5.1}$$

This is known as the *circumradius*.
Solution: See Fig. A.5.5.1.

A.5.6. Verify that

$$\angle BOC = 2\angle BAC. \tag{A.5.6.1}$$

Solution:

a) To find the value of $\angle BOC$:

$$\mathbf{B} - \mathbf{O} = \begin{pmatrix} \frac{5}{12} \\ \frac{67}{12} \end{pmatrix}, \ \mathbf{C} - \mathbf{O} = \begin{pmatrix} \frac{17}{12} \\ \frac{-65}{12} \end{pmatrix} \tag{A.5.6.2}$$

$$\implies (\mathbf{B} - \mathbf{O})^\top (\mathbf{C} - \mathbf{O}) = \frac{-4270}{144} \tag{A.5.6.3}$$

$$\implies \|\mathbf{B} - \mathbf{O}\| = \frac{\sqrt{4514}}{12}, \ \|\mathbf{C} - \mathbf{O}\| = \frac{\sqrt{4514}}{12} \tag{A.5.6.4}$$

Thus,

$$\cos BOC = \frac{(\mathbf{B}-\mathbf{O})^\top(\mathbf{C}-\mathbf{O})}{\|\mathbf{B}-\mathbf{O}\|\,\|\mathbf{C}-\mathbf{O}\|} = \frac{-4270}{4514} \tag{A.5.6.5}$$

$$\implies \angle BOC = \cos^{-1}\left(\frac{-4270}{4514}\right) \tag{A.5.6.6}$$

$$= 161.07536^\circ \text{ or } 198.92464^\circ \tag{A.5.6.7}$$

b) To find the value of $\angle BAC$:

$$\mathbf{B}-\mathbf{A} = \begin{pmatrix}-5\\7\end{pmatrix}, \mathbf{C}-\mathbf{A} = \begin{pmatrix}-4\\-4\end{pmatrix} \tag{A.5.6.8}$$

$$\implies (\mathbf{B}-\mathbf{A})^\top(\mathbf{C}-\mathbf{A}) = -8 \tag{A.5.6.9}$$

$$\|\mathbf{B}-\mathbf{A}\| = \sqrt{74}\,\|\mathbf{C}-\mathbf{A}\| = 4\sqrt{2} \tag{A.5.6.10}$$

Thus,

$$\cos BAC = \frac{(\mathbf{B}-\mathbf{A})^\top(\mathbf{C}-\mathbf{A})}{\|\mathbf{B}-\mathbf{A}\|\,\|\mathbf{C}-\mathbf{A}\|} = \frac{-8}{4\sqrt{148}} \tag{A.5.6.11}$$

$$\implies \angle BAC = \cos^{-1}\left(\frac{-8}{4\sqrt{148}}\right) \tag{A.5.6.12}$$

$$= 99.46232^\circ \tag{A.5.6.13}$$

From (A.5.6.13) and (A.5.6.7),

$$2 \times \angle BAC = \angle BOC \tag{A.5.6.14}$$

A.5.7. Let

$$\mathbf{P} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix} \tag{A.5.7.1}$$

where

$$\theta = \angle BOC \tag{A.5.7.2}$$

Verify that

$$\mathbf{B}-\mathbf{O} = \mathbf{P}(\mathbf{C}-\mathbf{O}) \tag{A.5.7.3}$$

All codes for this section are available at

```
codes/triangle/perp−bisect.py
```

A.6 Angle Bisector

A.6.1. Let $\mathbf{D}_3, \mathbf{E}_3, \mathbf{F}_3$, be points on AB, BC and CA respectively such that

$$BD_3 = BF_3 = m, CD_3 = CE_3 = n, AE_3 = AF_3 = p. \tag{A.6.1.1}$$

Obtain m, n, p in terms of a, b, c obtained in Problem A.1.2.
Solution: From the given information,

$$a = m + n, \tag{A.6.1.2}$$

$$b = n + p, \tag{A.6.1.3}$$

$$c = m + p \tag{A.6.1.4}$$

which can be expressed as

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} m \\ n \\ p \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{A.6.1.5}$$

$$\implies \begin{pmatrix} m \\ n \\ p \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{A.6.1.6}$$

Using row reduction,

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \tag{A.6.1.7}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - R_1} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \end{array}\right) \tag{A.6.1.8}$$

$$\xleftrightarrow[R_1 \leftarrow R_1 - R_2]{R_3 \leftarrow R_3 + R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array}\right) \tag{A.6.1.9}$$

$$\xleftrightarrow[R_1 \leftarrow 2R_1 + R_3]{R_2 \leftarrow 2R_2 - R_3} \left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & -1 & 1 \\ 0 & 2 & 0 & 1 & 1 & -1 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array}\right) \tag{A.6.1.10}$$

yielding

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix} \tag{A.6.1.11}$$

Therefore,

$$\begin{aligned} p &= \frac{c + b - a}{2} = \frac{\sqrt{74} + \sqrt{32} - \sqrt{122}}{2} \\ m &= \frac{a + c - b}{2} = \frac{\sqrt{74} + \sqrt{122} - \sqrt{32}}{2} \\ n &= \frac{a + b - c}{2} = \frac{\sqrt{122} + \sqrt{32} - \sqrt{74}}{2} \end{aligned} \tag{A.6.1.12}$$

upon substituting from (A.1.2.7), (A.1.2.9) and (A.1.2.11).

A.6.2. Using section formula, find

$$\mathbf{D}_3 = \frac{m\mathbf{C} + n\mathbf{B}}{m+n}, \mathbf{E}_3 = \frac{n\mathbf{A} + p\mathbf{C}}{n+p}, \mathbf{F}_3 = \frac{p\mathbf{B} + m\mathbf{A}}{p+m} \tag{A.6.2.1}$$

A.6.3. Find the circumcentre and circumradius of $\triangle D_3E_3F_3$. These are the *incentre* and *inradius* of $\triangle ABC$.

A.6.4. Draw the circumcircle of $\triangle D_3E_3F_3$. This is known as the *incircle* of $\triangle ABC$.
Solution: See Fig. A.6.4.1

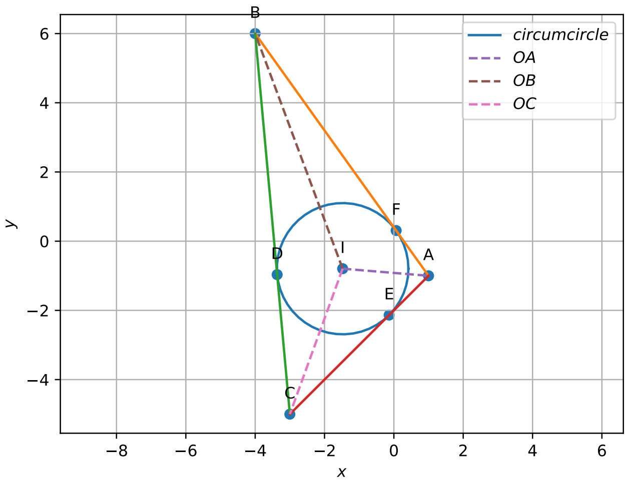

Fig. A.6.4.1: Incircle of $\triangle ABC$

A.6.5. Using (A.1.7.1) verify that

$$\angle BAI = \angle CAI. \tag{A.6.5.1}$$

AI is the bisector of $\angle A$.

A.6.6. Verify that BI, CI are also the angle bisectors of $\triangle ABC$. All codes for this section are available at

```
codes/triangle/ang−bisect.py
```

A.7 Eigenvalues and Eigenvectors

A.7.1. The equation of a circle is given by

$$\|\mathbf{x}\|^2 + 2\mathbf{u}^\top\mathbf{x} + f = 0 \tag{A.7.1.1}$$

for

$$\mathbf{u} = -\mathbf{O}, f = \|\mathbf{O}\| - r^2, \tag{A.7.1.2}$$

O being the incentre and r the inradius.

A.7.2. Compute

$$\boldsymbol{\Sigma} = (\mathbf{Vh}+\mathbf{u})(\mathbf{Vh}+\mathbf{u})^{\top} - g(\mathbf{h})\mathbf{V} \tag{A.7.2.1}$$

for $\mathbf{h} = \mathbf{A}$.

A.7.3. Find the roots of the equation

$$\left|\lambda\mathbf{I} - \boldsymbol{\Sigma}\right| = 0 \tag{A.7.3.1}$$

These are known as the eigenvalues of $\boldsymbol{\Sigma}$.

A.7.4. Find $\mathbf{p}$ such that

$$\boldsymbol{\Sigma}\mathbf{p} = \lambda\mathbf{p} \tag{A.7.4.1}$$

using row reduction. These are known as the eigenvectors of Σ.

A.7.5. Define

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \tag{A.7.5.1}$$

$$\mathbf{P} = \begin{pmatrix} \frac{\mathbf{p}_1}{\|\mathbf{p}_1\|} & \frac{\mathbf{p}_2}{\|\mathbf{p}_2\|} \end{pmatrix} \tag{A.7.5.2}$$

A.7.6. Verify that

$$\mathbf{P}^{\top} = \mathbf{P}^{-1}. \tag{A.7.6.1}$$

P is defined to be an orthogonal matrix.

