CHAPTER-11
CIRCLES

Excercise 11.1

Q10.Find the equation of the circle passing through the points $(4,1)$ and $(6,5)$ and whose centre is on the line $4x+y=16$.
Solution: The equation of the circle is given by

$$\|\mathbf{x}\|^2 + 2\mathbf{x}^\top\mathbf{u} + f = 0 \tag{1}$$

where

$$\mathbf{u} = -\mathbf{c} \tag{2}$$
$$f = \|\mathbf{c}\| - r^2 \tag{3}$$

Given points are

$$\mathbf{x}_1 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}, \mathbf{x}_2 = \begin{pmatrix} 6 \\ 5 \end{pmatrix} \tag{4}$$

And the line passing through the centre

$$\begin{pmatrix} 4 & 1 \end{pmatrix} \mathbf{x} = 16 \tag{5}$$

Substituting points from (4) into (1)

$$\left(4^2+1^2\right) + 2\begin{pmatrix} 4 & 1 \end{pmatrix}\mathbf{u} + f = 0 \tag{6}$$
$$\implies 2\begin{pmatrix} 4 & 1 \end{pmatrix}\mathbf{u} + f = -17 \tag{7}$$
$$\left(6^2+5^2\right) + 2\begin{pmatrix} 6 & 5 \end{pmatrix}\mathbf{u} + f = 0 \tag{8}$$
$$\implies 2\begin{pmatrix} 6 & 5 \end{pmatrix}\mathbf{u} + f = -61 \tag{9}$$

And since (5) passes through the centre

$$-\mathbf{n}^\top\mathbf{u} = c \tag{10}$$
$$-\begin{pmatrix} 4 & 1 \end{pmatrix}\mathbf{u} = 16 \tag{11}$$

Representing (7),(9) and (11) in matrix form

$$\begin{pmatrix} -4 & -1 & 0 \\ 12 & 10 & 1 \\ 8 & 2 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ f \end{pmatrix} = \begin{pmatrix} 16 \\ -61 \\ -17 \end{pmatrix} \tag{12}$$

The augmented matrix is expressed as

$$\left(\begin{array}{ccc|c} -4 & -1 & 0 & 16 \\ 12 & 10 & 1 & -61 \\ 8 & 2 & 1 & -17 \end{array}\right) \tag{13}$$

Performing sequence of row operations to transform into an Echelon form

$$\xleftrightarrow[R_2 \to R_2 + 3R_1]{R_3 \to R_3 + 2R_1} \left(\begin{array}{ccc|c} -4 & -1 & 0 & 16 \\ 0 & 7 & 1 & -13 \\ 0 & 0 & 1 & 15 \end{array}\right) \tag{14}$$

$$\xleftrightarrow{R_2 \to R_2 - R_3} \left(\begin{array}{ccc|c} -4 & -1 & 0 & 16 \\ 0 & 7 & 0 & -28 \\ 0 & 0 & 1 & 15 \end{array}\right) \tag{15}$$

$$\xleftrightarrow{R_2 \to \frac{R_2}{7}, R_1 \to \frac{-R_1}{4}} \left(\begin{array}{ccc|c} 1 & \frac{1}{4} & 0 & -4 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 15 \end{array}\right) \tag{16}$$

$$\xleftrightarrow{R_1 \to R_1 - \frac{1}{4} R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 15 \end{array}\right) \tag{17}$$

So, from (17)

$$\mathbf{u} = \begin{pmatrix} -3 \\ -4 \end{pmatrix} \tag{18}$$

$$f = 15 \tag{19}$$

Since $\mathbf{u} = -\mathbf{c}$

$$\mathbf{c} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \tag{20}$$

$$r^2 = \left(3^2 + 4^2\right) - 15 \tag{21}$$

$$r = \sqrt{10} \tag{22}$$

Hence, the equation of circle is

$$\|\mathbf{x}\|^2 + 2\mathbf{u}^\top \mathbf{x} + 15 = 0 \tag{23}$$

$$\text{where } \mathbf{u} = \begin{pmatrix} -3 \\ -4 \end{pmatrix} \tag{24}$$

The corresponding is shown in Figure 1

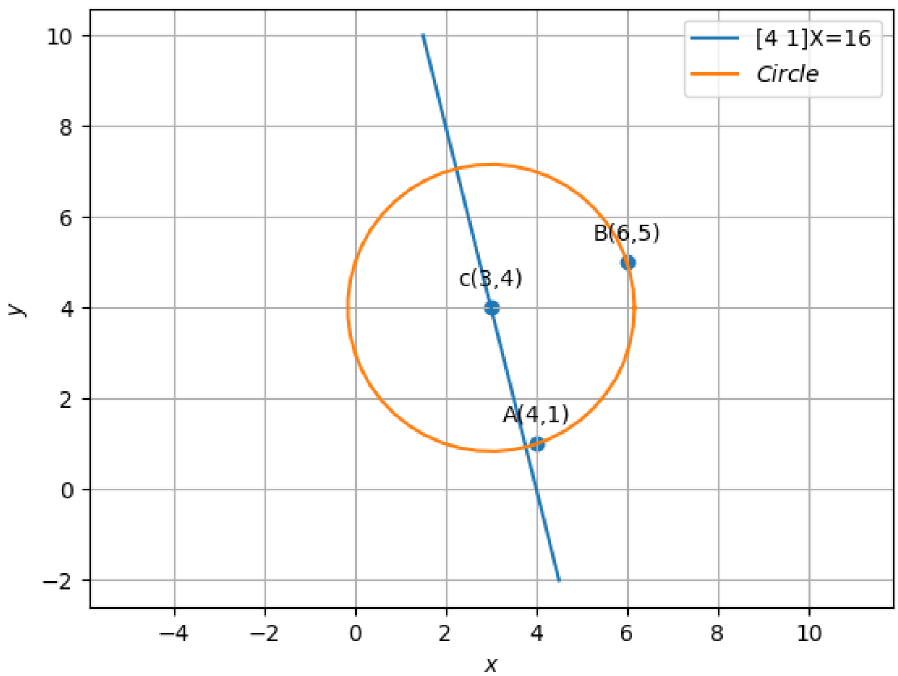

Figure 1: