Excercise 10.2

Q2.In fig 1, if TP and TQ are two tangents to a circle with centre O so that $\angle POQ = 110°$ then $\angle PTQ$ is equal to.
Solution: Let the output angle be ϕ. The input parameters are given as

Input Parameters	Value	Description
$\mathbf{O}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	Centre of the circle
r	1cm	radius of the circle
θ	110°	$\angle POQ$

Table 1:

Any point $\mathbf{X}$ on the circle is given as

$$\mathbf{X} = \mathbf{O} + r \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \tag{1}$$

So points $\mathbf{P}$ and $\mathbf{Q}$ can be calculated as

$$\mathbf{P} = \mathbf{O} + \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \tag{2}$$

$$\mathbf{Q} = \mathbf{e}_1 \tag{3}$$

For tangent TP

$$\mathbf{n}_1 = \mathbf{P} - \mathbf{O} \tag{4}$$

$$= \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \begin{pmatrix} 1 \\ \tan\theta \end{pmatrix} \tag{5}$$

$$\mathbf{m}_1 = \begin{pmatrix} 1 \\ -\cot\theta \end{pmatrix} \tag{6}$$

For tangent TQ

$$\mathbf{n}_2 = \mathbf{e}_1 - \mathbf{O} \tag{7}$$

$$= \mathbf{e}_1 \tag{8}$$

$$\mathbf{m}_2 = \mathbf{e}_2 \tag{9}$$

The equation of TP is given as

$$\mathbf{n}_1^\top (\mathbf{x} - \mathbf{P}) = 0 \tag{10}$$

$$\mathbf{n}_1^\top \left(\mathbf{x} - \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}\right) = 0 \tag{11}$$

$$\begin{pmatrix} \cos\theta & \sin\theta \end{pmatrix} \mathbf{x} = 1 \tag{12}$$

The equation of TQ is given as

$$\mathbf{n}_2^\top (\mathbf{x} - \mathbf{e}_1) = 0 \tag{13}$$

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 1 \tag{14}$$

The tangent point can be calculated by solving (12) and (14)

$$\begin{pmatrix} \cos\theta & \sin\theta \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{15}$$

$$\implies \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ \tan\frac{\theta}{2} \end{pmatrix} \tag{16}$$

Now, $\mathbf{T}$ =(16), since it is the intersection of TP and TQ. Hence, it is given as

$$\mathbf{T} = \begin{pmatrix} 1 \\ \tan 55^\circ \end{pmatrix} = \begin{pmatrix} 1 \\ 1.428 \end{pmatrix} \tag{17}$$

The angle between two lines with slope $\mathbf{m}_1$ and $\mathbf{m}_2$ is given as

$$\cos\phi = \frac{\mathbf{m}_1^\top \mathbf{m}_2}{\|\mathbf{m}_1\| \, \|\mathbf{m}_2\|} \tag{18}$$

$$= \frac{\begin{pmatrix} 1 & -\cot\theta \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}}{(\csc\theta)(1)} \tag{19}$$

$$= -\cos\theta \tag{20}$$

$$\implies \cos\phi = -\cos\theta \tag{21}$$

Hence,

$$\phi = \cos^{-1}\left(\cos\left(180^\circ - \theta\right)\right) \tag{22}$$

$$= 180^\circ - \theta = 70^\circ \tag{23}$$

Hence, $\angle PTQ = 70^\circ$. See Fig 1

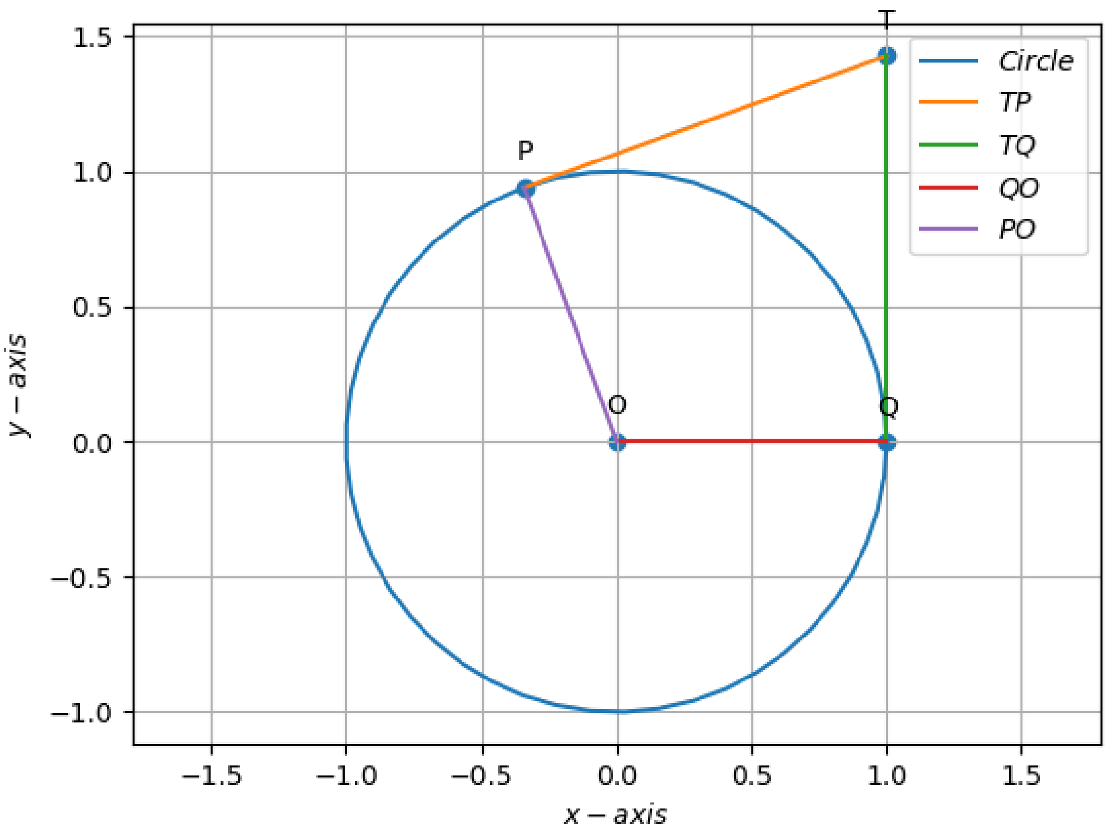
T
P
O
Q
Circle
TP
TQ
QO
PO
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
$y-axis$
$x-axis$

Figure 1: