Linear Algebra and Matrices

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Abstract—**This book provides a simple introduction to linear algebra and matrix analysis. The content and exercises are based on NCERT textbooks from Class 6-12.**

1 POINTS AND VECTORS

1.1 Vector Algebra

1.1.1. Find the distance between the points

$$\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}36\\15\end{pmatrix} \tag{1.1.1.1}$$

Solution: The desired distance is

$$\|\mathbf{A}-\mathbf{B}\| = \left\|\begin{pmatrix}0\\0\end{pmatrix}-\begin{pmatrix}36\\15\end{pmatrix}\right\| = \sqrt{36^2+15^2} = 39 \tag{1.1.1.2}$$

1.1.2. Find the distance between the following pairs of points

a)

$$\begin{pmatrix}2\\3\end{pmatrix}, \begin{pmatrix}4\\1\end{pmatrix} \tag{1.1.2.1}$$

b)

$$\begin{pmatrix}-5\\7\end{pmatrix}, \begin{pmatrix}-1\\3\end{pmatrix} \tag{1.1.2.2}$$

c)

$$\begin{pmatrix}a\\b\end{pmatrix}, \begin{pmatrix}-1\\b\end{pmatrix} \tag{1.1.2.3}$$

Solution: The distance between two vectors is given by

$$\|\mathbf{A}-\mathbf{B}\| \tag{1.1.2.4}$$

From (1.1.2.4),

a) $\left\|\begin{pmatrix}2\\3\end{pmatrix}-\begin{pmatrix}4\\1\end{pmatrix}\right\| = \sqrt{(2-4)^2+(3-1)^2 = 2\sqrt{2}}$

b) $\left\|\begin{pmatrix}-5\\7\end{pmatrix}-\begin{pmatrix}-1\\3\end{pmatrix}\right\| = \sqrt{(-5+1)^2+(7-3)^2 = 4\sqrt{2}}$

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c) $\left\|\begin{pmatrix}a\\b\end{pmatrix}-\begin{pmatrix}-1\\b\end{pmatrix}\right\| = a+1$

1.1.3. Name the type of quadrilateral formed, if any, by the following points, and give reasons for your answer.

a)

$$\mathbf{P}=\begin{pmatrix}-1\\-2\end{pmatrix},\mathbf{Q}=\begin{pmatrix}1\\0\end{pmatrix},\mathbf{R}=\begin{pmatrix}-1\\2\end{pmatrix},\mathbf{S}=\begin{pmatrix}-3\\0\end{pmatrix} \tag{1.1.3.1}$$

b)

$$\mathbf{P}=\begin{pmatrix}-3\\5\end{pmatrix},\mathbf{Q}=\begin{pmatrix}3\\1\end{pmatrix},\mathbf{R}=\begin{pmatrix}0\\3\end{pmatrix},\mathbf{S}=\begin{pmatrix}-1\\-4\end{pmatrix} \tag{1.1.3.2}$$

c)

$$\mathbf{P}=\begin{pmatrix}4\\5\end{pmatrix},\mathbf{Q}=\begin{pmatrix}7\\6\end{pmatrix}, \tag{1.1.3.3}$$

$$\mathbf{R}=\begin{pmatrix}4\\3\end{pmatrix},\mathbf{S}=\begin{pmatrix}1\\2\end{pmatrix} \tag{1.1.3.4}$$

Solution:

a) In Fig. 1.1.3

$$\mathbf{P}-\mathbf{S}=\mathbf{Q}-\mathbf{R}=\begin{pmatrix}2\\-2\end{pmatrix} \tag{1.1.3.5}$$

$$\mathbf{R}-\mathbf{S}=\mathbf{Q}-\mathbf{P}=\begin{pmatrix}2\\2\end{pmatrix} \tag{1.1.3.6}$$

Hence $PQRS$ is a ||gm $\because$ opposite sides are parallel. Also,

$$\|\mathbf{P}-\mathbf{S}\| = \|\mathbf{Q}-\mathbf{R}\| \tag{1.1.3.7}$$

$$= \|\mathbf{R}-\mathbf{S}\| = \|\mathbf{Q}-\mathbf{P}\| = 2\sqrt{2} \tag{1.1.3.8}$$

$\because$ all sides are equal, the ||gm is a rhombus. The angle between PS and RS is given by

$$\cos\theta = \frac{(\mathbf{S}-\mathbf{P})^\top(\mathbf{S}-\mathbf{R})}{\|\mathbf{S}-\mathbf{P}\|^\top\|\mathbf{S}-\mathbf{R}\|} \tag{1.1.3.9}$$

$\because$

$$(\mathbf{S}-\mathbf{P})^\top(\mathbf{S}-\mathbf{R}) = \begin{pmatrix}2 & -2\end{pmatrix}\begin{pmatrix}2\\2\end{pmatrix} = 0 \tag{1.1.3.10}$$

upon substituting from (1.1.3.5) and (1.1.3.6),

$$\cos\theta = 0 \implies PS \perp RS \tag{1.1.3.11}$$

Thus, the rhombus is actually a square.

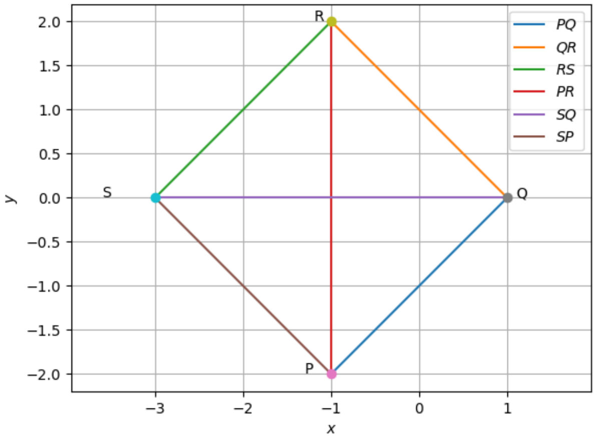

Fig. 1.1.3: quadrilateral1

codes/vectors/quad1.py

b) In Fig. 1.1.3

$$\mathbf{Q}-\mathbf{P}=\begin{pmatrix}6\\-4\end{pmatrix} \tag{1.1.3.12}$$

$$\mathbf{R}-\mathbf{P}=\begin{pmatrix}3\\-2\end{pmatrix} \tag{1.1.3.13}$$

$$\mathbf{Q}-\mathbf{R}=\begin{pmatrix}3\\-2\end{pmatrix} \tag{1.1.3.14}$$

$$(\mathbf{Q}-\mathbf{P}) = (\mathbf{R}-\mathbf{P}) + (\mathbf{Q}-\mathbf{R}) = \begin{pmatrix}6\\-4\end{pmatrix} \tag{1.1.3.15}$$

Hence, $\mathbf{P},\mathbf{Q}$ and $\mathbf{R}$ lie on a straight line, so $PQRS$ is not a quadrilateral.

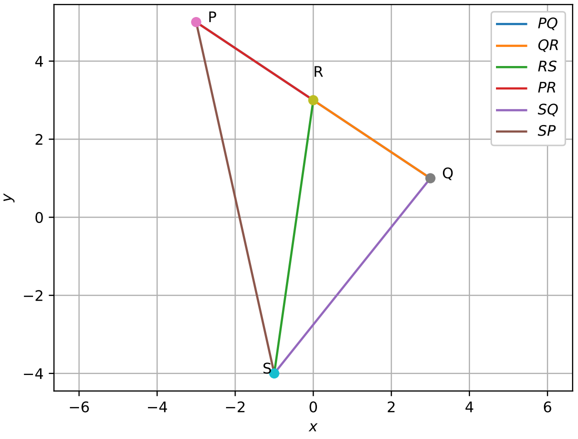

Fig. 1.1.3: quadrilateral2

c) See Fig. 1.1.3.

$$\because (\mathbf{Q}-\mathbf{P}) = (\mathbf{R}-\mathbf{S}) = \begin{pmatrix}3\\1\end{pmatrix} \quad (1.1.3.16)$$

$$(\mathbf{P}-\mathbf{S}) = (\mathbf{Q}-\mathbf{R}) = \begin{pmatrix}3\\3\end{pmatrix}, \quad (1.1.3.17)$$

PQRS is a parallelogram. Also,

$$\|\mathbf{Q}-\mathbf{P}\| \neq \|\mathbf{P}-\mathbf{S}\| \quad (1.1.3.18)$$

Hence, *PQRS* is neither a rhombus nor a square.

$$\because (\mathbf{Q}-\mathbf{P})^T(\mathbf{Q}-\mathbf{R}) = \begin{pmatrix}3 & 1\end{pmatrix}\begin{pmatrix}3\\3\end{pmatrix} \neq 0, \quad (1.1.3.19)$$

PQRS is not a rectangle.

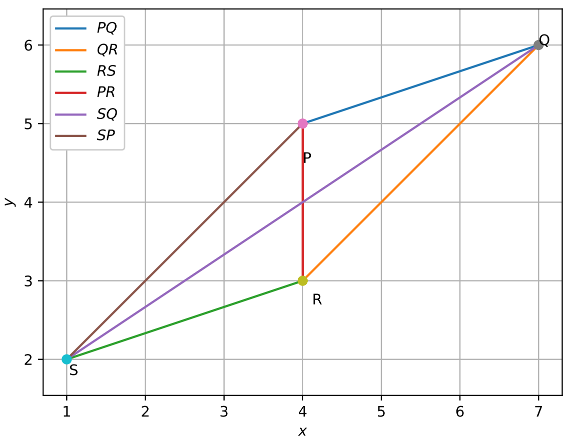

Fig. 1.1.3

1.2 Triangle

1.2.1. Draw Fig. 1.2.1 for $a = 4, c = 3$.
Solution: The vertices of $\triangle ABC$ are

$$\mathbf{A} = \begin{pmatrix}0\\c\end{pmatrix} = \begin{pmatrix}0\\3\end{pmatrix}, \mathbf{B} = \begin{pmatrix}0\\0\end{pmatrix}, \mathbf{C} = \begin{pmatrix}a\\0\end{pmatrix} = \begin{pmatrix}4\\0\end{pmatrix} \quad (1.2.1.1)$$

The python code for Fig. 1.2.1 is

codes/triangle/tri_right_angle.py

and the equivalent latex-tikz code is

figs/constr/triangle/tri_right_angle.tex

The above latex code can be compiled as a standalone document as

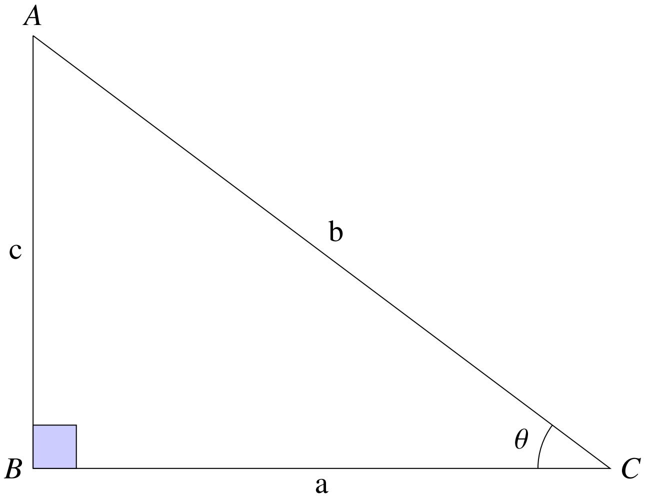

Fig. 1.2.1: Right Angled Triangle

figs/constr/triangle/tri_right_angle_alone.tex

1.2.2. Draw Fig. 1.2.2 for $a = 4, c = 3$.
Solution: The vertex **A** can be expressed in *polar coordinate form* as

$$\mathbf{A} = b\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} \quad (1.2.2.1)$$

where

$$b = \sqrt{a^2+c^2} = 5, \tan\theta = \frac{3}{4} \quad (1.2.2.2)$$

The python code for Fig. 1.2.2 is

codes/triangle/tri_polar.py

and the equivalent latex-tikz code is

figs/constr/triangle/tri_polar.tex

1.2.3. Draw Fig. 1.2.3 with $a = 6$, $b = 5$ and $c = 4$.
Solution: Let the vertices of $\triangle ABC$ and **D** be

$$\mathbf{A} = \begin{pmatrix}p\\q\end{pmatrix}, \mathbf{B} = \begin{pmatrix}0\\0\end{pmatrix}, \mathbf{C} = \begin{pmatrix}a\\0\end{pmatrix}, \mathbf{D} = \begin{pmatrix}p\\0\end{pmatrix} \quad (1.2.3.1)$$

Then

$$AB = \|\mathbf{A}-\mathbf{B}\|^2 = \|\mathbf{A}\|^2 = c^2 \quad \because \mathbf{B} = \mathbf{0} \quad (1.2.3.2)$$

$$BC = \|\mathbf{C}-\mathbf{B}\|^2 = \|\mathbf{C}\|^2 = a^2 \quad (1.2.3.3)$$

$$AC = \|\mathbf{A}-\mathbf{C}\|^2 = b^2 \quad (1.2.3.4)$$

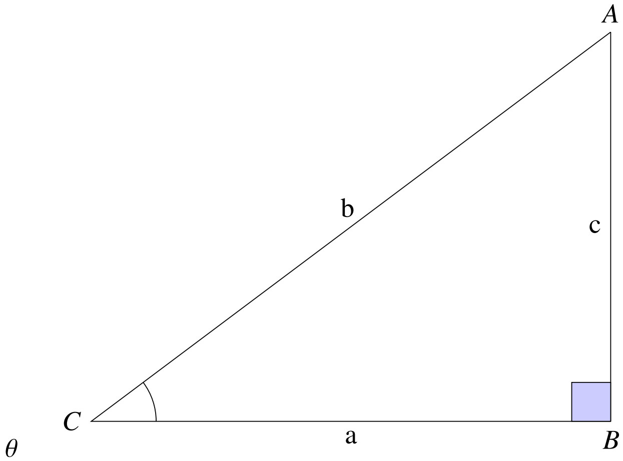

Fig. 1.2.2: Right Angled Triangle

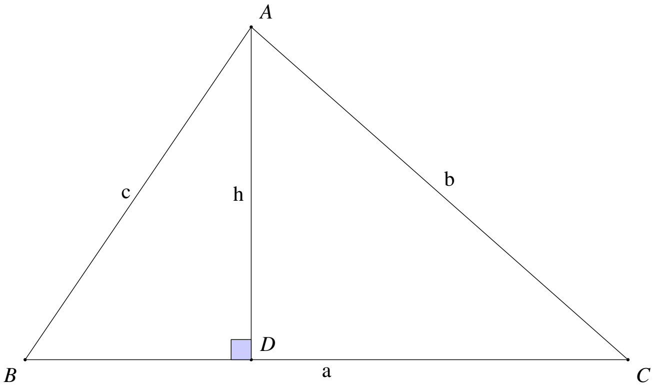

Fig. 1.2.3

From (1.2.3.4),

$$\begin{aligned} b^2 &= \left\|\mathbf{A}-\mathbf{C}\right\|^2 = \left\|\mathbf{A}-\mathbf{C}\right\|^T\left\|\mathbf{A}-\mathbf{C}\right\| && (1.2.3.5)\\ &= \mathbf{A}^T\mathbf{A}+\mathbf{C}^T\mathbf{C}-\mathbf{A}^T\mathbf{C}-\mathbf{C}^T\mathbf{A} && (1.2.3.6)\\ &= \left\|\mathbf{A}\right\|^2+\left\|\mathbf{C}\right\|^2-2\mathbf{A}^T\mathbf{C} \quad \left(\because \mathbf{A}^T\mathbf{C}=\mathbf{C}^T\mathbf{A}\right) && (1.2.3.7)\\ &= a^2+c^2-2ap && (1.2.3.8) \end{aligned}$$

yielding

$$p = \frac{a^2+c^2-b^2}{2a} \tag{1.2.3.9}$$

From (1.2.3.2),

$$\begin{aligned} \left\|\mathbf{A}\right\|^2 = c^2 &= p^2+q^2 && (1.2.3.10)\\ \implies q &= \pm\sqrt{c^2-p^2} && (1.2.3.11) \end{aligned}$$

The python code for Fig. 1.2.3 is

```
codes/triangle/tri_sss.py
```

and the equivalent latex-tikz code is

```
figs/constr/triangle/tri_sss.tex
```

1.3 Quadrilateral

1.3.1. Construct parallelogram $ABCD$ in Fig. 1.3.1 given that $BC = 5, AB = 6, \angle C = 85°$.

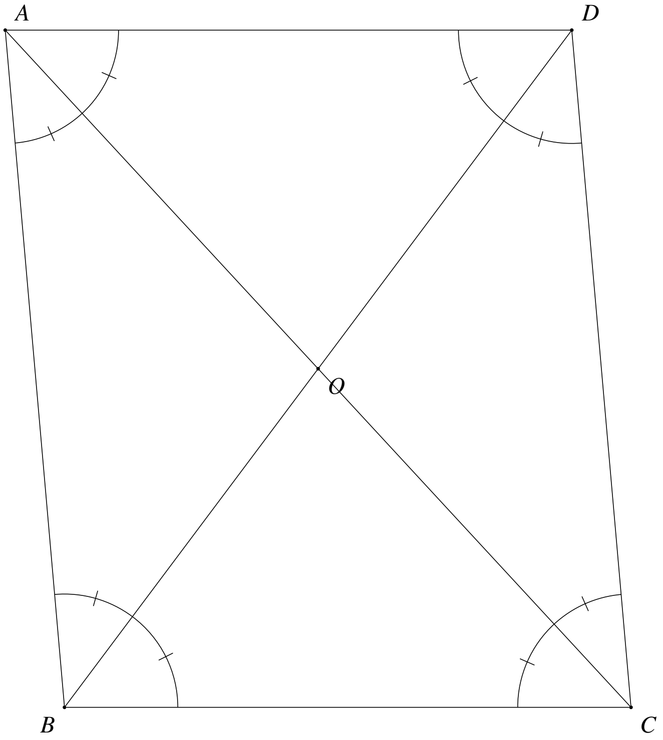

Fig. 1.3.1: Parallelogram Properties

Solution: BD is found using the cosine formula and $\triangle BDC$ is drawn using the approach in Construction 1.2.3 with

$$\mathbf{B} = \begin{pmatrix}0\\0\end{pmatrix}, \mathbf{C} = \begin{pmatrix}5\\0\end{pmatrix}, \tag{1.3.1.1}$$

Since the diagonals bisect each other,

$$\begin{aligned} \mathbf{O} &= \frac{\mathbf{B}+\mathbf{D}}{2} && (1.3.1.2)\\ \mathbf{A} &= 2\mathbf{O}-\mathbf{C}. && (1.3.1.3) \end{aligned}$$

AB and AD are then joined to complete the ||gm. The python code for Fig. 1.3.1 is

```
codes/quad/pgm_sas.py
```

and The equivalent latex-tikz code is

```
figs/constr/quad/pgm_sas.tex
```

1.3.2. Draw the ||gm $ABCD$ in Fig. 1.3.2 with $BC = 6, CD = 4.5$ and $BD = 7.5$. Show that it is a rectangle.

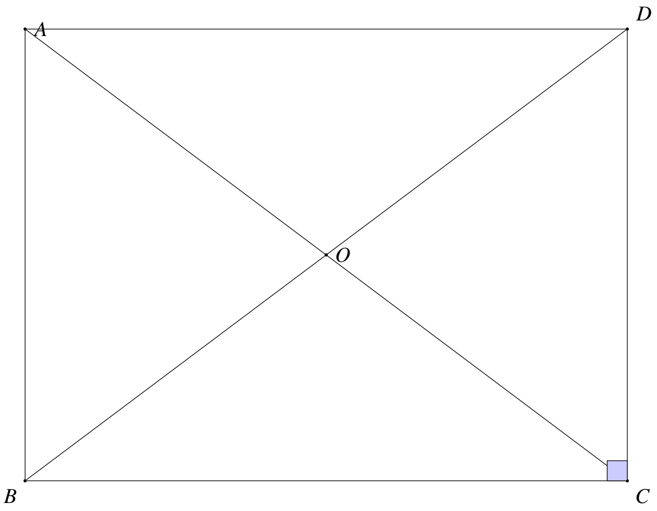

Fig. 1.3.2: Rectangle

Solution: It is easy to verify that

$$BD^2 = BC^2 + C^2 \tag{1.3.2.1}$$

Hence, using Baudhayana theorem,

$$\angle BCD = 90^\circ \tag{1.3.2.2}$$

and $ABCD$ is a rectangle.

$$\mathbf{A} = \begin{pmatrix} 0 \\ 4.5 \end{pmatrix} \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \mathbf{C} = \begin{pmatrix} 6 \\ 0 \end{pmatrix} \mathbf{D} = \begin{pmatrix} 6 \\ 4 \end{pmatrix} \tag{1.3.2.3}$$

The python code for Fig. 1.3.2 is

```
codes/quad/pgm_sss.py
```

and the equivalent latex-tikz code is

```
figs/constr/quad/pgm_sss.tex
```

1.3.3. Draw the rhombus $BEST$ with $BE = 4.5$ and $ET = 6$.
Solution: The coordinates of the various points in Fig. 1.3.3 are obtained as

$$\mathbf{O} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ -4.5 \end{pmatrix} \tag{1.3.3.1}$$

$$\mathbf{E} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \mathbf{S} = \begin{pmatrix} 4.5 \\ 0 \end{pmatrix}, \mathbf{T} = \begin{pmatrix} 0 \\ -3 \end{pmatrix} \tag{1.3.3.2}$$

1.3.4. A square is a rectangle whose sides are equal. Draw a square of side 4.5.

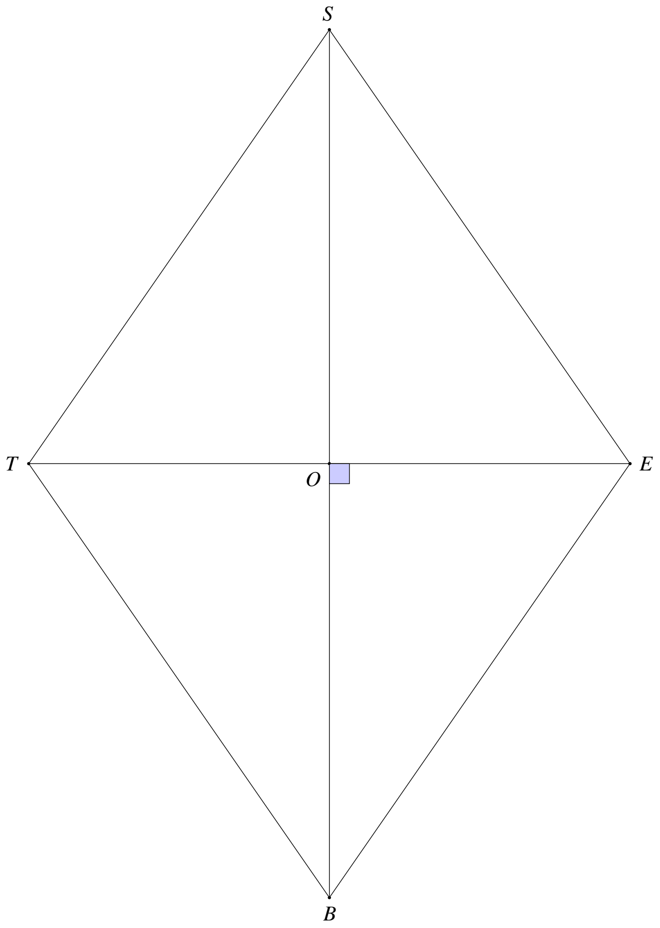

Fig. 1.3.3: Rhombus

Solution: The coordinates of the various points in Fig. 1.3.4 are obtained as

$$\mathbf{A} = \begin{pmatrix} 0 \\ 4.5 \end{pmatrix} \tag{1.3.4.1}$$

$$\mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 4.5 \\ 0 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 4.5 \\ 4.5 \end{pmatrix} \mathbf{O} = \frac{\mathbf{B} + \mathbf{C}}{2} \tag{1.3.4.2}$$

1.4 Circle

1.4.1. Draw a circle of radius 3 units.Take two points P and Q on one its extended diameter each at a distance of 7 units from its centre. Draw tangents to the circle from these two points P and Q.
Solution: The given parameters are listed in Table 1.4.1

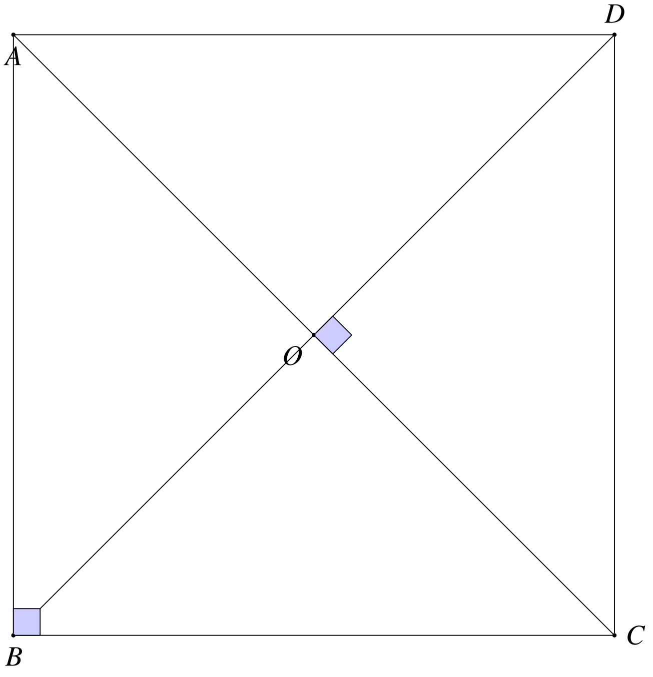

Fig. 1.3.4: Square

	Circle
Centre	$\mathbf{O}=\begin{pmatrix}0\\0\end{pmatrix}$
Radius	r=3
Radius	d=7

TABLE 1.4.1: Input values

Lemma 1.1. *The points of contact for the tangent drawn from a point*

$$\mathbf{P} = d\mathbf{e}_1, \ \textit{where}\ \mathbf{e}_1 = \begin{pmatrix}1\\0\end{pmatrix} \tag{1.4.1.1}$$

to the circle are given by

$$\mathbf{x} = \frac{r^2}{d}\mathbf{e}_1 \pm r\sqrt{1-\frac{r^2}{d^2}}\mathbf{e}_2 \tag{1.4.1.2}$$

If $\mathbf{x}$ be a point of contact for the tangent from $\mathbf{P}$,

$$PR \perp RO \tag{1.4.1.3}$$

$$\implies (\mathbf{O}-\mathbf{x})^\top(\mathbf{x}-\mathbf{P}) = 0 \tag{1.4.1.4}$$

$$\text{or, } \mathbf{P}^\top\mathbf{x} = \|\mathbf{x}\|^2 = r^2 \tag{1.4.1.5}$$

$$\implies \mathbf{e}_1^\top\mathbf{x} = \frac{r^2}{d} \tag{1.4.1.6}$$

$\because \mathbf{O} = 0$. The above equation can be expressed in parametric form as

$$\mathbf{x} = \frac{r^2}{d}\mathbf{e}_1 + \lambda\mathbf{e}_2 \tag{1.4.1.7}$$

Substituting the above in

$$\|\mathbf{x}\|^2 = r^2, \tag{1.4.1.8}$$

yields

$$\left\|\frac{r^2}{d}\mathbf{e}_1 + \lambda\mathbf{e}_2\right\|^2 = r^2 \tag{1.4.1.9}$$

$$\implies \lambda^2 = r^2\left[1-\frac{r^2}{d^2}\right] \tag{1.4.1.10}$$

$$\text{or, } \lambda = \pm r\sqrt{1-\frac{r^2}{d^2}} \tag{1.4.1.11}$$

Substituting λ in (1.4.1.7) yields (1.4.1.2). Fig. 1.4.1 shows all possible tangents and their points of contact after substituting the numerical values in (1.4.1.2).

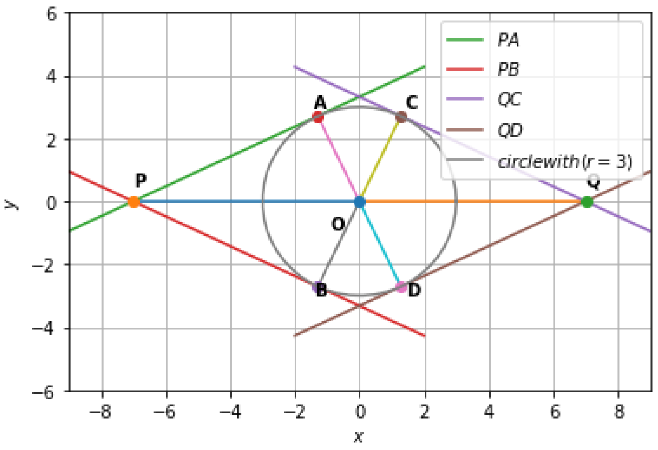

Fig. 1.4.1: Tangent lines to circle of radius 3 units.

1.4.2. Draw a pair of tangents to a circle of radius 5 units which are inclined to each other at an angle of 60°.

Solution: The angle between the tangents from $\mathbf{P}$ is given by

Lemma 1.2. *Given a circle of radius r and angle θ between the tangents, the intersection of the tangents and points of contact are given by Lemma 1.1 where*

$$\implies d = r\sin\frac{\theta}{2} \tag{1.4.2.1}$$

Proof. From Fig. 1.4.1,

$$\sin\frac{\theta}{2} = \frac{r}{d} \tag{1.4.2.2}$$

$$\implies d = r\sin\frac{\theta}{2} \tag{1.4.2.3}$$

□

1.5 Vector Calculus

1.5.1. *Definition:* Let $\mathbf{x} \in \mathbb{R}^2, f(\mathbf{x}) \in \mathbb{R}$. Then,

$$\frac{df(\mathbf{x})}{d\mathbf{x}} = \begin{pmatrix} \frac{\partial f(\mathbf{x})}{\partial x_1} \\ \frac{\partial f(\mathbf{x})}{\partial x_2} \end{pmatrix} \tag{1.5.1.1}$$

1.5.2.

$$\begin{aligned} \frac{d\mathbf{x}}{dx_1} &= \begin{pmatrix} \frac{dx_1}{dx_1} \\ \frac{dx_1}{dx_1} \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ m \end{pmatrix} = \mathbf{m} \end{aligned} \tag{1.5.2.1}$$

1.5.3. Show that

$$\begin{aligned} \frac{d\left(\mathbf{u}^T\mathbf{x}\right)}{d\mathbf{x}} &= \mathbf{u} \\ \frac{d\left(\mathbf{x}^T\mathbf{V}\mathbf{x}\right)}{d\mathbf{x}} &= 2\mathbf{V}^T\mathbf{x} \end{aligned} \tag{1.5.3.1}$$

1.5.4. Differentiating (3.1.2.1) with respect to x_1,

$$\left[\frac{d\left(\mathbf{x}^T\mathbf{V}\mathbf{x}\right)}{d\mathbf{x}}\right]^T \frac{d\mathbf{x}}{dx_1} + 2\frac{d\left(\mathbf{u}^T\mathbf{x}\right)}{\mathbf{x}}\frac{d\mathbf{x}}{dx_1} = 0 \tag{1.5.4.1}$$

$$\implies 2\left(\mathbf{V}^T\mathbf{x} + \mathbf{u}\right)\mathbf{m} = 0 \tag{1.5.4.2}$$

from (1.5.2.1) and (1.5.3.1). Substituting the point of contact $\mathbf{x} = \mathbf{q}$ and simplifying results in

$$\left(\mathbf{V}\mathbf{q} + \mathbf{u}\right)\mathbf{m} = 0 \tag{1.5.4.3}$$

which, upon taking the transpose, yields (3.2.1).

1.6 Vector Inequalities

1.6.1. *(Cauchy-Schwarz Inequality:)* Show that

$$\left|\mathbf{a}^T\mathbf{b}\right| \le \|\mathbf{a}\|\,\|\mathbf{b}\| \tag{1.6.1.1}$$

Proof. Using the definition of the inner product,

$$\cos\theta = \frac{\mathbf{a}^T\mathbf{b}}{\|\mathbf{a}\|\,\|\mathbf{b}\|} \tag{1.6.1.2}$$

$$\because |\cos\theta| \le 1, \left|\mathbf{a}^T\mathbf{b}\right| \le \|\mathbf{a}\|\,\|\mathbf{b}\| \tag{1.6.1.3}$$

(Triangle Inequality:) Show that

$$\|\mathbf{a} + \mathbf{b}\| \le \|\mathbf{a}\| + \|\mathbf{b}\| \tag{1.6.1.4}$$

Proof. Let $\mathbf{O}$ be the origin. In the triangle formed by $\mathbf{O}, \mathbf{a}$ and $-\mathbf{b}$, the lengths of the sides are

$$\|\mathbf{a}\|, \|\mathbf{b}\|, \|\mathbf{a} + \mathbf{b}\| \tag{1.6.1.5}$$

$\because$ the sum of two sides of a triangle is always greater than the third side,

$$\|\mathbf{a} + \mathbf{b}\| \le \|\mathbf{a}\| + \|\mathbf{b}\| \tag{1.6.1.6}$$

2 LINEAR FORMS

2.1 Line

2.1.1. Any point $\mathbf{P}$ in the 2-D plane can be expressed in terms of its coordinates (p_1, p_2) as the column vector

$$\mathbf{P} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \tag{2.1.1.1}$$

2.1.2. The *direction vector* of the line joining $\mathbf{P}, \mathbf{Q}$ is defined as

$$\mathbf{m} = \mathbf{P} - \mathbf{Q} = \begin{pmatrix} p_1 - q_1 \\ p_2 - q_2 \end{pmatrix} \tag{2.1.2.1}$$

$$= (p_1 - q_1)\begin{pmatrix} 1 \\ \frac{p_2 - q_2}{p_1 - q_1} \end{pmatrix} = (p_1 - q_1)\begin{pmatrix} 1 \\ m \end{pmatrix} \tag{2.1.2.2}$$

where

$$m = \frac{p_2 - q_2}{p_1 - q_1}. \tag{2.1.2.3}$$

Without loss of generality, $k\mathbf{m}$, for any real scalar k is also a direction vector. In the rest of the paper, $\mathbf{m}$ *and $k\mathbf{m}$ are interchanged for computational simplicity*. Thus, if m be the slope of the line PQ,

$$\mathbf{m} = \begin{pmatrix} 1 \\ m \end{pmatrix} \tag{2.1.2.4}$$

2.1.3. Let $\mathbf{P}, \mathbf{Q}$ be two points on a line. The vector equation of the line is given by

$$\mathbf{x} = \mathbf{P} + \lambda \mathbf{m}, \quad \lambda \in \mathbb{R} \tag{2.1.3.1}$$

$$\mathbf{m} = \mathbf{P} - \mathbf{Q} \tag{2.1.3.2}$$

(2.1.3.1) can be used in 3D as well.

2.1.4. The *normal vector* $\mathbf{n}$ to a line is orthogonal to the direction vector $\mathbf{m}$ so that

$$\mathbf{m}^T \mathbf{n} = 0 \tag{2.1.4.1}$$

If $\mathbf{P}$ be a point on the line, the equation of the line can be expressed as

$$\mathbf{n}^T (\mathbf{x} - \mathbf{P}) = 0 \tag{2.1.4.2}$$

$$\text{or, } \mathbf{n}^T \mathbf{x} = c, \tag{2.1.4.3}$$

where

$$c = \mathbf{n}^T \mathbf{P} \tag{2.1.4.4}$$

which is the desired equation of the straight line. By subsuming the c in (2.1.4.3) within $\mathbf{n}$, the equation of a line can also be expressed as

$$\mathbf{n}^T \mathbf{x} = 1 \tag{2.1.4.5}$$

Note that in 3D, (2.1.4.2) and (2.1.4.3) are used to represent the equation of a plane.

2.1.5. *Orthogonality:* Show that the points

$$\mathbf{A} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 3 \\ -4 \\ -4 \end{pmatrix} \tag{2.1.5.1}$$

are the vertices of a right angled triangle.

Solution: Let

$$\mathbf{v}_1 = \mathbf{A} - \mathbf{C} = \begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix} \tag{2.1.5.2}$$

$$\mathbf{v}_2 = \mathbf{B} - \mathbf{C} = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} \tag{2.1.5.3}$$

Then

$$\mathbf{v}_1^T \mathbf{v}_2 = \begin{pmatrix} -1 & 3 & 5 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} = 0 \tag{2.1.5.4}$$

$$\implies AC \perp BC \tag{2.1.5.5}$$

and $\mathbf{v}_1$ and $\mathbf{v}_2$ are said to be orthogonal.

2.1.6. Find the equation of the line through $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$ with slope - 4

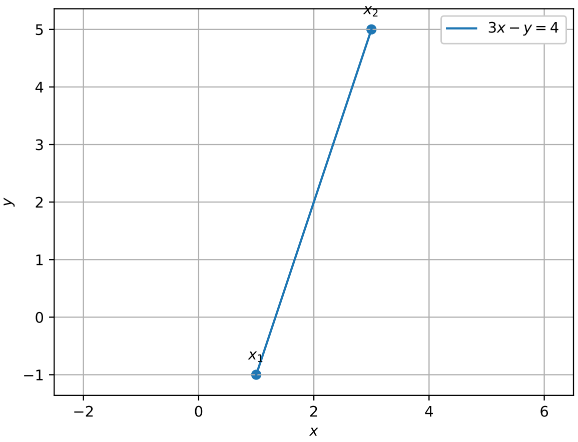

Fig. 2.1.6: Line obtained in Problem 2.1.6.

Solution: From (2.1.2.4), the direction vector is

$$\mathbf{m} = \begin{pmatrix} 1 \\ -4 \end{pmatrix} \tag{2.1.6.1}$$

and from (2.1.4.1), the normal vector is

$$\mathbf{n} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \tag{2.1.6.2}$$

Using (2.1.4.2), the equation of the line is

$$\begin{pmatrix} 4 & 1 \end{pmatrix} \left\{ \mathbf{x} - \begin{pmatrix} -2 \\ 3 \end{pmatrix} \right\} = 0 \tag{2.1.6.3}$$

$$\implies \begin{pmatrix} 4 & 1 \end{pmatrix} \mathbf{x} = -5 \tag{2.1.6.4}$$

Fig. 2.1.6 shows the line passing through the given point.

2.1.7. Write the equation of the line through the points $\mathbf{x}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\mathbf{x}_2 = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$.

Solution: From (2.1.4.5),

$$\mathbf{n}^T \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1 \tag{2.1.7.1}$$

$$\mathbf{n}^T \begin{pmatrix} 3 \\ 5 \end{pmatrix} = 1 \tag{2.1.7.2}$$

resulting in the the matrix equation

$$\begin{pmatrix} 1 & -1 \\ 3 & 5 \end{pmatrix} \mathbf{n} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.1.7.3}$$

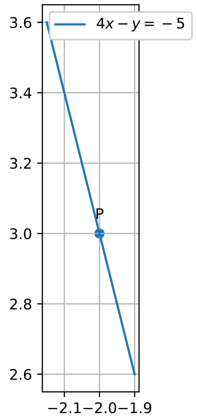

Fig. 2.1.7: Line obtained in Problem 2.1.7.

yielding the augmented matrix

$$\begin{pmatrix} 1 & -1 & 1 \\ 3 & 5 & 1 \end{pmatrix} \tag{2.1.7.4}$$

Performing row reduction,

$$\begin{pmatrix} 1 & -1 & 1 \\ 3 & 5 & 1 \end{pmatrix} \tag{2.1.7.5}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - 3R_1} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 8 & -2 \end{pmatrix} \tag{2.1.7.6}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{R_2}{2}} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 4 & -1 \end{pmatrix} \tag{2.1.7.7}$$

$$\xleftrightarrow{R_1 \leftarrow 4R_1 + R_2} \begin{pmatrix} 4 & 0 & 3 \\ 0 & 4 & -1 \end{pmatrix} \tag{2.1.7.8}$$

$$\xleftrightarrow[R_1 \leftarrow \frac{R_1}{4}]{R_2 \leftarrow \frac{R_2}{4}} \begin{pmatrix} 1 & 0 & \frac{3}{4} \\ 0 & 1 & -\frac{1}{4} \end{pmatrix} \tag{2.1.7.9}$$

From (2.1.7.9),

$$\mathbf{n} = \frac{1}{4}\begin{pmatrix} 3 \\ -1 \end{pmatrix} \tag{2.1.7.10}$$

Thus the equation of the desired line is

$$\frac{1}{4}\begin{pmatrix} 3 & -1 \end{pmatrix}\mathbf{x} = 1 \tag{2.1.7.11}$$

$$\text{or, } \begin{pmatrix} 3 & -1 \end{pmatrix}\mathbf{x} = 4 \tag{2.1.7.12}$$

Fig. 2.1.7 shows the line passing through the given points.

2.1.8. (*Linear Dependence*) Prove that the three points $\begin{pmatrix} 3 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ -2 \end{pmatrix}, \begin{pmatrix} 8 \\ 2 \end{pmatrix}$ are collinear

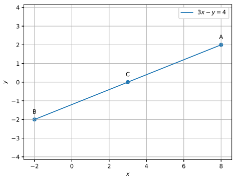

Fig. 2.1.8: Points on a line and points forming a triangle in Example 2.1.8.

Solution: Let

$$\begin{aligned} \mathbf{v}_1 &= \begin{pmatrix} 3 \\ 0 \end{pmatrix} - \begin{pmatrix} -2 \\ -2 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \\ \mathbf{v}_2 &= \begin{pmatrix} -2 \\ -2 \end{pmatrix} - \begin{pmatrix} 8 \\ 2 \end{pmatrix} = \begin{pmatrix} -10 \\ -4 \end{pmatrix} \end{aligned} \tag{2.1.8.1}$$

Then, the given points are collinear if

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = 0 \tag{2.1.8.2}$$

has a nontrivial solution as well, i.e.

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \neq \mathbf{0} \tag{2.1.8.3}$$

Substituting (2.1.8.1) in (2.1.8.2) results in the matrix equation

$$\begin{pmatrix} 5 & -10 \\ 2 & -4 \end{pmatrix}\mathbf{x} = 0 \tag{2.1.8.4}$$

Performing row operations on the matrix,

$$\begin{pmatrix} 5 & -10 \\ 2 & -4 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow 2R_1 - 5R_2} \begin{pmatrix} 5 & -10 \\ 0 & 0 \end{pmatrix} \tag{2.1.8.5}$$

which can be expressed as

$$\begin{pmatrix} 5 & -10 \end{pmatrix}\mathbf{x} = 0 \tag{2.1.8.6}$$

$$\text{or, } \mathbf{x} = x_1\begin{pmatrix} 1 \\ -2 \end{pmatrix} \tag{2.1.8.7}$$

Thus, there are infinite solutions. The vectors $\mathbf{v}_1, \mathbf{v}_2$ are are linearly dependent and the given points lie on a straight line.

