

Constructions using Python

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Abstract—This book introduces constructions in high school geometry using Python. The content and exercises are based on NCERT textbooks from Class 6-12. A simple introduction to Python and \LaTeX figures is provided in the process.

Download all python codes from

```
svn co https://github.com/gadepall/school/trunk/ncert/constructions/codes
```

and latex-tikz codes from

```
svn co https://github.com/gadepall/school/trunk/ncert/constructions/figs
```

1 EXAMPLES

1.1. Draw Fig. 1.1.1 for $a = 4, c = 3$.

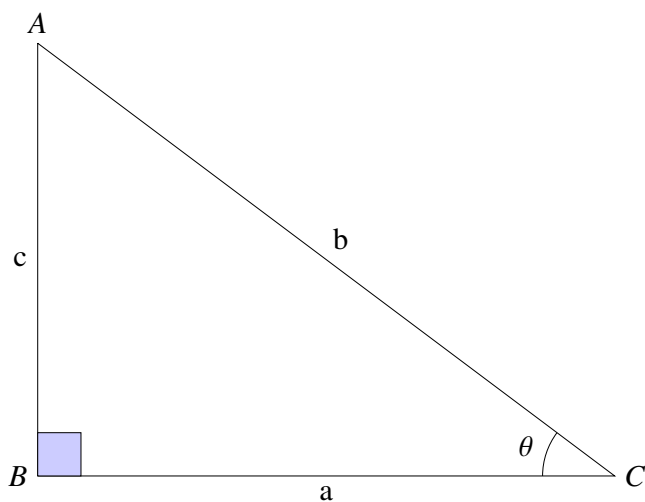


Fig. 1.1.1: Right Angled Triangle

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Solution: The vertices of $\triangle ABC$ are

$$\mathbf{A} = \begin{pmatrix} 0 \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} a \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \quad (1.1.1)$$

The python code for Fig. 1.1.1 is

```
codes/triangle/tri_right_angle.py
```

and the equivalent latex-tikz code is

```
figs/triangle/tri_right_angle.tex
```

The above latex code can be compiled as a standalone document as

```
figs/triangle/tri_right_angle_alone.tex
```

1.2. Draw Fig. 1.2.1 for $a = 4, c = 3$.

Solution: The vertex \mathbf{A} can be expressed in polar coordinate form as

$$\mathbf{A} = b \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \quad (1.2.1)$$

where

$$b = \sqrt{a^2 + c^2} = 5, \tan \theta = \frac{3}{4} \quad (1.2.2)$$

The python code for Fig. 1.2.1 is

```
codes/triangle/tri_polar.py
```

and the equivalent latex-tikz code is

```
figs/triangle/tri_polar.tex
```

1.3. Draw Fig. 1.3.1 with $a = 6, b = 5$ and $c = 4$.

Solution: Let the vertices of $\triangle ABC$ and \mathbf{D} be

$$\mathbf{A} = \begin{pmatrix} p \\ q \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} a \\ 0 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} p \\ 0 \end{pmatrix} \quad (1.3.1)$$

Then

$$AB = \|\mathbf{A} - \mathbf{B}\|^2 = \|\mathbf{A}\|^2 = c^2 \quad \because \mathbf{B} = \mathbf{0} \quad (1.3.2)$$

$$BC = \|\mathbf{C} - \mathbf{B}\|^2 = \|\mathbf{C}\|^2 = a^2 \quad (1.3.3)$$

$$AC = \|\mathbf{A} - \mathbf{C}\|^2 = b^2 \quad (1.3.4)$$

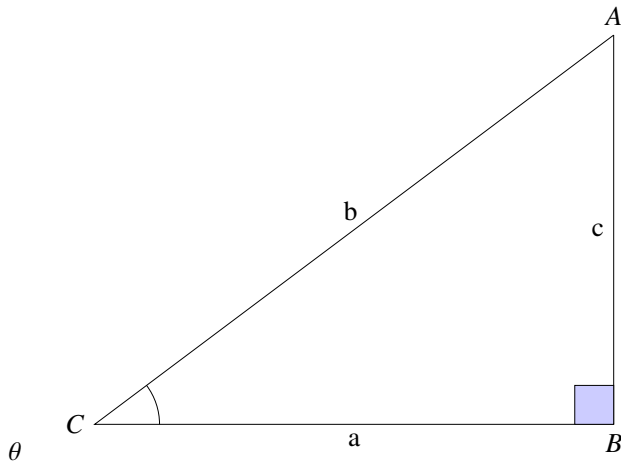


Fig. 1.2.1: Right Angled Triangle

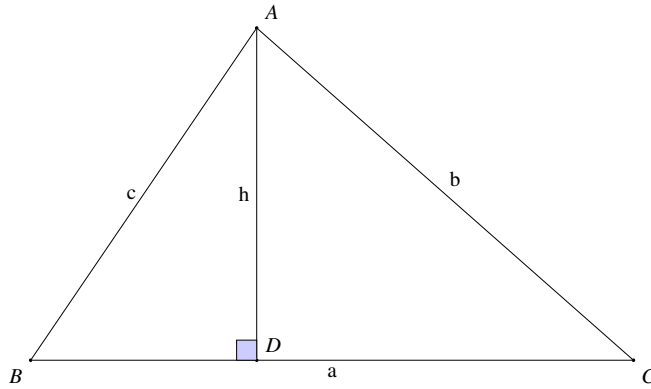


Fig. 1.3.1

From (1.3.4),

$$b^2 = \|\mathbf{A} - \mathbf{C}\|^2 = \|\mathbf{A} - \mathbf{C}\|^T \|\mathbf{A} - \mathbf{C}\| \quad (1.3.5)$$

$$= \mathbf{A}^T \mathbf{A} + \mathbf{C}^T \mathbf{C} - \mathbf{A}^T \mathbf{C} - \mathbf{C}^T \mathbf{A} \quad (1.3.6)$$

$$= \|\mathbf{A}\|^2 + \|\mathbf{C}\|^2 - 2\mathbf{A}^T \mathbf{C} \quad (\because \mathbf{A}^T \mathbf{C} = \mathbf{C}^T \mathbf{A}) \quad (1.3.7)$$

$$= a^2 + c^2 - 2ap \quad (1.3.8)$$

yielding

$$p = \frac{a^2 + c^2 - b^2}{2a} \quad (1.3.9)$$

From (1.3.2),

$$\|\mathbf{A}\|^2 = c^2 = p^2 + q^2 \quad (1.3.10)$$

$$\Rightarrow q = \pm \sqrt{c^2 - p^2} \quad (1.3.11)$$

The python code for Fig. 1.3.1 is

```
codes/triangle/tri_sss.py
```

and the equivalent latex-tikz code is

```
figs/triangle/tri_sss.tex
```

1.4. Construct parallelogram $ABCD$ in Fig. 1.4.1 given that $BC = 5, AB = 6, \angle C = 85^\circ$.

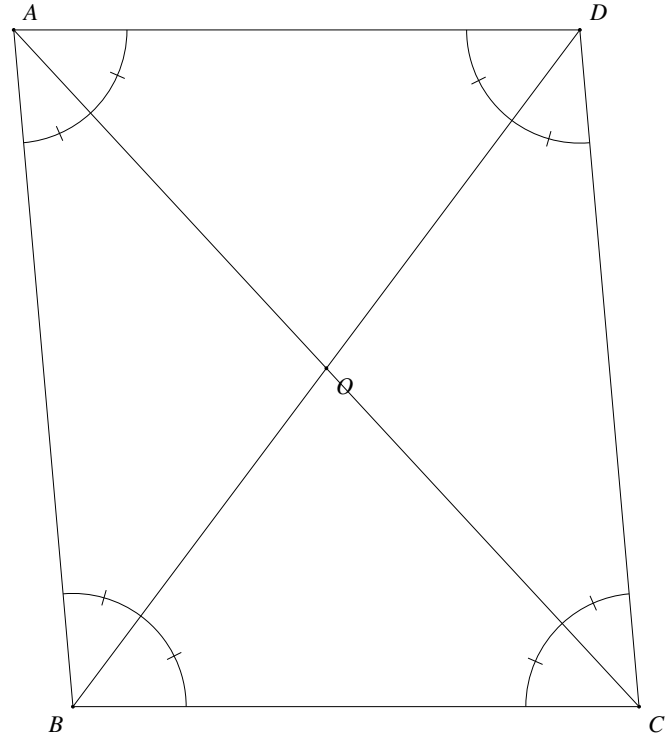


Fig. 1.4.1: Parallelogram Properties

Solution: BD is found using the cosine formula and $\triangle BDC$ is drawn using the approach in Construction 1.3 with

$$\mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}, \quad (1.4.1)$$

Since the diagonals bisect each other,

$$\mathbf{O} = \frac{\mathbf{B} + \mathbf{D}}{2} \quad (1.4.2)$$

$$\mathbf{A} = 2\mathbf{O} - \mathbf{C}. \quad (1.4.3)$$

AB and AD are then joined to complete the ||gm. The python code for Fig. 1.4.1 is

```
codes/quad/pgm_sas.py
```

and The equivalent latex-tikz code is

```
figs/quad/pgm_sas.tex
```

1.5. Draw the ||gm $ABCD$ in Fig. 1.5.1 with $BC = 6, CD = 4.5$ and $BD = 7.5$. Show that it is a rectangle.

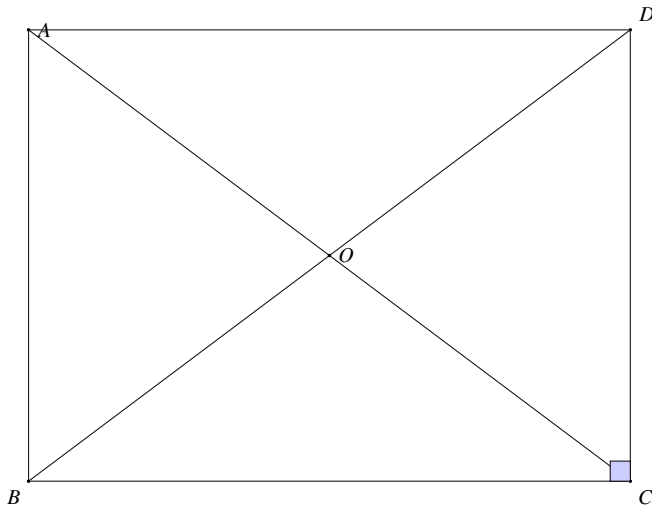


Fig. 1.5.1: Rectangle

Solution: It is easy to verify that

$$BD^2 = BC^2 + C^2 \quad (1.5.1)$$

Hence, using Baudhayana theorem,

$$\angle BCD = 90^\circ \quad (1.5.2)$$

and $ABCD$ is a rectangle.

$$\mathbf{A} = \begin{pmatrix} 0 \\ 4.5 \end{pmatrix} \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \mathbf{C} = \begin{pmatrix} 6 \\ 0 \end{pmatrix} \mathbf{D} = \begin{pmatrix} 6 \\ 4 \end{pmatrix} \quad (1.5.3)$$

The python code for Fig. 1.5.1 is

```
codes/quad/pgm_sss.py
```

and the equivalent latex-tikz code is

```
figs/quad/pgm_sss.tex
```

- 1.6. Draw the rhombus $BEST$ with $BE = 4.5$ and $ET = 6$.

Solution: The coordinates of the various points in Fig. 1.6.1 are obtained as

$$\mathbf{O} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ -4.5 \end{pmatrix} \quad (1.6.1)$$

$$\mathbf{E} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \mathbf{S} = \begin{pmatrix} 4.5 \\ 0 \end{pmatrix}, \mathbf{T} = \begin{pmatrix} 0 \\ -3 \end{pmatrix} \quad (1.6.2)$$

- 1.7. A square is a rectangle whose sides are equal. Draw a square of side 4.5.

