Conic Assignment

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***Abstract*—This document contains the solution to Question 14 of Exercise 4 in Chapter 11 of the class 11 NCERT textbook.**

1) Find the equation of the hyperbola eccentricity is $e = \frac{4}{3}$ and whose vertices are

$$\mathbf{P_1} = \begin{pmatrix} 7 \\ 0 \end{pmatrix}, \ \mathbf{P_2} = \begin{pmatrix} -7 \\ 0 \end{pmatrix} \tag{1}$$

Solution: Let the equation of the conic with focus $\mathbf{F}$, directrix $\mathbf{n}^\top \mathbf{x} = c$ and eccentricity e be

$$\mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{2}$$

where

$$\mathbf{V} \triangleq \|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{n}\mathbf{n}^\top \tag{3}$$

$$\mathbf{u} \triangleq ce^2 \mathbf{n} - \|\mathbf{n}\|^2 \mathbf{F} \tag{4}$$

$$f \triangleq \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2 e^2 \tag{5}$$

The major axis of a conic is the chord which passes through the vertices of the conic. The direction vector of the major axis in this case is

$$\mathbf{P_2} - \mathbf{P_1} = \begin{pmatrix} 14 \\ 0 \end{pmatrix} \tag{6}$$

Hence, the normal to the major axis $P_1 P_2$ is

$$\mathbf{n_M} = \mathbf{e_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{7}$$

Thus, the equation of the major axis is

$$\mathbf{e_2}^\top \mathbf{x} = \mathbf{e_2}^\top \begin{pmatrix} 7 \\ 0 \end{pmatrix} = 0 \tag{8}$$

which is clearly the x-axis.
Since the conic is a hyperbola whose vertices are given by (1) and the major axis is the x-axis, the directrix is parallel to the y-axis. Hence,

$$\mathbf{n} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{9}$$

Thus,

$$\mathbf{V} = \begin{pmatrix} 1 - e^2 & 0 \\ 0 & 1 \end{pmatrix} \tag{10}$$

$$\mathbf{u} = ce^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \mathbf{F} \tag{11}$$

$$f = \|\mathbf{F}\|^2 - c^2 e^2 \tag{12}$$

Substituting $\mathbf{P_1}$ and $\mathbf{P_2}$ in (2),

$$\mathbf{P_1}^\top \mathbf{V} \mathbf{P_1} + 2\mathbf{u}^\top \mathbf{P_1} + f = 0 \tag{13}$$

$$\mathbf{P_2}^\top \mathbf{V} \mathbf{P_2} + 2\mathbf{u}^\top \mathbf{P_2} + f = 0 \tag{14}$$

Subtracting (14) from (13), and noting that $\mathbf{P_2} = -\mathbf{P_1}$,

$$\mathbf{u}^\top \mathbf{P_1} = 0 \tag{15}$$

Hence, from (1), we see that $\mathbf{u}$ lies on the y-axis. The general expression of the centre of a conic is given by

$$\mathbf{c} = -\mathbf{V}^{-1} \mathbf{u} \tag{16}$$

$$= \frac{1}{e^2 - 1} \begin{pmatrix} 1 & 0 \\ 0 & 1 - e^2 \end{pmatrix} \mathbf{u} \tag{17}$$

We let $\mathbf{u} \triangleq \begin{pmatrix} 0 \\ u \end{pmatrix}$ and obtain from (17)

$$\mathbf{c} = \begin{pmatrix} 0 \\ -u \end{pmatrix} = -\mathbf{u} \tag{18}$$

Since the major axis of the hyperbola is the x-axis, we see that $\mathbf{c}$ lies on the x-axis. Thus, (18) implies $\mathbf{c} = -\mathbf{u} = \mathbf{0}$. Thus, from (11),

$$\mathbf{F} = \begin{pmatrix} ce^2 \\ 0 \end{pmatrix} \tag{19}$$

and so,

$$f = c^2 e^2 \left(e^2 - 1\right) \tag{20}$$

Putting $\mathbf{x} = \mathbf{P_1}$ or $\mathbf{x} = \mathbf{P_2}$ in (2) and using (19) and (20),

$$\begin{pmatrix} \pm 7 & 0 \end{pmatrix} \begin{pmatrix} 1-e^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \pm 7 \end{pmatrix} + f = 0 \tag{21}$$

$$\implies 49e^2 - f = 49 \tag{22}$$

Since $e = \frac{4}{3}$, (22) implies

$$f = 49\left(e^2 - 1\right) = \frac{343}{9} \tag{23}$$

Therefore, the equation of the conic is

$$\mathbf{x}^\top \begin{pmatrix} -\frac{7}{9} & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + \frac{343}{9} = 0 \tag{24}$$

The situation is illustrated in Fig. 1.

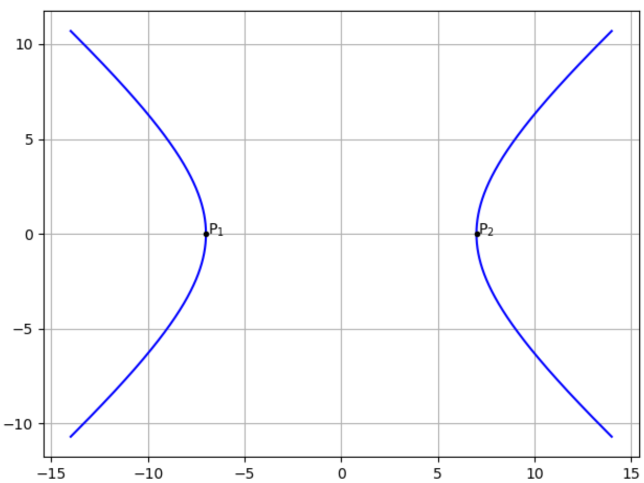

Fig. 1: Locus of the required hyperbola.