MATRIX ANALYSIS

Through Coordinate Geometry

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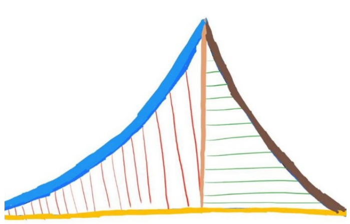


Contents

Introduction

This book links high school coordinate geometry to linear algebra and matrix analysis through solved problems.

Chapter 1

Vectors

1.1. Length

1.1.1 If $\overrightarrow{a}$ is a nonzero vector of magnitude 'a' and λ a nonzero scalar , then $\lambda\overrightarrow{a}$ is unit vector if

(a) $\lambda = 1$

(b) $\lambda = -1$

(c) $a = |\lambda|$

(d) $a = 1/|\lambda|$

1.1.2 Compute the magnitude of the following vectors:

$$\mathbf{a} = \hat{i} + \hat{j} + k; \mathbf{b} = 2\hat{i} - 7\hat{j} - 3\hat{k}; \mathbf{c} = \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} - \frac{1}{3}\hat{k}$$

Solution: Let

$$\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 2 \\ -7 \\ 3 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{pmatrix} \tag{1.1.2.1}$$

Then

$$\|\mathbf{a}\| = \sqrt{\mathbf{a}^\top \mathbf{a}} = \|\mathbf{a}\| = \sqrt{3}, \tag{1.1.2.2}$$

$$\|\mathbf{b}\| = \sqrt{\mathbf{b}^\top \mathbf{b}} = \|\mathbf{b}\| = \sqrt{62}, \tag{1.1.2.3}$$

$$\|\mathbf{c}\| = \sqrt{\mathbf{c}^\top \mathbf{c}} = \|\mathbf{c}\| = 1 \tag{1.1.2.4}$$

1.1.3 Find the unit vector in the direction of the vector $\mathbf{a} = \hat{i} + \hat{j} + 2\hat{k}$.

1.1.4 Find the unit vector in the direction of vector $\overrightarrow{PQ}$, where $\mathbf{P}$ and $\mathbf{Q}$ are the points (1, 2, 3) and (4, 5, 6), respectively.

1.1.5 For given vectors, $\mathbf{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\mathbf{b} = -\hat{i} + \hat{j} - \hat{k}$, find the unit vector in the direction of the vector $\mathbf{a} + \mathbf{b}$.

Solution: Since

$$\mathbf{a} + \mathbf{b} = \mathbf{u} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \tag{1.1.5.1}$$

the unit vector in direction of $\mathbf{u}$ is,

$$\hat{\mathbf{u}} = \frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{1}{\sqrt{2}}\mathbf{u} \tag{1.1.5.2}$$

$$\implies \hat{\mathbf{u}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \tag{1.1.5.3}$$

1.1.6 Find a vector in the direction of vector $5\hat{\imath}-\hat{\jmath}+2\hat{k}$ which has magnitude 8 units. has magnitude 8 units.

Solution: Let the required vector be $c\begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix}$, where $c \in \mathbb{R}$. Since this vector has magnitude 8,

$$\left\| c\begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix} \right\| = c\sqrt{5^2 + (-1)^2 + 2^2} = 8 \tag{1.1.6.1}$$

$$\implies c = \frac{8}{\sqrt{30}} = \frac{4\sqrt{30}}{15} \tag{1.1.6.2}$$

Thus, the required vector is $\frac{4\sqrt{30}}{15}\begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix}$.

1.1.7 Find the direction cosines of the vector $\hat{\imath} + 2\hat{\jmath} + 3\hat{k}$.

Solution: Let

$$\mathbf{A} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \tag{1.1.7.1}$$

The Directional vectors of x, y and z axes are given respectively as

$$\mathbf{e_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{e_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{e_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{1.1.7.2}$$

The magnitudes for $\mathbf{A}$ and directional vectors $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$ are

$$\|\mathbf{A}\| = \sqrt{14}, \|\mathbf{e_1}\| = 1, \|\mathbf{e_1}\| = 1, \|\mathbf{e_1}\| = 1 \tag{1.1.7.3}$$

So for different values of $\cos\theta_i$ the direction cosines of vector **A** are

$$\cos\theta_1 = \frac{\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}{\sqrt{14}} = \frac{1}{\sqrt{14}} \tag{1.1.7.4}$$

$$\cos\theta_2 = \frac{\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}{\sqrt{14}} = \frac{2}{\sqrt{14}} \tag{1.1.7.5}$$

$$\cos\theta_3 = \frac{\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}{\sqrt{14}} = \frac{3}{\sqrt{14}} \tag{1.1.7.6}$$

1.1.8 Find the direction cosines of the vector joining the points **A** (1, 2, –3) and **B**(–1, –2, 1), directed from **A** to **B**.

Solution: The direction vector is given by

$$\mathbf{m} = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \\ 4 \end{pmatrix} \tag{1.1.8.1}$$

and the corresponding unit vector is

$$\frac{\mathbf{m}}{\|\mathbf{m}\|} = \frac{1}{6}\begin{pmatrix} -2 \\ -4 \\ 4 \end{pmatrix} \tag{1.1.8.2}$$

1.1.9 Show that the vector $\hat{i} + \hat{j} + \hat{k}$ is equally inclined to the axes OX, OY and OZ.

Solution: Let

$$\mathbf{A} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{1.1.9.1}$$

Since all entries of the vector are equal, the vector is equally inclined to the axes.

1.1.10 If a line has the direction ratios –18, 12, –4, then what are its direction cosines?

Solution: Let

$$\mathbf{A} = \begin{pmatrix} -18 \\ 12 \\ -4 \end{pmatrix} \tag{1.1.10.1}$$

Then the unit direction vector is

$$\mathbf{B} = \frac{\mathbf{A}}{\|\mathbf{A}\|} = \begin{pmatrix} \frac{-9}{11} \\ \frac{6}{11} \\ \frac{-2}{11} \end{pmatrix} \tag{1.1.10.2}$$

1.1.11 Find the direction cosines of the sides of a triangle whose vertices are $\begin{pmatrix} 3 \\ 5 \\ -4 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -5 \\ -5 \\ -2 \end{pmatrix}$.

Solution: Let the vertices be

$$\mathbf{A} = \begin{pmatrix} 3 \\ 5 \\ -4 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -5 \\ -5 \\ -2 \end{pmatrix} \tag{1.1.11.1}$$

The direction vectors of the sides are,

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} 4 \\ 4 \\ -6 \end{pmatrix} = \mathbf{m_1}, \mathbf{B} - \mathbf{C} = \begin{pmatrix} 4 \\ 6 \\ 4 \end{pmatrix} = \mathbf{m_2}, \mathbf{C} - \mathbf{A} = \begin{pmatrix} -8 \\ -10 \\ 2 \end{pmatrix} = \mathbf{m_3}, \tag{1.1.11.2}$$

The direction vectors of the axes are,

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{1.1.11.3}$$

Direction cosines of a vector $\mathbf{m}$ are given by

$$\begin{pmatrix} \cos\theta_1 \\ \cos\theta_2 \\ \cos\theta_3 \end{pmatrix} = \begin{pmatrix} \frac{\mathbf{m}^\top \mathbf{e}_1}{\|\mathbf{m}\|\|\mathbf{e}_1\|} \\ \frac{\mathbf{m}^\top \mathbf{e}_1}{\|\mathbf{m}\|\|\mathbf{e}_1\|} \\ \frac{\mathbf{m}^\top \mathbf{e}_1}{\|\mathbf{m}\|\|\mathbf{e}_1\|} \end{pmatrix} = \frac{1}{\|\mathbf{m}\|} \begin{pmatrix} \mathbf{m}^\top \mathbf{e}_1 \\ \mathbf{m}^\top \mathbf{e}_2 \\ \mathbf{m}^\top \mathbf{e}_3 \end{pmatrix} = \frac{\mathbf{m}}{\|\mathbf{m}\|} \tag{1.1.11.4}$$

Direction cosines of side $\mathbf{m_1}$ are

$$\begin{pmatrix} \cos\theta_1 \\ \cos\theta_2 \\ \cos\theta_3 \end{pmatrix} = \frac{\mathbf{m_1}}{\|\mathbf{m_1}\|} = \begin{pmatrix} \frac{2}{\sqrt{17}} \\ \frac{2}{\sqrt{17}} \\ \frac{-3}{\sqrt{17}} \end{pmatrix} \tag{1.1.11.5}$$

Direction cosines of side $\mathbf{m_2}$ are,

$$\begin{pmatrix} \cos\theta_1 \\ \cos\theta_2 \\ \cos\theta_3 \end{pmatrix} = \frac{\mathbf{m_2}}{\|\mathbf{m_2}\|} = \begin{pmatrix} \frac{2}{\sqrt{17}} \\ \frac{3}{\sqrt{17}} \\ \frac{2}{\sqrt{17}} \end{pmatrix} \tag{1.1.11.6}$$

Direction cosines of side $\mathbf{m_3}$ are,

$$\begin{pmatrix} \cos\theta_1 \\ \cos\theta_2 \\ \cos\theta_3 \end{pmatrix} = \frac{\mathbf{m_3}}{\|\mathbf{m_3}\|} = \begin{pmatrix} \frac{-4}{\sqrt{42}} \\ \frac{-5}{\sqrt{42}} \\ \frac{1}{\sqrt{42}} \end{pmatrix} \tag{1.1.11.7}$$

1.1.12 If $\mathbf{a} = \mathbf{b} + \mathbf{c}$, then is it true that $|\mathbf{a}| = |\mathbf{b}| + |\mathbf{c}|$? Justify your answer.

Solution: Let

$$\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} \tag{1.1.12.1}$$

Then

$$\mathbf{a} = \mathbf{b} + \mathbf{c} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \tag{1.1.12.2}$$

$$\implies \|\mathbf{a}\| = \sqrt{11},\ \|\mathbf{b}\| = \sqrt{14},\ \|\mathbf{c}\| = 3. \tag{1.1.12.3}$$

Thus

$$\|\mathbf{a}\| \neq \|\mathbf{b}\| + \|\mathbf{c}\| \tag{1.1.12.4}$$

1.1.13 Find the value of x for which $x(\hat{i} + \hat{j} + \hat{k})$ is a unit vector.

Solution: Let

$$\mathbf{x} = x \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \implies \|\mathbf{x}\| = 1 \tag{1.1.13.1}$$

$$\implies x\sqrt{3} = 1, \text{ or, } x = \frac{1}{\sqrt{3}} \tag{1.1.13.2}$$

1.2. Distance

1.2.1 Find the distances between the following pairs of points:

(a) $(2,3),(4,1)$

(b) $(-5,7),(-1,3)$

(c) $(a,b),(-a,-b)$

Solution:

(a) The coordinates are given as

$$\mathbf{A} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \tag{1.2.1.1}$$

$$\implies \mathbf{A} - \mathbf{B} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix} \tag{1.2.1.2}$$

$$\implies (\mathbf{A} - \mathbf{B})^\top (\mathbf{A} - \mathbf{B}) = 8 \tag{1.2.1.3}$$

Thus, the desired distance is

$$d = \|\mathbf{A} - \mathbf{B}\| = \sqrt{8} \tag{1.2.1.4}$$

See Fig. 1.2.1.1.

(b) The coordinates are given as

$$\mathbf{C} = \begin{pmatrix} -5 \\ 7 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} -1 \\ 3 \end{pmatrix} \implies \mathbf{C} - \mathbf{D} = \begin{pmatrix} -5 \\ 7 \end{pmatrix} - \begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} -4 \\ 4 \end{pmatrix} \tag{1.2.1.5}$$

$$\implies (\mathbf{C} - \mathbf{D})^\top (\mathbf{C} - \mathbf{D}) = 32 \tag{1.2.1.6}$$

Thus,

$$d = \|\mathbf{C} - \mathbf{D}\| = 4\sqrt{2} \tag{1.2.1.7}$$

See Fig. 1.2.1.1.

(c) The coordinates are given as

$$\mathbf{E} = \begin{pmatrix} a \\ b \end{pmatrix}, \mathbf{F} = \begin{pmatrix} -a \\ -b \end{pmatrix} \implies \mathbf{E} - \mathbf{F} = \begin{pmatrix} a \\ b \end{pmatrix} - \begin{pmatrix} -a \\ -b \end{pmatrix} = \begin{pmatrix} 2a \\ 2b \end{pmatrix} \tag{1.2.1.8}$$

$$\implies (\mathbf{E} - \mathbf{F})^\top (\mathbf{E} - \mathbf{F}) = 4a^2 + 4b^2 \tag{1.2.1.9}$$

Thus,

$$d = \|\mathbf{E} - \mathbf{F}\| = 2\sqrt{a^2 + b^2} \tag{1.2.1.10}$$

See Fig. 1.2.1.1 for $a = 1, b = 2$

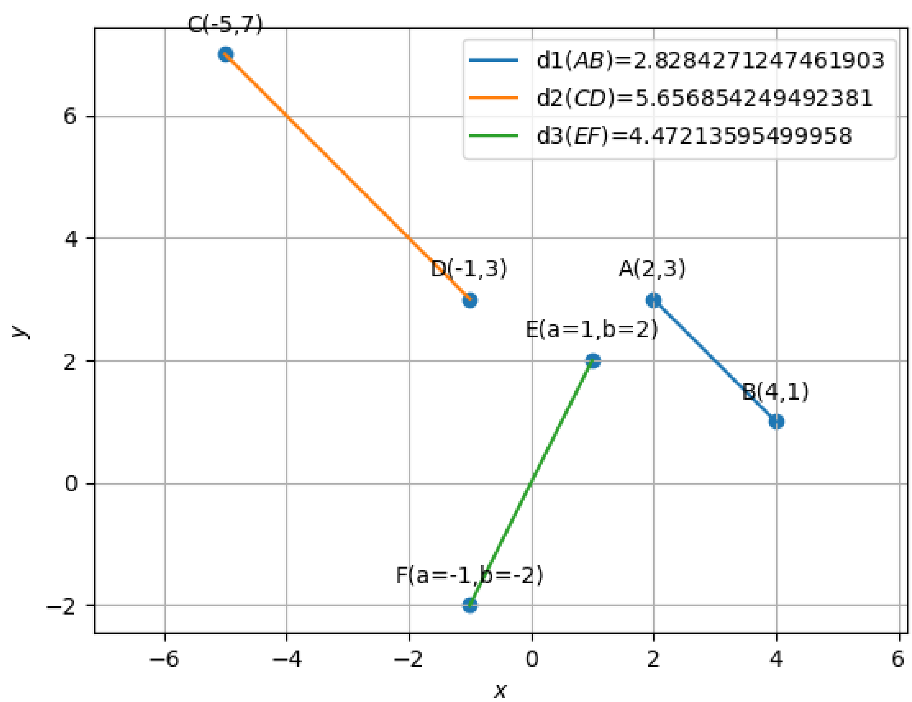


Figure 1.2.1.1:

1.2.2 Find the distance between the points $(0, 0)$ and $(36, 15)$.

Solution: Let

$$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \mathbf{B} = \begin{pmatrix} 36 \\ 15 \end{pmatrix} \tag{1.2.2.1}$$

$$\implies \mathbf{d} = \|\mathbf{A} - \mathbf{B}\| = \sqrt{\left(\mathbf{A} - \mathbf{B}\right)^T \left(\mathbf{A} - \mathbf{B}\right)} \tag{1.2.2.2}$$

$$= 39 \tag{1.2.2.3}$$

See Fig. 1.2.2.1.

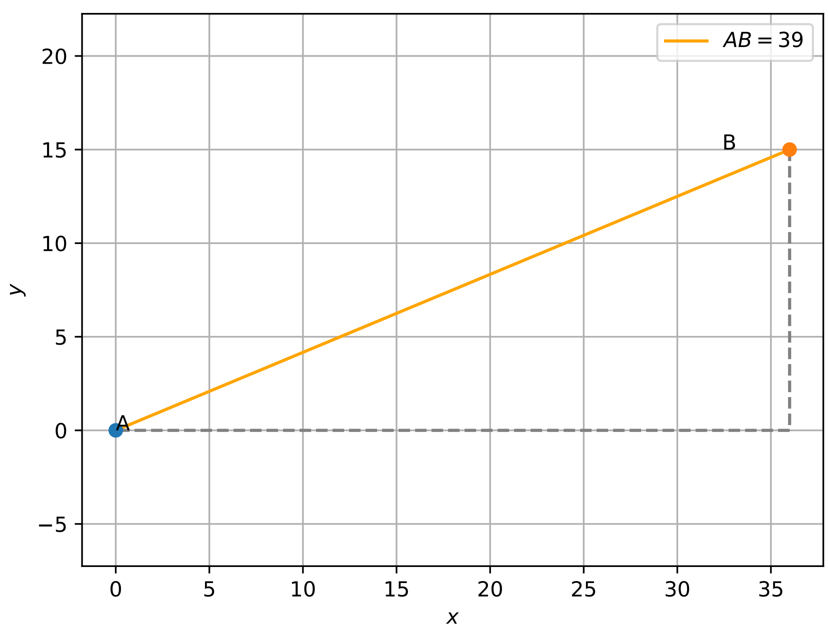


Figure 1.2.2.1:

1.2.3 Find the point on the x-axis which is equidistant from $(2, -5)$ and $(-2, 9)$.

Solution: The input parameters for this problem are available in Table 1.2.3.1 If $\mathbf{O}$ lies on the x-axis and is equidistant from the points $\mathbf{A}$ and

Symbol	Value	Description
$\mathbf{A}$	$\begin{pmatrix} 2 \\ -5 \end{pmatrix}$	First point
$\mathbf{B}$	$\begin{pmatrix} -2 \\ 9 \end{pmatrix}$	Second point
$\mathbf{O}$	?	Desired point

Table 1.2.3.1:

$\mathbf{B}$,

$$\|\mathbf{O} - \mathbf{A}\| = \|\mathbf{A} - \mathbf{B}\| \tag{1.2.3.1}$$

$$\implies \|\mathbf{O} - \mathbf{A}\|^2 = \|\mathbf{O} - \mathbf{B}\|^2 \tag{1.2.3.2}$$

which can be expressed as

$$\begin{aligned} &(\mathbf{O} - \mathbf{A})^\top (\mathbf{O} - \mathbf{A}) = (\mathbf{O} - \mathbf{B})^\top (\mathbf{O} - \mathbf{B}) \\ &\implies \|\mathbf{O}\|^2 - 2\mathbf{O}^\top\mathbf{A} + \|\mathbf{A}\|^2 \\ &= \|\mathbf{O}\|^2 - 2\mathbf{O}^\top\mathbf{B} + \|\mathbf{B}\|^2 \end{aligned} \tag{1.2.3.3}$$

which can be simplified to obtain

$$\mathbf{O} = x\mathbf{e}_1 \tag{1.2.3.4}$$

where

$$x = \frac{\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2}{2\left(\mathbf{A} - \mathbf{B}\right)^\top \mathbf{e}_1} \tag{1.2.3.5}$$

Substituting from Table (1.2.3.1) in (1.2.3.5),

$$\left(\mathbf{A} - \mathbf{B}\right)^\top = \left(\begin{pmatrix} 2 \\ -5 \end{pmatrix} - \begin{pmatrix} -2 \\ 9 \end{pmatrix}\right)^\top = \begin{pmatrix} 4 & -14 \end{pmatrix} \tag{1.2.3.6}$$

$$\|\mathbf{A}\|^2 = 21, \|\mathbf{B}\|^2 = 85 \tag{1.2.3.7}$$

yielding $x = -7$. Thus,

$$\mathbf{O} = \begin{pmatrix} -7 \\ 0 \end{pmatrix}. \tag{1.2.3.8}$$

See Fig. 1.2.3.1.

1.2.4 Find the values of y for which the distance between the points $\mathbf{P}(2, -3)$ and $\mathbf{Q}(10, y)$ is 10 units.

1.2.5 If $\mathbf{Q}(0, 1)$ is equidistant from $\mathbf{P}(5, -3)$ and $\mathbf{R}(x, 6)$, find the values of x. Also find the distances QR and PR.

1.2.6 Find a relation between x and y such that the point (x, y) is equidistant from the point $(3, 6)$ and $(-3, 4)$.

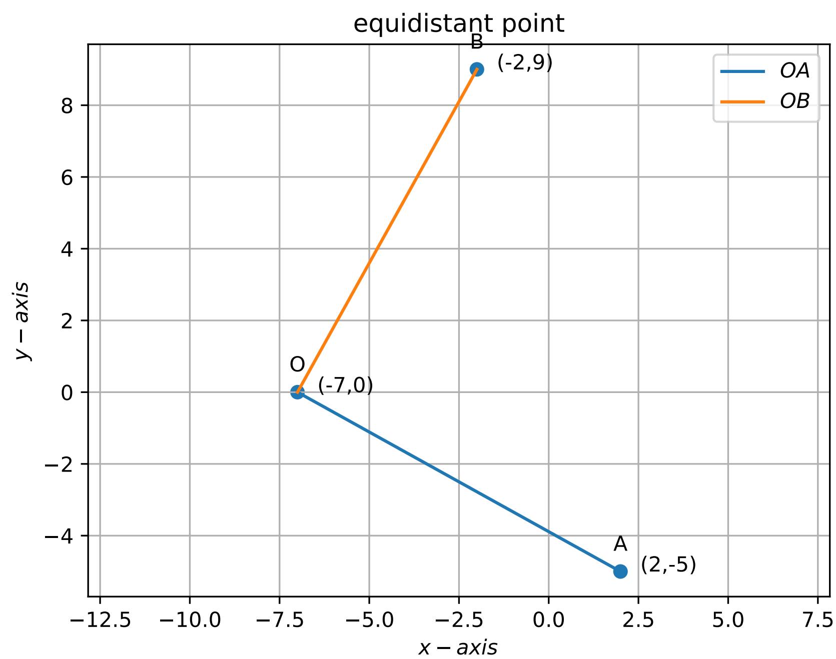


Figure 1.2.3.1:

Solution: The input parameters for this problem are given as

$$\mathbf{P} = \begin{pmatrix} x \\ y \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 3 \\ -4 \end{pmatrix} \tag{1.2.6.1}$$

$$\mathbf{P} = y\mathbf{e}_1 \tag{1.2.6.2}$$

where

$$y = \frac{\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2}{2\left(\mathbf{A} - \mathbf{B}\right)^\top \mathbf{e}_1} \tag{1.2.6.3}$$

Substituting the $\mathbf{A}, \mathbf{B}$ values in (1.2.6.3),

$$(\mathbf{A} - \mathbf{B}) = \begin{pmatrix} 6 \\ 2 \end{pmatrix}, \|\mathbf{A}\|^2 = 45, \|\mathbf{B}\|^2 = 25 \tag{1.2.6.4}$$

yielding $y = 5$. Hence,

$$\mathbf{P} = \begin{pmatrix} 0 \\ 5 \end{pmatrix} \tag{1.2.6.5}$$

See Fig. 1.2.6.1.

1.2.7 Find a point on the x-axis,which is equidistant from the points $\begin{pmatrix} 7 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

Solution: From the given information

$$\|\mathbf{x} - \mathbf{A}\|^2 = \|\mathbf{x} - \mathbf{B}\|^2 \tag{1.2.7.1}$$

$$\implies (\mathbf{x} - \mathbf{A})^\top (\mathbf{x} - \mathbf{A}) = (\mathbf{x} - \mathbf{B})^\top (\mathbf{x} - \mathbf{B}) \tag{1.2.7.2}$$

$$\implies \|\mathbf{x}\|^2 - 2\mathbf{A}^\top\mathbf{x} + \|\mathbf{A}\|^2 = \|\mathbf{x}\|^2 - 2\mathbf{B}^\top\mathbf{x} + \|\mathbf{B}\|^2 \tag{1.2.7.3}$$

$$\text{or, } (\mathbf{A} - \mathbf{B})^\top \mathbf{x} = \frac{\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2}{2} \tag{1.2.7.4}$$

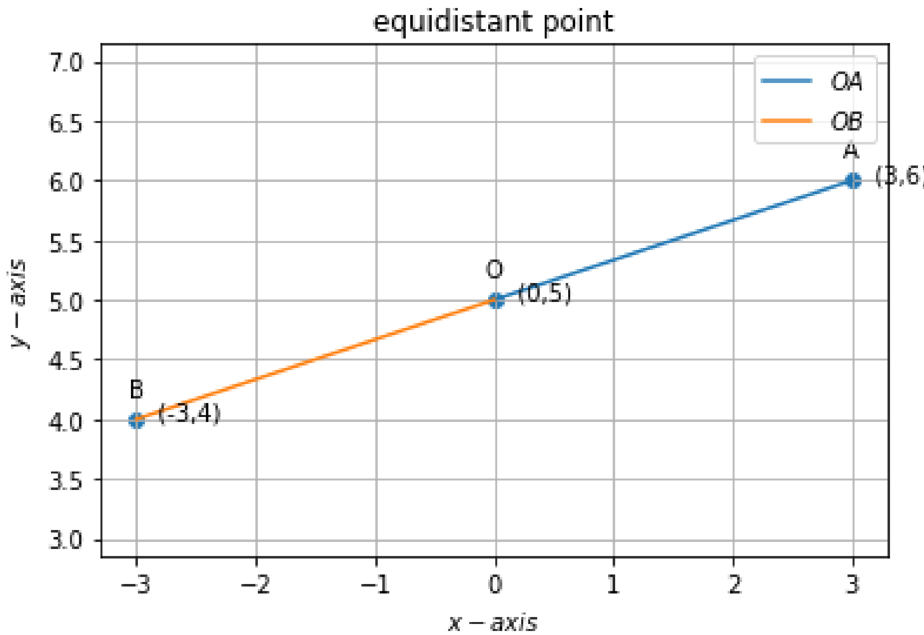


Figure 1.2.6.1:

Since $\mathbf{x}$ lies on the x-axis,

$$\mathbf{x} = k\mathbf{e}_1 \tag{1.2.7.5}$$

which, upon substituting in (1.2.7.4) yields

$$k = \frac{15}{2} \tag{1.2.7.6}$$

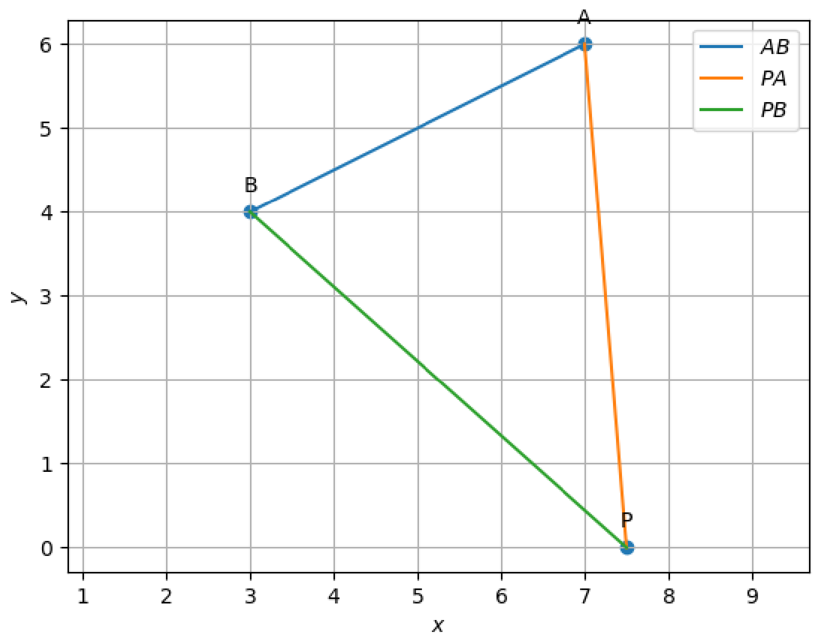


Figure 1.2.7.1:

1.3. Exercises

1.3.1 The distance between the points $\mathbf{A}(0, 6)$ and $\mathbf{B}(0, -2)$ is

(a) 6

(b) 8

(c) 4

(d) 2

1.3.2 The distance of the point $\mathbf{P}(-6, 8)$ from the origin is

(a) 8

(b) $2\sqrt{7}$

(c) 10

(d) 6

1.3.3 The distance between the points $(0,5)$ and $(-5,0)$ is

(a) 5

(b) 5

(c) 5

(d) 10

1.3.4 **AOBC** is a rectangle whose three vertices are vertices $\mathbf{A}(0,3), \mathbf{O}(0,0)$ and $\mathbf{B}(5,0)$. The length of its diagonal is

(a) 5

(b) 3

(c) 34

(d) 4

1.3.5 The perimeter of a triangle with vertices $(0,4),(0,0)$ and $(3,0)$ is

(a) 5

(b) 12

(c) 11

(d) 7

1.3.6 A circle drawn with origin as the centre passes through $(\frac{13}{2},0)$. The point which does not lie in the interior of the circle is

(a) $(\frac{-3}{4}, 1)$

(b) $(2, \frac{7}{3})$

(c) $(5, \frac{-1}{2})$

(d) $(-6, \frac{-5}{2})$

1.3.7 If the distance between the points $(4, P)$ and $(1, 0)$ is 5,then the value of **P** is

(a) 4 only

(b) +4 only

(c) -4 only

(d) 0

1.3.8 A circle has its centre at the origin and a point $\mathbf{P}(5, 0)$ lies on it The point $\mathbf{Q}(6, 8)$ lies outside the circle

1.3.9 The point $\mathbf{P}(-2, 4)$lies on circie of radius 6 and center $\mathbf{C}(3, 5)$

1.3.10 Find the points on the x-axis which are at a distance on $2\sqrt{5}$ from the point$(7, -4)$. How many such points are there?

1.3.11 Find the value of a, if the if the distance between the points $\mathbf{A}(-3, -14)$ and $B(a, -5)$ is 9 units.

1.3.12 Find a point which is equidistant from the points $\mathbf{A}(-5, 4)$ and $(-1, 6)$? How many such points are there ?

1.3.13 If the point $\mathbf{A}(2, -4)$ is equidistant from $\mathbf{P}(3, 8)$ and $\mathbf{Q}(-10, y)$, find the values of y, Also find distance **PQ**.

1.3.14 If (a, b) is the mid-point of the line segment joining the point $\mathbf{A}(10, -6)$ and $\mathbf{B}(k, 4)$ and $a - 2b = 18$, find the value of a, b and the distance $\mathbf{AB}$.

1.3.15 The centre of a circle is $(2a, a - 7)$. Find the values of a if the circle passes through the point $(11, -9)$ and has diameter $10\sqrt{2}$ units.

1.4. Section Formula

1.4.1 Find the coordinates of the point which divides the join of $(-1, 7)$ and $(4, -3)$ in the ratio 2:3.

Solution: The coordinates and ratio are given as

$$\mathbf{P} = \begin{pmatrix} -1 \\ 7 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}, n = \frac{3}{2} \tag{1.4.1.1}$$

Using section formula

$$\mathbf{R} = \frac{\mathbf{Q} + n\mathbf{P}}{1 + n} \tag{1.4.1.2}$$

$$= \frac{1}{1 + \frac{3}{2}} \left(\begin{pmatrix} 4 \\ -3 \end{pmatrix} + \frac{3}{2} \begin{pmatrix} -1 \\ 7 \end{pmatrix} \right) \tag{1.4.1.3}$$

$$= \begin{pmatrix} 1 \\ 3 \end{pmatrix} \tag{1.4.1.4}$$

See Fig. 1.4.1.1

1.4.2 Find the coordinates of the points of trisection of the line segment joining $(4, -1)$ and $(-2, 3)$.

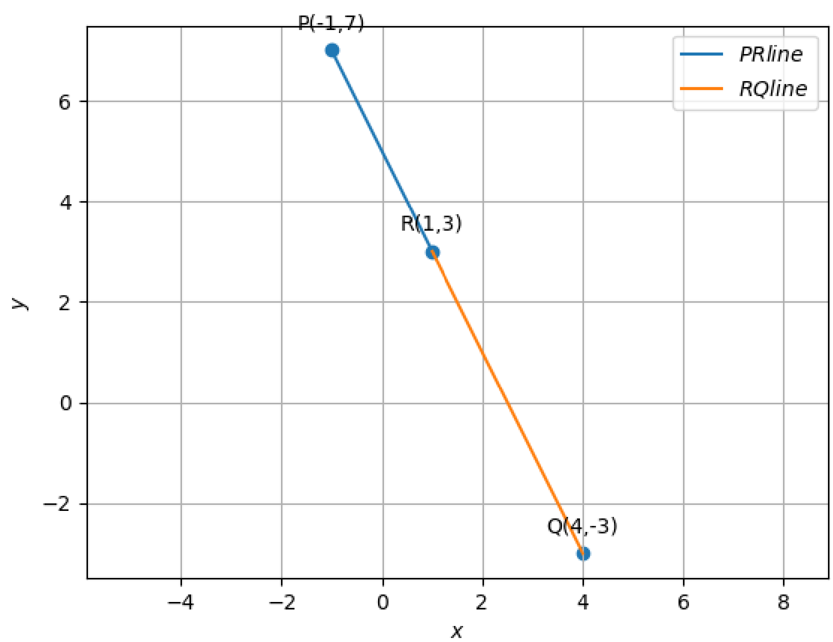


Figure 1.4.1.1:

Solution: Let the given points be

$$\mathbf{Q} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}, \mathbf{P} = \begin{pmatrix} -2 \\ -3 \end{pmatrix} \tag{1.4.2.1}$$

Using section formula

$$\mathbf{R} = \frac{\mathbf{Q} + n\mathbf{P}}{1+n} \tag{1.4.2.2}$$

Choosing $n = \frac{1}{2}$,

$$\mathbf{R} = \frac{1}{1+\frac{1}{2}}\left(\begin{pmatrix} 4 \\ -1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -2 \\ -3 \end{pmatrix}\right) \tag{1.4.2.3}$$

$$= \begin{pmatrix} 2 \\ \frac{-5}{3} \end{pmatrix} \tag{1.4.2.4}$$

$$\tag{1.4.2.5}$$

and choosing $n = 2$

$$\mathbf{S} = \frac{1}{1+\frac{2}{1}}\left(\begin{pmatrix} 4 \\ -1 \end{pmatrix} + \frac{2}{1}\begin{pmatrix} -2 \\ -3 \end{pmatrix}\right) \tag{1.4.2.6}$$

$$= \begin{pmatrix} 0 \\ \frac{-7}{3} \end{pmatrix} \tag{1.4.2.7}$$

which are the desired points of trisection. These are plotted in Fig. 1.4.2.1

1.4.3

1.4.4 Find the ratio in which the line segment joining the points $(-3, 10)$ and $(6, -8)$ is divided by $(-1, 6)$.

Solution: The input parameters for this problem are available in Table (1.4.4.1). Using section formula,

$$\mathbf{R} = \frac{\mathbf{Q} + n\mathbf{P}}{1+n} \tag{1.4.4.1}$$

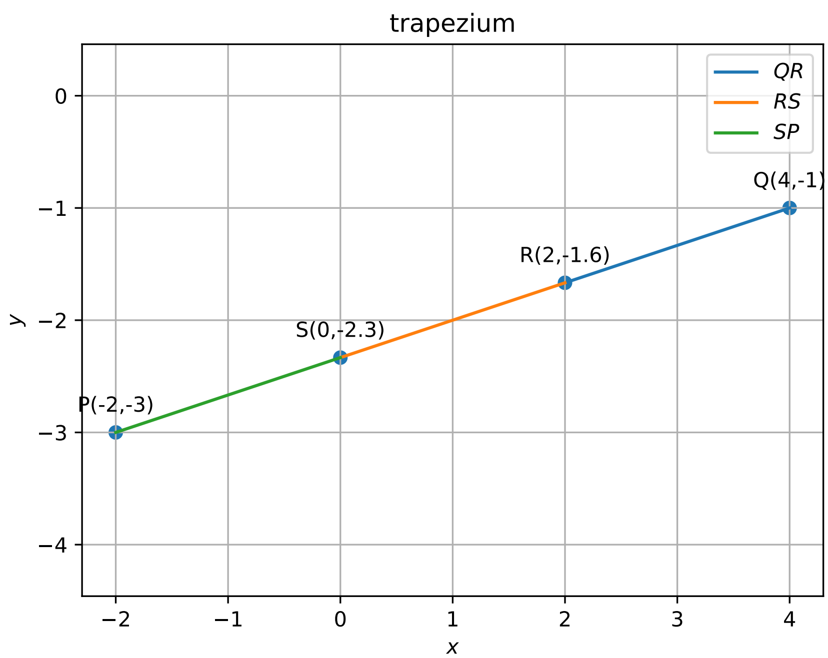


Figure 1.4.2.1:

Symbol	Value	Description
$\mathbf{P}$	$\begin{pmatrix}-3\\10\end{pmatrix}$	First point
$\mathbf{Q}$	$\begin{pmatrix}6\\-8\end{pmatrix}$	Second point
$\mathbf{R}$	$\begin{pmatrix}-1\\6\end{pmatrix}$	Desired point

Table 1.4.4.1:

Substituting the values of $\mathbf{P}, \mathbf{Q}$ and $\mathbf{R}$ in (1.4.4.1)

$$\begin{pmatrix}-1\\6\end{pmatrix} = \frac{\begin{pmatrix}-3\\10\end{pmatrix} + n\begin{pmatrix}6\\-8\end{pmatrix}}{1+n} \tag{1.4.4.2}$$

Simplifying (1.4.4.4) yeilds,

$$-1 = \frac{-3+6n}{1+n} \tag{1.4.4.5}$$

$$\implies n = \frac{2}{7} \tag{1.4.4.6}$$

Also,

$$6 = \frac{10-8n}{1+n} \tag{1.4.4.7}$$

$$\implies n = \frac{2}{7} \tag{1.4.4.8}$$

Hence the desired ratio is $\frac{2}{7}$.

1.4.5 Find the ratio in which the line segment joining $A(1,-5)$ and $B(-4,5)$ is divided by the x-axis. Also find the coordinates of the point of division.

1.4.6 If $(1,2),(4,y),(x,6),(3,5)$ are the vertices of a parallelogram taken in order, find x and y.

Solution:

The input parameters for this problem are available in 1.4.6.1. From the given information,

$$\mathbf{B}-\mathbf{A} = \begin{pmatrix} 4 \\ y \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ y-2 \end{pmatrix} \tag{1.4.6.1}$$

$$\mathbf{C}-\mathbf{D} = \begin{pmatrix} x \\ 6 \end{pmatrix} - \begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} x-3 \\ 1 \end{pmatrix} \tag{1.4.6.2}$$

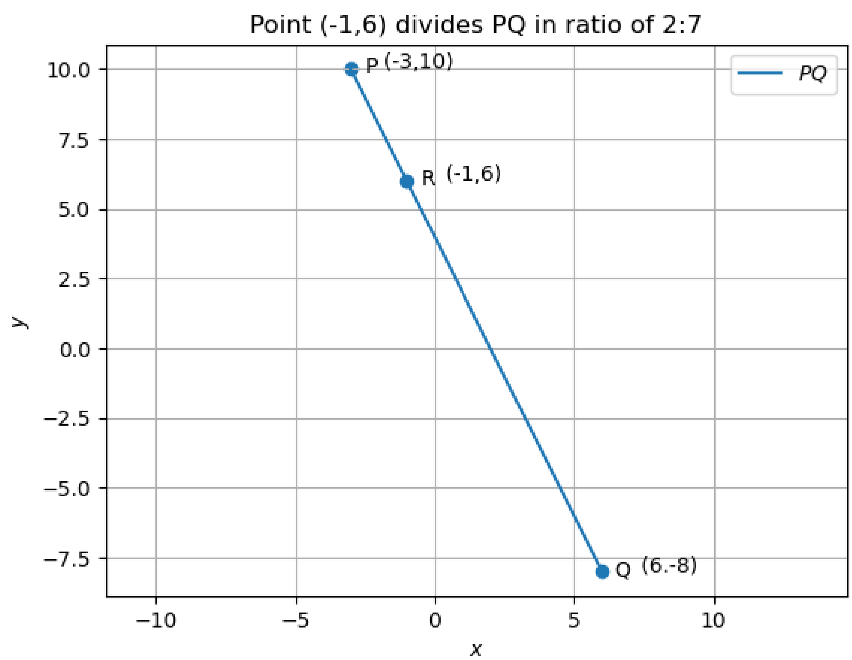


Figure 1.4.4.1:

Symbol	**Value**	**Description**
A	$\begin{pmatrix}1\\2\end{pmatrix}$	First point
B	$\begin{pmatrix}4\\y\end{pmatrix}$	Second point
C	$\begin{pmatrix}x\\6\end{pmatrix}$	Third point
D	$\begin{pmatrix}3\\5\end{pmatrix}$	Fourth point

Table 1.4.6.1:

Since $ABCD$ is a parallellogram,

$$\begin{pmatrix} 3 \\ y-2 \end{pmatrix} = \begin{pmatrix} x-3 \\ 1 \end{pmatrix} \tag{1.4.6.3}$$

$$\implies x = 6, y = 3 \tag{1.4.6.4}$$

Fig. 1.4.6.1 provides a verification.

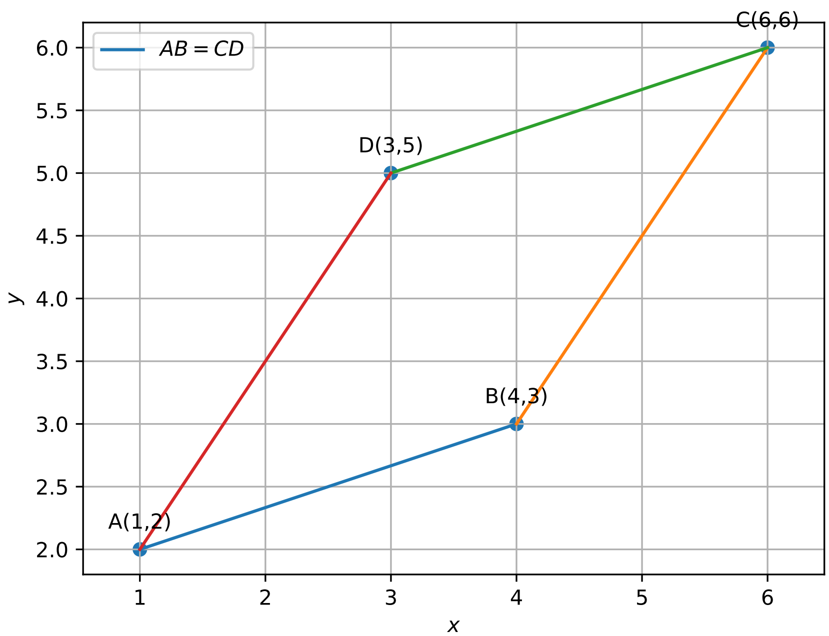


Figure 1.4.6.1:

1.4.7 Find the coordinates of a point A, where AB is the diameter of a circle whose centre is $(2, -3)$ and B is $(1, 4)$.

Solution: Let

$$\mathbf{B} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} \tag{1.4.7.1}$$

Hence,

$$\mathbf{C} = \frac{\mathbf{A} + \mathbf{B}}{2} \tag{1.4.7.2}$$

$$\implies 2\mathbf{C} = \mathbf{A} + \mathbf{B} \tag{1.4.7.3}$$

$$\text{or, } \mathbf{A} = 2\mathbf{C} - \mathbf{B} \tag{1.4.7.4}$$

$$= \begin{pmatrix} 3 \\ -10 \end{pmatrix} \tag{1.4.7.5}$$

See Fig. 1.4.7.1.

1.4.8 If A and B are $(-2, -2)$ and $(2, -4)$, respectively, find the coordinates of P such that AP= $\frac{3}{7}$AB and P lies on the line segment AB.

Solution: Using section formula,

$$\mathbf{P} = \frac{\mathbf{A} + n\mathbf{B}}{1 + n} \tag{1.4.8.1}$$

where

$$n = \frac{3}{4} \tag{1.4.8.2}$$

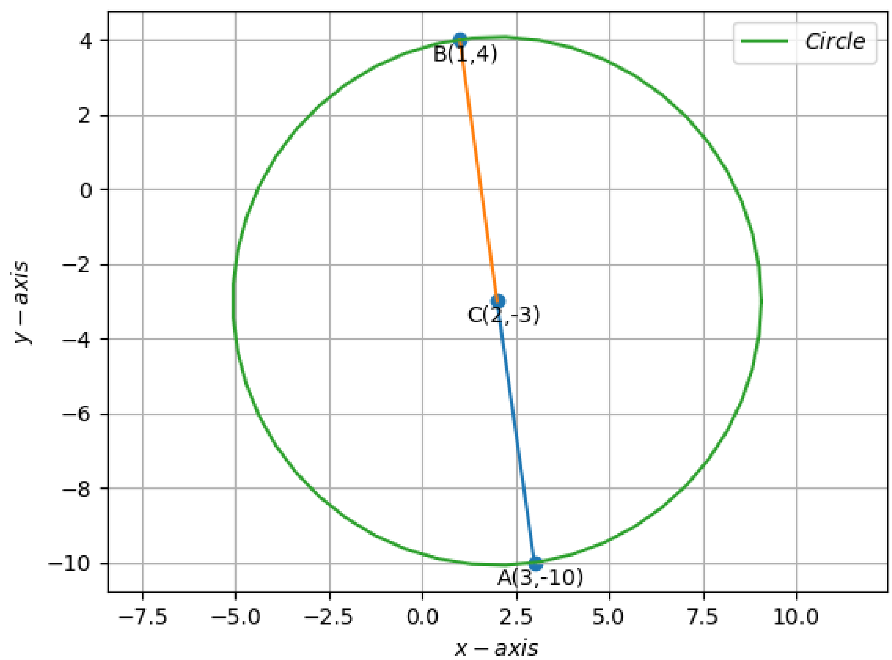


Figure 1.4.7.1:

Thus,

$$\mathbf{P} = \frac{1}{1+\frac{3}{4}}\left(\begin{pmatrix}-2\\-2\end{pmatrix} + \frac{3}{4}\begin{pmatrix}2\\-4\end{pmatrix}\right) \tag{1.4.8.3}$$

$$= \begin{pmatrix}\frac{-2}{7}\\\frac{-20}{7}\end{pmatrix} \tag{1.4.8.4}$$

See Fig. 1.4.8.1

1.4.9 Find the coordinates of the points which divide the line segment joining $A(-2,2)$ and $B(2,8)$ into four equal parts.

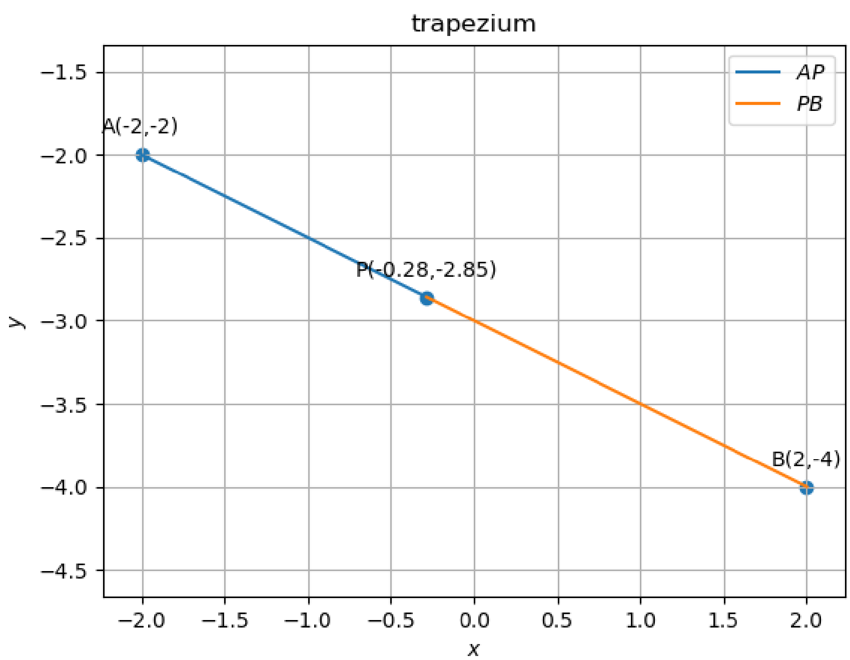

trapezium
AP
PB
A(-2,-2)
P(-0.28,-2.85)
B(2,-4)
−1.5
−2.0
−2.5
−3.0
−3.5
−4.0
−4.5
y
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0
x

Figure 1.4.8.1:

Solution: Let

$$\mathbf{A} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 \\ 8 \end{pmatrix}, \tag{1.4.9.1}$$

Using section formula

$$\mathbf{R}_i = \frac{\mathbf{B} + n\mathbf{A}}{1+n} \tag{1.4.9.2}$$

(a) $n = 3$

$$\mathbf{R}_1 = \begin{pmatrix} -1 \\ \\ \frac{7}{2} \end{pmatrix} \tag{1.4.9.3}$$

(b) $n = 1$

$$\mathbf{R}_2 = \begin{pmatrix} 0 \\ 5 \end{pmatrix} \tag{1.4.9.4}$$

(c) $n = \frac{1}{3}$

$$\mathbf{R}_3 = \begin{pmatrix} 1 \\ \\ \frac{13}{2} \end{pmatrix} \tag{1.4.9.5}$$

See Fig. 1.4.9.1

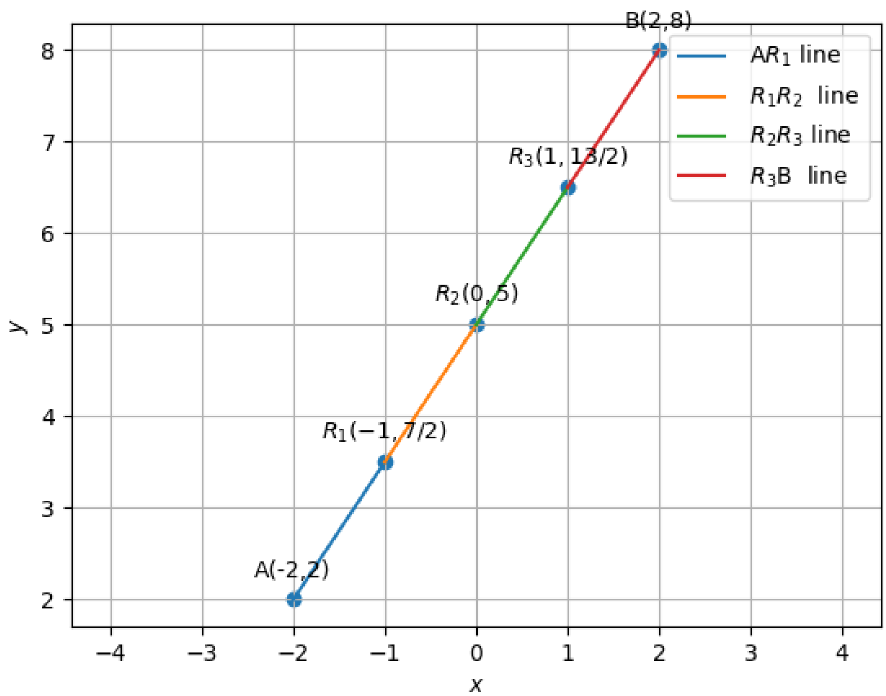


Figure 1.4.9.1:

1.4.10 Find the area of a rhombus if its vertices are $(3, 0), (4, 5), (-1, 4)$ and $(-2, -1)$ taken in order. [**Hint** : Area of rhombus $=\frac{1}{2}$(product of its diagonals)]

Solution: The input vertices for this problem are given as

$$\mathbf{A} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} -2 \\ -1 \end{pmatrix} \tag{1.4.10.1}$$

Since

$$\mathbf{A} - \mathbf{D} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} - \begin{pmatrix} -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix} \tag{1.4.10.2}$$

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} 4 \\ 5 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \tag{1.4.10.3}$$

the area of the rhombus is

$$\left\| \left(\mathbf{A} - \mathbf{D}\right) \times \left(\mathbf{B} - \mathbf{A}\right) \right\| = \begin{vmatrix} 5 & 1 \\ 1 & 5 \end{vmatrix} = 24 \tag{1.4.10.4}$$

See Fig. 1.4.10.1.

1.4.11 Find the position vector of a point R which divides the line joining two points **P** and **Q** whose position vectors are $\hat{i} + 2\hat{j} - \hat{k}$ and $-\hat{i} + \hat{j} + \hat{k}$ respectively, in the ratio 2 : 1

(a) internally

(b) externally

Solution:

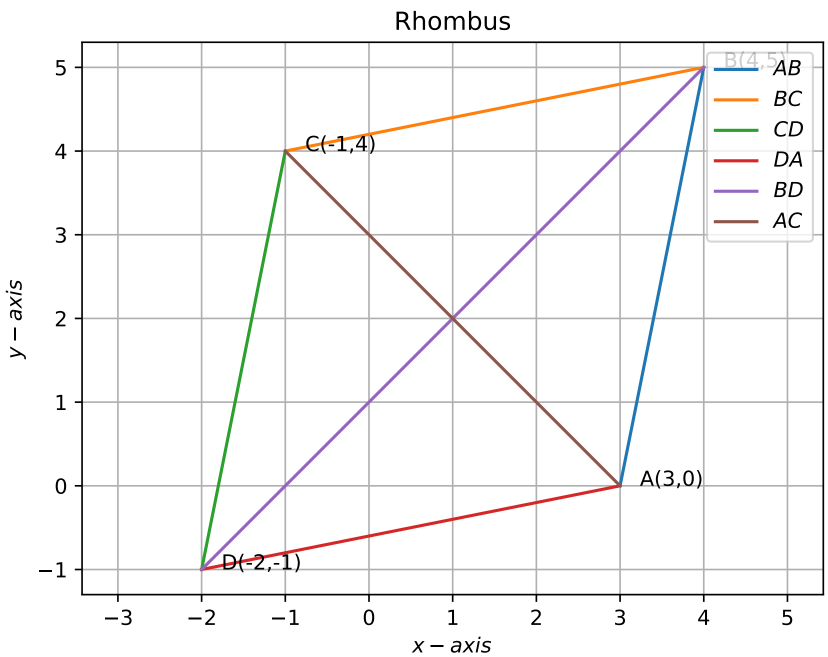


Figure 1.4.10.1:

(a) When $\mathbf{R}$ divides line segment joining $\mathbf{P}$ and $\mathbf{Q}$ internally,

$$\mathbf{R} = \frac{2\mathbf{P} + 1\mathbf{Q}}{3} \tag{1.4.11.1}$$

$$= \frac{2}{3}\mathbf{P} + \frac{1}{3}\mathbf{Q} \tag{1.4.11.2}$$

$$\mathbf{R} = \begin{pmatrix} \frac{1}{3} \\ \frac{5}{3} \\ \frac{-1}{3} \end{pmatrix} \tag{1.4.11.3}$$

(b) When $\mathbf{R}$ divides line segment joining $\mathbf{P}$ and $\mathbf{Q}$ externally,

$$\mathbf{R} = 2\mathbf{Q} - \mathbf{P} \tag{1.4.11.4}$$

$$= 2\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \tag{1.4.11.5}$$

$$\mathbf{R} = \begin{pmatrix} -3 \\ 0 \\ 3 \end{pmatrix} \tag{1.4.11.6}$$

See Fig. 1.4.11.1.

1.4.12 Find the position vector of the mid point of the vector joining the points $\mathbf{P}(2, 3, 4)$ and $\mathbf{Q}(4, 1, -2)$.

Solution: Let the midpoint of PQ be $\mathbf{R}$. Position vector of $\mathbf{R}$ is given by

$$\mathbf{R} = \frac{(\mathbf{P} + \mathbf{Q})}{2} \tag{1.4.12.1}$$

$$= \frac{1}{2}\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} \tag{1.4.12.2}$$

$$= \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \tag{1.4.12.3}$$

1.4.13 Determine the ratio in which the line $2x + y - 4 = 0$ divides the line

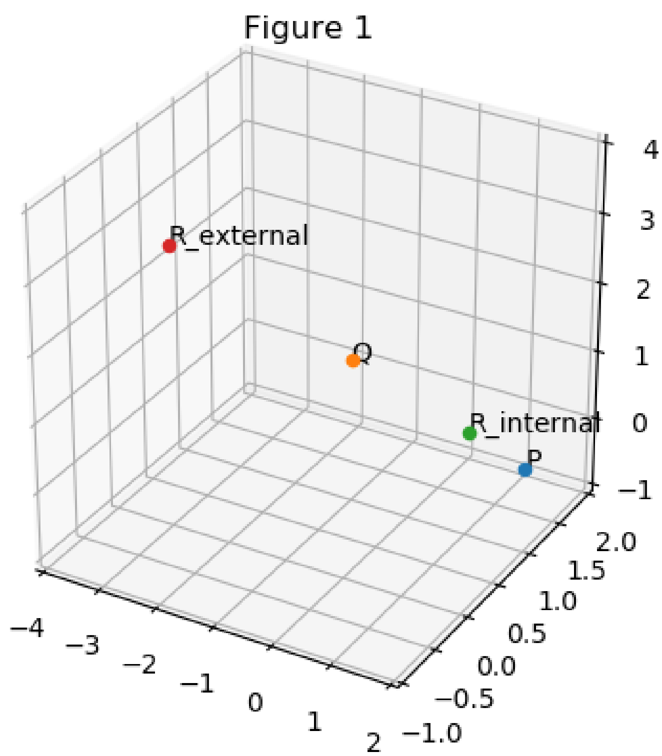


Figure 1.4.11.1:

segment joining the points $\mathbf{A}(2,-2)$ and $\mathbf{B}(3,7)$.

Solution: The given equation can be expressed as

$$\begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = 4 \tag{1.4.13.1}$$

$$\tag{1.4.13.2}$$

Using section formula, the point of division

$$\mathbf{P} = \frac{k\mathbf{B} + \mathbf{A}}{k+1} \tag{1.4.13.3}$$

which upon substitution in the equation of a line yields

$$\implies \mathbf{n}^\top \left(\frac{k\mathbf{B}+\mathbf{A}}{k+1}\right) = c \tag{1.4.13.4}$$

$$\implies k = \frac{c - \mathbf{n}^\top \mathbf{A}}{\mathbf{n}^\top \mathbf{B} - c} \tag{1.4.13.5}$$

$$\tag{1.4.13.6}$$

upon simplification. Substituting numerical values,

$$k = \frac{2}{9} \tag{1.4.13.7}$$

See Fig. 1.4.13.1.

1.4.14 Let $\mathbf{A}(4, 2), \mathbf{B}(6, 5)$ and $\mathbf{C}(1, 4)$ be the vertices of $\triangle ABC$.

(a) The median from $\mathbf{A}$ meets BC at $\mathbf{D}$. Find the coordinates of the point $\mathbf{D}$.

(b) Find the coordinates of the point $\mathbf{P}$ on AD such that $AP : PD = 2 : 1$.

(c) Find the coordinates of points $\mathbf{Q}$ and $\mathbf{R}$ on medians BE and CF respectively such that $BQ : QE = 2 : 1$ and $CR : RF = 2 : 1$.

(d) What do you observe?

(e) If $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ are the vertices of $\triangle ABC$, find the coordinates of

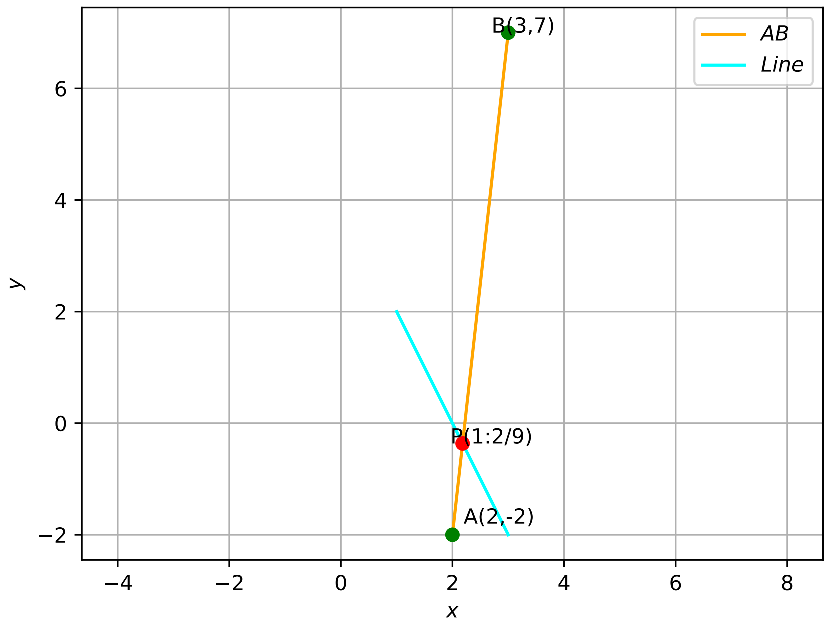


Figure 1.4.13.1:

the centroid of the triangle.

Solution:

(a)

$$\mathbf{D} = \frac{\mathbf{B} + \mathbf{C}}{2} \tag{1.4.14.1}$$

$$= \begin{pmatrix} \frac{7}{2} \\ \frac{9}{2} \end{pmatrix} \tag{1.4.14.2}$$

$$\mathbf{E} = \frac{\mathbf{A} + \mathbf{C}}{2} \tag{1.4.14.3}$$

$$= \begin{pmatrix} \frac{5}{2} \\ 3 \end{pmatrix} \tag{1.4.14.4}$$

$$\mathbf{F} = \frac{\mathbf{A} + \mathbf{B}}{2} \tag{1.4.14.5}$$

$$= \begin{pmatrix} 5 \\ \frac{7}{2} \end{pmatrix} \tag{1.4.14.6}$$

(b) For $n = 2$,

$$\mathbf{P} = \frac{1}{1+n} \left(\left(\mathbf{A} + n\mathbf{D} \right) \right) \tag{1.4.14.7}$$

$$= \frac{1}{3} \begin{pmatrix} 11 \\ 11 \end{pmatrix} \tag{1.4.14.8}$$

(c)

$$\mathbf{Q} = \frac{1}{1+n}\left(\left(\mathbf{B} + n\mathbf{E}\right)\right) \tag{1.4.14.9}$$

$$= \frac{1}{3}\begin{pmatrix} 11 \\ 11 \end{pmatrix} \tag{1.4.14.10}$$

$$\mathbf{R} = \frac{1}{1+n}\left(\left(\mathbf{C} + n\mathbf{F}\right)\right) \tag{1.4.14.11}$$

$$= \frac{1}{3}\begin{pmatrix} 11 \\ 11 \end{pmatrix} \tag{1.4.14.12}$$

$$\tag{1.4.14.13}$$

(d) $\mathbf{P}, \mathbf{Q}, \mathbf{R}$ are the same point.

(e)

$$\mathbf{G} = \frac{\mathbf{D} + \mathbf{E} + \mathbf{F}}{3} \tag{1.4.14.14}$$

$$= \frac{1}{3}\begin{pmatrix} 11 \\ 11 \end{pmatrix} \tag{1.4.14.15}$$

$$\tag{1.4.14.16}$$

See Fig. 1.4.14.1.

1.4.15 Find the slope of a line, which passes through the origin and the mid point of the line segment joining the points $\mathbf{P}$(0,-4) and $\mathbf{B}$(8,0).

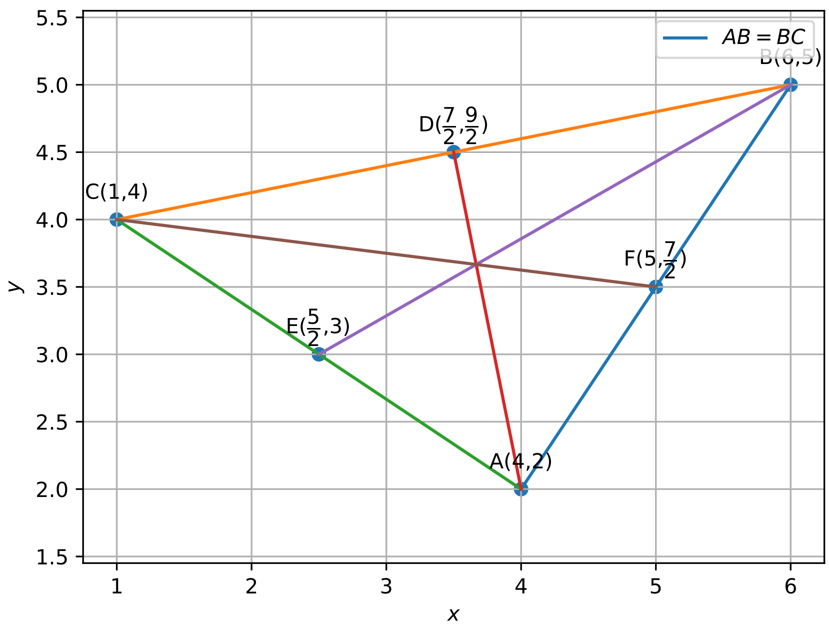


Figure 1.4.14.1:

Solution: The mid point of PB is

$$\mathbf{M} = \frac{1}{2}(\mathbf{P} + \mathbf{B}) = \begin{pmatrix} 4 \\ -2 \end{pmatrix} \tag{1.4.15.1}$$

The direction vector of line joining $\mathbf{O}, \mathbf{M}$ is

$$\mathbf{m} = \mathbf{O} - \mathbf{M} = -\mathbf{M} \tag{1.4.15.2}$$

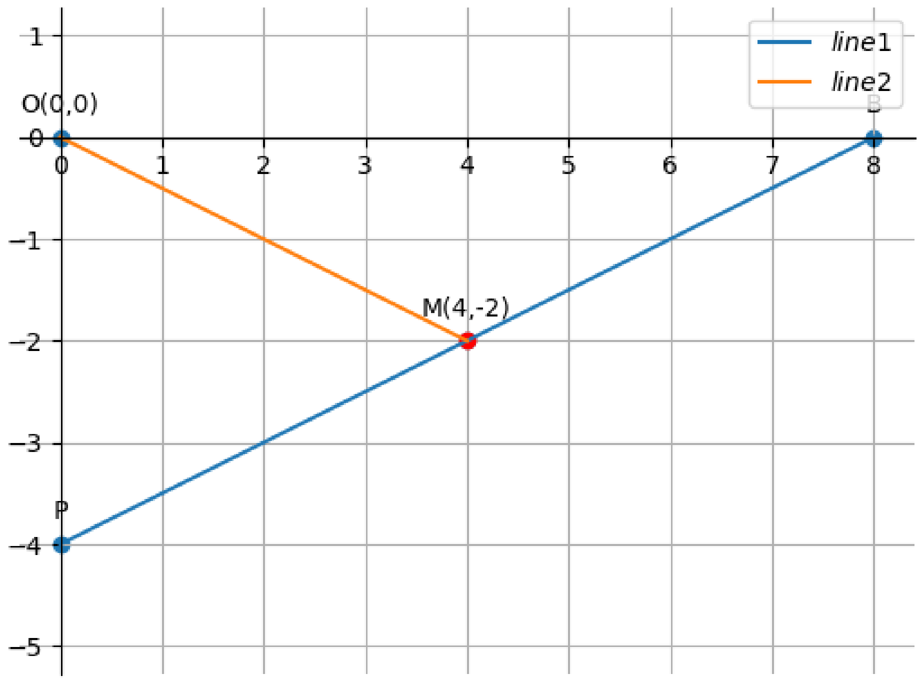


Figure 1.4.15.1:

which can be expressed as

$$\begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} \tag{1.4.15.3}$$

Thus the slope is

$$m = -\frac{1}{2} \tag{1.4.15.4}$$

1.4.16 Find the position vector of a point R which divides the line joining two points P and Q whose position vectors are $(2\mathbf{a} + \mathbf{b})$ and $(\mathbf{a} - 3\mathbf{b})$

externally in the ratio 1 : 2. Also, show that P is the mid point of the line segment RQ.

Solution:

1.5. Exercises

1.5.1 The point which divides the line segment joining the points $\mathbf{P}(7, –6)$ and $(3, 4)$ in ratio 1 : 2 internally lies in the

(a) I quadrant

(b) II quadrant

(c) III quadrant

(d) IV quadrant

1.5.2 If the point $\mathbf{P}(2, 1)$ lies on the line segment joining points$\mathbf{A}(4, 2)$ and $\mathbf{B}(8, 4)$, then

(a) $\mathbf{AP} = \frac{1}{3}\mathbf{AB}$

(b) $\mathbf{AP} = \mathbf{PE}$

(c) $\mathbf{PB} = \frac{1}{3}\mathbf{AB}$

(d) $\mathbf{AP} = \frac{1}{2}\mathbf{AB}$

1.5.3 If P $\frac{a}{3}$ is the mid-point of the line segment joining the points $\mathbf{Q}(–6, 5)$ and $(–2, 3)$, then the value of a is

(a) – 4

(b) – 12

(c) 12

(d) – 6

1.5.4 A line interects the y-axis and x-axis of the points **P** and **Q**, respectiveiy.lf $(2,5)$is the mid-point of **PQ**.then the coordinates of **P** and **Q** are,respectively

(a) $(0,-5)$ and $(2,0)$

(b) $(0,-10)$ and $(-4,0)$

(c) $(0,4)$ and $(-10,0)$

(d) $(0,-10)$ and $(4,0)$

1.5.5 Point $\mathbf{P}(5,-3)$ is one of the two points of trisection of line segment joining the points $\mathbf{A}(7,-2)$ and $\mathbf{B}(1,-5)$

1.5.6 Points $\mathbf{A}(-6,10), \mathbf{B}(-4,6)$ and $\mathbf{C}(3,-8)$ are collinear such that $\mathbf{AB} = \frac{2}{9}\mathbf{AC}$

1.5.7 In what ratio does the x-axis divide the line segment joining the points $(-4,-6)$ and $(-1,7)$? Find the coordinates of the point of division.

1.5.8 Find the ratio in which the point $\mathbf{P}\left(\frac{3}{4}, \frac{5}{12}\right)$ divides the line segment joining the points $\mathbf{A}\left(\frac{1}{2}, \frac{3}{2}\right)$ and $B(2,-5)$.

1.5.9 If $\mathbf{P}(9a-2,-b)$ divides line segment joining $\mathbf{A}(3a+1,-3)$ and $\mathbf{B}(8a,5)$ in the ratio 3:1,find the values of a and b.

1.5.10 The line segment joining the points $\mathbf{A}(3,2)$ and $\mathbf{B}(5,1)$ is divided at the point $\mathbf{P}$ in the ratio 1:2 and it lies $3x-18y+k=0$, Find the value of k

1.5.11 Find the coordinates of the point $\mathbf{R}$ on the line segment joining the points $\mathbf{P}(-1,3)$ and $\mathbf{Q}(2,5)$ such that $\mathbf{PR} = \frac{3}{5}\mathbf{PQ}$.

1.5.12 Find the ratio in which the line 2x+3y-5=0 divides the line segment joining the points $(8,-9)$ and $(2,1)$. Also find the coordinates of the point of division,

1.5.13 If **a** and **b** are the postion vectors of A and B, respectively, find the position vector of a point C in BA produced such that BC=1.5BA.

1.5.14 The position vector of the point which divides the join of points 2**a**-3**b** and $\mathbf{a}+\mathbf{b}$ in the ratio 3:1 is

(a) $\frac{3\mathbf{a}-2\mathbf{b}}{2}$

(b) $\frac{7\mathbf{a}-8\mathbf{b}}{4}$

(c) $\frac{3\mathbf{a}}{4}$

(d) $\frac{5\mathbf{a}}{4}$

1.6. Rank

1.6.1 Determine if the points $(1,5),(2,3)$ and $(-2,-11)$ are collinear.

Solution: We know that points $\mathbf{A},\mathbf{B}$ and $\mathbf{C}$ are collinear, if

$$\text{rank}\begin{pmatrix}\mathbf{A}^\top\\ \mathbf{B}^\top\\ \mathbf{C}^\top\end{pmatrix} = 1 \tag{1.6.1.1}$$

Since

$$\begin{pmatrix}\mathbf{A}^\top\\ \mathbf{B}^\top\\ \mathbf{C}^\top\end{pmatrix} = \begin{pmatrix}1 & 5\\ 2 & 3\\ -2 & -11\end{pmatrix} \tag{1.6.1.2}$$

$$\xleftrightarrow[R_3\to R_3+2R_1]{R_2\to R_2-2R_1}\begin{pmatrix}1 & 5\\ 0 & -7\\ 0 & -1\end{pmatrix}\xleftrightarrow{R_3\to R_3-\frac{1}{7}R_2}\begin{pmatrix}1 & 5\\ 0 & -7\\ 0 & 0\end{pmatrix}, \tag{1.6.1.3}$$

the rank of the matrix is 2. From (1.6.1.1), the points are not collinear. This is verified by Fig. 1.6.1.1, where the given points constitute a triangle and not a line.

1.6.2 Show that the points $\mathbf{A}(1,2,7),\mathbf{B}(2,6,3)$ and $\mathbf{C}(3,10,-1)$ are collinear.

Solution: Points $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are on a line if

$$\text{rank}\begin{pmatrix}\mathbf{A} & \mathbf{B} & \mathbf{C}\end{pmatrix} < 3 \tag{1.6.2.1}$$

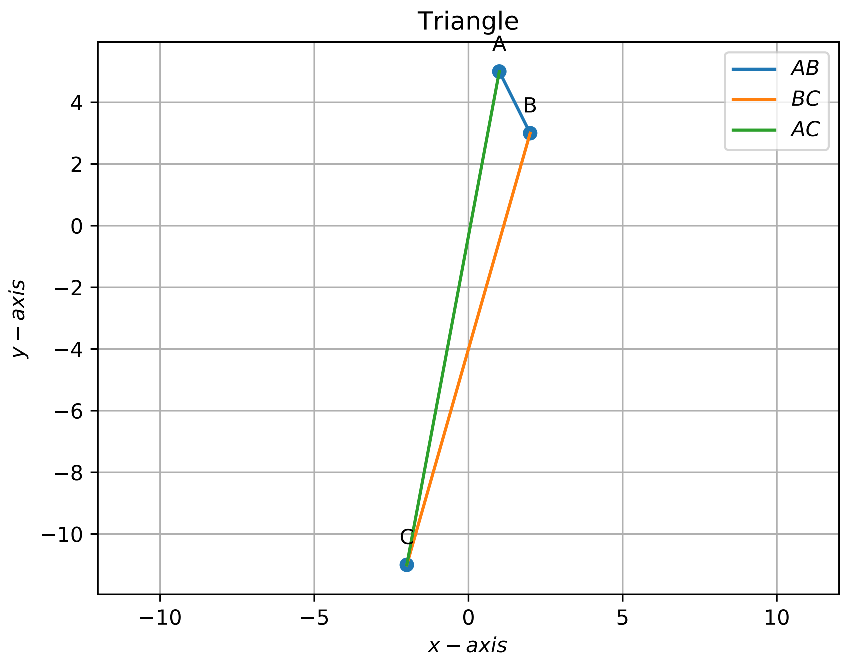


Figure 1.6.1.1:

Substituting, we must find the rank of

$$\mathbf{M} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 6 & 10 \\ 7 & 3 & -1 \end{pmatrix} \tag{1.6.2.2}$$

Using row reduction methods to bring $\mathbf{M}$ into row-reduced echelon form,

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 6 & 10 \\ 7 & 3 & -1 \end{pmatrix} \xleftrightarrow{R_2 \to R_2 - 2R_1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 7 & 3 & -1 \end{pmatrix} \tag{1.6.2.3}$$

$$\xleftrightarrow{R_3 \to R_3 - 7R_1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & -11 & -22 \end{pmatrix} \tag{1.6.2.4}$$

$$\xleftrightarrow{R_1 \to R_1 - R_2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 4 \\ 0 & -11 & -22 \end{pmatrix} \tag{1.6.2.5}$$

$$\xleftrightarrow{R_3 \to R_3 + \frac{11}{2} R_2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.6.2.6}$$

Clearly, the rank of $\mathbf{M}$ is 2, and hence the given points are collinear. Fig. 1.6.2.1 verifies that the three points are indeed collinear as claimed.

1.6.3 Show that the vectors $2\hat{i} - 3\hat{j} + 4\hat{k}$ and $-4\hat{i} + 6\hat{j} - 8\hat{k}$ are collinear.

Solution: Let

$$\mathbf{A} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -4 \\ 6 \\ -8 \end{pmatrix} \tag{1.6.3.1}$$

$$\tag{1.6.3.2}$$

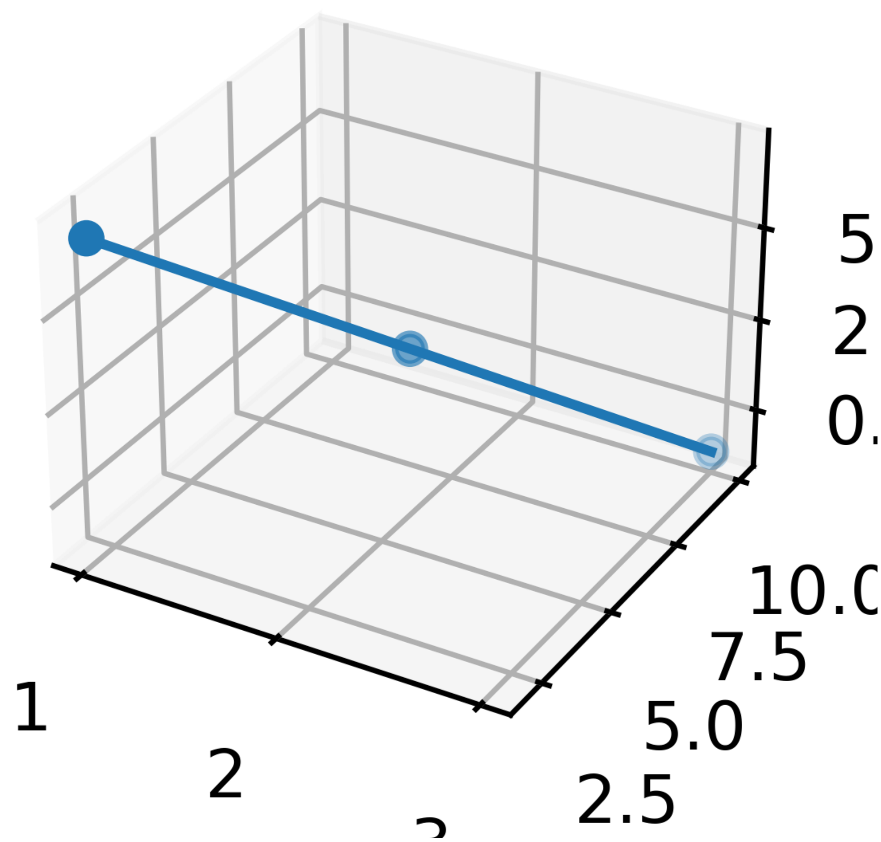


Figure 1.6.2.1: Points **A**, **B** and **C** are collinear.

Forming the collinearity matrix

$$\begin{pmatrix} \mathbf{A}^\top \\ \mathbf{B}^\top \end{pmatrix} = \begin{pmatrix} 2 & -3 & 4 \\ -4 & 6 & -8 \end{pmatrix} \xleftrightarrow{\frac{1}{2}R_1 \to R_1} \begin{pmatrix} 1 & -\frac{3}{2} & 2 \\ -4 & 6 & -8 \end{pmatrix} \tag{1.6.3.3}$$

$$\xleftrightarrow{-\frac{1}{4}R_2 \leftarrow R_2} \begin{pmatrix} 1 & -\frac{3}{2} & 2 \\ 1 & \frac{3}{2} & 2 \end{pmatrix} \xleftrightarrow{R_2 - 1R_1 \to R_2} \begin{pmatrix} 1 & -\frac{3}{2} & 2 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.6.3.4}$$

Thus, the rank of the matrix is 1 and the vectors are collinear.

1.6.4 Show that the points (2, 3, 4), (–1, –2, 1), (5, 8, 7) are collinear.

Solution: The points given are,

$$\mathbf{A} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 5 \\ 8 \\ 7 \end{pmatrix} \tag{1.6.4.1}$$

To check whether the given points are collinear, we find the rank of the matrix

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} = \begin{pmatrix} 2 & -1 & 5 \\ 3 & -2 & 8 \\ 4 & 1 & 7 \end{pmatrix} \xleftrightarrow[R_3 \leftarrow R_3 - 2R_1]{R_2 \leftarrow R_2 - \frac{3}{2}R_1} \begin{pmatrix} 2 & -1 & 5 \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 3 & -3 \end{pmatrix} \tag{1.6.4.2}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 + 6R_2} \begin{pmatrix} 2 & -1 & 5 \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix} \tag{1.6.4.3}$$

The matrix has a rank of 2. Hence the given points are collinear.

1.6.5 In each of the following, find the value of 'k', for which the points are collinear.

(a) $(7,-2),(5,1),(3,k)$

(b) $(8,1),(k,-4),(2,-5)$

Solution:

(a) Let

$$\mathbf{A} = \begin{pmatrix} 7 \\ -2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 3 \\ k \end{pmatrix} \tag{1.6.5.1}$$

Then

$$\mathbf{D} = (\mathbf{A} - \mathbf{B}) = \left(\begin{pmatrix} 7 \\ -2 \end{pmatrix} - \begin{pmatrix} 5 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 2 \\ -3 \end{pmatrix} \tag{1.6.5.2}$$

$$\mathbf{E} = (\mathbf{A} - \mathbf{C}) = \left(\begin{pmatrix} 7 \\ -2 \end{pmatrix} - \begin{pmatrix} 3 \\ k \end{pmatrix}\right) = \begin{pmatrix} 4 \\ -2 - k \end{pmatrix} \tag{1.6.5.3}$$

Forming the collinearlity matrix,

$$\mathbf{F} = \begin{pmatrix} \mathbf{D} \\ \mathbf{E} \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ 4 & -2 - k \end{pmatrix} \tag{1.6.5.4}$$

yielding

$$\xleftrightarrow{R_2 = R_2 - 2R_1} \begin{pmatrix} 2 & -3 \\ 0 & -k + 4 \end{pmatrix} \tag{1.6.5.5}$$

For the matrix to be rank 1,

$$-k + 4 = 0 \implies k = 4 \tag{1.6.5.6}$$

This is verified in Fig. 1.6.5.1.

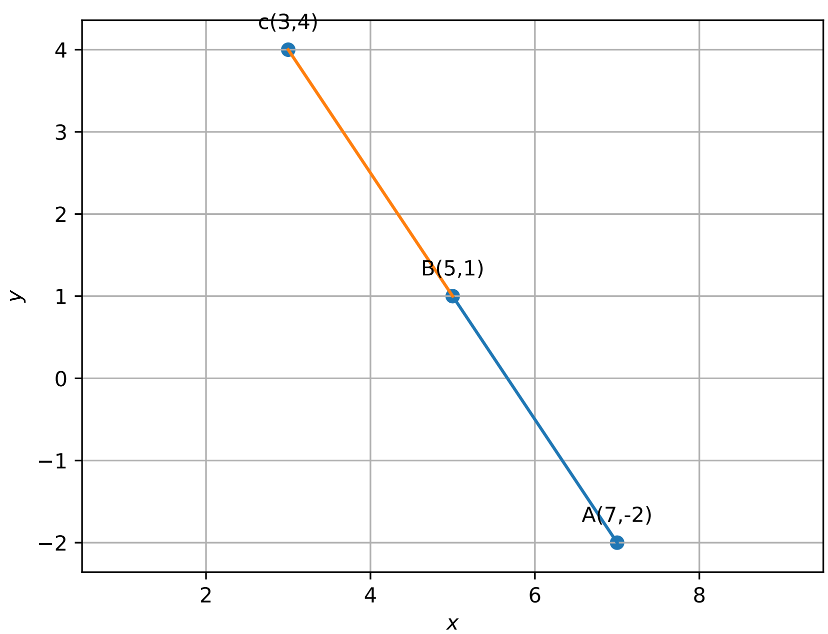


Figure 1.6.5.1:

(b) In this case,

$$\mathbf{A} = \begin{pmatrix} 8 \\ 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} k \\ -4 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 2 \\ -5 \end{pmatrix}. \tag{1.6.5.7}$$

Since

$$\mathbf{D} = (\mathbf{A} - \mathbf{B}) = \left(\begin{pmatrix} 8 \\ 1 \end{pmatrix} - \begin{pmatrix} k \\ -4 \end{pmatrix}\right) = \begin{pmatrix} 8-k \\ 5 \end{pmatrix} \tag{1.6.5.8}$$

$$\mathbf{E} = (\mathbf{A} - \mathbf{C}) = \left(\begin{pmatrix} 8 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ -5 \end{pmatrix}\right) = \begin{pmatrix} 6 \\ 6 \end{pmatrix} \tag{1.6.5.9}$$

the collinearity matrix is

$$\mathbf{F} = \begin{pmatrix} \mathbf{D} \\ \mathbf{E} \end{pmatrix} = \begin{pmatrix} 8-k & 5 \\ 6 & 6 \end{pmatrix} \tag{1.6.5.10}$$

yielding

$$\xleftrightarrow{R_1 = \frac{R_1}{8-k}} \begin{pmatrix} 1 & \frac{5}{8-k} \\ 6 & 6 \end{pmatrix} \tag{1.6.5.11}$$

$$\xleftrightarrow{R_2 = R_2 - 6R_1} \begin{pmatrix} 1 & \frac{5}{8-k} \\ 0 & 6 - \frac{30}{8-k} \end{pmatrix} \tag{1.6.5.12}$$

For the matrix to be rank 1,

$$6 - \frac{30}{8-k} = 0 \tag{1.6.5.13}$$

$$\implies k = 3 \tag{1.6.5.14}$$

This is verified in Fig. 1.6.5.2

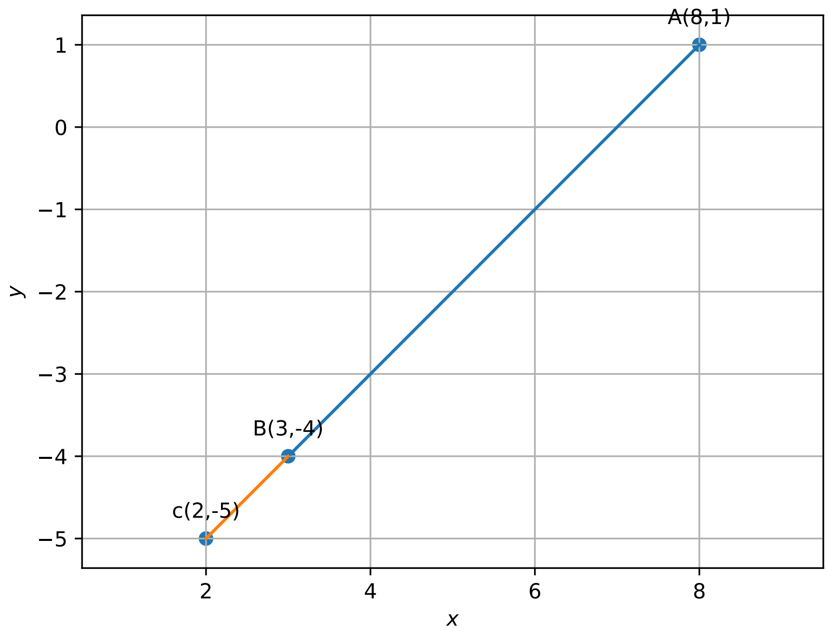


Figure 1.6.5.2:

1.6.6 Find a relation between x and y if the points $(x, y), (1, 2)$ and $(7, 0)$ are collinear.

Solution: Let

$$\mathbf{A} = \begin{pmatrix} x \\ y \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 7 \\ 0 \end{pmatrix} \tag{1.6.6.1}$$

Then

$$\mathbf{D} = (\mathbf{A} - \mathbf{B}) = \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right) = \begin{pmatrix} x-1 \\ y-2 \end{pmatrix} \tag{1.6.6.2}$$

$$\mathbf{E} = (\mathbf{A} - \mathbf{C}) = \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 7 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} x-7 \\ y \end{pmatrix} \tag{1.6.6.3}$$

Forming the collinearity matrix

$$\mathbf{F} = \begin{pmatrix} \mathbf{D}^\top \\ \mathbf{E}^\top \end{pmatrix} \tag{1.6.6.4}$$

and performing row reduction,

$$\begin{pmatrix} x-1 & y-2 \\ x-7 & y \end{pmatrix} \xleftrightarrow{R_2=R_2-R_1} \begin{pmatrix} x-1 & y-2 \\ -6 & 2 \end{pmatrix} \tag{1.6.6.5}$$

$$\xleftrightarrow{R_2=\frac{R_2}{-6}(x-1)-R_1} \begin{pmatrix} x-1 & y-2 \\ 0 & -\frac{1}{3}(x-1)-(y-2) \end{pmatrix} \tag{1.6.6.6}$$

For the rank of the matrix to be 1,

$$-\frac{1}{3}(x-1)-(y-2) = 0 \tag{1.6.6.7}$$

$$\implies \begin{pmatrix} 1 & 3 \end{pmatrix} \mathbf{x} = 7 \tag{1.6.6.8}$$

For $x = -2, y = 3$, see Fig. 1.6.6.1 verifying that the points are collinear.

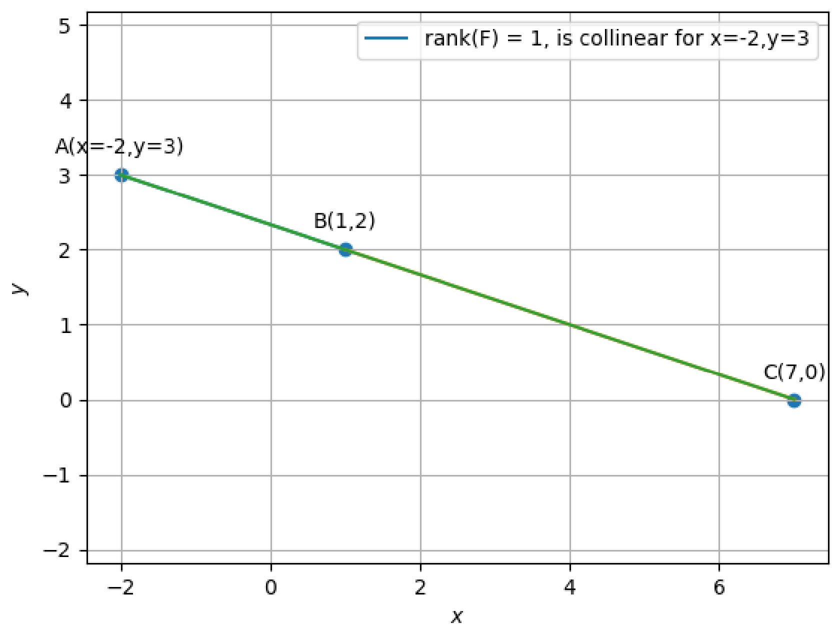


Figure 1.6.6.1:

1.6.7 If three points $(x, -1), (2, 1)$ and $(4, 5)$ are collinear, find the value of x. If three points $(x, -1), (2, 1)$ and $(4, 5)$ are collinear, find the value of x.

Solution: Let

$$\mathbf{A} = \begin{pmatrix} x \\ -1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}. \tag{1.6.7.1}$$

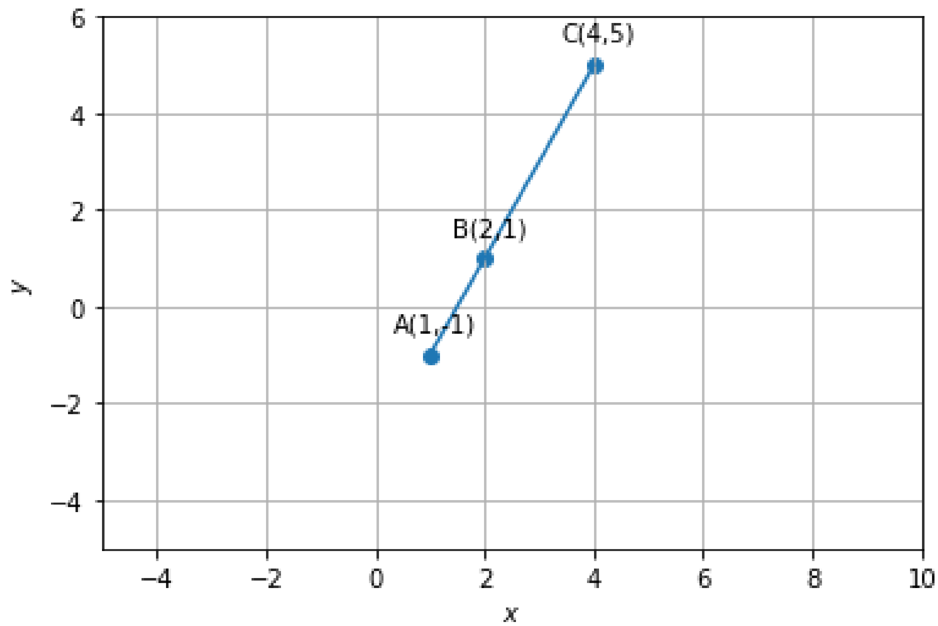


Figure 1.6.7.1:

Then

$$\mathbf{A}-\mathbf{B}=\begin{pmatrix}x-2\\-2\end{pmatrix} \tag{1.6.7.2}$$

$$\mathbf{A}-\mathbf{C}=\begin{pmatrix}4-x\\6\end{pmatrix} \tag{1.6.7.3}$$

Forming the collinearity matrix using (D.1.4.1),

$$\begin{pmatrix}x-2 & -2\\4-x & 6\end{pmatrix} \xleftrightarrow{R_1=3R_1+R-2} = \begin{pmatrix}2x-2 & 0\\4-x & 6\end{pmatrix} \tag{1.6.7.4}$$

If the rank of the matrix is 1, any one of the rows must be zero. So, making the first element in the above matrix 0,

$$x = 1 \tag{1.6.7.5}$$

1.6.8 If three points $(h, 0), (a, b)$ and $(0, k)$ lie on a line, show that

$$\frac{a}{h} + \frac{b}{k} = 1 \tag{1.6.8.1}$$

Solution: Let

$$\mathbf{A} = \begin{pmatrix} h \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} a \\ b \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 0 \\ k \end{pmatrix} \tag{1.6.8.2}$$

Forming the matrix in (D.1.4.1), we obtain, upon row reduction

$$\begin{pmatrix} h-a & -b \\ h & -k \end{pmatrix} \xleftrightarrow{\frac{R_1}{h-a}} \begin{pmatrix} 1 & \frac{-b}{h-a} \\ h & -k \end{pmatrix} \tag{1.6.8.3}$$

$$\xleftrightarrow{R_2 \to R_2 - hR_1} \begin{pmatrix} 1 & \frac{-b}{h-a} \\ 0 & -k + \frac{bh}{h-a} \end{pmatrix} \tag{1.6.8.4}$$

For obtaining a rank 1 matrix,

$$-k + \frac{bh}{h-a} = 0 \tag{1.6.8.5}$$

$$\implies \frac{a}{h} + \frac{b}{k} = 1 \tag{1.6.8.6}$$

upon simplification.

1.6.9 Show that the points A (1, -2, -8), B (5, 0, -2) and C (11, 3, 7) are collinear, and find the ratio in which B divides AC.

1.7. Exercises

1.7.1 lf the points $\mathbf{A}(1,2), \mathbf{0}(0,0)$ and $\mathbf{C}(a,b)$ are collinear,then

(a) a=b

(b) a=2b

(c) 2a=b

(d) a=-b

True/false

1.7.2 $\triangle\mathbf{ABC}$ with vertices $\mathbf{A}(-2,0), \mathbf{B}(2,0)$ and $\mathbf{C}(0,2)$ is similar to $\triangle\mathbf{DEF}$ with vertices $\mathbf{D}(-4,0), \mathbf{E}(4,0)$ and $\mathbf{F}(0,4)$

1.7.3 Point $(-4,2)$ lies on the line segment joining the points $\mathbf{A}(-4,6)$ and $\mathbf{B}(-4,-6)$

1.7.4 The points $(0,5), (0,-9)$ and $(3,6)$ are collinear

1.7.5 Points $\mathbf{A}(3,1), \mathbf{B}(12,-2)$ and $\mathbf{C}(0,2)$ cannot be the vertices of a triangle

1.7.6 Find the value of m if the points $(5,1), (-2,-3)$ and $(8,2m)$ are collinear.

1.7.7 Find the values of k if the points $\mathbf{A}(k+1,2k), \mathbf{B}(3k,2k+3)$ and $\mathbf{C}(5k-1,5k)$ are collinear

1.7.8 Using vectors, find the value of k such that the points $(k, -10, 3)$, $(1, -1, 3)$ and $(3, 5, 3)$ are collinear.

1.8. Scalar Product

1.8.1 Find the angle between two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ with magnitudes $\sqrt{3}$ and 2 respectively having $\overrightarrow{a}.\overrightarrow{b} = \sqrt{6}$.

Solution: From the given information,

$$\|\mathbf{a}\| = \sqrt{3} \tag{1.8.1.1}$$

$$\|\mathbf{b}\| = 2 \tag{1.8.1.2}$$

$$\mathbf{a}^\top \mathbf{b} = \sqrt{6} \tag{1.8.1.3}$$

Thus,

$$\cos\theta = \frac{\mathbf{a}^\top \mathbf{b}}{\|\mathbf{a}\| \|\mathbf{b}\|} \tag{1.8.1.4}$$

$$= \frac{1}{\sqrt{2}} \tag{1.8.1.5}$$

$$\implies \theta = 45° \tag{1.8.1.6}$$

1.8.2 Find the angle between the the vectors $\hat{i} - 2\hat{j} + 3\hat{k}$ and $3\hat{i} - 2\hat{j} + \hat{k}$.

Solution: Let the given points be

$$\mathbf{a} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}, \tag{1.8.2.1}$$

Since

$$\mathbf{a}^\top \mathbf{b} = 10, \ \|\mathbf{a}\| = \sqrt{14}, \ \|\mathbf{b}\| = \sqrt{14}, \tag{1.8.2.2}$$

$$\cos\theta = \frac{\mathbf{a}^\top \mathbf{b}}{\|\mathbf{a}\| \, \|\mathbf{b}\|} = \frac{5}{7} \tag{1.8.2.3}$$

$$\implies \theta = \cos^{-1}\frac{5}{7} \tag{1.8.2.4}$$

1.8.3 Find $|\overrightarrow{a}|$ and $\left|\overrightarrow{b}\right|$,if $(\overrightarrow{a} + \overrightarrow{b}).(\overrightarrow{a} - \overrightarrow{b}) = 8$ and $|\overrightarrow{a}| = 8\left|\overrightarrow{b}\right|$.

Solution: Since

$$(\mathbf{a} + \mathbf{b})^\top (\mathbf{a} - \mathbf{b}) = 8, \tag{1.8.3.1}$$

$$\|\mathbf{a}\|^2 - \|\mathbf{b}\|^2 = 8 \tag{1.8.3.2}$$

$$\implies \|8\mathbf{b}\|^2 - \|\mathbf{b}\|^2 = 8 \tag{1.8.3.3}$$

$$\implies \|\mathbf{b}\| = \frac{2\sqrt{2}}{3\sqrt{7}} \tag{1.8.3.4}$$

Thus,

$$\|\mathbf{a}\| = 8\,\|\mathbf{b}\| = \frac{16\sqrt{2}}{3\sqrt{7}} \tag{1.8.3.5}$$

1.8.4 Evaluate the product$(3\overrightarrow{a} - 5\overrightarrow{b}).(2\overrightarrow{a} + 7\overrightarrow{b})$.

Solution:

$$(3\mathbf{a}-5\mathbf{b})^\top(2\mathbf{a}+7\mathbf{b}) = \left(3\mathbf{a}^\top\right)(2\mathbf{a})+\left(3\mathbf{a}^\top\right)(7\mathbf{b})-\left(5\mathbf{b}^\top\right)(2\mathbf{a})-\left(5\mathbf{b}^\top\right)(7\mathbf{b})$$
$$= 6\|\mathbf{a}\|^2+21\mathbf{a}^\top\mathbf{b}-10\mathbf{b}^\top\mathbf{a}-35\|\mathbf{b}\|^2 = 6\|\mathbf{a}\|^2-35\|\mathbf{b}\|^2+11\mathbf{a}^\top\mathbf{b} \tag{1.8.4.1}$$

1.8.5 Find the magnitude of two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$, having the same magnitude and such that the angle between them is 60° and their scalar product is $\frac{1}{2}$

Solution: Given

$$\|\mathbf{a}\|\|\mathbf{b}\| = \frac{\mathbf{a}^\top\mathbf{b}}{\cos\theta} \tag{1.8.5.1}$$
$$\implies \|\mathbf{a}\| = \sqrt{\frac{\mathbf{a}^\top\mathbf{b}}{\cos\theta}} \tag{1.8.5.2}$$
$$\tag{1.8.5.3}$$

since

$$\|\mathbf{a}\| = \|\mathbf{b}\| \tag{1.8.5.4}$$

Substituting numerical values,

$$\|\mathbf{a}\| = \|\mathbf{b}\| = 1 \tag{1.8.5.5}$$

1.8.6 Find $|\overrightarrow{x}|$,if for a unit vector $\overrightarrow{a}$, $(\overrightarrow{x}-\overrightarrow{a}).(\overrightarrow{x}+\overrightarrow{a})$=12.

Solution: From the given information,

$$(\mathbf{x}-\mathbf{a})^{\top}(\mathbf{x}+\mathbf{a}) = 12 \tag{1.8.6.1}$$

$$\implies \mathbf{x}^{\top}\mathbf{x}-\mathbf{a}^{\top}\mathbf{x}+\mathbf{x}^{\top}\mathbf{a}-\mathbf{a}^{\top}\mathbf{a} = 12 \tag{1.8.6.2}$$

$$\implies \|\mathbf{x}\|^2-\|\mathbf{a}\|^2 = 12 \tag{1.8.6.3}$$

$$\implies \|\mathbf{x}\|^2-1 = 12 \tag{1.8.6.4}$$

$$\text{or, } \|\mathbf{x}\| = \sqrt{13} \tag{1.8.6.5}$$

1.8.7 If the vertices A,B,C of a triangle ABC are (1,2,3),(-1,0,0)(0,1,2), respectively , then find $\angle ABC$.[$\angle ABC$ is the angle between the vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$].

Solution: From the given information,

$$\mathbf{A}-\mathbf{B} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} \tag{1.8.7.1}$$

$$\mathbf{C}-\mathbf{B} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \tag{1.8.7.2}$$

$$\implies \angle ABC = \cos^{-1}\frac{(\mathbf{A}-\mathbf{B})^{T}(\mathbf{C}-\mathbf{B})}{\|\mathbf{A}-\mathbf{B}\|\,\|\mathbf{C}-\mathbf{B}\|} \tag{1.8.7.3}$$

$$= \cos^{-1}\frac{10}{\sqrt{102}} \tag{1.8.7.4}$$

$$\tag{1.8.7.5}$$

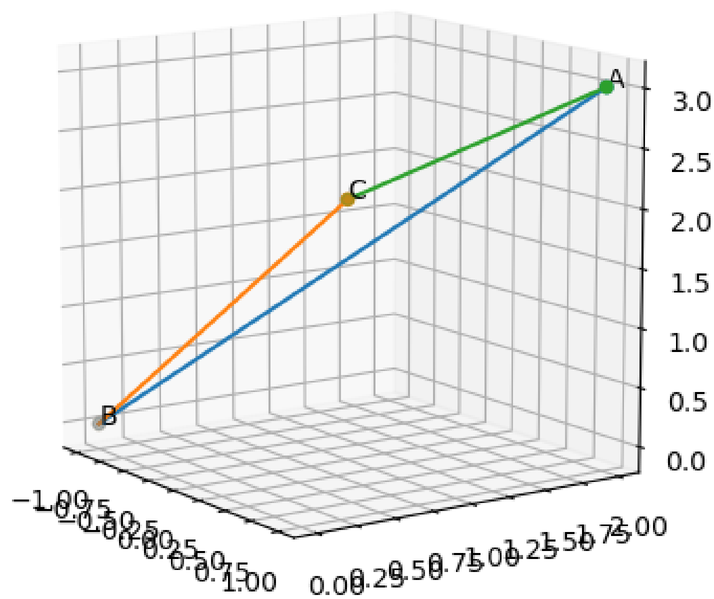


Figure 1.8.7.1:

1.8.8 Find the direction cosines of a line which makes equal angles with the coordinate axes.

Solution: The unit direction vector can be expressed as

$$\mathbf{x} = \begin{pmatrix} \cos\alpha \\ \cos\alpha \\ \cos\alpha \end{pmatrix} \tag{1.8.8.1}$$

$$\implies \|\mathbf{x}\| = \cos\alpha\sqrt{3} = 1 \implies \cos\alpha = \pm\frac{1}{\sqrt{3}} \tag{1.8.8.2}$$

Thus

$$\mathbf{x} = \pm\frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{1.8.8.3}$$

1.8.9 Find a unit vector perpendicular to each of the vector $\overrightarrow{a} + \overrightarrow{b}$ and $\overrightarrow{a} - \overrightarrow{b}$, where $\overrightarrow{a} = 3\hat{i} + 2\hat{j} + 2\hat{k}$ and $\overrightarrow{b} = \hat{i} + 2\hat{j} - 2\hat{k}$.

Solution: Since

$$\mathbf{a} + \mathbf{b} = \begin{pmatrix} 4 \\ 4 \\ 0 \end{pmatrix}, \mathbf{a} - \mathbf{b} = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} \tag{1.8.9.1}$$

the desired vector is obtained as

$$\begin{pmatrix} \mathbf{a}+\mathbf{b} & \mathbf{a}-\mathbf{b} \end{pmatrix}^\top \mathbf{x} = 0 \tag{1.8.9.2}$$

$$\implies \begin{pmatrix} 4 & 4 & 0 \\ 2 & 0 & 4 \end{pmatrix} \xleftrightarrow{R_1=\frac{R_1}{4}} \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 4 \end{pmatrix} \xleftrightarrow{R_2=\frac{R_2}{2}} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} \tag{1.8.9.3}$$

$$\xleftrightarrow{R_2=R_1-R_2} \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 2 \end{pmatrix} \xleftrightarrow{R_2=\frac{R_2}{-1}} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -2 \end{pmatrix} \xleftrightarrow{R_1=R_1-R_2} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \end{pmatrix} \tag{1.8.9.4}$$

yielding

$$\begin{aligned} x_1 + 2x_3 &= 0 \\ x_2 - 2x_3 &= 0 \end{aligned} \implies \mathbf{x} = x_3 \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} \tag{1.8.9.5}$$

1.8.10 If a unit vector $\overrightarrow{a}$ makes angles $\frac{\pi}{3}$ with $\hat{i}$, $\frac{\pi}{4}$ with $\hat{j}$ and an acute angle θ with $\hat{k}$, then find θ and hence, the components of $\overrightarrow{a}$.

Solution: Let

$$\mathbf{A} = \begin{pmatrix} \cos\theta_1 \\ \cos\theta_2 \\ \cos\theta_3 \end{pmatrix} \tag{1.8.10.1}$$

where

$$\cos\theta_1 = \cos\frac{\pi}{3} = \frac{1}{2} \tag{1.8.10.2}$$

$$\cos\theta_2 = \cos\frac{\pi}{4} \tag{1.8.10.3}$$

$$= \frac{1}{\sqrt{2}} \tag{1.8.10.4}$$

Since

$$\|\mathbf{A}\| = 1, \sqrt{\cos^2\theta_1 + \cos^2\theta_2 + \cos^2\theta_3} \quad = 1 \implies \sqrt{\frac{1}{2}^2 + \frac{1}{\sqrt{2}}^2 + \cos^2\theta_3} = 1 \tag{1.8.10.5}$$

$$\implies \cos\theta_3 = \pm\frac{1}{2} \tag{1.8.10.6}$$

Since θ_3 is an acute angle

$$\cos\theta_3 = \frac{1}{2} \tag{1.8.10.7}$$

Hence

$$\mathbf{A} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix} \tag{1.8.10.8}$$

1.8.11 Show that the direction cosines of a vector equally inclined to the axes OX, OY and OZ are $\pm\left[\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right]$.

Solution:

The direction vector can be expressed in terms of the direction cosines as

$$\mathbf{m} = \begin{pmatrix} \cos\theta \\ \cos\theta \\ \cos\theta \end{pmatrix} \tag{1.8.11.1}$$

Since

$$\|\mathbf{m}\| = 1, \left|\sqrt{3}\cos\theta\right| = 1 \implies \cos\theta = \pm\frac{1}{\sqrt{3}} \tag{1.8.11.2}$$

1.8.12 Write down a unit vector in XY-plane, making an angle of 30° with the positive direction of x-axis.

1.8.13 The scalar product of the vector $\hat{\imath}+\hat{\jmath}+\hat{k}$ with a unit vector along the sum of vectors $2\hat{\imath}+4\hat{\jmath}-5\hat{k}$ and $\lambda\hat{\imath}+2\hat{\jmath}+3\hat{k}$ is equal to one. Find the value of λ.

1.8.14 If θ is the angle between two vectors $\mathbf{a}$ and $\mathbf{b}$,then $\mathbf{a}.\mathbf{b} \geq 0$ only when

(a) $0 < \theta < \frac{\pi}{2}$

(b) $0 \leq \theta \leq \frac{\pi}{2}$

(c) $0 < \theta < \pi$

(d) $0 \leq \theta \leq \pi$

Solution: Since

$$\mathbf{a}^\top \mathbf{b} = \cos\theta \left\|\mathbf{a}\right\| \left\|\mathbf{b}\right\|, \tag{1.8.14.1}$$

$$\mathbf{a}^\top \mathbf{b} \geq 0 \implies \cos\theta \geq 0 \tag{1.8.14.2}$$

$$\therefore 0 \leq \theta \leq \frac{\pi}{2}, \frac{3\pi}{2} \leq \theta \leq 2\pi. \tag{1.8.14.3}$$

(a) $0 < \theta < \frac{\pi}{2}$: Comparing with (1.8.14.3), option 1.8.14a is incorrect.

(b) $0 \leq \theta \leq \frac{\pi}{2}$: Comparing with (1.8.14.3), option 1.8.14b is correct.

(c) $0 < \theta < \pi$: Comparing with (1.8.14.3), option 1.8.14c is incorrect.

(d) $0 \leq \theta \leq \pi$: Comparing with (1.8.14.3), option 1.8.14d is incorrect.

1.8.15 Find the slope of the line, which makes an angle of 30 degrees with the positive direction of y-axis measures anticlockwise.

Solution: Let the direction vector of the y-axis be

$$\mathbf{m_1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.8.15.1}$$

and the direction vector of the line be,

$$\mathbf{m_2} = \begin{pmatrix} 1 \\ m \end{pmatrix} \tag{1.8.15.2}$$

where m is the slope of the line. Then,

$$\mathbf{m_1}^\top \mathbf{m_2} = m, \left\|\mathbf{m_1}\right\| = 1, \left\|\mathbf{m_2}\right\| = \sqrt{1+m^2} \tag{1.8.15.3}$$

yielding the angle between the two as

$$\cos(\phi) = \frac{m}{\sqrt{1+m^2}} = \frac{\sqrt{3}}{2} \tag{1.8.15.4}$$

$$\implies m = \pm\sqrt{3} \tag{1.8.15.5}$$

Thus, $m = \sqrt{3}$ is the correct slope.

1.8.16 Find the angle between x-axis and the line joining points (3,-1) and (4,-2).

Solution: See Fig. 1.8.16.1.

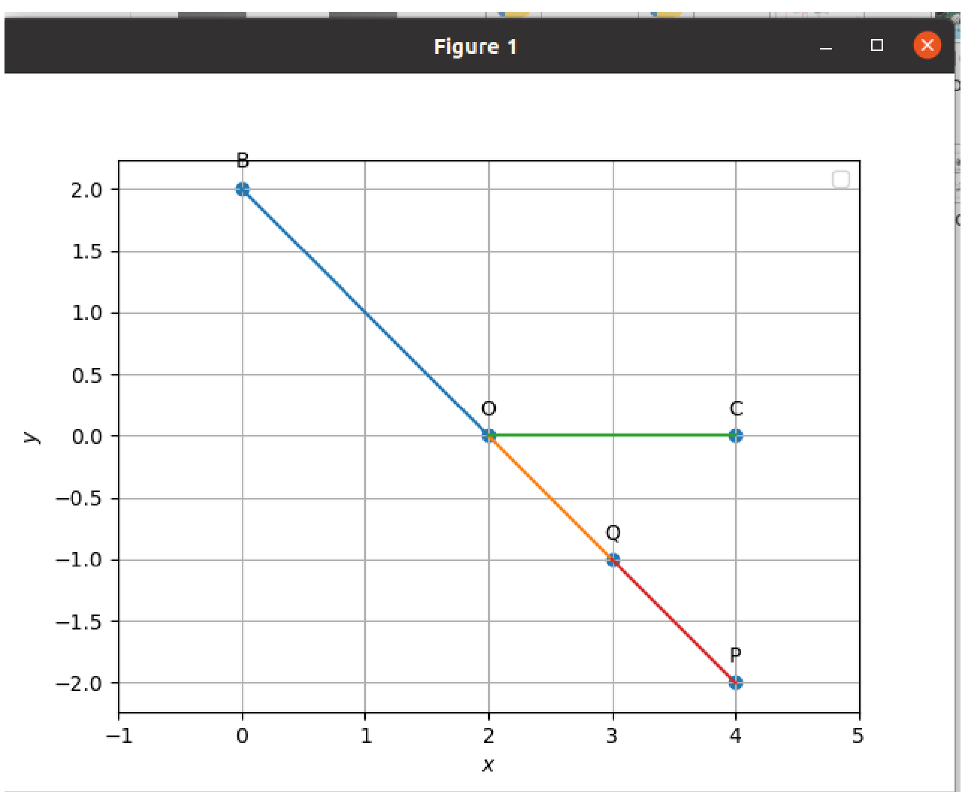


Figure 1.8.16.1:

Let

$$\mathbf{P} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} 4 \\ -2 \end{pmatrix} \tag{1.8.16.1}$$

Then

$$\mathbf{C} = \mathbf{P} - \mathbf{Q} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{1.8.16.2}$$

The desired angle is given by

$$\cos\theta = \frac{\mathbf{C}^T \mathbf{e}_1}{\|\mathbf{C}\| \, \|\mathbf{e}_1\|} \tag{1.8.16.3}$$

$$= -\frac{1}{\sqrt{2}} \tag{1.8.16.4}$$

$$\implies \theta = 135^\circ \tag{1.8.16.5}$$

1.8.17 The slope of a line is double of the slope of another line. If tangent of the angle between them is 1/3, find the slopes of the lines.

Solution: The direction vector of a line is expressed as

$$\mathbf{m} = \begin{pmatrix} 1 \\ m \end{pmatrix} \tag{1.8.17.1}$$

where m is defined to be the slope of the line. If the angle between the

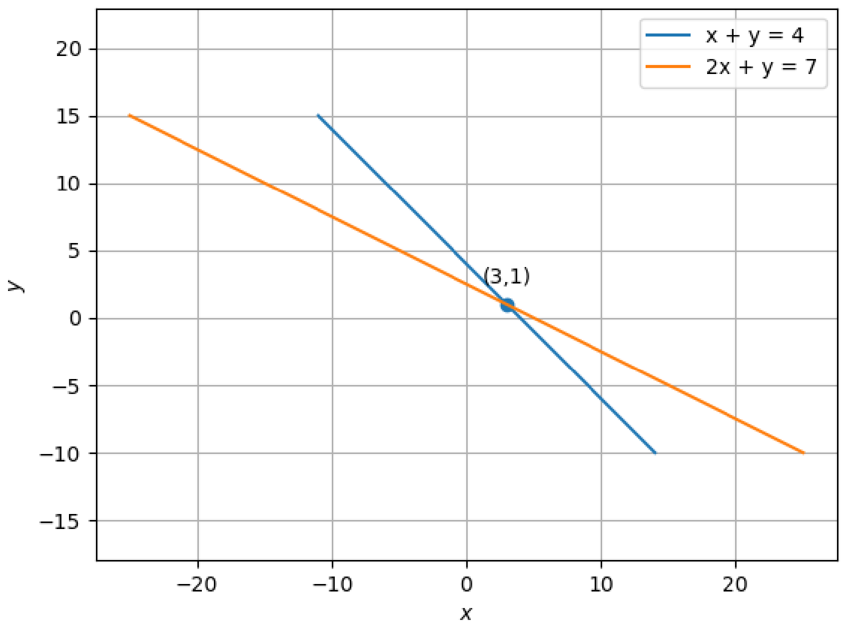


Figure 1.8.17.1:

lines be θ,

$$\tan\theta = \frac{1}{3} \implies \cos\theta = \frac{3}{\sqrt{10}} \tag{1.8.17.2}$$

The angle between two vectors is then expressed as

$$\frac{3}{\sqrt{10}} = \frac{\mathbf{m}_1^\top \mathbf{m}_2}{\|\mathbf{m}_1\| \|\mathbf{m}_2\|} \tag{1.8.17.3}$$

$$= \frac{\begin{pmatrix} 1 & m \end{pmatrix}\begin{pmatrix} 1 \\ 2m \end{pmatrix}}{\left\|\begin{pmatrix} 1 \\ m \end{pmatrix}\right\|\left\|\begin{pmatrix} 1 \\ 2m \end{pmatrix}\right\|} \tag{1.8.17.4}$$

$$= \frac{2m^2+1}{\sqrt{m^2+1}\sqrt{4m^2+1}} \tag{1.8.17.5}$$

$$\implies \frac{9}{10} = \frac{4m^4+4m^2+1}{4m^4+5m^2+1} \tag{1.8.17.6}$$

$$\text{or, } 4m^4 - 5m^2 + 1 = 0 \tag{1.8.17.7}$$

yielding

$$m = \pm\frac{1}{2}, \pm 1 \tag{1.8.17.8}$$

1.8.18 Find angle between the lines, $\sqrt{3}x + y = 1$ and $x + \sqrt{3}y$=1.

Solution: From the given equations, the normal vectors can be expressed as

$$\mathbf{n}_1 = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}, \mathbf{n}_2 = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} \tag{1.8.18.1}$$

The angle between the lines can then be expressed as

$$cos\theta = \frac{\mathbf{n}_1^T \mathbf{n}_2}{\left\|\mathbf{n}_1\right\| \left\|\mathbf{n}_2\right\|} \tag{1.8.18.2}$$

$$= \frac{\sqrt{3}}{2} \tag{1.8.18.3}$$

$$\text{or, } \theta = 30^\circ \tag{1.8.18.4}$$

1.9. Exercises

1.9.1 Let $\mathbf{a}$ and $\mathbf{b}$ be two unit vectors and θ is the angle between them.Then $\mathbf{a}+\mathbf{b}$ is a unit vector if

(a) $\theta = \frac{\pi}{4}$

(b) $\theta = \frac{\pi}{3}$

(c) $\theta = \frac{\pi}{2}$

(d) $\theta = \frac{2\pi}{3}$

1.9.2 The value of $\hat{i}.(\hat{j} \times \hat{k}) + \hat{j}.(\hat{i} \times \hat{k}) + \hat{k}.(\hat{i} \times \hat{j})$ is

(a) 0

(b) -1

(c) 1

(d) 3

1.9.3 If θ is the angle between any two vectors $\mathbf{a}$ and $\mathbf{b}$,then $|\mathbf{a}.\mathbf{b}| = |\mathbf{a} \times \mathbf{b}|$ when θ is equal to

(a) 0

(b) $\frac{\pi}{4}$

(c) $\frac{\pi}{2}$

(d) π

1.9.4 A vector $\mathbf{r}$ has a magnitude 14 and direction ratios 2,3,-6. Find the direction cosines and components of $\mathbf{r}$, given that $\mathbf{r}$ makes an acute angle with x-axis.

1.9.5 Find the angle between the vectors $2\hat{\imath} - \hat{\jmath} + \hat{k}$ and $3\hat{\imath} + 4\hat{\jmath} - \hat{k}$.

1.9.6 If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are the three vectors such that $\mathbf{a} + \mathbf{b} + \mathbf{c} = 0$ and $|\mathbf{a}| = 2$, $|\mathbf{b}|$=3, $|\mathbf{c}|$=5, the value of $\mathbf{a}.\mathbf{b} + \mathbf{b}.\mathbf{c} + \mathbf{c}.\mathbf{a}$ is

(a) 0

(b) 1

(c) -19

(d) 38

1.9.7 If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are unit vectors such that $\mathbf{a}$+$\mathbf{b}$+$\mathbf{c}$=0, then the value of $\mathbf{a}.\mathbf{b} + \mathbf{b}.\mathbf{c} + \mathbf{c}.\mathbf{a}$ is

(a) 1

(b) 3

(c) $\frac{-3}{2}$

(d) None of these

1.9.8 The angles between two vectors **a** and **b** with magnitude $\sqrt{3}$ and 4, respectively, and **a**, **b**= $2\sqrt{3}$ is

(a) $\frac{\pi}{6}$

(b) $\frac{\pi}{3}$

(c) $\frac{\pi}{2}$

(d) $\frac{5\pi}{2}$

1.9.9 The vector $\mathbf{a} + \mathbf{b}$ bisects the angle between the non-collinear vectors **a** and **b** if ______.

1.9.10 The vectors $\mathbf{a} = 3\hat{\imath} - 2\hat{\jmath} + 2\hat{k}$ and $\mathbf{b} = \hat{\imath} - 2\hat{k}$ are the adjancent sides of a parallelogram. The acute angle between its diagonals is ______.

1.9.11 If **a** is any non-zero vector, then $(\mathbf{a}.\hat{\imath})\hat{\imath}+(\mathbf{a}.\hat{\jmath})\hat{\jmath}+(\mathbf{a}.\hat{k})\ \hat{k}$ equals ______.

1.9.12 If **a** and **b** are adjacent sides of a rhombus, then **a**.**b**.=0.

1.9.13 Find the angle between the lines

$$\overrightarrow{r} = 3\hat{\imath} - 2\hat{\jmath} + 6\hat{k} + \lambda(2\hat{\imath} + \hat{\jmath} + 2\hat{k}) \text{ and } \overrightarrow{r} = (2\hat{\jmath} - 5\hat{k}) + \mu(6\hat{\imath} + 3\hat{\jmath} + 2\hat{k})$$

1.9.14 Find the angle between the lines whose direction cosines are given by the equations $l + m + n = 0$, $l^2 + m^2 - n^2 = 0$.

1.9.15 If a variable line in two adjacent positions has directions cosines l, m, n and $l + \delta l, m + \delta m, n + \delta n$, show that the small angle $\delta\theta$ between the two positions is given by

$$\delta\theta^2 = \delta l^2 + \delta m^2 + \delta n^2$$

1.9.16 The sine of the angle between the straight line $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$ and the plane $2x - 2y + z = 5$ is

(a) $\frac{10}{6\sqrt{5}}$

(b) $\frac{4}{5\sqrt{2}}$

(c) $\frac{2\sqrt{3}}{5}$

(d) $\frac{\sqrt{2}}{10}$

1.9.17 The plane $2x - 3y + 6z - 11 = 0$ makes an angle $\sin^{-1}(\alpha)$ with x-axis. The value of α is equal to

(a) $\frac{\sqrt{3}}{2}$

(b) $\frac{\sqrt{2}}{3}$

(c) $\frac{2}{7}$

(d) $\frac{3}{7}$

1.9.18 The angle between the line $\vec{r} = (5\hat{i} - \hat{j} - 4\hat{k}) + \lambda(2\hat{i} - \hat{j} + \hat{k})$ and the plane $\vec{r} \cdot (3\hat{i} - 4\hat{j} - \hat{k}) + 5 = 0$ is $\sin^{-1}\left(\frac{5}{2\sqrt{91}}\right)$.

1.9.19 The angle between the planes $\vec{r} \cdot (2\hat{i} - 3\hat{j} + \hat{k}) = 1$ and $\vec{r} \cdot (\hat{i} - \hat{j}) = 4$ is $\cos^{-1}\left(\frac{-5}{\sqrt{58}}\right)$.

1.9.20 Let **a** and **b** be two unit vectors and θ is the angle between them.Then $\mathbf{a} + \mathbf{b}$ is a unit vector if

(a) $\theta = \frac{\pi}{4}$

(b) $\theta = \frac{\pi}{3}$

(c) $\theta = \frac{\pi}{2}$

(d) $\theta = \frac{2\pi}{3}$

1.9.21 The value of $\hat{\imath}.(\hat{\jmath} \times \hat{k}) + \hat{\jmath}.(\hat{\imath} \times \hat{k}) + \hat{k}.(\hat{\imath} \times \hat{\jmath})$ is

(a) 0

(b) -1

(c) 1

(d) 3

1.9.22 If θ is the angle between any two vectors $\mathbf{a}$ and $\mathbf{b}$,then $|\mathbf{a}.\mathbf{b}| = |\mathbf{a} \times \mathbf{b}|$ when θ is equal to

(a) 0

(b) $\frac{\pi}{4}$

(c) $\frac{\pi}{2}$

(d) π

1.10. Orthogonality

1.10.1 Check whether$(5, -2), (6, 4)$ and $(7, -2)$ are the vertices of an isosceles triangle.

1.10.2 Name the type of quadrilateral formed, if any, by the following points,and give reasons for your answer

(a) $(-1, -2), (1, 0), (-1, 2), (-3, 0)$

(b) $(-3,5),(-3,1),(0,3),(-1,-4)$

(c) $(4,5),(7,6),(4,3),(1,2)$

Solution:

(a) The coordinates are given as

$$\mathbf{A} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \text{ and } \mathbf{D} = \begin{pmatrix} -3 \\ 0 \end{pmatrix} \tag{1.10.2.1}$$

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} -1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \tag{1.10.2.2}$$

$$\mathbf{C} - \mathbf{B} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix} \tag{1.10.2.3}$$

$$\mathbf{C} - \mathbf{D} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} - \begin{pmatrix} -3 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \tag{1.10.2.4}$$

$$\mathbf{D} - \mathbf{A} = \begin{pmatrix} -3 \\ 0 \end{pmatrix} - \begin{pmatrix} -1 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix} \tag{1.10.2.5}$$

$$\mathbf{C}-\mathbf{A}=\begin{pmatrix}-1\\2\end{pmatrix}-\begin{pmatrix}-1\\-2\end{pmatrix}=\begin{pmatrix}0\\4\end{pmatrix} \tag{1.10.2.6}$$

$$\mathbf{D}-\mathbf{B}=\begin{pmatrix}-3\\0\end{pmatrix}-\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}-4\\0\end{pmatrix} \tag{1.10.2.7}$$

$$\mathbf{B}-\mathbf{A}=\mathbf{C}-\mathbf{D} \text{ and } \mathbf{C}-\mathbf{B}=\mathbf{D}-\mathbf{A}. \tag{1.10.2.8}$$

Hence, $ABCD$ is a parallelogram.

i. Now checking if the adjacent sides are orthogonal to each other

$$(\mathbf{B}-\mathbf{A})^\top(\mathbf{C}-\mathbf{B})=\begin{pmatrix}2 & 2\end{pmatrix}\begin{pmatrix}-2\\2\end{pmatrix}=-4+4=0 \tag{1.10.2.9}$$

ii. Now checking if the diagonals are also orthogonal then it is a square else a rectangle.

$$(\mathbf{C}-\mathbf{A})^\top(\mathbf{D}-\mathbf{B})=\begin{pmatrix}0 & 4\end{pmatrix}\begin{pmatrix}-4\\0\end{pmatrix}=0 \tag{1.10.2.10}$$

Hence the diagonals are orthogonal to each other.

So, we can conclude that $ABCD$ is a square.

As shown in Figure 1.10.2.1 we can see that $ABCD$ is a square hence we can conclude that our theoritical result is verified.

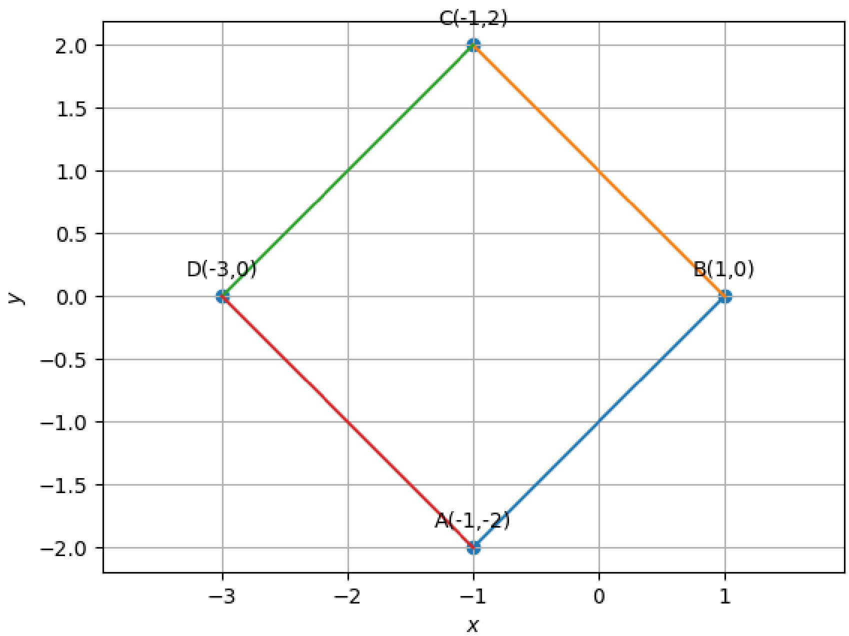


Figure 1.10.2.1:

(b) The coordinates are given as

$$\mathbf{A} = \begin{pmatrix} -3 \\ 5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} \text{ and } \mathbf{D} = \begin{pmatrix} -1 \\ -4 \end{pmatrix} \tag{1.10.2.11}$$

$$\mathbf{B}-\mathbf{A}=\begin{pmatrix}3\\1\end{pmatrix}-\begin{pmatrix}-3\\5\end{pmatrix}=\begin{pmatrix}6\\-4\end{pmatrix} \tag{1.10.2.12}$$

$$\mathbf{C}-\mathbf{B}=\begin{pmatrix}0\\3\end{pmatrix}-\begin{pmatrix}3\\1\end{pmatrix}=\begin{pmatrix}-3\\2\end{pmatrix} \tag{1.10.2.13}$$

$$\mathbf{C}-\mathbf{D}=\begin{pmatrix}0\\3\end{pmatrix}-\begin{pmatrix}-1\\-4\end{pmatrix}=\begin{pmatrix}1\\7\end{pmatrix} \tag{1.10.2.14}$$

$$\mathbf{D}-\mathbf{A}=\begin{pmatrix}-1\\-4\end{pmatrix}-\begin{pmatrix}-3\\5\end{pmatrix}=\begin{pmatrix}2\\-9\end{pmatrix} \tag{1.10.2.15}$$

$$\mathbf{C}-\mathbf{A}=\begin{pmatrix}0\\3\end{pmatrix}-\begin{pmatrix}-3\\5\end{pmatrix}=\begin{pmatrix}3\\-2\end{pmatrix} \tag{1.10.2.16}$$

$$\mathbf{D}-\mathbf{B}=\begin{pmatrix}-1\\-4\end{pmatrix}-\begin{pmatrix}3\\1\end{pmatrix}=\begin{pmatrix}-4\\-5\end{pmatrix} \tag{1.10.2.17}$$

$$\mathbf{B}-\mathbf{A}\neq\mathbf{C}-\mathbf{D} \text{ and } \mathbf{C}-\mathbf{B}\neq\mathbf{D}-\mathbf{A}, \tag{1.10.2.18}$$

Hence, $ABCD$ is not a parallelogram, it can be a irregular quadilateral.

i. Now to check if any three points are collinear,
   if rank of $\begin{pmatrix}\mathbf{B}-\mathbf{A} & \mathbf{C}-\mathbf{B}\end{pmatrix}=1$ then points are collinear

Forming the collinearity matrix

$$\begin{pmatrix} 6 & -3 \\ -4 & 2 \end{pmatrix} \xleftrightarrow{R_2 \to R_2 + \frac{2}{3}R_1} = \begin{pmatrix} 6 & -3 \\ 0 & 0 \end{pmatrix} \tag{1.10.2.19}$$

Hence, rank = 1

Since none of the opposite sides are parallel to each other and three points are collinear so these does not form a quadilateral.

As shown in Figure 1.10.2.2 we can see that $ABCD$ does not form a quadilateral and three points are collinear hence, our theoritical result is verified.

