

Line Assignment

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Abstract—This document contains the solution to Question 16 of Exercise 2 in Chapter 11 of the class 12 NCERT textbook.

- 1) Find the shortest distance between the lines whose vector equations are

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \quad (1)$$

and

$$\mathbf{x} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \quad (2)$$

Solution: Suppose there are two lines in general given by

$$\mathbf{x} = \mathbf{x}_1 + \lambda_1 \mathbf{m}_1 \quad (3)$$

$$\mathbf{x} = \mathbf{x}_2 + \lambda_2 \mathbf{m}_2 \quad (4)$$

If these lines intersect, then

$$\lambda_1 \mathbf{m}_1 - \lambda_2 \mathbf{m}_2 = \mathbf{x}_2 - \mathbf{x}_1 \quad (5)$$

$$\implies \mathbf{M}\lambda = \mathbf{x}_2 - \mathbf{x}_1 \quad (6)$$

where

$$\mathbf{M} \triangleq (\mathbf{m}_1 \quad \mathbf{m}_2) \quad (7)$$

$$\lambda \triangleq \begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} \quad (8)$$

In this case,

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \mathbf{x}_2 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \quad \mathbf{m}_1 = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \quad \mathbf{m}_2 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \quad (9)$$

To check whether (6) has a solution in λ , we

use the augmented matrix.

$$\begin{pmatrix} 1 & 2 & 3 \\ -3 & 3 & 3 \\ 2 & 1 & 3 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 + 3R_1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 9 & 12 \\ 2 & 1 & 3 \end{pmatrix} \quad (10)$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - 2R_1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 9 & 12 \\ 0 & -3 & -3 \end{pmatrix} \quad (11)$$

$$\xleftrightarrow{R_3 \leftarrow -3R_3 + R_2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 9 & 12 \\ 0 & 0 & 3 \end{pmatrix} \quad (12)$$

Clearly, the rank of this matrix is 3, and therefore, the lines are skew.

Now, suppose that the closest points on both lines are

$$\mathbf{A} = \mathbf{x}_1 + \lambda_1 \mathbf{m}_1 \quad (13)$$

$$\mathbf{B} = \mathbf{x}_2 + \lambda_2 \mathbf{m}_2 \quad (14)$$

Then, AB is perpendicular to both lines, hence

$$\mathbf{m}_1^\top (\mathbf{A} - \mathbf{B}) = 0 \quad (15)$$

$$\mathbf{m}_2^\top (\mathbf{A} - \mathbf{B}) = 0 \quad (16)$$

$$\implies \mathbf{M}^\top (\mathbf{A} - \mathbf{B}) = \mathbf{O} \quad (17)$$

Using (13) and (14) in (17),

$$\mathbf{M}^\top (\mathbf{x}_1 - \mathbf{x}_2 + \mathbf{M}\lambda) = \mathbf{O} \quad (18)$$

$$\implies \mathbf{M}^\top \mathbf{M}\lambda = \mathbf{M}^\top (\mathbf{x}_2 - \mathbf{x}_1) \quad (19)$$

Substituting from (9) in (19) and forming the

augmented matrix,

$$\begin{pmatrix} 14 & -5 & 0 \\ -5 & 14 & 18 \end{pmatrix} \xrightarrow{R_1 \leftarrow R_1 + R_2} \begin{pmatrix} 9 & 9 & 18 \\ -5 & 14 & 18 \end{pmatrix} \quad (20)$$

$$\xrightarrow{R_1 \leftarrow \frac{R_1}{9}} \begin{pmatrix} 1 & 1 & 2 \\ -5 & 14 & 18 \end{pmatrix} \quad (21)$$

$$\xrightarrow{R_2 \leftarrow R_2 + 5R_1} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 19 & 28 \end{pmatrix} \quad (22)$$

$$\xrightarrow{R_1 \leftarrow 19R_1 - R_2} \begin{pmatrix} 19 & 0 & 10 \\ 0 & 19 & 28 \end{pmatrix} \quad (23)$$

$$\xrightarrow{\begin{matrix} R_1 \leftarrow \frac{R_1}{19} \\ R_2 \leftarrow \frac{R_2}{19} \end{matrix}} \begin{pmatrix} 1 & 0 & \frac{10}{19} \\ 0 & 1 & \frac{28}{19} \end{pmatrix} \quad (24)$$

$$\Rightarrow \lambda = \frac{1}{19} \begin{pmatrix} 10 \\ 28 \end{pmatrix} \quad (25)$$

Hence, using (8) and substituting into (13) and (14),

$$\mathbf{A} = \frac{1}{19} \begin{pmatrix} 29 \\ 8 \\ 77 \end{pmatrix} \quad \mathbf{B} = \frac{1}{19} \begin{pmatrix} 20 \\ 11 \\ 86 \end{pmatrix} \quad (26)$$

Thus, the required distance is

$$\|\mathbf{B} - \mathbf{A}\| = \frac{\sqrt{9^2 + 3^2 + (-9)^2}}{19} = \frac{3}{\sqrt{19}} \quad (27)$$

The situation is depicted in Fig. 1.

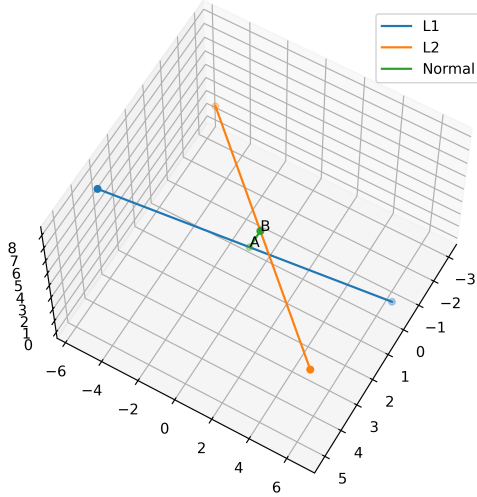


Fig. 1: AB is the required shortest distance.