

Points and Vectors

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CONTENTS

1	Examples	1
2	Exercises	46

Abstract—This book provides a computational approach to school geometry based on the NCERT textbooks from Class 6-12. Links to sample Python codes are available in the text.

Download python codes using

```
svn co https://github.com/gadepall/school/trunk/ncert/computation/codes
```

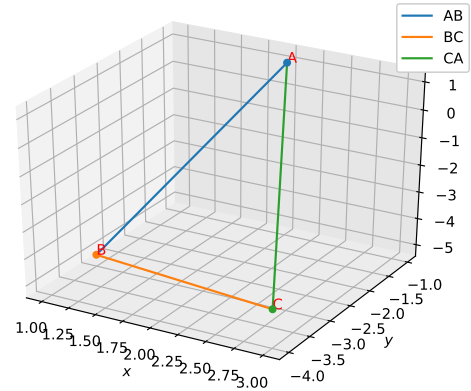


Fig. 1.1

1 EXAMPLES

1.1. Show that the points

$$\mathbf{A} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 3 \\ -4 \\ -4 \end{pmatrix} \quad (1.1.1)$$

are the vertices of a right angled triangle.

Solution: The following code plots Fig. 1.1

```
codes/triangle/triangle_3d.py
```

From the figure, it appears that $\triangle ABC$ is right angled at \mathbf{C} . Since

$$(\mathbf{A} - \mathbf{C})^T (\mathbf{B} - \mathbf{C}) = 0 \quad (1.1.2)$$

it is proved that the triangle is indeed right angled.

1.2. Do the points $\mathbf{A} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ form a triangle? If so, name the type of triangle formed.

Solution:

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The direction vectors of AB and BC are

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} -5 \\ -5 \end{pmatrix} \quad (1.2.1)$$

$$\mathbf{C} - \mathbf{A} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad (1.2.2)$$

If $\mathbf{A}, \mathbf{B}, \mathbf{C}$ form a line, then, AB and AC should have the same direction vector. Hence, there exists a k such that

$$\mathbf{B} - \mathbf{A} = k(\mathbf{C} - \mathbf{A}) \quad (1.2.3)$$

$$\Rightarrow \mathbf{B} = \frac{k\mathbf{C} + \mathbf{A}}{k+1} \quad (1.2.4)$$

Since

$$\mathbf{B} - \mathbf{A} \neq k(\mathbf{C} - \mathbf{A}), \quad (1.2.5)$$

the points are not collinear and form a triangle. An alternative method is to create the matrix

$$\mathbf{M} = (\mathbf{B} - \mathbf{A} \quad \mathbf{C} - \mathbf{A})^T \quad (1.2.6)$$

If $\text{rank}(\mathbf{M}) = 1$, the points are collinear. The rank of a matrix is the number of nonzero rows

left after doing row operations. In this problem,

$$\mathbf{M} = \begin{pmatrix} -5 & -5 \\ -1 & 1 \end{pmatrix} \xrightarrow{R_2 \leftarrow -5R_2 - R_1} \begin{pmatrix} -5 & -5 \\ 0 & 10 \end{pmatrix} \quad (1.2.7)$$

$$\Rightarrow \text{rank}(\mathbf{M}) = 2 \quad (1.2.8)$$

as the number of non zero rows is 2. The following code plots Fig. 1.2

```
codes/triangle/check_tri.py
```

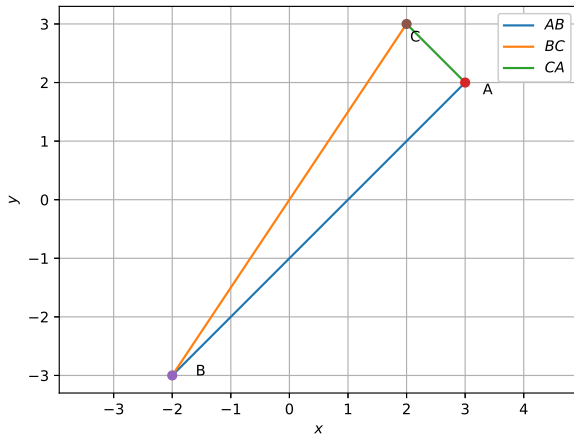


Fig. 1.2

From the figure, it appears that $\triangle ABC$ is right angled, with BC as the hypotenuse. From Baudhayana's theorem, this would be true if

$$\|\mathbf{B} - \mathbf{A}\|^2 + \|\mathbf{C} - \mathbf{A}\|^2 = \|\mathbf{B} - \mathbf{C}\|^2 \quad (1.2.9)$$

which can be expressed as

$$\begin{aligned} \|\mathbf{A}\|^2 + \|\mathbf{C}\|^2 - 2\mathbf{A}^T \mathbf{C} + \|\mathbf{A}\|^2 + \|\mathbf{B}\|^2 - 2\mathbf{A}^T \mathbf{B} \\ = \|\mathbf{B}\|^2 + \|\mathbf{C}\|^2 - 2\mathbf{B}^T \mathbf{C} \end{aligned} \quad (1.2.10)$$

to obtain

$$(\mathbf{B} - \mathbf{A})^T (\mathbf{C} - \mathbf{A}) = 0 \quad (1.2.11)$$

after simplification. From (1.2.1) and (1.2.2), it is easy to verify that

$$(\mathbf{B} - \mathbf{A})^T (\mathbf{C} - \mathbf{A}) = \begin{pmatrix} -5 & -5 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 0 \quad (1.2.12)$$

satisfying (1.2.11). Thus, $\triangle ABC$ is right angled at \mathbf{A} .

- 1.3. Find the area of a triangle whose vertices are $\mathbf{A} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -4 \\ 6 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} -3 \\ -5 \end{pmatrix}$.

Solution: Using Hero's formula, the following code computes the area of the triangle as 24.

```
codes/triangle/area_tri.py
```

- 1.4. Find the area of a triangle formed by the vertices $\mathbf{A} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 7 \\ -4 \end{pmatrix}$. **Solution:** The area of $\triangle ABC$ is also obtained in terms of the *magnitude* of the determinant of the matrix \mathbf{M} in (1.2.6) as

$$\frac{1}{2} |\mathbf{M}| \quad (1.4.1)$$

The computation is done in **area_tri.py**

- 1.5. Find the area of a triangle formed by the points $\mathbf{P} = \begin{pmatrix} -1.5 \\ 3 \end{pmatrix}$, $\mathbf{Q} = \begin{pmatrix} 6 \\ -2 \end{pmatrix}$, $\mathbf{R} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$.

Solution: Another formula for the area of $\triangle ABC$ is

$$\frac{1}{2} \begin{vmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{B} & \mathbf{C} \end{vmatrix} \quad (1.5.1)$$

- 1.6. Find the area of a triangle having the points

$$\mathbf{A} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \quad (1.6.1)$$

as its vertices.

Solution: The area of a triangle using the *vector product* is obtained as

$$\frac{1}{2} \|(\mathbf{B} - \mathbf{A}) \times (\mathbf{C} - \mathbf{A})\| \quad (1.6.2)$$

For any two vectors $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$,

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \quad (1.6.3)$$

The following code computes the area using the vector product.

```
codes/triangle/area_tri_vec.py
```

- 1.7. The centroid of a $\triangle ABC$ is at the point $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. If the coordinates of \mathbf{A} and \mathbf{B} are $\begin{pmatrix} 3 \\ -5 \\ 7 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 7 \\ -6 \end{pmatrix}$, respectively, find the coordinates of the point

C.

Solution: The centroid of $\triangle ABC$ is given by

$$\mathbf{O} = \frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} \quad (1.7.1)$$

Thus,

$$\mathbf{C} = 3\mathbf{O} - \mathbf{A} - \mathbf{B} \quad (1.7.2)$$

1.8. Without using the Pythagoras theorem, show that the points $\begin{pmatrix} 4 \\ 4 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -1 \end{pmatrix}$ are the vertices of a right angled triangle.

Solution: The direction vectors of $\mathbf{A} - \mathbf{B}$, $\mathbf{A} - \mathbf{C}$ and $\mathbf{B} - \mathbf{C}$ are

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad (1.8.1)$$

$$\mathbf{A} - \mathbf{C} = \begin{pmatrix} -5 \\ -5 \end{pmatrix} \quad (1.8.2)$$

$$\mathbf{B} - \mathbf{C} = \begin{pmatrix} -4 \\ -6 \end{pmatrix} \quad (1.8.3)$$

a)

$$(\mathbf{A} - \mathbf{B})^T(\mathbf{B} - \mathbf{C}) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \begin{pmatrix} -4 \\ -6 \end{pmatrix} = -2 \quad (1.8.4)$$

$$(\mathbf{A} - \mathbf{B})^T(\mathbf{B} - \mathbf{C}) = -2 \neq 0 \quad (1.8.5)$$

Sides $\mathbf{A} - \mathbf{B}$ and $\mathbf{B} - \mathbf{C}$ of triangle are not perpendicular.

b)

$$(\mathbf{A} - \mathbf{C})^T(\mathbf{B} - \mathbf{C}) = \begin{pmatrix} -5 \\ -5 \end{pmatrix} \begin{pmatrix} -4 \\ -6 \end{pmatrix} = 50 \quad (1.8.6)$$

$$(\mathbf{A} - \mathbf{C})^T(\mathbf{B} - \mathbf{C}) = 50 \neq 0 \quad (1.8.7)$$

Sides $\mathbf{A} - \mathbf{C}$ and $\mathbf{B} - \mathbf{C}$ of triangle are not perpendicular.

c)

$$(\mathbf{A} - \mathbf{B})^T(\mathbf{A} - \mathbf{C}) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \begin{pmatrix} -5 \\ -5 \end{pmatrix} = 0 \quad (1.8.8)$$

$$(\mathbf{A} - \mathbf{B})^T(\mathbf{A} - \mathbf{C}) = 0 \quad (1.8.9)$$

Sides $\mathbf{A} - \mathbf{B}$ and $\mathbf{A} - \mathbf{C}$ of triangle are perpendicular to each other and the right angle at vertex $\begin{pmatrix} 4 \\ 4 \end{pmatrix}$, and the following figure represents the triangle formed by given

points \mathbf{A} , \mathbf{B} and \mathbf{C} .

1.9. Draw the graphs of the equations

$$(1 \ -1)\mathbf{x} + 1 = 0 \quad (1.9.1)$$

$$(3 \ 2)\mathbf{x} - 12 = 0 \quad (1.9.2)$$

Determine the coordinates of the vertices of the triangle formed by these lines and the x-axis, and shade the triangular region.

Solution: Let

$$\mathbf{x} = \begin{pmatrix} a \\ 0 \end{pmatrix} \quad (1.9.3)$$

Substituting in (1.9.1),

$$(1 \ -1)\begin{pmatrix} a \\ 0 \end{pmatrix} = -1 \quad (1.9.4)$$

$$\Rightarrow a = -1 \quad (1.9.5)$$

Similarly, substituting

$$\mathbf{x} = \begin{pmatrix} 0 \\ b \end{pmatrix}, \quad (1.9.6)$$

in (1.9.1),

$$b = 1 \quad (1.9.7)$$

The intercepts on the x and y-axis from above are

$$\begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (1.9.8)$$

Similarly, the intercepts on x and y-axis for (1.9.2) are

$$\begin{pmatrix} 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 6 \end{pmatrix} \quad (1.9.9)$$

The intersection of the lines in (1.9.1), (1.9.1) is obtained from

$$\begin{pmatrix} 1 & -1 \\ 3 & 2 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -1 \\ 12 \end{pmatrix} \quad (1.9.10)$$

The augmented matrix for the above equation is row reduced as follows

$$\begin{pmatrix} 1 & -1 & -1 \\ 3 & 2 & 12 \end{pmatrix} \xrightarrow{R_2 \leftarrow \frac{R_2 - 3R_1}{5}} \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 3 \end{pmatrix} \quad (1.9.11)$$

$$\xrightarrow{R_1 \leftarrow R_1 + R_2} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix} \quad (1.9.12)$$

$$\Rightarrow \mathbf{x} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad (1.9.13)$$

The desired triangle is available in Fig. (1.9) with vertices

$$\mathbf{A} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \quad (1.9.14)$$

The equivalent python code for figure (1.9) is



Fig. 1.9

solutions/1/codes/triangle/shaded.py

- 1.10. In a $\triangle ABC$, $\angle C = 3\angle B = 2(\angle A + \angle B)$. Find the three angles.

Solution:

The given equations result in the matrix equation In vector form:

$$\begin{pmatrix} 6 & 0 & -1 \\ 0 & 3 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 180 \end{pmatrix} \quad (1.10.1)$$

which can be solved as

$$\begin{pmatrix} 6 & 0 & -1 & 0 \\ 0 & 3 & -1 & 0 \\ 1 & 1 & 1 & 180 \end{pmatrix} \xrightarrow{R_1 \leftarrow \frac{R_1}{6}} \begin{pmatrix} 1 & 0 & -\frac{1}{6} & 0 \\ 0 & 3 & -1 & 0 \\ 1 & 1 & 1 & 180 \end{pmatrix} \quad (1.10.2)$$

$$\xrightarrow{R_3 \leftarrow R_3 - R_1} \begin{pmatrix} 1 & 0 & -\frac{1}{6} & 0 \\ 0 & 3 & -1 & 0 \\ 0 & 1 & \frac{7}{6} & 180 \end{pmatrix} \xrightarrow{R_2 \leftarrow \frac{R_2}{3}} \begin{pmatrix} 1 & 0 & -\frac{1}{6} & 0 \\ 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{7}{6} & 180 \end{pmatrix} \quad (1.10.3)$$

$$\xrightarrow{R_3 \leftarrow R_3 - R_2} \begin{pmatrix} 1 & 0 & -\frac{1}{6} & 0 \\ 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & \frac{3}{2} & 180 \end{pmatrix} \xrightarrow{R_3 \leftarrow \frac{2R_3}{3}} \begin{pmatrix} 1 & 0 & -\frac{1}{6} & 0 \\ 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & 120 \end{pmatrix} \quad (1.10.4)$$

$$\xrightarrow{\begin{matrix} R_1 \leftarrow R_1 + \frac{R_3}{6} \\ R_2 \leftarrow R_2 + \frac{R_3}{3} \end{matrix}} \begin{pmatrix} 1 & 0 & 1 & 20 \\ 0 & 1 & 0 & 40 \\ 0 & 0 & 1 & 120 \end{pmatrix} \quad (1.10.5)$$

$$\therefore \angle C = 120^\circ \quad \angle A = 20^\circ \quad \angle B = 40^\circ \quad (1.10.6)$$

- 1.11. Draw the graphs of the equations $5x - y = 5$ and $3x - y = 3$. Determine the co-ordinates of the vertices of the triangle formed by these lines and the y axis.

Solution:

Line $5x - y = 5$ can be represented in vector form as,

$$(5 \ -1)\mathbf{x} = 5 \quad (1.11.1)$$

Line $3x - y = 3$ can be represented in vector form as,

$$(3 \ -1)\mathbf{x} = 3 \quad (1.11.2)$$

Also the equation of y axis is

$$(1 \ 0)\mathbf{x} = 0 \quad (1.11.3)$$

Let line (1.11.1) and line (1.11.2) meet at point A. Then,

$$\begin{pmatrix} 5 & -1 \\ 3 & -1 \end{pmatrix} \mathbf{A} = \begin{pmatrix} 5 \\ 3 \end{pmatrix} \quad (1.11.4)$$

$$\mathbf{A} = \begin{pmatrix} 5 & -1 \\ 3 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 5 \\ 3 \end{pmatrix} \quad (1.11.5)$$

$$\mathbf{A} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.11.6)$$

Let line (1.11.1) and line (1.11.3) meet at point

B. Then,

$$\begin{pmatrix} 5 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{B} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \quad (1.11.7)$$

$$\mathbf{B} = \begin{pmatrix} 5 & -1 \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 5 \\ 0 \end{pmatrix} \quad (1.11.8)$$

$$\mathbf{B} = \begin{pmatrix} 0 \\ -5 \end{pmatrix} \quad (1.11.9)$$

Let line (1.11.2) and line (1.11.3) meet at point C. Then,

$$\begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{C} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \quad (1.11.10)$$

$$\mathbf{C} = \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 3 \\ 0 \end{pmatrix} \quad (1.11.11)$$

$$\mathbf{C} = \begin{pmatrix} 0 \\ -3 \end{pmatrix} \quad (1.11.12)$$

So, $\triangle ABC$ is formed by intersection of (1.11.1), (1.11.2) and (1.11.3). The following Python code generates Fig. 1.11. The lines (1.11.1) and (1.11.2) and the triangle ABC formed by the two lines and y-axis are plotted in the figure below

codes/triangle/linesandtri.py



Fig. 1.11: Plot of lines and the Triangle ABC

1.12. The vertices of $\triangle PQR$ are $\mathbf{P} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\mathbf{Q} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$, $\mathbf{R} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$. Find the equation of the median through the vertex R.

Solution: In Fig. 1.12, RS is the median.

Hence,

$$\mathbf{S} = \frac{\mathbf{P} + \mathbf{Q}}{2} \quad (1.12.1)$$

Hence, the equation of the median going through points S and R can be given as

$$\mathbf{x} = \mathbf{R} + \lambda(\mathbf{S} - \mathbf{R}) \quad (1.12.2)$$

$$\mathbf{x} = \begin{pmatrix} 4 \\ 5 \end{pmatrix} + \lambda \left(\begin{pmatrix} 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 4 \\ 5 \end{pmatrix} \right) \quad (1.12.3)$$

$$\mathbf{x} = \begin{pmatrix} 4 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ -3 \end{pmatrix} \quad (1.12.4)$$



Fig. 1.12

solutions/4/codes/triangle/triangle.py

1.13. In the $\triangle ABC$ with vertices $\mathbf{A} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, find the equation and length of the altitude from the vertex A.

Solution: The following python code computes the length of the altitude AD in Fig.1.13.

./solutions/5/codes/triangle/q2.py

In $\triangle ABC$,

$$(\mathbf{A} - \mathbf{C})^T (\mathbf{B} - \mathbf{C}) = 0 \quad (1.13.1)$$

Hence, $\triangle ABC$ is a right triangle. The direction vector of BC is

$$(\mathbf{B} - \mathbf{C}) = \begin{pmatrix} 3 \\ -3 \end{pmatrix} \quad (1.13.2)$$



Fig. 1.13: Triangle of Q.1.2.5

Hence, the equation of AD is

$$(\mathbf{B} - \mathbf{C})^T (\mathbf{x} - \mathbf{A}) = 0 \quad (1.13.3)$$

$$\Rightarrow (1 \ -1)\mathbf{x} = -1 \quad (1.13.4)$$

The length of the altitude is obtained as $\|\mathbf{A} - \mathbf{D}\| = 1.414$

1.14. Find the area of the triangle whose vertices are

- a) $\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -4 \end{pmatrix}$
 b) $\begin{pmatrix} -5 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ -5 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \end{pmatrix}$

Solution:

- a) See Fig. 1.14 generated using the following python code

solutions/6/codes/triangle/triangle1.py

$$ar(\triangle ABC) = \frac{1}{2} \|(\mathbf{B} - \mathbf{A}) \times (\mathbf{C} - \mathbf{A})\| \quad (1.14.1)$$

$$= \frac{1}{2} \left\| \begin{pmatrix} -3 \\ -3 \end{pmatrix} \times \begin{pmatrix} 0 \\ -7 \end{pmatrix} \right\| = \frac{21}{2} \quad (1.14.2)$$

and verified by

solutions/6/codes/triangle/tri_area_ABC.py

- b) See $\triangle PQR$ in Fig. 1.14 generated using the following python code

solutions/6/codes/triangle/triangle2.py

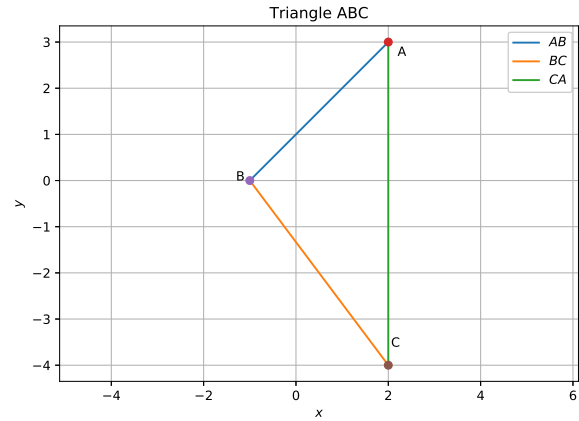


Fig. 1.14: Triangle ABC using python

$$ar(\triangle PQR) = \frac{1}{2} \|(\mathbf{Q} - \mathbf{P}) \times (\mathbf{R} - \mathbf{P})\| \quad (1.14.3)$$

$$= \frac{1}{2} \left\| \begin{pmatrix} 8 \\ -4 \end{pmatrix} \times \begin{pmatrix} 10 \\ 3 \end{pmatrix} \right\| = \frac{64}{2} \quad (1.14.4)$$

and verified by

solutions/6/codes/triangle/tri_area_PQR.py

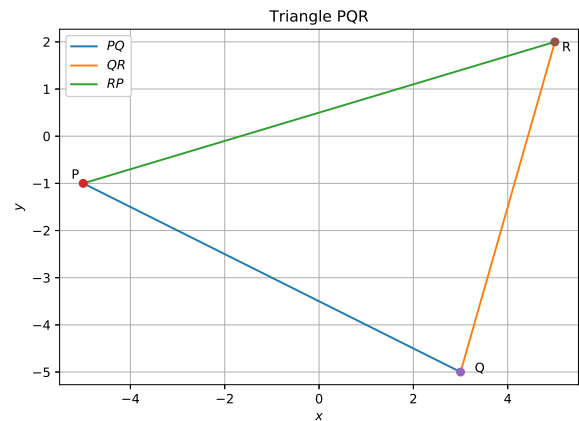


Fig. 1.14: Triangle PQR using python

- 1.15. Find the area of the triangle formed by joining the mid points of the sides of a triangle whose vertices are $\begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix}$.

Solution: See Fig. 1.15. Let the vertices be

A, B, C. The midpoints of each side are

$$\mathbf{D} = \frac{\mathbf{A} + \mathbf{B}}{2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (1.15.1)$$

$$\mathbf{E} = \frac{\mathbf{B} + \mathbf{C}}{2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.15.2)$$

$$\mathbf{F} = \frac{\mathbf{A} + \mathbf{C}}{2} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (1.15.3)$$

$$(1.15.4)$$

Area of a $\triangle ABC$ is given by

$$\begin{aligned} \frac{1}{2} \|(\mathbf{E} - \mathbf{D}) \times (\mathbf{F} - \mathbf{D})\| \\ = \frac{1}{2} \left\| \begin{pmatrix} 0 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\| \\ = 1 \end{aligned} \quad (1.15.5)$$



Fig. 1.15

Download the python code for finding a triangle's area from

`solutions/7/codes/triangle/area_tri_area.py`

and the figure from

`solutions/7/figs/triangle/draw_triangle.py`

- 1.16. Verify that the median of $\triangle ABC$ with vertices $\mathbf{A} = \begin{pmatrix} 4 \\ -6 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$ divides it into two triangles of equal areas.

Solution: The following Python code generates Fig. 1.16

`codes/triangle.py`

From the given information,

$$\mathbf{A} = \begin{pmatrix} 4 \\ -6 \end{pmatrix} \quad (1.16.1)$$

$$\mathbf{B} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}, \quad (1.16.2)$$

$$\mathbf{C} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \quad (1.16.3)$$

$\therefore \mathbf{M}$ is the midpoint of AB ,

$$\mathbf{M} = \frac{\mathbf{A} + \mathbf{B}}{2} = \frac{1}{2} \begin{pmatrix} 7 \\ -8 \end{pmatrix} \quad (1.16.4)$$

$\therefore \mathbf{N}$ is the midpoint of BC ,

$$\mathbf{N} = \frac{\mathbf{B} + \mathbf{C}}{2} = \frac{1}{2} \begin{pmatrix} 8 \\ 0 \end{pmatrix} \quad (1.16.5)$$

$\therefore \mathbf{P}$ is the midpoint of CA ,

$$\mathbf{P} = \frac{\mathbf{C} + \mathbf{A}}{2} = \frac{1}{2} \begin{pmatrix} 9 \\ -4 \end{pmatrix} \quad (1.16.6)$$

The following Python code verifies the determinant values.

