Problem: 11.11.3.9

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1 Problem

Find the co-ordinates of the foci, the vertices, the length of major axis, the minor axis, the eccentricity and the length of latus rectum of the ellipse $4x^2 + 9y^2 = 36$.

2 Solution

1) Given ellipse equation:

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{2.0.1}$$

$$here, \quad \mathbf{V} = \begin{pmatrix} \frac{4}{9} & 0 \\ 0 & 1 \end{pmatrix} \tag{2.0.2}$$

$$f = -4 \tag{2.0.3}$$

$$\mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.0.4}$$

2) points of intersection of a line $\mathbf{x} = \mathbf{h} + \mu\mathbf{m}$ with ellipse are given by:

$$\mu_i = \frac{1}{\mathbf{m}^\top \mathbf{V}\mathbf{m}} \left(-m^\top (\mathbf{V}\mathbf{h} + \mathbf{u}) \pm \sqrt{\left(\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u})\right)^2 - g(\mathbf{h})(\mathbf{m}^\top \mathbf{V}\mathbf{m})}\right) \tag{2.0.5}$$

where,

$$g(\mathbf{h}) = \mathbf{h}^\top \mathbf{V}\mathbf{h} + 2\mathbf{u}^\top \mathbf{h} + f \tag{2.0.6}$$

3) Center of the Ellipse

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \tag{2.0.7}$$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.0.8}$$

4) Major axis

$$\mathbf{p_2}^\top (\mathbf{x} - \mathbf{c}) = 0 \tag{2.0.9}$$

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{2.0.10}$$

$$i.e., \ \mathbf{x} = \mu \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.0.11}$$

5) Vertices

Vertices lie on major axis, therefore let

$$\mathbf{v} = \mu_i \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.0.12}$$

$$\mathbf{m}^\top \mathbf{V}\mathbf{m} = \frac{4}{9} \tag{2.0.13}$$

$$\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u}) = 0 \tag{2.0.14}$$

$$g(h) = -4 \tag{2.0.15}$$

$$\mu_i = \frac{0 \pm \sqrt{0 - (-4)\frac{4}{9}}}{\frac{4}{9}} \tag{2.0.16}$$

$$= \pm 3 \tag{2.0.17}$$

Vertices are

$$\mathbf{v_1} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \tag{2.0.18}$$

$$\mathbf{v_2} = \begin{pmatrix} -3 \\ 0 \end{pmatrix} \tag{2.0.19}$$

6) Length of major axis

$$\text{length of major axis} = \|\mathbf{v_1} - \mathbf{v_2}\| \tag{2.0.20}$$

$$= 6 \tag{2.0.21}$$

7) minor axis

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 0 \tag{2.0.22}$$

$$i.e., \quad \mathbf{x} = \mu \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.0.23}$$

$$\mathbf{m}^\top \mathbf{V}\mathbf{m} = 1 \tag{2.0.24}$$

$$\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u}) = 0 \tag{2.0.25}$$

$$g(h) = -4 \tag{2.0.26}$$

$$\mu_i = 0 \pm \sqrt{0 - (-4)} \tag{2.0.27}$$

$$= \pm 2 \tag{2.0.28}$$

Points of intersection of minor axis with ellipse be $\mu_i \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\mathbf{p_1} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \tag{2.0.29}$$

$$\mathbf{p_2} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} \tag{2.0.30}$$

8) Length of minor axis

$$\text{length of minor axis} = \|\mathbf{p_1} - \mathbf{p_2}\| \tag{2.0.31}$$

$$= 4 \tag{2.0.32}$$

Normal to directrix,

$$\mathbf{n} = \text{direction vector of major axis} \tag{2.0.33}$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.0.34}$$

$$\mathbf{V} = \|\mathbf{n}\|^2 \mathbf{I} - e^2\mathbf{n}\mathbf{n}^\top = \begin{pmatrix} \frac{4}{9} & 0 \\ 0 & 1 \end{pmatrix} \tag{2.0.35}$$

$$\mathbf{u} = ce^2\mathbf{n} - \|\mathbf{n}\|^2 \mathbf{F} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.0.36}$$

$$f = \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2e^2 = -4 \tag{2.0.37}$$

9) Eccentricity
Substituing value of $\mathbf{n}$ in (2.0.35),

$$e = \frac{\sqrt{5}}{3} \tag{2.0.38}$$

substituing (2.0.38) in (2.0.36) and (2.0.37),

$$\mathbf{F} = \frac{5c}{9}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.0.39}$$

$$\|\mathbf{F}\|^2 = c^2e^2 - 4 = \frac{25c^2}{81} \tag{2.0.40}$$

upon substituting (2.0.38) in (2.0.40),

$$c = \pm\frac{9}{\sqrt{5}} \tag{2.0.41}$$

10) Foci
substitung (2.0.41) in (2.0.39),

$$\mathbf{F} = \begin{pmatrix} \pm\sqrt{5} \\ 0 \end{pmatrix} \tag{2.0.42}$$

11) equation of Latus Recta

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{F}) = 0 \tag{2.0.43}$$

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 3 \tag{2.0.44}$$

$$i.e., \mathbf{x} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.0.45}$$

Let points of intersection of latus rectum and curve be,

$$\mathbf{x} = \mathbf{F} + \mu_i\mathbf{m} \tag{2.0.46}$$

here,

$$\mathbf{m}^\top\mathbf{V}\mathbf{m} = 1 \tag{2.0.47}$$

$$\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u}) = 0 \tag{2.0.48}$$

$$g(h) = -\frac{16}{9} \tag{2.0.49}$$

$$\mu_i = 0 \pm \sqrt{0 - (-1)\frac{16}{9}} \tag{2.0.50}$$

$$= \pm\frac{4}{3} \tag{2.0.51}$$

The points of intersection of Latus rectum with curve:

$$\mathbf{k_1} = \begin{pmatrix} 3 \\ \frac{4}{3} \end{pmatrix} \tag{2.0.52}$$

$$\mathbf{k_2} = \begin{pmatrix} 3 \\ -\frac{4}{3} \end{pmatrix} \tag{2.0.53}$$

12) Length of latus recta

$$\text{length of latus recta} = \|\mathbf{k_1} - \mathbf{k_2}\| \tag{2.0.54}$$

$$= \frac{8}{3} \tag{2.0.55}$$

Parameter	Value	Description
$\mathbf{V}$	$\begin{pmatrix} \frac{4}{9} & 0 \\ 0 & 1 \end{pmatrix}$	matrix **V**from ellipse equation
$\mathbf{u}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	vector **u** from ellipse equation
f	-4	constant f from ellipse equation

TABLE 12: Table 1

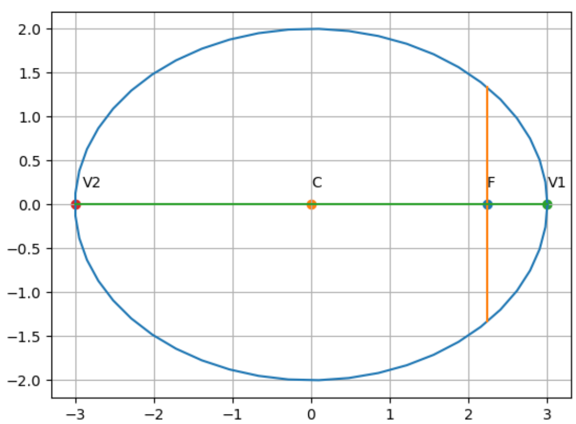

Fig. 12: Figure 1