A.7.7. Verify that

$$\mathbf{P}^{\top}\boldsymbol{\Sigma}\mathbf{P} = \mathbf{D}, \tag{A.7.7.1}$$

This is known as the spectral (eigenvalue) decomposition of a symmetric matrix

A.7.8. The direction vectors of the tangents from a point **h** to the circle in (7.1.1.1) are given by

$$\mathbf{m} = \mathbf{P}\begin{pmatrix} \sqrt{|\lambda_2|} \\ \pm\sqrt{|\lambda_1|} \end{pmatrix} \tag{A.7.8.1}$$

A.7.9. The points of contact of the pair of tangents to the circle in (7.1.1.1) from a point **h** are given by

$$\mathbf{x} = \mathbf{h} + \mu\mathbf{m} \tag{A.7.9.1}$$

where

$$\mu = -\frac{\mathbf{m}^{\top}(\mathbf{Vh}+\mathbf{u})}{\mathbf{m}^{\top}\mathbf{V}\mathbf{m}} \tag{A.7.9.2}$$

for **m** in (A.7.8.1). Compute the points of contact. You should get the same points that you obtained in the previous section.

All codes for this section are available at

```
codes/triangle/tangpair.py
```

A.8 Formulae

A.8.1 The equation of the *incircle* is given by

$$\|\mathbf{x}-\mathbf{O}\|^2 = r^2 \tag{A.8.1.1}$$

which can be expressed as (7.1.1.1) using (A.7.1.2).

A.8.2 In Fig. A.6.4.1, let (A.7.9.1) be the equation of AB. Then, the intersection of (A.7.9.1) and (7.1.1.1) can be expressed as

$$(\mathbf{h}+\mu\mathbf{m})^\top \mathbf{V}(\mathbf{h}+\mu\mathbf{m}) + 2\mathbf{u}^\top(\mathbf{h}+\mu\mathbf{m}) + f = 0 \tag{A.8.2.1}$$

$$\implies \mu^2\mathbf{m}^\top\mathbf{V}\mathbf{m} + 2\mu\mathbf{m}^\top(\mathbf{V}\mathbf{h}+\mathbf{u}) + g(\mathbf{h}) = 0 \tag{A.8.2.2}$$

For (A.8.2.2) to have exactly one root, the discriminant

$$\left\{\mathbf{m}^\top(\mathbf{V}\mathbf{h}+\mathbf{u})\right\}^2 - g(\mathbf{h})\,\mathbf{m}^\top\mathbf{V}\mathbf{m} = 0 \tag{A.8.2.3}$$

and (A.7.9.2) is obtained.

A.8.3 (A.8.2.3) can be expressed as

$$\mathbf{m}^\top(\mathbf{V}\mathbf{h}+\mathbf{u})^\top(\mathbf{V}\mathbf{h}+\mathbf{u})\,\mathbf{m} - g(\mathbf{h})\,\mathbf{m}^\top\mathbf{V}\mathbf{m} = 0 \tag{A.8.3.1}$$

$$\implies \mathbf{m}^\top\mathbf{\Sigma}\mathbf{m} = 0 \tag{A.8.3.2}$$

for $\mathbf{\Sigma}$ defined in (A.8.3.2). Substituting (A.7.7.1) in (A.8.3.2),

$$\mathbf{m}^\top\mathbf{P}\mathbf{D}\mathbf{P}^\top\mathbf{m} = 0 \tag{A.8.3.3}$$

$$\implies \mathbf{v}^\top\mathbf{D}\mathbf{v} = 0 \tag{A.8.3.4}$$

where

$$\mathbf{v} = \mathbf{P}^\top\mathbf{m} \tag{A.8.3.5}$$

(A.8.3.4) can be expressed as

$$\lambda_1 v_1^2 + \lambda_2 v_2^2 = 0 \tag{A.8.3.6}$$

$$\implies \mathbf{v} = \begin{pmatrix}\sqrt{|\lambda_2|}\\ \pm\sqrt{|\lambda_1|}\end{pmatrix} \tag{A.8.3.7}$$

after some algebra. From (A.8.3.7) and (A.8.3.5) we obtain (A.7.8.1).

A.9 Matrices

A.9.1. The matrix of the vertices of the triangle is defined as

$$\mathbf{P} = \begin{pmatrix}\mathbf{A} & \mathbf{B} & \mathbf{C}\end{pmatrix} \tag{A.9.1.1}$$

A.9.2. Obtain the direction matrix of the sides of $\triangle ABC$ defined as

$$\mathbf{M} = \begin{pmatrix}\mathbf{A}-\mathbf{B} & \mathbf{B}-\mathbf{C} & \mathbf{C}-\mathbf{A}\end{pmatrix} \tag{A.9.2.1}$$

Solution:

$$\mathbf{M} = \begin{pmatrix}\mathbf{A}-\mathbf{B} & \mathbf{B}-\mathbf{C} & \mathbf{C}-\mathbf{A}\end{pmatrix} \tag{A.9.2.2}$$

$$= \begin{pmatrix}\mathbf{A} & \mathbf{B} & \mathbf{C}\end{pmatrix}\begin{pmatrix}1 & 0 & -1\\ -1 & 1 & 0\\ 0 & -1 & 1\end{pmatrix} \tag{A.9.2.3}$$

where the second matrix above is known as a *circulant* matrix. Note that the 2nd and 3rd row of the above matrix are circular shifts of the 1st row.

A.9.3. Obtain the normal matrix of the sides of $\triangle ABC$
Solution: Considering the roation matrix

$$\mathbf{R} = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \tag{A.9.3.1}$$

the normal matrix is obtained as

$$\mathbf{N} = \mathbf{RM} \tag{A.9.3.2}$$

A.9.4. Obtain a, b, c.
Solution: The sides vector is obtained as

$$\mathbf{d} = \sqrt{\text{diag}(\mathbf{M}^\top\mathbf{M})} \tag{A.9.4.1}$$

A.9.5. Obtain the constant terms in the equations of the sides of the triangle.
Solution: The constants for the lines can be expressed in vector form as

$$\mathbf{c} = \text{diag}\left\{\left(\mathbf{N}^\top\mathbf{P}\right)\right\} \tag{A.9.5.1}$$

A.9.6. Obtain the mid point matrix for the sides of the triangle
Solution:

$$\begin{pmatrix}\mathbf{D} & \mathbf{E} & \mathbf{F}\end{pmatrix} = \frac{1}{2}\begin{pmatrix}\mathbf{A} & \mathbf{B} & \mathbf{C}\end{pmatrix}\begin{pmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{pmatrix} \tag{A.9.6.1}$$

A.9.7. Obtain the median direction matrix.
Solution: The median direction matrix is given by

$$\mathbf{M}_1 = \begin{pmatrix}\mathbf{A}-\mathbf{D} & \mathbf{B}-\mathbf{E} & \mathbf{C}-\mathbf{F}\end{pmatrix} \tag{A.9.7.1}$$

$$= \begin{pmatrix}\mathbf{A}-\frac{\mathbf{B}+\mathbf{C}}{2} & \mathbf{B}-\frac{\mathbf{C}+\mathbf{A}}{2} & \mathbf{C}-\frac{\mathbf{A}+\mathbf{B}}{2}\end{pmatrix} \tag{A.9.7.2}$$

$$= \begin{pmatrix}\mathbf{A} & \mathbf{B} & \mathbf{C}\end{pmatrix}\begin{pmatrix}1 & -\frac{1}{2} & -\frac{1}{2}\\ -\frac{1}{2} & 1 & -\frac{1}{2}\\ -\frac{1}{2} & -\frac{1}{2} & 1\end{pmatrix} \tag{A.9.7.3}$$

A.9.8. Obtain the median normal matrix.

A.9.9. Obtian the median equation constants.

A.9.10. Obtain the centroid by finding the intersection of the medians.

