Points and Vectors

G V V Sharma*

Contents

***Abstract*—This book provides a computational approach to school geometry based on the NCERT textbooks from Class 6-12. Links to sample Python codes are available in the text.**

Download python codes using

```
svn co https://github.com/gadepall/school/trunk/
    ncert/computation/codes
```

1 Examples

1.1. Show that the points

$$\mathbf{A} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 3 \\ -4 \\ -4 \end{pmatrix} \tag{1.1.1}$$

are the vertices of a right angled triangle.
Solution: The following code plots Fig. 1.1

```
codes/triangle/triangle_3d.py
```

From the figure, it appears that $\triangle ABC$ is right angled at $\mathbf{C}$. Since

$$(\mathbf{A} - \mathbf{C})^T (\mathbf{B} - \mathbf{C}) = 0 \tag{1.1.2}$$

it is proved that the triangle is indeed right angled.

1.2. Do the points $\mathbf{A} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ form a triangle? If so, name the type of triangle formed.
Solution:

*The author is with the Department of Electrical Engineering, Indian Institute of Technology, Hyderabad 502285 India e-mail: gadepall@iith.ac.in. All content in this manual is released under GNU GPL. Free and open source.

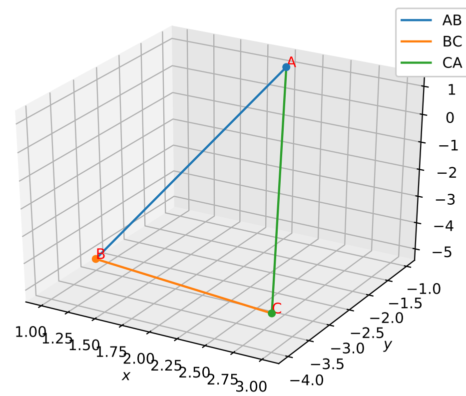

Fig. 1.1

The direction vectors of AB and BC are

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} -5 \\ -5 \end{pmatrix} \tag{1.2.1}$$

$$\mathbf{C} - \mathbf{A} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{1.2.2}$$

If $\mathbf{A}, \mathbf{B}, \mathbf{C}$ form a line, then, AB and AC should have the same direction vector. Hence, there exists a k such that

$$\mathbf{B} - \mathbf{A} = k(\mathbf{C} - \mathbf{B}) \tag{1.2.3}$$

$$\implies \mathbf{B} = \frac{k\mathbf{C} + \mathbf{A}}{k+1} \tag{1.2.4}$$

Since

$$\mathbf{B} - \mathbf{A} \neq k(\mathbf{C} - \mathbf{A}), \tag{1.2.5}$$

the points are not collinear and form a triangle. An alternative method is to create the matrix

$$\mathbf{M} = \begin{pmatrix} \mathbf{B} - \mathbf{A} & \mathbf{B} - \mathbf{A} \end{pmatrix}^T \tag{1.2.6}$$

If $rank(\mathbf{M}) = 1$, the points are collinear. The rank of a matrix is the number of nonzero rows left after doing row operations. In this problem,

$$\mathbf{M} = \begin{pmatrix} -5 & -5 \\ -1 & 1 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow 5R_2 - R_1} \begin{pmatrix} -5 & -5 \\ 0 & 10 \end{pmatrix} \quad (1.2.7)$$

$$\implies rank(\mathbf{M}) = 2 \quad (1.2.8)$$

as the number of non zero rows is 2. The following code plots Fig. 1.2

```
codes/triangle/check_tri.py
```

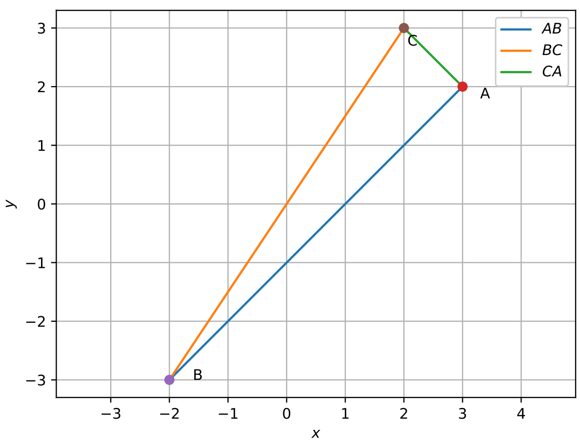

Fig. 1.2

From the figure, it appears that $\triangle ABC$ is right angled, with BC as the hypotenuse. From Baudhayana's theorem, this would be true if

$$\|\mathbf{B}-\mathbf{A}\|^2 + \|\mathbf{C}-\mathbf{A}\|^2 = \|\mathbf{B}-\mathbf{C}\|^2 \quad (1.2.9)$$

which can be expressed as

$$\|\mathbf{A}\|^2 + \|\mathbf{C}\|^2 - 2\mathbf{A}^T\mathbf{C} + \|\mathbf{A}\|^2 + \|\mathbf{B}\|^2 - 2\mathbf{A}^T\mathbf{B} = \|\mathbf{B}\|^2 + \|\mathbf{C}\|^2 - 2\mathbf{B}^T\mathbf{C} \quad (1.2.10)$$

to obtain

$$(\mathbf{B}-\mathbf{A})^T(\mathbf{C}-\mathbf{A}) = 0 \quad (1.2.11)$$

after simplification. From (1.2.1) and (1.2.2), it is easy to verify that

$$(\mathbf{B}-\mathbf{A})^T(\mathbf{C}-\mathbf{A}) = \begin{pmatrix} -5 & -5 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = 0 \quad (1.2.12)$$

satisfying (1.2.11). Thus, $\triangle ABC$ is right angled at $\mathbf{A}$.

1.3. Find the area of a triangle whose vertices are $\mathbf{A} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -4 \\ 6 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} -3 \\ -5 \end{pmatrix}$.

Solution: Using Hero's formula, the following code computes the area of the triangle as 24.

```
codes/triangle/area_tri.py
```

1.4. Find the area of a triangle formed by the vertices $\mathbf{A} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 7 \\ -4 \end{pmatrix}$. **Solution:** The area of $\triangle ABC$ is also obtained in terms of the *magnitude* of the determinant of the matrix $\mathbf{M}$ in (1.2.6) as

$$\frac{1}{2}\left|\mathbf{M}\right| \quad (1.4.1)$$

The computation is done in **area_tri.py**

1.5. Find the area of a triangle formed by the points $\mathbf{P} = \begin{pmatrix} -1.5 \\ 3 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} 6 \\ -2 \end{pmatrix}, \mathbf{R} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$.
Solution: Another formula for the area of $\triangle ABC$ is

$$\frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{B} & \mathbf{C} \end{vmatrix} \quad (1.5.1)$$

1.6. Find the area of a triangle having the points

$$\mathbf{A} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \quad (1.6.1)$$

as its vertices.
Solution: The area of a triangle using the *vector product* is obtained as

$$\frac{1}{2}\|(\mathbf{B}-\mathbf{A}) \times (\mathbf{C}-\mathbf{A})\| \quad (1.6.2)$$

For any two vectors $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$,

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \quad (1.6.3)$$

The following code computes the area using the vector product.

```
codes/triangle/area_tri_vec.py
```

1.7. The centroid of a $\triangle ABC$ is at the point $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. If the coordinates of $\mathbf{A}$ and $\mathbf{B}$ are $\begin{pmatrix} 3 \\ -5 \\ 7 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 7 \\ -6 \end{pmatrix}$, respectively, find the coordinates of the point

C.
Solution: The centroid of $\triangle ABC$ is given by

$$\mathbf{O} = \frac{\mathbf{A}+\mathbf{B}+\mathbf{C}}{3} \tag{1.7.1}$$

Thus,

$$\mathbf{C} = 3\mathbf{C} - \mathbf{A} - \mathbf{B} \tag{1.7.2}$$

1.8. Without using the Pythagoras theorem, show that the points $\begin{pmatrix}4\\4\end{pmatrix}$, $\begin{pmatrix}3\\5\end{pmatrix}$ and $\begin{pmatrix}-1\\-1\end{pmatrix}$ are the vertices of a right angled triangle.
Solution: The direction vectors of $\mathbf{A}-\mathbf{B}$, $\mathbf{A}-\mathbf{C}$ and $\mathbf{B}-\mathbf{C}$ are

$$\mathbf{A}-\mathbf{B} = \begin{pmatrix}-1\\1\end{pmatrix} \tag{1.8.1}$$

$$\mathbf{A}-\mathbf{C} = \begin{pmatrix}-5\\-5\end{pmatrix} \tag{1.8.2}$$

$$\mathbf{B}-\mathbf{C} = \begin{pmatrix}-4\\-6\end{pmatrix} \tag{1.8.3}$$

a)

$$(\mathbf{A}-\mathbf{B})^T(\mathbf{B}-\mathbf{C}) = \begin{pmatrix}-1\\1\end{pmatrix}\begin{pmatrix}-4\\-6\end{pmatrix} = -2 \tag{1.8.4}$$

$$(\mathbf{A}-\mathbf{B})^T(\mathbf{B}-\mathbf{C}) = -2 \neq 0 \tag{1.8.5}$$

Sides $\mathbf{A}-\mathbf{B}$ and $\mathbf{B}-\mathbf{C}$ of triangle are not perpendicular.

b)

$$(\mathbf{A}-\mathbf{C})^T(\mathbf{B}-\mathbf{C}) = \begin{pmatrix}-5\\-5\end{pmatrix}\begin{pmatrix}-4\\-6\end{pmatrix} = 50 \tag{1.8.6}$$

$$(\mathbf{A}-\mathbf{C})^T(\mathbf{B}-\mathbf{C}) = 50 \neq 0 \tag{1.8.7}$$

Sides $\mathbf{A}-\mathbf{C}$ and $\mathbf{B}-\mathbf{C}$ of triangle are not perpendicular.

c)

$$(\mathbf{A}-\mathbf{B})^T(\mathbf{A}-\mathbf{C}) = \begin{pmatrix}-1\\1\end{pmatrix}\begin{pmatrix}-5\\-5\end{pmatrix} = 0 \tag{1.8.8}$$

$$(\mathbf{A}-\mathbf{B})^T(\mathbf{A}-\mathbf{C}) = 0 \tag{1.8.9}$$

Sides $\mathbf{A}-\mathbf{B}$ and $\mathbf{A}-\mathbf{C}$ of triangle are perpendicular to each other and the right angle at vertex $\begin{pmatrix}4\\4\end{pmatrix}$, and the following figure represents the triangle formed by given points **A**, **B** and **C**.

1.9. Draw the graphs of the equations

$$\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} + 1 = 0 \tag{1.9.1}$$

$$\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x} - 12 = 0 \tag{1.9.2}$$

Determine the coordinates of the vertices of the triangle formed by these lines and the x-axis, and shade the triangular region.
Solution: Let

$$\mathbf{x} = \begin{pmatrix}a\\0\end{pmatrix} \tag{1.9.3}$$

Substituting in (1.9.1),

$$\begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}a\\0\end{pmatrix} = -1 \tag{1.9.4}$$

$$\implies a = -1 \tag{1.9.5}$$

Simiarly, substituting

$$\mathbf{x} = \begin{pmatrix}0\\b\end{pmatrix}, \tag{1.9.6}$$

in (1.9.1),

$$b = 1 \tag{1.9.7}$$

The intercepts on the x and y-axis from above are

$$\begin{pmatrix}-1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \tag{1.9.8}$$

Similarly, the intercepts on x and y-axis for (1.9.2) are

$$\begin{pmatrix}4\\0\end{pmatrix}, \begin{pmatrix}0\\6\end{pmatrix} \tag{1.9.9}$$

The interection of the lines in (1.9.1), (1.9.1) is obtained from

$$\begin{pmatrix}1 & -1\\3 & 2\end{pmatrix}\mathbf{x} = \begin{pmatrix}-1\\12\end{pmatrix} \tag{1.9.10}$$

The augmented matrix for the above equation is row reduced as follows

$$\begin{pmatrix}1 & -1 & -1\\3 & 2 & 12\end{pmatrix} \xleftrightarrow{R_2 \leftarrow \frac{R_2-3R_1}{5}} \begin{pmatrix}1 & -1 & -1\\0 & 1 & 3\end{pmatrix} \tag{1.9.11}$$

$$\xleftrightarrow{R_1 \leftarrow R_1+R_2} \begin{pmatrix}1 & 0 & 2\\0 & 1 & 3\end{pmatrix} \tag{1.9.12}$$

$$\implies \mathbf{x} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \tag{1.9.13}$$

The desired triangle is available in Fig. (1.9) with vertices

$$\mathbf{A} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \tag{1.9.14}$$

The equivalent python code for figure (1.9) is

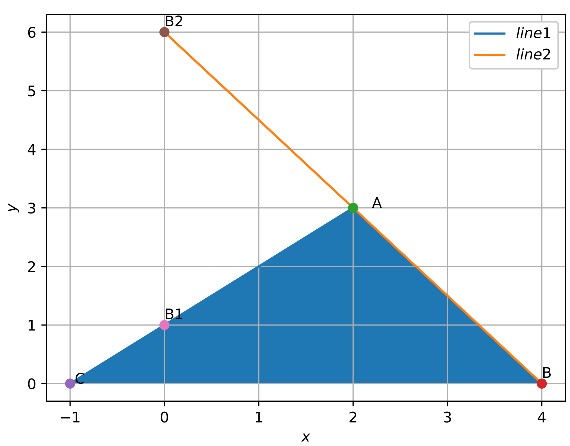

Fig. 1.9

```
solutions/1/codes/triangle/shaded.py
```

1.10. In a $\triangle ABC, \angle C = 3\angle B = 2(\angle A + \angle B)$. Find the three angles.
Solution:
The given equations result in the matrix equation In vector form:

$$\begin{pmatrix} 6 & 0 & -1 \\ 0 & 3 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 180 \end{pmatrix} \tag{1.10.1}$$

whcih can be solved as

$$\begin{pmatrix} 6 & 0 & -1 & 0 \\ 0 & 3 & -1 & 0 \\ 1 & 1 & 1 & 180 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{R_1}{6}} \begin{pmatrix} 1 & 0 & \frac{-1}{6} & 0 \\ 0 & 3 & -1 & 0 \\ 1 & 1 & 1 & 180 \end{pmatrix} \tag{1.10.2}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - R_1} \begin{pmatrix} 1 & 0 & \frac{-1}{6} & 0 \\ 0 & 3 & -1 & 0 \\ 0 & 1 & \frac{7}{6} & 180 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow \frac{R_2}{3}} \begin{pmatrix} 1 & 0 & -\frac{1}{6} & 0 \\ 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{7}{6} & 180 \end{pmatrix} \tag{1.10.3}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - R_2} \begin{pmatrix} 1 & 0 & -\frac{1}{6} & 0 \\ 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & \frac{3}{2} & 180 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow \frac{2R_3}{3}} \begin{pmatrix} 1 & 0 & -\frac{1}{6} & 0 \\ 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & 120 \end{pmatrix} \tag{1.10.4}$$

$$\xleftrightarrow[R_2 \leftarrow R_2 + \frac{R_3}{3}]{R_1 \leftarrow R_1 + \frac{R_3}{6}} \begin{pmatrix} 1 & 0 & 1 & 20 \\ 0 & 1 & 0 & 40 \\ 0 & 0 & 1 & 120 \end{pmatrix} \tag{1.10.5}$$

$$\therefore \angle C = 120^\circ \ \angle A = 20^\circ \ \angle B = 40^\circ \tag{1.10.6}$$

1.11. Draw the graphs of the equations $5x–y = 5$ and $3x–y = 3$. Determine the co-ordinates of the vertices of the triangle formed by these lines and the y axis.
Solution:
Line $5x - y = 5$ can be represented in vector form as,

$$\begin{pmatrix} 5 & -1 \end{pmatrix} \mathbf{x} = 5 \tag{1.11.1}$$

Line $3x - y = 3$ can be represented in vector form as,

$$\begin{pmatrix} 3 & -1 \end{pmatrix} \mathbf{x} = 3 \tag{1.11.2}$$

Also the equation of y axis is

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 0 \tag{1.11.3}$$

Let line (1.11.1) and line (1.11.2) meet at point **A**.Then,

$$\begin{pmatrix} 5 & -1 \\ 3 & -1 \end{pmatrix} \mathbf{A} = \begin{pmatrix} 5 \\ 3 \end{pmatrix} \tag{1.11.4}$$

$$\mathbf{A} = \begin{pmatrix} 5 & -1 \\ 3 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 5 \\ 3 \end{pmatrix} \tag{1.11.5}$$

$$\mathbf{A} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.11.6}$$

Let line (1.11.1) and line (1.11.3) meet at point

B. Then,

$$\begin{pmatrix}5 & -1\\1 & 0\end{pmatrix}\mathbf{B} = \begin{pmatrix}5\\0\end{pmatrix} \tag{1.11.7}$$

$$\mathbf{B} = \begin{pmatrix}5 & -1\\1 & 0\end{pmatrix}^{-1}\begin{pmatrix}5\\0\end{pmatrix} \tag{1.11.8}$$

$$\mathbf{B} = \begin{pmatrix}0\\-5\end{pmatrix} \tag{1.11.9}$$

Let line (1.11.2) and line (1.11.3) meet at point **C**. Then,

$$\begin{pmatrix}3 & -1\\1 & 0\end{pmatrix}\mathbf{C} = \begin{pmatrix}3\\0\end{pmatrix} \tag{1.11.10}$$

$$\mathbf{C} = \begin{pmatrix}3 & -1\\1 & 0\end{pmatrix}^{-1}\begin{pmatrix}3\\0\end{pmatrix} \tag{1.11.11}$$

$$\mathbf{C} = \begin{pmatrix}0\\-3\end{pmatrix} \tag{1.11.12}$$

So, $\triangle ABC$ is formed by intersection of (1.11.1),(1.11.2) and (1.11.3). The following Python code generates Fig. 1.11 The lines (1.11.1) and (1.11.2) and the triangle ABC formed by the two lines and y-axis are plotted in the figure below

```
codes/triangle/linesandtri.py
```

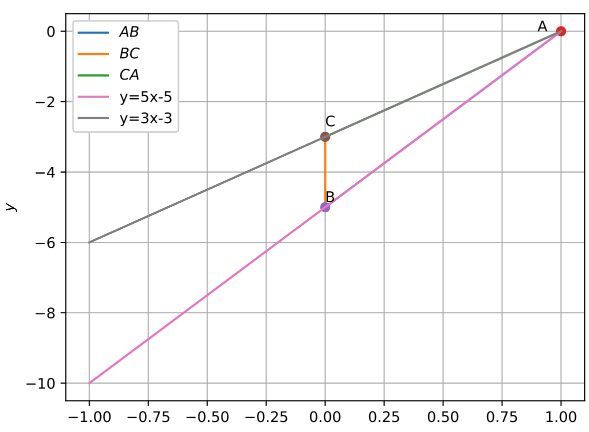

Fig. 1.11: Plot of lines and the Triangle ABC

1.12. The vertices of $\triangle PQR$ are $\mathbf{P} = \begin{pmatrix}2\\1\end{pmatrix}, \mathbf{Q} = \begin{pmatrix}-2\\3\end{pmatrix}, \mathbf{R} = \begin{pmatrix}4\\5\end{pmatrix}$. Find the equation of the median through the vertex **R**.
Solution: In Fig. 1.12, RS is the median. Hence,

$$\mathbf{S} = \frac{\mathbf{P}+\mathbf{Q}}{2} \tag{1.12.1}$$

Hence, the equation of the median going through points **S** and **R** can be given as

$$\mathbf{x} = \mathbf{R} + \lambda\left(\mathbf{S}-\mathbf{R}\right) \tag{1.12.2}$$

$$\mathbf{x} = \begin{pmatrix}4\\5\end{pmatrix} + \lambda\left(\begin{pmatrix}0\\2\end{pmatrix} - \begin{pmatrix}4\\5\end{pmatrix}\right) \tag{1.12.3}$$

$$\mathbf{x} = \begin{pmatrix}4\\5\end{pmatrix} + \lambda\left(\begin{pmatrix}-4\\-3\end{pmatrix}\right) \tag{1.12.4}$$

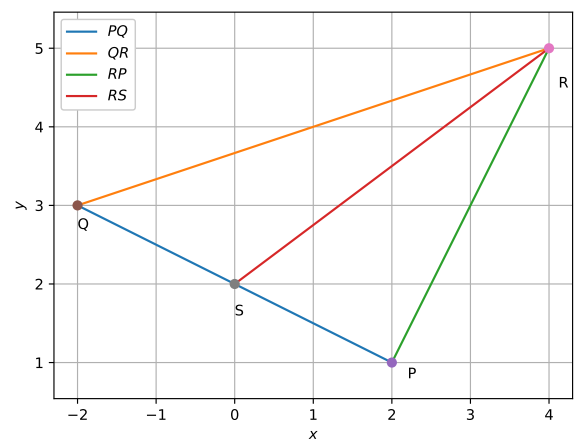

Fig. 1.12

```
solutions/4/codes/triangle/triangle.py
```

1.13. In the $\triangle ABC$ with vertices $\mathbf{A} = \begin{pmatrix}2\\3\end{pmatrix}, \mathbf{B} = \begin{pmatrix}4\\-1\end{pmatrix}, \mathbf{C} = \begin{pmatrix}1\\2\end{pmatrix}$, find the equation and length of the altitude from the vertex **A**.
Solution: The following python code computes the length of the altitude **AD** in Fig.1.13.

```
./solutions/5/codes/triangle/q2.py
```

In $\triangle ABC$,

$$\left(\mathbf{A}-\mathbf{C}\right)^T\left(\mathbf{B}-\mathbf{C}\right) = 0 \tag{1.13.1}$$

Hence, ABC is a right triangle. The direction vector of BC is

$$\left(\mathbf{B}-\mathbf{C}\right) = \begin{pmatrix}3\\-3\end{pmatrix} \tag{1.13.2}$$

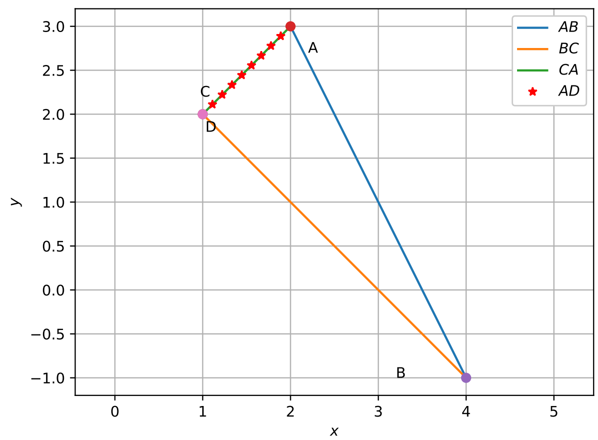

Fig. 1.13: Triangle of Q.1.2.5

Hence, the equation of AD is

$$(\mathbf{B}-\mathbf{C})^T(\mathbf{x}-\mathbf{A}) = 0 \tag{1.13.3}$$

$$\implies \begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = -1 \tag{1.13.4}$$

The length of the altitude is obtained as $\|\mathbf{A}-\mathbf{D}\| = 1.414$

1.14. Find the area of the triangle whose vertices are

a) $\begin{pmatrix}2\\3\end{pmatrix}, \begin{pmatrix}-1\\0\end{pmatrix}, \begin{pmatrix}2\\-4\end{pmatrix}$

b) $\begin{pmatrix}-5\\-1\end{pmatrix}, \begin{pmatrix}3\\-5\end{pmatrix}, \begin{pmatrix}5\\2\end{pmatrix}$

Solution:

a) See Fig. 1.14 generated using the following python code

solutions/6/codes/triangle/triangle1.py

$$ar(\triangle ABC) = \frac{1}{2}\|(\mathbf{B}-\mathbf{A})\times(\mathbf{C}-\mathbf{A})\| \tag{1.14.1}$$

$$= \frac{1}{2}\left\|\begin{pmatrix}-3\\-3\end{pmatrix}\times\begin{pmatrix}0\\-7\end{pmatrix}\right\| = \frac{21}{2} \tag{1.14.2}$$

and verified by

solutions/6/codes/triangle/tri_area_ABC.py

b) See $\triangle PQR$ in Fig. 1.14 generated using the following python code

solutions/6/codes/triangle/triangle2.py

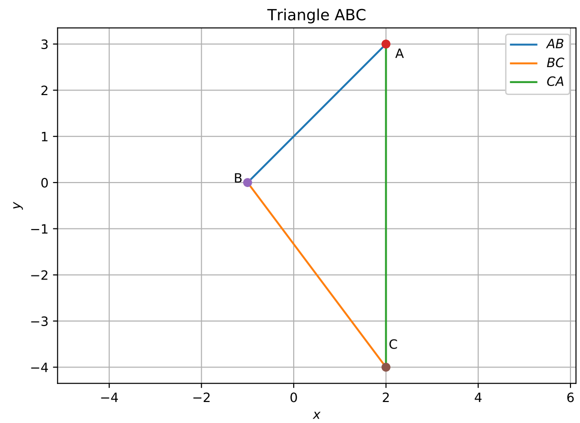

Fig. 1.14: Triangle ABC using python

$$ar(\triangle PQR) = \frac{1}{2}\|(\mathbf{Q}-\mathbf{P})\times(\mathbf{R}-\mathbf{P})\| \tag{1.14.3}$$

$$= \frac{1}{2}\left\|\begin{pmatrix}8\\-4\end{pmatrix}\times\begin{pmatrix}10\\3\end{pmatrix}\right\| = \frac{64}{2} \tag{1.14.4}$$

and verified by

solutions/6/codes/triangle/tri_area_PQR.py

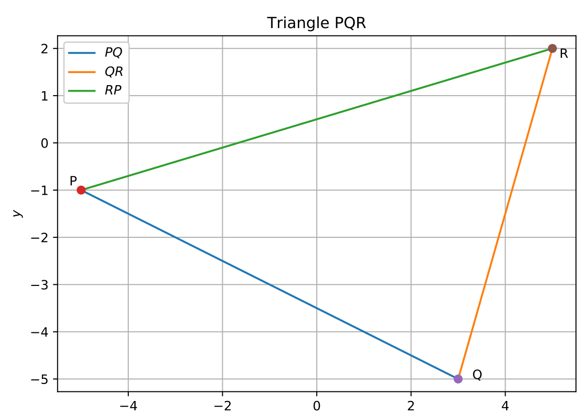

Fig. 1.14: Triangle PQR using python

1.15. Find the area of the triangle formed by joining the mid points of the sides of a triangle whose vertices are $\begin{pmatrix}0\\-1\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}0\\3\end{pmatrix}$.

Solution: See Fig. 1.15. Let the vertices be

$\mathbf{A}, \mathbf{B}, \mathbf{C}$. The midpoints of each side are

$$\mathbf{D} = \frac{\mathbf{A}+\mathbf{B}}{2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.15.1}$$

$$\mathbf{E} = \frac{\mathbf{B}+\mathbf{C}}{2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.15.2}$$

$$\mathbf{F} = \frac{\mathbf{A}+\mathbf{C}}{2} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{1.15.3}$$

$$\tag{1.15.4}$$

Area of a $\triangle$ ABC is given by

$$\frac{1}{2}\left\|(\mathbf{E}-\mathbf{D}) \times (\mathbf{F}-\mathbf{D})\right\|$$

$$= \frac{1}{2}\left\|\begin{pmatrix} 0 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\|$$

$$= 1 \tag{1.15.5}$$

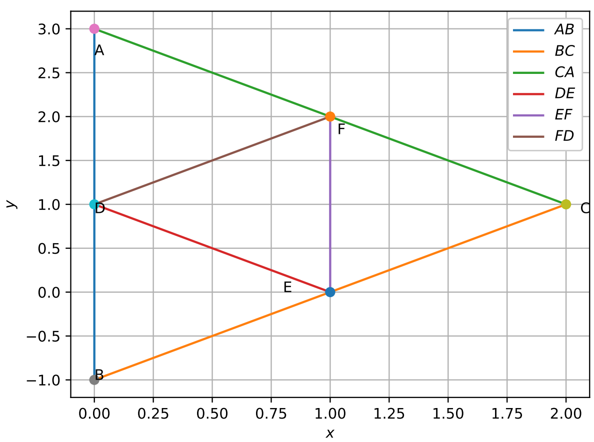

Fig. 1.15

Download the python code for finding a triangle's area from

solutions/7/codes/triangle/area_tri_area.py

and the figure from

solutions/7/figs/triangle/draw_triangle.py

1.16. Verify that the median of $\triangle ABC$ with vertices $\mathbf{A} = \begin{pmatrix} 4 \\ -6 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$ divides it into two triangles of equal areas.
Solution: The following Python code generates Fig. 1.16

codes/triangle.py

From the given information,

$$\mathbf{A} = \begin{pmatrix} 4 \\ -6 \end{pmatrix} \tag{1.16.1}$$

$$\mathbf{B} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}, \tag{1.16.2}$$

$$\mathbf{C} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \tag{1.16.3}$$

$\because$ $\mathbf{M}$ is the midpoint of AB,

$$\mathbf{M} = \frac{\mathbf{A}+\mathbf{B}}{2} = \frac{1}{2}\begin{pmatrix} 7 \\ -8 \end{pmatrix} \tag{1.16.4}$$

$\because$ $\mathbf{N}$ is the midpoint of BC,

$$\mathbf{N} = \frac{\mathbf{B}+\mathbf{C}}{2} = \frac{1}{2}\begin{pmatrix} 8 \\ 0 \end{pmatrix} \tag{1.16.5}$$

$\because$ $\mathbf{P}$ is the midpoint of CA,

$$\mathbf{P} = \frac{\mathbf{C}+\mathbf{A}}{2} = \frac{1}{2}\begin{pmatrix} 9 \\ -4 \end{pmatrix} \tag{1.16.6}$$

The following Python code verifies the determinant values.

codes/determinant_check.py

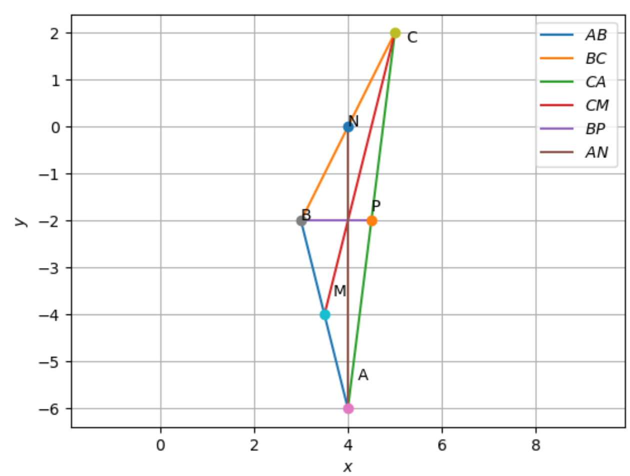

Fig. 1.16

For $\triangle ABC$, the vertices are $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$. So the area of the triangle $\triangle ABC$ by using determinant will be :

$$Area = \frac{1}{2}\begin{vmatrix}4 & -6 & 1\\3 & -2 & 1\\5 & 2 & 1\end{vmatrix} \xleftrightarrow{C_2\leftarrow\frac{C_2}{2}} \frac{2}{2}\begin{vmatrix}4 & -3 & 1\\3 & -1 & 1\\5 & 1 & 1\end{vmatrix}$$
$$\xleftrightarrow[R_3\leftarrow R_3-R_1]{R_2\leftarrow R_2-R_1} \begin{vmatrix}4 & -3 & 1\\-1 & 2 & 0\\1 & 4 & 0\end{vmatrix} \xleftrightarrow{R_3\leftarrow R_3+R_2} \begin{vmatrix}4 & -3 & 1\\-1 & 2 & 0\\0 & 6 & 0\end{vmatrix}$$
$$\xleftrightarrow{R_3\leftarrow\frac{R_3}{6}} 6\begin{vmatrix}4 & -3 & 1\\-1 & 2 & 0\\0 & 1 & 0\end{vmatrix}$$
$$= -6 \tag{1.16.7}$$

Now, we will consider the absolute value of area only. So, $Area = |-6| = 6$.

