# Line Assignment

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*Abstract*—**This document contains a general solution to Question 16 of Exercise 2 in Chapter 11 of the class 12 NCERT textbook.**

1) Find the shortest distance between the lines whose vector equations are

$$L_1 : \mathbf{x} = \mathbf{x_1} + \lambda_1 \mathbf{m_1} \tag{1}$$
$$L_2 : \mathbf{x} = \mathbf{x_2} + \lambda_2 \mathbf{m_2} \tag{2}$$

**Solution:** Let $\mathbf{A}$ and $\mathbf{B}$ be points on lines $L_1$ and $L_2$ respectively such that $AB$ is normal to both lines. Define

$$\mathbf{M} \triangleq \begin{pmatrix} \mathbf{m_1} & \mathbf{m_2} \end{pmatrix} \tag{3}$$
$$\boldsymbol{\lambda} \triangleq \begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} \tag{4}$$
$$\mathbf{x} \triangleq \mathbf{x_2} - \mathbf{x_1} \tag{5}$$

Then, we have the following equations:

$$\mathbf{A} = \mathbf{x_1} + \lambda_1 \mathbf{m_1} \tag{6}$$
$$\mathbf{B} = \mathbf{x_2} + \lambda_2 \mathbf{m_2} \tag{7}$$

From (6) and (7), define the real-valued function $f$ as

$$f(\boldsymbol{\lambda}) \triangleq \|\mathbf{A} - \mathbf{B}\|^2 \tag{8}$$
$$= \|\mathbf{M}\boldsymbol{\lambda} - \mathbf{x}\|^2 \tag{9}$$
$$= (\mathbf{M}\boldsymbol{\lambda} - \mathbf{x})^\top (\mathbf{M}\boldsymbol{\lambda} - \mathbf{x}) \tag{10}$$
$$= \boldsymbol{\lambda}^\top (\mathbf{M}^\top\mathbf{M}) \boldsymbol{\lambda} - 2\mathbf{x}^\top \mathbf{M}\boldsymbol{\lambda} + \|\mathbf{x}\|^2 \tag{11}$$

From (11), we see that $f$ is quadratic in $\boldsymbol{\lambda}$. We now prove a useful lemma here.

**Lemma 1.** The quadratic form

$$q(\mathbf{x}) \triangleq \mathbf{x}^\top \mathbf{A}\mathbf{x} + \mathbf{b}^\top \mathbf{x} + c \tag{12}$$

is convex iff $\mathbf{A}$ is positive semi-definite.

*Proof.* Consider two points $\mathbf{x_1}$ and $\mathbf{x_2}$, and a real constant $0 \le \mu \le 1$. Then,

$$\mu f(\mathbf{x_1}) + (1-\mu) f(\mathbf{x_2}) - f(\mu\mathbf{x_1} + (1-\mu)\mathbf{x_2})$$
$$= \left(\mu - \mu^2\right) \mathbf{x_1}^\top \mathbf{A}\mathbf{x_1} + \left(1 - \mu - (1-\mu)^2\right) \mathbf{x_2}^\top \mathbf{A}\mathbf{x_2}$$
$$- 2\mu(1-\mu)\mathbf{x_1}^\top \mathbf{A}\mathbf{x_2} \tag{13}$$
$$= \mu(1-\mu)\left(\mathbf{x_1}^\top \mathbf{A}\mathbf{x_1} - 2\mathbf{x_1}^\top \mathbf{A}\mathbf{x_2} + \mathbf{x_2}^\top \mathbf{A}\mathbf{x_2}\right) \tag{14}$$
$$= \mu(1-\mu)(\mathbf{x_1} - \mathbf{x_2})^\top \mathbf{A}(\mathbf{x_1} - \mathbf{x_2}) \tag{15}$$

Since $\mathbf{x_1}$ and $\mathbf{x_2}$ are arbitrary, it follows from (15) that

$$\mu f(\mathbf{x_1}) + (1-\mu) f(\mathbf{x_2}) \ge f(\mu\mathbf{x_1} + (1-\mu)\mathbf{x_2}) \tag{16}$$

iff $\mathbf{A}$ is positive semi-definite, as required. □

Using the above lemma, we show that $f$ is convex by showing that $\mathbf{M}^\top\mathbf{M}$ is positive semi-definite. Indeed, for any $\mathbf{p} \triangleq \begin{pmatrix} x \\ y \end{pmatrix}$,

$$\mathbf{p}^\top \mathbf{M}^\top \mathbf{M}\mathbf{p} = \|\mathbf{M}\mathbf{p}\|^2 \ge 0 \tag{17}$$

and thus, $f$ is convex.

