Linear Forms

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Contents

***Abstract*—This book provides a computational approach to school geometry based on the NCERT textbooks from Class 6-12. Links to sample Python codes are available in the text.**

Download python codes using

```
svn co https://github.com/gadepall/school/trunk/
    ncert/computation/codes
```

1 Examples

1.1. Check whether –2 and 2 are zeroes of the polynomial $x+2$.
Solution: Let

$$y = x+2 \implies \begin{pmatrix} -1 & 1 \end{pmatrix}\mathbf{x} = 2 \tag{1.1.1}$$

Thus,

$$y = 0 \tag{1.1.2}$$
$$\implies x+2 = 0 \tag{1.1.3}$$
$$\text{or, } x = -2 \tag{1.1.4}$$

Hence -2 is a zero. This is verified in Fig. 1.1.

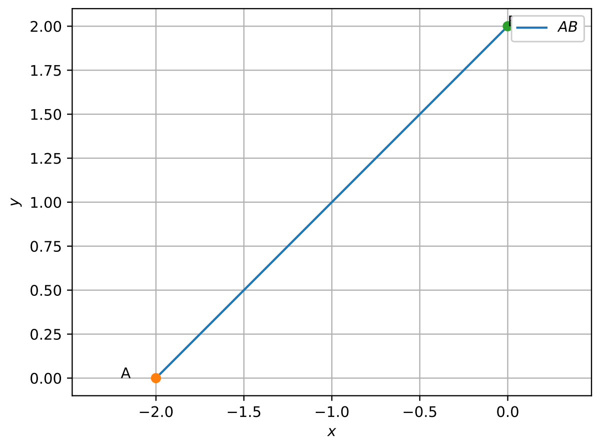

Fig. 1.1

1.2. Find a zero of the polynomial $p(x) = 2x+1$.
Solution: $p\left(-\frac{1}{2}\right) = 0$.

1.3. Find four different solutions of the equation

$$\begin{pmatrix} 1 & 2 \end{pmatrix}\mathbf{x} = 6 \tag{1.3.1}$$

Solution: Let

$$\mathbf{x} = \begin{pmatrix} a \\ 0 \end{pmatrix} \tag{1.3.2}$$

Substituting in (1.3.1),

$$\begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} a \\ 0 \end{pmatrix} = 6 \tag{1.3.3}$$
$$\implies a = 6 \tag{1.3.4}$$

Simiarly, substituting

$$\mathbf{x} = \begin{pmatrix} 0 \\ b \end{pmatrix}, \tag{1.3.5}$$

in (1.3.1),

$$b = 3 \tag{1.3.6}$$

More solutions can be obtained in a similar fashion.

1.4. Draw the graph of

$$\begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} = 7 \tag{1.4.1}$$

Solution: The intercepts on the x and y-axis can be obtained from Problem 1.3 as

$$\mathbf{A} = \begin{pmatrix} 7 \\ 0 \end{pmatrix}\mathbf{B} = \begin{pmatrix} 0 \\ 7 \end{pmatrix} \tag{1.4.2}$$

The following python code can be used to draw the graph in Fig. 1.4.

```
codes/line/line_icept.py
```

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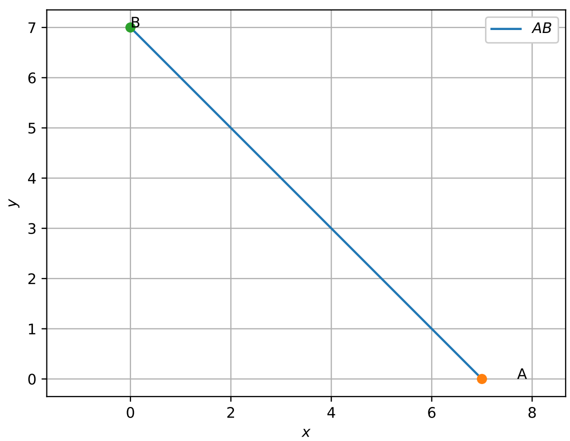

Fig. 1.4

1.5. Two rails are represented by the equations

$$\begin{aligned} \begin{pmatrix} 1 & 2 \end{pmatrix} \mathbf{x} &= 4 \text{ and} \\ \begin{pmatrix} 2 & 4 \end{pmatrix} \mathbf{x} &= 12. \end{aligned} \tag{1.5.1}$$

Will the rails cross each other?
Solution: The above equations can be expressed as the matrix equation

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 4 \\ 12 \end{pmatrix} \tag{1.5.2}$$

The augmented matrix for the above equation is row reduced as follows

$$\begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 12 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow \frac{R_2}{2}} \begin{pmatrix} 1 & 2 & 4 \\ 1 & 2 & 6 \end{pmatrix} \tag{1.5.3}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 2 & 4 \\ 0 & 0 & 2 \end{pmatrix} \tag{1.5.4}$$

$\because$ row reduction of the 2×3 matrix

$$\begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 12 \end{pmatrix} \tag{1.5.5}$$

results in a matrix with 2 nonzero rows, its rank is 2. Similarly, the rank of the matrix

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \tag{1.5.6}$$

is 1, from 1.5.4.

$$\because rank \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \neq rank \begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 12 \end{pmatrix}, \tag{1.5.7}$$

(1.5.1) has no solution. The equivalent python code is

```
codes/line/line_check_sol.py
```

which plots Fig. 1.5, which shows that the rails are parallel.

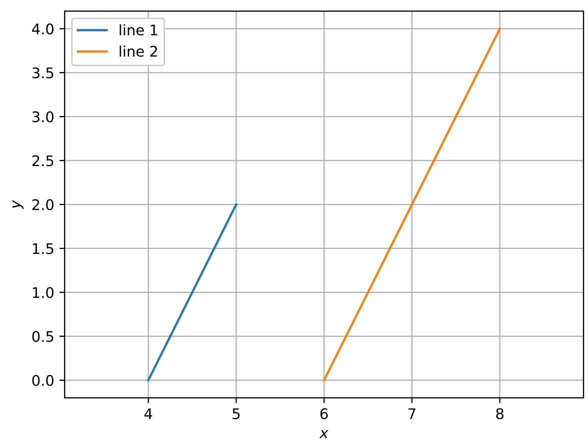

Fig. 1.5

1.6. Check whether the pair of equations

$$\begin{aligned} \begin{pmatrix} 1 & 3 \end{pmatrix} \mathbf{x} &= 6 \text{ and} \\ \begin{pmatrix} 2 & -3 \end{pmatrix} \mathbf{x} &= 12 \end{aligned} \tag{1.6.1}$$

is consistent.
Solution: The above equations can be expressed as the matrix equation

$$\begin{pmatrix} 1 & 3 \\ 2 & -3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 6 \\ 12 \end{pmatrix} \tag{1.6.2}$$

The augmented matrix for the above equation is row reduced as follows

$$\begin{pmatrix} 1 & 3 & 6 \\ 2 & -3 & 12 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow \frac{R_2 - 2R_1}{-9}} \tag{1.6.3}$$

$$\begin{pmatrix} 1 & 3 & 6 \\ 0 & 1 & 0 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 - 3R_2} \begin{pmatrix} 1 & 0 & 6 \\ 0 & 1 & 0 \end{pmatrix} \tag{1.6.4}$$

$$\implies \mathbf{x} = \begin{pmatrix} 6 \\ 0 \end{pmatrix} \tag{1.6.5}$$

which is the solution of 1.5.1. The python code in Problem 1.5 can be used to plot Fig. 1.6, which shows that the lines intersect.

1.7. Find whether the following pair of equations has no solution, unique solution or infinitely

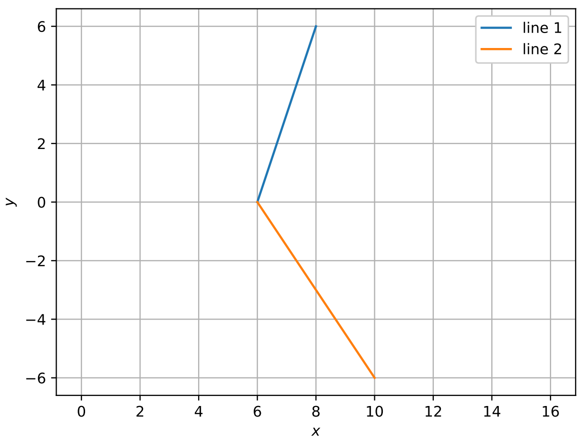

Fig. 1.6

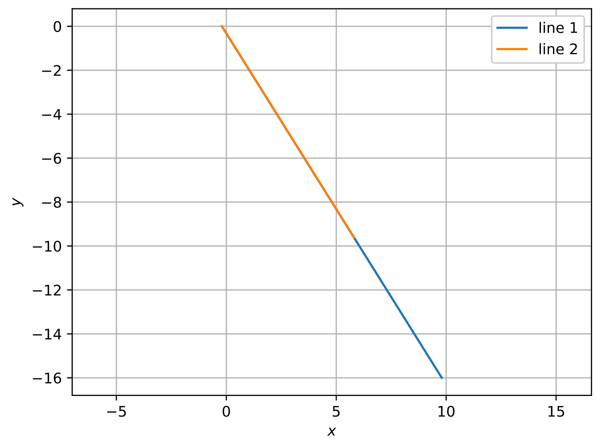

Fig. 1.7

many solutions:

$$\begin{aligned} \begin{pmatrix} 5 & -8 \end{pmatrix} \mathbf{x} = -1 \text{ and} \\ \begin{pmatrix} 3 & -\frac{24}{5} \end{pmatrix} \mathbf{x} = -\frac{3}{5} \end{aligned} \tag{1.7.1}$$

Solution: The above equations can be expressed as the matrix equation

$$\begin{pmatrix} 5 & -8 \\ 3 & -\frac{24}{5} \end{pmatrix} \mathbf{x} = -\begin{pmatrix} 1 \\ \frac{3}{5} \end{pmatrix} \tag{1.7.2}$$

The augmented matrix for the above equation is row reduced as follows

$$\left(\begin{array}{cc|c} 5 & -8 & -1 \\ 3 & -\frac{24}{5} & -\frac{3}{5} \end{array}\right) \xleftrightarrow{R_2 \leftarrow 5R_2} \left(\begin{array}{cc|c} 5 & -8 & 1 \\ 15 & -24 & -3 \end{array}\right) \tag{1.7.3}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - 3R_1} \left(\begin{array}{cc|c} 5 & -8 & 1 \\ 0 & 0 & 0 \end{array}\right) \tag{1.7.4}$$

$$\because rank \begin{pmatrix} 5 & -8 \\ 3 & -\frac{24}{5} \end{pmatrix} = rank \begin{pmatrix} 5 & -8 & 1 \\ 3 & -\frac{24}{5} & -\frac{3}{5} \end{pmatrix} \tag{1.7.5}$$

$$= 1 < dim \begin{pmatrix} 5 & -8 \\ 3 & -\frac{24}{5} \end{pmatrix} = 2, \tag{1.7.6}$$

(1.7.1) has infinitely many solutions. The python code in Problem 1.5 can be used to plot Fig. 1.7, which shows that the lines are the same.

1.8. Solve the following pair of equations

$$\begin{pmatrix} 7 & -15 \end{pmatrix} \mathbf{x} = 2 \tag{1.8.1}$$

$$\begin{pmatrix} 1 & 2 \end{pmatrix} \mathbf{x} = 3 \tag{1.8.2}$$

Solution: The above equations can be expressed as the matrix equation

$$\begin{pmatrix} 7 & -15 \\ 1 & 2 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \tag{1.8.3}$$

The augmented matrix for the above equation is row reduced as follows

$$\left(\begin{array}{cc|c} 7 & -15 & 2 \\ 1 & 2 & 3 \end{array}\right) \xleftrightarrow{R_2 \leftarrow 7R_2 - R_1} \left(\begin{array}{cc|c} 7 & -15 & 2 \\ 0 & 29 & 19 \end{array}\right) \tag{1.8.4}$$

$$\xleftrightarrow{R_1 \leftarrow \frac{15R_2 29 + 29R_1}{29}} \left(\begin{array}{cc|c} 7 & 0 & 2 \\ 0 & 29 & 19 \end{array}\right) \tag{1.8.5}$$

$$\implies \mathbf{x} = \begin{pmatrix} \frac{2}{7} \\ \frac{19}{29} \end{pmatrix} \tag{1.8.6}$$

The python code in Problem 1.5 can be used to plot Fig. 1.8, which shows that the lines are the same.

1.9. Find all possibe solutions of

$$\begin{aligned} \begin{pmatrix} 2 & 3 \end{pmatrix} \mathbf{x} = 8 \\ \begin{pmatrix} 4 & 6 \end{pmatrix} \mathbf{x} = 7 \end{aligned} \tag{1.9.1}$$

Solution: The above equations can be expressed as the matrix equation

$$\begin{pmatrix} 2 & -3 \\ 4 & 6 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 8 \\ 7 \end{pmatrix} \tag{1.9.2}$$

The augmented matrix for the above equation

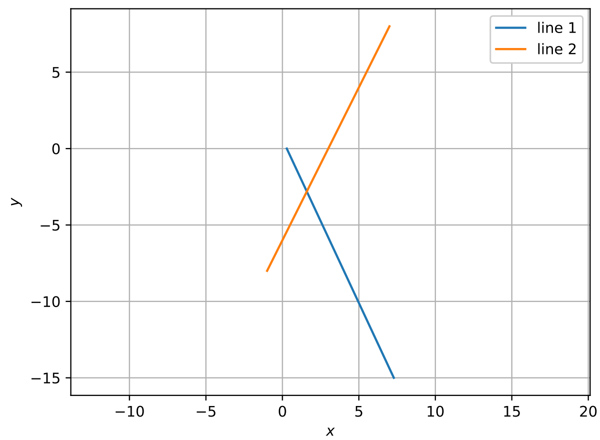

Fig. 1.8

is row reduced as follows

$$\begin{pmatrix} 2 & 3 & 8 \\ 4 & 6 & 7 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{pmatrix} 2 & -3 & 8 \\ 0 & 0 & -9 \end{pmatrix} \tag{1.9.3}$$

$$\implies rank \begin{pmatrix} 2 & -3 \\ 4 & 6 \end{pmatrix} \neq \begin{pmatrix} 2 & 3 & 8 \\ 4 & 6 & 7 \end{pmatrix}. \tag{1.9.4}$$

Hence, (1.9.1) has no solution. The python code in Problem 1.5 can be used to plot Fig. 1.9, which shows that the lines are parallel.

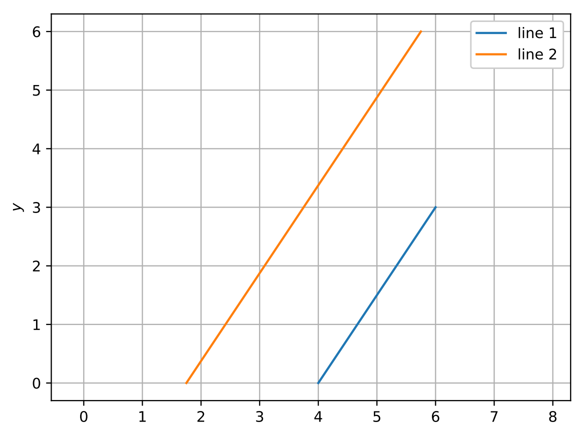

Fig. 1.9

1.10. For which values of p does the pair of equations given below has unique solution?

$$\begin{aligned} \begin{pmatrix} 4 & p \end{pmatrix} \mathbf{x} &= -8 \\ \begin{pmatrix} 2 & 2 \end{pmatrix} \mathbf{x} &= -2 \end{aligned} \tag{1.10.1}$$

Solution: (1.10.1) has a unique solution

$$\iff \begin{vmatrix} 4 & p \\ 2 & 2 \end{vmatrix} \neq 0 \tag{1.10.2}$$

$$\text{or, } p \neq 4 \tag{1.10.3}$$

1.11. For what values of k will the following pair of linear equations have infinitely many solutions?

$$\begin{aligned} \begin{pmatrix} k & 3 \end{pmatrix} \mathbf{x} &= k - 3 \\ \begin{pmatrix} 12 & k \end{pmatrix} \mathbf{x} &= k \end{aligned} \tag{1.11.1}$$

Solution: The first condition for (1.11.1) to have infinite solutions is

$$\begin{vmatrix} k & 3 \\ 12 & k \end{vmatrix} = 0 \tag{1.11.2}$$

$$\implies k^2 = 36, \text{or, } k = \pm 6 \tag{1.11.3}$$

For $k = 6$, the augmented matrix for the above equation is row reduced as follows

$$\begin{pmatrix} 6 & 3 & 3 \\ 12 & 6 & 6 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{pmatrix} 6 & 3 & 3 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.11.4}$$

indicating that (1.11.1) has infinite number of solutions. For $k = -6$, the augmented matrix is

$$\begin{pmatrix} 6 & 3 & -9 \\ 12 & 6 & -6 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{pmatrix} 6 & 3 & -9 \\ 0 & 0 & 12 \end{pmatrix} \tag{1.11.5}$$

indicating that (1.11.1) has no solution Thus, (1.11.2) is a necessary condition but not sufficient.

1.12. Find the condition for $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ to be equidistant from the points $\begin{pmatrix} 7 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \end{pmatrix}$.

Solution: From the given information,

$$\left\| \mathbf{x} - \begin{pmatrix} 7 \\ 1 \end{pmatrix} \right\|^2 = \left\| \mathbf{x} - \begin{pmatrix} 3 \\ 5 \end{pmatrix} \right\|^2 \tag{1.12.1}$$

$$\begin{aligned} \implies \|\mathbf{x}\|^2 + \left\| \begin{pmatrix} 7 \\ 1 \end{pmatrix} \right\|^2 - 2 \begin{pmatrix} 7 & 1 \end{pmatrix} \mathbf{x} \\ = \|\mathbf{x}\|^2 + \left\| \begin{pmatrix} 3 \\ 5 \end{pmatrix} \right\|^2 - 2 \begin{pmatrix} 3 & 5 \end{pmatrix} \mathbf{x} \end{aligned} \tag{1.12.2}$$

which can be simplified to obtain

$$\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 2 \tag{1.12.3}$$

which is the desired condition. The following code plots Fig. 1.12clearly showing that the above equation is the perpendicular bisector of AB.

```
codes/line/line_perp_bisect.py
```

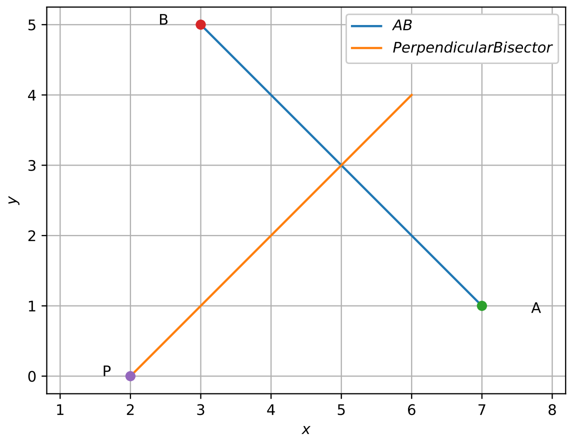

Fig. 1.12

1.13. Find the direction vectors and slopes of the lines passing through the points

a) $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 4 \end{pmatrix}$.

b) $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 7 \\ -2 \end{pmatrix}$.

c) $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

d) Making an inclination of $60°$ with the positive direction of the x-axis.