Solution: The coordinates of the various points

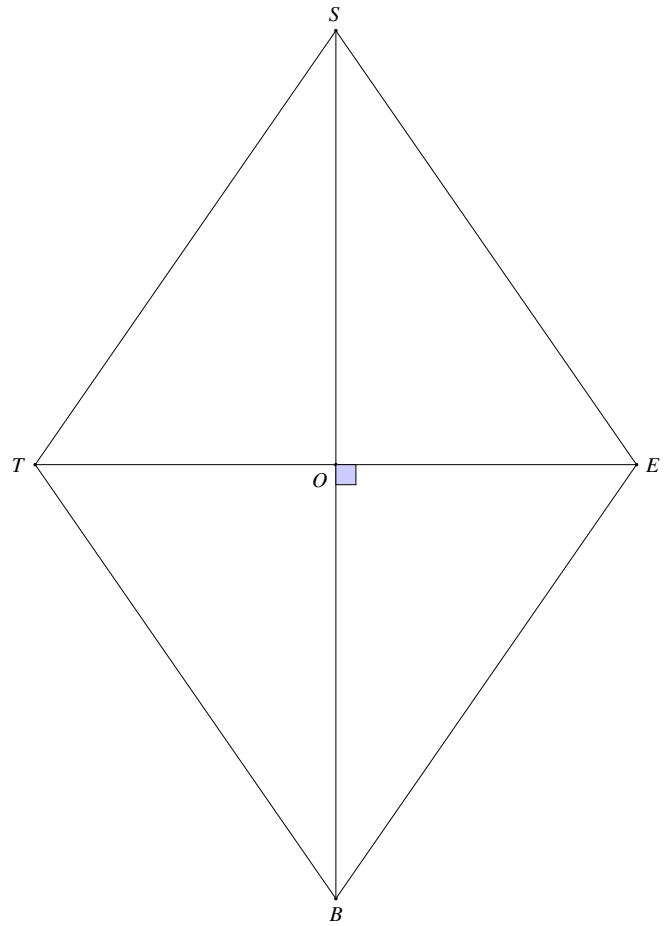


Fig. 1.6.1: Rhombus

in Fig. 1.7.1 are obtained as

$$\mathbf{A} = \begin{pmatrix} 0 \\ 4.5 \end{pmatrix} \quad (1.7.1)$$

$$\mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 4.5 \\ 0 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 4.5 \\ 4.5 \end{pmatrix} \mathbf{O} = \frac{\mathbf{B} + \mathbf{C}}{2} \quad (1.7.2)$$

2 EXERCISES

- 2.1. Construct a triangle of sides $a = 4$, $b = 5$ and $c = 6$.

Solution:

The vertex \mathbf{A} can be expressed in *polar coordinate form* as

$$\mathbf{A} = c \begin{pmatrix} \cos B \\ \sin B \end{pmatrix} \quad (2.1.1)$$

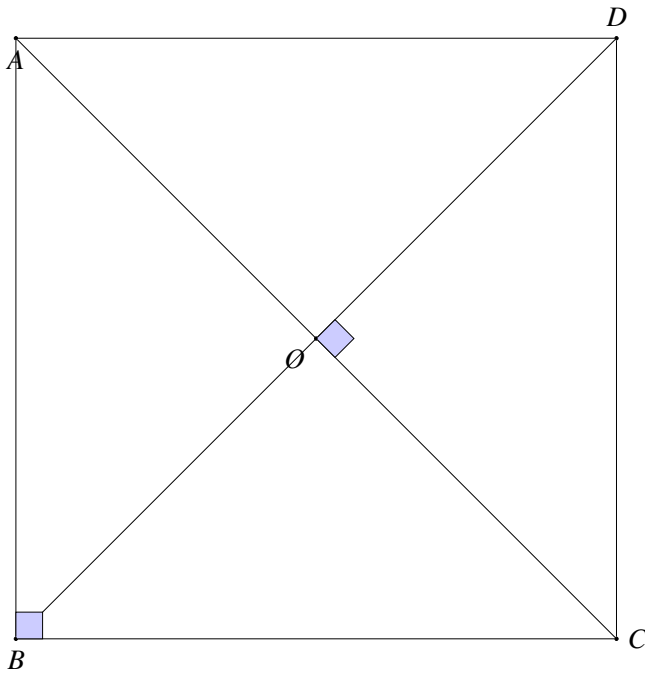


Fig. 1.7.1: Square

From $\triangle ABC$, we use the law of cosines:

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} \quad (2.1.2)$$

$$= 0.5625 \quad (2.1.3)$$

$$\Rightarrow B = 55.771^\circ \quad (2.1.4)$$

Thus,

$$\mathbf{A} = 6 \begin{pmatrix} \cos 55.771 \\ \sin 55.771 \end{pmatrix} \quad (2.1.5)$$

$$\mathbf{A} = \begin{pmatrix} 3.375 \\ 4.960 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}. \quad (2.1.6)$$

which are plotted in Fig. 2.1.1

- 2.2. Construct an isosceles triangle whose base is $a = 8\text{cm}$ and altitude $AD = h = 4\text{cm}$

Solution: From the given information,

$$\mathbf{A} = \begin{pmatrix} a/2 \\ h \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 8 \\ 0 \end{pmatrix} \quad (2.2.1)$$

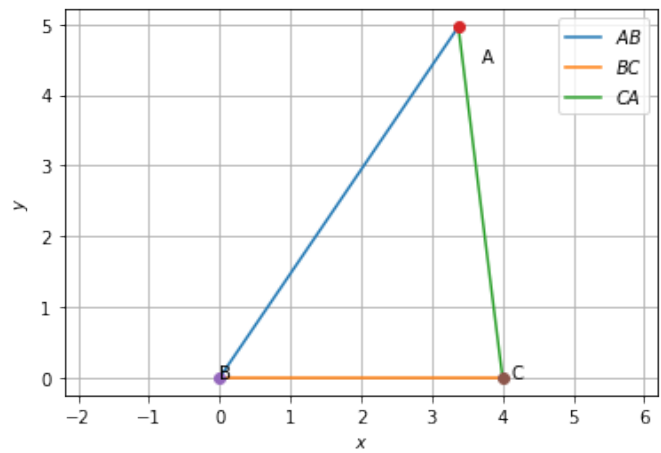
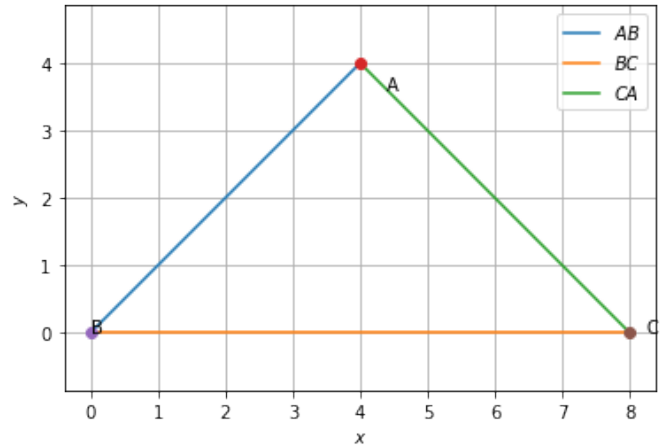
which are used to plot the triangle in Fig. 2.2.1

- 2.3. In $\triangle ABC$, given that $a+b+c = 11$, $\angle B = 45^\circ$ and $\angle C = 45^\circ$, find a, b, c and sketch the triangle.

Solution: Use sine formula,

$$b \sin 45 = c \sin 45 \quad (2.3.1)$$

$$\Rightarrow b = c \quad (2.3.2)$$

Fig. 2.1.1: $\triangle ABC$ Fig. 2.2.1: isosceles triangle $\triangle ABC$

$$a \sin 45 = b \sin 90 \quad (2.3.3)$$

$$\Rightarrow a = \sqrt{2}b \quad (2.3.4)$$

which can be expressed as the matrix equation

$$\begin{pmatrix} 0 & 1 & -1 \\ 1 & -\sqrt{2} & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 11 \end{pmatrix} \quad (2.3.5)$$

solving which yields

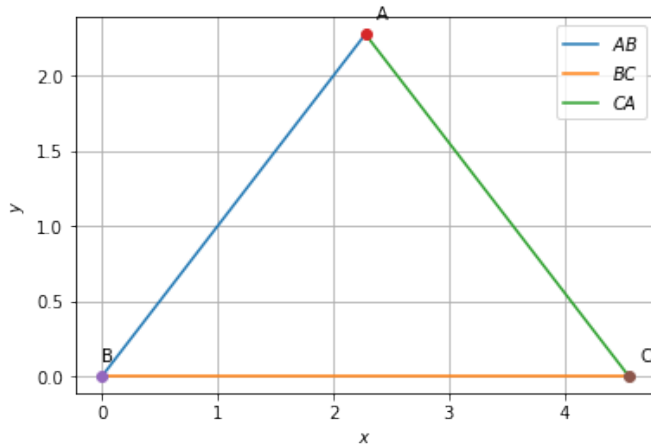
$$\mathbf{A} = \begin{pmatrix} 0 \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 3.22 \end{pmatrix} \quad (2.3.6)$$

$$\mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (2.3.7)$$

$$\mathbf{C} = \begin{pmatrix} a \\ 0 \end{pmatrix} = \begin{pmatrix} 4.55 \\ 0 \end{pmatrix} \quad (2.3.8)$$

resulting in $\triangle ABC$ plotted in Fig. 2.3.1.

- 2.4. Draw $\triangle ABC$ with $a = 6$, $c = 5$ and $\angle B = 60^\circ$.

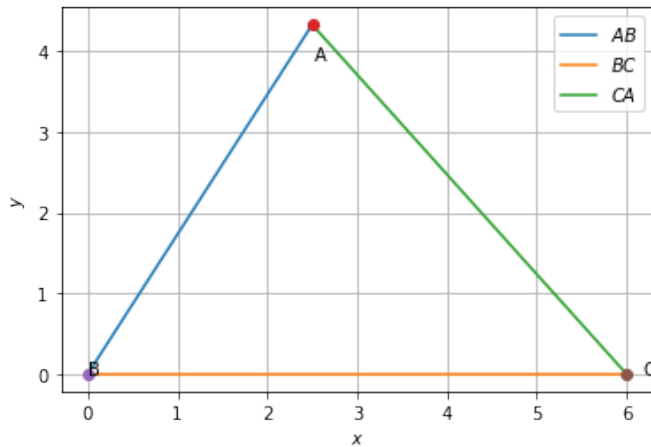
Fig. 2.3.1: $\triangle ABC$

Solution: The vertex **A** can be expressed in polar coordinate form as

$$\mathbf{A} = c \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} a \\ 0 \end{pmatrix}, \quad (2.4.1)$$

$$\Rightarrow \mathbf{A} = 5 \begin{pmatrix} \cos 60^\circ \\ \sin 60^\circ \end{pmatrix} = \begin{pmatrix} 2.5 \\ 2.5\sqrt{3} \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 6 \\ 0 \end{pmatrix} \quad (2.4.2)$$

upon substituting the given values. The triangle is plotted in Fig. 2.4.1.

Fig. 2.4.1: $\triangle ABC$

2.5. Draw $\triangle ABC$ with $a = 7$, $\angle B = 45^\circ$ and $\angle A = 105^\circ$.

Solution: Let

$$\mathbf{A} = c \begin{pmatrix} \cos B \\ \sin B \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} a \\ 0 \end{pmatrix} \quad (2.5.1)$$

$$\therefore \angle C = 30^\circ, \quad (2.5.2)$$

By law of sines

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \quad (2.5.3)$$

$$\Rightarrow c = \frac{7 \sin 30^\circ}{\sin 105^\circ} \quad (2.5.4)$$

$$c = 3.62 \quad (2.5.5)$$

and

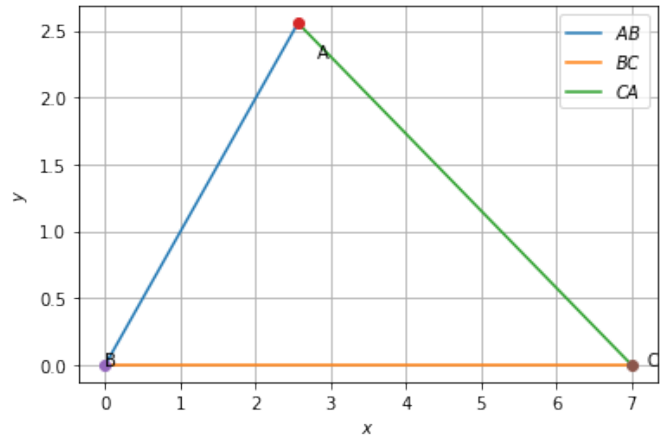
$$\mathbf{A} = c \begin{pmatrix} \cos B \\ \sin B \end{pmatrix} \quad (2.5.6)$$

$$= \begin{pmatrix} 2.55 \\ 2.55 \end{pmatrix} \quad (2.5.7)$$

Thus, the vertices of given $\triangle ABC$ are

$$\mathbf{A} = \begin{pmatrix} 2.55 \\ 2.55 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 7 \\ 0 \end{pmatrix} \quad (2.5.8)$$

and $\triangle ABC$ is plotted in Fig. 2.5.1.

Fig. 2.5.1: $\triangle ABC$

2.6. $\triangle ABC$ is right angled at **B**. If $a = 12$ and $b+c = 18$, find b, c and draw the triangle.

Solution: Let,

$$\mathbf{A} = \begin{pmatrix} 0 \\ c \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} a \\ 0 \end{pmatrix} \quad (2.6.1)$$

Given,

$$a = 12, b + c = 18 \quad (2.6.2)$$

From $\triangle ABC$, using the Baudhayana sutra,

$$b^2 = c^2 + a^2 \quad (2.6.3)$$

$$\Rightarrow b - c = 8 \quad (\because b + c = 18) \quad (2.6.4)$$

Now we have,

$$b + c = 18 \quad (2.6.5)$$

$$b - c = 8 \quad (2.6.6)$$

which can be expressed as the matrix equation

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} 18 \\ 8 \end{pmatrix} \quad (2.6.7)$$

Applying row reduction,

$$\begin{pmatrix} 1 & 1 & 18 \\ 1 & -1 & 8 \end{pmatrix} \xrightarrow{R_2 \rightarrow R_2 - R_1} \begin{pmatrix} 1 & 1 & 18 \\ 0 & -2 & -10 \end{pmatrix} \quad (2.6.8)$$

$$\xrightarrow{R_1 \rightarrow 2R_1 + R_2} \begin{pmatrix} 2 & 0 & 26 \\ 0 & -2 & -10 \end{pmatrix} \xrightarrow{\begin{matrix} R_1 \rightarrow \frac{R_1}{2} \\ R_2 \rightarrow -\frac{R_2}{2} \end{matrix}} \begin{pmatrix} 1 & 0 & 13 \\ 0 & 1 & 5 \end{pmatrix} \quad (2.6.9)$$

Therefore,

$$\begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} 13 \\ 5 \end{pmatrix} \quad (2.6.10)$$

Thus,

$$\mathbf{A} = \begin{pmatrix} 0 \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} a \\ 0 \end{pmatrix} = \begin{pmatrix} 12 \\ 0 \end{pmatrix} \quad (2.6.11)$$

and $\triangle ABC$ is plotted in Fig. 2.6.1

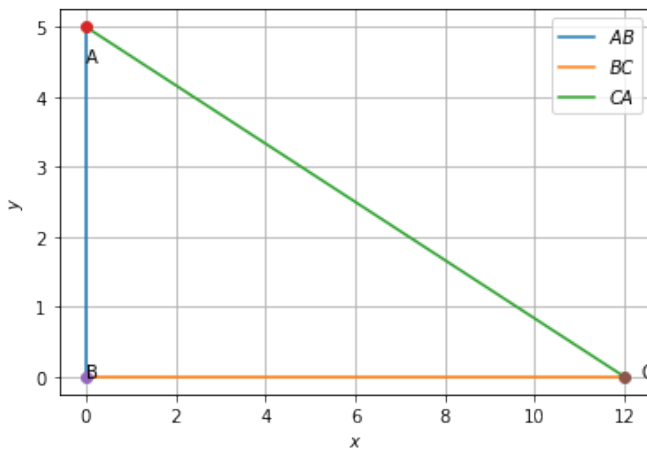


Fig. 2.6.1: Right Angle $\triangle ABC$

2.7. In $\triangle ABC$, $a = 8$, $\angle B = 45^\circ$ and $c - b = 3.5$. Sketch $\triangle ABC$.

Solution: Let

$$\mathbf{A} = c \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} a \\ 0 \end{pmatrix}, \quad (2.7.1)$$

Using the cosine formula in $\triangle ABC$,

$$b^2 = a^2 + c^2 - 2ac \cos B \quad (2.7.2)$$

$$\Rightarrow (c + b)(c - b) + 8^2 - 2 \times 8 \times \left(\frac{1}{\sqrt{2}} \right) c = 0 \quad (2.7.3)$$

$$\Rightarrow (7 - 16\sqrt{2})c + 7b = -128 \quad (2.7.4)$$

upon simplification. From the given information,

$$c - b = \frac{7}{2}, \quad (2.7.5)$$

and the above equations can be expressed as the matrix equation

$$\begin{pmatrix} 7 - 16\sqrt{2} & 7 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} c \\ b \end{pmatrix} = \begin{pmatrix} -128 \\ \frac{7}{2} \end{pmatrix} \quad (2.7.6)$$

yielding

$$\begin{pmatrix} c \\ b \end{pmatrix} = \begin{pmatrix} 11.99 \\ 8.49 \end{pmatrix} \quad (2.7.7)$$

Thus, the vertices of $\triangle ABC$ are

$$\mathbf{A} = 11.99 \begin{pmatrix} \cos 45^\circ \\ \sin 45^\circ \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 8 \\ 0 \end{pmatrix}. \quad (2.7.8)$$

which are used to plot Fig. 2.7.1.

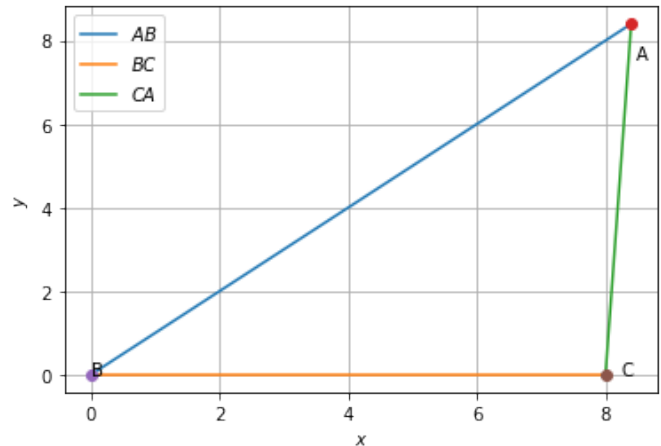


Fig. 2.7.1: $\triangle ABC$

2.8. In $\triangle ABC$, $a = 6$, $\angle B = 60^\circ$ and $b - c = 2$. Sketch $\triangle ABC$.

Let

$$\mathbf{A} = b \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} a \\ 0 \end{pmatrix}, \quad (2.8.1)$$

Using the cosine formula,

$$b^2 = a^2 + c^2 - 2ac \cos B \quad (2.8.2)$$

$$\Rightarrow (b + c)(b - c) = 6^2 - 2(6)\frac{1}{2}c \quad (\because \angle B = 60^\circ) \quad (2.8.3)$$

$$\Rightarrow (b + c)(2) = 36 - 6c \quad (\because b - c = 2) \quad (2.8.4)$$

$$\text{or, } b + 4c = 18 \quad (2.8.5)$$

From the above, we obtain the matrix equation

$$\begin{pmatrix} 1 & 4 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} 18 \\ 2 \end{pmatrix} \quad (2.8.6)$$

By applying row reduction:

$$\begin{pmatrix} 1 & 4 & 18 \\ 1 & -1 & 2 \end{pmatrix} \xrightarrow{R_2 \rightarrow R_2 - R_1} \begin{pmatrix} 1 & 4 & 18 \\ 0 & -5 & -16 \end{pmatrix} \quad (2.8.7)$$

$$\xrightarrow{R_1 \rightarrow 5R_1 + 4R_2} \begin{pmatrix} 5 & 0 & 26 \\ 0 & -5 & -16 \end{pmatrix} \quad (2.8.8)$$

$$\xrightarrow{\begin{matrix} R_1 \rightarrow \frac{R_1}{5} \\ R_2 \rightarrow -\frac{R_2}{5} \end{matrix}} \begin{pmatrix} 1 & 0 & \frac{26}{5} \\ 0 & 1 & \frac{16}{5} \end{pmatrix} \quad (2.8.9)$$

$$\therefore \begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} \frac{26}{5} \\ \frac{16}{5} \end{pmatrix} \quad (2.8.10)$$

Thus, the vertices of $\triangle ABC$ are

$$\mathbf{A} = \frac{26}{5} \begin{pmatrix} \cos 60^\circ \\ \sin 60^\circ \end{pmatrix} = \begin{pmatrix} 2.6 \\ 4.5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 6 \\ 0 \end{pmatrix} \quad (2.8.11)$$

and the plot of $\triangle ABC$ is obtained in Fig. 2.8.1

2.9. Draw $\triangle ABC$, given that $a + b + c = 11$, $\angle B = 30^\circ$ and $\angle C = 90^\circ$.

Solution: Using the sine formula,

$$b \sin C = c \sin B \quad (2.9.1)$$

$$\Rightarrow b \sin 90 = c \sin 30 \quad (2.9.2)$$

$$\text{or, } c = 2b \quad (2.9.3)$$

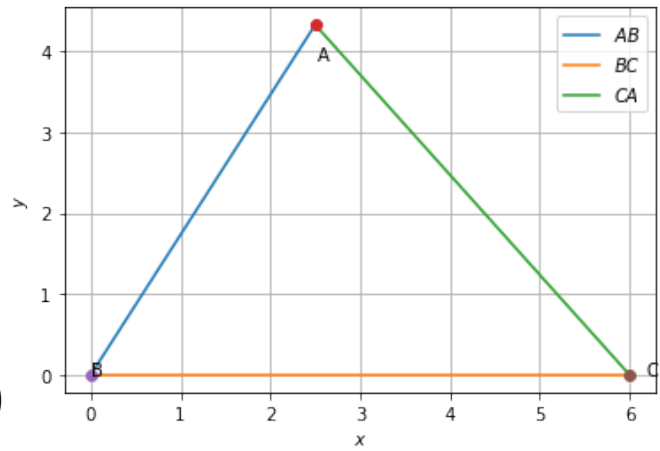


Fig. 2.8.1: $\triangle ABC$

Similarly,

$$a \sin B = b \sin A \quad (2.9.4)$$

$$\Rightarrow a = \sqrt{3}b \quad (2.9.5)$$

Formulating the above as a matrix equation

$$\begin{pmatrix} 0 & -2 & 1 \\ 1 & -\sqrt{3} & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 11 \end{pmatrix} \quad (2.9.6)$$

Solving the above,

$$a = 4.026, b = 2.32, c = 4.64 \quad (2.9.7)$$

which are used to obtain the vertices of $\triangle ABC$ using Problem 1.3.

$$\mathbf{A} = \begin{pmatrix} 0 \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 4.64 \end{pmatrix} \quad (2.9.8)$$

$$\mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (2.9.9)$$

$$\mathbf{C} = \begin{pmatrix} a \\ 0 \end{pmatrix} = \begin{pmatrix} 4.02 \\ 0 \end{pmatrix} \quad (2.9.10)$$

The desired triangle is plotted in Fig. 2.9.1.

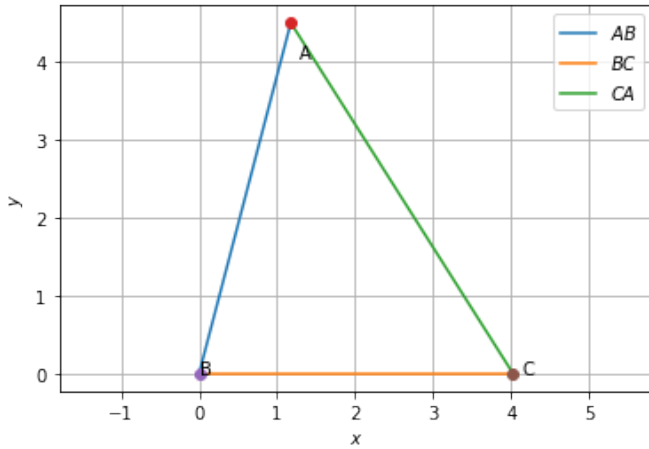
2.10. Construct $\triangle xyz$ where $xy = 4.5$, $yz = 5$ and $zx = 6$.

Solution: Let

$$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} c \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} p \\ q \end{pmatrix} \quad (2.10.1)$$

The vertex C can be expressed in polar coordinate form as

$$\mathbf{C} = b \begin{pmatrix} \cos A \\ \sin A \end{pmatrix} \quad (2.10.2)$$

Fig. 2.9.1: $\triangle ABC$

Using the cosine formula,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \quad (2.10.3)$$

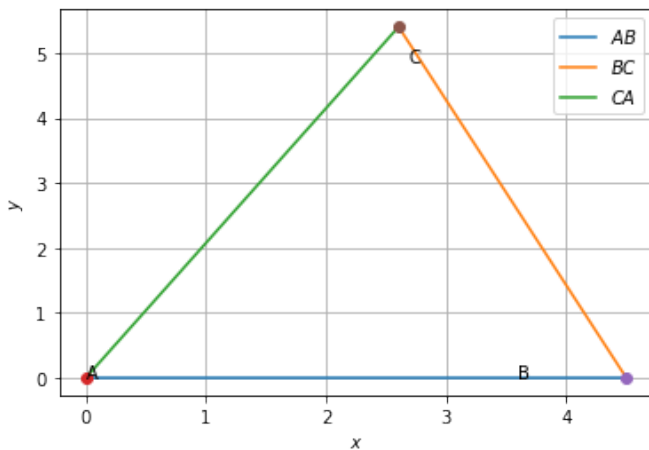
$$\Rightarrow A = 54.640^\circ \quad (2.10.4)$$

Hence,

$$\mathbf{C} = 6 \begin{pmatrix} \cos 54.640 \\ \sin 54.640 \end{pmatrix} = \mathbf{C} = \begin{pmatrix} 3.472 \\ 3.990 \end{pmatrix}, \quad (2.10.5)$$

$$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} c \\ 0 \end{pmatrix} = \begin{pmatrix} 4.5 \\ 0 \end{pmatrix} \quad (2.10.6)$$

which are plotted in Fig. 2.10.1

Fig. 2.10.1: $\triangle ABC$

2.11. Draw an equilateral triangle of side 5.5.

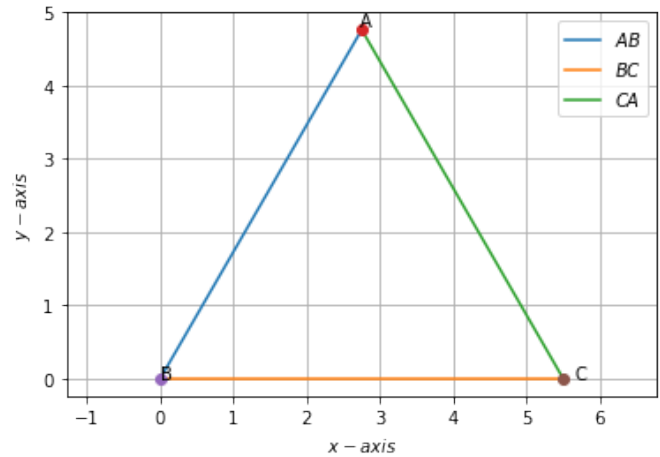
Solution:

Let,

$$\mathbf{A} = a \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} a \\ 0 \end{pmatrix} \quad (2.11.1)$$

$$= 5.5 \begin{pmatrix} \cos 60^\circ \\ \sin 60^\circ \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 5.5 \\ 0 \end{pmatrix} \quad (2.11.2)$$

after substituting $\theta = 60^\circ$ and $a = 5.5$. The triangle is then plotted in Fig. 2.11.1

Fig. 2.11.1: $\triangle ABC$

2.12. Draw $\triangle PQR$ with $PQ = 4$, $QR = 3.5$ and $PR = 4$. What type of triangle is this?

Solution: Let

$$\mathbf{P} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \mathbf{R} = PR \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \quad (2.12.1)$$

where,

$$PR \left(\frac{\sin \theta}{2} \right) = \frac{QR}{2} \quad (2.12.2)$$

$$\Rightarrow \theta = 2 \sin^{-1} \left(\frac{QR}{2PR} \right) \quad (2.12.3)$$

$$= 51.88 \quad (2.12.4)$$

Thus, the vertices of $\triangle PQR$ are

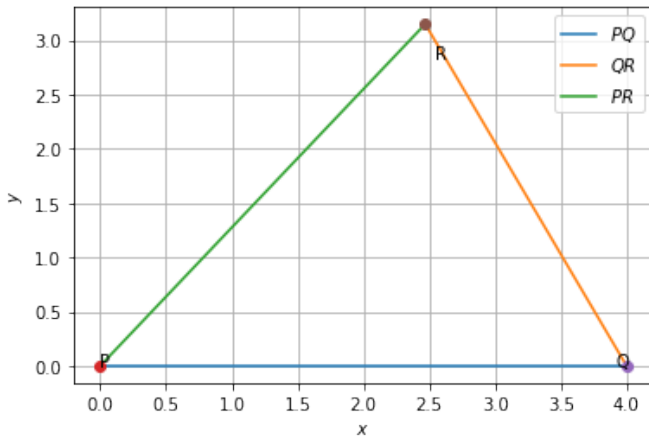
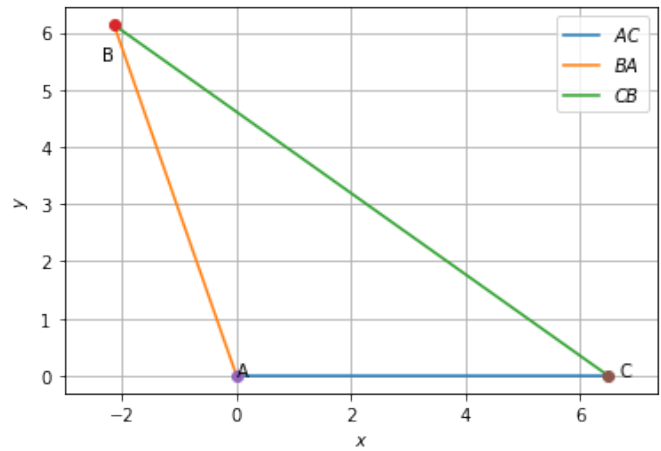
$$\mathbf{P} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \mathbf{R} = \begin{pmatrix} 2.47 \\ 3.15 \end{pmatrix} \quad (2.12.5)$$

which are used to plot $\triangle PQR$ in Fig. 2.12.1.

2.13. Construct $\triangle ABC$ such that $AB = 2.5$, $BC = 6$ and $AC = 6.5$. Find $\angle B$.

2.14. Construct $\triangle PQR$, given that $PQ = 3$, $QR = 5.5$ and $\angle PQR = 60^\circ$.

2.15. Construct $\triangle DEF$ such that $DE = 5$, $DF = 3$ and $\angle D = 90^\circ$.

Fig. 2.12.1: isosceles $\triangle PQR$ Fig. 2.16.1: Isosceles $\triangle ABC$

Solution: From the given information, the vertices of $\triangle DEF$ are

$$\mathbf{E} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{F} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \quad (2.15.1)$$

which are used to plot Fig. 2.15.1.

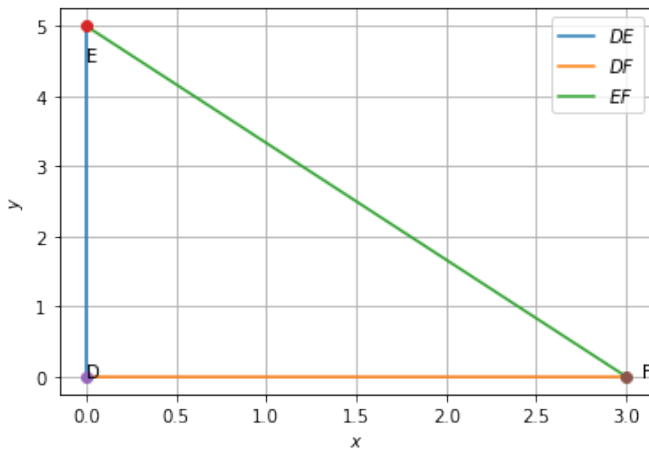


Fig. 2.15.1

- 2.16. Construct an isosceles triangle in which the lengths of the equal sides is 6.5 and the angle between them is 110° .

Solution: Let the vertices be

$$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} b \\ 0 \end{pmatrix}, \mathbf{B} = c \begin{pmatrix} \cos A \\ \sin A \end{pmatrix} \quad (2.16.1)$$

Then, the vertices of isosceles $\triangle ABC$ are

$$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 6.5 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -2.22313 \\ 6.10798 \end{pmatrix} \quad (2.16.2)$$

which are plotted in Fig. 2.16.1.

- 2.17. Construct $\triangle ABC$ with $BC = 7.5$, $AC = 5$ and $\angle C = 60^\circ$.
 2.18. Construct $\triangle XYZ$ if $XY = 6$, $\angle X = 30^\circ$ and $\angle Y = 100^\circ$.
 2.19. If $AC = 7$, $\angle A = 60^\circ$ and $\angle B = 50^\circ$, can you draw the triangle?
 2.20. Construct $\triangle ABC$ given that $\angle A = 60^\circ$, $\angle B = 30^\circ$ and $AB = 5.8$.

Solution: From the given information,

$$\angle C = 90^\circ \quad (2.20.1)$$

Hence,

$$\mathbf{A} = \begin{pmatrix} 0 \\ c \sin B \end{pmatrix} \quad (2.20.2)$$

$$= \begin{pmatrix} 0 \\ 2.9 \end{pmatrix} \quad (2.20.3)$$

$$\mathbf{B} = \begin{pmatrix} c \cos B \\ 0 \end{pmatrix} \quad (2.20.4)$$

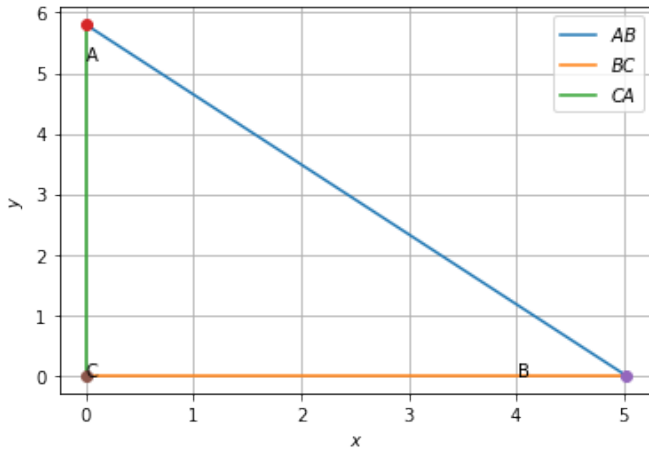
$$= \begin{pmatrix} 5.02294 \\ 0 \end{pmatrix} \quad (2.20.5)$$

$$\mathbf{C} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (2.20.6)$$

which are used to draw $\triangle ABC$ in Fig. 2.20.1.

- 2.21. Construct $\triangle PQR$ if $PQ = 5$, $\angle Q = 105^\circ$ and $\angle R = 40^\circ$.
 2.22. Can you construct $\triangle DEF$ such that $EF = 7.2$, $\angle E = 110^\circ$ and $\angle F = 180^\circ$?
 2.23. Construct $\triangle LMN$ right angled at M such that $LN = 5$ and $MN = 3$.

Solution:

Fig. 2.20.1: $\triangle ABC$

Let

$$\mathbf{L} = \begin{pmatrix} 0 \\ l \end{pmatrix}, \mathbf{M} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{N} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \quad (2.23.1)$$

From the given information,

$$\|\mathbf{N} - \mathbf{M}\|^2 = \|\mathbf{N}\|^2 = 3^2 = 9 \quad (2.23.2)$$

$$\|\mathbf{L} - \mathbf{M}\|^2 = \|\mathbf{L}\|^2 = l^2 \quad (2.23.3)$$

$$\|\mathbf{L} - \mathbf{N}\|^2 = 5^2 = 25 \quad (2.23.4)$$

which can be expressed as

$$\|\mathbf{L} - \mathbf{N}\|^2 = (\mathbf{L} - \mathbf{N})^T (\mathbf{L} - \mathbf{N}) \quad (2.23.5)$$

$$= \|\mathbf{L}\|^2 + \|\mathbf{N}\|^2 - 2\mathbf{L}^T \mathbf{N} \quad (2.23.6)$$

$$\Rightarrow l^2 + 9 = 25 \quad (2.23.7)$$

$$\text{or, } l = \pm 4 \quad (2.23.8)$$

For $l=4$, $\triangle LMN$ is plotted in the first quadrant in Fig. 2.23.1.

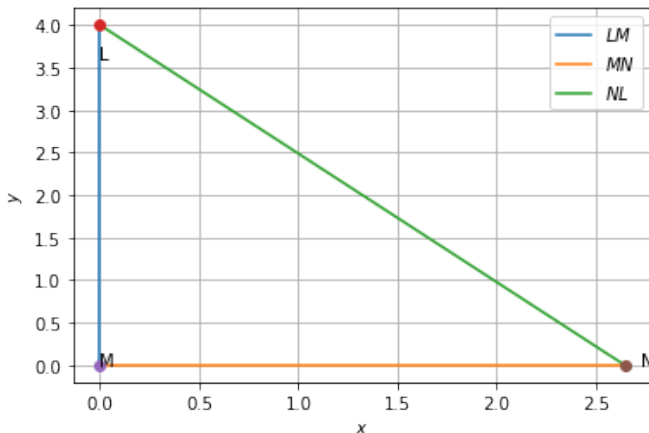


Fig. 2.23.1

2.24. Construct $\triangle PQR$ right angled at Q such that $QR = 8$ and $PR = 10$.

Solution: Let

$$\mathbf{Q} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{R} = \begin{pmatrix} 8 \\ 0 \end{pmatrix}, \mathbf{P} = \begin{pmatrix} 0 \\ p \end{pmatrix} \quad (2.24.1)$$

Then,

$$\|\mathbf{P} - \mathbf{R}\|^2 = (\mathbf{P} - \mathbf{R})^T (\mathbf{P} - \mathbf{R}) \quad (2.24.2)$$

$$= \|\mathbf{P}\|^2 + \|\mathbf{R}\|^2 \quad (2.24.3)$$

$$\therefore \mathbf{P}^T \mathbf{R} = \mathbf{R}^T \mathbf{P}, \mathbf{R}^T \mathbf{P} = 0 \quad (2.24.4)$$

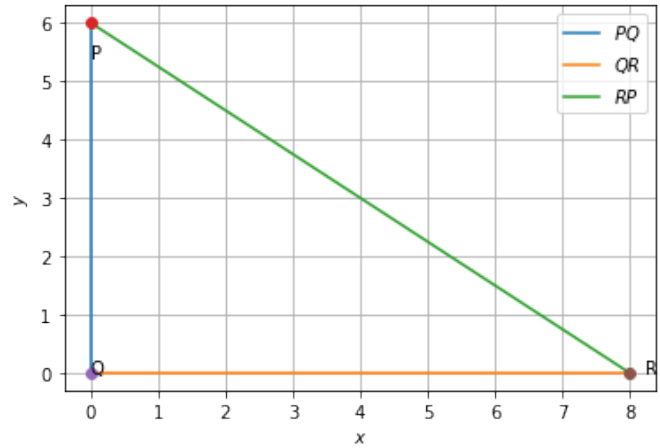
$$= p^2 + 64 = 10^2 \quad (2.24.5)$$

$$\Rightarrow p = \pm 6 \quad (2.24.6)$$

Since positive area is considered here, only $p = 6$ is taken into consideration. Thus,

$$\mathbf{P} = \begin{pmatrix} 0 \\ 6 \end{pmatrix} \quad (2.24.7)$$

and the desired triangle is plotted in Fig. 2.24.1

Fig. 2.24.1: Right Angle $\triangle PQR$

2.25. Construct right angled \triangle whose hypotenuse is 6 and one of the legs is 4.

Solution: Let us consider $\triangle PQR$ right angled at Q and assume that we are restricted to first quadrant such that

$$\mathbf{Q} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{R} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \mathbf{P} = \begin{pmatrix} 0 \\ p \end{pmatrix} \quad (2.25.1)$$

Then,

$$\|\mathbf{P} - \mathbf{R}\|^2 = 36 \quad (2.25.2)$$

$$\Rightarrow p^2 + 16 = 36 \quad (2.25.3)$$

$$\Rightarrow p = \pm 2\sqrt{5} \quad (2.25.4)$$

Since first quadrant was assumed here, only $p = +2\sqrt{5}$ is taken into consideration. So, the vertices of $\triangle PQR$ in Fig. 2.25.1 are

$$\mathbf{P} = \begin{pmatrix} 0 \\ 2\sqrt{5} \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{R} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \quad (2.25.5)$$

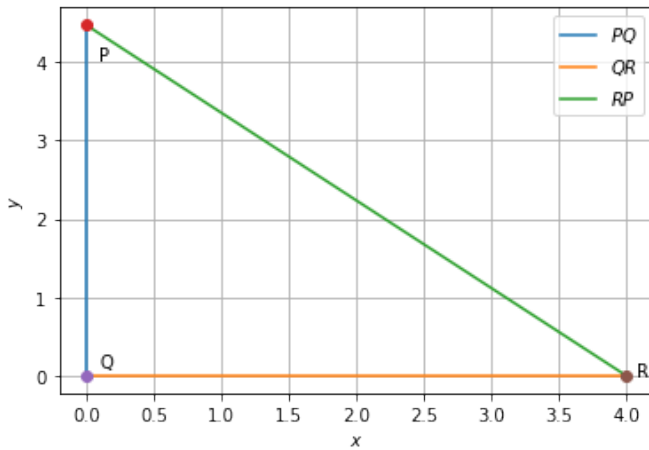


Fig. 2.25.1: Right Angled $\triangle PQR$

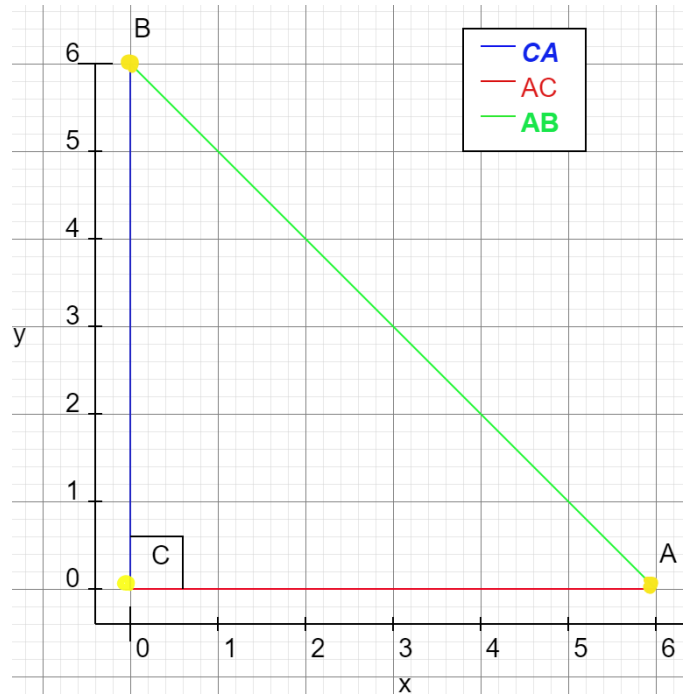


Fig. 2.26.1: Isosceles Right Angle $\triangle ABC$

S.No	Triangle	Given Measurements		
1	$\triangle ABC$	$\angle A = 85^\circ$	$\angle B = 115^\circ$	$AB = 5$
2	$\triangle PQR$	$\angle Q = 30^\circ$	$\angle R = 60^\circ$	$QR = 4.7$
3	$\triangle ABC$	$\angle A = 70^\circ$	$\angle B = 50^\circ$	$AC = 3$
4	$\triangle LMN$	$\angle L = 60^\circ$	$\angle N = 120^\circ$	$LM = 5$
5	$\triangle ABC$	$BC = 2$	$AB = 4$	$AC = 2$
6	$\triangle PQR$	$PQ = 2.5$	$QR = 4$	$PR = 3.5$
7	$\triangle XYZ$	$XY = 3$	$YZ = 4$	$XZ = 5$
8	$\triangle DEF$	$DE = 4.5$	$EF = 5.5$	$DF = 4$

TABLE 2.27.1

- 2.26. Construct an isosceles right angled $\triangle ABC$ right angled at C such $AC = 6$.

Solution:

$\therefore \triangle ABC$ is isosceles, its vertices are

$$\mathbf{C} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 6 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 6 \end{pmatrix} \quad (2.26.1)$$

which are used to plot the desired triangle in Fig. 2.26.1.

- 2.27. Construct the triangles in Table 2.27.1. **Solution:**

a)

- b) **Solution:** From the given information, $\triangle PQR$ is a right angled triangle. Let $QR = p$ and $\theta = 30^\circ$. Then the vertices of the triangle

are

$$\mathbf{P} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (2.27.1)$$

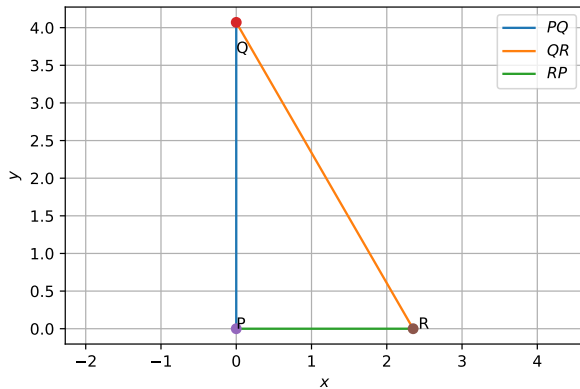
$$\mathbf{Q} = \begin{pmatrix} 0 \\ p \cos \theta \end{pmatrix} \quad (2.27.2)$$

$$= \begin{pmatrix} 0 \\ 4.07 \end{pmatrix} \quad (2.27.3)$$

$$\mathbf{R} = \begin{pmatrix} p \sin \theta \\ 0 \end{pmatrix} \quad (2.27.4)$$

$$= \begin{pmatrix} 2.35 \\ 0 \end{pmatrix} \quad (2.27.5)$$

The triangle is plotted in Fig. 2.27.1

Fig. 2.27.1: $\triangle PQR$ constructed using python

c) **Solution:** From the given information,

$$\angle C = 60^\circ \quad (2.27.6)$$

Using the sine formula,

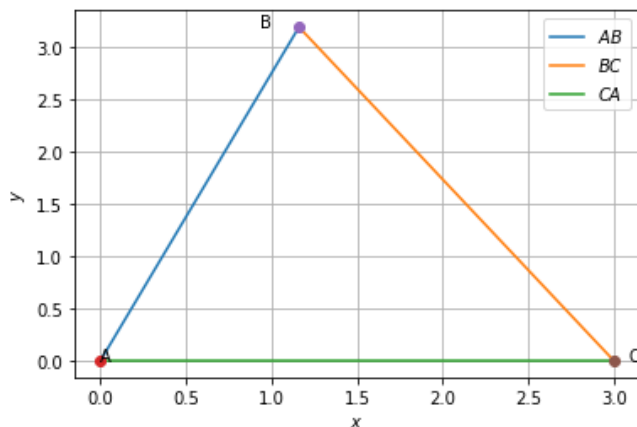
$$c = b \left(\frac{\sin C}{\sin B} \right) \quad (2.27.7)$$

$$= 3.3915 \quad (2.27.8)$$

the vertices of $\triangle ABC$ are

$$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{B} = c \begin{pmatrix} \cos 70^\circ \\ \sin 70^\circ \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \quad (2.27.9)$$

and plotted in Fig. 2.27.2.

Fig. 2.27.2: Plot of $\triangle ABC$

2.28. Construct a quadrilateral $ABCD$ such that $AB = 5$, $\angle A = 50^\circ$, $AC = 4$, $BD = 5$ and $AD = 6$.

Solution:

The rough figure of the expected quadrilateral $ABCD$ is given in Fig. 2.28.1

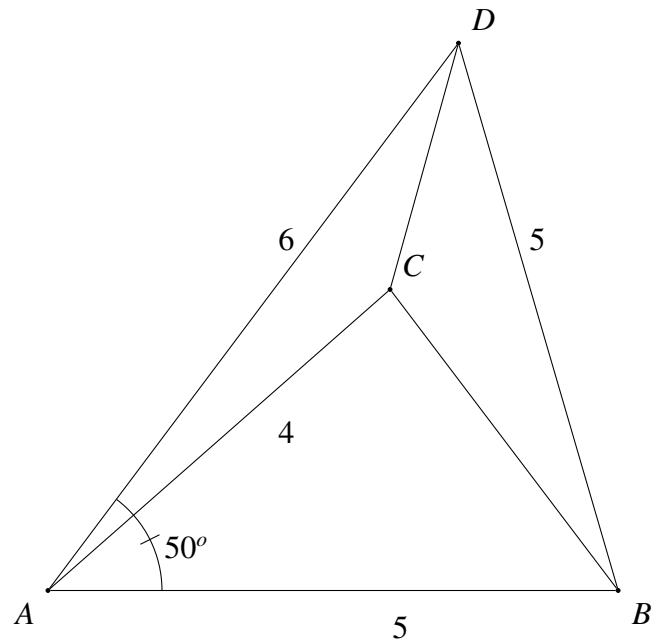


Fig. 2.28.1: Rough Figure

From the given information, in $\triangle ABD$,

$$\cos A = \frac{\|\mathbf{B} - \mathbf{A}\|^2 + \|\mathbf{D} - \mathbf{A}\|^2 - \|\mathbf{D} - \mathbf{B}\|^2}{2 \|\mathbf{B} - \mathbf{A}\| \|\mathbf{D} - \mathbf{A}\|} \quad (2.28.1)$$

$$\Rightarrow \angle A = \cos^{-1}(0.6) \approx 53.13^\circ \quad (2.28.2)$$

$$\neq 50^\circ \quad (2.28.3)$$

resulting in a contradiction. Therefore construction of quadrilateral with given measurements is not possible.

2.29. Construct $PQRS$ where $PQ = 4$, $QR = 6$, $RS = 5$, $PS = 5.5$ and $PR = 7$.

2.30. Draw $JUMP$ with $JU = 3.5$, $UM = 4$, $MP = 5$, $PJ = 4.5$ and $PU = 6.5$

2.31. Construct a quadrilateral $ABCD$ such that $BC = 4.5$, $AC = 5.5$, $CD = 5$, $BD = 7$ and $AD = 5.5$.

2.32. Can you construct a quadrilateral $PQRS$ with $PQ = 3$, $RS = 3$, $PS = 7.5$, $PR = 8$ and $SQ = 4$?

Solution: From the given information,

$$\|\mathbf{P} - \mathbf{Q}\| = 3 \quad (2.32.1)$$

$$\|\mathbf{R} - \mathbf{S}\| = 3 \quad (2.32.2)$$

$$\|\mathbf{P} - \mathbf{S}\| = 7.5 \quad (2.32.3)$$

$$\|\mathbf{P} - \mathbf{R}\| = 8 \quad (2.32.4)$$

$$\|\mathbf{S} - \mathbf{Q}\| = 4 \quad (2.32.5)$$

Let quadrilateral $PQRS$ be made up of two triangles $\triangle PSQ$ and $\triangle PSR$ on base PS .

a) In $\triangle PSR$,

$$\begin{aligned}\|P - S\| + \|R - S\| &= 7.5 + 3 = 10.5 \\ &> \|P - R\| \quad (2.32.6)\end{aligned}$$

$$\begin{aligned}\|P - R\| + \|R - S\| &= 8 + 3 = 11 > \|P - S\| \\ (2.32.7)\end{aligned}$$

$$\begin{aligned}\|P - S\| + \|P - R\| &= 7.5 + 8 = 15.5 \\ &> \|R - S\| \quad (2.32.8)\end{aligned}$$

\therefore using triangle inequality, construction of $\triangle PSR$ is possible.

b) In $\triangle PSQ$,

$$\begin{aligned}\|P - S\| + \|S - Q\| &= 7.5 + 4 = 11.5 \\ &> \|P - Q\| \quad (2.32.9)\end{aligned}$$

$$\begin{aligned}\|P - S\| + \|P - Q\| &= 7.5 + 3 = 10.5 \\ &> \|S - Q\| \quad (2.32.10)\end{aligned}$$

$$\begin{aligned}\|P - Q\| + \|S - Q\| &= 3 + 4 = 7 < \|P - S\| \\ (2.32.11)\end{aligned}$$

which violates triangle inequality. \therefore construction of $\triangle PSQ$ is not possible.

Fig. 2.32.1 highlights this.

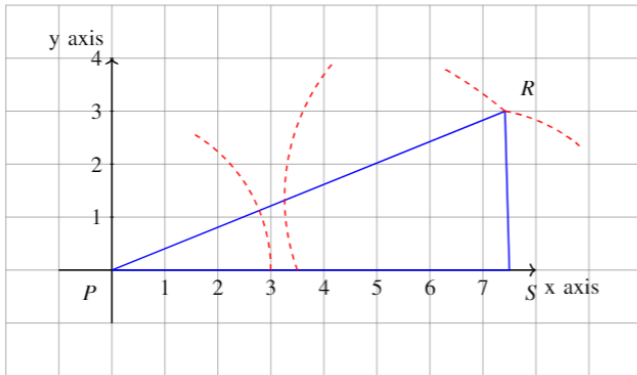


Fig. 2.32.1: Construction of quadrilateral $PQRS$

- 2.33. Construct $LIFT$ such that $LI = 4, IF = 3, TL = 2.5, LF = 4.5, IT = 4$.
- 2.34. Draw $GOLD$ such that $OL = 7.5, GL = 6, GD = 6, LD = 5, OD = 10$.

Solution: In $\triangle LDO$

$$\|O - L\| + \|O - D\| = 17.5 > \|L - D\| \quad (2.34.1)$$

$$\|O - D\| + \|L - D\| = 15 > \|O - L\| \quad (2.34.2)$$

$$\|O - L\| + \|L - D\| = 12.5 > \|O - D\| \quad (2.34.3)$$

and triangle inequality is satisfied. Similarly, in $\triangle LDG$

$$\|L - D\| + \|G - L\| = 11 > \|G - D\| \quad (2.34.4)$$

$$\|G - L\| + \|G - D\| = 12 > \|L - D\| \quad (2.34.5)$$

$$\|L - D\| + \|G - D\| = 11 > \|G - L\| \quad (2.34.6)$$

and triangle inequality is satisfied. \therefore the given sides form a quadrilateral which can be constructed by using the approach in Problem 1.3 to obtain the vertices of $\triangle LDO$ and $\triangle LDG$ as

$$L = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, D = \begin{pmatrix} 5 \\ 0 \end{pmatrix}, O = \begin{pmatrix} -1.875 \\ 7.26 \end{pmatrix}, G = \begin{pmatrix} 2.5 \\ 5.5 \end{pmatrix} \quad (2.34.7)$$

and plotting the quadrilateral $GOLD$ in Fig. 2.34.1

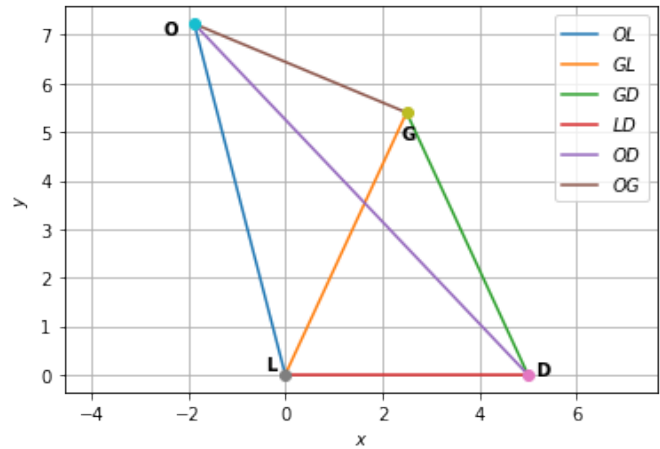


Fig. 2.34.1: Quadrilateral $GOLD$

- 2.35. DRAW rhombus $BEND$ such that $BN = 5.6, DE = 6.5$.
- 2.36. construct a quadrilateral $MIST$ where $MI = 3.5, IS = 6.5, \angle M = 75^\circ, \angle I = 105^\circ$ and $\angle S = 120^\circ$.
- 2.37. Can you construct the above quadrilateral $MIST$ if $\angle M = 100^\circ$ instead of 75° .
- 2.38. Can you construct the quadrilateral $PLAN$ if $PL = 6, LA = 9.5, \angle P = 75^\circ, \angle L = 150^\circ$ and $\angle A = 140^\circ$?

- 2.39. Construct *MORE* where $MO = 6, OR = 4.5, \angle M = 60^\circ, \angle O = 105^\circ, \angle R = 105^\circ$.
- 2.40. Construct *PLAN* where $PL = 4, LA = 6.5, \angle P = 90^\circ, \angle A = 110^\circ$ and $\angle N = 85^\circ$.
- 2.41. Draw rectangle *OKAY* with $OK = 7$ and $KA = 5$.
- 2.42. Construct *ABCD*, where $AB = 4, BC = 5, Cd = 6.5, \angle B = 105^\circ$ and $\angle C = 80^\circ$.

Solution:

Let

$$\angle B = 105^\circ = \theta \quad (2.42.1)$$

$$\angle C = 80^\circ = \alpha \quad (2.42.2)$$

$$\|\mathbf{A} - \mathbf{B}\| = 4 = p \quad (2.42.3)$$

$$\|\mathbf{C} - \mathbf{B}\| = 5 = q \quad (2.42.4)$$

$$\|\mathbf{D} - \mathbf{C}\| = 6.5 = r \quad (2.42.5)$$

and

$$\mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \quad (2.42.6)$$

Lemma 2.1.

$$\mathbf{A} = p\mathbf{b} \quad \left(\because \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) \quad (2.42.7)$$

$$\mathbf{D} = \mathbf{C} + r\mathbf{c} \quad (2.42.8)$$

where

$$\mathbf{b} = \begin{pmatrix} \cos B \\ \sin B \end{pmatrix}, \mathbf{c} = \begin{pmatrix} \cos C \\ \sin C \end{pmatrix} \quad (2.42.9)$$

Thus,

$$\mathbf{A} = 4 \begin{pmatrix} \cos 105^\circ \\ \sin 105^\circ \end{pmatrix} \quad (2.42.10)$$

$$= \begin{pmatrix} -1.03 \\ 3.86 \end{pmatrix} \quad (2.42.11)$$

and

$$\mathbf{D} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} + 6.5 \begin{pmatrix} \cos 80^\circ \\ \sin 80^\circ \end{pmatrix} \quad (2.42.12)$$

$$= \begin{pmatrix} 6.12 \\ 6.39 \end{pmatrix} \quad (2.42.13)$$

which are then used to plot Fig. 2.42.1

- 2.43. Construct *DEAR* with $DE = 4, EA = 5, AR = 4.5, \angle E = 60^\circ$ and $\angle A = 90^\circ$.