We need to minimize $f$ as a function of $\boldsymbol{\lambda}$. Thus, differentiating (11) using the chain rule,

$$\frac{df(\boldsymbol{\lambda})}{d\boldsymbol{\lambda}} = \mathbf{M}^\top (\mathbf{M}\boldsymbol{\lambda} - \mathbf{x}) + \mathbf{M}(\mathbf{M}\boldsymbol{\lambda} - \mathbf{x})^\top \tag{18}$$
$$= 2\mathbf{M}^\top (\mathbf{M}\boldsymbol{\lambda} - \mathbf{x}) \tag{19}$$

Setting (19) to zero gives

$$\mathbf{M}^\top \mathbf{M}\boldsymbol{\lambda} = \mathbf{M}^\top \mathbf{x} \tag{20}$$

We have the following cases:

a) There exists a $\boldsymbol{\lambda}$ satisfying

$$\mathbf{M}\boldsymbol{\lambda} = \mathbf{x} \tag{21}$$
$$\implies \lambda_1 \mathbf{m_1} - \lambda_2 \mathbf{m_2} = \mathbf{x_2} - \mathbf{x_1} \tag{22}$$
$$\implies \mathbf{x_1} + \lambda_1 \mathbf{m_1} = \mathbf{x_2} + \lambda_2 \mathbf{m_2} \tag{23}$$

Thus, both lines intersect at a point and the shortest distance between them is 0. To check for the existence of such a $\boldsymbol{\lambda}$, we can bring the augmented matrix $\begin{pmatrix} \mathbf{M} & \mathbf{x} \end{pmatrix}$

into row-reduced echelon form and check whether there is a pivot in the last column.

b) $\mathbf{M}^\top\mathbf{M}$ is singular. Since $\mathbf{M}^\top\mathbf{M}$ is a sqaure matrix of order 2, its rank must be 1. Further,

$$\det\left(\mathbf{M}^\top\mathbf{M}\right) = \begin{vmatrix} \mathbf{m_1}^\top\mathbf{m_1} & \mathbf{m_1}^\top\mathbf{m_2} \\ \mathbf{m_1}^\top\mathbf{m_2} & \mathbf{m_2}^\top\mathbf{m_2} \end{vmatrix} \tag{24}$$

$$= \left(\|\mathbf{m_1}\| \,.\, \|\mathbf{m_2}\|\right)^2 - \left(\mathbf{m_1}^\top\mathbf{m_2}\right)^2 \tag{25}$$

Thus, equating the determinant to zero gives

$$\|\mathbf{m_1}\| \,.\, \|\mathbf{m_2}\| = \left|\mathbf{m_1}^\top\mathbf{m_2}\right| \tag{26}$$

which implies that both lines are parallel to each other. Setting $\mathbf{m_2} = k\mathbf{m_1}, k \in \mathbb{R}\setminus\{0\}$, we obtain one equation from (20).

$$\mathbf{m_1}^\top\mathbf{m_1}\left(\lambda_1 - k\lambda_2\right) = \mathbf{m_1}^\top\mathbf{x} \tag{27}$$

$$\implies \lambda_1 - k\lambda_2 = \frac{\mathbf{m_1}^\top\mathbf{x}}{\|\mathbf{m_1}\|^2} \tag{28}$$

Therefore, the required shortest distance is

$$\|\mathbf{A} - \mathbf{B}\| = \left\|\frac{\mathbf{m_1}^\top\mathbf{x}\mathbf{m_1}}{\|\mathbf{m_1}\|^2} - \mathbf{x}\right\| \tag{29}$$

c) $\mathbf{M}^\top\mathbf{M}$ is nonsinglar. This implies that the lines are skew. From (20),

$$\boldsymbol{\lambda} = \left(\mathbf{M}^\top\mathbf{M}\right)^{-1}\mathbf{M}^\top\mathbf{x} \tag{30}$$

and therefore, the shortest distance is

$$\|\mathbf{A} - \mathbf{B}\| = \left\|\left(\mathbf{M}\left(\mathbf{M}^\top\mathbf{M}\right)^{-1}\mathbf{M}^\top - \mathbf{I_n}\right)\mathbf{x}\right\| \tag{31}$$

where $\mathbf{I_n}$ is the identity matrix of order $n$.