A.9.11. Find the normal matrix for the altitudes

Solution: The desired matrix is

$$\mathbf{M}_2 = \begin{pmatrix}\mathbf{B}-\mathbf{C} & \mathbf{C}-\mathbf{A} & \mathbf{A}-\mathbf{B}\end{pmatrix} \tag{A.9.11.1}$$

$$= \begin{pmatrix}\mathbf{A} & \mathbf{B} & \mathbf{C}\end{pmatrix}\begin{pmatrix}0 & -1 & 1\\ 1 & 0 & -1\\ -1 & 1 & 0\end{pmatrix} \tag{A.9.11.2}$$

A.9.12. Find the constants vector for the altitudes.
Solution: The desired vector is

$$\mathbf{c}_2 = \text{diag}\left\{\left(\mathbf{M}^\top \mathbf{P}\right)\right\} \tag{A.9.12.1}$$

A.9.13. Find the normal matrix for the perpendicular bisectors
Solution: The normal matrix is $\mathbf{M}_2$

A.9.14. Find the constants vector for the perpendicular bisectors.
Solution: The desired vector is

$$\mathbf{c}_3 = \text{diag}\left\{\mathbf{M}_2^\top \begin{pmatrix}\mathbf{D} & \mathbf{E} & \mathbf{F}\end{pmatrix}\right\} \tag{A.9.14.1}$$

A.9.15. Find the points of contact.
Solution: The points of contact are given by

$$\begin{pmatrix}\frac{m\mathbf{C}+n\mathbf{B}}{m+n} & \frac{n\mathbf{A}+p\mathbf{C}}{n+p} & \frac{p\mathbf{B}+m\mathbf{A}}{p+m}\end{pmatrix} = \begin{pmatrix}\mathbf{A} & \mathbf{B} & \mathbf{C}\end{pmatrix}\begin{pmatrix}0 & \frac{n}{b} & \frac{m}{c}\\ \frac{n}{a} & 0 & \frac{p}{c}\\ \frac{m}{a} & \frac{p}{b} & 0\end{pmatrix} \tag{A.9.15.1}$$

All codes for this section are available at

```
codes/triangle/mat−alg.py
```

Appendix B
Conic Section

B.1 Equation

B.1.1. Let $\mathbf{q}$ be a point such that the ratio of its distance from a fixed point $\mathbf{F}$ and the distance (d) from a fixed line

$$L : \mathbf{n}^\top \mathbf{x} = c \tag{B.1.1.1}$$

is constant, given by

$$\frac{\|\mathbf{q} - \mathbf{F}\|}{d} = e \tag{B.1.1.2}$$

The locus of $\mathbf{q}$ is known as a conic section. The line L is known as the directrix and the point $\mathbf{F}$ is the focus. e is defined to be the eccentricity of the conic.

a) For $e = 1$, the conic is a parabola
b) For $e < 1$, the conic is an ellipse
c) For $e > 1$, the conic is a hyperbola

B.1.2. The equation of a conic with directrix $\mathbf{n}^\top \mathbf{x} = c$, eccentricity e and focus $\mathbf{F}$ is given by

$$g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{B.1.2.1}$$

where

$$\mathbf{V} = \|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{n}\mathbf{n}^\top, \tag{B.1.2.2}$$

$$\mathbf{u} = ce^2 \mathbf{n} - \|\mathbf{n}\|^2 \mathbf{F}, \tag{B.1.2.3}$$

$$f = \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2 e^2 \tag{B.1.2.4}$$

Solution: Using Definition B.1.1 and (4.1.6.6), for any point $\mathbf{x}$ on the conic,

$$\|\mathbf{x} - \mathbf{F}\|^2 = e^2 \frac{(\mathbf{n}^\top \mathbf{x} - c)^2}{\|\mathbf{n}\|^2} \tag{B.1.2.5}$$

$$\implies \|\mathbf{n}\|^2 (\mathbf{x} - \mathbf{F})^\top (\mathbf{x} - \mathbf{F}) = e^2 \left(\mathbf{n}^\top \mathbf{x} - c\right)^2 \tag{B.1.2.6}$$

$$\implies \|\mathbf{n}\|^2 \left(\mathbf{x}^\top \mathbf{x} - 2\mathbf{F}^\top \mathbf{x} + \|\mathbf{F}\|^2\right) = e^2 \left(c^2 + \left(\mathbf{n}^\top \mathbf{x}\right)^2 - 2c\mathbf{n}^\top \mathbf{x}\right) \tag{B.1.2.7}$$

$$= e^2 \left(c^2 + \left(\mathbf{x}^\top \mathbf{n}\mathbf{n}^\top \mathbf{x}\right) - 2c\mathbf{n}^\top \mathbf{x}\right) \tag{B.1.2.8}$$

which can be expressed as (B.1.2.1) after simplification.

B.1.3. The eccentricity, directrices and foci of (B.1.2.1) are given by

$$e = \sqrt{1 - \frac{\lambda_1}{\lambda_2}} \tag{B.1.3.1}$$

$$\mathbf{n} = \sqrt{\lambda_2}\mathbf{p}_1,$$

$$c = \begin{cases} \frac{e\mathbf{u}^\top\mathbf{n} \pm \sqrt{e^2(\mathbf{u}^\top\mathbf{n})^2 - \lambda_2\left(e^2-1\right)\left(\|\mathbf{u}\|^2 - \lambda_2 f\right)}}{\lambda_2 e\left(e^2-1\right)} & e \neq 1 \\ \frac{\|\mathbf{u}\|^2 - \lambda_2 f}{2\mathbf{u}^\top\mathbf{n}} & e = 1 \end{cases} \tag{B.1.3.2}$$

$$\mathbf{F} = \frac{ce^2\mathbf{n} - \mathbf{u}}{\lambda_2} \tag{B.1.3.3}$$

Solution: From (B.1.2.2), using the fact that $\mathbf{V}$ is symmetric with $\mathbf{V} = \mathbf{V}^\top$,

$$\mathbf{V}^\top\mathbf{V} = \left(\|\mathbf{n}\|^2 \mathbf{I} - e^2\mathbf{n}\mathbf{n}^\top\right)^\top \left(\|\mathbf{n}\|^2 \mathbf{I} - e^2\mathbf{n}\mathbf{n}^\top\right) \tag{B.1.3.4}$$

$$\implies \mathbf{V}^2 = \|\mathbf{n}\|^4 \mathbf{I} + e^4\mathbf{n}\mathbf{n}^\top\mathbf{n}\mathbf{n}^\top - 2e^2 \|\mathbf{n}\|^2 \mathbf{n}\mathbf{n}^\top \tag{B.1.3.5}$$

$$= \|\mathbf{n}\|^4 \mathbf{I} + e^4 \|\mathbf{n}\|^2 \mathbf{n}\mathbf{n}^\top - 2e^2 \|\mathbf{n}\|^2 \mathbf{n}\mathbf{n}^\top \tag{B.1.3.6}$$

$$= \|\mathbf{n}\|^4 \mathbf{I} + e^2\left(e^2 - 2\right) \|\mathbf{n}\|^2 \mathbf{n}\mathbf{n}^\top \tag{B.1.3.7}$$

$$= \|\mathbf{n}\|^4 \mathbf{I} + \left(e^2 - 2\right) \|\mathbf{n}\|^2 \left(\|\mathbf{n}\|^2 \mathbf{I} - \mathbf{V}\right) \tag{B.1.3.8}$$

which can be expressed as

$$\mathbf{V}^2 + \left(e^2 - 2\right) \|\mathbf{n}\|^2 \mathbf{V} - \left(e^2 - 1\right) \|\mathbf{n}\|^4 \mathbf{I} = 0 \tag{B.1.3.9}$$

Using the Cayley-Hamilton theorem, (B.1.3.9) results in the characteristic equation,

$$\lambda^2 - \left(2 - e^2\right) \|\mathbf{n}\|^2 \lambda + \left(1 - e^2\right) \|\mathbf{n}\|^4 = 0 \tag{B.1.3.10}$$

which can be expressed as

$$\left(\frac{\lambda}{\|\mathbf{n}\|^2}\right)^2 - \left(2 - e^2\right)\left(\frac{\lambda}{\|\mathbf{n}\|^2}\right) + \left(1 - e^2\right) = 0 \tag{B.1.3.11}$$

$$\implies \frac{\lambda}{\|\mathbf{n}\|^2} = 1 - e^2, 1 \tag{B.1.3.12}$$

$$\text{or, } \lambda_2 = \|\mathbf{n}\|^2, \lambda_1 = \left(1 - e^2\right)\lambda_2 \tag{B.1.3.13}$$

From (B.1.3.13), the eccentricity of (B.1.2.1) is given by (B.1.3.1). Multiplying both sides of (B.1.2.2) by $\mathbf{n}$,

$$\mathbf{V}\mathbf{n} = \|\mathbf{n}\|^2 \mathbf{n} - e^2\mathbf{n}\mathbf{n}^\top\mathbf{n} \tag{B.1.3.14}$$

$$= \|\mathbf{n}\|^2 \left(1 - e^2\right)\mathbf{n} \tag{B.1.3.15}$$

$$= \lambda_1\mathbf{n} \tag{B.1.3.16}$$

$$\tag{B.1.3.17}$$

from (B.1.3.13). Thus, λ_1 is the corresponding eigenvalue for $\mathbf{n}$. From (A.7.5.2) and

(B.1.3.17), this implies that

$$\mathbf{p}_1 = \frac{\mathbf{n}}{\|\mathbf{n}\|} \tag{B.1.3.18}$$

$$\text{or, } \mathbf{n} = \|\mathbf{n}\| \, \mathbf{p}_1 = \sqrt{\lambda_2}\mathbf{p}_1 \tag{B.1.3.19}$$

from (B.1.3.13) . From (B.1.2.3) and (B.1.3.13),

$$\mathbf{F} = \frac{ce^2\mathbf{n} - \mathbf{u}}{\lambda_2} \tag{B.1.3.20}$$

$$\implies \|\mathbf{F}\|^2 = \frac{\left(ce^2\mathbf{n} - \mathbf{u}\right)^\top \left(ce^2\mathbf{n} - \mathbf{u}\right)}{\lambda_2^2} \tag{B.1.3.21}$$

$$\implies \lambda_2^2 \|\mathbf{F}\|^2 = c^2e^4\lambda_2 - 2ce^2\mathbf{u}^\top\mathbf{n} + \|\mathbf{u}\|^2 \tag{B.1.3.22}$$

Also, (B.1.2.4) can be expressed as

$$\lambda_2 \|\mathbf{F}\|^2 = f + c^2e^2 \tag{B.1.3.23}$$

From (B.1.3.22) and (B.1.3.23),

$$c^2e^4\lambda_2 - 2ce^2\mathbf{u}^\top\mathbf{n} + \|\mathbf{u}\|^2 = \lambda_2\left(f + c^2e^2\right) \tag{B.1.3.24}$$

$$\implies \lambda_2 e^2\left(e^2 - 1\right)c^2 - 2ce^2\mathbf{u}^\top\mathbf{n} + \|\mathbf{u}\|^2 - \lambda_2 f = 0 \tag{B.1.3.25}$$

yielding (B.1.3.3).