Solution:

a) If the direction vector is

$$\begin{pmatrix} 1 \\ m \end{pmatrix}, \tag{1.13.1}$$

the slope is m. Thus, the direction vector is

$$\begin{pmatrix} -1 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} -4 \\ 6 \end{pmatrix} = -\frac{1}{4}\begin{pmatrix} -4 \\ 6 \end{pmatrix} \tag{1.13.2}$$

$$= \begin{pmatrix} 1 \\ -\frac{3}{2} \end{pmatrix} \implies m = -\frac{3}{2} \tag{1.13.3}$$

b) The direction vector is

$$\begin{pmatrix} 7 \\ -2 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \tag{1.13.4}$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \implies m = 0 \tag{1.13.5}$$

c) The direction vector is

$$\begin{pmatrix} 3 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \end{pmatrix} \tag{1.13.6}$$

$$= \begin{pmatrix} 1 \\ \infty \end{pmatrix} \implies m = \infty \tag{1.13.7}$$

d) The slope is $m = \tan 60° = \sqrt{3}$ and the direction vector is

$$\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} \tag{1.13.8}$$

1.14. If the angle between two lines is $\frac{\pi}{4}$ and the slope of one of the lines is $\frac{1}{4}$ find the slope of the other line.
Solution: The angle θ between two lines is given by

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2} \tag{1.14.1}$$

$$\implies 1 = \frac{m_1 - \frac{1}{4}}{1 + \frac{m_1}{4}} \tag{1.14.2}$$

$$\text{or } m_1 = \frac{5}{3} \tag{1.14.3}$$

1.15. Two positions of time and distance are recorded as, when $T = 0, D = 2$ and when $T = 3, D = 8$. Using the concept of slope, find law of motion, i.e., how distance depends upon time.
Solution: The equation of the line joining the points $\mathbf{A} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 \\ 8 \end{pmatrix}$ is obtained as

$$\mathbf{x} = \mathbf{A} + \lambda(\mathbf{B} - \mathbf{A}) \tag{1.15.1}$$

$$\implies \begin{pmatrix} T \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} - \lambda\begin{pmatrix} -3 \\ -6 \end{pmatrix} \tag{1.15.2}$$

which can be expressed as

$$\begin{pmatrix} 2 & -1 \end{pmatrix}\begin{pmatrix} T \\ D \end{pmatrix} = \begin{pmatrix} 2 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \end{pmatrix} \tag{1.15.3}$$

$$\implies \begin{pmatrix} 2 & -1 \end{pmatrix}\begin{pmatrix} T \\ D \end{pmatrix} = -2 \tag{1.15.4}$$

$$\implies D = 2 + 2T \tag{1.15.5}$$

1.16. Find the equations of the lines parallel to the axes and passing through $\mathbf{A} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$.
Solution: The line parallel to the x-axis has direction vector $\mathbf{m} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Hence,its equation is

obtined as

$$\mathbf{x} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.16.1}$$

Similarly, the equation of the line parallel to the y-axis can be obtained as

$$\mathbf{x} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} + \lambda_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.16.2}$$

The following code plots Fig. 1.16

```
codes/line/line_parallel_axes.py
```

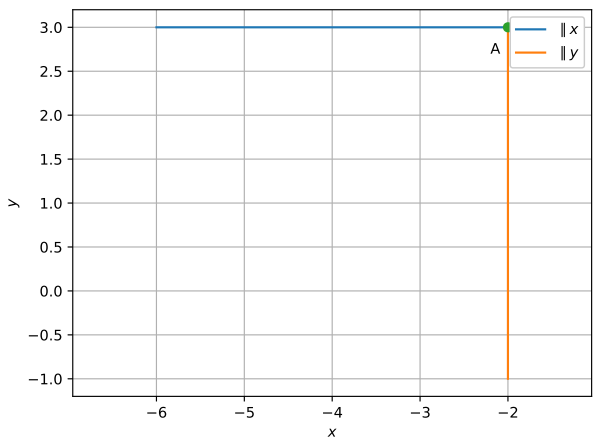

Fig. 1.16

1.17. Find the equation of the line through $\mathbf{A} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$ with slope –4.

Solution: The direction vector is $\mathbf{m} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$. Hence, the normal vector

$$\mathbf{n} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{m} \tag{1.17.1}$$

$$= \begin{pmatrix} 4 \\ 1 \end{pmatrix} \tag{1.17.2}$$

The equation of the line in terms of the normal vector is then obtained as

$$\mathbf{n}^T (\mathbf{x} - \mathbf{A}) = 0 \tag{1.17.3}$$

$$\implies \begin{pmatrix} 4 & 1 \end{pmatrix} \mathbf{x} = -5 \tag{1.17.4}$$

1.18. Write the equation of the line through the points $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$.

Solution: Use (1.16.1).

1.19. Write the equation of the lines for which $\tan\theta = \frac{1}{2}$, where θ is the inclination of the line and

a) y-intercept is $-\frac{3}{2}$
b) x-intercept is 4.

Solution: From the given information, $\tan\theta = \frac{1}{2} = m$.

a) y-intercept is $-\frac{3}{2} \implies$ the line cuts through the y-axis at $\begin{pmatrix} 0 \\ -\frac{3}{2} \end{pmatrix}$.
b) x-intercept is $4 \implies$ the line cuts through the x-axis at $\begin{pmatrix} 4 \\ 0 \end{pmatrix}$.

Use the above information get the equations for the lines.

1.20. Find the equation of a line through the point $\begin{pmatrix} 5 \\ 2 \\ -4 \end{pmatrix}$ and parallel to the vector $\begin{pmatrix} 3 \\ 2 \\ -8 \end{pmatrix}$.

Solution: The equation of the line is

$$\mathbf{x} = \begin{pmatrix} 5 & 2 \\ -4 & \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \\ -8 \end{pmatrix} \tag{1.20.1}$$

1.21. Find the equation of a line passing through the points $\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix}$.

Solution: Using (1.15.1), the desired equation of the line is

$$\mathbf{x} = \begin{pmatrix} -1 & 0 \\ 2 & \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix} \tag{1.21.1}$$

$$= \begin{pmatrix} -1 & 0 \\ 2 & \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{1.21.2}$$

1.22. If

$$\frac{x+3}{2} = \frac{y-5}{4} = \frac{z+6}{2} = \lambda \tag{1.22.1}$$

find the equation of the line.

Solution: The line can be expressed from

(1.22.1) as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 + 2\lambda \\ 5 + 4\lambda \\ -6 + 2\lambda \end{pmatrix} \tag{1.22.2}$$

$$\implies \mathbf{x} = \begin{pmatrix} -3 \\ 5 \\ -6 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix} \tag{1.22.3}$$

$$\implies \mathbf{x} = \begin{pmatrix} -3 \\ 5 \\ -6 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \tag{1.22.4}$$

1.23. Find the equation of the line, which makes intercepts -3 and 2 on the x and y axes respectively.
Solution:

1.24. Find the equation of the line whose perpendicular distance from the origin is 4 units and the angle which the normal makes with the positive direction of x-axis is 15°.
Solution: In Fig. 1.24, the foot of the perpendicular P is the intersection of the lines L and M. Thus,

$$\mathbf{n}^T\mathbf{P} = c \tag{1.24.1}$$

$$\mathbf{P} = \mathbf{A} + \lambda\mathbf{n} \tag{1.24.2}$$

$$\text{or, } \mathbf{n}^T\mathbf{P} = \mathbf{n}^T\mathbf{A} + \lambda \left\|\mathbf{n}\right\|^2 = c \tag{1.24.3}$$

$$\implies -\lambda = \frac{\mathbf{n}^T\mathbf{A} - c}{\left\|\mathbf{n}\right\|^2} \tag{1.24.4}$$

Also, the distance between **A** and L is obtained from

$$\mathbf{P} = \mathbf{A} + \lambda\mathbf{n} \tag{1.24.5}$$

$$\implies \left\|\mathbf{P} - \mathbf{A}\right\| = \left|\lambda\right| \left\|\mathbf{n}\right\| \tag{1.24.6}$$

From (1.24.4) and (1.24.6)

$$\left\|\mathbf{P} - \mathbf{A}\right\| = \frac{\left|\mathbf{n}^T\mathbf{A} - c\right|}{\left\|\mathbf{n}\right\|} \tag{1.24.7}$$

$$\mathbf{n} = \begin{pmatrix} 1 \\ \tan 15^\circ \end{pmatrix} \tag{1.24.8}$$

$\because \mathbf{A} = \mathbf{0}$,

$$4 = \frac{|c|}{\left\|\mathbf{n}\right\|} \implies c = \pm 4\sqrt{1 + \tan^2 15^\circ} \tag{1.24.9}$$

$$= \pm 4 \sec 15^\circ \tag{1.24.10}$$

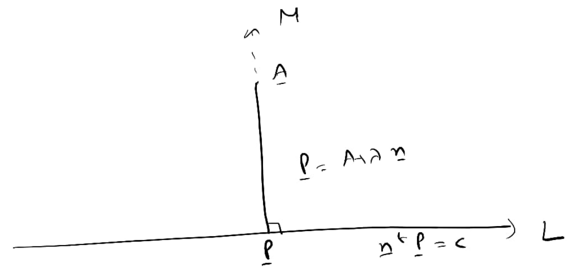

Fig. 1.24

where

$$\sec\theta = \frac{1}{\cos\theta} \tag{1.24.11}$$

This follows from the fact that

$$\cos^2\theta + \sin^2\theta = 1 \tag{1.24.12}$$

$$\implies 1 + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \tag{1.24.13}$$

It is easy to verify that

$$\frac{\sin\theta}{\cos\theta} = \tan\theta \tag{1.24.14}$$

$$\implies 1 + \tan^2\theta = \sec^2\theta \tag{1.24.15}$$

Thus, the equation of the line is

$$\begin{pmatrix} 1 & \tan 15^\circ \end{pmatrix} \mathbf{c} = \pm 4 \sec 15^\circ \tag{1.24.16}$$

1.25. The Farenheit temperature F and absolute temperature K satisfy a linear equation. Given $K = 273$ when $F = 32$ and that $K = 373$ when $F = 212$, express K in terms of F and find the value of F, when $K = 0$.
Solution: Let

$$\mathbf{x} = \begin{pmatrix} F & K \end{pmatrix} \tag{1.25.1}$$

Since the relation between F, K is linear, $\begin{pmatrix} 273 \\ 32 \end{pmatrix}$, $\begin{pmatrix} 373 \\ 21 \end{pmatrix}$ are on a line. The corresponding equation is obtained from (1.17.3) and (1.17.1) as

$$\begin{pmatrix} 11 & -100 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 11 & -100 \end{pmatrix} \begin{pmatrix} 273 \\ 32 \end{pmatrix} \tag{1.25.2}$$

$$\implies \begin{pmatrix} 11 & -100 \end{pmatrix} \mathbf{x} = -197 \tag{1.25.3}$$

If $\begin{pmatrix} F \\ 0 \end{pmatrix}$ is a point on the line,

$$\begin{pmatrix} 11 & -100 \end{pmatrix} \begin{pmatrix} F \\ 0 \end{pmatrix} = -197 \implies F = -\frac{197}{11} \tag{1.25.4}$$

1.26. Equation of a line is

$$\begin{pmatrix} 3 & -4 \end{pmatrix} \mathbf{x} + 10 = 0. \tag{1.26.1}$$

Find its

a) slope,

b) x - and y-intercepts.

Solution: From the given information,

$$\mathbf{n} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}, \tag{1.26.2}$$

$$\mathbf{m} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}, \tag{1.26.3}$$

a) $m = \frac{3}{4}$

b) x-intercept is $-\frac{10}{3}$ and y-intercept is $\frac{10}{4} = \frac{5}{2}$.

1.27. Find the angle between two vectors **a** and **b** where

$$\|\mathbf{a}\| = 1, \|\mathbf{b}\| = 2, \mathbf{a}^T \mathbf{b} = 1. \tag{1.27.1}$$

Solution: In Fig. 1.27, from the cosine formula,

$$\cos\theta = \frac{\|\mathbf{A}-\mathbf{B}\|^2 + \|\mathbf{B}-\mathbf{C}\|^2 - \|\mathbf{A}-\mathbf{C}\|^2}{2\,\|\mathbf{A}-\mathbf{B}\|\,\|\mathbf{B}-\mathbf{C}\|} \tag{1.27.2}$$

Letting $\mathbf{a} = \mathbf{A}-\mathbf{B}, \mathbf{b} = \mathbf{B}-\mathbf{C}$,

$$\cos\theta = \frac{\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - \|\mathbf{a}+\mathbf{b}\|^2}{2\,\|\mathbf{a}\|\,\|\mathbf{b}\|} \tag{1.27.3}$$

$$= \frac{\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - \left[\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\mathbf{a}^T\mathbf{b}\right]}{2\,\|\mathbf{a}\|\,\|\mathbf{b}\|} \tag{1.27.4}$$

$$\implies \cos\theta = \frac{\mathbf{a}^T\mathbf{b}}{\|\mathbf{a}\|\,\|\mathbf{b}\|} \tag{1.27.5}$$

Thus, the angle θ between two vectors is given by

$$\cos\theta = \frac{\mathbf{a}^T\mathbf{b}}{\|\mathbf{a}\|\,\|\mathbf{b}\|} \tag{1.27.6}$$

$$= \frac{1}{2} \tag{1.27.7}$$

$$\implies \theta = 60^\circ \tag{1.27.8}$$

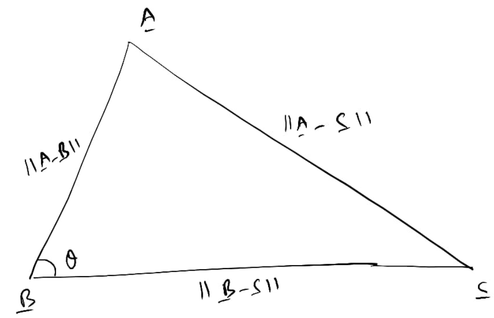

Fig. 1.27

1.28. Find the angle between the lines

$$\begin{pmatrix} 1 & -\sqrt{3} \end{pmatrix} \mathbf{x} = 5 \tag{1.28.1}$$

$$\begin{pmatrix} \sqrt{3} & -1 \end{pmatrix} \mathbf{x} = -6. \tag{1.28.2}$$

Solution: The angle between the lines can also be expressed in terms of the normal vectors as

$$\cos\theta = \frac{\mathbf{n}_1\mathbf{n}_2}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|} \tag{1.28.3}$$

$$= \frac{\sqrt{3}}{2} \implies \theta = 30^\circ \tag{1.28.4}$$

1.29. Find the equation of a line perpendicular to the line

$$\begin{pmatrix} 1 & -2 \end{pmatrix} \mathbf{x} = 3 \tag{1.29.1}$$

and passes through the point $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

Solution: The normal vector of the perpendicular line is

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{1.29.2}$$

Thus, the desired equation of the line is

$$\begin{pmatrix} 2 & 1 \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right) = 0 \tag{1.29.3}$$

$$\implies \begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{1.29.4}$$

1.30. Find the distance of the point $\begin{pmatrix} 3 \\ -5 \end{pmatrix}$ from the line

$$\begin{pmatrix} 3 & -4 \end{pmatrix} \mathbf{x} = 26 \tag{1.30.1}$$

Solution: Use (1.24.7).

1.31. If the lines

$$\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} = 3 \tag{1.31.1}$$

$$\begin{pmatrix}5 & k\end{pmatrix}\mathbf{x} = 3 \tag{1.31.2}$$

$$\begin{pmatrix}3 & -1\end{pmatrix}\mathbf{x} = 2 \tag{1.31.3}$$

are concurrent, find the value of k.

Solution: If the lines are concurrent, the *augmented* matrix should have a 0 row upon row reduction. Hence,

$$\begin{pmatrix}2 & 1 & 3\\5 & k & 3\\3 & -1 & 2\end{pmatrix} \xleftrightarrow{R_2\leftrightarrow R_3} \begin{pmatrix}2 & 1 & 3\\3 & -1 & 2\\5 & k & 3\end{pmatrix} \tag{1.31.4}$$

$$\xleftrightarrow[R_3\leftarrow 2R_3-5R_1]{R_2\leftrightarrow 2R_2-3R_1} \begin{pmatrix}2 & 1 & 3\\0 & -5 & -5\\0 & 2k-5 & -9\end{pmatrix} \tag{1.31.5}$$

$$\xleftrightarrow{R_2\leftarrow -\frac{R_2}{5}} \begin{pmatrix}2 & 1 & 3\\0 & 1 & 1\\0 & 2k-5 & -9\end{pmatrix} \tag{1.31.6}$$

$$\xleftrightarrow{R_3\leftarrow R_3-(2k-5)R_2} \begin{pmatrix}2 & 1 & 3\\0 & 1 & 1\\0 & 0 & -2k-4\end{pmatrix} \tag{1.31.7}$$

$$\implies k = -2 \tag{1.31.8}$$

1.32. Find the distance of the line

$$L_1 : \quad \begin{pmatrix}4 & 1\end{pmatrix}\mathbf{x} = 0 \tag{1.32.1}$$

from the point $\begin{pmatrix}4\\1\end{pmatrix}$ measured along the line L_2 making an angle of 135° with the positive x-axis.

Solution: Let P be the point of intersection of L_1 and L_2. The direction vector of L_2 is

$$\mathbf{m} = \begin{pmatrix}1\\\tan 135°\end{pmatrix} \tag{1.32.2}$$

Since $\begin{pmatrix}4\\1\end{pmatrix}$ lies on L_2, the equation of L_2 is

$$\mathbf{x} = \begin{pmatrix}4\\1\end{pmatrix} + \lambda\mathbf{m} \tag{1.32.3}$$

$$\implies \mathbf{P} = \begin{pmatrix}4\\1\end{pmatrix} + \lambda\mathbf{m} \tag{1.32.4}$$

$$\text{or,} \quad \left\|\mathbf{P} - \begin{pmatrix}4\\1\end{pmatrix}\right\| = d = |\lambda| \, \|\mathbf{m}\| \tag{1.32.5}$$

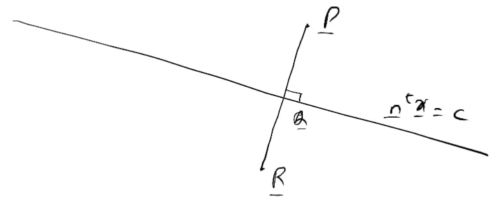

Fig. 1.33

Since $\mathbf{P}$ lies on L_1, from (1.32.1),

$$\begin{pmatrix}4 & 1\end{pmatrix}\mathbf{P} = 0 \tag{1.32.6}$$

Substituting from the above in (1.32.3),

$$\begin{pmatrix}4 & 1\end{pmatrix}\begin{pmatrix}4\\1\end{pmatrix} + \lambda\begin{pmatrix}4 & 1\end{pmatrix}\mathbf{m} = 0 \tag{1.32.7}$$

$$\implies \lambda = \frac{\begin{pmatrix}4 & 1\end{pmatrix}\mathbf{m}}{17} \tag{1.32.8}$$

substituting $|\lambda|$ in (1.32.5) gives the desired answer.