Solution: The given information can be ex-

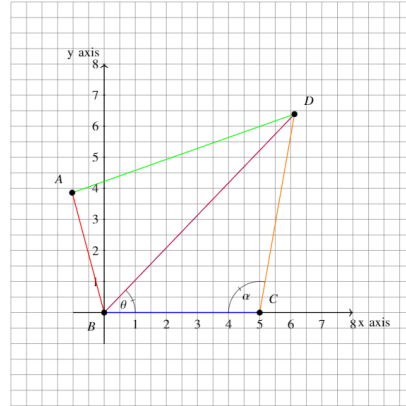


Fig. 2.42.1: Quadrilateral ABCD

pressed as

$$\angle E = 60^\circ = \theta \quad (2.43.1)$$

$$\angle A = 90^\circ = \alpha \quad (2.43.2)$$

$$\|\mathbf{D} - \mathbf{E}\| = 4 = a \quad (2.43.3)$$

$$\|\mathbf{E} - \mathbf{A}\| = 5 = b \quad (2.43.4)$$

$$\|\mathbf{A} - \mathbf{R}\| = 4.5 = c \quad (2.43.5)$$

Let,

$$\mathbf{E} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \quad (2.43.6)$$

Lemma 2.2.

$$\mathbf{D} = a\mathbf{e} \quad \left(\because \mathbf{E} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) \quad (2.43.7)$$

$$\mathbf{R} = \mathbf{A} + c\mathbf{a} \quad (2.43.8)$$

where

$$\mathbf{e} = \begin{pmatrix} \cos E \\ \sin E \end{pmatrix}, \mathbf{a} = \begin{pmatrix} \cos A \\ \sin A \end{pmatrix} \quad (2.43.9)$$

Thus, from (2.43.1) and (2.43.3) in (2.43.7),

$$\mathbf{D} = 4 \begin{pmatrix} \cos 60^\circ \\ \sin 60^\circ \end{pmatrix} \quad (2.43.10)$$

$$= \begin{pmatrix} 2 \\ 3.46 \end{pmatrix} \quad (2.43.11)$$

and from (2.43.2) and (2.43.5) in (2.43.8),

$$\mathbf{R} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} + 4.5 \begin{pmatrix} \cos 90^\circ \\ \sin 90^\circ \end{pmatrix} \quad (2.43.12)$$

$$= \begin{pmatrix} 5 \\ 1 \end{pmatrix} \quad (2.43.13)$$

Thus

$$\mathbf{D} = \begin{pmatrix} 2 \\ 3.46 \end{pmatrix}, \mathbf{E} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}, \mathbf{R} = \begin{pmatrix} 5 \\ 1 \end{pmatrix} \quad (2.43.14)$$

and the quadrilateral DEAR is plotted in Fig. 2.43.1.

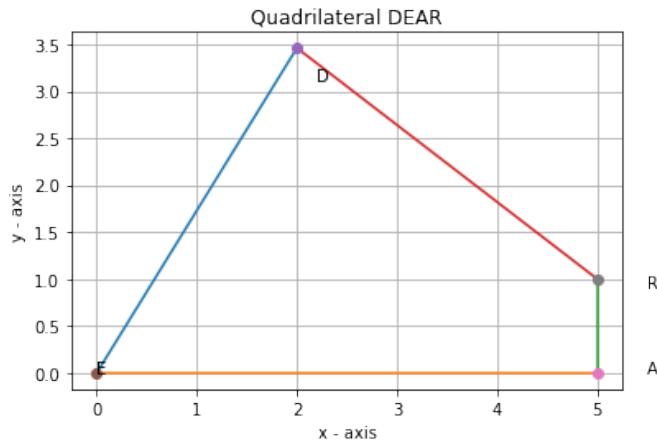


Fig. 2.43.1: Quadrilateral DEAR

2.44. Construct $TRUE$ with $TR = 3.5$, $RU = 3$, $UE = 4$, $\angle R = 75^\circ$ and $\angle U = 120^\circ$.

Solution: From the given information,

$$\angle R = 75^\circ = \theta \quad (2.44.1)$$

$$\angle U = 120^\circ = \alpha \quad (2.44.2)$$

$$\|\mathbf{T} - \mathbf{R}\| = 3.5 = a \quad (2.44.3)$$

$$\|\mathbf{U} - \mathbf{R}\| = 3 = b \quad (2.44.4)$$

$$\|\mathbf{E} - \mathbf{U}\| = 4 = c \quad (2.44.5)$$

Let,

$$\mathbf{R} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{U} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \quad (2.44.6)$$

Lemma 2.3.

$$\mathbf{T} = C\mathbf{u} \quad \left(\because \mathbf{R} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) \quad (2.44.7)$$

$$\mathbf{E} = \mathbf{U} + a\mathbf{r} \quad (2.44.8)$$

where

$$\mathbf{r} = \begin{pmatrix} \cos R \\ \sin R \end{pmatrix}, \mathbf{u} = \begin{pmatrix} \cos U \\ \sin U \end{pmatrix} \quad (2.44.9)$$

Thus,

$$\mathbf{T} = 4 \begin{pmatrix} \cos 120^\circ \\ \sin 120^\circ \end{pmatrix} \quad (2.44.10)$$

$$= \begin{pmatrix} -2 \\ 3.46 \end{pmatrix} \quad (2.44.11)$$

and

$$\mathbf{E} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} + 3.5 \begin{pmatrix} \cos 75^\circ \\ \sin 75^\circ \end{pmatrix} \quad (2.44.12)$$

$$= \begin{pmatrix} 3.39 \\ 3.38 \end{pmatrix} \quad (2.44.13)$$

The vertices of given quadrilateral TRUE can be written as,

$$\mathbf{T} = \begin{pmatrix} -2 \\ 3.46 \end{pmatrix}, \mathbf{R} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{U} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \mathbf{E} = \begin{pmatrix} 3.39 \\ 3.38 \end{pmatrix} \quad (2.44.14)$$

which are plotted in Fig. 2.44.1.

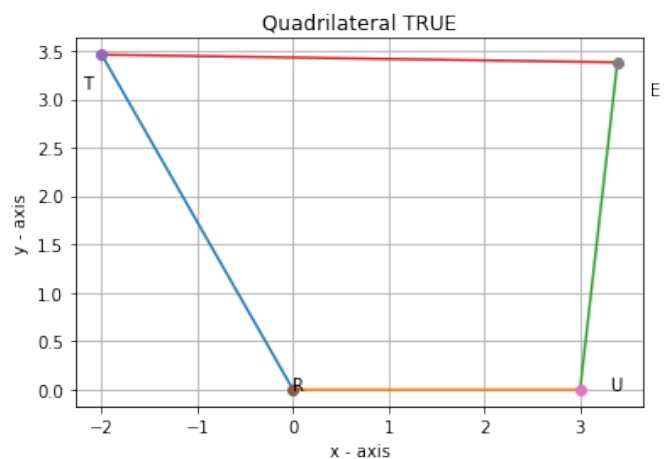


Fig. 2.44.1: Quadrilateral TRUE

2.45. Can you construct a rhombus $ABCD$ with $AC = 6$ and $BD = 7$?

Solution: We obtain the vertices of the rhom-

bus as follows

$$\mathbf{A} = \begin{pmatrix} -3 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ -3.5 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 0 \\ 3.5 \end{pmatrix} \quad (2.45.1)$$

which are plotted in Fig. 2.45.1.

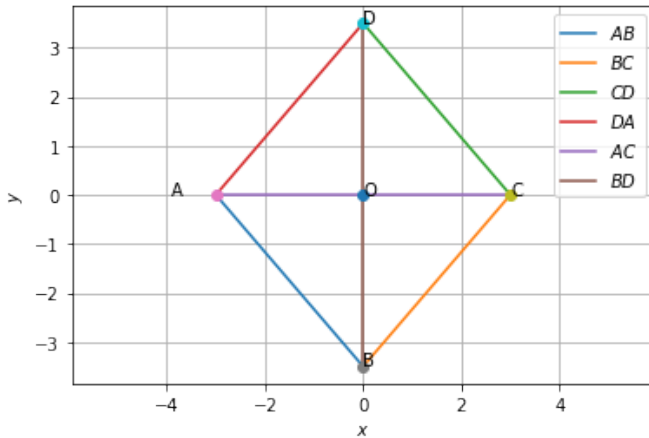


Fig. 2.45.1: Rhombus ABCD

2.46. Draw a square $READ$ with $RE = 5.1$.

Solution: The vertices are given by

$$\mathbf{R} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{E} = \begin{pmatrix} 5.1 \\ 0 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 5.1 \\ 5.1 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 0 \\ 5.1 \end{pmatrix} \quad (2.46.1)$$

The desired square is plotted in Fig. 2.46.1

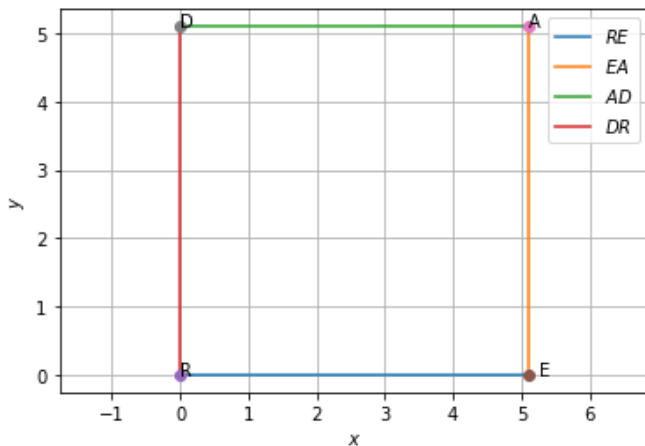


Fig. 2.46.1: Square $READ$

2.47. Draw a rhombus who diagonals are 5.2 and 6.4.

Solution: We obtain the vertices of the rhom-

bus as

$$\mathbf{A} = \begin{pmatrix} -2.6 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ -3.2 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 2.6 \\ 0 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 0 \\ 3.2 \end{pmatrix} \quad (2.47.1)$$

which are plotted in Fig. 2.47.1

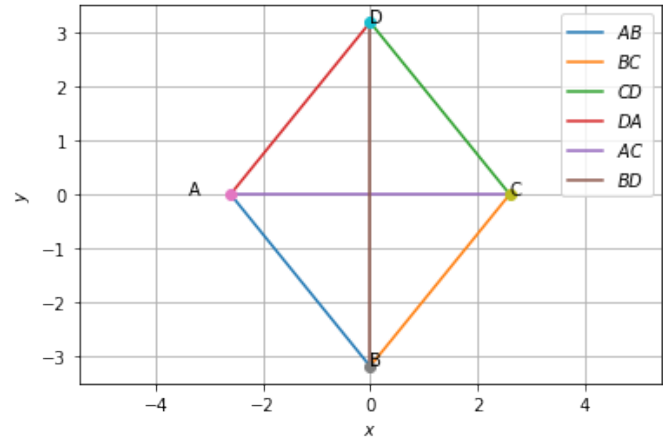


Fig. 2.47.1: Rhombus ABCD

2.48. Draw a rectangle with adjacent sides 5 and 4.

Solution: The vertices of rectangle $ABCD$ are

$$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} a \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 0 \\ c \end{pmatrix}, \mathbf{D} = \begin{pmatrix} a \\ c \end{pmatrix} \quad (2.48.1)$$

$$\Rightarrow \mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 5 \\ 4 \end{pmatrix} \quad (2.48.2)$$

where $a = 5$ and $c = 4$. The rectangle $ABCD$ is plotted in Fig. 2.48.1

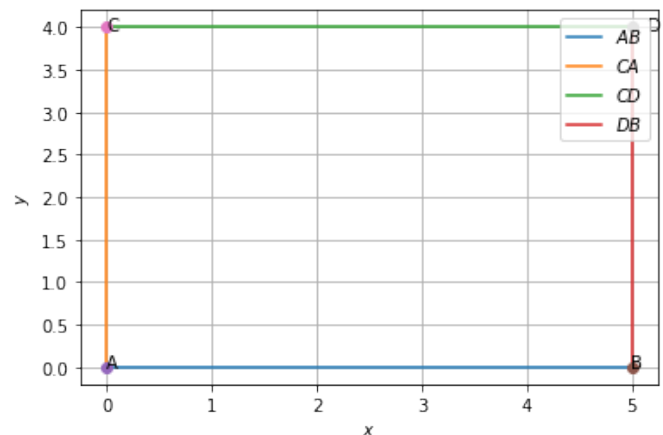


Fig. 2.48.1: Rectangle $ABCD$

2.49. Draw a parallelogram $OKAY$ with $OK = 5.5$ and $KA = 4.2$.

Solution: There are infinite number of parallelograms that can be drawn. For a unique parallelogram, one angle needs to be specified.