B.1.4. (B.1.2.1) represents

a) a parabola for $\left|\mathbf{V}\right| = 0$,
b) ellipse for $\left|\mathbf{V}\right| > 0$ and
c) hyperbola for $\left|\mathbf{V}\right| < 0$.

Solution: From (B.1.3.1),

$$\frac{\lambda_1}{\lambda_2} = 1 - e^2 \tag{B.1.4.1}$$

Also,

$$\left|\mathbf{V}\right| = \lambda_1\lambda_2 \tag{B.1.4.2}$$

yielding Table B.1.4.

Eccentricity	**Conic**	**Eigenvalue**	**Determinant**
$e = 1$	Parabola	$\lambda_1 = 0$	$\left|\mathbf{V}\right| = 0$
$e < 1$	Ellipse	$\lambda_1 > 0, \lambda_2 > 0$	$\left|\mathbf{V}\right| > 0$
$e > 1$	Hyperbola	$\lambda_1 < 0, \lambda_2 > 0$	$\left|\mathbf{V}\right| < 0$

TABLE B.1.4

B.1.5. Using the affine transformation in (2.1.15.1), the conic in (B.1.2.1) can be expressed in standard form as

$$\mathbf{y}^\top \left(\frac{\mathbf{D}}{f_0}\right)\mathbf{y} = 1 \qquad |\mathbf{V}| \neq 0 \tag{B.1.5.1}$$

$$\mathbf{y}^\top \mathbf{D}\mathbf{y} = -\eta \mathbf{e}_1^\top \mathbf{y} \qquad |\mathbf{V}| = 0 \tag{B.1.5.2}$$

where

$$f_0 = \mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f \neq 0 \tag{B.1.5.3}$$

$$\eta = 2\mathbf{u}^\top \mathbf{p}_1 \tag{B.1.5.4}$$

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{B.1.5.5}$$

Solution: Using (2.1.15.1), (B.1.2.1) can be expressed as

$$(\mathbf{P}\mathbf{y} + \mathbf{c})^\top \mathbf{V}(\mathbf{P}\mathbf{y} + \mathbf{c}) + 2\mathbf{u}^\top (\mathbf{P}\mathbf{y} + \mathbf{c}) + f = 0, \tag{B.1.5.6}$$

yielding

$$\mathbf{y}^\top \mathbf{P}^\top \mathbf{V}\mathbf{P}\mathbf{y} + 2(\mathbf{V}\mathbf{c} + \mathbf{u})^\top \mathbf{P}\mathbf{y} + \mathbf{c}^\top \mathbf{V}\mathbf{c} + 2\mathbf{u}^\top \mathbf{c} + f = 0 \tag{B.1.5.7}$$

From (B.1.5.7) and (A.7.7.1),

$$\mathbf{y}^\top \mathbf{D}\mathbf{y} + 2(\mathbf{V}\mathbf{c} + \mathbf{u})^\top \mathbf{P}\mathbf{y} + \mathbf{c}^\top (\mathbf{V}\mathbf{c} + \mathbf{u}) + \mathbf{u}^\top \mathbf{c} + f = 0 \tag{B.1.5.8}$$

When $\mathbf{V}^{-1}$ exists, choosing

$$\mathbf{V}\mathbf{c} + \mathbf{u} = \mathbf{0}, \quad \text{or}, \mathbf{c} = -\mathbf{V}^{-1}\mathbf{u}, \tag{B.1.5.9}$$

and substituting (B.1.5.9) in (B.1.5.8) yields (B.1.5.1). When $|\mathbf{V}| = 0, \lambda_1 = 0$ and

$$\mathbf{V}\mathbf{p}_1 = 0, \mathbf{V}\mathbf{p}_2 = \lambda_2 \mathbf{p}_2. \tag{B.1.5.10}$$

Substituting (7.1.3.2) in (B.1.5.8),

$$\begin{aligned}
&\mathbf{y}^\top \mathbf{D}\mathbf{y} + 2\left(\mathbf{c}^\top \mathbf{V} + \mathbf{u}^\top\right)\begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix}\mathbf{y} + \mathbf{c}^\top (\mathbf{V}\mathbf{c} + \mathbf{u}) + \mathbf{u}^\top \mathbf{c} + f = 0 \\
\implies &\mathbf{y}^\top \mathbf{D}\mathbf{y} + 2\left((\mathbf{c}^\top \mathbf{V} + \mathbf{u}^\top)\mathbf{p}_1 \; (\mathbf{c}^\top \mathbf{V} + \mathbf{u}^\top)\mathbf{p}_2\right)\mathbf{y} + \mathbf{c}^\top (\mathbf{V}\mathbf{c} + \mathbf{u}) + \mathbf{u}^\top \mathbf{c} + f = 0 \\
\implies &\mathbf{y}^\top \mathbf{D}\mathbf{y} + 2\begin{pmatrix} \mathbf{u}^\top \mathbf{p}_1 & (\lambda_2 \mathbf{c}^\top + \mathbf{u}^\top)\mathbf{p}_2 \end{pmatrix}\mathbf{y} + \mathbf{c}^\top (\mathbf{V}\mathbf{c} + \mathbf{u}) + \mathbf{u}^\top \mathbf{c} + f = 0
\end{aligned}$$

upon substituting from (B.1.5.10), yielding

$$\lambda_2 y_2^2 + 2\left(\mathbf{u}^\top \mathbf{p}_1\right) y_1 + 2y_2 (\lambda_2 \mathbf{c} + \mathbf{u})^\top \mathbf{p}_2 + \mathbf{c}^\top (\mathbf{V}\mathbf{c} + \mathbf{u}) + \mathbf{u}^\top \mathbf{c} + f = 0 \tag{B.1.5.11}$$

Thus, (B.1.5.11) can be expressed as (B.1.5.2) by choosing

$$\eta = 2\mathbf{u}^\top \mathbf{p}_1 \tag{B.1.5.12}$$

and $\mathbf{c}$ in (B.1.5.8) such that

$$2\mathbf{P}^\top (\mathbf{V}\mathbf{c} + \mathbf{u}) = \eta \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{B.1.5.13}$$

$$\mathbf{c}^\top (\mathbf{V}\mathbf{c} + \mathbf{u}) + \mathbf{u}^\top \mathbf{c} + f = 0 \tag{B.1.5.14}$$

B.1.6. The center/vertex of a conic section are given by

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \qquad \left|\mathbf{V}\right| \neq 0 \tag{B.1.6.1}$$

$$\begin{pmatrix} \mathbf{u}^\top + \frac{\eta}{2}\mathbf{p}_1^\top \\ \mathbf{V} \end{pmatrix}\mathbf{c} = \begin{pmatrix} -f \\ \frac{\eta}{2}\mathbf{p}_1 - \mathbf{u} \end{pmatrix} \qquad \left|\mathbf{V}\right| = 0 \tag{B.1.6.2}$$

Solution: $\because$ $\mathbf{P}^\top\mathbf{P} = \mathbf{I}$, multiplying (B.1.5.13) by $\mathbf{P}$ yields

$$(\mathbf{V}\mathbf{c} + \mathbf{u}) = \frac{\eta}{2}\mathbf{p}_1, \tag{B.1.6.3}$$

which, upon substituting in (B.1.5.14) results in

$$\frac{\eta}{2}\mathbf{c}^\top\mathbf{p}_1 + \mathbf{u}^\top\mathbf{c} + f = 0 \tag{B.1.6.4}$$

(B.1.6.3) and (B.1.6.4) can be clubbed together to obtain (B.1.6.2).

B.1.7. In (2.1.15.1), substituting $\mathbf{y} = \mathbf{0}$, the center/vertex for the quadratic form is obtained as

$$\mathbf{x} = \mathbf{c}, \tag{B.1.7.1}$$

where $\mathbf{c}$ is derived as (B.1.6.1) and (B.1.6.2) in Appendix B.1.5.

B.2 Standard Conic

B.2.1. For the standard conic,

$$\mathbf{P} = \mathbf{I} \tag{B.2.1.1}$$

$$\mathbf{u} = \begin{cases} 0 & e \neq 1 \\ \frac{\eta}{2}\mathbf{e}_1 & e = 1 \end{cases} \tag{B.2.1.2}$$

$$\lambda_1 \begin{cases} = 0 & e = 1 \\ \neq 0 & e \neq 1 \end{cases} \tag{B.2.1.3}$$

where

$$\mathbf{I} = \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 \end{pmatrix} \tag{B.2.1.4}$$

is the identity matrix.

