Matrix Analysis

G V V Sharma*

Contents

***Abstract*—This book provides a computational approach to school geometry based on the NCERT textbooks from Class 6-12. Links to sample Python codes are available in the text.**

Download python codes using

```
svn co https://github.com/gadepall/school/trunk/
    ncert/computation/codes
```

1 Examples

1.1. Balance the following chemical equation.

$$HNO_3 + Ca(OH)_2 \to Ca(NO_3)_2 + H_2O \tag{1.1.1}$$

Solution: Let the balanced version of (1.1.1) be

$$x_1 HNO_3 + x_2 Ca(OH)_2 \to x_3 Ca(NO_3)_2 + x_4 H_2O \tag{1.1.2}$$

which results in the following equations:

$$(x_1 + 2x_2 - 2x_4)H = 0 \tag{1.1.3}$$
$$(x_1 - 2x_3)N = 0 \tag{1.1.4}$$
$$(3x_1 + 2x_2 - 6x_3 - x_4)O = 0 \tag{1.1.5}$$
$$(x_2 - x_3)Ca = 0 \tag{1.1.6}$$

which can be expressed as

$$x_1 + 2x_2 + 0.x_3 - 2x_4 = 0 \tag{1.1.7}$$
$$x_1 + 0.x_2 - 2x_3 + 0.x_4 = 0 \tag{1.1.8}$$
$$3x_1 + 2x_2 - 6x_3 - x_4 = 0 \tag{1.1.9}$$
$$0.x_1 + x_2 - x_3 + 0.x_4 = 0 \tag{1.1.10}$$

resulting in the matrix equation

$$\begin{pmatrix} 1 & 2 & 0 & -2 \\ 1 & 0 & -2 & 0 \\ 3 & 2 & -6 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix} \mathbf{x} = \mathbf{0} \tag{1.1.11}$$

where,

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \tag{1.1.12}$$

*The author is with the Department of Electrical Engineering, Indian Institute of Technology, Hyderabad 502285 India e-mail: gadepall@iith.ac.in.

(1.1.11) can be reduced as follows:

$$\begin{pmatrix}1&2&0&-2\\1&0&-2&0\\3&2&-6&-1\\0&1&-1&0\end{pmatrix}\xleftrightarrow[R_3\leftarrow\frac{R_3}{3}-R_1]{R_2\leftarrow R_2-R_1}\begin{pmatrix}1&2&0&-2\\0&-2&-2&2\\0&-\frac{4}{3}&-2&\frac{5}{3}\\0&1&-1&0\end{pmatrix}\tag{1.1.13}$$

$$\xleftrightarrow{R_2\leftarrow-\frac{R_2}{2}}\begin{pmatrix}1&2&0&-2\\0&1&1&-1\\0&-\frac{4}{3}&-2&\frac{5}{3}\\0&1&-1&0\end{pmatrix}\tag{1.1.14}$$

$$\xleftrightarrow[R_4\leftarrow R_4-R_2]{R_3\leftarrow R_3+\frac{4}{3}R_2}\begin{pmatrix}1&2&0&-2\\0&1&1&-1\\0&0&-\frac{2}{3}&\frac{1}{3}\\0&0&-2&1\end{pmatrix}\tag{1.1.15}$$

$$\xleftrightarrow[R_3\leftarrow-\frac{3}{2}R_3]{R_1\leftarrow R_1-2R_2}\begin{pmatrix}1&0&-2&0\\0&1&1&-1\\0&0&1&-\frac{1}{2}\\0&0&-2&1\end{pmatrix}\tag{1.1.16}$$

$$\xleftrightarrow{R_4\leftarrow R_4+2R_3}\begin{pmatrix}1&0&-2&0\\0&1&1&-1\\0&0&1&-\frac{1}{2}\\0&0&0&0\end{pmatrix}\tag{1.1.17}$$

$$\xleftrightarrow[R_2\leftarrow R_2-R_3]{R_1\leftarrow R_1+2R_3}\begin{pmatrix}1&0&0&-1\\0&1&0&-\frac{1}{2}\\0&0&1&-\frac{1}{2}\\0&0&0&0\end{pmatrix}\tag{1.1.18}$$

Thus,

$$x_1 = x_4, x_2 = \frac{1}{2}x_4, x_3 = \frac{1}{2}x_4 \tag{1.1.19}$$

$$\implies \quad \mathbf{x} = x_4\begin{pmatrix}1\\\frac{1}{2}\\\frac{1}{2}\\1\end{pmatrix} = \begin{pmatrix}2\\1\\1\\2\end{pmatrix} \tag{1.1.20}$$

by substituting $x_4 = 2$.

Hence, (1.1.2) finally becomes

$$2HNO_3 + Ca(OH)_2 \rightarrow Ca(NO_3)_2 + 2H_2O \tag{1.1.21}$$

1.2. Balance the following chemical equation.

$$\text{Zinc + Silver nitrate} \rightarrow \text{Zinc nitrate + Silver} \tag{1.2.1}$$

Solution:
Equation 1.2.1 can be written as :

$$Zn + AgNO_3 \rightarrow Ag + Zn(NO_3)_2 \tag{1.2.2}$$

Suppose balance form of the equation is :

$$x_1Zn + x_2AgNO_3 \rightarrow x_3Ag + x_4Zn(NO_3)_2 \tag{1.2.3}$$

which results in the following equations:

$$(x_1 - 2x_4)Zn = 0 \tag{1.2.4}$$
$$(x_2 - x_3)Ag = 0 \tag{1.2.5}$$
$$(x_3 - 2x_4)N = 0 \tag{1.2.6}$$
$$(3x_3 - 6x_4)O = 0 \tag{1.2.7}$$

which can be expressed as

$$x_1 + 0.x_2 + 0.x_3 - x_4 = 0 \tag{1.2.8}$$
$$0.x_1 + x_2 - x_3 + 0.x_4 = 0 \tag{1.2.9}$$
$$0.x_1 + 0.x_2 + x_3 - 2.x_4 = 0 \tag{1.2.10}$$
$$0.x_1 + 0.x_2 + 3x_3 - 6.x_4 = 0 \tag{1.2.11}$$

resulting in the matrix equation

$$\begin{pmatrix}1&0&0&-1\\0&1&-1&0\\0&0&1&-2\\0&0&3&-6\end{pmatrix}\mathbf{x} = \mathbf{0} \tag{1.2.12}$$

where,

$$\mathbf{x} = \begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} \tag{1.2.13}$$

(1.2.12) can be reduced as follows:

$$\begin{pmatrix}1&0&0&-1\\0&1&-1&0\\0&0&1&-2\\0&0&3&-6\end{pmatrix}\xleftrightarrow{R_4\leftarrow R_4-3R_3}\begin{pmatrix}1&0&0&-1\\0&1&-1&0\\0&0&1&-2\\0&0&0&0\end{pmatrix}\tag{1.2.14}$$

Thus,

$$x_1 = x_4, x_2 = 2x_4, x_3 = 2x_4 \tag{1.2.15}$$

$$\implies \quad \mathbf{x} = \begin{pmatrix} x_4 \\ 2x_4 \\ 2x_4 \\ x_4 \end{pmatrix} = x_4 \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix} \tag{1.2.16}$$

by substituting $x_4 = 1$, we get :

$$\implies \quad \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix} \tag{1.2.17}$$

Hence, (1.2.3) finally becomes

$$Zn + 2AgNO_3 \rightarrow 2Ag + Zn(NO_3)_2 \tag{1.2.18}$$

1.3. Write the balanced chemical equations for the following reaction :

$$BaCl_2 + K_2SO_4 \rightarrow BaSO_4 + KCl \tag{1.3.1}$$

Solution: We know that the number of atoms of each element remains the same, before and after a chemical reaction.
Equation (1.3.1) can be written as:

$$x_1BaCl_2 + x_2K_2SO_4 \rightarrow x_3BaSO_4 + x_4KCl \tag{1.3.2}$$

Element wise contribution in forming the respective chemical compound can be written in the form of equation as :

$$Ba : x_1 + 0x_2 - x_3 - 0x_4 = 0 \tag{1.3.3}$$
$$Cl : 2x_1 + 0x_2 - 0x_3 - 1x_4 = 0 \tag{1.3.4}$$
$$K : 0x_1 + 2x_2 - 0x_3 - 1x_4 = 0 \tag{1.3.5}$$
$$S : 0x_1 + 1x_2 - 1x_3 - 0x_4 = 0 \tag{1.3.6}$$
$$O : 0x_1 + 4x_2 - 4x_3 - 0x_4 = 0 \tag{1.3.7}$$

In matrix form this can be written as:

$$A\mathbf{x} = 0 \tag{1.3.8}$$

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 2 & 0 & 0 & -1 \\ 0 & 2 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 4 & -4 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{1.3.9}$$

Using Gaussian Elimination method

$$\xleftrightarrow{R_2 \leftrightarrow R_5} \begin{pmatrix} 1 & 0 & -1 & 0 & : & 0 \\ 0 & 4 & -4 & 0 & : & 0 \\ 0 & 2 & 0 & -1 & : & 0 \\ 0 & 1 & -1 & 0 & : & 0 \\ 2 & 0 & 0 & -1 & : & 0 \end{pmatrix} \tag{1.3.10}$$

$$\xleftrightarrow{R_5 \leftarrow 2R_1 - R_5} \begin{pmatrix} 1 & 0 & -1 & 0 & : & 0 \\ 0 & 4 & -4 & 0 & : & 0 \\ 0 & 2 & 0 & -1 & : & 0 \\ 0 & 1 & -1 & 0 & : & 0 \\ 0 & 0 & -2 & 1 & : & 0 \end{pmatrix} \tag{1.3.11}$$

$$\xleftrightarrow[R_4 \leftarrow 4R_4 - R_2]{R_3 \leftarrow 2R_3 - R_2} \begin{pmatrix} 1 & 0 & -1 & 0 & : & 0 \\ 0 & 4 & -4 & 0 & : & 0 \\ 0 & 0 & 4 & -2 & : & 0 \\ 0 & 0 & 0 & 0 & : & 0 \\ 0 & 0 & -2 & 1 & : & 0 \end{pmatrix} \tag{1.3.12}$$

$$\xleftrightarrow{R_5 \leftrightarrow R_5} \begin{pmatrix} 1 & 0 & -1 & 0 & : & 0 \\ 0 & 4 & -4 & 0 & : & 0 \\ 0 & 0 & 4 & -2 & : & 0 \\ 0 & 0 & -2 & 1 & : & 0 \\ 0 & 0 & 0 & 0 & : & 0 \end{pmatrix} \tag{1.3.13}$$

$$\xleftrightarrow{R_4 \leftarrow 2R_4 - R_3} \begin{pmatrix} 1 & 0 & -1 & 0 & : & 0 \\ 0 & 4 & -4 & 0 & : & 0 \\ 0 & 0 & 4 & -2 & : & 0 \\ 0 & 0 & 0 & 0 & : & 0 \\ 0 & 0 & 0 & 0 & : & 0 \end{pmatrix} \tag{1.3.14}$$

Clearly the system is linearly dependent. Therefore by fixing the value of $x_4 = 2$, one of the possible vector $\mathbf{x}$ is:

$$\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \end{pmatrix} \tag{1.3.15}$$

Hence by putting the values of x_1, x_2, x_3, x_4 in equation (1.3.1) we get our balanced chemical equation as followes :

$$BaCl_2 + K_2SO_4 \rightarrow BaSO_4 + 2KCl \tag{1.3.16}$$

1.4. Balance the following chemical equation.

$$Fe + H_2O \rightarrow Fe_3O_4 + H_2 \tag{1.4.1}$$

Solution: Let the balanced version of (1.4.1) be

$$x_1Fe + x_2H_2O \rightarrow x_3Fe_3O_4 + x_4H_2 \tag{1.4.2}$$

which results in the following equations

$$\begin{aligned}(x_1 - 3x_3)\,Fe &= 0\\ (2x_2 - 2x_4)\,H &= 0\\ (x_2 - 4x_3)\,O &= 0\end{aligned} \tag{1.4.3}$$

which can be expressed as

$$\begin{aligned}x_1 + 0.x_2 - 3x_3 + 0.x_4 &= 0\\ 0.x_1 + 2x_2 + 0.x_3 - 2x_4 &= 0\\ 0.x_1 + x_2 - 4x_3 + 0.x_4 &= 0\end{aligned} \tag{1.4.4}$$

resulting in the matrix equation

$$\begin{pmatrix}1 & 0 & -3 & 0\\ 0 & 2 & 0 & -2\\ 0 & 1 & -4 & 0\end{pmatrix}\mathbf{x} = \mathbf{0} \tag{1.4.5}$$

where

$$\mathbf{x} = \begin{pmatrix}x_1\\ x_2\\ x_3\\ x_4\end{pmatrix} \tag{1.4.6}$$

(1.4.5) can be row reduced as follows

$$\begin{pmatrix}1 & 0 & -3 & 0\\ 0 & 2 & 0 & -2\\ 0 & 1 & -4 & 0\end{pmatrix} \xleftrightarrow{R_2 \leftarrow \frac{R_2}{2}} \begin{pmatrix}1 & 0 & -3 & 0\\ 0 & 1 & 0 & -1\\ 0 & 1 & -4 & 0\end{pmatrix} \tag{1.4.7}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - R_2} \begin{pmatrix}1 & 0 & -3 & 0\\ 0 & 1 & 0 & -1\\ 0 & 0 & -4 & 1\end{pmatrix} \tag{1.4.8}$$

$$\xleftrightarrow{R_1 \leftarrow 4R_1 - 3R_3} \begin{pmatrix}4 & 0 & 0 & -3\\ 0 & 1 & 0 & -1\\ 0 & 0 & -4 & 1\end{pmatrix} \tag{1.4.9}$$

$$\xleftrightarrow[R_3 \leftarrow -\frac{1}{4}R_3]{R_1 \leftarrow \frac{1}{4}} \begin{pmatrix}1 & 0 & 0 & -\frac{3}{4}\\ 0 & 1 & 0 & -1\\ 0 & 0 & 1 & -\frac{1}{4}\end{pmatrix} \tag{1.4.10}$$

Thus,

$$x_1 = \frac{3}{4}x_4, x_2 = x_4, x_3 = \frac{1}{4}x_4 \tag{1.4.11}$$

$$\tag{1.4.12}$$

$$\implies \mathbf{x} = x_4\begin{pmatrix}\frac{3}{4}\\ 1\\ \frac{1}{4}\\ 1\end{pmatrix} = \begin{pmatrix}3\\ 4\\ 1\\ 4\end{pmatrix} \tag{1.4.13}$$

upon substituting $x_4 = 4$. (1.4.2) then becomes

$$3Fe + 4H_2O \rightarrow Fe_3O_4 + 4H_2 \tag{1.4.14}$$

1.5. Consider the following information regarding the number of men and women workers in the three factories I,II and III

	Men Workers	Women Workers
I	30	25
II	25	31
III	27	26

Represent the above information in the form of a 3×2 matrix. What does the entry in the third row and second column represent?

1.6. If a matrix has 8 elements, what are the possible orders it can have?

1.7. Construct a 3×2 matrix whose elements are given by $a_{ij} = \frac{1}{2}|i - 3j|$

1.8. $\begin{pmatrix}x+3 & z+4 & 2y-7\\ -6 & a-1 & 0\\ b-3 & -21 & 0\end{pmatrix} = \begin{pmatrix}0 & 6 & 3y-2\\ -6 & -3 & 2c+2\\ 2b+4 & -21 & 0\end{pmatrix}$
Find the values of a,b,c,x,y and z.
Solution: As the two matrices are equal their corresponding entries are also equal. Hence

$$x + 3 = 0 \implies x = -3 \tag{1.8.1}$$

$$z + 4 = 6 \implies z = 2 \tag{1.8.2}$$

$$2y - 7 = 3y - 2 \implies y = -5 \tag{1.8.3}$$

$$a - 1 = -3 \implies a = -2 \tag{1.8.4}$$

$$2c + 2 = 0 \implies c = -1 \tag{1.8.5}$$

$$b - 3 = 2b + 4 \implies b = -7 \tag{1.8.6}$$

1.9. Find the values of a,b,c and d from the following equation:
$\begin{pmatrix}2a+b & a-2b\\ 5c-d & 4c+3d\end{pmatrix} = \begin{pmatrix}4 & -3\\ 11 & 24\end{pmatrix}$
Solution: These equations can be written as:

$$\begin{pmatrix}2 & 1 & 0 & 0\\ 1 & -2 & 0 & 0\\ 0 & 0 & 5 & -1\\ 0 & 0 & 4 & 3\end{pmatrix}\begin{pmatrix}a\\ b\\ c\\ d\end{pmatrix} = \begin{pmatrix}4\\ -3\\ 11\\ 24\end{pmatrix} \tag{1.9.1}$$

So the coefficient matrix A can be expressed as:

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 0 & 5 & -1 \\ 0 & 0 & 4 & 3 \end{pmatrix} \tag{1.9.2}$$

And the augmented matrix B can be expressed as:

$$B = \begin{pmatrix} 2 & 1 & 0 & 0 & 4 \\ 1 & -2 & 0 & 0 & -3 \\ 0 & 0 & 5 & -1 & 11 \\ 0 & 0 & 4 & 3 & 24 \end{pmatrix} \tag{1.9.3}$$

Now, if we express the augmented matrix as Echelon form, then it will be:

$$\begin{aligned} B &= \begin{pmatrix} 2 & 1 & 0 & 0 & 4 \\ 1 & -2 & 0 & 0 & -3 \\ 0 & 0 & 5 & -1 & 11 \\ 0 & 0 & 4 & 3 & 24 \end{pmatrix} \\ &\xleftrightarrow[R_4 \leftarrow R_4 - R_3]{R_2 \leftarrow 2R_2 - R_1} \begin{pmatrix} 2 & 1 & 0 & 0 & 4 \\ 0 & -5 & 0 & 0 & -10 \\ 0 & 0 & 5 & -1 & 11 \\ 0 & 0 & -1 & 4 & 13 \end{pmatrix} \\ &\xleftrightarrow[R_4 \leftarrow 5R_4 + R_3]{R_2 \leftarrow \frac{R_2}{(-5)}} \begin{pmatrix} 2 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 5 & -1 & 11 \\ 0 & 0 & 0 & 19 & 76 \end{pmatrix} \\ &\xleftrightarrow{R_4 \leftarrow \frac{R_4}{(19)}} \begin{pmatrix} 2 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 5 & -1 & 11 \\ 0 & 0 & 0 & 1 & 4 \end{pmatrix} \\ &\xleftrightarrow[R_1 \leftarrow R_1 - R_2]{R_3 \leftarrow R_3 + R_4} \begin{pmatrix} 2 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 5 & 0 & 15 \\ 0 & 0 & 0 & 1 & 4 \end{pmatrix} \\ &\xleftrightarrow[R_1 \leftarrow \frac{R_1}{2}]{R_3 \leftarrow \frac{R_3}{5}} \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{pmatrix} \end{aligned} \tag{1.9.4}$$

Thus,

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \tag{1.9.5}$$

1.10. Given A=$\begin{pmatrix} \sqrt{3} & 1 & -1 \\ 2 & 3 & 0 \end{pmatrix}$ and B=$\begin{pmatrix} 2 & \sqrt{5} & 1 \\ -2 & 3 & \frac{1}{2} \end{pmatrix}$, find A+B.
Solution:

$$A + B = \begin{pmatrix} \sqrt{3} + 2 & \sqrt{5} + 1 & 0 \\ 0 & 6 & \frac{1}{2} \end{pmatrix} \tag{1.10.1}$$

The python code for matrix addition can be downloaded from

```
solutions/7/codes/line/matrix/matrix_add.py
```

1.11. In the matrix A=$\begin{pmatrix} 2 & 5 & 19 & -7 \\ 35 & -2 & \frac{5}{2} & 12 \\ \sqrt{3} & 1 & -5 & 17 \end{pmatrix}$, write

a) The order of the matrix
b) The number of elements
c) Write the elements $a_{31}, a_{21}, a_{33}, a_{24}, a_{23}$.

Solution:

a) The order of matrix for above problem is 3×4.
b) The number of elements=12
c) The elements are

$$a_{31} = \sqrt{3}, a_{21} = 35 \tag{1.11.1}$$
$$a_{33} = -5, a_{24} = 12 \tag{1.11.2}$$
$$a_{23} = \frac{5}{2} \tag{1.11.3}$$

The python code is given in

```
solutions/1/codes/line/matrix.py
```

1.12. If a matrix has 24 elements,what are the possible orders it can have? What,if it has 13 elements?
Solution: The following Python code generates all possible dimensions for any matrix size

```
solutions/2/codes/line_ex/matrix/matrix.py
```

1.13. If a matrix has 18 elements,what are the possible orders it can have? What,if it has 5 elements?

Solution:

1.1. A matrix with n elements can be represented as a matrix of order $(r \times c)$ if and only if n,r and c are all natural numbers(Here r is the number of rows and c is the number of coloumns in the matrix.).This is possible only if r is a divisor of n.

1.2. So the total possible orders a matrix with n elements can have is equal to the total number of divisors of n.
The following python code finds the total possible orders (d) for a matrix of n elements.

```
solutions/3/codes/line/matrix/matrix.py
```

So a matrix of 18 elements has 6 possible orders and a matrix of 5 elements can have 2 possible orders.

1.14. Construct a 2×2 matrix,A=$[a_{ij}]$,whose elements are given by:
(i) a_{ij}=$\frac{(i+j)^2}{2}$ (ii) a_{ij}=$\frac{i}{j}$ (iii) a_{ij}=$\frac{(i+2j)^2}{2}$
Solution: From the following code,

```
solutions/4/codes/line/matrix/matrix.
py
```

a) $\mathbf{A} = \begin{pmatrix} 2 & 4.5 \\ 4.5 & 8 \end{pmatrix}$
b) $\mathbf{A} = \begin{pmatrix} 1 & 0.5 \\ 2 & 1 \end{pmatrix}$
c) $\mathbf{A} = \begin{pmatrix} 4.5 & 12.5 \\ 2 & 18 \end{pmatrix}$

1.15. Construct a 3×4 matrix,whose elements are given by:
(i) a_{ij}=$\frac{1}{2}\left|-3i + j\right|$ (ii) a_{ij}=2i-j
Solution:
The following python code computes the required matrix.

```
./codes/lines/q13.py
```

a) The matrix $A_{ij} = \frac{1}{2}\left| -3i + j\right|$ obtained is

$$\begin{pmatrix} 0 & 0.5 & 1 & 1.5 \\ 1.5 & 1 & 0.5 & 0 \\ 3 & 2.5 & 2 & 1.5 \end{pmatrix} \tag{1.15.1}$$

b) The matrix $A_{ij} = 2i - j$ obtained is

$$\begin{pmatrix} 0 & -1 & -2 & -3 \\ 2 & 1 & 0 & -1 \\ 4 & 3 & 2 & 1 \end{pmatrix} \tag{1.15.2}$$

1.16. Find the values of x,y and z from the following equations:
(i) $\begin{pmatrix} 4 & 3 \\ x & 5 \end{pmatrix} = \begin{pmatrix} y & z \\ 1 & 5 \end{pmatrix}$ (ii) $\begin{pmatrix} x+y & 2 \\ 5+z & xy \end{pmatrix} = \begin{pmatrix} 6 & 2 \\ 5 & 8 \end{pmatrix}$
(iii) $\begin{pmatrix} x+y+z \\ x+y \\ y+z \end{pmatrix}$=$\begin{pmatrix} 9 \\ 5 \\ 7 \end{pmatrix}$

Solution: This problem is solved by comparing the respective elements in both the matrices
a)

$$\begin{pmatrix} 4 & 3 \\ x & 5 \end{pmatrix} = \begin{pmatrix} y & z \\ 1 & 5 \end{pmatrix} \tag{1.16.1}$$
$$x = 1, y = 4, z = 3 \tag{1.16.2}$$

b)

$$\begin{pmatrix} x+y & 2 \\ 5+z & xy \end{pmatrix} = \begin{pmatrix} 6 & 2 \\ 5 & 8 \end{pmatrix} \tag{1.16.3}$$
$$5 + z = 5 \tag{1.16.4}$$
$$\implies z = 0 \tag{1.16.5}$$
$$x + y = 6 \tag{1.16.6}$$
$$xy = 8 \tag{1.16.7}$$
$$x = 4, y = 2 \tag{1.16.8}$$
$$x = 2, y = 4 \tag{1.16.9}$$

$x = 4, y = 2, z = 0$ or $x = 2, y = 4, z = 0$

c) $\begin{pmatrix} x+y+z \\ x+y \\ y+z \end{pmatrix} = \begin{pmatrix} 9 \\ 5 \\ 7 \end{pmatrix}$
Expressing it as $Ax = b$ and $x = A^{-1}b$,

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ 5 \\ 7 \end{pmatrix} \tag{1.16.10}$$
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 9 \\ 5 \\ 7 \end{pmatrix} \tag{1.16.11}$$
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \tag{1.16.12}$$
$$x = 2, y = 3, z = 4 \tag{1.16.13}$$

1.17. Compute the indicated products.
a) $\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$
Solution:

$$\mathbf{C} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} + \begin{pmatrix} a & b \\ b & a \end{pmatrix} \tag{1.17.1}$$
$$\mathbf{C} = \begin{pmatrix} a+a & b+b \\ -b+b & a+a \end{pmatrix} \tag{1.17.2}$$
$$\mathbf{C} = \begin{pmatrix} 2a & 2b \\ 0 & 2a \end{pmatrix} \tag{1.17.3}$$

b) $\begin{pmatrix}1\\2\\3\end{pmatrix}\begin{pmatrix}2 & 3 & 4\end{pmatrix}$

Solution: By applying matrix addition

$$= \begin{pmatrix}a^2+b^2+2ab & b^2+c^2+2bc\\ a^2+c^2-2ac & a^2+b^2-2ab\end{pmatrix} \quad (1.17.4)$$

$$= \begin{pmatrix}(a+b)^2 & (b+c)^2\\ (a-c)^2 & (a-b)^2\end{pmatrix} \quad (1.17.5)$$

c) $\begin{pmatrix}1 & -2\\2 & 3\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\2 & 3 & 1\end{pmatrix}$

1.18. Simplify $\cos\theta\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}+\sin\theta\begin{pmatrix}\sin\theta & -\cos\theta\\ \cos\theta & \sin\theta\end{pmatrix}$

Solution:

$$\cos\theta\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}+\sin\theta\begin{pmatrix}\sin\theta & -\cos\theta\\ \cos\theta & \sin\theta\end{pmatrix} \quad (1.18.1)$$

$$= \cos\theta\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix} + \sin\theta\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix} \quad (1.18.2)$$

$$= \left(\cos\theta\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}+\sin\theta\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\right) \times\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix} \quad (1.18.3)$$

$$= \left(\begin{pmatrix}\cos\theta & 0\\0 & \cos\theta\end{pmatrix}+\begin{pmatrix}0 & -\sin\theta\\ \sin\theta & 0\end{pmatrix}\right) \times\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix} \quad (1.18.4)$$

$$= \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix} \quad (1.18.5)$$

In (1.18.5), the left matrix rotates a vector by angle $+\theta$. Obviously the right matrix rotates a vector by angle -θ. Then the product of matrices is identity matrix.

$$= \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = \mathbf{I} \quad (1.18.6)$$

Hence it is simplified.

1.19. Find X and Y,if

(i)X+Y=$\begin{pmatrix}7 & 0\\2 & 5\end{pmatrix}$ and X-Y=$\begin{pmatrix}3 & 0\\0 & 3\end{pmatrix}$

(ii)2X+3Y=$\begin{pmatrix}2 & 3\\4 & 0\end{pmatrix}$ and 3X+2Y=$\begin{pmatrix}2 & -2\\-1 & 5\end{pmatrix}$

Solution: Let,

$$X+Y=\begin{pmatrix}7 & 0\\2 & 5\end{pmatrix}=A \quad (1.19.1)$$

$$X-Y=\begin{pmatrix}3 & 0\\0 & 3\end{pmatrix}=B \quad (1.19.2)$$

Now, expressing the matrices (1.19.1), (1.19.2) in vector form,

$$\begin{pmatrix}I & I\end{pmatrix}\begin{pmatrix}X\\Y\end{pmatrix}=A$$

$$\begin{pmatrix}I & -I\end{pmatrix}\begin{pmatrix}X\\Y\end{pmatrix}=B$$

Combining both the equations into a single matrix equation and constructing the augmented matrix,

$$\begin{pmatrix}I & I & A\\I & -I & B\end{pmatrix} \quad (1.19.3)$$

Transforming the equation (1.19.3) using row reduction,

$$\begin{pmatrix}I & I & A\\I & -I & B\end{pmatrix} \xleftrightarrow{R2\leftarrow\frac{R1-R2}{2}} \begin{pmatrix}I & I & A\\0 & I & \frac{A-B}{2}\end{pmatrix} \xleftrightarrow{R1\leftarrow R1-R2}$$

$$\begin{pmatrix}I & 0 & \frac{A+B}{2}\\0 & I & \frac{A-B}{2}\end{pmatrix} \quad (1.19.4)$$

From (1.19.4),

$$X=\frac{A+B}{2}=\begin{pmatrix}5 & 0\\1 & 4\end{pmatrix} \quad (1.19.5)$$

$$Y=\frac{A-B}{2}=\begin{pmatrix}2 & 0\\1 & 1\end{pmatrix} \quad (1.19.6)$$

Let,

$$2X+3Y=\begin{pmatrix}2 & 3\\4 & 0\end{pmatrix}=A \quad (1.19.7)$$

$$3X+2Y=\begin{pmatrix}2 & -2\\-1 & 5\end{pmatrix}=B \quad (1.19.8)$$

Now, expressing the matrices (1.19.7), (1.19.8) in vector form,

$$\begin{pmatrix}2I & 3I\end{pmatrix}\begin{pmatrix}X\\Y\end{pmatrix} = A$$

$$\begin{pmatrix}3I & 2I\end{pmatrix}\begin{pmatrix}X\\Y\end{pmatrix} = B$$

Combining both the equations into a single matrix equation and constructing the augmented matrix,

$$\begin{pmatrix}2I & 3I & A\\3I & 2I & B\end{pmatrix} \tag{1.19.9}$$

Transforming the equation (1.19.9) using row reduction,

$$\begin{pmatrix}2I & 3I & A\\3I & 2I & B\end{pmatrix} \xleftrightarrow{R2\leftarrow\frac{3R1-2R2}{5}} \begin{pmatrix}2I & 3I & A\\0 & I & \frac{3A-2B}{5}\end{pmatrix} \xleftrightarrow{R1\leftarrow\frac{R1-3R2}{2}}$$

$$\begin{pmatrix}I & 0 & \frac{3B-2A}{5}\\0 & I & \frac{3A-2B}{5}\end{pmatrix} \tag{1.19.10}$$

From (1.19.10),

$$X = \frac{3B-2A}{5} = \begin{pmatrix}\frac{2}{5} & \frac{-12}{5}\\\frac{-11}{5} & 3\end{pmatrix} \tag{1.19.11}$$

$$Y = \frac{3A-2B}{5} = \begin{pmatrix}\frac{2}{5} & \frac{13}{5}\\\frac{14}{5} & -2\end{pmatrix} \tag{1.19.12}$$

1.20. Find X if Y=$\begin{pmatrix}3 & 2\\1 & 4\end{pmatrix}$ and 2X+Y=$\begin{pmatrix}1 & 0\\-3 & 2\end{pmatrix}$

Solution: Let,

$$Y = \begin{pmatrix}3 & 2\\1 & 4\end{pmatrix} = M \tag{1.20.1}$$

$$2X + Y = \begin{pmatrix}1 & 0\\-3 & 2\end{pmatrix} = N \tag{1.20.2}$$

Expressing the matrices (1.20.1), (1.20.2) in the vector form,

$$\begin{pmatrix}0 & I\end{pmatrix}\begin{pmatrix}X\\Y\end{pmatrix} = M \tag{1.20.3}$$

$$\begin{pmatrix}2I & I\end{pmatrix}\begin{pmatrix}X\\Y\end{pmatrix} = N \tag{1.20.4}$$

Combining both the equations into a single matrix equation and constructing the augmented matrix,

$$\begin{pmatrix}0 & I & M\\2I & I & N\end{pmatrix} \tag{1.20.5}$$

Transforming (1.20.5) using row reduction,

$$\begin{pmatrix}0 & I & M\\2I & I & N\end{pmatrix} \xleftrightarrow{R1\leftrightarrow R2} \begin{pmatrix}2I & I & N\\0 & I & M\end{pmatrix} \tag{1.20.6}$$

$$\xleftrightarrow{R1\leftarrow R1-R2} \begin{pmatrix}2I & 0 & N-M\\0 & I & M\end{pmatrix} \xleftrightarrow{R1\leftarrow\frac{R1}{2}} \begin{pmatrix}I & 0 & \frac{N-M}{2}\\0 & I & M\end{pmatrix} \tag{1.20.7}$$

From (1.20.7),

$$X = \frac{N-M}{2} = \begin{pmatrix}-1 & -1\\-2 & -1\end{pmatrix} \tag{1.20.8}$$

1.21. If F(x)=$\begin{pmatrix}\cos x & -\sin x & 0\\\sin x & \cos x & 0\\0 & 0 & 1\end{pmatrix}$,show that F(x)F(y)=F(x+y)

Solution: Given,

$$F(x) = \begin{pmatrix}\cos x & -\sin x & 0\\\sin x & \cos x & 0\\0 & 0 & 1\end{pmatrix} \tag{1.21.1}$$

Replacing x with y in above equation 2.0.1 F(y) is given as,

$$F(y) = \begin{pmatrix}\cos y & -\sin y & 0\\\sin y & \cos y & 0\\0 & 0 & 1\end{pmatrix} \tag{1.21.2}$$

From the problem statement,
Consider the LHS,
F(x)F(y)

$$= \begin{pmatrix}\cos x & -\sin x & 0\\\sin x & \cos x & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}\cos y & -\sin y & 0\\\sin y & \cos y & 0\\0 & 0 & 1\end{pmatrix} \tag{1.21.3}$$

$$= \begin{pmatrix}\cos x\cos y - \sin x\sin y & -\cos x\sin y - \sin x\cos y & 0\\\sin x\cos y + \cos x\sin y & -\sin x\sin y + \cos x\cos y & 0\\0 & 0 & 1\end{pmatrix} \tag{1.21.4}$$

$$= \begin{pmatrix}\cos(x+y) & -\sin(x+y) & 0\\\sin(x+y) & \cos(x+y) & 0\\0 & 0 & 1\end{pmatrix} \tag{1.21.5}$$

$$= F(x+y) \tag{1.21.6}$$

Therefore,

$$F(x)F(y) = F(x+y) \tag{1.21.7}$$

1.22. Find $A^2 - 5A + 6I$,if A $= \begin{pmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{pmatrix}$

Solution: Splitting matrix as (A-3I)(A-5I) we have

$$A-3I = \begin{pmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{pmatrix} + \begin{pmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix} \quad (1.22.1)$$

$$A-5I = \begin{pmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{pmatrix} + \begin{pmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{pmatrix} \quad (1.22.2)$$

Multiplying the above,

$$A^2 - 5A + 6I = \begin{pmatrix} -1 & 0 & 1 \\ 2 & -2 & 3 \\ 1 & -1 & -3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 2 & -1 & 3 \\ 1 & -1 & -2 \end{pmatrix} \quad (1.22.3)$$

$$A^2 - 5A + 6I = \begin{pmatrix} 1 & -1 & -3 \\ -1 & -1 & -10 \\ -5 & 4 & 4 \end{pmatrix} \quad (1.22.4)$$

1.23. If A=$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{pmatrix}$,prove that $A^3 - 6A^2 + 7A + 2I = 0$

Solution: The characteristic equation is

$$|\mathbf{A} - \lambda \mathbf{I}| = 0 \quad (1.23.1)$$

$$\implies \det \begin{pmatrix} 1-\lambda & 0 & 2 \\ 0 & 2-\lambda & 1 \\ 2 & 0 & 3-\lambda \end{pmatrix} = 0 \quad (1.23.2)$$

$$\implies (1-\lambda)(2-\lambda)(3-\lambda) + 2(-2(2-\lambda)) = 0 \quad (1.23.3)$$

$$\implies \lambda^3 - 6\lambda^2 + 7\lambda + 2 = 0 \quad (1.23.4)$$

The above equation is similar to the equation to be proved.
By the Cayley-Hamilton theorem, every square matrix satisfies its own characteristic equation. Hence proved that $\mathbf{A^3 - 6A^2 + 7A + 2I = 0}$.

1.24. If A=$\begin{pmatrix} 0 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 0 \end{pmatrix}$ and I is the identity matrix of order 2,show that I+A=(I-A)$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$

Solution:

Since

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (1.24.1)$$

$$\mathbf{A} = \tan\frac{\alpha}{2} \begin{pmatrix} \cos 90 & -\sin 90 \\ \sin 90 & \cos 90 \end{pmatrix} \quad (1.24.2)$$

$$\mathbf{I} - \mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \tan\frac{\alpha}{2} \begin{pmatrix} \cos 90 & -\sin 90 \\ \sin 90 & \cos 90 \end{pmatrix} \quad (1.24.3)$$

$$= \frac{1}{\cos\frac{\alpha}{2}} \begin{pmatrix} \cos\frac{\alpha}{2} & 0 \\ 0 & \cos\frac{\alpha}{2} \end{pmatrix} - \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} \begin{pmatrix} \cos 90 & -\sin 90 \\ \sin 90 & \cos 90 \end{pmatrix} \quad (1.24.4)$$

$$= \frac{1}{\cos\frac{\alpha}{2}} \begin{pmatrix} \cos\frac{\alpha}{2} & 0 \\ 0 & \cos\frac{\alpha}{2} \end{pmatrix} - \frac{1}{\cos\frac{\alpha}{2}} \begin{pmatrix} 0 & -\sin\frac{\alpha}{2} \\ \sin\frac{\alpha}{2} & 0 \end{pmatrix} \quad (1.24.5)$$

$$= \frac{1}{\cos\frac{\alpha}{2}} \begin{pmatrix} \cos\frac{\alpha}{2} & \sin\frac{\alpha}{2} \\ -\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix} \quad (1.24.6)$$

The matrix $\mathbf{I} - \mathbf{A}$ is a rotational Matrix with rotation $-\frac{\alpha}{2}$

The Matrix $\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$ is also a rotational Matrix with an angle $+\alpha$.

Multiplying two rotational matrices gives the resultant rotational matrix $+\alpha - \frac{\alpha}{2} = +\frac{\alpha}{2}$

$$RHS = \left(I - A\right) \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & cos\alpha \end{pmatrix} \quad (1.24.7)$$

$$= \frac{1}{\cos\frac{\alpha}{2}} \begin{pmatrix} \cos\frac{\alpha}{2} & \sin\frac{\alpha}{2} \\ -\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix} \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & cos\alpha \end{pmatrix} \quad (1.24.8)$$

$$= \frac{1}{\cos\frac{\alpha}{2}} \begin{pmatrix} \cos\frac{\alpha}{2} & -\sin\frac{\alpha}{2} \\ \sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix} \quad (1.24.9)$$

$$(1.24.10)$$

Solving LHS= $\mathbf{I} + \mathbf{A}$

$$\mathbf{I} + \mathbf{A} = \begin{pmatrix} 1 & \tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 1 \end{pmatrix} \quad (1.24.11)$$

$$= \frac{1}{\cos\frac{\alpha}{2}} \begin{pmatrix} \cos\frac{\alpha}{2} & -\sin\frac{\alpha}{2} \\ \sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix} \quad (1.24.12)$$

This term is a rotational Matrix with angle $+\frac{\alpha}{2}$.Hence both sides evaluates to be a rotational matrix with angle $+\frac{\alpha}{2}$.

1.25. A trust fund has ₹30,000 that must be invested in two different types of bonds. The first bond pays 5% interest per year, and the second bond pays 7% interest per year. Using matrix multiplication, determine how to divide ₹ 30,000 among the two types of bonds. If the trust fund must obtain an annual total interest of:
(a) ₹1800 (b)₹2000
Solution: Let ₹30000 be divided into two part x_1 and x_2 in part **a**),and into two part y_1 and y_2 in part **b**). Then x_1, x_2, y_1, y_2 satisfies following equations

$$x_1 + x_2 = 30000 \tag{1.25.1}$$
$$0.05x_1 + 0.07x_2 = 1800 \tag{1.25.2}$$
$$y_1 + y_2 = 30000 \tag{1.25.3}$$
$$0.05y_1 + 0.07y_2 = 2000 \tag{1.25.4}$$

From (1.25.1) and (1.25.2) we get

$$\begin{pmatrix} 1 & 1 \\ 0.05 & 0.07 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 30000 \\ 1800 \end{pmatrix} \tag{1.25.5}$$

and from (1.25.3) and (1.25.4) we get

$$\begin{pmatrix} 1 & 1 \\ 0.05 & 0.07 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 30000 \\ 2000 \end{pmatrix} \tag{1.25.6}$$

Combining the two we get

$$\begin{pmatrix} 1 & 1 \\ 0.05 & 0.07 \end{pmatrix}\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} = \begin{pmatrix} 30000 & 30000 \\ 1800 & 2000 \end{pmatrix}$$
$$\xleftrightarrow{R_2=R_2-0.05R_1} \begin{pmatrix} 1 & 1 \\ 0 & 0.02 \end{pmatrix}\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} = \begin{pmatrix} 30000 & 30000 \\ 300 & 500 \end{pmatrix}$$
$$\xleftrightarrow{R_2=50R_2} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} = \begin{pmatrix} 30000 & 30000 \\ 15000 & 25000 \end{pmatrix}$$
$$\xleftrightarrow{R_1=R_1-R_2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} = \begin{pmatrix} 15000 & 5000 \\ 15000 & 25000 \end{pmatrix} \tag{1.25.7}$$

From (1.25.7) we get x_1=₹15000, x_2=₹15000, y_1=₹5000 and y_2=₹25000.Therefore to get an annual total interest of ₹1800 trust must invest ₹15000 in first bond and ₹15000 in second bond and to get an annual interest of ₹2000 trust must invest ₹5000 in first bond and ₹25000 in second bond.

1.26. For the matrices A and B,verify that $(AB)' = B'A'$,where
(i)A=$\begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix}$,B=$\begin{pmatrix} -1 & 2 & 1 \end{pmatrix}$
(ii)A=$\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$,B=$\begin{pmatrix} 1 & 5 & 7 \end{pmatrix}$

1.27. If (i) A=$\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$, then verify that $A'A = I$
(ii) If A=$\begin{pmatrix} \sin\alpha & \cos\alpha \\ -\cos\alpha & \sin\alpha \end{pmatrix}$,then verify that $A'A = I$
Solution:
a)

$$A = \begin{pmatrix} \cos\alpha & \sin\alpha \\ \text{-}\sin\alpha & \cos\alpha \end{pmatrix} \tag{1.27.1}$$
$$\implies A^T = \begin{pmatrix} \cos\alpha & \text{-}\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \tag{1.27.2}$$
$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a_1 & \text{-}a_2 \\ a_2 & a_1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.27.3}$$
$$i.e\ A = \begin{pmatrix} \cos\alpha \\ \text{-}\sin\alpha \end{pmatrix} \tag{1.27.4}$$
$$\implies A = e^{-j\alpha} \tag{1.27.5}$$
$$i.e.\ A^T = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} \tag{1.27.6}$$
$$\implies A^T = e^{j\alpha} \tag{1.27.7}$$
$$\implies A^T A = e^{j\alpha}e^{-j\alpha} = 1 \tag{1.27.8}$$
$$\implies A^T A = I \tag{1.27.9}$$

b)

$$A = \begin{pmatrix} \sin\alpha & \cos\alpha \\ \text{-}\cos\alpha & \sin\alpha \end{pmatrix} \tag{1.27.10}$$
$$\implies A^T = \begin{pmatrix} \sin\alpha & \text{-}\cos\alpha \\ \cos\alpha & \sin\alpha \end{pmatrix} \tag{1.27.11}$$

Using (3),

$$i.e\ A = \begin{pmatrix} \sin\alpha \\ \text{-}\cos\alpha \end{pmatrix} \quad (1.27.12)$$

$$\implies \quad A = e^{-j\left(\frac{n\pi}{2}-\alpha\right)} \quad (1.27.13)$$

$$A^T = \begin{pmatrix} \sin\alpha \\ \cos\alpha \end{pmatrix} \quad (1.27.14)$$

$$\implies \quad A^T = e^{j\left(\frac{n\pi}{2}-\alpha\right)} \quad (1.27.15)$$

$$\implies A^T A = e^{j\left(\frac{n\pi}{2}-\alpha\right)} e^{-j\left(\frac{n\pi}{2}-\alpha\right)} = 1 \quad (1.27.16)$$

$$\implies A^T A = I \quad (1.27.17)$$

Here, n in (1.27.13) and (1.27.15) is an odd number.

Hence proved for both Problems 1.27a and 1.27b.

