Continuous Math

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Contents

***Abstract*—This book provides a computational approach to continuous mathematics based on the NCERT textbooks from Class 6-12. Links to sample Python codes are available in the text.**

Download python codes using

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svn co https://github.com/gadepall/school/trunk/
    ncert/continuous/codes
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1 Curves

1.1 Examples

1. Find the value of each of the following polynomials at the indicated value of variables:
   a) $q(y) = 3y^3 - 4y + 11$ at $y = 2$.
   b) $p(t) = 4t^4 + 5t^3 - t2 + 6$ at $t = a$.
2. Find $p(0)$, $p(1)$ and $p(2)$ for each of the following polynomials:
   a) $p(t) = 2 + t + 2t^2 - t^3$
   b) $p(x) = x^3$
3. Find the remainder when $x^4 + x^3 - 2x^2 + x + 1$ is divided by $x - 1$.
4. Check whether the polynomial $q(t) = 4t^3+4t^2-t-1$ is a multiple of $2t+1$.
5. Examine whether $x+2$ is a factor of x^3+3x^2+5x+6 and of $2x+4$.
6. Find the remainder obtained on dividing $p(x) = x^3+1$ by $x+1$.
7. Factorize $x^3 - 23x^2 + 142x - 120$.
8. Verify that $3, -1, \frac{1}{3}$, are the zeroes of the cubic polynomial $p(x) = 3x^3-5x^2-11x-3$, and then verify the relationship between the zeroes and the coefficients.
9. Show that the function f given by
$$f(x) = \begin{cases} x^3 + 3 & x \neq 0 \\ 1, & x = 0 \end{cases} \tag{1.1.9.1}$$
is not continuous at $x = 0$.
10. Discuss the continuity of the function f defined by $f(x) = x^2 + x + 1$.
11. Discuss the continuity of the function f defined by $f(x) = \frac{1}{x}, x \neq 0$.
12. Show that every polynomial function is continuous.
13. Find all the points of discontinuity of the greatest integer function defined by $f(x) = [x]$, where $[x]$ denotes the greatest integer less than or equal to x.
14. Discuss the continuity of sine function.
15. Show that the function defined by $f(x) = \sin(x^2)$ is a continuous function.
16. Find the slope of the tangent to the curve $y = x^3 - x$ at $x = 2$
17. Find the equation of the tangent to the curve $y = \frac{x-7}{(x-2)(x-3)}$
18. Find the equations of the tangent and normal to the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}}$ at $\begin{pmatrix}1\\1\end{pmatrix}$.
19. Find the equation of the tangent to the curve

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$\begin{pmatrix} a\sin^3 t \\ b\cos^3 t \end{pmatrix}$ at $t = \frac{\pi}{2}$.

20. Find the equation of tangents to the curve $y = \cos(x+y), -2\pi \le x \le 2\pi$ that are parallel to the line $\begin{pmatrix} 1 & 2 \end{pmatrix} \mathbf{x} = 0$.
21. Find the area bounded by the curve $y = \cos x$ between $x = 0$ and $x = 2\pi$.
22. Find the area bounded by the curve $y = \sin x$ between $x = 0$ and $x = 2\pi$.
23. Show that the function f given by
$$f(x) = x^3 - 3x^2 + 4x, x \in \mathbf{R} \tag{1.1.23.1}$$
is increasing on $\mathbf{R}$.
24. Prove that the function given by $f(x) = \cos x$ is
   a) decreasing in $(0, \pi)$.
   b) increasing in $(\pi, 2\pi)$ and
25. Find the intervals in which the function
$$f(x) = 4x^3 - 6x^2 - 72x + 30 \tag{1.1.25.1}$$
is
   a) increasing
   b) decreasing.
26. Find the intervals in which the function given by
$$f(x) = \sin x, x \in \left[0, \frac{\pi}{2}\right] \tag{1.1.26.1}$$
is
   a) increasing
   b) decreasing.
27. Find the intervals in which the function given by
$$f(x) = \sin x + \cos x, x \in [0, 2\pi] \tag{1.1.27.1}$$
is increasing or decreasing.
28. Find all points of local maxima and local minima of the function f given by
$$f(x) = x^3 - 3x + 3 \tag{1.1.28.1}$$
29. Find all points of local maxima and local minima of the function f given by
$$f(x) = 2x^3 - 6x^2 + 6x + 5 \tag{1.1.29.1}$$
30. Find the local maxima and minima of the function f given by
$$f(x) = 3x^4 + 4x^3 - 12x^2 + 12 \tag{1.1.30.1}$$
31. Find the absolute maximum and minimum values of a function f given by
$$f(x) = 2x^3 - 15x^2 + 36x + 1, \quad x \in [1, 5]. \tag{1.1.31.1}$$
32. Find the absolute maximum and minimum values of a function f given by
$$f(x) = 12x^{\frac{4}{3}} - 6x^{\frac{1}{3}}, \quad x \in [1, 1]. \tag{1.1.32.1}$$
33. A car starts from a point P at time $t = 0$ seconds and stops at point Q. The distance x, in metres, covered by it, in t seconds is given by
$$x = t^2\left(2 - \frac{t}{3}\right). \tag{1.1.33.1}$$
Find the time taken by it to reach Q and also find the distance between P and Q.
34. A water tank has the shape of an inverted right circular cone with its axis vertical and vertex lowermost. Its semi-vertical angle is $\tan^{-1}(0.5)$. Water is poured into it at a constant rate of 5 cubic metre per hour. Find the rate at which the level of the water is rising at the instant when the depth of water in the tank is 4 m.
35. A man of height 2 metres walks at a uniform speed of 5 km/h away from a lamp post which is 6 metres high. Find the rate at which the length of his shadow increases.
36. Find intervals in which the function given by
$$f(x) = \frac{3}{10}x^4 - \frac{4}{5}x^3 - 3x^2 + \frac{36}{5} + 11 \tag{1.1.36.1}$$
is
   a) decreasing
   b) increasing
37. Show that the function f given by
$$f(x) = \tan^{-1}(\sin x + \cos x), \quad x > 0 \tag{1.1.37.1}$$
is always an increasing function in $\left(0, \frac{\pi}{4}\right)$.
38. A circular disc of radius 3 cm is being heated. Due to expansion, its radius increases at the rate of 0.05 cm/s. Find the rate at which its area is increasing when radius is 3.2 cm.
39. An open topped box is to be constructed by removing equal squares from each corner of a 3 metre by 8 metre rectangular sheet of

aluminium and folding up the sides. Find the volume of the largest such box.

40. A manufacturer can sell x items at a price of |$\left(5-\frac{x}{500}\right)$ each. The cost price of x items is |$\left(\frac{x}{5}+500\right)$. Find the number of items he should sell to earn maximum profit.
41. Find the limits
    a) $\lim_{x\to 1} x^3 - x^2 + 1$
    b) $\lim_{x\to 1} x(x+1)$
    c) $\lim_{x\to 1} 1 + x + x^2 + \cdots + x^10$
42. Find the limits
    a) $\lim_{x\to 1} \frac{x^2+1}{x+100}$
    b) $\lim_{x\to 2} \frac{x^3-4x^2+4x}{x^2-4}$
    c) $\lim_{x\to 1} \frac{x^2-4}{x^3-4x^2+4x}$
    d) $\lim_{x\to 1} \frac{x^3-2x^2}{x^2-5x+6}$
    e) $\lim_{x\to 1} \left[\frac{x-2}{x^2-x} - \frac{1}{x^3-3x^2+2x}\right]$
43. Evaluate
    a) $\lim_{x\to 1} \frac{x^15-1}{x^10-1}$
    b) $\lim_{x\to 2} \frac{\sqrt{1+x}}{x}$
44. Evaluate

$$\lim_{x\to 0} \frac{\sin 4x}{\sin 2x} \tag{1.1.44.1}$$

45. Evaluate

$$\int_{-1}^{\frac{3}{2}} |x \sin(\pi x)| \, dx \tag{1.1.45.1}$$

46. Evaluate

$$\int_0^{\pi} \frac{x}{a^2\cos^2 x + b^2 \sin^2 x} \, dx \tag{1.1.46.1}$$

47. Evaluate

$$\int_0^2 e^x \, dx \tag{1.1.47.1}$$

as a limit of a sum.

48. Evaluate the following integrals:
    a) $\int_2^3 x^2 \, dx$
    b) $\int_4^9 \frac{\sqrt{x}}{\left(30-x^{\frac{3}{2}}\right)^2} \, dx$
    c) $\int_1^2 \frac{x}{(x+1)(x+2)} \, dx$
    d) $\int_0^{\frac{\pi}{4}} \sin^3 2t \cos 2t \, dx$
49. Evaluate

$$\int_{-1}^{1} 5x^4 \sqrt{x^5+1} \, dx \tag{1.1.49.1}$$

50. Evaluate

$$\int_0^1 \frac{\tan^{-1} x}{1+x^2} \, dx \tag{1.1.50.1}$$

51. Evaluate

$$\int_{-1}^{2} \left|x^3 - x\right| \, dx \tag{1.1.51.1}$$

52. Evaluate

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^2 x \, dx \tag{1.1.52.1}$$

53. Evaluate

$$\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} \, dx \tag{1.1.53.1}$$

54. Evaluate

$$\int_{-1}^{1} \sin^5 x \cos^4 x \, dx \tag{1.1.54.1}$$

55. Evaluate

$$\int_0^{\frac{\pi}{2}} \frac{\sin^4 x}{\sin^4 + \cos^4 x} \, dx \tag{1.1.55.1}$$

56. Evaluate

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{1+\sqrt{\tan x}} \, dx \tag{1.1.56.1}$$

57. Evaluate

$$\int_0^{\frac{\pi}{2}} \log \sin x \, dx \tag{1.1.57.1}$$

58. Solve the differential equation

$$y_1 = -4xy^2, \quad y(0) = 1 \tag{1.1.58.1}$$

59. Find the equation of the curve passing through the point $\begin{pmatrix}1\\1\end{pmatrix}$, whose differential equation is $xdy = \left(2x^2+1\right) dx \, (x \neq 0)$.
60. Find the equaion of a curve passing through the point $\begin{pmatrix}-2\\3\end{pmatrix}$ given that the slope of the tangent to the curve at any point $\begin{pmatrix}x\\y\end{pmatrix}$ is $\frac{2x}{y^2}$.
61. In a bank, principal increases continuously at the rate of 5% per year. In how many years will |100 double itself?

62. Solve
$$2ye^{\frac{x}{y}}\,dx + \left(y - 2xe^{\frac{x}{y}}\right)dy = 0, \quad y(0) = 1 \tag{1.1.62.1}$$
63. Solve
$$y_1 + y\cot x = 2x + x^2\cot x\,(x \neq 0)\,, \quad y\left(\frac{\pi}{2}\right) = 0 \tag{1.1.63.1}$$
64. Find the equation of a curve passing through the point $\begin{pmatrix}0\\1\end{pmatrix}$. If the slope of the tangent to the curve at any point $\begin{pmatrix}x\\y\end{pmatrix}$ is equal to the sum of the x coordinate (abscissa) and the product of the x coordinate and y coordinate (ordinate) of that point.
65. Solve
$$\log y_1 = 3x + 4y, \quad y(0) = 0 \tag{1.1.65.1}$$
66. The position of an object moving along x-axis is given by $x = a + bt^2$ where $a = 8.5m, b = 2.5ms^{-2}$ and t is measured in seconds. What is its velocity at t = 0 s and t = 2.0 s?

1.2 Exercises

1. Find the remainder when $x^3 + 3x^2 + 3x + 1$ is divided by
   a) $x + 1$
   b) $x - \frac{1}{2}$
   c) x
   d) $x + \pi$
   e) $5 + 2x$
2. Check whether $7 + 3x$ is a factor of $3x^3 + 7x$.
3. Determine which of the following polynomials has $(x + 1)$ as a factor:
   a) $x^3 + x^2 + x + 1$
   b) $x^4 + x^3 + x^2 + x + 1$
   c) $x^4 + 3x^3 + 3x^2 + x + 1$
   d) $x^3 - x^2 - \left(2 + \sqrt{2}\right) + \sqrt{2}$.
4. Determine whether $g(x)$ is a factor of $p(x)$ in each of the following cases:
   a) $p(x) = 2x^3 + x^2 - 2x - 1, g(x) = x + 1$
   b) $p(x) = x^3 + 3x^2 + 3x + 1, g(x) = x + 2$
   c) $p(x) = x^4 - 4x^2 + x + 6, g(x) = x - 3$
5. Factorise :
   a) $x^3–2x^2–x + 2$
   b) $x^3–3x^2–9x–5$
   c) $x^3 + 13x^2 + 32x + 20$
   d) $2y^3 + y^2–2y–1$
6. Find the roots of the following equations:
   a) $x - \frac{1}{x} = 3, x \neq= 0$
   b) $\frac{1}{x+4} - \frac{1}{x-7} = \frac{11}{30}, x \neq= -4, 7$
7. Find the slope of the tangent to the curve $y = 3x^4 - 4x$ at $x = 4$.
8. Find the slope of the tangent to curve $y = x^3 - 3x + 2$ at the point whose x-coordinate is 2.
9. Find the slope of the tangent to the curve $y = x^3 - 3x + 2$ at the point whose x-coordinate is 3.
10. Find the slope of the normal to the curve $\mathbf{x} = a\begin{pmatrix}\cos^3\theta\\\sin^3\theta\end{pmatrix}$ at $\theta = \frac{\pi}{4}$.
11. Find the slope of the normal to the curve $\mathbf{x} = \begin{pmatrix}1 - a\sin\theta\\b\cos^2\theta\end{pmatrix}$ at $\theta = \frac{\pi}{2}$.
12. Find points at which the tangent to the curve $y = x^3 - 3x^2 - 9x + 7$ is parallel to th x-axis.
13. Find the point on the curve $y = x^3–11x + 5$ at which the tangent is $\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = 11$.
14. Find the equations of all lines having slope 0 which are tangent to the curve $y = \frac{1}{x^2-2x+3}$.
15. Find the equations of the tangent and normal to the given curves at the indicated points:
   a) $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at $\begin{pmatrix}0\\5\end{pmatrix}$.
   b) $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at $\begin{pmatrix}1\\3\end{pmatrix}$.
   c) $y = x^3$ at $\begin{pmatrix}1\\1\end{pmatrix}$.
16. Show that the tangents to the curve $y = 7x^3+11$ at the points where $x = 2$ and $x = -2$ are parallel.
17. Find the points on the curve $y = x^3$ at which the slope of the tangent is equal to the y-coordinate of the point.
18. For the curve $y = 4x^3–2x^5$ find all the points at which the tangent passes through the origin.
19. Find the equation of the normal at the point $\begin{pmatrix}am^2\\am^3\end{pmatrix}$ for the curve $ay^2 = x^3$
20. Find the equation of the normals to the curve $y = x^3 + 2x + 6$ which are parallel to the line $\begin{pmatrix}1 & 14\end{pmatrix} + 4 = 0$.
21. Find the slope of the normal to the curve $y = 2x^2 + 3\sin x$ at $x = 0$. Show that the normal at any point θ to the curve $\mathbf{x} = \begin{pmatrix}a\cos\theta + a\theta\sin\theta\\a\sin\theta–a\theta\cos\theta\end{pmatrix}$

is at a constant distance from the origin.

22. Find the slope of the tangent to the curve $\mathbf{x} = \begin{pmatrix} t^2+3t-8 \\ 2t^2-2t-5 \end{pmatrix}$ at the point $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$.
23. Find the points on the curve $9y^2 = x^3$, where the normal to the curve makes equal intercepts with the axes.
24. Find the area under $y = x^4, x = 1, x = 5$ and x-axis.
25. Find the area bounded by the curve $y = x^3, x = -2, x = 1$ and the x-axis.
26. Find the area bounded by the curve $y = x|x|, x = -1, x = 1$ and the x-axis.
27. Find the area bounded by the y-axis, $y = \cos x$ and $y = \sin x$ when $0 \le x \le \frac{\pi}{2}$.
28. Show that the function given by $f(x) = 3x+17$ is increasing on $\mathbf{R}$.
29. Show that the function given by $f(x) = e^{2x}$ is increasing on $\mathbf{R}$.
30. Show that the function given by
$$f(x) = \sin x \tag{1.2.30.1}$$
is
   a) increasing in $\left(0, \frac{\pi}{2}\right)$
   b) decreasing in $\left(\frac{\pi}{2}, \pi\right)$
31. Find the intervals in which the function given by
$$f(x) = 2x^3 - 3x^2 - 36x + 7 \tag{1.2.31.1}$$
is
   a) increasing
   b) decreasing.
32. Find the intervals in which the following functions are strictly increasing or decreasing
   a) $(x+1)^3(x-3)^3$
   b) $-2x^3 - 9x^2 - 12x + 1$
33. Show that
$$y = \log(1+x) - \frac{2x}{2+x}, x > -1, \tag{1.2.33.1}$$
is an increasing function of x throughout its domain.
34. Find the values of x for which $y = x(x-2)^2$ is an increasing function.
35. Prove that
$$y = \frac{4\sin\theta}{2+\cos\theta} - \theta \tag{1.2.35.1}$$
is an incresing function of θ in $\left[0, \frac{\pi}{2}\right]$.
36. Prove that the logarithmic function is increasing on $(0, \infty)$.
37. Which of the following functions are decreasing on $\left[0, \frac{\pi}{2}\right]$?
   a) $\cos x$
   b) $\cos 2x$
   c) $\cos 3x$
   d) $\tan x$
38. Find the intervals on which
$$f(x) = x^{100} + \sin x - 1 \tag{1.2.38.1}$$
is decreasing.
39. Let I be any interval disjoint from $[1, -1]$. Prove that the function f given by $f(x) = x + \frac{1}{x}$ is increasing on I.
40. Prove that the function f given by $f(x) = \log\sin x$ is increasing on $\left(0, \frac{\pi}{2}\right)$ and decreasing on $\left(\frac{\pi}{2}, \pi\right)$.
41. Prove that the function f given by $f(x) = \log|\cos x|$ is decreasing on $\left(0, \frac{\pi}{2}\right)$ and increasing on $\left(\frac{3\pi}{2}, 2\pi\right)$.
42. Prove that the function given by $f(x) = x^3 - 3x^2 + 3x - 100$ is increasing in $\mathbf{R}$.
43. Find the interval(s) in which $f(x) = x^2e^{-x}$ is increasing.
44. Find the maximum and minimum values, if any, of $g(x) = x^3 + 1$.
45. Find the maximum and minimum values, if any of the following functions given by
   a) $h(x) = \sin(2x) + 5$
   b) $f(x) = |\sin(4x) + 3|$
46. Find the local maximum and minima, if any, of the following functions. Find also the local maximum and local minimum values, as the case may be
   a) $g(x) = x^3 - 3x$
   b) $h(x) = \sin x + \cos x, x \in \left(0, \frac{\pi}{2}\right)$
   c) $f(x) = \sin x - \cos x, x \in (0, 2\pi)$
   d) $f(x) = x^3 - 6x^2 + 9x + 15$
   e) $g(x) = \frac{x}{2} + \frac{2}{x}, x > 0$
   f) $g(x) = \frac{1}{x^2+2}$
   g) $f(x) = x\sqrt{1-x}, 0 < x < 1.$
47. Prove that the following functions do not have maxima or minima:
   a) $f(x) = e^x$
   b) $g(x) = \log x$
   c) $h(x) = x^3 + x^2 + x + 1$

48. Find the absolute maximum and absoute minimum value of the following functions in the given intervals
    a) $f(x) = x^3, x \in (-2, 2)$
    b) $f(x) = \sin x + \cos x, x \in (0, \pi)$.
49. Find both the maximum value and the minimum value of
$$3x^4 - 8x^3 + 12x^2 - 48x + 25, x \in [0, 3]. \tag{1.2.49.1}$$
50. At what points in the interval $[0, 2\pi]$, does the function $\sin 2x$ attain its maximum value?
51. What is the maximum value of the function $\sin x + \cos x$?
52. Find the maximum value of $2x^3 - 24x + 107$ in the interval $[1, 3]$. Find the maximum value of the same function in $[-3, 1]$.
53. It is given that at $x = 1$, the function $x^4 - 62x^2 + ax + 9$ attains its maximum value on the interval $[0, 2]$. Find the value of a.
54. Find the maximum and minimum values of $x + \sin 2x$ on $[0, 2\pi]$.
55. For all real values of x, the minimum value of
$$\frac{1 - x + x^2}{1 + x + x^2}. \tag{1.2.55.1}$$
56. Find the maximum value of
$$[x(x-1)]^{\frac{1}{3}}. \tag{1.2.56.1}$$
57. Using differentials, find the approximate value of each of the following
    a) $\left(\frac{17}{81}\right)^{\frac{1}{4}}$
    b) $(33)^{-\frac{1}{5}}$
58. Show that the function given by $f(x) = \frac{\log x}{x}$ has maximum at $x = 3$.
59. Find the intervals in which the function f given by
$$f(x) = \frac{4\sin x - 2x - x\cos x}{2 + \cos x} \tag{1.2.59.1}$$
    is
    a) increasing
    b) decreasing
60. Find the interals in which the function f given by
$$f(x) = x^3 + \frac{1}{x^3}, \quad x \neq 0 \tag{1.2.60.1}$$
    is
    a) increasing
    b) decreasing
61. Find the absolute maximum and minimum values of the function f given by
$$f(x) = \cos^2 x + \sin x, \quad x \in [0, \pi] \tag{1.2.61.1}$$
62. Find the points at which the function f given by
$$f(x) = (x-2)^4 (x+1)^3 \tag{1.2.62.1}$$
    has
    a) local maxima
    b) local minima
    c) point of inflexion
63. Examine the following functions for continuity.
    a) $f(x) = \frac{1}{x-5}$
    b) $f(x) = \frac{x^2-25}{x+5}, x \neq -5$
64. Prove that the function $f(x) = x^n$ is continuous at $x = n$, where n is a positive integer.
    a) $f(x) = \begin{cases} x^3 - 3, & x \leq 2, \\ x^2 + 1, & x > 2 \end{cases}$
    b) $f(x) = \begin{cases} x^1 0 - 1, & x \leq 1, \\ x^2, & x > 1 \end{cases}$
65. Discuss the continuity of the following functions:
    a) $f(x) = \sin x + \cos x$
    b) $f(x) = \sin x - \cos x$
    c) $f(x) = \sin x \cos x$
66. Discuss the continuity of the cosine, cosecant, secant and cotangent functions.
67. Find all points of discontinuity of f, where
$$f(x) = \begin{cases} \frac{\sin x}{x}, & x < 0, \\ x + 1, & x \geq 0 \end{cases} \tag{1.2.67.1}$$
68. Determine if
$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0, \\ 0, & x = 0 \end{cases} \tag{1.2.68.1}$$
    is a continuous function.
69. Examine the continuity of
$$f(x) = \begin{cases} \sin x - \cos x, & x \neq 0, \\ -1, & x = 0 \end{cases} \tag{1.2.69.1}$$
70. Find values of k so that the following functions are continuous at the points indicated

a) $\begin{cases} \frac{k\cos x}{\pi-2x} & x \neq \frac{\pi}{2}, \\ 3, & x = \frac{\pi}{2} \end{cases}, \quad x = \frac{\pi}{2}$

b) $\begin{cases} kx+1 & x \leq \pi, \\ \cos x, & x > \pi, \end{cases} \quad x = \pi$

71. Show that the function defined by $f(x) = cos(x^2)$ is a continuous function.
72. Show that the function defined by $f(x) = |cosx|$ is a continuous function.
73. Examine that $sin\,|x|$ is a continuous function.
74. Find all the points of discontinuity of f defined by $f(x) = |x| - |x+1|$.
75. Evaluate the following limits
    a) $\lim_{x\to 4} \frac{4x+3}{x-2}$
    b) $\lim_{x\to -1} \frac{x^10+x^5+1}{x-1}$
    c) $\lim_{x\to 0} \frac{(x+1)^5-1}{x}$
    d) $\lim_{x\to 2} \frac{3x^2-x-10}{x^2-4}$
    e) $\lim_{x\to 3} \frac{x^4-81}{2x^2-5x-3}$
    f) $\lim_{x\to 0} \frac{ax+b}{cx+1}$
    g) $\lim_{z\to 1} \frac{z^{\frac{1}{3}}-1}{z^{\frac{1}{6}}-1}$
    h) $\lim_{x\to 1} \frac{ax^2+bx+3}{cx^2+bx+a}, \quad a+b+c \neq 0$
    i) $\lim_{x\to 2} \frac{\frac{1}{x}+\frac{1}{2}}{x+2}$
    j) $\lim_{x\to 0} \frac{\sin ax}{bx}$
    k) $\lim_{x\to 0} \frac{\sin ax}{\sin bx}, \quad a, b \neq 0$
    l) $\lim_{x\to \pi} \frac{\sin(\pi-x)}{\pi(\pi-x)}, \quad a, b \neq 0$
    m) $\lim_{x\to 0} \frac{\cos x}{\pi-x}$
    n) $\lim_{x\to 0} \frac{\cos 2x-1}{\cos x-1}$
    o) $\lim_{x\to 0} \frac{ax+x\cos x}{b\sin x}$
    p) $\lim_{x\to 0} x\sec x$
    q) $\lim_{x\to 0} \frac{\sin ax+bx}{ax+\sin bx}, \quad a, b, a+b \neq 0$
    r) $\lim_{x\to 0} \csc - \cot x$
    s) $\lim_{x\to \frac{\pi}{2}} \frac{\tan 2x}{x-\frac{\pi}{2}}$
76. Find $\lim_{x\to 0} f(x)$ and $\lim_{x\to 1} f(x)$where
$$f(x) = \begin{cases} 2x+3 & x \leq 0 \\ 3(x+1), & x > 0 \end{cases} \tag{1.2.76.1}$$
77. Let $a_1, a_2, \ldots a_n$ be fixed real numbers and define a function
$$f(x) = (x-a_1)(x-a_2)\ldots(x-a_n) \tag{1.2.77.1}$$
What is $\lim_{x\to a_1} f(x)$? For some $a \neq a_1, a_2, \ldots, a_n$, compute $\lim_{x\to a} f(x)$.
78. If
$$\lim_{x\to 1} \frac{f(x)-2}{x^2-1} = \pi, \tag{1.2.78.1}$$
evaluate $\lim_{x\to 1} f(x)$.
79. If
$$f(x) = \begin{cases} mx^2+n & x < 0 \\ nx+m, & 0 \leq x \leq 1 \\ nx^3+m, & x > 1, \end{cases} \tag{1.2.79.1}$$
for what integers m and n does both $\lim_{x\to 0} f(x)$ and $\lim_{x\to 1} f(x)$ exist ?
80. Integrate the following as limit of sums:
    (i) $\int_{-1}^{1} e^x\,dx$
    (ii) $\int_{-1}^{1} \left(x - e^{2x}\right)dx$
81. Evaluate the following definite integrals
    (i) $\int_2^3 \frac{1}{x}\,dx$
    (ii) $\int_1^2 \left(4x^3 - 5x^2 + 6x + 9\right)dx$
    (iii) $\int_0^{\frac{\pi}{4}} \sin 2x\,dx$
    (iv) $\int_0^{\frac{\pi}{2}} \cos 2x\,dx$
    (v) $\int_4^5 e^x\,dx$
    (vi) $\int_0^{\frac{\pi}{4}} \tan 2x\,dx$
    (vii) $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \csc 2x\,dx$
    (viii) $\int_0^1 \frac{dx}{\sqrt{1-x^2}}\,dx$
    (ix) $\int_0^1 \frac{dx}{1+x^2}\,dx$
    (x) $\int_2^3 \frac{dx}{x^2-1}\,dx$
    (xi) $\int_0^{\frac{\pi}{2}} \cos^2 x\,dx$
    (xii) $\int_2^3 \frac{x}{1+x^2}\,dx$
    (xiii) $\int_0^1 \frac{2x+3}{5x^2+1}\,dx$
    (xiv) $\int_0^1 xe^{x^2}\,dx$
    (xv) $\int_1^2 \frac{5x^2}{x^2+4x+3}\,dx$
    (xvi) $\int_0^{\frac{\pi}{4}} \left(2\sec^2 x + x^3 + 2\right)dx$
    (xvii) $\int_0^{\pi} \left(\sin^2\frac{x}{2} - \cos^2\frac{x}{2}\right)dx$
    (xviii) $\int_0^2 \frac{6x+3}{x^2+4}\,dx$
    (xix) $\int_0^1 \left(xe^x + \sin\frac{\pi x}{4}\right)dx$
82. Find $\int_1^{\sqrt{3}} \frac{dx}{1+x^2}\,dx$
83. Find $\int_0^{\frac{2}{3}} \frac{dx}{4+9x^2}\,dx$
84. Evaluate the following definite integrals
    (i) $\int_0^{\frac{\pi}{2}} \cos^2 x\,dx$
    (ii) $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$
    (iii) $\int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} x}{\sin^{\frac{3}{2}} x+\cos^{\frac{3}{2}} x}\,dx$
    (iv) $\int_0^{\frac{\pi}{2}} \frac{\cos^5 x}{\sin^5 x+\cos^5 x}\,dx$
    (v) $\int_0^1 x(1-x)^n\,dx$
    (vi) $\int_0^{\frac{\pi}{4}} \log(1+\tan x)\,dx$