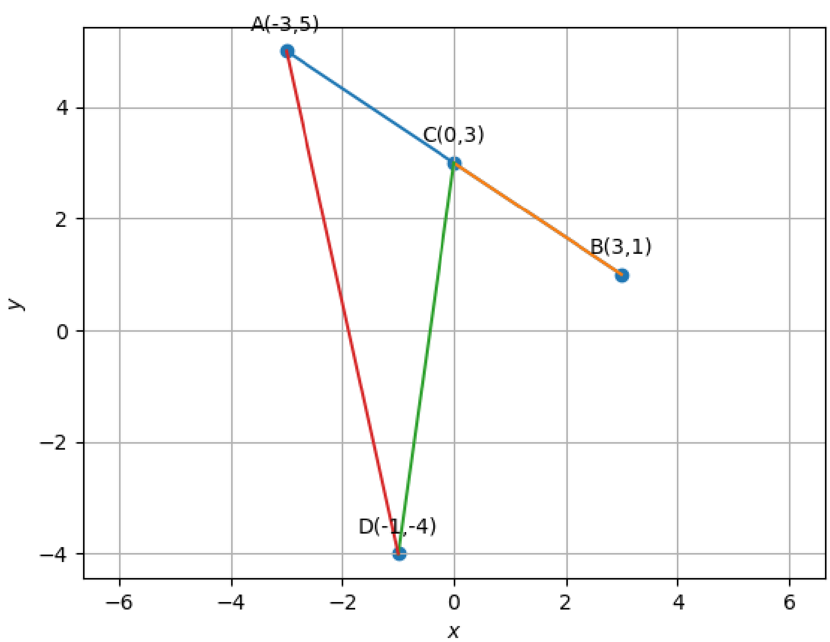


Figure 1.10.2.2:

(c) The coordinates are given as

$$\mathbf{A} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 7 \\ 6 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \text{ and } \mathbf{D} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{1.10.2.20}$$

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} 7 \\ 6 \end{pmatrix} - \begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \tag{1.10.2.21}$$

$$\mathbf{C} - \mathbf{B} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} - \begin{pmatrix} 7 \\ 6 \end{pmatrix} = \begin{pmatrix} -3 \\ -3 \end{pmatrix} \tag{1.10.2.22}$$

$$\mathbf{C} - \mathbf{D} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \tag{1.10.2.23}$$

$$\mathbf{D} - \mathbf{A} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} -3 \\ -3 \end{pmatrix} \tag{1.10.2.24}$$

$$\mathbf{C} - \mathbf{A} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} - \begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} \tag{1.10.2.25}$$

$$\mathbf{D} - \mathbf{B} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 7 \\ 6 \end{pmatrix} = \begin{pmatrix} -6 \\ -4 \end{pmatrix} \tag{1.10.2.26}$$

$$\mathbf{B} - \mathbf{A} = \mathbf{C} - \mathbf{D} \text{ and } \mathbf{C} - \mathbf{B} = \mathbf{D} - \mathbf{A}, \tag{1.10.2.27}$$

Hence, $ABCD$ is a parallelogram.

i. Now checking if the adjacent sides are orthogonal to each other

$$(\mathbf{B}-\mathbf{A})^\top(\mathbf{C}-\mathbf{B}) = \begin{pmatrix} 3 & 1 \end{pmatrix}\begin{pmatrix} -3 \\ -3 \end{pmatrix} = -9-3 = -12 \tag{1.10.2.28}$$

Since inner product is not zero so adjacent sides are not orthogonal.

Hence, we can say that $ABCD$ is neither a rectangle nor a square.

ii. Now checking if the diagonals are orthogonal then it is a Rhombus.

$$(\mathbf{C}-\mathbf{A})^\top(\mathbf{D}-\mathbf{B}) = \begin{pmatrix} 0 & -2 \end{pmatrix}\begin{pmatrix} -6 \\ -4 \end{pmatrix} = 0+8 = 8 \tag{1.10.2.29}$$

Hence the diagonals are also not orthogonal so we conclude that $ABCD$ is a parallelogram.

As shown in Figure 1.10.2.3 we can see that $ABCD$ forms a parallelogram hence, our theoritical result is verified.

1.10.3 Find the projection of the vector $\hat{i}-\hat{j}$ on the vector $\hat{i}+\hat{j}$.

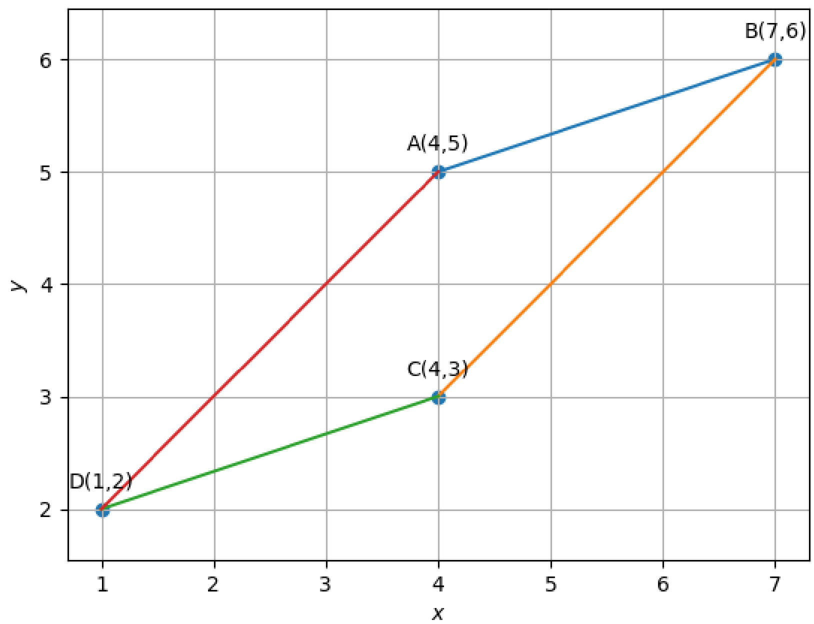


Figure 1.10.2.3:

Solution: The given points are

$$\mathbf{A} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.10.3.1}$$

Since

$$\mathbf{A}^\top\mathbf{B} = \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \times 1 \end{pmatrix} + \begin{pmatrix} -1 \times 1 \end{pmatrix} = 0 \tag{1.10.3.2}$$

$$\|\mathbf{B}\|^2 = (\mathbf{B}^\top\mathbf{B}) = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = (1 \times 1) + (1 \times 1) = 2, \tag{1.10.3.3}$$

and the project vector is given by

$$\mathbf{C} = \frac{\mathbf{A}^\top\mathbf{B}^2}{\|\mathbf{B}\|}\mathbf{B} \qquad = \frac{0}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1.10.3.4}$$

This is verfied in Fig. 1.10.3.1.

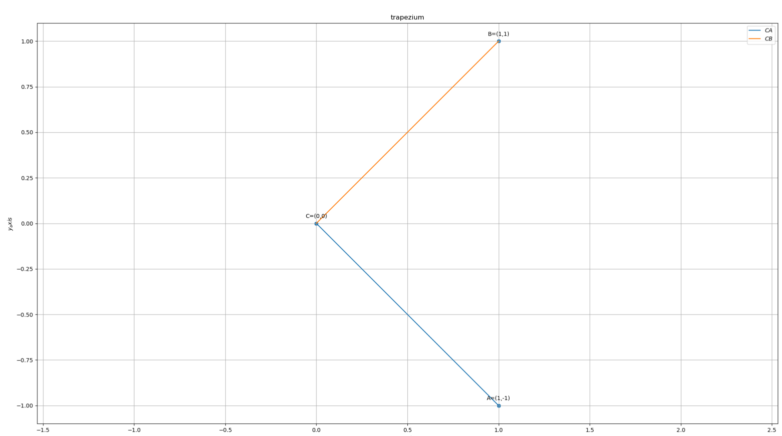


Figure 1.10.3.1:

1.10.4 Find the projection of the vector $\hat{i} + 3\hat{j} + 7\hat{k}$ on the vector $7\hat{i} - \hat{j} + 8\hat{k}$.

Solution: Let

$$\mathbf{A} = \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 7 \\ -1 \\ 8 \end{pmatrix} \tag{1.10.4.1}$$

The desired projection is given by

$$\mathbf{C} = \frac{\mathbf{A}^\top \mathbf{B}}{\left\|\mathbf{B}\right\|^2}\mathbf{B} = \frac{\begin{pmatrix} 1 & 3 & 7 \end{pmatrix}\begin{pmatrix} 7 \\ -1 \\ 8 \end{pmatrix}}{2}\begin{pmatrix} 7 \\ -1 \\ 8 \end{pmatrix} = \frac{10}{19}\begin{pmatrix} 7 \\ -1 \\ 8 \end{pmatrix} \tag{1.10.4.2}$$

1.10.5 Show that each of the given three vectors is a unit vector: $\frac{1}{7}(2\hat{i} + 3\hat{j} + 6\hat{k})$, $\frac{1}{7}(3\hat{i} - 6\hat{j} + 2\hat{k})$, $\frac{1}{7}(6\hat{i} + 2\hat{j} - 3\hat{k})$ Also,show that they are mutually perpendicular to each other.

Solution: Let

$$\mathbf{A} = \begin{pmatrix} \frac{2}{7} \\ \frac{3}{7} \\ \frac{6}{7} \end{pmatrix}, \mathbf{B} = \begin{pmatrix} \frac{3}{7} \\ -\frac{6}{7} \\ \frac{2}{7} \end{pmatrix}, \mathbf{C} = \begin{pmatrix} \frac{6}{7} \\ \frac{2}{7} \\ -\frac{3}{7} \end{pmatrix} \tag{1.10.5.1}$$

Then

$$\left\|\mathbf{A}\right\| = \left\|\mathbf{B}\right\| = \left\|\mathbf{C}\right\| = 1 \tag{1.10.5.2}$$

Also,

$$\mathbf{A}^\top\mathbf{B} = \begin{pmatrix} \frac{2}{7} & \frac{3}{7} & \frac{6}{7} \end{pmatrix} \begin{pmatrix} \frac{3}{7} \\ -\frac{6}{7} \\ \frac{2}{7} \end{pmatrix} = \frac{6}{49} - \frac{18}{49} + \frac{12}{49} = 0 \tag{1.10.5.3}$$

$$\mathbf{B}^\top\mathbf{C} = \begin{pmatrix} \frac{3}{7} & -\frac{6}{7} & \frac{2}{7} \end{pmatrix} \begin{pmatrix} \frac{6}{7} \\ \frac{2}{7} \\ -\frac{3}{7} \end{pmatrix} = \frac{18}{49} - \frac{12}{49} - \frac{6}{49} = 0 \tag{1.10.5.4}$$

$$\mathbf{C}^\top\mathbf{A} = \begin{pmatrix} \frac{6}{7} & \frac{2}{7} & -\frac{3}{7} \end{pmatrix} \begin{pmatrix} \frac{2}{7} \\ \frac{3}{7} \\ \frac{6}{7} \end{pmatrix} = \frac{12}{49} + \frac{6}{49} - \frac{18}{49} = 0 \tag{1.10.5.5}$$

1.10.6 If $\overrightarrow{a} = 2\hat{i} + 2\hat{j}3\hat{k}$, $\overrightarrow{b} = -\hat{i} + 2\hat{j} + \hat{k}$ and $\overrightarrow{c} = 3\hat{i} + \hat{j}$ are such that $\overrightarrow{a} + \lambda\overrightarrow{b}$ is perpendicular to $\overrightarrow{c}$,then find the value of λ.

Solution: Given that

$$(\mathbf{a} + \lambda\mathbf{b})^\top\mathbf{c} = 0 \tag{1.10.6.1}$$

$$\implies \mathbf{a}^\top\mathbf{c} + \lambda\mathbf{b}^\top\mathbf{c} = 0 \tag{1.10.6.2}$$

$$\implies \lambda\mathbf{b}^\top\mathbf{c} = -\mathbf{a}^\top\mathbf{c} \tag{1.10.6.3}$$

$$\implies \lambda(\mathbf{b}^\top\mathbf{c})(\mathbf{b}^\top\mathbf{c})^{-1} = -(\mathbf{a}^\top\mathbf{c})(\mathbf{b}^\top\mathbf{c})^{-1} \tag{1.10.6.4}$$

$$\implies \lambda = -(\mathbf{a}^\top\mathbf{c})(\mathbf{b}^\top\mathbf{c})^{-1} \tag{1.10.6.5}$$

Now substituting the values

$$\mathbf{a}^\top\mathbf{c} = \begin{pmatrix} 2 & 2 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} = 8 \tag{1.10.6.6}$$

$$\mathbf{b}^\top\mathbf{c} = \begin{pmatrix} -1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} = -1, \tag{1.10.6.7}$$

$$\lambda = -(\mathbf{a}^\top\mathbf{c})(\mathbf{b}^\top\mathbf{c})^{-1} \tag{1.10.6.8}$$

$$= -(8)(-1)^{-1} \tag{1.10.6.9}$$

$$= 8 \tag{1.10.6.10}$$

1.10.7 Show that $|\overrightarrow{a}|\,\overrightarrow{b} + \left|\overrightarrow{b}\right|\overrightarrow{a}$ is perpendicular to $|\overrightarrow{a}|\,\overrightarrow{b} - \left|\overrightarrow{b}\right|\overrightarrow{a}$, for any two nonzero vectors $\overrightarrow{a}$ and $\overrightarrow{b}$.

Solution:

$$(\|\mathbf{a}\|\,\mathbf{b} + \|\mathbf{b}\|\,\mathbf{a})^\top(\|\mathbf{a}\|\,\mathbf{b} - \|\mathbf{b}\|\,\mathbf{a}) =$$

$$\|\mathbf{a}\|^2\,\mathbf{b}^\top\mathbf{b} + \|\mathbf{a}\|\,\|\mathbf{b}\|\,\mathbf{a}^\top\mathbf{b} - \|\mathbf{a}\|\,\|\mathbf{b}\|\,\mathbf{b}^\top\mathbf{a} - \|\mathbf{b}\|^2\,\mathbf{a}^\top\mathbf{a} =$$

$$\|\mathbf{a}\|^2\,\|\mathbf{b}\|^2 + \|\mathbf{a}\|\,\|\mathbf{b}\|\,\mathbf{a}^\top\mathbf{b} - \|\mathbf{a}\|\,\|\mathbf{b}\|\,\mathbf{a}^\top\mathbf{b} - \|\mathbf{b}\|^2\,\|\mathbf{a}\|^2 = 0 \tag{1.10.7.1}$$

1.10.8 If $\overrightarrow{a}.\overrightarrow{a}$=0 and $\overrightarrow{a}.\overrightarrow{b}$=0, then what can be conculded about the vector $\overrightarrow{b}$?

1.10.9 If $\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}$ are unit vectors such that $\overrightarrow{a} + \overrightarrow{b} + \overrightarrow{c} = \overrightarrow{0}$, find the value of $\overrightarrow{a}.\overrightarrow{b} + \overrightarrow{b}.\overrightarrow{c} + \overrightarrow{c}.\overrightarrow{a}$.

Solution:

$$\|\mathbf{a}+\mathbf{b}+\mathbf{c}\|^2 = 0 \tag{1.10.9.1}$$

$$\implies \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 + \|\mathbf{c}\|^2 + 2(\mathbf{a}^\top\mathbf{b} + \mathbf{b}^\top\mathbf{c} + \mathbf{c}^\top\mathbf{a}) = 0 \tag{1.10.9.2}$$

$$\implies 3 + 2(\mathbf{a}^\top\mathbf{b} + \mathbf{b}^\top\mathbf{c} + \mathbf{c}^\top\mathbf{a}) = 0 \tag{1.10.9.3}$$

$$\implies \mathbf{a}^\top\mathbf{b} + \mathbf{b}^\top\mathbf{c} + \mathbf{c}^\top\mathbf{a} = -\frac{3}{2} \tag{1.10.9.4}$$

1.10.10 If either vector $\overrightarrow{a} = 0$ or $\overrightarrow{b} = 0$, then $\overrightarrow{a}.\overrightarrow{b}$=0. But the converse need not be true. Justify your answer with an example.

Solution: Let

$$\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \tag{1.10.10.1}$$

$$\implies \mathbf{a}^\top\mathbf{b} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \tag{1.10.10.2}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{1.10.10.3}$$

Here, $\mathbf{a} \neq 0$ and $\mathbf{b} \neq 0$ Therefore, the converse need not be true.

1.10.11 Show that the vectors $2\hat{i} - \hat{j} + \hat{k}, \hat{i} - 3\hat{j} - 5\hat{k}$ and $3\hat{i} - 4\hat{j} - 4\hat{k}$ from the vertices of a right angled triangle.

Solution: Let

$$\mathbf{A} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 3 \\ -4 \\ -4 \end{pmatrix} \tag{1.10.11.1}$$

Form the matrix

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} = \begin{pmatrix} 2 & 1 & 3 \\ -1 & -3 & -4 \\ 1 & -5 & -4 \end{pmatrix} \tag{1.10.11.2}$$

$$\xleftrightarrow[R_2 \leftarrow R_2 + \frac{1}{2}R_1]{R_3 \leftarrow R_3 - \frac{1}{2}R_1} \tag{1.10.11.3}$$

$$\begin{pmatrix} 2 & 1 & 3 \\ 0 & -\frac{5}{2} & -\frac{5}{2} \\ 0 & -\frac{11}{2} & -\frac{11}{2} \end{pmatrix} \tag{1.10.11.4}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - \frac{11}{5}R_2} \tag{1.10.11.5}$$

$$\begin{pmatrix} 2 & 1 & 3 \\ 0 & -\frac{5}{2} & -\frac{5}{2} \\ 0 & 0 & 0 \end{pmatrix}, \tag{1.10.11.6}$$

the rank of the matrix is 2 and the points are in 3-Dimensional space, so the points $\mathbf{A}, \mathbf{B}, \mathbf{C}$ form a triangle.

(a) checking whether the triangle is right angled at $\mathbf{A}$

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} -1 \\ -2 \\ -6 \end{pmatrix} \tag{1.10.11.7}$$

$$\mathbf{C} - \mathbf{A} = \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix} \tag{1.10.11.8}$$

$$(\mathbf{B} - \mathbf{A})^\top (\mathbf{C} - \mathbf{A}) = \begin{pmatrix} -1 & -2 & -6 \end{pmatrix} \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix} = 35 \neq 0 \tag{1.10.11.9}$$

The triangle is not right angled at $\mathbf{A}$.

(b) checking whether the triangle is right angled at $\mathbf{B}$

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} \tag{1.10.11.10}$$

$$\mathbf{C} - \mathbf{B} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \tag{1.10.11.11}$$

$$(\mathbf{A}-\mathbf{B})^{\top}(\mathbf{C}-\mathbf{B}) = \begin{pmatrix} 1 & 2 & 6 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = 6 \neq 0 \tag{1.10.11.12}$$

The triangle is not right angled at **B**.

(c) checking whether the triangle is right angled at **C**

$$\mathbf{A}-\mathbf{C} = \begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix} \tag{1.10.11.13}$$

$$\mathbf{B}-\mathbf{C} = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} \tag{1.10.11.14}$$

$$(\mathbf{A}-\mathbf{C})^{\top}(\mathbf{B}-\mathbf{C}) = \begin{pmatrix} -1 & 3 & 5 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} = 0 \tag{1.10.11.15}$$

$$\tag{1.10.11.16}$$

Hence the triangle is right angled at **C**.

1.10.12 Show that the points A, B and C with position vectors,$\mathbf{a} = 3\hat{\imath} - 4\hat{\jmath} - 4\hat{k}$,$\mathbf{b} = 2\hat{\imath} - \hat{\jmath} + \hat{k}$ and $\mathbf{c} = \hat{\imath} - 3\hat{\jmath} - 5\hat{k}$, respectively form the vertices of a right angled triangle.

Solution: We write the direction vectors of the three sides as

$$\mathbf{c} = \mathbf{B} - \mathbf{A} = \begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix}, \mathbf{a} \quad = \mathbf{C} - \mathbf{B} = \begin{pmatrix} -1 \\ -2 \\ -6 \end{pmatrix}, \mathbf{b} = \mathbf{C} - \mathbf{A} = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} \tag{1.10.12.1}$$

Taking the inner product of each pair of vectors,

$$\mathbf{c}^\top \mathbf{a} = -35, \mathbf{a}^\top \mathbf{b} \qquad = 6, \mathbf{b}^\top \mathbf{c} = 0 \tag{1.10.12.2}$$

From (1.10.12.2), $\mathbf{b}^\top \mathbf{c} = 0$, which implies that $\mathbf{b} \perp \mathbf{c}$. Hence, $\triangle ABC$ is right angled at $\mathbf{A}$.

1.10.13 Let $\mathbf{a} = \hat{i} + 4\hat{j} + 2\hat{k}$,$\mathbf{b} = 3\hat{i} - 2\hat{j} + 7\hat{k}$ and $\mathbf{c} = 2\hat{i} - \hat{j} + 4\hat{k}$.Find a vector **d** which is perpendicular to both **a** and **b**,and **c**.**d**=15.

Solution: Let

$$\mathbf{a} = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 3 \\ -2 \\ 7 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}. \tag{1.10.13.1}$$

From the given information.

$$\mathbf{a}^\top \mathbf{D} = 0 \tag{1.10.13.2}$$

$$\mathbf{b}^\top \mathbf{D} = 0 \tag{1.10.13.3}$$

$$\mathbf{c}^\top \mathbf{D} = 15 \tag{1.10.13.4}$$

Joining all the equations in matrix form gives,

$$\begin{pmatrix} \mathbf{a}^\top \\ \mathbf{b}^\top \\ \mathbf{c}^\top \end{pmatrix} \mathbf{d} = \begin{pmatrix} 0 \\ 0 \\ 15 \end{pmatrix} \tag{1.10.13.5}$$

$$\begin{pmatrix} 1 & 4 & 2 \\ 3 & -2 & 7 \\ 2 & -1 & 4 \end{pmatrix} \mathbf{D} = \begin{pmatrix} 0 \\ 0 \\ 15 \end{pmatrix} \tag{1.10.13.6}$$

The augmented matrix for the system equations in (1.10.13.6) is expressed as

$$\left(\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 3 & -2 & 7 & 0 \\ 2 & -1 & 4 & 15 \end{array}\right) \xleftrightarrow[R_3 \leftarrow R_3 - 2R_1]{R_2 \leftarrow R_2 - 3R_1} \left(\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 0 & -14 & 1 & 0 \\ 0 & -9 & 0 & 15 \end{array}\right) \tag{1.10.13.7}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - \frac{9}{14} R_2} \left(\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 0 & -14 & 1 & 0 \\ 0 & 0 & -\frac{9}{14} & 15 \end{array}\right) \tag{1.10.13.8}$$

The augmented matrix for the system equations is reduced to Row echelon form. From (1.10.13.8), we obtain

$$\mathbf{d} = \begin{pmatrix} \frac{160}{3} \\ \\ -\frac{5}{3} \\ \\ -\frac{70}{3} \end{pmatrix} \tag{1.10.13.9}$$

1.10.14 Prove that $(\mathbf{a}+\mathbf{b}).(\mathbf{a}+\mathbf{b})=|\mathbf{a}|^2+|\mathbf{b}|^2$,if and only if $\mathbf{a},\mathbf{b}$ are perpendicular, given $\mathbf{a} \neq \mathbf{0}$,$\mathbf{b} \neq \mathbf{0}$.

Solution:

$$(\mathbf{a}+\mathbf{b})^\top(\mathbf{a}+\mathbf{b}) = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 + 2\mathbf{a}^\top\mathbf{b} \tag{1.10.14.1}$$

Thus,

$$(\mathbf{a}+\mathbf{b})^\top(\mathbf{a}+\mathbf{b}) = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 \tag{1.10.14.2}$$

$$\iff \mathbf{a}^\top\mathbf{b} = 0 \tag{1.10.14.3}$$

1.10.15 $ABCD$ is a rectangle formed by the points $\mathbf{A}(-1,-1), \mathbf{B}(-1,4), \mathbf{C}(5,4)$ and $\mathbf{D}(5,-1)$. $\mathbf{P},\mathbf{Q},\mathbf{R}$ and $\mathbf{S}$ are the mid-points of AB, BC, CD and DA respectively. Is the quadrilateral $PQRS$ a square? a rectangle? or a rhombus? Justify your answer.

Solution: See Fig. 1.10.15.1.

$$\mathbf{P} = \frac{1}{2}(\mathbf{A}+\mathbf{B}) = \frac{1}{2}\left(\begin{pmatrix}-1\\-1\end{pmatrix}+\begin{pmatrix}-1\\4\end{pmatrix}\right) = \begin{pmatrix}-1\\\frac{3}{2}\end{pmatrix} \tag{1.10.15.1}$$

$$\mathbf{Q} = \frac{1}{2}(\mathbf{B}+\mathbf{C}) = \frac{1}{2}\left(\begin{pmatrix}-1\\4\end{pmatrix}+\begin{pmatrix}5\\4\end{pmatrix}\right) = \begin{pmatrix}2\\4\end{pmatrix} \tag{1.10.15.2}$$

$$\mathbf{R} = \frac{1}{2}(\mathbf{C}+\mathbf{D}) = \frac{1}{2}\left(\begin{pmatrix}5\\4\end{pmatrix}+\begin{pmatrix}5\\-1\end{pmatrix}\right) = \begin{pmatrix}5\\\frac{3}{2}\end{pmatrix} \tag{1.10.15.3}$$

$$\mathbf{S} = \frac{1}{2}(\mathbf{D}+\mathbf{A}) = \frac{1}{2}\left(\begin{pmatrix}5\\-1\end{pmatrix}+\begin{pmatrix}-1\\-1\end{pmatrix}\right) = \begin{pmatrix}2\\-1\end{pmatrix} \tag{1.10.15.4}$$

We know that PQRS is a parallelogram. To know, if it is a rectangle, we need to ascertain whether any of the two adjacent sides are perpendicular. That means $(\mathbf{Q}-\mathbf{P})^\top(\mathbf{R}-\mathbf{Q})$ should be equal to zero.

$$\mathbf{Q}-\mathbf{P} = \begin{pmatrix}2\\4\end{pmatrix} - \begin{pmatrix}-1\\\frac{3}{2}\end{pmatrix} = \begin{pmatrix}3\\\frac{5}{2}\end{pmatrix} \tag{1.10.15.5}$$

$$\mathbf{R}-\mathbf{Q} = \begin{pmatrix}5\\\frac{3}{2}\end{pmatrix} - \begin{pmatrix}2\\4\end{pmatrix} = \begin{pmatrix}3\\-\frac{5}{2}\end{pmatrix} \tag{1.10.15.6}$$

$$(\mathbf{Q}-\mathbf{P})^\top(\mathbf{R}-\mathbf{Q}) = \begin{pmatrix}3 & \frac{5}{2}\end{pmatrix}\begin{pmatrix}3\\-\frac{5}{2}\end{pmatrix} \neq 0 \tag{1.10.15.7}$$

Therefore PQRS is not a rectangle. Let us check if it is a rhombus. For a rhombus, the diagonals bisect perpendicularly. That means

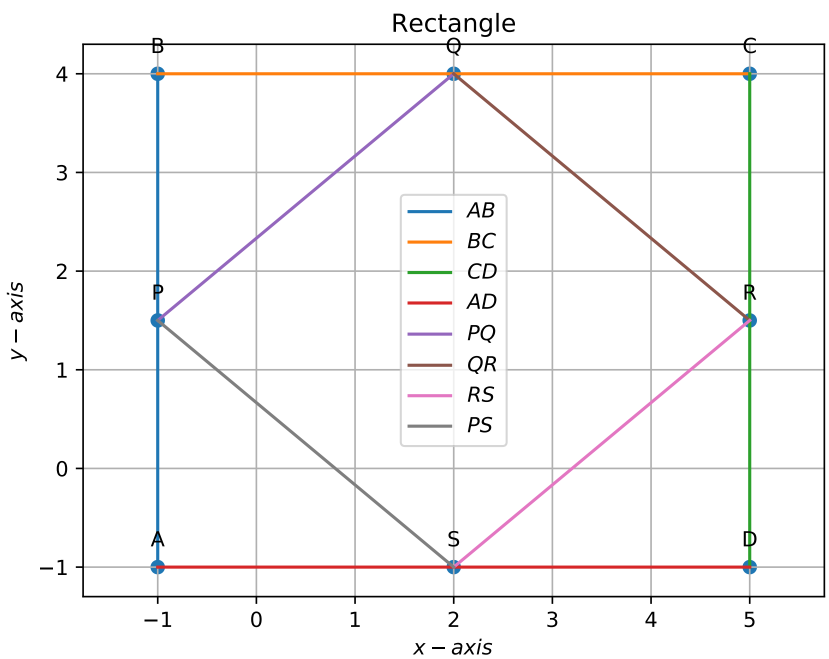


Figure 1.10.15.1:

$(\mathbf{R}-\mathbf{P})^{\top}(\mathbf{S}-\mathbf{Q})$ should be equal to zero.

$$\mathbf{R}-\mathbf{P} = \begin{pmatrix} 5 \\ \frac{3}{2} \end{pmatrix} - \begin{pmatrix} -1 \\ \frac{3}{2} \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \end{pmatrix} \tag{1.10.15.8}$$

$$\mathbf{S}-\mathbf{Q} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ -5 \end{pmatrix} \tag{1.10.15.9}$$

$$(\mathbf{R}-\mathbf{P})^{\top}(\mathbf{S}-\mathbf{Q}) = \begin{pmatrix} 6 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ -5 \end{pmatrix} = 0 \tag{1.10.15.10}$$

Therefore $PQRS$ is a rhombus.

1.10.16 Without using the Baudhayana theorem, show that the points $(4,4)$, $(3,5)$ and $(-1,-1)$ are the vertices of a right angled triangle.

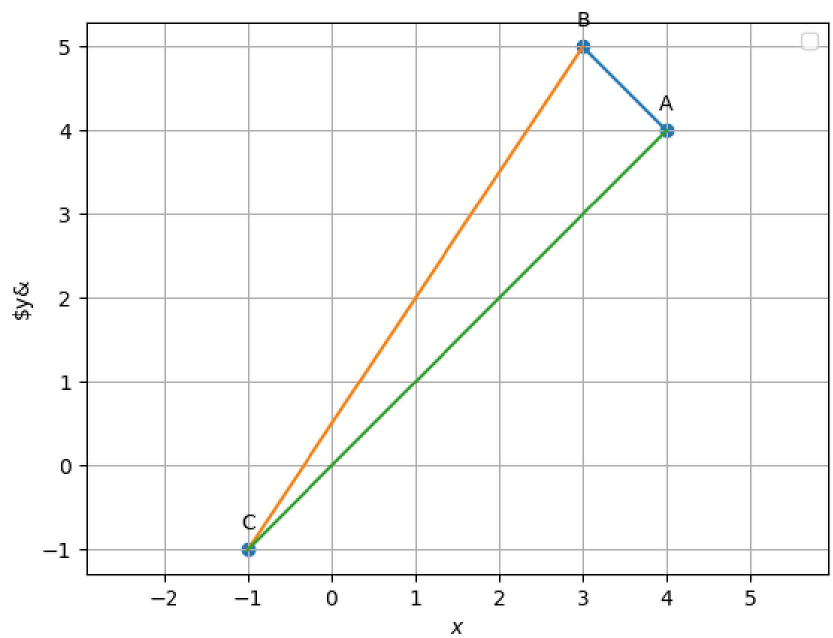


Figure 1.10.16.1:

$$\mathbf{C} - \mathbf{A} = \begin{pmatrix} -5 \\ -5 \end{pmatrix}, \tag{1.10.16.1}$$

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{1.10.16.2}$$

$$\implies (\mathbf{C} - \mathbf{A})^\top (\mathbf{A} - \mathbf{B}) = 0 \tag{1.10.16.3}$$

Thus, $AB \perp AC$.

1.10.17 The line through the points $(h, 3)$ and $(4, 1)$ intersects the line $7x - 9y - 19 = 0$ at right angle. Find the value of h.

Solution: Let

$$\mathbf{A} = \begin{pmatrix} h \\ 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \implies \mathbf{B} - \mathbf{A} = \begin{pmatrix} 4 - h \\ -2 \end{pmatrix} \tag{1.10.17.1}$$

The given line equation can be expressed as

$$\begin{pmatrix} 7 & -9 \end{pmatrix} \mathbf{x} = 19 \tag{1.10.17.2}$$

yielding

$$\mathbf{n} = \begin{pmatrix} 7 \\ -9 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} 9 \\ 7 \end{pmatrix} \tag{1.10.17.3}$$

Thus,

$$\mathbf{m}^\top (\mathbf{B} - \mathbf{A}) = 0 \tag{1.10.17.4}$$

$$\implies \begin{pmatrix} 9 & 7 \end{pmatrix} \begin{pmatrix} 4 - h \\ -2 \end{pmatrix} = 0 \tag{1.10.17.5}$$

$$\implies h = \frac{22}{9} \tag{1.10.17.6}$$

See Fig. 1.10.17.1.

1.10.18 In the following cases, determine whether the given planes are

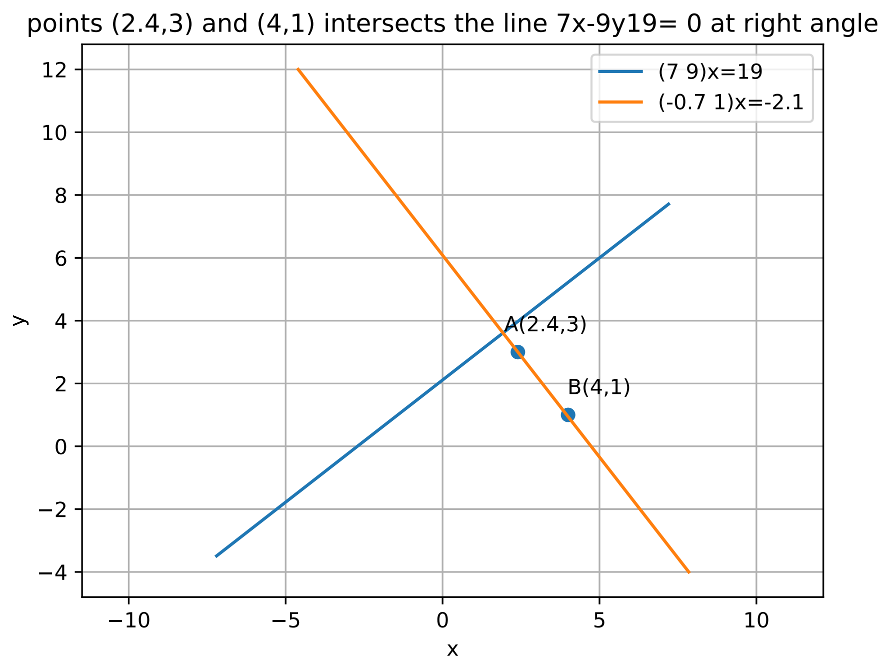


Figure 1.10.17.1:

parallel or perpendicular, and in case they are neither, find the angles between them.

(a) $7x + 5y + 6z + 30 = 0$ and $3x–y–10z + 4 = 0$

(b) $2x + y + 3z–2 = 0$ and $x–2y + 5 = 0$

(c) $2x–2y + 4z + 5 = 0$ and $3x–3y + 6z–1 = 0$

(d) $2x–y + 3z–1 = 0$ and $2x–y + 3z + 3 = 0$

(e) $4x + 8y + z–8 = 0$ and $y + z–4 = 0$

Solution: The angle between the planes is the angle between the normals of the given planes.

$$\mathbf{n_1}^\top \mathbf{x} = c_1,\ \mathbf{n_2}^\top \mathbf{x} = c_2 \tag{1.10.18.1}$$

The angle θ between the planes is given by,

$$\cos\theta = \frac{\mathbf{n_1}^\top \mathbf{n_2}}{\|\mathbf{n_1}\|\,\|\mathbf{n_2}\|} \tag{1.10.18.2}$$

(a)

$$\mathbf{n_1} = \begin{pmatrix} 7 \\ 5 \\ 6 \end{pmatrix},\ \mathbf{n_2} = \begin{pmatrix} 3 \\ -1 \\ -10 \end{pmatrix} \tag{1.10.18.3}$$

$$\implies \mathbf{n_1}^\top \mathbf{n_2} = \begin{pmatrix} 7 & 5 & 6 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \\ -10 \end{pmatrix} = -44 \tag{1.10.18.4}$$

$$\|\mathbf{n_1}\| = \sqrt{7^2 + 5^2 + 6^2} = \sqrt{110} \tag{1.10.18.5}$$

$$\|\mathbf{n_2}\| = \sqrt{3^2 + (-1)^2 + (-10)^2} = \sqrt{110} \tag{1.10.18.6}$$

$$\cos\theta = -\frac{44}{\sqrt{110}\sqrt{110}} = -\frac{2}{5} \tag{1.10.18.7}$$

The planes are inclined at an angle of $\arccos\left(-\frac{2}{5}\right)$ degrees.

(b)

$$\mathbf{n_1} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \mathbf{n_2} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} \tag{1.10.18.8}$$

$$\mathbf{n_1}^\top \mathbf{n_2} = \begin{pmatrix} 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} \tag{1.10.18.9}$$

$$= 0 \tag{1.10.18.10}$$

The planes are perpendicular.

(c)

$$\mathbf{n_1} = \begin{pmatrix} 2 \\ -2 \\ 4 \end{pmatrix}, \mathbf{n_2} = \begin{pmatrix} 3 \\ -3 \\ 6 \end{pmatrix} \tag{1.10.18.11}$$

$$\mathbf{n_1}^\top \mathbf{n_2} = \begin{pmatrix} 2 & -2 & 4 \end{pmatrix} \begin{pmatrix} 3 \\ -3 \\ 6 \end{pmatrix} = 36 \tag{1.10.18.12}$$

$$\|\mathbf{n_1}\| = \sqrt{2^2 + (-2)^2 + 4^2} = \sqrt{24} \tag{1.10.18.13}$$

$$\|\mathbf{n_2}\| = \sqrt{3^2 + (-3)^2 + 6^2} = \sqrt{54} \tag{1.10.18.14}$$

$$\cos\theta = \frac{36}{\sqrt{24}\sqrt{54}} \tag{1.10.18.15}$$

$$= 1 \tag{1.10.18.16}$$

The planes are parallel.

(d)

$$\mathbf{n_1} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}, \mathbf{n_2} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \tag{1.10.18.17}$$

$$\mathbf{n_1}^\top \mathbf{n_2} = \begin{pmatrix} 2 & -1 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = 14 \tag{1.10.18.18}$$

$$\|\mathbf{n_1}\| = \sqrt{2^2 + (-1)^2 + 3^2} \quad = \sqrt{14} \tag{1.10.18.19}$$

$$\|\mathbf{n_2}\| = \sqrt{2^2 + (-1)^2 + 3^2} = \sqrt{14} \tag{1.10.18.20}$$

$$\cos\theta = \frac{14}{\sqrt{14}\sqrt{14}} = 1 \tag{1.10.18.21}$$

The planes are parallel.

(e)

$$\mathbf{n_1} = \begin{pmatrix} 4 \\ 8 \\ 1 \end{pmatrix}, \mathbf{n_2} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \tag{1.10.18.22}$$

$$\mathbf{n_1}^\top \mathbf{n_2} = \begin{pmatrix} 4 & 8 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 9 \tag{1.10.18.23}$$

$$\|\mathbf{n_1}\| = \sqrt{4^2 + 8^2 + 1^2} = 9 \tag{1.10.18.24}$$

$$\|\mathbf{n_2}\| = \sqrt{0^2 + 1^2 + 1^2} = \sqrt{2} \tag{1.10.18.25}$$

$$\cos\theta = \frac{9}{9\sqrt{2}} = \frac{1}{\sqrt{2}} \tag{1.10.18.26}$$

The planes are inclined at an angle of 45 degrees.

1.10.19 Show that the line joining the origin to the point $(2, 1, 1)$ is perpendicular to the line determined by the points $(3, 5, -1), (4, 3, -1)$.

Solution: Let

$$\mathbf{P} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 4 \\ 3 \\ -1 \end{pmatrix} \tag{1.10.19.1}$$

Then

$$\mathbf{m} = \mathbf{A} - \mathbf{B} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} \tag{1.10.19.2}$$

and

$$\mathbf{m}^\top \mathbf{P} = \begin{pmatrix} -1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = 0 \tag{1.10.19.3}$$

This proves the result.

1.10.20 If l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of two mutually perpendicular lines, show that the direction cosines of the line perpendicular to both these are
$m_1n_2 - m_2n_1, n_1l_2 - n_2l_1, l_1m_2 - l_2m_1$.

Solution: Let

$$\mathbf{A} = \begin{pmatrix} l_1 \\ m_1 \\ n_1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} l_2 \\ m_2 \\ n_2 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} m_1n_2 - m_2n_1 \\ n_1l_2 - n_2l_1 \\ l_1m_2 - l_2m_1 \end{pmatrix}. \tag{1.10.20.1}$$

Given that

$$\mathbf{A}^\top\mathbf{B} = \mathbf{0},\ \mathbf{A}^\top\mathbf{A} = \mathbf{1},\ \mathbf{B}^\top\mathbf{B} = \mathbf{1} \tag{1.10.20.2}$$

Let

$$\mathbf{P} = \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} = \begin{pmatrix} l_1 & l_2 & m_1n_2 - m_2n_1 \\ m_1 & m_2 & n_1l_2 - n_2l_1 \\ n_1 & n_2 & l_1m_2 - l_2m_1 \end{pmatrix} \tag{1.10.20.3}$$

Then

$$\mathbf{P}^\top \mathbf{P} = \mathbf{I} \tag{1.10.20.4}$$

Hence, the three vectors are mutually perpendicular.

1.10.21 If the lines $\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}$ and $\frac{x-1}{3k} = \frac{y-1}{1} = \frac{z-6}{-5}$ are perpendicular, find the value of k.

Solution: From the given information,

$$\mathbf{m}_1 = \begin{pmatrix} -3 \\ 2k \\ 2 \end{pmatrix}, \mathbf{m}_2 = \begin{pmatrix} 3k \\ 1 \\ -5 \end{pmatrix} \implies \begin{pmatrix} -3 & 2k & 2 \end{pmatrix}^\top \begin{pmatrix} 3k \\ 1 \\ -5 \end{pmatrix} = 0 \tag{1.10.21.1}$$

$$\implies k = -\frac{10}{7} \tag{1.10.21.2}$$

See Fig.

1.10.22 If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are mutually perpendicular vectors of equal magnitudes, show that the vector **c.d**=15 is equally inclined to $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$.

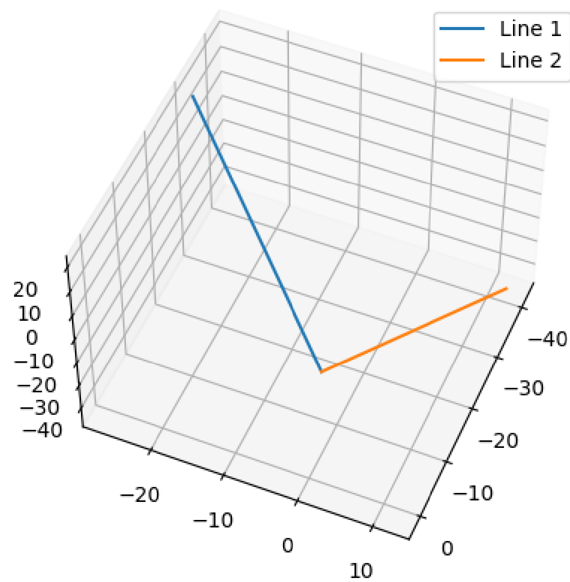


Figure 1.10.21.1: lines represented for the given points and direction vector with k=$\frac{-10}{7}$

1.11. Exercises

1.11.1 The perpendicular bisector of the line segment joining the points $\mathbf{A}(1,5)$ and $\mathbf{B}(4,6)$ cuts the y-axis at

(a) $(0,13)$

(b) $(0,-13)$

(c) $(0,12)$

(d) $(13,0)$

1.11.2 The point which lies on the perpendicular bisector of the line segment joining the points $\mathbf{A}(-2,-5)$ and $\mathbf{B}(2,5)$is

(a) $(0,0)$

(b) $(0,2)$

(c) $(2,0)$

(d) $(-2,0)$

1.11.3 The points $(-4,0),(4,0),(0,3)$are the vertices of

(a) right triangle

(b) isosceles triangle

(c) equilateral triangle

(d) scalene triangle

1.11.4 The point $\mathbf{A}(2,7)$ lies on the perpendicular bisector of line segment joining the points $\mathbf{P}(6,5)$ and $\mathbf{Q}(0,-4)$

1.11.5 The points $\mathbf{A}(-1,-2),\mathbf{B}(4,3),\mathbf{C}(2,5)$ and $\mathbf{D}(-3,0)$ in that order a rectangle

1.11.6 Name the type of triangle formed by the points $\mathbf{A}(-5,6),\mathbf{B}(-4,-2),$ and $\mathbf{C}(7,5)$.

1.11.7 What type of a quadrilateral do the points $\mathbf{A}(2,-2),\mathbf{B}(7,3),\mathbf{C}(11,-1),$ and $\mathbf{D}(6,-6)$ taken in that order,form?

1.11.8 Find the coordinates of the point $\mathbf{Q}$ on the x-axis which lies on the perpendicular bisector of the line segment joining the points $\mathbf{A}(-5,-2)$ and $B(4,-2)$.Name the type of triangle formed by points $\mathbf{Q},\mathbf{A}$ and $\mathbf{B}$.

1.11.9 The points $\mathbf{A}(2,9), \mathbf{B}(a,5)$ and $\mathbf{C}(5,5)$ are the verices of a triangle **ABC** right angled at **B**. Find the values of a and hence the area of $\triangle$**ABC**.

1.11.10 Find a vector of magnitude 6, which is perpendicular to both the vectors $2\hat{\imath} - \hat{\jmath}+2\hat{k}$ and $4\hat{\imath} - \hat{\jmath} + 3\hat{k}$.

1.11.11 If A,B,C,D are the points with position vectors $\hat{\imath} + \hat{\jmath} - \hat{k}$, $2\hat{\imath} - \hat{\jmath} + 3\hat{k}$, $2\hat{\imath} - 3\hat{k}$, $3\hat{\imath}$-$2\hat{\jmath}$+$\hat{k}$, respectively, find the projection of $\overline{AB}$ along $\overline{CD}$.

1.11.12 Find the value of λ such that the vectors $\mathbf{a} = 2\hat{\imath} + \lambda\hat{\jmath} + \hat{k}$ and $\mathbf{b} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$ are orthogonal.

(a) 0

(b) 1

(c) $\frac{3}{2}$

(d) $-\frac{5}{2}$

1.11.13 Projection vector of **a** on **b** is

(a) $\left(\dfrac{\mathbf{a.b}}{|\mathbf{b}|^2}\right)$

(b) $\frac{\mathbf{a.b}}{|\mathbf{b}|}$

(c) $\frac{\mathbf{a.b}}{|\mathbf{a}|}$

(d) $\left(\dfrac{\mathbf{a.b}}{|\mathbf{a}|^2}\right)$

1.11.14 The vectors $\lambda\hat{\imath} + \lambda\hat{\jmath} + 2\hat{k}$, $\hat{\imath} + \lambda\hat{\jmath} - \hat{k}$ and $2\hat{\imath} - \hat{\jmath} + \lambda\hat{k}$ are coplanar if

(a) $\lambda = -2$

(b) $\lambda = 0$

(c) $\lambda = 1$

(d) $\lambda = -1$

1.11.15 The number of vectors of unit length perpendicular to the vectors $\mathbf{a} = 2\hat{i} + \hat{j} + 2\hat{k}$ and $\mathbf{b} = \hat{j} + \hat{k}$ is

(a) one

(b) two

(c) three

(d) infinite

1.11.16 If $\mathbf{r}.\mathbf{a} = 0$, $\mathbf{r}.\mathbf{b} = 0$ and $\mathbf{r}.\mathbf{c} = 0$ for some non-zero vector $\mathbf{r}$, then the value of $\mathbf{a}.(\mathbf{b} \times \mathbf{c})$ is ______.

1.11.17 If $|\mathbf{a} + \mathbf{b}| = |\mathbf{a} - \mathbf{b}|$, then the vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal.

1.11.18 Prove that the lines $x = py + q, z = ry + s$ and $x = p'y + q', z = r'y + s'$ are perpendicular if $pp' + rr' + 1 = 0$.

1.11.19 Find the equation of a plane which bisects perpendicularly the line joining the points A$(2, 3, 4)$ and B$(4, 5, 8)$ at right angles.

1.11.20 $\overrightarrow{AB} = 3\hat{i} - \hat{j} + \hat{k}$ and $\overrightarrow{CD} = -3\hat{i} + 2\hat{j} + 4\hat{k}$ are two vectors. The position vectors of the points A and C are $6\hat{i} + 7\hat{j} + 4\hat{k}$ and $-9\hat{j} + 2\hat{k}$, respectively. Find the position vector of a point P on the line AB and a point Q on the line CD such that $\overrightarrow{PQ}$ is perpendicular to $\overrightarrow{AB}$ and $\overrightarrow{CD}$ both.

1.11.21 Show that the straight lines whose direction cosines are given by $2l + 2m - n = 0$ and $mn + nl + lm = 0$ are at right angles.

1.11.22 If $l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3$ are the direction cosines of the three mutually perpendcular lines, prove that the line whose direction cosines are propotional to $l_1 + l_2 + l_3, m_1 + m_2, m_3, n_1 + n_2 + n_3$ make angles with them.

1.11.23 The intercepts made by the plane $2x - 3y + 5z + 4 = 0$ on the co-ordinate axis are $\left(-2, \frac{4}{3}, -\frac{4}{5}\right)$.

1.11.24 The line $\overrightarrow{r} = 2\hat{i} - 3\hat{j} - \hat{k} + \lambda(\hat{i} - \hat{j} + 2\hat{k})$ lies in the plane $\overrightarrow{r} \cdot (3\hat{i} + \hat{j} - \hat{k}) + 2 = 0$.

1.12. Vector Product

1.12.1 Find $\left|\overrightarrow{a} \times \overrightarrow{b}\right|$, if $\overrightarrow{a} = \hat{i} - 7\hat{j} + 7\hat{k}$ and $\overrightarrow{b} = 3\hat{i} - 2\hat{j} + 2\hat{k}$.

Solution: Since

$$\begin{vmatrix} \mathbf{A}_{23} & \mathbf{B}_{23} \end{vmatrix} = \begin{vmatrix} -7 & -2 \\ 7 & 2 \end{vmatrix} = -14 + 14 = 0 \tag{1.12.1.1}$$

$$\begin{vmatrix} \mathbf{A}_{31} & \mathbf{B}_{31} \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 7 & 2 \end{vmatrix} = 2 - 21 = -19 \tag{1.12.1.2}$$

$$\begin{vmatrix} \mathbf{A}_{12} & \mathbf{B}_{12} \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ -7 & -2 \end{vmatrix} = -2 + 21 = 19, \tag{1.12.1.3}$$

$$\|\mathbf{a} \times \mathbf{b}\| = \begin{pmatrix} \begin{vmatrix} \mathbf{A}_{23} & \mathbf{B}_{23} \\ \mathbf{A}_{31} & \mathbf{B}_{31} \\ \mathbf{A}_{12} & \mathbf{B}_{12} \end{vmatrix} \end{pmatrix} = 19\sqrt{2} \tag{1.12.1.4}$$

1.12.2 Show that

$$(\overrightarrow{a} - \overrightarrow{b}) \times (\overrightarrow{a} + \overrightarrow{b}) = 2(\overrightarrow{a} \times \overrightarrow{b})$$

Solution: Consider

$$\mathbf{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{1.12.2.1}$$

Then

$$\left(\mathbf{a}-\mathbf{b}\right) = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \left(\mathbf{a}+\mathbf{b}\right) = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \tag{1.12.2.2}$$

$$\implies \left(\mathbf{a}-\mathbf{b}\right) \times \left(\mathbf{a}+\mathbf{b}\right) = \begin{vmatrix} -1 & 3 \\ 0 & 2 \end{vmatrix} = -2 \tag{1.12.2.3}$$

and

$$2\left(\mathbf{a}\times\mathbf{b}\right) = 2\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = -2 \tag{1.12.2.4}$$

1.12.3 Find λ and μ if $(2\hat{i}+6\hat{j}+27\hat{k})\times(\hat{i}+\lambda\hat{j}+\mu\hat{k}) = \overrightarrow{0}$.

Solution:

$$\text{Let } \mathbf{A} = \begin{pmatrix} 2 \\ 6 \\ 27 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 1 \\ \lambda \\ \mu \end{pmatrix} \tag{1.12.3.1}$$

$$\tag{1.12.3.2}$$

The cross product or vector product of $\mathbf{A}, \mathbf{B}$ is defined as

$$\mathbf{A}\times\mathbf{B} = \begin{pmatrix} \begin{vmatrix} \mathbf{A}_{23} & \mathbf{B}_{23} \\ \mathbf{A}_{31} & \mathbf{B}_{31} \\ \mathbf{A}_{12} & \mathbf{B}_{12} \end{vmatrix} \end{pmatrix} \tag{1.12.3.3}$$

Hence

$$\begin{vmatrix}\mathbf{A}_{23} & \mathbf{B}_{23}\end{vmatrix} = \begin{vmatrix}6 & \lambda \\ 27 & \mu\end{vmatrix} = 6\mu - 27\lambda \tag{1.12.3.4}$$

$$\begin{vmatrix}\mathbf{A}_{31} & \mathbf{B}_{31}\end{vmatrix} = \begin{vmatrix}27 & \mu \\ 2 & 1\end{vmatrix} = 27 - 2\mu \tag{1.12.3.5}$$

$$\begin{vmatrix}\mathbf{A}_{12} & \mathbf{B}_{12}\end{vmatrix} = \begin{vmatrix}2 & 1 \\ 6 & \lambda\end{vmatrix} = 2\lambda - 6 \tag{1.12.3.6}$$

Substituting the values

$$\mathbf{A} \times \mathbf{B} = \begin{pmatrix}6\mu - 27\lambda \\ 27 - 2\mu \\ 2\lambda - 6\end{pmatrix} \tag{1.12.3.7}$$

Since

$$\mathbf{A} \times \mathbf{B} = \mathbf{0}, \tag{1.12.3.8}$$

$$\begin{pmatrix}6\mu - 27\lambda \\ 27 - 2\mu \\ 2\lambda - 6\end{pmatrix} = \begin{pmatrix}0 \\ 0 \\ 0\end{pmatrix}, \tag{1.12.3.9}$$

which can be represented in matrix form as

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \\ 6 & -27 \end{pmatrix} \begin{pmatrix} \mu \\ \lambda \end{pmatrix} = \begin{pmatrix} 27 \\ 6 \\ 0 \end{pmatrix}. \tag{1.12.3.10}$$

The augmented matrix is given as

$$\left(\begin{array}{cc|c} 2 & 0 & 27 \\ 0 & 2 & 6 \\ 6 & -27 & 0 \end{array}\right) \xleftrightarrow{R_3 \to R_3 - 3R_1} \left(\begin{array}{cc|c} 2 & 0 & 27 \\ 0 & 2 & 6 \\ 0 & -27 & -81 \end{array}\right) \tag{1.12.3.11}$$

$$\xleftrightarrow{R_3 \to R_3 + \frac{27}{2} R_2} \left(\begin{array}{cc|c} 2 & 0 & 27 \\ 0 & 2 & 6 \\ 0 & 0 & 0 \end{array}\right) \tag{1.12.3.12}$$

yielding

$$\mu = 13.5, \lambda = 3 \tag{1.12.3.13}$$

1.12.4 Given that $\overrightarrow{a} \cdot \overrightarrow{b} = 0$ and $\overrightarrow{a} \times \overrightarrow{b} = \overrightarrow{0}$. What can you conclude about the vectors $\overrightarrow{a}$ and $\overrightarrow{b}$?

1.12.5 Let the vectors be given as $\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}$ be given as $a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k},\ b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k},\ c_1\hat{\imath} + c_2\hat{\jmath} + c_3\hat{k}$. Then show that $\overrightarrow{a} \times (\overrightarrow{b} + \overrightarrow{c}) = \overrightarrow{a} \times \overrightarrow{b} + \overrightarrow{a} \times \overrightarrow{c}$.

Solution: ‘ Let,

$$\mathbf{a} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{1.12.5.1}$$

$$\tag{1.12.5.2}$$

Then

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} \tag{1.12.5.3}$$

$$= \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} \tag{1.12.5.4}$$

Similarly,

$$(\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c}) = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} \tag{1.12.5.5}$$

1.12.6 If either $\overrightarrow{a} = \overrightarrow{0}$ or $\overrightarrow{b} = \overrightarrow{0}$, then $\overrightarrow{a} \times \overrightarrow{b} = \overrightarrow{0}$. Is the converse true? Justify your answer with an example.

Solution: Let $\mathbf{a} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$. Here neither of $\mathbf{a}$ or $\mathbf{b}$ is zero.

Since

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} \begin{vmatrix} \mathbf{a}_{23} & \mathbf{b}_{23} \\ \mathbf{a}_{31} & \mathbf{b}_{31} \\ \mathbf{a}_{12} & \mathbf{b}_{12} \end{vmatrix} \end{pmatrix} \tag{1.12.6.1}$$

and

$$\begin{vmatrix} \mathbf{a}_{23} & \mathbf{b}_{23} \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = 0 \tag{1.12.6.2}$$

$$\begin{vmatrix} \mathbf{a}_{31} & \mathbf{b}_{31} \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 1 & 2 \end{vmatrix} = 0 \tag{1.12.6.3}$$

$$\begin{vmatrix} \mathbf{a}_{12} & \mathbf{b}_{12} \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} = 0, \tag{1.12.6.4}$$

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{1.12.6.5}$$

So the converse is false

1.12.7 Find the area of the triangle with vertices $A(1, 1, 2)$, $B(2, 3, 5)$, and $C(1, 5, 5)$

Solution: Since

$$\mathbf{B}-\mathbf{A}=\begin{pmatrix}1\\2\\3\end{pmatrix},\mathbf{C}-\mathbf{A}=\begin{pmatrix}0\\4\\3\end{pmatrix} \tag{1.12.7.1}$$

$$\tag{1.12.7.2}$$

the desired area is given by

$$\frac{1}{2}\left\|\begin{pmatrix}1\\2\\3\end{pmatrix}\times\begin{pmatrix}0\\4\\3\end{pmatrix}\right\|=\frac{1}{2}\left\|\begin{pmatrix}-6\\3\\4\end{pmatrix}\right\|=\frac{\sqrt{61}}{2} \tag{1.12.7.3}$$

1.12.8 Find the area of the parallelogram whose adjacent sides are determined by the vectors $\overrightarrow{a}=\hat{i}-\hat{j}+3\hat{k}$ and $\overrightarrow{b}=2\hat{i}-7\hat{j}+\hat{k}$.

Solution: Let

$$\mathbf{A}=\begin{pmatrix}1\\-1\\3\end{pmatrix} \text{ and } \mathbf{B}=\begin{pmatrix}2\\-7\\1\end{pmatrix}. \tag{1.12.8.1}$$

Then

$$\mathbf{A}\times\mathbf{B}=\begin{pmatrix}20\\5\\-5\end{pmatrix} \tag{1.12.8.2}$$

$$\implies \|\mathbf{A}\times\mathbf{B}\|=15\sqrt{2} \tag{1.12.8.3}$$

1.12.9 Let the vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ be such that $|\overrightarrow{a}| = 3$ and $|\overrightarrow{b}| = \frac{\sqrt{2}}{3}$, then $\overrightarrow{a} \times \overrightarrow{b}$ is a unit vector, if the angle between $\overrightarrow{a}$ and $\overrightarrow{b}$ is

(a) $\frac{\pi}{6}$

(b) $\frac{\pi}{4}$

(c) $\frac{\pi}{3}$

(d) $\frac{\pi}{2}$

Solution: From the given information,

$$\mathbf{a} \times \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \sin\theta = 1 \tag{1.12.9.1}$$

$$\implies \sin\theta = \frac{1}{\|\mathbf{a}\| \, \|\mathbf{b}\|} = \frac{1}{\sqrt{2}} \tag{1.12.9.2}$$

$$\implies \theta = \frac{\pi}{4} \tag{1.12.9.3}$$

1.12.10 Area of a rectangle having vertices A, B, C and D with position vectors $-\hat{i} + \frac{1}{2}\hat{j} + 4\hat{k}, \hat{i} + \frac{1}{2}\hat{j} + 4\hat{k}, \hat{i} - \frac{1}{2}\hat{j} + 4\hat{k}$ and $-\hat{i} - \frac{1}{2}\hat{j} + 4\hat{k}$, respectively is

(a) $\frac{1}{2}$

(b) 1

(c) 2

(d) 4

Solution: Since

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix} \tag{1.12.10.1}$$

$$\mathbf{C} - \mathbf{B} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \tag{1.12.10.2}$$

area of the rectangle is

$$\|(\mathbf{A} - \mathbf{B}) \times (\mathbf{C} - \mathbf{D})\| = 2 \tag{1.12.10.3}$$

See Fig. 1.12.10.1

1.12.11 Find the area of the triangle whose vertices are

(a) $(2, 3), (-1, 0), (2, -4)$

(b) $(-5, -1), (3, -5), (5, 2)$

Solution:

(a) In this case, the area is given by

$$\frac{1}{2} \|(\mathbf{A} - \mathbf{B}) \times (\mathbf{A} - \mathbf{C})\| \tag{1.12.11.1}$$

$$\tag{1.12.11.2}$$

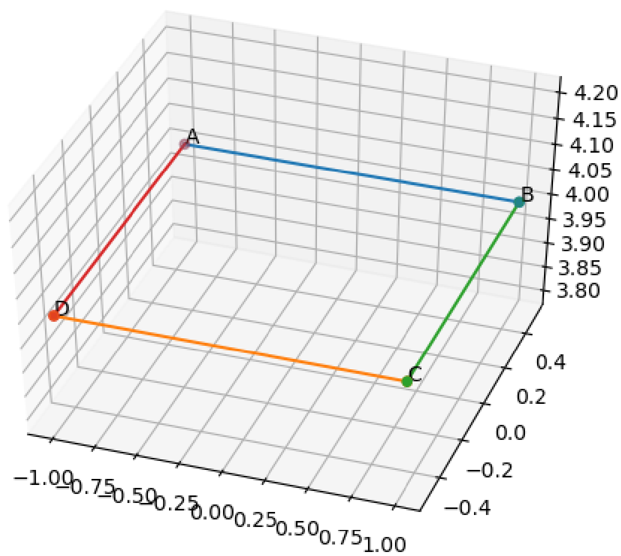


Figure 1.12.10.1:

Since

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} \tag{1.12.11.3}$$

$$\mathbf{A} - \mathbf{C} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ -4 \end{pmatrix} = \begin{pmatrix} 0 \\ 7 \end{pmatrix} \tag{1.12.11.4}$$

the desired area is given by

$$\frac{1}{2}\begin{vmatrix} 3 & 0 \\ 3 & 7 \end{vmatrix} = \frac{21}{2} \tag{1.12.11.5}$$

(b) In this case,

$$\mathbf{A}-\mathbf{B}=\begin{pmatrix}-5\\-1\end{pmatrix}-\begin{pmatrix}3\\-5\end{pmatrix}=\begin{pmatrix}-8\\4\end{pmatrix} \tag{1.12.11.6}$$

$$\mathbf{A}-\mathbf{C}=\begin{pmatrix}-5\\-1\end{pmatrix}-\begin{pmatrix}5\\2\end{pmatrix}=\begin{pmatrix}-10\\-3\end{pmatrix} \tag{1.12.11.7}$$

$$\implies \text{Area}=\frac{1}{2}\begin{vmatrix}-8 & -10\\ 4 & -3\end{vmatrix}=32 \tag{1.12.11.8}$$

1.12.12 Find the area of the triangle formed by joining the mid-points of the sides of the triangle whose vertices are $(0,-1),(2,1)$ and $(0,3)$. Find the ratio of this area to the area of the given triangle.

Solution: The coordinates are given as

$$\mathbf{A}=\begin{pmatrix}0\\-1\end{pmatrix},\mathbf{B}=\begin{pmatrix}2\\1\end{pmatrix},\mathbf{C}=\begin{pmatrix}0\\3\end{pmatrix} \tag{1.12.12.1}$$

Calculating midpoints,

$$\mathbf{P}=\frac{1}{2}(\mathbf{A}+\mathbf{B})=\frac{1}{2}\begin{pmatrix}2\\0\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix} \tag{1.12.12.2}$$

$$\mathbf{Q}=\frac{1}{2}(\mathbf{B}+\mathbf{C})=\frac{1}{2}\begin{pmatrix}2\\4\end{pmatrix}=\begin{pmatrix}1\\2\end{pmatrix} \tag{1.12.12.3}$$

$$\mathbf{R}=\frac{1}{2}(\mathbf{A}+\mathbf{C})=\frac{1}{2}\begin{pmatrix}0\\2\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix} \tag{1.12.12.4}$$

Since

$$\mathbf{P}-\mathbf{Q}=\begin{pmatrix}1\\0\end{pmatrix}-\begin{pmatrix}1\\2\end{pmatrix}=\begin{pmatrix}0\\-2\end{pmatrix} \tag{1.12.12.5}$$

$$\mathbf{Q}-\mathbf{R}=\begin{pmatrix}1\\2\end{pmatrix}-\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix} \tag{1.12.12.6}$$

the area is obtained as

$$ar(PQR)=\frac{1}{2}\|(\mathbf{P}-\mathbf{Q})\times(\mathbf{Q}-\mathbf{R})\| \tag{1.12.12.7}$$

$$=\frac{1}{2}\begin{vmatrix}0 & 1\\-2 & 1\end{vmatrix}=1 \tag{1.12.12.8}$$

Similarly,

$$\mathbf{A}-\mathbf{B}=\begin{pmatrix}0\\-1\end{pmatrix}-\begin{pmatrix}2\\1\end{pmatrix}=\begin{pmatrix}-2\\-2\end{pmatrix} \tag{1.12.12.9}$$

$$\mathbf{A}-\mathbf{C}=\begin{pmatrix}0\\-1\end{pmatrix}-\begin{pmatrix}0\\3\end{pmatrix}=\begin{pmatrix}0\\-4\end{pmatrix} \tag{1.12.12.10}$$

the area is obained as

$$ar(ABC)=\frac{1}{2}\|(\mathbf{A}-\mathbf{B})\times(\mathbf{A}-\mathbf{C})\| \tag{1.12.12.11}$$

$$=\frac{1}{2}\begin{vmatrix}-2 & 0\\-2 & -4\end{vmatrix}=4 \tag{1.12.12.12}$$

Thus, the resultant ratio of two areas is 1:4. See Fig. 1.12.12.1

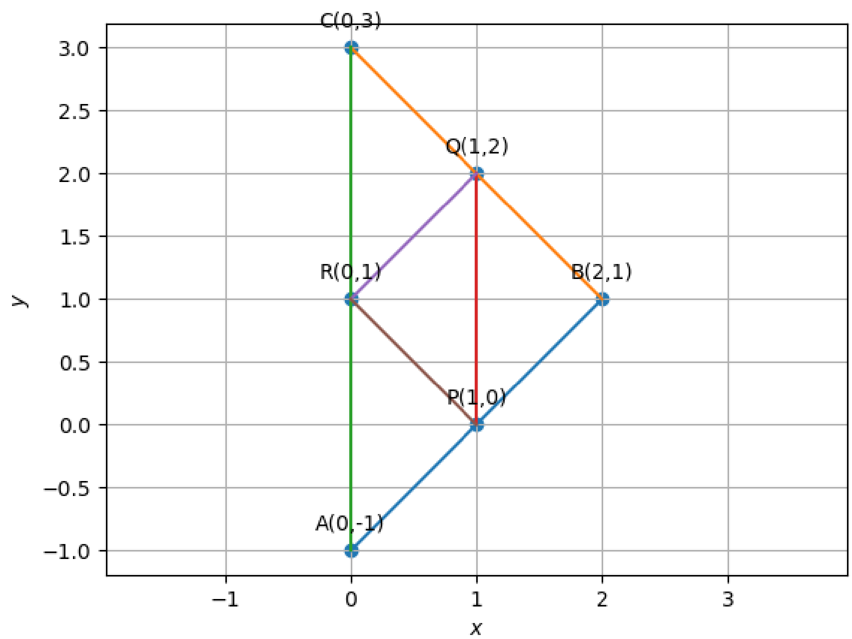


Figure 1.12.12.1:

1.12.13 Find the area of the quadrilateral whose vertices, taken in order, are $(-4,-2),(-3,-5),(3,-2)$ and $(2,3)$.

Solution: The input parameters for this problem are available in Table (1.12.13.1). By joining **B** to **D**, two triangles **ABD** and **BCD** are

Symbol	**Value**	**Description**
$\mathbf{A}$	$\begin{pmatrix}-4\\-2\end{pmatrix}$	First point
$\mathbf{B}$	$\begin{pmatrix}-3\\-5\end{pmatrix}$	Second point
$\mathbf{C}$	$\begin{pmatrix}3\\-2\end{pmatrix}$	Third point
$\mathbf{D}$	$\begin{pmatrix}2\\3\end{pmatrix}$	Fourth point

Table 1.12.13.1:

obtained. Since

$$\mathbf{A}-\mathbf{B}=\begin{pmatrix}-4\\-2\end{pmatrix}-\begin{pmatrix}-3\\-5\end{pmatrix}=\begin{pmatrix}-1\\3\end{pmatrix} \tag{1.12.13.1}$$

$$\mathbf{A}-\mathbf{D}=\begin{pmatrix}-4\\-2\end{pmatrix}-\begin{pmatrix}2\\3\end{pmatrix}=\begin{pmatrix}-6\\-5\end{pmatrix} \tag{1.12.13.2}$$

$$ar(ABD)=\frac{1}{2}\left\|(\mathbf{A}-\mathbf{B})\times(\mathbf{A}-\mathbf{D})\right\| \tag{1.12.13.3}$$

$$=\frac{1}{2}\begin{vmatrix}-1 & 3\\-6 & -5\end{vmatrix}=\frac{23}{2} \tag{1.12.13.4}$$

upon substituting the values of (1.12.13.1) and (1.12.13.2) in (1.12.13.3).

Similarly,

$$\mathbf{B}-\mathbf{C}=\begin{pmatrix}-3\\-5\end{pmatrix}-\begin{pmatrix}3\\-2\end{pmatrix}=\begin{pmatrix}-6\\-5\end{pmatrix} \tag{1.12.13.5}$$

$$\mathbf{B}-\mathbf{D}=\begin{pmatrix}-3\\-5\end{pmatrix}-\begin{pmatrix}2\\3\end{pmatrix}=\begin{pmatrix}-3\\-8\end{pmatrix} \tag{1.12.13.6}$$

yielding

$$ar(BCD)=\frac{1}{2}\left\|(\mathbf{B}-\mathbf{C})\times(\mathbf{B}-\mathbf{D})\right\| \tag{1.12.13.7}$$

$$=\frac{1}{2}\begin{vmatrix}-6 & -3\\-5 & -8\end{vmatrix}=\frac{33}{2} \tag{1.12.13.8}$$

upon substituting the values of (1.12.13.5) and (1.12.13.6) in (1.12.13.7) Thus,

$$ar(ABCD)=ar(ABD)+ar(BCD)=28 \tag{1.12.13.9}$$

See Fig. 1.12.13.1

1.12.14 Verify that a median of a triangle divides it into two triangles of equal areas for $\triangle ABC$ whose vertices are $\mathbf{A}(4,-6),\mathbf{B}(3,2),$ and $\mathbf{C}(5,2)$.

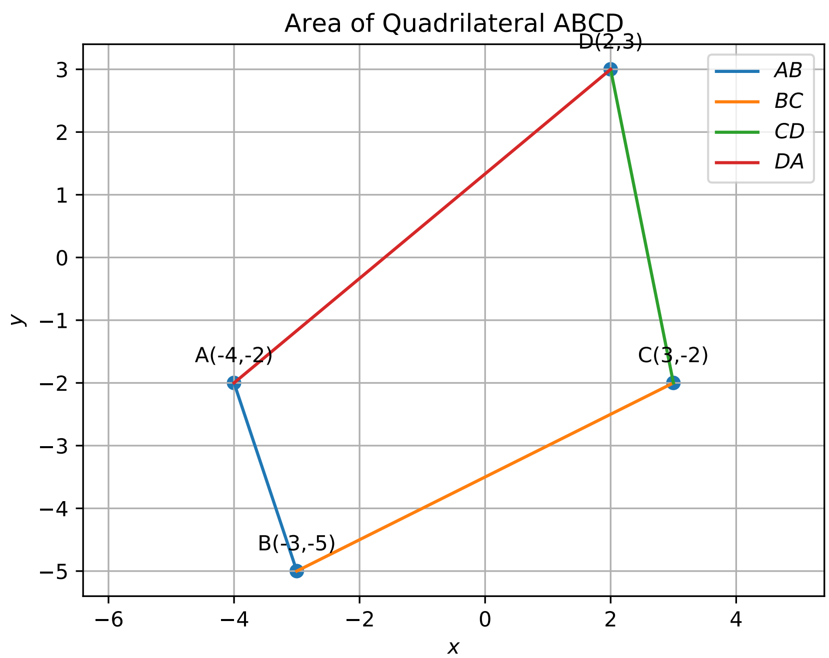


Figure 1.12.13.1:

Solution: The median of the triangle

$$\mathbf{D} = \frac{\mathbf{B}+\mathbf{C}}{2} \tag{1.12.14.1}$$

$$= \begin{pmatrix} 4 \\ 0 \end{pmatrix} \tag{1.12.14.2}$$

Since

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} 4 \\ -6 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \end{pmatrix} \tag{1.12.14.3}$$

$$\mathbf{A} - \mathbf{D} = \begin{pmatrix} 4 \\ -6 \end{pmatrix} - \begin{pmatrix} 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -6 \end{pmatrix} \tag{1.12.14.4}$$

$$ar(ABD) = \frac{1}{2} \left\| (\mathbf{A} - \mathbf{B}) \times (\mathbf{A} - \mathbf{D}) \right\| \tag{1.12.14.5}$$

$$= \frac{1}{2} \begin{vmatrix} 1 & 0 \\ -4 & -6 \end{vmatrix} = 3 \tag{1.12.14.6}$$

upon Substituting from (1.12.14.3) and (1.12.14.4) in (1.12.14.5). Similarly,

$$\mathbf{A} - \mathbf{C} = \begin{pmatrix} 4 \\ -6 \end{pmatrix} - \begin{pmatrix} 5 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -8 \end{pmatrix} \tag{1.12.14.7}$$

$$\mathbf{A} - \mathbf{D} = \begin{pmatrix} 4 \\ -6 \end{pmatrix} - \begin{pmatrix} 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -6 \end{pmatrix} \tag{1.12.14.8}$$

yielding

$$ar(ACD) = \frac{1}{2} \left\| (\mathbf{A} - \mathbf{C}) \times (\mathbf{A} - \mathbf{D}) \right\| \tag{1.12.14.9}$$

$$= \frac{1}{2} \begin{vmatrix} -1 & 0 \\ -8 & -6 \end{vmatrix} = 3 \tag{1.12.14.10}$$

upon substituting from (1.12.14.7) and (1.12.14.8) in (1.12.14.9). Thus,

$$ar(ABD) = ar(ACD) \tag{1.12.14.11}$$

See Fig. 1.12.14.1.

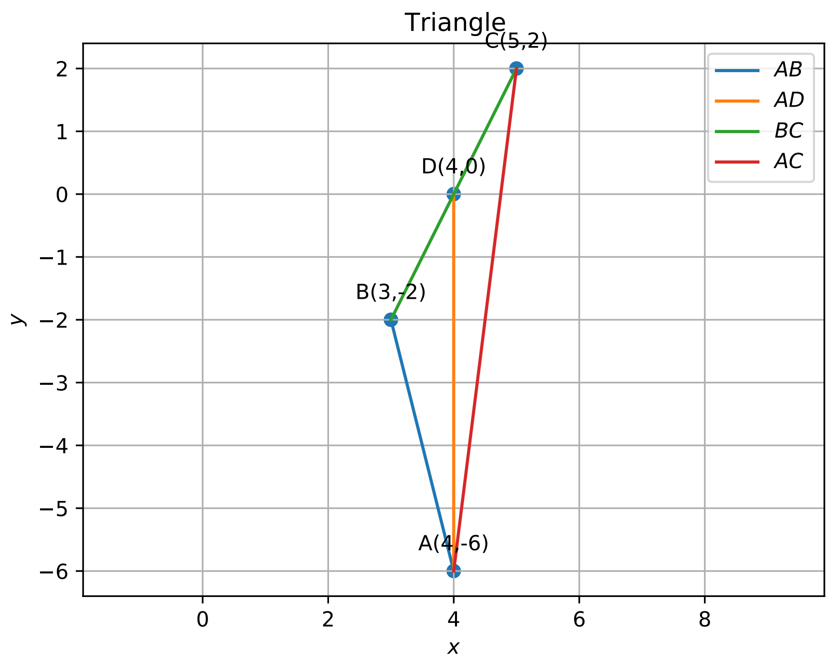


Figure 1.12.14.1:

1.12.15 The two adjacent sides of a parallelogram are $2\hat{i}-4\hat{j}+5\hat{k}$ and $\hat{i}-2\hat{j}-3\hat{k}$. Find the unit vector parallel to its diagonal. Also, find its area.

Solution: Let the sides of the parallelogram be

$$\mathbf{a} = \begin{pmatrix} 2 \\ -4 \\ 5 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}. \tag{1.12.15.1}$$

The diagonals of the parallelogram are given by

$$\mathbf{D}_1 = \mathbf{a} + \mathbf{b} = \begin{pmatrix} 3 \\ -6 \\ 2 \end{pmatrix} \tag{1.12.15.2}$$

$$\mathbf{D}_2 = \mathbf{a} - \mathbf{b} = \begin{pmatrix} 1 \\ -2 \\ 8 \end{pmatrix} \tag{1.12.15.3}$$

The unit vectors parallel to the diagonals are then given by

$$\hat{\mathbf{D}_1} = \frac{\mathbf{D}_1}{\|\mathbf{D}_1\|} = \begin{pmatrix} \frac{3}{\sqrt{45}} \\ -\frac{6}{\sqrt{45}} \\ \frac{2}{\sqrt{45}} \end{pmatrix} \tag{1.12.15.4}$$

$$\hat{\mathbf{D}_2} = \frac{\mathbf{D}_2}{\|\mathbf{D}_2\|} = \begin{pmatrix} \frac{1}{\sqrt{69}} \\ -\frac{2}{\sqrt{69}} \\ \frac{8}{\sqrt{69}} \end{pmatrix} \tag{1.12.15.5}$$

Since

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} 22 \\ -11 \\ 0 \end{pmatrix}, \tag{1.12.15.6}$$

he area of the parallelogram is given by

$$\|\mathbf{a} \times \mathbf{b}\| = \sqrt{605} \tag{1.12.15.7}$$

1.12.16 The vertices of a $\triangle ABC$ are $\mathbf{A}(4,6), \mathbf{B}(1,5)$ and $\mathbf{C}(7,2)$. A line is drawn to intersect sides AB and AC at $\mathbf{D}$ and $\mathbf{E}$ respectively, such that $\frac{AD}{AB} = \frac{AE}{AC} = \frac{1}{4}$. Calculate the area of $\triangle ADE$ and compare it with the area of the $\triangle ABC$.

Solution: The input parameters for this problem are available in Table (1.12.16.1). Given,

Symbol	Value	Description
$\mathbf{A}$	$\begin{pmatrix} 4 \\ 6 \end{pmatrix}$	First point
$\mathbf{B}$	$\begin{pmatrix} 1 \\ 5 \end{pmatrix}$	Second point
$\mathbf{C}$	$\begin{pmatrix} 7 \\ 2 \end{pmatrix}$	Third point
$\mathbf{D}$	?	Desired point
$\mathbf{E}$	?	Desired point

Table 1.12.16.1:

$$\frac{AD}{AB} = \frac{AE}{AC} = \frac{1}{4} \tag{1.12.16.1}$$

Using Section formula,

$$\mathbf{D} = \frac{\mathbf{A} + n\mathbf{B}}{1+n} \tag{1.12.16.2}$$

$$= \begin{pmatrix} \frac{13}{4} \\ \frac{23}{4} \end{pmatrix} \tag{1.12.16.3}$$

substituting $n = \frac{1}{3}$. Similarly,

$$\mathbf{E} = \frac{\mathbf{A} + n\mathbf{C}}{1+n} \qquad = \begin{pmatrix} \frac{19}{4} \\ \frac{20}{4} \end{pmatrix} \tag{1.12.16.4}$$

and

$$\mathbf{A} - \mathbf{D} = \begin{pmatrix} 4 \\ 6 \end{pmatrix} - \begin{pmatrix} \frac{13}{4} \\ \frac{23}{4} \end{pmatrix} = \begin{pmatrix} \frac{3}{4} \\ \frac{1}{4} \end{pmatrix} \tag{1.12.16.5}$$

$$\mathbf{A} - \mathbf{E} = \begin{pmatrix} 4 \\ 6 \end{pmatrix} - \begin{pmatrix} \frac{19}{4} \\ \frac{20}{4} \end{pmatrix} = \begin{pmatrix} \frac{-3}{4} \\ 1 \end{pmatrix} \tag{1.12.16.6}$$

yielding

$$ar(ABD) = \frac{1}{2}\left\|(\mathbf{A}-\mathbf{D}) \times (\mathbf{A}-\mathbf{E})\right\| \tag{1.12.16.7}$$

$$= \frac{1}{2}\begin{vmatrix} \frac{3}{4} & \frac{-3}{4} \\ \frac{1}{4} & 1 \end{vmatrix} \tag{1.12.16.8}$$

$$= \frac{15}{32} \tag{1.12.16.9}$$

Similarly,

$$\mathbf{A}-\mathbf{B} = \begin{pmatrix} 4 \\ 6 \end{pmatrix} - \begin{pmatrix} 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \tag{1.12.16.10}$$

$$\mathbf{B}-\mathbf{C} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} - \begin{pmatrix} 7 \\ 2 \end{pmatrix} = \begin{pmatrix} -6 \\ 3 \end{pmatrix} \tag{1.12.16.11}$$

yielding

$$ar(ABC) = \frac{1}{2}\left\|(\mathbf{A}-\mathbf{B}) \times (\mathbf{B}-\mathbf{C})\right\| \tag{1.12.16.12}$$

$$= \frac{1}{2}\begin{vmatrix} 3 & -6 \\ 1 & 3 \end{vmatrix} \tag{1.12.16.13}$$

$$= \frac{15}{2} \tag{1.12.16.14}$$

Thus,

$$\frac{ar\,(ADE)}{ar\,(ABC)} = \frac{1}{16} \tag{1.12.16.15}$$

See Fig. 1.12.16.1.

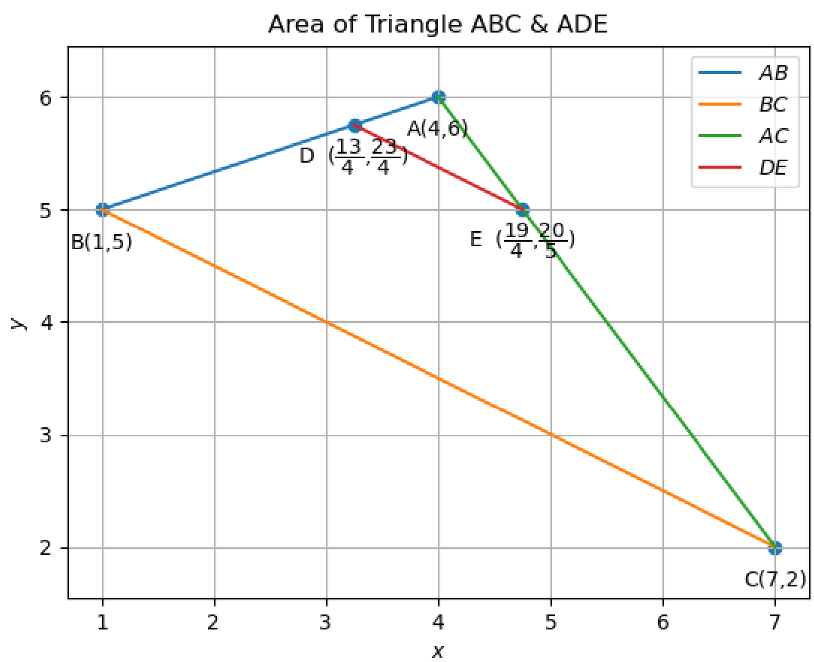


Figure 1.12.16.1:

1.12.17 Draw a quadrilateral in the Cartesian plane, whose vertices are

$$\mathbf{A} = \begin{pmatrix} -4 \\ 5 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 0 \\ 7 \end{pmatrix} \tag{1.12.17.1}$$

$$\mathbf{C} = \begin{pmatrix} 5 \\ -5 \end{pmatrix} \quad \mathbf{D} = \begin{pmatrix} -4 \\ -2 \end{pmatrix} \tag{1.12.17.2}$$

Also, find its area.

Solution: The points are plotted in Fig. 1.12.17.1. The plot is generated using the Python code `codes/quad.py`.

The area vector (denoted by $\mathbf{R_X}$ for region X) of the quadrilateral is perpendicular to the plane of the quadrilateral and its orientation is assumed to be in the positive z-direction here.

$$\mathbf{R_{ABCD}} = \mathbf{R_{ABC}} + \mathbf{R_{ACD}} \tag{1.12.17.3}$$
$$= \frac{1}{2}\left((\mathbf{B}-\mathbf{A}) \times (\mathbf{C}-\mathbf{A}) + (\mathbf{C}-\mathbf{A}) \times (\mathbf{D}-\mathbf{A})\right) \tag{1.12.17.4}$$
$$= \frac{1}{2}\left((\mathbf{C}-\mathbf{A}) \times (\mathbf{D}-\mathbf{A}+\mathbf{A}-\mathbf{B})\right) \tag{1.12.17.5}$$
$$= \frac{1}{2}\left((\mathbf{C}-\mathbf{A}) \times (\mathbf{D}-\mathbf{B})\right) \tag{1.12.17.6}$$
$$\tag{1.12.17.7}$$

Thus the area of quadrilateral ABCD is

$$\operatorname{ar}(ABCD) = \left\|\mathbf{R_{ABCD}}\right\| \tag{1.12.17.8}$$
$$= \frac{1}{2}\left\|(\mathbf{C}-\mathbf{A}) \times (\mathbf{D}-\mathbf{B})\right\| \tag{1.12.17.9}$$
$$= \frac{1}{2}\begin{vmatrix} 9 & -4 \\ -10 & -9 \end{vmatrix} \tag{1.12.17.10}$$
$$= 60.5 \text{ sq. units.} \tag{1.12.17.11}$$

1.12.18 Find the area of region bounded by the triangle whose vertices are $(1,0),(2,2)$ and $(3,1)$.

1.12.19 Find the area of region bounded by the triangle whose vertices are

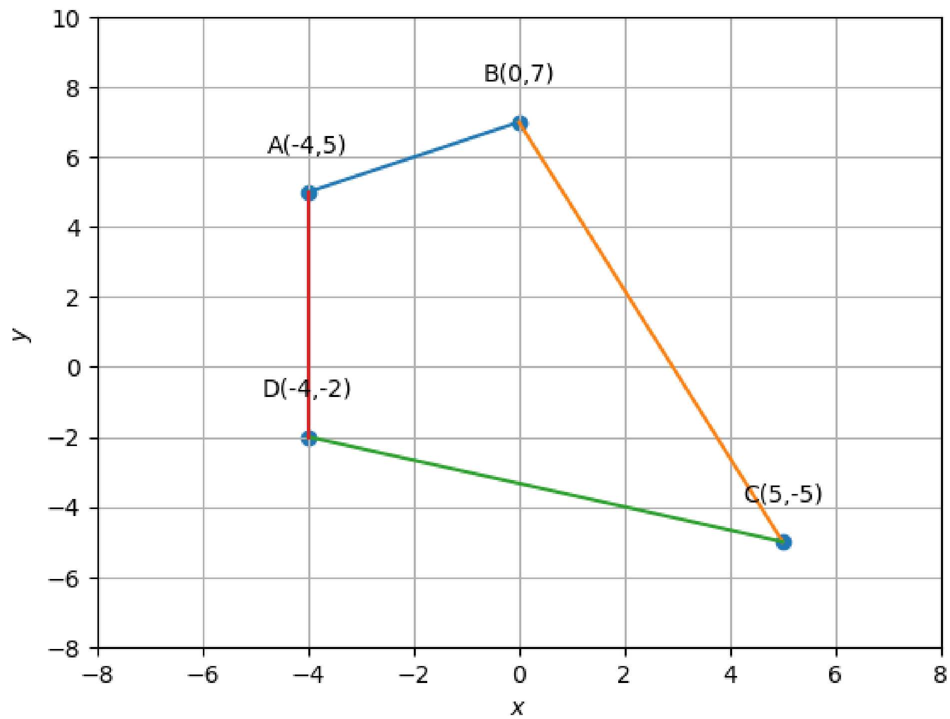


Figure 1.12.17.1: Plot of quadrilateral $ABCD$

$(-1, 0), (1, 3)$ and $(3, 2)$.

1.12.20 Find the area of the $\triangle ABC$, coordinates of whose vertices are $\mathbf{A}(2, 0), \mathbf{B}(4, 5),$ and $\mathbf{C}(6, 3)$.

1.13. Exercises

1.13.1 The area of a triangle with vertices $\mathbf{A}(3, 0), \mathbf{B}(7, 0)$ and $\mathbf{C}(8, 4)$ is

(a) 14

(b) 28

(c) 8

(d) 6

1.13.2 The area of a triangle with vertices $(a, b+c), (b, c+a)$ and $(c, a+b)$ is

(a) $(a+b+c)^2$

(b) 0

(c) a+b+c

(d) abc

1.13.3 Find the area of the triangle whose vertices are $(-8,4), (-6,6)$ and $(-3,9)$.

1.13.4 If $\mathbf{D}\left(\frac{-1}{2}, \frac{5}{2}\right)$, $\mathbf{E}(7,3)$ and $\mathbf{F}\left(\frac{7}{2}, \frac{7}{2}\right)$ are the midpoints of sides of $\triangle\mathbf{ABC}$, find the area of the $\triangle\mathbf{ABC}$.

1.13.5 If $\mathbf{a}+\mathbf{b}+\mathbf{c}$=0, show that $\mathbf{a}\times\mathbf{b}$=$\mathbf{b}\times\mathbf{c}$=$\mathbf{c}\times\mathbf{a}$. Interpret the result geometrically?

1.13.6 Find the sine of the angle between the vectors $\mathbf{a} = 3\hat{\imath} + \hat{\jmath} + 2\hat{k}$ and $\mathbf{b} = 2\hat{\imath} - 2\hat{\jmath} + 4\hat{k}$.

1.13.7 Using vectors, find the area of $\triangle ABC$ with vertices A(1,2,3), B(2,-1,4) and C(4,5,-1).

1.13.8 Using vectors, prove that the parallelogram on the same base and between the same parallels are equal in area.

1.13.9 If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$,determine the vertices of a triangle, show that $\frac{1}{2}\left[\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}+\mathbf{a}\times\mathbf{b}\right]$ gives the vector area of the trianlge. Hence deduce the condition that the three points $\mathbf{a}, \mathbf{b}, \mathbf{c}$, are collinear. Also find the unit vector normal to the plane of the triangle.

1.13.10 Show that area of the parallelogram whose diagonals are given by $\mathbf{a} \times \mathbf{b}$ is $\frac{|\mathbf{a} \times \mathbf{b}|}{2}$. Also find the area of the parallelogram whose diagonals are $2\hat{i} - \hat{j} + \hat{k}$ and $\hat{i} + 3\hat{j} - \hat{k}$.

1.13.11 The vector from origin to the points A and B are $\mathbf{a} = 2\hat{i} - 3\hat{j} + 2\hat{k}$ and $\mathbf{b} = 2\hat{i} + 3\hat{j} + \hat{k}$, respectively, then the area of $\triangle OAB$ is

(a) 340

(b) $\sqrt{25}$

(c) $\sqrt{229}$

(d) $\frac{1}{2}\sqrt{229}$

1.13.12 For any vector $\mathbf{a}$, the value of $(\mathbf{a} \times \hat{i})^2 + (\mathbf{a} \times \hat{j})^2 + (\mathbf{a} \times \hat{k})^2$is equal to

(a) a

(b) 3a

(c) 4a

(d) 2a

1.13.13 If $|\mathbf{a}|$=10, $|\mathbf{b}| = 2$ and $\mathbf{a}$, $\mathbf{b}$=12, then value of $|\mathbf{a} \times \mathbf{b}|$ is

(a) 5

(b) 10

(c) 14

(d) 16

1.13.14 If $\mathbf{a} = \hat{i} + \hat{j} + \hat{k}$ and $\mathbf{b} = \hat{j} - \hat{k}$, find a the vector $\mathbf{c}$ such that $\mathbf{a} \times \mathbf{c} = \mathbf{b}$ and $\mathbf{a.c} = 3$.

1.13.15 The formula $(\mathbf{a}+\mathbf{b}) = \mathbf{a}+\mathbf{b} + 2\mathbf{a} \times \mathbf{b}$ is valid for non-zero vectors $\mathbf{a}$ and $\mathbf{b}$.

1.13.16 The area of the quadrilateral ABCD, where A$(0, 4, 1)$, B$(2, 3, -1)$, C$(4, 5, 0)$ and D$(2, 6, 2)$, is equal to

(a) 9 sq. units

(b) 18 sq. units

(c) 27 sq. units

(d) 81 sq. units

1.13.17 Find the area of region bounded by the triangle whose vertices are (-1, 1), (0, 5) and (3, 2).

1.14. Miscellaneous

1.14.1 Find the sum of the vectors $\mathbf{a} = \hat{\imath} - 2\hat{\jmath} + \hat{k}$, $\mathbf{b} = -2\hat{\imath} + 4\hat{\jmath} + 5\hat{k}$ and $\mathbf{c} = \hat{\imath} - 6\hat{\jmath} - 7\hat{k}$.

1.14.2 In triangle ABC (Fig 10.18), which of the following is not true:

(a) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}=\mathbf{0}$

(b) $\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{CA}=\mathbf{0}$

(c) $\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{CA}=\mathbf{0}$

(d) $\overrightarrow{AB} - \overrightarrow{BC} + \overrightarrow{CA}=\mathbf{0}$

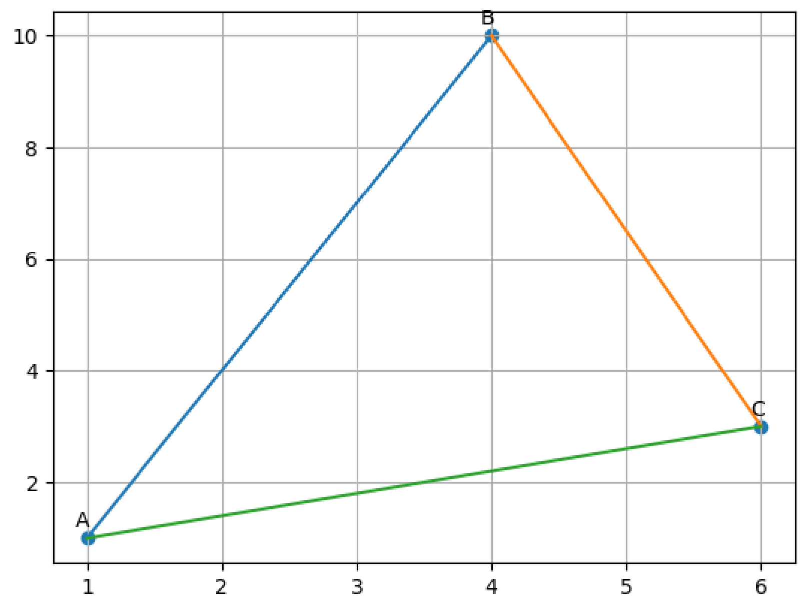


Figure 1.14.2.1:

Solution: We know that,

$$\overrightarrow{AB} = \mathbf{B} - \mathbf{A} \tag{1.14.2.1}$$

$$\overrightarrow{BC} = \mathbf{C} - \mathbf{B} \tag{1.14.2.2}$$

$$\overrightarrow{CA} = \mathbf{A} - \mathbf{C} \tag{1.14.2.3}$$

By usinig this we verify each of the given option

(a)

$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \mathbf{B} - \mathbf{A} + \mathbf{C} - \mathbf{B} + \mathbf{A} - \mathbf{C} \tag{1.14.2.4}$$

$$= 0 \tag{1.14.2.5}$$

Option A is correct.

(b)

$$\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{AC} = \mathbf{B} - \mathbf{A} + \mathbf{C} - \mathbf{B} - (\mathbf{C} - \mathbf{A}) \tag{1.14.2.6}$$

$$= 0 \tag{1.14.2.7}$$

Option B is correct.

(c)

$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{AC} = \mathbf{B} - \mathbf{A} + \mathbf{C} - \mathbf{B} + \mathbf{C} - \mathbf{A} \tag{1.14.2.8}$$

$$= 2(\mathbf{C} - \mathbf{A}) \tag{1.14.2.9}$$

Option C is incorrect.

(d)

$$\overrightarrow{AB} - \overrightarrow{CB} + \overrightarrow{CA} = \mathbf{B} - \mathbf{A} - (\mathbf{B} - \mathbf{C}) + \mathbf{A} - \mathbf{C} \tag{1.14.2.10}$$

$$= 0 \tag{1.14.2.11}$$

Option D is correct.

Verification: Let us take an example to verify

$$\mathbf{A} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 6 \\ 6 \end{pmatrix} \tag{1.14.2.12}$$

$$\overrightarrow{AB} = \mathbf{B} - \mathbf{A} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \overrightarrow{BC} = \mathbf{C} - \mathbf{B} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}, \overrightarrow{CA} = \mathbf{A} - \mathbf{C} = \begin{pmatrix} -5 \\ -5 \end{pmatrix} \tag{1.14.2.13}$$

Thus,

$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \begin{pmatrix} 2+3+(-5) \\ 0+5+(-5) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1.14.2.14}$$

Similarly other options can be verified.

1.14.3 If $\mathbf{a}$ and $\mathbf{b}$ are two collinear vectors, then which of the following are incorrect:

(a) $\mathbf{b} = \lambda\mathbf{a}$, for some scalar λ

(b) $\mathbf{a} = \pm\mathbf{b}$

(c) the respective components of $\mathbf{a}$ and $\mathbf{b}$ are not proportiona

(d) both the vectors $\mathbf{a}$ and $\mathbf{b}$ have same direction, but different magnitudes.

1.14.4 If a line makes angles $90°, 135°, 45°$ with x,y and z-axis respectivly. Find its direction cosines.

Solution: The direction vector is

$$\mathbf{A} = \begin{pmatrix} \cos 90^\circ \\ \cos 135^\circ \\ \cos 45^\circ \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \tag{1.14.4.1}$$

1.14.5 A girl walks 4 km towards west, then she walks 3 km in a direction 30° east of north and stops. Determine the girl's displacement from her initial point of departure.

Solution: Let

$$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{B} - \mathbf{A} = \begin{pmatrix} -4 \\ 0 \end{pmatrix}, \mathbf{C} - \mathbf{B} = \begin{pmatrix} 3\cos 60^\circ \\ 3\sin 60^\circ \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ \frac{3\sqrt{3}}{2} \end{pmatrix} \tag{1.14.5.1}$$

By triangle law of vector addition,

$$\mathbf{C} - \mathbf{A} = \mathbf{B} - \mathbf{A} + \mathbf{C} - \mathbf{B} \tag{1.14.5.2}$$

$$= \begin{pmatrix} -4 \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{3}{2} \\ \frac{3\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} \frac{-5}{2} \\ \frac{3\sqrt{3}}{2} \end{pmatrix} \tag{1.14.5.3}$$

which is the girl's displacement from her initial point of departure See Fig. 1.14.5.1.

1.14.6 If $\mathbf{a} = \hat{i} + \hat{j} + \hat{k}$,$\mathbf{b} = 2\hat{i} - \hat{j} + 3\hat{k}$ and $\mathbf{c} = \hat{i} - 2\hat{j} + \hat{k}$, find a unit vector parallel to the vector $2\mathbf{a} - \mathbf{b} + 3\mathbf{c}$.

Solution:

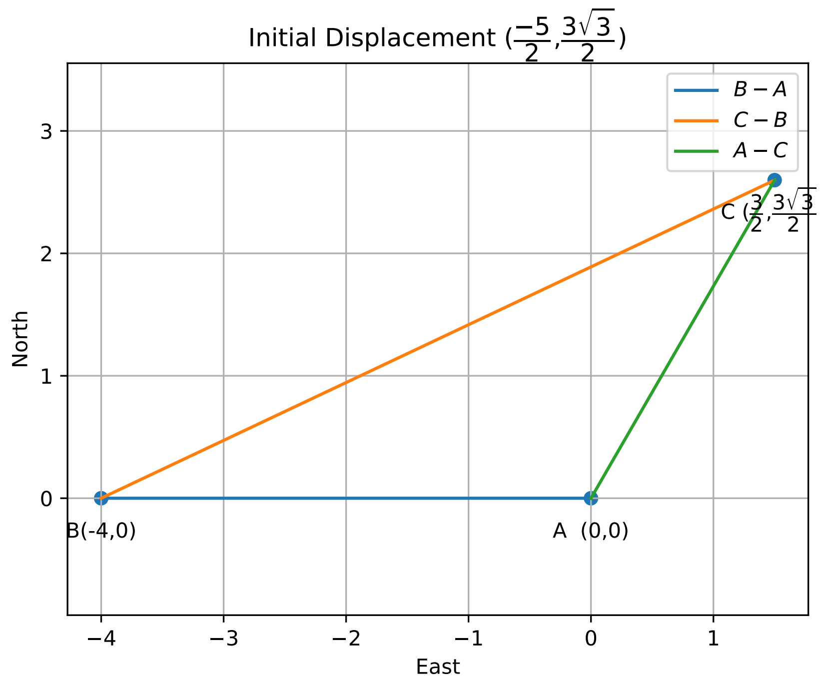


Figure 1.14.5.1:

$$\mathbf{u} = 2\mathbf{a} - \mathbf{b} + 3\mathbf{c} = \begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix} \implies \hat{\mathbf{u}} = \frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{1}{\sqrt{22}} \begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix} \tag{1.14.6.1}$$

1.14.7 The two opposite vertices of a square are $(-1, 2)$ and $(3, 2)$. Find the coordinates of the other two vertices.

Solution: Let

$$\mathbf{A} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \tag{1.14.7.1}$$

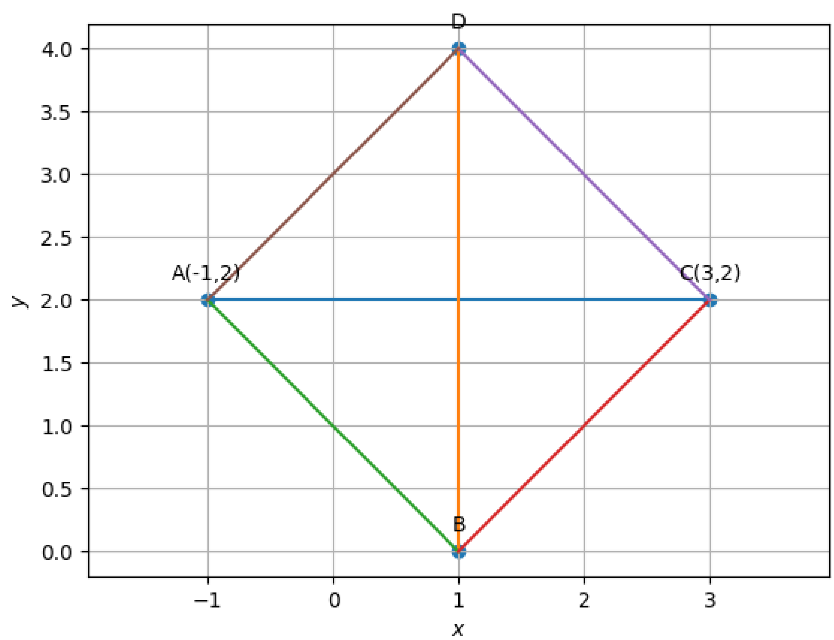


Figure 1.14.7.1:

Shifting $\mathbf{A}$ to origin with reference to Fig. 1.14.7.2,

$$\mathbf{A}' = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C}' = \mathbf{C} - \mathbf{A} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \tag{1.14.7.2}$$

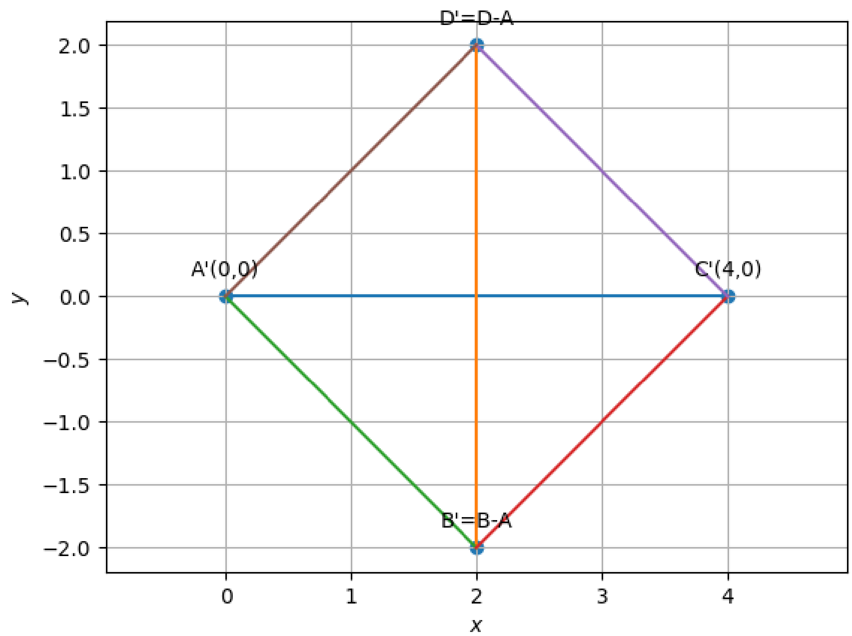


Figure 1.14.7.2:

Since

$$\mathbf{C}-\mathbf{A}=\begin{pmatrix}4\\0\end{pmatrix}\equiv\begin{pmatrix}1\\0\end{pmatrix}, \tan\theta=\frac{0}{4}\implies\theta=0^\circ \tag{1.14.7.3}$$

where θ is the angle made by AC with the x-axis. Considering the rotation matrix

$$\mathbf{P}=\begin{pmatrix}\cos\left(\frac{\pi}{4}-\theta\right) & -\sin\left(\frac{\pi}{4}-\theta\right)\\ \sin\left(\frac{\pi}{4}-\theta\right) & \cos\left(\frac{\pi}{4}-\theta\right)\end{pmatrix} \tag{1.14.7.4}$$

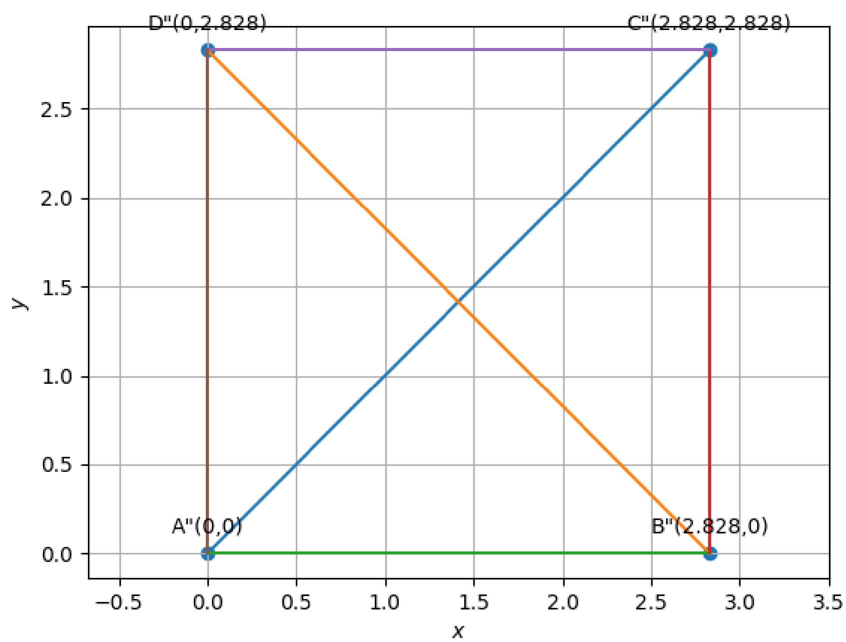


Figure 1.14.7.3:

from Figure 1.14.7.3,

$$\mathbf{C}'' = \mathbf{P}(\mathbf{C} - \mathbf{A}) \tag{1.14.7.5}$$

$$\mathbf{B}'' = \begin{pmatrix} \mathbf{e}_1 & \mathbf{0} \end{pmatrix} \mathbf{C}'' \tag{1.14.7.6}$$

$$\mathbf{D}'' = \begin{pmatrix} \mathbf{0} & \mathbf{e}_2 \end{pmatrix} \mathbf{C}'' \tag{1.14.7.7}$$

Now,

$$\mathbf{B} = \mathbf{P}^\top \mathbf{B}'' + \mathbf{A} \tag{1.14.7.8}$$

$$\mathbf{D} = \mathbf{P}^\top \mathbf{D}'' + \mathbf{A} \tag{1.14.7.9}$$

by reversing the process of translation and rotation. Thus, from (1.14.7.8) (1.14.7.6), (1.14.7.9) and (1.14.7.7)

$$\mathbf{B} = \mathbf{P}^\top \begin{pmatrix} \mathbf{e}_1 & \mathbf{0} \end{pmatrix} \mathbf{P}(\mathbf{C} - \mathbf{A}) + \mathbf{A} \tag{1.14.7.10}$$

$$\mathbf{D} = \mathbf{P}^\top \begin{pmatrix} \mathbf{0} & \mathbf{e}_2 \end{pmatrix} \mathbf{P}(\mathbf{C} - \mathbf{A}) + \mathbf{A} \tag{1.14.7.11}$$

yielding

$$\mathbf{B} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{D} \begin{pmatrix} 1 \\ 4 \end{pmatrix}. \tag{1.14.7.12}$$

1.14.8 The base of an equilateral triangle with side $2a$ lies along the y-axis such that the mid-point of the base is at the origin. Find vertices of the triangle.

Solution: Let the base be BC. From the given information,

$$\mathbf{B} = a\mathbf{e}_2, \mathbf{C} = -a\mathbf{e}_2 \tag{1.14.8.1}$$

Since $\mathbf{A}$ lies on the x-axis,

$$\mathbf{A} = k\mathbf{e}_1 \tag{1.14.8.2}$$

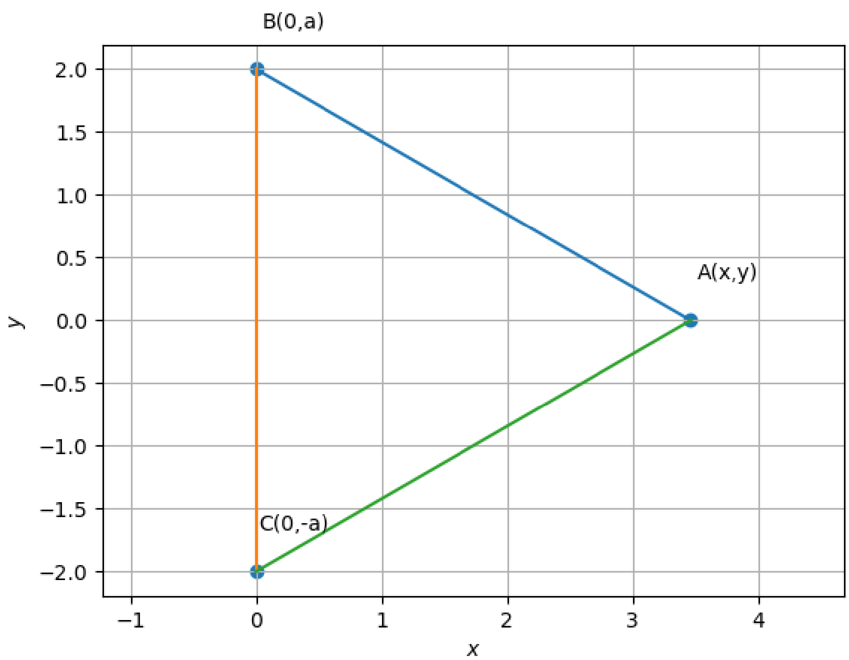


Figure 1.14.8.1:

and

$$\|\mathbf{A}-\mathbf{C}\|^2 = (2a)^2 \tag{1.14.8.3}$$

$$\implies \|\mathbf{A}\|^2 + \|\mathbf{C}\|^2 - 2\mathbf{A}^\top\mathbf{C} = 4a^2 \tag{1.14.8.4}$$

$$\implies k^2 + a^2 = 4a^2 \tag{1.14.8.5}$$

$$\text{or, } k = \pm a\sqrt{3} \tag{1.14.8.6}$$

Thus,

$$\mathbf{A} = \pm\sqrt{3}a\mathbf{e}_1 \tag{1.14.8.7}$$

Fig. 1.14.8.1 is plotted for $a = 2$.

1.14.9 Without using distance formula, show that points (– 2, – 1), (4, 0), (3, 3) and (–3, 2) are the vertices of a parallelogram.

To show that the given points are the vertices of a parallelogram, we need to show the following

(a) $\mathbf{A} - \mathbf{B} = \mathbf{D} - \mathbf{C}$

(b) $\mathbf{A} - \mathbf{D} = \mathbf{B} - \mathbf{C}$

So,

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} -2 \\ -1 \end{pmatrix} - \begin{pmatrix} 4 \\ 0 \end{pmatrix} \tag{1.14.9.1}$$

$$= \begin{pmatrix} -6 \\ -1 \end{pmatrix} \tag{1.14.9.2}$$

$$\mathbf{D} - \mathbf{C} = \begin{pmatrix} -3 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 3 \end{pmatrix} \tag{1.14.9.3}$$

$$= \begin{pmatrix} -6 \\ -1 \end{pmatrix} \tag{1.14.9.4}$$

$$\mathbf{A} - \mathbf{D} = \begin{pmatrix} -2 \\ -1 \end{pmatrix} - \begin{pmatrix} -3 \\ 2 \end{pmatrix} \tag{1.14.9.5}$$

$$= \begin{pmatrix} 1 \\ -3 \end{pmatrix} \tag{1.14.9.6}$$

$$\mathbf{B} - \mathbf{C} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} - \begin{pmatrix} 3 \\ 3 \end{pmatrix} \tag{1.14.9.7}$$

$$= \begin{pmatrix} 1 \\ -3 \end{pmatrix} \tag{1.14.9.8}$$

Therefore, we can now say that the given points are the vertices of a parallelogram since both the sides of the quadrilateral are parallel to each other. See Fig. 1.14.9.1.

1.14.10 A line passes through (x_1, y_1) and (h, k). If slope of the line is m show

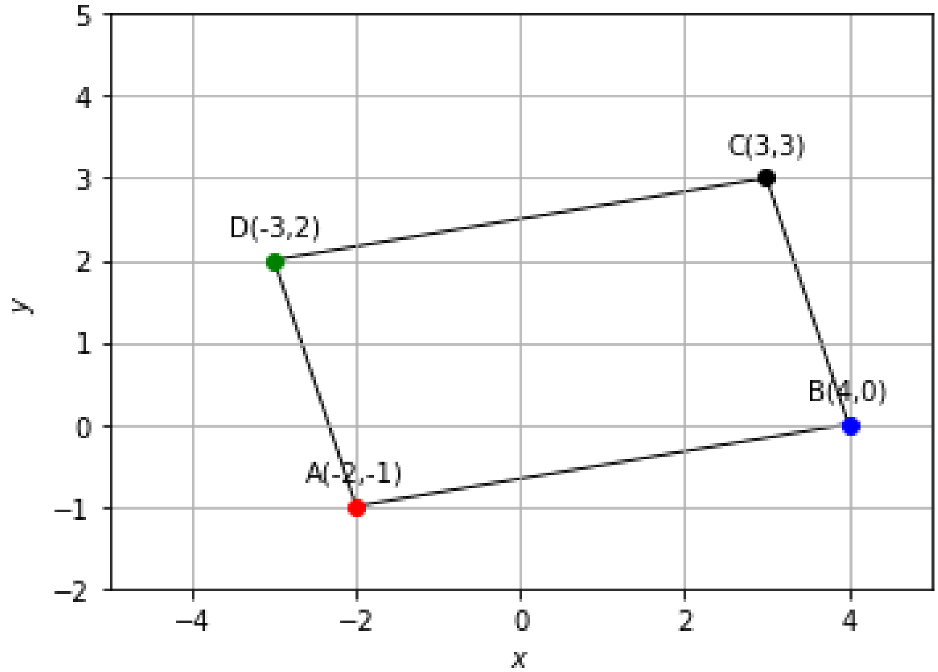


Figure 1.14.9.1:

that $(k - y_1) = m(h - x_1)$.

Solution: Given

$$\mathbf{A} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} h \\ k \end{pmatrix} \tag{1.14.10.1}$$

The direction vector

$$\mathbf{m} = \mathbf{B} - \mathbf{A} \tag{1.14.10.2}$$

$$= \begin{pmatrix} h - x_1 \\ k - y_1 \end{pmatrix} \equiv \begin{pmatrix} 1 \\ \frac{k-y_1}{h-x_1} \end{pmatrix} \tag{1.14.10.3}$$

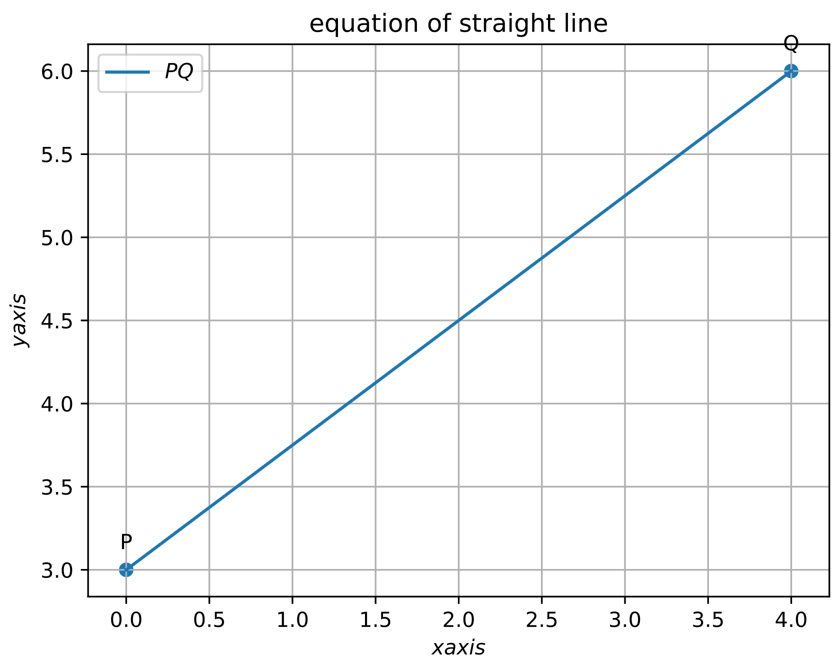


Figure 1.14.10.1:

which yields the desired relation from (A.1.18.1).

1.14.11 Consider the following population and year graph, Find the slope of the line AB and using it, find what will be the population in the year 2010?

Solution:

Given the points

$$\mathbf{A}\begin{pmatrix}1985\\92\end{pmatrix},\mathbf{B}\begin{pmatrix}1995\\97\end{pmatrix} \tag{1.14.11.1}$$

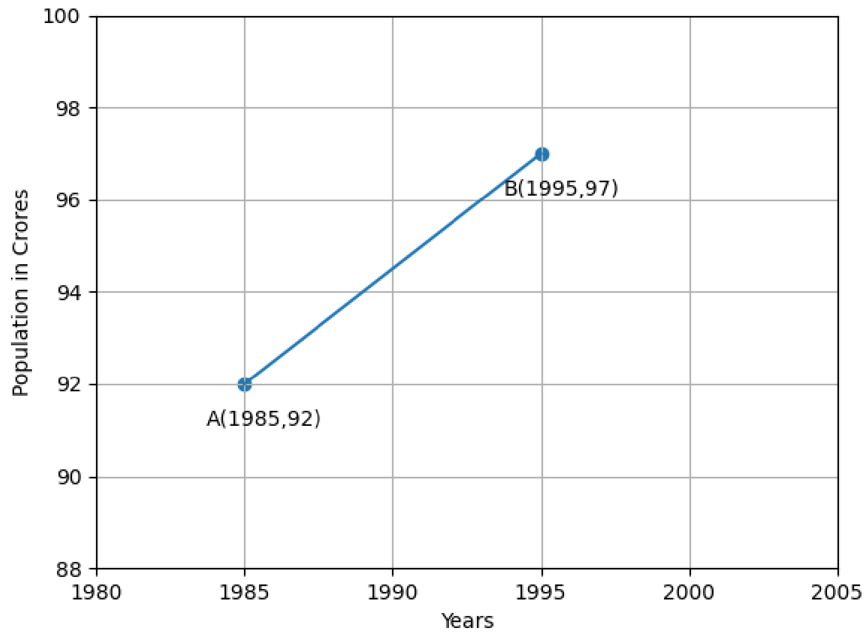


Figure 1.14.11.1:

The direction vector is given by,

$$\mathbf{m} = \mathbf{B} - \mathbf{A} \tag{1.14.11.2}$$

$$= \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{1.14.11.3}$$

The slope of the line is then obtained as

$$m = \frac{1}{2} \tag{1.14.11.4}$$

The normal of the line is given by,

$$\mathbf{n} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \tag{1.14.11.5}$$

Any point, $\mathbf{x}$ on the line can be written as

$$\mathbf{n}^\top(\mathbf{x} - \mathbf{A}) = 0 \tag{1.14.11.6}$$

$$\implies \begin{pmatrix} 1 & -2 \end{pmatrix} \mathbf{x} = 1801 \tag{1.14.11.7}$$

We need to find the population in the year 2010. For

$$\mathbf{x} = \begin{pmatrix} 2010 \\ y \end{pmatrix} \tag{1.14.11.8}$$

$$\begin{pmatrix} 1 & -2 \end{pmatrix} \begin{pmatrix} 2010 \\ y \end{pmatrix} = 1801 \tag{1.14.11.9}$$

$$\implies y = \frac{209}{2} \tag{1.14.11.10}$$

1.14.12 Find a vector of magnitude 5 units, and parallel to the resultant of the vectors $\mathbf{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\mathbf{b} = \hat{i} - 2\hat{j} + \hat{k}$.

1.15. Exercises

1.15.1 The fourth vertex **D** of a parallelogram **ABCD** whose three vertices are $\mathbf{A}(-2,3), \mathbf{B}(6,7)$ and $\mathbf{C}(8,3)$ is

(a) $(0,1)$

(b) $(0,-1)$

(c) $(-1,0)$

(d) $(1,0)$

1.15.2 Points $\mathbf{A}(4,3), \mathbf{B}(6,4), c(5,-6)$ and $\mathbf{D}(-3,5)$ are the vertices of a parallelogram

1.16. Triangle

1.16.1 Construct a triangle ABC in which $BC = 7cm, \angle B = 75°$ and $AB + AC = 13cm$.

1.16.2 Construct a triangle ABC in which $BC = 8cm, \angle B = 45°$ and $AB - AC = 3.5cm$.

1.16.3 Construct a triangle PQR in which $QR = 6cm, \angle Q = 60°$ and $PR - PQ = 2cm$.

1.16.4 Construct a triangle XYZ in which $\angle Y = 30°, \angle Z = 90°$ and $XY + YZ + ZX = 11cm$.

1.16.5 Construct a right triangle whose base is 12cm and sum of its hypotenuse and other side is 18cm.

1.16.6 In Fig. 1.16.6.1, $AC = AE, AB = AD$ and $\angle BAD = \angle EAC$. Show that $BC = DE$.

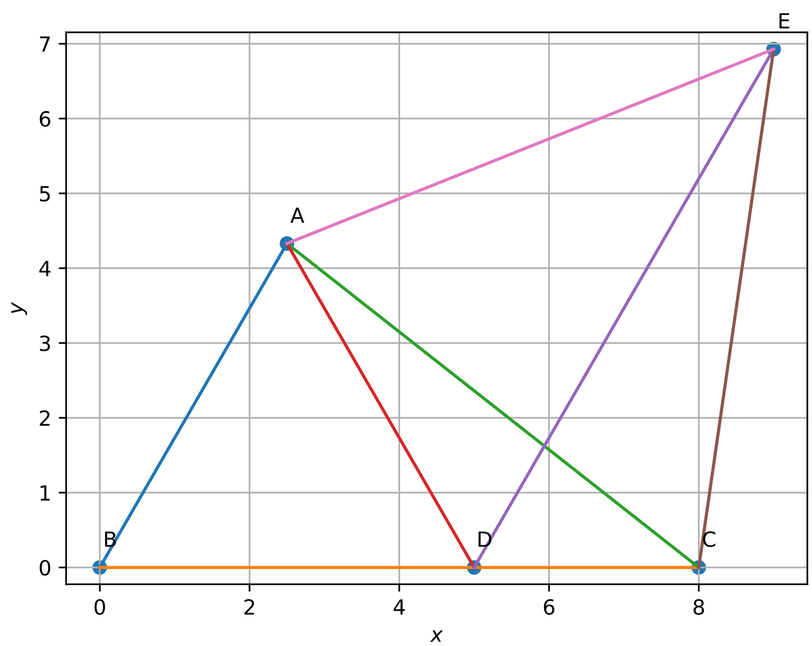


Figure 1.16.6.1:

Solution: The input parameters are available in Table 1.16.6. The

Symbol	**Values**	**Description**
θ	$60°$	$\angle BAD = \angle EAC$
a	8	BC
c	5	AB
$\mathbf{e_1}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	basis vector

vertices of $\triangle ABC$ are given by

$$\mathbf{A} = c\begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = a\mathbf{e}_1, \tag{1.16.6.1}$$

and

$$\mathbf{D} = \left(2c\sin\frac{\theta}{2}\right)\mathbf{e_1} \tag{1.16.6.2}$$

Then, using the cosine formula and the fact that $\triangle AEC$ is iscosceles,

$$AC = b = \sqrt{a^2 + c^2 - 2ac\cos\theta},\ EC = 2b\sin\frac{\theta}{2} \tag{1.16.6.3}$$

Also,

$$\angle BCA = \cos^{-1}\left(\frac{a^2 + b^2 - c^2}{2ab}\right), \angle ACE = 90^\circ - \frac{\theta}{2} \tag{1.16.6.4}$$

Let ϕ be the angle made by the vector EC with the x-axis. Then,

$$\phi = 180^\circ - (\angle BCA + \angle ACE) == 90^\circ - \frac{\theta}{2}\cos^{-1}\left(\frac{a^2 + b^2 - c^2}{2ab}\right) \tag{1.16.6.5}$$

Consequently, Using vector addition,

$$\mathbf{E} - \mathbf{C} = \left(2b \sin \frac{\theta}{2}\right) \begin{pmatrix} \cos \phi \\ \sin \phi \end{pmatrix} \tag{1.16.6.6}$$

$$\implies \mathbf{E} = \mathbf{C} + 2b \sin \frac{\theta}{2} \begin{pmatrix} \cos \phi \\ \sin \phi \end{pmatrix} \tag{1.16.6.7}$$

$$\tag{1.16.6.8}$$

Substituting numerical values,

$$\|\mathbf{B} - \mathbf{C}\| = \|\mathbf{D} - \mathbf{E}\| \tag{1.16.6.9}$$

1.16.7 AB is a line segment and $\mathbf{P}$ is its mid-point. $\mathbf{D}$ and $\mathbf{E}$ are points on the same side of AB such that $\angle BAD = \angle ABE$ and $\angle EPA = \angle DPB$. Show that

(a) $\triangle DAP \cong \triangle EBP$

(b) $AD = BE$.

Solution: The input parameters for this construction are available in Table 1.16.7.2. Let

Symbol	**Value**	**Description**
c	8	AB
b	8.2	AD
θ	$35°$	$\angle BAD$

Table 1.16.7.2:

$$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{e_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{B} = c\mathbf{e_1} \tag{1.16.7.1}$$

From the given information,

$$\mathbf{P} = \frac{\mathbf{A}+\mathbf{B}}{2}, \mathbf{D} = b \begin{pmatrix} \cos A \\ \sin A \end{pmatrix} \tag{1.16.7.2}$$

$$\tag{1.16.7.3}$$

Then, letting

$$\mathbf{Q} - \mathbf{A} = \mathbf{E} - \mathbf{B}, \tag{1.16.7.4}$$

and assuming that $AD = BE$,

$$\mathbf{Q} = b \begin{pmatrix} \cos(180 - A) \\ \sin(180 - A) \end{pmatrix} = b \begin{pmatrix} -\cos A \\ \sin A \end{pmatrix}, \tag{1.16.7.5}$$

$$\implies \mathbf{E} = \mathbf{B} + \mathbf{Q} - \mathbf{A} = c\mathbf{e_1} + b \begin{pmatrix} -\cos A \\ \sin A \end{pmatrix} \tag{1.16.7.6}$$

Substituting numerical values,

$$\mathbf{m_1} = \mathbf{D} - \mathbf{P} = \begin{pmatrix} 2.7 \\ 4.7 \end{pmatrix}, \mathbf{m_2} = \mathbf{B} - \mathbf{P} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \tag{1.16.7.7}$$

$$\implies \theta_1 = \cos^{-1} \frac{\mathbf{m_1}^\top \mathbf{m_2}}{\|\mathbf{m_1}\| \|\mathbf{m_2}\|} = 60^\circ \tag{1.16.7.8}$$

$$\tag{1.16.7.9}$$

and

$$\mathbf{n_1} = \mathbf{E} - \mathbf{P} = \begin{pmatrix} -2.7 \\ 4.7 \end{pmatrix}, \mathbf{n_2} = \mathbf{A} - \mathbf{P} = \begin{pmatrix} -4 \\ 0 \end{pmatrix} \tag{1.16.7.10}$$

$$\implies \theta_2 = \cos^{-1} \frac{\mathbf{n_1}^\top \mathbf{n_2}}{\|\mathbf{n_1}\| \|\mathbf{n_2}\|} \tag{1.16.7.11}$$

$$= 60^\circ \tag{1.16.7.12}$$

Thus,

$$\angle EPA = \angle DPB \tag{1.16.7.13}$$

See Fig. 1.16.7.1.

1.16.8 In right triangle ABC, right angled at C, M is the mid-point of hypotenuse AB. C is joined to M and produced to a point D such that DM = CM. Point D is joined to point B (see Figure 1.16.8.1). Show that:

(a) $\triangle AMC \cong \triangle BMD$

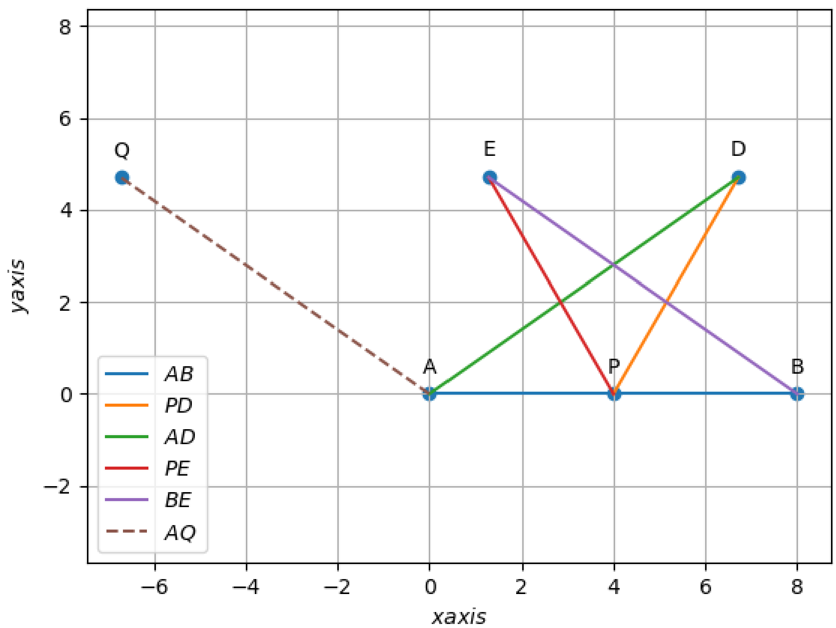


Figure 1.16.7.1:

(b) $\angle DBC$ is a right angle.

(c) $\triangle DBC \cong \triangle ACB$

(d) $CM = \frac{1}{2}AB$

The input parameters for construction are available in Table 1.16.8.2.

Thus,

Symbol	Values	Description
a	4	CB
b	3	AC

Table 1.16.8.2:

$$\mathbf{A} = \begin{pmatrix} 0 \\ b \end{pmatrix}, \mathbf{B} = \begin{pmatrix} a \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1.16.8.1}$$

Thus,

$$(\mathbf{D}-\mathbf{B})^{\top}(\mathbf{B}-\mathbf{C}) = \begin{pmatrix} 0 & b \end{pmatrix}\begin{pmatrix} a \\ 0 \end{pmatrix} = 0 \tag{1.16.8.5}$$

$$\implies BD \perp BC \tag{1.16.8.6}$$

$$\tag{1.16.8.7}$$

Also,

$$\left\|\mathbf{A}-\mathbf{B}\right\| = \left\|\begin{pmatrix} -a \\ b \end{pmatrix}\right\| \tag{1.16.8.8}$$

$$\left\|\mathbf{C}-\mathbf{D}\right\| = \left\|\begin{pmatrix} -a \\ -b \end{pmatrix}\right\| \tag{1.16.8.9}$$

$$\implies \left\|\mathbf{A}-\mathbf{B}\right\| = \left\|\mathbf{C}-\mathbf{D}\right\| \tag{1.16.8.10}$$

$$\text{or, } AB = CD \tag{1.16.8.11}$$

From (1.16.8.11)

$$\implies CM = \frac{1}{2}CD = \frac{1}{2}AB \tag{1.16.8.12}$$

See Fig. 1.16.8.1.

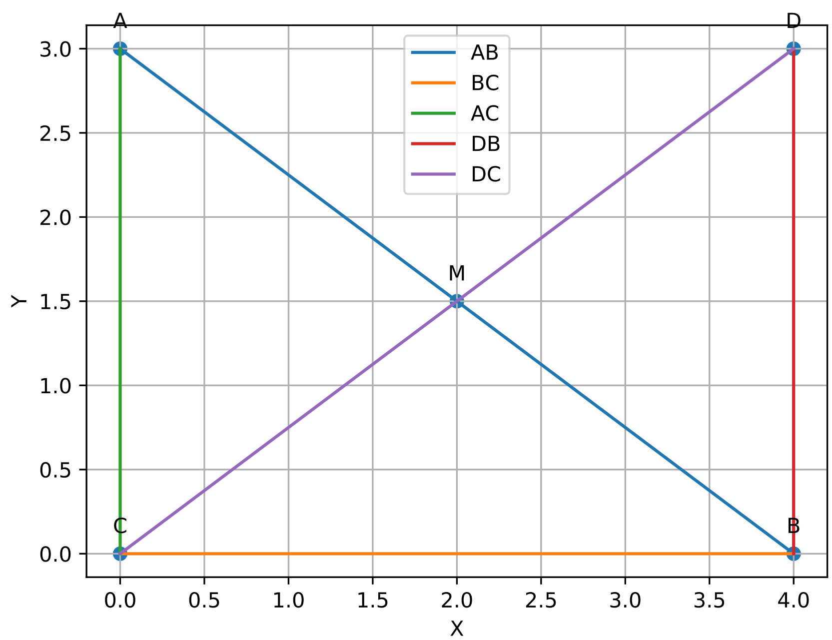


Figure 1.16.8.1:

1.17. Exercises

1.17.1 Draw a right triangle ABC in which BC = 12 cm, AB = 5 cm and $\angle B = 90°$.

1.17.2 Draw an isosceles triangle ABC in which AB=AC=6 cm and BC =6 cm.

1.17.3 Draw a triangle ABC in which AB=5 cm. $BC = 6cm$ and $\angle ABC = 60°$.

1.17.4 Draw a triangle ABC in which AB=4 cm, $BC = 6cm$ and $AC = 9$

1.17.5 Draw a triangle ABC in which $BC = 6$ cm, $CA = 5$ cm and $AB = 4$ cm.

1.17.6 Draw a parallelogram $ABCD$ in which $BC = 5$ cm, $AB = 3$ cm and $\angle ABC = 60°$, divide it into triangles ACB and ABD by the diagonal BD.

1.17.7 In triangles ABC and PQR, $\angle A = \angle Q$ and $\angle B = \angle R$.Which side of $\triangle PQR$ should be equal to side AB of $\triangle ABC$ so that the two triangles are congruent?Give reason for your answer.

1.17.8 In triangles ABC and PQR, $\angle A = \angle Q$ and $\angle B = \angle R$.Which side of $\triangle PQR$should be equal to side BC of $\triangle ABC$ so that the two triangles are congruent?Give reason for your answer.

1.17.9 "If two sides and an angle of one triangle are equal to two sides and an angle ofanother triangle,then two triangles must be congruent."Is the statement true?Why?

1.17.10 "If two angles and a side one triangle are equal to two angles and a side of another triangle,then the two triangles must be congruent."Is the statement true?Why?

1.17.11 Is it possible to construct a triangle with lengths of its sides as 4 cm, 3 cm and 7 cm?Give reason for your answer.

1.17.12 It is given that $\triangle ABC \cong \triangle RPQ$.Is it true to say that BC = QR?Why?

1.17.13 If $\triangle PQR \cong \triangle EDF$,then is it true to say that PR = EF?Give reason for your answer.

1.17.14 In $\triangle PQR$,$\angle P = 70°$ and $\angle R = 30°$.Which side of the triangle is the longest?Give reason for your answer.

1.17.15 AD is a median of the triangle ABC.Is it true that AB + BC + CA > 2AD?Give reason for your answer.

1.17.16 M is a point on side BC of a triangle ABC such that AM is the bisector of $\angle BAC$.Is it true to say that perimeter of the triangle is greater than 2AM?Give reason for your answer.

1.17.17 Is it possible to construct a triangle with lengths of its sides as 9 cm, 7 cm and 17 cm?Give reason for your answer.

1.17.18 Is it possible to cosntruct a triangle with lengths of its sides as 8 cm, 7 cm and 4 cm?Give reason for your answer.

1.17.19 **ABC** is an isosceles triangle with **AB** = **AC** and **BD** and **CE** are its two medians. Show that **BD** = **CE**.

1.17.20 In Fig. 1.17.20.1, **D** and **E** are the points on side **BC** of a $\triangle$**ABC** such that **BD** = **CE** and **AD** = **AE**. Show that $\triangle$**ABD** $\cong$ $\triangle$**ACE**.

1.17.21 **CDE** is an equilateral triangle formed on a side **CD** of a square **ABCD** (Fig. 1.17.21.1). Show that $\triangle$**ADE** $\cong$ $\triangle$**BCE**.

1.17.22 In Fig. 1.17.22.1, **BA** $\perp$ **AC**, **DE** $\perp$ **DF** such that **BA** = **DE** and **BF** = **EC**. Show that $\triangle$**ABC** $\cong$ $\triangle$**DEF**.

1.17.23 **Q** is a point on the side **SR** of $\triangle$**PSR** such that **PQ** = **PR**. Prove that **PS** > **PQ**.

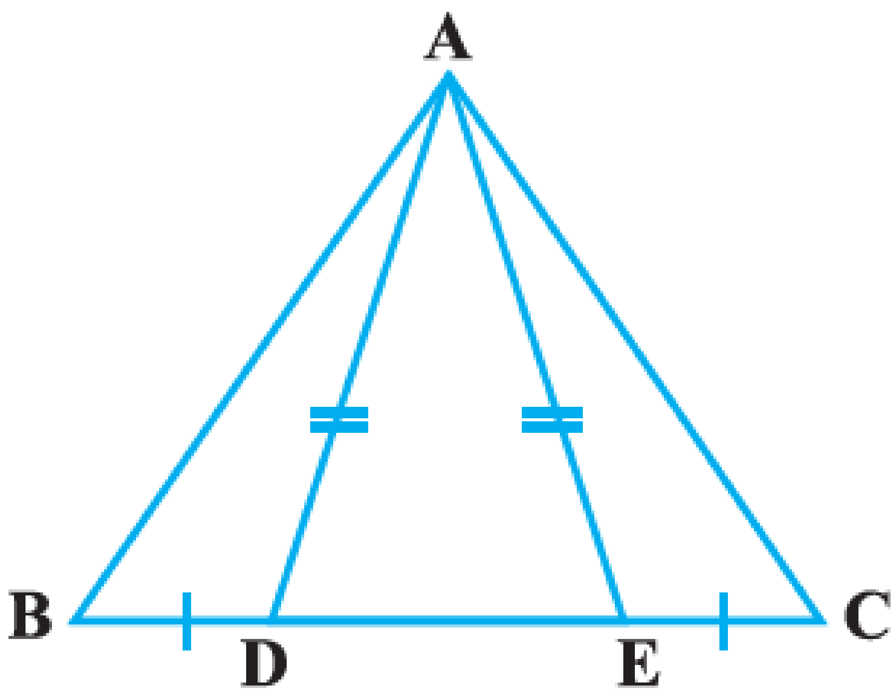


Figure 1.17.20.1:

1.17.24 **S** is any point on side **QR** of a $\triangle$**PQR**. Show that **PQ** + **QR** + **RP** > **2PS**.

1.17.25 **D** is any point on side **AC** of a $\triangle$**ABC** with **AB** = **AC**. Show that **CD** < **BD**.

1.17.26 In Fig. 1.17.26.1, **l**$\|$**m** an **M** is the mid-point of a line segment **AB**. Show that **M** is also the mid-point of any line segment **CD**, having its end points on **l** and **m**, respectively.

1.17.27 Bisectors of the $\angle$**B** and $\angle$**C** of an isosceles triangle with **AB** = **AC** intersect each other at **O**. **BO** is produced to a point **M**. Prove that $\angle$**MOC** = $\angle$**ABC**.

1.17.28 Bisectors of the $\angle\mathbf{B}$ and $\angle\mathbf{C}$ of an isosceles triangle $\mathbf{ABC}$ with $\mathbf{AB} = \mathbf{AC}$ intersect each other at $\mathbf{O}$. Show that the external angle adjacent to $\angle\mathbf{ABC}$ is equal to $\angle\mathbf{BOC}$.

1.17.29 In Fig. 1.17.29.1, $\mathbf{AD}$ is the bisector of $\angle\mathbf{BAC}$. Prove that $\mathbf{AB} > \mathbf{BD}$.

1.17.30 Find all the angles of an equilateral triangle.

1.17.31 P is a point on the bisector of $\angle ABC$. If the line through P, parallel to BA meet BC at Q, prove that BPQ is an isosceles triangle.

1.17.32 ABC is a right triangle with $AB = AC$. Bisector of $\angle A$ meets BC at D. Prove that $BC = 2AD$

1.17.33 ABC and DBC are two triangles on the same base BC such that A and D lie on the opposite sides of BC, $AB = AC$ and $DB = DC$. Show that AD is the perpendicular bisector of BC.

1.17.34 ABC is an isosceles triangle in which $AC = BC$. AD and BE are respectively two altitudes to sides BC and AC. Prove that $AE = BD$.

1.17.35 Prove that sum of any two sides of a triangle is greater than twice the median with respect to the third side.

1.17.36 In a triangle ABC, D is the mid-point of side AC such that $BD = \frac{1}{2}AC$. Show that $\angle ABC$ is a right angle.

1.17.37 In a right triangle, prove that the line-segment joining the mid-point of the hypotenuse to the opposite vertex is half the hypotenuse.

1.17.38 Two lines l and m intersect at the point O and P is a point on a line n passing through the point O such that P is equidistant from l and m. Prove that n is the bisector of the angle formed by l and m.

1.17.39 ABC is a right triangle such that $AB = AC$ and bisector of angle C intersects the side AB at D. Prove that $AC + AD = BC$.

1.17.40 Prove that in a triangle, other than an equilateral triangle, angle opposite the longest side is greater than $\frac{2}{3}$ of a right angle.

1.17.41 In △ABC,$AB = AC$ and $\angle B = 50°$.Then ∠C is equal to

(a) 40°

(b) 50°

(c) 80°

(d) 130°

1.17.42 In △ABC, $BC = AB$ and $\angle B = 80°$.Then ∠A is equal to

(a) 80°

(b) 40°

(c) 50°

(d) 100°

1.17.43 In △PQR, $\angle R$=$\angle P$ and $QR = 4cm$ and $PR = 5cm$. Then the length of PQ is

(a) $4cm$

(b) $5cm$

(c) $2cm$

(d) $2.5cm$

1.17.44 D is a Point on the side BC of a △ABC such that AD bisects ∠BAC. Then

(a) $BD = CD$

(b) $BA > BD$

(c) $BD > BA$

(d) $CD > CA$

1.17.45 Two sides of a triangle are of lengths 5cm and 1.5cm.The length of the third side of the triangle cannot be

(a) 3.6cm

(b) 4.1cm

(c) 3.8cm

(d) 3.4cm

1.17.46 In△PQR, if $\angle R > \angle Q$,then

(a) QR > PR

(b) PQ > PR

(c) PQ < PR

(d) QR < PR

1.17.47 The construction of a triangle ABC, given that $BC = 6cm, \angle B = 45°$ is not possible when difference of AB and AC is equal to

(a) 6.9 cm

(b) 5.2 cm

(c) 5.0 cm

(d) 4.0 cm

1.17.48 The construction of a triangle ABC, given that $BC = 6cm, \angle C = 60°$ is possible when difference of AB and AC is equal to

(a) 3.2 cm

(b) 3.1 cm

(c) 3 cm

(d) 2.8 cm

1.17.49 Construct a triangle whose sides are $3.6cm$, $3.0cm$ and $4.8cm$. Bisect the smallest angle and measure each part.

1.17.50 Construct a triangle ABC in which $BC = 5cm$, $\angle B = 60°$ and $AC + AB = 7.5cm$.

Construct each of the following and give justification :

1.17.51 A triangle if its perimeter is 10.4 cm and two angles are 45°and 120°.

1.17.52 A triangle PQR given that $QR =$ 3cm, $\angle PQR = 45°$ and $QP - PR = 2cm$.

1.17.53 A right triangle when one side is 3.5 cm and sum of other sides and the hypotenuse is 5.5 cm.

1.17.54 An equilateral triangle if its altitude is 3.2 cm.

Write true or false in each of the following. Give reasons for your answer:

1.17.55 A triangle ABC can be constructed in which AB = 5 cm, ∠A =45° and BC + AC = 5 cm.

1.17.56 A triangle ABC can be constructed in which BC = 6 cm, ∠30° and AC - AB=4 cm.

1.17.57 A triangle ABC can be constructed in which ∠B =105°,∠C =90° and AB + BC + AC = 10 cm.

1.17.58 A triangle ABC can be constructed in which ∠B =60°,∠C =45° and AB + BC + AC = 12 cm

1.18. Quadrilateral

1.18.1 In the Figure 1.18.1.1, $ABCD$ is a parallelogram, $AE \perp DC$ and $CF \perp AD$. If $AB = 16cm$, $AE = 8cm$, and $CF = 10cm$, find AD.

1.18.2 For a given Parallelogram $ABCD$, show that for any point **P** inside the parallelogram,

(a) $Ar(APD) + Ar(PBC) = \frac{1}{2}Ar(ABCD)$

(b) $Ar(APD) + Ar(PBC) = Ar(APB) + Ar(PCD)$

1.18.3 In Fig. 1.18.3.1, $PQRS$ and $ABRS$ are parallelograms and **X** is any point on side BR. Show that

(a) $ar(PQRS) = ar(ABRS)$

(b) $ar(AXS) = \frac{1}{2} ar(PQRS)$

Proof. (a) From Appendix A.1.24.1,

$$\mathbf{A} - \mathbf{B} = \mathbf{S} - \mathbf{R} = \mathbf{P} - \mathbf{Q} \tag{1.18.3.1}$$

and from Appendix A.1.26, using (1.18.3.1), we obtain Property 1.18.3a.

(b) Using section formula, let

$$\mathbf{X} = \frac{\mathbf{R} + k\mathbf{B}}{1+k}. \tag{1.18.3.2}$$

Then,

$$ar\left(AXS\right) = \frac{1}{2}\left\|\mathbf{S}\times\mathbf{X} + \mathbf{X}\times\mathbf{A} + \mathbf{A}\times\mathbf{S}\right\| \tag{1.18.3.3}$$

$$= \frac{1}{2}\left\|\frac{\mathbf{S}\times\mathbf{R} + k\mathbf{S}\times\mathbf{B} + \mathbf{R}\times\mathbf{A} + k\mathbf{B}\times\mathbf{A}}{k+1} + \mathbf{A}\times\mathbf{S}\right\| \tag{1.18.3.4}$$

Substituting for $\mathbf{B}$ from (1.18.3.1) in the above,

$$ar(AXS) = \frac{1}{2}\left\|\frac{\mathbf{S}\times\mathbf{R}+\mathbf{R}\times\mathbf{A}+k(\mathbf{S}-\mathbf{A})\times(\mathbf{A}-\mathbf{S}+\mathbf{R})}{k+1}+\mathbf{A}\times\mathbf{S}\right\| \tag{1.18.3.5}$$

$$= \frac{1}{2}\left\|\frac{\mathbf{S}\times\mathbf{R}+\mathbf{R}\times\mathbf{A}+k(\mathbf{S}-\mathbf{A})\times\mathbf{R}}{k+1}+\mathbf{A}\times\mathbf{S}\right\| \tag{1.18.3.6}$$

$$= \frac{1}{2}\|\mathbf{S}\times\mathbf{R}+\mathbf{R}\times\mathbf{A}+\mathbf{A}\times\mathbf{S}\| \tag{1.18.3.7}$$

$$= \frac{1}{2}ar(ABRS) \tag{1.18.3.8}$$

□

1.18.4 In quadrilateral $CBAD$,$CA = AD$ and BA bisect $\angle A$ shown in figure Table 1.18.4.1. Show that $\triangle CAB \cong \triangle DAB$. What can you say about BC and BD?

Solution: See Table 1.18.4.1. The vertices of the quadrilateral can

Symbol	Values	Description
θ	$30°$	$\angle BAD = \angle BAC$
a	9	AB
c	5	AC
$\mathbf{e}_1$	$\begin{pmatrix}1\\0\end{pmatrix}$	basis vector

Table 1.18.4.1: Parameters

be expressed as

$$\mathbf{A} = \begin{pmatrix}0\\0\end{pmatrix}, \mathbf{B} = a\mathbf{e_1}, \mathbf{C} = \begin{pmatrix}c\cos\theta\\c\sin\theta\end{pmatrix}, \mathbf{D} = \begin{pmatrix}c\cos\theta\\-c\sin\theta\end{pmatrix} \tag{1.18.4.1}$$

where

$$\mathbf{C}-\mathbf{A}=\mathbf{A}-\mathbf{D} \tag{1.18.4.2}$$

$$\angle CAB=\angle DAB \tag{1.18.4.3}$$

$$AB:\mathbf{n}^\top\mathbf{x}=0, \tag{1.18.4.4}$$

where

$$\mathbf{n}=\begin{pmatrix}0\\1\end{pmatrix} \tag{1.18.4.5}$$

Letting

$$\theta_1 =\angle CBA \tag{1.18.4.6}$$

and substituting numerical values,

$$\mathbf{m_1} =\mathbf{B}-\mathbf{C}=\begin{pmatrix}4.7\\-2.5\end{pmatrix}, \mathbf{m_2}=\mathbf{B}-\mathbf{A}=\begin{pmatrix}9\\0\end{pmatrix} \tag{1.18.4.7}$$

$$\implies \theta_1 = \cos^{-1}\frac{\mathbf{m_1}^\top\mathbf{m_2}}{\left\|\mathbf{m_1}\right\|\left\|\mathbf{m_2}\right\|} \tag{1.18.4.8}$$

$$= \cos^{-1}\frac{\begin{pmatrix}4.7 & -2.5\end{pmatrix}\begin{pmatrix}9\\0\end{pmatrix}}{(9.2)(9)} = 59.3^\circ \tag{1.18.4.9}$$

Similalry, for

$$\theta_2 = \angle ABD, \tag{1.18.4.10}$$

$$\mathbf{n_1} = \mathbf{D} - \mathbf{B} = \begin{pmatrix} -4.7 \\ 2.5 \end{pmatrix}, \mathbf{n_2} = \mathbf{A} - \mathbf{B} = \begin{pmatrix} -9 \\ 0 \end{pmatrix} \tag{1.18.4.11}$$

$$\implies \theta_2 = \cos^{-1} \frac{\mathbf{n_1}^\top \mathbf{n_2}}{\|\mathbf{n_1}\| \, \|\mathbf{n_2}\|} \tag{1.18.4.12}$$

$$= \cos^{-1} \frac{\begin{pmatrix} -4.7 & 2.5 \end{pmatrix} \begin{pmatrix} -9 \\ 0 \end{pmatrix}}{(9.2)(9)} = 59.3° \tag{1.18.4.13}$$

$$\tag{1.18.4.14}$$

From (1.18.4.9) and (1.18.4.13),

$$\angle BAC = \angle BAD \tag{1.18.4.15}$$

Similarly, equality can be shown for other sides and angles.

1.18.5 $ABCD$ is a quadrilateral in which $AD = BC$ and $\angle DAB = \angle CBA$ as shown in figure 1.18.5.1. Prove that

(a) $\triangle ABD \cong \triangle BAC$

(b) $BD = AC$

(c) $\angle ABD = \angle BAC$

Solution: The input parameters for construction are shown in Table 1.18.5.1 Let

Symbol	**Values**	**Description**
θ	$45°$	$\angle BAD = \angle ABC$
a	5	AB
c	9	BC
$\mathbf{e}_1$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	basis vector

Table 1.18.5.1: Parameters

$$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} a \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} c\cos\theta \\ c\sin\theta \end{pmatrix}, \mathbf{D} = \begin{pmatrix} -c\cos\theta \\ c\sin\theta \end{pmatrix} \tag{1.18.5.1}$$

$$AB : \mathbf{n}^\top \mathbf{x} = 0, \tag{1.18.5.2}$$

$$\tag{1.18.5.3}$$

where

$$\mathbf{n} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.18.5.4}$$

From the above assumptions, we get the coordinates of C and D as

$$\mathbf{C} = \begin{pmatrix} 4.3 \\ -2.5 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} -4.3 \\ -2.5 \end{pmatrix} \tag{1.18.5.5}$$

Let

$$\theta_1 = \angle ADB \tag{1.18.5.6}$$

Since

$$\mathbf{m_1} = \mathbf{D} - \mathbf{A} = \begin{pmatrix} -4.7 \\ -2.5 \end{pmatrix}, \mathbf{m_2} = \mathbf{D} - \mathbf{B} = \begin{pmatrix} -13.7 \\ -2.5 \end{pmatrix} \tag{1.18.5.7}$$

$$\theta_1 = \cos^{-1} \frac{\mathbf{m_1}^\top \mathbf{m_2}}{\left\|\mathbf{m_1}\right\| \left\|\mathbf{m_2}\right\|} \tag{1.18.5.8}$$

$$= \cos^{-1} \frac{\begin{pmatrix} -4.7 & -2.5 \end{pmatrix} \begin{pmatrix} -13.7 \\ -2.5 \end{pmatrix}}{(9.2)(15.8)} = 61^\circ \tag{1.18.5.9}$$

Similalrly, letting

$$\theta_2 = \angle ACB, \tag{1.18.5.10}$$

$$\mathbf{n_1} = \mathbf{C} - \mathbf{A} = \begin{pmatrix} 4.7 \\ -2.5 \end{pmatrix}, \mathbf{n_2} = \mathbf{C} - \mathbf{B} = \begin{pmatrix} 13.7 \\ -2.5 \end{pmatrix} \tag{1.18.5.11}$$

$$\theta_2 = \cos^{-1} \frac{\mathbf{n_1}^\top \mathbf{n_2}}{\left\|\mathbf{n_1}\right\| \left\|\mathbf{n_2}\right\|} \tag{1.18.5.12}$$

$$= \cos^{-1} \frac{\begin{pmatrix} 4.7 & -2.5 \end{pmatrix} \begin{pmatrix} 13.7 \\ -2.5 \end{pmatrix}}{(9.2)(15.8)} = 61^\circ \tag{1.18.5.13}$$

From (1.18.5.9) and (1.18.5.13),

$$\angle ABD = \angle CAB \tag{1.18.5.14}$$

Since all the angles and sides of triangles CAB and CAD are equal,

$$\triangle ACB \cong \triangle ADB \tag{1.18.5.15}$$

1.18.6 AD and BC are equal perpendiculars to a line segment AB. Show that CD bisects AB.

Solution: See Fig. 1.18.6.1 The input parameters for construction are shown in Table 1.18.6.1 Let

Symbol	Values	Description
a	3	$AD = BC$
b	8	AB
$\mathbf{e}_1$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	basis vector

Table 1.18.6.1: Parameters

$$\mathbf{A} = a\mathbf{e_1}, \mathbf{B} = \begin{pmatrix} a \\ b \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 2a \\ b \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1.18.6.1}$$

Solution: Given

$$\|\mathbf{D} - \mathbf{A}\| = \|\mathbf{B} - \mathbf{C}\| \tag{1.18.6.2}$$

$$\angle DAB = \angle CBA = 90^\circ \tag{1.18.6.3}$$

$$\frac{1}{2}(\mathbf{C}+\mathbf{D}) = \frac{1}{2}\begin{pmatrix} a \\ \frac{b}{2} \end{pmatrix} \tag{1.18.6.4}$$

$$\frac{1}{2}(\mathbf{A}+\mathbf{B}) = \frac{1}{2}\begin{pmatrix} a \\ \frac{b}{2} \end{pmatrix} \tag{1.18.6.5}$$

$$\tag{1.18.6.6}$$

Thus, CD bisects AB.

1.19. Exercises

1.19.1 $ABCD$ is a quadrilateral in which $AB = BC$ and $AD = CD$. Show that BD bisects both the angles ABC and ADC.

1.19.2 O is a point in the interior of a square $ABCD$ suchthat OAB is an equilateral triangle. Show that $\triangle OCD$ is an isosceles triangle.

1.19.3 Show that in a quadrilateral $ABCD$,

$$AB + BC + CD + DA < (BD + AC) \tag{1.19.3.1}$$

1.19.4 Show that in a quadrilateral $ABCD$,

$$AB + BC + CD + DA > AC + BD \tag{1.19.4.1}$$

1.19.5 Line segment joining the mid-points M and N of parallel sides AB and DC, respectively of a trapezium $ABCD$ is perpendicular to both the sides AB and DC. Prove that $AD = BC$.

1.19.6 $ABCD$ is a quadrilateral such that diagonal AC bisects the angles A and C. Prove that $AB = AD$ and $CB = CD$.

1.19.7 AB and CD are the smallest and largest sides of a quadrilateral $ABCD$. Out of $\angle B$ and $\angle D$ decide which is greater.

1.19.8 $ABCD$ is quadrilateral such that $AB = AD$ and $CB = CD$. Prove that AC is the perpendicular bisector of BD.

1.19.9 A point **E** is taken on the side BC of a parallelogram ABCD.AE and DC are produced to meet at **F**.Prove that $ar(ADF) = ar(ABFC)$.

1.19.10 The diagonals of a parallelogram ABCD intersect at a point **O**.Through **O**,a line is drawn to intersect AD at **P** and BC at **Q**.Show that PQ divides the parallelogram into two parts of equal area.

1.19.11 The medians BE and CF of a triangle ABC intersect at **G**.Prove that the area of △GBC= area of the quadrilateral AFGE.

1.19.12 In Fig.1.19.12.1,$CD \parallel AE$ and $CY \parallel BA$.Prove that $ar(CBX) = ar(AXY)$

1.19.13 ABCD is a trapezium in which $AB \parallel DC$,$DC = 30cm$ and $AB = 50cm$.If **X** and **Y** are,respectively the mid-points of AD and BC,prove that $ar(DCYX) = \frac{7}{9}ar(XYBA)$.

1.19.14 In △ABC,if **L** and **M** are the points on AB and AC,respectively such that $LM \parallel BC$.Prove that $ar(LOB) = ar(MOC)$.

1.19.15 In Fig.1.19.15.1,ABCDE is any pentagon.BP drawn parallel to AC meets DC produced at **P** and EQ drawn parallel to AD meets CD produced at **Q**.Prove that $ar(ABCDE) = ar(APQ)$.

1.19.16 If the medians of a $\triangle$ ABC intersect at **G**,show that

$$ar(AGB) = ar(AGC) = ar(BGC) = \frac{1}{3}ar(ABC) \qquad (1.19.16.1)$$

1.19.17 In Fig.1.19.17.1,**X** and **Y** are the mid-points of AC and AB respectively,$QP \parallel BC$ and CYQ and BXP are straight lines.Prove that $ar(ABP) = ar(ACQ)$.

1.19.18 In Fig.1.19.18.1,ABCD and AEFD are two parallelograms.Prove that $ar(PEA) = ar(QFD)$ [Hint:Join PD].

1.19.19 ABCD is a parallelogram and **X** is the mid-point of AB.If $ar(AXCD) = 24cm^2$,then $ar(ABC) = 24cm^2$.

1.19.20 PQRS is a rectangle inscribed in a quadrant of radius 13 cm.A is any point on PQ.If PS=5 cm,then $ar(PAS) = 30cm^2$

1.19.21 PQRS is a parallelogram whose area is $180cm^2$ and A is any point on the diagonal QS.The area of $\triangle ASR = 90cm^2$.

1.19.22 ABC and BDE are two equilateral triangles such that **D**is the mid-point of BC.Then ar(BDE)=$\frac{1}{4}ar(ABC)$.

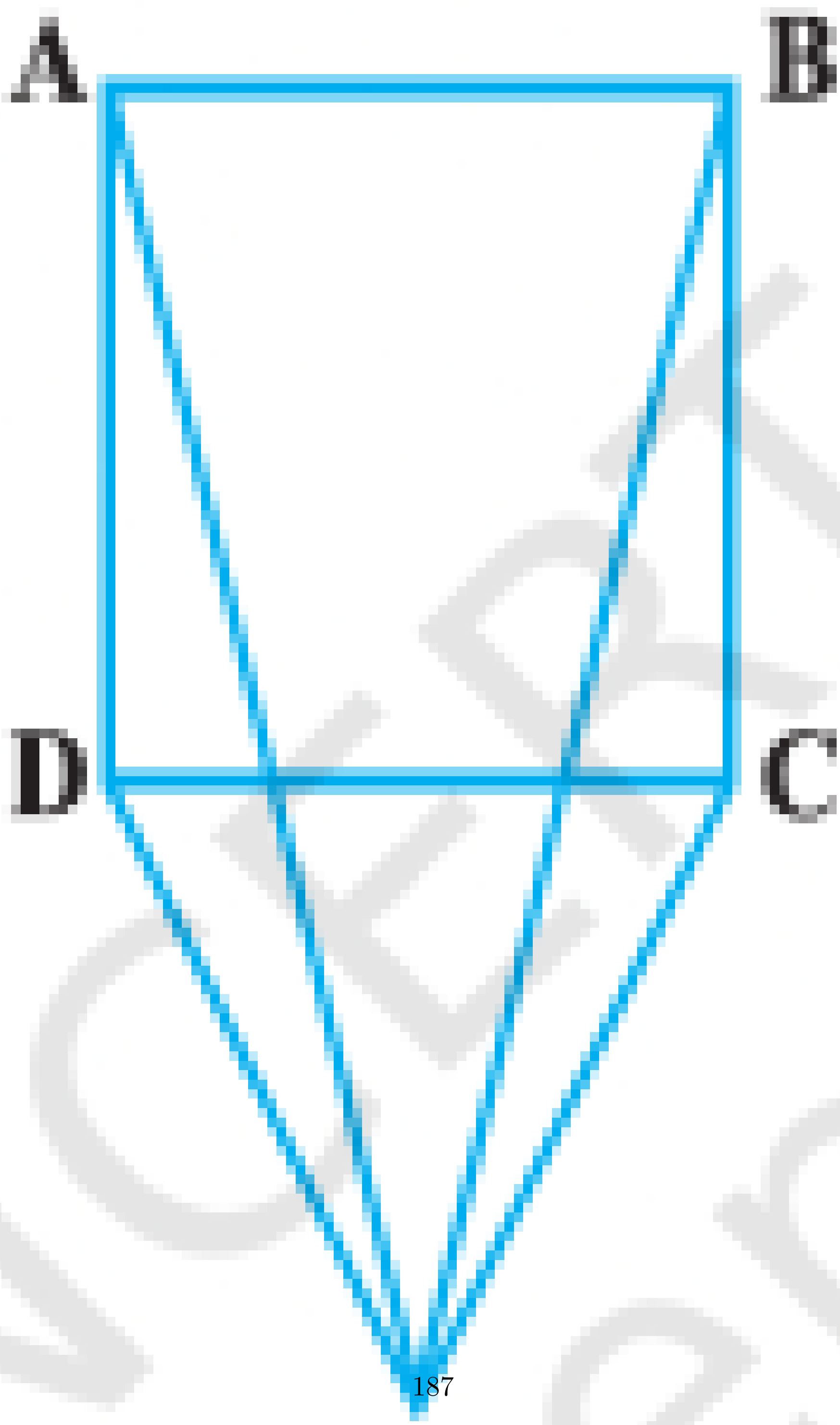
1.19.23 In Fig.1.19.23.1, $ABCD$ and $EFGD$ are two parallelograms and **G** is the mid-point of CD. Then$ar(DPC) = \frac{1}{2}ar(EFGD)$.

1.19.24 Construct a square of side $3cm$.

1.19.25 Construct a rectangle whose adjacent sides are of lengths $5cm$ and $3.5cm$.

1.19.26 Construct a rhombus whose side is of length $3.4cm$ and one of its angles is $45°$.

1.19.27 Construct a rhombus whose diagonals are 4 cm and 6 cm in lengths.


A
B
D
C

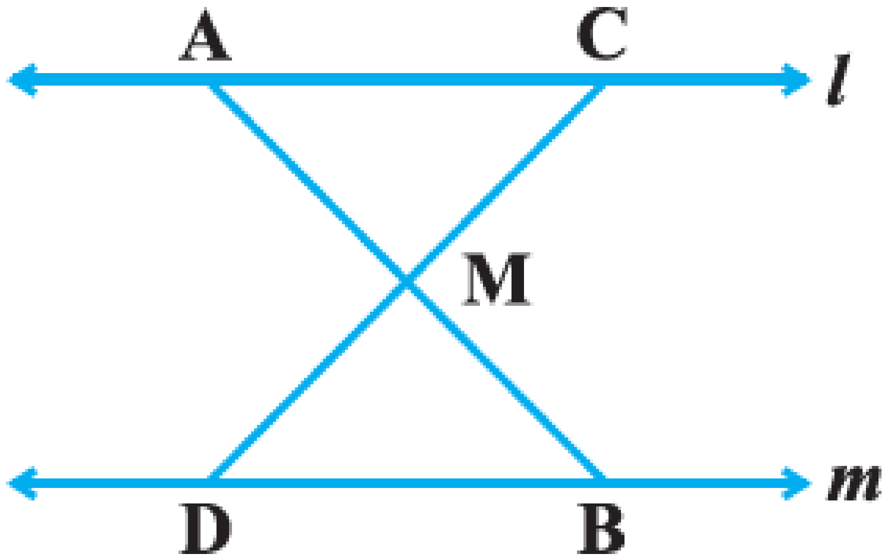


Figure 1.17.22.1:

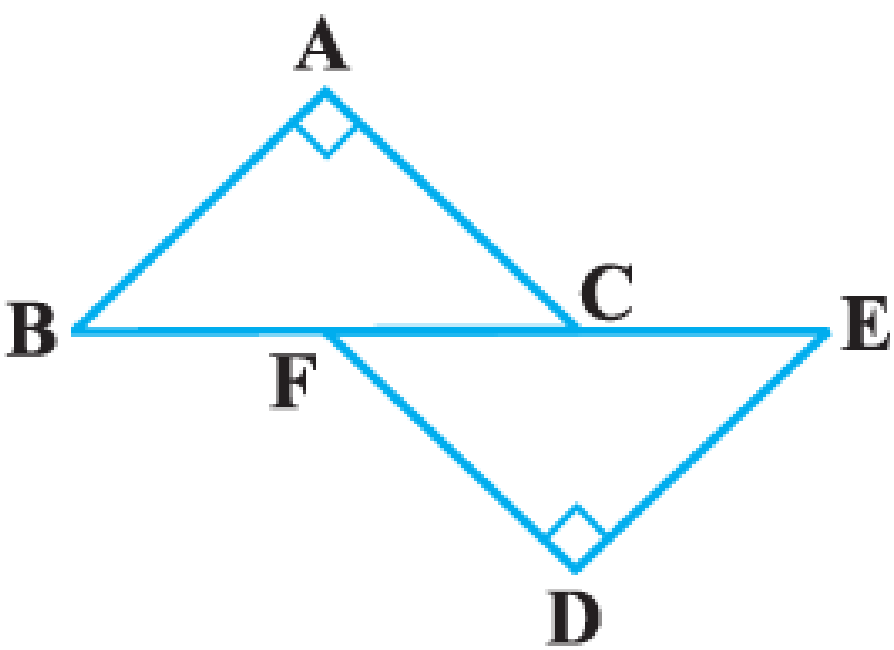


Figure 1.17.26.1:

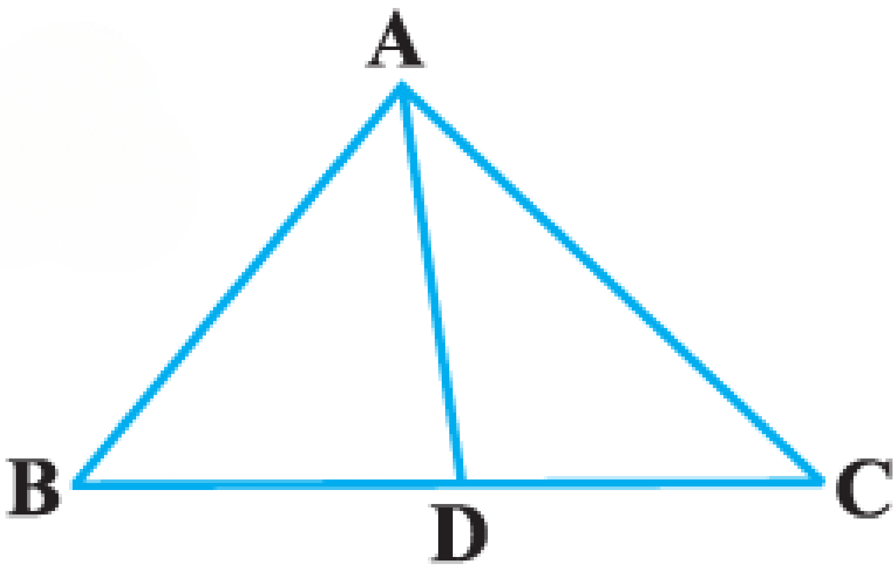


Figure 1.17.29.1:

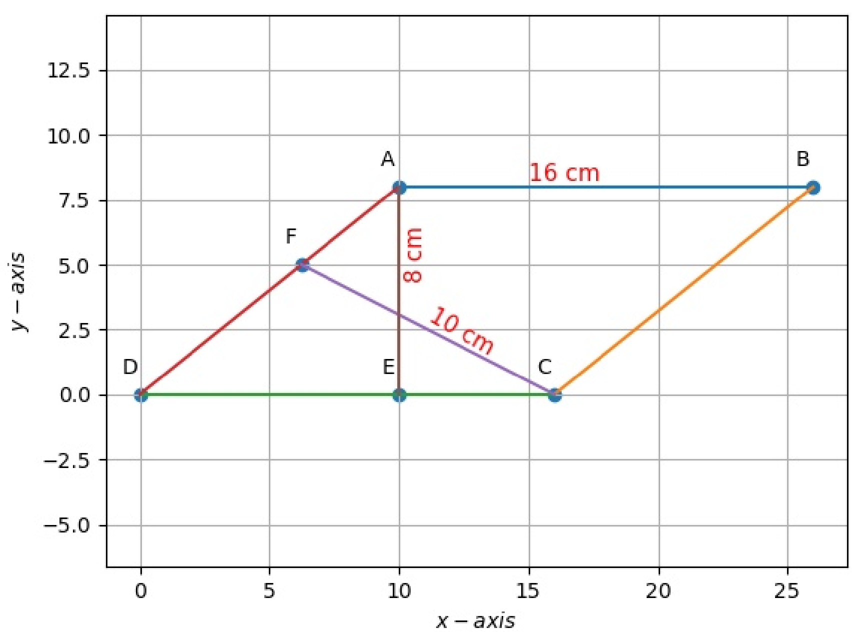


Figure 1.18.1.1:

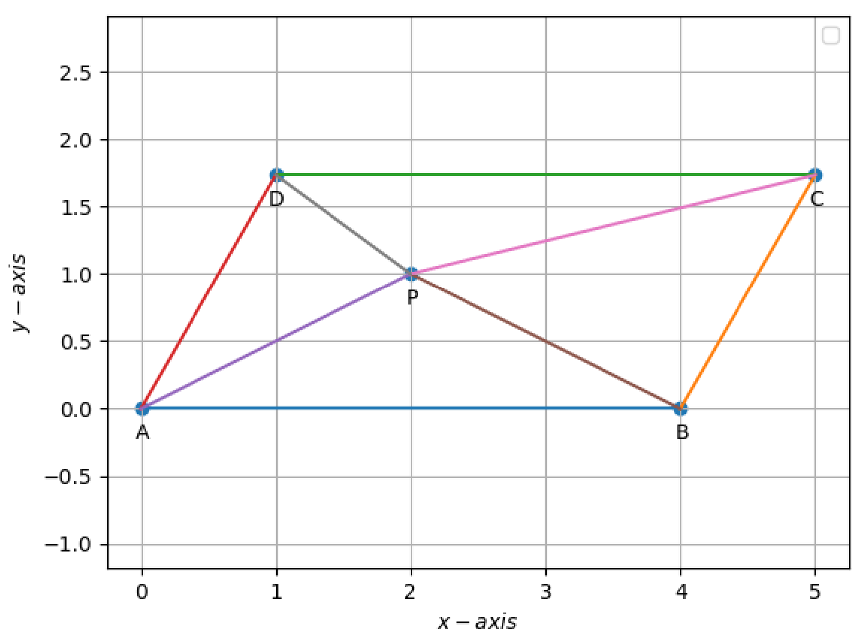


Figure 1.18.2.1:

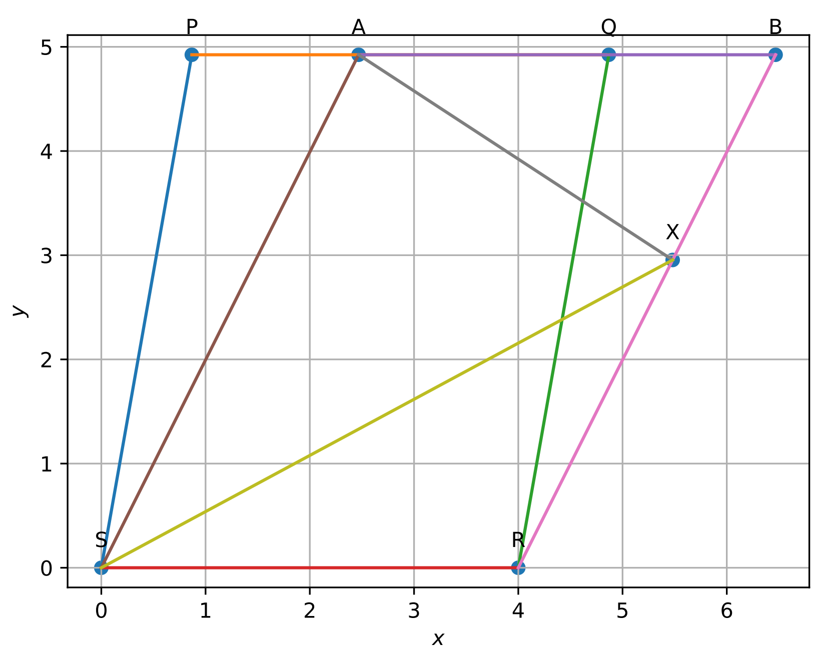


Figure 1.18.3.1:

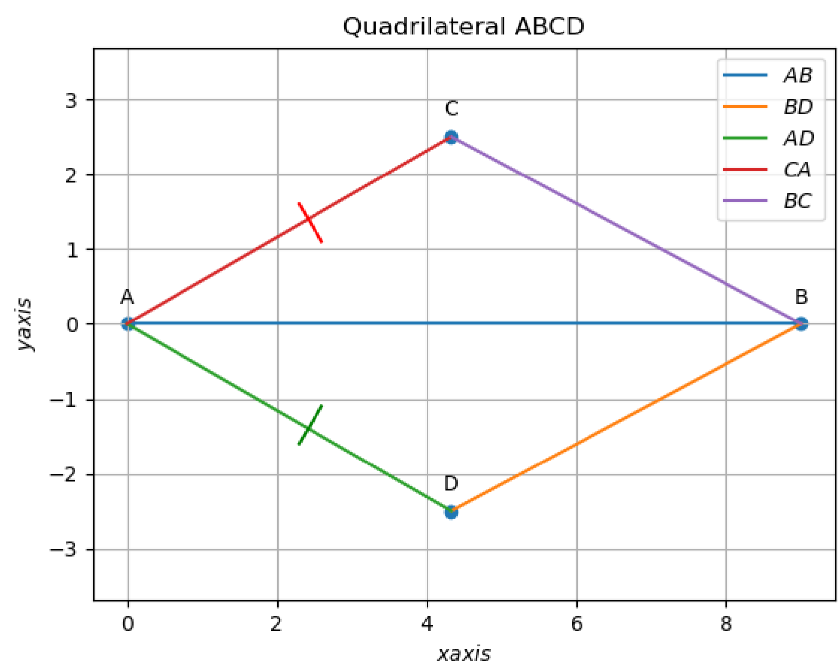


Figure 1.18.4.1: Quadrilateral CBAD

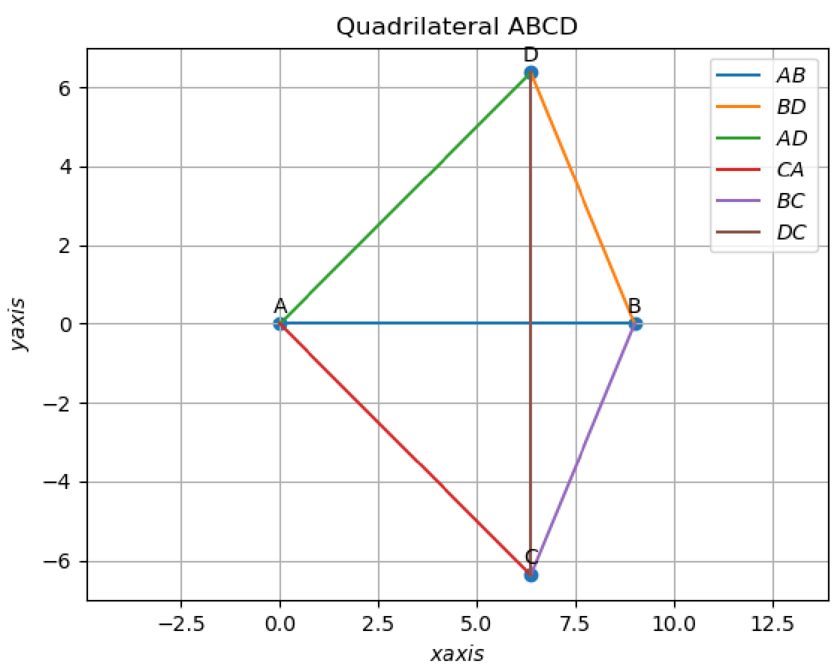


Figure 1.18.5.1: Quadrilateral ABCD

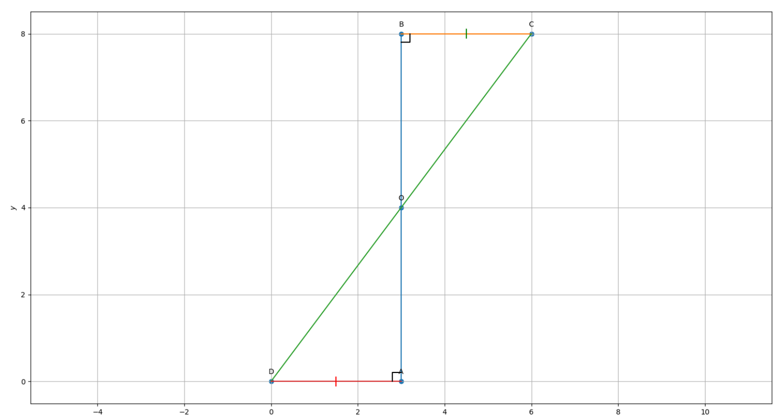


Figure 1.18.6.1:

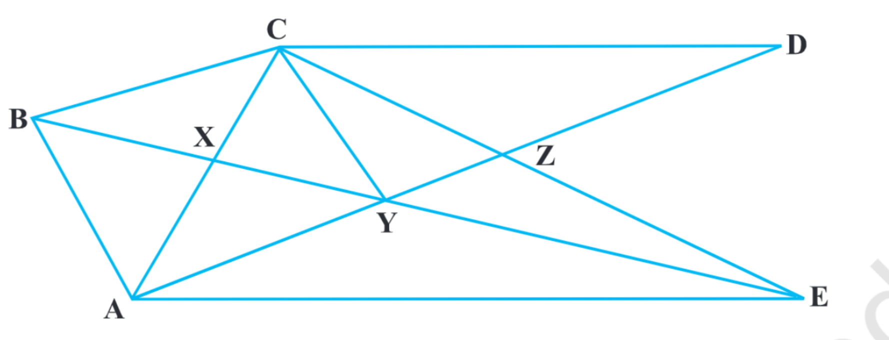


Figure 1.19.12.1:

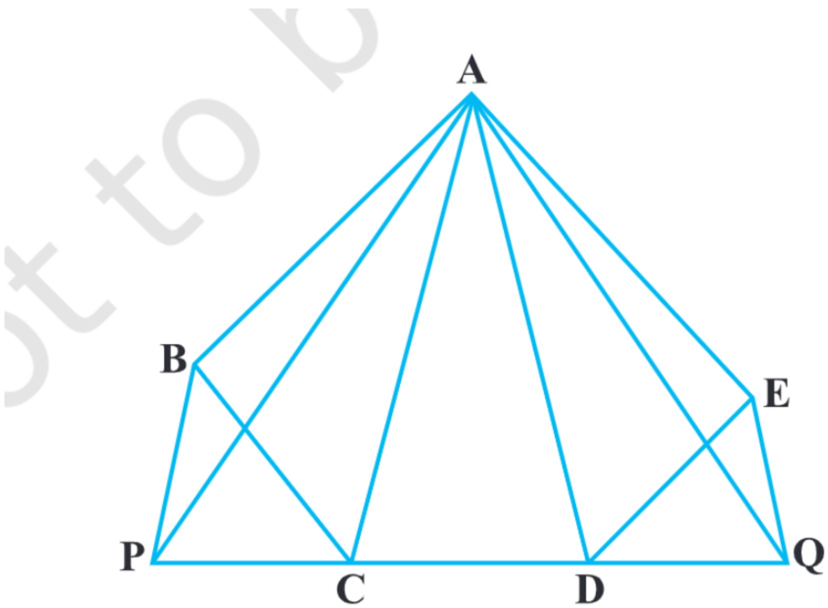


Figure 1.19.15.1:

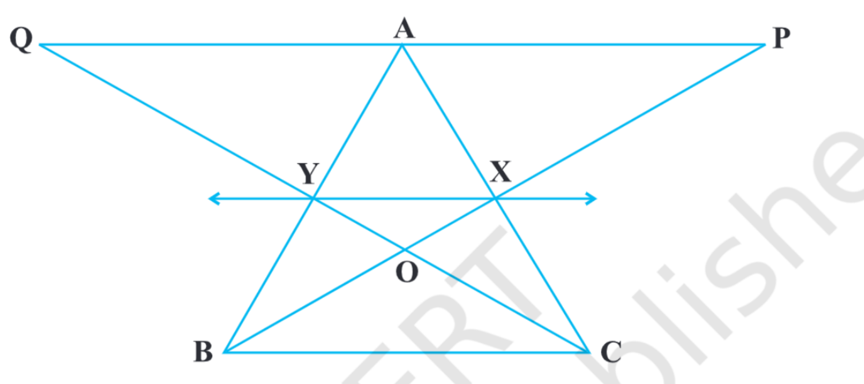


Figure 1.19.17.1:

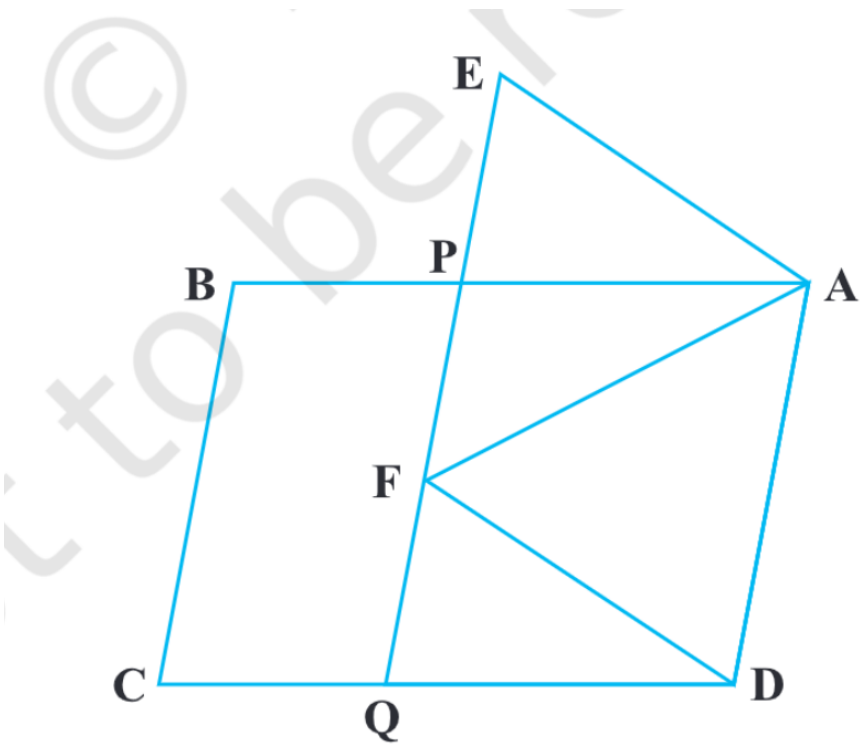


Figure 1.19.18.1:

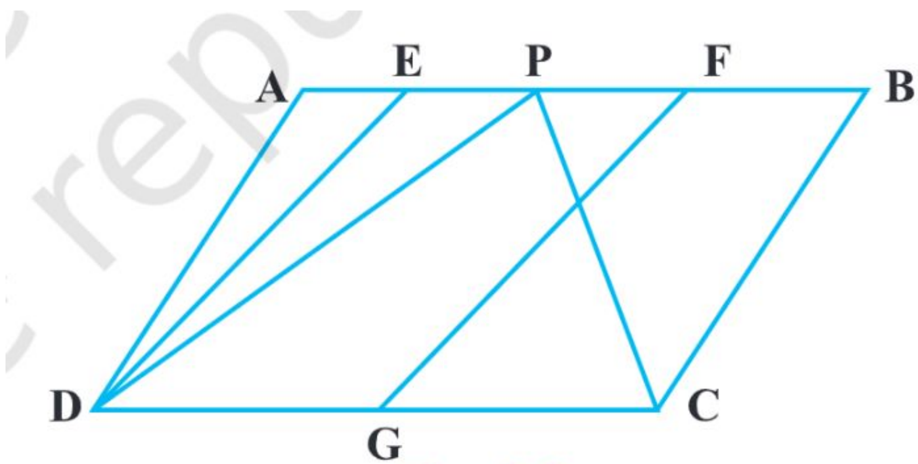


Fig. 9.8

Figure 1.19.23.1:

Chapter 2

Linear Forms

2.1. Equation of a Line

Find the equation of line

2.1.1

2.1.2 passing through the point (– 4, 3) with slope $\frac{1}{2}$. Let

$$\mathbf{P} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}, m = \frac{1}{2} \tag{2.1.2.1}$$

The directional vector is

$$\mathbf{m} = \begin{pmatrix} -1 \\ \frac{1}{2} \end{pmatrix} \tag{2.1.2.2}$$

The normal vector is

$$\mathbf{n} = \begin{pmatrix} \frac{1}{2} \\ -1 \end{pmatrix} \tag{2.1.2.3}$$

$$\tag{2.1.2.4}$$

The line equation is represented in the form

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{P}) = 0 \tag{2.1.2.5}$$

$$\implies \begin{pmatrix} \frac{1}{2} & -1 \end{pmatrix} \mathbf{x} = -5 \tag{2.1.2.6}$$

See Fig. 2.1.2.1.

2.1.3 passing through $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ with slope m.

Solution: Line passing through point $\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is given by,

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{A}) = 0 \tag{2.1.3.1}$$

Where,

$$\mathbf{n} = \begin{pmatrix} m \\ -1 \end{pmatrix} \tag{2.1.3.2}$$

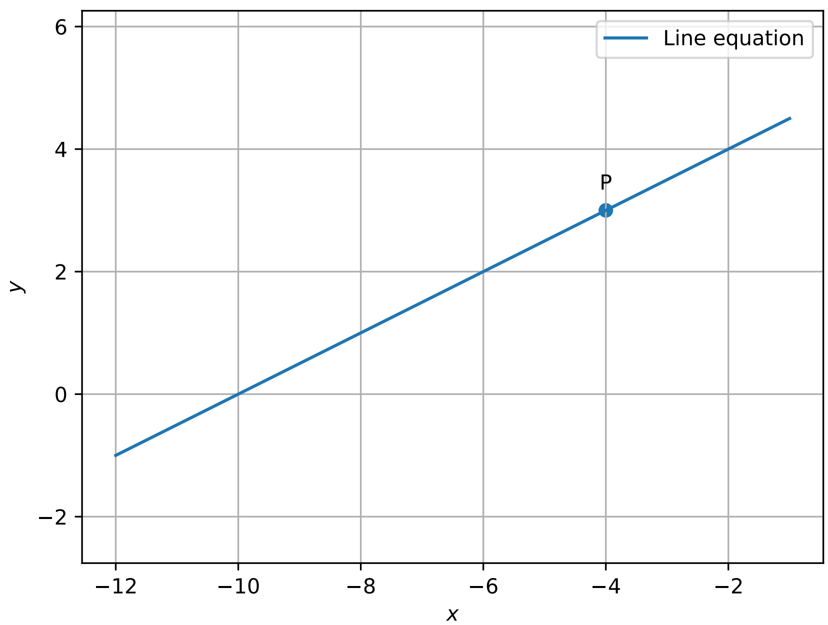


Figure 2.1.2.1: line

Substituting $\mathbf{A}$ and $\mathbf{n}$ in equation (2.1.3.1)

$$\begin{pmatrix} m & -1 \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) = 0 \tag{2.1.3.3}$$

$$\implies \begin{pmatrix} m & -1 \end{pmatrix} \mathbf{x} = 0 \tag{2.1.3.4}$$

Line segment passing through $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ with slope $m = 2$ is shown in Fig. 2.1.3.1

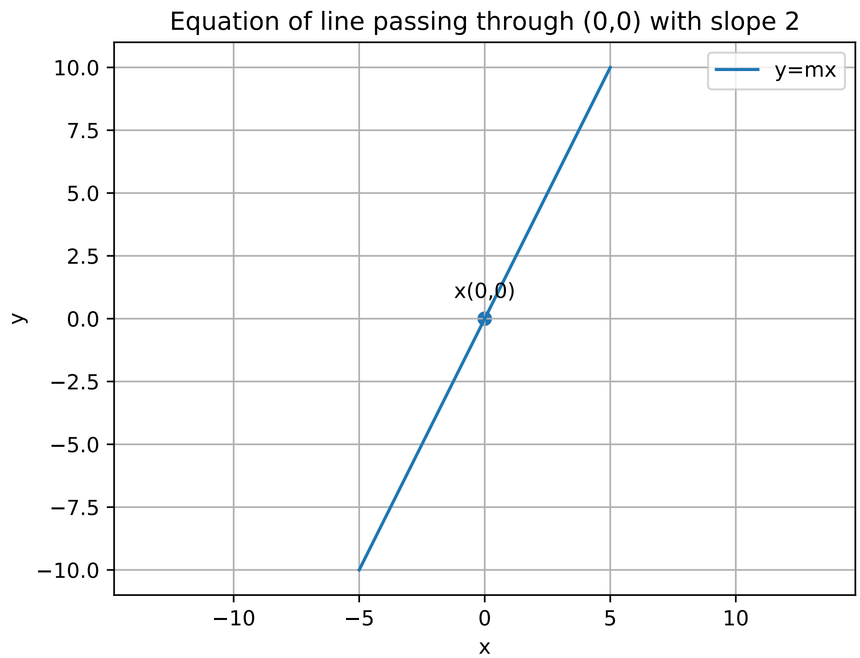


Figure 2.1.3.1:

2.1.4 passing through $\mathbf{A} = \begin{pmatrix} 2 \\ 2\sqrt{3} \end{pmatrix}$ and inclined with the x-axis at an angle of $75°$.

Solution: Since $\tan 75° = 2 + \sqrt{3}$, the direction vector of the line is

$$\mathbf{m} = \begin{pmatrix} 2+\sqrt{3} \\ 1 \end{pmatrix} \tag{2.1.4.1}$$

and hence the normal vector is

$$\mathbf{n} = \begin{pmatrix} -1 \\ 2+\sqrt{3} \end{pmatrix} \tag{2.1.4.2}$$

The equation of the line is

$$\mathbf{n}^\top \left(\mathbf{x} - \mathbf{A}\right) = 0 \tag{2.1.4.3}$$

$$\implies \mathbf{n}^\top \mathbf{x} = \mathbf{n}^\top \mathbf{A} = 4\left(\sqrt{3}+1\right) \tag{2.1.4.4}$$

$$\implies \begin{pmatrix} -1 & 2+\sqrt{3} \end{pmatrix} \mathbf{x} = 4\left(\sqrt{3}+1\right) \tag{2.1.4.5}$$

The line is plotted in Fig. 2.1.4.1.

2.1.5 intersecting the x-axis at a distance of 3 units to the left of origin with slope of -2.

Solution: From the given information,

$$\mathbf{A} = \begin{pmatrix} -3 \\ 0 \end{pmatrix} \tag{2.1.5.1}$$

$$m = -2 \implies \mathbf{m} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \tag{2.1.5.2}$$

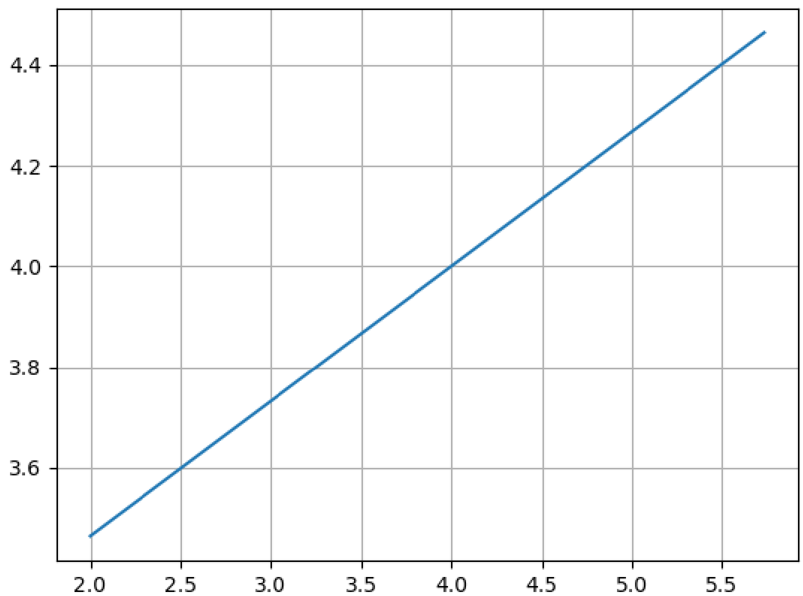


Figure 2.1.4.1: Line represented by (2.1.4.5).

Thus, the normal vector $\mathbf{n}$ to the line is given as

$$\mathbf{n} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} \tag{2.1.5.3}$$

$$= \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{2.1.5.4}$$

The desired equation of the line is

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{A}) = 0 \tag{2.1.5.5}$$

$$\implies \begin{pmatrix} 2 & 1 \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} -3 \\ 0 \end{pmatrix} \right) = 0 \tag{2.1.5.6}$$

$$\text{or, } \begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = -6 \tag{2.1.5.7}$$

The line segment is shown in Fig. 2.1.5.1.

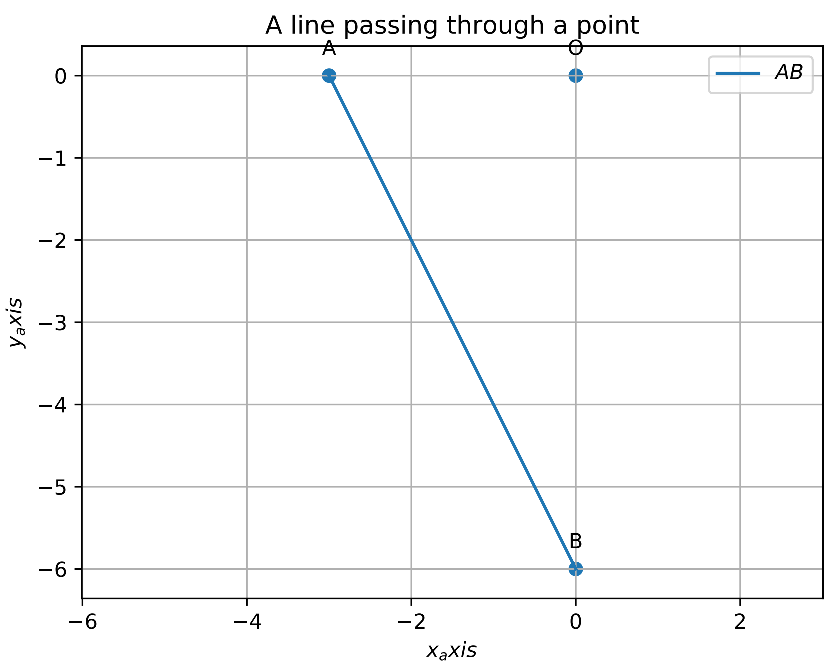


Figure 2.1.5.1:

2.1.6 Find the equation of the line which satisfy the given conditions: Intersecting the y-axis at a distance of 2 units above the origin and making an angle of 30° with positive direction of the x-axis.

Solution: The direction vector of the line is given by

$$\mathbf{m} = \begin{pmatrix} 1 \\ \tan(30°) \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{\sqrt{3}} \end{pmatrix} \tag{2.1.6.1}$$

The normal vector $\mathbf{n}$ to the line is given by

$$\mathbf{n} = \begin{pmatrix} -\frac{1}{\sqrt{3}} \\ 1 \end{pmatrix} \tag{2.1.6.2}$$

The line is passing through the point

$$\mathbf{A} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \tag{2.1.6.3}$$

Hence, the equation of the line is given by

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{A}) = 0 \tag{2.1.6.4}$$

$$\implies \begin{pmatrix} -\frac{1}{\sqrt{3}} & 1 \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right) = 0 \tag{2.1.6.5}$$

$$\text{or,} \begin{pmatrix} -\frac{1}{\sqrt{3}} & 1 \end{pmatrix} \mathbf{x} = 2 \tag{2.1.6.6}$$

2.1.7 Find the equation of line passing through the points $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -4 \end{pmatrix}$.

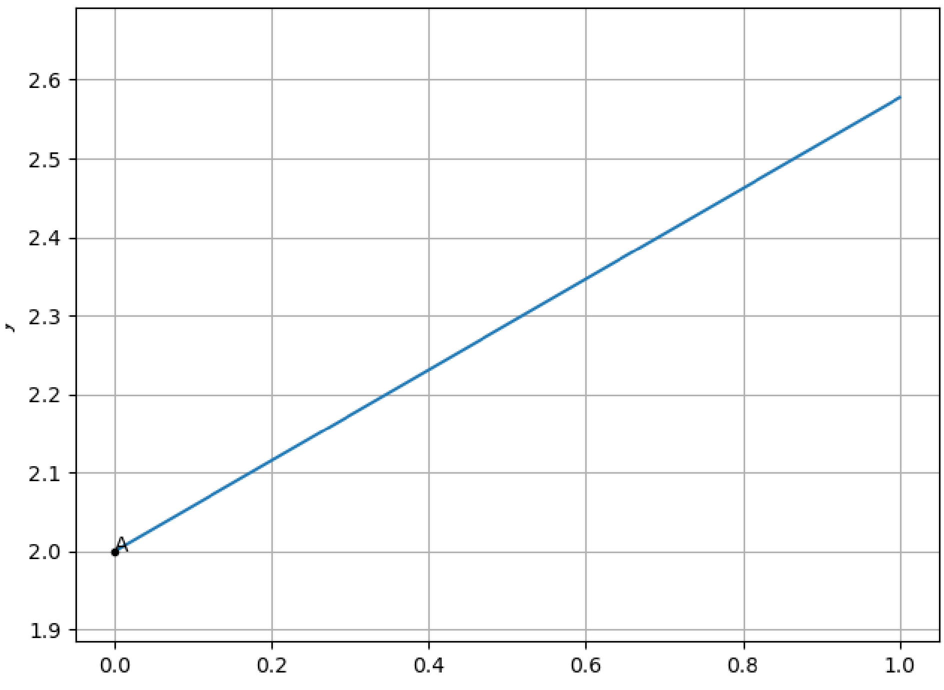


Figure 2.1.6.1:

Solution: Let

$$\mathbf{A} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \ \mathbf{B} = \begin{pmatrix} 2 \\ -4 \end{pmatrix} \tag{2.1.7.1}$$

The direction vector

$$\mathbf{m} = \mathbf{A} - \mathbf{B} = \begin{pmatrix} -3 \\ 5 \end{pmatrix} \tag{2.1.7.2}$$

The corresponding normal vector is

$$\mathbf{n} = \begin{pmatrix} 5 \\ 3 \end{pmatrix} \tag{2.1.7.3}$$

Thus, the equation of line is

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{A}) = 0 \tag{2.1.7.4}$$

$$\implies \begin{pmatrix} 5 & 3 \end{pmatrix} \mathbf{x} = -2 \tag{2.1.7.5}$$

See Fig. 2.1.7.1.

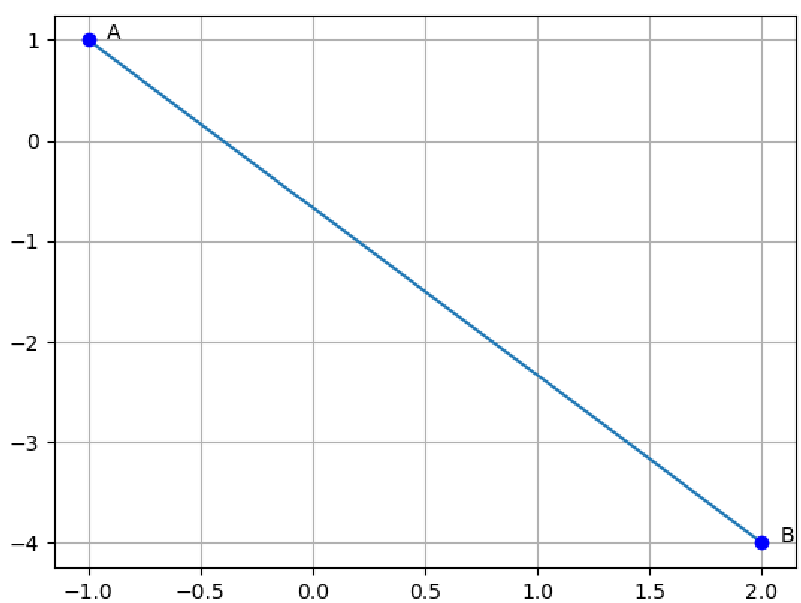


Figure 2.1.7.1:

2.1.8 Find the equation of line whose perpendicular distance from the origin is 5 units and the angle made by the perpendicular with the positive x-axis is $30°$.

Solution: From the given information, the normal vector of the line is

$$\mathbf{n} = \begin{pmatrix} \cos 30° \\ \sin 30° \end{pmatrix} \tag{2.1.8.1}$$

The distance from the origin to the line is given by

$$d = \frac{|c|}{\|\mathbf{n}\|} \implies c = \pm d = \pm 5 \tag{2.1.8.2}$$

since

$$\|\mathbf{n}\| = 1, \tag{2.1.8.3}$$

Thus, the equation of lines are

$$\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \mathbf{x} = \pm 5 \tag{2.1.8.4}$$

See Fig. 2.1.8.1.

2.1.9 The Vertices of Triangle PQR is $\mathbf{P}(2,1), \mathbf{Q}(-2,3), \mathbf{R}(4,5)$. Find the equation of the Median Through $\mathbf{R}$.

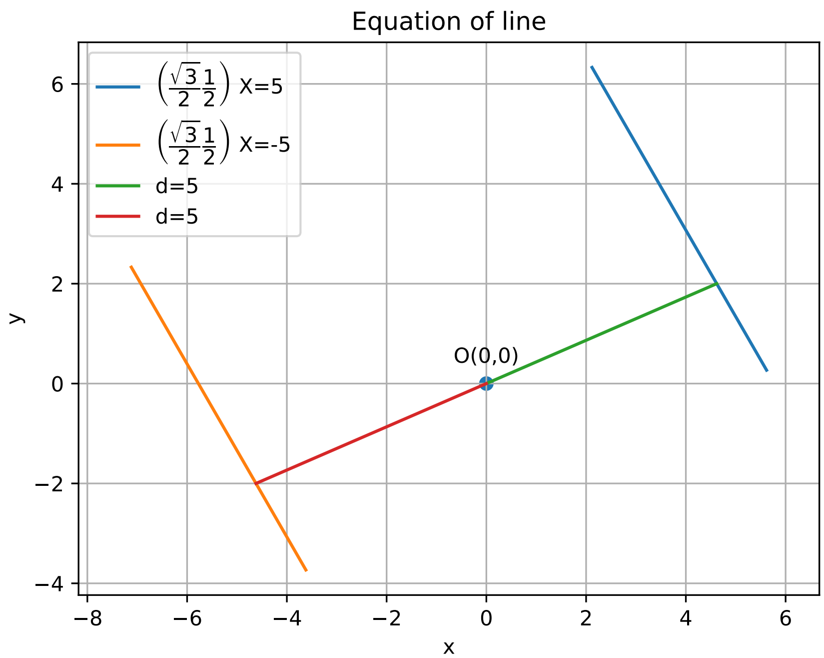


Figure 2.1.8.1:

Solution: See Fig. 2.1.9.1. Using Section Formula,

$$\mathbf{A} = \frac{\mathbf{P}+\mathbf{Q}}{2} \tag{2.1.9.1}$$

$$= \begin{pmatrix} 0 \\ 2 \end{pmatrix} \tag{2.1.9.2}$$

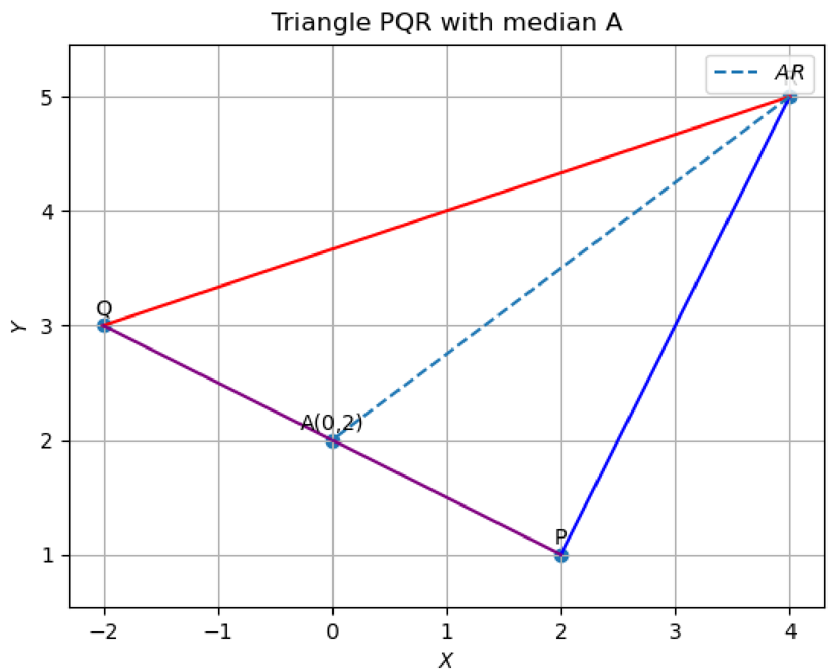


Figure 2.1.9.1:

So , the Direction Vector of AR is

$$\mathbf{m} = \mathbf{R} - \mathbf{A} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \tag{2.1.9.3}$$

$$\implies \mathbf{n} = \begin{pmatrix} 3 \\ -4 \end{pmatrix} \tag{2.1.9.4}$$

which is the normal vector. Thus, from (D.1.2.1), the equation of the line is

$$\begin{pmatrix} 3 & -4 \end{pmatrix} (\mathbf{x} - \mathbf{R}) = 0 \tag{2.1.9.5}$$

$$\implies \begin{pmatrix} 3 & -4 \end{pmatrix} \mathbf{x} = 8 \tag{2.1.9.6}$$

2.1.10 Find the equation of the line passing through (-3,5) and perpendicular to the line through the points (2,5) and (-3,6).

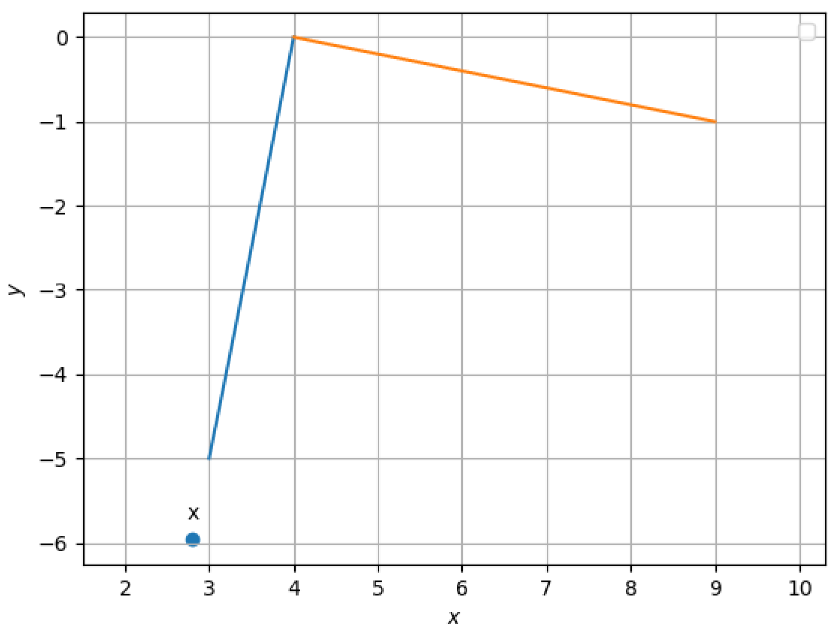


Figure 2.1.10.1:

Solution: Let

$$\mathbf{A} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -3 \\ 6 \end{pmatrix}, \mathbf{P} = \begin{pmatrix} -3 \\ 5 \end{pmatrix} \tag{2.1.10.1}$$

The normal vector of the desired line is then given by

$$\mathbf{n} = \mathbf{B} - \mathbf{A} = \begin{pmatrix} 5 \\ -1 \end{pmatrix} \tag{2.1.10.2}$$

Thus, the equation of the line is

$$\begin{pmatrix} 5 & -1 \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} -3 \\ 5 \end{pmatrix} \right) = 0 \tag{2.1.10.3}$$

$$\implies \begin{pmatrix} 5 & -1 \end{pmatrix} \mathbf{x} = -20 \tag{2.1.10.4}$$

2.1.11 A line perpendicular to the line segement joining the points (1,0) and (2,3) divides it in the ratio $1 : n$. Find the equation of the line.

Solution: Let

$$\mathbf{P} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \tag{2.1.11.1}$$

The direction vector of PQ is

$$\mathbf{m} = \mathbf{Q} - \mathbf{P} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \tag{2.1.11.2}$$

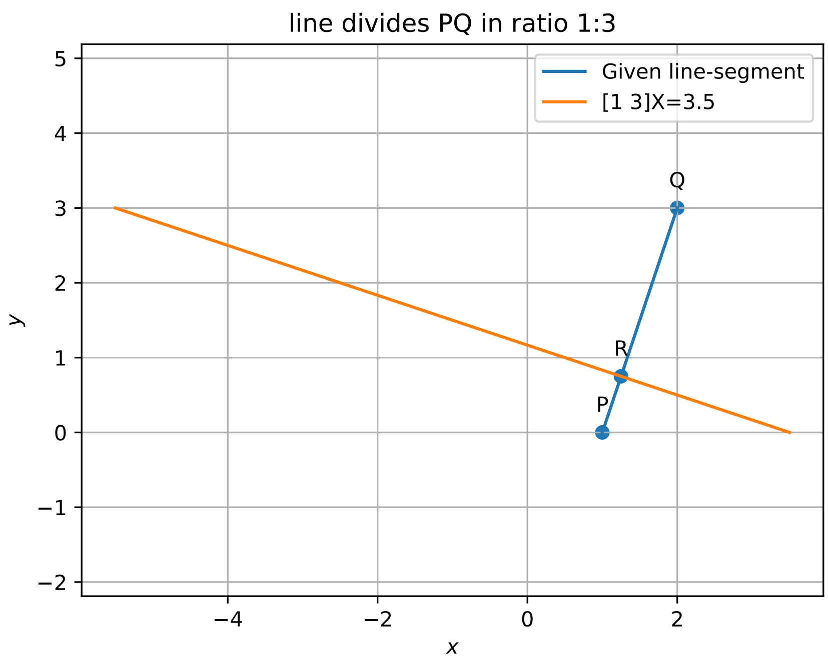


Figure 2.1.11.1:

Also, using section formula,

$$\mathbf{R} = \frac{\mathbf{Q} + n\mathbf{P}}{1+n} \tag{2.1.11.3}$$

and the equation of line passing through $\mathbf{R}$ is

$$\mathbf{m}^\top (\mathbf{x} - \mathbf{R}) = 0 \tag{2.1.11.4}$$

$$\implies \begin{pmatrix} 1 & 3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 \ 3 \end{pmatrix} \begin{pmatrix} \frac{2+n}{1+n} \\ \frac{3}{1+n} \end{pmatrix} \tag{2.1.11.5}$$

$$= \frac{11+n}{1+n} \tag{2.1.11.6}$$

2.1.12

2.1.13 Find the equation of a line that cuts off equal intercepts on the coordinate axes and passes through the point $(2, 3)$.

Solution: Let $\mathbf{P}(a, 0)$, and $\mathbf{Q}(0, a)$ be the 2 points on x and y-axes respectively having a as the intercept on both the axes. We know that the the direction vector $\mathbf{m}$ of the line joining two points $\mathbf{P}, \mathbf{Q}$ is given by

$$\mathbf{m} = \mathbf{P} - \mathbf{Q} \tag{2.1.13.1}$$

$$= \begin{pmatrix} a \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ a \end{pmatrix} = a \begin{pmatrix} 1 \\ -1 \end{pmatrix} \equiv \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{2.1.13.2}$$

Thus, the normal vector $\mathbf{n}$ to the line is given as

$$\mathbf{n} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.1.13.3}$$

The equation of a line with normal vector $\mathbf{n}$ and passing through a point $\mathbf{A}(2,3)$ is given by

$$\mathbf{n}^{\top}(\mathbf{x}-\mathbf{A}) = 0 \tag{2.1.13.4}$$

$$\begin{pmatrix}1 & 1\end{pmatrix}\left(\mathbf{x}-\begin{pmatrix}2\\3\end{pmatrix}\right) = 0 \tag{2.1.13.5}$$

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x}-5 = 0 \tag{2.1.13.6}$$

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 5 \tag{2.1.13.7}$$

To find the intercepts, we know that, since $\mathbf{P}$ and $\mathbf{Q}$ lie on the straight line, they should satisfy (2.1.13.7).

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{P} = 5 \tag{2.1.13.8}$$

$$\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}a\\0\end{pmatrix} = 5 \tag{2.1.13.9}$$

$$a+0 = 5 \tag{2.1.13.10}$$

$$a = 5 \tag{2.1.13.11}$$

Both $\mathbf{P}$ and $\mathbf{Q}$ have the same intercept value, hence the intercept on both x and y-axes is 5 units. The line segment is as shown in Fig. 2.1.13.1.

2.1.14 Find equation of a line passing trough a point (2,2) and cutting off intercepts on the axes whose sum is 9.

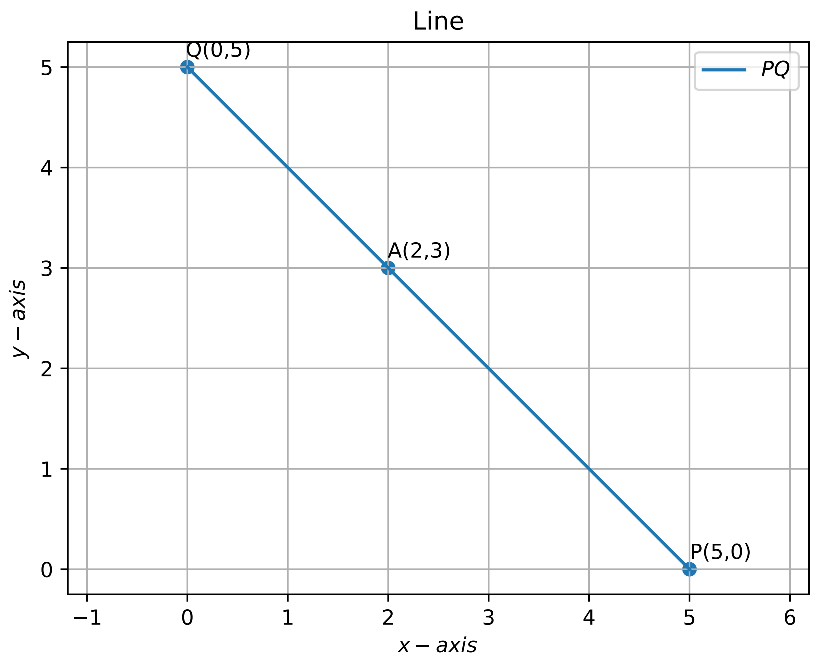


Figure 2.1.13.1:

Solution: Let the x intercept be a and the y intercept be b. Then

$$a + b = 9 \tag{2.1.14.1}$$

Let

$$\mathbf{P} = \begin{pmatrix} a \\ 0 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} 0 \\ b \end{pmatrix}, \mathbf{R} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \tag{2.1.14.2}$$

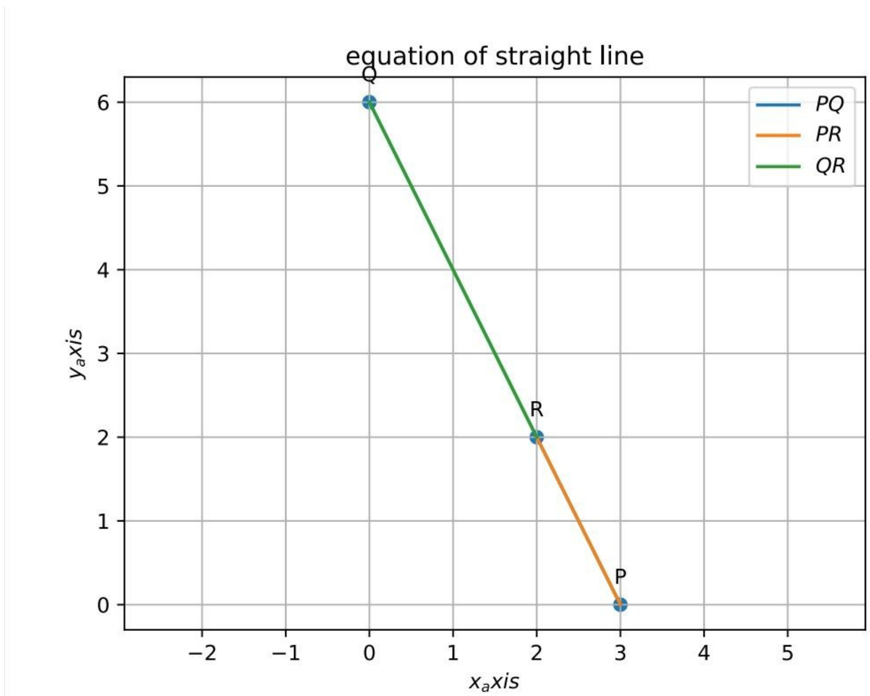


Figure 2.1.14.1:

Since the points are collinear, from (D.1.4.1), we obtain the matrix

$$\begin{pmatrix} \mathbf{P}-\mathbf{Q} & \mathbf{P}-\mathbf{R} \end{pmatrix} = \begin{pmatrix} a & a-2 \\ -b & -2 \end{pmatrix} \tag{2.1.14.3}$$

which is singular if the determinant

$$-2a + b(a-2) = ab - 2\left(a+b\right) = 0 \tag{2.1.14.4}$$

yielding

$$ab = 18 \tag{2.1.14.5}$$

upon substituting from (2.1.14.1). (2.1.14.5) and (2.1.14.1) form

$$x^2 - 9x + 18 = 0 \tag{2.1.14.6}$$

with roots

$$x = 6, 3 \tag{2.1.14.7}$$

$$\text{or,} \quad \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \end{pmatrix} \tag{2.1.14.8}$$

Since the direction vector of the line is

$$\mathbf{P} - \mathbf{Q} = \begin{pmatrix} a \\ -b \end{pmatrix}, \tag{2.1.14.9}$$

the normal vector is

$$\mathbf{n} = \begin{pmatrix} b \\ a \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{2.1.14.10}$$

Thus, the possible equations of the line are

$$\begin{pmatrix} 1 & 2 \end{pmatrix} \mathbf{x} = 6 \tag{2.1.14.11}$$

$$\begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = 6 \tag{2.1.14.12}$$

2.1.15 Find the equation of the line through the point (0,2) making an angle

$$2\pi/3 \tag{2.1.15.1}$$

with the positive X-axis. Also find the equation of the line parallel to it and crossing the Y-axis at a distance of 2 units below the origin

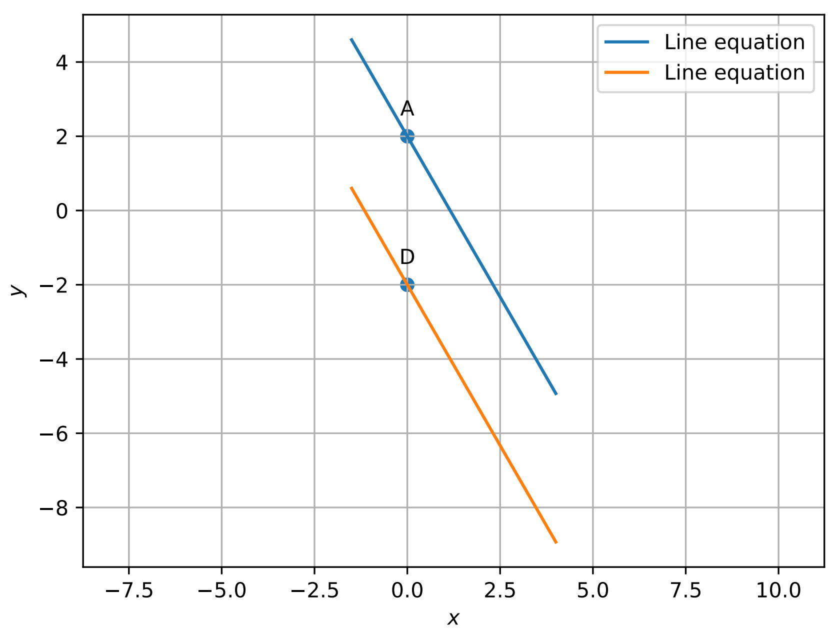


Figure 2.1.15.1:

Solution: From the given information, the direction vector is

$$\mathbf{m} = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} \tag{2.1.15.2}$$

Thus, the normal vector is

$$\mathbf{n} = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} \tag{2.1.15.3}$$

and the equation of the line is

$$\begin{pmatrix} \sqrt{3} & 1 \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right) = 0 \tag{2.1.15.4}$$

$$\implies \begin{pmatrix} \sqrt{3} & 1 \end{pmatrix} \mathbf{x} = 2 \tag{2.1.15.5}$$

The equation of the parallel crossing the Y-axis at a distance of 2 units below the origin is given by

$$\begin{pmatrix} \sqrt{3} & 1 \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} 0 \\ -2 \end{pmatrix} \right) = 0 \tag{2.1.15.6}$$

$$\implies \begin{pmatrix} \sqrt{3} & 1 \end{pmatrix} \mathbf{x} = -2 \tag{2.1.15.7}$$

2.1.16 The perpendicular from the origin to a line meets it at the point (-2,9). Find the equation of the line.

Solution:

Given

$$\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} -2 \\ 9 \end{pmatrix} \tag{2.1.16.1}$$

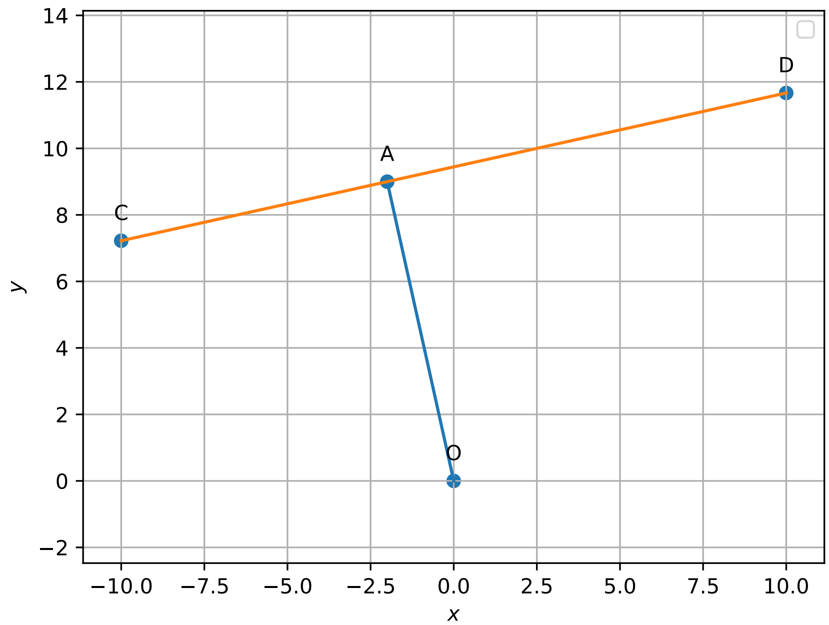


Figure 2.1.16.1:

The normal vector is

$$\mathbf{n} = \mathbf{O} - \mathbf{A} \tag{2.1.16.2}$$

$$= \begin{pmatrix} 2 \\ -9 \end{pmatrix} \tag{2.1.16.3}$$

yielding the equation of the line as

$$\begin{pmatrix} 2 & -9 \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} 2 \\ -9 \end{pmatrix} \right) = 0 \tag{2.1.16.4}$$

$$\implies \begin{pmatrix} 2 & -9 \end{pmatrix} \mathbf{x} = 85 \tag{2.1.16.5}$$

2.1.17 $P(a, b)$ is the mid-point of the line segment between axes. Show that the equation of the line is $\frac{x}{a} + \frac{y}{b} = 2$

Solution: Let

$$\mathbf{A} = x\mathbf{e_1}, \mathbf{B} = y\mathbf{e_2}, \mathbf{P} = \begin{pmatrix} a \\ b \end{pmatrix} \tag{2.1.17.1}$$

where

$$\mathbf{e_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \mathbf{e_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.1.17.2}$$

as shown in Fig. 2.1.17.1 Given that

$$\mathbf{P} = \frac{\mathbf{A} + \mathbf{B}}{2} = \frac{x\mathbf{e_1} + y\mathbf{e_2}}{2} \tag{2.1.17.3}$$

$$\implies 2\mathbf{P} = x\mathbf{e_1} + y\mathbf{e_2} \tag{2.1.17.4}$$

$$\mathbf{e_1}^\top (2\mathbf{P}) = \mathbf{e_1^\top} (x\mathbf{e_1} + y\mathbf{e_2}) = x \tag{2.1.17.5}$$

$$\text{and } \mathbf{e_2}^\top (2\mathbf{P}) = \mathbf{e_2^\top} (x\mathbf{e_1} + y\mathbf{e_2}) = y \tag{2.1.17.6}$$

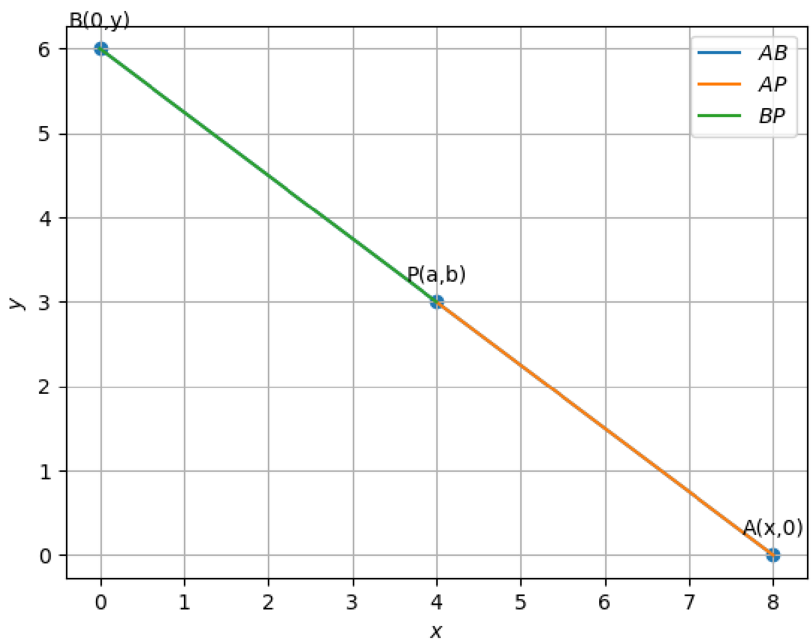


Figure 2.1.17.1:

Thus,

$$x = 2\mathbf{e_1}^\top \mathbf{P} = 2a \tag{2.1.17.7}$$

$$y = 2\mathbf{e_2}^\top \mathbf{P} = 2b \tag{2.1.17.8}$$

yielding

$$\mathbf{A} = 2a\mathbf{e_1}\mathbf{B} \qquad = 2b\mathbf{e_1} \tag{2.1.17.9}$$

Thus, the direction vector of the line is

$$\mathbf{m} = \mathbf{A} - \mathbf{B} \tag{2.1.17.10}$$

$$= \begin{pmatrix} a \\ -b \end{pmatrix} \tag{2.1.17.11}$$

and the normal vector is

$$\mathbf{n} = \begin{pmatrix} b \\ a \end{pmatrix} \tag{2.1.17.12}$$

The equation of line passing through $\mathbf{P}$ is then obtained as

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{P}) = 0 \tag{2.1.17.13}$$

$$\begin{pmatrix} b & a \end{pmatrix} \mathbf{x} = 2ab. \tag{2.1.17.14}$$

2.1.18 Point $\mathbf{R}(h, k)$ divides a line segment between the axes in the ratio 1: 2. Find the equation of the line. Let the line segment between the axes be AB, with point $\mathbf{A}$ on X-axis, $\mathbf{B}$ on Y-axis. Let the points $\mathbf{A}$, $\mathbf{B}$ be

$$\mathbf{A} = \begin{pmatrix} \alpha \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ \beta \end{pmatrix} \tag{2.1.18.1}$$

Given that $\frac{AR}{RB} = \frac{1}{2}$. By using section formula, we get

$$\mathbf{R} = \frac{2\mathbf{A} + \mathbf{B}}{3} \tag{2.1.18.2}$$

$$\begin{pmatrix} h \\ k \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2\alpha \\ \beta \end{pmatrix} \tag{2.1.18.3}$$

$$h = \frac{2\alpha}{3} \tag{2.1.18.4}$$

$$k = \frac{\beta}{3} \tag{2.1.18.5}$$

$$\mathbf{A} = \begin{pmatrix} \frac{3h}{2} \\ 0 \end{pmatrix}, \ \mathbf{B} = \begin{pmatrix} 0 \\ 3k \end{pmatrix} \tag{2.1.18.6}$$

The direction vector of the line is given by,

$$\mathbf{m} = \mathbf{R} - \mathbf{B} \tag{2.1.18.7}$$

$$\mathbf{m} = \begin{pmatrix} h \\ -2k \end{pmatrix} \tag{2.1.18.8}$$

The normal vector to the line is given by,

$$\mathbf{n} = \begin{pmatrix} 2k \\ h \end{pmatrix} \tag{2.1.18.9}$$

The equation of the line is given by,

$$\mathbf{n}^\top \mathbf{x} = \mathbf{n}^\top \mathbf{B} \tag{2.1.18.10}$$

$$\implies \begin{pmatrix} 2k & h \end{pmatrix} \mathbf{x} = \begin{pmatrix} 2k & h \end{pmatrix} \begin{pmatrix} 0 \\ 3k \end{pmatrix} \tag{2.1.18.11}$$

$$\text{or, } \begin{pmatrix} 2k & h \end{pmatrix} \mathbf{x} = 3hk \tag{2.1.18.12}$$

2.1.19 By using the concept of equation of a line, prove that the three points (3, 0), (− 2, − 2) and (8, 2) are collinear.

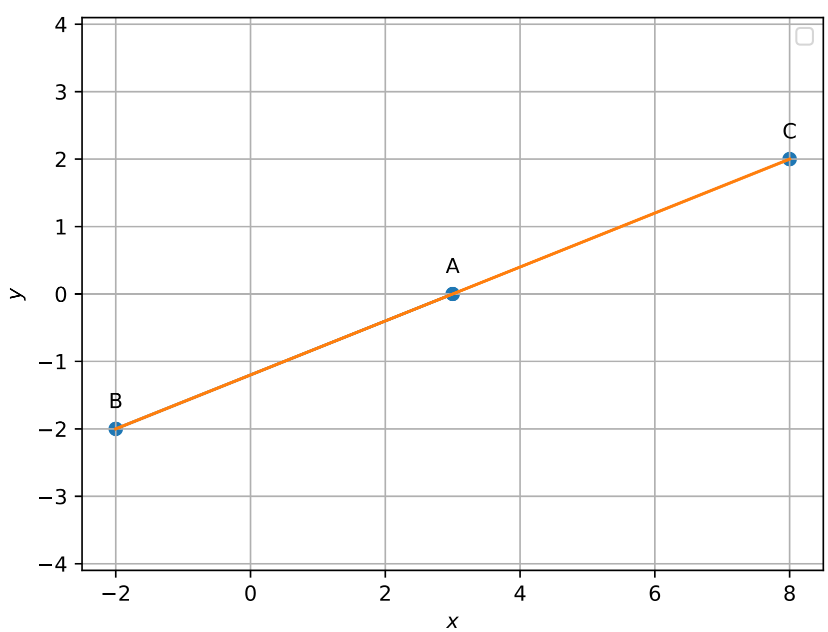


Figure 2.1.19.1:

Solution: The collinearity matrix can be expressed as

$$\begin{pmatrix} -5 & -2 \\ 5 & 2 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_1 + R_2} \begin{pmatrix} -5 & -2 \\ 0 & 0 \end{pmatrix} \tag{2.1.19.1}$$

which is a rank 1 matrix.

2.1.20 Find the equation of the line parallel to the line 3x-4y+2=0 and passing through the point (-2,3).

Solution: From the given information,

$$\mathbf{n} = \begin{pmatrix} 3 \\ -4 \end{pmatrix} \tag{2.1.20.1}$$

$$\implies \begin{pmatrix} 3 & -4 \end{pmatrix} \left\{ \mathbf{x} - \begin{pmatrix} -2 \\ 3 \end{pmatrix} \right\} = 0 \tag{2.1.20.2}$$

$$= -18 \tag{2.1.20.3}$$

which is the required equation of the line.

2.1.21 Find the equation of line perpendicular to the line $x - 7y + 5 = 0$ and having x intercept 3

Solution: The given line parameters are

$$\mathbf{n} = \begin{pmatrix} 1 \\ -7 \end{pmatrix}, c = -5 \tag{2.1.21.1}$$

yielding

$$\mathbf{m} = \begin{pmatrix} 7 \\ 1 \end{pmatrix} \tag{2.1.21.2}$$

The perpendicular passes through

$$\mathbf{A} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \tag{2.1.21.3}$$

Hence, the desired equation is

$$\begin{pmatrix} 7 & 1 \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} 3 \\ 0 \end{pmatrix} \right) = 0 \tag{2.1.21.4}$$

$$\implies \begin{pmatrix} 7 & 1 \end{pmatrix} \mathbf{x} = 21 \tag{2.1.21.5}$$

See Fig. 2.1.21.1.

2.1.22 Prove that the line through the point(x_1, y_1) and parallel to the line Ax+By+C=0 is A$(x - x_1)$+B$(y - y_1)$=0.

Solution: The given line parameters are

$$\mathbf{n} = \begin{pmatrix} A \\ B \end{pmatrix}, c = C \tag{2.1.22.1}$$

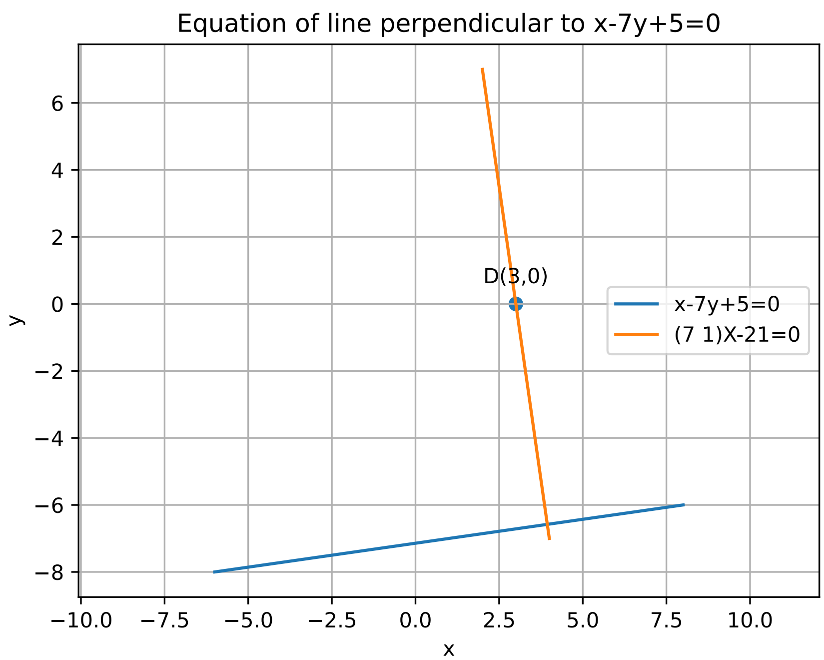


Figure 2.1.21.1:

Let

$$\mathbf{P} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \tag{2.1.22.2}$$

$$\tag{2.1.22.3}$$

Then the equation of the desired line is

$$\mathbf{n}^{\top} (\mathbf{x} - \mathbf{P}) = 0 \tag{2.1.22.4}$$

$$\implies A(x - x_1) + B(y - y_1) = 0 \tag{2.1.22.5}$$

2.1.23 Find the equation of the line passing through the point $(1, 2, -4)$ and perpendicular to the two lines

$$\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7} \text{ and} \tag{2.1.23.1}$$

$$\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5} \tag{2.1.23.2}$$

Solution: The direction vectors of the lines are

$$\mathbf{m_1} = \begin{pmatrix} 3 \\ -16 \\ 7 \end{pmatrix}, \mathbf{m_2} = \begin{pmatrix} 3 \\ 8 \\ -5 \end{pmatrix} \tag{2.1.23.3}$$

Let $\mathbf{m}$ denote the direction vector of the line perpendicular to the given two lines. Then,

$$\mathbf{m_1}^\top \mathbf{m} = 0 \tag{2.1.23.4}$$

$$\mathbf{m_2}^\top \mathbf{m} = 0 \tag{2.1.23.5}$$

$$\implies \begin{pmatrix} 3 & -16 & 7 \\ 3 & 8 & -5 \end{pmatrix} \mathbf{m} = 0 \tag{2.1.23.6}$$

Row reducing the augmented matrix,

$$\xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 3 & -16 & 7 \\ 0 & 24 & -12 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 + \frac{2}{3} R_2} \begin{pmatrix} 3 & 0 & -1 \\ 0 & 24 & -12 \end{pmatrix} \tag{2.1.23.7}$$

$$\xleftrightarrow{R_2 \leftarrow R_2/12} \begin{pmatrix} 3 & 0 & -1 \\ 0 & 2 & -1 \end{pmatrix} \tag{2.1.23.8}$$

yielding

$$\mathbf{m} = \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} \tag{2.1.23.9}$$

Hence the vector equation of the line passing through $(1, 2, -4)$ is,

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} \tag{2.1.23.10}$$

2.1.24 Find the vector equation of the line passing through $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and parallel to the planes $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}^\top \mathbf{x} = 5$ and $\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}^\top \mathbf{x} = 6$. **Solution:** The normal vectors of the planes are

$$\mathbf{n}_1 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \mathbf{n}_2 = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}. \tag{2.1.24.1}$$

Since the desired line is parallel to both the planes,

$$\mathbf{n}_1^\top \mathbf{m} = 0 \tag{2.1.24.2}$$

$$\mathbf{n}_2^\top \mathbf{m} = 0 \tag{2.1.24.3}$$

$$\implies \begin{pmatrix} 1 & -1 & 2 \\ 3 & 1 & 1 \end{pmatrix} \mathbf{m} = 0 \tag{2.1.24.4}$$

where $\mathbf{m}$ is the direction vector of the line. Let's reduce the matrix from equation (2.1.24.4) to row-echelon form

$$\begin{pmatrix} 1 & -1 & 2 \\ 3 & 1 & 1 \end{pmatrix} \xleftrightarrow{R_2 \to -\frac{3}{4}R_1 + \frac{1}{4}R_2} \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -\frac{5}{4} \end{pmatrix} \tag{2.1.24.5}$$

$$\begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -\frac{5}{4} \end{pmatrix} \xleftrightarrow{R_1 \to R_1 + R_2} \begin{pmatrix} 1 & 0 & \frac{3}{4} \\ 0 & 1 & -\frac{5}{4} \end{pmatrix} \tag{2.1.24.6}$$

Using (2.1.24.4), (2.1.24.5) and (2.1.24.6),

$$\begin{pmatrix} 1 & 0 & \frac{3}{4} \\ 0 & 1 & -\frac{5}{4} \end{pmatrix} \mathbf{m} = 0 \tag{2.1.24.7}$$

$$\implies \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = \begin{pmatrix} -\frac{3}{4}m_3 \\ \frac{5}{4}m_3 \\ m_3 \end{pmatrix} \tag{2.1.24.8}$$

$$\implies \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = m_3 \begin{pmatrix} -\frac{3}{4} \\ \frac{5}{4} \\ 1 \end{pmatrix} \tag{2.1.24.9}$$

$$\implies \mathbf{m} = \begin{pmatrix} -3 \\ 5 \\ 4 \end{pmatrix} \tag{2.1.24.10}$$

It is given that the line passes through the point $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, so the final equation of line is

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 5 \\ 4 \end{pmatrix} \tag{2.1.24.11}$$

2.2. Perpendicular

2.2.1

2.2.2

2.2.3 Reduce the following equations into normal form. Find their perpendicular distances from the origin and angle between perpendicular and the positive x-axis.

(a) $x - \sqrt{3}y + 8 = 0$

(b) $y - 2 = 0$

(c) $x - y = 4$

Solution:

(a) The given equation can be expressed as

$$\begin{pmatrix} 1 & -\sqrt{3} \end{pmatrix} \mathbf{x} = -8 \tag{2.2.3.1}$$

yielding

$$\mathbf{n} = \begin{pmatrix} 1 & -\sqrt{3} \end{pmatrix}, c = -8 \tag{2.2.3.2}$$

From the above, the angle between perpendicular and the positive x-axis is given by

$$\tan^{-1}\left(-\sqrt{3}\right) = \frac{2\pi}{3} \tag{2.2.3.3}$$

The perpendicular distance from the origin to the line is given by

$$d = \frac{|c|}{\|\mathbf{n}\|} = 4 \tag{2.2.3.4}$$

(b) In this case, the given equation becomes

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 2 \tag{2.2.3.5}$$

yielding

$$\mathbf{n} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, c = 2 \tag{2.2.3.6}$$

Angle between perpendicular and the positive x-axis is given by:

$$\tan^{-1} \infty = \frac{\pi}{2} \tag{2.2.3.7}$$

and the perpendicular distance from the origin to the line is given by

$$d = \frac{|c|}{\|\mathbf{n}\|} = 2 \tag{2.2.3.8}$$

(c) The given equation can be expressed as

$$\begin{pmatrix} -1 & 1 \end{pmatrix} \mathbf{x} = 4 \tag{2.2.3.9}$$

yielding

$$\mathbf{n} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, c = 4 \tag{2.2.3.10}$$

Angle between perpendicular and the positive x-axis is given by

$$\tan^{-1}(-1) = \frac{3\pi}{4} \tag{2.2.3.11}$$

The perpendicular distance from the origin to the line is given by

$$d = \frac{|c|}{\|\mathbf{n}\|} = \frac{4}{\sqrt{2}} = 2\sqrt{2} \tag{2.2.3.12}$$

2.2.4 Find the distance of the point $(-1, 1)$ from the line $12(x+6) = 5(y-2)$.

Solution:

(a) The equation of the line is $12(x+6) = 5(y-2)$. Rearranging the equation,

$$12x - 5y = -10 - 72 \tag{2.2.4.1}$$

$$12x - 5y = -82 \tag{2.2.4.2}$$

This can be equated to

$$\mathbf{n}^\top \mathbf{x} = c \tag{2.2.4.3}$$

$$\text{where } \mathbf{n} = \begin{pmatrix} 12 \\ -5 \end{pmatrix}, c = -82 \tag{2.2.4.4}$$

We need to compute the distance from a point $\mathbf{P}\begin{pmatrix}-1\\1\end{pmatrix}$ to the line. Without loss of generality, let $\mathbf{A}$ be the foot of the perpendicular from $\mathbf{P}$ to the line in Equation (2.2.4.3). The equation of the normal to Equation (2.2.4.3) can then be expressed as

$$\mathbf{x} = \mathbf{A} + \lambda\mathbf{n} \tag{2.2.4.5}$$

$$\implies \mathbf{P} - \mathbf{A} = \lambda\mathbf{n} \tag{2.2.4.6}$$

$\because$ $\mathbf{P}$ lies on (2.2.4.5). From the above, the desired distance can be expressed as

$$d = \|\mathbf{P} - \mathbf{A}\| = |\lambda| \|\mathbf{n}\| \tag{2.2.4.7}$$

From (2.2.4.6),

$$\mathbf{n}^\top (\mathbf{P} - \mathbf{A}) = \lambda\mathbf{n}^\top\mathbf{n} = \lambda \|\mathbf{n}\|^2 \tag{2.2.4.8}$$

$$\implies |\lambda| = \frac{\left|\mathbf{n}^\top (\mathbf{P} - \mathbf{A})\right|}{\|\mathbf{n}\|^2} \tag{2.2.4.9}$$

Substituting the above in (2.2.4.7) and using the fact that

$$\mathbf{n}^\top \mathbf{A} = c \tag{2.2.4.10}$$

from (2.2.4.3), yields

$$d = \frac{\left|\mathbf{n}^\top \mathbf{P} - c\right|}{\|\mathbf{n}\|} \tag{2.2.4.11}$$

$$= \frac{\left|\begin{pmatrix} 12 & -5 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} - (-82)\right|}{\sqrt{12^2 + (-5)^2}} \tag{2.2.4.12}$$

$$= \frac{|-17 + 82|}{\sqrt{169}} = \frac{|65|}{13} = 5 \text{ units} \tag{2.2.4.13}$$

(b) The foot of the perpendicular from $\mathbf{P}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ to line in (2.2.4.3) is expressed as

$$\begin{pmatrix} \mathbf{m} & \mathbf{n} \end{pmatrix}^\top \mathbf{A} = \begin{pmatrix} \mathbf{m}^\top \mathbf{P} \\ c \end{pmatrix} \tag{2.2.4.14}$$

where $\mathbf{m}$ is the direction vector of the given line

$$\because \mathbf{n} = \begin{pmatrix} 12 \\ -5 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} 5 \\ 12 \end{pmatrix} \tag{2.2.4.15}$$

$$(2.2.4.14) \implies \begin{pmatrix} 5 & 12 \\ 12 & -5 \end{pmatrix} \mathbf{A} = \begin{pmatrix} \begin{pmatrix} 5 & 12 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \\ -82 \end{pmatrix} \tag{2.2.4.16}$$

$$\begin{pmatrix} 5 & 12 \\ 12 & -5 \end{pmatrix} \mathbf{A} = \begin{pmatrix} 7 \\ -82 \end{pmatrix} \tag{2.2.4.17}$$

The augmented matrix for the system equations in (2.2.4.17) is expressed as

$$\left(\begin{array}{cc|c} 5 & 12 & 7 \\ 12 & -5 & -82 \end{array}\right) \tag{2.2.4.18}$$

Performing sequence of row operations to transform into RREF form

$$\xleftrightarrow{R_2 \to R_2 - \frac{12}{5} R_1} \left(\begin{array}{cc|c} 5 & 12 & 7 \\ 0 & -\frac{169}{5} & -\frac{494}{5} \end{array}\right) \tag{2.2.4.19}$$

$$\xleftrightarrow[R_1 \to \frac{1}{5} R_1]{R_2 \to \frac{-5}{169} R_2} \left(\begin{array}{cc|c} 1 & \frac{12}{5} & \frac{7}{5} \\ 0 & 1 & \frac{38}{13} \end{array}\right) \tag{2.2.4.20}$$

$$\xleftrightarrow{R_1 \to R_1 - \frac{12}{5} R_2} \left(\begin{array}{cc|c} 1 & 0 & -\frac{73}{13} \\ 0 & 1 & \frac{38}{13} \end{array}\right) \tag{2.2.4.21}$$

$$\mathbf{A} = \begin{pmatrix} -\frac{73}{13} \\ \frac{38}{13} \end{pmatrix} \tag{2.2.4.22}$$

The desired line and the perpendicular line from $\mathbf{P}$ is shown as in Fig. 2.2.4.1

2.2.5 Find the points on the x-axis, whose distances from the line $\frac{x}{3} + \frac{y}{4} = 1$ are 4 units.

Solution: The given line can be expressed as

$$\mathbf{n}^\top \mathbf{x} = c, \text{ where } \mathbf{n} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}, c = 12 \tag{2.2.5.1}$$

The distance formula is given by

$$d = \frac{\left|\mathbf{n}^\top \mathbf{P} - c\right|}{\|\mathbf{n}\|} \tag{2.2.5.2}$$

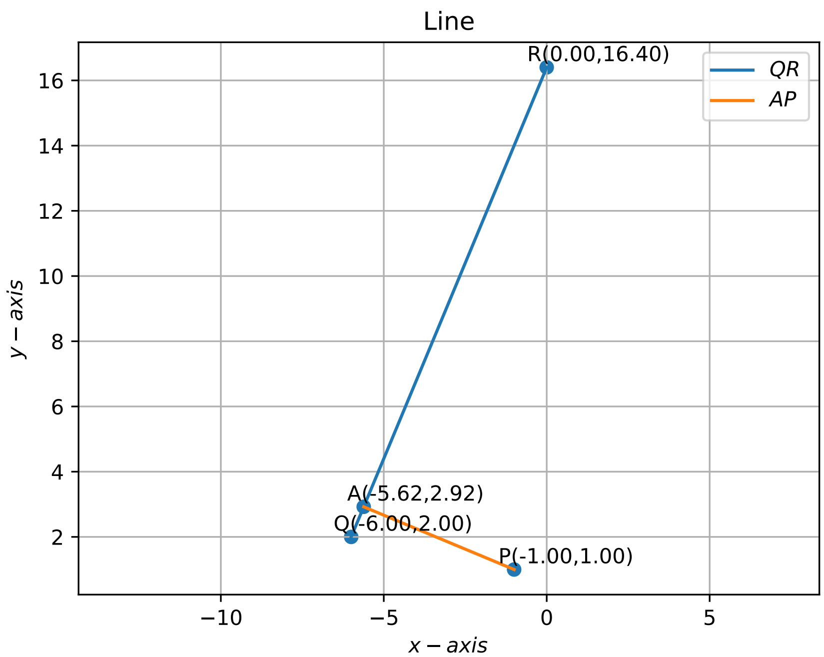


Figure 2.2.4.1:

Let the desired point be

$$\mathbf{P} = x\mathbf{e}_1 = \begin{pmatrix} x \\ 0 \end{pmatrix} \tag{2.2.5.3}$$

Substituting the values in the distance formula,

$$d = \frac{\left|\mathbf{n}^\top \mathbf{P} - c\right|}{\|\mathbf{n}\|} \tag{2.2.5.4}$$

$$= \frac{\left|x\mathbf{n}^\top \mathbf{e}_1 - c\right|}{\|\mathbf{n}\|} \tag{2.2.5.5}$$

$$\implies \left|x\mathbf{n}^\top \mathbf{e}_1 - c\right| = d\left\|\mathbf{n}\right\| \tag{2.2.5.6}$$

$$\text{or, } x = \frac{\pm d\left\|\mathbf{n}\right\| + c}{\mathbf{n}^\top \mathbf{e}_1} \tag{2.2.5.7}$$

Since

$$d = 4, \tag{2.2.5.8}$$

substituting numerical values,

$$x = 8, -2 \tag{2.2.5.9}$$

This is verified in Fig. 2.2.5.1.

2.2.6 Find the distance between parallel lines

(a) $15x + 8y - 34 = 0$ and $15x + 8y + 31 = 0$

(b) $l(x + y) + p = 0$ and $l(x + y) - r = 0$

Solution:

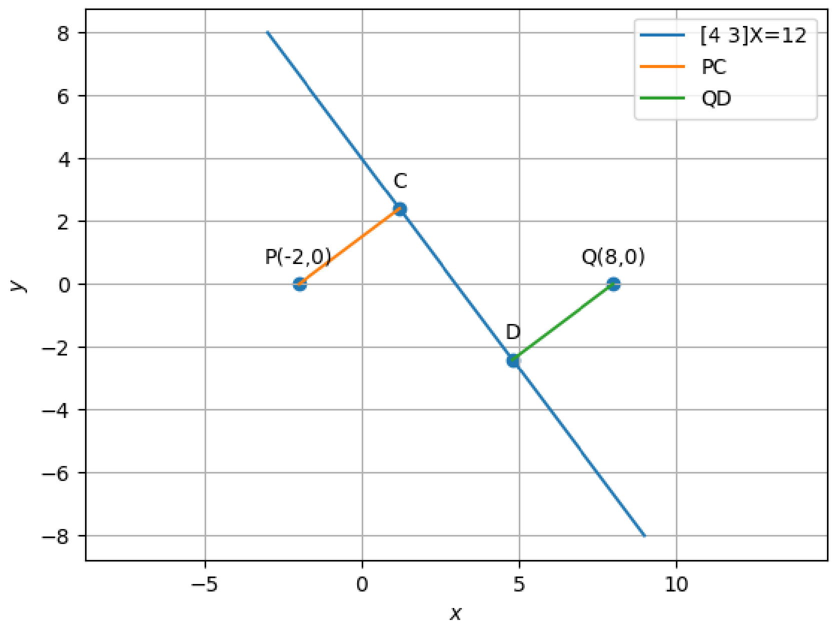


Figure 2.2.5.1:

(a) The given lines can be expressed as

$$\begin{pmatrix} 15 & 8 \end{pmatrix} \mathbf{x} = -34 \tag{2.2.6.1}$$

$$\begin{pmatrix} 15 & 8 \end{pmatrix} \mathbf{x} = 31 \tag{2.2.6.2}$$

$$\implies \mathbf{n} = \begin{pmatrix} 15 \\ 8 \end{pmatrix}, c_1 = -34, c_2 = 31 \tag{2.2.6.3}$$

The distance between them can then be expressed as

$$d = \frac{|c_1 - c_2|}{\|\mathbf{n}\|} = \frac{|-34 - 31|}{\sqrt{289}} = \frac{65}{17} \tag{2.2.6.4}$$

(b) The given lines can be expressed as

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = \frac{-p}{l} \tag{2.2.6.5}$$

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = \frac{-r}{l} \tag{2.2.6.6}$$

$$\implies \mathbf{n} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, c_1 = \frac{-p}{l}, c_2 \frac{-r}{l} \tag{2.2.6.7}$$

The distance between them is then obtained as

$$d = \frac{1}{l\sqrt{2}} |p - r| \tag{2.2.6.8}$$

2.2.7 Find the coordinates of the foot of the perpendicular from $(-1, 3)$ to the line $3x - 4y - 16 = 0$.

Solution: Let

$$\mathbf{P} = \begin{pmatrix} -1 \\ 3 \end{pmatrix} \tag{2.2.7.1}$$

The line parameters are

$$\mathbf{n} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}, c = 16 \tag{2.2.7.2}$$

The desired foot of the perpendicular is then given by

$$\begin{pmatrix} 4 & 3 \\ 3 & -4 \end{pmatrix} \mathbf{A} = \begin{pmatrix} \begin{pmatrix} 4 & 3 \end{pmatrix} \begin{pmatrix} -1 \\ 3 \end{pmatrix} \\ 16 \end{pmatrix} \tag{2.2.7.3}$$

$$= \begin{pmatrix} 5 \\ 16 \end{pmatrix} \tag{2.2.7.4}$$

The augmented matrix for the above system is

$$\begin{pmatrix} 4 & 3 & 5 \\ 3 & -4 & 16 \end{pmatrix} \xleftrightarrow{R_2 = R_2 - \frac{3}{4}R_1} \begin{pmatrix} 4 & 3 & 5 \\ 0 & \frac{-25}{4} & \frac{49}{4} \end{pmatrix} \tag{2.2.7.5}$$

$$\xleftrightarrow{R_2 = \frac{-4}{25}} \begin{pmatrix} 4 & 3 & 5 \\ 0 & 1 & \frac{-49}{25} \end{pmatrix} \xleftrightarrow{R_1 = \frac{1}{4}R_1} \begin{pmatrix} 1 & \frac{3}{4} & \frac{5}{4} \\ 0 & 1 & \frac{-49}{25} \end{pmatrix} \tag{2.2.7.6}$$

$$\xleftrightarrow{R_1 = R_1 - \frac{3}{4}R_2} \begin{pmatrix} 1 & 0 & \frac{68}{25} \\ 0 & 1 & \frac{-49}{25} \end{pmatrix} \tag{2.2.7.7}$$

yielding

$$\mathbf{A} = \begin{pmatrix} \frac{68}{25} \\ \frac{-49}{25} \end{pmatrix} \tag{2.2.7.8}$$

See Fig. 2.2.7.1.

2.2.8 If p and q are the lengths of perpendiculars from the origin to the lines $x\cos\theta - y\sin\theta = k\cos 2\theta$ and $x\sec\theta + y\,\text{cosec}\,\theta = k$, respectively, prove that $p^2 + 4q^2 = k^2$

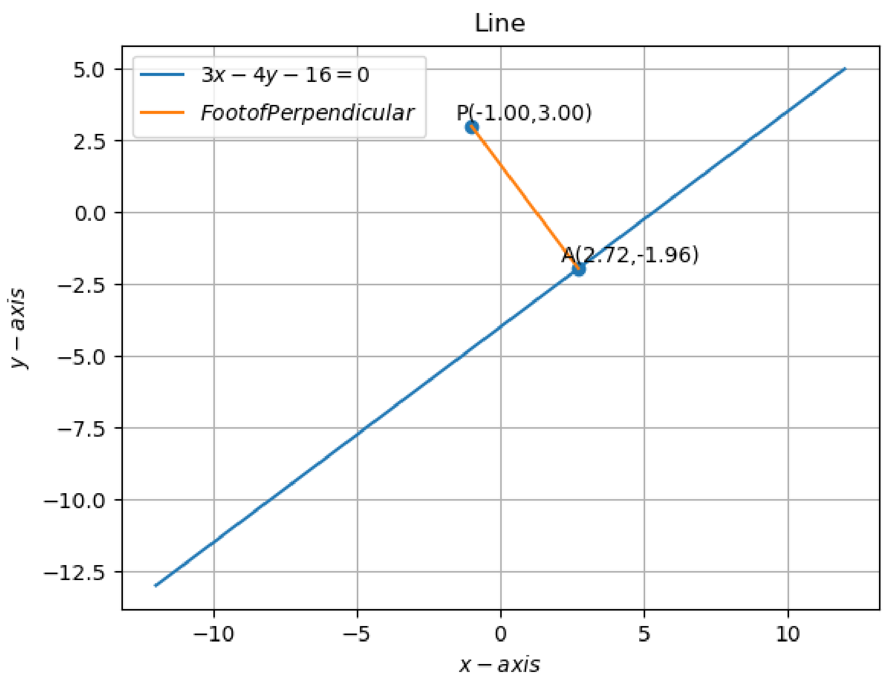


Figure 2.2.7.1:

Solution: The line parameters are

$$\mathbf{n}_1 = \begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix} \text{ and } c_1 = k\cos 2\theta \tag{2.2.8.1}$$

$$\mathbf{n}_2 = \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix} \text{ and } c_2 = \frac{1}{2}k\sin 2\theta \tag{2.2.8.2}$$

Then

$$p = \frac{\left|\mathbf{n}_1^\top \mathbf{x} - c_1\right|}{\|\mathbf{n}_1\|} = |k \cos 2\theta| \tag{2.2.8.3}$$

$$q = \frac{\left|\mathbf{n}_2^\top \mathbf{x} - c_2\right|}{\|\mathbf{n}_2\|} = \left|\frac{1}{2} k \sin 2\theta\right| \tag{2.2.8.4}$$

Therefore

$$p^2 + 4q^2 = k^2 \cos^2 2\theta + 4\left(\frac{1}{4}\right) k^2 \sin^2 2\theta = k^2 \tag{2.2.8.5}$$

2.2.9 In the triangle ABC with vertices $\mathbf{A}\,(2,3)$, $\mathbf{B}\,(4,-1)$ and $\mathbf{C}\,(1,2)$, find the equation and length of altitude from the vertex $\mathbf{A}$.

Solution:

(a) Let

$$\mathbf{A} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}. \tag{2.2.9.1}$$

The normal vector for the altitude from vertex $\mathbf{A}$ is the direction vector of the line BC. The direction vector of the line BC is given by,

$$\mathbf{m}_{BC} = \mathbf{B} - \mathbf{C}\mathbf{m}_{BC} \equiv \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{2.2.9.2}$$

$$\implies \mathbf{n}_{BC} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.2.9.3}$$

The equation of line BC is given by,

$$\mathbf{n}_{BC}^{\top}\mathbf{x} = \mathbf{n}_{BC}^{\top}\mathbf{B} \tag{2.2.9.4}$$

$$\implies \begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} = 3 \tag{2.2.9.5}$$

Using the distance formula, the length of the desired altitude is

$$d = \frac{\left|\mathbf{n}^{\top}\mathbf{A} - c\right|}{\|\mathbf{n}\|} = \sqrt{2} \tag{2.2.9.6}$$

(b) The normal vector of the altitude from $\mathbf{A}$ is,

$$\mathbf{n} = \mathbf{m}_{BC} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{2.2.9.7}$$

The equation of the altitide from vertex $\mathbf{A}$ is given by,

$$\mathbf{n}^{\top}\mathbf{x} = \mathbf{n}^{\top}\mathbf{A} \tag{2.2.9.8}$$

$$\implies \begin{pmatrix} 1 & -1 \end{pmatrix}\mathbf{x} = -1 \tag{2.2.9.9}$$

See Fig. 2.2.9.1.

2.2.10 If p is the length of perpendicular from origin to the line whose intercepts on the axes are a and b, then show that

$$\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2} \tag{2.2.10.1}$$

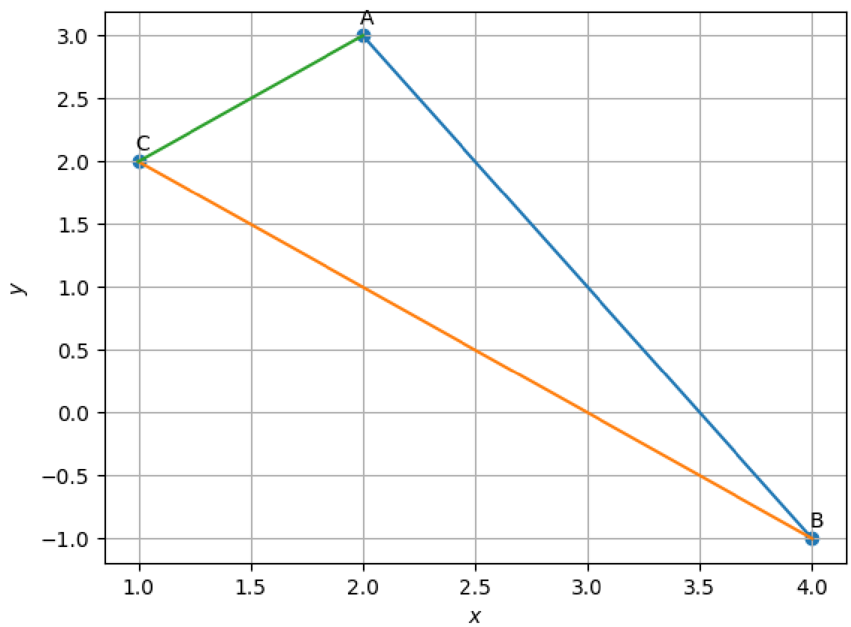


Figure 2.2.9.1:

The x-intercept of the line is $\mathbf{A} = \begin{pmatrix} a \\ 0 \end{pmatrix}$ and the y-intercept is $\mathbf{B} = \begin{pmatrix} 0 \\ b \end{pmatrix}$ The direction vector of the line is given by

$$\mathbf{m} = \begin{pmatrix} a \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ b \end{pmatrix} \tag{2.2.10.2}$$

$$= \begin{pmatrix} a \\ -b \end{pmatrix} \tag{2.2.10.3}$$

The normal vector is,

$$\mathbf{n} = \begin{pmatrix} b \\ a \end{pmatrix} \tag{2.2.10.4}$$

The line equation is,

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{A}) = 0 \tag{2.2.10.5}$$

$$\implies \begin{pmatrix} b & a \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} a \\ 0 \end{pmatrix}\right) = 0 \tag{2.2.10.6}$$

$$\implies \begin{pmatrix} b & a \end{pmatrix} \mathbf{x} = ab \tag{2.2.10.7}$$

Thus,

$$c = ab \tag{2.2.10.8}$$

and the perpendicular distance from the origin to the line is

$$p = \frac{\left|\mathbf{n}^\top \mathbf{O} - c\right|}{\|\mathbf{n}\|} \tag{2.2.10.9}$$

$$= \frac{ab}{\sqrt{a^2 + b^2}} \tag{2.2.10.10}$$

$$\implies \frac{1}{p^2} = \frac{a^2 + b^2}{a^2 b^2} \tag{2.2.10.11}$$

$$= \frac{1}{a^2} + \frac{1}{b^2} \tag{2.2.10.12}$$

2.2.11 What are the points on the y-axis whose distance from the line $\frac{x}{3} + \frac{y}{4} = 1$ is 4 units.

Solution: Given line parameters are

$$\mathbf{n} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}, c = 12. \tag{2.2.11.1}$$

The distance of the line from y-axis

$$d = \frac{\mathbf{n}^\top \mathbf{P} - c}{|n|} \tag{2.2.11.2}$$

$$\implies \pm 4 = \frac{\begin{pmatrix} 0 \\ 3y \end{pmatrix} - 12}{5} \tag{2.2.11.3}$$

$$\implies y = \frac{32}{3} \text{ or } y = \frac{-8}{3} \tag{2.2.11.4}$$

See Fig. 2.2.11.1.

2.2.12 Find perpendicular distance from the origin to the line joining the points$(\cos\theta, \sin\theta)$ and $(\cos\phi, \sin\phi)$.

Solution: Let

$$\mathbf{A} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \mathbf{B} = \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} \tag{2.2.12.1}$$

$$\implies \mathbf{m} = \mathbf{B} - \mathbf{A} = \begin{pmatrix} \cos\phi - \cos\theta \\ \sin\phi - \sin\theta \end{pmatrix} \tag{2.2.12.2}$$

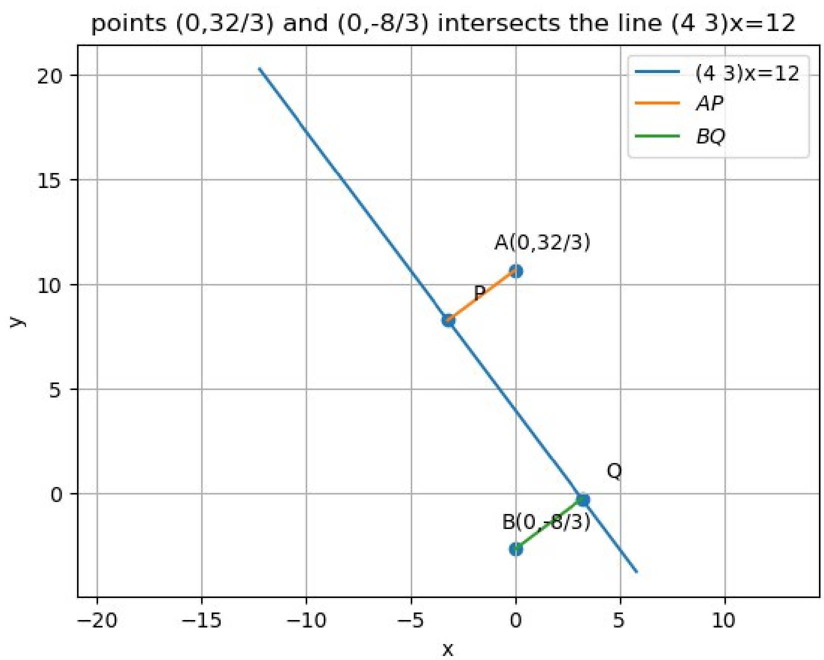


Figure 2.2.11.1:

The normal vector is then given by,

$$\mathbf{n} = \begin{pmatrix} \sin\phi - \sin\theta \\ \cos\theta - \cos\phi \end{pmatrix} \implies \|\mathbf{n}\| = 2\sin\left(\frac{\phi-\theta}{2}\right) \tag{2.2.12.3}$$

The equation of the line is

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{A}) = 0 \tag{2.2.12.4}$$

$$\implies \begin{pmatrix} \sin\phi - \sin\theta & \cos\theta - \cos\phi \end{pmatrix} \mathbf{x} = \sin(\phi - \theta) \tag{2.2.12.5}$$

Thus,

$$c = \sin(\phi - \theta) \tag{2.2.12.6}$$

The perpendicular distance from the origin to the line is

$$d = \frac{|c|}{\|\mathbf{n}\|} \tag{2.2.12.7}$$

$$\implies d = \frac{\sin(\phi - \theta)}{2\sin\left(\frac{\phi-\theta}{2}\right)} = \cos\left(\frac{\phi - \theta}{2}\right) \tag{2.2.12.8}$$

2.2.13 Find the equation of line which is equidistant from parallel lines $9x + 6y - 7 = 0$ and $3x + 2y + 6 = 0$.

Solution: The distance between two parallel lines is given by

$$d = \frac{|c_1 - c_2|}{\|\mathbf{n}\|} \tag{2.2.13.1}$$

We need to find c such that,

$$\frac{|c - c_1|}{\|\mathbf{n}\|} = \frac{|c - c_2|}{\|\mathbf{n}\|} \tag{2.2.13.2}$$

Since

$$c_1 = \frac{7}{3},\ c_2 = -6,\ \mathbf{n} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \tag{2.2.13.3}$$

$$\left|c - \frac{7}{3}\right| = |c - (-6)| \implies c = \frac{-11}{6} \tag{2.2.13.4}$$

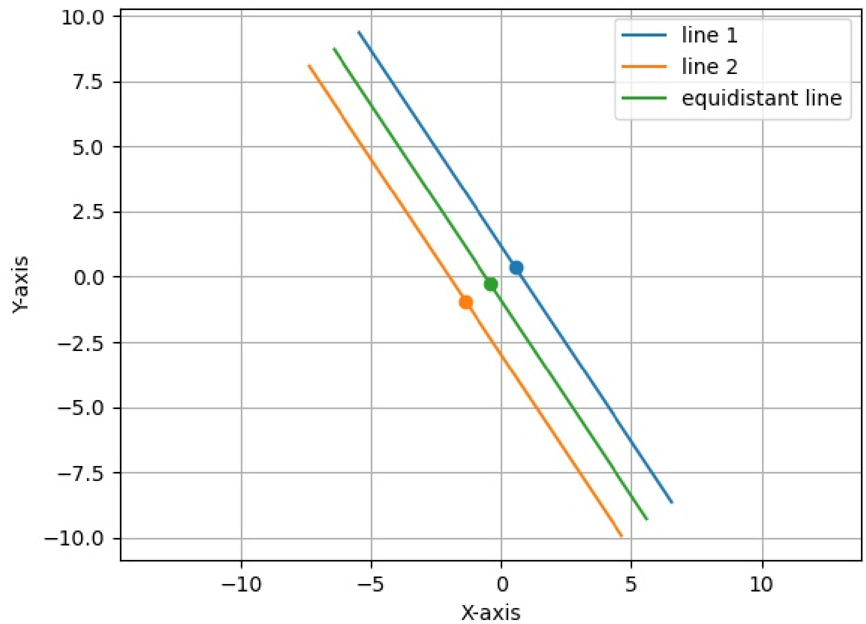


Figure 2.2.13.1:

Hence, the desired equation is

$$\begin{pmatrix} 3 & 2 \end{pmatrix} \mathbf{x} = -\frac{11}{6} \tag{2.2.13.5}$$

2.2.14 Prove that the products of the lengths of the perpendiculars drawn from the points $\begin{pmatrix} \sqrt{a^2 - b^2} \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -\sqrt{a^2 - b^2} \\ 0 \end{pmatrix}$ to the line $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$ is b^2.

Solution: Let

$$\mathbf{P} = \begin{pmatrix} \sqrt{a^2 - b^2} \\ 0 \end{pmatrix} \tag{2.2.14.1}$$

The parameters of the given line are

$$\mathbf{n} = \begin{pmatrix} \frac{\cos\theta}{a} \\ \frac{\sin\theta}{b} \end{pmatrix}, \ c = 1 \tag{2.2.14.2}$$

Hence, the distance from $\mathbf{P}$ to the line is

$$d_1 = \frac{\left|\mathbf{n}^\top \mathbf{P} - c\right|}{\|\mathbf{n}\|} \tag{2.2.14.3}$$

Similarly, the distance from $-\mathbf{P}$ to the line is

$$d_2 = \frac{\left|-\mathbf{n}^\top \mathbf{P} - c\right|}{\|\mathbf{n}\|} \tag{2.2.14.4}$$

$$= \frac{\left|\mathbf{n}^\top \mathbf{P} + c\right|}{\|\mathbf{n}\|} \tag{2.2.14.5}$$

yielding

$$d_1 d_2 = \frac{\left|\left(\mathbf{n}^\top \mathbf{P}\right)^2 - c^2\right|}{\|\mathbf{n}\|} = \frac{\left|\frac{\cos^2\theta\left(a^2 - b^2\right)}{a^2} - 1\right|}{\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}} \tag{2.2.14.6}$$

$$= \frac{\left(b^2\cos^2\theta + a^2\sin^2\theta\right)a^2b^2}{\left(b^2\cos^2\theta + a^2\sin^2\theta\right)a^2} = b^2 \tag{2.2.14.7}$$

2.2.15 Find the equation of line drawn perpendicular to the line $\frac{x}{4} + \frac{y}{6} = 1$ through the point where it meets the y-axis

Solution: The given equation can be arranged as

$$3x + 2y - 12 = 0 \tag{2.2.15.1}$$

with parameters

$$\mathbf{n} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, c = 12, \mathbf{m} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} \tag{2.2.15.2}$$

The given point is

$$\mathbf{A} = \begin{pmatrix} 0 \\ 6 \end{pmatrix} \tag{2.2.15.3}$$

Thus, the equation of the desired line is

$$\mathbf{m}^\top (\mathbf{x} - \mathbf{A}) = 0 \tag{2.2.15.4}$$

$$\implies \begin{pmatrix} -2 & 3 \end{pmatrix} \mathbf{x} = -18 \tag{2.2.15.5}$$

See Fig. 2.2.15.1.

2.2.16 In each of the following cases, determine the direction cosines of the normal to the plane and the distance from the origin.

(a) $z = 2$

(b) $x + y + z = 1$

(c) $2x + 3y – z = 5$

(d) $5y + 8 = 0$

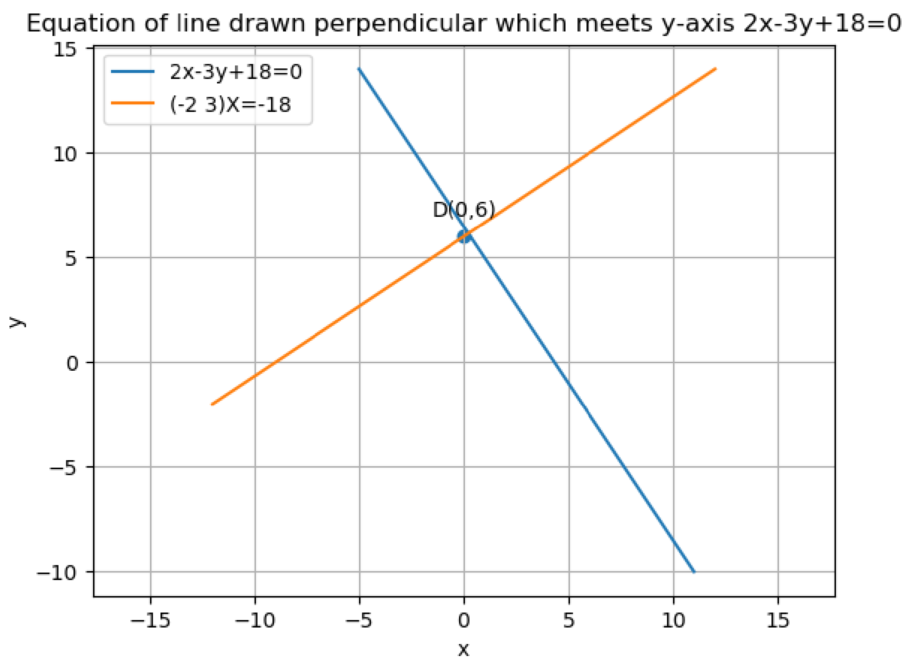


Figure 2.2.15.1:

Solution:

(a) From the given equation

$$\mathbf{n} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, c = 2 \tag{2.2.16.1}$$

The distance from the origin is given by:

$$d = \frac{|c|}{\|\mathbf{n}\|} = \frac{2}{1} = 2 \tag{2.2.16.2}$$

(b) From the given equation

$$\mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, c = 1 \tag{2.2.16.3}$$

The distance from the origin is given by

$$d = \frac{|c|}{\|\mathbf{n}\|} = \frac{1}{\sqrt{3}} \tag{2.2.16.4}$$

(c) From the given equation

$$\mathbf{n} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}, c = 5 \tag{2.2.16.5}$$

The distance from the origin is given by

$$d = \frac{|c|}{\|\mathbf{n}\|} = \frac{5}{\sqrt{14}} \tag{2.2.16.6}$$

(d) From the given equation

$$\mathbf{n} = \begin{pmatrix} 0 \\ -5 \\ 0 \end{pmatrix}, c = 8 \tag{2.2.16.7}$$

The distance from the origin is given by

$$d = \frac{|c|}{\|\mathbf{n}\|} = \frac{8}{5} \tag{2.2.16.8}$$

2.3. Plane

2.3.1 Find the vector equation of a plane which is at a distance of 7 units from the origin and normal to the vector $3\hat{\imath} + 5\hat{\jmath} - 6\hat{k}$.

Solution: From the given information,

$$\mathbf{n} = \begin{pmatrix} 3 \\ 5 \\ -6 \end{pmatrix}, \; d = \frac{|c|}{\|\mathbf{n}\|} = 7 \tag{2.3.1.1}$$

yielding

$$c = \pm 7\sqrt{70} \tag{2.3.1.2}$$

2.3.2 Find the equations of the planes that pass through the points

(a) $\mathbf{A} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 6 \\ 4 \\ -5 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -4 \\ -2 \\ 3 \end{pmatrix}$

(b) $\mathbf{A} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix}$

Solution:

(a) From (D.2.15.1),

$$\begin{pmatrix} 1 & 6 & -4 \\ 1 & 4 & -2 \\ -1 & -5 & 3 \end{pmatrix}^{\top} \mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \implies \begin{pmatrix} 1 & 1 & -1 \\ 6 & 4 & -5 \\ -4 & -2 & 3 \end{pmatrix} \mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{2.3.2.1}$$

the augmented matrix is given by,

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 6 & 4 & -5 & 1 \\ -4 & -2 & 3 & 1 \end{array}\right) \xleftrightarrow[R_3 \leftarrow R_3 + 4R_1]{R_2 \leftarrow R_2 - 6R_1} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & -2 & 1 & -5 \\ 0 & 2 & -1 & 5 \end{array}\right) \tag{2.3.2.2}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{-R_2}{2}} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & \frac{-1}{2} & \frac{5}{2} \\ 0 & 2 & -1 & 5 \end{array}\right) \xleftrightarrow[R_1 \leftarrow R_1 - R_2]{R_3 \leftarrow R_3 - 2R_2} \left(\begin{array}{ccc|c} 1 & 0 & \frac{-1}{2} & \frac{-3}{2} \\ 0 & 1 & \frac{-1}{2} & \frac{5}{2} \\ 0 & 0 & 0 & 0 \end{array}\right) \tag{2.3.2.3}$$

Since we obtain a 0 row, the given points are collinear. The direction vector of the line is

$$\mathbf{m} = \mathbf{B} - \mathbf{C} = \begin{pmatrix} 10 \\ 6 \\ -8 \end{pmatrix} \tag{2.3.2.4}$$

and the equation of a line is given by,

$$\mathbf{x} = \mathbf{p} + \lambda \mathbf{m} \tag{2.3.2.5}$$

$$\implies \mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 10 \\ 6 \\ -8 \end{pmatrix} \tag{2.3.2.6}$$

See Fig. 2.3.2.1.

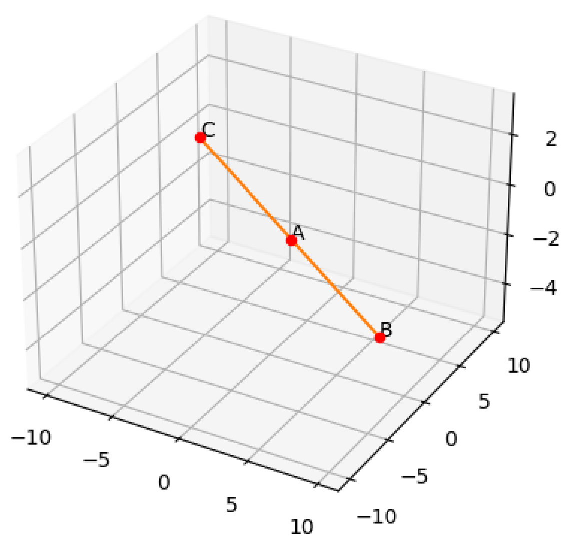


Figure 2.3.2.1: The figure shows that the given points are collinear

(b) In this case,

$$\begin{pmatrix} 1 & 1 & -2 \\ 1 & 2 & 2 \\ 0 & 1 & -1 \end{pmatrix}^{\top} \mathbf{n} = \mathbf{1} \implies \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ -2 & 2 & -1 \end{pmatrix} \mathbf{n} = \mathbf{1} \tag{2.3.2.7}$$

The augmented matrix is given by,

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 \\ -2 & 2 & -1 & 1 \end{array}\right) \xleftrightarrow[R_3 \leftarrow R_3 + 2R_1]{R_2 \leftarrow R_2 - R_1} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 4 & -1 & 3 \end{array}\right) \tag{2.3.2.8}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - R_2} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 4 & -1 & 3 \end{array}\right) \xleftrightarrow{R_3 \leftarrow R_3 - 4R_2} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -5 & 3 \end{array}\right) \tag{2.3.2.9}$$

$$\xleftrightarrow{R_3 \leftarrow \frac{-R_3}{5}} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & \frac{-3}{5} \end{array}\right) \xleftrightarrow[R_1 \leftarrow R_1 + R_3]{R_2 \leftarrow R_2 - R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{2}{5} \\ 0 & 1 & 0 & \frac{3}{5} \\ 0 & 0 & 1 & \frac{-3}{5} \end{array}\right) \tag{2.3.2.10}$$

Hence, the equation of the plane is

$$\begin{pmatrix} 2 & 3 & -3 \end{pmatrix} \mathbf{x} = 5 \tag{2.3.2.11}$$

See Fig. 2.3.2.2

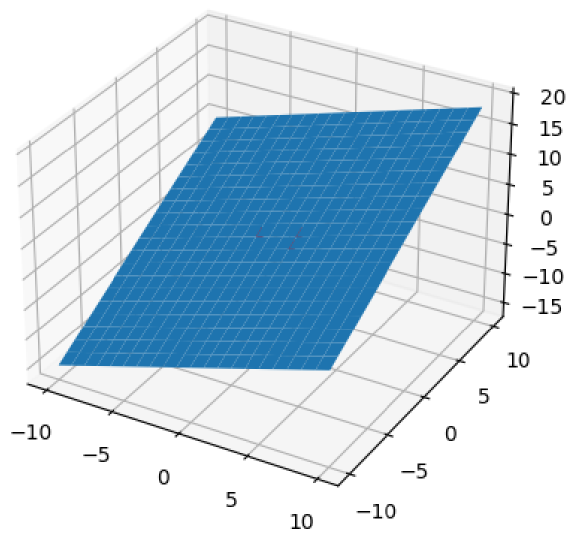


Figure 2.3.2.2: Plane passing through the given points

2.3.3 Find the equation of the plane with an intercept 3 on the Y-axis and parallel to ZOX-Plane.

Solution: The normal vector to the ZOX plane is

$$\mathbf{n} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \tag{2.3.3.1}$$

Since, Y-axis has the intercept 3, the desired plane passes through the point

$$\mathbf{P} = \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}. \tag{2.3.3.2}$$

Thus, the equation of the plane is given by,

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{P}) = 0 \tag{2.3.3.3}$$

$$\implies \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \mathbf{x} = 3 \tag{2.3.3.4}$$

See Fig. 2.3.3.1.

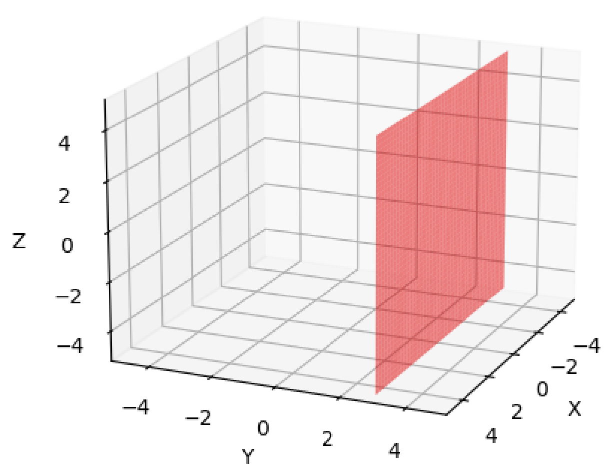


Figure 2.3.3.1:

2.3.4 Find the equation of the plane through the intersection of the planes $3x–y + 2z–4 = 0$ and $x + y + z–2 = 0$ and the point $\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$. **Solution:**

The parameters of the given planes are

$$\mathbf{n}_1 = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}, \mathbf{n}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, c_1 = 4, c_2 = 2. \tag{2.3.4.1}$$

The intersection of the planes is given as

$$\mathbf{n}_1^\top \mathbf{x} - c_1 + \lambda \left(\mathbf{n}_2^\top \mathbf{x} - c_2 \right) = 0 \tag{2.3.4.2}$$

where

$$\lambda = \frac{c_1 - n_1^\top \mathbf{P}}{n_2^\top \mathbf{P} - c_2} = -\frac{2}{3} \tag{2.3.4.3}$$

upon substituting

$$\mathbf{P} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \tag{2.3.4.4}$$

Thus, the equation of plane is

$$\begin{pmatrix} 7 & -5 & 4 \end{pmatrix} \mathbf{x} = 8 \tag{2.3.4.5}$$

2.4. Miscellaneous

2.4.1 Find the values of k for which the line

$$(k-3)x-(4-k^2)y+k^2-7k+6=0 \tag{2.4.1.1}$$

is

(a) Parallel to the x-axis

(b) Parallel to the y-axis

(c) Passing through the origin

Solution: The parameters of the given line are

$$\mathbf{n}^\top \mathbf{x} = c \tag{2.4.1.2}$$

This equation can be expressed in the form of

$$\mathbf{n} = \begin{pmatrix} k-3 \\ -4+k^2 \end{pmatrix}, c = -k^2+7k-6 \tag{2.4.1.3}$$

(a) In this case, equating $\mathbf{n}$ to the normal vector of x-axis,

$$\begin{pmatrix} k-3 \\ -4+k^2 \end{pmatrix} = \alpha \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.4.1.4}$$

$$\implies k = 3 \tag{2.4.1.5}$$

Substituting the value of k in (2.4.1.1), the desired equation is

$$\begin{pmatrix} 0 & 5 \end{pmatrix} \mathbf{x} = 6 \tag{2.4.1.6}$$

(b) In this case, equating $\mathbf{n}$ to the normal vector of y-axis,

$$\begin{pmatrix} k-3 \\ -4+k^2 \end{pmatrix} = \beta \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.4.1.7}$$

$$\implies k = \pm 2 \tag{2.4.1.8}$$

Substituting the value of k in (2.4.1.1), the desired equation is

$$\begin{pmatrix} -1 & 0 \end{pmatrix} \mathbf{x} = 4, \quad k = 2 \tag{2.4.1.9}$$

$$\begin{pmatrix} -5 & 0 \end{pmatrix} \mathbf{x} = -24, \quad k = -2 \tag{2.4.1.10}$$

(c) In this case,

$$c = 0 \implies -k^2 + 7k - 6 = 0 \tag{2.4.1.11}$$

$$\implies k = 1 \text{ or } k = 6 \tag{2.4.1.12}$$

Substituting the value of k in (2.4.1.1), the desired equations are

$$\begin{pmatrix} -2 & -3 \end{pmatrix} \mathbf{x} = 0, \quad k = 1 \tag{2.4.1.13}$$

$$\begin{pmatrix} 3 & 32 \end{pmatrix} \mathbf{x} = 0, \quad k = 6 \tag{2.4.1.14}$$

2.4.2 Find the values of θ and p, if the equation $x\cos\theta + y\sin\theta = p$ is the normal form of the line $\sqrt{3}x + y + 2 = 0$.

Solution: The parameters of the given line are

$$\mathbf{n} = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}, c = -2 \tag{2.4.2.1}$$

From the above,

$$\tan\theta = -\sqrt{3} \tag{2.4.2.2}$$

$$\implies \theta = -60^\circ \tag{2.4.2.3}$$

and

$$p = \frac{|c|}{\|\mathbf{n}\|} = \frac{2}{2} = 1 \tag{2.4.2.4}$$

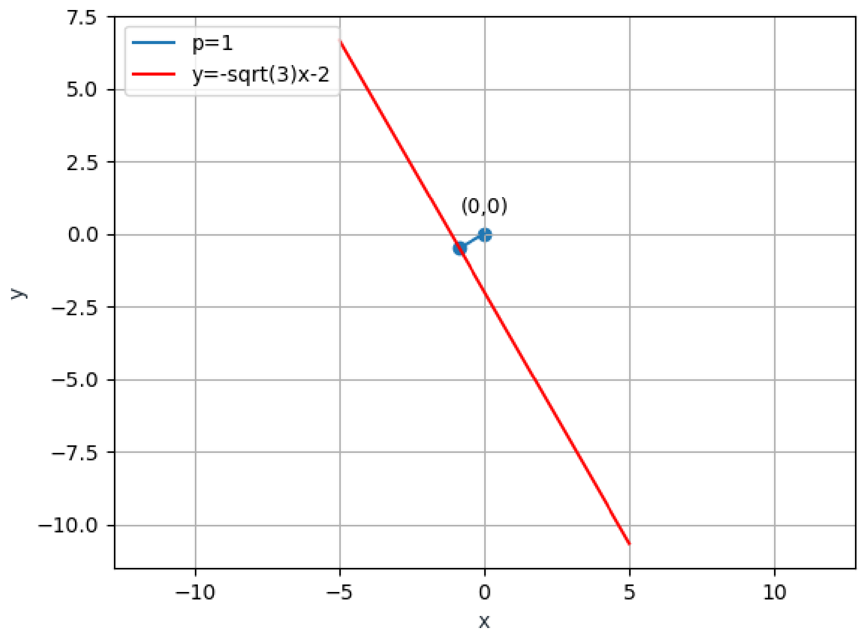


Figure 2.4.2.1:

2.4.3 Find the equations of the lines, which cutoff intercepts on the axes whose sum and product are 1 and -6 respectively.

Solution: Let the x intercept be a and the y intercept be b ,Then

$$a + b = 1 \tag{2.4.3.1}$$

$$ab = -6 \tag{2.4.3.2}$$

$$\implies a = 3, b = -2 \tag{2.4.3.3}$$

Thus, the possible intercepts are

$$\begin{pmatrix} 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix} \tag{2.4.3.4}$$

yielding

$$\mathbf{m} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \text{or,} \begin{pmatrix} -2 \\ 3 \end{pmatrix} \tag{2.4.3.5}$$

(a) For

$$\mathbf{n} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} \tag{2.4.3.6}$$

the equation of the line is

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{A}) = 0 \tag{2.4.3.7}$$

$$\begin{pmatrix} -2 & 3 \end{pmatrix} \mathbf{x} = 6 \tag{2.4.3.8}$$

(b) For

$$\mathbf{n} = \begin{pmatrix} -3 \\ -2 \end{pmatrix} \tag{2.4.3.9}$$

the equation of the line is

$$\mathbf{n}^{\top}(\mathbf{x}-\mathbf{B})=0 \tag{2.4.3.10}$$

$$\begin{pmatrix}-3 & -2\end{pmatrix}\mathbf{x}=6 \tag{2.4.3.11}$$

See Fig. 2.4.3.1.

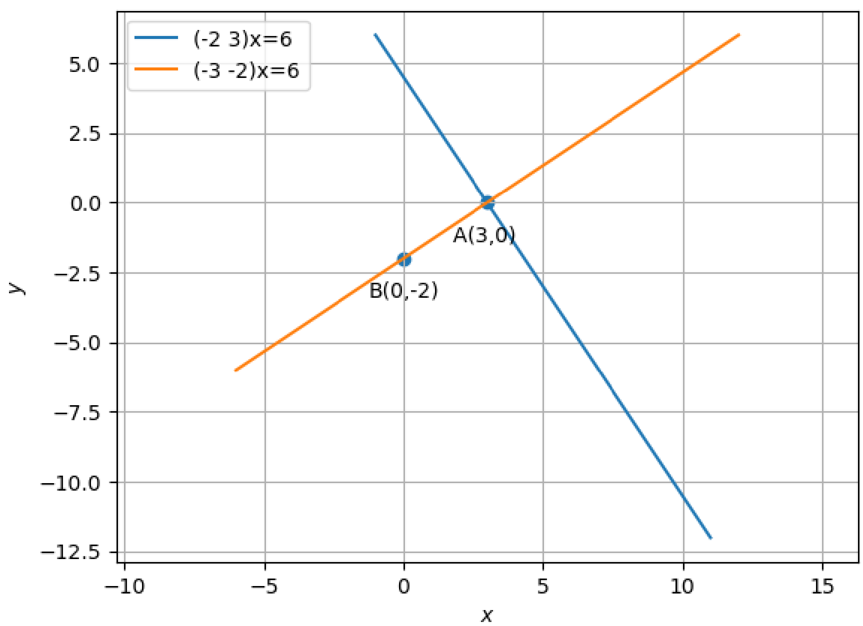


Figure 2.4.3.1:

2.4.4 Find the equation of the line parallel to y-axis and drawn through the point of intersection of the lines x − 7y + 5 = 0 and 3x + y = 0.

Solution: The given line can be expressed as

$$\begin{pmatrix} 1 & -7 \end{pmatrix} \mathbf{x} = -5 \tag{2.4.4.1}$$

$$\begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{2.4.4.2}$$

The intersection of two lines is given by row reducing the augmented matrix

$$\begin{pmatrix} 1 & -7 & 5 \\ 3 & 1 & 0 \end{pmatrix} \xleftrightarrow{R_2 = R_2 - 3R_1} \begin{pmatrix} 1 & -7 & 5 \\ 0 & 22 & -15 \end{pmatrix} \tag{2.4.4.3}$$

$$\xleftrightarrow{R_2 = \frac{R_2}{22}} \begin{pmatrix} 1 & -7 & 5 \\ 0 & 1 & -\frac{15}{22} \end{pmatrix} \xleftrightarrow{R_1 = R_1 + 7R_2} \begin{pmatrix} 1 & 0 & \frac{5}{22} \\ 0 & 1 & \frac{-15}{22} \end{pmatrix} \tag{2.4.4.4}$$

yielding

$$\mathbf{P} = \begin{pmatrix} -\frac{5}{22} \\ \frac{15}{22} \end{pmatrix} \tag{2.4.4.5}$$

The normal vector of the desired line is

$$\mathbf{n} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.4.4.6}$$

The desired equation is then given by

$$\mathbf{n}^\top \left(\mathbf{x} - \mathbf{P}\right) = 0 \tag{2.4.4.7}$$

$$\implies \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = -\frac{5}{22} \tag{2.4.4.8}$$

See Fig. 2.4.4.1

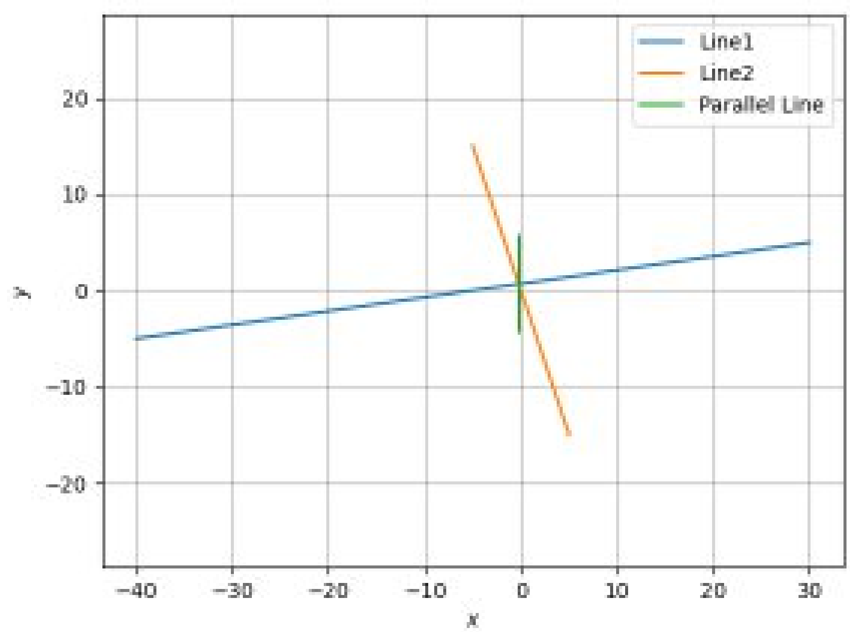


Figure 2.4.4.1:

2.4.5 Find the area of triangle formed by the lines $y - x = 0, x + y = 0,$ and $x - k = 0$. **Solution:** Given line equations represented in vector form

$$\begin{pmatrix} -1 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{2.4.5.1}$$

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{2.4.5.2}$$

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = k \tag{2.4.5.3}$$

The coordinates of the intersection of (2.4.5.1),(2.4.5.2)

$$\left(\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right) \xleftrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ -1 & 1 & 0 \end{array}\right) \tag{2.4.5.4}$$

$$\xleftrightarrow{R_2 \to R_2 + R_1} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 2 & 0 \end{array}\right) \tag{2.4.5.5}$$

$$\xleftrightarrow{R_2 \to \frac{R_2}{2}} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right) \tag{2.4.5.6}$$

$$\xleftrightarrow{R_1 \to R_1 - R_2} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right) \tag{2.4.5.7}$$

$$\text{The intersection of lines is} \tag{2.4.5.8}$$

$$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.4.5.9}$$

The coordinates of the intersection of (2.4.5.2),(2.4.5.3)

$$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 0 & k \end{array}\right) \xleftrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|c} 1 & 0 & k \\ 1 & 1 & 0 \end{array}\right) \tag{2.4.5.10}$$

$$\xleftrightarrow{R_2 \rightarrow R_2 - R_1} \left(\begin{array}{cc|c} 1 & 0 & k \\ 0 & 1 & -k \end{array}\right) \tag{2.4.5.11}$$

The intersection of lines is (2.4.5.12)

$$\mathbf{B} = \begin{pmatrix} k \\ -k \end{pmatrix} \tag{2.4.5.13}$$

The coordinates of the intersection of (2.4.5.3),(2.4.5.1)

$$\left(\begin{array}{cc|c} 1 & 0 & k \\ -1 & 1 & 0 \end{array}\right) \xleftrightarrow{R_2 \rightarrow R_2 + R_1} \left(\begin{array}{cc|c} 1 & 0 & k \\ 0 & 1 & k \end{array}\right) \tag{2.4.5.14}$$

The intersection of lines is (2.4.5.15)

$$\mathbf{C} = \begin{pmatrix} k \\ k \end{pmatrix} \tag{2.4.5.16}$$

We know that

$$ar(ABC) = \frac{1}{2} \left\| (\mathbf{A} - \mathbf{B}) \times (\mathbf{A} - \mathbf{C}) \right\| \tag{2.4.5.17}$$

$$= \frac{1}{2} \left\| \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} k \\ -k \end{pmatrix} \right) \times \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} k \\ k \end{pmatrix} \right) \right\| \tag{2.4.5.18}$$

$$= \frac{1}{2} \left\| \begin{pmatrix} -k \\ k \end{pmatrix} \times \begin{pmatrix} -k \\ -k \end{pmatrix} \right\| \tag{2.4.5.19}$$

$$= \frac{1}{2} \left\| 2k^2 \right\| \tag{2.4.5.20}$$

$$\implies = k^2 \tag{2.4.5.21}$$

2.4.6 A ray of light passing through the point $(1, 2)$ reflects on the x-axis at point $\mathbf{A}$ and the reflected ray passes through the point $(5, 3)$. Find the coordinates of $\mathbf{A}$.

Solution:

Let the points be,

$$\mathbf{P} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} 5 \\ 3 \end{pmatrix} \tag{2.4.6.1}$$

The equation of x-axis is given by,

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{2.4.6.2}$$

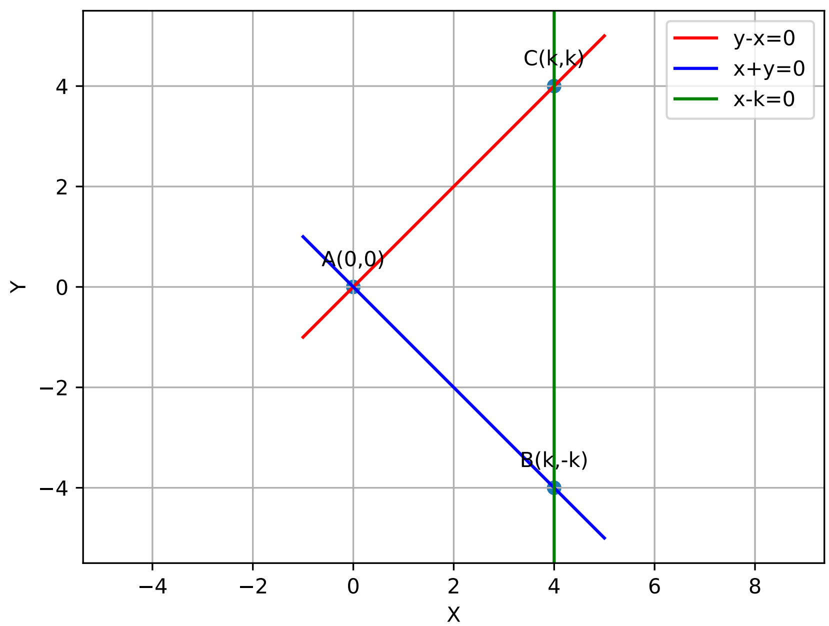


Figure 2.4.5.1:

The reflection of point $\mathbf{Q}$ in the x-axis is given by

$$\mathbf{R} = \mathbf{Q} - \frac{2\left(\mathbf{n}^\top \mathbf{Q} - c\right)}{\|\mathbf{n}\|}\mathbf{n} = \begin{pmatrix} 5 \\ -3 \end{pmatrix} \tag{2.4.6.3}$$

Direction vector of line PR is given by,

$$\mathbf{m} = \mathbf{R} - \mathbf{P} = \begin{pmatrix} 4 \\ -5 \end{pmatrix} \tag{2.4.6.4}$$

The corresponding normal vector is

$$\mathbf{n} = \begin{pmatrix} 5 \\ 4 \end{pmatrix} \tag{2.4.6.5}$$

Equation of line PR is given by

$$\begin{pmatrix} 5 & 4 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 5 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{2.4.6.6}$$

$$\implies \begin{pmatrix} 5 & 4 \end{pmatrix} \mathbf{x} = 13 \tag{2.4.6.7}$$

The point $\mathbf{A}$ is the point of intersection of the line PR and x-axis. Hence,

$$\mathbf{A} = \begin{pmatrix} x \\ 0 \end{pmatrix} \tag{2.4.6.8}$$

Since $\mathbf{A}$ satisfies (2.4.6.7),

$$5 \times x = 13 \implies x = \frac{13}{5} \tag{2.4.6.9}$$

yielding

$$\mathbf{A} = \begin{pmatrix} \frac{13}{5} \\ 0 \end{pmatrix} \tag{2.4.6.10}$$

See Fig. 2.4.6.1.

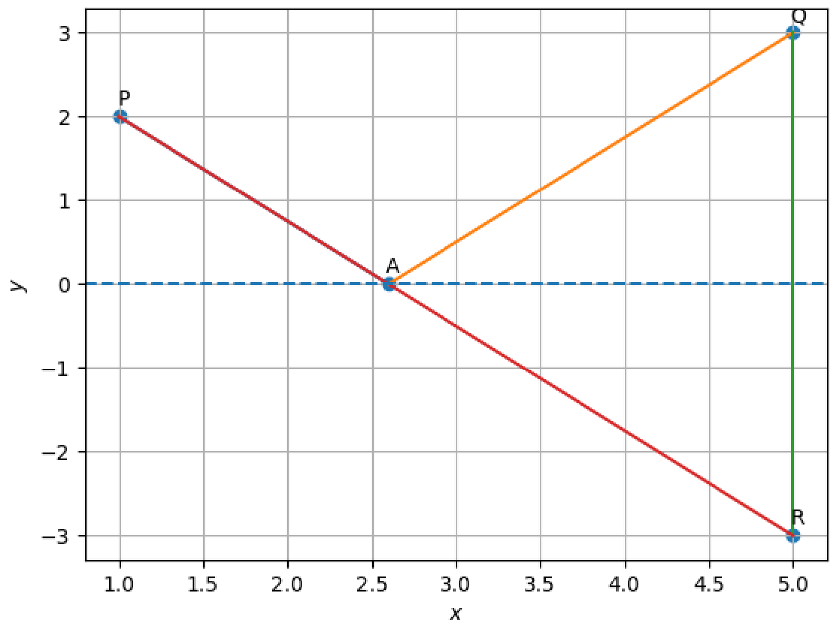


Figure 2.4.6.1:

2.4.7 A person standing at the junction (crossing) of two straight paths represented by the equations

$$\begin{pmatrix} 2 & -3 \end{pmatrix} \mathbf{x} = -4 \tag{2.4.7.1}$$

and

$$\begin{pmatrix} 3 & 4 \end{pmatrix} \mathbf{x} = 5 \tag{2.4.7.2}$$

wants to reach the path whose equation is

$$\begin{pmatrix} 6 & -7 \end{pmatrix} \mathbf{x} = -8 \tag{2.4.7.3}$$

Find equation of the path that he should follow.

Solution: We first find the coordinates of the intersection of (2.4.7.1) and (2.4.7.2). Using the augmented matrix and row reduction methods,

$$\left(\begin{array}{cc|c} 2 & -3 & -4 \\ 3 & 4 & 5 \end{array}\right) \xleftrightarrow{R_2 \to 2R_2 - 3R_1} \left(\begin{array}{cc|c} 2 & -3 & -4 \\ 0 & 17 & 22 \end{array}\right) \tag{2.4.7.4}$$

$$\xleftrightarrow{R_1 \to 17R_1 + 3R_2} \left(\begin{array}{cc|c} 17 & 0 & -1 \\ 0 & 17 & 22 \end{array}\right) \tag{2.4.7.5}$$

$$\xleftrightarrow[R_2 \to \frac{R_2}{17}]{R_1 \to \frac{R_1}{17}} \left(\begin{array}{cc|c} 1 & 0 & -\frac{1}{17} \\ 0 & 1 & \frac{22}{17} \end{array}\right) \tag{2.4.7.6}$$

the intersection of the lines is

$$\mathbf{A} = \frac{1}{17} \begin{pmatrix} -1 \\ 22 \end{pmatrix} \tag{2.4.7.7}$$

Clearly, the man should follow the path perpendicular to (2.4.7.3) from **A** to reach it in the shortest time. The normal vector of (2.4.7.3) is

$$\mathbf{m} = \begin{pmatrix} 6 \\ -7 \end{pmatrix} \tag{2.4.7.8}$$

which is consequently the direction vector of the required line. Therefore, the required normal vector is given by

$$\mathbf{n} = \begin{pmatrix} 7 \\ 6 \end{pmatrix} \tag{2.4.7.9}$$

and hence, the equation of the line is

$$\mathbf{n}^\top \mathbf{x} = \mathbf{n}^\top \mathbf{A} \tag{2.4.7.10}$$

$$\implies \begin{pmatrix} 7 & 6 \end{pmatrix} \mathbf{x} = \frac{1}{17} \begin{pmatrix} 7 & 6 \end{pmatrix} \begin{pmatrix} -1 \\ 22 \end{pmatrix} = \frac{125}{17} \tag{2.4.7.11}$$

See Fig. 2.4.7.1. In this figure **F** represents the foot of the prependicular drawn from **A** onto (2.4.7.3).

2.5. Exemplar

2.5.1 Find the equation of the straight line which passes through the point (1, -2) and cuts off equal intercepts from axes.

2.5.2 Find the equation of the line passing through the point (5,2) and perpendicular to the line joining the points (2,3) and (3, -1).

2.5.3 Find the angle between the lines $y(2-\sqrt{3})(x+5)$ and $y = (2+\sqrt{3})(x-7)$.

2.5.4 Find the equation of the lines which passes the point (3,4) and cuts off intercepts from the coordinate axes such that their sum is 14.

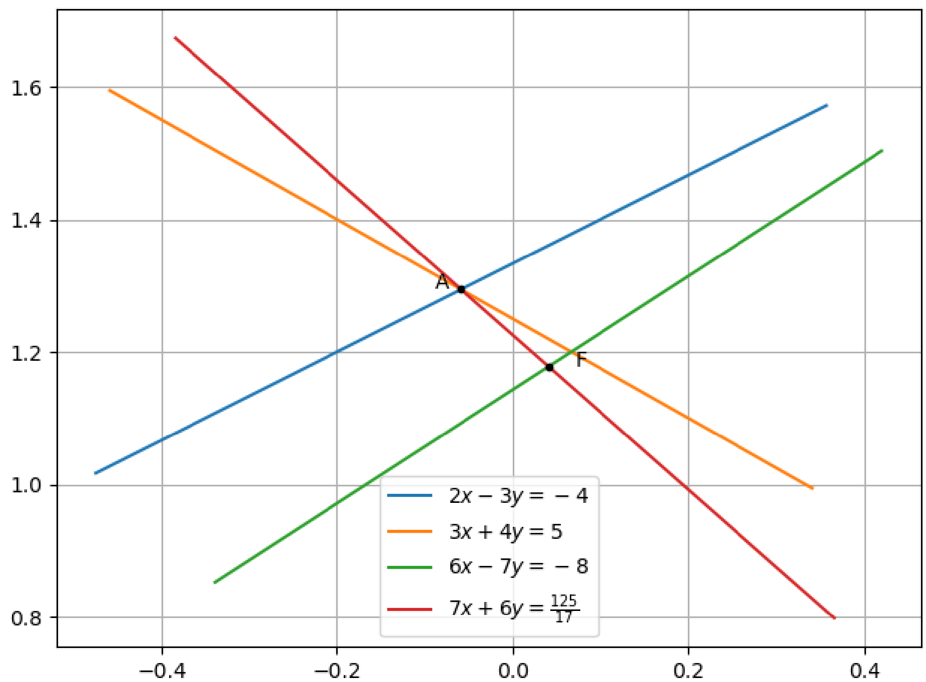


Figure 2.4.7.1: AF is the required line.

2.5.5 Find the points on the line $x+y=4$ which lie at a unit distance from the line $4x+3y=10$.

2.5.6 Show that the tangent of an angle between the lines $\frac{x}{a}+\frac{y}{b}=1$ and $\frac{x}{a}-\frac{y}{b}=1$ is $\frac{2ab}{a^2-b^2}$.

2.5.7 Find the equation of lines passing through (1,2) and making angle 30° with y-axis.

2.5.8 Find the equation of the line passing through the point of intersection of $2x+y=5$ and $x+3y+8=0$ and parallel the line $3x+4y=7$.

2.5.9 For what values of a and b the intercepts cut off on the coordinate axes by the line $ax + by + 8 = 0$are equal in length but opposite in signs to those cut off by the line $2x - 3y + = 0$ on the axes.

2.5.10 If the intercept of a line between the coordinate axes is divided by the point (-5,4) in the ratio 1:2 then find the equation of the line.

2.5.11 Find the equation of a straight line on which length of perpendicular from the origin is four units and the line makes on angle of $120°$ with the positive direction of x-axis. [**Hint** : Use normal form, here $\omega = 30°$.]

2.5.12 Find the equation of one of the sides of an isosceles right angled triangle whose hypotenuse is given by $3x + 4y = 4$ and the opposite vertex of the hypotenuse is (2,2).

Long Answer Type

13. If the equation of the base of an equilateral triangle is $x + y = 2$ and the vertex is (2,-1), then find the length of the side of the triangle. [**Hint** : Find length of perpendicular (p) from (2,-1) to the line and use $p = l \sin 60 degree$,where l is the length of the triangle].

14. A variable line passes through a fixed point **P**.The algebraic sum of the perpendiculars drawn from the points (2,0),(0,2) and (1,1) on the line is zero. Find the coordinates of the point **P**. [**Hint** : let the slope of the line be m. Then the equation of the line passing through the fixed point $\mathbf{P}(x_1, y_1) y - y_1 = m(x - x_1)$. Taking the algebraic sum of perpendicular distances equal to zero, we get $y - l = m(x - 1)$. Thus (x_1, y_1) is (1,1).]

15. In what direction should a line be drawn through the point (1,2) so that its point of intersection with line $x + y = 4$ is at a distance $\sqrt{6}3$ from the given equilateral

16. A straight line moves so that the sum of the reciprocals of its intercepts made on axes is constant. Show that the line passes through a fixed point. [**Hint** : $\frac{x}{a} + \frac{y}{b} = 1$ where$\frac{1}{a} + \frac{1}{b} =$ constant $= \frac{1}{k}$(say). This implies that $\frac{k}{a} + \frac{k}{b} = 1$ line passes through the fixed point (k, k).]

17. Find the equation of the line which passes through the point (-4,3) and the portion of the line intercepted between the axes is divided internally in ratio 5:3 by this point.

18. Find the equations of the lines through the point of intersection of the line $x - y + 1 = 0$ and $2x - 3y + 5 = 0$ and whose distance from the point (3,2) is $\frac{7}{5}$.

19. If the sum of the distances of a moving point in a plane from the axes is l, then finds the locus of the point. [**Hint** :Given that $|x| + |y| = 1$, which gives four sides of a square.]

20. $\mathbf{P}_1, \mathbf{P}_2$ are points on either of the two lines $y - \sqrt{3}\,|x| = 2$ at a distance of 5 units from their point of intersection. Find the coordinates of the root of perpendiculars drawn from P_1, P_2 on the bisector of the angle between the given lines. [**Hint** : Lines are $y = \sqrt{3}x + 2$ and $y = -\sqrt{3}x + 2$ according as $x \geq 0$ or $x0.y$-xis is the bisector of the angles between the lines. P_1, P_2 are the points on these lines at a distance of 5 units from the point of intersection of these lines which have a point

on y-axis as a common foot of perpendiculars from these points. The y-coordinate of the foot of the perpendicular is given by 2=5 $\cos 30°$.]

21. If p is the length of perpendicular from the origin on the lien $\frac{x}{a}+\frac{y}{b}=1$ and a^2, p^2, b^2 are in A.P, then show that $a^4+b^4=0$.

Objective Type Questions

choose the correct answer from the given four options in Exercises 22 to 41

22. A line cutting off intercept -3 from the tangent at angle to the x-axis is $\sqrt{3}5$, its equation is

    (a) $5y-3x+15=0$

    (b) $3y-5x+15=0$

    (c) $5y-3x-15=0$

    (d) none of these

23. Slope of a line which cuts off intercepts of equal length on the axes is

    (a) -1

    (b) -0

    (c) 2

    (d) $\sqrt{3}$

24. The equation of the straight line passing through the point (3,2) and perpendicular to the line $y=x$ is

    (a) $x-y=5$

    (b) $x+y=5$

(c) $x + y = 1$

(d) $x - y = 1$

25. The equation of the line passing through the point (1,2) and perpendicular to the line $x + y + 1 = 0$ is

(a) $y - x + 1 = 0$

(b) $y - x - 1 = 0$

(c) $y - x + 2 = 0$

(d) $y - x - 1 = 0$

26. The tangent of angle between the lines whose intercepts on the axes are $a, -b$ and $b, -a$, respectively, is

(a) $\frac{a^2-b^2}{ab}$

(b) $\frac{b^2-a^2}{2}$

(c) $\frac{b^2-a^2}{2ab}$

(d) none of these

27. If the line $\frac{x}{a} + \frac{y}{b} = 1$ passes the points (2,-3) and (4,-5), then (a, b) is

(a) (1,1)

(b) (-1,1)

(c) (1,-1)

(d) (-1,-1)

28. The distance of the point of intersection of the lines $2x - 3y + 5 = 0$ and $3x + 4y = 0$ from the line $5x - 2y = 0$ is

(a) $\frac{130}{17\sqrt{29}}$

(b) $\frac{13}{7\sqrt{29}}$

(c) $\frac{130}{7}$

(d) none of these

29. The equations of the lines which pass through the point (3, -2) and are inclined at $60°$ to the line $\sqrt{3}x + y = 1$ is

    (a) $y + 2 = 0,\ \sqrt{3}x - y - 2 - 3\sqrt{3} = 0$

    (b) $x - 2 = 0,\ \sqrt{3}x - y + 2 + 3\sqrt{3} = 0$

    (c) $\sqrt{3}x - y - 2 - 3\sqrt{3} = 0$

    (d) None of these

30. The equations of the lines passing through the point (1,0) and at a distance $\frac{\sqrt{3}}{2}$ from the origin, are

    (a) $\sqrt{3}x + y - \sqrt{3} = 0,\ \sqrt{3}x - y - \sqrt{3} = 0$

    (b) $\sqrt{3}x + y + \sqrt{3} = 0,\ \sqrt{3}x - y + \sqrt{3} = 0$

    (c) $x + \sqrt{3}y - \sqrt{3} = 0,\ \sqrt{3}y - \sqrt{3} = 0$

    (d) None of these.

31. The distance between the lines $y = mx + c$, and $y = mx + c^2$ is

    (a) $\frac{c_1 - c_2}{\sqrt{m+1}}$

    (b) $\frac{|c_1 - c_2|}{\sqrt{1+m^2}}$

    (c) $\frac{c^2 - c^1}{\sqrt{1+m^2}}$

    (d) 0

32. The coordinates of the foot of perpendiculars from the point (2,3) on the line $y = 3x + 4$ is given by

   (a) $\frac{37}{10}$, $\frac{-1}{10}$

   (b) $\frac{-1}{10}$, $\frac{37}{10}$

   (c) $\frac{10}{37}$, -10

   (d) $\frac{2}{3}$, $\frac{-1}{3}$

33. If the coordinates of middle point of the portion of a line intercepted between the coordinate axes is (3,2),then the equation of the line will be

   (a) $2x + 3y = 12$

   (b) $3x + 2y = 12$

   (c) $4x - 3y = 6$

   (d) $5x - 2y = 10$

34. Equation of the line passing through (1,2) and parallel to the line $y = 3x - 1$ is

   (a) $y + 2 = x + 1$

   (b) $y + 2 = 3(x + 1)$

   (c) $y - 2 = 3(x - 1)$

   (d) $y - 2 = x - 1$

35. Equations of diagonals of the square formed by the lines $x = 0$, $y = 0$, $x = 1$ and $y = 1$ are

(a) $y = x,\ y + x = 1$

(b) $y = x$,$x + y = 2$

(c) $2y = x,\ y + x = \frac{1}{3}$

(d) $y = 2x,\ y + 2x = 1$

36. For specifying a straight line, how many geometrical parameters should be known?

    (a) 1

    (b) 2

    (c) 4

    (d) 3

37. The point (4,1)undergoes the following two successive transformations :

    (a) Reflection about the line $y = x$

    (b) Translation through a distance 2 units along the positive x-axis

    Then the final coordinates of the point are

    (a) (4,3)

    (b) (3,4)

    (c) (1,4)

    (d) $\frac{7}{2}, \frac{7}{2}$

38. A point equidistant from the lines $4x + 3y + 10 = 0$, $5x - 12y + 26 = 0$ and $7x + 24y - 50 = 0$ is

(a) (1,-1)

(b) (1,1)

(c) (0,0)

(d) (0,1)

39. A line passes through (2,2) and is perpendicular to the line $3x+y=3$. Its y-intercept is

(a) $\frac{1}{3}$

(b) $\frac{2}{3}$

(c) 1

(d) $\frac{4}{3}$

40. The ratio in which the line $3x+4y+2=0$ divides the distance between the lines $3x+4y+5=0$ and $3x+4y-5=0$ is

(a) 1:2

(b) 3:7

(c) 2:3

(d) 2:5

41. One vertex of the equilateral with centroid at the origin and one side as $x+y-2=0$ is

(a) (-1,-1)

(b) (2,2)

(c) (-2-2)

(d) (2,-2)

[**Hint** : Let ABC be the equilateral triangle with vertex $\mathbf{A}(h,k)$ and let $\mathbf{D}(\alpha,\beta)$ be the point on BC. Then $\frac{2\alpha+h}{3}=0=\frac{2\beta+k}{3}$. Also $\alpha+\beta-2=0$ and $\frac{k-0}{h-o}x(-1)=-1$]

Fill in the blank in Exercises 42 to 47.

42. If a,b,c are is A.P.,then the straight lines $ax+by+c=0$ will always pass through ______.

43. The line which cuts off equal intercept from the axes and pass through the equilateral2) is ______.

44. Equations of the lines through the point (3,2) and making an angle of 40° with the line $x-2y=3$ are ______.

45. The points (3,4) and (2,-6)are situated on the ______ of the line $3x-4y-8=0$.

46. A point moves so that square of its distance from the point (3,-2) is numerically equal to its distance from the line $5x-12y=3$. The equation of its locus is

47. Locus of the mid-points of the portion of the line $x\sin\theta+y\cos\theta=p$ intercepted between the axes is ______. State whether the statements in Exercises 48 to 56 are true or false. Justify.

48. If the vertices of a triangle have integral coordinates, then the triangle can not be equilateral.

49. The points $\mathbf{A}(2,1)$, $\mathbf{B}(0,5)$, $\mathbf{C}(-1,2)$ are collinear.

50. Equation of the line passing through the point $(a\cos^3\theta, a\sin^3\theta)$ and perpendicular to the line $x\sec\theta + y\csc\theta = a$ is $x\cos\theta - y\sin\theta = \alpha\sin 2\theta$.

51. The straight line $5x+4y=0$ passes through the point of intersection of the straight lines $x+2y-10=0$ and $2x+y+5=0$.

52. The vertex of on equilateral triangle is (intercepted equation of the opposite side is $x+y=2$.then the other two sides are $y-3=(2\pm\sqrt{3})(x-2)$.

53. The equation of the line joining the point (3,5) to the point of intersection of the lines $4x+y-=0$ and $7x-3y-5=0$ is equidistant from the points (0,0) and (8,34).

54. The line $\frac{x}{a}+\frac{y}{b}=1$ moves in such a way that $\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{c^2}$, where c is a constant.The locus of the foot of the perpendicular from the origin on the given line is $x^2+y^2=c^2$.

55. The lines $ax+2y+1=0$, $bx=3y+1=0$ and $cx+4y+1=0$ are concurrent if a, b, c are in G.P.

56. Line joining the points (3,-4) and (-2,6) is perpendicular to the line joining the points (-3,6) and (9,-18).

Match the questions given under Column C_1 with their appropriate answers given under the Column C_2 is Exercises 57 to 59.

57. **Column C_1** **Column C_2**

Column C_1	Column C_2
1. The coordinates of the points P and Q on the line x + 5y = 13 which are at a distance of 2 units from the line 12x – 5y + 26 = 0 are	a) (3,1),(-7,11)
2. The coordinates of the point on the line x + y = 4, which are at a unit distance from the line 4x + 3y – 10 = 0 are	b) $-\frac{1}{11}, \frac{11}{3}$, $\frac{4}{3}, \frac{7}{3}$
3. The coordinates of the point on the line joining A (–2, 5) and B (3, 1) such that AP = PQ = QB are	c) $1, \frac{12}{5}$, $-3, \frac{16}{5}$

58. The value of the λ, if the lines $(2x + 3y + 4) + \lambda(6x - y + 12) = 0$ are

Column C_1	Column C_2
1. parallel to y-axis is	a) $\lambda = -\frac{3}{4}$
2. perpendicular to $7x + y - 4 = 0$ is	b) $\lambda = -\frac{1}{3}$
3. passes through (1,2) is	c) $\lambda = -\frac{17}{41}$
4. parallel to x axis is	d) $\lambda = 3$

59. The equation of the line through the intersection of the lines $2x - 3y = 0$ and $4x - 5y = 2$ and

Column C_1	Column C_2
1. through the point (2,1) is	a) $2x - y = 4$
2. perpendicular to the line	b) $x + y - 5 = 0$
3. parallel to the line $3x-4y+5=$ 0 is	c) $x - y - 1 = 0$
4. equally inclined to the axes is	d) $3x - 4y - 1 = 0$

2.6. Singular Value Decomposition

2.6.1 Find the shortest distance between the lines

$\overrightarrow{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k})$ and

$\overrightarrow{r} = 2\hat{i} - \hat{j} - \hat{k} + \mu(2\hat{i} + \hat{j} + 2\hat{k})$

2.6.2 Find the shortest distance between the lines

$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$ and $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$ **Solution:**

2.6.3 Find the shortest distance between the lines whose vector equations are

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \tag{2.6.3.1}$$

and

$$\mathbf{x} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \tag{2.6.3.2}$$

Solution: Let $\mathbf{A}$ and $\mathbf{B}$ be points on lines L_1 and L_2 respectively such that AB is normal to both lines. Define

$$\mathbf{M} \triangleq \begin{pmatrix} \mathbf{m_1} & \mathbf{m_2} \end{pmatrix} \tag{2.6.3.3}$$

$$\lambda \triangleq \begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} \tag{2.6.3.4}$$

$$\mathbf{x} \triangleq \mathbf{x_2} - \mathbf{x_1} \tag{2.6.3.5}$$

Then, we have the following equations:

$$\mathbf{A} = \mathbf{x_1} + \lambda_1 \mathbf{m_1} \tag{2.6.3.6}$$

$$\mathbf{B} = \mathbf{x_2} + \lambda_2 \mathbf{m_2} \tag{2.6.3.7}$$

From (2.6.3.6) and (2.6.3.7), define the real-valued function f as

$$f(\lambda) \triangleq \|\mathbf{A} - \mathbf{B}\|^2 \tag{2.6.3.8}$$

$$= \|\mathbf{M}\lambda - \mathbf{x}\|^2 \tag{2.6.3.9}$$

$$= (\mathbf{M}\lambda - \mathbf{x})^\top (\mathbf{M}\lambda - \mathbf{x}) \tag{2.6.3.10}$$

$$= \lambda^\top \left(\mathbf{M}^\top \mathbf{M}\right) \lambda - 2\mathbf{x}^\top \mathbf{M}\lambda + \|\mathbf{x}\|^2 \tag{2.6.3.11}$$

From (2.6.3.11), we see that f is quadratic in λ.

We now prove a useful lemma here.

Lemma 2.6.1. The quadratic form

$$q(\mathbf{x}) \triangleq \mathbf{x}^\top \mathbf{A}\mathbf{x} + \mathbf{b}^\top \mathbf{x} + c \tag{2.6.3.12}$$

is convex iff $\mathbf{A}$ is positive semi-definite.

Proof. Consider two points $\mathbf{x_1}$ and $\mathbf{x_2}$, and a real constant $0 \leq \mu \leq 1$. Then,

$$\begin{aligned}
&\mu f(\mathbf{x_1}) + (1-\mu) f(\mathbf{x_2}) - f(\mu\mathbf{x_1} + (1-\mu)\mathbf{x_2}) \\
&= \left(\mu - \mu^2\right) \mathbf{x_1}^\top \mathbf{A}\mathbf{x_1} + \left(1 - \mu - (1-\mu)^2\right) \mathbf{x_2}^\top \mathbf{A}\mathbf{x_2} \\
&- 2\mu(1-\mu)\mathbf{x_1}^\top \mathbf{A}\mathbf{x_2} && (2.6.3.13) \\
&= \mu(1-\mu)\left(\mathbf{x_1}^\top \mathbf{A}\mathbf{x_1} - 2\mathbf{x_1}^\top \mathbf{A}\mathbf{x_2} + \mathbf{x_2}^\top \mathbf{A}\mathbf{x_2}\right) && (2.6.3.14) \\
&= \mu(1-\mu)(\mathbf{x_1} - \mathbf{x_2})^\top \mathbf{A}(\mathbf{x_1} - \mathbf{x_2}) && (2.6.3.15)
\end{aligned}$$

Since $\mathbf{x_1}$ and $\mathbf{x_2}$ are arbitrary, it follows from (2.6.3.15) that

$$\mu f(\mathbf{x_1}) + (1-\mu) f(\mathbf{x_2}) \geq f(\mu\mathbf{x_1} + (1-\mu)\mathbf{x_2}) \tag{2.6.3.16}$$

iff $\mathbf{A}$ is positive semi-definite, as required. □

Using the above lemma, we show that f is convex by showing that

$\mathbf{M}^\top\mathbf{M}$ is positive semi-definite. Indeed, for any $\mathbf{p} \triangleq \begin{pmatrix} x \\ y \end{pmatrix}$,

$$\mathbf{p}^\top\mathbf{M}^\top\mathbf{M}\mathbf{p} = \|\mathbf{M}\mathbf{p}\|^2 \geq 0 \tag{2.6.3.17}$$

and thus, f is convex.

We need to minimize f as a function of λ. Differentiating (2.6.3.11) using the chain rule,

$$\frac{df(\lambda)}{d\lambda} = \mathbf{M}^\top(\mathbf{M}\lambda - \mathbf{x}) + \mathbf{M}(\mathbf{M}\lambda - \mathbf{x})^\top \tag{2.6.3.18}$$

$$= 2\mathbf{M}^\top(\mathbf{M}\lambda - \mathbf{x}) \tag{2.6.3.19}$$

Setting (2.6.3.19) to zero gives the equation

$$\mathbf{M}^\top\mathbf{M}\lambda = \mathbf{M}^\top\mathbf{x} \tag{2.6.3.20}$$

We use singular value decomposition here. Let

$$\mathbf{M} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^\top \tag{2.6.3.21}$$

where $\mathbf{U}, \mathbf{V}$ are orthogonal and $\boldsymbol{\Sigma}$ is diagonal with nonnegative diagonal entries. Substituting in (2.6.3.20),

$$\mathbf{V}\mathbf{\Sigma}\mathbf{U}^\top\mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top\lambda = \mathbf{V}\mathbf{\Sigma}\mathbf{U}^\top\mathbf{x} \tag{2.6.3.22}$$

$$\implies \mathbf{V}\mathbf{\Sigma}^2\mathbf{V}^\top\lambda = \mathbf{V}\mathbf{\Sigma}\mathbf{U}^\top\mathbf{x} \tag{2.6.3.23}$$

$$\implies \lambda = \left(\mathbf{V}\mathbf{\Sigma}^2\mathbf{V}^\top\right)^{-1}\mathbf{V}\mathbf{\Sigma}\mathbf{U}^\top\mathbf{x} \tag{2.6.3.24}$$

$$\implies \lambda = \mathbf{V}\mathbf{\Sigma}^{-2}\mathbf{V}^\top\mathbf{V}\mathbf{\Sigma}\mathbf{U}^\top\mathbf{x} \tag{2.6.3.25}$$

$$\implies \lambda = \mathbf{V}\mathbf{\Sigma}^{-1}\mathbf{U}^\top\mathbf{x} \tag{2.6.3.26}$$

where $\mathbf{\Sigma}^{-1}$ is obtained by inverting the nonzero elements of $\mathbf{\Sigma}$. Thus, the shortest distance is given by using (2.6.3.21) and (2.6.3.26) in (2.6.3.11), and is given by

$$d = \left\|\left(\mathbf{U}\left(\mathbf{\Sigma}\mathbf{\Sigma}^{-1}\right)\mathbf{U}^\top - \mathbf{I}\right)\mathbf{x}\right\| \tag{2.6.3.27}$$

For this problem,

$$\mathbf{x} = \mathbf{x_2} - \mathbf{x_1} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} \tag{2.6.3.28}$$

$$\mathbf{M} = \begin{pmatrix} \mathbf{m_1} & \mathbf{m_2} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -3 & 3 \\ 2 & 1 \end{pmatrix} \tag{2.6.3.29}$$

Thus,

$$\mathbf{M}^\top\mathbf{M} = \begin{pmatrix} 1 & -3 & 2 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -3 & 3 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 14 & -5 \\ -5 & 14 \end{pmatrix} \tag{2.6.3.30}$$

$$\mathbf{M}\mathbf{M}^\top = \begin{pmatrix} 1 & 2 \\ -3 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -3 & 2 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 3 & 4 \\ 3 & 18 & -3 \\ 4 & -3 & 5 \end{pmatrix} \tag{2.6.3.31}$$

We perform the eigendecompositions for each matrix and bring them into the form

$$\mathbf{M}\mathbf{M}^\top = \mathbf{P_1}\mathbf{D_1}\mathbf{P_1}^\top \tag{2.6.3.32}$$

$$\mathbf{M}^\top\mathbf{M} = \mathbf{P_2}\mathbf{D_2}\mathbf{P_2}^\top \tag{2.6.3.33}$$

(a) For $\mathbf{M}\mathbf{M}^\top$, the characteristic polynomial is

$$\text{char}\mathbf{M}\mathbf{M}^\top = \begin{vmatrix} x-5 & -3 & -4 \\ -3 & x-18 & 3 \\ -4 & 3 & x-5 \end{vmatrix} \tag{2.6.3.34}$$

$$= x\left(x-9\right)\left(x-19\right) \tag{2.6.3.35}$$

Thus, the eigenvalues are given by

$$\lambda_1 = 19,\ \lambda_2 = 9,\ \lambda_3 = 0 \tag{2.6.3.36}$$

For λ_1, the augmented matrix formed from the eigenvalue-eigenvector equation is

$$\begin{pmatrix} -14 & 3 & 4 & 0 \\ 3 & -1 & -3 & 0 \\ 4 & -3 & -14 & 0 \end{pmatrix}$$

$$\xleftrightarrow{R_1 \leftarrow \frac{R_1+R_3}{-10}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 3 & -1 & -3 & 0 \\ 4 & -3 & -14 & 0 \end{pmatrix} \tag{2.6.3.37}$$

$$\xleftrightarrow{R_2 \leftarrow -R_2+3R_1} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 6 & 0 \\ 4 & -3 & -14 & 0 \end{pmatrix} \tag{2.6.3.38}$$

$$\xleftrightarrow{R_3 \leftarrow R_3-4R_1} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & -6 & 0 \\ 0 & -3 & -18 & 0 \end{pmatrix} \tag{2.6.3.39}$$

$$\xleftrightarrow{R_3 \leftarrow R_3-3R_2} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.6.3.40}$$

Hence, the normalized eigenvector is

$$\mathbf{p_1} = \frac{1}{\sqrt{38}} \begin{pmatrix} -1 \\ -6 \\ 1 \end{pmatrix} \tag{2.6.3.41}$$

For λ_2, the augmented matrix formed from the eigenvalue-eigenvector

equation is

$$
\begin{pmatrix} -4 & 3 & 4 & 0 \\ 3 & 9 & -3 & 0 \\ 4 & 3 & -4 & 0 \end{pmatrix}
$$

$$
\xleftrightarrow{R_3 \leftarrow R_1 + R_3} \begin{pmatrix} -4 & 3 & 4 & 0 \\ 3 & 9 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.6.3.42}
$$

$$
\xleftrightarrow{R_2 \leftarrow \frac{4R_2 + 3R_1}{45}} \begin{pmatrix} -4 & 3 & 4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.6.3.43}
$$

$$
\xleftrightarrow{R_1 \leftarrow \frac{R_1 - 3R_2}{-4}} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.6.3.44}
$$

Hence, the normalized eigenvector is

$$
\mathbf{p_2} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \tag{2.6.3.45}
$$

For λ_3, the augmented matrix formed from the eigenvalue-eigenvector

equation is

$$\begin{pmatrix} 5 & 3 & 4 & 0 \\ 3 & 18 & -3 & 0 \\ 4 & -3 & 5 & 0 \end{pmatrix}$$

$$\xleftrightarrow{R_1 \leftarrow \frac{R_1+R_3}{9}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 3 & 18 & -3 & 0 \\ 4 & -3 & 5 & 0 \end{pmatrix} \tag{2.6.3.46}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - 3R_1} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 18 & -6 & 0 \\ 4 & -3 & 5 & 0 \end{pmatrix} \tag{2.6.3.47}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - 4R_1} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 18 & -6 & 0 \\ 0 & -3 & 1 & 0 \end{pmatrix} \tag{2.6.3.48}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{R_2}{6}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 3 & -1 & 0 \\ 0 & -3 & 1 & 0 \end{pmatrix} \tag{2.6.3.49}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 + R_2} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.6.3.50}$$

Hence, the normalized eigenvector is

$$\mathbf{p_3} = \frac{1}{\sqrt{19}} \begin{pmatrix} -3 \\ 1 \\ 3 \end{pmatrix} \tag{2.6.3.51}$$

Using (2.6.3.32), we see that

$$\mathbf{P_1} = \begin{pmatrix} -\frac{1}{\sqrt{38}} & \frac{1}{\sqrt{2}} & -\frac{3}{\sqrt{19}} \\ -\frac{6}{\sqrt{38}} & 0 & \frac{1}{\sqrt{19}} \\ \frac{1}{\sqrt{38}} & -\frac{1}{\sqrt{2}} & \frac{3}{\sqrt{19}} \end{pmatrix} \tag{2.6.3.52}$$

$$\mathbf{D_1} = \begin{pmatrix} 19 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.6.3.53}$$

(b) For $\mathbf{M}^\top\mathbf{M}$, the characteristic polynomial is

$$\text{char}\mathbf{M}^\top\mathbf{M} = \begin{vmatrix} x-14 & 5 \\ 5 & x-14 \end{vmatrix} \tag{2.6.3.54}$$

$$= (x-9)(x-19) \tag{2.6.3.55}$$

Thus, the eigenvalues are given by

$$\lambda_1 = 19, \ \lambda_2 = 9 \tag{2.6.3.56}$$

For λ_1, the augmented matrix formed from the eigenvalue-eigenvector

equation is

$$\begin{pmatrix} -5 & -5 & 0 \\ -5 & -5 & 0 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 - R_2} \begin{pmatrix} 0 & 0 & 0 \\ -5 & -5 & 0 \end{pmatrix} \tag{2.6.3.57}$$

Hence, the normalized eigenvector is

$$\mathbf{p_1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{2.6.3.58}$$

For λ_2, the augmented matrix formed from the eigenvalue-eigenvector equation is

$$\begin{pmatrix} 5 & -5 & 0 \\ -5 & 5 & 0 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 + R_2} \begin{pmatrix} 0 & 0 & 0 \\ 5 & -5 & 0 \end{pmatrix} \tag{2.6.3.59}$$

Hence, the normalized eigenvector is

$$\mathbf{p_2} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.6.3.60}$$

Thus, from (2.6.3.33),

$$\mathbf{P_2} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \ \mathbf{D_2} = \begin{pmatrix} 9 & 0 \\ 0 & 19 \end{pmatrix} \tag{2.6.3.61}$$

Therefore, from (2.6.3.21),

$$\mathbf{U} = \mathbf{P_1} \tag{2.6.3.62}$$

$$\mathbf{V} = \mathbf{P_2} \tag{2.6.3.63}$$

$$\mathbf{\Sigma} = \begin{pmatrix} \sqrt{19} & 0 \\ 0 & 3 \\ 0 & 0 \end{pmatrix} \tag{2.6.3.64}$$

and substituting into (2.6.3.26), we get

$$\lambda = \frac{1}{19}\begin{pmatrix} 10 \\ 28 \end{pmatrix} \tag{2.6.3.65}$$

which agrees with earlier solutions as well. See Fig. 2.6.3.1 depicting the situation.

2.6.4 Find the shortest distance between the lines l_1 and l_2 whose vector equations are $\overrightarrow{r} = \hat{i} + \hat{j} + \lambda(2\hat{i} - \hat{j} + \hat{k})$ and $\overrightarrow{r} = 2\hat{i} + \hat{j} - \hat{k} + \mu(3\hat{i} - 5\hat{j} + 2\hat{k})$.

Solution: The shortest distance between the lines whose vector equations are

$$L_1 : \mathbf{x} = \mathbf{x_1} + \lambda_1 \mathbf{m_1} \tag{2.6.4.1}$$

$$L_2 : \mathbf{x} = \mathbf{x_2} + \lambda_2 \mathbf{m_2} \tag{2.6.4.2}$$

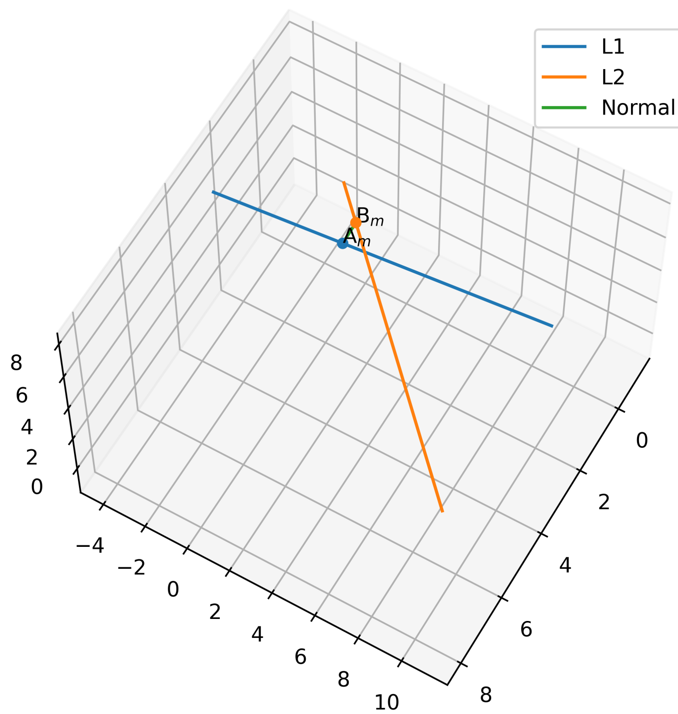


Figure 2.6.3.1: Finding the shortest distance between two lines using SVD.

is given by,

$$d = \left\| \left(\mathbf{U} \left(\mathbf{\Sigma} \mathbf{\Sigma}^{-1} \right) \mathbf{U}^{\top} - \mathbf{I} \right) \mathbf{x} \right\| \tag{2.6.4.3}$$

with the parameter λ given by

$$\boldsymbol{\lambda} = \mathbf{V} \mathbf{\Sigma}^{-1} \mathbf{U}^{\top} \mathbf{x} \tag{2.6.4.4}$$

where

$$\mathbf{M} \triangleq \begin{pmatrix} \mathbf{m_1} & \mathbf{m_2} \end{pmatrix} \tag{2.6.4.5}$$

$$\boldsymbol{\lambda} \triangleq \begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} \tag{2.6.4.6}$$

$$\mathbf{x} \triangleq \mathbf{x_2} - \mathbf{x_1} \tag{2.6.4.7}$$

We use singular value decomposition of the matrix $\mathbf{M}$

$$\mathbf{M} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top \tag{2.6.4.8}$$

where $\mathbf{U}, \mathbf{V}$ are orthogonal and $\mathbf{\Sigma}$ is diagonal with nonnegative diagonal entries.

(a) In this problem we have the lines l_1 and l_2 as

$$\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \tag{2.6.4.9}$$

$$\mathbf{x} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} \tag{2.6.4.10}$$

We first need to check whether the given lines are skew. The lines

(2.6.4.1), (2.6.4.2) intersect if

$$\mathbf{M}\boldsymbol{\lambda} = \mathbf{x_2} - \mathbf{x_1} \tag{2.6.4.11}$$

Here we have,

$$\mathbf{M} = \begin{pmatrix} 2 & 3 \\ -1 & -5 \\ 1 & 2 \end{pmatrix} \tag{2.6.4.12}$$

$$\mathbf{x} = \mathbf{x_2} - \mathbf{x_1} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \tag{2.6.4.13}$$

We check whether the equation (2.6.4.14) has a solution

$$\begin{pmatrix} 2 & 3 \\ -1 & -5 \\ 1 & 2 \end{pmatrix} \boldsymbol{\lambda} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \tag{2.6.4.14}$$

the augmented matrix is given by,

$$\left(\begin{array}{cc|c} 2 & 3 & 1 \\ -1 & -5 & 0 \\ 1 & 2 & -1 \end{array}\right) \tag{2.6.4.15}$$

$$\xleftrightarrow[R_3 \leftarrow R_3 - \frac{1}{2}R_1]{R_2 \leftarrow R_2 + \frac{1}{2}R_1} \left(\begin{array}{cc|c} 2 & 3 & 1 \\ 0 & -\frac{7}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{3}{2} \end{array}\right) \tag{2.6.4.16}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 + 7R_2} \left(\begin{array}{cc|c} 2 & 3 & 1 \\ 0 & -\frac{7}{2} & \frac{1}{2} \\ 0 & 0 & -10 \end{array}\right) \tag{2.6.4.17}$$

The rank of the matrix is 3. So the given lines are skew.

(b) From (2.6.4.12) we have

$$\mathbf{M}^\top \mathbf{M} = \begin{pmatrix} 2 & -1 & 1 \\ 3 & -5 & 2 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ -1 & -5 \\ 1 & 2 \end{pmatrix} \tag{2.6.4.18}$$

$$= \begin{pmatrix} 6 & 13 \\ 13 & 38 \end{pmatrix} \tag{2.6.4.19}$$

$$\mathbf{M}\mathbf{M}^\top = \begin{pmatrix} 2 & 3 \\ -1 & -5 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & -1 & 1 \\ 3 & -5 & 2 \end{pmatrix} \tag{2.6.4.20}$$

$$= \begin{pmatrix} 13 & -17 & 8 \\ -17 & 26 & -11 \\ 8 & -11 & 5 \end{pmatrix} \tag{2.6.4.21}$$

We perform the eigen decompositions for the matrics (2.6.4.21), (2.6.4.19) and write them in the form

$$\mathbf{M}\mathbf{M}^\top = \mathbf{P_1}\mathbf{D_1}\mathbf{P_1}^\top \tag{2.6.4.22}$$

$$\mathbf{M}^\top\mathbf{M} = \mathbf{P_2}\mathbf{D_2}\mathbf{P_2}^\top \tag{2.6.4.23}$$

The characteristic polynomial of the matrix $\mathbf{M}\mathbf{M}^\top$ is given by,

$$\text{char}\left(\mathbf{M}\mathbf{M}^\top\right) = \begin{vmatrix} 13-x & -17 & 8 \\ -17 & 26-x & -11 \\ 8 & -11 & 5-x \end{vmatrix} \tag{2.6.4.24}$$

$$= -x^3 + 44x^2 - 59x \tag{2.6.4.25}$$

Thus, the eigenvalues are given by

$$\lambda_1 = 22 + 5\sqrt{17},\ \lambda_2 = 22 - 5\sqrt{17},\ \lambda_3 = 0 \tag{2.6.4.26}$$

From the augmented matrix formed from the eigen value - eigen vector equation we get, the normalized eigen vectors as

$$\mathbf{p_1} = \frac{\sqrt{5}}{\sqrt{68-6\sqrt{17}}}\begin{pmatrix}\frac{12-\sqrt{17}}{5}\\ \frac{1-3\sqrt{17}}{5}\\ 1\end{pmatrix} \tag{2.6.4.27}$$

$$\mathbf{p_2} = \frac{\sqrt{5}}{\sqrt{68+6\sqrt{17}}}\begin{pmatrix}\frac{12+\sqrt{17}}{5}\\ \frac{1+3\sqrt{17}}{5}\\ 1\end{pmatrix} \tag{2.6.4.28}$$

$$\mathbf{p_3} = \frac{1}{\sqrt{59}}\begin{pmatrix}-3\\ 1\\ 7\end{pmatrix} \tag{2.6.4.29}$$

where $\mathbf{p_1}, \mathbf{p_2}, \mathbf{p_3}$ corresponds to the eigen values $\lambda_1, \lambda_2, \lambda_3$ respectively. Using (2.6.4.22), we get

$$\mathbf{P_1} = \begin{pmatrix}\frac{12-\sqrt{17}}{\sqrt{5}\sqrt{68-6\sqrt{17}}} & \frac{12+\sqrt{17}}{\sqrt{5}\sqrt{68+6\sqrt{17}}} & -\frac{3}{\sqrt{59}}\\ \frac{1-3\sqrt{17}}{\sqrt{5}\sqrt{68-6\sqrt{17}}} & \frac{1+3\sqrt{17}}{\sqrt{5}\sqrt{68+6\sqrt{17}}} & \frac{1}{\sqrt{59}}\\ \frac{\sqrt{5}}{\sqrt{68-6\sqrt{17}}} & \frac{\sqrt{5}}{\sqrt{68+6\sqrt{17}}} & \frac{7}{\sqrt{59}}\end{pmatrix} \tag{2.6.4.30}$$

$$\mathbf{D_1} = \begin{pmatrix}22+5\sqrt{17} & 0 & 0\\ 0 & 22-5\sqrt{17} & 0\\ 0 & 0 & 0\end{pmatrix} \tag{2.6.4.31}$$

For $\mathbf{M}^\top\mathbf{M}$, the characteristic polynomial is

$$\text{char}\left(\mathbf{M}^\top\mathbf{M}\right) = \begin{vmatrix} 6-x & 13 \\ 13 & 38-x \end{vmatrix} \tag{2.6.4.32}$$

$$= x^2 - 44x + 59 \tag{2.6.4.33}$$

Thus, the eigenvalues are given by

$$\lambda_1 = 22 + 5\sqrt{17},\ \lambda_2 = 22 - 5\sqrt{17} \tag{2.6.4.34}$$

From the augmented matrix formed from the eigen value - eigen vector equation we get, the normalized eigen vectors as

$$\mathbf{p_1} = \frac{13}{\sqrt{850 - 160\sqrt{17}}} \begin{pmatrix} \frac{-16+5\sqrt{17}}{13} \\ 1 \end{pmatrix} \tag{2.6.4.35}$$

$$\mathbf{p_2} = \frac{13}{\sqrt{850 + 160\sqrt{17}}} \begin{pmatrix} \frac{-16-5\sqrt{17}}{13} \\ 1 \end{pmatrix} \tag{2.6.4.36}$$

where $\mathbf{p_1}, \mathbf{p_2}$ corresponds to the eigen values λ_1, λ_2 respectively. Using (2.6.4.23), we get

$$\mathbf{P_2} = \begin{pmatrix} \frac{-16-5\sqrt{17}}{\sqrt{850+160\sqrt{17}}} & \frac{13}{\sqrt{850-160\sqrt{17}}} \\ \frac{13}{\sqrt{850+160\sqrt{17}}} & \frac{-16+5\sqrt{17}}{\sqrt{850-160\sqrt{17}}} \end{pmatrix} \tag{2.6.4.37}$$

$$\mathbf{D_2} = \begin{pmatrix} 22 - 5\sqrt{17} & 0 \\ 0 & 22 + 5\sqrt{17} \end{pmatrix} \tag{2.6.4.38}$$

Therefore, from (2.6.4.8) we have

$$\mathbf{U} = \mathbf{P_1} \tag{2.6.4.39}$$

$$\mathbf{V} = \mathbf{P_2} \tag{2.6.4.40}$$

$$\mathbf{\Sigma} = \begin{pmatrix} \sqrt{22+5\sqrt{17}} & 0 \\ 0 & \sqrt{22-5\sqrt{17}} \\ 0 & 0 \end{pmatrix} \tag{2.6.4.41}$$

and substituting into (2.6.4.4), we get

$$\boldsymbol{\lambda} = \begin{pmatrix} \frac{25}{59} \\ \\ -\frac{7}{59} \end{pmatrix} \tag{2.6.4.42}$$

The minimum distance between the lines is given by,

$$\|\mathbf{B} - \mathbf{A}\| = \left\| \frac{1}{59} \begin{pmatrix} 30 \\ -10 \\ -70 \end{pmatrix} \right\| \tag{2.6.4.43}$$

$$= \frac{\sqrt{30^2 + 10^2 + 70^2}}{59} \tag{2.6.4.44}$$

$$= \frac{10}{\sqrt{59}} \tag{2.6.4.45}$$

The shortest distance between the given lines is $\frac{10}{\sqrt{59}}$ units. See Fig. 2.6.4.1.

2.6.5 Find the shortest distance between the lines given by $\overrightarrow{r} = (8 + 3\lambda\hat{i} -$

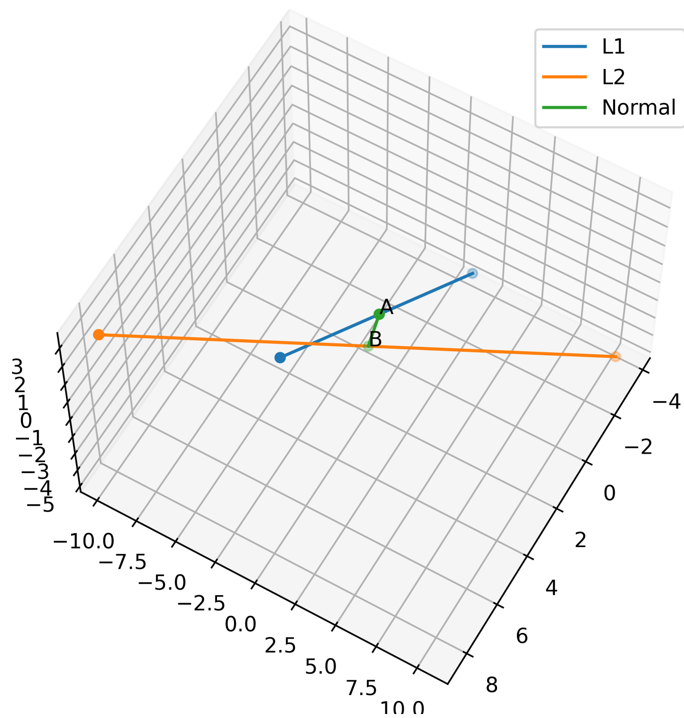


Figure 2.6.4.1: AB is the required shortest distance.

$(9+16\lambda)\hat{j}+(10+7\lambda)\hat{k}$ and $\overrightarrow{r}=15\hat{i}+29\hat{j}+5\hat{k}+\mu(3\hat{i}+8\hat{j}-5\hat{k})$.

Chapter 3

Conics

3.1. Circle

3.1.1 Find the equation of the circle passing through the points $(4, 1)$ and $(6, 5)$ and whose centre is on the line $4x + y = 16$.

Solution: The equation of the circle is given by

$$\|\mathbf{x}\|^2 + 2\mathbf{x}^\top \mathbf{u} + f = 0 \tag{3.1.1.1}$$

where

$$\mathbf{u} = -\mathbf{c} \tag{3.1.1.2}$$

$$f = \|\mathbf{c}\| - r^2 \tag{3.1.1.3}$$

Given points are

$$\mathbf{x}_1 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}, \mathbf{x}_2 = \begin{pmatrix} 6 \\ 5 \end{pmatrix} \tag{3.1.1.4}$$

And the line passing through the centre

$$\begin{pmatrix} 4 & 1 \end{pmatrix} \mathbf{x} = 16 \tag{3.1.1.5}$$

Substituting points from (3.1.1.4) into (3.1.1.1)

$$\left(4^2 + 1^2\right) + 2 \begin{pmatrix} 4 & 1 \end{pmatrix} \mathbf{u} + f = 0 \tag{3.1.1.6}$$

$$\implies 2 \begin{pmatrix} 4 & 1 \end{pmatrix} \mathbf{u} + f = -17 \tag{3.1.1.7}$$

$$\left(6^2 + 5^2\right) + 2 \begin{pmatrix} 6 & 5 \end{pmatrix} \mathbf{u} + f = 0 \tag{3.1.1.8}$$

$$\implies 2 \begin{pmatrix} 6 & 5 \end{pmatrix} \mathbf{u} + f = -61 \tag{3.1.1.9}$$

And since (3.1.1.5) passes through the centre

$$-\mathbf{n}^\top \mathbf{u} = c \tag{3.1.1.10}$$

$$-\begin{pmatrix} 4 & 1 \end{pmatrix} \mathbf{u} = 16 \tag{3.1.1.11}$$

Representing (3.1.1.7),(3.1.1.9) and (3.1.1.11) in matrix form

$$\begin{pmatrix} -4 & -1 & 0 \\ 12 & 10 & 1 \\ 8 & 2 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ f \end{pmatrix} = \begin{pmatrix} 16 \\ -61 \\ -17 \end{pmatrix} \tag{3.1.1.12}$$

The augmented matrix is expressed as

$$\left(\begin{array}{ccc|c} -4 & -1 & 0 & 16 \\ 12 & 10 & 1 & -61 \\ 8 & 2 & 1 & -17 \end{array}\right) \tag{3.1.1.13}$$

Performing a sequence of row operations to transform into an Echelon form

$$\xleftrightarrow[R_2 \to R_2+3R_1]{R_3 \to R_3+2R_1} \left(\begin{array}{ccc|c} -4 & -1 & 0 & 16 \\ 0 & 7 & 1 & -13 \\ 0 & 0 & 1 & 15 \end{array}\right) \tag{3.1.1.14}$$

$$\xleftrightarrow{R_2 \to R_2-R_3} \left(\begin{array}{ccc|c} -4 & -1 & 0 & 16 \\ 0 & 7 & 0 & -28 \\ 0 & 0 & 1 & 15 \end{array}\right) \tag{3.1.1.15}$$

$$\xleftrightarrow{R_2 \to \frac{R_2}{7}, R_1 \to \frac{-R_1}{4}} \left(\begin{array}{ccc|c} 1 & \frac{1}{4} & 0 & -4 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 15 \end{array}\right) \tag{3.1.1.16}$$

$$\xleftrightarrow{R_1 \to R_1-\frac{1}{4}R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 15 \end{array}\right) \tag{3.1.1.17}$$

So, from (3.1.1.17)

$$\mathbf{u} = \begin{pmatrix} -3 \\ -4 \end{pmatrix} \tag{3.1.1.18}$$

$$f = 15 \tag{3.1.1.19}$$

Since $\mathbf{u} = -\mathbf{c}$

$$\mathbf{c} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \tag{3.1.1.20}$$

$$r^2 = \left(3^2 + 4^2\right) - 15 \tag{3.1.1.21}$$

$$r = \sqrt{10} \tag{3.1.1.22}$$

Hence, the equation of circle is

$$\left\|\mathbf{x}\right\|^2 + 2\mathbf{u}^\top\mathbf{x} + 15 = 0 \tag{3.1.1.23}$$

$$\text{where } \mathbf{u} = \begin{pmatrix} -3 \\ -4 \end{pmatrix} \tag{3.1.1.24}$$

The circle is plotted in Fig. 3.1.1.1.

3.1.2 Find the equation of the circle passing through the points $(2, 3)$ and $(-1, 1)$ and whose centre is on the line $x - 3y - 11 = 0$.

Solution: See Fig. From (E.2.1.1), and the given information,

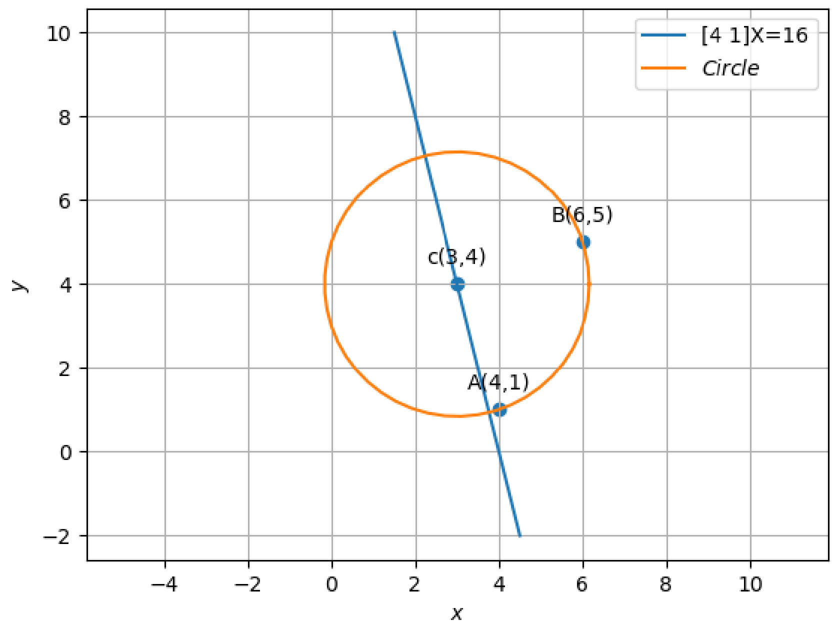


Figure 3.1.1.1:

$$\|\mathbf{P}\|^2 + 2\mathbf{u}^\top \mathbf{P} + f = 0 \tag{3.1.2.1}$$

$$\|\mathbf{Q}\|^2 + 2\mathbf{u}^\top \mathbf{Q} + f = 0 \tag{3.1.2.2}$$

$$-\mathbf{n}^\top \mathbf{u} = c \tag{3.1.2.3}$$

by noting that the centre of the circle is $-\mathbf{u}$. Substituting numerical

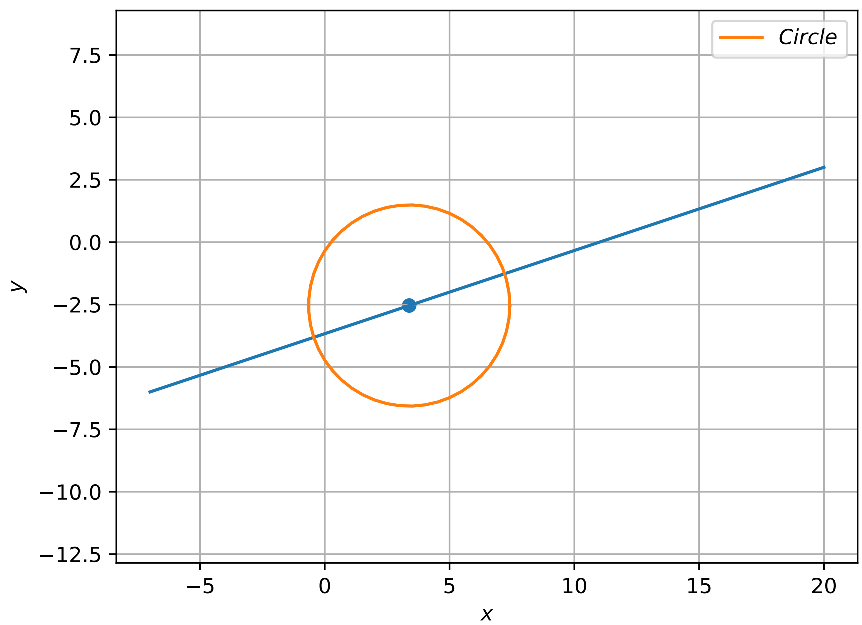


Figure 3.1.2.1:

values, we obtain the matrix equation

$$\begin{pmatrix} 4 & 6 & 1 \\ -2 & 2 & 1 \\ -1 & 3 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ f \end{pmatrix} = \begin{pmatrix} -13 \\ -2 \\ 11 \end{pmatrix} \tag{3.1.2.4}$$

$$\tag{3.1.2.5}$$

The augmented matrix for (3.1.2.4) can be expressed as

$$\xleftrightarrow{1/4R_1 \leftrightarrow R_1} \left(\begin{array}{ccc|c} 1 & 3/2 & .1/4 & -13/4 \\ -2 & 2 & 1 & -2 \\ -1 & 3 & 0 & 11 \end{array}\right) \tag{3.1.2.6}$$

which can be reduced to echelon form using row operations to obtain

$$\mathbf{u} = \begin{pmatrix} -7/2 \\ 5/2 \end{pmatrix}, f = -14 \tag{3.1.2.7}$$

3.1.3 Find the equation of the circle with radius 5 whose centre lies on x-axis and passes through the point $(2, 3)$.

Solution: See Fig. 3.1.3.1. From the given information, the following equations can be formulated using (E.2.1.1).

$$\|\mathbf{P}\|^2 + 2\mathbf{u}^\top \mathbf{P} + f = 0 \tag{3.1.3.1}$$

$$\mathbf{u} = k\mathbf{e}_1 \tag{3.1.3.2}$$

$$\|\mathbf{u}\|^2 - f = r^2 \tag{3.1.3.3}$$

where

$$\mathbf{P} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \text{ and } r = 5 \tag{3.1.3.4}$$

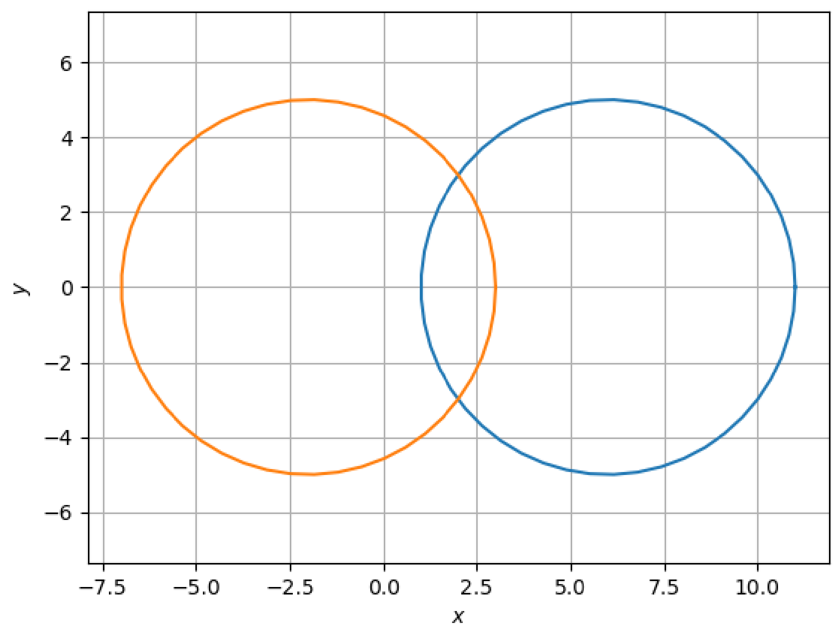


Figure 3.1.3.1:

From (3.1.3.1) and (3.1.3.3),

$$\|\mathbf{P}\|^2 + 2\mathbf{u}^\top \mathbf{P} + \|\mathbf{u}\|^2 = r^2 \tag{3.1.3.5}$$

Substituting from (3.1.3.2) in the above,

$$k^2 + 2k\mathbf{e}_1^\top \mathbf{P} + \|\mathbf{P}\|^2 - r^2 = 0 \tag{3.1.3.6}$$

resulting in

$$k = -\mathbf{e}_1^\top \mathbf{P} \pm \sqrt{\left(\mathbf{e}_1^\top \mathbf{P}\right)^2 + r^2 - \|\mathbf{P}\|^2} \tag{3.1.3.7}$$

Substituting numerical values,

$$k = 2, -6 \tag{3.1.3.8}$$

resulting in circles with centre

$$-\mathbf{u} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 6 \\ 0 \end{pmatrix}. \tag{3.1.3.9}$$

This is verified in Fig. (3.1.3.1).

3.1.4 Find the equation of the circle passing through $(0, 0)$ and making intercepts a and b on the coordinate axes.

3.1.5 Find the equation of a circle with centre $(2, 2)$ and passes through the point $(4, 5)$.

From the given information

$$\mathbf{u} = -\begin{pmatrix} 2 \\ 2 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 4 \\ 5 \end{pmatrix} \tag{3.1.5.1}$$

$$\implies \|\mathbf{A}\|^2 + 2\mathbf{u}^\top \mathbf{A} + f = 0 \tag{3.1.5.2}$$

$$\implies f = -\|\mathbf{A}\|^2 - 2\mathbf{u}^\top \mathbf{A} = -5 \tag{3.1.5.3}$$

Hence the equation of circle is

$$\|\mathbf{x}\|^2 + 2\begin{pmatrix} -2 & -2 \end{pmatrix}\mathbf{x} - 5 = 0 \tag{3.1.5.4}$$

See Fig. 3.1.5.1.

3.1.6 Does the point $(-2.5, 3.5)$ lie inside, outside or on the circle $x^2 + y^2 = 25$?

Solution: See Table 3.1.6.1. Given circle equation can be expressed

Condition	Inference
$\|\mathbf{x} - \mathbf{O}\|^2 < r^2$	point lies inside the circle
$\|\mathbf{x} - \mathbf{O}\|^2 > r^2$	point lies outside the circle
$\|\mathbf{x} - \mathbf{O}\|^2 = r^2$	point lies on the circle

Table 3.1.6.1: Table1

as,

$$\|\mathbf{x}\|^2 = 25 \tag{3.1.6.1}$$

Let,

$$\mathbf{P} = \begin{pmatrix} -2.5 \\ 3.5 \end{pmatrix} \tag{3.1.6.2}$$

Since

$$\|\mathbf{P} - \mathbf{O}\|^2 = 18.5 < 25, \tag{3.1.6.3}$$

$$\implies \|\mathbf{x} - \mathbf{O}\|^2 < r^2, \tag{3.1.6.4}$$

the point lies inside the given circle. See Fig. 3.1.6.1.

3.1.7 Find the centre of a circle passing though the points $(6, -6), (3, -7)$ and $(3, 3)$.

Solution: The equation of the circle is given by

$$\|\mathbf{x}\|^2 + 2\mathbf{x}^\top \mathbf{u} + f = 0 \tag{3.1.7.1}$$

where

$$\mathbf{u} = -\mathbf{c} \text{ and} \tag{3.1.7.2}$$

$$f = \|\mathbf{c}\|^2 - r^2 \tag{3.1.7.3}$$

Given points are

$$\mathbf{x_1} = \begin{pmatrix} 6 \\ -6 \end{pmatrix}, \mathbf{x_2} = \begin{pmatrix} 3 \\ -7 \end{pmatrix}, \mathbf{x_3} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} \tag{3.1.7.4}$$

Substituting points from (3.1.7.4) into (3.1.7.1)

$$\left(6^2 + (-6)^2\right) + 2\begin{pmatrix} 6 & -6 \end{pmatrix}\mathbf{u} + f = 0 \tag{3.1.7.5}$$

$$\implies 2\begin{pmatrix} 6 & -6 \end{pmatrix}\mathbf{u} + f = -72 \tag{3.1.7.6}$$

$$\left(3^2 + (-7)^2\right) + 2\begin{pmatrix} 3 & -7 \end{pmatrix}\mathbf{u} + f = 0 \tag{3.1.7.7}$$

$$\implies 2\begin{pmatrix} 3 & -7 \end{pmatrix}\mathbf{u} + f = -58 \tag{3.1.7.8}$$

$$(3^2 + 3^2) + 2\begin{pmatrix} 3 & 3 \end{pmatrix}\mathbf{u} + f = 0 \tag{3.1.7.9}$$

$$\implies 2\begin{pmatrix} 3 & 3 \end{pmatrix}\mathbf{u} + f = -18 \tag{3.1.7.10}$$

Representing the above system of equations in matrix form

$$\begin{pmatrix} 6 & -14 & 1 \\ 12 & -12 & 1 \\ 6 & 6 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ f \end{pmatrix} = \begin{pmatrix} -58 \\ -72 \\ -18 \end{pmatrix} \tag{3.1.7.11}$$

The augmented matrix is expressed as

$$\left(\begin{array}{ccc|c} 6 & -14 & 1 & -58 \\ 12 & -12 & 1 & -72 \\ 6 & 6 & 1 & -18 \end{array}\right) \tag{3.1.7.12}$$

Performing sequence of row operations to transform into an Echelon form

$$\xleftrightarrow{R_2 \to R_2 - 2R_1} \left(\begin{array}{ccc|c} 6 & -14 & 1 & -58 \\ 0 & 16 & -1 & 44 \\ 6 & 6 & 1 & -18 \end{array}\right) \tag{3.1.7.13}$$

$$\xleftrightarrow{R_3 \to R_3 - R_1} \left(\begin{array}{ccc|c} 6 & -14 & 1 & -58 \\ 0 & 16 & -1 & 44 \\ 0 & 20 & 0 & 40 \end{array}\right) \tag{3.1.7.14}$$

$$\xleftrightarrow{R_3 \to R_3 - \frac{20}{16}R_2} \left(\begin{array}{ccc|c} 6 & -14 & 1 & -58 \\ 0 & 16 & -1 & 44 \\ 0 & 0 & \frac{20}{16} & -15 \end{array}\right) \tag{3.1.7.15}$$

$$\xleftrightarrow[R_2 \to \frac{1}{16}R_2, R_3 \to \frac{16}{20}R_3]{R_1 \to \frac{1}{6}R_1} \left(\begin{array}{ccc|c} 1 & -\frac{14}{6} & \frac{1}{6} & -\frac{58}{6} \\ 0 & 1 & -\frac{1}{16} & \frac{44}{16} \\ 0 & 0 & 1 & -12 \end{array}\right) \tag{3.1.7.16}$$

$$\xleftrightarrow[R_2 \to R_2 + \frac{1}{16}R_3]{R_1 \to R_1 - \frac{1}{6}R_3} \left(\begin{array}{ccc|c} 1 & -\frac{14}{6} & 0 & -\frac{46}{6} \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -12 \end{array}\right) \tag{3.1.7.17}$$

$$\xleftrightarrow{R_1 \to R_1 + \frac{14}{6}R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -12 \end{array}\right) \tag{3.1.7.18}$$

So, from (3.1.7.18)

$$\mathbf{u} = \begin{pmatrix} -3 \\ 2 \end{pmatrix} \tag{3.1.7.19}$$

$$f = -12 \tag{3.1.7.20}$$

Since $\mathbf{u} = -\mathbf{c}$,

$$\mathbf{c} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \tag{3.1.7.21}$$

$$(3.1.7.3) \implies r^2 = \left(3^2 + (-2)^2\right) + 12 \tag{3.1.7.22}$$

$$r = 5 \tag{3.1.7.23}$$

Therefore, the equation of the circle is

$$\left\| \mathbf{x} - \begin{pmatrix} 3 \\ -2 \end{pmatrix} \right\| = 5 \tag{3.1.7.24}$$

The relevant diagram is shown in Figure 3.1.7.1

In each of the following exercises, find the equation of the circle with the following parameters

3.1.8 centre $(0, 2)$ and radius 2

Solution: The equation of the circle is given by

$$\|\mathbf{x}\|^2 + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{3.1.8.1}$$

From the given information,

$$\mathbf{c} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \text{ and } r = 2, \tag{3.1.8.2}$$

Since

$$\mathbf{u} = -\mathbf{c} \text{ and } f = \left\|\mathbf{u}\right\|^2 - r^2, \tag{3.1.8.3}$$

substituting numerical values,

$$\mathbf{u} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}, f = 0 \tag{3.1.8.4}$$

Thus, the equation of circle is obtained as

$$\left\|\mathbf{x}\right\|^2 + 2\begin{pmatrix} 0 & 2 \end{pmatrix}\mathbf{x} = 0 \tag{3.1.8.5}$$

See Fig. 3.1.8.1

3.1.9 centre $(-2, 3)$ and radius 4

Solution: Given

$$\mathbf{u} = -\begin{pmatrix} -2 \\ 3 \end{pmatrix} \text{ and } r = 4 \tag{3.1.9.1}$$

Hence,

$$f = \left\|\mathbf{u}\right\|^2 - r^2 = -3 \tag{3.1.9.2}$$

The equation of the circle is then obtained as

$$\left\|\mathbf{x}\right\|^2 + 2\begin{pmatrix} 2 & -3 \end{pmatrix}\mathbf{x} - 3 = 0 \tag{3.1.9.3}$$

See Fig. 3.1.9.1.

3.1.10 centre $\left(\frac{1}{2}, \frac{1}{4}\right)$ and radius $\frac{1}{12}$

Solution: Given,

$$\mathbf{c} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{4} \end{pmatrix} \text{ and } r = \frac{1}{12}. \tag{3.1.10.1}$$

Hence,

$$\mathbf{u} = -\mathbf{c},\ f = \|\mathbf{u}\|^2 - r^2 = \frac{11}{36} \tag{3.1.10.2}$$

Thus, the equation of the circle is

$$\|\mathbf{x}\|^2 + \begin{pmatrix} -1 & -\frac{1}{2} \end{pmatrix} \mathbf{x} + \frac{11}{36} = 0 \tag{3.1.10.3}$$

3.1.11 centre $(1, 1)$ and radius $\sqrt{2}$

Solution: Given

$$\mathbf{c} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } r = \sqrt{2}, \tag{3.1.11.1}$$

$$\mathbf{u} = -\mathbf{c} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \tag{3.1.11.2}$$

$$\tag{3.1.11.3}$$

$$f = \|\mathbf{u}\|^2 - r^2 = 0 \tag{3.1.11.4}$$

Thus, the equation of circle is

$$\|\mathbf{x}\|^2 - 2\begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} = 0 \tag{3.1.11.5}$$

See Fig. 3.1.11.1.

3.1.12 centre $(-a, -b)$ and radius $\sqrt{a^2 - b^2}$.

Solution: Since

$$\mathbf{c} = \begin{pmatrix} -a \\ -b \end{pmatrix} \text{ and } r = \sqrt{a^2 - b^2} \tag{3.1.12.1}$$

$$\mathbf{u} = \begin{pmatrix} a \\ b \end{pmatrix}, \ f = \|\mathbf{u}\|^2 - r^2 = 2b^2 \tag{3.1.12.2}$$

Thus, the equation of circle is

$$\|\mathbf{x}\|^2 + 2\begin{pmatrix} a & b \end{pmatrix}\mathbf{x} + 2b^2 = 0 \tag{3.1.12.3}$$

See Fig. 3.1.12.1 for

$$a = -3, b = -2 \tag{3.1.12.4}$$

In each of the following exercises, find the centre and radius of the circles.

3.1.13 $x^2 + y^2 + 10x - 6y - 2 = 0$.

Solution: The circle parameters are

$$\mathbf{u} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}, \ f = -2 \tag{3.1.13.1}$$

$$\implies \mathbf{c} = \begin{pmatrix} -5 \\ 3 \end{pmatrix}, \ r = \sqrt{\|\mathbf{u}\|^2 - f} = 6 \tag{3.1.13.2}$$

See Fig. 3.1.13.1.

3.1.14 $x^2 + y^2 - 4x - 8y - 45 = 0$

Solution: The given circle can be expressed as

$$\|\mathbf{x}\|^2 + 2\begin{pmatrix} -2 & -4 \end{pmatrix}\mathbf{x} - 45 = 0 \tag{3.1.14.1}$$

where

$$\mathbf{u} = \begin{pmatrix} -2 \\ -4 \end{pmatrix}, \ f = -45 \tag{3.1.14.2}$$

$$\implies \mathbf{c} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \ r = \sqrt{65}. \tag{3.1.14.3}$$

See Fig. 3.1.14.1.

3.1.15 $x^2 + y^2 - 8x + 10y - 12 = 0$

Solution: From the given informtion,

$$\mathbf{u} = \begin{pmatrix} -4 \\ 5 \end{pmatrix}, \ f = -12 \tag{3.1.15.1}$$

$$\implies \mathbf{c} = \begin{pmatrix} 4 \\ -5 \end{pmatrix}, \tag{3.1.15.2}$$

$$r = \sqrt{\|\mathbf{u}\|^2 - f} = \sqrt{53} \tag{3.1.15.3}$$

See Fig. 3.1.15.1.

3.1.16 $2x^2 + 2y^2 - x = 0$

Solution: The given equation can be expressed as

$$x^2 + y^2 - \frac{x}{2} = 0 \tag{3.1.16.1}$$

$$\implies \|\mathbf{x}\|^2 + 2\begin{pmatrix} \frac{-1}{4} & 0 \end{pmatrix}\mathbf{x} = 0 \tag{3.1.16.2}$$

The centre of circle is then given by

$$\mathbf{u} = -\mathbf{c} = \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix} \tag{3.1.16.3}$$

and the radius of circle is obtained as

$$r = \sqrt{\|\mathbf{u}\|^2 - f} = \frac{1}{4} \tag{3.1.16.4}$$

See Fig. 3.1.16.1.

3.2. Exercises

3.2.1 The area of the circle centred ot (1,2) and passing through (4,6) is

(a) 5μ

(b) 10μ

(c) 25μ

(d) none of these

3.2.2 equation of the circle with centre on the Y-axis and passing through the orgin and the point (2,3) is

(a) $x^2 + y^2 + 6x + 6y + 3 = 0$

(b) $x^2 + y^2 - 6x - 6y - 9 = 0$

(c) $x^2 + y^2 - 6x - 6y + 9 = 0$

(d) none of these

3.2.3 equation of the circle with centre on the y-axis and passing through the origin and the point (2,3) is

(a) $x^2 + y^2 + 13y = 0$

(b) $3x^2 + 3y^2 + 13x + 3 = 0$

(c) $6x^2 + 6y^2 - 13x = 0$

(d) $x^2 + y^2 + 13x + 3 = 0$

3.2.4 The equation of a circle with origin as centre and passing through the vertices of an equilateral triangle whose median is of lengh 3 a is

(a) $x^2 + y^2 = 9a^2$

(b) $x^2 + y^2 = 16a^2$

(c) $x^2 + y^2 = 4a^2$

(d) $x^2 + y^2 = a^2$ [Hint: centroid of the triangle caincdes with the centre of the circle and the redius of the circle is $\frac{2}{3}$of the length of the median]

In each of the following exercises, find the equation of the circle with the following parameters

3.2.5 Find the equation of a circle concentric with the circle $x^2 + y^2 - 6x + 12y + 15 = 0$ and has double of its area.

3.2.6 If one end of a diameter of the circle $x^2 + y^2 - 4x - 6y + 11 = 0$ is (3,4), then find the coordinate of the other end of the diameter.

3.2.7 Find the equation of the circle having (1,-2) as its centre and passing through $3x + y = 14, 2x + 5y = 18$.

3.2.8 Show that the point (x, y) given by $x = \frac{2at}{1+t^2}$ and $y = \frac{a(1-t^2)}{1+t^2}$ lies on a circle for all real values of t such that -1≤ t ≤1 where a is any given real numbers.

3.2.9 lf a circle passes through the point $(0, 0), (a, 0) and (0, b)$ then find the coordinates of its centre.

3.2.10 If the lines 2x-3y=5 and 3x-4y=7 are the diameters of a circie of area 154 squar untits, then obtian the equation of the circle.

3.2.11 Find the equation of the circle which passes through the points (2,3) and (4,5) and the centre lies on the straight line y-4x+3=0.

3.2.12 Find the equation of a circle whose centre is (3,1) and which cuts offachord of length 6 units on the line 2x-5y+18=0 [Hint:To detemine the radius of the circle, find the perpendicular distance from the centre to the given line]

3.2.13 Find the equation of a circle of redius 5 which is touching another circle $x^2 + y^2 - 2x - 4y - 20 = 0$ at (5,5).

3.2.14 Find the equation of a circle passing through the point (7,3) having radius 3 units and whose centre lies on the line y=x-1

State whether the statements are True or False

3.2.15 The line $x^2 + 3y = 0$ is a diameter of the circle $x^2 + y^2 + 6x + 2y = 0$.

3.2.16 The point (1,2) lies inside the circle $x^2 + y^2 - 2x + 6y + 1 = 0$,

Fill in the Blanks

3.2.17 The equation of the circle having centre at (3,-4) and touching the line 5x+12y-12=0 is ______ [Hint: To determine radius find the perpendicular distance from the centre of the circle to the line.]

3.2.18 The equation of the circle circumscribing the triangle whose sides are the lines y=x+2,3y=4x,2y=3x is ______

3.3. Construction

3.3.1 Two circles of radii 5cm and 3cm intersect at two points and the distance between their center is 4cm. Find the length of the common chord.

Solution: See Fig. 3.3.1.1. and

Parameter	Value	Description
$\mathbf{c}_1$	$\mathbf{0}$	Center of Circle 1
$\mathbf{c}_2$	$4\mathbf{e}_1$	Center of Circle 2
r_1	5	Radius of Circle 1
r_2	3	Radius of Circle 2

Table 3.3.1.2:

From Table 3.3.1.2, (E.2.1.1) and (E.2.2.1), the equations of the two circles are

$$\begin{aligned} \|\mathbf{x}\|^2 - 25 &= 0 \\ \|\mathbf{x}\|^2 - 8\mathbf{e}_1^\top \mathbf{x} + 7 &= 0 \end{aligned} \tag{3.3.1.1}$$

From (3.3.1.1) and (E.2.4.1) the equation of the common chord is

$$\mathbf{e}_1^\top \mathbf{x} = 4 \tag{3.3.1.2}$$

It is easy to verify that

$$\mathbf{q} = 4\mathbf{e}_1 \tag{3.3.1.3}$$

is a point on (3.3.1.2). Substituting

$$\mathbf{m} = \mathbf{e}_2, \mathbf{q} = 4\mathbf{e}_1, \mathbf{V} = \mathbf{I}, \mathbf{u} = \mathbf{0}, f = -25 \tag{3.3.1.4}$$

in (G.3.3.1), the length of the chord in (G.3.1.1) is given by

$$\frac{2\sqrt{\left[\mathbf{e}_2^\top\left(4\mathbf{e}_1\right)\right]^2 - \left(16\mathbf{e}_1^\top\mathbf{e}_1 - 25\right)\left(\mathbf{e}_2^\top\mathbf{e}_2\right)}}{\mathbf{e}_2^\top\mathbf{e}_2}\left\|\mathbf{e}_2\right\| = 6 \tag{3.3.1.5}$$

3.3.2 If two equal chords of a circle intersect within the circle, prove that the segments of one chord are equal to corresponding segments of the other chord.

Solution: See Table 3.3.2.1 for the input parameters. Consider

Symbol	**Value**	**Description**
$\mathbf{C}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	Circle point
d	1.5	Length of Chord
r	1	Radius
θ_1	$30°$	–
θ_3	$60°$	–
θ_2	$-67.1806°$	–
θ_4	$37.1806°$	–

Table 3.3.2.1:

$$\mathbf{P} = \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} \cos\theta_2 \\ \sin\theta_2 \end{pmatrix}, \mathbf{R} = \begin{pmatrix} \cos\theta_3 \\ \sin\theta_3 \end{pmatrix}, \mathbf{S} = \begin{pmatrix} \cos\theta_4 \\ \sin\theta_4 \end{pmatrix} \tag{3.3.2.1}$$

such that

$$\mathbf{P}-\mathbf{Q}=\begin{pmatrix}\cos\theta_1-\cos\theta_2\\ \sin\theta_1-\sin\theta_2\end{pmatrix} \tag{3.3.2.2}$$

$$\implies \left\|\mathbf{P}-\mathbf{Q}\right\|^2=\left(\cos\theta_1-\cos\theta_2\right)^2+\left(\sin\theta_1-\sin\theta_2\right)^2=d^2 \tag{3.3.2.3}$$

$$\tag{3.3.2.4}$$

yielding

$$\left(\frac{\theta_1-\theta_2}{2}\right)=\sin^{-1}\left(\frac{d}{2}\right). \tag{3.3.2.5}$$

Similarly,

$$\left(\frac{\theta_3-\theta_4}{2}\right)=\sin^{-1}\left(\frac{d}{2}\right). \tag{3.3.2.6}$$

The equations of PQ and RS are obtained using

$$\mathbf{n_1}^\top\left(\mathbf{x}-\mathbf{P}\right)=0 \tag{3.3.2.7}$$

$$\mathbf{n_2}^\top\left(\mathbf{x}-\mathbf{R}\right)=0 \tag{3.3.2.8}$$

where

$$\mathbf{n}_1 = \begin{pmatrix} \sin\theta_1 - \sin\theta_2 \\ \cos\theta_2 - \cos\theta_1 \end{pmatrix} \tag{3.3.2.9}$$

$$\mathbf{n}_2 = \begin{pmatrix} \sin\theta_3 - \sin\theta_4 \\ \cos\theta_4 - \cos\theta_3 \end{pmatrix} \tag{3.3.2.10}$$

Substiuting numerical values, the point of intersection of lines PQ, RS is

$$\mathbf{T} = \begin{pmatrix} 0.68341409 \\ -0.04288508 \end{pmatrix} \tag{3.3.2.11}$$

Thus,

$$\|\mathbf{P} - \mathbf{T}\| = \|\mathbf{S} - \mathbf{T}\| = 0.5727 \tag{3.3.2.12}$$

3.3.3 If a line intersects two concentric circles (circles with the same centre) with centre $\mathbf{O}$ at $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$, prove that $AB = CD$.

Solution: See Table 3.3.3.1. Let the equations of two concentric circles be,

$$\|\mathbf{x}\|^2 = 4 \tag{3.3.3.1}$$

$$\|\mathbf{x}\|^2 = 9 \tag{3.3.3.2}$$

Consider the line

$$\mathbf{x} = \mathbf{h} + \mu \mathbf{m} \tag{3.3.3.3}$$

$$\tag{3.3.3.4}$$

with parameters

$$\mathbf{h} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.3.3.5}$$

Then,

$$\mu^2 \mathbf{m}^\top \mathbf{V} \mathbf{m} + 2\mu \mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u}) + \mathrm{g}(\mathbf{h}) = 0 \tag{3.3.3.6}$$

with

$$\mathrm{g}(\mathbf{x}) = \mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{3.3.3.7}$$

Substituting

$$\mathbf{V} = \mathbf{I},\ \mathbf{u} = \mathbf{O},\ f = -4 \tag{3.3.3.8}$$

the points of intersection of circle (3.3.3.1) and the line (3.3.3.3) with parameters given by (3.3.3.5) $\mathbf{B}, \mathbf{C}$ are given by,

$$\mu^2 - 3 = 0 \implies \mu = \pm\sqrt{3} \tag{3.3.3.9}$$

yielding

$$\mathbf{B} = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} \tag{3.3.3.10}$$

Substituting

$$\mathbf{V} = \mathbf{I},\ \mathbf{u} = \mathbf{O},\ f = -9 \tag{3.3.3.11}$$

$$\tag{3.3.3.12}$$

The points of intersection of circle (3.3.3.2) and the line (3.3.3.3) with parameters given by (3.3.3.5) $\mathbf{A}, \mathbf{D}$ are given by,

$$\mu^2 - 8 = 0 \implies \mu = \pm 2\sqrt{2} \tag{3.3.3.13}$$

yielding

$$\mathbf{A} = \begin{pmatrix} 1 \\ 2\sqrt{2} \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 1 \\ -2\sqrt{2} \end{pmatrix} \tag{3.3.3.14}$$

Thus,

$$\|\mathbf{A} - \mathbf{B}\| = \|\mathbf{C} - \mathbf{D}\| = 2\sqrt{2} - \sqrt{3} \tag{3.3.3.15}$$

$$\implies AB = CD. \tag{3.3.3.16}$$

See Fig. 3.3.3.1

3.3.4 If a line intersects two concentric circles (circles with the same centre)

Parameter	**Description**	**Value**
$\mathbf{O}$	center of both circles	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
r_1	radius of smaller circle	2
r_2	radius of larger circle	3
$\mathbf{h}$	point on the line	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$
$\mathbf{m}$	direction vector of the line	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Table 3.3.3.1:

with centre $\mathbf{O}$ at $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$, prove that $AB = CD$ (see Fig. 3.3.4.1).

3.3.5 Three girls Reshma, Salma and Mandip are playing a game by standing on a circle of radius 5m drawn in a park. Reshma throws a ball to Salma, Salma to Mandip, Mandip to Reshma. If the distance between Reshma and Salma and between Salma and Mandip is 6m each, what is the distance between Reshma and Mandip?

Solution: Consider Reshma, Salma and Mandip be standing at $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ respectively, and the center the of the circle $\mathbf{O}$. The input parameters are listed in Table 3.3.5.1. Let

$$\mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{O} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \tag{3.3.5.1}$$

Therefore, the equation of the cicle is given by

$$\|\mathbf{x}-\mathbf{O}\|^2 = 25 \tag{3.3.5.2}$$

$$\implies \|\mathbf{x}\|^2 - 2\mathbf{O}^\top \mathbf{x} + \|\mathbf{O}\|^2 - 25 = 0 \tag{3.3.5.3}$$

$$\implies \|\mathbf{x}\|^2 - 2\begin{pmatrix} 5 & 0 \end{pmatrix}\mathbf{x} = 0 \tag{3.3.5.4}$$

Also, **A** and **C** are equidistant (6m) from **B**, we can say that they lie on the circle having **B** as center and radius 6m. Equation of this circle is given by

$$\|\mathbf{x}\|^2 - 2\mathbf{B}^\top \mathbf{x} + \|\mathbf{B}\|^2 - 36 = 0 \tag{3.3.5.5}$$

$$\|\mathbf{x}\|^2 = 36 \tag{3.3.5.6}$$

$$i.e., \mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ f = -36 \tag{3.3.5.7}$$

From (3.3.5.4) and (3.3.5.7), the line passing through **A** and **C** is

$$\begin{pmatrix} 5 & 0 \end{pmatrix}\mathbf{x} = 18 \tag{3.3.5.8}$$

$$\implies \mathbf{x} = \begin{pmatrix} \frac{18}{5} \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.3.5.9}$$

$$i.e., \ \mathbf{h} = \begin{pmatrix} \frac{18}{5} \\ 0 \end{pmatrix}, \ \mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.3.5.10}$$

For the circle,$\mathbf{V} = \mathbf{I}$

$$\mu_i = \frac{1}{\mathbf{m}^\top \mathbf{V} \mathbf{m}} \left(-m^\top \left(\mathbf{V}\mathbf{h} + \mathbf{u} \right) \pm \sqrt{\left(\mathbf{m}^\top \left(\mathbf{V}\mathbf{h} + \mathbf{u} \right) \right)^2 - \mathrm{g}\left(\mathbf{h} \right) \left(\mathbf{m}^\top \mathbf{V} \mathbf{m} \right)} \right) \tag{3.3.5.11}$$

where,

$$\mathrm{g}\left(\mathbf{h} \right) = \mathbf{h}^\top \mathbf{V} \mathbf{h} + 2\mathbf{u}^\top \mathbf{h} + f \tag{3.3.5.12}$$

yielding

$$\mu_i = \pm \frac{24}{5} \tag{3.3.5.13}$$

Therefore,

$$\mathbf{A} = \begin{pmatrix} \frac{18}{5} \\ \frac{24}{5} \end{pmatrix}, \mathbf{C} = \begin{pmatrix} \frac{18}{5} \\ -\frac{24}{5} \end{pmatrix} \tag{3.3.5.14}$$

and the distance between Reshma and Mandip is

$$\left\| \mathbf{A} - \mathbf{C} \right\| = \left\| \begin{pmatrix} 0 \\ \frac{48}{5} \end{pmatrix} \right\| = \frac{48}{5} \tag{3.3.5.15}$$

See Fig. 3.3.5.1.

3.3.6 A circular park of radius 20m is situated in a colony. Three boys Ankur, Syed and David are sitting at equal distance on its boundary each having a toy telephone in his hands to talk each other. Find the length

$\mathbf{O}$	$\begin{pmatrix} 5 \\ 0 \end{pmatrix}$	Center of the given circle
$\mathbf{B}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	Point where Salma is standing
r	5	radius of given circle
d	6	distance AB and BC

Table 3.3.5.1:

of the string of each phone.

3.3.7 If two equal chords of a circle intersect within the circle, prove that the line joining the point of intersection to the centre makes equal angles with the chords.

Solution: Let the position vectors of the boys be

$$\mathbf{A} = \begin{pmatrix} r \\ 0 \end{pmatrix}, \ \mathbf{S} = \begin{pmatrix} r\cos\beta \\ r\sin\beta \end{pmatrix}, \ \mathbf{D} = \begin{pmatrix} r\cos\gamma \\ r\sin\gamma \end{pmatrix} \tag{3.3.7.1}$$

where

$$\beta, \gamma \in (0, 2\pi) \tag{3.3.7.2}$$

We have,

$$\|\mathbf{A} - \mathbf{S}\|^2 = \|\mathbf{A} - \mathbf{D}\|^2 \tag{3.3.7.3}$$

$$\implies \mathbf{A}^\top \mathbf{S} = \mathbf{A}^\top \mathbf{D} \tag{3.3.7.4}$$

$$\implies \cos\beta = \cos\gamma \tag{3.3.7.5}$$

$$\implies \beta = 2n\pi \pm \gamma \tag{3.3.7.6}$$

where $n \in \mathbb{Z}$. From (3.3.7.2), we force $n = 1$ and consider the negative sign to get

$$\beta + \gamma = 2\pi \tag{3.3.7.7}$$

Therefore, using (3.3.7.7)

$$\|\mathbf{A} - \mathbf{S}\|^2 = \|\mathbf{S} - \mathbf{D}\|^2 \tag{3.3.7.8}$$

$$\implies \mathbf{A}^\top \mathbf{S} = \mathbf{D}^\top \mathbf{S} \tag{3.3.7.9}$$

$$\implies \cos\beta = \cos(\beta - \gamma) \tag{3.3.7.10}$$

$$\implies 2\beta - 2\pi = 2m\pi \pm \beta \tag{3.3.7.11}$$

$$\implies 2\beta \pm \beta = 2k\pi \tag{3.3.7.12}$$

where $k \in \mathbb{Z}$. From (3.3.7.2), we can only consider the plus sign and $k \in \{1, 2\}$ to get

$$\beta, \gamma \in \left\{\frac{2\pi}{3}, \frac{4\pi}{3}\right\} \tag{3.3.7.13}$$

Therefore, the length of the thread from (3.3.7.13) is

$$\|\mathbf{S} - \mathbf{D}\| = \left\| r \begin{pmatrix} \cos\beta - \cos\gamma \\ \sin\beta - \sin\gamma \end{pmatrix} \right\| \tag{3.3.7.14}$$

$$= r\sqrt{3} \tag{3.3.7.15}$$

Here, $r = 20$ m. Thus, the length is $20\sqrt{3}$ m.

The situation is demonstrated in Fig. 3.3.7.1, plotted by the Python code `codes/equilateral.py`. The values used for construction are shown in Table 3.3.7.1.

Parameter	**Value**
r	20
β	$\frac{2\pi}{3}$
γ	$\frac{4\pi}{3}$
$\mathbf{O}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Table 3.3.7.1: Parameters used in the construction of Fig. 3.3.7.1.

3.3.8 $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are the three points on a circle with centre $\mathbf{O}$ such that $\angle$BOC=30°and $\angle$AOB=60°. If $\mathbf{D}$ is a point on the circle other than the arc ABC, find $\angle$ADC.

Solution: The input parameters are available in Table 3.3.8 yielding

Symbol	**Values**	**Description**
r	1 unit	Radius of OA and OB
$\mathbf{O}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	Center of the circle
$\mathbf{C}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	Standard basis vector $\mathbf{e}_1$
α	30°	$\angle$BOC
β	60°	$\angle$AOB
γ	??	$\angle$ADC

$$\mathbf{C} = \mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} \cos(\alpha+\beta) \\ \sin(\alpha+\beta) \end{pmatrix}, \mathbf{D} = \begin{pmatrix} \cos\gamma \\ \sin\gamma \end{pmatrix}. \tag{3.3.8.1}$$

Since

$$\mathbf{A}-\mathbf{D} = \begin{pmatrix} \cos(\alpha+\beta)-\cos\gamma \\ \sin(\alpha+\beta)-\sin\gamma \end{pmatrix}, \mathbf{C}-\mathbf{D} = \begin{pmatrix} 1-\cos\gamma \\ -\sin\gamma \end{pmatrix}, \left\|\mathbf{A}-\mathbf{D}\right\| \left\|\mathbf{C}-\right. \tag{3.3.8.2}$$

$$\cos(\angle ADC) = \frac{(\mathbf{A}-\mathbf{D})^\top(\mathbf{C}-\mathbf{D})}{\left\|\mathbf{A}-\mathbf{D}\right\|\left\|\mathbf{C}-\mathbf{D}\right\|}, \tag{3.3.8.3}$$

$$= 4\sin\frac{\alpha+\gamma}{2}\sin\frac{\beta+\gamma}{2}\cos\frac{\alpha+\beta}{2} = \cos\frac{\alpha+\beta}{2} \tag{3.3.8.4}$$

Substituting α and β in (3.3.8.4)

$$\angle ADC = \frac{\alpha+\beta}{2} = \frac{(30^\circ+60^\circ)}{2} = 45^\circ \tag{3.3.8.5}$$

See Fig. 3.3.8.1.

3.3.9 A chord of a circle is equal to the radius of the circle. Find the angle subtended by the chord at a point on the minor arc and also at a point on the major arc.

Solution:

The input parameters are listed in Table 3.3.9.1. Take three points Q,R and P on a unit circle at angles $\theta, \alpha,$ and β. Then

$$\mathbf{Q} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \mathbf{R} = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}, \mathbf{S} = \begin{pmatrix} \cos\beta \\ \sin\beta \end{pmatrix} \tag{3.3.9.1}$$

$$\cos\angle QRP = \frac{(\mathbf{Q}-\mathbf{R})^\top(\mathbf{P}-\mathbf{R})}{\left\|\mathbf{Q}-\mathbf{R}\right\|\left\|\mathbf{P}-\mathbf{R}\right\|} \tag{3.3.9.2}$$

Symbol	**Value**	**Description**
r	1	Radius
$\mathbf{O}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	circle point
$\mathbf{P}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	standard basis vector$\mathbf{e_1}$
θ	$60°$	$\angle QOP$
α	$130°$	$\angle QRP$
β	$-40°$	$\angle QSP$

Table 3.3.9.1:

where

$$(\mathbf{Q} - \mathbf{R})^\top (\mathbf{P} - \mathbf{R}) = (\cos\alpha - \cos\theta)\cos\alpha + (\sin\theta - \sin\alpha) \tag{3.3.9.3}$$

and

$$\|\mathbf{Q} - \mathbf{R}\|^2 \|\mathbf{P} - \mathbf{R}\|^2 = (2 - 2\cos\theta\cos\alpha - 2\sin\theta\sin\alpha)(2 - \cos\alpha) \tag{3.3.9.4}$$

Substituing (3.3.9.3) and (3.3.9.4) in (3.3.9.2),

$$\cos\angle QRP \implies \angle QRP = 62° \tag{3.3.9.5}$$

Similarly,

$$(\mathbf{Q} - \mathbf{S})^\top (\mathbf{P} - \mathbf{S}) = (\cos\beta - \cos\theta)\cos\beta + (\sin\theta - \sin\beta) \tag{3.3.9.6}$$

and

$$\|\mathbf{Q}-\mathbf{S}\|^2 \|\mathbf{P}-\mathbf{S}\|^2 = (2-2\cos\theta\cos\beta-2\sin\theta\sin\beta)(2-\cos\beta) \tag{3.3.9.7}$$

Substituing (3.3.9.6) and (3.3.9.7) in (3.3.9.7),

$$\cos\angle QSP = \frac{1.048}{1.098} \tag{3.3.9.8}$$

$$\implies \angle QSP = 17^\circ \tag{3.3.9.9}$$

See Fig. 3.3.9.1.

3.3.10 $\angle PQR = 100^\circ$, where $\mathbf{P}, \mathbf{Q}$ and $\mathbf{R}$ are points on a circle with centre $\mathbf{O}$. Find $\angle OPR$

Solution:

3.3.11 If diagonals of a cyclic quadrilateral are diameters of the circle through the vertices of quadrilateral,prove that it is a rectangle.

Solution: The input parameters for construction are available in Table 3.3.11. From the given information,

Symbol	**Values**	**Description**
r	2	radius
$\mathbf{O}$	$\begin{pmatrix}0\\0\end{pmatrix}$	center

$$\mathbf{A} = r\begin{pmatrix}\cos 0\\ \sin 0\end{pmatrix} = \begin{pmatrix}2\\ 0\end{pmatrix} \tag{3.3.11.1}$$

$$\mathbf{B} = r\begin{pmatrix}\cos\frac{\pi}{3}\\ \sin\frac{\pi}{3}\end{pmatrix} = \begin{pmatrix}1\\ \sqrt{3}\end{pmatrix} \tag{3.3.11.2}$$

$$\mathbf{C} = 2\mathbf{O} - \mathbf{A} = \begin{pmatrix}-2\\ 0\end{pmatrix} \tag{3.3.11.3}$$

$$\mathbf{D} = 2\mathbf{O} - \mathbf{B} = \begin{pmatrix}-1\\ -\sqrt{3}\end{pmatrix} \tag{3.3.11.4}$$

Consider a circle of radius 2 units. Let AC and DB be diameters of circle which are diagonals of cyclic quadrilateral. Then, from the above equations,

$$\mathbf{A} - \mathbf{C} = \mathbf{D} - \mathbf{B} \tag{3.3.11.5}$$

(a) AB and DC are parellel to each other

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} \tag{3.3.11.6}$$

$$= \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} \tag{3.3.11.7}$$

$$\mathbf{D} - \mathbf{C} = \begin{pmatrix} -1 \\ -\sqrt{3} \end{pmatrix} - \begin{pmatrix} -2 \\ 0 \end{pmatrix} \tag{3.3.11.8}$$

$$= \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} \tag{3.3.11.9}$$

Thus, $ABCD$ is parallelogram.

(b) Let's check the angle between adjacent sides of this quadrilateral, AB and BC

$$(\mathbf{A} - \mathbf{B})^\top (\mathbf{B} - \mathbf{C}) = \begin{pmatrix} 1 & -\sqrt{3} \end{pmatrix} \begin{pmatrix} 3 \\ \sqrt{3} \end{pmatrix} \tag{3.3.11.10}$$

$$= 0 \tag{3.3.11.11}$$

$$\implies \angle ABC = 90^\circ \tag{3.3.11.12}$$

from 3.3.11a and 3.3.11b , Hence the quadrilateral $ABCD$ is rectangle.

See Fig. 3.3.11.1.

3.3.12 If circles are drawn taking two sides of a triangle as diameters, prove that the point of intersection of these circles lie on the third side.

Solution: The input parameters are available in Table 3.3.12.1. The

Parameter	**Description**	**Value**
$\mathbf{A}$	vertex of the triangle	$\begin{pmatrix}0\\4\end{pmatrix}$
$\mathbf{B}$	vertex of the triangle	$\begin{pmatrix}0\\-4\end{pmatrix}$
$\mathbf{C}$	vertex of the triangle	$\begin{pmatrix}6\\6\end{pmatrix}$

Table 3.3.12.1:

equation of circle taking AB as diameter is given by,

$$\|\mathbf{x}\|^2 + 2\mathbf{u_1}^\top \mathbf{x} + f_1 = 0 \tag{3.3.12.1}$$

$$\implies \|\mathbf{x}\|^2 - 16 = 0 \tag{3.3.12.2}$$

where

$$\mathbf{u_1} = -\left(\frac{\mathbf{A}+\mathbf{B}}{2}\right) = \begin{pmatrix}0\\0\end{pmatrix} \tag{3.3.12.3}$$

$$r_1 = \frac{\|\mathbf{A}-\mathbf{B}\|}{2} = 4 \tag{3.3.12.4}$$

$$f_1 = \|\mathbf{u_1}\|^2 - r_1^2 = -4 \tag{3.3.12.5}$$

The equation of circle taking AC as diameter is given by,

$$\|\mathbf{x}\|^2 + 2\mathbf{u_2}^\top \mathbf{x} + f_2 = 0 \tag{3.3.12.6}$$

$$\implies \|\mathbf{x}\|^2 - 2\begin{pmatrix}3 & 5\end{pmatrix}\mathbf{x} + 24 = 0 \tag{3.3.12.7}$$

where

$$\mathbf{u_2} = -\left(\frac{\mathbf{A}+\mathbf{C}}{2}\right) = -\begin{pmatrix} 3 \\ 5 \end{pmatrix} \tag{3.3.12.8}$$

$$r_2 = \frac{\|\mathbf{A}-\mathbf{C}\|}{2} = \sqrt{10} \tag{3.3.12.9}$$

$$f_2 = \|\mathbf{u_2}\|^2 - r_2^2 = 24 \tag{3.3.12.10}$$

Let the intersection of circles (3.3.12.2) and (3.3.12.7) be $\mathbf{P}$. The equation of the common chord of intersection of two circles, AP is given by,

$$2\mathbf{u_1}^\top \mathbf{x} - 2\mathbf{u_2}^\top \mathbf{x} + f_1 - f_2 = 0 \tag{3.3.12.11}$$

$$\implies 2\begin{pmatrix} 3 & 5 \end{pmatrix}\mathbf{x} - 16 - 24 = 0 \tag{3.3.12.12}$$

$$\begin{pmatrix} 3 & 5 \end{pmatrix}\mathbf{x} = 20 \tag{3.3.12.13}$$

(3.3.12.13) can be written in parametric form as,

$$\mathbf{x} = \mathbf{h} + \mu\mathbf{m},\ \text{where } \mathbf{h} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} -5 \\ 3 \end{pmatrix} \tag{3.3.12.14}$$

and μ is given by

$$\mu^2\mathbf{m}^\top\mathbf{V}\mathbf{m} + 2\mu\mathbf{m}^\top\left(\mathbf{V}\mathbf{h} + \mathbf{u}\right) + \mathrm{g}\left(\mathbf{h}\right) = 0 \tag{3.3.12.15}$$

with

$$
\mathrm{g}\left(\mathbf{h}\right) = \mathbf{h}^{\top}\mathbf{V}\mathbf{h} + 2\mathbf{u}^{\top}\mathbf{h} + f \tag{3.3.12.16}
$$

Substituting

$$
\mathbf{V} = \mathbf{I},\ \mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},\ f = 16\mathbf{h} = \begin{pmatrix} 0 \\ 4 \end{pmatrix},\ \mathbf{m} = \begin{pmatrix} -5 \\ 3 \end{pmatrix} \tag{3.3.12.17}
$$

$$
34\mu^2 + 24\mu = 0 \implies \mu = 0, -\frac{12}{17} \tag{3.3.12.18}
$$

where $\mu = 0$ corresponds to point $\mathbf{A}$. Thus,

$$
\mathbf{P} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} - \frac{12}{17}\begin{pmatrix} -5 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{60}{17} \\ \frac{32}{17} \end{pmatrix} \tag{3.3.12.19}
$$

The direction vector of BC is given by,

$$
\mathbf{m} = \mathbf{C} - \mathbf{B} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \implies \mathbf{n} = \begin{pmatrix} -5 \\ 3 \end{pmatrix} \tag{3.3.12.20}
$$

yielding the equation

$$\mathbf{n}^\top \mathbf{x} = \mathbf{n}^\top \mathbf{B} \tag{3.3.12.21}$$

$$\implies \begin{pmatrix} -5 & 3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -5 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ -4 \end{pmatrix} = -12 \tag{3.3.12.22}$$

It is clear that $\mathbf{P}$ satisfies the equation of BC in (3.3.12.22). Hence, the point of intersection of the circles drawn by taking two sides of a triangle as diameters lies on the third side. See Fig. 3.3.12.1.

3.3.13 Prove that a cyclic paralellogram is a rectangle.

Solution: Consider the points $\mathbf{P_i},\ 1 \le i \le 4$ in anticlockwise order on the unit circle. Thus, for $1 \le i \le 4$,

$$\mathbf{P_i} = \begin{pmatrix} \cos\theta_i \\ \sin\theta_i \end{pmatrix} \tag{3.3.13.1}$$

where

$$\theta_i \in [0, 2\pi),\ i \neq j \iff \theta_i \neq \theta_j \tag{3.3.13.2}$$

Without loss of generality, suppose that P_1P_2 and P_3P_4 are parallel to the x-axis. Since

$$\mathbf{P_1} - \mathbf{P_2} = \begin{pmatrix} \cos\theta_1 - \cos\theta_2 \\ \sin\theta_1 - \sin\theta_2 \end{pmatrix} \tag{3.3.13.3}$$

$$\mathbf{P_3} - \mathbf{P_4} = \begin{pmatrix} \cos\theta_4 - \cos\theta_4 \\ \sin\theta_3 - \sin\theta_4 \end{pmatrix} \tag{3.3.13.4}$$

we have

$$\sin\theta_1 = \sin\theta_2 \tag{3.3.13.5}$$

$$\implies \theta_1 = n\pi + (-1)^n \theta_2 \tag{3.3.13.6}$$

However, (3.3.13.2) forces $n \in \{1, 3\}$, thus

$$\theta_1 + \theta_2 \in \{\pi, 3\pi\} \tag{3.3.13.7}$$

Similarly,

$$\theta_3 + \theta_4 \in \{\pi, 3\pi\} \tag{3.3.13.8}$$

Since $P_1P_2P_3P_4$ is a parallelogram, its diagonals bisect each other.

Thus, using (3.3.13.7) and (3.3.13.8),

$$\frac{\mathbf{P_1} + \mathbf{P_3}}{2} = \frac{\mathbf{P_2} + \mathbf{P_4}}{2} \tag{3.3.13.9}$$

$$\implies \mathbf{P_1} + \mathbf{P_3} = \mathbf{P_2} + \mathbf{P_4} \tag{3.3.13.10}$$

$$\implies \begin{pmatrix} \cos\theta_1 + \cos\theta_3 \\ \sin\theta_1 + \sin\theta_3 \end{pmatrix} = \begin{pmatrix} \cos\theta_2 + \cos\theta_4 \\ \sin\theta_2 + \sin\theta_4 \end{pmatrix} \tag{3.3.13.11}$$

$$\implies \cos\theta_1 + \cos\theta_3 = \cos\theta_2 + \cos\theta_4 \tag{3.3.13.12}$$

$$= -\left(\cos\theta_1 + \cos\theta_3\right) \tag{3.3.13.13}$$

$$\implies \cos\theta_1 + \cos\theta_3 = \cos\theta_2 + \cos\theta_4 = 0 \tag{3.3.13.14}$$

Using (3.3.13.14), (3.3.13.7) and (3.3.13.8), we have

$$\cos\theta_1 = -\cos\theta_3 = \cos\theta_4 \tag{3.3.13.15}$$

$$\cos\theta_2 = -\cos\theta_4 = \cos\theta_3 \tag{3.3.13.16}$$

Thus,

$$\mathbf{P_1} - \mathbf{P_4} = \begin{pmatrix} \cos\theta_1 - \cos\theta_4 \\ \sin\theta_1 - \sin\theta_4 \end{pmatrix} \tag{3.3.13.17}$$

$$= \begin{pmatrix} 0 \\ \sin\theta_1 - \sin\theta_4 \end{pmatrix} \tag{3.3.13.18}$$

Thus, from (3.3.13.18),

$$\left(\mathbf{P_1} - \mathbf{P_2}\right)^\top \left(\mathbf{P_1} - \mathbf{P_4}\right)$$
$$= \begin{pmatrix} \cos\theta_1 - \cos\theta_2 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \sin\theta_1 - \sin\theta_4 \end{pmatrix} = 0 \qquad (3.3.13.19)$$

From (3.3.13.19), we see that $P_1P_2 \perp P_1P_4$. Hence, $P_1P_2P_3P_4$ is a rectangle.

The situation is demonstrated in Fig. 3.3.13.1, plotted by the Python code `codes/circle.py`. The various input parameters are shown in Table 3.3.13.1.

Parameter	**Value**
r	1
θ_1	$\frac{\pi}{6}$
θ_2	$\frac{5\pi}{6}$
θ_3	$\frac{7\pi}{6}$
θ_4	$\frac{11\pi}{6}$

Table 3.3.13.1: Parameters used in the construction of Fig. 3.3.13.1.

3.3.14 Prove that the line of centres of two intersecting circles subtends equal angles at the two points of intersection.

Solution: The input parameters are available in Table 3.3.14.

The two circle equations are given by

$$\|x\|^2 - 9 = 0 \qquad (3.3.14.1)$$

$$\|x\|^2 - 8\mathbf{e}_1 + 12 = 0 \qquad (3.3.14.2)$$

Symbol	Values	Description
$\mathbf{A}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	Center of circle 1
r_1	3 units	Radius of the circle 1
$\mathbf{B}$	$\begin{pmatrix} 4 \\ 0 \end{pmatrix}$	Center of circle 2
r_2	2 units	Radius of circle 2
$\mathbf{e}_1$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	Standard basis vector 1
$\mathbf{e}_2$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	Standard basis vector 2

yielding the intersection of the circles as the line

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = \frac{21}{8} \tag{3.3.14.3}$$

$$\tag{3.3.14.4}$$

(3.3.14.4) can be expressed as

$$\mathbf{x} = \mathbf{q} + \lambda \mathbf{m} \tag{3.3.14.5}$$

The distance form origin to point $\mathbf{x}$ is given by

$$\|\mathbf{x}\|^2 = d^2 \tag{3.3.14.6}$$

Then substituting (3.3.14.5) in (3.3.14.6) yeilds,

$$\left(\mathbf{q}+\lambda\mathbf{m}\right)^{\top}\left(\mathbf{q}+\lambda\mathbf{m}\right)=d^2 \tag{3.3.14.7}$$

$$\implies \lambda^2\left\|\mathbf{m}\right\|^2+2\lambda\mathbf{q}^{\top}\mathbf{m}+\left\|\mathbf{q}\right\|^2=d^2 \tag{3.3.14.8}$$

where

$$\mathbf{q}=\begin{pmatrix}\frac{21}{8}\\0\end{pmatrix},\mathbf{m}=\begin{pmatrix}0\\1\end{pmatrix} \text{ and } d=r_1=3 \tag{3.3.14.9}$$

Substituting the values in (3.3.14.9) in (3.3.14.8),

$$\lambda^2(1)+2\lambda\begin{pmatrix}\frac{21}{8} & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}+\frac{441}{64}=9 \tag{3.3.14.10}$$

$$\implies \lambda_i=\pm\frac{3\sqrt{5}}{8} \tag{3.3.14.11}$$

Thus, the intersecting points **C** and **D** are given by

$$\mathbf{C}=\mathbf{q}+\lambda_1\mathbf{m}=\begin{pmatrix}\frac{21}{8}\\-\frac{3\sqrt{5}}{8}\end{pmatrix} \tag{3.3.14.12}$$

$$\mathbf{D}=\mathbf{q}+\lambda_2\mathbf{m}=\begin{pmatrix}\frac{21}{8}\\\frac{3\sqrt{5}}{8}\end{pmatrix} \tag{3.3.14.13}$$

(a) Finding ∠ADB

$$\mathbf{A}-\mathbf{D} = \begin{pmatrix} -\frac{21}{8} \\ -\frac{3\sqrt{5}}{8} \end{pmatrix}, \mathbf{B}-\mathbf{D} = \begin{pmatrix} \frac{11}{8} \\ -\frac{3\sqrt{5}}{8} \end{pmatrix} \tag{3.3.14.14}$$

$$(\mathbf{A}-\mathbf{D})^\top(\mathbf{B}-\mathbf{D}) = -\frac{3}{2} \tag{3.3.14.15}$$

$$\|\mathbf{A}-\mathbf{D}\| \, \|\mathbf{C}-\mathbf{D}\| = 6 \tag{3.3.14.16}$$

$$\implies \cos(\angle ADB) = \frac{(\mathbf{A}-\mathbf{D})^\top(\mathbf{B}-\mathbf{D})}{\|\mathbf{A}-\mathbf{D}\| \, \|\mathbf{B}-\mathbf{D}\|} \tag{3.3.14.17}$$

$$\text{or, } \angle ADB = 104^\circ \tag{3.3.14.18}$$

(b) Finding ∠ACB

$$\mathbf{A}-\mathbf{C} = \begin{pmatrix} -\frac{21}{8} \\ \frac{3\sqrt{5}}{8} \end{pmatrix}, \mathbf{B}-\mathbf{C} = \begin{pmatrix} \frac{11}{8} \\ \frac{3\sqrt{5}}{8} \end{pmatrix} \tag{3.3.14.19}$$

$$(\mathbf{A}-\mathbf{C})^\top(\mathbf{B}-\mathbf{C}) = -\frac{3}{2} \tag{3.3.14.20}$$

$$\|\mathbf{A}-\mathbf{C}\| \, \|\mathbf{B}-\mathbf{C}\| = 6 \tag{3.3.14.21}$$

$$\implies \cos(\angle ACB) = \frac{(\mathbf{A}-\mathbf{C})^\top(\mathbf{B}-\mathbf{C})}{\|\mathbf{A}-\mathbf{C}\| \, \|\mathbf{B}-\mathbf{C}\|} \tag{3.3.14.22}$$

$$\text{or, } \angle ACB = 104^\circ = \angle ADB \tag{3.3.14.23}$$

See Fig. 3.3.14.1.

3.3.15 AC and BD are chords of a circle which bisect each other. Prove that

(a) AC and BD are diameters,

(b) $ABCD$ is a rectangle.

Solution: Consider a unit circle with center at origin. Let AC and BD be the diameters of the circle. The points on circle that we consider are available in Table (3.3.15.1).

A	$\begin{pmatrix} \cos 0 \\ \sin 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$
B	$\begin{pmatrix} \cos \frac{\pi}{2} \\ \sin \frac{\pi}{2} \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
C	$\begin{pmatrix} \cos \pi \\ \sin \pi \end{pmatrix}$	$\begin{pmatrix} -1 \\ 0 \end{pmatrix}$
D	$\begin{pmatrix} \cos \frac{3}{2}\pi \\ \sin \frac{3}{2}\pi \end{pmatrix}$	$\begin{pmatrix} 0 \\ -1 \end{pmatrix}$

Table 3.3.15.1:

(a) AC and BD are diameters of the circle. Let's check if they bisect each other,

$$\mathbf{A} + \mathbf{C} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \tag{3.3.15.1}$$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.3.15.2}$$

$$\mathbf{B} + \mathbf{D} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \tag{3.3.15.3}$$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.3.15.4}$$

From equation (3.3.15.2) and (3.3.15.4) AC and BD bisect each other. Hence, we can say that if two chords bisect each other then they are diameters.

(b) Let's check if $ABCD$ is a rectangle. The sides of a rectangle are parallel to each other. Let's check if AB and BC are parallel to each other.

$$\mathbf{A}-\mathbf{B}=\begin{pmatrix}1\\0\end{pmatrix}-\begin{pmatrix}0\\1\end{pmatrix} \tag{3.3.15.5}$$

$$=\begin{pmatrix}1\\-1\end{pmatrix} \tag{3.3.15.6}$$

$$\mathbf{D}-\mathbf{C}=\begin{pmatrix}0\\-1\end{pmatrix}-\begin{pmatrix}-1\\0\end{pmatrix} \tag{3.3.15.7}$$

$$=\begin{pmatrix}1\\-1\end{pmatrix} \tag{3.3.15.8}$$

From equation (3.3.15.6) and (3.3.15.8), AB and DC are parallel to each other. $\implies$ $ABCD$ is a parallelogram.

Now let's check if its a rectangle. Let's check the angle between adjacent sides of this quadrilateral, i.e. AB and BC.

$$(\mathbf{A}-\mathbf{B})^\top(\mathbf{B}-\mathbf{C})=\begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} \tag{3.3.15.9}$$

$$=0 \tag{3.3.15.10}$$

From equation (3.3.15.9), we can say that the angle between AB

and BC is $90°$. Hence, the quadrilateral $ABCD$ is a rectangle.

3.3.16 Bisectors of angles A, B and C of a triangle ABC intersect its circumcircle at D, E and F respectively. Prove that the angles of triangle DEF are $90° - \frac{A}{2}$, $90° - \frac{B}{2}$ and $90° - \frac{C}{2}$.

Solution: The input parameters are listed in Table 3.3.16.1. Let the

Parameter	Value	Desription
a	5	length of side opposite to Vertex $\mathbf{A}$
c	5	length of side opposite to vertex $\mathbf{C}$
θ	$60°$	$\angle ABC$

Table 3.3.16.1: Table

vertices of the triangle ABC be

$$\mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} a \\ 0 \end{pmatrix}, \mathbf{A} = c \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \tag{3.3.16.1}$$

(a) Circumcenter: From (E.2.9.1), the circumcentre is given by

$$\begin{pmatrix} a & 0 \\ c\cos\theta & c\sin\theta \end{pmatrix} \mathbf{O} = \begin{pmatrix} \frac{a^2}{2} \\ \frac{c^2}{2} \end{pmatrix} \tag{3.3.16.2}$$

and the circumradius is

$$R = \|\mathbf{A} - \mathbf{O}\| \tag{3.3.16.3}$$

(b) From (D.1.13.1), the equation of the angle bisector at $\mathbf{A}$ is given by

$$\mathbf{x} = \mathbf{A} + \mu\mathbf{m} \tag{3.3.16.4}$$

where

$$\mathbf{m} = \mathbf{B} + \mathbf{C} - 2\mathbf{A} \tag{3.3.16.5}$$

Thus, $\mathbf{D}$ is obtained from (G.3.1.2) using the parameters of the angle bisector at $\mathbf{A}$. Similarly, $\mathbf{E}, \mathbf{F}$ are found using above method and are available in Table 3.3.16.2.

$\mathbf{D}$	$\begin{pmatrix} 2.5 \\ -1.44 \end{pmatrix}$
$\mathbf{E}$	$\begin{pmatrix} 5 \\ 2.886 \end{pmatrix}$
$\mathbf{F}$	$\begin{pmatrix} 0 \\ 2.886 \end{pmatrix}$

Table 3.3.16.2: Table

(c) From (A.1.7.1),

$$\cos\left(\angle DEF\right) = \frac{\left(\mathbf{D} - \mathbf{E}\right)^{\top}\left(\mathbf{F} - \mathbf{E}\right)}{\left\|\mathbf{D} - \mathbf{E}\right\|\left\|\mathbf{F} - \mathbf{E}\right\|} = 60^{\circ} \tag{3.3.16.6}$$

3.3.17 Two congruent circles intersect each other at points $\mathbf{A}$ and $\mathbf{B}$. Through $\mathbf{A}$ any line segment $\mathbf{PAQ}$ is drawn so that $\mathbf{P}$, $\mathbf{Q}$ lie on the two circles. Prove that $BP = BQ$.

Solution: The parameters used in the construction are shown in Table

3.3.17.1. Let the equations of these circles be given by

$$\left\|\mathbf{x}\right\|^2 + 2\mathbf{u_1}^\top \mathbf{x} + f = 0 \tag{3.3.17.1}$$

$$\left\|\mathbf{x}\right\|^2 + 2\mathbf{u_2}^\top \mathbf{x} + f = 0 \tag{3.3.17.2}$$

$$\mathbf{u_1} = -\begin{pmatrix} 2 \\ 0 \end{pmatrix}, \mathbf{u_2} = -\begin{pmatrix} -2 \\ 0 \end{pmatrix} \tag{3.3.17.3}$$

$$f = -4. \tag{3.3.17.4}$$

The common chord of the circles is given by

$$2\mathbf{u_1}^\top \mathbf{x} - 2\mathbf{u_2}^\top \mathbf{x} + f_1 - f_2 = 0 \tag{3.3.17.5}$$

$$\implies \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 0 \tag{3.3.17.6}$$

(3.3.17.6) can be written in parametric form as

$$\mathbf{h} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.3.17.7}$$

$$\mathbf{x} = \mathbf{h} + \mu \mathbf{m} \tag{3.3.17.8}$$

The parameter μ for the points of intersection of the above line with the given conic is given by the equation

$$\mu^2 \mathbf{m}^\top \mathbf{V} \mathbf{m} + 2\mu \mathbf{m}^\top \left(\mathbf{V}\mathbf{h} + \mathbf{u}\right) + \mathrm{g}\left(\mathbf{h}\right) = 0 \tag{3.3.17.9}$$

Substituting numerical values,

$$\mathbf{V} = \mathbf{I}, \, \mathbf{u} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}, \, f = -4, \, \mathbf{h} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \, \mathbf{m} \quad = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.3.17.10}$$

$$\tag{3.3.17.11}$$

we obtain

$$\mu^2 - 4 = 0 \implies \mu = \pm 2 \tag{3.3.17.12}$$

From (3.3.17.8),

$$\mathbf{A} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \, \mathbf{B} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} \tag{3.3.17.13}$$

Since

$$\mathbf{m} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{3.3.17.14}$$

The equation of the line passing through the point $\mathbf{A}$ with the direction vector (3.3.17.14) in parametric form is given by,

$$\mathbf{x} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \alpha \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{3.3.17.15}$$

Substituing

$$\mathbf{V} = \mathbf{I},\ \mathbf{u} = \begin{pmatrix} -2 \\ 0 \end{pmatrix},\ f = -4\mathbf{h} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \tag{3.3.17.16}$$

the intersection of the line in 3.3.17.15 with one circle is given by

$$5\alpha^2 - 4\alpha = 0 \implies \alpha = 0, \frac{4}{5} \tag{3.3.17.17}$$

Thus,

$$\mathbf{P} = \begin{pmatrix} \frac{8}{5} \\ \frac{14}{5} \end{pmatrix} \tag{3.3.17.18}$$

Similarly, the intersection with the second circle is given by

$$5\beta^2 + 12\beta = 0, \implies \beta = 0, -\frac{12}{5} \tag{3.3.17.19}$$

$$\text{and } \mathbf{Q} = \begin{pmatrix} -\frac{24}{5} \\ -\frac{2}{5} \end{pmatrix} \tag{3.3.17.20}$$

Thus,

$$\|\mathbf{B} - \mathbf{P}\| = \|\mathbf{B} - \mathbf{Q}\| = \frac{8\sqrt{10}}{5} \tag{3.3.17.21}$$

$$\implies BP = BQ. \tag{3.3.17.22}$$

See Fig. 3.3.17.1.

Parameter	Description	Value
$\mathbf{C}_1$	center of circle 1	$\begin{pmatrix}-2\\0\end{pmatrix}$
r_1	radius of circle 1	$2\sqrt{2}$
$\mathbf{C}_2$	center of circle 2	$\begin{pmatrix}2\\0\end{pmatrix}$
r_2	radius of circle 2	$2\sqrt{2}$
$\mathbf{m}$	direction vector of the line passing through $\mathbf{A}$	$\begin{pmatrix}2\\1\end{pmatrix}$

Table 3.3.17.1:

3.3.18 In any $\triangle ABC$, if the angle bisector of $\angle A$ and perpendicular bisector of BC intersect, prove that they intersect on the circumcircle of $\triangle ABC$.

Solution: Let the position vectors of the points be

$$\mathbf{A} = \begin{pmatrix}1\\0\end{pmatrix}, \ \mathbf{B} = \begin{pmatrix}\cos\beta\\\sin\beta\end{pmatrix}, \ \mathbf{C} = \begin{pmatrix}\cos\gamma\\\sin\gamma\end{pmatrix} \tag{3.3.18.1}$$

Then, the equation of the perpendicular bisector of BC is given by

$$\left\|\mathbf{x}-\mathbf{B}\right\|^2 = \left\|\mathbf{x}-\mathbf{C}\right\|^2 \tag{3.3.18.2}$$

$$\implies (\mathbf{B}-\mathbf{C})^\top \mathbf{x} = 0 \tag{3.3.18.3}$$

and the equation of the angle bisector of $\angle A$ is given by

$$\frac{(\mathbf{B}-\mathbf{A})^\top(\mathbf{x}-\mathbf{A})}{\|\mathbf{B}-\mathbf{A}\|} = \frac{(\mathbf{C}-\mathbf{A})^\top(\mathbf{x}-\mathbf{A})}{\|\mathbf{C}-\mathbf{A}\|} \tag{3.3.18.4}$$

$$\left(\frac{\mathbf{B}-\mathbf{A}}{\|\mathbf{B}-\mathbf{A}\|} - \frac{\mathbf{C}-\mathbf{A}}{\|\mathbf{C}-\mathbf{A}\|}\right)^\top (\mathbf{x}-\mathbf{A}) = 0 \tag{3.3.18.5}$$

Note that from (3.3.18.1),

$$\frac{\mathbf{B}-\mathbf{A}}{\|\mathbf{B}-\mathbf{A}\|} = \frac{1}{\sqrt{2(1-\cos\beta)}}\begin{pmatrix}\cos\beta-1\\ \sin\beta\end{pmatrix} \tag{3.3.18.6}$$

$$= \begin{pmatrix}-\sin\frac{\beta}{2}\\ \cos\frac{\beta}{2}\end{pmatrix} \tag{3.3.18.7}$$

Therefore, using (3.3.18.1) in (3.3.18.3) and (3.3.18.7) in (3.3.18.5),

$$\begin{pmatrix}\cos\beta-\cos\gamma\\ \sin\beta-\sin\gamma\end{pmatrix}^\top \mathbf{x} = 0 \tag{3.3.18.8}$$

$$\begin{pmatrix}\sin\frac{\gamma}{2}-\sin\frac{\beta}{2}\\ \cos\frac{\beta}{2}-\cos\frac{\gamma}{2}\end{pmatrix}^\top \mathbf{x} = \sin\frac{\gamma}{2}-\sin\frac{\beta}{2} \tag{3.3.18.9}$$

Using (3.3.18.8) and (3.3.18.9), we form the matrix equation

$$\begin{pmatrix}\cos\beta-\cos\gamma & \sin\beta-\sin\gamma\\ \sin\frac{\gamma}{2}-\sin\frac{\beta}{2} & \cos\frac{\beta}{2}-\cos\frac{\gamma}{2}\end{pmatrix}\mathbf{x}$$

$$= \begin{pmatrix}0\\ \sin\frac{\gamma}{2}-\sin\frac{\beta}{2}\end{pmatrix} \tag{3.3.18.10}$$

Solving (3.3.18.10) using the augmented matrix, and letting $\theta \triangleq \frac{\beta+\gamma}{2}$,

$$\begin{pmatrix} \cos\beta - \cos\gamma & \sin\beta - \sin\gamma & 0 \\ \sin\frac{\gamma}{2} - \sin\frac{\beta}{2} & \cos\frac{\beta}{2} - \cos\frac{\gamma}{2} & \sin\frac{\gamma}{2} - \sin\frac{\beta}{2} \end{pmatrix} \tag{3.3.18.11}$$

$$\xleftrightarrow[]{\substack{R_1 \leftarrow \frac{R_1}{\cos\beta - \cos\gamma} \\ R_2 \leftarrow \frac{R_2}{\sin\frac{\gamma}{2} - \sin\frac{\beta}{2}}}} \begin{pmatrix} 1 & -\cot\theta & 0 \\ 1 & \tan\frac{\theta}{2} & 1 \end{pmatrix} \tag{3.3.18.12}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & -\cot\theta & 0 \\ 0 & \tan\frac{\theta}{2} + \cot\theta & 1 \end{pmatrix} \tag{3.3.18.13}$$

$$= \begin{pmatrix} 1 & -\cot\theta & 0 \\ 0 & \csc\theta & 1 \end{pmatrix} \tag{3.3.18.14}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + R_2\cos\theta} \begin{pmatrix} 1 & 0 & \cos\theta \\ 0 & \csc\theta & 1 \end{pmatrix} \tag{3.3.18.15}$$

$$\xleftrightarrow{R_2 \leftarrow R_2\sin\theta} \begin{pmatrix} 1 & 0 & \cos\theta \\ 0 & 1 & \sin\theta \end{pmatrix} \tag{3.3.18.16}$$

Thus, the intersection of the lines in (3.3.18.8) and (3.3.18.9) is

$$\mathbf{D} \triangleq \begin{pmatrix} \cos\frac{\beta+\gamma}{2} \\ \sin\frac{\beta+\gamma}{2} \end{pmatrix} \tag{3.3.18.17}$$

Hence, it is clear from (3.3.18.17) that $\mathbf{D}$ lies on the circumcircle of $\triangle ABC$, as required.

The situation is illustrated in Fig. 3.3.18.1. The parameters used in the construction are shown in Table 3.3.18.1.

Parameter	Value
r	1
β	$100°$
γ	$200°$

Table 3.3.18.1: Parameters used in the construction of Fig. 3.3.18.1.

3.4. Exercises

3.4.1 AD is a diameter of a circle and AB is a chord. If $AD = 34cm$, $AB = 30cm$, the distance of AB from the centre of the circle is:

(a) 17cm

(b) 15cm

(c) 4cm

(d) 8cm

3.4.2 In Fig. 3.4.2.1, if $OA = 5cm, AB = 8cm$ and OD is perpendicular to AB, then CD is equal to:

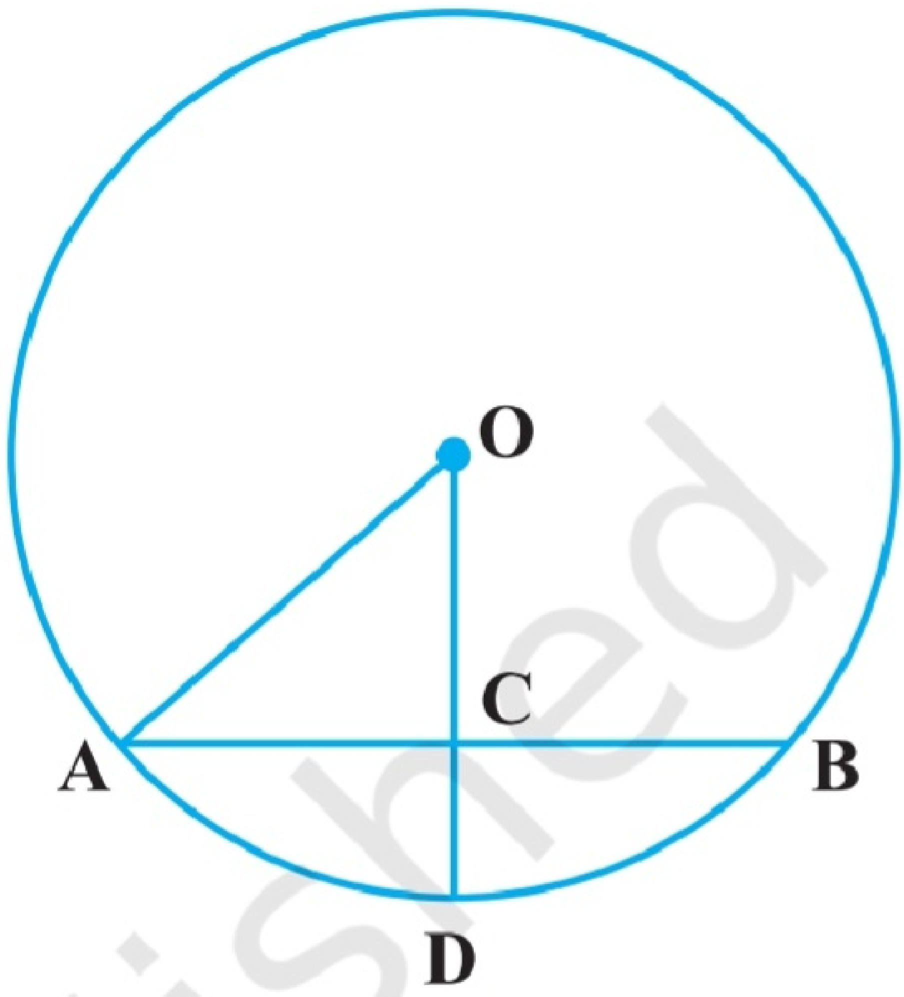


Figure 3.4.2.1:

(a) 2cm

(b) 3cm

(c) 4cm

(d) 5cm

3.4.3 If $AB = 12cm, BC = 16cm$ and AB is perpendicular to BC, then the radius of the circle passing through the points $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ is:

(a) 6cm

(b) 8cm

(c) 10cm

(d) 12cm

3.4.4 In Fig. 3.4.4.1, if $\angle ABC = 20°$, then $\angle AOC$ is equal to:

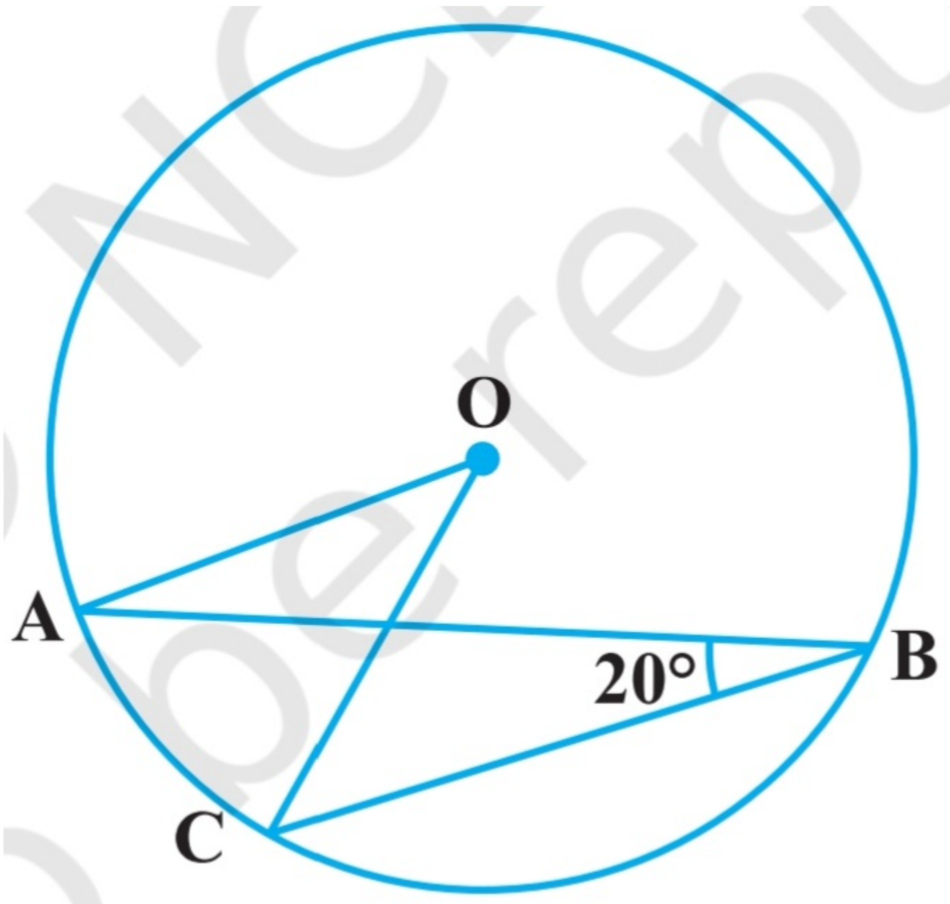


Figure 3.4.4.1:

(a) 20°

(b) 40°

(c) 60°

(d) 10°

3.4.5 In Fig. 3.4.5.1, if AOB is a diameter of the circle and $AC = BC$,then $\angle CAB$ is equal to:

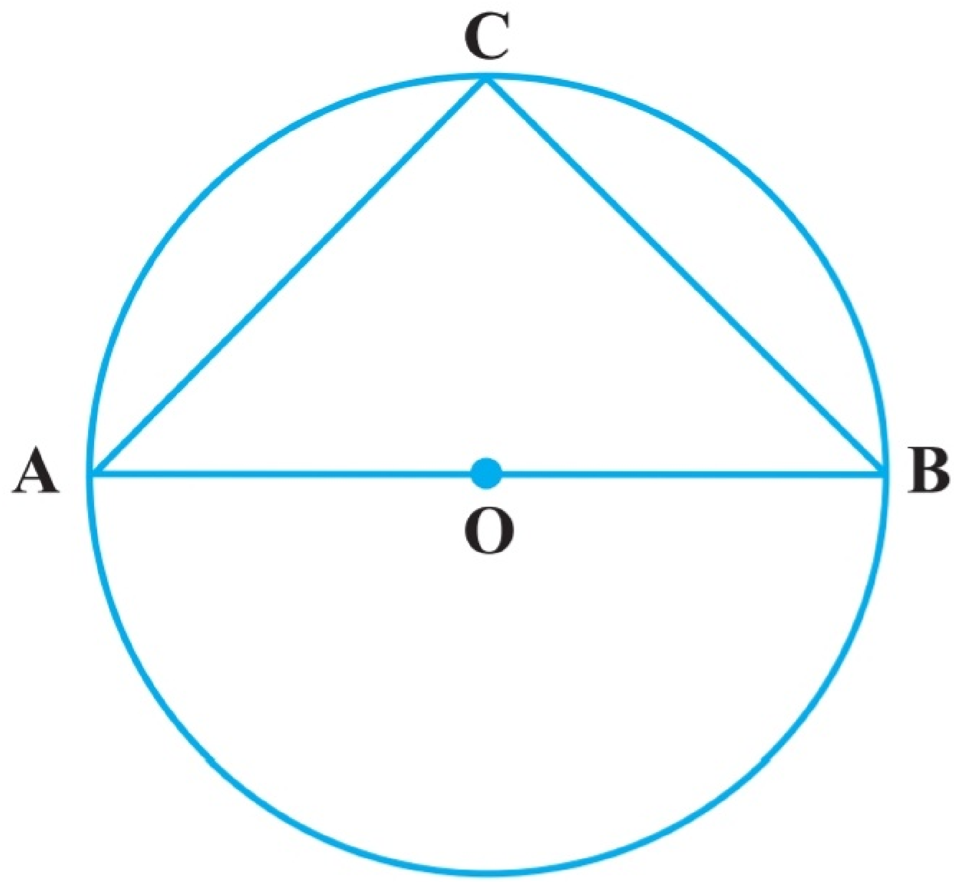


Figure 3.4.5.1:

(a) 30°

(b) 60°

(c) 90°

(d) 45°

3.4.6 In Fig. 3.4.6.1, if $\angle OAB = 40^\circ$, then $\angle ACB$ is equal to:

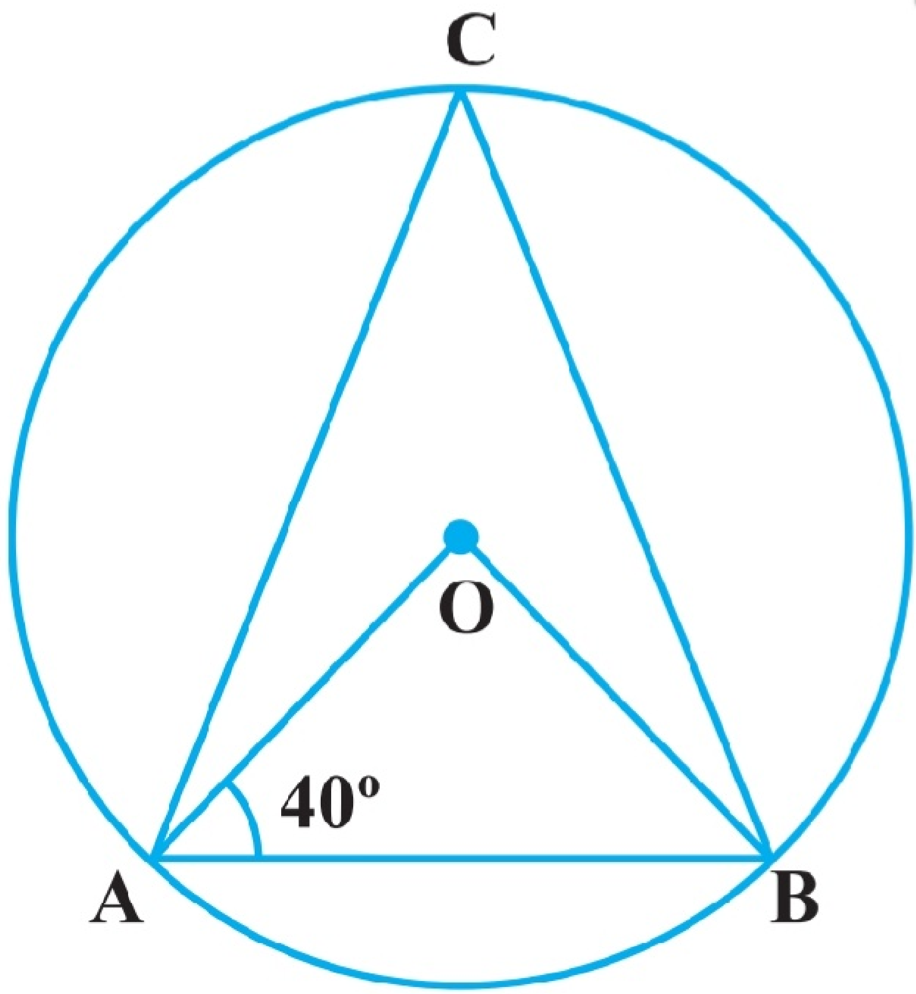


Figure 3.4.6.1:

(a) 50°

(b) 40°

(c) 60°

(d) 70°

3.4.7 In Fig. 3.4.7.1, if $\angle DAB = 60°, \angle ABD = 50°$, then $\angle ACB$ is equal to:

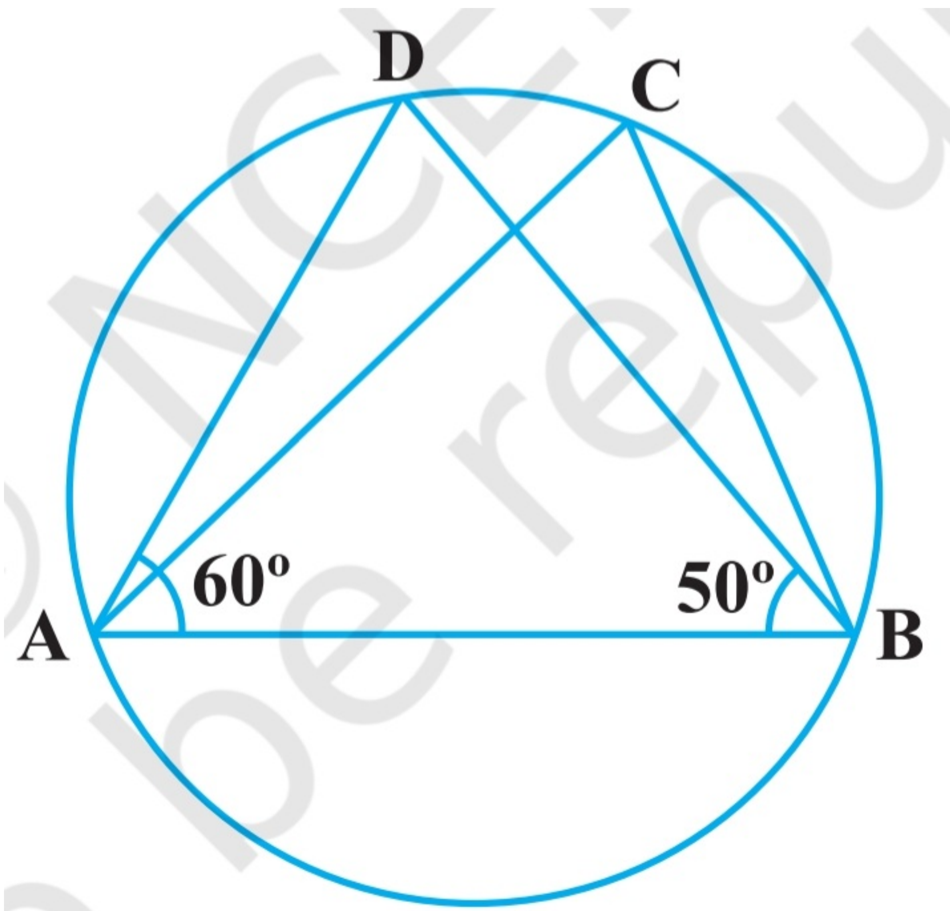


Figure 3.4.7.1:

(a) $60°$

(b) $50°$

(c) $70°$

(d) $80°$

3.4.8 $ABCD$ is a cyclic quadrilateral such that AB is a diameter of the circle circumscribing it and $\angle ADC = 140°$, then $\angle BAC$ is equal to:

(a) $80°$

(b) $50°$

(c) $40°$

(d) $30°$

3.4.9 In Fig. 3.4.9.1, BC is a diameter of the circle and $\angle BAO = 60°$. Then $\angle ADC$ is equal to:

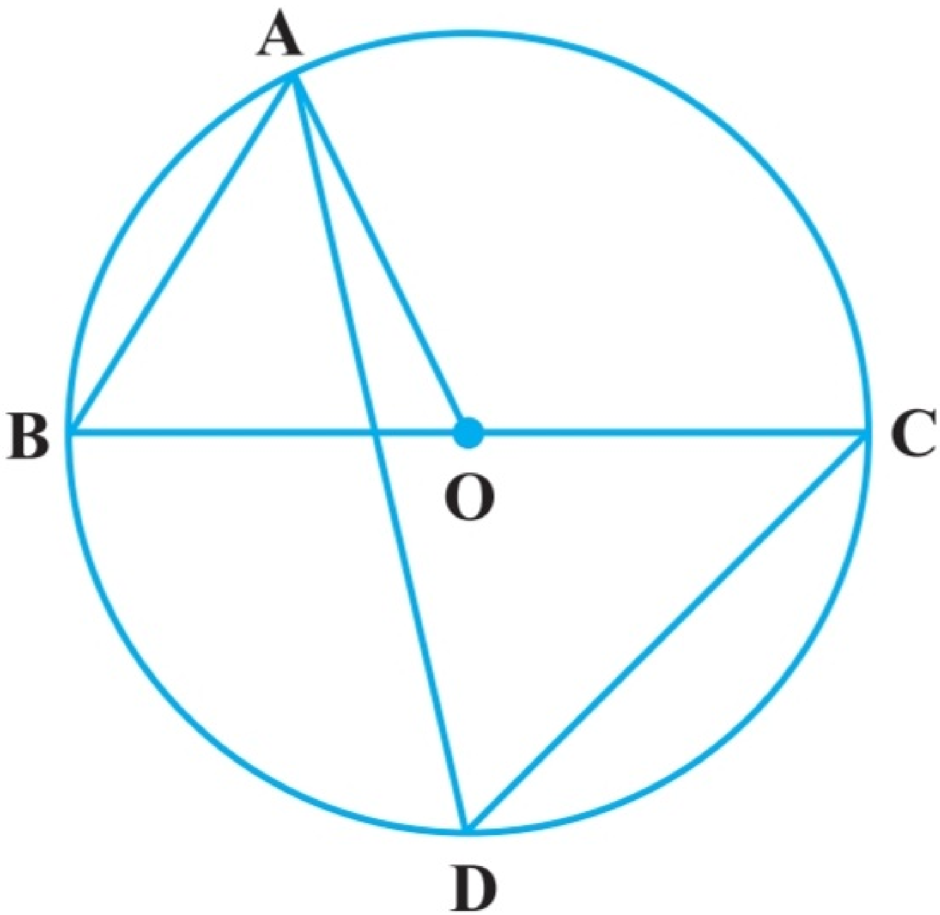


Figure 3.4.9.1:

(a) $30°$

(b) $45°$

(c) $60°$

(d) $120°$

3.4.10 In Fig. 3.4.10.1, $\angle AOB = 90°$ and $\angle ABC = 30°$, then $\angle CAO$ is equal to:

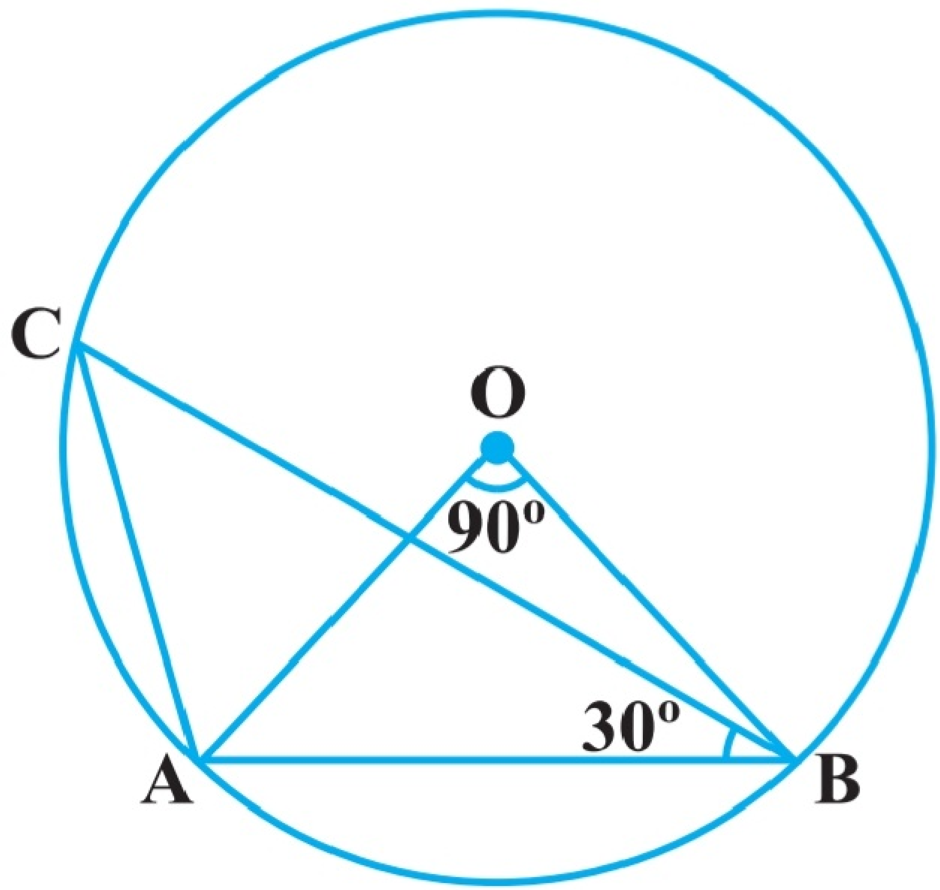


Figure 3.4.10.1:

(a) 30°

(b) 45°

(c) 90°

(d) 60°

3.4.11 Two chords AB and CD of a circle are each at distances $4cm$ from the centre The $AB = CD$.

3.4.12 Two chords AB and AC of a circle with centre O are an the opposite sides of OA Then $\angle 0AB = \angle 0AC$

3.4.13 Two congruent circles with centres 0 and 0 intersect at two points A and B Then $\angle AOB = \angle AOB$

3.4.14 Through three collinear points a circle can be drawn

3.4.15 A circle of radius $3cm$ can be drawn through two points A, B such that $AB = 6cm$

3.4.16 If AOB is a diameter of a circle and C is a point on the circle, then $AC^2 + B^2 = AB^2$.

3.4.17 $ABCD$ is a cyclic quadrilateral such that $\angle A = 90°, \angle B = 70°, \angle C = 95°$ and $\angle D = 105°$

3.4.18 If A,B,C,D are four points such that $\angle BAC = 30°, \angle BDC = 60°$, then D is the centre of the circle through A,B and C.

3.4.19 If A,B,C and D are four points such that $\angle BAC = 45°$ and $\angle BDC = 45°$, then A,B,C,D are concyclic.

3.4.20 In fig 3.4.20.1 if AOB is a diameter and $\angle ADC = 120°$ then $\angle CAB = 30°$

3.4.21 If two equal chords of a circle intersect prove that the parts of one chord are separately equal to the parts of the other chord.

3.4.22 If non-parallel sides of a trapezium are equal. Prove that it is cyclic

3.4.23 If $\mathbf{P}, \mathbf{Q}$ and $\mathbf{R}$ are the mid-points of the sides BC, CA and AB of a triangle and AD is the perpendicular from $\mathbf{A}$ on BC. Prove that $\mathbf{P}, \mathbf{Q}, \mathbf{R}$ and $\mathbf{D}$ are concyclic.

3.4.24 $ABCD$ is a parallelogram. A circle through $\mathbf{A}$, $\mathbf{B}$ is so drawn that it intersects AD at $\mathbf{P}$ and BC at $\mathbf{Q}$. Prove that $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{D}$ are concyclic.

3.4.25 Prove that angle bisector of any angle of a triangle and perpendicular bisector of the opposite side if intersect, they will intersent on the circumcircle of the triangle.

3.4.26 If two chords AB and CD of a circle AYDZBWCX intersect at right angles see Fig.3.4.26.1. Prove that

$$arc\,(CXA) + arc\,(DZB) = arc\,(AYD) + arc\,(AYD) + arc\,(BWC) \tag{3.4.26.1}$$

$$= semi - circle \tag{3.4.26.2}$$

3.4.27 If ABC is an equilateral triangle inscribed in a circle and $\mathbf{P}$ be any point on the minor arc BC which does not coincide with $\mathbf{B}$ or $\mathbf{C}$. Prove that PA is angle bisector of $\angle BPC$.

3.4.28 In Fig.3.4.28.1, AB and CD are two chords of a circle intersecting each other at point **E**. Prove that

$$\angle AEC = \frac{1}{2}(\text{Angle subtended by arc CXA at centre} \tag{3.4.28.1}$$

$$+\text{angle subtended by arc DYB at the centre}). \tag{3.4.28.2}$$

3.4.29 If bisectors of opposite angles of a cyclic quadrilateral $ABCD$ intersect the circle, circumscribing it at the points **P** and **Q**. Prove that PQ is a diameter of the circle.

3.4.30 A circle has radius $\sqrt{442}$ cm it is divided into two segments by a chord of length 2cm. Prove that the angle subtended by the chord at a point in major segment is 45°.

3.4.31 Two equal chords AB and CD of a circle when produced intersect at a point **P**. Prove that $PB = PD$

3.4.32 AB and AC are two chords of a circle of radius r such that $AB = 2AC$. If **P** and **Q** are the distances of AB and AC from the centre. Prove that $4q^2 = p^2 + 3r^2$.

3.4.33 In Fig.3.4.33.1, **O** is the centre of the circle, $\angle BCO = 30°$. Find x and y.

3.4.34 In Fig.3.4.34.1, **O** is the centre of the circle, $BD = OD$ and $CD \perp AB$. Find $\angle CAB$.

3.5. Parabola

3.5.1 An arch is in the form of a parabola with its axis vertical. The arch is 10m high and 5m wide at the base. How wide is it 2m from the vertex of the parabola?

Solution:

3.5.2 The cable of a uniformly loaded suspension bridge hangs in the form of a parabola. The roadway which is horizontal and 100 m long is supported by vertical wires attached to the cable, the longest wire being 30 m and the shortest being 6 m. Find the length of a supporting wire attached to the roadway 18 m from the middle.

Solution: Uniformly loaded suspension bridge cable hangs in the form of a parabola facing upwards. The parameters are then listed in Table 3.5.2.1. As conic is upward facing parabola,

O	Lowest point of cable	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
d	Length of the cable	100 m
d_1	Length of longest wire	30 m
d_2	Length of shortest wire	6 m
A	End point of cable	$\begin{pmatrix} \frac{d}{2} \\ d_1 - d_2 \end{pmatrix}$
B	End point of cable	$\begin{pmatrix} -\frac{d}{2} \\ d_1 - d_2 \end{pmatrix}$

Table 3.5.2.1: points

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.5.2.1}$$

Points $\mathbf{O}, \mathbf{A}$, and $\mathbf{B}$ are on conic, so we have

$$\mathbf{O}^\top \mathbf{V} \mathbf{O} + 2\mathbf{u}^\top \mathbf{O} + f = 0 \tag{3.5.2.2}$$

$$\mathbf{A}^\top \mathbf{V} \mathbf{A} + 2\mathbf{u}^\top \mathbf{A} + f = 0 \tag{3.5.2.3}$$

$$\mathbf{B}^\top \mathbf{V} \mathbf{B} + 2\mathbf{u}^\top \mathbf{B} + f = 0 \tag{3.5.2.4}$$

Rewrite the equations as

$$2\mathbf{O}^\top \mathbf{u} + f = -\mathbf{O}^\top \mathbf{V} \mathbf{O} \tag{3.5.2.5}$$

$$2\mathbf{A}^\top \mathbf{u} + f = -\mathbf{A}^\top \mathbf{V} \mathbf{A} \tag{3.5.2.6}$$

$$2\mathbf{B}^\top \mathbf{u} + f = -\mathbf{B}^\top \mathbf{V} \mathbf{B} \tag{3.5.2.7}$$

This can be formulated as the matrix equation

$$\begin{pmatrix} 2\mathbf{O}^\top & 1 \\ 2\mathbf{A}^\top & 1 \\ 2\mathbf{B}^\top & 1 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ f \end{pmatrix} = - \begin{pmatrix} \mathbf{O}^\top \mathbf{V} \mathbf{O} \\ \mathbf{A}^\top \mathbf{V} \mathbf{A} \\ \mathbf{B}^\top \mathbf{V} \mathbf{B} \end{pmatrix} \tag{3.5.2.8}$$

Substituting numerical values in the above equation,

$$\begin{pmatrix} 0 & 0 & 1 \\ 100 & 48 & 1 \\ -100 & 48 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ f \end{pmatrix} = - \begin{pmatrix} 0 \\ -2500 \\ -2500 \end{pmatrix} \tag{3.5.2.9}$$

$$\implies f = 0 \text{ and } \mathbf{u} = \begin{pmatrix} 0 \\ -\frac{625}{12} \end{pmatrix} \tag{3.5.2.10}$$

So, the equation of the parabola is

$$\mathbf{x}^\top \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} 0 & -\frac{625}{12} \end{pmatrix} \mathbf{x} = 0 \tag{3.5.2.11}$$

At a point λ_1 m from middle,

$$\mathbf{x} = \lambda_1 \mathbf{e}_1 + \lambda_2 \mathbf{e}_2 \tag{3.5.2.12}$$

Substituting this in the parabola equation,

$$\begin{aligned} &(\lambda_1 \mathbf{e}_1 + \lambda_2 \mathbf{e}_2)^\top \mathbf{V} (\lambda_1 \mathbf{e}_1 + \lambda_2 \mathbf{e}_2) + 2\mathbf{u}^\top (\lambda_1 \mathbf{e}_1 + \lambda_2 \mathbf{e}_2) + f = 0 \\ &\implies \lambda_1^2 \mathbf{e}_1^\top \mathbf{V} \mathbf{e}_1 + \lambda_2^2 \mathbf{e}_2^\top \mathbf{V} \mathbf{e}_2 + \lambda_1 \lambda_2 \mathbf{e}_1^\top \mathbf{V} \mathbf{e}_2 + \lambda_1 \lambda_2 \mathbf{e}_2^\top \mathbf{V} \mathbf{e}_1 + 2\lambda_1 \mathbf{u}^\top \mathbf{e}_1 \\ &\qquad + 2\lambda_2 \mathbf{u}^\top \mathbf{e}_2 + f = 0 \end{aligned} \tag{3.5.2.13}$$

Substituting numerical values in the above,

$$\lambda_1^2 - \frac{6}{625} \lambda_2 = 0 \tag{3.5.2.14}$$

Since $\lambda_1 = 18$,

$$\lambda_2 = \frac{6}{625}(18)^2 = \frac{1944}{625} \tag{3.5.2.15}$$

Thus, the length of a supporting wire attached to the roadway $18m$ from the middle is

$$\lambda_2 + d_2 = \frac{1944}{625} + 6 = \frac{5694}{625}m \tag{3.5.2.16}$$

See Fig. 3.5.2.1.

3.5.3 Find the area of the triangle formed by the lines joining the vertex of the parabola

$$x^2 = 12y \tag{3.5.3.1}$$

to the ends of its latus rectum.

Rewriting (3.5.3.1) in matrix form,

$$\mathbf{x}^\top \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + 2\begin{pmatrix} 0 & -6 \end{pmatrix} \mathbf{x} = 0 \tag{3.5.3.2}$$

Since the parabola is clearly symmetric about the y-axis, we see that the directrix is parallel to the x-axis, thus

$$\mathbf{n} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.5.3.3}$$

Using the standard definition of the conic and equating $\mathbf{u}$ and f,

$$\begin{pmatrix} 0 \\ -6 \end{pmatrix} = c \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \mathbf{F} \tag{3.5.3.4}$$

$$0 = \|\mathbf{F}\|^2 - c^2 \tag{3.5.3.5}$$

From (3.5.3.4), we have

$$\mathbf{F} = \begin{pmatrix} 0 \\ c+6 \end{pmatrix} \tag{3.5.3.6}$$

Using (3.5.3.6) in (3.5.3.5),

$$(c+6)^2 = c^2 \tag{3.5.3.7}$$

$$\implies c = -3 \tag{3.5.3.8}$$

Thus,

$$\mathbf{F} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} \tag{3.5.3.9}$$

The latus rectum of the parabola is the chord passing through the focus parallel to the directrix. Its equation is given by

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \end{pmatrix} = 3 \tag{3.5.3.10}$$

Hence, for $\lambda \in \mathbb{R}$,

$$\mathbf{x} = \lambda \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} \lambda \\ 3 \end{pmatrix} \tag{3.5.3.11}$$

Adding (3.5.3.2) to 12 times (3.5.3.10), and using (3.5.3.11)

$$\mathbf{x}^\top \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} = 36 \tag{3.5.3.12}$$

$$\implies \lambda^2 = 36 \tag{3.5.3.13}$$

$$\implies \lambda = \pm 6 \tag{3.5.3.14}$$

Thus, the ends of the latus rectum are

$$\mathbf{x} = \begin{pmatrix} \pm 6 \\ 3 \end{pmatrix} \tag{3.5.3.15}$$

Since the vertex of the parabola is at $\mathbf{P} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, we see that the area of the required triangle is

$$\text{ar}\left(\triangle PAB\right) = \frac{1}{2} \begin{vmatrix} 6 & 3 \\ -6 & 3 \end{vmatrix} = 18 \text{ sq. units} \tag{3.5.3.16}$$

The situation is illustrated in Fig. 3.5.3.1

3.5.4 An equilateral triangle is inscribed in the parabola $y^2 = 4ax$,where one vertex is at the vertex of the parabola. Find the length of the side of the triangle. An equilateral triangle is inscribed in the parabola $y^2 = 4ax$,where one vertex is at the vertex of the parabola. Find the length of the side of the triangle.

Solution:

In the each of the following Exercises, find the coordinates of the focus, axis of the parabola, the equation of the directrix and the length of the latus rectum.

3.5.5 y^2=12x

Solution: The given equation of the parabola can be rearranged as

$$y^2 - 12x = 0 \tag{3.5.5.1}$$

The above equation can be equated to the generic equation of conic sections

$$g(\mathbf{x}) = \mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{3.5.5.2}$$

Comparing coefficients of (3.5.5.1) and (3.5.5.2),

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.5.5.3}$$

$$\mathbf{u} = -\begin{pmatrix} 6 \\ 0 \end{pmatrix} \tag{3.5.5.4}$$

$$f = 0 \tag{3.5.5.5}$$

(a) From (3.5.5.3), since $\mathbf{V}$ is already diagonalized, the Eigen values λ_1 and λ_2 are given as

$$\lambda_1 = 0 \tag{3.5.5.6}$$

$$\lambda_2 = 1 \tag{3.5.5.7}$$

and the eigenvector matrix

$$\mathbf{P} = \mathbf{I}. \tag{3.5.5.8}$$

$$\therefore \mathbf{n} = \sqrt{\lambda_2}\mathbf{p_1} \tag{3.5.5.9}$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.5.5.10}$$

Since

$$c = \frac{\left\|\mathbf{u}^2\right\| - \lambda_2 f}{2\mathbf{u}^\top \mathbf{n}}, \tag{3.5.5.11}$$

Substituting values of $\mathbf{u}, \mathbf{n}, \lambda_2$ and f in (3.5.5.11)

$$c = \frac{6^2 - 1\,(0)}{-2\begin{pmatrix} 6 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}} = -3 \tag{3.5.5.12}$$

$$\tag{3.5.5.13}$$

The focus $\mathbf{F}$ of parabola is expressed as

$$\mathbf{F} = \frac{ce^2\mathbf{n} - \mathbf{u}}{\lambda_2} \tag{3.5.5.14}$$

$$= \frac{-3\left(1\right)^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 6 \\ 0 \end{pmatrix}}{1} \tag{3.5.5.15}$$

$$= \begin{pmatrix} 3 \\ 0 \end{pmatrix} \tag{3.5.5.16}$$

(b) The directrix is given by

$$\mathbf{n}^\top \mathbf{x} = c \tag{3.5.5.17}$$

$$\implies \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = -3 \tag{3.5.5.18}$$

(c) The equation for the axis of parabola passing through $\mathbf{F}$ and orthogonal to the directrix is given as

$$\mathbf{m}^\top \left(\mathbf{x} - \mathbf{F}\right) = 0 \tag{3.5.5.19}$$

where $\mathbf{m}$ is the normal vector to the axis and also the slope of

the directrix.

$$\because \mathbf{n} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.5.5.20}$$

$$(3.5.5.19) \implies \begin{pmatrix} 0 & 1 \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} 3 \\ 0 \end{pmatrix} \right) = 0 \tag{3.5.5.21}$$

$$\text{or, } \begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{3.5.5.22}$$

(d) The latus rectum of a parabola is given by

$$l = \frac{\eta}{\lambda_2} = \frac{2\mathbf{u}^\top \mathbf{p_1}}{\lambda_2} \tag{3.5.5.23}$$

$$= \frac{2 \begin{pmatrix} 6 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}}{1} \tag{3.5.5.24}$$

$$= 12 \text{ units} \tag{3.5.5.25}$$

The relevant diagram is shown in Fig. 3.5.5.1

3.5.6 x^2=6y

Solution: The given equation of the parabola can be rearranged as

$$x^2 - 6y = 0 \tag{3.5.6.1}$$

The above equation can be equated to the generic equation of conic sections

$$g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{3.5.6.2}$$

Comparing the coefficients of both equations (3.5.6.1) and (3.5.6.2)

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{3.5.6.3}$$

$$\mathbf{u} = -\begin{pmatrix} 0 \\ 3 \end{pmatrix} \tag{3.5.6.4}$$

$$f = 0 \tag{3.5.6.5}$$

(a) From equation (3.5.6.3), since $\mathbf{V}$ is already diagonalized, the Eigen values λ_1 and λ_2 are given as

$$\lambda_1 = 1 \tag{3.5.6.6}$$

$$\lambda_2 = 0 \tag{3.5.6.7}$$

And the corresponding eigen vector matrix $\mathbf{P}$ is indentity, so the

Eigen vector $\mathbf{p}_2$ corresponding to Eigen value λ_2 is

$$\mathbf{p}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.5.6.8}$$

$$\mathbf{n} = \sqrt{\lambda_1}\mathbf{p}_2 \tag{3.5.6.9}$$

$$= \sqrt{1}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.5.6.10}$$

$$= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.5.6.11}$$

Now,

$$c = \frac{\left\|\mathbf{u}\right\|^2 - \lambda_1 f}{2\mathbf{u}^\top \mathbf{n}} \tag{3.5.6.12}$$

Substituting values of $\mathbf{u}, \mathbf{n}, \lambda_1$ and f in (3.5.6.12)

$$c = \frac{3^2 - 1\left(0\right)}{-2\begin{pmatrix} 0 & 3 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}} = -\frac{3}{2} \tag{3.5.6.13}$$

The focus $\mathbf{F}$ of parabola is expressed as

$$\mathbf{F} = \frac{ce^2\mathbf{n} - \mathbf{u}}{\lambda_1} \tag{3.5.6.14}$$

$$= \frac{-\frac{3}{2}(1)^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \end{pmatrix}}{1} \tag{3.5.6.15}$$

$$= \begin{pmatrix} 0 \\ \frac{3}{2} \end{pmatrix} \tag{3.5.6.16}$$

(b) Equation of directrix is given as

$$\mathbf{n}^\top \mathbf{x} = c \tag{3.5.6.17}$$

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = -\frac{3}{2} \tag{3.5.6.18}$$

(c) The equation for the axis of parabola passing through $\mathbf{F}$ and orthogonal to the directrix is given as

$$\mathbf{m}^\top (\mathbf{x} - \mathbf{F}) = 0 \tag{3.5.6.19}$$

where $\mathbf{m}$ is the normal vector to the axis and also the slope of

the directrix. Now since

$$\mathbf{n} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.5.6.20}$$

$$\mathbf{m} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.5.6.21}$$

Substituting in (3.5.6.19)

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} 0 \\ \frac{3}{2} \end{pmatrix} \right) = 0 \tag{3.5.6.22}$$

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 0 \tag{3.5.6.23}$$

(d) The latus rectum of a parabola is given by

$$l = \frac{\eta}{\lambda_1} \tag{3.5.6.24}$$

$$= \frac{2\mathbf{u}^\top \mathbf{p}_2}{\lambda_1} \tag{3.5.6.25}$$

$$= \frac{2 \begin{pmatrix} 0 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}}{1} \tag{3.5.6.26}$$

$$= 6 \text{ units} \tag{3.5.6.27}$$

See Fig. 3.5.6.1

3.5.7 Find the coordinates of the focus, axis of the parabola, the equation of the directrix and the length of the latus rectum of $y^2 = -8x$

Solution: The given equation of the parabola can be rearranged as

$$y^2 + 8x = 0 \tag{3.5.7.1}$$

The above equation can be equated to the generic equation of conic sections

$$\text{g}\left(\mathbf{x}\right) = \mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{3.5.7.2}$$

Comparing coefficients of (3.5.7.1) and (3.5.7.2), we get the parameters as given in Table 3.5.7.1

Parameter	**Value**
$\mathbf{V}$	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
$\mathbf{u}$	$\begin{pmatrix} 4 \\ 0 \end{pmatrix}$
f	0

Table 3.5.7.1:

(a) Focus: Since $\mathbf{V}$ is already diagonalized, the Eigen values λ_1 and λ_2 are given as

$$\lambda_1 = 0, \lambda_2 = 1 \tag{3.5.7.3}$$

and the eigenvector matrix

$$\mathbf{P} = \mathbf{I} \implies \mathbf{p_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.5.7.4}$$

$$\text{and } \mathbf{n} = \sqrt{\lambda_2}\mathbf{p_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.5.7.5}$$

Since

$$c = \frac{\left\|\mathbf{u}^2\right\| - \lambda_2 f}{2\mathbf{u}^\top \mathbf{n}}, \tag{3.5.7.6}$$

Substituting values of $\mathbf{u}, \mathbf{n}, \lambda_2$ and f in (3.5.7.6), we get

$$c = \frac{4^2 - 1\,(0)}{2\begin{pmatrix} 4 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}} = 2 \tag{3.5.7.7}$$

The focus $\mathbf{F}$ of parabola is expressed as

$$\mathbf{F} = \frac{ce^2\mathbf{n} - \mathbf{u}}{\lambda_2} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \tag{3.5.7.8}$$

(b) Directrix: The directrix is given by

$$\mathbf{n}^\top \mathbf{x} = c \tag{3.5.7.9}$$

$$\implies \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 2 \tag{3.5.7.10}$$

(c) Axis: The equation for the axis of parabola, passing through $\mathbf{F}$ and orthogonal to the directrix is given as

$$\mathbf{m}^\top (\mathbf{x} - \mathbf{F}) = 0 \tag{3.5.7.11}$$

where $\mathbf{m}$ is the normal vector to the axis and also the slope of the directrix. Since

$$\mathbf{n} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{3.5.7.12}$$

the axis is given by

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} -2 \\ 0 \end{pmatrix} \right) = 0 \tag{3.5.7.13}$$

$$\implies \begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{3.5.7.14}$$

(d) Latus rectum: The latus rectum of a parabola is given by

$$l = \frac{\eta}{\lambda_2} = \frac{\mathbf{2u}^\top \mathbf{p_1}}{\lambda_2} \tag{3.5.7.15}$$

$$= 8 \text{ units} \tag{3.5.7.16}$$

See Fig. 3.5.7.1.

3.5.8 y^2=-8x

3.5.9 x^2=-16y

Solution: The given equation of the parabola can be written as

$$x^2 + 16y = 0 \tag{3.5.9.1}$$

The general equation for conic section is

$$g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{3.5.9.2}$$

Comparaing both equations (3.5.9.1) and (3.5.9.2) we get,

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{3.5.9.3}$$

$$\mathbf{u} = \begin{pmatrix} 0 \\ 8 \end{pmatrix} \tag{3.5.9.4}$$

$$f = 0 \tag{3.5.9.5}$$

(a) As $\mathbf{V}$ matrix is already diagonalized (3.5.9.3), the Eigen values λ_1 and λ_2 are given as

$$\lambda_1 = 1 \tag{3.5.9.6}$$

$$\lambda_2 = 0 \tag{3.5.9.7}$$

Eigen vector matrix $\mathbf{P}$ is identical the eigen vector $\mathbf{P}_2$ by eigen value λ_2 is

$$\mathbf{p}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.5.9.8}$$

$$\mathbf{n} = \sqrt{\lambda_1}\mathbf{p}_2 \tag{3.5.9.9}$$

$$= \sqrt{1}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.5.9.10}$$

$$= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.5.9.11}$$

So,

$$\frac{\|\mathbf{u}\|^2 - \lambda_1 f}{2\mathbf{u}^\top \mathbf{n}} = c \tag{3.5.9.12}$$

Substituting $\mathbf{u}, \mathbf{n}, \lambda_1$ and f values in (3.5.9.12) we get

$$c = \frac{8^2 - 1\,(0)}{2\begin{pmatrix} 0 & 8 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}} = 4 \tag{3.5.9.13}$$

The focus $\mathbf{F}$ of parabola is

$$\mathbf{F} = \frac{ce^2\mathbf{n} - \mathbf{u}}{\lambda_1} \tag{3.5.9.14}$$

$$= \frac{4(1)^2\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 8 \end{pmatrix}}{1} \tag{3.5.9.15}$$

$$= \begin{pmatrix} 0 \\ -4 \end{pmatrix} \tag{3.5.9.16}$$

(b) Equation of directrix is given as

$$\mathbf{n}^\top \mathbf{x} = c \tag{3.5.9.17}$$

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 4 \tag{3.5.9.18}$$

$$\mathbf{x} = 4 \tag{3.5.9.19}$$

(c) Equation for tha axis of parabola is

$$\mathbf{m}^\top (\mathbf{x} - \mathbf{F}) = 0 \tag{3.5.9.20}$$

where $\mathbf{m}$ is the normal vector to the axis and also the slope of

the directrix

$$\mathbf{n} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.5.9.21}$$

$$\mathbf{m} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.5.9.22}$$

Substituting in (3.5.9.20)

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} 0 \\ -4 \end{pmatrix} \right) = 0 \tag{3.5.9.23}$$

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 0 \tag{3.5.9.24}$$

$$\mathbf{x} = 0 \tag{3.5.9.25}$$

(d) Latus rectum of parabola is

$$l = \frac{\eta}{\lambda_1} \tag{3.5.9.26}$$

$$= \frac{2\mathbf{u}^\top \mathbf{p}_2}{\lambda_1} \tag{3.5.9.27}$$

$$= \frac{2\begin{pmatrix} 0 & 8 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}}{1} \tag{3.5.9.28}$$

$$= 16 \text{ units} \tag{3.5.9.29}$$

3.5.10 y^2=10x

3.5.11 x^2=-9y

Each of the Exercises, find the equation of the parabola, that satisfies the given conditions.

3.5.12 Focus(6,0); directrix x=-6

3.5.13 Focus(0,-3); directrix y=3

3.5.14 Vertex(0,0); Focus(3,0)

3.5.15 Vertex(0,0); Focus(-2,0)

3.5.16 Vertex(0,0) passing through(2,3) and axis is along x-axis

3.5.17 Vertex(0,0) passing through(5,2) symmetric with respect to y-axis

3.6. Exercises

3.6.1 If the focus of a parabola is (0,-3) and its directrix is y=3, then its equation is

(a) $x^2 = -12y$

(b) $x^2 = 12y$

(c) $y^2 = -12x$

(d) $y^2 = 12x$

3.6.2 If the parabola $y^2 = 4ax$ passes through the point (3,2), then the length of its latus rectum is

(a) 2±3

(b) 4±4

(c) 1±3

(d) 4

3.6.3 If the vertex of the parabola is the point (-3,0) and the directrix is the line x+5=0, then its equation is

(a) $y^2 = 8(x + 3)$

(b) $x^2 = 8(y + 3)$

(c) $y^2 = -8(x + 3)$

(d) $y^2 = 8(x + 5)$

3.6.4 Find the coordinates of a point on the parabla $y^2 = 8x$ whose focal distance is 4.

3.6.5 Find the length of the line-segment joining the vertex of the parabola $y^2 = 4ax$ and a point on the parabola where the line - segment makes an angle 0 to the x-axis.

3.6.6 If the points (0,4) and (0,2) are respectively the vertex and focus of a parabola. then find the equation of the parabola

Find the equation of each of the following parabolas

3.6.7 Directrix x=0. focus ot (6,0)

3.6.8 vertex ot (0,4), focus at (0,2)

3.6.9 Focus at (-1,2), directrix x-2y+3=0

Fill in the Blanks

3.6.10 The equation of the parabola having focus at (-1,-2) and the directrix x-2y+3=0 is ______

3.7. Ellipse

3.7.1 An arch is in the form of a semi-ellipse. It is 8 m wide and 2 m high at the centre. Find the height of the arch at a point 1.5 m from one end.

Solution:

3.7.2 A rod of length 12cm moves with its ends always touching the coordinate axes. Determine the equation of locus of a point P on the rod, which is 3cm from the end in contact with $x - axis$. Let the angle made by the rod with x-axis be θ. Then

(a) x-intercept:

$$\mathbf{A} = \begin{pmatrix} 12\cos\theta \\ 0 \end{pmatrix} \tag{3.7.2.1}$$

(b) y-intercept:

$$\mathbf{B} = \begin{pmatrix} 0 \\ 12\sin\theta \end{pmatrix} \tag{3.7.2.2}$$

(c) direction vector of rod:

$$\mathbf{A} - \mathbf{B} = 12 \begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix} \tag{3.7.2.3}$$

Unit vector along direction vector:

$$\mathbf{m} = \begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix} \tag{3.7.2.4}$$

(d) given point $\mathbf{P}$:

$$\mathbf{P} = \mathbf{A} - 3\mathbf{m} \tag{3.7.2.5}$$

$$= \begin{pmatrix} 9\cos\theta \\ 3\sin\theta \end{pmatrix} \tag{3.7.2.6}$$

(e) parametric form of locus:

$$\mathbf{x} = \begin{pmatrix} 9\cos\theta \\ 3\sin\theta \end{pmatrix} \tag{3.7.2.7}$$

$$\tag{3.7.2.8}$$

Consider $\mathbf{Q} = \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$

$$\mathbf{Q}\mathbf{x} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \tag{3.7.2.9}$$

$$\|\mathbf{Q}\mathbf{x}\|^2 = 1 \tag{3.7.2.10}$$

$$(\mathbf{Q}\mathbf{x})^\top (\mathbf{Q}\mathbf{x}) = 1 \tag{3.7.2.11}$$

$$\mathbf{x}^\top \mathbf{Q}^\top \mathbf{Q}\mathbf{x} = 1 \tag{3.7.2.12}$$

$$\mathbf{x}^T \begin{pmatrix} \frac{1}{81} & 0 \\ 0 & \frac{1}{9} \end{pmatrix} \mathbf{x} = 1 \tag{3.7.2.13}$$

The locus of point $\mathbf{P}$ is a conic

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{3.7.2.14}$$

where,

$$\mathbf{V} = \begin{pmatrix} \frac{1}{81} & 0 \\ 0 & \frac{1}{9} \end{pmatrix} \tag{3.7.2.15}$$

$$\mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.7.2.16}$$

$$f = -1 \tag{3.7.2.17}$$

See Table 3.7.2.1. See Fig. 3.7.2.1

Parameter	Value	Description
$\mathbf{A}$	$\begin{pmatrix} 12\cos\theta \\ 0 \end{pmatrix}$	x-intercept of rod
$\mathbf{C}$	$\begin{pmatrix} 0 \\ 12\sin\theta \end{pmatrix}$	y-intercept of the rod
$\mathbf{P}$	$\begin{pmatrix} 9\cos\theta \\ 3\sin\theta \end{pmatrix}$	Point on rod, at given distance from $\mathbf{A}$
θ	$\frac{\pi}{3}$	parameter θ for $\mathbf{A}, \mathbf{B}, \mathbf{P}$
length	12	Length of the rod
dist	3	Distance between $\mathbf{A}, \mathbf{P}$

Table 3.7.2.1:

3.7.3 A man running a racecourse notes that the sum of the distances from the two flag posts from him is always 10 m and the distance between the flag posts is 8 m. Find the equation of the posts traced by the man. The conic section for the given problem is an ellipse. Let $\mathbf{O}$ $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ be the centre of the Ellipse. Then, the focii are given by

$$\mathbf{F_1} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \tag{3.7.3.1}$$

$$\mathbf{F_2} = \begin{pmatrix} -4 \\ 0 \end{pmatrix} \tag{3.7.3.2}$$

The sum of the distances from two focii to the point on the locus of the ellipse is equal to $10m$. Let $\mathbf{P}$ $\begin{pmatrix} p \\ 0 \end{pmatrix}$ and $\mathbf{Q}$ $\begin{pmatrix} -q \\ 0 \end{pmatrix}$ be the vertices of the ellipse. Then

$$\|\mathbf{P} - \mathbf{F_1}\| + \|\mathbf{P} - \mathbf{F_2}\| = 10 \tag{3.7.3.3}$$

$$(p-4) + (p+4) = 10 \tag{3.7.3.4}$$

$$2p = 10 \tag{3.7.3.5}$$

$$p = 5 \tag{3.7.3.6}$$

$$\therefore \mathbf{P} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \tag{3.7.3.7}$$

Similarly

$$\|\mathbf{Q} - \mathbf{F_1}\| + \|\mathbf{Q} - \mathbf{F_2}\| = 10 \tag{3.7.3.8}$$

$$(q-4) + (q+4) = 10 \tag{3.7.3.9}$$

$$2q = 10 \tag{3.7.3.10}$$

$$q = 5 \tag{3.7.3.11}$$

$$\therefore \mathbf{Q} = \begin{pmatrix} -5 \\ 0 \end{pmatrix} \tag{3.7.3.12}$$

We know that the Vertex of a standard ellipse is given by

$$\mathbf{P} = \begin{pmatrix} \sqrt{\left|\frac{f_0}{\lambda_1}\right|} \\ 0 \end{pmatrix} \tag{3.7.3.13}$$

$$\begin{pmatrix} 5 \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{\left|\frac{f_0}{\lambda_1}\right|} \\ 0 \end{pmatrix} \tag{3.7.3.14}$$

$$\frac{f_0}{\lambda_1} = 25 \tag{3.7.3.15}$$

$$f_0 = 25\lambda_1 \tag{3.7.3.16}$$

We know that the Focii for standard Ellipse are given as

$$\mathbf{F} = \pm e\sqrt{\frac{|f_0|}{\lambda_2\left(1-e^2\right)}}\mathbf{e}_1 \tag{3.7.3.17}$$

Substituting values of $\mathbf{F_1}$ from (3.7.3.1) and f_0 from (3.7.3.16)

$$(3.7.3.17) \implies \begin{pmatrix} 4 \\ 0 \end{pmatrix} = e\sqrt{\frac{25\lambda_1}{\lambda_2\left(1-e^2\right)}}\mathbf{e}_1 \tag{3.7.3.18}$$

We know that

$$1 - e^2 = \frac{\lambda_1}{\lambda_2} \tag{3.7.3.19}$$

$$(3.7.3.18) \implies 4 = 5e \tag{3.7.3.20}$$

$$e = \frac{4}{5} \tag{3.7.3.21}$$

$$\therefore \frac{\lambda_1}{\lambda_2} = 1 - \left(\frac{4}{5}\right)^2 \tag{3.7.3.22}$$

$$= \frac{9}{25} \tag{3.7.3.23}$$

$$\mathbf{n} = \sqrt{\frac{\lambda_2}{f_0}}\mathbf{e}_1 \tag{3.7.3.24}$$

$$= \sqrt{\frac{\lambda_2}{25\lambda_1}}\mathbf{e}_1 \tag{3.7.3.25}$$

$$= \frac{1}{5} \times \frac{5}{3}\mathbf{e}_1 \tag{3.7.3.26}$$

$$= \frac{1}{3}\mathbf{e}_1 \tag{3.7.3.27}$$

$$c = \frac{1}{e\sqrt{1 - e^2}} = \frac{25}{12} \tag{3.7.3.28}$$

For the standard ellipse, f is given as

$$f = \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2 e^2 \tag{3.7.3.29}$$
$$= \left(\frac{1}{3}\right)^2 16 - \frac{25}{9} \tag{3.7.3.30}$$
$$= -1 \tag{3.7.3.31}$$
$$f_0 = -f = 1 \tag{3.7.3.32}$$
$$\lambda_1 = \frac{f_0}{25} = \frac{1}{25} \tag{3.7.3.33}$$
$$\lambda_2 = \frac{25\lambda_1}{9} = \frac{1}{9} \tag{3.7.3.34}$$
$$\therefore \mathbf{V} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{25} & 0 \\ 0 & \frac{1}{9} \end{pmatrix} \tag{3.7.3.35}$$

For a standard ellipse, $\mathbf{u} = 0$.

The generic equation of conic section is given as

$$g(\mathbf{x}) = \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \tag{3.7.3.36}$$
$$= \mathbf{x}^T \begin{pmatrix} \frac{1}{25} & 0 \\ 0 & \frac{1}{9} \end{pmatrix} \mathbf{x} - 1 = 0 \tag{3.7.3.37}$$

The relevant diagram is shown in Figure 3.7.3.1

In each of the following exercises, find the coordinates of the foci, the vertices,the length of major axis, the minor axis,the eccentricity and the length of the latus rectum of the ellipse.

3.7.4 $\frac{x^2}{36} + \frac{y^2}{16} = 1$

Solution: The given equation of ellipse can be rearranged as

$$4x^2 + 9y^2 - 144 = 0 \tag{3.7.4.1}$$

The above equation can be equated to the general equation of conic sections

$$g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{3.7.4.2}$$

From (3.7.4.1) and (3.7.4.2)

$$\mathbf{V} = \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix} \tag{3.7.4.3}$$

$$\mathbf{u} = \mathbf{0} \tag{3.7.4.4}$$

$$f = -144 \tag{3.7.4.5}$$

From (3.7.4.3) the eigen values λ_1 and λ_2 are given as

$$\lambda_1 = 4 \tag{3.7.4.6}$$

$$\lambda_2 = 9 \tag{3.7.4.7}$$

(a) The eccentricity of the ellipse is given as

$$e = \sqrt{1 - \frac{\lambda_2}{\lambda_1}} \tag{3.7.4.8}$$

$$= \sqrt{1 - \frac{4}{9}} \tag{3.7.4.9}$$

$$= \frac{\sqrt{5}}{3} \tag{3.7.4.10}$$

(b) Finding the coordinates of Focii

$$\mathbf{F} = \pm e\sqrt{\frac{|f_0|}{\lambda_2\left(1 - e^2\right)}}\mathbf{e_1} \tag{3.7.4.11}$$

$$\text{Where } f_0 = -f \tag{3.7.4.12}$$

$$\mathbf{F} = \pm 2\sqrt{5}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.7.4.13}$$

$$= \pm \begin{pmatrix} 2\sqrt{5} \\ 0 \end{pmatrix} \tag{3.7.4.14}$$

(c) The length of the major axis is given by

$$2\sqrt{\left|\frac{f_0}{\lambda_1}\right|} \tag{3.7.4.15}$$

$$2\sqrt{\left|\frac{144}{4}\right|} = 12 \tag{3.7.4.16}$$

(d) The length of minor axis is given by

$$2\sqrt{\left|\frac{f_0}{\lambda_2}\right|} \tag{3.7.4.17}$$

$$2\sqrt{\left|\frac{144}{9}\right|} = 8 \tag{3.7.4.18}$$

(e) The vertices of the ellipse are given by

$$\pm\begin{pmatrix} 0 \\ \sqrt{\left|\frac{f_0}{\lambda_2}\right|} \end{pmatrix} = \pm\begin{pmatrix} 0 \\ 4 \end{pmatrix} \tag{3.7.4.19}$$

(f) The length of latus rectum is given as

$$2\frac{\sqrt{|f_0\lambda_1|}}{\lambda_2} \tag{3.7.4.20}$$

$$= 2\frac{\sqrt{|144\,(4)|}}{9} \tag{3.7.4.21}$$

$$= \frac{16}{3} \tag{3.7.4.22}$$

See Fig. 3.7.4.1.

3.7.5 $\frac{x^2}{4} + \frac{y^2}{25} = 1$

Solution: The equation of the given ellipse can be rearranged as

$$25x^2 + 4y^2 - 100 = 0 \tag{3.7.5.1}$$

The above equation can be equated to the generic equation of conic sections

$$g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{3.7.5.2}$$

Comparing coefficients of both equations (3.7.5.1) and (3.7.5.2)

$$\mathbf{V} = \begin{pmatrix} 25 & 0 \\ 0 & 4 \end{pmatrix} \tag{3.7.5.3}$$

$$\mathbf{u} = \mathbf{0} \tag{3.7.5.4}$$

$$f = -100 \tag{3.7.5.5}$$

From equation (3.7.5.3), since $\mathbf{V}$ is already diagonalized, the eigen values λ_1 and λ_2 are given as

$$\lambda_1 = 25 \tag{3.7.5.6}$$

$$\lambda_2 = 4 \tag{3.7.5.7}$$

Since the given matrix $\mathbf{V}$ is diagonal, the Eigen vector matrix will be identity. It is given as

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix} \tag{3.7.5.8}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.7.5.9}$$

(a) The eccentricity of the ellipse is given as

$$e = \sqrt{1 - \frac{\lambda_2}{\lambda_1}} = \sqrt{1 - \frac{4}{25}} \tag{3.7.5.10}$$

$$= \frac{\sqrt{21}}{5} \tag{3.7.5.11}$$

(b) Finding the coordinates of Focii

$$\mathbf{n} = \sqrt{\lambda_1}\mathbf{p}_2 = \sqrt{25}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.7.5.12}$$

$$= \begin{pmatrix} 0 \\ 5 \end{pmatrix} \tag{3.7.5.13}$$

$$c = \frac{e\mathbf{u}^\top\mathbf{n} \pm \sqrt{e^2\left(\mathbf{u}^\top\mathbf{n}\right)^2 - \lambda_1\left(e^2 - 1\right)\left(\left\|\mathbf{u}\right\|^2 - \lambda_1 f\right)}}{\lambda_1 e\left(e^2 - 1\right)} \tag{3.7.5.14}$$

Substituting values of $e, \mathbf{u}, \mathbf{n}, \lambda_1$ and f in (3.7.5.14)

$$c = \frac{0 \pm \sqrt{0 - 25\left(\frac{21}{25} - 1\right)\left(0 + 25\left(100\right)\right)}}{25\frac{\sqrt{21}}{5}\left(\frac{21}{25} - 1\right)} \tag{3.7.5.15}$$

$$= \frac{\pm 125}{\sqrt{21}} \tag{3.7.5.16}$$

The focus $\mathbf{F}$ of the ellipse is expressed as

$$\mathbf{F} = \frac{ce^2\mathbf{n} - \mathbf{u}}{\lambda_1} \tag{3.7.5.17}$$

$$= \frac{\pm\frac{125}{\sqrt{21}}\left(\frac{21}{25}\right)\begin{pmatrix}0\\5\end{pmatrix}}{25} \tag{3.7.5.18}$$

$$= \begin{pmatrix}0\\ \pm\sqrt{21}\end{pmatrix} \tag{3.7.5.19}$$

(c) The length of the major axis is given by

$$2\sqrt{\left|\frac{f_0}{\lambda_2}\right|} \tag{3.7.5.20}$$

$$f_0 = \mathbf{u}^\top\mathbf{V}^{-1}\mathbf{u} - f \tag{3.7.5.21}$$

$$= 100 \tag{3.7.5.22}$$

$$(3.7.5.20) \implies 2\sqrt{\left|\frac{100}{4}\right|} = 10 \tag{3.7.5.23}$$

(d) The length of minor axis is given by

$$2\sqrt{\left|\frac{f_0}{\lambda_1}\right|} = 2\sqrt{\left|\frac{100}{25}\right|} \tag{3.7.5.24}$$

$$= 4 \tag{3.7.5.25}$$

(e) The vertices of the ellipse are given by

$$\pm\begin{pmatrix}0\\ \sqrt{\left|\frac{f_0}{\lambda_2}\right|}\end{pmatrix}=\pm\begin{pmatrix}0\\ 5\end{pmatrix} \tag{3.7.5.26}$$

(f) The length of latus rectum is given as

$$2\frac{\sqrt{|f_0\lambda_2|}}{\lambda_1}=2\frac{\sqrt{|100\,(4)|}}{25} \tag{3.7.5.27}$$

$$=\frac{8}{5} \tag{3.7.5.28}$$

The corresponding is shown in Fig. 3.7.5.1.

3.7.6 $\frac{x^2}{16}+\frac{y^2}{9}=1$

Solution: The given equation of ellipse can be rearranged as

$$9x^2+16y^2-144=0 \tag{3.7.6.1}$$

The above equation can be equated to the general equation of conic sections

$$g\left(\mathbf{x}\right)=\mathbf{x}^{\top}\mathbf{V}\mathbf{x}+2\mathbf{u}^{\top}\mathbf{x}+f=0 \tag{3.7.6.2}$$

From (3.7.6.1) and (3.7.6.2)

$$\mathbf{V} = \begin{pmatrix} 9 & 0 \\ 0 & 16 \end{pmatrix} \tag{3.7.6.3}$$

$$\mathbf{u} = \mathbf{0} \tag{3.7.6.4}$$

$$f = -144 \tag{3.7.6.5}$$

From (3.7.6.3) the eigen values λ_1 and λ_2 are given as

$$\lambda_1 = 9 \tag{3.7.6.6}$$

$$\lambda_2 = 16 \tag{3.7.6.7}$$

(a) The eccentricity of the ellipse is given as

$$e = \sqrt{1 - \frac{\lambda_2}{\lambda_1}} \tag{3.7.6.8}$$

$$= \sqrt{1 - \frac{9}{16}} \tag{3.7.6.9}$$

$$= \frac{\sqrt{7}}{4} \tag{3.7.6.10}$$

(b) Finding the coordinates of Focii

$$\mathbf{F} = \pm e\sqrt{\frac{|f_0|}{\lambda_2\left(1-e^2\right)}}\mathbf{e_1} \tag{3.7.6.11}$$

$$\text{Where } f_0 = -f \tag{3.7.6.12}$$

$$\mathbf{F} = \pm\sqrt{7}\begin{pmatrix}1\\0\end{pmatrix} \tag{3.7.6.13}$$

$$= \pm\begin{pmatrix}\sqrt{7}\\0\end{pmatrix} \tag{3.7.6.14}$$

(c) The length of the major axis is given by

$$2\sqrt{\left|\frac{f_0}{\lambda_1}\right|} \tag{3.7.6.15}$$

$$2\sqrt{\left|\frac{144}{9}\right|} = 8 \tag{3.7.6.16}$$

(d) The length of minor axis is given by

$$2\sqrt{\left|\frac{f_0}{\lambda_2}\right|} \tag{3.7.6.17}$$

$$2\sqrt{\left|\frac{144}{16}\right|} = 6 \tag{3.7.6.18}$$

(e) The vertices of the ellipse are given by

$$\pm\begin{pmatrix}0\\\sqrt{\left|\frac{f_0}{\lambda_2}\right|}\end{pmatrix} = \pm\begin{pmatrix}0\\4\end{pmatrix} \tag{3.7.6.19}$$

(f) The length of latus rectum is given as

$$2\frac{\sqrt{|f_0\lambda_1|}}{\lambda_2} \tag{3.7.6.20}$$
$$= 2\frac{\sqrt{|144\,(9)|}}{16} \tag{3.7.6.21}$$
$$= \frac{9}{2} \tag{3.7.6.22}$$

3.7.7 $\frac{x^2}{25} + \frac{y^2}{100} = 1$

3.7.8 $\frac{x^2}{49} + \frac{y^2}{36} = 1$

3.7.9 $\frac{x^2}{100} + \frac{y^2}{400} = 1$

3.7.10 $36x^2 + 4y^2 = 144$

3.7.11 $16x^2 + y^2 = 16$

3.7.12 $4x^2 + 9y^2 = 36$

In each of the following exercises, find the equation for the ellipse that satisfies the given conditions

3.7.13 vertices $(\pm 5, 0), \text{foci}(\pm 4, 0)$

3.7.14 vertices $(\pm 0, 13), \text{foci}(0, \pm 5)$

3.7.15 vertices $(\pm 6, 0), \text{foci}(\pm 4, 0)$

3.7.16 Ends of major axis $(\pm 3, 0)$, ends of minor axis$(0, \pm 2)$

3.7.17 ends of major axis $(0, \pm\sqrt{5})$, ends of minor axis$(\pm 1, 0)$

3.7.18 length of major axis $26, \text{foci}(\pm 5, 0)$

3.7.19 length of minor axis $16, \text{foci}(0, \pm 6)$

3.7.20 $\text{foci}(\pm 3, 0), a = 4$

3.7.21 b=3,c=4, centre at the origin; foci on the x axis

3.7.22 centre at$(0, 0)$, major axis on the y-axis and passes through the points(3,2) and (1,6).

Solution: The input parameters are listed in 3.7.22.1. The equation of the conic with focus $\mathbf{F}$, directrix $\mathbf{n}^\top \mathbf{x} = c$ and eccentricity e is given by

$$\mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{3.7.22.1}$$

where

$$\mathbf{V} \triangleq \|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{n}\mathbf{n}^\top \tag{3.7.22.2}$$

$$\mathbf{u} \triangleq ce^2 \mathbf{n} - \|\mathbf{n}\|^2 \mathbf{F} \tag{3.7.22.3}$$

$$f \triangleq \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2 e^2 \tag{3.7.22.4}$$

Given that the conic is an ellipse with major axis along the y-axis,

$$\mathbf{n} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.7.22.5}$$

Thus,

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 - e^2 \end{pmatrix} \tag{3.7.22.6}$$

$$\mathbf{u} = ce^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \mathbf{F} \tag{3.7.22.7}$$

$$f = \|\mathbf{F}\|^2 - c^2 e^2 \tag{3.7.22.8}$$

The centre of the conic is given by

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \tag{3.7.22.9}$$

Since $\mathbf{c} = \mathbf{0}$ and $\mathbf{V}^{-1} \neq \mathbf{0}$, it follows from (3.7.22.9) that

$$\mathbf{u} = \mathbf{0} \tag{3.7.22.10}$$

Thus, from (3.7.22.7),

$$\mathbf{F} = \begin{pmatrix} 0 \\ ce^2 \end{pmatrix} \tag{3.7.22.11}$$

and so,

$$f = c^2 e^2 \left(e^2 - 1\right) \tag{3.7.22.12}$$

Given that the conic passes through

$$\mathbf{P} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \tag{3.7.22.13}$$

putting $\mathbf{x} = \mathbf{P}$ in (3.7.22.1) we get,

$$\begin{pmatrix} 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1-e^2 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} + f = 0 \tag{3.7.22.14}$$

$$\implies 4e^2 - f = 13 \tag{3.7.22.15}$$

Given that the conic passes through

$$\mathbf{Q} = \begin{pmatrix} 1 \\ 6 \end{pmatrix}, \tag{3.7.22.16}$$

putting $\mathbf{x} = \mathbf{Q}$ in (3.7.22.1), we get

$$\begin{pmatrix} 1 & 6 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1-e^2 \end{pmatrix} \begin{pmatrix} 1 \\ 6 \end{pmatrix} + f = 0 \tag{3.7.22.17}$$

$$\implies 36e^2 - f = 37 \tag{3.7.22.18}$$

The equations (3.7.22.15) and (3.7.22.18) can be formulated as the following matrix equation

$$\begin{pmatrix} 4 & -1 \\ 36 & -1 \end{pmatrix} \begin{pmatrix} e^2 \\ f \end{pmatrix} = \begin{pmatrix} 13 \\ 37 \end{pmatrix} \tag{3.7.22.19}$$

The augmented matrix is given by,

$$\left(\begin{array}{cc|c} 4 & -1 & 13 \\ 36 & -1 & 37 \end{array}\right) \tag{3.7.22.20}$$

yielding

$$\xleftrightarrow{R_1 \leftarrow R_1 - R_2} \left(\begin{array}{cc|c} -32 & 0 & -24 \\ 36 & -1 & 37 \end{array}\right) \tag{3.7.22.21}$$

$$\xleftrightarrow{R_1 \leftarrow -\frac{R_1}{8}} \left(\begin{array}{cc|c} 4 & 0 & 3 \\ 36 & -1 & 37 \end{array}\right) \tag{3.7.22.22}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - 9R_1} \left(\begin{array}{cc|c} 4 & 0 & 3 \\ 0 & -1 & 10 \end{array}\right) \tag{3.7.22.23}$$

$$\xleftrightarrow[R_2 \leftarrow -R_2]{R_1 \leftarrow \frac{R_1}{4}} \left(\begin{array}{cc|c} 1 & 0 & \frac{3}{4} \\ 0 & 1 & -10 \end{array}\right) \tag{3.7.22.24}$$

Thus,

$$e^2 = \frac{3}{4}, \ f = -10 \tag{3.7.22.25}$$

and the equation of the conic is given by

$$\mathbf{x}^\top \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{pmatrix} \mathbf{x} - 10 = 0 \tag{3.7.22.26}$$

See Fig. 3.7.22.1.

Parameter	Description	Value
$\mathbf{C}$	center of the ellipse	$\begin{pmatrix}0\\0\end{pmatrix}$
$\mathbf{m}$	Direction vector of major axis	$\begin{pmatrix}0\\1\end{pmatrix}$
$\mathbf{P}$	Point on the ellipse	$\begin{pmatrix}3\\2\end{pmatrix}$
$\mathbf{Q}$	Point on the ellipse	$\begin{pmatrix}1\\6\end{pmatrix}$

Table 3.7.22.1:

3.7.23 major axis on the x-axis and passes through the points (4,3) and (6,2)

Solution: Let the equation of the conic with focus $\mathbf{F}$, directrix $\mathbf{n}^\top\mathbf{x} = c$ and eccentricity e be

$$\mathbf{x}^\top\mathbf{V}\mathbf{x} + 2\mathbf{u}^\top\mathbf{x} + f = 0 \tag{3.7.23.1}$$

where

$$\mathbf{V} \triangleq \|\mathbf{n}\|^2 \mathbf{I} - e^2\mathbf{n}\mathbf{n}^\top \tag{3.7.23.2}$$

$$\mathbf{u} \triangleq ce^2\mathbf{n} - \|\mathbf{n}\|^2 \mathbf{F} \tag{3.7.23.3}$$

$$f \triangleq \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2e^2 \tag{3.7.23.4}$$

Since the conic is an ellipse whose major axis is along the x-axis, we have

$$\mathbf{n} = \begin{pmatrix}1\\0\end{pmatrix} \tag{3.7.23.5}$$

Thus,

$$\mathbf{V} = \begin{pmatrix} 1 - e^2 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.7.23.6}$$

$$\mathbf{u} = ce^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \mathbf{F} \tag{3.7.23.7}$$

$$f = \|\mathbf{F}\|^2 - c^2 e^2 \tag{3.7.23.8}$$

The centre of the conic is given by

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \tag{3.7.23.9}$$

Since $\mathbf{c} = \mathbf{0}$ and $\mathbf{V}^{-1} \neq \mathbf{0}$, it follows from (3.7.23.9) that $\mathbf{u} = \mathbf{0}$. Thus, from (3.7.23.7),

$$\mathbf{F} = \begin{pmatrix} ce^2 \\ 0 \end{pmatrix} \tag{3.7.23.10}$$

and so,

$$f = c^2 e^2 \left(e^2 - 1\right) \tag{3.7.23.11}$$

Putting $\mathbf{x} = \mathbf{P}$ in (3.7.23.1) and using (3.7.23.10) and (3.7.23.11),

$$\begin{pmatrix} 4 & 3 \end{pmatrix} \begin{pmatrix} 1-e^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} + f = 0 \tag{3.7.23.12}$$

$$\implies 16e^2 - f = 25 \tag{3.7.23.13}$$

Putting $\mathbf{x} = \mathbf{Q}$ in (3.7.23.1), we get

$$\begin{pmatrix} 6 & 2 \end{pmatrix} \begin{pmatrix} 1-e^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 6 \\ 2 \end{pmatrix} + f = 0 \tag{3.7.23.14}$$

$$\implies 36e^2 - f = 40 \tag{3.7.23.15}$$

The equations (3.7.23.13) and (3.7.23.15) can be formulated as a matrix equation

$$\begin{pmatrix} 16 & -1 \\ 36 & -1 \end{pmatrix} \begin{pmatrix} e^2 \\ f \end{pmatrix} = \begin{pmatrix} 25 \\ 40 \end{pmatrix} \tag{3.7.23.16}$$

and can be solved using the augmented matrix.

$$\begin{pmatrix} 16 & -1 & 25 \\ 36 & -1 & 40 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 - R_2} \begin{pmatrix} -20 & 0 & -15 \\ 36 & -1 & 40 \end{pmatrix} \tag{3.7.23.17}$$

$$\xleftrightarrow[R_2 \leftarrow -R_2]{R_1 \leftarrow \frac{R_1}{-5}} \begin{pmatrix} 4 & 0 & 3 \\ -36 & 1 & -40 \end{pmatrix} \tag{3.7.23.18}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 + 9R_1} \begin{pmatrix} 4 & 0 & 3 \\ 0 & 1 & -13 \end{pmatrix} \tag{3.7.23.19}$$

$$\xleftrightarrow{R_1 \leftarrow \frac{R_1}{4}} \begin{pmatrix} 1 & 0 & \frac{3}{4} \\ 0 & 1 & -13 \end{pmatrix} \tag{3.7.23.20}$$

Thus,

$$e^2 = \frac{3}{4}, \ f = -13 \tag{3.7.23.21}$$

And the equation of the conic is given by

$$\mathbf{x}^\top \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - 13 = 0 \tag{3.7.23.22}$$

The situation is illustrated in Fig. 3.7.23.1

3.8. Exercises

3.8.1 If the latus rectum of an ellipse is equal to half of minor axis, then find its eccentricity.

3.8.2 Given the ellipse with equation $9x^2 + 25y^2 = 225$, find the eccentricity and focl.

3.8.3 If the eccentricity of an ellipse is $\frac{5}{8}$ and the distance between its foci is 10 then find latus rectum of the ellipse.

3.8.4 Find the equation of ellipse whose eccentricity is $\frac{2}{3}$,latus rectum is 5 and the centre is(0,0).

3.8.5 Find the distance between the directrices of the ellipse $\frac{x^2}{36} + \frac{y^2}{20}$

3.8.6 Find the equation of the set of all points the sum of whose distances from the points (3,0) and (9,0) is 12.

3.8.7 Find the equation of the set of all pints whose distance from (0,4) are 2±3 of their distance from the line y=9.

3.8.8 The equation of the ellipse whose focus is (1,-1), the directrix the line x-y-3 =0 and eccentricity 1pm2 is

(a) $7x^2 + 2xy + 7y^2 - 10x + 10y + 7 = 0$

(b) $7x^2 + 2xy + 7y^2 + 7 = 0$

(c) $7x^2 + 2xy + 7y^2 + 10x - 10y - 7 = 0$

(d) none

3.8.9 The length of the latus rectum of the ellipes $3x^2 + y^2 = 12$ is

(a) 4

(b) 3

(c) 8

(d) $4\sqrt{3}$

3.8.10 lf e is the eccentricity of the ellipes $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 (a < b)$,then

(a) $b^2 = a^2(1 - e^2)$

(b) $a^2 = b^2(1 - e^2)$

(c) $a^2 = b^2(e^-1)$

(d) $b^2 = a^2(e^2 - 1)$

State whether the statements are True or False

3.8.11 If P is a point on the ellipse $\frac{x^2}{16} + \frac{y^2}{25} = 1$ whose foci are s and s' then Ps +Ps'=8.

Fill in the Blanks

3.8.12 An ellipse is described by using an endless string which is passed over two pins lf the oxes are 6cm and 4cm, the length of the string and distance between the pins are ______

3.8.13 The equation of the ellipse having foci (0,1),(0,1) and minor axis of length is ______

3.9. Hyperbola

3.9.1 Find the coordinates of the focii, the vertices, the eccentricity and the length of the latus rectum of a hyperbola whose equation is given by $\frac{x^2}{16} - \frac{y^2}{9} = 1$.

Solution: The given equation can be equated to the generic equation of conic sections a

$$\mathbf{V} = \begin{pmatrix} 9 & 0 \\ 0 & -16 \end{pmatrix} \tag{3.9.1.1}$$

$$\mathbf{u} = 0 \tag{3.9.1.2}$$

$$f = -144 \tag{3.9.1.3}$$

From equation (3.9.1.1), since $\mathbf{V}$ is already diagonalized, the Eigen values λ_1 and λ_2 are given as

$$\lambda_1 = 9 \tag{3.9.1.4}$$

$$\lambda_2 = -16 \tag{3.9.1.5}$$

(a) The eccentricity of the hyperbola is given as

$$e = \sqrt{1 - \frac{\lambda_1}{\lambda_2}} \tag{3.9.1.6}$$

$$= \sqrt{1 - \frac{9}{-16}} \tag{3.9.1.7}$$

$$= \frac{5}{4} \tag{3.9.1.8}$$

(b) For the standard hyperbola, the coordinates of Focii are given as:

$$\mathbf{F} = \pm \frac{\left(\frac{1}{e\sqrt{1-e^2}}\right)\left(e^2\right)\sqrt{\frac{\lambda_2}{f_0}}}{\frac{\lambda_2}{f_0}} \mathbf{e1} \tag{3.9.1.9}$$

where

$$f_0 = -f \tag{3.9.1.10}$$

$$(3.9.1.9) \implies = \pm \frac{\left(\frac{1}{\frac{5}{4}\sqrt{1-\frac{25}{16}}}\right)\left(\frac{25}{16}\right)\sqrt{\frac{-16}{144}}\mathbf{e_1}}{\frac{-16}{144}} \tag{3.9.1.11}$$

$$= \pm \begin{pmatrix} 5 \\ 0 \end{pmatrix} \tag{3.9.1.12}$$

(c) The vertices of the hyperbola are given by

$$\pm \begin{pmatrix} a \\ 0 \end{pmatrix} \tag{3.9.1.13}$$

$$= \pm \begin{pmatrix} 4 \\ 0 \end{pmatrix} \tag{3.9.1.14}$$

(d) The length of the latus rectum is given as

$$2\frac{\sqrt{|f_0\lambda_1|}}{\lambda_2} \tag{3.9.1.15}$$

$$= 2\frac{\sqrt{|144\,(9)|}}{-16} \tag{3.9.1.16}$$

$$= \frac{9}{2} \tag{3.9.1.17}$$

as length can't be negative. The relevant diagram is shown in Figure 3.9.1.1

3.9.2 Find the coordinates of the focii, the vertices, the eccentricity and the length of the latus rectum of a hyperbola whose equation is given by $\frac{y^2}{9} - \frac{x^2}{27} = 1$.

Solution:

The equation of the hyperbola can be rearranged as

$$-x^2 + 3y^2 - 27 = 0 \tag{3.9.2.1}$$

The above equation can be equaded to the generic equation of conic sections

$$g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{3.9.2.2}$$

Comparing coefficients of both equations (3.9.2.1) and (3.9.2.2)

$$\mathbf{V} = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix} \tag{3.9.2.3}$$

$$\mathbf{u} = \mathbf{0} \tag{3.9.2.4}$$

$$f = -27 \tag{3.9.2.5}$$

From equation (3.9.2.3), since $\mathbf{V}$ is already diagonalized, the eigen values λ_1 and λ_2 are given as

$$\lambda_1 = -1 \tag{3.9.2.6}$$

$$\lambda_2 = 3 \tag{3.9.2.7}$$

(a) The eccentricity of the hyperbola is given as

$$e = \sqrt{1 - \frac{\lambda_2}{\lambda_1}} = \sqrt{1 + \frac{3}{1}} \tag{3.9.2.8}$$

$$= 2 \tag{3.9.2.9}$$

(b) For the standard hyperbola, the coordinates of Focii are given as

$$\mathbf{F} = \pm \frac{\left(\frac{1}{e\sqrt{1-e^2}}\right)\left(e^2\right)\sqrt{\frac{\lambda_1}{f_0}}}{\frac{\lambda_1}{f_0}} \mathbf{e}_2 \tag{3.9.2.10}$$

where

$$f_0 = -f \tag{3.9.2.11}$$

$$(3.9.2.10) \implies = \pm \frac{\left(\frac{1}{2\sqrt{1-4}}\right)(4)\sqrt{\frac{-1}{27}}}{\frac{-1}{27}} \mathbf{e}_2 \tag{3.9.2.12}$$

$$= \pm \begin{pmatrix} 0 \\ 6 \end{pmatrix} \tag{3.9.2.13}$$

(c) The vertices of the hyperbola are given by

$$\pm \begin{pmatrix} 0 \\ \sqrt{\left|\frac{f_0}{\lambda_2}\right|} \end{pmatrix} = \pm \begin{pmatrix} 0 \\ 3 \end{pmatrix} \tag{3.9.2.14}$$

(d) The length of latus rectum is given as

$$2\frac{\sqrt{|f_0\lambda_2|}}{\lambda_1} = 2\frac{\sqrt{|27\,(3)|}}{-1} \tag{3.9.2.15}$$

$$= 18 \tag{3.9.2.16}$$

as length cannot be negative.

See Fig. 3.9.2.1

3.9.3 Find the coordinates of the foci and the vertices, the eccentricity and the length of the latus rectum of the hyperbolas, whose equation is given by $5y^2 - 9x^2 = 36$.

Solution:

The equation of the hyperbola can be rearranged as

$$-x^2 + \frac{5}{9}y^2 - 4 = 0 \tag{3.9.3.1}$$

The above equation can be equaded to the generic equation of conic sections

$$g\left(\mathbf{x}\right) = \mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{3.9.3.2}$$

Comparing coefficients of both equations (3.9.3.1) and (3.9.3.2)

$$\mathbf{V} = \begin{pmatrix} -1 & 0 \\ 0 & \frac{5}{9} \end{pmatrix} \tag{3.9.3.3}$$

$$\mathbf{u} = \mathbf{0} \tag{3.9.3.4}$$

$$f = -4 \tag{3.9.3.5}$$

From equation (3.9.3.3), since $\mathbf{V}$ is already diagonalized, the eigen values λ_1 and λ_2 are given as

$$\lambda_1 = -1 \tag{3.9.3.6}$$

$$\lambda_2 = \frac{5}{9} \tag{3.9.3.7}$$

(a) The eccentricity of the hyperbola is given as

$$e = \sqrt{1 - \frac{\lambda_2}{\lambda_1}} = \sqrt{1 + \frac{5}{9}} \tag{3.9.3.8}$$

$$= \frac{\sqrt{14}}{3} \tag{3.9.3.9}$$

(b) For the standard hyperbola, the coordinates of Focii are given as

$$\mathbf{F} = \pm \frac{\left(\frac{1}{e\sqrt{1-e^2}}\right)\left(e^2\right)\sqrt{\frac{\lambda_1}{f_0}}}{\frac{\lambda_1}{f_0}} \mathbf{e}_2 \tag{3.9.3.10}$$

where

$$f_0 = -f \tag{3.9.3.11}$$

$$(3.9.3.10) \implies = \pm \frac{\left(\frac{1}{\frac{\sqrt{14}}{3}\sqrt{1-\frac{14}{9}}}\right)\left(\frac{14}{9}\right)\sqrt{\frac{-1}{4}}}{\frac{-1}{4}} \mathbf{e}_2 \tag{3.9.3.12}$$

$$= \pm \begin{pmatrix} 0 \\ \frac{6}{2\sqrt{\frac{14}{5}}} \end{pmatrix} \tag{3.9.3.13}$$

(c) The vertices of the hyperbola are given by

$$\pm \begin{pmatrix} 0 \\ \sqrt{\left|\frac{f_0}{\lambda_2}\right|} \end{pmatrix} = \pm \begin{pmatrix} 0 \\ \frac{6}{\sqrt{5}} \end{pmatrix} \tag{3.9.3.14}$$

(d) The length of latus rectum is given as

$$2\frac{\sqrt{|f_0\lambda_2|}}{\lambda_1} = 2\frac{\sqrt{\left|14\left(\frac{5}{9}\right)\right|}}{-1} \tag{3.9.3.15}$$

$$= 4\frac{\sqrt{5}}{3} \tag{3.9.3.16}$$

as length cannot be negative.

3.9.4 Find the equation of the hyperbola whose foci is $(0, \pm 8)$ and vertices $(0, \pm 5)$.

Solution:

3.9.5 Find the equations of hyperbola having Vertices $\begin{pmatrix} 0 \\ \pm 3 \end{pmatrix}$ and Foci

$$\begin{pmatrix} 0 \\ \pm 5 \end{pmatrix}$$

Solution:

(a) Transverse axis: Line joining two foci

$$\mathbf{m} = \mathbf{F}_1 - \mathbf{F}_2 \tag{3.9.5.1}$$

$$= \begin{pmatrix} 0 \\ 10 \end{pmatrix} \tag{3.9.5.2}$$

$$\begin{pmatrix} 1 & 0 \end{pmatrix} (\mathbf{x} - \mathbf{F}_1) = 0 \tag{3.9.5.3}$$

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 0 \tag{3.9.5.4}$$

(b) Center of hyperbola, **O** is given by:

$$\mathbf{O} = \frac{\mathbf{F}_1 + \mathbf{F}_2}{2} \tag{3.9.5.5}$$

$$\mathbf{O} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.9.5.6}$$

(c) Normal vector of directrix

$$\mathbf{n} = \text{direction vector of transverse axis} \tag{3.9.5.7}$$

$$= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.9.5.8}$$

$$
\begin{aligned}
\mathbf{V} &= \|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{n}\mathbf{n}^\top && (3.9.5.9)\\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - e^2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} && (3.9.5.10)\\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 - e^2 \end{pmatrix} && (3.9.5.11)
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{u} &= ce^2 \mathbf{n} - \|\mathbf{n}\|^2 \mathbf{F} && (3.9.5.12)\\
&= \begin{pmatrix} 0 \\ ce^2 - 5 \end{pmatrix} && (3.9.5.13)
\end{aligned}
$$

$$
\begin{aligned}
f &= \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2 e^2 && (3.9.5.14)\\
&= 25 - c^2 e^2 && (3.9.5.15)
\end{aligned}
$$

Equation of the hyperbola

$$
\mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{3.9.5.16}
$$

Vertex lies on this curve,

$$
\begin{aligned}
\mathbf{v_1}^\top \mathbf{V} \mathbf{v_1} + 2\mathbf{u}^\top \mathbf{v_1} + f &= 0 && (3.9.5.17)\\
9\left(1 - e^2\right) + 6\left(ce^2 - 5\right) - c^2 e^2 + 25 &= 0 && (3.9.5.18)\\
4 - 9e^2 + 6ce^2 - c^2 e^2 &= 0 && (3.9.5.19)
\end{aligned}
$$

Also, the center is given by,

$$\mathbf{O} = -\mathbf{V}^{-1}\mathbf{u} \tag{3.9.5.20}$$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{ce^2-5}{1-e^2} \end{pmatrix} \tag{3.9.5.21}$$

$$ce^2 = 5 \tag{3.9.5.22}$$

Solving (3.9.5.19) and (3.9.5.22),

$$c = \frac{9}{5} \tag{3.9.5.23}$$

$$e = \frac{5}{3} \tag{3.9.5.24}$$

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{16}{9} \end{pmatrix} \tag{3.9.5.25}$$

$$\mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.9.5.26}$$

$$f = 16 \tag{3.9.5.27}$$

Equation of the Hyperbola,

$$\mathbf{x}^\top \begin{pmatrix} 1 & 0 \\ 0 & -\frac{16}{9} \end{pmatrix} \mathbf{x} + 16 = 0 \tag{3.9.5.28}$$

Parameter	Value	Description
$\mathbf{F}_1$	$\begin{pmatrix}0\\5\end{pmatrix}$	Focus
$\mathbf{F}_2$	$\begin{pmatrix}0\\-5\end{pmatrix}$	Focus
$\mathbf{v}_1$	$\begin{pmatrix}0\\3\end{pmatrix}$	Vertex
$\mathbf{v}_2$	$\begin{pmatrix}0\\-3\end{pmatrix}$	Vertex

Table 3.9.5.1:

Given

$$\mathbf{F} = \begin{pmatrix}0\\ \pm 8\end{pmatrix}, \mathbf{V} = \begin{pmatrix}0\\ \pm 5\end{pmatrix} \tag{3.9.5.29}$$

(a) We know the vertex is given as

$$\mathbf{V} = \pm \begin{pmatrix}0\\ \sqrt{\frac{f_0}{\lambda_2}}\end{pmatrix} = \pm \begin{pmatrix}0\\ 5\end{pmatrix} \tag{3.9.5.30}$$

$$\implies f_0 = 25\lambda_2 \tag{3.9.5.31}$$

(b) We know the Focii is given as

$$\mathbf{F} = \pm \frac{\left(\frac{1}{e\sqrt{1-e^2}}\right)\left(e^2\right)\sqrt{\frac{\lambda_1}{f_0}}}{\frac{\lambda_1}{f_0}}\mathbf{e}_2 \tag{3.9.5.32}$$

$$= \frac{\frac{e}{\sqrt{1-e^2}}}{\sqrt{\frac{\lambda_1}{f_0}}}\mathbf{e}_2 \tag{3.9.5.33}$$

Substituting (3.9.5.31) we get

$$\mathbf{F} = 5e\mathbf{e}_2 \tag{3.9.5.34}$$

$$\begin{pmatrix} 0 \\ \\ 8 \end{pmatrix} = 5e\mathbf{e}_2 \tag{3.9.5.35}$$

$$\implies e = \frac{8}{5} \tag{3.9.5.36}$$

(c) Now we know the eccentricity is given as

$$e = \sqrt{1 - \frac{\lambda_2}{\lambda_1}} \tag{3.9.5.37}$$

$$\implies \frac{\lambda_2}{\lambda_1} = -\frac{39}{25} \tag{3.9.5.38}$$

(d) Now we know from the standard equation

$$f = \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2e^2 \tag{3.9.5.39}$$

Calculating $\mathbf{n}$ and c

$$\mathbf{n} = \sqrt{\frac{\lambda_1}{f_0}}\mathbf{e}_2 = \frac{1}{5}\sqrt{\frac{\lambda_1}{\lambda_2}}\mathbf{e}_2 \tag{3.9.5.40}$$

$$= \frac{1}{\sqrt{-39}}\mathbf{e}_2 \tag{3.9.5.41}$$

$$c = \frac{1}{e\sqrt{1-e^2}} = \frac{25}{8\sqrt{-39}} \tag{3.9.5.42}$$

Now

$$\|\mathbf{n}\|^2 = -\frac{1}{39} \tag{3.9.5.43}$$

$$\|\mathbf{F}\|^2 = 64 \tag{3.9.5.44}$$

Substituting all the values in (3.9.5.39) we get

$$f = -\left(\frac{1}{39}\right)(64) + \left(\frac{25}{8}\right)^2 \left(\frac{1}{39}\right)\left(\frac{64}{25}\right) \tag{3.9.5.45}$$

$$= -1 \tag{3.9.5.46}$$

$$f_0 = -f = 1 \tag{3.9.5.47}$$

substituting (3.9.5.47) in (3.9.5.31) we get

$$\lambda_2 = \frac{1}{25} \tag{3.9.5.48}$$

Substituting (3.9.5.48) in (3.9.5.38) we get

$$\lambda_1 = -\frac{1}{39} \tag{3.9.5.49}$$

Therefore the equation of the hyperbola is given as

$$g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{3.9.5.50}$$

where

$$\mathbf{V} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{39} & 0 \\ 0 & \frac{1}{25} \end{pmatrix} \tag{3.9.5.51}$$

$$\mathbf{u} = \mathbf{0} \tag{3.9.5.52}$$

$$f = -1 \tag{3.9.5.53}$$

See Fig. 3.9.5.2.

