Conic Sections - Ellipse

1 11^{th} Maths - Chapter 11

This is Problem-7 from Exercise 11.5

1. A man running a racecourse notes that the sum of the distances from the two flag posts from him is always 10 m and the distance between the flag posts is 8 m.Find the equation of the posts traced by the man.

   Solution: The conic section for the given problem is Ellipse. Let $\mathbf{O}\begin{pmatrix}0\\0\end{pmatrix}$ be the centre of the Ellipse. Then, the focii are given by

$$\mathbf{F_1} = \begin{pmatrix}4\\0\end{pmatrix} \tag{1}$$

$$\mathbf{F_2} = \begin{pmatrix}-4\\0\end{pmatrix} \tag{2}$$

   The sum of the distances from two focii to the point on the locus of the ellipse is equal to $10m$. Let $\mathbf{P}\begin{pmatrix}p\\0\end{pmatrix}$ and $\mathbf{Q}\begin{pmatrix}-q\\0\end{pmatrix}$ be the vertices of the ellipse. Then

$$\|\mathbf{P}-\mathbf{F_1}\| + \|\mathbf{P}-\mathbf{F_2}\| = 10 \tag{3}$$

$$(p-4)+(p+4) = 10 \tag{4}$$

$$2p = 10 \tag{5}$$

$$p = 5 \tag{6}$$

$$\therefore \mathbf{P} = \begin{pmatrix}5\\0\end{pmatrix} \tag{7}$$

Similarly

$$
\begin{aligned}
\|\mathbf{Q}-\mathbf{F_1}\| + \|\mathbf{Q}-\mathbf{F_2}\| &= 10 && (8)\\
(q-4)+(q+4) &= 10 && (9)\\
2q &= 10 && (10)\\
q &= 5 && (11)\\
\therefore \mathbf{Q} &= \begin{pmatrix}-5\\0\end{pmatrix} && (12)
\end{aligned}
$$

We know that the Vertex of a standard ellipse is given by

$$
\begin{aligned}
\mathbf{P} &= \begin{pmatrix}\sqrt{\left|\frac{f_0}{\lambda_1}\right|}\\0\end{pmatrix} && (13)\\
\begin{pmatrix}5\\0\end{pmatrix} &= \begin{pmatrix}\sqrt{\left|\frac{f_0}{\lambda_1}\right|}\\0\end{pmatrix} && (14)\\
\frac{f_0}{\lambda_1} &= 25 && (15)\\
f_0 &= 25\lambda_1 && (16)
\end{aligned}
$$

We know that the Focii for standard Ellipse are given as

$$
\mathbf{F} = \pm e\sqrt{\frac{|f_0|}{\lambda_2\left(1-e^2\right)}}\mathbf{e}_1 \tag{17}
$$

Substituting values of $\mathbf{F_1}$ from (1) and f_0 from (16)

$$
(17) \implies \begin{pmatrix}4\\0\end{pmatrix} = e\sqrt{\frac{25\lambda_1}{\lambda_2\left(1-e^2\right)}}\mathbf{e}_1 \tag{18}
$$

We know that

$$1 - e^2 = \frac{\lambda_1}{\lambda_2} \tag{19}$$

$$(18) \implies 4 = 5e \tag{20}$$

$$e = \frac{4}{5} \tag{21}$$

$$\therefore \frac{\lambda_1}{\lambda_2} = 1 - \left(\frac{4}{5}\right)^2 \tag{22}$$

$$= \frac{9}{25} \tag{23}$$

$$\mathbf{n} = \sqrt{\frac{\lambda_2}{f_0}}\mathbf{e}_1 \tag{24}$$

$$= \sqrt{\frac{\lambda_2}{25\lambda_1}}\mathbf{e}_1 \tag{25}$$

$$= \frac{1}{5} \times \frac{5}{3}\mathbf{e}_1 \tag{26}$$

$$= \frac{1}{3}\mathbf{e}_1 \tag{27}$$

$$c = \frac{1}{e\sqrt{1 - e^2}} = \frac{25}{12} \tag{28}$$

For the standard ellipse, f is given as

$$f = \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2 e^2 \tag{29}$$

$$= \left(\frac{1}{3}\right)^2 16 - \frac{25}{9} \tag{30}$$

$$= -1 \tag{31}$$

$$f_0 = -f = 1 \tag{32}$$

$$\lambda_1 = \frac{f_0}{25} = \frac{1}{25} \tag{33}$$

$$\lambda_2 = \frac{25\lambda_1}{9} = \frac{1}{9} \tag{34}$$

$$\therefore \mathbf{V} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{25} & 0 \\ 0 & \frac{1}{9} \end{pmatrix} \tag{35}$$

For a standard ellipse, $\mathbf{u} = 0$.

The generic equation of conic section is given as

$$g(\mathbf{x}) = \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \tag{36}$$

$$= \mathbf{x}^T \begin{pmatrix} \frac{1}{25} & 0 \\ 0 & \frac{1}{9} \end{pmatrix} \mathbf{x} - 1 = 0 \tag{37}$$

The relevant diagram is shown in Figure 1

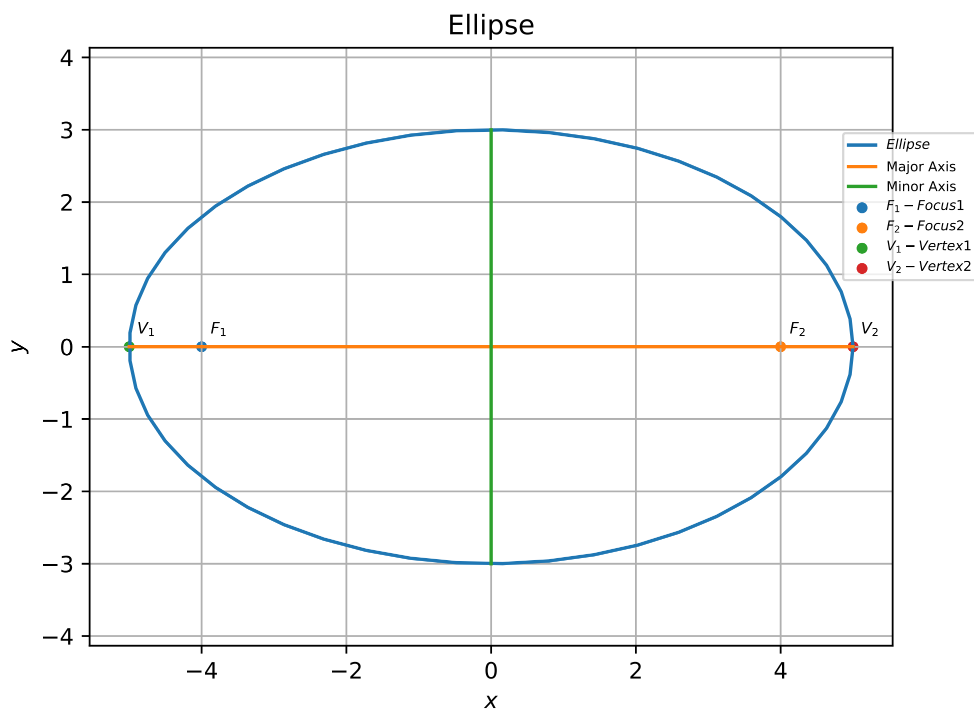

Figure 1