12.10.3.13

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Class 12, Chapter 10, Exercise 3.13

13) If are $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are unit vectors such that $\mathbf{a}+\mathbf{b}+\mathbf{c} = 0$, find the value of $\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a}$.
Solution: We have given that $\mathbf{a}+\mathbf{b}+\mathbf{c} = 0$, and $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are unit vectors.

$$\implies \|\mathbf{a}+\mathbf{b}+\mathbf{c}\|^2 = 0 \tag{1}$$

$$\text{and } \|\mathbf{a}\|^2 = \|\mathbf{b}\|^2 = \|\mathbf{c}\|^2 = 1 \tag{2}$$

$$\implies \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 + \|\mathbf{c}\|^2 + 2(\mathbf{a}^\top\mathbf{b} + \mathbf{b}^\top\mathbf{c} + \mathbf{c}^\top\mathbf{a}) = 0 \tag{3}$$

$$\implies 1+1+1+2(\mathbf{a}^\top\mathbf{b} + \mathbf{b}^\top\mathbf{c} + \mathbf{c}^\top\mathbf{a}) = 0 \tag{4}$$

$$\implies \mathbf{a}^\top\mathbf{b} + \mathbf{b}^\top\mathbf{c} + \mathbf{c}^\top\mathbf{a} = -\frac{3}{2} \tag{5}$$

Let's verify with a numerical example.

$$\mathbf{a} = \begin{pmatrix}\cos(0)\\ \sin(0)\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} \tag{6}$$

$$\mathbf{b} = \begin{pmatrix}\cos(2\pi/3)\\ \sin(2\pi/3)\end{pmatrix} = \begin{pmatrix}-\frac{1}{2}\\ \frac{\sqrt{3}}{2}\end{pmatrix} \tag{7}$$

$$\mathbf{c} = \begin{pmatrix}\cos(4\pi/3)\\ \sin(4\pi/3)\end{pmatrix} = \begin{pmatrix}-\frac{1}{2}\\ -\frac{\sqrt{3}}{2}\end{pmatrix} \tag{8}$$

$$\implies \mathbf{a}+\mathbf{b}+\mathbf{c} = \begin{pmatrix}0\\0\end{pmatrix} \tag{9}$$

$$\mathbf{a}^\top\mathbf{b} = \begin{pmatrix}1\\0\end{pmatrix}^\top \begin{pmatrix}-\frac{1}{2}\\ \frac{\sqrt{3}}{2}\end{pmatrix} = -\frac{1}{2} \tag{10}$$

$$\mathbf{b}^\top\mathbf{c} = \begin{pmatrix}-\frac{1}{2}\\ \frac{\sqrt{3}}{2}\end{pmatrix}^\top \begin{pmatrix}-\frac{1}{2}\\ -\frac{\sqrt{3}}{2}\end{pmatrix} = -\frac{1}{2} \tag{11}$$

$$\mathbf{c}^\top\mathbf{a} = \begin{pmatrix}-\frac{1}{2}\\ -\frac{\sqrt{3}}{2}\end{pmatrix}^\top \begin{pmatrix}1\\0\end{pmatrix} = -\frac{1}{2} \tag{12}$$

$$\implies \mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a} = -\frac{3}{2} \tag{13}$$

Verified.