Conic Assignment

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***Abstract*—This document contains the solution to Question 27 of Exercise 5 in Chapter 6 of the class 12 NCERT textbook.**

1) The point on the curve
$$x^2 = 2y \tag{1}$$
which is nearest to the point $\mathbf{P} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$ is

a) $\begin{pmatrix} 2\sqrt{2} \\ 4 \end{pmatrix}$
b) $\begin{pmatrix} 2\sqrt{2} \\ 0 \end{pmatrix}$
c) $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
d) $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$

Solution: We rewrite the conic (1) in matrix form.
$$\mathbf{x}^\top \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + 2\begin{pmatrix} 0 & -1 \end{pmatrix} \mathbf{x} = 0 \tag{2}$$
Comparing with the general equation of the conic,
$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{3}$$
$$\mathbf{u} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \tag{4}$$
$$f = 0 \tag{5}$$
Therefore, the equation of the normal where $\mathbf{u}$ is the point of contact and $\mathbf{R} \triangleq \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is
$$(\mathbf{V}\mathbf{q} + \mathbf{u})^\top \mathbf{R}\left(\begin{pmatrix} 0 \\ 5 \end{pmatrix} - \mathbf{q}\right) = 0 \tag{6}$$
Substituting the appropriate values and simplifying, we get the equation
$$\mathbf{q}^\top \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mathbf{q} + 2\mathbf{q}^\top \begin{pmatrix} -2 \\ 0 \end{pmatrix} = 0 \tag{7}$$
Comparing with the general equation of the conic,
$$\mathbf{V'} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{8}$$
$$\mathbf{u'} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \tag{9}$$
$$f' = 0 \tag{10}$$
To solve (7), we must make $\mathbf{V'}$ symmetric. Thus, substituting $\mathbf{V'} \leftarrow \frac{\mathbf{V'} + \mathbf{V'}^\top}{2}$, the equation becomes
$$\mathbf{q}^\top \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \mathbf{q} + 2\begin{pmatrix} -2 \\ 0 \end{pmatrix} \mathbf{q} = 0 \tag{11}$$
Note that
$$\mathbf{V'}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{12}$$
$$\mathbf{V'}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{13}$$
and hence the eigenparameters of $\mathbf{V'}$ are
$$\mathbf{P} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \ \mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{14}$$
Applying the affine transformation and since $\det V' = -\frac{1}{4} \neq 0$, (11) becomes
$$\mathbf{y}^\top \mathbf{D}\mathbf{y} = f_0 \tag{15}$$
where
$$\mathbf{q} = \mathbf{P}\mathbf{y} + \mathbf{c} \tag{16}$$
$$\mathbf{c} = -\mathbf{V'}^{-1}\mathbf{u} \tag{17}$$
$$= -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -4 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} \tag{18}$$
$$f_0 = \mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f \tag{19}$$
$$= \begin{pmatrix} -4 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -4 \\ 0 \end{pmatrix} = 0 \tag{20}$$
Since $f_0 = 0$, we see that (15) represents a pair

of straight lines. Expressing $\mathbf{y} \triangleq \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, we get

$$y_1^2 - y_2^2 = 0 \tag{21}$$
$$\implies y_1 = \pm y_2 \tag{22}$$
$$\implies \mathbf{y} = \begin{pmatrix} a \\ \pm a \end{pmatrix}, \quad a \in \mathbb{R} \tag{23}$$

Hence, using (23),

$$\mathbf{q} = \mathbf{P}\mathbf{y} + \mathbf{c} \tag{24}$$
$$= \begin{pmatrix} a \pm a \\ a \mp a + 4 \end{pmatrix} \tag{25}$$
$$\implies \mathbf{q} \in \left\{ \begin{pmatrix} 2a \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 2a + 4 \end{pmatrix} \right\} \tag{26}$$

In the first case, (1) implies $a^2 = 2$. In the second case, we have $2a + 4 = 0$. Thus, the points of contact are

$$\mathbf{N} \in \left\{ \begin{pmatrix} \pm 2\sqrt{2} \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} \tag{27}$$

The nearest point out of these three candidates for $\mathbf{N}$ is $\begin{pmatrix} \pm 2\sqrt{2} \\ 4 \end{pmatrix}$. Thus, the correct answer is **a)**.

The situation is depicted in Fig. 1 plotted by the Python code `codes/normal.py`.

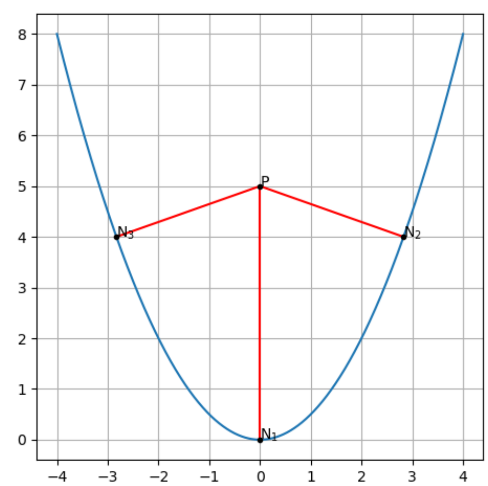

Fig. 1: N_1, N_2, N_3 are the points of contact of the normal from P to the parabola.