1.33. Assuming that straight lines work as a plane mirror for a point, find the image of the point $\begin{pmatrix}1\\2\end{pmatrix}$ in the line

$$\begin{pmatrix}1 & -3\end{pmatrix}\mathbf{x} = -4. \tag{1.33.1}$$

Solution: Since $\mathbf{R}$ is the reflection of $\mathbf{P}$ and $\mathbf{Q}$ lies on L, $\mathbf{Q}$ bisects PR. This leads to the following equations

$$2\mathbf{Q} = \mathbf{P} + \mathbf{R} \tag{1.33.2}$$

$$\mathbf{n}^T\mathbf{Q} = c \tag{1.33.3}$$

$$\mathbf{m}^T\mathbf{R} = \mathbf{m}^T\mathbf{P} \tag{1.33.4}$$

where $\mathbf{m}$ is the direction vector of L. From (1.33.2) and (1.33.3),

$$\mathbf{n}^T\mathbf{R} = 2c - \mathbf{n}^T\mathbf{P} \tag{1.33.5}$$

From (1.33.5) and (1.33.4),

$$\begin{pmatrix}\mathbf{m} & \mathbf{n}\end{pmatrix}^T\mathbf{R} = \begin{pmatrix}\mathbf{m} & -\mathbf{n}\end{pmatrix}^T\mathbf{P} + \begin{pmatrix}0\\2c\end{pmatrix} \tag{1.33.6}$$

Letting

$$\mathbf{V} = \begin{pmatrix}\mathbf{m} & \mathbf{n}\end{pmatrix} \tag{1.33.7}$$

with the condition that $\mathbf{m}, \mathbf{n}$ are orthonormal, i.e.

$$\mathbf{V}^T\mathbf{V} = \mathbf{I} \tag{1.33.8}$$

Noting that

$$\begin{pmatrix}\mathbf{m} & -\mathbf{n}\end{pmatrix} = \begin{pmatrix}\mathbf{m} & \mathbf{n}\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}, \tag{1.33.9}$$

(1.33.6) can be expressed as

$$\mathbf{V}^T\mathbf{R} = \left[\mathbf{V}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\right]^T\mathbf{P} + \begin{pmatrix}0\\2c\end{pmatrix} \tag{1.33.10}$$

$$\implies \mathbf{R} = \left[\mathbf{V}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\mathbf{V}^{-1}\right]^T\mathbf{P} + \mathbf{V}\begin{pmatrix}0\\2c\end{pmatrix} \tag{1.33.11}$$

$$= \mathbf{V}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\mathbf{V}^T\mathbf{P} + 2c\mathbf{n} \tag{1.33.12}$$

It can be verified that the reflection is also given by

$$\frac{\mathbf{R}}{2} = \frac{\mathbf{m}\mathbf{m}^T - \mathbf{n}\mathbf{n}^T}{\mathbf{m}^T\mathbf{m} + \mathbf{n}^T\mathbf{n}}\mathbf{P} + c\frac{\mathbf{n}}{\|\mathbf{n}\|^2} \tag{1.33.13}$$

The following code plots Fig. 1.33 while computing the reflection

```
codes/line/line_reflect.py
```

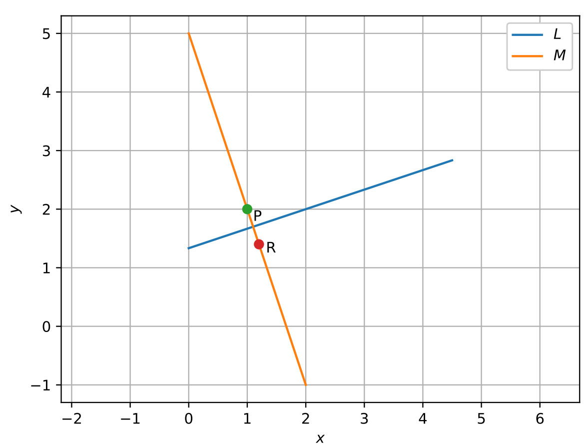

Fig. 1.33

1.34. A line L is such that its segment between the lines is bisected at the point $\mathbf{P} = \begin{pmatrix}1\\5\end{pmatrix}$. Obtain its equation.

$$L_1: \quad \begin{pmatrix}5 & -1\end{pmatrix}\mathbf{x} = -4 \tag{1.34.1}$$

$$L_2: \quad \begin{pmatrix}3 & 4\end{pmatrix}\mathbf{x} = 4 \tag{1.34.2}$$

Solution: Let

$$L: \quad \mathbf{x} = \mathbf{P} + \lambda\mathbf{m} \tag{1.34.3}$$

If L intersects L_1 and L_2 at $\mathbf{A}$ and $\mathbf{B}$ respectively,

$$\mathbf{A} = \mathbf{P} + \lambda\mathbf{m} \tag{1.34.4}$$

$$\mathbf{B} = \mathbf{P} - \lambda\mathbf{m} \tag{1.34.5}$$

since $\mathbf{P}$ bisects AB. Note that λ is a measure of the distance from P along the line L. From (1.34.1), (1.34.4) and (1.34.5),

$$\begin{pmatrix}5 & -1\end{pmatrix}\mathbf{A} = \begin{pmatrix}5 & -1\end{pmatrix}\begin{pmatrix}1\\5\end{pmatrix} + \lambda\begin{pmatrix}5 & -1\end{pmatrix}\mathbf{m} = -4 \tag{1.34.6}$$

$$\begin{pmatrix}3 & 4\end{pmatrix}\mathbf{B} = \begin{pmatrix}3 & 4\end{pmatrix}\begin{pmatrix}1\\5\end{pmatrix} - \lambda\begin{pmatrix}3 & 4\end{pmatrix}\mathbf{m} = 4 \tag{1.34.7}$$

yielding

$$19\begin{pmatrix}5 & -1\end{pmatrix}\mathbf{m} = -4\begin{pmatrix}3 & 4\end{pmatrix}\mathbf{m} \tag{1.34.8}$$

$$\implies \begin{pmatrix}107 & -3\end{pmatrix}\mathbf{m} = 0 \tag{1.34.9}$$

$$\text{or, } \mathbf{n} = \begin{pmatrix}107\\-3\end{pmatrix} \tag{1.34.10}$$

after simplification. Thus, the equation of the line is

$$\mathbf{n}^T(\mathbf{x} - \mathbf{P}) = 0 \tag{1.34.11}$$

1.35. Show that the path of a moving point such that its distances from two lines

$$\begin{pmatrix}3 & -2\end{pmatrix}\mathbf{x} = 5 \tag{1.35.1}$$

$$\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x} = 5 \tag{1.35.2}$$

are equal is a straight line.

Solution: Using (1.24.7) the point $\mathbf{x}$ satisfies

$$\frac{\left|\begin{pmatrix}3 & -2\end{pmatrix}\mathbf{x}-5\right|}{\left\|\begin{pmatrix}3\\-2\end{pmatrix}\right\|} = \frac{\left|\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x}-5\right|}{\left\|\begin{pmatrix}3\\2\end{pmatrix}\right\|} \tag{1.35.3}$$

$$\implies \left|\begin{pmatrix}3 & -2\end{pmatrix}\mathbf{x}-5\right| = \left|\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x}-5\right| \tag{1.35.4}$$

resulting in

$$\begin{pmatrix}3 & -2\end{pmatrix}\mathbf{x}-5 = \pm\left(\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x}-5\right) \tag{1.35.5}$$

leading to the possible lines

$$L_1 : \quad \begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} = 0 \tag{1.35.6}$$

$$L_2 : \quad \begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = \frac{5}{3} \tag{1.35.7}$$

1.36. Find the angle between the vectors $\mathbf{a} = \begin{pmatrix}1\\1\\-1\end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix}1\\-1\\1\end{pmatrix}$.

Solution: The angle between 2 vectors is given by

$$\cos\theta = \frac{\mathbf{a}^T\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|}$$

Computing the numerator

$$\mathbf{a}^T\mathbf{b} = \begin{pmatrix}1 & 1 & -1\end{pmatrix}\begin{pmatrix}1\\-1\\1\end{pmatrix}$$

$$\implies \mathbf{a}^T\mathbf{b} = -1 \tag{1.36.1}$$

Computing the denominator

$$\|\mathbf{a}\|\|\mathbf{b}\| = \sqrt{(1)^2+(1)^2+(-1)^2} \tag{1.36.2}$$

$$\times\sqrt{(1)^2+(-1)^2+(1)^2} \tag{1.36.3}$$

$$\implies \|\mathbf{a}\|\|\mathbf{b}\| = (\sqrt{3})^2 \tag{1.36.4}$$

$$\implies \|\mathbf{a}\|\|\mathbf{b}\| = 3 \tag{1.36.5}$$

So, we get $\cos\theta$ to be,

$$\cos\theta = \frac{-1}{3} \tag{1.36.6}$$

$$\tag{1.36.7}$$

Therefore,

$$\theta = \cos^{-1}\left(\frac{-1}{3}\right)$$

$$\implies \theta = 1.9106^c$$

$$\implies \theta = 109.47°$$

Therefore, the angle between the 2 vectors is **109.47**°.

1.37. Find the angle between the pair of lines given by

$$\mathbf{x} = \begin{pmatrix}3\\2\\-4\end{pmatrix} + \lambda_1\begin{pmatrix}1\\2\\2\end{pmatrix} \tag{1.37.1}$$

$$\mathbf{x} = \begin{pmatrix}5\\-2\\0\end{pmatrix} + \lambda_2\begin{pmatrix}3\\2\\6\end{pmatrix} \tag{1.37.2}$$

Solution: Looking at the directions of the lines,

$$\mathbf{a} = \begin{pmatrix}1\\2\\2\end{pmatrix} \tag{1.37.3}$$

$$\mathbf{b} = \begin{pmatrix}3\\2\\6\end{pmatrix} \tag{1.37.4}$$

Clearly over here,

$$\|\mathbf{a}\| = \sqrt{(1)^2 + (2)^2 + (2)^2} = \sqrt{9} = 3 \tag{1.37.5}$$

(1.37.6)

$$\|\mathbf{b}\| = \sqrt{(3)^2 + (2)^2 + (6)^2} = \sqrt{49} = 7 \tag{1.37.7}$$

(1.37.8)

$$\mathbf{a}^T\mathbf{b} = (1)(3) + (2)(2) + (2)(6) = 19 \tag{1.37.9}$$

(1.37.10)

$$\implies \cos\theta = \frac{\mathbf{a}^T\mathbf{b}}{\|\mathbf{a}\|\,\|\mathbf{b}\|} = \frac{19}{(3)(7)} = \frac{19}{21} \tag{1.37.11}$$

(1.37.12)

$$\implies \theta = \arccos\left(\frac{19}{21}\right) \tag{1.37.13}$$

(1.37.14)

$$\implies \theta \approx 25.22^\circ \tag{1.37.15}$$

1.38. Find the angle between the pair of lines

$$\frac{x+3}{3} = \frac{y-1}{5} = \frac{z+3}{4}, \tag{1.38.1}$$

$$\frac{x+1}{1} = \frac{y-4}{1} = \frac{z-5}{2} \tag{1.38.2}$$

Solution: Using the definition of a line in coordinate geometry, we see from the above two equations, the direction vectors **a** and **b** of the two lines are

$$\mathbf{a} = \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix} \tag{1.38.3}$$

$$\mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \tag{1.38.4}$$

respectively. In order to find the angle between the two direction vectors, we use the definition of dot product,

$$\cos\theta = \frac{\mathbf{a}^T\mathbf{b}}{\|\mathbf{a}\|\,\|\mathbf{b}\|} \tag{1.38.5}$$

Which gives us,

$$\mathbf{a}^T\mathbf{b} = 16 \tag{1.38.6}$$

$$\|\mathbf{a}\| = \sqrt{50} \tag{1.38.7}$$

$$\|\mathbf{b}\| = \sqrt{6} \tag{1.38.8}$$

Which gives us

$$\cos\theta = \frac{8}{5\sqrt{3}} \tag{1.38.9}$$

$$\implies \theta = \arccos\frac{8}{5\sqrt{3}} \tag{1.38.10}$$

$$\implies \theta = 22.517^\circ \tag{1.38.11}$$

1.39. Find a unit vector that makes an angle of $90°, 60°$ and $30°$ with the positive x, y and z axis respectively.
Solution: The direction vector is

$$\mathbf{x} = \begin{pmatrix} \cos 90^\circ \\ \cos 60^\circ \\ \cos 30^\circ \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} \tag{1.39.1}$$

$\because \|\mathbf{x}\| = 1$, it is the desired unit vector.

1.40. Find the distance between the lines

$$L_1 : \quad \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} \tag{1.40.1}$$

$$L_2 : \quad \mathbf{x} = \begin{pmatrix} 3 \\ 3 \\ -5 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} \tag{1.40.2}$$

Solution: Both the lines have the same direction vector, so the lines are parallel. The following code plots

```
codes/line/line_dist_parallel.py
```

Fig. 1.40 From Fig. 1.40, the distance is

$$\|\mathbf{A_2} - \mathbf{A_1}\| \sin\theta = \frac{\|\mathbf{m} \times (\mathbf{A}_2 - \mathbf{A}_1)\|}{\|\mathbf{m}\|} \tag{1.40.3}$$

where

$$\mathbf{A}_1 = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}, \mathbf{A}_2 = \begin{pmatrix} 3 \\ 3 \\ -5 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} \tag{1.40.4}$$

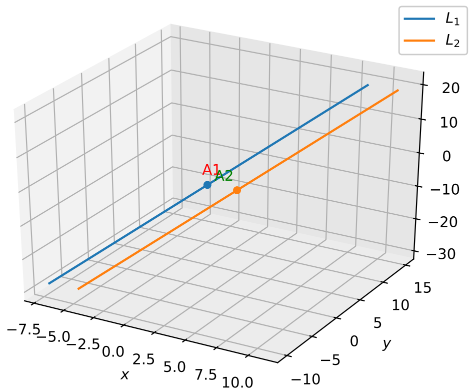

Fig. 1.40

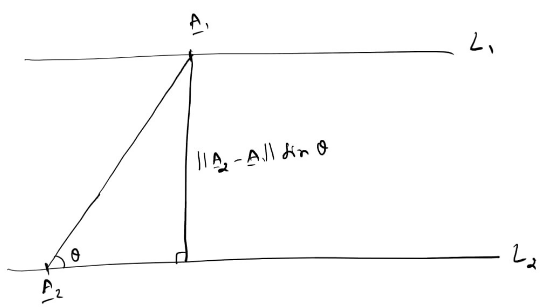

Fig. 1.40

1.41. Find the shortest distance between the lines

$$L_1: \quad \mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \tag{1.41.1}$$

$$L_2: \quad \mathbf{x} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} \tag{1.41.2}$$

Solution: In the given problem

$$\mathbf{A}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{m}_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \mathbf{A}_2 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \mathbf{m}_2 = \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix}. \tag{1.41.3}$$

The lines will intersect if

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} \tag{1.41.4}$$

$$\implies \lambda_1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} - \lambda_2 \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \tag{1.41.5}$$

$$\implies \begin{pmatrix} 2 & 3 \\ -1 & -5 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \tag{1.41.6}$$

Row reducing the augmented matrix,

$$\begin{pmatrix} 2 & 3 & 1 \\ -1 & -5 & 0 \\ 1 & 2 & -1 \end{pmatrix} \xleftrightarrow{R_3 \leftrightarrow R_1} \begin{pmatrix} 1 & 2 & -1 \\ -1 & -5 & 0 \\ 2 & 3 & 1 \end{pmatrix} \tag{1.41.7}$$

$$\xleftrightarrow[R_3 = 2R_1 - R_3]{R_2 = R_1 + R_2} \begin{pmatrix} 1 & 2 & -1 \\ 0 & -3 & -1 \\ 0 & 1 & -3 \end{pmatrix} \xleftrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -3 \\ 0 & -3 & -1 \end{pmatrix} \tag{1.41.8}$$

$$\xleftrightarrow{R_3 = 3R_2 + R_3} \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & -10 \end{pmatrix} \tag{1.41.9}$$

The above matrix has *rank* = 3. Hence, the lines do not intersect. Note that the lines are not parallel but they lie on parallel planes. Such lines are known as *skew* lines. The following code plots Fig. 1.41

```
codes/line/line_dist_skew.py
```

The normal to both the lines (and corresponding planes) is

$$\mathbf{n} = \mathbf{m}_1 \times \mathbf{m}_2 \tag{1.41.10}$$

The equation of the second plane is then obtained as

$$\mathbf{n}^T \mathbf{x} = \mathbf{n}^T \mathbf{A}_2 \tag{1.41.11}$$

The distance from $\mathbf{A}_1$ to the above line is then obtained using (1.24.7) as

$$\frac{\left|\mathbf{n}^T (\mathbf{A}_2 - \mathbf{A}_1)\right|}{\|\mathbf{n}\|} = \frac{\left|(\mathbf{A}_2 - \mathbf{A}_1)^T (\mathbf{m}_1 \times \mathbf{m}_2)\right|}{\|\mathbf{m}_1 \times \mathbf{m}_2\|} \tag{1.41.12}$$

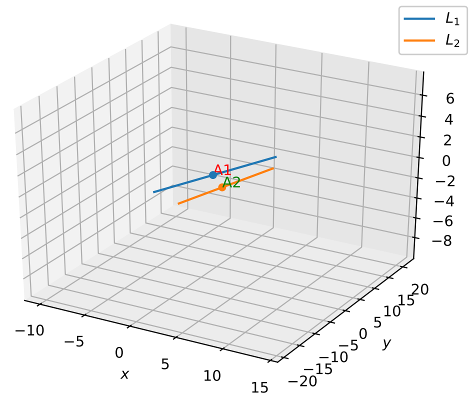

Fig. 1.41

1.42. Find the distance of the plane

$$\begin{pmatrix} 2 & -3 & 4 \end{pmatrix} \mathbf{x} - 6 = 0 \tag{1.42.1}$$

from the origin.
Solution: From (1.24.7), the distance is obtained as

$$\frac{|c|}{\|\mathbf{n}\|} = \frac{6}{\sqrt{2^2 + 3^2 + 4^2}} \tag{1.42.2}$$

$$= \frac{6}{\sqrt{29}} \tag{1.42.3}$$

1.43. Find the equation of a plane which is at a distance of $\frac{6}{\sqrt{29}}$ from the origin and has normal vector $\mathbf{n} = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$.
Solution: From the previous problem, the desired equation is

$$\begin{pmatrix} 2 & -3 & 4 \end{pmatrix} \mathbf{x} - 6 = 0 \tag{1.43.1}$$

1.44. Find the unit normal vector of the plane

$$\begin{pmatrix} 6 & -3 & -2 \end{pmatrix} \mathbf{x} = 1. \tag{1.44.1}$$

Solution: The normal vector is

$$\mathbf{n} = \begin{pmatrix} 6 & -3 & -2 \end{pmatrix} \tag{1.44.2}$$

$$\because \|\mathbf{n}\| = 7, \tag{1.44.3}$$

the unit normal vector is

$$\frac{\mathbf{n}}{\|\mathbf{n}\|} = \frac{1}{7}\begin{pmatrix} 6 & -3 & -2 \end{pmatrix} \tag{1.44.4}$$

1.45. Find the coordinates of the foot of the perpendicular drawn from the origin to the plane

$$\begin{pmatrix} 2 & -3 & 4 \end{pmatrix} \mathbf{x} - 6 = 0 \tag{1.45.1}$$

Solution: The normal vector is

$$\mathbf{n} = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} \tag{1.45.2}$$

Hence, the foot of the perpendicular from the origin is $\lambda \mathbf{n}$. Substituting in (1.45.1),

$$\lambda \|\mathbf{n}\|^2 = 6 \implies \lambda = \frac{6}{\|\mathbf{n}\|^2} = \frac{6}{29} \tag{1.45.3}$$

Thus, the foot of the perpendicular is

$$\frac{6}{29}\begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} \tag{1.45.4}$$

1.46. Find the equation of the plane which passes through the point $\mathbf{A} = \begin{pmatrix} 5 \\ 2 \\ -4 \end{pmatrix}$ and perpendicular to the line with direction vector $\mathbf{n} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$.
Solution: The normal vector to the plane is $\mathbf{n}$. Hence from (1.17.3), the equation of the plane is

$$\mathbf{n}^T (\mathbf{x} - \mathbf{A}) = 0 \tag{1.46.1}$$

$$\implies \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 2 & 3 & -1 \end{pmatrix} \begin{pmatrix} 5 \\ 2 \\ -4 \end{pmatrix} \tag{1.46.2}$$

$$= 20 \tag{1.46.3}$$

1.47. Find the equation of the plane passing through $\boldsymbol{R} = \begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix}, \boldsymbol{S} = \begin{pmatrix} -2 \\ -3 \\ 5 \end{pmatrix}$ and $\boldsymbol{T} = \begin{pmatrix} 5 \\ 3 \\ -3 \end{pmatrix}$.
Solution: If the equation of the plane be

$$\mathbf{n}^T \mathbf{x} = c, \tag{1.47.1}$$

$$\mathbf{n}^T \mathbf{R} = \mathbf{n}^T \mathbf{S} = \mathbf{n}^T \mathbf{T} = c, \tag{1.47.2}$$

$$\implies \begin{pmatrix} \mathbf{R} - \mathbf{S} & \mathbf{S} - \mathbf{T} \end{pmatrix}^T \mathbf{n} = 0 \tag{1.47.3}$$

after some algebra. Using row reduction on the above matrix,

$$\begin{pmatrix} 4 & 8 & -8 \\ -7 & -6 & 8 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{R_1}{4}} \begin{pmatrix} 1 & 2 & -2 \\ -7 & -6 & 8 \end{pmatrix} \tag{1.47.4}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 + 7R_1} \begin{pmatrix} 1 & 2 & -2 \\ 0 & 8 & -6 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow \frac{R_2}{2}} \begin{pmatrix} 1 & 2 & -2 \\ 0 & 4 & -3 \end{pmatrix} \tag{1.47.5}$$

$$\xleftrightarrow{R_1 \leftarrow 2R_1 - R_2} \begin{pmatrix} 2 & 0 & -1 \\ 0 & 4 & -3 \end{pmatrix} \tag{1.47.6}$$

Thus,

$$\mathbf{n} = \begin{pmatrix} \frac{1}{2} \\ \frac{3}{4} \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \text{ and} \tag{1.47.7}$$

$$c = \mathbf{n}^T \mathbf{T} = 7 \tag{1.47.8}$$

Thus, the equation of the plane is

$$\begin{pmatrix} 2 & 3 & 4 \end{pmatrix} \mathbf{n} = 7 \tag{1.47.9}$$

Alternatively, the normal vector to the plane can be obtained as

$$\mathbf{n} = (\mathbf{R} - \mathbf{S}) \times (\mathbf{S} - \mathbf{T}) \tag{1.47.10}$$

The equation of the plane is then obtained from (1.17.3) as

$$\mathbf{n}^T (\mathbf{x} - \mathbf{T}) = [(\mathbf{R} - \mathbf{S}) \times (\mathbf{S} - \mathbf{T})]^T (\mathbf{x} - \mathbf{T}) = 0 \tag{1.47.11}$$

1.48. Find the equation of the plane with intercepts 2, 3 and 4 on the x, y and z axis respectively.
Solution: From the given information, the plane passes through the points $\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$ respectively. The equation can be obtained using Problem 1.47.

1.49. Find the equation of the plane passing through the intersection of the planes

$$\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \mathbf{x} = 6 \tag{1.49.1}$$

$$\begin{pmatrix} 2 & 3 & 4 \end{pmatrix} \mathbf{x} = -5 \tag{1.49.2}$$

and the point $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Solution: The intersection of the planes is obtained by row reducing the augmented matrix as

$$\begin{pmatrix} 1 & 1 & 1 & 6 \\ 2 & 3 & 4 & -5 \end{pmatrix} \xleftrightarrow{R_2 = R_2 - 2R_1} \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & -17 \end{pmatrix} \tag{1.49.3}$$

$$\xleftrightarrow{R_1 = R_1 - R_2} \begin{pmatrix} 1 & 0 & -1 & 23 \\ 0 & 1 & 2 & -17 \end{pmatrix} \tag{1.49.4}$$

$$\implies \mathbf{x} = \begin{pmatrix} 23 \\ -17 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \tag{1.49.5}$$

Thus, $\begin{pmatrix} 23 \\ -17 \\ 0 \end{pmatrix}$ is another point on the plane. The normal vector to the plane is then obtained as The normal vector to the plane is then obtained as

$$\left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 23 \\ -17 \\ 0 \end{pmatrix} \right) \times \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \tag{1.49.6}$$

which can be obtained by row reducing the matrix

$$\begin{pmatrix} 1 & -2 & 1 \\ -22 & 18 & 1 \end{pmatrix} \xleftrightarrow{R_2 = R_2 + 22R_1} \begin{pmatrix} 1 & -2 & 1 \\ 0 & -26 & 23 \end{pmatrix} \tag{1.49.7}$$

$$\xleftrightarrow{R_1 = 13R_1 - R_2} \begin{pmatrix} 13 & 0 & -10 \\ 0 & -26 & 23 \end{pmatrix} \tag{1.49.8}$$

$$\implies \mathbf{n} = \begin{pmatrix} \frac{10}{13} \\ \frac{23}{26} \\ 1 \end{pmatrix} = \begin{pmatrix} 20 \\ 23 \\ 26 \end{pmatrix} \tag{1.49.9}$$

Since the plane passes through $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, using (1.17.3),

$$\begin{pmatrix} 20 & 23 & 26 \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right) = 0 \tag{1.49.10}$$

$$\implies \begin{pmatrix} 20 & 23 & 26 \end{pmatrix} \mathbf{x} = 69 \tag{1.49.11}$$

Alternatively, the plane passing through the intersection of (1.49.1) and (1.49.2) has the form

$$\begin{pmatrix}1 & 1 & 1\end{pmatrix}\mathbf{x} + \lambda\begin{pmatrix}2 & 3 & 4\end{pmatrix}\mathbf{x} = 6 - 5\lambda \tag{1.49.12}$$

Substituting $\begin{pmatrix}1\\1\\1\end{pmatrix}$ in the above,

$$\begin{pmatrix}1 & 1 & 1\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix} + \lambda\begin{pmatrix}2 & 3 & 4\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix} = 6 - 5\lambda \tag{1.49.13}$$

$$\implies 3 + 9\lambda = 6 - 5\lambda \tag{1.49.14}$$

$$\implies \lambda = \frac{3}{14} \tag{1.49.15}$$

Substituting this value of λ in (1.49.12) yields the equation of the plane.

1.50. Show that the lines

$$\frac{x+3}{-3} = \frac{y-1}{1} = \frac{z-5}{5}, \tag{1.50.1}$$

$$\frac{x+1}{-1} = \frac{y-2}{2} = \frac{z-5}{5} \tag{1.50.2}$$

are coplanar.

Solution: Since the given lines have different direction vectors, they are not parallel. From Problem (1.41), the lines are coplanar if the distance between them is 0, i.e. they intersect. This is possible if

$$(\mathbf{A}_2 - \mathbf{A}_1)^T (\mathbf{m}_1 \times \mathbf{m}_2) = 0 \tag{1.50.3}$$

From the given information,

$$\mathbf{A}_2 - \mathbf{A}_1 = \begin{pmatrix}-3\\1\\5\end{pmatrix} - \begin{pmatrix}-1\\2\\5\end{pmatrix} = \begin{pmatrix}-2\\-1\\0\end{pmatrix} \tag{1.50.4}$$

$\mathbf{m}_1 \times \mathbf{m}_2$ is obtained by row reducing the matrix

$$\begin{pmatrix}-1 & 2 & 5\\-3 & 1 & 5\end{pmatrix} \xleftrightarrow{R_2 = \frac{R_2 - 3R_1}{5}} \begin{pmatrix}-1 & 2 & 5\\0 & 1 & 2\end{pmatrix} \tag{1.50.5}$$

$$\xleftrightarrow{R_1 = -R_1 + 2R_2} \begin{pmatrix}1 & 0 & -1\\0 & 1 & 2\end{pmatrix} \implies \begin{pmatrix}-1\\2\\5\end{pmatrix} \times \begin{pmatrix}-3\\1\\5\end{pmatrix} = \begin{pmatrix}1\\-2\\1\end{pmatrix} \tag{1.50.6}$$

The LHS of (1.50.3) is

$$\begin{pmatrix}-2 & -1 & 0\end{pmatrix}\begin{pmatrix}1\\-2\\1\end{pmatrix} = 0 \tag{1.50.7}$$

which completes the proof. Alternatively, the lines are coplanar if

$$\left|\mathbf{A}_1 - \mathbf{A}_2 \quad \mathbf{m}_1 \quad \mathbf{m}_2\right| = 0 \tag{1.50.8}$$

1.51. Find the angle between the two planes

$$\begin{pmatrix}2 & 1 & -2\end{pmatrix}\mathbf{x} = 5 \tag{1.51.1}$$

$$\begin{pmatrix}3 & -6 & -2\end{pmatrix}\mathbf{x} = 7. \tag{1.51.2}$$

Solution: The angle between two planes is the same as the angle between their normal vectors. This can be obtained from (1.27.6).

1.52. Find the angle between the two planes

$$\begin{pmatrix}2 & 2 & -2\end{pmatrix}\mathbf{x} = 5 \tag{1.52.1}$$

$$\begin{pmatrix}3 & -6 & 2\end{pmatrix}\mathbf{x} = 7. \tag{1.52.2}$$

Solution: See Problem (1.51).

1.53. Find the distance of a point $\begin{pmatrix}2\\5\\-3\end{pmatrix}$ from the plane

$$\begin{pmatrix}6 & -3 & 2\end{pmatrix}\mathbf{x} = 4 \tag{1.53.1}$$

Solution: Use (1.24.7).

1.54. Find the angle between the line

$$L: \quad \frac{x+1}{2} = \frac{y}{3} = \frac{z-3}{6} \tag{1.54.1}$$

and the plane

$$P: \quad \begin{pmatrix}10 & 2 & -11\end{pmatrix}\mathbf{x} = 3 \tag{1.54.2}$$

Solution: The angle between the direction vector of L and normal vector of P is

$$\cos\theta = \frac{\left|\begin{pmatrix}10 & 2 & -11\end{pmatrix}\begin{pmatrix}2\\3\\6\end{pmatrix}\right|}{\sqrt{225} \times \sqrt{49}} = \frac{8}{21} \tag{1.54.3}$$

Thus, the desired angle is $90° - \theta$.

1.55. Find the equation of the plane that contains the point $\begin{pmatrix}1\\-1\\2\end{pmatrix}$ and is perpedicular to each of the planes

$$\begin{pmatrix}2 & 3 & -2\end{pmatrix}\mathbf{x} = 5 \tag{1.55.1}$$

$$\begin{pmatrix}1 & 2 & -3\end{pmatrix}\mathbf{x} = 8 \tag{1.55.2}$$

Solution: The normal vector to the desired plane is $\perp$ the normal vectors of both the given planes. Thus,

$$\mathbf{n} = \begin{pmatrix}2\\3\\-2\end{pmatrix} \times \begin{pmatrix}1\\2\\-3\end{pmatrix} \tag{1.55.3}$$

The equation of the plane is then obtained as

$$\mathbf{n}^T(\mathbf{x} - \mathbf{A}) = 0 \tag{1.55.4}$$

1.56. Find the distance between the point $\mathbf{P} = \begin{pmatrix}6\\5\\9\end{pmatrix}$ and the plane determined by the points $\boldsymbol{A} = \begin{pmatrix}3\\-1\\2\end{pmatrix}, \boldsymbol{B} = \begin{pmatrix}5\\2\\4\end{pmatrix}$ and $\boldsymbol{C} = \begin{pmatrix}-1\\-1\\6\end{pmatrix}$.
Solution: Find the equation of the plane using Problem 1.47. Find the distance using (1.24.7).

1.57. Find the coordinates of the point where the line through the points $\mathbf{A} = \begin{pmatrix}3\\4\\1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}5\\1\\6\end{pmatrix}$ crosses the XY plane.
Solution: The equation of the line is

$$\mathbf{x} = \mathbf{A} + \lambda(\mathbf{B} - \mathbf{A}) \tag{1.57.1}$$

$$= \begin{pmatrix}3\\4\\1\end{pmatrix} + \lambda\begin{pmatrix}2\\-3\\5\end{pmatrix} \tag{1.57.2}$$

The line crosses the XY plane for $x_3 = 0 \implies \lambda = -\frac{1}{5}$. Thus, the desired point is

$$\begin{pmatrix}3\\4\\1\end{pmatrix} - \frac{1}{5}\begin{pmatrix}2\\-3\\5\end{pmatrix} = \frac{1}{5}\begin{pmatrix}13\\23\\0\end{pmatrix} \tag{1.57.3}$$

1.58. Show that the function given by $f(x) = 7x–3$ is increasing on $\mathbf{R}$.
Solution: A function is said to be increasing if

$$x_2 > x_1 \implies f(x_2) > f(x_1) \tag{1.58.1}$$

$$\implies \frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0 \tag{1.58.2}$$

Letting $x_1 = x, x_2 = x+h$ in (1.58.2), this results in

$$\frac{f(x+h) - f(x)}{h} > 0 \tag{1.58.3}$$

In the given problem,

$$\frac{f(x+h) - f(x)}{h} = 7 > 0 \tag{1.58.4}$$

Hence, the given function is increasing.

1.59. A funcion $f(x)$ is increasing if

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} > 0 \tag{1.59.1}$$

$f'(x)$ is defined as the *derivative* of $f(x)$. The function is decreasing if $f'(x) < 0$.

1.60. The funcion $f(x) = ax + b, a \neq 0$ is increasing if $f'(x) = a > 0$. Else, it is decreasing.

1.61. Find the maximum and minimum values, if any, of the function given by

$$f(x) = x, x \in (0, 1)\,. \tag{1.61.1}$$

Solution: It is easy to verify that $f'(x) = 1 > 0$ in the given interval. Hence, the function is increasing. The maximum value in the given interval is 1 and the minimum value is 0.

1.62. Find all points of local maxima and local minima of the function f given by

$$f(x) = 3 + |x|, \quad x \in \mathbf{R} \tag{1.62.1}$$

Solution: (1.62.1) can be expressed as

$$f(x) = \begin{cases} 3 + x & x > 0 \\ 0 & x = 0 \\ 3 - x & < 0 \end{cases} \tag{1.62.2}$$

From Theorem 1.60,

$$f'(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases} \tag{1.62.3}$$

Thus, $f(x)$ is increasing in $(0, \infty)$ and decreasing in $(-\infty, 0)$. It is obvious that the minimum value of $f(x) = 0$.

1.63. Sketch the graph of $y = |x + 3|$ and evaluate its

area for $-6 \le x \le 0$.
Solution: Fig. shows

$$y_1 = |x+3|, -6 \le x \le 0 \quad (1.63.1)$$
$$y_2 = |x|, -3 \le x \le 3 \quad (1.63.2)$$
$$y_3 = x, 0 < x < 3 \quad (1.63.3)$$
$$\implies ar(y_1) = ar(y_2) = 2ar(y_3) \quad (1.63.4)$$

From Fig. ,

$$ar(y_3) = h(h + 2h + 3h + \cdots + nh), \quad nh = 3 \quad (1.63.5)$$
$$= h^2(1 + 2 + 3 + \ldots n) = h^2 \sum_{k=1}^{n} k \quad (1.63.6)$$

Let

$$S_n = 1 + 2 + 3 + \ldots n \quad (1.63.7)$$
$$\implies S_n = n + n - 1 + n - 2 + \ldots 1 \quad (1.63.8)$$
$$\implies 2S_n = (n+1) + (n+1) + \cdots + (n+1) \quad n \text{ times} \quad (1.63.9)$$

$$\implies 2S_n = n(n+1) \quad (1.63.10)$$
$$\text{or, } S_n = \frac{n(n+1)}{2} \quad (1.63.11)$$

Substituting (1.63.11) in (1.63.6),

$$ar(y_3) = \frac{nh(nh+h)}{2} \quad (1.63.12)$$
$$= \frac{3(3+h)}{2} \quad (1.63.13)$$
$$\text{or, } ar(y_3) = \lim_{h \to 0} ar(y_3) = \frac{9}{2} \quad (1.63.14)$$

This result agrees with the area of a triangle calculated using the base and altitude. Thus, from (1.63.14) and (1.63.1),

$$ar(y_1) = 2ar(y_3) = 9 \quad (1.63.15)$$

1.64. Check the continuity of the function f given by $f(x) = 2x + 3$ at $x = 1$.
Solution: See Fig.

$$\because f(1+h) = 2(1+h) + 3 = 5 + h \quad (1.64.1)$$
$$f(1) = 5 \quad (1.64.2)$$
$$f(1-h) = 2(1+h) + 3 = 5 - h \quad (1.64.3)$$
$$\lim_{h \to 0} f(1+h) = f(1) = f(1-h) = 5 \quad (1.64.4)$$

Hence, the function is continuous.

1.65. A function $f(x)$ is defined to be *continuous* at $x = a$ if

$$\lim_{h \to 0} f(a+h) = f(a) = f(a-h) \quad (1.65.1)$$

It is possible to draw a continuious function $f(x)$ without lifting a pencil.

1.66. Discuss the continuity of the function f given by $f(x) = |x|$ at $x = 0$.
Solution: See Fig. . It appears to be continuous. To prove this, we note that

$$\lim_{h \to 0} f(0+h) = f(0) = f(0-h) = 0 \quad (1.66.1)$$

1.67. Check the points where the constant function $f(x) = k$ is continuous.
Solution: $f(x)$ is continuous everywhere.

1.68. Find all the points of discontinuity of the function f defined by

$$f(x) = \begin{cases} x+2 & x < 1 \\ 0 & x = 1 \\ x-2 & x > 1 \end{cases} \quad (1.68.1)$$

Solution: From Fig. , the discontinuity appears to be at $x = 1$ and verified by the fact that

$$\because f(1-h) = 3 \neq f(1) \neq f(1+h), \quad (1.68.2)$$

1.69. Discuss the continuity of the function f defined by

$$f(x) = \begin{cases} x+2 & x < 0 \\ -x+2 & x > 0 \end{cases} \quad (1.69.1)$$

The function is not defined at $x = 0$, so it is discontinuous at that point. At all other points it is continuious.

1.70. Show that the function f defined by

$$f(x) = |1 - x + |x||, \quad (1.70.1)$$

where x is any real number, is a continuous function.
Solution: The sum of continuous functions is

continuous.

1.71. If

$$y = f(x), \frac{dy}{dx} = f'(x) \tag{1.71.1}$$

1.72. Find $\frac{dy}{dx}$ if $x - y = \pi$.
Solution:

$$\because y = f(x) = x - \pi, \tag{1.72.1}$$

from Theorem 1.60,

$$\frac{dy}{dx} = 1. \tag{1.72.2}$$

1.73. Find the derivative at $x = 2$ of the function $f(x) = 3$.
Solution: The derivative is 0.

1.74. Find the derivative of $f(x) = 3$ at $x = 0$ and $x = 3$.
Solution: $\frac{dy}{dx} = 0$.

1.75. Find the derivative of $f(x) = 10x$.
Solution: $\frac{dy}{dx} = 10$.

1.76. Find the derivative of $f(x) = a$ for a fixed real number a.
Solution: $\frac{dy}{dx} = 0$.

1.77. Form the *differential equation* representing the family of curves $y = mx$, where, m is an arbitrary constant. **Solution:** The desired equation is

$$\frac{dy}{dx} = m \tag{1.77.1}$$

1.78. Verify whether the following are zeroes of the polynomial, indicated against them.

a) $p(x) = 3x + 1, x = \frac{1}{3}$
b) $p(x) = 5x - \pi, x = \frac{4}{5}$
c) $p(x) = 5lx + m, x = -\frac{m}{l}$
d) $p(x) = 2x + 1, x = \frac{1}{2}$

Solution:

1. Let

$$y = 3x + 1 \tag{1.78.1}$$

$$\implies \begin{pmatrix} 3 & -1 \end{pmatrix} \mathbf{x} = -1 \tag{1.78.2}$$

For $x = \frac{1}{3}$ to be a zero,

$$\mathbf{x} = \begin{pmatrix} \frac{1}{3} \\ 0 \end{pmatrix} \tag{1.78.3}$$

should satisfy (1.78.2).

$$\because \begin{pmatrix} 3 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{3} \\ 0 \end{pmatrix} = 1 \neq -1, \tag{1.78.4}$$

$\mathbf{x} = \frac{1}{3}$ is not a zero. This is verified in Fig. 1.78.1.

