11.10.3.16

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Class 11, Chapter 10, Exercise 3.16

Q16. If p and q are the lengths of perpendiculars from the origin to the lines $x\cos\theta - y\sin\theta = k\cos 2\theta$ and $x\sec\theta + y\,\mathrm{cosec}\,\theta = k$, respectively, prove that $p^2 + 4q^2 = k^2$

Solution: Equation of lines are as follows:

$$L_1 : x\cos\theta - y\sin\theta = k\cos 2\theta \tag{1}$$

$$\implies \mathbf{n}_1 = \begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix} \text{ and } c_1 = k\cos 2\theta \tag{2}$$

$$L_2 : x\sec\theta + y\,\mathrm{cosec}\,\theta = k \tag{3}$$

$$L_2 : x\sin\theta + y\cos\theta = k\cos\theta\sin\theta \tag{4}$$

$$L_2 : x\sin\theta + y\cos\theta = \frac{1}{2}k\sin 2\theta \tag{5}$$

$$\implies \mathbf{n}_2 = \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix} \text{ and } c_2 = \frac{1}{2}k\sin 2\theta \tag{6}$$

The lengths of perpendiculars from origin can be found by using the following formula:

$$p = \frac{\left|\mathbf{n}_1^\top \mathbf{x} - c_1\right|}{\|\mathbf{n}_1\|} \tag{7}$$

$$p = \frac{\left|\begin{pmatrix} \cos\theta & -\sin\theta \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} - k\cos 2\theta\right|}{\sqrt{\cos\theta^2 + \sin\theta^2}} \tag{8}$$

$$p = |k\cos 2\theta| \tag{9}$$

$$\implies p^2 = k^2\cos^2 2\theta \tag{10}$$

$$q = \frac{\left|\mathbf{n}_2^\top \mathbf{x} - c_2\right|}{\|\mathbf{n}_2\|} \tag{11}$$

$$q = \frac{\left|\begin{pmatrix} \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} - \frac{1}{2}k\sin 2\theta\right|}{\sqrt{\sin\theta^2 + \cos\theta^2}} \tag{12}$$

$$q = \left|\frac{1}{2}k\sin 2\theta\right| \tag{13}$$

$$\implies q^2 = \frac{1}{4}k^2\sin^2 2\theta \tag{14}$$

Therefore using (10) and (14), we get:

$$p^2 + 4q^2 = k^2\cos^2 2\theta + 4\left(\frac{1}{4}\right)k^2\sin^2 2\theta \tag{15}$$

$$\implies k^2\left(\cos^2 2\theta + \sin^2 2\theta\right) = k^2 \tag{16}$$