

Question: 12.11.2.15

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1 PROBLEM

Find the shortest distance between the lines $\frac{x+1}{-6} = \frac{y+1}{1}$ and $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$

2 SOLUTION

The lines l_1 and l_2 in vector form can be written as

$$\mathbf{x} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 7 \\ -6 \\ 1 \end{pmatrix} \quad (2.0.1)$$

$$\mathbf{x} = \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \quad (2.0.2)$$

$$\mathbf{x}_1 = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}, \mathbf{x}_2 = \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix}, \mathbf{m}_1 = \begin{pmatrix} 7 \\ -6 \\ 1 \end{pmatrix}, \mathbf{m}_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \quad (2.0.3)$$

We first check whether the given lines are skew. The lines

$$\mathbf{x} = \mathbf{x}_1 + \lambda_1 \mathbf{m}_1, \mathbf{x} = \mathbf{x}_2 + \lambda_2 \mathbf{m}_2 \quad (2.0.4)$$

intersect if

$$\mathbf{M}\lambda = \mathbf{x}_2 - \mathbf{x}_1 \quad (2.0.5)$$

$$\mathbf{M} \triangleq (\mathbf{m}_1 \quad \mathbf{m}_2) \quad (2.0.6)$$

$$\lambda \triangleq \begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} \quad (2.0.7)$$

$$(2.0.8)$$

Here we have,

$$\mathbf{M} = \begin{pmatrix} 7 & 1 \\ -6 & -2 \\ 1 & 1 \end{pmatrix} \quad (2.0.9)$$

$$\mathbf{x}_2 - \mathbf{x}_1 = \begin{pmatrix} 4 \\ 6 \\ 8 \end{pmatrix} \quad (2.0.10)$$

We check whether the equation (2.0.11) has a solution

$$\begin{pmatrix} 7 & 1 \\ -6 & -2 \\ 1 & 1 \end{pmatrix} \lambda = \begin{pmatrix} 4 \\ 6 \\ 8 \end{pmatrix} \quad (2.0.11)$$

the augmented matrix is given by,

$$\left(\begin{array}{cc|c} 7 & 1 & 4 \\ -6 & -2 & 6 \\ 1 & 1 & 8 \end{array} \right) \quad (2.0.12)$$

$$\begin{array}{l} \xleftrightarrow{R_2 \leftarrow R_2 + \frac{6}{7}R_1} \\ \xleftrightarrow{R_3 \leftarrow R_3 - \frac{1}{7}R_1} \end{array} \quad (2.0.13)$$

$$\left(\begin{array}{cc|c} 7 & 1 & 4 \\ 0 & -\frac{8}{7} & \frac{66}{7} \\ 0 & \frac{6}{7} & -\frac{52}{7} \end{array} \right) \quad (2.0.14)$$

$$\xleftrightarrow{R_3 \leftarrow R_3 + \frac{3}{4}R_2} \quad (2.0.15)$$

$$\left(\begin{array}{cc|c} 2 & 3 & 1 \\ 0 & -\frac{7}{2} & \frac{1}{2} \\ 0 & 0 & -\frac{5}{14} \end{array} \right) \quad (2.0.16)$$

The rank of the matrix is 3. So the given lines are skew. The closest points on two skew lines defined by (2.0.4) are given by

$$\mathbf{M}^T \mathbf{M} \lambda = \mathbf{M}^T (\mathbf{x}_2 - \mathbf{x}_1) \quad (2.0.17)$$

$$\begin{pmatrix} 7 & -6 & 1 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 7 & 1 \\ -6 & -2 \\ 1 & 1 \end{pmatrix} \lambda = \begin{pmatrix} 7 & -6 & 1 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 6 \\ 8 \end{pmatrix} \quad (2.0.18)$$

$$\begin{pmatrix} 86 & 20 \\ 20 & 6 \end{pmatrix} \lambda = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (2.0.19)$$

The augmented matrix of the above equation

(2.0.19) is given by,

$$\left(\begin{array}{cc|c} 86 & 20 & 0 \\ 20 & 6 & 0 \end{array} \right) \quad (2.0.20)$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - \frac{10}{43} R_1} \quad (2.0.21)$$

$$\left(\begin{array}{cc|c} 86 & 20 & 0 \\ 0 & \frac{58}{43} & 0 \end{array} \right) \quad (2.0.22)$$

$$\xleftrightarrow{R_1 \leftarrow \frac{1}{86} (R_1 - \frac{430}{29} R_2)} \quad (2.0.23)$$

$$\xleftrightarrow{R_2 \leftarrow \frac{43}{58} R_2} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right) \quad (2.0.24)$$

So, we get

$$\begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (2.0.25)$$

The closest points **A** on line l_1 and **B** on line l_2 are given by,

$$\mathbf{A} = \mathbf{x}_1 + \lambda_1 \mathbf{m}_1 \quad (2.0.26)$$

$$= \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} + 0 \mathbf{m}_1 \quad (2.0.27)$$

$$= \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \quad (2.0.28)$$

$$\mathbf{B} = \mathbf{x}_2 + \lambda_2 \mathbf{m}_2 \quad (2.0.29)$$

$$= \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix} + 0 \mathbf{m}_2 \quad (2.0.30)$$

$$= \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix} \quad (2.0.31)$$

The minimum distance between the lines is given by,

$$\|\mathbf{B} - \mathbf{A}\| = \left\| \begin{pmatrix} 4 \\ 6 \\ 8 \end{pmatrix} \right\| \quad (2.0.32)$$

$$= 2\sqrt{29} \quad (2.0.33)$$

The shortest distance between the given lines is $2\sqrt{29}$ units.

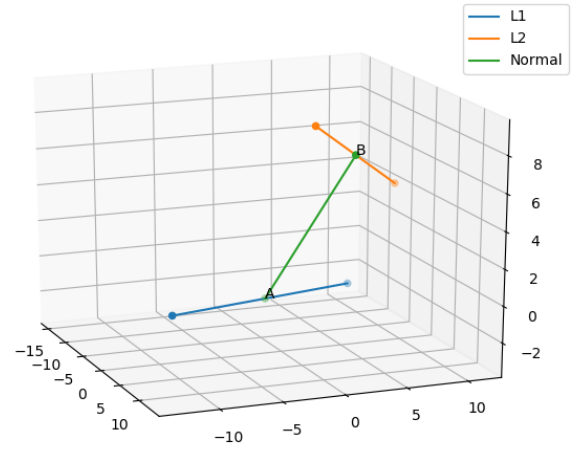


Fig. 0: AB is the required shortest distance.