Constructions using Python

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Abstract—**This book introduces constructions in high school geometry using Python. The content and exercises are based on NCERT textbooks from Class 6-12. A simple introduction to Python and LaTeXfigures is provided in the process.**

Download all python codes from

```
svn co https://github.com/gadepall/school/trunk/
    ncert/constructions/codes
```

and latex-tikz codes from

```
svn co https://github.com/gadepall/school/trunk/
    ncert/constructions/figs
```

1 Examples

1.1. Draw Fig. 1.1.1 for $a = 4, c = 3$.

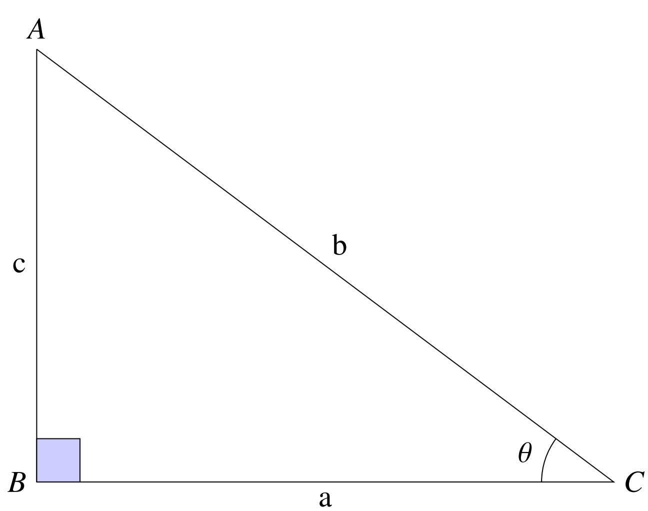

Fig. 1.1.1: Right Angled Triangle

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Solution: The vertices of $\triangle ABC$ are

$$\mathbf{A} = \begin{pmatrix} 0 \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} a \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \tag{1.1.1}$$

The python code for Fig. 1.1.1 is

```
codes/triangle/tri_right_angle.py
```

and the equivalent latex-tikz code is

```
figs/triangle/tri_right_angle.tex
```

The above latex code can be compiled as a standalone document as

```
figs/triangle/tri_right_angle_alone.tex
```

1.2. Draw Fig. 1.2.1 for $a = 4, c = 3$.
Solution: The vertex $\mathbf{A}$ can be expressed in *polar coordinate form* as

$$\mathbf{A} = b \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \tag{1.2.1}$$

where

$$b = \sqrt{a^2 + c^2} = 5, \tan\theta = \frac{3}{4} \tag{1.2.2}$$

The python code for Fig. 1.2.1 is

```
codes/triangle/tri_polar.py
```

and the equivalent latex-tikz code is

```
figs/triangle/tri_polar.tex
```

1.3. Draw Fig. 1.3.1 with $a = 6$, $b = 5$ and $c = 4$.
Solution: Let the vertices of $\triangle ABC$ and $\mathbf{D}$ be

$$\mathbf{A} = \begin{pmatrix} p \\ q \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} a \\ 0 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} p \\ 0 \end{pmatrix} \tag{1.3.1}$$

Then

$$AB = \|\mathbf{A} - \mathbf{B}\|^2 = \|\mathbf{A}\|^2 = c^2 \quad \because \mathbf{B} = \mathbf{0} \tag{1.3.2}$$

$$BC = \|\mathbf{C} - \mathbf{B}\|^2 = \|\mathbf{C}\|^2 = a^2 \tag{1.3.3}$$

$$AC = \|\mathbf{A} - \mathbf{C}\|^2 = b^2 \tag{1.3.4}$$

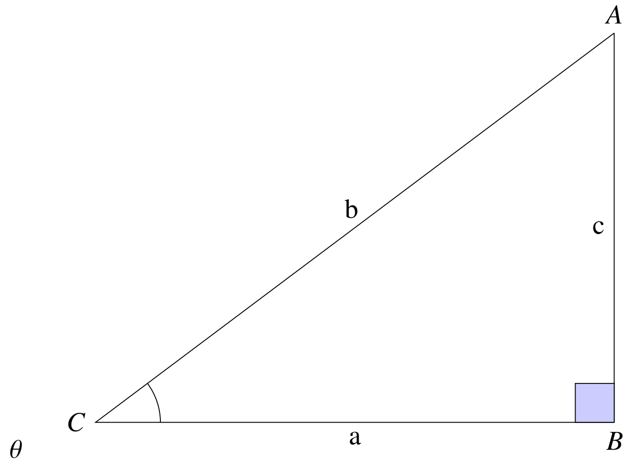

Fig. 1.2.1: Right Angled Triangle

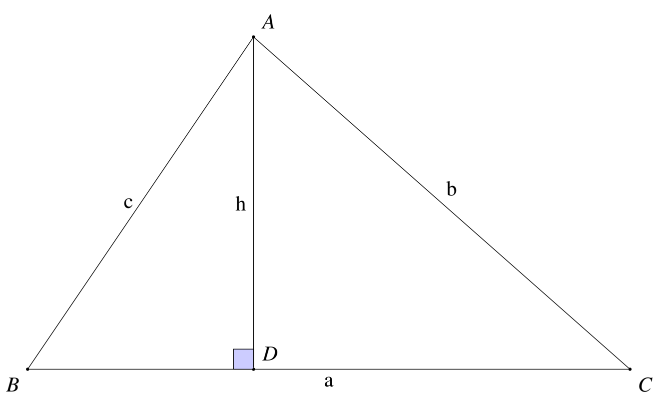

Fig. 1.3.1

From (1.3.4),

$$b^2 = \|\mathbf{A} - \mathbf{C}\|^2 = \|\mathbf{A} - \mathbf{C}\|^T \|\mathbf{A} - \mathbf{C}\| \tag{1.3.5}$$

$$= \mathbf{A}^T\mathbf{A} + \mathbf{C}^T\mathbf{C} - \mathbf{A}^T\mathbf{C} - \mathbf{C}^T\mathbf{A} \tag{1.3.6}$$

$$= \|\mathbf{A}\|^2 + \|\mathbf{C}\|^2 - 2\mathbf{A}^T\mathbf{C} \quad \left(\because \mathbf{A}^T\mathbf{C} = \mathbf{C}^T\mathbf{A}\right) \tag{1.3.7}$$

$$= a^2 + c^2 - 2ap \tag{1.3.8}$$

yielding

$$p = \frac{a^2 + c^2 - b^2}{2a} \tag{1.3.9}$$

From (1.3.2),

$$\|\mathbf{A}\|^2 = c^2 = p^2 + q^2 \tag{1.3.10}$$

$$\implies q = \pm\sqrt{c^2 - p^2} \tag{1.3.11}$$

The python code for Fig. 1.3.1 is

```
codes/triangle/tri_sss.py
```

and the equivalent latex-tikz code is

```
figs/triangle/tri_sss.tex
```

1.4. Construct parallelogram $ABCD$ in Fig. 1.4.1 given that $BC = 5, AB = 6, \angle C = 85°$.

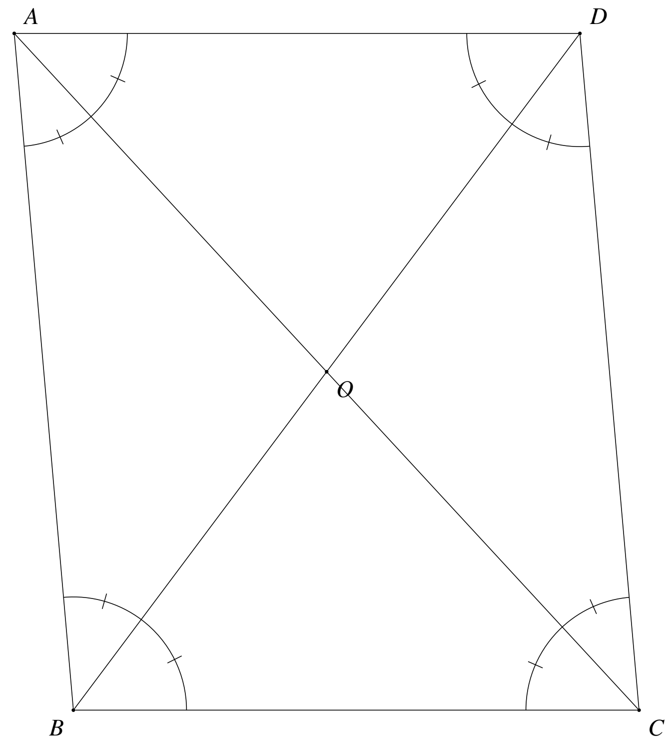

Fig. 1.4.1: Parallelogram Properties

Solution: BD is found using the cosine formula and $\triangle BDC$ is drawn using the approach in Construction 1.3 with

$$\mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}, \tag{1.4.1}$$

Since the diagonals bisect each other,

$$\mathbf{O} = \frac{\mathbf{B} + \mathbf{D}}{2} \tag{1.4.2}$$

$$\mathbf{A} = 2\mathbf{O} - \mathbf{C}. \tag{1.4.3}$$

AB and AD are then joined to complete the ||gm. The python code for Fig. 1.4.1 is

```
codes/quad/pgm_sas.py
```

and The equivalent latex-tikz code is

```
figs/quad/pgm_sas.tex
```

1.5. Draw the ||gm $ABCD$ in Fig. 1.5.1 with $BC = 6, CD = 4.5$ and $BD = 7.5$. Show that it is a rectangle.

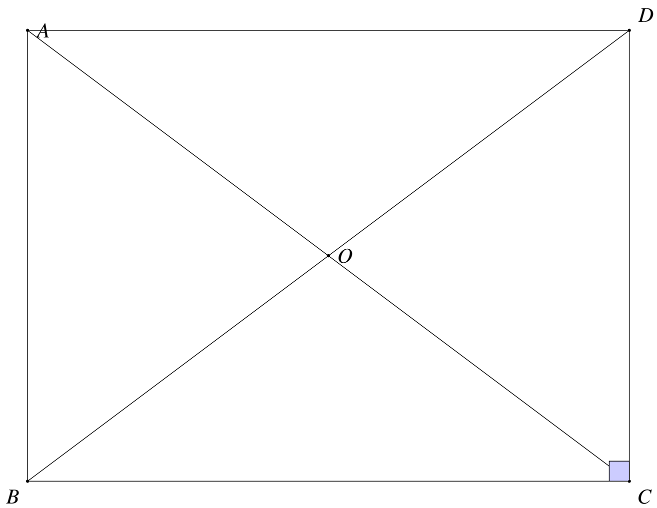

Fig. 1.5.1: Rectangle

Solution: It is easy to verify that

$$BD^2 = BC^2 + C^2 \tag{1.5.1}$$

Hence, using Baudhayana theorem,

$$\angle BCD = 90^\circ \tag{1.5.2}$$

and $ABCD$ is a rectangle.

$$\mathbf{A} = \begin{pmatrix} 0 \\ 4.5 \end{pmatrix} \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \mathbf{C} = \begin{pmatrix} 6 \\ 0 \end{pmatrix} \mathbf{D} = \begin{pmatrix} 6 \\ 4 \end{pmatrix} \tag{1.5.3}$$

The python code for Fig. 1.5.1 is

```
codes/quad/pgm_sss.py
```

and the equivalent latex-tikz code is

```
figs/quad/pgm_sss.tex
```

1.6. Draw the rhombus $BEST$ with $BE = 4.5$ and $ET = 6$.
Solution: The coordinates of the various points in Fig. 1.6.1 are obtained as

$$\mathbf{O} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ -4.5 \end{pmatrix} \tag{1.6.1}$$

$$\mathbf{E} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \mathbf{S} = \begin{pmatrix} 4.5 \\ 0 \end{pmatrix}, \mathbf{T} = \begin{pmatrix} 0 \\ -3 \end{pmatrix} \tag{1.6.2}$$

1.7. A square is a rectangle whose sides are equal. Draw a square of side 4.5.
Solution: The coordinates of the various points

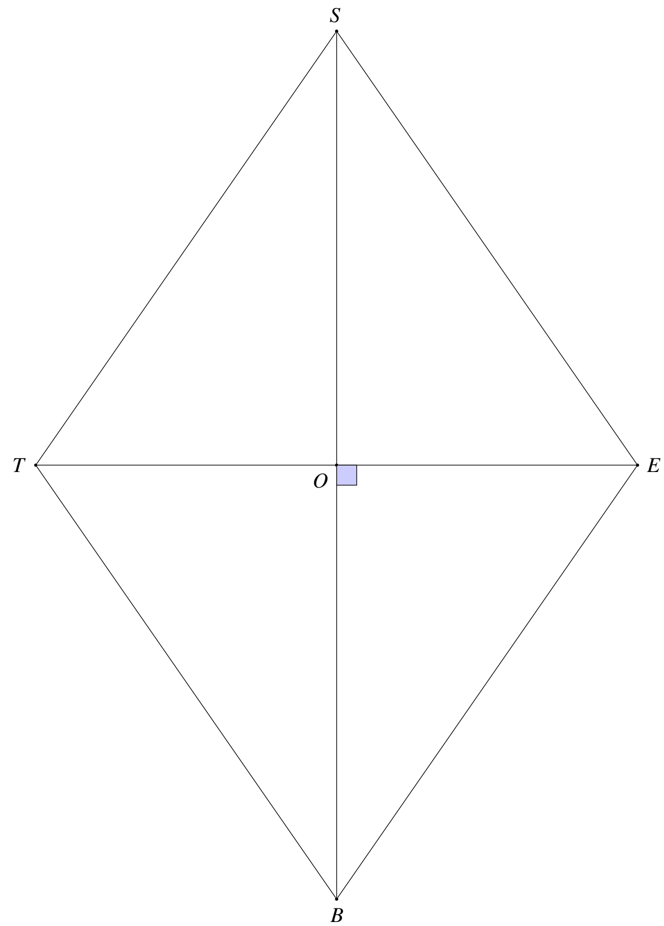

Fig. 1.6.1: Rhombus

in Fig. 1.7.1 are obtained as

$$\mathbf{A} = \begin{pmatrix} 0 \\ 4.5 \end{pmatrix} \tag{1.7.1}$$

$$\mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 4.5 \\ 0 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 4.5 \\ 4.5 \end{pmatrix} \mathbf{O} = \frac{\mathbf{B} + \mathbf{C}}{2} \tag{1.7.2}$$

2 Exercises

2.1. Construct a triangle of sides $a = 4$, $b = 5$ and $c = 6$.
Solution:
The vertex $\mathbf{A}$ can be expressed in *polar coordinate form* as

$$\mathbf{A} = c \begin{pmatrix} \cos B \\ \sin B \end{pmatrix} \tag{2.1.1}$$

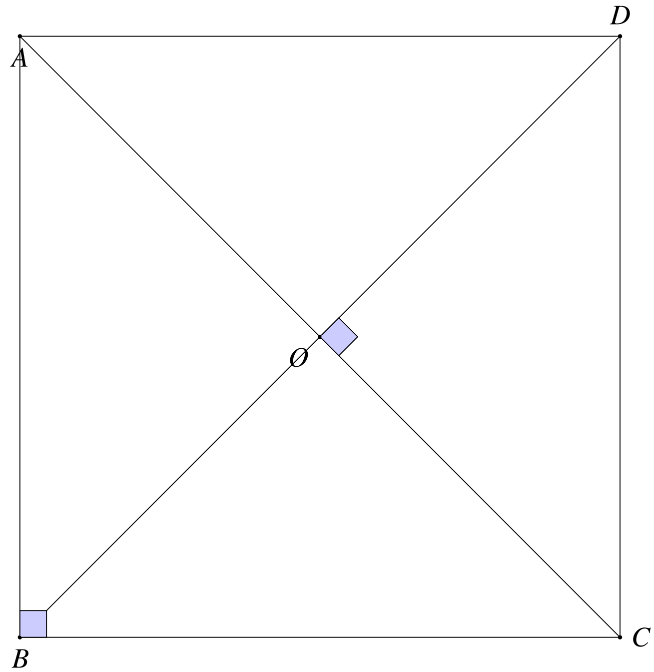

Fig. 1.7.1: Square

From $\triangle ABC$,we use the law of cosines:

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} \quad (2.1.2)$$
$$= 0.5625 \quad (2.1.3)$$
$$\implies B = 55.771^\circ \quad (2.1.4)$$

Thus,

$$\mathbf{A} = 6\begin{pmatrix}\cos 55.771\\ \sin 55.771\end{pmatrix} \quad (2.1.5)$$
$$\mathbf{A} = \begin{pmatrix}3.375\\ 4.960\end{pmatrix}, \mathbf{B} = \begin{pmatrix}0\\ 0\end{pmatrix}, \mathbf{C} = \begin{pmatrix}4\\ 0\end{pmatrix}. \quad (2.1.6)$$

which are plotted in Fig. 2.1.1

2.2. Construct an isosceles triangle whose base is $a = 8$cm and altitude $AD = h = 4$cm
Solution: From the given infromation,

$$\mathbf{A} = \begin{pmatrix}a/2\\ h\end{pmatrix} = \begin{pmatrix}4\\ 4\end{pmatrix}, \mathbf{B} = \begin{pmatrix}0\\ 0\end{pmatrix}, \mathbf{C} = \begin{pmatrix}8\\ 0\end{pmatrix} \quad (2.2.1)$$

which are used to plot the triangle in Fig. 2.2.1

2.3. In $\triangle ABC$, given that $a+b+c = 11$, $\angle B = 45^\circ$ and $\angle C = 45^\circ$, find a, b, c and sketch the triangle.
Solution: Use sine formula,

$$b \sin 45 = c \sin 45 \quad (2.3.1)$$
$$\implies b = c \quad (2.3.2)$$

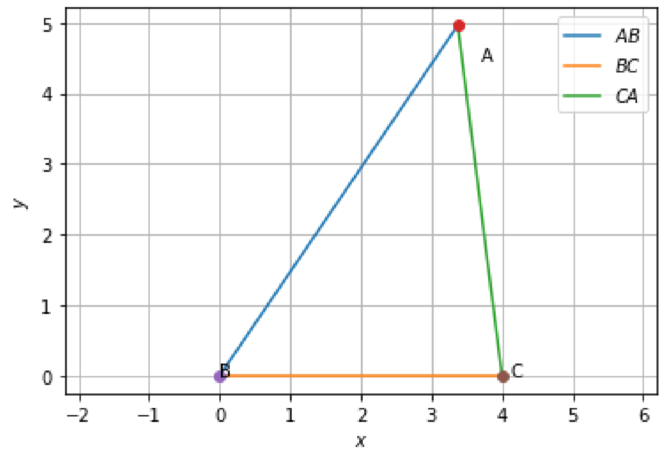

Fig. 2.1.1: $\triangle ABC$

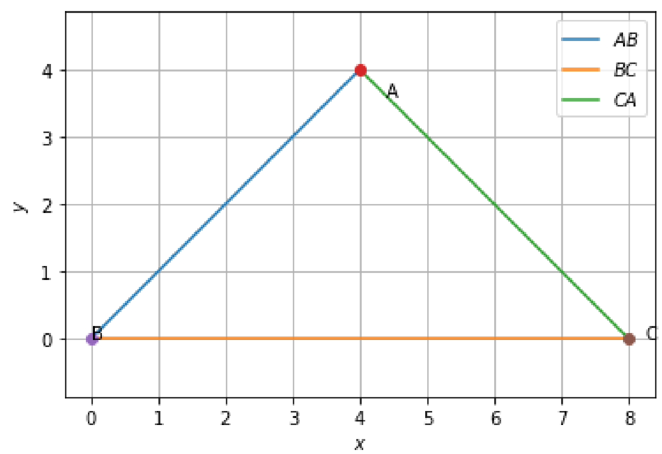

Fig. 2.2.1: isosceles triangle$\triangle ABC$

$$a \sin 45 = b \sin 90 \quad (2.3.3)$$
$$\implies a = \sqrt{2}b \quad (2.3.4)$$

which can be expressed as the matrix equation

$$\begin{pmatrix}0 & 1 & -1\\ 1 & -\sqrt{2} & 0\\ 1 & 1 & 1\end{pmatrix}\begin{pmatrix}a\\ b\\ c\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 11\end{pmatrix} \quad (2.3.5)$$

solving which yields

$$\mathbf{A} = \begin{pmatrix}0\\ c\end{pmatrix} = \begin{pmatrix}0\\ 3.22\end{pmatrix} \quad (2.3.6)$$
$$\mathbf{B} = \begin{pmatrix}0\\ 0\end{pmatrix} \quad (2.3.7)$$
$$\mathbf{C} = \begin{pmatrix}a\\ 0\end{pmatrix} = \begin{pmatrix}4.55\\ 0\end{pmatrix} \quad (2.3.8)$$

resulting in $\triangle ABC$ plotted in Fig. 2.3.1.

2.4. Draw $\triangle ABC$ with $a = 6, c = 5$ and $\angle B = 60^\circ$.

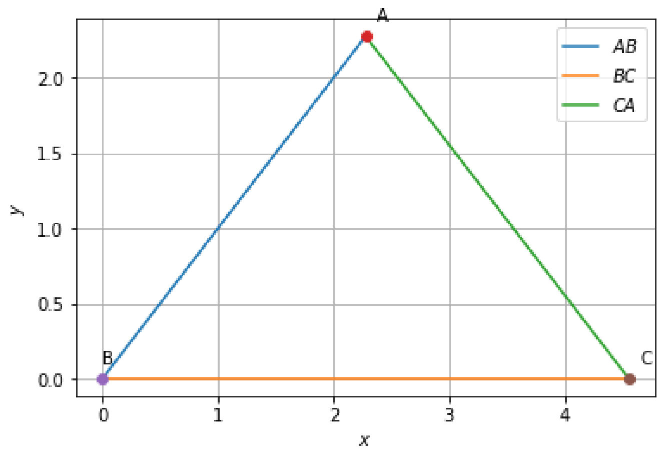

Fig. 2.3.1: $\triangle ABC$

Solution: The vertex **A** can be expressed in *polar coordinate form* as

$$\mathbf{A} = c\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}, \mathbf{B} = \begin{pmatrix}0\\0\end{pmatrix}, \mathbf{C} = \begin{pmatrix}a\\0\end{pmatrix}, \tag{2.4.1}$$

$$\implies \mathbf{A} = 5\begin{pmatrix}\cos 60\\ \sin 60\end{pmatrix} = \begin{pmatrix}2.5\\ 2.5\sqrt{3}\end{pmatrix}, \mathbf{B} = \begin{pmatrix}0\\0\end{pmatrix}, \mathbf{C} = \begin{pmatrix}6\\0\end{pmatrix} \tag{2.4.2}$$

upon substituting the given values. The triangle is plotted in Fig. 2.4.1.

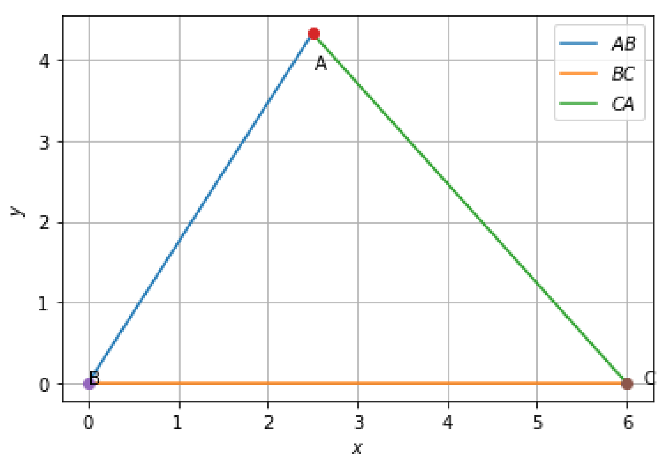

Fig. 2.4.1: $\triangle ABC$

2.5. Draw $\triangle ABC$ with $a = 7, \angle B = 45°$ and $\angle A = 105°$.
Solution: Let

$$\mathbf{A} = c\begin{pmatrix}\cos B\\ \sin B\end{pmatrix}, \mathbf{B} = \begin{pmatrix}0\\0\end{pmatrix}, \mathbf{C} = \begin{pmatrix}a\\0\end{pmatrix} \tag{2.5.1}$$

$$\because \angle C = 30^{\circ}, \tag{2.5.2}$$

By law of sines

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \tag{2.5.3}$$

$$\implies c = \frac{7\sin 30^{\circ}}{\sin 105^{\circ}} \tag{2.5.4}$$

$$c = 3.62 \tag{2.5.5}$$

and

$$\mathbf{A} = c\begin{pmatrix}\cos B\\ \sin B\end{pmatrix} \tag{2.5.6}$$

$$= \begin{pmatrix}2.55\\ 2.55\end{pmatrix} \tag{2.5.7}$$

Thus, the vertices of given $\triangle ABC$ are

$$\mathbf{A} = \begin{pmatrix}2.55\\ 2.55\end{pmatrix}, \mathbf{B} = \begin{pmatrix}0\\0\end{pmatrix}, \mathbf{C} = \begin{pmatrix}7\\0\end{pmatrix} \tag{2.5.8}$$

and $\triangle ABC$ is plotted in Fig. 2.5.1.

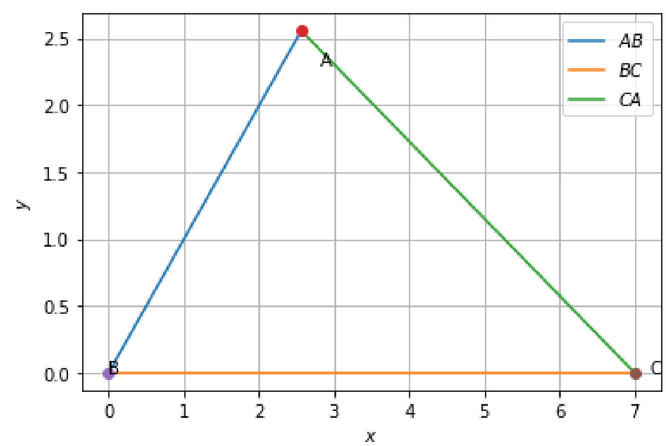

Fig. 2.5.1: $\triangle ABC$

2.6. $\triangle ABC$ is right angled at **B**. If $a = 12$ and $b+c = 18$, find b, c and draw the triangle.
Solution: Let,

$$\mathbf{A} = \begin{pmatrix}0\\c\end{pmatrix}, \mathbf{B} = \begin{pmatrix}0\\0\end{pmatrix}, \mathbf{C} = \begin{pmatrix}a\\0\end{pmatrix} \tag{2.6.1}$$

Given,

$$a = 12, \ b + c = 18 \tag{2.6.2}$$

From $\triangle ABC$, using the Baudhayana sutra,

$$b^2 = c^2 + a^2 \tag{2.6.3}$$
$$\implies b - c = 8 \quad (\because b + c = 18) \tag{2.6.4}$$

Now we have,

$$b + c = 18 \tag{2.6.5}$$
$$b - c = 8 \tag{2.6.6}$$

which can be expressed as the matrix equation

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} 18 \\ 8 \end{pmatrix} \tag{2.6.7}$$

Applying row reduction,

$$\begin{pmatrix} 1 & 1 & 18 \\ 1 & -1 & 8 \end{pmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{pmatrix} 1 & 1 & 18 \\ 0 & -2 & -10 \end{pmatrix} \tag{2.6.8}$$

$$\xrightarrow{R_1 \to 2R_1 + R_2} \begin{pmatrix} 2 & 0 & 26 \\ 0 & -2 & -10 \end{pmatrix} \xrightarrow[R_2 \to -\frac{R_2}{2}]{R_1 \to \frac{R_1}{2}} \begin{pmatrix} 1 & 0 & 13 \\ 0 & 1 & 5 \end{pmatrix} \tag{2.6.9}$$

Therefore,

$$\begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} 13 \\ 5 \end{pmatrix} \tag{2.6.10}$$

Thus,

$$\mathbf{A} = \begin{pmatrix} 0 \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} a \\ 0 \end{pmatrix} = \begin{pmatrix} 12 \\ 0 \end{pmatrix} \tag{2.6.11}$$

and $\triangle ABC$ is plotted in Fig. 2.6.1

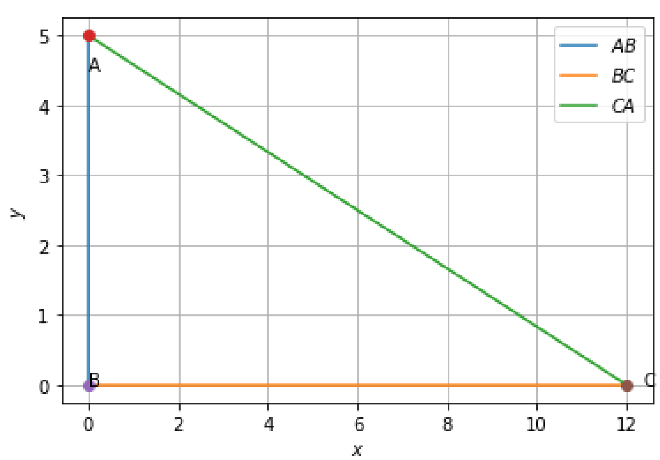

Fig. 2.6.1: Right Angle $\triangle ABC$

2.7. In $\triangle ABC$, $a = 8, \angle B = 45^{\circ}$ and $c - b = 3.5$. Sketch $\triangle ABC$.

Solution: Let

$$\mathbf{A} = c\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} a \\ 0 \end{pmatrix}, \tag{2.7.1}$$

Using the cosine formula in $\triangle ABC$,

$$b^2 = a^2 + c^2 - 2ac\cos B \tag{2.7.2}$$

$$\implies (c + b)(c - b) + 8^2 - 2 \times 8 \times \left(\frac{1}{\sqrt{2}}\right)c = 0 \tag{2.7.3}$$

$$\implies (7 - 16\sqrt{2})c + 7b = -128 \tag{2.7.4}$$

upon simplification. From the given information,

$$c - b = \frac{7}{2}, \tag{2.7.5}$$

and teh above equations can be expressed as the matrix equation

$$\begin{pmatrix} 7 - 16\sqrt{2} & 7 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} c \\ b \end{pmatrix} = \begin{pmatrix} -128 \\ \frac{7}{2} \end{pmatrix} \tag{2.7.6}$$

yielding

$$\begin{pmatrix} c \\ b \end{pmatrix} = \begin{pmatrix} 11.99 \\ 8.49 \end{pmatrix} \tag{2.7.7}$$

Thus, the vertices of $\triangle ABC$ are

$$\mathbf{A} = 11.99\begin{pmatrix} \cos 45 \\ \sin 45 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 8 \\ 0 \end{pmatrix}. \tag{2.7.8}$$

which are used to plot Fig. 2.7.1.

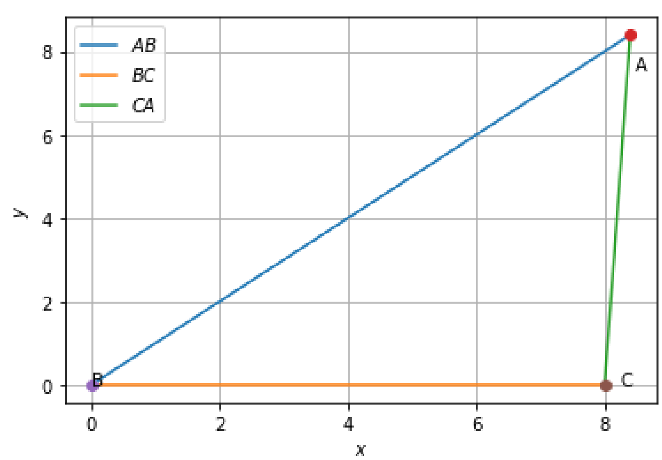

Fig. 2.7.1: $\triangle ABC$

2.8. In $\triangle ABC$, $a = 6$, $\angle B = 60^\circ$ and $b - c = 2$. Sketch $\triangle ABC$.
Let

$$\mathbf{A} = b\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}, \mathbf{B} = \begin{pmatrix}0\\0\end{pmatrix}, \mathbf{C} = \begin{pmatrix}a\\0\end{pmatrix}, \tag{2.8.1}$$

Using the cosine formula,

$$b^2 = a^2 + c^2 - 2ac\cos B \tag{2.8.2}$$

$$\implies (b+c)(b-c) = 6^2 - 2(6)\frac{1}{2}c \quad (\because \angle B = 60^\circ) \tag{2.8.3}$$

$$\implies (b+c)(2) = 36 - 6c \quad (\because b - c = 2) \tag{2.8.4}$$

$$\text{or, } b + 4c = 18 \tag{2.8.5}$$

From the above, we obtain the matrix equation

$$\begin{pmatrix}1 & 4\\ 1 & -1\end{pmatrix}\begin{pmatrix}b\\c\end{pmatrix} = \begin{pmatrix}18\\2\end{pmatrix} \tag{2.8.6}$$

By applying row reduction:

$$\begin{pmatrix}1 & 4 & 18\\ 1 & -1 & 2\end{pmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{pmatrix}1 & 4 & 18\\ 0 & -5 & -16\end{pmatrix} \tag{2.8.7}$$

$$\xrightarrow{R_1 \to 5R_1 + 4R_2} \begin{pmatrix}5 & 0 & 26\\ 0 & -5 & -16\end{pmatrix} \tag{2.8.8}$$

$$\xrightarrow[R_2 \to -\frac{R_2}{5}]{R_1 \to \frac{R_1}{5}} \begin{pmatrix}1 & 0 & \frac{26}{5}\\ 0 & 1 & \frac{16}{5}\end{pmatrix} \tag{2.8.9}$$

$$\therefore \begin{pmatrix}b\\c\end{pmatrix} = \begin{pmatrix}\frac{26}{5}\\ \frac{16}{5}\end{pmatrix} \tag{2.8.10}$$

Thus, the vertices of $\triangle ABC$ are

$$\mathbf{A} = \frac{26}{5}\begin{pmatrix}\cos 60\\ \sin 60\end{pmatrix} = \begin{pmatrix}2.6\\4.5\end{pmatrix}, \mathbf{B} = \begin{pmatrix}0\\0\end{pmatrix}, \mathbf{C} = \begin{pmatrix}6\\0\end{pmatrix} \tag{2.8.11}$$

and the plot of $\triangle ABC$ is obtained in Fig. 2.8.1

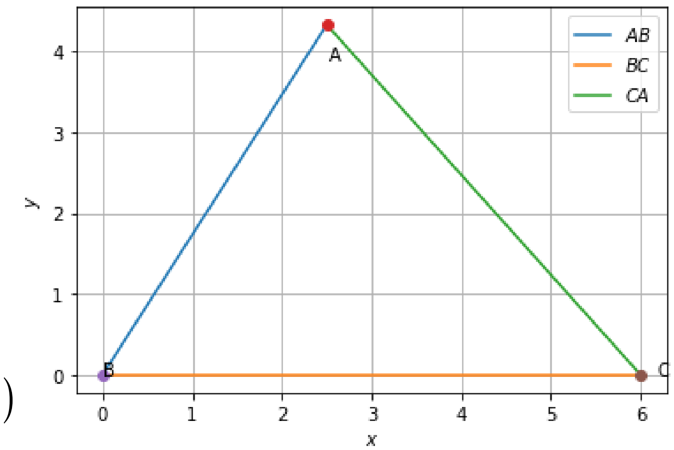

Fig. 2.8.1: $\triangle ABC$

2.9. Draw $\triangle ABC$, given that $a+b+c = 11$, $\angle B = 30^\circ$ and $\angle C = 90^\circ$.
Solution: Using the sine formula,

$$b\sin C = c\sin B \tag{2.9.1}$$

$$\implies b\sin 90 = c\sin 30 \tag{2.9.2}$$

$$\text{or, } c = 2b \tag{2.9.3}$$

Similarly,

$$a\sin B = b\sin A \tag{2.9.4}$$

$$\implies a = \sqrt{3}b \tag{2.9.5}$$

Formulating the above as a matrix equation

$$\begin{pmatrix}0 & -2 & 1\\ 1 & -\sqrt{3} & 0\\ 1 & 1 & 1\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}0\\0\\11\end{pmatrix} \tag{2.9.6}$$

Solving the above,

$$a = 4.026, b = 2.32, c = 4.64 \tag{2.9.7}$$

which are used to obtain the vertices of $\triangle ABC$ using Problem 1.3.

$$\mathbf{A} = \begin{pmatrix}0\\c\end{pmatrix} = \begin{pmatrix}0\\4.64\end{pmatrix} \tag{2.9.8}$$

$$\mathbf{B} = \begin{pmatrix}0\\0\end{pmatrix} \tag{2.9.9}$$

$$\mathbf{C} = \begin{pmatrix}a\\0\end{pmatrix} = \begin{pmatrix}4.02\\0\end{pmatrix} \tag{2.9.10}$$

The desired triangle is plotted in Fig. 2.9.1.

2.10. Construct $\triangle xyz$ where $xy = 4.5, yz = 5$ and $zx = 6$.
Solution: Let

$$\mathbf{A} = \begin{pmatrix}0\\0\end{pmatrix}, \mathbf{B} = \begin{pmatrix}c\\0\end{pmatrix}, \mathbf{C} = \begin{pmatrix}p\\q\end{pmatrix} \tag{2.10.1}$$

The vertex C can be expressed in polar coordinate form as

$$\mathbf{C} = b\begin{pmatrix}\cos A\\ \sin A\end{pmatrix} \tag{2.10.2}$$

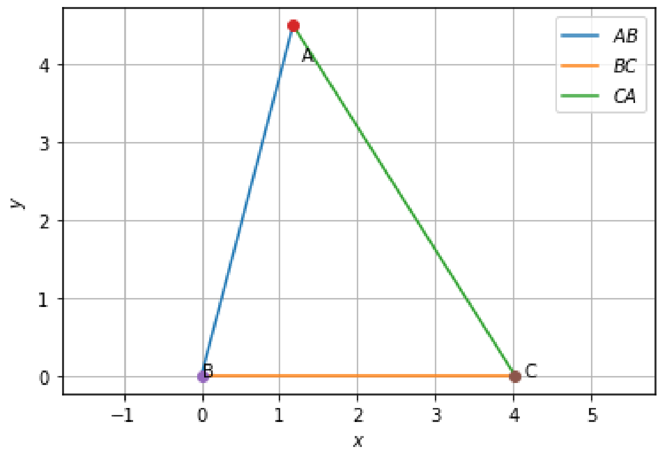

Fig. 2.9.1: $\triangle ABC$

Using the cosine formula,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \tag{2.10.3}$$
$$\implies A = 54.640° \tag{2.10.4}$$

Hence,

$$\mathbf{C} = 6\begin{pmatrix} \cos 54.640 \\ \sin 54.640 \end{pmatrix} = \mathbf{C} = \begin{pmatrix} 3.472 \\ 3.990 \end{pmatrix}, \tag{2.10.5}$$
$$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} c \\ 0 \end{pmatrix} = \begin{pmatrix} 4.5 \\ 0 \end{pmatrix} \tag{2.10.6}$$

which are plotted in Fig. 2.10.1

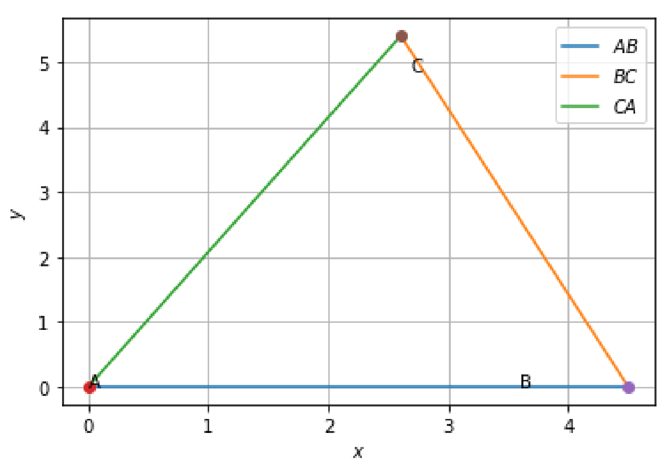

Fig. 2.10.1: $\triangle ABC$

2.11. Draw an equilateral triangle of side 5.5.
Solution: Let,

$$\mathbf{A} = a\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} a \\ 0 \end{pmatrix} \tag{2.11.1}$$
$$= 5.5\begin{pmatrix} \cos 60° \\ \sin 60° \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 5.5 \\ 0 \end{pmatrix} \tag{2.11.2}$$

after substituting $\theta = 60°$ and $a = 5.5$. The triangle is then plotted in Fig. 2.11.1

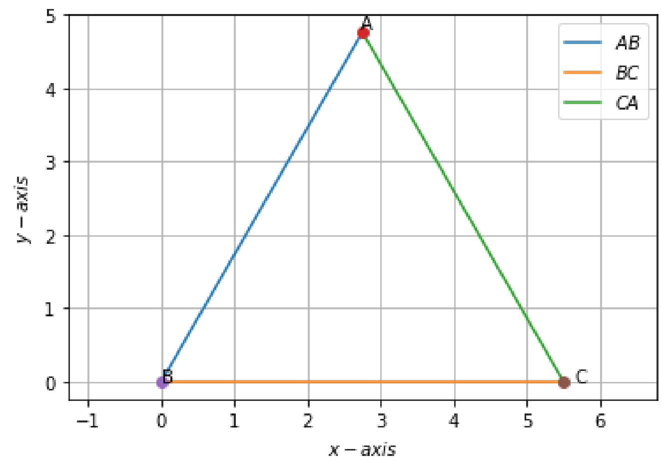

Fig. 2.11.1: $\triangle ABC$

2.12. Draw $\triangle PQR$ with $PQ = 4, QR = 3.5$ and $PR = 4$. What type of triangle is this?
Solution: Let

$$\mathbf{P} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \mathbf{R} = PR\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \tag{2.12.1}$$

where,

$$PR\left(\frac{\sin\theta}{2}\right) = \frac{QR}{2} \tag{2.12.2}$$
$$\implies \theta = 2\sin^{-1}\left(\frac{QR}{2PR}\right) \tag{2.12.3}$$
$$= 51.88 \tag{2.12.4}$$

Thus, the vertices of $\triangle PQR$ are

$$\mathbf{P} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \mathbf{R} = \begin{pmatrix} 2.47 \\ 3.15 \end{pmatrix} \tag{2.12.5}$$

which are used to plot $\triangle PQR$ in Fig. 2.12.1.

2.13. Construct $\triangle ABC$ such that $AB = 2.5, BC = 6$ and $AC = 6.5$. Find $\angle B$.

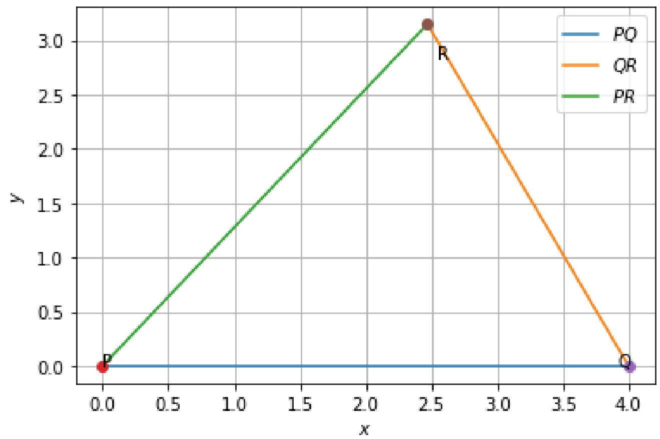

Fig. 2.12.1: isosceles $\triangle PQR$

Solution: From the given information,

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} \tag{2.13.1}$$

$$\implies \cos B = 0 \tag{2.13.2}$$

$$\text{or, } \angle B = 90^\circ \tag{2.13.3}$$

Thus, the vertices of $\triangle ABC$ are

$$\mathbf{A} = \begin{pmatrix} 0 \\ 2.5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 6 \\ 0 \end{pmatrix} \tag{2.13.4}$$

and plotted in Fig. 2.13.1.

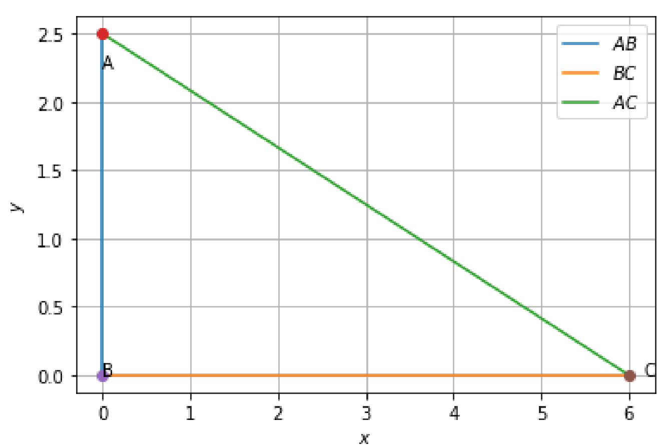

Fig. 2.13.1: $\triangle ABC$

2.14. Construct $\triangle PQR$, given that $PQ = 3, QR = 5.5$ and $\angle PQR = 60^\circ$.

2.15. Construct $\triangle DEF$ such that $DE = 5, DF = 3$ and $\angle D = 90^\circ$.
Solution: From the given information, the vertices of $\triangle DEF$ are

$$\mathbf{E} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{F} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \tag{2.15.1}$$

which are used to plot Fig. 2.15.1.

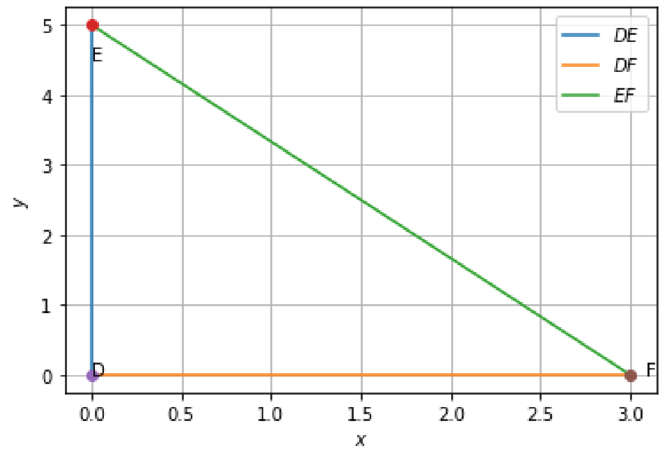

Fig. 2.15.1

2.16. Construct an isosceles triangle in which the lengths of the equal sides is 6.5 and the angle between them is 110°.
Solution: Let the vertices be

$$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} b \\ 0 \end{pmatrix}, \mathbf{B} = c \begin{pmatrix} \cos A \\ \sin A \end{pmatrix} \tag{2.16.1}$$

Then, the vertices of isosceles $\triangle ABC$ are

$$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 6.5 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -2.22313 \\ 6.10798 \end{pmatrix} \tag{2.16.2}$$

which are plotted in Fig. 2.16.1.

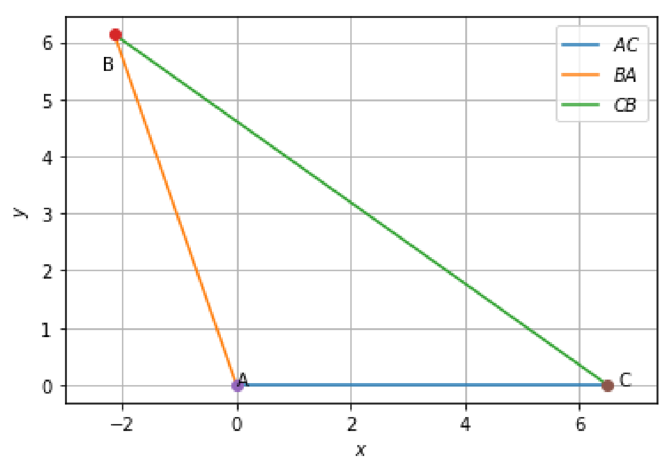

Fig. 2.16.1: Isosceles $\triangle ABC$

2.17. Construct $\triangle ABC$ with $BC = 7.5, AC = 5$ and $\angle C = 60^\circ$.

2.18. Construct $\triangle XYZ$ if $XY = 6, \angle X = 30°$ and $\angle Y = 100°$.

2.19. If $AC = 7, \angle A = 60°$ and $\angle B = 50°$, can you draw the triangle?

2.20. Construct $\triangle ABC$ given that $\angle A = 60°, \angle B = 30°$ and $AB = 5.8$.
Solution: From the given information,

$$\angle C = 90^{\circ} \tag{2.20.1}$$

Hence,

$$\mathbf{A} = \begin{pmatrix} 0 \\ c \sin B \end{pmatrix} \tag{2.20.2}$$
$$= \begin{pmatrix} 0 \\ 2.9 \end{pmatrix} \tag{2.20.3}$$
$$\mathbf{B} = \begin{pmatrix} c \cos B \\ 0 \end{pmatrix} \tag{2.20.4}$$
$$= \begin{pmatrix} 5.02294 \\ 0 \end{pmatrix} \tag{2.20.5}$$
$$\mathbf{C} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.20.6}$$

which are used to draw $\triangle ABC$ in Fig. 2.20.1.

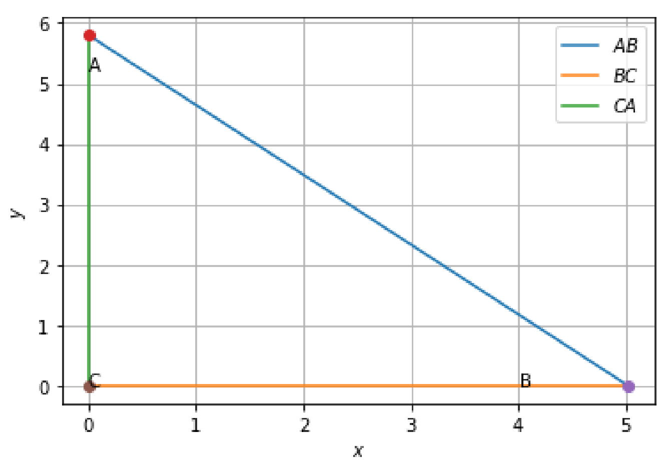

Fig. 2.20.1: $\triangle ABC$

2.21. Construct $\triangle PQR$ if $PQ = 5, \angle Q = 105°$ and $\angle R = 40°$.

2.22. Can you construct $\triangle DEF$ such that $EF = 7.2, \angle E = 110°$ and $\angle F = 180°$?

2.23. Construct $\triangle LMN$ right angled at M such that $LN = 5$ and $MN = 3$.
Solution:
Let

$$\mathbf{L} = \begin{pmatrix} 0 \\ l \end{pmatrix}, \mathbf{M} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{N} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \tag{2.23.1}$$

From the given information,

$$\|\mathbf{N} - \mathbf{M}\|^2 = \|\mathbf{N}\|^2 = 3^2 = 9 \tag{2.23.2}$$
$$\|\mathbf{L} - \mathbf{M}\|^2 = \|\mathbf{L}\|^2 = l^2 \tag{2.23.3}$$
$$\|\mathbf{L} - \mathbf{N}\|^2 = 5^2 = 25 \tag{2.23.4}$$

which can be expressed as

$$\|\mathbf{L} - \mathbf{N}\|^2 = (\mathbf{L} - \mathbf{N})^T (\mathbf{L} - \mathbf{N}) \tag{2.23.5}$$
$$= \|\mathbf{L}\|^2 + \|\mathbf{N}\|^2 - 2\mathbf{L}^T\mathbf{N} \tag{2.23.6}$$
$$\implies l^2 + 9 = 25 \tag{2.23.7}$$
$$\text{or, } l = \pm 4 \tag{2.23.8}$$

For l=4, $\triangle LMN$ is plotted in the first quadrant in Fig. 2.23.1.

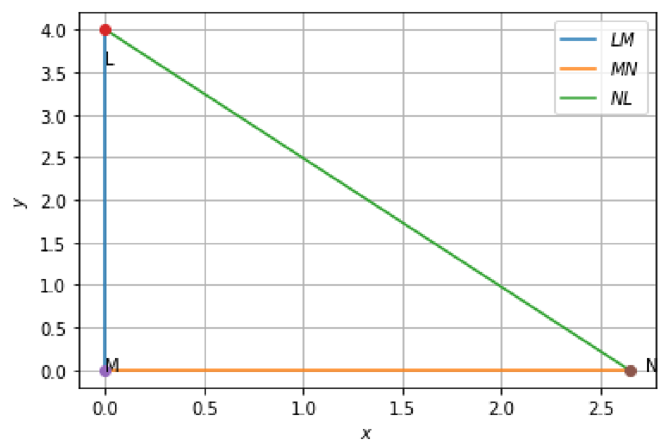

Fig. 2.23.1

2.24. Construct $\triangle PQR$ right angled at Q such that $QR = 8$ and $PR = 10$.
Solution: Let

$$\mathbf{Q} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{R} = \begin{pmatrix} 8 \\ 0 \end{pmatrix}, \mathbf{P} = \begin{pmatrix} 0 \\ p \end{pmatrix} \tag{2.24.1}$$

Then,

$$\|\mathbf{P} - \mathbf{R}\|^2 = (\mathbf{P} - \mathbf{R})^T (\mathbf{P} - \mathbf{R}) \tag{2.24.2}$$
$$= \|\mathbf{P}\|^2 + \|\mathbf{R}\|^2 \tag{2.24.3}$$
$$\because \mathbf{P}^T\mathbf{R} = \mathbf{R}^T\mathbf{P}, \mathbf{R}^T\mathbf{P} = 0 \tag{2.24.4}$$
$$= p^2 + 64 = 10^2 \tag{2.24.5}$$
$$\implies p = \pm 6 \tag{2.24.6}$$

Since positive area is considered here,only $p = 6$ is taken into consideration. Thus,

$$\mathbf{P} = \begin{pmatrix} 0 \\ 6 \end{pmatrix} \tag{2.24.7}$$

and the desired traingle is plotted in Fig. 2.24.1

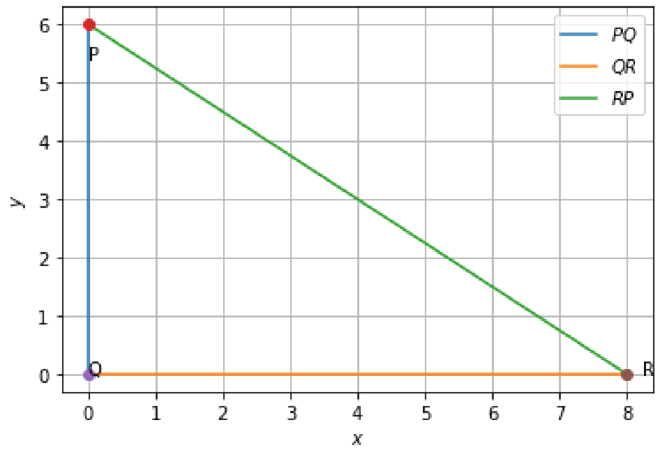

Fig. 2.24.1: Right Angle $\triangle PQR$

2.25. Construct right angled $\triangle$ whose hypotenuse is 6 and one of the legs is 4.
Solution: Let us consider $\triangle PQR$ right angled at Q and assume that we are restricted to first quadrant such that

$$\mathbf{Q} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{R} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \mathbf{P} = \begin{pmatrix} 0 \\ p \end{pmatrix} \tag{2.25.1}$$

Then,

$$\|\mathbf{P} - \mathbf{R}\|^2 = 36 \tag{2.25.2}$$

$$\implies p^2 + 16 = 36 \tag{2.25.3}$$

$$\implies p = \pm 2\sqrt{5} \tag{2.25.4}$$

Since first quadrant was assumed here,only $p = +2\sqrt{5}$ is taken into consideration. So,the vertices of $\triangle PQR$ in Fig. 2.25.1 are

$$\mathbf{P} = \begin{pmatrix} 0 \\ 2\sqrt{5} \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{R} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \tag{2.25.5}$$

2.26. Construct an isosceles right angled $\triangle ABC$ right angled at C such $AC = 6$.
Solution:
$\because \triangle ABC$ is isosceles, its vertices are

$$\mathbf{C} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 6 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 6 \end{pmatrix} \tag{2.26.1}$$

which are used to plot the desired triangle in Fig. 2.26.1.

2.27. Construct the triangles in Table 2.27.1. **Solution:**

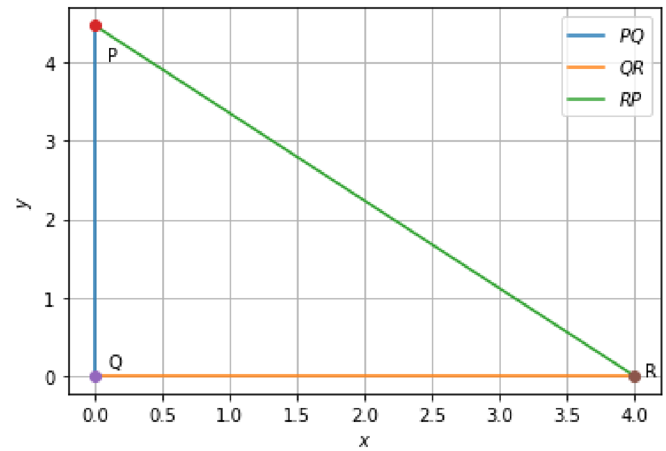

Fig. 2.25.1: Right Angled $\triangle PQR$

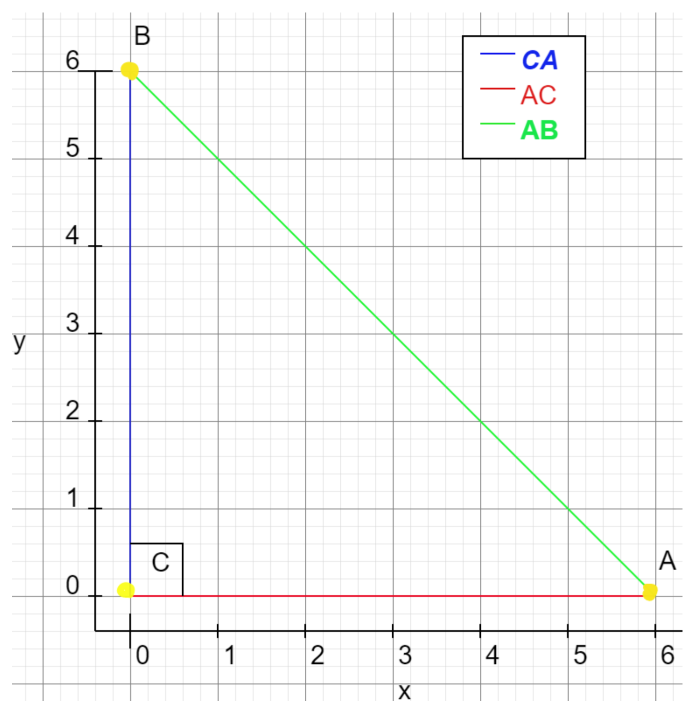

Fig. 2.26.1: Isosceles Right Angle $\triangle ABC$

a)
b) **Solution:** From the given information, $\triangle PQR$ is a right angled triangle. Let $QR = p$ and θ=30°. Then the vertices of the triangle

S.No	Triangle	Given Measurements		
1	△ABC	$\angle A = 85°$	$\angle B = 115°$	AB = 5
2	△PQR	$\angle Q = 30°$	$\angle R = 60°$	QR = 4.7
3	△ABC	$\angle A = 70°$	$\angle B = 50°$	AC = 3
4	△LMN	$\angle L = 60°$	$\angle N = 120°$	LM = 5
5	△ABC	BC = 2	AB = 4	AC = 2
6	△PQR	PQ = 2.5	QR = 4	PR = 3.5
7	△XYZ	XY = 3	YZ = 4	XZ = 5
8	△DEF	DE = 4.5	EF = 5.5	DF = 4

TABLE 2.27.1

are

$$\mathbf{P} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.27.1}$$

$$\mathbf{Q} = \begin{pmatrix} 0 \\ p\cos\theta \end{pmatrix} \tag{2.27.2}$$

$$= \begin{pmatrix} 0 \\ 4.07 \end{pmatrix} \tag{2.27.3}$$

$$\mathbf{R} = \begin{pmatrix} p\sin\theta \\ 0 \end{pmatrix} \tag{2.27.4}$$

$$= \begin{pmatrix} 2.35 \\ 0 \end{pmatrix} \tag{2.27.5}$$

The triangle is plotted in Fig. 2.27.1

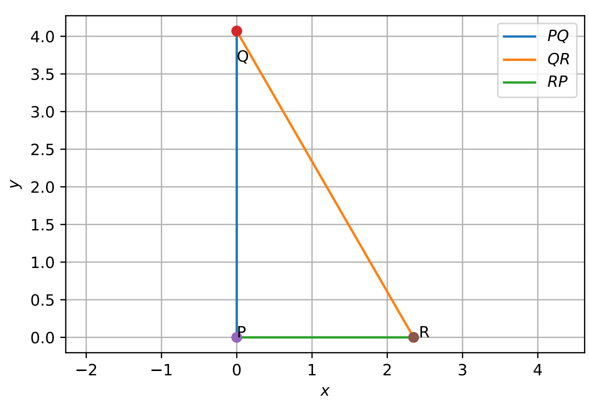

Fig. 2.27.1: $\triangle PQR$ constructed using python

c) **Solution:** From the given information,

$$\angle C = 60° \tag{2.27.6}$$

Using the sine formula,

$$c = b\left(\frac{\sin C}{\sin B}\right) \tag{2.27.7}$$

$$= 3.3915 \tag{2.27.8}$$

the vertices of $\triangle ABC$ are

$$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{B} = c\begin{pmatrix} \cos 70° \\ \sin 70° \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \tag{2.27.9}$$

and plotted in Fig. 2.27.2.

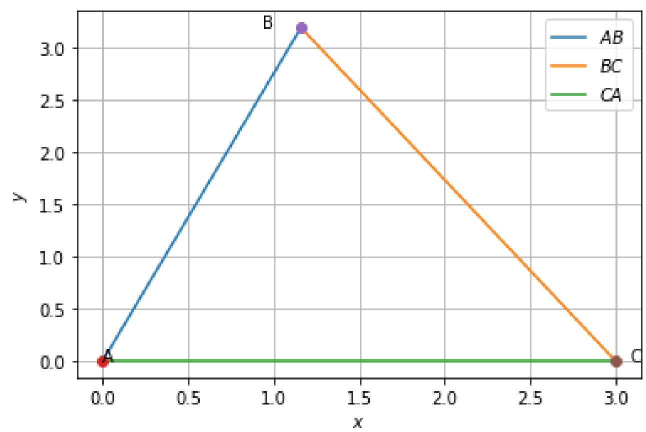

Fig. 2.27.2: Plot of $\triangle ABC$

2.28. Construct a quadrilateral $ABCD$ such that $AB = 5, \angle A = 50°, AC = 4, BD = 5$ and $AD = 6$.
Solution:
The rough figure of the expected quadrilateral ABCD is given in Fig. 2.28.1
From the given information, in $\triangle ABD$,

$$\cos A = \frac{\|\mathbf{B}-\mathbf{A}\|^2 + \|\mathbf{D}-\mathbf{A}\|^2 - \|\mathbf{D}-\mathbf{B}\|^2}{2\|\mathbf{B}-\mathbf{A}\|\|\mathbf{D}-\mathbf{A}\|} \tag{2.28.1}$$

$$\implies \angle A = \cos^{-1}(0.6) \approx 53.13° \tag{2.28.2}$$

$$\neq 50° \tag{2.28.3}$$

resulting in a contradiction. Therefore construction of quadrilateral with given measurements is not possible.

2.29. Construct $PQRS$ where $PQ = 4, QR = 6, RS = 5, PS = 5.5$ and $PR = 7$.

2.30. Draw $JUMP$ with $JU = 3.5, UM = 4, MP = 5, PJ = 4.5$ and $PU = 6.5$

2.31. Construct a quadrilateral $ABCD$ such that $BC = 4.5, AC = 5.5, CD = 5, BD = 7$ and $AD = 5.5$.

2.32. Can you construct a quadrilateral $PQRS$ with

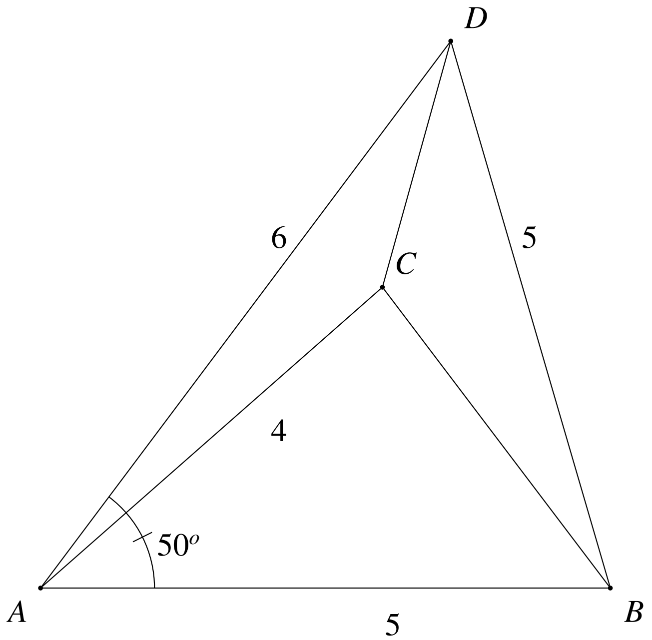

Fig. 2.28.1: Rough Figure

$PQ = 3, RS = 3, PS = 7.5, PR = 8$ and $SQ = 4$?

Solution: From the given information,

$$\|\mathbf{P}-\mathbf{Q}\| = 3 \tag{2.32.1}$$

$$\|\mathbf{R}-\mathbf{S}\| = 3 \tag{2.32.2}$$

$$\|\mathbf{P}-\mathbf{S}\| = 7.5 \tag{2.32.3}$$

$$\|\mathbf{P}-\mathbf{R}\| = 8 \tag{2.32.4}$$

$$\|\mathbf{S}-\mathbf{Q}\| = 4 \tag{2.32.5}$$

Let quadrilateral $PQRS$ be made up of two triangles $\triangle PSQ$ and $\triangle PSR$ on base PS.

a) In $\triangle PSR$,

$$\|\mathbf{P}-\mathbf{S}\| + \|\mathbf{R}-\mathbf{S}\| = 7.5 + 3 = 10.5 > \|\mathbf{P}-\mathbf{R}\| \tag{2.32.6}$$

$$\|\mathbf{P}-\mathbf{R}\| + \|\mathbf{R}-\mathbf{S}\| = 8 + 3 = 11 > \|\mathbf{P}-\mathbf{S}\| \tag{2.32.7}$$

$$\|\mathbf{P}-\mathbf{S}\| + \|\mathbf{P}-\mathbf{R}\| = 7.5 + 8 = 15.5 > \|\mathbf{R}-\mathbf{S}\| \tag{2.32.8}$$

$\therefore$ using triangle inequality, construction of $\triangle PSR$ is possible.

b) In $\triangle PSQ$,

$$\|\mathbf{P}-\mathbf{S}\| + \|\mathbf{S}-\mathbf{Q}\| = 7.5 + 4 = 11.5 > \|\mathbf{P}-\mathbf{Q}\| \tag{2.32.9}$$

$$\|\mathbf{P}-\mathbf{S}\| + \|\mathbf{P}-\mathbf{Q}\| = 7.5 + 3 = 10.5 > \|\mathbf{S}-\mathbf{Q}\| \tag{2.32.10}$$

$$\|\mathbf{P}-\mathbf{Q}\| + \|\mathbf{S}-\mathbf{Q}\| = 3 + 4 = 7 < \|\mathbf{P}-\mathbf{S}\| \tag{2.32.11}$$

which violates triangle inequality. $\therefore$ construction of $\triangle PSQ$ is not possible.

Fig. 2.32.1 highlights this.

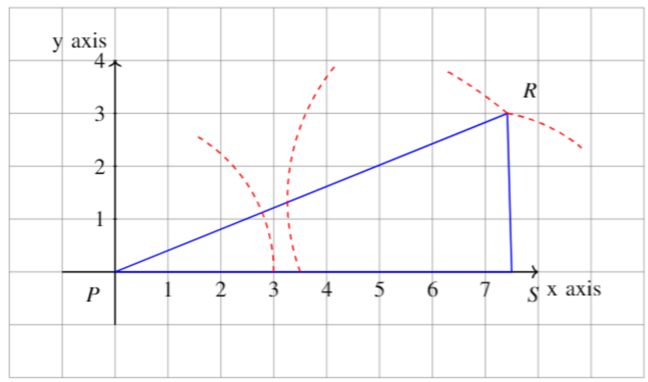

Fig. 2.32.1: Construction of quadrilateral $PQRS$

2.33. Construct $LIFT$ such that $LI = 4, IF = 3, TL = 2.5, LF = 4.5, IT = 4$.

2.34. Draw $GOLD$ such that $OL = 7.5, GL = 6, GD = 6, LD = 5, OD = 10$.

Solution: In $\triangle LDO$

$$\|\mathbf{O}-\mathbf{L}\| + \|\mathbf{O}-\mathbf{D}\| = 17.5 > \|\mathbf{L}-\mathbf{D}\| \tag{2.34.1}$$

$$\|\mathbf{O}-\mathbf{D}\| + \|\mathbf{L}-\mathbf{D}\| = 15 > \|\mathbf{O}-\mathbf{L}\| \tag{2.34.2}$$

$$\|\mathbf{O}-\mathbf{L}\| + \|\mathbf{L}-\mathbf{D}\| = 12.5 > \|\mathbf{O}-\mathbf{D}\| \tag{2.34.3}$$

and triangle inequality is satisfied. Similarly, in $\triangle LDG$

$$\|\mathbf{L}-\mathbf{D}\| + \|\mathbf{G}-\mathbf{L}\| = 11 > \|\mathbf{G}-\mathbf{D}\| \tag{2.34.4}$$

$$\|\mathbf{G}-\mathbf{L}\| + \|\mathbf{G}-\mathbf{D}\| = 12 > \|\mathbf{L}-\mathbf{D}\| \tag{2.34.5}$$

$$\|\mathbf{L}-\mathbf{D}\| + \|\mathbf{G}-\mathbf{D}\| = 11 > \|\mathbf{G}-\mathbf{L}\| \tag{2.34.6}$$

and triangle inequality is satisfied. $\therefore$ the given sides form a quadrilateral which can be constructed by using the approach in Problem 1.3

to obtain the vertices of $\triangle LDO$ and $\triangle LDG$ as

$$\mathbf{L} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}, \mathbf{O} = \begin{pmatrix} -1.875 \\ 7.26 \end{pmatrix}, \mathbf{G} = \begin{pmatrix} 2.5 \\ 5.5 \end{pmatrix} \tag{2.34.7}$$

and plotting the quadrilateral GOLD in Fig. 2.34.1

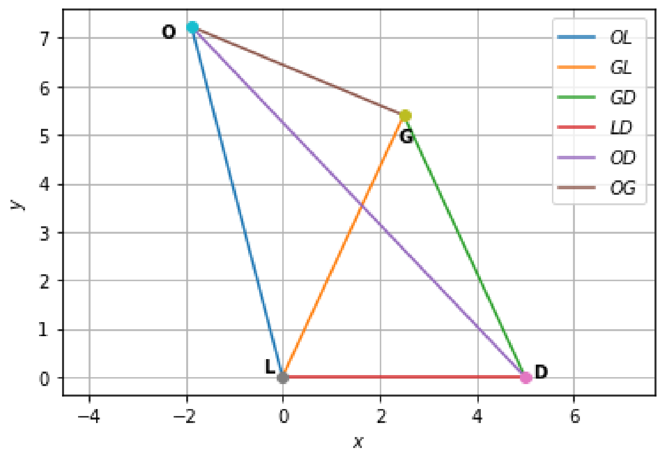

Fig. 2.34.1: Quadrilateral GOLD

2.35. DRAW rhombus $BEND$ such that $BN = 5.6$, $DE = 6.5$.

2.36. construct a quadrilateral MIST where $MI = 3.5, IS = 6.5, \angle M = 75°, \angle I = 105°$ and $\angle S = 120°$.

2.37. Can you construct the above quadrilateral MIST if $\angle M = 100°$ instead of $75°$.

2.38. Can you constrcut the quadrilateral PLAN if $PL = 6, LA = 9.5, \angle P = 75°, \angle L = 150°$ and $\angle A = 140°$?

2.39. Construct $MORE$ where $MO = 6, OR = 4.5, \angle M = 60°, \angle O = 105°, \angle R = 105°$.

2.40. Construct $PLAN$ where $PL = 4, LA = 6.5, \angle P = 90°, \angle A = 110°$ and $\angle N = 85°$.

2.41. Draw rectangle $OKAY$ with $OK = 7$ and $KA = 5$.

2.42. Construct $ABCD$, where $AB = 4, BC = 5, Cd = 6.5, \angle B = 105°$ and $\angle C = 80°$.
Solution:
Let

$$\angle B = 105° = \theta \tag{2.42.1}$$
$$\angle C = 80° = \alpha \tag{2.42.2}$$
$$\|\mathbf{A} - \mathbf{B}\| = 4 = p \tag{2.42.3}$$
$$\|\mathbf{C} - \mathbf{B}\| = 5 = q \tag{2.42.4}$$
$$\|\mathbf{D} - \mathbf{C}\| = 6.5 = r \tag{2.42.5}$$

and

$$\mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \tag{2.42.6}$$

Lemma 2.1.

$$\mathbf{A} = p\mathbf{b} \quad \left(\because \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) \tag{2.42.7}$$
$$\mathbf{D} = \mathbf{C} + r\mathbf{c} \tag{2.42.8}$$

where

$$\mathbf{b} = \begin{pmatrix} \cos B \\ \sin B \end{pmatrix}, \mathbf{c} = \begin{pmatrix} \cos C \\ \sin C \end{pmatrix} \tag{2.42.9}$$

Thus,

$$\mathbf{A} = 4\begin{pmatrix} \cos 105 \\ \sin 105 \end{pmatrix} \tag{2.42.10}$$
$$= \begin{pmatrix} -1.03 \\ 3.86 \end{pmatrix} \tag{2.42.11}$$

and

$$\mathbf{D} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} + 6.5\begin{pmatrix} \cos 80 \\ \sin 80 \end{pmatrix} \tag{2.42.12}$$
$$= \begin{pmatrix} 6.12 \\ 6.39 \end{pmatrix} \tag{2.42.13}$$

which are then used to plot Fig. 2.42.1

2.43. Construct $DEAR$ with $DE = 4, EA = 5, AR = 4.5, \angle E = 60°$ and $\angle A = 90°$.
Solution: The given information can be expressed as

$$\angle E = 60° = \theta \tag{2.43.1}$$
$$\angle A = 90° = \alpha \tag{2.43.2}$$
$$\|\mathbf{D} - \mathbf{E}\| = 4 = a \tag{2.43.3}$$
$$\|\mathbf{E} - \mathbf{A}\| = 5 = b \tag{2.43.4}$$
$$\|\mathbf{A} - \mathbf{R}\| = 4.5 = c \tag{2.43.5}$$

Let,

$$\mathbf{E} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \tag{2.43.6}$$

Lemma 2.2.

$$\mathbf{D} = a\mathbf{e} \quad \left(\because \mathbf{E} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) \tag{2.43.7}$$
$$\mathbf{R} = \mathbf{A} + c\mathbf{a} \tag{2.43.8}$$

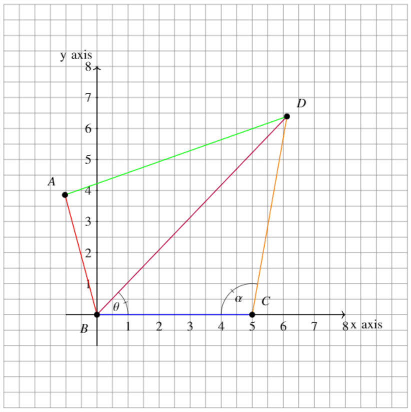

Fig. 2.42.1: Quadrilateral ABCD

where

$$\mathbf{e} = \begin{pmatrix} \cos E \\ \sin E \end{pmatrix}, \mathbf{a} = \begin{pmatrix} \cos A \\ \sin A \end{pmatrix} \tag{2.43.9}$$

Thus, from (2.43.1) and (2.43.3) in (2.43.7),

$$\mathbf{D} = 4\begin{pmatrix} \cos 60^\circ \\ \sin 60^\circ \end{pmatrix} \tag{2.43.10}$$

$$= \begin{pmatrix} 2 \\ 3.46 \end{pmatrix} \tag{2.43.11}$$

and from (2.43.2) and (2.43.5) in (2.43.8),

$$\mathbf{R} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} + 4.5\begin{pmatrix} \cos 90^\circ \\ \sin 90^\circ \end{pmatrix} \tag{2.43.12}$$

$$= \begin{pmatrix} 5 \\ 1 \end{pmatrix} \tag{2.43.13}$$

Thus

$$\mathbf{D} = \begin{pmatrix} 2 \\ 3.46 \end{pmatrix}, \mathbf{E} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}, \mathbf{R} = \begin{pmatrix} 5 \\ 1 \end{pmatrix} \tag{2.43.14}$$

and the quadrilateral DEAR is the plotted in Fig. 2.43.1.

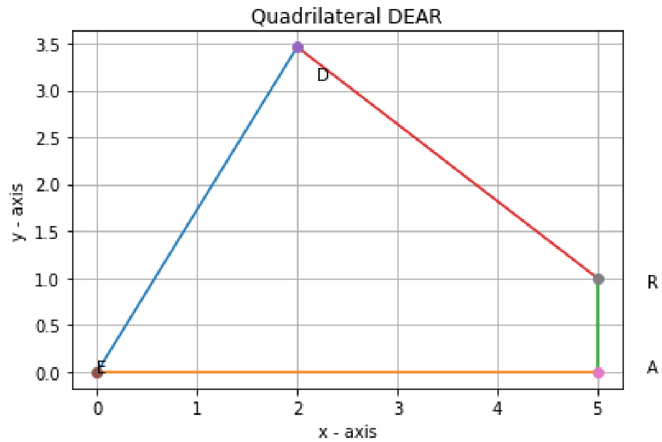

Fig. 2.43.1: Quadrilateral DEAR

2.44. Construct $TRUE$ with $TR = 3.5, RU = 3, UE = 4 \angle R = 75^\circ$ and $\angle U = 120^\circ$.
Solution: From the given information,

$$\angle R = 75^\circ = \theta \tag{2.44.1}$$

$$\angle U = 120^\circ = \alpha \tag{2.44.2}$$

$$\|\mathbf{T} - \mathbf{R}\| = 3.5 = a \tag{2.44.3}$$

$$\|\mathbf{U} - \mathbf{R}\| = 3 = b \tag{2.44.4}$$

$$\|\mathbf{E} - \mathbf{U}\| = 4 = c \tag{2.44.5}$$

Let,

$$\mathbf{R} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{U} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \tag{2.44.6}$$

Lemma 2.3.

$$\mathbf{T} = C\mathbf{u} \quad \left(\because \mathbf{R} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) \tag{2.44.7}$$

$$\mathbf{E} = \mathbf{U} + a\mathbf{r} \tag{2.44.8}$$

where

$$\mathbf{r} = \begin{pmatrix} \cos R \\ \sin R \end{pmatrix}, \mathbf{u} = \begin{pmatrix} \cos U \\ \sin U \end{pmatrix} \tag{2.44.9}$$

Thus,

$$\mathbf{T} = 4\begin{pmatrix} \cos 120 \\ \sin 120 \end{pmatrix} \tag{2.44.10}$$

$$= \begin{pmatrix} -2 \\ 3.46 \end{pmatrix} \tag{2.44.11}$$

and

$$\mathbf{E} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} + 3.5 \begin{pmatrix} \cos 75 \\ \sin 75 \end{pmatrix} \tag{2.44.12}$$

$$= \begin{pmatrix} 3.39 \\ 3.38 \end{pmatrix} \tag{2.44.13}$$

The vertices of given quadrilateral TRUE can be written as,

$$\mathbf{T} = \begin{pmatrix} -2 \\ 3.46 \end{pmatrix}, \mathbf{R} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{U} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \mathbf{E} = \begin{pmatrix} 3.39 \\ 3.38 \end{pmatrix} \tag{2.44.14}$$

which are plotted in Fig. 2.44.1.

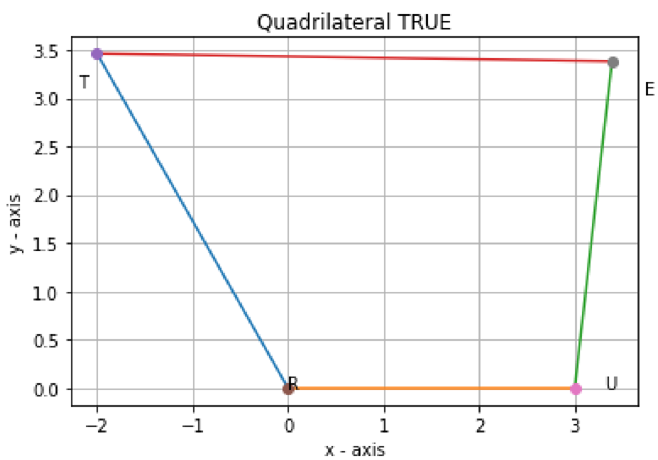

Fig. 2.44.1: Quadrilateral TRUE

2.45. Can you construct a rhombus $ABCD$ with $AC = 6$ and $BD = 7$?
Solution: We obtain the vertices of the rhombus as follows

$$\mathbf{A} = \begin{pmatrix} -3 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ -3.5 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 0 \\ 3.5 \end{pmatrix} \tag{2.45.1}$$

which are plotted in Fig. 2.45.1.

2.46. Draw a square $READ$ with $RE = 5.1$.
Solution: The vertices are given by

$$\mathbf{R} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{E} = \begin{pmatrix} 5.1 \\ 0 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 5.1 \\ 5.1 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 0 \\ 5.1 \end{pmatrix} \tag{2.46.1}$$

The desired square is plotted in Fig. 2.46.1

2.47. Draw a rhombus who diagonals are 5.2 and 6.4.
Solution: We obtain the vertices of the rhombus as

$$\mathbf{A} = \begin{pmatrix} -2.6 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ -3.2 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 2.6 \\ 0 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 0 \\ 3.2 \end{pmatrix} \tag{2.47.1}$$

which are plotted in Fig. 2.47.1

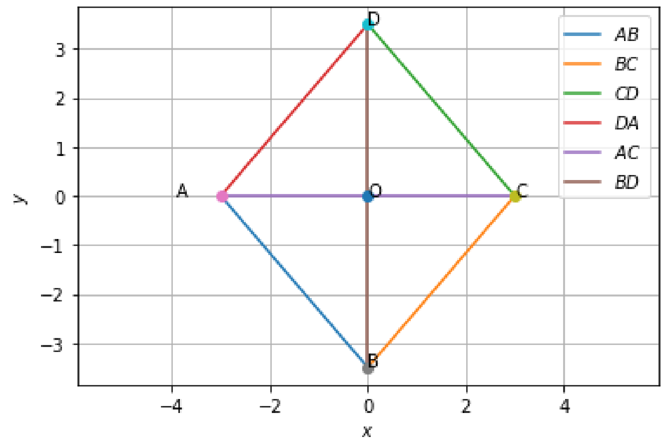

Fig. 2.45.1: Rhombus ABCD

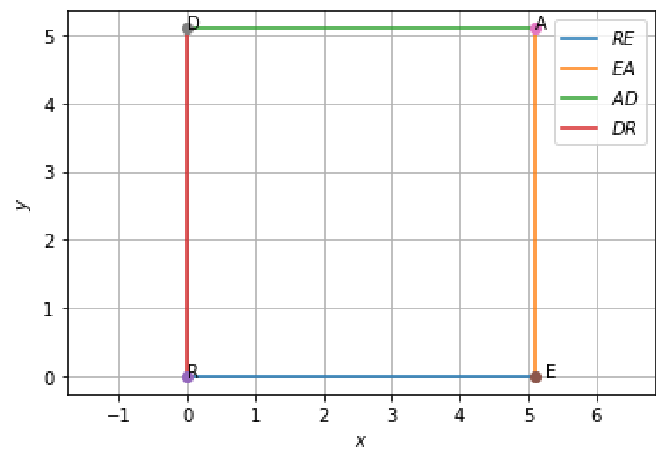

Fig. 2.46.1: Square $READ$

2.48. Draw a rectangle with adjacent sides 5 and 4.
Solution: The vertices of rectangle $ABCD$ are

$$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} a \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 0 \\ c \end{pmatrix}, \mathbf{D} = \begin{pmatrix} a \\ c \end{pmatrix} \tag{2.48.1}$$

$$\implies \mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 5 \\ 4 \end{pmatrix} \tag{2.48.2}$$

where $a = 5$ and $c = 4$. The rectangle $ABCD$ is plotted in Fig. 2.48.1

2.49. Draw a parallelogram $OKAY$ with $OK = 5.5$ and $KA = 4.2$.

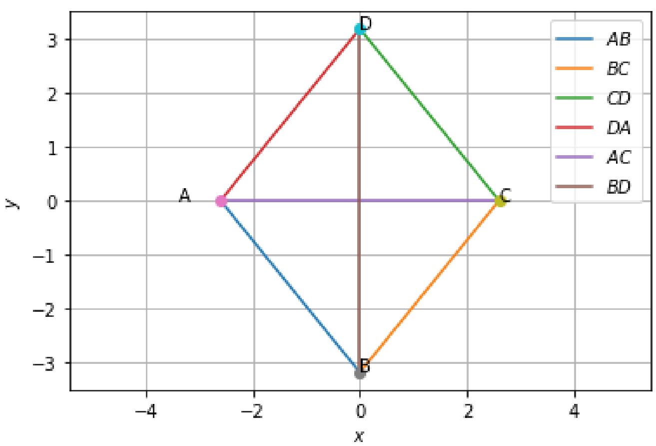

Fig. 2.47.1: Rhombus ABCD

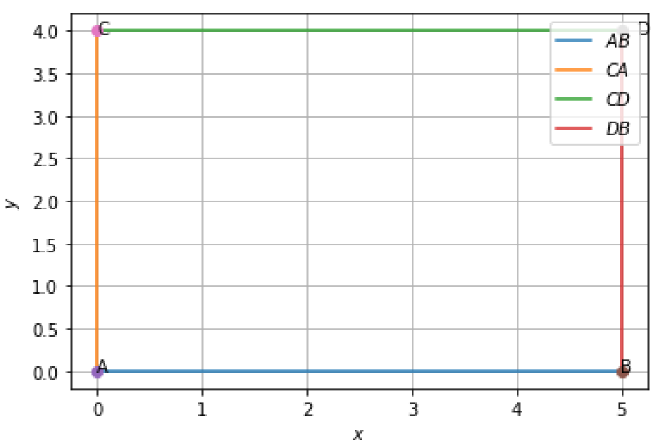

Fig. 2.48.1: Rectangle $ABCD$

Solution: There are infinite number of parallelograms that can be draw. For a unique parallelogram, one angle needs to be specified.

2.50. Construct a kite $EASY$ if $AY = 8, EY = 4$ and $SY = 6$.

2.51. Draw a circle of diameter 6.1

2.52. With the same centre $\mathbf{O}$, draw two circles of radii 4 and 2.5

Solution:

All input values required to plot Fig. 2.52.1 are given in Table 2.52.1 as shown below

	Symbols	Circle1	Circle2
Centre	$\mathbf{O}$	$\begin{pmatrix}0\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\end{pmatrix}$
Radius	r_1,r_2	2.5	4
Polar coordinate	$\mathbf{C}_1$,$\mathbf{C}_2$	$2.5\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$	$4\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$
Angle	θ	0-2π	0-2π

TABLE 2.52.1: Input values

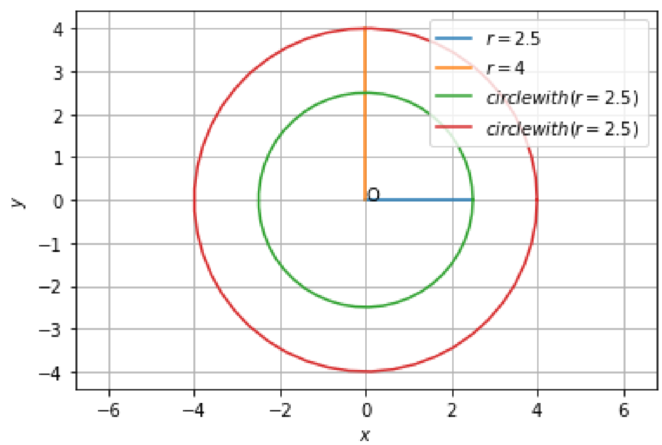

Fig. 2.52.1: Concentric circles with centre as origin and radii 2.5 and 4 respectively

2.53. Draw a circle with centre $\mathbf{B}$ and radius 6. If $\mathbf{C}$ be a point 10 units away from its centre, construct the pair of tangents AC and CD to the circle.

2.54. Draw a circle of radius 3 and any two of its diameters. Draw the ends of these diameters. What figure do you get?

2.55. Let $\mathbf{A}$ and $\mathbf{B}$ be the centres of two circles of equal radii 3 such that each one of them passes through the centre of the other. Let them intersect at $\mathbf{C}$ and $\mathbf{D}$. Is $AB \perp CD$?

Solution: The centers and radii of the two circles without any loss of generality are given in Table 2.55.1

	Circle 1	Circle 2
Centre	$\mathbf{A}$=$\begin{pmatrix}0\\0\end{pmatrix}$	$\mathbf{B}$=$\begin{pmatrix}3\\0\end{pmatrix}$
Radius	$r_1 = r_2 = 3$	

TABLE 2.55.1: Input values

Let

$$\mathbf{u} = \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}, \theta \in [0, 2\pi]. \tag{2.55.1}$$

Then on Circle 1 and Circle 2 are given by

$$\mathbf{x} = \mathbf{A} + r\mathbf{u} \tag{2.55.2}$$

$$\mathbf{x} = \mathbf{B} + r\mathbf{u} \tag{2.55.3}$$

Fig. 2.55.1 is plotted using the above equations. Fig. 2.55.1

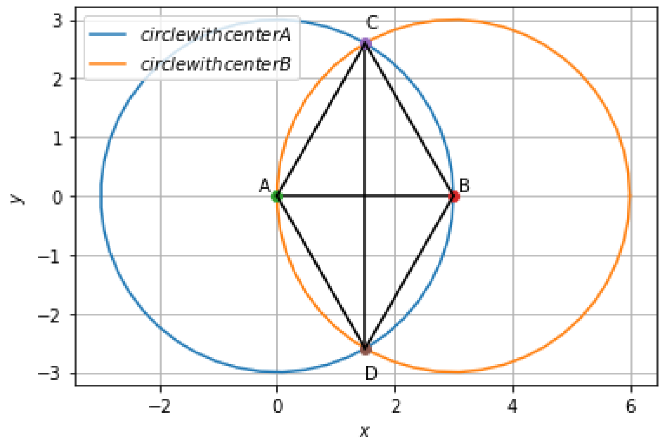

Fig. 2.55.1: Circles with their points of intersection

The general equation of Circle 1 is given by

$$\|\mathbf{x}-\mathbf{A}\|^2 = r^2 \tag{2.55.4}$$

$$\mathbf{x}^\top\mathbf{x} - 2\mathbf{A}^\top\mathbf{x} + \|\mathbf{A}\|^2 - r_1^2 = 0 \tag{2.55.5}$$

Similarly, for Circle 2,

$$\mathbf{x}^\top\mathbf{x} - 2\mathbf{B}^\top\mathbf{x} + \|\mathbf{B}\|^2 - r_2^2 = 0 \tag{2.55.6}$$

Subtracting (2.55.6) from (2.55.5),

$$2\mathbf{B}^\top\mathbf{x} = \|\mathbf{B}\|^2 \tag{2.55.7}$$

$$\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = \frac{3}{2} \tag{2.55.8}$$

which can be expressed as

$$\mathbf{x} = \frac{1}{2}\begin{pmatrix}3\\0\end{pmatrix} + \lambda\begin{pmatrix}0\\1\end{pmatrix} \tag{2.55.9}$$

$$= \mathbf{q} + \lambda\mathbf{m} \text{ where} \tag{2.55.10}$$

$$\mathbf{q} = \begin{pmatrix}1.5\\0\end{pmatrix}, \mathbf{m} = \begin{pmatrix}0\\1\end{pmatrix} \tag{2.55.11}$$

Substituting (2.55.10) in (2.55.5)

$$\|\mathbf{x}\|^2 = r^2 \quad (\because \mathbf{A} = 0) \tag{2.55.12}$$

$$\|\mathbf{q}+\lambda\mathbf{m}\|^2 = r^2 \tag{2.55.13}$$

$$(\mathbf{q}+\lambda\mathbf{m})^\top(\mathbf{q}+\lambda\mathbf{m}) = r^2 \tag{2.55.14}$$

$$\implies \mathbf{q}^\top(\mathbf{q}+\lambda\mathbf{m}) + \lambda\mathbf{m}^\top(\mathbf{q}+\lambda\mathbf{m}) = r^2 \tag{2.55.15}$$

$$\implies \|\mathbf{q}\|^2 + \lambda\mathbf{q}^\top\mathbf{m} + \lambda\mathbf{m}^\top\mathbf{q} + \lambda^2\|\mathbf{m}\|^2 = r^2 \tag{2.55.16}$$

$$\implies \|\mathbf{q}\|^2 + 2\lambda\mathbf{q}^\top\mathbf{m} + \lambda^2\|\mathbf{m}\|^2 = r^2 \tag{2.55.17}$$

$$\implies \lambda = \pm\sqrt{\frac{9-\|\mathbf{q}\|^2}{\|\mathbf{m}\|^2}} \quad \because \mathbf{q}^\top\mathbf{m} = 0 \tag{2.55.18}$$

Substituting the value of λ in (2.55.10),

$$\mathbf{C} = \mathbf{q} + \lambda\mathbf{m} \tag{2.55.19}$$

$$\mathbf{D} = \mathbf{q} - \lambda\mathbf{m} \tag{2.55.20}$$

$$\implies (\mathbf{A}-\mathbf{B})^\top(\mathbf{C}-\mathbf{D}) = 2\begin{pmatrix}-3 & 0\end{pmatrix}\begin{pmatrix}0\\\sqrt{6.75}\end{pmatrix} \tag{2.55.21}$$

$$= 0 \tag{2.55.22}$$

$$\implies AB \perp CD \tag{2.55.23}$$

2.56. Construct a tangent to a circle of radius 4 units from a point on the concentric circle of radius 6 units.
Solution: The given information is summarised in Table 2.56.1. See Fig. 2.56.1. Let P be a point on Circle 2 with radius 6. Then

	Symbols	Circle1	Circle2
Centre	$\mathbf{O}$	$\begin{pmatrix}0\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\end{pmatrix}$
Radius	r_1,r_2	4	6

TABLE 2.56.1

$$\mathbf{P} = \begin{pmatrix}6\\0\end{pmatrix} \tag{2.56.1}$$

Let PQ and PR be tangents from point $\mathbf{P}$ on circle with radius 6 to the points $\mathbf{Q}$ and $\mathbf{R}$ on circle with radius 4 . Now,

$$(\mathbf{O}-\mathbf{Q})^T(\mathbf{Q}-\mathbf{P}) = 0 \quad (\because OQ \perp QP) \tag{2.56.2}$$

$$\implies \mathbf{P}^T\mathbf{Q} = 16 \quad \left(\because \|\mathbf{Q}\|^2 = 16\right) \tag{2.56.3}$$

$$\text{or, } \begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{Q} = \frac{8}{3} \tag{2.56.4}$$

$$\implies \mathbf{Q} = \begin{pmatrix} \frac{8}{3} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.56.5}$$

$$= \mathbf{q} + \lambda\mathbf{m} \tag{2.56.6}$$

$$\text{where } \mathbf{q} = \begin{pmatrix} \frac{8}{3} \\ 0 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.56.7}$$

We know,

$$\|\mathbf{q} + \lambda\mathbf{m}\|^2 = r_1^2 \tag{2.56.8}$$

$$(\mathbf{q} + \lambda\mathbf{m})^T(\mathbf{q} + \lambda\mathbf{m}) = r_1^2 \tag{2.56.9}$$

$$\lambda^2 = \frac{r_1^2 - \|\mathbf{q}\|^2}{\|\mathbf{m}\|^2} \tag{2.56.10}$$

$$\lambda = \pm 2.98 \tag{2.56.11}$$

Substituting the above in (2.56.5),

$$\mathbf{Q} = \begin{pmatrix} \frac{8}{3} \\ 2.98 \end{pmatrix}, \mathbf{R} = \begin{pmatrix} \frac{8}{3} \\ -2.98 \end{pmatrix} \tag{2.56.12}$$

The circles as well as the tangents are plotted in Fig. 2.56.1

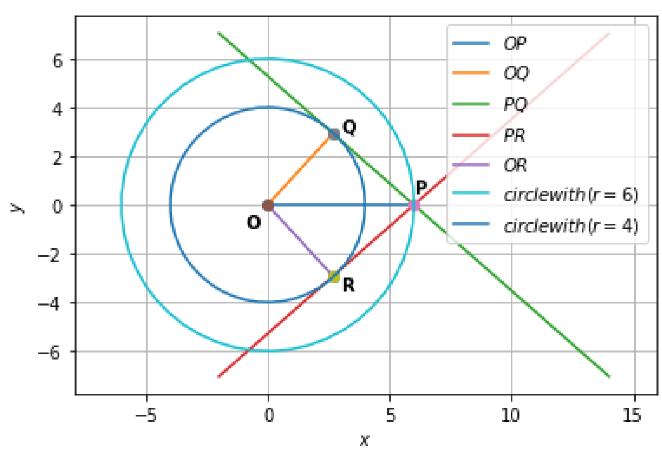

Fig. 2.56.1: Tangent lines to circle of radius 4 units.

2.57. Draw a circle of radius 3 units. Take two points **P** and **Q** on one of its extended diameter each at a distance of 7 units from its centre. Draw tangents to the circle from these two points **P** and **Q**.

Solution: Take the diameter to be on the x-axis.

2.58. Draw a pair of tangents to a circle of radius 5 units which are inclined to each other at an angle of 60°.

Solution: The tangent is perpendicular to the radius.

2.59. Draw a line segment AB of length 8 units. Taking **A** as centre, draw a circle of radius 4 units and taking **B** as centre, draw another circle of radius 3 units. Construct tangents to each circle from the centre of the other circle.

Solution: Let

$$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 8 \\ 0 \end{pmatrix}. \tag{2.59.1}$$

2.60. Let ABC be a right triangle in which $a = 8, c = 6$ and $\angle B = 90^\circ$. BD is the perpendicular from **B** on AC (altitude). The circle through $\mathbf{B}, \mathbf{C}, \mathbf{D}$ (circumcircle of $\triangle BCD$) is drawn. Construct the tangents from **A** to this circle.

2.61. Draw a circle with centre **C** and radius 3.4. Draw any chord. Construct the perpendicular bisector of the chord and examine if it passes through **C**