`codes/determinant_check.py`



Fig. 1.16

For $\triangle ABC$, the vertices are \mathbf{A} , \mathbf{B} and \mathbf{C} . So the area of the triangle $\triangle ABC$ by using determinant

will be :

$$\begin{aligned}
 \text{Area} &= \frac{1}{2} \begin{vmatrix} 4 & -6 & 1 \\ 3 & -2 & 1 \\ 5 & 2 & 1 \end{vmatrix} \xrightarrow{C_2 \leftarrow \frac{C_2}{-2}} \frac{1}{2} \begin{vmatrix} 4 & -3 & 1 \\ 3 & -1 & 1 \\ 5 & 1 & 1 \end{vmatrix} \\
 &\xrightarrow{\begin{matrix} R_2 \leftarrow R_2 - R_1 \\ R_3 \leftarrow R_3 - R_1 \end{matrix}} \begin{vmatrix} 4 & -3 & 1 \\ -1 & 2 & 0 \\ 1 & 4 & 0 \end{vmatrix} \xrightarrow{R_3 \leftarrow R_3 + R_2} \begin{vmatrix} 4 & -3 & 1 \\ -1 & 2 & 0 \\ 0 & 6 & 0 \end{vmatrix} \\
 &\xrightarrow{R_3 \leftarrow \frac{R_3}{6}} \begin{vmatrix} 4 & -3 & 1 \\ -1 & 2 & 0 \\ 0 & 1 & 0 \end{vmatrix} \\
 &= -6 \\
 &\quad (1.16.7)
 \end{aligned}$$

Now, we will consider the absolute value of area only. So, $\text{Area} = |-6| = 6$.

To verify the problem statement we have to check 3 cases:

Case 1: When **BP** is median, we will consider $\triangle ABP$ triangle. In that case, the vertices will be **A**, **B** and **P**.

Now, the area of $\triangle ABP$ will be :

$$\begin{aligned}
 A1 &= \frac{1}{2} \begin{vmatrix} 4 & -6 & 1 \\ 3 & -2 & 1 \\ 4.5 & -2 & 1 \end{vmatrix} \xrightarrow{C_2 \leftarrow \frac{C_2}{(-2)}} \frac{(-2)}{2} \begin{vmatrix} 4 & 3 & 1 \\ 3 & 1 & 1 \\ 4.5 & 1 & 1 \end{vmatrix} \\
 &\xrightarrow{\begin{matrix} R_2 \leftarrow R_2 - R_1 \\ R_3 \leftarrow R_3 - R_1 \end{matrix}} \begin{vmatrix} 4 & 3 & 1 \\ -1 & -2 & 0 \\ 0.5 & -2 & 0 \end{vmatrix} \\
 &\xrightarrow{R_3 \leftarrow R_3 - R_2} \begin{vmatrix} 4 & 3 & 1 \\ -1 & -2 & 0 \\ 1.5 & 0 & 0 \end{vmatrix} \\
 &= -3 \\
 &\quad (1.16.8)
 \end{aligned}$$

But, we will consider the absolute value of area only. So, $A1 = |-3| = 3$.

or, $A1 = \frac{1}{2}(\text{Area of } \triangle ABC)$

Case 2: When **AN** is median, we will consider $\triangle ABN$ triangle. In that case, the vertices will be **A**, **B** and **N**.

Now, the area of $\triangle ABN$ will be :

$$\begin{aligned}
 A2 &= \frac{1}{2} \begin{vmatrix} 4 & -6 & 1 \\ 3 & -2 & 1 \\ 4 & 0 & 1 \end{vmatrix} \xrightarrow{C_2 \leftarrow \frac{C_2}{(-2)}} \frac{(-2)}{2} \begin{vmatrix} 4 & 3 & 1 \\ 3 & 1 & 1 \\ 4 & 0 & 1 \end{vmatrix} \\
 &\xrightarrow{\begin{matrix} R_2 \leftarrow R_2 - R_1 \\ R_3 \leftarrow R_3 - R_1 \end{matrix}} \begin{vmatrix} 4 & 3 & 1 \\ -1 & -2 & 0 \\ 0 & -3 & 0 \end{vmatrix} \\
 &\xrightarrow{R_3 \leftarrow \frac{R_3}{(-3)}} \begin{vmatrix} 4 & 3 & 1 \\ -1 & -2 & 0 \\ 0 & 1 & 0 \end{vmatrix} \\
 &= -3 \\
 &\quad (1.16.9)
 \end{aligned}$$

But, we will consider the absolute value of area only. So, $A2 = |-3| = 3$.

or, $A2 = \frac{1}{2}(\text{Area of } \triangle ABC)$

Case 3: When **CM** is median, we will consider $\triangle CAM$ triangle. In that case, the vertices will be **A**, **C** and **M**.

Now, the area of $\triangle CAM$ will be :

$$\begin{aligned}
 A3 &= \frac{1}{2} \begin{vmatrix} 5 & 2 & 1 \\ 4 & -6 & 1 \\ 3.5 & -4 & 1 \end{vmatrix} \xrightarrow{C_2 \leftarrow \frac{C_2}{-2}} \frac{2}{2} \begin{vmatrix} 5 & 1 & 1 \\ 4 & -3 & 1 \\ 3.5 & -2 & 1 \end{vmatrix} \\
 &\xrightarrow{\begin{matrix} R_2 \leftarrow R_2 - R_1 \\ R_3 \leftarrow R_3 - R_1 \end{matrix}} \begin{vmatrix} 5 & 1 & 1 \\ -1 & -4 & 0 \\ -1.5 & -3 & 0 \end{vmatrix} \\
 &\xrightarrow{\begin{matrix} R_2 \leftarrow \frac{R_2}{(-1)} \\ R_3 \leftarrow \frac{R_3}{(-1.5)} \end{matrix}} \begin{vmatrix} 5 & 1 & 1 \\ 1 & 4 & 0 \\ 1 & 2 & 0 \end{vmatrix} \\
 &\xrightarrow{R_3 \leftarrow R_3 - R_2} \begin{vmatrix} 5 & 1 & 1 \\ 1 & 4 & 0 \\ 0 & -2 & 0 \end{vmatrix} \\
 &= -3 \\
 &\quad (1.16.10)
 \end{aligned}$$

But, we will consider the absolute value of area only. So, $A3 = |-3| = 3$.

or, $A3 = \frac{1}{2}(\text{Area of } \triangle ABC)$

Hence, the above problem statement is verified.

1.17. Let $\mathbf{A} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 6 \\ 5 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ be the vertices of $\triangle ABC$.

- The median from **A** meets **BC** at **D**. Find the coordinates of the point **D**.
- Find the coordinates of the point **P** on **AD** such that $AP : PD = 2 : 1$.
- Find the coordinates of the points **Q** and **R** on medians **BE** and **CF** respectively such that $BQ : QE = 2 : 1$ and $CR : RF = 2 : 1$.

Solution:

a. Given $\triangle ABC$ with vertices

$$\mathbf{A} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 6 \\ 5 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \quad (1.17.1)$$

Given that the median from \mathbf{A} meets BC at \mathbf{D} , now the coordinate of \mathbf{D} is given as,

$$\mathbf{D} = \frac{\mathbf{B} + \mathbf{C}}{2} = \frac{\begin{pmatrix} 6 \\ 5 \end{pmatrix} + \begin{pmatrix} 1 \\ 4 \end{pmatrix}}{2} \quad (1.17.2)$$

$$\Rightarrow \mathbf{D} = \begin{pmatrix} \frac{7}{2} \\ \frac{9}{2} \end{pmatrix} \quad (1.17.3)$$

b. Result :The coordinates of point \mathbf{C} dividing the line AB in the ratio $m : n$ is given by

$$\frac{n\mathbf{A} + m\mathbf{B}}{m + n} \quad (1.17.4)$$

Given that the point \mathbf{P} divides AD in the ratio $2 : 1$, now to find \mathbf{P} we use (1.17.4),

$$\mathbf{P} = \frac{1 \begin{pmatrix} 4 \\ 2 \end{pmatrix} + 2 \begin{pmatrix} \frac{7}{2} \\ \frac{9}{2} \end{pmatrix}}{3} = \begin{pmatrix} \frac{11}{3} \\ \frac{11}{3} \end{pmatrix} \quad (1.17.5)$$

c. Given that the point \mathbf{Q} on the median BE divides it in the ratio $2 : 1$, first we find \mathbf{E} ,

$$\mathbf{E} = \frac{\mathbf{A} + \mathbf{C}}{2} = \frac{\begin{pmatrix} 4 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 4 \end{pmatrix}}{2} \quad (1.17.6)$$

$$\Rightarrow \mathbf{E} = \begin{pmatrix} \frac{5}{2} \\ 3 \end{pmatrix}. \quad (1.17.7)$$

Now we find \mathbf{Q} using (1.17.4)

$$\mathbf{Q} = \frac{1 \begin{pmatrix} 6 \\ 5 \end{pmatrix} + 2 \begin{pmatrix} \frac{5}{2} \\ 3 \end{pmatrix}}{3} = \begin{pmatrix} \frac{11}{3} \\ \frac{11}{3} \end{pmatrix} \quad (1.17.8)$$

Similarly, Given that the point \mathbf{R} on the median CF divides it in the ratio $2 : 1$, first we find \mathbf{F} ,

$$\mathbf{F} = \frac{\mathbf{A} + \mathbf{B}}{2} = \frac{\begin{pmatrix} 4 \\ 2 \end{pmatrix} + \begin{pmatrix} 6 \\ 5 \end{pmatrix}}{2} \quad (1.17.9)$$

$$\Rightarrow \mathbf{F} = \begin{pmatrix} 5 \\ \frac{7}{2} \end{pmatrix}. \quad (1.17.10)$$

Now we find \mathbf{R} using (1.17.4)

$$\mathbf{R} = \frac{1 \begin{pmatrix} 1 \\ 4 \end{pmatrix} + 2 \begin{pmatrix} 5 \\ \frac{7}{2} \end{pmatrix}}{3} = \begin{pmatrix} \frac{11}{3} \\ \frac{11}{3} \end{pmatrix} \quad (1.17.11)$$

The plot of the $\triangle ABC$ is given in Fig. 1.17.



Fig. 1.17: Plot of $\triangle ABC$

1.18. Show that the points

$$\mathbf{A} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 3 \\ -4 \\ -4 \end{pmatrix} \quad (1.18.1)$$

are the vertices of a right angled triangle.

Solution:

$$\therefore (\mathbf{A} - \mathbf{C})^T (\mathbf{B} - \mathbf{C}) = \begin{pmatrix} -1 & 3 & 5 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} \quad (1.18.2)$$

$$= 0 \quad (1.18.3)$$

the triangle in Fig. 1.18 is right angled.

1.19. In $\triangle ABC$, $\mathbf{A} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$. Find

$\angle B$.

Solution: Let

$$\mathbf{A} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}. \quad (1.19.1)$$

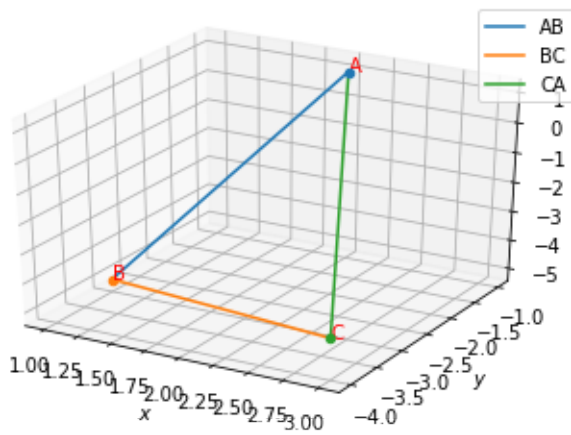


Fig. 1.18

Then,

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} \quad (1.19.2)$$

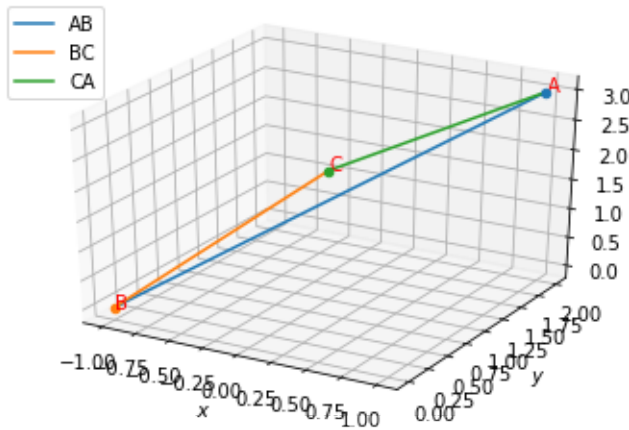
$$\mathbf{C} - \mathbf{B} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \quad (1.19.3)$$

Thus,

$$\mathbf{B} = \cos^{-1} \left(\frac{(\mathbf{A}-\mathbf{B})^T(\mathbf{C}-\mathbf{B})}{\|\mathbf{A}-\mathbf{B}\| \|\mathbf{C}-\mathbf{B}\|} \right) = \cos^{-1} \left(\frac{10}{\sqrt{17} \sqrt{6}} \right) \quad (1.19.4)$$

$$= 66.15 \quad (1.19.5)$$

See Fig. 1.19

Fig. 1.19: $\triangle ABC$

1.20. Find the area of a triangle having the points

$$\mathbf{A} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \text{ and } \mathbf{C} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \text{ as its vertices.}$$

Solution: From the given information,

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \quad (1.20.1)$$

$$\mathbf{C} - \mathbf{A} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \quad (1.20.2)$$

$$(1.20.3)$$

The area of a triangle using the vector product is then obtained as

$$\frac{1}{2} \|(\mathbf{B} - \mathbf{A}) \times (\mathbf{C} - \mathbf{A})\| \quad (1.20.4)$$

$$\frac{1}{2} \left\| \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \right\| \quad (1.20.5)$$

$$= 1 \quad (1.20.6)$$

1.21. Find the area of a triangle with vertices $\mathbf{A} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} 1 \\ 5 \\ 5 \end{pmatrix}$

Solution: From the given information,

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad (1.21.1)$$

$$\mathbf{C} - \mathbf{A} = \begin{pmatrix} 1 \\ 5 \\ 5 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix} \quad (1.21.2)$$

The area of a triangle using the vector product is then obtained as

$$\frac{1}{2} \|(\mathbf{B} - \mathbf{A}) \times (\mathbf{C} - \mathbf{A})\| \quad (1.21.3)$$

$$= \frac{1}{2} \left\| \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \times \begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix} \right\| \quad (1.21.4)$$

$$= \frac{17}{2} \quad (1.21.5)$$

1.22. Show that $\mathbf{A} = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 3 \\ 8 \\ -11 \end{pmatrix}$ are collinear.

Solution: Let

$$\mathbf{A} = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 3 \\ 8 \\ -11 \end{pmatrix} \quad (1.22.1)$$

Then

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} -1 \\ -5 \\ 7 \end{pmatrix}, \mathbf{C} - \mathbf{A} = \begin{pmatrix} 1 \\ 5 \\ -7 \end{pmatrix} \quad (1.22.2)$$

and

$$\mathbf{M} = (\mathbf{B} - \mathbf{A} \quad \mathbf{C} - \mathbf{A})^T \quad (1.22.3)$$

$$= \begin{pmatrix} -1 & -5 & 7 \\ 1 & 5 & -7 \end{pmatrix} \xrightarrow{R_1 \rightarrow -R_1} \begin{pmatrix} 1 & 5 & -7 \\ 1 & 5 & -7 \end{pmatrix} \quad (1.22.4)$$

$$\xrightarrow{R_2 \rightarrow R_2 - R_1} \begin{pmatrix} 1 & 5 & -7 \\ 0 & 0 & 0 \end{pmatrix} \quad (1.22.5)$$

$$\Rightarrow \text{rank}(\mathbf{M}) = 1 \quad (1.22.6)$$

Thus, the points are collinear as can be verified in Fig. 1.22.

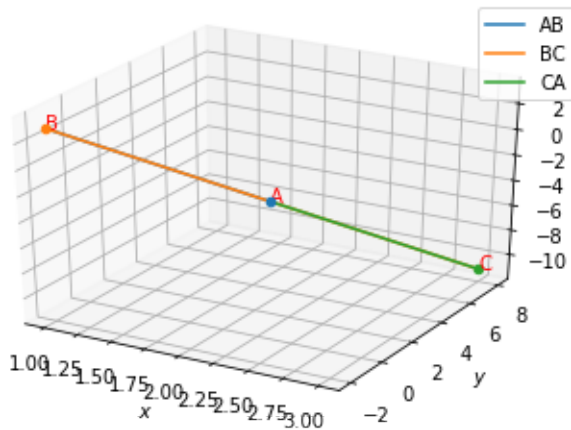


Fig. 1.22: collinear

1.23. Find the equation of set of points \mathbf{P} such that

$$PA^2 + PB^2 = 2k^2, \quad (1.23.1)$$

$$\mathbf{A} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -1 \\ 3 \\ -7 \end{pmatrix}, \quad (1.23.2)$$

respectively. **Solution:** Let,

$$\mathbf{P} = \mathbf{X}; \quad (1.23.3)$$

so,

$$(\mathbf{PA})^2 = \|\mathbf{P} - \mathbf{A}\|^2 \quad (1.23.4)$$

$$= \|\mathbf{X} - \mathbf{A}\|^2 \quad (1.23.5)$$

$$= \|\mathbf{X}\|^2 + \|\mathbf{A}\|^2 - 2\mathbf{X}^T \mathbf{A} \quad (1.23.6)$$

and

$$(\mathbf{PB})^2 = \|\mathbf{P} - \mathbf{B}\|^2 \quad (1.23.7)$$

$$= \|\mathbf{X} - \mathbf{B}\|^2 \quad (1.23.8)$$

$$= \|\mathbf{X}\|^2 + \|\mathbf{B}\|^2 - 2\mathbf{X}^T \mathbf{B} \quad (1.23.9)$$

The given equation is

$$(\mathbf{PA})^2 + (\mathbf{PB})^2 = 2k^2 \quad (1.23.10)$$

Sub (1.23.6) and (1.23.9) values in (1.23.10)

$$\|\mathbf{X}\|^2 + \|\mathbf{A}\|^2 - 2\mathbf{X}^T \mathbf{A} + \|\mathbf{X}\|^2 + \|\mathbf{B}\|^2 - 2\mathbf{X}^T \mathbf{B} = 2k^2 \quad (1.23.11)$$

$$\Rightarrow 2\|\mathbf{X}\|^2 + \|\mathbf{A}\|^2 + \|\mathbf{B}\|^2 - 2\mathbf{X}^T (\mathbf{A} + \mathbf{B}) = 2k^2 \quad (1.23.12)$$

sub \mathbf{A}, \mathbf{B} values in equation (1.23.12), we get

$$2\|\mathbf{X}\|^2 + \left\| \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} \right\|^2 + \left\| \begin{pmatrix} -1 \\ 3 \\ -7 \end{pmatrix} \right\|^2 - 2\mathbf{X}^T \left(\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \\ -7 \end{pmatrix} \right) = 2k^2 \quad (1.23.13)$$

\therefore the required equation is

$$2\|\mathbf{X}\|^2 - 2\mathbf{X}^T \begin{pmatrix} 2 \\ 7 \\ -2 \end{pmatrix} + 109 - 2k^2 = 0 \quad (1.23.14)$$

1.24. Find the coordinates of a point which divides

the line segment joining the points $\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ and

$\begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix}$ in the ratio 2 : 3

- a) internally, and
- b) externally.

Solution:

a) The coordinates of point \mathbf{P} dividing the line

AB in the ratio $m : n$ is given by

$$\mathbf{P} = \frac{m\mathbf{B} + n\mathbf{A}}{m + n} \quad (1.24.1)$$

$$= \frac{2 \begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}}{(2 + 3)} \quad (1.24.2)$$

$$= \begin{pmatrix} \frac{9}{5} \\ \frac{2}{5} \\ \frac{-1}{5} \end{pmatrix} \quad (1.24.3)$$

which is verified in Fig. 1.24

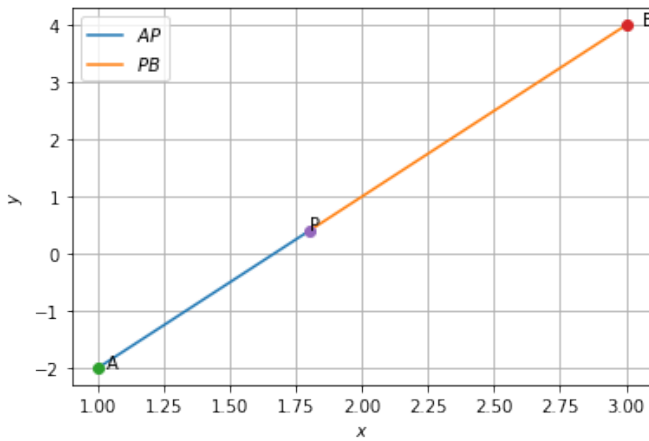


Fig. 1.24: INTERNALLY

- b) The coordinates of point \mathbf{Q} dividing the line AB in the ratio $m : n$ is given by

$$\mathbf{Q} = \frac{m\mathbf{B} - n\mathbf{A}}{m - n} \quad (1.24.4)$$

$$= \frac{2 \begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix} - 3 \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}}{(2 - 3)} \quad (1.24.5)$$

$$= \begin{pmatrix} -3 \\ -14 \\ 19 \end{pmatrix} \quad (1.24.6)$$

which is verified in Fig. 1.24

- 1.25. Prove that the three points $\begin{pmatrix} -4 \\ 6 \\ 10 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} 14 \\ 0 \\ -2 \end{pmatrix}$

are collinear.

Solution: Let

$$\mathbf{A} = \begin{pmatrix} -4 \\ 6 \\ 10 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 14 \\ 0 \\ -2 \end{pmatrix} \quad (1.25.1)$$



Fig. 1.24: EXTERNALLY

Then

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} 6 \\ -2 \\ -4 \end{pmatrix}, \mathbf{C} - \mathbf{A} = \begin{pmatrix} 18 \\ -6 \\ -12 \end{pmatrix} \quad (1.25.2)$$

$$\Rightarrow \mathbf{M} = (\mathbf{B} - \mathbf{A} \quad \mathbf{C} - \mathbf{A})^T \quad (1.25.3)$$

$$= \begin{pmatrix} 6 & -2 & -4 \\ 18 & -6 & -12 \end{pmatrix} \xrightarrow{R_2 \rightarrow R_2 - R_1} \begin{pmatrix} 6 & -2 & -4 \\ 12 & -4 & -8 \end{pmatrix} \quad (1.25.4)$$

$$\xrightarrow{R_2 \rightarrow R_2 - 2R_1} \begin{pmatrix} 6 & -2 & -4 \\ 0 & 0 & 0 \end{pmatrix} \quad (1.25.5)$$

$$\Rightarrow \text{rank}(\mathbf{M}) = 1 \quad (1.25.6)$$

Thus, the points are collinear as can be seen in Fig. 1.25

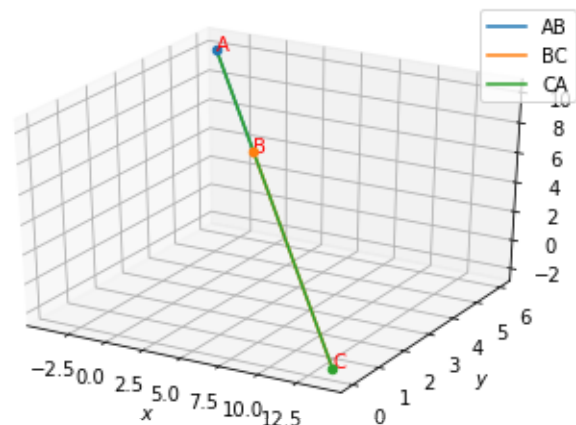


Fig. 1.25: collinear

- 1.26. Find the equation of the set of points \mathbf{P} such that its distances from the points $\mathbf{A} = \begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$ are equal.