2.50. Construct a kite $EASY$ if $AY = 8$, $EY = 4$ and $SY = 6$.

2.51. Draw a circle of diameter 6.1

2.52. With the same centre O , draw two circles of radii 4 and 2.5

Solution:

All input values required to plot Fig. 2.52.1 are given in Table 2.52.1 as shown below

	Symbols	Circle1	Circle2
Centre	O	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
Radius	r_1, r_2	2.5	4
Polar coordinate	C_1, C_2	$2.5 \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$	$4 \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$
Angle	θ	$0-2\pi$	$0-2\pi$

TABLE 2.52.1: Input values

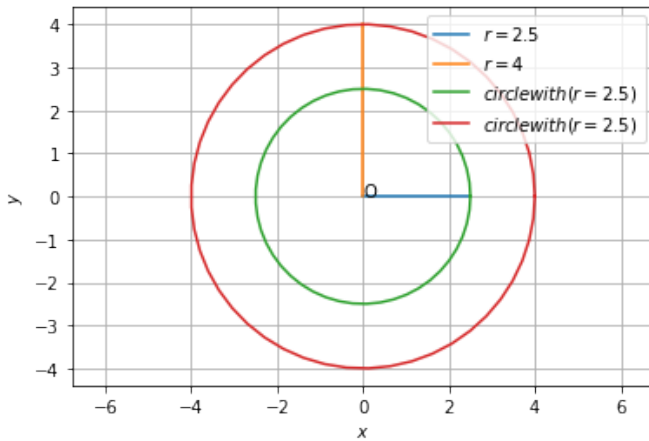


Fig. 2.52.1: Concentric circles with centre as origin and radii 2.5 and 4 respectively

2.53. Draw a circle with centre B and radius 6. If C be a point 10 units away from its centre, construct the pair of tangents AC and CD to the circle.

2.54. Draw a circle of radius 3 and any two of its diameters. Draw the ends of these diameters. What figure do you get?

2.55. Let A and B be the centres of two circles of equal radii 3 such that each one of them passes through the centre of the other. Let them intersect at C and D . Is $AB \perp CD$?

Solution: The centers and radii of the two circles without any loss of generality are given in Table 2.55.1

	Circle 1	Circle 2
Centre	$A = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$B = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$
Radius	$r_1 = r_2 = 3$	

TABLE 2.55.1: Input values

Let

$$\mathbf{u} = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \theta \in [0, 2\pi]. \quad (2.55.1)$$

Then on Circle 1 and Circle 2 are given by

$$\mathbf{x} = \mathbf{A} + r\mathbf{u} \quad (2.55.2)$$

$$\mathbf{x} = \mathbf{B} + r\mathbf{u} \quad (2.55.3)$$

Fig. 2.55.1 is plotted using the above equations. Fig. 2.55.1

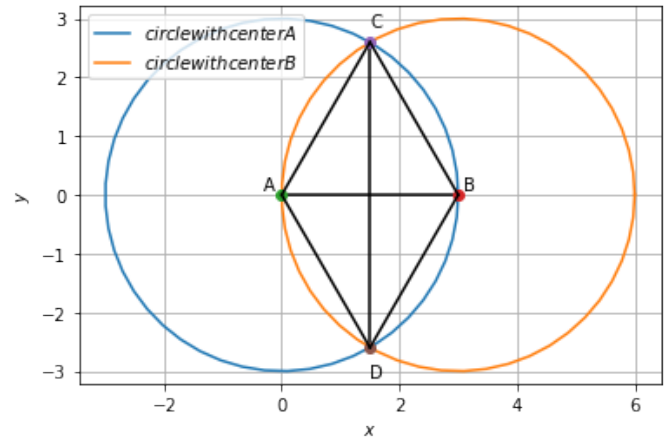


Fig. 2.55.1: Circles with their points of intersection

The general equation of Circle 1 is given by

$$\|\mathbf{x} - \mathbf{A}\|^2 = r^2 \quad (2.55.4)$$

$$\mathbf{x}^T \mathbf{x} - 2\mathbf{A}^T \mathbf{x} + \|\mathbf{A}\|^2 - r^2 = 0 \quad (2.55.5)$$

Similarly, for Circle 2,

$$\mathbf{x}^T \mathbf{x} - 2\mathbf{B}^T \mathbf{x} + \|\mathbf{B}\|^2 - r^2 = 0 \quad (2.55.6)$$

Subtracting (2.55.6) from (2.55.5),

$$2\mathbf{B}^T \mathbf{x} = \|\mathbf{B}\|^2 \quad (2.55.7)$$

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = \frac{3}{2} \quad (2.55.8)$$

which can be expressed as

$$\mathbf{x} = \frac{1}{2} \begin{pmatrix} 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (2.55.9)$$

$$= \mathbf{q} + \lambda \mathbf{m} \text{ where } \quad (2.55.10)$$

$$\mathbf{q} = \begin{pmatrix} 1.5 \\ 0 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (2.55.11)$$

Substituting (2.55.10) in (2.55.5)

$$\|\mathbf{x}\|^2 = r^2 \quad (\because \mathbf{A} = 0) \quad (2.55.12)$$

$$\|\mathbf{q} + \lambda \mathbf{m}\|^2 = r^2 \quad (2.55.13)$$

$$(\mathbf{q} + \lambda \mathbf{m})^T (\mathbf{q} + \lambda \mathbf{m}) = r^2 \quad (2.55.14)$$

$$\Rightarrow \mathbf{q}^T (\mathbf{q} + \lambda \mathbf{m}) + \lambda \mathbf{m}^T (\mathbf{q} + \lambda \mathbf{m}) = r^2 \quad (2.55.15)$$

$$\Rightarrow \|\mathbf{q}\|^2 + \lambda \mathbf{q}^T \mathbf{m} + \lambda \mathbf{m}^T \mathbf{q} + \lambda^2 \|\mathbf{m}\|^2 = r^2 \quad (2.55.16)$$

$$\Rightarrow \|\mathbf{q}\|^2 + 2\lambda \mathbf{q}^T \mathbf{m} + \lambda^2 \|\mathbf{m}\|^2 = r^2 \quad (2.55.17)$$

$$\Rightarrow \lambda = \pm \sqrt{\frac{9 - \|\mathbf{q}\|^2}{\|\mathbf{m}\|^2}} \quad \because \mathbf{q}^T \mathbf{m} = 0 \quad (2.55.18)$$

Substituting the value of λ in (2.55.10),

$$\mathbf{C} = \mathbf{q} + \lambda \mathbf{m} \quad (2.55.19)$$

$$\mathbf{D} = \mathbf{q} - \lambda \mathbf{m} \quad (2.55.20)$$

$$\Rightarrow (\mathbf{A} - \mathbf{B})^T (\mathbf{C} - \mathbf{D}) = 2 \begin{pmatrix} -3 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \sqrt{6.75} \end{pmatrix} \quad (2.55.21)$$

$$= 0 \quad (2.55.22)$$

$$\Rightarrow AB \perp CD \quad (2.55.23)$$

2.56. Construct a tangent to a circle of radius 4 units from a point on the concentric circle of radius 6 units.

Solution: The given information is summarised in Table 2.56.1. See Fig. 2.56.1. Let \mathbf{P} be a point on Circle 2 with radius 6. Then

$$\mathbf{P} = \begin{pmatrix} 6 \\ 0 \end{pmatrix} \quad (2.56.1)$$

Let PQ and PR be tangents from point \mathbf{P} on circle with radius 6 to the points \mathbf{Q} and \mathbf{R} on

	Symbols	Circle1	Circle2
Centre	\mathbf{O}	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
Radius	r_1, r_2	4	6

TABLE 2.56.1

circle with radius 4 . Now,

$$(\mathbf{O} - \mathbf{Q})^T (\mathbf{Q} - \mathbf{P}) = 0 \quad (\because OQ \perp QP) \quad (2.56.2)$$

$$\Rightarrow \mathbf{P}^T \mathbf{Q} = 16 \quad (\because \|\mathbf{Q}\|^2 = 16) \quad (2.56.3)$$

$$\text{or, } \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{Q} = \frac{8}{3} \quad (2.56.4)$$

$$\Rightarrow \mathbf{Q} = \begin{pmatrix} \frac{8}{3} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (2.56.5)$$

$$= \mathbf{q} + \lambda \mathbf{m} \quad (2.56.6)$$

$$\text{where } \mathbf{q} = \begin{pmatrix} \frac{8}{3} \\ 0 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (2.56.7)$$

We know,

$$\|\mathbf{q} + \lambda \mathbf{m}\|^2 = r_1^2 \quad (2.56.8)$$

$$(\mathbf{q} + \lambda \mathbf{m})^T (\mathbf{q} + \lambda \mathbf{m}) = r_1^2 \quad (2.56.9)$$

$$\lambda^2 = \frac{r_1^2 - \|\mathbf{q}\|^2}{\|\mathbf{m}\|^2} \quad (2.56.10)$$

$$\lambda = \pm 2.98 \quad (2.56.11)$$

Substituting the above in (2.56.5),

$$\mathbf{Q} = \begin{pmatrix} \frac{8}{3} \\ 2.98 \end{pmatrix}, \mathbf{R} = \begin{pmatrix} \frac{8}{3} \\ -2.98 \end{pmatrix} \quad (2.56.12)$$

The circels as well as the tangents are plotted in Fig. 2.56.1

2.57. Draw a circle of radius 3 units. Take two points \mathbf{P} and \mathbf{Q} on one of its extended diameter each at a distance of 7 units from its centre. Draw tangents to the circle from these two points \mathbf{P} and \mathbf{Q} .

Solution: Take the diameter to be on the x -axis.

2.58. Draw a pair of tangents to a circle of radius 5 units which are inclined to each other at an angle of 60° .

Solution: The tangent is perpendicular to the radius.

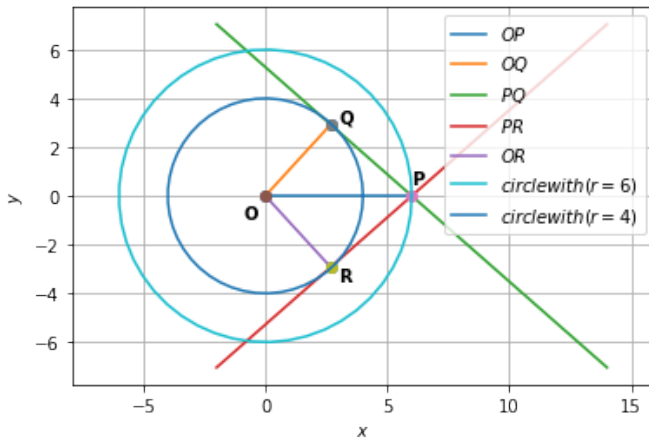


Fig. 2.56.1: Tangent lines to circle of radius 4 units.

- 2.59. Draw a line segment AB of length 8 units. Taking A as centre, draw a circle of radius 4 units and taking B as centre, draw another circle of radius 3 units. Construct tangents to each circle from the centre of the other circle.

Solution: Let

$$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 8 \\ 0 \end{pmatrix}. \quad (2.59.1)$$

- 2.60. Let ABC be a right triangle in which $a = 8$, $c = 6$ and $\angle B = 90^\circ$. BD is the perpendicular from B on AC (altitude). The circle through B, C, D (circumcircle of $\triangle BCD$) is drawn. Construct the tangents from A to this circle.
- 2.61. Draw a circle with centre C and radius 3.4. Draw any chord. Construct the perpendicular bisector of the chord and examine if it passes through C