Excercise 11.2.4

Find the coordinates of the focus, axis of the parabola, the equation of the directrix and the length of the latus rectum of a parabola whose equation is given by $x^2 = -16y$.

Solution: The given equation of the parabola can be written as

$$x^2 + 16y = 0 \tag{1}$$

The genereal equation for conic section is

$$g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{2}$$

Comparaing both equations (1) and (2) we get,

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{3}$$

$$\mathbf{u} = \begin{pmatrix} 0 \\ 8 \end{pmatrix} \tag{4}$$

$$f = 0 \tag{5}$$

1. As $\mathbf{V}$ matrix is already diagonalized (3), the Eigen values λ_1 and λ_2 are given as

$$\lambda_1 = 1 \tag{6}$$

$$\lambda_2 = 0 \tag{7}$$

Eigen vector matrix $\mathbf{P}$ is identical the eigen vector $\mathbf{P}_2$ by eigen value λ_2 is

$$\mathbf{p}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{8}$$

$$\mathbf{n} = \sqrt{\lambda_1}\mathbf{p}_2 \tag{9}$$

$$= \sqrt{1}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{10}$$

$$= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{11}$$

So,

$$\frac{\|\mathbf{u}\|^2 - \lambda_1 f}{2\mathbf{u}^\top \mathbf{n}} = c \tag{12}$$

Substituting $\mathbf{u}, \mathbf{n}, \lambda_1$ and f values in (12) we get

$$c = \frac{8^2 - 1\,(0)}{2\begin{pmatrix}0 & 8\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}} = 4 \tag{13}$$

The focus $\mathbf{F}$ of parabola is

$$\mathbf{F} = \frac{ce^2\mathbf{n} - \mathbf{u}}{\lambda_1} \tag{14}$$

$$= \frac{4\,(1)^2\begin{pmatrix}0\\1\end{pmatrix} - \begin{pmatrix}0\\8\end{pmatrix}}{1} \tag{15}$$

$$= \begin{pmatrix}0\\-4\end{pmatrix} \tag{16}$$

2. Equation of directrix is given as

$$\mathbf{n}^\top \mathbf{x} = c \tag{17}$$

$$\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} = 4 \tag{18}$$

$$\mathbf{x} = 4 \tag{19}$$

3. Equation for tha axis of parabola is

$$\mathbf{m}^\top (\mathbf{x} - \mathbf{F}) = 0 \tag{20}$$

where $\mathbf{m}$ is the normal vector to the axis and also the slope of the directrix

$$\mathbf{n} = \begin{pmatrix}0\\1\end{pmatrix} \tag{21}$$

$$\mathbf{m} = \begin{pmatrix}1\\0\end{pmatrix} \tag{22}$$

Substituting in (20)

$$\begin{pmatrix}1 & 0\end{pmatrix}\left(\mathbf{x}-\begin{pmatrix}0\\-4\end{pmatrix}\right)=0 \tag{23}$$

$$\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x}=0 \tag{24}$$

$$\mathbf{x}=0 \tag{25}$$

4. Latus rectum of parabola is

$$l=\frac{\eta}{\lambda_1} \tag{26}$$

$$=\frac{2\mathbf{u}^\top\mathbf{p}_2}{\lambda_1} \tag{27}$$

$$=\frac{2\begin{pmatrix}0 & 8\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}}{1} \tag{28}$$

$$=16 \text{ units} \tag{29}$$

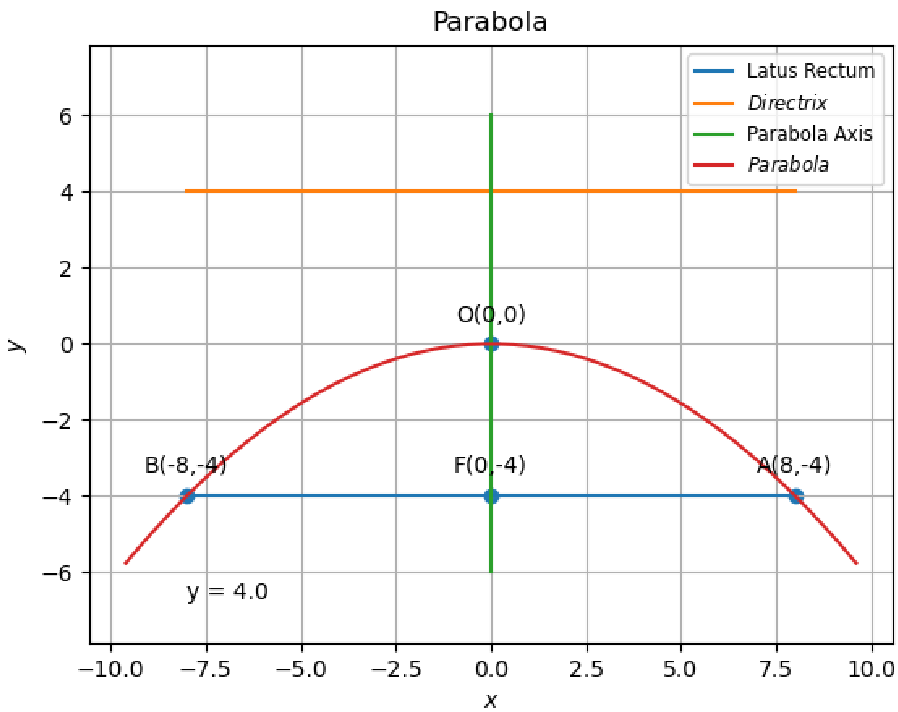
Parabola
Latus Rectum
Directrix
Parabola Axis
Parabola
O(0,0)
B(-8,-4)
F(0,-4)
A(8,-4)
y = 4.0
6
4
2
0
−2
−4
−6
−10.0
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0
x
y

Figure 1: