

# Challenge Problem 3

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Download all python codes from

<https://github.com/ka-raja-babu/Matrix-Theory/tree/main/ChallengeProblem3/Codes>

and latex-tikz codes from

<https://github.com/ka-raja-babu/Matrix-Theory/tree/main/ChallengeProblem3>

## 1 CHALLENGE QUESTION 3

For the quadratic equation to not have any real roots, the y coordinate should always be either positive or negative. Express this in terms of the matrix/vector parameters of the parabola.

## 2 SOLUTION

The general form of a quadratic equation is

$$y = ax^2 + bx + c \quad (2.0.1)$$

$$\implies ax^2 + bx + c - y = 0 \quad (2.0.2)$$

which can be written in vector form as

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (2.0.3)$$

where

$$\mathbf{V} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \quad (2.0.4)$$

$$\mathbf{u} = \begin{pmatrix} \frac{b}{2} \\ -\frac{1}{2} \end{pmatrix} \quad (2.0.5)$$

$$f = c \quad (2.0.6)$$

Using eigenvalue decomposition,

$$\mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} \quad (2.0.7)$$

$$\mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (2.0.8)$$

Also,

$$\eta = \mathbf{u}^T \mathbf{p}_1 \quad (2.0.9)$$

$$= \frac{-1}{2} \quad (2.0.10)$$

Now, for y coordinate to be always positive, two conditions need to be satisfied:

- 1) y-coordinate of vertex  $\mathbf{c}$  of parabola needs to be always positive.
- 2) Coefficient  $a$  of  $x^2$  needs to be always positive.

$\therefore$  For condition 1:

$$\begin{pmatrix} \mathbf{u}^T + \eta \mathbf{p}_1^T \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \eta \mathbf{p}_1 - \mathbf{u} \end{pmatrix} \quad (2.0.11)$$

$$\mathbf{p}_1^T \mathbf{c} > 0 \quad (2.0.12)$$

For condition 2:

$$\mathbf{p}_2^T \mathbf{V} \mathbf{p}_2 > 0 \quad (2.0.13)$$

And, for y coordinate to be always negative, two conditions need to be satisfied:

- 1) y-coordinate of vertex  $\mathbf{c}$  of parabola needs to be always negative.
- 2) Coefficient  $a$  of  $x^2$  needs to be always negative.

$\therefore$  For condition 1:

$$\begin{pmatrix} \mathbf{u}^T + \eta \mathbf{p}_1^T \\ -\mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \eta \mathbf{p}_1 - \mathbf{u} \end{pmatrix} \quad (2.0.14)$$

$$\mathbf{p}_1^T \mathbf{c} < 0 \quad (2.0.15)$$

For condition 2:

$$\mathbf{p}_2^T \mathbf{V} \mathbf{p}_2 < 0 \quad (2.0.16)$$

Hence, condition for non-real roots in terms of matrix/vector parameters of the parabola is

$$(\mathbf{p}_1^T \mathbf{c})(\mathbf{p}_2^T \mathbf{V} \mathbf{p}_2) > 0 \quad (2.0.17)$$

## 3 EXAMPLES

1)

$$y = 21x^2 - 28x + 10 \quad (3.0.1)$$

Here,

$$\mathbf{V} = \begin{pmatrix} 21 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = -\begin{pmatrix} 14 \\ \frac{1}{2} \end{pmatrix}, f = 10 \quad (3.0.2)$$

Using eigenvalue decomposition,

$$\mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & 21 \end{pmatrix}, \mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (3.0.3)$$

$\therefore$  Vertex  $\mathbf{c}$  is given by

$$\begin{pmatrix} -14 & -1 \\ 21 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -10 \\ 14 \\ 0 \end{pmatrix} \quad (3.0.4)$$

$$\Rightarrow \begin{pmatrix} -14 & -1 \\ 21 & 0 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -10 \\ 14 \end{pmatrix} \quad (3.0.5)$$

$$\Rightarrow \mathbf{c} = \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \end{pmatrix} \quad (3.0.6)$$

Now,

$$\mathbf{p}_1^T \mathbf{c} = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \end{pmatrix} \quad (3.0.7)$$

$$= \frac{2}{3} \quad (3.0.8)$$

and,

$$\mathbf{p}_2^T \mathbf{V} \mathbf{p}_2 = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 21 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.0.9)$$

$$= 21 \quad (3.0.10)$$

$\therefore$

$$(\mathbf{p}_1^T \mathbf{c})(\mathbf{p}_2^T \mathbf{V} \mathbf{p}_2) = \left(\frac{2}{3}\right)(21) = \frac{42}{3} > 0 \quad (3.0.11)$$

Hence, the given equation does not have any real roots.