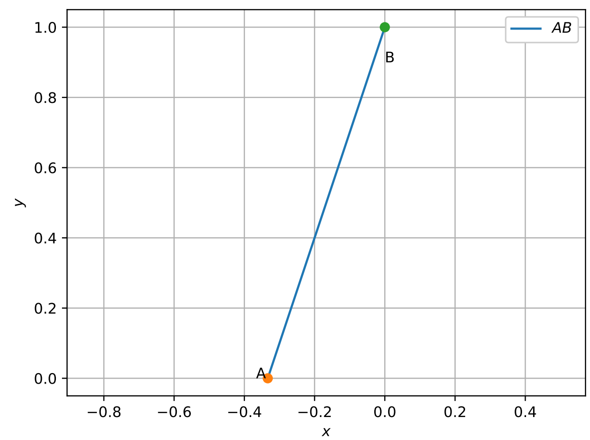

Fig. 1.78.1

2. Let

$$y = 5x - \pi \tag{1.78.5}$$

$$\implies \begin{pmatrix} 5 & -1 \end{pmatrix} \mathbf{x} = \pi \tag{1.78.6}$$

$$\begin{pmatrix} 5 & -1 \end{pmatrix} \begin{pmatrix} \frac{4}{5} \\ 0 \end{pmatrix} = 4 \neq \pi \tag{1.78.7}$$

Hence $\mathbf{x} = \frac{4}{5}$ is not a zero. This is verified in Fig. 1.78.2.

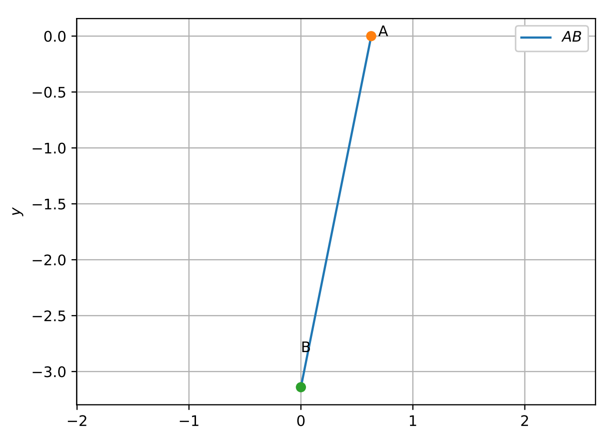

Fig. 1.78.2

3. Let

$$y = 5lx + m \tag{1.78.8}$$

$$\implies \begin{pmatrix} 5l & -1 \end{pmatrix} \mathbf{x} = -m \tag{1.78.9}$$

Thus,

$$\begin{pmatrix} 5l & -1 \end{pmatrix} \begin{pmatrix} -\frac{m}{l} \\ 0 \end{pmatrix} = -5m \neq -m \tag{1.78.10}$$

Hence $\mathbf{x} = -\frac{m}{l}$ is not a zero. This is verified in Fig. 1.78.3.

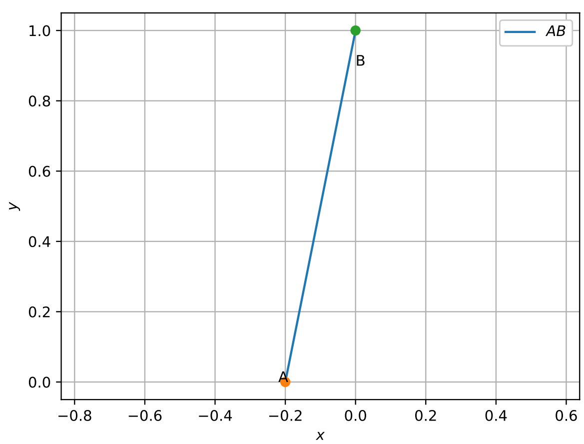

Fig. 1.78.3

4. Let

$$y = 2x + 1 \tag{1.78.11}$$

$$\implies \begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} = -1 \tag{1.78.12}$$

Thus,

$$\begin{pmatrix} 2 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} = 2 \neq -1 \tag{1.78.13}$$

Hence $\mathbf{x} = \frac{1}{2}$ is not a zero. This is verified in Fig. 1.78.4.

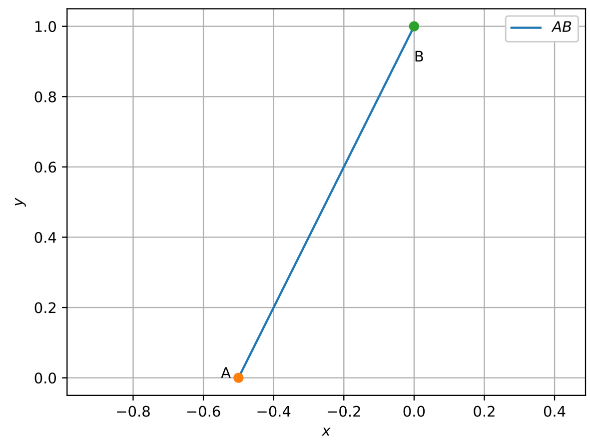

Fig. 1.78.4

1.79. Find the zero of the polynomial in each of the following cases:

a) $p(x) = x + 5$
b) $p(x) = x – 5$
c) $p(x) = 2x + 5$
d) $p(x) = 3x – 2$
e) $p(x) = 3x$
f) $p(x) = ax, a \neq 0$
g) $p(x) = cx + d, c \neq 0, c, d$ are real numbers.

Solution:

a) For $p(x) = x + 5$

The given equation can be represented as follows in the vector form:

$$\begin{pmatrix} 5 & -1 \end{pmatrix} \mathbf{x} + 5 = 0 \tag{1.79.1}$$

To find the roots $y = 0$:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \tag{1.79.2}$$

$$x_1 + 5 = 0 \tag{1.79.3}$$

$$x_1 = -5 \tag{1.79.4}$$

b) For $p(x) = x - 5$

The given equation can be represented as follows in the vector form:

$$\begin{pmatrix} 5 & -1 \end{pmatrix} \mathbf{x} - 5 = 0 \tag{1.79.5}$$

To find the roots $y = 0$:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \tag{1.79.6}$$

$$x_1 - 5 = 0 \tag{1.79.7}$$

$$x_1 = 5 \tag{1.79.8}$$

c) For $p(x) = 2x + 5$

The given equation can be represented as follows in the vector form:

$$\begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} + 5 = 0 \tag{1.79.9}$$

To find the roots $y = 0$:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \tag{1.79.10}$$

$$2x_1 + 5 = 0 \tag{1.79.11}$$

$$x_1 = \frac{-5}{2} \tag{1.79.12}$$

d) For $p(x) = 3x - 2$
The given equation can be represented as follows in the vector form:

$$\begin{pmatrix} 3 & -1 \end{pmatrix} \mathbf{x} - 2 = 0 \tag{1.79.13}$$

To find the roots $y = 0$:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \tag{1.79.14}$$

$$3x_1 - 2 = 0 \tag{1.79.15}$$

$$x_1 = \frac{2}{3} \tag{1.79.16}$$

e) For $p(x) = 3x$
The given equation can be represented as follows in the vector form:

$$\begin{pmatrix} 3 & -1 \end{pmatrix} \mathbf{x} = 0 \tag{1.79.17}$$

To find the roots $y = 0$:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \tag{1.79.18}$$

$$3x_1 = 0 \tag{1.79.19}$$

$$x_1 = 0 \tag{1.79.20}$$

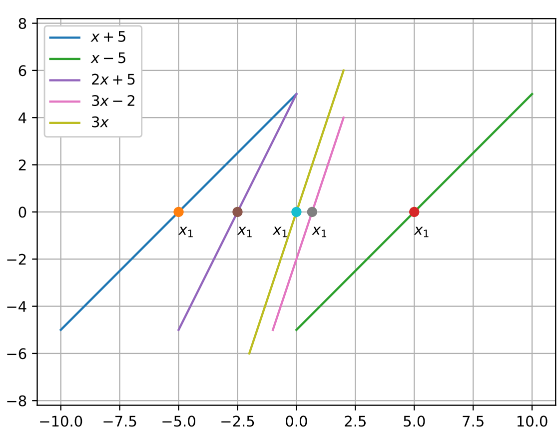

Fig. 1.79.1

The following Python code generates Fig 1.79.1

```
solutions/2/codes/line_ex/
    lines_and_planes/linear_eq_roots.py
```

1.80. Find two solutions for each of the following equations:

a) $\begin{pmatrix} 4 & 3 \end{pmatrix} \mathbf{x} = 12$

b) $\begin{pmatrix} 2 & 5 \end{pmatrix} \mathbf{x} = 0$

c) $\begin{pmatrix} 0 & 3 \end{pmatrix} \mathbf{x} = 4$

Solution:

1.1. A point $\mathbf{c}$ lying on the line

$$\begin{pmatrix} a & b \end{pmatrix} \mathbf{x} = d \tag{1.80.1}$$

at a distance λ from point $\mathbf{x}$ lying on the same line is given as

$$\mathbf{c} = \mathbf{x} + \frac{\lambda}{\sqrt{a^2 + b^2}} \begin{pmatrix} b \\ -a \end{pmatrix} \tag{1.80.2}$$

$$\lambda = \sqrt{a^2 + b^2} \implies \mathbf{c} = \mathbf{x} + \begin{pmatrix} b \\ -a \end{pmatrix} \tag{1.80.3}$$

Equation of y axis is

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 0 \tag{1.80.4}$$

$$(a) \begin{pmatrix} 4 & 3 \end{pmatrix} \mathbf{x} = 12 \tag{1.80.5}$$

The line meets y-axis at point $\mathbf{y_1}$ given using 1.80.4 as,

$$\begin{pmatrix} 4 & 3 \\ 1 & 0 \end{pmatrix} \mathbf{y_1} = \begin{pmatrix} 12 \\ 0 \end{pmatrix} \tag{1.80.6}$$

$$\mathbf{y_1} = \begin{pmatrix} 4 & 3 \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 12 \\ 0 \end{pmatrix} \tag{1.80.7}$$

$$\mathbf{y_1} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} \tag{1.80.8}$$

Another point $\mathbf{c_1}$ on the line is found using equation 1.80.3

$$\mathbf{c_1} = \mathbf{y_1} + \begin{pmatrix} 3 \\ -4 \end{pmatrix} \tag{1.80.9}$$

$$\implies \mathbf{c_1} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \tag{1.80.10}$$

$$(b)\begin{pmatrix}2 & 5\end{pmatrix}\mathbf{x} = 0 \tag{1.80.11}$$

The line meets y-axis at point $\mathbf{y_2}$ given using 1.80.4 as,

$$\begin{pmatrix}2 & 5\\1 & 0\end{pmatrix}\mathbf{y_1} = \begin{pmatrix}0\\0\end{pmatrix} \tag{1.80.12}$$

$$\mathbf{y_1} = \begin{pmatrix}2 & 5\\1 & 0\end{pmatrix}^{-1}\begin{pmatrix}0\\0\end{pmatrix} \tag{1.80.13}$$

$$\mathbf{y_1} = \begin{pmatrix}0\\0\end{pmatrix} \tag{1.80.14}$$

Another point $\mathbf{c_2}$ on the line is found using equation 1.80.3

$$\mathbf{c_2} = \mathbf{y_2} + \begin{pmatrix}5\\-2\end{pmatrix} \tag{1.80.15}$$

$$\implies \mathbf{c_2} = \begin{pmatrix}5\\-2\end{pmatrix} \tag{1.80.16}$$

$$(c)\begin{pmatrix}0 & 3\end{pmatrix}\mathbf{x} = 4 \tag{1.80.17}$$

The line meets y-axis at point $\mathbf{y_2}$ given using 1.80.4 as,

$$\begin{pmatrix}0 & 3\\1 & 0\end{pmatrix}\mathbf{y_1} = \begin{pmatrix}4\\0\end{pmatrix} \tag{1.80.18}$$

$$\mathbf{y_1} = \begin{pmatrix}0 & 3\\1 & 0\end{pmatrix}^{-1}\begin{pmatrix}4\\0\end{pmatrix} \tag{1.80.19}$$

$$\mathbf{y_1} = \begin{pmatrix}0\\\frac{4}{3}\end{pmatrix} \tag{1.80.20}$$

Another point $\mathbf{c_2}$ on the line is found using equation 1.80.3

$$\mathbf{c_2} = \mathbf{y_2} + \begin{pmatrix}3\\0\end{pmatrix} \tag{1.80.21}$$

$$\implies \mathbf{c_2} = \begin{pmatrix}3\\\frac{4}{3}\end{pmatrix} \tag{1.80.22}$$

The python code for the above problem , plotting the figure 1.80.1 is available at

```
solutions/3/codes/line/pointonline2/
    pointonline.py
```

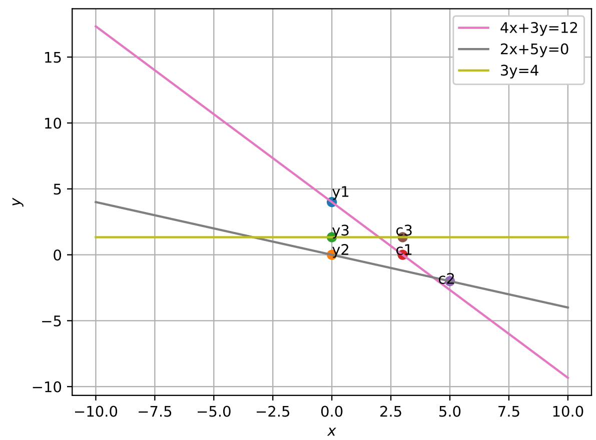

Fig. 1.80.1: Plot of the three lines and the points on them

1.81. Sketch the following lines

a) $\begin{pmatrix}2 & 3\end{pmatrix}\mathbf{x} = 9.35$

b) $\begin{pmatrix}1 & -\frac{1}{5}\end{pmatrix}\mathbf{x} = 10$

c) $\begin{pmatrix}-2 & 3\end{pmatrix}\mathbf{x} = 6$

d) $\begin{pmatrix}1 & -3\end{pmatrix}\mathbf{x} = 0$

e) $\begin{pmatrix}2 & 5\end{pmatrix}\mathbf{x} = 0$

f) $\begin{pmatrix}3 & 0\end{pmatrix}\mathbf{x} = -2$

g) $\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} = 2$

h) $\begin{pmatrix}2 & 0\end{pmatrix}\mathbf{x} = 5$

Solution:

a) put $\mathbf{x} = \begin{pmatrix}x\\0\end{pmatrix}$ in equation

$$\begin{pmatrix}2 & 3\end{pmatrix}\begin{pmatrix}x\\0\end{pmatrix} = \frac{187}{20} \tag{1.81.1}$$

$$x = \frac{187}{40} \tag{1.81.2}$$

put $\mathbf{x}\begin{pmatrix}0\\y\end{pmatrix}$ in equation

$$\begin{pmatrix}2 & 3\end{pmatrix}\begin{pmatrix}0\\y\end{pmatrix} = \frac{187}{20} \tag{1.81.3}$$

$$y = \frac{187}{60} \tag{1.81.4}$$

$$\mathbf{P1} = \begin{pmatrix}\frac{187}{40}\\0\end{pmatrix}, \mathbf{Q1} = \begin{pmatrix}0\\\frac{187}{60}\end{pmatrix} \tag{1.81.5}$$

b) put $\mathbf{x}\begin{pmatrix}x\\0\end{pmatrix}$ in equation

$$\begin{pmatrix}1 & -\frac{1}{5}\end{pmatrix}\begin{pmatrix}x\\0\end{pmatrix} = 10 \tag{1.81.6}$$

$$x = 10 \tag{1.81.7}$$

put $\mathbf{x}\begin{pmatrix}0\\y\end{pmatrix}$ in equation

$$\begin{pmatrix}1 & -\frac{1}{5}\end{pmatrix}\begin{pmatrix}0\\y\end{pmatrix} = 10 \tag{1.81.8}$$

$$y = -50 \tag{1.81.9}$$

$$\mathbf{P2} = \begin{pmatrix}10\\0\end{pmatrix}, \mathbf{Q2} = \begin{pmatrix}0\\-50\end{pmatrix} \tag{1.81.10}$$

c) put $\mathbf{x}\begin{pmatrix}x\\0\end{pmatrix}$ in equation

$$\begin{pmatrix}-2 & 3\end{pmatrix}\begin{pmatrix}x\\0\end{pmatrix} = 6 \tag{1.81.11}$$

$$x = -3 \tag{1.81.12}$$

put $\mathbf{x}\begin{pmatrix}0\\y\end{pmatrix}$ in equation

$$\begin{pmatrix}-2 & 3\end{pmatrix}\begin{pmatrix}0\\y\end{pmatrix} = 6 \tag{1.81.13}$$

$$y = 2 \tag{1.81.14}$$

$$\mathbf{P3} = \begin{pmatrix}-3\\0\end{pmatrix}, \mathbf{Q3} = \begin{pmatrix}0\\2\end{pmatrix} \tag{1.81.15}$$

d) there is no constant in the line equation thus it passes through the origin.
put $\mathbf{x}\begin{pmatrix}3\\y\end{pmatrix}$ in equation

$$\begin{pmatrix}1 & -3\end{pmatrix}\begin{pmatrix}3\\y\end{pmatrix} = 0 \tag{1.81.16}$$

$$y = 1 \tag{1.81.17}$$

$$\mathbf{P4} = \begin{pmatrix}0\\0\end{pmatrix}, \mathbf{Q4} = \begin{pmatrix}3\\1\end{pmatrix} \tag{1.81.18}$$

e) there is no constant in the line equation thus it passes through the origin
put $\mathbf{x}\begin{pmatrix}1\\y\end{pmatrix}$ in equation

$$\begin{pmatrix}2 & -1\end{pmatrix}\begin{pmatrix}1\\y\end{pmatrix} = 0 \tag{1.81.19}$$

$$y = 1 \tag{1.81.20}$$

$$\mathbf{P5} = \begin{pmatrix}0\\0\end{pmatrix}, \mathbf{Q5} = \begin{pmatrix}1\\2\end{pmatrix} \tag{1.81.21}$$

f) put $\mathbf{x}\begin{pmatrix}x\\0\end{pmatrix}$ in equation

$$\begin{pmatrix}3 & 0\end{pmatrix}\begin{pmatrix}x\\0\end{pmatrix} = -2 \tag{1.81.22}$$

$$x = -\frac{2}{3} \tag{1.81.23}$$

we can see in this equation the value of x coordinate does not depend on the y coordinate so we can say that it is parallel to the y-axis.

g) put $\mathbf{x}\begin{pmatrix}x\\0\end{pmatrix}$ in equation

$$\begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}0\\y\end{pmatrix} = 2 \tag{1.81.24}$$

$$y = 2 \tag{1.81.25}$$

we can see in this equation the value of y coordinate does not depend on the x coordinate so we can say that it is parallel to the x-axis.

h) put $\mathbf{x}\begin{pmatrix}x\\0\end{pmatrix}$ in equation

$$\begin{pmatrix}2 & 0\end{pmatrix}\begin{pmatrix}x\\0\end{pmatrix} = 5 \tag{1.81.26}$$

$$x = -\frac{5}{2} \tag{1.81.27}$$

we can see in this equation the value of x coordinate does not depend on the y coordinate so we can say that it is parallel to the y-axis.

```
solutions/4/codes/line/lines_and_planes/
    plane_and_line.py
```

1.82. Draw the graphs of the following equations

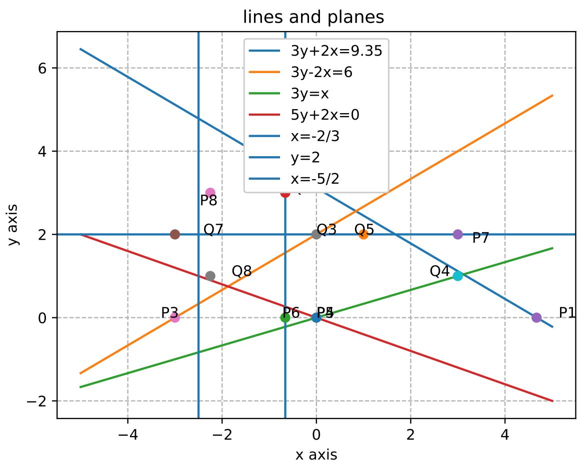

Fig. 1.81.1: lines

a) $\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 0$

b) $\begin{pmatrix}2 & -1\end{pmatrix}\mathbf{x} = 0$

c) $\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = 0$

d) $\begin{pmatrix}2 & -1\end{pmatrix}\mathbf{x} = -1$

e) $\begin{pmatrix}2 & -1\end{pmatrix}\mathbf{x} = 4$

f) $\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = 4$

Solution: The following python codes draw the graphs which are represented in Fig.1.82.1 and Fig.1.82.2.

```
./solutions/5/codes/lines/q11a.py
./solutions/5/codes/lines/q11b.py
```

Solution:

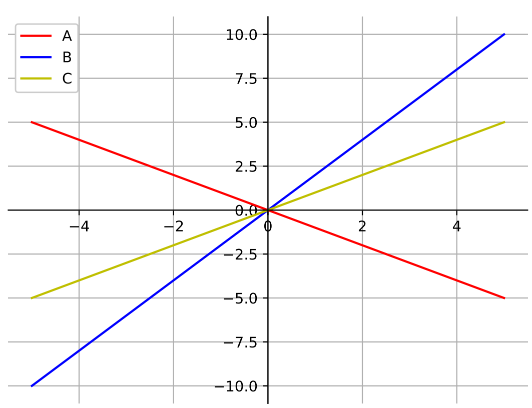

Fig. 1.82.1

1.83. Write four solutions for each of the following equations

a) $\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} = 7$

b) $\begin{pmatrix}\pi & 1\end{pmatrix}\mathbf{x} = 9$

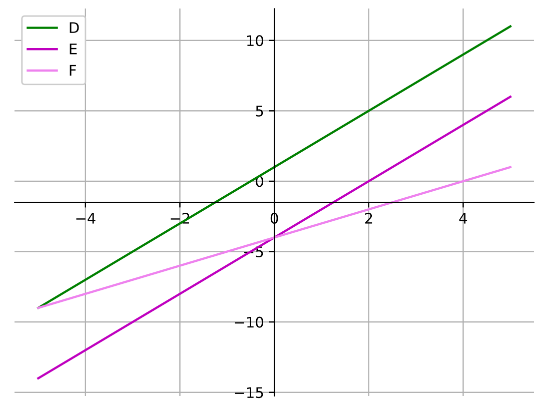

Fig. 1.82.2

c) $\begin{pmatrix}1 & -4\end{pmatrix}\mathbf{x} = 0$

Solution: The points are obtained by substituting

$$\mathbf{x} = \begin{pmatrix}a\\0\end{pmatrix}, \mathbf{x} = \begin{pmatrix}0\\b\end{pmatrix}, \mathbf{x} = \begin{pmatrix}c\\1\end{pmatrix}, \mathbf{x} = \begin{pmatrix}1\\d\end{pmatrix} \tag{1.83.1}$$

a)

$$\begin{pmatrix}2 & 1\end{pmatrix}\begin{pmatrix}a\\0\end{pmatrix} = 7 \tag{1.83.2}$$

$$\implies a = \frac{7}{2} \tag{1.83.3}$$

$$\begin{pmatrix}2 & 1\end{pmatrix}\begin{pmatrix}0\\b\end{pmatrix} = 7 \tag{1.83.4}$$

$$\implies b = 7 \tag{1.83.5}$$

ii)

$$\begin{pmatrix}2 & 1\end{pmatrix}\begin{pmatrix}c\\1\end{pmatrix} = 7 \tag{1.83.6}$$

$$\implies c = 3 \tag{1.83.7}$$

iii)

$$\begin{pmatrix}2 & 1\end{pmatrix}\begin{pmatrix}1\\d\end{pmatrix} = 7 \tag{1.83.8}$$

$$\implies d = 5 \tag{1.83.9}$$

1.84. Find m if

$$\begin{pmatrix}2 & 3\end{pmatrix}\mathbf{x} = 11$$
$$\begin{pmatrix}2 & -4\end{pmatrix}\mathbf{x} = -24 \tag{1.84.1}$$
$$\begin{pmatrix}m & -1\end{pmatrix}\mathbf{x} = -3$$

Solution: Given, the system of equations in matrix equation format are as below

$$\begin{pmatrix}2 & 3\\ 2 & -4\\ m & -1\end{pmatrix}\mathbf{x} = \begin{pmatrix}11\\ -24\\ -3\end{pmatrix} \tag{1.84.2}$$

Step1: Assuming the system of equations are consistent, lets reduce the augmented matrix [A'b], to find the value of m.

$$\begin{pmatrix}2 & 3 & 11\\ 2 & -4 & -24\\ m & -1 & -3\end{pmatrix}$$
$$\xleftrightarrow{R_2\leftarrow R_2-R_1}$$
$$\begin{pmatrix}2 & 3 & 11\\ 0 & -7 & -35\\ m & -1 & -3\end{pmatrix}$$
$$\xleftrightarrow{R_3\leftarrow 2R_3+R_1} \tag{1.84.3}$$
$$\begin{pmatrix}2 & 3 & 11\\ 0 & -7 & -35\\ 2m+2 & 1 & 5\end{pmatrix}$$
$$\xleftrightarrow{R_3\leftarrow R_2+7R_3}$$
$$\begin{pmatrix}2 & 3 & 11\\ 0 & -7 & -35\\ 14m+14 & 0 & 0\end{pmatrix}$$

Since the system of equations are assumed consistent,

$$\Rightarrow 14m + 14 = 0$$
$$\Rightarrow m = -1 \tag{1.84.4}$$

Step2:The system of equations can be represented as vectors as below:

1.85. Solve the following

a)

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 5$$
$$\begin{pmatrix}2 & -3\end{pmatrix}\mathbf{x} = 4 \tag{1.85.1}$$

b)

$$\begin{pmatrix}3 & 4\end{pmatrix}\mathbf{x} = 10$$
$$\begin{pmatrix}2 & -2\end{pmatrix}\mathbf{x} = 2 \tag{1.85.2}$$

b) i)

$$\begin{pmatrix}\pi & 1\end{pmatrix}\begin{pmatrix}a\\0\end{pmatrix} = 9 \tag{1.83.10}$$

$$\implies a = \frac{9}{\pi} \tag{1.83.11}$$

ii)

$$\begin{pmatrix}\pi & 1\end{pmatrix}\begin{pmatrix}0\\b\end{pmatrix} = 9 \tag{1.83.12}$$

$$\implies b = 9 \tag{1.83.13}$$

iii)

$$\begin{pmatrix}\pi & 1\end{pmatrix}\begin{pmatrix}c\\1\end{pmatrix} = 9 \tag{1.83.14}$$

$$\implies c = \frac{8}{\pi} \tag{1.83.15}$$

iv)

$$\begin{pmatrix}\pi & 1\end{pmatrix}\begin{pmatrix}1\\d\end{pmatrix} = 9 \tag{1.83.16}$$

$$\implies d = 9 - \pi \tag{1.83.17}$$

c) i)

$$\begin{pmatrix}1 & -4\end{pmatrix}\begin{pmatrix}a\\0\end{pmatrix} = 0 \tag{1.83.18}$$

$$\implies a = 0 \tag{1.83.19}$$

ii)

$$\begin{pmatrix}1 & -4\end{pmatrix}\begin{pmatrix}0\\b\end{pmatrix} = 0 \tag{1.83.20}$$

$$\implies b = 0 \tag{1.83.21}$$

iii)

$$\begin{pmatrix}1 & -4\end{pmatrix}\begin{pmatrix}c\\1\end{pmatrix} = 0 \tag{1.83.22}$$

$$\implies c = 4 \tag{1.83.23}$$

iv)

$$\begin{pmatrix}1 & -4\end{pmatrix}\begin{pmatrix}1\\d\end{pmatrix} = 0 \tag{1.83.24}$$

$$\implies d = \frac{1}{4} \tag{1.83.25}$$

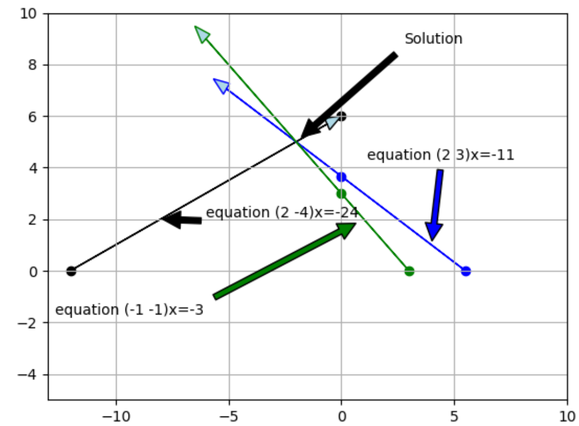

Fig. 1.84.1: System of Equations displaying intersecting at a point (-2 5).

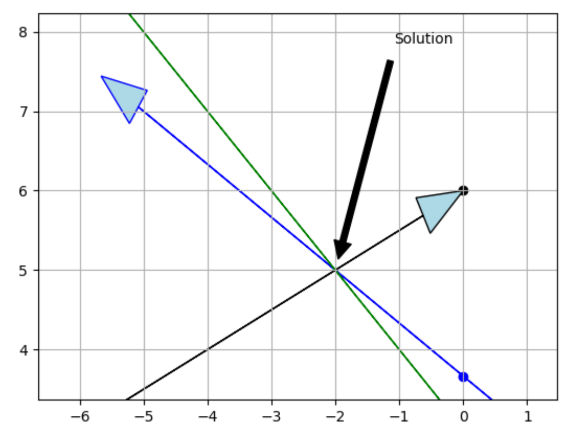

Fig. 1.84.2: A zoomed in view of System of Equations displaying intersecting at a point (-2 5).

c)

$$\begin{pmatrix}3 & -5\end{pmatrix}\mathbf{x} = 4$$
$$\begin{pmatrix}9 & -2\end{pmatrix}\mathbf{x} = 7 \tag{1.85.3}$$

$$\begin{pmatrix}\frac{1}{2} & \frac{2}{3}\end{pmatrix}\mathbf{x} = -1$$
$$\begin{pmatrix}1 & -\frac{1}{3}\end{pmatrix}\mathbf{x} = 3 \tag{1.85.4}$$

Solution:

a) We converted these line vectors in augmented matrix form:

$$\left(\begin{array}{cc|c}1 & 1 & 14\\1 & -1 & 4\end{array}\right) \tag{1.85.5}$$

$$\tag{1.85.6}$$

Now We will apply Row elementary operation to convert left part of matrix to identity matrix.

$$\xleftrightarrow{R_2=R_2-R_1}\left(\begin{array}{cc|c}1 & 1 & 14\\ 1 & -2 & -10\end{array}\right) \tag{1.85.7}$$

$$\xleftrightarrow{R_2=\frac{R_2}{-2}}\left(\begin{array}{cc|c}1 & 1 & 14\\ 0 & 1 & 5\end{array}\right) \tag{1.85.8}$$

$$\xleftrightarrow{R_1=R_1-R_2}\left(\begin{array}{cc|c}1 & 0 & 9\\ 0 & 1 & 5\end{array}\right) \tag{1.85.9}$$

As left part is converted into a identity matrix the intersection vector is $\begin{pmatrix}9\\5\end{pmatrix}$

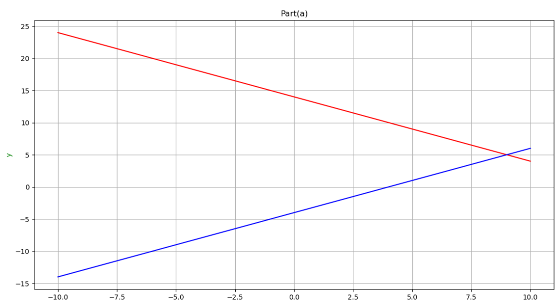

Fig. 1.85.1: part(a)

b) We converted these line vectors in augmented matrix form:

$$\left(\begin{array}{cc|c}1 & -1 & 3\\ \frac{1}{3} & \frac{1}{2} & 6\end{array}\right) \tag{1.85.10}$$

$$\xleftrightarrow{R_2=6\times R_2}\left(\begin{array}{cc|c}1 & -1 & 3\\ 2 & 3 & 36\end{array}\right) \tag{1.85.11}$$

$$\xleftrightarrow{R_2=R_2-2\times R_1}\left(\begin{array}{cc|c}1 & -1 & 3\\ 0 & 5 & 30\end{array}\right) \tag{1.85.12}$$

$$\xleftrightarrow{R_2=\frac{R_2}{5}}\left(\begin{array}{cc|c}1 & -1 & 3\\ 0 & 1 & 6\end{array}\right) \tag{1.85.13}$$

$$\xleftrightarrow{R_1=R_1+R_2}\left(\begin{array}{cc|c}1 & 0 & 9\\ 0 & 1 & 6\end{array}\right) \tag{1.85.14}$$

As left part is converted into a identity matrix the intersection vector is $\begin{pmatrix}9\\6\end{pmatrix}$

c) We converted these line vectors in augmented matrix form:

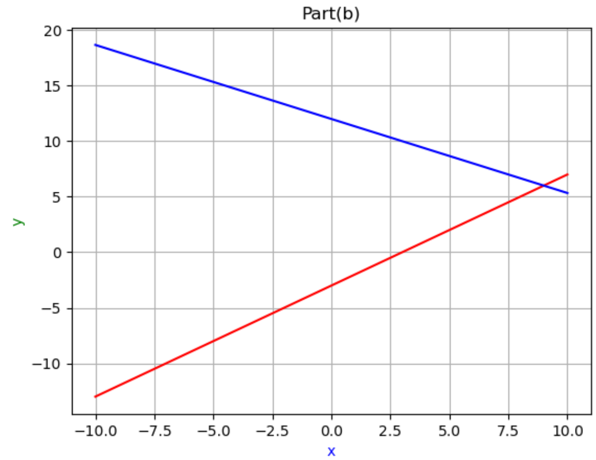

Fig. 1.85.2: part(b)

$$\left(\begin{array}{cc|c}3 & -1 & 3\\ 9 & -3 & 9\end{array}\right) \tag{1.85.15}$$

$$\xleftrightarrow{R_2=\frac{R_2}{3}}\left(\begin{array}{cc|c}1 & -1 & 3\\ 1 & -1 & 3\end{array}\right) \tag{1.85.16}$$

As $R_1 = R_2$, left part can never be converted into a identity matrix , and we can see now both row are same that means both lines are same they intersect at infinitely many points.

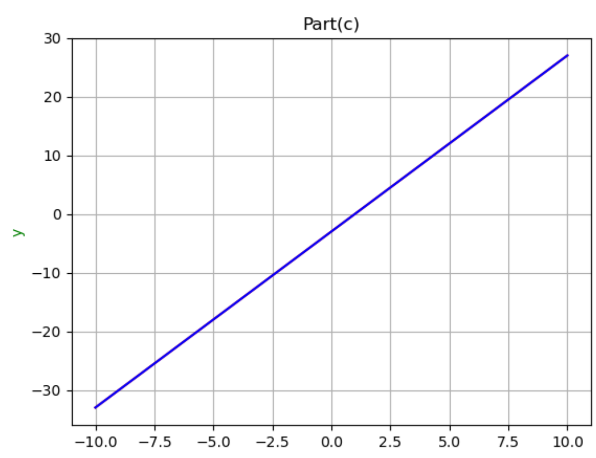

Fig. 1.85.3: part(c)

d) We converted these line vectors in augmented matrix form:

$$\left(\begin{array}{cc|c} 0.2 & 0.3 & 1.3 \\ 0.4 & 0.5 & 2.3 \end{array}\right) \tag{1.85.17}$$

$$\xleftrightarrow{R_2=R_2-2\times R_1} \left(\begin{array}{cc|c} 0.2 & 0.3 & 1.3 \\ 0 & -0.1 & -0.3 \end{array}\right) \tag{1.85.18}$$

$$\xleftrightarrow{R_2=\frac{R_2}{-0.1}} \left(\begin{array}{cc|c} 0.2 & 0.3 & 1.3 \\ 0 & 1 & 3 \end{array}\right) \tag{1.85.19}$$

$$\xleftrightarrow{R_1=R_1-0.3\times R_2} \left(\begin{array}{cc|c} 0.2 & 0 & 0.4 \\ 0 & 1 & 3 \end{array}\right) \tag{1.85.20}$$

$$\xleftrightarrow{R_1=\frac{R_1}{0.2}} \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \end{array}\right) \tag{1.85.21}$$

As left part is converted into a identity matrix the intersection vector is $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

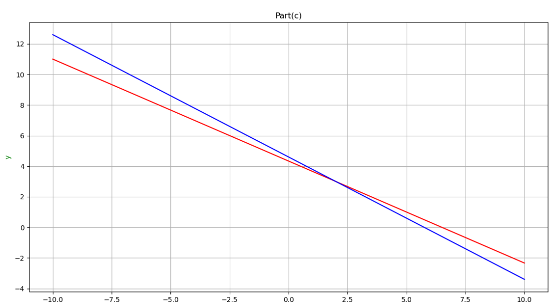

Fig. 1.85.4: part(d)

e) We converted these line vectors in augmented matrix form:

$$\left(\begin{array}{cc|c} \sqrt{2} & \sqrt{3} & 0 \\ \sqrt{3} & \sqrt{8} & 0 \end{array}\right) \tag{1.85.22}$$

$$\xleftrightarrow{R_2=R_2-\frac{\sqrt{3}}{\sqrt{2}}\times R_1} \left(\begin{array}{cc|c} \sqrt{2} & \sqrt{3} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{array}\right) \tag{1.85.23}$$

As we see whatever operation we are applying last column of our augmented matrix remains zero.So the lines are homogeneous lines and they always pass through origin , the intersection vector is $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

f) We converted these line vectors in augmented matrix form:

