

# Conic Sections - Parabola

## 1 11<sup>th</sup> Maths - Chapter 11

This is Problem-1 from Exercise 11.2

1. Find the coordinates of the focus, equation of the directrix, axis and length of the latus rectum of a parabola whose equation is given by  $y^2 = 12x$ .

**Solution:** The given equation of the parabola can be rearranged as

$$y^2 - 12x = 0 \quad (1)$$

The above equation can be equated to the generic equation of conic sections

$$g(\mathbf{x}) = \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (2)$$

Comparing coefficients of both equations (1) and (2)

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (3)$$

$$\mathbf{u} = - \begin{pmatrix} 6 \\ 0 \end{pmatrix} \quad (4)$$

$$f = 0 \quad (5)$$

- (a) From equation (3), since  $\mathbf{V}$  is already diagonalized, the Eigen values  $\lambda_1$  and  $\lambda_2$  are given as

$$\lambda_1 = 0 \quad (6)$$

$$\lambda_2 = 1 \quad (7)$$

The Eigen vector  $\mathbf{p}_1$  corresponding to Eigen value  $\lambda_1$  is computed as shown below

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (8)$$

$$\mathbf{V} - \lambda_1 \mathbf{I} = \begin{pmatrix} 0 - \lambda_1 & 0 \\ 0 & 1 - \lambda_1 \end{pmatrix} \quad (9)$$

Substituting value of  $\lambda_1$  from (6) in (9)

$$(9) \implies \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (10)$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (11)$$

$x_1$  is free variable and  $x_2 = 0$ .

$$\therefore \mathbf{p}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (12)$$

$$\mathbf{n} = \sqrt{\lambda_2} \mathbf{p}_1 \quad (13)$$

$$= \sqrt{1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (14)$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (15)$$

$$c = \frac{\|\mathbf{u}^2\| - \lambda_2 f}{2\mathbf{u}^\top \mathbf{n}} \quad (16)$$

Substituting values of  $\mathbf{u}$ ,  $\mathbf{n}$ ,  $\lambda_2$  and  $f$  in (16)

$$c = \frac{6^2 - 1(0)}{-2 \begin{pmatrix} 6 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}} = -3 \quad (17)$$

$$(18)$$

The focus  $\mathbf{F}$  of parabola is expressed as

$$\mathbf{F} = \frac{ce^2\mathbf{n} - \mathbf{u}}{\lambda_2} \quad (19)$$

$$= \frac{-3(1)^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 6 \\ 0 \end{pmatrix}}{1} \quad (20)$$

$$= \begin{pmatrix} 3 \\ 0 \end{pmatrix} \quad (21)$$

(b) Equation of directrix is given as

$$\mathbf{n}^\top \mathbf{x} = c \quad (22)$$

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = -3 \quad (23)$$

(c) The equation for the axis of parabola passing through  $\mathbf{F}$  and orthogonal to the directrix is given as

$$\mathbf{m}^\top (\mathbf{x} - \mathbf{F}) = 0 \quad (24)$$

where  $\mathbf{m}$  is the normal vector to the axis and also the slope of the directrix.

$$\because \mathbf{n} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (25)$$

$$(24) \implies \begin{pmatrix} 0 & 1 \end{pmatrix} \left( \mathbf{x} - \begin{pmatrix} 3 \\ 0 \end{pmatrix} \right) \quad (26)$$

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 0 \quad (27)$$

(d) The latus rectum of a parabola is given by

$$l = \frac{\eta}{\lambda_2} \quad (28)$$

$$= \frac{2\mathbf{u}^\top \mathbf{p}_1}{\lambda_2} \quad (29)$$

$$= \frac{2 \begin{pmatrix} 6 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}}{1} \quad (30)$$

$$= 12 \text{ units} \quad (31)$$

The relevant diagram is shown in Figure 1

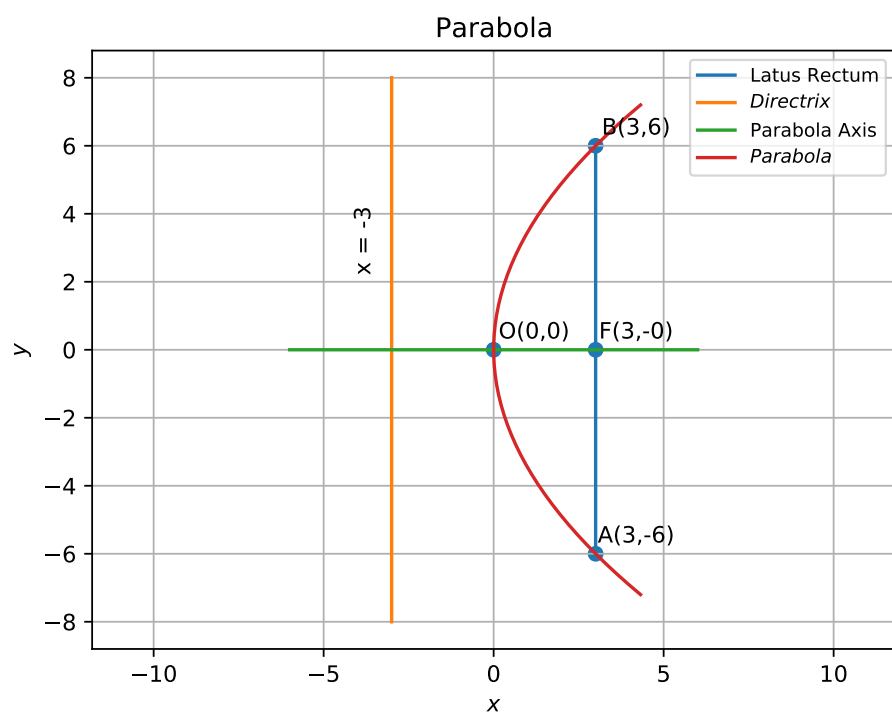


Figure 1