Line Assignment

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***Abstract*—This document contains a general solution to Question 16 of Exercise 2 in Chapter 11 of the class 12 NCERT textbook.**

1) Find the shortest distance between the lines whose vector equations are

$$L_1 : \mathbf{x} = \mathbf{x_1} + \lambda_1 \mathbf{m_1} \tag{1}$$
$$L_2 : \mathbf{x} = \mathbf{x_2} + \lambda_2 \mathbf{m_2} \tag{2}$$

Solution: Let $\mathbf{A}$ and $\mathbf{B}$ be points on lines L_1 and L_2 respectively such that AB is normal to both lines. Define

$$\mathbf{M} \triangleq \begin{pmatrix} \mathbf{m_1} & \mathbf{m_2} \end{pmatrix} \tag{3}$$
$$\boldsymbol{\lambda} \triangleq \begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} \tag{4}$$
$$\mathbf{x} \triangleq \mathbf{x_2} - \mathbf{x_1} \tag{5}$$

Then, we have the following equations:

$$\mathbf{A} = \mathbf{x_1} + \lambda_1 \mathbf{m_1} \tag{6}$$
$$\mathbf{B} = \mathbf{x_2} + \lambda_2 \mathbf{m_2} \tag{7}$$

From (6) and (7), define the real-valued function f as

$$f(\boldsymbol{\lambda}) \triangleq \|\mathbf{A} - \mathbf{B}\|^2 \tag{8}$$
$$= \|\mathbf{M}\boldsymbol{\lambda} - \mathbf{x}\|^2 \tag{9}$$
$$= (\mathbf{M}\boldsymbol{\lambda} - \mathbf{x})^\top (\mathbf{M}\boldsymbol{\lambda} - \mathbf{x}) \tag{10}$$
$$= \boldsymbol{\lambda}^\top \left(\mathbf{M}^\top \mathbf{M}\right) \boldsymbol{\lambda} - 2\mathbf{x}^\top \mathbf{M}\boldsymbol{\lambda} + \|\mathbf{x}\|^2 \tag{11}$$

From (11), we see that f is quadratic in $\boldsymbol{\lambda}$. We now prove a useful lemma here.

Lemma 1. The quadratic form

$$q(\mathbf{x}) \triangleq \mathbf{x}^\top \mathbf{A}\mathbf{x} + \mathbf{b}^\top \mathbf{x} + c \tag{12}$$

is convex iff $\mathbf{A}$ is positive semi-definite.

Proof. Consider two points $\mathbf{x_1}$ and $\mathbf{x_2}$, and a real constant $0 \le \mu \le 1$. Then,

$$\mu f(\mathbf{x_1}) + (1-\mu) f(\mathbf{x_2}) - f(\mu\mathbf{x_1} + (1-\mu)\mathbf{x_2})$$
$$= \left(\mu - \mu^2\right) \mathbf{x_1}^\top \mathbf{A}\mathbf{x_1} + \left(1 - \mu - (1-\mu)^2\right) \mathbf{x_2}^\top \mathbf{A}\mathbf{x_2} - 2\mu(1-\mu)\mathbf{x_1}^\top \mathbf{A}\mathbf{x_2} \tag{13}$$
$$= \mu(1-\mu)\left(\mathbf{x_1}^\top \mathbf{A}\mathbf{x_1} - 2\mathbf{x_1}^\top \mathbf{A}\mathbf{x_2} + \mathbf{x_2}^\top \mathbf{A}\mathbf{x_2}\right) \tag{14}$$
$$= \mu(1-\mu)(\mathbf{x_1} - \mathbf{x_2})^\top \mathbf{A}(\mathbf{x_1} - \mathbf{x_2}) \tag{15}$$

