Challenge 6

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***Abstract*—This document is to prove that convolution is a unique map.**

Download all python codes from

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https://github.com/Zeeshan−IITH/IITH−EE5609/
    new/master/codes
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and latex-tikz codes from

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https://github.com/Zeeshan−IITH/IITH−EE5609
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1 Problem

Given two signals $(x_0, \dots, x_n 1)$ and $(h_0, \dots, h_m 1)$, the (linear) convolution of the two is an m+n1-length signal defined as

$$y(t) = (h * x)_t = \sum_{\tau=0}^{\tau=n-1} x_\tau h_{(t-\tau)} \tag{1.0.1}$$

$$0 \le t < m + n - 1$$

If

$$y(n) = (h_1 * x) \tag{1.0.2}$$

$$y(n) = (h_2 * x) \tag{1.0.3}$$

then prove that $h_1 = h_2$

2 Construction

Writing the convolution operation in matrix form

$$Y = \begin{pmatrix} h_0 & 0 & 0 & . & . & 0 & 0 \\ h_1 & h_0 & 0 & . & . & 0 & 0 \\ h_2 & h_1 & h_0 & . & . & 0 & 0 \\ . & . & . & . & . & . & . \\ h_{n-1} & h_{n-2} & n_{n-3} & . & . & h_1 & h_0 \\ . & . & . & . & . & . & . \\ h_{m-1} & h_{m-2} & h_{m-3} & . & . & h_{m-n+1} & h_{m-n} \\ 0 & h_{m-1} & h_{m-2} & . & . & h_{m-n+2} & h_{m-n+1} \\ 0 & 0 & h_{m-1} & . & . & h_{m-n+3} & h_{m-n+2} \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & 0 & h_{m-1} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2. \\ . \\ . \\ x_{n-1} \end{pmatrix} \tag{2.0.1}$$

3 Explanation

Therefore we can write equation (1.0.1) in matrix form as $\mathbf{Y} = \mathbf{HX}$ where

$$Y = \begin{pmatrix} h_0 & 0 & 0 & . & . & 0 & 0 \\ h_1 & h_0 & 0 & . & . & 0 & 0 \\ h_2 & h_1 & h_0 & . & . & 0 & 0 \\ . & . & . & . & . & . & . \\ h_{n-1} & h_{n-2} & n_{n-3} & . & . & h_1 & h_0 \\ . & . & . & . & . & . & . \\ h_{m-1} & h_{m-2} & h_{m-3} & . & . & h_{m-n+1} & h_{m-n} \\ 0 & h_{m-1} & h_{m-2} & . & . & h_{m-n+2} & h_{m-n+1} \\ 0 & 0 & h_{m-1} & . & . & h_{m-n+3} & h_{m-n+2} \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & 0 & h_{m-1} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2. \\ . \\ . \\ x_{n-1} \end{pmatrix} \tag{3.0.1}$$

So the equations (1.0.2) and (1.0.3) can be written in matrix form where

$$\mathbf{H_1} = \begin{pmatrix} h_{10} & 0 & 0 & . & . & 0 & 0 \\ h_{11} & h_{10} & 0 & . & . & 0 & 0 \\ h_{12} & h_{11} & h_{10} & . & . & 0 & 0 \\ . & . & . & . & . & . & . \\ h_{1n-1} & h_{1n-2} & n_{1n-3} & . & . & h_{11} & h_{10} \\ . & . & . & . & . & . & . \\ h_{1m-1} & h_{1m-2} & h_{1m-3} & . & . & h_{1m-n+1} & h_{1m-n} \\ 0 & h_{1m-1} & h_{1m-2} & . & . & h_{1m-n+2} & h_{1m-n+1} \\ 0 & 0 & h_{1m-1} & . & . & h_{1m-n+3} & h_{1m-n+2} \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & 0 & h_{1m-1} \end{pmatrix} \tag{3.0.2}$$

$$\mathbf{H_1} = \begin{pmatrix} h_{20} & 0 & 0 & . & . & 0 & 0 \\ h_{21} & h_{20} & 0 & . & . & 0 & 0 \\ h_{22} & h_{21} & h_{20} & . & . & 0 & 0 \\ . & . & . & . & . & . & . \\ h_{2n-1} & h_{2n-2} & n_{2n-3} & . & . & h_{21} & h_{20} \\ . & . & . & . & . & . & . \\ h_{2m-1} & h_{2m-2} & h_{2m-3} & . & . & h_{2m-n+1} & h_{2m-n} \\ 0 & h_{2m-1} & h_{2m-2} & . & . & h_{2m-n+2} & h_{2m-n+1} \\ 0 & 0 & h_{2m-1} & . & . & h_{2m-n+3} & h_{2m-n+2} \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & 0 & h_{2m-1} \end{pmatrix} \tag{3.0.3}$$

