Problem: 9.10.6.8

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1 Problem

Bisector of angles A,B and C of a triangle ABC intersect its circumcircle at D,E,and F respectively. Prove that the angles of triangle DEF are $90° - \frac{A}{2}$, $90° - \frac{B}{2}$ and $90° - \frac{C}{2}$.

2 Solution

1) Let the vertices of the triangle ABC be:

$$\mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.0.1}$$

$$\mathbf{C} = \begin{pmatrix} a \\ 0 \end{pmatrix} \tag{2.0.2}$$

$$\mathbf{A} = c \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \tag{2.0.3}$$

For ease of calculation let's assume $\theta = 60°, a = c = 5$.

2) side length b is given by:

$$b = \|A - C\| \tag{2.0.4}$$

$$= \left\| \begin{pmatrix} c\cos\theta - a \\ c\sin\theta \end{pmatrix} \right\| \tag{2.0.5}$$

$$= \sqrt{a^2 + c^2 - 2ac\cos\theta} \tag{2.0.6}$$

$$\tag{2.0.7}$$

3) Circumcenter of triangle ABC:
Circumcenter of triangle can be found by calculating point of intersection of perpedicular bisectors of two sides.
   a) perpendicular bisector of side BC:

$$(\mathbf{C} - \mathbf{B})^\top \left(\mathbf{x} - \frac{\mathbf{C} + \mathbf{B}}{2} \right) = 0 \tag{2.0.8}$$

$$(\mathbf{C} - \mathbf{B})^\top \mathbf{x} = \frac{\|\mathbf{C}\|^2 - \|\mathbf{B}\|^2}{2} \tag{2.0.9}$$

$$\mathbf{C}^\top \mathbf{x} = \frac{a^2}{2} \tag{2.0.10}$$

   b) perpendicular bisector of side AB:

$$\mathbf{A}^\top \mathbf{x} = \frac{c^2}{2} \tag{2.0.11}$$

Therefore, the circumcenter is given by:

$$\begin{pmatrix} a & 0 \\ c\cos\theta & c\sin\theta \end{pmatrix} \mathbf{O} = \begin{pmatrix} \frac{a^2}{2} \\ \frac{c^2}{2} \end{pmatrix} \tag{2.0.12}$$

4) Circumradius of triangle ABC,

$$R = \|\mathbf{A} - \mathbf{O}\| \tag{2.0.13}$$

5) For Circumcircle of triangle ABC, Writing equation of circle in form $\mathbf{x}\mathbf{V}\mathbf{x}^\top\mathbf{u}^\top\mathbf{x} + f = 0$,

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.0.14}$$

$$\mathbf{u} = -\mathbf{O} \tag{2.0.15}$$

$$f = \|\mathbf{O}\|^2 - R^2 \tag{2.0.16}$$

$$= 2\mathbf{O}^\top\mathbf{A} - \|\mathbf{A}\|^2 \tag{2.0.17}$$

6) angular bisectors: Expressing the line equations in form $\mathbf{x} = \mathbf{h} + \mu\mathbf{m}$,
   a) Angular Bisector of angle A:

$$\mathbf{m} = (\mathbf{C} - \mathbf{A}) + (\mathbf{B} - \mathbf{A}) \tag{2.0.18}$$

$$= \mathbf{B} + \mathbf{C} - 2\mathbf{A} \tag{2.0.19}$$

$$\mathbf{h} = \mathbf{A} \tag{2.0.20}$$

   b) Angular Bisector of angle B:

$$\mathbf{m} = (\mathbf{C} - \mathbf{B}) + (\mathbf{A} - \mathbf{B}) \tag{2.0.21}$$

$$= \mathbf{A} + \mathbf{C} - 2\mathbf{B} \tag{2.0.22}$$

$$\mathbf{h} = \mathbf{A} \tag{2.0.23}$$

   c) Angular Bisector of angle C:

$$\mathbf{m} = (\mathbf{A} - \mathbf{C}) + (\mathbf{B} - \mathbf{C}) \tag{2.0.24}$$

$$= \mathbf{A} + \mathbf{B} - 2\mathbf{C} \tag{2.0.25}$$

$$\mathbf{h} = \mathbf{C} \tag{2.0.26}$$

7) The point of intersection of any line $\mathbf{x} = \mathbf{h} + \mu\mathbf{m}$ and conic $\mathbf{x}^\top\mathbf{V}\mathbf{x} + 2\mathbf{u}^\top\mathbf{x} + f = 0$:

$$\mu_i = \frac{1}{\mathbf{m}^\top \mathbf{V}\mathbf{m}} \left(-m^\top (\mathbf{V}\mathbf{h} + \mathbf{u}) \pm \sqrt{(\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u}))^2 - \mathrm{g}(\mathbf{h}) (\mathbf{m}^\top \mathbf{V}\mathbf{m})} \right) \tag{2.0.27}$$

where,

$$\mathrm{g}(\mathbf{h}) = \mathbf{h}^\top \mathbf{V}\mathbf{h} + 2\mathbf{u}^\top \mathbf{h} + f \tag{2.0.28}$$

8) The points of intersection D, E and F are found using above method. Refer `codes/9.10.6.8.py`. The resulting $\mathbf{D}, \mathbf{E}, \mathbf{F}$ are given in table 8.

$\mathbf{D}$	$\begin{pmatrix} 2.5 \\ -1.44 \end{pmatrix}$
$\mathbf{E}$	$\begin{pmatrix} 5 \\ 2.886 \end{pmatrix}$
$\mathbf{F}$	$\begin{pmatrix} 0 \\ 2.886 \end{pmatrix}$

TABLE 8: Table

9) The desired angle i.e $\angle DEF$ is found by using,

$$\cos(\angle DEF) = \frac{(\mathbf{D} - \mathbf{E})^\top (\mathbf{F} - \mathbf{E})}{\|\mathbf{D} - \mathbf{E}\| \, \|\mathbf{F} - \mathbf{E}\|} \tag{2.0.29}$$

10) In `codes/9.10.6.8.py`, the problem is solved for $a = c = 5$ and $\theta = 60°$.
It is seen that the angle $\angle DEF$ is $60°$ which satsifies the statement to be proved.

Parameter	Value	Desription
a	5	length of side opposite to Verte:
c	5	length of side opposite to vertex C
θ	$60°$	$\angle ABC$

TABLE 10: Table

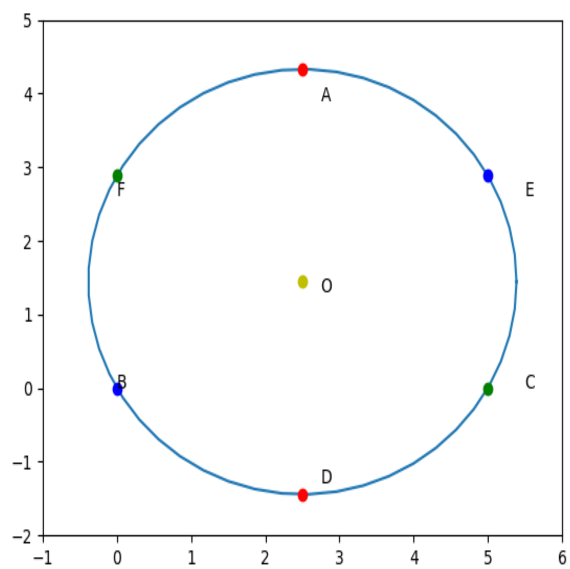

Fig. 10: Figure