Assignment 1

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1) Find the shortest distance between the lines l_1 and l_2 whose vector equations are $\overrightarrow{r} = \hat{i} + \hat{j} + \lambda(2\hat{i} - \hat{j} + \hat{k})$ and $\overrightarrow{r} = 2\hat{i} + \hat{j} - \hat{k} + \mu(3\hat{i} - 5\hat{j} + 2\hat{k})$

Solution: The shortest distance between the lines whose vector equations are

$$L_1 : \mathbf{x} = \mathbf{x_1} + \lambda_1 \mathbf{m_1} \tag{0.0.1}$$

$$L_2 : \mathbf{x} = \mathbf{x_2} + \lambda_2 \mathbf{m_2} \tag{0.0.2}$$

is given by,

$$d = \left\| \left(\mathbf{U} \left(\mathbf{\Sigma} \mathbf{\Sigma}^{-1} \right) \mathbf{U}^\top - \mathbf{I} \right) \mathbf{x} \right\| \tag{0.0.3}$$

with the parameter λ given by

$$\boldsymbol{\lambda} = \mathbf{V} \mathbf{\Sigma}^{-1} \mathbf{U}^\top \mathbf{x} \tag{0.0.4}$$

where

$$\mathbf{M} \triangleq \begin{pmatrix} \mathbf{m_1} & \mathbf{m_2} \end{pmatrix} \tag{0.0.5}$$

$$\boldsymbol{\lambda} \triangleq \begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} \tag{0.0.6}$$

$$\mathbf{x} \triangleq \mathbf{x_2} - \mathbf{x_1} \tag{0.0.7}$$

We use singular value decomposition of the matrix $\mathbf{M}$

$$\mathbf{M} = \mathbf{U} \mathbf{\Sigma} \mathbf{V}^\top \tag{0.0.8}$$

where $\mathbf{U}, \mathbf{V}$ are orthogonal and $\mathbf{\Sigma}$ is diagonal with nonnegative diagonal entries.

a) In this problem we have the lines l_1 and l_2 as

$$\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \tag{0.0.9}$$

$$\mathbf{x} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} \tag{0.0.10}$$

We first need to check whether the given lines are skew. The lines (0.0.1), (0.0.2) intersect if

$$\mathbf{M} \boldsymbol{\lambda} = \mathbf{x_2} - \mathbf{x_1} \tag{0.0.11}$$

Here we have,

$$\mathbf{M} = \begin{pmatrix} 2 & 3 \\ -1 & -5 \\ 1 & 2 \end{pmatrix} \tag{0.0.12}$$

$$\mathbf{x} = \mathbf{x_2} - \mathbf{x_1} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \tag{0.0.13}$$

We check whether the equation (0.0.14) has a solution

$$\begin{pmatrix} 2 & 3 \\ -1 & -5 \\ 1 & 2 \end{pmatrix} \boldsymbol{\lambda} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \tag{0.0.14}$$

the augmented matrix is given by,

$$\left(\begin{array}{cc|c} 2 & 3 & 1 \\ -1 & -5 & 0 \\ 1 & 2 & -1 \end{array}\right) \tag{0.0.15}$$

$$\xleftrightarrow[R_3 \leftarrow R_3 - \frac{1}{2}R_1]{R_2 \leftarrow R_2 + \frac{1}{2}R_1} \left(\begin{array}{cc|c} 2 & 3 & 1 \\ 0 & -\frac{7}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{3}{2} \end{array}\right) \tag{0.0.16}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 + 7R_2} \left(\begin{array}{cc|c} 2 & 3 & 1 \\ 0 & -\frac{7}{2} & \frac{1}{2} \\ 0 & 0 & -10 \end{array}\right) \tag{0.0.17}$$

The rank of the matrix is 3. So the given lines are skew.