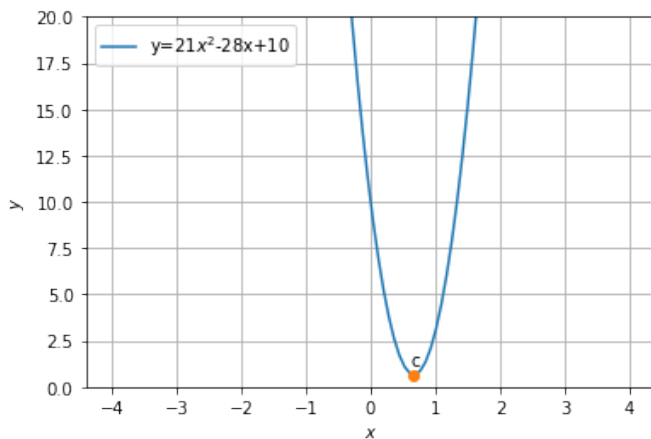


Fig. 3.1:  $y = 21x^2 - 28x + 10$

2)

$$y = 6x^2 - x - 2 \quad (3.0.12)$$

Here,

$$\mathbf{V} = \begin{pmatrix} 6 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = -\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, f = -2 \quad (3.0.13)$$

Using eigenvalue decomposition,

$$\mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & 6 \end{pmatrix}, \mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (3.0.14)$$

$\therefore$  Vertex  $\mathbf{c}$  is given by

$$\begin{pmatrix} \frac{-1}{2} & -1 \\ 6 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{c} = \begin{pmatrix} 2 \\ \frac{1}{2} \\ 0 \end{pmatrix} \quad (3.0.15)$$

$$\Rightarrow \begin{pmatrix} \frac{-1}{2} & -1 \\ 6 & 0 \end{pmatrix} \mathbf{c} = \begin{pmatrix} 2 \\ \frac{1}{2} \end{pmatrix} \quad (3.0.16)$$

$$\Rightarrow \mathbf{c} = \begin{pmatrix} \frac{1}{12} \\ \frac{49}{24} \end{pmatrix} \quad (3.0.17)$$

Now,

$$\mathbf{p}_1^T \mathbf{c} = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{12} \\ \frac{49}{24} \end{pmatrix} \quad (3.0.18)$$

$$= \frac{-49}{24} \quad (3.0.19)$$

and,

$$\mathbf{p}_2^T \mathbf{V} \mathbf{p}_2 = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 6 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.0.20)$$

$$= 6 \quad (3.0.21)$$

$\therefore$

$$(\mathbf{p}_1^T \mathbf{c})(\mathbf{p}_2^T \mathbf{V} \mathbf{p}_2) = \left(\frac{-49}{24}\right)(6) = \frac{-49}{4} < 0 \quad (3.0.22)$$

Hence, the given equation has real roots.

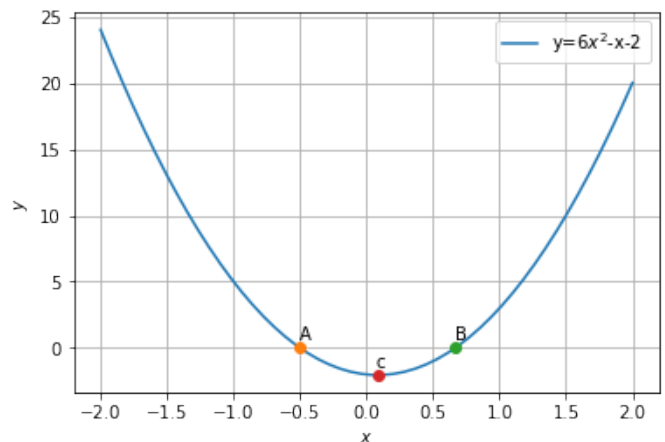


Fig. 3.2:  $y = 6x^2 - x - 2$

3)

$$y = -x^2 - 4x - 5 \quad (3.0.23)$$

Here,

$$\mathbf{V} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = -\begin{pmatrix} 2 \\ \frac{1}{2} \end{pmatrix}, f = -5 \quad (3.0.24)$$