(vii) $\int_0^1 x\sqrt{2-x}\,dx$
(viii) $\int_0^{\frac{\pi}{2}} (2\log\sin x - \log\sin 2x)\,dx$
(ix) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x\,dx$
(x) $\int_0^{\pi} \frac{x}{1+\sin x}\,dx$
(xi) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x\,dx$
(xii) $\int_0^{2\pi} \cos^5 x\,dx$
(xiii) $\int_0^{\frac{\pi}{2}} \frac{\sin x-\cos x}{1+\sin x\cos x}\,dx$
(xiv) $\int_0^{\pi} \log(1+\cos x)\,dx$
(xv) $\int_0^{a} \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}\,dx$

85. Find the value of

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(x^3 + x\cos x + \tan^5 x + 1\right)\,dx \quad (1.2.85.1)$$

86. Find the value of

$$\int_0^{\frac{\pi}{2}} \log\left(\frac{4+3\sin x}{4+3\cos x}\right)\,dx \quad (1.2.86.1)$$

87. Evaluate the following definite integrals
(i) $\int_{\frac{\pi}{2}}^{\pi} \left(\frac{1-\sin x}{1-\cos x}\right)\,dx$
(ii) $\int_0^{\frac{\pi}{4}} \frac{\sin x\cos x}{\cos^4 x+\sin^4 x}\,dx$
(iii) $\int_0^{\frac{\pi}{4}} \frac{\cos^2 x}{\cos^2 x+4\sin^2 x}\,dx$
(iv) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x+\cos x}{\sqrt{\sin 2x}}\,dx$
(v) $\int_0^1 \frac{dx}{\sqrt{1+x}-\sqrt{x}}\,dx$
(vi) $\int_0^{\frac{\pi}{4}} \frac{\sin x+\cos x}{9+16\sin 2x}\,dx$
(vii) $\int_{\frac{\pi}{2}}^{\pi} \sin 2x\tan^{-1}(\sin x)\,dx$
(viii) $\int_0^{\pi} \frac{x\tan x}{\sec x+\tan x}\,dx$

88. Prove that
(i) $\int_1^3 \frac{dx}{x^2(x+1)} = \frac{2}{3} + \log\frac{2}{3}$
(ii) $\int_0^1 e^x\,dx = 1$
(iii) $\int_{-1}^1 x^{17}\cos^4\,dx = 0$
(iv) $\int_{\frac{\pi}{2}}^{\pi} \sin^3 x\,dx = \frac{2}{3}$
(v) $\int_{\frac{\pi}{4}}^{\pi} 2\tan^3 x\,dx = 1-\log 2$
(vi) $\int_0^1 \sin^{-1} x\,dx = \frac{\pi}{2} - 1$

89. Evaluate $\int_0^1 e^{2-3x}\,dx$ as a limit of a sum.

90. Find the value of $\int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right)\,dx = \frac{\pi}{2} - 1$

91. Solve
(i) $\left(x^3+x^2+x+1\right)y_1 = 2x^2+x \quad y(0)=1$
(ii) $\left(x\left(x^2-1\right)\right)y_1 = 2x^2+x \quad y(2)=0$
(iii) $\cos(y_1) = y\tan x; y(0) = 1$

92. Find the equation of a curve passing through the origin and whose differential equation is $y_1 = e^x\sin x$

93. For the differential equation $xyy_1 = (x+2)(y+2)$, find the solution curve passing through the point $\begin{pmatrix}1\\-1\end{pmatrix}$

94. Find the equation of a curve passing through the point $\left(0,-2\right)$ given that at any point $\left(x,y\right)$ on the curve, the product of the slope of its tangent and y coordinate of the point is equal to the x coordinate of the point.

95. At any point $\left(x,y\right)$ of a curve, the slope of the tangent is twice the slope of the line segment joining the point of contact to the point $\left(-4,-3\right)$. Find the equation of the curve given that it passes through $\left(-2,1\right)$.

96. The volume of spherical balloon being inflated changes at a constant rate. If initially its radius is 3 units and after 3 seconds it is 6 units. Find the radius of balloon after t seconds.

97. In a bank, principal increases continuously at the rate of r% per year. Find the value of r if ₹100 double itself in 10 years.

98. In a bank, principal increases continuously at the rate of 5% per year. An amount of ₹1000 is deposited with this bank, how much will it worth after 10 years.

99. In a culture, the bacteria count is 1,00,000. The number is increased by 10% in 2 hours. In how many hours will the count reach 2,00,000, if the rate of growth of bacteria is proportional to the number present?

100. Solve
(i) $(x+y)\,dy + (x-y)\,dx = 0, y(1) = 1$
(ii) $x^2\,dy + \left(xy+y^2\right)\,dx = 0, y(1) = 1$
(iii) $\left[x\sin^2\left(\frac{y}{x}-y\right)\right]dx + x\,dy = 0, y(1) = \frac{\pi}{4}$
(iv) $y_1 - \frac{y}{x} + \csc\left(\frac{y}{x}\right) = 0, y(1) = 0$
(v) $2xy + y^2 - 2x^2y_1 = 0, y(1) = 2$

101. Solve
(i) $y_1 + 2y\tan x = \sin x, y\left(\frac{\pi}{3}\right) = 0$
(ii) $\left(1+x^2\right)y_1 + 2xy = \frac{1}{1+x^2}, y(0) = 1$
(iii) $y_1 - 3y\cot x = \sin 2x, y\left(\frac{\pi}{2}\right) = 2$

102. Find the equation of a curve passing through the origin given that the slope of the tangent to the curve at any point $\begin{pmatrix}x\\y\end{pmatrix}$ is equal to the sum of the coordinates of the point.

103. Find the equation of a curve passing through

the point $\begin{pmatrix}0\\2\end{pmatrix}$ given that the sum of the coordinates of any point on the curve exceeds the magnitude of the slope of the tangent to the curve at that point by 5.

104. Find the equation of the curve passing through the point $\begin{pmatrix}0\\\frac{\pi}{4}\end{pmatrix}$ whose differential equation is $\sin x \cos y\, dx + \cos x \sin y\, dy = 0$

105. Solve

$$\left(1 + e^{2x}\right) dy + \left(1 + y^2\right) e^x\, dx = 0, \quad y(0) = 1 \qquad (1.2.105.1)$$

106. Solve

$$(x - y)(dx - dy) = dx - dy, \quad y(0) = -1 \qquad (1.2.106.1)$$

107. Solve

$$y_1 + y \cot x = 4x \csc x \quad y\left(\frac{\pi}{2}\right) = 0 \quad (1.2.107.1)$$

108. Solve

$$(x + 1)\, y_1 = 2e^{-y} - 1 \quad y(0) = 0 \quad (1.2.108.1)$$

109. The population of a village increases continuously at the rate proportional to the number of its inhabitants present at any time. If the population of the village was 20, 000 in 1999 and 25000 in the year 2004, what will be the population of the village in 2009?

110. A three-wheeler starts from rest, accelerates uniformly with 1 ms^{-2} on a straight road for 10 s, and then moves with uniform velocity. Plot the distance covered by the vehicle during the nth second (n = 1,2,3. . . .) versus n. What do you expect this plot to be during accelerated motion : a straight line or a parabola ?

111. Two stones are thrown up simultaneously from the edge of a cliff 200 m high with initial speeds of 15 ms^{-1} and 30 ms^{-1} . Verify that the graph shown in Fig. 1.2.111 correctly represents the time variation of the relative position of the second stone with respect to the first. Neglect air resistance and assume that the stones do not rebound after hitting the ground. Take g = 10 ms^{-2}. Give the equations for the linear and curved parts of the plot.

112. Figure 1.2.112 gives the x-t plot of a particle executing one-dimensional simple harmonic motion. Give the signs of position, velocity and acceleration variables of the particle at t = 0.3 s, 1.2 s, – 1.2 s.

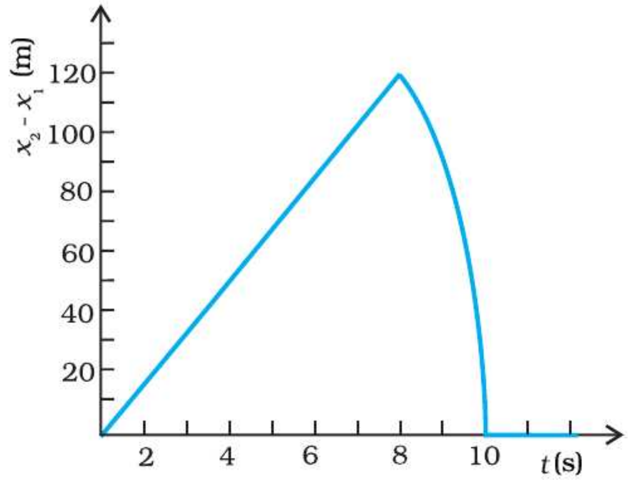

Fig. 1.2.111

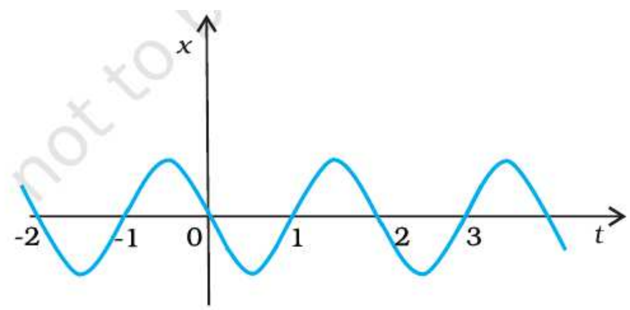

Fig. 1.2.112

113. The speed-time graph of a particle moving along a fixed direction is shown in Fig. 1.2.113. Obtain the distance traversed by the particle between (a) t = 0 s to 10 s, (b) t = 2 s to 6 s.

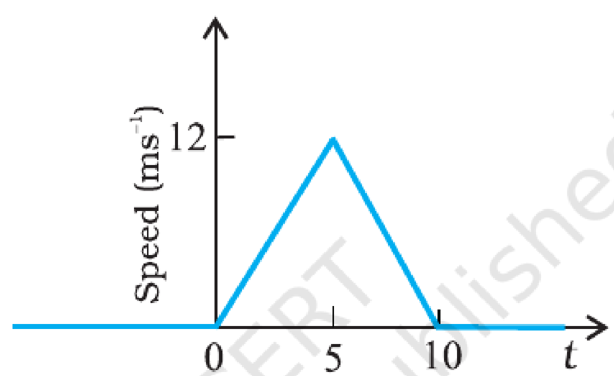

Fig. 1.2.113

114. Figure 1.2.114 shows the x-t plot of one-dimensional motion of a particle. Is it correct

to say from the graph that the particle moves in a straight line for $t < 0$ and on a parabolic path for $t > 0$? If not, suggest a suitable physical context for this graph.