To verify the problem statement we have to check 3 cases:

Case 1: When **BP** is median, we will consider $\triangle ABP$ triangle. In that case, the vertices will be **A**, **B** and **P**.

Now, the area of $\triangle ABP$ will be :

$$A1 = \frac{1}{2}\begin{vmatrix}4 & -6 & 1\\3 & -2 & 1\\4.5 & -2 & 1\end{vmatrix} \xleftrightarrow{C_2\leftarrow\frac{C_2}{(-2)}} \frac{(-2)}{2}\begin{vmatrix}4 & 3 & 1\\3 & 1 & 1\\4.5 & 1 & 1\end{vmatrix}$$
$$\xleftrightarrow[R_3\leftarrow R_3-R_1]{R_2\leftarrow R_2-R_1} (-1)\begin{vmatrix}4 & 3 & 1\\-1 & -2 & 0\\0.5 & -2 & 0\end{vmatrix}$$
$$\xleftrightarrow{R_3\leftarrow R_3-R_2} (-1)\begin{vmatrix}4 & 3 & 1\\-1 & -2 & 0\\1.5 & 0 & 0\end{vmatrix}$$
$$= -3 \tag{1.16.8}$$

But, we will consider the absolute value of area only. So, **A1** $= |-3| = 3$.

or, **A1** $= \frac{1}{2}$(Area of $\triangle ABC$)

Case 2: When **AN** is median,we will consider $\triangle ABN$ triangle. In that case, the vertices will be **A**, **B** and **N**.

Now, the area of $\triangle ABN$ will be :

$$A2 = \frac{1}{2}\begin{vmatrix}4 & -6 & 1\\3 & -2 & 1\\4 & 0 & 1\end{vmatrix} \xleftrightarrow{C_2\leftarrow\frac{C_2}{(-2)}} \frac{(-2)}{2}\begin{vmatrix}4 & 3 & 1\\3 & 1 & 1\\4 & 0 & 1\end{vmatrix}$$
$$\xleftrightarrow[R_3\leftarrow R_3-R_1]{R_2\leftarrow R_2-R_1} (-1)\begin{vmatrix}4 & 3 & 1\\-1 & -2 & 0\\0 & -3 & 0\end{vmatrix}$$
$$\xleftrightarrow{R_3\leftarrow\frac{R_3}{(-3)}} 3\begin{vmatrix}4 & 3 & 1\\-1 & -2 & 0\\0 & 1 & 0\end{vmatrix}$$
$$= -3 \tag{1.16.9}$$

But, we will consider the absolute value of area only. So, **A2** $= |-3| = 3$.

or, **A2** $= \frac{1}{2}$(Area of $\triangle ABC$)

Case 3: When **CM** is median,we will consider $\triangle CAM$ triangle. In that case, the vertices will be **A**, **C** and **M**.

Now, the area of $\triangle CAM$ will be :

$$A3 = \frac{1}{2}\begin{vmatrix}5 & 2 & 1\\4 & -6 & 1\\3.5 & -4 & 1\end{vmatrix} \xleftrightarrow{C_2\leftarrow\frac{C_2}{2}} \frac{2}{2}\begin{vmatrix}5 & 1 & 1\\4 & -3 & 1\\3.5 & -2 & 1\end{vmatrix}$$
$$\xleftrightarrow[R_3\leftarrow R_3-R_1]{R_2\leftarrow R_2-R_1} \begin{vmatrix}5 & 1 & 1\\-1 & -4 & 0\\-1.5 & -3 & 0\end{vmatrix}$$
$$\xleftrightarrow[R_3\leftarrow\frac{R_3}{(-1.5)}]{R_2\leftarrow\frac{R_2}{(-1)}} 1.5\begin{vmatrix}5 & 1 & 1\\1 & 4 & 0\\1 & 2 & 0\end{vmatrix}$$
$$\xleftrightarrow{R_3\leftarrow R_3-R_2} 1.5\begin{vmatrix}5 & 1 & 1\\1 & 4 & 0\\0 & -2 & 0\end{vmatrix}$$
$$= -3 \tag{1.16.10}$$

But, we will consider the absolute value of area only. So, **A3** $= |-3| = 3$.

or, **A3** $= \frac{1}{2}$(Area of $\triangle ABC$)

Hence, the above problem statement is verified.

1.17. Let $\mathbf{A} = \begin{pmatrix}4\\2\end{pmatrix}, \mathbf{B} = \begin{pmatrix}6\\5\end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix}1\\4\end{pmatrix}$ be the vertices of $\triangle ABC$.

a) The median from **A** meets BC at **D**. Find the coordinates of the point **D**.
b) Find the coordinates of the point **P** on AD such that $AP : PD = 2 : 1$.
c) Find the coordinates of the points **Q** and **R** on medians BE and CF respectively such that $BQ : QE = 2 : 1$ and $CR : RF = 2 : 1$.

Solution:

a. Given $\triangle ABC$ with vertices

$$\mathbf{A} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 6 \\ 5 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \tag{1.17.1}$$

Given that the median from $\mathbf{A}$ meets BC at $\mathbf{D}$, now the coordinate of $\mathbf{D}$ is given as,

$$\mathbf{D} = \frac{\mathbf{B}+\mathbf{C}}{2} = \frac{\begin{pmatrix} 6 \\ 5 \end{pmatrix} + \begin{pmatrix} 1 \\ 4 \end{pmatrix}}{2} \tag{1.17.2}$$

$$\implies \mathbf{D} = \begin{pmatrix} \frac{7}{2} \\ \frac{9}{2} \end{pmatrix} \tag{1.17.3}$$

b. Result :The coordinates of point $\mathbf{C}$ dividing the line AB in the ratio $m : n$ is given by

$$\frac{n\mathbf{A} + m\mathbf{B}}{m+n} \tag{1.17.4}$$

Given that the point $\mathbf{P}$ divides AD in the ratio $2 : 1$, now to find $\mathbf{P}$ we use (1.17.4),

$$\mathbf{P} = \frac{1\begin{pmatrix} 4 \\ 2 \end{pmatrix} + 2\begin{pmatrix} \frac{7}{2} \\ \frac{9}{2} \end{pmatrix}}{3} = \begin{pmatrix} \frac{11}{3} \\ \frac{11}{3} \end{pmatrix} \tag{1.17.5}$$

c. Given that the point $\mathbf{Q}$ on the median BE divides it in the ratio $2 : 1$, first we find $\mathbf{E}$,

$$\mathbf{E} = \frac{\mathbf{A}+\mathbf{C}}{2} = \frac{\begin{pmatrix} 4 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 4 \end{pmatrix}}{2} \tag{1.17.6}$$

$$\implies \mathbf{E} = \begin{pmatrix} \frac{5}{2} \\ 3 \end{pmatrix}. \tag{1.17.7}$$

Now we find $\mathbf{Q}$ using (1.17.4)

$$\mathbf{Q} = \frac{1\begin{pmatrix} 6 \\ 5 \end{pmatrix} + 2\begin{pmatrix} \frac{5}{2} \\ 3 \end{pmatrix}}{3} = \begin{pmatrix} \frac{11}{3} \\ \frac{11}{3} \end{pmatrix} \tag{1.17.8}$$

Similarly,Given that the point $\mathbf{R}$ on the median CF divides it in the ratio $2 : 1$, first we find $\mathbf{F}$,

$$\mathbf{F} = \frac{\mathbf{A}+\mathbf{B}}{2} = \frac{\begin{pmatrix} 4 \\ 2 \end{pmatrix} + \begin{pmatrix} 6 \\ 5 \end{pmatrix}}{2} \tag{1.17.9}$$

$$\implies \mathbf{F} = \begin{pmatrix} 5 \\ \frac{7}{2} \end{pmatrix}. \tag{1.17.10}$$

Now we find $\mathbf{R}$ using (1.17.4)

$$\mathbf{R} = \frac{1\begin{pmatrix} 1 \\ 4 \end{pmatrix} + 2\begin{pmatrix} 5 \\ \frac{7}{2} \end{pmatrix}}{3} = \begin{pmatrix} \frac{11}{3} \\ \frac{11}{3} \end{pmatrix} \tag{1.17.11}$$

The plot of the $\triangle ABC$ is given in Fig. 1.17.

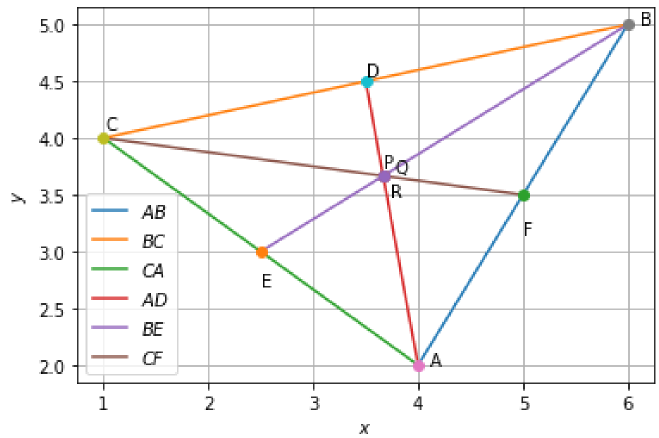

Fig. 1.17: Plot of $\triangle ABC$

1.18. Show that the points

$$\mathbf{A} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 3 \\ -4 \\ -4 \end{pmatrix} \tag{1.18.1}$$

are the vertices of a right angled triangle.
Solution:

$$\because \left(\mathbf{A}-\mathbf{C}\right)^\top \left(\mathbf{B}-\mathbf{C}\right) = \begin{pmatrix} -1 & 3 & 5 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} \tag{1.18.2}$$

$$= 0 \tag{1.18.3}$$

the triangle in Fig. 1.18 is right angled.

1.19. In $\triangle ABC$, $\mathbf{A} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$. Find $\angle B$.

Solution: Let

$$\mathbf{A} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}. \tag{1.19.1}$$

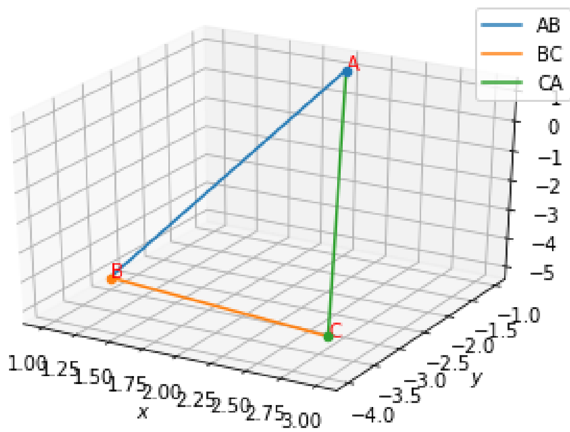

Fig. 1.18

Then,

$$\mathbf{A}-\mathbf{B} = \begin{pmatrix}2\\2\\3\end{pmatrix} \tag{1.19.2}$$

$$\mathbf{C}-\mathbf{B} = \begin{pmatrix}1\\1\\2\end{pmatrix} \tag{1.19.3}$$

Thus,

$$\mathbf{B} = \cos^{-1}\left(\frac{(\mathbf{A}-\mathbf{B})^T(\mathbf{C}-\mathbf{B})}{\|\mathbf{A}-\mathbf{B}\|\|\mathbf{C}-\mathbf{B}\|}\right) = \cos^{-1}\left(\frac{10}{\sqrt{17}\sqrt{6}}\right) \tag{1.19.4}$$

$$= 66.15 \tag{1.19.5}$$

See Fig. 1.19

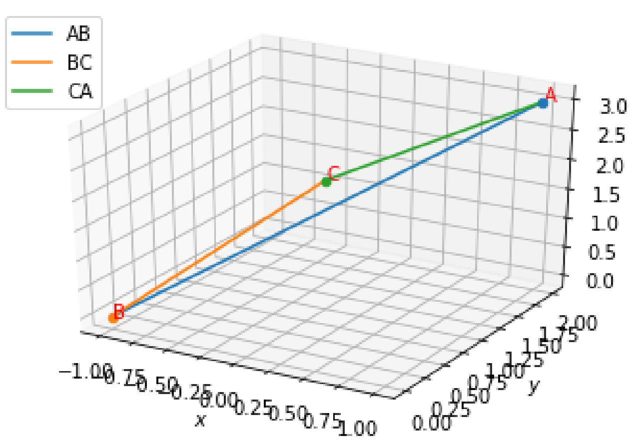

Fig. 1.19: $\triangle$ABC

1.20. Find the area of a triangle having the points $\mathbf{A} = \begin{pmatrix}1\\1\\1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}1\\2\\3\end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix}2\\3\\1\end{pmatrix}$ as its vertices.

Solution: From the given information,

$$\mathbf{B}-\mathbf{A} = \begin{pmatrix}1\\2\\3\end{pmatrix} - \begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}0\\1\\2\end{pmatrix} \tag{1.20.1}$$

$$\mathbf{C}-\mathbf{A} = \begin{pmatrix}2\\3\\1\end{pmatrix} - \begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\2\\0\end{pmatrix} \tag{1.20.2}$$

$$\tag{1.20.3}$$

The area of a triangle using the vector product is then obtained as

$$\frac{1}{2}\left\|\left(\mathbf{B}-\mathbf{A}\right)\times\left(\mathbf{C}-\mathbf{A}\right)\right\| \tag{1.20.4}$$

$$\frac{1}{2}\left\|\begin{pmatrix}0\\1\\2\end{pmatrix}\times\begin{pmatrix}1\\2\\0\end{pmatrix}\right\| \tag{1.20.5}$$

$$= 1 \tag{1.20.6}$$

1.21. Find the area of a triangle with vertices $\mathbf{A} = \begin{pmatrix}1\\1\\2\end{pmatrix}, \mathbf{B} = \begin{pmatrix}2\\3\\5\end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix}1\\5\\5\end{pmatrix}$

Solution: From the given information,

$$\mathbf{B}-\mathbf{A} = \begin{pmatrix}2\\3\\5\end{pmatrix} - \begin{pmatrix}1\\1\\2\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix} \tag{1.21.1}$$

$$\mathbf{C}-\mathbf{A} = \begin{pmatrix}1\\5\\5\end{pmatrix} - \begin{pmatrix}1\\1\\2\end{pmatrix} = \begin{pmatrix}0\\4\\3\end{pmatrix} \tag{1.21.2}$$

The area of a triangle using the vector product is then obtained as

$$\frac{1}{2}\left\|\left(\mathbf{B}-\mathbf{A}\right)\times\left(\mathbf{C}-\mathbf{A}\right)\right\| \tag{1.21.3}$$

$$= \frac{1}{2}\left\|\begin{pmatrix}1\\2\\3\end{pmatrix}\times\begin{pmatrix}0\\4\\3\end{pmatrix}\right\| \tag{1.21.4}$$

$$= \frac{17}{2} \tag{1.21.5}$$

1.22. Show that $\mathbf{A} = \begin{pmatrix}2\\3\\-4\end{pmatrix}, \mathbf{B} = \begin{pmatrix}1\\-2\\3\end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix}3\\8\\-11\end{pmatrix}$ are collinear.

Solution: Let

$$\mathbf{A} = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 3 \\ 8 \\ -11 \end{pmatrix} \tag{1.22.1}$$

Then

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} -1 \\ -5 \\ 7 \end{pmatrix}, \mathbf{C} - \mathbf{A} = \begin{pmatrix} 1 \\ 5 \\ -7 \end{pmatrix} \tag{1.22.2}$$

and

$$\mathbf{M} = \begin{pmatrix} B - A & C - A \end{pmatrix}^T \tag{1.22.3}$$

$$= \begin{pmatrix} -1 & -5 & 7 \\ 1 & 5 & -7 \end{pmatrix} \xleftrightarrow{R_1 \to -R_1} \begin{pmatrix} 1 & 5 & -7 \\ 1 & 5 & -7 \end{pmatrix} \tag{1.22.4}$$

$$\xleftrightarrow{R_2 \to R_2 - R_1} \begin{pmatrix} 1 & 5 & -7 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.22.5}$$

$$\implies \text{rank}\left(M\right) = 1 \tag{1.22.6}$$

Thus, the points are collinear as can be verified in Fig. 1.22.

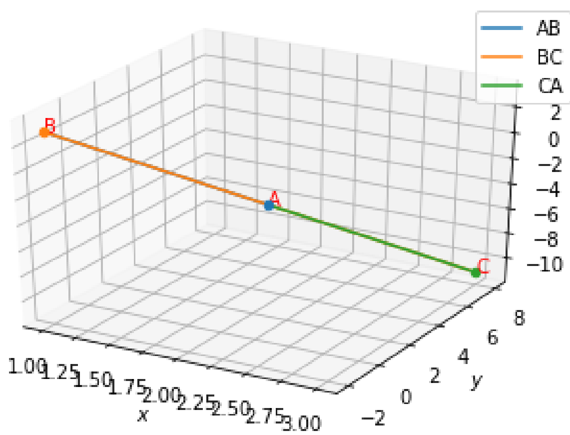

Fig. 1.22: collinear

1.23. Find the equation of set of points $\mathbf{P}$ such that

$$PA^2 + PB^2 = 2k^2, \tag{1.23.1}$$

$$\mathbf{A} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -1 \\ 3 \\ -7 \end{pmatrix}, \tag{1.23.2}$$

respectively. **Solution:** Let,

$$\mathbf{P} = \mathbf{X}; \tag{1.23.3}$$

so,

$$(\mathbf{PA})^2 = \|\mathbf{P} - \mathbf{A}\|^2 \tag{1.23.4}$$

$$= \|\mathbf{X} - \mathbf{A}\|^2 \tag{1.23.5}$$

$$= \|\mathbf{X}\|^2 + \|\mathbf{A}\|^2 - 2\mathbf{X}^T\mathbf{A} \tag{1.23.6}$$

and

$$(\mathbf{PB})^2 = \|\mathbf{P} - \mathbf{B}\|^2 \tag{1.23.7}$$

$$= \|\mathbf{X} - \mathbf{B}\|^2 \tag{1.23.8}$$

$$= \|\mathbf{X}\|^2 + \|\mathbf{B}\|^2 - 2\mathbf{X}^T\mathbf{B} \tag{1.23.9}$$

The given equation is

$$(\mathbf{PA})^2 + (\mathbf{PB})^2 = 2k^2 \tag{1.23.10}$$

Sub (1.23.6) and (1.23.9) values in (1.23.10)

$$\|\mathbf{X}\|^2 + \|\mathbf{A}\|^2 - 2\mathbf{X}^T\mathbf{A} + \|\mathbf{X}\|^2 + \|\mathbf{B}\|^2 - 2\mathbf{X}^T\mathbf{B} = 2k^2 \tag{1.23.11}$$

$$\implies 2\|\mathbf{X}\|^2 + \|\mathbf{A}\|^2 + \|\mathbf{B}\|^2 - 2\mathbf{X}^T(\mathbf{A} + \mathbf{B}) = 2k^2 \tag{1.23.12}$$

sub $\mathbf{A}$,$\mathbf{B}$ values in equation (1.23.12) ,we get

$$2\|\mathbf{X}\|^2 + \left\|\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}\right\|^2 + \left\|\begin{pmatrix} -1 \\ 3 \\ -7 \end{pmatrix}\right\|^2 - 2\mathbf{X}^T\left(\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \\ -7 \end{pmatrix}\right) = 2k^2 \tag{1.23.13}$$

$\therefore$ the required equation is

$$2\|\mathbf{X}\|^2 - 2\mathbf{X}^T\begin{pmatrix} 2 \\ 7 \\ -2 \end{pmatrix} + 109 - 2k^2 = 0 \tag{1.23.14}$$

1.24. Find the coordinates of a point which divides the line segment joining the points $\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix}$ in the ratio 2 : 3

a) internally, and
b) externally.

Solution:

a) The coordinates of point $\mathbf{P}$ dividing the line

AB in the ratio $m : n$ is given by

$$\mathbf{P} = \frac{m\mathbf{B} + n\mathbf{A}}{m + n} \tag{1.24.1}$$

$$= \frac{2\begin{pmatrix}3\\4\\-5\end{pmatrix} + 3\begin{pmatrix}1\\-2\\3\end{pmatrix}}{(2+3)} \tag{1.24.2}$$

$$= \begin{pmatrix}\frac{9}{5}\\\frac{2}{5}\\\frac{-1}{5}\end{pmatrix} \tag{1.24.3}$$

which is verified in Fig. 1.24

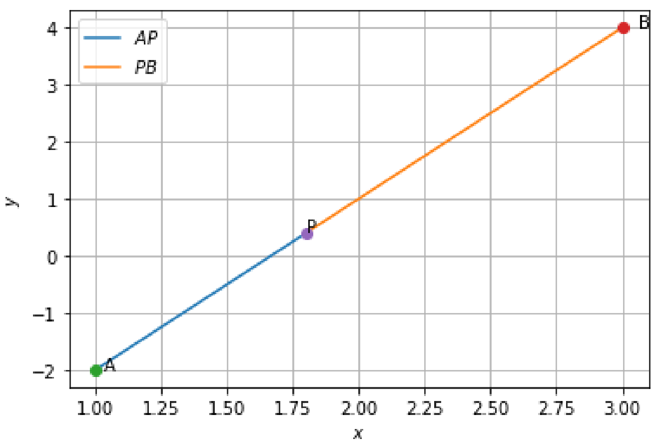

Fig. 1.24: INTERNALLY

b) The coordinates of point $\mathbf{Q}$ dividing the line AB in the ratio $m : n$ is given by

$$\mathbf{Q} = \frac{m\mathbf{B} - n\mathbf{A}}{m + n} \tag{1.24.4}$$

$$= \frac{2\begin{pmatrix}3\\4\\-5\end{pmatrix} - 3\begin{pmatrix}1\\-2\\3\end{pmatrix}}{(2-3)} \tag{1.24.5}$$

$$= \begin{pmatrix}-3\\-14\\19\end{pmatrix} \tag{1.24.6}$$

which is verified in Fig. 1.24

1.25. Prove that the three points $\begin{pmatrix}-4\\6\\10\end{pmatrix}$, $\begin{pmatrix}2\\4\\6\end{pmatrix}$ and $\begin{pmatrix}14\\0\\-2\end{pmatrix}$ are collinear.

Solution: Let

$$\mathbf{A} = \begin{pmatrix}-4\\6\\10\end{pmatrix}, \mathbf{B} = \begin{pmatrix}2\\4\\6\end{pmatrix}, \mathbf{C} = \begin{pmatrix}14\\0\\-2\end{pmatrix} \tag{1.25.1}$$

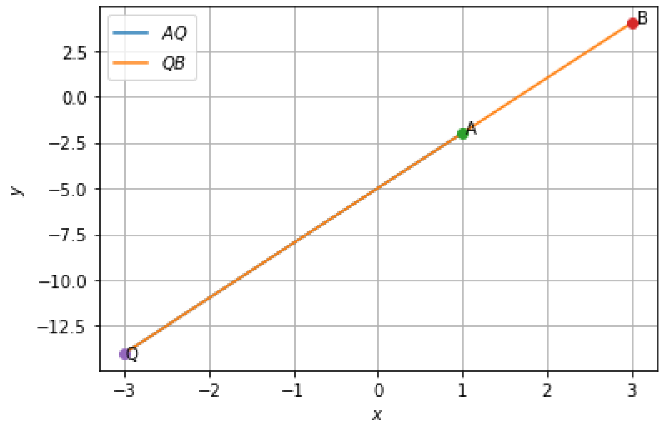

Fig. 1.24: EXTERNALLY

Then

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix}6\\-2\\-4\end{pmatrix}, \mathbf{C} - \mathbf{A} = \begin{pmatrix}18\\-6\\-12\end{pmatrix} \tag{1.25.2}$$

$$\implies \mathbf{M} = \begin{pmatrix}B - A & C - A\end{pmatrix}^T \tag{1.25.3}$$

$$= \begin{pmatrix}6 & -2 & -4\\18 & -6 & -12\end{pmatrix} \xleftrightarrow{R_2 \to R_2 - R_1} \begin{pmatrix}6 & -2 & -4\\12 & -4 & -8\end{pmatrix} \tag{1.25.4}$$

$$\xleftrightarrow{R_2 \to R_2 - 2R_1} \begin{pmatrix}6 & -2 & -4\\0 & 0 & 0\end{pmatrix} \tag{1.25.5}$$

$$\implies \text{rank}\left(M\right) = 1 \tag{1.25.6}$$

Thus, the points are collinear as can be seen in Fig. 1.25

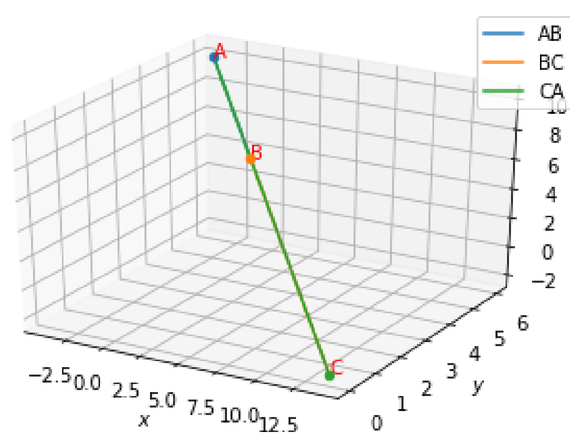

Fig. 1.25: collinear

1.26. Find the equation of the set of points $\mathbf{P}$ such that its distances from the points $\mathbf{A} = \begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$ are equal.

Solution:

a) From the given information,

$$||\mathbf{P} - \mathbf{A}||^2 = ||\mathbf{P} - \mathbf{B}||^2 \tag{1.26.1}$$

$$\implies ||\mathbf{P}||^2 + ||\mathbf{A}||^2 - 2\mathbf{A}^T\mathbf{P} \tag{1.26.2}$$

$$= ||\mathbf{P}||^2 + ||\mathbf{B}||^2 - 2\mathbf{B}^T\mathbf{P} \tag{1.26.3}$$

$$\implies 2\mathbf{A}^T\mathbf{P} - 2\mathbf{B}^T\mathbf{P} = ||\mathbf{A}||^2 - ||\mathbf{B}||^2 \tag{1.26.4}$$

b) Equation of plane is $\mathbf{n}^T\mathbf{P} = \mathbf{d}$
where,$\mathbf{n}^T$ is the normal vector to the plane

- From (1.26.4),

$$\left(2\mathbf{A}^T - 2\mathbf{B}^T\right)\mathbf{P} = ||\mathbf{A}||^2 - ||\mathbf{B}||^2 \tag{1.26.5}$$

$\mathbf{P}$ is a plane and it is perpendicular bisector to $\mathbf{A} - \mathbf{B}$
$\because$ $\mathbf{P}$ is perpendicular to line joining $\mathbf{A}$ and $\mathbf{B}$

- Midpoint of $\mathbf{A}$ and $\mathbf{B}$

$$\mathbf{M} = \frac{\mathbf{A} + \mathbf{B}}{2} \tag{1.26.6}$$

- Substitute in (1.26.5),

$$\left(2\mathbf{A}^T - 2\mathbf{B}^T\right)\left(\frac{\mathbf{A} + \mathbf{B}}{2}\right) = \left(\mathbf{A}^T - \mathbf{B}^T\right)\left(\mathbf{A} + \mathbf{B}\right) \tag{1.26.7}$$

$$= \mathbf{A}^T\mathbf{A} + \mathbf{A}^T\mathbf{B} - \mathbf{B}^T\mathbf{A} - \mathbf{B}^T\mathbf{B} \tag{1.26.8}$$

$$\because \mathbf{A}^T\mathbf{A} = ||\mathbf{A}||^2, \tag{1.26.9}$$

$$\mathbf{B}^T\mathbf{B} = ||\mathbf{B}||^2, \tag{1.26.10}$$

$$\mathbf{A}^T\mathbf{B} = \mathbf{B}^T\mathbf{A} \tag{1.26.11}$$

$$\implies \left(2\mathbf{A}^T - 2\mathbf{B}^T\right)\left(\frac{\mathbf{A} + \mathbf{B}}{2}\right) = ||\mathbf{A}||^2 - ||\mathbf{B}||^2 \tag{1.26.12}$$

$\implies \frac{\mathbf{A}+\mathbf{B}}{2}$ satisfies (1.26.4)

- $\therefore$ $\mathbf{P}$ is the plane that is perpendicular bisector of the line joining the given points

c) Putting given values $\mathbf{A}$ and $\mathbf{B}$ in (1.26.4),we get

$$2\begin{pmatrix} 3 & 4 & -5 \end{pmatrix}\mathbf{P} - 2\begin{pmatrix} -2 & 1 & 4 \end{pmatrix}\mathbf{P} \tag{1.26.13}$$

$$= \left\|\begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix}\right\|^2 - \left\|\begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}\right\|^2 \tag{1.26.14}$$

$$\implies \begin{pmatrix} 6 & 8 & -10 \end{pmatrix}\mathbf{P} + \begin{pmatrix} 4 & -2 & -8 \end{pmatrix}\mathbf{P} \tag{1.26.15}$$

$$= 50 - 21 \tag{1.26.16}$$

$$\implies \begin{pmatrix} 10 & 6 & -18 \end{pmatrix}\mathbf{P} = 29 \tag{1.26.17}$$

$\therefore$ The required equation is

$$\begin{pmatrix} 10 & 6 & -18 \end{pmatrix}\mathbf{P} = 29 \tag{1.26.18}$$

1.27. The line through the points $\begin{pmatrix} -2 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 8 \end{pmatrix}$ is perpendicular to the line through the points $\begin{pmatrix} 8 \\ 12 \end{pmatrix}$ and $\begin{pmatrix} x \\ 24 \end{pmatrix}$. Find the value of x.

Solution: Let

$$\mathbf{n_1} = \begin{pmatrix} 4 \\ 8 \end{pmatrix} - \begin{pmatrix} -2 \\ 6 \end{pmatrix} \tag{1.27.1}$$

$$= \begin{pmatrix} 6 \\ 2 \end{pmatrix} \tag{1.27.2}$$

and

$$\mathbf{n_2} = \begin{pmatrix} x \\ 24 \end{pmatrix} - \begin{pmatrix} 8 \\ 12 \end{pmatrix} \tag{1.27.3}$$

$$= \begin{pmatrix} x - 8 \\ 12 \end{pmatrix} \tag{1.27.4}$$

From the given information,

$$\mathbf{n_1}^\top\mathbf{n_2} = 0 \tag{1.27.5}$$

$$\implies \begin{pmatrix} 6 & 2 \end{pmatrix}\begin{pmatrix} x - 8 \\ 12 \end{pmatrix} = 0 \tag{1.27.6}$$

$$\text{or, } x = 4 \tag{1.27.7}$$

Fig. 1.27 verifies the result.

1.28. Show that the line joining the origin to the

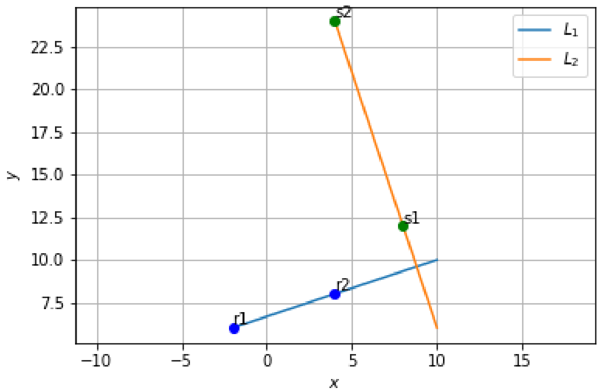

Fig. 1.27: Lines L_1 and L_2

point $\begin{pmatrix}2\\1\\1\end{pmatrix}$ is perpendicular to the line determined by the points $\begin{pmatrix}3\\5\\-1\end{pmatrix}, \begin{pmatrix}4\\3\\-1\end{pmatrix}$.

Solution: Let

$$\mathbf{O} = \begin{pmatrix}0\\0\\0\end{pmatrix}, \mathbf{P} = \begin{pmatrix}2\\1\\1\end{pmatrix}, \mathbf{A} = \begin{pmatrix}3\\5\\-1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}4\\3\\-1\end{pmatrix} \tag{1.28.1}$$

Then,

$$\mathbf{O} - \mathbf{P} = \begin{pmatrix}2\\1\\1\end{pmatrix} \tag{1.28.2}$$

$$= \begin{pmatrix}4\\3\\-1\end{pmatrix} - \begin{pmatrix}3\\5\\-1\end{pmatrix} \tag{1.28.3}$$

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix}1\\-2\\0\end{pmatrix} \tag{1.28.4}$$

and

$$(\mathbf{O} - \mathbf{P})^T (\mathbf{A} - \mathbf{B}) = 0 \tag{1.28.5}$$

$$\implies (\mathbf{O} - \mathbf{P}) \perp (\mathbf{A} - \mathbf{B}) \tag{1.28.6}$$

1.29. Are the points

$$\mathbf{A} = \begin{pmatrix}3\\6\\9\end{pmatrix}, \mathbf{B} = \begin{pmatrix}10\\20\\30\end{pmatrix}, \mathbf{C} = \begin{pmatrix}25\\-41\\5\end{pmatrix}, \tag{1.29.1}$$

the vertices of a right angled triangle?

Solution: Let

$$\mathbf{A} = \begin{pmatrix}3\\6\\9\end{pmatrix}, \mathbf{B} = \begin{pmatrix}10\\20\\30\end{pmatrix}, \mathbf{C} = \begin{pmatrix}25\\-41\\5\end{pmatrix} \tag{1.29.2}$$

$$(\mathbf{B} - \mathbf{A})^\top (\mathbf{C} - \mathbf{A}) = \begin{pmatrix}7 & 14 & 21\end{pmatrix} \begin{pmatrix}22\\-47\\-4\end{pmatrix} \tag{1.29.3}$$

$$= -521 \neq 0 \tag{1.29.4}$$

$$(\mathbf{A} - \mathbf{B})^\top (\mathbf{C} - \mathbf{B}) = \begin{pmatrix}-7 & -14 & -21\end{pmatrix} \begin{pmatrix}15\\-61\\-25\end{pmatrix} \tag{1.29.5}$$

$$= 1274 \neq 0 \tag{1.29.6}$$

$$(\mathbf{A} - \mathbf{B})^\top (\mathbf{C} - \mathbf{B}) = \begin{pmatrix}-7 & -14 & -21\end{pmatrix} \begin{pmatrix}15\\-61\\-25\end{pmatrix} \tag{1.29.7}$$

$$= 3397 \neq 0 \tag{1.29.8}$$

Hence, $\triangle ABC$ is not right angled as can be seen in Fig. 1.29.

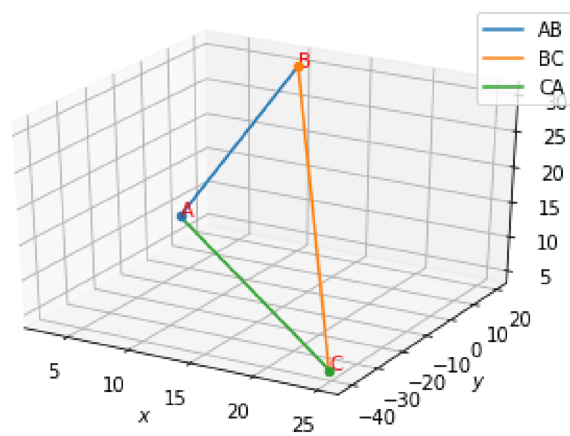

Fig. 1.29: Plot of the triangle

1.30. Find a condition on $\mathbf{x}$ such that the points $\mathbf{x}, \begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}7\\0\end{pmatrix}$ are collinear.

Solution: Let

$$\mathbf{A} = \begin{pmatrix}1\\2\end{pmatrix}, \mathbf{B} = \begin{pmatrix}7\\0\end{pmatrix} \tag{1.30.1}$$

The parametric equation of the line is

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m} \tag{1.30.2}$$

where

$$\mathbf{m} = \mathbf{B} - \mathbf{A} = \begin{pmatrix} 6 \\ -2 \end{pmatrix} \tag{1.30.3}$$

is the direction vector. Substituting values in (1.30.2)

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 6 \\ -2 \end{pmatrix} \tag{1.30.4}$$

1.31. Find a unit vector in the direction of the line passing through $\begin{pmatrix} -2 \\ 4 \\ -5 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

Solution: Given,

$$\mathbf{A} = \begin{pmatrix} -2 \\ 4 \\ -5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \tag{1.31.1}$$

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} -2 \\ 4 \\ -5 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 8 \end{pmatrix} \tag{1.31.2}$$

$$\implies \|\mathbf{A} - \mathbf{B}\| = \sqrt{77} \tag{1.31.3}$$

The unit vector is then calculated as

$$\frac{\mathbf{A} - \mathbf{B}}{\|\mathbf{A} - \mathbf{B}\|} = \frac{1}{\sqrt{77}} \begin{pmatrix} 3 \\ -2 \\ 8 \end{pmatrix} \tag{1.31.4}$$

1.32. Find a point on the y-axis which is equidistant from the points $\mathbf{A} = \begin{pmatrix} 6 \\ 5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$.

Solution: Let $\mathbf{x}$ be the point on y-axis.Then

$$\mathbf{x} = y \begin{pmatrix} 0 \\ 1 \end{pmatrix} = y\mathbf{e}_2 \tag{1.32.1}$$

and from the given information,

$$\|\mathbf{x} - \mathbf{A}\|^2 = \|\mathbf{x} - \mathbf{B}\|^2 \tag{1.32.2}$$

$$2\mathbf{x}(\mathbf{A} - \mathbf{B})^\top = \|\mathbf{A}\|^2 - \|\mathbf{B}\|^2 \tag{1.32.3}$$

$$\text{or, } 2y\mathbf{e}_2(\mathbf{A}^\top - \mathbf{B}^\top) = \|\mathbf{A}\|^2 - \|\mathbf{B}\|^2 \tag{1.32.4}$$

$$\implies \mathbf{y} = \frac{\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2}{2\mathbf{e}_2(\mathbf{A} - \mathbf{B})^\top} \tag{1.32.5}$$

$$= 9 \tag{1.32.6}$$

upon substituting numerical values. This is verified in Fig. 1.32

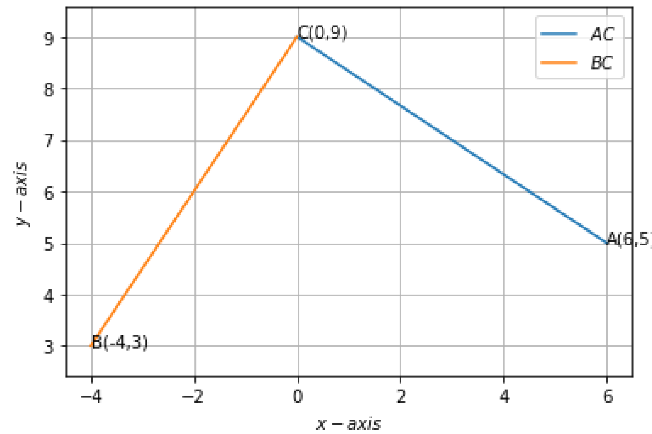

Fig. 1.32

1.33. Find the direction vectors of the sides of a triangle with vertices $\mathbf{A} = \begin{pmatrix} 3 \\ 5 \\ -4 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} -5 \\ -5 \\ -2 \end{pmatrix}$

Solution: The desired direction vectors are

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} -4 \\ -4 \\ 6 \end{pmatrix} \tag{1.33.1}$$

$$\mathbf{C} - \mathbf{B} = \begin{pmatrix} -4 \\ -6 \\ -4 \end{pmatrix} \tag{1.33.2}$$

$$\mathbf{A} - \mathbf{C} = \begin{pmatrix} 8 \\ 10 \\ -2 \end{pmatrix} \tag{1.33.3}$$

1.34. Show that the vectors $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix}, \begin{pmatrix} 3 \\ -4 \\ -4 \end{pmatrix}$ form the vertices of a right angled triangle.

Solution:

Let

$$\mathbf{A} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 3 \\ -4 \\ -4 \end{pmatrix} \tag{1.34.1}$$

$$(\mathbf{B}-\mathbf{A})^\top(\mathbf{C}-\mathbf{A}) = \begin{pmatrix}-1 & -2 & -6\end{pmatrix}\begin{pmatrix}1\\-3\\-5\end{pmatrix} \tag{1.34.2}$$

$$= 35 \neq 0 \tag{1.34.3}$$

$$(\mathbf{A}-\mathbf{B})^\top(\mathbf{C}-\mathbf{B}) = \begin{pmatrix}1 & 2 & 6\end{pmatrix}\begin{pmatrix}2\\-1\\1\end{pmatrix} \tag{1.34.4}$$

$$= 6 \neq 0 \tag{1.34.5}$$

$$(\mathbf{A}-\mathbf{C})^\top(\mathbf{B}-\mathbf{C}) = \begin{pmatrix}-1 & 3 & 5\end{pmatrix}\begin{pmatrix}-2\\1\\-1\end{pmatrix} \tag{1.34.6}$$

$$= 0 \tag{1.34.7}$$

Hence, $\triangle ABC$ is right angled at C as shown in Fig. 1.34.

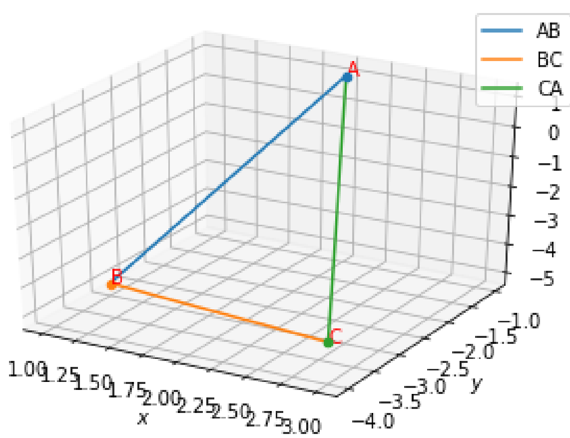

Fig. 1.34: Plot of the triangle

1.35. Find a unit vector in the direction of

$$\begin{pmatrix}1\\1\\-2\end{pmatrix}. \tag{1.35.1}$$

Solution: From the given info,

$$\|\mathbf{a}\| = \sqrt{(1)^2+(1)^2+(-2)^2}// \quad = \sqrt{6} \tag{1.35.2}$$

The unit vector is then calculated as

$$\frac{\mathbf{a}}{\|\mathbf{a}\|} = \frac{1}{\sqrt{6}}\begin{pmatrix}1\\1\\-2\end{pmatrix} \tag{1.35.3}$$

1.36. Find a unit vector in the direction of $\begin{pmatrix}2\\-1\\-2\end{pmatrix}$.

Solution: Let $\mathbf{U}$ be the unit vector in the direction of given vector and

$$\mathbf{V} = \begin{pmatrix}2\\-1\\-2\end{pmatrix} \tag{1.36.1}$$

Then

$$\|\mathbf{V}\| = \sqrt{(2)^2+(-1)^2+(-2)^2} \tag{1.36.2}$$

$$\implies \|\mathbf{V}\| = 3 \tag{1.36.3}$$

and

$$\mathbf{U} = \frac{\mathbf{V}}{\|\mathbf{V}\|} = \frac{1}{3}\begin{pmatrix}2\\-1\\-2\end{pmatrix} \tag{1.36.4}$$

1.37. Show that the points $\mathbf{A} = \begin{pmatrix}1\\7\end{pmatrix}, \mathbf{B} = \begin{pmatrix}4\\2\end{pmatrix}, \mathbf{C} = \begin{pmatrix}-1\\-1\end{pmatrix}, \mathbf{D} = \begin{pmatrix}-4\\4\end{pmatrix}$ are the vertices of a square.
Solution: By inspection,

$$\frac{\mathbf{A}+\mathbf{C}}{2} = \frac{\mathbf{B}+\mathbf{D}}{2} = \begin{pmatrix}0\\3\end{pmatrix} \tag{1.37.1}$$

Hence, the diagonals AC and BD bisect each other. Also,

$$(\mathbf{A}-\mathbf{C})^T(\mathbf{B}-\mathbf{D}) = 0 \tag{1.37.2}$$

$\implies AC \perp BD$. Hence $ABCD$ is a square.

1.38. If the points $\mathbf{A} = \begin{pmatrix}6\\1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}8\\2\end{pmatrix}, \mathbf{C} = \begin{pmatrix}9\\4\end{pmatrix}, \mathbf{D} = \begin{pmatrix}p\\3\end{pmatrix}$ are the vertices of a parallelogram, taken in order, find the value of p.
Solution: In the parallelogram $ABCD$, AC and BD bisect each other. This can be used to find p.

1.39. If $\mathbf{A} = \begin{pmatrix}-5\\7\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-4\\-5\end{pmatrix}, \mathbf{C} = \begin{pmatrix}-1\\-6\end{pmatrix}, \mathbf{D} = \begin{pmatrix}4\\5\end{pmatrix}$, find the area of the quadrilateral $ABCD$.
Solution: The area of $ABCD$ is the sum of the areas of trianges ABD and CBD and is given by

$$\frac{1}{2}\|(\mathbf{A}-\mathbf{B})\times(\mathbf{A}-\mathbf{D})\| + \frac{1}{2}\|(\mathbf{C}-\mathbf{B})\times(\mathbf{C}-\mathbf{D})\| \tag{1.39.1}$$

1.40. Show that the points $\mathbf{A} = \begin{pmatrix}1\\2\\3\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-1\\-2\\-1\end{pmatrix}, \mathbf{C} = \begin{pmatrix}2\\3\\2\end{pmatrix}, \mathbf{D} = \begin{pmatrix}4\\7\\6\end{pmatrix}$. are the vertices of a parallelogram $ABCD$ but it is not a rectangle.
Solution: Since the direction vectors

$$\mathbf{A} - \mathbf{B} = \mathbf{D} - \mathbf{C} \tag{1.40.1}$$
$$\mathbf{A} - \mathbf{D} = \mathbf{B} - \mathbf{C} \tag{1.40.2}$$

$AB \parallel CD$ and $AD \parallel BC$. Hence $ABCD$ is a parallelogram. However,

$$(\mathbf{A} - \mathbf{B})^T (\mathbf{A} - \mathbf{D}) \neq 0 \tag{1.40.3}$$

Hence, it is not a rectangle. The following code plots Fig. 1.40

```
codes/triangle/quad_3d.py
```

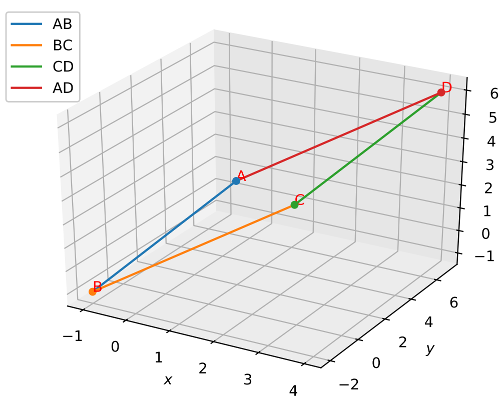

Fig. 1.40

1.41. Find the area of a parallelogram whose adjacent sides are given by the vectors $\begin{pmatrix}3\\1\\4\end{pmatrix}$ and $\begin{pmatrix}1\\-1\\1\end{pmatrix}$.
Solution: The area is given by

$$\frac{1}{2}\left\|\begin{pmatrix}3\\1\\4\end{pmatrix} \times \begin{pmatrix}1\\-1\\1\end{pmatrix}\right\| \tag{1.41.1}$$

1.42. $ABCD$ is a rectangle formed by the points $\mathbf{A} = \begin{pmatrix}-1\\-1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-1\\4\end{pmatrix}, \mathbf{C} = \begin{pmatrix}5\\4\end{pmatrix}, \mathbf{D} = \begin{pmatrix}5\\-1\end{pmatrix}$. $\mathbf{P}, \mathbf{Q}, \mathbf{R}, \mathbf{S}$ are the mid points of AB, BC, CD, DA respectively. Is the quadrilateral $PQRS$ a

a) square?
b) rectangle?
c) rhombus?

Solution:

$$\begin{aligned}\mathbf{P} &= \frac{\mathbf{A}+\mathbf{B}}{2} = \begin{pmatrix}-1 & \frac{3}{2}\end{pmatrix}\\ \mathbf{Q} &= \frac{\mathbf{B}+\mathbf{C}}{2} = \begin{pmatrix}2 & 4\end{pmatrix}\\ \mathbf{R} &= \frac{\mathbf{C}+\mathbf{D}}{2} = \begin{pmatrix}5 & \frac{3}{2}\end{pmatrix}\\ \mathbf{S} &= \frac{\mathbf{A}+\mathbf{D}}{2} = \begin{pmatrix}2 & -1\end{pmatrix}\end{aligned} \tag{1.42.1}$$

$\because$

$$\frac{\mathbf{P}+\mathbf{R}}{2} = \frac{\mathbf{Q}+\mathbf{S}}{2} = \frac{1}{2}\begin{pmatrix}4\\3\end{pmatrix} \tag{1.42.2}$$

PQRS is a parallelogram.

$$(\mathbf{P}-\mathbf{R}) = \begin{pmatrix}-6 & 0\end{pmatrix} (\mathbf{Q}-\mathbf{S}) = \begin{pmatrix}0\\5\end{pmatrix} \tag{1.42.3}$$
$$\tag{1.42.4}$$

$$(\mathbf{P}-\mathbf{R})^T(\mathbf{Q}-\mathbf{S}) = \begin{pmatrix}-6 & 0\end{pmatrix}\begin{pmatrix}0\\5\end{pmatrix} \tag{1.42.5}$$
$$(\mathbf{P}-\mathbf{R})^T(\mathbf{Q}-\mathbf{S}) = \begin{pmatrix}0\end{pmatrix} \tag{1.42.6}$$
$$\tag{1.42.7}$$

Diagonal bisect orthogonally. Thus, PQRS is a rhombus. Se Fig. 1.42
Step4: We will check whether Parallelogram PQRS is Square or not.

$$(\mathbf{P}-\mathbf{Q}) = \frac{1}{2}\begin{pmatrix}-6\\-5\end{pmatrix} \tag{1.42.8}$$
$$(\mathbf{P}-\mathbf{S}) = \frac{1}{2}\begin{pmatrix}-6\\5\end{pmatrix} \tag{1.42.9}$$
$$\tag{1.42.10}$$

If adjacent side of parallelogram are orthogonal to each other then PQRS is a Square.

$$(\mathbf{P}-\mathbf{Q})^T(\mathbf{P}-\mathbf{S}) = \frac{1}{4}\begin{pmatrix}-6 & -5\end{pmatrix}\begin{pmatrix}-6\\5\end{pmatrix} \neq= 0 \tag{1.42.11}$$

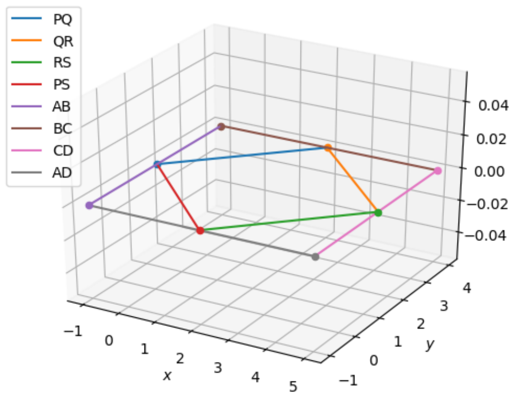

Fig. 1.42: Simulation of midpoint of ABCD forms PQRS.

Here the angle between adjacent side is not 90 °. Hence, PQRS is not a Square.

1.43. $ABCD$ is a cyclic quadrilateral with

$$\angle A = 4y + 20 \tag{1.43.1}$$
$$\angle B = 3y - 5 \tag{1.43.2}$$
$$\angle C = -4x \tag{1.43.3}$$
$$\angle D = -7x + 5 \tag{1.43.4}$$

Find its angles.
Solution: From the given information,

$$\angle A + \angle C = 180^\circ \tag{1.43.5}$$
$$\angle B + \angle D = 180^\circ \tag{1.43.6}$$

which can be expressed as

$$\begin{pmatrix} -4 & 4 \\ -7 & 3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 160 \\ 180 \end{pmatrix} \tag{1.43.7}$$

and solved as

$$\begin{pmatrix} -4 & 4 & 160 \\ -7 & 3 & 180 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{-R_1}{4}} \begin{pmatrix} 1 & -1 & -40 \\ -7 & 3 & 180 \end{pmatrix} \tag{1.43.8}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 + 7R_1} \begin{pmatrix} 1 & -1 & -40 \\ 0 & -4 & -100 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow \frac{-R_2}{4}} \begin{pmatrix} 1 & -1 & -40 \\ 0 & 1 & 25 \end{pmatrix} \tag{1.43.9}$$

$$\xleftrightarrow{R_1 \leftarrow \ R_1 + R_2} \begin{pmatrix} 1 & 0 & -15 \\ 0 & 1 & 25 \end{pmatrix} \tag{1.43.10}$$

Thus,

$$x = -15, y = 25 \tag{1.43.11}$$
$$\implies \angle A = 120^\circ, \angle B = 70^\circ, \tag{1.43.12}$$
$$\implies \angle C = 60^\circ, \angle D = 110^\circ \tag{1.43.13}$$

1.44. Draw a quadrilateral in the Cartesian plane, whose vertices are $\begin{pmatrix} -4 \\ 5 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 7 \end{pmatrix}$, $\begin{pmatrix} 5 \\ -5 \end{pmatrix}$ and $\begin{pmatrix} -4 \\ -2 \end{pmatrix}$. Also, find its area.
Solution: Let

$$\mathbf{A} = \begin{pmatrix} -4 \\ 5 \end{pmatrix} \mathbf{B} = \begin{pmatrix} 0 \\ 7 \end{pmatrix} \mathbf{C} = \begin{pmatrix} 5 \\ -5 \end{pmatrix} \mathbf{D} = \begin{pmatrix} -4 \\ -2 \end{pmatrix} \tag{1.44.1}$$

Quadrilateral ABCD is drawn by joining its vertices **A** and **B**,**B** and **C**, **C** and **D**, **D** and **A**. The following Python code generates Fig. 1.44

```
codes/quad/quad.py
```

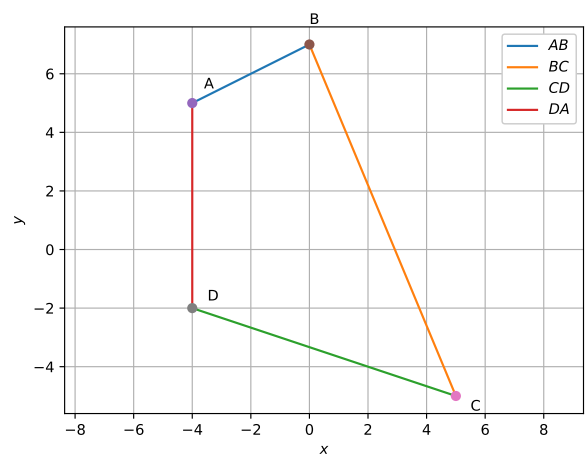

Fig. 1.44: Quadrilateral ABCD

From Figure 1.44 Area of the Quadrilateral ABCD can be given as

$$Ar\left(\triangle ABC\right) + Ar\left(\triangle BCD\right) \tag{1.44.2}$$

$$\frac{1}{2}\left\|(\mathbf{A}-\mathbf{B})\times(\mathbf{A}-\mathbf{D})\right\| + \frac{1}{2}\left\|(\mathbf{C}-\mathbf{B})\times(\mathbf{C}-\mathbf{D})\right\| \tag{1.44.3}$$

For two vectors $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

$$\|\mathbf{a} \times \mathbf{b}\| = |a_1 b_2 - a_2 b_1| \tag{1.44.4}$$

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} -4 \\ -2 \end{pmatrix} \tag{1.44.5}$$

$$\mathbf{A} - \mathbf{D} = \begin{pmatrix} 0 \\ 7 \end{pmatrix} \tag{1.44.6}$$

$$\mathbf{C} - \mathbf{B} = \begin{pmatrix} 5 \\ -12 \end{pmatrix} \tag{1.44.7}$$

$$\mathbf{C} - \mathbf{D} = \begin{pmatrix} 9 \\ -3 \end{pmatrix} \tag{1.44.8}$$

Using (1.44.4)

$$\frac{1}{2}\|(\mathbf{A} - \mathbf{B}) \times (\mathbf{A} - \mathbf{D})\| = \frac{1}{2}|(-28)| \tag{1.44.9}$$

$$= 14 \tag{1.44.10}$$

$$\frac{1}{2}\|(\mathbf{C} - \mathbf{B}) \times (\mathbf{C} - \mathbf{D})\| = \frac{1}{2}|(-15 + 108)| \tag{1.44.11}$$

$$= 46.5 \tag{1.44.12}$$

Substituting the above values in equation (1.44.3),We get

$$Area = 14 + 46.5 = 60.5 sq.units \tag{1.44.13}$$

1.45. Find the area of a rhombus if its vertices are

$$\mathbf{P} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}, \tag{1.45.1}$$

$$\mathbf{R} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}, \mathbf{S} = \begin{pmatrix} -2 \\ -1 \end{pmatrix} \tag{1.45.2}$$

taken in order.
Solution: In Fig. 1.45,

$$\mathbf{P} - \mathbf{S} = \begin{pmatrix} 3+2 \\ 0+1 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix} \tag{1.45.3}$$

$$\mathbf{Q} - \mathbf{P} = \begin{pmatrix} 4-3 \\ 5-0 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \tag{1.45.4}$$

Thus, the area of the rhombus can be calculated as

$$\|(\mathbf{P} - \mathbf{S}) \times (\mathbf{Q} - \mathbf{P})\| = \left\|\begin{pmatrix} 5 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 5 \end{pmatrix}\right\| \tag{1.45.5}$$

$$\|\Delta\| = 5 \times 5 - 1 \times 1 = 24 \tag{1.45.6}$$

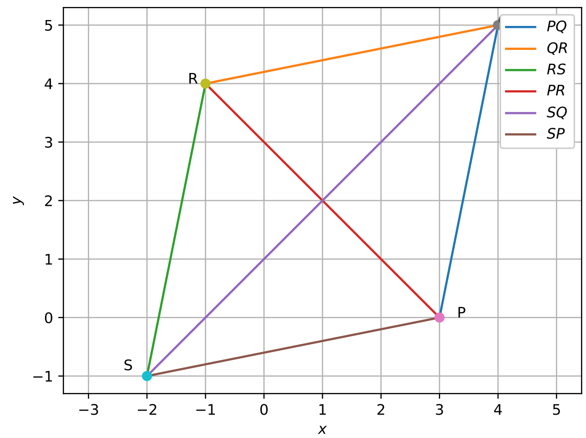

Fig. 1.45

```
solutions/4/codes/quadrilateral/quad.py
```

1.46. Without using distance formula, show that points $\begin{pmatrix} -2 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 4 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$ are the vertices of a parallelogram.
Solution: The following python code plots Fig.1.46.

```
./solutions/5/codes/quadrilateral/q4.py
```

$$\because \mathbf{A} - \mathbf{B} = \mathbf{D} - \mathbf{C} \tag{1.46.1}$$

$$\mathbf{A} - \mathbf{D} = \mathbf{B} - \mathbf{C}, \tag{1.46.2}$$

$AB \parallel CD$ and $AD \parallel BC$. Hence, $ABCD$ is a ||gm.

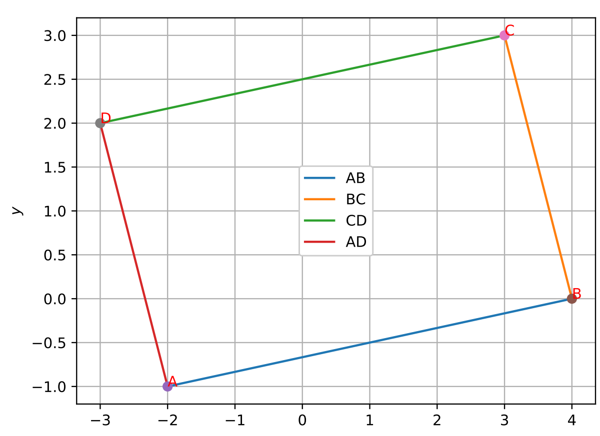

Fig. 1.46

1.47. Find the area of the quadrilateral whose vertices, taken in order, are $\begin{pmatrix}-4\\2\end{pmatrix}, \begin{pmatrix}-3\\-5\end{pmatrix}, \begin{pmatrix}3\\-2\end{pmatrix}, \begin{pmatrix}2\\3\end{pmatrix}$.
Solution: See quadrilateral $ABCD$ in Fig.1.47 is generated using the following python code

```
solutions/6/codes/quadrilateral/quad.py
```

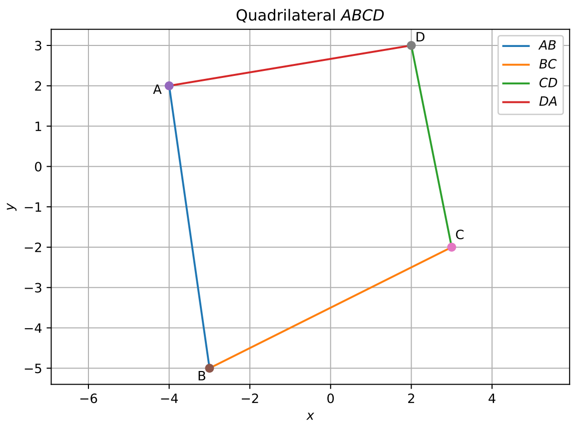

Fig. 1.47: Quadrilateral $ABCD$ using python

$$ar(ABCD) = ar(\triangle ABC) + ar(\triangle ACD) \tag{1.47.1}$$
$$= \frac{1}{2}\left\|(\mathbf{B}-\mathbf{A})\times(\mathbf{C}-\mathbf{A})\right\| \tag{1.47.2}$$
$$+ \frac{1}{2}\left\|(\mathbf{C}-\mathbf{A})\times(\mathbf{D}-\mathbf{A})\right\| \tag{1.47.3}$$
$$= \frac{1}{2}\left\|\begin{pmatrix}1\\-7\end{pmatrix}\times\begin{pmatrix}7\\-4\end{pmatrix}\right\| \tag{1.47.4}$$
$$+ \frac{1}{2}\left\|\begin{pmatrix}7\\-4\end{pmatrix}\times\begin{pmatrix}6\\1\end{pmatrix}\right\| \tag{1.47.5}$$
$$= 38 \tag{1.47.6}$$

and verified using the following codes

```
solutions/6/codes/tri_area_ABC.py
```

```
solutions/6/codes/tri_area_ACD.py
```

1.48. The two opposite vertices of a square are $\begin{pmatrix}-1\\2\end{pmatrix}$, $\begin{pmatrix}3\\2\end{pmatrix}$. Find the coordinates of the other two vertices.
Solution: See Fig. 1.48.

