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Assignment 1

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1) Find the equation of the ellipse that satisfies the conditions - Centre at (0,0), major axis on the y-axis and passes through the points (3,2) and (1,6).

Solution: The equation of the conic with focus **F**, directrix $\mathbf{n}^{\mathsf{T}}\mathbf{x} = c$ and eccentricity e is given by

$$\mathbf{x}^{\mathsf{T}}\mathbf{V}\mathbf{x} + 2\mathbf{u}^{\mathsf{T}}\mathbf{x} + f = 0 \tag{0.0.1}$$

where

$$\mathbf{V} \triangleq ||\mathbf{n}||^2 \mathbf{I} - e^2 \mathbf{n} \mathbf{n}^{\top} \tag{0.0.2}$$

$$\mathbf{u} \triangleq ce^2\mathbf{n} - ||\mathbf{n}||^2\mathbf{F} \tag{0.0.3}$$

$$f \triangleq ||\mathbf{n}||^2 ||\mathbf{F}||^2 - c^2 e^2$$
 (0.0.4)

Given that the conic is an ellipse with major axis along the y-axis, we get

$$\mathbf{n} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{0.0.5}$$

Thus,

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 - e^2 \end{pmatrix} \tag{0.0.6}$$

$$\mathbf{u} = ce^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \mathbf{F} \tag{0.0.7}$$

$$f = ||\mathbf{F}||^2 - c^2 e^2 \tag{0.0.8}$$

The centre of the conic is given by

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \tag{0.0.9}$$

Since $\mathbf{c} = \mathbf{0}$ and $\mathbf{V}^{-1} \neq \mathbf{0}$, it follows from (0.0.9) that

$$\mathbf{u} = \mathbf{0} \tag{0.0.10}$$

Thus, from (0.0.7),

$$\mathbf{F} = \begin{pmatrix} 0 \\ ce^2 \end{pmatrix} \tag{0.0.11}$$

and so.

$$f = c^2 e^2 \left(e^2 - 1 \right) \tag{0.0.12}$$

Given that the conic passes throught point,

$$\mathbf{P} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \tag{0.0.13}$$

Putting $\mathbf{x} = \mathbf{P}$ in (0.0.1) we get,

$$(3 2) \begin{pmatrix} 1 & 0 \\ 0 & 1 - e^2 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} + f = 0 (0.0.14)$$

$$\Rightarrow 4e^2 - f = 13 (0.0.15)$$

Given that the conic passes throught point,

$$\mathbf{Q} = \begin{pmatrix} 1 \\ 6 \end{pmatrix} \tag{0.0.16}$$

Putting $\mathbf{x} = \mathbf{Q}$ in (0.0.1), we get

$$(1 6)\begin{pmatrix} 1 & 0 \\ 0 & 1 - e^2 \end{pmatrix}\begin{pmatrix} 1 \\ 6 \end{pmatrix} + f = 0 (0.0.17)$$

$$\implies 36e^2 - f = 37 \tag{0.0.18}$$

The equations (0.0.15) and (0.0.18) can be formulated as the following matrix equation

$$\begin{pmatrix} 4 & -1 \\ 36 & -1 \end{pmatrix} \begin{pmatrix} e^2 \\ f \end{pmatrix} = \begin{pmatrix} 13 \\ 37 \end{pmatrix} \tag{0.0.19}$$

The augmented matrix is given by,

$$\begin{pmatrix} 4 & -1 & | & 13 \\ 36 & -1 & | & 37 \end{pmatrix} \tag{0.0.20}$$

$$\stackrel{R_1 \leftarrow R_1 - R_2}{\longleftrightarrow} \begin{pmatrix} -32 & 0 & | & -24 \\ 36 & -1 & | & 37 \end{pmatrix} \tag{0.0.21}$$

$$\stackrel{R_1 \leftarrow -\frac{R_1}{8}}{\longleftrightarrow} \begin{pmatrix} 4 & 0 & | & 3\\ 36 & -1 & | & 37 \end{pmatrix} \tag{0.0.22}$$

$$\stackrel{R_2 \leftarrow R_2 - 9R_1}{\longleftrightarrow} \begin{pmatrix} 4 & 0 & 3 \\ 0 & -1 & 10 \end{pmatrix} \tag{0.0.23}$$

$$\stackrel{R_1 \leftarrow \frac{R_1}{4}}{\longleftrightarrow} \begin{pmatrix} 1 & 0 & \begin{vmatrix} \frac{3}{4} \\ 0 & 1 & -10 \end{pmatrix}$$
(0.0.24)

Thus,

$$e^2 = \frac{3}{4}, \ f = -10$$
 (0.0.25)

The equation of the conic is given by

$$\mathbf{x}^{\mathsf{T}} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{pmatrix} \mathbf{x} - 10 = 0 \tag{0.0.26}$$

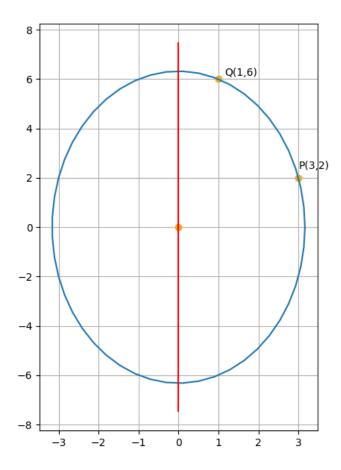


Fig. 1: Graph

Parameter	Description	Value
C	center of the ellipse	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
m	Direction vector of major axis	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
P	Point on the ellipse	$\begin{pmatrix} 3 \\ 2 \end{pmatrix}$
Q	Point on the ellipse	$\begin{pmatrix} 1 \\ 6 \end{pmatrix}$

TABLE 1