Solution:

- a) From the given information,

$$\|\mathbf{P} - \mathbf{A}\|^2 = \|\mathbf{P} - \mathbf{B}\|^2 \quad (1.26.1)$$

$$\Rightarrow \|\mathbf{P}\|^2 + \|\mathbf{A}\|^2 - 2\mathbf{A}^T \mathbf{P} \quad (1.26.2)$$

$$= \|\mathbf{P}\|^2 + \|\mathbf{B}\|^2 - 2\mathbf{B}^T \mathbf{P} \quad (1.26.3)$$

$$\Rightarrow 2\mathbf{A}^T \mathbf{P} - 2\mathbf{B}^T \mathbf{P} = \|\mathbf{A}\|^2 - \|\mathbf{B}\|^2 \quad (1.26.4)$$

- b) Equation of plane is $\mathbf{n}^T \mathbf{P} = \mathbf{d}$
where, \mathbf{n}^T is the normal vector to the plane

- From (1.26.4),

$$(2\mathbf{A}^T - 2\mathbf{B}^T) \mathbf{P} = \|\mathbf{A}\|^2 - \|\mathbf{B}\|^2 \quad (1.26.5)$$

\mathbf{P} is a plane and it is perpendicular bisector to $\mathbf{A} - \mathbf{B}$

$\therefore \mathbf{P}$ is perpendicular to line joining \mathbf{A} and \mathbf{B}

- Midpoint of \mathbf{A} and \mathbf{B}

$$\mathbf{M} = \frac{\mathbf{A} + \mathbf{B}}{2} \quad (1.26.6)$$

- Substitute in (1.26.5),

$$(2\mathbf{A}^T - 2\mathbf{B}^T) \left(\frac{\mathbf{A} + \mathbf{B}}{2} \right) = (\mathbf{A}^T - \mathbf{B}^T) (\mathbf{A} + \mathbf{B}) \quad (1.26.7)$$

$$= \mathbf{A}^T \mathbf{A} + \mathbf{A}^T \mathbf{B} - \mathbf{B}^T \mathbf{A} - \mathbf{B}^T \mathbf{B} \quad (1.26.8)$$

$$\therefore \mathbf{A}^T \mathbf{A} = \|\mathbf{A}\|^2, \quad (1.26.9)$$

$$\mathbf{B}^T \mathbf{B} = \|\mathbf{B}\|^2, \quad (1.26.10)$$

$$\mathbf{A}^T \mathbf{B} = \mathbf{B}^T \mathbf{A} \quad (1.26.11)$$

$$\Rightarrow (2\mathbf{A}^T - 2\mathbf{B}^T) \left(\frac{\mathbf{A} + \mathbf{B}}{2} \right) = \|\mathbf{A}\|^2 - \|\mathbf{B}\|^2 \quad (1.26.12)$$

$$\Rightarrow \frac{\mathbf{A} + \mathbf{B}}{2} \text{ satisfies (1.26.4)}$$

- $\therefore \mathbf{P}$ is the plane that is perpendicular bisector of the line joining the given

points

- c) Putting given values \mathbf{A} and \mathbf{B} in (1.26.4), we get

$$2 \begin{pmatrix} 3 & 4 & -5 \end{pmatrix} \mathbf{P} - 2 \begin{pmatrix} -2 & 1 & 4 \end{pmatrix} \mathbf{P} \quad (1.26.13)$$

$$= \left\| \begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix} \right\|^2 - \left\| \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} \right\|^2 \quad (1.26.14)$$

$$\Rightarrow \begin{pmatrix} 6 & 8 & -10 \end{pmatrix} \mathbf{P} + \begin{pmatrix} 4 & -2 & -8 \end{pmatrix} \mathbf{P} \quad (1.26.15)$$

$$= 50 - 21 \quad (1.26.16)$$

$$\Rightarrow \begin{pmatrix} 10 & 6 & -18 \end{pmatrix} \mathbf{P} = 29 \quad (1.26.17)$$

\therefore The required equation is

$$\begin{pmatrix} 10 & 6 & -18 \end{pmatrix} \mathbf{P} = 29 \quad (1.26.18)$$

- 1.27. The line through the points $\begin{pmatrix} 4 \\ 8 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 8 \\ 12 \end{pmatrix}$ is perpendicular to the line through the points $\begin{pmatrix} 8 \\ 12 \\ 24 \end{pmatrix}$ and $\begin{pmatrix} x \\ 24 \end{pmatrix}$. Find the value of x .

Solution: Let

$$\mathbf{n}_1 = \begin{pmatrix} 4 \\ 8 \\ 6 \end{pmatrix} - \begin{pmatrix} -2 \\ 6 \end{pmatrix} \quad (1.27.1)$$

$$= \begin{pmatrix} 6 \\ 2 \end{pmatrix} \quad (1.27.2)$$

and

$$\mathbf{n}_2 = \begin{pmatrix} x \\ 24 \\ 12 \end{pmatrix} - \begin{pmatrix} 8 \\ 12 \end{pmatrix} \quad (1.27.3)$$

$$= \begin{pmatrix} x-8 \\ 12 \end{pmatrix} \quad (1.27.4)$$

From the given information,

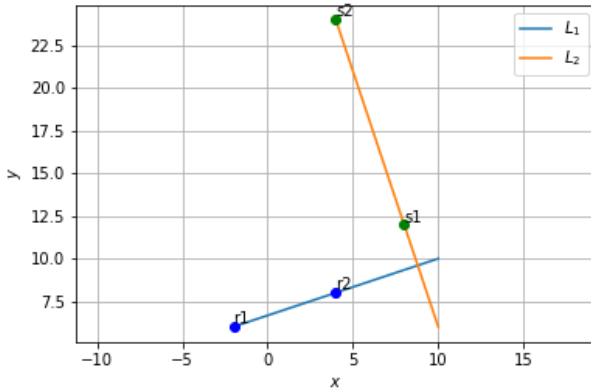
$$\mathbf{n}_1^T \mathbf{n}_2 = 0 \quad (1.27.5)$$

$$\Rightarrow \begin{pmatrix} 6 & 2 \end{pmatrix} \begin{pmatrix} x-8 \\ 12 \end{pmatrix} = 0 \quad (1.27.6)$$

$$\text{or, } x = 4 \quad (1.27.7)$$

Fig. 1.27 verifies the result.

- 1.28. Show that the line joining the origin to the

Fig. 1.27: Lines L_1 and L_2

point $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ is perpendicular to the line deter-

mined by the points $\begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \\ -1 \end{pmatrix}$.

Solution: Let

$$\mathbf{O} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{P} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 4 \\ 3 \\ -1 \end{pmatrix} \quad (1.28.1)$$

Then,

$$\mathbf{O} - \mathbf{P} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \quad (1.28.2)$$

$$= \begin{pmatrix} 4 \\ 3 \\ -1 \end{pmatrix} - \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix} \quad (1.28.3)$$

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} \quad (1.28.4)$$

and

$$(\mathbf{O} - \mathbf{P})^T (\mathbf{A} - \mathbf{B}) = 0 \quad (1.28.5)$$

$$\Rightarrow (\mathbf{O} - \mathbf{P}) \perp (\mathbf{A} - \mathbf{B}) \quad (1.28.6)$$

1.29. Show that the points $\mathbf{A} = \begin{pmatrix} 1 \\ 7 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} -4 \\ 4 \end{pmatrix}$ are the vertices of a square.

Solution: By inspection,

$$\frac{\mathbf{A} + \mathbf{C}}{2} = \frac{\mathbf{B} + \mathbf{D}}{2} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} \quad (1.29.1)$$

Hence, the diagonals AC and BD bisect each other. Also,

$$(\mathbf{A} - \mathbf{C})^T (\mathbf{B} - \mathbf{D}) = 0 \quad (1.29.2)$$

$\Rightarrow AC \perp BD$. Hence $ABCD$ is a square.

1.30. If the points $\mathbf{A} = \begin{pmatrix} 6 \\ 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 8 \\ 2 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 9 \\ 4 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} p \\ 3 \end{pmatrix}$ are the vertices of a parallelogram, taken in order, find the value of p .

Solution: In the parallelogram $ABCD$, AC and BD bisect each other. This can be used to find p .

1.31. If $\mathbf{A} = \begin{pmatrix} -5 \\ 7 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -4 \\ -5 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -1 \\ -6 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$, find the area of the quadrilateral $ABCD$.

Solution: The area of $ABCD$ is the sum of the areas of triangles ABD and CBD and is given by

$$\frac{1}{2} \|(\mathbf{A} - \mathbf{B}) \times (\mathbf{A} - \mathbf{D})\| + \frac{1}{2} \|(\mathbf{C} - \mathbf{B}) \times (\mathbf{C} - \mathbf{D})\| \quad (1.31.1)$$

1.32. Show that the points $\mathbf{A} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 4 \\ 7 \\ 6 \end{pmatrix}$ are the vertices of a parallelo-

gram $ABCD$ but it is not a rectangle.

Solution: Since the direction vectors

$$\mathbf{A} - \mathbf{B} = \mathbf{D} - \mathbf{C} \quad (1.32.1)$$

$$\mathbf{A} - \mathbf{D} = \mathbf{B} - \mathbf{C} \quad (1.32.2)$$

$AB \parallel CD$ and $AD \parallel BC$. Hence $ABCD$ is a parallelogram. However,

$$(\mathbf{A} - \mathbf{B})^T (\mathbf{A} - \mathbf{D}) \neq 0 \quad (1.32.3)$$

Hence, it is not a rectangle. The following code plots Fig. 1.32

codes/triangle/quad_3d.py

1.33. Find the area of a parallelogram whose adja-

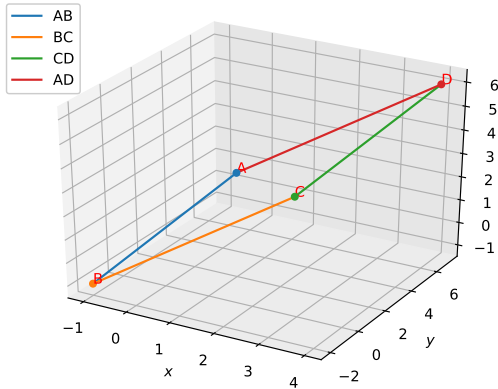


Fig. 1.32

cent sides are given by the vectors $\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$ and

$$\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

Solution: The area is given by

$$\frac{1}{2} \left\| \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\| \quad (1.33.1)$$

- 1.34. $ABCD$ is a rectangle formed by the points $A = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$, $B = \begin{pmatrix} -1 \\ 4 \\ 4 \end{pmatrix}$, $C = \begin{pmatrix} 5 \\ 4 \\ 4 \end{pmatrix}$, $D = \begin{pmatrix} 5 \\ -1 \\ -1 \end{pmatrix}$. P, Q, R, S are the mid points of AB, BC, CD, DA respectively. Is the quadrilateral $PQRS$ a

- square?
- rectangle?
- rhombus?

Solution:

$$\begin{aligned} P &= \frac{A+B}{2} = \begin{pmatrix} -1 & \frac{3}{2} \end{pmatrix} \\ Q &= \frac{B+C}{2} = \begin{pmatrix} 2 & 4 \end{pmatrix} \\ R &= \frac{C+D}{2} = \begin{pmatrix} 5 & \frac{3}{2} \end{pmatrix} \\ S &= \frac{A+D}{2} = \begin{pmatrix} 2 & -1 \end{pmatrix} \end{aligned} \quad (1.34.1)$$

\therefore

$$\frac{P+R}{2} = \frac{Q+S}{2} = \frac{1}{2} \begin{pmatrix} 4 \\ 3 \end{pmatrix} \quad (1.34.2)$$

$PQRS$ is a parallelogram.

$$(P-R) = \begin{pmatrix} -6 & 0 \end{pmatrix} (Q-S) = \begin{pmatrix} 0 \\ 5 \end{pmatrix} \quad (1.34.3)$$

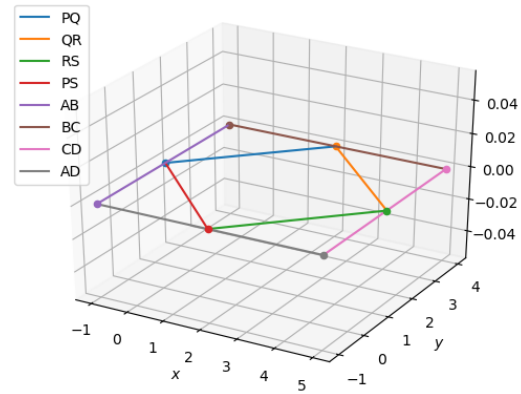
$$(1.34.4)$$

$$(P-R)^T(Q-S) = \begin{pmatrix} -6 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 5 \end{pmatrix} \quad (1.34.5)$$

$$(P-R)^T(Q-S) = \begin{pmatrix} 0 \end{pmatrix} \quad (1.34.6)$$

$$(1.34.7)$$

Diagonal bisect orthogonally. Thus, $PQRS$ is a rhombus. Se Fig. 1.34

Fig. 1.34: Simulation of midpoint of $ABCD$ forms $PQRS$.

Step4: We will check whether Parallelogram $PQRS$ is Square or not.

$$(P-Q) = \frac{1}{2} \begin{pmatrix} -6 \\ -5 \end{pmatrix} \quad (1.34.8)$$

$$(P-S) = \frac{1}{2} \begin{pmatrix} -6 \\ 5 \end{pmatrix} \quad (1.34.9)$$

$$(1.34.10)$$

If adjacent side of parallelogram are orthogonal to each other then $PQRS$ is a Square.

$$(P-Q)^T(P-S) = \frac{1}{4} \begin{pmatrix} -6 & -5 \end{pmatrix} \begin{pmatrix} -6 \\ 5 \end{pmatrix} \neq 0 \quad (1.34.11)$$

Here the angle between adjacent side is not 90° . Hence, $PQRS$ is not a Square.

1.35. $ABCD$ is a cyclic quadrilateral with

$$\angle A = 4y + 20 \quad (1.35.1)$$

$$\angle B = 3y - 5 \quad (1.35.2)$$

$$\angle C = -4x \quad (1.35.3)$$

$$\angle D = -7x + 5 \quad (1.35.4)$$

Find its angles.

Solution: From the given information,

$$\angle A + \angle C = 180^\circ \quad (1.35.5)$$

$$\angle B + \angle D = 180^\circ \quad (1.35.6)$$

which can be expressed as

$$\begin{pmatrix} -4 & 4 \\ -7 & 3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 160 \\ 180 \end{pmatrix} \quad (1.35.7)$$

and solved as

$$\begin{pmatrix} -4 & 4 & 160 \\ -7 & 3 & 180 \end{pmatrix} \xrightarrow{R_1 \leftarrow \frac{-R_1}{4}} \begin{pmatrix} 1 & -1 & -40 \\ -7 & 3 & 180 \end{pmatrix} \quad (1.35.8)$$

$$\xrightarrow{R_2 \leftarrow R_2 + 7R_1} \begin{pmatrix} 1 & -1 & -40 \\ 0 & -4 & -100 \end{pmatrix} \xrightarrow{R_2 \leftarrow \frac{-R_2}{4}} \begin{pmatrix} 1 & -1 & -40 \\ 0 & 1 & 25 \end{pmatrix} \quad (1.35.9)$$

$$\xrightarrow{R_1 \leftarrow R_1 + R_2} \begin{pmatrix} 1 & 0 & -15 \\ 0 & 1 & 25 \end{pmatrix} \quad (1.35.10)$$

Thus,

$$x = -15, y = 25 \quad (1.35.11)$$

$$\Rightarrow \angle A = 120^\circ, \angle B = 70^\circ, \quad (1.35.12)$$

$$\Rightarrow \angle C = 60^\circ, \angle D = 110^\circ \quad (1.35.13)$$

1.36. Draw a quadrilateral in the Cartesian plane, whose vertices are $\begin{pmatrix} -4 \\ 5 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 7 \end{pmatrix}$, $\begin{pmatrix} 5 \\ -5 \end{pmatrix}$ and $\begin{pmatrix} -4 \\ -2 \end{pmatrix}$.

Also, find its area.

Solution: Let

$$\mathbf{A} = \begin{pmatrix} -4 \\ 5 \end{pmatrix} \mathbf{B} = \begin{pmatrix} 0 \\ 7 \end{pmatrix} \mathbf{C} = \begin{pmatrix} 5 \\ -5 \end{pmatrix} \mathbf{D} = \begin{pmatrix} -4 \\ -2 \end{pmatrix} \quad (1.36.1)$$

Quadrilateral $ABCD$ is drawn by joining its vertices \mathbf{A} and \mathbf{B} , \mathbf{B} and \mathbf{C} , \mathbf{C} and \mathbf{D} , \mathbf{D} and \mathbf{A} . The following Python code generates Fig. 1.36

```
codes/quad/quad.py
```

From Figure 1.36 Area of the Quadrilateral

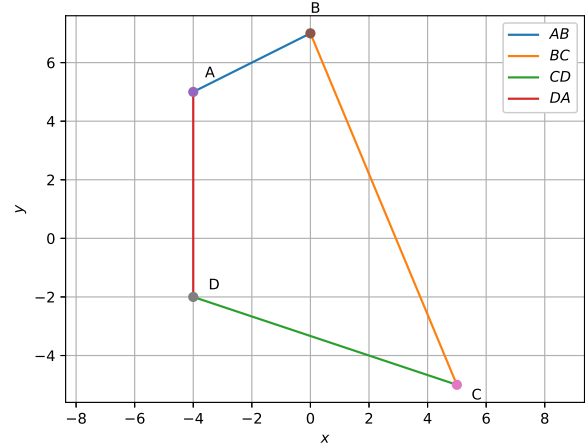


Fig. 1.36: Quadrilateral $ABCD$

$ABCD$ can be given as

$$Ar(\triangle ABC) + Ar(\triangle BCD) \quad (1.36.2)$$

$$\frac{1}{2} \|(\mathbf{A} - \mathbf{B}) \times (\mathbf{A} - \mathbf{D})\| + \frac{1}{2} \|(\mathbf{C} - \mathbf{B}) \times (\mathbf{C} - \mathbf{D})\| \quad (1.36.3)$$

For two vectors $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

$$\|\mathbf{a} \times \mathbf{b}\| = |a_1 b_2 - a_2 b_1| \quad (1.36.4)$$

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} -4 \\ -2 \end{pmatrix} \quad (1.36.5)$$

$$\mathbf{A} - \mathbf{D} = \begin{pmatrix} 0 \\ 7 \end{pmatrix} \quad (1.36.6)$$

$$\mathbf{C} - \mathbf{B} = \begin{pmatrix} 5 \\ -12 \end{pmatrix} \quad (1.36.7)$$

$$\mathbf{C} - \mathbf{D} = \begin{pmatrix} 9 \\ -3 \end{pmatrix} \quad (1.36.8)$$

Using (1.36.4)

$$\frac{1}{2} \|(\mathbf{A} - \mathbf{B}) \times (\mathbf{A} - \mathbf{D})\| = \frac{1}{2} |(-28)| \quad (1.36.9)$$

$$= 14 \quad (1.36.10)$$

$$\frac{1}{2} \|(\mathbf{C} - \mathbf{B}) \times (\mathbf{C} - \mathbf{D})\| = \frac{1}{2} |(-15 + 108)| \quad (1.36.11)$$

$$= 46.5 \quad (1.36.12)$$

Substituting the above values in equation

(1.36.3), We get

$$\text{Area} = 14 + 46.5 = 60.5 \text{ sq. units} \quad (1.36.13)$$

1.37. Find the area of a rhombus if its vertices are

$$\mathbf{P} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}, \quad (1.37.1)$$

$$\mathbf{R} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}, \mathbf{S} = \begin{pmatrix} -2 \\ -1 \end{pmatrix} \quad (1.37.2)$$

taken in order.

Solution: In Fig. 1.37,

$$\mathbf{P} - \mathbf{S} = \begin{pmatrix} 3+2 \\ 0+1 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix} \quad (1.37.3)$$

$$\mathbf{Q} - \mathbf{P} = \begin{pmatrix} 4-3 \\ 5-0 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \quad (1.37.4)$$

Thus, the area of the rhombus can be calculated as

$$\|(\mathbf{P} - \mathbf{S}) \times (\mathbf{Q} - \mathbf{P})\| = \left\| \begin{pmatrix} 5 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 5 \end{pmatrix} \right\| \quad (1.37.5)$$

$$\|\Delta\| = 5 \times 5 - 1 \times 1 = 24 \quad (1.37.6)$$

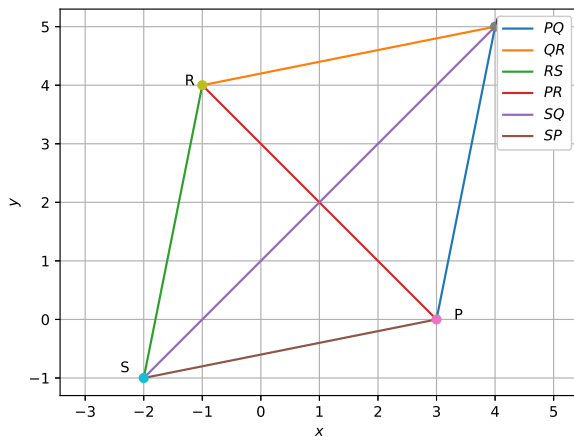


Fig. 1.37

solutions/4/codes/quadrilateral/quad.py

1.38. Without using distance formula, show that points $\begin{pmatrix} -2 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 4 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$ are the vertices of a parallelogram.

Solution: The following python code plots Fig.1.38.

./solutions/5/codes/quadrilateral/q4.py

$$\therefore \mathbf{A} - \mathbf{B} = \mathbf{D} - \mathbf{C} \quad (1.38.1)$$

$$\mathbf{A} - \mathbf{D} = \mathbf{B} - \mathbf{C}, \quad (1.38.2)$$

$AB \parallel CD$ and $AD \parallel BC$. Hence, $ABCD$ is a ||gm.

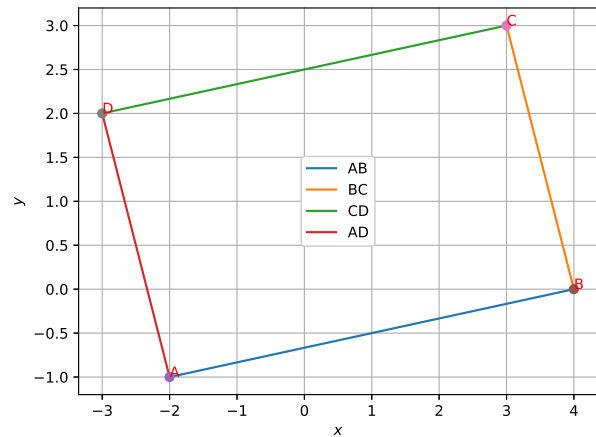


Fig. 1.38

1.39. Find the area of the quadrilateral whose vertices, taken in order, are $\begin{pmatrix} -4 \\ 2 \end{pmatrix}$, $\begin{pmatrix} -3 \\ -5 \end{pmatrix}$, $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

Solution: See quadrilateral $ABCD$ in Fig.1.39 is generated using the following python code

solutions/6/codes/quadrilateral/quad.py

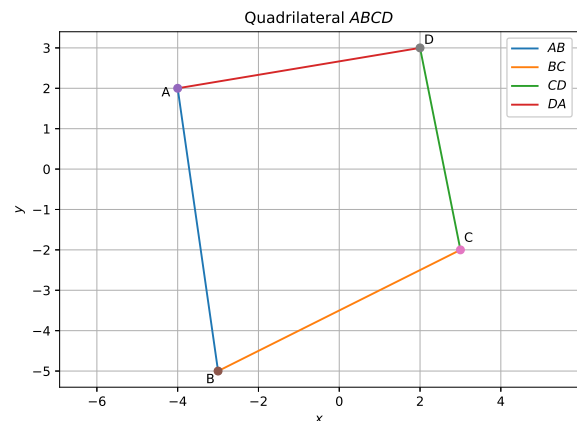


Fig. 1.39: Quadrilateral $ABCD$ using python

$$ar(ABCD) = ar(\triangle ABC) + ar(\triangle ACD) \quad (1.39.1)$$

$$= \frac{1}{2} \|(\mathbf{B} - \mathbf{A}) \times (\mathbf{C} - \mathbf{A})\| \quad (1.39.2)$$

$$+ \frac{1}{2} \|(\mathbf{C} - \mathbf{A}) \times (\mathbf{D} - \mathbf{A})\| \quad (1.39.3)$$

$$= \frac{1}{2} \left\| \begin{pmatrix} 1 \\ -7 \end{pmatrix} \times \begin{pmatrix} 7 \\ -4 \end{pmatrix} \right\| \quad (1.39.4)$$

$$+ \frac{1}{2} \left\| \begin{pmatrix} 7 \\ -4 \end{pmatrix} \times \begin{pmatrix} 6 \\ 1 \end{pmatrix} \right\| \quad (1.39.5)$$

$$= 38 \quad (1.39.6)$$

and verified using the following codes

```
solutions/6/codes/tri_area_ABC.py
```

```
solutions/6/codes/tri_area_ACD.py
```

- 1.40. The two opposite vertices of a square are $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$. Find the coordinates of the other two vertices.

Solution: See Fig. 1.40.

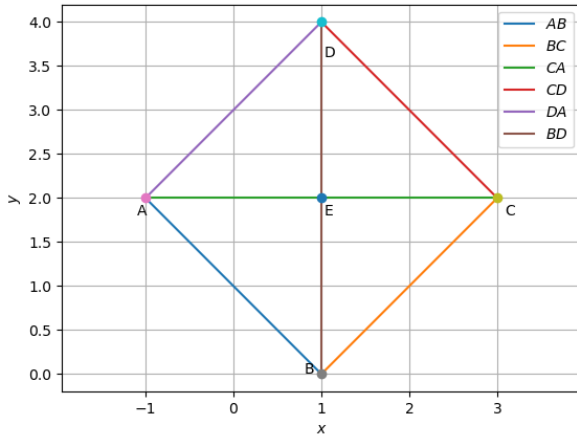


Fig. 1.40: Square ABCD

- a) From inspection we see that the opposite vertices forms a diagonal which is parallel to x-axis. Then the diagonal formed by other two vertices is parallel to y-axis(i.e. their x coordinates are equal). Let $\mathbf{A} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

- b) Diagonals bisect each other at 90° . Let \mathbf{B} and \mathbf{D} be other two vertices.
c) Using the property that diagonals bisect each other at 90° , we can obtain other vertices by rotating diagonal AC by 90° about \mathbf{E} in clockwise or anticlockwise direction.
d) The rotation matrix for a rotation of angle 90° about origin in anticlockwise direction is given by

$$\begin{pmatrix} \cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (1.40.1)$$

The \mathbf{E} is given by

$$\mathbf{E} = \frac{\mathbf{A} + \mathbf{C}}{2} \quad (1.40.2)$$

$$= \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (1.40.3)$$

- e) To make the rotation we need to shift the \mathbf{E} to origin. So the change in other vectors are

$$\mathbf{A} - \mathbf{E} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \quad (1.40.4)$$

$$\mathbf{C} - \mathbf{E} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad (1.40.5)$$

The required matrix now is $\begin{pmatrix} -2 & 2 \\ 0 & 0 \end{pmatrix}$. Multiplying this with rotation matrix

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -2 & 2 \\ 0 & 0 \end{pmatrix} \quad (1.40.6)$$

$$= \begin{pmatrix} 0 & 0 \\ -2 & 2 \end{pmatrix} \quad (1.40.7)$$

Now we obtained the coordinates as $\begin{pmatrix} 0 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$. To obtain the final coordinates we will add \mathbf{E} to shift to the actual position.

$$\mathbf{B} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (1.40.8)$$

$$\mathbf{D} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (1.40.9)$$

Thus

$$\mathbf{B} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.40.10)$$

$$\mathbf{D} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \quad (1.40.11)$$

f) The python code for the figure can be downloaded from

solutions/7/codes/quad/quad.py

1.41. Find the area of a parallelogram whose adjacent sides are given by the vectors $\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$ and

$$\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

Solution: The area of a parallelogram is defined as

$$\|\mathbf{a} \times \mathbf{b}\| \quad (1.41.1)$$

where

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \quad (1.41.2)$$

$$= \begin{pmatrix} 0 & -4 & 1 \\ 4 & 0 & -3 \\ -1 & 3 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ -4 \end{pmatrix} \quad (1.41.3)$$

Thus, the desired area is

$$\|\mathbf{a} \times \mathbf{b}\| = \sqrt{5^2 + 1^2 + (-1)^2} \quad (1.41.4)$$

$$= 3\sqrt{3} \quad (1.41.5)$$

The following Python code generates Fig. 1.41

codes/parallelogram.py

The following Python code verifies the cross-product value.

codes/cross_product_check.py

1.42. Find the area of a rectangle $ABCD$ with vertices $\mathbf{A} = \begin{pmatrix} -1 \\ \frac{1}{2} \\ 4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 \\ \frac{1}{2} \\ 4 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ 4 \end{pmatrix}$, $\mathbf{D} =$

$$\begin{pmatrix} -1 \\ -\frac{1}{2} \\ 4 \end{pmatrix}.$$

Solution: Area of rectangle = cross product of

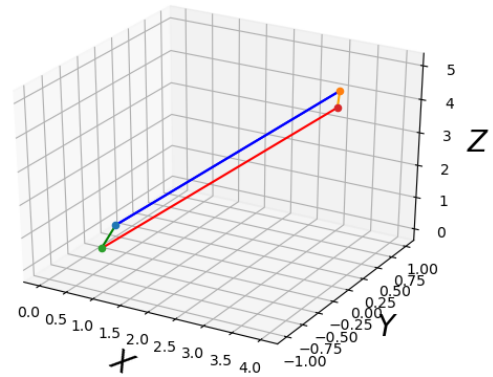


Fig. 1.41: Parallelogram generated using python 3D-plot

vectors of adjacent sides

$$\mathbf{A} - \mathbf{D} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \quad \mathbf{B} - \mathbf{A} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \quad (1.42.1)$$

Area = cross product of vectors

$$\|(\mathbf{A} - \mathbf{D}) \times (\mathbf{B} - \mathbf{A})\| \quad (1.42.2)$$

$$= \left\| \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \right\| \quad (1.42.3)$$

$$= \left\| \begin{pmatrix} 0 & -0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \right\| \quad (1.42.4)$$

$$= 2 \quad (1.42.5)$$

Area = 2

1.43. A town B is located 36km east and 15 km north of the town A. How would you find the distance from town A to town B without actually measuring it?

Solution: See Fig. 1.43.

$$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 36 \\ 15 \end{pmatrix} \quad (1.43.1)$$

The distance d between A and B is given by

$$\|\mathbf{B} - \mathbf{A}\| = \|\mathbf{B}\| \quad (1.43.2)$$

$$= 39\text{km} \quad (1.43.3)$$

The following Python code generates Fig. 1.43.

solutions/3/codes/line/towns/towns.py

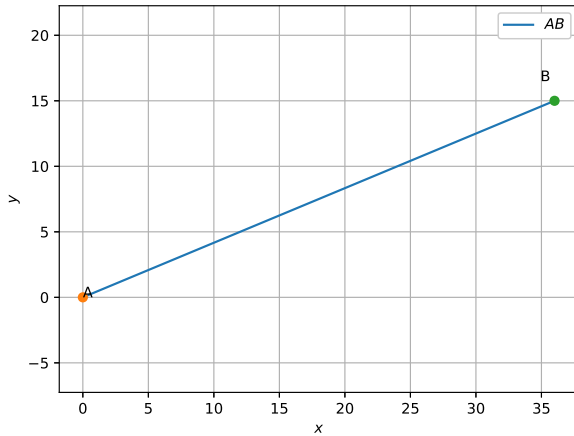


Fig. 1.43: Position of Towns A and B

1.44. Find the angle between the x-axis and the line joining the points $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ -2 \end{pmatrix}$. **Solution:**

$$\frac{(\mathbf{A} - \mathbf{B})^T \begin{pmatrix} 1 \\ 0 \end{pmatrix}}{\|\mathbf{A} - \mathbf{B}\| \left\| \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\|} = \frac{(-1 \ 1)^T \begin{pmatrix} 1 \\ 0 \end{pmatrix}}{\left\| \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\| \left\| \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\|} \quad (1.44.1)$$

$$= -\frac{1}{\sqrt{2}} = \cos^{-1}(135^\circ) \quad (1.44.2)$$

Thus, the desired angle is 135° . The following python code generates Fig. 1.44.

```
./solutions/5/codes/lines/q9.py
```

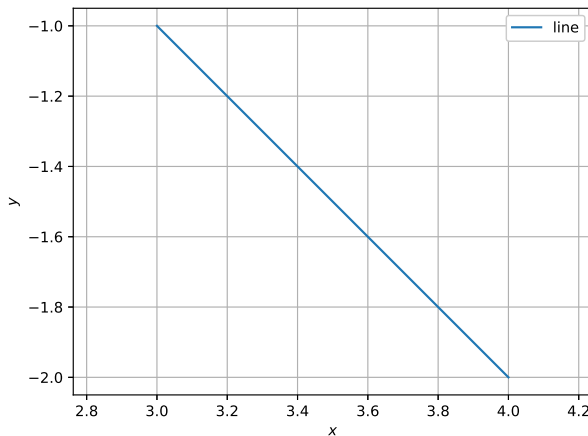


Fig. 1.44

1.45. Find the point on the x-axis which is equidistant from

$$\begin{pmatrix} 2 \\ -5 \end{pmatrix}, \begin{pmatrix} -2 \\ 9 \end{pmatrix}, \quad (1.45.1)$$

Solution: From the given information,

$$\left\| \mathbf{x} - \begin{pmatrix} 2 \\ -5 \end{pmatrix} \right\|^2 = \left\| \mathbf{x} - \begin{pmatrix} -2 \\ 9 \end{pmatrix} \right\|^2 \quad (1.45.2)$$

$$\begin{aligned} \Rightarrow \|\mathbf{x}\|^2 + \left\| \begin{pmatrix} 2 \\ -5 \end{pmatrix} \right\|^2 - 2 \begin{pmatrix} 2 & -5 \end{pmatrix} \mathbf{x} \\ = \|\mathbf{x}\|^2 + \left\| \begin{pmatrix} -2 \\ 9 \end{pmatrix} \right\|^2 - 2 \begin{pmatrix} -2 & 9 \end{pmatrix} \mathbf{x} \end{aligned} \quad (1.45.3)$$

which can be simplified to obtain

$$\begin{pmatrix} 8 & -28 \end{pmatrix} \mathbf{x} = -56 \quad (1.45.4)$$

Choose $\mathbf{x} = \begin{pmatrix} x \\ 0 \end{pmatrix}$ as the point lies on the x-axis

$$\begin{pmatrix} 8 & -28 \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} = -56 \quad (1.45.5)$$

$$\Rightarrow x = -7 \quad (1.45.6)$$

The desired point is $\begin{pmatrix} -7 \\ 0 \end{pmatrix}$.

See Fig. 1.45 generated by the following python code

```
solutions/6/codes/line/point_vector/
point_vector.py
```

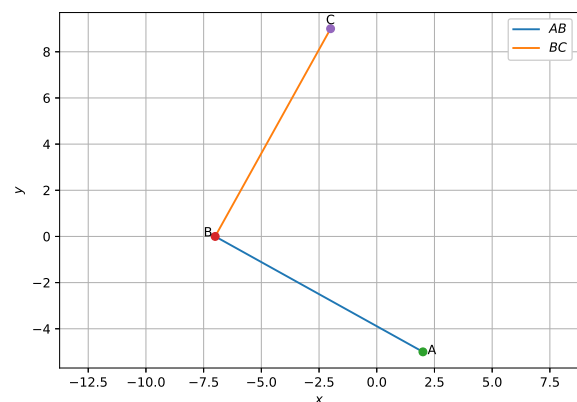


Fig. 1.45

1.46. Find the values of y for which the distance between the points

$$\mathbf{P} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} 10 \\ y \end{pmatrix} \quad (1.46.1)$$

is 10 units. **Solution:** The distance between two points is given by equation

$$(\mathbf{P} - \mathbf{Q})^T (\mathbf{P} - \mathbf{Q}) = 10^2 \quad (1.46.2)$$

$$\Rightarrow \|\mathbf{P}\|^2 - \mathbf{P}^T \mathbf{Q} - \mathbf{Q}^T \mathbf{P} + \|\mathbf{Q}\|^2 = 100 \quad (1.46.3)$$

which, upon substituting the values yields

$$y^2 + 6y - 27 = 0 \quad (1.46.4)$$

$$(y + 9)(y - 3) = 0 \Rightarrow y = -9, 3 \quad (1.46.5)$$

and

$$\mathbf{Q} = \begin{pmatrix} 10 \\ 3 \end{pmatrix}, \begin{pmatrix} 10 \\ -9 \end{pmatrix} \quad (1.46.6)$$

The python code to find the roots of the quadratic equation can be downloaded from

`solutions/7/codes/line/point_vec/roots.py`

The python code for Fig. 1.46 can be downloaded from

`solutions/7/codes/line/point_vec/point_vec.py`

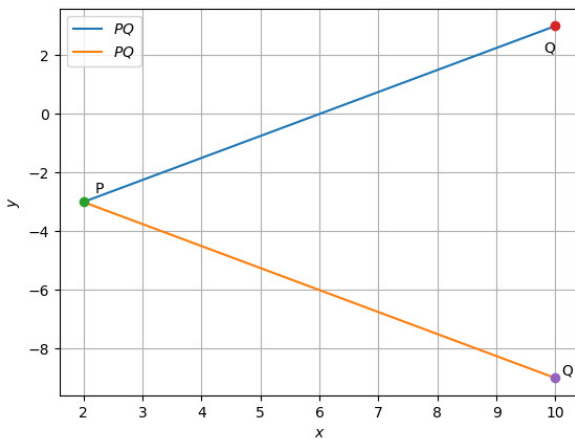


Fig. 1.46

unit vector

$$\frac{1}{7} \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}, \frac{1}{7} \begin{pmatrix} 3 \\ -6 \\ 2 \end{pmatrix}, \frac{1}{7} \begin{pmatrix} 6 \\ 2 \\ -3 \end{pmatrix}. \quad (1.47.1)$$

Also, show that they are mutually perpendicular to each other.

Solution: Let $\mathbf{A} = \frac{1}{7} \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}, \mathbf{B} = \frac{1}{7} \begin{pmatrix} 3 \\ -6 \\ 2 \end{pmatrix}, \mathbf{C} = \frac{1}{7} \begin{pmatrix} 6 \\ 2 \\ -3 \end{pmatrix}$

$$\|\mathbf{A}\| = \frac{1}{7} \sqrt{2^2 + 3^2 + 6^2} = 1 \quad (1.47.2)$$

$$\|\mathbf{B}\| = \frac{1}{7} \sqrt{3^2 + (-6)^2 + 2^2} = 1 \quad (1.47.3)$$

$$\|\mathbf{C}\| = \frac{1}{7} \sqrt{6^2 + 2^2 + (-3)^2} = 1 \quad (1.47.4)$$

When two vectors are perpendicular to each other their dot product is zero. The dot product of \mathbf{A}, \mathbf{B} and \mathbf{C} with each other is

$$\mathbf{A}^T \mathbf{B} = \frac{1}{7} \times \frac{1}{7} (2 \times 3 + 3 \times -6 + 6 \times 2) = 0 \quad (1.47.5)$$

$$\mathbf{B}^T \mathbf{C} = \frac{1}{7} \times \frac{1}{7} (2 \times 3 + 3 \times -6 + 6 \times 2) = 0 \quad (1.47.6)$$

$$\mathbf{C}^T \mathbf{A} = \frac{1}{7} \times \frac{1}{7} (6 \times 2 + 2 \times 3 + -3 \times 6) = 0 \quad (1.47.7)$$

Hence, the three unit vectors are mutually perpendicular to each other.

1.48. For

$$\mathbf{a} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \quad (1.48.1)$$

$(\mathbf{a} + k\mathbf{b}) \perp \mathbf{c}$. Find λ . **Solution:**

The two vectors are perpendicular to each other if their dot product is zero.

So,

$$\mathbf{c}^T (\mathbf{a} + k\mathbf{b}) = 0 \quad (1.48.2)$$

$$\mathbf{c}^T \mathbf{a} + k\mathbf{c}^T \mathbf{b} = 0 \quad (1.48.3)$$

$$k\mathbf{c}^T \mathbf{b} = -\mathbf{c}^T \mathbf{a} \quad (1.48.4)$$

$$\Rightarrow k = \frac{-\mathbf{c}^T \mathbf{a}}{\mathbf{c}^T \mathbf{b}} \quad (1.48.5)$$

1.47. Show that each of the given three vectors is a

On solving the matrix multiplication,

$$\mathbf{c}^T \mathbf{b} = -1, \quad (1.48.6)$$

$$\mathbf{c}^T \mathbf{a} = 8 \quad (1.48.7)$$

So,

$$\Rightarrow k = \frac{-8}{-1} \quad (1.48.8)$$

$$k = 8 \quad (1.48.9)$$

1.49. Find $\mathbf{a} \times \mathbf{b}$ if

$$\mathbf{a} = \begin{pmatrix} 1 \\ -7 \\ 7 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}. \quad (1.49.1)$$

Solution: Cross product of two vectors is determined by spanning a vector into skew symmetric matrix

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 & -7 & -7 \\ 7 & 0 & -1 \\ 7 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 19 \\ 19 \end{pmatrix} \quad (1.49.2)$$

1.50. Find a unit vector perpendicular to each of the vectors $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$, where

$$\mathbf{a} = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}. \quad (1.50.1)$$

Solution: Let $\mathbf{A} = \mathbf{a} + \mathbf{b}$ and $\mathbf{B} = \mathbf{a} - \mathbf{b}$

$$\mathbf{A} = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 0 \end{pmatrix} \quad (1.50.2)$$

$$\mathbf{B} = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} \quad (1.50.3)$$

Let \mathbf{n} be a vector Perpendicular to \mathbf{A} and \mathbf{B} both

$$\mathbf{A}^T \mathbf{n} = 0 \quad (1.50.4)$$

$$\mathbf{B}^T \mathbf{n} = 0 \quad (1.50.5)$$

The augmented matrix can be represented as follows:

$$\left(\begin{array}{ccc|c} 4 & 4 & 0 & 0 \\ 2 & 0 & 4 & 0 \end{array} \right) \quad (1.50.6)$$

Using row reduction to find an expression for

n.

$$\xrightarrow[R_2 \leftarrow R_2 - 2R_1]{R_1 \leftarrow \frac{R_1}{4}} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -2 & 4 & 0 \end{array} \right) \quad (1.50.7)$$

$$\xrightarrow[R_1 \leftarrow R_1 - R_2]{R_2 \leftarrow \frac{R_2}{-2}} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -2 & 0 \end{array} \right) \quad (1.50.8)$$

From above equations we get,

$$\therefore \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} -2n_3 \\ 2n_3 \\ n_3 \end{pmatrix} = n_3 \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} \quad (1.50.9)$$

Let us consider n_3 to be 1 which gives us:

$$\therefore \mathbf{n} = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} \quad (1.50.10)$$

$\|\mathbf{n}\| = \sqrt{(-2)^2 + 2^2 + 1^2} = 3$ (1.50.11)
Let \mathbf{u} be the unit vector of \mathbf{n} which can be found as follows:

$$\mathbf{u} = \frac{\mathbf{n}}{\|\mathbf{n}\|} \quad (1.50.12)$$

Solving the above equation gives the unit vector \mathbf{u} which is perpendicular to vectors \mathbf{A} and \mathbf{B}

$$\therefore \mathbf{u} = \frac{1}{3} \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix} \quad (1.50.13)$$

1.51. If $\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$, find a unit vector parallel to the vector $2\mathbf{a} - \mathbf{b} + 3\mathbf{c}$.

Solution:

$$\mathbf{d} = 2\mathbf{a} - \mathbf{b} + 3\mathbf{c} \quad (1.51.1)$$

$$2\mathbf{a} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \quad (1.51.2)$$

$$-\mathbf{b} = \begin{pmatrix} -2 \\ 1 \\ -3 \end{pmatrix} \quad (1.51.3)$$

$$3\mathbf{c} = \begin{pmatrix} 3 \\ -6 \\ 3 \end{pmatrix} \quad (1.51.4)$$

From the above,

$$\mathbf{d} = \begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix} \quad (1.51.5)$$

$$\|\mathbf{d}\| = \sqrt{3^2 + (-3)^2 + 2^2} = \sqrt{22} \quad (1.51.6)$$

$$\mathbf{e} = \frac{\mathbf{d}}{\|\mathbf{d}\|} \quad (1.51.7)$$

\mathbf{e} is the unit vector parallel to given vector

Thus,

$$\boxed{\mathbf{e} = \frac{1}{\sqrt{22}} \begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix}} \quad (1.51.8)$$

1.52. Find a vector of magnitude 5 units, and parallel

to the resultant of the vectors $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$, $\mathbf{b} =$

$$\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix},$$

Solution: First find resultant \mathbf{R} of $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$

$$\text{and } \mathbf{b} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$\mathbf{R} = \mathbf{a} + \mathbf{b} \quad (1.52.1)$$

$$\Rightarrow \mathbf{R} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \quad (1.52.2)$$

$$\Rightarrow \mathbf{R} = \begin{pmatrix} 2+1 \\ 3-2 \\ -1+1 \end{pmatrix} \quad (1.52.3)$$

$$\Rightarrow \mathbf{R} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}. \quad (1.52.4)$$

Magnitude of \mathbf{R} is

$$\|\mathbf{R}\| = \sqrt{3^2 + 1^2 + 0^2} \quad (1.52.5)$$

$$\Rightarrow \|\mathbf{R}\| = \sqrt{10} \quad (1.52.6)$$

$$(1.52.7)$$

Then unit vector \mathbf{r} along \mathbf{R} is

$$\mathbf{r} = \frac{\mathbf{R}}{\|\mathbf{R}\|} \quad (1.52.8)$$

$$\Rightarrow \mathbf{r} = \frac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \quad (1.52.9)$$

Then vector of magnitude 5 units parallel to resultant \mathbf{R} is given by

$$\mathbf{u} = 5\mathbf{r} \quad (1.52.10)$$

$$\Rightarrow \mathbf{u} = \frac{5}{\sqrt{10}} \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \quad (1.52.11)$$

$$\Rightarrow \mathbf{u} = \begin{pmatrix} 4.7434 \\ 1.5811 \\ 0 \end{pmatrix} \quad (1.52.12)$$

1.53. Show that the unit direction vector inclined equally to the coordinate axes is $\begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$.

Solution: Let \mathbf{m} be a unit vector such that $\mathbf{m} = \begin{pmatrix} m_x \\ m_y \\ m_z \end{pmatrix}$. Let $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ be the direction vectors of the coordinate axes.

As \mathbf{m} is a unit vector, so $\|\mathbf{m}\| = 1$ and also we are given is that \mathbf{m} is inclined equally to the coordinate axis,

$$\mathbf{e}_1^T \mathbf{m} = \mathbf{e}_2^T \mathbf{m} = \mathbf{e}_3^T \mathbf{m} \quad (1.53.1)$$

Now, 1.53.1 implies

$$(\mathbf{e}_1 - \mathbf{e}_2)^T \mathbf{m} = 0 \quad (1.53.2)$$

$$(\mathbf{e}_2 - \mathbf{e}_3)^T \mathbf{m} = 0 \quad (1.53.3)$$

$$(\mathbf{e}_3 - \mathbf{e}_1)^T \mathbf{m} = 0 \quad (1.53.4)$$

Thus, converting above system of equations into matrix form, we get

$$\mathbf{A}\mathbf{m} = 0 \quad (1.53.5)$$

To find the solution of 1.53.5, we find the

echelon form of \mathbf{A} .

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix} \xleftrightarrow{r_3 \leftarrow r_1 + r_3} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \quad (1.53.6)$$

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \xleftrightarrow{r_3 \leftarrow r_2 + r_3} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \quad (1.53.7)$$

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \xleftrightarrow{r_1 \leftarrow r_1 + r_2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \quad (1.53.8)$$

From 1.53.8, we find out that

$$m_x = m_y = m_z \quad (1.53.9)$$

$$\mathbf{m} = \begin{pmatrix} m_z \\ m_z \\ m_z \end{pmatrix} \Rightarrow \mathbf{m} = m_z \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad (1.53.10)$$

Taking $m_z = 1$, then $\|\mathbf{m}\| = \frac{1}{\sqrt{3}}$ and for \mathbf{m} to be a unit vector, we need to divide each element of \mathbf{m} by $\|\mathbf{m}\|$.

Thus, we see that

$$\mathbf{m} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} \quad (1.53.11)$$

is the unit direction vector inclined equally to the coordinate axes.

1.54. Let $\mathbf{a} = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 3 \\ -2 \\ 7 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$. Find a

vector \mathbf{d} such that $\mathbf{d} \perp \mathbf{a}$, $\mathbf{d} \perp \mathbf{b}$ and $\mathbf{d}^T \mathbf{c} = 15$.

Solution: From the given information

$$\mathbf{d}^T \mathbf{a} = 0 \quad (1.54.1)$$

Similarly, as $\mathbf{d} \perp \mathbf{b}$

$$\mathbf{d}^T \mathbf{b} = 0 \quad (1.54.2)$$

It is given that

$$\mathbf{d}^T \mathbf{c} = 15 \quad (1.54.3)$$

Using equations 1.54.1, 1.54.2, 1.54.3, we can represent them in a Matrix Representation of Linear Equations $Ax=B$ form as:

$$\begin{pmatrix} \mathbf{a}^T \\ \mathbf{b}^T \\ \mathbf{c}^T \end{pmatrix} \mathbf{d} = \begin{pmatrix} 0 \\ 0 \\ 15 \end{pmatrix} \quad (1.54.4)$$

Numerically, using \mathbf{a} , \mathbf{b} , \mathbf{c} the above equation 1.54.4 can be written as,

$$\begin{pmatrix} 1 & 4 & 2 \\ 3 & -2 & 7 \\ 2 & -1 & 4 \end{pmatrix} \mathbf{d} = \begin{pmatrix} 0 \\ 0 \\ 15 \end{pmatrix} \quad (1.54.5)$$

we can use Gaussian Elimination Method in order to find the coordinate values of \mathbf{d} .

$$\left(\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 3 & -2 & 7 & 0 \\ 2 & -1 & 4 & 15 \end{array} \right) \quad (1.54.6)$$

$$\xleftrightarrow{\begin{matrix} R_3 \leftarrow R_3 - 2R_1 \\ R_2 \leftarrow R_2 - 3R_1 \end{matrix}} \left(\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 0 & -14 & 1 & 0 \\ 0 & -9 & 0 & 15 \end{array} \right) \quad (1.54.7)$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - \frac{9}{14}R_2} \left(\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 0 & -14 & 1 & 0 \\ 0 & 0 & \frac{-9}{14} & 15 \end{array} \right) \quad (1.54.8)$$

$$\xleftrightarrow{\begin{matrix} R_3 \leftarrow \frac{-14}{9}R_3 \\ R_2 \leftarrow \frac{-1}{14}R_2 \end{matrix}} \left(\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 0 & 1 & \frac{-1}{14} & 0 \\ 0 & 0 & 1 & \frac{-210}{9} \end{array} \right) \quad (1.54.9)$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + \frac{1}{14}R_3} \left(\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 0 & 1 & 0 & \frac{-210}{126} \\ 0 & 0 & 1 & \frac{-210}{9} \end{array} \right) \quad (1.54.10)$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - 4R_3} \left(\begin{array}{ccc|c} 1 & 0 & 2 & \frac{840}{126} \\ 0 & 1 & 0 & \frac{-210}{126} \\ 0 & 0 & 1 & \frac{-210}{9} \end{array} \right) \quad (1.54.11)$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - 2R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{6720}{126} \\ 0 & 1 & 0 & \frac{-210}{126} \\ 0 & 0 & 1 & \frac{-210}{9} \end{array} \right) \quad (1.54.12)$$

By using Gaussian Elimination Method, we

were able to get the vector \mathbf{d} as $\begin{pmatrix} \frac{6720}{126} \\ \frac{-210}{126} \\ \frac{-210}{9} \end{pmatrix}$

1.55. The scalar product of $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ with a unit vector along the sum of the vectors $\begin{pmatrix} 2 \\ 4 \\ -5 \end{pmatrix}$ and $\begin{pmatrix} \lambda \\ 2 \\ 3 \end{pmatrix}$ is unity. Find the value of λ .

1.56. The value of

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}^T \left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}^T \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}^T \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right) \quad (1.56.1)$$

is

- a) 0 c) 1
b) -1 d) 3

Solution: Given

$$\mathbf{a} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad (1.56.2)$$

Using scalar triple product property we deduce

$$\mathbf{a}^T (\mathbf{b} \times \mathbf{c}) = \mathbf{b}^T (\mathbf{c} \times \mathbf{a}) = \mathbf{c}^T (\mathbf{a} \times \mathbf{b}) \quad (1.56.3)$$

Note: Cross product is given by:

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \quad (1.56.4)$$

Equating (1.56.2) with problem statement we deduce the following:

$$\mathbf{a}^T (\mathbf{b} \times \mathbf{c}) + \mathbf{b}^T (\mathbf{a} \times \mathbf{c}) + \mathbf{c}^T (\mathbf{a} \times \mathbf{b}) \quad (1.56.5)$$

As Cross Product is anti-commutative we get:

$$\mathbf{a}^T (\mathbf{b} \times \mathbf{c}) - \mathbf{b}^T (\mathbf{c} \times \mathbf{a}) + \mathbf{c}^T (\mathbf{a} \times \mathbf{b}) \quad (1.56.6)$$

$$= \mathbf{a}^T (\mathbf{b} \times \mathbf{c}) - \mathbf{c}^T (\mathbf{a} \times \mathbf{b}) + \mathbf{c}^T (\mathbf{a} \times \mathbf{b}) \quad (1.56.7)$$

$$= \mathbf{a}^T (\mathbf{b} \times \mathbf{c}) \quad (1.56.8)$$

So instead of calculating each step we just calculate one iteration by referring (1.56.4) and

(1.56.8) i.e.

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad (1.56.9)$$

$$\Rightarrow (1 \ 0 \ 0) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = (1) \quad (1.56.10)$$

1.57. Find a unit vector that makes an angle of 90° , 135° and 45° with the positive x, y and z axis respectively. **Solution:**

$$\mathbf{m} = \begin{pmatrix} \cos 90^\circ \\ \cos 135^\circ \\ \cos 45^\circ \end{pmatrix} \quad (1.57.1)$$

we know that,

$$\mathbf{m} = \frac{\mathbf{m}}{\|\mathbf{m}\|} \quad (1.57.2)$$

Also,

$$\|\mathbf{m}\| = \sqrt{0^2 + \left(\frac{-1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} \Rightarrow \|\mathbf{m}\| = 1 \quad (1.57.3)$$

Hence, From (1.57.1) and (1.57.3) we have the unit vector:

$$\mathbf{m} = \begin{pmatrix} 0 \\ \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \quad (1.57.4)$$

1.58. Show that the lines with direction vectors $\begin{pmatrix} 12 \\ -3 \\ -4 \end{pmatrix}$,

$\begin{pmatrix} 4 \\ 12 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -4 \\ 12 \end{pmatrix}$ are mutually perpendicular.

1.59. Show that the line through the points $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$,

$\begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix}$ is perpendicular to the line through the points $\begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 5 \\ 6 \end{pmatrix}$.

Solution: Let the points be $\mathbf{P} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$, $\mathbf{Q} = \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix}$,

$\mathbf{R} = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}$ and $\mathbf{S} = \begin{pmatrix} 3 \\ 5 \\ 6 \end{pmatrix}$. The direction vector for the line through the points \mathbf{P} and \mathbf{Q} is

$$\mathbf{A} = \mathbf{P} - \mathbf{Q} \quad (1.59.1)$$

$$\Rightarrow \mathbf{A} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix} \quad (1.59.2)$$

$$\Rightarrow \mathbf{A} = \begin{pmatrix} -2 \\ -5 \\ 4 \end{pmatrix} \quad (1.59.3)$$

The direction vector for the line through the points \mathbf{R} and \mathbf{S} is

$$\mathbf{B} = \mathbf{R} - \mathbf{S} \quad (1.59.4)$$

$$\Rightarrow \mathbf{B} = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 5 \\ 6 \end{pmatrix} \quad (1.59.5)$$

$$\Rightarrow \mathbf{B} = \begin{pmatrix} -3 \\ -2 \\ -4 \end{pmatrix} \quad (1.59.6)$$

$$(1.59.7)$$

To check if the two lines are perpendicular, we perform scalar product of the two direction vectors \mathbf{A} and \mathbf{B} as follows

$$\mathbf{AB} = \mathbf{A}^T \mathbf{B} \quad (1.59.8)$$

$$= \begin{pmatrix} -2 & -5 & 4 \end{pmatrix} \begin{pmatrix} -3 \\ -2 \\ -4 \end{pmatrix} \quad (1.59.9)$$

$$= 6 + 10 - 16 \quad (1.59.10)$$

$$= 0 \quad (1.59.11)$$

Thus, the lines are **perpendicular**.

1.60. Show that the line through the points $\begin{pmatrix} 4 \\ 7 \\ 8 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$

is parallel to the line through the points $\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix},$

$$\begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}.$$

Solution: Let the lines be parallel and the first two points pass through $\mathbf{n}^T \mathbf{x} = c_1$. i.e.

$$\mathbf{n}^T \mathbf{x}_1 = c_1 \Rightarrow \mathbf{x}_1^T \mathbf{n} = c_1 \quad (1.60.1)$$

$$\mathbf{n}^T \mathbf{x}_2 = c_2 \Rightarrow \mathbf{x}_2^T \mathbf{n} = c_2 \quad (1.60.2)$$

and the second two points pass through $\mathbf{n}^T \mathbf{x} = c_2$. Then

$$\mathbf{n}^T \mathbf{x}_3 = c_3 \Rightarrow \mathbf{x}_3^T \mathbf{n} = c_3 \quad (1.60.3)$$

$$\mathbf{n}^T \mathbf{x}_4 = c_4 \Rightarrow \mathbf{x}_4^T \mathbf{n} = c_4 \quad (1.60.4)$$

Putting all the equations together, we obtain

$$\begin{pmatrix} \mathbf{x}_1^T \\ \mathbf{x}_2^T \\ \mathbf{x}_3^T \\ \mathbf{x}_4^T \end{pmatrix} \mathbf{n} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} \quad (1.60.5)$$

Now if this equation has a solution, then \mathbf{n} exists and the lines will be parallel. Given

the points, $\mathbf{A} = \begin{pmatrix} 4 \\ 7 \\ 8 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}, \text{ and } \mathbf{C} =$

$$\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$$

Applying the row reduction procedure on the coefficient matrix:

$$\begin{pmatrix} 4 & 7 & 8 \\ 2 & 3 & 4 \\ -1 & -2 & 1 \\ 1 & 2 & 5 \end{pmatrix} \quad (1.60.6)$$

$$\begin{matrix} R_2 \leftarrow R_1 - 2R_2 \\ R_4 \leftarrow R_3 + R_4 \end{matrix} \begin{pmatrix} 4 & 7 & 8 \\ 0 & 1 & 0 \\ -1 & -2 & 1 \\ 0 & 0 & 6 \end{pmatrix} \quad (1.60.7)$$

$$\begin{matrix} R_1 \leftarrow R_1 - 7R_2 \\ R_3 \leftarrow R_3 - 6R_4 \end{matrix} \begin{pmatrix} 4 & 0 & 8 \\ 0 & 1 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & 6 \end{pmatrix} \quad (1.60.8)$$

$$\begin{matrix} R_4 \leftarrow R_4 / 6 \\ R_1 \leftarrow R_1 - 8R_4 \end{matrix} \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad (1.60.9)$$

$$\begin{matrix} R_3 \leftarrow (-R_3 - 2R_2) \\ R_3 \leftarrow R_3 + R_4 \end{matrix} \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (1.60.10)$$

$$\begin{matrix} R_1 \leftarrow R_1 - 4R_3 \end{matrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (1.60.11)$$

Here, the number of non-zero rows are three and hence the rank of the matrix is 3 which

implies that the solution exists. Therefore the lines passing through **A, B** and **C, D** are parallel.

- 1.61. Find a point on the x-axis, which is equidistant from the points $\begin{pmatrix} 7 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

Solution: Given,

$$\mathbf{P} = \begin{pmatrix} 7 \\ 6 \end{pmatrix} \quad \mathbf{Q} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \quad (1.61.1)$$

A vector on the X-axis **X** is equidistant to both **P** and **Q**.

$$\text{i.e. } \mathbf{X} = \frac{\mathbf{P} + \mathbf{Q}}{2} \quad (1.61.2)$$

Need to find k. Let $\mathbf{X} = k \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ be the vector on the X-axis.

$$\Rightarrow \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{X} = k \quad (1.61.3)$$

$$\Rightarrow \mathbf{X} = \frac{\begin{pmatrix} 7 \\ 6 \end{pmatrix} + \begin{pmatrix} 3 \\ 4 \end{pmatrix}}{2} \quad (1.61.4)$$

$$\Rightarrow \mathbf{X} = \begin{pmatrix} 5 \\ 5 \end{pmatrix} \quad (1.61.5)$$

$$\Rightarrow \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{X} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 5 \\ 5 \end{pmatrix} \quad (1.61.6)$$

$$(1.61.7)$$

Therefore, $k = 5$ i.e. $\mathbf{X} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$ See Fig. 1.61

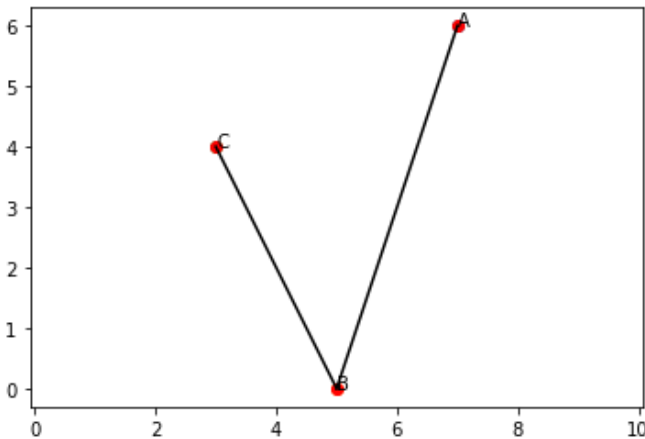


Fig. 1.61: Plot representing the Points

- 1.62. Find the angle between the vectors

$$\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \quad (1.62.1)$$

Solution: Let

$$\mathbf{a} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \quad (1.62.2)$$

Angle between the vectors is given by,

$$\theta = \cos^{-1} \left(\frac{\mathbf{a}^T \mathbf{b}}{\|\mathbf{a}\| \|\mathbf{b}\|} \right) \quad (1.62.3)$$

$$\|\mathbf{a}\| = \sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{14} \quad (1.62.4)$$

$$\|\mathbf{b}\| = \sqrt{3^2 + (-2)^2 + 1^2} = \sqrt{14} \quad (1.62.5)$$

$$\mathbf{a}^T \mathbf{b} = (1)(3) + (-2)(-2) + (3)(1) = 10 \quad (1.62.6)$$

$$\theta = \cos^{-1} \left(\frac{10}{(\sqrt{14})(\sqrt{14})} \right) \quad (1.62.7)$$

$$= \cos^{-1} \left(\frac{10}{14} \right) \quad (1.62.8)$$

$$(1.62.9)$$

- 1.63. Find the projection of the vector

$$\begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix} \quad (1.63.1)$$

on the vector

$$\begin{pmatrix} 7 \\ -1 \\ 8 \end{pmatrix} \quad (1.63.2)$$

Solution:

We have,

$$\mathbf{u} = \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} 7 \\ -1 \\ 8 \end{pmatrix}$$

$$\mathbf{p} = \left[\frac{\begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix}^T \begin{pmatrix} 7 \\ -1 \\ 8 \end{pmatrix}}{\left\| \begin{pmatrix} 7 \\ -1 \\ 8 \end{pmatrix} \right\|^2} \right] \begin{pmatrix} 7 \\ -1 \\ 8 \end{pmatrix} \quad (1.63.3)$$

$$\mathbf{p} = \left[\frac{(7 - 3 + 56)}{(\sqrt{7^2 + (-1)^2 + 8^2})^2} \right] \begin{pmatrix} 7 \\ -1 \\ 8 \end{pmatrix} \quad (1.63.4)$$

$$\mathbf{p} = \frac{13}{25} \begin{pmatrix} 7 \\ -1 \\ 8 \end{pmatrix} = \begin{pmatrix} \frac{92}{25} \\ -\frac{13}{25} \\ \frac{21}{5} \end{pmatrix} \quad (1.63.5)$$

Hence the projection of \mathbf{u} on \mathbf{v} is

$$\mathbf{p} = \begin{pmatrix} \frac{92}{25} \\ -\frac{13}{25} \\ \frac{21}{5} \end{pmatrix}$$

- 1.64. Write down a unit vector in the xy-plane, making an angle of 30° with the positive direction of the x-axis.

Solution:

$$\because m = \tan 30^\circ = \frac{1}{\sqrt{3}}, \quad (1.64.1)$$

the direction vector is

$$\mathbf{a} = \begin{pmatrix} 1 \\ \frac{1}{\sqrt{3}} \end{pmatrix} \quad (1.64.2)$$

and the unit vector

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{\|\mathbf{a}\|} \quad (1.64.3)$$

$$\Rightarrow \hat{\mathbf{a}} = \begin{pmatrix} \frac{1}{\frac{2}{\sqrt{3}}} \\ \frac{\frac{1}{\sqrt{3}}}{\frac{2}{\sqrt{3}}} \end{pmatrix} \quad (1.64.4)$$

$$\hat{\mathbf{a}} = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix} \quad (1.64.5)$$

$$\Rightarrow \hat{\mathbf{a}} = \frac{1}{2} \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} \quad (1.64.6)$$

- 1.65. Find the value of x for which $x \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is a unit vector.

Solution:

$$\left\| x \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\| = 1 \quad (1.65.1)$$

$$\Rightarrow x \left\| \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\| = 1 \quad (1.65.2)$$

$$\text{or, } \sqrt{3}x^2 = 1 \Rightarrow x = \pm \frac{1}{\sqrt{3}} \quad (1.65.3)$$

- 1.66. Find the angle between the force $\mathbf{F} = \begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix}$ and displacement $\mathbf{d} = \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix}$.

Solution: Let the angle between \mathbf{F} and $\mathbf{d} = \theta$
Then,

$$\cos(\theta) = \frac{\mathbf{F}^T \mathbf{d}}{\|\mathbf{F}\| \|\mathbf{d}\|} \quad (1.66.1)$$

where $\mathbf{F}^T \mathbf{d}$ is scalar product of vectors \mathbf{F} and \mathbf{d}

And, $\|\mathbf{F}\|$ and $\|\mathbf{d}\|$ are their respective magnitudes So,

$$\mathbf{F}^T \mathbf{d} = \begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix}^T \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix} \quad (1.66.2)$$

$$\Rightarrow \mathbf{F}^T \mathbf{d} = \begin{pmatrix} 3 & 4 & -5 \end{pmatrix} \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix} \quad (1.66.3)$$

$$= 16 \quad (1.66.4)$$

$$\|\mathbf{F}\| = \sqrt{3^2 + 4^2 + (-5)^2} = 5\sqrt{2} \quad (1.66.5)$$

$$\|\mathbf{d}\| = \sqrt{5^2 + 4^2 + 3^2} = 5\sqrt{2} \quad (1.66.6)$$

Substituting these values in Equation 1.66.1,

$$\cos(\theta) = \frac{16}{(5\sqrt{2})(5\sqrt{2})} \quad (1.66.7)$$

$$= \frac{8}{25} \quad (1.66.8)$$

$$\Rightarrow \theta = \arccos\left(\frac{8}{25}\right) \quad (1.66.9)$$

$$\Rightarrow \theta \approx 71.3^\circ \quad (1.66.10)$$

1.67. A body constrained to move along the z-axis of a coordinate system is subject to a constant force

$$\mathbf{F} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} \quad (1.67.1)$$

What is the work done by this force in moving the body a distance of 4 m along the z-axis ?

Solution: Work done in moving an object by a distance \mathbf{s} using an external force \mathbf{F} is given by:

$$W = \mathbf{F}^T \mathbf{s} \quad (1.67.2)$$

As seen above, work done is the scalar product (dot product) of Force and distance. Here,

$$\mathbf{s} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} \quad (1.67.3)$$

The scalar product of the variables is given by:

$$\mathbf{F}^T \mathbf{s} = \begin{pmatrix} 1 & -2 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} = 12 \quad (1.67.4)$$

The work done by the force \mathbf{F} is 12 J

1.68. Find the scalar and vector products of the two vectors

$$\mathbf{a} = \begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} -2 \\ 1 \\ -3 \end{pmatrix} \quad (1.68.1)$$

Solution:

$$\mathbf{a}^T \mathbf{b} = \begin{pmatrix} 3 & -4 & 5 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ -3 \end{pmatrix} \quad (1.68.2)$$

$$= (3 \times -2) + (-4 \times 1) + (5 \times -3) \quad (1.68.3)$$

$$= -25 \quad (1.68.4)$$

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} 0 & 5 & -4 \\ 5 & 0 & -3 \\ -(-4) & 3 & 0 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} \quad (1.68.5)$$

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} (0 \times -2) + (-5 \times 1) + (-4 \times -3) \\ (5 \times -2) + (0 \times 1) + (-3 \times -3) \\ (4 \times -2) + (3 \times 1) + (0 \times -3) \end{pmatrix} \quad (1.68.6)$$

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} 7 \\ -1 \\ 5 \end{pmatrix} \quad (1.68.7)$$

1.69. Find the torque of a force $\begin{pmatrix} 7 \\ 3 \\ -5 \end{pmatrix}$ about the

origin. The force acts on a particle whose position vector is $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

Solution: The torque \mathbf{T} is given by the cross product (vector product) of the position (or distance) vector \mathbf{r} and the force vector \mathbf{F} .

$$\mathbf{T} = \mathbf{r} \times \mathbf{F} \quad (1.69.1)$$

And the vector cross product of vectors

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \quad (1.69.2)$$

$$\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \quad (1.69.3)$$

can be expressed as the product of a skew-symmetric matrix and a vector:

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \quad (1.69.4)$$

Torque at the origin is given by,

$$\mathbf{F} \times \mathbf{r} = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 7 \\ 3 \\ -5 \end{pmatrix} \quad (1.69.5)$$

$$\Rightarrow \mathbf{F} \times \mathbf{r} = \begin{pmatrix} (0 \times 7) + (-1 \times 3) + (-1 \times -5) \\ (1 \times 7) + (0 \times 3) + (-1 \times -5) \\ (1 \times 7) + (1 \times 3) + (0 \times -5) \end{pmatrix} \quad (1.69.6)$$

$$\Rightarrow \mathbf{T} = \begin{pmatrix} 2 \\ 12 \\ 10 \end{pmatrix} \quad (1.69.7)$$

1.70. Find the values of x, y, z such that

$$\begin{pmatrix} x \\ 2 \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ y \\ 1 \end{pmatrix} \quad (1.70.1)$$

Solution: $x = 2, y = 2, z = 1$.

1.71. If

$$\mathbf{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad (1.71.1)$$

verify if

a) $\|\mathbf{a}\| = \|\mathbf{b}\|$

b) $\mathbf{a} = \mathbf{b}$

Solution:

a) $\|\mathbf{a}\| = \|\mathbf{b}\|, \mathbf{a} \neq \mathbf{b}$.

1.72. Find a unit vector in the direction of $\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$.

Solution: The unit vector is given by

$$\frac{\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}}{\left\| \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \right\|} = \frac{1}{\sqrt{14}} \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \quad (1.72.1)$$

1.73. Find the distance between the points

$$\mathbf{P} = \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} -4 \\ 1 \\ 2 \end{pmatrix} \quad (1.73.1)$$

Solution:

The distance between the two points is given

by or,

$$\begin{aligned} d &= \|\mathbf{P} - \mathbf{Q}\| \\ &= \left\| \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix} - \begin{pmatrix} -4 \\ 1 \\ 2 \end{pmatrix} \right\| = \left\| \begin{pmatrix} 5 \\ -4 \\ 2 \end{pmatrix} \right\| \quad (1.73.2) \\ \Rightarrow d &= \sqrt{5^2 + (-4)^2 + 2^2} \\ &= 3\sqrt{5} \end{aligned}$$

The following Python code generates Fig. 1.73

```
solutions/line/geometry/examples/54/codes/
point_distance.py
```

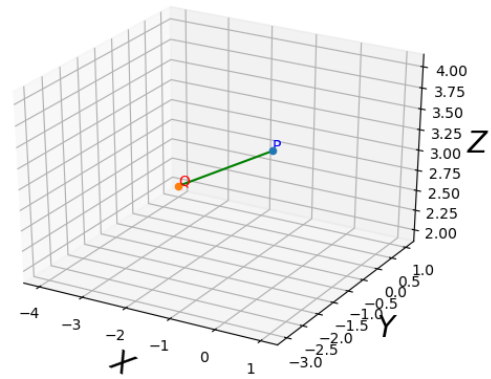


Fig. 1.73: Two points and distance between them.

The distance is given by $\|\mathbf{P} - \mathbf{Q}\|$

1.74. Show that the points $\mathbf{A} = \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and

$\mathbf{C} = \begin{pmatrix} 7 \\ 0 \\ -1 \end{pmatrix}$ are collinear.

Solution: Forming the matrix in (1.2.6)

$$\mathbf{M} = \begin{pmatrix} 3 & -1 & -2 \\ 9 & -3 & -6 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 - 3R_1} \begin{pmatrix} 3 & -1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \quad (1.74.1)$$

$\Rightarrow \text{rank}(\mathbf{M}) = 1$. The following code plots Fig. 1.74 showing that the points are collinear.

```
codes/line/draw_lines_3d.py
```

1.75. If $\mathbf{a} = \begin{pmatrix} 5 \\ -1 \\ -3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix}$, then show that the vectors $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$ are perpendicular.

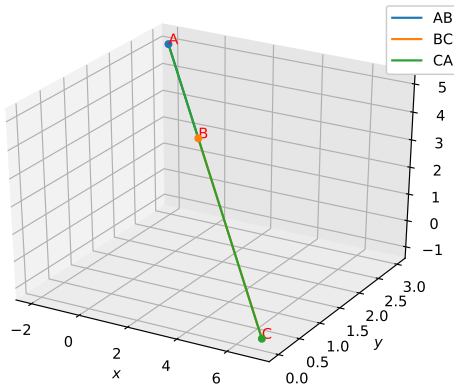


Fig. 1.74

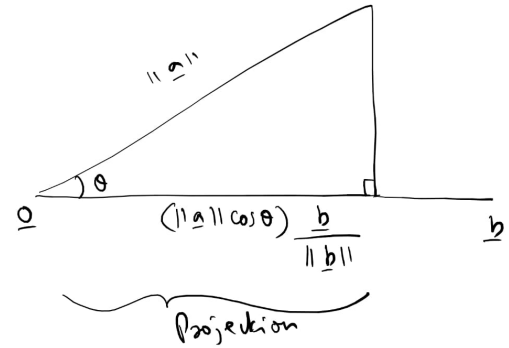


Fig. 1.76

Solution:

$$\mathbf{A}^T \mathbf{B} = 0 \quad (1.75.1)$$

$$\mathbf{A}^T \mathbf{B} = (\mathbf{a} + \mathbf{b})^T (\mathbf{a} - \mathbf{b}) \quad (1.75.2)$$

The transpose of a sum is the sum of transposes so,

$$(\mathbf{a} + \mathbf{b})^T = (\mathbf{a}^T + \mathbf{b}^T) \quad (1.75.3)$$

$$\mathbf{A}^T \mathbf{B} = (\mathbf{a}^T + \mathbf{b}^T)(\mathbf{a} - \mathbf{b}) \quad (1.75.4)$$

$$\mathbf{a}^T (\mathbf{a} - \mathbf{b}) + \mathbf{b}^T (\mathbf{a} - \mathbf{b}) \quad (1.75.5)$$

$$\Rightarrow \mathbf{a}^T \mathbf{a} - \mathbf{a}^T \mathbf{b} + \mathbf{b}^T \mathbf{a} - \mathbf{b}^T \mathbf{b} \quad (1.75.6)$$

$$\because \mathbf{a}^T \mathbf{a} = \|\mathbf{a}\|^2 \quad (1.75.7)$$

$$\because \mathbf{b}^T \mathbf{b} = \|\mathbf{b}\|^2 \quad (1.75.8)$$

$$\because \mathbf{a}^T \mathbf{b} = \mathbf{b}^T \mathbf{a} \quad (1.75.9)$$

Using (1.75.7), (1.75.8) and (1.75.9)

$$\mathbf{A}^T \mathbf{B} = \|\mathbf{a}\|^2 - \mathbf{a}^T \mathbf{b} + \mathbf{a}^T \mathbf{b} - \|\mathbf{b}\|^2 \quad (1.75.10)$$

$$\|\mathbf{a}\|^2 = 5^2 + (-1)^2 + (-3)^2 = 35 \quad (1.75.11)$$

$$\|\mathbf{b}\|^2 = 1^2 + (3)^2 + (-5)^2 = 35 \quad (1.75.12)$$

$$\mathbf{A}^T \mathbf{B} = \|\mathbf{a}\|^2 - \|\mathbf{b}\|^2 \quad (1.75.13)$$

Using (1.75.11) and (1.75.12)

$$\Rightarrow \mathbf{A}^T \mathbf{B} = 35 - 35 = 0 \quad (1.75.14)$$

Thus the direction vectors of the two lines satisfies the equation 1.75.1, hence proved that the lines are **perpendicular**.

1.76. Find the projection of the vector

$$\mathbf{a} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} \quad (1.76.1)$$

on the vector

$$\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}. \quad (1.76.2)$$

Solution: The projection of \mathbf{a} on \mathbf{b} is shown in Fig. 1.76. It has magnitude $\|\mathbf{a}\| \cos \theta$ and is in the direction of \mathbf{b} . Thus, the projection is defined as

$$(\|\mathbf{a}\| \cos \theta) \frac{\mathbf{b}}{\|\mathbf{b}\|} = \frac{(\mathbf{a}^T \mathbf{b}) \|\mathbf{a}\|}{\|\mathbf{b}\|} \mathbf{b} \quad (1.76.3)$$

1.77. Find $\|\mathbf{a} - \mathbf{b}\|$, if

$$\|\mathbf{a}\| = 2, \|\mathbf{b}\| = 3, \mathbf{a}^T \mathbf{b} = 4. \quad (1.77.1)$$

Solution:

$$\begin{aligned} \|\mathbf{a} - \mathbf{b}\|^2 &= \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\mathbf{a}^T \mathbf{b} \\ \Rightarrow \|\mathbf{a} - \mathbf{b}\|^2 &= 2^2 + 3^2 - 2 \times 4 \\ \Rightarrow \|\mathbf{a} - \mathbf{b}\|^2 &= 5 \\ \Rightarrow \|\mathbf{a} - \mathbf{b}\| &= \sqrt{5} \end{aligned} \quad (1.77.2)$$

1.78. If \mathbf{a} is a unit vector and

$$(\mathbf{x} - \mathbf{a})(\mathbf{x} + \mathbf{a}) = 8, \quad (1.78.1)$$

then find \mathbf{x} .

Solution:

$$(\mathbf{x} - \mathbf{a})(\mathbf{x} + \mathbf{a}) = \|\mathbf{x}\|^2 - \|\mathbf{a}\|^2 \quad (1.78.2)$$

$$\implies \|\mathbf{x}\|^2 = 9 \text{ or, } \|\mathbf{x}\| = 3. \quad (1.78.3)$$

1.79. Given

$$\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix}, \quad (1.79.1)$$

find $\|\mathbf{a} \times \mathbf{b}\|$.

Solution: Use (1.6.3).

1.80. Find a unit vector perpendicular to each of the vectors $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$, where

$$\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}. \quad (1.80.1)$$

Solution: If \mathbf{x} is the desired vector,

$$(\mathbf{a} + \mathbf{b})^T \mathbf{x} = 0 \quad (1.80.2)$$

$$(\mathbf{a} - \mathbf{b})^T \mathbf{x} = 0 \quad (1.80.3)$$

resulting in the matrix equation

$$\begin{pmatrix} 2 & 3 & 4 \\ 0 & -1 & -2 \end{pmatrix} \mathbf{x} = 0 \quad (1.80.4)$$

Performing row operations,

$$\begin{pmatrix} 2 & 3 & 4 \\ 0 & -1 & -2 \end{pmatrix} \xrightarrow[R_2 \leftarrow -R_2]{R_1 \leftarrow R_1 + 3R_2} \begin{pmatrix} 2 & 0 & -2 \\ 0 & -1 & -2 \end{pmatrix} \quad (1.80.5)$$

$$\xrightarrow{R_1 \leftarrow \frac{R_1}{2}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix} \implies \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \quad (1.80.6)$$

The desired unit vector is then obtained as

$$\mathbf{x} = \frac{\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}}{\left\| \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\|} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \quad (1.80.7)$$

1.81. Show that $\mathbf{A} = \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 7 \\ 0 \\ -1 \end{pmatrix}$, are

collinear.

Solution: See Problem 1.74.

1.82. If $\mathbf{A} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 3 \\ 2 \\ -3 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 1 \\ -6 \\ -1 \end{pmatrix}$, show that $\mathbf{A} - \mathbf{B}$ and $\mathbf{C} - \mathbf{D}$ are collinear.

Solution:

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} -1 \\ -4 \\ 1 \end{pmatrix} \quad (1.82.1)$$

$$\mathbf{C} - \mathbf{D} = \begin{pmatrix} 2 \\ 8 \\ -2 \end{pmatrix} \quad (1.82.2)$$

$$\therefore -2(\mathbf{A} - \mathbf{B}) = \mathbf{C} - \mathbf{D}, \quad (1.82.3)$$

$\mathbf{A} - \mathbf{B}$ and $\mathbf{C} - \mathbf{D}$ are collinear.

1.83. Let $\|\mathbf{a}\| = 3, \|\mathbf{b}\| = 4, \|\mathbf{c}\| = 5$ such that each vector is perpendicular to the other two. Find $\|\mathbf{a} + \mathbf{b} + \mathbf{c}\|$.

Solution: Given that

$$\mathbf{a}^T \mathbf{b} = \mathbf{b}^T \mathbf{c} = \mathbf{c}^T \mathbf{a} = 0. \quad (1.83.1)$$

Then,

$$\begin{aligned} \|\mathbf{a} + \mathbf{b} + \mathbf{c}\|^2 &= \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 + \|\mathbf{c}\|^2 \\ &\quad + \mathbf{a}^T \mathbf{b} + \mathbf{b}^T \mathbf{c} + \mathbf{c}^T \mathbf{a}. \end{aligned} \quad (1.83.2)$$

which reduces to

$$\|\mathbf{a} + \mathbf{b} + \mathbf{c}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 + \|\mathbf{c}\|^2 \quad (1.83.3)$$

using (1.83.1)

1.84. Given

$$\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}, \quad (1.84.1)$$

evaluate

$$\mathbf{a}^T \mathbf{b} + \mathbf{b}^T \mathbf{c} + \mathbf{c}^T \mathbf{a}, \quad (1.84.2)$$

given that $\|\mathbf{a}\| = 3, \|\mathbf{b}\| = 4$ and $\|\mathbf{c}\| = 2$.

Solution: Multiplying (1.84.1) with $\mathbf{a}, \mathbf{b}, \mathbf{c}$,

$$\|\mathbf{a}\|^2 + \mathbf{a}^T \mathbf{b} + \mathbf{a}^T \mathbf{c} = 0 \quad (1.84.3)$$

$$\mathbf{a}^T \mathbf{b} + \|\mathbf{b}\|^2 + \mathbf{b}^T \mathbf{c} = 0 \quad (1.84.4)$$

$$+\mathbf{c}^T \mathbf{a} + \mathbf{b}^T \mathbf{c} + \|\mathbf{c}\|^2 = 0 \quad (1.84.5)$$

Adding all the above equations and rearranging,

$$\mathbf{a}^T \mathbf{b} + \mathbf{b}^T \mathbf{c} + \mathbf{c}^T \mathbf{a} = -\frac{\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 + \|\mathbf{c}\|^2}{2} \quad (1.84.6)$$

- 1.85. Let $\alpha = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}, \beta = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$. Find β_1, β_2 such that $\beta = \beta_1 + \beta_2, \beta_1 \parallel \alpha$ and $\beta_2 \perp \alpha$.

Solution: Let $\beta_1 = k\alpha$. Then,

$$\beta = k\alpha + \beta_2 \quad (1.85.1)$$

$$\Rightarrow k = \frac{\alpha^T \beta}{\|\alpha\|^2} \quad (1.85.2)$$

and

$$\beta_2 = \beta - k\alpha \quad (1.85.3)$$

This process is known as *Gram-Schmidt orthogonalization*.

- 1.86. Find a vector \mathbf{x} in the direction of $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ such that $\|\mathbf{x}\| = 7$. **Solution:** Let $\mathbf{x} = k \begin{pmatrix} 1 \\ -2 \end{pmatrix}$. Then

$$\|\mathbf{x}\| = |k| \left\| \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\| = 7 \quad (1.86.1)$$

$$\Rightarrow |k| = \frac{7}{\sqrt{5}} \quad (1.86.2)$$

$$\text{or, } \mathbf{x} = \frac{7}{\sqrt{5}} \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad (1.86.3)$$

- 1.87. Find the direction vector of PQ , where

$$\mathbf{P} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} -1 \\ -2 \\ -4 \end{pmatrix} \quad (1.87.1)$$

Solution: The direction vector of PQ is

$$\mathbf{P} - \mathbf{Q} = \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix}, \quad (1.87.2)$$

- 1.88. Draw a line segment of length 7.6 cm and divide it in the ratio 5 : 8.

Solution: Let the end points of the line be

$$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 7.6 \\ 0 \end{pmatrix} \quad (1.88.1)$$

Using section formula, the point \mathbf{C}

$$\mathbf{C} = \frac{k\mathbf{B} + \mathbf{A}}{k + 1} \quad (1.88.2)$$

If \mathbf{C} divides AB in the ratio

$$m = \frac{5}{8}, \quad (1.88.3)$$

then,

$$\frac{\|\mathbf{C} - \mathbf{A}\|^2}{\|\mathbf{B} - \mathbf{C}\|^2} = m^2 \quad (1.88.4)$$

$$\Rightarrow \frac{\frac{k^2 \|\mathbf{B} - \mathbf{A}\|^2}{(k+1)^2}}{\frac{\|\mathbf{B} - \mathbf{A}\|^2}{(k+1)^2}} = m^2 \quad (1.88.5)$$

$$\Rightarrow k = m \quad (1.88.6)$$

upon substituting from (1.88.4) and simplifying. (1.88.2) is known as the section formula. The following code plots Fig. 1.88

```
codes/line/draw_section.py
```

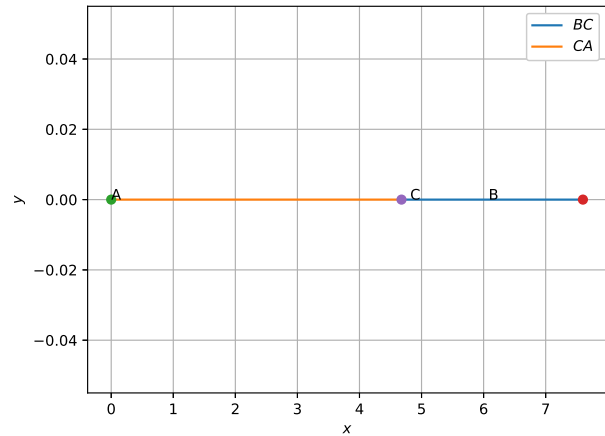


Fig. 1.88

- 1.89. Find the coordinates of the point which divides the line segment joining the points $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 8 \\ 5 \end{pmatrix}$ in the ratio 3 : 1 internally.

Solution: Using (1.88.2), the desired point is

$$\mathbf{P} = \frac{3 \begin{pmatrix} 4 \\ -3 \end{pmatrix} + \begin{pmatrix} 8 \\ 5 \end{pmatrix}}{4} \quad (1.89.1)$$

- 1.90. In what ratio does the point $\begin{pmatrix} -4 \\ 6 \end{pmatrix}$ divide the line segment joining the points

$$\mathbf{A} = \begin{pmatrix} -6 \\ 10 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 3 \\ -8 \end{pmatrix} \quad (1.90.1)$$

Solution: Use (1.88.2).

- 1.91. Find the coordinates of the points of trisection of the line segment joining the points

$$\mathbf{A} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -7 \\ 4 \end{pmatrix} \quad (1.91.1)$$

Solution: Using (1.88.2), the coordinates are

$$\mathbf{P} = \frac{2\mathbf{A} + \mathbf{B}}{3} \quad (1.91.2)$$

$$\mathbf{Q} = \frac{\mathbf{A} + 2\mathbf{B}}{3} \quad (1.91.3)$$

- 1.92. Find the ratio in which the y-axis divides the line segment joining the points $\begin{pmatrix} 5 \\ -6 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -4 \end{pmatrix}$.

Solution: Let the corresponding point on the y-axis be $\begin{pmatrix} 0 \\ y \end{pmatrix}$. If the ratio be $k : 1$, using (1.88.2), the coordinates are

$$\begin{pmatrix} 0 \\ y \end{pmatrix} = k \begin{pmatrix} 5 \\ -6 \end{pmatrix} + \begin{pmatrix} -1 \\ -4 \end{pmatrix} \quad (1.92.1)$$

$$\Rightarrow 0 = 5k - 1 \Rightarrow k = \frac{1}{5} \quad (1.92.2)$$

- 1.93. Find the value of k if the points $\mathbf{A} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 4 \\ k \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 6 \\ -3 \end{pmatrix}$ are collinear.

Solution: Forming the matrix

$$\mathbf{M} = (\mathbf{B} - \mathbf{A} \quad \mathbf{B} - \mathbf{A})^T = \begin{pmatrix} 2 & k-3 \\ 4 & -6 \end{pmatrix} \quad (1.93.1)$$

$$\xleftrightarrow{R_2 \leftarrow \frac{R_2}{2}} \begin{pmatrix} 2 & k-3 \\ 2 & -3 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 2 & k-3 \\ 0 & -k \end{pmatrix} \quad (1.93.2)$$

$$\Rightarrow \text{rank}(\mathbf{M}) = 1 \iff R_2 = \mathbf{0}, \text{ or } k = 0 \quad (1.93.3)$$

- 1.94. Find the coordinates of the point which divides the join of

$$\begin{pmatrix} -1 \\ 7 \end{pmatrix}, \begin{pmatrix} 4 \\ -3 \end{pmatrix} \quad (1.94.1)$$

in the ratio $2 : 3$.

Solution:

$$1. \mathbf{A} = \begin{pmatrix} -1 \\ 7 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$$

Then \mathbf{C} that divides \mathbf{A}, \mathbf{B} in the ratio $k : 1$ is

$$\mathbf{C} = \frac{k\mathbf{A} + \mathbf{B}}{k+1} \quad (1.94.2)$$

For the given problem $k=2 : 3$

Using the equation 1.94.2, the desired point is

$$\mathbf{C} = \frac{\frac{2}{3} \begin{pmatrix} -1 \\ 7 \end{pmatrix} + \begin{pmatrix} 4 \\ -3 \end{pmatrix}}{\frac{2}{3} + 1} \quad (1.94.3)$$

$$\therefore \mathbf{C} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad (1.94.4)$$

The following code plots Fig. 1.94

```
codes/line/section.py
```

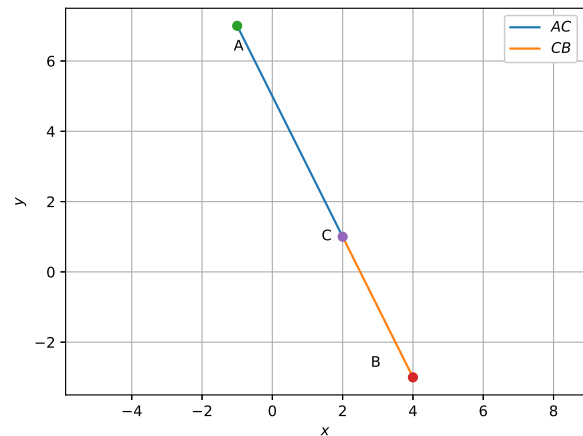


Fig. 1.94

- 1.95. Find the coordinates of the points of trisection of the line segment joining $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ -3 \end{pmatrix}$.

Solution: The points of trisection are

$$\mathbf{C} = \frac{0.5\mathbf{A} + \mathbf{B}}{0.5 + 1} \quad (1.95.1)$$

$$\mathbf{D} = \frac{2\mathbf{A} + \mathbf{B}}{2 + 1} \quad (1.95.2)$$

$$\Rightarrow \therefore \mathbf{C} = \begin{pmatrix} 0 \\ -2.33 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 2 \\ -1.66 \end{pmatrix} \quad (1.95.3)$$

The following Python code generates Fig. 1.95

```
solutions/2/codes/line_ex/pts_on_a_line/
trisection.py
```

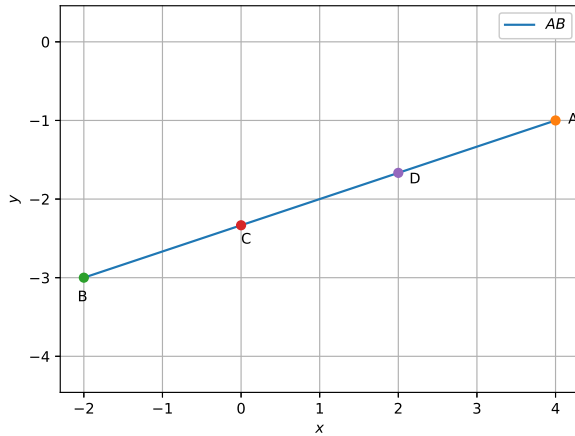


Fig. 1.95

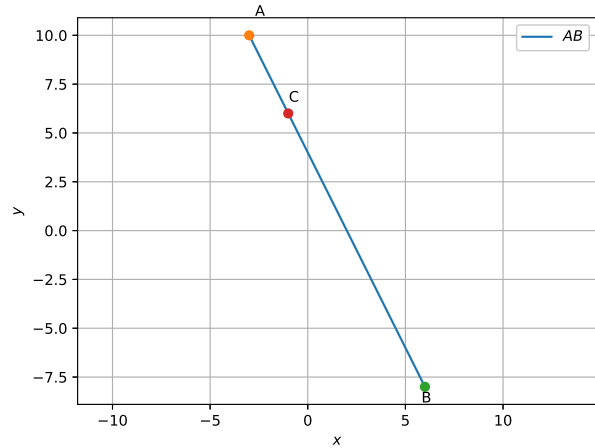


Fig. 1.96: C divides AB in ratio k:1

- 1.96. Find the ratio in which the line segment joining the points $\begin{pmatrix} -3 \\ 10 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ -8 \end{pmatrix}$ is divided by $\begin{pmatrix} -1 \\ 6 \end{pmatrix}$.

Solution: Let

$$\mathbf{A} = \begin{pmatrix} -3 \\ 10 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 6 \\ -8 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -1 \\ 6 \end{pmatrix} \quad (1.96.1)$$

Then by section formula,

$$\mathbf{C} = \frac{k\mathbf{B} + \mathbf{A}}{k+1} \quad (1.96.2)$$

$$\begin{pmatrix} -1 \\ 6 \end{pmatrix} = \frac{1}{k+1} \begin{pmatrix} 6k-3 \\ -8k+10 \end{pmatrix} \quad (1.96.3)$$

$$\Rightarrow k = \frac{2}{7} \quad (1.96.4)$$

The following Python code generates Fig. 1.96

```
solutions/3/codes/line/section/section.py
```

- 1.97. Find the ratio in which the line segment joining $\mathbf{A} = \begin{pmatrix} 1 \\ -5 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -4 \\ 5 \end{pmatrix}$ is divided by the x -axis. Also find the coordinates of the point of division.

Solution: Let

$$\mathbf{C} \begin{pmatrix} x \\ 0 \end{pmatrix} \quad (1.97.1)$$

divide \mathbf{AB} in $k:1$ ratio. Then,

$$(k+1) \begin{pmatrix} x \\ 0 \end{pmatrix} = k \begin{pmatrix} 1 \\ -5 \end{pmatrix} + \begin{pmatrix} -4 \\ 5 \end{pmatrix} \quad (1.97.2)$$

$$\Rightarrow 0 = -5k + 5 \quad (1.97.3)$$

$$\text{or, } k = 1 \quad (1.97.4)$$

$$\mathbf{C} = \frac{\begin{pmatrix} -3 \\ 0 \end{pmatrix}}{2} = \begin{pmatrix} -1.5 \\ 0 \end{pmatrix} \quad (1.97.5)$$

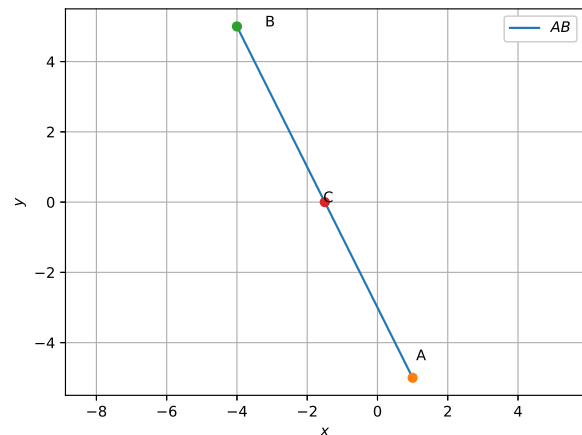


Fig. 1.97: line

The following code plots Fig. 1.97

```
solutions/4/codes/line/point_on_line/
points_on_line.py
```

- 1.98. If $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 4 \\ y \end{pmatrix}$, $\begin{pmatrix} x \\ 6 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$ are the vertices of a parallelogram taken in order, find x and y .