B.2.2. The center of the standard ellipse/hyperbola, defined to be the mid point of the line joining the foci, is the origin.

B.2.3. The principal (major) axis of the standard ellipse/hyperbola, defined to be the line joining the two foci is the x-axis.

Proof. From (B.2.7.3), it is obvious that the line joining the foci passes through the origin. Also, the direction vector of this line is $\mathbf{e}_1$. Thus, the principal axis is the x-axis. □

B.2.4. The minor axis of the standard ellipse/hyperbola, defined to be the line orthogonal to the x-axis is the y-axis.

B.2.5. The axis of symmetry of the standard parabola, defined to be the line perpendicular to the directrix and passing through the focus, is the x- axis.

Proof. From (B.2.7.7) and (B.2.7.3), the axis of the parabola can be expressed as

$$\mathbf{e}_2^\top\left(\mathbf{y} + \frac{\eta}{4\lambda_2}\mathbf{e}_1\right) = 0 \tag{B.2.5.1}$$

$$\implies \mathbf{e}_2^\top\mathbf{y} = 0, \tag{B.2.5.2}$$

which is the equation of the x-axis. □

B.2.6. The point where the parabola intersects its axis of symmetry is called the vertex. For the standard parabola, the vertex is the origin.

Proof. (B.2.5.2) can be expressed as

$$\mathbf{y} = \alpha\mathbf{e}_1. \tag{B.2.6.1}$$

Substituting (B.2.6.1) in (B.1.5.2),

$$\alpha^2\mathbf{e}_1^\top\mathbf{D}\mathbf{e}_1 = -\eta\alpha\mathbf{e}_1^\top\mathbf{e}_1 \tag{B.2.6.2}$$

$$\implies \alpha = 0, \text{ or, } \mathbf{y} = \mathbf{0}. \tag{B.2.6.3}$$

□

B.2.7.

a) The directrices for the standard conic are given by

$$\mathbf{e}_1^\top\mathbf{y} = \pm\frac{1}{e}\sqrt{\frac{|f_0|}{\lambda_2\left(1-e^2\right)}} \qquad e \neq 1 \tag{B.2.7.1}$$

$$\mathbf{e}_1^\top\mathbf{y} = \frac{\eta}{2\lambda_2} \qquad e = 1 \tag{B.2.7.2}$$

b) The foci of the standard ellipse and hyperbola are given by

$$\mathbf{F} = \begin{cases} \pm e\sqrt{\frac{|f_0|}{\lambda_2\left(1-e^2\right)}}\mathbf{e}_1 & e \neq 1 \\ -\frac{\eta}{4\lambda_2}\mathbf{e}_1 & e = 1 \end{cases} \tag{B.2.7.3}$$

Proof. a) For the standard hyperbola/ellipse in (B.1.5.1), from (B.2.1.1), (B.1.3.2) and (B.2.1.2),

$$\mathbf{n} = \sqrt{\frac{\lambda_2}{f_0}}\mathbf{e}_1 \tag{B.2.7.4}$$

$$c = \pm\frac{\sqrt{-\frac{\lambda_2}{f_0}\left(e^2-1\right)\left(\frac{\lambda_2}{f_0}\right)}}{\frac{\lambda_2}{f_0}e\left(e^2-1\right)} \tag{B.2.7.5}$$

$$= \pm\frac{1}{e\sqrt{1-e^2}} \tag{B.2.7.6}$$

yielding (B.2.7.1) upon substituting from (B.1.3.1) and simplifying. For the standard parabola in (B.1.5.2), from (B.2.1.1), (B.1.3.2) and (B.2.1.2), noting that $f = 0$,

$$\mathbf{n} = \sqrt{\lambda_2}\mathbf{e}_1 \tag{B.2.7.7}$$

$$c = \frac{\left\|\frac{\eta}{2}\mathbf{e}_1\right\|^2}{2\left(\frac{\eta}{2}\right)(\mathbf{e_1})^\top \mathbf{n}} \tag{B.2.7.8}$$

$$\tag{B.2.7.9}$$

$$= \frac{\eta}{4\sqrt{\lambda_2}} \tag{B.2.7.10}$$

yielding (B.2.7.2).

b) For the standard ellipse/hyperbola, substituting from (B.2.7.6), (B.2.7.4), (B.2.1.2) and (B.1.3.1) in (B.1.3.3),

$$\mathbf{F} = \pm\frac{\left(\frac{1}{e\sqrt{1-e^2}}\right)\left(e^2\right)\sqrt{\frac{\lambda_2}{f_0}}\mathbf{e}_1}{\frac{\lambda_2}{f_0}} \tag{B.2.7.11}$$

yielding (B.2.7.3) after simplification. For the standard parabola, substituting from (B.2.7.10), (B.2.7.7), (B.2.1.2) and (B.1.3.1) in (B.1.3.3),

$$\mathbf{F} = \frac{\left(\frac{\eta}{4\sqrt{\lambda_2}}\right)\sqrt{\lambda_2}\mathbf{e}_1 - \frac{\eta}{2}\mathbf{e_1}}{\lambda_2} \tag{B.2.7.12}$$

$$\tag{B.2.7.13}$$

yielding (B.2.7.3) after simplification.

□

B.2.8. The *focal length* of the standard parabola, , defined to be the distance between the vertex and the focus, measured along the axis of symmetry, is $\left|\frac{\eta}{4\lambda_2}\right|$

B.3 Conic Lines

B.3.1. The points of intersection of the line

$$L: \quad \mathbf{x} = \mathbf{h} + \kappa\mathbf{m} \quad \kappa \in \mathbb{R} \tag{B.3.1.1}$$

with the conic section in (B.1.2.1) are given by

$$\mathbf{x}_i = \mathbf{h} + \kappa_i\mathbf{m} \tag{B.3.1.2}$$

where

$$\kappa_i = \frac{1}{\mathbf{m}^\top\mathbf{V}\mathbf{m}}\left(-\mathbf{m}^\top(\mathbf{V}\mathbf{h}+\mathbf{u}) \pm \sqrt{\left[\mathbf{m}^\top(\mathbf{V}\mathbf{h}+\mathbf{u})\right]^2 - \mathrm{g}(\mathbf{h})\left(\mathbf{m}^\top\mathbf{V}\mathbf{m}\right)}\right) \tag{B.3.1.3}$$

Solution: Substituting (B.3.1.1) in (B.1.2.1),

$$(\mathbf{h}+\kappa\mathbf{m})^\top \mathbf{V}(\mathbf{h}+\kappa\mathbf{m}) + 2\mathbf{u}^\top(\mathbf{h}+\kappa\mathbf{m}) + f = 0 \tag{B.3.1.4}$$

$$\implies \kappa^2\mathbf{m}^\top\mathbf{V}\mathbf{m} + 2\kappa\mathbf{m}^\top(\mathbf{V}\mathbf{h}+\mathbf{u}) + \mathbf{h}^\top\mathbf{V}\mathbf{h} + 2\mathbf{u}^\top\mathbf{h} + f = 0 \tag{B.3.1.5}$$

$$\text{or, } \kappa^2\mathbf{m}^\top\mathbf{V}\mathbf{m} + 2\kappa\mathbf{m}^\top(\mathbf{V}\mathbf{h}+\mathbf{u}) + \mathrm{g}(\mathbf{h}) = 0 \tag{B.3.1.6}$$

for g defined in (B.1.2.1). Solving the above quadratic in (B.3.1.6) yields (B.3.1.3).

B.3.2. The length of the chord in (B.3.1.1) is given by

$$\frac{2\sqrt{\left[\mathbf{m}^\top\left(\mathbf{V}\mathbf{h}+\mathbf{u}\right)\right]^2-\left(\mathbf{h}^\top\mathbf{V}\mathbf{h}+2\mathbf{u}^\top\mathbf{h}+f\right)\left(\mathbf{m}^\top\mathbf{V}\mathbf{m}\right)}}{\mathbf{m}^\top\mathbf{V}\mathbf{m}}\left\|\mathbf{m}\right\| \tag{B.3.2.1}$$

Proof. The distance between the points in (B.3.1.2) is given by

$$\left\|\mathbf{x}_1-\mathbf{x}_2\right\| = \left|\kappa_1-\kappa_2\right|\left\|\mathbf{m}\right\| \tag{B.3.2.2}$$

Substituing κ_i from (B.3.1.3) in (B.3.2.2) yields (B.3.2.1). □

B.3.3. The affine transform for the conic section, preserves the norm. This implies that the length of any chord of a conic is invariant to translation and/or rotation.