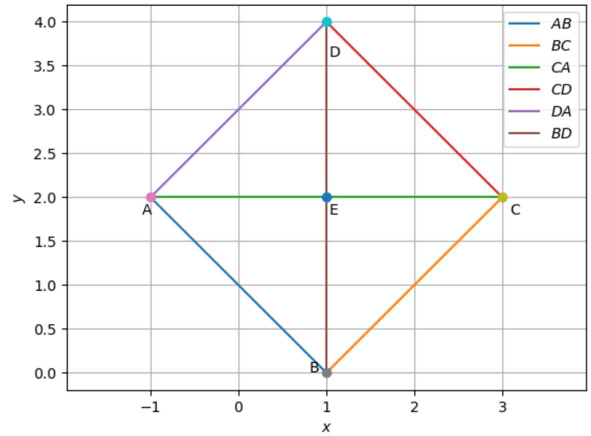

Fig. 1.48: Square ABCD

a) From inspection we see that the opposite vertices forms a diagonal which is parallel to x-axis. Then the diagonal formed by other two vertices is parallel to y-axis(i.e. their x coordinates are equal). Let $\mathbf{A} = \begin{pmatrix}-1\\2\end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix}3\\2\end{pmatrix}$.
b) Diagonals bisect each other at 90°. Let $\mathbf{B}$ and $\mathbf{D}$ be other two vertices.
c) Using the property that diagonals bisect each other at 90°, we can obtain other vertices by rotating diagonal AC by 90°about $\mathbf{E}$ in clockwise or anticlockwise direction.
d) The rotation matrix for a rotation of angle 90°about origin in anticlockwise direction is given by

$$\begin{pmatrix}\cos 90^\circ & -\sin 90^\circ\\ \sin 90^\circ & \cos 90^\circ\end{pmatrix} = \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix} \tag{1.48.1}$$

The $\mathbf{E}$ is given by

$$\mathbf{E} = \frac{\mathbf{A}+\mathbf{C}}{2} \tag{1.48.2}$$
$$= \begin{pmatrix}1\\2\end{pmatrix} \tag{1.48.3}$$

e) To make the rotation we need to shift the $\mathbf{E}$

to origin. So the change in other vectors are

$$\mathbf{A} - \mathbf{E} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \tag{1.48.4}$$

$$\mathbf{C} - \mathbf{E} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \tag{1.48.5}$$

The required matrix now is $\begin{pmatrix} -2 & 2 \\ 0 & 0 \end{pmatrix}$. Multiplying this with rotation matrix

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -2 & 2 \\ 0 & 0 \end{pmatrix} \tag{1.48.6}$$

$$= \begin{pmatrix} 0 & 0 \\ -2 & 2 \end{pmatrix} \tag{1.48.7}$$

Now we obtained the coordinates as $\begin{pmatrix} 0 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$. To obtain the final coordinates we will add $\mathbf{E}$ to shift to the actual position.

$$\mathbf{B} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{1.48.8}$$

$$\mathbf{D} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{1.48.9}$$

Thus

$$\mathbf{B} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.48.10}$$

$$\mathbf{D} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \tag{1.48.11}$$

f) The python code for the figure can be downloaded from

```
solutions/7/codes/quad/quad.py
```

1.49. Find the area of a parallelogram whose adjacent sides are given by the vectors $\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

Solution: The area of a parallelogram is defined as

$$\|\mathbf{a} \times \mathbf{b}\| \tag{1.49.1}$$

where

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \tag{1.49.2}$$

$$= \begin{pmatrix} 0 & -4 & 1 \\ 4 & 0 & -3 \\ -1 & 3 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ -4 \end{pmatrix} \tag{1.49.3}$$

Thus, the desired area is

$$\|\mathbf{a} \times \mathbf{b}\| = \sqrt{5^2 + 1^2 + (-1)^2} \tag{1.49.4}$$

$$= 3\sqrt{3} \tag{1.49.5}$$

The following Python code generates Fig. 1.49

```
codes/parallelogram.py
```

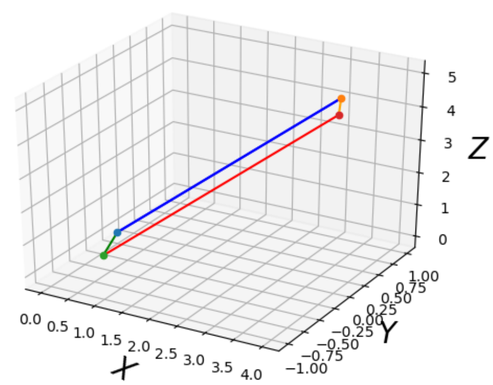

Fig. 1.49: Parallelogram generated using python 3D-plot

The following Python code verifies the cross-product value.

```
codes/cross_product_check.py
```

1.50. Find the area of a rectangle $ABCD$ with vertices $\mathbf{A} = \begin{pmatrix} -1 \\ \frac{1}{2} \\ 4 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ \frac{1}{2} \\ 4 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ 4 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} -1 \\ -\frac{1}{2} \\ 4 \end{pmatrix}$.

Solution: Area of rectangle = cross product of vectors of adjacent sides

$$\mathbf{A} - \mathbf{D} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \mathbf{B} - \mathbf{A} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \tag{1.50.1}$$

Area = cross product of vectors

$$\|(\mathbf{A}-\mathbf{D})\times(\mathbf{B}-\mathbf{A})\| \tag{1.50.2}$$

$$= \left\| \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \right\| \tag{1.50.3}$$

$$= \left\| \begin{pmatrix} 0 & -0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \right\| \tag{1.50.4}$$

$$= 2 \tag{1.50.5}$$

Area = 2

1.51. The two adjacent sides of a parallelogram are $\begin{pmatrix} 2 \\ -4 \\ -5 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}$. Find the unit vector parallel to its diagonal. Also, find its area.

Solution:

Let

$$\mathbf{A} = \begin{pmatrix} 2 \\ -4 \\ -5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} \tag{1.51.1}$$

be the adjacent sides of the parallelogram. Let $\mathbf{D}$ be the diagonal of the parallelogram. Then,

$$\mathbf{D} = \mathbf{A} + \mathbf{B} \tag{1.51.2}$$

$$= \begin{pmatrix} 3 \\ -6 \\ -8 \end{pmatrix} \tag{1.51.3}$$

$$\|\mathbf{D}\| = \sqrt{(3)^2 + (-6)^2 + (-8)^2} = \sqrt{109} \tag{1.51.4}$$

Let $\mathbf{U}$ be the unit vector of $\mathbf{D}$ which can be found as follows:

$$\mathbf{U} = \frac{\mathbf{D}}{\|\mathbf{D}\|} \tag{1.51.5}$$

Solving the above equation gives the unit vector $\mathbf{U}$ which is parallel to the diagonal $\mathbf{D}$.

$$\therefore \boxed{\mathbf{U} = \frac{1}{\sqrt{109}} \begin{pmatrix} 3 \\ -6 \\ -8 \end{pmatrix}} \tag{1.51.6}$$

$$\because \mathbf{A}\times\mathbf{B} = \begin{pmatrix} 0 & -A_3 & A_2 \\ A_3 & 0 & -A_1 \\ -A_2 & A_1 & 0 \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B3 \end{pmatrix} \tag{1.51.7}$$

$$= \begin{pmatrix} 0 & 5 & -4 \\ -5 & 0 & -2 \\ 4 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \tag{1.51.8}$$

$$\|\mathbf{A}\times\mathbf{B}\| = \sqrt{(-2)^2 + (1)^2 + (0)^2} \tag{1.51.9}$$

$$= \sqrt{5} \tag{1.51.10}$$

which is the desired area.

1.52. A town B is located 36km east and 15 km north of the town A. How would you find the distance from town A to town B without actually measuring it?

Solution: See Fig. 1.52.

$$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 36 \\ 15 \end{pmatrix} \tag{1.52.1}$$

The distance d between A and B is given by

$$\|\mathbf{B}-\mathbf{A}\| = \|\mathbf{B}\| \tag{1.52.2}$$

$$= 39km \tag{1.52.3}$$

The following Python code generates Fig. 1.52.

```
solutions/3/codes/line/towns/towns.py
```

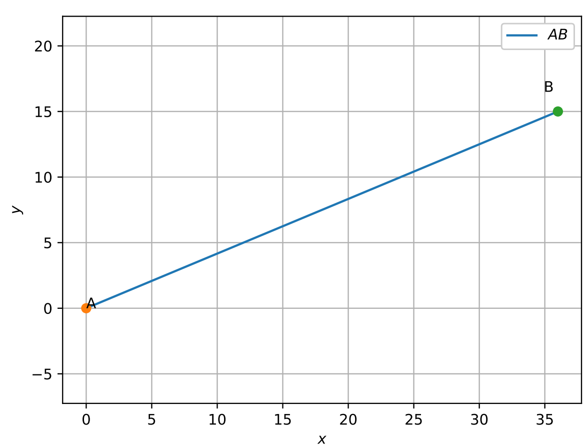

Fig. 1.52: Position of Towns A and B

1.53. Find the angle between the x-axis and the line joining the points $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ -2 \end{pmatrix}$. **Solution:**

$$\frac{(\mathbf{A}-\mathbf{B})^T\begin{pmatrix}1\\0\end{pmatrix}}{\|\mathbf{A}-\mathbf{B}\|\left\|\begin{pmatrix}1\\0\end{pmatrix}\right\|} = \frac{\begin{pmatrix}-1 & 1\end{pmatrix}^T\begin{pmatrix}1\\0\end{pmatrix}}{\left\|\begin{pmatrix}-1\\1\end{pmatrix}\right\|\left\|\begin{pmatrix}-1\\1\end{pmatrix}\right\|} \tag{1.53.1}$$

$$= -\frac{1}{\sqrt{2}} = \cos^{-1}(135°) \tag{1.53.2}$$

Thus, the desired angle is 135°. The following python code generates Fig. 1.53.

```
./solutions/5/codes/lines/q9.py
```

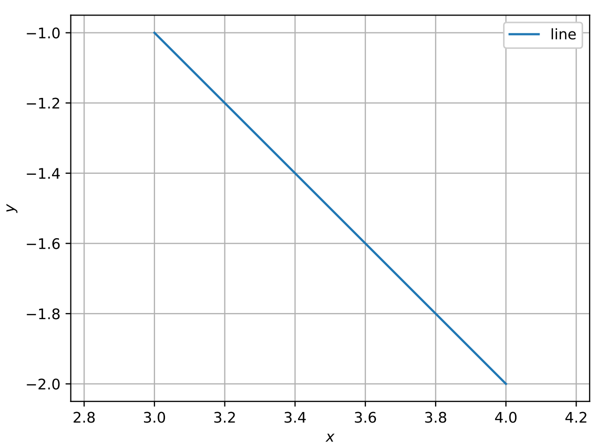

Fig. 1.53

1.54. Find the point on the x-axis which is equidistant from

$$\begin{pmatrix}2\\-5\end{pmatrix}, \begin{pmatrix}-2\\9\end{pmatrix}, \tag{1.54.1}$$

Solution: From the given information,

$$\left\|\mathbf{x} - \begin{pmatrix}2\\-5\end{pmatrix}\right\|^2 = \left\|\mathbf{x} - \begin{pmatrix}-2\\9\end{pmatrix}\right\|^2 \tag{1.54.2}$$

$$\implies \|\mathbf{x}\|^2 + \left\|\begin{pmatrix}2\\-5\end{pmatrix}\right\|^2 - 2\begin{pmatrix}2 & -5\end{pmatrix}\mathbf{x}$$

$$= \|\mathbf{x}\|^2 + \left\|\begin{pmatrix}-2\\9\end{pmatrix}\right\|^2 - 2\begin{pmatrix}-2 & 9\end{pmatrix}\mathbf{x} \tag{1.54.3}$$

which can be simplified to obtain

$$\begin{pmatrix}8 & -28\end{pmatrix}\mathbf{x} = -56 \tag{1.54.4}$$

Choose $\mathbf{x} = \begin{pmatrix}x\\0\end{pmatrix}$ as the point lies on the x-axis

$$\begin{pmatrix}8 & -28\end{pmatrix}\begin{pmatrix}x\\0\end{pmatrix} = -56 \tag{1.54.5}$$

$$\implies x = -7 \tag{1.54.6}$$

The desired point is $\begin{pmatrix}-7\\0\end{pmatrix}$.

See Fig. 1.54 generated by the following python code

```
solutions/6/codes/line/point_vector/
    point_vector.py
```

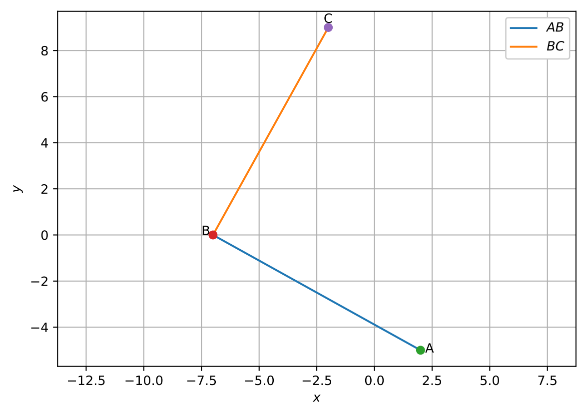

Fig. 1.54

1.55. Find the values of y for which the distance between the points

$$\mathbf{P} = \begin{pmatrix}2\\-3\end{pmatrix}, \mathbf{Q} = \begin{pmatrix}10\\y\end{pmatrix} \tag{1.55.1}$$

is 10 units. **Solution:** The distance between two points is given by equation

$$(\mathbf{P}-\mathbf{Q})^T(\mathbf{P}-\mathbf{Q}) = 10^2 \tag{1.55.2}$$

$$\implies \|P\|^2 - \mathbf{P}^T\mathbf{Q} - \mathbf{Q}^T\mathbf{P} + \|Q\|^2 = 100 \tag{1.55.3}$$

which, upon subsituting the values yields

$$y^2 + 6y - 27 = 0 \tag{1.55.4}$$

$$(y+9)(y-3) = 0 \implies y = -9, 3 \tag{1.55.5}$$

and

$$\mathbf{Q} = \begin{pmatrix}10\\3\end{pmatrix}, \begin{pmatrix}10\\-9\end{pmatrix} \tag{1.55.6}$$

The python code to find the roots of the quadratic equation can be downloaded from

```
solutions/7/codes/line/point_vec/roots.py
```

The python code for Fig. 1.55 can be downloaded from

```
solutions/7/codes/line/point_vec/point_vec.py
```

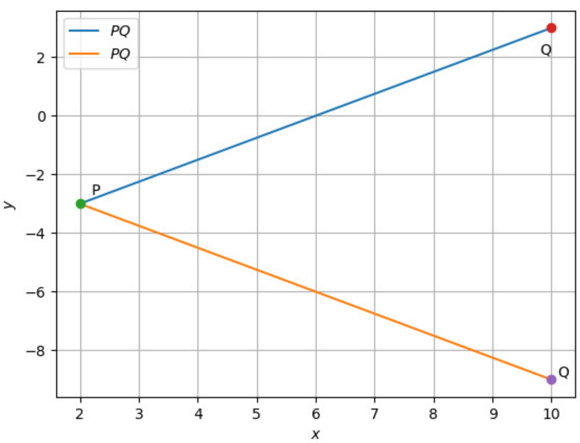

Fig. 1.55

1.56. Show that each of the given three vectors is a unit vector

$$\frac{1}{7}\begin{pmatrix}2\\3\\6\end{pmatrix}, \frac{1}{7}\begin{pmatrix}3\\-6\\2\end{pmatrix}, \frac{1}{7}\begin{pmatrix}6\\2\\-3\end{pmatrix}. \tag{1.56.1}$$

Also, show that they are mutually perpendicular to each other.

Solution: Let $\boldsymbol{A}=\frac{1}{7}\begin{pmatrix}2\\3\\6\end{pmatrix}, \boldsymbol{B}=\frac{1}{7}\begin{pmatrix}3\\-6\\2\end{pmatrix}, \boldsymbol{C}=\frac{1}{7}\begin{pmatrix}6\\2\\-3\end{pmatrix}$

$$\|\boldsymbol{A}\| = \frac{1}{7}\sqrt{2^2+3^2+6^2} = 1 \tag{1.56.2}$$

$$\|\boldsymbol{B}\| = \frac{1}{7}\sqrt{3^2+-6^2+2^2} = 1 \tag{1.56.3}$$

$$\|\boldsymbol{C}\| = \frac{1}{7}\sqrt{6^2+2^2+-3^2} = 1 \tag{1.56.4}$$

When two vectors are perpendicular to each other their dot product is zero. The dot product of $\boldsymbol{A}$,$\boldsymbol{B}$ and $\boldsymbol{C}$ with each other is

$$\boldsymbol{A}^T\boldsymbol{B} = \frac{1}{7}\times\frac{1}{7}(2\times3+3\times-6+6\times2) = 0 \tag{1.56.5}$$

$$\boldsymbol{B}^T\boldsymbol{C} = \frac{1}{7}\times\frac{1}{7}(2\times3+3\times-6+6\times2) = 0 \tag{1.56.6}$$

$$\boldsymbol{C}^T\boldsymbol{A} = \frac{1}{7}\times\frac{1}{7}(6\times2+2\times3+-3\times6) = 0 \tag{1.56.7}$$

Hence, the three unit vectors are mutually perpendicular to each other.

1.57. For

$$\mathbf{a} = \begin{pmatrix}2\\2\\3\end{pmatrix}, \mathbf{b} = \begin{pmatrix}-1\\2\\1\end{pmatrix}, \mathbf{c} = \begin{pmatrix}3\\1\\0\end{pmatrix}, \tag{1.57.1}$$

$(\mathbf{a}+k\mathbf{b}) \perp \mathbf{c}$. Find λ. **Solution:**
The two vectors are perpendicular to each other if their dot product is zero.
So,

$$\mathbf{c}^T(\mathbf{a}+k\mathbf{b}) = 0 \tag{1.57.2}$$

$$\mathbf{c}^T\mathbf{a}+k\mathbf{c}^T\mathbf{b} = 0 \tag{1.57.3}$$

$$k\mathbf{c}^T\mathbf{b} = -\mathbf{c}^T\mathbf{a} \tag{1.57.4}$$

$$\implies k = \frac{-\mathbf{c}^T\mathbf{a}}{\mathbf{c}^T\mathbf{b}} \tag{1.57.5}$$

On solving the matrix multiplication,

$$\mathbf{c}^T\mathbf{b} = -1, \tag{1.57.6}$$

$$\mathbf{c}^T\mathbf{a} = 8 \tag{1.57.7}$$

So,

$$\implies k = \frac{-8}{-1} \tag{1.57.8}$$

$$k = 8 \tag{1.57.9}$$

1.58. Find $\mathbf{a}\times\mathbf{b}$ if

$$\mathbf{a} = \begin{pmatrix}1\\-7\\7\end{pmatrix}, \mathbf{b} = \begin{pmatrix}3\\-2\\2\end{pmatrix}. \tag{1.58.1}$$

Solution: Cross product of two vectors is determined by spanning a vector into skew symmetric matrix

$$\mathbf{a}\times\mathbf{b} = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 & -7 & -7 \\ 7 & 0 & -1 \\ 7 & 1 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 19 \\ 19 \end{pmatrix} \tag{1.58.2}$$

1.59. Find a unit vector perpendicular to each of the vectors $\mathbf{a}+\mathbf{b}$ and $\mathbf{a}-\mathbf{b}$, where

$$\mathbf{a} = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}. \tag{1.59.1}$$

Solution: Let $\mathbf{A} = \mathbf{a}+\mathbf{b}$ *and* $\mathbf{B} = \mathbf{a}-\mathbf{b}$

$$\mathbf{A} = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 0 \end{pmatrix} \tag{1.59.2}$$

$$\mathbf{B} = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} \tag{1.59.3}$$

Let $\mathbf{n}$ be a vector Perpendicular to $\mathbf{A}$ and $\mathbf{B}$ both

$$\mathbf{A}^T\mathbf{n} = 0 \tag{1.59.4}$$

$$\mathbf{B}^T\mathbf{n} = 0 \tag{1.59.5}$$

The augmented matrix can be represented as follows:

$$\left(\begin{array}{ccc|c} 4 & 4 & 0 & 0 \\ 2 & 0 & 4 & 0 \end{array}\right) \tag{1.59.6}$$

Using row reduction to find an expression for $\mathbf{n}$.

$$\xleftrightarrow[R_2\leftarrow R_2-2R_1]{R_1\leftarrow\frac{R_1}{4}} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -2 & 4 & 0 \end{array}\right) \tag{1.59.7}$$

$$\xleftrightarrow[R_1\leftarrow R_1-R_2]{R_2\leftarrow\frac{R_2}{-2}} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right) \tag{1.59.8}$$

From above equations we get,

$$\therefore \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} -2n_3 \\ 2n_3 \\ n_3 \end{pmatrix} = n_3\begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} \tag{1.59.9}$$

Let us consider n_3 to be 1 which gives us:

$$\therefore \mathbf{n} = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} \tag{1.59.10}$$

$$\|\mathbf{n}\| = \sqrt{(-2)^2+2^2+1^2} = 3 \tag{1.59.11}$$

Let $\mathbf{u}$ be the unit vector of $\mathbf{n}$ which can be found as follows:

$$\mathbf{u} = \frac{\mathbf{n}}{\|\mathbf{n}\|} \tag{1.59.12}$$

Solving the above equation gives the unit vector $\mathbf{u}$ which is perpendicular to vectors $\mathbf{A}$ *and* $\mathbf{B}$

$$\boxed{\therefore \mathbf{u} = \frac{1}{3}\begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}} \tag{1.59.13}$$

1.60. If $\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$, find a unit vector parallel to the vector $2\mathbf{a}-\mathbf{b}+3\mathbf{c}$.

Solution:

$$\boldsymbol{d} = \mathbf{2}\boldsymbol{a} - \boldsymbol{b} + \mathbf{3}\boldsymbol{c} \tag{1.60.1}$$

$$\mathbf{2}\boldsymbol{a} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \tag{1.60.2}$$

$$-\boldsymbol{b} = \begin{pmatrix} -2 \\ 1 \\ -3 \end{pmatrix} \tag{1.60.3}$$

$$\mathbf{3}\boldsymbol{c} = \begin{pmatrix} 3 \\ -6 \\ 3 \end{pmatrix} \tag{1.60.4}$$

From the above,

$$\boldsymbol{d} = \begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix} \tag{1.60.5}$$

$$\|\boldsymbol{d}\| = \sqrt{3^2+(-3)^2+2^2} = \sqrt{22} \tag{1.60.6}$$

$$\boldsymbol{e} = \frac{\boldsymbol{d}}{\|\boldsymbol{d}\|} \tag{1.60.7}$$

$\boldsymbol{e}$ is the unit vector parallel to given vector Thus,

$$\boxed{\boldsymbol{e} = \frac{1}{\sqrt{22}}\begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix}} \tag{1.60.8}$$

1.61. Find a vector of magnitude 5 units, and parallel to the resultant of the vectors $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}, \mathbf{b} =$

$\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$,

Solution: First find resultant $\mathbf{R}$ of $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

$$\mathbf{R} = \mathbf{a} + \mathbf{b} \tag{1.61.1}$$

$$\implies \mathbf{R} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \tag{1.61.2}$$

$$\implies \mathbf{R} = \begin{pmatrix} 2+1 \\ 3-2 \\ -1+1 \end{pmatrix} \tag{1.61.3}$$

$$\implies \mathbf{R} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}. \tag{1.61.4}$$

Magnitude of $\mathbf{R}$ is

$$\|\mathbf{R}\| = \sqrt{3^2 + 1^2 + 0^2} \tag{1.61.5}$$

$$\implies \|\mathbf{R}\| = \sqrt{10} \tag{1.61.6}$$

$$\tag{1.61.7}$$

Then unit vector $\mathbf{r}$ along $\mathbf{R}$ is

$$\mathbf{r} = \frac{\mathbf{R}}{\|\mathbf{R}\|} \tag{1.61.8}$$

$$\implies \mathbf{r} = \frac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \tag{1.61.9}$$

Then vector of magnitude 5 units parallel to resultant $\mathbf{R}$ is given by

$$\mathbf{u} = 5\mathbf{r} \tag{1.61.10}$$

$$\implies \mathbf{u} = \frac{5}{\sqrt{10}} \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \tag{1.61.11}$$

$$\implies \mathbf{u} = \begin{pmatrix} 4.7434 \\ 1.5811 \\ 0 \end{pmatrix} \tag{1.61.12}$$

1.62. Show that the unit direction vector inclined equally to the coordinate axes is $\begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$.

Solution: Let $\mathbf{m}$ be a unit vector such that $\mathbf{m} = \begin{pmatrix} m_x \\ m_y \\ m_z \end{pmatrix}$. Let $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ be the direction vectors of the coordinate axes.
As $\mathbf{m}$ is a unit vector, so $\|\mathbf{m}\| = 1$ and also we are given is that $\mathbf{m}$ is inclined equally to the coordinate axis,

$$\mathbf{e}_1^T\mathbf{m} = \mathbf{e}_2^T\mathbf{m} = \mathbf{e}_3^T\mathbf{m} \tag{1.62.1}$$

Now, 1.62.1 implies

$$(\mathbf{e}_1 - \mathbf{e}_2)^T\mathbf{m} = 0 \tag{1.62.2}$$

$$(\mathbf{e}_2 - \mathbf{e}_3)^T\mathbf{m} = 0 \tag{1.62.3}$$

$$(\mathbf{e}_3 - \mathbf{e}_1)^T\mathbf{m} = 0 \tag{1.62.4}$$

Thus, converting above system of equations into matrix form, we get

$$\mathbf{A}\mathbf{m} = 0 \tag{1.62.5}$$

To find the solution of 1.62.5, we find the echelon form of $\mathbf{A}$.

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix} \xleftrightarrow{r_3 \leftarrow r_1 + r_3} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \tag{1.62.6}$$

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \xleftrightarrow{r_3 \leftarrow r_2 + r_3} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.62.7}$$

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \xleftrightarrow{r_1 \leftarrow r_1 + r_2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.62.8}$$

From 1.62.8, we find out that

$$m_x = m_y = m_z \tag{1.62.9}$$

$$\mathbf{m} = \begin{pmatrix} m_z \\ m_z \\ m_z \end{pmatrix} \implies \mathbf{m} = m_z \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{1.62.10}$$

Taking $m_z = 1$, then $\|\mathbf{m}\| = \frac{1}{\sqrt{3}}$ and for $\mathbf{m}$ to be a unit vector, we need to divide each element of $\mathbf{m}$ by $\|\mathbf{m}\|$.
Thus, we see that

$$\mathbf{m} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} \tag{1.62.11}$$

is the unit direction vector inclined equally to the coordinate axes.

1.63. Let $\mathbf{a} = \begin{pmatrix}1\\4\\2\end{pmatrix}, \mathbf{b} = \begin{pmatrix}3\\-2\\7\end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix}2\\-1\\4\end{pmatrix}$. Find a vector $\mathbf{d}$ such that $\mathbf{d} \perp \mathbf{a}, \mathbf{d} \perp \mathbf{b}$ and $\mathbf{d}^T\mathbf{c} = 15$.

Solution: From the given information

$$\mathbf{d}^T\mathbf{a} = 0 \tag{1.63.1}$$

Similarly, as $\mathbf{d} \perp \mathbf{b}$

$$\mathbf{d}^T\mathbf{b} = 0 \tag{1.63.2}$$

It is given that

$$\mathbf{d}^T\mathbf{c} = 15 \tag{1.63.3}$$

Using equations 1.63.1, 1.63.2, 1.63.3, we can represent them in a Matrix Representation of Linear Equations Ax=B form as:

$$\begin{pmatrix}\mathbf{a}^T\\\mathbf{b}^T\\\mathbf{c}^T\end{pmatrix}\mathbf{d} = \begin{pmatrix}0\\0\\15\end{pmatrix} \tag{1.63.4}$$

Numerically, using $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ the above equation 1.63.4 can be written as,

$$\begin{pmatrix}1 & 4 & 2\\3 & -2 & 7\\2 & -1 & 4\end{pmatrix}\mathbf{d} = \begin{pmatrix}0\\0\\15\end{pmatrix} \tag{1.63.5}$$

we can use Guassian Elimination Method in order to find the coordinate values of $\mathbf{d}$.

$$\left(\begin{array}{ccc|c}1 & 4 & 2 & 0\\3 & -2 & 7 & 0\\2 & -1 & 4 & 15\end{array}\right) \tag{1.63.6}$$

$$\xleftrightarrow[R_2\leftarrow R_2-3R_1]{R_3\leftarrow R_3-2R_1}\left(\begin{array}{ccc|c}1 & 4 & 2 & 0\\0 & -14 & 1 & 0\\0 & -9 & 0 & 15\end{array}\right) \tag{1.63.7}$$

$$\xleftrightarrow{R_3\leftarrow R_3-\frac{9}{14}R_2}\left(\begin{array}{ccc|c}1 & 4 & 2 & 0\\0 & -14 & 1 & 0\\0 & 0 & \frac{-9}{14} & 15\end{array}\right) \tag{1.63.8}$$

$$\xleftrightarrow[R_2\leftarrow\frac{-1}{14}R_2]{R_3\leftarrow\frac{-14}{9}R_2}\left(\begin{array}{ccc|c}1 & 4 & 2 & 0\\0 & 1 & \frac{-1}{14} & 0\\0 & 0 & 1 & \frac{-210}{9}\end{array}\right) \tag{1.63.9}$$

$$\xleftrightarrow{R_1\leftarrow R_1+\frac{1}{14}R_3}\left(\begin{array}{ccc|c}1 & 4 & 2 & 0\\0 & 1 & 0 & \frac{-210}{126}\\0 & 0 & 1 & \frac{-210}{9}\end{array}\right) \tag{1.63.10}$$

$$\xleftrightarrow{R_1\leftarrow R_1-4R_3}\left(\begin{array}{ccc|c}1 & 0 & 2 & \frac{840}{126}\\0 & 1 & 0 & \frac{-210}{126}\\0 & 0 & 1 & \frac{-210}{9}\end{array}\right) \tag{1.63.11}$$

$$\xleftrightarrow{R_1\leftarrow R_1-2R_3}\left(\begin{array}{ccc|c}1 & 0 & 0 & \frac{6720}{126}\\0 & 1 & 0 & \frac{-210}{126}\\0 & 0 & 1 & \frac{-210}{9}\end{array}\right) \tag{1.63.12}$$

By using Guassian Elimination Method, we were able to get the vector $\mathbf{d}$ as $\begin{pmatrix}\frac{6720}{126}\\\frac{-210}{126}\\\frac{-210}{9}\end{pmatrix}$

1.64. The scalar product of $\begin{pmatrix}1\\1\\1\end{pmatrix}$ with a unit vector along the sum of the vectors $\begin{pmatrix}2\\4\\-5\end{pmatrix}$ and $\begin{pmatrix}\lambda\\2\\3\end{pmatrix}$ is unity. Find the value of λ.

1.65. The value of

$$\begin{pmatrix}1\\0\\0\end{pmatrix}^T\left(\begin{pmatrix}0\\1\\0\end{pmatrix}\times\begin{pmatrix}0\\0\\1\end{pmatrix}\right)+\begin{pmatrix}0\\1\\0\end{pmatrix}^T\left(\begin{pmatrix}1\\0\\0\end{pmatrix}\times\begin{pmatrix}0\\0\\1\end{pmatrix}\right) + \begin{pmatrix}0\\0\\1\end{pmatrix}^T\left(\begin{pmatrix}1\\0\\0\end{pmatrix}\times\begin{pmatrix}0\\1\\0\end{pmatrix}\right) \tag{1.65.1}$$

is

a) 0 c) 1
b) -1 d) 3

Solution: Given

$$\mathbf{a} = \begin{pmatrix}1\\0\\0\end{pmatrix}, \mathbf{b} = \begin{pmatrix}0\\1\\0\end{pmatrix}, \mathbf{c} = \begin{pmatrix}0\\0\\1\end{pmatrix} \tag{1.65.2}$$

Using scalar triple product property we deduce

$$\mathbf{a}^T (\mathbf{b} \times \mathbf{c}) = \mathbf{b}^T (\mathbf{c} \times \mathbf{a}) = \mathbf{c}^T (\mathbf{a} \times \mathbf{b}) \tag{1.65.3}$$

Note: Cross product is given by:

$$\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix} \times \begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix} = \begin{pmatrix}0 & -a_3 & a_2\\ a_3 & 0 & -a_1\\ -a_2 & a_1 & 0\end{pmatrix}\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix} \tag{1.65.4}$$

Equating (1.65.2) with problem statement we deduce the following:

$$\mathbf{a}^T (\mathbf{b} \times \mathbf{c}) + \mathbf{b}^T (\mathbf{a} \times \mathbf{c}) + \mathbf{c}^T (\mathbf{a} \times \mathbf{b}) \tag{1.65.5}$$

As Cross Product is anti-commutative we get:

$$\mathbf{a}^T (\mathbf{b} \times \mathbf{c}) - \mathbf{b}^T (\mathbf{c} \times \mathbf{a}) + \mathbf{c}^T (\mathbf{a} \times \mathbf{b}) \tag{1.65.6}$$

$$= \mathbf{a}^T (\mathbf{b} \times \mathbf{c}) - \mathbf{c}^T (\mathbf{a} \times \mathbf{b}) + \mathbf{c}^T (\mathbf{a} \times \mathbf{b}) \tag{1.65.7}$$

$$= \mathbf{a}^T (\mathbf{b} \times \mathbf{c}) \tag{1.65.8}$$

So instead of calculating each step we just calculate one iteration by referring (1.65.4) and (1.65.8) i.e.

$$\begin{pmatrix}0\\1\\0\end{pmatrix} \times \begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix} \tag{1.65.9}$$

$$\implies \begin{pmatrix}1 & 0 & 0\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\end{pmatrix} \tag{1.65.10}$$

1.66. Find a unit vector that makes an angle of $90°, 135°$ and $45°$ with the positive x, y and z axis respectively. **Solution:**

$$\mathbf{m} = \begin{pmatrix}\cos 90°\\ \cos 135°\\ \cos 45°\end{pmatrix} \tag{1.66.1}$$

we know that,

$$\mathbf{m} = \frac{\mathbf{m}}{\|\mathbf{m}\|} \tag{1.66.2}$$

Also,

$$\|\mathbf{m}\| = \sqrt{0^2 + \left(\frac{-1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} \implies \|\mathbf{m}\| = 1 \tag{1.66.3}$$

Hence,From (1.66.1) and (1.66.3) we have the unit vector:

$$\mathbf{m} = \begin{pmatrix}0\\ \frac{-1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix} \tag{1.66.4}$$

1.67. Show that the lines with direction vectors $\begin{pmatrix}12\\-3\\-4\end{pmatrix}$, $\begin{pmatrix}4\\12\\3\end{pmatrix}$ and $\begin{pmatrix}3\\-4\\12\end{pmatrix}$ are mutually perpendicular.