3.9.6 Find the equation of the hyperbola that satisfies the conditions - Foci $(\pm 4, 0)$, the latus rectum is of length 12.

Solution: The equation of the conic with focus $\mathbf{F}$, directrix $\mathbf{n}^\top \mathbf{x} = c$ and eccentricity e is given by

$$\mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{3.9.6.1}$$

where

$$\mathbf{V} \triangleq \|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{n}\mathbf{n}^\top \tag{3.9.6.2}$$

$$\mathbf{u} \triangleq ce^2\mathbf{n} - \|\mathbf{n}\|^2 \mathbf{F} \tag{3.9.6.3}$$

$$f \triangleq \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2 e^2 \tag{3.9.6.4}$$

also

$$f_0 = \mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f \tag{3.9.6.5}$$

$$l = 2\frac{\sqrt{|f_0\lambda_2|}}{\lambda_1} \tag{3.9.6.6}$$

(a) **n**: Given that the conic has foci as

$$\mathbf{F_1} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \tag{3.9.6.7}$$

$$\mathbf{F_2} = \begin{pmatrix} -4 \\ 0 \end{pmatrix} \tag{3.9.6.8}$$

The direction vector of F_1F_2 is given by

$$\mathbf{m} = \mathbf{F_1} - \mathbf{F_2} \tag{3.9.6.9}$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.9.6.10}$$

Hence the normal to the directrix is given by,

$$\mathbf{n} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.9.6.11}$$

(b) **u**: The centre of the conic is given by

$$\mathbf{c} = \frac{\mathbf{F_1} + \mathbf{F_2}}{2} \tag{3.9.6.12}$$

$$\mathbf{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.9.6.13}$$

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \tag{3.9.6.14}$$

Since $\mathbf{c} = \mathbf{0}$ and $\mathbf{V}^{-1} \neq \mathbf{0}$, it follows from (3.9.6.14) that

$$\mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.9.6.15}$$

From the above expressions we get

$$\mathbf{V} = \begin{pmatrix} 1 - e^2 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.9.6.16}$$

$$\mathbf{F} = \begin{pmatrix} ce^2 \\ 0 \end{pmatrix} \tag{3.9.6.17}$$

$$f = c^2 e^2 \left(e^2 - 1\right) \tag{3.9.6.18}$$

$$f_0 = -f \tag{3.9.6.19}$$

$$l = 2\frac{\sqrt{|f\lambda_2|}}{\lambda_1} \tag{3.9.6.20}$$

From equation (3.9.6.16) the eigen values of matrix $\mathbf{V}$ - λ_1, λ_2 are given by,

$$\lambda_1 = 1 - e^2 \tag{3.9.6.21}$$

$$\lambda_2 = 1 \tag{3.9.6.22}$$

From equation (3.9.6.17) we get,

$$ce^2 = 4 \tag{3.9.6.23}$$

(c) Eccentricity: Given that the conic has the latus rectum length 12. Substituting the expressions of λ_1, λ_2 from the equations (3.9.6.21), (3.9.6.22) in (3.9.6.20) gives

$$l = \frac{2ce}{\sqrt{e^2 - 1}} = 12 \tag{3.9.6.24}$$

$$\frac{ce}{\sqrt{e^2 - 1}} = 6 \tag{3.9.6.25}$$

Substitute the expression of c from (3.9.6.23) gives,

$$\frac{4}{e\sqrt{e^2 - 1}} = 6 \tag{3.9.6.26}$$

Squaring on both sides gives,

$$9e^2\left(e^2 - 1\right) = 4 \tag{3.9.6.27}$$

$$9e^4 - 9e^2 - 4 = 0 \tag{3.9.6.28}$$

The equation (3.9.6.28) is a quadratic equation in e^2. Solving it gives two roots one of which is negative, as e^2 is positive we have

$$e^2 = \frac{4}{3} \tag{3.9.6.29}$$

From equation (3.9.6.18), (3.9.6.23), we get

$$f = 4 \tag{3.9.6.30}$$

The equation of the conic is given by

$$\mathbf{x}^\top \begin{pmatrix} -\frac{1}{3} & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + 4 = 0 \tag{3.9.6.31}$$

Parameter	**Description**	**Value**
$\mathbf{F_1}$	Focus 1 of hyperbola	$\begin{pmatrix} 4 \\ 0 \end{pmatrix}$
$\mathbf{F_2}$	Focus 2 of hyperbola	$\begin{pmatrix} -4 \\ 0 \end{pmatrix}$
l	Length of latus rectum	12

Table 3.9.6.1:

3.9.7 Find the equation of the hyperbola whose eccentricity is $e = \frac{4}{3}$ and whose vertices are

$$\mathbf{P_1} = \begin{pmatrix} 7 \\ 0 \end{pmatrix}, \ \mathbf{P_2} = \begin{pmatrix} -7 \\ 0 \end{pmatrix} \tag{3.9.7.1}$$

Solution: Let the equation of the conic with focus $\mathbf{F}$, directrix $\mathbf{n}^\top \mathbf{x} = c$ and eccentricity e be

$$\mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{3.9.7.2}$$

where

$$\mathbf{V} \triangleq \|\mathbf{n}\|^2 \mathbf{I} - e^2\mathbf{n}\mathbf{n}^\top \tag{3.9.7.3}$$

$$\mathbf{u} \triangleq ce^2\mathbf{n} - \|\mathbf{n}\|^2 \mathbf{F} \tag{3.9.7.4}$$

$$f \triangleq \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2e^2 \tag{3.9.7.5}$$

The major axis of a conic is the chord which passes through the vertices of the conic. The direction vector of the major axis in this case is

$$\mathbf{P_2} - \mathbf{P_1} = \begin{pmatrix} 14 \\ 0 \end{pmatrix} \tag{3.9.7.6}$$

Hence, the normal to the major axis P_1P_2 is

$$\mathbf{n_M} = \mathbf{e_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.9.7.7}$$

Thus, the equation of the major axis is

$$\mathbf{e_2}^\top\mathbf{x} = \mathbf{e_2}^\top \begin{pmatrix} 7 \\ 0 \end{pmatrix} = 0 \tag{3.9.7.8}$$

which is clearly the x-axis.

Since the conic is a hyperbola whose vertices are given by (3.9.7.1) and the major axis is the x-axis, the directrix is parallel to the y-axis.

Hence,

$$\mathbf{n} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.9.7.9}$$

Thus,

$$\mathbf{V} = \begin{pmatrix} 1 - e^2 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.9.7.10}$$

$$\mathbf{u} = ce^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \mathbf{F} \tag{3.9.7.11}$$

$$f = \|\mathbf{F}\|^2 - c^2e^2 \tag{3.9.7.12}$$

Substituting $\mathbf{P_1}$ and $\mathbf{P_2}$ in (3.9.7.2),

$$\mathbf{P_1}^\top \mathbf{V} \mathbf{P_1} + 2\mathbf{u}^\top \mathbf{P_1} + f = 0 \tag{3.9.7.13}$$

$$\mathbf{P_2}^\top \mathbf{V} \mathbf{P_2} + 2\mathbf{u}^\top \mathbf{P_2} + f = 0 \tag{3.9.7.14}$$

Subtracting (3.9.7.14) from (3.9.7.13), and noting that $\mathbf{P_2} = -\mathbf{P_1}$,

$$\mathbf{u}^\top \mathbf{P_1} = 0 \tag{3.9.7.15}$$

Hence, from (3.9.7.1), we see that $\mathbf{u}$ lies on the y-axis. The general expression of the centre of a conic is given by

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \tag{3.9.7.16}$$

$$= \frac{1}{e^2-1}\begin{pmatrix} 1 & 0 \\ 0 & 1-e^2 \end{pmatrix}\mathbf{u} \tag{3.9.7.17}$$

We let $\mathbf{u} \triangleq \begin{pmatrix} 0 \\ u \end{pmatrix}$ and obtain from (3.9.7.17)

$$\mathbf{c} = \begin{pmatrix} 0 \\ -u \end{pmatrix} = -\mathbf{u} \tag{3.9.7.18}$$

Since the major axis of the hyperbola is the x-axis, we see that $\mathbf{c}$ lies on the x-axis. Thus, (3.9.7.18) implies $\mathbf{c} = -\mathbf{u} = \mathbf{0}$. Thus, from (3.9.7.11),

$$\mathbf{F} = \begin{pmatrix} ce^2 \\ 0 \end{pmatrix} \tag{3.9.7.19}$$

and so,

$$f = c^2e^2\left(e^2-1\right) \tag{3.9.7.20}$$

Putting $\mathbf{x} = \mathbf{P_1}$ or $\mathbf{x} = \mathbf{P_2}$ in (3.9.7.2) and using (3.9.7.19) and

(3.9.7.20),

$$\begin{pmatrix} \pm 7 & 0 \end{pmatrix} \begin{pmatrix} 1-e^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \pm 7 \end{pmatrix} + f = 0 \tag{3.9.7.21}$$

$$\implies 49e^2 - f = 49 \tag{3.9.7.22}$$

Since $e = \frac{4}{3}$, (3.9.7.22) implies

$$f = 49\left(e^2 - 1\right) = \frac{343}{9} \tag{3.9.7.23}$$

Therefore, the equation of the conic is

$$\mathbf{x}^\top \begin{pmatrix} -\frac{7}{9} & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + \frac{343}{9} = 0 \tag{3.9.7.24}$$

The situation is illustrated in Fig. 3.9.7.1.

3.10. Exercises

3.10.1 show that the set of all points such that the difference of their distances from (4,0) and (-4,0) is always equal to 2 represent a hyperbola .

3.10.2 If the distance between the foci of a hyperbola is 16 and its eccentricity is $\sqrt{2}$, then obtain the equation of the hyperbola.

3.10.3 Find the eccentricity of the hyperbola $9y^2 - 4x^2 = 36$.

3.10.4 Find the equation of the hyperbola with eccentricity $\frac{3}{2}$ and foci at

$(\pm 2, 0)$.

3.10.5 The eccentricity of the hyperbola whose latus rectum is 8 and conjugate axis is equal to half of th distance between the foci is

(a) 4 ± 3

(b) $\frac{4}{\sqrt{3}}$

(c) $\frac{2}{\sqrt{3}}$

(d) none of these

3.10.6 The distance between the foci of a hyperbola is 16 and its eccentricity is ≤ 2. lts equation is

(a) $x^2 - y^2 = 3^2$

(b) $\frac{x^2}{4-}\frac{y^2}{9} = 1$

(c) $2x - 3y^2 = 7$

(d) none of these

3.10.7 Equation of the hyperbola with eccentricty 3 ± 2 and foci at $(\pm 2, 0)$ is

(a) $\frac{x^2}{4} - \frac{y^2}{5} = \frac{4}{9}$

(b) $\frac{x^2}{9} - \frac{y^2}{9} = \frac{4}{9}$

(c) $\frac{x^2}{4} - \frac{y^2}{9} = 1$

(d) none of these.

Find the equation of the hyperbola with

3.10.8 vertices $(\pm 5, 0)$, focic $(\pm 7, 0)$

3.10.9 vertices (0 ± 7) ,e $=\frac{4}{3}$

3.10.10 Foci (0,$\pm\sqrt{10}$). passing through (2,3)

State whether the statements are True or False

3.10.11 The locus of the point of intersecton of lines $\sqrt{3}x + y - 4\sqrt{3}k = 0$ and $\sqrt{3} = 0\sqrt{3}kx + ky - 4\sqrt{3} = 0$ for different value of K is a hyperboia whose eccentricity is 2. [Hint: Eliminate k between the given equations]

Fill in the Blanks

3.10.12 The equation of the hyperbola with vertices ot $(0, \pm 6)$ and eccntricity $\frac{5}{3}$ is and its foci are ______

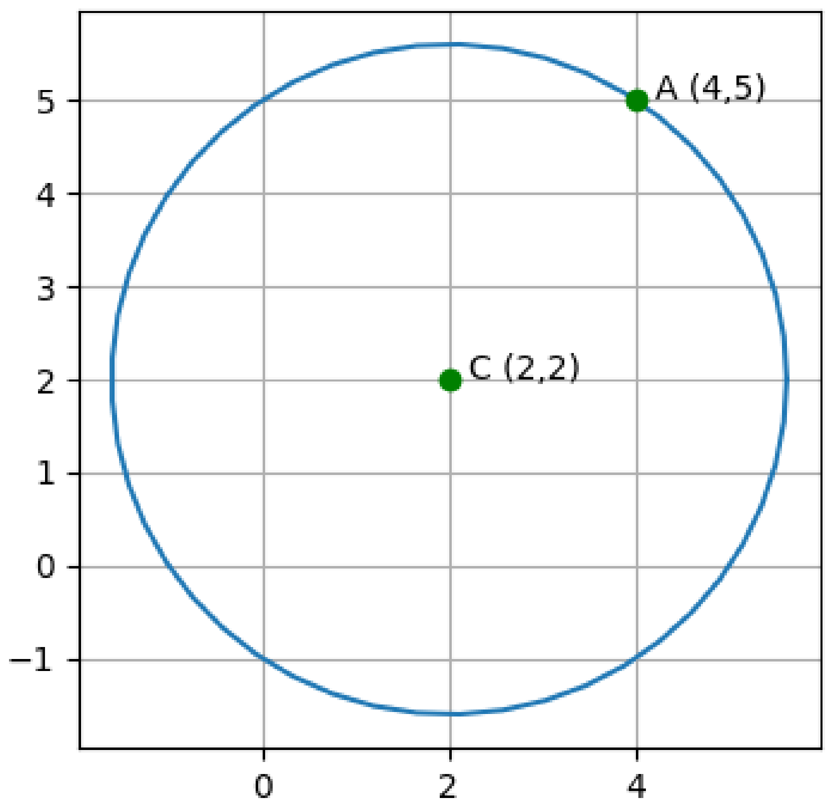


Figure 3.1.5.1:

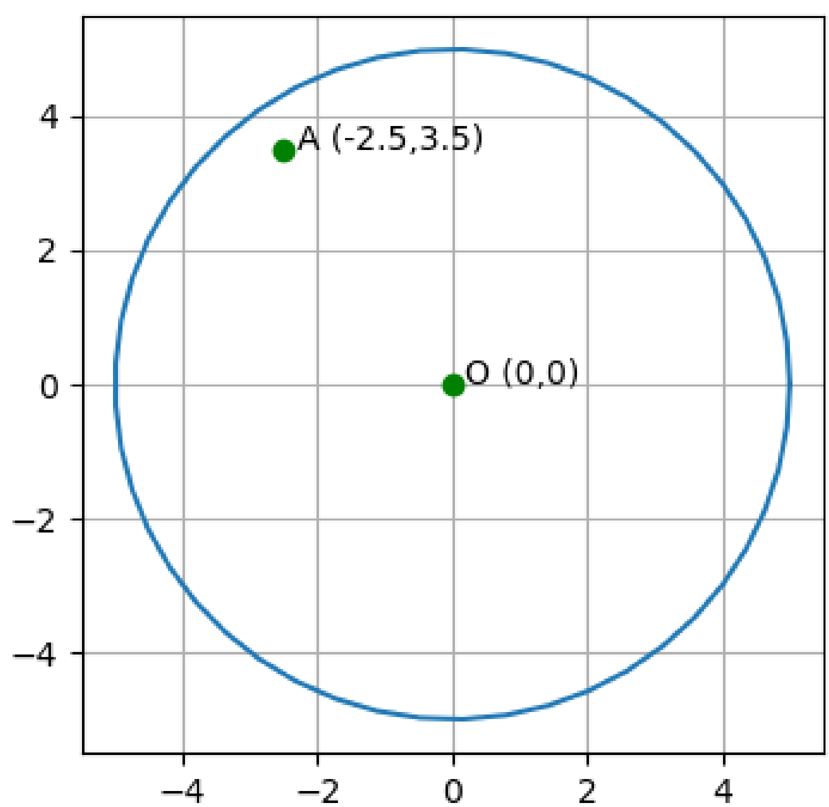


Figure 3.1.6.1: Figure 1

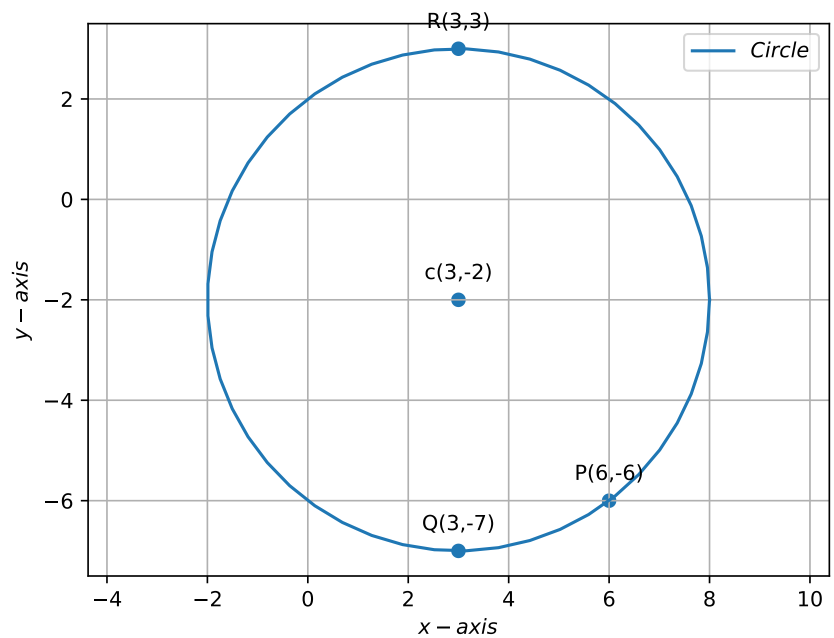


Figure 3.1.7.1:

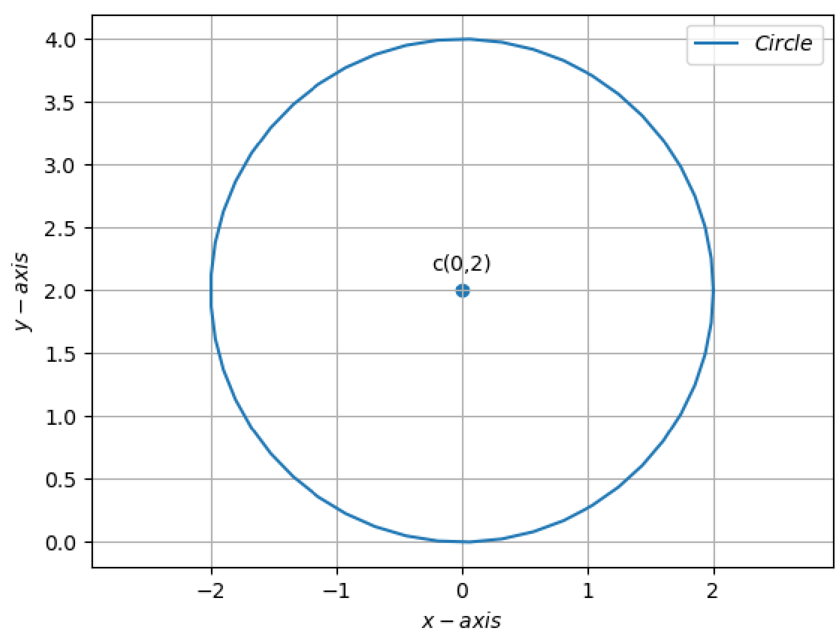


Figure 3.1.8.1:

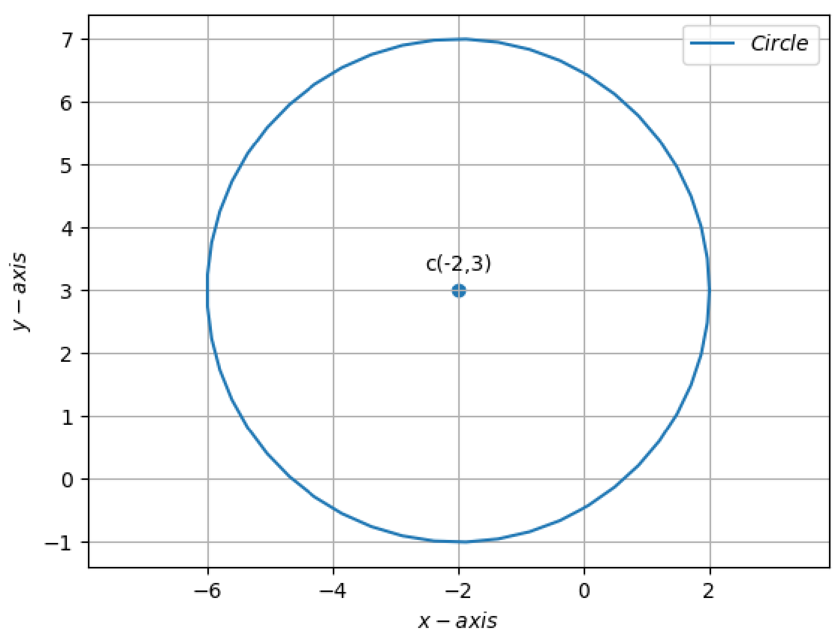
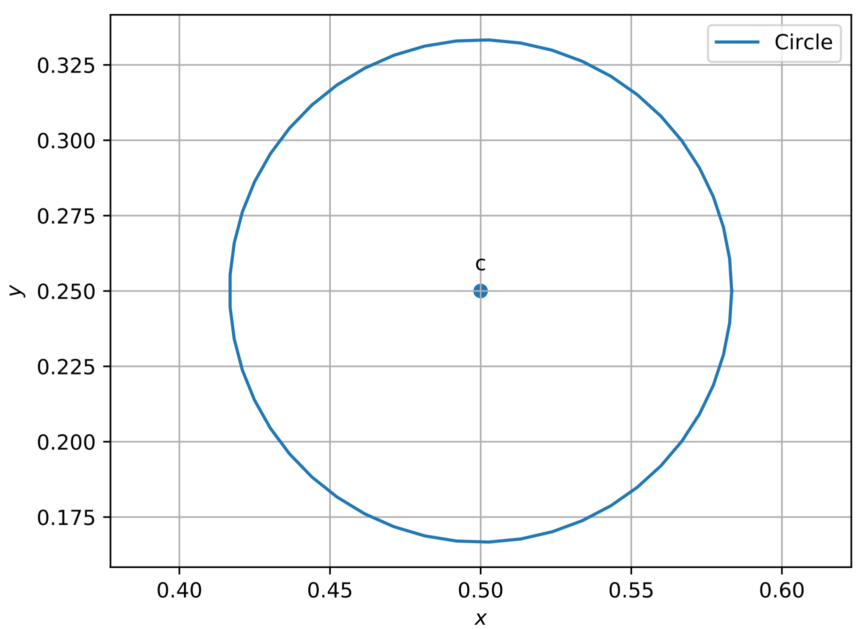


Figure 3.1.9.1:


Circle
c
0.325
0.300
0.275
0.250
0.225
0.200
0.175
y
0.40
0.45
0.50
0.55
0.60
x

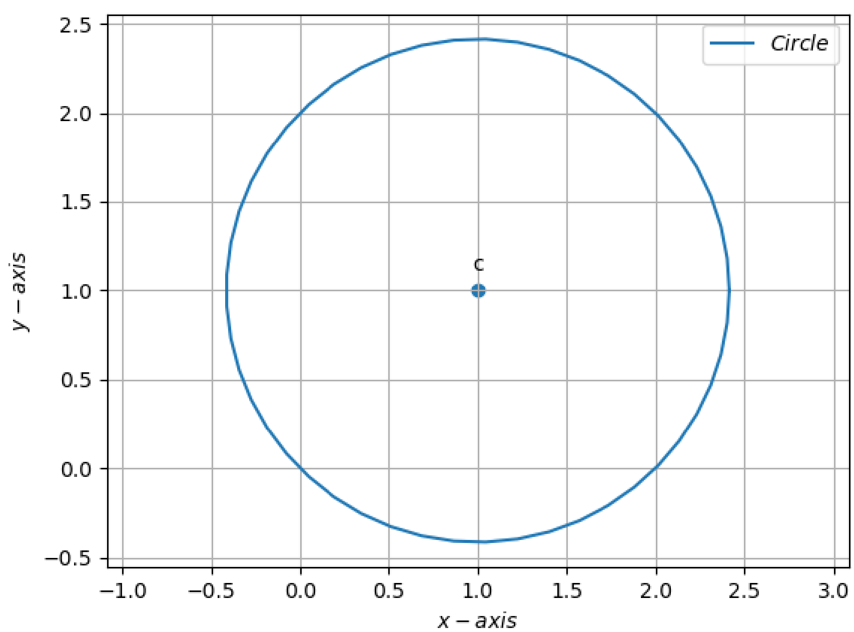


Figure 3.1.11.1:

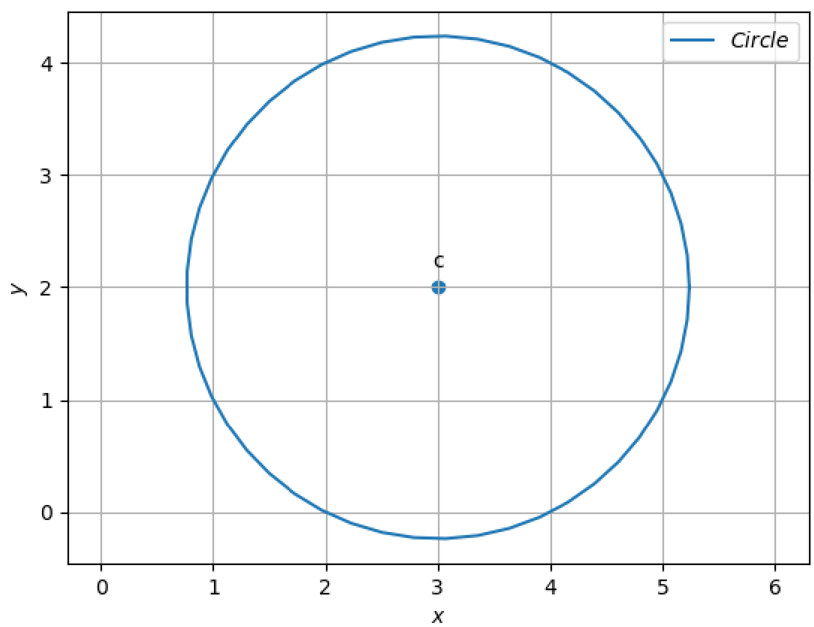


Figure 3.1.12.1: circle

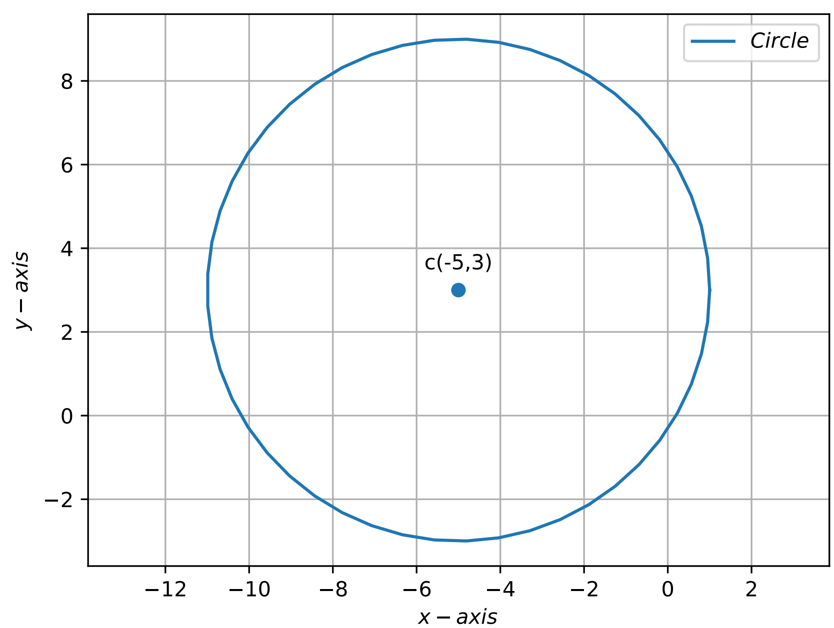


Figure 3.1.13.1:

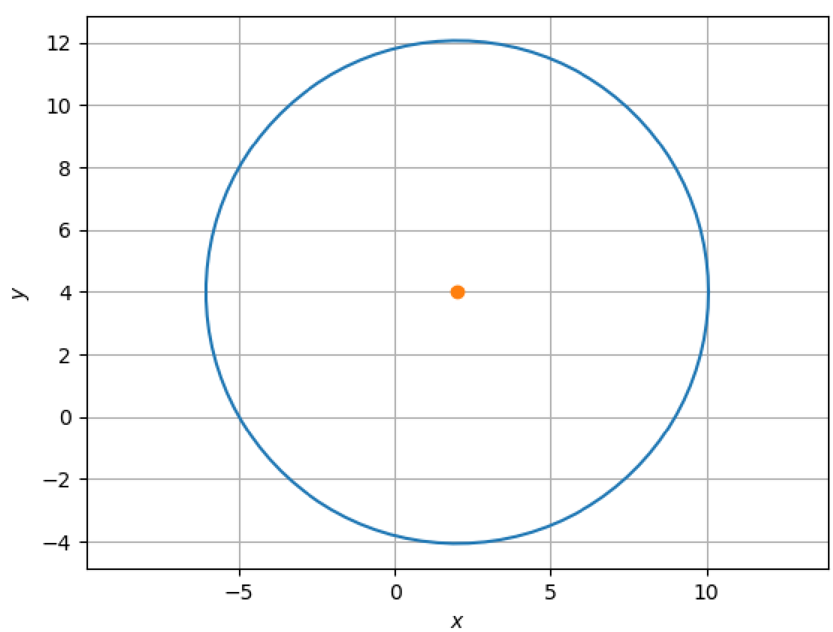


Figure 3.1.14.1:

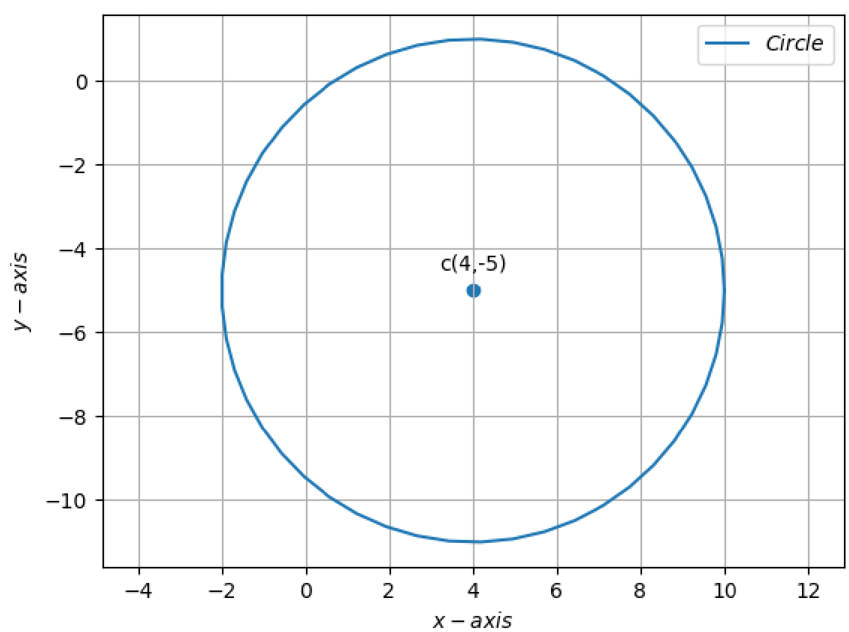


Figure 3.1.15.1:

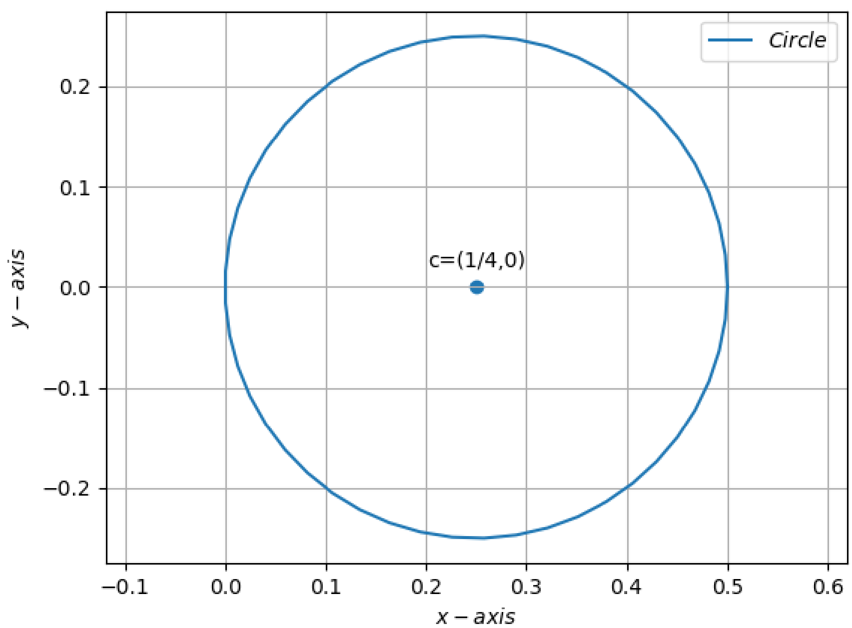


Figure 3.1.16.1:

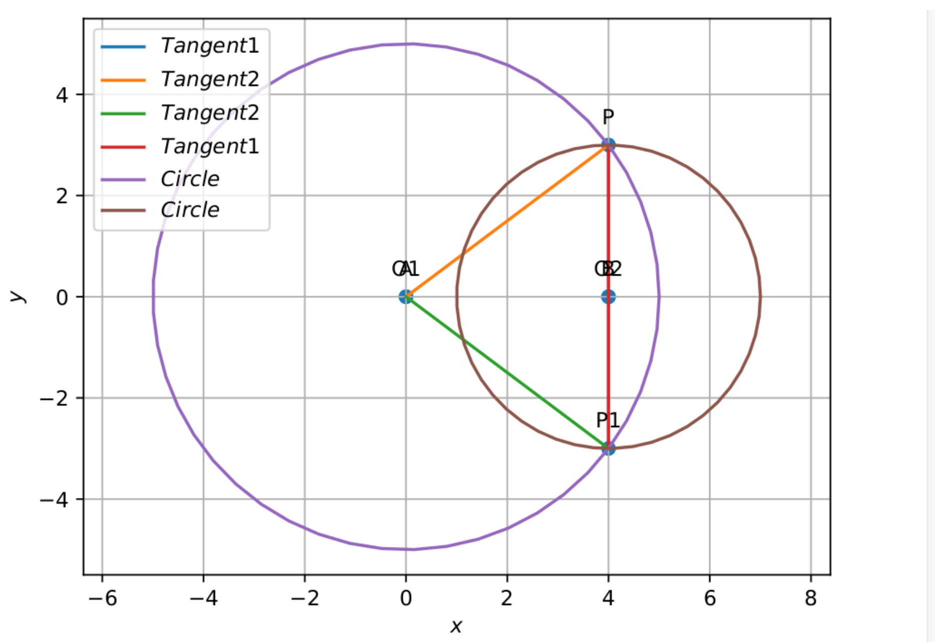

Tangent1
Tangent2
Tangent2
Tangent1
Circle
Circle
P
O A1
O B2
P1
x
y

Figure 3.3.1.1:

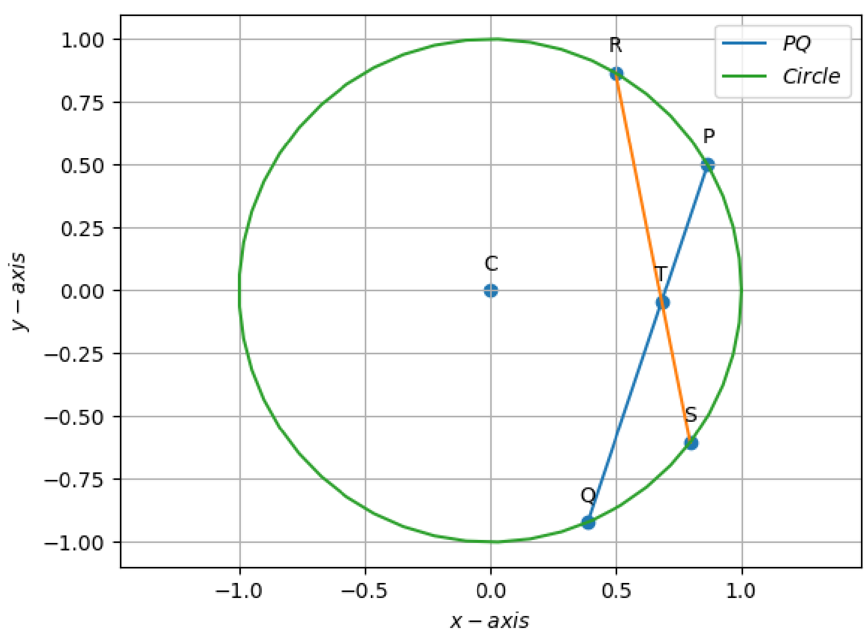


Figure 3.3.2.1: Two equal chords intersecting in a circle

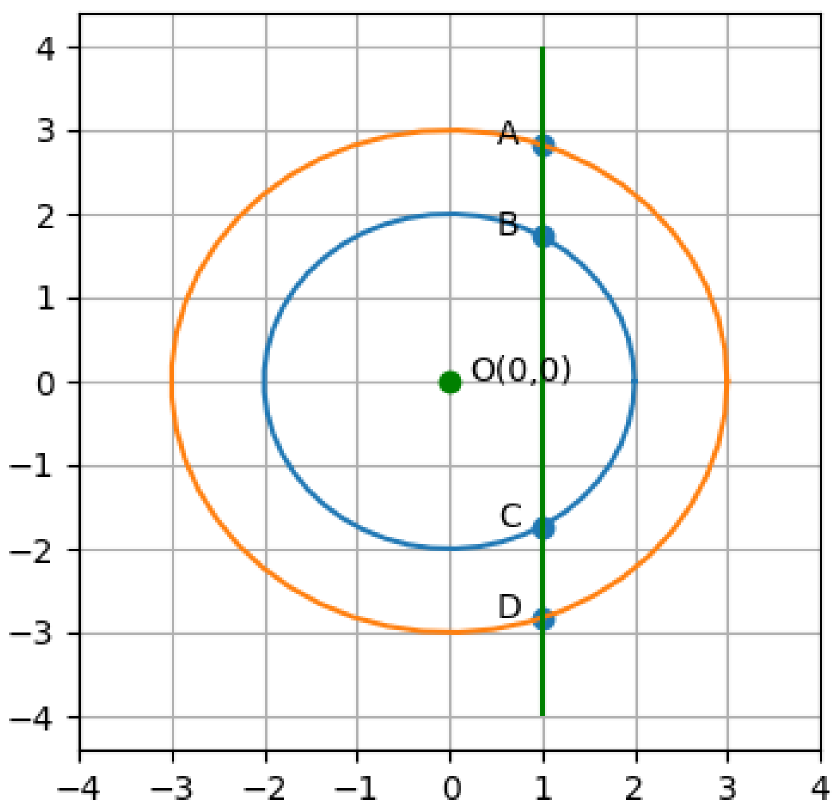


Figure 3.3.3.1:

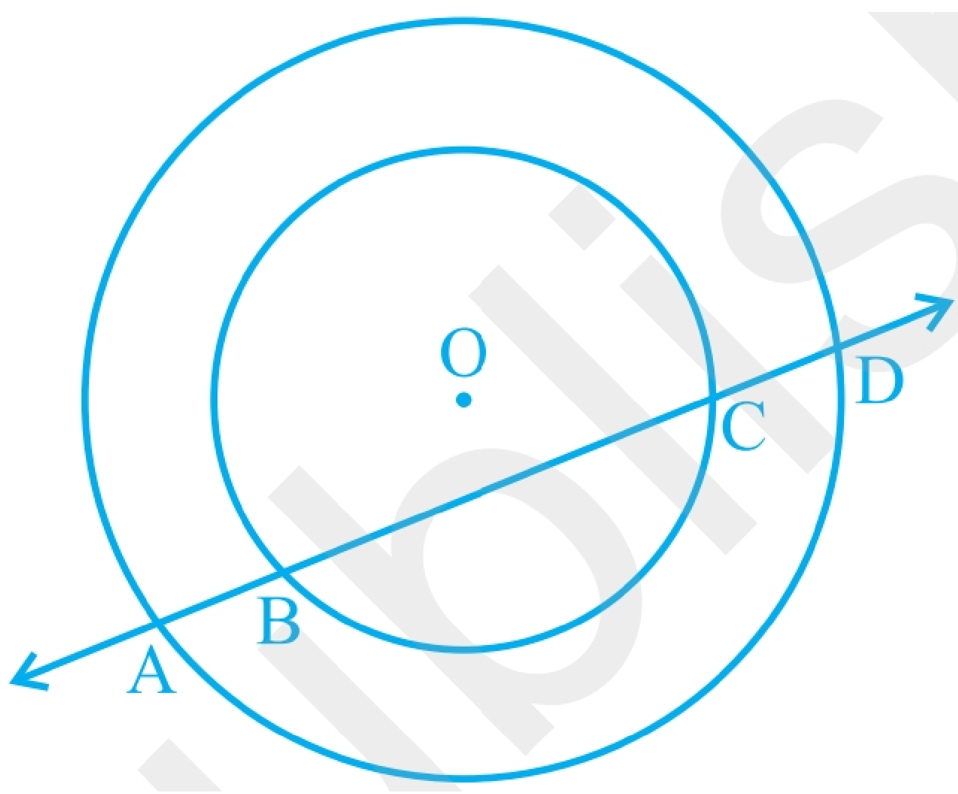


Figure 3.3.4.1:

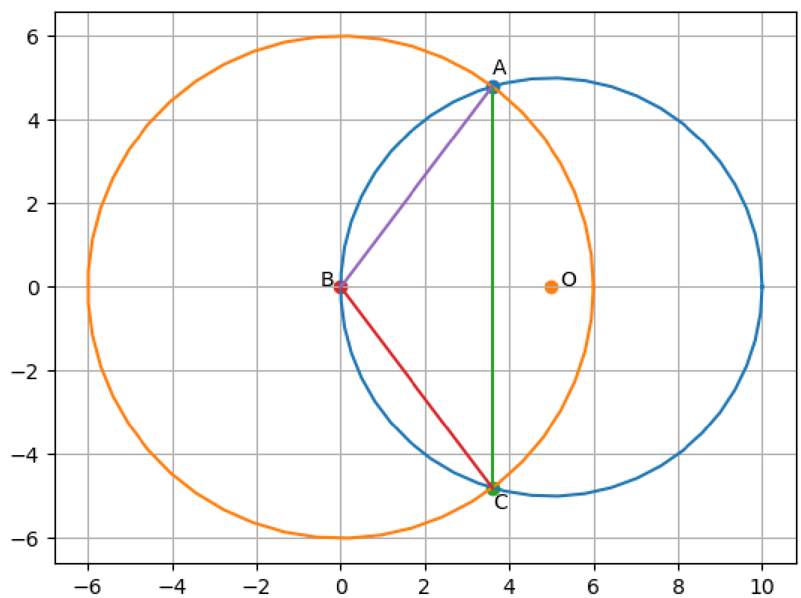


Figure 3.3.5.1:

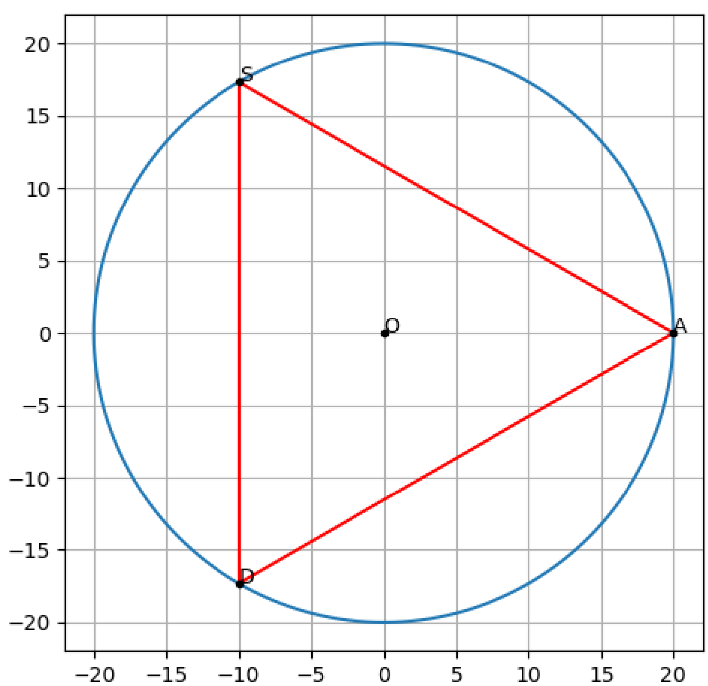
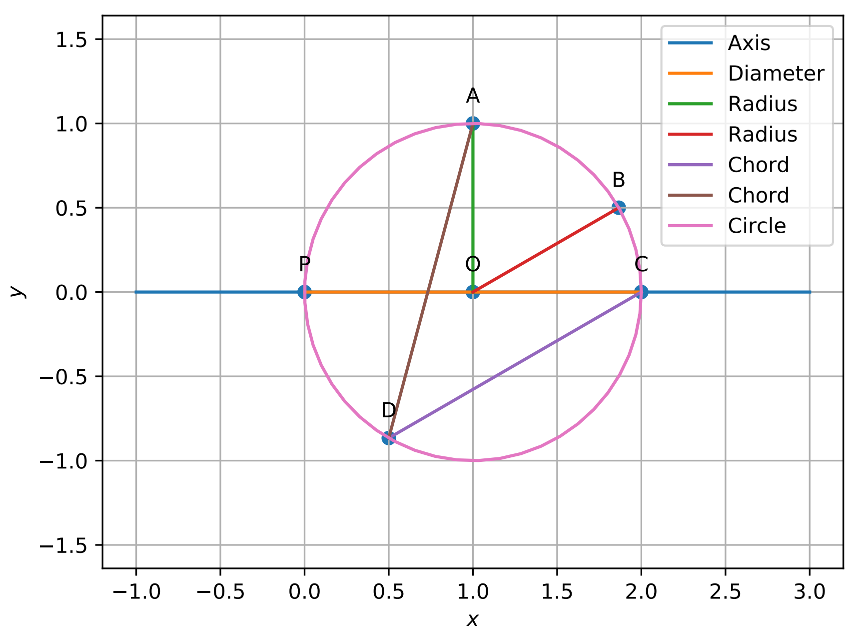


Figure 3.3.7.1: ASD is an equilateral triangle of side $20\sqrt{3}$ m.



Axis
Diameter
Radius
Radius
Chord
Chord
Circle
A
B
P
O
C
D
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
y
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0
2.5
3.0
x

Figure 3.3.8.1:

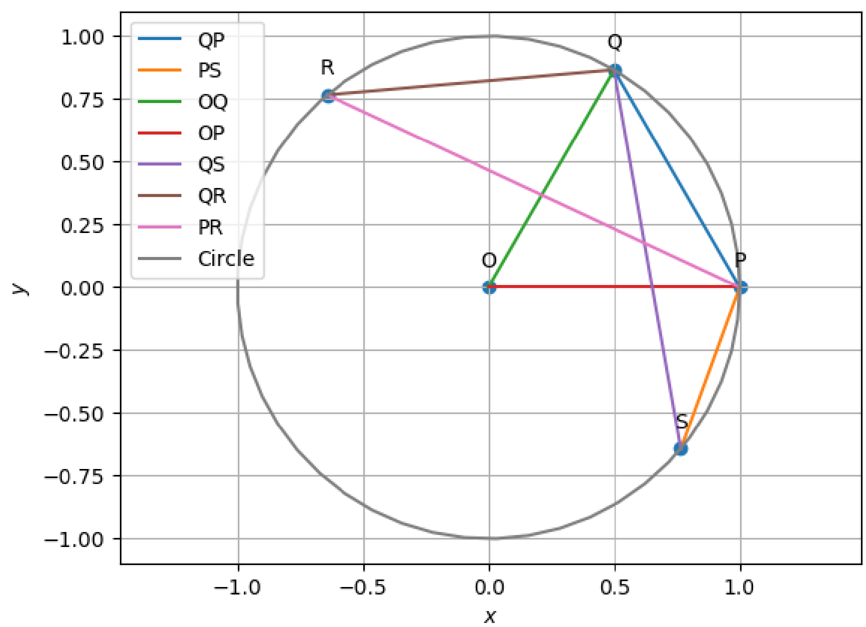

QP
PS
OQ
OP
QS
QR
PR
Circle
R
Q
O
P
S
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
y
−1.0
−0.5
0.0
0.5
1.0
x

Figure 3.3.9.1:

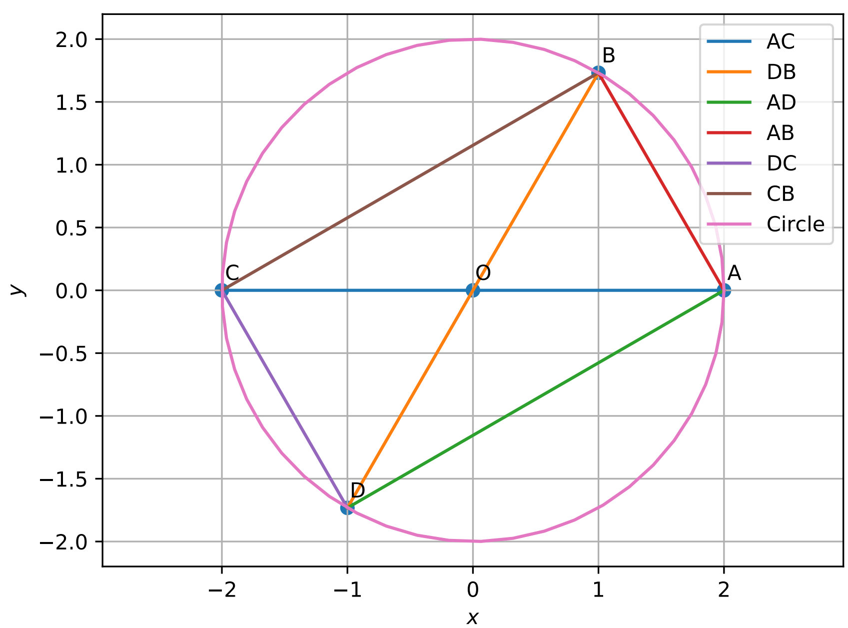


Figure 3.3.11.1:

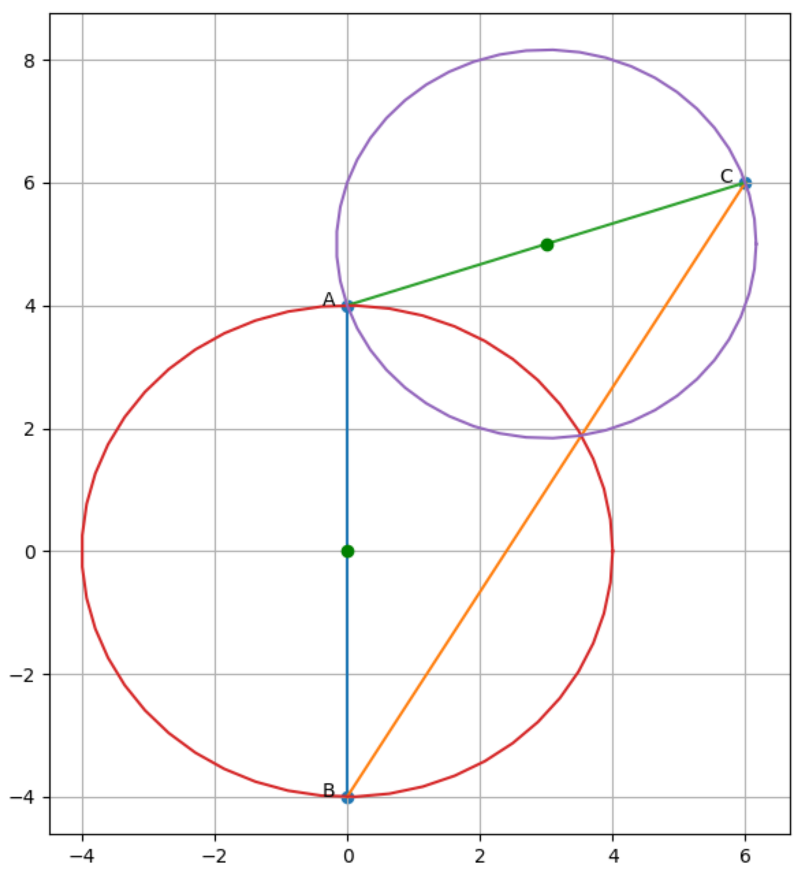


Figure 3.3.12.1: Graph

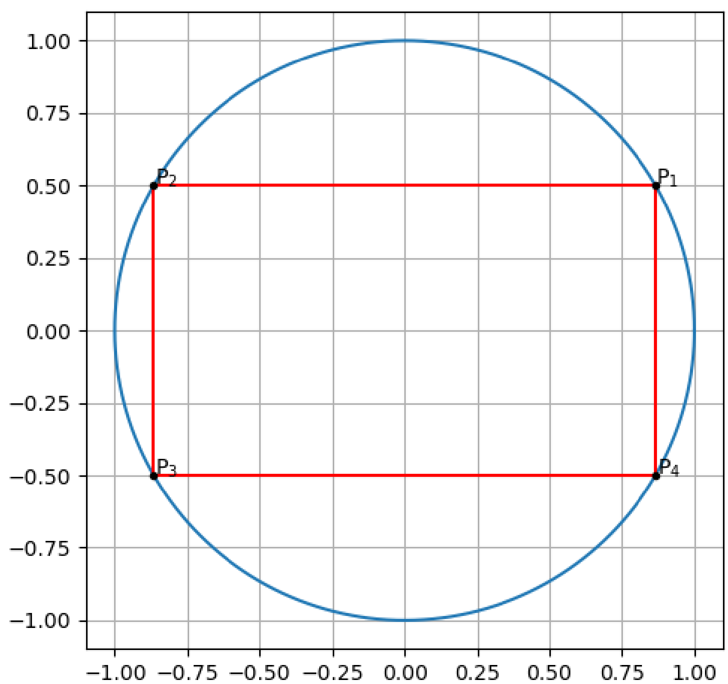


Figure 3.3.13.1: $P_1P_2P_3P_4$ is a rectangle.

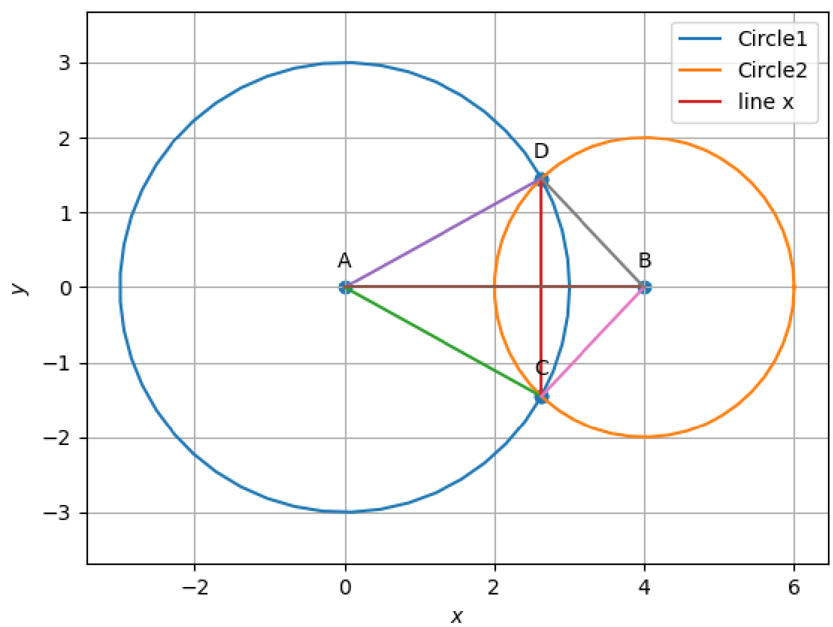


Figure 3.3.14.1:

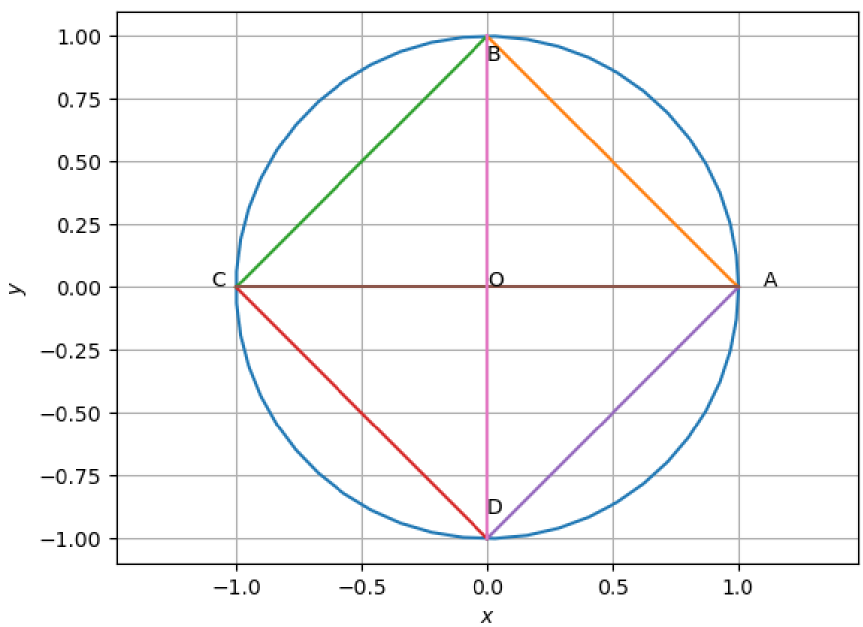


Figure 3.3.15.1: circle

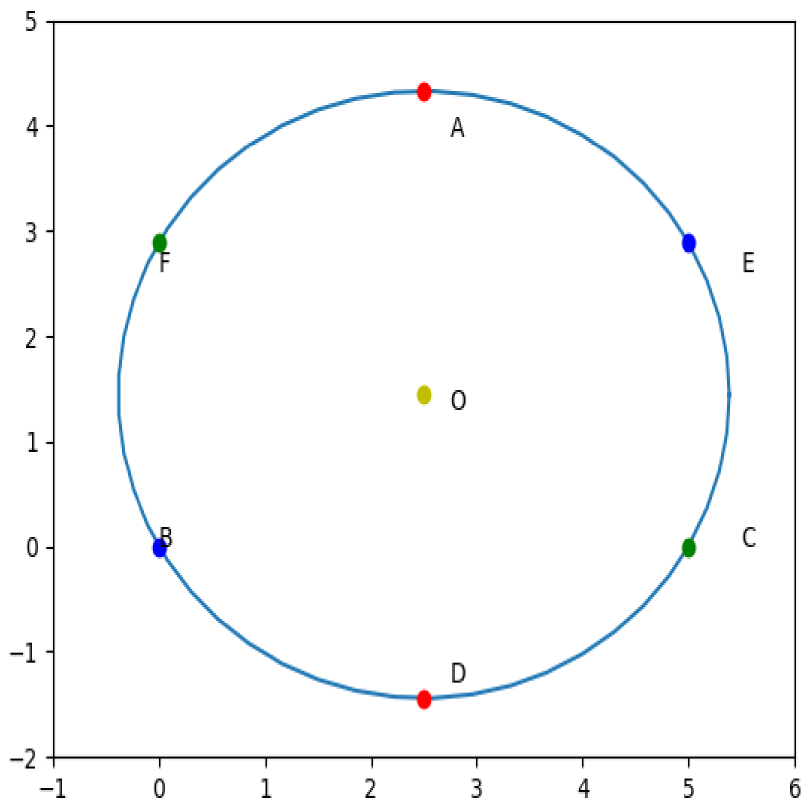


Figure 3.3.16.1: Figure

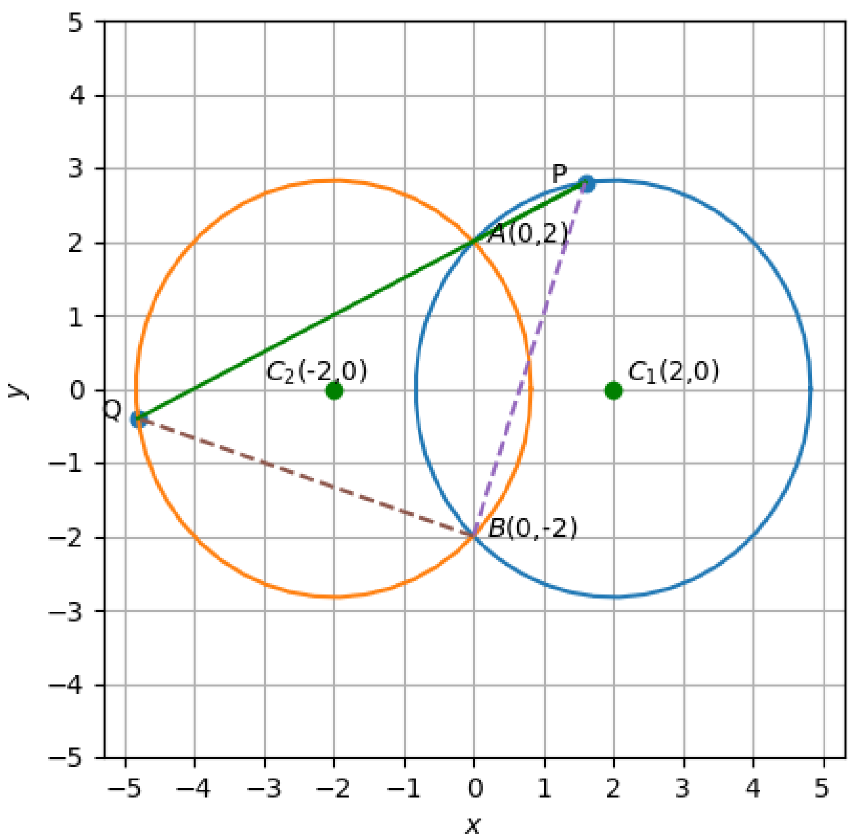


Figure 3.3.17.1:

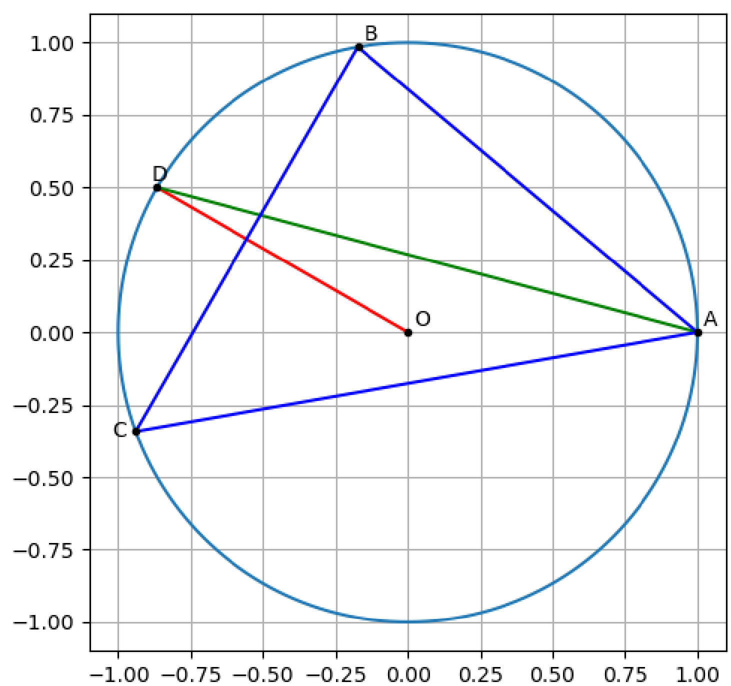


Figure 3.3.18.1: The bisector of $\angle A$ and of BC meet on the circumcircle at D.

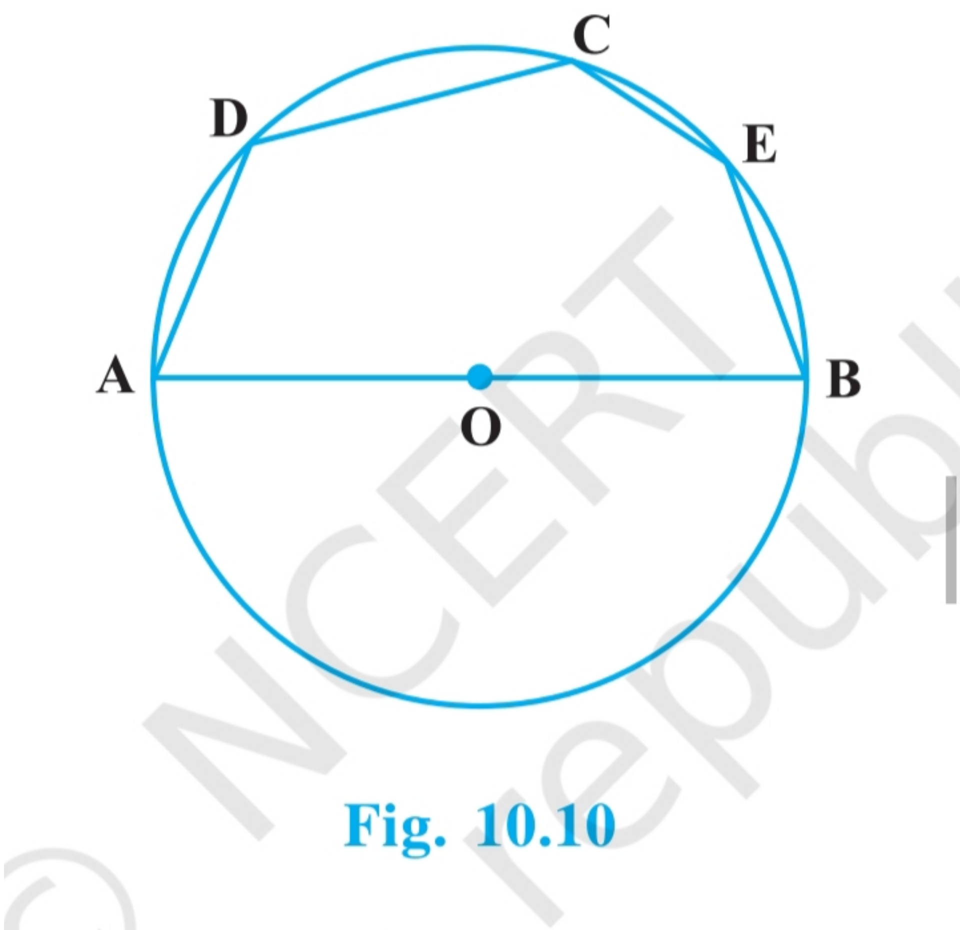


Figure 3.4.20.1:

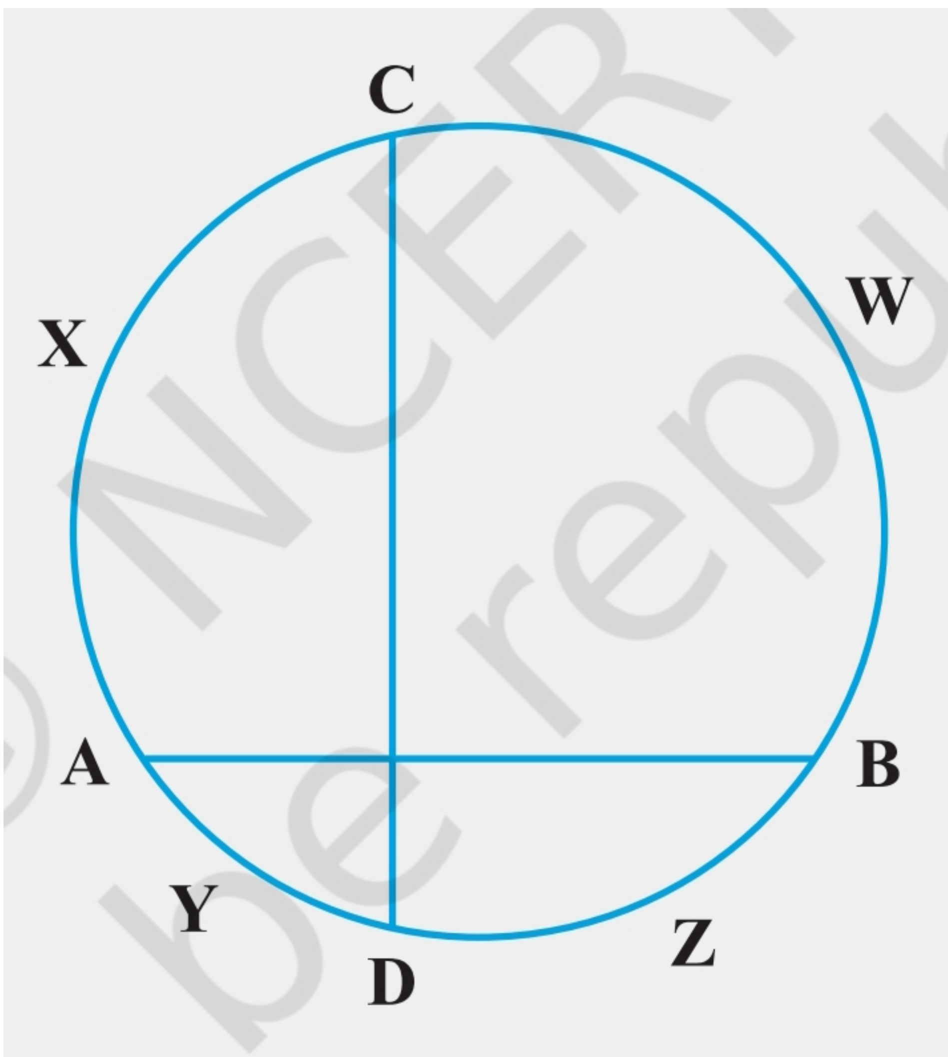


Figure 3.4.26.1:

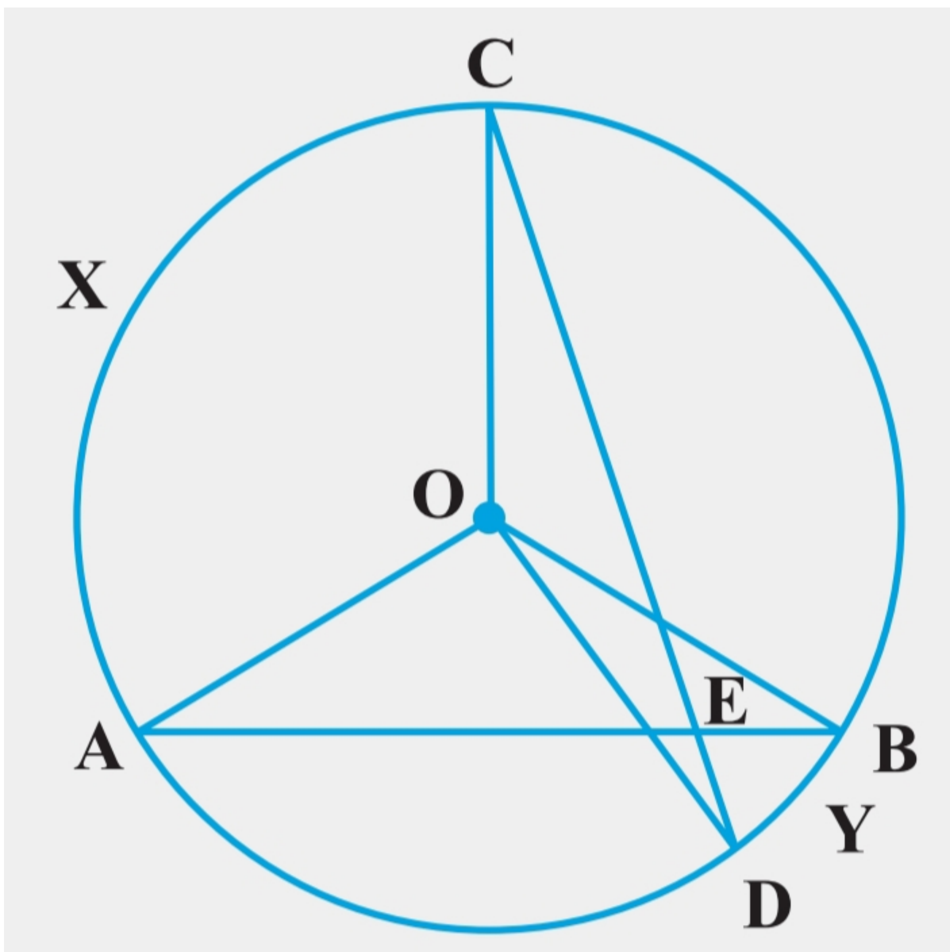


Figure 3.4.28.1:

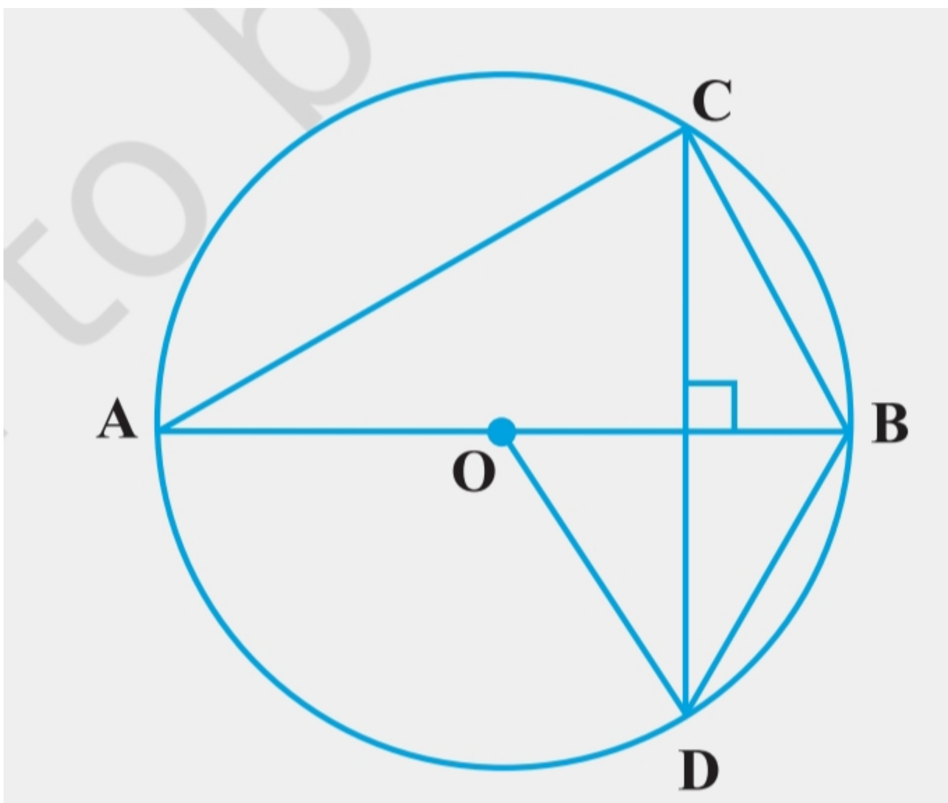


Figure 3.4.33.1:

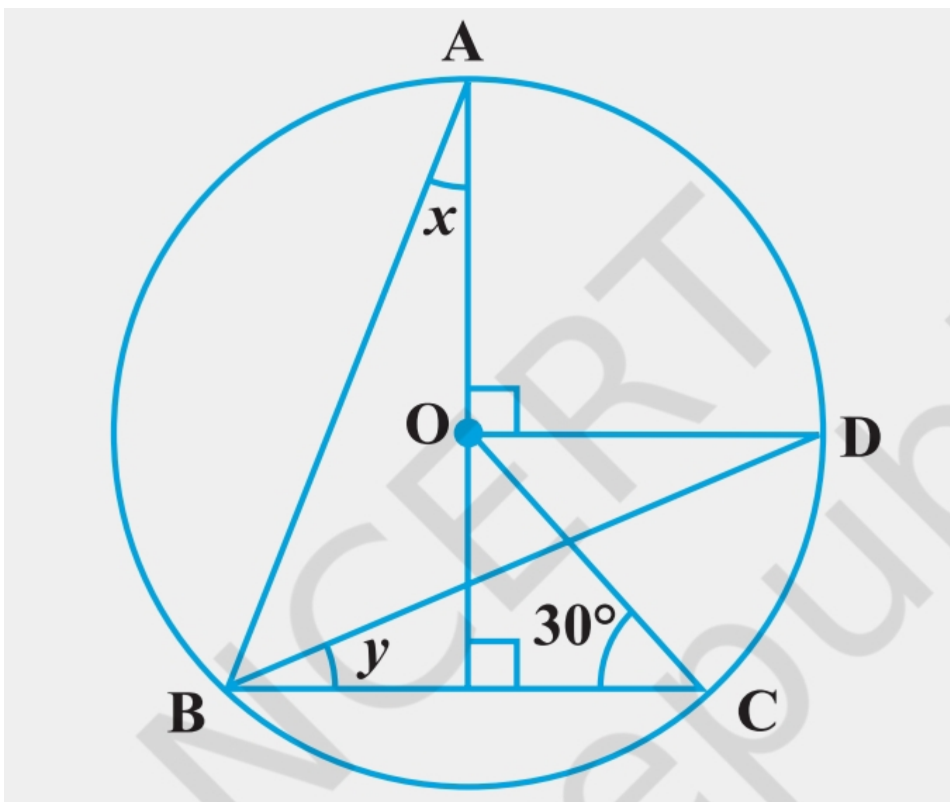


Figure 3.4.34.1:

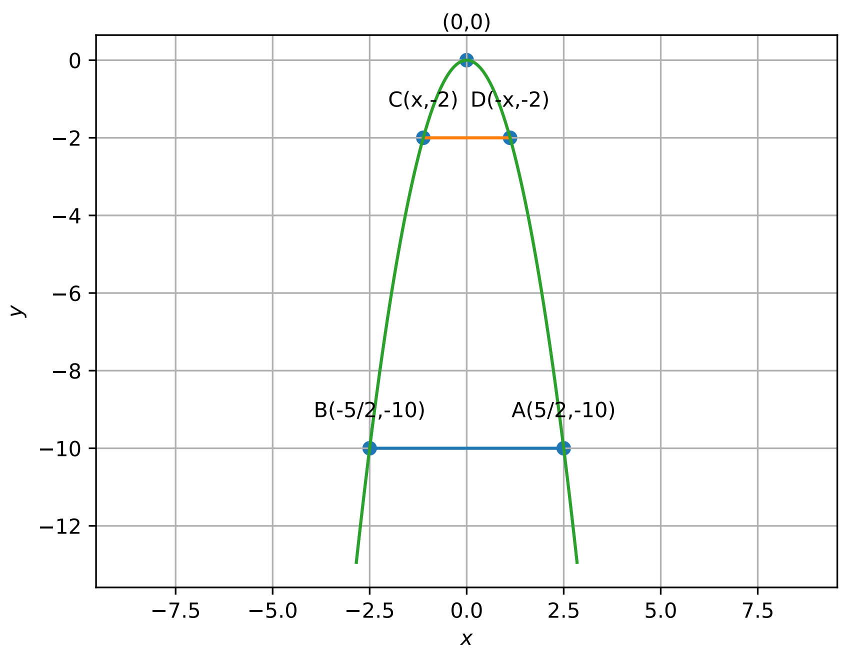


Figure 3.5.1.1:

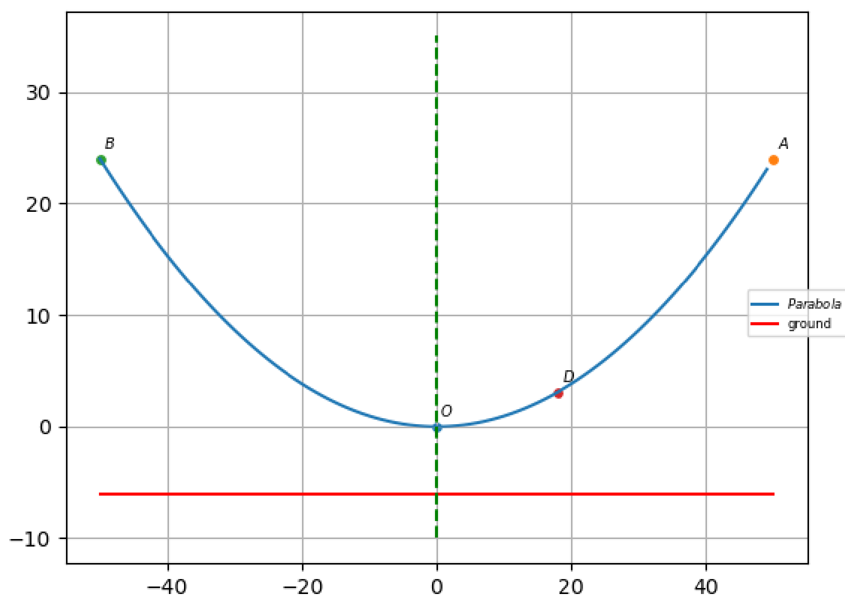


Figure 3.5.2.1: Parabola

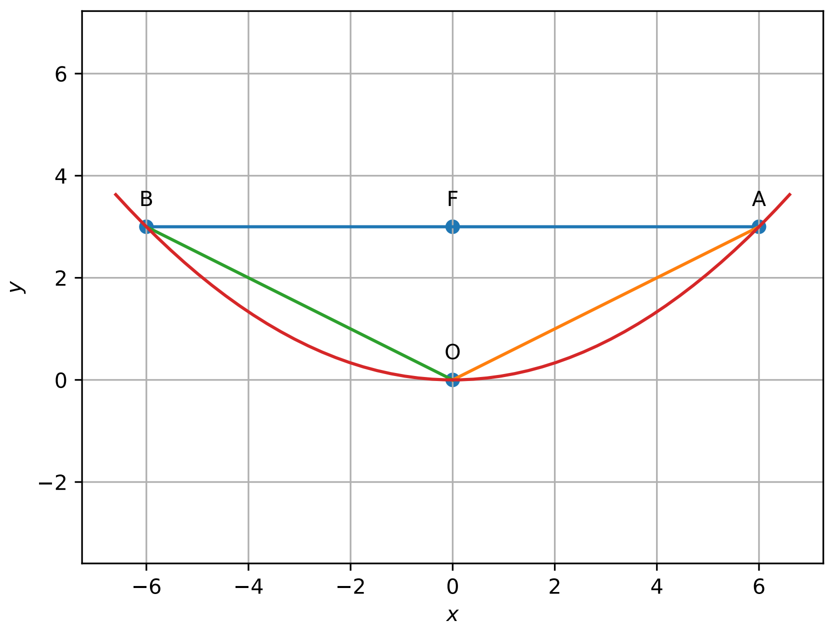


Figure 3.5.3.1: PAB is the triangle whose area is to be found.

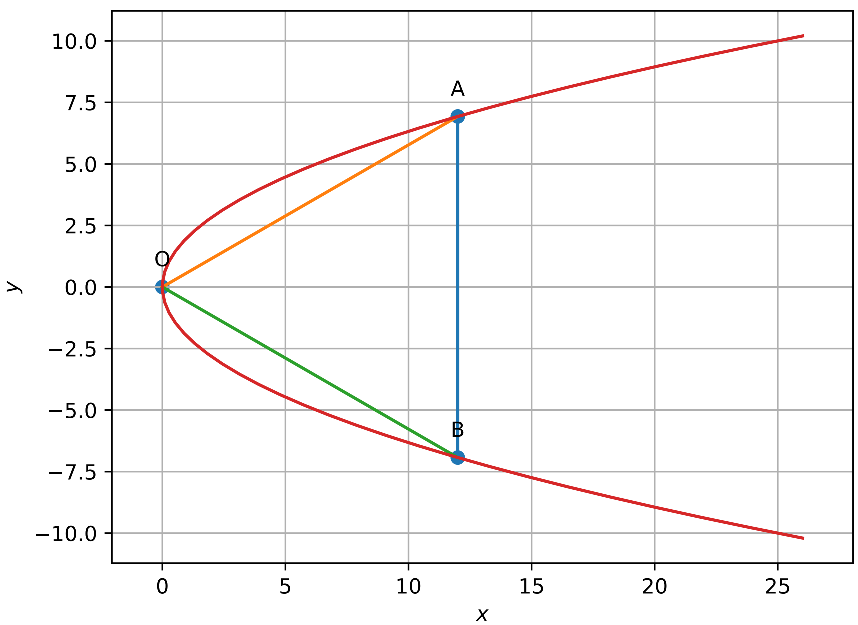
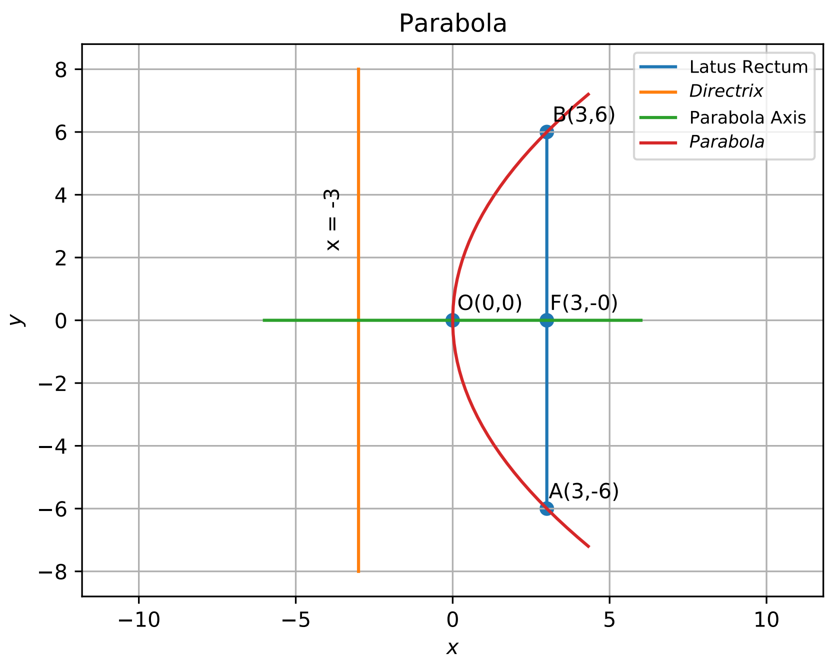


Figure 3.5.4.1:



Parabola
Latus Rectum
Directrix
Parabola Axis
Parabola
B(3,6)
x = -3
O(0,0)
F(3,-0)
A(3,-6)
8
6
4
2
0
−2
−4
−6
−8
y
−10
−5
0
5
10
x

Figure 3.5.5.1:

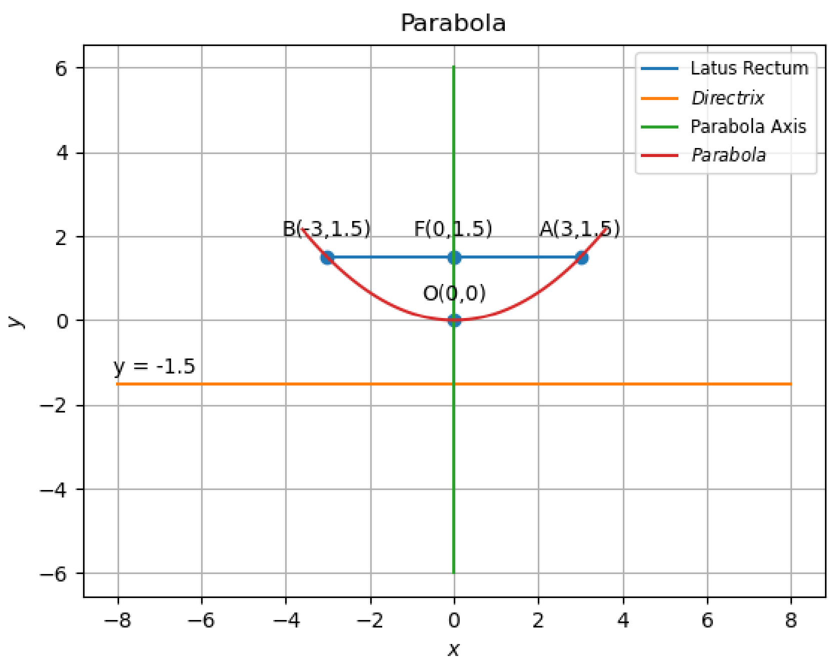

Parabola
Latus Rectum
Directrix
Parabola Axis
Parabola
B(-3,1.5)
F(0,1.5)
A(3,1.5)
O(0,0)
y = -1.5
y
x
6
4
2
0
−2
−4
−6
−8
8

Figure 3.5.6.1:

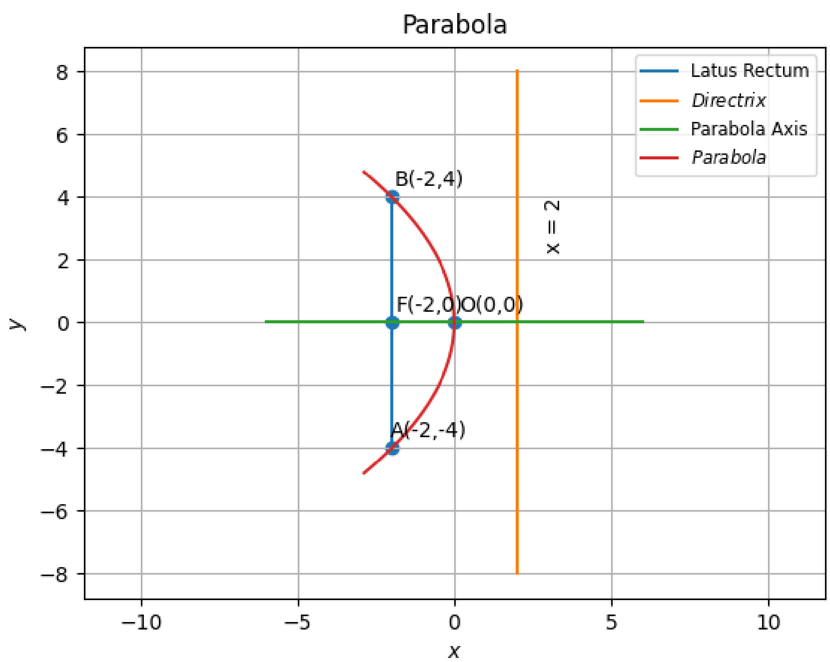


Figure 3.5.7.1: Graph

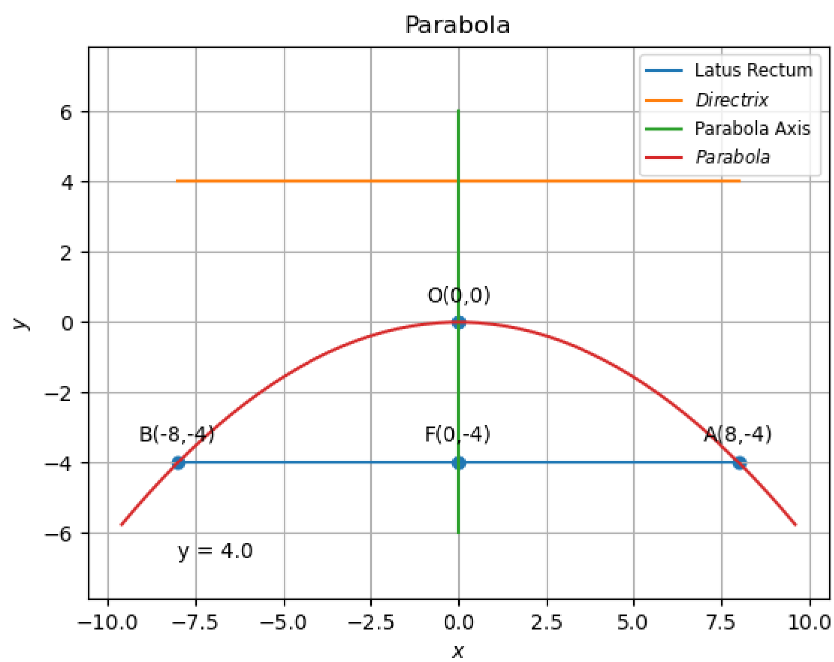
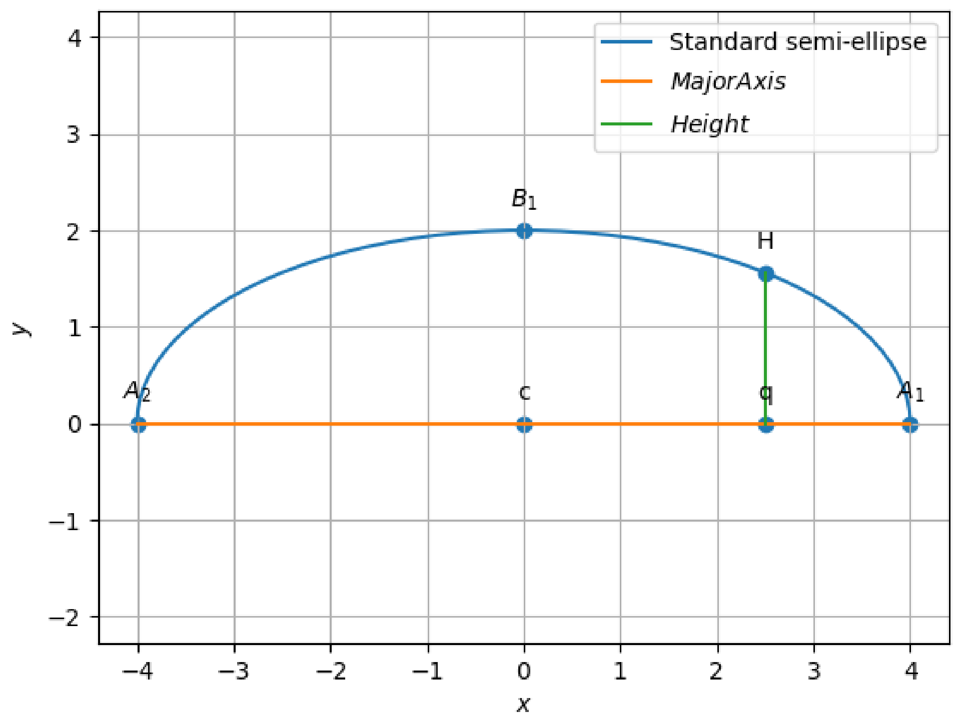

Parabola
Latus Rectum
Directrix
Parabola Axis
Parabola
O(0,0)
B(-8,-4)
F(0,-4)
A(8,-4)
y = 4.0
x
y

Figure 3.5.9.1:



Standard semi-ellipse
MajorAxis
Height
B_1
H
A_2
c
q
A_1
y
x
4
3
2
1
0
−1
−2
−4
−3
−2
−1
0
1
2
3
4

Figure 3.7.1.1:

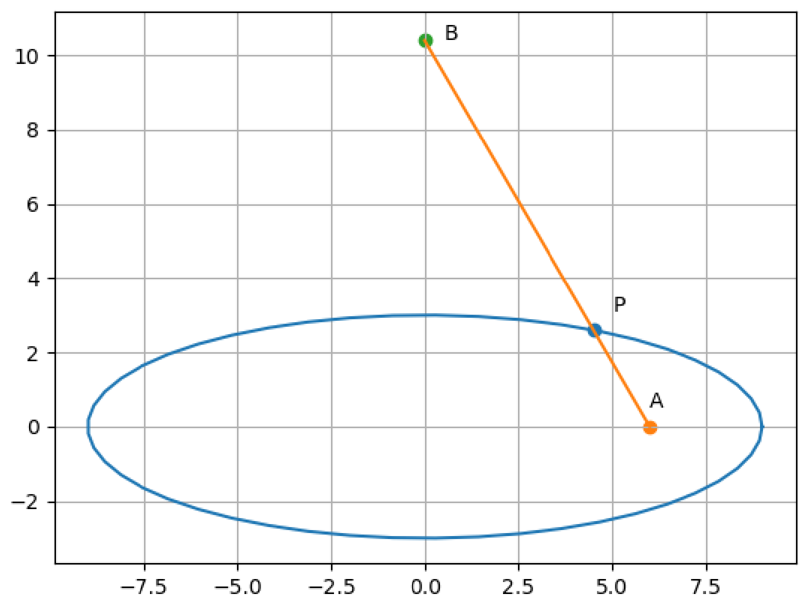
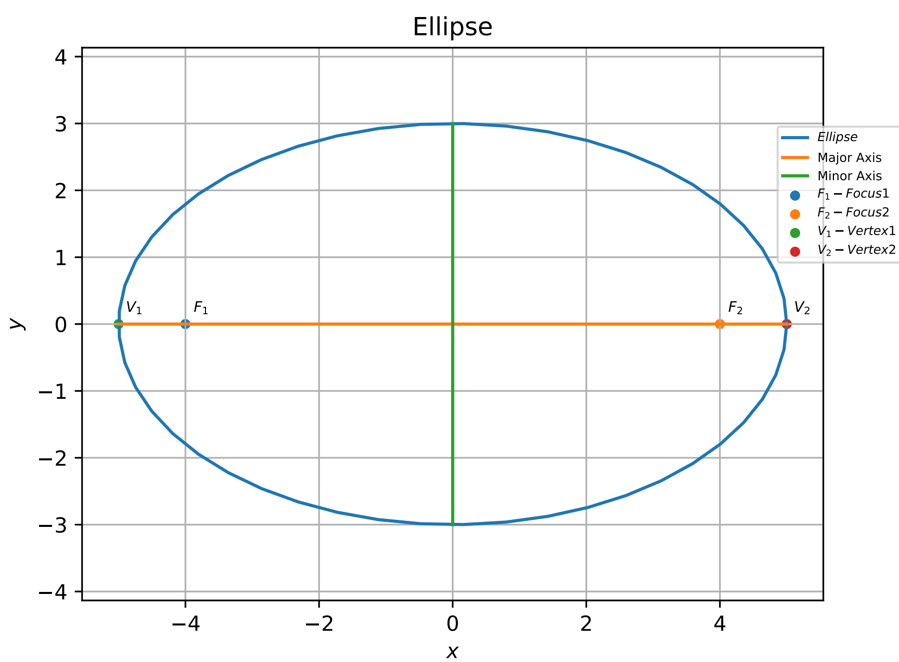
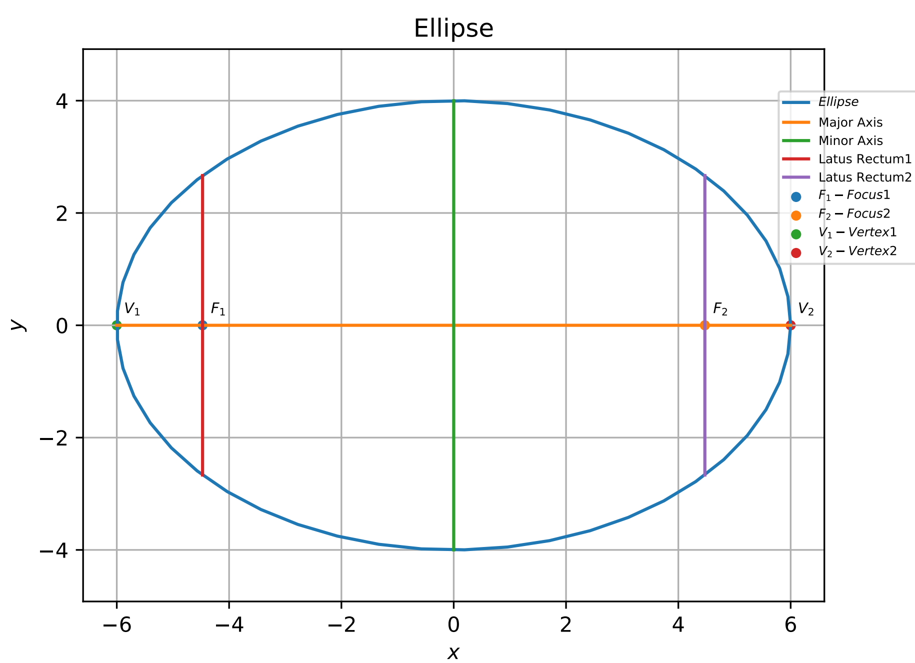


Figure 3.7.2.1:



Ellipse
4
3
2
1
0
−1
−2
−3
−4
y
−4
−2
0
2
4
x
V_1
F_1
F_2
V_2
Ellipse
Major Axis
Minor Axis
$F_1 - Focus1$
$F_2 - Focus2$
$V_1 - Vertex1$
$V_2 - Vertex2$

Figure 3.7.3.1:



Ellipse
Ellipse
Major Axis
Minor Axis
Latus Rectum1
Latus Rectum2
$F_1 - Focus1$
$F_2 - Focus2$
$V_1 - Vertex1$
$V_2 - Vertex2$
V_1
F_1
F_2
V_2
x
y

Figure 3.7.4.1:

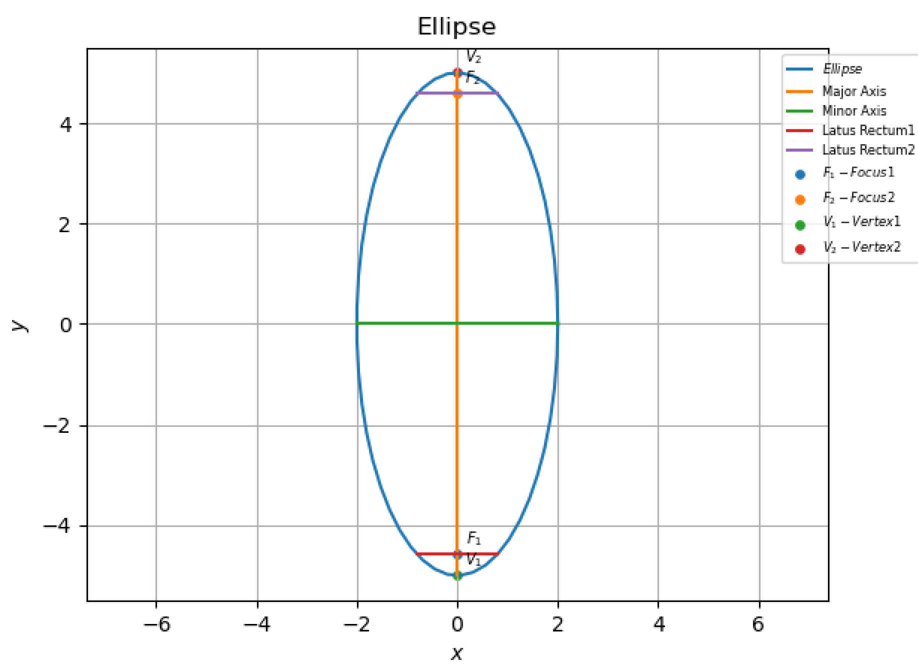
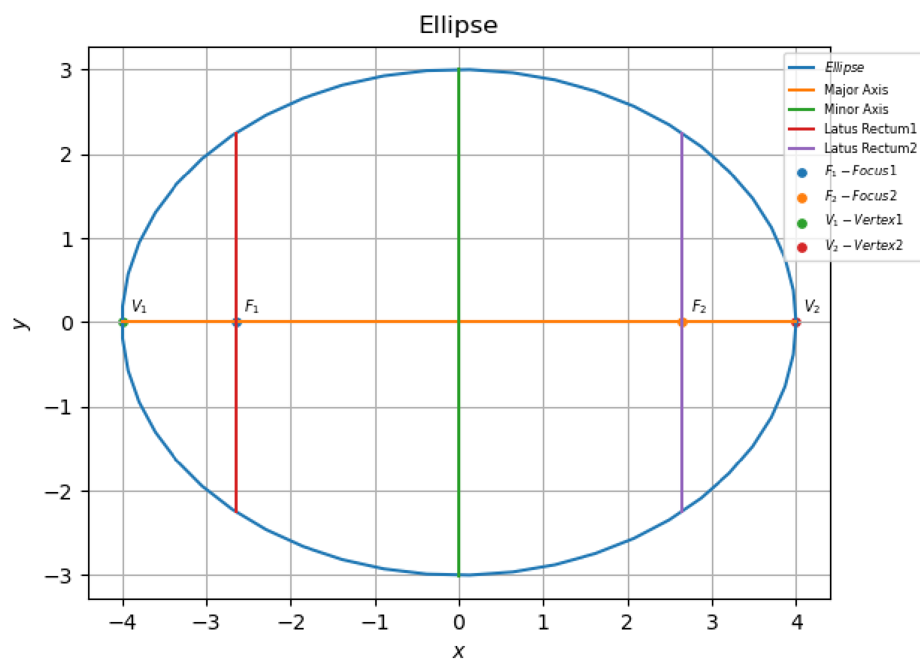

Ellipse
V_2
F_2
F_1
V_1
Ellipse
Major Axis
Minor Axis
Latus Rectum1
Latus Rectum2
$F_1 - Focus1$
$F_2 - Focus2$
$V_1 - Vertex1$
$V_2 - Vertex2$
x
y

Figure 3.7.5.1:



Ellipse
Ellipse
Major Axis
Minor Axis
Latus Rectum1
Latus Rectum2
$F_1 - Focus1$
$F_2 - Focus2$
$V_1 - Vertex1$
$V_2 - Vertex2$
V_1
F_1
F_2
V_2
3
2
1
0
−1
−2
−3
−4
−3
−2
−1
0
1
2
3
4
x
y

Figure 3.7.6.1:

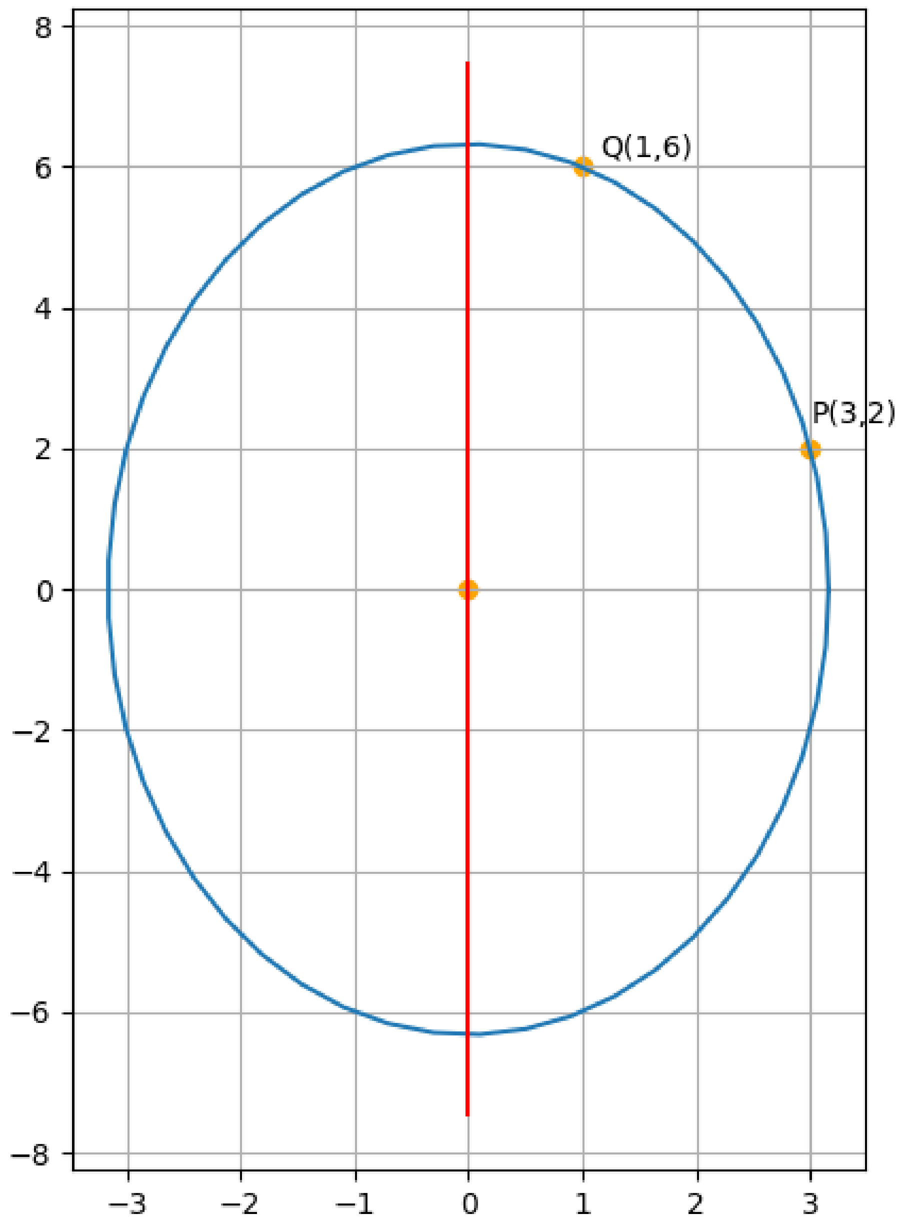


Figure 3.7.22.1: Graph

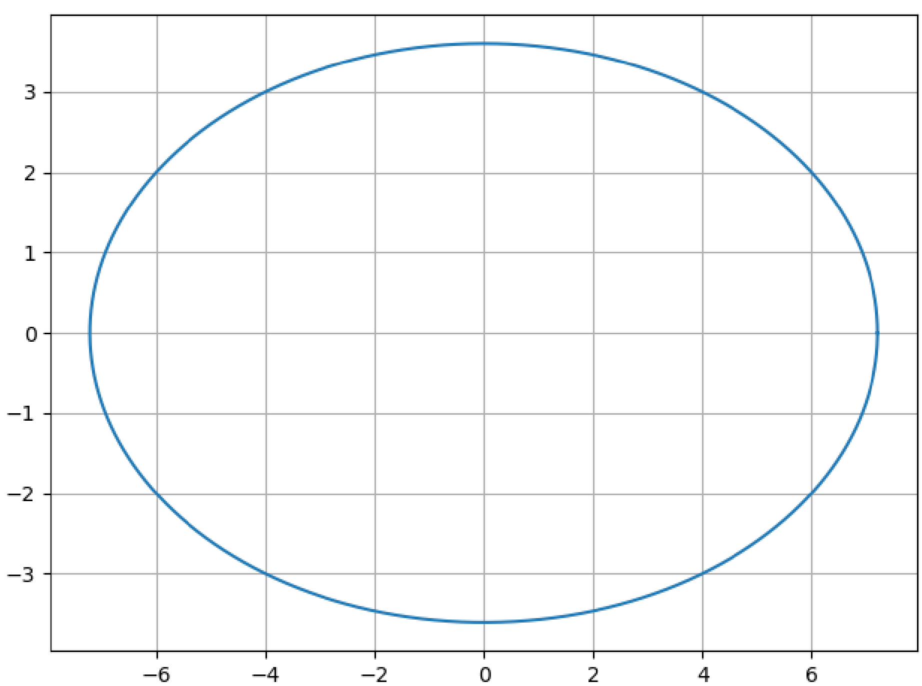
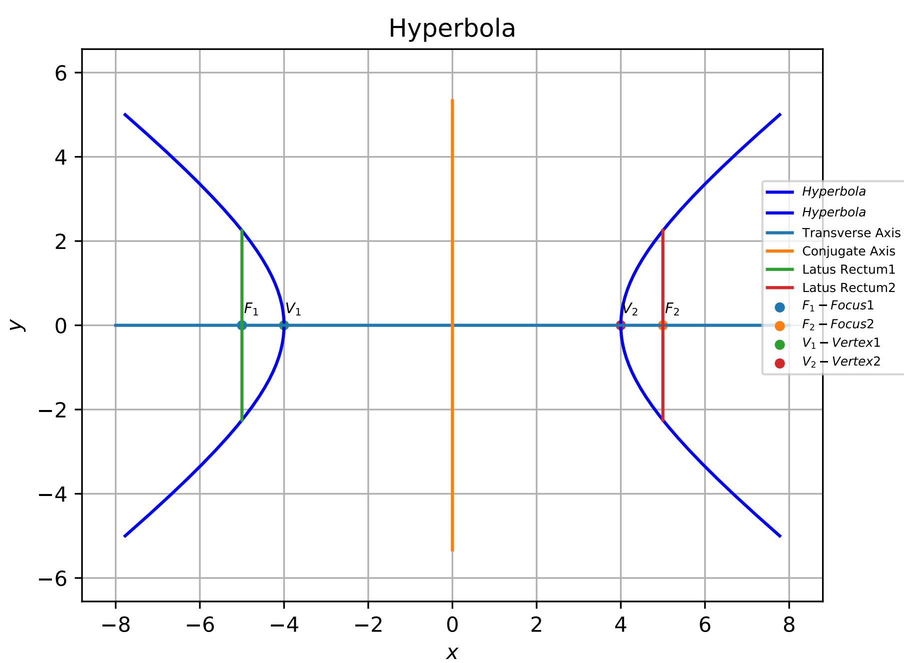


Figure 3.7.23.1: Locus of the required ellipse.



Hyperbola
6
4
2
0
−2
−4
−6
y
−8
−6
−4
−2
0
2
4
6
8
x
F_1
V_1
V_2
F_2
Hyperbola
Hyperbola
Transverse Axis
Conjugate Axis
Latus Rectum1
Latus Rectum2
$F_1 - Focus1$
$F_2 - Focus2$
$V_1 - Vertex1$
$V_2 - Vertex2$

Figure 3.9.1.1:

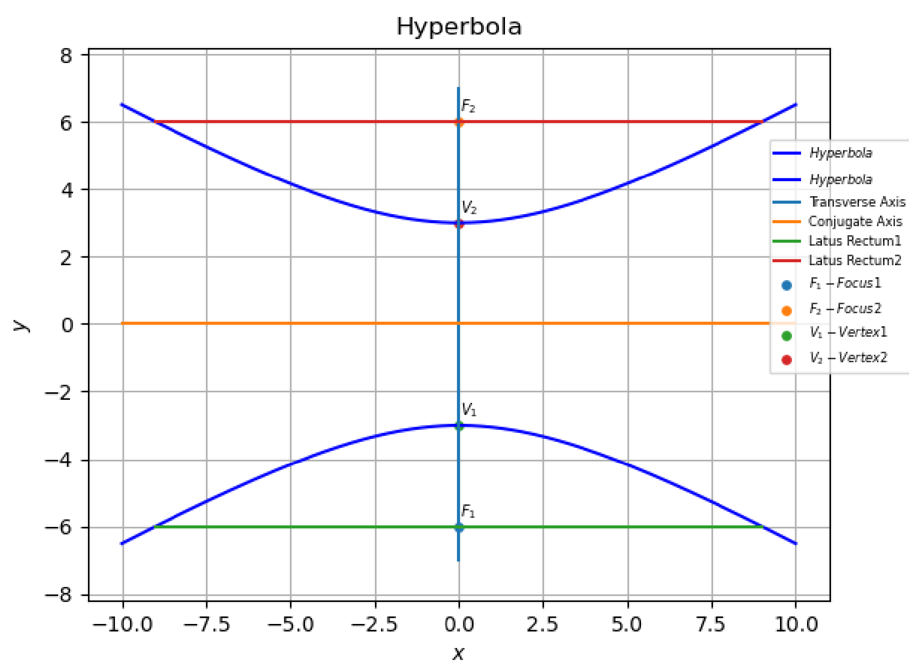
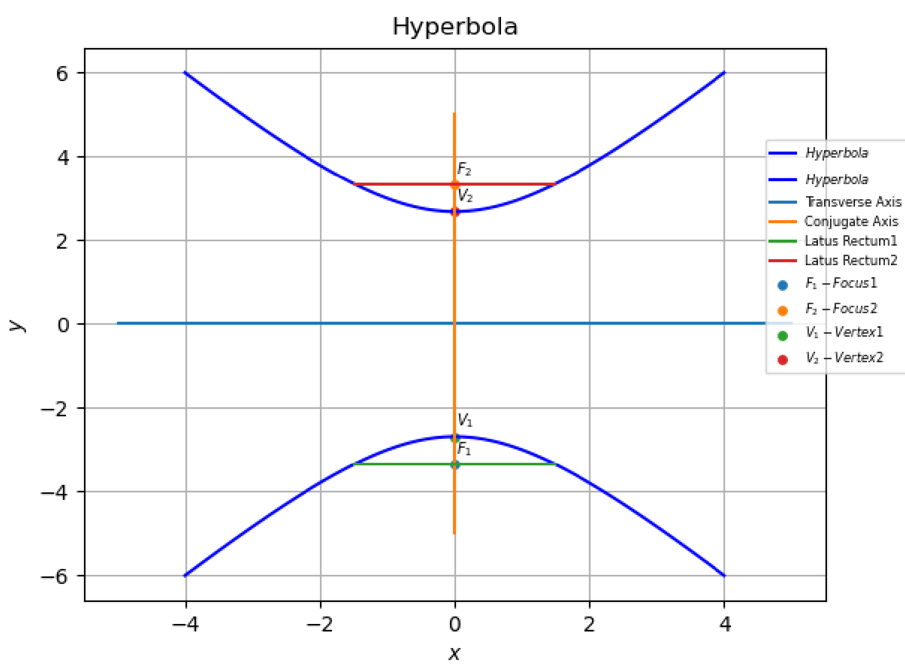

Hyperbola
8
6
4
2
0
−2
−4
−6
−8
y
−10.0
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0
x
F₂
V₂
V₁
F₁
Hyperbola
Hyperbola
Transverse Axis
Conjugate Axis
Latus Rectum1
Latus Rectum2
F₁ − Focus1
F₂ − Focus2
V₁ − Vertex1
V₂ − Vertex2

Figure 3.9.2.1:



Hyperbola
F_2
V_2
V_1
F_1
Hyperbola
Hyperbola
Transverse Axis
Conjugate Axis
Latus Rectum1
Latus Rectum2
$F_1 - Focus1$
$F_2 - Focus2$
$V_1 - Vertex1$
$V_2 - Vertex2$
x
y

Figure 3.9.3.1:

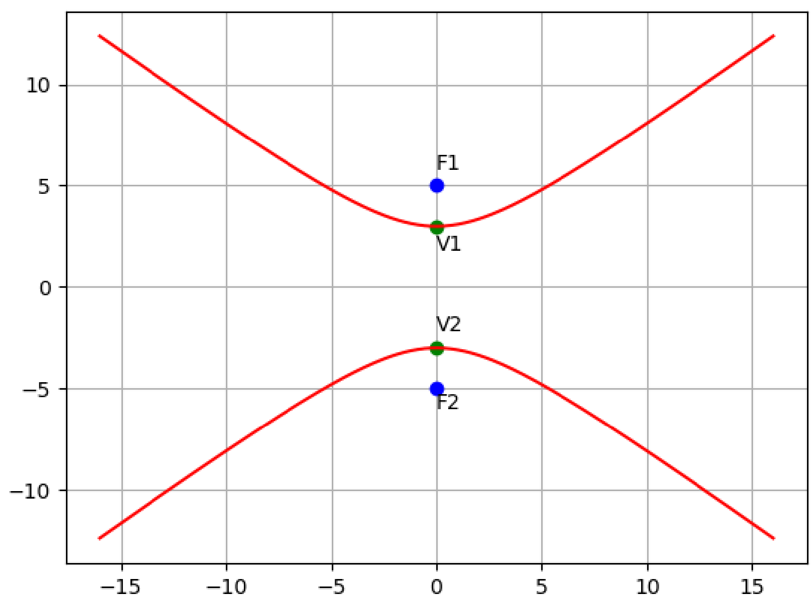
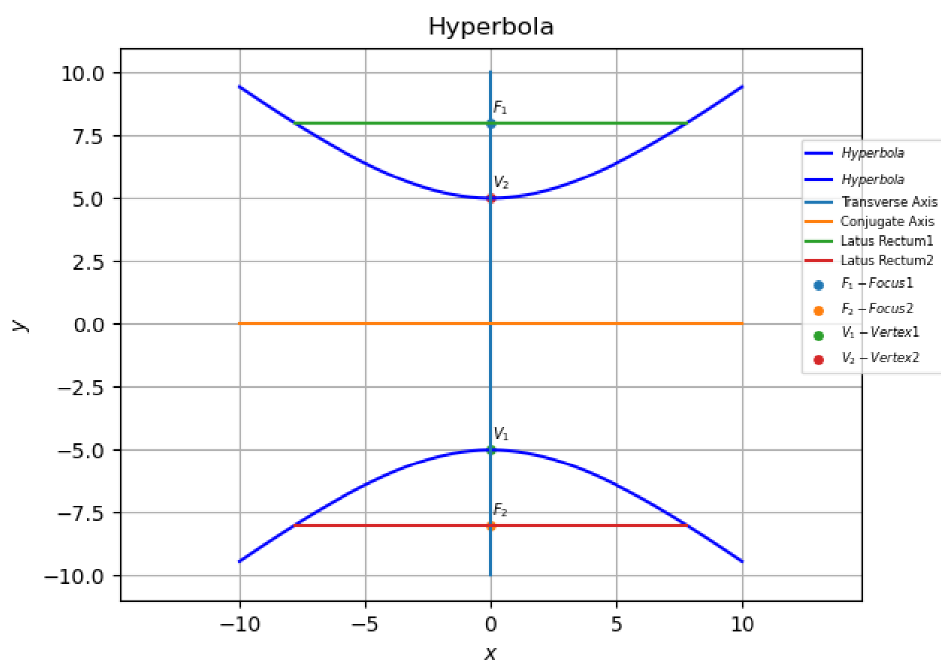


Figure 3.9.5.1: Figure 1



Hyperbola
10.0
7.5
5.0
2.5
0.0
−2.5
−5.0
−7.5
−10.0
y
−10
−5
0
5
10
x
F1
V2
V1
F2
Hyperbola
Hyperbola
Transverse Axis
Conjugate Axis
Latus Rectum1
Latus Rectum2
F1 − Focus1
F2 − Focus2
V1 − Vertex1
V2 − Vertex2

Figure 3.9.5.2:

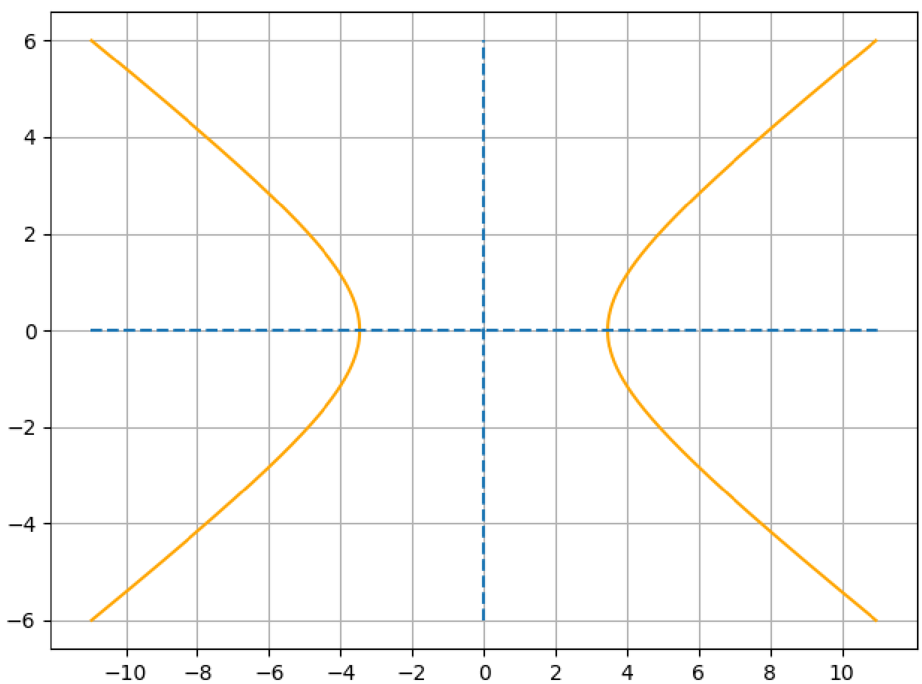


Figure 3.9.6.1: Graph

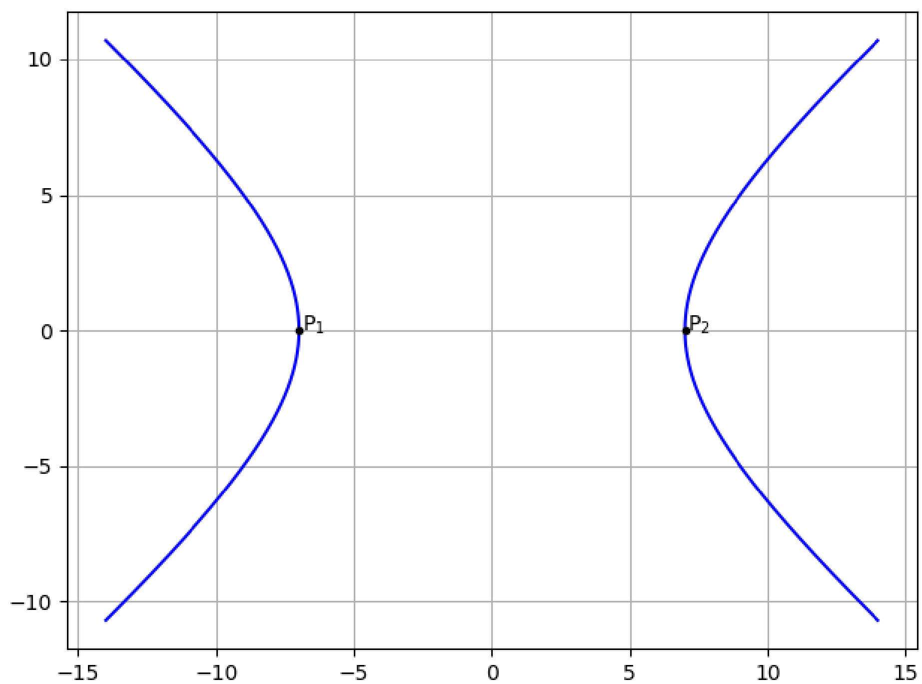


Figure 3.9.7.1: Locus of the required hyperbola.

Chapter 4

Intersection of Conics

4.1. Chords

4.1.1 Find the area of the region bounded by the curve $y^2 = x$ and the lines $x = 1$ and $x = 4$ and the axis in the first quadrant.

Solution:

The parameters of the conic are

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u} = -\frac{1}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, f = 0 \tag{4.1.1.1}$$

For the line $x - 1 = 0$, the parameters are

$$\mathbf{q}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{m}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{4.1.1.2}$$

Substituting from the above in (G.3.1.3),

$$\mu_i = 1, -1 \tag{4.1.1.3}$$

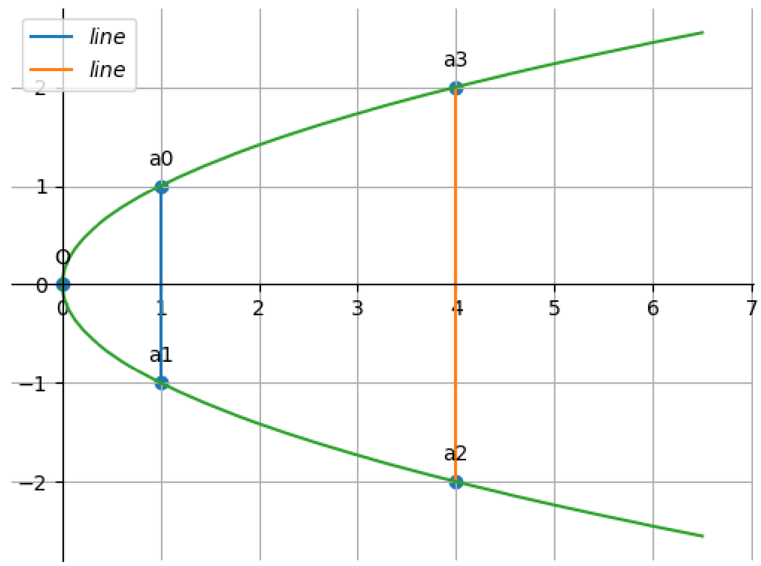


Figure 4.1.1.1:

yilelding the points of intersection

$$\mathbf{a}_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \mathbf{a}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{4.1.1.4}$$

Similarly, for the line $x - 4 = 0$

$$\mathbf{q_1} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \mathbf{m_1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{4.1.1.5}$$

yielding

$$\mu_i = 2, -2 \tag{4.1.1.6}$$

from which, the points of intersection are

$$\mathbf{a_3} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \mathbf{a_2} = \begin{pmatrix} 4 \\ -2 \end{pmatrix} \tag{4.1.1.7}$$

Thus, the area of the parabola in between the lines $x = 1$ and $x = 4$ is given by

$$\int_0^4 \sqrt{x}\, dx - \int_0^1 \sqrt{x}\, dx = 14/3 \tag{4.1.1.8}$$

4.1.2 Find the area of the region bounded by the curve $y^2 = 9x$ and the lines $x = 2$ and $x = 4$ and the axis in the first quadrant.

Solution: The parameters of the conic are

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u} = \frac{9}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, f = 0. \tag{4.1.2.1}$$

The parameters of the line $x - 2 = 0$ are

$$\mathbf{q_2} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \mathbf{m_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{4.1.2.2}$$

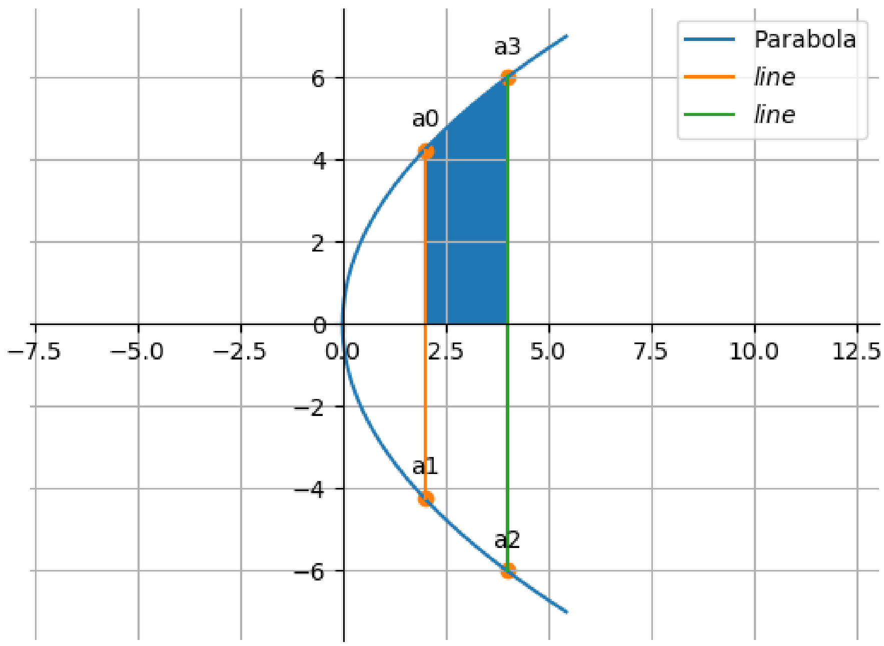


Figure 4.1.2.1:

Substituting in (G.3.1.3),

$$\mu_i = \pm 3\sqrt{2} \tag{4.1.2.3}$$

yielding

$$\mathbf{a_0} = \begin{pmatrix} 2 \\ 3\sqrt{2} \end{pmatrix}, \mathbf{a_1} = \begin{pmatrix} 2 \\ -3\sqrt{2} \end{pmatrix}. \tag{4.1.2.4}$$

Similarly, for the line $x - 4 = 0$,

$$\mathbf{q_1} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \mathbf{m_1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{4.1.2.5}$$

yielding

$$\mu_i = \pm 6. \tag{4.1.2.6}$$

Thus,

$$\mathbf{a_3} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}, \mathbf{a_2} = \begin{pmatrix} 4 \\ -6 \end{pmatrix} \tag{4.1.2.7}$$

and the desired area of the parabola is

$$\int_0^4 3\sqrt{x}\,dx - \int_0^2 3\sqrt{x}\,dx = 16 - 4\sqrt{2} \tag{4.1.2.8}$$

4.1.3 Find the area of the region bounded by $x^2 = 4y$, $y = 2$, $y = 4$ and the y-axis in the first quadrant.

4.1.4 Find the area of the region bounded by the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$

4.1.5 Find the area of the region bounded by the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$

4.1.6 Find the area of the region in the first quadrant enclosed by the x-axis, line $x = \sqrt{3}y$ and circle $x^2 + y^2 = 4$.

Solution: From the given information, the parameters of the circle

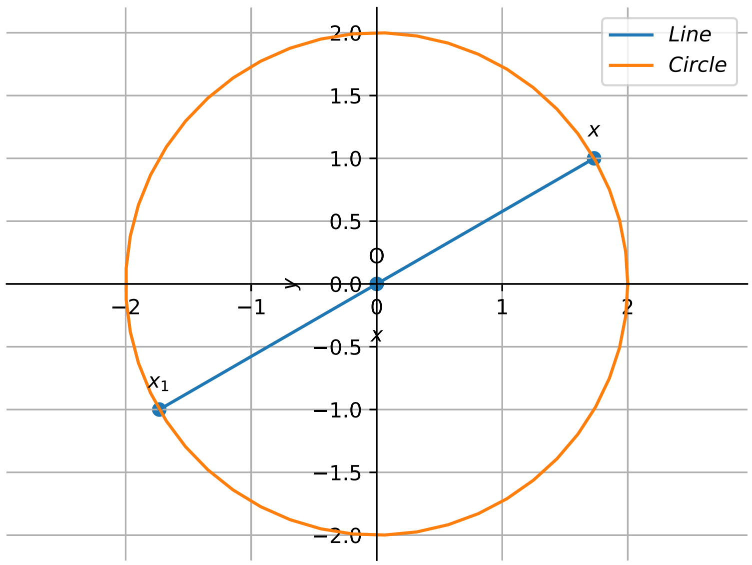


Figure 4.1.6.1:

and line are

$$f = -4, \mathbf{u} = \mathbf{0}, \mathbf{V} = \mathbf{I}, \mathbf{m} = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}, \mathbf{h} = \mathbf{0} \tag{4.1.6.1}$$

Substituting the above parameters in (G.3.1.3),

$$\mu = \sqrt{3} \tag{4.1.6.2}$$

yielding the desired point of intersection as

$$\mathbf{x} = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} \tag{4.1.6.3}$$

From (4.1.6.1), the angle between the given line and the x axis is

$$\theta = 30^\circ \tag{4.1.6.4}$$

and the area of the sector is

$$\frac{\theta}{360}\pi r^2 = \frac{\pi}{3} \tag{4.1.6.5}$$

4.1.7 Find the area of the smaller part of the circle $x^2 + y^2 = a^2$ cut off by the line x=$\frac{a}{\sqrt{2}}$.

Solution: The given circle can be expressed as a conic with parameters

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u} = 0, f = -a^2 \tag{4.1.7.1}$$

The given line parameters are

$$\mathbf{h} = \begin{pmatrix} \frac{a}{\sqrt{2}} \\ 0 \end{pmatrix}, \mathbf{m} = \mathbf{e}_2. \tag{4.1.7.2}$$

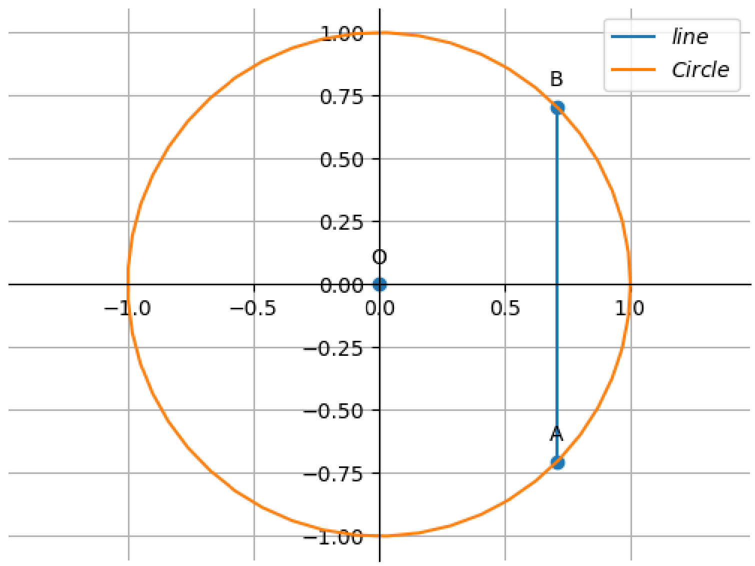


Figure 4.1.7.1:

Substituting the above in (G.3.1.3),

$$\mu = \pm \frac{a}{\sqrt{2}} \tag{4.1.7.3}$$

yielding the points of intersection of the line with circle as

$$\mathbf{A} = \begin{pmatrix} \frac{a}{\sqrt{2}} \\ -\frac{a}{\sqrt{2}} \end{pmatrix}, \mathbf{B} = \begin{pmatrix} \frac{a}{\sqrt{2}} \\ \frac{a}{\sqrt{2}} \end{pmatrix} \tag{4.1.7.4}$$

From Fig. 4.1.7.1, the total area of the portion is given by

$$ar(APQ) = 2ar(APR) \tag{4.1.7.5}$$

$$= 2\int_0^{\frac{a}{\sqrt{2}}} \sqrt{a^2 - x^2}\, dx \tag{4.1.7.6}$$

$$= \frac{a^2}{2}\left(1 + \frac{\pi}{2}\right) \tag{4.1.7.7}$$

4.1.8 The area between $x = y^2$ and $x = 4$ is divided into two equal parts by the line $x = a$, find the value of a.

Solution: The given conic parameters are

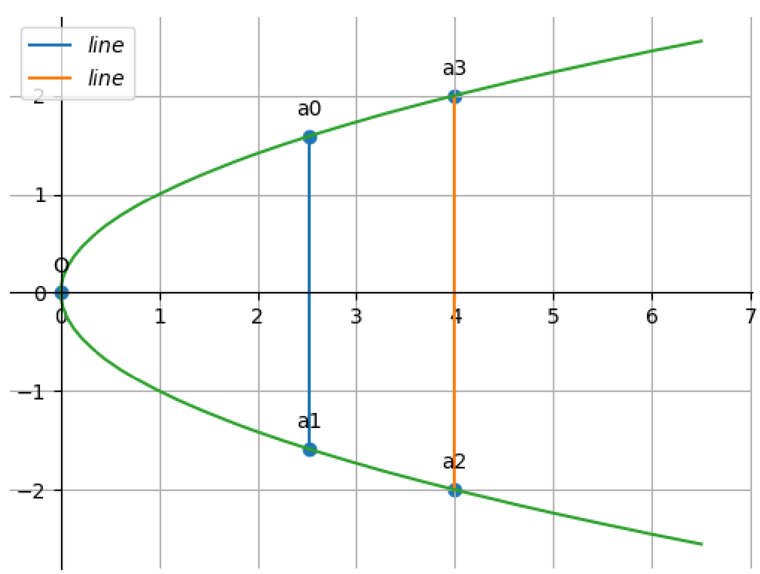


Figure 4.1.8.1:

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u} = -\frac{1}{2}\mathbf{e}_1 f = 0 \tag{4.1.8.1}$$

The parameters of the lines are

$$\mathbf{q}_2 = \begin{pmatrix} a \\ 0 \end{pmatrix}, \mathbf{m}_2 = \mathbf{e}_2 \tag{4.1.8.2}$$

Substituting the above values in (G.3.1.3),

$$\mu_i = a, -a \tag{4.1.8.3}$$

yielding the points of intersection as

$$\mathbf{a_0} = \begin{pmatrix} a \\ a \end{pmatrix}, \mathbf{a_1} = \begin{pmatrix} a \\ -a \end{pmatrix} \tag{4.1.8.4}$$

Similarly, for the line $x - 4 = 0$,

$$\mathbf{q_1} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \mathbf{m_1} = \mathbf{e}_2 \tag{4.1.8.5}$$

yielding

$$\mu_i = 2, -2 \tag{4.1.8.6}$$

and

$$\mathbf{a}_3 = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \mathbf{a}_2 = \begin{pmatrix} 4 \\ -2 \end{pmatrix}. \tag{4.1.8.7}$$

Area between parabola and the line $x = 4$ is divided equally by the line $x = a$. Thus,

$$A_1 = \int_0^a \sqrt{x}\, dx \tag{4.1.8.8}$$

$$A_2 = \int_a^4 \sqrt{x}\, dx \tag{4.1.8.9}$$

$$\text{and } A_1 = A_2 \tag{4.1.8.10}$$

$$\implies a = 4^{\frac{2}{3}} \tag{4.1.8.11}$$

4.1.9 Find the area of the region bounded by the parabola $y = x^2$ and $y = |x|$.

Solution:

4.1.10 Find the area bounded by the curve $x^2 = 4y$ and the line $x = 4y - 2$.

Solution: The given curve can be expressed as a conic with parameters

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}, f = 0 \tag{4.1.10.1}$$

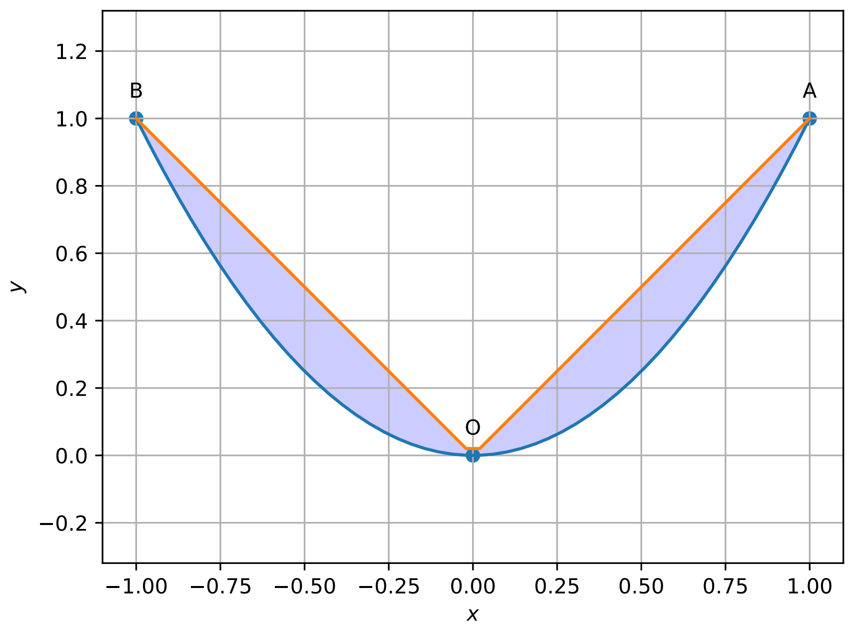


Figure 4.1.9.1:

The parameters of the given line are

$$\mathbf{q} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \tag{4.1.10.2}$$

The points of intersection can then be obtained from (G.3.1.3) as

$$\therefore \mathbf{x}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \mathbf{x}_2 = \begin{pmatrix} -1 \\ \frac{1}{4} \end{pmatrix} \tag{4.1.10.3}$$

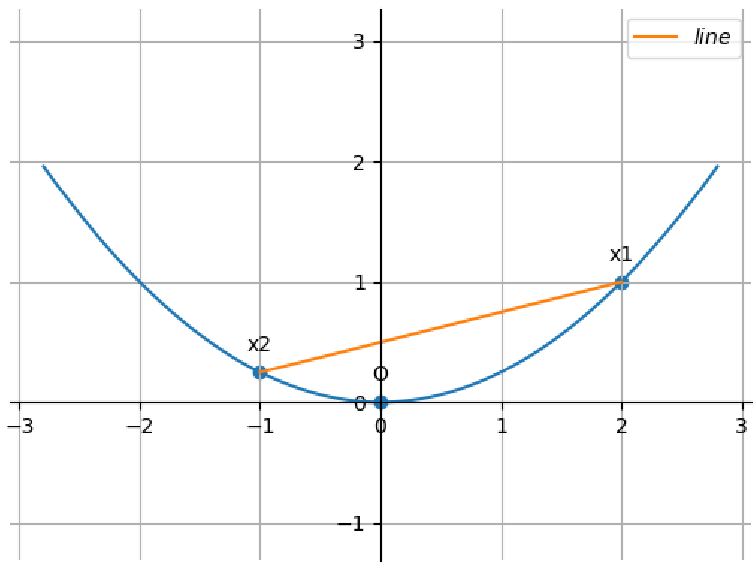


Figure 4.1.10.1:

The desired area is then obtained as

$$A = \int_{x_2}^{x_1} [f(x) - g(x)]\, dx \tag{4.1.10.4}$$

$$= \int_{-1}^{2} \left(\frac{x+2}{4} - \frac{x^2}{4} \right) dx \tag{4.1.10.5}$$

$$= \frac{9}{8} \tag{4.1.10.6}$$

4.1.11 Find the area of the region bounded by the curve $y^2 = 4x$ and the line $x = 3$.

Choose the correct answer in the following Exercises 12 and 13.

12. Area lying in the first quadrant and bounded by the circle $x^2 + y^2 = 4$ and the lines $x = 0$ and $x = 2$ is

(a) π

(b) $\frac{\pi}{2}$

(c) $\frac{\pi}{3}$

(d) $\frac{\pi}{4}$

13. Find the area of the region bounded by the curve $y^2 = 4x$, y-axis and the line $y = 3$.

Solution: The given equation of the curve can be rearranged as

$$y^2 - 4x = 0 \tag{13.1}$$

$$\implies \mathbf{x}^\top \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} -2 & 0 \end{pmatrix} \mathbf{x} + 0 = 0 \tag{13.2}$$

The above equation can be equated to the generic equation of conic sections

$$g(\mathbf{x}) = \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \tag{13.3}$$

Comparing coefficients of both equations (13.2) and (13.3)

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{13.4}$$

$$\mathbf{u} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \tag{13.5}$$

$$f = 0 \tag{13.6}$$

For the given line $y = 3$, the parameters are

$$\mathbf{h} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{13.7}$$

To calculate the point of contact of line with the conic, we use

$$\mu^2 \mathbf{m}^\top \mathbf{V} \mathbf{m} + 2\mu \mathbf{m}^\top \left(\mathbf{V}\mathbf{h} + \mathbf{u}\right) + g\left(\mathbf{h}\right) = 0 \tag{13.8}$$

$$\begin{aligned} g\left(\mathbf{h}\right) = \begin{pmatrix} 0 & 3 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \end{pmatrix} \\ + 2 \begin{pmatrix} -2 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \end{pmatrix} + 0 \\ \implies g\left(\mathbf{h}\right) = \begin{pmatrix} 0 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \end{pmatrix} + 2 \begin{pmatrix} 0 \end{pmatrix} \\ \implies g\left(\mathbf{h}\right) = 9 \end{aligned} \tag{13.9}$$

$$(13.8) \implies \mu^2 \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$+ 2\mu \begin{pmatrix} 1 & 0 \end{pmatrix} \left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \end{pmatrix} + \begin{pmatrix} -2 \\ 0 \end{pmatrix} \right) + 9 = 0$$

$$\implies \mu^2 (0) + 2\mu \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} -2 \\ 3 \end{pmatrix} + 9 = 0$$

$$\implies -4\mu + 9 = 0$$

$$\implies \mu = \frac{9}{4} \quad (13.10)$$

The point of contact is given as

$$\mathbf{a}_0 = \begin{pmatrix} \frac{9}{4} \\ \\ 3 \end{pmatrix} \quad (13.11)$$

The desired area of the region is given as

$$\int_0^3 \frac{y^2}{4}\, dy = \frac{1}{12} \left[y^3 \right]_0^3 \quad (13.12)$$

$$= \frac{1}{12} (27 - 0) \quad (13.13)$$

$$= \frac{9}{4} \text{ sq.units} \quad (13.14)$$

The relevant diagram is shown in Figure 13.1

14. Find the area of the region bounded by the curve $x^2 = 4y$ and the lines y=2 and y=4 and the y-axis in the first quadrant.

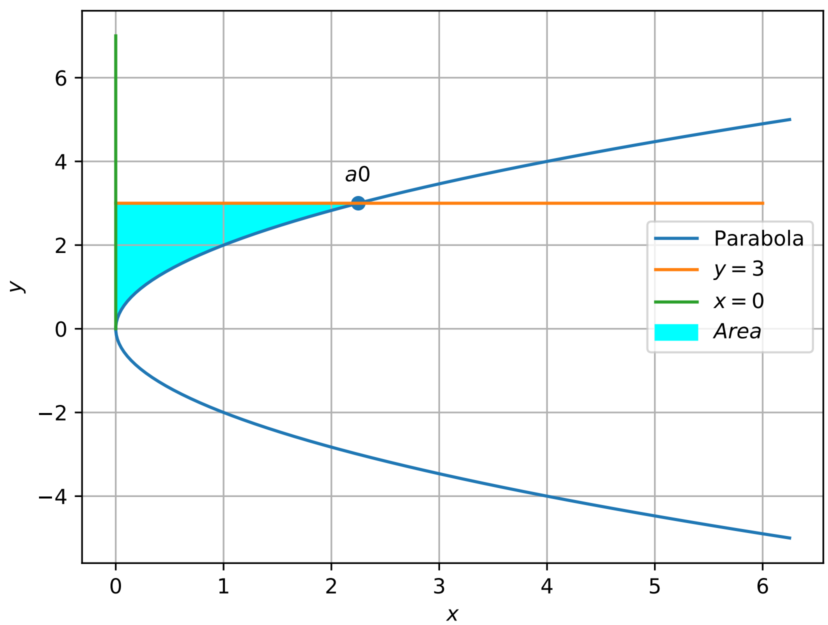


Figure 13.1:

Solution: The conic parameters are

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}, f = 0 \tag{14.1}$$

The vector parameters of $y - 4 = 0$ are

$$\mathbf{h}_1 = \begin{pmatrix} 0 \\ 4 \end{pmatrix}, \mathbf{m}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{14.2}$$

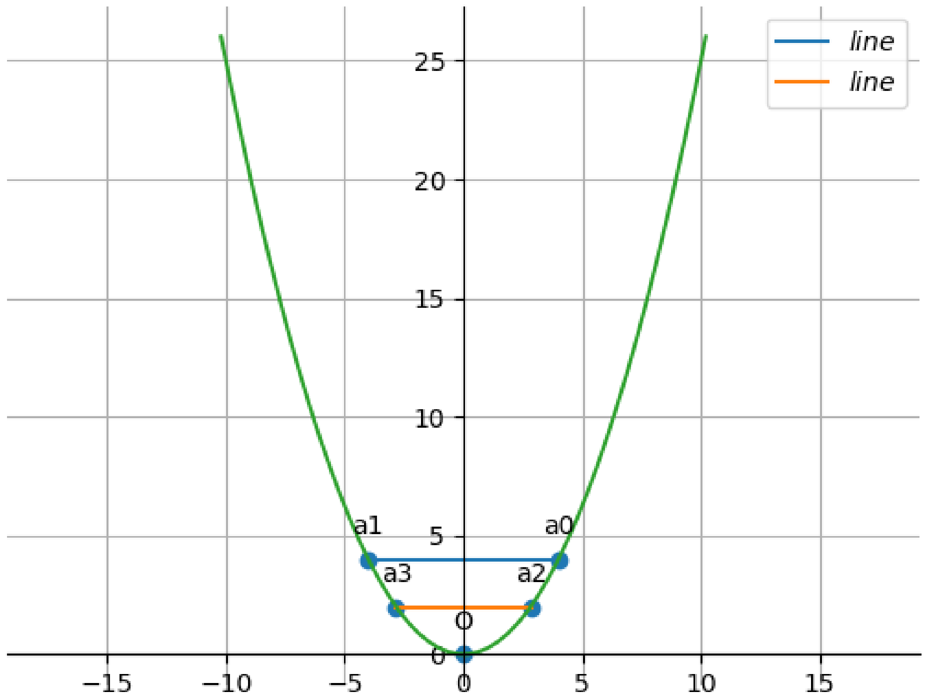


Figure 14.1:

Substituting the above in (G.3.1.3),

$$\mu_i = 4, -4 \tag{14.3}$$

yielding the points of intersection with the parabola as

$$\mathbf{a}_0 = \begin{pmatrix} 4 \\ 4 \end{pmatrix}, \mathbf{a}_1 = \begin{pmatrix} -4 \\ 4 \end{pmatrix} \tag{14.4}$$

Similarly, for the line $y - 2 = 0$, the vector parameters are

$$\mathbf{h}_2 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \mathbf{m}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{14.5}$$

yielding

$$\mu_i = 2.8, -2.8 \tag{14.6}$$

and the points of intersection

$$\mathbf{a}_2 = \begin{pmatrix} 2.8 \\ 2 \end{pmatrix}, \mathbf{a}_3 = \begin{pmatrix} -2.8 \\ 2 \end{pmatrix} \tag{14.7}$$

From Fig. 14.1, the area of the parabola between the lines $y = 2$ and $y = 4$ is given by

$$\int_0^4 2\sqrt{y}\,dy - \int_0^2 2\sqrt{y}\,dy = 6.895 \tag{14.8}$$

15. Find the area enclosed by the parabola $4y = 3x^2$ and the line $2y = 3x + 12$.

Solution: The parameters of the given conic are

$$\mathbf{V} = \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}, f = 0. \tag{15.1}$$

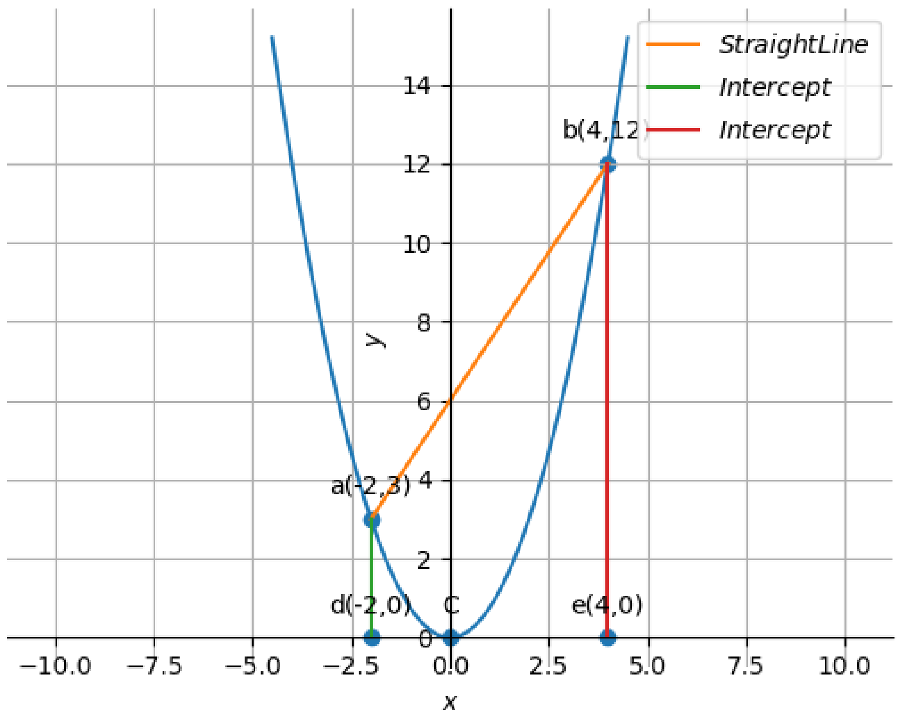


Figure 15.1:

For the line, the parameters are

$$\mathbf{h} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \tag{15.2}$$

yielding

$$\mu = -2.5, 2.7 \tag{15.3}$$

upon substitution in (G.3.1.3) resulting in the points of intersection

$$\mathbf{A} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 4 \\ 12 \end{pmatrix}. \tag{15.4}$$

From Fig. 15.1, the desired area is

$$\int_{-2}^{4} \frac{3x+12}{2}\, dx - \int_{-2}^{4} \frac{3x^2}{4}\, dx = 27 \tag{15.5}$$

16. Find the area of the smaller region bounded by the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and the line $\frac{x}{3} + \frac{y}{2} = 1$.

Solution: The given ellipse can be expressed as conics with parameters

$$\mathbf{V} = \begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix}, \mathbf{u} = 0, f = -(a^2b^2). \tag{16.1}$$

The line parameters are

$$\mathbf{h} = \begin{pmatrix} a \\ 0 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} \frac{1}{b} \\ -\frac{1}{a} \end{pmatrix}. \tag{16.2}$$

Substituting the given parameters in (G.3.1.3),

$$\mu = 0, -6 \tag{16.3}$$

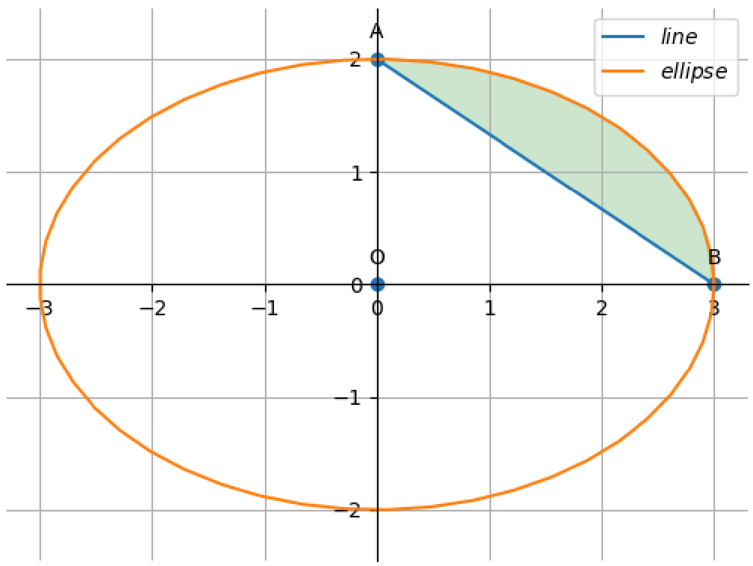


Figure 16.1:

yielding the points of intersection

$$\mathbf{A} = \begin{pmatrix} a \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ b \end{pmatrix}. \tag{16.4}$$

From Fig. 16.1, the desired area is

$$\int_0^3 \frac{2}{3}\sqrt{9 - x^2}\, dx - \int_0^3 \frac{2}{3}(3 - x)\, dx = 3\left(\frac{\pi}{2} - 1\right) \tag{16.5}$$

17. Find the area of the smaller region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the line $\frac{x}{a} + \frac{y}{b} = 1$.

Solution: The given ellipse can be expressed as a conic with param-

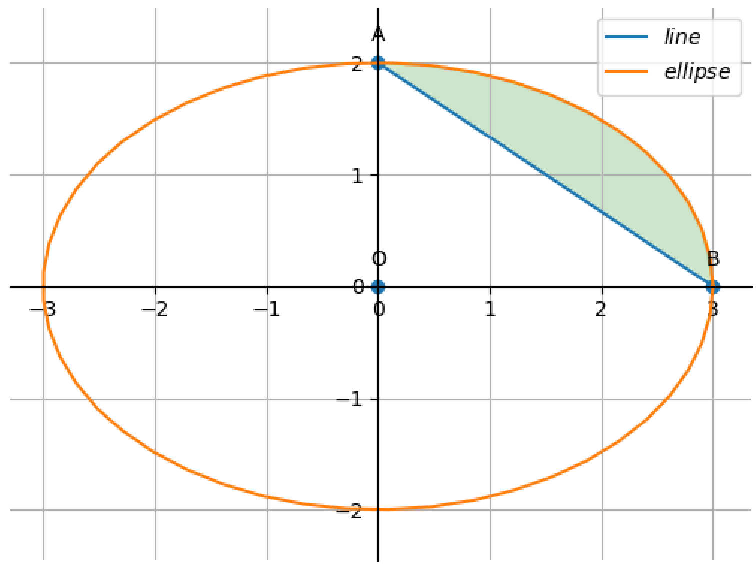


Figure 17.1:

eters

$$\mathbf{V} = \begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix}, \mathbf{u} = 0, f = -(a^2b^2). \tag{17.1}$$

The given line parameters are

$$\mathbf{h} = \begin{pmatrix} a \\ 0 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} \frac{1}{b} \\ -\frac{1}{a} \end{pmatrix}. \tag{17.2}$$

Substituting the given parameters in (G.3.1.3)

$$\mu = 0, -6 \tag{17.3}$$

yielding the points of intersection

$$\mathbf{A} = \begin{pmatrix} a \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ b \end{pmatrix} \tag{17.4}$$

From Fig. 17.1, the desired area is

$$\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\, dx - \int_0^a \frac{b}{a}(a - x)\, dx = \frac{ab}{2}\left(\frac{\pi}{2} - 1\right) \tag{17.5}$$

18. Find the area of the region bounded by the curve $x^2 = y$ and the lines $y = x + 2$ and the x axis.

    Solution:

19. Find the area bounded by the curve $y = x|x|$, x-axis and the ordinates x=-1 and x=1.

    Solution: The parameters of the given conics are

$$\mathbf{V}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u_1} = \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix}, f_1 = 0 \tag{19.1}$$

$$\mathbf{V}_2 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u_2} = \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix}, f_2 = 0 \tag{19.2}$$

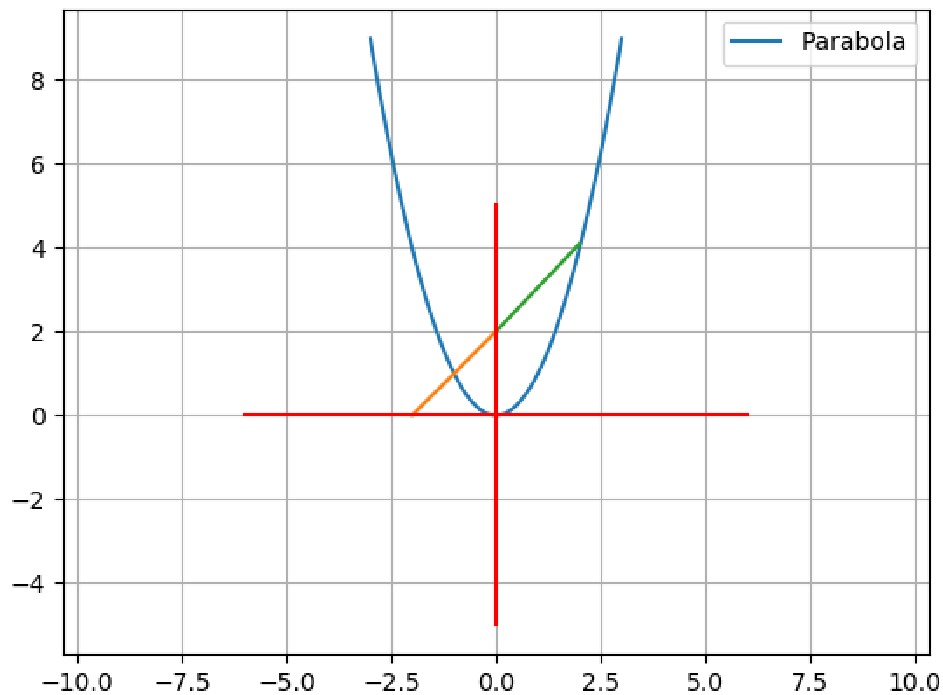


Figure 18.1:

The determinant equation for the intersection of two conics is

$$\begin{vmatrix} 1-\mu & 0 & 0 \\ 0 & 0 & -\frac{1}{2}-\frac{\mu}{2} \\ 0 & -\frac{1}{2}-\frac{\mu}{2} & 0 \end{vmatrix} = 0 \tag{19.3}$$

yielding,

$$\mu^3 + \mu^2 - \mu - 1 = 0 \tag{19.4}$$

$$\implies \mu = -1, 1, 1 \tag{19.5}$$

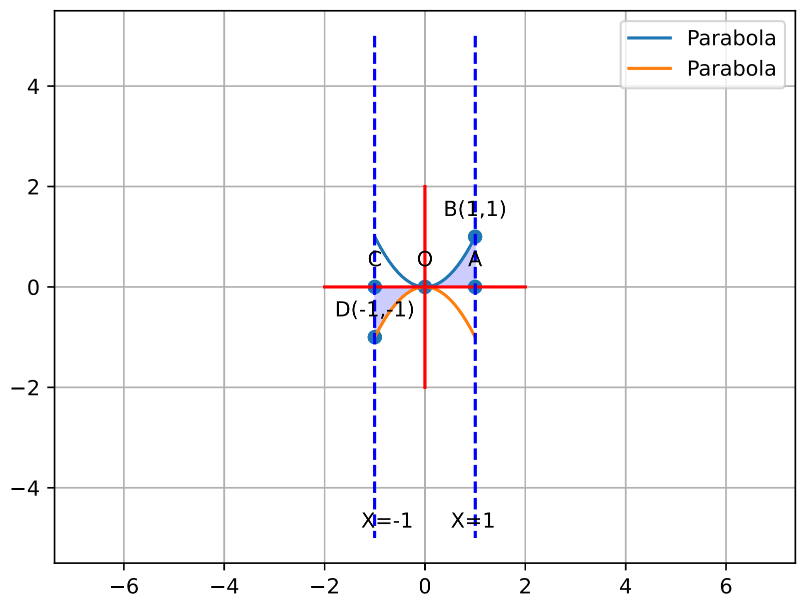


Figure 19.1:

20. Find the area of the region bounded by the curves $y = x^2 + 2$, $y = x$, $x = 0$ and $x = 3$.

    Solution: The conic parameters are

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 0 \\ -1/2 \end{pmatrix}, f = 2. \tag{20.1}$$

21. Find the smaller area enclosed by the circle $x^2 + y^2 = 4$ and the line $x + y = 2$.

    Solution: The given circle can be expressed as conics with parameters,

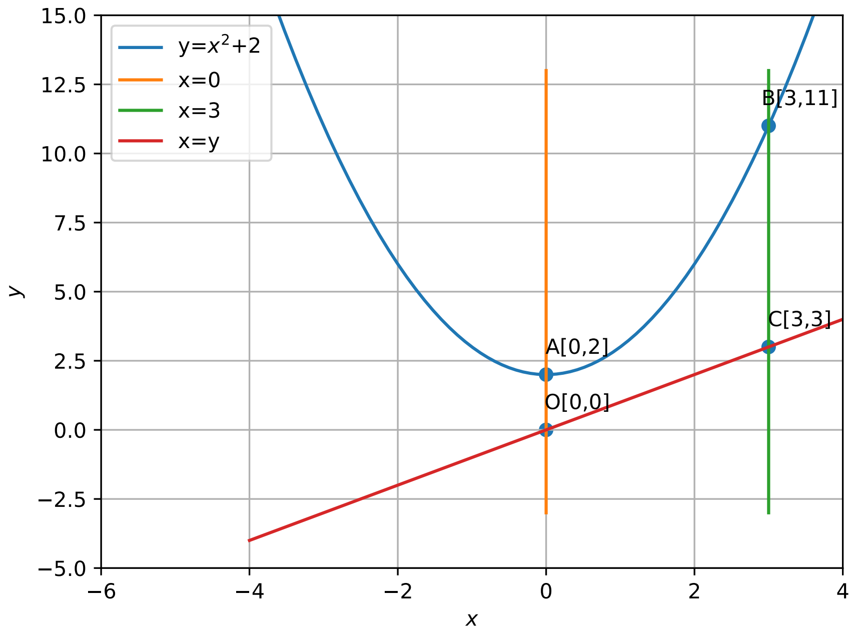


Figure 20.1:

$$\mathbf{V} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}, \mathbf{u} = 0, f = -16 \tag{21.1}$$

The line parameters are

$$\mathbf{h} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} \tag{21.2}$$

Substituting the parameters in (G.3.1.3),

$$\mu = 0, -4 \tag{21.3}$$

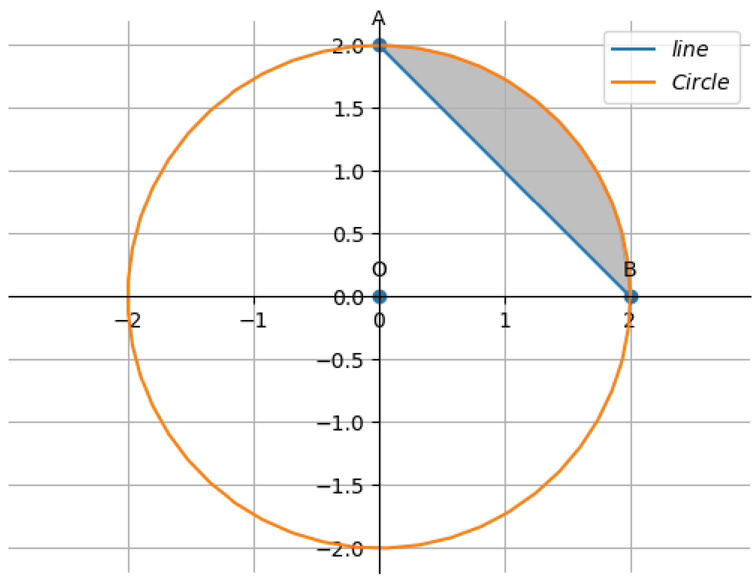


Figure 21.1:

yielding the points of intersection as

$$\mathbf{A} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \tag{21.4}$$

From Fig. 21.1, the desired area is

$$\int_0^2 \sqrt{4-x^2}\, dx - \int_0^2 (2-x)\, dx = \pi - 2 \tag{21.5}$$

4.2. Curves

4.2.1 Find the area of the circle $4x^2 + 4y^2 = 9$ which is interior to the parabola $x^2 = 4y$.

Solution: The given circle and parabola can be expressed as conics

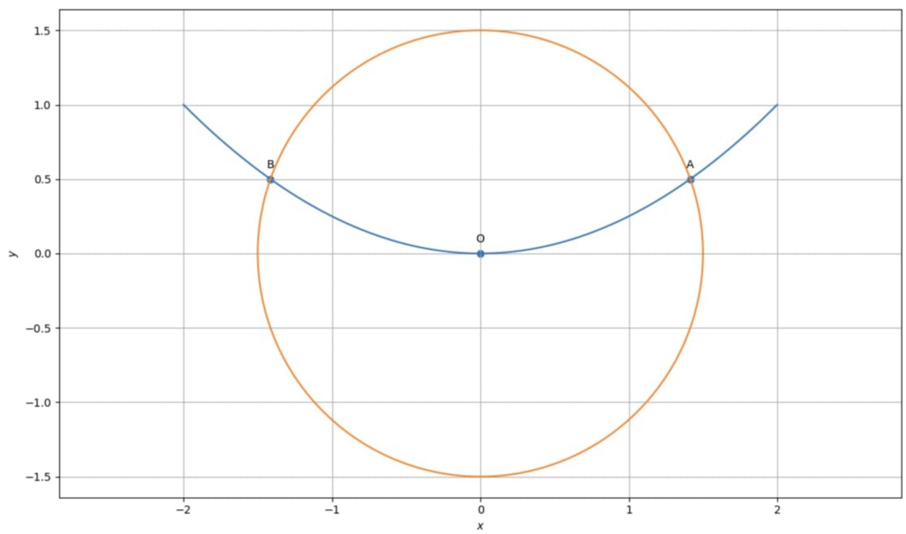


Figure 4.2.1.1:

with parameters

$$\mathbf{V}_1 = 4\mathbf{I}, \mathbf{u_1} = \mathbf{0}, f_1 = -9 \tag{4.2.1.1}$$

$$\mathbf{V}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u_2} = -\begin{pmatrix} 0 \\ 2 \end{pmatrix}, f_2 = 0 \tag{4.2.1.2}$$

The intersection of the given conics is obtained as

$$\mathbf{x}^\top \left(\mathbf{V}_1 + \mu\mathbf{V}_2\right)\mathbf{x} + 2\left(\mathbf{u}_1 + \mu\mathbf{u}_2\right)^\top \mathbf{x} + \left(f_1 + \mu f_2\right) = 0 \tag{4.2.1.3}$$

This conic represents a pair of straight lines if

$$\begin{vmatrix} \mathbf{V}_1 + \mu \mathbf{V}_2 & \mathbf{u}_1 + \mu \mathbf{u}_2 \\ (\mathbf{u}_1 + \mu \mathbf{u}_2)^\top & f_1 + \mu f_2 \end{vmatrix} = 0 \tag{4.2.1.4}$$

which can be expressed as

$$\implies \begin{vmatrix} \mu + 4 & 0 & 0 \\ 0 & 4 & -2\mu \\ 0 & -2\mu & -9 \end{vmatrix} = 0 \tag{4.2.1.5}$$

Solving the above equation we get,

$$\mu^3 + 4\mu^2 + 9\mu + 36 = 0 \tag{4.2.1.6}$$

yielding

$$\mu = -4. \tag{4.2.1.7}$$

Thus, the parameters for the pair of straight lines can be expressed as

$$\mathbf{V} = \mathbf{V}_1 + \mu\mathbf{V}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix}, \tag{4.2.1.8}$$

$$\mathbf{u} = \mathbf{u}_1 + \mu\mathbf{u}_2 = \begin{pmatrix} 0 \\ 8 \end{pmatrix} \tag{4.2.1.9}$$

$$f = -9, \tag{4.2.1.10}$$

$$\implies \mathbf{D} = \mathbf{V}, \mathbf{P} = \mathbf{I} \tag{4.2.1.11}$$

4.2.2 Find the area bounded by the curves $(x-1)^2+y^2 = 1$ and $x^2+y^2 = 1$.

Solution: The general equation of a conic is given as

$$g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{4.2.2.1}$$

The first curve equation can be rearranged as

$$x^2 + y^2 - 2x = 0 \tag{4.2.2.2}$$

Comparing (4.2.2.1) and (4.2.2.2) we get

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4.2.2.3}$$

$$\mathbf{u} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \tag{4.2.2.4}$$

$$f = 0 \tag{4.2.2.5}$$

The second curve equation can be rearranged as

$$x^2 + y^2 - 1 = 0 \tag{4.2.2.6}$$

Comparing (4.2.2.1) and (4.2.2.6) we get

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4.2.2.7}$$

$$\mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{4.2.2.8}$$

$$f = -1 \tag{4.2.2.9}$$

The intersection of conics is obtained as

$$\mathbf{x}^\top \left(\mathbf{V}_1 + \mu\mathbf{V}_2\right)\mathbf{x} + 2\left(\mathbf{u}_1 + \mu\mathbf{u}_2\right)^\top \mathbf{x} + \left(f_1 + \mu f_2\right) = 0 \tag{4.2.2.10}$$

The locus of the intersection is a pair of straight lines if

$$\begin{vmatrix} \mathbf{V}_1 + \mu \mathbf{V}_2 & \mathbf{u}_1 + \mu \mathbf{u}_2 \\ (\mathbf{u}_1 + \mu \mathbf{u}_2) & f_1 + \mu f_2 \end{vmatrix} = 0 \tag{4.2.2.11}$$

On substituting values we get

$$\begin{vmatrix} 1+\mu & 0 & -1 \\ 0 & 1+\mu & 0 \\ -1 & 0 & -\mu \end{vmatrix} = 0 \tag{4.2.2.12}$$

solving the determinant we get

$$\mu^3 + 2\mu^2 + 2\mu + 1 = 0 \tag{4.2.2.13}$$

$$\implies \mu = -1 \tag{4.2.2.14}$$

Thus, the parametrs for straight line cann be expressed as

$$\mathbf{V} = \mathbf{V}_1 + \mu \mathbf{V}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{4.2.2.15}$$

$$\mathbf{u} = \mathbf{u}_1 + \mu \mathbf{u}_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \tag{4.2.2.16}$$

$$f = 1 \tag{4.2.2.17}$$

Substituting these values we get

$$\mathbf{x}^\top \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} -1 & 0 \end{pmatrix} \mathbf{x} + 1 = 0 \tag{4.2.2.18}$$

$$\begin{pmatrix} -2 & 0 \end{pmatrix} \mathbf{x} = -1 \tag{4.2.2.19}$$

Therefore

$$\mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ and } \mathbf{h} = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} \tag{4.2.2.20}$$

Now intersection of line with a conic is given by

$$\mathbf{x}_i = \mathbf{h} + \mu_i \mathbf{m} \tag{4.2.2.21}$$

where

$$\mu_i = \frac{1}{\mathbf{m}^\top \mathbf{V} \mathbf{m}} \left(-\mathbf{m}^\top \left(\mathbf{V}\mathbf{h} + \mathbf{u} \right) \pm \sqrt{\left[\mathbf{m}^\top \left(\mathbf{V}\mathbf{h} + \mathbf{u} \right) \right]^2 - g\left(\mathbf{h} \right) \left(\mathbf{m}^\top \mathbf{V} \mathbf{m} \right)} \right) \tag{4.2.2.22}$$

Now

$$g(\mathbf{h}) = \begin{pmatrix} \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} - 1 \tag{4.2.2.23}$$

$$= -\frac{3}{4} \tag{4.2.2.24}$$

$$\mathbf{m}^\top \mathbf{V} \mathbf{m} = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1 \tag{4.2.2.25}$$

$$\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u}) = 0 \tag{4.2.2.26}$$

substituting in (4.2.2.22) we get

$$\mu_i = \pm \frac{\sqrt{3}}{2} \tag{4.2.2.27}$$

Hence the point of intersection are

$$\mathbf{a}_0 = \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}, \mathbf{a}_2 = \begin{pmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix} \tag{4.2.2.28}$$

The desired area of region is given as

$$= 2\left(\int_0^{\frac{1}{2}} \sqrt{1-(x-1)^2}dx + \int_{\frac{1}{2}}^1 \sqrt{1-x^2}dx\right) \tag{4.2.2.29}$$

$$\begin{aligned} &= 2\left[\frac{1}{2}(x-1)\sqrt{1-(x-1)^2} + \frac{1}{2}\sin^{-1}(x-1)\right]_0^{\frac{1}{2}} \\ &+ 2\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right]_{\frac{1}{2}}^1 \end{aligned} \tag{4.2.2.30}$$

$$= \frac{2\pi}{3} - \frac{\sqrt{3}}{2} \tag{4.2.2.31}$$

See figure 4.2.2.1

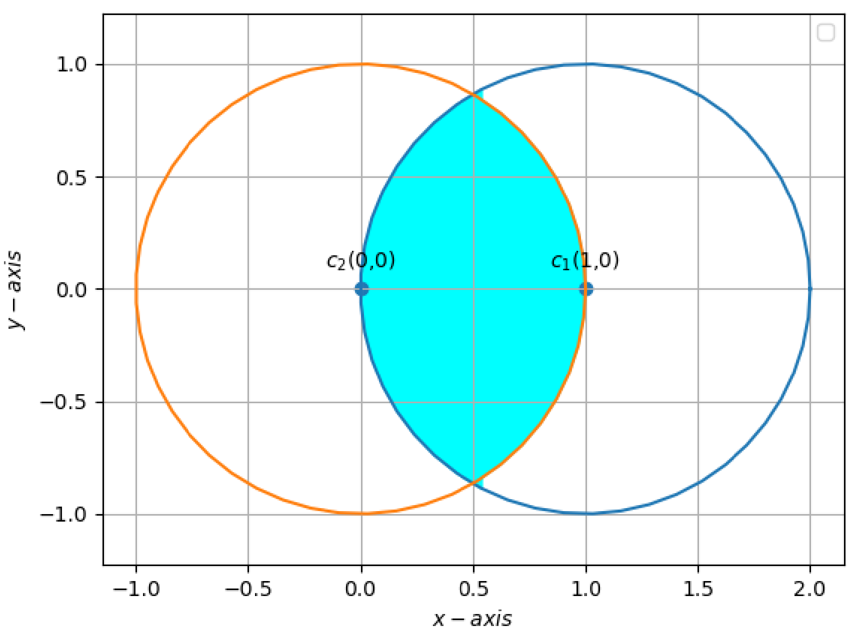


Figure 4.2.2.1:

4.2.3 Find the area between the curves $y = x$ and $y = x^2$.

Solution: The given curve can be expressed as a conic with parameters

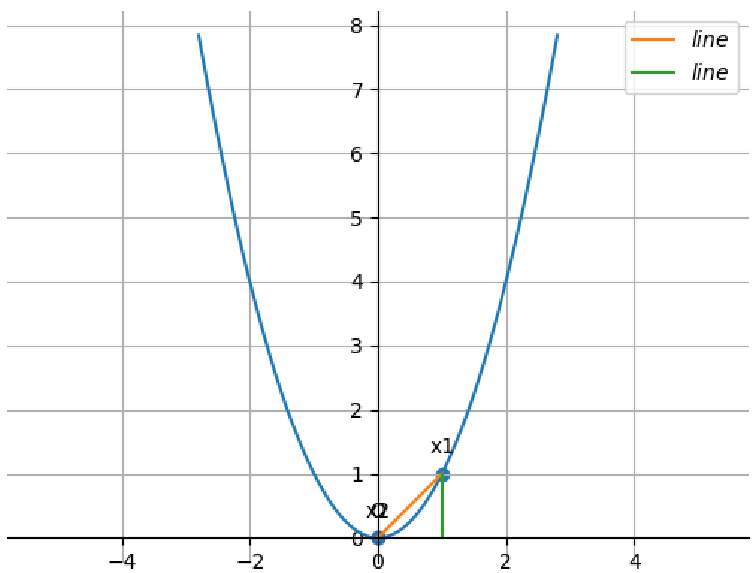


Figure 4.2.3.1:

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix}, f = 0 \tag{4.2.3.1}$$

The given line parameters are

$$\mathbf{h} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{4.2.3.2}$$

Substituting the given parameters in (G.3.1.3),

$$\mathbf{x_1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{x_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{4.2.3.3}$$

From Fig. 4.2.3.1, the area bounded by the curve $y = x^2$ and line $y = x$ is given by

$$\int_0^1 \left(x - \frac{x^2}{2}\right) dx = \frac{1}{6} \tag{4.2.3.4}$$

4.2.4 Find the area of the circle $x^2 + y^2 = 16$ exterior to the parabola $y^2 = 6x$.

Solution: The given circle and parabola can be expressed as conics with respectiveparameters

$$\mathbf{V}_1 = \mathbf{I}, \mathbf{u_1} = 0, f_1 = -16, \tag{4.2.4.1}$$

$$\mathbf{V}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u_2} = -\begin{pmatrix} 3 \\ 0 \end{pmatrix}, f_2 = 0 \tag{4.2.4.2}$$

The determinant of the intersection of the given conics is

$$\implies \begin{vmatrix} 1 & 0 & -3\mu \\ 0 & 1+\mu & 0 \\ -3\mu & 0 & -16 \end{vmatrix} = 0 \tag{4.2.4.3}$$

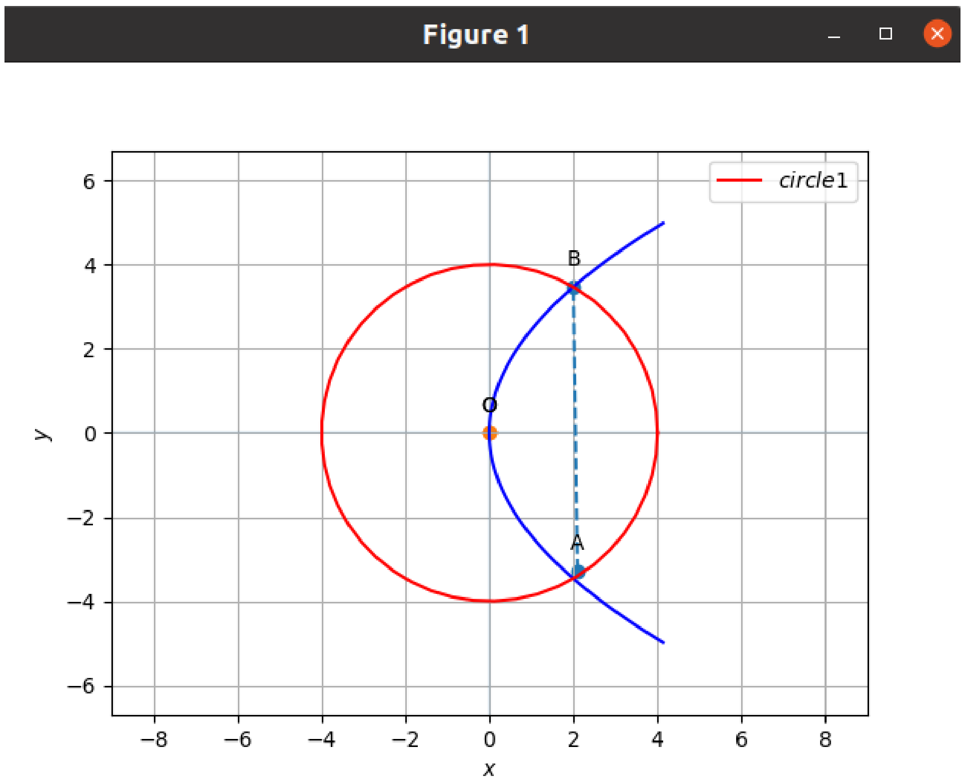


Figure 4.2.4.1:

yielding

$$9\mu^3 + 9\mu^2 + 16\mu + 16 = 0 \tag{4.2.4.4}$$

$$\text{or, } \mu = -1 \tag{4.2.4.5}$$

Chapter 5

Tangent And Normal

5.1. Examples

5.1.1 Find the slope of the tangent to the curve $y = \frac{x-1}{x-2}$, $x \neq 2$ at $x = 10$.

Solution:

(a) The given equation of the curve can be rearranged as

$$xy - x - 2y + 1 = 0 \tag{5.1.1.1}$$

$$\implies \mathbf{x}^\top \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -1 & -2 \end{pmatrix} \mathbf{x} + 1 = 0 \tag{5.1.1.2}$$

The above equation can be equated to the generic equation of conic sections

$$g(\mathbf{x}) = \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \tag{5.1.1.3}$$

Comparing coefficients of both equations (5.1.1.2) and (5.1.1.3)

$$\mathbf{V} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \tag{5.1.1.4}$$

$$\mathbf{u} = -\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} \tag{5.1.1.5}$$

$$f = 1 \tag{5.1.1.6}$$

Given the point of contact $\mathbf{q}$, the normal vector of the tangent to (5.1.1.3) is

$$\kappa\mathbf{n} = \mathbf{V}\mathbf{q} + \mathbf{u}, \kappa \in \mathbb{R} \tag{5.1.1.7}$$

For the given point of contact with $\mathbf{q}_1 = 10$,

$$\mathbf{q}_2 = \frac{10-1}{10-2} = \frac{9}{8} \tag{5.1.1.8}$$

$$\therefore \mathbf{q} = \begin{pmatrix} 10 \\ \frac{9}{8} \end{pmatrix} \tag{5.1.1.9}$$

$$(5.1.1.7) \implies \kappa\mathbf{n} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 10 \\ \frac{9}{8} \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} \tag{5.1.1.10}$$

$$= \left(\begin{pmatrix} \frac{9}{16} \\ 5 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} \right) \tag{5.1.1.11}$$

$$\therefore \mathbf{n} = \alpha \begin{pmatrix} 1 \\ 64 \end{pmatrix} \tag{5.1.1.12}$$

$$\mathbf{m} = \alpha \begin{pmatrix} 1 \\ \frac{-1}{64} \end{pmatrix} \tag{5.1.1.13}$$

(b) Now, we have to determine the nature of the conic. The matrix($\mathbf{A}$) of the quadratic equation (5.1.1.3) is represented as

$$\mathbf{A} = \begin{pmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^\top & f \end{pmatrix} \tag{5.1.1.14}$$

$$= \begin{pmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & 0 & -1 \\ -\frac{1}{2} & -1 & 1 \end{pmatrix} \tag{5.1.1.15}$$

$$\left| A \right| = \frac{1}{4} \tag{5.1.1.16}$$

$$\left| A_{33} \right| = -\frac{1}{4} \tag{5.1.1.17}$$

$\because \left| A \right| \neq 0$ and $\left| A_{33} \right| < 0$, the conic is a hyperbola. Moreover, the Eigen vectors for $\mathbf{V}$, which are given as below, indicate that the axes of hyperbola are rotated by 45°.

$$\begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{5.1.1.18}$$

To summarize, the conic is a 45° rotated hyperbola.

The relevant diagram is shown in Figure 5.1.1.1

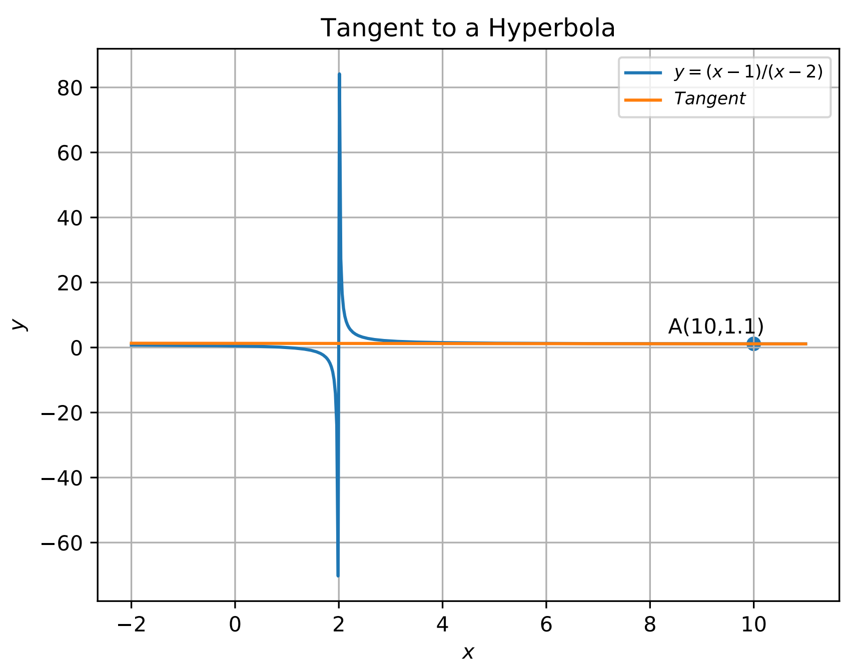


Figure 5.1.1.1:

5.1.2 Find a point on the curve

$$y = (x-2)^2 \tag{5.1.2.1}$$

at which a tangent is parallel to the chord joining the points (2,0) and (4,4).

Solution: The equation of the conic can be represented as

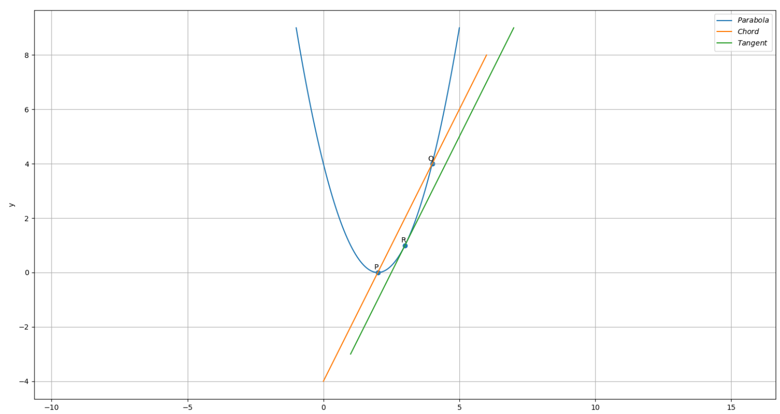


Figure 5.1.2.1:

$$\mathbf{x}^\top \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} -2 & \frac{-1}{2} \end{pmatrix} \mathbf{x} + 4 = 0 \tag{5.1.2.2}$$

So,

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u}^\top = \begin{pmatrix} -2 & \frac{-1}{2} \end{pmatrix}, f = 4 \tag{5.1.2.3}$$

The direction vector of the line passing through (2,0) and (4,4) is

$$\mathbf{m} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \implies \mathbf{n} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}. \tag{5.1.2.4}$$

From (G.4.7.1), the point of contact to parabola is given by

$$\begin{pmatrix} (\mathbf{u} + \kappa\mathbf{n})^\top \\ \mathbf{V} \end{pmatrix} \mathbf{q} = \begin{pmatrix} -f \\ \kappa\mathbf{n} - \mathbf{u} \end{pmatrix} \tag{5.1.2.5}$$

$$\text{where } \kappa = \frac{\mathbf{p}_1^\top \mathbf{u}}{\mathbf{p}_1^\top \mathbf{n}}, \quad \mathbf{V}\mathbf{p}_1 = 0 \tag{5.1.2.6}$$

The eigenvector corresponding to the zero eigenvalue is

$$\mathbf{p}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{5.1.2.7}$$

from which,

$$\kappa = \frac{\begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} -2 \\ \frac{-1}{2} \end{pmatrix}}{\begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix}} \tag{5.1.2.8}$$

$$= \frac{1}{2} \tag{5.1.2.9}$$

Substituting κ in (5.1.2.5),

$$\begin{pmatrix} \left[\begin{pmatrix} -2 \\ \frac{-1}{2} \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 2 \\ -1 \end{pmatrix}\right]^{\top} \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \mathbf{q} = \begin{pmatrix} -4 \\ \frac{1}{2}\begin{pmatrix} 2 \\ -1 \end{pmatrix} - \begin{pmatrix} -2 \\ \frac{-1}{2} \end{pmatrix} \end{pmatrix} \tag{5.1.2.10}$$

$$\implies \begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{q} = \begin{pmatrix} -4 \\ 3 \\ 0 \end{pmatrix} \tag{5.1.2.11}$$

As the last row elements are all zero, we can eliminate that row

$$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{q} = \begin{pmatrix} -4 \\ 3 \end{pmatrix} \tag{5.1.2.12}$$

For applying row reduction method the augmented matrix is written as

$$\left(\begin{array}{cc|c} -1 & -1 & -4 \\ 1 & 0 & 3 \end{array}\right) \tag{5.1.2.13}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + 2R_2} \left(\begin{array}{cc|c} 1 & -1 & 2 \\ 1 & 0 & 3 \end{array}\right) \tag{5.1.2.14}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - R_1} \left(\begin{array}{cc|c} 1 & -1 & 2 \\ 0 & 1 & 1 \end{array}\right) \tag{5.1.2.15}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + R_2} \left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 1 \end{array}\right) \tag{5.1.2.16}$$

$$\implies \mathbf{q} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \tag{5.1.2.17}$$

which is the desired point of contact. See Fig. 5.1.2.1.

5.1.3 Find the equation of all lines having slope -1 that are tangents to the curve

$$y = \frac{1}{x-1}, x \neq 1 \tag{5.1.3.1}$$

Solution: From the given information,

$$\mathbf{V} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix}, f = -1, m = -1 \tag{5.1.3.2}$$

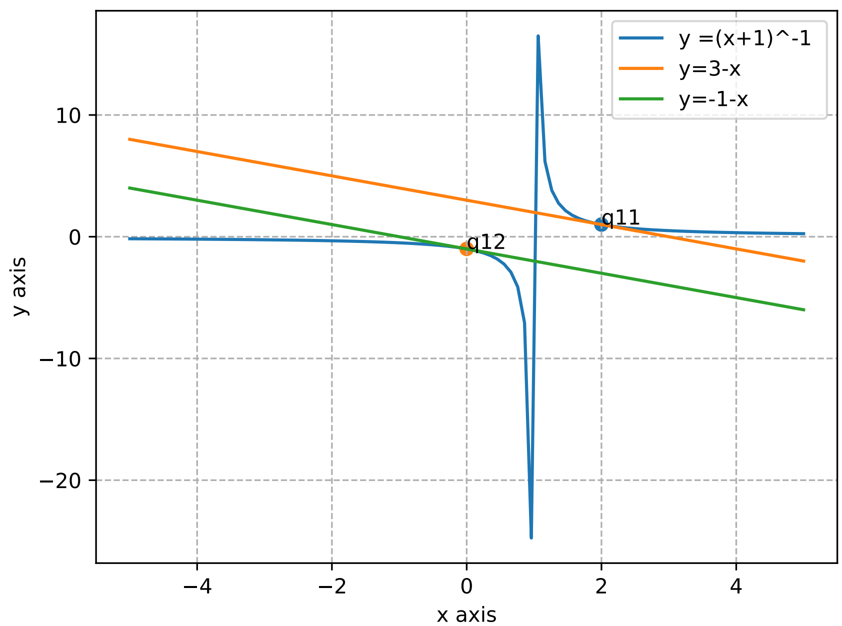


Figure 5.1.3.1:

From the above, the normal vector is

$$\mathbf{n} = \begin{pmatrix} -m \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{5.1.3.3}$$

From (G.4.4.1), the point(s) of contact are given by

$$\mathbf{q} = \mathbf{V}^{-1}(k_i\mathbf{n} - \mathbf{u}) \text{ where,} \tag{5.1.3.4}$$

$$k_i = \pm\sqrt{\frac{f_0}{\mathbf{n}^\top \mathbf{V}^{-1}\mathbf{n}}} \tag{5.1.3.5}$$

$$f_0 = f + \mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} \tag{5.1.3.6}$$

Substituting from (5.1.3.3) and (5.1.3.2) in the above,

$$\mathbf{q} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}. \tag{5.1.3.7}$$

From (G.4.1.1), the equations of tangents are given by

$$(\mathbf{V}\mathbf{q} + \mathbf{u})^\top \mathbf{x} + \mathbf{u}^\top \mathbf{q} + f = 0 \tag{5.1.3.8}$$

yielding

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} + 1 = 0 \tag{5.1.3.9}$$

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} - 3 = 0 \tag{5.1.3.10}$$

$$\tag{5.1.3.11}$$

See Fig. 5.1.3.1.

5.1.4 Find the equation of all lines having slope 2 which are tangents to the curve

$$y = \frac{1}{x-3}, x \neq 3 \tag{5.1.4.1}$$

Solution: From the given information

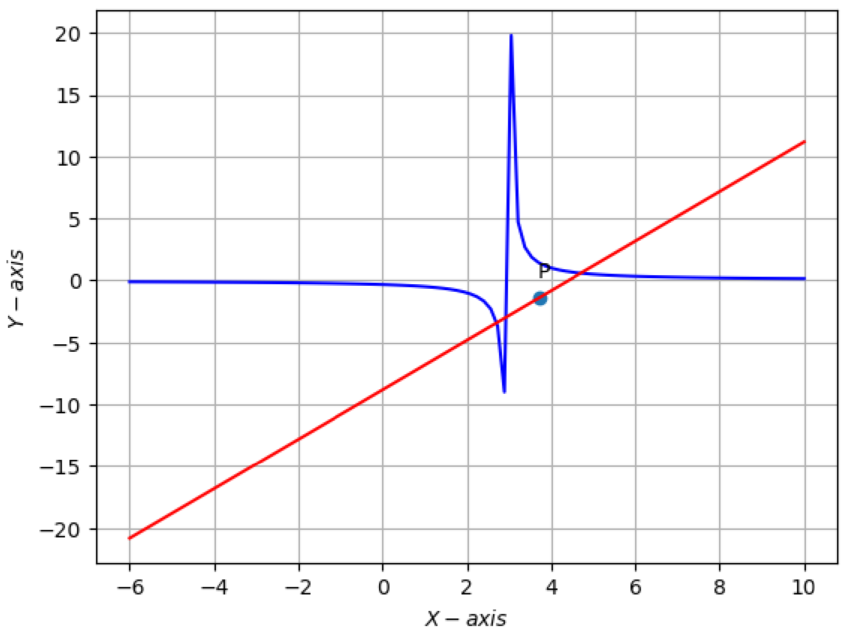


Figure 5.1.4.1:

$$\mathbf{V} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 0 \\ -\frac{3}{2} \end{pmatrix}, f = -1, m = 2 \tag{5.1.4.2}$$

$$\implies \mathbf{n} = \begin{pmatrix} -m \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \tag{5.1.4.3}$$

$$\tag{5.1.4.4}$$

Hence, the given curve is a hyperbola. Substituting numerical values, we obtain the condition in (G.4.5), which implies that the line with slope 2 is not a tangent. This can be verified from Fig. 5.1.4.1.

5.1.5 Find points on the curve $\frac{x^2}{9} + \frac{y^2}{16} = 1$ at which the tangents are

(a) parallel to x-axis

(b) parallel to y-axis

Solution: The parameters of the given conic are

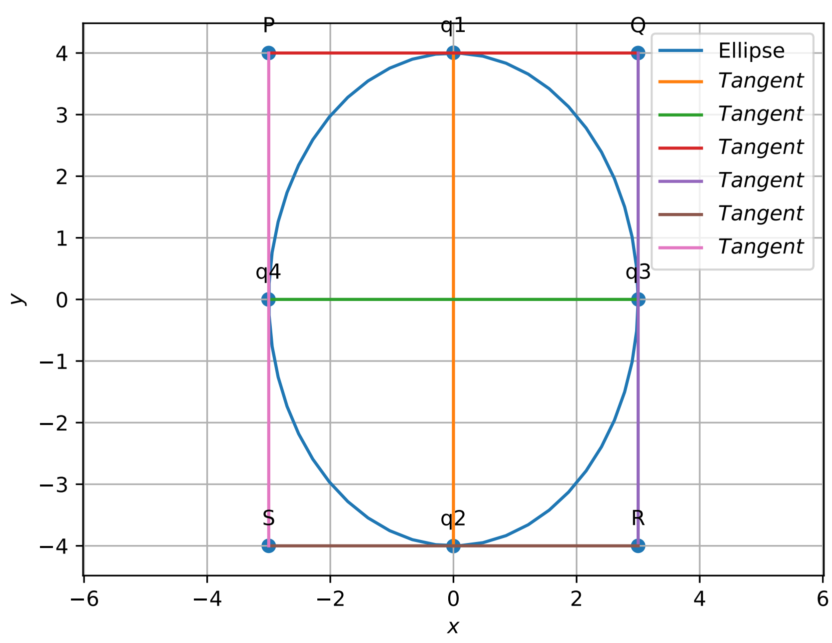


Figure 5.1.5.1:

$$\lambda_1 = 16, \lambda_2 = 9 \tag{5.1.5.1}$$

$$\mathbf{V} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, f = -144 \tag{5.1.5.2}$$

(a) The normal vector in this case is

$$\mathbf{n_1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{5.1.5.3}$$

which can be used along with the parameters in (5.1.5.2) to obtain

$$\mathbf{q_1} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}, \mathbf{q_2} = \begin{pmatrix} 0 \\ -4 \end{pmatrix} \tag{5.1.5.4}$$

using (G.4.4.1).

(b) Simlarly, choosing

$$\mathbf{n_2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{5.1.5.5}$$

$$\mathbf{q_3} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \mathbf{q_4} = \begin{pmatrix} -3 \\ 0 \end{pmatrix} \tag{5.1.5.6}$$

5.1.6 Find the equation of the tangent line to the curve

$$y = x^2 - 2x + 7 \tag{5.1.6.1}$$

(a) parallel to the line $2x - y + 9 = 0$.

(b) perpendicular to the line $5y - 15x = 13$.

Solution: The parameters of the given conic are

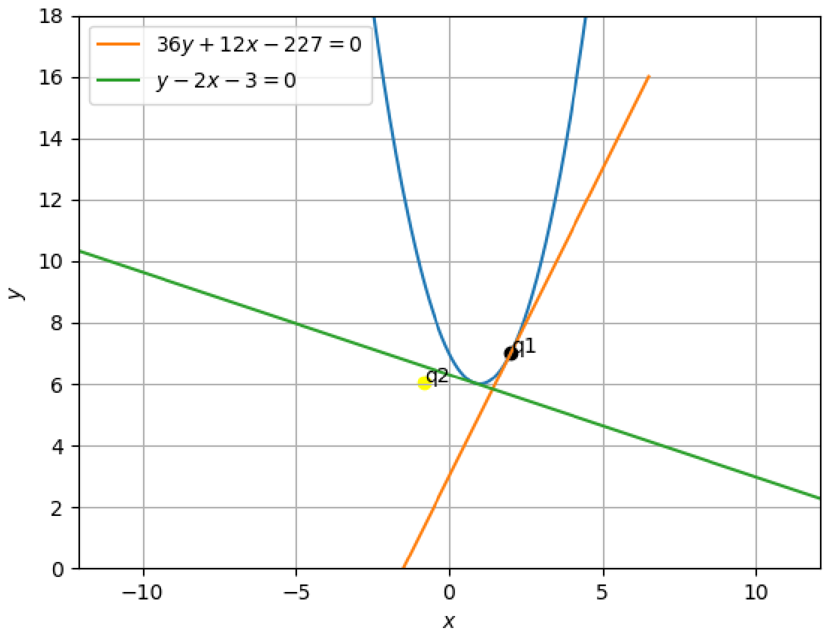


Figure 5.1.6.1:

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = -\begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix}, f = 7 \tag{5.1.6.2}$$

(a) In this case, the normal vector

$$\mathbf{n}_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \tag{5.1.6.3}$$

Since $\mathbf{V}$ is not invertible, the point of contact is given by (G.4.7.1) resulting in

$$\begin{pmatrix} \left(\begin{pmatrix} -1 \\ -\frac{1}{2} \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 2 \\ -1 \end{pmatrix}\right)^{\top} \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \mathbf{q}_1 = \begin{pmatrix} -7 \\ \frac{1}{2}\begin{pmatrix} 2 \\ -1 \end{pmatrix} - \begin{pmatrix} -1 \\ -\frac{1}{2} \end{pmatrix} \end{pmatrix} \tag{5.1.6.4}$$

By solving the above equation, we can get the point of contact as

$$\mathbf{q}_1 = \begin{pmatrix} 2 \\ 7 \end{pmatrix} \tag{5.1.6.5}$$

The tangent equation is then obtained as

$$\mathbf{n}_1^{\top}(\mathbf{x} - \mathbf{q}_1) = 0 \tag{5.1.6.6}$$

$$\implies \begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} + 3 = 0 \tag{5.1.6.7}$$

(b) In this case,

$$\mathbf{n}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \tag{5.1.6.8}$$

resulting in

$$\begin{pmatrix} \left(\begin{pmatrix} -1 \\ -\frac{1}{2} \end{pmatrix} + -\frac{1}{6}\begin{pmatrix} 1 \\ 3 \end{pmatrix}\right)^\top \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \mathbf{q}_2 = \begin{pmatrix} -7 \\ -\frac{1}{6}\begin{pmatrix} 1 \\ 3 \end{pmatrix} - \begin{pmatrix} -1 \\ -\frac{1}{2} \end{pmatrix} \end{pmatrix} \tag{5.1.6.9}$$

$$\text{or, } \mathbf{q}_2 = \begin{pmatrix} \frac{5}{6} \\ \frac{217}{36} \end{pmatrix} \tag{5.1.6.10}$$

The tangent equation is

$$\mathbf{n}_2^\top (\mathbf{x} - \mathbf{q}_2) = 0 \tag{5.1.6.11}$$

$$\text{or, } \begin{pmatrix} 1 & 3 \end{pmatrix} \mathbf{x} = \frac{227}{12} \tag{5.1.6.12}$$

5.1.7 Find the points on the curve $x^2 + y^2 - 2x - 3 = 0$ at which the tangents are parallel to the x-axis. Given that

$$\mathbf{u} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, f = -3 \tag{5.1.7.1}$$

Hence, the centre and radius are given as

$$\mathbf{c} = -\mathbf{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, r = \sqrt{\|\mathbf{u}\|^2 - f} = 2 \tag{5.1.7.2}$$

For a circle, the point of contact of tangent are given by

$$\mathbf{q}_{ij} = \left(\pm r\frac{\mathbf{n}_j}{\|\mathbf{n}_j\|} - \mathbf{u}\right) \text{ i,j} = 1, 2 \tag{5.1.7.3}$$

Since, tangents are parallel to the x-axis, the normal is given as

$$\mathbf{n} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{5.1.7.4}$$

Substituting in (5.1.7.3) we get

$$\mathbf{q}_{11} = \left(\pm 2\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \end{pmatrix}\right) \tag{5.1.7.5}$$

$$= \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{5.1.7.6}$$

Hence, the two points of contact are

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ -2 \end{pmatrix} \tag{5.1.7.7}$$

See Fig.5.1.7.1.

5.1.8

5.1.9 Find the equation of the tangent to the curve

$$y = \sqrt{3x - 2} \tag{5.1.9.1}$$

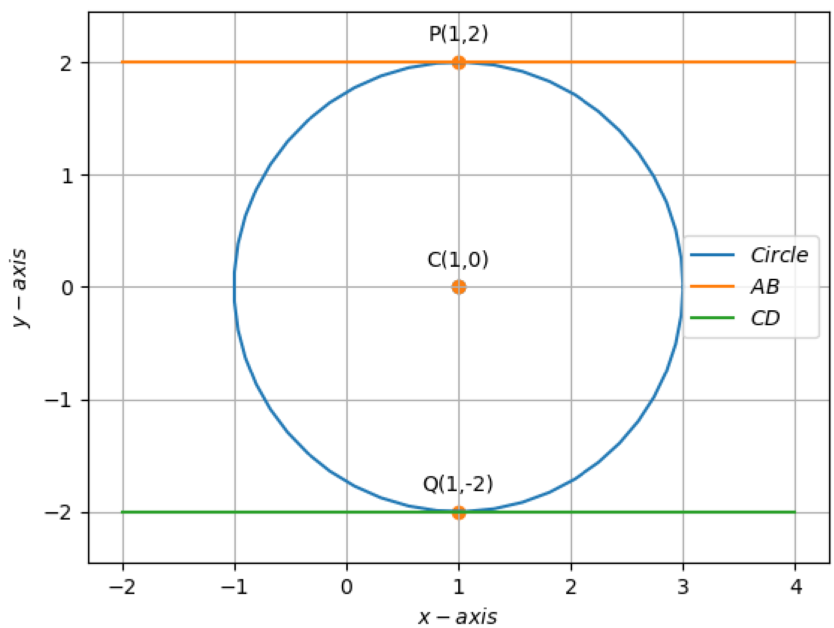


Figure 5.1.7.1:

which is parallel to the line

$$4x - 2y + 5 = 0 \tag{5.1.9.2}$$

Solution: The parameters for the given conic are

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{5.1.9.3}$$

$$\mathbf{u} = \begin{pmatrix} -3/2 \\ 0 \end{pmatrix}, \tag{5.1.9.4}$$

$$f = 2 \tag{5.1.9.5}$$

which represent a parabola. Following the approach in problem 5.1.6,

$$\mathbf{p_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{5.1.9.6}$$

$$\mathbf{n} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \tag{5.1.9.7}$$

yielding the matrix equation

$$\begin{pmatrix} -3 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{q} = \begin{pmatrix} -41/16 \\ 0 \\ 3/4 \end{pmatrix} \tag{5.1.9.8}$$

$$\tag{5.1.9.9}$$

The augmented matrix for (5.1.9.8) can be expressed as

$$\xleftrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{cc|c} -3 & 0 & -41/16 \\ 0 & 1 & 0 \\ 0 & 0 & 3/4 \end{array}\right) \tag{5.1.9.10}$$

$$\xleftrightarrow{-\frac{R_1}{-3} \leftarrow R_2} \left(\begin{array}{cc|c} 1 & 0 & 41/48 \\ 0 & 1 & 0 \\ 0 & 0 & 3/4 \end{array}\right) \tag{5.1.9.11}$$

$$\implies \mathbf{q} = \begin{pmatrix} \frac{41}{48} \\ \frac{3}{4} \end{pmatrix} \tag{5.1.9.12}$$

The equation of tangent is then obtained as

$$\begin{pmatrix} -2 & 1 \end{pmatrix} \mathbf{x} + \frac{23}{24} = 0 \tag{5.1.9.13}$$

See Fig. 5.1.9.1.

5.1.10 Find the point at which the line $y = x + 1$ is a tangent to the curve $y^2 = 4x$.

Solution: The parameters of the conic are

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} -2 & 0 \end{pmatrix}, f = 0 \tag{5.1.10.1}$$

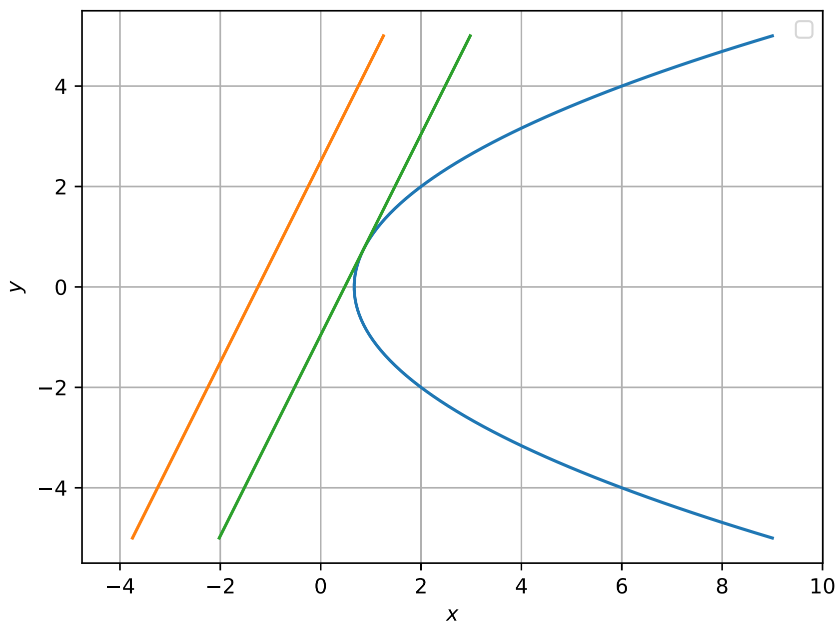


Figure 5.1.9.1:

Following the approach in Problem 5.1.6, since

$$\mathbf{n} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{5.1.10.2}$$

we obtain

$$\mathbf{q} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{5.1.10.3}$$

See Fig. 5.1.10.1,

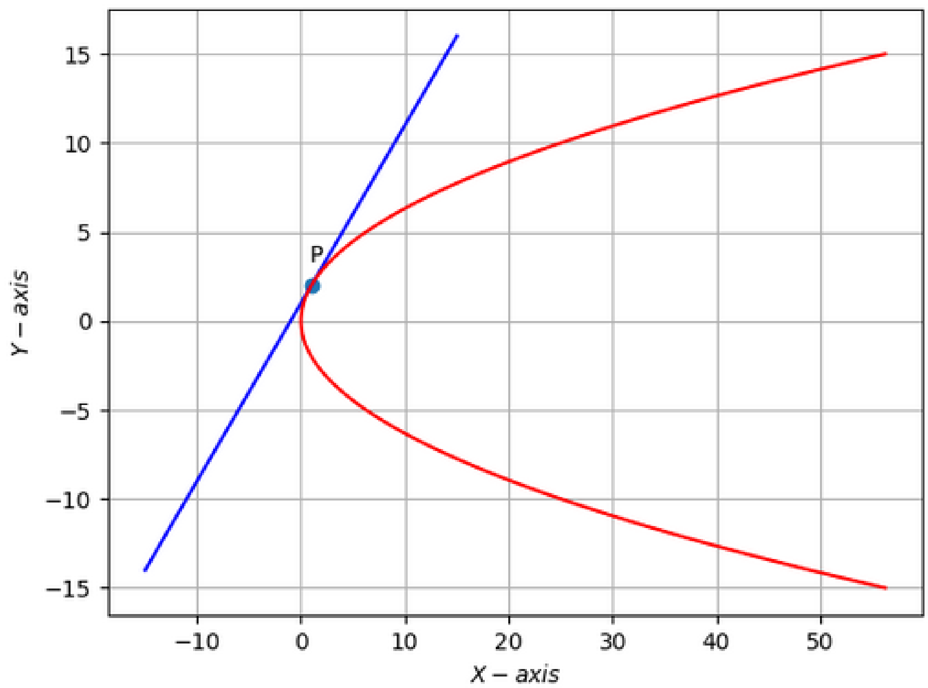


Figure 5.1.10.1:

5.1.11 The point on the curve

$$x^2 = 2y \tag{5.1.11.1}$$

which is nearest to the point $\mathbf{P} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$ is

(a) $\begin{pmatrix} 2\sqrt{2} \\ 4 \end{pmatrix}$

(b) $\begin{pmatrix} 2\sqrt{2} \\ 0 \end{pmatrix}$

(c) $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

(d) $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$

Solution: We rewrite the conic (5.1.11.1) in matrix form.

$$\mathbf{x}^\top \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} 0 & -1 \end{pmatrix} \mathbf{x} = 0 \tag{5.1.11.2}$$

Comparing with the general equation of the conic,

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{5.1.11.3}$$

$$\mathbf{u} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \tag{5.1.11.4}$$

$$f = 0 \tag{5.1.11.5}$$

Therefore, the equation of the normal where $\mathbf{u}$ is the point of contact and $\mathbf{R} \triangleq \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is

$$(\mathbf{Vq} + \mathbf{u})^\top \mathbf{R} \left(\begin{pmatrix} 0 \\ 5 \end{pmatrix} - \mathbf{q} \right) = 0 \tag{5.1.11.6}$$

Substituting the appropriate values and simplifying, we get the equation

$$\mathbf{q}^\top \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mathbf{q} + 2\mathbf{q}^\top \begin{pmatrix} -2 \\ 0 \end{pmatrix} = 0 \tag{5.1.11.7}$$

Comparing with the general equation of the conic,

$$\mathbf{V}' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{5.1.11.8}$$

$$\mathbf{u}' = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \tag{5.1.11.9}$$

$$f' = 0 \tag{5.1.11.10}$$

To solve (5.1.11.7), we must make $\mathbf{V}'$ symmetric. Thus, substituting $\mathbf{V}' \leftarrow \frac{\mathbf{V}'+\mathbf{V}'^\top}{2}$, the equation becomes

$$\mathbf{q}^\top \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \mathbf{q} + 2 \begin{pmatrix} -2 \\ 0 \end{pmatrix} \mathbf{q} = 0 \tag{5.1.11.11}$$

Note that

$$\mathbf{V}' \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{5.1.11.12}$$

$$\mathbf{V}' \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{5.1.11.13}$$

and hence the eigenparameters of $\mathbf{V'}$ are

$$\mathbf{P} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \ \mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{5.1.11.14}$$

Applying the affine transformation and since $\det V' = -\frac{1}{4} \neq 0$, (5.1.11.11) becomes

$$\mathbf{y}^\top \mathbf{D} \mathbf{y} = f_0 \tag{5.1.11.15}$$

where

$$\mathbf{q} = \mathbf{P}\mathbf{y} + \mathbf{c} \tag{5.1.11.16}$$

$$\mathbf{c} = -\mathbf{V'}^{-1}\mathbf{u} \tag{5.1.11.17}$$

$$= -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -4 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} \tag{5.1.11.18}$$

$$f_0 = \mathbf{u}^\top \mathbf{V}^{-1} \mathbf{u} - f \tag{5.1.11.19}$$

$$= \begin{pmatrix} -4 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -4 \\ 0 \end{pmatrix} = 0 \tag{5.1.11.20}$$

Since $f_0 = 0$, we see that (5.1.11.15) represents a pair of straight lines.

Expressing $\mathbf{y} \triangleq \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, we get

$$y_1^2 - y_2^2 = 0 \tag{5.1.11.21}$$

$$\implies y_1 = \pm y_2 \tag{5.1.11.22}$$

$$\implies \mathbf{y} = \begin{pmatrix} a \\ \pm a \end{pmatrix}, \quad a \in \mathbb{R} \tag{5.1.11.23}$$

Hence, using (5.1.11.23),

$$\mathbf{q} = \mathbf{P}\mathbf{y} + \mathbf{c} \tag{5.1.11.24}$$

$$= \begin{pmatrix} a \pm a \\ a \mp a + 4 \end{pmatrix} \tag{5.1.11.25}$$

$$\implies \mathbf{q} \in \left\{ \begin{pmatrix} 2a \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 2a + 4 \end{pmatrix} \right\} \tag{5.1.11.26}$$

In the first case, (5.1.11.1) implies $a^2 = 2$. In the second case, we have $2a + 4 = 0$. Thus, the points of contact are

$$\mathbf{N} \in \left\{ \begin{pmatrix} \pm 2\sqrt{2} \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} \tag{5.1.11.27}$$

The nearest point out of these three candidates for $\mathbf{N}$ is $\begin{pmatrix} \pm 2\sqrt{2} \\ 4 \end{pmatrix}$.

Thus, the correct answer is **a)**.

The situation is depicted in Fig. 5.1.11.1 plotted by the Python code `codes/normal.py`.

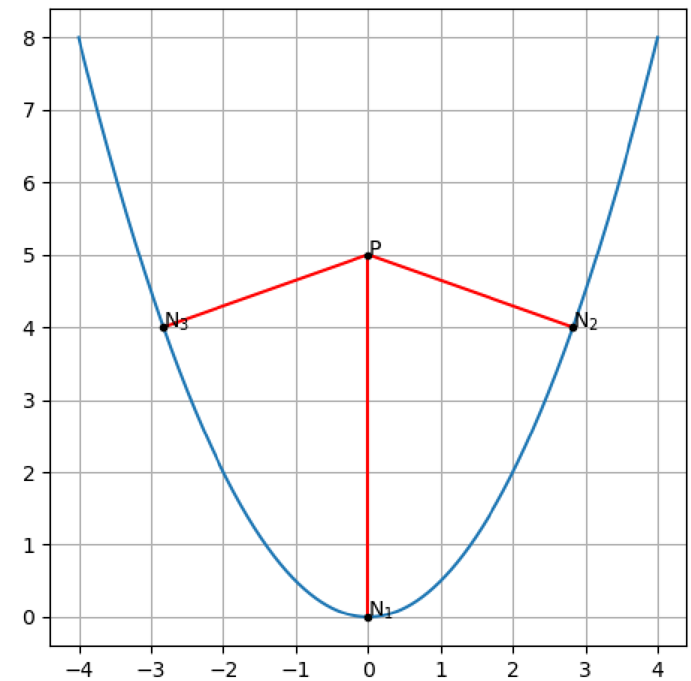


Figure 5.1.11.1: $N_1,\ N_2,\ N_3$ are the points of contact of the normal from P to the parabola.

5.1.12 Find the equation of the normal to curve $x^2 = 4y$ which passes through the point (1, 2).

Solution: The conic parameters are

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}, f = 0 \tag{5.1.12.1}$$

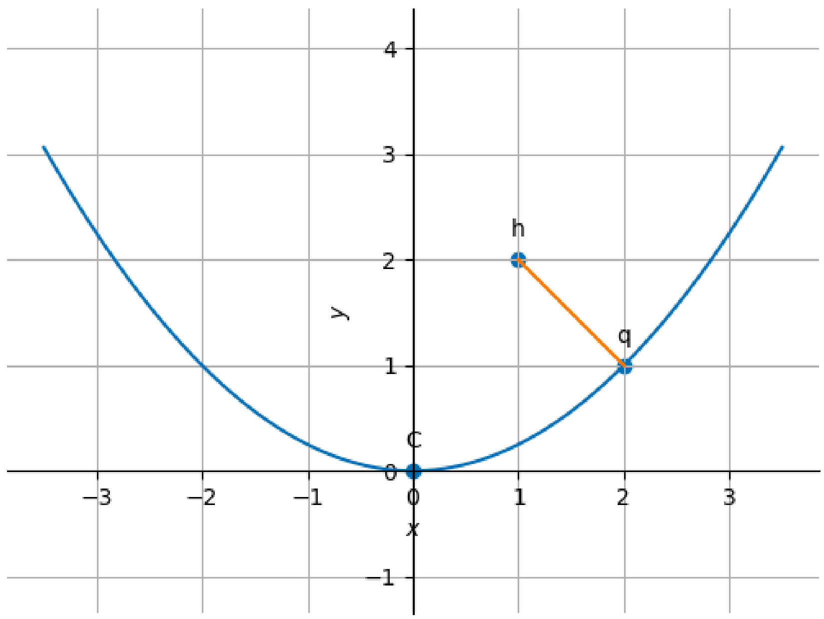


Figure 5.1.12.1:

Substituting these values in (G.4.10.1), we obtain

$$m = 1 \tag{5.1.12.2}$$

as the only real solution. Thus,

$$\mathbf{m} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{5.1.12.3}$$

and the equation of the normal is then obtained as

$$\mathbf{m}^{\top}(\mathbf{x}-\mathbf{h}) = 0 \tag{5.1.12.4}$$

$$\implies \begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{5.1.12.5}$$

$$= 3 \tag{5.1.12.6}$$

5.1.13 The line $y = mx + 1$ is a tangent to the curve $y^2 = 4x$, find the value of m.

Solution: The parameters for the given conic are

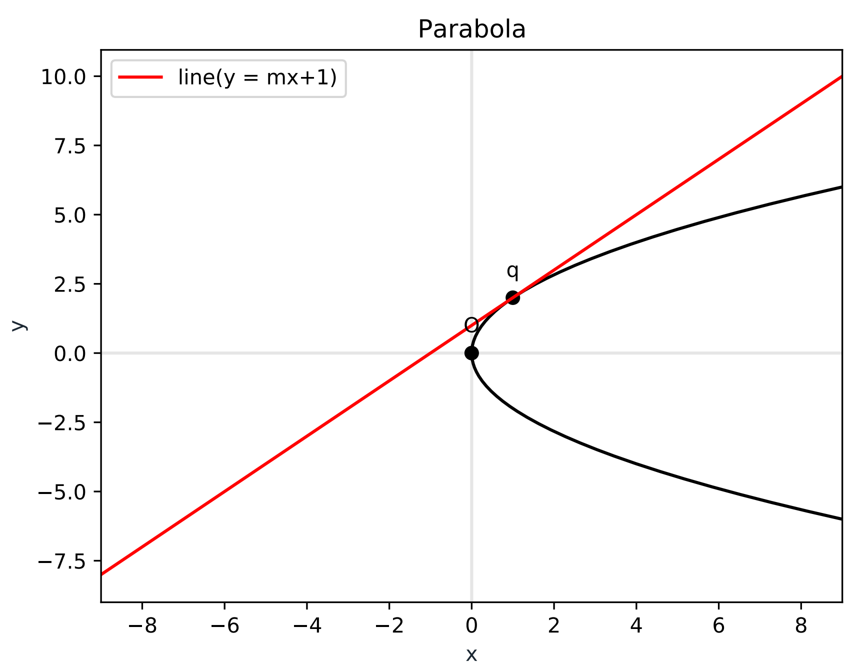


Figure 5.1.13.1:

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}, f = 0 \tag{5.1.13.1}$$

The given tangent can be expressed in parametric form as

$$\mathbf{x} = \mathbf{e}_2 + \mu \mathbf{m} \tag{5.1.13.2}$$

Substituting from (5.1.13.2) and (5.1.13.1) in (G.4.8.1) and solving, we obtain

$$m = 1. \tag{5.1.13.3}$$

5.1.14 Find the normal at the point (1,1) on the curve

$$2y + x^2 = 3 \tag{5.1.14.1}$$

Solution: Use (G.3.2.1) with

$$\mathbf{m} = \begin{pmatrix} 1 \\ m \end{pmatrix} \tag{5.1.14.2}$$

5.2. Exercises

5.2.1 Find the equation of the circle which touches both the axes in first quadrant and whose radius is a.

5.2.2 Find the equation of the circle which touches x-axis and whose centre is $(1, 2)$

5.2.3 If the lines $3x - 4y + 4 = 0$ and $6x - 8y - 7 = 0$ are tangents to a circle, then find the radius of the circle. [Hint:Distance between given parallel lines gives the diameter of the circle.]

5.2.4 Find the equation of a circle which touches both the axes and the line $3x - 4y + 8 = 0$ and lies in the third quadrant. [Hint:Let a be the radius of the circle, then $(-a, -a)$ will be centre and perpendicular distance from the centre to the given line gives the radius of the circle.]

5.2.5 If the line $y = \sqrt{3}x + K$ touches the circle $x^2 = 16y$, then find the value of K. [Hint:Equate perpendicular distance from the centre of the cirle to its radius].

5.2.6 If the line y=mx+1 is tangent to the parabolay $y^2 = 4x$ then find the value of m. [Hint:solving the equation of line and parbola, we obtain a quadratic . equation and then apply the tangency condition giving the value of m]

5.2.7 Find the condition that the curves $2x = y^2$ and $2xy = k$ intersect orthogonally.

5.2.8 Prove that the curves $xy = 4$ and $x^2 + y^2 = 8$ touch each other.

5.2.9 Find the co-ordinates of the point on the curve $\sqrt{x} + \sqrt{y}$=4 ot which tangent is equally inclined to the axes.

5.2.10 Find the angle of intersection of the curves $y = 4 - x^2$ and $y = x^2$

5.2.11 Prove that the curves $y^2 = 4x$ and $x^2 + y^2 - 6x + 1 = 0$ touch each each other ot the point (1,2).

5.2.12 Find the equation of the normal lines to the curve $3x^2 - y^2 = 8$ which are parallel to the line $x + 3y = 4$.

5.2.13 At what points on the curve $x^2 + y^2 - 2x - 4y + 1 = 0$, the tangents are parallel to the y-axis?

5.2.14 Show that the line $\frac{x}{a} + \frac{y}{b} = 1$ touches the curve $y = be - \frac{x}{a}$ at the point where the curve intersects the axis of y.

5.2.15 The equation of the normal to the curve $3x^2 - y^2 = 8$ which is parallel to the line $x + 3y = 8$ is

(a) $3x - y = 8$

(b) $3x + y + 8 = 0$

(c) $x + 3y + 8 = 0$

(d) $x + 3y = 0$

5.2.16 The equation of the tangent to the curve $(1 + x^2) = 2 - x$, where it crosses x-axis

(a) $x + 5y = 2$

(b) $x - 5y = 2$

(c) $5x - y = 2$

(d) $5x + y = 2$

State whether the statements are True or False

5.2.17 State whether the statements in each of the exercis from 33 to 40 are Trueor False justify

5.2.18 The shortest distance from the point (2,7) to the circle $x^2 + y^2$- 14x-10y-151=0 is equal to 5. [Hint:The shortest distance is equal to the difference of the redius and the distance between the cntre and the given point.]

5.2.19 lf the line lx+my=1 is a tangent to the circle $x^2 + y^2 = a^2$, then the ponit (1,m) lies an a circle. [Hint:use that distance from tne centre of the centre of the circle to the given line is equal to radius of the circle.]

5.2.20 The line 1x+my+n=0 will touch the parabola $y^2 = 4ax$ if $ln = am^2$,

5.2.21 The line 2x+3y=12 touches the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 2$ at the point (3,2).

5.3. Construction

5.3.1 Draw a circle of radius 6 cm. From a point 10 cm away from its centre, construct the pair of tangents to the circle and measure their lengths.

Solution: Follow the approach in Problem 5.3.14.

5.3.2 Construct a tangent to a circle of radius 4cm from a point on the concentric circle of radius 6cm and measure its length. Also verify the measurement by actual calculation.

Solution: See Fig. 5.3.2.1.

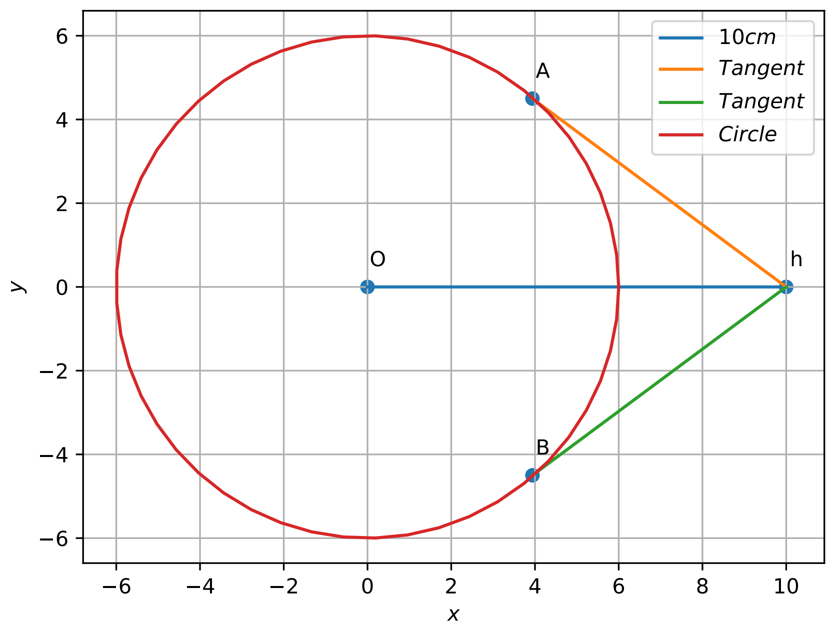


Figure 5.3.1.1:

5.3.3 Draw a circle of radius 3 cm. Take two points **P** and **Q** on one of its extended diameter each at a distance of 7 cm from its centre. Draw tangents to the circle from these two points **P** and **Q**.

Solution: See Fig. 5.3.3.1.

5.3.4 Draw a pair of tangents to a circle of radius 5 cm which are inclined to each other at an angle of 60°.

Solution: See Fig. 5.3.4.1.

5.3.5 Draw a line segment AB of length 8cm. Taking **A** as centre, draw a circle of radius 4cm and taking **B** as centre, draw another circle of radius 3cm. Construct tangents to each circle from the centre of the

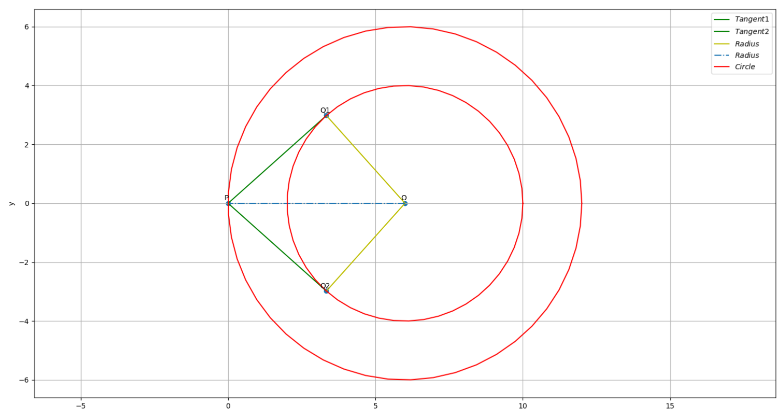


Figure 5.3.2.1:

circle.

Solution: See Fig. 5.3.5.1.

5.3.6 Let ABC be a right triangle in which $AB = 6cm, BC = 8cm$ and $\angle B = 90°$. BD is the perpendicular from $\mathbf{B}$ on AC. The circle through $\mathbf{B}, \mathbf{C}, \mathbf{D}$ is drawn. Construct the tangents from $\mathbf{A}$ to this circle.

Solution: See Fig. 5.3.6.1.

$$BD \perp AC \implies \mathbf{O} = \frac{\mathbf{B}+\mathbf{C}}{2} \tag{5.3.6.1}$$

From (D.1.11.1), the coordinates of $\mathbf{D}$ can be obtained.

5.3.7 A tangent PQ at a point of a circle of radius 5cm meets a line through the centre O at a point Q so that OQ=12cm then length of PQ is

Solution: The input parameters are available in Table 5.3.7.1. From

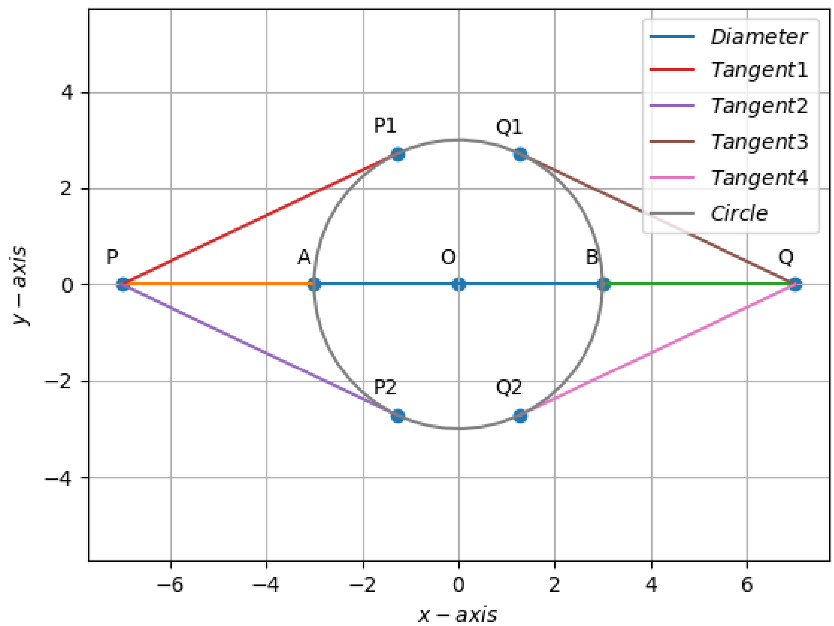


Figure 5.3.3.1:

Symbol	Value	Description
r	5	Radius
$\mathbf{O}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	Centre $\mathbf{O}$
$\mathbf{P}$	$r\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$	Point of contact
d	12	Lenght of OQ

Table 5.3.7.1:

the given information,

$$\|\mathbf{Q}\|^2 = d^2 \tag{5.3.7.1}$$

$$(\mathbf{Q} - \mathbf{P})^\top \mathbf{P} = 0 \implies \mathbf{P}^\top \mathbf{Q} = \|\mathbf{P}\|^2 = r^2 \tag{5.3.7.2}$$

$$\text{or, } \mathbf{P}^\top \mathbf{Q} = 25 \tag{5.3.7.3}$$

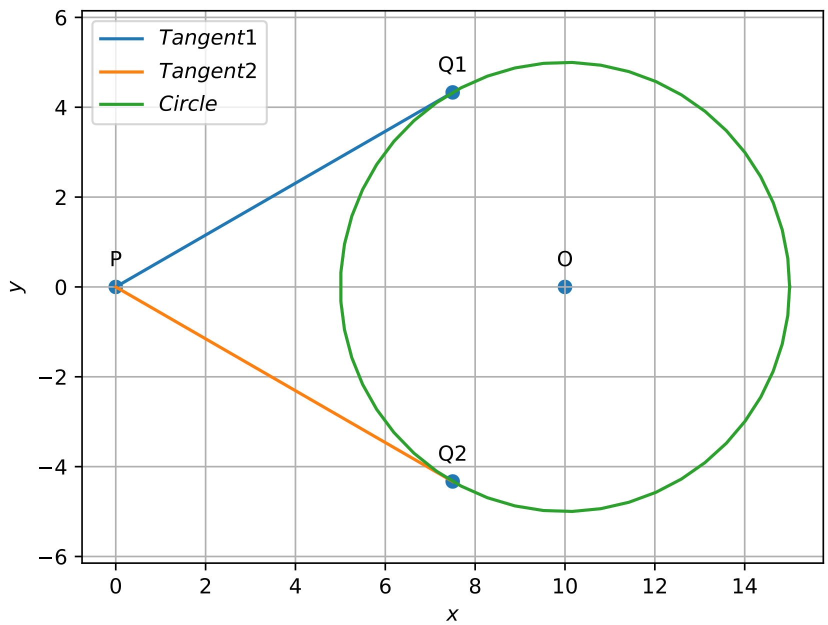


Figure 5.3.4.1:

For $\theta = 0°$

$$\mathbf{P} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \tag{5.3.7.4}$$

Substituting the above in (5.3.7.3),

$$\begin{pmatrix} 5 & 0 \end{pmatrix} \mathbf{Q} = 25 \tag{5.3.7.5}$$

$$\implies \mathbf{Q} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{5.3.7.6}$$

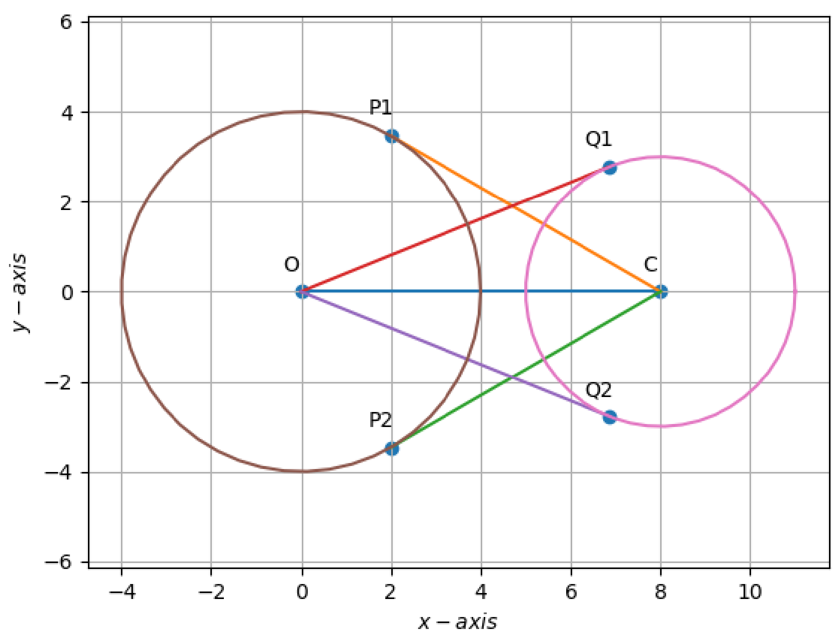


Figure 5.3.5.1:

which is of the form

$$\mathbf{Q} = \mathbf{A} + \mu\mathbf{m} \tag{5.3.7.7}$$

Then substituting (5.3.7.7) in (5.3.7.1) yeilds,

$$\implies (\mathbf{A} + \mu\mathbf{m})^\top (\mathbf{A} + \mu\mathbf{m}) = d^2 \tag{5.3.7.8}$$

$$\implies \mu^2 \|\mathbf{m}\|^2 + 2\mu\mathbf{A}^\top\mathbf{m} + \|\mathbf{A}\|^2 = d^2 \tag{5.3.7.9}$$

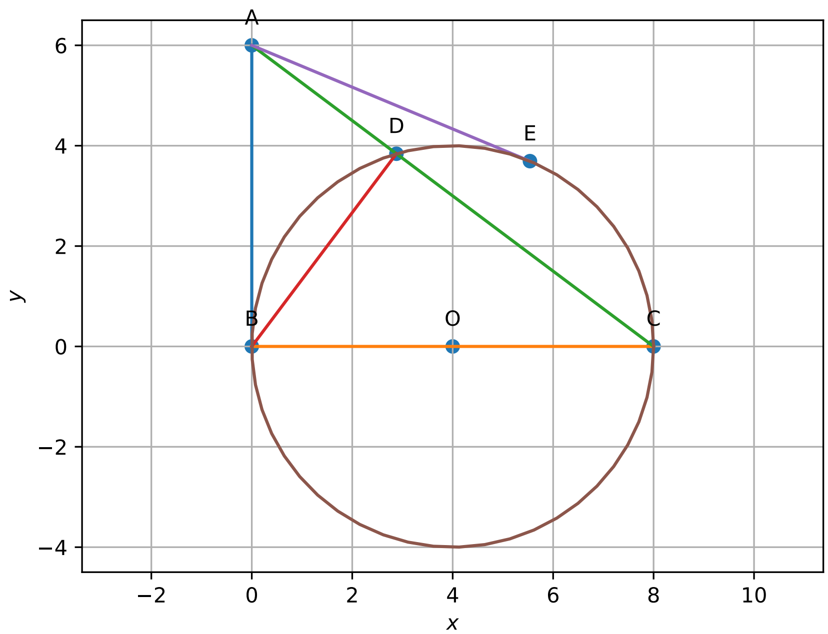


Figure 5.3.6.1:

where

$$\mathbf{A} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \text{ and } \mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{5.3.7.10}$$

Substituting numerical values,

$$\mu^2 = 119 \implies \mu = \pm\sqrt{119} \tag{5.3.7.11}$$

yeilding

$$\mathbf{Q_1} = \begin{pmatrix} 5 \\ \sqrt{119} \end{pmatrix}, \mathbf{Q_2} = \begin{pmatrix} 5 \\ -\sqrt{119} \end{pmatrix} \tag{5.3.7.12}$$

Thus,

$$PQ_1 = PQ_2 = \sqrt{119} \tag{5.3.7.13}$$

See Fig. 5.3.7.1.

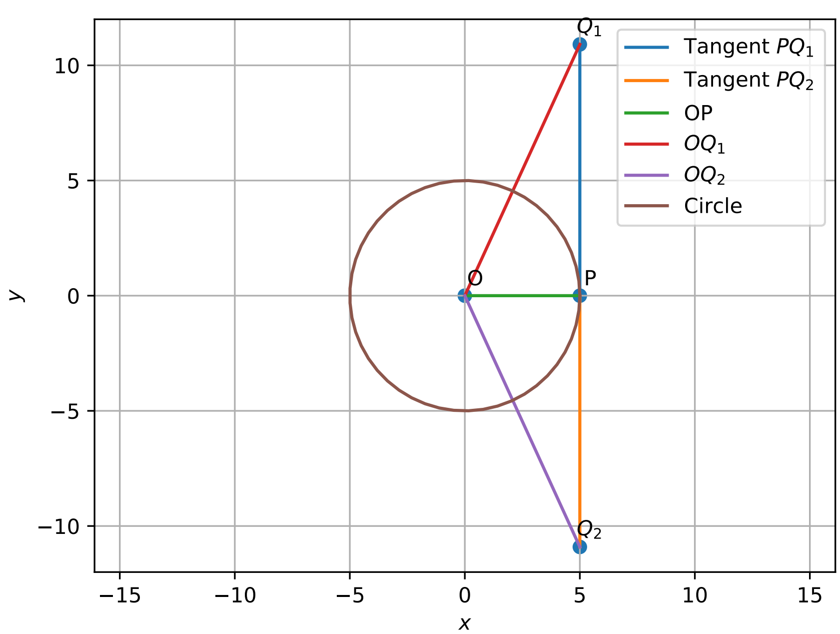


Figure 5.3.7.1:

5.3.8 Draw a circle and two lines parallel to a given line such that one is a tangent and the other is a secant to the circle

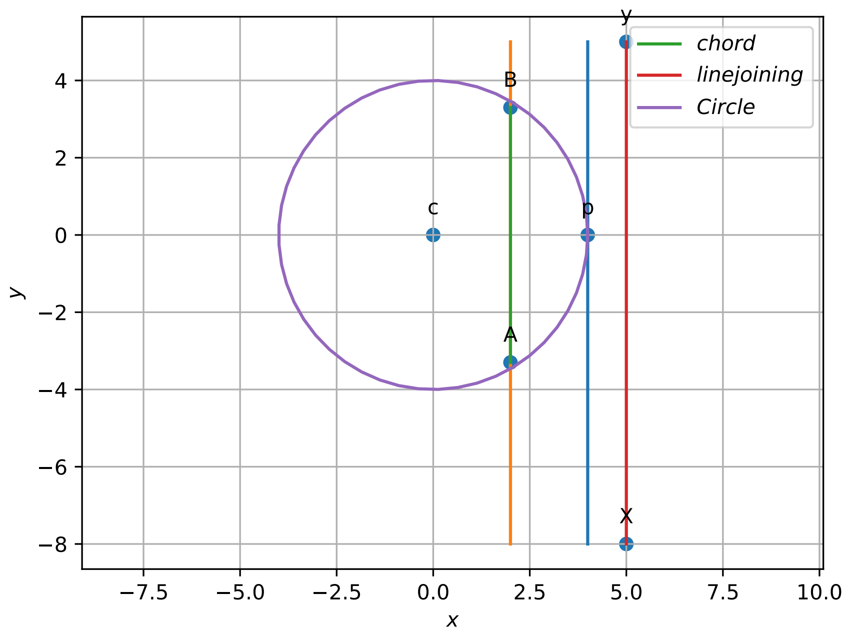


Figure 5.3.8.1:

Solution: The parameters of the circle in Fig. 5.3.8.1 are

$$\mathbf{u} = \mathbf{0}, f = -16 \tag{5.3.8.1}$$

Considering the given line to be

$$\mathbf{e}_1^\top \mathbf{x} = 5 \tag{5.3.8.2}$$

the tangent to the circle will be

$$\mathbf{e}_1^\top \mathbf{x} = 4 \tag{5.3.8.3}$$

and the secant will be

$$\mathbf{e}_1^\top \mathbf{x} = c \tag{5.3.8.4}$$

where

$$|c| < 4 \tag{5.3.8.5}$$

5.3.9 From a point $\mathbf{Q}$, the length of the tangent to a circle is $24cm$ and the distance of $\mathbf{Q}$ from the centre is $25cm$. Find the radius of the circle. Draw the circle and the tangents.

Solution: Let

$$\mathbf{Q} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{5.3.9.1}$$

and $\mathbf{O}$ be the centre of the circle. Let $\mathbf{R}_1$ and $\mathbf{R}_2$ be the two points on the circle such that R_1Q and R_2Q are tangents to the circle from the point $\mathbf{Q}$. Given that,

$$OQ = 25, R_1Q = R_2Q = 24 \tag{5.3.9.2}$$

$$\therefore \mathbf{O} = \begin{pmatrix} 25 \\ 0 \end{pmatrix} \tag{5.3.9.3}$$

$$r = OR_1 = \sqrt{OQ^2 - R_1Q^2} \tag{5.3.9.4}$$

$$= \sqrt{25^2 - 24^2} \tag{5.3.9.5}$$

$$= 7 \tag{5.3.9.6}$$

We have to find points $\mathbf{R}_1$ and $\mathbf{R}_2$. We know that the equation to the circle is given as

$$\|\mathbf{x}\|^2 + 2\mathbf{x}^\top \mathbf{u} + f = 0 \tag{5.3.9.7}$$

where

$$\mathbf{u} = -\mathbf{O} = -\begin{pmatrix} 25 \\ 0 \end{pmatrix} \text{ and} \tag{5.3.9.8}$$

$$f = \|\mathbf{O}\|^2 - r^2 = 576 \tag{5.3.9.9}$$

The matrix

$$\mathbf{\Sigma} = (\mathbf{Q}+\mathbf{u})(\mathbf{Q}+\mathbf{u})^\top - \left(\|\mathbf{Q}\|^2 + 2\mathbf{u}^\top\mathbf{Q} + f\right)\mathbf{I} \tag{5.3.9.10}$$

$$= \left(\begin{pmatrix}0\\0\end{pmatrix} - \begin{pmatrix}25\\0\end{pmatrix}\right)\left(\begin{pmatrix}0\\0\end{pmatrix} - \begin{pmatrix}25\\0\end{pmatrix}\right)^\top - \left(0 - 2\begin{pmatrix}25 & 0\end{pmatrix}\begin{pmatrix}0\\0\end{pmatrix} + 576\right)\mathbf{I} \tag{5.3.9.11}$$

$$= \begin{pmatrix}-25\\0\end{pmatrix}\begin{pmatrix}-25 & 0\end{pmatrix} - (576)\,\mathbf{I} \tag{5.3.9.12}$$

$$= \begin{pmatrix}625 & 0\\0 & 0\end{pmatrix} - \begin{pmatrix}576 & 0\\0 & 576\end{pmatrix} \tag{5.3.9.13}$$

$$= \begin{pmatrix}49 & 0\\0 & -576\end{pmatrix} \tag{5.3.9.14}$$

From (5.3.9.14), we can deduce Eigen pairs as follows

$$\lambda_1 = 49, \lambda_2 = -576 \tag{5.3.9.15}$$

$$\mathbf{p_1} = \begin{pmatrix}1\\0\end{pmatrix}, \mathbf{p_2} = \begin{pmatrix}0\\1\end{pmatrix} \tag{5.3.9.16}$$

Then

$$\mathbf{n_1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{|\lambda_1|} \\ \sqrt{|\lambda_2|} \end{pmatrix} = \begin{pmatrix} 7 \\ 24 \end{pmatrix} \tag{5.3.9.17}$$

$$\mathbf{n_2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{|\lambda_1|} \\ -\sqrt{|\lambda_2|} \end{pmatrix} = \begin{pmatrix} 7 \\ -24 \end{pmatrix} \tag{5.3.9.18}$$

The points of contact of a tangent on a circle from an external point is given by

$$\mathbf{q_{ij}} = \left(\pm r \frac{\mathbf{n_j}}{\|\mathbf{n_j}\|} - \mathbf{u} \right), \quad i, j = 1, 2 \tag{5.3.9.19}$$

$$\mathbf{q_{i1}} = \left(\pm r \frac{\mathbf{n_1}}{\|\mathbf{n_1}\|} - \mathbf{u} \right) \tag{5.3.9.20}$$

$$= \left(\pm \frac{7}{25} \begin{pmatrix} 7 \\ 24 \end{pmatrix} + \begin{pmatrix} 25 \\ 0 \end{pmatrix} \right) \tag{5.3.9.21}$$

$$= \left(\pm \begin{pmatrix} \frac{49}{25} \\ \frac{168}{25} \end{pmatrix} + \begin{pmatrix} 25 \\ 0 \end{pmatrix} \right) \tag{5.3.9.22}$$

$$= \begin{pmatrix} \frac{674}{25} \\ \frac{168}{25} \end{pmatrix}, \begin{pmatrix} \frac{576}{25} \\ -\frac{168}{25} \end{pmatrix} \tag{5.3.9.23}$$

$$\mathbf{q_{i2}} = \left(\pm r \frac{\mathbf{n_2}}{\|\mathbf{n_2}\|} - \mathbf{u}\right) \tag{5.3.9.24}$$

$$= \left(\pm \frac{7}{25}\begin{pmatrix} 7 \\ -24 \end{pmatrix} + \begin{pmatrix} 25 \\ 0 \end{pmatrix}\right) \tag{5.3.9.25}$$

$$= \left(\pm \begin{pmatrix} \frac{49}{25} \\ \frac{-168}{25} \end{pmatrix} + \begin{pmatrix} 25 \\ 0 \end{pmatrix}\right) \tag{5.3.9.26}$$

$$= \begin{pmatrix} \frac{674}{25} \\ \frac{-168}{25} \end{pmatrix}, \begin{pmatrix} \frac{576}{25} \\ \frac{168}{25} \end{pmatrix} \tag{5.3.9.27}$$

$$\therefore \mathbf{R}_1 = \mathbf{q}_{22} = \begin{pmatrix} \frac{576}{25} \\ \frac{168}{25} \end{pmatrix} \tag{5.3.9.28}$$

$$\mathbf{R}_2 = \mathbf{q}_{12} = \begin{pmatrix} \frac{576}{25} \\ -\frac{168}{25} \end{pmatrix} \tag{5.3.9.29}$$

The figure is as shown in 5.3.9.1

5.3.10 In Fig. 5.3.10.1, if TP and TQ are two tangents to a circle with centre O so that $\angle POQ = 110°$ then find $\angle PTQ$.

Solution: Let the output angle be ϕ. The input parameters are given as

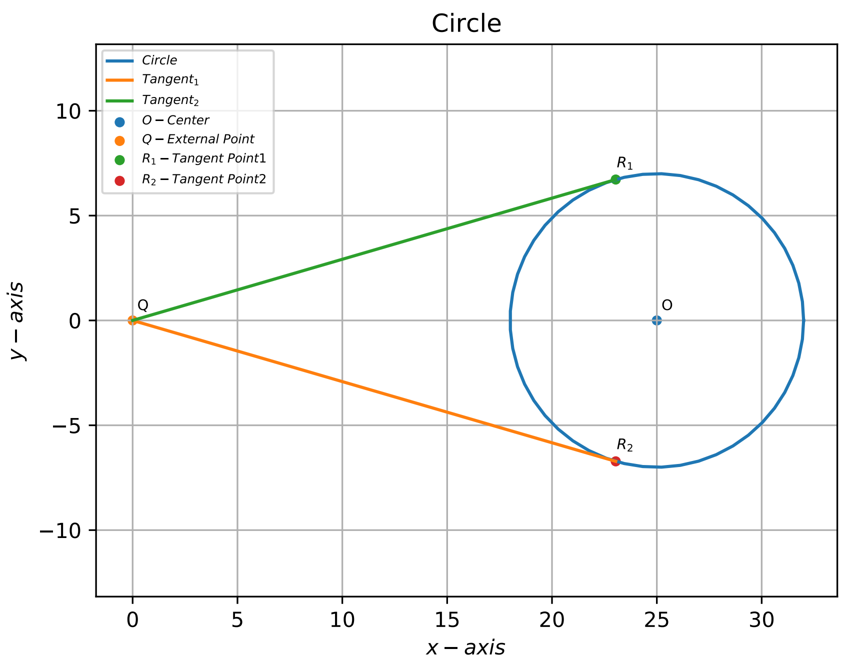


Figure 5.3.9.1:

Input Parameters	**Value**	**Description**
$\mathbf{O}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	Centre of the circle
r	1cm	radius of the circle
θ	$110°$	$\angle POQ$

Table 5.3.10.1:

Any point $\mathbf{X}$ on the circle is given as

$$\mathbf{X} = \mathbf{O} + r \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \tag{5.3.10.1}$$

So points $\mathbf{P}$ and $\mathbf{Q}$ can be calculated as

$$\mathbf{P} = \mathbf{O} + \begin{pmatrix} \cos\theta \\ \\ \sin\theta \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \\ \sin\theta \end{pmatrix} \tag{5.3.10.2}$$

$$\mathbf{Q} = \mathbf{e}_1 \tag{5.3.10.3}$$

For tangent TP

$$\mathbf{n}_1 = \mathbf{P} - \mathbf{O} \tag{5.3.10.4}$$

$$= \begin{pmatrix} \cos\theta \\ \\ \sin\theta \end{pmatrix} = \begin{pmatrix} 1 \\ \\ \tan\theta \end{pmatrix} \tag{5.3.10.5}$$

$$\mathbf{m}_1 = \begin{pmatrix} 1 \\ \\ -\cot\theta \end{pmatrix} \tag{5.3.10.6}$$

For tangent TQ

$$\mathbf{n}_2 = \mathbf{e}_1 - \mathbf{O} \tag{5.3.10.7}$$

$$= \mathbf{e}_1 \tag{5.3.10.8}$$

$$\mathbf{m}_2 = \mathbf{e}_2 \tag{5.3.10.9}$$

The equation of TP is given as

$$\mathbf{n}_1^\top (\mathbf{x} - \mathbf{P}) = 0 \tag{5.3.10.10}$$

$$\mathbf{n}_1^\top \left(\mathbf{x} - \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}\right) = 0 \tag{5.3.10.11}$$

$$\begin{pmatrix} \cos\theta & \sin\theta \end{pmatrix} \mathbf{x} = 1 \tag{5.3.10.12}$$

The equation of TQ is given as

$$\mathbf{n}_2^\top (\mathbf{x} - \mathbf{e}_1) = 0 \tag{5.3.10.13}$$

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 1 \tag{5.3.10.14}$$

The tangent point can be calculated by solving (5.3.10.12) and (5.3.10.14)

$$\begin{pmatrix} \cos\theta & \sin\theta \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{5.3.10.15}$$

$$\implies \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ \tan\frac{\theta}{2} \end{pmatrix} \tag{5.3.10.16}$$

Now, $\mathbf{T}$ =(5.3.10.16), since it is the intersection of TP and TQ. Hence, it is given as

$$\mathbf{T} = \begin{pmatrix} 1 \\ \tan 55° \end{pmatrix} = \begin{pmatrix} 1 \\ 1.428 \end{pmatrix} \tag{5.3.10.17}$$

The angle between two lines with slope $\mathbf{m}_1$ and $\mathbf{m}_2$ is given as

$$\cos\phi = \frac{\mathbf{m}_1^\top \mathbf{m}_2}{\left\|\mathbf{m}_1\right\| \left\|\mathbf{m}_2\right\|} \tag{5.3.10.18}$$

$$= \frac{\begin{pmatrix} 1 & -\cot\theta \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}}{(\csc\theta)(1)} \tag{5.3.10.19}$$

$$= -\cos\theta \tag{5.3.10.20}$$

$$\implies \cos\phi = -\cos\theta \tag{5.3.10.21}$$

Hence,

$$\phi = \cos^{-1}\left(\cos\left(180^\circ - \theta\right)\right) \tag{5.3.10.22}$$

$$= 180^\circ - \theta = 70^\circ \tag{5.3.10.23}$$

Hence, $\angle PTQ = 70^\circ$. See Fig 5.3.10.1

5.3.11 If the tangents PA and PB from a point $\mathbf{P}$ to a circle with center $\mathbf{O}$ are inclined to each other at 80°, find $\angle POA$.

Solution: The input parameters are listed in Table 5.3.11.1. Since

$$\angle APB = 80^\circ \tag{5.3.11.1}$$

$$\angle APO = \frac{1}{2}\angle APB \tag{5.3.11.2}$$

$$= 40^\circ \quad = \theta\ (say) \tag{5.3.11.3}$$

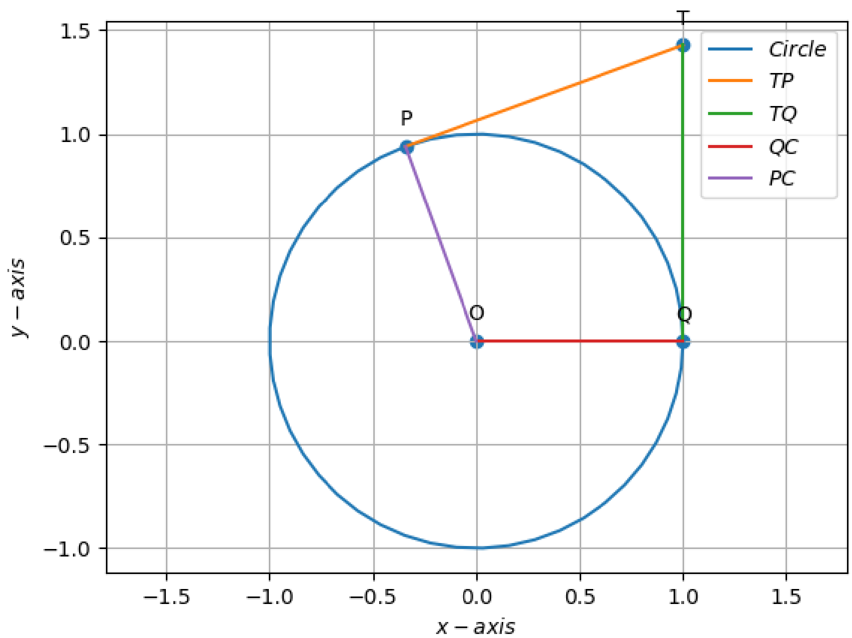


Figure 5.3.10.1:

Therefore, it can be said that **P** lies on the line

$$\begin{pmatrix} -\sin\theta & \cos\theta \end{pmatrix} \mathbf{x} = 0 \tag{5.3.11.4}$$

Let the circle be $\|\mathbf{x}\|^2 = r^2$ and $\mathbf{A}$ be $\begin{pmatrix} 0 \\ r \end{pmatrix}$. Therefore, the tangent that P lies on is given by

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = r \tag{5.3.11.5}$$

The point $\mathbf{P}$ is given by:

$$\begin{pmatrix} -\cos\theta & \sin\theta \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 0 \\ r \end{pmatrix} \tag{5.3.11.6}$$

The augmented matrix for the above equation is

$$\left(\begin{array}{cc|c} -\sin\theta & \cos\theta & 0 \\ 0 & 1 & r \end{array}\right) \tag{5.3.11.7}$$

$$\xleftrightarrow{R_1 \leftarrow \frac{R_1}{-\sin\theta} + \cot\theta R_2} \tag{5.3.11.8}$$

$$\left(\begin{array}{cc|c} 1 & 0 & r\cot\theta \\ 0 & 1 & r \end{array}\right) \tag{5.3.11.9}$$

yielding

$$\mathbf{P} = \begin{pmatrix} r\cot\theta \\ r \end{pmatrix} \tag{5.3.11.10}$$

Let $\angle POA = \phi$. Then,

$$\cos\phi = \frac{(\mathbf{P}-\mathbf{O})^\top(\mathbf{A}-\mathbf{O})}{\|\mathbf{P}-\mathbf{O}\|\,\|\mathbf{A}-\mathbf{O}\|} \tag{5.3.11.11}$$

$$= \frac{\begin{pmatrix} r\cot\theta & r \end{pmatrix}\begin{pmatrix} 0 \\ r \end{pmatrix}}{r^2\operatorname{cosec}\theta} \tag{5.3.11.12}$$

$$\cos\phi = \sin\theta \tag{5.3.11.13}$$

$$\implies \phi = 90^\circ - \theta \tag{5.3.11.14}$$

$$\phi = 50^\circ \tag{5.3.11.15}$$

See Fig. 5.3.11.1.

$\mathbf{O}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	Center of the given circle
$\mathbf{A}$	$\begin{pmatrix} 0 \\ 5 \end{pmatrix}$	Point where tangent is taken
r	5	radius of given circle
$\angle APB$	80°	Angle between tangents

Table 5.3.11.1: Table 1

5.3.12 Show that the tangents of circle drawn at the ends of diameter are parallel.

Solution: See Fig. 5.3.12.1. Let $\mathbf{A}, \mathbf{B}$ be the end points of the diameter

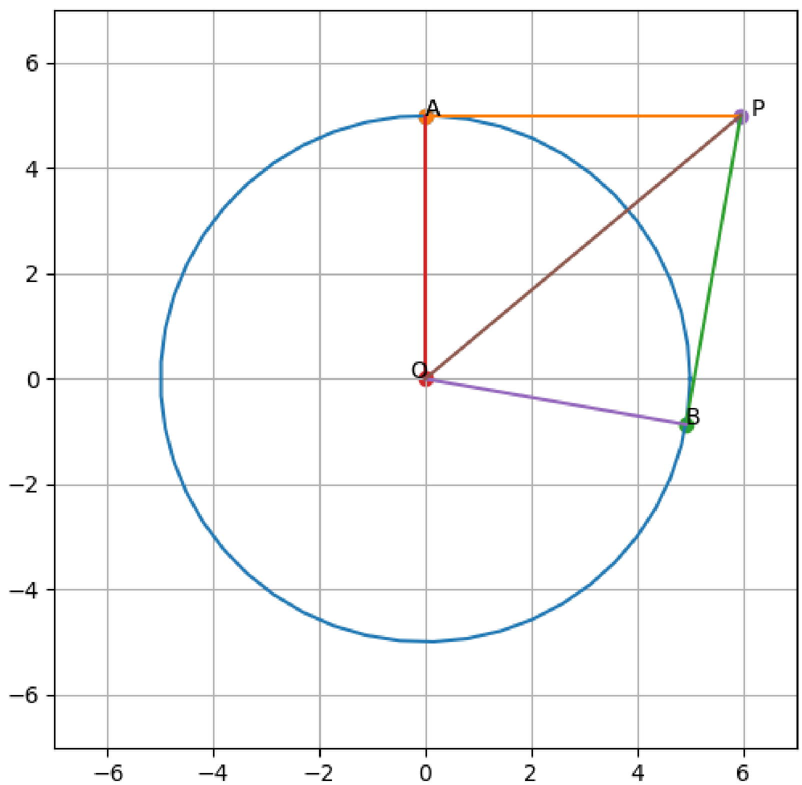


Figure 5.3.11.1: Figure 1

of the circle through which the tangents are drawn. From (E.2.2.1),

$$\frac{\mathbf{A}+\mathbf{B}}{2} = -\mathbf{u} \tag{5.3.12.1}$$

$$\implies \mathbf{A}+\mathbf{B} = -2\mathbf{u} \tag{5.3.12.2}$$

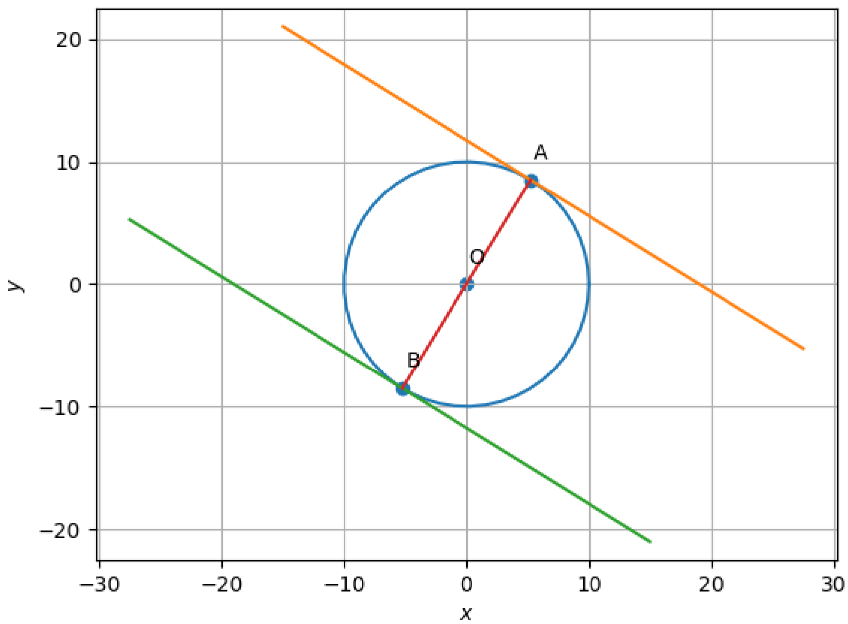


Figure 5.3.12.1:

From (G.3.2.1),

$$\mathbf{m}_1^\top (\mathbf{A} + \mathbf{u}) = 0 \tag{5.3.12.3}$$

$$\mathbf{m}_2^\top (\mathbf{B} + \mathbf{u}) = 0 \tag{5.3.12.4}$$

where $\mathbf{m_1}, \mathbf{m_2}$ are the direction vectors of the tangents at $\mathbf{A}, \mathbf{B}$ respectively. Then, the normal vectors at the point of contact of tangets are

$$\mathbf{A} + \mathbf{u} = k_1\mathbf{n_1} \tag{5.3.12.5}$$

$$\mathbf{B} + \mathbf{u} = k_2\mathbf{n_2} \tag{5.3.12.6}$$

Adding (5.3.12.5) and (5.3.12.6),

$$k_1\mathbf{n_1} + k_2\mathbf{n_2} = \mathbf{A} + \mathbf{B} + 2\mathbf{u} \tag{5.3.12.7}$$

$$= \mathbf{0} \tag{5.3.12.8}$$

from (5.3.12.2), (5.3.12.8) can be expressed as

$$k_1\mathbf{n_1} + k_2\mathbf{n_2} = 0 \tag{5.3.12.9}$$

$$k_1\mathbf{n_1} = -k_2\mathbf{n_2} \tag{5.3.12.10}$$

Since

$$\mathbf{n_1} \times \mathbf{n_2} = \mathbf{0}, \tag{5.3.12.11}$$

$$\mathbf{n_1} \parallel \mathbf{n_2} \implies \mathbf{m_1} \parallel \mathbf{m_2} \tag{5.3.12.12}$$

5.3.13

5.3.14 The length of a tangent from a point $\mathbf{A}$ at distance 5 cm from the centre of the circle is 4 cm. Find the radius of the circle.