Proof. Let

$$\mathbf{x}_i = \mathbf{P}\mathbf{y}_i + \mathbf{c} \tag{B.3.3.1}$$

be any two points on the conic. Then the distance between the points is given by

$$\left\|\mathbf{x}_1-\mathbf{x}_2\right\| = \left\|\mathbf{P}\left(\mathbf{y}_1-\mathbf{y}_2\right)\right\| \tag{B.3.3.2}$$

which can be expressed as

$$\left\|\mathbf{x}_1-\mathbf{x}_2\right\|^2 = \left(\mathbf{y}_1-\mathbf{y}_2\right)^\top\mathbf{P}^\top\mathbf{P}\left(\mathbf{y}_1-\mathbf{y}_2\right) \tag{B.3.3.3}$$

$$= \left\|\mathbf{y}_1-\mathbf{y}_2\right\|^2 \tag{B.3.3.4}$$

since

$$\mathbf{P}^\top\mathbf{P} = \mathbf{I} \tag{B.3.3.5}$$

□

B.3.4. For the standard hyperbola/ellipse, the length of the major axis is

$$2\sqrt{\left|\frac{f_0}{\lambda_1}\right|} \tag{B.3.4.1}$$

and the minor axis is

$$2\sqrt{\left|\frac{f_0}{\lambda_2}\right|} \tag{B.3.4.2}$$

Solution: Since the major axis passes through the origin,

$$\mathbf{q} = \mathbf{0} \tag{B.3.4.3}$$

Further, from Corollary (B.2.3),

$$\mathbf{m} = \mathbf{e}_2, \tag{B.3.4.4}$$

and from (B.1.5.1),

$$\mathbf{V} = \frac{\mathbf{D}}{f_0}, \mathbf{u} = 0, f = -1 \tag{B.3.4.5}$$

Substituting the above in (B.3.2.1),

$$\frac{2\sqrt{\mathbf{e}_1^\top \frac{\mathbf{D}}{f_0}\mathbf{e}_1}}{\mathbf{e}_1^\top \frac{\mathbf{D}}{f_0}\mathbf{e}_1}\left\|\mathbf{e}_1\right\| \tag{B.3.4.6}$$

yielding (B.3.4.1). Similarly, for the minor axis, the only different parameter is

$$\mathbf{m} = \mathbf{e}_2, \tag{B.3.4.7}$$

Substituting the above in (B.3.2.1),

$$\frac{2\sqrt{\mathbf{e}_2^\top \frac{\mathbf{D}}{f_0}\mathbf{e}_2}}{\mathbf{e}_2^\top \frac{\mathbf{D}}{f_0}\mathbf{e}_2}\left\|\mathbf{e}_2\right\| \tag{B.3.4.8}$$

yielding (B.3.4.2).

B.3.5. The equation of the minor and major axes for the ellipse/hyperbola are respectively given by

$$\mathbf{p}_i^\top (\mathbf{x} - \mathbf{c}) = 0, i = 1, 2 \tag{B.3.5.1}$$

The axis of symmetry for the parabola is also given by (B.3.5.1).

Proof. From (B.2.3), the major/symmetry axis for the hyperbola/ellipse/parabola can be expressed using (2.1.15.1) as

$$\mathbf{e}_2^\top \mathbf{P}^\top (\mathbf{x} - \mathbf{c}) = 0 \tag{B.3.5.2}$$

$$\implies (\mathbf{P}\mathbf{e}_2)^\top (\mathbf{x} - \mathbf{c}) = 0 \tag{B.3.5.3}$$

yielding (B.3.5.1), and the proof for the minor axis is similar. □

B.3.6. The latus rectum of a conic section is the chord that passes through the focus and is perpendicular to the major axis. The length of the latus rectum for a conic is given by

$$l = \begin{cases} 2\frac{\sqrt{|f_0\lambda_1|}}{\lambda_2} & e \neq 1 \\ \frac{\eta}{\lambda_2} & e = 1 \end{cases} \tag{B.3.6.1}$$

Solution: The latus rectum is perpendicular to the major axis for the standard conic. Hence, from Corollary (B.2.3),

$$\mathbf{m} = \mathbf{e}_2, \tag{B.3.6.2}$$

Since it passes through the focus, from (B.2.7.3)

$$\mathbf{q} = \mathbf{F} = \pm e\sqrt{\frac{f_0}{\lambda_2\left(1 - e^2\right)}}\mathbf{e}_1 \tag{B.3.6.3}$$

for the standard hyperbola/ellipse. Also, from (B.1.5.1),

$$\mathbf{V} = \frac{\mathbf{D}}{f_0}, \mathbf{u} = 0, f = -1 \tag{B.3.6.4}$$

Substituting the above in (B.3.2.1), we obtain

$$\frac{2\sqrt{\left[\mathbf{e}_2^\top\left(\frac{\mathbf{D}}{f_0}e\sqrt{\frac{f_0}{\lambda_2(1-e^2)}}\mathbf{e}_1\right)\right]^2-\left(e\sqrt{\frac{f_0}{\lambda_2(1-e^2)}}\mathbf{e}_1^\top\frac{\mathbf{D}}{f_0}e\sqrt{\frac{f_0}{\lambda_2(1-e^2)}}\mathbf{e}_1-1\right)\left(\mathbf{e}_2^\top\frac{\mathbf{D}}{f_0}\mathbf{e}_2\right)}}{\mathbf{e}_2^\top\frac{\mathbf{D}}{f_0}\mathbf{e}_2}\|\mathbf{e}_2\| \tag{B.3.6.5}$$

Since

$$\mathbf{e}_2^\top\mathbf{D}\mathbf{e}_1=0, \mathbf{e}_1^\top\mathbf{D}\mathbf{e}_1=\lambda_1, \mathbf{e}_1^\top\mathbf{e}_1=1, \|\mathbf{e}_2\|=1, \mathbf{e}_2^\top\mathbf{D}\mathbf{e}_2=\lambda_2, \tag{B.3.6.6}$$

(B.3.6.5) can be expressed as

$$\frac{2\sqrt{\left(1-\frac{\lambda_1e^2}{\lambda_2(1-e^2)}\right)\left(\frac{\lambda_2}{f_0}\right)}}{\frac{\lambda_2}{f_0}} \tag{B.3.6.7}$$

$$=2\frac{\sqrt{f_0\lambda_1}}{\lambda_2} \qquad \left(\because e^2=1-\frac{\lambda_1}{\lambda_2}\right) \tag{B.3.6.8}$$

For the standard parabola, the parameters in (B.3.2.1) are

$$\mathbf{q}=\mathbf{F}=-\frac{\eta}{4\lambda_2}\mathbf{e}_1, \mathbf{m}=\mathbf{e}_1, \mathbf{V}=\mathbf{D}, \mathbf{u}=\frac{\eta}{2}\mathbf{e}_1^\top, f=0 \tag{B.3.6.9}$$

Substituting the above in (B.3.2.1), the length of the latus rectum can be expressed as

$$\frac{2\sqrt{\left[\mathbf{e}_2^\top\left(\mathbf{D}\left(-\frac{\eta}{4\lambda_2}\mathbf{e}_1\right)+\frac{\eta}{2}\mathbf{e}_1\right)\right]^2-\left(\left(-\frac{\eta}{4\lambda_2}\mathbf{e}_1\right)^\top\mathbf{D}\left(-\frac{\eta}{4\lambda_2}\mathbf{e}_1\right)+2\frac{\eta}{2}\mathbf{e}_1^\top\left(-\frac{\eta}{4\lambda_2}\mathbf{e}_1\right)\right)\left(\mathbf{e}_2^\top\mathbf{D}\mathbf{e}_2\right)}}{\mathbf{e}_2^\top\mathbf{D}\mathbf{e}_2}\|\mathbf{e}_2\| \tag{B.3.6.10}$$

Since

$$\mathbf{e}_2^\top\mathbf{D}\mathbf{e}_1=0, \mathbf{e}_2^\top\mathbf{e}_2=0, \mathbf{e}_1^\top\mathbf{D}\mathbf{e}_1=0, \tag{B.3.6.11}$$

$$\mathbf{e}_1^\top\mathbf{e}_1=1, \|\mathbf{e}_1\|=1, \mathbf{e}_2^\top\mathbf{D}\mathbf{e}_2=\lambda_2, \tag{B.3.6.12}$$

(B.3.6.10) can be expressed as

$$2\frac{\sqrt{\frac{\eta^2}{4\lambda_2}\lambda_2}}{\lambda_2}=\frac{\eta}{\lambda_2} \tag{B.3.6.13}$$

B.4 Tangent and Normal

B.4.1. If L in (B.3.1.1) touches (B.1.2.1) at exactly one point $\mathbf{q}$,

$$\mathbf{m}^\top(\mathbf{V}\mathbf{q}+\mathbf{u})=0 \tag{B.4.1.1}$$

Proof. In this case, (B.3.1.6) has exactly one root. Hence, in (B.3.1.3)

$$\left[\mathbf{m}^\top(\mathbf{V}\mathbf{q}+\mathbf{u})\right]^2-\left(\mathbf{m}^\top\mathbf{V}\mathbf{m}\right)g(\mathbf{q})=0 \tag{B.4.1.2}$$

$\because$ $\mathbf{q}$ is the point of contact,

$$g(\mathbf{q}) = 0 \tag{B.4.1.3}$$

Substituting (B.4.1.3) in (B.4.1.2) and simplifying, we obtain (B.4.1.1). □

B.4.2. Given the point of contact $\mathbf{q}$, the equation of a tangent to (B.1.2.1) is

$$(\mathbf{V}\mathbf{q} + \mathbf{u})^\top \mathbf{x} + \mathbf{u}^\top \mathbf{q} + f = 0 \tag{B.4.2.1}$$