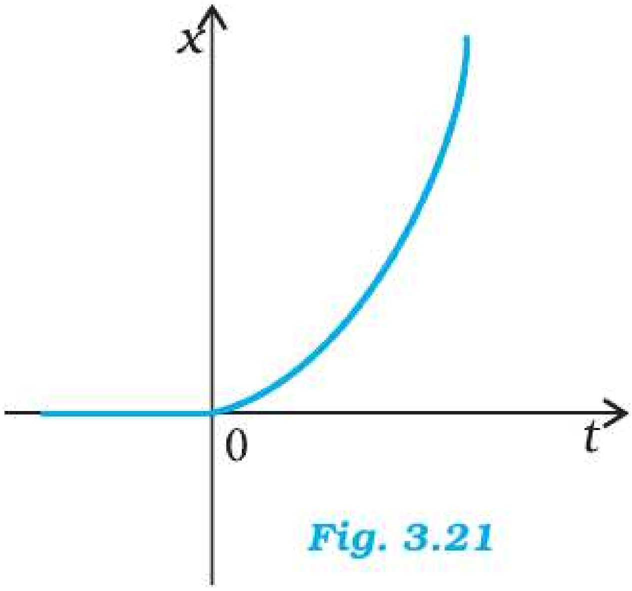

Fig. 1.2.114

2 Trigonometry

2.1 Examples

1. Convert $40°20'$ into radian measure.
2. Convert 6 radians into radian measure.
3. Find the radius of the circle in which a central angle of 60^o intercepts an arc of length 37.4 cm (use $\pi = \frac{22}{7}$).
4. The minute hand of watch is 1.5 cm long. How far does its tip move in 40 minutes? ($\pi = 3.14$)
5. If the arcs of the same lengths in two circles subtend angles 65^o and 110^o at the centre, find the ratio of their radii.
6. If $\cos x = -\frac{3}{5}$, x lies in the third quadrant, find the values of other five trigonometric function.
7. If $\cot x = -\frac{5}{12}$, x lies in the second quadrant, find the values of other five trigonometric function.
8. Find the value of $\sin\frac{31\pi}{3}$.
9. Find the value of $\cos(-1710^o)$.
10. Prove that $3\sin\frac{\pi}{6}\sec\frac{\pi}{3} - 4\sin\frac{5\pi}{6}\cot\frac{\pi}{4} = 1$.
11. Find the value of $\sin 15^o$.
12. Find the value of $\tan\frac{13\pi}{12}$.
13. Prove that $\frac{\sin(x+y)}{\sin(x-y)} = \frac{\tan x+\tan y}{\tan x-\tan y}$
14. Show that

    $$\tan 3x \tan 2x \tan x = \tan 3x - \tan 2x - \tan x.$$

15. Prove that

    $$\cos(\tfrac{\pi}{4} + x) + \cos(\tfrac{\pi}{4} - x) = \sqrt{2}\cos x$$

16. Prove that $\frac{\cos 7x+\cos 5x}{\cos 7x-\cos 5x} = \cot x$
17. Prove that $\frac{\sin 5x-2\sin 3x+\sin x}{\cos 5x-\cos x} = \tan x$
18. Find the principal solutions of the equation $\sin x = \frac{\sqrt{3}}{2}$.
19. Find the principal solutions of the equation $\tan x = -\frac{1}{\sqrt{3}}$.
20. Find the solution of $\sin x = -\frac{\sqrt{3}}{2}$.
21. Solve $\cos x = \frac{1}{2}$.
22. Solve $\tan 2x = -\cot(x + \frac{\pi}{3})$.
23. Solve $\sin 2x - \sin 4x + \sin 6x = 0$.
24. Solve $2\cos^2 x + 3\sin x = 0$
25. If $\sin x = \frac{3}{5}, \cos y = -\frac{12}{13}$, where x and y both lies in second quadrant, find the value of $\sin(x + y)$.
26. Prove that
    $\cos 2x \cos\frac{x}{2} - \cos 3x \cos\frac{9x}{2} = \sin 5x \sin\frac{5x}{2}$
    .
27. Find the value of $\tan\frac{\pi}{8}$.
28. If $\tan x = \frac{3}{4}, \pi < x < \frac{3\pi}{2}$, find the value of $\sin\frac{x}{2}, \cos\frac{x}{2}$ and $\tan\frac{x}{2}$
29. Prove that
    $\cos^2 x + \cos^2(x + \frac{\pi}{3}) + \cos^2(x - \frac{\pi}{3}) = \frac{3}{2}$

2.2 Exercises

1. Find the radian measures corresponding to the following meausres:
   (i) $25°$
   (ii) $-47°30'$
   (iii) $240°$
   (iv) $520°$
2. Find the degree measures corresponding to the following radian measures(use π=3.14)
   (i) $\frac{11}{16}$
   (ii) -4
   (iii) $\frac{5\pi}{3}$
   (iv) $\frac{7\pi}{6}$

3. A wheel makes 360 revolutions in one minute. Through how many radians does it turn in one second?
4. Find the degree measure of the angle subtended at the centre of a circle of radius 100 cm by an arc of length 22 cm?
5. In a circle of diameter 40 cm, the length of a chord is 20 cm. Find the length of minor arc of the chord.
6. If in two circles, arcs of the same length subtend angles 60° and 75° at the centre, find the ratio of their radii?
7. Find the angle in radian through which a pendulum swings if its length is 75 cm and the tip describes an arc of length
   (i) 10 cm
   (ii) 15 cm
   (iii) 21 cm
8. Find the values of other five trigonometric functions
   1. $\cos x = -\frac{1}{2}$, x lies in third quadrant.
   2. $\sin x = \frac{3}{5}$, x lies in second quadrant.
   3. $\cot x = \frac{3}{4}$, x lies in third quadrant.
   4. $\sec x = \frac{13}{5}$, x lies in fourth quadrant.
   5. $\tan x = -\frac{5}{12}$, x lies in second quadrant.
9. Find the values of the trigonometric functions
   1. $\sin 765^o$
   2. $cosec(-1410^o)$
   3. $\tan\frac{19\pi}{3}$
   4. $\sin\frac{-11\pi}{3}$
   5. $\cot\frac{-15\pi}{4}$
10. Prove that
    1. $\sin^2\frac{\pi}{6} + \cos^2\frac{\pi}{3} - \tan^2\frac{\pi}{4} = -\frac{1}{2}$

    2. $2\sin^2\frac{\pi}{6} + cosec^2\frac{7\pi}{6}\cos^2\frac{\pi}{3} = -\frac{3}{2}$

    3. $\cot^2\frac{\pi}{6} + cosec^2\frac{5\pi}{6} + 3\tan^2\frac{\pi}{6}$=6

    4. $2\sin^2\frac{3\pi}{4} + 2\cos^2\frac{\pi}{4} + 2\sec^2\frac{\pi}{3}$=10

11. Find the value of
    (i) $\sin 75^o$
    (ii) $\tan 15^o$

12. Prove that
    $\cos(\frac{\pi}{4} - x)\cos(\frac{\pi}{4} - y) - \sin(\frac{\pi}{4} - x)\sin(\frac{\pi}{4} - y) = \sin(x + y)$

13. Prove that

    $\frac{\tan(\frac{\pi}{4}+x)}{\tan(\frac{\pi}{4}-x)} = (\frac{1+\tan x}{1-\tan x})^2$

14. Prove that

    $\frac{\cos(\pi+x)\cos(-x)}{\sin(\pi-x)\cos(\frac{\pi}{2}+x)} = \cot^2 x$

15. Prove that

    $\cos(\frac{3\pi}{2}+x)\cos(2\pi+x)[\cot(\frac{3\pi}{2}-x)+\cot(2\pi+x)] = 1$
16. Prove that

    $\sin(n+1)x\sin(n+2)x+\cos(n+1)x\cos(n+2)x = \cos x$

17. Prove that

    $\cos(\frac{3\pi}{4} + x) - \cos(\frac{3\pi}{4} - x) = -\sqrt{2}\sin x$

18. Prove that

    $\sin^2 6x - \sin^2 4x = \sin 2x \sin 10x$

19. Prove that

    $\cos^2 2x - \cos^2 6x = \sin 4x \sin 8x$

20. Prove that

    $\sin 2x + 2\sin 4x + \sin 6x = 4\cos^2 x \sin 4x$

21. Prove that

    $\cot 4x(\sin 5x + \sin 3x) = \cot x(\sin 5x - \sin 3x)$

22. Prove that

$$\frac{\cos 9x - \cos 5x}{\sin 17x - \sin 3x} = -\frac{\sin 2x}{\cos 10x}$$

23. Prove that

$$\frac{\sin 5x + \sin 3x}{\cos 5x + \cos 3x} = \tan 4x$$

24. Prove that

$$\frac{\sin x + \sin y}{\cos x + \cos y} = \tan(\frac{x-y}{2})$$

25. Prove that

$$\frac{\sin x + \sin 3x}{\cos x + \cos 3x} = \tan 2x$$

26. Prove that

$$\frac{\sin x - \sin 3x}{\sin^2 x - \cos^2 x} = 2\sin x$$

27. Prove that

$$\frac{\cos 4x + \cos 3x + \cos 2x}{\sin 4x + \sin 3x + \sin 2x} = \cot 3x$$

28. Prove that

$$\cot x \cot 2x - \cot 2x \cot 3x - \cot 3x \cot x = 1$$

29. Prove that

$$\tan 4x = \frac{4\tan x(1-\tan^2 x)}{1-6\tan^2 x + \tan^4 x}$$

30. Prove that

$$\cos 4x = 1 - 8\sin^2 x \cos^2 x$$

31. Prove that

$$\cos 6x = 32\cos^6 x - 48\cos^4 x + 18\cos^2 x - 1$$

32. Find the principle and general solutions of the following equations:
    1. $\tan x = \sqrt{3}$
    2. $\sec x = 2$
    3. $\cot x = -\sqrt{3}$
    4. $cosecx = -2$

33. Find the general solution for each of the following equations:
    1. $\cos 4x = \cos 2x$
    2. $\cos 3x + \cos x - \cos 2x = 0$
    3. $\sin 2x + \cos x = 0$
    4. $\sec^2 2x = 1 - \tan 2x$
    5. $\sin x + \sin 3x + \sin 5x = 0$

34. Prove that
    1. $2\cos\frac{\pi}{13}\cos\frac{9\pi}{13} + \cos\frac{3\pi}{13} + \cos\frac{5\pi}{13} = 0$
    2. $(\sin 3x + \sin x)\sin x + (\cos 3x - \cos x)\cos x = 0$
    3. $(\cos x + \cos y)^2 + (\sin x - \sin y)^2 = 4\cos^2(\frac{x+y}{2})$
    4. $(\cos x - \cos y)^2 + (\sin x - \sin y)^2 = 4\sin^2(\frac{x-y}{2})$
    5. $\sin x + \sin 3x + \sin 5x + \sin 7x = 4\cos x \cos 2x \sin 4x$
    6. $\frac{(\sin 7x + \sin 5x) + (\sin 9x + \sin 3x)}{(\cos 7x + \cos 5x) + (\cos 9x + \cos 3x)} = \tan 6x$
    7. $\sin 3x + \sin 2x - \sin x = 4\sin x \cos\frac{x}{2\cos\frac{3x}{2}}$

35. Find $\sin\frac{x}{2}, \cos\frac{x}{2}$ and $\tan\frac{x}{2}$ in each of the following:
    1. $\tan x = -\frac{4}{3}$, x in second quadrant.
    2. $\sin x = \frac{1}{4}$, x in second quadrant.
    3. $\cos x = -\frac{1}{3}$, x in third quadrant.

3 Calculus

3.1 Examples

1. Find the derivative of the function given by $f(x) = \sin\left(x^2\right)$.
2. Find the derivative of $\tan(2x+3)$.
3. Find $\frac{dy}{dx}$ if $y + \sin y = \cos x$.
4. Find the derivative of $f(x) = \sin^{-1} x$ assuming it exists.
5. Find the derivative of $f(x) = \tan^{-1} x$ assuming it exists.
6. Differentiate the following with respect to x.
   a) e^x
   b) $\sin(\log x), x > 0$
   c) $\cos^{-1}(e^x)$
   d) $e^{\cos x}$.
7. Differentiate

$$\sqrt{\frac{(x-3)(x^2+4)}{3x^2+4x+5}} \qquad (3.1.7.1)$$

8. Differentiate a^x w.r.t. x, where a is a positive constant.
9. Differentiate $x^{\sin x}, x > 0$ w.r.t. x.
10. Find $\frac{dy}{dx}$, if $Y^x + x^y + x^x = a^b$.
11. Find $\frac{dy}{dx}$, if $x = a\cos\theta, y = a\sin\theta$.
12. Find $\frac{dy}{dx}$, if $x = at^2, y = 2at$.

