Assignment 5

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1) An arch is in the form of a parabola with its axis vertical. The arch is 10 m high and 5 m wide at the base. How wide is it 2 m from the vertex of the parabola?

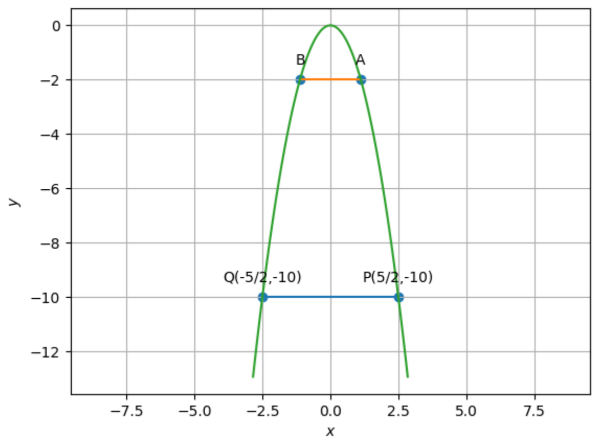

Fig. 1: Graph

Solution: The equation of the conic with focus $\mathbf{F}$, directrix $\mathbf{n}^\top \mathbf{x} = c$ and eccentricity e is given by

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{0.0.1}$$

where

$$\mathbf{V} \triangleq \|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{n}\mathbf{n}^\top \tag{0.0.2}$$
$$\mathbf{u} \triangleq ce^2\mathbf{n} - \|\mathbf{n}\|^2 \mathbf{F} \tag{0.0.3}$$
$$f \triangleq \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2e^2 \tag{0.0.4}$$

a) $\mathbf{V}$: Given that the arch is in the form of parabola, axis is vertical

$$e = 1 \tag{0.0.5}$$
$$\mathbf{n} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{0.0.6}$$

Substituting (0.0.5), (0.0.6) in the equaiton (0.0.2), it gives

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{0.0.7}$$

b) f: Without loss of generality, we can assume that the vertex $\mathbf{v}$ to be at origin,

$$\mathbf{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{0.0.8}$$

As the point $\mathbf{v}$ satisfies (0.0.1), and this gives

$$f = 0 \tag{0.0.9}$$

c) $\mathbf{u}$: Given that arch is 10m high and 5m wide at the base, since the parabola is symmetric to the axis of the parabola, The points

$$\mathbf{P} = \begin{pmatrix} \frac{5}{2} \\ -10 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} -\frac{5}{2} \\ -10 \end{pmatrix} \tag{0.0.10}$$

satisfies the equation (0.0.1).
Substituting the point $\mathbf{P}$ in (0.0.1) gives,

$$\frac{25}{4} + 2\mathbf{u}^\top \begin{pmatrix} \frac{5}{2} \\ -10 \end{pmatrix} = 0 \tag{0.0.11}$$
$$\implies \begin{pmatrix} 4 & -16 \end{pmatrix} \mathbf{u} = -5 \tag{0.0.12}$$

Substituting the point $\mathbf{Q}$ in (0.0.1) gives,

$$\frac{25}{4} + 2\mathbf{u}^\top \begin{pmatrix} -\frac{5}{2} \\ -10 \end{pmatrix} = 0 \tag{0.0.13}$$
$$\implies \begin{pmatrix} -4 & -16 \end{pmatrix} \mathbf{u} = -5 \tag{0.0.14}$$

Writing the equations (0.0.12), (0.0.14) in matrix form gives,

$$\begin{pmatrix} 4 & -16 \\ -4 & -16 \end{pmatrix} \mathbf{u} = \begin{pmatrix} -5 \\ -5 \end{pmatrix} \tag{0.0.15}$$

The augmented matrix for the system equations in (0.0.15) is expressed as

$$\left(\begin{array}{cc|c} 4 & -16 & -5 \\ -4 & -16 & -5 \end{array}\right) \tag{0.0.16}$$
$$\xleftrightarrow{R_2 \leftarrow R_2 + R_1} \left(\begin{array}{cc|c} 4 & -16 & -5 \\ 0 & -32 & -10 \end{array}\right) \tag{0.0.17}$$

The augmented matrix for the system equations is reduced to Row echelon form, From the above equation (0.0.17) we get the vector

$\mathbf{u}$ as

$$\mathbf{u} = \begin{pmatrix} 0 \\ \frac{5}{16} \end{pmatrix} \tag{0.0.18}$$

To find the the how wide the arch at 2m from the vertex of the parabola, we first find the points of intersection $\mathbf{A}, \mathbf{B}$ of the line and the parabola

$$\mathbf{x} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{0.0.19}$$

$$\mathbf{x}^\top \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} 0 \\ \frac{5}{16} \end{pmatrix}^\top \mathbf{x} = 0 \tag{0.0.20}$$

The parameter μ of the points of intersection of line (0.0.21) with the conic section (0.0.22)

$$\mathbf{x} = \mathbf{h} + \mu \mathbf{m} \tag{0.0.21}$$

$$g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{0.0.22}$$

is given by the equation

$$\mu^2 \mathbf{m}^\top \mathbf{V} \mathbf{m} + 2\mu \mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u}) + g(\mathbf{h}) = 0 \tag{0.0.23}$$

Here from (0.0.19), (0.0.20) we get,

$$\mathbf{m}^\top \mathbf{V} \mathbf{m} = 1 \tag{0.0.24}$$

$$\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u}) = 0 \tag{0.0.25}$$

$$g(\mathbf{h}) = -\frac{5}{4} \tag{0.0.26}$$

From (0.0.23) we get,

$$\mu^2 - \frac{5}{4} = 0 \tag{0.0.27}$$

$$\mu = \pm \frac{\sqrt{5}}{2} \tag{0.0.28}$$

$$\mathbf{A} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} + \frac{\sqrt{5}}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{0.0.29}$$

$$= \begin{pmatrix} \frac{\sqrt{5}}{2} \\ -2 \end{pmatrix} \tag{0.0.30}$$

$$\mathbf{B} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} - \frac{\sqrt{5}}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{0.0.31}$$

$$= \begin{pmatrix} -\frac{\sqrt{5}}{2} \\ -2 \end{pmatrix} \tag{0.0.32}$$

The required width is given by,

$$w = \|\mathbf{A} - \mathbf{B}\| \tag{0.0.33}$$

$$= \left\| \begin{pmatrix} \sqrt{5} \\ 0 \end{pmatrix} \right\| \tag{0.0.34}$$

$$= \sqrt{5} \tag{0.0.35}$$

Parameter	Description	Value
$\mathbf{n}$	Direction vector of axis of parabola	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
$\mathbf{v}$	Vertex of the parabola	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
$\mathbf{P}$	Point on the parabola	$\begin{pmatrix} \frac{5}{2} \\ -10 \end{pmatrix}$
$\mathbf{Q}$	Point on the parabola	$\begin{pmatrix} -\frac{5}{2} \\ -10 \end{pmatrix}$

TABLE 1