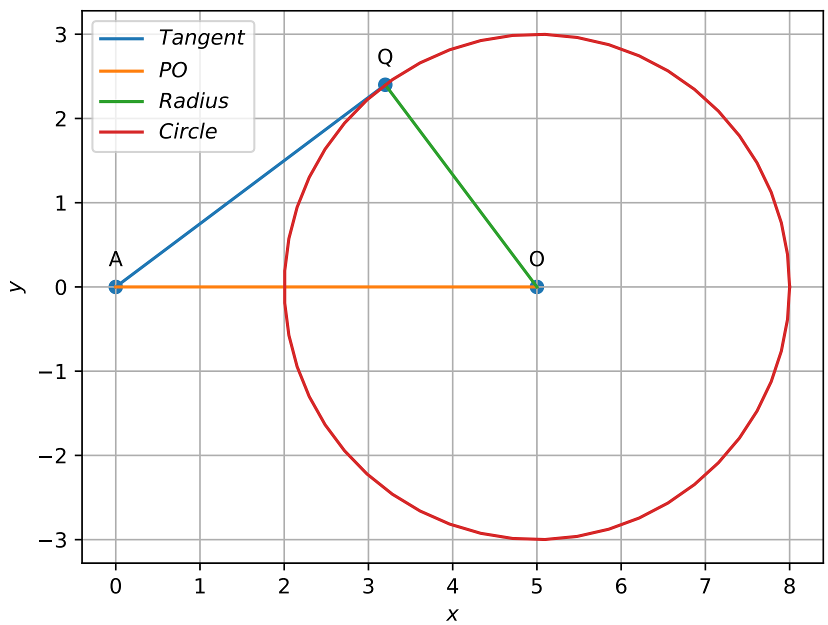


Figure 5.3.14.1:

Solution: From the Baudhayana theorem, the radius

$$r = 3 \tag{5.3.14.1}$$

Let

$$\mathbf{A} = \mathbf{O} \text{ and } \mathbf{O} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \tag{5.3.14.2}$$

The equation of the circle can then be expressed as

$$\|\mathbf{x}\|^2 + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{5.3.14.3}$$

where

$$\mathbf{u} = -\mathbf{O} = -\begin{pmatrix} 5 \\ 0 \end{pmatrix} \tag{5.3.14.4}$$

$$f = \|\mathbf{u}\|^2 - r^2 = 16 \tag{5.3.14.5}$$

From (G.4.9.2),

$$\mathbf{\Sigma} = (\mathbf{A} + \mathbf{u})(\mathbf{A} + \mathbf{u})^\top - \left(\mathbf{A}^\top \mathbf{A} + 2\mathbf{u}^\top \mathbf{A} + f\right)\mathbf{I} \tag{5.3.14.6}$$

$$= \begin{pmatrix} 9 & 0 \\ 0 & -16 \end{pmatrix} \tag{5.3.14.7}$$

Thus, from (G.4.9.1),

$$\mathbf{P} = \mathbf{I}, \lambda_1 = 9, \lambda_2 = -16 \tag{5.3.14.8}$$

$$\implies \mathbf{n}_1 = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \quad \text{and } \mathbf{n}_2 = \begin{pmatrix} 3 \\ -4 \end{pmatrix} \tag{5.3.14.9}$$

Substituting from the above in (G.4.6.1),

$$\mathbf{q}_{22} = \frac{1}{5}\begin{pmatrix} 16 \\ 12 \end{pmatrix} = \mathbf{Q} \tag{5.3.14.10}$$

in Fig. 5.3.14.1.

5.3.15 Two concentric circles are of radii 5cm and 3cm. Find the length of the chord of the larger circle which touches the smaller circle.

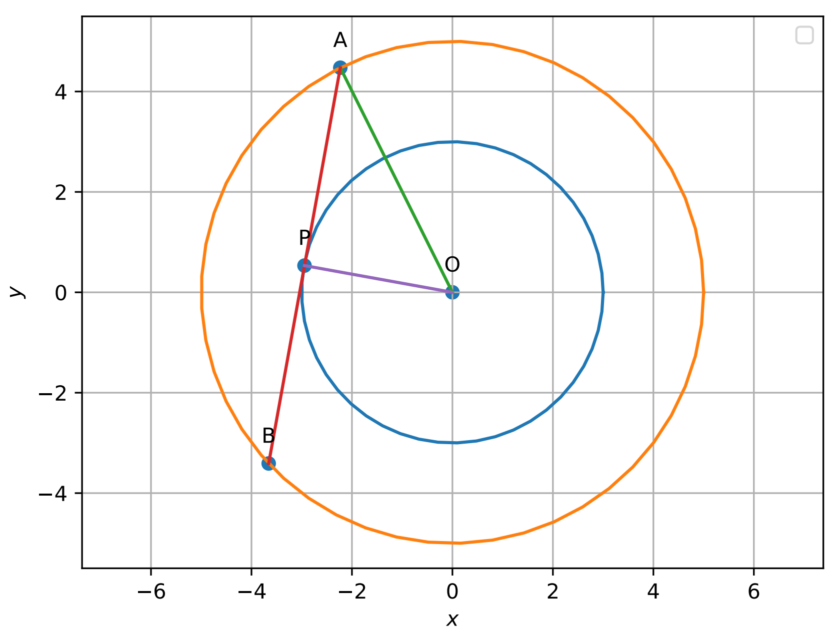


Figure 5.3.15.1:

Solution: See Fig. 5.3.15.1. Let

$$\mathbf{O} = \mathbf{0} \tag{5.3.15.1}$$

$$r_1 = 5, r_2 = 3. \tag{5.3.15.2}$$

Choosing

$$\mathbf{A} = r_1 \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \tag{5.3.15.3}$$

$\mathbf{P}$ can be obtained following the approach in Problem 5.3.15. From Appendix E.2.5, $\mathbf{P}$ is the mid point of AB. This can be used to obtain $\mathbf{B}$.

5.3.16 A quadrilateral $ABCD$ is drawn to circumscribe a circle. Show that $AB + CD$ is equal to $BC + AD$

Solution:

(a) Draw the circle.

(b) Choose the point $\mathbf{A}$.

(c) Draw the tangents from $\mathbf{A}$ to the circle.

(d) Choose points $\mathbf{B}, \mathbf{D}$ on the tangents.

(e) From $\mathbf{B}, \mathbf{D}$, draw tangents to the circle intersecting at $\mathbf{C}$.

5.3.17 In Fig. 5.3.17.1, XY and EF are two parallel tangents to a circle with centre $\mathbf{O}$ and another tangent AB with point of contact $\mathbf{C}$ intersecting XY at $\mathbf{A}$ and EF at $\mathbf{B}$. Prove that $\angle AOB = 90°$.

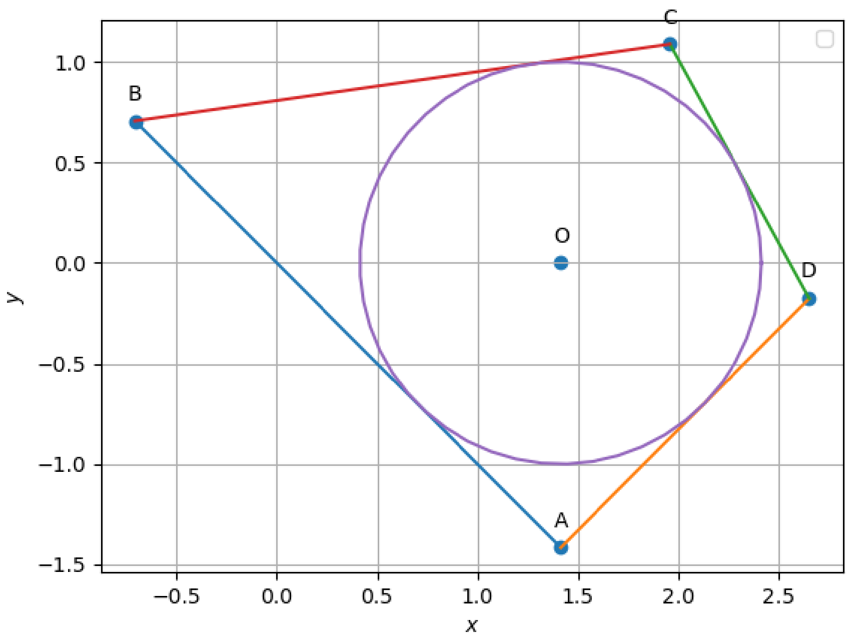


Figure 5.3.16.1:

Solution:

5.3.18 Prove that the angle between the two tangents drawn from an external point to a circle is supplementary to the angle subtended by the line-segment joining the points of contact at the centre.

Solution: Follow the approach in Problem 5.3.14 for constructing the tangents to the circle.

5.3.19

5.3.20 A triangle ABC is drawn to circumscribe a circle of radius 4cm such that the segments BD and DC into which BC is divided by the point

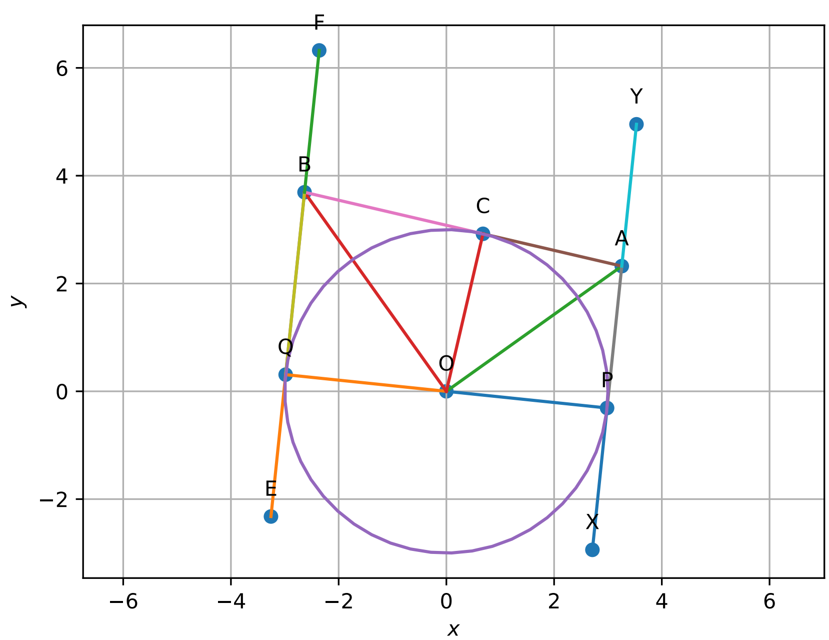


Figure 5.3.17.1:

of contact D are of lengths 8cm and 6cm respectively. Find the sides AB and AC.

5.3.21 Prove that opposite sides of a quadrilateral circumscribing a circle subtend supplementary angles at the centre of the circle.

Solution: We begin by proving a useful lemma.

Lemma 5.3.1. The line joining the centre of the circle to an external point bisects the angle subtended by the tangent chord at the centre.

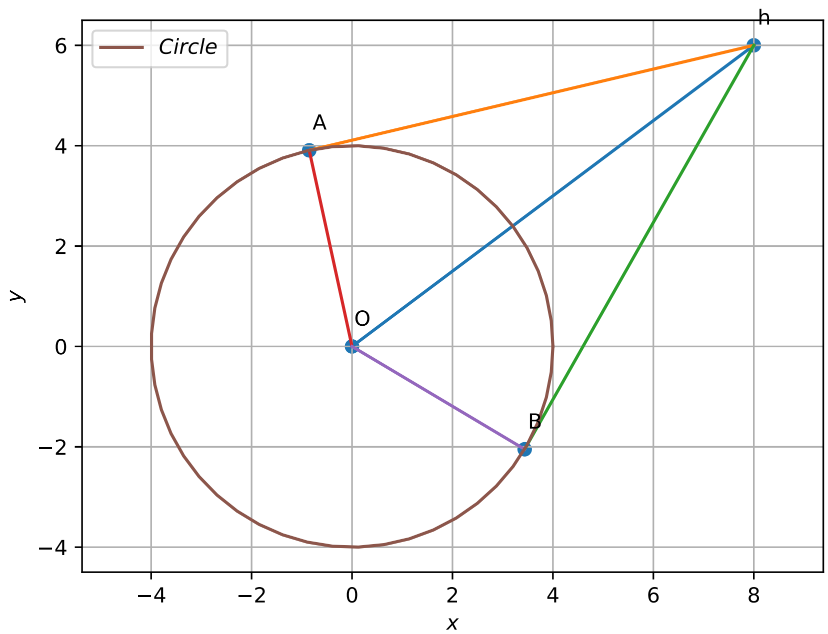


Figure 5.3.18.1:

Proof. Refer to Fig. 5.3.21.1. Set $\mathbf{O}$ to be the origin. Since $OA \perp AP$,

$$\mathbf{A}^\top (\mathbf{A} - \mathbf{P}) = 0 \tag{5.3.21.1}$$

$$\implies \mathbf{A}^\top \mathbf{P} = \|\mathbf{A}\|^2 \tag{5.3.21.2}$$

Similarly,

$$\mathbf{B}^\top \mathbf{P} = \|\mathbf{B}\|^2 \tag{5.3.21.3}$$

Since $\mathbf{A}$ and $\mathbf{B}$ lie on the circle, their norms are equal. Thus, from

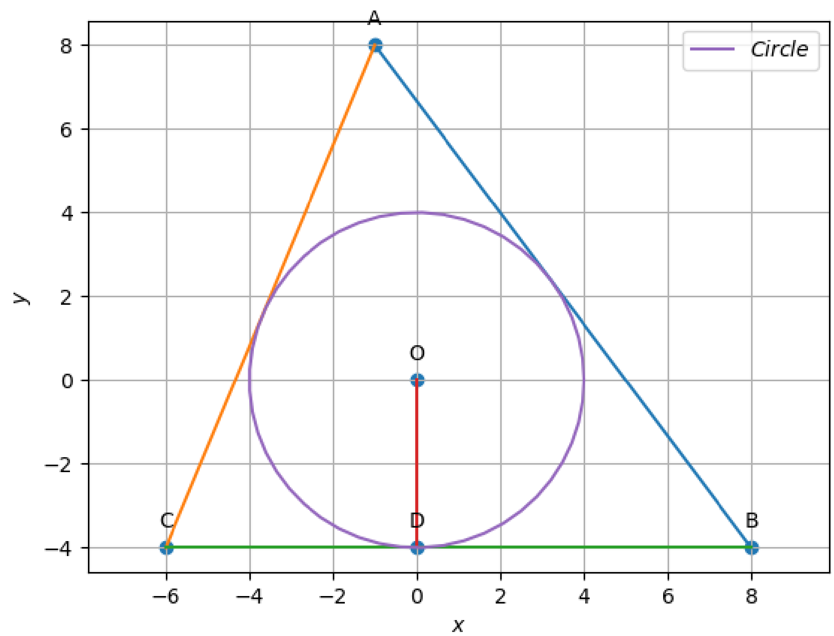


Figure 5.3.20.1:

(5.3.21.2) and (5.3.21.3),

$$\mathbf{A}^{\top}\mathbf{P} = \mathbf{B}^{\top}\mathbf{P} \tag{5.3.21.4}$$

and the lemma follows. $\square$

Call the quadrilateral $ABCD$, where

$$\mathbf{A} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}, \ \mathbf{C} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{5.3.21.5}$$

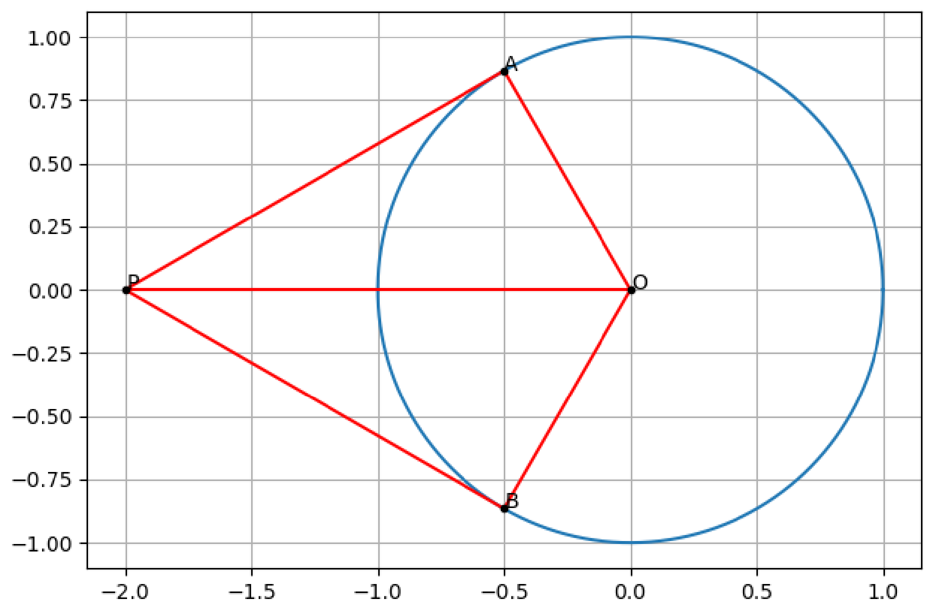


Figure 5.3.21.1: OP bisects $\angle AOB$.

Suppose that $ABCD$ circumscribes the unit circle, given by

$$\mathbf{x}^\top \mathbf{x} - 1 = 0 \tag{5.3.21.6}$$

Comparing (5.3.21.6) with the general equation of the circle,

$$\mathbf{u} = \mathbf{0}, \;\; f = -1 \tag{5.3.21.7}$$

To find the points of contact from $\mathbf{A}$, we have

$$\mathbf{\Sigma_A} = (\mathbf{A} + \mathbf{u})(\mathbf{A} + \mathbf{u})^\top - \left(\mathbf{A}^\top\mathbf{A} + 2\mathbf{u}^\top\mathbf{A} + f\right)\mathbf{I} \tag{5.3.21.8}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix} \tag{5.3.21.9}$$

The eigenvalues of $\mathbf{\Sigma_A}$ are

$$\lambda_1 = 1,\ \lambda_2 = -3 \tag{5.3.21.10}$$

and since the eigenvector matrix $\mathbf{P_A} = \mathbf{I}$,

$$\mathbf{n_1} = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix},\ \mathbf{n_2} = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} \tag{5.3.21.11}$$

Thus, the points of contact are given by

$$\mathbf{E} = \frac{1}{2}\begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix},\ \mathbf{H} = \frac{1}{2}\begin{pmatrix} -1 \\ -\sqrt{3} \end{pmatrix} \tag{5.3.21.12}$$

Similarly for $\mathbf{C}$,

$$\mathbf{\Sigma_C} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - \mathbf{I} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{5.3.21.13}$$

Notice that

$$\mathbf{\Sigma_C}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} \tag{5.3.21.14}$$

$$\mathbf{\Sigma_C}\begin{pmatrix}1\\-1\end{pmatrix} = -\begin{pmatrix}1\\-1\end{pmatrix} \tag{5.3.21.15}$$

$$\tag{5.3.21.16}$$

Thus, the eigenvalues and the corresponding eigenvector matrix is

$$\mu_1 = 1,\ \mu_2 = -1,\ \mathbf{P_C} = \begin{pmatrix}1 & 1\\1 & -1\end{pmatrix} \tag{5.3.21.17}$$

and thus

$$\mathbf{m_1} = \mathbf{P_C}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}2\\0\end{pmatrix} \tag{5.3.21.18}$$

$$\mathbf{m_2} = \mathbf{P_C}\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}0\\2\end{pmatrix} \tag{5.3.21.19}$$

Therefore, the points of contact of $\mathbf{C}$ are

$$\mathbf{F} = \begin{pmatrix}1\\0\end{pmatrix},\ \mathbf{G} = \begin{pmatrix}0\\1\end{pmatrix} \tag{5.3.21.20}$$

Using the lemma we proved above, the direction vectors of $\mathbf{B}$ and $\mathbf{D}$

are

$$\mathbf{d_B} = \mathbf{E} + \mathbf{F} = \frac{\sqrt{3}}{2}\begin{pmatrix}\sqrt{3}\\ \\ 1\end{pmatrix} \tag{5.3.21.21}$$

$$\mathbf{d_D} = \mathbf{G} + \mathbf{H} = \frac{1}{2}\begin{pmatrix}-1\\ \\ 2-\sqrt{3}\end{pmatrix} \tag{5.3.21.22}$$

Clearly,

$$\|\mathbf{d_B}\| = \sqrt{3} \tag{5.3.21.23}$$

$$\|\mathbf{d_D}\| = \sqrt{2-\sqrt{3}} \tag{5.3.21.24}$$

and from (5.3.21.5), (5.3.21.21) and (5.3.21.22),

$$\cos\angle AOD = \frac{\mathbf{A}^\top \mathbf{d_D}}{\|A\|\,\|\mathbf{d_D}\|} \tag{5.3.21.25}$$

$$= \frac{-1}{2\sqrt{2\sqrt{3}}} \tag{5.3.21.26}$$

$$= -\frac{\sqrt{2+\sqrt{3}}}{2} \tag{5.3.21.27}$$

$$= -\frac{\sqrt{3}+1}{2\sqrt{2}} \tag{5.3.21.28}$$

$$\cos\angle BOC = \frac{\mathbf{C}^\top \mathbf{d_B}}{\|C\|\,\|\mathbf{d_B}\|} \tag{5.3.21.29}$$

$$= \frac{\sqrt{3}+1}{2\sqrt{2}} \tag{5.3.21.30}$$

Hence, from (5.3.21.24), (5.3.21.28) and (5.3.21.30),

$$\cos \angle AOD + \cos \angle BOC = 0 \tag{5.3.21.31}$$

and hence, $\angle AOD + \angle BOC = \pi$, as required.

The situation is illustrated in Fig. 5.3.21.2. The numerical parameters used in the construction are shown in Table 5.3.21.1.

Parameter	**Value**
r	1
$\mathbf{A}$	$\begin{pmatrix} -2 \\ 0 \end{pmatrix}$
$\mathbf{C}$	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Table 5.3.21.1: Parameters used in the construction of Fig. 5.3.21.2.

5.4. Exercises

5.4.1 To draw a pair of tangents to a circle which are inclined to each other at an angle of 60°, it is required to draw tangents at end points of those two radii of the circle, the angle between them should be

(a) 135°

(b) 90°

(c) 60°

(d) 120°

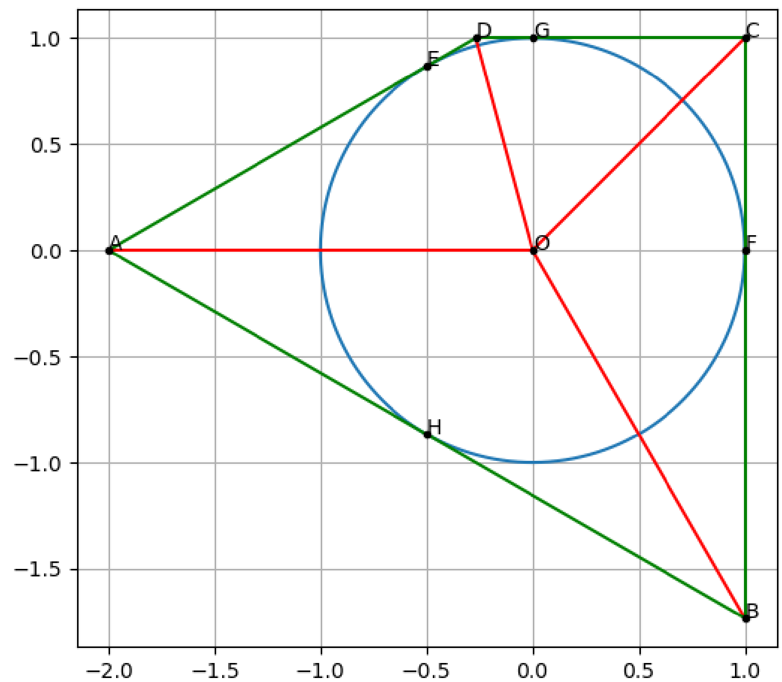


Figure 5.3.21.2: Angles subtended by the opposite sides of a circumscribing quadrilateral at the center of its incircle are supplementary.

5.4.2 Draw two concentric circles of radii 3 cm and 5 cm. Taking a point on outer circle construct the pair of tangents to the other. Measure the length of a tangent and verify it by actual calculation.

5.4.3 Draw a circle of radius 4 cm .Construct a pair of tangents to it, the angle detween which is 60°. Also justify the construction. Measure the distance between the centre of the circle and the point of intersection of tangents.

5.4.4 Construct a tangent to a circle of radius 4 cm from a point which is at a distance of 6 cm from its centre.

Write True or False and give reasons for your answer in each of the following

5.4.5 A pair of tangents can be constructed from a point **p** to a circle of radius 3.5 cm situated at a distance of 3 cm from the centre.

5.4.6 A pair of tangents can be constructed to a circle inclined at an angle of 170°.

Appendix A

Vectors

A.1. 2×1 vectors

A.1.1. Let

$$\mathbf{A} \equiv \overrightarrow{A} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \tag{A.1.1.1}$$

$$\equiv a_1 \overrightarrow{i} + a_2 \overrightarrow{j}, \tag{A.1.1.2}$$

$$\mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \tag{A.1.1.3}$$

be 2×1 vectors. Then, the determinant of the 2×2 matrix

$$\mathbf{M} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} \tag{A.1.1.4}$$

is defined as

$$\left|\mathbf{M}\right| = \left|\mathbf{A} \quad \mathbf{B}\right| \tag{A.1.1.5}$$

$$= \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1 \tag{A.1.1.6}$$

A.1.2. The value of the cross product of two vectors is given by (A.1.1.5).

A.1.3. The area of the triangle with vertices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ is given by

$$\frac{1}{2}\left\|(\mathbf{A}-\mathbf{B}) \times (\mathbf{A}-\mathbf{C})\right\| = \frac{1}{2}\left\|\mathbf{A}\times\mathbf{B} + \mathbf{B}\times\mathbf{C} + \mathbf{C}\times\mathbf{A}\right\| \tag{A.1.3.1}$$

A.1.4. If

$$\left\|\mathbf{A}\times\mathbf{B}\right\| = \left\|\mathbf{C}\times\mathbf{D}\right\|, \quad \text{then} \tag{A.1.4.1}$$

$$\mathbf{A}\times\mathbf{B} = \pm\left(\mathbf{C}\times\mathbf{D}\right) \tag{A.1.4.2}$$

where the sign depends on the orientation of the vectors.

A.1.5. The median divides the triangle into two triangles of equal area.

A.1.6. The transpose of $\mathbf{A}$ is defined as

$$\mathbf{A}^\top = \begin{pmatrix} a_1 & a_2 \end{pmatrix} \tag{A.1.6.1}$$

A.1.7. The inner product or dot product is defined as

$$\mathbf{A}^\top \mathbf{B} \equiv \mathbf{A} \cdot \mathbf{B} \tag{A.1.7.1}$$

$$= \begin{pmatrix} a_1 & a_2 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = a_1 b_1 + a_2 b_2 \tag{A.1.7.2}$$

A.1.8. norm of $\mathbf{A}$ is defined as

$$\|A\| \equiv \left|\overrightarrow{A}\right| \tag{A.1.8.1}$$

$$= \sqrt{\mathbf{A}^\top \mathbf{A}} = \sqrt{a_1^2 + a_2^2} \tag{A.1.8.2}$$

Thus,

$$\|\lambda \mathbf{A}\| \equiv \left|\lambda \overrightarrow{A}\right| \tag{A.1.8.3}$$

$$= |\lambda| \, \|\mathbf{A}\| \tag{A.1.8.4}$$

A.1.9. The distance betwen the points $\mathbf{A}$ and $\mathbf{B}$ is given by

$$\|\mathbf{A} - \mathbf{B}\| \tag{A.1.9.1}$$

A.1.10. Let $\mathbf{x}$ be equidistant from the points $\mathbf{A}$ and $\mathbf{B}$. Then

$$(\mathbf{A} - \mathbf{B})^\top \mathbf{x} = \frac{\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2}{2} \tag{A.1.10.1}$$

Solution:

$$\|\mathbf{x} - \mathbf{A}\| = \|\mathbf{A} - \mathbf{B}\| \tag{A.1.10.2}$$

$$\implies \|\mathbf{x} - \mathbf{A}\|^2 = \|\mathbf{x} - \mathbf{B}\|^2 \tag{A.1.10.3}$$

which can be expressed as

$$\begin{aligned}(\mathbf{x} - \mathbf{A})^\top (\mathbf{x} - \mathbf{A}) = (\mathbf{x} - \mathbf{B})^\top (\mathbf{x} - \mathbf{B}) \\ \implies \|\mathbf{x}\|^2 - 2\mathbf{x}^\top\mathbf{A} + \|\mathbf{A}\|^2 \\ = \|\mathbf{x}\|^2 - 2\mathbf{x}^\top\mathbf{B} + \|\mathbf{B}\|^2\end{aligned} \tag{A.1.10.4}$$

which can be simplified to obtain (A.1.10.1).

A.1.11. If $\mathbf{x}$ lies on the x-axis and is equidistant from the points $\mathbf{A}$ and $\mathbf{B}$,

$$\mathbf{x} = x\mathbf{e}_1 \tag{A.1.11.1}$$

where

$$x = \frac{\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2}{2(\mathbf{A} - \mathbf{B})^\top \mathbf{e}_1} \tag{A.1.11.2}$$

Solution: From (A.1.10.1).

$$x(\mathbf{A} - \mathbf{B})^\top \mathbf{e}_1 = \frac{\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2}{2} \tag{A.1.11.3}$$

yielding (A.1.11.2).

A.1.12. The angle between two vectors is given by

$$\theta = \cos^{-1} \frac{\mathbf{A}^\top \mathbf{B}}{\|A\| \, \|B\|} \tag{A.1.12.1}$$

A.1.13. If two vectors are orthogonal (perpendicular),

$$\mathbf{A}^\top \mathbf{B} = 0 \tag{A.1.13.1}$$

A.1.14. For an isoceles triangle ABC ith $AB = AC$, the median $AD \perp BC$.

A.1.15. The direction vector of the line joining two points $\mathbf{A}, \mathbf{B}$ is given by

$$\mathbf{m} = \mathbf{A} - \mathbf{B} \tag{A.1.15.1}$$

A.1.16. The points **AAA**

A.1.17. The unit vector in the direction of $\mathbf{m}$ is defined as

$$\frac{\mathbf{m}}{\|\mathbf{m}\|} \tag{A.1.17.1}$$

A.1.18. If the direction vector of a line is expressed as

$$\mathbf{m} = \begin{pmatrix} 1 \\ m \end{pmatrix}, \tag{A.1.18.1}$$

the m is defined to be the slope of the line.

A.1.19. $AB \parallel CD$ if

$$\mathbf{A} - \mathbf{B} = k\left(\mathbf{C} - \mathbf{D}\right) \tag{A.1.19.1}$$

A.1.20. The normal vector to $\mathbf{m}$ is defined by

$$\mathbf{m}^\top \mathbf{n} = 0 \tag{A.1.20.1}$$

A.1.21. If

$$\mathbf{m}^\top \mathbf{n}_1 = 0 \tag{A.1.21.1}$$

$$\mathbf{m}^\top \mathbf{n}_2 = 0, \tag{A.1.21.2}$$

$$\mathbf{n}_1 \parallel \mathbf{n}_2 \tag{A.1.21.3}$$

A.1.22. The point $\mathbf{P}$ that divides the line segment AB in the ratio $k : 1$ is given by

$$\mathbf{P} = \frac{k\mathbf{B} + \mathbf{A}}{k+1} \tag{A.1.22.1}$$

A.1.23. The standard basis vectors are defined as

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ \\ 0 \end{pmatrix}, \tag{A.1.23.1}$$

$$\mathbf{e}_2 = \begin{pmatrix} 0 \\ \\ 1 \end{pmatrix}. \tag{A.1.23.2}$$

A.1.24. If $ABCD$ be a parallelogram,

$$\mathbf{B} - \mathbf{A} = \mathbf{C} - \mathbf{D} \tag{A.1.24.1}$$

A.1.25. Diagonals of a parallelogram bisect each other.

A.1.26. The area of the parallelogram with vertices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{D}$ is given by

$$\|(\mathbf{A} - \mathbf{B}) \times (\mathbf{A} - \mathbf{D})\| = \|\mathbf{A} \times \mathbf{B} + \mathbf{B} \times \mathbf{C} + \mathbf{C} \times \mathbf{A}\| \tag{A.1.26.1}$$

A.1.27. Points $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ form a triangle if

$$p\left(\mathbf{A} - \mathbf{B}\right) + q\left(\mathbf{A} - \mathbf{C}\right) = 0 \tag{A.1.27.1}$$

$$\text{or,}\;\; (p+q)\,\mathbf{A} - p\mathbf{B} - q\mathbf{C} = 0 \tag{A.1.27.2}$$

$$\implies p = 0, q = 0 \tag{A.1.27.3}$$

are linearly independent.

A.1.28. In $\triangle ABC$, if $\mathbf{D}, \mathbf{E}$ divide the lines AB, AC in the ratio $k:1$ respectively, then $DE \parallel BC$.

Proof. From (A.1.22.1),

$$\mathbf{D} = \frac{k\mathbf{B} + \mathbf{A}}{k+1} \tag{A.1.28.1}$$

$$\mathbf{E} = \frac{k\mathbf{C} + \mathbf{A}}{k+1} \tag{A.1.28.2}$$

$$\implies \mathbf{D} - \mathbf{E} = \frac{k}{k+1}\left(\mathbf{B} - \mathbf{C}\right) \tag{A.1.28.3}$$

Thus, from Appendix A.1.18, $DE \parallel BC$.

□

A.1.29. In $\triangle ABC$, if $DE \parallel BC$, $\mathbf{D}$ and $\mathbf{E}$ divide the lines AB, AC in the same ratio.

Proof. If $DE \parallel BC$, from (A.1.19.1)

$$\left(\mathbf{B} - \mathbf{C}\right) = k\left(\mathbf{D} - \mathbf{E}\right) \tag{A.1.29.1}$$

Using (A.1.22.1), let

$$\mathbf{D} = \frac{k_1\mathbf{B} + \mathbf{A}}{k_1+1} \tag{A.1.29.2}$$

$$\mathbf{E} = \frac{k_2\mathbf{C} + \mathbf{A}}{k_2+1} \tag{A.1.29.3}$$

Subtituting the above in (A.1.29.1), after some algebra, we obtain

$$(p+q)\,\mathbf{A} - p\mathbf{B} - q\mathbf{C} = 0 \tag{A.1.29.4}$$

where

$$p = \frac{1}{k} - \frac{k_1}{k_1+1}, q = \frac{1}{k} - \frac{k_1}{k_1+1} \tag{A.1.29.5}$$

From (A.1.27.2),

$$p = q = 0 \tag{A.1.29.6}$$

$$\implies k_1 = k_2 = \frac{1}{k-1} \tag{A.1.29.7}$$

□

A.2. 3×1 vectors

A.2.1. Let

$$\mathbf{A} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \equiv a_1 \overrightarrow{i} + a_2 \overrightarrow{j} + a_3 \overrightarrow{j}, \tag{A.2.1.1}$$

$$\mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \tag{A.2.1.2}$$

and

$$\mathbf{A}_{ij} = \begin{pmatrix} a_i \\ a_j \end{pmatrix}, \tag{A.2.1.3}$$

$$\mathbf{B}_{ij} = \begin{pmatrix} b_i \\ b_j \end{pmatrix}. \tag{A.2.1.4}$$

A.2.2. The <u>cross product</u> or <u>vector product</u> of $\mathbf{A}, \mathbf{B}$ is defined as

$$\mathbf{A} \times \mathbf{B} = \begin{pmatrix} \begin{vmatrix} \mathbf{A}_{23} & \mathbf{B}_{23} \end{vmatrix} \\ \begin{vmatrix} \mathbf{A}_{31} & \mathbf{B}_{31} \end{vmatrix} \\ \begin{vmatrix} \mathbf{A}_{12} & \mathbf{B}_{12} \end{vmatrix} \end{pmatrix} \tag{A.2.2.1}$$

A.2.3. Verify that

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \tag{A.2.3.1}$$

A.2.4. The area of a triangle is given by

$$\frac{1}{2} \|\mathbf{A} \times \mathbf{B}\| \tag{A.2.4.1}$$

A.2.5. (Cauchy-Schwarz Inequality)

$$\left|\mathbf{a}^\top \mathbf{b}\right| \leq \|\mathbf{a}\| \, \|\mathbf{b}\| \tag{A.2.5.1}$$

Solution:

$$\left\| \mathbf{a} - \frac{\mathbf{a}^\top \mathbf{b}}{\|\mathbf{b}\|^2} \mathbf{b} \right\|^2 \geq 0 \tag{A.2.5.2}$$

$$\implies \|\mathbf{a}\|^2 - 2\frac{\left(\mathbf{a}^\top \mathbf{b}\right)^2}{\|\mathbf{b}\|^2} + \frac{\left(\mathbf{a}^\top \mathbf{b}\right)^2}{\|\mathbf{b}\|^2} \geq 0 \tag{A.2.5.3}$$

$$\implies \|\mathbf{a}\|^2 - \frac{\left(\mathbf{a}^\top \mathbf{b}\right)^2}{\|\mathbf{b}\|^2} \geq 0 \tag{A.2.5.4}$$

$$\implies \|\mathbf{a}\|^2 \|\mathbf{b}\|^2 \geq \left(\mathbf{a}^\top \mathbf{b}\right)^2 \tag{A.2.5.5}$$

$$\tag{A.2.5.6}$$

yielding (A.2.5.1).

A.2.6. (Triangle Inequality)

$$\|\mathbf{a} + \mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\| \tag{A.2.6.1}$$

Solution: Using (A.2.5.1),

$$\mathbf{a}^\top \mathbf{b} \leq \|\mathbf{a}\| \|\mathbf{b}\| \tag{A.2.6.2}$$

$$\implies \|\mathbf{a}\|^2 + 2\mathbf{a}^\top \mathbf{b} + \|\mathbf{b}\|^2 \leq \|\mathbf{a}\|^2 + 2\|\mathbf{a}\| \|\mathbf{b}\| + \|\mathbf{b}\|^2 \tag{A.2.6.3}$$

$$\implies \|\mathbf{a} + \mathbf{b}\|^2 \leq (\|\mathbf{a}\| + \|\mathbf{b}\|)^2 \tag{A.2.6.4}$$

yielding (A.2.6.1).

Appendix B

Matrices

B.1. Eigenvalues and Eigenvectors

B.1.1. The eigenvalue λ and the eigenvector $\mathbf{x}$ for a matrix $\mathbf{A}$ are defined as,

$$\mathbf{Ax} = \lambda \mathbf{x} \tag{B.1.1.1}$$

B.1.2. The eigenvalues are calculated by solving the equation

$$f(\lambda) = \left| \lambda \mathbf{I} - \mathbf{A} \right| = 0 \tag{B.1.2.1}$$

The above equation is known as the characteristic equation.

B.1.3. According to the Cayley-Hamilton theorem,

$$f(\lambda) = 0 \implies f(\mathbf{A}) = 0 \tag{B.1.3.1}$$

B.1.4. The trace of a square matrix is defined to be the sum of the diagonal elements.

$$\operatorname{tr}(\mathbf{A}) = \sum_{i=1}^{N} a_{ii}. \tag{B.1.4.1}$$

where a_{ii} is the ith diagonal element of the matrix $\mathbf{A}$.

B.1.5. The trace of a matrix is equal to the sum of the eigenvalues

$$\operatorname{tr}(\mathbf{A}) = \sum_{i=1}^{N} \lambda_i \tag{B.1.5.1}$$

B.2. Determinants

B.2.1. Let

$$\mathbf{A} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}. \tag{B.2.1.1}$$

be a 3×3 matrix. Then,

$$\begin{vmatrix}\mathbf{A}\end{vmatrix} = a_1 \begin{pmatrix} b_2 & c_2 \\ b_3 & c_3 \end{pmatrix} - a_2 \begin{pmatrix} b_1 & c_1 \\ b_3 & c_3 \end{pmatrix} + a_3 \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}. \tag{B.2.1.2}$$

B.2.2. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of a matrix $\mathbf{A}$. Then, the product of the eigenvalues is equal to the determinant of **A**.

$$\left|\mathbf{A}\right| = \prod_{i=1}^{n} \lambda_i \tag{B.2.2.1}$$

B.2.3.

$$\left|\mathbf{AB}\right| = \left|\mathbf{A}\right|\left|\mathbf{B}\right| \tag{B.2.3.1}$$

B.2.4. If **A** be an $n \times n$ matrix,

$$\left|k\mathbf{A}\right| = k^n \left|\mathbf{A}\right| \tag{B.2.4.1}$$

B.3. Rank of a Matrix

B.3.1. The rank of a matrix is defined as the number of linearly independent rows. This is also known as the row rank.

B.3.2. Row rank = Column rank.

B.3.3. The rank of a matrix is obtained as the number of nonzero rows obtained after row reduction.

B.3.4. An $n \times n$ matrix is invertible if and only if its rank is n.

B.3.5. Points $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are on a line if

$$\operatorname{rank}\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\\ \mathbf{C}\end{pmatrix} = 1 \tag{B.3.5.1}$$

B.3.6. Points $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ form a paralelogram if

$$\operatorname{rank}\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\\ \mathbf{C}\\ \mathbf{D}\end{pmatrix} = 1, \operatorname{rank}\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\\ \mathbf{C}\end{pmatrix} = 2 \tag{B.3.6.1}$$

B.4. Inverse of a Matrix

B.4.1. For a 2×2 matrix

$$\mathbf{A} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}, \tag{B.4.1.1}$$

the inverse is given by

$$\mathbf{A}^{-1} = \frac{1}{\left|\mathbf{A}\right|}\begin{pmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{pmatrix}, \tag{B.4.1.2}$$

B.4.2. For higher order matrices, the inverse should be calculated using row operations.

B.5. Orthogonality

B.5.1. The rotation matrix is defined as

$$\mathbf{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad \theta \in [0, 2\pi] \tag{B.5.1.1}$$

B.5.2. The rotation matrix is <u>orthogonal</u>

$$\mathbf{R}_\theta^\top \mathbf{R}_\theta = \mathbf{R}_\theta \mathbf{R}_\theta^\top = \mathbf{I} \tag{B.5.2.1}$$

B.5.3. If the angle of rotation is $\frac{\pi}{2}$,

$$\mathbf{m}^\top \mathbf{n} = 0 \implies \mathbf{n} = \mathbf{R}_{\frac{\pi}{2}} \mathbf{m} \tag{B.5.3.1}$$

B.5.4.

$$\mathbf{n}^\top \mathbf{h} = 1 \implies \mathbf{n} = \frac{\mathbf{e}_1}{\mathbf{e}_1^\top \mathbf{h}} + \mu \mathbf{R}_{\frac{\pi}{2}} \mathbf{h}, \quad \mu \in \mathbb{R}. \tag{B.5.4.1}$$

B.5.5. The <u>affine</u> transformation is given by

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c} \quad \text{(Affine Transformation)} \tag{B.5.5.1}$$

where $\mathbf{P}$ is invertible.

B.5.6. The eigenvalue decomposition of a symmetric matrix $\mathbf{V}$ is given by

$$\mathbf{P}^\top \mathbf{V}\mathbf{P} = \mathbf{D}. \quad \text{(Eigenvalue Decomposition)} \tag{B.5.6.1}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \tag{B.5.6.2}$$

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix}, \quad \mathbf{P}^\top = \mathbf{P}^{-1}, \tag{B.5.6.3}$$

B.6. Singular Value Decomposition

B.6.1.

B.6.2. Consider the rectangular equation

$$\mathbf{M}^\top \mathbf{x} = \mathbf{b} \tag{B.6.2.1}$$

B.6.3. Find $\mathbf{M}^T\mathbf{M}$ and $\mathbf{M}\mathbf{M}^T$.

B.6.4. Obtain the eigen decomposition

$$\mathbf{M}^T\mathbf{M} = \mathbf{P}_1\mathbf{D}_1\mathbf{P}_1^T \tag{B.6.4.1}$$

and

$$\mathbf{M}\mathbf{M}^T = \mathbf{P}_2\mathbf{D}_2\mathbf{P}_2^T \tag{B.6.4.2}$$

B.6.5. Perform the QR decompositions

$$\mathbf{P}_1 = \mathbf{U}\mathbf{R}_1,\ \mathbf{P}_2 = \mathbf{V}\mathbf{R}_2 \tag{B.6.5.1}$$

B.6.6. The singular value decomposition is the given by

$$\mathbf{M} = \mathbf{U}\Sigma\mathbf{V}^T, \tag{B.6.6.1}$$

where Σ has the same shape as $\mathbf{M}$ and

$$\Sigma = \begin{pmatrix} \mathbf{D}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \tag{B.6.6.2}$$

B.6.7. (B.6.2.1) can then be expressed as

$$\mathbf{U}\Sigma\mathbf{V}^T\mathbf{x} = \mathbf{b} \tag{B.6.7.1}$$

$$\implies \mathbf{x} = \mathbf{V}\Sigma^{-1}\mathbf{U}^T\mathbf{b} \tag{B.6.7.2}$$

where $\mathbf{\Sigma}^{-1}$ is obtained by inverting only the non-zero elements of $\mathbf{\Sigma}$.

Appendix C

Triangle Constructions

C.0.1 Construct a triangle ABC in which $a, \angle B$ and $c + b = K$ are given.

Solution: Using the cosine formula in $\triangle ABC$,

$$b^2 = a^2 + c^2 - 2ac\cos B \tag{C.0.1.1}$$

$$\implies (b+c)(b-c) = a^2 - 2ac\cos B \tag{C.0.1.2}$$

$$\text{or, } K(b-c) = a^2 - 2ac\cos B \tag{C.0.1.3}$$

where

$$K = b + c \tag{C.0.1.4}$$

From (C.0.1.3) and (C.0.1.4),

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} \frac{a^2-2ac\cos B}{K} \\ K \end{pmatrix} \tag{C.0.1.5}$$

$$\implies \begin{pmatrix} b \\ c \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{a^2-2ac\cos B}{K} \\ K \end{pmatrix} \tag{C.0.1.6}$$

$$\because \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = 2\mathbf{I} \tag{C.0.1.7}$$

From (C.0.1.6)

$$c = \frac{1}{2}\mathbf{e}_2^\top \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{a^2}{K} \\ K \end{pmatrix} - \frac{2ac\cos B}{K} \tag{C.0.1.8}$$

$$\implies c = \frac{1}{2\left(1 + \frac{2a\cos B}{K}\right)}\mathbf{e}_2^\top \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{a^2}{K} \\ K \end{pmatrix} \tag{C.0.1.9}$$

The coordinates of $\triangle ABC$ can then be expressed as

$$\mathbf{A} = c \begin{pmatrix} \cos B \\ \sin B \end{pmatrix}, \mathbf{B} = \mathbf{0}, \mathbf{C} = \begin{pmatrix} a \\ 0 \end{pmatrix}. \tag{C.0.1.10}$$

C.0.2 Construct a triangle ABC in which $\angle B, \angle C$ and $a+b+c = K$ are given.

Solution: From the given information,

$$a + b + c = K \tag{C.0.2.1}$$

$$b\cos C + c\cos B - a = 0 \tag{C.0.2.2}$$

$$b\sin C - c\sin B = 0 \tag{C.0.2.3}$$

resulting in the matrix equation

$$\begin{pmatrix} 1 & 1 & 1 \\ \cos C & \cos B & -1 \\ \sin C & -\sin B & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = K\mathbf{e}_1 \tag{C.0.2.4}$$

which can be solved to obtain all the sides.

Appendix D

Linear Forms

D.1. Two Dimensions

D.1.1. The equation of a line is given by

$$\mathbf{n}^\top \mathbf{x} = c \tag{D.1.1.1}$$

where $\mathbf{n}$ is the normal vector of the line.

D.1.2. The equation of a line with normal vector $\mathbf{n}$ and passing through a point $\mathbf{A}$ is given by

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{A}) = 0 \tag{D.1.2.1}$$

D.1.3. The equation of a line L is also given by

$$\mathbf{n}^\top \mathbf{x} = \begin{cases} 0 & \mathbf{0} \in L \\ 1 & \text{otherwise} \end{cases} \tag{D.1.3.1}$$

D.1.4. Points $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are collinear if

$$\operatorname{rank}\begin{pmatrix}\mathbf{B}-\mathbf{A} & \mathbf{C}-\mathbf{A}\end{pmatrix} < 2 \tag{D.1.4.1}$$

Proof. From (D.1.1.1),

$$\mathbf{n}^\top \mathbf{A} = c \tag{D.1.4.2}$$

$$\mathbf{n}^\top \mathbf{B} = c \tag{D.1.4.3}$$

$$\mathbf{n}^\top \mathbf{C} = c \tag{D.1.4.4}$$

which can be expressed as

$$\begin{pmatrix}\mathbf{A} & \mathbf{B} & \mathbf{C}\end{pmatrix}^\top \mathbf{n} = c\begin{pmatrix}1\\1\\1\end{pmatrix} \tag{D.1.4.5}$$

The above set of equations are consistent if

$$\operatorname{rank}\begin{pmatrix}1 & 1 & 1\\ \mathbf{A} & \mathbf{B} & \mathbf{C}\end{pmatrix} < 3 \tag{D.1.4.6}$$

$$\implies \operatorname{rank}\begin{pmatrix}1 & 0 & 0\\ \mathbf{A} & \mathbf{B}-\mathbf{A} & \mathbf{C}-\mathbf{A}\end{pmatrix} < 3 \tag{D.1.4.7}$$

using the fact that row rank = column rank. The above condition can then be expressed as (D.1.4.1).

$\square$

D.1.5. The parametric equation of a line is given by

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m} \tag{D.1.5.1}$$

where $\mathbf{m}$ is the direction vector of the line and $\mathbf{A}$ is any point on the line.

D.1.6. Let $\mathbf{A}$ and $\mathbf{B}$ be two points on a straight line and let $\mathbf{P} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$ be any point on it. If p_2 is known, then

$$\mathbf{P} = \mathbf{A} + \frac{p_2 - \mathbf{e}_2^\top \mathbf{A}}{\mathbf{e}_2^\top (\mathbf{B} - \mathbf{A})} (\mathbf{B} - \mathbf{A}) \tag{D.1.6.1}$$

Solution: The equation of the line can be expressed in parametric from as

$$\mathbf{x} = \mathbf{A} + \lambda (\mathbf{B} - \mathbf{A}) \tag{D.1.6.2}$$

$$\implies \mathbf{P} = \mathbf{A} + \lambda (\mathbf{B} - \mathbf{A}) \tag{D.1.6.3}$$

$$\implies \mathbf{e}_2^\top \mathbf{P} = \mathbf{e}_2^\top \mathbf{A} + \lambda \mathbf{e}_2^\top (\mathbf{B} - \mathbf{A}) \tag{D.1.6.4}$$

$$\implies p_2 = \mathbf{e}_2^\top \mathbf{A} + \lambda \mathbf{e}_2^\top (\mathbf{B} - \mathbf{A}) \tag{D.1.6.5}$$

$$\text{or, } \lambda = \frac{p_2 - \mathbf{e}_2^\top \mathbf{A}}{\mathbf{e}_2^\top (\mathbf{B} - \mathbf{A})} \tag{D.1.6.6}$$

yielding (D.1.6.1).

D.1.7. The distance from a point $\mathbf{P}$ to the line in (D.1.1.1) is given by

$$d = \frac{\left|\mathbf{n}^\top \mathbf{P} - c\right|}{\|\mathbf{n}\|} \tag{D.1.7.1}$$

Solution: Without loss of generality, let $\mathbf{A}$ be the foot of the perpendicular from $\mathbf{P}$ to the line in (D.1.5.1). The equation of the normal to (D.1.1.1) can then be expressed as

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{n} \tag{D.1.7.2}$$

$$\implies \mathbf{P} - \mathbf{A} = \lambda \mathbf{n} \tag{D.1.7.3}$$

$\because$ $\mathbf{P}$ lies on (D.1.7.2). From the above, the desired distance can be expressed as

$$d = \left\|\mathbf{P} - \mathbf{A}\right\| = \left|\lambda\right| \left\|\mathbf{n}\right\| \tag{D.1.7.4}$$

From (D.1.7.3),

$$\mathbf{n}^\top \left(\mathbf{P} - \mathbf{A}\right) = \lambda \mathbf{n}^\top \mathbf{n} = \lambda \left\|\mathbf{n}\right\|^2 \tag{D.1.7.5}$$

$$\implies \left|\lambda\right| = \frac{\left|\mathbf{n}^\top \left(\mathbf{P} - \mathbf{A}\right)\right|}{\left\|\mathbf{n}\right\|^2} \tag{D.1.7.6}$$

Substituting the above in (D.1.7.4) and using the fact that

$$\mathbf{n}^\top \mathbf{A} = c \tag{D.1.7.7}$$

from (D.1.1.1), yields (D.1.7.1)

D.1.8. The distance from the origin to the line in (D.1.1.1) is given by

$$d = \frac{\left|c\right|}{\left\|\mathbf{n}\right\|} \tag{D.1.8.1}$$

D.1.9. The distance between the parallel lines

$$\mathbf{n}^\top \mathbf{x} = c_1$$
$$\mathbf{n}^\top \mathbf{x} = c_2 \tag{D.1.9.1}$$

is given by

$$d = \frac{|c_1 - c_2|}{\|\mathbf{n}\|} \tag{D.1.9.2}$$

D.1.10. The equation of the line perpendicular to (D.1.1.1) and passing through the point $\mathbf{P}$ is given by

$$\mathbf{m}^\top (\mathbf{x} - \mathbf{P}) = 0 \tag{D.1.10.1}$$

D.1.11. The foot of the perpendicular from $\mathbf{P}$ to the line in (D.1.1.1) is given by

$$\begin{pmatrix} \mathbf{m} & \mathbf{n} \end{pmatrix}^\top \mathbf{x} = \begin{pmatrix} \mathbf{m}^\top \mathbf{P} \\ c \end{pmatrix} \tag{D.1.11.1}$$

Solution: From (D.1.1.1) and (D.1.2.1) the foot of the perpendicular satisfies the equations

$$\mathbf{n}^\top \mathbf{x} = c \tag{D.1.11.2}$$

$$\mathbf{m}^\top (\mathbf{x} - \mathbf{P}) = 0 \tag{D.1.11.3}$$

where $\mathbf{m}$ is the direction vector of the given line. Combining the above into a matrix equation results in (D.1.11.1).

D.1.12. The equations of the angle bisectors of the lines

$$\mathbf{n}_1^\top \mathbf{x} = c_1 \tag{D.1.12.1}$$

$$\mathbf{n}_2^\top \mathbf{x} = c_2 \tag{D.1.12.2}$$

are given by

$$\frac{\mathbf{n}_1^\top \mathbf{x} - c_1}{\left\|\mathbf{n}_1\right\|} = \pm \frac{\mathbf{n}_2^\top \mathbf{x} - c_2}{\left\|\mathbf{n}_2\right\|} \tag{D.1.12.3}$$

Proof. Any point on the angle bisector is equidistant from the lines. □

D.1.13. In $\triangle ABC$, the direction of the angle bisector of A is given by

$$\mathbf{m} = \mathbf{B} + \mathbf{C} - 2\mathbf{A} \tag{D.1.13.1}$$

Solution: Since the direction vectors of AB and AC are

$$\mathbf{B} - \mathbf{A}, \mathbf{C} - \mathbf{A} \tag{D.1.13.2}$$

using the parallelogram law, we obtain (D.1.13.1).

D.1.14. The perpendicular bisector of BC is given by

$$\left(\mathbf{B} - \mathbf{C}\right)^\top \mathbf{x} = \frac{\left\|\mathbf{B}\right\|^2 - \left\|\mathbf{C}\right\|^2}{2} \tag{D.1.14.1}$$

Solution: The perpendicular bisector passes through the mid point

$$\frac{\mathbf{B}+\mathbf{C}}{2} \tag{D.1.14.2}$$

and has normal vector

$$\mathbf{B}-\mathbf{C} \tag{D.1.14.3}$$

Hence, the equation of the perpendicular bisector is

$$(\mathbf{B}-\mathbf{C})^{\top}\left[\mathbf{x}-\frac{\mathbf{B}+\mathbf{C}}{2}\right]=0 \tag{D.1.14.4}$$

yielding (D.1.14.1).

mid point of BC is

D.2. Three Dimensions

D.2.1. Points $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are on a line if

$$\operatorname{rank}\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\\ \mathbf{C}\end{pmatrix}=1 \tag{D.2.1.1}$$

D.2.2. Points $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ form a paralelogram if

$$\operatorname{rank}\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\\ \mathbf{C}\\ \mathbf{D}\end{pmatrix} = 1, \operatorname{rank}\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\\ \mathbf{C}\end{pmatrix} = 2 \tag{D.2.2.1}$$

D.2.3. The equation of a line is given by (D.1.5.1)

D.2.4. The equation of a plane is given by (D.1.1.1)

D.2.5. The distance from the origin to the line in (D.1.1.1) is given by (D.1.8.1)

D.2.6. The distance from a point $\mathbf{P}$ to the line in (D.1.5.1) is given by

$$d = \|\mathbf{A} - \mathbf{P}\|^2 - \frac{\left\{\mathbf{m}^\top (\mathbf{A} - \mathbf{P})\right\}^2}{\|\mathbf{m}\|^2} \tag{D.2.6.1}$$

Solution:

$$d(\lambda) = \|\mathbf{A} + \lambda\mathbf{m} - \mathbf{P}\| \tag{D.2.6.2}$$

$$\implies d^2(\lambda) = \|\mathbf{A} + \lambda\mathbf{m} - \mathbf{P}\|^2 \tag{D.2.6.3}$$

which can be simplified to obtain

$$d^2(\lambda) = \lambda^2 \|\mathbf{m}\|^2 + 2\lambda\mathbf{m}^\top (\mathbf{A} - \mathbf{P}) + \|\mathbf{A} - \mathbf{P}\|^2 \tag{D.2.6.4}$$

which is of the form

$$d^2(\lambda) = a\lambda^2 + 2b\lambda + c \tag{D.2.6.5}$$

$$= a\left\{\left(\lambda + \frac{b}{a}\right)^2 + \left[\frac{c}{a} - \left(\frac{b}{a}\right)^2\right]\right\} \tag{D.2.6.6}$$

with

$$a = \|\mathbf{m}\|^2, b = \mathbf{m}^\top(\mathbf{A} - \mathbf{P}), c = \|\mathbf{A} - \mathbf{P}\|^2 \tag{D.2.6.7}$$

which can be expressed as From the above, $d^2(\lambda)$ is smallest when upon substituting from (D.2.6.7)

$$\lambda + \frac{b}{2a} = 0 \implies \lambda = -\frac{b}{2a} \tag{D.2.6.8}$$

$$= -\frac{\mathbf{m}^\top(\mathbf{A} - \mathbf{P})}{\|\mathbf{m}\|^2} \tag{D.2.6.9}$$

and consequently,

$$d_{\min}(\lambda) = a\left(\frac{c}{a} - \left(\frac{b}{a}\right)^2\right) \tag{D.2.6.10}$$

$$= c - \frac{b^2}{a} \tag{D.2.6.11}$$

yielding (D.2.6.1) after substituting from (D.2.6.7).

D.2.7. The distance between the parallel planes (D.1.9.1) is given by (D.1.9.2).

D.2.8. The plane

$$\mathbf{n}^\top \mathbf{x} = c \tag{D.2.8.1}$$

contains the line

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m} \tag{D.2.8.2}$$

if

$$\mathbf{m}^\top \mathbf{n} = 0 \tag{D.2.8.3}$$

Solution: Any point on the line (D.2.8.2) should also satisfy (D.2.8.1). Hence,

$$\mathbf{n}^\top \left(\mathbf{A} + \lambda \mathbf{m}\right) = \mathbf{n}^\top \mathbf{A} = c \tag{D.2.8.4}$$

which can be simplified to obtain (D.2.8.3)

D.2.9. The foot of the perpendicular from a point $\mathbf{P}$ to the plane

$$\mathbf{n}^\top \mathbf{x} = c \tag{D.2.9.1}$$

is given by

$$\mathbf{x} = \mathbf{P} + \frac{c - \mathbf{n}^\top \mathbf{P}}{\left\|\mathbf{n}\right\|^2} \mathbf{n} \tag{D.2.9.2}$$

Solution: The equation of the line perpendicular to the given plane and passing through $\mathbf{P}$ is

$$\mathbf{x} = \mathbf{P} + \lambda\mathbf{n} \tag{D.2.9.3}$$

From (D.2.12.1), the intersection of the above line with the given plane is (D.2.9.2).

D.2.10. The image of a point $\mathbf{P}$ with respect to the plane

$$\mathbf{n}^\top\mathbf{x} = c \tag{D.2.10.1}$$

is given by

$$\mathbf{R} = \mathbf{P} + 2\frac{c - \mathbf{n}^\top\mathbf{P}}{\|\mathbf{n}\|^2} \tag{D.2.10.2}$$

Solution: Let $\mathbf{R}$ be the desired image. Then, subtituting the expression for the foot of the perpendicular from $\mathbf{P}$ to the given plane using (D.2.9.2),

$$\frac{\mathbf{P} + \mathbf{R}}{2} = \mathbf{P} + \frac{c - \mathbf{n}^\top\mathbf{P}}{\|\mathbf{n}\|^2} \tag{D.2.10.3}$$

D.2.11. Let a plane pass through the points $\mathbf{A}, \mathbf{B}$ and be perpendicular to the plane

$$\mathbf{n}^\top\mathbf{x} = c \tag{D.2.11.1}$$

Then the equation of this plane is given by

$$\mathbf{p}^\top \mathbf{x} = 1 \tag{D.2.11.2}$$

where

$$\mathbf{p} = \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{n} \end{pmatrix}^{-\top} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \tag{D.2.11.3}$$

Solution: From the given information,

$$\mathbf{p}^\top \mathbf{A} = d \tag{D.2.11.4}$$

$$\mathbf{p}^\top \mathbf{B} = d \tag{D.2.11.5}$$

$$\mathbf{p}^\top \mathbf{n} = 0 \tag{D.2.11.6}$$

$\because$ the normal vectors to the two planes will also be perpendicular. The system of equations in (D.2.11.6) can be expressed as the matrix equation

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{n} \end{pmatrix}^{\top} \mathbf{p} = d \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \tag{D.2.11.7}$$

which yields (D.2.11.3) upon normalising with d.

D.2.12. The intersection of the line represented by (D.1.5.1) with the plane represented by (D.1.1.1) is given by

$$\mathbf{x} = \mathbf{A} + \frac{c - \mathbf{n}^\top \mathbf{A}}{\mathbf{n}^\top \mathbf{m}} \mathbf{m} \tag{D.2.12.1}$$

Solution: From (D.1.5.1) and (D.1.1.1),

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m} \tag{D.2.12.2}$$

$$\mathbf{n}^\top \mathbf{x} = c \tag{D.2.12.3}$$

$$\implies \mathbf{n}^\top \left(\mathbf{A} + \lambda \mathbf{m}\right) = c \tag{D.2.12.4}$$

which can be simplified to obtain

$$\mathbf{n}^\top \mathbf{A} + \lambda \mathbf{n}^\top \mathbf{m} = c \tag{D.2.12.5}$$

$$\implies \lambda = \frac{c - \mathbf{n}^\top \mathbf{A}}{\mathbf{n}^\top \mathbf{m}} \tag{D.2.12.6}$$

Substituting the above in (D.2.12.4) yields (D.2.12.1).

D.2.13. The foot of the perpendicular from the point $\mathbf{P}$ to the line represented by (D.1.5.1) is given by

$$\mathbf{x} = \mathbf{A} + \frac{\mathbf{m}^\top \left(\mathbf{P} - \mathbf{A}\right)}{\left\|\mathbf{m}\right\|^2} \mathbf{m} \tag{D.2.13.1}$$

Solution: Let the equation of the line be

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m} \tag{D.2.13.2}$$

The equation of the plane perpendicular to the given line passing through $\mathbf{P}$ is given by

$$\mathbf{m}^{\top}(\mathbf{x}-\mathbf{P})=0 \tag{D.2.13.3}$$

$$\implies \mathbf{m}^{\top}\mathbf{x}=\mathbf{m}^{\top}\mathbf{P} \tag{D.2.13.4}$$

The desired foot of the perpendicular is the intersection of (D.2.13.2) with (D.2.13.3) which can be obtained from (D.2.12.1) as (D.2.13.1)

D.2.14. The foot of the perpendicular from a point $\mathbf{P}$ to a plane is $\mathbf{Q}$. The equation of the plane is given by

$$(\mathbf{P}-\mathbf{Q})^{\top}(\mathbf{x}-\mathbf{Q})=0 \tag{D.2.14.1}$$

Solution: The normal vector to the plane is given by

$$\mathbf{n}=\mathbf{P}-\mathbf{Q} \tag{D.2.14.2}$$

Hence, the equation of the plane is (D.2.14.1).

D.2.15. Let $\mathbf{A},\mathbf{B},\mathbf{C}$ be points on a plane. The equation of the plane is then given by

$$\begin{pmatrix}\mathbf{A} & \mathbf{B} & \mathbf{C}\end{pmatrix}^{\top}\mathbf{n}=\begin{pmatrix}1\\1\\1\end{pmatrix} \tag{D.2.15.1}$$

Solution: Let the equation of the plane be

$$\mathbf{n}^\top \mathbf{x} = 1 \tag{D.2.15.2}$$

Then

$$\mathbf{n}^\top \mathbf{A} = 1 \tag{D.2.15.3}$$

$$\mathbf{n}^\top \mathbf{B} = 1 \tag{D.2.15.4}$$

$$\mathbf{n}^\top \mathbf{C} = 1 \tag{D.2.15.5}$$

which can be combined to obtain (D.2.15.1).

D.2.16. The lines

$$\mathbf{x} = \mathbf{x_1} + \lambda_1 \mathbf{m_1} \tag{D.2.16.1}$$

$$\mathbf{x} = \mathbf{x_2} + \lambda_2 \mathbf{m_2} \tag{D.2.16.2}$$

intersect if

$$\mathbf{M}\boldsymbol{\lambda} = \mathbf{x_2} - \mathbf{x_1} \tag{D.2.16.3}$$

where

$$\mathbf{M} \triangleq \begin{pmatrix} \mathbf{m_1} & \mathbf{m_2} \end{pmatrix} \tag{D.2.16.4}$$

$$\boldsymbol{\lambda} \triangleq \begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} \tag{D.2.16.5}$$

D.2.17. The closest points on two skew lines are given by

$$\mathbf{M}^\top \mathbf{M}\boldsymbol{\lambda} = \mathbf{M}^\top (\mathbf{x_2} - \mathbf{x_1}) \tag{D.2.17.1}$$

Solution: For the lines defined in (D.2.16.1) and (D.2.16.2), Suppose the closest points on both lines are

$$\mathbf{A} = \mathbf{x_1} + \lambda_1 \mathbf{m_1} \tag{D.2.17.2}$$

$$\mathbf{B} = \mathbf{x_2} + \lambda_2 \mathbf{m_2} \tag{D.2.17.3}$$

Then, AB is perpendicular to both lines, hence

$$\mathbf{m_1}^\top (\mathbf{A} - \mathbf{B}) = 0 \tag{D.2.17.4}$$

$$\mathbf{m_2}^\top (\mathbf{A} - \mathbf{B}) = 0 \tag{D.2.17.5}$$

$$\implies \mathbf{M}^\top (\mathbf{A} - \mathbf{B}) = \mathbf{O} \tag{D.2.17.6}$$

Using (D.2.17.2) and (D.2.17.3) in (D.2.17.6),

$$\mathbf{M}^\top (\mathbf{x_1} - \mathbf{x_2} + \mathbf{M}\boldsymbol{\lambda}) = \mathbf{0} \tag{D.2.17.7}$$

$$\tag{D.2.17.8}$$

yielding D.2.17.1.

D.2.18. (Parallelogram Law) Let $\mathbf{A}, \mathbf{B}, \mathbf{D}$ be three vertices of a parallelogram.

Then the vertex $\mathbf{C}$ is given by

$$\mathbf{C} = \mathbf{B} + \mathbf{C} - \mathbf{A} \tag{D.2.18.1}$$

Solution: Shifting $\mathbf{A}$ to the origin, we obtain a parallelogram with corresponding vertices

$$\mathbf{0}, \mathbf{B} - \mathbf{A}, \mathbf{D} - \mathbf{A} \tag{D.2.18.2}$$

The fourth vertex of this parallelogram is then obtained as

$$(\mathbf{B} - \mathbf{A}) + (\mathbf{D} - \mathbf{A}) = \mathbf{D} + \mathbf{B} - 2\mathbf{A} \tag{D.2.18.3}$$

Shifting the origin to $\mathbf{A}$, the fourth vertex is obtained as

$$\mathbf{C} = \mathbf{D} + \mathbf{B} - 2\mathbf{A} + \mathbf{A} \tag{D.2.18.4}$$

$$= \mathbf{D} + \mathbf{B} - \mathbf{A} \tag{D.2.18.5}$$

D.2.19. (Affine Transformation) Let $\mathbf{A}, \mathbf{C}$, be opposite vertices of a square. The other two points can be obtained as

$$\mathbf{B} = \frac{\|\mathbf{A} - \mathbf{C}\|}{\sqrt{2}}\mathbf{P}\mathbf{e}_1 + \mathbf{A} \tag{D.2.19.1}$$

$$\mathbf{D} = \frac{\|\mathbf{A} - \mathbf{C}\|}{\sqrt{2}}\mathbf{P}\mathbf{e}_2 + \mathbf{A} \tag{D.2.19.2}$$

where

$$\mathbf{P} = \begin{pmatrix} \cos\left(\theta - \frac{\pi}{4}\right) & \sin\left(\theta - \frac{\pi}{4}\right) \\ \sin\left(\theta - \frac{\pi}{4}\right) & \cos\left(\theta - \frac{\pi}{4}\right) \end{pmatrix} \tag{D.2.19.3}$$

and

$$\cos\theta = \frac{(\mathbf{C} - \mathbf{A})^\top \mathbf{e}_1}{\|\mathbf{A} - \mathbf{C}\| \, \|\mathbf{e}_1\|} \tag{D.2.19.4}$$

Appendix E

Quadratic Forms

E.1. Conic equation

E.1.1. Let $\mathbf{q}$ be a point such that the ratio of its distance from a fixed point $\mathbf{F}$ and the distance (d) from a fixed line

$$L : \mathbf{n}^\top \mathbf{x} = c \tag{E.1.1.1}$$

is constant, given by

$$\frac{\|\mathbf{q} - \mathbf{F}\|}{d} = e \tag{E.1.1.2}$$

The locus of $\mathbf{q}$ is known as a conic section. The line L is known as the directrix and the point $\mathbf{F}$ is the focus. e is defined to be the eccentricity of the conic.

(a) For $e = 1$, the conic is a parabola

(b) For $e < 1$, the conic is an ellipse

(c) For $e > 1$, the conic is a hyperbola

E.1.2. The equation of a conic with directrix $\mathbf{n}^\top \mathbf{x} = c$, eccentricity e and focus $\mathbf{F}$ is given by

$$\mathrm{g}\left(\mathbf{x}\right) = \mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{E.1.2.1}$$

where

$$\mathbf{V} = \left\|\mathbf{n}\right\|^2 \mathbf{I} - e^2 \mathbf{n}\mathbf{n}^\top, \tag{E.1.2.2}$$

$$\mathbf{u} = ce^2\mathbf{n} - \left\|\mathbf{n}\right\|^2 \mathbf{F}, \tag{E.1.2.3}$$

$$f = \left\|\mathbf{n}\right\|^2 \left\|\mathbf{F}\right\|^2 - c^2e^2 \tag{E.1.2.4}$$

Proof. Using Definition E.1.1 and Lemma D.1.7.1, for any point $\mathbf{x}$ on the conic,

$$\left\|\mathbf{x} - \mathbf{F}\right\|^2 = e^2 \frac{\left(\mathbf{n}^\top \mathbf{x} - c\right)^2}{\left\|\mathbf{n}\right\|^2} \tag{E.1.2.5}$$

$$\implies \left\|\mathbf{n}\right\|^2 \left(\mathbf{x} - \mathbf{F}\right)^\top \left(\mathbf{x} - \mathbf{F}\right) = e^2 \left(\mathbf{n}^\top \mathbf{x} - c\right)^2 \tag{E.1.2.6}$$

$$\implies \left\|\mathbf{n}\right\|^2 \left(\mathbf{x}^\top \mathbf{x} - 2\mathbf{F}^\top \mathbf{x} + \left\|\mathbf{F}\right\|^2\right) = e^2 \left(c^2 + \left(\mathbf{n}^\top \mathbf{x}\right)^2 - 2c\mathbf{n}^\top \mathbf{x}\right) \tag{E.1.2.7}$$

$$= e^2 \left(c^2 + \left(\mathbf{x}^\top \mathbf{n}\mathbf{n}^\top \mathbf{x}\right) - 2c\mathbf{n}^\top \mathbf{x}\right) \tag{E.1.2.8}$$

which can be expressed as (E.1.2.1) after simplification.

□

E.1.3. The eccentricity, directrices and foci of (E.1.2.1) are given by

$$e = \sqrt{1 - \frac{\lambda_1}{\lambda_2}} \tag{E.1.3.1}$$

$$\mathbf{n} = \sqrt{\lambda_2}\mathbf{p}_1,$$

$$c = \begin{cases} \frac{e\mathbf{u}^\top \mathbf{n} \pm \sqrt{e^2\left(\mathbf{u}^\top \mathbf{n}\right)^2 - \lambda_2(e^2-1)\left(\|\mathbf{u}\|^2 - \lambda_2 f\right)}}{\lambda_2 e(e^2-1)} & e \neq 1 \\ \frac{\|\mathbf{u}\|^2 - \lambda_2 f}{2\mathbf{u}^\top \mathbf{n}} & e = 1 \end{cases} \tag{E.1.3.2}$$

$$\mathbf{F} = \frac{ce^2\mathbf{n} - \mathbf{u}}{\lambda_2} \tag{E.1.3.3}$$

Proof. From (E.1.2.2), using the fact that $\mathbf{V}$ is symmetric with $\mathbf{V} = \mathbf{V}^\top$,

$$\mathbf{V}^\top \mathbf{V} = \left(\|\mathbf{n}\|^2 \mathbf{I} - e^2\mathbf{n}\mathbf{n}^\top\right)^\top \left(\|\mathbf{n}\|^2 \mathbf{I} - e^2\mathbf{n}\mathbf{n}^\top\right) \tag{E.1.3.4}$$

$$\implies \mathbf{V}^2 = \|\mathbf{n}\|^4 \mathbf{I} + e^4\mathbf{n}\mathbf{n}^\top\mathbf{n}\mathbf{n}^\top - 2e^2 \|\mathbf{n}\|^2 \mathbf{n}\mathbf{n}^\top \tag{E.1.3.5}$$

$$= \|\mathbf{n}\|^4 \mathbf{I} + e^4 \|\mathbf{n}\|^2 \mathbf{n}\mathbf{n}^\top - 2e^2 \|\mathbf{n}\|^2 \mathbf{n}\mathbf{n}^\top \tag{E.1.3.6}$$

$$= \|\mathbf{n}\|^4 \mathbf{I} + e^2\left(e^2 - 2\right) \|\mathbf{n}\|^2 \mathbf{n}\mathbf{n}^\top \tag{E.1.3.7}$$

$$= \|\mathbf{n}\|^4 \mathbf{I} + \left(e^2 - 2\right) \|\mathbf{n}\|^2 \left(\|\mathbf{n}\|^2 \mathbf{I} - \mathbf{V}\right) \tag{E.1.3.8}$$

which can be expressed as

$$\mathbf{V}^2 + \left(e^2 - 2\right) \|\mathbf{n}\|^2 \mathbf{V} - \left(e^2 - 1\right) \|\mathbf{n}\|^4 \mathbf{I} = 0 \tag{E.1.3.9}$$

Using the Cayley-Hamilton theorem, (E.1.3.9) results in the characteristic equation,

$$\lambda^2 - \left(2 - e^2\right) \|\mathbf{n}\|^2 \lambda + \left(1 - e^2\right) \|\mathbf{n}\|^4 = 0 \tag{E.1.3.10}$$

which can be expressed as

$$\left(\frac{\lambda}{\|\mathbf{n}\|^2}\right)^2 - \left(2 - e^2\right) \left(\frac{\lambda}{\|\mathbf{n}\|^2}\right) + \left(1 - e^2\right) = 0 \tag{E.1.3.11}$$

$$\implies \frac{\lambda}{\|\mathbf{n}\|^2} = 1 - e^2, 1 \tag{E.1.3.12}$$

$$\text{or, } \lambda_2 = \|\mathbf{n}\|^2, \lambda_1 = \left(1 - e^2\right) \lambda_2 \tag{E.1.3.13}$$

From (E.1.3.13), the eccentricity of (E.1.2.1) is given by (E.1.3.1). Multiplying both sides of (E.1.2.2) by $\mathbf{n}$,

$$\mathbf{V}\mathbf{n} = \|\mathbf{n}\|^2 \mathbf{n} - e^2 \mathbf{n}\mathbf{n}^\top \mathbf{n} \tag{E.1.3.14}$$

$$= \|\mathbf{n}\|^2 \left(1 - e^2\right) \mathbf{n} \tag{E.1.3.15}$$

$$= \lambda_1 \mathbf{n} \tag{E.1.3.16}$$

$$\tag{E.1.3.17}$$

from (E.1.3.13). Thus, λ_1 is the corresponding eigenvalue for $\mathbf{n}$. From (B.5.6.3) and (E.1.3.17), this implies that

$$\mathbf{p}_1 = \frac{\mathbf{n}}{\|\mathbf{n}\|} \tag{E.1.3.18}$$

$$\text{or, } \mathbf{n} = \|\mathbf{n}\| \mathbf{p}_1 = \sqrt{\lambda_2} \mathbf{p}_1 \tag{E.1.3.19}$$

from (E.1.3.13) . From (E.1.2.3) and (E.1.3.13),

$$\mathbf{F} = \frac{ce^2\mathbf{n} - \mathbf{u}}{\lambda_2} \tag{E.1.3.20}$$

$$\implies \|\mathbf{F}\|^2 = \frac{\left(ce^2\mathbf{n} - \mathbf{u}\right)^\top \left(ce^2\mathbf{n} - \mathbf{u}\right)}{\lambda_2^2} \tag{E.1.3.21}$$

$$\implies \lambda_2^2 \|\mathbf{F}\|^2 = c^2e^4\lambda_2 - 2ce^2\mathbf{u}^\top\mathbf{n} + \|\mathbf{u}\|^2 \tag{E.1.3.22}$$

Also, (E.1.2.4) can be expressed as

$$\lambda_2 \|\mathbf{F}\|^2 = f + c^2e^2 \tag{E.1.3.23}$$

From (E.1.3.22) and (E.1.3.23),

$$c^2e^4\lambda_2 - 2ce^2\mathbf{u}^\top\mathbf{n} + \|\mathbf{u}\|^2 = \lambda_2 \left(f + c^2e^2\right) \tag{E.1.3.24}$$

$$\implies \lambda_2 e^2 \left(e^2 - 1\right) c^2 - 2ce^2\mathbf{u}^\top\mathbf{n} + \|\mathbf{u}\|^2 - \lambda_2 f = 0 \tag{E.1.3.25}$$

yielding (E.1.3.3). □

E.1.4. (E.1.2.1) represents

(a) a parabola for $\left|\mathbf{V}\right| = 0$,

(b) ellipse for $\left|\mathbf{V}\right| > 0$ and

(c) hyperbola for $\left|\mathbf{V}\right| < 0$.

Proof. From (E.1.3.1),

$$\frac{\lambda_1}{\lambda_2} = 1 - e^2 \tag{E.1.4.1}$$

Also,

$$\left|\mathbf{V}\right| = \lambda_1\lambda_2 \tag{E.1.4.2}$$

yielding Table E.1.4.2 □

Eccentricity	**Conic**	**Eigenvalue**	**Determinant**
$e = 1$	Parabola	$\lambda_1 = 0$	$\left\|\mathbf{V}\right\| = 0$
$e < 1$	Ellipse	$\lambda_1 > 0, \lambda_2 > 0$	$\left\|\mathbf{V}\right\| > 0$
$e > 1$	Hyperbola	$\lambda_1 < 0, \lambda_2 > 0$	$\left\|\mathbf{V}\right\| < 0$

Table E.1.4.2:

E.2. Circles

E.2.1. The equation of a circle is given by

$$\|\mathbf{x}\|^2 + 2\mathbf{u}^\top\mathbf{x} + f = 0 \tag{E.2.1.1}$$

E.2.2. For a circle with centre $\mathbf{c}$ and radius r,

$$\mathbf{u} = -\mathbf{c}, f = \|\mathbf{u}\|^2 - r^2 \tag{E.2.2.1}$$

E.2.3. Any point $\mathbf{x}$ on a circle can be expressed as

$$\mathbf{x} = \mathbf{c} + r \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}. \tag{E.2.3.1}$$

E.2.4. The equation of the common chord of intersection of two circles is given by

$$2\left(\mathbf{u}_1 - \mathbf{u}_2\right)^\top \mathbf{x} + f_1 - f_2 = 0 \tag{E.2.4.1}$$

E.2.5. The line joining the centre of a circle to the mid point of any chord is perpendicular to the chord.

Proof. Let AB be any chord of a circle with centre $\mathbf{O} = \mathbf{0}$ and radius r. Then,

$$\left\|\mathbf{A}\right\|^2 = \left\|\mathbf{B}\right\|^2 = r^2 \tag{E.2.5.1}$$

$$\implies \left\|\mathbf{A}\right\|^2 - \left\|\mathbf{B}\right\|^2 = \mathbf{0} \tag{E.2.5.2}$$

$$\text{or, } \left(\mathbf{A} - \mathbf{B}\right)^\top \left(\mathbf{A} + \mathbf{B}\right) = \mathbf{0} \tag{E.2.5.3}$$

which can be expressed as

$$\left(\mathbf{A} - \mathbf{B}\right)^\top \left(\frac{\mathbf{A} + \mathbf{B}}{2} - \mathbf{O}\right) = \mathbf{0} \tag{E.2.5.4}$$

□

E.2.6. Let

$$\mathbf{A} = \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} \cos\theta_2 \\ \sin\theta_2 \end{pmatrix}, \tag{E.2.6.1}$$

be points on a unit circle with centre $\mathbf{O}$ at the origin. Then

$$\cos AOB = \mathbf{A}^\top \mathbf{B} \tag{E.2.6.2}$$

E.2.7. Let

$$\mathbf{A} = \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} \cos\theta_2 \\ \sin\theta_2 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \tag{E.2.7.1}$$

be points on a unit circle. Then

$$\cos ACB = \frac{(\mathbf{C} - \mathbf{A})^\top (\mathbf{C} - \mathbf{B})}{\|\mathbf{C} - \mathbf{A}\| \, \|\mathbf{C} - \mathbf{B}\|} \tag{E.2.7.2}$$

$$= \cos\left(\frac{\theta_1 - \theta_2}{2}\right) \tag{E.2.7.3}$$

Proof. Since

$$
\begin{aligned}
(\mathbf{C}-\mathbf{A})^\top(\mathbf{C}-\mathbf{B}) &= \|\mathbf{C}\|^2 - \mathbf{C}^\top(\mathbf{A}+\mathbf{B}) + \mathbf{A}^\top\mathbf{B} && \text{(E.2.7.4)}\\
&= 1-\cos(\theta-\theta_1)-\cos(\theta-\theta_2)+\cos(\theta_1-\theta_2) && \text{(E.2.7.5)}\\
&= 2\cos^2\left(\frac{\theta_1-\theta_2}{2}\right) - 2\cos\left(\frac{\theta_1-\theta_2}{2}\right)\cos\left(\theta-\frac{\theta_1+\theta_2}{2}\right) && \text{(E.2.7.6)}\\
&= 4\cos\left(\frac{\theta_1-\theta_2}{2}\right)\sin\left(\frac{\theta-\theta_1}{2}\right)\sin\left(\frac{\theta-\theta_2}{2}\right), && \text{(E.2.7.7)}
\end{aligned}
$$

and

$$
\begin{aligned}
\|\mathbf{C}-\mathbf{A}\|^2 &= \|\mathbf{C}\|^2 + \|\mathbf{A}\|^2 - 2\mathbf{C}^\top\mathbf{A}, && \text{(E.2.7.8)}\\
&= 4\sin^2\left(\frac{\theta-\theta_1}{2}\right), && \text{(E.2.7.9)}\\
\|\mathbf{C}-\mathbf{B}\|^2 &= \|\mathbf{C}\|^2 + \|\mathbf{B}\|^2 - 2\mathbf{C}^\top\mathbf{B}, && \text{(E.2.7.10)}\\
&= 4\sin^2\left(\frac{\theta-\theta_2}{2}\right), && \text{(E.2.7.11)}
\end{aligned}
$$

(E.2.7.2) can be expressed as

$$
\frac{\cos\left(\frac{\theta_1-\theta_2}{2}\right)\sin\left(\frac{\theta-\theta_1}{2}\right)\sin\left(\frac{\theta-\theta_2}{2}\right)}{\sin\left(\frac{\theta-\theta_1}{2}\right)\sin\left(\frac{\theta-\theta_1}{2}\right)} \tag{E.2.7.12}
$$

yielding (E.2.7.3) □

E.2.8. From (E.2.6.2) and (E.2.7.3),

$$\angle AOB = 2\angle AOC \tag{E.2.8.1}$$

E.2.9. The circumcentre $\mathbf{O}$ of $\triangle ABC$ is given by the matrix equation

$$\begin{pmatrix} \mathbf{A}-\mathbf{B} & \mathbf{B}-\mathbf{C} \end{pmatrix}^\top \mathbf{O} = \frac{1}{2}\begin{pmatrix} \|\mathbf{A}\|^2 - \|\mathbf{B}\|^2 \\ \|\mathbf{B}\|^2 - \|\mathbf{C}\|^2 \end{pmatrix} \tag{E.2.9.1}$$

Solution: Since $\mathbf{A}, \mathbf{B}, \mathbf{C}$ lie on the circle, from (E.2.1.1)

$$\|\mathbf{A}\|^2 + 2\mathbf{u}^\top \mathbf{A} + f = 0 \, \|\mathbf{B}\|^2 + 2\mathbf{u}^\top \mathbf{B} + f = 0 \, \|\mathbf{C}\|^2 + 2\mathbf{u}^\top \mathbf{C} + f = 0 \tag{E.2.9.2}$$

Subtracting equations and simplifying,

$$(\mathbf{A}-\mathbf{B})^\top \mathbf{u} = -\frac{\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2}{2} \tag{E.2.9.3}$$

$$(\mathbf{B}-\mathbf{C})^\top \mathbf{u} = -\frac{\|\mathbf{B}\|^2 - \|\mathbf{C}\|^2}{2} \tag{E.2.9.4}$$

which can be stacked to obtain (E.2.9.1).

E.3. Standard Form

E.3.1. Using the affine transformation in (B.5.5.1), the conic in (E.1.2.1) can be expressed in standard form as

$$\mathbf{y}^\top \left(\frac{\mathbf{D}}{f_0}\right) \mathbf{y} = 1 \qquad |\mathbf{V}| \neq 0 \tag{E.3.1.1}$$

$$\mathbf{y}^\top \mathbf{D} \mathbf{y} = -\eta \mathbf{e}_1^\top \mathbf{y} \qquad |\mathbf{V}| = 0 \tag{E.3.1.2}$$

where

$$f_0 = \mathbf{u}^\top \mathbf{V}^{-1} \mathbf{u} - f \neq 0 \tag{E.3.1.3}$$

$$\eta = 2\mathbf{u}^\top \mathbf{p}_1 \tag{E.3.1.4}$$

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{E.3.1.5}$$

Proof. Using (B.5.5.1) (E.1.2.1) can be expressed as

$$(\mathbf{P}\mathbf{y} + \mathbf{c})^\top \mathbf{V} (\mathbf{P}\mathbf{y} + \mathbf{c}) + 2\mathbf{u}^\top (\mathbf{P}\mathbf{y} + \mathbf{c}) + f = 0, \tag{E.3.1.6}$$

yielding

$$\mathbf{y}^\top \mathbf{P}^\top \mathbf{V} \mathbf{P} \mathbf{y} + 2 (\mathbf{V}\mathbf{c} + \mathbf{u})^\top \mathbf{P}\mathbf{y} + \mathbf{c}^\top \mathbf{V} \mathbf{c} + 2\mathbf{u}^\top \mathbf{c} + f = 0 \tag{E.3.1.7}$$

From (E.3.1.7) and (B.5.6.1),

$$\mathbf{y}^\top \mathbf{D}\mathbf{y} + 2\left(\mathbf{V}\mathbf{c} + \mathbf{u}\right)^\top \mathbf{P}\mathbf{y} + \mathbf{c}^\top \left(\mathbf{V}\mathbf{c} + \mathbf{u}\right) + \mathbf{u}^\top \mathbf{c} + f = 0 \quad \text{(E.3.1.8)}$$

When $\mathbf{V}^{-1}$ exists, choosing

$$\mathbf{V}\mathbf{c} + \mathbf{u} = \mathbf{0}, \quad \text{or}, \mathbf{c} = -\mathbf{V}^{-1}\mathbf{u}, \quad \text{(E.3.1.9)}$$

and substituting (E.3.1.9) in (E.3.1.8) yields (E.3.1.1). When $|\mathbf{V}| = 0, \lambda_1 = 0$ and

$$\mathbf{V}\mathbf{p}_1 = 0, \mathbf{V}\mathbf{p}_2 = \lambda_2 \mathbf{p}_2. \quad \text{(E.3.1.10)}$$

where $\mathbf{p}_1, \mathbf{p}_2$ are the eigenvectors of $\mathbf{V}$ such that (B.5.6.1)

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix}, \quad \text{(E.3.1.11)}$$

Substituting (E.3.1.11) in (E.3.1.8),

$$\mathbf{y}^\top \mathbf{D}\mathbf{y} + 2\left(\mathbf{c}^\top \mathbf{V} + \mathbf{u}^\top\right) \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix} \mathbf{y} + \mathbf{c}^\top \left(\mathbf{V}\mathbf{c} + \mathbf{u}\right) + \mathbf{u}^\top \mathbf{c} + f = 0 \quad \text{(E.3.1.12)}$$

$$\implies \mathbf{y}^\top \mathbf{D}\mathbf{y} + 2\left(\left(\mathbf{c}^\top \mathbf{V} + \mathbf{u}^\top\right) \mathbf{p}_1 \left(\mathbf{c}^\top \mathbf{V} + \mathbf{u}^\top\right) \mathbf{p}_2\right) \mathbf{y} + \mathbf{c}^\top \left(\mathbf{V}\mathbf{c} + \mathbf{u}\right) + \mathbf{u}^\top \mathbf{c} + f = 0 \quad \text{(E.3.1.13)}$$

$$\implies \mathbf{y}^\top \mathbf{D}\mathbf{y} + 2\begin{pmatrix} \mathbf{u}^\top \mathbf{p}_1 & \left(\lambda_2 \mathbf{c}^\top + \mathbf{u}^\top\right) \mathbf{p}_2 \end{pmatrix} \mathbf{y} + \mathbf{c}^\top \left(\mathbf{V}\mathbf{c} + \mathbf{u}\right) + \mathbf{u}^\top \mathbf{c} + f = 0 \quad \text{(E.3.1.14)}$$

upon substituting from (E.3.1.10) yielding

$$\lambda_2 y_2^2 + 2\left(\mathbf{u}^\top \mathbf{p}_1\right) y_1 + 2y_2 \left(\lambda_2 \mathbf{c} + \mathbf{u}\right)^\top \mathbf{p}_2 + \mathbf{c}^\top \left(\mathbf{V}\mathbf{c} + \mathbf{u}\right) + \mathbf{u}^\top \mathbf{c} + f = 0 \tag{E.3.1.15}$$

Thus, (E.3.1.15) can be expressed as (E.3.1.2) by choosing

$$\eta = 2\mathbf{u}^\top \mathbf{p}_1 \tag{E.3.1.16}$$

and $\mathbf{c}$ in (E.3.1.8) such that

$$2\mathbf{P}^\top \left(\mathbf{V}\mathbf{c} + \mathbf{u}\right) = \eta \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{E.3.1.17}$$

$$\mathbf{c}^\top \left(\mathbf{V}\mathbf{c} + \mathbf{u}\right) + \mathbf{u}^\top \mathbf{c} + f = 0 \tag{E.3.1.18}$$

$\because \mathbf{P}^\top \mathbf{P} = \mathbf{I}$, multiplying (E.3.1.17) by $\mathbf{P}$ yields

$$\left(\mathbf{V}\mathbf{c} + \mathbf{u}\right) = \frac{\eta}{2}\mathbf{p}_1, \tag{E.3.1.19}$$

which, upon substituting in (E.3.1.18) results in

$$\frac{\eta}{2}\mathbf{c}^\top \mathbf{p}_1 + \mathbf{u}^\top \mathbf{c} + f = 0 \tag{E.3.1.20}$$

(E.3.1.19) and (E.3.1.20) can be clubbed together to obtain (F.2.1.2).

□

E.3.2. For the standard conic,

$$\mathbf{P} = \mathbf{I} \tag{E.3.2.1}$$

$$\mathbf{u} = \begin{cases} 0 & e \neq 1 \\ \frac{\eta}{2}\mathbf{e}_1 & e = 1 \end{cases} \tag{E.3.2.2}$$

$$\lambda_1 \begin{cases} = 0 & e = 1 \\ \neq 0 & e \neq 1 \end{cases} \tag{E.3.2.3}$$

where

$$\mathbf{I} = \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 \end{pmatrix} \tag{E.3.2.4}$$

is the identity matrix.

E.3.3.

(a) The directrices for the standard conic are given by

$$\mathbf{e}_1^\top \mathbf{y} = \pm\frac{1}{e}\sqrt{\frac{|f_0|}{\lambda_2\left(1-e^2\right)}} \qquad e \neq 1 \tag{E.3.3.1}$$

$$\mathbf{e}_1^\top \mathbf{y} = \frac{\eta}{2\lambda_2} \qquad e = 1 \tag{E.3.3.2}$$

(b) The foci of the standard ellipse and hyperbola are given by

$$\mathbf{F} = \begin{cases} \pm e\sqrt{\frac{|f_0|}{\lambda_2(1-e^2)}}\mathbf{e}_1 & e \neq 1 \\ -\frac{\eta}{4\lambda_2}\mathbf{e}_1 & e = 1 \end{cases} \tag{E.3.3.3}$$

Proof. (a) For the standard hyperbola/ellipse in (E.3.1.1), from (E.3.2.1), (E.1.3.2) and (E.3.2.2),

$$\mathbf{n} = \sqrt{\frac{\lambda_2}{f_0}}\mathbf{e}_1 \tag{E.3.3.4}$$

$$c = \pm\frac{\sqrt{-\frac{\lambda_2}{f_0}\left(e^2-1\right)\left(\frac{\lambda_2}{f_0}\right)}}{\frac{\lambda_2}{f_0}e\left(e^2-1\right)} \tag{E.3.3.5}$$

$$= \pm\frac{1}{e\sqrt{1-e^2}} \tag{E.3.3.6}$$

yielding (E.3.3.1) upon substituting from (E.1.3.1) and simplifying. For the standard parabola in (E.3.1.2), from (E.3.2.1), (E.1.3.2) and (E.3.2.2), noting that $f = 0$,

$$\mathbf{n} = \sqrt{\lambda_2}\mathbf{e}_1 \tag{E.3.3.7}$$

$$c = \frac{\left\|\frac{\eta}{2}\mathbf{e}_1\right\|^2}{2\left(\frac{\eta}{\mathbf{2}}\right)\left(\mathbf{e_1}\right)^\top \mathbf{n}} \tag{E.3.3.8}$$

$$\tag{E.3.3.9}$$

$$= \frac{\eta}{4\sqrt{\lambda_2}} \tag{E.3.3.10}$$

yielding (E.3.3.2).

(b) For the standard ellipse/hyperbola, substituting from (E.3.3.6), (E.3.3.4), (E.3.2.2) and (E.1.3.1) in (E.1.3.3),

$$\mathbf{F} = \pm\frac{\left(\frac{1}{e\sqrt{1-e^2}}\right)\left(e^2\right)\sqrt{\frac{\lambda_2}{f_0}}\mathbf{e}_1}{\frac{\lambda_2}{f_0}} \tag{E.3.3.11}$$

yielding (E.3.3.3) after simplification. For the standard parabola, substituting from (E.3.3.10), (E.3.3.7), (E.3.2.2) and (E.1.3.1) in (E.1.3.3),

$$\mathbf{F} = \frac{\left(\frac{\eta}{4\sqrt{\lambda_2}}\right)\sqrt{\lambda_2}\mathbf{e}_1 - \frac{\eta}{\mathbf{2}}\mathbf{e_1}}{\lambda_2} \tag{E.3.3.12}$$

$$\tag{E.3.3.13}$$

yielding (E.3.3.3) after simplification.

□

Appendix F

Conic Parameters

F.1. Standard Form

F.1.1. The center of the standard ellipse/hyperbola, defined to be the mid point of the line joining the foci, is the origin.

F.1.2. The principal (major) axis of the standard ellipse/hyperbola, defined to be the line joining the two foci is the x-axis.

Proof. From (E.3.3.3), it is obvious that the line joining the foci passes through the origin. Also, the direction vector of this line is $\mathbf{e}_1$. Thus, the principal axis is the x-axis. $\square$

F.1.3. The minor axis of the standard ellipse/hyperbola, defined to be the line orthogonal to the x-axis is the y-axis.

F.1.4. The axis of symmetry of the standard parabola, defined to be the line perpendicular to the directrix and passing through the focus, is the x-axis.

Proof. From (E.3.3.7) and (E.3.3.3), the axis of the parabola can be expressed using (D.1.2.1) as

$$\mathbf{e}_2^\top \left(\mathbf{y} + \frac{\eta}{4\lambda_2}\mathbf{e}_1\right) = 0 \tag{F.1.4.1}$$

$$\implies \mathbf{e}_2^\top \mathbf{y} = 0, \tag{F.1.4.2}$$

which is the equation of the x-axis. □

F.1.5. The point where the parabola intersects its axis of symmetry is called the vertex. For the standard parabola, the vertex is the origin.

Proof. (F.1.4.2) can be expressed as

$$\mathbf{y} = \alpha \mathbf{e}_1, \tag{F.1.5.1}$$

using (D.1.2.1). Substituting (F.1.5.1) in (E.3.1.2),

$$\alpha^2 \mathbf{e}_1^\top \mathbf{D} \mathbf{e}_1 = -\eta \alpha \mathbf{e}_1^\top \mathbf{e}_1 \tag{F.1.5.2}$$

$$\implies \alpha = 0, \text{ or, } \mathbf{y} = \mathbf{0}. \tag{F.1.5.3}$$

□

F.1.6. The focal length of the standard parabola, , defined to be the distance between the vertex and the focus, measured along the axis of symmetry, is $\left|\frac{\eta}{4\lambda_2}\right|$

F.2. Quadratic Form

F.2.1. The center/vertex of a conic section are given by

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \qquad \left|\mathbf{V}\right| \neq 0 \tag{F.2.1.1}$$

$$\begin{pmatrix} \mathbf{u}^\top + \frac{\eta}{2}\mathbf{p}_1^\top \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \frac{\eta}{2}\mathbf{p}_1 - \mathbf{u} \end{pmatrix} \qquad \left|\mathbf{V}\right| = 0 \tag{F.2.1.2}$$

Proof. In (B.5.5.1), substituting $\mathbf{y} = \mathbf{0}$, the center/vertex for the quadratic form is obtained as

$$\mathbf{x} = \mathbf{c}, \tag{F.2.1.3}$$

where $\mathbf{c}$ is derived as (F.2.1.1) and (F.2.1.2) in Appendix E.3.1. □

F.2.2. The equation of the minor and major axes for the ellipse/hyperbola are respectively given by

$$\mathbf{p}_i^\top (\mathbf{x} - \mathbf{c}) = 0, i = 1, 2 \tag{F.2.2.1}$$

The axis of symmetry for the parabola is also given by (F.2.2.1).

Proof. From (F.1.2), the major/symmetry axis for the hyperbola/ellipse/parabola can be expressed using (B.5.5.1) as

$$\mathbf{e}_2^\top \mathbf{P}^\top (\mathbf{x} - \mathbf{c}) = 0 \tag{F.2.2.2}$$

$$\implies (\mathbf{P}\mathbf{e}_2)^\top (\mathbf{x} - \mathbf{c}) = 0 \tag{F.2.2.3}$$

yielding (F.2.2.1), and the proof for the minor axis is similar. $\square$

Appendix G

Conic Lines

G.1. Pair of Straight Lines

G.1.1. The asymptotes of the hyperbola in (E.3.1.1), defined to be the lines that do not intersect the hyperbola, are given by

$$\left(\sqrt{|\lambda_1|} \quad \pm\sqrt{|\lambda_2|}\right)\mathbf{y} = 0 \tag{G.1.1.1}$$

Proof. From (E.3.1.1), it is obvious that the pair of lines represented by

$$\mathbf{y}^\top \mathbf{D}\mathbf{y} = 0 \tag{G.1.1.2}$$

do not intersect the conic

$$\mathbf{y}^\top \mathbf{D}\mathbf{y} = f_0 \tag{G.1.1.3}$$

Thus, (G.1.1.2) represents the asysmptotes of the hyperbola in (E.3.1.1) and can be expressed as

$$\lambda_1 y_1^2 + \lambda_2 y_1^2 = 0, \tag{G.1.1.4}$$

which can then be simplified to obtain (G.1.1.1).

$\square$

G.1.2. (E.1.2.1) represents a pair of straight lines if

$$\mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f = 0 \tag{G.1.2.1}$$

G.1.3. (E.1.2.1) represents a pair of straight lines if the matrix

$$\begin{pmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^\top & f \end{pmatrix} \tag{G.1.3.1}$$

is singular.

Proof. Let

$$\begin{pmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^\top & f \end{pmatrix} \mathbf{x} = \mathbf{0} \tag{G.1.3.2}$$

Expressing

$$\mathbf{x} = \begin{pmatrix} \mathbf{y} \\ y_3 \end{pmatrix}, \tag{G.1.3.3}$$

$$\begin{pmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^\top & f \end{pmatrix} \begin{pmatrix} \mathbf{y} \\ y_3 \end{pmatrix} = \mathbf{0} \tag{G.1.3.4}$$

$$\implies \mathbf{V}\mathbf{y} + y_3\mathbf{u} = \mathbf{0} \quad \text{and} \tag{G.1.3.5}$$

$$\mathbf{u}^\top\mathbf{y} + fy_3 = 0 \tag{G.1.3.6}$$

From (G.1.3.5) we obtain,

$$\mathbf{y}^\top\mathbf{V}\mathbf{y} + y_3\mathbf{y}^\top\mathbf{u} = \mathbf{0} \tag{G.1.3.7}$$

$$\implies \mathbf{y}^\top\mathbf{V}\mathbf{y} + y_3\mathbf{u}^\top\mathbf{y} = \mathbf{0} \tag{G.1.3.8}$$

yielding (G.1.2.1) upon substituting from (G.1.3.6). $\square$

G.1.4. Using the affine transformation, (G.1.1.1) can be expressed as the lines

$$\begin{pmatrix} \sqrt{|\lambda_1|} & \pm\sqrt{|\lambda_2|} \end{pmatrix} \mathbf{P}^\top (\mathbf{x} - \mathbf{c}) = 0 \tag{G.1.4.1}$$

G.1.5. The angle between the asymptotes can be expressed as

$$\cos\theta = \frac{|\lambda_1| - |\lambda_2|}{|\lambda_1| + |\lambda_2|} \tag{G.1.5.1}$$

Proof. The normal vectors of the lines in (G.1.4.1) are

$$\begin{aligned}\mathbf{n}_1 &= \mathbf{P}\begin{pmatrix}\sqrt{|\lambda_1|}\\ \\ \sqrt{|\lambda_2|}\end{pmatrix}\\ \mathbf{n}_2 &= \mathbf{P}\begin{pmatrix}\sqrt{|\lambda_1|}\\ \\ -\sqrt{|\lambda_2|}\end{pmatrix}\end{aligned} \tag{G.1.5.2}$$

The angle between the asymptotes is given by

$$\cos\theta = \frac{\mathbf{n_1}^\top \mathbf{n_2}}{\|\mathbf{n_1}\|\,\|\mathbf{n_2}\|} \tag{G.1.5.3}$$

The orthogonal matrix $\mathbf{P}$ preserves the norm, i.e.

$$\|\mathbf{n_1}\| = \left\|\mathbf{P}\begin{pmatrix}\sqrt{|\lambda_1|}\\ \\ \sqrt{|\lambda_2|}\end{pmatrix}\right\| = \left\|\begin{pmatrix}\sqrt{|\lambda_1|}\\ \\ \sqrt{|\lambda_2|}\end{pmatrix}\right\| \tag{G.1.5.4}$$

$$= \sqrt{|\lambda_1| + |\lambda_2|} = \|\mathbf{n_2}\| \tag{G.1.5.5}$$

It is easy to verify that

$$\mathbf{n_1}^\top \mathbf{n_2} = |\lambda_1| - |\lambda_2| \tag{G.1.5.6}$$

Thus, the angle between the asymptotes is obtained from (G.1.5.3) as (G.1.5.1). $\square$

G.2. Intersection of Conics

G.2.1. Let

$$\mathbf{x}^\top \mathbf{V}_i \mathbf{x} + 2\mathbf{u}_i^\top \mathbf{x} + f_i = 0, \quad i = 1, 2 \tag{G.2.1.1}$$

be the equation of two conics. The locus of their intersection is a pair of straight lines if

$$\begin{vmatrix} \mathbf{V}_1 + \mu \mathbf{V}_2 & \mathbf{u}_1 + \mu \mathbf{u}_2 \\ (\mathbf{u}_1 + \mu \mathbf{u}_2)^\top & f \end{vmatrix} = 0, \begin{vmatrix} \mathbf{V}_1 + \mu \mathbf{V}_2 \end{vmatrix} < 0 \tag{G.2.1.2}$$

Proof. The intersection of the conics in (G.2.1.1) is given by the curve

$$\mathbf{x}^\top (\mathbf{V}_1 + \mu \mathbf{V}_2) \mathbf{x} + 2 (\mathbf{u}_1 + \mu \mathbf{u}_2)^\top \mathbf{x} + f_1 + \mu f_2 = 0, \tag{G.2.1.3}$$

which, from Theorem G.1.3 represents a pair of straight lines if (G.2.1.2) is satisfied. □

G.2.2. The points of intersection of the conics in (G.2.1.1) are the points of the intersection of the lines in (G.2.1.3).

G.3. Chords of a Conic

G.3.1. The points of intersection of the line

$$L : \quad \mathbf{x} = \mathbf{h} + \mu \mathbf{m} \quad \mu \in \mathbb{R} \tag{G.3.1.1}$$

with the conic section in (E.1.2.1) are given by

$$\mathbf{x}_i = \mathbf{h} + \mu_i \mathbf{m} \tag{G.3.1.2}$$

where

$$\mu_i = \frac{1}{\mathbf{m}^\top \mathbf{V} \mathbf{m}} \left(-\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u}) \pm \sqrt{\left[\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u})\right]^2 - \mathrm{g}(\mathbf{h}) \left(\mathbf{m}^\top \mathbf{V} \mathbf{m}\right)} \right) \tag{G.3.1.3}$$

Proof. Substituting (G.3.1.1) in (E.1.2.1),

$$(\mathbf{h} + \mu \mathbf{m})^\top \mathbf{V} (\mathbf{h} + \mu \mathbf{m}) + 2\mathbf{u}^\top (\mathbf{h} + \mu \mathbf{m}) + f = 0 \tag{G.3.1.4}$$

$$\implies \mu^2 \mathbf{m}^\top \mathbf{V} \mathbf{m} + 2\mu \mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u}) + \mathbf{h}^\top \mathbf{V} \mathbf{h} + 2\mathbf{u}^\top \mathbf{h} + f = 0 \tag{G.3.1.5}$$

$$\text{or, } \mu^2 \mathbf{m}^\top \mathbf{V} \mathbf{m} + 2\mu \mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u}) + \mathrm{g}(\mathbf{h}) = 0 \tag{G.3.1.6}$$

for g defined in (E.1.2.1). Solving the above quadratic in (G.3.1.6) yields (G.3.1.3). □

G.3.2. If L in (G.3.1.1) touches (E.1.2.1) at exactly one point $\mathbf{q}$,

$$\mathbf{m}^\top (\mathbf{V}\mathbf{q} + \mathbf{u}) = 0 \tag{G.3.2.1}$$

Proof. In this case, (G.3.1.6) has exactly one root. Hence, in (G.3.1.3)

$$\left[\mathbf{m}^\top (\mathbf{V}\mathbf{q}+\mathbf{u})\right]^2 - \left(\mathbf{m}^\top \mathbf{V}\mathbf{m}\right) \mathrm{g}(\mathbf{q}) = 0 \tag{G.3.2.2}$$

$\because$ $\mathbf{q}$ is the point of contact,

$$\mathrm{g}(\mathbf{q}) = 0 \tag{G.3.2.3}$$

Substituting (G.3.2.3) in (G.3.2.2) and simplifying, we obtain (G.3.2.1).

$\square$

G.3.3. The length of the chord in (G.3.1.1) is given by

$$\frac{2\sqrt{\left[\mathbf{m}^\top (\mathbf{V}\mathbf{h}+\mathbf{u})\right]^2 - \left(\mathbf{h}^\top \mathbf{V}\mathbf{h} + 2\mathbf{u}^\top\mathbf{h} + f\right)\left(\mathbf{m}^\top \mathbf{V}\mathbf{m}\right)}}{\mathbf{m}^\top \mathbf{V}\mathbf{m}} \left\|\mathbf{m}\right\| \tag{G.3.3.1}$$

Proof. The distance between the points in (G.3.1.2) is given by

$$\left\|\mathbf{x}_1 - \mathbf{x}_2\right\| = \left|\mu_1 - \mu_2\right| \left\|\mathbf{m}\right\| \tag{G.3.3.2}$$

Substituing μ_i from (G.3.1.3) in (G.3.3.2) yields (G.3.3.1). $\square$

G.3.4. The affine transform for the conic section, preserves the norm. This implies that the length of any chord of a conic is invariant to translation and/or rotation.

Proof. Let

$$\mathbf{x}_i = \mathbf{P}\mathbf{y}_i + \mathbf{c} \tag{G.3.4.1}$$

be any two points on the conic. Then the distance between the points is given by

$$\|\mathbf{x}_1 - \mathbf{x}_2\| = \|\mathbf{P}(\mathbf{y}_1 - \mathbf{y}_2)\| \tag{G.3.4.2}$$

which can be expressed as

$$\|\mathbf{x}_1 - \mathbf{x}_2\|^2 = (\mathbf{y}_1 - \mathbf{y}_2)^\top \mathbf{P}^\top\mathbf{P}(\mathbf{y}_1 - \mathbf{y}_2) \tag{G.3.4.3}$$

$$= \|\mathbf{y}_1 - \mathbf{y}_2\|^2 \tag{G.3.4.4}$$

since

$$\mathbf{P}^\top\mathbf{P} = \mathbf{I} \tag{G.3.4.5}$$

□

G.3.5. For the standard hyperbola/ellipse, the length of the major axis is

$$2\sqrt{\left|\frac{f_0}{\lambda_1}\right|} \tag{G.3.5.1}$$

and the minor axis is

$$2\sqrt{\left|\frac{f_0}{\lambda_2}\right|} \tag{G.3.5.2}$$

Proof. Since the major axis passes through the origin,

$$\mathbf{q} = \mathbf{0} \tag{G.3.5.3}$$

Further, from Corollary (F.1.2),

$$\mathbf{m} = \mathbf{e}_2, \tag{G.3.5.4}$$

and from (E.3.1.1),

$$\mathbf{V} = \frac{\mathbf{D}}{f_0}, \mathbf{u} = 0, f = -1 \tag{G.3.5.5}$$

Substituting the above in (G.3.3.1),

$$\frac{2\sqrt{\mathbf{e}_1^\top \frac{\mathbf{D}}{f_0}\mathbf{e}_1}}{\mathbf{e}_1^\top \frac{\mathbf{D}}{f_0}\mathbf{e}_1}\left\|\mathbf{e}_1\right\| \tag{G.3.5.6}$$

yielding (G.3.5.1). Similarly, for the minor axis, the only different parameter is

$$\mathbf{m} = \mathbf{e}_2, \tag{G.3.5.7}$$

Substituting the above in (G.3.3.1),

$$\frac{2\sqrt{\mathbf{e}_2^\top \frac{\mathbf{D}}{f_0}\mathbf{e}_2}}{\mathbf{e}_2^\top \frac{\mathbf{D}}{f_0}\mathbf{e}_2}\left\|\mathbf{e}_2\right\| \tag{G.3.5.8}$$

yielding (G.3.5.2).

□

G.3.6. The latus rectum of a conic section is the chord that passes through the focus and is perpendicular to the major axis. The length of the latus rectum for a conic is given by

$$l = \begin{cases} 2\frac{\sqrt{|f_0\lambda_1|}}{\lambda_2} & e \neq 1 \\ \frac{\eta}{\lambda_2} & e = 1 \end{cases} \tag{G.3.6.1}$$

Proof. The latus rectum is perpendicular to the major axis for the standard conic. Hence, from Corollary (F.1.2),

$$\mathbf{m} = \mathbf{e}_2, \tag{G.3.6.2}$$

Since it passes through the focus, from (E.3.3.3)

$$\mathbf{q} = \mathbf{F} = \pm e\sqrt{\frac{f_0}{\lambda_2\left(1-e^2\right)}}\mathbf{e}_1 \tag{G.3.6.3}$$

for the standard hyperbola/ellipse. Also, from (E.3.1.1),

$$\mathbf{V} = \frac{\mathbf{D}}{f_0}, \mathbf{u} = 0, f = -1 \tag{G.3.6.4}$$

Substituting the above in (G.3.3.1),

$$\frac{2\sqrt{\left[\mathbf{e}_2^\top\left(\frac{\mathbf{D}}{f_0}e\sqrt{\frac{f_0}{\lambda_2(1-e^2)}}\mathbf{e}_1\right)\right]^2-\left(e\sqrt{\frac{f_0}{\lambda_2(1-e^2)}}\mathbf{e}_1^\top\frac{\mathbf{D}}{f_0}e\sqrt{\frac{f_0}{\lambda_2(1-e^2)}}\mathbf{e}_1-1\right)\left(\mathbf{e}_2^\top\frac{\mathbf{D}}{f_0}\mathbf{e}_2\right)}}{\mathbf{e}_2^\top\frac{\mathbf{D}}{f_0}\mathbf{e}_2}\|\mathbf{e}_2\| \tag{G.3.6.5}$$

Since

$$\mathbf{e}_2^\top\mathbf{D}\mathbf{e}_1=0,\mathbf{e}_1^\top\mathbf{D}\mathbf{e}_1=\lambda_1,\mathbf{e}_1^\top\mathbf{e}_1=1,\|\mathbf{e}_2\|=1,\mathbf{e}_2^\top\mathbf{D}\mathbf{e}_2=\lambda_2, \tag{G.3.6.6}$$

(G.3.6.5) can be expressed as

$$\frac{2\sqrt{\left(1-\frac{\lambda_1e^2}{\lambda_2(1-e^2)}\right)\left(\frac{\lambda_2}{f_0}\right)}}{\frac{\lambda_2}{f_0}} \tag{G.3.6.7}$$

$$=2\frac{\sqrt{f_0\lambda_1}}{\lambda_2} \qquad \left(\because e^2=1-\frac{\lambda_1}{\lambda_2}\right) \tag{G.3.6.8}$$

For the standard parabola, the parameters in (G.3.3.1) are

$$\mathbf{q}=\mathbf{F}=-\frac{\eta}{4\lambda_2}\mathbf{e}_1,\mathbf{m}=\mathbf{e}_1,\mathbf{V}=\mathbf{D},\mathbf{u}=\frac{\eta}{2}\mathbf{e}_1^\top,f=0 \tag{G.3.6.9}$$

Substituting the above in (G.3.3.1), the length of the latus rectum can be expressed as

$$\frac{2\sqrt{\left[\mathbf{e}_2^\top\left(\mathbf{D}\left(-\frac{\eta}{4\lambda_2}\mathbf{e}_1\right)+\frac{\eta}{2}\mathbf{e}_1\right)\right]^2-\left(\left(-\frac{\eta}{4\lambda_2}\mathbf{e}_1\right)^\top\mathbf{D}\left(-\frac{\eta}{4\lambda_2}\mathbf{e}_1\right)+2\frac{\eta}{2}\mathbf{e}_1^\top\left(-\frac{\eta}{4\lambda_2}\mathbf{e}_1\right)\right)\left(\mathbf{e}_2^\top\mathbf{D}\mathbf{e}_2\right)}}{\mathbf{e}_2^\top\mathbf{D}\mathbf{e}_2}\|\mathbf{e}_2\| \tag{G.3.6.10}$$

Since

$$\mathbf{e}_2^\top \mathbf{D}\mathbf{e}_1 = 0, \mathbf{e}_2^\top \mathbf{e}_2 = 0, \mathbf{e}_1^\top \mathbf{D}\mathbf{e}_1 = 0, \mathbf{e}_1^\top \mathbf{e}_1 = 1, \|\mathbf{e}_1\| = 1, \mathbf{e}_2^\top \mathbf{D}\mathbf{e}_2 = \lambda_2, \tag{G.3.6.11}$$

(G.3.6.10) can be expressed as

$$2\frac{\sqrt{\frac{\eta^2}{4\lambda_2}\lambda_2}}{\lambda_2} = \frac{\eta}{\lambda_2} \tag{G.3.6.12}$$

$\square$

G.4. Tangent and Normal

G.4.1. Given the point of contact $\mathbf{q}$, the equation of a tangent to (E.1.2.1) is

$$(\mathbf{V}\mathbf{q} + \mathbf{u})^\top \mathbf{x} + \mathbf{u}^\top \mathbf{q} + f = 0 \tag{G.4.1.1}$$

Proof. The normal vector is obtained from (G.3.2.1) and (A.1.20.1) as

$$\kappa \mathbf{n} = \mathbf{V}\mathbf{q} + \mathbf{u}, \kappa \in \mathbb{R} \tag{G.4.1.2}$$

From (G.4.1.2) and (D.1.2.1), the equation of the tangent is

$$(\mathbf{V}\mathbf{q} + \mathbf{u})^\top (\mathbf{x} - \mathbf{q}) = 0 \tag{G.4.1.3}$$

$$\implies (\mathbf{V}\mathbf{q} + \mathbf{u})^\top \mathbf{x} - \mathbf{q}^\top \mathbf{V}\mathbf{q} - \mathbf{u}^\top \mathbf{q} = 0 \tag{G.4.1.4}$$

which, upon substituting from (G.3.2.3) and simplifying yields (G.4.1.1)

□

G.4.2. Given the point of contact $\mathbf{q}$, the equation of the normal to (E.1.2.1) is

$$(\mathbf{V}\mathbf{q}+\mathbf{u})^\top \mathbf{R}\left(\mathbf{x}-\mathbf{q}\right)=0 \tag{G.4.2.1}$$

Proof. The direction vector of the tangent is obtained from (G.4.1.2) as as

$$\mathbf{m}=\mathbf{R}\left(\mathbf{V}\mathbf{q}+\mathbf{u}\right), \tag{G.4.2.2}$$

where $\mathbf{R}$ is the rotation matrix. From (G.4.2.2) and (D.1.2.1), the equation of the normal is given by (G.4.2.1) □

G.4.3. Given the tangent

$$\mathbf{n}^\top \mathbf{x}=c, \tag{G.4.3.1}$$

the point of contact to the conic in (E.1.2.1) is given by

$$\begin{pmatrix}\mathbf{n}^\top \\ \mathbf{m}^\top \mathbf{V}\end{pmatrix}\mathbf{q}=\begin{pmatrix}c \\ -\mathbf{m}^\top \mathbf{u}\end{pmatrix} \tag{G.4.3.2}$$

Proof. From (G.3.2.1),

$$\mathbf{m}^\top (\mathbf{V}\mathbf{q} + \mathbf{u}) = 0 \tag{G.4.3.3}$$

$$\implies \mathbf{m}^\top \mathbf{V}\mathbf{q} = -\mathbf{m}^\top \mathbf{u} \tag{G.4.3.4}$$

Combining (G.4.3.1) and (G.4.3.4), (G.4.3.2) is obtained.

□

G.4.4. If $\mathbf{V}^{-1}$ exists, given the normal vector $\mathbf{n}$, the tangent points of contact to (E.1.2.1) are given by

$$\begin{aligned} \mathbf{q}_i &= \mathbf{V}^{-1}\left(\kappa_i \mathbf{n} - \mathbf{u}\right), i = 1, 2 \\ \text{where } \kappa_i &= \pm\sqrt{\frac{f_0}{\mathbf{n}^\top \mathbf{V}^{-1}\mathbf{n}}} \end{aligned} \tag{G.4.4.1}$$

Proof. From (G.4.1.2),

$$\mathbf{q} = \mathbf{V}^{-1}\left(\kappa \mathbf{n} - \mathbf{u}\right), \quad \kappa \in \mathbb{R} \tag{G.4.4.2}$$

Substituting (G.4.4.2) in (G.3.2.3),

$$\left(\kappa \mathbf{n} - \mathbf{u}\right)^\top \mathbf{V}^{-1}\left(\kappa \mathbf{n} - \mathbf{u}\right) + 2\mathbf{u}^\top \mathbf{V}^{-1}\left(\kappa \mathbf{n} - \mathbf{u}\right) + f = 0 \tag{G.4.4.3}$$

$$\implies \kappa^2 \mathbf{n}^\top \mathbf{V}^{-1}\mathbf{n} - \mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} + f = 0 \tag{G.4.4.4}$$

$$\text{or, } \kappa = \pm\sqrt{\frac{f_0}{\mathbf{n}^\top \mathbf{V}^{-1}\mathbf{n}}} \tag{G.4.4.5}$$

Substituting (G.4.4.5) in (G.4.4.2) yields (G.4.4.1). □

G.4.5. For a conic/hyperbola, a line with normal vector $\mathbf{n}$ cannot be a tangent if

$$\frac{\mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f}{\mathbf{n}^\top \mathbf{V}^{-1}\mathbf{n}} < 0 \tag{G.4.5.1}$$

G.4.6. For a circle,

$$\mathbf{q}_{ij} = \left(\pm r \frac{\mathbf{n}_j}{\|\mathbf{n}_j\|} - \mathbf{u}\right), \quad i, j = 1, 2 \tag{G.4.6.1}$$

Proof. From (G.4.4.1), and (E.2.2.1),

$$\kappa_{ij} = \pm \frac{r}{\|\mathbf{n}_j\|} \tag{G.4.6.2}$$

□

G.4.7. If $\mathbf{V}$ is not invertible, given the normal vector $\mathbf{n}$, the point of contact to (E.1.2.1) is given by the matrix equation

$$\begin{pmatrix} (\mathbf{u} + \kappa \mathbf{n})^\top \\ \mathbf{V} \end{pmatrix} \mathbf{q} = \begin{pmatrix} -f \\ \kappa \mathbf{n} - \mathbf{u} \end{pmatrix} \tag{G.4.7.1}$$

$$\text{where } \kappa = \frac{\mathbf{p}_1^\top \mathbf{u}}{\mathbf{p}_1^\top \mathbf{n}}, \quad \mathbf{V}\mathbf{p}_1 = 0 \tag{G.4.7.2}$$

Proof. If $\mathbf{V}$ is non-invertible, it has a zero eigenvalue. If the corresponding eigenvector is $\mathbf{p}_1$, then,

$$\mathbf{V}\mathbf{p}_1 = 0 \tag{G.4.7.3}$$

From (G.4.1.2),

$$\kappa\mathbf{n} = \mathbf{V}\mathbf{q} + \mathbf{u}, \quad \kappa \in \mathbb{R} \tag{G.4.7.4}$$

$$\implies \kappa\mathbf{p}_1^\top\mathbf{n} = \mathbf{p}_1^\top\mathbf{V}\mathbf{q} + \mathbf{p}_1^\top\mathbf{u} \tag{G.4.7.5}$$

$$\text{or, } \kappa\mathbf{p}_1^\top\mathbf{n} = \mathbf{p}_1^\top\mathbf{u}, \quad \because \mathbf{p}_1^\top\mathbf{V} = 0, \quad (\text{ from (G.4.7.3)}) \tag{G.4.7.6}$$

yielding κ in (G.4.7.2). From (G.4.7.4),

$$\kappa\mathbf{q}^\top\mathbf{n} = \mathbf{q}^\top\mathbf{V}\mathbf{q} + \mathbf{q}^\top\mathbf{u} \tag{G.4.7.7}$$

$$\implies \kappa\mathbf{q}^\top\mathbf{n} = -f - \mathbf{q}^\top\mathbf{u} \quad \text{from (G.3.2.3)}, \tag{G.4.7.8}$$

$$\text{or, } \left(\kappa\mathbf{n} + \mathbf{u}\right)^\top\mathbf{q} = -f \tag{G.4.7.9}$$

(G.4.7.4) can be expressed as

$$\mathbf{V}\mathbf{q} = \kappa\mathbf{n} - \mathbf{u}. \tag{G.4.7.10}$$

(G.4.7.9) and (G.4.7.10) clubbed together result in (G.4.7.1). □

G.4.8. A point $\mathbf{h}$ lies on a tangent to the conic in (E.1.2.1) if

$$\mathbf{m}^\top\left[\left(\mathbf{V}\mathbf{h} + \mathbf{u}\right)\left(\mathbf{V}\mathbf{h} + \mathbf{u}\right)^\top - \mathbf{V}\mathrm{g}\left(\mathbf{h}\right)\right]\mathbf{m} = 0 \tag{G.4.8.1}$$

Proof. From (G.3.1.3) and (G.3.2.2)

$$\left[\mathbf{m}^\top\left(\mathbf{V}\mathbf{h} + \mathbf{u}\right)\right]^2 - \left(\mathbf{m}^\top\mathbf{V}\mathbf{m}\right)\mathrm{g}\left(\mathbf{h}\right) = 0 \tag{G.4.8.2}$$

yielding (G.4.8.1). □

G.4.9. The normal vectors of the tangents to the conic in (E.1.2.1) from a point $\mathbf{h}$ are given by

$$\begin{aligned}\mathbf{n}_1 &= \mathbf{P}\begin{pmatrix}\sqrt{|\lambda_1|}\\ \sqrt{|\lambda_2|}\end{pmatrix}\\ \mathbf{n}_2 &= \mathbf{P}\begin{pmatrix}\sqrt{|\lambda_1|}\\ -\sqrt{|\lambda_2|}\end{pmatrix}\end{aligned} \tag{G.4.9.1}$$

where $\lambda_i, \mathbf{P}$ are the eigenparameters of

$$\mathbf{\Sigma} = (\mathbf{V}\mathbf{h}+\mathbf{u})(\mathbf{V}\mathbf{h}+\mathbf{u})^\top - (\mathrm{g}(\mathbf{h}))\,\mathbf{V}. \tag{G.4.9.2}$$

Proof. From (G.4.8.1) we obtain (G.4.9.2). Consequently, from (G.1.5.2), (G.4.9.1) can be obtained. □

G.4.10. A point $\mathbf{h}$ lies on a normal to the conic in (E.1.2.1) if

$$\begin{aligned}\left(\mathbf{m}^\top(\mathbf{V}\mathbf{h}+\mathbf{u})\right)^2\left(\mathbf{n}^\top\mathbf{V}\mathbf{n}\right) - 2\left(\mathbf{m}^\top\mathbf{V}\mathbf{n}\right)\left(\mathbf{m}^\top(\mathbf{V}\mathbf{h}+\mathbf{u})\mathbf{n}^\top(\mathbf{V}\mathbf{h}+\mathbf{u})\right)\\ + \mathrm{g}(\mathbf{h})\left(\mathbf{m}^\top\mathbf{V}\mathbf{n}\right)^2 = 0\end{aligned} \tag{G.4.10.1}$$

Proof. The point of contact for the normal passing through a point $\mathbf{h}$ is given by

$$\mathbf{q} = \mathbf{h} + \mu\mathbf{n} \tag{G.4.10.2}$$

From (G.3.2.1), the tangent at $\mathbf{q}$ satisfies

$$\mathbf{m}^{\top}(\mathbf{V}\mathbf{q}+\mathbf{u})=0 \tag{G.4.10.3}$$

Substituting (G.4.10.2) in (G.4.10.3),

$$\mathbf{m}^{\top}(\mathbf{V}(\mathbf{h}+\mu\mathbf{n})+\mathbf{u})=0 \tag{G.4.10.4}$$

$$\implies \mu\mathbf{m}^{\top}\mathbf{V}\mathbf{n}=-\mathbf{m}^{\top}(\mathbf{V}\mathbf{h}+\mathbf{u}) \tag{G.4.10.5}$$

yielding

$$\mu=-\frac{\mathbf{m}^{\top}(\mathbf{V}\mathbf{h}+\mathbf{u})}{\mathbf{m}^{\top}\mathbf{V}\mathbf{n}}, \tag{G.4.10.6}$$

From (G.3.1.6),

$$\mu^{2}\mathbf{n}^{\top}\mathbf{V}\mathbf{n}+2\mu\mathbf{n}^{\top}\left(\mathbf{V}\mathbf{h}+\mathbf{u}\right)+\mathrm{g}\left(\mathbf{h}\right)=0 \tag{G.4.10.7}$$

From (G.4.10.6), (G.4.10.7) can be expressed as

$$\left(-\frac{\mathbf{m}^{\top}(\mathbf{V}\mathbf{h}+\mathbf{u})}{\mathbf{m}^{\top}\mathbf{V}\mathbf{n}}\right)^{2}\mathbf{n}^{\top}\mathbf{V}\mathbf{n}+2\left(-\frac{\mathbf{m}^{\top}(\mathbf{V}\mathbf{h}+\mathbf{u})}{\mathbf{m}^{\top}\mathbf{V}\mathbf{n}}\right)\mathbf{n}^{\top}\left(\mathbf{V}\mathbf{h}+\mathbf{u}\right)+\mathrm{g}\left(\mathbf{h}\right)=0 \tag{G.4.10.8}$$

yielding (G.4.10.1). □