So

$$(h_1 * x) - (h_2 * x) = y(n) - y(n) = 0 \tag{3.0.4}$$

$$(\mathbf{H_1} - \mathbf{H_2})\mathbf{X} = 0 \tag{3.0.5}$$

For the sake of simplicity lets assume that $m = n$,then Toeplitz matrix is of the form

$$\mathbf{H} = \begin{pmatrix} L \\ U \end{pmatrix} \tag{3.0.6}$$

Because $\mathbf{H_1}, \mathbf{H_2}$ are in toeplitz form,their difference is also in toeplitz form.

$$\mathbf{H} = \mathbf{H_1} - \mathbf{H_2} \tag{3.0.7}$$

$$\mathbf{H} = \begin{pmatrix} h_0 & 0 & 0 & . & . & 0 & 0 \\ h_1 & h_0 & 0 & . & . & 0 & 0 \\ h_2 & h_1 & h_0 & . & . & 0 & 0 \\ . & . & . & . & . & . & . \\ h_{m-2} & h_{m-3} & . & . & h_1 & h_0 & 0 \\ h_{m-1} & h_{m-2} & h_{m-3} & . & . & h_1 & h_0 \\ 0 & h_{m-1} & h_{m-2} & . & . & h_2 & h_1 \\ 0 & 0 & h_{m-1} & . & . & h_3 & h_2 \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & 0 & h_{m-1} \end{pmatrix} \tag{3.0.8}$$

Suppose

$$L = \begin{pmatrix} h_0 & 0 & 0 & . & . & 0 & 0 \\ h_1 & h_0 & 0 & . & . & 0 & 0 \\ h_2 & h_1 & h_0 & . & . & 0 & 0 \\ . & . & . & . & . & . & . \\ h_{n-2} & h_{n-3} & . & . & h_1 & h_0 & 0 \\ h_{n-1} & h_{n-2} & h_{n-3} & . & . & h_1 & h_0 \end{pmatrix} \tag{3.0.9}$$

$$LX = 0 \tag{3.0.10}$$

For $LX = 0$ to have a non-trivial solution i.e nullity$\neq 0$, the rank of the matrix $R(L) < n$, which implies $h_0 = 0$.

$$L = \begin{pmatrix} 0 & 0 & 0 & . & . & 0 & 0 \\ h_1 & 0 & 0 & . & . & 0 & 0 \\ h_2 & h_1 & 0 & . & . & 0 & 0 \\ . & . & . & . & . & . & . \\ h_{n-2} & h_{n-3} & . & . & h_1 & 0 & 0 \\ h_{n-1} & h_{n-2} & h_{n-3} & . & . & h_1 & 0 \end{pmatrix} \tag{3.0.11}$$

Similarly if you consider the submatrix of L

$$L^1 = \begin{pmatrix} h_1 & 0 & 0 & . & . & 0 & 0 \\ h_2 & h_1 & 0 & . & . & 0 & 0 \\ h_3 & h_2 & h_1 & . & . & 0 & 0 \\ . & . & . & . & . & . & . \\ h_{n-2} & h_{n-3} & . & . & h_2 & h_2 & 0 \\ h_{n-1} & h_{n-2} & h_{n-3} & . & . & h_2 & h_1 \end{pmatrix} \tag{3.0.12}$$

where

$$\begin{pmatrix} h_1 & 0 & 0 & . & . & 0 & 0 \\ h_2 & h_1 & 0 & . & . & 0 & 0 \\ h_3 & h_2 & h_1 & . & . & 0 & 0 \\ . & . & . & . & . & . & . \\ h_{n-2} & h_{n-3} & . & . & h_2 & h_1 & 0 \\ h_{n-1} & h_{n-2} & h_{n-3} & . & . & h_2 & h_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2. \\ . \\ . \\ x_{n-2} \end{pmatrix} = 0 \tag{3.0.13}$$

By using the similar logic,$h_1 = 0$. And all the elements of the lower triangular matrix are zero i.e $L = 0$,rank(L)=0 which implies $H = 0$.

