Assignment 1

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1) Find the equation of the hyperbola that satisfies the conditions - Foci $(\pm 4, 0)$, the latus rectum is of length 12.
Solution: The equation of the conic with focus $\mathbf{F}$, directrix $\mathbf{n}^\top \mathbf{x} = c$ and eccentricity e is given by

$$\mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \qquad (0.0.1)$$

where

$$\mathbf{V} \triangleq \|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{n}\mathbf{n}^\top \qquad (0.0.2)$$

$$\mathbf{u} \triangleq ce^2 \mathbf{n} - \|\mathbf{n}\|^2 \mathbf{F} \qquad (0.0.3)$$

$$f \triangleq \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2 e^2 \qquad (0.0.4)$$

also

$$f_0 = \mathbf{u}^\top \mathbf{V}^{-1} \mathbf{u} - f \qquad (0.0.5)$$

$$l = 2\frac{\sqrt{|f_0 \lambda_2|}}{\lambda_1} \qquad (0.0.6)$$

a) $\mathbf{n}$: Given that the conic has foci as

$$\mathbf{F_1} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \qquad (0.0.7)$$

$$\mathbf{F_2} = \begin{pmatrix} -4 \\ 0 \end{pmatrix} \qquad (0.0.8)$$

The direction vector of $F_1 F_2$ is given by

$$\mathbf{m} = \mathbf{F_1} - \mathbf{F_2} \qquad (0.0.9)$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad (0.0.10)$$

Hence the normal to the directrix is given by,

$$\mathbf{n} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad (0.0.11)$$

b) $\mathbf{u}$: The centre of the conic is given by

$$\mathbf{c} = \frac{\mathbf{F_1} + \mathbf{F_2}}{2} \qquad (0.0.12)$$

$$\mathbf{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (0.0.13)$$

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \qquad (0.0.14)$$

Since $\mathbf{c} = \mathbf{0}$ and $\mathbf{V}^{-1} \neq \mathbf{0}$, it follows from (0.0.14) that

$$\mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (0.0.15)$$

From the above expressions we get

$$\mathbf{V} = \begin{pmatrix} 1 - e^2 & 0 \\ 0 & 1 \end{pmatrix} \qquad (0.0.16)$$

$$\mathbf{F} = \begin{pmatrix} ce^2 \\ 0 \end{pmatrix} \qquad (0.0.17)$$

$$f = c^2 e^2 \left(e^2 - 1 \right) \qquad (0.0.18)$$

$$f_0 = -f \qquad (0.0.19)$$

$$l = 2\frac{\sqrt{|f \lambda_2|}}{\lambda_1} \qquad (0.0.20)$$

From equation (0.0.16) the eigen values of matrix $\mathbf{V}$ - λ_1, λ_2 are given by,

$$\lambda_1 = 1 - e^2 \qquad (0.0.21)$$

$$\lambda_2 = 1 \qquad (0.0.22)$$

From equation (0.0.17) we get,

$$ce^2 = 4 \qquad (0.0.23)$$

c) Eccentricity: Given that the conic has the latus rectum length 12. Substituting the expressions of λ_1, λ_2 from the equations (0.0.21), (0.0.22) in (0.0.20) gives

$$l = \frac{2ce}{\sqrt{e^2 - 1}} = 12 \qquad (0.0.24)$$

$$\frac{ce}{\sqrt{e^2 - 1}} = 6 \qquad (0.0.25)$$

Substitute the expression of c from (0.0.23) gives,

$$\frac{4}{e\sqrt{e^2 - 1}} = 6 \qquad (0.0.26)$$

Squaring on both sides gives,

$$9e^2\left(e^2 - 1\right) = 4 \qquad (0.0.27)$$
$$9e^4 - 9e^2 - 4 = 0 \qquad (0.0.28)$$

The equation (0.0.28) is a quadratic equation in e^2. Solving it gives two roots one of which is negative, as e^2 is positive we have

$$e^2 = \frac{4}{3} \qquad (0.0.29)$$

From equation (0.0.18), (0.0.23), we get

$$f = 4 \qquad (0.0.30)$$

The equation of the conic is given by

$$\mathbf{x}^\top \begin{pmatrix} -\frac{1}{3} & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + 4 = 0 \qquad (0.0.31)$$
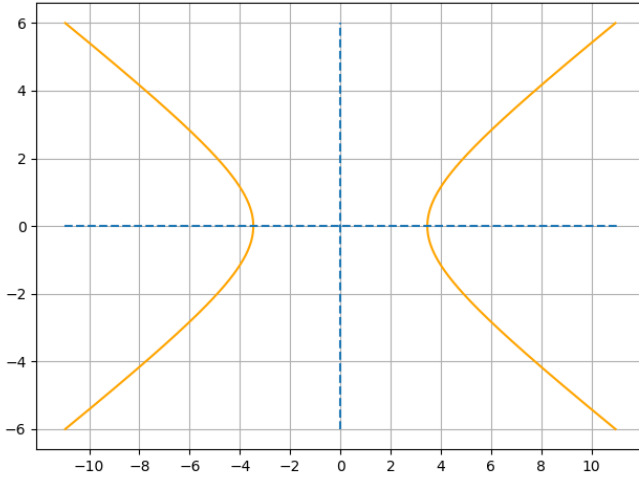

Fig. 1: Graph

Parameter	Description	Value
$\mathbf{F_1}$	Focus 1 of hyperbola	$\begin{pmatrix} 4 \\ 0 \end{pmatrix}$
$\mathbf{F_2}$	Focus 2 of hyperbola	$\begin{pmatrix} -4 \\ 0 \end{pmatrix}$
l	Length of latus rectum	12

TABLE 1