Equation of Line

1 11^{th} Maths - Chapter 10

This is Problem-4 from Exercise 10.3

1. Find the distance of the point $(-1, 1)$ from the line $12(x+6) = 5(y-2)$.
   Solution:

   (a) The equation of the line is $12(x+6) = 5(y-2)$. Rearranging the equation,

   $$12x - 5y = -10 - 72 \tag{1}$$
   $$12x - 5y = -82 \tag{2}$$

   This can be equated to

   $$\mathbf{n}^\top \mathbf{x} = c \tag{3}$$

   $$\text{where } \mathbf{n} = \begin{pmatrix} 12 \\ -5 \end{pmatrix}, c = -82 \tag{4}$$

   We need to compute the distance from a point $\mathbf{P}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ to the line. Without loss of generality, let $\mathbf{A}$ be the foot of the perpendicular from $\mathbf{P}$ to the line in Equation (3). The equation of the normal to Equation (3) can then be expressed as

   $$\mathbf{x} = \mathbf{A} + \lambda \mathbf{n} \tag{5}$$
   $$\implies \mathbf{P} - \mathbf{A} = \lambda \mathbf{n} \tag{6}$$

$\because$ $\mathbf{P}$ lies on (5). From the above, the desired distance can be expressed as

$$d = \|\mathbf{P} - \mathbf{A}\| = |\lambda| \, \|\mathbf{n}\| \tag{7}$$

From (6),

$$\mathbf{n}^\top (\mathbf{P} - \mathbf{A}) = \lambda \mathbf{n}^\top \mathbf{n} = \lambda \|\mathbf{n}\|^2 \tag{8}$$

$$\implies |\lambda| = \frac{\left|\mathbf{n}^\top (\mathbf{P} - \mathbf{A})\right|}{\|\mathbf{n}\|^2} \tag{9}$$

Substituting the above in (7) and using the fact that

$$\mathbf{n}^\top \mathbf{A} = c \tag{10}$$

from (3), yields

$$d = \frac{\left|\mathbf{n}^\top \mathbf{P} - c\right|}{\|\mathbf{n}\|} \tag{11}$$

$$= \frac{\left|\begin{pmatrix} 12 & -5 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} - (-82)\right|}{\sqrt{12^2 + (-5)^2}} \tag{12}$$

$$= \frac{|-17 + 82|}{\sqrt{169}} = \frac{|65|}{13} = 5 \text{ units} \tag{13}$$

(b) The foot of the perpendicular from $\mathbf{P}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ to line in (3) is expressed as

$$\begin{pmatrix} \mathbf{m} & \mathbf{n} \end{pmatrix}^\top \mathbf{A} = \begin{pmatrix} \mathbf{m}^\top \mathbf{P} \\ c \end{pmatrix} \tag{14}$$

where $\mathbf{m}$ is the direction vector of the given line

$$\because \mathbf{n} = \begin{pmatrix} 12 \\ -5 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} 5 \\ 12 \end{pmatrix} \tag{15}$$

$$(14) \implies \begin{pmatrix} 5 & 12 \\ 12 & -5 \end{pmatrix} \mathbf{A} = \begin{pmatrix} \begin{pmatrix} 5 & 12 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \\ -82 \end{pmatrix} \tag{16}$$

$$\begin{pmatrix} 5 & 12 \\ 12 & -5 \end{pmatrix} \mathbf{A} = \begin{pmatrix} 7 \\ -82 \end{pmatrix} \tag{17}$$

The augmented matrix for the system equations in (17) is expressed as

$$\left(\begin{array}{cc|c} 5 & 12 & 7 \\ 12 & -5 & -82 \end{array}\right) \tag{18}$$

Performing sequence of row operations to transform into RREF form

$$\xleftrightarrow{R_2 \to R_2 - \frac{12}{5} R_1} \left(\begin{array}{cc|c} 5 & 12 & 7 \\ 0 & -\frac{169}{5} & -\frac{494}{5} \end{array}\right) \tag{19}$$

$$\xleftrightarrow[R_1 \to \frac{1}{5} R_1]{R_2 \to \frac{-5}{169} R_2} \left(\begin{array}{cc|c} 1 & \frac{12}{5} & \frac{7}{5} \\ 0 & 1 & \frac{38}{13} \end{array}\right) \tag{20}$$

$$\xleftrightarrow{R_1 \to R_1 - \frac{12}{5} R_2} \left(\begin{array}{cc|c} 1 & 0 & -\frac{73}{13} \\ 0 & 1 & \frac{38}{13} \end{array}\right) \tag{21}$$

$$\mathbf{A} = \begin{pmatrix} -\frac{73}{13} \\ \frac{38}{13} \end{pmatrix} \tag{22}$$

The desired line and the perpendicular line from $\mathbf{P}$ is shown as in Figure 1

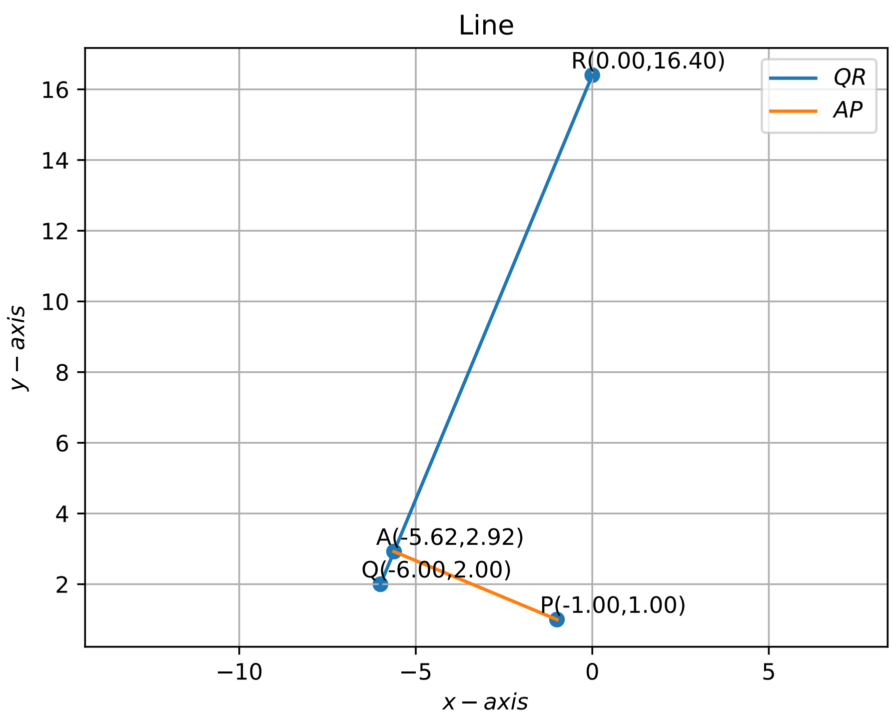
Line
R(0.00,16.40)
QR
AP
A(-5.62,2.92)
Q(-6.00,2.00)
P(-1.00,1.00)
y − axis
x − axis
16
14
12
10
8
6
4
2
−10
−5
0
5

Figure 1