1.68. Show that the line through the points $\begin{pmatrix}1\\-1\\2\end{pmatrix}$, $\begin{pmatrix}3\\4\\-2\end{pmatrix}$ is perpendicular to the line through the points $\begin{pmatrix}0\\3\\2\end{pmatrix}$, $\begin{pmatrix}3\\5\\6\end{pmatrix}$.

Solution: Let the points be $\mathbf{P}=\begin{pmatrix}1\\-1\\2\end{pmatrix}$, $\mathbf{Q}=\begin{pmatrix}3\\4\\-2\end{pmatrix}$, $\mathbf{R}=\begin{pmatrix}0\\3\\2\end{pmatrix}$ and $\mathbf{S}=\begin{pmatrix}3\\5\\6\end{pmatrix}$. The direction vector for the line through the points $\mathbf{P}$ and $\mathbf{Q}$ is

$$\mathbf{A} = \mathbf{P} - \mathbf{Q} \tag{1.68.1}$$

$$\implies \mathbf{A} = \begin{pmatrix}1\\-1\\2\end{pmatrix} - \begin{pmatrix}3\\4\\-2\end{pmatrix} \tag{1.68.2}$$

$$\implies \mathbf{A} = \begin{pmatrix}-2\\-5\\4\end{pmatrix} \tag{1.68.3}$$

The direction vector for the line through the points **R** and **S** is

$$\mathbf{B} = \mathbf{R} - \mathbf{S} \tag{1.68.4}$$

$$\implies \mathbf{B} = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 5 \\ 6 \end{pmatrix} \tag{1.68.5}$$

$$\implies \mathbf{B} = \begin{pmatrix} -3 \\ -2 \\ -4 \end{pmatrix} \tag{1.68.6}$$

$$\tag{1.68.7}$$

To check if the two lines are perpendicular, we perform scalar product of the two direction vectors **A** and **B** as follows

$$\mathbf{AB} = \mathbf{A}^T\mathbf{B} \tag{1.68.8}$$

$$= \begin{pmatrix} -2 & -5 & 4 \end{pmatrix} \begin{pmatrix} -3 \\ -2 \\ -4 \end{pmatrix} \tag{1.68.9}$$

$$= 6 + 10 - 16 \tag{1.68.10}$$

$$= 0 \tag{1.68.11}$$

Thus, the lines are **perpendicular**.

1.69. Show that the line through the points $\begin{pmatrix} 4 \\ 7 \\ 8 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$ is parallel to the line through the points $\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$.

Solution: Let the lines be parallel and the first two points pass through $\mathbf{n}^T\mathbf{x} = c1$. i.e.

$$\mathbf{n}^T\mathbf{x}_1 = c_1 => \mathbf{x}_1^T\mathbf{n} = c_1 \tag{1.69.1}$$

$$\mathbf{n}^T\mathbf{x}_2 = c_2 => \mathbf{x}_2^T\mathbf{n} = c_2 \tag{1.69.2}$$

and the second two points pass through $\mathbf{n}^T\mathbf{x} = c2$ Then

$$\mathbf{n}^T\mathbf{x}_3 = c_3 => \mathbf{x}_3^T\mathbf{n} = c_3 \tag{1.69.3}$$

$$\mathbf{n}^T\mathbf{x}_4 = c_4 => \mathbf{x}_4^T\mathbf{n} = c_4 \tag{1.69.4}$$

Putting all the equations together, we obtain

$$\begin{pmatrix} \mathbf{x}_1^T \\ \mathbf{x}_2^T \\ \mathbf{x}_3^T \\ \mathbf{x}_4^T \end{pmatrix} \mathbf{n} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} \tag{1.69.5}$$

Now if this equation has a solution, then **n** exists and the lines will be parallel. Given the points, $\mathbf{A} = \begin{pmatrix} 4 \\ 7 \\ 8 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$, *and* $\mathbf{C} = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$

Applying the row reduction procedure on the coefficient matrix:

$$\begin{pmatrix} 4 & 7 & 8 \\ 2 & 3 & 4 \\ -1 & -2 & 1 \\ 1 & 2 & 5 \end{pmatrix} \tag{1.69.6}$$

$$\xleftrightarrow[R_4 \leftarrow R_3 + R_4]{R_2 \leftarrow R_1 - 2R_2} \begin{pmatrix} 4 & 7 & 8 \\ 0 & 1 & 0 \\ -1 & -2 & 1 \\ 0 & 0 & 6 \end{pmatrix} \tag{1.69.7}$$

$$\xleftrightarrow[R_3 \leftarrow R_3 - 6R_4]{R_1 \leftarrow R_1 - 7R_2} \begin{pmatrix} 4 & 0 & 8 \\ 0 & 1 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & 6 \end{pmatrix} \tag{1.69.8}$$

$$\xleftrightarrow[R_1 \leftarrow R_1 - 8R_4]{R_4 \leftarrow R_4/6} \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.69.9}$$

$$\xleftrightarrow[R_3 \leftarrow R_3 + R_4]{R_3 \leftarrow (-R_3 - 2R_2)} \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.69.10}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - 4R_3} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.69.11}$$

Here, the number of non-zero rows are three and hence the rank of the matrix is 3 which implies that the solution exists. Therefore the lines passing through $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}, \mathbf{D}$ are parallel.

1.70. Find a point on the x-axis, which is equidistant from the points $\begin{pmatrix} 7 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

Solution: Given,

$$\mathbf{P} = \begin{pmatrix} 7 \\ 6 \end{pmatrix} \quad \mathbf{Q} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \tag{1.70.1}$$

A vector on the X-axis **X** is equidistant to both

P and **Q**.

$$i.e.\ \mathbf{X} = \frac{\mathbf{P}+\mathbf{Q}}{2} \tag{1.70.2}$$

Need to find k. Let $\mathbf{X} = k\begin{pmatrix}1\\0\end{pmatrix}$ be the vector on the X-axis.

$$\implies \begin{pmatrix}1 & 0\end{pmatrix}\mathbf{X} = k \tag{1.70.3}$$

$$\implies \mathbf{X} = \frac{\begin{pmatrix}7\\6\end{pmatrix}+\begin{pmatrix}3\\4\end{pmatrix}}{2} \tag{1.70.4}$$

$$\implies \mathbf{X} = \begin{pmatrix}5\\5\end{pmatrix} \tag{1.70.5}$$

$$\implies \begin{pmatrix}1 & 0\end{pmatrix}\mathbf{X} = \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}5\\5\end{pmatrix} \tag{1.70.6}$$

$$\tag{1.70.7}$$

Therefore, k = 5 i.e. $\mathbf{X} = \begin{pmatrix}5\\0\end{pmatrix}$ See Fig. 1.70

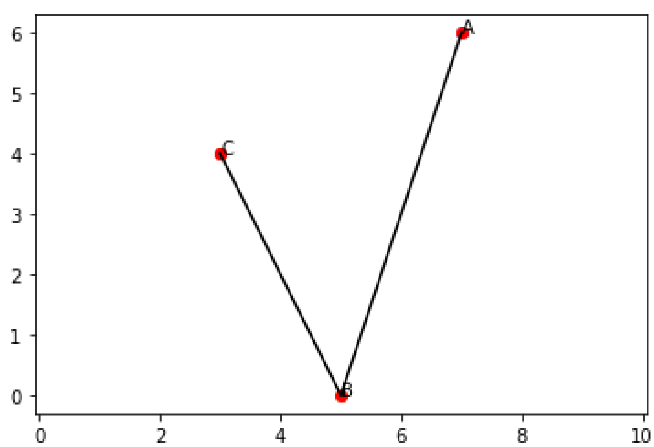

Fig. 1.70: Plot representing the Points

1.71. Find the angle between the vectors

$$\begin{pmatrix}1\\-2\\3\end{pmatrix}, \begin{pmatrix}3\\-2\\1\end{pmatrix} \tag{1.71.1}$$

Solution: Let

$$\mathbf{a} = \begin{pmatrix}1\\-2\\3\end{pmatrix} \mathbf{b} = \begin{pmatrix}3\\-2\\1\end{pmatrix} \tag{1.71.2}$$

Angle between the vectors is given by,

$$\theta = \cos^{-1}\left(\frac{\mathbf{a}^T\mathbf{b}}{\|\mathbf{a}\|\,\|\mathbf{b}\|}\right) \tag{1.71.3}$$

$$\|\mathbf{a}\| = \sqrt{1^2+(-2)^2+3^2} = \sqrt{14} \tag{1.71.4}$$

$$\|\mathbf{b}\| = \sqrt{3^2+(-2)^2+1^2} = \sqrt{14} \tag{1.71.5}$$

$$\mathbf{a}^T\mathbf{b} = (1)(3)+(-2)(-2)+(3)(1) = 10 \tag{1.71.6}$$

$$\theta = \cos^{-1}\left(\frac{10}{(\sqrt{14})(\sqrt{14})}\right) \tag{1.71.7}$$

$$= \cos^{-1}\left(\frac{10}{14}\right) \tag{1.71.8}$$

$$\tag{1.71.9}$$

1.72. Find the projection of the vector

$$\begin{pmatrix}1\\3\\7\end{pmatrix} \tag{1.72.1}$$

on the vector

$$\begin{pmatrix}7\\-1\\8\end{pmatrix} \tag{1.72.2}$$

Solution:
We have,

$$\mathbf{u} = \begin{pmatrix}1\\3\\7\end{pmatrix}, \mathbf{v} = \begin{pmatrix}7\\-1\\8\end{pmatrix}$$

$$\mathbf{p} = \left[\frac{\begin{pmatrix}1\\3\\7\end{pmatrix}^T\begin{pmatrix}7\\-1\\8\end{pmatrix}}{\left\|\begin{matrix}7\\-1\\8\end{matrix}\right\|^2}\right]\begin{pmatrix}7\\-1\\8\end{pmatrix} \tag{1.72.3}$$

$$\mathbf{p} = \left[\frac{(7-3+56)}{\left(\sqrt{7^2+(-1)^2+8^2}\right)^2}\right]\begin{pmatrix}7\\-1\\8\end{pmatrix} \tag{1.72.4}$$

$$\mathbf{p} = \frac{13}{25}\begin{pmatrix}7\\-1\\8\end{pmatrix} = \begin{pmatrix}\frac{92}{25}\\-\frac{13}{25}\\\frac{21}{5}\end{pmatrix} \tag{1.72.5}$$

Hence the projection of $\mathbf{u}$ on $\mathbf{v}$ is

$$\mathbf{p} = \begin{pmatrix} \frac{92}{25} \\ -\frac{13}{25} \\ \frac{21}{5} \end{pmatrix}$$

1.73. Write down a unit vector in the xy-plane, makeing an angle of 30° with the positive direction of the x-axis.
Solution:

$$\because m = \tan 30^\circ = \frac{1}{\sqrt{3}}, \tag{1.73.1}$$

the direction vector is

$$\mathbf{a} = \begin{pmatrix} 1 \\ \frac{1}{\sqrt{3}} \end{pmatrix} \tag{1.73.2}$$

and the unit vector

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{||a||} \tag{1.73.3}$$

$$\implies \hat{\mathbf{a}} = \begin{pmatrix} \frac{\frac{1}{2}}{\sqrt{3}} \\ \frac{\frac{1}{\sqrt{3}}}{\frac{2}{\sqrt{3}}} \end{pmatrix} \tag{1.73.4}$$

$$\hat{\mathbf{a}} = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix} \tag{1.73.5}$$

$$\implies \boxed{\hat{\mathbf{a}} = \frac{1}{2}\begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}} \tag{1.73.6}$$

1.74. Find the value of x for which $x\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is a unit vector.
Solution:

$$\left\| x\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\| = 1 \tag{1.74.1}$$

$$\implies x\left\| \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\| = 1 \tag{1.74.2}$$

$$\text{or, } \sqrt{3x^2} = 1 \implies x = \pm\frac{1}{\sqrt{3}} \tag{1.74.3}$$

1.75. Find the angle between the force $\mathbf{F} = \begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix}$ and displacement $\mathbf{d} = \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix}$.
Solution: Let the angle between $\mathbf{F}$ and $\mathbf{d} = \theta$ Then,

$$\cos(\theta) = \frac{\mathbf{F}^T\mathbf{d}}{||\mathbf{F}||\,||\mathbf{d}||} \tag{1.75.1}$$

where $\mathbf{F}^T\mathbf{d}$ is scalar product of vectors $\mathbf{F}$ and $\mathbf{d}$
And, $||\mathbf{F}||$ and $||\mathbf{d}||$ are their respective magnitudes So,

$$\mathbf{F}^T\mathbf{d} = \begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix}^T \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix} \tag{1.75.2}$$

$$\implies \mathbf{F}^T\mathbf{d} = \begin{pmatrix} 3 & 4 & -5 \end{pmatrix}\begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix} \tag{1.75.3}$$

$$= 16 \tag{1.75.4}$$

$$||\mathbf{F}|| = \sqrt{3^2 + 4^2 + (-5)^2} = 5\sqrt{2} \tag{1.75.5}$$

$$||\mathbf{d}|| = \sqrt{5^2 + 4^2 + 3^2} = 5\sqrt{2} \tag{1.75.6}$$

Substituting these values in Equation 1.75.1,

$$\cos(\theta) = \frac{16}{(5\sqrt{2})(5\sqrt{2})} \tag{1.75.7}$$

$$= \frac{8}{25} \tag{1.75.8}$$

$$\implies \theta = \arccos\left(\frac{8}{25}\right) \tag{1.75.9}$$

$$\implies \theta \approx 71.3^\circ \tag{1.75.10}$$

1.76. A body constrained to move along the z-axis of a coordinate system is subject to a constant force

$$\mathbf{F} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} \tag{1.76.1}$$

What is the work done by this force in moving the body a distance of 4 m along the z-axis ?
Solution: Work done in moving an object by a distance $\mathbf{s}$ using an external force $\mathbf{F}$ is given

by:

$$W = \mathbf{F^T s} \tag{1.76.2}$$

As seen above, work done is the scalar product (dot product) of Force and distance. Here,

$$\mathbf{s} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} \tag{1.76.3}$$

The scalar product of the variables is given by:

$$\mathbf{F^T s} = \begin{pmatrix} 1 & -2 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} = 12 \tag{1.76.4}$$

The work done by the force $\mathbf{F}$ is 12 J

1.77. Find the scalar and vector products of the two vectors

$$\mathbf{a} = \begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} -2 \\ 1 \\ -3 \end{pmatrix} \tag{1.77.1}$$

Solution:

$$\mathbf{a^T b} = \begin{pmatrix} 3 & -4 & 5 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ -3 \end{pmatrix} \tag{1.77.2}$$

$$= (3 \times -2) + (-4 \times 1) + (5 \times -3) \tag{1.77.3}$$

$$= -25 \tag{1.77.4}$$

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} 0 & 5 & -4 \\ 5 & 0 & -3 \\ -(-4) & 3 & 0 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} \tag{1.77.5}$$

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} (0 \times -2) + (-5 \times 1) + (-4 \times -3) \\ (5 \times -2) + (0 \times 1) + (-3 \times -3) \\ (4 \times -2) + (3 \times 1) + (0 \times -3) \end{pmatrix} \tag{1.77.6}$$

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} 7 \\ -1 \\ 5 \end{pmatrix} \tag{1.77.7}$$

1.78. Find the torque of a force $\begin{pmatrix} 7 \\ 3 \\ -5 \end{pmatrix}$ about the origin. The force acts on a particle whose position vector is $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

Solution: The torque $\mathbf{T}$ is given by the cross product (vector product) of the position (or distance) vector $\mathbf{r}$ and the force vector $\mathbf{F}$.

$$\mathbf{T} = \mathbf{r} \times \mathbf{F} \tag{1.78.1}$$

And the vector cross product of vectors

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \tag{1.78.2}$$

$$\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \tag{1.78.3}$$

can be expressed as the product of a skew-symmetric matrix and a vector:

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \tag{1.78.4}$$

Torque at the origin is given by,

$$\mathbf{F} \times \mathbf{r} = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 7 \\ 3 \\ -5 \end{pmatrix} \tag{1.78.5}$$

$$\implies \mathbf{F} \times \mathbf{r} = \begin{pmatrix} (0 \times 7) + (-1 \times 3) + (-1 \times -5) \\ (1 \times 7) + (0 \times 3) + (-1 \times -5) \\ (1 \times 7) + (1 \times 3) + (0 \times -5) \end{pmatrix} \tag{1.78.6}$$

$$\implies \mathbf{T} = \begin{pmatrix} 2 \\ 12 \\ 10 \end{pmatrix} \tag{1.78.7}$$

1.79. Find the values of x, y, z such that

$$\begin{pmatrix} x \\ 2 \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ y \\ 1 \end{pmatrix} \tag{1.79.1}$$

Solution: $x = 2, y = 2, z = 1$.

1.80. If

$$\mathbf{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \tag{1.80.1}$$

verify if

a) $\|\mathbf{a}\| = \|\mathbf{b}\|$

b) $\mathbf{a} = \mathbf{b}$

Solution:

a) $\|\mathbf{a}\| = \|\mathbf{b}\|, \mathbf{a} \neq \mathbf{b}$.

1.81. Find a unit vector in the direction of $\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$.

Solution: The unit vector is given by

$$\frac{\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}}{\left\|\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}\right\|} = \frac{1}{\sqrt{14}} \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \tag{1.81.1}$$

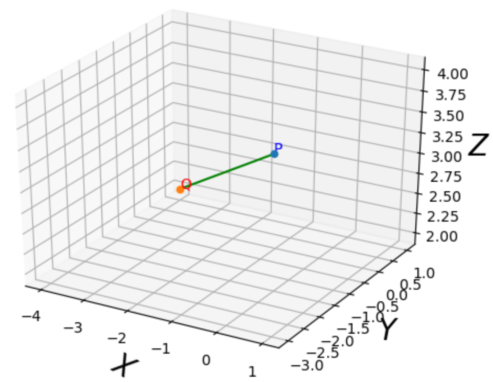

Fig. 1.82: Two points and distance between them.

1.82. Find the distance between the points

$$\mathbf{P} = \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} -4 \\ 1 \\ 2 \end{pmatrix} \tag{1.82.1}$$

Solution:

The distance between the two points is given by or,

$$\begin{aligned} d &= \|\mathbf{P} - \mathbf{Q}\| \\ &= \left\|\begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix} - \begin{pmatrix} -4 \\ 1 \\ 2 \end{pmatrix}\right\| = \left\|\begin{pmatrix} 5 \\ -4 \\ 2 \end{pmatrix}\right\| \\ \implies d &= \sqrt{5^2 + (-4)^2 + 2^2} \\ &= 3\sqrt{5} \end{aligned} \tag{1.82.2}$$

The following Python code generates Fig. 1.82

```
solutions/line/geometry/examples/54/codes/
    point_distance.py
```

The distance is given by $\|\mathbf{P} - \mathbf{Q}\|$

1.83. Show that the points $\mathbf{A} = \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 7 \\ 0 \\ -1 \end{pmatrix}$ are collinear.

Solution: Forming the matrix in (1.2.6)

$$\mathbf{M} = \begin{pmatrix} 3 & -1 & -2 \\ 9 & -3 & -6 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - 3R_1} \begin{pmatrix} 3 & -1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.83.1}$$

$\implies$ $rank(\mathbf{M}) = 1$. The following code plots Fig. 1.83 showing that the points are collinear.

```
codes/line/draw_lines_3d.py
```

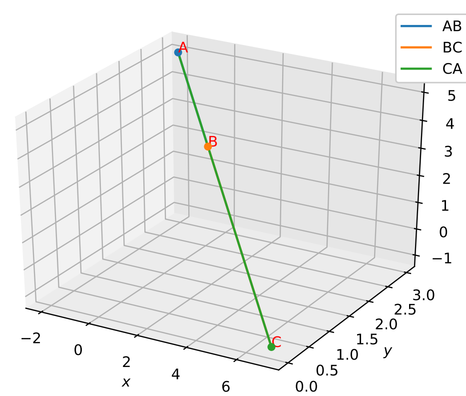

Fig. 1.83

1.84. If $\mathbf{a} = \begin{pmatrix} 5 \\ -1 \\ -3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix}$, then show that the vectors $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$ are perpendicular.

Solution:

$$\mathbf{A}^{\mathbf{T}}\mathbf{B} = 0 \tag{1.84.1}$$

$$\mathbf{A}^T\mathbf{B} = (\mathbf{a} + \mathbf{b})^T (\mathbf{a} - \mathbf{b}) \tag{1.84.2}$$

The transpose of a sum is the sum of transposes so,

$$\left(\mathbf{a}+\mathbf{b}\right)^T = \left(\mathbf{a}^T + \mathbf{b}^T\right) \tag{1.84.3}$$

$$\mathbf{A}^T\mathbf{B} = \left(\mathbf{a}^T + \mathbf{b}^T\right)\left(\mathbf{a}-\mathbf{b}\right) \tag{1.84.4}$$

$$\mathbf{a}^T\left(\mathbf{a}-\mathbf{b}\right) + \mathbf{b}^T\left(\mathbf{a}-\mathbf{b}\right) \tag{1.84.5}$$

$$\implies \mathbf{a}^T\mathbf{a} - \mathbf{a}^T\mathbf{b} + \mathbf{b}^T\mathbf{a} - \mathbf{b}^T\mathbf{b} \tag{1.84.6}$$

$$\because \mathbf{a}^T\mathbf{a} = \|\mathbf{a}\|^2 \tag{1.84.7}$$

$$\because \mathbf{b}^T\mathbf{b} = \|\mathbf{b}\|^2 \tag{1.84.8}$$

$$\because \mathbf{a}^T\mathbf{b} = \mathbf{b}^T\mathbf{a} \tag{1.84.9}$$

Using (1.84.7), (1.84.8) and (1.84.9)

$$\mathbf{A}^T\mathbf{B} = \|\mathbf{a}\|^2 - \mathbf{a}^T\mathbf{b} + \mathbf{a}^T\mathbf{b} - \|\mathbf{b}\|^T \tag{1.84.10}$$

$$\|\mathbf{a}\|^2 = 5^2 + (-1)^2 + (-3)^2 = 35 \tag{1.84.11}$$

$$\|\mathbf{b}\|^2 = 1^2 + (3)^2 + (-5)^2 = 35 \tag{1.84.12}$$

$$\mathbf{A}^T\mathbf{B} = \|\mathbf{a}\|^2 - \|\mathbf{b}\|^2 \tag{1.84.13}$$

Using (1.84.11) and (1.84.12)

$$\implies \mathbf{A}^T\mathbf{B} = 35 - 35 = 0 \tag{1.84.14}$$

Thus the direction vectors of the two lines satisfies the equation 1.84.1, hence proved that the lines are **perpendicular**.

1.85. Find the projection of the vector

$$\mathbf{a} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} \tag{1.85.1}$$

on the vector

$$\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}. \tag{1.85.2}$$

Solution: The projection of $\mathbf{a}$ on $\mathbf{b}$ is shown in Fig. 1.85. It has magnitude $\|\mathbf{a}\|\cos\theta$ and is in the direction of $\mathbf{b}$. Thus, the projection is defined as

$$\left(\|\mathbf{a}\|\cos\theta\right)\frac{\mathbf{b}}{\|\mathbf{b}\|} = \frac{\left(\mathbf{a}^T\mathbf{b}\right)\|\mathbf{a}\|}{\|\mathbf{b}\|}\mathbf{b} \tag{1.85.3}$$

1.86. Find $\|\mathbf{a}-\mathbf{b}\|$, if

$$\|\mathbf{a}\| = 2, \|\mathbf{b}\| = 3, \mathbf{a}^T\mathbf{b} = 4. \tag{1.86.1}$$

Solution:

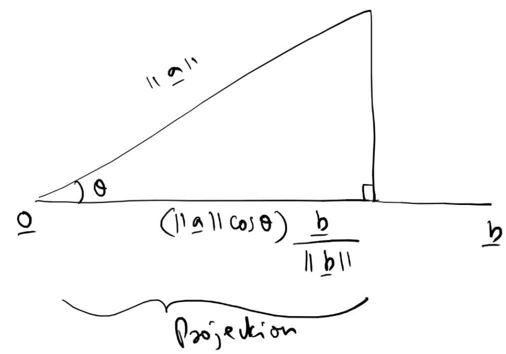

Fig. 1.85

$$\begin{aligned} \|\mathbf{a}-\mathbf{b}\|^2 &= \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\mathbf{a}^T\mathbf{b} \\ \implies \|\mathbf{a}-\mathbf{b}\|^2 &= 2^2 + 3^2 - 2\times 4 \\ \implies \|\mathbf{a}-\mathbf{b}\|^2 &= 5 \\ \implies \|\mathbf{a}-\mathbf{b}\| &= \sqrt{5} \end{aligned} \tag{1.86.2}$$

1.87. If $\mathbf{a}$ is a unit vector and

$$\left(\mathbf{x}-\mathbf{a}\right)\left(\mathbf{x}+\mathbf{a}\right) = 8, \tag{1.87.1}$$

then find $\mathbf{x}$.
Solution:

$$\left(\mathbf{x}-\mathbf{a}\right)\left(\mathbf{x}+\mathbf{a}\right) = \|\mathbf{x}\|^2 - \|\mathbf{a}\|^2 \tag{1.87.2}$$

$$\implies \|\mathbf{x}\|^2 = 9 \text{ or, } \|\mathbf{x}\| = 3. \tag{1.87.3}$$

1.88. Given

$$\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix}, \tag{1.88.1}$$

find $\|\mathbf{a}\times\mathbf{b}\|$.
Solution: Use (1.6.3).

1.89. Find a unit vector perpendicular to each of the vectors $\mathbf{a}+\mathbf{b}$ and $\mathbf{a}-\mathbf{b}$, where

$$\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}. \tag{1.89.1}$$

Solution: If $\mathbf{x}$ is the desired vector,

$$\left(\mathbf{a}+\mathbf{b}\right)^T\mathbf{x} = 0 \tag{1.89.2}$$

$$\left(\mathbf{a}-\mathbf{b}\right)^T\mathbf{x} = 0 \tag{1.89.3}$$

resulting in the matrix equation

$$\begin{pmatrix}2 & 3 & 4\\0 & -1 & -2\end{pmatrix}\mathbf{x} = 0 \tag{1.89.4}$$

Performing row operations,

$$\begin{pmatrix}2 & 3 & 4\\0 & -1 & -2\end{pmatrix} \xleftrightarrow[R_2\leftarrow -R_2]{R_1\leftarrow R_1+3R_2} \begin{pmatrix}2 & 0 & -2\\0 & -1 & -2\end{pmatrix} \tag{1.89.5}$$

$$\xleftrightarrow{R_1\leftarrow \frac{R_1}{2}} \begin{pmatrix}1 & 0 & -1\\0 & 1 & 2\end{pmatrix} \implies \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = x_3\begin{pmatrix}1\\-2\\1\end{pmatrix} \tag{1.89.6}$$

The desired unit vector is then obtained as

$$\mathbf{x} = \frac{\begin{pmatrix}1\\-2\\1\end{pmatrix}}{\left\|\begin{pmatrix}1\\-2\\1\end{pmatrix}\right\|} = \frac{1}{\sqrt{6}}\begin{pmatrix}1\\-2\\1\end{pmatrix} \tag{1.89.7}$$

1.90. Show that $\mathbf{A} = \begin{pmatrix}-2\\3\\5\end{pmatrix}, \mathbf{B} = \begin{pmatrix}1\\2\\3\end{pmatrix}, \mathbf{C} = \begin{pmatrix}7\\0\\-1\end{pmatrix}$, are collinear.
Solution: See Problem 1.83.

1.91. If $\mathbf{A} = \begin{pmatrix}1\\1\\1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}2\\5\\0\end{pmatrix}, \mathbf{C} = \begin{pmatrix}3\\2\\-3\end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix}1\\-6\\-1\end{pmatrix}$, show that $\mathbf{A}-\mathbf{B}$ and $\mathbf{C}-\mathbf{D}$ are collinear.
Solution:

$$\mathbf{A}-\mathbf{B} = \begin{pmatrix}-1\\-4\\1\end{pmatrix} \tag{1.91.1}$$

$$\mathbf{C}-\mathbf{D} = \begin{pmatrix}2\\8\\-2\end{pmatrix} \tag{1.91.2}$$

$$\because -2(\mathbf{A}-\mathbf{B}) = \mathbf{C}-\mathbf{D}, \tag{1.91.3}$$

$\mathbf{A}-\mathbf{B}$ and $\mathbf{C}-\mathbf{D}$ are collinear.

1.92. Let $\|\mathbf{a}\| = 3, \|\mathbf{b}\| = 4, \|\mathbf{c}\| = 5$ such that each vector is perpendicular to the other two. Find $\|\mathbf{a}+\mathbf{b}+\mathbf{c}\|$.
Solution: Given that

$$\mathbf{a}^T\mathbf{b} = \mathbf{b}^T\mathbf{c} = \mathbf{c}^T\mathbf{a} = 0. \tag{1.92.1}$$

Then,

$$\|\mathbf{a}+\mathbf{b}+\mathbf{c}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 + \|\mathbf{c}\|^2 + \mathbf{a}^T\mathbf{b} + \mathbf{b}^T\mathbf{c} + \mathbf{c}^T\mathbf{a}. \tag{1.92.2}$$

which reduces to

$$\|\mathbf{a}+\mathbf{b}+\mathbf{c}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 + \|\mathbf{c}\|^2 \tag{1.92.3}$$

using (1.92.1)

1.93. Given

$$\mathbf{a}+\mathbf{b}+\mathbf{c} = \mathbf{0}, \tag{1.93.1}$$

evaluate

$$\mathbf{a}^T\mathbf{b} + \mathbf{b}^T\mathbf{c} + \mathbf{c}^T\mathbf{a}, \tag{1.93.2}$$

given that $\|\mathbf{a}\| = 3, \|\mathbf{b}\| = 4$ and $\|\mathbf{c}\| = 2$.
Solution: Multiplying (1.93.1) with $\mathbf{a}, \mathbf{b}, \mathbf{c}$,

$$\|\mathbf{a}\|^2 + \mathbf{a}^T\mathbf{b} + \mathbf{a}^T\mathbf{c} = 0 \tag{1.93.3}$$

$$\mathbf{a}^T\mathbf{b} + \|\mathbf{b}\|^2 + \mathbf{b}^T\mathbf{c} = 0 \tag{1.93.4}$$

$$+\mathbf{c}^T\mathbf{a} + \mathbf{b}^T\mathbf{c} + \|\mathbf{c}\|^2 = 0 \tag{1.93.5}$$

Adding all the above equations and rearranging,

$$\mathbf{a}^T\mathbf{b} + \mathbf{b}^T\mathbf{c} + \mathbf{c}^T\mathbf{a} = -\frac{\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 + \|\mathbf{c}\|^2}{2} \tag{1.93.6}$$

1.94. Let $\boldsymbol{\alpha} = \begin{pmatrix}3\\-1\\0\end{pmatrix}, \boldsymbol{\beta} = \begin{pmatrix}2\\1\\-3\end{pmatrix}$. Find $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2$ such that $\boldsymbol{\beta} = \boldsymbol{\beta}_1 + \boldsymbol{\beta}_2, \boldsymbol{\beta}_1 \parallel \boldsymbol{\alpha}$ and $\boldsymbol{\beta}_2 \perp \boldsymbol{\alpha}$.
Solution: Let $\beta_1 = k\alpha$. Then,

$$\boldsymbol{\beta} = k\boldsymbol{\alpha} + \boldsymbol{\beta}_2 \tag{1.94.1}$$

$$\implies k = \frac{\boldsymbol{\alpha}^T\boldsymbol{\beta}}{\|\boldsymbol{\alpha}\|^2} \tag{1.94.2}$$

and

$$\boldsymbol{\beta}_2 = \boldsymbol{\beta} - k\boldsymbol{\alpha} \tag{1.94.3}$$

This process is known as *Gram-Schmidth orthogonalization*.