Since $\mathbf{x_1}$ and $\mathbf{x_2}$ are arbitrary, it follows from (15) that

$$\mu f(\mathbf{x_1}) + (1-\mu) f(\mathbf{x_2}) \ge f(\mu\mathbf{x_1} + (1-\mu)\mathbf{x_2}) \tag{16}$$

iff $\mathbf{A}$ is positive semi-definite, as required. $\square$

Using the above lemma, we show that f is convex by showing that $\mathbf{M}^\top\mathbf{M}$ is positive semi-definite. Indeed, for any $\mathbf{p} \triangleq \begin{pmatrix} x \\ y \end{pmatrix}$,

$$\mathbf{p}^\top \mathbf{M}^\top \mathbf{M}\mathbf{p} = \|\mathbf{M}\mathbf{p}\|^2 \ge 0 \tag{17}$$

and thus, f is convex.
We need to minimize f as a function of $\boldsymbol{\lambda}$. Differentiating (11) using the chain rule,

$$\frac{df(\boldsymbol{\lambda})}{d\boldsymbol{\lambda}} = \mathbf{M}^\top (\mathbf{M}\boldsymbol{\lambda} - \mathbf{x}) + \mathbf{M}(\mathbf{M}\boldsymbol{\lambda} - \mathbf{x})^\top \tag{18}$$
$$= 2\mathbf{M}^\top (\mathbf{M}\boldsymbol{\lambda} - \mathbf{x}) \tag{19}$$

Setting (19) to zero gives the equation

$$\mathbf{M}^\top \mathbf{M}\boldsymbol{\lambda} = \mathbf{M}^\top \mathbf{x} \tag{20}$$

We use singular value decomposition here. Let

$$\mathbf{M} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^\top \tag{21}$$

where $\mathbf{U}, \mathbf{V}$ are orthogonal and $\boldsymbol{\Sigma}$ is diagonal with nonnegative diagonal entries. Substituting in (20),

$$\mathbf{V\Sigma U}^\top\mathbf{U\Sigma V}^\top\lambda = \mathbf{V\Sigma U}^\top\mathbf{x} \tag{22}$$
$$\implies \mathbf{V\Sigma}^2\mathbf{V}^\top\lambda = \mathbf{V\Sigma U}^\top\mathbf{x} \tag{23}$$
$$\implies \lambda = \left(\mathbf{V\Sigma}^2\mathbf{V}^\top\right)^{-1}\mathbf{V\Sigma U}^\top\mathbf{x} \tag{24}$$
$$\implies \lambda = \mathbf{V\Sigma}^{-2}\mathbf{V}^\top\mathbf{V\Sigma U}^\top\mathbf{x} \tag{25}$$
$$\implies \lambda = \mathbf{V\Sigma}^{-1}\mathbf{U}^\top\mathbf{x} \tag{26}$$

where $\mathbf{\Sigma}^{-1}$ is obtained by inverting the nonzero elements of $\mathbf{\Sigma}$. Thus, the shortest distance is given by using (21) and (26) in (11), and is given by

$$d = \left\|\left(\mathbf{U}\left(\mathbf{\Sigma\Sigma}^{-1}\right)\mathbf{U}^\top - \mathbf{I}\right)\mathbf{x}\right\| \tag{27}$$

For this problem,

$$\mathbf{x} = \mathbf{x_2} - \mathbf{x_1} = \begin{pmatrix}3\\3\\3\end{pmatrix} \tag{28}$$

$$\mathbf{M} = \begin{pmatrix}\mathbf{m_1} & \mathbf{m_2}\end{pmatrix} = \begin{pmatrix}1 & 2\\-3 & 3\\2 & 1\end{pmatrix} \tag{29}$$

Thus,

$$\mathbf{M}^\top\mathbf{M} = \begin{pmatrix}1 & -3 & 2\\2 & 3 & 1\end{pmatrix}\begin{pmatrix}1 & 2\\-3 & 3\\2 & 1\end{pmatrix} = \begin{pmatrix}14 & -5\\-5 & 14\end{pmatrix} \tag{30}$$

$$\mathbf{M}\mathbf{M}^\top = \begin{pmatrix}1 & 2\\-3 & 3\\2 & 1\end{pmatrix}\begin{pmatrix}1 & -3 & 2\\2 & 3 & 1\end{pmatrix} = \begin{pmatrix}5 & 3 & 4\\3 & 18 & -3\\4 & -3 & 5\end{pmatrix} \tag{31}$$