1.28. If A=$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$,and $A + A' = I$,then the value of α is
(A) $\frac{\pi}{6}$
(B)$\frac{\pi}{3}$
(C) π
(D)$\frac{3\pi}{2}$
Solution:
The Complex number equivalent to the matrix is:

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \implies \begin{pmatrix} a \\ b \end{pmatrix} \quad (1.28.1)$$

$$\mathbf{A} = \begin{pmatrix} a \\ b \end{pmatrix} \implies \mathbf{A}^\top = \begin{pmatrix} a \\ -b \end{pmatrix} \quad (1.28.2)$$

And addition of $\mathbf{A}$ with $\mathbf{A}^\top$ results in :

$$\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} a \\ -b \end{pmatrix} = \begin{pmatrix} 2a \\ 0 \end{pmatrix} \quad (1.28.3)$$

So, According to the given question $\mathbf{A} + \mathbf{A}^\top$ is :

$$\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} + \begin{pmatrix} \cos\alpha \\ -\sin\alpha \end{pmatrix} = \begin{pmatrix} 2\cos\alpha \\ 0 \end{pmatrix} \quad (1.28.4)$$

Given that $\mathbf{A} + \mathbf{A}^\top = \mathbf{I}$:

$$\begin{pmatrix} 2\cos\alpha \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.28.5)$$

That Implies,

$$2\cos\alpha = 1 \implies \cos\alpha = \frac{1}{2} \quad (1.28.6)$$

As per the cosine values, the angle α is :

$$\alpha = \frac{\pi}{3} = 1.047 \quad (1.28.7)$$

1.29. $\begin{pmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{pmatrix}$
Solution:

$$\left(\begin{array}{ccc|ccc} 2 & -3 & 3 & 1 & 0 & 0 \\ 2 & 2 & 3 & 0 & 1 & 0 \\ 3 & -2 & 2 & 0 & 0 & 1 \end{array}\right) \quad (1.29.1)$$

$$\xleftrightarrow{C_2 \leftarrow C_2 + C_1} \left(\begin{array}{ccc|ccc} 2 & 0 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 3 & 0 & 2 & 0 & 1 & 1 \end{array}\right) \quad (1.29.2)$$

$$\xleftrightarrow{C_1 \leftarrow C_3 - C_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & -1 & 0 & 0 \\ 1 & 5 & 3 & 0 & 1 & 0 \\ -1 & 0 & 2 & 1 & 1 & 1 \end{array}\right) \quad (1.29.3)$$

$$\xleftrightarrow{C_3 \leftarrow C_3 - 3C_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 3 \\ 1 & 5 & 0 & 0 & 1 & 0 \\ -1 & 0 & 5 & 1 & 1 & -2 \end{array}\right) \quad (1.29.4)$$

$$\xleftrightarrow{C_3 \leftarrow \frac{1}{5}C_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 3/5 \\ 0 & 5 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 1 & 1 & -2/5 \end{array}\right) \quad (1.29.5)$$

$$\xleftrightarrow{C_2 \leftarrow \frac{1}{5}C_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 3/5 \\ 0 & 1 & 0 & 0 & 1/5 & 0 \\ -1 & 0 & 1 & 1 & 1/5 & -2/5 \end{array}\right) \quad (1.29.6)$$

$$\xleftrightarrow{C_1 \leftarrow C_1 - C_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 3/5 \\ 0 & 1 & 0 & -1/5 & 1/5 & 0 \\ -1 & 0 & 1 & 4/5 & 1/5 & -2/5 \end{array}\right) \quad (1.29.7)$$

$$\xleftrightarrow{C_1 \leftarrow C_1 + C_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2/5 & 0 & 3/5 \\ 0 & 1 & 0 & -1/5 & 1/5 & 0 \\ 0 & 0 & 1 & 2/5 & 1/5 & -2/5 \end{array}\right) \quad (1.29.8)$$

$$\mathbf{A^{-1}} = \begin{pmatrix} -2/5 & 0 & 3/5 \\ -1/5 & 1/5 & 0 \\ 2/5 & 1/5 & -2/5 \end{pmatrix} \tag{1.29.9}$$

1.30. $\begin{pmatrix} 1 & 3 & -2 \\ -3 & 0 & -5 \\ 2 & 5 & 0 \end{pmatrix}$

Solution:

$$Let\ \mathbf{A} = \begin{pmatrix} 1 & 3 & -2 \\ -3 & 0 & -5 \\ 2 & 5 & 0 \end{pmatrix} \tag{1.30.1}$$

Therefore the augmented matrix can be represented as follows :

$$\left(\begin{array}{ccc|ccc} 1 & 3 & -2 & 1 & 0 & 0 \\ -3 & 0 & -5 & 0 & 1 & 0 \\ 2 & 5 & 0 & 0 & 0 & 1 \end{array}\right) \tag{1.30.2}$$

Applying elementary transformations on $\mathbf{A}$ as follows:

$$\xleftrightarrow[R_3 \leftarrow R_3 - 2R_1]{R_2 \leftarrow R_2 + 3R_1} \left(\begin{array}{ccc|ccc} 1 & 3 & -2 & 1 & 0 & 0 \\ 0 & 9 & -11 & 3 & 1 & 0 \\ 0 & -1 & 4 & -2 & 0 & 1 \end{array}\right) \tag{1.30.3}$$

$$\xleftrightarrow[R_1 \leftarrow R_1 - 3R_2]{R_2 \leftrightarrow -R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 10 & -5 & 0 & 3 \\ 0 & 1 & -4 & 2 & 0 & -1 \\ 0 & 9 & -11 & 3 & 1 & 0 \end{array}\right) \tag{1.30.4}$$

$$\xleftrightarrow[R_3 \leftarrow \frac{R_3}{25}]{R_3 \leftarrow R_3 - 4R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 10 & -5 & 0 & 3 \\ 0 & 1 & -4 & 2 & 0 & -1 \\ 0 & 0 & 1 & \frac{-3}{5} & \frac{1}{25} & \frac{9}{25} \end{array}\right) \tag{1.30.5}$$

$$\xleftrightarrow[R_2 \leftarrow R_2 + 4R_3]{R_1 \leftarrow R_1 - 10R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & \frac{-2}{5} & \frac{-3}{5} \\ 0 & 1 & 0 & \frac{-2}{5} & \frac{4}{25} & \frac{11}{25} \\ 0 & 0 & 1 & \frac{-3}{5} & \frac{1}{25} & \frac{9}{25} \end{array}\right) \tag{1.30.6}$$

Therefore $\mathbf{A^{-1}}$ is as follows:

$$\begin{pmatrix} 1 & \frac{-2}{5} & \frac{-3}{5} \\ \frac{-2}{5} & \frac{4}{25} & \frac{11}{25} \\ \frac{-3}{5} & \frac{1}{25} & \frac{9}{25} \end{pmatrix} \tag{1.30.7}$$

1.31. $\begin{pmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix}$

Solution: The augmented matrix $[A|I]$ is as given below:-

$$\begin{pmatrix} 2 & 0 & -1 & 1 & 0 & 0 \\ 5 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{pmatrix} \tag{1.31.1}$$

We apply the elementary row operations on $[A|I]$ as follows :-

$$[A|I] = \begin{pmatrix} 2 & 0 & -1 & 1 & 0 & 0 \\ 5 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{pmatrix} \tag{1.31.2}$$

$$\xrightarrow{R_2 \leftarrow 2R_2 - 5R_1} \begin{pmatrix} 2 & 0 & -1 & 1 & 0 & 0 \\ 0 & 2 & 5 & -5 & 2 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{pmatrix} \tag{1.31.3}$$

$$\xrightarrow{R_3 \leftarrow 2R_3 - R_2} \begin{pmatrix} 2 & 0 & -1 & 1 & 0 & 0 \\ 0 & 2 & 5 & -5 & 2 & 0 \\ 0 & 0 & 1 & 5 & -2 & 2 \end{pmatrix} \tag{1.31.4}$$

$$\xleftrightarrow[R_2 \leftarrow \frac{R_2}{2}]{R_1 \leftarrow \frac{R_1}{2}} \begin{pmatrix} 1 & 0 & \frac{-1}{2} & \frac{-1}{2} & 0 & 0 \\ 0 & 1 & \frac{5}{2} & \frac{-5}{2} & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & 2 \end{pmatrix} \tag{1.31.5}$$

$$\xleftrightarrow[R_1 \leftarrow R_1 + \frac{R_3}{2}]{R_2 \leftarrow R_2 - \frac{5}{2}R_3} \begin{pmatrix} 1 & 0 & 0 & 3 & -1 & 1 \\ 0 & 1 & 0 & -15 & 6 & -5 \\ 0 & 0 & 1 & 5 & -2 & 2 \end{pmatrix} \tag{1.31.6}$$

By performing elementary transormations on augmented matrix$[A|I]$, we obtained the augmented matrix in the form $[I|B]$. Hence we can conclude that the matrix A is invertible and inverse of the matrix is:-

$$A^{-1} = \begin{pmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix} \tag{1.31.7}$$

1.32. If A=$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$,

prove that $A^n = \begin{pmatrix} 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \end{pmatrix}, n \epsilon N$

Solution: The above problem can be proven by the method of induction

$$A^2 = AA = \begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix} = \begin{pmatrix} 3^1 & 3^1 & 3^1 \\ 3^1 & 3^1 & 3^1 \\ 3^1 & 3^1 & 3^1 \end{pmatrix} \tag{1.32.1}$$

$$A^3 = A^2A = \begin{pmatrix} 3^2 & 3^2 & 3^2 \\ 3^2 & 3^2 & 3^2 \\ 3^2 & 3^2 & 3^2 \end{pmatrix} \tag{1.32.2}$$

let this be true for n=k,then

$$A^k = \begin{pmatrix} 3^{k-1} & 3^{k-1} & 3^{k-1} \\ 3^{k-1} & 3^{k-1} & 3^{k-1} \\ 3^{k-1} & 3^{k-1} & 3^{k-1} \end{pmatrix} \tag{1.32.3}$$

$$A^{k+1} = A^kA = \begin{pmatrix} 3^{k-1} & 3^{k-1} & 3^{k-1} \\ 3^{k-1} & 3^{k-1} & 3^{k-1} \\ 3^{k-1} & 3^{k-1} & 3^{k-1} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \tag{1.32.4}$$

$$A^{k+1} = \begin{pmatrix} 3.3^{k-1} & 3.3^{k-1} & 3.3^{k-1} \\ 3.3^{k-1} & 3.3^{k-1} & 3.3^{k-1} \\ 3.3^{k-1} & 3.3^{k-1} & 3.3^{k-1} \end{pmatrix} \tag{1.32.5}$$

$$A^{k+1} = \begin{pmatrix} 3^k & 3^k & 3^k \\ 3^k & 3^k & 3^k \\ 3^k & 3^k & 3^k \end{pmatrix} \tag{1.32.6}$$

therefore it holds for all n$\in \mathbb{N}$

$$A^n = \begin{pmatrix} 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \end{pmatrix}, n \in \mathbb{N} \tag{1.32.7}$$

1.33. If A=$\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$,

then prove that A^n=$\begin{pmatrix} 1+2n & -4n \\ n & 1-2n \end{pmatrix}$,where n is any positive integer

Solution: Characteristic equation of A is $\lambda^2 - 2\lambda + 1 = 0$. By using Cayley Hamolton theorem, given matrix A is a solution of its own characteristic equation. So $A^2 - 2A + I = 0$.

$$A^2 = 2A - I \tag{1.33.1}$$

$$A^3 = 3A - 2I \tag{1.33.2}$$

$$A^4 = 4A - 3I \tag{1.33.3}$$

$$A^5 = 5A - 4I \tag{1.33.4}$$

By the substitution,

$$A^n = nA - (n-1)I \tag{1.33.5}$$

a) Proving above equation is true for n=2,

$$A^2 = 2A - I \tag{1.33.6}$$

b) Assume it is true for n = k

$$A^k = kA - (k-1)I \tag{1.33.7}$$

c) Proving it is true for n = k+1

$$A^{k+1} = A^kA \tag{1.33.8}$$

$$= (kA - (k-1)I)A \tag{1.33.9}$$

$$= kA^2 - (k-1)A \tag{1.33.10}$$

$$= k(2A - I) - (k-1)A \tag{1.33.11}$$

$$= (2k - k + 1)A - kI \tag{1.33.12}$$

$$= (k+1)A - kI \tag{1.33.13}$$

So, $A^n = nA - (n-1)I$

Substituting matrix A = $\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$ in the equation5,

$$A^n = n\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} - (n-1)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.33.14}$$

$$= \begin{pmatrix} 3n & -4n \\ n & -n \end{pmatrix} - \begin{pmatrix} (n-1) & 0 \\ 0 & (n-1) \end{pmatrix} \tag{1.33.15}$$

$$= \begin{pmatrix} 2n+1 & -4n \\ n & 1-2n \end{pmatrix} \tag{1.33.16}$$

So, $A^n = \begin{pmatrix} 2n+1 & -4n \\ n & 1-2n \end{pmatrix}$

1.34. For what values of x:

$\begin{pmatrix} 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ x \end{pmatrix}$=0?

Solution: Below is the solution :

$$\begin{pmatrix} 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ x \end{pmatrix} = 0 \tag{1.34.1}$$

$$\implies \begin{pmatrix} \begin{pmatrix} 1 & 2 \end{pmatrix} & \begin{pmatrix} 1 \end{pmatrix} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \end{pmatrix} & \begin{pmatrix} 2 \end{pmatrix} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} 0 \\ 2 \end{pmatrix} \\ \begin{pmatrix} x \end{pmatrix} \end{pmatrix} = 0 \tag{1.34.2}$$

$$\implies \left(\left(\begin{matrix}\begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}1 & 2\\ 2 & 0\end{pmatrix}\\ + \\ \begin{pmatrix}1\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix}\end{matrix}\right) \quad \left(\begin{matrix}\begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix}\\ + \\ \begin{pmatrix}1\end{pmatrix}\begin{pmatrix}2\end{pmatrix}\end{matrix}\right)\right)\begin{pmatrix}\begin{pmatrix}0\\ 2\end{pmatrix}\\ \begin{pmatrix}x\end{pmatrix}\end{pmatrix} = 0 \tag{1.34.3}$$

$$\implies \left(\begin{pmatrix}5 & 2\end{pmatrix} + \begin{pmatrix}1 & 0\end{pmatrix} \quad \begin{pmatrix}2+2\end{pmatrix}\right)\begin{pmatrix}\begin{pmatrix}0\\ 2\end{pmatrix}\\ \begin{pmatrix}x\end{pmatrix}\end{pmatrix} = 0 \tag{1.34.4}$$

$$\implies \left(\begin{pmatrix}6 & 2\end{pmatrix} \quad \begin{pmatrix}4\end{pmatrix}\right)\begin{pmatrix}\begin{pmatrix}0\\ 2\end{pmatrix}\\ \begin{pmatrix}x\end{pmatrix}\end{pmatrix} = 0 \tag{1.34.5}$$

$$\implies \left(\begin{pmatrix}6 & 2\end{pmatrix}\begin{pmatrix}0\\ 2\end{pmatrix} + \begin{pmatrix}4\end{pmatrix}\begin{pmatrix}x\end{pmatrix}\right) = 0 \tag{1.34.6}$$

$$\implies \left(4 + 4 \times x\right) = 0 \tag{1.34.7}$$

$$\implies 4 \times x = -4 \tag{1.34.8}$$

$$\implies x = -1 \tag{1.34.9}$$

1.35. If A=$\begin{pmatrix}3 & 1\\ -1 & 2\end{pmatrix}$,show that A^2-5A+7I=0
Solution: The characteristic equation is

$$|\mathbf{A} - \lambda\mathbf{I}| = 0 \tag{1.35.1}$$

$$\implies det\begin{pmatrix}3-\lambda & 1\\ -1 & 2-\lambda\end{pmatrix} = 0 \tag{1.35.2}$$

$$\implies (3-\lambda)(2-\lambda) + 1 = 0 \tag{1.35.3}$$

$$\lambda^2 - 5\lambda + 7 = 0 \tag{1.35.4}$$

By Cayley-Hamilton theorem, every square matrix satisfies its characteristic equation. Hence, proved

$$\mathbf{A}^2 - 5\mathbf{A} + 7\mathbf{I} = 0 \tag{1.35.5}$$

1.36. Find $\begin{vmatrix}2 & 4\\ -5 & -1\end{vmatrix}$

Solution: For a 2×2 matrix, the determinant

$$\begin{vmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} \tag{1.36.1}$$

$$\implies \begin{vmatrix}2 & 4\\ -5 & -1\end{vmatrix} = 18 \tag{1.36.2}$$

1.37. (i) $\begin{vmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{vmatrix}$ (ii) $\begin{vmatrix}x^2-x+1 & x-1\\ x+1 & x+1\end{vmatrix}$
Solution:

a)

$$\begin{vmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{vmatrix} = 1 \tag{1.37.1}$$

b) The following python code calculates the determinant

```
solutions/2/codes/line_ex/determinants/det.
    py
```

$$\begin{vmatrix}x^2-x+1 & x-1\\ x+1 & x+1\end{vmatrix} = x^3 - x^2 + 2 \tag{1.37.2}$$

1.38. If$\mathbf{A} = \begin{vmatrix}1 & 2\\ 4 & 2\end{vmatrix}$,then show that $|2\mathbf{A}| = 4\,|\mathbf{A}|$
Solution:
Determinant of a (2×2) matrix is calculated as follows

$$\begin{vmatrix}a & b\\ c & d\end{vmatrix} = ad - bc \tag{1.38.1}$$

$$\mathbf{A} = \begin{pmatrix}1 & 2\\ 4 & 2\end{pmatrix} \tag{1.38.2}$$

$$\mathbf{2A} = \begin{pmatrix}2 & 4\\ 8 & 4\end{pmatrix} \tag{1.38.3}$$

Using (1.38.1) ,(1.38.2), (1.38.3)

$$|A| = 2 - 8 = -6 \implies 4|A| = -24 \tag{1.38.4}$$

$$|2A| = 8 - 32 = -24 \tag{1.38.5}$$

$$\implies |2A| = 4|A| \tag{1.38.6}$$

1.39. If $\mathbf{A} = \begin{vmatrix}1 & 0 & 1\\ 0 & 1 & 2\\ 0 & 0 & 4\end{vmatrix}$, then show that $|3\mathbf{A}| = 27\,|\mathbf{A}|$
Solution:

$$\left|3\mathbf{A}\right| = \begin{vmatrix}3 & 0 & 3\\ 0 & 3 & 6\\ 0 & 0 & 12\end{vmatrix} = 108 \tag{1.39.1}$$

and

$$\left|\mathbf{A}\right| = 4 \tag{1.39.2}$$

Hence,

$$\left|3\mathbf{A}\right| = 27\left|\mathbf{A}\right| \tag{1.39.3}$$

In general, for an $n \times n$ matrix $\mathbf{A}$,

$$\left|k\mathbf{A}\right| = k^n\left|\mathbf{A}\right| \tag{1.39.4}$$

1.40. Evaluate the determinants

a) $\begin{vmatrix} 3 & -1 & -2 \\ 0 & 0 & -1 \\ 3 & -5 & 0 \end{vmatrix}$

b) $\begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix}$

Solution:
The following python code computes the required determinant value.

./solutions/5/codes/lines/q14.py

i) $\begin{vmatrix} 3 & -1 & -2 \\ 0 & 0 & -1 \\ 3 & -5 & 0 \end{vmatrix} = -12$

ii) $\begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix} = -46$

iii) $\begin{vmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{vmatrix} = 0$

iv) $\begin{vmatrix} 2 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{vmatrix} = 5$

c) $\begin{vmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{vmatrix}$

d) $\begin{vmatrix} 2 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{vmatrix}$

1.41. If A=$\begin{vmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & -9 \end{vmatrix}$, find $|A|$

Solution:

$$\begin{pmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & -9 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{pmatrix} 1 & 1 & -2 \\ 0 & -1 & 1 \\ 5 & 4 & -9 \end{pmatrix} \tag{1.41.1}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - 5R_1} \begin{pmatrix} 1 & 1 & -2 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 - R_2} \begin{pmatrix} 1 & 1 & -2 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.41.2}$$

$$\implies \left|\mathbf{A}\right| = 0 \tag{1.41.3}$$

This is verified by the following python code

codes/line/determinants/det.py

1.42. Find the values of x,If
(i)$\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$ (ii)$\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} x & 3 \\ 2x & 5 \end{vmatrix}$

Solution:

a) Expanding the determinant,

$$-18 = 2x^2 - 24 \tag{1.42.1}$$

$$\implies x = \pm\sqrt{3} \tag{1.42.2}$$

b) Following the same steps as above we get,

$$-2 = 5x - 6x \tag{1.42.3}$$

$$\implies x = 2 \tag{1.42.4}$$

1.43. If $\begin{vmatrix} x & 2 \\ 18 & x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix}$, then x is equal to

a) 6
b) ± 6
c) -6
d) 0

1.44. $\begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix} = 0$

Solution: Given determinant: $\begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix}$=0

Applying transformation:

$$\begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix} \xleftrightarrow{C_3 \leftarrow C_3 - C_2} \begin{vmatrix} x & a & x \\ y & b & y \\ z & c & z \end{vmatrix} \tag{1.44.1}$$

$$\begin{vmatrix} x & a & x \\ y & b & y \\ z & c & z \end{vmatrix} \xleftrightarrow{C_3 \leftarrow C_3 - C_1} \begin{vmatrix} x & a & 0 \\ y & b & 0 \\ z & c & 0 \end{vmatrix} \tag{1.44.2}$$

From 1.44.2 If any row or column of determinant is zero ,than it's value is zero

$$\begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix} = \begin{vmatrix} x & a & 0 \\ y & b & 0 \\ z & c & 0 \end{vmatrix} = 0 \qquad (1.44.3)$$

1.45. $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$

Solution:

$$\text{Let, } |\mathbf{A}| = \begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} \qquad (1.45.1)$$

Applying row transformation in above determinant we get,

$$\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} \xleftrightarrow{R_1 \leftarrow R_1+R_2+R_3} \begin{vmatrix} 0 & 0 & 0 \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} \qquad (1.45.2)$$

From equation (1.45.2) one of the row of $|\mathbf{A}|$ is zero

$$\implies |\mathbf{A}| = 0 \qquad (1.45.3)$$

1.46. $\begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix} = 0$

Solution:

$$\begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix} \xleftrightarrow{C_3 \leftarrow C_3-9C_2} \qquad (1.46.1)$$

$$\begin{vmatrix} 2 & 7 & 2 \\ 3 & 8 & 3 \\ 5 & 9 & 5 \end{vmatrix} \xleftrightarrow{C_3 \leftarrow C_3-C_1} \begin{vmatrix} 2 & 7 & 0 \\ 3 & 8 & 0 \\ 5 & 9 & 0 \end{vmatrix} = 0 \qquad (1.46.2)$$

1.47. $\begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix} = 0$

Solution: Applying transformation:

$$\Delta = \begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix} \qquad (1.47.1)$$

$$= \begin{vmatrix} 1 & bc & ab+ac \\ 1 & ca & bc+ab \\ 1 & ab & ac+bc \end{vmatrix} \qquad (1.47.2)$$

$$\xleftrightarrow{C_3 \leftarrow C_3+C_2} \begin{vmatrix} 1 & bc & ab+ac+bc \\ 1 & ca & bc+ab+ca \\ 1 & ab & ac+bc+ab \end{vmatrix} \qquad (1.47.3)$$

Taking (ab+bc+ac) common from C3:

$$\Delta = (ab+bc+ac) \begin{vmatrix} 1 & bc & 1 \\ 1 & ca & 1 \\ 1 & ab & 1 \end{vmatrix} \qquad (1.47.4)$$

If any two row or column of determinant is same, then the value of determinant is zero:

$$\Delta = (ab+bc+ac) \begin{vmatrix} 1 & bc & 1 \\ 1 & ca & 1 \\ 1 & ab & 1 \end{vmatrix} = 0 \qquad (1.47.5)$$

$$\therefore \begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix} = 0 \qquad (1.47.6)$$

Hence proved.

1.48. $\begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix} = 2 \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$

Solution:

$$\begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix} \qquad (1.48.1)$$

$$\xleftrightarrow{R_1 \leftarrow R_1+R_2+R_3} \qquad (1.48.2)$$

$$2 \begin{vmatrix} (a+b+c) & (p+q+r) & (x+y+z) \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix} \qquad (1.48.3)$$

$$\xleftrightarrow[R_2 \leftarrow R_2-R_1]{R_3 \leftarrow R_3-R_1} 2 \begin{vmatrix} (a+b+c) & (p+q+r) & (x+y+z) \\ -b & -q & -y \\ -c & -r & -z \end{vmatrix} \qquad (1.48.4)$$

$$\xleftrightarrow{R_1 \leftarrow R_1+R_2+R_3} (-1) \times (-1) \times 2 \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} \qquad (1.48.5)$$

1.49. $\begin{vmatrix} -a^2 & ab & ab \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix} = 4a^2b^2c^2$

Solution:

$$\begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix} = abc \begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix} \tag{1.49.1}$$

$$= abc \begin{vmatrix} -a & a & a \\ b & -b & b \\ c & c & -c \end{vmatrix} \tag{1.49.2}$$

$$= a^2b^2c^2 \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} \tag{1.49.3}$$

$$\xleftrightarrow[R_2 \leftrightarrow R_3]{R_1 \leftrightarrow R_2} (-1)(-1)a^2b^2c^2 \begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{vmatrix} \tag{1.49.4}$$

$$\xleftrightarrow[R_3 \leftarrow R_3 - R_1]{R_2 \leftarrow R_2 - R_1} a^2b^2c^2 \begin{vmatrix} 1 & -1 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 2 \end{vmatrix} \tag{1.49.5}$$

$$= a^2b^2c^2 \times (1 \times 2 \times 2) \tag{1.49.6}$$

$$= 4a^2b^2c^2 \tag{1.49.7}$$

By Using properties of determinants, in Exercises 16 to 22,Show that;

1.50. (i) $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$=(a-b)(b-c)(c-a)

(ii) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$=(a-b)(b-c)(c-a)(a+b+c)

Solution: Using column operations to simplify the equation, we get:

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} \xleftrightarrow{C_1 \to C_1 - C_2} \begin{vmatrix} 0 & 1 & 1 \\ a-b & b & c \\ a^3-b^3 & b^3 & c^3 \end{vmatrix} \tag{1.50.1}$$

$$\xleftrightarrow{C_2 \to C_2 - C_3} \begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ a^3-b^3 & b^3-c^3 & c^3 \end{vmatrix} \tag{1.50.2}$$

$$(a-b)(b-c) \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c \\ a^2+ab+b^2 & b^2+bc+c^2 & c^3 \end{vmatrix} \tag{1.50.3}$$

$$\xleftrightarrow{C_1 \to C_1 - C_2} (a-b)(b-c) \begin{vmatrix} 0 & 0 \\ 0 & 1 \\ (a^2-c^2)+b(a-c) & b^2+bc+c^2 \end{vmatrix} \tag{1.50.4}$$

$$\implies (a-b)(b-c) \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & c \\ k & b^2+bc+c^2 & c^3 \end{vmatrix} \tag{1.50.5}$$

Where

$$k = (a-c)(a+c) + b(a-c) \tag{1.50.6}$$

$$(a-b)(b-c)(a-c) \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & c \\ a+b+c & b^2+bc+c^2 & c^3 \end{vmatrix} \tag{1.50.7}$$

$$= (a-b)(b-c)(a-c)(-1)(a+b+c) \tag{1.50.8}$$

$$= (a-b)(b-c)(c-a)(a+b+c) \tag{1.50.9}$$

1.51. $\begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix}$=(x-y)(y-z)(z-x)(xy+yz+zx)

1.52. (i) $\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$=$(5x+4)(4-x)^2$

Solution:

$$\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} \xleftrightarrow{R_1 \leftarrow R_1 + R_2 + R_3} \tag{1.52.1}$$

$$(5x+4) \begin{vmatrix} 1 & 1 & 1 \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} \xleftrightarrow[C_2 \leftarrow C_2 - C_3]{C_1 \leftarrow C_1 + C_2} \tag{1.52.2}$$

$$(5x+4) \begin{vmatrix} 0 & 0 & 1 \\ x-4 & 4-x & 2x \\ 0 & x-4 & x+4 \end{vmatrix} \tag{1.52.3}$$

$$(5x+4)(4-x)^2 \begin{vmatrix} 0 & 0 & 1 \\ -1 & 1 & 2x \\ 0 & -1 & x+4 \end{vmatrix} \tag{1.52.4}$$

Therefore finding the determinant along Row1:

$$= (5x+4)(4-x)^2 \tag{1.52.5}$$

(ii) $\begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & xy+k \end{vmatrix} = k^2(3y+k)$

Solution: Given determinant:

$$\Delta = \begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & xy+k \end{vmatrix} \tag{1.52.6}$$

Applying transformation:

$$\xleftrightarrow{R_1 \leftarrow R_1+R_2+R_3} \begin{vmatrix} 3y+k & 3y+k & 3y+k \\ y & y+k & y \\ y & y & y+k \end{vmatrix} \tag{1.52.7}$$

$$\xleftrightarrow{C_2 \leftarrow C_2-C_1} \begin{vmatrix} 3y+k & 0 & 3y+k \\ y & k & y \\ y & 0 & y+k \end{vmatrix} \tag{1.52.8}$$

$$\xleftrightarrow{C_3 \leftarrow C_3-C_1} \begin{vmatrix} 3y+k & 0 & 0 \\ y & k & 0 \\ y & 0 & k \end{vmatrix} \tag{1.52.9}$$

Taking determinant

$$\implies \Delta = (3y+k)k^2 \tag{1.52.10}$$

$$\begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & xy+k \end{vmatrix} = k^2(3y+k) \tag{1.52.11}$$

Hence proved.

1.53. $\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix} = 1+a^2+b^2+c^2$

Solution:

$$\begin{vmatrix} 1+a^2 & ab & ac \\ ab & 1+b^2 & bc \\ ac & bc & 1+c^2 \end{vmatrix} \xleftrightarrow[R_3 \leftarrow cR_3]{R_1 \leftarrow aR_1; R_2 \leftarrow bR_2}$$

$$\frac{1}{abc}\begin{vmatrix} a(1+a^2) & a^2b & a^2c \\ ab^2 & b(1+b^2) & b^2c \\ ac^2 & bc^2 & c(1+c^2) \end{vmatrix} \xleftrightarrow[C_3 \leftarrow \frac{C_3}{c}]{C_1 \leftarrow \frac{C_1}{a}; C_2 \leftarrow \frac{C_2}{b}}$$

$$\begin{vmatrix} 1+a^2 & a^2 & a^2 \\ b^2 & 1+b^2 & b^2 \\ c^2 & c^2 & 1+c^2 \end{vmatrix} \xleftrightarrow{R_1 \leftarrow R_1+R_2+R_3}$$

$$\begin{vmatrix} 1+a^2+b^2+c^2 & 1+a^2+b^2+c^2 & 1+a^2+b^2+c^2 \\ b^2 & 1+b^2 & b^2 \\ c^2 & c^2 & 1+c^2 \end{vmatrix} \tag{1.53.1}$$

Taking $1+a^2+b^2+c^2$ out from (1.53.1),

$$\implies (1+a^2+b^2+c^2)\begin{vmatrix} 1 & 1 & 1 \\ b^2 & 1+b^2 & b^2 \\ c^2 & c^2 & 1+c^2 \end{vmatrix}$$

$$\xleftrightarrow[C_3 \leftarrow C_3-C_1]{C_2 \leftarrow C_2-C_1} (1+a^2+b^2+c^2)\begin{vmatrix} 1 & 0 & 0 \\ b^2 & 1 & 0 \\ c^2 & 0 & 1 \end{vmatrix}$$

$= 1+a^2+b^2+c^2$ ($\because$ Determinant of a lower triangle matrix is the product of it's diagonal elements) Choose the correct answer in Exercises 23 and 24.

1.54. Let A be a square matrix of order 3X3, then $|kA|$ is equal to

a) $k\,|A|$
b) $k^2\,|A|$
c) $k^3\,|A|$
d) $3k\,|A|$

1.55. Which of the following is correct

a) Determinant is a square matrix.
b) Determinant is a number associated to a matrix.
c) Determinant is a number associated to a square matrix.
d) None of these.

1.56. Find area of the triangle with vertices at the point given in each of the following :

(i) $\begin{pmatrix}1 & 0\end{pmatrix}, \begin{pmatrix}6 & 0\end{pmatrix}, \begin{pmatrix}4 & 3\end{pmatrix}$
(ii) $\begin{pmatrix}2 & 7\end{pmatrix}, \begin{pmatrix}1 & 1\end{pmatrix}, \begin{pmatrix}10 & 8\end{pmatrix}$
(iii) $\begin{pmatrix}-2 & -3\end{pmatrix}, \begin{pmatrix}3 & 2\end{pmatrix}, \begin{pmatrix}-1 & -8\end{pmatrix}$

1.57. Show that points A=$\begin{pmatrix}a & b+c\end{pmatrix}$, B=$\begin{pmatrix}b & c+a\end{pmatrix}$, C=$\begin{pmatrix}c & a+b\end{pmatrix}$ are collinear.

Solution: The points **A**, **B** and **C** will be collinear if

$$\begin{pmatrix} \mathbf{A}^T \\ \mathbf{B}^T \\ \mathbf{C}^T \end{pmatrix}\mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{1.57.1}$$

$$\implies \begin{pmatrix} a & b+c \\ b & c+a \\ c & a+b \end{pmatrix}\mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{1.57.2}$$

So the augmented matrix of (1.57.2) is given by

$$\begin{pmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{pmatrix} \tag{1.57.3}$$

Using row reduction we get,

$$\begin{pmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{pmatrix} \tag{1.57.4}$$

$$\xleftrightarrow{R_3 = cR_3 - bR_2} \begin{pmatrix} a & b+c & 1 \\ b & c+a & 1 \\ 0 & (b-c)(a+b+c) & b-c \end{pmatrix} \tag{1.57.5}$$

$$\xleftrightarrow{R_3 = \frac{1}{(b-c)}R_3} \begin{pmatrix} a & b+c & 1 \\ b & c+a & 1 \\ 0 & a+b+c & 1 \end{pmatrix} \tag{1.57.6}$$

$$\xleftrightarrow{R_2 = aR_2 - bR_1} \begin{pmatrix} a & b+c & 1 \\ 0 & (a-b)(a+b+c) & a-b \\ 0 & a+b+c & 1 \end{pmatrix} \tag{1.57.7}$$

$$\xleftrightarrow{R_2 = \frac{1}{(a-b)}R_2} \begin{pmatrix} a & b+c & 1 \\ 0 & a+b+c & 1 \\ 0 & a+b+c & 1 \end{pmatrix} \tag{1.57.8}$$

$$\xleftrightarrow{R_3 = R_3 - R_2} \begin{pmatrix} a & b+c & 1 \\ 0 & a+b+c & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.57.9}$$

$$\tag{1.57.10}$$

From (1.57.10) we see that the rank of the augmented matrix is less than 3, hence **A**,**B** and **C** are colinear. We illustrate the concept by an example. Let a=1, b=2 and c=3. The points are **A**=$\begin{pmatrix}1 & 5\end{pmatrix}$,**B**=$\begin{pmatrix}2 & 4\end{pmatrix}$ and **C**=$\begin{pmatrix}3 & 3\end{pmatrix}$. Below is the diagram of the line passing through the points **A**, **B** and **C**.

1.58. Find values of k if area of triangle is 4sq.units and vertices are
(i) $\begin{pmatrix}k & 0\end{pmatrix}$, $\begin{pmatrix}4 & 0\end{pmatrix}$, $\begin{pmatrix}0 & 2\end{pmatrix}$
(ii) $\begin{pmatrix}-2 & 0\end{pmatrix}$, $\begin{pmatrix}0 & 4\end{pmatrix}$, $\begin{pmatrix}0 & k\end{pmatrix}$

1.59. (i) Find equation of line joining $\begin{pmatrix}1 & 2\end{pmatrix}$ and $\begin{pmatrix}3 & 6\end{pmatrix}$ using determinants.
(ii) Find equation of line joining $\begin{pmatrix}3 & 1\end{pmatrix}$ and $\begin{pmatrix}9 & 3\end{pmatrix}$ using determinants.

1.60. If the area of triangle is 35 sq.units with vertices $\begin{pmatrix}2 & -6\end{pmatrix}$, $\begin{pmatrix}5 & 4\end{pmatrix}$ and $\begin{pmatrix}k & 4\end{pmatrix}$.then k is
a) 12
b) -2
c) -12,-2
d) 12,-2

Write Minors and Coafactors of the elements of following determinants:

1.61. (i) $\begin{vmatrix} 2 & -4 \\ 0 & 3 \end{vmatrix}$
(ii) $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$

1.62. (i) $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$
(ii) $\begin{vmatrix} 1 & 0 & 4 \\ 3 & 5 & -1 \\ 0 & 1 & 2 \end{vmatrix}$

1.63. Using Cofactors of elements of second row,evaluate $\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$.

1.64. Using Cofactors of elements of third column ,evaluate $\Delta = \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}$.

1.65. If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ and A_{ij} is Cofactors of a_{ij} then value of Δ is given by
a) $a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33}$
b) $a_{11}A_{11} + a_{12}A_{21} + a_{13}A_{31}$
c) $a_{21}A_{11} + a_{22}A_{12} + a_{23}A_{13}$
d) $a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$

Find adjoint of each of the matrices

1.66. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

1.67. $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$ Verify A(adjA)=(adjA)A=$|A|\, I$

1.68. $\begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}$

1.69. $\begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}$

1.70. $\begin{bmatrix} 2 & -2 \\ 4 & 3 \end{bmatrix}$

1.71. $\begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix}$

1.72. $\begin{bmatrix}1 & 2 & 3\\0 & 2 & 4\\0 & 0 & 5\end{bmatrix}$

1.73. $\begin{bmatrix}1 & 0 & 0\\3 & 3 & 0\\5 & 2 & -1\end{bmatrix}$

1.74. $\begin{bmatrix}2 & 1 & 3\\4 & -1 & 0\\-7 & 2 & 1\end{bmatrix}$

1.75. $\begin{bmatrix}1 & -1 & 2\\0 & 2 & -3\\3 & -2 & 4\end{bmatrix}$

1.76. $\begin{bmatrix}1 & 0 & 0\\0 & \cos\alpha & \sin\alpha\\0 & \sin\alpha & -\cos\alpha\end{bmatrix}$

1.77. Let A= $\begin{bmatrix}3 & 7\\2 & 5\end{bmatrix}$ and B= $\begin{bmatrix}6 & 8\\7 & 9\end{bmatrix}$. Verify that $(AB)^{-1} = B^{-1}A^{-1}$

1.78. Let A =$\begin{bmatrix}3 & 1\\-1 & 2\end{bmatrix}$, show that $A^2 - 5A + 7I = O$. Hence find A^{-1} **Solution:**
Multiplying above equation with $\mathbf{A}^{-1}$ on both sides, We get :

$$\implies \mathbf{A}^2\mathbf{A}^{-1} - 5\mathbf{A}\mathbf{A}^{-1} + 7\mathbf{I}\mathbf{A}^{-1} = 0 \quad (1.78.1)$$

$$\implies \mathbf{A}\mathbf{A}\mathbf{A}^{-1} - 5\mathbf{I} + 7\mathbf{A}^{-1} = 0 \quad (1.78.2)$$

$$\implies \mathbf{A}\mathbf{I} - 5\mathbf{I} + 7\mathbf{A}^{-1} = 0 \quad (1.78.3)$$

$$\implies \mathbf{A}^{-1} = \frac{1}{7}(5\mathbf{I} - \mathbf{A}) \quad (1.78.4)$$

Solving for $\mathbf{A}^{-1}$, we get :

$$\mathbf{A}^{-1} = \frac{1}{7}\left(5\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} - \begin{pmatrix}3 & 1\\-1 & 2\end{pmatrix}\right) \quad (1.78.5)$$

$$\implies \mathbf{A}^{-1} = \frac{1}{7}\left(\begin{pmatrix}2 & -1\\1 & 3\end{pmatrix}\right) \quad (1.78.6)$$

1.79. For the matrix A= $\begin{bmatrix}3 & 2\\1 & 1\end{bmatrix}$, find the numbers a and b such that $A^2 + aA + bI = O$.
Solution: For a general square matrix (A), the characteristic equation in variable λ is defined by

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0 \quad (1.79.1)$$

$$\implies \begin{vmatrix}3-\lambda & 2\\1 & 1-\lambda\end{vmatrix} = 0 \quad (1.79.2)$$

$$\implies (3-\lambda)(1-\lambda) - 2 = 0 \quad (1.79.3)$$

$$\implies \lambda^2 - 4\lambda + 3 - 2 = 0 \quad (1.79.4)$$

$$\implies \lambda^2 - 4\lambda + 1 = 0 \quad (1.79.5)$$

Now by Cayley-Hamilton Theorem **A** satisfies (1.79.5),then replacing λ with **A** we get

$$\mathbf{A}^2 - 4\mathbf{A} + \mathbf{I} = 0 \quad (1.79.6)$$

We need to find a and b in

$$\mathbf{A}^2 + a\mathbf{A} + b\mathbf{I} = 0 \quad (1.79.7)$$

Comparing (1.79.6) and (1.79.7) we get a = -4 and b = 1.

1.80. For the matrix A=$\begin{pmatrix}1 & 1 & 1\\1 & 2 & -3\\2 & -1 & 3\end{pmatrix}$.Show that

$$A^3 - 6A^2 + 5A + 11I = 0 \quad (1.80.1)$$

and hence find A^{-1}. **Solution:** Given matrix is

$$A = \begin{pmatrix}1 & 1 & 1\\1 & 2 & -3\\2 & -1 & 3\end{pmatrix} \quad (1.80.2)$$

The characteristic polynomial of the matrix will be According to cayley-hamilton theorem every matrix satisfies it's own characteristic equation. So the matrix A satifies the equation (1.80.1)

$$A^3 - 6A^2 + 5A + 11I = 0 \quad (1.80.3)$$

The matrix A satisifes the characteristic equation,so

$$A^3 - 6A^2 + 5A + 11I = 0 \quad (1.80.4)$$

Multiplying with A^{-1} we get

$$A^2 - 6A + 5I + 11A^{-1} = 0 \quad (1.80.5)$$

$$A^{-1} = \frac{1}{11}\left(6A - A^2 - 5I\right) \quad (1.80.6)$$

$$A^{-1} = \frac{1}{11}\left[6\begin{pmatrix}1 & 1 & 1\\ 1 & 2 & -3\\ 2 & -1 & 3\end{pmatrix} - \begin{pmatrix}4 & 2 & 1\\ -3 & 8 & -14\\ 7 & -3 & 14\end{pmatrix} \quad (1.80.7)$$

$$-5\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\right] \quad (1.80.8)$$

$$A^{-1} = \frac{1}{11}\begin{pmatrix}-3 & 4 & 5\\ 9 & -1 & -4\\ 5 & -3 & -1\end{pmatrix} \quad (1.80.9)$$

$$\implies A^{-1} = \begin{pmatrix}\frac{-3}{11} & \frac{4}{11} & \frac{5}{11}\\ \frac{9}{11} & \frac{-1}{11} & \frac{-4}{11}\\ \frac{5}{11} & \frac{-3}{11} & \frac{-1}{11}\end{pmatrix} \quad (1.80.10)$$

1.81. Let A be a nonsingular square matrix of order 3X3 .Then $|adjA|$ is equal to
   a) $|A|$
   b) $|A|^2$
   c) $|A|^3$
   d) $3\,|A|$

1.82. If A is an invertible matrix of order 2, then $\det(A^{-1})$ is equal to
   a) det(A)
   b) $\frac{1}{det(A)}$
   c) 1
   d) 0

Examine the consistency of the system of given Equations.

1.83. $x + 3y = 5$

$2x + 6y = 8$

Solution: If solution exists for the given system of linear equations then they are said to be consistent, otherwise they are inconsistent. we can represent the given lines

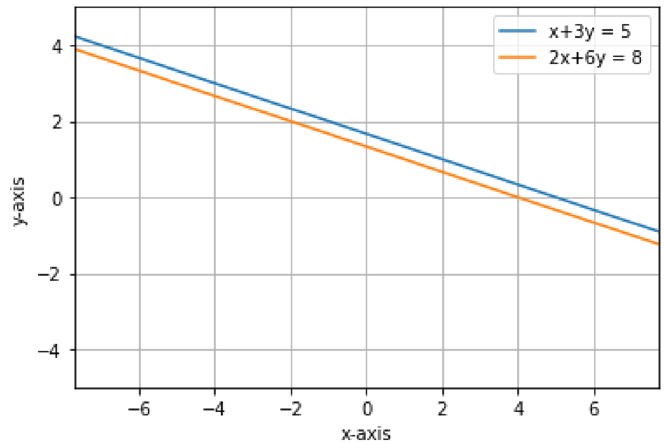

Fig. 1.83: Plot showing the given two lines are parallel

in the form of

$$\begin{pmatrix}1 & 3\end{pmatrix}\mathbf{x} = 5 \quad (1.83.1)$$

$$\begin{pmatrix}2 & 6\end{pmatrix}\mathbf{x} = 8 \quad (1.83.2)$$

writing the above equations in matrix form

$$\begin{pmatrix}1 & 3 & -5\\ 2 & 6 & -8\end{pmatrix}\mathbf{x} = 0 \quad (1.83.3)$$

The matrix equation is row reduced as follows

$$\begin{pmatrix}1 & 3 & -5\\ 2 & 6 & -8\end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{pmatrix}1 & 3 & -5\\ 0 & 0 & 2\end{pmatrix} \quad (1.83.4)$$

Thus, from the above row reduced form we can conclude that the given system of lines has no solution. Therefore, they are inconsistent. See 1.83

1.84. x+y+z=1
2x+3y+2z=2
ax+ay+2az=4

Solution: Assume that a is any real number. The above system of equations can be expressed in the form of matrix:

$$\begin{pmatrix}1 & 1 & 1\\ 2 & 3 & 2\\ a & a & 2a\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}1\\ 2\\ 4\end{pmatrix} \quad (1.84.1)$$

This is in the form of:

$$\mathbf{Ax} = \mathbf{B} \quad (1.84.2)$$

The system defined above is consistent and has a solution only when

$$rank(\mathbf{A}|\mathbf{B}) = rank(\mathbf{A}) = dim(\mathbf{A}) \quad (1.84.3)$$

Reducing the augmented matrix to row echelon form, we get:

$$\begin{pmatrix} 1 & 1 & 1 & | & 1 \\ 2 & 3 & 2 & | & 2 \\ a & a & 2a & | & 4 \end{pmatrix} \xleftrightarrow{R_2 \to R_2 - 2R_1} \begin{pmatrix} 1 & 1 & 1 & | & 1 \\ 0 & 1 & 0 & | & 0 \\ a & a & 2a & | & 4 \end{pmatrix} \quad (1.84.4)$$

$$\begin{pmatrix} 1 & 1 & 1 & | & 1 \\ 0 & 1 & 0 & | & 0 \\ a & a & 2a & | & 4 \end{pmatrix} \xleftrightarrow{R_3 \to R_3 - aR_1} \begin{pmatrix} 1 & 1 & 1 & | & 1 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & a & | & 4-a \end{pmatrix} \quad (1.84.5)$$

$$\implies rank(\mathbf{A}) = rank(\mathbf{A}|\mathbf{B}) = dim(\mathbf{A}) \quad (1.84.6)$$

The system of equations is consistent and has a unique solution except at $a = 0$.

1.85. 3x-y-2z=2
2y-z=-1
3x-5y=3

Solution:
The given system of equations can be represented as:

$$\mathbf{Ax} = \mathbf{B} \quad (1.85.1)$$

Coefficient Matrix **A**

$$\mathbf{A} = \begin{pmatrix} 3 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{pmatrix} \quad (1.85.2)$$

Constant Vector **B**

$$\mathbf{B} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \quad (1.85.3)$$

Augumented Matrix-

$$\mathbf{A}|\mathbf{B} = \left(\begin{array}{ccc|c} 3 & -1 & -2 & 2 \\ 0 & 2 & -1 & -1 \\ 3 & -5 & 0 & 3 \end{array}\right) \quad (1.85.4)$$

Applying Row operations on (2.4)

$$\left(\begin{array}{ccc|c} 3 & -1 & -2 & 2 \\ 0 & 2 & -1 & -1 \\ 3 & -5 & 0 & 3 \end{array}\right) \xleftrightarrow{R3 \to R1 - R3} \left(\begin{array}{ccc|c} 3 & -1 & -2 & 2 \\ 0 & 2 & -1 & -1 \\ 0 & 4 & -2 & -1 \end{array}\right) \quad (1.85.5)$$

$$\left(\begin{array}{ccc|c} 3 & -1 & -2 & 2 \\ 0 & 2 & -1 & -1 \\ 0 & 4 & -2 & -1 \end{array}\right) \xleftrightarrow{R3 \to R3 - R2} \left(\begin{array}{ccc|c} 3 & -1 & -2 & 2 \\ 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{array}\right) \quad (1.85.6)$$

$$\mathbf{A}|\mathbf{B} = \left(\begin{array}{ccc|c} 3 & -1 & -2 & 2 \\ 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{array}\right) \mathbf{A} = \begin{pmatrix} 3 & -1 & -2 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix} \quad (1.85.7)$$

$$R(\mathbf{A}|\mathbf{B}) = 3 \quad (1.85.8)$$

$$R(\mathbf{A} = 2 \quad (1.85.9)$$

$$R(\mathbf{A}|\mathbf{B}) \neq R(\mathbf{A}) \quad (1.85.10)$$

The given system of equations are inconsistent.

1.86. 5x-y+4z=5
2x+3y+5z=2
5x-2y+6z=-1

Solution: The given equations can be written as

$$\mathbf{Ax} = \mathbf{b}$$

where

$$\mathbf{A} = \begin{pmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ 5 & -2 & 6 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix} \quad (1.86.1)$$

By row reducing the augmented matrix :

$$\left(\begin{array}{ccc|c} 5 & -1 & 4 & 5 \\ 2 & 3 & 5 & 2 \\ 5 & -2 & 6 & -1 \end{array}\right) \quad (1.86.2)$$

$$\xleftrightarrow[R_3 \leftarrow R_3 - R_1]{R_2 \leftarrow 5R_2 - (R_1 + R_3)} \left(\begin{array}{ccc|c} 5 & -1 & 4 & 5 \\ 0 & 18 & 15 & 6 \\ 0 & -1 & 2 & -6 \end{array}\right) \quad (1.86.3)$$

$$\xleftrightarrow{R_3 \leftarrow 18R_3 + R_2} \left(\begin{array}{ccc|c} 5 & -1 & 4 & 5 \\ 0 & 18 & 15 & 6 \\ 0 & 0 & 51 & -102 \end{array}\right) \quad (1.86.4)$$

$$\xleftrightarrow{R_3 \leftarrow \frac{R_3}{51}} \left(\begin{array}{ccc|c} 5 & -1 & 4 & 5 \\ 0 & 18 & 15 & 6 \\ 0 & 0 & 1 & -2 \end{array}\right) \quad (1.86.5)$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - 15R_3} \left(\begin{array}{ccc|c} 5 & -1 & 4 & 5 \\ 0 & 18 & 0 & 36 \\ 0 & 0 & 1 & -2 \end{array}\right) \quad (1.86.6)$$

$$\xleftrightarrow{R_2 \leftarrow \frac{R_2}{18}} \left(\begin{array}{ccc|c} 5 & -1 & 4 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \end{array}\right) \quad (1.86.7)$$

$$\xleftrightarrow{R_1 \leftarrow R_1+R_2-4R_3} \left(\begin{array}{ccc|c} 5 & 0 & 0 & 15 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \end{array}\right) \quad (1.86.8)$$

$$\xleftrightarrow{R_1 \leftarrow \frac{R_1}{5}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \end{array}\right) \quad (1.86.9)$$

$$\implies rank\begin{pmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ 5 & -2 & 6 \end{pmatrix} = rank\left(\begin{array}{ccc|c} 5 & -1 & 4 & 5 \\ 2 & 3 & 5 & 2 \\ 5 & -2 & 6 & -1 \end{array}\right) = 3 = dim\begin{pmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ 5 & -2 & 6 \end{pmatrix} \quad (1.86.10)$$

i.e., the $rank(\mathbf{A}) = rank(\mathbf{A} : \mathbf{b}) = 3$, which is equal to the row size of $\mathbf{x}$. Hence the system of linear equations is consistent, with a unique solution.

The unique solution is

$$\mathbf{x} = \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} \quad (1.86.11)$$

Solve the system linear equations,using matrix method.

1.87. $5x + 2y = 4$

$7x + 3y = 5$

Solution: The above equations can be expressed in vector form as,

$$\begin{pmatrix} 5 & 2 \end{pmatrix} \mathbf{x} = 4 \quad (1.87.1)$$

$$\begin{pmatrix} 7 & 3 \end{pmatrix} \mathbf{x} = 5 \quad (1.87.2)$$

Now, writing it in the matrix form as,

$$\begin{pmatrix} 5 & 2 \\ 7 & 3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 4 \\ 5 \end{pmatrix} \quad (1.87.3)$$

The augmented matrix of above equation is solved by row reduction as follows

$$\begin{pmatrix} 5 & 2 & 4 \\ 7 & 3 & 5 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - \left(\frac{7}{5}\right) R_1} \begin{pmatrix} 5 & 2 & 4 \\ 0 & \frac{1}{5} & \frac{-3}{5} \end{pmatrix} \quad (1.87.4)$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - 10R_2} \begin{pmatrix} 5 & 0 & 10 \\ 0 & \frac{1}{5} & \frac{-3}{5} \end{pmatrix} \quad (1.87.5)$$

$$\implies \mathbf{x} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} \quad (1.87.6)$$

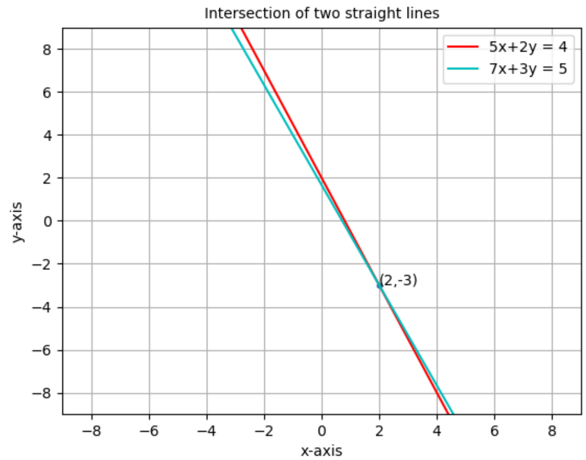

Fig. 1.87: Intersection of 2 lines

See Fig. 1.87

1.88. $2x - y = -2$

$3x + 4y = 3$

Solution:

Given equation can be repsented in a matrix form as

$$\begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} \quad (1.88.1)$$

The corresponding augmented matrix is

$$\left(\begin{array}{cc|c} 2 & -1 & -2 \\ 3 & 4 & 3 \end{array}\right) \quad (1.88.2)$$

We use the Guass Jordan Elimination method as:

$$\left(\begin{array}{cc|c} 2 & -1 & -2 \\ 3 & 4 & 3 \end{array}\right) \quad (1.88.3)$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - \frac{3}{2} R_1} \left(\begin{array}{cc|c} 2 & -1 & -2 \\ 0 & \frac{11}{2} & 6 \end{array}\right) \quad (1.88.4)$$

$$\xleftrightarrow{R_2 \leftarrow \frac{2}{11} R_2} \left(\begin{array}{cc|c} 2 & -1 & -2 \\ 0 & 1 & \frac{12}{11} \end{array}\right) \quad (1.88.5)$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + R_2} \left(\begin{array}{cc|c} 2 & 0 & \frac{-10}{11} \\ 0 & 1 & \frac{12}{11} \end{array}\right) \quad (1.88.6)$$

$$\xleftrightarrow{R_1 \leftarrow \frac{1}{2} R_1} \left(\begin{array}{cc|c} 1 & 0 & \frac{-5}{11} \\ 0 & 1 & \frac{12}{11} \end{array}\right) \quad (1.88.7)$$

Therefore, the values of x and y are:

$$x = \frac{-5}{11} \tag{1.88.8}$$

$$y = \frac{12}{11} \tag{1.88.9}$$

1.89. $4x - 3y = 3$

$3x - 5y = 7$

Solution: Writing both euations in matrix form

$$\begin{pmatrix} 4 & -3 \end{pmatrix} \mathbf{x} = 3 \tag{1.89.1}$$

$$\begin{pmatrix} 3 & -5 \end{pmatrix} \mathbf{x} = 7 \tag{1.89.2}$$

Forming the augmented matrix and reducing the matrix to row echelon form:

$$\begin{pmatrix} 4 & -3 & 3 \\ 3 & -5 & 7 \end{pmatrix} \tag{1.89.3}$$

$$\xleftrightarrow{R_1 \leftarrow R_1/4} \begin{pmatrix} 1 & -3/4 & 3/4 \\ 3 & -5 & 7 \end{pmatrix} \tag{1.89.4}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - 3R_1} \begin{pmatrix} 1 & -3/4 & 3/4 \\ 0 & -11/4 & 19/4 \end{pmatrix} \tag{1.89.5}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 \times -4/11} \begin{pmatrix} 1 & -3/4 & 3/4 \\ 0 & 1 & -19/11 \end{pmatrix} \tag{1.89.6}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + 3/4 \times R_2} \begin{pmatrix} 1 & 0 & -6/11 \\ 0 & 1 & -19/11 \end{pmatrix} \tag{1.89.7}$$

Here, $Rank(A) = Rank(A|B)$. Therefore, the system is consistent. Also, there exist a unique solution as $Rank(A) = n$ (number of unknown). From equation 1.89.7, we get:

$$\mathbf{x} = \frac{1}{11} \begin{pmatrix} -6 \\ -19 \end{pmatrix} \tag{1.89.8}$$

Plotting the lines and the intersection point in Fig. 1.89

1.90. $5x + 2y = 3$

$3x + 2y = 5$

Solution: he given set of equations can be represented in the matrix equation form as

$$\begin{pmatrix} 5 & 2 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \tag{1.90.1}$$

The augmented matrix for this system becomes

$$\begin{pmatrix} 5 & 2 & 3 \\ 3 & 2 & 5 \end{pmatrix} \tag{1.90.2}$$

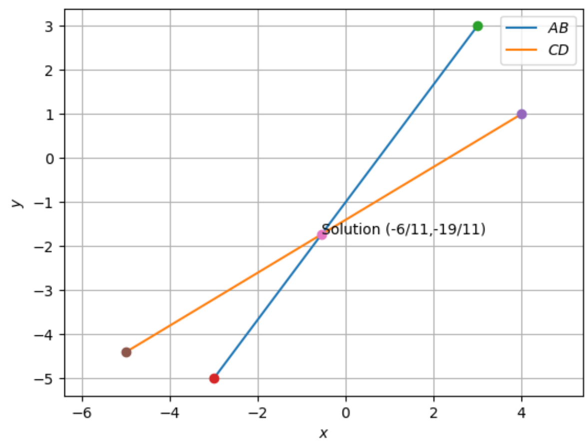

Fig. 1.89: Lines and their intersection denoting the solution

Row reducing the matrix

$$\begin{pmatrix} 5 & 2 & 3 \\ 3 & 2 & 5 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 \times \frac{5}{3} - R_1} \begin{pmatrix} 5 & 2 & 3 \\ 0 & \frac{4}{3} & \frac{16}{3} \end{pmatrix} \tag{1.90.3}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 \times \frac{3}{4}} \begin{pmatrix} 5 & 2 & 3 \\ 0 & 1 & 4 \end{pmatrix} \tag{1.90.4}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - 2 \times R_2} \begin{pmatrix} 5 & 0 & -5 \\ 0 & 1 & 4 \end{pmatrix} \tag{1.90.5}$$

$$\xleftrightarrow{R_1 \leftarrow \frac{R_1}{5}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 4 \end{pmatrix} \tag{1.90.6}$$

$$\implies \text{Rank} \begin{pmatrix} 5 & 2 \\ 3 & 2 \end{pmatrix} = \text{Rank} \begin{pmatrix} 5 & 2 & 3 \\ 3 & 2 & 5 \end{pmatrix} = 2 \tag{1.90.7}$$

$$= dim \begin{pmatrix} 5 & 2 \\ 3 & 2 \end{pmatrix} \tag{1.90.8}$$

So, the given system of equations are consistent with a unique solution of

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix} \tag{1.90.9}$$

1.91. 2x+y+z = 1

x-2y-z = $\frac{3}{2}$

3y- 5z = 9

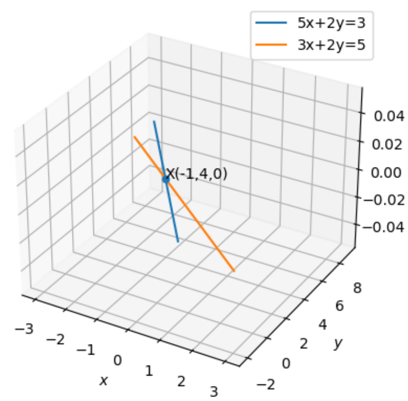

Fig. 1.90: plot showing intersection of lines

1.92. x-y+z = 4
2x+y-3z = 0
x+y+z = 2

1.93. 2x+3y+3z = 5
x-2y+z = -4
3x-y-2z = 3

1.94. x-y+2z = 7
3x+4y-5z = -5
2x-y+3z = 12

1.95. If A=$\begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$, find A^{-1}. Using A^{-1} solve the system of equations
2x-3y+5z = 11,
3x+2y-4z = -5,
x+y-2z =-3.

1.96. The cost of 4 kg onion, 3 kg wheat and 2 kg rice is ₹60. The cost of 2 kg onion,4 kg wheat and 6 kg rice is ₹90.The cost of 6kg onion 2kg wheat and 3kg rice is ₹70.Find the cost of each item per kg by matrix mathod.

1.97. Prove that the determinant $\begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$ is independent of θ

Solution:

$$\begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix} \tag{1.97.1}$$

Now, Solving the determinant:-

$$\begin{aligned} &= x(-x^2-1) - \sin\theta(-x\sin\theta - \cos\theta) \\ &\quad + \cos\theta(-\sin\theta + x\cos\theta) \\ &= -x^3 - x + x\sin^2\theta + \sin\theta\cos\theta \\ &\quad - \sin\theta\cos\theta + x\cos^2\theta \end{aligned} \tag{1.97.2}$$

$$= -x^3 - x + x\sin^2\theta + x\cos^2\theta \tag{1.97.3}$$

$$= -x^3(\because \sin^2\theta + \cos^2\theta = 1) \tag{1.97.4}$$

Hence, the determinant is independent of θ.

1.98. Without expanding the determinant, prove that $\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$.

Solution:

1.99. Evaluate $\begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{vmatrix}$.

Solution:

We first multiply either the rows or the columns, and then try taking the common element out.

$$\begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{vmatrix} \xleftrightarrow[C_3\leftarrow(\sin\alpha)C_3]{C_3\leftarrow(\cos\alpha)C_3}$$

$$\left(\frac{1}{\sin\alpha\cos\alpha}\right)\begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin^2\alpha\cos\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos^2\alpha\sin\alpha \end{vmatrix} \tag{1.99.1}$$

From (1.99.1) R_1 we take out common element $\cos\alpha$. And from row R_2 we take out common element $\sin\alpha$

$$\begin{vmatrix} \cos\beta & \sin\beta & -\sin^2\alpha \\ -\sin\beta & \cos\beta & 0 \\ \cos\beta & \sin\beta & \cos^2\alpha \end{vmatrix} \xleftrightarrow{R_1\leftarrow R_1-R_3}$$

$$\begin{vmatrix} 0 & 0 & -\sin^2\alpha-\cos^2\alpha \\ -\sin\beta & \cos\beta & 0 \\ \cos\beta & \sin\beta & \cos^2\alpha \end{vmatrix} \tag{1.99.2}$$

Now, we can expand the determinant from row 1 in (1.99.2), and we get

$$\left(-\sin^2\alpha-\cos^2\alpha\right)\left(-\sin^2\beta-\cos^2\beta\right)$$
$$\implies \left(\sin^2\alpha+\cos^2\alpha\right)\left(\sin^2\beta+\cos^2\beta\right) = 1 \tag{1.99.3}$$

Therefore, the determinant of the matrix is 1.

1.100. If a,b and c are real numbers, and $\Delta = \begin{vmatrix} b+c & c=a & a=b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0$, Show that either a+b+c=0 or a=b=c.

Solution: Given,

$$\triangle = \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} \tag{1.100.1}$$

$$\xleftrightarrow{C_1\leftarrow C_1+C_2+C_3} \begin{vmatrix} 2(a+b+c) & c+a & a+b \\ 2(a+b+c) & a+b & b+c \\ 2(a+b+c) & b+c & c+a \end{vmatrix} \tag{1.100.2}$$

$$= 2(a+b+c)\begin{vmatrix} 1 & c+a & a+b \\ 1 & a+b & b+c \\ 1 & b+c & c+a \end{vmatrix} \tag{1.100.3}$$

$$\xleftrightarrow{R_1\leftarrow R_1-R_2;R_2\leftarrow R_2-R_3} 2(a+b+c)\begin{vmatrix} 0 & c-b & a-c \\ 0 & a-c & b-a \\ 1 & b+c & c+a \end{vmatrix} = 0 \tag{1.100.4}$$

On expanding determinant along first column from equation (1.100.4),

$$\implies 2(a+b+c)[(c-b)(b-a)-(a-c)^2] = 0$$
$$\implies 2(a+b+c)(a^2+b^2+c^2 - ab-bc-ca) = 0$$
$$\implies (a+b+c)(2a^2+2b^2+2c^2 - 2ab-2bc-2ca) = 0$$
$$\implies (a+b+c) [(a-b)^2+(b-c)^2+(c-a)^2] = 0 \tag{1.100.5}$$

From equation (1.100.5) we get 2 equations,

$$\implies \boxed{(a+b+c)=0} \tag{1.100.6}$$

or,

$$\implies (a-b)^2+(b-c)^2+(c-a)^2 = 0 \tag{1.100.7}$$

Equation (1.100.7) is possible only when, $a = b = c$

$$\implies \boxed{a=b=c} \tag{1.100.8}$$

From equation (1.100.6) and (1.100.8) we can say that, $\triangle = 0$ if $a+b+c=0$ or $a=b=c$. From equation (1.100.6) and (1.100.8) we can say that, $\triangle = 0$ if $a+b+c=0$ or $a=b=c$.

1.101. Solve the equation $\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0, a \neq 0$

Solution: Given,

$$\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix} \tag{1.101.1}$$

$$\xleftrightarrow{R_1\leftarrow R_1+R_2+R_3} \begin{vmatrix} 3x+a & 3x+a & 3x+a \\ x & x+a & x \\ x & x & x+a \end{vmatrix} \tag{1.101.2}$$

$$(3x+a)\begin{vmatrix} 1 & 1 & 1 \\ x & x+a & x \\ x & x & x+a \end{vmatrix} \tag{1.101.3}$$

$$\xleftrightarrow[C_3\leftarrow C_3-C_1]{C_2\leftarrow C_2-C_1} (3x+a)\begin{vmatrix} 1 & 0 & 0 \\ x & a & 0 \\ x & 0 & a \end{vmatrix} \tag{1.101.4}$$

$$= (3x+a)(a^2) \tag{1.101.5}$$

Since a cannot be equal to zero, 3x+a should be zero for determinant to be zero

$$3x+a = 0 \tag{1.101.6}$$
$$a = -3x \tag{1.101.7}$$

1.102. Prove that

$$\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2$$

Solution:

$$\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix}$$

$$\xleftrightarrow{R1\leftarrow\frac{R1+R2+R3}{2}} 2\begin{vmatrix} a^2+ab & b^2+bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix}$$

$$\xleftrightarrow{R1\leftarrow R1-R2} 2\begin{vmatrix} 0 & bc & c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix}$$

$$\xleftrightarrow[C3\leftarrow\frac{C3}{c}]{C1\leftarrow\frac{C1}{a};C2\leftarrow\frac{C2}{b}} 2abc\begin{vmatrix} 0 & c & c \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$$

$$\xleftrightarrow{R3\leftarrow R3-R1} 2abc\begin{vmatrix} 0 & c & c \\ a+b & b & a \\ b & b & 0 \end{vmatrix}$$

$$\xleftrightarrow{R2\leftarrow R2-R3} 2abc\begin{vmatrix} 0 & c & c \\ a & 0 & a \\ b & b & 0 \end{vmatrix}$$

$$\xleftrightarrow[R3\leftarrow\frac{R3}{b}]{R1\leftarrow\frac{R1}{c};R2\leftarrow\frac{R2}{a}} 2a^2b^2c^2\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} \quad (1.102.1)$$

$$\xleftrightarrow{R1\leftarrow R1+R2-R3} 2a^2b^2c^2\begin{vmatrix} 0 & 0 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} \quad (1.102.2)$$

$$\xleftrightarrow{R1\leftarrow\frac{R1}{2}} 4a^2b^2c^2\begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 4a^2b^2c^2(1) \quad (1.102.3)$$

$$= 4a^2b^2c^2 \quad (1.102.4)$$

1.103. If $A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$ and B=$\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$, find $(AB)^{-1}$

1.104. Let A=$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$. Verify that

(i) $[adjA]^{-1} = adj(A)^{-1}$

(ii) $(A^{-1})^{-1} = A$

1.105. Evaluate $\begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$

Solution: Given,

$$\left|\mathbf{A}\right| = \begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix} \quad (1.105.1)$$

$$\xleftrightarrow{R_1\leftarrow R_1+R_2+R_3} \begin{vmatrix} 2(x+y) & 2(x+y) & 2(x+y) \\ y & x+y & x \\ x+y & x & y \end{vmatrix} \quad (1.105.2)$$

$$= 2(x+y)\begin{vmatrix} 1 & 1 & 1 \\ y & x+y & x \\ x+y & x & y \end{vmatrix} \quad (1.105.3)$$

$$\xleftrightarrow[C_3\leftarrow C_3-C_1]{C_2\leftarrow C_2-C_1} 2(x+y)\begin{vmatrix} 1 & 0 & 0 \\ y & x & x-y \\ x+y & -y & -x \end{vmatrix} \quad (1.105.4)$$

Expanding the determinant from (1.105.4), we get

$$= 2\left(x+y\right)\left[-x^2 - \{(-y)\left(x-y\right)\}\right] \quad (1.105.5)$$

$$= 2\left(x+y\right)\left(-x^2 + xy - y^2\right) \quad (1.105.6)$$

$$= -2x^3 + 2x^2y - 2xy^2 - 2x^2y + 2xy^2 - 2y^3 \quad (1.105.7)$$

$$= -2\left(x^3 + y^3\right) \quad (1.105.8)$$

$$\therefore \begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix} = -2\left(x^3+y^3\right) \quad (1.105.9)$$

1.106. Evaluate $\begin{vmatrix} 1 & x & y \\ 1 & x+y & y \\ 1 & x & x+y \end{vmatrix}$ **Solution:**

$$\Delta = \begin{vmatrix} 1 & x & y \\ 1 & x+y & y \\ 1 & x & x+y \end{vmatrix} \quad (1.106.1)$$

Performing transformations

$$\begin{vmatrix}1 & x & y\\1 & x+y & y\\1 & x & x+y\end{vmatrix} \xleftrightarrow[R_3=R_3-R_1]{R_2=R_2-R_1} xy\begin{vmatrix}1 & x & y\\0 & 1 & 0\\1 & x & 1\end{vmatrix} \tag{1.106.2}$$

Finding determinant along Row 3

$$\Delta = xy \tag{1.106.3}$$

Using properties of determinants ,prove that:

1.107. $\begin{vmatrix}\alpha & \alpha^2 & \beta+\gamma\\\beta & \beta^2 & \gamma+\alpha\\\gamma & \gamma^2 & \alpha+\beta\end{vmatrix} = (\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)(\alpha+\beta+\gamma)$

Solution: Using transformations and properties of determinants:

$$\begin{vmatrix}\alpha & \alpha^2 & \beta+\gamma\\\beta & \beta^2 & \gamma+\alpha\\\gamma & \gamma^2 & \alpha+\beta\end{vmatrix} \tag{1.107.1}$$

$$\xleftrightarrow{C_3\leftarrow C_1+C_3} \begin{vmatrix}\alpha & \alpha^2 & \alpha+\beta+\gamma\\\beta & \beta^2 & \alpha+\beta+\gamma\\\gamma & \gamma^2 & \alpha+\beta+\gamma\end{vmatrix} \tag{1.107.2}$$

$$\implies \quad (\alpha+\beta+\gamma)\begin{vmatrix}\alpha & \alpha^2 & 1\\\beta & \beta^2 & 1\\\gamma & \gamma^2 & 1\end{vmatrix} \tag{1.107.3}$$

$$\xleftrightarrow[R_2\leftarrow R_2-R_1]{R_3\leftarrow R_3-R_1} (\alpha+\beta+\gamma)\begin{vmatrix}\alpha & \alpha^2 & 1\\\beta-\alpha & \beta^2-\alpha^2 & 0\\\gamma-\alpha & \gamma^2-\alpha^2 & 0\end{vmatrix} \tag{1.107.4}$$

$$\implies \quad (\beta-\alpha)(\gamma-\alpha)(\alpha+\beta+\gamma)\begin{vmatrix}\alpha & \alpha^2 & 1\\1 & \beta+\alpha & 0\\1 & \gamma+\alpha & 0\end{vmatrix} \tag{1.107.5}$$

$$\implies (\beta-\alpha)(\gamma-\alpha)(\alpha+\beta+\gamma)(-1)^{1+3}\begin{vmatrix}1 & \beta+\alpha\\1 & \gamma+\alpha\end{vmatrix} \tag{1.107.6}$$

$$\implies \quad (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)(\alpha+\beta+\gamma) \tag{1.107.7}$$

1.108. $\begin{vmatrix}x & x^2 & 1+px^3\\y & y^2 & 1+py^3\\z & z^2 & 1+pz^3\end{vmatrix} = (1+pxyz)(x-y)(y-z)(z-x),$ where p is any scalar.

Solution:

$$LHS = \begin{vmatrix}x & x^2 & 1+px^3\\y & y^2 & 1+py^3\\z & z^2 & 1+pz^3\end{vmatrix} \tag{1.108.1}$$

By expanding using sum property

$$= \begin{vmatrix}x & x^2 & 1\\y & y^2 & 1\\z & z^2 & 1\end{vmatrix} + \begin{vmatrix}x & x^2 & px^3\\y & y^2 & py^3\\z & z^2 & pz^3\end{vmatrix} \tag{1.108.2}$$

By using switching of rows(or columns) property

$$= (-1)\begin{vmatrix}1 & x^2 & x\\1 & y^2 & y\\1 & z^2 & z\end{vmatrix} + \begin{vmatrix}x & x^2 & px^3\\y & y^2 & py^3\\z & z^2 & pz^3\end{vmatrix} \tag{1.108.3}$$

$$= (-1)^2\begin{vmatrix}1 & x & x^2\\1 & y & y^2\\1 & z & z^2\end{vmatrix} + \begin{vmatrix}x & x^2 & px^3\\y & y^2 & py^3\\z & z^2 & pz^3\end{vmatrix} \tag{1.108.4}$$

By using scalar multiplication property

$$= \begin{vmatrix}1 & x & x^2\\1 & y & y^2\\1 & z & z^2\end{vmatrix} + (pxyz)\begin{vmatrix}1 & x & x^2\\1 & y & y^2\\1 & z & z^2\end{vmatrix} \tag{1.108.5}$$

$$= (1+pxyz)\begin{vmatrix}1 & x & x^2\\1 & y & y^2\\1 & z & z^2\end{vmatrix} \tag{1.108.6}$$

By applying row reduction

$$= (1+pxyz)\begin{vmatrix}1 & x & x^2\\1 & y & y^2\\1 & z & z^2\end{vmatrix} \tag{1.108.7}$$

$$\xleftrightarrow{R_2\leftarrow R_2-R_3} (1+pxyz)\begin{vmatrix}1 & x & x^2\\0 & y-z & y^2-z^2\\1 & z & z^2\end{vmatrix} \tag{1.108.8}$$

$$\xleftrightarrow{R_3\leftarrow R_3-R_1} (1+pxyz)\begin{vmatrix}1 & x & x^2\\0 & y-z & y^2-z^2\\0 & z-x & z^2-x^2\end{vmatrix} \tag{1.108.9}$$

By using scalar multiplication property

$$= (1+pxyz)(y-z)(z-x)\begin{vmatrix}1 & x & x^2\\0 & 1 & y+z\\0 & 1 & z+x\end{vmatrix} \tag{1.108.10}$$

By applying the determinant formula

$$= (1 + pxyz)(y - z)(z - x)(z + x - y - z) \quad (1.108.11)$$

$$= (1 + pxyz)(x - y)(y - z)(z - x) \quad (1.108.12)$$

$$= RHS \quad (1.108.13)$$

Hence Proved.

1.109. $\begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix}$=3(a+b+c)(ab+bc+ca)

Solution: Applying transformation:

$$\begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix} \quad (1.109.1)$$

$$\xleftrightarrow{C_1 \leftarrow C1+C_2+C_3} \begin{vmatrix} a+b+c & -a+b & -a+c \\ a+b+c & 3b & -b+c \\ a+b+c & -c+b & 3c \end{vmatrix} \quad (1.109.2)$$

$$\xleftrightarrow[R_2 \leftarrow R_2 - R_1]{R_3 \leftarrow R_3 - R_2} (a+b+c) \begin{vmatrix} 1 & -a+b & -a+c \\ 0 & a+2b & a-b \\ 0 & -2b-c & b+2c \end{vmatrix} \quad (1.109.3)$$

$$= (a+b+c)\{(a+2b)(b+2c)+(2b+c)(a-b)\}$$

$$= (a+b+c)\{ab+2ac+2b^2+4bc+2ab-2b^2+ca-cb\}$$

$$= (a+b+c)(3ab+3bc+3ca)$$

$$= 3(a+b+c)(ab+bc+ca) \quad (1.109.4)$$

1.110. Find the QR decomposition of $\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$

Solution: Let

$$\alpha = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (1.110.1)$$

$$\beta = \begin{pmatrix} -1 \\ 3 \end{pmatrix} \quad (1.110.2)$$

We can express these as

$$\alpha = k_1 \mathbf{u}_1 \quad (1.110.3)$$

$$\beta = r_1 \mathbf{u}_1 + k_2 \mathbf{u}_2 \quad (1.110.4)$$

where

$$k_1 = \|\alpha\| \quad (1.110.5)$$

$$\mathbf{u}_1 = \frac{\alpha}{k_1} \quad (1.110.6)$$

$$r_1 = \frac{\mathbf{u}_1^T \beta}{\|\mathbf{u}_1\|^2} \quad (1.110.7)$$

$$\mathbf{u}_2 = \frac{\beta - r_1 \mathbf{u}_1}{\|\beta - r_1 \mathbf{u}_1\|} \quad (1.110.8)$$

$$k_2 = \mathbf{u}_2^T \beta \quad (1.110.9)$$

From (1.110.3) and (1.110.4),

$$\begin{pmatrix} \alpha & \beta \end{pmatrix} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{pmatrix} \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \quad (1.110.10)$$

$$\begin{pmatrix} \alpha & \beta \end{pmatrix} = \mathbf{QR} \quad (1.110.11)$$

From above we can see that $\mathbf{R}$ is an upper triangular matrix and

$$\mathbf{Q}^T \mathbf{Q} = \mathbf{I} \quad (1.110.12)$$

Now by using equations (1.110.5) to (1.110.9)

$$k_1 = \sqrt{5} \quad (1.110.13)$$

$$\mathbf{u}_1 = \sqrt{\frac{1}{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad (1.110.14)$$

$$r_1 = \sqrt{5} \quad (1.110.15)$$

$$\mathbf{u}_2 = \sqrt{\frac{1}{5}} \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad (1.110.16)$$

$$k_2 = \sqrt{5} \quad (1.110.17)$$

Thus obtained QR decomposition is

$$\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \sqrt{5} & \sqrt{5} \\ 0 & \sqrt{5} \end{pmatrix} \quad (1.110.18)$$

1.111. Find QR decomposition of $\begin{pmatrix} 2 & 3 \\ 3 & -4 \end{pmatrix}$

Solution: Let $\mathbf{a}$ and $\mathbf{b}$ be the column vectors of the given matrix.

$$\mathbf{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad (1.111.1)$$

$$\mathbf{b} = \begin{pmatrix} 3 \\ -4 \end{pmatrix} \quad (1.111.2)$$

The column vectors can be expressed as follows,

$$\mathbf{a} = k_1\mathbf{u}_1 \tag{1.111.3}$$
$$\mathbf{b} = r_1\mathbf{u}_1 + k_2\mathbf{u}_2 \tag{1.111.4}$$

Here,

$$k_1 = \|\mathbf{a}\| \tag{1.111.5}$$
$$\mathbf{u}_1 = \frac{\mathbf{a}}{k_1} \tag{1.111.6}$$
$$r_1 = \frac{\mathbf{u}_1^T\mathbf{b}}{\|\mathbf{u}_1\|^2} \tag{1.111.7}$$
$$\mathbf{u}_2 = \frac{\mathbf{b} - r_1\mathbf{u}_1}{\|\mathbf{b} - r_1\mathbf{u}_1\|} \tag{1.111.8}$$
$$k_2 = \mathbf{u}_2^T\mathbf{b} \tag{1.111.9}$$

The (1.111.3) and (1.111.4) can be written as,

$$\begin{pmatrix}\mathbf{a} & \mathbf{b}\end{pmatrix} = \begin{pmatrix}\mathbf{u}_1 & \mathbf{u}_2\end{pmatrix}\begin{pmatrix}k_1 & r_1 \\ 0 & k_2\end{pmatrix} \tag{1.111.10}$$
$$\begin{pmatrix}\mathbf{a} & \mathbf{b}\end{pmatrix} = \mathbf{QR} \tag{1.111.11}$$

Now, $\mathbf{R}$ is an upper triangular matrix and also,

$$\mathbf{Q}^T\mathbf{Q} = \mathbf{I} \tag{1.111.12}$$

Now using equations (1.111.5) to (1.111.9) we get,

$$k_1 = \sqrt{2^2 + 3^2} = \sqrt{13} \tag{1.111.13}$$
$$\mathbf{u}_1 = \frac{1}{\sqrt{13}}\begin{pmatrix}2 \\ 3\end{pmatrix} \tag{1.111.14}$$
$$r_1 = \begin{pmatrix}\frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}}\end{pmatrix}\begin{pmatrix}3 \\ -4\end{pmatrix} = -\frac{6}{\sqrt{13}} \tag{1.111.15}$$
$$\mathbf{u}_2 = \frac{1}{\sqrt{13}}\begin{pmatrix}3 \\ -2\end{pmatrix} \tag{1.111.16}$$
$$k_2 = \begin{pmatrix}\frac{3}{\sqrt{13}} & -\frac{2}{\sqrt{13}}\end{pmatrix}\begin{pmatrix}3 \\ -4\end{pmatrix} = \frac{17}{\sqrt{13}} \tag{1.111.17}$$

Thus putting the values from (1.111.13) to (1.111.17) in (1.111.11) we obtain QR decomposition,

$$\begin{pmatrix}2 & 3 \\ 3 & -4\end{pmatrix} = \begin{pmatrix}\frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & -\frac{2}{\sqrt{13}}\end{pmatrix}\begin{pmatrix}\sqrt{13} & -\frac{6}{\sqrt{13}} \\ 0 & \frac{17}{\sqrt{13}}\end{pmatrix} \tag{1.111.18}$$

1.112. Find the QR decomposition of $\begin{pmatrix}3 & 2 \\ 1 & 4\end{pmatrix}$

Solution:
Let $\mathbf{c_1}$ and $\mathbf{c_2}$ be the column vectors of the given matrix.

$$\mathbf{c_1} = \begin{pmatrix}3 \\ 1\end{pmatrix} \tag{1.112.1}$$
$$\mathbf{c_2} = \begin{pmatrix}2 \\ 4\end{pmatrix} \tag{1.112.2}$$

The column vectors can be represented as,

$$\mathbf{c_1} = k_1\mathbf{u}_1 \tag{1.112.3}$$
$$\mathbf{c_2} = r_1\mathbf{u}_1 + k_2\mathbf{u}_2 \tag{1.112.4}$$

where,

$$k_1 = \|\mathbf{c_1}\| \tag{1.112.5}$$
$$\mathbf{u_1} = \frac{\mathbf{c_1}}{k_1} \tag{1.112.6}$$
$$r_1 = \frac{\mathbf{u}_1^T\mathbf{c_2}}{\|\mathbf{u}_1\|^2} \tag{1.112.7}$$
$$\mathbf{u_2} = \frac{\mathbf{c_2} - r_1\mathbf{u_1}}{\|\mathbf{c_2} - r_1\mathbf{u_1}\|} \tag{1.112.8}$$
$$k_2 = \mathbf{u_2}^T\mathbf{c_2} \tag{1.112.9}$$

From (1.112.3) and (1.112.4),

$$\begin{pmatrix}\mathbf{c_1} & \mathbf{c_2}\end{pmatrix} = \begin{pmatrix}\mathbf{u}_1 & \mathbf{u}_2\end{pmatrix}\begin{pmatrix}k_1 & r_1 \\ 0 & k_2\end{pmatrix} \tag{1.112.10}$$
$$\begin{pmatrix}\mathbf{c_1} & \mathbf{c_2}\end{pmatrix} = \mathbf{QR} \tag{1.112.11}$$

Where $\mathbf{R}$ is an upper triangular matrix and

$$\mathbf{Q}^T\mathbf{Q} = \mathbf{I} \tag{1.112.12}$$

Using equations (1.112.5) to (1.112.9) we get,

$$k_1 = \sqrt{3^2 + 1^2} = \sqrt{10} \tag{1.112.13}$$
$$\mathbf{u_1} = \frac{1}{\sqrt{10}}\begin{pmatrix}3 \\ 1\end{pmatrix} = \begin{pmatrix}\frac{3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}}\end{pmatrix} \tag{1.112.14}$$
$$r_1 = \begin{pmatrix}\frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}}\end{pmatrix}\begin{pmatrix}2 \\ 4\end{pmatrix} = \sqrt{10} \tag{1.112.15}$$
$$\mathbf{u_2} = \begin{pmatrix}\frac{-1}{\sqrt{10}} \\ \frac{3}{\sqrt{10}}\end{pmatrix} \tag{1.112.16}$$
$$k_2 = \begin{pmatrix}\frac{-1}{\sqrt{10}} & \frac{3}{\sqrt{10}}\end{pmatrix}\begin{pmatrix}2 \\ 4\end{pmatrix} = \sqrt{10} \tag{1.112.17}$$

Now putting the values from (1.112.13) to (1.112.17), we obtain the QR decomposition of given matrix,

$$\begin{pmatrix}3 & 2 \\ 1 & 4\end{pmatrix} = \begin{pmatrix}\frac{3}{\sqrt{10}} & \frac{-1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}}\end{pmatrix}\begin{pmatrix}\sqrt{10} & \sqrt{10} \\ 0 & \sqrt{10}\end{pmatrix} \tag{1.112.18}$$

1.113. Find QR decomposition of $\begin{pmatrix} 4 & 3 \\ 5 & -2 \end{pmatrix}$

Solution: The QR decomposition of a matrix is a decomposition of the matrix into an orthogonal matrix and an upper triangular matrix. A QR decomposition of a real square matrix A is a decomposition of A as

$$\mathbf{A} = \mathbf{QR} \tag{1.113.1}$$

where $\mathbf{Q}$ is an orthogonal matrix and $\mathbf{R}$ is an upper triangular matrix Given

$$\mathbf{A} = \begin{pmatrix} 4 & 3 \\ 5 & -2 \end{pmatrix} \tag{1.113.2}$$

Let $\mathbf{a}$ and $\mathbf{b}$ be the column vectors of the given matrix

$$\mathbf{a} = \begin{pmatrix} 4 \\ 5 \end{pmatrix} \tag{1.113.3}$$

$$\mathbf{b} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \tag{1.113.4}$$

The above column vectors (1.113.3) ,(1.113.4) can be expressed as ,

$$\mathbf{a} = t_1\mathbf{u}_1 \tag{1.113.5}$$

$$\mathbf{b} = s_1\mathbf{u}_1 + t_2\mathbf{u}_2 \tag{1.113.6}$$

Where,

$$t_1 = \|\mathbf{a}\| \tag{1.113.7}$$

$$\mathbf{u}_1 = \frac{\mathbf{a}}{t_1} \tag{1.113.8}$$

$$s_1 = \frac{\mathbf{u}_1^T\mathbf{b}}{\|\mathbf{u}_1\|^2} \tag{1.113.9}$$

$$\mathbf{u}_2 = \frac{\mathbf{b} - s_1\mathbf{u}_1}{\|\mathbf{b} - s_1\mathbf{u}_1\|} \tag{1.113.10}$$

$$t_2 = \mathbf{u}_2^T\mathbf{b} \tag{1.113.11}$$

The (1.113.5) and (1.113.6) can be written as,

$$\begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{pmatrix}\begin{pmatrix} t_1 & s_1 \\ 0 & t_2 \end{pmatrix} \tag{1.113.12}$$

$$\begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} = \mathbf{QR} \tag{1.113.13}$$

Here, $\mathbf{R}$ is an upper triangular matrix and $\mathbf{Q}$ is an orthogonal matrix such that

$$\mathbf{Q}^T\mathbf{Q} = \mathbf{I} \tag{1.113.14}$$

Now using equations from (1.113.7) to (1.113.11) we get,

$$t_1 = \sqrt{4^2 + 5^2} = \sqrt{41} \tag{1.113.15}$$

$$\mathbf{u}_1 = \frac{1}{\sqrt{41}}\begin{pmatrix} 4 \\ 5 \end{pmatrix} \tag{1.113.16}$$

$$s_1 = \begin{pmatrix} \frac{4}{\sqrt{41}} & \frac{5}{\sqrt{41}} \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix} = \frac{2}{\sqrt{41}} \tag{1.113.17}$$

$$\mathbf{u}_2 = \frac{1}{\sqrt{41}}\begin{pmatrix} 5 \\ -4 \end{pmatrix} \tag{1.113.18}$$

$$t_2 = \begin{pmatrix} \frac{5}{\sqrt{41}} & \frac{-4}{\sqrt{41}} \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix} = \frac{23}{\sqrt{41}} \tag{1.113.19}$$

Substituting the values from (1.113.15) to (1.113.19) in (1.113.13) we obtain QR decomposition as,

$$\begin{pmatrix} 4 & 3 \\ 5 & -2 \end{pmatrix} = \begin{pmatrix} \frac{4}{\sqrt{41}} & \frac{5}{\sqrt{41}} \\ \frac{5}{\sqrt{41}} & \frac{-4}{\sqrt{41}} \end{pmatrix}\begin{pmatrix} \sqrt{41} & \frac{2}{\sqrt{41}} \\ 0 & \frac{23}{\sqrt{41}} \end{pmatrix} \tag{1.113.20}$$

1.114. Perform the QR decomposition of matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} \tag{1.114.1}$$

Solution:
If α and β are the columns of a (2×2) matrix **A**,
then **A** can be decomposed as

$$\mathbf{A} = \mathbf{QR} \tag{1.114.2}$$

$$\text{where, } \mathbf{U} = \begin{pmatrix} \mathbf{u_1} & \mathbf{u_2} \end{pmatrix}, \tag{1.114.3}$$

$$\text{uppertriangular matrix } \mathbf{R} = \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \tag{1.114.4}$$

$$k_1 = \|\alpha\|, \mathbf{u_1} = \frac{\alpha}{k_1} \tag{1.114.5}$$

$$r_1 = \frac{\mathbf{u_1}^T\beta}{\|\mathbf{u_1}\|^2} \tag{1.114.6}$$

$$\mathbf{u_2} = \frac{\beta - r_1\mathbf{u_1}}{\|\beta - r_1\mathbf{u_1}\|}, k_2 = \mathbf{u_2}^T\beta \tag{1.114.7}$$

$$\alpha = \begin{pmatrix}1\\3\end{pmatrix}, \beta = \begin{pmatrix}2\\1\end{pmatrix} \tag{1.114.8}$$

$$\text{From, (1.114.5), } k_1 = ||\alpha|| = \sqrt{10} \tag{1.114.9}$$

$$\text{and } \mathbf{u_1} = \frac{1}{\sqrt{10}}\begin{pmatrix}1\\3\end{pmatrix} \tag{1.114.10}$$

$$\text{From (1.114.6), } r_1 = \frac{1}{\sqrt{10}}\begin{pmatrix}1 & 3\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix} = \frac{5}{\sqrt{10}} \tag{1.114.11}$$

$$\beta - r_1\mathbf{u_1} = \begin{pmatrix}2\\1\end{pmatrix} - \frac{5}{\sqrt{10}}\frac{1}{\sqrt{10}}\begin{pmatrix}1\\3\end{pmatrix} \tag{1.114.12}$$

$$= \begin{pmatrix}\frac{3}{2}\\\frac{-1}{2}\end{pmatrix} \tag{1.114.13}$$

$$\text{From (1.114.7), } \mathbf{u_2} = \frac{\begin{pmatrix}\frac{3}{2}\\\frac{-1}{2}\end{pmatrix}}{\sqrt{\frac{9}{4}+\frac{1}{4}}} \tag{1.114.14}$$

$$\implies \mathbf{u_2} = \begin{pmatrix}\frac{3}{\sqrt{10}}\\\frac{-1}{\sqrt{10}}\end{pmatrix}, \tag{1.114.15}$$

$$k_2 = \begin{pmatrix}\frac{3}{\sqrt{10}} & \frac{-1}{\sqrt{10}}\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix} = \frac{5}{\sqrt{10}} \tag{1.114.16}$$

Note that,

$$\mathbf{Q}^T\mathbf{Q} = \begin{pmatrix}\frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}}\\\frac{3}{\sqrt{10}} & \frac{-1}{\sqrt{10}}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}}\\\frac{3}{\sqrt{10}} & \frac{-1}{\sqrt{10}}\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = \mathbf{I} \tag{1.114.17}$$

The matrix $\mathbf{A}$ can now be rewritten using (1.114.2) as

$$\begin{pmatrix}1 & 2\\3 & 1\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}}\\\frac{3}{\sqrt{10}} & \frac{-1}{\sqrt{10}}\end{pmatrix}\begin{pmatrix}\sqrt{10} & \frac{5}{\sqrt{10}}\\0 & \frac{5}{\sqrt{10}}\end{pmatrix} \tag{1.114.18}$$

1.115. Find the QR decomposition of the given matrix.

$$\begin{pmatrix}1 & 2\\2 & -2\end{pmatrix} \tag{1.115.1}$$

Solution: QR decomposition of a square matrix is given by,

$$\mathbf{A} = \mathbf{QR} \tag{1.115.2}$$

where $\mathbf{Q}$ is an orthogonal matrix and $\mathbf{R}$ is an upper triangular matrix.

Given matrix,

$$\mathbf{A} = \begin{pmatrix}1 & 2\\2 & -2\end{pmatrix} \tag{1.115.3}$$

The column vectors of the matrix is given by,

$$\mathbf{a} = \begin{pmatrix}1\\2\end{pmatrix} \quad \mathbf{b} = \begin{pmatrix}2\\-2\end{pmatrix} \tag{1.115.4}$$

Equation (1.115.3) can be written in form of (1.115.4) as,

$$\begin{pmatrix}\mathbf{a} & \mathbf{b}\end{pmatrix} = \begin{pmatrix}\mathbf{q_1} & \mathbf{q_2}\end{pmatrix}\begin{pmatrix}u_1 & u_3\\0 & u_2\end{pmatrix} = \mathbf{QR} \tag{1.115.5}$$

where,

$$u_1 = ||\mathbf{a}|| = \sqrt{1^2+2^2} = \sqrt{5} \tag{1.115.6}$$

$$\mathbf{q_1} = \frac{\mathbf{a}}{u_1} = \begin{pmatrix}\frac{1}{\sqrt{5}}\\\frac{2}{\sqrt{5}}\end{pmatrix} \tag{1.115.7}$$

$$u_3 = \frac{\mathbf{q_1}^T\mathbf{b}}{||\mathbf{q_1}||^2} = \begin{pmatrix}\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}}\end{pmatrix}\begin{pmatrix}2\\-2\end{pmatrix} = \frac{-2}{\sqrt{5}} \tag{1.115.8}$$

$$\mathbf{q_2} = \frac{\mathbf{b} - u_3\mathbf{q_1}}{||\mathbf{b} - u_3\mathbf{q_1}||} = \begin{pmatrix}\frac{2}{\sqrt{5}}\\-\frac{1}{\sqrt{5}}\end{pmatrix} \tag{1.115.9}$$

$$u_2 = \mathbf{q_2}^T\mathbf{b} = \begin{pmatrix}\frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}}\end{pmatrix}\begin{pmatrix}2\\-2\end{pmatrix} = \frac{6}{\sqrt{5}} \tag{1.115.10}$$

Substituting equation (1.115.6) to (1.115.10) in (1.115.5),

$$\begin{pmatrix}1 & 2\\2 & -2\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}}\\\frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}}\end{pmatrix}\begin{pmatrix}\sqrt{5} & -\frac{2}{\sqrt{5}}\\0 & \frac{6}{\sqrt{5}}\end{pmatrix} \tag{1.115.11}$$

The QR decomposition is,

$$\begin{pmatrix}1 & 2\\2 & -2\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}}\\\frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}}\end{pmatrix}\begin{pmatrix}\sqrt{5} & -\frac{2}{\sqrt{5}}\\0 & \frac{6}{\sqrt{5}}\end{pmatrix} \tag{1.115.12}$$

1.116. Find the QR decomposition on a given 2×2 matrix.

$$\begin{pmatrix}2 & 1\\1 & -2\end{pmatrix} \tag{1.116.1}$$

Solution: The QR decomposition of a matrix is a decomposition of the matrix into an orthogonal matrix and an upper triangular matrix. QR decomposition of a square matrix is given by,

$$\mathbf{A} = \mathbf{QR} \tag{1.116.2}$$

Here $\mathbf{Q}$ is an orthogonal matrix and $\mathbf{R}$ is an upper triangular matrix.

Given matrix,

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \tag{1.116.3}$$

The column vectors of the matrix is given by,

$$\mathbf{a} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \tag{1.116.4}$$

Equation (1.116.3) can be written in **QR** form as:

$$\mathbf{QR} = \begin{pmatrix} \mathbf{q_1} & \mathbf{q_2} \end{pmatrix} \begin{pmatrix} u_1 & u_3 \\ 0 & u_2 \end{pmatrix} \tag{1.116.5}$$

Now,

$$u_1 = \|\mathbf{a}\| = \sqrt{1^2 + 2^2} = \sqrt{5} \tag{1.116.6}$$

$$\mathbf{q_1} = \frac{\mathbf{a}}{u_1} = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} \tag{1.116.7}$$

$$u_3 = \frac{\mathbf{q_1}^T \mathbf{b}}{\|\mathbf{q_1}\|^2} = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = 0 \tag{1.116.8}$$

$$\mathbf{q_2} = \frac{\mathbf{b} - u_3 \mathbf{q_1}}{\|\mathbf{b} - u_3 \mathbf{q_1}\|} = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{pmatrix} \tag{1.116.9}$$

$$u_2 = \mathbf{q_2}^T \mathbf{b} = \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \sqrt{5} \tag{1.116.10}$$

Substituting equation (1.116.6) to (1.116.10) in (1.116.5),to obtain the QR Decomposition of the given matrix as:

$$\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \sqrt{5} & 0 \\ 0 & \sqrt{5} \end{pmatrix} \tag{1.116.11}$$

In equation (1.116.11) $\mathbf{R}$ is diagonal because the columns and rows are orthogonal to each other.

1.117. Perform QR decomposition on matrix $\mathbf{A}$

$$\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 3 & -5 \end{pmatrix} \tag{1.117.1}$$

Solution:
The columns of matrix $\mathbf{A}$ can be represented in α and β as

$$\implies \alpha = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \tag{1.117.2}$$

$$\beta = \begin{pmatrix} 4 \\ -5 \end{pmatrix} \tag{1.117.3}$$

For QR decomposition, matrix $\mathbf{A}$ can be expressed as

$$\mathbf{A} = \mathbf{QR} \tag{1.117.4}$$

where, $\mathbf{Q}$ and $\mathbf{R}$ are expressed as

$$\mathbf{Q} = \begin{pmatrix} \mathbf{u_1} & \mathbf{u_2} \end{pmatrix} \tag{1.117.5}$$

$$\mathbf{R} = \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \tag{1.117.6}$$

Note that $\mathbf{R}$ is an upper triangular matrix.
Now,we calculate

$$k_1 = \|\alpha\| = \sqrt{10} \tag{1.117.7}$$

$$\mathbf{u_1} = \frac{\alpha}{k_1} = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \tag{1.117.8}$$

$$r_1 = \frac{\mathbf{u_1}^T \beta}{\|\mathbf{u_1}\|^2} = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & 3 \end{pmatrix} \begin{pmatrix} 4 \\ -5 \end{pmatrix} \tag{1.117.9}$$

$$\implies r_1 = -\frac{11}{\sqrt{10}} \tag{1.117.10}$$

$$\mathbf{u_2} = \frac{\beta - r_1 \mathbf{u_1}}{\|\beta - r_1 \mathbf{u_1}\|} \tag{1.117.11}$$

Consider

$$\beta - r_1 \mathbf{u_1} = \begin{pmatrix} 4 \\ -5 \end{pmatrix} + \frac{11}{\sqrt{10}} \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \tag{1.117.12}$$

$$\implies \beta - r_1 \mathbf{u_1} = \begin{pmatrix} \frac{51}{10} \\ -\frac{17}{10} \end{pmatrix} \tag{1.117.13}$$

$$\|\beta - r_1 \mathbf{u_1}\| = \frac{17}{\sqrt{10}} \tag{1.117.14}$$

Substitute (1.117.13),(1.117.14) in (1.117.11), we get

$$\mathbf{u_2} = \begin{pmatrix} \frac{3}{\sqrt{10}} \\ -\frac{1}{\sqrt{10}} \end{pmatrix} \tag{1.117.15}$$

$$k_2 = \mathbf{u_2}^T \beta = \begin{pmatrix} \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \end{pmatrix} \begin{pmatrix} 4 \\ -5 \end{pmatrix} \tag{1.117.16}$$

$$\implies k_2 = \frac{17}{\sqrt{10}} \tag{1.117.17}$$

Therefore, from (1.117.5) and (1.117.6)

$$\mathbf{Q} = \begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \end{pmatrix} \tag{1.117.18}$$

$$\mathbf{R} = \begin{pmatrix} \sqrt{10} & -\frac{11}{\sqrt{10}} \\ 0 & \frac{17}{\sqrt{10}} \end{pmatrix} \tag{1.117.19}$$

Note that,

$$\mathbf{Q}^T\mathbf{Q} = \begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I} \tag{1.117.20}$$

Now matrix $\mathbf{A}$ can be written as (1.117.4)

$$\begin{pmatrix} 1 & 4 \\ 3 & -5 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \end{pmatrix}\begin{pmatrix} \sqrt{10} & -\frac{11}{\sqrt{10}} \\ 0 & \frac{17}{\sqrt{10}} \end{pmatrix} \tag{1.117.21}$$

1.118. Perform QR decomposition on matrix $\mathbf{A}$

$$\mathbf{A} = \begin{pmatrix} 1 & -7 \\ 3 & 1 \end{pmatrix} \tag{1.118.1}$$

Solution: The columns of matrix $\mathbf{A}$ can be represented in α and β as

$$\implies \alpha = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \tag{1.118.2}$$

$$\beta = \begin{pmatrix} -7 \\ 1 \end{pmatrix} \tag{1.118.3}$$

For QR decomposition, matrix $\mathbf{A}$ can be expressed as

$$\mathbf{A} = \mathbf{QR} \tag{1.118.4}$$

where, $\mathbf{Q}$ and $\mathbf{R}$ are expressed as

$$\mathbf{Q} = \begin{pmatrix} \mathbf{u_1} & \mathbf{u_2} \end{pmatrix} \tag{1.118.5}$$

$$\mathbf{R} = \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \tag{1.118.6}$$

Note that $\mathbf{R}$ is an upper triangular matrix.

Now,we calculate

$$k_1 = \|\alpha\| = \sqrt{10} \tag{1.118.7}$$

$$\mathbf{u_1} = \frac{\alpha}{k_1} = \frac{1}{\sqrt{10}}\begin{pmatrix} 1 \\ 3 \end{pmatrix} \tag{1.118.8}$$

$$r_1 = \frac{\mathbf{u_1}^T\beta}{\|\mathbf{u_1}\|^2} = \frac{1}{\sqrt{10}}\begin{pmatrix} 1 & 3 \end{pmatrix}\begin{pmatrix} -7 \\ 1 \end{pmatrix} \tag{1.118.9}$$

$$\implies r_1 = -\frac{4}{\sqrt{10}} \tag{1.118.10}$$

$$\mathbf{u_2} = \frac{\beta - r_1\mathbf{u_1}}{\|\beta - r_1\mathbf{u_1}\|} \tag{1.118.11}$$

Consider

$$\beta - r_1\mathbf{u_1} = \begin{pmatrix} -7 \\ 1 \end{pmatrix} + \frac{4}{\sqrt{10}}\frac{1}{\sqrt{10}}\begin{pmatrix} 1 \\ 3 \end{pmatrix} \tag{1.118.12}$$

$$\implies \beta - r_1\mathbf{u_1} = \begin{pmatrix} \frac{-66}{10} \\ \frac{22}{10} \end{pmatrix} \tag{1.118.13}$$

$$\|\beta - r_1\mathbf{u_1}\| = \frac{22}{\sqrt{10}} \tag{1.118.14}$$

Substitute (1.118.13),(1.118.14) in (1.118.11), we get

$$\mathbf{u_2} = \begin{pmatrix} -\frac{3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{pmatrix} \tag{1.118.15}$$

$$k_2 = \mathbf{u_2}^T\beta = \begin{pmatrix} -\frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{pmatrix}\begin{pmatrix} -7 \\ 1 \end{pmatrix} \tag{1.118.16}$$

$$\implies k_2 = \frac{22}{\sqrt{10}} \tag{1.118.17}$$

Therefore, from (1.118.5) and (1.118.6)

$$\mathbf{Q} = \begin{pmatrix} \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{pmatrix} \tag{1.118.18}$$

$$\mathbf{R} = \begin{pmatrix} \sqrt{10} & -\frac{4}{\sqrt{10}} \\ 0 & \frac{22}{\sqrt{10}} \end{pmatrix} \tag{1.118.19}$$

Note that,

$$\mathbf{Q}^T\mathbf{Q} = \begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ -\frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I} \tag{1.118.20}$$

Now matrix $\mathbf{A}$ can be written as (1.118.4)

$$\begin{pmatrix} 1 & -7 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{pmatrix}\begin{pmatrix} \sqrt{10} & -\frac{4}{\sqrt{10}} \\ 0 & \frac{22}{\sqrt{10}} \end{pmatrix} \tag{1.118.21}$$

1.119. Given a matrix $\mathbf{A} = \begin{pmatrix} 3 & -2 \\ 4 & -2 \end{pmatrix}$, find its **QR** decomposition **Solution:**
Given

$$\mathbf{A} = \begin{pmatrix} 3 & -2 \\ 4 & -2 \end{pmatrix} \tag{1.119.1}$$

Let us use the Gram-Schmidt approach to obtain QR decomposition of **A**. Consider column vectors say $\mathbf{a}_1$ and $\mathbf{a}_2$ of **A** which is given by

$$\mathbf{a}_1 = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \tag{1.119.2}$$

$$\mathbf{a}_2 = \begin{pmatrix} -2 \\ -2 \end{pmatrix} \tag{1.119.3}$$

$$\mathbf{u}_1 = \mathbf{a}_1 = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \tag{1.119.4}$$

$$\mathbf{e}_1 = \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} = \begin{pmatrix} \frac{3}{5} \\ \frac{4}{5} \end{pmatrix} \tag{1.119.5}$$

$$\mathbf{u}_2 = \mathbf{a}_2 - \left(\mathbf{a}_2^T.\mathbf{e}_1\right)\mathbf{e}_1 \tag{1.119.6}$$

$$= \begin{pmatrix} -2 \\ -2 \end{pmatrix} - \left(-\frac{14}{5}\right)\begin{pmatrix} \frac{3}{5} \\ \frac{4}{5} \end{pmatrix} \tag{1.119.7}$$

$$= \begin{pmatrix} -2 \\ -2 \end{pmatrix} - \begin{pmatrix} -\frac{42}{25} \\ -\frac{56}{25} \end{pmatrix} = \begin{pmatrix} -\frac{8}{25} \\ \frac{6}{25} \end{pmatrix} \tag{1.119.8}$$

$$\mathbf{e}_2 = \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} = \begin{pmatrix} -\frac{4}{5} \\ \frac{3}{5} \end{pmatrix} \tag{1.119.9}$$

The matrix **Q** and **R** is given by,

$$\mathbf{Q} = \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix} \tag{1.119.10}$$

$$\mathbf{R} = \begin{pmatrix} \mathbf{a}_1^T.\mathbf{e}_1 & \mathbf{a}_2^T.\mathbf{e}_1 \\ 0 & \mathbf{a}_2^T.\mathbf{e}_2 \end{pmatrix} = \begin{pmatrix} 5 & -\frac{14}{5} \\ 0 & \frac{2}{5} \end{pmatrix} \tag{1.119.11}$$

Hence, the **QR** decomposition of matrix **A** is as follows:

$$\begin{pmatrix} 3 & -2 \\ 4 & -2 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}\begin{pmatrix} 5 & -\frac{14}{5} \\ 0 & \frac{2}{5} \end{pmatrix} \tag{1.119.12}$$

1.120. Perform the QR decomposition of the matrix $\mathbf{A} = \begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix}$. **Solution:**
Let **a** and **b** are the columns of matrix **A**. The matrix **A** can be decomposed in the form

$$\mathbf{A} = \mathbf{QR} \tag{1.120.1}$$

$$\mathbf{U} = \begin{pmatrix} \mathbf{u_1} & \mathbf{u_2} \end{pmatrix} \tag{1.120.2}$$

$$\mathbf{R} = \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \tag{1.120.3}$$

where

$$k_1 = \|\mathbf{a}\| \tag{1.120.4}$$

$$\mathbf{u_1} = \frac{\mathbf{a}}{k_1} \tag{1.120.5}$$

$$r_1 = \frac{\mathbf{u_1}^T\mathbf{b}}{\|u_1\|^2} \tag{1.120.6}$$

$$\mathbf{u_2} = \frac{\mathbf{b} - r_1\mathbf{u_1}}{\|\mathbf{b} - r_1\mathbf{u_1}\|} \tag{1.120.7}$$

$$k_2 = \mathbf{u_2}^T\mathbf{b} \tag{1.120.8}$$

Then the given matrix can be represented as,

$$\begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} = \begin{pmatrix} \mathbf{u_1} & \mathbf{u_2} \end{pmatrix}\begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \tag{1.120.9}$$

The the columns of matrix $\mathbf{A} = \begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix}$ are **a** and **b** where

$$\mathbf{a} = \begin{pmatrix} 3 \\ -4 \end{pmatrix} \tag{1.120.10}$$

$$\mathbf{b} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \tag{1.120.11}$$

Now for the given matrix, From (1.120.4) and (1.120.5)

$$k_1 = \|\mathbf{a}\| = 5 \tag{1.120.12}$$

$$\mathbf{u_1} = \frac{1}{5}\begin{pmatrix} 3 \\ -4 \end{pmatrix} \tag{1.120.13}$$

From (1.120.6)

$$r_1 = \frac{1}{5}\begin{pmatrix} 3 & -4 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \frac{-11}{5} \tag{1.120.14}$$

From (1.120.7)

$$\mathbf{b} - r_1\mathbf{u_1} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} - \frac{-11}{5}\begin{pmatrix} \frac{3}{5} \\ \frac{-4}{5} \end{pmatrix} \tag{1.120.15}$$

$$\|\mathbf{b} - r_1\mathbf{u_1}\| = \frac{2}{5} \tag{1.120.16}$$

$$\implies \mathbf{u_2} = \frac{5}{2}\begin{pmatrix} \frac{8}{25} \\ \frac{6}{25} \end{pmatrix} \tag{1.120.17}$$

From (1.120.8)

$$k_2 = \mathbf{u_2}^T\mathbf{b} = \begin{pmatrix} \frac{4}{5} & \frac{3}{5} \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \frac{2}{5} \tag{1.120.18}$$

Now we can observe that $\mathbf{Q}^T\mathbf{Q} = \mathbf{Q}\mathbf{Q}^T = I$

$$\begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{-4}{5} & \frac{3}{5} \end{pmatrix}\begin{pmatrix} \frac{3}{5} & \frac{-4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.120.19}$$

From (1.120.9), The matrix $\mathbf{A}$ can now be written as,

$$\mathbf{A} = \begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{-4}{5} & \frac{3}{5} \end{pmatrix}\begin{pmatrix} -5 & \frac{-11}{5} \\ 0 & \frac{2}{5} \end{pmatrix} \tag{1.120.20}$$

1.121. Perform QR decomposition on matrix A given by

$$A = \begin{pmatrix} 3 & -4 \\ -4 & 3 \end{pmatrix}$$

Solution: Representing matrix A in terms of its column vectors as

$$A = \begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} \tag{1.121.1}$$

Let

$$\mathbf{q_1} = \frac{\mathbf{a}}{\|\mathbf{a}\|} \tag{1.121.2}$$

An orthonormal vector to $\mathbf{q_1}$ can be obtained by subtracting the projection of $\mathbf{b}$ on $\mathbf{q_1}$ from $\mathbf{b}$. Thus

$$\mathbf{q_2} = \frac{\mathbf{b} - k\mathbf{q_1}}{\|\mathbf{b} - k\mathbf{q_1}\|} \tag{1.121.3}$$

where

$$k = \frac{\mathbf{b}^T\mathbf{q_1}}{\|\mathbf{q_1}\|^2} \tag{1.121.4}$$

From (1.121.2) and (1.121.3)

$$\mathbf{a} = \|\mathbf{a}\|\,\mathbf{q_1} \tag{1.121.5}$$

$$\mathbf{b} = k\mathbf{q_1} + \|\mathbf{b} - k\mathbf{q_1}\|\,\mathbf{q_2} \tag{1.121.6}$$

$$\implies \begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} = \begin{pmatrix} \mathbf{q_1} & \mathbf{q_2} \end{pmatrix}\begin{pmatrix} \|\mathbf{a}\| & k \\ 0 & \|\mathbf{b} - k\mathbf{q_1}\| \end{pmatrix} \tag{1.121.7}$$

$$\implies \mathbf{A} = \mathbf{QR} \tag{1.121.8}$$

QR decomposition of a matrix A is essentially representation of column vectors of matrix A in terms of linear combination of orthonormal basis of column space of A. For matrix A

$$\mathbf{a} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}, \ \mathbf{b} = \begin{pmatrix} -4 \\ 3 \end{pmatrix} \tag{1.121.9}$$

$$\tag{1.121.10}$$

Let

$$\mathbf{q_1} = \frac{1}{\sqrt{5}}\begin{pmatrix} 3 \\ -4 \end{pmatrix} \tag{1.121.11}$$

$$\tag{1.121.12}$$

From (1.121.3) and (1.121.4)

$$\mathbf{q_2} = \frac{1}{\sqrt{5}}\begin{pmatrix} -4 \\ 3 \end{pmatrix} \tag{1.121.13}$$

$$\implies \mathbf{Q} = \frac{1}{\sqrt{5}}\begin{pmatrix} 3 & -4 \\ -4 & 3 \end{pmatrix} \tag{1.121.14}$$

$$\mathbf{R} = \begin{pmatrix} \sqrt{5} & 0 \\ 0 & \sqrt{5} \end{pmatrix} \tag{1.121.15}$$

Therefore the matrix A can be decomposed as

$$\mathbf{A} = \begin{pmatrix} \frac{3}{\sqrt{5}} & -\frac{4}{\sqrt{5}} \\ -\frac{4}{\sqrt{5}} & \frac{3}{\sqrt{5}} \end{pmatrix}\begin{pmatrix} \sqrt{5} & 0 \\ 0 & \sqrt{5} \end{pmatrix} \tag{1.121.16}$$

1.122. Find the QR Decomposition of matrix,

$$\mathbf{A} = \begin{pmatrix} 2 & -6 \\ 1 & -2 \end{pmatrix} \tag{1.122.1}$$

Solution: Let c_1 and c_2 be the column vectors of given matrix $\mathbf{A}$

$$c_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{1.122.2}$$

$$c_2 = \begin{pmatrix} -6 \\ -2 \end{pmatrix} \tag{1.122.3}$$

We can express the matrix $\mathbf{A}$ as,

$$\mathbf{A} = \mathbf{QR} \tag{1.122.4}$$

Where, $\mathbf{Q}$ is an orthogonal matrix given as,

$$\mathbf{Q} = \begin{pmatrix} \mathbf{u_1} & \mathbf{u_2} \end{pmatrix} \tag{1.122.5}$$

and $\mathbf{R}$ is an upper triangular matrix given as,

$$\mathbf{R} = \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \tag{1.122.6}$$

Now, we can express α and β as,

$$c_1 = k_1\mathbf{u_1} \tag{1.122.7}$$

$$c_2 = r_1\mathbf{u_1} + k_2\mathbf{u_2} \tag{1.122.8}$$

$$\text{where, } k_1 = \|c_1\| = \sqrt{2^2 + (1^2} = \sqrt{5} \tag{1.122.9}$$

Solving equation (1.122.7) for $\mathbf{u_1}$,

$$\mathbf{u_1} = \frac{c_1}{k_1} = \frac{1}{\sqrt{5}}\begin{pmatrix}2\\1\end{pmatrix} \tag{1.122.10}$$

$$\text{Now, } r_1 = \frac{\mathbf{u_1}^T c_2}{\|\mathbf{u_1}\|^2} \tag{1.122.11}$$

$$\implies \frac{\frac{1}{\sqrt{5}}\begin{pmatrix}2 & 1\end{pmatrix}\begin{pmatrix}-6\\-2\end{pmatrix}}{1} \tag{1.122.12}$$

$$\text{Hence, } r_1 = -\frac{-14}{\sqrt{5}} \tag{1.122.13}$$

$$\mathbf{u_2} = \frac{c_2 - r_1\mathbf{u_1}}{\|c_2 - r_1\mathbf{u_1}\|} \tag{1.122.14}$$

$$\implies \frac{\begin{pmatrix}-6\\-2\end{pmatrix} - \left(\frac{-14}{\sqrt{5}}\right)\left(\frac{1}{\sqrt{5}}\begin{pmatrix}2\\1\end{pmatrix}\right)}{\left\|\begin{pmatrix}-6\\-2\end{pmatrix} - \left(-\frac{14}{\sqrt{5}}\frac{1}{\sqrt{5}}\begin{pmatrix}2\\1\end{pmatrix}\right)\right\|} \tag{1.122.15}$$

$$\implies \mathbf{u_2} = \frac{1}{\sqrt{5}}\begin{pmatrix}-1\\2\end{pmatrix} \tag{1.122.16}$$

$$\text{Now, } k_2 = u_2^T c_2 \tag{1.122.17}$$

$$\implies \frac{1}{\sqrt{5}}\begin{pmatrix}-1 & 2\end{pmatrix}\begin{pmatrix}-6\\-2\end{pmatrix} \tag{1.122.18}$$

$$\implies k_2 = \frac{2}{\sqrt{5}} \tag{1.122.19}$$

Hence substituting the values of unknown parameter from equations (1.122.9), (1.122.19), (1.122.10), (1.122.16) and (1.122.13) to equation (1.122.5) and (1.122.6) we get,

$$\mathbf{Q} = \frac{1}{\sqrt{5}}\begin{pmatrix}2 & -1\\1 & 2\end{pmatrix} \tag{1.122.20}$$

$$\mathbf{R} = \begin{pmatrix}\sqrt{5} & \frac{-14}{\sqrt{5}}\\0 & \frac{2}{\sqrt{5}}\end{pmatrix} \tag{1.122.21}$$

1.123. Find the QR decompostion of

$$\mathbf{A} = \begin{pmatrix}4 & 7\\3 & 5\end{pmatrix} \tag{1.123.1}$$

Solution: If $\mathbf{A} \in \mathbf{R}^{m\times n}$ has linearly independent columns then it can be factored as

$$\mathbf{A} = \mathbf{QR}$$

where $\mathbf{Q}$ is a orthogonal matrix and $\mathbf{R}$ is a upper triangular matrix with non zero diagonal elements

$$\mathbf{A} = \begin{pmatrix}4 & 7\\3 & 5\end{pmatrix} \tag{1.123.2}$$

The column vectors of $\mathbf{A}$ are,

$$\mathbf{a} = \begin{pmatrix}4\\7\end{pmatrix}; \quad \mathbf{b} = \begin{pmatrix}3\\5\end{pmatrix} \tag{1.123.3}$$

(1.123.2) can be written as,

$$\mathbf{QR} = \begin{pmatrix}\mathbf{p_1} & \mathbf{p_2}\end{pmatrix}\begin{pmatrix}u_1 & u_3\\0 & u_2\end{pmatrix} \tag{1.123.4}$$

Now,

$$u_1 = \|a\| = \sqrt{4^2 + 7^2} = \sqrt{65} \tag{1.123.5}$$

$$\mathbf{q_1} = \frac{\mathbf{a}}{u_1} = \begin{pmatrix}\frac{4}{\sqrt{65}}\\\frac{7}{\sqrt{65}}\end{pmatrix} \tag{1.123.6}$$

$$u_3 = \frac{\mathbf{q_1}^T\mathbf{b}}{\|\mathbf{q_1}\|^2} = \begin{pmatrix}\frac{4}{\sqrt{65}} & \frac{7}{\sqrt{65}}\end{pmatrix}\begin{pmatrix}3\\5\end{pmatrix} = \frac{47}{\sqrt{65}} \tag{1.123.7}$$

$$\mathbf{q_2} = \frac{\mathbf{b} - u_3\mathbf{q_1}}{\|\mathbf{b} - u_3\mathbf{q_1}\|} = \begin{pmatrix}\frac{7}{\sqrt{65}}\\-\frac{4}{\sqrt{65}}\end{pmatrix} \tag{1.123.8}$$

$$u_2 = \mathbf{q_2}^T\mathbf{b} = \begin{pmatrix}\frac{7}{\sqrt{65}} & -\frac{4}{\sqrt{65}}\end{pmatrix}\begin{pmatrix}3\\5\end{pmatrix} = \frac{1}{\sqrt{65}} \tag{1.123.9}$$

Substituting (1.123.5) to (1.123.9) in (1.123.4),

$$\begin{pmatrix}4 & 7\\3 & 5\end{pmatrix} = \begin{pmatrix}\frac{4}{\sqrt{65}} & \frac{7}{\sqrt{65}}\\\frac{7}{\sqrt{65}} & -\frac{4}{\sqrt{65}}\end{pmatrix}\begin{pmatrix}\sqrt{65} & \frac{47}{\sqrt{65}}\\0 & \frac{1}{\sqrt{65}}\end{pmatrix} \tag{1.123.10}$$

Which can also be written as,

$$\begin{pmatrix}4 & 7\\3 & 5\end{pmatrix} = \begin{pmatrix}-\frac{4}{\sqrt{65}} & -\frac{7}{\sqrt{65}}\\-\frac{7}{\sqrt{65}} & \frac{4}{\sqrt{65}}\end{pmatrix}\begin{pmatrix}-\sqrt{65} & -\frac{47}{\sqrt{65}}\\0 & -\frac{1}{\sqrt{65}}\end{pmatrix} \tag{1.123.11}$$

1.124. Find the QR Decomposition of matrix,

$$\mathbf{A} = \begin{pmatrix}4 & -3\\6 & -2\end{pmatrix} \tag{1.124.1}$$

Solution: Let c_1 and c_2 be the column vectors of given matrix **A**

$$c_1 = \begin{pmatrix}4\\6\end{pmatrix} \tag{1.124.2}$$

$$c_2 = \begin{pmatrix}-3\\-2\end{pmatrix} \tag{1.124.3}$$

We can express the matrix **A** as,

$$\mathbf{A} = \mathbf{QR} \tag{1.124.4}$$

Where, **Q** is an orthogonal matrix given as,

$$\mathbf{Q} = \begin{pmatrix}\mathbf{u_1} & \mathbf{u_2}\end{pmatrix} \tag{1.124.5}$$

and **R** is an upper triangular matrix given as,

$$\mathbf{R} = \begin{pmatrix}k_1 & r_1\\0 & k_2\end{pmatrix} \tag{1.124.6}$$

Now, we can express α and β as,

$$c_1 = k_1\mathbf{u_1} \tag{1.124.7}$$

$$c_2 = r_1\mathbf{u_1} + k_2\mathbf{u_2} \tag{1.124.8}$$

$$\text{where, } k_1 = \|c_1\| = \sqrt{4^2 + (6^2} = \sqrt{52} \tag{1.124.9}$$

Solving equation (1.124.7) for $\mathbf{u_1}$,

$$\mathbf{u_1} = \frac{c_1}{k_1} = \frac{1}{\sqrt{52}}\begin{pmatrix}4\\6\end{pmatrix} \tag{1.124.10}$$

$$\text{Now, } r_1 = \frac{\mathbf{u_1}^T c_2}{\|\mathbf{u_1}\|^2} \tag{1.124.11}$$

$$\implies \frac{\frac{1}{\sqrt{52}}\begin{pmatrix}4 & 6\end{pmatrix}\begin{pmatrix}-3\\-2\end{pmatrix}}{1} \tag{1.124.12}$$

$$\text{Hence, } r_1 = -\frac{-24}{\sqrt{52}} \tag{1.124.13}$$

$$\mathbf{u_2} = \frac{c_2 - r_1\mathbf{u_1}}{\|c_2 - r_1\mathbf{u_1}\|} \tag{1.124.14}$$

$$\implies \frac{\begin{pmatrix}-3\\-2\end{pmatrix} - \left(\frac{-24}{\sqrt{52}}\right)\left(\frac{1}{\sqrt{52}}\begin{pmatrix}4\\6\end{pmatrix}\right)}{\left\|\begin{pmatrix}-3\\-2\end{pmatrix} - \left(-\frac{24}{\sqrt{52}}\frac{1}{\sqrt{52}}\begin{pmatrix}4\\6\end{pmatrix}\right)\right\|} \tag{1.124.15}$$

$$\implies \mathbf{u_2} = \frac{1}{\sqrt{335}}\begin{pmatrix}-15\\10\end{pmatrix} \tag{1.124.16}$$

$$\text{Now, } k_2 = u_2^T c_2 \tag{1.124.17}$$

$$\implies \frac{1}{\sqrt{335}}\begin{pmatrix}-15 & 10\end{pmatrix}\begin{pmatrix}-3\\-2\end{pmatrix} \tag{1.124.18}$$

$$\implies k_2 = \frac{25}{\sqrt{335}} \tag{1.124.19}$$

Hence substituting the values of unknown parameter from equations (1.124.9), (1.124.19), (1.124.10), (1.124.16) and (1.124.13) to equation (1.124.5) and (1.124.6) we get,

$$\mathbf{Q} = \begin{pmatrix}\frac{4}{\sqrt{52}} & \frac{-15}{\sqrt{335}}\\ \frac{6}{\sqrt{52}} & \frac{10}{\sqrt{335}}\end{pmatrix} \tag{1.124.20}$$

$$\mathbf{R} = \begin{pmatrix}\sqrt{52} & \frac{-24}{\sqrt{52}}\\ 0 & \frac{25}{\sqrt{335}}\end{pmatrix} \tag{1.124.21}$$

1.125. Perform QR decomposition on the matrix **A**

$$\mathbf{A} = \begin{pmatrix}1 & 3\\2 & 4\end{pmatrix} \tag{1.125.1}$$

Solution: The columns of the matrix **A** can be represented as:

$$\alpha = \begin{pmatrix}1\\2\end{pmatrix} \tag{1.125.2}$$

$$\beta = \begin{pmatrix}3\\4\end{pmatrix} \tag{1.125.3}$$

For QR decomposition, matrix A is represented in the form:

$$\mathbf{A} = \mathbf{QR} \tag{1.125.4}$$

where **Q** and **R** are:

$$\mathbf{Q} = \begin{pmatrix}\mathbf{u_1} & \mathbf{u_2}\end{pmatrix} \tag{1.125.5}$$

$$\mathbf{R} = \begin{pmatrix}k_1 & r_1\\0 & k_2\end{pmatrix} \tag{1.125.6}$$

Here **R** is a upper triangular matrix and Q is a orthogonal matrix such that,

$$\mathbf{Q^TQ} = \mathbf{I} \tag{1.125.7}$$

Now we calculate the above values,

$$k_1 = \|\alpha\| \tag{1.125.8}$$
$$\mathbf{u_1} = \frac{\alpha}{k_1} \tag{1.125.9}$$
$$r_1 = \frac{\mathbf{u_1^T}\beta}{\|\mathbf{u_1}\|^2} \tag{1.125.10}$$
$$\mathbf{u_2} = \frac{\beta - r_1\mathbf{u_1}}{\|\beta - r_1\mathbf{u_1}\|} \tag{1.125.11}$$
$$k_2 = \mathbf{u_2^T}\beta \tag{1.125.12}$$

Substituting (1.125.2) and (1.125.3) in the above equations, we get

$$k_1 = \sqrt{1^2 + 2^2} = \sqrt{5} \tag{1.125.13}$$
$$\mathbf{u_1} = \frac{1}{\sqrt{5}}\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{5}}\\\frac{2}{\sqrt{5}}\end{pmatrix} \tag{1.125.14}$$
$$r_1 = \frac{1}{\left(\sqrt{\frac{1}{5}+\frac{4}{5}}\right)^2}\begin{pmatrix}\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}}\end{pmatrix}\begin{pmatrix}3\\4\end{pmatrix} \tag{1.125.15}$$
$$\implies r_1 = \frac{11}{\sqrt{5}} \tag{1.125.16}$$
$$\beta - r_1\mathbf{u_1} = \begin{pmatrix}3\\4\end{pmatrix} - \begin{pmatrix}\frac{11}{5}\\\frac{22}{5}\end{pmatrix} = \begin{pmatrix}\frac{4}{5}\\\frac{-2}{5}\end{pmatrix} \tag{1.125.17}$$
$$\mathbf{u_2} = \frac{\begin{pmatrix}\frac{4}{5}\\\frac{-2}{5}\end{pmatrix}}{\sqrt{\frac{4}{5}^2 + \frac{-2}{5}^2}} = \begin{pmatrix}\frac{2}{\sqrt{5}}\\\frac{-1}{\sqrt{5}}\end{pmatrix} \tag{1.125.18}$$
$$k_2 = \begin{pmatrix}\frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}}\end{pmatrix}\begin{pmatrix}3\\4\end{pmatrix} = \frac{2}{\sqrt{5}} \tag{1.125.19}$$

Therefore, from (1.125.5) and (1.125.6) we get,

$$\mathbf{Q} = \begin{pmatrix}\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}}\\\frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}}\end{pmatrix} \tag{1.125.20}$$
$$\mathbf{R} = \begin{pmatrix}\sqrt{5} & \frac{11}{\sqrt{5}}\\0 & \frac{2}{\sqrt{5}}\end{pmatrix} \tag{1.125.21}$$

where

$$\mathbf{Q}^T\mathbf{Q} = \begin{pmatrix}\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}}\\\frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}}\\\frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}}\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = \mathbf{I} \tag{1.125.22}$$

Therefore matrix **A** in QR decomposed form is,

$$\begin{pmatrix}1 & 3\\2 & 4\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}}\\\frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}}\end{pmatrix}\begin{pmatrix}\sqrt{5} & \frac{11}{\sqrt{5}}\\0 & \frac{2}{\sqrt{5}}\end{pmatrix} \tag{1.125.23}$$

1.126. Find the QR Decomposition of matrix,

$$\mathbf{A} = \begin{pmatrix}3 & -1\\-4 & 2\end{pmatrix} \tag{1.126.1}$$

Solution: Let α and β be the column vectors of given matrix **A**

$$\alpha = \begin{pmatrix}3\\-4\end{pmatrix} \tag{1.126.2}$$
$$\beta = \begin{pmatrix}-1\\2\end{pmatrix} \tag{1.126.3}$$

We can express these as,

$$\alpha = k_1\mathbf{u_1} \tag{1.126.4}$$
$$\beta = r_1\mathbf{u_1} + k_2\mathbf{u_2} \tag{1.126.5}$$

Where,

$$k_1 = \|\alpha\| \tag{1.126.6}$$
$$\mathbf{u_1} = \frac{\alpha}{k_1} \tag{1.126.7}$$
$$r_1 = \frac{\mathbf{u_1}^T\beta}{\|\mathbf{u_1}\|^2} \tag{1.126.8}$$
$$\mathbf{u_2} = \frac{\beta - r_1\mathbf{u_1}}{\|\beta - r_1\mathbf{u_1}\|} \tag{1.126.9}$$
$$k_2 = \mathbf{u_2}^T\beta \tag{1.126.10}$$

From (1.126.4) and (1.126.5)

$$\begin{pmatrix}\alpha & \beta\end{pmatrix} = \begin{pmatrix}\mathbf{u_1} & \mathbf{u_2}\end{pmatrix}\begin{pmatrix}k_1 & r_1\\0 & k_2\end{pmatrix} \tag{1.126.11}$$
$$\mathbf{A} = \mathbf{QR} \tag{1.126.12}$$

From the above equation we can see that **R** is an upper triangular matrix and **Q** is an orthogonal matrix

$$\mathbf{Q}^T\mathbf{Q} = \mathbf{I} \tag{1.126.13}$$

Now by using equations (1.126.2) to (1.126.10)

$$k_1 = \sqrt{9+16} = 5 \qquad (1.126.14)$$

$$\mathbf{u_1} = \frac{1}{5}\begin{pmatrix}3\\-4\end{pmatrix} = \begin{pmatrix}\frac{3}{5}\\ \frac{-4}{5}\end{pmatrix} \qquad (1.126.15)$$

$$r_1 = \frac{\begin{pmatrix}\frac{3}{5} & \frac{-4}{5}\end{pmatrix}\begin{pmatrix}-1\\2\end{pmatrix}}{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = \frac{-11}{5} \qquad (1.126.16)$$

$$\mathbf{u_2} = \frac{\begin{pmatrix}-1\\2\end{pmatrix} - \frac{-11}{5}\begin{pmatrix}\frac{3}{5}\\ \frac{-4}{5}\end{pmatrix}}{\left\|\begin{pmatrix}-1\\2\end{pmatrix} - \frac{-11}{5}\begin{pmatrix}\frac{3}{5}\\ \frac{-4}{5}\end{pmatrix}\right\|} = \begin{pmatrix}\frac{4}{5}\\ \frac{3}{5}\end{pmatrix} \qquad (1.126.17)$$

$$k_2 = \begin{pmatrix}\frac{4}{5} & \frac{3}{5}\end{pmatrix}\begin{pmatrix}-1\\2\end{pmatrix} = \frac{2}{5} \qquad (1.126.18)$$

From equations (1.126.11) and (1.126.12) the obtained **QR** decomposition is

$$\begin{pmatrix}3 & -1\\-4 & 2\end{pmatrix} = \begin{pmatrix}\frac{3}{5} & \frac{4}{5}\\ \frac{-4}{5} & \frac{3}{5}\end{pmatrix}\begin{pmatrix}5 & \frac{-11}{5}\\0 & \frac{2}{5}\end{pmatrix} \qquad (1.126.19)$$

1.127. Perform QR decomposition of matrix $\begin{pmatrix}6 & 1\\-8 & 2\end{pmatrix}$

Solution: Let **a** and **b** are the columns of matrix **A**. The matrix **A** can be decomposed in the form

$$\mathbf{A} = \mathbf{QR} \qquad (1.127.1)$$

$$\mathbf{U} = \begin{pmatrix}\mathbf{u_1} & \mathbf{u_2}\end{pmatrix} \qquad (1.127.2)$$

$$\mathbf{R} = \begin{pmatrix}k_1 & r_1\\0 & k_2\end{pmatrix} \qquad (1.127.3)$$

where

$$k_1 = \|\mathbf{a}\| \qquad (1.127.4)$$

$$\mathbf{u_1} = \frac{\mathbf{a}}{k_1} \qquad (1.127.5)$$

$$r_1 = \frac{\mathbf{u_1}^T\mathbf{b}}{\|u_1\|^2} \qquad (1.127.6)$$

$$\mathbf{u_2} = \frac{\mathbf{b} - r_1\mathbf{u_1}}{\|\mathbf{b} - r_1\mathbf{u_1}\|} \qquad (1.127.7)$$

$$k_2 = \mathbf{u_2}^T\mathbf{b} \qquad (1.127.8)$$

The given matrix can be represented as,

$$\begin{pmatrix}\mathbf{a} & \mathbf{b}\end{pmatrix} = \begin{pmatrix}\mathbf{u_1} & \mathbf{u_2}\end{pmatrix}\begin{pmatrix}k_1 & r_1\\0 & k_2\end{pmatrix} \qquad (1.127.9)$$

The columns of matrix $\mathbf{A} = \begin{pmatrix}6 & 1\\-8 & 2\end{pmatrix}$ are **a** and **b** where

$$\mathbf{a} = \begin{pmatrix}6\\-8\end{pmatrix} \qquad (1.127.10)$$

$$\mathbf{b} = \begin{pmatrix}1\\2\end{pmatrix} \qquad (1.127.11)$$

Now for the given matrix, From (1.127.4) and (1.127.5)

$$k_1 = \|\mathbf{a}\| = 10 \qquad (1.127.12)$$

$$\mathbf{u_1} = \frac{1}{10}\begin{pmatrix}6\\-8\end{pmatrix} \qquad (1.127.13)$$

From (1.127.6)

$$r_1 = \frac{1}{10}\begin{pmatrix}6 & -8\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix} = -1 \qquad (1.127.14)$$

From (1.127.7)

$$\mathbf{b} - r_1\mathbf{u_1} = \begin{pmatrix}1\\2\end{pmatrix} + \begin{pmatrix}\frac{6}{10}\\ \frac{-8}{10}\end{pmatrix} = \begin{pmatrix}\frac{16}{10}\\ \frac{12}{10}\end{pmatrix} \qquad (1.127.15)$$

$$\|\mathbf{b} - r_1\mathbf{u_1}\| = \frac{20}{10} = 2 \qquad (1.127.16)$$

$$\implies \mathbf{u_2} = \frac{1}{10}\begin{pmatrix}8\\6\end{pmatrix} \qquad (1.127.17)$$

From (1.127.8)

$$k_2 = \mathbf{u_2}^T\mathbf{b} = \begin{pmatrix}\frac{8}{10} & \frac{6}{10}\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix} = \frac{20}{10} = 2 \qquad (1.127.18)$$

Now we can observe that $\mathbf{Q}^T\mathbf{Q} = \mathbf{Q}\mathbf{Q}^T = I$

$$\begin{pmatrix}\frac{6}{10} & \frac{8}{10}\\ \frac{-8}{10} & \frac{6}{10}\end{pmatrix}\begin{pmatrix}\frac{6}{10} & \frac{-8}{10}\\ \frac{8}{10} & \frac{6}{10}\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \qquad (1.127.19)$$

From (1.127.9), The matrix **A** can now be written as,

$$\mathbf{A} = \begin{pmatrix}6 & 1\\-8 & 2\end{pmatrix} = \begin{pmatrix}\frac{6}{10} & \frac{8}{10}\\ \frac{-8}{10} & \frac{6}{10}\end{pmatrix}\begin{pmatrix}10 & -1\\0 & 2\end{pmatrix} \qquad (1.127.20)$$

1.128. Find QR decomposition of $\begin{pmatrix}3 & 1\\-4 & 1\end{pmatrix}$ **Solution:** Let α and β be transpose of column vectors of the given matrix.

$$\alpha = \begin{pmatrix}3\\-4\end{pmatrix} \qquad (1.128.1)$$

$$\beta = \begin{pmatrix}1\\1\end{pmatrix} \qquad (1.128.2)$$

We can express these as

$$\alpha = k_1 \mathbf{u}_1 \tag{1.128.3}$$
$$\beta = r_1 \mathbf{u}_1 + k_2 \mathbf{u}_2 \tag{1.128.4}$$

where

$$k_1 = \|\alpha\| \tag{1.128.5}$$
$$\mathbf{u}_1 = \frac{\alpha}{k_1} \tag{1.128.6}$$
$$r_1 = \frac{\mathbf{u}_1^T \beta}{\|\mathbf{u}_1\|^2} \tag{1.128.7}$$
$$\mathbf{u}_2 = \frac{\beta - r_1 \mathbf{u}_1}{\|\beta - r_1 \mathbf{u}_1\|} \tag{1.128.8}$$
$$k_2 = \mathbf{u}_2^T \beta \tag{1.128.9}$$

From (1.128.3) and (1.128.4),

$$\begin{pmatrix} \alpha & \beta \end{pmatrix} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{pmatrix} \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \tag{1.128.10}$$
$$\begin{pmatrix} \alpha & \beta \end{pmatrix} = \mathbf{QR} \tag{1.128.11}$$

From above we can see that $\mathbf{R}$ is an upper triangular matrix and

$$\mathbf{Q}^T \mathbf{Q} = \mathbf{I} \tag{1.128.12}$$

Now by using equations (1.128.5) to (1.128.9)

$$k_1 = 5 \tag{1.128.13}$$
$$\mathbf{u}_1 = \frac{1}{5} \begin{pmatrix} 3 \\ -4 \end{pmatrix}, \tag{1.128.14}$$
$$r_1 = \frac{-1}{5} \tag{1.128.15}$$
$$\mathbf{u}_2 = \frac{5}{7} \begin{pmatrix} \frac{28}{25} \\ \frac{21}{25} \end{pmatrix} \tag{1.128.16}$$
$$k_2 = \frac{7}{5} \tag{1.128.17}$$

Thus obtained QR decomposition is

$$\begin{pmatrix} 3 & 1 \\ -4 & 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{pmatrix} \begin{pmatrix} 5 & -\frac{1}{5} \\ 0 & \frac{7}{5} \end{pmatrix} \tag{1.128.18}$$

1.129. Find the QR decomposition of $\begin{pmatrix} 55 & -60 \\ -60 & 20 \end{pmatrix}$

Solution:
Let $\mathbf{a}$ and $\mathbf{b}$ are the column vectors of the given matrix. So,

$$\mathbf{a} = \begin{pmatrix} 55 \\ -60 \end{pmatrix} \tag{1.129.1}$$
$$\mathbf{b} = \begin{pmatrix} -60 \\ 20 \end{pmatrix} \tag{1.129.2}$$

$\mathbf{a}$ and $\mathbf{b}$ can be expressed as:

$$\mathbf{a} = K_1 \mathbf{u_1} \tag{1.129.3}$$
$$\mathbf{b} = r_1 \mathbf{u_1} + K_2 \mathbf{u_2} \tag{1.129.4}$$

Where,

$$K_1 = \|\mathbf{a}\| \tag{1.129.5}$$
$$\implies \mathbf{u_1} = \frac{\mathbf{a}}{\|\mathbf{a}\|} \tag{1.129.6}$$
$$r_1 = \frac{\mathbf{u_1}^T \mathbf{b}}{\|\mathbf{u_1}\|^2} \tag{1.129.7}$$
$$\mathbf{u_2} = \frac{\mathbf{b} - r_1 \mathbf{u_1}}{\|\mathbf{b} - r_1 \mathbf{u_1}\|} \tag{1.129.8}$$
$$K_2 = \mathbf{u_2}^T \mathbf{b} \tag{1.129.9}$$

Then we can express the given matrix as:

$$\begin{pmatrix} a & b \end{pmatrix} = \begin{pmatrix} u_1 & u_2 \end{pmatrix} \begin{pmatrix} K_1 & r_1 \\ 0 & K_2 \end{pmatrix} \tag{1.129.10}$$
$$\text{or,} \begin{pmatrix} a & b \end{pmatrix} = \mathbf{QR} \tag{1.129.11}$$

Now,

$$K_1 = \|\mathbf{a}\| = \sqrt{55^2 + (-60)^2} = 5\sqrt{265} \tag{1.129.12}$$
$$\mathbf{u_1} = \frac{\mathbf{a}}{5\sqrt{265}} = \frac{1}{\sqrt{265}} \begin{pmatrix} 11 \\ -12 \end{pmatrix} \tag{1.129.13}$$
$$\|\mathbf{u_1}\| = 1 \tag{1.129.14}$$
$$r_1 = \mathbf{u_1}^T \mathbf{b} = \frac{1}{\sqrt{265}} \begin{pmatrix} 11 & -12 \end{pmatrix} \begin{pmatrix} -60 \\ 20 \end{pmatrix} \tag{1.129.15}$$
$$\implies r_1 = -\frac{900}{\sqrt{265}} \tag{1.129.16}$$

So,

$$r_1\mathbf{u_1} = -\frac{900}{\sqrt{265}}\frac{1}{\sqrt{265}}\begin{pmatrix}11\\-12\end{pmatrix} \quad (1.129.17)$$

$$\implies r_1\mathbf{u_1} = \begin{pmatrix}-\frac{1980}{53}\\\frac{2160}{53}\end{pmatrix} \quad (1.129.18)$$

$$\mathbf{b} - r_1\mathbf{u_1} = \begin{pmatrix}-60\\20\end{pmatrix} - \begin{pmatrix}-\frac{1980}{53}\\\frac{2160}{53}\end{pmatrix} \quad (1.129.19)$$

$$\implies \mathbf{b} - r_1\mathbf{u_1} = -\frac{100}{53}\begin{pmatrix}12\\11\end{pmatrix} \quad (1.129.20)$$

$$\|\mathbf{b} - r_1\mathbf{u_1}\| = \frac{100\sqrt{5}}{\sqrt{53}} \quad (1.129.21)$$

Similarly, we can obtain:

$$\mathbf{u_2} = -\frac{1}{\sqrt{265}}\begin{pmatrix}12\\11\end{pmatrix} = \begin{pmatrix}-\frac{12}{\sqrt{265}}\\-\frac{11}{\sqrt{265}}\end{pmatrix} \quad (1.129.22)$$

$$K_2 = \mathbf{u_2}^T\mathbf{b} = -\frac{1}{\sqrt{265}}\begin{pmatrix}12 & 11\end{pmatrix}\begin{pmatrix}-60\\20\end{pmatrix} \quad (1.129.23)$$

$$= \frac{500}{\sqrt{265}} \quad (1.129.24)$$

Now, after QR decomposition of the given matrix we can get:

$$\begin{pmatrix}55 & -60\\-60 & 20\end{pmatrix} = \begin{pmatrix}\frac{11}{\sqrt{265}} & \frac{-12}{\sqrt{265}}\\\frac{-12}{\sqrt{265}} & \frac{-11}{\sqrt{265}}\end{pmatrix}\begin{pmatrix}5\sqrt{265} & \frac{-900}{\sqrt{265}}\\0 & \frac{500}{\sqrt{265}}\end{pmatrix} \quad (1.129.25)$$

1.130. Find QR decomposition for matrix

$$\mathbf{V} = \begin{pmatrix}6 & \frac{17}{2}\\\frac{17}{2} & 12\end{pmatrix} \quad (1.130.1)$$

Solution:

$$\text{Let } \mathbf{V} = \begin{pmatrix}\mathbf{c_1} & \mathbf{c_2}\end{pmatrix} = \begin{pmatrix}a & h\\h & b\end{pmatrix} \quad (1.130.2)$$

Let $\mathbf{Q}$ be an orthogonal and $\mathbf{R}$ be an upper triangular matrix such that,

$$\mathbf{V} = \mathbf{QR} \quad (1.130.3)$$

$$= \begin{pmatrix}\mathbf{q_1} & \mathbf{q_2}\end{pmatrix}\begin{pmatrix}r_{11} & r_{12}\\0 & r_{22}\end{pmatrix} \quad (1.130.4)$$

$$\begin{pmatrix}\mathbf{c_1} & \mathbf{c_2}\end{pmatrix} = \begin{pmatrix}\mathbf{q_1}r_{11} & \mathbf{q_1}r_{12} + \mathbf{q_2}r_{22}\end{pmatrix} \quad (1.130.5)$$

$$\implies \mathbf{c_1} = \mathbf{q_1}r_{11} \quad (1.130.6)$$

$$r_{11} = \|\mathbf{c_1}\| = \sqrt{a^2 + h^2} \quad (1.130.7)$$

$$\mathbf{q_1} = \frac{\mathbf{c_1}}{r_{11}} \quad (1.130.8)$$

$$\mathbf{c_2} = \mathbf{q_1}r_{12} + \mathbf{q_2}r_{22} \quad (1.130.9)$$

$$r_{21} = \mathbf{q_1}^T c_2 \quad (1.130.10)$$

$$r_{22} = \|\mathbf{c_2} - \mathbf{q_1}r_{12}\| \quad (1.130.11)$$

$$\mathbf{q_2} = \frac{\mathbf{c_2} - \mathbf{q_1}r_{12}}{r_{22}} \quad (1.130.12)$$

Given,

$$\mathbf{V} = \begin{pmatrix}6 & \frac{17}{2}\\\frac{17}{2} & 12\end{pmatrix} \quad (1.130.13)$$

From (1.130.7),

$$r_{11} = \sqrt{6^2 + \left(\frac{17}{2}\right)^2} = \frac{\sqrt{433}}{2} \quad (1.130.14)$$

Substitute (1.130.14) in (1.130.8)

$$\mathbf{q_1} = \begin{pmatrix}\frac{12}{\sqrt{433}}\\\frac{17}{\sqrt{433}}\end{pmatrix} \quad (1.130.15)$$

Substitute (1.130.15) in (1.130.10)

$$r_{21} = \frac{306}{\sqrt{433}} \quad (1.130.16)$$

Substitute (1.130.11) in (1.130.16)

$$r_{22} = \frac{\sqrt{433}}{866} \quad (1.130.17)$$

Substitute (1.130.17) in (1.130.12)

$$\mathbf{q_2} = \begin{pmatrix}\frac{17}{\sqrt{433}}\\-\frac{12}{\sqrt{433}}\end{pmatrix} \quad (1.130.18)$$

Hence QR decomposition of V is,

$$\mathbf{V} = \begin{pmatrix}\frac{12}{\sqrt{433}} & \frac{17}{\sqrt{433}}\\\frac{17}{\sqrt{433}} & -\frac{12}{\sqrt{433}}\end{pmatrix}\begin{pmatrix}\frac{\sqrt{433}}{2} & \frac{306}{\sqrt{433}}\\0 & \frac{\sqrt{433}}{866}\end{pmatrix} \quad (1.130.19)$$

1.131. Find the QR decomposition of

$$\mathbf{A} = \begin{pmatrix}7 & 3\\2 & 4\end{pmatrix} \quad (1.131.1)$$

Solution: Let $\mathbf{x}$ and $\mathbf{y}$ be the column vectors of the given matrix.

$$\mathbf{x} = \begin{pmatrix}7\\2\end{pmatrix} \tag{1.131.2}$$

$$\mathbf{y} = \begin{pmatrix}3\\4\end{pmatrix} \tag{1.131.3}$$

The column vectors can be expressed as follows,

$$\mathbf{x} = k_1\mathbf{u}_1 \tag{1.131.4}$$

$$\mathbf{y} = r_1\mathbf{u}_1 + k_2\mathbf{u}_2 \tag{1.131.5}$$

Here,

$$k_1 = \|\mathbf{x}\| \tag{1.131.6}$$

$$\mathbf{u}_1 = \frac{\mathbf{x}}{k_1} \tag{1.131.7}$$

$$r_1 = \frac{\mathbf{u}_1^T\mathbf{y}}{\|\mathbf{u}_1\|^2} \tag{1.131.8}$$

$$\mathbf{u}_2 = \frac{\mathbf{y} - r_1\mathbf{u}_1}{\|\mathbf{y} - r_1\mathbf{u}_1\|} \tag{1.131.9}$$

$$k_2 = \mathbf{u}_2^T\mathbf{y} \tag{1.131.10}$$

The (1.131.4) and (1.131.5) can be written as,

$$\begin{pmatrix}\mathbf{x} & \mathbf{y}\end{pmatrix} = \begin{pmatrix}\mathbf{u}_1 & \mathbf{u}_2\end{pmatrix}\begin{pmatrix}k_1 & r_1\\0 & k_2\end{pmatrix} \tag{1.131.11}$$

$$\begin{pmatrix}\mathbf{x} & \mathbf{y}\end{pmatrix} = \mathbf{QR} \tag{1.131.12}$$

Now, $\mathbf{R}$ is an upper triangular matrix and also,

$$\mathbf{Q}^T\mathbf{Q} = \mathbf{I} \tag{1.131.13}$$

Now using equations (1.131.6) to (1.131.10) we get,

$$k_1 = \sqrt{7^2 + 2^2} = \sqrt{53} \tag{1.131.14}$$

$$\mathbf{u}_1 = \frac{1}{\sqrt{53}}\begin{pmatrix}7\\2\end{pmatrix} \tag{1.131.15}$$

$$\mathbf{u}_1 = \begin{pmatrix}\frac{7}{\sqrt{53}}\\\frac{2}{\sqrt{53}}\end{pmatrix} \tag{1.131.16}$$

$$r_1 = \begin{pmatrix}\frac{7}{\sqrt{53}} & \frac{2}{\sqrt{53}}\end{pmatrix}\begin{pmatrix}3\\4\end{pmatrix} = \frac{29}{\sqrt{53}} \tag{1.131.17}$$

$$\mathbf{u}_2 = \frac{1}{\sqrt{53}}\begin{pmatrix}2\\-7\end{pmatrix} \tag{1.131.18}$$

$$\mathbf{u}_2 = \begin{pmatrix}\frac{2}{\sqrt{53}}\\-\frac{7}{\sqrt{53}}\end{pmatrix} \tag{1.131.19}$$

$$k_2 = \begin{pmatrix}\frac{2}{\sqrt{53}} & -\frac{7}{\sqrt{53}}\end{pmatrix}\begin{pmatrix}3\\4\end{pmatrix} = -\frac{22}{\sqrt{53}} \tag{1.131.20}$$

Thus putting the values from (1.131.14) to (1.131.20) in (1.131.12) we obtain QR decomposition,

$$\begin{pmatrix}7 & 3\\2 & 4\end{pmatrix} = \begin{pmatrix}\frac{7}{\sqrt{53}} & \frac{2}{\sqrt{53}}\\\frac{2}{\sqrt{53}} & -\frac{7}{\sqrt{53}}\end{pmatrix}\begin{pmatrix}\sqrt{53} & \frac{29}{\sqrt{53}}\\0 & -\frac{22}{\sqrt{53}}\end{pmatrix} \tag{1.131.21}$$

Which can also be written as,

$$\begin{pmatrix}7 & 3\\2 & 4\end{pmatrix} = \begin{pmatrix}-\frac{7}{\sqrt{53}} & -\frac{2}{\sqrt{53}}\\-\frac{2}{\sqrt{53}} & \frac{7}{\sqrt{53}}\end{pmatrix}\begin{pmatrix}-\sqrt{53} & -\frac{29}{\sqrt{53}}\\0 & \frac{22}{\sqrt{53}}\end{pmatrix} \tag{1.131.22}$$

1.132. Find the QR decomposition of

$$\mathbf{V} = \begin{pmatrix}14 & -2\\-2 & 11\end{pmatrix} \tag{1.132.1}$$

Solution:

$$\text{If, } \mathbf{V} = \begin{pmatrix}a_1 & b_1\\a_2 & b_2\end{pmatrix} \text{ Consider, } \mathbf{V} = \begin{pmatrix}\mathbf{a} & \mathbf{b}\end{pmatrix}$$

$$\text{Where, } \mathbf{a} = \begin{pmatrix}a1\\a2\end{pmatrix} \text{and } \mathbf{b} = \begin{pmatrix}b1\\b2\end{pmatrix} \tag{1.132.2}$$

$$\text{Then, } \mathbf{u_1} = \mathbf{a}\ ,\ \mathbf{e_1} = \frac{\mathbf{u_1}}{\|\mathbf{u_1}\|}$$

$$\mathbf{u_2} = \mathbf{b} - (\mathbf{b}\cdot\mathbf{e_1})\,\mathbf{e_1}\ ,\ \mathbf{e_2} = \frac{\mathbf{u_2}}{\|\mathbf{u_2}\|} \tag{1.132.3}$$

$$\text{and, } \mathbf{V} = \begin{pmatrix}\mathbf{e_1} & \mathbf{e_2}\end{pmatrix}\begin{pmatrix}\mathbf{a}\cdot\mathbf{e_1} & \mathbf{b}\cdot\mathbf{e_1}\\0 & \mathbf{b}\cdot\mathbf{e_2}\end{pmatrix} = \mathbf{QR} \tag{1.132.4}$$

Performing QR decomposition on $\mathbf{V}$ we get,

$$\mathbf{e_1} = \frac{1}{10\sqrt{2}}\begin{pmatrix}14\\-2\end{pmatrix} \tag{1.132.5}$$

$$\mathbf{e_2} = \frac{1}{\sqrt{50}}\begin{pmatrix}1\\7\end{pmatrix} \tag{1.132.6}$$

$$\mathbf{Q} = \begin{pmatrix}\mathbf{e_1} & \mathbf{e_2}\end{pmatrix} \tag{1.132.7}$$

$$\mathbf{R} = \begin{pmatrix}10\sqrt{2} & -5\sqrt{2}\\0 & \frac{75}{\sqrt{50}}\end{pmatrix} \tag{1.132.8}$$

It is easy to verify that $\mathbf{QR} = \mathbf{V}$ and $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$. Thus, $\mathbf{V}$ is decomposed into an orthogonal matrix and an upper triangular matrix.

1.133. Perform QR Decomposition on matrix $\begin{pmatrix}1 & 3\\2 & 4\end{pmatrix}$

Solution: Let **a** and **b** be columns of a **A**. Then, the matrix **A** can be decomposed in the form as:

$$\mathbf{A} = \mathbf{QR} \tag{1.133.1}$$

such that

$$\mathbf{QQ}^T = \mathbf{Q}^T\mathbf{Q} = \mathbf{I} \tag{1.133.2}$$

$$\mathbf{Q} = \begin{pmatrix} \mathbf{u_1} & \mathbf{u_2} \end{pmatrix} \tag{1.133.3}$$

$$\mathbf{R} = \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \tag{1.133.4}$$

where

$$k_1 = \|\mathbf{a}\| \tag{1.133.5}$$

$$\mathbf{u_1} = \frac{\mathbf{a}}{k_1} \tag{1.133.6}$$

$$r_1 = \frac{\mathbf{u_1}^T\mathbf{b}}{\|\mathbf{u_1}\|^2} \tag{1.133.7}$$

$$\mathbf{u_2} = \frac{\mathbf{b} - r_1\mathbf{u_1}}{\|\mathbf{b} - r_1\mathbf{u_1}\|} \tag{1.133.8}$$

$$k_2 = \mathbf{u_2}^T\mathbf{b} \tag{1.133.9}$$

Then, the matrix can be represented as

$$\begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} = \begin{pmatrix} \mathbf{u_1} & \mathbf{u_2} \end{pmatrix}\begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \tag{1.133.10}$$

Let **A** be the given matrix. Then $\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$ and columns of **A** are **a** and **b**, where

$$\mathbf{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{1.133.11}$$

$$\mathbf{b} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \tag{1.133.12}$$

Now, for given matrix from (1.133.5) and (1.133.6), we have

$$k_1 = \|\mathbf{a}\| = \sqrt{5} \tag{1.133.13}$$

$$\mathbf{u_1} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{1.133.14}$$

By, (1.133.7), we find

$$r_1 = \frac{\frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix}}{1} = \frac{11}{\sqrt{5}} \tag{1.133.15}$$

Now, by (1.133.8)

$$\mathbf{u_2} = \frac{\begin{pmatrix} 3 \\ 4 \end{pmatrix} - \frac{11}{5}\begin{pmatrix} 1 \\ 2 \end{pmatrix}}{\left\|\begin{pmatrix} 3 \\ 4 \end{pmatrix} - \frac{11}{5}\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\|} = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} \end{pmatrix} \tag{1.133.16}$$

From (1.133.9),

$$k_2 = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & -1 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \frac{2}{\sqrt{5}} \tag{1.133.17}$$

Now,

$$\mathbf{Q} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{pmatrix} \tag{1.133.18}$$

Now, we observe that $\mathbf{QQ}^T = \mathbf{Q}^T\mathbf{Q} = \mathbf{I}$

$$\begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.133.19}$$

Now, by (1.133.10) we can wrie matrix **A** as

$$\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{pmatrix}\begin{pmatrix} \sqrt{5} & \frac{11}{\sqrt{5}} \\ 0 & \frac{2}{\sqrt{5}} \end{pmatrix} \tag{1.133.20}$$

which is the required **QR** decomposition of **A**.

1.134. Find QR decomposition for matrix

$$\mathbf{V} = \begin{pmatrix} 12 & \frac{7}{2} \\ \frac{7}{2} & -10 \end{pmatrix} \tag{1.134.1}$$

Solution: Let **x** and **y** be the column vectors of the given matrix.

$$\mathbf{x} = \begin{pmatrix} 12 \\ \frac{7}{2} \end{pmatrix} \tag{1.134.2}$$

$$\mathbf{y} = \begin{pmatrix} \frac{7}{2} \\ -10 \end{pmatrix} \tag{1.134.3}$$

The column vectors can be expressed as follows,

$$\mathbf{x} = k_1\mathbf{u}_1 \tag{1.134.4}$$

$$\mathbf{y} = r_1\mathbf{u}_1 + k_2\mathbf{u}_2 \tag{1.134.5}$$

Here,

$$k_1 = ||\mathbf{x}|| \tag{1.134.6}$$

$$\mathbf{u}_1 = \frac{\mathbf{x}}{k_1} \tag{1.134.7}$$

$$r_1 = \frac{\mathbf{u}_1^T \mathbf{y}}{||\mathbf{u}_1||^2} \tag{1.134.8}$$

$$\mathbf{u}_2 = \frac{\mathbf{y} - r_1 \mathbf{u}_1}{||\mathbf{y} - r_1 \mathbf{u}_1||} \tag{1.134.9}$$

$$k_2 = \mathbf{u}_2^T \mathbf{y} \tag{1.134.10}$$

The (1.134.4) and (1.134.5) can be written as,

$$\begin{pmatrix}\mathbf{x} & \mathbf{y}\end{pmatrix} = \begin{pmatrix}\mathbf{u}_1 & \mathbf{u}_2\end{pmatrix}\begin{pmatrix}k_1 & r_1 \\ 0 & k_2\end{pmatrix} \tag{1.134.11}$$

$$\begin{pmatrix}\mathbf{x} & \mathbf{y}\end{pmatrix} = \mathbf{QR} \tag{1.134.12}$$

Now, $\mathbf{R}$ is an upper triangular matrix and also,

$$\mathbf{Q}^T\mathbf{Q} = \mathbf{I} \tag{1.134.13}$$

Now using equations (1.134.6) to (1.134.10) we get,

$$k_1 = \sqrt{\left(\frac{7}{2}\right)^2 + 12^2} = \frac{25}{2} \tag{1.134.14}$$

$$\mathbf{u}_1 = \begin{pmatrix}\frac{24}{25} \\ \frac{7}{25}\end{pmatrix} \tag{1.134.15}$$

$$r_1 = \begin{pmatrix}\frac{24}{25} & \frac{7}{25}\end{pmatrix}\begin{pmatrix}\frac{7}{2} \\ -10\end{pmatrix} = \frac{14}{25} \tag{1.134.16}$$

$$\mathbf{u}_2 = \begin{pmatrix}\frac{7}{25} \\ -\frac{24}{25}\end{pmatrix} \tag{1.134.17}$$

$$k_2 = \begin{pmatrix}\frac{7}{25} & -\frac{24}{25}\end{pmatrix}\begin{pmatrix}\frac{7}{2} \\ -10\end{pmatrix} = \frac{529}{50} \tag{1.134.18}$$

Thus putting the values from (1.134.14) to (1.134.18) in (1.134.11) we obtain QR decomposition,

$$\begin{pmatrix}12 & \frac{7}{2} \\ \frac{7}{2} & -10\end{pmatrix} = \begin{pmatrix}\frac{24}{25} & \frac{7}{25} \\ \frac{7}{25} & -\frac{24}{25}\end{pmatrix}\begin{pmatrix}\frac{25}{2} & \frac{14}{25} \\ 0 & \frac{529}{50}\end{pmatrix} \tag{1.134.19}$$

which can also be written as,

$$\begin{pmatrix}12 & \frac{7}{2} \\ \frac{7}{2} & -10\end{pmatrix} = \begin{pmatrix}-\frac{24}{25} & -\frac{7}{25} \\ -\frac{7}{25} & \frac{24}{25}\end{pmatrix}\begin{pmatrix}-\frac{25}{2} & -\frac{14}{25} \\ 0 & -\frac{529}{50}\end{pmatrix} \tag{1.134.20}$$

1.135. Find QR decomposition of matrix

$$\mathbf{V} = \begin{pmatrix}12 & -5 \\ -5 & 2\end{pmatrix} \tag{1.135.1}$$

Solution:
Let $\mathbf{x}$ and $\mathbf{y}$ be the column vectors of the given matrix.

$$\mathbf{x} = \begin{pmatrix}12 \\ -5\end{pmatrix} \tag{1.135.2}$$

$$\mathbf{y} = \begin{pmatrix}-5 \\ 2\end{pmatrix} \tag{1.135.3}$$

The column vectors can be expressed as follows,

$$\mathbf{x} = k_1 \mathbf{u}_1 \tag{1.135.4}$$

$$\mathbf{y} = r_1 \mathbf{u}_1 + k_2 \mathbf{u}_2 \tag{1.135.5}$$

$$k_1 = ||\mathbf{x}|| \tag{1.135.6}$$

$$\mathbf{u}_1 = \frac{\mathbf{x}}{k_1} \tag{1.135.7}$$

$$r_1 = \frac{\mathbf{u}_1^T \mathbf{y}}{||\mathbf{u}_1||^2} \tag{1.135.8}$$

$$\mathbf{u}_2 = \frac{\mathbf{y} - r_1 \mathbf{u}_1}{||\mathbf{y} - r_1 \mathbf{u}_1||} \tag{1.135.9}$$

$$k_2 = \mathbf{u}_2^T \mathbf{y} \tag{1.135.10}$$

The (1.135.4) and (1.135.5) can be written as,

$$\begin{pmatrix}\mathbf{x} & \mathbf{y}\end{pmatrix} = \begin{pmatrix}\mathbf{u}_1 & \mathbf{u}_2\end{pmatrix}\begin{pmatrix}k_1 & r_1 \\ 0 & k_2\end{pmatrix} \tag{1.135.11}$$

$$\begin{pmatrix}\mathbf{x} & \mathbf{y}\end{pmatrix} = \mathbf{QR} \tag{1.135.12}$$

Now, $\mathbf{R}$ is an upper triangular matrix and also,

$$\mathbf{Q}^T\mathbf{Q} = \mathbf{I} \tag{1.135.13}$$

Now using equations (1.135.6) to (1.135.10) we get,

$$k_1 = \sqrt{12^2 + 5^2} = 13 \tag{1.135.14}$$

$$\mathbf{u}_1 = \begin{pmatrix}\frac{12}{13} \\ \frac{-5}{13}\end{pmatrix} \tag{1.135.15}$$

$$r_1 = \begin{pmatrix}\frac{12}{13} & \frac{-5}{13}\end{pmatrix}\begin{pmatrix}-5 \\ 2\end{pmatrix} = -\frac{70}{13} \tag{1.135.16}$$

$$\mathbf{u}_2 = \begin{pmatrix}-\frac{5}{13} \\ -\frac{12}{13}\end{pmatrix} \tag{1.135.17}$$

$$k_2 = \begin{pmatrix}-\frac{5}{13} & -\frac{12}{13}\end{pmatrix}\begin{pmatrix}-5 \\ 2\end{pmatrix} = \frac{1}{13} \tag{1.135.18}$$

Thus putting the values from (1.135.14) to (1.135.18) in (1.135.11) we obtain QR decomposition,

$$\begin{pmatrix}12 & -5\\-5 & 2\end{pmatrix} = \begin{pmatrix}\frac{12}{13} & -\frac{5}{13}\\ -\frac{5}{13} & -\frac{12}{13}\end{pmatrix}\begin{pmatrix}13 & -\frac{70}{13}\\ 0 & \frac{1}{13}\end{pmatrix} \tag{1.135.19}$$

which can also be written as,

$$\begin{pmatrix}12 & -5\\-5 & 2\end{pmatrix} = \begin{pmatrix}-\frac{12}{13} & \frac{5}{13}\\ \frac{5}{13} & \frac{12}{13}\end{pmatrix}\begin{pmatrix}-13 & \frac{70}{13}\\ 0 & -\frac{1}{13}\end{pmatrix} \tag{1.135.20}$$

1.136. Perform QR decomposition of the matrix

$$\mathbf{V} = \begin{pmatrix}19 & 12\\12 & 1\end{pmatrix} \tag{1.136.1}$$

1.137. Find the shortest distance between the lines

$$\mathbf{x} = \begin{pmatrix}1\\1\\0\end{pmatrix} + \lambda_1\begin{pmatrix}2\\-1\\1\end{pmatrix} \tag{1.137.1}$$

$$\mathbf{x} = \begin{pmatrix}2\\1\\-1\end{pmatrix} + \lambda_2\begin{pmatrix}3\\-5\\2\end{pmatrix} \tag{1.137.2}$$

Solution:
The lines will intersect if

$$\begin{pmatrix}1\\1\\0\end{pmatrix} + \lambda_1\begin{pmatrix}2\\-1\\1\end{pmatrix} = \begin{pmatrix}2\\1\\-1\end{pmatrix} + \lambda_2\begin{pmatrix}3\\-5\\2\end{pmatrix} \tag{1.137.3}$$

$$\begin{pmatrix}2 & 3\\-1 & -5\\1 & 2\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}1\\0\\-1\end{pmatrix} \tag{1.137.4}$$

$$\mathbf{Mx} = \mathbf{b} \tag{1.137.5}$$

Since the rank of augmented matrix will be 3. We can say that lines do not intersect.

$$\mathbf{M} = \mathbf{USV}^T \tag{1.137.6}$$

Where the columns of $\mathbf{V}$ are the eigenvectors of $\mathbf{A}^T\mathbf{A}$,the columns of $\mathbf{U}$ are the eigenvectors of $\mathbf{AA}^T$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{A}^T\mathbf{A}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix}6 & 13\\13 & 38\end{pmatrix} \tag{1.137.7}$$

$$\mathbf{MM}^T = \begin{pmatrix}13 & -17 & 8\\-17 & 26 & -11\\8 & -11 & 5\end{pmatrix} \tag{1.137.8}$$

Calculating eigen value of $\mathbf{M}^T\mathbf{M}$.

$$\begin{vmatrix}6-\lambda & 13\\13 & 38-\lambda\end{vmatrix}\lambda^2 - 44\lambda + 59 = 0 \tag{1.137.9}$$

$$\lambda_2 = -5\sqrt{17} + 22, \lambda_1 = 5\sqrt{17} + 22 \tag{1.137.10}$$

Eigen vectors of $\mathbf{MM}^T$.

$$\begin{vmatrix}13-\lambda & -17 & 8\\17 & 26-\lambda & -11\\8 & -11 & 5-\lambda\end{vmatrix} - \lambda^3 + 44\lambda^2 - 59\lambda = 0 \tag{1.137.11}$$

$$\lambda_4 = -5\sqrt{17} + 22, \lambda_3 = 5\sqrt{17} + 22, \lambda_5 = 0, \tag{1.137.12}$$

Hence,The eigenvectors will be

$$\mathbf{u}_2 = \begin{pmatrix}\frac{\sqrt{17}+12}{5}\\ \frac{3\sqrt{17}+1}{5}\\ 1\end{pmatrix}\mathbf{u}_1 = \begin{pmatrix}\frac{-\sqrt{17}+12}{5}\\ \frac{-3\sqrt{17}+1}{5}\\ 1\end{pmatrix}\mathbf{u}_3 = \begin{pmatrix}\frac{-3}{7}\\ \frac{1}{7}\\ 1\end{pmatrix} \tag{1.137.13}$$

Normalising the eigenvectors

$$l_1 = \sqrt{\left(\frac{12-\sqrt{17}}{5}\right)^2 + \left(\frac{1-3\sqrt{17}}{5}\right)^2 + 1^2} \tag{1.137.14}$$

$$\mathbf{u}_1 = \begin{pmatrix}\frac{-\sqrt{17}+12}{\sqrt{340-30\sqrt{17}}}\\ \frac{-3\sqrt{17}+1}{\sqrt{340-30\sqrt{17}}}\\ \frac{5}{\sqrt{340-30\sqrt{17}}}\end{pmatrix} \tag{1.137.15}$$

$$\tag{1.137.16}$$

$$l_2 = \sqrt{\left(\frac{\sqrt{17}+12}{5}\right)^2 + \left(\frac{3\sqrt{17}+1}{5}\right)^2 + 1^2} \tag{1.137.17}$$

$$\mathbf{u}_2 = \frac{5}{\sqrt{340+30\sqrt{7}}}\begin{pmatrix} \frac{\sqrt{17}+12}{5} \\ \frac{3\sqrt{17}+1}{5} \\ 1 \end{pmatrix} \tag{1.137.18}$$

$$\mathbf{u}_2 = \begin{pmatrix} \frac{\sqrt{17}+12}{\sqrt{340+30\sqrt{17}}} \\ \frac{3\sqrt{17}+1}{\sqrt{340+30\sqrt{17}}} \\ \frac{5}{\sqrt{340+30\sqrt{17}}} \end{pmatrix} \tag{1.137.19}$$

$$l_3 = \sqrt{\left(\frac{-3}{7}\right)^2 + \left(\frac{1}{7}\right)^2 + 1^2} \tag{1.137.20}$$

$$\mathbf{u}_3 = \frac{7}{\sqrt{59}}\begin{pmatrix} \frac{-3}{7} \\ \frac{1}{7} \end{pmatrix} \tag{1.137.21}$$

$$\mathbf{u}_3 = \begin{pmatrix} \frac{-3}{\sqrt{59}} \\ \frac{1}{\sqrt{59}} \\ \frac{7}{\sqrt{59}} \\ \frac{7}{\sqrt{59}} \end{pmatrix} \tag{1.137.22}$$

$$\mathbf{U} = \begin{pmatrix} \frac{-\sqrt{17}+12}{\sqrt{340-30\sqrt{17}}} & \frac{\sqrt{17}+12}{\sqrt{340+30\sqrt{17}}} & \frac{-3}{\sqrt{59}} \\ \frac{-3\sqrt{17}+1}{\sqrt{340-30\sqrt{17}}} & \frac{3\sqrt{17}+1}{\sqrt{340+30\sqrt{17}}} & \frac{1}{\sqrt{59}} \\ \frac{5}{\sqrt{340-30\sqrt{17}}} & \frac{5}{\sqrt{340+30\sqrt{17}}} & \frac{7}{\sqrt{59}} \end{pmatrix} \tag{1.137.23}$$

Now,

$$\mathbf{S} = \begin{pmatrix} \sqrt{5\sqrt{17}+22} & 0 \\ 0 & \sqrt{-5\sqrt{17}+22} \\ 0 & 0 \end{pmatrix} \tag{1.137.24}$$

Now, $\mathbf{V} = \mathbf{M}^T \frac{\mathbf{u_i}}{\sqrt{\lambda_i}}$

$$\mathbf{V} = \begin{pmatrix} \frac{\sqrt{17}+28}{\sqrt{340-30\sqrt{17}}\sqrt{5\sqrt{17}+22}} & \frac{-\sqrt{17}+28}{\sqrt{340+30\sqrt{17}}\sqrt{-5\sqrt{17}+22}} \\ \frac{12\sqrt{17}+41}{\sqrt{340-30\sqrt{17}}\sqrt{5\sqrt{17}+22}} & \frac{-12\sqrt{17}+41}{\sqrt{340+30\sqrt{17}}\sqrt{-5\sqrt{17}+22}} \end{pmatrix} \tag{1.137.25}$$

So,from equation (1.137.6)

$$\begin{pmatrix} 2 & 3 \\ -1 & -5 \\ 1 & 2 \end{pmatrix} = \tag{1.137.26}$$

$$\begin{pmatrix} \frac{-\sqrt{17}+12}{\sqrt{340-30\sqrt{17}}} & \frac{\sqrt{17}+12}{\sqrt{340+30\sqrt{17}}} & \frac{-3}{\sqrt{59}} \\ \frac{-3\sqrt{17}+1}{\sqrt{340-30\sqrt{17}}} & \frac{3\sqrt{17}+1}{\sqrt{340+30\sqrt{17}}} & \frac{1}{\sqrt{59}} \\ \frac{5}{\sqrt{340-30\sqrt{17}}} & \frac{5}{\sqrt{340+30\sqrt{17}}} & \frac{7}{\sqrt{59}} \end{pmatrix} \tag{1.137.27}$$

$$\begin{pmatrix} \sqrt{5\sqrt{17}+22} & 0 \\ 0 & \sqrt{-5\sqrt{17}+22} \\ 0 & 0 \end{pmatrix} \tag{1.137.28}$$

$$\begin{pmatrix} \frac{\sqrt{17}+28}{\sqrt{340-30\sqrt{17}}\sqrt{5\sqrt{17}+22}} & \frac{-\sqrt{17}+28}{\sqrt{340+30\sqrt{17}}\sqrt{-5\sqrt{17}+22}} \\ \frac{12\sqrt{17}+41}{\sqrt{340-30\sqrt{17}}\sqrt{5\sqrt{17}+22}} & \frac{-12\sqrt{17}+41}{\sqrt{340+30\sqrt{17}}\sqrt{-5\sqrt{17}+22}} \end{pmatrix}^T \tag{1.137.29}$$

Now, Finding Moore-Penrose Pseudo inverse of **S**

$$\mathbf{S}_+ = \begin{pmatrix} \frac{1}{\sqrt{5\sqrt{17}+22}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{-5\sqrt{17}+22}} & 0 \end{pmatrix} \tag{1.137.30}$$

We,know that, $\mathbf{x} = \mathbf{V}(\mathbf{S}_+(\mathbf{U}^T\mathbf{b}))$

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{-\sqrt{17}+7}{\sqrt{340-30\sqrt{17}}} \\ \frac{\sqrt{17}+7}{\sqrt{340+0\sqrt{17}}} \\ \frac{-10}{\sqrt{59}} \end{pmatrix} \tag{1.137.31}$$

$$\mathbf{S}_+(\mathbf{U}^T\mathbf{b}) = \begin{pmatrix} \frac{-\sqrt{17}+7}{\sqrt{340-30\sqrt{17}}\sqrt{5\sqrt{17}+22}} \\ \frac{\sqrt{17}+7}{\sqrt{340+30\sqrt{17}}\sqrt{-5\sqrt{17}+22}} \end{pmatrix} \tag{1.137.32}$$

$$\mathbf{x} = \begin{pmatrix} \frac{\sqrt{17}+28}{\sqrt{340-30\sqrt{17}}\sqrt{5\sqrt{17}+22}} & \frac{-\sqrt{17}+28}{\sqrt{340+30\sqrt{17}}\sqrt{-5\sqrt{17}+22}} \\ \frac{12\sqrt{17}+41}{\sqrt{340-30\sqrt{17}}\sqrt{5\sqrt{17}+22}} & \frac{-12\sqrt{17}+41}{\sqrt{340+30\sqrt{17}}\sqrt{-5\sqrt{17}+22}} \end{pmatrix} \tag{1.137.33}$$

$$\begin{pmatrix} \frac{-\sqrt{17}+7}{\sqrt{340-30\sqrt{17}}\sqrt{5\sqrt{17}+22}} \\ \frac{\sqrt{17}+7}{\sqrt{340+30\sqrt{17}}\sqrt{-5\sqrt{17}+22}} \end{pmatrix} \tag{1.137.34}$$

$$\mathbf{x} = \begin{pmatrix} \frac{2507500}{(4930-1040\sqrt{17})(4930+1040\sqrt{17})} \\ \frac{-702100}{(4930-1040\sqrt{17})(4930+1040\sqrt{17})} \end{pmatrix} \tag{1.137.35}$$

Simplifying the values of x_1 and x_2

$$x_2 = \frac{-702100}{(4930 - 1040\sqrt{17})(4930 + 1040\sqrt{17})} \tag{1.137.36}$$

$$= \frac{-702100}{591700} \tag{1.137.37}$$

$$= -\frac{7}{59} \tag{1.137.38}$$

$$x_1 = \frac{2507500}{(4930 - 1040\sqrt{17})(4930 + 1040\sqrt{17})} \tag{1.137.39}$$

$$= \frac{2507500}{591700} \tag{1.137.40}$$

$$= \frac{25}{59} \tag{1.137.41}$$

Now, Verifying the values using

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \tag{1.137.42}$$

Solving R.H.S

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.137.43}$$

Now using equation (1.137.7) in (1.137.43)

$$\begin{pmatrix} 6 & 13 \\ 13 & 38 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.137.44}$$

Solving the augmented matrix.

$$\begin{pmatrix} 6 & 13 & 1 \\ 13 & 38 & 1 \end{pmatrix} \xleftrightarrow{R_2 - \frac{13}{6}R_1} \begin{pmatrix} 6 & 13 & 1 \\ 0 & \frac{59}{6} & -\frac{7}{6} \end{pmatrix} \tag{1.137.45}$$

$$\frac{59}{6}x_2 = -\frac{7}{6} \tag{1.137.46}$$

$$6x_1 + 13x_2 = 1 \tag{1.137.47}$$

$$x_1 = \frac{25}{59}, x_2 = -\frac{7}{59} \tag{1.137.48}$$

$$\mathbf{x} = \begin{pmatrix} \frac{25}{59} \\ -\frac{7}{59} \end{pmatrix} \tag{1.137.49}$$

1.138. Find the distance of the point $\begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix}$ from the plane $\begin{pmatrix} 6 & -3 & 2 \end{pmatrix}\mathbf{x} = 4$

Solution:

First we find orthogonal vectors $\mathbf{m_1}$ and $\mathbf{m_2}$ to the given normal vector $\mathbf{n}$. Let, $\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, then

$$\mathbf{m^T n} = 0 \tag{1.138.1}$$

$$\implies \begin{pmatrix} a & b & c \end{pmatrix}\begin{pmatrix} 6 \\ -3 \\ 2 \end{pmatrix} = 0 \tag{1.138.2}$$

$$\implies 6a - 3b + 2c = 0 \tag{1.138.3}$$

Putting a=1 and b=0 we get,

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \tag{1.138.4}$$

Putting a=0 and b=1 we get,

$$\mathbf{m_2} = \begin{pmatrix} 0 \\ 1 \\ \frac{3}{2} \end{pmatrix} \tag{1.138.5}$$

Now we solve the equation,

$$\mathbf{M}\mathbf{x} = \mathbf{b} \tag{1.138.6}$$

Putting values in (1.138.6),

$$\begin{pmatrix}1 & 0\\ 0 & 1\\ 3 & \frac{3}{2}\end{pmatrix}\mathbf{x} = \begin{pmatrix}2\\ 5\\ -3\end{pmatrix} \tag{1.138.7}$$

Now, to solve (1.138.7), we perform Singular Value Decomposition on $\mathbf{M}$ as follows,

$$\mathbf{M} = \mathbf{U}\mathbf{S}\mathbf{V}^T \tag{1.138.8}$$

Where the columns of $\mathbf{V}$ are the eigen vectors of $\mathbf{M}^T\mathbf{M}$,the columns of $\mathbf{U}$ are the eigen vectors of $\mathbf{M}\mathbf{M}^T$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix}10 & \frac{9}{2}\\ \frac{9}{2} & \frac{13}{4}\end{pmatrix} \tag{1.138.9}$$

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix}1 & 0 & 3\\ 0 & 1 & \frac{3}{2}\\ 3 & \frac{3}{2} & \frac{45}{4}\end{pmatrix} \tag{1.138.10}$$

From (1.138.6) putting (1.138.8) we get,

$$\mathbf{U}\mathbf{S}\mathbf{V}^T\mathbf{x} = \mathbf{b} \tag{1.138.11}$$

$$\implies \mathbf{x} = \mathbf{V}\mathbf{S}_{+}\mathbf{U}^{\mathbf{T}}\mathbf{b} \tag{1.138.12}$$

Where $\mathbf{S}_{+}$ is Moore-Penrose Pseudo-Inverse of $\mathbf{S}$.Now, calculating eigen value of $\mathbf{M}\mathbf{M}^T$,

$$\left|\mathbf{M}\mathbf{M}^T - \lambda\mathbf{I}\right| = 0 \tag{1.138.13}$$

$$\implies \begin{vmatrix}1-\lambda & 0 & 3\\ 0 & 1-\lambda & \frac{3}{2}\\ 3 & \frac{3}{2} & \frac{45}{4}-\lambda\end{vmatrix} = 0 \tag{1.138.14}$$

$$\implies \lambda^3 - \frac{53}{4}\lambda^2 + \frac{49}{4}\lambda = 0 \tag{1.138.15}$$

Hence eigen values of $\mathbf{M}\mathbf{M}^T$ are,

$$\lambda_1 = \frac{49}{4} \tag{1.138.16}$$

$$\lambda_2 = 1 \tag{1.138.17}$$

$$\lambda_3 = 0 \tag{1.138.18}$$

Hence the eigen vectors of $\mathbf{M}\mathbf{M}^T$ are,

$$\mathbf{u}_1 = \begin{pmatrix}\frac{4}{15}\\ \frac{2}{15}\\ 1\end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix}-\frac{1}{2}\\ 1\\ 0\end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix}-3\\ -\frac{3}{2}\\ 1\end{pmatrix} \tag{1.138.19}$$

Normalizing the eigen vectors we get,

$$\mathbf{u}_1 = \begin{pmatrix}\frac{4}{7\sqrt{5}}\\ \frac{2}{7\sqrt{5}}\\ \frac{3\sqrt{5}}{7}\end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix}-\frac{1}{\sqrt{5}}\\ \frac{2}{\sqrt{5}}\\ 0\end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix}-\frac{6}{7}\\ -\frac{3}{7}\\ \frac{2}{7}\end{pmatrix} \tag{1.138.20}$$

Hence we obtain $\mathbf{U}$ of (1.138.8) as follows,

$$\mathbf{U} = \begin{pmatrix}\frac{4}{7\sqrt{5}} & -\frac{1}{\sqrt{5}} & -\frac{6}{7}\\ \frac{2}{7\sqrt{5}} & \frac{2}{\sqrt{5}} & -\frac{3}{7}\\ \frac{3\sqrt{5}}{7} & 0 & \frac{2}{7}\end{pmatrix} \tag{1.138.21}$$

After computing the singular values from eigen values $\lambda_1, \lambda_2, \lambda_3$ we get $\mathbf{S}$ of (1.138.8) as follows,

$$\mathbf{S} = \begin{pmatrix}\frac{7}{2} & 0\\ 0 & 1\\ 0 & 0\end{pmatrix} \tag{1.138.22}$$

Now, calculating eigen value of $\mathbf{M}^T\mathbf{M}$,

$$\left|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}\right| = 0 \tag{1.138.23}$$

$$\implies \begin{vmatrix}10-\lambda & \frac{9}{2}\\ \frac{9}{2} & \frac{13}{4}-\lambda\end{vmatrix} = 0 \tag{1.138.24}$$

$$\implies \lambda^2 - \frac{53}{4}\lambda + \frac{49}{4} = 0 \tag{1.138.25}$$

Hence eigen values of $\mathbf{M}^T\mathbf{M}$ are,

$$\lambda_4 = \frac{49}{4} \tag{1.138.26}$$

$$\lambda_5 = 1 \tag{1.138.27}$$

Hence the eigen vectors of $\mathbf{M}^T\mathbf{M}$ are,

$$\mathbf{v}_1 = \begin{pmatrix}2\\ 1\end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix}-\frac{1}{2}\\ 1\end{pmatrix} \tag{1.138.28}$$

Normalizing the eigen vectors we get,

$$\mathbf{v}_1 = \begin{pmatrix}\frac{2}{\sqrt{5}}\\ \frac{1}{\sqrt{5}}\end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix}-\frac{1}{\sqrt{5}}\\ \frac{2}{\sqrt{5}}\end{pmatrix} \tag{1.138.29}$$

Hence we obtain $\mathbf{V}$ of (1.138.8) as follows,

$$\mathbf{V} = \begin{pmatrix}\frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}}\\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}}\end{pmatrix} \tag{1.138.30}$$

Finally from (1.138.8) we get the Singualr

Value Decomposition of $\mathbf{M}$ as follows,

$$\mathbf{M} = \begin{pmatrix} \frac{4}{7\sqrt{5}} & -\frac{1}{\sqrt{5}} & -\frac{6}{7} \\ \frac{2}{7\sqrt{5}} & \frac{2}{\sqrt{5}} & -\frac{3}{7} \\ \frac{3\sqrt{5}}{7} & 0 & \frac{2}{7} \end{pmatrix} \begin{pmatrix} \frac{7}{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}^T \tag{1.138.31}$$

Now, Moore-Penrose Pseudo inverse of $\mathbf{S}$ is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{2}{7} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{1.138.32}$$

From (1.138.12) we get,

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} -\frac{27}{7\sqrt{5}} \\ \frac{8}{7\sqrt{5}} \\ -\frac{33}{7} \end{pmatrix} \tag{1.138.33}$$

$$\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} -\frac{54}{49\sqrt{5}} \\ \frac{8}{7\sqrt{5}} \end{pmatrix} \tag{1.138.34}$$

$$\mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} -\frac{100}{49} \\ \frac{146}{49} \end{pmatrix} \tag{1.138.35}$$

Verifying the solution of (1.138.35) using,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \tag{1.138.36}$$

Evaluating the R.H.S in (1.138.36) we get,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \begin{pmatrix} -7 \\ \frac{1}{2} \end{pmatrix} \tag{1.138.37}$$

$$\implies \begin{pmatrix} 10 & \frac{9}{2} \\ \frac{9}{2} & \frac{13}{4} \end{pmatrix} \mathbf{x} = \begin{pmatrix} -7 \\ \frac{1}{2} \end{pmatrix} \tag{1.138.38}$$

Solving the augmented matrix of (1.138.38) we get,

$$\begin{pmatrix} 10 & \frac{9}{2} & -7 \\ \frac{9}{2} & \frac{13}{4} & \frac{1}{2} \end{pmatrix} \xleftrightarrow{R_1=\frac{1}{10}R_1} \begin{pmatrix} 1 & \frac{9}{20} & -\frac{7}{10} \\ \frac{9}{2} & \frac{13}{4} & \frac{1}{2} \end{pmatrix} \tag{1.138.39}$$

$$\xleftrightarrow{R_2=R_2-\frac{9}{2}R_1} \begin{pmatrix} 1 & \frac{9}{20} & -\frac{7}{10} \\ 0 & \frac{49}{40} & \frac{73}{20} \end{pmatrix} \tag{1.138.40}$$

$$\xleftrightarrow{R_2=\frac{40}{49}R_2} \begin{pmatrix} 1 & \frac{9}{20} & -\frac{7}{10} \\ 0 & 1 & \frac{146}{49} \end{pmatrix} \tag{1.138.41}$$

$$\xleftrightarrow{R_1=R_1-\frac{9}{20}R_1} \begin{pmatrix} 1 & 0 & -\frac{100}{49} \\ 0 & 1 & \frac{146}{49} \end{pmatrix} \tag{1.138.42}$$

Hence, Solution of (1.138.36) is given by,

$$\mathbf{x} = \begin{pmatrix} -\frac{100}{49} \\ \frac{146}{49} \end{pmatrix} \tag{1.138.43}$$

Comparing results of $\mathbf{x}$ from (1.138.35) and (1.138.43) we conclude that the solution is verified.

1.139. Check whether the given line equations intersect. If they do not intersect find the closest points on the lines

$$L_1 : \quad \mathbf{x} = \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} \tag{1.139.1}$$

$$L_2 : \quad \mathbf{x} = \begin{pmatrix} 7 \\ -6 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \tag{1.139.2}$$

Solution:
Given

$$L_1 : \quad \mathbf{x} = \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} \tag{1.139.3}$$

$$L_2 : \quad \mathbf{x} = \begin{pmatrix} 7 \\ -6 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \tag{1.139.4}$$

The above equations (1.139.3), (1.139.4) are in the form

$$L_1 : \quad \mathbf{x} = \mathbf{a_1} + \lambda_1\mathbf{b_1} \tag{1.139.5}$$

$$L_2 : \quad \mathbf{x} = \mathbf{a_2} + \lambda_2\mathbf{b_2} \tag{1.139.6}$$

Here ,

$$\mathbf{a_1} = \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} \tag{1.139.7}$$

$$\mathbf{a_2} = \begin{pmatrix} 7 \\ -6 \\ 0 \end{pmatrix} \tag{1.139.8}$$

$$\mathbf{b_1} = \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} \tag{1.139.9}$$

$$\mathbf{b_2} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \tag{1.139.10}$$

Now let us assume the lines L_1 and L_2 are intersecting at a point. Therefore ,

$$\begin{pmatrix}2\\-5\\1\end{pmatrix}+\lambda_1\begin{pmatrix}3\\2\\6\end{pmatrix}=\begin{pmatrix}7\\-6\\0\end{pmatrix}+\lambda_2\begin{pmatrix}1\\2\\2\end{pmatrix} \quad (1.139.11)$$

$$\lambda_1\begin{pmatrix}3\\2\\6\end{pmatrix}+\lambda_2\begin{pmatrix}-1\\-2\\-2\end{pmatrix}=\begin{pmatrix}5\\-1\\-1\end{pmatrix} \quad (1.139.12)$$

$$\begin{pmatrix}3&-1\\2&-2\\6&-2\end{pmatrix}\begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix}=\begin{pmatrix}5\\-1\\-1\end{pmatrix} \quad (1.139.13)$$

The augumented matrix of (1.139.13) is given by

$$\left(\begin{array}{cc|c}3&-1&5\\2&-2&-1\\6&-2&-1\end{array}\right) \quad (1.139.14)$$

$$\left(\begin{array}{cc|c}3&-1&5\\2&-2&-1\\6&-2&-1\end{array}\right)\xleftrightarrow{R_2=R_2-\frac{2}{3}R_1}\left(\begin{array}{cc|c}3&-1&5\\0&-\frac{4}{3}&-\frac{13}{3}\\6&-2&-1\end{array}\right) \quad (1.139.15)$$

$$\left(\begin{array}{cc|c}3&-1&5\\0&-\frac{4}{3}&-\frac{13}{3}\\6&-2&-1\end{array}\right)\xleftrightarrow{R_3=R_3-2R_1}\left(\begin{array}{cc|c}3&-1&5\\0&-\frac{4}{3}&-\frac{13}{3}\\0&0&-11\end{array}\right) \quad (1.139.16)$$

Since the rank of augmented matrix will be 3. We can say that lines do not intersect.Hence our assumptions is wrong
Equation (1.139.13) can be expressed as

$$\mathbf{Mx}=\mathbf{b} \quad (1.139.17)$$

By singular value decomposition $\mathbf{M}$ can be expressed as

$$\mathbf{M}=\mathbf{USV}^T \quad (1.139.18)$$

Where the columns of $\mathbf{V}$ are the eigenvectors of $\mathbf{M}^T\mathbf{M}$,the columns of $\mathbf{U}$ are the eigenvectors of $\mathbf{MM}^T$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M}=\begin{pmatrix}49&-19\\-19&9\end{pmatrix} \quad (1.139.19)$$

$$\mathbf{MM}^T=\begin{pmatrix}10&8&20\\8&8&16\\20&16&40\end{pmatrix} \quad (1.139.20)$$

The characteristic equation of $\mathbf{M}^T\mathbf{M}$ is obtained by evaluating the determinant

$$\begin{vmatrix}49-\lambda&-19\\-19&9-\lambda\end{vmatrix}=0 \quad (1.139.21)$$

$$\implies \lambda^2-58\lambda+80=0 \quad (1.139.22)$$

The eigenvalues are the roots of equation 1.139.22 is given by

$$\lambda_{11}=29+\sqrt{761} \quad (1.139.23)$$

$$\lambda_{12}=29-\sqrt{761} \quad (1.139.24)$$

The eigen vectors comes out to be ,

$$\mathbf{u_{11}}=\begin{pmatrix}\frac{-20-\sqrt{761}}{19}\\1\end{pmatrix},\mathbf{u_{12}}=\begin{pmatrix}\frac{-20+\sqrt{761}}{19}\\1\end{pmatrix} \quad (1.139.25)$$

Normalising the eigen vectors,

$$l_{11}=\sqrt{\left(\frac{-20-\sqrt{761}}{19}\right)^2+1^2} \quad (1.139.26)$$

$$\implies l_{11}=\frac{\sqrt{1522+40\sqrt{761}}}{19} \quad (1.139.27)$$

$$\mathbf{u_{11}}=\begin{pmatrix}\frac{-20-\sqrt{761}}{\sqrt{1522+40\sqrt{761}}}\\\frac{19}{\sqrt{1522+40\sqrt{761}}}\end{pmatrix} \quad (1.139.28)$$

$$l_{12}=\sqrt{\left(\frac{-20+\sqrt{761}}{19}\right)^2+1^2} \quad (1.139.29)$$

$$\implies l_{12}=\frac{\sqrt{1522-40\sqrt{761}}}{19} \quad (1.139.30)$$

$$\mathbf{u_{12}}=\begin{pmatrix}\frac{-20+\sqrt{761}}{\sqrt{1522-40\sqrt{761}}}\\\frac{19}{\sqrt{1522-40\sqrt{761}}}\end{pmatrix} \quad (1.139.31)$$

$$\mathbf{V}=\begin{pmatrix}\frac{-20-\sqrt{761}}{\sqrt{1522+40\sqrt{761}}}&\frac{-20+\sqrt{761}}{\sqrt{1522-40\sqrt{761}}}\\\frac{19}{\sqrt{1522+40\sqrt{761}}}&\frac{19}{\sqrt{1522-40\sqrt{761}}}\end{pmatrix} \quad (1.139.32)$$

$\mathbf{S}$ is given by

$$\mathbf{S} = \begin{pmatrix} \sqrt{29+\sqrt{761}} & 0 \\ 0 & \sqrt{29-\sqrt{761}} \\ 0 & 0 \end{pmatrix} \tag{1.139.33}$$

The characteristic equation of $\mathbf{MM}^T$ is obtained by evaluating the determinant

$$\begin{vmatrix} 10-\lambda & 8 & 20 \\ 8 & 8-\lambda & 16 \\ 20 & 16 & 40-\lambda \end{vmatrix} = 0 \tag{1.139.34}$$

$$\implies \lambda^3 - 58\lambda^2 + 80\lambda = 0 \tag{1.139.35}$$

The eigenvalues are the roots of equation 1.139.35 is given by

$$\lambda_{21} = 29 + \sqrt{761} \tag{1.139.36}$$

$$\lambda_{22} = 29 - \sqrt{761} \tag{1.139.37}$$

$$\lambda_{23} = 0 \tag{1.139.38}$$

The eigen vectors comes out to be ,

$$\mathbf{u_{21}} = \begin{pmatrix} \frac{-1}{2} \\ \frac{-\sqrt{761}+21}{16} \\ -1 \end{pmatrix}, \mathbf{u_{22}} = \begin{pmatrix} \frac{1}{2} \\ \frac{-\sqrt{761}-21}{16} \\ 1 \end{pmatrix}, \mathbf{u_{23}} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \tag{1.139.39}$$

Normalising the eigen vectors,

$$l_{21} = \sqrt{\left(\frac{-1}{2}\right)^2 + \left(\frac{21-\sqrt{761}}{16}\right)^2 + (-1)^2} \tag{1.139.40}$$

$$\implies l_{21} = \frac{\sqrt{1522 - 42\sqrt{761}}}{16} \tag{1.139.41}$$

$$\mathbf{u_{21}} = \begin{pmatrix} \frac{-8}{\sqrt{1522-42\sqrt{761}}} \\ \frac{21-\sqrt{761}}{\sqrt{1522-42\sqrt{761}}} \\ \frac{-16}{\sqrt{1522-42\sqrt{761}}} \end{pmatrix} \tag{1.139.42}$$

$$l_{22} = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{-21-\sqrt{761}}{16}\right)^2 + 1^2} \tag{1.139.43}$$

$$\implies l_{22} = \frac{\sqrt{1522 + 42\sqrt{761}}}{16} \tag{1.139.44}$$

$$\mathbf{u_{22}} = \begin{pmatrix} \frac{8}{\sqrt{1522+42\sqrt{761}}} \\ \frac{-21-\sqrt{761}}{\sqrt{1522+42\sqrt{761}}} \\ \frac{16}{\sqrt{1522+42\sqrt{761}}} \end{pmatrix} \tag{1.139.45}$$

$$l_{23} = \sqrt{(-2)^2 + 1^2} = \sqrt{5} \tag{1.139.46}$$

$$\mathbf{u_{23}} = \begin{pmatrix} \frac{-2}{\sqrt{5}} \\ 0 \\ \frac{1}{\sqrt{5}} \end{pmatrix} \tag{1.139.47}$$

$$\mathbf{U} = \begin{pmatrix} \frac{-8}{\sqrt{1522-42\sqrt{761}}} & \frac{8}{\sqrt{1522+42\sqrt{761}}} & \frac{-2}{\sqrt{5}} \\ \frac{21-\sqrt{761}}{\sqrt{1522-42\sqrt{761}}} & \frac{-21-\sqrt{761}}{\sqrt{1522+42\sqrt{761}}} & 0 \\ \frac{-16}{\sqrt{1522-42\sqrt{761}}} & \frac{16}{\sqrt{1522+42\sqrt{761}}} & \frac{1}{\sqrt{5}} \end{pmatrix} \tag{1.139.48}$$

From equation (1.139.18) we rewrite $\mathbf{M}$ as follows,

$$\begin{pmatrix} 3 & -1 \\ 2 & -2 \\ 6 & -2 \end{pmatrix} = \begin{pmatrix} \frac{-8}{\sqrt{1522-42\sqrt{761}}} & \frac{8}{\sqrt{1522+42\sqrt{761}}} & \frac{-2}{\sqrt{5}} \\ \frac{21-\sqrt{761}}{\sqrt{1522-42\sqrt{761}}} & \frac{-21-\sqrt{761}}{\sqrt{1522+42\sqrt{761}}} & 0 \\ \frac{-16}{\sqrt{1522-42\sqrt{761}}} & \frac{16}{\sqrt{1522+42\sqrt{761}}} & \frac{1}{\sqrt{5}} \end{pmatrix} \tag{1.139.49}$$

$$\begin{pmatrix} \sqrt{29+\sqrt{7 \ldots}} \\ 0 \\ 0 \end{pmatrix} \tag{1.139.50}$$

$$\begin{pmatrix} \frac{-20-\sqrt{761}}{\sqrt{1522+40\sqrt{76 \ldots}}} \\ \frac{19}{\sqrt{1522+40\sqrt{76 \ldots}}} \end{pmatrix} \tag{1.139.51}$$

By substituting the equation (1.139.18) in equation (1.139.17) we get

$$\mathbf{USV}^T\mathbf{x} = \mathbf{b} \tag{1.139.52}$$

$$\implies \mathbf{x} = \mathbf{VS_+U^Tb} \tag{1.139.53}$$

Where $\mathbf{S}_+$ is Moore-Penrose Pseudo-Inverse of $\mathbf{S}$

$$\mathbf{S}_+ = \begin{pmatrix} \frac{1}{\sqrt{29+\sqrt{761}}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{29-\sqrt{761}}} & 0 \end{pmatrix} \tag{1.139.54}$$

From (1.139.53) we get,

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{\sqrt{761}-45}{\sqrt{1522-42\sqrt{761}}} \\ \frac{45+\sqrt{761}}{\sqrt{1522+42\sqrt{761}}} \\ -\frac{11}{\sqrt{5}} \end{pmatrix} \tag{1.139.55}$$

$$\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{761\sqrt{15}-761-45\sqrt{11415}+45\sqrt{761}}{10654} \\ \frac{45\sqrt{11415}+45\sqrt{761}+761\sqrt{15}+761}{10654} \end{pmatrix} \tag{1.139.56}$$

$$\mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{11}{20} \\ \frac{21}{20} \end{pmatrix} \tag{1.139.57}$$

Verifying the solution of (1.139.57) using,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \tag{1.139.58}$$

Evaluating the R.H.S in (1.139.58) we get,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \begin{pmatrix} 7 \\ -1 \end{pmatrix} \tag{1.139.59}$$

$$\implies \begin{pmatrix} 49 & -19 \\ -19 & 9 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 7 \\ -1 \end{pmatrix} \tag{1.139.60}$$

Solving the augmented matrix of (1.139.60) we get,

$$\begin{pmatrix} 49 & -19 & 7 \\ -19 & 9 & -1 \end{pmatrix} \xleftrightarrow{R_2=R_2+\frac{19}{49}R_1} \begin{pmatrix} 49 & -19 & 7 \\ 0 & \frac{80}{49} & \frac{12}{7} \end{pmatrix} \tag{1.139.61}$$

$$\xleftrightarrow{R_1=\frac{1}{49}R_1} \begin{pmatrix} 1 & \frac{-19}{49} & \frac{7}{49} \\ 0 & \frac{80}{49} & \frac{12}{7} \end{pmatrix} \tag{1.139.62}$$

$$\xleftrightarrow{R_2=\frac{80}{49}R_2} \begin{pmatrix} 1 & \frac{-19}{49} & \frac{7}{49} \\ 0 & 1 & \frac{21}{20} \end{pmatrix} \tag{1.139.63}$$

$$\xleftrightarrow{R_1=R_1+\frac{19}{49}R_2} \begin{pmatrix} 1 & 0 & \frac{11}{20} \\ 0 & 1 & \frac{21}{20} \end{pmatrix} \tag{1.139.64}$$

Hence, Solution of (1.139.58) is given by,

$$\mathbf{x} = \begin{pmatrix} \frac{11}{20} \\ \frac{21}{20} \end{pmatrix} \tag{1.139.65}$$

Comparing results of $\mathbf{x}$ from (1.139.57) and (1.139.65) we conclude that the solution is verified.

1.140. Check if the lines L_1, L_2 are skew.If so, find the closest points on those lines using Singular Value Decomposition(SVD)

$$L_1 : \mathbf{x} = \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} \tag{1.140.1}$$

$$L_2 : \mathbf{x} = \begin{pmatrix} 7 \\ -6 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \tag{1.140.2}$$

Solution:
The matrix $\mathbf{M}$ of dimensions $(m \times n)$ can be decomposed using SVD as

$$\mathbf{M} = \mathbf{U}\mathbf{S}\mathbf{V}^T \tag{1.140.3}$$

where,columns of $\mathbf{U}_{(m\times m)}$ are eigen vectors of $\mathbf{M}\mathbf{M}^T$
columns of $\mathbf{V}_{(n\times n)}$ are eigen vectors of $\mathbf{M}^T\mathbf{M}$
$\mathbf{S}$ is a diagonal matrix containing singular values of $\mathbf{M}$. Also, $\mathbf{U}$ and $\mathbf{V}$ are orthogonal matrices

$$\mathbf{U}\mathbf{U}^T = \mathbf{U}^T\mathbf{U} = \mathbf{I} \tag{1.140.4}$$

$$\mathbf{V}\mathbf{V}^T = \mathbf{V}^T\mathbf{V} = \mathbf{I} \tag{1.140.5}$$

Given line equations intersect if

$$\begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} = \begin{pmatrix} 7 \\ -6 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \tag{1.140.6}$$

This can be written as

$$\begin{pmatrix} 3 & 1 \\ 2 & 2 \\ 6 & 2 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 5 \\ -1 \\ -1 \end{pmatrix} \tag{1.140.7}$$

$$\mathbf{M}\mathbf{x} = \mathbf{b} \tag{1.140.8}$$

$$\text{where, } \mathbf{x} = \begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} \tag{1.140.9}$$

The augmented matrix is

$$\begin{pmatrix} 3 & 1 & 5 \\ 2 & 2 & -1 \\ 6 & 2 & -1 \end{pmatrix} \xleftrightarrow[R_2\leftarrow R_2-R_1\times\frac{2}{3}]{R_3\leftarrow R_3-2\times R_1} \begin{pmatrix} 3 & 1 & 5 \\ 0 & \frac{5}{3} & -\frac{13}{3} \\ 0 & 0 & -11 \end{pmatrix} \tag{1.140.10}$$

So, the given pair of lines do not intersect and also their direction vectors are not parallel. Hence they are skew lines.

To find **U**,

$$\mathbf{MM}^T = \begin{pmatrix} 3 & 1 \\ 2 & 2 \\ 6 & 2 \end{pmatrix} \begin{pmatrix} 3 & 2 & 6 \\ 1 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 10 & 8 & 20 \\ 8 & 8 & 16 \\ 20 & 16 & 40 \end{pmatrix} \tag{1.140.11}$$

To calculate its Eigen values,

$$\begin{vmatrix} 10-\lambda & 8 & 20 \\ 8 & 8-\lambda & 16 \\ 20 & 16 & 40-\lambda \end{vmatrix} = 0 \tag{1.140.12}$$

$$\implies \lambda^3 + 58\lambda^2 + 80\lambda = 0 \tag{1.140.13}$$

$$\lambda_1 = 29 - \sqrt{761}, \lambda_2 = 29 + \sqrt{761}, \lambda_3 = 0 \tag{1.140.14}$$

with corresponding Eigen vectors as

$$\mathbf{u_1} = \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2 + 1 + \left(\frac{21+\sqrt{761}}{16}\right)^2}} \begin{pmatrix} \frac{1}{2} \\ \frac{-21-\sqrt{761}}{16} \\ 1 \end{pmatrix} \tag{1.140.15}$$

$$\mathbf{u_2} = \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2 + 1 + \left(\frac{-21+\sqrt{761}}{16}\right)^2}} \begin{pmatrix} \frac{1}{2} \\ \frac{-21+\sqrt{761}}{16} \\ 1 \end{pmatrix} \tag{1.140.16}$$

$$\mathbf{u_3} = \frac{1}{\sqrt{(-2)^2 + 1}} \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \tag{1.140.17}$$

Solving, the **U** matrix becomes

$$\mathbf{U} = \begin{pmatrix} \frac{8}{\sqrt{1522+42\sqrt{761}}} & \frac{8}{\sqrt{1522-42\sqrt{761}}} & -\frac{2}{\sqrt{5}} \\ \frac{-21-\sqrt{761}}{\sqrt{1522+42\sqrt{761}}} & \frac{-21+\sqrt{761}}{\sqrt{1522-42\sqrt{761}}} & 0 \\ \frac{16}{\sqrt{1522+42\sqrt{761}}} & \frac{16}{\sqrt{1522-42\sqrt{761}}} & \frac{1}{\sqrt{5}} \end{pmatrix} \tag{1.140.18}$$

Also, from the obtained Eigen values,the **S** matrix becomes

$$\mathbf{S} = \begin{pmatrix} \sqrt{29 - \sqrt{761}} & 0 \\ 0 & \sqrt{29 + \sqrt{761}} \\ 0 & 0 \end{pmatrix} \tag{1.140.19}$$

The Moore-Penrose pseudo inverse of **S** is given by

$$\mathbf{S}_+ = \begin{pmatrix} \frac{1}{\sqrt{29-\sqrt{761}}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{29+\sqrt{761}}} & 0 \end{pmatrix} \tag{1.140.20}$$

Now to find **V**,
Rewriting (1.140.3)

$$\mathbf{V} = (\mathbf{M}^T\mathbf{U})\mathbf{S}_+{}^T \tag{1.140.21}$$

$\mathbf{M}^T\mathbf{U}$ becomes

$$\begin{pmatrix} 3 & 2 & 6 \\ 1 & 2 & 2 \end{pmatrix} \tag{1.140.22}$$

$$\begin{pmatrix} \frac{8}{\sqrt{1522+42\sqrt{761}}} & \frac{8}{\sqrt{1522-42\sqrt{761}}} & -\frac{2}{\sqrt{5}} \\ \frac{-21-\sqrt{761}}{\sqrt{1522+42\sqrt{761}}} & \frac{-21+\sqrt{761}}{\sqrt{1522-42\sqrt{761}}} & 0 \\ \frac{16}{\sqrt{1522+42\sqrt{761}}} & \frac{16}{\sqrt{1522-42\sqrt{761}}} & \frac{1}{\sqrt{5}} \end{pmatrix} \tag{1.140.23}$$

$$= \begin{pmatrix} \frac{78-2\sqrt{761}}{\sqrt{1522+42\sqrt{761}}} & \frac{78+2\sqrt{761}}{\sqrt{1522-42\sqrt{761}}} & 0 \\ \frac{-2-2\sqrt{761}}{\sqrt{1522+42\sqrt{761}}} & \frac{-2+2\sqrt{761}}{\sqrt{1522-42\sqrt{761}}} & 0 \end{pmatrix} \tag{1.140.24}$$

Therefore from (1.140.20),(1.140.21),(1.140.24),

$$\mathbf{V} = \begin{pmatrix} \frac{78-2\sqrt{761}}{\sqrt{1522+42\sqrt{761}}\sqrt{29-\sqrt{761}}} & \frac{78+2\sqrt{761}}{\sqrt{1522-42\sqrt{761}}\sqrt{29+\sqrt{761}}} \\ \frac{-2-2\sqrt{761}}{\sqrt{1522+42\sqrt{761}}\sqrt{29-\sqrt{761}}} & \frac{-2+2\sqrt{761}}{\sqrt{1522-42\sqrt{761}}\sqrt{29+\sqrt{761}}} \end{pmatrix} \tag{1.140.25}$$

Now, to calculate **x**

$$\mathbf{Mx} = \mathbf{b} \tag{1.140.26}$$

$$\implies \mathbf{USV}^T\mathbf{x} = \mathbf{b} \tag{1.140.27}$$

$$\implies \mathbf{SV}^T\mathbf{x} = \mathbf{U}^T\mathbf{b} \tag{1.140.28}$$

$$\implies \mathbf{x} = \mathbf{V}(\mathbf{S}_+(\mathbf{U}^T\mathbf{b})) \tag{1.140.29}$$

Calculating $\mathbf{U}^T\mathbf{b}$, we have

$$\begin{pmatrix} \frac{45+\sqrt{761}}{\sqrt{1522+42\sqrt{761}}} \\ \frac{45-\sqrt{761}}{\sqrt{1522-42\sqrt{761}}} \end{pmatrix} \tag{1.140.30}$$

$$\mathbf{S}_+(\mathbf{U}^T\mathbf{b}) = \begin{pmatrix} \frac{45+\sqrt{761}}{\sqrt{1522+42\sqrt{761}}\sqrt{29-\sqrt{761}}} \\ \frac{45-\sqrt{761}}{\sqrt{1522-42\sqrt{761}}\sqrt{29+\sqrt{761}}} \end{pmatrix} \tag{1.140.31}$$

$\mathbf{V}(\mathbf{S}_+(\mathbf{U}^T\mathbf{b}))$

$$= \begin{pmatrix} \frac{78-2\sqrt{761}}{\sqrt{1522+42\sqrt{761}}\sqrt{29-\sqrt{761}}} & \frac{78+2\sqrt{761}}{\sqrt{1522-42\sqrt{761}}\sqrt{29+\sqrt{761}}} \\ \frac{-2-2\sqrt{761}}{\sqrt{1522+42\sqrt{761}}\sqrt{29-\sqrt{761}}} & \frac{-2+2\sqrt{761}}{\sqrt{1522-42\sqrt{761}}\sqrt{29+\sqrt{761}}} \end{pmatrix} \tag{1.140.32}$$

$$\begin{pmatrix} \frac{45+\sqrt{761}}{\sqrt{1522+42\sqrt{761}}\sqrt{29-\sqrt{761}}} \\ \frac{45-\sqrt{761}}{\sqrt{1522-42\sqrt{761}}\sqrt{29+\sqrt{761}}} \end{pmatrix} \tag{1.140.33}$$

Solving,

$$\mathbf{x} = \begin{pmatrix} \frac{8371}{15220} \\ \frac{-15981}{15220} \end{pmatrix} = \begin{pmatrix} \frac{11}{20} \\ \frac{-21}{20} \end{pmatrix} \tag{1.140.34}$$

Verifying the solution,

$$\mathbf{M}\mathbf{x} = \mathbf{b} \tag{1.140.35}$$

$$\implies \mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \tag{1.140.36}$$

$$\mathbf{M}^T\mathbf{b} = \begin{pmatrix} 3 & 2 & 6 \\ 1 & 2 & 2 \end{pmatrix}\begin{pmatrix} 5 \\ -1 \\ -1 \end{pmatrix} \tag{1.140.37}$$

$$= \begin{pmatrix} 7 \\ 1 \end{pmatrix} \tag{1.140.38}$$

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 3 & 2 & 6 \\ 1 & 2 & 2 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 2 & 2 \\ 6 & 2 \end{pmatrix} \tag{1.140.39}$$

$$= \begin{pmatrix} 49 & 19 \\ 19 & 9 \end{pmatrix} \tag{1.140.40}$$

$$\text{From, (1.140.36)} \begin{pmatrix} 49 & 19 \\ 19 & 9 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 7 \\ 1 \end{pmatrix} \tag{1.140.41}$$

Solving for $\mathbf{x}$

$$\begin{pmatrix} 49 & 19 & 7 \\ 19 & 9 & 1 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - R_1 \times \frac{19}{49}} \begin{pmatrix} 49 & 19 & 7 \\ 0 & \frac{80}{49} & \frac{-84}{49} \end{pmatrix} \tag{1.140.42}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 \times \frac{1}{49}} \begin{pmatrix} 1 & \frac{19}{49} & \frac{7}{49} \\ 0 & \frac{80}{49} & \frac{-84}{49} \end{pmatrix} \tag{1.140.43}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - R_2 \times \frac{19}{80}} \begin{pmatrix} 1 & 0 & \frac{11}{20} \\ 0 & \frac{80}{49} & \frac{-84}{49} \end{pmatrix} \tag{1.140.44}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 \times \frac{49}{80}} \begin{pmatrix} 1 & 0 & \frac{11}{20} \\ 0 & 1 & \frac{-21}{20} \end{pmatrix} \tag{1.140.45}$$

$$\implies \mathbf{x} = \begin{pmatrix} \frac{11}{20} \\ \frac{-21}{20} \end{pmatrix} \tag{1.140.46}$$

1.141. Find the point on the plane closest to the point

$\begin{pmatrix}6\\5\\9\end{pmatrix}$ and the plane is determined by the points

$$\mathbf{A} = \begin{pmatrix}3\\-1\\2\end{pmatrix}, \mathbf{B} = \begin{pmatrix}5\\2\\4\end{pmatrix}, \mathbf{C} = \begin{pmatrix}-1\\-1\\6\end{pmatrix}$$

Solution: The equation of plane is given by,

$$\mathbf{n}^T\mathbf{x} = c \tag{1.141.1}$$

$$\mathbf{n}^T\mathbf{A} = \mathbf{n}^T\mathbf{B} = \mathbf{n}^T\mathbf{C} = c \tag{1.141.2}$$

$$\implies \begin{pmatrix}\mathbf{A}-\mathbf{B} & \mathbf{B}-\mathbf{C}\end{pmatrix}^T \mathbf{n} = 0 \tag{1.141.3}$$

Using row reduction on above matrix,

$$\begin{pmatrix}-2 & -3 & -2\\6 & 3 & -2\end{pmatrix} \xleftrightarrow{R_1\leftarrow\frac{R_1}{-2}} \begin{pmatrix}1 & \frac{3}{2} & 1\\6 & 3 & -2\end{pmatrix} \tag{1.141.4}$$

$$\xleftrightarrow{R_2\leftarrow R_2-6R_1} \begin{pmatrix}1 & \frac{3}{2} & 1\\0 & -6 & -8\end{pmatrix} \xleftrightarrow{R_2\leftarrow\frac{R_2}{-2}} \begin{pmatrix}1 & \frac{3}{2} & 1\\0 & 3 & 4\end{pmatrix} \tag{1.141.5}$$

$$\xleftrightarrow{R_1\leftarrow R_1-\frac{R_2}{2}} \begin{pmatrix}1 & 0 & -1\\0 & 3 & 4\end{pmatrix} \tag{1.141.6}$$

Thus,

$$\mathbf{n} = \begin{pmatrix}1\\\frac{-4}{3}\\1\end{pmatrix} = \begin{pmatrix}3\\-4\\3\end{pmatrix} \tag{1.141.7}$$

$$c = \mathbf{n}^T\mathbf{A} = 19 \tag{1.141.8}$$

Thus the equation of the plane is,

$$\begin{pmatrix}3 & -4 & 3\end{pmatrix}\mathbf{x} = 19 \tag{1.141.9}$$

Let $\mathbf{m_1}$ and $\mathbf{m_2}$ be the two orthogonal vectors to the given normal. Let, $\mathbf{m} = \begin{pmatrix}a\\b\\c\end{pmatrix}$, then

$$\mathbf{m}^T\mathbf{n} = 0 \tag{1.141.10}$$

$$\implies \begin{pmatrix}a & b & c\end{pmatrix}\begin{pmatrix}3\\-4\\3\end{pmatrix} = 0 \tag{1.141.11}$$

$$\implies 3a - 4b + 3c = 0 \tag{1.141.12}$$

Let $a = 1$, $b = 0$ we get,

$$\mathbf{m_1} = \begin{pmatrix}1\\0\\-1\end{pmatrix} \tag{1.141.13}$$

Let $a = 0$, $b = 1$ we get,

$$\mathbf{m_2} = \begin{pmatrix}0\\1\\\frac{4}{3}\end{pmatrix} \tag{1.141.14}$$

Solving the equation,

$$\mathbf{Mx} = \mathbf{b} \tag{1.141.15}$$

Putting the values in (1.141.15),

$$\begin{pmatrix}1 & 0\\0 & 1\\-1 & \frac{4}{3}\end{pmatrix}\mathbf{x} = \begin{pmatrix}6\\5\\9\end{pmatrix} \tag{1.141.16}$$

To solve (1.141.16), we perform Singular Value Decomposition on $\mathbf{M}$,

$$\mathbf{M} = \mathbf{USV}^T \tag{1.141.17}$$

Where the columns of $\mathbf{V}$ are the eigen vectors of $\mathbf{M}^T\mathbf{M}$,the columns of $\mathbf{U}$ are the eigen vectors of $\mathbf{MM}^T$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix}2 & \frac{-4}{3}\\\frac{-4}{3} & \frac{25}{9}\end{pmatrix} \tag{1.141.18}$$

$$\mathbf{MM}^T = \begin{pmatrix}1 & 0 & -1\\0 & 1 & \frac{4}{3}\\-1 & \frac{4}{3} & \frac{25}{9}\end{pmatrix} \tag{1.141.19}$$

Putting (1.141.17) in (1.141.15) we get,

$$\mathbf{USV}^T\mathbf{x} = \mathbf{b} \tag{1.141.20}$$

$$\implies \mathbf{x} = \mathbf{VS_+U^Tb} \tag{1.141.21}$$

Where $\mathbf{S_+}$ is Moore-Penrose Pseudo-Inverse of $\mathbf{S}$. Now, calculating eigen values of $\mathbf{MM}^T$,

$$\left|\mathbf{MM}^T - \lambda\mathbf{I}\right| = 0 \tag{1.141.22}$$

$$\implies \begin{pmatrix}1-\lambda & 0 & -1\\0 & 1-\lambda & \frac{4}{3}\\-1 & \frac{4}{3} & \frac{25}{9}-\lambda\end{pmatrix} = 0 \tag{1.141.23}$$

$$\implies \lambda^3 - \frac{43}{9}\lambda^2 + \frac{34}{9}\lambda = 0 \tag{1.141.24}$$

Thus the eigen values of $\mathbf{MM}^T$ are,

$$\lambda_1 = \frac{34}{9} \tag{1.141.25}$$

$$\lambda_2 = 1 \tag{1.141.26}$$

$$\lambda_3 = 0 \tag{1.141.27}$$

The eigen vectors comes out to be,

$$\mathbf{u_1} = \begin{pmatrix} \frac{-9}{25} \\ \frac{12}{25} \\ 1 \end{pmatrix}, \mathbf{u_2} = \begin{pmatrix} \frac{4}{3} \\ 1 \\ 0 \end{pmatrix}, \mathbf{u_3} = \begin{pmatrix} 1 \\ \frac{-4}{3} \\ 1 \end{pmatrix} \tag{1.141.28}$$

Normalising the eigen vectors,

$$\mathbf{u_1} = \begin{pmatrix} \frac{-9}{5\sqrt{34}} \\ \frac{12}{5\sqrt{34}} \\ \frac{5}{\sqrt{34}} \end{pmatrix}, \mathbf{u_2} = \begin{pmatrix} \frac{4}{5} \\ \frac{3}{5} \\ 0 \end{pmatrix}, \mathbf{u_3} = \begin{pmatrix} \frac{3}{\sqrt{34}} \\ \frac{-4}{\sqrt{34}} \\ \frac{3}{\sqrt{34}} \end{pmatrix} \tag{1.141.29}$$

Hence we obtain $\mathbf{U}$ matrix as,

$$\mathbf{U} = \begin{pmatrix} \frac{-9}{5\sqrt{34}} & \frac{4}{5} & \frac{3}{\sqrt{34}} \\ \frac{12}{5\sqrt{34}} & \frac{3}{5} & \frac{-4}{\sqrt{34}} \\ \frac{5}{\sqrt{34}} & 0 & \frac{3}{\sqrt{34}} \end{pmatrix} \tag{1.141.30}$$

Now,

$$\mathbf{S} = \begin{pmatrix} \frac{\sqrt{34}}{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{1.141.31}$$

Calculating the eigen values of $\mathbf{M}^T\mathbf{M}$,

$$\left|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}\right| = 0 \tag{1.141.32}$$

$$\implies \begin{pmatrix} 2 - \lambda & \frac{-4}{3} \\ \frac{-4}{3} & \frac{25}{9} - \lambda \end{pmatrix} = 0 \tag{1.141.33}$$

$$\implies \lambda^2 - \frac{43}{9}\lambda + \frac{34}{9} = 0 \tag{1.141.34}$$

The eigen values are,

$$\lambda_1 = \frac{34}{9} \tag{1.141.35}$$

$$\lambda_2 = 1 \tag{1.141.36}$$

The eigen vectors are,

$$\mathbf{v_1} = \begin{pmatrix} \frac{-3}{4} \\ 1 \end{pmatrix}, \mathbf{v_2} = \begin{pmatrix} \frac{4}{3} \\ 1 \end{pmatrix} \tag{1.141.37}$$

Normalising the eigen vectors,

$$\mathbf{v_1} = \begin{pmatrix} \frac{-3}{5} \\ \frac{4}{5} \end{pmatrix}, \mathbf{v_2} = \begin{pmatrix} \frac{4}{5} \\ \frac{3}{5} \end{pmatrix} \tag{1.141.38}$$

Hence we obtain $\mathbf{V}$ matrix as,

$$\mathbf{V} = \begin{pmatrix} \frac{-3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix} \tag{1.141.39}$$

Thus we get the Singular Value Decomposition of $\mathbf{M}$ as,

$$\mathbf{M} = \begin{pmatrix} \frac{-9}{5\sqrt{34}} & \frac{4}{5} & \frac{3}{\sqrt{34}} \\ \frac{12}{5\sqrt{34}} & \frac{3}{5} & \frac{-4}{\sqrt{34}} \\ \frac{5}{\sqrt{34}} & 0 & \frac{3}{\sqrt{34}} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{34}}{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{-3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}^T \tag{1.141.40}$$

The Moore-Penrose Pseudo inverse of $\mathbf{S}$ is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{3}{\sqrt{34}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{1.141.41}$$

From (1.141.21) we get,

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{231}{5\sqrt{34}} \\ \frac{39}{5} \\ \frac{25}{\sqrt{34}} \end{pmatrix} \tag{1.141.42}$$

$$\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{693}{170} \\ \frac{39}{5} \end{pmatrix} \tag{1.141.43}$$

$$\mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{129}{34} \\ \frac{135}{17} \end{pmatrix} \tag{1.141.44}$$

Verifying the solution of (1.141.44) using,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \tag{1.141.45}$$

Evaluating the R.H.S in (1.141.45) we get,

$$\mathbf{M}^T\mathbf{b} = \begin{pmatrix} -3 \\ 17 \end{pmatrix} \tag{1.141.46}$$

$$\implies \begin{pmatrix} 2 & \frac{-4}{3} \\ \frac{-4}{3} & \frac{25}{9} \end{pmatrix}\mathbf{x} = \begin{pmatrix} -3 \\ 17 \end{pmatrix} \tag{1.141.47}$$

Solving the augmented matrix of (1.141.47) we get,

$$\begin{pmatrix} 2 & \frac{-4}{3} & -3 \\ \frac{-4}{3} & \frac{25}{9} & 17 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{R_1}{2}} \begin{pmatrix} 1 & \frac{-2}{3} & \frac{-3}{2} \\ \frac{-4}{3} & \frac{25}{9} & 17 \end{pmatrix} \tag{1.141.48}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 + \frac{4}{3}R-1} \begin{pmatrix} 1 & \frac{-2}{3} & \frac{-3}{2} \\ 0 & \frac{17}{9} & 15 \end{pmatrix} \tag{1.141.49}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + \frac{6}{17}R_2} \begin{pmatrix} 1 & 0 & \frac{129}{34} \\ 0 & \frac{17}{9} & 15 \end{pmatrix} \tag{1.141.50}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{9}{17}R_2} \begin{pmatrix} 1 & 0 & \frac{129}{34} \\ 0 & 1 & \frac{135}{17} \end{pmatrix} \tag{1.141.51}$$

Hence, solution of (1.141.45) is given by,

$$\mathbf{x} = \begin{pmatrix} \frac{129}{34} \\ \frac{135}{17} \end{pmatrix} \tag{1.141.52}$$

Comparing results of $\mathbf{x}$ from (1.141.44) and (1.141.52) we conclude that the solution is verified. Find the foot of the perpendicular using svd drawn from $\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$ to the plane

$$\begin{pmatrix} 2 & -1 & 2 \end{pmatrix} \mathbf{x} + 3 = 0 \tag{1.141.53}$$

Solution:

1.142. Find the distance of the given point $\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$ from the plane $\begin{pmatrix} 2 & -1 & 2 \end{pmatrix} \mathbf{x} = 3$.

Solution: Let us consider orthogonal vectors $\mathbf{m_1}$ and $\mathbf{m_2}$ to the given normal vector $\mathbf{n}$. Let, $\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, then

$$\mathbf{m^T n} = 0 \tag{1.142.1}$$

$$\implies \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = 0 \tag{1.142.2}$$

$$\implies 2a - b + 2c = 0 \tag{1.142.3}$$

Let a=1 and b=0 we get,

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \tag{1.142.4}$$

Let a=0 and b=1 we get,

$$\mathbf{m_2} = \begin{pmatrix} 0 \\ 1 \\ \frac{1}{2} \end{pmatrix} \tag{1.142.5}$$

Let us solve the equation,

$$\mathbf{Mx} = \mathbf{b} \tag{1.142.6}$$

Substituting (1.142.4) and (1.142.5) in (1.142.6),

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & \frac{1}{2} \end{pmatrix} \mathbf{x} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \tag{1.142.7}$$

To solve (1.142.7), we will perform Singular Value Decomposition on $\mathbf{M}$ as follows,

$$\mathbf{M} = \mathbf{USV}^T \tag{1.142.8}$$

Where the columns of $\mathbf{V}$ are the eigen vectors of $\mathbf{M}^T\mathbf{M}$,the columns of $\mathbf{U}$ are the eigen vectors of $\mathbf{MM}^T$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 2 & \frac{-1}{2} \\ \frac{-1}{2} & \frac{5}{4} \end{pmatrix} \tag{1.142.9}$$

$$\mathbf{MM}^T = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & \frac{1}{2} \\ -1 & \frac{1}{2} & \frac{5}{4} \end{pmatrix} \tag{1.142.10}$$

Substituting (1.142.8) in (1.142.6),

$$\mathbf{USV}^T\mathbf{x} = \mathbf{b} \tag{1.142.11}$$

$$\implies \mathbf{x} = \mathbf{VS_+U^Tb} \tag{1.142.12}$$

Where $\mathbf{S}_+$ is Moore-Penrose Pseudo-Inverse of $\mathbf{S}$.

Let us calculate eigen values of $\mathbf{MM}^T$,

$$\left|\mathbf{MM}^T - \lambda\mathbf{I}\right| = 0 \tag{1.142.13}$$

$$\implies \begin{pmatrix} 1-\lambda & 0 & -1 \\ 0 & 1-\lambda & \frac{1}{2} \\ -1 & \frac{1}{2} & \frac{5}{4}-\lambda \end{pmatrix} = 0 \tag{1.142.14}$$

$$\implies \lambda^3 - \frac{13}{4}\lambda^2 + \frac{9}{4}\lambda = 0 \tag{1.142.15}$$

From equation (1.142.15) eigen values of $\mathbf{MM}^T$ are,

$$\lambda_1 = \frac{9}{4} \quad \lambda_2 = 1 \quad \lambda_3 = 0 \tag{1.142.16}$$

The eigen vectors of $\mathbf{MM}^T$ are,

$$\mathbf{u}_1 = \begin{pmatrix} -\frac{4}{5} \\ \frac{2}{5} \\ 1 \end{pmatrix} \quad \mathbf{u}_2 = \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix} \quad \mathbf{u}_3 = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ 1 \end{pmatrix} \tag{1.142.17}$$

Normalizing the eigen vectors in equation (1.142.17)

$$\mathbf{u}_1 = \begin{pmatrix} -\frac{4}{3\sqrt{5}} \\ \frac{2}{3\sqrt{5}} \\ \frac{\sqrt{5}}{3} \end{pmatrix} \quad \mathbf{u}_2 = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \\ 0 \end{pmatrix} \quad \mathbf{u}_3 = \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{pmatrix} \tag{1.142.18}$$

Hence we obtain $\mathbf{U}$ as follows,

$$\mathbf{U} = \begin{pmatrix} -\frac{4}{3\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{2}{3} \\ \frac{2}{3\sqrt{5}} & \frac{2}{\sqrt{5}} & -\frac{1}{3} \\ \frac{\sqrt{5}}{3} & 0 & \frac{2}{3} \end{pmatrix} \quad (1.142.19)$$

After computing the singular values from eigen values $\lambda_1, \lambda_2, \lambda_3$ we get $\mathbf{S}$ as follows,

$$\mathbf{S} = \begin{pmatrix} \frac{9}{4} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (1.142.20)$$

Now, lets calculate eigen values of $\mathbf{M}^T\mathbf{M}$,

$$\left|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}\right| = 0 \quad (1.142.21)$$

$$\implies \begin{pmatrix} 2-\lambda & -\frac{1}{2} \\ -\frac{1}{2} & \frac{5}{4}-\lambda \end{pmatrix} = 0 \quad (1.142.22)$$

$$\implies \lambda^2 - \frac{13}{4}\lambda + \frac{9}{4} = 0 \quad (1.142.23)$$

Hence eigen values of $\mathbf{M}^T\mathbf{M}$ are,

$$\lambda_1 = \frac{9}{4} \quad \lambda_2 = 1 \quad (1.142.24)$$

Hence the eigen vectors of $\mathbf{M}^T\mathbf{M}$ are,

$$\mathbf{v}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} \quad (1.142.25)$$

Normalizing the eigen vectors,

$$\mathbf{v}_1 = \begin{pmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix} \quad (1.142.26)$$

Hence we obtain $\mathbf{V}$ as,

$$\mathbf{V} = \begin{pmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \quad (1.142.27)$$

From (1.142.6), the Singular Value Decomposition of $\mathbf{M}$ is as follows,

$$\mathbf{M} = \begin{pmatrix} -\frac{4}{3\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{2}{3} \\ \frac{2}{3\sqrt{5}} & \frac{2}{\sqrt{5}} & -\frac{1}{3} \\ \frac{\sqrt{5}}{3} & 0 & \frac{2}{3} \end{pmatrix} \begin{pmatrix} \frac{9}{4} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}^T \quad (1.142.28)$$

Now, Moore-Penrose Pseudo inverse of $\mathbf{S}$ is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (1.142.29)$$

From (1.142.12) we get,

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} -\frac{11}{3\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \\ \frac{10}{3} \end{pmatrix} \quad (1.142.30)$$

$$\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} -\frac{22}{9\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{pmatrix} \quad (1.142.31)$$

$$\mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{7}{9} \\ -\frac{8}{9} \end{pmatrix} \quad (1.142.32)$$

Verifying the solution of (1.142.32) using,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \quad (1.142.33)$$

Evaluating the R.H.S in (1.142.33) we get,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \begin{pmatrix} 2 \\ -\frac{3}{2} \end{pmatrix} \quad (1.142.34)$$

$$\implies \begin{pmatrix} 2 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{5}{4} \end{pmatrix}\mathbf{x} = \begin{pmatrix} 2 \\ -\frac{3}{2} \end{pmatrix} \quad (1.142.35)$$

Solving the augmented matrix of (1.142.35) we get,

$$\begin{pmatrix} 2 & -\frac{1}{2} & 2 \\ -\frac{1}{2} & \frac{5}{4} & -\frac{3}{2} \end{pmatrix} \xleftrightarrow{R_1=\frac{R_1}{2}} \begin{pmatrix} 1 & -\frac{1}{4} & 1 \\ -\frac{1}{2} & \frac{5}{4} & -\frac{3}{2} \end{pmatrix} \quad (1.142.36)$$

$$\xleftrightarrow{R_2=R_2+\frac{R_1}{2}} \begin{pmatrix} 1 & -\frac{1}{4} & 1 \\ 0 & \frac{9}{8} & -1 \end{pmatrix} \quad (1.142.37)$$

$$\xleftrightarrow{R_2=\frac{8}{9}R_2} \begin{pmatrix} 1 & -\frac{1}{4} & 1 \\ 0 & 1 & -\frac{8}{9} \end{pmatrix} \quad (1.142.38)$$

$$\xleftrightarrow{R_1=R_1+\frac{R_2}{4}} \begin{pmatrix} 1 & 0 & \frac{7}{9} \\ 0 & 1 & -\frac{8}{9} \end{pmatrix} \quad (1.142.39)$$

From equation (1.142.39), solution is given by,

$$\mathbf{x} = \begin{pmatrix} \frac{7}{9} \\ -\frac{8}{9} \end{pmatrix} \quad (1.142.40)$$

Comparing results of $\mathbf{x}$ from (1.142.32) and (1.142.40), we can say that the solution is verified.

1.143. Find the distance of the point $\begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix}$ from the plane $\begin{pmatrix} 1 & 2 & -2 \end{pmatrix}\mathbf{x} = 9$

Solution:

Find the distance of the point $\begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix}$ from the plane $\begin{pmatrix} 1 & 2 & -2 \end{pmatrix}\mathbf{x} = 9$ First we find orthogonal vectors $\mathbf{m_1}$ and $\mathbf{m_2}$ to the given normal vector $\mathbf{n}$. Let, $\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, then

$$\mathbf{m^T n} = 0 \tag{1.143.1}$$

$$\implies \begin{pmatrix} a & b & c \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = 0 \tag{1.143.2}$$

$$\implies a + 2b - 2c = 0 \tag{1.143.3}$$

Putting a=1 and b=0 we get,

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ 0 \\ \frac{1}{2} \end{pmatrix} \tag{1.143.4}$$

Putting a=0 and b=1 we get,

$$\mathbf{m_2} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \tag{1.143.5}$$

Now we solve the equation,

$$\mathbf{Mx} = \mathbf{b} \tag{1.143.6}$$

Putting values in (1.143.6),

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{2} & 1 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix} \tag{1.143.7}$$

In order to solve (1.143.7), perform Singular Value Decomposition on $\mathbf{M}$ as follows,

$$\mathbf{M} = \mathbf{USV}^T \tag{1.143.8}$$

Where the columns of $\mathbf{V}$ are the eigen vectors of $\mathbf{M}^T\mathbf{M}$,the columns of $\mathbf{U}$ are the eigen vectors of $\mathbf{MM}^T$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} \frac{5}{4} & \frac{1}{2} \\ \frac{1}{2} & 2 \end{pmatrix} \tag{1.143.9}$$

$$\mathbf{MM}^T = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 1 \\ \frac{1}{2} & 1 & \frac{5}{4} \end{pmatrix} \tag{1.143.10}$$

From (1.143.6) putting (1.143.8) we get,

$$\mathbf{USV}^T\mathbf{x} = \mathbf{b} \tag{1.143.11}$$

$$\implies \mathbf{x} = \mathbf{VS_+U^Tb} \tag{1.143.12}$$

Where $\mathbf{S}_+$ is Moore-Penrose Pseudo-Inverse of $\mathbf{S}$.Now, calculating eigen value of $\mathbf{MM}^T$,

$$\left|\mathbf{MM}^T - \lambda\mathbf{I}\right| = 0 \tag{1.143.13}$$

$$\implies \begin{pmatrix} 1-\lambda & 0 & \frac{1}{2} \\ 0 & 1-\lambda & 1 \\ \frac{1}{2} & 1 & \frac{5}{2}-\lambda \end{pmatrix} = 0 \tag{1.143.14}$$

$$\implies -4\lambda^3 + 13\lambda^2 - 9\lambda = 0 \tag{1.143.15}$$

Hence eigen values of $\mathbf{MM}^T$ are,

$$\lambda_1 = \frac{9}{4} \tag{1.143.16}$$

$$\lambda_2 = 1 \tag{1.143.17}$$

$$\lambda_3 = 0 \tag{1.143.18}$$

Hence the eigen vectors of $\mathbf{MM}^T$ are,

$$\mathbf{u}_1 = \begin{pmatrix} \frac{2}{5} \\ \frac{4}{5} \\ 1 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} \frac{-1}{2} \\ -1 \\ 1 \end{pmatrix} \tag{1.143.19}$$

Normalizing the eigen vectors we get,

$$\mathbf{u}_1 = \begin{pmatrix} \frac{2}{\sqrt{45}} \\ \frac{4}{\sqrt{45}} \\ \frac{5}{\sqrt{45}} \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ 0 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} -\frac{1}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{pmatrix} \tag{1.143.20}$$

Hence we obtain $\mathbf{U}$ of (1.143.8) as follows,

$$\begin{pmatrix} \frac{2}{\sqrt{45}} & -\frac{2}{\sqrt{5}} & -\frac{1}{3} \\ \frac{4}{\sqrt{45}} & \frac{1}{\sqrt{5}} & -\frac{2}{3} \\ \frac{5}{\sqrt{45}} & 0 & \frac{2}{3} \end{pmatrix} \tag{1.143.21}$$

After computing the singular values from eigen values $\lambda_1, \lambda_2, \lambda_3$ we get $\mathbf{S}$ of (1.143.8) as follows,

$$\mathbf{S} = \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{1.143.22}$$

Now, calculating eigen value of $\mathbf{M}^T\mathbf{M}$,

$$\left|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}\right| = 0 \tag{1.143.23}$$

$$\implies \begin{pmatrix} \frac{5}{4} - \lambda & \frac{1}{2} \\ \frac{1}{2} & 2 - \lambda \end{pmatrix} = 0 \tag{1.143.24}$$

$$\implies \lambda^2 - \frac{13}{4}\lambda + \frac{9}{4} = 0 \tag{1.143.25}$$

Hence eigen values of $\mathbf{M}^T\mathbf{M}$ are,

$$\lambda_4 = \frac{9}{4} \tag{1.143.26}$$

$$\lambda_5 = 1 \tag{1.143.27}$$

Hence the eigen vectors of $\mathbf{M}^T\mathbf{M}$ are,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \tag{1.143.28}$$

Normalizing the eigen vectors we get,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} \tag{1.143.29}$$

Hence we obtain $\mathbf{V}$ of (1.143.8) as follows,

$$\mathbf{V} = \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \tag{1.143.30}$$

From (1.143.8) we get the Singular Value Decomposition of $\mathbf{M}$,

$$\mathbf{M} = \begin{pmatrix} \frac{2}{\sqrt{45}} & -\frac{2}{\sqrt{5}} & -\frac{1}{3} \\ \frac{4}{\sqrt{45}} & \frac{1}{\sqrt{5}} & \frac{-2}{3} \\ \frac{5}{\sqrt{45}} & 0 & \frac{2}{3} \end{pmatrix} \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}^T \tag{1.143.31}$$

Moore-Penrose Pseudo inverse of $\mathbf{S}$ is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{1.143.32}$$

From (1.143.11) we get,

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} -\frac{3\sqrt{5}}{5} \\ -\frac{\sqrt{5}}{5} \\ -6 \end{pmatrix} \tag{1.143.33}$$

$$\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} -\frac{2\sqrt{5}}{5} \\ -\frac{\sqrt{5}}{5} \end{pmatrix} \tag{1.143.34}$$

$$\mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \tag{1.143.35}$$

Verifying the solution of (1.143.35) using,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \tag{1.143.36}$$

Evaluating the R.H.S in (1.143.36) we get,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \begin{pmatrix} -\frac{1}{2} \\ -2 \end{pmatrix} \tag{1.143.37}$$

$$\implies \begin{pmatrix} \frac{5}{4} & \frac{1}{2} \\ \frac{1}{2} & 2 \end{pmatrix}\mathbf{x} = \begin{pmatrix} -\frac{1}{2} \\ -2 \end{pmatrix} \tag{1.143.38}$$

Solving the augmented matrix of (1.143.38) we get,

$$\begin{pmatrix} \frac{5}{4} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & 2 & -2 \end{pmatrix} \xleftrightarrow{R_1=\frac{4}{5}R_1} \begin{pmatrix} 1 & \frac{2}{5} & \frac{-2}{5} \\ \frac{1}{2} & 2 & -2 \end{pmatrix} \tag{1.143.39}$$

$$\xleftrightarrow{R_2=R_2-\frac{1}{2}R_2} \begin{pmatrix} 1 & -\frac{2}{5} & -\frac{2}{5} \\ 0 & \frac{9}{5} & -\frac{9}{5} \end{pmatrix} \tag{1.143.40}$$

$$\xleftrightarrow{R_2=\frac{5}{9}R_2} \begin{pmatrix} 1 & \frac{2}{5} & -\frac{2}{5} \\ 0 & 1 & -1 \end{pmatrix} \tag{1.143.41}$$

$$\xleftrightarrow{R_1=R_1-\frac{2}{5}R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix} \tag{1.143.42}$$

From equation (1.143.42), solution is given by,

$$\mathbf{x} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \tag{1.143.43}$$

Comparing results of $\mathbf{x}$ from (1.143.35) and (1.143.43), we can say that the solution is verified.

1.144. If the lines

$$\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}, \tag{1.144.1}$$

$$\frac{x-3}{3k} = \frac{y-1}{1} = \frac{z-6}{-5}, \tag{1.144.2}$$

are perpendicular, find the value of k.

Solution: In the given problem,

$$\mathbf{A}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{m}_1 = \begin{pmatrix} -3 \\ 2k \\ 2 \end{pmatrix}, \mathbf{A}_2 = \begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix}, \mathbf{m}_2 = \begin{pmatrix} 3k \\ 1 \\ -5 \end{pmatrix} \tag{1.144.3}$$

To find the value of k, let's assume that the given lines are perpendicular to each other. Then the dot product of their direction vectors should be 0. i.e.,

$$\mathbf{m}_1\mathbf{m}_2 = 0 \tag{1.144.4}$$

$$\implies \begin{pmatrix} -3 \\ 2k \\ 2 \end{pmatrix}\begin{pmatrix} 3k \\ 1 \\ -5 \end{pmatrix} = 0 \tag{1.144.5}$$

$$\implies k = -\frac{10}{7} \tag{1.144.6}$$

The lines will intersect if

$$\mathbf{A}_1 + \lambda_1\mathbf{m}_1 = \mathbf{A}_2 + \lambda_2\mathbf{m}_2 \tag{1.144.7}$$

$$\implies \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda_1\begin{pmatrix} -3 \\ 2k \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix} + \lambda_2\begin{pmatrix} 3k \\ 1 \\ -5 \end{pmatrix} \tag{1.144.8}$$

$$\implies \lambda_1\begin{pmatrix} -3 \\ 2k \\ 2 \end{pmatrix} - \lambda_2\begin{pmatrix} 3k \\ 1 \\ -5 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \tag{1.144.9}$$

$$\implies \begin{pmatrix} -3 & 3k \\ 2k & 1 \\ 2 & -5 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \tag{1.144.10}$$

$$\implies \begin{pmatrix} -3 & -\frac{30}{7} \\ -\frac{20}{7} & 1 \\ 2 & -5 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \tag{1.144.11}$$

Row reducing the augmented matrix,

$$\begin{pmatrix} -3 & -\frac{30}{7} & 2 \\ -\frac{20}{7} & 1 & -1 \\ 2 & -5 & 3 \end{pmatrix} \xleftrightarrow[R_2 \leftarrow R_2 + \frac{20}{7}R_1]{R_1 \leftarrow -\frac{R_1}{3}} \begin{pmatrix} 1 & \frac{10}{7} & -\frac{2}{3} \\ 0 & \frac{249}{49} & -\frac{61}{21} \\ 2 & -5 & 3 \end{pmatrix} \tag{1.144.12}$$

$$\xleftrightarrow[R_2 \leftarrow \frac{49}{249}R_2]{R_3 \leftarrow R_3 - 2R_1} \begin{pmatrix} 1 & \frac{10}{7} & -\frac{2}{3} \\ 0 & \frac{249}{49} & -\frac{61}{21} \\ 2 & -5 & 3 \end{pmatrix} \tag{1.144.13}$$

$$\xleftrightarrow[R_2 \leftarrow \frac{49}{249}R_2]{R_3 \leftarrow R_3 - 2R_1} \begin{pmatrix} 1 & \frac{10}{7} & -\frac{2}{3} \\ 0 & 1 & -\frac{427}{747} \\ 0 & -\frac{55}{7} & \frac{13}{3} \end{pmatrix} \tag{1.144.14}$$

$$\xleftrightarrow[R_3 \leftarrow -\frac{747}{118}R_3]{R_3 \leftarrow R_3 + \frac{55}{7}R_2} \begin{pmatrix} 1 & \frac{10}{7} & -\frac{2}{3} \\ 0 & 1 & -\frac{427}{747} \\ 0 & 0 & 1 \end{pmatrix} \tag{1.144.15}$$

$$\xleftrightarrow[R_1 \leftarrow R_1 + \frac{2}{3}R_3 - \frac{10}{7}R_2]{R_2 \leftarrow R_2 + \frac{427}{47}R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.144.16}$$

The above matrix has $rank = 3$. Hence, the lines do not intersect which implies that the given lines are skew lines. To find the closest points using SVD, consider the equation (1.144.11) which can be expressed as

$$\mathbf{M}\mathbf{x} = \mathbf{b} \tag{1.144.17}$$

By singular value decomposition $\mathbf{M}$ can be expressed as

$$\mathbf{M} = \mathbf{U}\mathbf{S}\mathbf{V}^T \tag{1.144.18}$$

where the columns of $\mathbf{V}$ are the eigenvectors of $\mathbf{M}^T\mathbf{M}$, the columns of $\mathbf{U}$ are the eigenvectors of $\mathbf{M}\mathbf{M}^T$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} \frac{1037}{49} & 0 \\ 0 & \frac{2174}{49} \end{pmatrix} \tag{1.144.19}$$

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} \frac{1341}{49} & \frac{30}{7} & \frac{108}{7} \\ \frac{30}{7} & \frac{449}{49} & -\frac{75}{7} \\ \frac{108}{7} & -\frac{75}{7} & 29 \end{pmatrix} \tag{1.144.20}$$

To get $\mathbf{V}$ and $\mathbf{S}$ The characteristic equation of

$\mathbf{M}^T\mathbf{M}$ is obtained by evaluating the determinant

$$\begin{vmatrix} \frac{1037}{49} - \lambda & 0 \\ 0 & \frac{2174}{49} - \lambda \end{vmatrix} = 0 \tag{1.144.21}$$

$$\implies \lambda^2 - \frac{286699}{637}\lambda + \left[\frac{1037 \times 2174}{49^2}\right] = 0 \tag{1.144.22}$$

The eigenvalues are the roots of equation 1.144.22 is given by

$$\lambda_{11} = \frac{2174}{49} \tag{1.144.23}$$

$$\lambda_{12} = \frac{1037}{49} \tag{1.144.24}$$

The corresponding eigen vectors are,

$$\mathbf{u}_{11} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.144.25}$$

$$\mathbf{u}_{12} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.144.26}$$

$$\therefore \mathbf{V} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{1.144.27}$$

$\mathbf{S}$ is given by

$$\mathbf{S} = \begin{pmatrix} \frac{\sqrt{2174}}{7} & 0 \\ 0 & \frac{\sqrt{1037}}{7} \\ 0 & 0 \end{pmatrix} \tag{1.144.28}$$

To get U The characteristic equation of $\mathbf{M}\mathbf{M}^T$ is obtained by evaluating the determinant

$$\begin{vmatrix} \frac{1341}{49} - \lambda & \frac{30}{7} & \frac{108}{7} \\ \frac{30}{7} & \frac{449}{49} - \lambda & -\frac{75}{7} \\ \frac{108}{7} & -\frac{75}{7} & 29 - \lambda \end{vmatrix} = 0 \tag{1.144.29}$$

$$\implies -\lambda^3 + \frac{3211}{49}\lambda^2 - \frac{2254438}{2401}\lambda = 0 \tag{1.144.30}$$

The eigenvalues are the roots of equation 1.144.30 is given by

$$\lambda_{21} = \frac{2174}{49} \tag{1.144.31}$$

$$\lambda_{22} = \frac{1037}{49} \tag{1.144.32}$$

$$\lambda_{23} = 0 \tag{1.144.33}$$

The corresponding eigen vectors are ,

$$\mathbf{u}_{21} = \begin{pmatrix} -\frac{6}{7} \\ \frac{1}{5} \\ -1 \end{pmatrix}, \mathbf{u}_{22} = \begin{pmatrix} -\frac{3}{2} \\ -\frac{10}{7} \\ 1 \end{pmatrix}, \mathbf{u}_{23} = \begin{pmatrix} -\frac{602}{747} \\ \frac{384}{249} \\ 1 \end{pmatrix} \tag{1.144.34}$$

Normalizing the eigen vectors,

$$\|\mathbf{u}_{21}\| = \sqrt{\left(\frac{-6}{7}\right)^2 + \left(\frac{1}{5}\right)^2 + 1} = \frac{\sqrt{2174}}{35} \tag{1.144.35}$$

$$\implies \mathbf{u}_{21} = \begin{pmatrix} -\frac{210}{7\sqrt{2174}} \\ \frac{35}{5\sqrt{2176}} \\ -\frac{35}{\sqrt{2174}} \end{pmatrix} \tag{1.144.36}$$

$$\|\mathbf{u}_{22}\| = \sqrt{\left(\frac{-3}{2}\right)^2 + \left(\frac{-10}{7}\right)^2 + 1} = \frac{\sqrt{1037}}{14} \tag{1.144.37}$$

$$\implies \mathbf{u}_{22} = \begin{pmatrix} -\frac{42}{2\sqrt{1037}} \\ -\frac{20}{\sqrt{1037}} \\ \frac{14}{\sqrt{1037}} \end{pmatrix} \tag{1.144.38}$$

$$\|\mathbf{u}_{23}\| = \sqrt{\left(\frac{-602}{747}\right)^2 + \left(\frac{384}{249}\right)^2 + 1} = \frac{\sqrt{4027743}}{1000} \tag{1.144.39}$$

$$\implies \mathbf{u}_{23} = \begin{pmatrix} -\frac{602000}{747\sqrt{4027743}} \\ \frac{384000}{249\sqrt{4027743}} \\ \frac{1000}{\sqrt{4027743}} \end{pmatrix} \tag{1.144.40}$$

$$\mathbf{U} = \begin{pmatrix} \frac{-210}{7\sqrt{2174}} & \frac{-42}{2\sqrt{1037}} & \frac{-602000}{747\sqrt{4027743}} \\ \frac{35}{5\sqrt{2174}} & \frac{-20}{\sqrt{1037}} & \frac{384000}{249\sqrt{4027743}} \\ \frac{-35}{\sqrt{2174}} & \frac{14}{\sqrt{1037}} & \frac{1000}{\sqrt{4027743}} \end{pmatrix} \tag{1.144.41}$$

To get x Using (1.144.18) we rewrite **M** as follows,

$$\begin{pmatrix}-3 & -\frac{30}{7}\\ -\frac{20}{7} & 1\\ 2 & -5\end{pmatrix} = \begin{pmatrix}\frac{-210}{7\sqrt{2174}} & \frac{-42}{2\sqrt{1037}} & \frac{-602000}{747\sqrt{4027743}}\\ \frac{35}{5\sqrt{2174}} & \frac{-20}{\sqrt{1037}} & \frac{384000}{249\sqrt{4027743}}\\ \frac{-35}{\sqrt{2174}} & \frac{14}{\sqrt{1037}} & \frac{1000}{\sqrt{4027743}}\end{pmatrix}$$

$$\begin{pmatrix}\frac{\sqrt{2174}}{7} & 0\\ 0 & \frac{\sqrt{1037}}{7}\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}^T \tag{1.144.42}$$

By substituting the equation (1.144.18) in equation (1.144.17) we get

$$\mathbf{USV}^T\mathbf{x} = \mathbf{b} \tag{1.144.43}$$

$$\implies \mathbf{x} = \mathbf{VS_+U^Tb} \tag{1.144.44}$$

where $\mathbf{S}_+$ is Moore-Penrose Pseudo-Inverse of $\mathbf{S}$

$$\mathbf{S}_+ = \begin{pmatrix}\frac{7}{\sqrt{2174}} & 0 & 0\\ 0 & \frac{7}{\sqrt{1037}} & 0\end{pmatrix} \tag{1.144.45}$$

From (1.144.44) we get,

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix}\frac{-172}{\sqrt{2174}}\\ \frac{20}{\sqrt{1037}}\\ \frac{-115000}{747\sqrt{4027743}}\end{pmatrix} \tag{1.144.46}$$

$$\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix}\frac{-602}{1087}\\ \frac{140}{1037}\end{pmatrix} \tag{1.144.47}$$

$$\mathbf{x} = \mathbf{VS}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix}\frac{140}{1037}\\ \frac{-602}{1087}\end{pmatrix} \tag{1.144.48}$$

Verification of $\mathbf{x}$ Verifying the solution of (1.144.48) using,

$$\mathbf{M}^T\mathbf{Mx} = \mathbf{M}^T\mathbf{b} \tag{1.144.49}$$

Evaluating the R.H.S in (1.144.49) we get,

$$\mathbf{M}^T\mathbf{Mx} = \begin{pmatrix}\frac{20}{7}\\ -\frac{172}{7}\end{pmatrix} \tag{1.144.50}$$

$$\implies \begin{pmatrix}\frac{1037}{49} & 0\\ 0 & \frac{2174}{49}\end{pmatrix}\mathbf{x} = \begin{pmatrix}\frac{20}{7}\\ -\frac{172}{7}\end{pmatrix} \tag{1.144.51}$$

Solving the augmented matrix of (1.144.51) we get,

$$\begin{pmatrix}\frac{1037}{49} & 0 & \frac{20}{7}\\ 0 & \frac{2174}{49} & -\frac{172}{7}\end{pmatrix} \xleftrightarrow[R_2\leftarrow\frac{49}{2174}R_2]{R_1\leftarrow\frac{49}{1037}R_1} \begin{pmatrix}1 & 0 & \frac{140}{1037}\\ 0 & 1 & -\frac{602}{1087}\end{pmatrix} \tag{1.144.52}$$

Hence, Solution of (1.144.49) is given by,

$$\mathbf{x} = \begin{pmatrix}\frac{140}{1037}\\ \frac{-602}{1087}\end{pmatrix} \tag{1.144.53}$$

Comparing results of $\mathbf{x}$ from (1.144.48) and (1.144.53) we conclude that the solution is verified.

1.145. Find the distance of the given point $\begin{pmatrix}-6\\ 0\\ 0\end{pmatrix}$ from the plane $\begin{pmatrix}2 & -3 & 6\end{pmatrix}\mathbf{x} = 2$.

Solution: Let us consider orthogonal vectors $\mathbf{m_1}$ and $\mathbf{m_2}$ to the given normal vector $\mathbf{n}$. Let, $\mathbf{m} = \begin{pmatrix}a\\ b\\ c\end{pmatrix}$, then

$$\mathbf{m^Tn} = 0 \tag{1.145.1}$$

$$\implies \begin{pmatrix}a & b & c\end{pmatrix}\begin{pmatrix}2\\ -3\\ 6\end{pmatrix} = 0 \tag{1.145.2}$$

$$\implies 2a - 3b + 6c = 0 \tag{1.145.3}$$

Let a=1 and b=0 we get,

$$\mathbf{m_1} = \begin{pmatrix}1\\ 0\\ -\frac{1}{3}\end{pmatrix} \tag{1.145.4}$$

Let a=0 and b=1 we get,

$$\mathbf{m_2} = \begin{pmatrix}0\\ 1\\ \frac{1}{2}\end{pmatrix} \tag{1.145.5}$$

Now we solve the equation,

$$\mathbf{Mx} = \mathbf{b} \tag{1.145.6}$$

Substituting (1.145.4) and (1.145.5) in (1.145.6),

$$\begin{pmatrix}1 & 0\\ 0 & 1\\ -\frac{1}{3} & \frac{1}{2}\end{pmatrix}\mathbf{x} = \begin{pmatrix}-6\\ 0\\ 0\end{pmatrix} \tag{1.145.7}$$

To solve (1.145.7), we will perform Singular Value Decomposition on $\mathbf{M}$ as follows,

$$\mathbf{M} = \mathbf{USV}^T \tag{1.145.8}$$

Where the columns of $\mathbf{V}$ are the eigen vectors of $\mathbf{M}^T\mathbf{M}$,the columns of $\mathbf{U}$ are the eigen vectors of $\mathbf{MM}^T$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}\mathbf{M}^T$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} \frac{10}{9} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{5}{4} \end{pmatrix} \tag{1.145.9}$$

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & \frac{1}{2} \\ -\frac{1}{3} & \frac{1}{2} & \frac{13}{36} \end{pmatrix} \tag{1.145.10}$$

Substituting (1.145.8) in (1.145.6),

$$\mathbf{U}\mathbf{S}\mathbf{V}^T\mathbf{x} = \mathbf{b} \tag{1.145.11}$$

$$\implies \mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^\mathbf{T}\mathbf{b} \tag{1.145.12}$$

Where $\mathbf{S}_+$ is Moore-Penrose Pseudo-Inverse of $\mathbf{S}$.
Let us calculate eigen values of $\mathbf{M}\mathbf{M}^T$,

$$\left|\mathbf{M}\mathbf{M}^T - \lambda\mathbf{I}\right| = 0 \tag{1.145.13}$$

$$\implies \begin{pmatrix} 1-\lambda & 0 & -\frac{1}{3} \\ 0 & 1-\lambda & \frac{1}{2} \\ -\frac{1}{3} & \frac{1}{2} & \frac{13}{36}-\lambda \end{pmatrix} = 0 \tag{1.145.14}$$

$$\implies \lambda^3 - \frac{85}{36}\lambda^2 + \frac{49}{36}\lambda = 0 \tag{1.145.15}$$

From equation (1.145.15) eigen values of $\mathbf{M}\mathbf{M}^T$ are,

$$\lambda_1 = \frac{49}{36} \quad \lambda_2 = 1 \quad \lambda_3 = 0 \tag{1.145.16}$$

The eigen vectors of $\mathbf{M}\mathbf{M}^T$ are,

$$\mathbf{u}_1 = \begin{pmatrix} -\frac{12}{13} \\ \frac{18}{13} \\ 1 \end{pmatrix} \quad \mathbf{u}_2 = \begin{pmatrix} \frac{3}{2} \\ 1 \\ 0 \end{pmatrix} \quad \mathbf{u}_3 = \begin{pmatrix} \frac{1}{3} \\ -\frac{1}{2} \\ 1 \end{pmatrix} \tag{1.145.17}$$

Normalizing the eigen vectors in equation (1.145.17)

$$\mathbf{u}_1 = \begin{pmatrix} -\frac{12}{7\sqrt{13}} \\ \frac{18}{7\sqrt{13}} \\ \frac{13}{7\sqrt{13}} \end{pmatrix} \quad \mathbf{u}_2 = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \\ 0 \end{pmatrix} \quad \mathbf{u}_3 = \begin{pmatrix} \frac{2}{7} \\ -\frac{3}{7} \\ \frac{6}{7} \end{pmatrix} \tag{1.145.18}$$

Hence we obtain $\mathbf{U}$ as follows,

$$\mathbf{U} = \begin{pmatrix} -\frac{12}{7\sqrt{13}} & \frac{3}{\sqrt{13}} & \frac{2}{7} \\ \frac{18}{7\sqrt{13}} & \frac{2}{\sqrt{13}} & -\frac{3}{7} \\ \frac{13}{7\sqrt{13}} & 0 & \frac{6}{7} \end{pmatrix} \tag{1.145.19}$$

After computing the singular values from eigen values $\lambda_1, \lambda_2, \lambda_3$ we get $\mathbf{S}$ as follows,

$$\mathbf{S} = \begin{pmatrix} \frac{7}{6} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{1.145.20}$$

Now, lets calculate eigen values of $\mathbf{M}^T\mathbf{M}$,

$$\left|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}\right| = 0 \tag{1.145.21}$$

$$\implies \begin{pmatrix} \frac{10}{9}-\lambda & -\frac{1}{6} \\ -\frac{1}{6} & \frac{5}{4}-\lambda \end{pmatrix} = 0 \tag{1.145.22}$$

$$\implies \lambda^2 - \frac{85}{36}\lambda + \frac{49}{36} = 0 \tag{1.145.23}$$

Hence eigen values of $\mathbf{M}^T\mathbf{M}$ are,

$$\lambda_1 = \frac{49}{36} \quad \lambda_2 = 1 \tag{1.145.24}$$

Hence the eigen vectors of $\mathbf{M}^T\mathbf{M}$ are,

$$\mathbf{v}_1 = \begin{pmatrix} -\frac{2}{3} \\ 1 \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix} \tag{1.145.25}$$

Normalizing the eigen vectors,

$$\mathbf{v}_1 = \begin{pmatrix} -\frac{2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \end{pmatrix} \tag{1.145.26}$$

Hence we obtain $\mathbf{V}$ as,

$$\mathbf{V} = \begin{pmatrix} -\frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{pmatrix} \tag{1.145.27}$$

From (1.145.6), the Singular Value Decomposition of $\mathbf{M}$ is as follows,

$$\mathbf{M} = \begin{pmatrix} -\frac{12}{7\sqrt{13}} & \frac{3}{\sqrt{13}} & \frac{2}{7} \\ \frac{18}{7\sqrt{13}} & \frac{2}{\sqrt{13}} & -\frac{3}{7} \\ \frac{13}{7\sqrt{13}} & 0 & \frac{6}{7} \end{pmatrix} \begin{pmatrix} \frac{7}{6} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -\frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{pmatrix}^T \tag{1.145.28}$$

Now, Moore-Penrose Pseudo inverse of $\mathbf{S}$ is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{6}{7} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{1.145.29}$$

From (1.145.12) we get,

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{72}{7\sqrt{13}} \\ -\frac{18}{\sqrt{13}} \\ -\frac{12}{7} \end{pmatrix} \tag{1.145.30}$$

$$\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{432}{49\sqrt{13}} \\ -\frac{18}{\sqrt{13}} \end{pmatrix} \tag{1.145.31}$$

$$\mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} -\frac{270}{49} \\ -\frac{36}{49} \end{pmatrix} \tag{1.145.32}$$

Verifying the solution of (1.145.32) using,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \tag{1.145.33}$$

Evaluating the R.H.S in (1.145.33) we get,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \begin{pmatrix} -6 \\ 0 \end{pmatrix} \tag{1.145.34}$$

$$\implies \begin{pmatrix} \frac{10}{9} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{5}{4} \end{pmatrix}\mathbf{x} = \begin{pmatrix} -6 \\ 0 \end{pmatrix} \tag{1.145.35}$$

Solving the augmented matrix of (1.145.35) we get,

$$\begin{pmatrix} \frac{10}{9} & -\frac{1}{6} & -6 \\ -\frac{1}{6} & \frac{5}{4} & 0 \end{pmatrix} \xleftrightarrow{R_1=\frac{9}{10}R_1} \begin{pmatrix} 1 & -\frac{3}{20} & -\frac{54}{10} \\ -\frac{1}{6} & \frac{5}{4} & 0 \end{pmatrix} \tag{1.145.36}$$

$$\xleftrightarrow{R_2=R_2+\frac{1}{6}R_1} \begin{pmatrix} 1 & -\frac{3}{20} & -\frac{54}{10} \\ 0 & \frac{49}{40} & -\frac{9}{10} \end{pmatrix} \tag{1.145.37}$$

$$\xleftrightarrow{R_2=\frac{40}{49}R_2} \begin{pmatrix} 1 & -\frac{3}{20} & -\frac{54}{10} \\ 0 & 1 & -\frac{36}{49} \end{pmatrix} \tag{1.145.38}$$

$$\xleftrightarrow{R_1=R_1+\frac{3}{20}R_2} \begin{pmatrix} 1 & 0 & -\frac{270}{49} \\ 0 & 1 & -\frac{36}{49} \end{pmatrix} \tag{1.145.39}$$

From equation (1.145.39), solution is given by,

$$\mathbf{x} = \begin{pmatrix} -\frac{270}{49} \\ -\frac{36}{49} \end{pmatrix} \tag{1.145.40}$$

Comparing results of $\mathbf{x}$ from (1.145.32) and (1.145.40), we can say that the solution is verified.

2 Exercises

2.1. A=$[a_{ij}]_{mxn}$ is a square matrix,if
(A) m<n (B)m>n (C) m=n (D) None of these

2.2. Which of the given values of x and y make the following pair of matrices equal $\begin{pmatrix} 3x+7 & 5 \\ y+1 & 2-3x \end{pmatrix}$,$\begin{pmatrix} 0 & y-2 \\ 8 & 4 \end{pmatrix}$
(A)x=$\frac{-1}{3}$,y=7
(B) Not possible to find
(C) y=7, x=$\frac{-2}{3}$
(D) x=$\frac{-1}{3}$,y=$\frac{-2}{3}$

2.3. The number of all possible matrices of order 3X3 with each entry 0 or 1 is:
(A) 27 (B)18 (C)81 (D)512

2.4. Let A=$\begin{pmatrix} 2 & 4 \\ 3 & 2 \end{pmatrix}$,B=$\begin{pmatrix} 1 & 3 \\ -2 & 5 \end{pmatrix}$,C=$\begin{pmatrix} -2 & 5 \\ 3 & 4 \end{pmatrix}$ Find each of the following:
(i) A+B (ii)A-B (iii)3A-C (iv)AB (v)BA

2.5. Compute the following:
(i)$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}+\begin{pmatrix} a & b \\ b & a \end{pmatrix}$
(ii)$\begin{pmatrix} a^2+b^2 & b^2+c^2 \\ a^2+c^2 & a^2+b^2 \end{pmatrix}+\begin{pmatrix} 2ab & 2bc \\ -2ac & -2ab \end{pmatrix}$
(iii) $\begin{pmatrix} -1 & 4 & -6 \\ 8 & 5 & 16 \\ 2 & 8 & 5 \end{pmatrix}+\begin{pmatrix} 12 & 7 & 6 \\ 8 & 0 & 5 \\ 3 & 2 & 4 \end{pmatrix}$
(iv) $\begin{pmatrix} cos^2x & sin^2x \\ sin^2x & cos^2x \end{pmatrix}+\begin{pmatrix} sin^2x & cos^2x \\ cos^2x & sin^2x \end{pmatrix}$

2.6. Compute the indicated products.
a) $\begin{pmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{pmatrix}\begin{pmatrix} 1 & -3 & 5 \\ 0 & 2 & 4 \\ 3 & 0 & 5 \end{pmatrix}$
b) $\begin{pmatrix} 2 & 1 \\ 3 & 2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \end{pmatrix}$
c) $\begin{pmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{pmatrix}\begin{pmatrix} 2 & -3 \\ 1 & 0 \\ 3 & 1 \end{pmatrix}$

2.7. If,A=$\begin{pmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{pmatrix}$,B=$\begin{pmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{pmatrix}$and C=$\begin{pmatrix} 4 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & -2 & 3 \end{pmatrix}$,then compute (A+B) and (B-C).Also ,verify that A+(B-C)=(A+B)-C.

2.8. If A=$\begin{pmatrix} \frac{2}{3} & 1 & \frac{5}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{4}{3} \\ \frac{7}{3} & 2 & \frac{2}{3} \end{pmatrix}$ and B=$\begin{pmatrix} \frac{2}{5} & \frac{3}{5} & 1 \\ \frac{1}{5} & \frac{2}{5} & \frac{4}{5} \\ \frac{7}{5} & \frac{6}{5} & \frac{2}{5} \end{pmatrix}$,then compute 3A-5B.

2.9. Find x and y,if $2\begin{pmatrix}1 & 3\\0 & x\end{pmatrix}+\begin{pmatrix}y & 0\\1 & 2\end{pmatrix}=\begin{pmatrix}5 & 6\\1 & 8\end{pmatrix}$

2.10. Solve the equation for x,y,z and t,if
$2\begin{pmatrix}x & z\\y & t\end{pmatrix}+3\begin{pmatrix}1 & -1\\0 & 2\end{pmatrix}=3\begin{pmatrix}3 & 5\\4 & 6\end{pmatrix}$

2.11. If $x\begin{pmatrix}2\\3\end{pmatrix}+y\begin{pmatrix}-1\\1\end{pmatrix}=\begin{pmatrix}10\\5\end{pmatrix}$,find the values of x and y.

2.12. Given $3\begin{pmatrix}x & y\\z & w\end{pmatrix}=\begin{pmatrix}x & 6\\-1 & 2w\end{pmatrix}+\begin{pmatrix}4 & x+y\\z+w & 3\end{pmatrix}$,find the values of x,y,z and w.

2.13. The bookshop of a particular school has 10 dozen chemistry books, 8 dozen physics books, 10 dozen economics books. Their selling prices are ₹80, ₹60 and ₹40 each respectively. Find the total amount the bookshop will receive from selling all the books using matrix algebra.
Solution: The total books in the bookshop can be expressed as

$$\mathbf{A} = 12\begin{pmatrix}10\\8\\10\end{pmatrix} \tag{2.13.1}$$

and their selling price as

$$\mathbf{B} = \begin{pmatrix}80\\60\\40\end{pmatrix}. \tag{2.13.2}$$

The total amount received by the bookshop after selling the books is given by,

$$\mathbf{A}^\top\mathbf{B} = 12\begin{pmatrix}10 & 8 & 10\end{pmatrix}\begin{pmatrix}80\\60\\40\end{pmatrix} = 20160 \tag{2.13.3}$$

Assume X,Y,Z,W and P are matrices of orders $2\times n$,$3\times k$,$2\times p$,$n\times 3$ and $p\times k$,respectively. Choose the correct answer in Exercise 31 and 32.

2.14. The restriction on n,k and p so that PY+WY will be defined are:
(A)k=3,p=n
(B)k is arbitrary,p=2
(C)p is arbitrary,k=3
(D)k=2,p=3

2.15. If n=p,then the order of the matrix 7X-5Z is:
(A)$p\times 2$ (B)$2\times n$ (C)$n\times 3$ (D)$p\times n$

2.16. Find the transpose of each of the following matrices:
(i)$\begin{pmatrix}5\\\frac{1}{2}\\-1\end{pmatrix}$
(ii)$\begin{pmatrix}1 & -1\\2 & 3\end{pmatrix}$
(iii)$\begin{pmatrix}-1 & 5 & 6\\\sqrt{3} & 5 & 6\\2 & 3 & -1\end{pmatrix}$

2.17. If $A=\begin{pmatrix}-1 & 2 & 3\\5 & 7 & 9\\-1 & 1 & 1\end{pmatrix}$ and $B=\begin{pmatrix}-4 & 1 & -5\\1 & 2 & 0\\1 & 3 & 1\end{pmatrix}$,then verify that
(i)$(A+B)^{'}=A^{'}+B^{'}$
(ii) $(A-B)^{'}=A^{'}-B^{'}$

2.18. If $A^{'}=\begin{pmatrix}3 & 4\\-1 & 2\\0 & 1\end{pmatrix}$ and $B=\begin{pmatrix}-1 & 2 & 1\\1 & 2 & 3\end{pmatrix}$,then verify that
(i) $(A+B)^{'}=A^{'}+B^{'}$ (ii)$(A-B)^{'}=A^{'}-B^{'}$

2.19. If$A^{'}=\begin{pmatrix}-2 & 3\\1 & 2\end{pmatrix}$ and $B=\begin{pmatrix}-1 & 0\\1 & 2\end{pmatrix}$,then find that $(A+2B)^{'}$

2.20. (i) Show that the matrix $A=\begin{pmatrix}1 & -1 & 5\\-1 & 2 & 1\\5 & 1 & 3\end{pmatrix}$ is a symmetric matrix.
(ii) Show that the matrix $A=\begin{pmatrix}0 & 1 & -1\\-1 & 0 & 1\\1 & -1 & 0\end{pmatrix}$ is a skew symmetric matrix.

2.21. For the matrix $A=\begin{pmatrix}1 & 5\\6 & 7\end{pmatrix}$,verify that
(i)$(A+A^{'})$ is a symmetric matrix
(ii)$(A-A^{'})$ is a skew symmetric matrix

2.22. Find $\frac{1}{2}(A+A^{'})$and $\frac{1}{2}(A-A^{'})$,when $A=\begin{pmatrix}0 & a & b\\-a & 0 & c\\-b & -c & 0\end{pmatrix}$

2.23. Express the following matrices as the sum of a symmetric and a skew symmetric matrix:
(i) $\begin{pmatrix}3 & 5\\1 & 1\end{pmatrix}$

(ii) $\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$

(iii) $\begin{pmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{pmatrix}$

(iv) $\begin{pmatrix} 1 & 5 \\ -1 & 2 \end{pmatrix}$

Choose the correct answer in question number 43 and 44

2.24. If A,B are symmetric matrices of same order,then AB-BA is a
(A)Skew symmetric matrix
(B)Symmetric matrix
(C)Zero matrix
(D)Identity matrix

Using elementary transforamtions,find the inverse of each of the matrices,if it exists questions 45-61

2.25. $\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$

Solution:

a) Given that

$$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} \tag{2.25.1}$$

The augmented matrix $[A|I]$ is as given below:-

$$\left(\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right) \tag{2.25.2}$$

We apply the elementary row operations on $[A|I]$ as follows :-

$$[A|I] = \left(\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right) \tag{2.25.3}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - 2R_1} \left(\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & 5 & -2 & 1 \end{array}\right) \tag{2.25.4}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{R_2}{5}} \left(\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & 1 & \frac{-2}{5} & \frac{1}{5} \end{array}\right) \tag{2.25.5}$$

$$\xleftrightarrow{R_2 \leftarrow R_1 + R_2} \left(\begin{array}{cc|cc} 1 & 0 & \frac{3}{5} & \frac{1}{5} \\ 0 & 1 & \frac{-2}{5} & \frac{1}{5} \end{array}\right) \tag{2.25.6}$$

By performing elementary transformations on augmented matrix$[A|I]$, we obtained the augmented matrix in the form $[I|A]$. Hence we can conclude that the matrix A is invertible and inverse of the matrix is:-

$$\therefore \mathbf{A^{-1}} = \begin{pmatrix} \frac{3}{5} & \frac{1}{5} \\ \frac{-2}{5} & \frac{1}{5} \end{pmatrix} \tag{2.25.7}$$

b) QR decomposition of $\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$

$$\mathbf{v_1} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \mathbf{v_2} = \begin{pmatrix} -1 \\ 3 \end{pmatrix} \tag{2.25.8}$$

$$\mathbf{r_{11}} = \sqrt{5} \tag{2.25.9}$$

$$\mathbf{q_1} = \frac{1}{||\mathbf{v_1}||}\mathbf{v_1} = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix} \tag{2.25.10}$$

$$\mathbf{r_{12}} = \mathbf{v_1}^T\mathbf{v_2} \tag{2.25.11}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}\begin{pmatrix} -1 \\ 3 \end{pmatrix} = \sqrt{5} \tag{2.25.12}$$

$$\mathbf{v_2} = \mathbf{v_2} - \mathbf{r_{12}q_1} \tag{2.25.13}$$

$$= \begin{pmatrix} -1 \\ 3 \end{pmatrix} - \sqrt{5}\begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} \tag{2.25.14}$$

$$= \begin{pmatrix} -2 \\ 1 \end{pmatrix} \tag{2.25.15}$$

$$\mathbf{r_{22}} = ||\mathbf{v_2}|| = \sqrt{5} \tag{2.25.16}$$

$$\mathbf{q_2} = \frac{1}{||v_2||}\mathbf{v_2} \tag{2.25.17}$$

$$= \frac{1}{\sqrt{5}}\begin{pmatrix} \frac{-2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} = \begin{pmatrix} \frac{-2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} \tag{2.25.18}$$

Thus obtained QR decomposition is

$$\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}\begin{pmatrix} \sqrt{5} & \sqrt{5} \\ 0 & \sqrt{5} \end{pmatrix} \tag{2.25.19}$$

2.26. $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

Solution:

a) Given that

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \tag{2.26.1}$$

The augmented matrix $[A|I]$ is as given below:-

$$\left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right) \tag{2.26.2}$$

We apply the elementary row operations on

$[A|I]$ as follows :-

$$[A|I] = \left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right) \tag{2.26.3}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - R_2} \left(\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 1 & 1 & 0 & 1 \end{array}\right) \tag{2.26.4}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - R_1} \left(\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 2 \end{array}\right) \tag{2.26.5}$$

By performing elementary transformations on augmented matrix$[A|I]$, we obtained the augmented matrix in the form $[I|A]$. Hence we can conclude that the matrix A is invertible and inverse of the matrix is:-

$$\therefore \mathbf{A^{-1}} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \tag{2.26.6}$$

b) QR decomposition of $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be the column vectors of given matrix $\mathbf{A}$,

$$\boldsymbol{\alpha} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \boldsymbol{\beta} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.26.7}$$

QR decomposition of matrix form is:

$$\begin{pmatrix} \boldsymbol{\alpha} & \boldsymbol{\beta} \end{pmatrix} = \begin{pmatrix} \mathbf{u_1} & \mathbf{u_2} \end{pmatrix} \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \tag{2.26.8}$$

Finding values of the above equation, we get:

$$\implies k_1 = \|\boldsymbol{\alpha}\| = \left\| \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\| \tag{2.26.9}$$

$$\therefore k_1 = \sqrt{5} \tag{2.26.10}$$

$$\implies \mathbf{u_1} = \frac{\boldsymbol{\alpha}}{k_1} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{2.26.11}$$

$$\therefore \mathbf{u_1} = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} \tag{2.26.12}$$

$$\implies r_1 = \frac{\mathbf{u_1}^T \boldsymbol{\beta}}{\|\mathbf{u_1}\|^2} = \frac{\frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}}{1} \tag{2.26.13}$$

$$\therefore r_1 = \frac{3}{\sqrt{5}} \tag{2.26.14}$$

$$\implies \mathbf{u_2} = \frac{\boldsymbol{\beta} - r_1 \mathbf{u_1}}{\|\boldsymbol{\beta} - r_1 \mathbf{u_1}\|} = \frac{\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{3}{\sqrt{5}} \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}}{\left\| \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{3}{\sqrt{5}} \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} \right\|} \tag{2.26.15}$$

$$\therefore \mathbf{u_2} = \begin{pmatrix} \frac{-1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix} \tag{2.26.16}$$

$$\implies k_2 = \mathbf{u_2}^T \boldsymbol{\beta} = \begin{pmatrix} \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.26.17}$$

$$\therefore k_2 = \frac{1}{\sqrt{5}} \tag{2.26.18}$$

From equations (2.26.10), (2.26.12), (2.26.14), (2.26.16), (2.26.18) and using (2.26.7) the obtained **QR** decomposition is

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \sqrt{5} & \frac{3}{\sqrt{5}} \\ 0 & \frac{1}{\sqrt{5}} \end{pmatrix} \tag{2.26.19}$$

2.27. $\begin{pmatrix} 1 & 3 \\ 2 & 7 \end{pmatrix}$

2.28. $\begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix}$

2.29. $\begin{pmatrix} 2 & 1 \\ 7 & 4 \end{pmatrix}$

2.30. $\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$

2.31. $\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$

2.32. $\begin{pmatrix} 4 & 5 \\ 3 & 4 \end{pmatrix}$

2.33. $\begin{pmatrix} 3 & 10 \\ 2 & 7 \end{pmatrix}$

2.34. $\begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix}$

2.35. $\begin{pmatrix} 2 & -6 \\ 1 & -2 \end{pmatrix}$

2.36. $\begin{pmatrix} 6 & -3 \\ -2 & 1 \end{pmatrix}$

2.37. $\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$

2.38. $\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}$

2.39. Matrices Aand B will be inverse of each other only if
(A)AB=BA (B)AB=BA=0
(C)AB=0,BA=I (D)AB=BA=I

2.40. Let A=$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$,show that $(aI + bA)^n = a^nI + na^{n-1}bA$,where I is the identity matrix of order 2 and $n\epsilon N$

2.41. If A and B are symmetric matrices,prove that AB-BA is a skew symmetric matrix.

2.42. Show that the matrix $B^{\top}AB$ is symmetric or skew symmetric according as A is symmetric or skew symmetric
Solution:
a) Let

$$\mathbf{A}^{\top} = \mathbf{A} \tag{2.42.1}$$

then

$$(\mathbf{B}^{\top}\mathbf{A}\mathbf{B})^{\top} = \mathbf{B}^{\top}\mathbf{A}^{\top}\mathbf{B} \tag{2.42.2}$$
$$= \mathbf{B}^{\top}\mathbf{A}\mathbf{B} \tag{2.42.3}$$
$$\tag{2.42.4}$$

using 2.42.1 Hence

$$\mathbf{B}^{\top}\mathbf{A}\mathbf{B} \tag{2.42.5}$$

is symmetric.

b) If

$$\mathbf{A}^{\top} = -\mathbf{A} \tag{2.42.6}$$

then

$$(\mathbf{B}^{\top}\mathbf{A}\mathbf{B})^{\top} = \mathbf{B}^{\top}\mathbf{A}^{\top}\mathbf{B} \tag{2.42.7}$$
$$= -\mathbf{B}^{\top}\mathbf{A}\mathbf{B} \tag{2.42.8}$$
$$\tag{2.42.9}$$

$$\therefore \mathbf{B}^{\top}\mathbf{A}\mathbf{B} \tag{2.42.10}$$

is skew symmetric.

2.43. Find the values of x,y,z if the matrix A=$\begin{pmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{pmatrix}$ satisfy the equation $A^{\top}A$=I
Solution:

$$A^{\top}A = I \tag{2.43.1}$$

$$\implies \begin{pmatrix} 0 & x & x \\ 2y & y & -y \\ z & -z & z \end{pmatrix}\begin{pmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.43.2}$$

$$\text{or, } \begin{pmatrix} 2x^2 & 0 & 0 \\ 0 & 6y^2 & 0 \\ 0 & 0 & 3z^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.43.3}$$

Hence,

$$2x^2 = 1 \implies x = \pm\frac{1}{\sqrt{2}}, \tag{2.43.4}$$
$$6y^2 = 1 \implies y = \pm\frac{1}{\sqrt{6}} \tag{2.43.5}$$
$$3z^2 = 1 \implies z = \pm\frac{1}{\sqrt{3}} \tag{2.43.6}$$

2.44. Find x, if $\begin{pmatrix} x & -5 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ 4 \\ 1 \end{pmatrix} = 0$

Solution: Given ,

$$\begin{pmatrix} x & -5 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ 4 \\ 1 \end{pmatrix} = 0 \tag{2.44.1}$$

$$\begin{pmatrix} x-2 & -10 & 2x-8 \end{pmatrix}\begin{pmatrix} x \\ 4 \\ 1 \end{pmatrix} = 0 \tag{2.44.2}$$

$$\left(x^2 - 2x - 40 + 2x - 8\right) = 0 \tag{2.44.3}$$
$$x^2 - 48 = 0 \tag{2.44.4}$$
$$x = 6.92 \tag{2.44.5}$$

2.45. A manufactrer produces three products x,y,z which he sells in two markets. Annual sales are indicated below:

Market	Products		
I	10,000	2,000	18,000
II	6,000	20,000	8,000

(a) If unit sale prices of x,y and z are ₹2.50,₹1.50 and ₹1.00 respectively,find the total revenue in each market with the help of matrix algebra.
(b) If the unit cost of the above three commodities are ₹2.00,₹1.00 and 50 paise respectively.Find the gross profit.
Solution: Let the sales of the product x,y and z per market be denoted by matrix A

$$\begin{array}{ccc} x & y & z \\ \end{array}$$
$$\begin{pmatrix} 10000 & 2000 & 18000 \\ 6000 & 20000 & 8000 \end{pmatrix} \begin{pmatrix} \text{Market-I} \\ \text{Market-II} \end{pmatrix} \tag{2.45.1}$$

a) Let the unit sale price of the products x,y and z per market be denoted by matrix B

$$\mathbf{B} = \begin{pmatrix} 2.50 \\ 1.50 \\ 1.00 \end{pmatrix} \tag{2.45.2}$$

Total Revenue in Market-I and Market-II

$$\mathbf{AB} = \begin{pmatrix} 10000 & 2000 & 18000 \\ 6000 & 20000 & 8000 \end{pmatrix} \begin{pmatrix} 2.50 \\ 1.50 \\ 1.00 \end{pmatrix} \tag{2.45.3}$$

$$= \begin{pmatrix} 46000 \\ 53000 \end{pmatrix} \tag{2.45.4}$$

b) Let the unit cost price of the products x,y and z per market be denoted by matrix C

$$\mathbf{C} = \begin{pmatrix} 2.00 \\ 1.00 \\ 0.50 \end{pmatrix} \tag{2.45.5}$$

Total cost of Market-I and Market-II

$$\mathbf{AC} = \begin{pmatrix} 10000 & 2000 & 18000 \\ 6000 & 20000 & 8000 \end{pmatrix} \begin{pmatrix} 2.00 \\ 1.00 \\ 0.50 \end{pmatrix} \tag{2.45.6}$$

$$= \begin{pmatrix} 31000 \\ 36000 \end{pmatrix} \tag{2.45.7}$$

$\therefore$ Gross Profit = Total revenue - Total cost

$$\mathbf{AB} - \mathbf{AC} = \begin{pmatrix} 46000 \\ 53000 \end{pmatrix} - \begin{pmatrix} 31000 \\ 36000 \end{pmatrix} \tag{2.45.8}$$

$$= \begin{pmatrix} 15000 \\ 17000 \end{pmatrix} \tag{2.45.9}$$

$\therefore$ Total profit in Market-I = 15000 Total profit in Market-II =17000

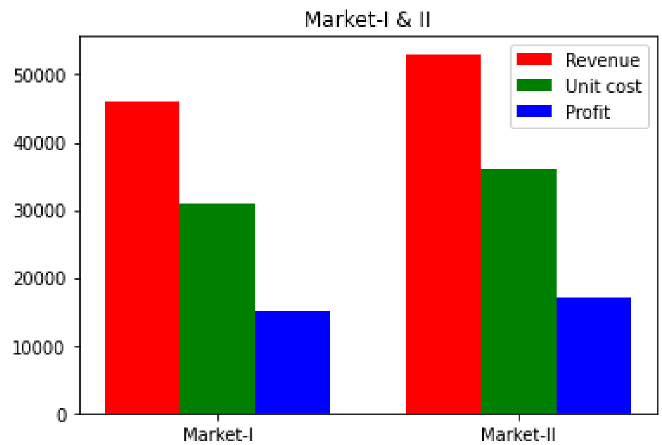

Fig. 2.45: Revenue,Sales & Profit Of Market-I & II

2.46. If A and B are square matrices of the same order such that AB=BA,then prove by indication that $AB^n = B^nA$.Further prove that $(AB)^n = A^nB^n$ for all $n \epsilon N$.
Choose the correct answer in the following questions:

2.47. If A=$\begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}$ is such that $A^2 = I$,then
(A)$1 + \alpha^2 + \beta\gamma = 0$ (B)$1 - \alpha 2 + \beta\gamma = 0$
(C)$1 - \alpha^2 - \beta\gamma = 0$ (D)$1 + \alpha^2 - \beta\gamma = 0$

Solution: If

$$\mathbf{A} = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}, \ \mathbf{A}^2 = I \tag{2.47.1}$$

The characteristic equation is

$$|\mathbf{A} - \lambda \mathbf{I}| = 0 \implies \det \begin{pmatrix} \alpha - \lambda & \beta \\ \gamma & -\alpha - \lambda \end{pmatrix} = 0 \tag{2.47.2}$$

$$\implies (\alpha - \lambda)(-\alpha - \lambda) - \gamma\beta = 0 \tag{2.47.3}$$

$$\implies \lambda^2 - \alpha^2 - \gamma\beta = 0 \tag{2.47.4}$$

By the Cayley-Hamilton theorem, every square matrix satisfies its own characteristic equation. Hence,on substituting from (2.47.1) in (2.47.4)

$$\implies \mathbf{A^2} - \alpha^2 - \gamma\beta = 0. \quad (2.47.5)$$
$$\implies \mathbf{I} - \alpha^2 - \gamma\beta = 0 \quad (2.47.6)$$

Hence, (c) is the correct answer

2.48. If the matrix A is both symmetric and skew symmetric,then
(A) A is a diagonal matrix
(B) A is a zero matriz
(C)A is a square matrix
(D)None of these

Solution: If **A** is symmetric, then

$$\mathbf{A}^T = \mathbf{A} \quad (2.48.1)$$

If matrix A is skew symmetric, then

$$\mathbf{A}^T = -\mathbf{A} \quad (2.48.2)$$

From (2.48.1) and (2.48.2),

$$\mathbf{A} = -\mathbf{A} \quad (2.48.3)$$
$$\implies 2\mathbf{A} = 0 \quad (2.48.4)$$
$$\text{or, } \mathbf{A} = 0 \quad (2.48.5)$$

$\therefore$ **A** is zero matrix.

2.49. If A is square matrix such that $A^2 = A$,then $(I + A)^3 - 7A$ is equal to
(A)A
(B)I-A
(C)I
(D)3A
Solution:
Given

$$A^2 = A \implies A^2 - A = 0 \quad (2.49.1)$$

Let λ is the eigen value then the every eigen value satisfies its very own characteristic equation

$$\implies \lambda^2 = \lambda \implies \lambda^2 - \lambda = 0 \quad (2.49.2)$$

Then $(I+A)^3-7A$ can be written as $(I+\lambda)^3-7\lambda$

$$(I + \lambda)^3 - 7\lambda = I^3 + \lambda^3 + 3I^2\lambda + 3\lambda^2 I - 7\lambda \quad (2.49.3)$$

we know that

$$I^3 = I^2 = 1 \quad (2.49.4)$$

$$= 1 + \lambda^3 + 3(1)\lambda + 3\lambda^2(1) - 7\lambda \quad (2.49.5)$$

$$= 1 + \lambda^3 + 3\lambda + 3\lambda^2 - 7\lambda \quad (2.49.6)$$

$$= 1 + (\lambda^2\lambda) + 3\lambda + 3\lambda - 7\lambda \quad (2.49.7)$$

$$= 1 + (\lambda\lambda) - \lambda \quad (2.49.8)$$

$$= 1 + \lambda^2 - \lambda \quad (2.49.9)$$

From the equation (2.49.2) $\lambda^2 - \lambda = 0$

$$= 1 + 0 \quad (2.49.10)$$

$$= 1 \quad (2.49.11)$$

Option C is the valid answer.

2.50. Balance the following chemical equation.

$$NaOH + H_2SO_4 \rightarrow Na_2SO_4 + H_2O \quad (2.50.1)$$

Solution: Let the balanced version of (2.50.1) be

$$x_1NaOH + x_2H_2SO_4 \rightarrow x_3Na_2SO_4 + x_4H_2O \quad (2.50.2)$$

which results in the following equations:

$$(x_1 - 2x_3)Na = 0 \quad (2.50.3)$$
$$(x_1 + 4x_2 - 4x_3 - x_4)O = 0 \quad (2.50.4)$$
$$(x_1 + 2x_2 - 2x_4)H = 0 \quad (2.50.5)$$
$$(x_2 - x_3)S = 0 \quad (2.50.6)$$

which can be expressed as

$$x_1 + 0.x_2 - 2x_3 + 0.x_4 = 0 \quad (2.50.7)$$
$$x_1 + 4x_2 - 4x_3 - x_4 = 0 \quad (2.50.8)$$
$$x_1 + 2x_2 + 0.x_3 - 2x_4 = 0 \quad (2.50.9)$$
$$0.x_1 + x_2 - x_3 + 0.x_4 = 0 \quad (2.50.10)$$

resulting in the matrix equation

$$\begin{pmatrix} 1 & 0 & -2 & 0 \\ 1 & 4 & -4 & -1 \\ 1 & 2 & 0 & -2 \\ 0 & 1 & -1 & 0 \end{pmatrix} \mathbf{x} = \mathbf{0} \quad (2.50.11)$$

where,

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \tag{2.50.12}$$

(2.50.11) can be reduced as follows:

$$\begin{pmatrix} 1 & 0 & -2 & 0 \\ 1 & 4 & -4 & -1 \\ 1 & 2 & 0 & -2 \\ 0 & 1 & -1 & 0 \end{pmatrix} \xleftrightarrow[R_3 \leftarrow R_3 - R_1]{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 4 & -2 & -1 \\ 0 & 2 & 2 & -2 \\ 0 & 1 & -1 & 0 \end{pmatrix} \tag{2.50.13}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{R_2}{4}} \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{4} \\ 0 & 2 & 2 & -2 \\ 0 & 1 & -1 & 0 \end{pmatrix} \tag{2.50.14}$$

$$\xleftrightarrow[R_4 \leftarrow R_4 - R_2]{R_3 \leftarrow R_3 - 2R_2} \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{4} \\ 0 & 0 & 3 & -\frac{3}{2} \\ 0 & 0 & -\frac{1}{2} & \frac{1}{4} \end{pmatrix} \tag{2.50.15}$$

$$\xleftrightarrow{R_3 \leftarrow \frac{R_3}{3}} \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{4} \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & \frac{1}{4} \end{pmatrix} \tag{2.50.16}$$

$$\xleftrightarrow[R_4 \leftarrow R_4 + \frac{R_3}{2}]{R_2 \leftarrow R_2 + \frac{R_3}{2}} \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.50.17}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + 2R_3} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.50.18}$$

Thus,

$$x_1 = x_4, x_2 = \frac{1}{2}x_4, x_3 = \frac{1}{2}x_4 \tag{2.50.19}$$

$$\implies \quad \mathbf{x} = x_4 \begin{pmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix} \tag{2.50.20}$$

by substituting $x_4 = 2$

$$\mathbf{x} = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 2 \end{pmatrix} \tag{2.50.21}$$

Hence, (2.50.2) finally becomes

$$2NaOH + H_2SO_4 \rightarrow Na_2SO_4 + 2H_2O \tag{2.50.22}$$

2.51. Balance the following chemical equation

$$BaCl_2 + H_2SO_4 \rightarrow BaSO_4 + HCl \tag{2.51.1}$$

2.52. If A=$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ and B=$\begin{pmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{pmatrix}$, then find 2A-B.

2.53. If A=$\begin{pmatrix} 8 & 0 \\ 4 & -2 \\ 3 & 6 \end{pmatrix}$ and B=$\begin{pmatrix} 2 & -2 \\ 4 & 2 \\ -5 & 1 \end{pmatrix}$, then find the matrix X, such that 2A+3X=5B.

2.54. Find X and Y, if X+Y=$\begin{pmatrix} 5 & 2 \\ 0 & 9 \end{pmatrix}$ and X-Y=$\begin{pmatrix} 3 & 6 \\ 0 & -1 \end{pmatrix}$.

2.55. Find the values of x and y from the following equation:
$2\begin{pmatrix} x & 5 \\ 7 & y-3 \end{pmatrix} + \begin{pmatrix} 3 & -4 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 7 & 6 \\ 15 & 14 \end{pmatrix}$

2.56. Two farmers Ramkishan and Gurcharan Singh cultivate only three varieties of rice namely Basmati, Permal and Naura. The sale (in Rupees) of these varieties of rice by both the farmers in the month of September and October are given by the following matrices **A** and **B** .

September Sales(in Rupees)

$$\mathbf{A} = \begin{array}{c} \begin{array}{ccc} \text{Basmati} & \text{Permal} & \text{Naura} \end{array} \\ \begin{pmatrix} 10000 & 20000 & 30000 \\ 50000 & 30000 & 10000 \end{pmatrix} \end{array} \begin{pmatrix} \text{Ramkishan} \\ \text{Gurucharan} \end{pmatrix} \tag{2.56.1}$$

October Sales(in Rupees)

$$\mathbf{B} = \begin{array}{c} \\ \\ \end{array}\begin{array}{ccc} \text{Basmati} & \text{Permal} & \text{Naura} \end{array}$$
$$\mathbf{B} = \begin{pmatrix} 5000 & 10000 & 6000 \\ 20000 & 10000 & 10000 \end{pmatrix} \begin{pmatrix} \text{Ramkishan} \\ \text{Gurucharan} \end{pmatrix} \tag{2.56.2}$$

a) Find the combined sales in September and October for each farmer in each variety.
b) Find the decrease in sales from September to October.
c) If both farmers receive 2% profit on gross sales, compute the profit for each farmer and for each variety sold in October.

Solution:

a) Combined sales in September and October is given by

$$\begin{array}{c} \\ \mathbf{A}+\mathbf{B} = \end{array} \begin{array}{c} \begin{array}{ccc} \text{Basmati} & \text{Permal} & \text{Naura} \end{array} \\ \begin{pmatrix} 15000 & 30000 & 36000 \\ 70000 & 40000 & 20000 \end{pmatrix} \end{array} \begin{array}{c} \\ \begin{pmatrix} \text{Ram} \\ \text{Guru} \end{pmatrix} \end{array} \tag{2.56.3}$$

b) Decrease in sales from September to October is given by

$$\begin{array}{c} \\ \mathbf{A}-\mathbf{B} = \end{array} \begin{array}{c} \begin{array}{ccc} \text{Basmati} & \text{Permal} & \text{Naura} \end{array} \\ \begin{pmatrix} 5000 & 10000 & 24000 \\ 30000 & 20000 & 0 \end{pmatrix} \end{array} \begin{array}{c} \\ \begin{pmatrix} \text{Ram} \\ \text{Guru} \end{pmatrix} \end{array} \tag{2.56.4}$$

c) Profit for sales in October is given by

$$\begin{array}{c} \\ \frac{2}{100}\mathbf{B} = \end{array} \begin{array}{c} \begin{array}{ccc} \text{Basmati} & \text{Permal} & \text{Naura} \end{array} \\ \begin{pmatrix} 100 & 200 & 120 \\ 400 & 200 & 200 \end{pmatrix} \end{array} \begin{array}{c} \\ \begin{pmatrix} \text{Ram} \\ \text{Guru} \end{pmatrix} \end{array} \tag{2.56.5}$$

2.57. Find AB, if $A=\begin{pmatrix} 6 & 9 \\ 2 & 3 \end{pmatrix}$ and $B=\begin{pmatrix} 2 & 6 & 0 \\ 7 & 9 & 8 \end{pmatrix}$.

2.58. If $A=\begin{pmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{pmatrix}$ and $B=\begin{pmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{pmatrix}$, then find AB,BA.Show that AB$\neq$BA

2.59. If $A=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $B=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then find AB,BA. Show that AB$\neq$BA

2.60. Find AB, if $A=\begin{pmatrix} 0 & -1 \\ 0 & 2 \end{pmatrix}$ and $B=\begin{pmatrix} 3 & 5 \\ 0 & 0 \end{pmatrix}$

2.61. If $A=\begin{pmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 3 & -1 & 2 \end{pmatrix}$, $B=\begin{pmatrix} 1 & 3 \\ 0 & 2 \\ -1 & 4 \end{pmatrix}$ and

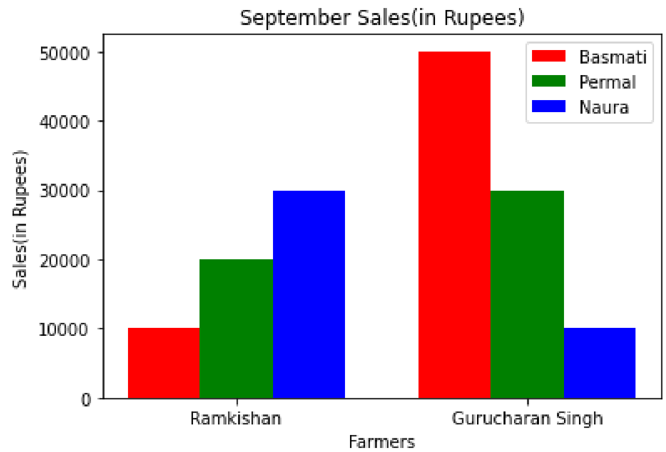

Fig. 2.56: September Sales(in Rupees)

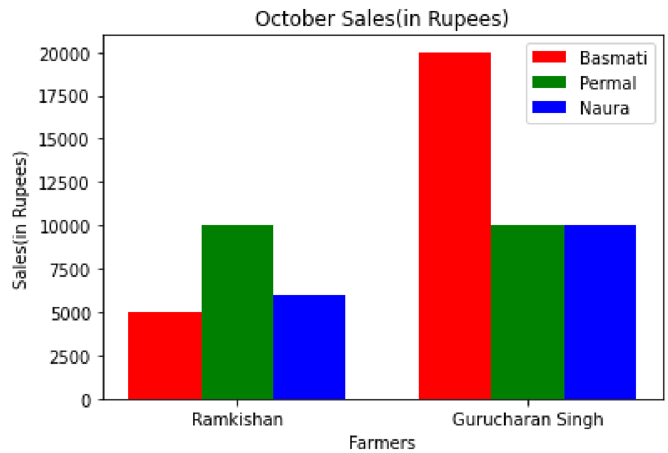

Fig. 2.56: October Sales(in Rupees)

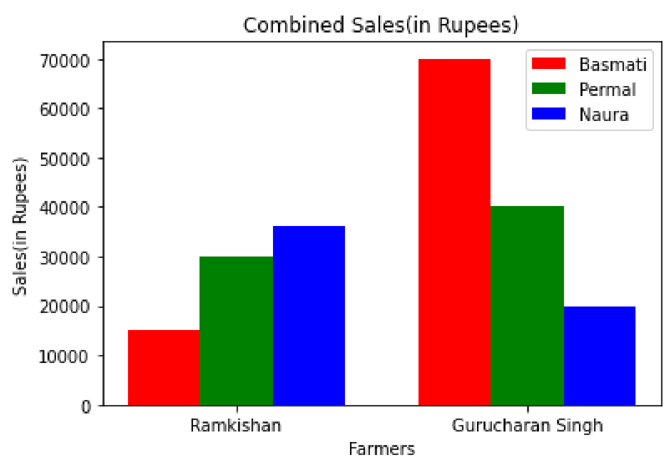

Fig. 2.56: Combined Sales(in Rupees)

$C=\begin{pmatrix} 1 & 2 & 3 & -4 \\ 2 & 0 & -2 & 1 \end{pmatrix}$, find A(BC),(AB)C and show that (AB)C=A(BC)

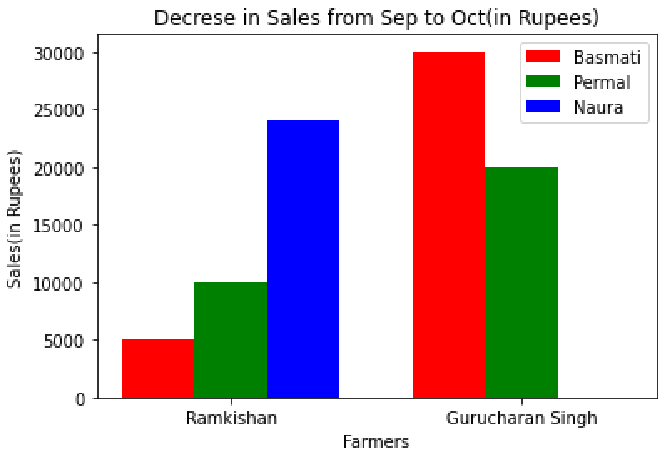

Fig. 2.56: Decrease in Sales from Sep to Oct(in Rupees)

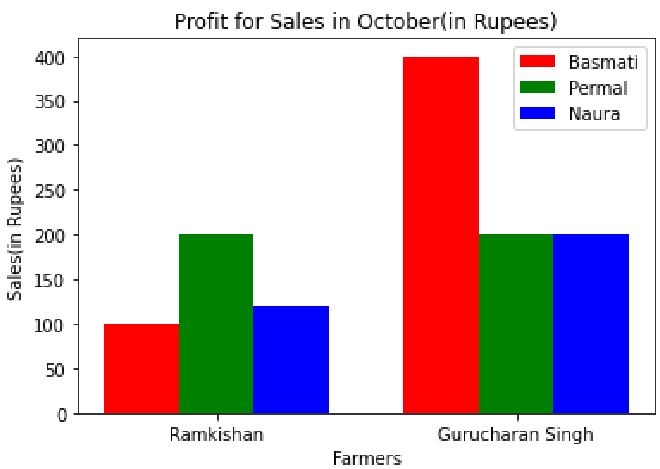

Fig. 2.56: Profit for Sales in October(in Rupees)

2.62. If A=$\begin{pmatrix} 0 & 6 & 7 \\ -6 & 0 & 8 \\ 7 & -8 & 0 \end{pmatrix}$, B=$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$,C=$\begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$ Calculate AC,BC and (A+B)C=AC+BC

2.63. If A=$\begin{pmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{pmatrix}$,then show that $A^3 - 23A - 40I = 0$

Solution: Given that $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{pmatrix}$.

a) **The Characteristic equation** is given by:

$$\implies |\mathbf{A} - \lambda \mathbf{I}| = 0 \quad (2.63.1)$$

$$\implies \begin{vmatrix} 1-\lambda & 2 & 3 \\ 3 & -2-\lambda & 1 \\ 4 & 2 & 1-\lambda \end{vmatrix} = 0 \quad (2.63.2)$$

$$\implies (1-\lambda)\left((-2-\lambda)(1-\lambda)-2\right) - 2\left(3(1-\lambda)-4\right) + 3\left(6+4(2+\lambda)\right) = 0 \quad (2.63.3)$$

$$\implies \lambda^3 - 23\lambda - 40 = 0 \quad (2.63.4)$$

The above equation is similar to equation to be proved.

b) According to **Cayley-Hamilton Theorem:** Every square matrix satisfies its own **characteristic equation**.

$$\therefore \mathbf{A}^3 - 23\mathbf{A} - 40\mathbf{I} = 0 \quad (2.63.5)$$

Hence Proved.

2.64. In a legislative assembly election, a political group hired a public relations firm to promote its candidate in three ways: telephone, house calls, and letters. The cost per contact (in paise) is given in matrix **A** as

Cost per Contact(in Paise)

$$\mathbf{A} = \begin{array}{c} \text{cost} \\ \begin{pmatrix} 40 \\ 100 \\ 50 \end{pmatrix} \end{array} \begin{pmatrix} \text{Telephone} \\ \text{Housecall} \\ \text{Letter} \end{pmatrix} \quad (2.64.1)$$

The number of contacts of each type made in two cities X and Y is given by matrix **B**

$$\mathbf{B} = \begin{array}{c} \begin{array}{ccc} \text{Telephone} & \text{Housecall} & \text{Letter} \end{array} \\ \begin{pmatrix} 1000 & 500 & 5000 \\ 3000 & 1000 & 10000 \end{pmatrix} \end{array} \begin{pmatrix} X \\ Y \end{pmatrix} \quad (2.64.2)$$

Find the total amount spent by the group in the two cities X and Y **Solution:**
The total amount spent is given by=

BA

$$= \begin{pmatrix} 1000 & 500 & 5000 \\ 3000 & 1000 & 10000 \end{pmatrix} \begin{pmatrix} 40 \\ 100 \\ 50 \end{pmatrix}$$

$$= \begin{pmatrix} 40000 + 50000 + 250000 \\ 120000 + 100000 + 500000 \end{pmatrix}$$

$$\begin{array}{c} \text{TotalCost} \\ \begin{pmatrix} 340000 \\ 720000 \end{pmatrix} \end{array} \begin{pmatrix} X \\ Y \end{pmatrix} \quad (2.64.3)$$

∴ the total amount spent in city X and city Y is 3400 and 7200 Rupees respectively. See Fig. 2.64

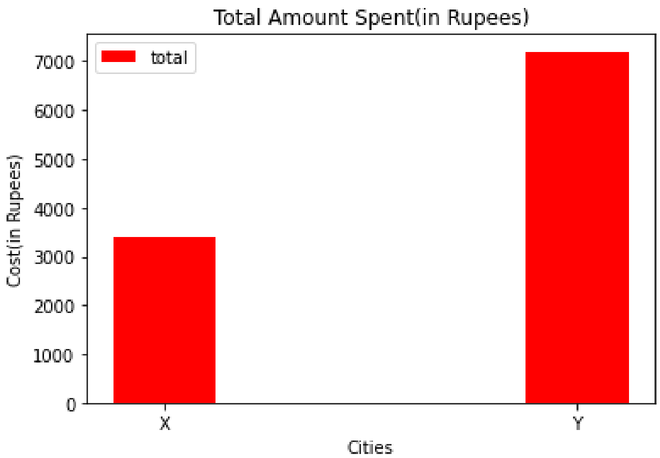

Fig. 2.64: Total Amount Spent by the group in cities X and Y

2.65. If A=$\begin{pmatrix} 3 & \sqrt{3} & 2 \\ 4 & 2 & 0 \end{pmatrix}$ and B=$\begin{pmatrix} 2 & -1 & 2 \\ 1 & 2 & 4 \end{pmatrix}$, verify that
(i) $(A')' = A$
(ii)$(A+B)' = A' + B'$,
(iii) $(kB)' = kB'$,where k is any constant.

2.66. If A=$\begin{pmatrix} -2 \\ 4 \\ 5 \end{pmatrix}$,B=$\begin{pmatrix} 1 & 3 & -6 \end{pmatrix}$, verify that $(AB)' = B'A'$

2.67. Express the matrix B=$\begin{pmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{pmatrix}$ as the sum of a symmetric and a skew symmetric matrix.
Solution:
Given

$$\mathbf{B} = \begin{pmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{pmatrix} \tag{2.67.1}$$

and

$$\mathbf{B}^\top = \begin{pmatrix} 2 & -1 & 1 \\ -2 & 3 & -2 \\ -4 & 4 & -3 \end{pmatrix} \tag{2.67.2}$$

$$\mathbf{C} = \frac{\mathbf{B}+\mathbf{B}^\top}{2} = \begin{pmatrix} 2 & -\frac{3}{2} & -\frac{3}{2} \\ -\frac{3}{2} & 3 & 1 \\ -\frac{3}{2} & 1 & -3 \end{pmatrix} \tag{2.67.3}$$

$$= \mathbf{C}^\top \tag{2.67.4}$$

Also,

$$\mathbf{D} = \frac{\mathbf{B}-\mathbf{B}^\top}{2} = \begin{pmatrix} 0 & -\frac{1}{2} & -\frac{5}{2} \\ \frac{1}{2} & 0 & 3 \\ \frac{5}{2} & -3 & 0 \end{pmatrix} \tag{2.67.5}$$

$$= -\mathbf{D}^\top \tag{2.67.6}$$

Hence, $\mathbf{C}$ is a symmetric matrix and $\mathbf{D}$ is skew symmetric. and $\mathbf{C}+\mathbf{D} = \mathbf{B}$.

2.68. By using elementary operations,find the inverse of the matrix
A=$\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$.

2.69. Obtain the inverse of the following matrix using elementary operations
A=$\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix}$.
Solution:

a) Given that

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix} \tag{2.69.1}$$

The augmented matrix $[A|I]$ is as given below:-

$$\left(\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \tag{2.69.2}$$

We apply the elementary row operations on

$[A|I]$ as follows :-

$$[A|I] = \left(\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \tag{2.69.3}$$

$$\xleftrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \tag{2.69.4}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - 3R_1} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -5 & -8 & 0 & -3 & 1 \end{array}\right) \tag{2.69.5}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - 2R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & -2 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -5 & -8 & 0 & -3 & 1 \end{array}\right) \tag{2.69.6}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 + 5R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & -2 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 5 & -3 & 1 \end{array}\right) \tag{2.69.7}$$

$$\xleftrightarrow{R_3 \leftarrow R_3/2} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & -2 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{array}\right) \tag{2.69.8}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{array}\right) \tag{2.69.9}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - 2R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & -4 & 3 & -1 \\ 0 & 0 & 1 & \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{array}\right) \tag{2.69.10}$$

By performing elementary transformations on augmented matrix$[A|I]$, we obtained the augmented matrix in the form $[I|A]$. Hence we can conclude that the matrix A is invertible and inverse of the matrix is:-

$$\therefore \mathbf{A^{-1}} = \begin{pmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{pmatrix} \tag{2.69.11}$$

b) QR decomposition of $\begin{pmatrix} 0 & 1 & 3 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$

Let us use the Gram-schmidt approach to obtain QR decomposition of$\mathbf{A}$. Consider rows vectors say $\mathbf{a_1}$,$\mathbf{a_2}$ and $\mathbf{a_3}$ of $\mathbf{A}$ which is given by

$$\mathbf{a_1} = \begin{pmatrix} 0 & 1 & 2 \end{pmatrix} \tag{2.69.12}$$

$$\mathbf{a_2} = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \tag{2.69.13}$$

$$\mathbf{a_3} = \begin{pmatrix} 3 & 1 & 1 \end{pmatrix} \tag{2.69.14}$$

we can express these as

$$\mathbf{u_1} = \mathbf{a_1} = \begin{pmatrix} 0 & 1 & 2 \end{pmatrix} \tag{2.69.15}$$

$$\mathbf{e_1} = \frac{\mathbf{u_1}}{||\mathbf{u_1}||} \tag{2.69.16}$$

$$\mathbf{e_1} = \frac{\begin{pmatrix} 0 & 1 & 2 \end{pmatrix}}{\sqrt{0+1+4}} \tag{2.69.17}$$

$$\mathbf{e_1} = \begin{pmatrix} 0 & \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \tag{2.69.18}$$

$$\mathbf{u_2} = \mathbf{a_2} - (\mathbf{a_2}\mathbf{e_1})\mathbf{e_1} \tag{2.69.19}$$

$$= \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} - (\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 0 & \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix})\begin{pmatrix} 0 & \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \tag{2.69.20}$$

$$= \begin{pmatrix} 1 & \frac{2}{5} & \frac{-1}{5} \end{pmatrix} \tag{2.69.21}$$

$$\mathbf{e_2} = \frac{\mathbf{u_2}}{||\mathbf{u_2}||} = \frac{\begin{pmatrix} 1 & \frac{2}{5} & \frac{-1}{5} \end{pmatrix}}{\sqrt{1 + \frac{4}{25} + \frac{1}{25}}} \tag{2.69.22}$$

$$\mathbf{e_2} = \begin{pmatrix} \frac{5}{\sqrt{30}} & \frac{2}{\sqrt{30}} & \frac{-1}{\sqrt{30}} \end{pmatrix} \tag{2.69.23}$$

$$\mathbf{u_3} = \mathbf{a_3} - (\mathbf{a_3}\mathbf{e_1})\mathbf{e_1} - (\mathbf{a_3}\mathbf{e_2})\mathbf{e_2} \tag{2.69.24}$$

$$\mathbf{u_3} = \begin{pmatrix} \frac{1}{3} & \frac{-2}{3} & \frac{1}{3} \end{pmatrix} \tag{2.69.25}$$

$$\mathbf{e_3} = \frac{\mathbf{u_3}}{||\mathbf{u_3}||} \tag{2.69.26}$$

$$= \frac{\begin{pmatrix} \frac{1}{3} & \frac{-2}{3} & \frac{1}{4} \end{pmatrix}}{\sqrt{\frac{1}{9} + \frac{4}{9} + \frac{1}{9}}} \tag{2.69.27}$$

$$\mathbf{e_3} = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix} \tag{2.69.28}$$

Thus,

$$\mathbf{Q} = (e_1|e_2| - - - -|e_n) \tag{2.69.29}$$

$$= \begin{pmatrix} 0 & \frac{5}{\sqrt{30}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{30}} & \frac{-2}{\sqrt{6}} \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{30}} & \frac{1}{\sqrt{6}} \end{pmatrix} \tag{2.69.30}$$

Then

$$\mathbf{R} = \begin{pmatrix} a_1e_1 & a_2e_1 & a_3e_1 \\ 0 & a_2e_2 & a_3e_2 \\ 0 & 0 & a_3e_3 \end{pmatrix} \tag{2.69.31}$$

$$= \begin{pmatrix} \frac{5}{\sqrt{5}} & \frac{8}{\sqrt{5}} & \frac{3}{\sqrt{5}} \\ 0 & \frac{6}{\sqrt{30}} & \frac{16}{\sqrt{30}} \\ 0 & 0 & \frac{2}{\sqrt{6}} \end{pmatrix} \tag{2.69.32}$$

From equations (2.69.30) and (2.69.32) the obtained **QR** Decomposition is

$$\begin{pmatrix} 0 & 1 & 3 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 0 & \frac{5}{\sqrt{30}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{30}} & \frac{-2}{\sqrt{6}} \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{30}} & \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} \frac{5}{\sqrt{5}} & \frac{8}{\sqrt{5}} & \frac{3}{\sqrt{5}} \\ 0 & \frac{6}{\sqrt{30}} & \frac{16}{\sqrt{30}} \\ 0 & 0 & \frac{2}{\sqrt{6}} \end{pmatrix} \tag{2.69.33}$$

2.70. Find P^{-1}, if it exists, given $P=\begin{pmatrix} 10 & -2 \\ -5 & 1 \end{pmatrix}$.

Solution: Using row reduction,

$$\begin{pmatrix} 10 & -2 \\ -5 & 1 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 + \frac{R_1}{2}} \begin{pmatrix} 10 & -2 \\ 0 & 0 \end{pmatrix} \tag{2.70.1}$$

Hence, $\mathbf{P}^{-1}$ does not exist.

2.71. If $A=\begin{pmatrix} \cos\theta & \sin\theta \\ sin\theta & \cos\theta \end{pmatrix}$,

then prove that $A^n = \begin{pmatrix} \cos\theta & \sin n\theta \\ sinn\theta & \cos n\theta \end{pmatrix}$, n $\in$ N.

2.72. If A and B are symmetric matrices of the same order, then show that AB is symmetric if and only if A and B commute,that AB = BA.

2.73. Let $A=\begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}$, $B=\begin{pmatrix} 5 & 2 \\ 7 & 4 \end{pmatrix}$, $C=\begin{pmatrix} 2 & 5 \\ 3 & 8 \end{pmatrix}$. Find a matrix D such that CD-AB=0.

2.74. Find the values of a,b,c and d from the equations: $\begin{pmatrix} a-b & 2a+c \\ 2a-b & 3c+d \end{pmatrix} = \begin{pmatrix} -1 & 5 \\ 0 & 13 \end{pmatrix}$

2.75. Show that

(i) $\begin{pmatrix} 5 & -1 \\ 6 & 7 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} \neq \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 5 & -1 \\ 6 & 7 \end{pmatrix}$

(ii) $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 3 & 4 \end{pmatrix} \neq \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$

2.76. If $A=\begin{pmatrix} 3 & -2 \\ 4 & -2 \end{pmatrix}$ and $I=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$,find k so that $A^2 = kA - 2I$

2.77. Find the matrix X so that $X\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}=\begin{pmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{pmatrix}$

2.78. (i) $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

(ii) $\begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix} = 2(x+y+z)^3$

2.79. $\begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} = (1-x^3)^2$

2.80. $\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = (1+a^2+b^2)^3$

2.81. Let $A=\begin{bmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{bmatrix}$, where $0 \le \theta \le 2\Pi$. Then

a) Det(A)=0
b) Det(A)$\in (2, \infty)$
c) Det(A)$\in (2, 4)$
d) Det(A)$\in [2, 4]$

2.82. $\begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 4+3p+2q \\ 3 & 6+3p & 10+6p+3q \end{vmatrix} = 1$

2.83. $\begin{vmatrix} \sin\alpha & \cos\alpha & \cos(\alpha+\delta) \\ \sin\beta & \cos\beta & \cos(\beta+\delta) \\ \sin\gamma & \cos\gamma & \cos(\gamma+\delta) \end{vmatrix} = 0$

2.84. Solve the system of equations

$$\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 4$$
$$\frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 1$$
$$\frac{6}{x} + \frac{9}{y} - \frac{20}{z} = 2$$

2.85. If a,b,c are in A.P, then the determinant $\begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}$ is

a) 0
b) 1
c) x
d) 2x

2.86. If x,y,z are nonzero real numbers, then the inverse of matrix $A=\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$ is

a) $\begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$

b) $xyz\begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$

c) $\frac{1}{xyz}\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$

d) $\frac{1}{xyz}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Examine the consistency of the system of given Equations.

2.87. $x + 2y = 2$

$2x + 3y = 3$

2.88. $2x - y = 5$

$x + y = 4$

2.89. Evaluate the determinant $\begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix} = 0$

Find the inverse and QR decomposition of the following.

2.90. $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

2.91. $\begin{pmatrix} 1 & 3 \\ 2 & 7 \end{pmatrix}$

2.92. $\begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix}$

2.93. $\begin{pmatrix} 2 & 1 \\ 7 & 4 \end{pmatrix}$

2.94. $\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$

2.95. $\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$

2.96. $\begin{pmatrix} 4 & 5 \\ 3 & 4 \end{pmatrix}$

2.97. $\begin{pmatrix} 3 & 10 \\ 2 & 7 \end{pmatrix}$

2.98. $\begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix}$

2.99. $\begin{pmatrix} 2 & -6 \\ 1 & -2 \end{pmatrix}$

2.100. $\begin{pmatrix} 6 & -3 \\ -2 & 1 \end{pmatrix}$

2.101. $\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$

2.102. $\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}$

2.103. Find the QR decomposition of

$$\mathbf{A} = \begin{pmatrix} 8 & 5 \\ 3 & 2 \end{pmatrix} \tag{2.103.1}$$

2.104. Find the QR decomposition of

$$\mathbf{A} = \begin{pmatrix} 2 & 5 \\ 1 & 4 \end{pmatrix} \tag{2.104.1}$$