1.95. Find a vector $\mathbf{x}$ in the direction of $\begin{pmatrix}1\\-2\end{pmatrix}$ such that $\|\mathbf{x}\| = 7$. **Solution:** Let $\mathbf{x} = k\begin{pmatrix}1\\-2\end{pmatrix}$. Then

$$\|\mathbf{x}\| = |k|\left\|\begin{pmatrix}1\\-2\end{pmatrix}\right\| = 7 \tag{1.95.1}$$

$$\implies |k| = \frac{7}{\sqrt{5}} \tag{1.95.2}$$

$$\text{or, } \mathbf{x} = \frac{7}{\sqrt{5}}\begin{pmatrix}1\\-2\end{pmatrix} \tag{1.95.3}$$

1.96. Find the direction vector of PQ, where

$$\mathbf{P} = \begin{pmatrix}2\\3\\0\end{pmatrix}, \mathbf{Q} = \begin{pmatrix}-1\\-2\\-4\end{pmatrix} \tag{1.96.1}$$

Solution: The direction vector of PQ is

$$\mathbf{P} - \mathbf{Q} = \begin{pmatrix}3\\5\\4\end{pmatrix}, \tag{1.96.2}$$

1.97. Draw a line segement of length 7.6 cm and divide it in the ratio 5 : 8.
Solution: Let the end points of the line be

$$\mathbf{A} = \begin{pmatrix}0\\0\end{pmatrix}, \mathbf{B} = \begin{pmatrix}7.6\\0\end{pmatrix} \tag{1.97.1}$$

Using section formula, the point $\mathbf{C}$

$$\mathbf{C} = \frac{k\mathbf{B} + \mathbf{A}}{k+1} \tag{1.97.2}$$

If $\mathbf{C}$ divides AB in the ratio

$$m = \frac{5}{8}, \tag{1.97.3}$$

then,

$$\frac{\|\mathbf{C} - \mathbf{A}\|^2}{\|\mathbf{B} - \mathbf{C}\|^2} = m^2 \tag{1.97.4}$$

$$\implies \frac{\frac{k^2\|\mathbf{B}-\mathbf{A}\|^2}{(k+1)^2}}{\frac{\|\mathbf{B}-\mathbf{A}\|^2}{(k+1)^2}} = m^2 \tag{1.97.5}$$

$$\implies k = m \tag{1.97.6}$$

upon substituting from (1.97.4) and simplifying. (1.97.2) is known as the section formula. The following code plots Fig. 1.97

```
codes/line/draw_section.py
```

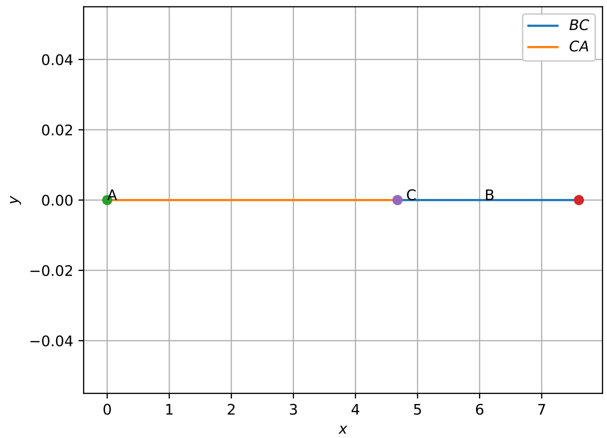

Fig. 1.97

1.98. Find the coordinates of the point which divides the line segment joining the points $\begin{pmatrix}4\\-3\end{pmatrix}$ and $\begin{pmatrix}8\\5\end{pmatrix}$ in the ratio 3 : 1 internally.
Solution: Using (1.97.2), the desired point is

$$\mathbf{P} = \frac{3\begin{pmatrix}4\\-3\end{pmatrix} + \begin{pmatrix}8\\5\end{pmatrix}}{4} \tag{1.98.1}$$

1.99. In what ratio does the point $\begin{pmatrix}-4\\6\end{pmatrix}$ divide the line segment joining the points

$$\mathbf{A} = \begin{pmatrix}-6\\10\end{pmatrix}, \mathbf{B} = \begin{pmatrix}3\\-8\end{pmatrix} \tag{1.99.1}$$

Solution: Use (1.97.2).

1.100. Find the coordinates of the points of trisection of the line segement joining the points

$$\mathbf{A} = \begin{pmatrix}2\\-2\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-7\\4\end{pmatrix} \tag{1.100.1}$$

Solution: Using (1.97.2), the coordinates are

$$\mathbf{P} = \frac{2\mathbf{A} + \mathbf{B}}{3} \tag{1.100.2}$$

$$\mathbf{Q} = \frac{\mathbf{A} + 2\mathbf{B}}{3} \tag{1.100.3}$$

1.101. Find the ratio in which the y-axis divides the line segment joining the points $\begin{pmatrix}5\\-6\end{pmatrix}$ and $\begin{pmatrix}-1\\-4\end{pmatrix}$.
Solution: Let the corresponding point on the y-axis be $\begin{pmatrix}0\\y\end{pmatrix}$. If the ratio be $k : 1$, using (1.97.2),

the coordinates are

$$\begin{pmatrix} 0 \\ y \end{pmatrix} = k\begin{pmatrix} 5 \\ -6 \end{pmatrix} + \begin{pmatrix} -1 \\ -4 \end{pmatrix} \tag{1.101.1}$$

$$\implies 0 = 5k - 1 \implies k = \frac{1}{5} \tag{1.101.2}$$

1.102. Find the value of k if the points $\mathbf{A} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 4 \\ k \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 6 \\ -3 \end{pmatrix}$ are collinear.

Solution: Forming the matrix

$$\mathbf{M} = \begin{pmatrix} \mathbf{B} - \mathbf{A} & \mathbf{B} - \mathbf{A} \end{pmatrix}^T = \begin{pmatrix} 2 & k-3 \\ 4 & -6 \end{pmatrix} \tag{1.102.1}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{R_2}{2}} \begin{pmatrix} 2 & k-3 \\ 2 & -3 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 2 & k-3 \\ 0 & -k \end{pmatrix} \tag{1.102.2}$$

$$\implies rank(\mathbf{M}) = 1 \iff R_2 = \mathbf{0}, \text{or } k = 0 \tag{1.102.3}$$

1.103. Find the coordinates of the point which divides the join of

$$\begin{pmatrix} -1 \\ 7 \end{pmatrix}, \begin{pmatrix} 4 \\ -3 \end{pmatrix} \tag{1.103.1}$$

in the ratio 2 : 3.

Solution:

1. $\mathbf{A} = \begin{pmatrix} -1 \\ 7 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$

   Then $\mathbf{C}$ that divides $\mathbf{A}, \mathbf{B}$ in the ratio $k : 1$ is

$$\mathbf{C} = \frac{k\mathbf{A} + \mathbf{B}}{k+1} \tag{1.103.2}$$

   For the given problem k=2 : 3

   Using the equation 1.103.2, the desired point is

$$\mathbf{C} = \frac{\frac{2}{3}\begin{pmatrix} -1 \\ 7 \end{pmatrix} + \begin{pmatrix} 4 \\ -3 \end{pmatrix}}{\frac{2}{3} + 1} \tag{1.103.3}$$

$$\therefore \mathbf{C} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{1.103.4}$$

   The following code plots Fig. 1.103

```
codes/line/section.py
```

1.104. Find the coordinates of the points of trisection of the line segment joining $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ -3 \end{pmatrix}$.

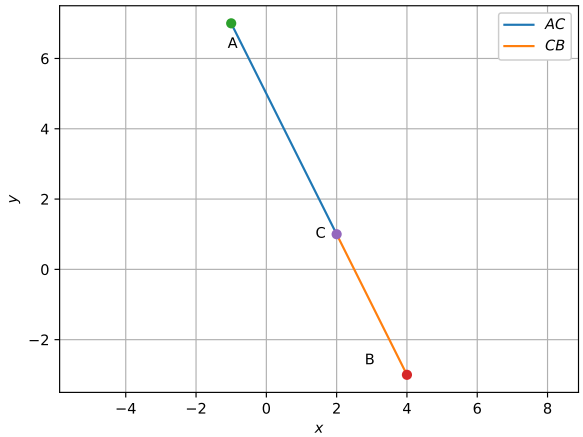

Fig. 1.103

Solution: The points of trisection are

$$\mathbf{C} = \frac{0.5\mathbf{A} + \mathbf{B}}{0.5 + 1} \tag{1.104.1}$$

$$\mathbf{D} = \frac{2\mathbf{A} + \mathbf{B}}{2 + 1} \tag{1.104.2}$$

$$\implies \therefore \mathbf{C} = \begin{pmatrix} 0 \\ -2.33 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 2 \\ -1.66 \end{pmatrix} \tag{1.104.3}$$

The following Python code generates Fig. 1.104

```
solutions/2/codes/line_ex/pts_on_a_line/
    trisection.py
```

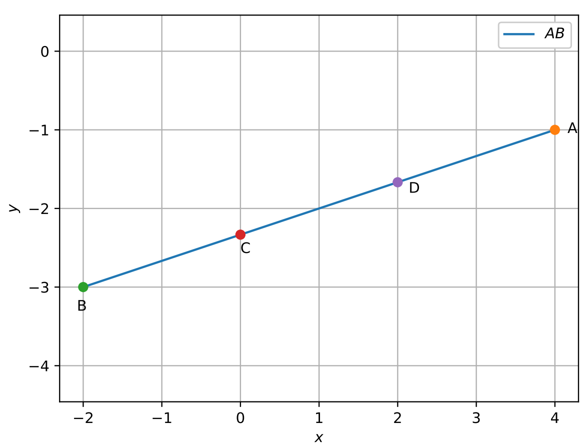

Fig. 1.104

1.105. Find the ratio in which the line segment joining the points $\begin{pmatrix} -3 \\ 10 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ -8 \end{pmatrix}$ is divided by $\begin{pmatrix} -1 \\ 6 \end{pmatrix}$.

Solution: Let

$$\mathbf{A} = \begin{pmatrix} -3 \\ 10 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 6 \\ -8 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -1 \\ 6 \end{pmatrix} \tag{1.105.1}$$

Then by section formula,

$$\mathbf{C} = \frac{k\mathbf{B} + \mathbf{A}}{k+1} \tag{1.105.2}$$

$$\begin{pmatrix} -1 \\ 6 \end{pmatrix} = \frac{1}{k+1} \begin{pmatrix} 6k - 3 \\ -8k + 10 \end{pmatrix} \tag{1.105.3}$$

$$\implies k = \frac{2}{7} \tag{1.105.4}$$

The following Python code generates Fig. 1.105

```
solutions/3/codes/line/section/section.py
```

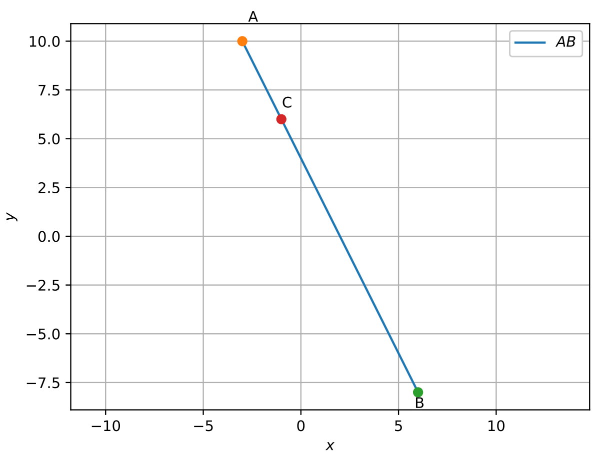

Fig. 1.105: C divides AB in ratio k:1

1.106. Find the ratio in whcih the line segment joining $\mathbf{A} = \begin{pmatrix} 1 \\ -5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -4 \\ 5 \end{pmatrix}$ is divided by the x-axis. Also find the coordinates of the point of division.
Solution: Let

$$\mathbf{C} \begin{pmatrix} x \\ 0 \end{pmatrix} \tag{1.106.1}$$

divide **AB** in k:1 ratio. Then,

$$(k+1) \begin{pmatrix} x \\ 0 \end{pmatrix} = k \begin{pmatrix} 1 \\ -5 \end{pmatrix} + \begin{pmatrix} -4 \\ 5 \end{pmatrix} \tag{1.106.2}$$

$$\implies 0 = -5k + 5 \tag{1.106.3}$$

$$\text{or, } k = 1 \tag{1.106.4}$$

$$\mathbf{C} = \frac{\begin{pmatrix} -3 \\ 0 \end{pmatrix}}{2} = \begin{pmatrix} -1.5 \\ 0 \end{pmatrix} \tag{1.106.5}$$

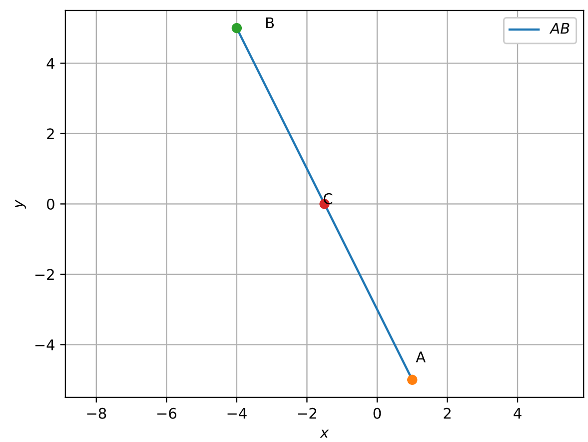

Fig. 1.106: line

The following code plots Fig. 1.106

```
solutions/4/codes/line/point_on_line/
    points_on_line.py
```

1.107. If $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 4 \\ y \end{pmatrix}$, $\begin{pmatrix} x \\ 6 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$ are the vertices of a parallelogram taken in order, find x and y.
Solution: See Fig. 1.107. In a parallelogram, the diagonals bisect each other. Hence

$$\frac{\mathbf{A} + \mathbf{C}}{2} = \frac{\mathbf{B} + \mathbf{D}}{2} \tag{1.107.1}$$

$$\therefore \frac{1+x}{2} = \frac{7}{2}, \frac{8}{2} = \frac{y+5}{2} \tag{1.107.2}$$

$$\implies x = 6, y = 3 \tag{1.107.3}$$

The following python code computes the value of x and y used in Fig. 1.107.

```
./solutions/5/codes/lines/q10.py
```

1.108. If $\mathbf{A} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$ respectively, find the coordinates of **P** such that $AP = \frac{3}{7}AB$ and **P**

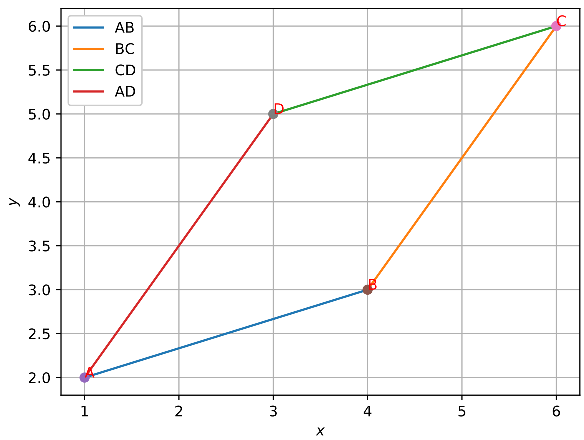

Fig. 1.107: Parallelogram of Q.3.6.5

lies on the line segment AB.
Solution: The desired point is

$$\mathbf{P} = \frac{\frac{3}{4}\begin{pmatrix}2\\-4\end{pmatrix} + 1\begin{pmatrix}-2\\-2\end{pmatrix}}{\frac{3}{4}+1} \tag{1.108.1}$$

$$\mathbf{P} = \begin{pmatrix}-2/7\\-20/7\end{pmatrix} \tag{1.108.2}$$

The following python code plots the Fig. 1.108

```
solutions/6/codes/point_line/int_sec.py
```

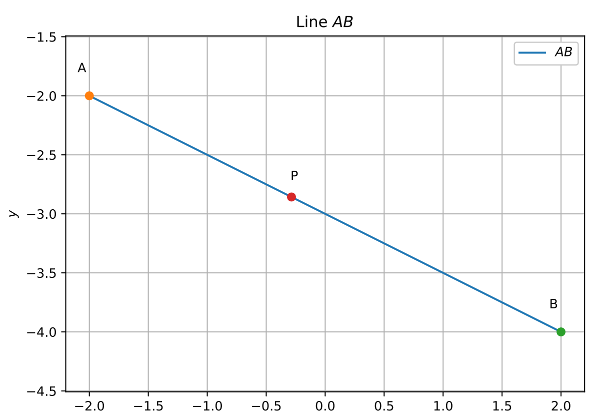

Fig. 1.108

1.109. Find the coordinates of the points which divide the line segment joining $\mathbf{A} = \begin{pmatrix}-2\\2\end{pmatrix}, \mathbf{B} = \begin{pmatrix}2\\8\end{pmatrix}$ into four equal parts.

Solution: The desired coordinates are

$$\mathbf{D} = \frac{1\mathbf{B} + 3\mathbf{A}}{4} = \begin{pmatrix}-1\\7/2\end{pmatrix} \tag{1.109.1}$$

$$\mathbf{E} = \frac{2\mathbf{B} + 2\mathbf{A}}{4} = \begin{pmatrix}0\\5\end{pmatrix} \tag{1.109.2}$$

$$\mathbf{F} = \frac{3\mathbf{B} + 1\mathbf{A}}{4} = \begin{pmatrix}1\\13/2\end{pmatrix} \tag{1.109.3}$$

The following code plots Fig. 1.109

```
solutions/7/codes/line/point_line/
    line_division.py
```

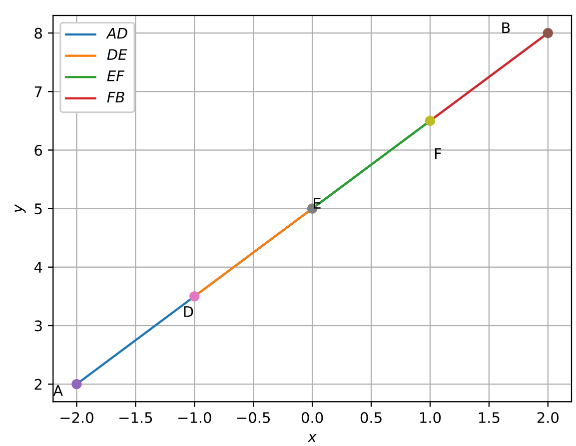

Fig. 1.109

1.110. Find $\begin{pmatrix}5\\-3\end{pmatrix}^3$

Solution: In general, the complex number $\begin{pmatrix}a_1\\a_2\end{pmatrix}$ has the matrix representation

$$\begin{pmatrix}a_1\\a_2\end{pmatrix} = \begin{pmatrix}a_1 & -a_2\\a_2 & a_1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} \tag{1.110.1}$$

$$= \mathbf{T}_a\begin{pmatrix}1\\0\end{pmatrix} \tag{1.110.2}$$

$$\implies \begin{pmatrix}5\\-3\end{pmatrix} = \begin{pmatrix}5 & 3\\-3 & 5\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} \tag{1.110.3}$$

Then,

$$\begin{pmatrix}5\\-3\end{pmatrix}^3 \triangleq \begin{pmatrix}5 & 3\\-3 & 5\end{pmatrix}^3\begin{pmatrix}1\\0\end{pmatrix} \tag{1.110.4}$$

$$= \begin{pmatrix}-10 & 198\\-198 & -10\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} \tag{1.110.5}$$

$$= \begin{pmatrix}-10\\-198\end{pmatrix} \tag{1.110.6}$$

The python code for above problem is

```
codes/line/comp.py
```

1.111. Find $\begin{pmatrix}-\sqrt{3}\\\sqrt{2}\end{pmatrix}\begin{pmatrix}2\sqrt{3}\\-1\end{pmatrix}$.

Solution: Using the equivalent matrices for the complex numbers,

$$\begin{pmatrix}-\sqrt{3}\\\sqrt{2}\end{pmatrix}\begin{pmatrix}2\sqrt{3}\\-1\end{pmatrix} = \begin{pmatrix}-\sqrt{3} & -\sqrt{2}\\\sqrt{2} & -\sqrt{3}\end{pmatrix}\begin{pmatrix}2\sqrt{3} & 1\\-1 & 2\sqrt{3}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$$
$$= \begin{pmatrix}\sqrt{2}-6 & -\sqrt{3}-2\sqrt{6}\\\sqrt{3}+2\sqrt{6} & \sqrt{2}-6\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\sqrt{2}-6\\\sqrt{3}+2\sqrt{6}\end{pmatrix} \tag{1.111.1}$$

The following code verifies the result.

```
codes/line_ex/complex_ex/complex_ex.py
```

1.112. Find the multiplicative inverse of $\begin{pmatrix}2\\-3\end{pmatrix}$.

Solution: Let $\mathbf{T}_a$ be the matrix for the complex number $\mathbf{a}$. $\mathbf{b}$ is defined to be the multiplicative inverse of $\mathbf{a}$ if

$$\mathbf{T}_a\mathbf{T}_b = \mathbf{T}_b\mathbf{T}_a = \mathbf{I} \tag{1.112.1}$$

Then, from (1.110.1)

$$\mathbf{b} = \mathbf{a}^{-1} = \begin{pmatrix}a_1 & -a_2\\a_2 & a_1\end{pmatrix}^{-1}\begin{pmatrix}1\\0\end{pmatrix} \tag{1.112.2}$$

$$= \frac{1}{\|\mathbf{a}\|^2}\begin{pmatrix}a_1\\-a_2\end{pmatrix} \tag{1.112.3}$$

Thus,

$$\begin{pmatrix}2\\-3\end{pmatrix}^{-1} = \frac{1}{13}\begin{pmatrix}2\\3\end{pmatrix} \tag{1.112.4}$$

The python code for above problem is

```
solutions/3/codes/line/comp/comp.py
```

Note that

$$\mathbf{T}_b = \mathbf{T}_a^{-1} = \frac{\mathbf{T}_a^T}{\left\|\mathbf{a}^2\right\|} \tag{1.112.5}$$

1.113. Find

a) $\begin{pmatrix}5\\\sqrt{2}\end{pmatrix}\begin{pmatrix}1\\-2\sqrt{3}\end{pmatrix}$.

b) $\begin{pmatrix}0\\1\end{pmatrix}^{-35}$.

c) Show that the polar representation of $\begin{pmatrix}1\\\sqrt{3}\end{pmatrix}$ is $2\angle 60°$.

1.114. Simplify the complex number $-\frac{16}{\begin{pmatrix}1\\\sqrt{3}\end{pmatrix}}$

Solution: Using the polar form,

$$\begin{pmatrix}1\\\sqrt{3}\end{pmatrix} = 2\begin{pmatrix}\cos 60°\\\sin 60°\end{pmatrix} = 2\underline{/60°} \tag{1.114.1}$$

$$\implies \frac{-16}{\begin{pmatrix}1\\\sqrt{3}\end{pmatrix}} = -8\underline{/-60°} = 4\begin{pmatrix}-1\\\sqrt{3}\end{pmatrix} \tag{1.114.2}$$

The following python code gives the desired answer

```
./solutions/5/codes/lines/q8.py
```

1.115. Find the conjugate of $\frac{\begin{pmatrix}3\\-2\end{pmatrix}\begin{pmatrix}2\\3\end{pmatrix}}{\begin{pmatrix}1\\2\end{pmatrix}\begin{pmatrix}2\\-1\end{pmatrix}}$.

Solution: Using the matrix form,

$$\frac{\begin{pmatrix}3\\-2\end{pmatrix}\begin{pmatrix}2\\3\end{pmatrix}}{\begin{pmatrix}1\\2\end{pmatrix}\begin{pmatrix}2\\-1\end{pmatrix}}$$
$$= \begin{pmatrix}3 & 2\\-2 & 3\end{pmatrix}\begin{pmatrix}2 & -3\\3 & 2\end{pmatrix}\left[\begin{pmatrix}1 & -2\\2 & 1\end{pmatrix}\begin{pmatrix}2 & 1\\-1 & 2\end{pmatrix}\right]^{-1}\begin{pmatrix}1\\0\end{pmatrix}$$
$$= \frac{1}{25}\begin{pmatrix}63\\-16\end{pmatrix} \tag{1.115.1}$$

The conjugate is given by

$$\frac{1}{25}\begin{pmatrix}63\\16\end{pmatrix} \tag{1.115.2}$$

1.116. Find the modulus and argument of the complex numbers

a) $\frac{\begin{pmatrix}1\\1\end{pmatrix}}{\begin{pmatrix}1\\-1\end{pmatrix}}$.

b) $\frac{1}{\begin{pmatrix}1\\1\end{pmatrix}}$.

Solution:

$$\begin{pmatrix}1\\1\end{pmatrix} = \sqrt{2}\begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{pmatrix} \tag{1.116.1}$$

$$= \sqrt{2}\begin{pmatrix}\cos 45^\circ\\\sin 45^\circ\end{pmatrix} \tag{1.116.2}$$

In the above, the modulus is $\left\|\begin{pmatrix}1\\1\end{pmatrix}\right\| = \sqrt{2}$ and the argument is 45°. Similarly,

$$\begin{pmatrix}1\\-1\end{pmatrix} = \sqrt{2}\begin{pmatrix}\cos 45^\circ\\-\sin 45^\circ\end{pmatrix} \tag{1.116.3}$$

$$\implies \begin{pmatrix}1\\-1\end{pmatrix}^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix}\cos 45^\circ\\\sin 45^\circ\end{pmatrix} \tag{1.116.4}$$

Using the matrix representation,

$$\frac{\begin{pmatrix}1\\1\end{pmatrix}}{\begin{pmatrix}1\\-1\end{pmatrix}} = \begin{pmatrix}\cos 45^\circ & -\sin 45^\circ\\\sin 45^\circ & \cos 45^\circ\end{pmatrix}$$

$$\times\begin{pmatrix}\cos 45^\circ & -\sin 45^\circ\\\sin 45^\circ & \cos 45^\circ\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} \tag{1.116.5}$$

$$= \begin{pmatrix}\cos 90^\circ\\\sin 90^\circ\end{pmatrix} = 1\underline{/90^\circ} \tag{1.116.6}$$

In general, if

$$\mathbf{z}_1 = r_1\begin{pmatrix}\cos\theta_1\\\sin\theta_1\end{pmatrix}, \mathbf{z}_2 = r_2\begin{pmatrix}\cos\theta_2\\\sin\theta_2\end{pmatrix}, \tag{1.116.7}$$

$$\mathbf{z}_1\mathbf{z}_2 = r_1r_2\begin{pmatrix}\cos(\theta_1+\theta_2)\\\sin(\theta_1+\theta_2)\end{pmatrix}. \tag{1.116.8}$$

Similarly, from (1.116.2),

$$\frac{1}{\begin{pmatrix}1\\1\end{pmatrix}} = \frac{1}{\sqrt{2}}\begin{pmatrix}\cos 45^\circ\\-\sin 45^\circ\end{pmatrix} \tag{1.116.9}$$

$$= \frac{1}{\sqrt{2}}\underline{/-45^\circ} \tag{1.116.10}$$

1.117. Find θ such that

$$\frac{\begin{pmatrix}3\\2\sin\theta\end{pmatrix}}{\begin{pmatrix}1\\-2\sin\theta\end{pmatrix}} \tag{1.117.1}$$

is purely real.

1.118. Convert the complex number

$$\mathbf{z} = \frac{\begin{pmatrix}-1\\1\end{pmatrix}}{\begin{pmatrix}\cos\frac{\pi}{3}\\\sin\frac{\pi}{3}\end{pmatrix}} \tag{1.118.1}$$

in the polar form.

1.119. Simplify

$$\mathbf{z} = \left(\frac{1}{\begin{pmatrix}1\\-4\end{pmatrix}} - \frac{2}{\begin{pmatrix}2\\1\end{pmatrix}}\right)\frac{\begin{pmatrix}3\\-4\end{pmatrix}}{\begin{pmatrix}5\\1\end{pmatrix}} \tag{1.119.1}$$

Solution: Using equivalent matrices for the complex numbers and matrix multiplication,

$$= \left(\begin{pmatrix}1 & 4\\-4 & 1\end{pmatrix}^{-1} - 2\begin{pmatrix}2 & -1\\1 & 2\end{pmatrix}^{-1}\right)\begin{pmatrix}3 & 4\\-4 & 3\end{pmatrix}\begin{pmatrix}5 & -1\\1 & 5\end{pmatrix}^{-1}$$

$$= \left(\frac{1}{1^2+4^2}\begin{pmatrix}1 & -4\\4 & 1\end{pmatrix} - 2\left(\frac{1}{2^2+1^2}\right)\begin{pmatrix}2 & 1\\-1 & 2\end{pmatrix}\right)\begin{pmatrix}3 & 4\\-4 & 3\end{pmatrix}$$

$$\frac{1}{5^2+1^2}\begin{pmatrix}5 & 1\\-1 & 5\end{pmatrix}$$

$$= \left(\frac{1}{1+16}\begin{pmatrix}1 & -4\\4 & 1\end{pmatrix} - \frac{2}{4+1}\begin{pmatrix}2 & 1\\-1 & 2\end{pmatrix}\right)\begin{pmatrix}3 & 4\\-4 & 3\end{pmatrix}$$

$$\frac{1}{25+1}\begin{pmatrix}5 & 1\\-1 & 5\end{pmatrix}$$

$$= \left(\frac{1}{17}\begin{pmatrix}1 & -4\\4 & 1\end{pmatrix} - \frac{2}{5}\begin{pmatrix}2 & 1\\-1 & 2\end{pmatrix}\right)\begin{pmatrix}3 & 4\\-4 & 3\end{pmatrix}\frac{1}{26}\begin{pmatrix}5 & 1\\-1 & 5\end{pmatrix}$$

$$= \left(\begin{pmatrix}\frac{1}{17} & \frac{-4}{17}\\\frac{4}{17} & \frac{1}{17}\end{pmatrix} - \begin{pmatrix}\frac{4}{5} & \frac{2}{5}\\\frac{-2}{5} & \frac{4}{5}\end{pmatrix}\right)\begin{pmatrix}3 & 4\\-4 & 3\end{pmatrix}\frac{1}{26}\begin{pmatrix}5 & 1\\-1 & 5\end{pmatrix}$$

$$= \begin{pmatrix}\frac{1}{17}-\frac{4}{5} & \frac{-4}{17}-\frac{2}{5}\\\frac{4}{17}+\frac{2}{5} & \frac{1}{17}-\frac{4}{5}\end{pmatrix}\begin{pmatrix}3 & 4\\-4 & 3\end{pmatrix}\frac{1}{26}\begin{pmatrix}5 & 1\\-1 & 5\end{pmatrix}$$

$$= \begin{pmatrix}\frac{-63}{85} & \frac{-54}{85}\\\frac{54}{85} & \frac{-63}{85}\end{pmatrix}\begin{pmatrix}3 & 4\\-4 & 3\end{pmatrix}\frac{1}{26}\begin{pmatrix}5 & 1\\-1 & 5\end{pmatrix}$$

$$= \frac{1}{85}\left(\begin{pmatrix}-63 & -54\\54 & -63\end{pmatrix}\begin{pmatrix}3 & 4\\-4 & 3\end{pmatrix}\right)\frac{1}{26}\begin{pmatrix}5 & 1\\-1 & 5\end{pmatrix}$$

$$= \frac{1}{2210}\left(\begin{pmatrix}-63 & -54\\54 & -63\end{pmatrix}\begin{pmatrix}3 & 4\\-4 & 3\end{pmatrix}\right)\begin{pmatrix}5 & 1\\-1 & 5\end{pmatrix}$$

$$= \frac{1}{2210}\begin{pmatrix}-189+216 & -162-252\\162+252 & 216-189\end{pmatrix}\begin{pmatrix}5 & 1\\-1 & 5\end{pmatrix}$$

$$= \frac{1}{2210}\begin{pmatrix}27 & -414\\414 & 27\end{pmatrix}\begin{pmatrix}5 & 1\\-1 & 5\end{pmatrix}$$

$$= \frac{1}{2210}\begin{pmatrix}27 & -414\\414 & 27\end{pmatrix}\begin{pmatrix}5 & 1\\-1 & 5\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$$

$$= \frac{1}{2210}\begin{pmatrix}135+414 & 27-2070\\2070-27 & 414+135\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$$

$$= \frac{1}{2210}\begin{pmatrix}549 & -2043\\2043 & 549\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$$

$$= \frac{1}{2210}\begin{pmatrix}549\\2043\end{pmatrix}$$

$$\Longrightarrow \mathbf{z} = \begin{pmatrix}\frac{549}{2210}\\\frac{2043}{2210}\end{pmatrix} \quad (1.119.2)$$

1.120. Convert the following in the polar form:

a) $\dfrac{\begin{pmatrix}1\\7\end{pmatrix}}{\begin{pmatrix}2\\-1\end{pmatrix}^2}$.

b) $\dfrac{\begin{pmatrix}1\\3\end{pmatrix}}{\begin{pmatrix}1\\-2\end{pmatrix}}$.

