Conic Assignment

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Abstract—**This document contains the solution to Question 20 of Exercise 3 in Chapter 11 of the class 11 NCERT textbook.**

1) Find the equation of the ellipse whose major axis is the x-axis and passes through the points

$$\mathbf{P} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}, \ \mathbf{Q} = \begin{pmatrix} 6 \\ 2 \end{pmatrix} \tag{1}$$

Solution: Let the equation of the conic with focus $\mathbf{F}$, directrix $\mathbf{n}^\top \mathbf{x} = c$ and eccentricity e be

$$\mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{2}$$

where

$$\mathbf{V} \triangleq \|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{n}\mathbf{n}^\top \tag{3}$$
$$\mathbf{u} \triangleq ce^2 \mathbf{n} - \|\mathbf{n}\|^2 \mathbf{F} \tag{4}$$
$$f \triangleq \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2 e^2 \tag{5}$$

Since the conic is an ellipse whose major axis is along the x-axis, we have

$$\mathbf{n} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{6}$$

Thus,

$$\mathbf{V} = \begin{pmatrix} 1 - e^2 & 0 \\ 0 & 1 \end{pmatrix} \tag{7}$$
$$\mathbf{u} = ce^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \mathbf{F} \tag{8}$$
$$f = \|\mathbf{F}\|^2 - c^2 e^2 \tag{9}$$

The centre of the conic is the point which bisects all chords passing through it. Suppose that the centre of the conic is $\mathbf{c}$. Then, if $\mathbf{c} + \mathbf{p}$ lies on the conic, so does $\mathbf{c} - \mathbf{p}$. Substituting these points in (2),

$$(\mathbf{c} + \mathbf{p})^\top \mathbf{V} (\mathbf{c} + \mathbf{p}) + 2\mathbf{u}^\top (\mathbf{c} + \mathbf{p}) + f = 0 \tag{10}$$
$$(\mathbf{c} - \mathbf{p})^\top \mathbf{V} (\mathbf{c} - \mathbf{p}) + 2\mathbf{u}^\top (\mathbf{c} - \mathbf{p}) + f = 0 \tag{11}$$

Subtracting (11) from (10), and noting from (3) that $\mathbf{V}$ is symmetric, we get

$$\mathbf{c}^\top \mathbf{V} \mathbf{p} + \mathbf{u}^\top \mathbf{p} = 0 \tag{12}$$
$$\implies (\mathbf{V}\mathbf{c} + \mathbf{u})^\top \mathbf{p} = 0 \tag{13}$$
$$\implies \mathbf{V}\mathbf{c} + \mathbf{u} = \mathbf{0} \tag{14}$$
$$\implies \mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \tag{15}$$

where (14) follows since $\mathbf{p}$ can be arbitrary. Since $\mathbf{V} \neq \mathbf{0}$, it follows from (15) that $\mathbf{u} = \mathbf{0}$. Thus, from (8),

$$\mathbf{F} = \begin{pmatrix} ce^2 \\ 0 \end{pmatrix} \tag{16}$$

and so,

$$f = c^2 e^2 \left(e^2 - 1\right) \tag{17}$$

Putting $\mathbf{x} = \mathbf{P}$ in (2) and using (16) and (17),

$$\begin{pmatrix} 4 & 3 \end{pmatrix} \begin{pmatrix} 1 - e^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} + f = 0 \tag{18}$$
$$\implies 16e^2 - f = 25 \tag{19}$$

Putting $\mathbf{x} = \mathbf{Q}$ in (2), we get

$$\begin{pmatrix} 6 & 2 \end{pmatrix} \begin{pmatrix} 1 - e^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 6 \\ 2 \end{pmatrix} + f = 0 \tag{20}$$
$$\implies 36e^2 - f = 40 \tag{21}$$

The equations (19) and (21) can be formulated as a matrix equation

$$\begin{pmatrix} 16 & -1 \\ 36 & -1 \end{pmatrix} \begin{pmatrix} e^2 \\ f \end{pmatrix} = \begin{pmatrix} 25 \\ 40 \end{pmatrix} \tag{22}$$

and can be solved using the augmented matrix.

$$\begin{pmatrix} 16 & -1 & 25 \\ 36 & -1 & 40 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 - R_2} \begin{pmatrix} -20 & 0 & -15 \\ 36 & -1 & 40 \end{pmatrix} \tag{23}$$

$$\xleftrightarrow[R_2 \leftarrow -R_2]{R_1 \leftarrow \frac{R_1}{-5}} \begin{pmatrix} 4 & 0 & 3 \\ -36 & 1 & -40 \end{pmatrix} \tag{24}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 + 9R_1} \begin{pmatrix} 4 & 0 & 3 \\ 0 & 1 & -13 \end{pmatrix} \tag{25}$$

$$\xleftrightarrow{R_1 \leftarrow \frac{R_1}{4}} \begin{pmatrix} 1 & 0 & \frac{3}{4} \\ 0 & 1 & -13 \end{pmatrix} \tag{26}$$

$$\tag{27}$$

Thus,

$$e^2 = \frac{3}{4}, \ f = -13 \tag{28}$$

And the equation of the conic is given by

$$\mathbf{x}^\top \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - 13 = 0 \tag{29}$$

The situation is illustrated in Fig. 1

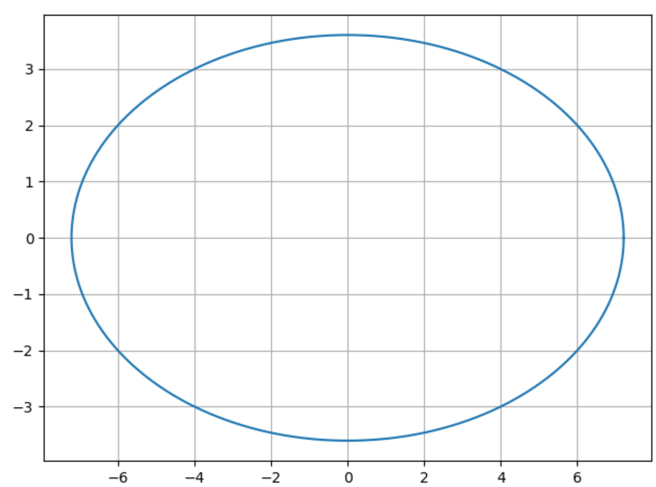

Fig. 1: Locus of the required ellipse.