2.1.9. Alternatively, if the given points are collinear, from (2.1.4.5),

$$\begin{pmatrix} 3 & 0 \\ -2 & -2 \\ 8 & 2 \end{pmatrix}\mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{2.1.9.1}$$

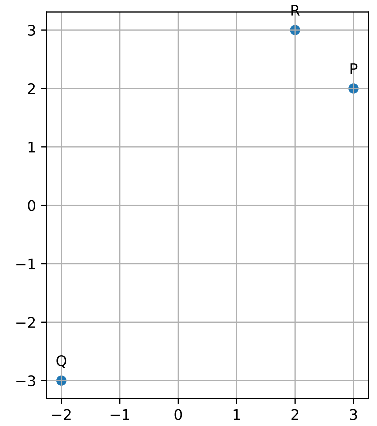

Fig. 2.1.10: Points on a triangle in Problem 2.1.10.

Row reducing the augmented matrix,

$$\begin{pmatrix} 3 & 0 & 1 \\ -2 & -2 & 1 \\ 8 & 2 & 1 \end{pmatrix} \tag{2.1.9.2}$$

$$\xleftrightarrow[R_2 \leftarrow 3R_2+2R_1]{R_3 \leftarrow 3R_3-8R_1} \begin{pmatrix} 3 & 0 & 1 \\ 0 & -6 & 5 \\ 0 & 6 & -5 \end{pmatrix} \tag{2.1.9.3}$$

$$\xleftrightarrow{R_3 \leftarrow R_3+R_2} \begin{pmatrix} 3 & 0 & 1 \\ 0 & 6 & -5 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.1.9.4}$$

The above matrix has a zero row in echelon form, hence (2.1.9.1) is consistent and the given points are on a straight line. Also,

$$\mathbf{n} = \frac{1}{6}\begin{pmatrix} 2 \\ -5 \end{pmatrix} \tag{2.1.9.5}$$

2.1.10. (*Linear Independence*) Do the points $\begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ -3 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ form a triangle?

Solution: In this case

$$\mathbf{v}_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix} - \begin{pmatrix} -2 \\ -3 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix} \tag{2.1.10.1}$$

$$\mathbf{v}_2 = \begin{pmatrix} -2 \\ -3 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -4 \\ -6 \end{pmatrix} \tag{2.1.10.2}$$

Thus,

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = 0 \tag{2.1.10.3}$$

$$\implies \begin{pmatrix} 5 & -4 \\ 5 & -6 \end{pmatrix}\mathbf{x} = 0 \tag{2.1.10.4}$$

Using row operations,

$$\begin{pmatrix} 5 & -4 \\ 5 & -6 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_1-R_2} \begin{pmatrix} 5 & -4 \\ 0 & 2 \end{pmatrix} \tag{2.1.10.5}$$

$$\xleftrightarrow{R_1 \leftarrow R_1+2R_2} \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix} \tag{2.1.10.6}$$

resulting in a *full rank* matrix. Hence,

$$\mathbf{x} = 0 \tag{2.1.10.7}$$

and $\mathbf{v}_1$ and $\mathbf{v}_2$ are *linearly independent*. The points lie on a triangle.

2.1.11. Alternatively, from (2.1.4.5), row reducing the augmented matrix

$$\begin{pmatrix} 3 & 2 & 1 \\ -2 & -3 & 1 \\ 2 & 3 & 1 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3+R_2} \begin{pmatrix} 3 & 2 & 1 \\ -2 & -3 & 1 \\ 0 & 0 & 2 \end{pmatrix} \tag{2.1.11.1}$$

The above matrix has a nonzero row in echelon form, hence the given points do not lie on a straight line. So they lie on a triangle.

2.1.12. Find the angle between the lines

$$\begin{aligned} \begin{pmatrix} 1 & -\sqrt{3} \end{pmatrix}\mathbf{x} &= 5 \\ \begin{pmatrix} \sqrt{3} & -1 \end{pmatrix}\mathbf{x} &= -6. \end{aligned} \tag{2.1.12.1}$$

Solution: The angle between the lines can be expressed in terms of the normal vectors

$$\mathbf{n}_1 = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}, \mathbf{n}_2 = \begin{pmatrix} \sqrt{3} \\ -1 \end{pmatrix} \tag{2.1.12.2}$$

as

$$\cos\theta = \frac{\mathbf{n}_1^T\mathbf{n}_2}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|} \tag{2.1.12.3}$$

$$= \frac{\sqrt{3}}{2} \implies \theta = 30^\circ \tag{2.1.12.4}$$

2.1.13. Find the projection of the vector

$$\mathbf{a} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} \tag{2.1.13.1}$$

on the vector

$$\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}. \tag{2.1.13.2}$$

Solution: If the angle between the vectors be

θ, the projection is defined as

$$\mathbf{proj_b a} = (\|\mathbf{a}\| \cos\theta) \frac{\mathbf{b}}{\|\mathbf{b}\|} = \frac{\left(\mathbf{a}^T\mathbf{b}\right)}{\|\mathbf{b}\|^2}\mathbf{b} \tag{2.1.13.3}$$

Substituting the values from (2.1.13.1) and (2.1.13.2),

$$\mathbf{proj_b a} = \frac{5}{3}\begin{pmatrix}1\\2\\1\end{pmatrix} \tag{2.1.13.4}$$

2.1.14. (*Reflection*) Assuming that straight lines work as a plane mirror for a point, find the image of the point $\mathbf{P} = \begin{pmatrix}1\\2\end{pmatrix}$ in the line

$$L: \quad \begin{pmatrix}1 & -3\end{pmatrix}\mathbf{x} = -4. \tag{2.1.14.1}$$

Solution: From the given equation, the line parameters are

$$\mathbf{n} = \begin{pmatrix}1\\-3\end{pmatrix}, c = -4, \mathbf{m} = \begin{pmatrix}3\\-1\end{pmatrix} \tag{2.1.14.2}$$

Let $\mathbf{R}$ be the reflection of $\mathbf{P}$ such that PR bisects the line L at $\mathbf{Q}$. Then $\mathbf{Q}$ bisects PR. This leads to the following equations

$$2\mathbf{Q} = \mathbf{P} + \mathbf{R} \tag{2.1.14.3}$$

$$\mathbf{n}^T\mathbf{Q} = c \quad \because \mathbf{Q} \text{ lies on the given line} \tag{2.1.14.4}$$

$$\mathbf{m}^T\mathbf{R} = \mathbf{m}^T\mathbf{P} \quad \because \mathbf{m} \perp \mathbf{P} - \mathbf{R} \tag{2.1.14.5}$$

From (2.1.14.3) and (2.1.14.4),

$$\mathbf{n}^T\mathbf{R} = 2c - \mathbf{n}^T\mathbf{P} \tag{2.1.14.6}$$

From (2.1.14.6) and (2.1.14.5),

$$\begin{pmatrix}\mathbf{m} & \mathbf{n}\end{pmatrix}^T \mathbf{R} = \begin{pmatrix}\mathbf{m} & -\mathbf{n}\end{pmatrix}^T \mathbf{P} + \begin{pmatrix}0\\2c\end{pmatrix} \tag{2.1.14.7}$$

Letting

$$\mathbf{V} = \begin{pmatrix}\mathbf{m} & \mathbf{n}\end{pmatrix} \tag{2.1.14.8}$$

with the condition that $\mathbf{m}, \mathbf{n}$ are orthonormal, i.e.

$$\mathbf{V}^T\mathbf{V} = \mathbf{I} \tag{2.1.14.9}$$

Noting that

$$\begin{pmatrix}\mathbf{m} & -\mathbf{n}\end{pmatrix} = \begin{pmatrix}\mathbf{m} & \mathbf{n}\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}, \tag{2.1.14.10}$$

(2.1.14.7) can be expressed as

$$\mathbf{V}^T\mathbf{R} = \left[\mathbf{V}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\right]^T \mathbf{P} + \begin{pmatrix}0\\2c\end{pmatrix} \tag{2.1.14.11}$$

$$\implies \mathbf{R} = \left[\mathbf{V}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\mathbf{V}^{-1}\right]^T \mathbf{P} + \mathbf{V}\begin{pmatrix}0\\2c\end{pmatrix} \tag{2.1.14.12}$$

$$= \mathbf{V}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\mathbf{V}^T\mathbf{P} + 2c\mathbf{n} \tag{2.1.14.13}$$

upon substituting from (2.1.14.8) in (2.1.14.13). It can be verified that the reflection is also given by

$$\mathbf{R} = \begin{pmatrix}\mathbf{m} & \mathbf{n}\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}\mathbf{m} & \mathbf{n}\end{pmatrix}^T \mathbf{P} + 2c\mathbf{n} \tag{2.1.14.14}$$

$$= \begin{pmatrix}\mathbf{m} & -\mathbf{n}\end{pmatrix}\begin{pmatrix}\mathbf{m}^T\\\mathbf{n}^T\end{pmatrix}\mathbf{P} + 2c\mathbf{n} \tag{2.1.14.15}$$

$$\implies \mathbf{R} = \left(\mathbf{m}\mathbf{m}^T - \mathbf{n}\mathbf{n}^T\right)\mathbf{P} + 2c\mathbf{n} \tag{2.1.14.16}$$

If $\mathbf{m}, \mathbf{n}$ are not orthonormal, (2.1.14.16) can be expressed as

$$\frac{\mathbf{R}}{2} = \frac{\mathbf{m}\mathbf{m}^T - \mathbf{n}\mathbf{n}^T}{\mathbf{m}^T\mathbf{m} + \mathbf{n}^T\mathbf{n}}\mathbf{P} + c\frac{\mathbf{n}}{\|\mathbf{n}\|^2} \tag{2.1.14.17}$$

2.1.15. (*Gram-schmidt orthogonalization*) Let

$$\alpha = \begin{pmatrix}3\\-1\\0\end{pmatrix} \tag{2.1.15.1}$$

$$\beta = \begin{pmatrix}2\\1\\-3\end{pmatrix} \tag{2.1.15.2}$$

Find β_1,β_2 such that

$$\beta = \beta_1 + \beta_2, \quad \beta_1 \parallel \alpha, \beta_2 \perp \alpha \tag{2.1.15.3}$$

Solution: Let $\beta_1 = k\alpha$. Then, $\beta_1 \parallel \alpha$ and

$$\beta = k\alpha + \beta_2 \tag{2.1.15.4}$$

$$\implies \alpha^T\beta = k\|\alpha\|^2 + k\beta_1^T\beta_2 \tag{2.1.15.5}$$

$$\text{or, } k = \frac{\alpha^T\beta}{\|\alpha\|^2}, \quad \because \beta_1 \perp \beta_2 \tag{2.1.15.6}$$

Thus,

$$\beta_1 = \frac{\alpha^T\beta}{\|\alpha\|^2}\alpha = \frac{1}{2}\begin{pmatrix}3\\-1\\0\end{pmatrix} \tag{2.1.15.7}$$

$$\beta_2 = \beta - \beta_1 = \begin{pmatrix}2\\1\\-3\end{pmatrix} - \frac{1}{2}\begin{pmatrix}3\\-1\\0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\3\\-6\end{pmatrix} \tag{2.1.15.8}$$

Thus, any given set of vectors can be expressed as a linear combination of another set of orthogonal vectors.

2.2 Plane

2.2.1. Find the equation of a plane passing through the points $\mathbf{a} = \begin{pmatrix}2\\5\\-3\end{pmatrix}, \mathbf{b} = \begin{pmatrix}-2\\-3\\5\end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix}5\\3\\-3\end{pmatrix}$

Solution: The equation of plane is also given by (2.1.4.5) in 3D. Following the approach in the previous example results in the matrix equation,

$$\begin{pmatrix}2 & 5 & -3\\-2 & -3 & 5\\5 & 3 & -3\end{pmatrix}\mathbf{n} = \begin{pmatrix}1\\1\\1\end{pmatrix} \tag{2.2.1.1}$$

Row reducing the augmented matrix,

$$\begin{pmatrix}2 & 5 & -3 & 1\\-2 & -3 & 5 & 1\\5 & 3 & -3 & 1\end{pmatrix} \tag{2.2.1.2}$$

$$\xleftrightarrow[R_3\leftarrow 2R_3-5R_1]{R_2\leftarrow\frac{R_2+R_1}{2}} \begin{pmatrix}2 & 5 & -3 & 1\\0 & 1 & 1 & 1\\0 & -19 & 9 & -3\end{pmatrix} \tag{2.2.1.3}$$

$$\xleftrightarrow[R_3\leftarrow\frac{R_3+19R_2}{4}]{R_1\leftarrow R_1-5R_2} \begin{pmatrix}2 & 0 & -8 & -4\\0 & 1 & 1 & 1\\0 & 0 & 7 & 4\end{pmatrix} \tag{2.2.1.4}$$

$$\xleftrightarrow[R_3\leftarrow 7R2-R_3]{R_1\leftarrow\frac{7R_1+8R_3}{2}} \begin{pmatrix}7 & 0 & 0 & 2\\0 & 7 & 0 & 3\\0 & 0 & 7 & 4\end{pmatrix} \tag{2.2.1.5}$$

$$\implies \mathbf{n} = \frac{1}{7}\begin{pmatrix}2\\3\\4\end{pmatrix} \tag{2.2.1.6}$$

Thus, the equation of the plane passing through the given points is

$$\begin{pmatrix}2 & 3 & 4\end{pmatrix}\mathbf{x} = 7 \tag{2.2.1.7}$$

2.2.2. Find the angle between the two planes

$$\begin{pmatrix}2 & 1 & -2\end{pmatrix}\mathbf{x} = 5 \tag{2.2.2.1}$$

$$\begin{pmatrix}3 & -6 & -2\end{pmatrix}\mathbf{x} = 7 \tag{2.2.2.2}$$

Solution: The angle between two planes is the same as the angle between their normal vectors. For

$$\mathbf{n}_1 = \begin{pmatrix}2\\1\\-2\end{pmatrix}\mathbf{n}_2 = \begin{pmatrix}3\\-6\\-2\end{pmatrix} \tag{2.2.2.3}$$

using (2.1.12.3),

$$\cos\theta = \frac{6-6+4}{\sqrt{9}\sqrt{49}} = \frac{4}{21} \tag{2.2.2.4}$$

2.3 Pseudo Inverse

2.3.1. To find the shortest distance between the lines

$$L_1 : \mathbf{x} = \begin{pmatrix}1\\2\\1\end{pmatrix} + \lambda_1\begin{pmatrix}1\\-1\\1\end{pmatrix} \tag{2.3.1.1}$$

$$L_2 : \mathbf{x} = \begin{pmatrix}2\\-1\\-1\end{pmatrix} + \lambda_2\begin{pmatrix}2\\1\\2\end{pmatrix} \tag{2.3.1.2}$$

2.3.2. If the two lines intersect,

$$\mathbf{x}_1 + \lambda_1\mathbf{m}_1 = \mathbf{x}_2 + \lambda_2\mathbf{m}_2 \tag{2.3.2.1}$$

$$\implies \begin{pmatrix}\mathbf{m}_1 & \mathbf{m}_2\end{pmatrix}\begin{pmatrix}\lambda_1\\-\lambda_2\end{pmatrix} = \mathbf{x}_2 - \mathbf{x}_1 \tag{2.3.2.2}$$

$$\text{or, } \mathbf{M}\begin{pmatrix}\lambda_1\\-\lambda_2\end{pmatrix} = \mathbf{x}_2 - \mathbf{x}_1 \tag{2.3.2.3}$$

where

$$\mathbf{x}_1 = \begin{pmatrix}1\\2\\1\end{pmatrix}, \mathbf{x}_2 = \begin{pmatrix}2\\-1\\-1\end{pmatrix}, \mathbf{m}_1 = \begin{pmatrix}1\\-1\\1\end{pmatrix}, \mathbf{m}_2 = \begin{pmatrix}2\\1\\2\end{pmatrix}. \tag{2.3.2.4}$$

$$\mathbf{M} = \begin{pmatrix}\mathbf{m}_1 & \mathbf{m}_2\end{pmatrix} \tag{2.3.2.5}$$

(2.3.2.3) can be expressed as the matrix equation

$$\begin{pmatrix}1 & 2\\-1 & 1\\1 & 2\end{pmatrix}\mathbf{x} = \begin{pmatrix}1\\-3\\-2\end{pmatrix} \tag{2.3.2.6}$$

From the augmented matrix in (2.3.2.3),

$$\begin{pmatrix} 1 & -2 & 1 \\ -1 & -1 & -3 \\ 1 & -2 & -2 \end{pmatrix} \tag{2.3.2.7}$$

$$\begin{pmatrix} 1 & -2 & 1 \\ -1 & -1 & -3 \\ 1 & -2 & -2 \end{pmatrix} \xleftrightarrow{R_1=R_1-R_2} \begin{pmatrix} 0 & 0 & 3 \\ -1 & -1 & -3 \\ 1 & -2 & -2 \end{pmatrix} \tag{2.3.2.8}$$

The above matrix has a *rank* = 3. Hence the lines do not intersect.

2.3.3. Let

$$\mathbf{A} = \mathbf{x}_1 + \lambda_1 \mathbf{m}_1 \tag{2.3.3.1}$$

$$\mathbf{B} = \mathbf{x}_2 + \lambda_2 \mathbf{m}_2 \tag{2.3.3.2}$$

be the closest points on L_1 and L_2 respectively. Then the shortest distance between two skew lines will be the length of line perpendicular to both the lines L_1,L_2 and passing through A and B. Thus,

$$\mathbf{m_1}^T (\mathbf{A} - \mathbf{B}) = 0 \tag{2.3.3.3}$$

$$\mathbf{m_2}^T (\mathbf{A} - \mathbf{B}) = 0 \tag{2.3.3.4}$$

$$\implies \mathbf{M}^T (\mathbf{A} - \mathbf{B}) = 0 \tag{2.3.3.5}$$

From (2.3.3.2) and (2.3.2.5)

$$\mathbf{A} - \mathbf{B} = \mathbf{x_1} - \mathbf{x_2} + \mathbf{M} \begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} \tag{2.3.3.6}$$

and using (2.3.3.5), in the above,

$$\mathbf{M}^T \mathbf{M} \begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} = \mathbf{M}^T (\mathbf{x}_2 - \mathbf{x}_1) \tag{2.3.3.7}$$

2.3.4. Substituting the values from (2.3.2.4) in (2.3.3.7) and forming the augmented matrix,

$$\begin{pmatrix} 3 & 3 & 2 \\ 3 & 9 & -5 \end{pmatrix} \tag{2.3.4.1}$$

$$\begin{pmatrix} 3 & 3 & 2 \\ 3 & 9 & -5 \end{pmatrix} \xleftrightarrow{R_2=R_2-R_1} \begin{pmatrix} 3 & 3 & 2 \\ 0 & 6 & -7 \end{pmatrix} \tag{2.3.4.2}$$

$$\begin{pmatrix} 3 & 3 & 2 \\ 0 & 6 & -7 \end{pmatrix} \xleftrightarrow{R_1=2R_1-R_2} \begin{pmatrix} 6 & 0 & 11 \\ 0 & 6 & -7 \end{pmatrix} \tag{2.3.4.3}$$

$$\begin{pmatrix} 6 & 0 & 11 \\ 0 & 6 & -7 \end{pmatrix} \xleftrightarrow{R_1=\frac{R_1}{6}, R_2=\frac{R_2}{6}} \begin{pmatrix} 1 & 0 & \frac{11}{6} \\ 0 & 1 & \frac{-7}{6} \end{pmatrix} \tag{2.3.4.4}$$

$$\lambda_1 = \frac{11}{6}, \lambda_2 = \frac{7}{6} \tag{2.3.4.5}$$

yielding

$$A = \frac{1}{6} \begin{pmatrix} 17 \\ 1 \\ 17 \end{pmatrix}, B = \frac{1}{6} \begin{pmatrix} 26 \\ 1 \\ 8 \end{pmatrix}. \tag{2.3.4.6}$$

2.3.5. The distance is then obtained as

$$\|\mathbf{B} - \mathbf{A}\| = \frac{3}{\sqrt{2}} \tag{2.3.5.1}$$

Fig. 2.3.5 shows the various points and distances between the lines.