Using eigenvalue decomposition,

$$\mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (3.0.25)$$

 $\therefore$  Vertex  $\mathbf{c}$  is given by

$$\begin{pmatrix} -2 & -1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{c} = \begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix} \quad (3.0.26)$$

$$\Rightarrow \begin{pmatrix} -2 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{c} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \quad (3.0.27)$$

$$\Rightarrow \mathbf{c} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad (3.0.28)$$

Now,

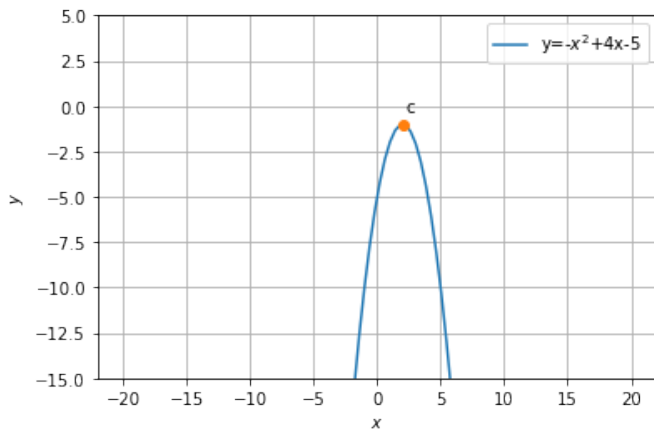
$$\mathbf{p}_1^T \mathbf{c} = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad (3.0.29)$$

$$= -1 \quad (3.0.30)$$

and,

$$\mathbf{p}_2^T \mathbf{V} \mathbf{p}_2 = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.0.31)$$

$$= -1 \quad (3.0.32)$$

Fig. 3.3:  $y = -x^2 + 4x - 5$  $\therefore$ 

$$(\mathbf{p}_1^T \mathbf{c})(\mathbf{p}_2^T \mathbf{V} \mathbf{p}_2) = (-1)(-1) = 1 > 0 \quad (3.0.33)$$

Hence, the given equation does not have real roots.

4)

$$y = -x^2 - 4x + 9 \quad (3.0.34)$$

Here,

$$\mathbf{V} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = -\begin{pmatrix} 2 \\ \frac{1}{2} \end{pmatrix}, f = 9 \quad (3.0.35)$$

Using eigenvalue decomposition,

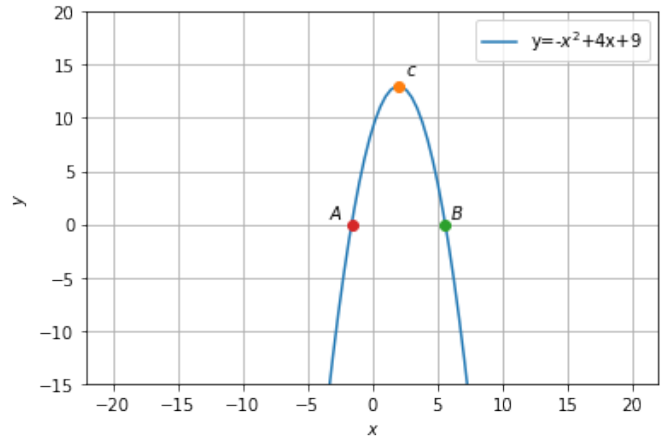
$$\mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (3.0.36)$$

 $\therefore$  Vertex  $\mathbf{c}$  is given by

$$\begin{pmatrix} -2 & -1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -9 \\ 2 \\ 0 \end{pmatrix} \quad (3.0.37)$$

$$\Rightarrow \begin{pmatrix} -2 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -9 \\ 2 \end{pmatrix} \quad (3.0.38)$$

$$\Rightarrow \mathbf{c} = \begin{pmatrix} 2 \\ 13 \end{pmatrix} \quad (3.0.39)$$

Fig. 3.4:  $y = -x^2 + 4x + 9$ 

Now,

$$\mathbf{p}_1^T \mathbf{c} = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 13 \end{pmatrix} \quad (3.0.40)$$

$$= 13 \quad (3.0.41)$$

and,

$$\mathbf{p}_2^T \mathbf{V} \mathbf{p}_2 = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.0.42)$$

$$= -1 \quad (3.0.43)$$

$\therefore$

$$(\mathbf{p}_1^T \mathbf{c})(\mathbf{p}_2^T \mathbf{V} \mathbf{p}_2) = (13)(-1) = -13 < 0$$

(3.0.44)

Hence, the given equation has real roots.