Solution: See Fig. 1.98. In a parallelogram, the diagonals bisect each other. Hence

$$\frac{\mathbf{A} + \mathbf{C}}{2} = \frac{\mathbf{B} + \mathbf{D}}{2} \quad (1.98.1)$$

$$\therefore \frac{1+x}{2} = \frac{7}{2}, \frac{8}{2} = \frac{y+5}{2} \quad (1.98.2)$$

$$\Rightarrow x = 6, y = 3 \quad (1.98.3)$$

The following python code computes the value of x and y used in Fig. 1.98.

```
./solutions/5/codes/lines/q10.py
```

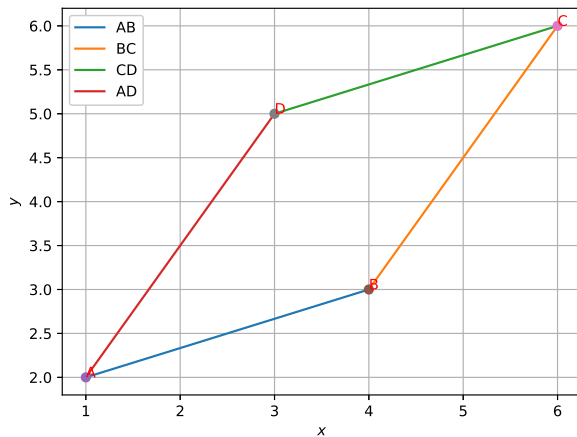


Fig. 1.98: Parallelogram of Q.3.6.5

- 1.99. If $\mathbf{A} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$ respectively, find the coordinates of \mathbf{P} such that $AP = \frac{3}{7}AB$ and \mathbf{P} lies on the line segment AB .

Solution: The desired point is

$$\mathbf{P} = \frac{\frac{3}{4} \begin{pmatrix} 2 \\ -4 \end{pmatrix} + 1 \begin{pmatrix} -2 \\ -2 \end{pmatrix}}{\frac{3}{4} + 1} \quad (1.99.1)$$

$$\mathbf{P} = \begin{pmatrix} -2/7 \\ -20/7 \end{pmatrix} \quad (1.99.2)$$

The following python code plots the Fig. 1.99

```
solutions/6/codes/point_line/int_sec.py
```

- 1.100. Find the coordinates of the points which divide the line segment joining $\mathbf{A} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 \\ 8 \end{pmatrix}$

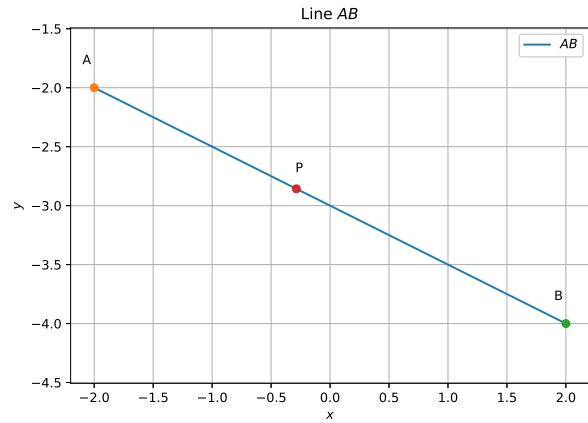


Fig. 1.99

into four equal parts.

Solution: The desired coordinates are

$$\mathbf{D} = \frac{1\mathbf{B} + 3\mathbf{A}}{4} = \begin{pmatrix} -1 \\ 7/2 \end{pmatrix} \quad (1.100.1)$$

$$\mathbf{E} = \frac{2\mathbf{B} + 2\mathbf{A}}{4} = \begin{pmatrix} 0 \\ 5 \end{pmatrix} \quad (1.100.2)$$

$$\mathbf{F} = \frac{3\mathbf{B} + 1\mathbf{A}}{4} = \begin{pmatrix} 1 \\ 13/2 \end{pmatrix} \quad (1.100.3)$$

The following code plots Fig. 1.100

```
solutions/7/codes/line/point_line/line_division.py
```

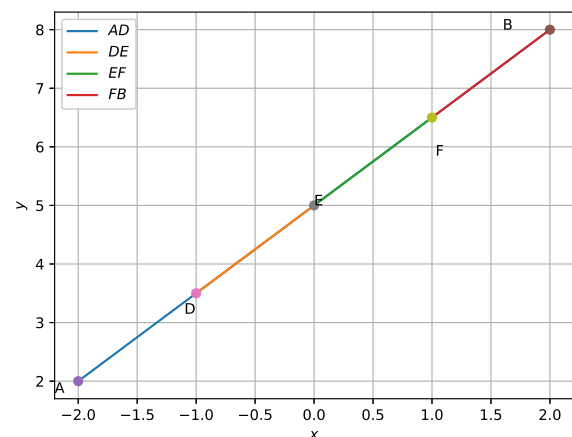


Fig. 1.100

1.101. Find $\begin{pmatrix} 5 \\ -3 \end{pmatrix}^3$

Solution: In general, the complex number $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$

has the matrix representation

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.101.1)$$

$$= \mathbf{T}_a \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.101.2)$$

$$\Rightarrow \begin{pmatrix} 5 \\ -3 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ -3 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.101.3)$$

Then,

$$\begin{pmatrix} 5 \\ -3 \end{pmatrix}^3 \triangleq \begin{pmatrix} 5 & 3 \\ -3 & 5 \end{pmatrix}^3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.101.4)$$

$$= \begin{pmatrix} -10 & 198 \\ -198 & -10 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.101.5)$$

$$= \begin{pmatrix} -10 \\ -198 \end{pmatrix} \quad (1.101.6)$$

The python code for above problem is

```
codes/line/comp.py
```

1.102. Find $\begin{pmatrix} -\sqrt{3} \\ \sqrt{2} \end{pmatrix} \begin{pmatrix} 2\sqrt{3} \\ -1 \end{pmatrix}$.

Solution: Using the equivalent matrices for the complex numbers,

$$\begin{aligned} \begin{pmatrix} -\sqrt{3} \\ \sqrt{2} \end{pmatrix} \begin{pmatrix} 2\sqrt{3} \\ -1 \end{pmatrix} &= \begin{pmatrix} -\sqrt{3} & -\sqrt{2} \\ \sqrt{2} & -\sqrt{3} \end{pmatrix} \begin{pmatrix} 2\sqrt{3} & 1 \\ -1 & 2\sqrt{3} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{2}-6 & -\sqrt{3}-2\sqrt{6} \\ \sqrt{3}+2\sqrt{6} & \sqrt{2}-6 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{2}-6 \\ \sqrt{3}+2\sqrt{6} \end{pmatrix} \end{aligned} \quad (1.102.1)$$

The following code verifies the result.

```
codes/line_ex/complex_ex/complex_ex.py
```

1.103. Find the multiplicative inverse of $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$.

Solution: Let \mathbf{T}_a be the matrix for the complex number \mathbf{a} . \mathbf{b} is defined to be the multiplicative inverse of \mathbf{a} if

$$\mathbf{T}_a \mathbf{T}_b = \mathbf{T}_b \mathbf{T}_a = \mathbf{I} \quad (1.103.1)$$

Then, from (1.101.1)

$$\mathbf{b} = \mathbf{a}^{-1} = \begin{pmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.103.2)$$

$$= \frac{1}{\|\mathbf{a}\|^2} \begin{pmatrix} a_1 \\ -a_2 \end{pmatrix} \quad (1.103.3)$$

Thus,

$$\begin{pmatrix} 2 \\ -3 \end{pmatrix}^{-1} = \frac{1}{13} \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad (1.103.4)$$

The python code for above problem is

```
solutions/3/codes/line/comp/comp.py
```

Note that

$$\mathbf{T}_b = \mathbf{T}_a^{-1} = \frac{\mathbf{T}_a^T}{\|\mathbf{a}\|^2} \quad (1.103.5)$$

1.104. Find

a) $\begin{pmatrix} 5 \\ \sqrt{2} \end{pmatrix} \begin{pmatrix} 1 \\ -2\sqrt{3} \end{pmatrix}$.

b) $\begin{pmatrix} 0 \\ 1 \end{pmatrix}^{-35}$.

c) Show that the polar representation of $\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}$ is $2\angle 60^\circ$.

1.105. Simplify the complex number $-\frac{16}{\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}}$

Solution: Using the polar form,

$$\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} = 2 \begin{pmatrix} \cos 60^\circ \\ \sin 60^\circ \end{pmatrix} = 2\angle 60^\circ \quad (1.105.1)$$

$$\Rightarrow \frac{-16}{\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}} = -8\angle -60^\circ = 4 \begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix} \quad (1.105.2)$$

The following python code gives the desired answer

```
./solutions/5/codes/lines/q8.py
```

1.106. Find the conjugate of $\frac{\begin{pmatrix} 3 \\ -2 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix}}{\begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix}}$.

Solution: Using the matrix form,

$$\begin{aligned} &\frac{\begin{pmatrix} 3 \\ -2 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix}}{\begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix}} \\ &= \begin{pmatrix} 3 & 2 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix} \left[\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \right]^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \frac{1}{25} \begin{pmatrix} 63 \\ -16 \end{pmatrix} \end{aligned} \quad (1.106.1)$$

The conjugate is given by

$$\frac{1}{25} \begin{pmatrix} 63 \\ 16 \end{pmatrix} \quad (1.106.2)$$

1.107. Find the modulus and argument of the complex numbers

a) $\frac{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}.$

b) $\frac{1}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}.$

Solution:

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \sqrt{2} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \quad (1.107.1)$$

$$= \sqrt{2} \begin{pmatrix} \cos 45^\circ \\ \sin 45^\circ \end{pmatrix} \quad (1.107.2)$$

In the above, the modulus is $\left\| \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\| = \sqrt{2}$ and the argument is 45° . Similarly,

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \sqrt{2} \begin{pmatrix} \cos 45^\circ \\ -\sin 45^\circ \end{pmatrix} \quad (1.107.3)$$

$$\Rightarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix}^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos 45^\circ \\ \sin 45^\circ \end{pmatrix} \quad (1.107.4)$$

Using the matrix representation,

$$\frac{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 \\ -1 \end{pmatrix}} = \begin{pmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{pmatrix} \times \begin{pmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.107.5)$$

$$= \begin{pmatrix} \cos 90^\circ \\ \sin 90^\circ \end{pmatrix} = 1 \angle 90^\circ \quad (1.107.6)$$

In general, if

$$\mathbf{z}_1 = r_1 \begin{pmatrix} \cos \theta_1 \\ \sin \theta_1 \end{pmatrix}, \mathbf{z}_2 = r_2 \begin{pmatrix} \cos \theta_2 \\ \sin \theta_2 \end{pmatrix}, \quad (1.107.7)$$

$$\mathbf{z}_1 \mathbf{z}_2 = r_1 r_2 \begin{pmatrix} \cos(\theta_1 + \theta_2) \\ \sin(\theta_1 + \theta_2) \end{pmatrix}. \quad (1.107.8)$$

Similarly, from (1.107.2),

$$\frac{1}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos 45^\circ \\ -\sin 45^\circ \end{pmatrix} \quad (1.107.9)$$

$$= \frac{1}{\sqrt{2}} \angle -45^\circ \quad (1.107.10)$$

1.108. Find θ such that

$$\frac{\begin{pmatrix} 3 \\ 2 \sin \theta \end{pmatrix}}{\begin{pmatrix} 1 \\ -2 \sin \theta \end{pmatrix}} \quad (1.108.1)$$

is purely real.

1.109. Convert the complex number

$$\mathbf{z} = \frac{\begin{pmatrix} -1 \\ 1 \end{pmatrix}}{\begin{pmatrix} \cos \frac{\pi}{3} \\ \sin \frac{\pi}{3} \end{pmatrix}} \quad (1.109.1)$$

in the polar form.

1.110. Simplify

$$\mathbf{z} = \left(\frac{1}{\begin{pmatrix} 1 \\ -4 \end{pmatrix}} - \frac{2}{\begin{pmatrix} 2 \\ 1 \end{pmatrix}} \right) \frac{\begin{pmatrix} 3 \\ -4 \end{pmatrix}}{\begin{pmatrix} 5 \\ 1 \end{pmatrix}} \quad (1.110.1)$$

Solution: Using equivalent matrices for the

complex numbers and matrix multiplication,

$$\begin{aligned}
 &= \left(\begin{pmatrix} 1 & 4 \\ -4 & 1 \end{pmatrix}^{-1} - 2 \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}^{-1} \right) \begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix} \begin{pmatrix} 5 & -1 \\ 1 & 5 \end{pmatrix}^{-1} \\
 &= \left(\frac{1}{1^2 + 4^2} \begin{pmatrix} 1 & -4 \\ 4 & 1 \end{pmatrix} - 2 \left(\frac{1}{2^2 + 1^2} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \right) \right) \begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix} \\
 &= \left(\frac{1}{17} \begin{pmatrix} 1 & -4 \\ 4 & 1 \end{pmatrix} - \frac{2}{5} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \right) \begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix} \\
 &= \left(\left(\frac{1}{17} \begin{pmatrix} 1 & -4 \\ 4 & 1 \end{pmatrix} \right) - \left(\frac{4}{5} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \right) \right) \begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix} \frac{1}{26} \begin{pmatrix} 5 & 1 \\ -1 & 5 \end{pmatrix} \\
 &= \left(\left(\frac{1}{17} \begin{pmatrix} 1 & -4 \\ 4 & 1 \end{pmatrix} - \frac{4}{5} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \right) \begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix} \right) \frac{1}{26} \begin{pmatrix} 5 & 1 \\ -1 & 5 \end{pmatrix} \\
 &= \left(\left(\frac{1}{17} \begin{pmatrix} 1 & -4 \\ 4 & 1 \end{pmatrix} - \frac{4}{5} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \right) \begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix} \right) \frac{1}{26} \begin{pmatrix} 5 & 1 \\ -1 & 5 \end{pmatrix} \\
 &= \left(\left(\frac{1}{17} \begin{pmatrix} 1 & -4 \\ 4 & 1 \end{pmatrix} - \frac{4}{5} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \right) \begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix} \right) \frac{1}{26} \begin{pmatrix} 5 & 1 \\ -1 & 5 \end{pmatrix} \\
 &= \frac{1}{85} \left(\begin{pmatrix} -63 & -54 \\ 54 & -63 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix} \right) \frac{1}{26} \begin{pmatrix} 5 & 1 \\ -1 & 5 \end{pmatrix} \\
 &= \frac{1}{2210} \left(\begin{pmatrix} -63 & -54 \\ 54 & -63 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix} \right) \begin{pmatrix} 5 & 1 \\ -1 & 5 \end{pmatrix} \\
 &= \frac{1}{2210} \begin{pmatrix} -189 + 216 & -162 - 252 \\ 162 + 252 & 216 - 189 \end{pmatrix} \begin{pmatrix} 5 & 1 \\ -1 & 5 \end{pmatrix} \\
 &= \frac{1}{2210} \begin{pmatrix} 27 & -414 \\ 414 & 27 \end{pmatrix} \begin{pmatrix} 5 & 1 \\ -1 & 5 \end{pmatrix} \\
 &= \frac{1}{2210} \begin{pmatrix} 27 & -414 \\ 414 & 27 \end{pmatrix} \begin{pmatrix} 5 & 1 \\ -1 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
 &= \frac{1}{2210} \begin{pmatrix} 135 + 414 & 27 - 2070 \\ 2070 - 27 & 414 + 135 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
 &= \frac{1}{2210} \begin{pmatrix} 549 & -2043 \\ 2043 & 549 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
 &= \frac{1}{2210} \begin{pmatrix} 549 \\ 2043 \end{pmatrix} \\
 &\Rightarrow \mathbf{z} = \begin{pmatrix} \frac{549}{2210} \\ \frac{2043}{2210} \end{pmatrix} \quad (1.110.2)
 \end{aligned}$$

b) $\frac{\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}}.$

Solution:

a) Below is the solution :

$$\begin{aligned}
 &\frac{\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}} \quad (1.111.1) \\
 &\left(\frac{2}{-1} \right)^2 = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.111.2) \\
 &\Rightarrow \left(\frac{2}{-1} \right)^2 = \begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.111.3) \\
 &\Rightarrow \left(\frac{2}{-1} \right)^2 = \begin{pmatrix} 3 \\ -4 \end{pmatrix} \quad (1.111.4) \\
 &= \begin{pmatrix} 1 \\ 7 \end{pmatrix} \begin{pmatrix} 3 \\ -4 \end{pmatrix}^{-1} \quad (1.111.5) \\
 &= \frac{1}{25} \begin{pmatrix} 1 & -7 \\ 7 & 1 \end{pmatrix} \begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.111.6) \\
 &= \frac{1}{25} \begin{pmatrix} -25 & -25 \\ 25 & -25 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.111.7) \\
 &= \frac{25}{25} \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.111.8) \\
 &= \sqrt{2} \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} \quad (1.111.9) \\
 &= \sqrt{2} \begin{pmatrix} \cos 135^\circ & -\sin 135^\circ \\ \sin 135^\circ & \cos 135^\circ \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.111.10) \\
 &= \sqrt{2} \begin{pmatrix} \cos 135^\circ \\ \sin 135^\circ \end{pmatrix} \quad (1.111.11) \\
 &= \sqrt{2} \angle 135^\circ \quad (1.111.12)
 \end{aligned}$$

1.111. Convert the following in the polar form:

a) $\frac{\begin{pmatrix} 1 \\ 7 \end{pmatrix}}{\begin{pmatrix} 2 \\ -1 \end{pmatrix}^2}.$

b) Below is the solution:

$$\frac{\begin{pmatrix} 1 \\ 3 \end{pmatrix}}{\begin{pmatrix} 1 \\ -2 \end{pmatrix}} \quad (1.111.13)$$

$$\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.111.14)$$

$$= \begin{pmatrix} 1 \\ 7 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix}^{-1} \quad (1.111.15)$$

$$= \frac{1}{5} \begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.111.16)$$

$$= \frac{1}{5} \begin{pmatrix} -5 & -5 \\ 5 & -5 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.111.17)$$

$$= \frac{5}{5} \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.111.18)$$

$$= \sqrt{2} \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} \quad (1.111.19)$$

$$= \sqrt{2} \begin{pmatrix} \cos 135^\circ & -\sin 135^\circ \\ \sin 135^\circ & \cos 135^\circ \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.111.20)$$

$$= \sqrt{2} \begin{pmatrix} \cos 135^\circ \\ \sin 135^\circ \end{pmatrix} \quad (1.111.21)$$

$$= \sqrt{2} \angle 135^\circ \quad (1.111.22)$$

1.112. If $\mathbf{z}_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, $\mathbf{z}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, find $\left\| \frac{\mathbf{z}_1 + \mathbf{z}_2 + 1}{\mathbf{z}_1 - \mathbf{z}_2 + 1} \right\|$

Solution: Let us consider $\frac{\mathbf{z}_1 + \mathbf{z}_2 + 1}{\mathbf{z}_1 - \mathbf{z}_2 + 1}$, then

$$\mathbf{z}_1 + \mathbf{z}_2 + 1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.112.1)$$

$$= \begin{pmatrix} 5 \\ -2 \end{pmatrix} \quad (1.112.2)$$

$$\mathbf{z}_1 - \mathbf{z}_2 + 1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.112.3)$$

$$= \begin{pmatrix} 2 \\ -2 \end{pmatrix} \quad (1.112.4)$$

$$\frac{\mathbf{z}_1 + \mathbf{z}_2 + 1}{\mathbf{z}_1 - \mathbf{z}_2 + 1} = \frac{\begin{pmatrix} 5 \\ -2 \end{pmatrix}}{\begin{pmatrix} 2 \\ -2 \end{pmatrix}} \quad (1.112.5)$$

The modulus of a complex number $\begin{pmatrix} a \\ b \end{pmatrix}$ is defined as $\sqrt{a^2 + b^2}$. Therefore,

$$\|\mathbf{z}_1 + \mathbf{z}_2 + 1\| = \sqrt{5^2 + (-2)^2} \quad (1.112.6)$$

$$= \sqrt{29} \quad (1.112.7)$$

$$\|\mathbf{z}_1 - \mathbf{z}_2 + 1\| = \sqrt{2^2 + (-2)^2} \quad (1.112.8)$$

$$= \sqrt{8} \quad (1.112.9)$$

Putting together (1.112.7) and (1.112.9), we have

$$\left\| \frac{\mathbf{z}_1 + \mathbf{z}_2 + 1}{\mathbf{z}_1 - \mathbf{z}_2 + 1} \right\| = \frac{\sqrt{29}}{\sqrt{8}} \quad (1.112.10)$$

1.113. Let $\mathbf{z}_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, $\mathbf{z}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$. Find

a) $\operatorname{Re} \left(\frac{\mathbf{z}_1 \mathbf{z}_2}{\mathbf{z}_1^*} \right)$.

b) $\operatorname{Im} \left(\frac{1}{\mathbf{z}_1 \mathbf{z}_1^*} \right)$.

Solution:

$$\left(\frac{\mathbf{z}_1 \mathbf{z}_2}{\mathbf{z}_1^*} \right) = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} -2 & -1 \\ 1 & -2 \end{pmatrix} \left[\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \right]^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.113.1)$$

$$\left(\frac{\mathbf{z}_1 \mathbf{z}_2}{\mathbf{z}_1^*} \right) = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} -2 & -1 \\ 1 & -2 \end{pmatrix} \left[\frac{1}{5} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \right] \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.113.2)$$

$$\left(\frac{\mathbf{z}_1 \mathbf{z}_2}{\mathbf{z}_1^*} \right) = \frac{1}{5} \begin{pmatrix} -2 & -11 \\ 11 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.113.3)$$

$$\left(\frac{\mathbf{z}_1 \mathbf{z}_2}{\mathbf{z}_1^*} \right) = \frac{1}{5} \begin{pmatrix} -2 \\ 11 \end{pmatrix} \quad (1.113.4)$$

Hence, the real part of $\left(\frac{z_1 z_2}{z_1^*}\right) = -\frac{2}{5}$

$$\left(\frac{1}{z_1 z_1^*}\right) = (z_1 z_1^*)^{-1} \quad (1.113.5)$$

$$\left(\frac{1}{z_1 z_1^*}\right) = \left[\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}\right]^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.113.6)$$

$$\left(\frac{1}{z_1 z_1^*}\right) = \left[\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}\right]^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.113.7)$$

$$\left(\frac{1}{z_1 z_1^*}\right) = \frac{1}{25} \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.113.8)$$

$$\left(\frac{1}{z_1 z_1^*}\right) = \frac{1}{25} \begin{pmatrix} 5 \\ 0 \end{pmatrix} \quad (1.113.9)$$

Hence, the imaginary part of $\left(\frac{1}{z_1 z_1^*}\right) = 0$.

1.114. Find the modulus and argument of the complex

number $\frac{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}{\begin{pmatrix} 1 \\ -3 \end{pmatrix}}$.

Solution: In general, any complex number can be expressed in matrix representation as follows:

$$\begin{pmatrix} a1 \\ a2 \end{pmatrix} = \begin{pmatrix} a1 & -a2 \\ a2 & a1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.114.1)$$

Converting complex number to matrix form:

$$\frac{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}{\begin{pmatrix} 1 \\ -3 \end{pmatrix}} = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.114.2)$$

$$\begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1/10 & -3/10 \\ 3/10 & 1/10 \end{pmatrix} \quad (1.114.3)$$

Sub (1.114.3) in (1.114.2),

$$\frac{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}{\begin{pmatrix} 1 \\ -3 \end{pmatrix}} = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1/10 & -3/10 \\ 3/10 & 1/10 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.114.4)$$

$$= \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1/10 \\ 3/10 \end{pmatrix} \quad (1.114.5)$$

$$= \begin{pmatrix} -5/10 \\ 5/10 \end{pmatrix} \quad (1.114.6)$$

$$\Rightarrow \boxed{\frac{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}{\begin{pmatrix} 1 \\ -3 \end{pmatrix}} = \begin{pmatrix} -1/2 \\ 1/2 \end{pmatrix}} \quad (1.114.7)$$

From (1.114.7), The modulus and argument of the complex number is,

$$r = \left\| \begin{pmatrix} -1/2 \\ 1/2 \end{pmatrix} \right\| = \frac{1}{\sqrt{2}} \quad (1.114.8)$$

$$\tan \theta = -1 \Rightarrow \theta = 180^\circ - 45^\circ = 135^\circ \quad (1.114.9)$$

1.115. Find the real numbers x, y such that $\begin{pmatrix} x \\ -y \end{pmatrix} \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ is the conjugate of $\begin{pmatrix} -6 \\ -24 \end{pmatrix}$.

Solution: The conjugate of $\begin{pmatrix} -6 \\ -24 \end{pmatrix}$ is $\begin{pmatrix} -6 \\ 24 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} x \\ -y \end{pmatrix} \begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} -6 \\ 24 \end{pmatrix} \quad (1.115.1)$$

$$\Rightarrow \begin{pmatrix} x \\ -y \end{pmatrix} = \frac{\begin{pmatrix} -6 \\ 24 \end{pmatrix}}{\begin{pmatrix} 3 \\ 5 \end{pmatrix}} \quad (1.115.2)$$

Using equivalent matrices for complex num-

bers, we have

$$\begin{pmatrix} x \\ -y \end{pmatrix} = \begin{pmatrix} -6 & -24 \\ 24 & -6 \end{pmatrix} \begin{pmatrix} 3 & -5 \\ 5 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.115.3)$$

$$= \frac{1}{34} \begin{pmatrix} -6 & -24 \\ 24 & -6 \end{pmatrix} \begin{pmatrix} 3 & 5 \\ -5 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.115.4)$$

$$= \frac{1}{34} \begin{pmatrix} 102 & -102 \\ 102 & 102 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.115.5)$$

$$= \begin{pmatrix} 3 & -3 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.115.6)$$

$$\Rightarrow \begin{pmatrix} x \\ -y \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} \quad (1.115.7)$$

$$\text{Therefore, } x = 3, \quad (1.115.8)$$

$$y = -3 \quad (1.115.9)$$

1.116. Find the modulus of $\frac{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 \\ -1 \end{pmatrix}} - \frac{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$.

Solution: In our case,

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.116.1)$$

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.116.2)$$

Now,

$$\frac{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 \\ -1 \end{pmatrix}} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.116.3)$$

$$= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (1.116.4)$$

Similarly,

$$\frac{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.116.5)$$

$$= \begin{pmatrix} 0 \\ -1 \end{pmatrix} \quad (1.116.6)$$

So,

$$\frac{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 \\ -1 \end{pmatrix}} - \frac{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ -1 \end{pmatrix} \quad (1.116.7)$$

$$= \begin{pmatrix} 0 \\ 2 \end{pmatrix} \quad (1.116.8)$$

Now, according to the problem statement:

$$\frac{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 \\ -1 \end{pmatrix}} - \frac{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} \quad (1.116.9)$$

$$= \begin{pmatrix} 0 \\ 2 \end{pmatrix} \quad (1.116.10)$$

\therefore

$$\left\| \frac{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 \\ -1 \end{pmatrix}} - \frac{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} \right\| \quad (1.116.11)$$

$$= \left\| \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right\| = \sqrt{0^2 + 2^2} = 2 \quad (1.116.12)$$

So, we can say that the modulus value of

$$\frac{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 \\ -1 \end{pmatrix}} - \frac{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} \quad (1.116.13)$$

is 2.

1.117. Rain is falling vertically with a speed of 35 ms^{-1} . Winds starts blowing after sometime with a speed of 12 ms^{-1} in east to west direction. In which direction should a boy waiting at a bus stop hold his umbrella?

Solution: See Fig. 1.117. From the given information, the rain velocity is

$$\mathbf{u} = \begin{pmatrix} 0 \\ 35 \end{pmatrix} \quad (1.117.1)$$

and the wind velocity is

$$\mathbf{v} = -\begin{pmatrix} 12 \\ 0 \end{pmatrix} \quad (1.117.2)$$

The resulting rain velocity is

$$\mathbf{u} + \mathbf{v} = \begin{pmatrix} -12 \\ 35 \end{pmatrix} \quad (1.117.3)$$

The desired angle is

$$-\tan^{-1} \frac{\mathbf{u} + \mathbf{v}}{35} = \tan^{-1} \frac{12}{35} \quad (1.117.4)$$

$$\approx 20.04^\circ \quad (1.117.5)$$

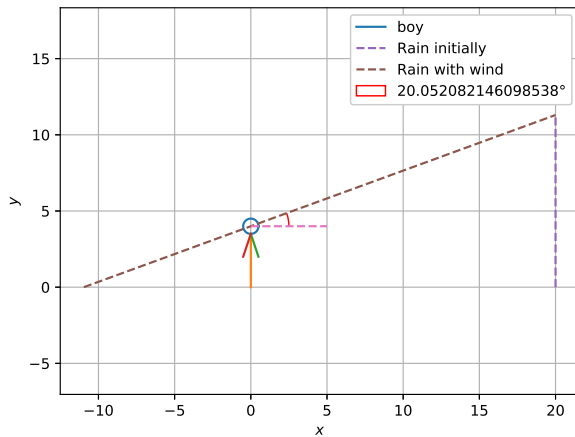


Fig. 1.117

- 1.118. A motorboat is racing towards north at 25 km/h and the water current in that region is 10 km/h in the direction of 60° east of south. Find the resultant velocity of the boat.

Solution: In Fig. 1.118, **A** denotes the velocity of the boat, **B** denotes the water current and **C** represents the resultant velocity.

$$\mathbf{A} = \begin{pmatrix} 0 \\ 25 \end{pmatrix} \quad (1.118.1)$$

$$\mathbf{B} = 10 \begin{pmatrix} \cos 30^\circ \\ -\sin 30^\circ \end{pmatrix} \quad (1.118.2)$$

$$\mathbf{C} = \mathbf{A} + \mathbf{B} \quad (1.118.3)$$

$$= 5 \begin{pmatrix} \sqrt{3} \\ 4 \end{pmatrix} \quad (1.118.4)$$

The following Python code generates Fig. 1.118.

```
solutions/2/codes/line_ex/
motion_in_a_plane/motion_plane.py
```

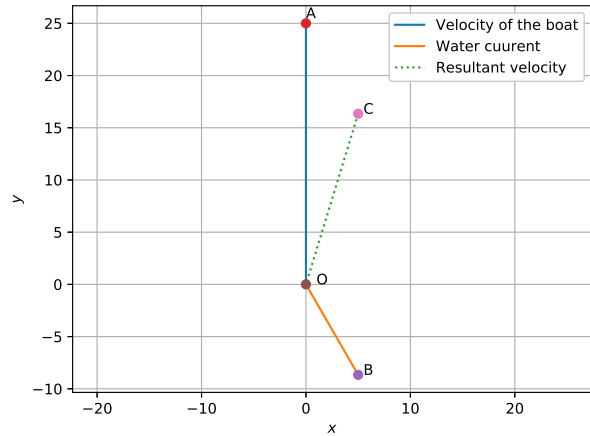


Fig. 1.118

- 1.119. Rain is falling vertically with a speed of 35 ms^{-1} . A woman rides a bicycle with a speed of 12 ms^{-1} in east to west direction. What is the direction in which she should hold her umbrella?

Solution: See Fig. 1.119. The velocity of rain and velocity of woman are

$$\mathbf{v}_r = \begin{pmatrix} 0 \\ -35 \end{pmatrix} \quad (1.119.1)$$

$$\mathbf{v}_w = \begin{pmatrix} -12 \\ 0 \end{pmatrix} \quad (1.119.2)$$

The relative velocity of rain w.r.t woman is given as

$$\mathbf{v}_{r_w} = \mathbf{v}_r - \mathbf{v}_w \quad (1.119.3)$$

$$= \begin{pmatrix} 12 \\ -35 \end{pmatrix} \quad (1.119.4)$$

So the woman must hold the umbrella along the direction of $-\mathbf{v}_{r_w}$. Thus, the desired angle is

$$\theta = \tan^{-1} \left(\frac{12}{35} \right) \quad (1.119.5)$$

The following python code generates Fig. 1.119.

```
solutions/3/codes/line/rain/rain.py
```

- 1.120. A hiker stands on the edge of a cliff 490 m above the ground and throws a stone horizontally with an initial speed of 15 ms^{-1} . Neglecting air resistance, find the time taken by the stone to reach the ground, and the speed

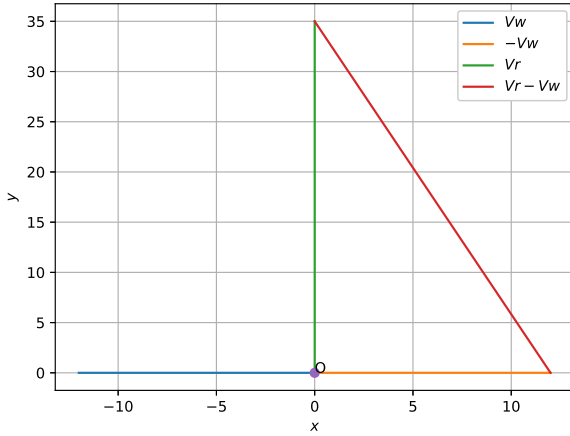


Fig. 1.119: Direction of umbrella

with which it hits the ground. (Take $g = 9.8 \text{ ms}^{-2}$).

Solution: From the given information, the hicker's position vector is

$$\mathbf{A} = \begin{pmatrix} 0 \\ 490 \end{pmatrix} \quad (1.120.1)$$

the acceleration of the stone is

$$\mathbf{a} = \begin{pmatrix} 0 \\ -9.8 \end{pmatrix} \quad (1.120.2)$$

and the initial velocity of the stone is

$$\mathbf{v}_A = \begin{pmatrix} 1.5 \\ 0 \end{pmatrix} \quad (1.120.3)$$

If \mathbf{B} be the final position of the stone,

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{a}t \quad (1.120.4)$$

$$\mathbf{B} = \mathbf{A} + \mathbf{v}_A t + \frac{1}{2} \mathbf{a} t^2 \quad (1.120.5)$$

$$\Rightarrow \mathbf{B} = \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 490 \end{pmatrix} + \begin{pmatrix} 1.5 \\ 0 \end{pmatrix} t + \frac{1}{2} \begin{pmatrix} 0 \\ -9.8 \end{pmatrix} t^2 \quad (1.120.6)$$

\therefore the stone finally comes to rest. Thus,

$$490 = \frac{1}{2} 9.8 t^2 \quad (1.120.7)$$

$$\Rightarrow t = 10 \quad (1.120.8)$$

Substituting in (1.120.4),

$$\mathbf{v}_B = \begin{pmatrix} 1.5 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 9.8 \end{pmatrix} 10 \quad (1.120.9)$$

$$= \begin{pmatrix} 1.5 \\ 98 \end{pmatrix} \quad (1.120.10)$$

The final speed is given by $\|\mathbf{v}_B\|$. The motion of the stone is plotted in Fig. 1.120 using (1.120.6) by varying t through the following code.

solutions/4/codes/line/motion/motion.py

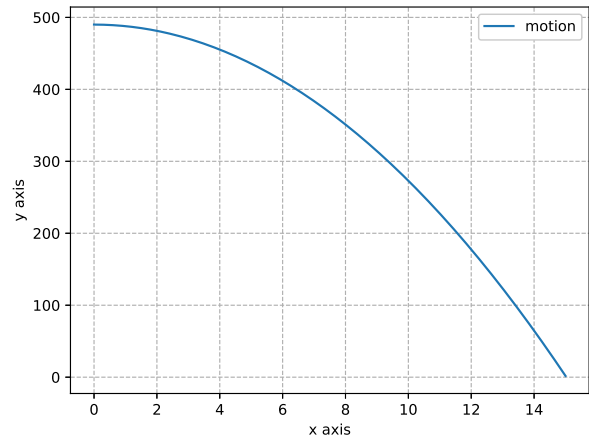


Fig. 1.120

1.121. Rain is falling vertically with a speed of 30 ms^{-1} . A woman rides a bicycle with a speed of 10 ms^{-1} in the north to south direction. What is the direction in which she should hold her umbrella?

Solution: See Fig. 1.121. The velocity of rain and velocity of woman are

$$\mathbf{v}_r = \begin{pmatrix} 0 \\ -30 \end{pmatrix} \quad (1.121.1)$$

$$\mathbf{v}_w = \begin{pmatrix} -10 \\ 0 \end{pmatrix} \quad (1.121.2)$$

The relative velocity of rain w.r.t woman is given as

$$\mathbf{v}_{r_w} = \mathbf{v}_r - \mathbf{v}_w \quad (1.121.3)$$

$$= \begin{pmatrix} 10 \\ -30 \end{pmatrix} \quad (1.121.4)$$

So the woman must hold the umbrella along the direction of $-\mathbf{v}_{rw}$. Thus, the desired angle is

$$\theta = \tan^{-1}\left(\frac{10}{30}\right) \quad (1.121.5)$$

The following python code plots Fig. 1.121.

```
./solutions/5/codes/lines/q12.py
```

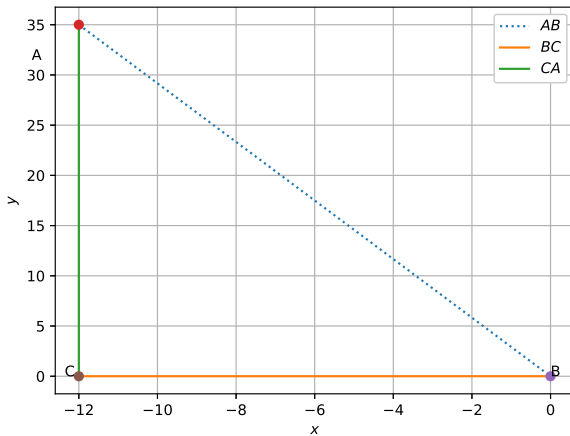


Fig. 1.121

- 1.122. A man can swim with a speed of 4.0 km/h in still water. How long does he take to cross a river 1.0 km wide if the river flows steadily at 3.0 km/h and he makes his strokes normal to the river current? How far down the river does he go when he reaches the other bank ?

Solution: The following code plots Fig. 1.122

```
solutions/6/codes/line/motion_plane/
man_river.py
```

In Fig. 1.122, let the man be at

$$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (1.122.1)$$

The opposite bank of the river is at

$$\mathbf{B} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.122.2)$$

River current

$$\mathbf{v} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \quad (1.122.3)$$

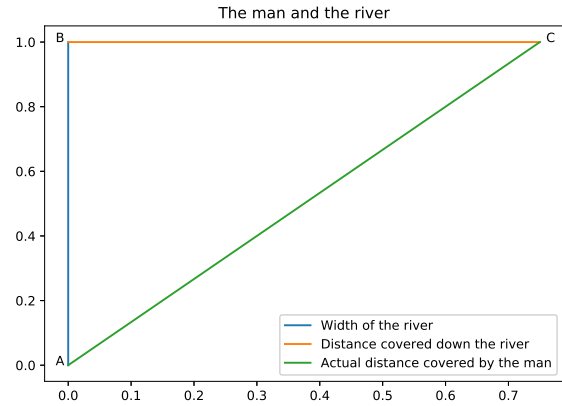


Fig. 1.122

Initial velocity of the man is

$$\mathbf{u} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} \quad (1.122.4)$$

The resultant velocity of the man is

$$\mathbf{u} + \mathbf{v} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \quad (1.122.5)$$

If the time taken by the man to cross the river be t , then

$$\mathbf{C} = (\mathbf{u} + \mathbf{v})t = \begin{pmatrix} 3 \\ 4 \end{pmatrix} t \quad (1.122.6)$$

$$= \mathbf{A} + \mathbf{B} = \begin{pmatrix} BC \\ 1 \end{pmatrix} \quad (1.122.7)$$

Thus,

$$\begin{pmatrix} 3 \\ 4 \end{pmatrix} t = \begin{pmatrix} BC \\ 1 \end{pmatrix} \quad (1.122.8)$$

$$\Rightarrow 4t = 1 \text{ or, } t = \frac{1}{4} \quad (1.122.9)$$

Distance traveled down the river

$$BC = 3t = \frac{3}{4} \quad (1.122.10)$$

- 1.123. In a harbour, wind is blowing at the speed of 72 km/h and the flag on the mast of a boat anchored in the harbour flutters along the N-E direction. If the boat starts moving at a speed of 51 km/h to the north, what is the direction of the flag on the mast of the boat ?

Solution: The velocity of wind and boat are

respectively,

$$\mathbf{v}_w = 72 \begin{pmatrix} \cos 45^\circ \\ \sin 45^\circ \end{pmatrix} \quad (1.123.1)$$

$$\mathbf{v}_b = \begin{pmatrix} 0 \\ 51 \end{pmatrix} \quad (1.123.2)$$

The resulting wind velocity is

$$\mathbf{v}_w - \mathbf{v}_b = \begin{pmatrix} 36\sqrt{2} \\ 36\sqrt{2} - 51 \end{pmatrix} \quad (1.123.3)$$

The direction of the flag is

$$\tan^{-1} \left(\frac{36\sqrt{2} - 51}{36\sqrt{2}} \right) \quad (1.123.4)$$

$$= -0.1^\circ \quad (1.123.5)$$

The python code for Fig. 1.123 is

```
solutions/7/codes/line/motion/motion.py
```

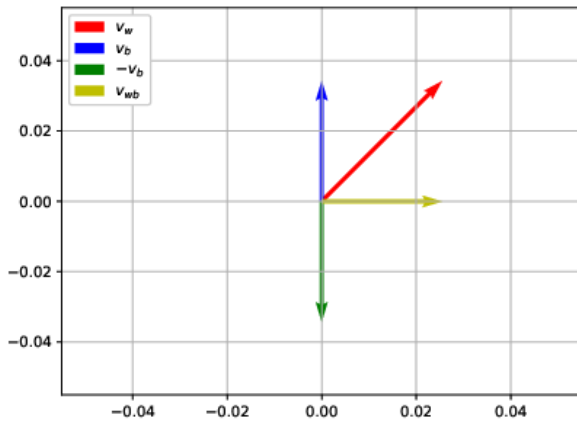


Fig. 1.123

2 EXERCISES

- 2.1. The vertices of $\triangle ABC$ are $\mathbf{A} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$. A line is drawn to intersect sides AB and AC at D and E respectively, such that

$$\frac{AD}{AB} = \frac{AE}{AC} = \frac{1}{4} \quad (2.1.1)$$

Find

$$\frac{\text{area of } \triangle ADE}{\text{area of } \triangle ABC}. \quad (2.1.2)$$

- 2.2. In $\triangle ABC$, Show that the centroid

$$\mathbf{O} = \frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} \quad (2.2.1)$$

- 2.3. Show that the vectors $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix}$, $\begin{pmatrix} 3 \\ -4 \\ -4 \end{pmatrix}$ form the vertices of a right angled triangle.

- 2.4. Find the direction vectors of the sides of a triangle with vertices $\mathbf{A} = \begin{pmatrix} 3 \\ 5 \\ -4 \end{pmatrix}$, $\mathbf{B} =$

$$\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}, \text{ and } \mathbf{C} = \begin{pmatrix} -5 \\ -5 \\ -2 \end{pmatrix}$$

- 2.5. Check whether

$$\begin{pmatrix} 5 \\ -2 \end{pmatrix}, \begin{pmatrix} 6 \\ 4 \end{pmatrix}, \begin{pmatrix} 7 \\ -2 \end{pmatrix} \quad (2.5.1)$$

are the vertices of an isosceles triangle.

- 2.6. Are the points

$$\mathbf{A} = \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 10 \\ 20 \\ 30 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 25 \\ -41 \\ 5 \end{pmatrix}, \quad (2.6.1)$$

the vertices of a right angled triangle?

- 2.7. Determine if the points

$$\begin{pmatrix} 1 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ -11 \end{pmatrix} \quad (2.7.1)$$

are collinear.

- 2.8. By using the concept of equation of a line, prove that the three points $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$, $\begin{pmatrix} -2 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 8 \\ 2 \end{pmatrix}$ are collinear.

- 2.9. Find the value of x for which the points $\begin{pmatrix} x \\ -1 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 5 \end{pmatrix}$ are collinear.

- 2.10. In each of the following, find the value of k for which the points are collinear

a) $\begin{pmatrix} 7 \\ -2 \end{pmatrix}, \begin{pmatrix} 5 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ k \end{pmatrix}$

b) $\begin{pmatrix} 8 \\ 1 \end{pmatrix}, \begin{pmatrix} k \\ -4 \end{pmatrix}, \begin{pmatrix} 2 \\ -5 \end{pmatrix}$

- 2.11. Find a condition on \mathbf{x} such that the points $\mathbf{x}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 7 \\ 0 \end{pmatrix}$ are collinear.

- 2.12. Show that the points $\mathbf{A} = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 \\ 6 \\ 3 \end{pmatrix}$ and

$\mathbf{C} = \begin{pmatrix} 3 \\ 10 \\ -1 \end{pmatrix}$ are collinear.

2.13. Show that the points $\mathbf{A} = \begin{pmatrix} 1 \\ -2 \\ 8 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 5 \\ 0 \\ -2 \end{pmatrix}$ and

$\mathbf{C} = \begin{pmatrix} 11 \\ 3 \\ 7 \end{pmatrix}$ are collinear, and find the ratio in which \mathbf{B} divides AC .

2.14. Show that $\mathbf{A} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 5 \\ 8 \\ 7 \end{pmatrix}$ are collinear.

2.15. A bullet fired at an angle of 30° with the horizontal hits the ground 3.0 km away. By adjusting its angle of projection, can one hope to hit a target 5.0 km away? Assume the muzzle speed to be fixed, and neglect air resistance.

2.16. A fighter plane flying horizontally at an altitude of 1.5 km with speed 720 km/h passes directly overhead an anti-aircraft gun. At what angle from the vertical should the gun be fired for the shell with muzzle speed 600 ms^{-1} to hit the plane? At what minimum altitude should the pilot fly the plane to avoid being hit? (Take $g = 10 \text{ ms}^{-2}$).

2.17. Give the magnitude and direction of the net force acting on a stone of mass 0.1 kg,

- just after it is dropped from the window of a stationary train,
- just after it is dropped from the window of a train running at a constant velocity of 36 km/h,
- just after it is dropped from the window of a train accelerating with 1 ms^{-2}
- lying on the floor of a train which is accelerating with 1 ms^{-2} , the stone being at rest relative to the train.

Neglect air resistance throughout.

2.18. Consider the collision depicted in Fig. 2.18 to be between two billiard balls with equal masses $m_1 = m_2$. The first ball is called the cue while the second ball is called the target. The billiard player wants to 'sink' the target ball in a corner pocket, which is at an angle $\theta_2 = 37^\circ$. Assume that the collision is elastic and that friction and rotational motion are not important. Obtain θ_1 .

2.19. Find the ratio in which the line segment joining

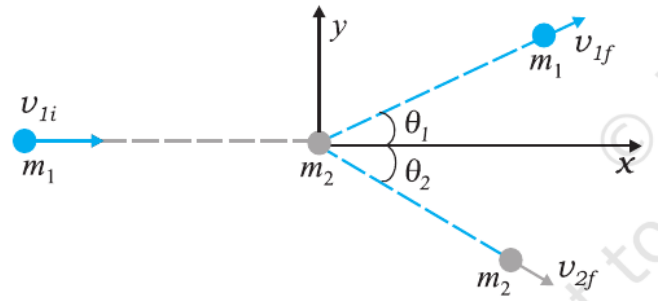


Fig. 2.18

the points $\begin{pmatrix} 4 \\ 8 \\ 10 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ 10 \\ -8 \end{pmatrix}$ is divided by the YZ-plane.

2.20. If

$$\mathbf{P} = 3\mathbf{a} - 2\mathbf{b} \quad (2.20.1)$$

$$\mathbf{Q} = \mathbf{a} + \mathbf{b} \quad (2.20.2)$$

find \mathbf{R} , which divides PQ in the ratio 2 : 1

- internally,
- externally.

2.21. Find a unit vector in the direction of $\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$.

2.22. Find a unit vector in the direction of the line passing through $\begin{pmatrix} -2 \\ 4 \\ -5 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

2.23. Find a unit vector in the direction of $\mathbf{a} + \mathbf{b}$, where

$$\mathbf{a} = \begin{pmatrix} 2 \\ 2 \\ -5 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}. \quad (2.23.1)$$

2.24. Find a unit vector in the direction of

$$\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}. \quad (2.24.1)$$

2.25. Find a point on the y-axis which is equidistant from the points $\mathbf{A} = \begin{pmatrix} 6 \\ 5 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$.

2.26. The two adjacent sides of a parallelogram are $\begin{pmatrix} 2 \\ -4 \\ -5 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}$. Find the unit vector parallel to its diagonal. Also, find its area.

2.27. Find the area of a parallelogram whose adjacent sides are determined by the vectors

$$\mathbf{a} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 2 \\ -7 \\ 1 \end{pmatrix}.$$

2.28. Verify if $\mathbf{A} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 8 \\ 6 \end{pmatrix}$ are points on a line.