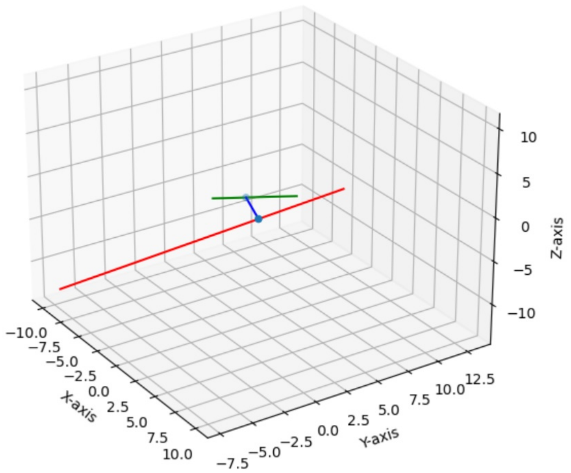

Fig. 2.3.5: This is the plot of the given skew lines and the blue line indicates the normal to the given lines

3 Quadratic Forms

3.1 Introduction

3.1.1. Let $\mathbf{P}$ be a point such that the ratio of its distance from a fixed point $\mathbf{F}$ and the distance (d) from a fixed line $L : \mathbf{n}^T \mathbf{x} = c$ is constant, given by

$$\frac{\|\mathbf{P} - \mathbf{F}\|}{d} = e \tag{3.1.1.1}$$

The locus of $\mathbf{P}$ such is known as a conic section. The line L is known as the directrix and the point $\mathbf{F}$ is the focus. e is defined to be the eccentricity of the conic.

a) For $e = 1$, the conic is a parabola
b) For $e < 1$, the conic is an ellipse
c) For $e > 1$, the conic is a hyperbola

3.1.2. The equation of a conic is given by

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \tag{3.1.2.1}$$

where

$$\mathbf{V} = t\mathbf{I} - \mathbf{n}\mathbf{n}^T, \tag{3.1.2.2}$$

$$\mathbf{u} = c\mathbf{n} - t\mathbf{F}, \tag{3.1.2.3}$$

$$f = t\left\|\mathbf{F}\right\|^2 - c^2 \tag{3.1.2.4}$$

and

$$t = \frac{\left\|\mathbf{n}\right\|^2}{e^2} \tag{3.1.2.5}$$

Solution: See Appendix A

3.1.3. *(Affine Transformation and Eigenvalue Decompostion)* Using

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c} \quad \text{(Affine Transformation)} \tag{3.1.3.1}$$

such that

$$\mathbf{P}^T\mathbf{V}\mathbf{P} = \mathbf{D}. \quad \text{(Eigenvalue Decomposition)} \tag{3.1.3.2}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \tag{3.1.3.3}$$

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix}, \quad \mathbf{P}^T = \mathbf{P}^{-1} \tag{3.1.3.4}$$

(3.1.2.1) can be expressed as

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f \qquad |V| \neq 0 \tag{3.1.3.5}$$

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = -2\eta\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{y} \qquad |V| = 0 \tag{3.1.3.6}$$

with

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \qquad |V| \neq 0 \tag{3.1.3.7}$$

$$\begin{pmatrix} \mathbf{u}^T + \eta\mathbf{p}_1^T \\ \mathbf{V} \end{pmatrix}\mathbf{c} = \begin{pmatrix} -f \\ \eta\mathbf{p}_1 - \mathbf{u} \end{pmatrix} \qquad |V| = 0 \tag{3.1.3.8}$$

$$\text{where } \eta = \mathbf{n}^T\mathbf{p}_1 \tag{3.1.3.9}$$

Solution: Proofs for (3.1.3.5), (3.1.3.6), (3.1.3.7) and (3.1.3.8) are available in Appendix B.

3.1.4. *(Centre/Vertex)* The centre/vertex of the conic section in (3.1.2.1) is given by **c** in (3.1.3.7) or (3.1.3.8). This is because from (3.1.3.1),

$$\mathbf{y} = \mathbf{P}^T\left(\mathbf{x} - \mathbf{c}\right) \tag{3.1.4.1}$$

and

$$\mathbf{y} = \mathbf{0} \implies \mathbf{x} = \mathbf{c} \tag{3.1.4.2}$$

3.1.5. *(Circle)* For a circle,

$$\mathbf{V} = \mathbf{D} = \mathbf{P} = \mathbf{I} \tag{3.1.5.1}$$

and the centre is obtained from (3.1.3.7), (3.1.4.2) as

$$\mathbf{c} = -\mathbf{u} \tag{3.1.5.2}$$

(3.1.3.5) becomes

$$\mathbf{y}^T\mathbf{y} = \left\|\mathbf{y}\right\|^2 = \left(\sqrt{\mathbf{u}^T\mathbf{u} - f}\right)^2 \tag{3.1.5.3}$$

and the radius is

$$\sqrt{\mathbf{u}^T\mathbf{u} - f} \tag{3.1.5.4}$$

3.1.6. *(Ellipse)* For

$$|\mathbf{V}| > 0, \quad \text{or, } \lambda_1 > 0, \lambda_2 > 0 \tag{3.1.6.1}$$

and (3.1.3.5) becomes

$$\lambda_1 y_1^2 + \lambda_2 y_1^2 = \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f \tag{3.1.6.2}$$

which is the equation of an ellipse with major and minor axes parameters

$$\sqrt{\frac{\lambda_1}{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}}, \sqrt{\frac{\lambda_2}{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}}. \tag{3.1.6.3}$$

The centre is obtained from (3.1.4.2) as (3.1.3.7).

3.1.7. *(Hyperbola)* For

$$|\mathbf{V}| < 0, \quad \text{or, } \lambda_1 > 0, \lambda_2 < 0 \tag{3.1.7.1}$$

and (3.1.3.5) becomes

$$\lambda_1 y_1^2 - \left(-\lambda_2\right) y_1^2 = \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f \tag{3.1.7.2}$$

with major and minor axes parameters

$$\sqrt{\frac{\lambda_1}{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}}, \sqrt{\frac{\lambda_2}{f - \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}}}, \tag{3.1.7.3}$$

The centre is obtained from (3.1.4.2) as (3.1.3.7).

3.1.8. (*Pair of straight lines:*) The *asymptotes* of the hyperbola (3.1.2.1) are defined as the pair of intersecting straight lines

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} = 0 \tag{3.1.8.1}$$

such that

$$|\mathbf{V}| < 0 \tag{3.1.8.2}$$

From (3.1.8.1), the equation of the asymptotes for (3.1.7.2) is

$$\left(\sqrt{|\lambda_1|} \quad \pm\sqrt{|\lambda_2|}\right)\mathbf{y} = 0 \tag{3.1.8.3}$$

and the asymptotes for the hyperbola are obtained using (3.1.3.1) as

$$\left(\sqrt{|\lambda_1|} \quad \pm \sqrt{|\lambda_2|}\right)\mathbf{P}^T(\mathbf{x}-\mathbf{c}) = 0 \tag{3.1.8.4}$$

Thus, $\mathbf{c}$ is the point of intersection of the lines and the normal vectors of the lines in (3.1.8.4) are

$$\begin{aligned}\mathbf{n}_1 &= \mathbf{P}\begin{pmatrix}\sqrt{|\lambda_1|}\\ \sqrt{|\lambda_2|}\end{pmatrix}\\ \mathbf{n}_2 &= \mathbf{P}\begin{pmatrix}\sqrt{|\lambda_1|}\\ -\sqrt{|\lambda_2|}\end{pmatrix}\end{aligned} \tag{3.1.8.5}$$

3.1.9. The angle between the asymptotes is given by

$$\cos\theta = \frac{\mathbf{n_1}^T\mathbf{n_2}}{\|\mathbf{n_1}\|\,\|\mathbf{n_2}\|} \tag{3.1.9.1}$$

The orthogonal matrix $\mathbf{P}$ preserves the norm, i.e.

$$\|\mathbf{n_1}\| = \left\|\mathbf{P}\begin{pmatrix}\sqrt{|\lambda_1|}\\ \sqrt{|\lambda_2|}\end{pmatrix}\right\| \tag{3.1.9.2}$$

$$= \left\|\begin{pmatrix}\sqrt{|\lambda_1|}\\ \sqrt{|\lambda_2|}\end{pmatrix}\right\| = \sqrt{|\lambda_1| + |\lambda_2|} = \|\mathbf{n_2}\| \tag{3.1.9.3}$$

It is easy to verify that

$$\mathbf{n_1}^T\mathbf{n_2} = |\lambda_1| - |\lambda_2| \tag{3.1.9.4}$$

Thus, the angle between the asymptotes is obtained from (3.1.9.1) as

$$\cos\theta = \frac{|\lambda_1| - |\lambda_2|}{|\lambda_1| + |\lambda_2|} \tag{3.1.9.5}$$

3.1.10. (*Conjugate Hyperbola:*) Another hyperbola with the same asymptotes as (3.1.8.4) can be obtained from (3.1.2.1) and (3.1.8.1) as

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + 2\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f = 0 \tag{3.1.10.1}$$

3.1.11. Another condition for (3.1.2.1) to represent a pair of straight lines is

$$\begin{vmatrix}\mathbf{V} & \mathbf{u}\\ \mathbf{u}^T & f\end{vmatrix} = 0 \tag{3.1.11.1}$$

3.1.12. *(Parabola)* For

$$|\mathbf{V}| = 0, \quad \text{or, } \lambda_1 = 0. \tag{3.1.12.1}$$

The vertex of the parabola is obtained using (3.1.3.8) and the focal length is

$$\left|\frac{2\mathbf{p}_1^T\mathbf{u}}{\lambda_2}\right| \tag{3.1.12.2}$$

3.2 *Tangents and Normals*

3.1. *Secant:* The points of intersection of the line

$$L: \quad \mathbf{x} = \mathbf{q} + \mu\mathbf{m} \quad \mu \in \mathbb{R} \tag{3.1.1}$$

with the conic section in (3.1.2.1) are given by

$$\mathbf{x}_i = \mathbf{q} + \mu_i\mathbf{m} \tag{3.1.2}$$

where

$$\begin{aligned}\mu_i = \frac{1}{\mathbf{m}^T\mathbf{V}\mathbf{m}}\Big(&-\mathbf{m}^T(\mathbf{V}\mathbf{q}+\mathbf{u})\\ &\pm\sqrt{[\mathbf{m}^T(\mathbf{V}\mathbf{q}+\mathbf{u})]^2 - (\mathbf{q}^T\mathbf{V}\mathbf{q} + 2\mathbf{u}^T\mathbf{q} + f)(\mathbf{m}^T\mathbf{V}\mathbf{m})}\Big)\end{aligned} \tag{3.1.3}$$

Solution: Substituting (3.1.1) in (3.1.2.1),

$$\begin{aligned}(\mathbf{q}+\mu\mathbf{m})^T\mathbf{V}(\mathbf{q}+\mu\mathbf{m}) + 2\mathbf{u}^T(\mathbf{q}+\mu\mathbf{m}) + f = 0\\ \implies \mu^2\mathbf{m}^T\mathbf{V}\mathbf{m} + 2\mu\mathbf{m}^T(\mathbf{V}\mathbf{q}+\mathbf{u})\\ + \mathbf{q}^T\mathbf{V}\mathbf{q} + 2\mathbf{u}^T\mathbf{q} + f = 0\end{aligned} \tag{3.1.4}$$

Solving the above quadratic in (3.1.4) yields (3.1.3).

3.2. *Tangent:* If L in (3.1.1) touches (3.1.2.1) at exactly one point $\mathbf{q}$,

$$\mathbf{m}^T(\mathbf{V}\mathbf{q}+\mathbf{u}) = 0 \tag{3.2.1}$$

Solution: In this case, (3.1.4) has exactly one root. Hence, in (3.1.3)

$$\begin{aligned}&\left[\mathbf{m}^T(\mathbf{V}\mathbf{q}+\mathbf{u})\right]^2\\ &\quad - \left(\mathbf{m}^T\mathbf{V}\mathbf{m}\right)\left(\mathbf{q}^T\mathbf{V}\mathbf{q} + 2\mathbf{u}^T\mathbf{q} + f\right) = 0\end{aligned} \tag{3.2.2}$$

$\because$ $\mathbf{q}$ is the point of contact, $\mathbf{q}$ satisfies (3.1.2.1) and

$$\mathbf{q}^T\mathbf{V}\mathbf{q} + 2\mathbf{u}^T\mathbf{q} + f = 0 \tag{3.2.3}$$

Substituting (3.2.3) in (3.2.2) and simplifying, we obtain (3.2.1).

3.3. The normal vector is obtained from (3.2.1) and (2.1.4.1) as

$$\mathbf{n} = \mathbf{V}\mathbf{q} + \mathbf{u} \tag{3.3.1}$$

3.4. Given the point of contact $\mathbf{q}$, the equation of a tangent is

$$(\mathbf{V}\mathbf{q}+\mathbf{u})^T\mathbf{x}+\mathbf{u}^T\mathbf{q}+f=0 \tag{3.4.1}$$

Solution: From (3.3.1) and (2.1.4.2), the equation of the tangent is

$$(\mathbf{V}\mathbf{q}+\mathbf{u})^T(\mathbf{x}-\mathbf{q})=0 \tag{3.4.2}$$

$$\implies (\mathbf{V}\mathbf{q}+\mathbf{u})^T\mathbf{x}-\mathbf{q}^T\mathbf{V}\mathbf{q}-\mathbf{u}^T\mathbf{q}=0 \tag{3.4.3}$$

which, upon substituting from (3.2.3) and simplifying yields (3.1.1).

3.5. If $\mathbf{V}^{-1}$ exists, given the normal vector $\mathbf{n}$, the tangent points of contact to (3.1.2.1) are given by

$$\mathbf{q}_i=\mathbf{V}^{-1}(\kappa_i\mathbf{n}-\mathbf{u}), i=1,2 \tag{3.5.1}$$

$$\text{where } \kappa_i=\pm\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}-f}{\mathbf{n}^T\mathbf{V}^{-1}\mathbf{n}}} \tag{3.5.2}$$

Solution: From (3.3.1),

$$\mathbf{q}=\mathbf{V}^{-1}(\kappa\mathbf{n}-\mathbf{u}), \quad \kappa\in\mathbb{R} \tag{3.5.3}$$

Substituting (3.5.3) in (3.2.3),

$$\begin{aligned}(\kappa\mathbf{n}-\mathbf{u})^T\mathbf{V}^{-1}(\kappa\mathbf{n}-\mathbf{u})&\\ +2\mathbf{u}^T\mathbf{V}^{-1}(\kappa\mathbf{n}-\mathbf{u})+f&=0\\ \implies \kappa^2\mathbf{n}^T\mathbf{V}^{-1}\mathbf{n}-\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}+f&=0\end{aligned}$$

$$\text{or, } \kappa=\pm\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}-f}{\mathbf{n}^T\mathbf{V}^{-1}\mathbf{n}}} \tag{3.5.4}$$

Substituting (3.5.4) in (3.5.3) yields (3.5.2).

3.6. If $\mathbf{V}$ is not invertible, given the normal vector $\mathbf{n}$, the point of contact to (3.1.2.1) is given by the matrix equation

$$\begin{pmatrix}\mathbf{u}+\kappa\mathbf{n}^T\\ \mathbf{V}\end{pmatrix}\mathbf{q}=\begin{pmatrix}-f\\ \kappa\mathbf{n}-\mathbf{u}\end{pmatrix} \tag{3.6.1}$$

$$\text{where } \kappa=\frac{\mathbf{p}_1^T\mathbf{u}}{\mathbf{p}_1^T\mathbf{n}}, \quad \mathbf{V}\mathbf{p}_1=0 \tag{3.6.2}$$

Solution: If $\mathbf{V}$ is non-invertible, it has a zero eigenvalue. If the corresponding eigenvector is $\mathbf{p}_1$, then,

$$\mathbf{V}\mathbf{p}_1=0 \tag{3.6.3}$$

From (3.3.1),

$$\kappa\mathbf{n}=\mathbf{V}\mathbf{q}+\mathbf{u}, \quad \kappa\in\mathbb{R} \tag{3.6.4}$$

$$\implies \kappa\mathbf{p}_1^T\mathbf{n}=\mathbf{p}_1^T\mathbf{V}\mathbf{q}+\mathbf{p}_1^T\mathbf{u} \tag{3.6.5}$$

$$\text{or, } \kappa\mathbf{p}_1^T\mathbf{n}=\mathbf{p}_1^T\mathbf{u}, \quad \because \mathbf{p}_1^T\mathbf{V}=0, \tag{3.6.6}$$

$$\text{from (3.6.3)} \tag{3.6.7}$$

yielding κ in (3.6.2). From (3.6.4),

$$\kappa\mathbf{q}^T\mathbf{n}=\mathbf{q}^T\mathbf{V}\mathbf{q}+\mathbf{q}^T\mathbf{u} \tag{3.6.8}$$

$$\implies \kappa\mathbf{q}^T\mathbf{n}=-f-\mathbf{q}^T\mathbf{u} \quad \text{from (3.2.3)}, \tag{3.6.9}$$

$$\text{or, } (\kappa\mathbf{n}+\mathbf{u})\mathbf{q}=-f \tag{3.6.10}$$

(3.6.4) can be expressed as

$$\mathbf{V}\mathbf{q}=\kappa\mathbf{n}-\mathbf{u}. \tag{3.6.11}$$

(3.6.10) and (3.6.11) clubbed together result in (3.6.1).

3.7. All the results related to conics are summarized in Table 3.7.

3.3 Circle

3.3.1. Find the centre and radius of the circle

$$x^2+y^2+8x+10y-8=0 \tag{3.3.1.1}$$

Solution: (3.3.1.1) can be expressed as

$$\mathbf{x}^T\mathbf{x}+2\begin{pmatrix}4 & 5\end{pmatrix}\mathbf{x}-8=0 \tag{3.3.1.2}$$

which is of the form (3.1.2.1) with

$$\mathbf{u}=\begin{pmatrix}4\\5\end{pmatrix}, f=-8 \tag{3.3.1.3}$$

From Table 3.7, the center and radius are given by

$$\mathbf{c}=-\mathbf{u}=\begin{pmatrix}-4\\-5\end{pmatrix}, r=\sqrt{\|u\|^2-f}=7 \tag{3.3.1.4}$$

3.3.2. Find the equation of a circle which passes through the points $\mathbf{P}=\begin{pmatrix}2\\-2\end{pmatrix}$ and $\mathbf{Q}=\begin{pmatrix}3\\4\end{pmatrix}$ and whose centre lies on the line

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x}=2 \tag{3.3.2.1}$$

Solution: From (3.1.2.1) and Table 3.7, the equation of a circle can be expressed as

$$\mathbf{x}^T\mathbf{x}-2\mathbf{c}^T\mathbf{x}+f=0 \tag{3.3.2.2}$$

Conic	Property	Standard Form	Standard Parameters	Point(s) of Contact
Circle	$\mathbf{V} = \mathbf{I}$	$\frac{\mathbf{y}^T\mathbf{D}\mathbf{y}}{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}-f} = 1$ $\mathbf{D} = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}$ $\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T$ $\mathbf{P} = \begin{pmatrix}\mathbf{p}_1 & \mathbf{p}_2\end{pmatrix}$	$\mathbf{c} = -\mathbf{u},$ $r = \sqrt{\mathbf{u}^T\mathbf{u} - f}$	$\mathbf{q} = \mathbf{V}^{-1}\left(\kappa\mathbf{n} - \mathbf{u}\right)$ $\kappa = \pm\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\mathbf{n}^T\mathbf{V}^{-1}\mathbf{n}}}$
Ellipse	$\lvert\mathbf{V}\rvert > 0$ $\lambda_1 > 0, \lambda_2 < 0$		$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u},$ $axes = \begin{cases}\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}-f}{\lambda_1}}\\ \sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}-f}{\lambda_2}}\end{cases}$	
Hyperbola	$\lvert\mathbf{V}\rvert < 0$ $\lambda_1 > 0, \lambda_2 < 0$		$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u},$ $axes = \begin{cases}\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}-f}{\lambda_1}}\\ \sqrt{\frac{f-\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}}{\lambda_2}}\end{cases}$	
Parabola	$\lvert\mathbf{V}\rvert = 0$ $\lambda_1 = 0$	$\mathbf{y}^T\mathbf{D}\mathbf{y} = -2\eta\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{y}$	focal length $= \left\lvert\frac{2\eta}{\lambda_2}\right\rvert$ $\begin{pmatrix}\mathbf{u}^T + \eta\mathbf{p}_1^T\\ \mathbf{V}\end{pmatrix}\mathbf{c} = \begin{pmatrix}-f\\ \eta\mathbf{p}_1 - \mathbf{u}\end{pmatrix}$ $\eta = \mathbf{p}_1^T\mathbf{u}$	$\begin{pmatrix}\mathbf{u} + \kappa\mathbf{n}^T\\ \mathbf{V}\end{pmatrix}\mathbf{q} = \begin{pmatrix}-f\\ \kappa\mathbf{n} - \mathbf{u}\end{pmatrix}$ $\kappa = \frac{\mathbf{p_1}^T\mathbf{u}}{\mathbf{p_1}^T\mathbf{n}}$

TABLE 3.7: $\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0$ can be expressed in the above standard form for various conics. $\mathbf{c}$ represents the centre/vertex of the conic. $\mathbf{q}$ is/are the point(s) of contact for the tangent(s).

where $\mathbf{c}$ is the centre. Substituting the given points in (3.3.2.2) and using (3.3.2.1), the following equations are obtained

$$2\begin{pmatrix}2 & -2\end{pmatrix}\mathbf{c} - f = 8 \tag{3.3.2.3}$$

$$2\begin{pmatrix}3 & 4\end{pmatrix}\mathbf{c} - f = 25 \tag{3.3.2.4}$$

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{c} = 2 \tag{3.3.2.5}$$

which can be expressed as the matrix equation

$$\begin{pmatrix}1 & 1 & 0\\ 4 & -4 & -1\\ 6 & 8 & -1\end{pmatrix}\begin{pmatrix}\mathbf{c}\\ f\end{pmatrix} = \begin{pmatrix}2\\ 8\\ 25\end{pmatrix} \tag{3.3.2.6}$$

Row reducing the augmented matrix

$$\begin{pmatrix}1 & 1 & 0 & 2\\ 4 & -4 & -1 & 8\\ 6 & 8 & -1 & 25\end{pmatrix} \tag{3.3.2.7}$$

$$\xleftrightarrow[R_3\leftarrow R_3-6R_1]{R_2\leftarrow -R_2+4R_1}\begin{pmatrix}1 & 1 & 0 & 2\\ 0 & 8 & 1 & 0\\ 0 & 2 & -1 & 13\end{pmatrix} \tag{3.3.2.8}$$

$$\xleftrightarrow[R_3\leftarrow -\frac{4R_3-R_2}{2}]{R_1\leftarrow 8R_1-R_3}\begin{pmatrix}8 & 0 & -1 & 16\\ 0 & 8 & 1 & 0\\ 0 & 0 & 5 & -52\end{pmatrix} \tag{3.3.2.9}$$

$$\xleftrightarrow[R_2\leftarrow\frac{5R_2-R_3}{4}]{R_1\leftarrow\frac{5R_1+R_3}{4}}\begin{pmatrix}10 & 0 & 0 & 7\\ 0 & 10 & 0 & 13\\ 0 & 0 & 5 & -52\end{pmatrix} \tag{3.3.2.10}$$

Thus,

$$\mathbf{c} = \frac{1}{10}\begin{pmatrix}7\\ 13\end{pmatrix} \tag{3.3.2.11}$$

$$f = -\frac{52}{5} \tag{3.3.2.12}$$

which give the desired equation of the circle. From Table 3.7,

$$r = \sqrt{\lVert\mathbf{c}\rVert^2 - f} = \frac{1}{10}\sqrt{1258} \tag{3.3.2.13}$$

Fig. 3.3.2 verifies the above results.