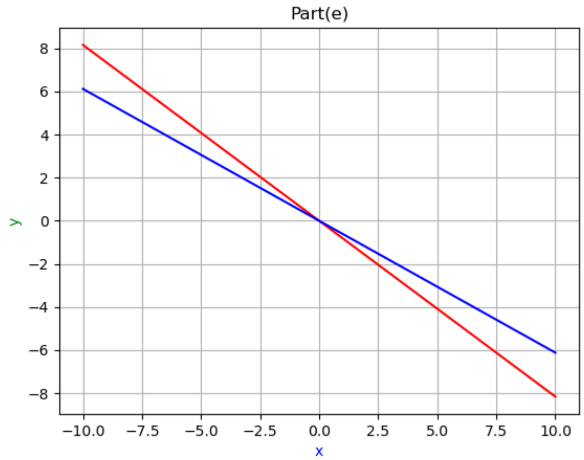

Fig. 1.85.5: part(e)

$$\left(\begin{array}{cc|c} \frac{3}{2} & \frac{-5}{3} & -2 \\ \frac{1}{3} & \frac{1}{2} & \frac{13}{6} \end{array}\right) \tag{1.85.24}$$

$$\xleftrightarrow{R_1=6\times R_1}\xleftrightarrow{R_2=6\times R_2} \left(\begin{array}{cc|c} 9 & -10 & -12 \\ 2 & 3 & 13 \end{array}\right) \tag{1.85.25}$$

$$\xleftrightarrow{R_1=R_1-4\times R_2} \left(\begin{array}{cc|c} 1 & -22 & -64 \\ 2 & 3 & 13 \end{array}\right) \tag{1.85.26}$$

$$\xleftrightarrow{R_2=R_2-2\times R_1} \left(\begin{array}{cc|c} 1 & -22 & -64 \\ 0 & 47 & 141 \end{array}\right) \tag{1.85.27}$$

$$\xleftrightarrow{R_2=\frac{R_2}{47}} \left(\begin{array}{cc|c} 1 & -22 & -64 \\ 0 & 1 & 3 \end{array}\right) \tag{1.85.28}$$

$$\xleftrightarrow{R_1=R_1+22\times R_2} \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \end{array}\right) \tag{1.85.29}$$

As left part is converted into a identity matrix the intersection vector is $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

1.86. For which values of a and b does the following pair of linear equations have an infinite number of solutions?

$$\begin{aligned} \begin{pmatrix} 2 & 3 \end{pmatrix}\mathbf{x} &= 7 \\ \begin{pmatrix} a-b & a+b \end{pmatrix}\mathbf{x} &= 3a+b-2 \end{aligned} \tag{1.86.1}$$

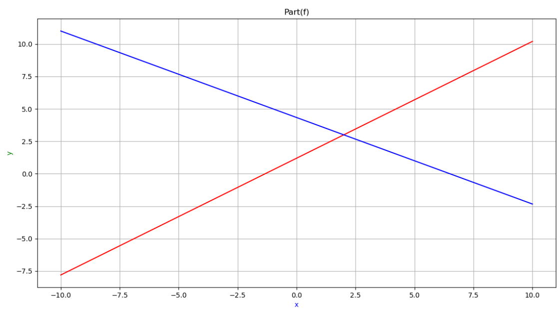

Fig. 1.85.6: part(f)

Solution: Constructing the augmented matrix

$$\begin{pmatrix} 2 & 3 & 7 \\ a-b & a+b & 3a+b-2 \end{pmatrix}$$

Transforming the matrix into row-echelon form

$$\begin{pmatrix} 2 & 3 & 7 \\ a-b & a+b & 3a+b-2 \end{pmatrix} \xleftrightarrow{R2 \leftarrow R1\times\frac{a-b}{2}-R2} \begin{pmatrix} 2 & 3 & 7 \\ 0 & \frac{3a-b}{2}-(a-b) & \frac{7a-b}{2}-(3a+b-2) \end{pmatrix} \tag{1.86.2}$$

For the linear equations to have infinite solution,
Rank(Coefficient matrix) =Rank(Augmented matrix)
and both ≠ Rank(Full matrix)

$$\begin{pmatrix} 1 & -5 & 0 \\ 1 & -9 & -4 \end{pmatrix} \xleftrightarrow{R2\leftarrow R2-R1} \begin{pmatrix} 1 & -5 & 0 \\ 0 & -4 & -4 \end{pmatrix} \tag{1.86.3}$$

$$\begin{pmatrix} 1 & -5 & 0 \\ 0 & -4 & -4 \end{pmatrix} \xleftrightarrow{R1\leftarrow 4\times R1} \begin{pmatrix} 4 & -20 & 0 \\ 0 & -4 & -4 \end{pmatrix} \tag{1.86.4}$$

$$\begin{pmatrix} 4 & -20 & 0 \\ 0 & -4 & -4 \end{pmatrix} \xleftrightarrow{R2\leftarrow 5\times R2} \begin{pmatrix} 4 & -20 & 0 \\ 0 & -20 & -20 \end{pmatrix} \tag{1.86.5}$$

$$\begin{pmatrix} 4 & -20 & 0 \\ 0 & -20 & -20 \end{pmatrix} \xleftrightarrow{R1\leftarrow R2-R1} \begin{pmatrix} -4 & 0 & -20 \\ 0 & -20 & -20 \end{pmatrix} \tag{1.86.6}$$

$$\begin{pmatrix} -4 & 0 & -20 \\ 0 & -20 & -20 \end{pmatrix} \xleftrightarrow{R1\leftarrow R1\div -4} \begin{pmatrix} 1 & 0 & 5 \\ 0 & -20 & -20 \end{pmatrix} \tag{1.86.7}$$

$$\begin{pmatrix} 1 & 0 & 5 \\ 0 & -20 & -20 \end{pmatrix} \xleftrightarrow{R2\leftarrow R2\div -20} \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 1 \end{pmatrix} \tag{1.86.8}$$

Now writing matrix in the form AX=B to obtain solution we have

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix} \tag{1.86.9}$$

Solving the above equation

$$\implies a = 5 \quad \& \quad b = 1 \tag{1.86.10}$$

1.87. For which value of k will the following pair of linear equations have no solution?

$$\begin{aligned} \begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} &= 1 \\ \begin{pmatrix} 2k-1 & k-1 \end{pmatrix} \mathbf{x} &= 2k+1 \end{aligned} \tag{1.87.1}$$

Solution: Constructing the augmented matrix

$$\begin{pmatrix} 3 & 1 & 1 \\ 2k-1 & k-1 & 2k+1 \end{pmatrix}$$

Transforming the matrix into row-echelon form

$$\begin{pmatrix} 3 & 1 & 1 \\ 2k-1 & k-1 & 2k+1 \end{pmatrix} \xleftrightarrow{R2\leftarrow R1*\frac{2k-1}{3}-R2} \begin{pmatrix} 3 & 1 & 1 \\ 0 & \frac{2k-1}{3}-(k-1) & \frac{2k-1}{3}-(2k+1) \end{pmatrix} \tag{1.87.2}$$

For the linear equations to have no solution,
Rank(Coefficient matrix) ≠ Rank(Augmented matrix)

$$\implies \frac{2k-1}{3} - (k-1) = 0 \tag{1.87.3}$$

and

$$\frac{2k-1}{3} - (2k+1) \neq 0 \tag{1.87.4}$$

Solving the above equations,

$$\implies k = 2 \quad \cap \quad k \neq -1 \tag{1.87.5}$$

From (1.87.5), it is clear that k = 2
Hence, for k = 2, the given set of linear equations will have no solution.

1.88. Solve the following pair of linear equations

$$\begin{pmatrix} 8 & 5 \end{pmatrix} \mathbf{x} = 9$$
$$\begin{pmatrix} 3 & 2 \end{pmatrix} \mathbf{x} = 4 \tag{1.88.1}$$

Solution: Step 1: Construct the Augmented Matrix

$$\begin{pmatrix} 8 & 5 & 9 \\ 3 & 2 & 4 \end{pmatrix} \tag{1.88.2}$$

Step 2: Perform row opertions to get a Row Echelon form

$$\begin{pmatrix} 8 & 5 & 9 \\ 3 & 2 & 4 \end{pmatrix} \xleftrightarrow{R_2 \to 8R_2 - 3R_1} \begin{pmatrix} 8 & 5 & 9 \\ 0 & 1 & 5 \end{pmatrix} \tag{1.88.3}$$

$$\begin{pmatrix} 8 & 5 & 9 \\ 0 & 1 & 5 \end{pmatrix} \xleftrightarrow{R_1 \to R_1 - 5R_2} \begin{pmatrix} 8 & 0 & -16 \\ 0 & 1 & 5 \end{pmatrix} \tag{1.88.4}$$

$$\begin{pmatrix} 8 & 0 & -16 \\ 0 & 1 & 5 \end{pmatrix} \xleftrightarrow{R_1 \to \frac{R_1}{8}} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 5 \end{pmatrix} \tag{1.88.5}$$

Above final matrix is in the reduced Echelon form and from this matrix we get the solution. Last column represents the solution of the given linear equation. Hence the solution is: $\begin{pmatrix} -2 \\ 5 \end{pmatrix}$ See Fig. 1.88.1

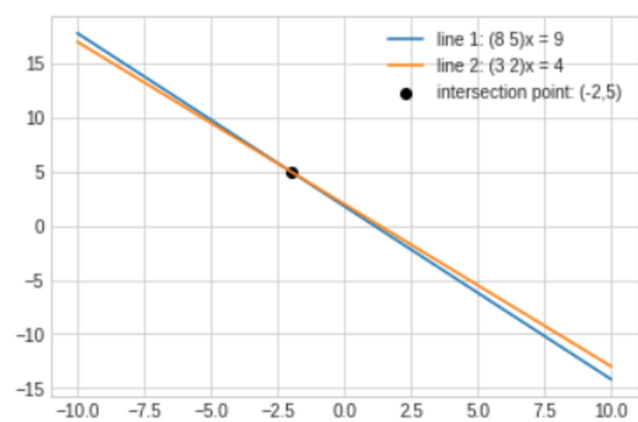

Fig. 1.88.1: Linear equations plot generated using python

1.89. Solve the following pair of linear equations

$$\begin{pmatrix} 158 & -378 \end{pmatrix} \mathbf{x} = -74$$
$$\begin{pmatrix} -378 & 152 \end{pmatrix} \mathbf{x} = -604 \tag{1.89.1}$$

Solution: Let

$$\mathbf{A} = \begin{pmatrix} 158 & -378 \\ -378 & 152 \end{pmatrix} \tag{1.89.2}$$

$$\mathbf{b} = \begin{pmatrix} -74 \\ -604 \end{pmatrix} \tag{1.89.3}$$

$$\mathbf{A} = \begin{pmatrix} 158 & -378 \\ -378 & 152 \end{pmatrix} \tag{1.89.4}$$

$$\mathbf{b} = \begin{pmatrix} -74 \\ -604 \end{pmatrix} \tag{1.89.5}$$

$$\mathbf{R} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{1.89.6}$$

$$\mathbf{RARAx} = \mathbf{RARb} \tag{1.89.7}$$

$$\mathbf{RARA} = k\mathbf{I} \tag{1.89.8}$$

I is the Identity Matrix and k is the constant which is equal to 118868 which is shown below

$$k\mathbf{Ix} = \mathbf{RARb} \tag{1.89.9}$$

$$k\mathbf{x} = \mathbf{RARb} \tag{1.89.10}$$

Finally divide the LHS and RHS by constant k in order to get the value of x. So, by putting the values of $\mathbf{R}, \mathbf{A}$ and $\mathbf{b}$ in equation 1.89.7 we can easily find out the value of x as follows:

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 158 & -378 \\ -378 & -152 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 158 & -378 \\ -378 & -152 \end{pmatrix} \mathbf{x}$$
$$= \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 158 & -378 \\ -378 & -152 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -74 \\ -604 \end{pmatrix} \tag{1.89.11}$$

After doing matrix multiplication in LHS and RHS,

$$\begin{pmatrix} 118868 & 0 \\ 0 & 118868 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 239560 \\ 123404 \end{pmatrix} \tag{1.89.12}$$

Now, divide both the rows by 118868:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 239560/118868 \\ 123404/118868 \end{pmatrix} \tag{1.89.13}$$

After further calculations in the fractional result:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 59890/29717 \\ 30851/29717 \end{pmatrix} \tag{1.89.14}$$

1.90. Find the slope of a line, which passes through the origin, and the mid-point of the line segment joining the points $\mathbf{P} = \begin{pmatrix} 0 \\ -4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 8 \\ 0 \end{pmatrix}$.

Solution: We are given two points $\mathbf{P}$ and $\mathbf{B}$. Let their mid-point be denoted by $\mathbf{Q}$.

$$\therefore \mathbf{Q} = \frac{\mathbf{P}+\mathbf{B}}{2} \tag{1.90.1}$$
$$= \frac{1}{2}\left[\begin{pmatrix} 0 \\ -4 \end{pmatrix} + \begin{pmatrix} 8 \\ 0 \end{pmatrix}\right] \tag{1.90.2}$$
$$= \begin{pmatrix} 4 \\ -2 \end{pmatrix} \tag{1.90.3}$$

We know, $\mathbf{O}$= Origin= (0,0)
Hence, the directional vector is:

$$\mathbf{m} = \mathbf{Q} - \mathbf{O} \tag{1.90.4}$$
$$= \mathbf{Q}, \quad \because \mathbf{O} = \mathbf{0} \tag{1.90.5}$$

The direction vector can be expressed in terms of the slope as

$$\mathbf{m} = \begin{pmatrix} 1 \\ m \end{pmatrix} \tag{1.90.6}$$

Now using (1.90.3), (1.90.5) and (1.90.6),

$$\begin{pmatrix} 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ m \end{pmatrix} \tag{1.90.7}$$
$$\implies \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ m \end{pmatrix} \tag{1.90.8}$$
$$\implies \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 \\ m \end{pmatrix} \tag{1.90.9}$$

Thus, by comparing, we have,

$$m = -\frac{1}{2} \tag{1.90.10}$$

1.91. The slope of a line is double of the slope of another line. If the tangent of the angle between them is $\frac{1}{3}$, find the slopes of the lines. **Solution:** The direction vector can be represented as below:-

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ m \end{pmatrix} \mathbf{m_2} = \begin{pmatrix} 1 \\ 2m \end{pmatrix} \tag{1.91.1}$$

The dot product of the vectors is given by:-

$$\mathbf{m_1}^T\mathbf{m_2} = \|\mathbf{m_1}\| \|\mathbf{m_2}\| \cos\theta \tag{1.91.2}$$

Given that

$$\tan\theta = \frac{1}{3} \tag{1.91.3}$$

By Baudhayana's theorem, we can obtain

$$\cos\theta = \frac{3}{\sqrt{10}} \tag{1.91.4}$$

Therefore,

$$\cos\theta = \frac{\mathbf{m_1}^T\mathbf{m_2}}{\|\mathbf{m_1}\| \|\mathbf{m_2}\|} \tag{1.91.5}$$
$$\frac{3}{\sqrt{10}} = \frac{1\times 1 + m\times 2m}{\sqrt{1+m^2}\sqrt{1+4m^2}} \tag{1.91.6}$$

Applying square on both sides:-

$$9\times(1+m^2)(1+4m^2) = 10(1+2m^2)^2 \tag{1.91.7}$$
$$4m^4 - 5m^2 + 1 = 0 \tag{1.91.8}$$
$$m_1 = m = 1, -1, \frac{1}{2}, \frac{-1}{2} \tag{1.91.9}$$

Substituting the value of m_1 we get value of $m_2 = 2, -2, 1, -1$.

1.92. Find the intercepts of the following lines on the axes.

a) $\begin{pmatrix} 3 & 2 \end{pmatrix}\mathbf{x} = 12.$
b) $\begin{pmatrix} 4 & -3 \end{pmatrix}\mathbf{x} = 6.$
c) $\begin{pmatrix} 3 & 2 \end{pmatrix}\mathbf{x} = 0.$

Solution:

a) Normal vector $\mathbf{n}$ is

$$\mathbf{n} = \begin{pmatrix} 1 \\ 7 \end{pmatrix} \tag{1.92.1}$$

Direction Vector

$$\mathbf{m} = \begin{pmatrix} 7 \\ -1 \end{pmatrix} \tag{1.92.2}$$

Y-intercept = 0

b) Normal vector $\mathbf{n}$ is

$$\mathbf{n} = \begin{pmatrix} 6 \\ 3 \end{pmatrix} \tag{1.92.3}$$

Direction Vector

$$\mathbf{m} = \begin{pmatrix} 3 \\ -6 \end{pmatrix} \tag{1.92.4}$$

Y-intercept = 5/3

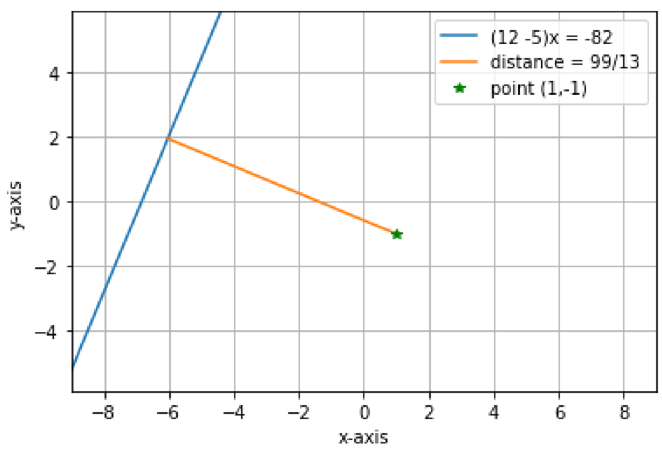

Fig. 1.93.1: Plot showing the distance between the point and the line

c) Normal vector $\mathbf{n}$ is

$$\mathbf{n} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.92.5}$$

Direction Vector

$$\mathbf{m} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.92.6}$$

Y-intercept = 0

1.93. Find the distance of the point $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ from the line $\begin{pmatrix} 12 & -5 \end{pmatrix} \mathbf{x} = -82$.

Solution:

The formula for calculating the distance between the point and the given line is

$$d = \frac{\left| c - \mathbf{n}^{\mathbf{T}}\mathbf{A} \right|}{\|\mathbf{n}\|} \tag{1.93.1}$$

By substituting the given values

$$\mathbf{A} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \mathbf{n} = \begin{pmatrix} 12 \\ -5 \end{pmatrix} \quad c = -82 \tag{1.93.2}$$

we get

$$\left| c - \mathbf{n}^{\mathbf{T}}\mathbf{A} \right| = 99 \tag{1.93.3}$$

Thus, the distance between the point and the line is

$$d = \frac{99}{13} \tag{1.93.4}$$

See Fig. 1.93.1

1.94. Find the points on the x-axis, whose distances from the line

$$\begin{pmatrix} 4 & 3 \end{pmatrix} \mathbf{x} = 12 \tag{1.94.1}$$

are 4 units.

Solution:

First we can find the lines at a distance of 4 from the given line and then it's intersection with the x-axis.

$$\mathbf{n} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$$

The parallel lines must have the same slope but different intercepts. Hence the lines must be of the form:

$$\begin{pmatrix} 4 & 3 \end{pmatrix} \mathbf{x} = c_1 \tag{1.94.2}$$

$$\begin{pmatrix} 4 & 3 \end{pmatrix} \mathbf{x} = c_2 \tag{1.94.3}$$

These c_1 and c_2 can be easily found by evaluating the distance between the parallel lines:

$$\frac{|(c - 12)|}{\sqrt{4^2 + 3^2}} = 4 \tag{1.94.4}$$

$$c = 12 \pm 20 \tag{1.94.5}$$

The two parallel lines at a distance of 4 thus obtained are:

$$\begin{pmatrix} 4 & 3 \end{pmatrix} \mathbf{x} = 32 \tag{1.94.6}$$

$$\begin{pmatrix} 4 & 3 \end{pmatrix} \mathbf{x} = -8 \tag{1.94.7}$$

Finally the points on x-axis are:

$$\mathbf{x} = 8 \tag{1.94.8}$$

$$\mathbf{x} = -2 \tag{1.94.9}$$

See Fig. 1.94.1

1.95. Find the equation of a line perpendicular to the line

$$\begin{pmatrix} 1 & -7 \end{pmatrix} \mathbf{x} = -5 \tag{1.95.1}$$

and having x intercept 3.

