12.11.3.9

Lokesh Surana

Class 12, Chapter 11, Exercise 4.19

Q. Find the vector equation of the line passing through $\begin{pmatrix}1\\2\\3\end{pmatrix}$ and parallel to the planes $\begin{pmatrix}1\\-1\\2\end{pmatrix}^\top \mathbf{r} = 5$ and $\begin{pmatrix}3\\1\\1\end{pmatrix}^\top \mathbf{r} = 6$.

Solution: The line equations are given as

$$\mathbf{r} = \mathbf{A} + \lambda\mathbf{m} \tag{1}$$

where $\mathbf{m}$ is the direction vector of the line and $\mathbf{A}$ is any point on the line.

The planes are given as

$$P_1 : \begin{pmatrix}1 & -1 & 2\end{pmatrix}\mathbf{r} = 5 \tag{2}$$

$$\implies \mathbf{n}_1 = \begin{pmatrix}1\\-1\\2\end{pmatrix} \tag{3}$$

$$P_2 : \begin{pmatrix}3 & 1 & 1\end{pmatrix}\mathbf{r} = 6 \tag{4}$$

$$\implies \mathbf{n}_2 = \begin{pmatrix}3\\1\\1\end{pmatrix} \tag{5}$$

The expected line is parallel to both the planes, then the direction vector of the line must be perpendicular to both the normal vectors. This means that

$$\mathbf{n}_1^\top\mathbf{m} = 0 \tag{6}$$

$$\mathbf{n}_2^\top\mathbf{m} = 0 \tag{7}$$

$$\implies \begin{pmatrix}1 & -1 & 2\\3 & 1 & 1\end{pmatrix}\mathbf{m} = 0 \tag{8}$$

Let's reduce the matrix from equation (8) to row-echelon form:

$$\begin{pmatrix}1 & -1 & 2\\3 & 1 & 1\end{pmatrix} \xleftrightarrow{R_2\to-\frac{3}{4}R_1+\frac{1}{4}R_2} \begin{pmatrix}1 & -1 & 2\\0 & 1 & -\frac{5}{4}\end{pmatrix} \tag{9}$$

$$\begin{pmatrix}1 & -1 & 2\\0 & 1 & -\frac{5}{4}\end{pmatrix} \xleftrightarrow{R_1\to R_1+R_2} \begin{pmatrix}1 & 0 & \frac{3}{4}\\0 & 1 & -\frac{5}{4}\end{pmatrix} \tag{10}$$

Using (8), (9) and (10), we get:

$$\implies \begin{pmatrix}1 & 0 & \frac{3}{4}\\0 & 1 & -\frac{5}{4}\end{pmatrix}\mathbf{m} = 0 \tag{11}$$

$$\implies \begin{pmatrix}m_1\\m_2\\m_3\end{pmatrix} = \begin{pmatrix}-\frac{3}{4}m_3\\\frac{5}{4}m_3\\m_3\end{pmatrix} \tag{12}$$

$$\implies \begin{pmatrix}m_1\\m_2\\m_3\end{pmatrix} = m_3\begin{pmatrix}-\frac{3}{4}\\\frac{5}{4}\\1\end{pmatrix} \tag{13}$$

$$\implies \mathbf{m} = \begin{pmatrix}-3\\5\\4\end{pmatrix} \tag{14}$$

It is given that line passes through point $\begin{pmatrix}1\\2\\3\end{pmatrix}$, so the final equation of line implies

$$\mathbf{r} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda\begin{pmatrix}-3\\5\\4\end{pmatrix} \tag{15}$$

$$\tag{16}$$