Excercise 11.5

Q8.An equilateral triangle is inscribed inthe parabola $y^2 = 4ax$, where one vertex is at the vertex of the parabola. Find the length of the side of the triangle.

Solution: The given equation of the parabola can be rearranged as

$$y^2 - 4ax = 0 \tag{1}$$

The above equation can be equated to the generic equation of conic sections

$$g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{2}$$

Comparing the coefficients of both equations (1) and (2)

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{3}$$

$$\mathbf{u} = -\begin{pmatrix} -2a \\ 0 \end{pmatrix} \tag{4}$$

$$f = 0 \tag{5}$$

Let length of side of triangle = r. Since,triangle is inscribed in the parabola and one of the vertex of the triangle lies at the vertex of the parabola, the other vertices are given as

$$\mathbf{P} = \begin{pmatrix} r\cos 30^\circ \\ r\sin 30^\circ \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2}r \\ \frac{1}{2}r \end{pmatrix} \tag{6}$$

$$\mathbf{Q} = \begin{pmatrix} r\cos 30^\circ \\ -r\sin 30^\circ \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2}r \\ -\frac{1}{2}r \end{pmatrix} \tag{7}$$

Now these point will satisfy the equation of the parabola. So substituting

(6) in (2)

$$\mathbf{P}^\top\mathbf{V}\mathbf{P} + 2\mathbf{u}^\top\mathbf{P} + f = 0 \tag{8}$$

$$\begin{pmatrix}\frac{\sqrt{3}}{2}r & \frac{1}{2}r\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\frac{\sqrt{3}}{2}r\\ \frac{1}{2}r\end{pmatrix} + 2\begin{pmatrix}-2a & 0\end{pmatrix}\begin{pmatrix}\frac{\sqrt{3}}{2}r\\ \frac{1}{2}r\end{pmatrix} = 0 \tag{9}$$

$$\frac{1}{4}r^2 - 2a\sqrt{3}r = 0 \tag{10}$$

$$r\left(\frac{r}{4} - 2a\sqrt{3}\right) = 0 \tag{11}$$

Hence, the length of side of triangle is

$$r = 8a\sqrt{3} \tag{12}$$

The result will be the same for substituting $\mathbf{Q}$. See Fig.1.

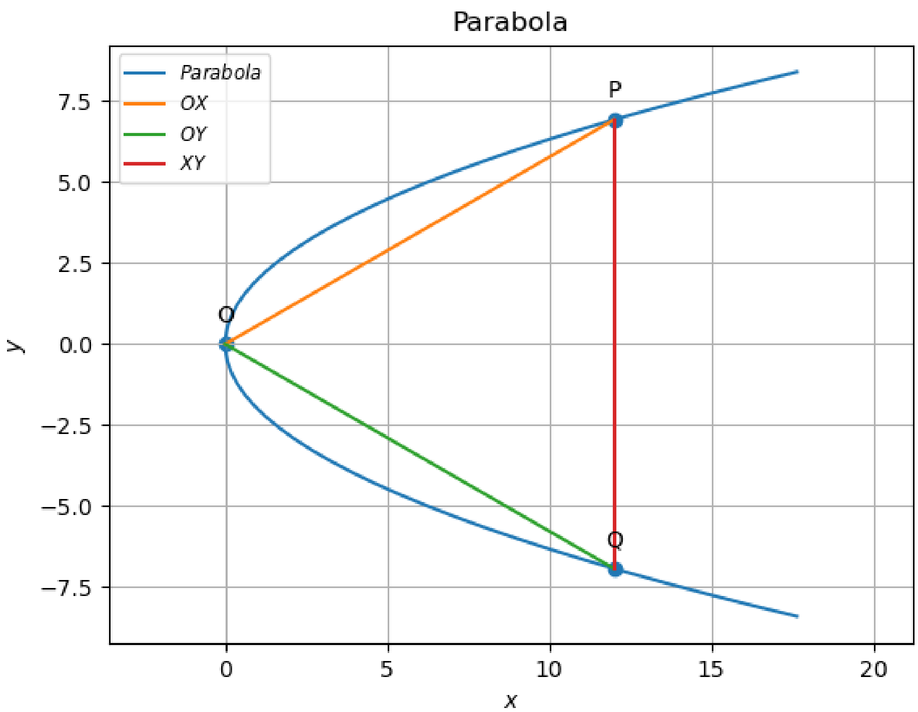

Figure 1: