Probability

G V V Sharma*

Contents

***Abstract*—This book provides a computational approach to probability and statistics based on the NCERT textbooks from Class 6-12. Links to sample Python codes are available in the text.**

Download python codes using

```
svn co https://github.com/gadepall/school/trunk/
    ncert/probability/codes
```

1 Bernoulli Distribution

1.1. A jar contains 24 marbles, some are green and others are blue. If a marble is drawn at random from the jar, the probability that it is green is $\frac{2}{3}$. Find the number of blue balls in the jar.
Solution: Let the random variable $X = \{0, 1\}$ denote the outcome of the given experiment.
$X = 1$ if the marble picked turns out *Green*.
$X = 0$ if the marble picked turns out *Blue*.
It is given that,

$$P(X = 1) = \frac{2}{3} \tag{1.1.1}$$
$$\implies P(X = 0) = 1 - P(X = 1) \tag{1.1.2}$$
$$\implies P(X = 0) = 1 - \frac{2}{3} \tag{1.1.3}$$
$$\implies P(X = 0) = \frac{1}{3} \tag{1.1.4}$$

Now

$$n(X = 0) + n(X = 1) = 24 \tag{1.1.5}$$
$$\because P(X = 0) = \frac{n(X = 0)}{n(X = 0) + n(X = 1)}, \tag{1.1.6}$$
$$n(X = 0) = P(X = 0)\left(n(X = 0) + n(X = 1)\right) \tag{1.1.7}$$
$$\implies n(X = 0) = \frac{(1) \times (24)}{3} \tag{1.1.8}$$
$$\implies n(X = 0) = 8 \tag{1.1.9}$$

$\therefore$ the number of blue balls is 8.

1.2. A bag contains lemon flavoured candies only. Malini takes out one candy without looking into the bag. What is the probability that she takes out
(i) an orange flavoured candy?
(ii) a lemon flavoured candy? **Solution:** Let the random variable $X = \{0, 1\}$ represent the outcome of the flavour of the candy Malini picks. $X = 0$ denotes an orange flavoured candy, while $X = 1$ denotes a lemon flavoured candy. Then

$$\Pr(X = 0) = 1, \tag{1.2.1}$$
$$\Pr(X = 1) = 0 \tag{1.2.2}$$

1.3.
1.4.
1.5.
1.6.
1.7.
1.8. A person buys a lottery ticket in 50 lotteries, in each of which his chance of winning a prize is $\frac{1}{100}$.What is the probability that he will win a prize
(a) at least once
(b) exactly once
(c) at least twice?
From the given information, the random vari-

*The author is with the Department of Electrical Engineering, Indian Institute of Technology, Hyderabad 502285 India e-mail: gadepall@iith.ac.in.

able representing the trials is

$$X \sim B\left(50, \frac{1}{100}\right) \tag{1.8.1}$$

Hence the desired probabilites are

a)

$$\Pr(X \geq 1) = 1 - \Pr(X = 0) = 1 - \left(\frac{99}{100}\right)^{50} \tag{1.8.2}$$

b)

$$\Pr(X = 1) = 50\left(\frac{99}{100}\right)^{49}\left(\frac{1}{100}\right) \tag{1.8.3}$$

c)

$$\Pr(X \geq 2) = 1 - \Pr(X \leq 1) \tag{1.8.4}$$
$$= 1 - \Pr(X = 0) - \Pr(X = 1) \tag{1.8.5}$$
$$= 1 - \left(\frac{149}{100}\right)\left(\frac{99}{100}\right)^{49} \tag{1.8.6}$$
$$= 0.0894 \tag{1.8.7}$$

1.9. In an examination, 20 questions of true-false type are asked. Suppose a student tosses a fair coin to determine his answer to each question. If the coin falls heads, he answers 'true'; if it falls tails, he answers 'false'. Find the probability that he answers at least 12 questions correctly.
Solution: Let $X_i \in \{0, 1\}$ represent the answer to the ith question. From the given information,

$$\Pr(X = 1) = p = \frac{1}{2} \tag{1.9.1}$$
$$\Pr(X = 0) = q = 1 - p \tag{1.9.2}$$

Defining

$$Y = \sum_{i=1}^{n} X_i, \tag{1.9.3}$$

the desire probability is

$$\Pr(Y \geq 12) = \sum_{i=12} 20^{20}C_i p^i q^{(20-i)} \tag{1.9.4}$$
$$= 0.251722 \tag{1.9.5}$$

$\because$ Y is a Bernoulli distribution with parameters $\left(20, \frac{1}{2}\right)$.

1.10. There are 5% defective items in a large bulk of items. What is the probability that a sample of 10 items will include not more than one defective item?
Solution: Let X be the random variable representing all the items. Then,

$$X = \sum_{i=1}^{10} X_i \tag{1.10.1}$$

has a Binomial distribution with $X_i \in \{0, 1\}$ being a Bernoulli r.v. representing the item condition. From the given information, the probability of an item being defective is given by

$$\Pr(X_i = 0) = \frac{1}{20} = p \tag{1.10.2}$$
$$\implies \Pr(X_i = 1) = q = 1 - p = 1 - \frac{1}{20} = \frac{19}{20} \tag{1.10.3}$$

$\because$

$$X \sim B(n = 10, p = 0.5), \tag{1.10.4}$$
$$\Pr(X = r) = {}^nC_r p^r q^{n-r} \tag{1.10.5}$$

$$\implies Pr(X \leq 1) = Pr(X = 0) + Pr(X = 1)$$
$$= {}^{10}C_0\left(\frac{1}{20}\right)^0\left(\frac{19}{20}\right)^{10} + {}^{10}C_1\left(\frac{1}{20}\right)^1\left(\frac{19}{20}\right)^9$$
$$= \left(\frac{29}{20}\right) \times \left(\frac{19}{20}\right)^9 = 0.9138 \tag{1.10.6}$$

1.11. In a meeting, 70% of the members favour and 30% oppose a certain proposal. A member is selected at random and we take X = 0 if he opposed, and X = 1 if he is in favour. Find E(X) and Var (X).
Solution: From the given information,

$$\Pr(X = 0) = 70\% = 0.7 \tag{1.11.1}$$
$$\Pr(X = 1) = 30\% = 0.3 \tag{1.11.2}$$

Hence,

$$E(X) = 1 \times 0.7 + 0 \times 0.3 = 0.7 \tag{1.11.3}$$
$$E(X^2) = 1^2 \times 0.7 + 0^2 \times 0.3 = 0.7 \tag{1.11.4}$$
$$\implies Var(X) = E(X^2) - [E(X)]^2 \tag{1.11.5}$$
$$= 0.7 - 0.7^2 = 0.21 \tag{1.11.6}$$

1.12. A coin is biased so that the head is 3 times as likely to occur as tail. If the coin is tossed twice, find the probability distribution of number of tails.

1.13. From a lot of 30 bulbs which include 6 defectives, a sample of 4 bulbs is drawn at random with replacement. Find the probability distribution of the number of defective bulbs.

1.14. Probability that A speaks truth is $\frac{4}{5}$. A coin is tossed. A reports that a head appears. The probability that actually there was head is

a) $\frac{4}{5}$
b) $\frac{1}{2}$
c) $\frac{1}{5}$
d) $\frac{2}{5}$

Solution: Let X $\in \{0, 1\}$ be the random variable denoting that A tells truth when X=1

$$\Pr(X = 1) = \frac{4}{5} \tag{1.14.1}$$

$$\Pr(X = 0) = 1 - \Pr(X = 1)$$

$$\Pr(X = 0) = \frac{1}{5} \tag{1.14.2}$$

Let Y $\in \{0, 1\}$ be the random variable denoting that Head appears on the coin when Y=1
As the coin is unbiased,

$$\Pr(Y = 1|X = 1) = \frac{1}{2} \tag{1.14.3}$$

$$\Pr(Y = 1|X = 0) = \frac{1}{2} \tag{1.14.4}$$

Probability that actually there was a head given that A reports a Head
$=\Pr(X = 1|Y = 1)$
From Bayes Theorm,

$$\Pr(X = 1|Y = 1) = \frac{\Pr(X = 1) \times \Pr(Y = 1|X = 1)}{\sum_{i=0}^{1} \Pr(X = i) \times \Pr(Y = 1|X = i)}$$

$$= \frac{\frac{4}{5} \times \frac{1}{2}}{\frac{4}{5} \times \frac{1}{2} + \frac{1}{5} \times \frac{1}{2}}$$

$$= \frac{4}{5}$$

Probability that actually there was a head given that A reports a Head=$\frac{4}{5}$
So, option a) is correct.

1.15. A box of oranges is inspected by examining three randomly selected oranges drawn without replacement. If all the three oranges are good, the box is approved for sale, otherwise, it is rejected. Find the probability that a box containing 15 oranges out of which 12 are good and 3 are bad ones will be approved for sale.
Solution: Let the ith inspection be $X_i \in \{0, 1\}$, where 1 represents a good orange. From the given information,

$$\Pr(X_1 = 1) = \left(\frac{12}{15}\right) \tag{1.15.1}$$

$$\Pr(X_2 = 1|X_1 = 1) = \left(\frac{11}{14}\right) \tag{1.15.2}$$

$$\Pr(X_3 = 1|X_1 = 1, X_2 = 1) = \left(\frac{10}{13}\right) \tag{1.15.3}$$

The probability that the box will be approved for sale is

$$\begin{aligned}
&\Pr(X_1 = 1, X_2 = 1, X_3 = 1) \\
&= \Pr(X_1 = 1) \times \Pr(X_2 = 1|X_1 = 1) \\
&\quad \times \Pr(X_3 = 1|X_1 = 1, X_2 = 1) \\
&= \frac{12}{15} \times \frac{11}{14} \times \frac{10}{13} \\
&= \frac{1320}{2730} = 0.483
\end{aligned} \tag{1.15.4}$$

1.16. Determine P(E/F), if a coin is tossed three times
(i) E : head on third toss , F : heads on first two tosses
(ii) E : at least two heads , F : at most two heads
(iii) E : at most two tails , F : at least one tail
Solution: In an experiment of tossing a coin $n(= 3)$ times, random variable $X \in \{0, 1, 2, 3\}$

follows binomial distribution.
The binomial distribution formula is:

$$\Pr(X = k) = {}^{n}C_{k} \times p^{k} \times (1 - p)^{n-k}$$

Where:

k	total number of "successes"
p	probability of a success on an individual trial
n	number of trials = 3

TABLE 1.16: The binomial distribution formula

(i) From table 1.16, $\Pr(E|F) = 0.5$
(ii) X denotes number of heads. From table 1.16, $\Pr(E|F) = 0.428$
(iii) X denotes number of tails. From table 1.16, $\Pr(E|F) = 0.857$

Pr(Event)	Calculation
$\Pr(F)$	From product rule , $= \frac{1}{2} \times \frac{1}{2}$ $= 0.25$
$\Pr(EF)$	From product rule, $= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$ $= 0.125$
$\Pr(E\|F)$	$= \frac{\Pr(EF)}{\Pr(F)}$ $= 0.5$

TABLE 1.16: Part(i)

Pr(Event)	Calculation
$\Pr(F)$	$= \Pr(X \leq 2)$ $= \Pr(X = 0) + \Pr(X = 1) + \Pr(X = 2)$ $= {}^{3}C_{0}\left(\frac{1}{2}\right)^{3} + {}^{3}C_{1}\left(\frac{1}{2}\right)^{3} + {}^{3}C_{2}\left(\frac{1}{2}\right)^{3}$ $= 0.875$
$\Pr(EF)$	$= \Pr(X = 2)$ $= 0.375$
$\Pr(E\|F)$	$=\frac{\Pr(EF)}{\Pr(F)}$ $= 0.428$

TABLE 1.16: Part(ii)

Pr(Event)	Calculation
$\Pr(F)$	$= \Pr(X \geq 1)$ $= 1 - \Pr(0)$ $= 0.875$
$\Pr(EF)$	$= \Pr(X = 1) + \Pr(X = 2)$ $= 0.75$
$\Pr(E\|F)$	$= \frac{\Pr(EF)}{\Pr(F)}$ $= 0.857$

TABLE 1.16: Part(iii)

1.17. Determine P(E/F), if two coins are tossed once, where
(i) E : tail appears on one coin, F : one coin shows head
(ii) E : no tail appears, F : no head appears
Solution: Let X denote the number of heads shown on the coins, where n = 2 and p = 0.5, q = 1-p

$$p(x) = \Pr(X = x) = \binom{n}{x} \times p^{x} \times q^{n-x} \quad (1.17.1)$$

X	0	1	2
P(X)	$\binom{2}{0}(0.5)^2 = \frac{1}{4}$	$\binom{2}{1}(0.5)^2 = \frac{1}{2}$	$\binom{2}{2}(0.5)^2 = \frac{1}{4}$

TABLE 1.17: Probability of number of heads shown on the coins

a)

$$\Pr(F) = \Pr(X \geq 1) \quad (1.17.2)$$
$$\Pr(F) = \Pr(X = 1) + \Pr(X = 2)$$
$$= \frac{1}{2} + \frac{1}{4} = \frac{3}{4} \quad (1.17.3)$$
$$\Pr(EF) = \Pr(X = 1) = \frac{1}{2} \quad (1.17.4)$$
$$\Pr(E/F) = \frac{\Pr(EF)}{\Pr(F)} = \frac{2}{3} \quad (1.17.5)$$

b)

$$\Pr(F) = \Pr(X = 0) = \frac{1}{4} \quad (1.17.6)$$
$$\Pr(EF) = 0 \quad (1.17.7)$$
$$\Pr(E/F) = \frac{\Pr(EF)}{\Pr(F)} = 0 \quad (1.17.8)$$

1.18. Two players, Sangeeta and Reshma, play a tennis match. It is known that the probability of Sangeeta winning the match is 0.62. What is

the probability of Reshma winning the match?
Solution: The desired probability is $1-0.62 = 0.38$.

1.19. Harpreet tosses two different coins simultaneously (say, one is of rupee 1 and other of rupee 2). What is the probability that she gets at least one head?

1.20. In a cricket match, a batswoman hits a boundary 6 times out of 30 balls she plays. Find the probability that she did not hit a boundary.
Solution: Let the sample space be $X \in \{0, 1\}$. From the given information, the probability of hitting a boundary is

$$\Pr(X = 1) = \frac{6}{30} \tag{1.20.1}$$
$$= \frac{1}{5} \tag{1.20.2}$$

Hence, the probability of not hitting the boundary is

$$\Pr(X = 0) = 1 - \Pr(X = 1) = 1 - \frac{1}{5} \tag{1.20.3}$$
$$= \frac{4}{5} \tag{1.20.4}$$

1.21. A coin is tossed 1000 times with the following frequencies:
Head : 455, Tail : 545
Compute the probability for each event.
Solution: Let $X \in \{0, 1\}$ represent the random variable, where 0 represents head and 1 represents tail. From the given information,

$$\Pr(X = 0) = \frac{455}{1000} \tag{1.21.1}$$
$$= 0.45 \tag{1.21.2}$$
$$\Pr(X = 1) = 1 - \Pr(X = 0) \tag{1.21.3}$$
$$= 0.545 \tag{1.21.4}$$

Codes for the above are available in

```
solutions/1−10/codes/probexm/probexm1.py
```

1.22. Two coins are tossed simultaneously 500 times, and we get
Two heads : 105 times
One head : 275 times
No head : 120 times
Find the probability of occurrence of each of these events.
Solution: Let $X_1 \{0, 1\}$ represent the first coin and $X_2 \{0, 1\}$ represent the second coin, where 0 represents tail and 1 represents head. Define

$$X = X_1 + X_2, \tag{1.22.1}$$

Hence $X \in \{0, 1, 2\}$. From the given information,

$$\Pr(X = 1) = \frac{105}{500} \tag{1.22.2}$$
$$= 0.21 \tag{1.22.3}$$
$$\Pr(X = 2) = \frac{275}{500} \tag{1.22.4}$$
$$= 0.55 \tag{1.22.5}$$
$$\Pr(X = 0) = \frac{120}{500} \tag{1.22.6}$$
$$= 0.24 \tag{1.22.7}$$

1.23. A die is thrown 1000 times with the frequencies for the outcomes 1, 2, 3, 4, 5 and 6 as given in the following Table 1.23. Find the probability of getting each outcome.

Outcome	1	2	3	4	5	6
Frequency	179	150	157	149	175	190

TABLE 1.23

Solution: Let $X \in \{i\}_{i=1}^{6}$ and f_i be the correspnding frequency. Then,

$$\Pr(X = i) = \frac{f_i}{1000} \tag{1.23.1}$$

The following code computes the probabilities

```
solutions/1−10/codes/probexm/probexm3.py
```

1.24. The record of a weather station shows that out of the past 250 consecutive days, its weather forecasts were correct 175 times.
(i) What is the probability that on a given day it was correct?
(ii) What is the probability that it was not correct on a given day?
Solution: Let $X \in \{0, 1\}$ be the random variable with 1 denoting correct forecast. From the given information,

$$\Pr(X = 1) = \frac{175}{250} \tag{1.24.1}$$
$$= 0.7 \tag{1.24.2}$$
$$\Pr(X = 0) = 1 - \Pr(X = 1) \tag{1.24.3}$$
$$= 0.3 \tag{1.24.4}$$

1.25. *: Random Process*A and B throw a die alternatively till one of them gets a '6' and wins the game. Find their respective probabilities of winning, if A starts first.

Solution: Let $X_k \in \{1, 2, 3, 4, 5, 6\}$ be the discrete random process representing the trials. Then, the odd trials belong to A and the even trials belong to B. Then, the probability that someone wins at the nth trial is

$$\Pr(X_n = 6 | X_k \neq 6, k = 1, 2, \ldots, n-1) = \frac{1}{6}\left(\frac{5}{6}\right)^{n-1}. \tag{1.25.1}$$

The probability that A wins is obtained by summing up over the even probabilities

$$\sum_{m=0}^{\infty} \Pr(X_{2m+1} = 6 | X_k \neq 6, k = 1, 2, \ldots, n-1) = \frac{1}{6}\sum_{m=0}^{\infty}\left(\frac{5}{6}\right)^{2m} = \frac{\frac{1}{6}}{1 - \left(\frac{5}{6}\right)^2} = \frac{6}{11} \tag{1.25.2}$$

The probabilty that B wins is then given by

$$1 - \frac{6}{11} = \frac{5}{11} \tag{1.25.3}$$

The python code for the above problem is,

solutions/20−10/prob/codes/exam46.py

In the above code 1000000 random outputs of a die are generated for A and B each.The probabilities are calculated using the total number of times A gets a six first and the total number of times B get a six first.

1.26. To know the opinion of the students about the subject statistics, a survey of 200 students was conducted. The data is recorded in Table 1.26 Find the probability that a student chosen at random

a) likes statistics,
b) does not like it.

Opinion	**Number of students**
like	135
dislike	65

TABLE 1.26

Solution: Let $X \in \{0, 1\}$ be the random variable denoting dislikes and likes.

a)

$$\Pr(X = 1) = \frac{135}{200} \tag{1.26.1}$$
$$= 0.675 \tag{1.26.2}$$

b)

$$\Pr(X = 0) = \frac{65}{200} = 0.325 \tag{1.26.3}$$

1.27. Assume that each born child is equally likely to be a boy or a girl. If a family has two children, what is the conditional probability that both are girls given that
(i) the youngest is a girl,
(ii) at least one is a girl?

Solution: Let $X \in \{0, 1\}$ represent the gender where 1 represents a girl. Let $Y_1, Y_2 \in \{0, 1\}$ represent the child in the family, where Y_1 denotes the older child.

a) Since Y_1, Y_2 are independent,

$$\Pr(Y_1 = 1, Y_2 = 1 | Y_2 = 1) = \frac{1}{2} \tag{1.27.1}$$

b)

$$\begin{aligned}
&\Pr(Y_1 = 1, Y_2 = 1 | 1 - \{Y_2 = 0, Y_1 = 0\}) \\
&= \frac{\Pr(\{Y_1 = 1\}\{Y_2 = 1\}[1 - \{Y_2 = 0\}\{Y_1 = 0\}])}{1 - \Pr(Y_2 = 0, Y_1 = 0)} \\
&= \frac{\Pr(\{Y_1 = 1\}\{Y_2 = 1\}) - \Pr(\{Y_1 = 1\}\{Y_2 = 1\}\{Y_1 = 0\}\{Y_2 = 0\})}{1 - \Pr(Y_2 = 0, Y_1 = 0)} \\
&= \frac{\Pr(\{Y_1 = 1\}\{Y_2 = 1\})}{1 - \Pr(Y_2 = 0, Y_1 = 0)} = \frac{\frac{1}{4}}{1 - \frac{1}{4}} = \frac{1}{3}
\end{aligned} \tag{1.27.2}$$

1.28. An instructor has a question bank consisting of 300 easy True / False questions, 200 difficult True / False questions, 500 easy multiple choice questions and 400 difficult multiple choice questions. If a question is selected at random from the question bank, what is the probability that it will be an easy question given that it is a multiple choice question?

Solution: Let $X \in \{0, 1\}$ where 0 represents an easy question. Let $Y \in \{0, 1\}$ where 1 denotes multiple choice questions. From the given information,

$$\Pr(X = 0, Y = 0) = \frac{300}{1400} = \frac{3}{14} \tag{1.28.1}$$
$$\Pr(X = 1, Y = 0) = \frac{200}{1400} = \frac{2}{14} \tag{1.28.2}$$
$$\Pr(X = 0, Y = 1) = \frac{500}{1400} = \frac{5}{14} \tag{1.28.3}$$
$$\Pr(X = 1, Y = 1) = \frac{400}{1400} = \frac{4}{14} \tag{1.28.4}$$

Then,

$$\Pr(X = 0|Y = 1) = \frac{\Pr(X = 0, Y = 1)}{\Pr(Y = 1)} \tag{1.28.5}$$

$$= \frac{\Pr(X = 0, Y = 1)}{\sum_i \Pr(X = i, Y = 1)} \tag{1.28.6}$$

$$= \frac{\frac{5}{14}}{\frac{5}{14} + \frac{4}{14}} \tag{1.28.7}$$

$$= \frac{5}{9} \tag{1.28.8}$$

1.29. Two cards are drawn at random and without replacement from a pack of 52 playing cards. Find the probability that both the cards are black.
Solution: Let $X_1, X_2 \in \{0, 1\}$ represent the colour, where 0 denotes black and 1 denotes red. From the given information,

$$\Pr(X_1 = 0) = \frac{26}{52} = \frac{1}{2} \tag{1.29.1}$$

$$\Pr(X_2 = 0|X_1 = 0) = \frac{25}{51} \tag{1.29.2}$$

Then,

$$\Pr(X_1 = 0, X_2 = 0)$$
$$= \Pr(X_2 = 0|X_1 = 0)\Pr(X_1 = 0) = \frac{25}{102} \tag{1.29.3}$$

1.30. Two balls are drawn at random with replacement from a box containing 10 black and 8 red balls. Find the probability that
(i) both balls are red.
(ii) first ball is black and second is red.
(iii) one of them is black and other is red.
Solution: Let $X \in \{0, 1\}$ where 0 represents black.

a) Probability of picking a black ball

$$\Pr(X = 0) = \frac{10}{18} = \frac{5}{9} \tag{1.30.1}$$

b) Probability of picking a red ball

$$\Pr(X = 1) = 1 - \Pr(X = 0) = \frac{4}{9} \tag{1.30.2}$$

c) Two balls are drawn with replacement. So each event is independent of each other. Probability that both balls are red

$$\Pr(X_1 = 1, X_2 = 1) = \left(\frac{4}{9}\right)^2 \tag{1.30.3}$$

$$= \frac{16}{81} \tag{1.30.4}$$

d) Probability that first ball is black and second is red

$$\Pr(X_1 = 0, X_2 = 1) = \frac{5}{9} \times \frac{4}{9} \tag{1.30.5}$$

$$= \frac{20}{81} \tag{1.30.6}$$

e) Probability that one ball is black and other is red

$$\Pr(X_1 = 0, X_2 = 1) + \Pr(X_1 = 1, X_2 = 0)$$
$$= \frac{16}{81} + \frac{20}{81} = \frac{4}{9} \tag{1.30.7}$$

f) The python code for finding probability using a sample size of 10000 can be downloaded from

```
solutions/40−50/probability/codes/Q41.py
```

1.31. Probability of solving specific problem independently by A and B are $\frac{1}{2}$ and $\frac{1}{3}$ respectively. If both try to solve the problem independently, find the probability that
(i) the problem is solved
(ii) exactly one of them solves the problem.
Solution: Let $A, B \in \{0, 1\}$ where 1 indicates solving a problem. Given that

$$\Pr(A = 1) = \frac{1}{2}, \Pr(B = 1) = \frac{1}{3} \tag{1.31.1}$$

a) A problem is solved when either A or B solves the problem or both solve the problem. So the probability that problem is solved

$$\Pr(A = 1, B = 0) + \Pr(A = 0, B = 1)$$
$$+ \Pr(A = 1, B = 1)$$
$$= \frac{1}{2} \times \frac{2}{3} + \frac{1}{2} \times \frac{1}{3} + \frac{1}{2} \times \frac{1}{3}$$
$$= \frac{2}{3} \tag{1.31.2}$$

b) Probability that exactly one of them solves the problem is

$$\Pr(A=1,B=0)+\Pr(A=0,B=1)$$
$$=\frac{1}{2}\times\frac{2}{3}+\frac{1}{2}\times\frac{1}{3}$$
$$=\frac{1}{2} \quad (1.31.3)$$

solutions/40−50/probability/codes/Q42.py

1.32. (i) A lot of 20 bulbs contain 4 defective ones. One bulb is drawn at random from the lot. What is the probability that this bulb is defective?
(ii) Suppose the bulb drawn in (i) is not defective and is not replaced. Now one bulb is drawn at random from the rest. What is the probability that this bulb is not defective ?
Solution: Let $X \in \{0,1\}$.

a) The probability of drawing a defective bulb is

$$\Pr(X=0)=\frac{4}{20}=\frac{1}{5} \quad (1.32.1)$$

b) After drawing a non defective bulb, The probability of drawing a non-defective bulb is

$$\Pr(X=1)=\frac{15}{19} \quad (1.32.2)$$

The python code for the above solution is

solutions/20−10/prob/codes/exer128.py

1.33. 12 defective pens are accidentally mixed with 132 good ones. It is not possible to just look at a pen and tell whether or not it is defective. One pen is taken out at random from this lot. Determine the probability that the pen taken out is a good one.
Solution: Let $X\{0,1\}$ represent the good and bad pens respectively. The probability of taking out a good pen is

$$\Pr(X=0)=\frac{132}{144}=\frac{11}{12} \quad (1.33.1)$$

The python code for the above solution is

solutions/20−10/prob/codes/pens127.py

1.34. Gopi buys a fish from a shop for his aquarium. The shopkeeper takes out one fish at random from a tank containing 5 male fish and 8 female fish (see Fig. 1.34). What is the probability that the fish taken out is a male fish?

Fig. 1.34

Solution: Let $X \in \{0,1\}$ represent the male and female fish respectively. Then the desired probability is

$$\Pr(X=0)=\frac{5}{5+8}=\frac{5}{13} \quad (1.34.1)$$

The python code for the distribution is

solutions/20−10/prob/codes/fish.py

The code checks how many times a male fish is picked out of the total times (taken as 100,000 in the given code) a fish is picked up from the tank with replacement.

1.35. A lot consists of 144 ball pens of which 20 are defective and the others are good. Nuri will buy a pen if it is good, but will not buy if it is defective. The shopkeeper draws one pen at random and gives it to her. What is the probability that
(i) She will buy it ?

(ii) She will not buy it ?
Solution: The sample size

$$S = 144 \tag{1.35.1}$$

The number of bad pens is

$$B = 20 \tag{1.35.2}$$

The probability that she doesn't buy a pen is

$$\Pr(B) = \frac{B}{S} = \frac{20}{144} \tag{1.35.3}$$

$$= \frac{5}{36} \tag{1.35.4}$$

The probability that she buys a pen is

$$\Pr(G) = 1 - \Pr(B) = \frac{31}{36} \tag{1.35.5}$$

The python code for the distribution is

```
solutions/10−1/prob/codes/prob2_a.py
```

1.36. A bag contains 5 red balls and some blue balls. If the probability of drawing a blue ball is double that of a red ball, determine the number of blue balls in the bag.
Solution: Let $X \in \{0, 1\}$ where 0 represents red. From the given information, if the number of blue balls is x,

$$\Pr(X = 1) = 2\Pr(X = 0) \tag{1.36.1}$$

$$\implies \frac{x}{x+5} = 2 \times \frac{5}{x+5} \tag{1.36.2}$$

$$\implies x = 10 \tag{1.36.3}$$

1.37. A box contains 12 balls out of which x are black. If one ball is drawn at random from the box,what is the probability that it will be a black ball?
If 6 more black balls are put in the box, the probability of drawing a black ball is now double of what it was before. Find x.
Solution: Let $X \in \{0, 1\}$ such that 0 represents black. Then,

$$\Pr(X = 0) = \frac{x}{12} \tag{1.37.1}$$

If 6 more black balls are put in the bag,

$$\Pr(X = 0) = \frac{x+6}{12+6} \tag{1.37.2}$$

From the given information

$$\frac{x+6}{12+6} = \frac{2x}{12} \tag{1.37.3}$$

$$\implies x = 3 \tag{1.37.4}$$

1.38. There are 40 students in Class X of a school of whom 25 are girls and 15 are boys. The class teacher has to select one student as a class representative. She writes the name of each student on a separate card, the cards being identical.Then she puts cards in a bag and stirs them thoroughly. She then draws one card from the bag. What is the probability that the name written on the card is the name of
(i) a girl?
(ii) a boy?
Solution: Let the random variable X = {0,1} represent the outcome whether the picked card has a girl's name or a boy's name. Then

$$\implies \Pr(X = 0) = \frac{25}{40} \tag{1.38.1}$$

$$\implies \Pr(X = 1) = \frac{15}{40} \tag{1.38.2}$$

1.39. A bag contains 3 red balls and 5 black balls. A ball is drawn at random from the bag. What is the probability that the ball drawn is
(i) red ?
(ii) not red? Total number of marbles = 3 + 5 = 8 marbles
Let $X \in \{0, 1\}$ represent the random variable, where 0 represents a red marble, 1 represents a black marble. From the given information,

a) Probability that the ball taken out will be red $= \Pr(X = 0)$

$$\Pr(X = 0) = \frac{\text{number of red balls}}{\text{total number of balls}} \tag{1.39.1}$$

$$\Pr(X = 0) = \frac{3}{8} = 0.375 \tag{1.39.2}$$

b) Probability that the marble taken out will not be red $= \Pr(X = 1)$
Because the complementary of $\Pr(X = 0)$ is $\Pr(X = 1)$
We know that the sum of probabilities of every random variable is 1. So,

$$\Pr(X = 1) + \Pr(X = 0) = 1 \tag{1.39.3}$$

$$\implies \Pr(X=1) = 1 - \Pr(X=0) \quad (1.39.4)$$
$$= 1 - 0.375 \quad (1.39.5)$$
$$= 0.625 \quad (1.39.6)$$

$$\implies \Pr(X=1) = 1 - 0.375 = 0.625 \quad (1.39.7)$$

2 Bayes Rule

2.1. Bag I contains 3 red and 4 black balls and Bag II contains 4 red and 5 black balls. One ball is transferred from Bag I to Bag II and then a ball is drawn from Bag II. The ball so drawn is found to be red in colour. Find the probability that the transferred ball is black.

Solution: Let the input variables $X \in \{0,1\}$ and $Y \in \{0,1\}$ be defined according to the table 2.1 .

Input Variable	Value	Description
X	0	Ball drawn from Bag II is Red
	1	Ball drawn from Bag II is Black
Y	0	Transferred ball from Bag I is Red
	1	Transferred ball from Bag I is Black

TABLE 2.1: Input Variables

Given data of the question in terms of probability is presented in the table 2.1 .

S.No.	Expression	Value
1.	$\Pr(X=0\|Y=0)$	$\frac{1}{2}$
2.	$\Pr(X=0\|Y=1)$	$\frac{4}{10}=\frac{2}{5}$
3.	Pr(Y=0)	$\frac{3}{7}$
4.	Pr(Y=1)	$\frac{4}{7}$

TABLE 2.1: Given Data

Hence,probability that the transferred ball from Bag I is Black given that the ball drawn from Bag II is Red is,

$$\Pr(Y=1|X=0) = \frac{\Pr(X=0|Y=1)\Pr(Y=1)}{\sum_{i=0}^{1}\Pr(X=0|Y=i)\Pr(Y=i)} \quad (2.1.1)$$
$$= \frac{\frac{2}{5}\times\frac{4}{7}}{\frac{1}{2}\times\frac{3}{7}+\frac{2}{5}\times\frac{4}{7}} \quad (2.1.2)$$
$$= \frac{16}{31} \quad (2.1.3)$$

2.2. Suppose we have four boxes A,B,C and D containing coloured marbles as given below:

Box	Red	White	Black
A	1	6	3
B	6	2	2
C	8	1	1
D	0	6	4

One of the boxes has been selected at random and a single marble is drawn from it. If the marble is red, what is the probability that it was drawn from box A?, box B?, box C?

Solution: The description of the random variables is available in Table 2.2. From the given information,

	A	B	C	D
X	0	1	2	3

	Red	White	Black
Y	0	1	2

TABLE 2.2

$$\Pr(X=0) = \Pr(X=1) = \Pr(X=2) \quad (2.2.1)$$
$$= \Pr(X=3) = \frac{1}{4} \quad (2.2.2)$$
$$\Pr(Y=0|X=0) = \frac{1}{10} \quad (2.2.3)$$
$$(2.2.4)$$
$$\Pr(Y=0|X=1) = \frac{6}{10} \quad (2.2.5)$$
$$(2.2.6)$$
$$\Pr(Y=0|X=2) = \frac{8}{10} \quad (2.2.7)$$
$$\Pr(Y=0|X=3) = 0 \quad (2.2.8)$$

Thus,

$$\Pr(Y=0) = \sum_{i=0}^{3}\Pr(Y=0|X=i)\Pr(X=i) \quad (2.2.9)$$
$$= \frac{3}{8} \quad (2.2.10)$$

a)

$$Pr(X = 0|Y = 0) = \frac{Pr(Y = 0|X = 0)Pr(X = 0)}{Pr(Y = 0)} = \frac{1}{15} \quad (2.2.11)$$

b)

$$Pr(X = 1|Y = 0) = \frac{Pr(Y = 0|X = 1)Pr(X = 1)}{Pr(Y = 0)} = \frac{2}{5} \quad (2.2.12)$$

c)

$$Pr(X = 2|Y = 0) = \frac{Pr(Y = 0|X = 2)Pr(X = 2)}{Pr(Y = 0)} = \frac{8}{15} \quad (2.2.13)$$

2.3. Assume that the chances of a patient having a heart attack is 40%. It is also assumed that a meditation and yoga course reduce the risk of heart attack by 30% and prescription of certain drug reduces its chances by 25%. At a time a patient can choose any one of the two options with equal probabilities. It is given that after going through one of the two options the patient selected at random suffers a heart attack. Find the probability that the patient followed a course of meditation and yoga?
Solution: Let $H \in \{0, 1\}$ denote the random variable of the patient having a heart attack, $A \in \{0, 1\}$ denote the random variable of the patient taking a meditation and yoga course, or the patient taking the drug. ($A = 0$ if the patient took a meditation and yoga course, and $A = 1$ if the patient took the prescription of the drug.)

Given that,

$$\Pr(H = 1) = 0.4 \quad (2.3.1)$$

$$\Pr(A = 0) = \Pr(A = 1) \quad (2.3.2)$$

$$\Pr(H = 1|A = 0) = \Pr(H = 1)(1 - 0.30) \quad (2.3.3)$$

$$= 0.28 \quad (2.3.4)$$

$$\Pr(H = 1|A = 1) = \Pr(H = 1)(1 - 0.25) \quad (2.3.5)$$

$$= 0.3 \quad (2.3.6)$$

$$(2.3.7)$$

Therefore, by Bayes' Theorem

$$\Pr(A = 0|H = 1) = \frac{\Pr(H = 1|A = 0)\Pr(A = 0)}{\sum_{i=0}^{1}\Pr(H = 1|A = i)\Pr(A = i)} \quad (2.3.8)$$

$$(2.3.9)$$

We can cancel $\Pr(A = 1)$ and $\Pr(A = 0)$ from the numerator and denominator as they are given to be equal.

$$\therefore \Pr(A = 0|H = 1) = \frac{\Pr(H = 1|A = 0)}{\Pr(H = 1|A = 0) + \Pr(H = 1|A = 1)} \quad (2.3.10)$$

$$= \frac{0.28}{0.28 + 0.3} \quad (2.3.11)$$

$$(2.3.12)$$

$$= \frac{0.28}{0.58} \quad (2.3.13)$$

$$(2.3.14)$$

$$= \frac{14}{29} \quad (2.3.15)$$

$$\approx 0.48275862069 \quad (2.3.16)$$

$$(2.3.17)$$

2.4. Suppose that 5% of men and 0.25% of women have grey hair. A grey haired person is selected at random. What is the probability of this person being male? Assume that there are equal number of males and females.
Solution: Let A=0,1 represent the random variable for being male or female and G=0,1 represent having grey hair or not. Then,

$$P(A=0) = 50\% = \frac{1}{2} \tag{2.4.1}$$
$$P(A=1) = 50\% = \frac{1}{2} \tag{2.4.2}$$
$$P(G=1|A=0) = 5\% = \frac{1}{20} \tag{2.4.3}$$
$$P(G=1|A=1) = 0.25\% = \frac{1}{400} \tag{2.4.4}$$

By Bayes rules,

$$P(A=0|G=1) = \frac{P(0) \times P(G=1|0)}{\{\sum_{i=0}^{1} \Pr(G)} \tag{2.4.5}$$

$$\because \sum_{i=0}^{1} \Pr(G) = P(0) \times P(G=1|0) + P(1) \times P(G=1|1) \tag{2.4.6}$$

$$P(A=0|G=1) = \frac{\frac{1}{2} \times \frac{1}{20}}{\frac{1}{2} \times \frac{1}{20} + \frac{1}{2} \times \frac{1}{400}} \tag{2.4.7}$$

$$P(A=0|G=1) = \frac{20}{21} \tag{2.4.8}$$

which is the desired probability.

2.5. A couple has two children,
(i) Find the probability that both children are males, if it is known that at least one of the children is male.
(ii) Find the probability that both children are females, if it is known that the elder child is a female.

2.6. A manufacturer has three machine operators A, B and C. The first operator A produces 1% defective items, where as the other two operators B and C produce 5% and 7% defective items respectively. A is on the job for 50% of the time, B is on the job for 30% of the time and C is on the job for 20% of the time. A defective item is produced, what is the probability that it was produced by A?
Solution: Let X $\in \{0, 1, 2\}$ be the random variable denoting that item was produced by operator A when X=0, X=1 denoting that item was produced by operator B ,X=2 denoting that item was produced by operator C, and random variable Y $\in \{0, 1\}$ be the random variable deoting that item produced was defective when Y=1.

$$\Pr(X=0) = 0.5 \tag{2.6.1}$$
$$\Pr(X=1) = 0.3 \tag{2.6.2}$$
$$\Pr(X=2) = 0.2 \tag{2.6.3}$$
$$\Pr(Y=1/X=0) = 0.01 \tag{2.6.4}$$
$$\Pr(Y=1/X=1) = 0.05 \tag{2.6.5}$$
$$\Pr(Y=1/X=2) = 0.07 \tag{2.6.6}$$

From conditional probability we say that

$$\Pr(X=0/Y=1) = \frac{\Pr(Y=1/X=0)\Pr(X=0)}{\sum_{i=0}^{i=2} \Pr(Y=1/X=i)\Pr(X=i)}$$
$$= \frac{5}{34}$$

2.7. A factory has two machines A and B. Past record shows that machine A produced 60% of the items of output and machine B produced 40% of the items. Further, 2% of the items produced by machine A and 1% produced by machine B were defective. All the items are put into one stockpile and then one item is chosen at random from this and is found to be defective. What is the probability that it was produced by machine B?
Solution: Let $A \in \{0, 1\}$ denote the random variables of an item produced and $D\{0, 1\}$ denote it being defective From the given information,

$$P(A=0) = 60\% = 0.6 \tag{2.7.1}$$
$$P(A=1) = 40\% = 0.4 \tag{2.7.2}$$

$$P(D=1|A=0) = P(1|0) = 2\% = 0.02 \tag{2.7.3}$$
$$P(D=1|A=1) = P(1|1) = 1\% = 0.01 \tag{2.7.4}$$

By Baye's rule,

$$P(A = 1|D = 1) = \frac{P(1) \times P(1|1)}{P(1) \times P(1|1) + P(0) \times P(1|0)}$$

$$P(A = 1|D = 1) = \frac{0.4 \times 0.01}{0.4 \times 0.01 + 0.6 \times 0.02}$$

$$P(A = 1|D = 1) = 0.25 \quad (2.7.5)$$

The probability that the defective item selected at random is produced by machine B is 25%

2.8. Two groups are competing for the position on the Board of directors of a corporation. The probabilities that the first and the second groups will win are 0.6 and 0.4 respectively. Further, if the first group wins, the probability of introducing a new product is 0.7 and the corresponding probability is 0.3 if the second group wins. Find the probability that the new product introduced was by the second group.
Solution:

- Let X and Y be the input variables which can be referred from the table 2.11:-

X	X=1 : New product is introduced
	X=0 : No New product
Y	Y=1 : First group wins
	Y=0 : Second group wins

TABLE 2.11: Assumed Variables

- Furthermore, Data given is tabularised in the table 2.12 :-

	Expression	Value
a.)	Pr(Y=1)	0.6
b.)	Pr(Y=0)	0.4
c.)	$\Pr(X = 1\|Y = 1)$	0.7
d.)	$\Pr(X = 1\|Y = 0)$	0.3

TABLE 2.12: Data Given

- So, the probability that new product is introduced by the second group can be given using **Baye's Theorem** as:-

$$\implies \Pr(Y = 0|X = 1) = \frac{\Pr(X = 1|Y = 0)\Pr(Y = 0)}{\sum_{i=0}^{1} \Pr(X = 1|Y = i)\Pr(Y = i)} \quad (2.8.1)$$

$$\implies \Pr(Y = 0|X = 1) = \frac{0.3 \times 0.4}{0.3 \times 0.4 + 0.7 \times 0.6} \quad (2.8.2)$$

$$\implies \Pr(Y = 0|X = 1) = \frac{0.12}{0.54} = \frac{2}{9} \quad (2.8.3)$$

$$\boxed{\therefore \Pr(Y = 0|X = 1) = \frac{2}{9}} \quad (2.8.4)$$

2.9. A laboratory blood test is 99% effective in detecting a certain disease when it is in fact, present. However, the test also yields a false positive result for 0.5% of the healthy person tested (i.e. if a healthy person is tested, then, with probability 0.005, the test will imply he has the disease). If 0.1 percent of the population actually has the disease, what is the probability that a person has the disease given that his test result is positive?
Solution: Let the input variables $X \in \{0, 1\}$ and $Y \in \{0, 1\}$ be defined according to the table 2.13 .

Input Variable	Value	Description
X	0	Absence of disease in test
	1	Presence of disease in test
Y	0	Absence of disease in reality
	1	Presence of disease in reality

TABLE 2.13: Input Variables

Given data of the question is presented in the table 2.14 .

S.No.	Expression	Value
1.	$\Pr(X = 1\|Y = 1)$	$\frac{99}{100}$
2.	$\Pr(X = 1\|Y = 0)$	$\frac{0.5}{100} = \frac{1}{200}$
3.	Pr(Y=1)	$\frac{0.1}{100} = \frac{1}{1000}$
4.	Pr(Y=0)	$\frac{99.9}{100} = \frac{999}{1000}$

TABLE 2.14: Given Data

Hence,probability that a person has the disease given that his test result is positive ,is given by

$$\Pr(Y = 1|X = 1) = \frac{\Pr(X = 1|Y = 1)\Pr(Y = 1)}{\sum_{i=0}^{1} \Pr(X = 1|Y = i)\Pr(Y = i)} \tag{2.9.1}$$

$$= \frac{\frac{99}{100} \times \frac{1}{1000}}{\frac{1}{200} \times \frac{999}{1000} + \frac{99}{100} \times \frac{1}{1000}} \tag{2.9.2}$$

$$= \frac{22}{133} \tag{2.9.3}$$

2.10. A family has two children. What is the probability that both the children are boys given that at least one of them is a boy?
Solution: X - Random variable for number of boys.

$$X = \{0, 1, 2\}$$

where $n = 2$ and $p = \frac{1}{2}$

X = x	$\Pr(X = x)$
X = 0	${}^2C_0 \times q^2$
X = 1	${}^2C_1 \times q \times p$
X = 2	${}^2C_2 \times p^2$

To find $\Pr(X = 2 \,|\, X \geq 1)$.

$$\Pr(X = 2 \,|\, X \geq 1) = \frac{\Pr(X = 2)}{\Pr(X \geq 1)} \tag{2.10.1}$$

$$= \frac{\frac{1}{4}}{\frac{3}{4}} \tag{2.10.2}$$

$$= \frac{1}{3} \tag{2.10.3}$$

2.11. Ten cards numbered 1 to 10 are placed in a box, mixed up thoroughly and then one card is drawn randomly. If it is known that the number on the drawn card is more than 3, what is the probability that it is an even number?
Solution: The set of sample space which contains cards numbered from 1 to 10 be

$$S \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \tag{2.11.1}$$

Now probability of picking a random card from ten cards is be $Pr(x)$. Then,

$$Pr(x \in S) = \frac{1}{10} \tag{2.11.2}$$

Let the Set of Even numbered cards be E and Set of cards numbered greater than 3 is A. and we know that the set of even numbers from 1 to 10 is {2,4,6,8,10} . Then,

$$E \in \{2, 4, 6, 8, 10\} \tag{2.11.3}$$

Now here, The Set of numbers greater than 3 is {4,5,6,7,8,9,10}. Then,

$$A \in \{4, 5, 6, 7, 8, 9, 10\} \tag{2.11.4}$$

$$Pr(A) = \frac{No.\ of\ elements\ in\ (A).}{Number\ of\ cards.} \tag{2.11.5}$$

$$Pr(A) = \frac{7}{10} = 0.7 \tag{2.11.6}$$

Now, the favoured outcomes are set of EA and Here the set EA contains the cards which are even and numbered greater than 3.

$$Pr(EA) = \frac{No.\ of\ elements\ in\ EA.}{Number\ of\ cards.} \tag{2.11.7}$$

$$Now, Pr(EA) = Pr(4) + Pr(6) + Pr(8) + Pr(10) \tag{2.11.8}$$

$$Pr(EA) = \frac{4}{10} = 0.4 \tag{2.11.9}$$

The probability that the card drawn is even number which is greater than 3 is

$$Pr(E|A) = \frac{Pr(EA)}{Pr(A)} \tag{2.11.10}$$

Now,using (2.11.6) and (2.11.9).

$$(\because Pr(EA) = 0.4\ and\ Pr(A) = 0.7) \tag{2.11.11}$$

$$And, Pr(E|A) = \frac{Pr(EA)}{Pr(A)} \tag{2.11.12}$$

$$= \frac{0.4}{0.7} \tag{2.11.13}$$

$$\therefore Pr(E|A) = \frac{4}{7} \tag{2.11.14}$$

$$\therefore Pr(E|A) = 0.5714285714285714 \tag{2.11.15}$$

2.12. In a school, there are 1000 students, out of which 430 are girls. It is known that out of 430, 10 percentage of the girls study in class XII.

What is the probability that a student chosen randomly studies in Class XII given that the chosen student is a girl?
Solution: Total number of students : 1000
Total number of girls : 430
Total number of girls in Class XII : 10 % of total girls

$$= \frac{10}{100} \times 430 = 43 \tag{2.12.1}$$

Let X ∈ {0, 1} be the random variable such that 1 represents girl, 0 represents boy.

TABLE 2.15: Probability distribution for values of X

X	P(X)
1	430/1000
0	570/100

Let Y ∈ {0, 1} be the random variable such that 1 represents chosen student is in Class XII, 0 represents chosen student is not in Class XII.

TABLE 2.16: Probability distribution for values of Y

Y	P(Y)
1	$\frac{1}{2}$
0	$\frac{1}{2}$

We require P(Y=1 | X=1) (using Baye's theorem)

$$= \frac{P(X = 1|Y = 1) \cdot P(Y = 1)}{\sum_{i=0}^{1} P(X = 1|Y = i) \cdot P(Y = i)} \tag{2.12.2}$$

$$= \frac{P(X=1|Y=1)P(()Y=1)}{P(X = 1|Y = 0)P(Y = 0) + P(X = 1|Y = 1)P(Y = 1)}$$

TABLE 2.17: Probability for different values of X,Y

Probability	Chosen Student	Value
$P(X = 1\|Y = 0)$	girl not in Class XII	$\frac{387}{1000}$
$P(X = 1\|Y = 1)$	girl in Class XII	$\frac{43}{1000}$

$$= \left[\frac{(()\frac{43}{1000} \times \frac{1}{2})}{(()\frac{387}{1000}) \times \frac{1}{2}) + (()\frac{43}{1000} \times \frac{1}{2})}\right] \tag{2.12.3}$$

$$= \frac{1}{10} \tag{2.12.4}$$

$$= 0.1 \tag{2.12.5}$$

2.13. A die is thrown three times. Events A and B are defined as below:
A : 4 on the third throw.
B : 6 on the first and 5 on the second throw.
Find the probability of A given that B has already occurred?
Solution: Let $X_i \in \{1, 2, 3, 4, 5, 6\}$ where $i = 1, 2, 3$ be the random variables representing the outcomes of throwing a die three times.

a) Probability of event A happening=Probability of $X_3 = 4$

$$\Pr(A) = \Pr(X_3 = 4) \tag{2.13.1}$$

Since all the outcomes are equally likely their probabilities are same
so

$$\Pr(A) = \Pr(X_3 = 4) = \frac{1}{6} \tag{2.13.2}$$

b) Probability of event B happening=Probability of $X_1 = 6, X_2 = 5$.
so

$$\Pr(B) = \Pr(X_1 = 6, X_2 = 5) \tag{2.13.3}$$

Random variable X_1 depends on first throw of die and random variable X_2 depends on second throw of die so X_1 and X_2 are independent.
so

$$\Pr(X_1 = 6, X_2 = 5) = \Pr(X_1 = 6)\Pr(X_2 = 5) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36} \tag{2.13.4}$$

$$\Pr(B) = \Pr(X_1 = 6, X_2 = 5) = \frac{1}{36} \tag{2.13.5}$$

Also A,B are also independent events therefore from (2.13.2) and (2.13.5)

$$\Pr(AB) = \Pr(A)\Pr(B) = \frac{1}{6} \times \frac{1}{36} \tag{2.13.6}$$

$$\implies \Pr(AB) = \frac{1}{216} \tag{2.13.7}$$

Since we have to find probability of A given that B has already happened.
so $\Pr(A|B)$

c) By formula of conditional probability

$$\Pr(A|B) = \frac{\Pr(AB)}{\Pr(B)} \tag{2.13.8}$$

From (2.13.5) and (2.13.7)

$$\implies \Pr(A|B) = \frac{\frac{1}{216}}{\frac{1}{36}} \tag{2.13.9}$$

$$\implies \Pr(A|B) = \frac{1}{6} \tag{2.13.10}$$

So the probability of A given that B has already happened = $\Pr(A|B) = \frac{1}{6}$

2.14. A die is thrown twice and the sum of the numbers appearing is observed to be 6. What is the conditional probability that the number 4 has appeared at least once?
Solution: Let $A \in \{2,3,4,5,6,7,8,9,10,11,12\}$ be a random variable representing the sum of outcomes when a die is thrown twice.
Let $B \in \{0,1,2\}$ be a random variable that represents the number of times 4 occurs in two throws.
We need the conditional probability of event $(B \geq 1)$ given that $(A = 6)$ has occurred.

$$\Pr((B \geq 1) \,|\, (A = 6)) = \frac{\Pr((A = 6) \cap (B \geq 1))}{\Pr(A = 6)} \tag{2.14.1}$$

We have that,

$$\Pr(A = n) = \begin{cases} 0 & n \leq 1 \\ \frac{n-1}{36} & 2 \leq n \leq 7 \\ \frac{13-n}{36} & 8 \leq n \leq 12 \\ 0 & n \geq 13 \end{cases} \tag{2.14.2}$$

Therefore using equation (0.0.2) we can write that,

$$\Pr(A = 6) = \frac{5}{36} \tag{2.14.3}$$

From binomial distribution we can write ,

$$\Pr(B \geq 1) = \Pr(B = 1) + \Pr(B = 2) \tag{2.14.4}$$

$$= \binom{2}{1}\left(\frac{1}{6}\right)\left(\frac{5}{6}\right) + \binom{2}{2}\left(\frac{1}{6}\right)^2 \tag{2.14.5}$$

$$= \frac{11}{36} \tag{2.14.6}$$

The event $((A = 6) \cap (B \geq 1))$ is such that the sum should be six 6 with atleast one 4.
There are only two possible cases {4,2},{2,4} out of 36 possible cases.
Hence,

$$\Pr((A = 6) \cap (B \geq 1)) = \frac{2}{36}. \tag{2.14.7}$$

Substituting equations (2.14.3),(2.14.7) in (2.14.1) , we get

$$\begin{aligned} \Pr((B \geq 1)|(A = 6)) &= \frac{\frac{2}{36}}{\frac{5}{36}} \\ &= \frac{2}{5}. \end{aligned} \tag{2.14.8}$$

Hence the probability of occurring atleast one 4 when the sum of the numbers is 6 when a die is thrown twice is $\frac{2}{5}$.

2.15. Consider the experiment of tossing a coin. If the coin shows head, toss it again but if it shows tail, then throw a die. Find the conditional probability of the event that ”the die shows a number greater than 4” given that ”there is at least one tail”.
Solution: Given that a coin is tossed. If coin shows head, it is tossed again. If it shows tail, then a die is thrown.
Let X $\in \{0,1\}$ be the random variable such that 1 represents occurrence of tail,0 represents occurrence of head when coin is tossed.

TABLE 2.18: Probability distribution for values of X

X	P(X)
1	$\frac{1}{2}$
0	$\frac{1}{2}$

Let Y denotes random variable for the getting a number on the die thrown, then the probability distribution is

TABLE 2.19: Probability distribution for values of Y

Y	1	2	3	4	5	6
P(Y)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$$\Pr(X = 1) = \sum_{i=1}^{6} \Pr(X = 1, Y = i) + \Pr(X = 0, X = 1) \tag{2.15.1}$$

$$= \frac{3}{4} \tag{2.15.2}$$

$$\Pr(X = 1, Y > 4) = \Pr(X = 1, Y = 5) + \Pr(X = 1, Y = 6) \tag{2.15.3}$$

$$= \frac{1}{6} \tag{2.15.4}$$

$$\Pr(Y > 4|X = 1) = \frac{\Pr(Y > 4, X = 1)}{\Pr(X = 1)} \tag{2.15.5}$$

$$= \frac{2}{9} \tag{2.15.6}$$

2.16. An urn contains 10 black and 5 white balls. Two balls are drawn from the urn one after the other without replacement. What is the probability that both drawn balls are black?

2.17. Three cards are drawn successively, without replacement from a pack of 52 well shuffled cards. What is the probability that first two cards are kings and the third card drawn is an ace?

2.18. A man is known to speak truth 3 out of 4 times. He throws a die and reports that it is a six. Find the probability that it is actually a six.
Solution: Let $X \in \{0, 1\}$ and $Y \in \{0, 1\}$ where, From the given information we have,

X	0	Man says truth
	1	Man lie
Y	0	Die shows six
	1	Die shows number other than 6

TABLE 2.20

$$Pr(X = 0) = \frac{3}{4} \tag{2.18.1}$$

$$Pr(X = 1) = \frac{1}{4} \tag{2.18.2}$$

$$Pr(Y = 0|X = 0) = \frac{1}{6} \tag{2.18.3}$$

$$Pr(Y = 0|X = 1) = \frac{5}{6} \tag{2.18.4}$$

The probability that the die shows 6 given the man reports it is six,

$$Pr(X = 0|Y = 0) = \frac{Pr(Y = 0|X = 0)Pr(X = 0)}{\sum_{i=0}^{1} Pr(Y = 0|X = i)Pr(X = i)} \tag{2.18.5}$$

Substituting the given values in (2.18.5),

$$Pr(X = 0|Y = 0) = \frac{\frac{1}{6}\frac{3}{4}}{\frac{1}{6}\frac{3}{4} + \frac{5}{6}\frac{1}{4}} = \frac{3}{8} \tag{2.18.6}$$

2.19. A person has undertaken a construction job. The probabilities are 0.65 that there will be strike, 0.80 that the construction job will be completed on time if there is no strike, and 0.32 that the construction job will be completed on time if there is a strike. Determine the probability that the construction job will be completed on time.
Solution: Let S denote strike and J denote job. From the given information,

$$\Pr(S) = 0.65, \Pr(J|S') = 0.8, \Pr(J|S) = 0.32 \tag{2.19.1}$$

Then,

$$\Pr(J) = \Pr(JS) + \Pr(JS') \tag{2.19.2}$$

$$= \Pr(J|S)\Pr(S) + \Pr(J|S')\Pr(S') \tag{2.19.3}$$

$$= \Pr(J|S)\Pr(S) + \Pr(J|S')(1 - \Pr(S)) \tag{2.19.4}$$

$$= (0.65)(0.32) + (0.35)(0.80) = 0.488 \tag{2.19.5}$$

2.20. Bag I contains 3 red and 4 black balls while another Bag II contains 5 red and 6 black balls. One ball is drawn at random from one of the bags and it is found to be red. Find the probability that it was drawn from Bag II.
Solution: Let $X \in 1, 2$ represent the Bag and $Y \in \{0, 1\}$ represent the colour, where 1 denotes red. From the given information,

$$\Pr(X = 1) = \Pr(X = 2) = \frac{1}{2} \tag{2.20.1}$$

$$\Pr(Y = 1|X = 1) = \frac{3}{7} \tag{2.20.2}$$

$$\Pr(Y = 1|X = 2) = \frac{5}{11} \tag{2.20.3}$$

Thus,

$$\Pr(X=2|Y=1) = \frac{\Pr(X=2, Y=1)}{\Pr(Y=1)} \quad (2.20.4)$$

$$= \frac{\Pr(Y=1|X=2)\Pr(X=2)}{\Pr(Y=1|X=1)\Pr(X=1) + \Pr(Y=1|X=2)\Pr(X=2)} \quad (2.20.5)$$

$$= \frac{\frac{5}{11} \times \frac{1}{2}}{\frac{3}{7} \times \frac{1}{2} + \frac{5}{11} \times \frac{1}{2}} \quad (2.20.6)$$

$$= \frac{35}{68} \quad (2.20.7)$$

2.21. Given three identical boxes I, II and III, each containing two coins. In box I, both coins are gold coins, in box II, both are silver coins and in the box III, there is one gold and one silver coin. A person chooses a box at random and takes out a coin. If the coin is of gold, what is the probability that the other coin in the box is also of gold?

Solution: Let $X \in \{1, 2, 3\}$ represent the box and $Y_1, Y_2 \in \{0, 1\}$ represent the coins, 1 representing gold. Then,

$$\Pr(X=1) = \Pr(X=2) = \Pr(X=3) = \frac{1}{3} \quad (2.21.1)$$

$$\Pr(Y_1 = 1, Y_2 = 1|X=1) = 1, \quad (2.21.2)$$

$$\Pr(Y_1 = 1, Y_2 = 1|X=2) = 0 \quad (2.21.3)$$

$$\Pr(Y = 1, Y_2 = 0|X=3) = \Pr(Y_1 = 1, Y_2 = 0|X=3) = \frac{1}{2} \quad (2.21.4)$$

Then

$$\Pr(Y_1 = 1|Y_2 = 1) = \frac{\Pr(Y_1 = 1, Y_2 = 1)}{\Pr(Y_2 = 1)} \quad (2.21.5)$$

Now,

$$\Pr(Y_1 = 1, Y_2 = 1) = \sum_i \Pr(Y_1 = 1, Y_2 = 1, X = i)$$

$$= \sum_i \Pr(Y_1 = 1, Y_2 = 1|X = i)\Pr(X = i) = \frac{1}{3} \quad (2.21.6)$$

and

$$\Pr(Y_2 = 1) = \Pr(Y_1 = 1, Y_2 = 1) + \Pr(Y_1 = 0, Y_2 = 1)$$
$$= \sum_i \Pr(Y_1 = 1, Y_2 = 1|X = i)\Pr(X = i) + \sum_i \Pr(Y_1 = 0, Y_2 = 1|X = i)\Pr(X = i)$$
$$= \frac{1}{3} + \frac{1}{6} = \frac{1}{2} \quad (2.21.7)$$

Substituting from (2.21.6) and (2.21.7) in (2.21.5),

$$\Pr(Y_1 = 1|Y_2 = 1) = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3} \quad (2.21.8)$$

2.22. Suppose that the reliability of a HIV test is specified as follows: Of people having HIV, 90% of the test detect the disease but 10% go undetected. Of people free of HIV, 99% of the test are judged HIV –ve but 1% are diagnosed as showing HIV +ve. From a large population of which only 0.1% have HIV, one person is selected at random, given the HIV test, and the pathologist reports him/her as HIV +ve. What is the probability that the person actually has HIV?

Solution: Let $X, Y \in \{0, 1\}$ represent HIV with 1 being positive. From the given information,

$$\Pr(X=1|Y=1) = \frac{9}{10}, \Pr(X=0|Y=1) = \frac{1}{10} \quad (2.22.1)$$

$$\Pr(X=1|Y=0) = \frac{1}{100}, \Pr(X=0|Y=0) = \frac{99}{100} \quad (2.22.2)$$

$$\Pr(Y=1) = \frac{1}{1000}, \Pr(Y=0) = \frac{999}{1000} \quad (2.22.3)$$

Then,

$$\Pr(Y=1|X=1) = \frac{\Pr(X=1|Y=1)\Pr(Y=1)}{\Pr(X=1|Y=1)\Pr(Y=1) + \Pr(X=1|Y=0)\Pr(Y=0)}$$
$$= \frac{\frac{9}{10} \times \frac{1}{1000}}{\frac{9}{10} \times \frac{1}{1000} + \frac{1}{100} \times \frac{999}{1000}} = \frac{10}{121} \quad (2.22.4)$$

2.23. In a factory which manufactures bolts, machines A, B and C manufacture respectively 25%, 35% and 40% of the bolts. Of their outputs, 5, 4 and 2 percent are respectively defective bolts. A bolt is drawn at random from the product and is found to be defective. What is the probability that it is manufactured by the machine B?

Solution: Let $X \in \{1, 2, 3\}$ represent the machines and $Y \in \{0, 1\}$ represent the bolt quality, 0 denoting defective bolt. From the given information,

$$\Pr(X = 1) = \frac{25}{100} \tag{2.23.1}$$

$$\Pr(X = 2) = \frac{35}{100} \tag{2.23.2}$$

$$\Pr(X = 3) = \frac{40}{100} \tag{2.23.3}$$

and

$$\Pr(Y = 0|X = 1) = \frac{5}{100} \tag{2.23.4}$$

$$\Pr(Y = 0|X = 2) = \frac{4}{100} \tag{2.23.5}$$

$$\Pr(Y = 0|X = 3) = \frac{2}{100} \tag{2.23.6}$$

Then,

$$\Pr(X = 2|Y = 0)$$

$$= \frac{\Pr(Y = 0|X = 2)\Pr(X = 2)}{\Pr(Y = 0|X = 1)\Pr(X = 1) + \Pr(Y = 0|X = 2)\Pr(X = 2) + \Pr(Y = 0|X = 3)\Pr(X = 3)}$$

$$= \frac{\frac{4}{100} \times \frac{35}{100}}{\frac{5}{100} \times \frac{25}{100} + \frac{4}{100} \times \frac{35}{100} + \frac{2}{100} \times \frac{40}{100}}$$

$$= \frac{28}{69} \tag{2.23.7}$$

2.24. A doctor is to visit a patient. From the past experience, it is known that the probabilities that he will come by train, bus, scooter or by other means of transport are respectively $\frac{3}{10}, \frac{1}{5}, \frac{1}{10}$ and $\frac{2}{5}$. The probabilities that he will be late are $\frac{1}{4}, \frac{1}{3}$ and $\frac{1}{12}$, if he comes by train, bus and scooter respectively, but if he comes by other means of transport, then he will not be late. When he arrives, he is late. What is the probability that he comes by train?
Solution: Let $X \in \{0, 1, 2, 3\}$ represent the mode of travel and $Y \in \{0, 1\}$ represent the time, where 0 denotes being late. From the given information,

$$\Pr(X = 1) = \frac{3}{10} \tag{2.24.1}$$

$$\Pr(X = 2) = \frac{1}{5} \tag{2.24.2}$$

$$\Pr(X = 3) = \frac{1}{10} \tag{2.24.3}$$

$$\Pr(X = 4) = \frac{2}{5} \tag{2.24.4}$$

and

$$\Pr(Y = 0|X = 1) = \frac{1}{4} \tag{2.24.5}$$

$$\Pr(Y = 0|X = 2) = \frac{1}{3} \tag{2.24.6}$$

$$\Pr(Y = 0|X = 3) = \frac{1}{12} \tag{2.24.7}$$

$$\Pr(Y = 0|X = 4) = 0 \tag{2.24.8}$$

Then,

$$\Pr(X = 1|Y = 0)$$

$$= \frac{\Pr(Y = 0|X = 1)\Pr(X = 1)}{\sum_{i=1}^{4} \Pr(Y = 0|X = i)\Pr(X = i)}$$

$$= \frac{\frac{3}{10} \times \frac{1}{4}}{\frac{1}{4} \times \frac{3}{10} + \frac{1}{3} \times \frac{1}{5} + \frac{1}{12} \times \frac{1}{10}} = \frac{1}{2} \tag{2.24.9}$$

2.25. Coloured balls are distributed in four boxes as shown in Table 2.21

Box	Black	White	Red	Blue
I	3	4	5	6
II	2	2	2	2
III	1	2	3	1
IV	4	3	1	5

TABLE 2.21: Distribution of the balls in the boxes

A box is selected at random and then a ball is randomly drawn from the selected box. The colour of the ball is black, what is the probability that ball drawn is from the box III?
Solution: Let $B \in \{1, 2, 3, 4\}$ denote the box number in sequence and $C \in \{1, 2, 3, 4\}$ denote the colours Black, White, Red and Blue respectively. Given that a black ball is selected, the probability that it is picked from box III is

$$\Pr(B = 3|C = 1)$$

$$= \frac{\Pr(C = 1|B = 3)\Pr(B = 3)}{\sum_{j=1}^{4} \Pr(C = 1|B = j)\Pr(B = j)} \tag{2.25.1}$$

From Table 2.21,

$$\Pr(C = 1|B = 1) = \frac{1}{6} \tag{2.25.2}$$

$$\Pr(C = 1|B = 2) = \frac{1}{4} \tag{2.25.3}$$

$$\Pr(C = 1|B = 3) = \frac{1}{7} \tag{2.25.4}$$

$$\Pr(C = 1|B = 4) = \frac{4}{13} \tag{2.25.5}$$

and

$$\Pr(B = 1) = \Pr(B = 2) = \Pr(B = 3) = \Pr(B = 4) = \frac{1}{4} \tag{2.25.6}$$

Substituting from (2.25.5) and (2.25.6) in (2.25.1),

$$\Pr(B = 3|C = 1) = \frac{156}{947} \tag{2.25.7}$$

The python code for the above problem is,

solutions/20−10/prob/codes/exam43.py

2.26. If a machine is correctly set up, it produces 90% acceptable items. If it is incorrectly set up, it produces only 40% acceptable items. Past experience shows that 80% of the set ups are correctly done. If after a certain set up, the machine produces 2 acceptable items, find the probability that the machine is correctly setup.
Solution: Let $X \in \{0, 1\}$ denote the setup. Let $Y_1, Y_2 \in \{0, 1\}$ denote the item production such that Y_1 and Y_2 are independent. Then, from the given information,

$$\Pr(Y_1 = 1|X = 1) = \Pr(Y_2 = 1|X = 1) \tag{2.26.1}$$

$$= \frac{90}{100} = \frac{9}{10} \tag{2.26.2}$$

$$\Pr(Y_1 = 1|X = 0) = \Pr(Y_2 = 1|X = 0) \tag{2.26.3}$$

$$= \frac{40}{10} = \frac{2}{5} \tag{2.26.4}$$

$$\Pr(X = 1) = \frac{80}{100} = \frac{4}{5} \tag{2.26.5}$$

Then

$$\Pr(X = 1|Y_1 = 1, Y_2 = 1) = \frac{\Pr(X = 1, Y_1 = 1, Y_2 = 1)}{\Pr(Y_1 = 1, Y_2 = 1)}$$
$$= \frac{\Pr(Y_1 = 1, Y_2 = 1|X = 1)\Pr(X = 1)}{\Pr(Y_1 = 1, Y_2 = 1|X = 1)\Pr(X = 1) + \Pr(Y_1 = 1, Y_2 = 1|X = 0)\Pr(X = 0)} \tag{2.26.6}$$

which can be expressed as

$$\frac{\prod_{k=1}^{2}\Pr(Y_k = 1|X = 1)\Pr(X = 1)}{\sum_{i=0}^{1}\prod_{k=1}^{2}\Pr(Y_k = 1|X = i)\Pr(X = i)}$$
$$= \frac{\left(\frac{9}{10}\right)^2\left(\frac{4}{5}\right)}{\left(\frac{9}{10}\right)^2\left(\frac{4}{5}\right) + \left(\frac{2}{5}\right)^2\left(\frac{1}{5}\right)} = \frac{81}{85} \tag{2.26.7}$$

The python code for the above problem is,

solutions/20−10/prob/codes/exam47.py

2.27. A bag contains a red ball, a blue ball and a yellow ball, all the balls being of the same size.Kritika takes out a ball from the bag without looking into it. What is the probability that she takes out the (i) yellow ball?
(ii) red ball?
(iii) blue ball?
Solution: Let the random variable representing the events be $X \in \{0, 1, 2\}$ Then

$$\Pr(X = i) = \frac{1}{3}, \quad i = 0, 1, 2. \tag{2.27.1}$$

The python code for the distribution is

solution/20−10/prob/codes/exam49.py

2.28. An urn contains 5 red and 5 black balls. A ball is drawn at random, its colour is noted and is returned to the urn. Moreover, 2 additional balls of the colour drawn are put in the urn and then a ball is drawn at random. What is the probability that the second ball is red?
Solution: Let $X \in 0, 1$ where 0 represents black. Let X_1 represent the event representing drawing the first ball. X_2 represent the event of drawing the second ball. Then probability of the second ball being red is

$$\Pr(X_2 = 1)$$
$$= \Pr(X_2 = 1, X_1 = 1) + \Pr(X_2 = 1, X_1 = 0)$$
$$= \Pr(X_2 = 1|X_1 = 1)\Pr(X_1 = 1) + \Pr(X_2 = 1|X_1 = 0)\Pr(X_1 = 0) \tag{2.28.1}$$

From the given information,

$$\Pr(X_1 = 0) = \Pr(X_1 = 1) = \frac{5}{10} = \frac{1}{2}. \quad (2.28.2)$$

Also,

$$\Pr(X_2 = 1|X_1 = 0) = \frac{5}{5+2+7} = \frac{5}{12} \quad (2.28.3)$$

$$\Pr(X_2 = 1|X_1 = 1) = \frac{5+2}{5+2+5} = \frac{7}{12} \quad (2.28.4)$$

Thus,

$$\Pr(X_2 = 1) = \frac{7}{12} \times \frac{1}{2} + \frac{5}{12} \times \frac{1}{2} = \frac{1}{2} \quad (2.28.5)$$

The python code for finding probability using a sample size of 10000 can be downloaded from

```
solutions/40−50/probability/codes/Q47.py
```

2.29. A bag contains 4 red and 4 black balls, another bag contains 2 red and 6 black balls. One of the two bags is selected at random and a ball is drawn from the bag which is found to be red. Find the probability that the ball is drawn from the first bag.
Solution: Let $X \in \{0,1\}$ represent the bags, where 0 represents the first bag and $Y \in \{0,1\}$ represent the colour, 0 being black. The desired probability is

$$\Pr(X = 0|Y = 1) = \frac{\Pr(Y = 1|X = 0)\Pr(X = 0)}{\Pr(Y = 1|X = 0)\Pr(X = 0) + \Pr(Y = 1|X = 0)\Pr(X = 0)} \quad (2.29.1)$$

From the given information,

$$\Pr(X = 0) = \Pr(X = 1) = \frac{1}{2} \quad (2.29.2)$$

$$\Pr(Y = 1|X = 0) = \frac{1}{2} \quad (2.29.3)$$

$$\Pr(Y = 1|X = 1) = \frac{1}{4} \quad (2.29.4)$$

Hence,

$$\Pr(X = 0|Y = 1) = \frac{\frac{1}{2} \times \frac{1}{2}}{\frac{1}{2} \times \frac{1}{2} + \frac{1}{4} \times \frac{1}{2}} \quad (2.29.5)$$

$$= \frac{2}{3} \quad (2.29.6)$$

```
solutions/40−50/probability/codes/Q48.py
```

2.30. Of the students in a college, it is known that 60% reside in hostel and 40% are day scholars (not residing in hostel). Previous year results report that 30% of all students who reside in hostel attain A grade and 20% of day scholars attain A grade in their annual examination. At the end of the year, one student is chosen at random from the college and he has an A grade, what is the probability that the student is a hostelier?
Solution: Let $X \in \{0,1\}$ represent student residence, 0 being a hostel residence. Let $Y\{0,1\}$ represent the grade, 0 being A grade. The objective is to find

$$\Pr(X = 0|Y = 0) = \frac{\Pr(Y = 0|X = 0)\Pr(X = 0)}{\Pr(Y = 0|X = 0)\Pr(X = 0) + \Pr(Y = 0|X = 1)\Pr(X = 1)} \quad (2.30.1)$$

From the given information,

$$\Pr(Y = 0|X = 0) = \frac{3}{10} \quad (2.30.2)$$

$$\Pr(Y = 0|X = 1) = \frac{2}{10} \quad (2.30.3)$$

$$\Pr(X = 0) = \frac{6}{10} \quad (2.30.4)$$

$$\Pr(X = 1) = \frac{4}{10} \quad (2.30.5)$$

Hence,

$$\Pr(X = 0|Y = 0) = \frac{\frac{3}{10} \times \frac{6}{10}}{\frac{3}{10} \times \frac{6}{10} + \frac{2}{10} \times \frac{4}{10}} = \frac{9}{13} \quad (2.30.6)$$

The python code for finding probability using a sample size of 10000 can be downloaded from

```
probability/codes/Q49.py
```

2.31. In answering a question on a multiple choice test, a student either knows the answer or guesses. Let $\frac{3}{4}$ be the probability that he knows the answer and $\frac{1}{4}$ be the probability that he guesses. Assuming that a student who guesses at the answer will be correct with probability $\frac{1}{4}$. What is the probability that the student knows the answer given that he answered it correctly?
Solution: Let $X \in \{0,1\}$ represent student knowledge where 0 denotes a guess. Let $Y = \{0,1\}$ represent the correctness of the answer, with 0 being the case when the answer is incorrect. Then, we need to find

$$\Pr(X = 1|Y = 1) = \frac{\Pr(Y = 1|X = 1)\Pr(X = 1)}{\Pr(Y = 1|X = 1)\Pr(X = 1) + \Pr(Y = 1|X = 0)\Pr(X = 0)} \quad (2.31.1)$$

From the given information,

$$\Pr(Y = 1|X = 0) = \frac{1}{4} \quad (2.31.2)$$
$$\Pr(Y = 1|X = 1) = 1 \quad (2.31.3)$$
$$\Pr(X = 0) = \frac{3}{4} \quad (2.31.4)$$
$$\Pr(X = 1) = \frac{1}{4} \quad (2.31.5)$$

$\because$ if the student knows the answer, she will definitely be correct. Hence,

$$\Pr(X = 1|Y = 1) = \frac{\frac{1}{4}}{\frac{1}{4} + \frac{1}{4} \times \frac{3}{4}} = \frac{4}{7} \quad (2.31.6)$$

solutions/40−50/probability/codes/Q50.py

3 Binomial Distribution

3.1. How many times must a man toss a fair coin so that the probability of having at least one head is more than 90%?

3.2. An experiment succeeds twice as often as it fails. Find the probability that in the next six trials, there will be at least 4 successes.
Solution:
As per question,

$$p = 2(1 - p) \quad (3.2.1)$$
$$\implies p = 2/3 \quad (3.2.2)$$

For a binomial distribution,

$$\Pr(X = k) = {}^{n}C_{k} p^{k} (1 - p)^{n-k} \quad (3.2.3)$$

For the given question, From (3.2.3) we have,

Variable	n	p
Value	6	2/3

TABLE 3.1: Value of variables

$$\Pr(X \geq 4) = \sum_{i=4}^{6} {}^{6}C_{i} p^{i} (1 - p)^{6-i} \quad (3.2.4)$$
$$= \frac{240}{729} + \frac{192}{729} + \frac{64}{729} \quad (3.2.5)$$
$$= \frac{496}{729} \quad (3.2.6)$$

3.3. Suppose X has a binomial distribution . Show that X = 3 is the most likely outcome.
Solution: Let X be a binomial random variable which has probability $p = \frac{1}{2}$

$$p(X) = \frac{1}{2} \quad (3.3.1)$$

Given number of times event(X) is performed(n) = 6

$$n(x) = 6 \quad (3.3.2)$$

Given probability of event(p) = $\frac{1}{2}$
Probability that event(X) does not occur is $(1 - p) = 1 - \frac{1}{2} = \frac{1}{2}$

$$1 - p(x) = \frac{1}{2} \quad (3.3.3)$$

We know that binomial probability

$$\Pr(X = k) = {}^{n}C_{k} p^{k} (1 - p)^{n-k} \quad (3.3.4)$$

For $\Pr(X = k)$ to be most likely outcome(highest probability), $\Pr(X = k)$ should be maximum ,where
$k = \{0, 1, 2, 3, 4, 5, 6\}$
To find maximum of $\Pr(X = k)$,let us apply logarithm on both sides for equation (3.3.4) and then diffenrentiate it with respect to p.

$$\log \Pr(X = k) = \log {}^{n}C_{k} \times p^{k} \times (1 - p)^{n-k} \quad (3.3.5)$$
$$= \log {}^{n}C_{k} + k \times \log p + (n - k) \times log(1 - p) \quad (3.3.6)$$

Differentaiate eq (3.3.6) with respect to p

$$\frac{\mathrm{d} \log \Pr(X = k)}{\mathrm{d}p} = \frac{\mathrm{d} \log {}^{n}C_{k}}{\mathrm{d}p} + k \times \frac{\mathrm{d} \log p}{\mathrm{d}p} \quad (3.3.7)$$
$$+ (n - k) \times \frac{\mathrm{d} \log(1 - p)}{\mathrm{d}p}$$
$$= 0 + \frac{k}{p} - \frac{n - k}{1 - p} \quad (3.3.8)$$

To find maximum ,substitute $\frac{\mathrm{d} \log \Pr(X=k)}{\mathrm{d}p} = 0$ in

(3.3.8)

$$\frac{k}{p} = \frac{n-k}{1-p} \quad (3.3.9)$$

$$\frac{n-k}{k} = \frac{1-p}{p} \quad (3.3.10)$$

$$\frac{n}{k} - 1 = \frac{1}{p} - 1 \quad (3.3.11)$$

$$\frac{n}{k} = \frac{1}{tp} \quad (3.3.12)$$

$$k = n \times p \quad (3.3.13)$$

substituting $n = 6, p = 1 - p = \frac{1}{2}$ in (3.3.13)
We get $k = 6 \times \frac{1}{2} = 3$ $\Pr(X = 3)$ is maximum,
$\therefore \Pr(X = 3)$ is most likely
outcome.
(Hint : P(X = 3) is the maximum among all P(x_i), x_i= 0,1,2,3,4,5,6)

3.4. The probability that a bulb produced by a factory will fuse after 150 days of use is 0.05. Find the probability that out of 5 such bulbs
(i) none
(ii) not more than one
(iii) more than one
(iv) at least one
will fuse after 150 days of use.
Solution: Let X be random variable which denoting number of bulbs fuses after 150 days of use,among the 5 bulbs.Then by Binomial Distribution.

$$\Pr(X = r) = \binom{n}{r} p^r q^{n-r} \quad (3.4.1)$$

$$\Pr(X \geq k) = \sum_{r=k}^{n} \binom{n}{r} p^r q^{n-r} \quad (3.4.2)$$

$$\Pr(X \leq k) = \sum_{r=0}^{k} \binom{n}{r} p^r q^{n-r} \quad (3.4.3)$$

$$\Pr(X > k) = \sum_{r=k+1}^{n} \binom{n}{r} p^r q^{n-r} \quad (3.4.4)$$

$$n = 5, \quad p = 0.05, \quad q = 0.95 \quad (3.4.5)$$

n	5	5	5	5
Condition	$\Pr(X = 0)$	$\Pr(X \leq 1)$	$\Pr(X > 1)$	$\Pr(X \geq 1)$
Value	0.77378	0.97740	0.02259	0.22621
Case	(i)	(ii)	(iii)	(iv)

TABLE 3.2: Probability Vs Condition

3.5. Find the mean number of heads in three tosses of a fair coin.
Solution: By observing we can get 4 cases which are 0 heads, 1 heads, 2 heads, 3 heads respectively when 3 fair coins are tossed simultaneously.
Let X_i be the number of heads in ith case .
So we can get the probability of number of heads each case as

$$\Pr(X = k) = \begin{cases} {}^3C_k \left(\frac{1}{2}\right)^k \left(1 - \frac{1}{2}\right)^{3-k} & 0 \leq k \leq 3 \\ 0 & otherwise \end{cases} \quad (3.5.1)$$

$$\Pr(X = k) = \begin{cases} {}^3C_k \left(\frac{1}{2}\right)^3 & 0 \leq k \leq 3 \\ 0 & otherwise \end{cases} \quad (3.5.2)$$

The probability distribution table is

i	1	2	3	4
X_i	0	1	2	3
$\Pr(X = X_i)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

TABLE 3.3: Probability distribution table

Hence by substituing values of $n = 3$ and $p = \frac{1}{2}$, we get

$$E(X) = np \quad (3.5.3)$$

$$= 3 \times \frac{1}{2} \quad (3.5.4)$$

$$= \frac{3}{2} \quad (3.5.5)$$

$$= 1.5 \quad (3.5.6)$$

3.6. Find the probability distribution of
(i) number of heads in two tosses of a coin.
(ii) number of tails in the simultaneous tosses of three coins.
(iii) number of heads in four tosses of a coin.
Solution:

$$\Pr(X = k) = \begin{cases} {}^nC_k p^k (1-p)^{n-k} & 0 \leq k \leq n \\ 0 & \text{otherwise} \end{cases} \quad (3.6.1)$$

Table (3.4) presents the solution to each case.

TABLE 3.4: Table of Probability distribution of different cases

Case	n (no. of coins)	k (no. of required outcomes)	0	1	2	3	4
(i)	2	$\Pr(X=k) = {}^2C_k\left(\frac{1}{2}\right)^2$	1/2	1/4	1/2	0	0
(ii)	3	$\Pr(X=k) = {}^3C_k\left(\frac{1}{2}\right)^3$	1/8	3/8	3/8	1/8	0
(iii)	4	$\Pr(X=k) = {}^4C_k\left(\frac{1}{2}\right)^4$	1/16	1/4	3/8	1/4	1/16

3.7. Let X represent the difference between the number of heads and the number of tails obtained when a coin is tossed 6 times. What are possible values of X?

3.8. Six balls are drawn successively from an urn containing 7 red and 9 black balls. Tell whether or not the trials of drawing balls are Bernoulli trials when after each draw the ball drawn is
(i) replaced
(ii) not replaced in the urn.
Solution: Properties to be satisfied if a trial needs to be a bernoulli trial:

a) Number of trials should be finite.
b) each trial should have utcomes of success and failure.
c) if P is the success probability then failure probability should be 1-P
d) probability of success should not vary with trial

Case(i):Replaced
Number of red balls = 7
Number of black balls = 9
let X be the random variable and

- X=1 is success which is Drawing red ball
- X=0 is Failure which is Drawing black ball

Success Probability

$$P(X=1) = \frac{7}{16} \tag{3.8.1}$$

Failure Probability

$$P(X=0) = \frac{9}{16} = 1 - P(X=1) \tag{3.8.2}$$

Success Probability is constant for all Trials. as X satisfies all properties of Bernoulli therefore Trials are Bernoulli Trials.

Case(ii):Not Replaced

In this case Success Probability is

$$P(X=1) = \frac{7}{16} \tag{3.8.3}$$

for Second Trial

$$P(X=1) = \frac{6}{15} \tag{3.8.4}$$

Corresponding Failure Probabilities are

$$P(X=0) = \frac{9}{16} \tag{3.8.5}$$

and for 2nd trial

$$P(X=0) = \frac{8}{15} \tag{3.8.6}$$

probability of success and corresponding failure is varying with trials therefore these are not Bernoulli Trials.

- Case(i):Trials are Bernoulli Trials
- Case(ii): Trials are not Bernoulli Trials

3.9. If a fair coin is tossed 10 times, find the probability of
a) exactly six heads
b) at least six heads
c) at most six heads

Solution: Let X be the random variable denoting the number of times head is obtained when a coin is tossed n times. Then by Binomial distribution,

$$\Pr(X=1) = p \tag{3.9.1}$$
$$\Pr(X=k) = {}^nC_k p^k (1-p)^{n-k} \tag{3.9.2}$$
$$k = 0, \dots, n \tag{3.9.3}$$

For the given problem, $n = 10$ and $p = 1 - p = \frac{1}{2}$ for a fair coin

a) From (3.9.3),

$$\Pr(X=6) = {}^{10}C_6\left(\frac{1}{2}\right)^{10} = \frac{105}{512} \tag{3.9.4}$$

b) Similarly,

$$\Pr(X \geq 6) = \sum_{k=6}^{10} \Pr(X = k) \tag{3.9.5}$$

$$= \sum_{k=6}^{10} {}^{10}C_k \left(\frac{1}{2}\right)^{10} \tag{3.9.6}$$

$$= \frac{193}{512} \tag{3.9.7}$$

c)

$$\Pr(X \leq 6) = 1 - \Pr(X \geq 6) + \Pr(X = 6) \tag{3.9.8}$$

$$= 1 - \frac{193}{512} + \frac{105}{512} \tag{3.9.9}$$

$$= \frac{53}{64} \tag{3.9.10}$$

upon substituting (3.9.4) and (3.9.4),

The python code for the above problem is,

```
./solutions/20−10/prob/codes/exam41.py
```

Experimental probability is calculated using the number of heads obtained in each of the 1,000,000 random experiments of tossing of 10 coins. The code compares the experimental probability to the theoretical probability. As number of experiments increase, the experimental probability approaches the theoritical probability.

3.10. Ten eggs are drawn successively with replacement from a lot containing 10% defective eggs. Find the probability that there is at least one defective egg.

Solution: Let X be the random variable representing the number of defective eggs from the ten eggs picked. X follows a binomial distribution. Since the probability of an egg being defective is 10%, substituting n=10, p= 0.1 and k=0 in equation (3.9.3), probability that there is atleast one defective egg is

$$\Pr(X \geq 1) = 1 - \Pr(X = 0) = 1 - (0.9)^{10} \tag{3.10.1}$$

$$= 0.6513215599 \tag{3.10.2}$$

The python code for the above problem is,

```
.solutions/20−10/prob/codes/exam42.py
```

3.11. Find the mean of the Binomial distribution B(4,$\frac{1}{3}$).

Solution: For a Binomial distribution $X \sim B(n, p)$

$$\Pr(X = 1) = p, \tag{3.11.1}$$

$$\Pr(X = k) = {}^nC_k p^k (1 - p)^{n-k}, \quad k = 0, \ldots, n \tag{3.11.2}$$

The mean is given by

$$E[X] = \sum_{k=0}^{n} k \Pr(X = k) \tag{3.11.3}$$

$$= \sum_{k=0}^{n} k^n C_k p^k (1 - p)^{n-k} \tag{3.11.4}$$

$$= np \tag{3.11.5}$$

Here $p = \frac{1}{3}$ and $n = 4$. Hence

$$E[X] = \frac{4}{3} \tag{3.11.6}$$

The python code for the above problem is,

```
solutions/20−10/prob/codes/exam44.py
```

3.12. The probability of a shooter hitting a target is $\frac{3}{4}$. How many minimum number of times must he/she fire so that the probability of hitting the target at least once is more than 0.99?

Solution: Let X be the random variable representing the number of times the shooter hits the target. Let n be the total number of times that the shoter fires. Then from the given information,

$$X \sim B(n, p), p = \frac{3}{4} \tag{3.12.1}$$

$$\Pr(X \geq 1) \geq 0.99 \tag{3.12.2}$$

Then from (3.9.3) probability of hitting target atleast once is

$$\Pr(X \geq 1) = 1 - \Pr(X = 0) = 1 - {}^nC_0 \left(\frac{1}{4}\right)^n \tag{3.12.3}$$

$$\geq 0.99 \tag{3.12.4}$$

$$\implies 1 - \left(\frac{1}{4}\right)^n \geq 0.99 \tag{3.12.5}$$

$$\implies \left(\frac{1}{4}\right)^n \leq 0.01 \tag{3.12.6}$$

$$\text{or, } n = 4 \tag{3.12.7}$$

The python code for the above problem is, solutions/20−10/prob/codes/exam45.py

3.13. Three coins are tossed simultaneously. Consider the event E ”three heads or three tails”, F ”at least two heads” and G ”at most two heads”. Of the pairs (E,F), (E,G) and (F,G), which are independent? which are dependent?
Solution: Let $X_i \in \{0,1\}$ represent the toss of each coin, with 1 being a head Let

$$X = X_1 + X_2 + X_3 \tag{3.13.1}$$

Then,

$$\Pr(E) = \Pr(\{X=3\} + \{X=0\}) \tag{3.13.2}$$
$$= \Pr(X=3) + \Pr(X=0) \tag{3.13.3}$$
$$= {}^3C_3\left(\frac{1}{2}\right)^3 + {}^3C_0\left(\frac{1}{2}\right)^3 \tag{3.13.4}$$
$$= \frac{1}{4} \tag{3.13.5}$$

$$\Pr(F) = \Pr(X \ge 2) \tag{3.13.6}$$
$$= {}^3C_2\left(\frac{1}{2}\right)^3 + {}^3C_3\left(\frac{1}{2}\right)^3 \tag{3.13.7}$$
$$= \frac{1}{2} \tag{3.13.8}$$

$$\Pr(G) = \Pr(X \le 2) \tag{3.13.9}$$
$$= 1 - \Pr(X > 2) \tag{3.13.10}$$
$$= 1 - {}^3C_3\left(\frac{1}{2}\right)^3 \tag{3.13.11}$$
$$= \frac{7}{8} \tag{3.13.12}$$

Now,

$$\Pr(EF) = \Pr([\{X=3\} + \{X=0\}]\{X \ge 2\}) \tag{3.13.13}$$
$$= \Pr(\{X=3\}\{X \ge 2\} \tag{3.13.14}$$
$$+ \{X=0\}\{X \ge 2\}) \tag{3.13.15}$$
$$= \Pr(X=3) = \frac{1}{8} \tag{3.13.16}$$

Similarly,

$$\Pr(EG) = \Pr([\{X=3\} + \{X=0\}]\{X \le 2\}) \tag{3.13.17}$$
$$= \Pr(\{X=3\}\{X \le 2\} \tag{3.13.18}$$
$$+ \{X=0\}\{X \le 2\}) \tag{3.13.19}$$
$$= \Pr(X=0) = \frac{1}{8} \tag{3.13.20}$$

and

$$\Pr(FG) = \Pr(\{X \ge 2\}\{X \le 2\}) \tag{3.13.21}$$
$$= \Pr(\{X=2\}) \tag{3.13.22}$$
$$= {}^3C_2\left(\frac{1}{2}\right)^3 = \frac{3}{8} \tag{3.13.23}$$

From the above equations we see that

$$P(EF) = P(E)P(F) \tag{3.13.24}$$
$$P(GF) \ne P(G)P(F) \tag{3.13.25}$$
$$P(EG) \ne P(E)P(G) \tag{3.13.26}$$

Hence only the pair (E,F) are independent events. The pairs (F,G) and (G,E) are dependent events.

3.14. A die is tossed thrice. Find the probability of getting an odd number at least once.

Solution: $X_i \in \{0,1\}$, where 0 represents an even number. The 3 trials are represented by

$$X = X_1 + X_2 + X_3 \tag{3.14.1}$$

Then $X \sim B\left(3, \frac{1}{2}\right)$ The probability of getting only an even number is

$$\Pr(X=0) = {}^nC_0\left(\frac{1}{2}\right)^3 \tag{3.14.2}$$

Thus, the probability of getting at least one odd number is

$$1 - \Pr(X=0) = 1 - \frac{1}{8} \tag{3.14.3}$$
$$= \frac{7}{8} \tag{3.14.4}$$

solutions/40−50/probability/codes/Q40binom.
py

3.15. A game consists of tossing a one rupee coin 3 times and noting its outcome each time. Hanif wins if all the tosses give the same result i.e.,

three heads or three tails, and loses otherwise. Calculate the probability that Hanif will lose the game.
Solution: Let $X_i \in \{0,1\}, i = 1,2,3$ represent a coin toss, or, the Bernoulli random variable. Then the outcome of the game is

$$X = X_1 + X_2 + X_3 \tag{3.15.1}$$

If

$$\Pr(X_i = 1) = p, \tag{3.15.2}$$

$$\Pr(X = k) = {}^nC_k p^k (1-p)^k, \quad k = 0,\dots,n \tag{3.15.3}$$

X is known as a Binomial random variable. For the given problem, $n = 3, p = \frac{1}{2}$ and the probability of a win is

$$\Pr(X = 3) + \Pr(X = 0) = \frac{1}{8} + \frac{1}{8} \tag{3.15.4}$$

$$= \frac{1}{4} \tag{3.15.5}$$

The loss probability is then

$$1 - \frac{1}{4} = \frac{3}{4} \tag{3.15.6}$$

The python code for the distribution of data,

solutions/10−1/prob/codes/prob4.py

3.16. A die is thrown twice. What is the probability that

(i) 5 will not come up either time?

(ii) 5 will come up at least once?

Hint : Throwing a die twice and throwing two dice simultaneously are treated as the same experiment
Solution:

(i) Let $X_i \in \{1,2,\dots,6\}$.

$$\Pr(X_1 \neq 5, X_2 \neq 5) = \Pr(X_1 \neq 5)\Pr(X_2 \neq 5) \tag{3.16.1}$$

$\because$ the two events are independent. Also,

$$\Pr(X_1 = 5) = \Pr(X_2 = 5) = \frac{1}{6} \tag{3.16.2}$$

$$\implies \Pr(X_1 \neq 5) = \Pr(X_2 \neq 5) \tag{3.16.3}$$

$$= 1 - \frac{1}{6} = \frac{5}{6} \tag{3.16.4}$$

From (3.16.1),

$$\Pr(X_1 \neq 5, X_2 \neq 5)$$
$$= [1 - \Pr(X_1 = 5)]\,[1 - \Pr(X_2 = 5)]$$
$$= \frac{25}{36} \tag{3.16.5}$$

upon substituting from (3.16.4)

(ii) The probability that 5 doesn't come at all is

$$1 - \Pr(X_1 \neq 5, X_2 \neq 5) = 1 - \frac{25}{36} = \frac{11}{36} \tag{3.16.6}$$

The python code for the problem is

solutions/10−1/prob/codes/prob5.py

3.17. A die is thrown again and again until three sixes are obtained. Find the probability of obtaining the third six in the sixth throw of the die.
Solution: For the 3rd six to occur in the 6th trial, 2 sixes should compulsorily occur in 5 trials. Defining $X \sim B\left(5, \frac{1}{6}\right)$, this probability is given by

$$\Pr(X = 2) = {}^5C_2 \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^3 \tag{3.17.1}$$

The desired probability is then obtained as

$$\Pr(X = 2) \times \frac{1}{6} = \frac{625}{23328} \tag{3.17.2}$$

3.18. Find the probability of throwing at most 2 sixes in 6 throws of a single die.
Solution: Defining $X \sim B\left(6, \frac{1}{5}\right)$, the desired probability is given by

$$\Pr(X \le 2) = \sum_{k=0}^{2} {}^6C_k \left(\frac{1}{6}\right)^k \left(\frac{5}{6}\right)^{6-k} \quad (3.18.1)$$
$$= \left(\frac{5}{6}\right)^4 \times \frac{35}{18} \quad (3.18.2)$$
$$= 0.9377 \quad (3.18.3)$$

3.19. In a box containing 100 bulbs, 10 are defective. The probability that out of a sample of 5 bulbs, none is defective is

a) 10^{-1}
b) $\left(\frac{1}{2}\right)^5$
c) $\left(\frac{9}{10}\right)^5$
d) $\frac{9}{10}$

Solution: From the given information, the probability of a bulb being defective is

$$p = \frac{10}{100} = \frac{1}{10} \quad (3.19.1)$$

Defining $X \sim B\left(5, \frac{1}{10}\right)$ the desired probability is

$$\Pr(X = 0) = {}^5C_0 \left(\frac{1}{10}\right)^0 \left(\frac{9}{10}\right)^{5-0} \quad (3.19.2)$$
$$= \left(\frac{9}{10}\right)^5 \quad (3.19.3)$$

3.20. In a hurdle race, a player has to cross 10 hurdles. The probability that he will clear each hurdle is $\frac{5}{6}$. What is the probability that he will knock down fewer than 2 hurdles?

Solution: Let $X_i \in \{0, 1\}$ represent the *ith* hurdle where 1 denotes a hurdle being knocked down and let

$$X = \sum_{i=1}^{n} X_i \quad (3.20.1)$$

where n is the total number of hurdles. Then X has a binomial distribution with

$$\Pr(X = k) = {}^nC_k p^{n-k} (1-p)^k \quad (3.20.2)$$

where

$$p = \frac{5}{6} = \frac{1}{6} \quad (3.20.3)$$
$$n = 10 \quad (3.20.4)$$

from the given information. Then,

$$\Pr(X < 2) = \Pr(X = 0) + \Pr(X = 1) \quad (3.20.5)$$
$$= {}^{10}C_0 \left(\frac{5}{6}\right)^{10} \left(\frac{1}{6}\right)^0 + {}^{10}C_1 \left(\frac{5}{6}\right)^9 \left(\frac{1}{6}\right)^1 \quad (3.20.6)$$
$$= \left(\frac{5}{6}\right)^9 \left\{\frac{5}{6} + \frac{10}{6}\right\} \quad (3.20.7)$$
$$= 3\left(\frac{5}{6}\right)^{10} \quad (3.20.8)$$

which is the desired probability.

3.21. Suppose that 90% of people are right-handed. What is the probability that at most 6 of a random sample of 10 people are right-handed?

Solution: From the given information, the random variable in question is $X \sim B\left(10, \frac{9}{10}\right)$. The desired probability is then given by

$$\Pr(X \le 6) = 1 - P(X > 6) \quad (3.21.1)$$
$$= 1 - \sum_{k=7}^{10} {}^{10}C_k \left(\frac{9}{10}\right)^k \left(\frac{1}{10}\right)^{10-k} \quad (3.21.2)$$
$$= 1 - \frac{9^7 \times 2064}{10^{10}} \quad (3.21.3)$$
$$= .0128 \quad (3.21.4)$$

3.22. The probability that a student is not a swimmer is $\frac{1}{5}$. Then the probability that out of five students, four are swimmers is

a) ${}^5C_4 \left(\frac{4}{5}\right)^4 \frac{1}{5}$
b) $\left(\frac{4}{5}\right)^4 \frac{1}{5}$
c) ${}^5C_1 \left(\frac{4}{5}\right)^4 \frac{1}{5}$
d) None of these

Solution: Let X be the number of swimmers and the probability that student is swimmer is

$$p = \frac{4}{5} \quad (3.22.1)$$

Then the desired probability is

$$\Pr(X = 4) = {}^5C_4 \left(\frac{4}{5}\right)^4 \left(\frac{1}{5}\right) \quad (3.22.2)$$

3.23. A die is thrown 6 times. If ‘getting an odd number’ is a success, what is the probability of

(i) 5 successes?
(ii) at least 5 successes?

(iii) at most 5 successes?
Solution: let $X \sim B\left(5, \frac{1}{2}\right)$ denote the random variable in question. Then

a)

$$\Pr(X = 5) = {}^6C_5 \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^{6-5} \tag{3.23.1}$$

$$= \frac{3}{32} \tag{3.23.2}$$

b)

$$\Pr(X \geq 5) = \sum_{k=5}^{6} {}^6C_5 \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{6-k} \tag{3.23.3}$$

$$= 0.109375 \tag{3.23.4}$$

c)

$$\Pr(X \leq 5) = 1 - \Pr(X = 6) \tag{3.23.5}$$

$$= 1 - {}^6C_6 \left(\frac{1}{2}\right)^6 \left(\frac{1}{2}\right)^{6-6} \tag{3.23.6}$$

$$= \frac{63}{64} \tag{3.23.7}$$

3.24. A bag consists of 10 balls each marked with one of the digits 0 to 9. If four balls are drawn successively with replacement from the bag, what is the probability that none is marked with the digit 0?
Solution: Let X be number marked on ball drawn. Since the balls are drawn with replacement, the trials are Bernoulli trials.
So X has Binomial Distribution

$$\Pr(X = k) = {}^nC_k \times q^{n-k} \times p^k \tag{3.24.1}$$

Here,

$$n = \text{number of times we pick the ball} \tag{3.24.2}$$

$$p = \text{Probability of getting ball marked as 0} \tag{3.24.3}$$

$$q = 1 - p \tag{3.24.4}$$

Variables	n	p	q	k
Values	4	1/10	9/10	0

TABLE 3.5: Variables and their values

Now,

$$\Pr(X = 0) = {}^4C_0 \times \left(\frac{9}{10}\right)^{(4-0)} \times \left(\frac{1}{10}\right)^0 \tag{3.24.5}$$

$$= \frac{4!}{(4-0)!0!} \times 1 \times \left(\frac{9}{10}\right)^4 \tag{3.24.6}$$

$$= \left(\frac{9}{10}\right)^4 \tag{3.24.7}$$

$$= 0.6561 \tag{3.24.8}$$

3.25. An urn contains 25 balls of which 10 balls bear a mark 'X' and the remaining 15 bear a mark 'Y'. A ball is drawn at random from the urn, its mark is noted down and it is replaced. If 6 balls are drawn in this way, find the probability that
(i) all will bear 'X' mark.
(ii) not more than 2 will bear 'Y' mark.
(iii) at least one ball will bear 'Y' mark.
(iv) the number of balls with 'X' mark and 'Y' mark will be equal.
Solution: Let X be the number of balls which have 'X' mark on them
Using the expression of binomial distribution

$$P(X = r) = \binom{n}{r} p^r q^{n-r} \tag{3.25.1}$$

$$P(X \geq k) = \sum_{r=k}^{n} \binom{n}{r} p^r q^{n-r} \tag{3.25.2}$$

$$P(X \leq k) = \sum_{r=0}^{k} \binom{n}{r} p^r q^{n-r} \tag{3.25.3}$$

$$n = 6, \quad p = 0.4, \quad q = 0.6 \tag{3.25.4}$$

n	6	6	6	6
Condition	$P(X = 6)$	$P(X \geq 4)$	$P(X \leq 5)$	$P(X = 3)$
Value	0.004096	0.1792	0.995904	0.27648
Case	(*i*)	(*ii*)	(*iii*)	(*iv*)

TABLE 3.6: Probabilities of each case

3.26. It is known that 10% of certain articles manufactured are defective. What is the probability that in a random sample of 12 such articles, 9 are defective?
Solution: $X = B(n, p)$ with $n = 12$ and $p = \frac{1}{10}$.

Hence, the desired probability is

$$\implies \Pr(X=9) = \binom{12}{9} \times \left(\frac{1}{10}\right)^9 \times \left(\frac{9}{10}\right)^3 \tag{3.26.1}$$

$$= \frac{16038}{10^{11}} \tag{3.26.2}$$

3.27. On a multiple choice examination with three possible answers for each of the five questions, what is the probability that a candidate would get four or more correct answers just by guessing ?
Solution: Let $X_i \in (0,1)$ be a random variable where $X_i = 1$ represents a successful guess and $X_i = 0$ represents unsuccessful guess on the i^{th} question.

$$p = \frac{1}{3}$$

$$X = \sum_{i=1}^{n} X_i \tag{3.27.1}$$

where n is the total number of questions. So, X has a binomial distribution.

$$\Pr(X \geq r) = \sum_{k=r}^{n} \binom{n}{r} p^k (1-p)^{n-k} \tag{3.27.2}$$

Now, in this case n=5 and r=4. From (3.27.2)

$$\Pr(X=4) = \frac{10}{243}$$

$$\Pr(X=5) = \frac{1}{243}$$

Hence, required probability= $\frac{11}{243}$

3.28. Five cards are drawn successively with replacement from a well shuffled deck of 52 cards. What is the probability that
(i) all the five cards are spades?
(ii) only 3 cards are spades?
(iii) none is a spade?
Solution: Let $X_i \in (0,1)$ be a random variable which denotes whether spade is drawn at the i^{th} draw.

$$X = \sum_{i=1}^{i=5} X_i \tag{3.28.1}$$

where X denotes the number of spades obtained.

$$Since, \Pr(x) = \frac{\text{number of favourable outcome}}{\text{total number of outcomes}}$$

$$\Pr(x) = \frac{{}^{13}C_x \times {}^{39}C_{5-x}}{{}^{52}C_5} \tag{3.28.2}$$

X	0	1	2	3	4	5
P(X)	$\frac{{}^{13}C_0 \times {}^{39}C_5}{{}^{52}C_5}$	$\frac{{}^{13}C_1 \times {}^{39}C_4}{{}^{52}C_5}$	$\frac{{}^{13}C_2 \times {}^{39}C_3}{{}^{52}C_5}$	$\frac{{}^{13}C_3 \times {}^{39}C_2}{{}^{52}C_5}$	$\frac{{}^{13}C_4 \times {}^{39}C_1}{{}^{52}C_5}$	$\frac{{}^{13}C_5 \times {}^{39}C_0}{{}^{52}C_5}$

TABLE 3.7: Probabilities of each case

3.29. Two dice are thrown simultaneously. If X denotes the number of sixes, find the expectation of X.
Solution: Two dice are thrown simultaneously. If X denotes the number of sixes, find the expectation of X

SOLUTION :

When 2 fair dice are thrown simultaneously we know that each die has 6 possible outcomes and outcome of one dice is independent of the outcome of other dice.
$\therefore$ Total possible outcomes are ${}^6C_1 \times {}^6C_1 = 36$

Let X be a random variable denoting number of sixes in the above case. Then by Binomial Distribution

$$\Pr(X=k) = \binom{n}{k} p^k \ (1-p)^{n-k} \tag{3.29.1}$$

$$k = 0, \ldots, n \tag{3.29.2}$$

$$\text{Where } k = 0, 1, 2 \tag{3.29.3}$$

$$n = 2 \tag{3.29.4}$$

$$p = \text{Probability of outcome 6 on a dice} \tag{3.29.5}$$

$$p = \frac{1}{6} \tag{3.29.6}$$

From equation (3.29.1) we obtain the following

$$\Pr(X=0) = \binom{2}{0}\left(\frac{1}{6}\right)^0\left(1-\frac{1}{6}\right)^2 = \frac{25}{36} \tag{3.29.7}$$

$$\Pr(X=1) = \binom{2}{1}\left(\frac{1}{6}\right)^1\left(1-\frac{1}{6}\right)^1 = \frac{10}{36} \tag{3.29.8}$$

$$\Pr(X=2) = \binom{2}{2}\left(\frac{1}{6}\right)^2\left(1-\frac{1}{6}\right)^0 = \frac{1}{36} \tag{3.29.9}$$

(3.29.10)

The probability distribution table is

X	0	1	2
$\Pr(X=k)$	$\frac{25}{36}$	$\frac{10}{36}$	$\frac{1}{36}$

$$\mathbb{E}(X=k) = \sum_{k=0}^{n} k \Pr(k) \tag{3.29.11}$$

$$= \sum_{k=0}^{n} k\binom{n}{k} p^k\ (1-p)^{n-k} \tag{3.29.12}$$

$$= n \cdot p \sum_{k=1}^{n-1}\binom{n-1}{k-1} p^{k-1}\ (1-p)^{(n-1)-(k-1)} \tag{3.29.13}$$

$$= n \cdot p\ (1+(1-p))^{n-1} \tag{3.29.14}$$

$$= n \cdot p \tag{3.29.15}$$

$$\mathbb{E}(X=k) = n \cdot p = 2 \times \frac{1}{6} = \frac{1}{3} \tag{3.29.16}$$

3.30. Find the variance of the number obtained on a throw of an unbiased die.
Let $X \in \{1,2,3,4,5,6\}$, be the random variable representing outcome of the die.The probability mass function(pmf) can be expressed as

$$p_X(n) = P(X=n) = \begin{cases} \frac{1}{6}, & \text{if } 1 \le n \le 6 \\ 0, & \text{otherwise} \end{cases} \tag{3.30.1}$$

The variance (Var(X)) of this distribution can be found by definition,

$$Var(X) = E\left(X^2\right) - (E(X))^2 \tag{3.30.2}$$

where,

$$E(X) = \sum_{k=1}^{k=6} k p_X(k) \tag{3.30.3}$$

$$E(X) = \frac{1}{6}\sum_{k=1}^{k=6} k \tag{3.30.4}$$

We know that, sum of natural numbers from 1 to n is,

$$\sum_{k=1}^{k=n} k = \frac{n(n+1)}{2} \tag{3.30.5}$$

By substituting the formula from (3.30.5) in (3.30.4) and n=6, We get,

$$E(X) = \frac{1}{6} \times \frac{6 \times 7}{2} \tag{3.30.6}$$

$$E(X) = \frac{7}{2} \tag{3.30.7}$$

And,

$$E\left(X^2\right) = \sum_{k=1}^{k=6} k^2 p_X(k) \tag{3.30.8}$$

$$E\left(X^2\right) = \frac{1}{6}\sum_{k=1}^{k=6} k^2 \tag{3.30.9}$$

We know that, sum of squares of natural numbers from 1 to n is,

$$\sum_{k=1}^{k=n} k^2 = \frac{n(n+1)(2n+1)}{6} \tag{3.30.10}$$

By substituting the formula from (3.30.10) in (3.30.9) and n=6, We get,

$$E\left(X^2\right) = \frac{1}{6} \times \frac{6 \times 7 \times 13}{6} \tag{3.30.11}$$

$$E\left(X^2\right) = \frac{91}{6} \tag{3.30.12}$$

By substituting the values from (3.30.12) and (3.30.7) in (3.30.2)

$$Var(X) = E\left(X^2\right) - (E(X))^2 \tag{3.30.13}$$

$$Var(X) = \frac{91}{6} - \frac{49}{4} \tag{3.30.14}$$

$$Var(X) = \frac{70}{12} \tag{3.30.15}$$

$$Var(X) = 2.9167 \tag{3.30.16}$$

3.31. A bag contains 2 white and 1 red balls. One ball is drawn at random and then put back in the box after noting its colour. The process is repeated again. If X denotes the number of red balls recorded in the two draws, describe X.
Given, a bag containing 2 white and 1 red balls. Let the random variable $X_i \in \{0, 1\}, i = 1, 2$, represent the outcome of the colour of the ball drawn in the first, second attempts. $X_i = 0, X_i = 1$ denote a white ball, red ball being drawn respectively, in the i^{th} attempt.

Define

$$X = X_1 + X_2 \tag{5.25.1}$$

so that $X \in \{0, 1, 2\}$ represents a random variable denoting the number of red balls drawn in both the attempts. Then, X has a binomial distribution with

$$Pr(X = k) = {}^nC_k p^k q^{n-k} \tag{5.25.2}$$

where,

$$n = 2 \tag{5.25.3}$$

p = probability of success = probability of drawing a red ball = $Pr(X_i = 1)$

$$p = \frac{1}{3} \tag{5.25.4}$$

q = probability of failure = $1 - p$

$$q = 1 - p = 1 - \frac{1}{3} = \frac{2}{3} \tag{5.25.5}$$

Hence, on substituting and simplifying, we get

$$Pr(X = 0) = \frac{4}{9} \tag{5.25.6}$$

$$Pr(X = 1) = \frac{4}{9} \tag{5.25.7}$$

$$Pr(X = 2) = \frac{1}{9} \tag{5.25.8}$$

Using (5.25.2), we get the following probability distribution.

3.32. Find the probability of throwing at most 2 sixes in 6 throws of a single die.
Solution: Let X represent the number of sixes in six throws of a dice
X ∈ {0,1,2,3,4,5,6}

TABLE 3.8: Probability distribution of X

Condition	$X = 0$	$X = 1$	$X = 2$
Probability	${}^2C_0 p^0 q^2$	${}^2C_1 p^1 q^1$	${}^2C_2 p^2 q^0$

By Binomial distribution formula,

$$P(X = k) = {}^nC_k p^k (1 - p)^{n-k} \tag{3.32.1}$$

Here,

Symbol	Meaning
k	no. of sixes in six throws of a dice
n	no. of throws = 6
p	Pr of getting 6 in single throw=$\frac{1}{6}$

TABLE 3.9: This table gives the meaning of each symbol used in the formula

$$\Pr(X \le 2) = \Pr(X = 0) + \Pr(X = 1) + \Pr(X = 2) \tag{3.32.2}$$

$$\Pr(X = 0) = {}^6C_0 \times \left(\frac{1}{6}\right)^0 \times \left(\frac{5}{6}\right)^{6-0} \tag{3.32.3}$$

$$\Pr(X = 1) = {}^6C_1 \times \left(\frac{1}{6}\right)^1 \times \left(\frac{5}{6}\right)^{6-1} \tag{3.32.4}$$

$$\Pr(X = 2) = {}^6C_2 \times \left(\frac{1}{6}\right)^2 \times \left(\frac{5}{6}\right)^{6-2} \tag{3.32.5}$$

$$\Pr(X \le 2) = \left(\frac{5^6}{6^6}\right) \times 1 + \left(\frac{5^5}{6^6}\right) \times 6 + \left(\frac{5^4}{6^6}\right) \times 15 \tag{3.32.6}$$

$$= 0.937714 \tag{3.32.7}$$

3.33. A person plays a game of tossing a coin thrice. For each head, he is given Rs 2 by the organiser of the game and for each tail, he has to give Rs 1.50 to the organiser. Let X denote the amount gained or lost by the person. Show that X is a random variable and exhibit it as a function on the sample space of the experiment.
Solution:

4 Uniform Distribution

4.1. Two dice, one blue and one grey, are thrown at the same time.

a) Complete Table 4.1.1.
b) A student argues that there are 11 possible outcomes 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 and 12. Therefore, each of them has a probability $\frac{1}{11}$. Do you agree with this argument? Justify your answer.

Event	**Value**
2	1/36
3	-
4	-
5	-
6	-
7	-
8	5/36
9	-
10	-
11	-
12	1/36

TABLE 4.1.1: Input Values

4.2. A die is thrown once. Find the probability of getting
(i) a prime number;
(ii) a number lying between 2 and 6;
(iii) an odd number.
Solution: Let X

$$X \epsilon \{1, 2, 3, 4, 5, 6\} \tag{4.2.1}$$

Since all events are equally likely,

$$\Pr(X = i) = \begin{cases} \frac{1}{6} & 1 \le i \le 6 \\ 0 & otherwise \end{cases} \tag{4.2.2}$$

a) The probability that the outcome is a prime number is

$$\Pr(X = 2) + \Pr(X = 3) + \Pr(X = 5) = \frac{1}{2} \tag{4.2.3}$$

b) Probability of occurance of number between 2 and 6 is

$$\Pr(2 < X < 6) = \frac{1}{2} \tag{4.2.4}$$

c) Probability of occurance of odd number is

$$\Pr(X = 1) + \Pr(X = 3) + \Pr(X = 5) = \frac{1}{2} \tag{4.2.5}$$

The python code for the distribution is

```
solutions/20−10/prob/codes/dice123.py
```

The above code checks number of times each of the above events occur when the dice is thrown 100,000 times.

4.3. In a game, a man wins a rupee for a six and loses a rupee for any other number when a fair die is thrown. The man decided to throw a die thrice but to quit as and when he gets a six. Find the expected value of the amount he wins / loses.

4.4.

4.5. Find the probability of getting 5 exactly twice in 7 throws of a die.

Solution: There are 6 outcomes when we throw a die, which are independent of one another. The probability of getting 5 on the die

$$p = \frac{1}{6} \tag{4.5.1}$$

The die is thrown 7 times and are not dependent on one another

$$n = 7 \tag{4.5.2}$$

Let the Random Variable be X denote the number of 5s in 7 throws By Binomial Distribution, we have

$$\Pr(X = k) = \binom{n}{k} p^k (1 - p)^{n-k} \tag{4.5.3}$$

We should get 5 exactly twice, so $k = 2$

TABLE 4.5.1: Definition of the variables

Variables	
p	Probability that outcome is 5 when we throw the die
n	No of times the die is thrown
X	Random Variable denoting the number of 5s out of n number of throws
k	Required number of times 5s appear on the die which is 2

The probability of getting 5 exactly twice in 7 throws of a die is given by

$$\Pr(X = 2) = {}^{7}C_{C2} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^5 \tag{4.5.4}$$

$$\Pr(X = 2) = 0.234428 \tag{4.5.5}$$

4.6.

4.7.

4.8.

4.9. Let X denote the sum of the numbers obtained when two fair dice are rolled. Find the variance and standard deviation of X.

Solution: When two fare dice are rolled. The sum of the numbers obtained can have the values 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12.

$\Pr(X)$ = probability of obtaining X as the sum and let us represent the case when first dice shows the number x_1 and the second dice shows the number x_2 as (x_1, x_2). For the above problem,we know that.

x	2	3	4	5	6	7	8	9	10	11	12
$\Pr(X)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

TABLE 4.9.1: Probability Distribution Table of X

$$p_x(n) = \begin{cases} 0 & \text{if } n \le 1, \\ \frac{n-1}{36} & \text{if } 2 \le n \le 7, \\ \frac{13-n}{36} & \text{if } 7 < n \le 12, \\ 0 & \text{if } n > 12. \end{cases} \tag{4.9.1}$$

$Mean, E(X)$

$$= \sum_{k=1}^{12}(k \times p_x(k)) \tag{4.9.2}$$

$$= \sum_{k=1}^{6} k \times \frac{1}{36}[k-1] + \sum_{k=7}^{12} k \times \frac{1}{36}[13-k] \tag{4.9.3}$$

$$= \frac{1}{36}\left[\sum_{k=1}^{6} k(k-1) + \sum_{k=7-6}^{12-6}(k+6) \times [13-(k+6)]\right] \tag{4.9.4}$$

$$= \frac{1}{36}\left[\sum_{k=1}^{6} k(k-1) + \sum_{k=1}^{6}(k+6) \times [13-(k+6)]\right] \tag{4.9.5}$$

$$= \frac{1}{36}\sum_{k=1}^{6}(k(k-1)+(k+6)(7-k)) \tag{4.9.6}$$

$$= \frac{1}{36}\sum_{k=1}^{6}\left((k^2-k)+(7k-k^2+42-6k)\right) \tag{4.9.7}$$

$$= \frac{1}{36}\sum_{k=1}^{6}(42) \tag{4.9.8}$$

$$= \frac{1}{36}[42 \times 6] \tag{4.9.9}$$

Therefore, Mean, $E(X) = 7$

$$Variance, \sigma^2 = E(X-E(X))^2 \tag{4.9.10}$$

$$= E\left(X^2\right) - (E(X))^2 \tag{4.9.11}$$

Let us consider $E\left(X^2\right)$,

$E\left(X^2\right)$

$$= \left(\sum_{k=1}^{12}(k^2 \times p_x(k))\right) \tag{4.9.12}$$

$$= \sum_{k=1}^{6} k^2 \times \frac{1}{36}[k-1] + \sum_{k=7}^{12}(k)^2 \times [13-k] \tag{4.9.13}$$

by rearrangement we get (4.9.14)

$$= \frac{1}{36}\left[\sum_{k=1}^{6} k^2(k-1) + \sum_{k=7-6}^{12-6}(k+6)^2 \times [13-(k+6)]\right] \tag{4.9.15}$$

$$= \frac{1}{36}\left[\sum_{k=1}^{6} k^2(k-1) + \sum_{k=1}^{6}(k+6)^2(7-k)\right] \tag{4.9.16}$$

$$= \frac{1}{36}\sum_{k=1}^{6}\left(k^2(k-1)+(k^2+36+12k)(7-k)\right) \tag{4.9.17}$$

$$= \frac{1}{36}\sum_{k=1}^{6}\left((k^3-k^2)+(-k^3-5k^2+48k+252)\right) \tag{4.9.18}$$

$$= \frac{1}{36}\sum_{k=1}^{6}(-6k^2+48k+252) \tag{4.9.19}$$

$$= \frac{1}{6}\sum_{k=1}^{6}(-k^2+8k+42) \tag{4.9.20}$$

$$= \frac{1}{6}\left[-\frac{(6)(7)(13)}{6} + 8 \times \frac{(6)(7)}{2} + (42)(6)\right] \tag{4.9.21}$$

$$= \frac{1}{6}[-91+168+252] \tag{4.9.22}$$

$$= \frac{329}{6} \tag{4.9.23}$$

$$\text{Variance, } \sigma^2 = E\left(X^2\right) - (E(X))^2 \tag{4.9.24}$$

$$= \frac{329}{6} - ((7)^2) \tag{4.9.25}$$

$$= \frac{329}{6} - 49 \tag{4.9.26}$$

$$\sigma^2 = \frac{35}{6} \tag{4.9.27}$$

4.10. Two numbers are selected at random (without replacement) from the first six positive integers. Let X denote the larger of the two numbers obtained. Find E(X).

Solution: The question can be seen as choosing a number first from 1 to 6 numbers and then choosing one more from the remaining 5 numbers, Let X_1 be the 1^{st} numbers drawn randomly from 1 to 6 and X_2be the 2^{nd} number drawn from remaining and $X = \max\ (X_1, X_2)$

$$Pr(X_1 = n_1) = \begin{cases} \frac{1}{6}, & \text{if } 1 \le n_1 \le 6 \\ 0, & \text{otherwise} \end{cases} \tag{4.10.1}$$

$$Pr(X_2 = n_2) = \begin{cases} \frac{1}{5}, & \text{if } 1 \le n_2 \le 6 \text{ and } n_2 \neq n_1 \\ 0, & \text{otherwise} \end{cases} \tag{4.10.2}$$

let $\max\ (X_1, X_2) = i$ and $Pr(X = i)$ denotes the probability that $X = \max\ (X_1, X_2) = i$

$$\begin{aligned} Pr(X = i) = Pr(X_1 = i \text{ and } X_2 < i) \\ + Pr(X_2 = i \text{ and } X_1 < i) \end{aligned} \tag{4.10.3}$$

since choosing of X_1, X_2 are independent events, so we can write

$$Pr(X_1 \text{ and } X_2) = Pr(X_1)Pr(X_2)$$

Substituting this in (4.10.3) gives us

$$\begin{aligned} Pr(X = i) = Pr(X_1 = i)Pr(X_2 < i)+ \\ Pr(X_2 = i)Pr(X_1 < i) \end{aligned} \tag{4.10.4}$$

$$\implies Pr(X = i) = \frac{1}{6} \times \frac{(i-1)}{5} + \frac{(i-1)}{6} \times \frac{1}{5} \tag{4.10.5}$$

$$\implies Pr(X = n) = \frac{(i-1)}{15} \tag{4.10.6}$$

The expectation value of X represented by E(X) is given by

$$E(X) = \sum_{i=1}^{6} Pr(X = i) \times i$$

$$\implies E(X) = \sum_{i=1}^{6} \frac{(i-1)}{15} \times i \tag{4.10.7}$$

$$\implies E(X) = \sum_{i=1}^{6} \frac{(i^2-i)}{15} \tag{4.10.8}$$

$$\implies E(X) = \frac{1}{15}\sum_{i=1}^{6} i^2 - \frac{1}{15}\sum_{i=1}^{6} i \tag{4.10.9}$$

$$\implies E(X) = \frac{1}{15} \times 91 - \frac{1}{15} \times 21 \tag{4.10.10}$$

$$\implies E(X) = \mathbf{4.6667} \tag{4.10.11}$$

4.11.

4.12. Determine P(E/F), if a die is thrown three times,
E : 4 appears on the third toss, F : 6 and 5 appears respectively on first two tosses

4.13. In a musical chair game, the person playing the music has been advised to stop playing the music at any time within 2 minutes after she starts playing.What is the probability that the music will stop within the first half-minute after starting? **Solution:** Let the random variable $X \in \mathbb{R}^+$ represent the time between starting the music and stopping in minutes For a uniform probability distribution, the Probability Density Function(pdf) is given by

$$p_X(x) = \begin{cases} \frac{1}{b-a} = \frac{1}{2} & \text{if } 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases} \tag{4.13.1}$$

The probability that the music will stop within the first half-minute after starting

$$\Pr\left(X = x \mid 0 \le x \le 0.5\right) = \int_0^{\frac{1}{2}} \frac{1}{2} dx = \frac{1}{4} = 0.25 \tag{4.13.2}$$

4.14. A missing helicopter is reported to have crashed somewhere in the rectangular region shown in Fig. 15.2. What is the probability that it crashed inside the lake shown in the figure?
Solution:

$$l = (9-6)kms \tag{4.14.1}$$
$$= 3kms \tag{4.14.2}$$

$$w = (4.5-2)kms \tag{4.14.3}$$
$$= 2.5kms \tag{4.14.4}$$

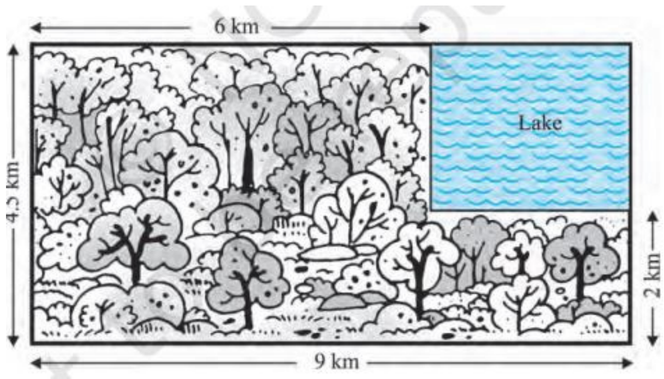


Fig. 4.14

TABLE 4.14.1: Dimensions

variables	Description
l	Length of the lake
w	Width of the lake
a	Area of the lake
L	Length of the whole region
W	Width of the whole region
A	Area of the whole region

As the lake is rectangular;

$$a = l \times w \tag{4.14.5}$$
$$= 3kms \times 2.5kms \tag{4.14.6}$$
$$= 7.5sq.kms \tag{4.14.7}$$

$$L = 9kms$$

$$W = 4.5kms$$

The whole region is of rectangular shape. Hence the area of the whole region is;

$$A = L \times W \tag{4.14.8}$$
$$= 9kms \times 4.5kms \tag{4.14.9}$$
$$= 40.5sq.kms \tag{4.14.10}$$

TABLE 4.14.2: Events and Probabilities

variables	Description
X	Helicopter getting crashed inside lake
Y	Helicopter getting crashed outside lake
P(X)	Probability of occurrence of X
P(Y)	Probability of occurrence of Y

$$P(X) = \frac{a}{A} \tag{4.14.11}$$
$$= \frac{7.5sq.kms}{40.5sq.kms} \tag{4.14.12}$$
$$= \frac{5}{27} = 0.185 \tag{4.14.13}$$

4.15. On one page of a telephone directory, there were 200 telephone numbers. The frequency distribution of their unit place digit (for example, in the number 25828573, the unit place digit is 3) is given in Table 4.15.1 below

Digit	0	1	2	3	4	5	6	7	8	9
Frequency	22	26	22	22	20	10	14	28	16	20

TABLE 4.15.1

Without looking at the page, the pencil is placed on one of these numbers, i.e., the number is chosen at random. What is the probability that the digit in its unit place is 6?
Solution:

$$P_r(X = i) = \frac{f_i}{200} \tag{4.15.1}$$

From Table 4.15.1

$$P_r(X = 6) = \frac{14}{200} \tag{4.15.2}$$
$$= 0.07 \tag{4.15.3}$$

The outputs of Python program are attached below: The **Law Of Large Numbers** is a

TABLE 2: For 200 randomly generated numbers

Digit	**Frequency**	**Probability**
0	21	0.105
1	13	0.065
2	20	0.1
3	21	0.105
4	20	0.1
5	25	0.125
6	15	0.075
7	24	0.12
8	20	0.1
9	21	0.105

fundamental concept for probability and statistics. It states that that as the number of trials increase, the experimental probability will get

TABLE 3: For 10000 randomly generated numbers

Digit	Frequency	Probability
0	1007	0.1007
1	988	0.0988
2	997	0.0997
3	1010	0.101
4	1005	0.1005
5	1018	0.1018
6	1000	0.1
7	984	0.0984
8	1019	0.1019
9	972	0.0972

closer and closer to the theoretical probability. From the output tables 2 and 3, we can deduce that as the number of trials increase, the ratio of the number of successful occurrences to the number of trials will tend to approach the theoretical probability of the outcome for an individual trial. Since all the digits are equiprobable, ideally each probability should be 1/10=0.1 In Table 3, when number of trials are 10,000, probability of each digit is approximately 0.1 with very little deviation. eg. 0.1005.

With 200 samples, Tables 2 and 3 are slightly different because the number of simulations is not sufficient for convergence in the probabilities.

4.16. Suppose we throw a die once. (i) What is the probability of getting a number greater than 4 ? (ii) What is the probability of getting a number less than or equal to 4 ?

Solution: Let

$$X \epsilon \{1, 2, 3, 4, 5, 6\} \tag{4.16.1}$$

For a fair dice,

$$\Pr(X = k) = \begin{cases} \frac{1}{6} & k = 1, 2, 3, 4, 5, 6 \\ 0 & otherwise \end{cases} \tag{4.16.2}$$

a)

$$\Pr(X > 4) = \Pr(X = 5) + \Pr(X = 6) = \frac{1}{3} \tag{4.16.3}$$

b)

$$\Pr(X \le 4) = 1 - \Pr(X > 4 =) = \frac{2}{3} \tag{4.16.4}$$

```
solutions/20−10/prob/codes/exam50.py
```

4.17. Given that the two numbers appearing on throwing two dice are different. Find the probability of the event 'the sum of numbers on the dice is 4'.

Solution: Let $X_1, X_2 \in \{1, 2, 3, 4, 5, 6\}$ represent the two dice.

$$\Pr(X_1 \neq X_2) = \frac{6 \times 5}{6 \times 6} = \frac{5}{6} \tag{4.17.1}$$

Then,

$$\begin{aligned} &\Pr(X_1 + X_2 = 4 | X_1 \neq X_2) \\ &= \frac{\Pr(X_1 + X_2 = 4, X_1 \neq X_2)}{\Pr(X_1 \neq X_2)} \\ &= \frac{\frac{2}{36}}{\frac{5}{6}} = \frac{1}{15} \end{aligned} \tag{4.17.2}$$

4.18. A game of chance consists of spinning an arrow which comes to rest pointing at one of the numbers 1, 2, 3, 4, 5, 6, 7, 8 (see Fig. 4.18), and these are equally likely outcomes. What is the probability that it will point at

(i) 8 ?

(ii) an odd number?

(iii) a number greater than 2?

(iv) a number less than 9?

Solution: Let

$$X \epsilon \{1, 2, 3, 4, 5, 6, 7, 8\} \tag{4.18.1}$$

Since all events are equally likely,

$$\Pr(X = i) = \frac{1}{8} \quad i = 1, 2, \ldots, 8. \tag{4.18.2}$$

a)

$$\Pr(X = 8) = \frac{1}{8} \tag{4.18.3}$$

b) Probability of occurance of odd numbers is

$$\begin{aligned} &\Pr(X = 1) + \Pr(X = 3) \\ &\quad + \Pr(X = 5) + \Pr(X = 7) = \frac{1}{2} \end{aligned} \tag{4.18.4}$$

Fig. 4.18

c)

$$\Pr(X > 2) = 1 - \Pr(X < 2) = \frac{3}{4} \quad (4.18.5)$$

d)

$$\Pr(X < 9) = \frac{8}{8} = 1 \quad (4.18.6)$$

The python code for the distribution is

```
solutions/chance/prob/codes/chance.py
```

The above code checks occurance of each of these events when the arrow is spinned 100,000 times.

4.19. Find the variance of the number obtained on a throw of an unbiased die. **Solution:**

5 Miscellaneous Distributions

5.1. It is given that in a group of 3 students, the probability of 2 students not having the same birthday is 0.992. What is the probability that the 2 students have the same birthday?

Solution: We know that two students either have birthday on same date or they don't have same birthday. No other cases are possible. Therefore, we can consider this as a bernoulli distribution, by defining a random variable X such that,if $X = 0$, then they don't have same birthday, if $X = 1$, then they have same birthday. Therefore,

$$Pr(X = 0) + Pr(X = 1) = 1 \quad (5.1.1)$$

$$Pr(X = 0) = 0.992 \quad (5.1.2)$$

$$Pr(X = 1) = 1 - Pr(X = 0) \quad (5.1.3)$$

$$Pr(X = 0) = 1 - 0.992 \quad (5.1.4)$$

$$Pr(X = 0) = 0.008 \quad (5.1.5)$$

5.2.

5.3. A box contains 5 red marbles, 8 white marbles and 4 green marbles. One marble is taken out of the box at random. What is the probability that the marble taken out will be
(i) red ?
(ii) white ?
(iii) not green?
Solution: Total number of marbles = 5 + 8 + 4 = 17 marbles. Let $X \in \{0, 1, 2\}$ represent the random variable, where 0 represents a red marble, 1 represents a white marble, and 2 represents a green marble. From the given information,

$$Pr(X = 0) = \frac{5}{17} \quad (5.3.1)$$

$$Pr(X = 1) = \frac{8}{17} \quad (5.3.2)$$

$$Pr(X = 2) = \frac{4}{17} \implies \Pr(X \neq 2) = \frac{13}{17} \quad (5.3.3)$$

5.4. A piggy bank contains hundred 50p coins, fifty rupee 1 coins, twenty rupee 2 coins and ten rupee 5 coins. If it is equally likely that one of the coins will fall out when the bank is turned upside down, what is the probability that the coin
(i) will be a 50 p coin ?
(ii) will not be a rupee5 coin?

5.5. If each element of a second order determinant is either zero or one, what is the probability that the value of the determinant is positive? (Assume that the individual entries of the determinant are chosen independently, each value being assumed with probability $\frac{1}{2}$).

5.6. If a leap year is selected at random, what is the chance that it will contain 53 Tuesdays?
Solution: Number of days in a leap year can be written as:

$366 = 52 \times 7 + 2$

Hence a leap year has 52 weeks and an extra two days.

Define a random variable $X = \{0, 1\}$ as shown in below table such that $X = 0$ and $X = 1$ denote the leap year has 52 and 53 Tuesdays respectively.

Let us set the number of leap years one chooses from as 4900.

$$\therefore n(Year) = 4900 \tag{5.6.1}$$

TABLE 5.6.1

S.No	X	2 Extra Days	$n(X)$
1)	0	(Sun,Mon)	700
2)	1	(Mon,Tue)	700
3)	1	(Tue,Wed)	700
4)	0	(Wed,Thu)	700
5)	0	(Thu,Fri)	700
6)	0	(Fri,Sat)	700
7)	0	(Sat,Sun)	700

$$\therefore n(X = 1) = 700 \times 2 = 1400 \tag{5.6.2}$$

Probability for the occurrence of the event $X = 1$ is given by: (from (5.6.1) and (5.6.2))

$$\therefore \Pr(X = 1) = \frac{n(X = 1)}{n(Year)} = \frac{1400}{4900} = \frac{2}{7} \quad \text{(Ans)}$$

5.7.

5.8.

5.9. Suppose that two cards are drawn at random from a deck of cards. Let X be the number of aces obtained. Then the value of E(X) is

a) $\frac{37}{221}$
b) $\frac{5}{13}$
c) $\frac{1}{13}$
d) $\frac{2}{13}$

Solution: Total number of cards =52 with 4 aces,48 non-ace's and we need to select 2 cards so X can be 0 ,1 or 2

Let $A \in \{0, 1\}$ represent the random variable, where 0 represents first card being an non ace, 1 represents first card being ace.
Let $B \in \{0, 1\}$ represent the random variable, where 0 represents second card being an non-ace, 1 represents second card being ace

TABLE 5.9.1: Probability for random variables

$\Pr(A = 0)$	48/52	$\Pr(A = 1)$	4/52
$\Pr(B = 0\|A = 0)$	47/51	$\Pr(B = 0\|A = 1)$	48/51
$\Pr(B = 1\|A = 0)$	4/51	$\Pr(B = 1\|A = 1)$	3/51

if A=1 then 3 aces left and if A=0 then 4 aces left in remaining 51 cards

Case 1: $X = 0$

$$\begin{aligned}\implies \Pr(X = 0) &= \Pr(A = 0, B = 0)\\ &= \Pr(A = 0) \times \Pr(B = 0|A = 0)\\ \implies \Pr(X = 0) &= 188/221\end{aligned} \tag{5.9.1}$$

Case 2: $X = 1$

$$\Pr(X = 1) = \Pr(A = 1, B = 0) + \Pr(A = 0, B = 1)$$
$$\Pr(A = 1, B = 0) = \Pr(A = 1) \times \Pr(B = 0|A = 1)$$
$$\Pr(A = 1, B = 0) = 16/221$$
$$\Pr(A = 0, B = 1) = \Pr(A = 0) \times \Pr(B = 1|A = 0)$$
$$\Pr(A = 0, B = 1) = 16/221$$
$$\implies \Pr(X = 1) = \frac{32}{221} \tag{5.9.2}$$

Case 3: $X = 2$

$$\begin{aligned}\implies \Pr(X = 2) &= \Pr(A = 1, B = 1)\\ &= \Pr(A = 1) \times \Pr(B = 1|A = 1)\\ \implies \Pr(X = 2) &= 1/221\end{aligned} \tag{5.9.3}$$

Now we know that E(X) denotes the average or expectation value which means that E(X) is the weighted average of all values X can take,each value being weighted by the probability of that

particular event/value of X occurring
i.e E(X) is given by

$$E(X) = \sum_{i=0}^{2} x_i \times \Pr(x_i) \tag{5.9.4}$$

TABLE 5.9.2: Probability for various X

X	0	1	2
$\Pr(X)$	188/221	32/221	1/221
$X\times \Pr(X)$	0	32/221	2/221

$$\implies E(X) = \frac{32+2}{221} = \frac{2}{13} \tag{5.9.5}$$

Final answer E(x) = 2/13 or option 4

5.10. The mean of the numbers obtained on throwing a die having written 1 on three faces, 2 on two faces and 5 on one face is

a) 1
b) 2
c) 5
d) $\frac{8}{3}$

Solution: Total number of faces in a die = 6
Let $X \in \{0, 1, 2\}$ represent the random variable, where X being 0 represents faces of die in which 1 is written, 1 represents faces of die in which 2 is written, and 2 represents face of die in which 5 is written.
From the given information,

a) Probability that faces of die in which 1 is written is obtained = Pr(X=0)

$$Pr(X = 0) = \frac{\text{Number of faces with 1}}{\text{Total number of faces}} \tag{5.10.1}$$

$$Pr(X = 0) = \frac{3}{6} = \frac{1}{2} = 0.5 \tag{5.10.2}$$

b) Probability that faces of die in which 2 is written is obtained = Pr(X=1)

$$Pr(X = 1) = \frac{\text{Number of faces with 2}}{\text{Total number of faces}} \tag{5.10.3}$$

$$Pr(X = 1) = \frac{2}{6} = \frac{1}{3} = 0.\overline{3} \tag{5.10.4}$$

c) Probability that faces of die in which 5 is written is obtained = Pr(X=2)

$$Pr(X = 2) = \frac{\text{Number of faces with 5}}{\text{Total number of faces}} \tag{5.10.5}$$

$$Pr(X = 2) = \frac{1}{6} = 0.1\overline{6} \tag{5.10.6}$$

Random Variable [X]	Probability [Pr(X)]
0	$\frac{1}{2}$
1	$\frac{1}{3}$
2	$\frac{1}{6}$

TABLE 5.10.1: This table shows probability associated with each value that the random variable X can take.

The mean of the numbers obtained on throwing a die = Expected value on face of die

$$E(X) = \sum_{i=0}^{2} Pr(X = i) \times x_i \tag{5.10.7}$$

where: $Pr(X = i)$ represents the probability that X=i and x_i represents value obtained in face of the die when X=i

$$= \frac{1}{2} \times 1 + \frac{1}{3} \times 2 + \frac{1}{6} \times 5 = 2 \tag{5.10.8}$$

5.11. A class has 15 students whose ages are 14, 17, 15, 14, 21, 17, 19, 20, 16, 18, 20, 17, 16, 19 and 20 years. One student is selected in such a manner that each has the same chance of being chosen and the age X of the selected student is recorded. What is the probability distribution of the random variable X? Find mean, variance and standard deviation of X.
Solution: Table 5.11.1 summarizes the given info. using which,

X	0	1	2	3	4	5	6	7
No. of students	2	1	2	3	1	2	3	1
P(X)	$\frac{2}{15}$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{1}{15}$

TABLE 5.11.1

$$E(X) = \sum_{i=1}^{n} x_i \Pr(X = i) = \frac{263}{15} \tag{5.11.1}$$

$$E(X^2) = \sum_{i=1}^{n} x_i^2 \Pr(X = i) = \frac{4683}{15} \tag{5.11.2}$$

$$\implies Var(X) = \frac{4683}{15} - (\frac{263}{15})^2 = 4.78 \tag{5.11.3}$$

5.12. A random variable X has the following probability distribution:

X	0	1	2	3	4	5	6	7
P(X)	0	k	2k	2k	3k	k^2	$2k^2$	$7k^2$+k

Determine
(i) k
(ii) P(X ¡ 3)
(iii) P(X ¿ 6)
(iv) P(0 ¡ X ¡ 3)

Solution:

$$\Pr(X) = \begin{cases} 0, & \text{for } X = 0 \\ k, & \text{for } X = 1 \\ 2k, & \text{for } X = 2 \\ 2k, & \text{for } X = 3 \\ 3k, & \text{for } X = 4 \\ k^2, & \text{for } X = 5 \\ 2k^2, & \text{for } X = 6 \\ 7k^2 + k, & \text{for } X = 7 \end{cases} \tag{5.12.1}$$

a) It is known that the sum of probabilities of a probability distribution is always one.

$$\therefore 0 + k + 2k + 3k + k^2 + 2k^2 + (7k^2 + k) = 1 \tag{5.12.2}$$

$$\implies 10k^2 + 9k - 1 = 0 \implies (10k - 1)(k + 1) = 0 \tag{5.12.3}$$

$$\implies k = -1, \frac{1}{10} \tag{5.12.4}$$

$$\therefore k = \frac{1}{10} (\because k \geq 0) \tag{1}$$

We know that $\Pr(X \leq x)$ = F(x)
and $\Pr(x < X \leq y)$ = F(y) - F(x)

X	0	1	2	3	4	5	6	7
F(X)	0	0.1	0.3	0.5	0.8	0.81	0.83	1

TABLE 5.12.1: CDF of X

b) $\Pr(X < 3) = \Pr(X \leq 3)$ - $\Pr(X = 3)$

$$\implies \Pr(X < 3) = F(3) - \Pr(X = 3) \tag{5.12.5}$$

$$\implies \Pr(X < 3) = \frac{5}{10} - \frac{2}{10} \tag{5.12.6}$$

$$\therefore \Pr(X < 3) = \frac{3}{10} \tag{2}$$

c) $\Pr(X > 6) = 1$ - $\Pr(X \leq 6) = 1$ - F(6)

$$\implies \Pr(X > 6) = 1 - \frac{83}{100} \tag{5.12.7}$$

$$\therefore \Pr(X > 6) = \frac{17}{100} \tag{3}$$

d) $\Pr(0 < X < 3) = \Pr(0 < X \leq 3)$ - $\Pr(X = 3)$

$$\implies \Pr(0 < X < 3) = F(3) - F(0) - \Pr(X = 3) \tag{5.12.8}$$

$$\implies \Pr(0 < X < 3) = \frac{5}{10} - 0 - \frac{2}{10} \tag{5.12.9}$$

$$\therefore \Pr(0 < X < 3) = \frac{3}{10} \tag{4}$$

5.13. Find the probability distribution of the number of successes in two tosses of a die, where a success is defined as
(i) number greater than 4
(ii) six appears on at least one die

5.14. An urn contains 5 red and 2 black balls. Two balls are randomly drawn. Let X represent the number of black balls. What are the possible values of X? Is X a random variable ?

5.15. State which of the following are not the probability distributions of a random variable. Give reasons for your answer.
(i)

X	0	1	2
P(X)	0.4	0.4	0.2

(ii)

X	0	1	2	3	4
P(X)	0.1	0.5	0.2	-0.1	0.3

(iii)

X	-1	0	1
P(X)	0.6	0.1	0.2

(iv)

X	3	2	1	0	-1
P(X)	0.3	0.2	0.4	0.1	0.05

Solution: Only (i) is valid. The remaining do not satisfy one of the following conditions.

$$0 \le \Pr(X = i) \le 1 \tag{5.15.1}$$
$$\sum_i \Pr(X = i) = 1 \tag{5.15.2}$$

5.16. A card from a pack of 52 cards is lost. From the remaining cards of the pack, two cards are drawn and are found to be both diamonds. Find the probability of the lost card being a diamond.
Solution: Let **X** $\in \{0, 1\}$ be a random variable where 0 represents a diamond card getting lost and 1 reperesent a card which is not a diamond becoming lost.
Let **Y** $\in \{0, 1\}$ be a random variable where 0 represents both cards drawn being diamonds and 1 represents the case where atleast 1 of the 2 cards drawn is not a diamond.
The required probability is pr(X=0|Y=0).

Since there are 13 diamond cards,

$$\Pr(X = 0) = \frac{13}{52} = \frac{1}{4} \tag{5.16.1}$$
$$\Pr(X = 1) = \frac{39}{52} = \frac{3}{4} \tag{5.16.2}$$

$(X = 0 \cap Y = 0)$ is the event of a diamond card getting lost and getting 2 diamond cards in the 2 draws.
Hence,

$$\Pr(X = 0 \cap Y = 0) = \frac{^{13}C_3}{^{52}C_3} \tag{5.16.3}$$

Using Total probability theorem,

$$\Pr(F) = \sum_{i=1}^{n} \Pr(F|E_i)\Pr(E_i) \tag{5.16.4}$$

$\Pr(Y = 0|X = 0)$ is probability of selecting 2 diamond cards given that one diamond card is lost.

$$\implies \Pr(Y = 0|X = 0) = \frac{^{12}C_2}{^{51}C_2} \tag{5.16.5}$$

$\Pr(Y = 0|X = 1)$ is probability of selecting 2 diamond cards given that the card lost is not a diamond.

$$\implies \Pr(Y = 0|X = 1) = \frac{^{13}C_2}{^{51}C_2} \tag{5.16.6}$$

Thus,

$$\Pr(Y = 0) = \frac{^{12}C_2}{^{51}C_2} + \frac{^{13}C_2}{^{51}C_2} \tag{5.16.7}$$

by definition,

$$\Pr(X = 0|Y = 0) = \frac{\Pr(X = 0 \cap Y = 0)}{\Pr(Y = 0)}$$
$$= \frac{11}{50}$$
$$= 0.22 \tag{5.16.8}$$

5.17. Suppose a girl throws a die. If she gets a 5 or 6, she tosses a coin three times and notes the number of heads. If she gets 1, 2, 3 or 4, she tosses a coin once and notes whether a head or tail is obtained. If she obtained exactly one head, what is the probability that she threw 1, 2, 3 or 4 with the die?
Solution: Let $X \in \{0, 1\}$ where X=0 represents that we get 1,2,3 or 4 when a die is rolled and X=1 represents that we get 5 or 6 when a die is rolled.
Let $Y \in \{0, 1, 2, 3\}$ where Y=1 represents that we get exactly one head.Here Y represents the number of heads obtained.

We are required to find probability of getting X=0 when Y=1.
Here we use Bayes' theorem.

$$\Pr(X = 0|Y = 1) = \frac{\Pr(X = 0)\ \ \Pr(Y = 1|X = 0)}{\sum_{i=0}^{1} \Pr(X = i)\ \ \Pr(Y = 1|X = i)} \tag{5.17.1}$$

Note that

$$\Pr(X = 0) = \frac{4}{6} = \frac{2}{3} = 0.6666666667 \quad (5.17.2)$$

$$\Pr(X = 1) = \frac{2}{6} = \frac{1}{3} = 0.3333333333 \quad (5.17.3)$$

Also we get

$$\Pr(Y = 1|X = 0) = \frac{1}{2} = 0.5 \quad (5.17.4)$$

$$\Pr(Y = 1|X = 1) = \frac{3}{8} = 0.375 \quad (5.17.5)$$

Substituting values, we get

$$\Pr(X = 0|Y = 1) = \frac{\frac{2}{3} \times \frac{1}{2}}{\frac{2}{3} \times \frac{1}{2} + \frac{1}{3} \times \frac{3}{8}} \quad (5.17.6)$$

$$\implies \Pr(X = 0|Y = 1) = \frac{8}{11} = 0.7272727273 \quad (5.17.7)$$

5.18. An insurance company insured 2000 scooter drivers, 4000 car drivers and 6000 truck drivers. The probability of an accidents are 0.01, 0.03 and 0.15 respectively. One of the insured persons meets with an accident. What is the probability that he is a scooter driver?
Solution: By definition

$$\Pr(A|B) = \frac{\Pr(AB)}{\Pr(B)} \quad (5.18.1)$$

Also, by Bayes' Theorem

$$\Pr(A) = \sum_{i=1}^{n} \Pr(A|E_i)\Pr(E_i) \quad (5.18.2)$$

where $E_1, E_2 \ldots E_n$ are partitions of the complete sample set.

Let X be a random variable taking the following values in Table 5.18.1. where X $\in \{0, 1, 2\}$

X = 0	Scooter Drivers
X = 1	Car Drivers
X = 2	Truck Drivers

TABLE 5.18.1

represent all the partitions of the sample set.
Let Y be a random variable taking the following values in Table 5.18.2.

Y = 0	Involved in an accident
Y = 1	Not involved in an accident

TABLE 5.18.2

Also, the following values are known:

$$\Pr(X = 0) = \frac{2000}{2000 + 4000 + 6000} = \frac{1}{6} \quad (5.18.3)$$

$$\Pr(X = 1) = \frac{4000}{2000 + 4000 + 6000} = \frac{1}{3} \quad (5.18.4)$$

$$\Pr(X = 2) = \frac{6000}{2000 + 4000 + 6000} = \frac{1}{2} \quad (5.18.5)$$

$$\Pr(Y = 0|X = 0) = 0.01 \quad (5.18.6)$$

$$\Pr(Y = 0|X = 1) = 0.03 \quad (5.18.7)$$

$$\Pr(Y = 0|X = 2) = 0.15 \quad (5.18.8)$$

We have to find:

$$\Pr(X = 0|Y = 0) = \frac{\Pr(X = 0 \cap Y = 0)}{\Pr(Y = 0)} \quad (5.18.9)$$

Using (6.3.4) and (6.3.5), we get:

$$\Pr(X = 0|Y = 0) = \frac{\Pr(Y = 0|X = 0)\Pr(X = 0)}{\sum_{i=0}^{i=2} \Pr(Y = 0|X = i)\Pr(X = i)} = \frac{\frac{0.01}{6}}{\frac{0.01}{6} + \frac{0.03}{3} + \frac{0.15}{2}} = \frac{1}{52} \quad (5.18.10)$$

5.19. A carton consists of 100 shirts of which 88 are good, 8 have minor defects and 4 have major defects.Jimmy, a trader, will only accept the shirts which are good, but Sujatha, another trader, will only reject the shirts which have major defects.One shirt is drawn at random from the carton. What is the probability that
(i) it is acceptable to Jimmy?
(ii) it is acceptable to Sujatha?
Solution: Let random variable $X \in \{0, 1, 2\}$ denote the outcomes of experiment of drawing a shirt from the carton as shown in Table 5.19.1

TABLE 5.19.1

Type of shirt	X	number	$\Pr(X)$
good	0	n(X=0) = 88	$\frac{22}{25}$
minor defect	1	n(X=1) = 8	$\frac{2}{25}$
major defect	2	n(X=2) = 4	$\frac{1}{25}$

i) The required probability is

$$p = \Pr(X = 0) \tag{5.19.1}$$
$$= \frac{88}{100} \tag{5.19.2}$$
$$= 0.88 \tag{5.19.3}$$

ii) The required probability is

$$p = \Pr(X = 0) + \Pr(X = 1) \tag{5.19.4}$$
$$= \frac{88}{100} + \frac{8}{100} \tag{5.19.5}$$
$$= 0.96 \tag{5.19.6}$$

5.20. Two dice, one blue and one grey, are thrown at the same time. Write down all the possible outcomes.What is the probability that the sum of the two numbers appearing on the top of the dice is
(i) 8?
(ii) 13?
(iii) less than or equal to 12?
Solution: Let $X_i \in \{1, 2, 3, 4, 5, 6\}$, $i = 1, 2$ be the random variables representing the outcomes of each die. The probability mass function is given below.

$$p_{X_i}(n) = \Pr(X_i = n) = \begin{cases} \frac{1}{6} & 1 \le n \le 6 \\ 0 & otherwise \end{cases} \tag{5.20.1}$$

Desired outcomes

$$X = X_1 + X_2 = n \tag{5.20.2}$$

We have the following expression for probability

$$p_X(n) = \begin{cases} 0 & n < 1 \\ \frac{n-1}{36} & 2 \le n \le 7 \\ \frac{13-n}{36} & 7 < n \le 12 \\ 0 & n > 12 \end{cases} \tag{5.20.3}$$

Using (5.20.3) we get the following answers

TABLE 5.20.1: answers

case	X=8	X=13	$X \le 12$
$p_X(n)$	$\frac{5}{36}$	0	1

5.21. Savita and Hamida are friends. What is the probability that both will have
(i) different birthdays?
(ii) the same birthday? (ignoring a leap year).
Solution: Let the Bernoulli random variable $X = \{0, 1\}$ denote the outcome of the given experiment.
$X = 0$ denotes the outcome that Savita and Hamida have their birthdays on a *same day* of the year.
$X = 1$ denotes the outcome that Savita and Hamida have their birthdays on *different days* of the year.

$$\Pr(X = 0) = \frac{1}{365} \tag{5.21.1}$$
$$\therefore \Pr(X = 0) = 0.00273972 \tag{5.21.2}$$
$$\because \Pr(X = 0) + \Pr(X = 1) = 1 \tag{5.21.3}$$
$$\therefore \Pr(X = 1) = 1 - \Pr(X = 0) \tag{5.21.4}$$

Putting the value of $\Pr(X = 0)$ from (5.21.1) in (5.21.4)

$$\Pr(X = 1) = 1 - \frac{1}{365} \tag{5.21.5}$$
$$\Pr(X = 1) = \frac{364}{365} \tag{5.21.6}$$
$$\therefore \Pr(X = 1) = 0.99726027 \tag{5.21.7}$$

5.22.

5.23. A box contains 3 blue, 2 white, and 4 red marbles. If a marble is drawn at random from the box, what is the probability that it will be
(i) white? (ii) blue? (iii) red?
Solution: inputsolutions/5/5.23.tex

5.24. One card is drawn from a well-shuffled deck of 52 cards. Calculate the probability that the card will
(i) be an ace,
(ii) not be an ace.
Solution: It is known that the total number of cards in the deck is 52, out of which there

are four aces. Let random variable $X \in \{0, 1\}$ denote the possible outcomes of the experiment of drawing a card from the shuffled deck.

Card	X	Number
Ace	0	$n(X = 0) = 4$
Not an Ace	1	$n(X = 1) = 48$

TABLE 5.24.1: Outcome of the Experiment

$$p(X = 0) = \frac{n(X = 0)}{n(X = 0) + n(X = 1)} = \frac{4}{52} \quad (5.24.1)$$

$$\Rightarrow p(X = 0) = 0.076923 \quad (5.24.2)$$

Similarly,

$$p(X = 1) = \frac{n(X = 1)}{n(X = 0) + n(X = 1)} = \frac{48}{52} \quad (5.24.3)$$

$$\Rightarrow p(X = 1) = 0.923077 \quad (5.24.4)$$

Hence, the required probabilities are:

(i) 0.076923

(ii) 0.923077

5.25.

5.26. Two cards are drawn successively with replacement from a well shuffled deck of 52 cards. Find the probability distribution of the number of aces.

Solution:
Let $X \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ represent the random variable, where 0 represents an ace card, 1 represents a card numbered '2', 2 represents a card numbered '3'... 9 represents a card numbered '10', 10 represents the J card, 11 represents the Q card, and 12 represents the K card. These are independent of the suit.

$$n(X = i) = 4, i \in \{0, 1, 2 \ldots 10, 11, 12\} \quad (5.26.1)$$

$$\Pr(X = i) = \begin{cases} \frac{4}{52} = \frac{1}{13} & i \in \{0, 1, 2 \ldots 10, 11, 12\} \\ 0 & \text{otherwise} \end{cases} \quad (5.26.2)$$

Let Y $\in \{0, 1, 2\}$ represent the random variable, where 0 represents the case where no aces are selected, 1 represents the case where one ace is selected, 2 represents the case where 2 aces are selected.

Let Z $\in \{0, 1\}$ represent the random variable, where 0 represents an ace card is picked while 1 represents a non-ace card is picked.

$$\Pr(Z = 0) = \frac{1}{13} \quad (5.26.3)$$

$$\Pr(Z = 1) = \Sigma_{i=1}^{12} Pr(X = i) \quad (5.26.4)$$

$$= \Sigma_{i=1}^{12} \frac{1}{13} \quad (5.26.5)$$

$$= \frac{12}{13} \quad (5.26.6)$$

Now, for finding the probability distribution of the number of aces,

a)

$$\Pr(Y = 0) = \Pr(Z = 1) \times \Pr(Z = 1) \quad (5.26.7)$$

$$= \frac{12}{13} \times \frac{12}{13} = \frac{144}{169} \quad (5.26.8)$$

$$= 0.852071 \quad (5.26.9)$$

b)

$$\Pr(Y = 1) = \Pr(Z = 0) \times \Pr(Z = 1) + \Pr(Z = 1) \times \Pr(Z = 0) \quad (5.26.10)$$

$$\Pr(Y = 1) = \frac{1}{13} \times \frac{12}{13} + \frac{12}{13} \times \frac{1}{13} \quad (5.26.11)$$

$$= \frac{24}{169} = 0.142012 \quad (5.26.12)$$

c)

$$\Pr(Y = 2) = \Pr(Z = 0) \times \Pr(Z = 0) \quad (5.26.13)$$

$$= \frac{1}{13} \times \frac{1}{13} = \frac{1}{169} \quad (5.26.14)$$

$$= 0.005917 \quad (5.26.15)$$

5.27. Find the probability distribution of number of doublets in three throws of a pair of dice?
Solution: Let the number of doublets in three throws of a pair of dice be represented by a random variable, X

Serial number	Case	Probability of the case
1	$\Pr(Y = 0)$	$\frac{144}{169}$
2	$\Pr(Y = 1)$	$\frac{24}{169}$
3	$\Pr(Y = 2)$	$\frac{1}{169}$

TABLE 5.26.1: Probability distribution table

When a pair of dice is thrown three times the number of doublets can be 0,1,2 and 3 respectively. So X can take these values

$$\Pr(X = 0) = \Pr(X_1 \neq X_2)^3 \tag{5.27.1}$$
$$= \frac{125}{216} \tag{5.27.2}$$

Similarly we have,

$$\Pr(X = 1) = 3 \times \Pr(X_1 \neq X_2)^2 \times \Pr(X_1 = X_2) \tag{5.27.3}$$
$$= \frac{75}{216} \tag{5.27.4}$$

Note : 3 is multiplied as we have to select which dice will have doublet

$$\Pr(X = 2) = 3 \times \Pr(X_1 \neq X_2) \times \Pr(X_1 = X_2)^2 \tag{5.27.5}$$
$$= \frac{15}{216} \tag{5.27.6}$$

And lastly,

$$\Pr(X = 2) = \Pr(X_1 = X_2)^3 \tag{5.27.7}$$
$$= \frac{1}{216} \tag{5.27.8}$$

The probability distribution of number of doublets in three throws of a pair of dice can be found at table 5.27.1

No. of doublets	**Probability**
0	125/216
1	75/216
2	15/216
3	1/216

TABLE 5.27.1: Probability of doublets

5.28. Let X denote the number of hours you study during a randomly selected school day. The probability that X can take the values x, has the following form, where k is some unknown constant.

$$P(X = x) = \begin{cases} 0.1, & \text{if } x = 0 \\ kx, & \text{if } x = 1 \text{ or } 2 \\ k(5 - x) & \text{if } x = 3 \text{ or } 4 \\ 0, & \text{otherwise} \end{cases} \tag{5.28.1}$$

a) Find the value of k.
b) What is the probability that you study at least two hours ? Exactly two hours? At most two hours?

Solution:
If we expand the probabilities given further more by substituting the value of x and only considering 0 to 4 hours as the probability of studying in the remaining hours is zero, we get

x	0	1	2	3	4
$\Pr(X = x)$	0.1	k	2k	2k	k

TABLE 5.28.1: Given probabilities

we also know that,

$$\sum_{k=0}^{4} \Pr(X = k) = 1 \tag{5.28.2}$$

By substituting the probabilities in (5.28.2)

$$\implies 0.1 + k + 2k + 2k + k = 1 \tag{5.28.3}$$
$$\implies 6k = 0.9 \tag{5.28.4}$$

Therefore, from (5.28.4)

$$k = 0.15 \tag{5.28.5}$$

So from 5.28.1

x	0	1	2	3	4
$\Pr(X = x)$	0.1	0.15	0.3	0.3	0.15

TABLE 5.28.2: Probabilities after finding k

We know that, Cumulative Distributive Function (CDF)

$$F(x) = \Pr(X \leq x) \tag{5.28.6}$$

x	0	1	2	3	4
$F(X)$	0.1	0.25	0.55	0.85	1

TABLE 5.28.3: CDF

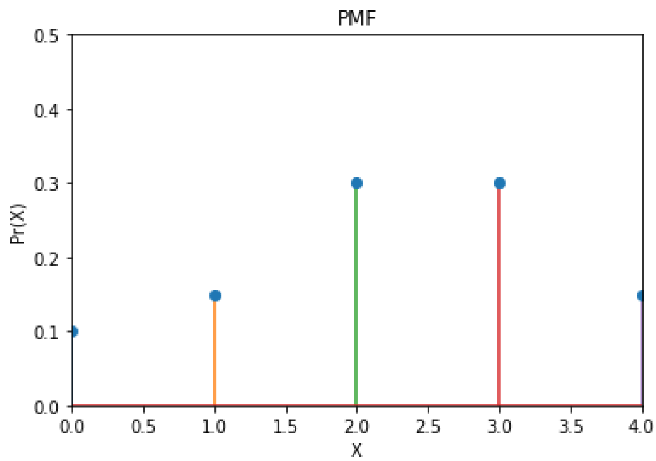


Fig. 5.28: Probability Mass Function (PMF)

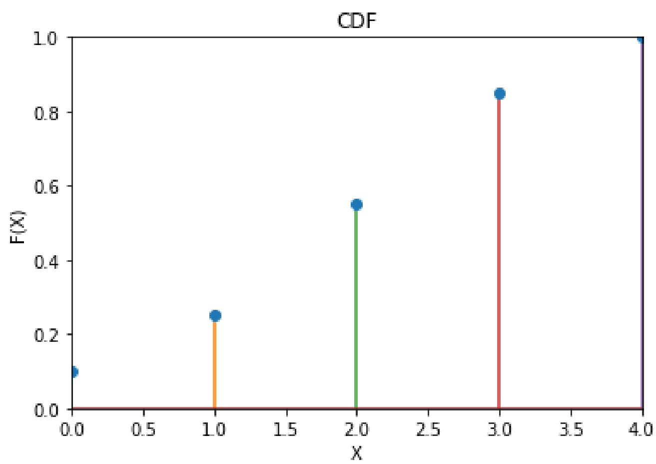


Fig. 5.28: Cumulative Distributive Function (CDF)

And also,

$$\Pr(x < X \leq y) = F(y) - F(x) \quad (5.28.7)$$

a) Probability of studying at least two hours

$$\implies \sum_{k=2}^{4} \Pr(X = k) = \Pr(X \geq 2) \quad (5.28.8)$$

$$\implies \Pr(1 < X \leq 4) \quad (5.28.9)$$

From (5.28.7) and (5.28.3)

$$= F(4) - F(1) \quad (5.28.10)$$

$$= 1 - 0.25 \quad (5.28.11)$$

$$= 0.75 \quad (5.28.12)$$

b) Probability of studying exactly two hours

$$= \Pr(X = 2) \quad (5.28.13)$$

$$= 0.3 \quad (5.28.14)$$

c) Probability of studying at most two hours

$$\implies \sum_{k=0}^{2} \Pr(X = k) = \Pr(X \leq 2) \quad (5.28.15)$$

From (5.28.3)

$$= F(2) \quad (5.28.16)$$

$$= 0.55 \quad (5.28.17)$$

$\Pr(X \geq 2)$	$\Pr(X = 2)$	$\Pr(X \leq 2)$
0.75	0.3	0.55
Case 1	Case 2	Case 3

TABLE 5.28.4: Final solution

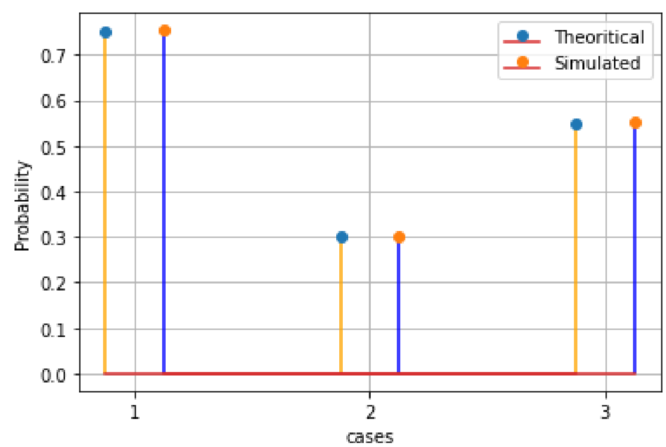


Fig. 5.28: Simulation and Theoretical Comparison

5.29. Let a pair of dice be thrown and the random variable X be the sum of the numbers that appear on the two dice. Find the mean or expectation of X.
Solution: Let X_1,X_2 $\in \{1, 2, 3, 4, 5, 6\}$ be two random variables associated with event.
$X = X_1 + X_2$, representing sum of outcomes of two dices.

$$\therefore X \in \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$$

Now

$$\Pr(X = n) = \Pr(X_1 + X_2 = n) \quad (5.29.1)$$

$$P_X(n) = \begin{cases} 0 & n < 2 \\ \frac{n-1}{36} & 2 \le n \le 7 \\ \frac{13-n}{36} & 7 < n \le 12 \\ 0 & 12 < n \end{cases} \quad (5.29.2)$$

For mean

n	2	3	4	5	6	7	8	9	10	11	12
$\Pr(X = n)$	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{9}$	$\frac{5}{36}$	$\frac{1}{6}$	$\frac{5}{36}$	$\frac{1}{9}$	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$

TABLE 5.29.1: Probability as a function of n

$$\hat{X} = \sum_{n=2}^{12} n \times \Pr(X = n) \quad (5.29.3)$$

$$= \sum_{n=2}^{7} n \times \left(\frac{n-1}{36}\right) + \sum_{n=8}^{12} n \times \left(\frac{13-n}{36}\right) \quad (5.29.4)$$

$$= \frac{112}{36} + \frac{140}{36} \quad = \frac{252}{36} \quad (5.29.5)$$

$$= 7.0 \quad (5.29.6)$$

5.30.

5.31. Two cards are drawn simultaneously (or successively without replacement) from a well shuffled pack of 52 cards. Find the mean, variance and standard deviation of the number of kings.

Solution: Let $X\{0, 1, 2\}$ be the random variable representing the number of kings present in the two cards. Then,

$$\Pr(X = 0) = \frac{{}^{48}C_2}{{}^{52}C_2} = \frac{188}{221} \quad (5.31.1)$$

$$\Pr(X = 1) = \frac{{}^{4}C_1 \times {}^{48}C_1}{{}^{52}C_2} = \frac{32}{221} \quad (5.31.2)$$

$$\Pr(X = 2) = \frac{{}^{4}C_2}{{}^{52}C_2} = \frac{1}{221} \quad (5.31.3)$$

and

$$E(X) = \sum_{i=0}^{2} i \Pr(X = i) \quad (5.31.4)$$

$$= 0 \times \frac{188}{221} + \frac{32}{221} + 2 \times \frac{1}{221} \quad (5.31.5)$$

$$= \frac{36}{221} \quad (5.31.6)$$

Similarly,

$$E\left(X^2\right) = \sum_{i=0}^{2} i^2 \Pr(X = i) = \frac{34}{221} \quad (5.31.7)$$

$$\implies Var(X) = E\left(X^2\right) - (E(X))^2 = \frac{6800}{48841} \quad (5.31.8)$$

5.32. A tyre manufacturing company kept a record of the distance covered before a tyre needed to be replaced. Table 5.32.1 shows the results of 1000 cases. If you buy a tyre of this company,

Distance(in km)	> 4000	4000-9000	9001-14000	<14000
Frequency	20	210	325	445

TABLE 5.32.1

what is the probability that :
(i) it will need to be replaced before it has covered 4000 km?
(ii) it will last more than 9000 km?
(iii) it will need to be replaced after it has covered somewhere between 4000 km and 14000 km?

Solution: From the given information,

a)

$$\Pr(X > 9000) = \frac{325 + 445}{1000} \quad (5.32.1)$$

$$= 0.77 \quad (5.32.2)$$

b)

$$\Pr(4000 < X < 14000) = \frac{20 + 210 + 325}{1000} \quad (5.32.3)$$

$$= 0.0.555 \quad (5.32.4)$$

c)

$$\Pr(X < 4000) = \frac{20}{1000} \quad (5.32.5)$$

$$= 0.02 \quad (5.32.6)$$

Related codes are available in

solutions/1−10/codes/probexm/probexm6.py

5.33. The percentage of marks obtained by a student in the monthly unit tests are given in Table 5.33.1 below. Based on this data, find the probability that the student gets more than

70% marks in a unit test.

Unit test	I	II	III	IV	V
Frequency	69	71	73	68	74

TABLE 5.33.1

Solution: From the given information,

$$\Pr(X > 70) = \frac{3}{5} \tag{5.33.1}$$
$$= 0.6 \tag{5.33.2}$$

5.34. Consider the frequency distribution in Table 5.34.1 below which gives the weights of 38 students of a class. (i) Find the probability that the weight of a student in the class lies in the interval 46-50 kg.
(ii) Give two events in this context, one having probability 0 and the other having probability 1.

Weights (in kg)	Number of students
31-35	9
36-40	5
41-45	14
46-50	3
51-55	1
56-60	2
61-65	2
66-70	1
71-75	1
Total	38

TABLE 5.34.1

Solution:

a) From the given information,

$$\Pr(46 < X < 50) = \frac{3}{38} \tag{5.34.1}$$
$$= 0.079 \tag{5.34.2}$$

b) There is no student whose weight is less than 31 kg thus the probability of a student to have the weight less than 31 kg = 0

All of the student in this context have the weight between 31-75 so we can say that the probability of the students to have the weight in the range 31-75 = 1

5.35. Fifty seeds were selected at random from each of 5 bags of seeds, and were kept under standardised conditions favourable to germination. After 20 days, the number of seeds which had germinated in each collection were counted and recorded in Table 5.35.1
What is the probability of germination of
(i)more than 40 seeds in a bag?
(ii) 49 seeds in a bag?
(iii) more that 35 seeds in a bag?

Bag	1	2	3	4	5
No.of seeds germinated	40	48	42	39	41

TABLE 5.35.1

Solution: Let X represent the seeds and Y represent the bags.

a)

$$\Pr(X > 40) = \frac{3}{5} \tag{5.35.1}$$
$$= 0.6 \tag{5.35.2}$$

b)

$$\Pr(X = 49) = \frac{0}{5} \tag{5.35.3}$$
$$= 0 \tag{5.35.4}$$

c)

$$\Pr(X > 35) = \frac{5}{5} \tag{5.35.5}$$
$$= 1 \tag{5.35.6}$$

Related code is available in

```
solutions/1−10/codes/probexm/probexm10.
    py
```

5.36. 1500 families with 2 children were selected randomly, and the following data in Table 5.36.1 were recorded. Compute the probability of a family, chosen at random, having

a) 2 girls
b) 1 girl
c) No girl

Also check whether the sum of these probabilities is 1.

No.of girls in a family	2	1	0
No. of families	475	814	211

TABLE 5.36.1

Solution: Let X be the random variable representing the number of girls.

a)

$$\Pr(X = 2) = \frac{475}{1500} \quad (5.36.1)$$
$$= 0.316 \quad (5.36.2)$$

b)

$$\Pr(X = 1) = \frac{814}{1500} \quad (5.36.3)$$
$$= 0.5427 \quad (5.36.4)$$

c)

$$\Pr(X = 0) = \frac{211}{1500} \quad (5.36.5)$$
$$= 0.1407 \quad (5.36.6)$$

It is easy to verify that

$$\Pr(X = 0) + \Pr(X = 1) + \Pr(X = 2) = 1 \quad (5.36.7)$$

5.37. In a particular section of Class IX, 40 students were asked about the months of their birth and the following graph in Fig. 5.37 was prepared for the data so obtained. Find the probability that a student of the class was born in August.

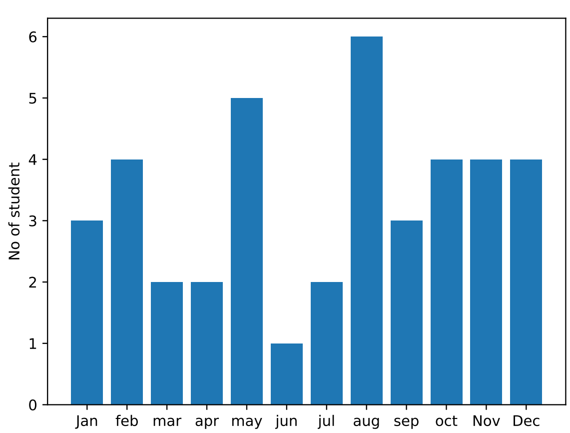


Fig. 5.37: student birth figure

Solution: Total no of the studend in a year = 40 no of student of class August = 6 let probabilaty of a student to be of august class be P(A)

$$P(A) = \frac{6}{40} = 0.15 \quad (5.37.1)$$

codes for the above equation can be get from here

```
solutions/1−10/codes/prob/prob3.py
```

5.38. Three coins are tossed simultaneously 200 times with the following frequencies of different outcomeslisted in Table 5.38.1. If the three coins are simultaneously tossed again, compute the probability of 2 heads coming up.

Outcome	3 heads	2 heads	1 head	No head
Frequency	23	72	77	28

TABLE 5.38.1

Solution:

a) From the given information,

$$\Pr(X < 20) = \frac{7}{90} \quad (5.38.1)$$
$$= 0.07 \quad (5.38.2)$$

b)

$$\Pr(X \geq 60) = \frac{15 + 8}{90} \quad (5.38.3)$$
$$= 0.256 \quad (5.38.4)$$

5.39. Refer to Table 5.39.1.

a) Find the probability that a student obtained less than 20% in the mathematics test.
b) Find the probability that a student obtained marks 60 or above.

Marks	Number of students
0-20	7
20-30	10
30-40	10
40-50	20
50-60	20
60-70	15
70-above	8
Total	90

TABLE 5.39.1

Solution:

a) From the given information,

$$\Pr(X < 20) = \frac{7}{90} \tag{5.39.1}$$
$$= 0.07 \tag{5.39.2}$$

b)

$$\Pr(X \geq 60) = \frac{15+8}{90} \tag{5.39.3}$$
$$= 0.256 \tag{5.39.4}$$

5.40. The distance (in kms) of 40 engineers from their residence to their place of work were found as follows in Table 5.40.1. What is the empirical probability that an engineer lives

a) less than 7 km from her place of work?
b) more than or equal to 7 km from her place of work?
c) within $\frac{1}{2}$km from her place of work?

5	3	10	20	25	11	13	7	12	31
19	10	12	17	18	11	32	17	16	2
7	9	7	8	3	5	12	15	18	3
12	14	2	9	6	15	15	7	6	12

TABLE 5.40.1

Solution:

a) total no of people working at the work place = 40 no of people live less than 7km from the work place = 9 let probability of a emgineer livinf less than 7 km from workplace = P(A)

$$P(A) = \frac{9}{40} \tag{5.40.1}$$

b) no of people live more than or equal 7km from the work place = 31 let probability of a emgineer livinf less than 7 km from workplace = P(B)

$$P(B) = \frac{31}{40} \tag{5.40.2}$$

c) there is no one who live within $\frac{1}{2}$ km from the work place so the probability will be 0.

solutions/1−10/figs/prob/prob7.py

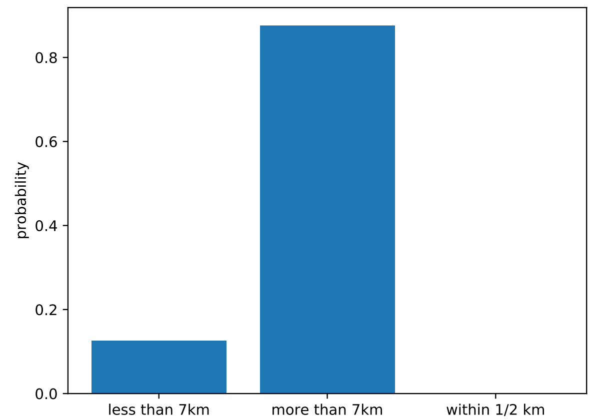


Fig. 5.40: probabilities a man to be near from work place

5.41. An organisation selected 2400 families at random and surveyed them to determine a relationship between income level and the number of vehicles in a family. The information gathered is listed in the Table 5.41.1 Suppose a family is chosen. Find the probability that the family chosen is

a) earning ₹10000 – ₹13000 per month and owning exactly 2 vehicles.
b) earning ₹16000 or more per month and owning exactly 1 vehicle.
c) earning less than ₹7000 per month and does not own any vehicle.
d) earning ₹13000 – ₹16000 per month and owning more than 2 vehicles.
e) owning not more than 1 vehicle.

Monthly income (in ₹)	**vehicles per family**			
	0	**1**	**2**	**Above 2**
Less than 7000	10	160	25	0
7000-10000	0	305	27	2
10000-13000	1	535	29	1
13000-16000	2	469	59	25
16000 or more	1	579	82	88

TABLE 5.41.1

Solution: Let X be the random variable denoting the number of vehicles and Y be the income.

a) no of total families chosen for survey = 2400

$$\Pr(X = 2, 10000 < Y < 13000) = \frac{29}{2400} \tag{5.41.1}$$
$$= 0.012 \tag{5.41.2}$$

b)

$$\Pr(X = 1, Y > 16000) = \frac{579}{2400} \quad (5.41.3)$$
$$= 0.241 \quad (5.41.4)$$

c)

$$\Pr(X = 0, Y < 7000) = \frac{10}{2400} \quad (5.41.5)$$
$$= 0.0042 \quad (5.41.6)$$

d)

$$\Pr(X > 2, 13000 < Y < 16000) = \frac{25}{2400} \quad (5.41.7)$$
$$= 0.0104 \quad (5.41.8)$$

e) The number of families is given by the sum of columns 0 and 1 in Table 5.41.1. Hence,

$$\Pr(X < 2) = \frac{1892}{2400} \quad (5.41.9)$$
$$= 0.78833 \quad (5.41.10)$$

5.42. Eleven bags of wheat flour, each marked 5 kg, actually contained the following weights of flour (in kg)
4.97 5.05 5.08 5.03 5.00 5.06 5.08 4.98 5.04 5.07 5.00
Find the probability that any of these bags chosen at random contains more than 5 kg of flour.
Solution:

5.43. From Table 5.43.1, prepare a frequency distribution table, regarding the concentration of sulphur dioxide in the air in parts per million of a certain city for 30 days. Using this table, find the probability of the concentration of sulphur dioxide in the interval 0.12 - 0.16 on any of these days. **Solution:**

0.03	0.08	0.08	0.09	0.04	0.17
0.16	0.05	0.02	0.06	0.18	0.20
0.11	0.08	0.12	0.13	0.22	0.07
0.08	0.01	0.10	0.06	0.09	0.18
0.11	0.07	0.05	0.07	0.01	10.04

TABLE 5.43.1

a) P(A) be the prbability of concentration of sulpher

concentration of sulphur	**friquency**
0.01	2
0.02	1
0.03	1
0.04	2
0.05	2
0.06	2
0.07	3
0.08	4
0.09	2
0.10	1
0.11	2
0.12	1
0.13	1
0.16	1
0.17	1
0.18	2
0.20	1
0.22	1

TABLE 5.43.2

$$p(A) = \frac{1+1+1}{30} \quad (5.43.1)$$
$$= 0.1 \quad (5.43.2)$$

codes for the above equation can be get from here

solutions/1−10/codes/prob/prob10.py

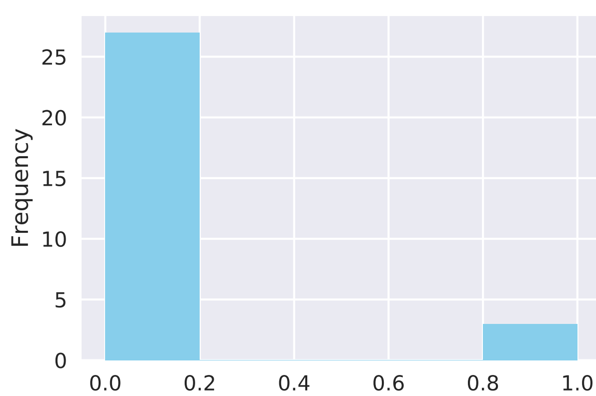


Fig. 5.43: probability of SO_2 0.12 to 0.16

5.44. A, B, O, O, AB, O, A, O, B, A, O, B, A, O, O, A, AB, O, A, A, O, O, AB, B, A, O, B, A, B, O.
prepare a frequency distribution table regarding the blood groups of 30 students of a class. Use this table to determine the probability that a student of this class, selected at random, has blood group AB.

5.45. Determine P(E/F), if mother, father and son line up at random for a family picture
E : son on one end, F : father in middle

5.46. Consider the experiment of throwing a die, if a multiple of 3 comes up, throw the die again and if any other number comes, toss a coin. Find the conditional probability of the event 'the coin shows a tail', given that 'at least one die shows a 3'.
Solution: Let $X_k \in \{-1, 0, 3, 6, r\}, k = 1, 2, \ldots$ represent the described process, where $r, 3, 6$ denote the outcome of the die and -1,0 denote the outcome of the coin, 0 representing a tail. In general, the transition probabilities for the Markov Chain are

$$\Pr(X_n = 0|X_{n-1} = r) = \Pr(X_n = -1|X_{n-1} = r) \tag{5.46.1}$$

$$= \frac{1}{2} \tag{5.46.2}$$

$$\Pr(X_n = 0|X_{n-1} = 3) = \Pr(X_n = -1|X_{n-1} = 3) \tag{5.46.3}$$

$$= 0 \tag{5.46.4}$$

$$\Pr(X_n = 0|X_{n-1} = 6) = \Pr(X_n = -1|X_{n-1} = 6) \tag{5.46.5}$$

$$= 0 \tag{5.46.6}$$

$$\Pr(X_n = 3|X_{n-1} = r) = 0 \tag{5.46.7}$$

$$\Pr(X_n = 6|X_{n-1} = r) = 0 \tag{5.46.8}$$

$$\Pr(X_n = r|X_{n-1} = r) = 0 \tag{5.46.9}$$

$$\Pr(X_n = 3|X_{n-1} = 6) = \Pr(X_n = 6|X_{n-1} = 3) \tag{5.46.10}$$

$$\Pr(X_n = 3|X_{n-1} = 3) = \Pr(X_n = 6|X_{n-1} = 6) \tag{5.46.11}$$

$$= \frac{1}{6} \tag{5.46.12}$$

$$\Pr(X_n = r|X_{n-1} = 3) = \Pr(X_n = r|X_{n-1} = 6) \tag{5.46.13}$$

$$= \frac{4}{6} \tag{5.46.14}$$

Thus,

$$\Pr(X_2 = 0|X_1 = 3) = 0 \tag{5.46.15}$$

5.47. One card is drawn from a well-shuffled deck of 52 cards. Find the probability of getting (i) a king of red colour
(ii) a face card
(iii) a red face card
(iv) the jack of hearts
(v) a spade
(vi) the queen of diamonds
Solution: Let $X \in \{0, 1, 2, 3\}$ be the card type, $Y \in 0, 1$ be the colour and $\mathbf{Z} \in \{0, 1, \ldots, 12\}$ be the card number. The sample size = total number of cards ina deck

a) The probability of drawing a king of red colour

$$\Pr(Y = 0, Z = 11) = \Pr(Y = 0)\Pr(Z = 11) \quad (5.47.1)$$

$$= \frac{1}{2} \times \frac{4}{52} = \frac{1}{23} \quad (5.47.2)$$

b) The probability of drawing a face card is

$$\Pr(8 \le Z \le 11) = \frac{12}{52} = \frac{3}{13} \quad (5.47.3)$$

c) The probability of drawing a red face card from the deck is

$$\Pr(Y = 0, 8 \le Z \le 11)$$
$$= \Pr(Y = 0)\Pr(8 \le Z \le 11)$$
$$= \frac{1}{2} \times \frac{3}{13} = \frac{3}{23} \quad (5.47.4)$$

d) The probability of drawing a jack of hearts is

$$\Pr(Z = 9, X = 0) = \frac{1}{13} \times \frac{1}{4} = \frac{1}{52} \quad (5.47.5)$$

e) The probability of drawing a spade is

$$\Pr(X = 1) = \frac{13}{52} = \frac{1}{4} \quad (5.47.6)$$

f) The probability of drawing a queen of diamond is

$$\Pr(X = 2, Z = 10) = \Pr(X = 2)\Pr(Z = 10) \quad (5.47.7)$$

$$= \frac{1}{4} \times \frac{1}{13} = \frac{1}{52} \quad (5.47.8)$$

The python code for the distribution is

solutions/20−10/prob/codes/cards125.py

5.48. Five cards—the ten, jack, queen, king and ace of diamonds, are well-shuffled with their face downwards. One card is then picked up at random.
(i) What is the probability that the card is the queen?
(ii) If the queen is drawn and put aside, what is the probability that the second card picked up is (a) an ace? (b) a queen?
Solution:

a) The probability that a queen is picked is

$$\Pr(Z = 10 | Z \in \{10, 11, 12, 13, 14\}) = \frac{1}{5} \quad (5.48.1)$$

b) After a queen is drawn and put aside The probability that an ace is picked is

$$\Pr(Z = 14 | Z \in 10, 11, 13, 14) = \frac{1}{4} \quad (5.48.2)$$

The probability that a queen is picked from the remaining cards is

$$\Pr(Z = 12 | Z \in 10, 11, 13, 14) = 0 \quad (5.48.3)$$

The python code below calculates the above probabilities for 100000 picks

solutions/20−10/prob/codes/cards126.py

5.49. A box contains 90 discs which are numbered from 1 to 90. If one disc is drawn at random from the box, find the probability that it bears (i) a two-digit number (ii) a perfect square number (iii) a number divisible by 5.
Solution: (i)The sample size

$$S = 90 \quad (5.49.1)$$

(i)number of discs bearing a two digit number is

$$T = 81 \quad (5.49.2)$$

The probability of drawing a disc bearing two digit number is

$$\Pr(T) = \frac{T}{S} = \frac{81}{90} \quad (5.49.3)$$

$$= \frac{9}{10} \quad (5.49.4)$$

(ii)number of discs bearing a perfect square is

$$Sq = 9 \quad (5.49.5)$$

The probability of drawing a disc bearing perfect square is

$$\Pr(Sq) = \frac{Sq}{S} = \frac{9}{90} \quad (5.49.6)$$

$$= \frac{1}{10} \quad (5.49.7)$$

(iii)number of discs bearing number divisible by 5 is

$$F = 18 \tag{5.49.8}$$

The probability of drawing a disc bearing number divisible by 5 is

$$\Pr(F) = \frac{F}{S} = \frac{18}{90} \tag{5.49.9}$$

$$= \frac{1}{5} \tag{5.49.10}$$

The python code for the above solution is

```
solutions/20−10/prob/codes/exer129.py
```

5.50. A child has a die whose six faces show the letters as given in Fig. 5.50 .

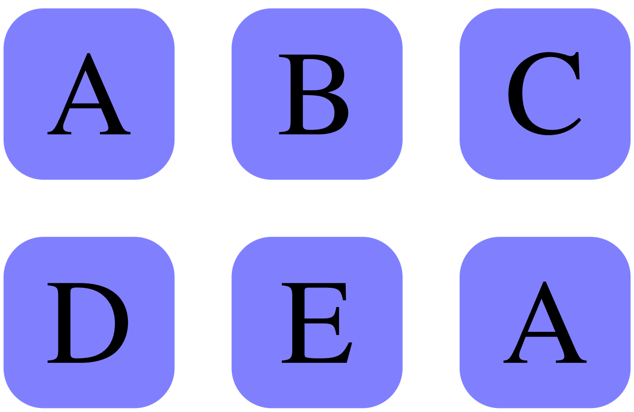


Fig. 5.50

The die is thrown once. What is the probability of getting (i) A? (ii) D?
Solution: The sample size= total faces of a die

$$S = 6 \tag{5.50.1}$$

(i)number of faces on which letter A appears

$$A = 2 \tag{5.50.2}$$

The probability of getting an A

$$\Pr(A) = \frac{A}{S} = \frac{2}{6} \tag{5.50.3}$$

$$= \frac{1}{3} \tag{5.50.4}$$

(ii)number of faces on which letter D appears

$$D = 1 \tag{5.50.5}$$

The probability of getting an A

$$\Pr(D) = \frac{D}{S} = \frac{1}{6} \tag{5.50.6}$$

The python code for the above solution is

```
./prob/codes/exer130.py
```

5.51. Which of the following arguments are correct and which are not correct? Give reasons for your answer.
(i) If two coins are tossed simultaneously there are three possible outcomes—two heads, two tails or one of each. Therefore, for each of these outcomes, the probability is $\frac{1}{3}$
(ii) If a die is thrown, there are two possible outcomes—an odd number or an even number. Therefore, the probability of getting an odd number is $\frac{1}{2}$.
Solution:

1. In the given question,
   The sample size = Total number of possibilities(S)=6

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix} \tag{5.51.1}$$

   Event size= Odd number =3

$$\begin{pmatrix} 1 & 3 & 5 \end{pmatrix} \tag{5.51.2}$$

   Probability for this event is = $\frac{1}{2}$
   The python code for the distribution of data,

```
prob/codes/prob6_b.py
```

   This shows the diagrametic representation of dice with the live update of probability with the role of dice.

5.52. Two customers Shyam and Ekta are visiting a particular shop in the same week (Tuesday to Saturday). Each is equally likely to visit the shop on any day as on another day. What is the probability that both will visit the shop on
(i) the same day?
(ii) consecutive days?
(iii) different days?
Solution: In the given question,

a) The sample size = Total number of possibilities(S)=25
   The possibilities are shown in the below table 5.52.1 Event size=Both same day=5
   Possibilities are given in table 5.52.2 Probability =

$$P = \frac{1}{5} \tag{5.52.1}$$

a) Event size = On consequitive days=8

Possibilities	
Shyam	Ekta
Tu	Tu,W,Th,F,Sa
W	Tu,W,Th,F,Sa
Th	Tu,W,Th,F,Sa
F	Tu,W,Th,F,Sa
Sa	Tu,W,Th,F,Sa

TABLE 5.52.1: Input Values

Possibilities	
Shyam	Ekta
Tu	Tu
W	W
Th	Th
F	F
Sa	Sa

TABLE 5.52.2: Event Values

Possibilities are given in the table 5.52.3
Probability =

Possibilities	
Shyam	Ekta
Tu	W
W	Tu,Th
Th	W,F
F	Th,Sa
Sa	F

TABLE 5.52.3: Event Values

$$P = \frac{8}{25} \tag{5.52.2}$$

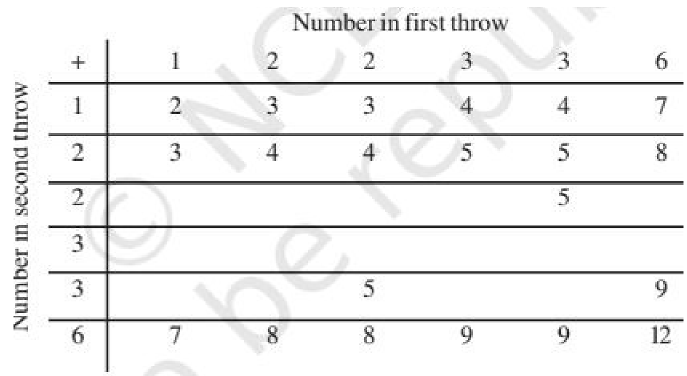
a) Event size= On different days=20
Possibilities are given in the table 5.52.4
Probability =

$$P = \frac{4}{5} \tag{5.52.3}$$

5.53. A die is numbered in such a way that its faces show the numbers 1, 2, 2, 3, 3, 6. It is thrown two times and the total score in two throws is noted. Complete the following table which gives a few values of the total score on the two throws:

Possibilities	
Shyam	Ekta
Tu	W,Th,F,Sa
W	Tu,Th,F,Sa
Th	Tu,W,F,Sa
F	Tu,W,Th,Sa
Sa	Tu,W,Th,F

TABLE 5.52.4: Event Values



+	1	2	2	3	3	6
1	2	3	3	4	4	7
2	3	4	4	5	5	8
2					5	
3						
3			5			9
6	7	8	8	9	9	12

What is the probability that the total score is (i) even? (ii) 6? (iii) at least 6?

Solution: In the given question,
The total number of possibilities=36
The Table 5.53.1 shows the possibilities

+	1	2	2	3	3	6
1	2	3	3	4	4	7
2	3	4	4	5	5	8
2	3	4	4	5	5	8
3	4	5	5	6	6	9
3	4	5	5	6	6	9
6	7	8	8	9	9	12

TABLE 5.53.1

a) Event size= No. of even numbers= 18
Probability=

$$P = \frac{1}{2} \tag{5.53.1}$$

a) Event size= No. of six=4
Probability=

$$P = \frac{1}{9} \tag{5.53.2}$$

a) Event size= Atleast six=15

Probability=

$$P = \frac{5}{12} \tag{5.53.3}$$

The python code for the calculation and completion of the excel file is at

```
solutions/10−1/prob/codes/prob8.py
```

6 Axioms of Probability

6.1. Which of the following cannot be the probability of an event?
(A)$\frac{2}{3}$(B) −1.5 (C) 15 (D) 0.7
Solution: $0 < \Pr(E) < 1$. Hence, (B) and (C) are the right answer.

6.2. If P(E) = 0.05, what is the probability of 'not E'? **Solution:** The desired probability is

$$\Pr(E') = 1 - \Pr(E) = 0.95 \tag{6.2.1}$$

6.3. If A and B are two events such that P(A) ≠ 0 and P(B/A) = 1, then (A) A ⊂ B
(B) B ⊂ A
(C) B = ϕ
(D) A = ϕ
Solution: Given

$$Pr(B|A) = 1. \tag{6.3.1}$$

By definition,

$$Pr(B|A) = \frac{Pr(AB)}{Pr(A)} \tag{6.3.2}$$

$$\implies \frac{Pr(AB)}{Pr(A)} = 1 \tag{6.3.3}$$
$$\implies Pr(AB) = Pr(A) \tag{6.3.4}$$
$$\implies AB = A \tag{6.3.5}$$

A) Take any

$$X \in A \tag{6.3.6}$$

. From (6.3.5), we get

$$X \in AB \tag{6.3.7}$$

is also true.
Therefore, for any

$$X \in A \tag{6.3.8}$$
$$\implies X \in B \tag{6.3.9}$$

$$A \subseteq B \tag{6.3.10}$$

is also true.
But, since A and B are two events,

$$A \neq B \tag{6.3.11}$$

. Hence,

$$A \subset B \tag{6.3.12}$$

Therefore, option (A) is correct.

B) If

$$B \subset A \tag{6.3.13}$$

Then,

$$AB = B. \tag{6.3.14}$$

$$\implies Pr(AB) = Pr(B) \tag{6.3.15}$$

But, from (6.3.4), we have,

$$Pr(AB) = Pr(A) \tag{6.3.16}$$
$$\implies Pr(AB) = Pr(A) = Pr(B) \tag{6.3.17}$$

But,since A and B are two events,

$$A \neq B \tag{6.3.18}$$

. Hence, option (B) is incorrect.

C) If

$$B = \phi \tag{6.3.19}$$

$$\implies Pr(AB) = 0 \tag{6.3.20}$$

From (6.3.4), we know that,

$$Pr(AB) = Pr(A) \tag{6.3.21}$$
$$\implies Pr(AB) = Pr(A) = 0 \tag{6.3.22}$$

But,from the given data, we know that

$$Pr(A) \neq 0 \tag{6.3.23}$$

Therefore,option C is incorrect.

D) If

$$A = \phi \tag{6.3.24}$$

$$\implies Pr(A) = 0 \tag{6.3.25}$$

But,from the given data, we know that

$$Pr(A) \neq 0 \tag{6.3.26}$$

Therefore,option D is incorrect.

6.4. If P(A/B) > P(A), then which of the following

is correct : (A) P(B/A) < P(B)
(B) P(A ∩ B) < P(A) . P(B)
(C) P(B/A) > P(B)
(D) P(B/A) = P(B)
Solution: From the given information,

$$\frac{\Pr(AB)}{\Pr(B)} > \Pr(A) \quad (6.4.1)$$

$$\implies \Pr(AB) > \Pr(A)\,.\Pr(B) \quad (6.4.2)$$

Hence, option (B) is false. Now, dividing the equation by $\Pr(A)$ on both sides i.e.,

$$\frac{\Pr(AB)}{\Pr(A)} > \Pr(B) \quad (6.4.3)$$

But $\frac{\Pr(AB)}{\Pr(A)} = \Pr(B|A)$. Therefore, from (6.4.3)

$$\Pr(B|A) > \Pr(B) \quad (6.4.4)$$

Hence, option (C) is correct.

6.5. If A and B are any two events such that $P(A) + P(B) - P(AB) = P(A)$, then
(A) $P(B/A) = 1$
(B) $P(A/B) = 1$
(C) $P(B/A) = 0$
(D) $P(A/B) = 0$
Solution:

$$\because \Pr(A) + \Pr(B) - \Pr(AB) = \Pr(A)\,, \quad (6.5.1)$$

$$\Pr(B) = \Pr(AB) \quad (6.5.2)$$

Also,

$$\Pr(A|B)\Pr(B) = \Pr(AB) \quad (6.5.3)$$

From (6.5.2),

$$\Pr(A|B)\Pr(B) = \Pr(B) \quad (6.5.4)$$

$$\implies \Pr(A|B) = 1, \quad \Pr(B) \neq 0 \quad (6.5.5)$$

6.6. Complete the following statements:
(i) Probability of an event E + Probability of the event ‘not E’ =——— .
(ii) The probability of an event that cannot happen is———- . Such an event is called——— .
(iii) The probability of an event that is certain to happen is ———.
(iv) The sum of the probabilities of all the elementary events of an experiment is———-.
(v) The probability of an event is greater than or equal to and less than or equal to ————–.

Solution:
a) 1
b) 0, null
c) 1
d) 1
e) $0 \leq \Pr(E) \leq 1$.

6.7. An electronic assembly consists of two subsystems, say, A and B. From previous testing procedures, the following probabilities are assumed to be known:
P(A fails) = 0.2
P(B fails alone) = 0.15
P(A and B fail) = 0.15

Evaluate the following probabilities
(i) P(A fails—B has failed)
(ii) P(A fails alone)
Solution: Given,

$$\Pr(\text{A fails}) = \Pr(A) = 0.2$$
$$\Pr(\text{B fails alone}) = \Pr(B - A) = 0.15$$
$$\Pr(\text{A and B fails}) = \Pr(AB) = 0.15$$

Now,we need to find $\Pr(\text{A fails alone}) = \Pr(A - B)$

a)

$$\Pr(A) = \Pr(A - B) + \Pr(AB) \quad (6.7.1)$$

$$\implies \Pr(A - B) = \Pr(A) - \Pr(AB) \quad (6.7.2)$$

$$\implies \Pr(A - B) = 0.20 - 0.15 \quad (6.7.3)$$

$$\implies \Pr(A - B) = 0.05 \quad (6.7.4)$$

Therefore, $\Pr(\text{A fails alone}) = \Pr(A - B) = 0.05$

b) Now,finding the probability of B

$$\Pr(B - A) = \Pr(B) - \Pr(AB) \quad (6.7.5)$$

$$\implies \Pr(B) = \Pr(B - A) + \Pr(AB) \quad (6.7.6)$$

$$\implies \Pr(B) = 0.15 + 0.15 \quad (6.7.7)$$

$$\implies \Pr(B) = 0.30 \quad (6.7.8)$$

Now, we need to find

Pr (A fails|B has failed)=Pr $(A|B)$

$$\Pr(A|B) = \frac{\Pr(AB)}{\Pr(B)} \quad (6.7.9)$$

$$\implies \Pr(A|B) = \frac{0.15}{0.30} \quad (6.7.10)$$

$$\implies \Pr(A|B) = 0.5 \quad (6.7.11)$$

Therefore, Pr (A fails| B has failed)=Pr $(A|B)$=0.5

6.8. A and B are two events such that P (A) $\neq$ 0. Find P(B/A), if
(i) A is a subset of B
(ii) A $\cap$ B = ϕ
Solution: By definition,

$$\Pr(B|A) = \frac{\Pr(AB)}{\Pr(A)} \quad (6.8.1)$$

a)

$$A \subset B \implies \Pr(AB) = \Pr(A) \quad (6.8.2)$$

$$\implies \Pr(B|A) = 1 \quad (6.8.3)$$

upon substituting in (6.8.1)

b)

$$A \cup B = \phi \implies \Pr(AB) = 0 \quad (6.8.4)$$

$$\implies \Pr(B|A) = 0 \quad (6.8.5)$$

upon substituting in (6.8.1)

6.9. If A and B are two events such that A $\subset$ B and P(B) $\neq$ 0, then which of the following is correct?

a) P(A/B) = $\frac{P(B)}{P(A)}$
b) $P(A/B) < P(A)$
c) P(A/B) $\geq$ P(A)
d) None of these

Solution: We know that A is the subset of B.
$\Rightarrow$ Every element of A is an element of B.

$$\therefore AB = A \quad (6.9.1)$$

We know that

$$\Pr(A|B) = \frac{\Pr(AB)}{\Pr(B)} = \frac{\Pr(A)}{\Pr(B)} \quad (6.9.2)$$

Given $0 < \Pr(B) \leq 1$

$$\Rightarrow \frac{1}{\Pr(B)} \geq 1 \quad (6.9.3)$$

By multiplying with $\Pr(A)$ on both sides of the inequality, we get

$$\frac{\Pr(A)}{\Pr(B)} \geq \Pr(A) \quad (6.9.4)$$

Using (6.9.2), we have

$$\Pr(A|B) \geq \Pr(A)$$

6.10. Let E and F be events with $\Pr(E) = \frac{3}{5}$, $\Pr(F) = \frac{3}{10}$ and $\Pr(E \cap F) = \frac{1}{5}$. Are E and F independent?
Solution:

$$\Pr(E)\Pr(F) = \frac{9}{50} \neq \Pr(E \cap F) \quad (6.10.1)$$

Hence E and F are not independent.

6.11. Given that the events A and B are such that P(A) = $\frac{1}{2}$, P(A $\cup B$)= $\frac{3}{5}$ and P(B) = p. Find p if they are
(i) mutually exclusive
(ii) independent.
Solution:

i) Since the events are mutually exclusive, by definition

$$\Pr(AB) = 0 \quad (6.11.1)$$

$$\implies \Pr(A+B) = \Pr(A) + \Pr(B) \quad (6.11.2)$$

On substituting the values of $\Pr(A)$, $\Pr(B)$ and $\Pr(A+B)$ in (6.11.2), we get

$$\frac{3}{5} = \frac{1}{2} + p \quad (6.11.3)$$

$$\implies p = \frac{1}{10} \quad (6.11.4)$$

ii) Since the events are independent

$$\Pr(AB) = \Pr(A)\Pr(B) \quad (6.11.5)$$

We know

$$\Pr(A+B) = \Pr(A) + \Pr(B) - \Pr(AB) \quad (6.11.6)$$

$$\implies \Pr(A+B) = \Pr(A) + \Pr(B) - \Pr(A)\Pr(B) \quad (6.11.7)$$

On substituting the values of $\Pr(A)$, $\Pr(B)$

and $\Pr(A+B)$ in (6.11.7), we get

$$\frac{3}{5} = \frac{1}{2} + p - \frac{1}{2}p \tag{6.11.8}$$

$$\implies p = \frac{1}{5} \tag{6.11.9}$$

6.12. Let A and B be independent events with P(A) = 0.3 and P(B) = 0.4. Find
(i) P(A ∩ B)
(ii) P(A ∪ B)
(iii) P(A/B)
(iv) P(B/A)
Solution: Given A and B are Independent events and

$$\Pr(A) = 0.3 \tag{6.12.1}$$

$$\Pr(B) = 0.4 \tag{6.12.2}$$

a) By definition,

$$\Pr(AB) = \Pr(A)\Pr(B) \tag{6.12.3}$$

$$\Pr(AB) = (0.3)(0.4) \tag{6.12.4}$$

$$\therefore \Pr(AB) = 0.12 \tag{6.12.5}$$

b) By definition,

$$\Pr(A+B) = \Pr(A) + \Pr(B) - \Pr(AB) \tag{6.12.6}$$

From (6.12.5),

$$\Pr(A+B) = 0.3 + 0.4 - (0.12) \tag{6.12.7}$$

$$\therefore \Pr(A+B) = 0.58 \tag{6.12.8}$$

c) From the definition of Independent Events,

$$\Pr(A/B) = \Pr(A) \tag{6.12.9}$$

$$\therefore \Pr(A/B) = 0.3 \tag{6.12.10}$$

d) From the definition of Independent Events,

$$\Pr(B/A) = \Pr(B) \tag{6.12.11}$$

$$\therefore \Pr(B/A) = 0.4 \tag{6.12.12}$$

6.13. If A and B are two events such that P(A) = $\frac{1}{4}$, P(B) = $\frac{1}{2}$ and P(A ∩ B) = $\frac{1}{8}$. find P (not A and not B).
Solution: $\Pr(notAandnotB)$ is equivalent to $\Pr(A'B')$.
from De-morgan's law,

$$(A'B') = (A+B)' \tag{6.13.1}$$

$$So, \Pr(A'B') = \Pr((A+B)') \tag{6.13.2}$$

$$\Pr((A+B)') = 1 - \Pr(AB) \tag{6.13.3}$$

$$= 1 - (\Pr(A) + \Pr(B) - \Pr(AB)) \tag{6.13.4}$$

$$= 1 - \left(\frac{1}{4} + \frac{1}{2} - \frac{1}{8}\right) \tag{6.13.5}$$

$$= \frac{3}{8} \tag{6.13.6}$$

$$\tag{6.13.7}$$

Therefore,

$$\Pr((A+B)') = \frac{3}{8} \tag{6.13.8}$$

$$\implies \Pr(A'B') = \frac{3}{8} \tag{6.13.9}$$

$$\tag{6.13.10}$$

6.14. Events A and B are such that P (A) = $\frac{1}{2}$, P(B) = $\frac{7}{12}$ and P(not A or not B) = $\frac{1}{4}$. State whether A and B are independent ?
Solution:

$$\Pr(not\ A\ or\ not\ B) = \Pr(A' + B') \tag{6.14.1}$$

We know that,

$$\Pr(A' + B') = \Pr((AB)') \tag{6.14.2}$$

As,

$$(AB)(AB)' = 0 \tag{6.14.3}$$

$$\Pr(AB) + \Pr((AB)') = 1 \tag{6.14.4}$$

$$\Pr(AB) = 1 - \Pr((AB)') \tag{6.14.5}$$

Using 6.14.2 in 6.14.5, We get

$$\Pr(AB) = 1 - \Pr(A' + B') \tag{6.14.6}$$

On substituting the value of $\Pr(A' + B')$ in 6.14.6, we get

$$\Pr(AB) = 1 - \frac{1}{4} \tag{6.14.7}$$

$$\implies \Pr(AB) = \frac{3}{4} \tag{6.14.8}$$

Given, $\Pr(A) = \frac{1}{2}$ *and* $\Pr(B) = \frac{7}{12}$

$$\implies \Pr(A)\Pr(B) = \frac{7}{24} \tag{6.14.9}$$

From 6.14.8 and 6.14.9,

$$\Pr(AB) \neq \Pr(A)\Pr(B) \tag{6.14.10}$$

As the events A and B does not satisfy the definition of independent events, $\therefore$ Events A and B are dependent.

6.15. Given two independent events A and B such that P(A) = 0.3, P(B) = 0.6. Find
(i) P(A and B)
(ii) P(A and not B)
(iii) P(A or B)
(iv) P(neither A nor B)

Solution:

i) Since the events A and B are independent events, by definition

$$\Pr(A\, and\, B) = \Pr(AB) = \Pr(A)\Pr(B) \tag{6.15.1}$$

On substituting the values of $\Pr(A)$,$\Pr(B)$ in (6.15.1), we get

$$\Pr(A\, and\, B) = \Pr(A)\Pr(B) \tag{6.15.2}$$
$$= (0.3)(0.6) \tag{6.15.3}$$
$$\implies \Pr(A\, and\, B) = 0.18 \tag{6.15.4}$$

ii) As the events A and B are independent, then A and B’ are also independent.

$$\implies \Pr(A\, and\, not\, B) = \Pr(AB') \tag{6.15.5}$$
$$= \Pr(A)\Pr(B') \tag{6.15.6}$$
$$\therefore \Pr(A\, and\, not\, B) = \Pr(A)\Pr(B') \tag{6.15.7}$$

And we know that,

$$\Pr(B') = 1 - \Pr(B) \tag{6.15.8}$$

Using (6.15.8) in (6.15.7) we will get,

$$\Pr(A\, and\, not\, B) = \Pr(AB') \tag{6.15.9}$$
$$= \Pr(A)\Pr(B') \tag{6.15.10}$$
$$\Pr(A\, and\, not\, B) = \Pr(A)(1 - \Pr(B)) \tag{6.15.11}$$

On substituting the values of $\Pr(A)$,$\Pr(B)$ in (6.15.11), we get

$$\Pr(A\, and\, not\, B) = 0.3(1 - 0.6) \tag{6.15.12}$$
$$= (0.3)(0.4) \tag{6.15.13}$$
$$\implies \Pr(A\, and\, not\, B) = 0.12 \tag{6.15.14}$$

iii)

$$\Pr(A\, or\, B) = \Pr(A + B) \tag{6.15.15}$$

We know that,

$$\Pr(A + B) = \Pr(A) + \Pr(B) - \Pr(AB) \tag{6.15.16}$$

As events A and B are independent events,

$$\Pr(AB) = \Pr(A)\Pr(B) \tag{6.15.17}$$

Using (6.15.17) and (6.15.16) in (6.15.15), We get

$$\Pr(A + B) = \Pr(A) + \Pr(B) - \Pr(A)\Pr(B) \tag{6.15.18}$$

On substituting the values of $\Pr(A)$,$\Pr(B)$ in (6.15.18), we get

$$\Pr(A\, or\, B) = 0.3 + 0.6 - (0.3)(0.6) \tag{6.15.19}$$
$$= 0.9 - 0.18 \tag{6.15.20}$$
$$\implies \Pr(A\, or\, B) = 0.72 \tag{6.15.21}$$

iv)

$$\Pr(neither\, A\, nor\, B) = \Pr(A'B') \tag{6.15.22}$$
$$= \Pr((A + B)') \tag{6.15.23}$$
$$\Pr(neither\, A\, nor\, B) = 1 - \Pr(A + B) \tag{6.15.24}$$

From (6.15.21),

$$\Pr(A\, or\, B) = \Pr(A + B) = 0.72 \tag{6.15.25}$$

Using (6.15.25) in (6.15.24), We get

$$\Pr(neither\, A\, nor\, B) = 1 - 0.72 \tag{6.15.26}$$
$$\implies \Pr(neither\, A\, nor\, B) = 0.28 \tag{6.15.27}$$

6.16. A die marked 1, 2, 3 in red and 4, 5, 6 in green is tossed. Let A be the event, ’the number is even,’ and B be the event, ’the number is red’.

Are A and B independent?
Solution: Let

$$X \in \{1, 2, 3, 4, 5, 6\} \quad (6.16.1)$$

$\because$

$$\Pr(X = i) = \begin{cases} \frac{1}{6} & 1 \le i \le 6 \\ 0 & otherwise \end{cases}, \quad (6.16.2)$$

$$\Pr(A) = \sum_{i=2,4,6} \Pr(X = i) = \frac{1}{2} \quad (6.16.3)$$

$$\Pr(B) = \sum_{i=1}^{3} \Pr(X = i) = \frac{1}{2} \quad (6.16.4)$$

$$\Pr(AB) = \Pr(X = 2) = \frac{1}{6} \quad (6.16.5)$$

$$\Pr(A) \times \Pr(B) = \frac{1}{4} \quad (6.16.6)$$

$\because$

$$\Pr(A) \times \Pr(B) \neq \Pr(AB), \quad (6.16.7)$$

A and B are not independent.

6.17. A person plays a game of tossing a coin thrice. For each head, he is given Rs 2 by the organiser of the game and for each tail, he has to give Rs 1.50 to the organiser. Let X denote the amount gained or lost by the person. Show that X is a random variable and exhibit it as a function on the sample space of the experiment.
Solution: Let Ω be the sample space.
Let X_0 be a random variable where, $X_0 \in \{2, -1.5\}$

$$X_1 = X_0 + Y \quad (6.17.1)$$

$$X_2 = X_1 + Y \quad (6.17.2)$$

Here, $Y \in \{2, -1.5\}$. X_1, X_2, Y are random variables.

$$X = X_2 \quad (6.17.3)$$

The value of X is obtained from a random process. So, X is a random variable.
Let c be the number of heads.

$$c = \frac{(X_0 + 1.5)}{3.5} + \frac{(Y + 1.5)}{3.5} + \frac{(Y + 1.5)}{3.5} \quad (6.17.4)$$

Y in eqs. (6.17.1) and (6.17.2) can have different values.

We can relate X with c,

$$X = 2c - 1.5(3 - c) \quad (6.17.5)$$

$$X = 3.5c - 4.5 \quad (6.17.6)$$

c	X
0	-4.5
1	-1
2	2.5
3	6

6.18. If $\Pr(A) = \frac{7}{13}, \Pr(B) = \frac{9}{13}$ and $\Pr(AB) = \frac{4}{13}$, evaluate $\Pr(A|B)$.
Solution:

$$\Pr(A|B) = \frac{\Pr(AB)}{\Pr(B)} = \frac{4/13}{9/13} = \frac{4}{9} \quad (6.18.1)$$

6.19. A die is thrown. If E is the event "the number appearing is a multiple of 3" and F be the event "the number appearing is even" then find whether E and F are independent ?
Solution:

Lemma 6.1. *If A and B are independent events then the property can be expressed as*

$$\Pr(A|B) = \Pr(A). \quad (6.19.1)$$

Let the random variable representing the events be $X \in \{0, 1\}$,where

X	0	Number appearing is a multiple of 3
	1	Number appearing is even

TABLE 6.19.1

From the given information we have,

$$\Pr(X = 0) = \frac{1}{3} \quad (6.19.2)$$

$$\Pr(X = 1) = \frac{1}{2} \quad (6.19.3)$$

$$\Pr(X = 0, X = 1) = \frac{1}{6} \quad (6.19.4)$$

Now to check whether the events are independent we use Lemma 6.1

$$\Pr(X = 0|X = 1) = \frac{\Pr(X = 0, X = 1)}{\Pr(X = 1)} \quad (6.19.5)$$

$$= \frac{1}{3} \quad (6.19.6)$$

$$= \Pr(X = 0) \quad (6.19.7)$$

Thus $\Pr(X = 0|X = 1) = \Pr(X = 0)$ which implies the events are independent.

6.20. An unbiased die is thrown twice. Let the event A be "odd number on the first throw" and B the event "odd number on the second throw". Check the independence of the events A and B.
Solution: Events A and B are independent.

6.21. Prove that if E and F are independent events, then so are the events E and $F^{'}$.
Solution: From the given information,

$$\Pr(EF) = \Pr(E)\Pr(F) \tag{6.21.1}$$

Then,

$$\Pr(EF') = \Pr(E(1-F)) = \Pr(E - EF) \tag{6.21.2}$$
$$= \Pr(E) - \Pr(E \cap F) \tag{6.21.3}$$
$$= \Pr(E) - \Pr(E)\Pr(F) \tag{6.21.4}$$
$$= \Pr(E)(1 - \Pr(F)) \tag{6.21.5}$$
$$= \Pr(E)\Pr(1-F) \tag{6.21.6}$$
$$= \Pr(E)\Pr(F') \tag{6.21.7}$$

$\therefore E$ and F' are independent events.

6.22. If A and B are two independent events, then the probability of occurrence of at least one of A and B is given by 1- $P(A^{'})P(B^{'})$
Solution: From the given information, using the fact that A, B are independent,

$$\Pr(A+B) = \Pr(A) + \Pr(B) - \Pr(AB) \tag{6.22.1}$$
$$= \Pr(A) + \Pr(B - AB) \tag{6.22.2}$$
$$= \Pr(A) + \Pr(A'B) \tag{6.22.3}$$
$$= \Pr(A) + \Pr(A')\Pr(B) \tag{6.22.4}$$
$$= \Pr(A) + \Pr(A')(1 - \Pr(B')) \tag{6.22.5}$$
$$= \Pr(A) + \Pr(A') - \Pr(A')\Pr(B') \tag{6.22.6}$$
$$= 1 - \Pr(A')\Pr(B') \tag{6.22.7}$$

6.23. Given that E and F are events such that $\Pr(E) = 0.6, \Pr(F) = 0.3$ and $\Pr(EF) = 0.2$, find $\Pr(E|F)$ and $\Pr(F|E)$.
Solution: By definition,

$$\Pr(A|B) = \frac{\Pr(AB)}{\Pr(B)} \tag{6.23.1}$$

Thus, we can write:

$$\Pr(E|F) = \frac{\Pr(EF)}{\Pr(F)} = \frac{0.2}{0.3} = \frac{2}{3} \tag{6.23.2}$$

In a similar manner:

$$\Pr(F|E) = \frac{\Pr(EF)}{\Pr(E)} = \frac{0.2}{0.6} = \frac{1}{3} \tag{6.23.3}$$

6.24. Compute P(A/B), if P(B) = 0.5 and P (A ∩ B) = 0.32.
Solution: For two events A and B, by definition,

$$\Pr(A|B) = \frac{\Pr(AB)}{\Pr(B)} \tag{6.24.1}$$

Given $\Pr(B) = 0.5$ and $\Pr(AB) = 0.32$

$$\Pr(A|B) = \frac{0.32}{0.5} \tag{6.24.2}$$

Thus,

$$\Pr(A|B) = 0.64 \tag{6.24.3}$$

6.25. If P(A) = 0.8, P(B) = 0.5 and P(B/A) = 0.4, find
(i) P(A ∩ B)
(ii) P(A/B)
(iii) P(A ∪ B)
Solution:

a) By the definition

$$\Pr(B/A) = \frac{\Pr(AB)}{\Pr(A)} \tag{6.25.1}$$

Substituting the values,

$$\Pr(AB) = \frac{2}{5} \times \frac{4}{5} \tag{6.25.2}$$
$$\Pr(AB) = \frac{8}{25} \tag{6.25.3}$$

b) By the definition

$$\Pr(A/B) = \frac{\Pr(AB)}{\Pr(B)} \tag{6.25.4}$$

Substituting the values,

$$\Pr(A/B) = \frac{\frac{8}{25}}{\frac{1}{2}} \tag{6.25.5}$$
$$\Pr(A/B) = \frac{16}{25} \tag{6.25.6}$$

c) By the result

$$\Pr(A+B) = \Pr(A) + \Pr(B) - \Pr(AB) \quad (6.25.7)$$

Substituting the values,

$$\Pr(A+B) = \frac{4}{5} + \frac{1}{2} - \frac{8}{25} \quad (6.25.8)$$

$$\Pr(A+B) = \frac{98}{100} \quad (6.25.9)$$

6.26. Evaluate P(A ∪ B), if 2P(A) = P(B) = $\frac{5}{13}$ and P(A/B) = $\frac{2}{5}$.
Solution: By definition,

$$\Pr(A/B) = \frac{\Pr(AB)}{\Pr(B)} \quad (6.26.1)$$

Substituting the values,

$$\frac{2}{5} = \frac{\Pr(AB)}{5/13} \quad (6.26.2)$$

$$\Pr(AB) = \frac{2}{13} \quad (6.26.3)$$

$$\Pr(A+B) = \Pr(A) + \Pr(B) - \Pr(AB) \quad (6.26.4)$$

Substituting the values,

$$\Pr(A+B) = \frac{5}{26} + \frac{5}{13} - \frac{2}{13} = \frac{11}{26} \quad (6.26.5)$$

6.27. If P(A) = $\frac{6}{11}$, P(B) = $\frac{5}{11}$ and P(A ∪ B) = $\frac{11}{7}$ find
(i) P(A ∩ B)
(ii) P(A/B)
(iii) P(B/A)
Solution:

a)

$$\Pr(A+B) = \Pr(A) + \Pr(B) - \Pr(AB) \quad (6.27.1)$$

Substituting the values,

$$\frac{7}{11} = \frac{6}{11} + \frac{5}{11} - \Pr(AB) \quad (6.27.2)$$

$$\Pr(AB) = 1 - \frac{7}{11} = \frac{4}{11} \quad (6.27.3)$$

b)

$$\Pr(A|B) = \frac{\Pr(AB)}{\Pr(B)} = \frac{4/11}{5/11} = \frac{4}{5} \quad (6.27.4)$$

c)

$$\Pr(B|A) = \frac{\Pr(AB)}{\Pr(A)} = \frac{4/11}{6/11} = \frac{2}{3} \quad (6.27.5)$$

6.28. A fair die is rolled. Consider the events E = (1, 3, 5), F = (2, 3) and G = (2, 3, 4, 5) Find
(i) P(E/F) and P(F/E)
(ii) P(E/G) and P(G/E)
(iii) P((E ∪ F)/G) and P((E ∩ F)/G)
Solution: From the given information,

$$\Pr(E) = \frac{3}{6} = \frac{1}{2} \quad (6.28.1)$$

$$\Pr(F) = \frac{2}{6} = \frac{1}{3} \quad (6.28.2)$$

$$\Pr(G) = \frac{4}{6} = \frac{2}{3} \quad (6.28.3)$$

$$\Pr(EF) = \frac{1}{6} \quad (6.28.4)$$

$$\Pr(EG) = \frac{2}{6} = \frac{1}{3} \quad (6.28.5)$$

$$\Pr(FG) = \frac{2}{6} = \frac{1}{3} \quad (6.28.6)$$

$$\Pr(EFG) = \frac{1}{6} \quad (6.28.7)$$

a)

$$\Pr(E|F) = \frac{\Pr(EF)}{\Pr(F)} \quad (6.28.8)$$

$$\Pr(E|F) = \frac{\frac{1}{6}}{\frac{1}{3}} = \frac{1}{2} \quad (6.28.9)$$

$$\Pr(F|E) = \frac{\Pr(FE)}{\Pr(E)} \quad (6.28.10)$$

$$\Pr(F|E) = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3} \quad (6.28.11)$$

b)

$$\Pr(E|G) = \frac{\Pr(EG)}{\Pr(G)} \quad (6.28.12)$$

$$\Pr(E|G) = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2} \quad (6.28.13)$$

$$\Pr(G|E) = \frac{\Pr(GE)}{\Pr(G)} \quad (6.28.14)$$

$$\Pr(G|E) = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3} \quad (6.28.15)$$

$$(6.28.16)$$

c)

$$\begin{aligned}\Pr(E+F|G) &= \frac{\Pr(\{E+F\}G)}{\Pr(G)} \\ &= \frac{\Pr(EG+FG)}{\Pr(G)} \\ &= \frac{\Pr(EG)+\Pr(FG)-\Pr(EFG)}{\Pr(G)} \\ &= \frac{3}{4} \end{aligned} \tag{6.28.17}$$

and

$$\Pr(EF|G) = \frac{\Pr(EFG)}{\Pr(G)} = \frac{1}{4} \tag{6.28.18}$$

6.29. Choose the correct answer, if P(A) = $\frac{1}{2}$, P(B) = 0, then P(A/B) is

a) 0
b) $\frac{1}{2}$
c) not defined
d) 1

Solution:

$$\Pr(A|B) = \frac{\Pr(AB)}{\Pr(B)} \tag{6.29.1}$$

$$\because \Pr(B) = 0, B = 0, \implies AB = 0 \tag{6.29.2}$$

$$\text{or, } \Pr(AB) = 0 \tag{6.29.3}$$

$$\implies \Pr(A|B) = 0 \tag{6.29.4}$$

6.30. If A and B are events such that P(A/B) = P(B/A), then

a) A ⊂ B but A ≠ B
b) A = B
c) A ∩ B = ϕ
d) P(A) = P(B)

Solution:

$$\Pr(A|B) = \Pr(B|A) \tag{6.30.1}$$

$$\implies \frac{\Pr(AB)}{\Pr(A)} = \frac{\Pr(AB)}{\Pr(B)} \tag{6.30.2}$$

$$\implies \Pr(AB) = 0 \implies AB = 0 \tag{6.30.3}$$

$$\text{or, } \Pr(A) = \Pr(B) \tag{6.30.4}$$

6.31. If P(A) = $\frac{3}{5}$ and P(B) = $\frac{1}{5}$, find P (A ∩ B) if A and B are independent events.

Solution:

$$\Pr(AB) = \Pr(A)\Pr(B) = \frac{3}{25} \tag{6.31.1}$$

6.32. One card is drawn at random from a well shuffled deck of 52 cards. In which of the following cases are the events E and F independent?
(i) E : 'the card drawn is a spade' F : 'the card drawn is an ace'
(ii) E : 'the card drawn is black' F : 'the card drawn is a king'
(iii) E : 'the card drawn is a king or queen' F : 'the card drawn is a queen or jack'.
Solution: Two events E and F are said to be independent if they satisfy the criterion:

$$P(E \cap F) = P(E)P(F) \tag{6.32.1}$$

a) There are 13 cards of spades, 4 cards of aces and 1 card of ace of spades.

$$P(E) = \frac{13}{52} \tag{6.32.2}$$

$$P(F) = \frac{4}{52} \tag{6.32.3}$$

$$P(E \cap F) = \frac{1}{52} \tag{6.32.4}$$

Clearly, $P(E \cap F) = P(E)P(F)$. Therefore E and F are independent events.

b) There are 26 black cards, 4 king cards and 2 black and king cards.

$$P(E) = \frac{26}{52} \tag{6.32.5}$$

$$P(F) = \frac{4}{52} \tag{6.32.6}$$

$$P(E \cap F) = \frac{2}{52} \tag{6.32.7}$$

Clearly, $P(E \cap F) = P(E)P(F)$. Therefore E and F are independent events.

c) There are 8 kings or queens, 8 queens or jacks. In both of these, common is the quuen cards.

$$P(E) = \frac{8}{52} \tag{6.32.8}$$

$$P(F) = \frac{8}{52} \tag{6.32.9}$$

$$P(E \cap F) = \frac{4}{52} \tag{6.32.10}$$

Clearly, $P(E \cap F) \neq P(E)P(F)$. Therefore E and F are not independent events.

6.33. Two events A and B will be independent, if

a) A and B are mutually exclusive
b) $P(A'B') = [1 - P(A)]\ [1 - P(B)]$
c) P(A) = P(B)
d) P(A) + P(B) = 1

Solution:

a) A and B are not mutually exclusive because $P(A \cap B) = P(A) \times P(B)$ and it is not zero.
b) Also $P(A) = P(B)$ is not necessariy true.
c) $P(A) + P(B)$ is not always equal to 1.
d) If A and B are independent,

$$P(A'B') = P(A')P(B')$$
$$= (1 - P(A))\,(1 - P(B)) \quad (6.33.1)$$

e) Answer= option(b)