Quiz 6

S Nithish

Abstract—**This document contains the solution of the question from NCERT 12th standard chapter 10 exercise 10.5 problem 15**

1 Exercise 10.5

1) Prove that $(\mathbf{a}+\mathbf{b})^\top(\mathbf{a}+\mathbf{b}) = ||\mathbf{a}||^2 + ||\mathbf{b}||^2$, if and only if $\mathbf{a}$,$\mathbf{b}$ are perpendicular, given $\mathbf{a} \neq 0, \mathbf{b} \neq 0$.

$$(\mathbf{a}+\mathbf{b})^\top(\mathbf{a}+\mathbf{b}) = \mathbf{a}^\top\mathbf{a} + \mathbf{a}^\top\mathbf{b} + \mathbf{b}^\top\mathbf{a} + \mathbf{b}^\top\mathbf{b} \tag{1.0.1}$$
$$= ||\mathbf{a}||^2 + ||\mathbf{b}||^2 + 2\mathbf{a}^\top\mathbf{b} \tag{1.0.2}$$

If

$$(\mathbf{a}+\mathbf{b})^\top(\mathbf{a}+\mathbf{b}) = ||\mathbf{a}||^2 + ||\mathbf{b}||^2 \tag{1.0.3}$$

Then,

$$(\mathbf{a}+\mathbf{b})^\top(\mathbf{a}+\mathbf{b}) = ||\mathbf{a}||^2 + ||\mathbf{b}||^2 \tag{1.0.4}$$
$$||\mathbf{a}||^2 + ||\mathbf{b}||^2 + 2\mathbf{a}^\top\mathbf{b} = ||\mathbf{a}||^2 + ||\mathbf{b}||^2 \tag{1.0.5}$$
$$2\mathbf{a}^\top\mathbf{b} = 0 \tag{1.0.6}$$

$\mathbf{a}^\top\mathbf{b} = 0$ implies that the two vectors are perpendicular.
If

$$(\mathbf{a}+\mathbf{b})^\top(\mathbf{a}+\mathbf{b}) = ||\mathbf{a}||^2 + ||\mathbf{b}||^2 \tag{1.0.7}$$

then $\mathbf{a}$ and $\mathbf{b}$ are perpendicular.
If $\mathbf{a}$ and $\mathbf{b}$ are perpendicular then,

$$\mathbf{a}^\top\mathbf{b} = 0 \tag{1.0.8}$$
$$(\mathbf{a}+\mathbf{b})^\top(\mathbf{a}+\mathbf{b}) = ||\mathbf{a}||^2 + ||\mathbf{b}||^2 + 2\mathbf{a}^\top\mathbf{b} \tag{1.0.9}$$
$$(\mathbf{a}+\mathbf{b})^\top(\mathbf{a}+\mathbf{b}) = ||\mathbf{a}||^2 + ||\mathbf{b}||^2 \tag{1.0.10}$$

Hence, if $\mathbf{a}$ and $\mathbf{b}$ are perpendicular then,

$$(\mathbf{a}+\mathbf{b})^\top(\mathbf{a}+\mathbf{b}) = ||\mathbf{a}||^2 + ||\mathbf{b}||^2 \tag{1.0.11}$$

Hence, $(\mathbf{a}+\mathbf{b})^\top(\mathbf{a}+\mathbf{b}) = ||\mathbf{a}||^2 + ||\mathbf{b}||^2$
, if and only if $\mathbf{a}$,$\mathbf{b}$ are perpendicular.

Example:
Let us take,

$$\mathbf{a} = \begin{pmatrix}1\\2\end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix}-2\\1\end{pmatrix} \tag{1.0.12}$$
$$||\mathbf{a}||^2 = 1^2 + 2^2 = 1 + 4 = 5 \tag{1.0.13}$$
$$||\mathbf{b}||^2 = (-2)^2 + 1^2 = 4 + 1 = 5 \tag{1.0.14}$$
$$\mathbf{a}+\mathbf{b} = \begin{pmatrix}1\\2\end{pmatrix} + \begin{pmatrix}-2\\1\end{pmatrix} = \begin{pmatrix}-1\\3\end{pmatrix} \tag{1.0.15}$$
$$(\mathbf{a}+\mathbf{b})^\top(\mathbf{a}+\mathbf{b}) = \begin{pmatrix}-1 & 3\end{pmatrix}^\top \begin{pmatrix}-1\\3\end{pmatrix} \tag{1.0.16}$$
$$= 1 + 9 = 10 \tag{1.0.17}$$
$$||\mathbf{a}||^2 + ||\mathbf{b}||^2 = 5 + 5 = 10 \tag{1.0.18}$$
$$\mathbf{a}^\top\mathbf{b} = \begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}-2\\1\end{pmatrix} \tag{1.0.19}$$
$$= -2 + 2 = 0 \tag{1.0.20}$$