3.3.3. Find the points on the curve

$$x^2 + y^2 - 2x - 3 = 0 \tag{3.3.3.1}$$

at which the tangents are parallel to the x-axis.
Solution: (3.3.3.1) can be expressed as

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}-2 & 0\end{pmatrix}\mathbf{x} - 3 = 0 \tag{3.3.3.2}$$

$$\implies \mathbf{V} = \mathbf{I}, \mathbf{u} = \begin{pmatrix}-1\\ 0\end{pmatrix}, f = -3 \tag{3.3.3.3}$$

From Table 3.7, the centre and radius are

$$\mathbf{c} = -\mathbf{u} = \begin{pmatrix}-1\\ 0\end{pmatrix}, r = \sqrt{\lVert\mathbf{u}\rVert^2 - f} = 2 \tag{3.3.3.4}$$

$\because$ the tangents are parallel to the x-axis, their

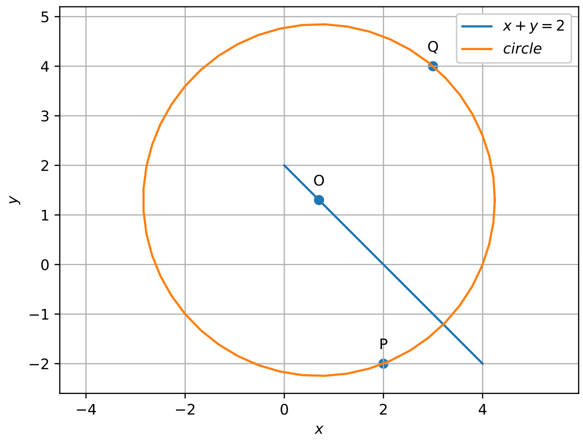

Fig. 3.3.2: Circle passing through $\begin{pmatrix}2\\-2\end{pmatrix}, \begin{pmatrix}3\\4\end{pmatrix}$. Center is on line $\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 2$.

direction and normal vectors are respectively,

$$\mathbf{m} = \begin{pmatrix}1\\0\end{pmatrix}, \mathbf{n} = \begin{pmatrix}0\\1\end{pmatrix}. \tag{3.3.3.5}$$

From Table 3.7,

$$\kappa = \pm\sqrt{\frac{\mathbf{u}^T\mathbf{u} - f}{\mathbf{n}^T\mathbf{n}}} = \pm\sqrt{\frac{4}{1}} = \pm 2 \tag{3.3.3.6}$$

and the desired points of contact are

$$\mathbf{q}_1, \mathbf{q}_2 = \begin{pmatrix}1\\0\end{pmatrix} \pm 2\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}1\\-2\end{pmatrix} \tag{3.3.3.7}$$

Fig. 3.3.2 verifies the above results.

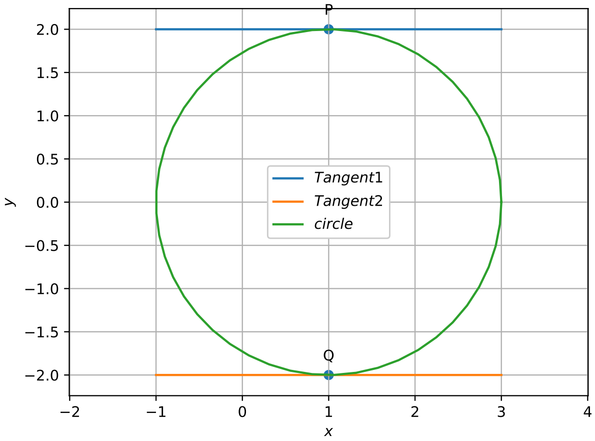

Fig. 3.3.3: Tangents are parallel to the x-axis.

3.4 Ellipse

3.4.1. Find $\frac{dy}{dx}$ if

$$E_1 : x^2 + xy + y^2 = 100 \tag{3.4.1.1}$$

Solution: Expressing (3.4.1.1) as (3.1.2.1),

$$\mathbf{V} = \begin{pmatrix}1 & \frac{1}{2}\\ \frac{1}{2} & 1\end{pmatrix}, \mathbf{u} = \mathbf{0}, f = -100. \tag{3.4.1.2}$$

$$\because |V| = \begin{vmatrix}1 & \frac{1}{2}\\ \frac{1}{2} & 1\end{vmatrix} > 0, \tag{3.4.1.3}$$

(3.4.1.1) is the equation of an ellipse. To verify that this is indeed the case, we do the following exercise. The characteristic equation of $\mathbf{V}$ is obtained by evaluating the determinant

$$\left|\lambda\mathbf{I} - \mathbf{V}\right| = \begin{vmatrix}\lambda - 1 & \frac{1}{2}\\ \frac{1}{2} & \lambda - 1\end{vmatrix} = 0 \tag{3.4.1.4}$$

$$\implies \lambda^2 - 2\lambda + \frac{3}{4} = 0 \tag{3.4.1.5}$$

The eigenvalues are the roots of (3.4.1.5) given by

$$\lambda_1 = \frac{1}{2}, \lambda_2 = \frac{3}{2} \tag{3.4.1.6}$$

The eigenvector $\mathbf{p}$ is defined as

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \tag{3.4.1.7}$$

$$\implies (\lambda\mathbf{I} - \mathbf{V})\,\mathbf{p} = 0 \tag{3.4.1.8}$$

where λ is the eigenvalue. For $\lambda_1 = \frac{3}{2}$,

$$(\lambda_1\mathbf{I} - \mathbf{V}) = \begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\\ -\frac{1}{2} & \frac{1}{2}\end{pmatrix} \xleftrightarrow[R_1\leftarrow 2R_1]{R_2\leftarrow R_2-R_1} \begin{pmatrix}1 & -1\\ 0 & 0\end{pmatrix} \tag{3.4.1.9}$$

$$\implies \mathbf{p}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \tag{3.4.1.10}$$

such that $\|\mathbf{p}_1\| = 1$. Similarly, the eigenvector corresponding to λ_2 can be obtained as

$$\mathbf{p}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\end{pmatrix} \tag{3.4.1.11}$$

It is easy to verify that

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1} = \mathbf{P}\mathbf{D}\mathbf{P}^T \quad \because \mathbf{P}^{-1} = \mathbf{P}^T \tag{3.4.1.12}$$

$$\text{or, } \mathbf{D} = \mathbf{P}^T\mathbf{V}\mathbf{P} \tag{3.4.1.13}$$

where

$$\mathbf{P} = \begin{pmatrix}\mathbf{p}_1 & \mathbf{p}_2\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix} \tag{3.4.1.14}$$

$$\mathbf{D} = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix} = \begin{pmatrix}\frac{3}{2} & 0\\ 0 & \frac{1}{2}\end{pmatrix} \tag{3.4.1.15}$$

From Table 3.7, ellipse parameters are given by

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} = \mathbf{0} \tag{3.4.1.16}$$

$$\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}} = 10\sqrt{\frac{2}{3}} \tag{3.4.1.17}$$

$$\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_2}} = 10\sqrt{2} \tag{3.4.1.18}$$

In Fig. 3.4.1 the actual ellipse ellipse in (3.4.1.1) is obtained from (3.1.3.5) using (3.1.3.1). The anticlockwise 45°rotation is due to the fact that (3.4.1.14) can be expressed as

$$\mathbf{P} = \begin{pmatrix}\cos 45^\circ & -\sin 45^\circ\\ \sin 45^\circ & \cos 45^\circ\end{pmatrix} \tag{3.4.1.19}$$

Coming back to the original question of finding $\frac{dy}{dx}$, if the point of contact

$$\mathbf{q} = \begin{pmatrix}q_1\\ q_2\end{pmatrix}, \tag{3.4.1.20}$$

from (3.4.1.2), (2.1.2.4) and (3.2.1),

$$\begin{pmatrix}1 & m\end{pmatrix}\begin{pmatrix}1 & \frac{1}{2}\\ \frac{1}{2} & 1\end{pmatrix}\begin{pmatrix}q_1\\ q_2\end{pmatrix} = 0 \tag{3.4.1.21}$$

$$\implies \begin{pmatrix}1 + \frac{m}{2} & \frac{1}{2} + m\end{pmatrix}\begin{pmatrix}q_1\\ q_2\end{pmatrix} = 0 \tag{3.4.1.22}$$

$$\implies \frac{m}{2}(q_1 + 2q_2) + q_1 + \frac{q_2}{2} = 0 \tag{3.4.1.23}$$

$$\text{or, } m = \frac{dy}{dx} = -\frac{2q_1 + q_2}{q_1 + 2q_2} \tag{3.4.1.24}$$

$\because \frac{dy}{dx}$ is the slope of the tangent. Note that no results from differential calculus were used to obtain (3.4.1.24).

3.4.2. Find the equation of the ellipse, with major axis along the x-axis and passing through the points $\mathbf{a} = \begin{pmatrix}4\\ 3\end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix}-1\\ 4\end{pmatrix}$

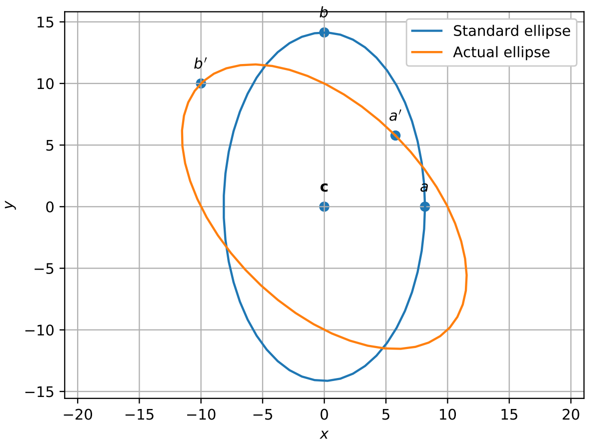

Fig. 3.4.1: Actual ellipse and transformed ellipse.

Solution: This is a standard ellipse given by

$$\mathbf{x}^T\mathbf{D}\mathbf{x} = 1, \quad \mathbf{D} = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}, \lambda_1, \lambda_2 > 0 \tag{3.4.2.1}$$

$\because$ $\mathbf{a}, \mathbf{b}$ satisfy (3.4.2.1),

$$\mathbf{a}^T\mathbf{D}\mathbf{a} = 1, \tag{3.4.2.2}$$

$$\mathbf{b}^T\mathbf{D}\mathbf{b} = 1 \tag{3.4.2.3}$$

which can be expressed as

$$\begin{aligned}\mathbf{a}^T\mathbf{A}\mathbf{d} &= 1,\\ \mathbf{b}^T\mathbf{B}\mathbf{d} &= 1\end{aligned} \tag{3.4.2.4}$$

where

$$\mathbf{d} = \begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix}, \mathbf{A} = \begin{pmatrix}4 & 0\\ 0 & 3\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-1 & 0\\ 0 & 4\end{pmatrix}. \tag{3.4.2.5}$$

(3.4.2.4) can then be expressed as the matrix equation

$$\begin{pmatrix}\mathbf{a}^T\mathbf{A}\\ \mathbf{b}^T\mathbf{B}\end{pmatrix}\mathbf{d} = \begin{pmatrix}1\\ 1\end{pmatrix}, \tag{3.4.2.6}$$

which, after substituing the appropriate values can be expressed as

$$\begin{pmatrix}16 & 9\\ 1 & 16\end{pmatrix}\mathbf{d} = \begin{pmatrix}1\\ 1\end{pmatrix}, \tag{3.4.2.7}$$

Forming the augmented matrix and performing row reduction,

$$\begin{pmatrix}16 & 9 & 1\\ 1 & 16 & 1\end{pmatrix} \xleftrightarrow[R_2\leftarrow -R_2]{R_2\leftarrow R_1} \begin{pmatrix}1 & 16 & 1\\ 0 & 247 & 15\end{pmatrix} \tag{3.4.2.8}$$

$$\xleftrightarrow{R_1\leftarrow 247R_1-16R_2} \begin{pmatrix}247 & 0 & 7\\ 0 & 247 & 15\end{pmatrix} \tag{3.4.2.9}$$

$$\implies \mathbf{d} = \frac{1}{247}\begin{pmatrix}7\\15\end{pmatrix}, \text{ or, } \mathbf{D} = \frac{1}{247}\begin{pmatrix}7 & 0\\ 0 & 15\end{pmatrix} \tag{3.4.2.10}$$

The ellipse parameters are obtained from Table 3.7 as

$$\mathbf{c} = \mathbf{0}, \frac{1}{\sqrt{\lambda_1}} = \sqrt{\frac{247}{7}}, \frac{1}{\sqrt{\lambda_2}} = \sqrt{\frac{247}{15}}. \tag{3.4.2.11}$$

Fig. 3.4.2 verifies the above results.

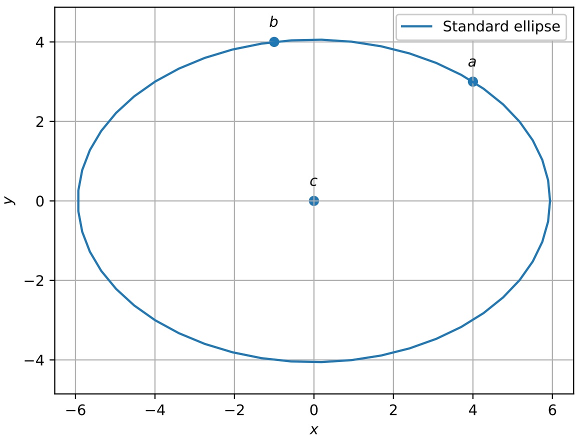

Fig. 3.4.2: Ellipse through the given points $\mathbf{a} = \begin{pmatrix}4\\3\end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix}-1\\4\end{pmatrix}$.

3.5 Hyperbola

3.5.1. Find the equation of all lines having slope 2 and being tangent to the curve

$$y + \frac{2}{x-3} = 0 \tag{3.5.1.1}$$

Solution: (3.5.1.1) can be expressed as

$$xy - 3y + 2 = 0 \tag{3.5.1.2}$$

which is of the same form as (3.1.2.1) with

$$\mathbf{V} = \frac{1}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \mathbf{u} = -\frac{3}{2}\begin{pmatrix}0\\1\end{pmatrix}, f = 2 \tag{3.5.1.3}$$

Using the approach in Example 3.4.1,

$$\mathbf{D} = \begin{pmatrix}\frac{1}{2} & 0\\ 0 & -\frac{1}{2}\end{pmatrix}, \mathbf{P} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix} \tag{3.5.1.4}$$

$$\because \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f = -2 < 0, \tag{3.5.1.5}$$

the major and minor axis are swapped and from Table 3.7 the hyperbola parameters are given by

$$\mathbf{c} = 3\begin{pmatrix}1\\0\end{pmatrix}, \sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_2}} = 2, \tag{3.5.1.6}$$

$$\sqrt{\frac{f - \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}}{\lambda_1}} = 2 \tag{3.5.1.7}$$

with the standard hyperbola equation becoming

$$\frac{y_2^2}{4} - \frac{y_1^2}{4} = 1, \tag{3.5.1.8}$$

Fig. 3.5.1 shows the actual hyperbola in (3.5.1.1) obtained from (3.5.1.8) using (3.1.3.1). The direction and normal vectors of the tangent with slope 2 are given by (2.1.2.4) and (2.1.4.1) as

$$\mathbf{m} = \begin{pmatrix}1\\2\end{pmatrix}, \mathbf{n} = \begin{pmatrix}2\\-1\end{pmatrix} \tag{3.5.1.9}$$

From (3.5.2) and (3.3.3.3), using (3.5.1.3),

$$\kappa = \frac{1}{2}, \mathbf{q}_1 = \begin{pmatrix}2\\2\end{pmatrix}, \mathbf{q}_2 = \begin{pmatrix}4\\-2\end{pmatrix}. \tag{3.5.1.10}$$

The desired tangents are

$$\begin{pmatrix}2 & -1\end{pmatrix}\left\{\mathbf{x} - \begin{pmatrix}2\\2\end{pmatrix}\right\} = 0 \implies \begin{pmatrix}2 & -1\end{pmatrix}\mathbf{x} = 2 \tag{3.5.1.11}$$

$$\begin{pmatrix}2 & -1\end{pmatrix}\left\{\mathbf{x} - \begin{pmatrix}4\\-2\end{pmatrix}\right\} = 0 \implies \begin{pmatrix}2 & -1\end{pmatrix}\mathbf{x} = 10 \tag{3.5.1.12}$$

All the above results are verified in Fig. 3.5.1. As we can see, the hyperbola in (3.5.1.1) is obtained by rotating the standard hyperbola by **P** and then translating it by **c**.