We perform the eigendecompositions for each matrix and bring them into the form

$$\mathbf{MM}^\top = \mathbf{P_1D_1P_1}^\top \tag{32}$$
$$\mathbf{M}^\top\mathbf{M} = \mathbf{P_2D_2P_2}^\top \tag{33}$$

a) For $\mathbf{MM}^\top$, the characteristic polynomial is

$$\operatorname{char}\left(\mathbf{MM}^\top\right) = \begin{vmatrix}x-5 & -3 & -4\\-3 & x-18 & 3\\-4 & 3 & x-5\end{vmatrix} \tag{34}$$
$$= x\left(x-9\right)\left(x-19\right) \tag{35}$$

Thus, the eigenvalues are given by

$$\lambda_1 = 19,\ \lambda_2 = 9,\ \lambda_3 = 0 \tag{36}$$

For λ_1, the augmented matrix formed from the eigenvalue-eigenvector equation is

$$\begin{pmatrix}-14 & 3 & 4 & 0\\3 & -1 & -3 & 0\\4 & -3 & -14 & 0\end{pmatrix}$$
$$\xleftrightarrow{R_1\leftarrow\frac{R_1+R_3}{-10}}\begin{pmatrix}1 & 0 & 1 & 0\\3 & -1 & -3 & 0\\4 & -3 & -14 & 0\end{pmatrix} \tag{37}$$
$$\xleftrightarrow{R_2\leftarrow -R_2+3R_1}\begin{pmatrix}1 & 0 & 1 & 0\\0 & 1 & 6 & 0\\4 & -3 & -14 & 0\end{pmatrix} \tag{38}$$
$$\xleftrightarrow{R_3\leftarrow R_3-4R_1}\begin{pmatrix}1 & 0 & 1 & 0\\0 & -1 & -6 & 0\\0 & -3 & -18 & 0\end{pmatrix} \tag{39}$$
$$\xleftrightarrow{R_3\leftarrow R_3-3R_2}\begin{pmatrix}1 & 0 & 1 & 0\\0 & -1 & -6 & 0\\0 & 0 & 0 & 0\end{pmatrix} \tag{40}$$

Hence, the normalized eigenvector is

$$\mathbf{p_1} = \frac{1}{\sqrt{38}}\begin{pmatrix}-1\\-6\\1\end{pmatrix} \tag{41}$$

For λ_2, the augmented matrix formed from the eigenvalue-eigenvector equation is

$$\begin{pmatrix}-4 & 3 & 4 & 0\\3 & 9 & -3 & 0\\4 & 3 & -4 & 0\end{pmatrix}$$
$$\xleftrightarrow{R_3\leftarrow R_1+R_3}\begin{pmatrix}-4 & 3 & 4 & 0\\3 & 9 & -3 & 0\\0 & 0 & 0 & 0\end{pmatrix} \tag{42}$$
$$\xleftrightarrow{R_2\leftarrow\frac{4R_2+3R_1}{45}}\begin{pmatrix}-4 & 3 & 4 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 0\end{pmatrix} \tag{43}$$
$$\xleftrightarrow{R_1\leftarrow\frac{R_1-3R_2}{-4}}\begin{pmatrix}1 & 0 & -1 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 0\end{pmatrix} \tag{44}$$

Hence, the normalized eigenvector is

$$\mathbf{p_2} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix} \tag{45}$$

For λ_3, the augmented matrix formed from the eigenvalue-eigenvector equation is

$$\begin{pmatrix} 5 & 3 & 4 & 0 \\ 3 & 18 & -3 & 0 \\ 4 & -3 & 5 & 0 \end{pmatrix}$$

$$\xleftrightarrow{R_1 \leftarrow \frac{R_1+R_3}{9}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 3 & 18 & -3 & 0 \\ 4 & -3 & 5 & 0 \end{pmatrix} \tag{46}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - 3R_1} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 18 & -6 & 0 \\ 4 & -3 & 5 & 0 \end{pmatrix} \tag{47}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - 4R_1} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 18 & -6 & 0 \\ 0 & -3 & 1 & 0 \end{pmatrix} \tag{48}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{R_2}{6}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 3 & -1 & 0 \\ 0 & -3 & 1 & 0 \end{pmatrix} \tag{49}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 + R_2} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{50}$$

