Que: 11.11.4.9

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1 Problem

Find the equations of hyperbola having Vertices $\begin{pmatrix}0\\\pm3\end{pmatrix}$ and Foci $\begin{pmatrix}0\\\pm5\end{pmatrix}$

2 Solution

1) Transverse axis: Line joining two foci

$$\mathbf{m} = \mathbf{F}_1 - \mathbf{F}_2 \tag{2.0.1}$$

$$= \begin{pmatrix}0\\10\end{pmatrix} \tag{2.0.2}$$

$$\begin{pmatrix}1 & 0\end{pmatrix}(\mathbf{x} - \mathbf{F}_1) = 0 \tag{2.0.3}$$

$$\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = 0 \tag{2.0.4}$$

2) Center of hyperbola, $\mathbf{O}$ is given by:

$$\mathbf{O} = \frac{\mathbf{F}_1 + \mathbf{F}_2}{2} \tag{2.0.5}$$

$$\mathbf{O} = \begin{pmatrix}0\\0\end{pmatrix} \tag{2.0.6}$$

3) Normal vector of directrix

$$\mathbf{n} = \text{direction vector of transverse axis} \tag{2.0.7}$$

$$= \begin{pmatrix}0\\1\end{pmatrix} \tag{2.0.8}$$

$$\mathbf{V} = \|\mathbf{n}\|^2 \mathbf{I} - e^2\mathbf{n}\mathbf{n}^\top \tag{2.0.9}$$

$$= \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} - e^2\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} \tag{2.0.10}$$

$$= \begin{pmatrix}1 & 0\\0 & 1-e^2\end{pmatrix} \tag{2.0.11}$$

$$\mathbf{u} = ce^2\mathbf{n} - \|\mathbf{n}\|^2 \mathbf{F} \tag{2.0.12}$$

$$= \begin{pmatrix}0\\ce^2 - 5\end{pmatrix} \tag{2.0.13}$$

$$f = \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2e^2 \tag{2.0.14}$$

$$= 25 - c^2e^2 \tag{2.0.15}$$

Equation of the hyperbola:

$$\mathbf{x}^\top\mathbf{V}\mathbf{x} + 2\mathbf{u}^\top\mathbf{x} + f = 0 \tag{2.0.16}$$

$$\tag{2.0.17}$$

Vertex lies on this curve,

$$\mathbf{v_1}^\top\mathbf{V}\mathbf{v_1} + 2\mathbf{u}^\top\mathbf{v_1} + f = 0 \tag{2.0.18}$$

$$9\left(1 - e^2\right) + 6\left(ce^2 - 5\right) - c^2e^2 + 25 = 0 \tag{2.0.19}$$

$$4 - 9e^2 + 6ce^2 - c^2e^2 = 0 \tag{2.0.20}$$

Also, the center is given by,

$$\mathbf{O} = -\mathbf{V}^{-1}\mathbf{u} \tag{2.0.21}$$

$$\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}0\\\frac{ce^2-5}{1-e^2}\end{pmatrix} \tag{2.0.22}$$

$$ce^2 = 5 \tag{2.0.23}$$

Solving (2.0.20) and (2.0.23),

$$c = \frac{9}{5} \tag{2.0.24}$$

$$e = \frac{5}{3} \tag{2.0.25}$$

$$\tag{2.0.26}$$

$$\mathbf{V} = \begin{pmatrix}1 & 0\\0 & -\frac{16}{9}\end{pmatrix} \tag{2.0.27}$$

$$\mathbf{u} = \begin{pmatrix}0\\0\end{pmatrix} \tag{2.0.28}$$

$$f = 16 \tag{2.0.29}$$

$$\tag{2.0.30}$$

Equation of the Hyperbola,

$$\mathbf{x}^{\top}\begin{pmatrix}1 & 0\\ 0 & -\frac{16}{9}\end{pmatrix}\mathbf{x} + 16 = 0 \qquad (2.0.31)$$

Parameter	Value	Description
$\mathbf{F}_1$	$\begin{pmatrix}0\\5\end{pmatrix}$	Focus
$\mathbf{F}_2$	$\begin{pmatrix}0\\-5\end{pmatrix}$	Focus
$\mathbf{v}_1$	$\begin{pmatrix}0\\3\end{pmatrix}$	Vertex
$\mathbf{v}_2$	$\begin{pmatrix}0\\-3\end{pmatrix}$	Vertex

TABLE 3: Table1

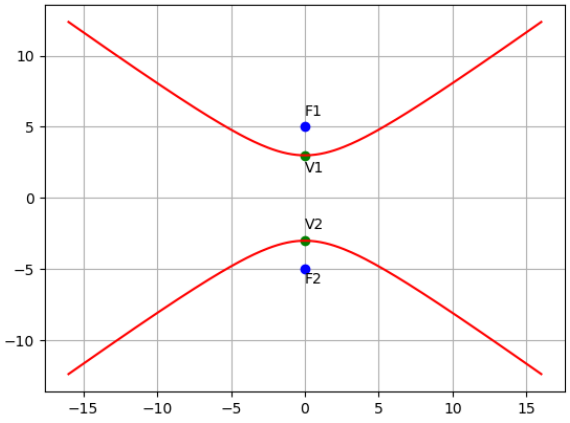

Fig. 3: Figure 1