Solution:

a) Below is the solution :

$$\frac{\begin{pmatrix}1\\7\end{pmatrix}}{\begin{pmatrix}2\\-1\end{pmatrix}^2} \quad (1.120.1)$$

$$\begin{pmatrix}2\\-1\end{pmatrix}^2 = \begin{pmatrix}2 & 1\\-1 & 2\end{pmatrix}\begin{pmatrix}2 & 1\\-1 & 2\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} \quad (1.120.2)$$

$$\Longrightarrow \begin{pmatrix}2\\-1\end{pmatrix}^2 = \begin{pmatrix}3 & 4\\-4 & 3\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} \quad (1.120.3)$$

$$\Longrightarrow \begin{pmatrix}2\\-1\end{pmatrix}^2 = \begin{pmatrix}3\\-4\end{pmatrix} \quad (1.120.4)$$

$$= \begin{pmatrix}1\\7\end{pmatrix}\begin{pmatrix}3\\-4\end{pmatrix}^{-1} \quad (1.120.5)$$

$$= \frac{1}{25}\begin{pmatrix}1 & -7\\7 & 1\end{pmatrix}\begin{pmatrix}3 & -4\\4 & 3\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} \quad (1.120.6)$$

$$= \frac{1}{25}\begin{pmatrix}-25 & -25\\25 & -25\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} \quad (1.120.7)$$

$$= \frac{25}{25}\begin{pmatrix}-1 & -1\\1 & -1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} \quad (1.120.8)$$

$$= \sqrt{2}\begin{pmatrix}\frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\\\frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{pmatrix} \quad (1.120.9)$$

$$= \sqrt{2}\begin{pmatrix}\cos 135^\circ & -\sin 135^\circ\\\sin 135^\circ & \cos 135^\circ\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} \quad (1.120.10)$$

$$= \sqrt{2}\begin{pmatrix}\cos 135^\circ\\\sin 135^\circ\end{pmatrix} \quad (1.120.11)$$

$$= \sqrt{2}\angle 135^\circ \quad (1.120.12)$$

b) Below is the solution:

$$\frac{\begin{pmatrix}1\\3\end{pmatrix}}{\begin{pmatrix}1\\-2\end{pmatrix}} \tag{1.120.13}$$

$$\begin{pmatrix}1\\-2\end{pmatrix} = \begin{pmatrix}1 & 2\\-2 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} \tag{1.120.14}$$

$$= \begin{pmatrix}1\\7\end{pmatrix}\begin{pmatrix}1\\-2\end{pmatrix}^{-1} \tag{1.120.15}$$

$$= \frac{1}{5}\begin{pmatrix}1 & -3\\3 & 1\end{pmatrix}\begin{pmatrix}1 & -2\\2 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} \tag{1.120.16}$$

$$= \frac{1}{5}\begin{pmatrix}-5 & -5\\5 & -5\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} \tag{1.120.17}$$

$$= \frac{5}{5}\begin{pmatrix}-1 & -1\\1 & -1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} \tag{1.120.18}$$

$$= \sqrt{2}\begin{pmatrix}\frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{pmatrix} \tag{1.120.19}$$

$$= \sqrt{2}\begin{pmatrix}\cos 135^\circ & -\sin 135^\circ\\ \sin 135^\circ & \cos 135^\circ\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} \tag{1.120.20}$$

$$= \sqrt{2}\begin{pmatrix}\cos 135^\circ\\ \sin 135^\circ\end{pmatrix} \tag{1.120.21}$$

$$= \sqrt{2}\angle 135^\circ \tag{1.120.22}$$

1.121. If $\mathbf{z}_1 = \begin{pmatrix}2\\-1\end{pmatrix}, \mathbf{z}_2 = \begin{pmatrix}1\\1\end{pmatrix}$, find $\left\|\frac{\mathbf{z}_1+\mathbf{z}_1+1}{\mathbf{z}_1-\mathbf{z}_2+1}\right\|$

Solution: Let us consider $\frac{\mathbf{z}_1+\mathbf{z}_1+1}{\mathbf{z}_1-\mathbf{z}_2+1}$, then

$$\mathbf{z}_1 + \mathbf{z}_1 + 1 = \begin{pmatrix}2\\-1\end{pmatrix} + \begin{pmatrix}2\\-1\end{pmatrix} + \begin{pmatrix}1\\0\end{pmatrix} \tag{1.121.1}$$

$$= \begin{pmatrix}5\\-2\end{pmatrix} \tag{1.121.2}$$

$$\mathbf{z}_1 - \mathbf{z}_2 + 1 = \begin{pmatrix}2\\-1\end{pmatrix} - \begin{pmatrix}1\\1\end{pmatrix} + \begin{pmatrix}1\\0\end{pmatrix} \tag{1.121.3}$$

$$= \begin{pmatrix}2\\-2\end{pmatrix} \tag{1.121.4}$$

$$\frac{\mathbf{z}_1 + \mathbf{z}_1 + 1}{\mathbf{z}_1 - \mathbf{z}_2 + 1} = \frac{\begin{pmatrix}5\\-2\end{pmatrix}}{\begin{pmatrix}2\\-2\end{pmatrix}} \tag{1.121.5}$$

The modulus of a complex number $\begin{pmatrix}a\\b\end{pmatrix}$ is defined as $\sqrt{a^2+b^2}$. Therefore,

$$\|\mathbf{z}_1 + \mathbf{z}_1 + 1\| = \sqrt{5^2 + (-2)^2} \tag{1.121.6}$$

$$= \sqrt{29} \tag{1.121.7}$$

$$\|\mathbf{z}_1 - \mathbf{z}_2 + 1\| = \sqrt{2^2 + (-2)^2} \tag{1.121.8}$$

$$= \sqrt{8} \tag{1.121.9}$$

Putting together (1.121.7) and (1.121.9), we have

$$\left\|\frac{\mathbf{z}_1 + \mathbf{z}_1 + 1}{\mathbf{z}_1 - \mathbf{z}_2 + 1}\right\| = \frac{\sqrt{29}}{\sqrt{8}} \tag{1.121.10}$$

1.122. Let $\mathbf{z}_1 = \begin{pmatrix}2\\-1\end{pmatrix}, \mathbf{z}_2 = \begin{pmatrix}-2\\1\end{pmatrix}$. Find

a) $\mathrm{Re}\left(\frac{\mathbf{z}_1\mathbf{z}_2}{\mathbf{z}_1^*}\right)$.

b) $\mathrm{Im}\left(\frac{1}{\mathbf{z}_1\mathbf{z}_1^*}\right)$.

Solution:

$$\left(\frac{\mathbf{z_1 z_2}}{\mathbf{z_1}^*}\right) = \begin{pmatrix}2 & 1\\-1 & 2\end{pmatrix}\begin{pmatrix}-2 & -1\\1 & -2\end{pmatrix}\left[\begin{pmatrix}2 & -1\\1 & 2\end{pmatrix}\right]^{-1}\begin{pmatrix}1\\0\end{pmatrix} \tag{1.122.1}$$

$$\left(\frac{\mathbf{z_1 z_2}}{\mathbf{z_1}^*}\right) = \begin{pmatrix}2 & 1\\-1 & 2\end{pmatrix}\begin{pmatrix}-2 & -1\\1 & -2\end{pmatrix}\left[\frac{1}{5}\begin{pmatrix}2 & 1\\-1 & 2\end{pmatrix}\right]\begin{pmatrix}1\\0\end{pmatrix} \tag{1.122.2}$$

$$\left(\frac{\mathbf{z_1 z_2}}{\mathbf{z_1}^*}\right) = \frac{1}{5}\begin{pmatrix}-2 & -11\\11 & -2\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} \tag{1.122.3}$$

$$\left(\frac{\mathbf{z_1 z_2}}{\mathbf{z_1}^*}\right) = \frac{1}{5}\begin{pmatrix}-2\\11\end{pmatrix} \tag{1.122.4}$$

Hence, the real part of $\left(\frac{\mathbf{z_1 z_2}}{\mathbf{z_1}^*}\right) = -\frac{2}{5}$

$$\left(\frac{1}{\mathbf{z_1 z_1}^*}\right) = (\mathbf{z_1 z_1}^*)^{-1} \tag{1.122.5}$$

$$\left(\frac{1}{\mathbf{z_1 z_1}^*}\right) = \left[\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}\right]^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.122.6}$$

$$\left(\frac{1}{\mathbf{z_1 z_1}^*}\right) = \left[\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}\right]^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.122.7}$$

$$\left(\frac{1}{\mathbf{z_1 z_1}^*}\right) = \frac{1}{25}\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.122.8}$$

$$\left(\frac{1}{\mathbf{z_1 z_1}^*}\right) = \frac{1}{25}\begin{pmatrix} 5 \\ 0 \end{pmatrix} \tag{1.122.9}$$

Hence, the imaginary part of $\left(\frac{1}{\mathbf{z_1 z_1}^*}\right) = 0$.

1.123. Find the modulus and argument of the complex number $\frac{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}{\begin{pmatrix} 1 \\ -3 \end{pmatrix}}$.

Solution: In general, any complex number can be expressed in matrix representation as follows:

$$\begin{pmatrix} a1 \\ a2 \end{pmatrix} = \begin{pmatrix} a1 & -a2 \\ a2 & a1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.123.1}$$

Converting complex number to matrix form:

$$\frac{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}{\begin{pmatrix} 1 \\ -3 \end{pmatrix}} = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.123.2}$$

$$\begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1/10 & -3/10 \\ 3/10 & 1/10 \end{pmatrix} \tag{1.123.3}$$

Sub (1.123.3) in (1.123.2),

$$\frac{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}{\begin{pmatrix} 1 \\ -3 \end{pmatrix}} = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1/10 & -3/10 \\ 3/10 & 1/10 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.123.4}$$

$$= \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1/10 \\ 3/10 \end{pmatrix} \tag{1.123.5}$$

$$= \begin{pmatrix} -5/10 \\ 5/10 \end{pmatrix} \tag{1.123.6}$$

$$\implies \boxed{\frac{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}{\begin{pmatrix} 1 \\ -3 \end{pmatrix}} = \begin{pmatrix} -1/2 \\ 1/2 \end{pmatrix}} \tag{1.123.7}$$

From (1.123.7), The modulus and argument of the complex number is,

$$r = \left\|\begin{pmatrix} -1/2 \\ 1/2 \end{pmatrix}\right\| = \frac{1}{\sqrt{2}} \tag{1.123.8}$$

$$\tan\theta = -1 \implies \theta = 180^\circ - 45^\circ = 135^\circ \tag{1.123.9}$$

1.124. Find the real numbers x, y such that $\begin{pmatrix} x \\ -y \end{pmatrix}\begin{pmatrix} 3 \\ 5 \end{pmatrix}$ is the conjugate of $\begin{pmatrix} -6 \\ -24 \end{pmatrix}$.

Solution: The conjugate of $\begin{pmatrix} -6 \\ -24 \end{pmatrix}$ is $\begin{pmatrix} -6 \\ 24 \end{pmatrix}$

$$\implies \begin{pmatrix} x \\ -y \end{pmatrix}\begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} -6 \\ 24 \end{pmatrix} \tag{1.124.1}$$

$$\implies \begin{pmatrix} x \\ -y \end{pmatrix} = \frac{\begin{pmatrix} -6 \\ 24 \end{pmatrix}}{\begin{pmatrix} 3 \\ 5 \end{pmatrix}} \tag{1.124.2}$$

Using equivalent matrices for complex numbers, we have

$$\begin{pmatrix} x \\ -y \end{pmatrix} = \begin{pmatrix} -6 & -24 \\ 24 & -6 \end{pmatrix} \begin{pmatrix} 3 & -5 \\ 5 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.124.3}$$

$$= \frac{1}{34} \begin{pmatrix} -6 & -24 \\ 24 & -6 \end{pmatrix} \begin{pmatrix} 3 & 5 \\ -5 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.124.4}$$

$$= \frac{1}{34} \begin{pmatrix} 102 & -102 \\ 102 & 102 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.124.5}$$

$$= \begin{pmatrix} 3 & -3 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.124.6}$$

$$\implies \begin{pmatrix} x \\ -y \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} \tag{1.124.7}$$

$$\text{Therefore,} x = 3, \tag{1.124.8}$$

$$y = -3 \tag{1.124.9}$$

1.125. Find the modulus of $\dfrac{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 \\ -1 \end{pmatrix}} - \dfrac{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$.

Solution: In our case,

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.125.1}$$

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.125.2}$$

Now,

$$\frac{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 \\ -1 \end{pmatrix}} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.125.3}$$

$$= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.125.4}$$

Similarly,

$$\frac{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.125.5}$$

$$= \begin{pmatrix} 0 \\ -1 \end{pmatrix} \tag{1.125.6}$$

So,

$$\frac{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 \\ -1 \end{pmatrix}} - \frac{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ -1 \end{pmatrix} \tag{1.125.7}$$

$$= \begin{pmatrix} 0 \\ 2 \end{pmatrix} \tag{1.125.8}$$

Now, according to the problem statement:

$$\frac{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 \\ -1 \end{pmatrix}} - \frac{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} \tag{1.125.9}$$

$$= \begin{pmatrix} 0 \\ 2 \end{pmatrix} \tag{1.125.10}$$

$\therefore$

$$\left\| \frac{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 \\ -1 \end{pmatrix}} - \frac{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} \right\| \tag{1.125.11}$$

$$= \left\| \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right\| = \sqrt{0^2 + 2^2} = 2 \tag{1.125.12}$$

So, we can say that the modulus value of

$$\frac{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 \\ -1 \end{pmatrix}} - \frac{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} \tag{1.125.13}$$

is 2.

1.126. Rain is falling vertically with a speed of 35 ms^{-1} . Winds starts blowing after sometime with a speed of 12 ms^{-1} in east to west direction. In which direction should a boy waiting at a bus stop hold his umbrella ?

Solution: See Fig. 1.126. From the given information, the rain velocity is

$$\mathbf{u} = \begin{pmatrix} 0 \\ 35 \end{pmatrix} \tag{1.126.1}$$

and the wind velocity is

$$\mathbf{v} = -\begin{pmatrix} 12 \\ 0 \end{pmatrix} \tag{1.126.2}$$

The resulting rain velocity is

$$\mathbf{u}+\mathbf{v} = \begin{pmatrix} -12 \\ 35 \end{pmatrix} \tag{1.126.3}$$

The desired angle is

$$-\tan^{-1} \underline{/\mathbf{u}+\mathbf{v}} = \tan^{-1}\frac{12}{35} \tag{1.126.4}$$
$$\approx 20.04^\circ \tag{1.126.5}$$

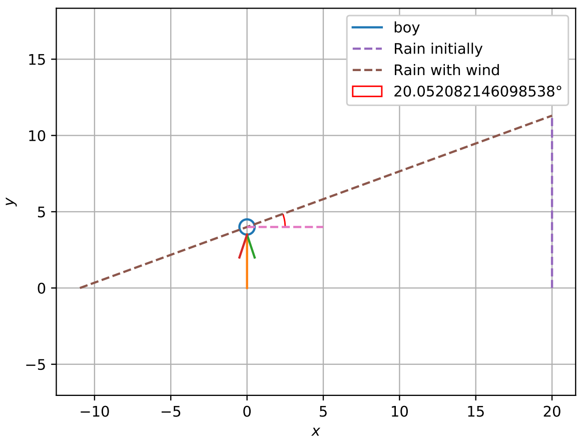

Fig. 1.126

1.127. A motorboat is racing towards north at 25 km/h and the water current in that region is 10 km/h in the direction of 60° east of south. Find the resultant velocity of the boat.
Solution: In Fig. 1.127, **A** denotes the velocity of the boat, **B** denotes the water current and **C** represents the resultant velocity.

$$\mathbf{A} = \begin{pmatrix} 0 \\ 25 \end{pmatrix} \tag{1.127.1}$$
$$\mathbf{B} = 10\begin{pmatrix} \cos 30^\circ \\ -\sin 30^\circ \end{pmatrix} \tag{1.127.2}$$
$$\mathbf{C} = \mathbf{A}+\mathbf{B} \tag{1.127.3}$$
$$= 5\begin{pmatrix} \sqrt{3} \\ 4 \end{pmatrix} \tag{1.127.4}$$

The following Python code generates Fig. 1.127

```
solutions/2/codes/line_ex/
    motion_in_a_plane/motion_plane.py
```

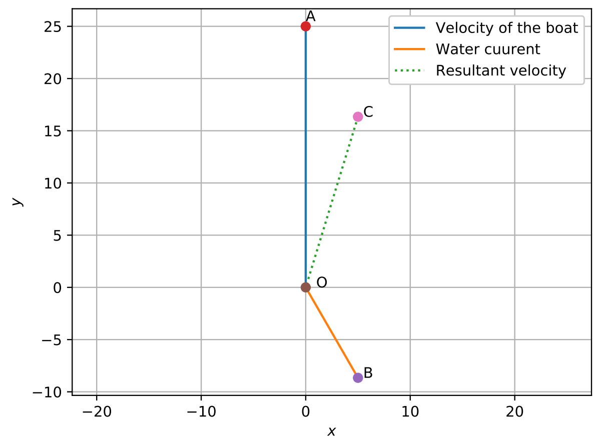

Fig. 1.127

1.128. Rain is falling vertically with a speed of 35 ms^{-1} . A woman rides a bicycle with a speed of 12 ms^{-1} in east to west direction. What is the direction in which she should hold her umbrella ?
Solution: See Fig. 1.128. The velocity of rain and velocity of woman are

$$\mathbf{v_r} = \begin{pmatrix} 0 \\ -35 \end{pmatrix} \tag{1.128.1}$$
$$\mathbf{v_w} = \begin{pmatrix} -12 \\ 0 \end{pmatrix} \tag{1.128.2}$$

The relative velocity of rain w.r.t woman is given as

$$\mathbf{v_{r_w}} = \mathbf{v_r} - \mathbf{v_w} \tag{1.128.3}$$
$$= \begin{pmatrix} 12 \\ -35 \end{pmatrix} \tag{1.128.4}$$

So the woman must hold the umbrella along the direction of $-\mathbf{v_{r_w}}$ Thus, the desired angle is

$$\theta = \tan^{-1}\left(\frac{12}{35}\right) \tag{1.128.5}$$

The following python code generates Fig. 1.128.

```
solutions/3/codes/line/rain/rain.py
```

1.129. A hiker stands on the edge of a cliff 490 m above the ground and throws a stone horizontally with an initial speed of 15 ms^{-1} . Neglecting air resistance, find the time taken by the stone to reach the ground, and the speed

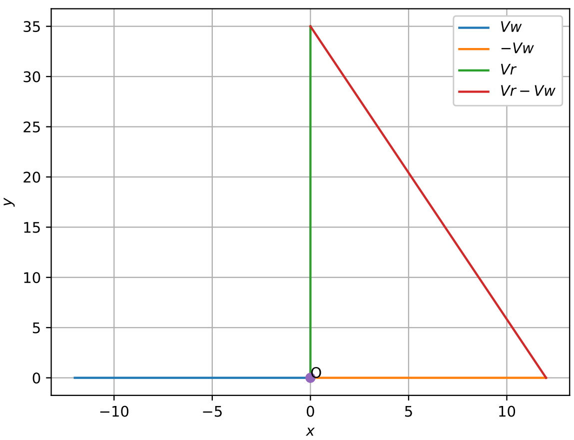

Fig. 1.128: Direction of umbrella

with which it hits the ground. (Take g = 9.8 ms^{-2}).

Solution: From the given information, the hicker's position vector is

$$\mathbf{A} = \begin{pmatrix} 0 \\ 490 \end{pmatrix} \tag{1.129.1}$$

the acceleration of the stone is

$$\mathbf{a} = \begin{pmatrix} 0 \\ -9.8 \end{pmatrix} \tag{1.129.2}$$

and the initial velocity of the stone is

$$\mathbf{v}_A = \begin{pmatrix} 1.5 \\ 0 \end{pmatrix} \tag{1.129.3}$$

If **B** be the final position of the stone,

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{a}t \tag{1.129.4}$$

$$\mathbf{B} = \mathbf{A} + \mathbf{v}_A t + \frac{1}{2}\mathbf{a}t^2 \tag{1.129.5}$$

$$\implies \mathbf{B} = \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 490 \end{pmatrix} + \begin{pmatrix} 1.5 \\ 0 \end{pmatrix} t + \frac{1}{2}\begin{pmatrix} 0 \\ -9.8 \end{pmatrix} t^2 \tag{1.129.6}$$

$\because$ the stone finally comes to rest. Thus,

$$490 = \frac{1}{2} 9.8 t^2 \tag{1.129.7}$$

$$\implies t = 10 \tag{1.129.8}$$

Substituting in (1.129.4),

$$\mathbf{v}_B = \begin{pmatrix} 1.5 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 9.8 \end{pmatrix} 10 \tag{1.129.9}$$

$$= \begin{pmatrix} 1.5 \\ 98 \end{pmatrix} \tag{1.129.10}$$

The final speed is given by $\|\mathbf{v}_B\|$. The motion of the stone is plotted in Fig. 1.129 using (1.129.6) by varying t through the following code.

```
solutions/4/codes/line/motion/motion.py
```

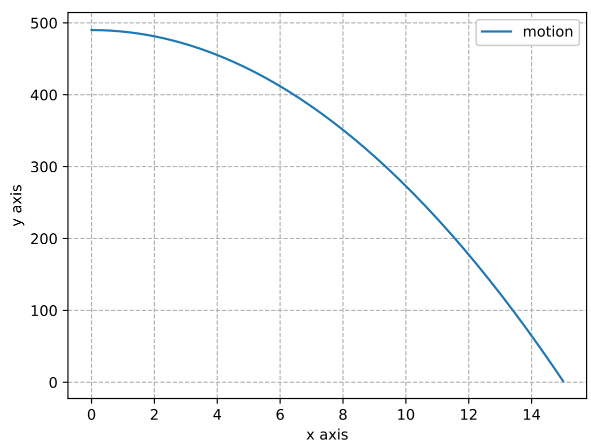

Fig. 1.129

1.130. Rain is falling vertically with a speed of 30 ms^{-1}. A woman rides a bicycle with a speed of 10 ms^{-1} in the north to south direction. What is the direction in which she should hold her umbrella?

Solution: See Fig. 1.130. The velocity of rain and velocity of woman are

$$\mathbf{v_r} = \begin{pmatrix} 0 \\ -30 \end{pmatrix} \tag{1.130.1}$$

$$\mathbf{v_w} = \begin{pmatrix} -10 \\ 0 \end{pmatrix} \tag{1.130.2}$$

The relative velocity of rain w.r.t woman is given as

$$\mathbf{v_{r_w}} = \mathbf{v_r} - \mathbf{v_w} \tag{1.130.3}$$

$$= \begin{pmatrix} 10 \\ -30 \end{pmatrix} \tag{1.130.4}$$

So the woman must hold the umbrella along the direction of $-\mathbf{v}_{\mathbf{r_w}}$ Thus, the desired angle is

$$\theta = \tan^{-1}\left(\frac{10}{30}\right) \tag{1.130.5}$$

The following python code plots Fig. 1.130.

```
./solutions/5/codes/lines/q12.py
```

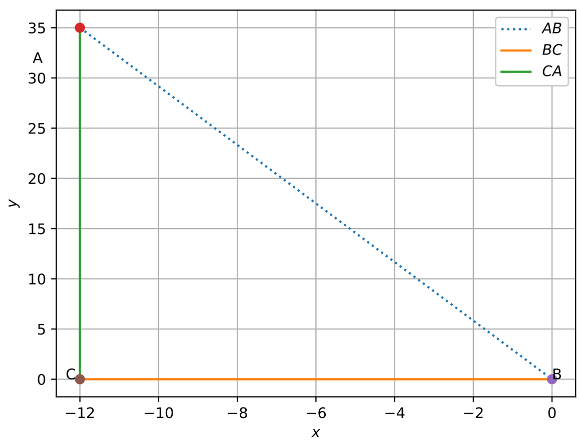

Fig. 1.130

1.131. A man can swim with a speed of 4.0 km/h in still water. How long does he take to cross a river 1.0 km wide if the river flows steadily at 3.0 km/h and he makes his strokes normal to the river current? How far down the river does he go when he reaches the other bank ?
Solution: The following code plots Fig. 1.131

```
solutions/6/codes/line/motion_plane/
    man_river.py
```

In Fig. 1.131, let the man be at

$$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1.131.1}$$

The opposite bank of the river is at

$$\mathbf{B} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.131.2}$$

River current

$$\mathbf{v} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \tag{1.131.3}$$

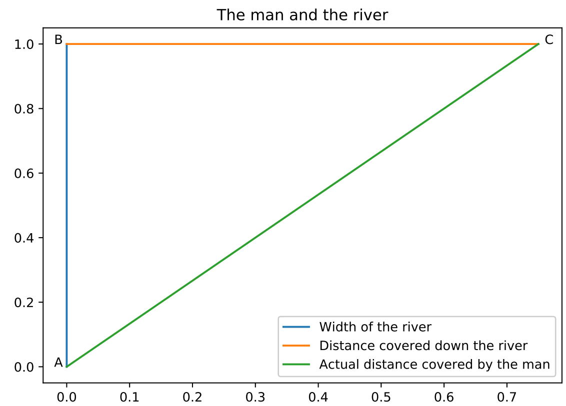

Fig. 1.131

Initial velocity of the man is

$$\mathbf{u} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} \tag{1.131.4}$$

The resultant velocity of the man is

$$\mathbf{u} + \mathbf{v} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \tag{1.131.5}$$

If the time taken by the man to cross the river be t, then

$$\mathbf{C} = (\mathbf{u} + \mathbf{v})\, t = \begin{pmatrix} 3 \\ 4 \end{pmatrix} t \tag{1.131.6}$$

$$= \mathbf{A} + \mathbf{B} = \begin{pmatrix} BC \\ 1 \end{pmatrix} \tag{1.131.7}$$

Thus,

$$\begin{pmatrix} 3 \\ 4 \end{pmatrix} t = \begin{pmatrix} BC \\ 1 \end{pmatrix} \tag{1.131.8}$$

$$\implies 4t = 1 \text{or, } t = \frac{1}{4} \tag{1.131.9}$$

Distance traveled down the river

$$BC = 3t = \frac{3}{4} \tag{1.131.10}$$

1.132. In a harbour, wind is blowing at the speed of 72 km/h and the flag on the mast of a boat anchored in the harbour flutters along the N-E direction. If the boat starts moving at a speed of 51 km/h to the north, what is the direction of the flag on the mast of the boat ?
Solution: The velocity of wind and boat are

respectively,

$$\mathbf{v_w} = 72\begin{pmatrix}\cos 45^\circ \\ \sin 45^\circ\end{pmatrix} \tag{1.132.1}$$

$$\mathbf{v_b} = \begin{pmatrix}0 \\ 51\end{pmatrix} \tag{1.132.2}$$

The resulting wind velocity is

$$\mathbf{v_w} - \mathbf{v_b} = \begin{pmatrix}36\sqrt{2} \\ 36\sqrt{2} - 51\end{pmatrix} \tag{1.132.3}$$

The direction of the flag is

$$\tan^{-1}\left(\frac{36\sqrt{2} - 51}{36\sqrt{2}}\right) \tag{1.132.4}$$

$$= -0.1^\circ \tag{1.132.5}$$

The python code for Fig. 1.132 is

```
solutions/7/codes/line/motion/motion.py
```

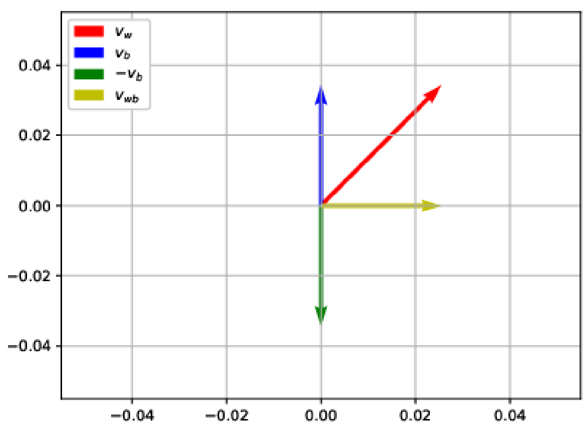

Fig. 1.132

2 Exercises

2.1. The vertices of $\triangle ABC$ are $\mathbf{A} = \begin{pmatrix}4 \\ 6\end{pmatrix}, \mathbf{B} = \begin{pmatrix}1 \\ 5\end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix}7 \\ 2\end{pmatrix}$. A line is drawn to intersect sides AB and AC at D and E respectively, such that

$$\frac{AD}{AB} = \frac{AE}{AC} = \frac{1}{4} \tag{2.1.1}$$

Find

$$\frac{\text{area of } \triangle ADE}{\text{area of } \triangle ABC}. \tag{2.1.2}$$

Solution: From the given information,

$$\frac{AE}{EC} = \frac{AD}{DB} = \frac{1}{3} \tag{2.1.3}$$

and **D** divides AB in the ratio 1 : 3 internally. **E** divides AE in the ratio 1 : 3 internally. Hence,

$$\implies \mathbf{D} = \frac{3\mathbf{A} + \mathbf{B}}{4} \tag{2.1.4}$$

$$= \begin{pmatrix}\frac{13}{4} \\ \\ \frac{23}{4}\end{pmatrix} \tag{2.1.5}$$

$$\mathbf{E} = \frac{3\mathbf{A} + \mathbf{C}}{4} \tag{2.1.6}$$

$$= \begin{pmatrix}\frac{19}{4} \\ \\ \frac{20}{4}\end{pmatrix} \tag{2.1.7}$$

$$\text{Area of } \triangle ABC = \frac{1}{2}\left\|(\mathbf{B} - \mathbf{A}) \times (\mathbf{C} - \mathbf{A})\right\| \tag{2.1.8}$$

$$= \frac{1}{2}\left\|\begin{pmatrix}-3 \\ -1\end{pmatrix} \times \begin{pmatrix}3 \\ -4\end{pmatrix}\right\| \tag{2.1.9}$$

$$= \frac{1}{2}\begin{vmatrix}-3 & 3 \\ -1 & -4\end{vmatrix} \tag{2.1.10}$$

$$= \frac{1}{2}\left[(-3 \times -4) - (-1 \times 3)\right] \tag{2.1.11}$$

$$= \frac{15}{2} \tag{2.1.12}$$

$$\text{Area of } \triangle ADE = \frac{1}{2} \|(\mathbf{D}-\mathbf{A}) \times (\mathbf{E}-\mathbf{A})\| \tag{2.1.13}$$

$$= \frac{1}{2} \left\| \begin{pmatrix} \frac{-3}{4} \\ \frac{-1}{4} \end{pmatrix} \times \begin{pmatrix} \frac{3}{4} \\ \frac{-4}{4} \end{pmatrix} \right\| \tag{2.1.14}$$

$$= \frac{1}{2} \begin{vmatrix} \frac{-3}{4} & \frac{3}{4} \\ \frac{-1}{4} & \frac{-4}{4} \end{vmatrix} \tag{2.1.15}$$

$$= \frac{1}{2} \left[\left(\frac{-3}{4} \times \frac{-4}{4} \right) - \left(\frac{-1}{4} \times \frac{3}{4} \right) \right] \tag{2.1.16}$$

$$= \frac{15}{2 \times 16} \tag{2.1.17}$$

$$\frac{\text{area of } \triangle ADE}{\text{area of } \triangle ABC} = \frac{1}{16} \tag{2.1.18}$$

See Fig. 2.1.