Proof. The normal vector is obtained from (B.4.1.1) as

$$\kappa\mathbf{n} = \mathbf{V}\mathbf{q} + \mathbf{u}, \kappa \in \mathbb{R} \tag{B.4.2.2}$$

From (B.4.2.2), the equation of the tangent is

$$(\mathbf{V}\mathbf{q} + \mathbf{u})^\top (\mathbf{x} - \mathbf{q}) = 0 \tag{B.4.2.3}$$

$$\implies (\mathbf{V}\mathbf{q} + \mathbf{u})^\top \mathbf{x} - \mathbf{q}^\top\mathbf{V}\mathbf{q} - \mathbf{u}^\top\mathbf{q} = 0 \tag{B.4.2.4}$$

which, upon substituting from (B.4.1.3) and simplifying yields (B.4.2.1) □

B.4.3. Given the point of contact $\mathbf{q}$, the equation of the normal to (B.1.2.1) is

$$(\mathbf{V}\mathbf{q} + \mathbf{u})^\top \mathbf{R}(\mathbf{x} - \mathbf{q}) = 0 \tag{B.4.3.1}$$

Proof. The direction vector of the tangent is obtained from (B.4.2.2) as as

$$\mathbf{m} = \mathbf{R}(\mathbf{V}\mathbf{q} + \mathbf{u}), \tag{B.4.3.2}$$

where $\mathbf{R}$ is the rotation matrix. From (B.4.3.2), the equation of the normal is given by (9.1.3.1) □

B.4.4. Given the tangent

$$\mathbf{n}^\top\mathbf{x} = c, \tag{B.4.4.1}$$

the point of contact to the conic in (B.1.2.1) is given by

$$\begin{pmatrix}\mathbf{n}^\top \\ \mathbf{m}^\top\mathbf{V}\end{pmatrix}\mathbf{q} = \begin{pmatrix} c \\ -\mathbf{m}^\top\mathbf{u}\end{pmatrix} \tag{B.4.4.2}$$

Proof. From (B.4.1.1),

$$\mathbf{m}^\top(\mathbf{V}\mathbf{q} + \mathbf{u}) = 0 \tag{B.4.4.3}$$

$$\implies \mathbf{m}^\top\mathbf{V}\mathbf{q} = -\mathbf{m}^\top\mathbf{u} \tag{B.4.4.4}$$

Combining (B.4.4.1) and (B.4.4.4), (B.4.4.2) is obtained.

□

B.4.5. If $\mathbf{V}^{-1}$ exists, given the normal vector $\mathbf{n}$, the tangent points of contact to (B.1.2.1) are given by

$$\mathbf{q}_i = \mathbf{V}^{-1}(\kappa_i\mathbf{n} - \mathbf{u}), i = 1, 2$$
$$\text{where } \kappa_i = \pm\sqrt{\frac{f_0}{\mathbf{n}^\top\mathbf{V}^{-1}\mathbf{n}}} \tag{B.4.5.1}$$

Proof. From (B.4.2.2),

$$\mathbf{q} = \mathbf{V}^{-1}(\kappa\mathbf{n} - \mathbf{u}), \quad \kappa \in \mathbb{R} \tag{B.4.5.2}$$

Substituting (B.4.5.2) in (B.4.1.3),

$$(\kappa\mathbf{n} - \mathbf{u})^\top \mathbf{V}^{-1}(\kappa\mathbf{n} - \mathbf{u}) + 2\mathbf{u}^\top\mathbf{V}^{-1}(\kappa\mathbf{n} - \mathbf{u}) + f = 0 \tag{B.4.5.3}$$

$$\implies \kappa^2\mathbf{n}^\top\mathbf{V}^{-1}\mathbf{n} - \mathbf{u}^\top\mathbf{V}^{-1}\mathbf{u} + f = 0 \tag{B.4.5.4}$$

$$\text{or, } \kappa = \pm\sqrt{\frac{f_0}{\mathbf{n}^\top\mathbf{V}^{-1}\mathbf{n}}} \tag{B.4.5.5}$$

Substituting (B.4.5.5) in (B.4.5.2) yields (B.4.5.1). □

B.4.6. For a conic/hyperbola, a line with normal vector $\mathbf{n}$ cannot be a tangent if

$$\frac{\mathbf{u}^\top\mathbf{V}^{-1}\mathbf{u} - f}{\mathbf{n}^\top\mathbf{V}^{-1}\mathbf{n}} < 0 \tag{B.4.6.1}$$

B.4.7. If $\mathbf{V}$ is not invertible, given the normal vector $\mathbf{n}$, the point of contact to (B.1.2.1) is given by the matrix equation

$$\begin{pmatrix}(\mathbf{u} + \kappa\mathbf{n})^\top \\ \mathbf{V}\end{pmatrix}\mathbf{q} = \begin{pmatrix}-f \\ \kappa\mathbf{n} - \mathbf{u}\end{pmatrix} \tag{B.4.7.1}$$

$$\text{where } \kappa = \frac{\mathbf{p}_1^\top\mathbf{u}}{\mathbf{p}_1^\top\mathbf{n}}, \quad \mathbf{V}\mathbf{p}_1 = 0 \tag{B.4.7.2}$$

Proof. If $\mathbf{V}$ is non-invertible, it has a zero eigenvalue. If the corresponding eigenvector is $\mathbf{p}_1$, then,

$$\mathbf{V}\mathbf{p}_1 = 0 \tag{B.4.7.3}$$

From (B.4.2.2),

$$\kappa\mathbf{n} = \mathbf{V}\mathbf{q} + \mathbf{u}, \quad \kappa \in \mathbb{R} \tag{B.4.7.4}$$

$$\implies \kappa\mathbf{p}_1^\top\mathbf{n} = \mathbf{p}_1^\top\mathbf{V}\mathbf{q} + \mathbf{p}_1^\top\mathbf{u} \tag{B.4.7.5}$$

$$\text{or, } \kappa\mathbf{p}_1^\top\mathbf{n} = \mathbf{p}_1^\top\mathbf{u}, \quad \because \mathbf{p}_1^\top\mathbf{V} = 0, \quad (\text{ from (B.4.7.3)}) \tag{B.4.7.6}$$

yielding κ in (B.4.7.2). From (B.4.7.4),

$$\kappa\mathbf{q}^\top\mathbf{n} = \mathbf{q}^\top\mathbf{V}\mathbf{q} + \mathbf{q}^\top\mathbf{u} \tag{B.4.7.7}$$

$$\implies \kappa\mathbf{q}^\top\mathbf{n} = -f - \mathbf{q}^\top\mathbf{u} \quad \text{from (B.4.1.3),} \tag{B.4.7.8}$$

$$\text{or, } (\kappa\mathbf{n} + \mathbf{u})^\top\mathbf{q} = -f \tag{B.4.7.9}$$

(B.4.7.4) can be expressed as

$$\mathbf{V}\mathbf{q} = \kappa\mathbf{n} - \mathbf{u}. \tag{B.4.7.10}$$

(B.4.7.9) and (B.4.7.10) clubbed together result in (B.4.7.1). □

B.4.8. The asymptotes of the hyperbola in (B.1.5.1), defined to be the lines that do not intersect the hyperbola, are given by

$$\left(\sqrt{|\lambda_1|} \quad \pm \sqrt{|\lambda_2|}\right)\mathbf{y} = 0 \tag{B.4.8.1}$$

Proof. From (B.1.5.1), it is obvious that the pair of lines represented by

$$\mathbf{y}^\top \mathbf{D}\mathbf{y} = 0 \tag{B.4.8.2}$$

do not intersect the conic

$$\mathbf{y}^\top \mathbf{D}\mathbf{y} = f_0 \tag{B.4.8.3}$$

Thus, (B.4.8.2) represents the asysmptotes of the hyperbola in (B.1.5.1) and can be expressed as

$$\lambda_1 y_1^2 + \lambda_2 y_1^2 = 0, \tag{B.4.8.4}$$

which can then be simplified using the steps in (A.8.3.4)- (A.8.3.7) to obtain (B.4.8.1). □

B.4.9. (B.1.2.1) represents a pair of straight lines if

$$\mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f = 0 \tag{B.4.9.1}$$

B.4.10. (B.1.2.1) represents a pair of straight lines if the matrix

$$\begin{pmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^\top & f \end{pmatrix} \tag{B.4.10.1}$$

is singular.