13. Find $\frac{dy}{dx}$, if $x = a(\theta + \sin\theta), y = a(1 - \cos\theta)$.
14. Find $\frac{dy}{dx}$, if $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.
15. Find $\frac{d^2y}{dx^2}$, if $y = x^3 + \tan x$.
16. If $y = A\sin x + B\cos x$, then prove that $\frac{d^2y}{dx^2} + y = 0$.
17. If $y = 3e^{2x} + 2e^{3x}$, prove that $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$.
18. If $y = \sin^{-1} x$, show that $\left(1 - x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0$.
19. Differentiate the following with respect to x.
    a) $\sqrt{3x+2} + \frac{1}{\sqrt{2x^2+4}}$
    b) $e^{\sec^2 x} + 3\cos^{-1} x$
    c) $\log_7(\log x)$
    d) $\cos^{-1}(\sin x)$
    e) $\tan^{-1}\left(\frac{1}{1+\cos x}\right)$
    f) $\sin^{-1}\left(\frac{2^{x+1}}{1+4^x}\right)$
20. Find $f'(x)$ if $f(x) = (\sin x)^{\sin x}$ for all $x \in (0, \pi)$.
21. For a positive constant a, find $\frac{dy}{dx}$, where

$$y = a^{t+\frac{1}{t}}, x = \left(t + \frac{1}{t}\right)^a \tag{3.1.21.1}$$

22. Differentiate $\sin^2 x$ w.r.t. $e^{\cos x}$.
23. Find the derivative of $\sin x$ at $x = 0$.
24. Find the derivative of $f(x) = \frac{1}{x}$.
25. Find the derivative of $f(x) = 1 + x + x^2 + x^3 + \cdots + x^50$ at $x = 1$.
26. Find the derivative of $f(x) = \frac{x+1}{x}$.
27. Find the derivative of $\sin x$.
28. Find the derivative of $\tan x$.
29. Find the derivative of $f(x) = \sin^2 x$.
30. Find the derivative of f from the first principle, where f is given by
    a) $f(x) = \frac{2x+3}{x-2}$
    b) $f(x) = x + \frac{1}{x}$
31. Find the derivative of f from the first principle, where $f(x)$ is
    a) $\sin x + \cos x$
    b) $x\sin x$
32. Compute the derivative of
    a) $f(x) = \sin 2x$
    b) $g(x) = \cot x$
33. Find the derivative of
    a) $\frac{x^5 - \cos x}{\sin x}$
    b) $\frac{x + \cos x}{\sin x}$
34. Write an an anti-derivative for each of the following functions using the method of inspection:
    a) $\cos 2x$
    b) $3x^2 + 4x^3$
    c) $\frac{1}{x}, x \neq 0$
35. Find the following integrals:
    a) $\int \frac{x^3-1}{x^2}\, dx$
    b) $\int \left(x^{\frac{2}{3}} + 1\right) x^2\, dx$
    c) $\int \left(x^{\frac{2}{3}} + 2e^x - \frac{1}{x}\right) x^2\, dx$
36. Find the following integrals:
    a) $\int (\sin x + \cos x)\, dx$
    b) $\int \csc x(\csc + \cot x)\, dx$
    c) $\int \frac{1-\sin x}{\cos^2 x}\, dx$
37. Find an anti-derivative F of f defined by $f(x) = 4x^3 - 6$, where $F(0) = 3$.
38. Integrate the following functions w.r.t x:
    a) $\sin mx$
    b) $2x\sin\left(x^2 + 1\right)$
    c) $\frac{\tan^4\sqrt{x}\sec^2\sqrt{x}}{\sqrt{x}}$
    d) $\frac{\sin\left(\tan^{-1} x\right)}{1+x^2}$
39. Find the following integrals:
    a) $\int \sin^3 x \cos^2 x\, dx$
    b) $\int \frac{\sin x}{\sin(x+a)}\, dx$
    c) $\int \frac{1}{1+\tan x}\, dx$
40. Find
    a) $\int \cos^2 x\, dx$
    b) $\int \sin 2x \cos 3x\, dx$
    c) $\int \sin^3 x\, dx$
41. Find the following integrals
    a) $\int \frac{dx}{x^2-16}$
    b) $\int \frac{dx}{\sqrt{2x-x^2}}$
42. Find the following integrals
    a) $\int \frac{dx}{x^2-6x+13}$
    b) $\int \frac{dx}{3x^2+13x-10}$
    c) $\int \frac{dx}{\sqrt{5x^2-2x}}$
43. Find the following integrals
    a) $\int \frac{x+2}{2x^2+6x+5}\, dx$
    b) $\int \frac{x+3}{\sqrt{5-4x-x^2}}\, dx$
44. Find

$$\int \frac{dx}{(x+1)(x+2)} \tag{3.1.44.1}$$

45. Find

$$\int \frac{x^2+1}{x^2-5x+6}\, dx \tag{3.1.45.1}$$

46. Find

$$\int \frac{3x-2}{(x+1)^2(x+3)}\,dx \tag{3.1.46.1}$$

47. Find

$$\int \frac{x^2}{(x^2+1)^2(x^2+4)}\,dx \tag{3.1.47.1}$$

48. Find

$$\int \frac{(3\sin\phi-2)\cos\phi}{5-\cos^2\phi-4\sin\phi}\,dx \tag{3.1.48.1}$$

49. Find

$$\int \frac{x^2+x+1}{(x+2)(x^2+1)}\,dx \tag{3.1.49.1}$$

50. Find

$$\int x\cos x\,dx \tag{3.1.50.1}$$

51. Find

$$\int \log x\,dx \tag{3.1.51.1}$$

52. Find

$$\int xe^x\,dx \tag{3.1.52.1}$$

53. Find

$$\int \frac{x\sin^{-1}x}{\sqrt{1-x^2}}\,dx \tag{3.1.53.1}$$

54. Find

$$\int e^x\sin x\,dx \tag{3.1.54.1}$$

55. Find
    a) $\int e^x\left(\tan^{-1}x+\frac{1}{1+x^2}\right)\,dx$
    b) $\int \frac{(x^2+1)e^x}{(x+1)^2}\,dx$
56. Find

$$\int \sqrt{x^2+2x+5}\,dx \tag{3.1.56.1}$$

57. Find

$$\int \sqrt{3-2x-x^2}\,dx \tag{3.1.57.1}$$

58. Find

$$\int \cos 6x\sqrt{1+\sin 6x}\,dx \tag{3.1.58.1}$$

59. Find

$$\int \frac{\left(x^4-x\right)^{\frac{1}{4}}}{x^5}\,dx \tag{3.1.59.1}$$

60. Find

$$\int \frac{x^4}{(x-1)(x^2+1)}\,dx \tag{3.1.60.1}$$

61. Find

$$\int [\log(\log x)]+\frac{1}{(\log x)^2}\,dx \tag{3.1.61.1}$$

62. Find

$$\int \left[\sqrt{\cot x}+\sqrt{\tan x}\right]\,dx \tag{3.1.62.1}$$

63. Find

$$\int \frac{\sin 2x\cos 2x}{\sqrt{9-\cos^4(2x)}}\,dx \tag{3.1.63.1}$$

64. Verify that $y=e^{-3x}$ is a solution of the differential equation

$$y_2+y_1-6y=0 \tag{3.1.64.1}$$

65. Verify that $y=a\cos x+b\sin x$ is a solution of the differential equation

$$y_2+y=0 \tag{3.1.65.1}$$

66. Form the differential equation representing the family of curves $y=a\sin(x+b)$, where a,b are arbitrary constants.
67. Find the general solution of the differential equation

$$y_1=\frac{x+1}{2-y} \tag{3.1.67.1}$$

68. Find the general solution of the differential equation

$$y_1=\frac{1+y^2}{1+x^2} \tag{3.1.68.1}$$

69. Show that the differential equation $(x-y)\,y_1=x+2y$ is homogeneous and solve it.
70. Solve $x\cos\left(\frac{x}{y}\right)y_1=y\cos\left(\frac{y}{x}\right)+x$.
71. Show that the family of curves for which the slope of the tangent at any point $\begin{pmatrix}x\\y\end{pmatrix}$ on it is $\frac{x^2+y^2}{xy}$, is given by $x^2-y^2=c$.

72. Solve

$$y_1 - y = \cos x \tag{3.1.72.1}$$

73. Solve

$$xy_1 + 2y = x^2 \tag{3.1.73.1}$$

74. Solve

$$y\,dx - \left(x + 2y^2\right)\,dy = 0 \tag{3.1.74.1}$$

75. Solve

$$y\,dx - \left(x + 2y^2\right)\,dy = 0 \tag{3.1.75.1}$$

76. Verify that $y = c_1e^{ax}\cos bx + c_2e^{ax}\sin bx$, where c_1, c_2 are arbitrary constants is a solution of the differential equation

$$y_2 - 2ay_1 + \left(a^2 + b^2\right)y = 0 \tag{3.1.76.1}$$

77. Solve

$$(x\,dy - y\,dx)\,y\sin\left(\frac{y}{x}\right) = (y\,dx + x\,dy)\,x\cos\left(\frac{y}{x}\right) \tag{3.1.77.1}$$

78. Solve the differential equation

$$\left(\tan^{-1}x - x\right)\,dy = \left(1 + y^2\right)\,dx \tag{3.1.78.1}$$

79. The position of a particle is given by

$$\mathbf{r} = \begin{pmatrix} 3t \\ 2t^2 \\ 5 \end{pmatrix} \tag{3.1.79.1}$$

where t is in seconds and the coefficients have the proper units for r to be in metres.

a) Find $\mathbf{v}(t)$ and $\mathbf{a}(t)$ of the particle.
b) Find the magnitude and direction of $\mathbf{v}(t)$ at t = 1.0 s.

80. A particle starts from origin at t = 0 with a velocity $\begin{pmatrix} 5.0 \\ 0 \end{pmatrix} m/s$ and moves in x-y plane under action of a force which produces a constant acceleration of $\begin{pmatrix} 3 \\ 2 \end{pmatrix} ms^{-2}$.

a) What is the y-coordinate of the particle at the instant its x-coordinate is 84 m ?
b) What is the speed of the particle at this time ?

3.2 Exercises

1. Differentiate the following functions with respect to x

a) $\sin\left(x^2 + 5\right)$
b) $\cos(\sin x)$
c) $\sin(ax + b)$
d) $\sec\left(\tan\sqrt{x}\right)$
e) $\frac{\sin(ax+b)}{\cos(cx+d)}$
f) $\cos x^3 \sin^2\left(x^5\right)$
g) $2\sqrt{\cot\left(x^2\right)}$
h) $\cos\left(\sqrt{x}\right)$

2. Find $\frac{dy}{dx}$ in the following:

a) $2x + 3y = \sin x$
b) $2x + 3y = \sin y$
c) $ax + by^2 = \cos y$
d) $xy + y^2 = \tan x + y$
e) $x^3 + x^2y + xy^2 + y^3 = 81$
f) $\sin^2 y + \cos xy = \kappa$
g) $\sin^2 x + \cos^2 y = 1$
h) $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$
i) $y = \tan^{-1}\left(\frac{3x-x^2}{1-3x^2}\right), x \in \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$
j) $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right), 0 < x < 1$
k) $y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right) < x < 1$
l) $y = \cos^{-1}\left(\frac{2x}{1+x^2}\right), -1 < x < 1$
m) $y = \sin^{-1}\left(2x\sqrt{1-x^2}\right), -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$
n) $y = \sec^{-1}\left(\frac{1}{2x^2-1}\right), 0 < x < \frac{1}{\sqrt{2}}$

3. Differentiate the following w.r.t. x:

a) $\frac{e^x}{\sin x}$
b) $e^{\sin^{-1}x}$
c) e^{x^3}
d) $\sin\left(\tan^{-1}e^{-x}\right)$
e) $\log(\cos e^x)$
f) $e^x + e^{x^2} + \cdots + e^{x^5}$
g) $\sqrt{e^{\sqrt{x}}}, x > 0$
h) $\log(\log x), x > 1$
i) $\frac{\cos x}{\log x}, x > 0$
j) $\cos(\log x + e^x), x > 0$

4. Differentiate the following w.r.t. x

a) $\cos x \cos 2x \cos 3x$
b) $\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$
c) $(\log x)^{\cos x}$
d) $x^x - 2^{\sin x}$
e) $(x+3)^2(x+4)^3(x+5)^4$
f) $\left(x + \frac{1}{x}\right)^x + x^{1+\frac{1}{x}}$
g) $(\log x)^x + (\sin x)^{\cos x}$
h) $(\sin x)^x + \sin^{-1}\sqrt{x}$
i) $x^{\sin x} + (\sin x)^{\cos x}$
j) $x^{\cos x} + \frac{x^2+1}{x^2-1}$

k) $(x\cos x)^x + (x\sin x)^{\frac{1}{x}}$
l) $x^y + y^x = 1$
m) $y^x = x^y$
n) $(\cos x)^y = (\cos y)^x$
o) $xy = e^{x-y}$

5. Find the derivative of the function given by $f(x) = (1+x)\left(1+x^2\right)\left(1+x^4\right)\left(1+x^8\right)$ and hence find $f'1$.
6. Differentiate $\left(x^2-5x+8\right)\left(x^3+7x+9\right)$ in three ways mentioned below:
   a) by using product rule
   b) by expanding the product to obtain a single polynomial
   c) by logarithmic differentiation.

   Do they all give the same answer?
7. Without eliminating the parameter, find $\frac{dy}{dx}$ in the following
   a) $x = 2at^2, y = at^4$
   b) $x = a\cos\theta, y = b\cos\theta$
   c) $x = \sin t, y = \cos t$
   d) $x = 4t, y = \frac{4}{t}$
   e) $x = \cos\theta - \cos 2\theta, y = \sin\theta - \sin 2\theta$
   f) $x = a(\theta - \sin\theta), y = a(1+\cos\theta)$
   g) $x = \frac{\sin^3 t}{\sqrt{\cos 2t}}, y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$
   h) $x = a\left(\cos t + \log\tan\frac{t}{2}\right), y = a\sin t$
   i) $x = a\sec\theta, y = b\tan\theta$
   j) $x = a(\cos\theta + \theta\sin\theta), y = a(\sin\theta - \theta\cos\theta)$
   k) If $x = \sqrt{a^{\sin^{-1} t}}, y = \sqrt{a^{\cos^{-1} t}}$

   show that $\frac{dy}{dx} = -\frac{y}{x}$.
8. Find the second order derivatives of the following functions
   a) x^2+3x+2
   b) x^{20}
   c) $x\cos x$
   d) $\log x$
   e) $x^3\log x$
   f) $x^x\sin 5x$
   g) $e^{6x}\cos 3x$
   h) $\tan^{-1} x$
   i) $\log(\log x)$
   j) $\sin(\log x)$
9. If $y = 5\cos x - 3\sin x$, prove that $\frac{d^2y}{dx^2} + y = 0$
10. If $y = \cos^{-1} x$, find $\frac{d^2y}{dx^2}$ in terms of y.
11. If $y = 3\cos(\log x) + 4\sin(\log x)$, show that $x^2y_2 + xy_1 + y = 0$
12. If $y = Ae^{mx} + Be^{nx}$, show that $y_2 - (m+n)y_1 + mny = 0$.
13. If $y = 500e^{7x} + 600e^{-7x}$, show that $y_2 = 49y$
14. If $e^y(x+1) = 1$, show that $y_2 = y_1^2$
15. If $y = \left(\tan^{-1} x\right)^2$, show that $\left(x^2+1\right)y_2 + 2x\left(x^2+1\right)y_1 = 2$
16. If $f : [-5,5] \to \mathbf{R}$ is a differentiable function and if $f'(x)$ does not vanish anywhere, then prove that $f(-5) \neq f(5)$.
17. Verify mean value theorem, if $f(x) = x^3 - 5x^2 - 3x, x \in [a,b]$ where $a = 1, b = 3$. Find all $c \in (1,3)$ for which $f'(c) = 0$
18. Differentiate the following functions w.r.t x
    a) $\left(3x^2-9x+5\right)^9$
    b) $\sin^3 x + \cos^6 x$
    c) $(5x)^{3\cos 2x}$
    d) $\sin^{-1}\left(x\sqrt{x}\right), \quad 0 \le x \le 1$
    e) $\frac{\cos^{-1}\frac{x}{2}}{\sqrt{2x+7}}, -2 < x < 2$
    f) $\cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right], 0 < x < \frac{\pi}{2}$
    g) $(\log x)^{\log x}, x > 1$
    h) $\cos(a\cos x + b\sin x)$, for some constant a and b.
    i) $(\sin x - \cos x)^{\sin x - \cos x}, \quad \frac{\pi}{4}, < x < \frac{3\pi}{4}$
    j) $x^x + x^a + a^x + a^a$, for some fixed $a > 0$ and $x > 0$.
    k) $x_+^{x^2-3}(x-3)^{x^2}$, for $x > 3$.
19. Find $\frac{dy}{dx}$, if $y = 12(1-\cos t), x = 10(t - \sin t), -\frac{\pi}{2} < x < \frac{\pi}{2}$.
20. Find $\frac{dy}{dx}$, if $y = \sin^{-1} x + \sin^{-1}\sqrt{1-x^2}, \quad 0 < x < 1$
21. If $x\sqrt{1+y} + y\sqrt{1+x} = 0$, for $-1 < x < 1$, prove that

$$\frac{dy}{dx} = -\frac{1}{(1+x)^2} \tag{3.2.21.1}$$

22. If $\cos y = x\cos(a+y)$, with $\cos a \neq \pm 1$, prove that $\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$
23. if $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$, find y_2
24. If $f(x) = |x|^3$, show that $f''(x)$ exists for all real x and find it.
25. Using mathematical induction, prove that $\frac{d}{dx}(x^n) = nx^{n-1}$ for all positive integers n.
26. Using the fact that

$$\sin(x+y) = \sin x\cos y + \cos x\sin y, \tag{3.2.26.1}$$

show that

$$\cos(x+y) = \cos x \cos y - \sin x \sin y \tag{3.2.26.2}$$

27. If

$$y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}, \tag{3.2.27.1}$$

prove that

$$\frac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix} \tag{3.2.27.2}$$

28. If $y = e^{a\cos^{-1}x}, -1 \le x \le 1$, show that

$$\left(1-x^2\right)y_2 - xy_1 - ay^2 = 0. \tag{3.2.28.1}$$

29. Find the derivative of the following functions from the first principle:
 a) $x^3 - 27$
 b) $\frac{1}{x^2}$
 c) $\frac{x+1}{x-1}$
30. For the function

$$f(x) = \frac{x^100}{100} + \frac{x^99}{99} + \cdots + \frac{x^2}{2} + x + 1. \tag{3.2.30.1}$$

prove that $f'(1) = 100f'(0)$.
31. Find the derivative of

$$x^n + ax^{n-1} + a^2x^{n-2} + \cdots + a^n \tag{3.2.31.1}$$

for some fixed real number a.
32. For some constans a and b, find the derivative of
 a) $\left(ax^2+b\right)^2$
 b) $\frac{x-a}{x-b}$
33. Find the derivative of $\frac{x^n-a^n}{x-a}$ for some constant a.
34. Find the derivative of
 a) $2x - \frac{3}{4}$
 b) $\left(5x^3+3x-1\right)(x-1)$
 c) $x^{-3}\left(3-4x^{-5}\right)$.
 d) $x^5\left(x-6x^{-9}\right)$
 e) $x^{-4}\left(3-4x^{-5}\right)$
 f) $\frac{2}{x+1} - \frac{x^2}{3x-1}$
35. Find the derivative of $\cos x$ from the first principle.
36. Find the derivative of the following functions:
 a) $\sin x \cos x$
 b) $\sec x$
 c) $5\sec x + 4\cos x$.
 d) $\csc x$
 e) $3\cot x + 5\csc x$
 f) $5\sin x - 6\cos x + 7$
 g) $2\tan x - 7\sec x$
37. Find the derivative of the following functions:
 (i) $(-x)^{-1}$
 (ii) $\sin(x+1)$
 (iii) $\cos\left(x - \frac{\pi}{8}\right)$
 (iv) $\frac{ax+b}{cx+d}$
 (v) $(px+q)\left(\frac{r}{x}+s\right)$
 (vi) $\frac{1+\frac{1}{x}}{1-\frac{1}{x}}$
 (vii) $\frac{1}{ax^2+bx+c}$
 (viii) $\frac{ax+b}{px^2+qx+r}$
 (ix) $\frac{px^2+qx+r}{ax+b}$
 (x) $\frac{a}{x^4} - \frac{b}{x^2} + \cos x$
 (xi) $4\sqrt{x} - 2$
 (xii) $(ax+b)^n$
 (xiii) $(ax+b)^n(cx+d)^m$
 (xiv) $\sin(x+a)$
 (xv) $\csc x \cot x$
 (xvi) $\frac{\cos x}{1+\sin x}$
 (xvii) $\frac{\sin x + \cos x}{\sin x - \cos x}$
 (xviii) $\frac{\sec x - 1}{\sec x + 1}$
 (xix) $\sin^n x$
 (xx) $\frac{a+b\sin x}{c+d\cos x}$
 (xxi) $\frac{\sin(x+a)}{\cos x}$
 (xxii) $x^4(5\sin x - 3\cos x)$
 (xxiii) $\left(x^2+1\right)\cos x$
 (xxiv) $\left(ax^2+\sin x\right)(p+q\cos x)$
 (xxv) $(x-\tan x)(x+\cos x)$
 (xxvi) $\frac{4x+5\sin x}{3x+7\cos x}$
 (xxvii) $\frac{x^2\cos\left(\frac{\pi}{4}\right)}{\sin x}$
 (xxviii) $(x)(1+\tan x)$
 (xxix) $(x+\sec x)(x-\tan x)$
 (xxx) $\frac{x}{\sin^n x}$
38. Find anti-derivative of each of the following functions
 a) $\sin 2x$
 b) $\cos 2x$
 c) e^{2x}
 d) $(ax+b)^2$
 e) $\sin 2x - 4e^{2x}$

39. Find the following integrals:
   a) $\int 4e^{3x} + 1, dx$
   b) $\int x^2 \left(1 - \frac{1}{x^2}\right), dx$
   c) $\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 dx$
   d) $\int \left(ax^2 + bx + c\right), dx$
   e) $\int \left(2x^2 + e^x\right), dx$
   f) $\int \frac{x^3+5x^2-4}{x^2}, dx$
   g) $\int \frac{x^3-x^2+x-1}{x-1}, dx$
   h) $\int (1 - x)\sqrt{x}, dx$
   i) $\int \sqrt{x}\left(3x^2 + 2x + 3\right), dx$
   j) $\int (2x - 3\cos x + e^x), dx$
   k) $\int \left(2x^2 - 3\sin x + 5\sqrt{x}\right), dx$
   l) $\int \sec x(\sec x + \tan x), dx$
   m) $\int \frac{\sec^2 x}{\csc^2 x}, dx$
40. Find anti-derivative of

$$\sqrt{x} + \frac{1}{\sqrt{x}} \tag{3.2.40.1}$$

41. If

$$\frac{d}{dx}f(x) = 4x^3 - \frac{3}{x^4}, \quad f(2) = 0 \tag{3.2.41.1}$$

   Find $f(x)$.
42. Integrate the following functions:
   (i) $\frac{2x}{1+x^2}$
   (ii) $\frac{(\log x)^2}{x}$
   (iii) $\frac{1}{x+x\log x}$
   (iv) $\sin x \sin(\cos x)$
   (v) $\sin(ax + b)\cos ax + b$
   (vi) $\sqrt{ax + b}$
   (vii) $x\sqrt{x + 2}$
   (viii) $x\sqrt{1 + 2x^2}$
   (ix) $(4x + 2)\sqrt{x^2 + x + 1}$
   (x) $\frac{1}{x-\sqrt{x}}$
   (xi) $\frac{x}{\sqrt{x+4}}, \quad x > 0$
   (xii) $\left(x^3 - 1\right)^{\frac{1}{3}} x^5$
   (xiii) $\frac{x^2}{(2+3x^3)^3}$
   (xiv) $\frac{1}{x(\log x)^m} \quad x > 0, m \neq 1$
   (xv) $\frac{x}{9-4x^2}$
   (xvi) e^{2x+3}
   (xvii) $\frac{x}{e^{x^2}}$
   (xviii) $e^{\frac{\tan^{-1} x}{1+x^2}}$
   (xix) $\frac{e^{2x}-1}{e^2x+1}$
   (xx) $\frac{e^{2x}-e^{-2x}}{e^2x+e^{-2x}}$
   (xxi) $\tan^2(2x - 3)$
   (xxii) $\sec^2(7 - 4x)$
   (xxiii) $\frac{\sin^{-1} x}{\sqrt{1-x^2}}$
   (xxiv) $\frac{2\cos x-3\sin x}{6\cos x+4\sin x}$
   (xxv) $\frac{1}{\cos^2 x(1-\tan x)^2}$
   (xxvi) $\frac{\cos\sqrt{x}}{\sqrt{x}}$
   (xxvii) $\sqrt{\sin 2x}\cos 2x$
   (xxviii) $\frac{\cos x}{\sqrt{1+\sin x}}$
   (xxix) $\cot x \log x \sin x$
   (xxx) $\frac{\sin x}{1+\cos x}$
   (xxxi) $\frac{\sin x}{(1+\cos x)^2}$
   (xxxii) $\frac{1}{1+\cot x}$
   (xxxiii) $\frac{1}{1-\tan x}$
   (xxxiv) $\frac{\sqrt{\tan x}}{\sin x\cos x}$
   (xxxv) $\frac{(1+\log x)^2}{x}$
   (xxxvi) $\frac{(x+1)(x+\log x)^2}{x}$
   (xxxvii) $\frac{x^3\sin\left(\tan^{-1} x^4\right)}{1+x^8}$
43. Find $\int \frac{10x^9+10^x \ln 10}{x^10+10^x}, dx$
44. Find $\int \frac{dx}{\sin^2 x\cos^2 x}, dx$
45. Find the integrals of the following functions:
   (i) $\sin^2(2x + 5)$
   (ii) $\sin 3x\cos 4x$
   (iii) $\cos 2x\cos 4x\cos 6x$
   (iv) $\sin^3(2x + 1)$
   (v) $\sin^3 x\cos^3 x$
   (vi) $\sin x\sin 2x\sin 3x$
   (vii) $\sin 4x\sin 8x$
   (viii) $\frac{1-\cos x}{1+\cos x}$
   (ix) $\frac{\cos x}{1+\cos x}$
   (x) $\sin^4 x$
   (xi) $\cos^4 x$
   (xii) $\frac{\sin^2 x}{1+\cos x}$
   (xiii) $\frac{\cos 2x-\cos 2\alpha}{\cos x-\cos\alpha}$
   (xiv) $\frac{\cos x-\sin x}{1+\sin 2x}$
   (xv) $\tan^3 2x\sec 2x$
   (xvi) $\tan^4 x$
   (xvii) $\frac{\sin^3 x+\cos^3 x}{\sin^2 x\cos^2 x}$
   (xviii) $\frac{\cos 2x+2\sin^2 x}{\cos^2 x}$
   (xix) $\frac{\cos 2x}{(\cos x+\sin x)^2}$
   (xx) $\sin^{-1}(\cos x)$
   (xxi) $\frac{1}{\cos(x-a)\cos(x-b)}$
46. Find $\frac{\sin^2 x-\cos^2 x}{\sin^2 x\cos^2 x}$
47. Find $\frac{e^x(1+x)}{\cos^2(e^x)}$
48. Integrate the following functions:
   (i) $\frac{3x^2}{x^6+1}$
   (ii) $\frac{1}{\sqrt{1+4x^2}}$

