

Tangents and Normals

1 12th Maths - Chapter 6

This is Problem-2 from Exercise 6.3

1. Find the slope of the tangent to the curve $y = \frac{x-1}{x-2}$, $x \neq 2$ at $x = 10$.

Solution:

- (a) The given equation of the curve can be rearranged as

$$xy - x - 2y + 1 = 0 \quad (1)$$

$$\implies \mathbf{x}^\top \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \mathbf{x} + (-1 \quad -2) \mathbf{x} + 1 = 0 \quad (2)$$

The above equation can be equated to the generic equation of conic sections

$$g(\mathbf{x}) = \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (3)$$

Comparing coefficients of both equations (2) and (3)

$$\mathbf{V} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \quad (4)$$

$$\mathbf{u} = - \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} \quad (5)$$

$$f = 1 \quad (6)$$

Given the point of contact \mathbf{q} , the normal vector of the tangent to (3) is

$$\kappa \mathbf{n} = \mathbf{V} \mathbf{q} + \mathbf{u}, \kappa \in \mathbb{R} \quad (7)$$

For the given point of contact with $\mathbf{q}_1 = 10$,

$$\mathbf{q}_2 = \frac{10 - 1}{10 - 2} = \frac{9}{8} \quad (8)$$

$$\therefore \mathbf{q} = \begin{pmatrix} 10 \\ \frac{9}{8} \end{pmatrix} \quad (9)$$

$$(7) \implies \kappa \mathbf{n} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 10 \\ \frac{9}{8} \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} \quad (10)$$

$$= \left(\begin{pmatrix} \frac{9}{16} \\ 5 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} \right) \quad (11)$$

$$\therefore \mathbf{n} = \alpha \begin{pmatrix} 1 \\ 64 \end{pmatrix} \quad (12)$$

$$\mathbf{m} = \alpha \begin{pmatrix} 1 \\ -\frac{1}{64} \end{pmatrix} \quad (13)$$

(b) Now, we have to determine the nature of the conic. The matrix(\mathbf{A}) of the quadratic equation (3) is represented as

$$\mathbf{A} = \begin{pmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^\top & f \end{pmatrix} \quad (14)$$

$$= \begin{pmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & 0 & -1 \\ -\frac{1}{2} & -1 & 1 \end{pmatrix} \quad (15)$$

$$|A| = \frac{1}{4} \quad (16)$$

$$|A_{33}| = -\frac{1}{4} \quad (17)$$

$\therefore |A| \neq 0$ and $|A_{33}| < 0$, the conic is a hyperbola. Moreover, the Eigen vectors for \mathbf{V} , which are given as below, indicate that the axes of hyperbola are rotated by 45° .

$$(\mathbf{p}_1 \quad \mathbf{p}_2) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad (18)$$

To summarize, the conic is a 45° rotated hyperbola.

The relevant diagram is shown in Figure 1

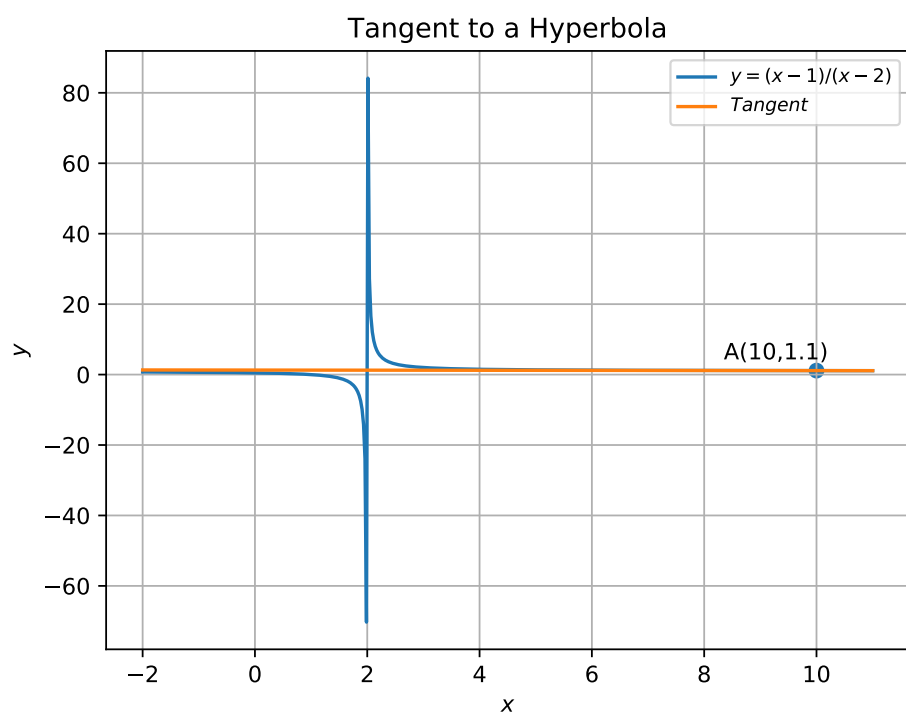


Figure 1