Line Assignment

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January 2023

***Problem Statement* - If l_1,m_1,n_1 and l_2,m_2,n_2 are the direction cosines of two mutually perpendicular lines, show that the direction cosines of the line perpendicular to both these are $m_1n_2-m_2n_1$,$n_1l_2-n_2l_1$,$l_1m_2-l_2m_1$**

Solution

Let us Consider

$$\mathbf{A} = \begin{pmatrix} l_1 \\ m_1 \\ n_1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} l_2 \\ m_2 \\ n_2 \end{pmatrix} \mathbf{C} = \begin{pmatrix} m_1n_2-m_2n_1 \\ n_1l_2-n_2l_1 \\ l_1m_2-l_2m_1 \end{pmatrix} \tag{1}$$

$$\mathbf{A}^\top\mathbf{B} = \mathbf{0} \tag{2}$$

$$\mathbf{A}^\top\mathbf{A} = \mathbf{1} \tag{3}$$

$$\mathbf{B}^\top\mathbf{B} = \mathbf{1} \tag{4}$$

In order to proove the vector $m_1n_2-m_2n_1$, $n_1l_2-n_2l_1$, $l_1m_2-l_2m_1$ is perpendicular to both the vectors l_1,m_1,n_1 and l_2,m_2,n_2. Let us consider the matrix $\mathbf{P}$.

$$\mathbf{P} = \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} \tag{5}$$

$$= \begin{pmatrix} l_1 & l_2 & m_1n_2-m_2n_1 \\ m_1 & m_2 & n_1l_2-n_2l_1 \\ n_1 & n_2 & l_1m_2-l_2m_1 \end{pmatrix} \tag{6}$$

$$\mathbf{P}^\top = \begin{pmatrix} \mathbf{A} \\ \mathbf{B} \\ \mathbf{C} \end{pmatrix} \tag{7}$$

$$= \begin{pmatrix} l_1 & m_1 & n_1 & & \\ l_2 & m_2 & n_2 & & \\ m_1n_2-m_2n_1 & m_2 & n_1l_2-n_2l_1 & n_2 & l_1m_2-l_2m_1 \end{pmatrix} \tag{8}$$

If the three vectors are mutually perpendicular then

$$\mathbf{P}^\top\mathbf{P} = \mathbf{I} \tag{9}$$

$$\begin{pmatrix} l_1^2+m_1^2+n_1^2 & l_1l_2+m_1m_2+n_1n_2 & l_1(m_1n_2-m_2n_1)+m_1(n_1l_2-n_2l_1)+n_1(l_1m_2-l_2m_1) \\ l_1l_2+m_1n_2+n_1n_2 & l_2^2+m_2^2+n_2^2 & l_2(m_1n_2-m_2n_1)+m_2(n_1l_2-n_2l_1)+n_2(l_1m_2-l_2m_1) \\ \begin{matrix} l_1(m_1n_2-n_2m_1)+m_1(n_1l_2-n_2l_1) \\ +n_1(l_1m_2-l_2m_1) \end{matrix} & l_2(m_1n_2-m_2n_1)+m_2(n_1l_2-n_2l_1)+n_2(l_1m_2-l_2m_1) & (l_1m_2-l_2m_1)^2+(n_1l_2-n_2l_1)^2+(m_1n_2-m_2n_1)^2 \end{pmatrix} \tag{10}$$

$$\mathbf{P}^\top\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{11}$$

Hences,The three vectors are mutually perpendicular So we prooved that If l_1,m_1,n_1 and l_2,m_2,n_2 are the direction cosines of two mutually perpendicular lines, show that the direction cosines of the line perpendicular to both these are $m_1n_2-m_2n_1$, $n_1l_2-n_2l_1$, $l_1m_2-l_2m_1$.