(iii) $\frac{1}{\sqrt{1+(2-x)^2}}$
(iv) $\frac{1}{\sqrt{9-25x^2}}$
(v) $\frac{3x}{1+2x^4}$
(vi) $\frac{x^2}{1-x^6}$
(vii) $\frac{x-1}{\sqrt{x^2-1}}$
(viii) $\frac{x^2}{\sqrt{x^6+a^6}}$
(ix) $\frac{\sec^2 x}{\sqrt{\tan^2 x+4}}$
(x) $\frac{1}{\sqrt{x^2+2x+2}}$
(xi) $\frac{1}{9x^2+6x+5}$
(xii) $\frac{1}{\sqrt{7-6x-x^2}}$
(xiii) $\frac{1}{\sqrt{(x-1)(x-2)}}$
(xiv) $\frac{1}{\sqrt{8+3x-x^2}}$
(xv) $\frac{1}{\sqrt{(x-a)(x-b)}}$
(xvi) $\frac{4x+1}{\sqrt{2x^2+x-3}}$
(xvii) $\frac{x+2}{\sqrt{x^2-1}}$
(xviii) $\frac{5x-2}{1+2x+3x^2}$
(xix) $\frac{6x+7}{\sqrt{(x-5)(x-4)}}$
(xx) $\frac{x+2}{\sqrt{4x-x^2}}$
(xxi) $\frac{x+2}{\sqrt{x^2+2x+3}}$
(xxii) $\frac{x+3}{3x^2-2x-5}$
(xxiii) $\frac{5x+3}{\sqrt{x^2+4x+10}}$

49. Find $\int \frac{dx}{x^2+2x+2}\,dx$
50. Find $\int \frac{dx}{\sqrt{9x-4x^2}}\,dx$
51. Integrate the following:
(i) $\frac{x}{(x+1)(x+2)}$
(ii) $\frac{1}{x^2-9}$
(iii) $\frac{3x-1}{(x-1)(x-2)(x-3)}$
(iv) $\frac{x}{(x-1)(x-2)(x-3)}$
(v) $\frac{x}{x^2+3x+2}$
(vi) $\frac{1-x^2}{x(1-2x)}$
(vii) $\frac{x}{\left(x^2+1\right)(x-1)}$
(viii) $\frac{x}{(x+2)(x-1)^2}$
(ix) $\frac{3x+5}{x^3-x^2-x+1}$
(x) $\frac{2x-3}{\left(x^2-1\right)(2x+3)}$
(xi) $\frac{5x}{\left(x^2-4\right)(x+1)}$
(xii) $\frac{x^3+x+1}{x^2-1}$
(xiii) $\frac{2}{(1-x)\left(1+x^2\right)}$
(xiv) $\frac{3x-1}{(x+2)^2}$
(xv) $\frac{1}{x^4-1}$
(xvi) $\frac{x}{x^n+1}$
(xvii) $\frac{\cos x}{(1\ \sin x)(2-\sin x)}$
(xviii) $\frac{\left(x^2+1\right)\left(x^2+2\right)}{\left(x^2+3\right)\left(x^2+4\right)}$
(xix) $\frac{1}{e^x-1}$

52. Find $\int \frac{x\,dx}{(x-1)(x-2)}$
53. Find $\int \frac{dx}{x\left(x^2+1\right)}$
54. Integrate the following functions:
(i) $x \sin x$
(ii) $x \sin 3x$
(iii) $x^2 e^x$
(iv) $x \log x$
(v) $x \log 2x$
(vi) $x^2 \log x$
(vii) $x \sin^{-1} x$
(viii) $x \tan^{-1} x$
(ix) $x \cos^{-1} x$
(x) $\left(\sin^{-1} x\right)^2$
(xi) $\frac{\cos^{-1} x}{\sqrt{1-x^2}}$
(xii) $x \sec^2 x$
(xiii) $\tan^{-1} x$
(xiv) $x\,(\log x)^2$
(xv) $\left(x^2+1\right)\log x$
(xvi) $e^x\,(\sin x + \cos x)$
(xvii) $\frac{xe^x}{(1+x)^2}$
(xviii) $e^x\left(\frac{1+\sin x}{1+\cos x}\right)$
(xix) $e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)$
(xx) $\frac{(x-3)e^x}{(x-1)^3}$
(xxi) $e^2 x \sin x$
(xxii) $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

55. Find $\int x^2 e^{x^3}\,dx$
56. Find $\int e^x \sec x\,(1+\tan x)\,dx$
57. Integrate the following functions:
(i) $\sqrt{4-x^2}$
(ii) $\sqrt{1-4x^2}$
(iii) $\sqrt{x^2+4x+6}$
(iv) $\sqrt{x^2+4x+1}$
(v) $\sqrt{1-4x-x^2}$
(vi) $\sqrt{x^2+4x-5}$
(vii) $\sqrt{1+3x-x^2}$
(viii) $\sqrt{x^2+3x}$
(ix) $\sqrt{1+\frac{x^2}{9}}$

58. Integrate $\int \sqrt{1+x^2}\,dx$
59. Integrate $\int \sqrt{x^2-8x+7}\,dx$
60. Show that

$$\int_0^a f(x)g(x)\,dx = 2\int_0^a f(x)\,dx \qquad (3.2.60.1)$$

if

$$f(x) = f(a-x)g(x) + g(a-x) \quad = 4 \tag{3.2.60.2}$$

61. Integrate the following functions:
    (i) $\frac{1}{x-x^3}$
    (ii) $\frac{1}{\sqrt{x+a}+\sqrt{x+b}}$
    (iii) $\frac{1}{x\sqrt{ax-x^2}}$
    (iv) $\frac{1}{x^2\left(x^4+1\right)^{\frac{3}{4}}}$
    (v) $\frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}$
    (vi) $\frac{5x}{(x+1)\left(x^2+9\right)}$
    (vii) $\frac{\sin x}{\sin(x-a)}$
    (viii) $\frac{e^{5\log x}-e^{4\log x}}{e^{3\log x}-e^{2\log x}}$
    (ix) $\frac{\cos x}{\sqrt{4-\sin^2 x}}$
    (x) $\frac{\sin^8 x-\cos^8 x}{1-2\sin^2 x\cos^2 x}$
    (xi) $\frac{1}{\cos(x+a)\cos(x+b)}$
    (xii) $\frac{x^3}{\sqrt{1-x^8}}$
    (xiii) $\frac{e^x}{\sqrt{2+e^x}}$
    (xiv) $\frac{1}{\left(x^2+1\right)\left(x^2+4\right)}$
    (xv) $\cos^3 xe^{\log\sin x}$
    (xvi) $e^{3\log x}\left(x^4+1\right)^{-1}$
    (xvii) $\frac{1}{\sqrt{\sin^3 x\sin(x+\alpha)}}$
    (xviii) $\frac{\sin^{-1}\sqrt{x}-\cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}}, x \in [0,1]$
    (xix) $\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}$
    (xx) $\frac{2+\sin 2x}{1+\cos 2x}e^x$
    (xxi) $\frac{x^2+x+1}{(x+1)^2(x+2)}$
    (xxii) $\tan^{-1}\sqrt{\frac{1-x}{1+x}}$
    (xxiii) $\frac{\sqrt{x^2+1}\log\left(x^2+1\right)-2\log x}{x^4}$
62. Find $\int \frac{dx}{e^x+e^{-x}}\,dx$
63. Find $\int \frac{\cos 2x}{\sin x+\cos x}\,dx$
64. Verify that the given functions is a solution of the corresponding differential equation:
    (i) $y = e^x + 1; y_2 - y_1 = 0$
    (ii) $y = x^2 + 2x + C; y_1 - 2x - 2 = 0$
    (iii) $y = \cos x + C; y_1 + \sin x = 0$
    (iv) $y = \sqrt{1+x^2}; y_1 = \frac{xy}{1+x^2}$
    (v) $y = Ax; xy_1 = y, x \neq 0$
    (vi) $y = x\sin x;$

    $$xy_1 = y + x\sqrt{x^2-y^2}, (x \neq 0, x > y \text{ or } x < -y)$$

    (vii) $xy = \log y + C; y_1 = \frac{y^2}{1-xy}, (xy \neq 1)$
    (viii) $y - \cos y = x; y^2y_1 + y^2 + 1 = 0$
    (ix) $y = \sqrt{a^2-x^2}, x \in (a,-a); x+yy_1 = 0, (y \neq 0)$
65. Form the differential equation representing the following family of curves where a, b are arbitrary constants.
    (i) $y = ae^{3x} + be^{-2x}$
    (ii) $y = e^{2x}(a+bx)$
    (iii) $y = e^x(a\cos x + b\sin x)$
66. Find the general solution for each of the following differntial equations
    (i) $y_1 = \frac{1-\cos x}{1+\cos x}$
    (ii) $y_1 = \sqrt{4-y^2} \quad (|y| < 2)$
    (iii) $y_1 + y = 1 \quad (y \neq 1)$
    (iv) $\sec^2 x\tan y\,dx + \sec^2 y\tan x\,dy = 0$
    (v) $\left(e^x + e^{-x}\right)dy - \left(e^y + e^{-y}\right)dx = 0$
    (vi) $y_1 = \left(1+x^2\right)\left(1+y^2\right)$
    (vii) $y\log y\,dx - x\,dy = 0$
    (viii) $x^5y_1 = -y^5$
    (ix) $y_1 = \sin^{-1} x$
    (x) $e^x\tan y\,dx + (1-e^x)\sec^2 y\,dy = 0$
67. Find the general solution of $y_1 = e^{x+y}$
68. Solve
    (i) $\left(x^2+xy\right)dy = \left(x^2+y^2\right)$
    (ii) $y_1 = \frac{x+y}{x}$
    (iii) $(x-y)\,dy - (x+y)\,dx = 0$
    (iv) $\left(x^2-y^2\right)dx + 2xy\,dy = 0$
    (v) $x^2y_1 = x^2 - 2y^2 + xy$
    (vi) $x\,dy - y\,dx = \sqrt{x^2+y^2}\,dx$
    (vii) $\left\{x\cos\left(\frac{y}{x}\right) + y\sin\left(\frac{y}{x}\right)\right\}y\,dx = \left\{y\sin\left(\frac{y}{x}\right) - x\cos\left(\frac{y}{x}\right)\right\}x\,dx$
    (viii) $xy_1 - y + x\sin\left(\frac{y}{x}\right) = 0$
    (ix) $y\,dx + x\log\left(\frac{y}{x}\right)dy - 2x\,dy = 0$
    (x) $\left(1+e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy = 0$
69. Solve
    (i) $y_1 + 2y = \sin x$
    (ii) $y_1 + 3y = e^{-2x}$
    (iii) $y_1 + \frac{y}{x} = x^2$
    (iv) $y_1 + y\sec x = \tan x \quad \left(0 \leq x \leq \frac{\pi}{2}\right)$
    (v) $\cos^2 xy_1 + y = \tan x \quad \left(0 \leq x \leq \frac{\pi}{2}\right)$
    (vi) $xy_1 + 2y = x^2\log x$
    (vii) $x\log xy_1 + y = \frac{2}{x}\log x$
    (viii) $\left(1+x^2\right)dy + 2xy\,dx = \cot x\,dx$
    (ix) $xy_1 + y - x + xy\cot x = 0$
    (x) $(x+y)y_1 = 1$
    (xi) $y\,dx + \left(x-y^2\right)dy = 0$
    (xii) $\left(x+3y^2\right)y_1 = y, \quad y > 0$

70. Solve

$$xy_1 - y = 2x^2 \tag{3.2.70.1}$$

71. Solve

$$\left(1 - y^2\right) y_1 + xy = ay \quad (-1 < y < 1) \tag{3.2.71.1}$$

72. For each of the exercises below, erify that the given function is a solution of the corresponding diferential equation:
73. Solve
   (i) $xy = ae^x + be^{-x} + x^2; \quad xy_2 + 2y_1 - xy + x^2 - 2 = 0$
   (ii) $y = e^x (a\cos x + b\sin x); \quad y_2 - 2y_1 + 2y = 0$
   (iii) $y = x\sin 3x; \quad y_2 + 9y_1 - 6\cos 3x = 0$
   (iv) $x^2 = 2y^2 \log y; \quad \left(x^2 + y^2\right) y_1 - xy = 0$
74. Prove that $x^2 - y^2 = c\left(x^+y^2\right)^2$ is the general solution of differential equation

$$\left(x^3 - 3xy^2\right) dx = \left(y^3 - 3x^2y\right) dy, \tag{3.2.74.1}$$

where c is a parameter.

75. Find the general solution of the differential equation

$$y_1 + \sqrt{\frac{1 - y^2}{1 - x^2}} = 0 \tag{3.2.75.1}$$

76. Show that the general solution of the differential equation

$$y_1 + \frac{y^2 + y + 1}{x^2 + x + 1} = 0 \tag{3.2.76.1}$$

is

$$(x + y + 1) = A(1 - x - y - 2xy), \tag{3.2.76.2}$$

where A is a parameter.

77. Solve

$$ye^{\frac{x}{y}} dx = \left(xe^{\frac{x}{y}} + y^2\right) dy \tag{3.2.77.1}$$

78. Solve

$$\left[\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right] \tag{3.2.78.1}$$

79. Solve

$$\frac{y\,dx - x\,dy}{y} = 0 \tag{3.2.79.1}$$

80. Solve

$$e^x dy + (ye^x + 2x) dx = 0 \tag{3.2.80.1}$$

81. The position of a particle is given by

$$\mathbf{r} = \begin{pmatrix} 3t \\ -2t^2 \\ 4 \end{pmatrix} \tag{3.2.81.1}$$

where t is in seconds and the coefficients have the proper units for r to be in metres.
   a) Find the **v** and **a** of the particle?
   b) What is the magnitude and direction of velocity of the particle at t = 2.0 s ?
82. A particle starts from the origin at t = 0 s with a velocity of $\begin{pmatrix} 0 \\ 10 \end{pmatrix} m/s$ and moves in the x-y plane with a constant acceleration of $\begin{pmatrix} 8 \\ 2 \end{pmatrix} ms^{-2}$.
   a) At what time is the x- coordinate of the particle 16 m? What is the y-coordinate of the particle at that time?
   b) What is the speed of the particle at the time ?
83. Figure 3.2.83 shows the position-time graph of a particle of mass 4 kg. What is the
   a) force on the particle for $t < 0, t > 4s, 0 < t < 4s$?
   b) impulse at t = 0 and t = 4 s ? (Consider one-dimensional motion only).

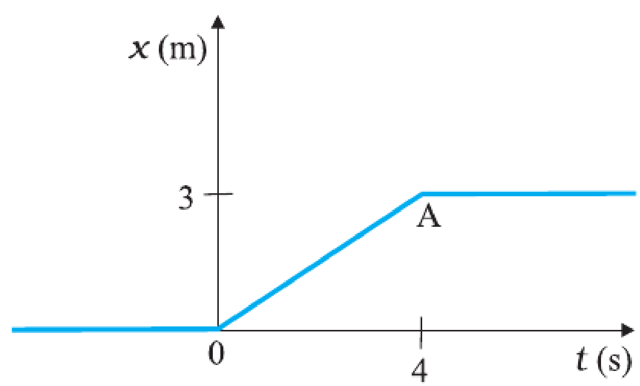

Fig. 3.2.83

4 Miscellaneous Exercises

1. If a parabolic reflector is 20 cm in diameter and 5 cm deep, find the focus.
2. An arch is in the form of a parabola with its axis vertical. The arch is 10 m high and 5 m wide at the base. How wide is it 2 m from the vertex of the parabola?

3. The cable of a uniformly loaded suspension bridge hangs in the form of a parabola. The roadway which is horizontal and 100 m long is supported by vertical wires attached to the cable, the longest wire being 30 m and the shortest being 6 m. Find the length of a supporting wire attached to the roadway 18 m from the middle.
4. An arch is in the form of a semi-ellipse. It is 8 m wide and 2 m high at the centre. Find the height of the arch at a point 1.5 m from one end.
5. A rod of length 12 cm moves with its ends always touching the coordinate axes. Determine the equation of the locus of a point P on the rod, which is 3 cm from the end in contact with the x-axis.
6. Find the area of the triangle formed by the lines joining the vertex of the parabola $x^2 = 12y$ to the ends of its latus rectum.
7. A man running a racecourse notes that the sum of the distances from the two flag posts from him is always 10 m and the distance between the flag posts is 8 m. Find the equation of the posts traced by the man.
8. An equilateral triangle is inscribed in the parabola $y^2 = 4ax$, where one vertex is at the vertex of the parabola. Find the length of the side of the triangle.
9. Prove that the curves $x = y^2$ and $kx = y$ cut at right angles if $8k^2 = 1$
10. Find the equations of the tangent and normal to the parabola $y^2 = 4ax$ at the point $\begin{pmatrix} at^2 \\ 2at \end{pmatrix}$.
11. Find the equations of the tangent and normal to the hyperbola $\mathbf{x}^T \begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & -\frac{1}{b^2} \end{pmatrix} \mathbf{x} = 1$ at the point $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$.
12. Find the area of the smaller part of the circle $\mathbf{x}^{\mathbf{x}} = a^2$ cut off by the line $x = \frac{a}{\sqrt{2}}$.
13. Find the area enclosed between the parabola $y^2 = 4ax$ ad the line $y = mx$.
14. The focus of a parabolic mirror is at a distance of 5 cm from its vertex. If the mirror is 45 cm deep, find the distance AB .
15. A beam is supported at its ends by supports which are 12 metres apart. Since the load is concentrated at its centre, there is a deflection of 3 cm at the centre and the deflected beam is in the shape of a parabola. How far from the centre is the deflection 1 cm?
16. 19 A rod AB of length 15 cm rests in between two coordinate axes in such a way that the end point **A** lies on x-axis and end point **B** lies on y-axis. A point **P** is taken on the rod in such a way that $AP = 6$ cm. Show that the locus of P is an ellipse
17. Find the area of the parabola $y^2 = 4ax$ bounded by its latus rectum.
18. Find the rate of change of the area of a circle per second with respect to its radius when $r =$ 5cm.
19. The volume of a cube is increasing at a rate of 9 cu cm per second. How fast is the surface area increasing when the length of an edge is 10 cm?
20. A stone is dropped into a quiet lake and waves move in circles at a speed of 4cm per second. At the instant, when the radius of the circular wave is 10 cm, how fast is the enclosed area increasing?
21. The length x of a rectangle is decreasing at the rate of 3 cm/minute and the width y is increasing at the rate of 2cm/minute. When x =10cm and y =6cm, find the rates of change of (a) the perimeter and (b) the area of the rectangle.
22. The total cost $C(x)$ in Rupees, associated with the production of x units of an item is given by $C(x) = 0.005x^3 – 0.02x^2 + 30x + 5000$ Find the marginal cost when 3 units are produced, where by marginal cost we mean the instantaneous rate of change of total cost at any level of output.
23. The total revenue in Rupees received from the sale of x units of a product is given by $R(x) = 3x^2 + 36x + 5$. Find the marginal revenue, when $x = 5$, where by marginal revenue we mean the rate of change of total revenue with respect to the number of items sold at an instant.
24. Find the rate of change of the area of a circle with respect to its radius r when (a) $r = 3$ cm (b) $r = 4$ cm
25. The volume of a cube is increasing at the rate of 8 cm^3/s. How fast is the surface area increasing when the length of an edge is 12 cm?
26. The radius of a circle is increasing uniformly at the rate of 3 cm/s. Find the rate at which the

area of the circle is increasing when the radius is 10 cm.

27. An edge of a variable cube is increasing at the rate of 3 cm/s. How fast is the volume of the cube increasing when the edge is 10 cm long?
28. A stone is dropped into a quiet lake and waves move in circles at the speed of 5 cm/s. At the instant when the radius of the circular wave is 8 cm, how fast is the enclosed area increasing?
29. The radius of a circle is increasing at the rate of 0.7 cm/s. What is the rate of increase of its circumference?
30. The length x of a rectangle is decreasing at the rate of 5 cm/minute and the width y is increasing at the rate of 4 cm/minute. When $x = 8$cm and $y = 6$cm, find the rates of change of (a) the perimeter, and (b) the area of the rectangle.
31. A balloon, which always remains spherical on inflation, is being inflated by pumping in 900 cubic centimetres of gas per second. Find the rate at which the radius of the balloon increases when the radius is 15 cm.
32. A balloon, which always remains spherical has a variable radius. Find the rate at which its volume is increasing with the radius when the later is 10 cm.
33. A ladder 5 m long is leaning against a wall. The bottom of the ladder is pulled along the ground, away from the wall, at the rate of 2cm/s. How fast is its height on the wall decreasing when the foot of the ladder is 4 m away from the wall ?
34. A particle moves along the curve $6y = x3 + 2$. Find the points on the curve at which the y-coordinate is changing 8 times as fast as the x-coordinate.
35. The radius of an air bubble is increasing at the rate of 12cm/s. At what rate is the volume of the bubble increasing when the radius is 1 cm?
36. A balloon, which always remains spherical, has a variable diameter $\frac{3}{2}2x + 1$. Find the rate of change of its volume with respect to x.
37. Sand is pouring from a pipe at the rate of 12 cm^3/s. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of the base. How fast is the height of the sand cone increasing when the height is 4 cm?
38. The total cost $C(x)$ in Rupees associated with the production of x units of an item is given by $C(x) = 0.007x^3–0.003x^2 + 15x + 4000$. Find the marginal cost when 17 units are produced.
39. The total revenue in Rupees received from the sale of x units of a product is given by $R(x) = 13x2 + 26x + 15$. Find the marginal revenue when x = 7.
40. Find the rate of change of the area of a circle with respect to its radius r at r = 6 cm.
41. The total revenue in | received from the sale of x units of a product is given by $R(x) = 3x^2 + 36x + 5$. Find the marginal revenue, when $x = 15$.
42. For what vaues of a the function given by $f(x) = x^2 + ax + 1$ is increasing on $[1, 2]$?
43. Let AP and BQ be two vertical poles at points A and B respectively. If $AP = 16$m, $BQ = 22$m, and $AB = 20$m, then find the distance of a point R on AB from the point A such that $RP^2 + RQ^2$ is minimum.
44. If length of three sides of a trapezium other than base are equal to 10cm, then find the area of the trapezium when it is maximum.
45. Prove that the radius of the right circular cylinder of greatest curved surface area which can be inscribed in a given cone is half of that of the cone.
46. Find two positive numbers x and y such that $x + y = 60$ and xy^3 is maximum.
47. Find two positive numbers x and y such that their sum is 35 and the product x^2y^5 is a maximum.
48. A square piece of tin of side 18 cm is to be made into a box without top, by cutting a square from each corner and folding up the flaps to form the box. What should be the side of the square to be cut off so that the volume of the box is the maximum possible.
49. A rectangular sheet of tin 45 cm by 24 cm is to be made into a box without top, by cutting off square from each corner and folding up the flaps. What should be the side of the square to be cut off so that the volume of the box is maximum ?
50. Show that of all the rectangles inscribed in a given fixed circle, the square has the maximum area.
51. Show that the right circular cylinder of given surface and maximum volume is such that its height is equal to the diameter of the base.

52. Of all the closed cylindrical cans (right circular), of a given volume of 100 cubic centimetres, find the dimensions of the can which has the minimum surface area.
53. A wire of length 28 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a circle. What should be the length of the two pieces so that the combined area of the square and the circle is minimum?
54. Prove that the volume of the largest cone that can be inscribed in a sphere of radius R is $\frac{8}{27}$ of the volume of the sphere.
55. Show that the right circular cone of least curved surface and given volume has an altitude equal to $\sqrt{2}$ time the radius of the base.
56. Show that the semi-vertical angle of the cone of the maximum volume and of given slant height is $\tan^{-1}\sqrt{2}$.
57. Show that semi-vertical angle of right circular cone of given surface area and maximum volume is $\sin^{-1}\frac{1}{3}$.
58. Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is $\frac{4r}{3}$.
59. Show that height of the cylinder of greatest volume which can be inscribed in a right circular cone of height h and semi vertical angle α is one-third that of the cone and the greatest volume of cylinder is $\frac{4}{27}\pi h^3 \tan^2\alpha$.
60. A cylindrical tank of radius 10 m is being filled with wheat at the rate of 314 cubic metre per hour. Find the rate at which the depth of the wheat is increasing.
61. Let f be a function defined on $[a, b]$ such that $f'(x) = 0$, for all $x \in (a, b)$. Then prove that f is an increasing function on (a, b).
62. Prove that every rational function is continuous.
63. Prove that the function defined by $f(x) = \tan x$ is a continuous function.
64. A cyclist is riding with a speed of 27 km/h. As he approaches a circular turn on the road of radius 80 m, he applies brakes and reduces his speed at the constant rate of 0.50 m/s every second. What is the magnitude and direction of the net acceleration of the cyclist on the circular turn ?
65. A block of mass m = 1 kg, moving on a horizontal surface with speed $v_i = 2ms^{-1}$ enters a rough patch ranging from x = 0.10 m to x = 2.01 m. The retarding force F_r on the block in this range is inversely proportional to x over this range,

$$F = \begin{cases} 0 & x < 0 \\ -\frac{k}{x}, & 0.1 < x < 2.01 \\ 0 & x > 2.01 \end{cases} \tag{4.0.65.1}$$

where k = 0.5 J. What is the final kinetic energy and speed v_f crosses this patch ?
66. The potential energy function for a particle executing linear simple harmonic motion is given by $V(x) = \frac{kx^2}{2}$, where k is the force constant of the oscillator. For $k = 0.5Nm^{-1}$, the graph of $V(x)$ versus x is shown in Fig. **??**. Show that a particle of total energy 1 J moving under this potential must 'turn back' when it reaches $x = \pm 2$ m.

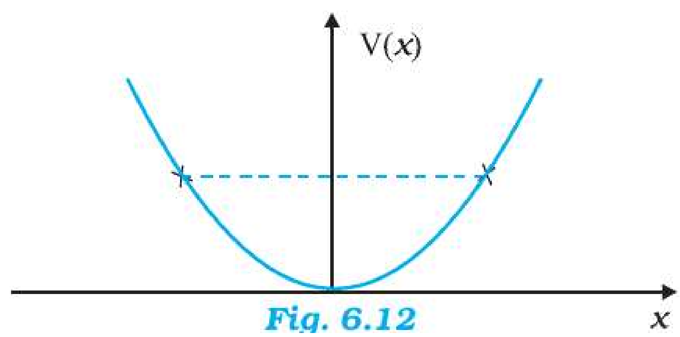

Fig. 4.0.66