Matrix Theory (EE5609) Challenging Problem 3

Arkadipta De
MTech Artificial Intelligence
AI20MTECH14002

Abstract—**This document proves the Cayley-Hamilton theorem.**

Download latex codes from

https://github.com/Arko98/EE5609/tree/master/Challenge_2

1 Problem

Prove Cayley-Hamilton Theorem.

2 Theorem Statement

Every Square matrix satisfies its own characteristic equation.
Let, $\mathbf{A}$ be a square matrix of order n and $p(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I})$ be the characteristic equation of A in λ and $\mathbf{I}$ is the identity matrix of order n which is the same order of the matrix $\mathbf{A}$ then the charcteristic equation of $\mathbf{A}$ is given by,

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0 \tag{2.0.1}$$

$$\implies \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} & \cdots & a_{nn} \\ a_{21} & a_{22} - \lambda & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0 \tag{2.0.2}$$

$$\implies a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_n\lambda^n = 0 \tag{2.0.3}$$

From Cayley-Hamilton theorem, the matrix $\mathbf{A}$ will satisfy (2.0.3),

$$a_0 + a_1\mathbf{A} + a_2\mathbf{A^2} + \cdots + a_n\mathbf{A^n} = 0 \tag{2.0.4}$$

3 Proof

If adj($\mathbf{A}$) is the adjoint matrix of the matrix $\mathbf{A}$ of order n which is the transpose of the cofactors of the matrix $\mathbf{A}$ then,

$$\mathbf{A}(adj(\mathbf{A})) = \det(\mathbf{A}) \tag{3.0.1}$$

Replacing $\mathbf{A}$ with $(\mathbf{A} - \lambda\mathbf{I})$ in (3.0.1) we obtain,

$$(\mathbf{A} - \lambda\mathbf{I})adj(\mathbf{A} - \lambda\mathbf{I}) = \det(\mathbf{A} - \lambda\mathbf{I})\mathbf{I} \tag{3.0.2}$$

As $\mathbf{A}$ has a polynomial of degree n for variable λ, then $adj(\mathbf{A}-\lambda\mathbf{I})$ has a polynomial of degree $n-1$ for variable λ. Expanding $adj(\mathbf{A} - \lambda\mathbf{I})$ with coefficients $b_0, b_1, \cdots, b_n - 1$ we get,

$$adj(\mathbf{A} - \lambda\mathbf{I}) = b_0 + b_1\lambda + b_2\lambda^2 + \cdots + b_{n-1}\lambda^{n-1} \tag{3.0.3}$$

Hence, from (3.0.2), putting the value of $adj(\mathbf{A}-\lambda\mathbf{I})$ we get,

$$(\mathbf{A} - \lambda\mathbf{I})adj(\mathbf{A} - \lambda\mathbf{I}) \tag{3.0.4}$$

$$= (\mathbf{A} - \lambda\mathbf{I})(b_0 + b_1\lambda + b_2\lambda^2 + \cdots + b_{n-1}\lambda^{n-1}) \tag{3.0.5}$$

$$= \mathbf{A}b_0 + \mathbf{A}b_1\lambda + \cdots + \mathbf{A}b_{n-1}\lambda^{n-1} - b_0\lambda \tag{3.0.6}$$

$$- b_1\lambda^2 - \cdots - b_{n-1}\lambda^n \tag{3.0.7}$$

$$= \mathbf{A}b_0 + \lambda(\mathbf{A}b_1 - b_0) + \lambda^2(\mathbf{A}b_2 - b_1) + \cdots - b_{n-1}\lambda^n \tag{3.0.8}$$

Putting values from (2.0.3) and (3.0.8) in (3.0.2),

$$\mathbf{A}b_0 + \lambda(\mathbf{A}b_1 - b_0) + \lambda^2(\mathbf{A}b_2 - b_1) + \cdots - b_{n-1}\lambda^n \tag{3.0.9}$$

$$= a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_n\lambda^n \tag{3.0.10}$$

Comparing coefficients of equal powers of λ in both sides of (3.0.10),

$$\mathbf{A}b_0 = a_0 \tag{3.0.11}$$

$$\mathbf{A}b_1 - b_0 = a_1 \tag{3.0.12}$$

$$\vdots$$

$$\mathbf{A}b_{n-1} - b_{n-2} = a_{n-1} \tag{3.0.13}$$

$$-b_{n-1} = a_n \tag{3.0.14}$$

Now, multiplying both sides of (3.0.11) by $\mathbf{I}$, both sides of (3.0.12) by $\mathbf{A}$ and so on upto both sides of

(3.0.13) by $\mathbf{A^{n-1}}$ and both sides of (3.0.14) by $\mathbf{A^n}$ we obtain the following,

$$\mathbf{A}b_0 = a_0\mathbf{I} \tag{3.0.15}$$

$$\mathbf{A^2}b_1 - \mathbf{A}b_0 = \mathbf{A}a_1 \tag{3.0.16}$$

$$\vdots$$

$$\mathbf{A^n}b_n - 1 - \mathbf{A^{n-1}}b_n - 2 = a_n - 1\mathbf{A^{n-1}} \tag{3.0.17}$$

$$-\mathbf{A^n}b_n - 1 = a_n\mathbf{A^n} \tag{3.0.18}$$

Adding the equations,

$$\mathbf{A}b_0 + \cdots - \mathbf{A^n}b_n - 1 = a_0 + a_1\mathbf{A} + \cdots + a_n\mathbf{A^n} \tag{3.0.19}$$

$$\implies a_0 + a_1\mathbf{A} + a_2\mathbf{A^2} + \cdots + a_n\mathbf{A^n} = 0 \tag{3.0.20}$$

(3.0.20) together with (2.0.3) proves the statement of Cayley-Hamilton theorem.