# Probability and Random Variables

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Contents



***Abstract*—This book provides a simple introduction to probability and random variables. The contents are largely based on NCERT textbooks from Class 9-12.**

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## 1 Axioms of Probability

### 1.1 Boolean Logic

If A and B are two events such that $\Pr(A) = \frac{1}{4}, \Pr(B) = \frac{1}{2}$ and $\Pr(AB) = \frac{1}{8}$. find $\Pr(\text{not A and not B})$.

1.1.1.

$$A'B' = (A+B)' \tag{1.1.1.1}$$
$$\implies \Pr(A'B') = \Pr((A+B)') \tag{1.1.1.2}$$
$$= 1 - \Pr(A+B) \tag{1.1.1.3}$$

1.1.2.

$$\because A + B = A(B+B') + B \tag{1.1.2.1}$$
$$= B(A+1) + AB' \tag{1.1.2.2}$$
$$= B + AB' \tag{1.1.2.3}$$
$$\implies \Pr(A+B) = \Pr(B + AB') \tag{1.1.2.4}$$
$$= \Pr(B) + \Pr(AB') \tag{1.1.2.5}$$
$$\because B(AB') = 0 \tag{1.1.2.6}$$

1.1.3.

$$A = A(B+B') = AB + AB' \tag{1.1.3.1}$$

and

$$(AB)(AB') = 0, \because BB' = 0 \tag{1.1.3.2}$$

Hence, $AB$ and $AB'$ are mutually exclusive and

$$\Pr(A) = \Pr(AB) + \Pr(AB') \tag{1.1.3.3}$$
$$\implies \Pr(AB') = \Pr(A) - \Pr(AB) \tag{1.1.3.4}$$

1.1.4. Substituting (1.1.3.4) in (1.1.2.6),

$$\Pr(A+B) = \Pr(A) + \Pr(B) - \Pr(AB) \tag{1.1.4.1}$$

1.1.5. Substituting (1.1.4.1) in (1.1.1.3)

$$\Pr(A'B') = 1 - \{\Pr(A) + \Pr(B) - \Pr(AB)\} \quad (1.1.5.1)$$

$$= 1 - \left(\frac{1}{4} + \frac{1}{2} - \frac{1}{8}\right) \quad (1.1.5.2)$$

$$= \frac{3}{8} \quad (1.1.5.3)$$

### *1.2 Independent Events*

1.2.1. Prove that if $E$ and $F$ are independent events, then so are the events $E$ and $F'$.
**Solution:** If $E$ and $F$ are independent,

$$\Pr(EF) = \Pr(E)\Pr(F) \quad (1.2.1.1)$$

From (1.1.3.2)

$$\Pr(EF') = \Pr(E) - \Pr(EF) \quad (1.2.1.2)$$

Substituting from (1.2.1.1) in (1.2.1.2),

$$\Pr(EF') = \Pr(E)(1 - \Pr(F)) \quad = \Pr(E)\Pr(F') \quad (1.2.1.3)$$

$$\because FF' = 0, F + F' = 1 \quad (1.2.1.4)$$

$$\implies \Pr(F) + \Pr(F') = 1 \quad (1.2.1.5)$$

By definition, from (1.2.1.3), we conclude that $E$ and $F'$ are independent.

1.2.2. If A and B are two independent events, then the probability of occurrence of at least one of A and B is given by 1- $P(A')P(B')$
**Solution:**

$$\because (A + B)(A + B)' = 0 \quad (1.2.2.1)$$

$$\implies 1 = \Pr(A + B) + \Pr((A + B)') \quad (1.2.2.2)$$

$$\implies \Pr(A + B) = 1 - \Pr(A'B') \quad (1.2.2.3)$$

$$= 1 - \Pr(A')\Pr(B') \quad (1.2.2.4)$$

using the definition of independence.

### *1.3 Conditional Probability*

1.3.1. Given that E and F are events such that P(E) = 0.6, P(F) = 0.3 and P(E F) = 0.2, find $\Pr(E|F)$ and $\Pr(F|E)$
**Solution:** By definition,

$$\Pr(E|F) = \frac{\Pr(EF)}{\Pr(F)} = \frac{0.2}{0.3} = \frac{2}{3} \quad (1.3.1.1)$$

Similarly,

$$\Pr(F|E) = \frac{\Pr(EF)}{\Pr(E)} = \frac{1}{3} \quad (1.3.1.2)$$

1.3.2. A fair die is rolled. Consider the events E = (1, 3, 5), F = (2, 3) and G = (2, 3, 4, 5) Find

a) $\Pr(E|F)$ and $\Pr(F|E)$
b) $\Pr(E|G)$ and $\Pr(F|E)$
c) $\Pr((E + F)|G)$ and $\Pr(EF|G)$

**Solution:**
From the given information,

$$\Pr(E) = \frac{3}{6} = \frac{1}{2} \quad (1.3.2.1)$$

$$\Pr(F) = \frac{2}{6} = \frac{1}{3} \quad (1.3.2.2)$$

$$\Pr(G) = \frac{4}{6} = \frac{2}{3} \quad (1.3.2.3)$$

$$\Pr(EF) = \frac{1}{6} \quad (1.3.2.4)$$

$$\Pr(EG) = \frac{2}{6} = \frac{1}{3} \quad (1.3.2.5)$$

$$\Pr(FG) = \frac{2}{6} = \frac{1}{3} \quad (1.3.2.6)$$

$$\Pr(EFG) = \frac{1}{6} \quad (1.3.2.7)$$

a)

$$\Pr(E|F) = \frac{\Pr(EF)}{\Pr(F)} \quad (1.3.2.8)$$

$$\Pr(E|F) = \frac{\frac{1}{6}}{\frac{1}{3}} = \frac{1}{2} \quad (1.3.2.9)$$

$$\Pr(F|E) = \frac{\Pr(FE)}{\Pr(E)} \quad (1.3.2.10)$$

$$\Pr(F|E) = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3} \quad (1.3.2.11)$$

b)

$$\Pr(E|G) = \frac{\Pr(EG)}{\Pr(G)} \tag{1.3.2.12}$$

$$\Pr(E|G) = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2} \tag{1.3.2.13}$$

$$\Pr(G|E) = \frac{\Pr(GE)}{\Pr(G)} \tag{1.3.2.14}$$

$$\Pr(G|E) = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3} \tag{1.3.2.15}$$

$$\tag{1.3.2.16}$$

c)

$$\Pr(E+F|G) = \frac{\Pr(\{E+F\}G)}{\Pr(G)} = \frac{\Pr(EG+FG)}{\Pr(G)} = \frac{\Pr(EG)+\Pr(FG)-\Pr(EFG)}{\Pr(G)} = \frac{3}{4} \tag{1.3.2.17}$$

and

$$\Pr(EF|G) = \frac{\Pr(EFG)}{\Pr(G)} = \frac{1}{4} \tag{1.3.2.18}$$

## 2 Sum of Independent Random Variables

### 2.1 The Uniform Distribution

Two dice, one blue and one grey, are thrown at the same time. The event defined by the sum of the two numbers appearing on the top of the dice can have 11 possible outcomes 2, 3, 4, 5, 6, 6, 8, 9, 10, 11 and 12. A student argues that each of these outcomes has a probability $\frac{1}{11}$. Do you agree with this argument? Justify your answer.

2.1.1. *The Uniform Distribution:* Let $X_i \in \{1,2,3,4,5,6\}, i = 1,2$, be the random variables representing the outcome for each die. Assuming the dice to be fair, the probability mass function (pmf) is expressed as

$$p_{X_i}(n) = \Pr(X_i = n) = \begin{cases} \frac{1}{6} & 1 \le n \le 6 \\ 0 & otherwise \end{cases} \tag{2.1.1.1}$$

The desired outcome is

$$X = X_1 + X_2, \tag{2.1.1.2}$$

$$\implies X \in \{1,2,\dots,12\} \tag{2.1.1.3}$$

The objective is to show that

$$p_X(n) \neq \frac{1}{11} \tag{2.1.1.4}$$

2.1.2. *Convolution:* From (2.1.1.2),

$$p_X(n) = \Pr(X_1 + X_2 = n) = \Pr(X_1 = n - X_2) \tag{2.1.2.1}$$

$$= \sum_k \Pr(X_1 = n-k|X_2 = k)\, p_{X_2}(k) \tag{2.1.2.2}$$

after unconditioning. $\because$ $X_1$ and $X_2$ are independent,

$$\Pr(X_1 = n-k|X_2 = k) = \Pr(X_1 = n-k) = p_{X_1}(n-k) \tag{2.1.2.3}$$

From (2.1.2.2) and (2.1.2.3),

$$p_X(n) = \sum_k p_{X_1}(n-k)p_{X_2}(k) = p_{X_1}(n) * p_{X_2}(n) \tag{2.1.2.4}$$

where $*$ denotes the convolution operation. Substituting from (2.1.1.1) in (2.1.2.4),

$$p_X(n) = \frac{1}{6}\sum_{k=1}^{6} p_{X_1}(n-k) = \frac{1}{6}\sum_{k=n-6}^{n-1} p_{X_1}(k) \tag{2.1.2.5}$$

$$\because p_{X_1}(k) = 0, \quad k \le 1, k \ge 6. \tag{2.1.2.6}$$

From (2.1.2.5),

$$p_X(n) = \begin{cases} 0 & n < 1 \\ \frac{1}{6}\sum_{k=1}^{n-1} p_{X_1}(k) & 1 \le n-1 \le 6 \\ \frac{1}{6}\sum_{k=n-6}^{6} p_{X_1}(k) & 1 < n-6 \le 6 \\ 0 & n > 12 \end{cases} \tag{2.1.2.7}$$

Substituting from (2.1.1.1) in (2.1.2.7),

$$p_X(n) = \begin{cases} 0 & n < 1 \\ \frac{n-1}{36} & 2 \le n \le 7 \\ \frac{13-n}{36} & 7 < n \le 12 \\ 0 & n > 12 \end{cases} \quad (2.1.2.8)$$

satisfying (2.1.1.4).

2.1.3. *The Z-transform:* The $Z$-transform of $p_X(n)$ is defined as

$$P_X(z) = \sum_{n=-\infty}^{\infty} p_X(n) z^{-n}, \quad z \in \mathbb{C} \quad (2.1.3.1)$$

From (2.1.1.1) and (2.1.3.1),

$$P_{X_1}(z) = P_{X_2}(z) = \frac{1}{6}\sum_{n=1}^{6} z^{-n} \quad (2.1.3.2)$$

$$= \frac{z^{-1}\left(1 - z^{-6}\right)}{6\left(1 - z^{-1}\right)}, \quad |z| > 1 \quad (2.1.3.3)$$

upon summing up the geometric progression.

$$\because p_X(n) = p_{X_1}(n) * p_{X_2}(n), \quad (2.1.3.4)$$

$$P_X(z) = P_{X_1}(z) P_{X_2}(z) \quad (2.1.3.5)$$

The above property follows from Fourier analysis and is fundamental to signal processing. From (2.1.3.3) and (2.1.3.5),

$$P_X(z) = \left\{\frac{z^{-1}\left(1 - z^{-6}\right)}{6\left(1 - z^{-1}\right)}\right\}^2 \quad (2.1.3.6)$$

$$= \frac{1}{36}\frac{z^{-2}\left(1 - 2z^{-6} + z^{-12}\right)}{\left(1 - z^{-1}\right)^2} \quad (2.1.3.7)$$

Using the fact that

$$p_X(n-k) \overset{\mathcal{H}}{\longleftrightarrow} Z P_X(z) z^{-k}, \quad (2.1.3.8)$$

$$n u(n) \overset{\mathcal{H}}{\longleftrightarrow} Z \frac{z^{-1}}{\left(1 - z^{-1}\right)^2} \quad (2.1.3.9)$$

after some algebra, it can be shown that

$$\frac{1}{36}\left[(n-1)\, u(n-1) - 2\,(n-7)\, u(n-7) + (n-13)\, u(n-13)\right] \overset{\mathcal{H}}{\longleftrightarrow} Z \frac{1}{36}\frac{z^{-2}\left(1 - 2z^{-6} + z^{-12}\right)}{\left(1 - z^{-1}\right)^2} \quad (2.1.3.10)$$

where

$$u(n) = \begin{cases} 1 & n \ge 0 \\ 0 & n < 0 \end{cases} \quad (2.1.3.11)$$

From (2.1.3.1), (2.1.3.7) and (2.1.3.10)

$$p_X(n) = \frac{1}{36}\left[(n-1)\, u(n-1) - 2\,(n-7)\, u(n-7) + (n-13)\, u(n-13)\right] \quad (2.1.3.12)$$

which is the same as (2.1.2.8). Note that (2.1.2.8) can be obtained from (2.1.3.10) using contour integration as well.

2.1.4. The experiment of rolling the dice was simulated using Python for 10000 samples. These were generated using Python libraries for uniform distribution. The frequencies for each outcome were then used to compute the resulting pmf, which is plotted in Figure 2.1.4.1. The theoretical pmf obtained in (2.1.2.8) is plotted for comparison.

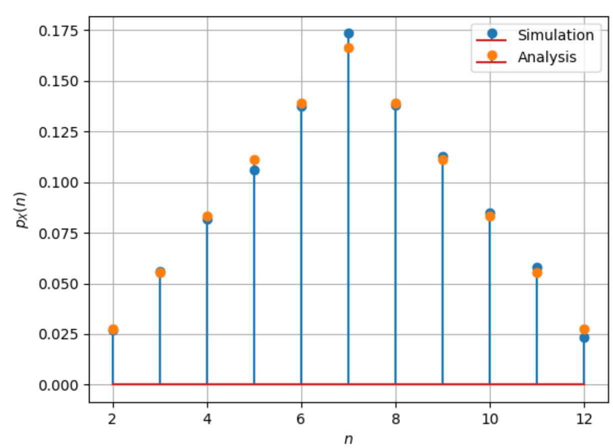


Fig. 2.1.4.1: Plot of $p_X(n)$. Simulations are close to the analysis.

2.1.5. The python code is available in

```
/codes/sum/dice.py
```

## 3 Cumulative Distribution Function

### 3.1 The Bernoulli Distribution

3.1.1. Find the probability of getting a head when a coin is tossed once. Also find the probability of getting a tail.

TABLE 3.1.3.1

| Colour | X | Number |
|---|---|---|
| Blue | 0 | $n(X = 0)$ |
| Green | 1 | $n(X = 1)$ |

**Solution:** Let the random variable be $X \in \{0, 1\}$. Then

$$\Pr(X = 0) = \Pr(X = 1) = \frac{1}{2} \tag{3.1.1.1}$$

The following code simulates the event for 100 coin tosses

```
codes/bernoulli/coin.py
```

3.1.2. *Bernoulli Distribution:* In general the binomial distribution is defined using the PMF

$$p_X(n) = \begin{cases} p & n = 1 \\ 1 - p & n = 0 \\ \text{otherwise} \end{cases} \tag{3.1.2.1}$$

3.1.3. A jar contains 24 marbles, some are green and others are blue. If a marble is drawn at random from the jar, the probability that it is green is $\frac{2}{3}$. Find the number of blue balls (marbles) in the jar.
**Solution:** Let random variable $X \in \{0, 1\}$ denote the outcomes of the experiment of drawing a marble from a jar as shown in Table 3.1.3.1 From the given information,

$$p_X(1) = \frac{2}{3} \tag{3.1.3.1}$$

$$\implies p = 1 - p_X(1) = \frac{1}{3} \tag{3.1.3.2}$$

$$n(X = 0) + n(X = 1) = 24 \tag{3.1.3.3}$$

$\because$

$$p = \frac{n(X = 0)}{n(X = 0) + n(X = 1)}, \tag{3.1.3.4}$$

from (3.1.3.4) and (3.1.3.3),

$$n(X = 0) = p\{n(X = 0) + n(X = 1)\} \tag{3.1.3.5}$$

$$= \frac{1}{3} \times 24 = 8. \tag{3.1.3.6}$$

The following code generates the number of blue marbles

```
codes/bernoulli/bernoulli.py
```

### 3.2 The Binomial Distribution

In a hurdle race, a player has to cross 10 hurdles. The probability that he will clear each hurdle is $\frac{5}{6}$. What is the probability that he will knock down fewer than 2 hurdles?