Hence, the normalized eigenvector is

$$\mathbf{p_3} = \frac{1}{\sqrt{19}} \begin{pmatrix} -3 \\ 1 \\ 3 \end{pmatrix} \tag{51}$$

Using (32), we see that

$$\mathbf{P_1} = \begin{pmatrix} -\frac{1}{\sqrt{38}} & \frac{1}{\sqrt{2}} & -\frac{3}{\sqrt{19}} \\ -\frac{6}{\sqrt{38}} & 0 & \frac{1}{\sqrt{19}} \\ \frac{1}{\sqrt{38}} & -\frac{1}{\sqrt{2}} & \frac{3}{\sqrt{19}} \end{pmatrix} \tag{52}$$

$$\mathbf{D_1} = \begin{pmatrix} 19 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{53}$$

b) For $\mathbf{M}^\top\mathbf{M}$, the characteristic polynomial is

$$\operatorname{char}\left(\mathbf{M}^\top\mathbf{M}\right) = \begin{vmatrix} x-14 & 5 \\ 5 & x-14 \end{vmatrix} \tag{54}$$

$$= (x-9)(x-19) \tag{55}$$

Thus, the eigenvalues are given by

$$\lambda_1 = 19, \ \lambda_2 = 9 \tag{56}$$

For λ_1, the augmented matrix formed from the eigenvalue-eigenvector equation is

$$\begin{pmatrix} -5 & -5 & 0 \\ -5 & -5 & 0 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 - R_2} \begin{pmatrix} 0 & 0 & 0 \\ -5 & -5 & 0 \end{pmatrix} \tag{57}$$

Hence, the normalized eigenvector is

$$\mathbf{p_1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{58}$$

For λ_2, the augmented matrix formed from the eigenvalue-eigenvector equation is

$$\begin{pmatrix} 5 & -5 & 0 \\ -5 & 5 & 0 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 + R_2} \begin{pmatrix} 0 & 0 & 0 \\ 5 & -5 & 0 \end{pmatrix} \tag{59}$$

Hence, the normalized eigenvector is

$$\mathbf{p_2} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{60}$$

Thus, from (33),

$$\mathbf{P_2} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \ \mathbf{D_2} = \begin{pmatrix} 9 & 0 \\ 0 & 19 \end{pmatrix} \tag{61}$$

Therefore, from (21),

$$\mathbf{U} = \mathbf{P_1} \tag{62}$$

$$\mathbf{V} = \mathbf{P_2} \tag{63}$$

$$\mathbf{\Sigma} = \begin{pmatrix} \sqrt{19} & 0 \\ 0 & 3 \\ 0 & 0 \end{pmatrix} \tag{64}$$

and substituting into (26), we get

$$\boldsymbol{\lambda} = \frac{1}{19} \begin{pmatrix} 10 \\ 28 \end{pmatrix} \tag{65}$$

which agrees with earlier solutions as well. The Python code `codes/svd.py` plots Fig. 1 depicting the situation.

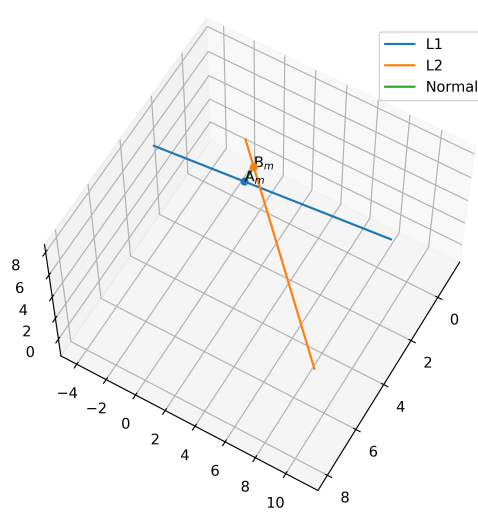

Fig. 1: Finding the shortest distance between two lines using SVD.