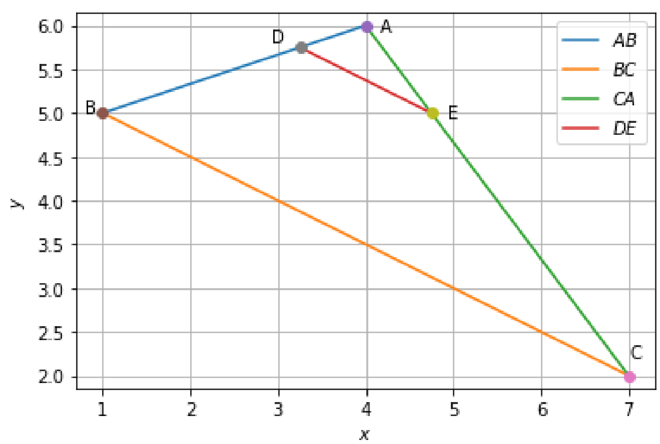

Fig. 2.1: Plot of the triangles

2.2. In $\triangle ABC$, Show that the centroid

$$\mathbf{O} = \frac{\mathbf{A}+\mathbf{B}+\mathbf{C}}{3} \tag{2.2.1}$$

2.3. Check whether

$$\begin{pmatrix} 5 \\ -2 \end{pmatrix}, \begin{pmatrix} 6 \\ 4 \end{pmatrix}, \begin{pmatrix} 7 \\ -2 \end{pmatrix} \tag{2.3.1}$$

are the vertices of an isosceles triangle.
Solution: Let,

$$\mathbf{A} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 7 \\ -2 \end{pmatrix} \tag{2.3.2}$$

$$\|\mathbf{A}-\mathbf{B}\|^2 = (-1)^2 + (-6)^2 = 37 \tag{2.3.3}$$

$$\|\mathbf{B}-\mathbf{C}\|^2 = (-1)^2 + 6^2 = 37 \tag{2.3.4}$$

$$\implies AB = BC \tag{2.3.5}$$

Hence, $\triangle ABC$ is isosceles. See Fig.

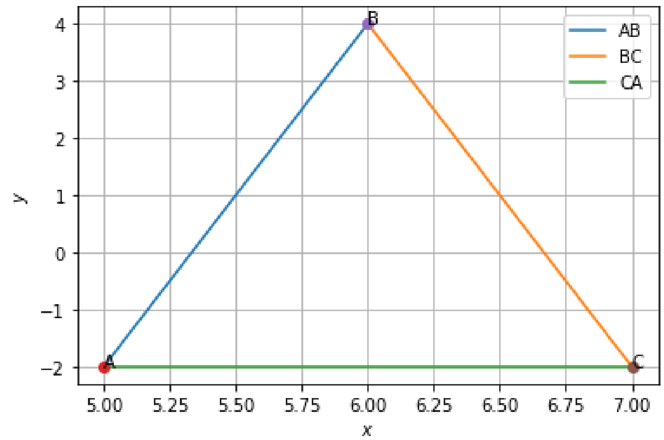

Fig. 2.3: ΔABC

2.4. Determine if the points

$$\begin{pmatrix} 1 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ -11 \end{pmatrix} \tag{2.4.1}$$

are collinear.
Solution:
Let

$$\mathbf{A} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}, \tag{2.4.2}$$

$$\mathbf{B} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \tag{2.4.3}$$

$$\mathbf{C} = \begin{pmatrix} -2 \\ -11 \end{pmatrix} \tag{2.4.4}$$

and

$$\mathbf{M} = \begin{pmatrix} \mathbf{B}-\mathbf{A} & \mathbf{C}-\mathbf{A} \end{pmatrix}^T \tag{2.4.5}$$

If $rank(\mathbf{M}) = 1$, the points are collinear. The rank of a matrix is the number of nonzero rows left after doing row operations. In this problem,

$$\mathbf{M} = \begin{pmatrix} 1 & -2 \\ -3 & -16 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow -\frac{R_2}{3} - R_1} \begin{pmatrix} 1 & -2 \\ 0 & \frac{22}{3} \end{pmatrix} \tag{2.4.6}$$

$$\implies rank(\mathbf{M}) = 2 \tag{2.4.7}$$

Therefore, the points are not collinear. This is verified in Fig. 2.4.

2.5. By using the concept of equation of a line,

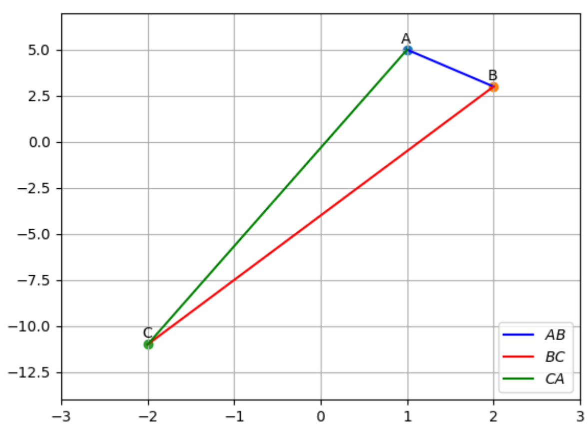

Fig. 2.4: Plot of the points

prove that the three points $\begin{pmatrix}3\\0\end{pmatrix}$, $\begin{pmatrix}-2\\-2\end{pmatrix}$ and $\begin{pmatrix}8\\2\end{pmatrix}$ are collinear.
Solution:
Let,

$$\mathbf{A} = \begin{pmatrix}3\\0\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-2\\-2\end{pmatrix}, \mathbf{C} = \begin{pmatrix}8\\2\end{pmatrix} \tag{2.5.1}$$

Then,

$$\mathbf{M} = \begin{pmatrix}\mathbf{B}-\mathbf{A} & \mathbf{B}-\mathbf{C}\end{pmatrix}^\top \tag{2.5.2}$$
$$= \begin{pmatrix}-5 & 5\\-2 & 2\end{pmatrix}^\top \tag{2.5.3}$$
$$= \begin{pmatrix}-5 & -2\\5 & 2\end{pmatrix} \tag{2.5.4}$$

Using matrix transformation,

$$\mathbf{M} = \begin{pmatrix}-5 & -2\\5 & 2\end{pmatrix} \xleftrightarrow{R_1\to -R_1} \begin{pmatrix}5 & 2\\5 & 2\end{pmatrix} \tag{2.5.5}$$
$$\xleftrightarrow{R_2\to R_2-R_1} \begin{pmatrix}5 & 2\\0 & 0\end{pmatrix} \tag{2.5.6}$$

$\implies$ rank($\mathbf{M}$) = 1. Thus, the given points are collinear, as can be verified from Fig. 2.5.

2.6. Find the value of x for which the points $\begin{pmatrix}x\\-1\end{pmatrix}$, $\begin{pmatrix}2\\1\end{pmatrix}$ and $\begin{pmatrix}4\\5\end{pmatrix}$ are collinear.
Solution: Let

$$\mathbf{A} = \begin{pmatrix}x\\-1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}2\\1\end{pmatrix}, \mathbf{C} = \begin{pmatrix}4\\5\end{pmatrix} \tag{2.6.1}$$

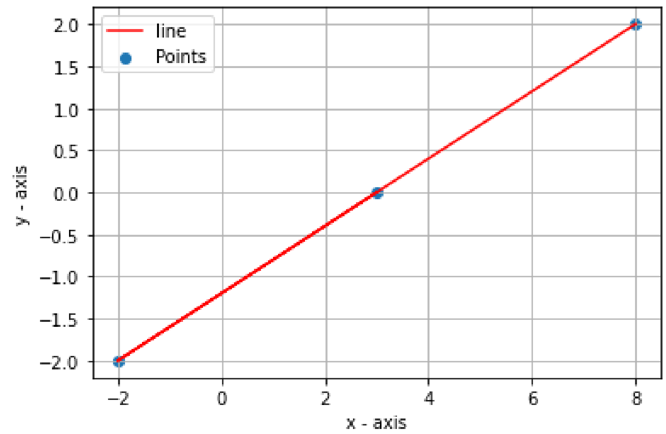

Fig. 2.5: Plot of the points

Now,

$$\mathbf{B}-\mathbf{A} = \begin{pmatrix}2-x\\1-(-1)\end{pmatrix} \tag{2.6.2}$$
$$= \begin{pmatrix}2-x\\2\end{pmatrix} \tag{2.6.3}$$

$$\mathbf{B}-\mathbf{C} = \begin{pmatrix}2-4\\1-5\end{pmatrix} \tag{2.6.4}$$
$$= \begin{pmatrix}-2\\-4\end{pmatrix} \tag{2.6.5}$$

Forming the matrix $\mathbf{M}$,

$$\mathbf{M} = \begin{pmatrix}\mathbf{B}-\mathbf{A} & \mathbf{B}-\mathbf{C}\end{pmatrix}^\top \tag{2.6.6}$$
$$= \begin{pmatrix}2-x & 2\\2 & -4\end{pmatrix}^\top \tag{2.6.7}$$
$$= \begin{pmatrix}2-x & 2\\-2 & -4\end{pmatrix} \tag{2.6.8}$$

Using matrix transformation,

$$\mathbf{M} = \begin{pmatrix}2-x & 2\\-2 & -4\end{pmatrix} \xleftrightarrow{R_2\to R_2/2} \begin{pmatrix}2-x & 2\\-1 & -2\end{pmatrix} \tag{2.6.9}$$

$$\xleftrightarrow{R_2\to R_2+R_1} \begin{pmatrix}2-x & 2\\1-x & 0\end{pmatrix} \tag{2.6.10}$$

$$rank(\mathbf{M}) = 1 \implies R_2 = 0 \tag{2.6.11}$$
$$\text{or, } x = 1 \tag{2.6.12}$$

See Fig. 2.6.

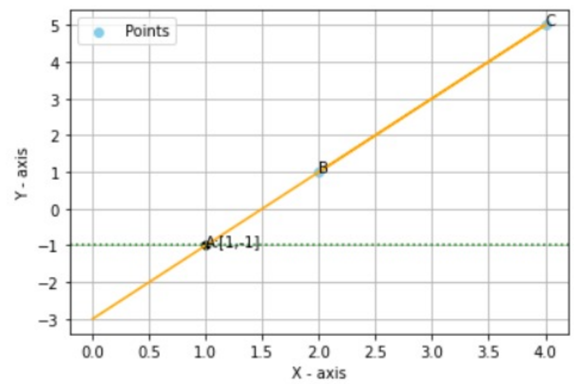

Fig. 2.6: Plot of the line

2.7. In each of the following, find the value of k for which the points are collinear

a) $\begin{pmatrix}7\\-2\end{pmatrix}, \begin{pmatrix}5\\1\end{pmatrix}, \begin{pmatrix}3\\k\end{pmatrix}$

b) $\begin{pmatrix}8\\1\end{pmatrix}, \begin{pmatrix}k\\-4\end{pmatrix}, \begin{pmatrix}2\\-5\end{pmatrix}$

Solution:

a) Let $\mathbf{A} = \begin{pmatrix}7\\-2\end{pmatrix}, \mathbf{B} = \begin{pmatrix}5\\1\end{pmatrix}, \mathbf{C} = \begin{pmatrix}3\\k\end{pmatrix}$

The direction vectors of AB and AC are

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix}-2\\3\end{pmatrix} \tag{2.7.1}$$

$$\mathbf{C} - \mathbf{A} = \begin{pmatrix}-4\\k+2\end{pmatrix} \tag{2.7.2}$$

$$\mathbf{M} = \begin{pmatrix}\mathbf{B} - \mathbf{A} & \mathbf{C} - \mathbf{A}\end{pmatrix}^\top \tag{2.7.3}$$

Substituting (2.7.1) and (2.7.2) in (2.7.3), we get

$$\mathbf{M} = \begin{pmatrix}-2 & 3\\-4 & k+2\end{pmatrix} \tag{2.7.4}$$

We know that if $rank\,(\mathbf{M}) = 1$, the points are collinear. Finding the rank of the matrix in the problem,

$$\mathbf{M} = \begin{pmatrix}-2 & 3\\-4 & k+2\end{pmatrix} \xleftrightarrow{R_2 \to R_2 - 2R_1} \begin{pmatrix}-2 & 3\\0 & k-4\end{pmatrix} \tag{2.7.5}$$

Since $rank\,(\mathbf{M}) = 1$, the number of non zero rows left after doing row operations should be equal to 1. Since row 1 in (2.7.5) is non zero, elements row 2 should be equal to 0.

$$\therefore k = 4 \tag{2.7.6}$$

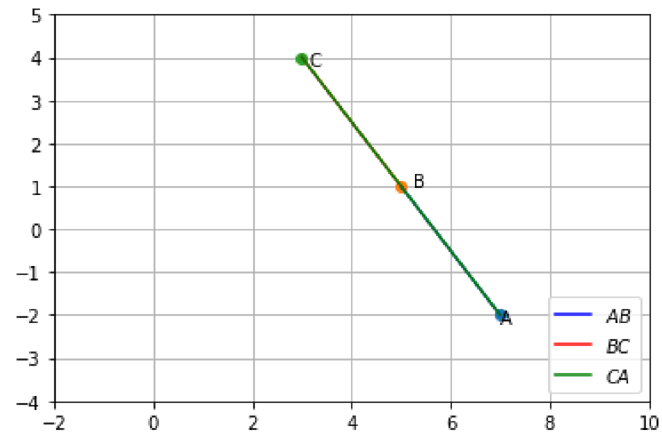

Fig. 2.7: Plot of the line

b) Let $\mathbf{A} = \begin{pmatrix}8\\1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}k\\-4\end{pmatrix}, \mathbf{C} = \begin{pmatrix}2\\-5\end{pmatrix}$

The direction vectors of AB and AC are

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix}k-8\\-5\end{pmatrix} \tag{2.7.7}$$

$$\mathbf{C} - \mathbf{A} = \begin{pmatrix}-6\\-6\end{pmatrix} \tag{2.7.8}$$

$$\mathbf{M} = \begin{pmatrix}\mathbf{B} - \mathbf{A} & \mathbf{C} - \mathbf{A}\end{pmatrix}^\top \tag{2.7.9}$$

Substituting (2.7.7) and (2.7.8) in (2.7.9), we get

$$\mathbf{M} = \begin{pmatrix}k-8 & -5\\-6 & -6\end{pmatrix} \tag{2.7.10}$$

We know that if $rank\,(\mathbf{M}) = 1$, the points are collinear. Finding the rank of the matrix in the problem,

$$\mathbf{M} = \begin{pmatrix}k-8 & -6\\-5 & -6\end{pmatrix} \xleftrightarrow{R_2 \to 5R_2 - 6R_1} \begin{pmatrix}k-8 & -5\\18-6k & 0\end{pmatrix} \tag{2.7.11}$$

Since $rank\,(\mathbf{M}) = 1$, the number of non zero rows left after doing row operations should be equal to 1. Since row 1 in (2.7.11) is non zero for any value of k , elements row 2 should be equal to 0.

$$\therefore k = 3 \tag{2.7.12}$$

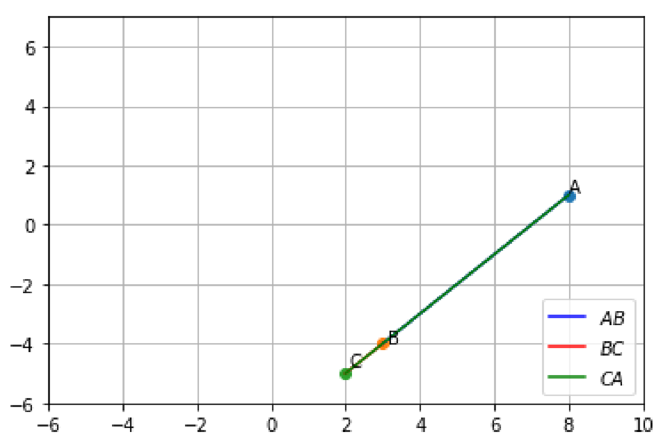

Fig. 2.7: Plot of the line

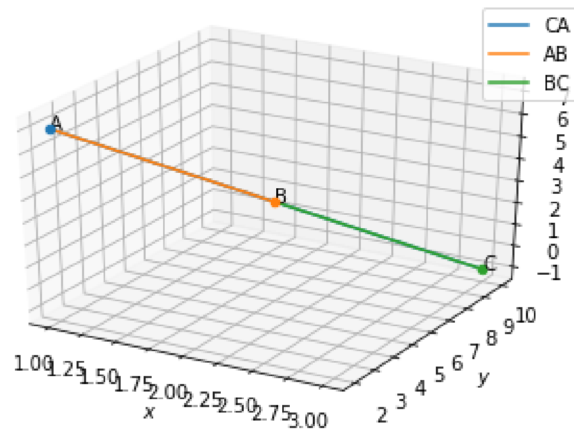

Fig. 2.8: Plot

2.8. Show that the points $\mathbf{A} = \begin{pmatrix}1\\2\\7\end{pmatrix}, \mathbf{B} = \begin{pmatrix}2\\6\\3\end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix}3\\10\\-1\end{pmatrix}$ are collinear.
Solution:

$$\mathbf{B}-\mathbf{A} = \begin{pmatrix}1\\4\\-4\end{pmatrix}, \mathbf{C}-\mathbf{A} = \begin{pmatrix}2\\8\\-8\end{pmatrix} \tag{2.8.1}$$

Forming the matrix $\mathbf{M}$,

$$\mathbf{M} = \begin{pmatrix}\mathbf{B}-\mathbf{A} & \mathbf{C}-\mathbf{A}\end{pmatrix}^{\top} \tag{2.8.2}$$

$$= \begin{pmatrix}1 & 4 & -4\\2 & 8 & -8\end{pmatrix} \tag{2.8.3}$$

Using matrix transformation,

$$\mathbf{M} = \begin{pmatrix}1 & 4 & -4\\2 & 8 & -8\end{pmatrix} \xleftrightarrow{R_2 \to R_2 - 2R_1} \begin{pmatrix}1 & 4 & -4\\0 & 0 & 0\end{pmatrix} \tag{2.8.4}$$

$$\implies rank(\mathbf{M}) = 1 \tag{2.8.5}$$

Thus $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are collinear as can be seen in Fig. 2.8.

2.9. Show that the points $\mathbf{A} = \begin{pmatrix}1\\-2\\-8\end{pmatrix}, \mathbf{B} = \begin{pmatrix}5\\0\\-2\end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix}11\\3\\7\end{pmatrix}$ are collinear, and find the ratio in which $\mathbf{B}$ divides AC.
Solution:

$$\mathbf{B}-\mathbf{A} = \begin{pmatrix}4\\2\\6\end{pmatrix}, \mathbf{C}-\mathbf{A} = \begin{pmatrix}10\\5\\15\end{pmatrix} \tag{2.9.1}$$

Forming the matrix

$$\mathbf{M} = \begin{pmatrix}\mathbf{B}-\mathbf{A} & \mathbf{C}-\mathbf{A}\end{pmatrix}^{\top} \tag{2.9.2}$$

$$= \begin{pmatrix}4 & 2 & 6\\10 & 5 & 15\end{pmatrix} \tag{2.9.3}$$

Using matrix transformation,

$$\mathbf{M} = \begin{pmatrix}4 & 2 & 6\\10 & 5 & 15\end{pmatrix} \xleftrightarrow{R_2 \to R_2 - \frac{5}{2}R_1} \begin{pmatrix}4 & 2 & 6\\0 & 0 & 0\end{pmatrix} \tag{2.9.4}$$

$$\implies rank(\mathbf{M}) = 1 \tag{2.9.5}$$

Thus $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are collinear.
Let $\mathbf{B}$ divide AC in the ratio $\lambda : 1$.

$$\implies \frac{\lambda}{1} = \frac{AB}{BC} \tag{2.9.6}$$

$$\implies \|\mathbf{B}-\mathbf{A}\| = \lambda \ \|\mathbf{C}-\mathbf{B}\| \tag{2.9.7}$$

$$\implies \lambda = \frac{2}{3} \tag{2.9.8}$$

Thus $\mathbf{B}$ divides AC in the ratio 2:3. See Fig. 2.9

2.10. Show that $\mathbf{A} = \begin{pmatrix}2\\3\\4\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-1\\-2\\1\end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix}5\\8\\7\end{pmatrix}$ are collinear.
Solution:

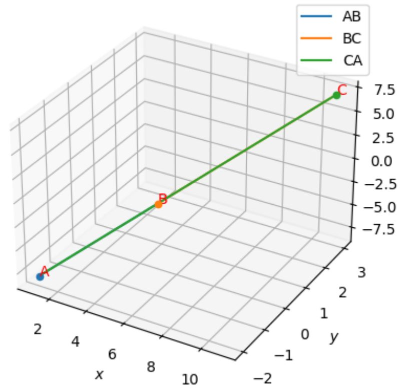

Fig. 2.9: Plot of the line

$$\mathbf{B}-\mathbf{A}=\begin{pmatrix}-3\\-5\\-3\end{pmatrix}, \mathbf{C}-\mathbf{A}=\begin{pmatrix}3\\5\\13\end{pmatrix} \tag{2.10.1}$$

Forming the matrix

$$\mathbf{M}=\begin{pmatrix}\mathbf{B}-\mathbf{A} & \mathbf{C}-\mathbf{A}\end{pmatrix}^\top \tag{2.10.2}$$

$$=\begin{pmatrix}-3 & -5 & -3\\ 3 & 5 & 3\end{pmatrix} \tag{2.10.3}$$

Using matrix transformation,

$$\mathbf{M}=\begin{pmatrix}-3 & -5 & -3\\ 3 & 5 & 3\end{pmatrix} \xleftrightarrow{R_2\to R_2+R_1} \begin{pmatrix}-3 & -5 & -3\\ 0 & 0 & 0\end{pmatrix} \tag{2.10.4}$$

$$\implies rank(\mathbf{M})=1 \tag{2.10.5}$$

Thus **A**, **B** and **C** are collinear as can be seen from Fig. 2.10

2.11. A bullet fired at an angle of 30° with the horizontal hits the ground 3.0 km away. By adjusting its angle of projection, can one hope to hit a target 5.0 km away ? Assume the muzzle speed to be fixed, and neglect air resistance.

2.12. A fighter plane flying horizontally at an altitude of 1.5 km with speed 720 km/h passes directly overhead an anti-aircraft gun. At what angle from the vertical should the gun be fired for the shell with muzzle speed 600 ms^{-1} to hit the plane ? At what minimum altitude should the pilot fly the plane to avoid being hit ? (Take g = $10ms^{-2}$).

2.13. Give the magnitude and direction of the net force acting on a stone of mass 0.1 kg,

a) just after it is dropped from the window of a stationary train,
b) just after it is dropped from the window of a train running at a constant velocity of 36 km/h,
c) just after it is dropped from the window of a train accelerating with $1ms^{-2}$
d) lying on the floor of a train which is accelerating with 1 ms^{-2}, the stone being at rest relative to the train.

Neglect air resistance throughout.

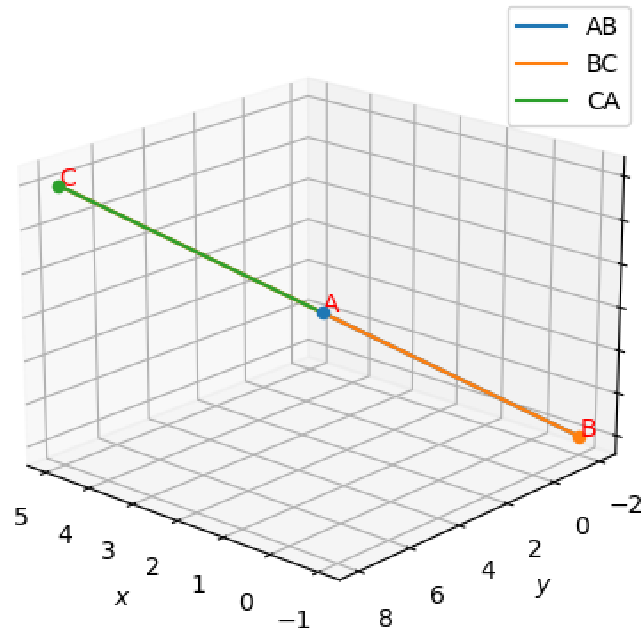

Fig. 2.10: Plot of the line

2.14. Consider the collision depicted in Fig. 2.14 to be between two billiard balls with equal masses $m_1 = m_2$. The first ball is called the cue while the second ball is called the target. The billiard player wants to 'sink' the target ball in a corner pocket, which is at an angle $\theta_2 = 37°$. Assume that the collosion is elastic and that friction and rotational motion are not important. Obtain θ_1.

2.15. Find the ratio in which the line segment joining the points $\begin{pmatrix}4\\8\\10\end{pmatrix}$ and $\begin{pmatrix}6\\10\\-8\end{pmatrix}$ is divided by the YZ-plane.

Solution:

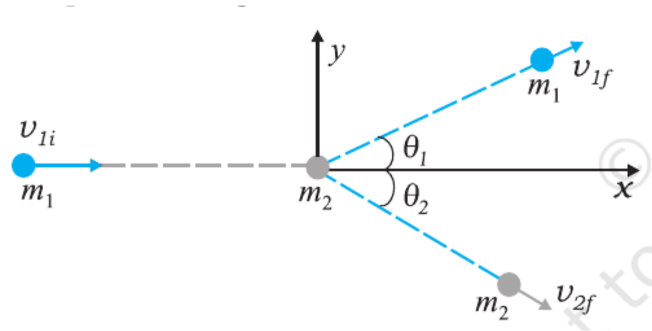

Fig. 2.14

Let

$$\mathbf{A} = \begin{pmatrix} 4 \\ 8 \\ 10 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 6 \\ 10 \\ -8 \end{pmatrix}. \tag{2.15.1}$$

and

$$\mathbf{P} = \frac{\mathbf{A} + k\mathbf{B}}{k+1} \tag{2.15.2}$$

Let the equation of the YZ plane be

$$\mathbf{n}^\top \mathbf{x} = d \tag{2.15.3}$$

Since $\mathbf{P}$ lies on YZ plane,

$$\mathbf{n}^\top \mathbf{P} = d \tag{2.15.4}$$

$$\implies \mathbf{n}^\top \left(\frac{\mathbf{A} + k\mathbf{B}}{k+1}\right) = d \tag{2.15.5}$$

$$\implies k = \frac{d - \mathbf{n}^\top \mathbf{A}}{\mathbf{n}^\top \mathbf{B} - d} \tag{2.15.6}$$

For YZ plane, $\mathbf{n} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $d = 0$. So,

$$k = -2/3 \tag{2.15.7}$$

Also, using (2.15.2)

$$\mathbf{P} = \frac{\mathbf{A} - (2/3)\mathbf{B}}{(-2/3) + 1} = 3\mathbf{A} - 2\mathbf{B} \tag{2.15.8}$$

$$= \begin{pmatrix} 0 \\ 4 \\ 46 \end{pmatrix} \tag{2.15.9}$$

See Fig. 2.15.

2.16. If

$$\mathbf{P} = 3\mathbf{a} - 2\mathbf{b} \tag{2.16.1}$$

$$\mathbf{Q} = \mathbf{a} + \mathbf{b} \tag{2.16.2}$$

find $\mathbf{R}$, which divides PQ in the ratio $2 : 1$

a) internally,
b) externally.

2.17. Find a unit vector in the direction of $\mathbf{A} + \mathbf{B}$, where

$$\mathbf{A} = \begin{pmatrix} 2 \\ 2 \\ -5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}. \tag{2.17.1}$$

Solution: Let $\mathbf{C}$ be the vector $\mathbf{A} + \mathbf{B}$

$$\mathbf{C} = \mathbf{A} + \mathbf{B} \tag{2.17.2}$$

$$\therefore \mathbf{C} = \begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix} \tag{2.17.3}$$

Now,

$$\|\mathbf{C}\| = \sqrt{(4)^2 + (3)^2 + (-2)^2} \tag{2.17.4}$$

$$\therefore \|\mathbf{C}\| = \sqrt{29} \tag{2.17.5}$$

Let $\mathbf{H}$ be the unit vector in the direction of $\mathbf{C}$.

$$\mathbf{H} = \frac{\mathbf{C}}{\|\mathbf{C}\|} \tag{2.17.6}$$

$$\therefore \mathbf{H} = \frac{1}{\sqrt{29}} \begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix} \tag{2.17.7}$$

Hence, $\mathbf{H}$ is the required unit vector.

2.18. Find the area of a parallelogram whose adjacent sides are determined by the vectors $\mathbf{a} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ -7 \\ 1 \end{pmatrix}$.

Solution:

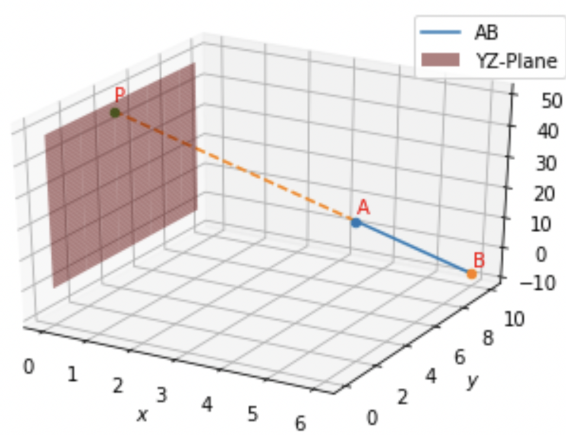

Fig. 2.15: 3D plot

The area of the required parallelogram is defined as

$$\mathbf{a}\times\mathbf{b} = \begin{pmatrix} 0 & -3 & -1 \\ 3 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ -7 \\ 1 \end{pmatrix} = \begin{pmatrix} 20 \\ 5 \\ -5 \end{pmatrix} \tag{2.18.1}$$

Thus, the desired area is

$$\|\mathbf{a}\times\mathbf{b}\| = \sqrt{(20)^2 + (5)^2 + (-5)^2} \tag{2.18.2}$$

$$= 15\sqrt{2} \tag{2.18.3}$$

2.19. Verify if $\mathbf{A} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 8 \\ 6 \end{pmatrix}$ are points on a line.