The solution for $LX = 0$ or $UX = 0$ when $X \neq 0$, is that each of the elements of the matrices L, U is zero irrespective of X because X, the input signal does not depend on the system.So,$H = 0$, $\mathbf{H_1} - \mathbf{H_2} = 0$ which means $\mathbf{H_1} = \mathbf{H_2}$

4 Generalization

In a toeplitz matrix, each column can be obtained by a shift of the previous column. For example consider

$$\mathbf{H} = \begin{pmatrix} \mathbf{c_0} & \mathbf{c_1} & \mathbf{c_2} & . & . & . & \mathbf{c_{n-1}} \end{pmatrix} \tag{4.0.1}$$

The dimension of the column vector $\mathbf{c_0}$ is $(m + n - 1) \times 1$.Here each of the column vector $c_2, c_3..c_n$ can be obtained from

$$\mathbf{c_i} = \mathbf{S^i c_0} \tag{4.0.2}$$

where

$$\mathbf{S} = \begin{pmatrix} 0 & 0 & 0 & . & . & 0 & 0 \\ 1 & 0 & 0 & . & . & 0 & 0 \\ 0 & 1 & 0 & . & . & 0 & 0 \\ 0 & 0 & 1 & . & . & 0 & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & 1 & 0 \end{pmatrix} \tag{4.0.3}$$

$$\mathbf{S^{m+n-1}} = 0 \tag{4.0.4}$$

The dimension of the matrix S is $(m + n - 1) \times (m + n - 1)$.

From equation (3.0.8),we can write it in the form

$$\mathbf{H} = \begin{pmatrix} \mathbf{c_0} & \mathbf{Sc_0} & \mathbf{S^2c_0} & . & . & \mathbf{S^{n-1}c_0} \end{pmatrix} \tag{4.0.5}$$

We can write the equation $\mathbf{HX} = 0$ as

$$\begin{pmatrix} \mathbf{c_0} & \mathbf{Sc_0} & \mathbf{S^2c_0} & . & . & \mathbf{S^{n-1}c_0} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ . \\ . \\ x_{n-1} \end{pmatrix} = 0 \tag{4.0.6}$$

$$\mathbf{Ic_0}x_0 + \mathbf{Sc_0}x_1 + \mathbf{S^2c_0}x_2 + ... + \mathbf{S^{n-1}c_0}x_{n-1} = 0 \tag{4.0.7}$$

$$\left(\mathbf{I}x_0 + \mathbf{S}x_1 + \mathbf{S^2}x_2 + ... + \mathbf{S^{n-1}}x_{n-1}\right)\mathbf{c_0} = 0 \tag{4.0.8}$$

where the dimension if I, S are $(m+n-1) \times (m+n-1)$.S^i can be written in general form as

$$\mathbf{S^i} = \begin{pmatrix} 0_{i\times(m+n-i-1)} & 0_{i\times i} \\ I_{(m+n-i-1)\times(m+n-i-1)} & 0_{(m+n-i-1)\times i} \end{pmatrix} \tag{4.0.9}$$

From equation (4.0.8) we get

$$\begin{pmatrix} x_0 & 0 & 0 & 0 & . & . & 0 & 0 \\ x_1 & x_0 & 0 & 0 & . & . & 0 & 0 \\ x_2 & x_1 & x_0 & 0 & . & . & 0 & 0 \\ x_3 & x_2 & x_1 & x_0 & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . \\ 0 & 0 & x_{n-1} & x_{n-2} & . & . & x_0 & 0 \\ 0 & 0 & 0 & x_{n-1} & . & . & x_1 & x_0 \end{pmatrix} \mathbf{c_0} = 0 \tag{4.0.10}$$

Since,the equation (4.0.10),should hold for any value of X, the matrix has a nullity space zero,because x_0 can be non zero.So,$\mathbf{c_0} = 0$ is the only solution.

$$\mathbf{H} = \begin{pmatrix} \mathbf{c_0} & \mathbf{Sc_0} & \mathbf{S^2c_0} & . & . & \mathbf{S^{n-1}c_0} \end{pmatrix} \tag{4.0.11}$$

$$\mathbf{H} = 0 \tag{4.0.12}$$

Therefore $H_1 = H_2$ or we can say that $h_1 = h_2$.