Circles

1 11^{th} Maths - Chapter 10

This is Problem-3 from Exercise 10.4

1. Find the centre of a circle passing though the points $(6, -6), (3, -7)$ and $(3, 3)$.
   Solution: The equation of the crcle is given by

$$\|\mathbf{x}\|^2 + 2\mathbf{x}^\top \mathbf{u} + f = 0 \tag{1}$$

where

$$\mathbf{u} = -\mathbf{c} \text{ and} \tag{2}$$
$$f = \|\mathbf{c}\|^2 - r^2 \tag{3}$$

Given points are

$$\mathbf{x_1} = \begin{pmatrix} 6 \\ -6 \end{pmatrix}, \mathbf{x_2} = \begin{pmatrix} 3 \\ -7 \end{pmatrix}, \mathbf{x_3} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} \tag{4}$$

Substituting points from (4) into (1)

$$\left(6^2 + (-6)^2\right) + 2\begin{pmatrix} 6 & -6 \end{pmatrix}\mathbf{u} + f = 0 \tag{5}$$
$$\implies 2\begin{pmatrix} 6 & -6 \end{pmatrix}\mathbf{u} + f = -72 \tag{6}$$
$$\left(3^2 + (-7)^2\right) + 2\begin{pmatrix} 3 & -7 \end{pmatrix}\mathbf{u} + f = 0 \tag{7}$$
$$\implies 2\begin{pmatrix} 3 & -7 \end{pmatrix}\mathbf{u} + f = -58 \tag{8}$$
$$\left(3^2 + 3^2\right) + 2\begin{pmatrix} 3 & 3 \end{pmatrix}\mathbf{u} + f = 0 \tag{9}$$
$$\implies 2\begin{pmatrix} 3 & 3 \end{pmatrix}\mathbf{u} + f = -18 \tag{10}$$

Representing the above system of equations in matrix form

$$\begin{pmatrix} 6 & -14 & 1 \\ 12 & -12 & 1 \\ 6 & 6 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ f \end{pmatrix} = \begin{pmatrix} -58 \\ -72 \\ -18 \end{pmatrix} \tag{11}$$

The augmented matrix is expressed as

$$\left(\begin{array}{ccc|c} 6 & -14 & 1 & -58 \\ 12 & -12 & 1 & -72 \\ 6 & 6 & 1 & -18 \end{array}\right) \tag{12}$$

Performing sequence of row operations to transform into an Echelon form

$$\xleftrightarrow{R_2 \to R_2 - 2R_1} \left(\begin{array}{ccc|c} 6 & -14 & 1 & -58 \\ 0 & 16 & -1 & 44 \\ 6 & 6 & 1 & -18 \end{array}\right) \tag{13}$$

$$\xleftrightarrow{R_3 \to R_3 - R_1} \left(\begin{array}{ccc|c} 6 & -14 & 1 & -58 \\ 0 & 16 & -1 & 44 \\ 0 & 20 & 0 & 40 \end{array}\right) \tag{14}$$

$$\xleftrightarrow{R_3 \to R_3 - \frac{20}{16} R_2} \left(\begin{array}{ccc|c} 6 & -14 & 1 & -58 \\ 0 & 16 & -1 & 44 \\ 0 & 0 & \frac{20}{16} & -15 \end{array}\right) \tag{15}$$

$$\xleftrightarrow[R_2 \to \frac{1}{16} R_2, R_3 \to \frac{16}{20} R_3]{R_1 \to \frac{1}{6} R_1} \left(\begin{array}{ccc|c} 1 & -\frac{14}{6} & \frac{1}{6} & -\frac{58}{6} \\ 0 & 1 & -\frac{1}{16} & \frac{44}{16} \\ 0 & 0 & 1 & -12 \end{array}\right) \tag{16}$$

$$\xleftrightarrow[R_2 \to R_2 + \frac{1}{16} R_3]{R_1 \to R_1 - \frac{1}{6} R_3} \left(\begin{array}{ccc|c} 1 & -\frac{14}{6} & 0 & -\frac{46}{6} \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -12 \end{array}\right) \tag{17}$$

$$\xleftrightarrow{R_1 \to R_1 + \frac{14}{6} R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -12 \end{array}\right) \tag{18}$$

So, from (18)

$$\mathbf{u} = \begin{pmatrix} -3 \\ 2 \end{pmatrix} \tag{19}$$

$$f = -12 \tag{20}$$

Since $\mathbf{u} = -\mathbf{c}$,

$$\mathbf{c} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \tag{21}$$

$$(3) \implies r^2 = \left(3^2 + (-2)^2\right) + 12 \tag{22}$$

$$r = 5 \tag{23}$$

Therefore, the equation of the circle is

$$\left\| \mathbf{x} - \begin{pmatrix} 3 \\ -2 \end{pmatrix} \right\| = 5 \tag{24}$$

The relevant diagram is shown in Figure 1

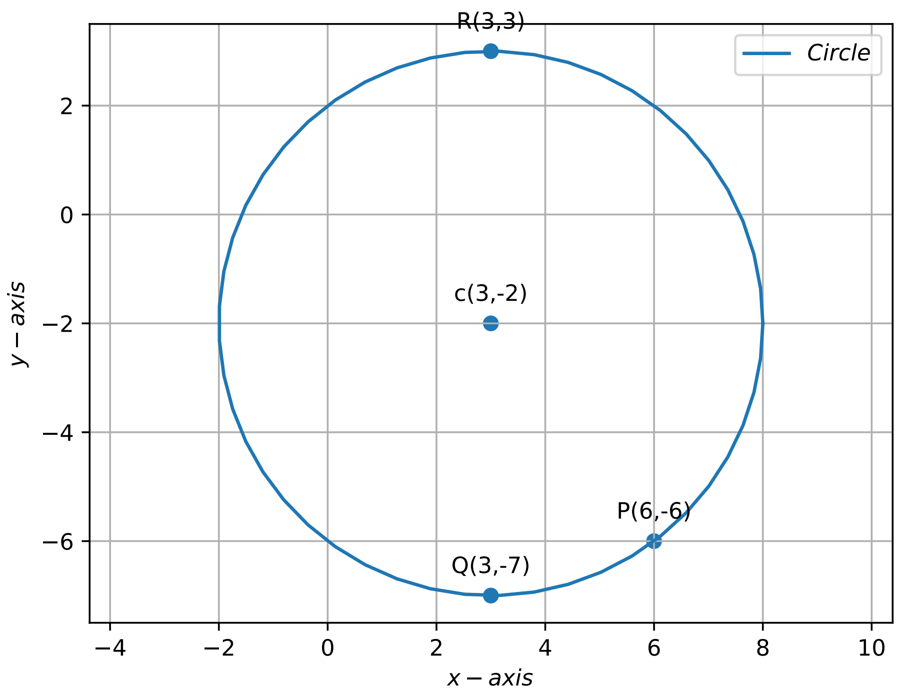

Figure 1