Proof. Let

$$\begin{pmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^\top & f \end{pmatrix}\mathbf{x} = \mathbf{0} \tag{B.4.10.2}$$

Expressing

$$\mathbf{x} = \begin{pmatrix} \mathbf{y} \\ y_3 \end{pmatrix}, \tag{B.4.10.3}$$

$$\begin{pmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^\top & f \end{pmatrix}\begin{pmatrix} \mathbf{y} \\ y_3 \end{pmatrix} = \mathbf{0} \tag{B.4.10.4}$$

$$\implies \mathbf{V}\mathbf{y} + y_3\mathbf{u} = \mathbf{0} \quad \text{and} \tag{B.4.10.5}$$

$$\mathbf{u}^\top \mathbf{y} + f y_3 = 0 \tag{B.4.10.6}$$

From (B.4.10.5) we obtain,

$$\mathbf{y}^\top \mathbf{V}\mathbf{y} + y_3\mathbf{y}^\top \mathbf{u} = \mathbf{0} \tag{B.4.10.7}$$

$$\implies \mathbf{y}^\top \mathbf{V}\mathbf{y} + y_3\mathbf{u}^\top \mathbf{y} = \mathbf{0} \tag{B.4.10.8}$$

yielding (B.4.9.1) upon substituting from (B.4.10.6). □

B.4.11. Using the affine transformation, (B.4.8.1) can be expressed as the lines

$$\left(\sqrt{|\lambda_1|} \quad \pm\sqrt{|\lambda_2|}\right)\mathbf{P}^\top(\mathbf{x}-\mathbf{c}) = 0 \tag{B.4.11.1}$$

B.4.12. The angle between the asymptotes can be expressed as

$$\cos\theta = \frac{|\lambda_1| - |\lambda_2|}{|\lambda_1| + |\lambda_2|} \tag{B.4.12.1}$$

Proof. The normal vectors of the lines in (B.4.11.1) are

$$\begin{aligned}\mathbf{n}_1 &= \mathbf{P}\begin{pmatrix}\sqrt{|\lambda_1|}\\ \sqrt{|\lambda_2|}\end{pmatrix}\\ \mathbf{n}_2 &= \mathbf{P}\begin{pmatrix}\sqrt{|\lambda_1|}\\ -\sqrt{|\lambda_2|}\end{pmatrix}\end{aligned} \tag{B.4.12.2}$$

The angle between the asymptotes is given by

$$\cos\theta = \frac{\mathbf{n_1}^\top\mathbf{n_2}}{\|\mathbf{n_1}\|\,\|\mathbf{n_2}\|} \tag{B.4.12.3}$$

The orthogonal matrix $\mathbf{P}$ preserves the norm, i.e.

$$\|\mathbf{n_1}\| = \left\|\mathbf{P}\begin{pmatrix}\sqrt{|\lambda_1|}\\ \sqrt{|\lambda_2|}\end{pmatrix}\right\| = \left\|\begin{pmatrix}\sqrt{|\lambda_1|}\\ \sqrt{|\lambda_2|}\end{pmatrix}\right\| \tag{B.4.12.4}$$

$$= \sqrt{|\lambda_1| + |\lambda_2|} = \|\mathbf{n_2}\| \tag{B.4.12.5}$$

It is easy to verify that

$$\mathbf{n_1}^\top\mathbf{n_2} = |\lambda_1| - |\lambda_2| \tag{B.4.12.6}$$

Thus, the angle between the asymptotes is obtained from (B.4.12.3) as (B.4.12.1). □

B.4.13. For a circle, the points of contact are

$$\mathbf{q}_{ij} = \left(\pm r\frac{\mathbf{n}_j}{\left\|\mathbf{n}_j\right\|} - \mathbf{u}\right), \quad i, j = 1, 2 \tag{B.4.13.1}$$

Proof. From (B.4.5.1), and (6.1.1.1),

$$\kappa_{ij} = \pm\frac{r}{\left\|\mathbf{n}_j\right\|} \tag{B.4.13.2}$$

□

B.4.14. A point $\mathbf{h}$ lies on a normal to the conic in (B.1.2.1) if

$$\left(\mathbf{m}^\top(\mathbf{V}\mathbf{h}+\mathbf{u})\right)^2\left(\mathbf{n}^\top\mathbf{V}\mathbf{n}\right) - 2\left(\mathbf{m}^\top\mathbf{V}\mathbf{n}\right)\left(\mathbf{m}^\top(\mathbf{V}\mathbf{h}+\mathbf{u})\mathbf{n}^\top(\mathbf{V}\mathbf{h}+\mathbf{u})\right) + g(\mathbf{h})\left(\mathbf{m}^\top\mathbf{V}\mathbf{n}\right)^2 = 0 \tag{B.4.14.1}$$

Proof. The point of contact for the normal passing through a point $\mathbf{h}$ is given by

$$\mathbf{q} = \mathbf{h} + \mu\mathbf{n} \tag{B.4.14.2}$$

From (B.4.1.1), the tangent at $\mathbf{q}$ satisfies

$$\mathbf{m}^\top(\mathbf{V}\mathbf{q}+\mathbf{u}) = 0 \tag{B.4.14.3}$$

Substituting (B.4.14.2) in (B.4.14.3),

$$\mathbf{m}^\top(\mathbf{V}(\mathbf{h}+\mu\mathbf{n})+\mathbf{u}) = 0 \tag{B.4.14.4}$$

$$\implies \mu\mathbf{m}^\top\mathbf{V}\mathbf{n} = -\mathbf{m}^\top(\mathbf{V}\mathbf{h}+\mathbf{u}) \tag{B.4.14.5}$$

yielding

$$\mu = -\frac{\mathbf{m}^\top(\mathbf{V}\mathbf{h}+\mathbf{u})}{\mathbf{m}^\top\mathbf{V}\mathbf{n}}, \tag{B.4.14.6}$$

From (B.3.1.6),

$$\mu^2\mathbf{n}^\top\mathbf{V}\mathbf{n} + 2\mu\mathbf{n}^\top\left(\mathbf{V}\mathbf{h}+\mathbf{u}\right) + \mathrm{g}\left(\mathbf{h}\right) = 0 \tag{B.4.14.7}$$

From (B.4.14.6), (B.4.14.7) can be expressed as

$$\left(-\frac{\mathbf{m}^\top(\mathbf{V}\mathbf{h}+\mathbf{u})}{\mathbf{m}^\top\mathbf{V}\mathbf{n}}\right)^2\mathbf{n}^\top\mathbf{V}\mathbf{n} + 2\left(-\frac{\mathbf{m}^\top(\mathbf{V}\mathbf{h}+\mathbf{u})}{\mathbf{m}^\top\mathbf{V}\mathbf{n}}\right)\mathbf{n}^\top\left(\mathbf{V}\mathbf{h}+\mathbf{u}\right) + \mathrm{g}\left(\mathbf{h}\right) = 0 \tag{B.4.14.8}$$

yielding (B.4.14.1). □

B.4.15. A point $\mathbf{h}$ lies on a tangent to the conic in (B.1.2.1) if

$$\mathbf{m}^\top\left[\left(\mathbf{V}\mathbf{h}+\mathbf{u}\right)\left(\mathbf{V}\mathbf{h}+\mathbf{u}\right)^\top - \mathbf{V}\mathrm{g}\left(\mathbf{h}\right)\right]\mathbf{m} = 0 \tag{B.4.15.1}$$

Proof. From (B.3.1.3) and (B.4.1.2)

$$\left[\mathbf{m}^\top\left(\mathbf{V}\mathbf{h}+\mathbf{u}\right)\right]^2 - \left(\mathbf{m}^\top\mathbf{V}\mathbf{m}\right)\mathrm{g}\left(\mathbf{h}\right) = 0 \tag{B.4.15.2}$$

yielding (B.4.15.1). □

B.4.16. The normal vectors of the tangents to the conic in (B.1.2.1) from a point $\mathbf{h}$ are given by

$$\begin{aligned}\mathbf{n}_1 &= \mathbf{P}\begin{pmatrix}\sqrt{|\lambda_1|}\\ \sqrt{|\lambda_2|}\end{pmatrix}\\ \mathbf{n}_2 &= \mathbf{P}\begin{pmatrix}\sqrt{|\lambda_1|}\\ -\sqrt{|\lambda_2|}\end{pmatrix}\end{aligned} \tag{B.4.16.1}$$

where $\lambda_i, \mathbf{P}$ are the eigenparameters of

$$\mathbf{\Sigma} = \left(\mathbf{V}\mathbf{h}+\mathbf{u}\right)\left(\mathbf{V}\mathbf{h}+\mathbf{u}\right)^\top - \left(\mathrm{g}\left(\mathbf{h}\right)\right)\mathbf{V}. \tag{B.4.16.2}$$

Proof. From (B.4.15.1) we obtain (B.4.16.2). Consequently, from (B.4.12.2), (B.4.16.1) can be obtained. □

B.4.17. (B.1.2.1) represents a pair of straight lines if the matrix

$$\begin{pmatrix}\mathbf{V} & \mathbf{u}\\ \mathbf{u}^\top & f\end{pmatrix} \tag{B.4.17.1}$$

is singular.

B.4.18. The intersection of two conics with parameters $\mathbf{V}_i, \mathbf{u}_i, f_i,\ i = 1, 2$ is defined as

$$\mathbf{x}^\top (\mathbf{V}_1 + \mu \mathbf{V}_2)\, \mathbf{x} + 2\, (\mathbf{u}_1 + \mu \mathbf{u}_2)^\top \mathbf{x} + (f_1 + \mu f_2) = 0 \tag{B.4.18.1}$$

B.4.19. From (B.4.17.1), (B.4.18.1) represents a pair of straight lines if

$$\begin{vmatrix} \mathbf{V}_1 + \mu \mathbf{V}_2 & \mathbf{u}_1 + \mu \mathbf{u}_2 \\ (\mathbf{u}_1 + \mu \mathbf{u}_2)^\top & f_1 + \mu f_2 \end{vmatrix} = 0 \tag{B.4.19.1}$$