b) From (0.0.12) we have

$$\mathbf{M}^\top \mathbf{M} = \begin{pmatrix} 2 & -1 & 1 \\ 3 & -5 & 2 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ -1 & -5 \\ 1 & 2 \end{pmatrix} \tag{0.0.18}$$

$$= \begin{pmatrix} 6 & 13 \\ 13 & 38 \end{pmatrix} \tag{0.0.19}$$

$$\mathbf{M}\mathbf{M}^\top = \begin{pmatrix} 2 & 3 \\ -1 & -5 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 & 1 \\ 3 & -5 & 2 \end{pmatrix} \tag{0.0.20}$$

$$= \begin{pmatrix} 13 & -17 & 8 \\ -17 & 26 & -11 \\ 8 & -11 & 5 \end{pmatrix} \tag{0.0.21}$$

We perform the eigen decompositions for the matrics (0.0.21), (0.0.19) and write them in the form

$$\mathbf{M}\mathbf{M}^\top = \mathbf{P_1}\mathbf{D_1}\mathbf{P_1}^\top \tag{0.0.22}$$

$$\mathbf{M}^\top\mathbf{M} = \mathbf{P_2}\mathbf{D_2}\mathbf{P_2}^\top \tag{0.0.23}$$

The characteristic polynomial of the matrix $\mathbf{M}\mathbf{M}^\top$ is given by,

$$\text{char}\left(\mathbf{M}\mathbf{M}^\top\right) = \begin{vmatrix} 13-x & -17 & 8 \\ -17 & 26-x & -11 \\ 8 & -11 & 5-x \end{vmatrix} \tag{0.0.24}$$

$$= -x^3 + 44x^2 - 59x \tag{0.0.25}$$

Thus, the eigenvalues are given by

$$\lambda_1 = 22 + 5\sqrt{17},\ \lambda_2 = 22 - 5\sqrt{17},\ \lambda_3 = 0 \tag{0.0.26}$$

From the augmented matrix formed from the eigen value - eigen vector equation we get, the normalized eigen vectors as

$$\mathbf{p_1} = \frac{\sqrt{5}}{\sqrt{68 - 6\sqrt{17}}}\begin{pmatrix} \frac{12-\sqrt{17}}{5} \\ \frac{1-3\sqrt{17}}{5} \\ 1 \end{pmatrix} \tag{0.0.27}$$

$$\mathbf{p_2} = \frac{\sqrt{5}}{\sqrt{68 + 6\sqrt{17}}}\begin{pmatrix} \frac{12+\sqrt{17}}{5} \\ \frac{1+3\sqrt{17}}{5} \\ 1 \end{pmatrix} \tag{0.0.28}$$

$$\mathbf{p_3} = \frac{1}{\sqrt{59}}\begin{pmatrix} -3 \\ 1 \\ 7 \end{pmatrix} \tag{0.0.29}$$

where $\mathbf{p_1}, \mathbf{p_2}, \mathbf{p_3}$ corresponds to the eigen values $\lambda_1, \lambda_2, \lambda_3$ respectively. Using (0.0.22), we get

$$\mathbf{P_1} = \begin{pmatrix} \frac{12-\sqrt{17}}{\sqrt{5}\sqrt{68-6\sqrt{17}}} & \frac{12+\sqrt{17}}{\sqrt{5}\sqrt{68+6\sqrt{17}}} & -\frac{3}{\sqrt{59}} \\ \frac{1-3\sqrt{17}}{\sqrt{5}\sqrt{68-6\sqrt{17}}} & \frac{1+3\sqrt{17}}{\sqrt{5}\sqrt{68+6\sqrt{17}}} & \frac{1}{\sqrt{59}} \\ \frac{\sqrt{5}}{\sqrt{68-6\sqrt{17}}} & \frac{\sqrt{5}}{\sqrt{68+6\sqrt{17}}} & \frac{7}{\sqrt{59}} \end{pmatrix} \tag{0.0.30}$$

$$\mathbf{D_1} = \begin{pmatrix} 22+5\sqrt{17} & 0 & 0 \\ 0 & 22-5\sqrt{17} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{0.0.31}$$

For $\mathbf{M}^\top\mathbf{M}$, the characteristic polynomial is

$$\text{char}\left(\mathbf{M}^\top\mathbf{M}\right) = \begin{vmatrix} 6-x & 13 \\ 13 & 38-x \end{vmatrix} \tag{0.0.32}$$

$$= x^2 - 44x + 59 \tag{0.0.33}$$

Thus, the eigenvalues are given by

$$\lambda_1 = 22 + 5\sqrt{17},\ \lambda_2 = 22 - 5\sqrt{17} \tag{0.0.34}$$

From the augmented matrix formed from the eigen value - eigen vector equation we get, the normalized eigen vectors as

$$\mathbf{p_1} = \frac{13}{\sqrt{850 - 160\sqrt{17}}}\begin{pmatrix} \frac{-16+5\sqrt{17}}{13} \\ 1 \end{pmatrix} \tag{0.0.35}$$

$$\mathbf{p_2} = \frac{13}{\sqrt{850 + 160\sqrt{17}}}\begin{pmatrix} \frac{-16-5\sqrt{17}}{13} \\ 1 \end{pmatrix} \tag{0.0.36}$$

where $\mathbf{p_1}, \mathbf{p_2}$ corresponds to the eigen values λ_1, λ_2 respectively. Using (0.0.23), we get

$$\mathbf{P_2} = \begin{pmatrix} \frac{-16-5\sqrt{17}}{\sqrt{850+160\sqrt{17}}} & \frac{13}{\sqrt{850-160\sqrt{17}}} \\ \frac{13}{\sqrt{850+160\sqrt{17}}} & \frac{-16+5\sqrt{17}}{\sqrt{850-160\sqrt{17}}} \end{pmatrix} \tag{0.0.37}$$

$$\mathbf{D_2} = \begin{pmatrix} 22-5\sqrt{17} & 0 \\ 0 & 22+5\sqrt{17} \end{pmatrix} \tag{0.0.38}$$

Therefore, from (0.0.8) we have

$$\mathbf{U} = \mathbf{P_1} \tag{0.0.39}$$

$$\mathbf{V} = \mathbf{P_2} \tag{0.0.40}$$

$$\mathbf{\Sigma} = \begin{pmatrix} \sqrt{22+5\sqrt{17}} & 0 \\ 0 & \sqrt{22-5\sqrt{17}} \\ 0 & 0 \end{pmatrix} \tag{0.0.41}$$

and substituting into (0.0.4), we get

$$\boldsymbol{\lambda} = \begin{pmatrix} \frac{25}{59} \\ \\ -\frac{7}{59} \end{pmatrix} \tag{0.0.42}$$

The minimum distance between the lines is given by,

$$\|\mathbf{B}-\mathbf{A}\| = \left\|\frac{1}{59}\begin{pmatrix}30\\-10\\-70\end{pmatrix}\right\| \tag{0.0.43}$$

$$= \frac{\sqrt{30^2+10^2+70^2}}{59} \tag{0.0.44}$$

$$= \frac{10}{\sqrt{59}} \tag{0.0.45}$$

The shortest distance between the given lines is $\frac{10}{\sqrt{59}}$ units.

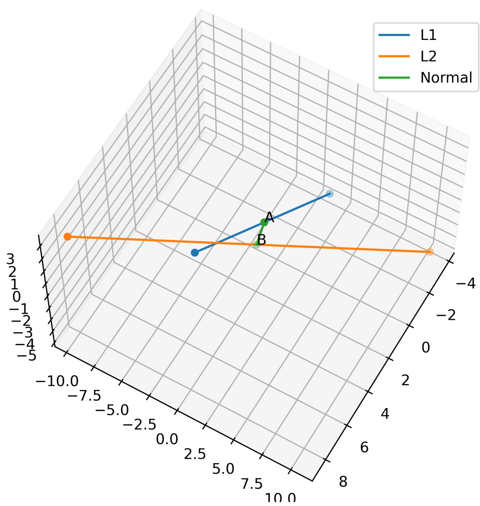

Fig. 1: AB is the required shortest distance.