3.5.2. Find the asymptotes of the hyperbola given

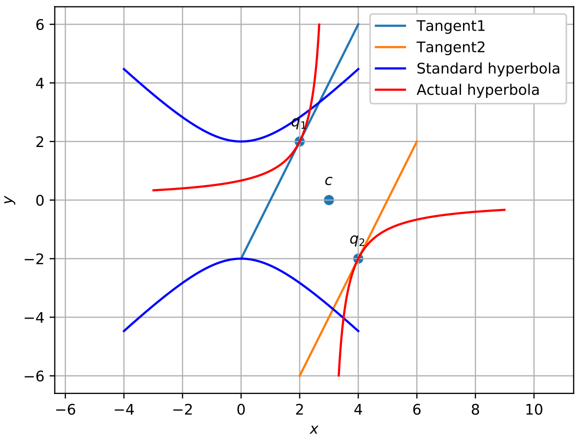

Fig. 3.5.1: Standard and actual hyperbola.

below and also the equations to their conjugate hyperbolas.

$$8x^2 + 10xy - 3y^2 - 2x + 4y - 2 = 0 \tag{3.5.2.1}$$

a) (3.5.2.1) can be expressed as (3.1.2.1) with

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 8 & 5 \\ 5 & -3 \end{pmatrix} \tag{3.5.2.2}$$

$$\mathbf{u} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \tag{3.5.2.3}$$

$$f = -2 \tag{3.5.2.4}$$

Expanding the Determinant of $\mathbf{V}$,

$$\Delta_V = \begin{vmatrix} 8 & 5 \\ 5 & -3 \end{vmatrix} = -49 < 0 \tag{3.5.2.5}$$

Hence from (3.1.7.1) and (3.5.2.5), (3.5.2.1) represents a hyperbola. The characteristic equation of $\mathbf{V}$ is obtained by evaluating the determinant

$$\left| V - \lambda\mathbf{I} \right| = 0 \tag{3.5.2.6}$$

$$\begin{vmatrix} 8 - \lambda & 5 \\ 5 & -3 - \lambda \end{vmatrix} = 0 \tag{3.5.2.7}$$

$$(8 - \lambda)(-3 - \lambda) - 25 = 0 \tag{3.5.2.8}$$

$$\implies \lambda^2 - 5\lambda - 49 = 0 \tag{3.5.2.9}$$

$$\lambda_1 = \frac{5 + \sqrt{221}}{2} \tag{3.5.2.10}$$

$$\lambda_2 = \frac{5 - \sqrt{221}}{2} \tag{3.5.2.11}$$

The eigenvector $\mathbf{p}$ is defined as

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \tag{3.5.2.12}$$

$$\implies (\mathbf{V} - \lambda\mathbf{I})\mathbf{p} = 0 \tag{3.5.2.13}$$

For $\lambda_1 = \frac{5+\sqrt{221}}{2}$,

$$(\mathbf{V} - \lambda_1\mathbf{I}) = \begin{pmatrix} \frac{11-\sqrt{221}}{2} & 5 \\ 5 & \frac{-11-\sqrt{221}}{2} \end{pmatrix} \tag{3.5.2.14}$$

By row reduction,

$$\begin{pmatrix} \frac{11-\sqrt{221}}{2} & 5 \\ 5 & \frac{-11-\sqrt{221}}{2} \end{pmatrix} \tag{3.5.2.15}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 + \frac{11+\sqrt{221}}{10} R_1} \begin{pmatrix} 5 & \frac{-11-\sqrt{221}}{2} \\ 0 & 0 \end{pmatrix} \tag{3.5.2.16}$$

Substituting (3.5.2.16) in (3.5.2.13) we get

$$\begin{pmatrix} 5 & \frac{-11-\sqrt{221}}{2} \end{pmatrix} \mathbf{p}_1 = \mathbf{0} \tag{3.5.2.17}$$

$$\implies \mathbf{p}_1 = k \begin{pmatrix} \frac{11+\sqrt{221}}{2} \\ 5 \end{pmatrix} \tag{3.5.2.18}$$

For $\lambda_2 = \frac{5-\sqrt{221}}{2}$,

$$(\mathbf{V} - \lambda_2\mathbf{I}) = \begin{pmatrix} \frac{11+\sqrt{221}}{2} & 5 \\ 5 & \frac{-11+\sqrt{221}}{2} \end{pmatrix} \tag{3.5.2.19}$$

By row reduction ,

$$\begin{pmatrix} \frac{11+\sqrt{221}}{2} & 5 \\ 5 & \frac{-11+\sqrt{221}}{2} \end{pmatrix} \tag{3.5.2.20}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 + \frac{11-\sqrt{221}}{10} R_1} \begin{pmatrix} \frac{11+\sqrt{221}}{2} & 5 \\ 0 & 0 \end{pmatrix} \tag{3.5.2.21}$$

Substiuting (3.5.2.21) in (3.5.2.13) we get

$$\begin{pmatrix} \frac{11+\sqrt{221}}{2} & 5 \end{pmatrix} \mathbf{p}_2 = \mathbf{0} \tag{3.5.2.22}$$

$$\implies \mathbf{p}_2 = k \begin{pmatrix} -5 \\ \frac{11+\sqrt{221}}{2} \end{pmatrix} \tag{3.5.2.23}$$

Thus, we obtain

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix} = k \begin{pmatrix} \frac{11+\sqrt{221}}{2} & -5 \\ 5 & \frac{11+\sqrt{221}}{2} \end{pmatrix} \tag{3.5.2.24}$$

For

$$\mathbf{P}^\top\mathbf{P} = \mathbf{I}, k = \sqrt{\frac{221 + 11\sqrt{221}}{2}} \tag{3.5.2.25}$$

$$\implies \mathbf{P} = \sqrt{\frac{2}{221 + 11\sqrt{221}}} \times \begin{pmatrix} \frac{11+\sqrt{221}}{2} & -5 \\ 5 & \frac{11+\sqrt{221}}{2} \end{pmatrix} \quad (3.5.2.26)$$

and

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T \quad (3.5.2.27)$$

where

$$\mathbf{D} = \begin{pmatrix} \frac{5+\sqrt{221}}{2} & 0 \\ 0 & \frac{5-\sqrt{221}}{2} \end{pmatrix} \quad (3.5.2.28)$$

Centre of the hyperbola is given by

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \quad (3.5.2.29)$$

$$\implies \mathbf{c} = -\begin{pmatrix} \frac{3}{49} & \frac{5}{49} \\ \frac{5}{49} & \frac{-8}{49} \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad (3.5.2.30)$$

$$\implies \mathbf{c} = \begin{pmatrix} \frac{-3}{49} & \frac{-5}{49} \\ \frac{-5}{49} & \frac{8}{49} \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad (3.5.2.31)$$

$$\implies \mathbf{c} = \begin{pmatrix} \frac{-1}{7} \\ \frac{3}{7} \end{pmatrix} \quad (3.5.2.32)$$

Since,

$$\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f = 1 > 0 \quad (3.5.2.33)$$

there isn’t a need to swap axes In hyperbola,

$$axes = \begin{cases} \sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}} \\ \sqrt{\frac{f - \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}}{\lambda_2}} \end{cases} \quad (3.5.2.34)$$

From above equations we can say that,

$$\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}} = \sqrt{\frac{2}{5 + \sqrt{221}}} \quad (3.5.2.35)$$

$$\sqrt{\frac{f - \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}}{\lambda_2}} = \sqrt{\frac{2}{5 - \sqrt{221}}} \quad (3.5.2.36)$$

The equation of the hyperbola at the origin is then given by (3.1.3.5) as

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f \quad (3.5.2.37)$$

$$\implies \mathbf{y}^T \begin{pmatrix} \frac{5+\sqrt{221}}{2} & 0 \\ 0 & \frac{5-\sqrt{221}}{2} \end{pmatrix} \mathbf{y} = 1 \quad (3.5.2.38)$$

b) (*Asymptotes of hyperbola:*) The equation for the asymptotes of (3.5.2.1) is given by (3.1.8.1) with

$$K = \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} \quad (3.5.2.39)$$

$$= \begin{pmatrix} -1 & 2 \end{pmatrix}\begin{pmatrix} 8 & 5 \\ 5 & -3 \end{pmatrix}^{-1}\begin{pmatrix} -1 \\ 2 \end{pmatrix} = -1 \quad (3.5.2.40)$$

From (3.1.8.5), (3.5.2.10) and (3.5.2.11),

$$\mathbf{n}_1 = \sqrt{\frac{2}{221 + 11\sqrt{221}}} \times \begin{pmatrix} \frac{11+\sqrt{221}}{2} & 5 \\ -5 & \frac{11+\sqrt{221}}{2} \end{pmatrix} \times \begin{pmatrix} \sqrt{\frac{\sqrt{221}+5}{2}} \\ \sqrt{\frac{\sqrt{221}-5}{2}} \end{pmatrix} \quad (3.5.2.41)$$

which can be expressed as

$$\mathbf{n}_1 = \frac{1}{\sqrt{(\lambda + 3)^2 + 5^2}} \times \begin{pmatrix} \lambda_1 + 3 & 5 \\ -5 & \lambda_1 + 3 \end{pmatrix}\begin{pmatrix} \sqrt{\lambda_1} \\ \frac{7}{\sqrt{\lambda_1}} \end{pmatrix} \quad (3.5.2.42)$$

which is equivalent to

$$\mathbf{n}_1 = \begin{pmatrix} \lambda_1 + 3 & 5 \\ -5 & \lambda_1 + 3 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ 7 \end{pmatrix} \quad (3.5.2.43)$$

$$= \begin{pmatrix} \lambda_1^2 + 3\lambda_1 + 35 \\ 2\lambda_1 + 21 \end{pmatrix} \quad (3.5.2.44)$$

$$= \begin{pmatrix} 8\lambda_1 + 84 \\ 2\lambda_1 + 21 \end{pmatrix} \quad (3.5.2.45)$$

using (3.5.2.9), which is equivalent to

$$\mathbf{n}_1 = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \quad (3.5.2.46)$$

Similarly, it can be shown that

$$\mathbf{n_2} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad (3.5.2.47)$$

Fig. 3.5.2 plots the hyperbola in (3.5.2.1) along with the asymptotes obtained using (3.1.8.4).

3.6 Parabola

3.6.1. Find the point at which the tangent to the curve

$$y = \sqrt{4x - 3} - 1 \quad (3.6.1.1)$$

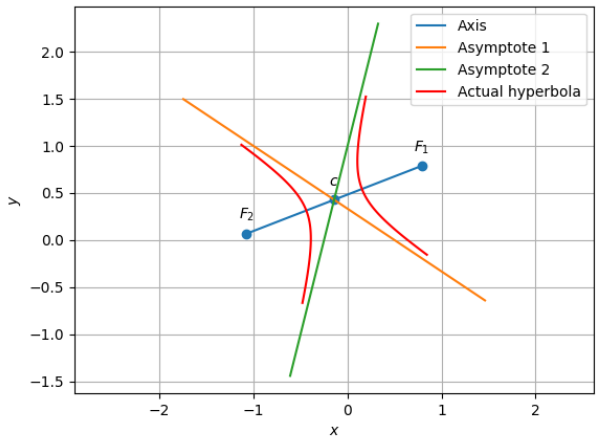

Fig. 3.5.2: Hyperbola with assymptotes and its conjugate

has slope $\frac{2}{3}$.
Solution: (3.6.1.1) can be expressed as

$$(y+1)^2 = 4x - 3 \tag{3.6.1.2}$$

$$\text{or, } y^2 - 4x + 2y + 4 = 0 \tag{3.6.1.3}$$

which has the form (3.1.2.1) with parameters

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, f = 4. \tag{3.6.1.4}$$

Thus, the given curve is a parabola. $\because$ $\mathbf{V}$ is diagonal and in standard form,

$$\mathbf{P} = \mathbf{I} \implies \mathbf{p}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.6.1.5}$$

From Table 3.7, the focus is 4 and the vertex $\mathbf{c}$ is

$$\begin{pmatrix} -4 & 1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -4 \\ 0 \\ -1 \end{pmatrix} \tag{3.6.1.6}$$

$$\implies \begin{pmatrix} -4 & 1 \\ 0 & 1 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -4 \\ -1 \end{pmatrix} \tag{3.6.1.7}$$

$$\text{or, } \mathbf{c} = \begin{pmatrix} \frac{3}{4} \\ -1 \end{pmatrix} \tag{3.6.1.8}$$

The direction vector and normal vectors are

$$\mathbf{m} = \begin{pmatrix} 1 \\ \frac{2}{3} \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \mathbf{n} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}. \tag{3.6.1.9}$$

Also,

$$\mathbf{V}\mathbf{p} = \mathbf{0} \tag{3.6.1.10}$$

$$\implies \mathbf{p} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.6.1.11}$$

From (3.6.2), (3.6.1.9) and (3.6.1.11),

$$\kappa = -1 \tag{3.6.1.12}$$

which, upon substitution in (3.6.1) and simplification yields the matrix equation

$$\begin{pmatrix} -4 & 4 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{q} = \begin{pmatrix} -4 \\ 0 \\ 2 \end{pmatrix} \tag{3.6.1.13}$$

$$\implies \begin{pmatrix} -4 & 4 \\ 0 & 1 \end{pmatrix} \mathbf{q} = \begin{pmatrix} -4 \\ 2 \end{pmatrix} \tag{3.6.1.14}$$

$$\text{or, } \mathbf{q} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \tag{3.6.1.15}$$

Fig. 3.6.1 verifies the above results.

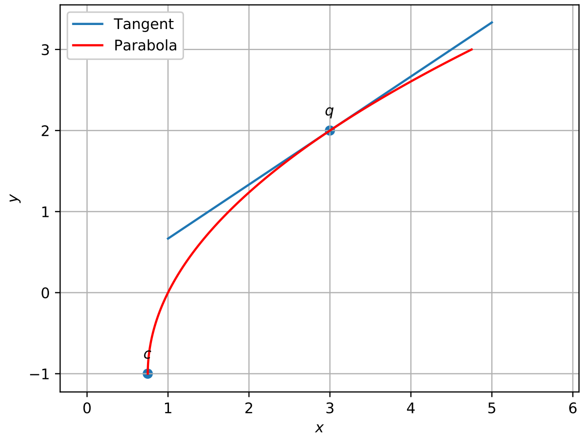

Fig. 3.6.1: Tangent to parabola in (3.6.1.1) with slope $\frac{2}{3}$.

3.6.2. Find a point on the curve

$$y = (x-2)^2 \tag{3.6.2.1}$$

at which the tangent is parallel to the chord joining the points (2, 0) and (4, 4).
Solution: (3.6.2.1) can be expressed as

$$x^2 - 4x - y + 4 = 0 \tag{3.6.2.2}$$

which has the form (3.1.2.1) with parameters

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = -\begin{pmatrix} 2 \\ \frac{1}{2} \end{pmatrix}, f = 4. \tag{3.6.2.3}$$

Using eigenvalue decomposition,

$$\mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.6.2.4}$$

Hence, the eigenvector of $\mathbf{V}$ corresponding to the zero eigenvalue is

$$\mathbf{p}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{3.6.2.5}$$

Substituting the above parameters in the equation for the vertex of the parabola in Table 3.7,

$$\begin{pmatrix} -2 & -\frac{5}{2} \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -4 \\ 2 \\ 0 \end{pmatrix} \tag{3.6.2.6}$$

$$\implies \begin{pmatrix} -1 & -\frac{5}{2} \\ 1 & 0 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -4 \\ 2 \end{pmatrix} \tag{3.6.2.7}$$

$$\text{or, } \mathbf{c} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \tag{3.6.2.8}$$

The direction vector is

$$\mathbf{m} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{3.6.2.9}$$

and normal vector is

$$\mathbf{n} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \tag{3.6.2.10}$$

From the equation for the point of contact for the parabola in Table 3.7,

$$\kappa = \frac{1}{2} \tag{3.6.2.11}$$

resulting in the matrix equation

$$\begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{q} = \begin{pmatrix} -4 \\ 3 \\ 0 \end{pmatrix} \tag{3.6.2.12}$$

$$\implies \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{q} = \begin{pmatrix} -4 \\ 3 \end{pmatrix} \tag{3.6.2.13}$$

$$\text{or, } \mathbf{q} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \tag{3.6.2.14}$$

Fig. 3.6.2 verifies the above results. Note that $\mathbf{P}$ rotates the standard parabola by 90°.

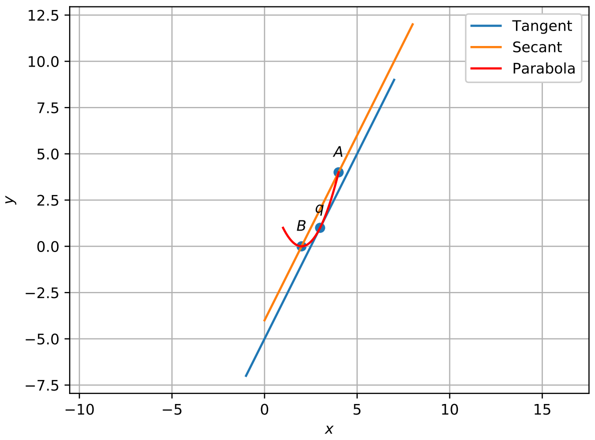

Fig. 3.6.2: Tangent to parabola in (3.6.2.1) is parallel to the line joining the points $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 4 \end{pmatrix}$.

3.6.3. What conic does the following equation represent.

$$9x^2 - 24xy + 16y^2 - 18x - 101y + 19 = 0 \tag{3.6.3.1}$$

Find the center.

Solution: From (3.6.3.1) and (3.1.2.1),

$$\mathbf{V} = \begin{pmatrix} 9 & -12 \\ -12 & 16 \end{pmatrix} \tag{3.6.3.2}$$

$$\mathbf{u} = \begin{pmatrix} -9 \\ -\frac{101}{2} \end{pmatrix} \tag{3.6.3.3}$$

$$f = 4 \tag{3.6.3.4}$$

a) Expanding the determinant of $\mathbf{V}$ we observe,

$$\begin{vmatrix} 9 & -12 \\ -12 & 16 \end{vmatrix} = 0 \tag{3.6.3.5}$$

Also

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = \begin{vmatrix} 9 & -12 & -9 \\ -12 & 16 & -\frac{101}{2} \\ -9 & -\frac{101}{2} & 4 \end{vmatrix} \tag{3.6.3.6}$$

$$\neq 0 \tag{3.6.3.7}$$

Hence from (3.6.3.5) and (3.6.3.7) we conclude that given equation is an parabola. The characteristic equation of $\mathbf{V}$ is given as follows,

$$\left|\lambda\mathbf{I} - \mathbf{V}\right| = \begin{vmatrix} \lambda - 9 & 12 \\ 12 & \lambda - 16 \end{vmatrix} = 0 \tag{3.6.3.8}$$

$$\implies \lambda^2 - 25\lambda = 0 \tag{3.6.3.9}$$

Hence the characteristic equation of $\mathbf{V}$ is given by (3.6.3.9). The roots of (3.6.3.9) i.e the eigenvalues are given by

$$\lambda_1 = 0, \lambda_2 = 25 \tag{3.6.3.10}$$

b) For $\lambda_1 = 0$, the eigen vector $\mathbf{p}$ is given by

$$\mathbf{V}\mathbf{p} = 0 \tag{3.6.3.11}$$

Row reducing $\mathbf{V}$ yields

$$\implies \begin{pmatrix} -9 & 12 \\ 12 & -16 \end{pmatrix} \xleftrightarrow[R_2 = R_2 + 4R_1]{R_1 = -\frac{R_1}{3}} \begin{pmatrix} 3 & -4 \\ 0 & 0 \end{pmatrix} \tag{3.6.3.12}$$

$$\implies \mathbf{p}_1 = \frac{1}{5}\begin{pmatrix} -4 \\ -3 \end{pmatrix} \tag{3.6.3.13}$$

Similarly,

$$\mathbf{p}_2 = \frac{1}{5}\begin{pmatrix} -3 \\ 4 \end{pmatrix} \tag{3.6.3.14}$$

Thus, the eigenvector rotation matrix and the eigenvalue matrix are

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -4 & -3 \\ -3 & 4 \end{pmatrix} \tag{3.6.3.15}$$

$$\mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & 25 \end{pmatrix} \tag{3.6.3.16}$$

Table 3.7, the focal length of the parabola is given by

$$\frac{\left|2\mathbf{u}^T\mathbf{p}_1\right|}{\lambda_2} = \frac{75}{25} = 3 \tag{3.6.3.17}$$

and its equation is

$$\mathbf{y}^{\mathbf{T}}\mathbf{D}\mathbf{y} = -2\eta\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{y} \tag{3.6.3.18}$$

where

$$\eta = \mathbf{u}^T\mathbf{p}_1 = \frac{75}{2} \tag{3.6.3.19}$$

$$\begin{pmatrix} \mathbf{u}^{\mathbf{T}} + \eta\mathbf{p}_1^{\mathbf{T}} \\ \mathbf{V} \end{pmatrix}\mathbf{c} = \begin{pmatrix} -f \\ \eta\mathbf{p}_1 - \mathbf{u} \end{pmatrix} \tag{3.6.3.20}$$

using equations (3.6.3.3),(3.6.3.4) and (3.6.3.13)

$$\begin{pmatrix} -39 & -73 \\ 9 & -12 \\ -12 & 16 \end{pmatrix}\mathbf{c} = \begin{pmatrix} -19 \\ -21 \\ 28 \end{pmatrix} \tag{3.6.3.21}$$

Forming the augmented matrix and row reducing it:

$$\begin{pmatrix} -39 & -73 & -19 \\ 9 & -12 & -21 \\ -12 & 16 & 28 \end{pmatrix} \tag{3.6.3.22}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 + (4/3)R_2} \begin{pmatrix} -39 & -73 & -19 \\ 9 & -12 & -21 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.6.3.23}$$

$$\xleftrightarrow{R_1 \leftarrow R_1/(-39)} \begin{pmatrix} 1 & 73/39 & 19/39 \\ 9 & -12 & -21 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.6.3.24}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - 9R_1} \begin{pmatrix} 1 & 73/39 & 19/39 \\ 0 & -1125/39 & -990/39 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.6.3.25}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 \times (-39/1125)} \begin{pmatrix} 1 & 73/39 & 19/39 \\ 0 & 1 & 22/25 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.6.3.26}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - (73/39)R_2} \begin{pmatrix} 1 & 0 & -29/25 \\ 0 & 1 & 22/25 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.6.3.27}$$

Thus the vertex $\mathbf{c}$ is:

$$\mathbf{c} = \begin{pmatrix} -29/25 \\ 22/25 \end{pmatrix} = \begin{pmatrix} -1.16 \\ 0.88 \end{pmatrix} \tag{3.6.3.28}$$

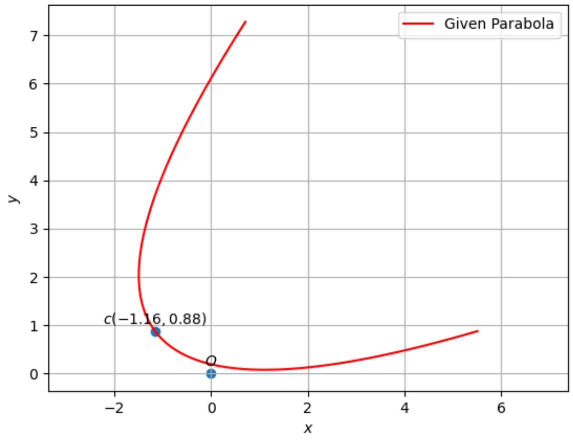

Fig. 3.6.3: Parabola with the center c

4 Matrix Decompositions

4.1 QR Decomposition

4.1.1. Revisiting Problem (2.1.15),

$$\alpha = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}, \beta = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} \tag{4.1.1.1}$$

we can express

$$\begin{aligned} \alpha &= k_1 \mathbf{u}_1 \\ \beta &= r_1 \mathbf{u}_1 + k_2 \mathbf{u}_2 \end{aligned} \tag{4.1.1.2}$$

where

$$k_1 = \left\|\alpha\right\|, \mathbf{u}_1 = \frac{\alpha}{k_1} \tag{4.1.1.3}$$

$$r_1 = \frac{\mathbf{u}_1^T \beta}{\left\|\mathbf{u}_1\right\|^2}, \mathbf{u}_2 = \frac{\beta - r_1 \mathbf{u}_1}{\left\|\beta - r_1 \mathbf{u}_1\right\|} \tag{4.1.1.4}$$

$$k_2 = \mathbf{u}_2^T \beta \tag{4.1.1.5}$$

From (4.1.1.2),

$$\begin{pmatrix} \alpha & \beta \end{pmatrix} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{pmatrix} \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \tag{4.1.1.6}$$

This is known as **QR** decomposition, where

$$\mathbf{R} = \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \tag{4.1.1.7}$$

$$\mathbf{Q} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{pmatrix} \tag{4.1.1.8}$$

Note that **R** is an upper triangular matrix and

$$\mathbf{Q}^T \mathbf{Q} = \mathbf{I}. \tag{4.1.1.9}$$

4.1.2. From (4.1.1.1),

$$k_1 = \sqrt{10}, \mathbf{u}_1 = \frac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}, \tag{4.1.2.1}$$

$$r_1 = \frac{1}{2}, \mathbf{u}_2 = \frac{1}{\sqrt{46}} \begin{pmatrix} 1 \\ 3 \\ -6 \end{pmatrix} \tag{4.1.2.2}$$

$$k_2 = \sqrt{\frac{23}{2}} \tag{4.1.2.3}$$

Thus, we obtain the **QR** decompositon

$$\begin{pmatrix} 3 & 2 \\ -1 & 1 \\ 0 & -3 \end{pmatrix} = \begin{pmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{46}} \\ \frac{-1}{\sqrt{10}} & \frac{3}{\sqrt{46}} \\ 0 & \frac{-6}{\sqrt{46}} \end{pmatrix} \begin{pmatrix} \sqrt{10} & \frac{1}{2} \\ 0 & \sqrt{\frac{23}{2}} \end{pmatrix} \tag{4.1.2.4}$$

4.2 Singular Value Decomposition

4.2.1. We revisit (2.3.2.6)

$$\begin{pmatrix} 1 & 2 \\ -1 & 1 \\ 1 & 2 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix} \tag{4.2.1.1}$$

4.2.2. Find $\mathbf{M}^T\mathbf{M}$ and $\mathbf{M}\mathbf{M}^T$.

4.2.3. Obtain the eigen decomposition

$$\mathbf{M}^T\mathbf{M} = \mathbf{P}_1 \mathbf{D}_1 \mathbf{P}_1^T \tag{4.2.3.1}$$

and

$$\mathbf{M}\mathbf{M}^T = \mathbf{P}_2 \mathbf{D}_2 \mathbf{P}_2^T \tag{4.2.3.2}$$

4.2.4. Perform the *QR* decompositions

$$\mathbf{P}_1 = \mathbf{U}\mathbf{R}_1 \mathbf{P}_2 = \mathbf{V}\mathbf{R}_2 \tag{4.2.4.1}$$

4.2.5. The singular value decomposition is the given by

$$\mathbf{M} = \mathbf{U}\Sigma\mathbf{V}^T, \tag{4.2.5.1}$$

where Σ has the same shape as **M** and

$$\Sigma = \begin{pmatrix} \mathbf{D}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \tag{4.2.5.2}$$

4.2.6. (2.3.2.6) can then be expressed as

$$\mathbf{U}\Sigma\mathbf{V}^T\mathbf{x} = \mathbf{b} \tag{4.2.6.1}$$

$$\implies \mathbf{x} = \mathbf{V}\Sigma^{-1}\mathbf{U}^T\mathbf{b} \tag{4.2.6.2}$$

where $\mathbf{\Sigma}^{-1}$ is obtained by inverting only the non-zero elements of $\mathbf{\Sigma}$.

4.2.7. The relevant codes are available at

codes/line/skew_builtin.py
codes/line/skew_svd.py

5 Optimization

5.1 Introduction

5.1.1. Express the problem of finding the distance of the point $\mathbf{P} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$ from the line

$$L: \quad \begin{pmatrix} 3 & -4 \end{pmatrix} \mathbf{x} = 26 \tag{5.1.1.1}$$

as an optimization problem.

Solution: The given problem can be expressed as

$$\min_{\mathbf{x}} g(\mathbf{x}) = \|\mathbf{x} - \mathbf{P}\|^2 \tag{5.1.1.2}$$

$$\text{s.t.} \quad \mathbf{n}^T\mathbf{x} = c \tag{5.1.1.3}$$

where

$$\mathbf{n} = \begin{pmatrix} 3 \\ -4 \end{pmatrix} \tag{5.1.1.4}$$

$$c = 26 \tag{5.1.1.5}$$

5.1.2. Explain Problem 5.1.1 through a plot and find a graphical solution.

5.1.3. Solve (5.1.1.2) using cvxpy.
Solution: The following code yields

codes/opt/line_dist_cvx.py

$$\mathbf{x}_{\min} = \begin{pmatrix} 2.64 \\ -4.52 \end{pmatrix}, \tag{5.1.3.1}$$

$$g\left(\mathbf{x}_{\min}\right) = 0.6 \tag{5.1.3.2}$$

5.1.4. Convert (5.1.1.2) to an *unconstrained* optimization problem.
Solution: L in (5.1.1.1) can be expressed in terms of the direction vector $\mathbf{m}$ as

$$\mathbf{x} = \mathbf{A} + \lambda\mathbf{m}, \tag{5.1.4.1}$$

where $\mathbf{A}$ is any point on the line and

$$\mathbf{m}^T\mathbf{n} = 0 \tag{5.1.4.2}$$

Substituting (5.1.4.1) in (5.1.1.2), an unconstrained optimization problem

$$\min_{\lambda} f(\lambda) = \|\mathbf{A} + \lambda\mathbf{m} - \mathbf{P}\|^2 \tag{5.1.4.3}$$

is obtained.

5.1.5. Solve (5.1.4.3).
Solution:

$$f(\lambda) = (\lambda\mathbf{m} + \mathbf{A} - \mathbf{P})^T (\lambda\mathbf{m} + \mathbf{A} - \mathbf{P}) \tag{5.1.5.1}$$

$$\begin{aligned} &= \lambda^2 \|\mathbf{m}\|^2 + 2\lambda\mathbf{m}^T (\mathbf{A} - \mathbf{P}) \\ &\quad + \|\mathbf{A} - \mathbf{P}\|^2 \end{aligned} \tag{5.1.5.2}$$

$$\because f^{(2)}\lambda = 2\|\mathbf{m}\|^2 > 0 \tag{5.1.5.3}$$

the minimum value of $f(\lambda)$ is obtained when

$$f^{(1)}(\lambda) = 2\lambda\|\mathbf{m}\|^2 + 2\mathbf{m}^T (\mathbf{A} - \mathbf{P}) = 0 \tag{5.1.5.4}$$

$$\implies \lambda_{\min} = -\frac{\mathbf{m}^T (\mathbf{A} - \mathbf{P})}{\|\mathbf{m}\|^2} \tag{5.1.5.5}$$

Choosing $\mathbf{A}$ such that

$$\mathbf{m}^T (\mathbf{A} - \mathbf{P}) = 0, \tag{5.1.5.6}$$

substituting in (5.1.5.5),

$$\lambda_{\min} = 0 \quad \text{and} \tag{5.1.5.7}$$

$$\mathbf{A} - \mathbf{P} = \mu\mathbf{n} \tag{5.1.5.8}$$

for some constant μ. (5.1.5.8) is a consequence of (5.1.4.2) and (5.1.5.6). Also, from (5.1.5.8),

$$\mathbf{n}^T (\mathbf{A} - \mathbf{P}) = \mu\|\mathbf{n}\|^2 \tag{5.1.5.9}$$

$$\implies \mu = \frac{\mathbf{n}^T\mathbf{A} - \mathbf{n}^T\mathbf{P}}{\|\mathbf{n}\|^2} = \frac{c - \mathbf{n}^T\mathbf{P}}{\|\mathbf{n}\|^2} \tag{5.1.5.10}$$

from (5.1.1.3). Substituting $\lambda_{\min} = 0$ in (5.1.4.3),

$$\min_{\lambda} f(\lambda) = \|\mathbf{A} - \mathbf{P}\|^2 = \mu^2 \|\mathbf{n}\|^2 \tag{5.1.5.11}$$

upon substituting from (5.1.5.8). The distance between $\mathbf{P}$ and L is then obtained from (5.1.5.11) as

$$\|\mathbf{A} - \mathbf{P}\| = |\mu| \|\mathbf{n}\| \tag{5.1.5.12}$$

$$= \frac{\left|\mathbf{n}^T\mathbf{P} - c\right|}{\|\mathbf{n}\|} \tag{5.1.5.13}$$

after substituting for μ from (5.1.5.10). Using the corresponding values from Problem (5.1.1) in (5.1.5.13),

$$\min_{\lambda} f(\lambda) = 0.6 \tag{5.1.5.14}$$

5.2 Convex Function

5.2.1. The following python script plots

$$f(\lambda) = a\lambda^2 + b\lambda + d \tag{5.2.1.1}$$

for

$$a = \|\mathbf{m}\|^2 > 0 \tag{5.2.1.2}$$

$$b = \mathbf{m}^T (\mathbf{A} - \mathbf{P}) \tag{5.2.1.3}$$

$$c = \|\mathbf{A} - \mathbf{P}\|^2 \tag{5.2.1.4}$$

where **A** is the intercept of the line L in (5.1.1.1) on the x-axis and the points

$$\mathbf{U} = \begin{pmatrix} \lambda_1 \\ f(\lambda_1) \end{pmatrix}, \mathbf{V} = \begin{pmatrix} \lambda_2 \\ f(\lambda_2) \end{pmatrix} \tag{5.2.1.5}$$

$$\mathbf{X} = \begin{pmatrix} t\lambda_1 + (1-t)\lambda_2 \\ f\left[t\lambda_1 + (1-t)\lambda_2\right] \end{pmatrix}, \tag{5.2.1.6}$$

$$\mathbf{Y} = \begin{pmatrix} t\lambda_1 + (1-t)\lambda_2 \\ tf(\lambda_1) + (1-t)f(\lambda_2) \end{pmatrix} \tag{5.2.1.7}$$

for

$$\lambda_1 = -3, \lambda_2 = 4, t = 0.3 \tag{5.2.1.8}$$

in Fig. 5.2.1. Geometrically, this means that any point **Y** between the points **U**, **V** on the line UV is always above the point **X** on the curve $f(\lambda)$. Such a function f is defined to be *convex* function

codes/opt/1.2.py

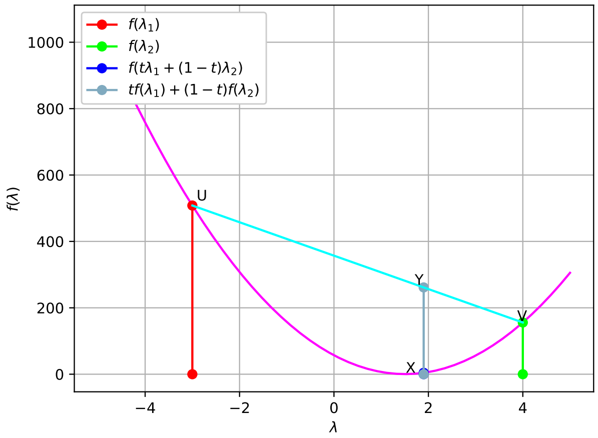

Fig. 5.2.1: $f(\lambda)$ versus λ

.

5.2.2. Show that

$$f\left[t\lambda_1 + (1-t)\lambda_2\right] \le tf(\lambda_1) + (1-t)f(\lambda_2) \tag{5.2.2.1}$$

for $0 < t < 1$. This is true for any convex function.

5.2.3. Show that

$$(5.2.2.1) \implies f^{(2)}(\lambda) > 0 \tag{5.2.3.1}$$

5.2.4. Show that a covex function has a unique minimum.

5.3 *Gradient Descent*

5.3.1. Find a numerical solution for (3.6.1.1)

Solution: A numerical solution for (3.6.1.1) is obtained as

$$\lambda_{n+1} = \lambda_n - \mu f'(\lambda_n) \tag{5.3.1.1}$$

$$= \lambda_n - \mu(2a\lambda_n + b) \tag{5.3.1.2}$$

where λ_0 is an inital guess and μ is a variable parameter. The choice of these parameters is very important since they decide how fast the algorithm converges.

5.3.2. Write a program to implement (5.3.1.2).

Solution: Download and execute

codes/opt/gd.py

5.3.3. Find a closed form solution for λ_n in (5.3.1.2) using the one sided Z transform.

5.3.4. Find the condition for which (5.3.1.2) converges, i.e.

$$\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0 \tag{5.3.4.1}$$

5.4 *Lagrange Multipliers*

5.4.1. Find

$$\min_{\mathbf{x}} g(\mathbf{x}) = \|\mathbf{x} - \mathbf{P}\|^2 = r^2 \tag{5.4.1.1}$$

$$\text{s.t.} \quad h(\mathbf{x}) = \mathbf{n}^T\mathbf{x} - c = 0 \tag{5.4.1.2}$$

by plotting the circles $g(\mathbf{x})$ for different values of r along with the line $g(\mathbf{x})$.

Solution: The following code plots Fig. 5.4.1

codes/opt/concirc.py

5.4.2. By solving the quadratic equation obtained from (5.4.1.1), show that

$$\min_{\mathbf{x}} r = \frac{3}{5}, \mathbf{x}_{\min} = \mathbf{Q} = \begin{pmatrix} 2.64 \\ -4.52 \end{pmatrix} \tag{5.4.2.1}$$

In Fig. 5.4.1, it can be seen that **Q** is the point of contact of the line L with the circle of minimum radius $r = \frac{3}{5}$.

5.4.3. Show that

$$\nabla h(\mathbf{x}) = \begin{pmatrix} 3 \\ -4 \end{pmatrix} = \mathbf{n} \tag{5.4.3.1}$$

where

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \end{pmatrix} \tag{5.4.3.2}$$

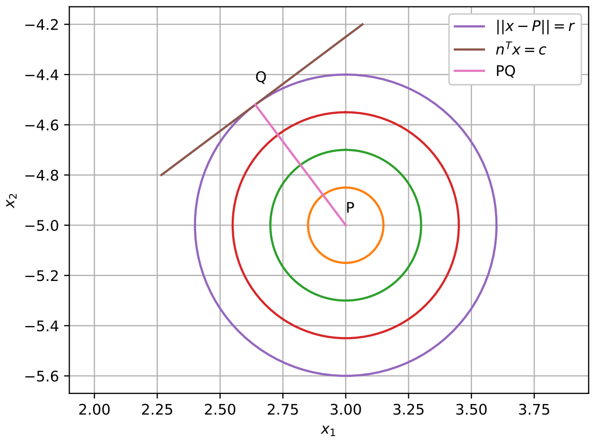

Fig. 5.4.1: Finding $\min_{\mathbf{x}} g(\mathbf{x})$

.

5.4.4. Show that

$$\nabla g(\mathbf{x}) = 2\left\{\mathbf{x} - \begin{pmatrix} 3 \\ -5 \end{pmatrix}\right\} = 2\{\mathbf{x} - \mathbf{P}\} \quad (5.4.4.1)$$

5.4.5. From Fig. 5.4.1, show that

$$\nabla g(\mathbf{Q}) = \lambda \nabla h(\mathbf{Q}), \quad (5.4.5.1)$$

Solution: In Fig. 5.4.1, PQ is the normal to the line L, represented by $h(\mathbf{x})$. $\because$ the normal vector of L is in the same direction as PQ, for some constant k,

$$(\mathbf{Q} - \mathbf{P}) = k\mathbf{n} \quad (5.4.5.2)$$

which is the same as (5.4.5.1) after substituting from (5.4.3.1). and (5.4.4.1).

5.4.6. Use (5.4.5.1) and $\mathbf{h}(\mathbf{Q}) = 0$ from (5.4.1.2) to obtain $\mathbf{Q}$.
Solution: From the given equations, we obtain

$$(\mathbf{Q} - \mathbf{P}) - \lambda\mathbf{n} = 0 \quad (5.4.6.1)$$

$$\mathbf{n}^T\mathbf{Q} - c = 0 \quad (5.4.6.2)$$

which can be simplifed to obtain

$$\begin{pmatrix} \mathbf{I} & -\mathbf{n} \\ \mathbf{n}^T & 0 \end{pmatrix}\begin{pmatrix} \mathbf{Q} \\ \lambda \end{pmatrix} = \begin{pmatrix} \mathbf{P} \\ c \end{pmatrix} \quad (5.4.6.3)$$

The following code computes the solution to (5.4.6.3)

```
codes/opt/lagmul.py
```

5.4.7. Define

$$C(\mathbf{x}, \lambda) = g(\mathbf{x}) - \lambda h(\mathbf{x}) \quad (5.4.7.1)$$

and show that $\mathbf{Q}$ can also be obtained by solving the equations

$$\nabla C(\mathbf{x}, \lambda) = 0. \quad (5.4.7.2)$$

What is the sign of λ? C is known as the Lagrangian and the above technique is known as the Method of Lagrange Multipliers.

5.4.8. Obtain $\mathbf{Q}$ using gradient descent.
Solution:

```
codes/opt/gd_lagrange.py
```

5.5 Quadratic Programming

5.5.1. An apache helicopter of the enemy is flying along the curve given by

$$y = x^2 + 7 \quad (5.5.1.1)$$

A soldier, placed at

$$\mathbf{P} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}. \quad (5.5.1.2)$$

wants to shoot the heicopter when it is nearest to him. Express this as an optimization problem.
Solution: The given problem can be expressed as

$$\min_{\mathbf{x}} \|\mathbf{x} - \mathbf{P}\|^2 \quad (5.5.1.3)$$

$$\text{s.t. } \mathbf{x}^T\mathbf{V}\mathbf{x} + \mathbf{u}^T\mathbf{x} + d = 0 \quad (5.5.1.4)$$

where

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (5.5.1.5)$$

$$\mathbf{u} = -\begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (5.5.1.6)$$

$$d = 7 \quad (5.5.1.7)$$

5.5.2. Show that the constraint in 5.5.1.3 is nonconvex.

5.5.3. Show that the following *relaxation* makes (5.5.1.3) a convex optimization problem.

$$\min_{\mathbf{x}} (\mathbf{x} - \mathbf{P})^T(\mathbf{x} - \mathbf{P}) \quad (5.5.3.1)$$

$$\text{s.t. } \mathbf{x}^T\mathbf{V}\mathbf{x} + \mathbf{u}^T\mathbf{x} \le 0 \quad (5.5.3.2)$$

5.5.4. Solve (5.5.3.1) using cvxpy.
Solution: The following code yields the min-

imum distance as 2.236 and the nearest point on the curve as

$$\mathbf{Q} = \begin{pmatrix} 1 \\ 8 \end{pmatrix} \tag{5.5.4.1}$$

```
codes/opt/qp_cvx.py
```

5.5.5. Solve (5.5.3.1) using the method of Lagrange multipliers.

5.5.6. Graphically verify the solution to Problem 5.5.1.
Solution: The following code plots Fig. 5.5.6

```
codes/opt/qp_parab.py
```

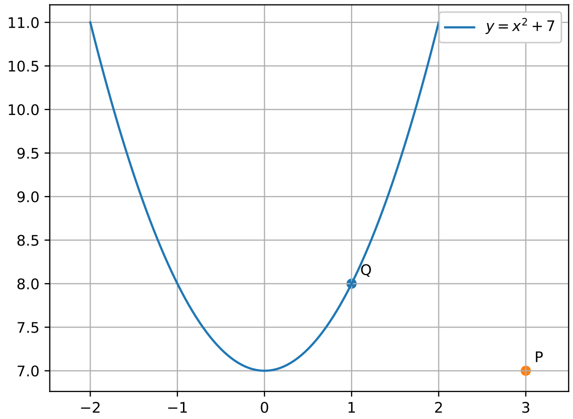

Fig. 5.5.6: **Q** is closest to **P**

.

5.5.7. Solve (5.5.3.1) using gradient descent.

5.6 Semi Definite Programming

5.6.1. Express the problem of finding the point on the curve

$$x^2 = 2y \tag{5.6.1.1}$$

nearest to the point

$$\mathbf{P} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}. \tag{5.6.1.2}$$

as an optimization problem.
Solution: The given problem can be expressed as

$$\min_{\mathbf{x}} \mathbf{x}^T \mathbf{Q}_0 \mathbf{x} + \mathbf{q}_0^T \mathbf{x} + c_0 \tag{5.6.1.3}$$

$$\text{s.t. } \mathbf{x}^T \mathbf{Q}_1 \mathbf{x} + \mathbf{q}_1^T \mathbf{x} + c_1 \le 0 \tag{5.6.1.4}$$

where

$$\mathbf{Q}_0 = \mathbf{I}, \mathbf{Q}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{5.6.1.5}$$

$$\mathbf{q}_0 = -2\mathbf{P}, \mathbf{q}_1 = -2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{5.6.1.6}$$

$$c_0 = \|\mathbf{P}\|^2, c_1 = 0 \tag{5.6.1.7}$$

5.6.2. Show that (5.6.1.3) is equivalent to

$$\min_{\mathbf{x},\theta} \theta$$

$$\text{s.t. } \begin{pmatrix} \mathbf{I} & \mathbf{M}_0 \mathbf{x} \\ \mathbf{x}^T \mathbf{M}_0^T & -c_0 - q_0^T \mathbf{x} + \theta \end{pmatrix} \ge 0 \tag{5.6.2.1}$$

$$\begin{pmatrix} \mathbf{I} & \mathbf{M}_1 \mathbf{x} \\ \mathbf{x}^T \mathbf{M}_1^T & -c_1 - q_1^T \mathbf{x} \end{pmatrix} \ge 0$$

where

$$\mathbf{Q}_i = \mathbf{M}_i^T \mathbf{M}_i, i = 0, 1 \tag{5.6.2.2}$$

5.6.3. Solve (5.6.2.1) using cvxpy.

5.6.4. Graphically verify the solution to Problem 5.6.1.

5.6.5. Solve (5.6.1.3) using the method of Lagrange multipliers.

5.7 Linear Programming

5.7.1. Solve

$$\max_{\mathbf{x}} Z = \begin{pmatrix} 4 & 1 \end{pmatrix} \mathbf{x} \tag{5.7.1.1}$$

$$s.t. \quad \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix} \mathbf{x} \le \begin{pmatrix} 50 \\ 90 \end{pmatrix} \tag{5.7.1.2}$$

$$\mathbf{x} \ge \mathbf{0} \tag{5.7.1.3}$$

using cvxpy.
Solution: The given problem can be expressed in general as

$$\max_{\mathbf{x}} \mathbf{c}^T \mathbf{x} \tag{5.7.1.4}$$

$$s.t. \quad \mathbf{A}\mathbf{x} \le \mathbf{b}, \tag{5.7.1.5}$$

$$\mathbf{x} \ge \mathbf{0} \tag{5.7.1.6}$$

where

$$\mathbf{c} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \tag{5.7.1.7}$$

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix} \tag{5.7.1.8}$$

$$\mathbf{b} = \begin{pmatrix} 50 \\ 90 \end{pmatrix} \tag{5.7.1.9}$$

and can be solved using *cvxpy* through the following code

```
codes/opt/lp_cvx.py
```

to obtain

$$\mathbf{x} = \begin{pmatrix} 30 \\ 0 \end{pmatrix}, Z = 120 \tag{5.7.1.10}$$

5.7.2. Graphically, show that the feasible region in Problem 5.7.1 result in the interior of a convex polygon and the optimal point is one of the vertices. **Solution:** The following code plots Fig. 5.7.2.

```
codes/opt/lp_cvx.py
```

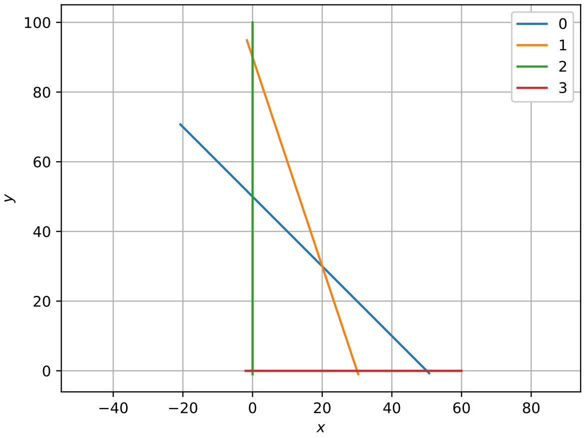

Fig. 5.7.2

5.7.3. Solve

$$\min_{\mathbf{x}} Z = \begin{pmatrix} 3 & 9 \end{pmatrix} \mathbf{x} \tag{5.7.3.1}$$

$$s.t. \quad \begin{pmatrix} 1 & 3 \\ -1 & -1 \\ 1 & -1 \end{pmatrix} \mathbf{x} \preceq \begin{pmatrix} 60 \\ -10 \\ 0 \end{pmatrix} \tag{5.7.3.2}$$

$$\mathbf{x} \succeq \mathbf{0} \tag{5.7.3.3}$$

Solution: The following code

```
codes/opt/lp_cvx_mult.py
```

is used to obtain

$$\mathbf{x} = \begin{pmatrix} 15 \\ 15 \end{pmatrix}, Z = 180 \tag{5.7.3.4}$$

5.7.4. Solve

$$\min_{\mathbf{x}} Z = \begin{pmatrix} -50 & 20 \end{pmatrix} \mathbf{x} \tag{5.7.4.1}$$

$$s.t. \quad \begin{pmatrix} -2 & 1 \\ -3 & -1 \\ 2 & -3 \end{pmatrix} \mathbf{x} \preceq \begin{pmatrix} 5 \\ -3 \\ 12 \end{pmatrix} \tag{5.7.4.2}$$

$$\mathbf{x} \succeq \mathbf{0} \tag{5.7.4.3}$$

Solution: The following code

```
codes/opt/lp_cvx_nosol.py
```

shows that the given problem has no solution.

5.7.5. Verify all the above solutions using Lagrange multipliers.

5.7.6. Repeat the above exercise using the Simplex method.

5.7.7. (**Diet problem**): A dietician wishes to mix two types of foods in such a way that vitamin contents of the mixture contain atleast 8 units of vitamin A and 10 units of vitamin C. Food 'I' contains 2 units/kg of vitamin A and 1 unit/kg of vitamin C. Food 'II' contains 1 unit/kg of vitamin A and 2 units/kg of vitamin C. It costs Rs 50 per kg to purchase Food 'I' and Rs 70 per kg to purchase Food 'II'. Formulate this problem as a linear programming problem to minimise the cost of such a mixture.
Solution: Let the mixture contain x kg of food I and y kg of food II.
The given problem can be expressed as

Resources	Food		Requirement
	I	II	
Vitamin A	2	1	Atleast 8 Units
Vitamin C	1	2	Atleast 10 Units
Cost	50	70	

$$\min_{\mathbf{x}} Z = \begin{pmatrix} 50 & 70 \end{pmatrix} \mathbf{x} \tag{5.7.7.1}$$

$$s.t. \quad \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \mathbf{x} \succeq \begin{pmatrix} 8 \\ 10 \end{pmatrix} \tag{5.7.7.2}$$

$$\mathbf{x} \succeq \mathbf{0} \tag{5.7.7.3}$$

The corner points of the feasible region are available in Table 5.7.7 and plotted in Fig. 5.7.7.
The smallest value of Z is 380 at the point (2,4). But the feasible region is unbounded

Corner Point	$Z = 50x + 70y$
(0,8)	560
(2,4)	380
(10,0)	500

TABLE 5.7.7

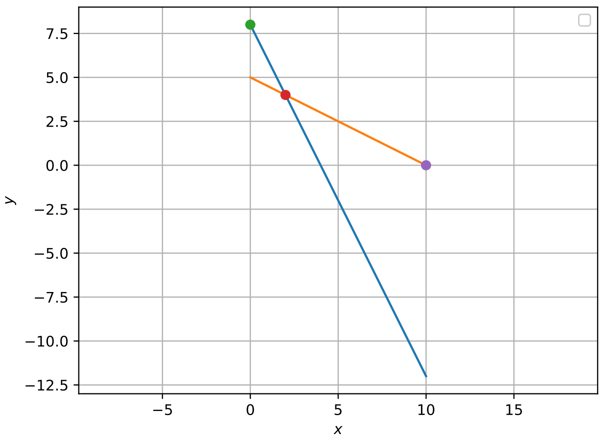

Fig. 5.7.7

therefore we draw the graph of the inequality

$$50x + 70y < 380 \tag{5.7.7.4}$$

to check whether the resulting open half has any point common with the feasible region but on checking it doesn't have any points in common. Thus the minimum value of Z is 380 attained at $\begin{pmatrix}2\\4\end{pmatrix}$. Hence optimal mixing strategy for the dietician would be to mix 2 Kg of Food I and 4 Kg of Food II. The following code provides the solution to (5.7.7.3).

```
codes/opt/diet.py
```

5.7.8. **(Allocation problem)** A cooperative society of farmers has 50 hectare of land to grow two crops X and Y. The profit from crops X and Y per hectare are estimated as Rs 10,500 and Rs 9,000 respectively. To control weeds, a liquid herbicide has to be used for crops X and Y at rates of 20 litres and 10 litres per hectare. Further, no more than 800 litres of herbicide should be used in order to protect fish and wild life using a pond which collects drainage from this land. How much land should be allocated to each crop so as to maximise the total profit of the society?

Solution: The given problem can be formulated as

$$\max_{\mathbf{x}} Z = \begin{pmatrix}10500 & 9000\end{pmatrix}\mathbf{x} \tag{5.7.8.1}$$

$$s.t. \quad \begin{pmatrix}20 & 10\end{pmatrix}\mathbf{x} \leq 800 \tag{5.7.8.2}$$

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 50 \tag{5.7.8.3}$$

Fig 5.7.8 shows the intersection of various lines and the optimal point as indicated.

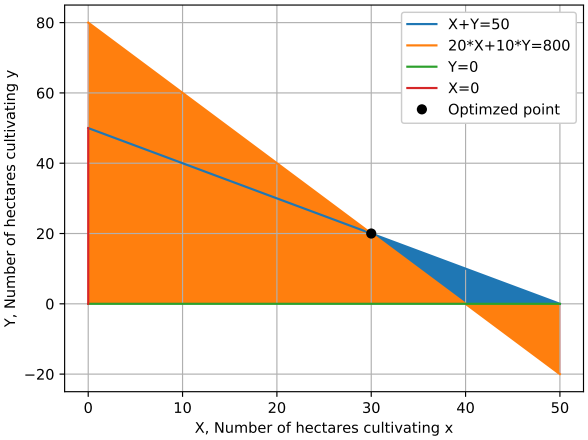

Fig. 5.7.8: Feasible region for allocation Problem

Fig. 5.7.8

The following code provides the solution to (5.7.8.3) at $\begin{pmatrix}30\\20\end{pmatrix}$.

```
codes/opt/allocation.py
```

5.7.9. **(Manufacturing problem)** A manufacturer has three machines I, II and III installed in his factory. Machines I and II are capable of being operated for at most 12 hours whereas machine III must be operated for atleast 5 hours a day. She produces only two items M and N each requiring the use of all the three machines. The number of hours required for producing 1 unit of each of M and N on the three machines are given in the following table:

Number of hours required on machines			
Items	**I**	**II**	**III**
M	1	2	1
N	2	1	1.25

She makes a profit of Rs 600 and Rs 400 on items M and N respectively. How many of each item should she produce so as to maximise her profit assuming that she can sell all the items that she produced? What will be the maximum profit?

Solution: The given problem can be formulated as

$$\max_{\mathbf{x}} Z = \begin{pmatrix}80000 & 12000\end{pmatrix}\mathbf{x} \tag{5.7.9.1}$$

$$s.t. \quad \begin{pmatrix}3 & 4\\1 & 3\end{pmatrix}\mathbf{x} \le \begin{pmatrix}60\\30\end{pmatrix} \tag{5.7.9.2}$$

Fig 5.7.9 shows the intersection of various lines and the optimal point as indicated.

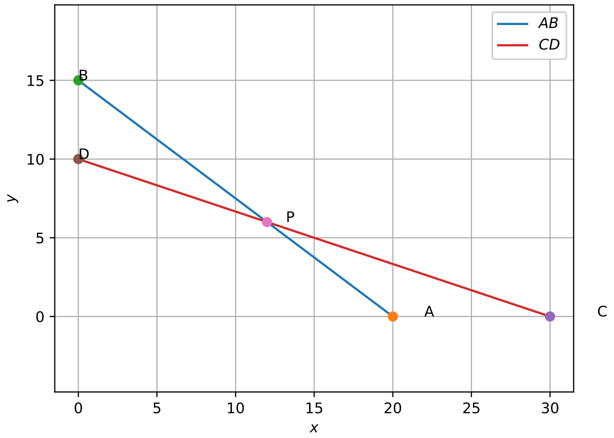

Fig. 5.7.9: Feasible region for manufacturing Problem

Fig. 5.7.9

The following code provides the solution to (5.7.9.2) at $\begin{pmatrix}12\\6\end{pmatrix}$.

codes/opt/Manufacturing.py

5.7.10. **(Transportation problem)** There are two factories located one at place P and the other at place Q. From these locations, a certain commodity is to be delivered to each of the three depots situated at A, B and C. The weekly requirements of the depots are respectively 5, 5 and 4 units of the commodity while the production capacity of the factories at P and Q are respectively 8 and 6 units. The cost of transportation per unit is given below where A,B,C are cost in ruppes:

From/To	A	B	C
P	160	100	150
Q	100	120	100

How many units should be transported from each factory to each depot in order that the transportation cost is minimum. What will be the minimum transportation cost?

Solution: The given problem can be formulated as

$$\min_{\mathbf{x}} Z = \begin{pmatrix}10 & -70\end{pmatrix}\mathbf{x} \tag{5.7.10.1}$$

$$s.t. \quad \begin{pmatrix}1 & 1\\-1 & -1\end{pmatrix}\mathbf{x} \le \begin{pmatrix}8\\-4\end{pmatrix} \tag{5.7.10.2}$$

$$\mathbf{x} \le \begin{pmatrix}5\\5\end{pmatrix} \tag{5.7.10.3}$$

Fig 5.7.10 shows the intersection of various lines and the optimal point indicated as OPT PT.

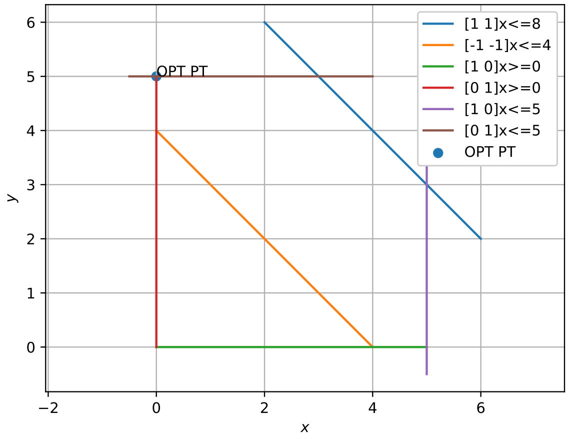

Fig. 5.7.10: Feasible region for Transportation Problem

Fig. 5.7.10

The following code provides the solution to (5.7.10.3) at $\begin{pmatrix}0\\5\end{pmatrix}$.

codes/opt/Transportation.py

Appendix A

Proof for the vector form of a conic section

Lemma A.1. *The distance of a point* $\mathbf{P}$ *from a line* $L : \mathbf{n}^T\mathbf{x} = c$ *is given by:*

$$d = \frac{\left|c - \mathbf{P}^T\mathbf{n}\right|}{\|\mathbf{n}\|} \tag{10.1}$$

Using Definition 3.1.1 and Lemma A.1, for any point $\mathbf{x}$ on the conic,

$$\|\mathbf{x}-\mathbf{F}\|^2 = e^2\frac{(c-\mathbf{x}^T\mathbf{n})^2}{\|\mathbf{n}\|^2} \tag{10.2}$$

$$t(\mathbf{x}-\mathbf{F})^T(\mathbf{x}-\mathbf{F}) = (c-\mathbf{x}^T\mathbf{n})^2 \tag{10.3}$$

$$t(\mathbf{x}^T\mathbf{x}-2\mathbf{F}^T\mathbf{x}+\|\mathbf{F}\|^2) = c^2+(\mathbf{x}^T\mathbf{n})^2-2c\mathbf{x}^T\mathbf{n} \tag{10.4}$$

$$t\mathbf{x}^T\mathbf{x}-(\mathbf{x}^T\mathbf{n})^2-2t\mathbf{F}^T\mathbf{x}+2c\mathbf{n}^T\mathbf{x} = c^2-t\|\mathbf{F}\|^2 \tag{10.5}$$

$$t\mathbf{x}^T\mathbf{I}\mathbf{x}-\mathbf{x}^T\mathbf{n}\mathbf{n}^T\mathbf{x}+2(c\mathbf{n}-t\mathbf{F})^T\mathbf{x} = c^2-t\|\mathbf{F}\|^2 \tag{10.6}$$

$$\mathbf{x}^T(t\mathbf{I}-\mathbf{n}\mathbf{n}^T)\mathbf{x}+2(c\mathbf{n}-t\mathbf{F})^T\mathbf{x}+t\|\mathbf{F}\|^2-c^2 = 0 \tag{10.7}$$

Appendix B
Proofs for the Parabola

B.1. Substituting (3.1.3.1) in (3.1.2.1)

$$(\mathbf{P}\mathbf{y}+\mathbf{c})^T\mathbf{V}(\mathbf{P}\mathbf{y}+\mathbf{c})+2\mathbf{u}^T(\mathbf{P}\mathbf{y}+\mathbf{c})+f=0, \tag{B.1.1}$$

which can be expressed as

$$\begin{aligned}\mathbf{y}^T\mathbf{P}^T\mathbf{V}\mathbf{P}\mathbf{y}&+2(\mathbf{V}\mathbf{c}+\mathbf{u})^T\mathbf{P}\mathbf{y}\\&+\mathbf{c}^T\mathbf{V}\mathbf{c}+2\mathbf{u}^T\mathbf{c}+f=0\end{aligned} \tag{B.1.2}$$

From (B.1.2) and (3.1.3.2),

$$\begin{aligned}\mathbf{y}^T\mathbf{D}\mathbf{y}&+2(\mathbf{V}\mathbf{c}+\mathbf{u})^T\mathbf{P}\mathbf{y}\\&+\mathbf{c}^T(\mathbf{V}\mathbf{c}+\mathbf{u})+\mathbf{u}^T\mathbf{c}+f=0\end{aligned} \tag{B.1.3}$$

When $\mathbf{V}^{-1}$ exists,

$$\mathbf{V}\mathbf{c}+\mathbf{u}=\mathbf{0}, \quad \text{or}, \mathbf{c}=-\mathbf{V}^{-1}\mathbf{u}, \tag{B.1.4}$$

and substituting (B.1.4) in (B.1.3) yields (3.1.3.5).

B.2. When $|\mathbf{V}|=0, \lambda_1=0$ and

$$\mathbf{V}\mathbf{p}_1=0, \mathbf{V}\mathbf{p}_2=\lambda_2\mathbf{p}_2. \tag{B.2.1}$$

where $\mathbf{p}_1, \mathbf{p}_2$ are the eigenvectors of $\mathbf{V}$such that (3.1.3.2)

$$\mathbf{P}=\begin{pmatrix}\mathbf{p}_1 & \mathbf{p}_2\end{pmatrix}, \tag{B.2.2}$$

Substituting (B.2.2) in (B.1.3),

$$\begin{aligned}&\mathbf{y}^T\mathbf{D}\mathbf{y}+2\left(\mathbf{c}^T\mathbf{V}+\mathbf{u}^T\right)\begin{pmatrix}\mathbf{p}_1 & \mathbf{p}_2\end{pmatrix}\mathbf{y}\\&\qquad+\mathbf{c}^T(\mathbf{V}\mathbf{c}+\mathbf{u})+\mathbf{u}^T\mathbf{c}+f=0\\&\implies \mathbf{y}^T\mathbf{D}\mathbf{y}\\&+2\begin{pmatrix}\left(\mathbf{c}^T\mathbf{V}+\mathbf{u}^T\right)\mathbf{p}_1 & \left(\mathbf{c}^T\mathbf{V}+\mathbf{u}^T\right)\mathbf{p}_2\end{pmatrix}\mathbf{y}\\&\qquad+\mathbf{c}^T(\mathbf{V}\mathbf{c}+\mathbf{u})+\mathbf{u}^T\mathbf{c}+f=0\\&\implies \mathbf{y}^T\mathbf{D}\mathbf{y}\\&+2\begin{pmatrix}\mathbf{u}^T\mathbf{p}_1 & \left(\lambda_2\mathbf{c}^T+\mathbf{u}^T\right)\mathbf{p}_2\end{pmatrix}\mathbf{y}\\&\qquad+\mathbf{c}^T(\mathbf{V}\mathbf{c}+\mathbf{u})+\mathbf{u}^T\mathbf{c}+f=0\\&\qquad\text{from (B.2.1)}\\&\implies \lambda_2y_2^2+2\left(\mathbf{u}^T\mathbf{p}_1\right)y_1+2y_2(\lambda_2\mathbf{c}+\mathbf{u})^T\mathbf{p}_2\\&\qquad+\mathbf{c}^T(\mathbf{V}\mathbf{c}+\mathbf{u})+\mathbf{u}^T\mathbf{c}+f=0\end{aligned} \tag{B.2.3}$$

which is the equation of a parabola. From (B.2.3), by comparing the coefficients of y_2^2 and y_1, the focal length of the parabola is obtained as

$$\left|\frac{2\mathbf{u}^T\mathbf{p}_1}{\lambda_2}\right|. \tag{B.2.4}$$

Thus, (B.2.3) can be expressed as (3.1.3.6) by choosing

$$\eta=\mathbf{u}^T\mathbf{p}_1 \tag{B.2.5}$$

and $\mathbf{c}$ in (B.1.3) such that

$$\mathbf{P}^T(\mathbf{V}\mathbf{c}+\mathbf{u})=\eta\begin{pmatrix}1\\0\end{pmatrix} \tag{B.2.6}$$

$$\mathbf{c}^T(\mathbf{V}\mathbf{c}+\mathbf{u})+\mathbf{u}^T\mathbf{c}+f=0 \tag{B.2.7}$$

Multiplying (B.2.6) by $\mathbf{P}$ yields

$$(\mathbf{V}\mathbf{c}+\mathbf{u})=\eta\mathbf{p}_1, \tag{B.2.8}$$

which, upon substituting in (B.2.7) results in

$$\eta\mathbf{c}^T\mathbf{p}_1+\mathbf{u}^T\mathbf{c}+f=0 \tag{B.2.9}$$

(B.2.8) and (B.2.9) can be clubbed together to obtain (3.1.3.8).