3.2.1. Let $X_i \in \{0, 1\}$ represent the *ith* hurdle where 1 denotes a hurdle being knocked down. Then, $X_i$ has a bernoulli distribution with parameter

$$p = 1 - \frac{5}{6} = \frac{1}{6} \tag{3.2.1.1}$$

3.2.2. *The Binomial Distribution:* Let

$$X = \sum_{i=1}^{n} X_i \tag{3.2.2.1}$$

where $n$ is the total number of hurdles. Then $X$ has a binomial distribution. Then, for

$$p_{X_i}(n) \overset{\mathcal{Z}}{\rightleftharpoons} P_{X_i}(z), \tag{3.2.2.2}$$

yielding

$$P_{X_i}(z) = 1 - p + pz^{-1} \tag{3.2.2.3}$$

with Using the fact that $X_i$ are i.i.d.,

$$P_X(z) = \left(1 - p + pz^{-1}\right)^n \tag{3.2.2.4}$$

$$= \sum_{k=0}^{n} {}^nC_k p^k (1 - p)^{n-k} z^{-k} \tag{3.2.2.5}$$

$$\implies p_X(k) = \begin{cases} {}^nC_k p^{n-k} (1 - p)^k & 0 \le k \le n \\ 0 & \text{otherwise} \end{cases} \tag{3.2.2.6}$$

The cumulative distribution function of $X$ is defined as

$$F_X(r) = \Pr(X \le r) = \sum_{k=0}^{r} {}^nC_k p^k (1 - p)^{n-k} \tag{3.2.2.7}$$

upon substituting from (3.2.2.6).

3.2.3. *Evaluationg the Probability:* Substituting from (3.2.1.1) in (3.2.2.7),

$$\Pr(X < 2) = F_X(1) \tag{3.2.3.1}$$
$$= \sum_{k=0}^{1} {}^{n}C_k \left(\frac{5}{6}\right)^{10-k} \left(\frac{1}{6}\right)^{k} \tag{3.2.3.2}$$
$$= 3\left(\frac{5}{6}\right)^{10} = 0.4845167486695371 \tag{3.2.3.3}$$

which is the desired probability.

3.2.4. The following code verifies the above result.

```
codes/binomial/binomial.py
```

## 4 Central Limit Theorem: Gaussian Distribution

### 4.1 Bernoulli to Gaussian

4.1.1 *Mean :* The mean of the bernoulli distribution is

$$\mu = E(X_i) = \sum_{k=0}^{1} k p_{X_i}(k) = p = \frac{1}{6} \tag{4.1.1}$$

4.1.2 *Moment:* The moment of the distribution is defined as

$$E(X_i^r) = \sum_{k=0}^{1} k^r p_{X_i}(k) = p = \frac{1}{6} \tag{4.2.1}$$

4.1.3 *Variance :* The variance of the bernoulli distribution is defined as

$$\sigma^2 = E(X - E(X))^2 = E\left(X^2\right) - E^2(X) \tag{4.3.1}$$
$$= p - p^2 = p(1-p) = \frac{5}{36} \tag{4.3.2}$$

The standard deviation

$$\sigma = \sqrt{p(1-p)} \tag{4.3.3}$$

4.1.4 *The Gaussian Distribution:* Define

$$G = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \frac{X_i - \mu}{\sigma} \tag{4.4.1}$$

4.1.5 *Approximating Binomial Using Gaussian:* From (4.4.1) and (3.2.2.1),

$$X \approx \sigma\sqrt{n}G + n\mu \tag{4.5.1}$$
$$\implies F_X(k) = \Pr\left(\sigma\sqrt{n}G + n\mu \le k\right) \tag{4.5.2}$$
$$= F_G\left(\frac{k - n\mu}{\sigma\sqrt{n}}\right) \approx \phi\left(\frac{k - n\mu}{\sigma\sqrt{n}}\right) \tag{4.5.3}$$

where

$$\phi_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, -\infty < x < \infty \tag{4.5.4}$$

4.1.6 The probability density function (PDF) of $G$ is

$$p_G(x) = \frac{d}{dx} F_X(x) \tag{4.6.1}$$
$$= \frac{1}{\sigma\sqrt{n}} \phi'\left(\frac{k - n\mu}{\sigma\sqrt{n}}\right) \tag{4.6.2}$$

For large $n$, $G$ is a continuous distribution with probability density function (PDF)

$$p_G(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{4.6.3}$$

4.1.7 *Evaluationg the Probability:* From 4.5.3 and 4.6.2,

$$\Pr(X \le 1) = F_G(1) = p_G(0) + p_G(1) \tag{4.7.1}$$
$$\approx 0.41299463887797094 \tag{4.7.2}$$

which is close to (3.2.3.3).

### 4.2 Uniform to Gaussian

4.2.1 Generate $10^6$ samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{4.1.1}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

**Solution:** Download the following files and execute the C program.

```
codes/cdf/exrand.c
codes/cdf/coeffs.h
```

4.2.2 Load gau.dat in python and plot the empirical CDF of $X$ using the samples in gau.dat. What properties does a CDF have?

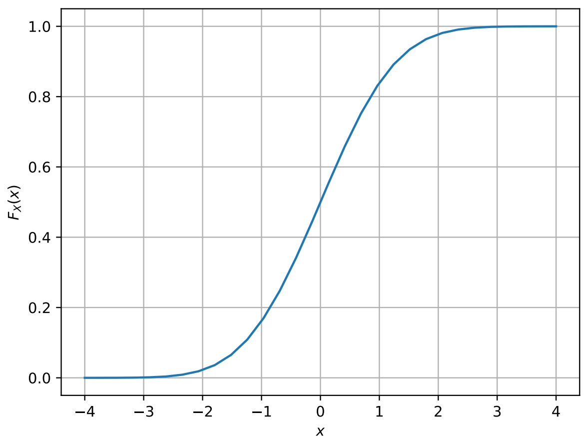


Fig. 4.2: The CDF of $X$

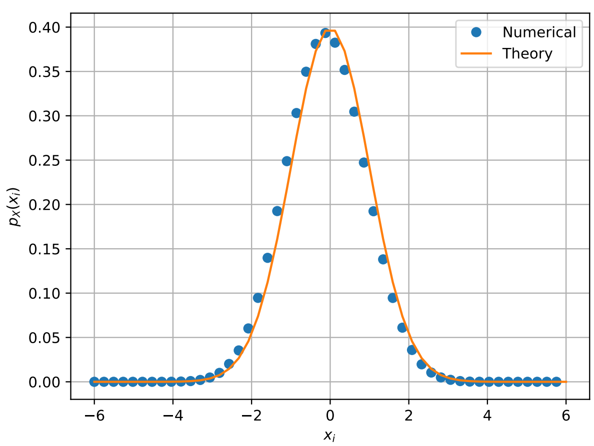


Fig. 4.3: The PDF of $X$

**Solution:** The CDF of $X$ is plotted in Fig. 4.2

4.2.3 Load gau.dat in python and plot the empirical PDF of $X$ using the samples in gau.dat. The PDF of $X$ is defined as

$$p_X(x) = \frac{d}{dx}F_X(x) \tag{4.3.1}$$

What properties does the PDF have?
**Solution:** The PDF of $X$ is plotted in Fig. 4.3 using the code below

```
codes/clt/pdf_plot.py
```

4.2.4 Find the mean and variance of $X$ by writing a C program.
**Solution:** Execute

```
codes/clt/gaussian_numbers.c
```

4.2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{4.5.1}$$

repeat the above exercise theoretically.
**Solution:**

$$E(X) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} xe^{-\frac{x^2}{2}}dx \tag{4.5.2}$$
$$= 0 \quad (\text{ odd function}) \tag{4.5.3}$$

$$E\left(X^2\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x^2e^{-\frac{x^2}{2}}dx \quad (evenfunction) \tag{4.5.4}$$
$$= \frac{2}{\sqrt{2\pi}}\int_{0}^{\infty} x^2e^{-\frac{x^2}{2}}dx \tag{4.5.5}$$
$$= \frac{2}{\sqrt{2\pi}}\int_{0}^{\infty} \sqrt{2u}e^{-u}du \quad \left(Let\frac{x^2}{2} = u\right) \tag{4.5.6}$$
$$= \frac{2}{\sqrt{\pi}}\int_{0}^{\infty} e^{-u}u^{\frac{3}{2}-1}du \tag{4.5.7}$$
$$= \frac{2}{\sqrt{\pi}}\Gamma\left(\frac{3}{2}\right) \tag{4.5.8}$$
$$= \frac{1}{\sqrt{\pi}}\Gamma\left(\frac{1}{2}\right) \tag{4.5.9}$$
$$= 1 \tag{4.5.10}$$

where we have used the fact that

$$\because \Gamma(n) = (n-1)\Gamma(n-1); \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \tag{4.5.11}$$

Thus, the variance is

$$\sigma^2 = E\left(X\right)^2 - E^2\left(X\right) = 1 \tag{4.5.12}$$

4.2.6 Let $U$ be a uniform random variable between 0 and 1.

4.2.7 Load the uni.dat file into python and plot the empirical CDF of $U$ using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr\left(U \le x\right) \tag{4.7.1}$$

**Solution:** The following code plots Fig. 4.7

```
codes/cdf/cdf_plot.py
```

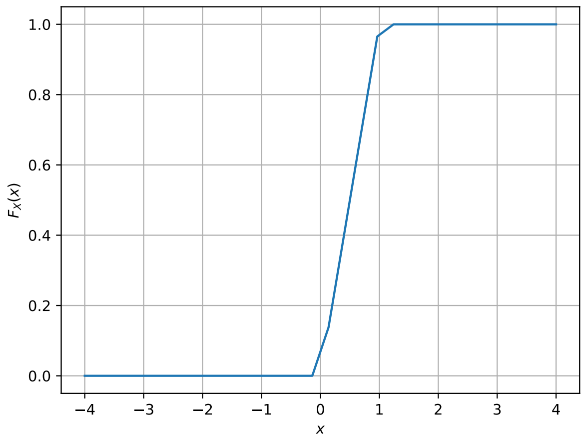


Fig. 4.7: The CDF of $U$

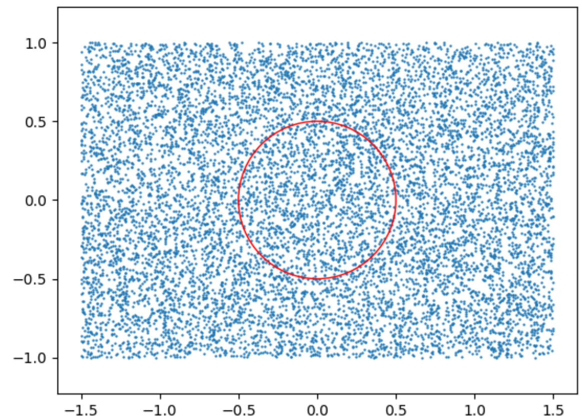

Fig. 5.1.1

4.2.8 Find a theoretical expression for $F_U(x)$.

4.2.9 The mean of $U$ is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{4.9.1}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{4.9.2}$$

Write a C program to find the mean and variance of $U$.

4.2.10 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{4.10.1}$$

## 5 Stochastic Geometry

Suppose you drop a die at random on the rectangular region shown in Fig. 5.1.1. What is the probability that it will land inside the circle with diameter 1m?

5.1. In Fig. 5.1.1, the sample size $S$ is the area of the rectangle given by

$$S = 3 \times 2 = 6m^2 \tag{5.1.1}$$

The event size is the area of the circle given by

$$E = \pi\left(\frac{1}{2}\right)^2 = \frac{\pi}{4}m^2 \tag{5.1.2}$$

The probabilty of the dice landing in the circle is

$$\Pr(E) = \frac{E}{S} = \frac{\pi}{24} \tag{5.1.3}$$

5.2. The python code is available in

```
/codes/stochastic/rect.py
```

The python code generates 10,000 points uniformly within the rectangle of dimensions 3×2 and checks for the number of points within the circle of radius 0.5. The ratio of these is close to $\frac{\pi}{24}$. Note that each time the code is run, the ratio will change, but will still be close to $\frac{\pi}{24}$.

## 6 Transformation of Variables

### 6.1 Using Definition

6.1.1. Let $X_1 \sim \mathcal{N}(0,1)$ and $X_2 \sim \mathcal{N}(0,1)$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \tag{6.1.1.1}$$

**Solution:** The following codes

```
codes/trans/6.1.1_CDF.py
codes/trans/6.1.1_PDf.py
```

generate the CDF of $V$ in Fig. 6.1.1.1 and the PDf of $V$ in Fig. 6.1.2 respectively.

6.1.2. If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases} \tag{6.1.2.1}$$

find $\alpha$.

**Solution:** For the value $\alpha = 0.5$, the theory

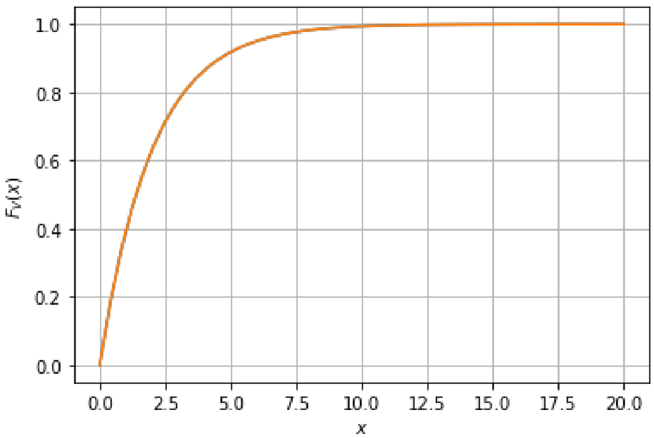


Fig. 6.1.1.1: CDF of $V$

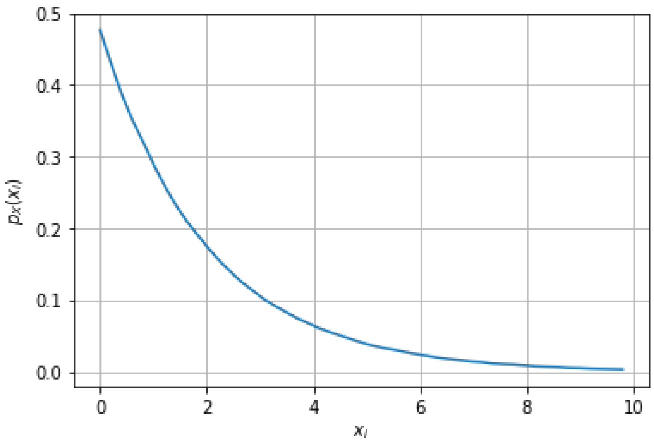


Fig. 6.1.1.2: PDf of $V$

matches the simulation. The following code generates the CDF of $V$ in Fig. 6.1.2.1 Fig.

```
codes/trans/6.1.2.py
```

