EE5609: Matrix Theory Assignment-5

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***Abstract*—This document contains solution to determine the conic representing the given equation.**

Download the python codes from latex-tikz codes from

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https://github.com/pavanmanesh/EE5609/tree/
    master/Assignment5
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1 Problem

What conic does the following equation represent.

$$9x^2 - 24xy + 16y^2 - 18x - 101y + 19 = 0$$

Find the center.

2 Solution

The general second degree equation can be expressed as follows,

$$\mathbf{x^T V x} + 2\mathbf{u^T x} + f = 0 \tag{2.0.1}$$

From the given second degree equation we get,

$$\mathbf{V} = \begin{pmatrix} 9 & -12 \\ -12 & 16 \end{pmatrix} \tag{2.0.2}$$

$$\mathbf{u} = \begin{pmatrix} -9 \\ -\frac{101}{2} \end{pmatrix} \tag{2.0.3}$$

$$f = 4 \tag{2.0.4}$$

Expanding the determinant of $\mathbf{V}$ we observe,

$$\begin{vmatrix} 9 & -12 \\ -12 & 16 \end{vmatrix} = 0 \tag{2.0.5}$$

Also

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = \begin{vmatrix} 9 & -12 & -9 \\ -12 & 16 & -\frac{101}{2} \\ -9 & -\frac{101}{2} & 4 \end{vmatrix} \tag{2.0.6}$$

$$\neq 0 \tag{2.0.7}$$

Hence from (2.0.5) and (2.0.7) we conclude that given equation is an parabola. The characteristic equation of $\mathbf{V}$ is given as follows,

$$\left|\lambda\mathbf{I} - \mathbf{V}\right| = \begin{vmatrix} \lambda - 9 & 12 \\ 12 & \lambda - 16 \end{vmatrix} = 0 \tag{2.0.8}$$

$$\implies \lambda^2 - 25\lambda = 0 \tag{2.0.9}$$

Hence the characteristic equation of $\mathbf{V}$ is given by (2.0.9). The roots of (2.0.9) i.e the eigenvalues are given by

$$\lambda_1 = 0, \lambda_2 = 25 \tag{2.0.10}$$

The eigen vector $\mathbf{p}$ is defined as,

$$\mathbf{Vp} = \lambda\mathbf{p} \tag{2.0.11}$$

$$\implies (\lambda\mathbf{I} - \mathbf{V})\,\mathbf{p} = 0 \tag{2.0.12}$$

for $\lambda_1 = 0$,

$$(\lambda_1\mathbf{I} - \mathbf{V}) = \begin{pmatrix} -9 & 12 \\ 12 & -16 \end{pmatrix} \xleftrightarrow[R_1 = \frac{1}{-9}R_1]{R_2 = 3R_1 - R_2} \begin{pmatrix} 1 & -\frac{4}{3} \\ 0 & 0 \end{pmatrix} \tag{2.0.13}$$

$$\implies \mathbf{p_1} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ 1 \end{pmatrix} \tag{2.0.14}$$

Substiuting equation 2.0.14 in equation 2.0.12 we get

$$\begin{pmatrix} 1 & -\frac{4}{3} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.0.15}$$

Where, $\mathbf{p} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ Let $v_2 = t$

$$v_1 = \frac{4}{3}t \tag{2.0.16}$$

Eigen vector $\mathbf{p_1}$ is given by

$$\mathbf{p_1} = \begin{pmatrix} \frac{4}{3}t \\ t \end{pmatrix} \tag{2.0.17}$$

Let $t = -\frac{6}{10}$, we get

$$\mathbf{p_1} = \begin{pmatrix} -\frac{8}{10} \\ -\frac{6}{10} \end{pmatrix} \tag{2.0.18}$$

Again, for $\lambda_2 = 25$,

$$(\lambda_2 \mathbf{I} - \mathbf{V}) = \begin{pmatrix} 16 & 12 \\ 12 & 9 \end{pmatrix} \xleftrightarrow[R_1 = \frac{1}{16}R_1]{R_2 = 12R_1 - R_2} \begin{pmatrix} 1 & \frac{3}{4} \\ 0 & 0 \end{pmatrix} \tag{2.0.19}$$

Substiuting equation 2.0.19 in equation 2.0.12 we get

$$\begin{pmatrix} 1 & \frac{3}{4} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.0.20}$$

Where, $\mathbf{p} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ Let $v_2 = t$

$$v_1 = -\frac{3}{4}t \tag{2.0.21}$$

Eigen vector $\mathbf{p_2}$ is given by

$$\mathbf{p_2} = \begin{pmatrix} -\frac{3}{4}t \\ t \end{pmatrix} \tag{2.0.22}$$

Let $t = -\frac{8}{10}$, we get

$$\mathbf{p_2} = \begin{pmatrix} \frac{6}{10} \\ -\frac{8}{10} \end{pmatrix} \tag{2.0.23}$$

The matrix P,

$$\mathbf{P} = \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} \end{pmatrix} = \begin{pmatrix} -\frac{8}{10} & \frac{6}{10} \\ -\frac{6}{10} & -\frac{8}{10} \end{pmatrix} \tag{2.0.24}$$

$$\mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & 25 \end{pmatrix} \tag{2.0.25}$$

$$\eta = 2\mathbf{u}^T \mathbf{p_1} = 75 \tag{2.0.26}$$

The focal length of the parabola is given by:

$$\left| \frac{\eta}{\lambda_2} \right| = \frac{75}{25} = 3 \tag{2.0.27}$$

When $\left| \mathbf{V} \right| = 0$,(2.0.1) can be written as

$$\mathbf{y^T D y} = -\eta \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{y} \tag{2.0.28}$$

And the vertex $\mathbf{c}$ is given by

$$\begin{pmatrix} \mathbf{u^T} + \eta \mathbf{p_1^T} \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \eta \mathbf{p_1} - \mathbf{u} \end{pmatrix} \tag{2.0.29}$$

using equations (2.0.3),(2.0.4) and (2.0.14)

$$\begin{pmatrix} -69 & -\frac{191}{2} \\ 9 & -12 \\ -12 & 16 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -4 \\ -51 \\ \frac{11}{2} \end{pmatrix} \tag{2.0.30}$$

This is in the form of

$$A\mathbf{c} = \mathbf{b} \tag{2.0.31}$$

using least squares solution of linear system

$$A^T A \mathbf{c} = A^T \mathbf{b} \implies \mathbf{c} = (A^T A)^{-1} A^T \mathbf{b} \tag{2.0.32}$$

$$A^T A = \begin{pmatrix} -69 & 9 & -12 \\ -\frac{191}{2} & -12 & 16 \end{pmatrix} \begin{pmatrix} -69 & -\frac{191}{2} \\ 9 & -12 \\ -12 & 16 \end{pmatrix}$$

$$\implies A^T A = \begin{pmatrix} 4986 & \frac{12579}{2} \\ \frac{12579}{2} & \frac{38081}{4} \end{pmatrix} \tag{2.0.33}$$

The inverse can be written as

$$(A^T A)^{-1} = \frac{31640625}{4} \begin{pmatrix} \frac{38081}{4} & -\frac{12579}{2} \\ -\frac{12579}{2} & 4986 \end{pmatrix} \tag{2.0.34}$$

$$A^T \mathbf{b} = \begin{pmatrix} -69 & 9 & -12 \\ -\frac{191}{2} & -12 & 16 \end{pmatrix} \begin{pmatrix} -4 \\ -51 \\ \frac{11}{2} \end{pmatrix}$$

$$A^T \mathbf{b} = \begin{pmatrix} -249 \\ 1082 \end{pmatrix} \tag{2.0.35}$$

using 2.0.34 and 2.0.35 in 2.0.32,the center $\mathbf{c}$

$$\mathbf{c} = \frac{31640625}{4} \begin{pmatrix} \frac{38081}{4} & -\frac{12579}{2} \\ -\frac{12579}{2} & 4986 \end{pmatrix} \begin{pmatrix} -249 \\ 1082 \end{pmatrix} \tag{2.0.36}$$

$$\implies \mathbf{c} = \begin{pmatrix} -\frac{29}{25} \\ \frac{22}{25} \end{pmatrix} \tag{2.0.37}$$

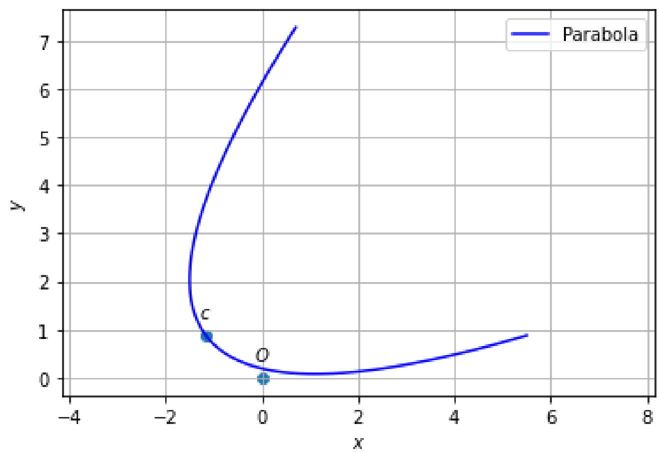

Fig. 1: Parabola with the center c