

Assignment 1

Jaswanth Chowdary Madala

- 1) Find the shortest distance between the lines l_1 and l_2 whose vector equations are $\vec{r} = \hat{i} + \hat{j} + \lambda(2\hat{i} - \hat{j} + \hat{k})$ and $\vec{r} = 2\hat{i} + \hat{j} - \hat{k} + \mu(3\hat{i} - 5\hat{j} + 2\hat{k})$

Solution: The lines l_1 and l_2 in vector form can be written as

$$\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \quad (0.0.1)$$

$$\mathbf{x} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} \quad (0.0.2)$$

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{x}_2 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \mathbf{m}_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \mathbf{m}_2 = \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} \quad (0.0.3)$$

We first check whether the given lines are skew. The lines

$$\mathbf{x} = \mathbf{x}_1 + \lambda_1 \mathbf{m}_1, \mathbf{x} = \mathbf{x}_2 + \lambda_2 \mathbf{m}_2 \quad (0.0.4)$$

intersect if

$$\mathbf{M}\lambda = \mathbf{x}_2 - \mathbf{x}_1 \quad (0.0.5)$$

$$\mathbf{M} \triangleq (\mathbf{m}_1 \quad \mathbf{m}_2) \quad (0.0.6)$$

$$\lambda \triangleq \begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} \quad (0.0.7)$$

$$(0.0.8)$$

Here we have,

$$\mathbf{M} = \begin{pmatrix} 2 & 3 \\ -1 & -5 \\ 1 & 2 \end{pmatrix} \quad (0.0.9)$$

$$\mathbf{x}_2 - \mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad (0.0.10)$$

We check whether the equation (0.0.11) has a

solution

$$\begin{pmatrix} 2 & 3 \\ -1 & -5 \\ 1 & 2 \end{pmatrix} \lambda = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad (0.0.11)$$

the augmented matrix is given by,

$$\left(\begin{array}{cc|c} 2 & 3 & 1 \\ -1 & -5 & 0 \\ 1 & 2 & -1 \end{array} \right) \quad (0.0.12)$$

$$\begin{array}{l} R_2 \leftarrow R_2 + \frac{1}{2}R_1 \\ R_3 \leftarrow R_3 - \frac{1}{2}R_1 \end{array} \quad (0.0.13)$$

$$\left(\begin{array}{cc|c} 2 & 3 & 1 \\ 0 & -\frac{7}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{3}{2} \end{array} \right) \quad (0.0.14)$$

$$R_3 \leftarrow R_3 + 7R_2 \quad (0.0.15)$$

$$\left(\begin{array}{cc|c} 2 & 3 & 1 \\ 0 & -\frac{7}{2} & \frac{1}{2} \\ 0 & 0 & -10 \end{array} \right) \quad (0.0.16)$$

The rank of the matrix is 3. So the given lines are skew. The closest points on two skew lines defined by (0.0.4) are given by

$$\mathbf{M}^T \mathbf{M} \lambda = \mathbf{M}^T (\mathbf{x}_2 - \mathbf{x}_1) \quad (0.0.17)$$

$$\begin{pmatrix} 2 & -1 & 1 \\ 3 & -5 & 2 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ -1 & -5 \\ 1 & 2 \end{pmatrix} \lambda = \begin{pmatrix} 2 & -1 & 1 \\ 3 & -5 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad (0.0.18)$$

$$\begin{pmatrix} 6 & 13 \\ 13 & 38 \end{pmatrix} \lambda = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (0.0.19)$$

The augmented matrix of the above equation

(0.0.19) is given by,

$$\left(\begin{array}{cc|c} 6 & 13 & 1 \\ 13 & 38 & 1 \end{array} \right) \quad (0.0.20)$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - \frac{13}{6}R_1} \quad (0.0.21)$$

$$\left(\begin{array}{cc|c} 6 & 13 & 1 \\ 0 & \frac{59}{6} & -\frac{7}{6} \end{array} \right) \quad (0.0.22)$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - \frac{78}{59}R_2} \quad (0.0.23)$$

$$\left(\begin{array}{cc|c} 6 & 0 & \frac{150}{59} \\ 0 & \frac{59}{6} & -\frac{7}{6} \end{array} \right) \quad (0.0.24)$$

So, we get

$$\begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} = \begin{pmatrix} \frac{25}{59} \\ -\frac{7}{59} \end{pmatrix} \quad (0.0.25)$$

The closest points **A** on line l_1 and **B** on line l_2 are given by,

$$\mathbf{A} = \mathbf{x}_1 + \lambda_1 \mathbf{m}_1 \quad (0.0.26)$$

$$= \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \frac{25}{59} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \quad (0.0.27)$$

$$= \frac{1}{59} \begin{pmatrix} 109 \\ 34 \\ 25 \end{pmatrix} \quad (0.0.28)$$

$$\mathbf{B} = \mathbf{x}_2 + \lambda_2 \mathbf{m}_2 \quad (0.0.29)$$

$$= \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \frac{7}{59} \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} \quad (0.0.30)$$

$$= \frac{1}{59} \begin{pmatrix} 139 \\ 24 \\ -45 \end{pmatrix} \quad (0.0.31)$$

The minimum distance between the lines is given by,

$$\|\mathbf{B} - \mathbf{A}\| = \left\| \frac{1}{59} \begin{pmatrix} 30 \\ -10 \\ -70 \end{pmatrix} \right\| \quad (0.0.32)$$

$$= \frac{\sqrt{30^2 + 10^2 + 70^2}}{59} \quad (0.0.33)$$

$$= \frac{10}{\sqrt{59}} \quad (0.0.34)$$

The shortest distance between the given lines is $\frac{10}{\sqrt{59}}$ units.

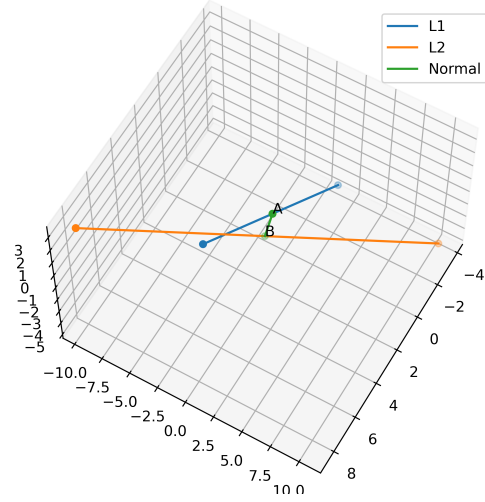


Fig. 1: AB is the required shortest distance.