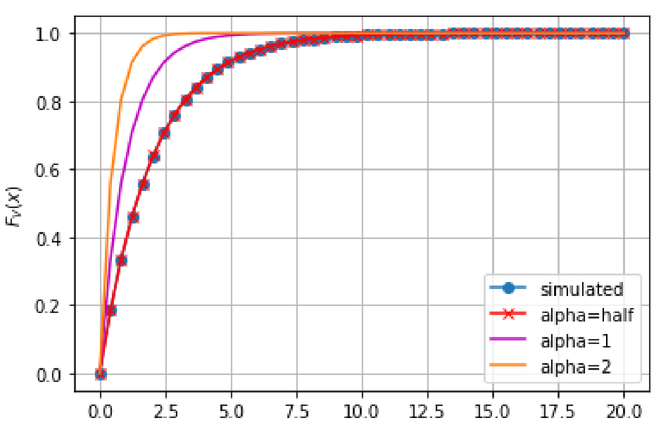


Fig. 6.1.2.1: CDF of $V$

6.1.3. Plot the CDF and PDf of

$$A = \sqrt{V} \tag{6.1.3.1}$$

**Solution:** The CDF and PDF of A are plotted in Figs. 6.1.3.1 and 6.1.3.2 using the codes below.

```
codes/trans/6.1.3_CDF.py
codes/trans/6.1.3_PDf.py
```

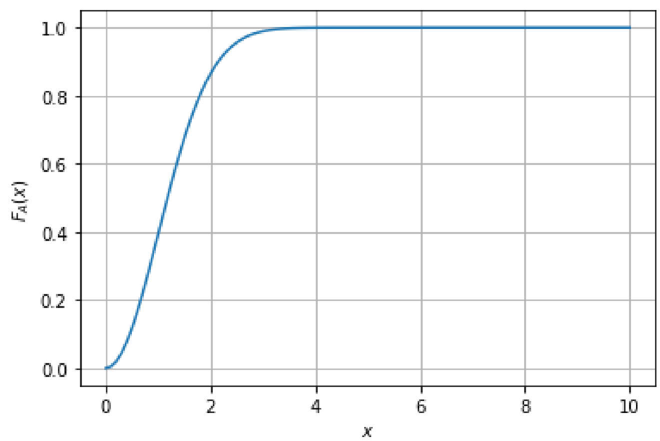


Fig. 6.1.3.1: CDF of $A$

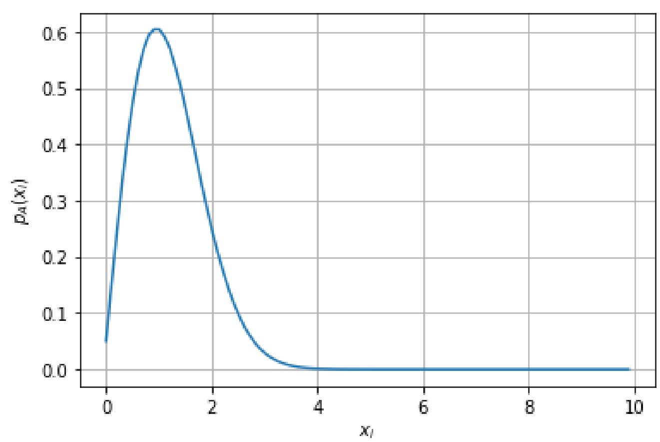


Fig. 6.1.3.2: PDf of $V$

6.1.4. Find an expression for $F_A(x)$ using the definition. Plot this expression and compare with the result of problem 6.1.3.
**Solution:**

$$F_A(x) = \Pr(A \leq x) = \Pr\left(\sqrt{V} \leq x\right) \tag{6.1.4.1}$$

$$= \Pr\left(V \leq x^2\right) = F_V\left(x^2\right) \tag{6.1.4.2}$$

From (6.1.2.1),

$$F_V\left(x^2\right) = \begin{cases} 1 - e^{-\alpha x^2} & x \geq 0 \\ 0 & x < 0, \end{cases} \quad (6.1.4.3)$$

Substituting

$$\alpha = \frac{1}{2} \quad (6.1.4.4)$$

$$F_V\left(x^2\right) = \begin{cases} 1 - e^{-\frac{x^2}{2}} & x \geq 0 \\ 0 & x < 0, \end{cases} \quad (6.1.4.5)$$

The CDF of A is plotted in Fig. 6.1.4.1 using the code below.

| codes/trans/6.1.4.py |
|---|

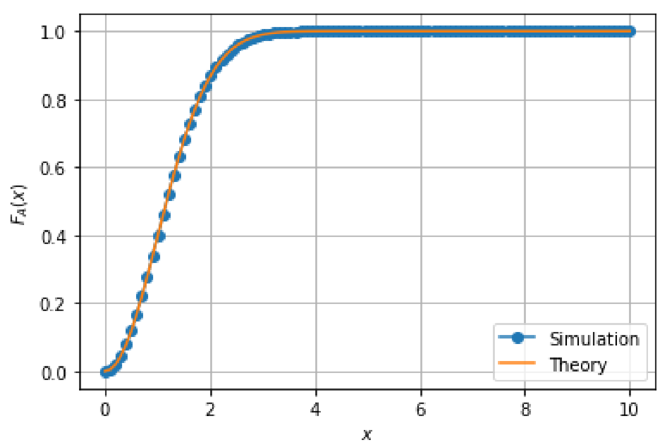


Fig. 6.1.4.1: CDF of $A$

6.1.5. Find an expression for $p_A(x)$.
**Solution:** The PDf is obtained as

$$f_V\left(x^2\right) = \frac{d}{dx} F_V\left(x^2\right) \quad (6.1.5.1)$$

$$= \begin{cases} xe^{-\frac{x^2}{2}} & x \geq 0 \\ 0 & x < 0, \end{cases} \quad (6.1.5.2)$$

The PDf of A is plotted in 6.1.5.1 using the code below.

| codes/trans/6.1.5.py |
|---|

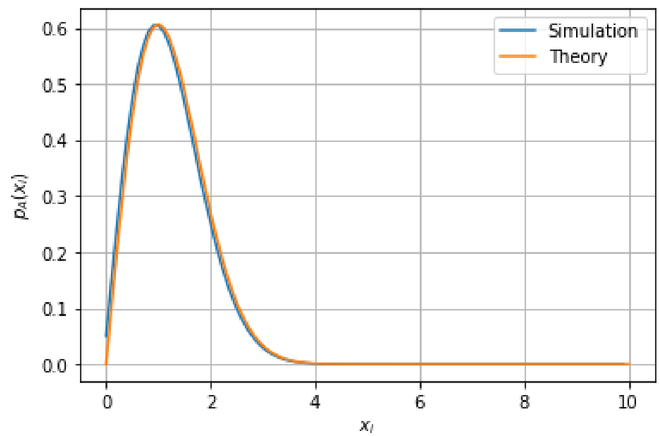


Fig. 6.1.5.1: PDf of $A$

*6.2 Using Jacobian*

6.2.1. Evaluate the joint PDF of $X_1, X_2$, given by

$$p_{X_1,X_2}(x_1, x_2) = p_{X_1}(x_1)\, p_{X_2}(x_2) \quad (6.2.1.1)$$

**Solution:** From (4.5.1)

$$p_{X_1}(x_1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x_1^2}{2}} \quad (6.2.1.2)$$

$$p_{X_2}(x_2) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x_2^2}{2}} \quad (6.2.1.3)$$

$$\implies p_{X_1,X_2}(x_1, x_2) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x_1^2+x_2^2}{2}} \quad (6.2.1.4)$$

6.2.2. Let

$$X_1 = \sqrt{V} \cos\theta \quad (6.2.2.1)$$

$$X_2 = \sqrt{V} \sin\theta. \quad (6.2.2.2)$$

Evaluate the Jacobian

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial v} & \frac{\partial x_2}{\partial v} \\ \frac{\partial x_1}{\partial \theta} & \frac{\partial x_2}{\partial \theta} \end{vmatrix} \quad (6.2.2.3)$$

**Solution:**

$$J = \begin{vmatrix} \frac{1}{2\sqrt{v}} \cos\theta & \frac{1}{2\sqrt{v}} \sin\theta \\ -v\sin\theta & v\cos\theta \end{vmatrix} = \frac{1}{2} \quad (6.2.2.4)$$

6.2.3. Find

$$p_{V,\Theta}(v, \theta) = |J|\, p_{X_1,X_2}(x_1, x_2) \quad (6.2.3.1)$$

**Solution:** From (6.2.2.4) and (6.2.1.4),

$$p_{V,\Theta}(v, \theta) = \frac{1}{4\pi} \exp\left(-\frac{v}{2}\right), v \geq 0, 0 < \theta < 2\pi \quad (6.2.3.2)$$

6.2.4. Find $p_V(v)$.

**Solution:** For $v \ge 0$, from (6.2.3.2),

$$p_V(v) = \int_0^{2\pi} \frac{1}{4\pi} \exp\left(-\frac{v}{2}\right) d\theta \quad (6.2.4.1)$$
$$= (2\pi) \times \frac{1}{4\pi} \exp\left(-\frac{v}{2}\right) \quad (6.2.4.2)$$
$$= \frac{1}{2} \exp\left(-\frac{v}{2}\right) \quad (6.2.4.3)$$
$$\therefore p_V(v) = \begin{cases} \frac{1}{2} \exp\left(-\frac{v}{2}\right) & v \ge 0 \\ 0 & v < 0 \end{cases} \quad (6.2.4.4)$$

6.2.5. Find $p_\Theta(\theta)$.
**Solution:** For $0 \le \theta \le 2\pi$, from (6.2.3.2),

$$p_\Theta(\theta) = \int_0^{\infty} \frac{1}{4\pi} \exp\left(-\frac{v}{2}\right) dv \quad (6.2.5.1)$$
$$= \frac{1}{2\pi} \left[1 - e^{-\frac{x}{2}}\right]_0^\infty \quad (6.2.5.2)$$
$$= \frac{1}{2\pi} \quad (6.2.5.3)$$
$$\therefore p_V(v) = \begin{cases} \frac{1}{2\pi} & 0 \le \theta \le 2\pi \\ 0 & \text{otherwise} \end{cases} \quad (6.2.5.4)$$

6.2.6. Are $V$ and $\Theta$ independent?
**Solution:** Yes,

$$\because p_V(v) p_\Theta(\theta) = \frac{1}{2} \exp\left(-\frac{v}{2}\right) \times \frac{1}{2\pi} \quad (6.2.6.1)$$
$$= \frac{1}{4\pi} \exp\left(-\frac{v}{2}\right) \quad (6.2.6.2)$$
$$= p_{V,\Theta}(v, \theta) \quad (6.2.6.3)$$

6.2.7. Find $p_A(x)$ using the Jacobian.
**Solution:**

$$p_A(x) = \Pr(A = x) = \Pr(\sqrt{V} = x) \quad (6.2.7.1)$$
$$= \Pr(V = x^2) = p_V(x^2) \quad (6.2.7.2)$$

From (6.2.4.4), as $x^2 \ge 0$,

$$p_V(x^2) = \frac{1}{2} \exp\left(-\frac{x^2}{2}\right) \quad (6.2.7.3)$$

## 7 Conditional Probability

7.1. Plot

$$P_e = \Pr\left(\hat{X} = -1 | X = 1\right) \quad (7.1.1)$$

for

$$Y = AX + N, \quad (7.1.2)$$

where $A$ is Raleigh with $E\left[A^2\right] = \gamma, N \sim \mathcal{N}(0,1), X \in (-1,1)$ for $0 \le \gamma \le 10$ dB.
**Solution:** See Fig. 7.4.1

7.2. Assuming that $N$ is a constant, find an expression for $P_e$. Call this $P_e(N)$.
**Solution:** The estimated value $\hat{X}$ is given by

$$\hat{X} = \begin{cases} +1 & Y > 0 \\ -1 & Y < 0 \end{cases} \quad (7.2.1)$$

For $X = 1$,

$$Y = A + N \quad (7.2.2)$$
$$P_e = \Pr\left(\hat{X} = -1 | X = 1\right) \quad (7.2.3)$$
$$= \Pr(Y < 0 | X = 1) \quad (7.2.4)$$
$$= \Pr(A < -N) \quad (7.2.5)$$
$$= F_A(-N) \quad (7.2.6)$$
$$= \int_{-\infty}^{-N} f_A(x) dx \quad (7.2.7)$$

By definition

$$f_A(x) = \begin{cases} \frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right) & x \ge 0 \\ 0 & otherwise \end{cases} \quad (7.2.8)$$

If $N > 0$, $f_A(x) = 0$. Then,

$$P_e = 0 \quad (7.2.9)$$

If $N < 0$. Then,

$$P_e(N) = \int_{-\infty}^{-N} f_A(x) dx \quad (7.2.10)$$
$$= \int_{-\infty}^{0} 0 dx + \int_{0}^{-N} f_A(x) dx \quad (7.2.11)$$
$$= \int_{0}^{-N} \frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx \quad (7.2.12)$$
$$= 1 - \exp\left(-\frac{N^2}{2\sigma^2}\right) \quad (7.2.13)$$

Therefore,

$$P_e(N) = \begin{cases} 1 - \exp\left(-\frac{N^2}{2\sigma^2}\right) & N < 0 \\ 0 & otherwise \end{cases} \quad (7.2.14)$$

7.3. For a function $g$,

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) p_X(x)\, dx \quad (7.3.1)$$

Find $P_e = E[P_e(N)]$.

**Solution:** Since $N \sim \mathcal{N}(0,1)$ ,

$$p_N(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \tag{7.3.2}$$

$$\tag{7.3.3}$$

And from (7.2.14)

$$P_e(x) = \begin{cases} 1 - \exp\left(-\frac{x^2}{2\sigma^2}\right) & x < 0 \\ 0 & otherwise \end{cases} \tag{7.3.4}$$

$$P_e = E\left[P_e(N)\right] = \int_{-\infty}^{\infty} P_e(x) p_N(x)\, dx \tag{7.3.5}$$

If $x < 0, P_e(x) = 0$ and using the fact that for an even function

$$\int_{-\infty}^{\infty} f(x) = 2\int_{-\infty}^{0} f(x) \tag{7.3.6}$$

we get

$$P_e = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \exp\left(-\frac{x^2}{2}\right)\left(1 - \exp\left(-\frac{x^2}{2\sigma^2}\right)\right) dx \tag{7.3.7}$$

$$= \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2}\right) dx - \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{(1+\sigma^2)x^2}{2\sigma^2}\right) dx \tag{7.3.8}$$

$$= \frac{\sqrt{2\pi} - \sqrt{\frac{\pi(2\sigma^2)}{1+\sigma^2}}}{2\sqrt{2\pi}} \tag{7.3.9}$$

$$= \frac{1}{2} - \frac{1}{2}\sqrt{\frac{\sigma^2}{1+\sigma^2}} \tag{7.3.10}$$

For a Rayleigh Distribution with scale $= \sigma$,

$$E\left[A^2\right] = 2\sigma^2 \tag{7.3.11}$$

$$\gamma = 2\sigma^2 \tag{7.3.12}$$

$$\therefore P_e = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{\gamma}{2+\gamma}} \tag{7.3.13}$$

7.4. Plot $P_e$ in problems 7.1 and 7.3 on the same graph w.r.t $\gamma$. Comment.
**Solution:** $P_e$ is plotted w.r.t $\gamma$ in 7.4.1 using the code below.

```
codes/cond/7.4.py
```

## 8 Two Dimensions

8.1. Let

$$\mathbf{y} = A\mathbf{x} + \mathbf{n}, \tag{8.1.1}$$

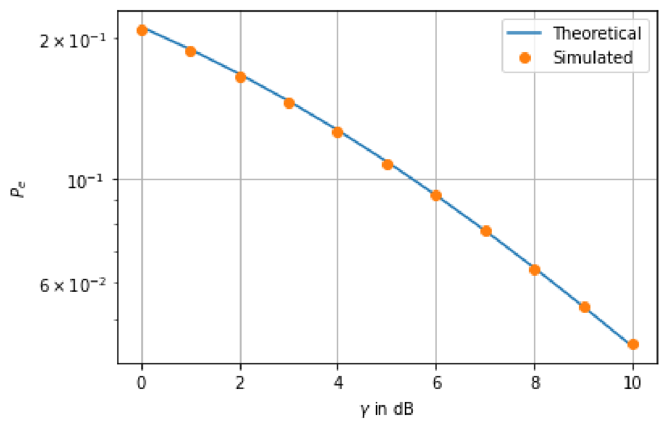


Fig. 7.4.1: $P_e$ w.r.t $\gamma$

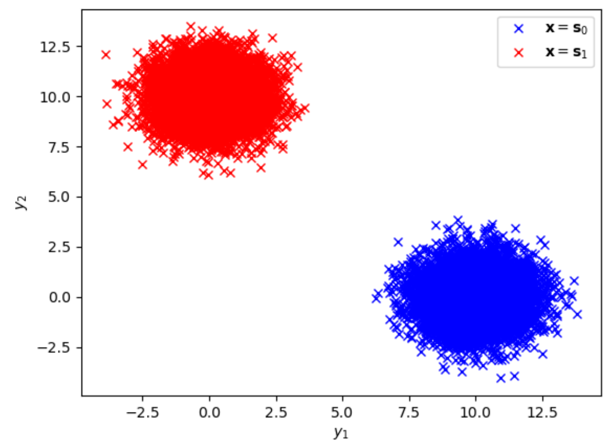


Fig. 8.2.1: Scatter plot of $\mathbf{y}$ for $A = 10$

where

$$x \in (\mathbf{s}_0, \mathbf{s}_1), \mathbf{s}_0 = \begin{pmatrix}1\\0\end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix}0\\1\end{pmatrix} \tag{8.1.2}$$

$$\mathbf{n} = \begin{pmatrix}n_1\\n_2\end{pmatrix}, n_1, n_2 \sim \mathcal{N}(0,1). \tag{8.1.3}$$

8.2. Plot

$$\mathbf{y}|\mathbf{s}_0 \text{ and } \mathbf{y}|\mathbf{s}_1 \tag{8.2.1}$$

on the same graph using a scatter plot. **Solution:** The following python code plots the scatter plot when $\mathbf{x} = \mathbf{s}_0$ and $\mathbf{x} = \mathbf{s}_1$ in Fig. 8.2.1

```
codes/twoD/scatter_plot.py
```

8.3. For the above problem, find a decision rule for detecting the symbols $\mathbf{s}_0$ and $\mathbf{s}_1$.
**Solution:** The multivariate Gaussian distribu-

tion is defined as

$$p_{\mathbf{x}}(x_1, \ldots, x_k) = \frac{1}{\sqrt{(2\pi)^k |\boldsymbol{\Sigma}|}} \exp\left\{-\frac{1}{2}(\mathbf{x}-\mu)^T \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\mu)\right\} \tag{8.3.1}$$

where $\mu$ is the mean vector, $\boldsymbol{\Sigma} = E\left[(\mathbf{x}-\mu)(\mathbf{x}-\mu)^T\right]$ is the covariance matrix and $|\boldsymbol{\Sigma}|$ is the determinant of $\boldsymbol{\Sigma}$. For a bivariate gaussian distribution,

$$p(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)} \times \left\{\frac{(x-\mu_x)^2}{\sigma_x^2} + \frac{\left(y-\mu_y\right)^2}{\sigma_y^2} - \frac{2\rho(x-\mu_x)\left(y-\mu_y\right)}{\sigma_x\sigma_y}\right\}\right] \tag{8.3.2}$$

where

$$\mu = \begin{pmatrix}\mu_x\\ \mu_y\end{pmatrix}, \boldsymbol{\Sigma} = \begin{pmatrix}\sigma_x^2 & \rho\sigma_x\sigma_y\\ \rho\sigma_x\sigma_y & \sigma_y^2\end{pmatrix}, \tag{8.3.3}$$

$$\rho = \frac{E\left[(x-\mu_x)\left(y-\mu_y\right)\right]}{\sigma_x\sigma_y}. \tag{8.3.4}$$

$$\mathbf{y}|s_0 = \begin{pmatrix}A+n_1\\ n_2\end{pmatrix} \tag{8.3.5}$$

$$\mathbf{y}|s_1 = \begin{pmatrix}n_1\\ A+n_2\end{pmatrix} \tag{8.3.6}$$

Substituting these values in (8.3.2),

$$p(\mathbf{y}|s_0) = \frac{1}{2\pi\sigma_{y_1}\sigma_{y_2}\sqrt{1-\rho_1^2}} \exp\left[-\frac{1}{2\left(1-\rho_1^2\right)} \times \left\{\frac{(y_1-A)^2}{\sigma_{y_1}^2} + \frac{(y_2)^2}{\sigma_{y_2}^2} - \frac{2\rho_1(y_1-A)(y_2)}{\sigma_{y_1}\sigma_{y_2}}\right\}\right] \tag{8.3.7}$$

$$p(\mathbf{y}|s_1) = \frac{1}{2\pi\sigma_{y_1}\sigma_{y_2}\sqrt{1-\rho_2^2}} \exp\left[-\frac{1}{2\left(1-\rho_2^2\right)} \times \left\{\frac{(y_1)^2}{\sigma_{y_1}^2} + \frac{(y_2-A)^2}{\sigma_{y_2}^2} - \frac{2\rho_2(y_1)(y_2-A)}{\sigma_{y_1}\sigma_{y_2}}\right\}\right] \tag{8.3.8}$$

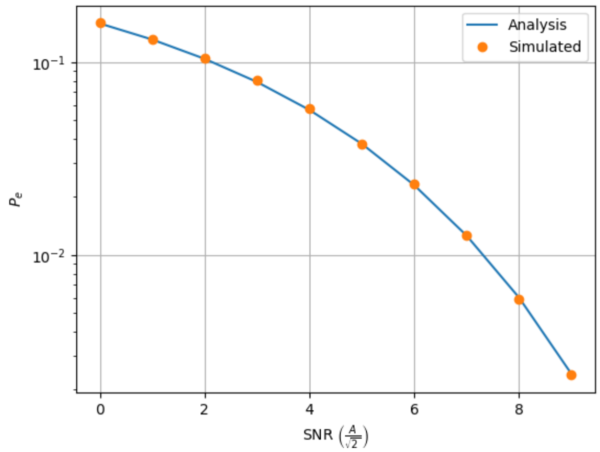


Fig. 8.4.1: $P_e$ with respect to SNR from 0 to 10 dB

where,

$$\rho_1 = E[(y_1-A)(y_2)] = E[n_1n_2] = 0,$$
$$\rho_2 = E[(y_1)(y_2-A)] = E[n_1n_2] = 0,$$
$$\sigma_{y_1} = \sigma_{y_2} = 1 \tag{8.3.9}$$

For equiprobably symbols, the MAP criterion is defined as

$$p(\mathbf{y}|s_0) \underset{s_1}{\overset{s_0}{\gtrless}} p(\mathbf{y}|s_1) \tag{8.3.10}$$

Using (8.3.7) and (8.3.8) and substituting the values from (8.3.9), we get

$$(y_1-A)^2 + y_2^2 \underset{s_0}{\overset{s_1}{\gtrless}} y_1^2 + (y_2-A)^2 \tag{8.3.11}$$

On simplifying, we get the decision rule is

$$y_1 \underset{s_1}{\overset{s_0}{\gtrless}} y_2 \tag{8.3.12}$$

8.4. Plot

$$P_e = \Pr(\hat{\mathbf{x}} = \mathbf{s}_1|\mathbf{x} = \mathbf{s}_0) \tag{8.4.1}$$

with respect to the SNR from 0 to 10 dB.
**Solution:**

8.5. Obtain an expression for $P_e$. Verify this by comparing the theory and simulation plots on the same graph.
**Solution:**

$$P_e = \Pr(\hat{\mathbf{x}} = \mathbf{s}_1|\mathbf{x} = \mathbf{s}_0) \tag{8.5.1}$$

Given that $\mathbf{s}_0$ was transmitted, the received signal is

$$\mathbf{y}|\mathbf{s}_0 = \begin{pmatrix} A \\ 0 \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \tag{8.5.2}$$

From (8.3.12), the probability of error is given by

$$P_e = \Pr\left(y_1 < y_2|\mathbf{s}_0\right) = \Pr\left(A + n_1 < n_2\right) \tag{8.5.3}$$
$$= \Pr\left(n_2 - n_1 > A\right) \tag{8.5.4}$$

Note that $n_2 - n_1 \sim \mathcal{N}(0, 2)$. Thus,

$$P_e = \Pr\left(\sqrt{2}w > A\right) \tag{8.5.5}$$
$$\Pr\left(w > \frac{A}{\sqrt{2}}\right) \tag{8.5.6}$$
$$\Rightarrow P_e = Q\left(\frac{A}{\sqrt{2}}\right) \tag{8.5.7}$$

where $w \sim \mathcal{N}(0, 1)$. The following code plots the $P_e$ curve in Fig. (8.4.1).

```
codes/twoD/ber_snr_plot.py
```

## 9 Transform Domain: Moment Generating Function

Let $X \sim \mathcal{N}\left(\mu, \sigma^2\right)$.

9.1. Find $M_X(s) = E\left[e^{-sX}\right]$.
**Solution:** The MGF of $X$ is

$$M_X(s) = \int_{-\infty}^{\infty} e^{-sX} p_X(x) dx \tag{9.1.1}$$
$$= e^{-s\mu} e^{-\frac{s^2\sigma^2}{2}} \tag{9.1.2}$$

9.2. Let

$$N = n_1 - n_2, \quad n_1, n_2 \sim \mathcal{N}(0, 1). \tag{9.2.1}$$

Find $M_N(s)$, assuming that $n_1$ and $n_2$ are independent.
**Solution:** Substituting from (9.1.2) and using independence,

$$M_N(s) = E[e^{-(n_1-n_2)s}] = M_{n_1}(s) M_{n_2}(-s) \tag{9.2.2}$$
$$= e^{-s^2\sigma^2} = e^{-s^2} \quad (\because \sigma = 1) \tag{9.2.3}$$

9.3. Show that $N$ is Gaussian. Find its mean and variance. Comment.
**Solution:** From (9.2.3) and (9.1.2), it is obvious that $X \sim \mathcal{N}(0, 2)$. Thus, the difference of two Gaussian random variables is also a Gaussian random variable.

## 10 Uniform to Other: Quantile function

10.1. Generate samples of

$$V = -2\ln(1 - U) \tag{10.1.1}$$

and plot its CDF. Comment.
**Solution:** The CDF of $V$ is defined as

$$F_V(v) = \Pr(V \le v) \tag{10.1.2}$$
$$= \Pr(-2\ln(1 - U) \le v) \tag{10.1.3}$$
$$= \Pr\left(\ln(1 - U) \ge -\frac{v}{2}\right) \tag{10.1.4}$$
$$= \Pr\left(1 - U \ge \exp\left(-\frac{v}{2}\right)\right) \tag{10.1.5}$$
$$= \Pr\left(U \le 1 - \exp\left(-\frac{v}{2}\right)\right) \tag{10.1.6}$$
$$= F_U\left(1 - \exp\left(-\frac{v}{2}\right)\right) \tag{10.1.7}$$

From (**??**),

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & x > 1 \end{cases} \tag{10.1.8}$$

Substituting the above in (10.1.7),

$$F_U\left(1 - \exp\left(-\frac{v}{2}\right)\right) = \begin{cases} 0 & 1 - \exp\left(-\frac{v}{2}\right) < 0 \\ 1 - \exp\left(-\frac{v}{2}\right) & 0 \le 1 - \exp\left(-\frac{v}{2}\right) \le 1 \\ 1 & 1 - \exp\left(-\frac{v}{2}\right) > 1 \end{cases} \tag{10.1.9}$$

After some algebra, the above conditions yield

$$F_V(v) = \begin{cases} 0 & v < 0 \\ 1 - exp\left(-\frac{v}{2}\right) & v \ge 0 \end{cases} \tag{10.1.10}$$

which is the CDF of the exponential distribution with parameter $\frac{1}{2}$. The following code generates the CDF obtained using (10.1.1) and (10.1.10) in Fig. 10.1.1.

```
codes/quantile/ASSIGNMENT_1.py
```

*Comment:* For the uniform distribution, $0 < U < 1$. For any random variable $V, 0 < F_V(x) < 1$. This similarity between $U$ and $F_V(x)$ is used to generate the random variable $V$ from $U$. Thus,

$$V = F_V^{-1}(U) \tag{10.1.11}$$

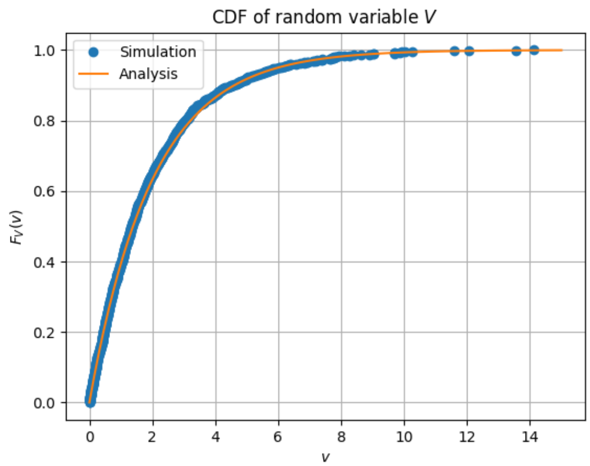


Fig. 10.1.1

where $F_V^{-1}$ is defined to be a *quantile* function.

10.2. Generate the Rayleigh distribution from Uniform. Verify your result through graphical plots.

## 11 Miscellaneous Distributions

11.1. A carton consists of 100 shirts of which 88 are good, 8 have minor defects and 4 have major defects.Jimmy, a trader, will only accept the shirts which are good, but Sujatha, another trader, will only reject the shirts which have major defects.One shirt is drawn at random from the carton. What is the probability that
(i) it is acceptable to Jimmy?
(ii) it is acceptable to Sujatha?
**Solution:** From the given information, Table 11.1 can be generated. Then

| **A Random variable which has 3 possible values** | |
|---|---|
| $\Pr(X = 0) = \frac{4}{100} = 0.04$ | out of 100, 4 have major defects shirts |
| $\Pr(X = 1) = \frac{8}{100} = 0.08$ | out of 100,8 are accepted minor defected shirts |
| $\Pr(X = 2) = \frac{88}{100} = 0.88$ | out of 100,88 are accepted good shirts |

TABLE 11.1: Random variables

a) The probability that the shirt is acceptable to Jimmy is

$$\Pr(X = 2) = \frac{88}{100} \tag{11.1.1}$$

b) The probability that the shirt is acceptable to Sujatha is

$$1 - \Pr(X = 0) = 1 - \frac{4}{100} = \frac{96}{100} \tag{11.1.2}$$

The following code simulates the probability

```
codes/misc/discrete.ipynb
```