Solution: The normal vector of the perpendicular line is

$$\begin{pmatrix} 7 \\ 1 \end{pmatrix} \tag{1.95.2}$$

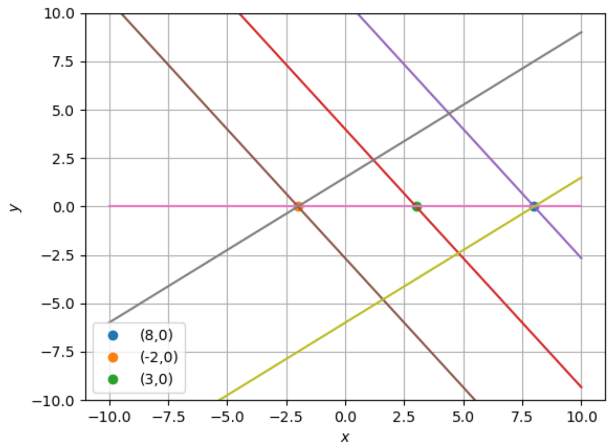

Fig. 1.94.1: Points on x-axis at a distance of 4 from the given line

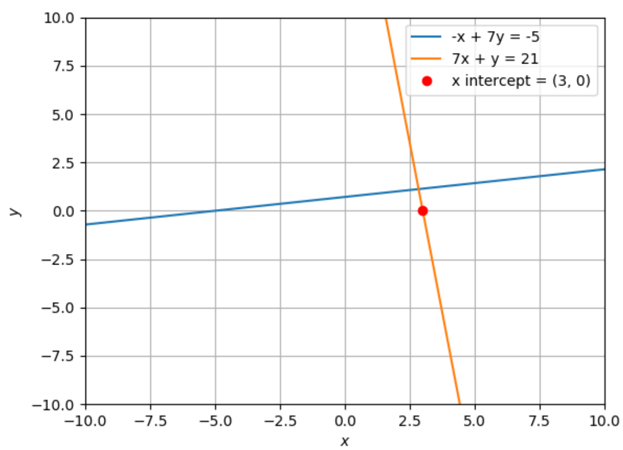

Fig. 1.95.1: Plot showing intersection

Thus, the desired equation of the line is

$$\begin{pmatrix}7 & 1\end{pmatrix}\left(\mathbf{x}-\begin{pmatrix}3\\0\end{pmatrix}\right)=0$$

$$\implies \begin{pmatrix}7 & 1\end{pmatrix}\mathbf{x}=21$$

See Fig. 1.95.1

1.96. Find angles between the lines

$$\begin{pmatrix}\sqrt{3} & 1\end{pmatrix}\mathbf{x}=1 \tag{1.96.1}$$

$$\begin{pmatrix}1 & \sqrt{3}\end{pmatrix}\mathbf{x}=1 \tag{1.96.2}$$

Solution: We will make direction vectors from these line vectors form:

$$\mathbf{m}_1=\begin{pmatrix}-\sqrt{3} & 1\end{pmatrix} \tag{1.96.3}$$

$$\mathbf{m}_2=\begin{pmatrix}-1 & \sqrt{3}\end{pmatrix} \tag{1.96.4}$$

Now we will find out magnitudes of each vectors $\mathbf{m}_1$,$\mathbf{m}_2$:

$$\|\mathbf{m}_1\|=\sqrt{3+1}=2 \tag{1.96.5}$$

$$\|\mathbf{m}_2\|=\sqrt{1+3}=2 \tag{1.96.6}$$

Thus angle between 2 vectors $\mathbf{m}_1$,$\mathbf{m}_2$ can be found using dot-product using the formula below, Let θ be angle between vectors $\mathbf{m}_1$,$\mathbf{m}_2$ then,

$$\theta=\cos^{-1}\left(\frac{\mathbf{m}_1\mathbf{m}_2^T}{\|\mathbf{m}_1\|\,\|\mathbf{m}_2\|}\right) \tag{1.96.7}$$

By,Putting values into above equation we get,

$$\theta=\cos^{-1}\left(\frac{\begin{pmatrix}-\sqrt{3} & 1\end{pmatrix}\begin{pmatrix}-1\\\sqrt{3}\end{pmatrix}}{4}\right) \tag{1.96.8}$$

$$=\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) \tag{1.96.9}$$

$$\implies \theta=30^\circ \tag{1.96.10}$$

1.97. The line through the points $\begin{pmatrix}h\\3\end{pmatrix}$ and $\begin{pmatrix}4\\1\end{pmatrix}$ intersects the line

$$\begin{pmatrix}7 & -9\end{pmatrix}\mathbf{x}=19 \tag{1.97.1}$$

at right angle. Find the value of h.
Solution:
Let the given points

$$\mathbf{A}=\begin{pmatrix}h\\3\end{pmatrix} \tag{1.97.2}$$

$$\mathbf{B}=\begin{pmatrix}4\\1\end{pmatrix} \tag{1.97.3}$$

Directional vector of line passing through points $\mathbf{A}$ and $\mathbf{B}$ is

$$\mathbf{P}=\mathbf{B}-\mathbf{A} \tag{1.97.4}$$

$$\mathbf{P}=\begin{pmatrix}h-4\\2\end{pmatrix} \tag{1.97.5}$$

Directional vector of the line $\begin{pmatrix} a & b \end{pmatrix}\mathbf{x} = c$ is

$$\mathbf{Q} = \begin{pmatrix} b \\ -a \end{pmatrix} \tag{1.97.6}$$

From (1.97.6) direction vector of line $\begin{pmatrix} 7 & -9 \end{pmatrix}\mathbf{x} = 19$ is

$$\mathbf{Q} = \begin{pmatrix} -9 \\ -7 \end{pmatrix} \tag{1.97.7}$$

If two straight lines intersects at right angles then inner product of their directional vectors is zero.

$$\mathbf{P}^T\mathbf{Q} = 0 \tag{1.97.8}$$

$$\begin{pmatrix} h-4 \\ 2 \end{pmatrix}^T \begin{pmatrix} -9 \\ -7 \end{pmatrix} = 0 \tag{1.97.9}$$

$$\begin{pmatrix} h-4 & 2 \end{pmatrix} \begin{pmatrix} -9 \\ -7 \end{pmatrix} = 0 \tag{1.97.10}$$

$$(h-4)(-9) + 2(-7) = 0 \tag{1.97.11}$$

$$h = \frac{22}{9} \tag{1.97.12}$$

Python Plot used to verify the result obtained from (1.97.12).

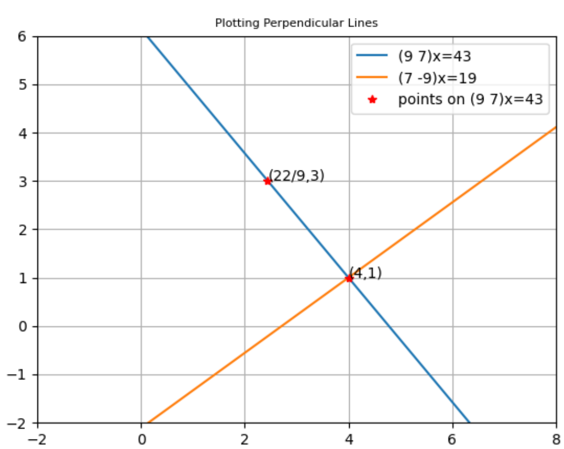

Fig. 1.97.1: Figure showing given data and corresponding results

According to the problem statement, equation of line passing through the point $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ and perpendicular to the line $\begin{pmatrix} 7 & -9 \end{pmatrix}\mathbf{x} = 19$ is

$$\begin{pmatrix} 9 & 7 \end{pmatrix}\mathbf{x} = 43 \tag{1.97.13}$$

Fig. 1.97.1 shows that equation (1.97.13) passes through the point $\begin{pmatrix} \frac{22}{9} \\ 3 \end{pmatrix}$

1.98. Two lines passing through the point $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ intersect each other at angle of 60°. If the slope of one line is 2, find the equation of the other line.
Solution: Directional vector of a line1 having slope 2 is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ Hence normal vector of line1 is given as

$$\mathbf{n_1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{1.98.1}$$

$$= \begin{pmatrix} -2 \\ 1 \end{pmatrix} \tag{1.98.2}$$

Similarly, normal vector for line 2

$$\mathbf{n_2} = \begin{pmatrix} -m_2 \\ 1 \end{pmatrix} \tag{1.98.3}$$

Angle between two lines θ can be given by

$$\cos\theta = \frac{\mathbf{n_1}^T\mathbf{n_2}}{\|\mathbf{n_1}\|\,\|\mathbf{n_2}\|} \tag{1.98.4}$$

$$\implies \cos 60^\circ = \frac{1}{2} \tag{1.98.5}$$

$$= \frac{2m_2 + 1}{\sqrt{5} \times \sqrt{1 + m_2}} \tag{1.98.6}$$

$$\implies 11m_2^2 + 16m_2 - 1 = 0 \tag{1.98.7}$$

Solving, m_2 yields values $\frac{-8+5\sqrt{3}}{11}$ and $\frac{-8-5\sqrt{3}}{11}$
Equation of line with normal vector $\mathbf{n}$ and passing through point A is given by

$$\mathbf{n}^T(\mathbf{x} - A) = 0 \tag{1.98.8}$$

Hence,equation of line with slope $\frac{-8+5\sqrt{3}}{11}$ passing through $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ is

$$\begin{pmatrix} \frac{8-5\sqrt{3}}{11} & 1 \end{pmatrix}\left(\mathbf{x} - \begin{pmatrix} 2 \\ 3 \end{pmatrix}\right) = 0 \tag{1.98.9}$$

$$\implies \begin{pmatrix} \frac{8-5\sqrt{3}}{11} & 1 \end{pmatrix}\mathbf{x} = \frac{49 - 10\sqrt{3}}{11} \tag{1.98.10}$$

Similarly,equation of line with slope $\frac{-8-5\sqrt{3}}{11}$

passing through $\begin{pmatrix}2\\3\end{pmatrix}$ is

$$\begin{pmatrix}\frac{8+5\sqrt{3}}{11} & 1\end{pmatrix}\left(\mathbf{x}-\begin{pmatrix}2\\3\end{pmatrix}\right)=0 \quad (1.98.11)$$

$$\implies \begin{pmatrix}\frac{8+5\sqrt{3}}{11} & 1\end{pmatrix}\mathbf{x}=\frac{49+10\sqrt{3}}{11} \quad (1.98.12)$$

Thus, the required line equations are

$$\begin{pmatrix}\frac{8-5\sqrt{3}}{11} & 1\end{pmatrix}\mathbf{x}=\frac{49-10\sqrt{3}}{11} \quad (1.98.13)$$

$$\begin{pmatrix}\frac{8+5\sqrt{3}}{11} & 1\end{pmatrix}\mathbf{x}=\frac{49+10\sqrt{3}}{11} \quad (1.98.14)$$

See Fig. 1.98.1

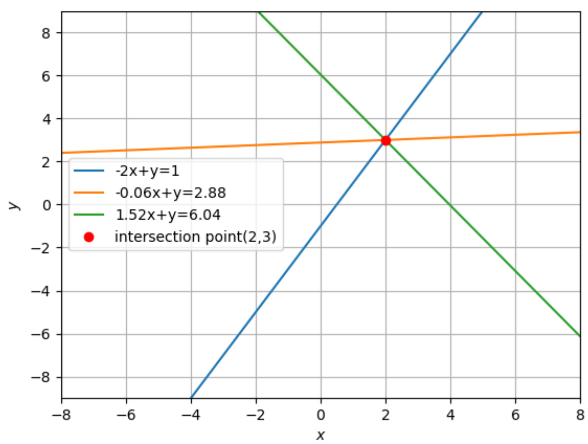

Fig. 1.98.1: plot showing intersection of lines

1.99. Find the equation of the right bisector of the line segment joining the points $\begin{pmatrix}3\\4\end{pmatrix}$ and $\begin{pmatrix}-1\\2\end{pmatrix}$.

Solution:
Let $\mathbf{M}$ be the midpoint of two points $\mathbf{A}=\begin{pmatrix}3\\4\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}-1\\2\end{pmatrix}$.

$$\mathbf{M}=\frac{\mathbf{A}+\mathbf{B}}{2}=\frac{1}{2}\begin{pmatrix}2\\6\end{pmatrix} \quad (1.99.1)$$

$$\implies \mathbf{M}=\begin{pmatrix}1\\3\end{pmatrix}$$

The direction vector of line AB is

$$\begin{pmatrix}-1\\2\end{pmatrix}-\begin{pmatrix}3\\4\end{pmatrix}=\begin{pmatrix}-4\\-2\end{pmatrix} \quad (1.99.2)$$

The direction vector of line AB is normal vector of right bisector. Then

$$\mathbf{n}=\begin{pmatrix}-4\\-2\end{pmatrix} \quad (1.99.3)$$

The equation of line in terms of normal vector is then obtained as

$$\mathbf{n}^T(\mathbf{x}-\mathbf{M})=0 \quad (1.99.4)$$

$$\implies \begin{pmatrix}-4 & -2\end{pmatrix}\left(\mathbf{x}-\begin{pmatrix}1\\3\end{pmatrix}\right)=0 \quad (1.99.5)$$

$$\implies \begin{pmatrix}-4 & -2\end{pmatrix}\mathbf{x}=-10 \quad (1.99.6)$$

$$\implies \begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x}=5 \quad (1.99.7)$$

We got equation of the right bisector of line segment joining points **A** and **B**. The line also passes through point **M**.
See Fig. 1.99.1 for plot of line segment and right bisector.

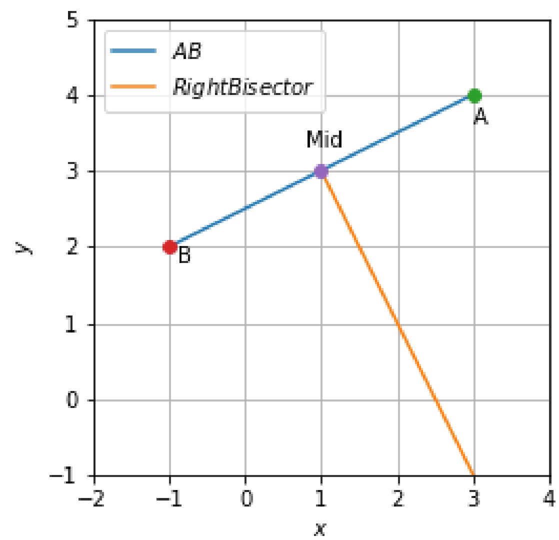

Fig. 1.99.1: Right bisector of line AB

1.100. Find the coordinates of the foot of the perpendicular from the point $\begin{pmatrix}-1\\3\end{pmatrix}$ to the line

$$\begin{pmatrix}3 & -4\end{pmatrix}\mathbf{x}=16. \quad (1.100.1)$$

Solution:
The normal vector to the perpendicular drawn from point (-1 3) is same as the direction vector of the given line:

$$\mathbf{n}=\begin{pmatrix}4\\3\end{pmatrix} \quad (1.100.2)$$

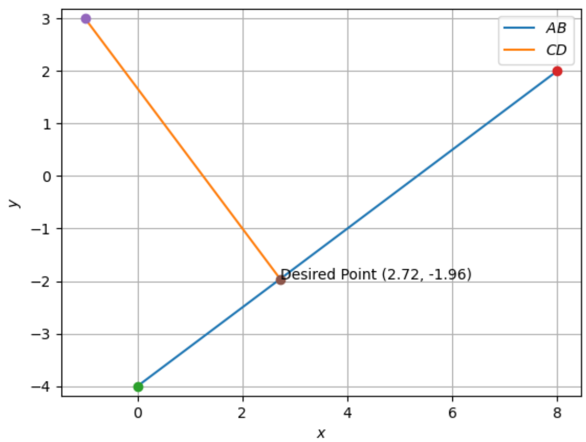

Fig. 1.100.1

The equation of the drawn perpendicular in terms of the normal vector is then obtained as

$$\mathbf{n}^T(\mathbf{x}-\mathbf{A}) = 0 \tag{1.100.3}$$

$$\begin{pmatrix}4 & 3\end{pmatrix}\mathbf{x} = 5 \tag{1.100.4}$$

The above two line equations can be expressed as the matrix equation

$$\begin{pmatrix}3 & -4\\4 & 3\end{pmatrix}\mathbf{x} = \begin{pmatrix}16\\5\end{pmatrix} \tag{1.100.5}$$

The augmented matrix for the above equation is row reduced as follows

$$\begin{pmatrix}3 & -4 & 16\\4 & 3 & 5\end{pmatrix} \xleftrightarrow{R_1\leftarrow R_1/3} \begin{pmatrix}1 & -4/3 & 16/3\\4 & 3 & 5\end{pmatrix} \tag{1.100.6}$$

$$\xleftrightarrow{R_2\leftarrow R_2-4R_1} \begin{pmatrix}1 & -4/3 & 16/3\\0 & 25/3 & -49/3\end{pmatrix} \tag{1.100.7}$$

$$\xleftrightarrow{R_2\leftarrow R_2\times 3/25} \begin{pmatrix}1 & -4/3 & 16/3\\0 & 1 & -49/25\end{pmatrix} \tag{1.100.8}$$

$$\xleftrightarrow{R_1\leftarrow R_1+4/3\times R_1} \begin{pmatrix}1 & 0 & 68/25\\0 & 1 & -49/25\end{pmatrix} \tag{1.100.9}$$

Thus, The foot of the perpendicular is at point (68/25 , -49/25) i.e. (2.72, -1.96) See Fig. 1.100.1

1.101. The perpendicular from the origin to the line

$$\begin{pmatrix}-m & 1\end{pmatrix}\mathbf{x} = c \tag{1.101.1}$$

meets it at the point $\begin{pmatrix}-1\\2\end{pmatrix}$. Find the values of m and c.

Solution: The line

$$\begin{pmatrix}-m & 1\end{pmatrix}\mathbf{x} = c \tag{1.101.2}$$

meets it at the point $\mathbf{P} = \begin{pmatrix}-1\\2\end{pmatrix}$ Since,

$$\mathbf{P}-\mathbf{0} = \mathbf{P} \tag{1.101.3}$$

is the normal vector, where $\mathbf{0}$ is the origin, then

$$\mathbf{m} = \begin{pmatrix}1\\m\end{pmatrix} \tag{1.101.4}$$

is the direction vector, Hence

$$\mathbf{m}^T\mathbf{P} = 0 \tag{1.101.5}$$

$$\implies \begin{pmatrix}1 & m\end{pmatrix}\begin{pmatrix}-1\\2\end{pmatrix} = 0$$

$$\implies (-1+2m) = 0$$

$$\implies m = \frac{1}{2} \tag{1.101.6}$$

now, the line

$$\begin{pmatrix}-m & 1\end{pmatrix}\mathbf{x} = c$$

meets it at the point $\mathbf{P}=\begin{pmatrix}-1\\2\end{pmatrix}$and using the value of m from 1.101.6 we get,

$$\begin{pmatrix}\frac{-1}{2} & 1\end{pmatrix}\mathbf{P} = c$$

$$\implies \begin{pmatrix}\frac{-1}{2} & 1\end{pmatrix}\begin{pmatrix}-1\\2\end{pmatrix} = c$$

$$\implies c = \frac{5}{2}$$

Hence, the value of m and c are obtained as

$$m = \frac{1}{2},\ c = \frac{5}{2}$$

respectively. See Fig. 1.101.1.

1.102. Find the equations of the lines, which cut-off intercepts on the axes whose sum and product are 1 and -6 respectively.

Solution: The equation of line in terms of vector notations can be written as

$$\mathbf{n}^T\mathbf{x} = c$$

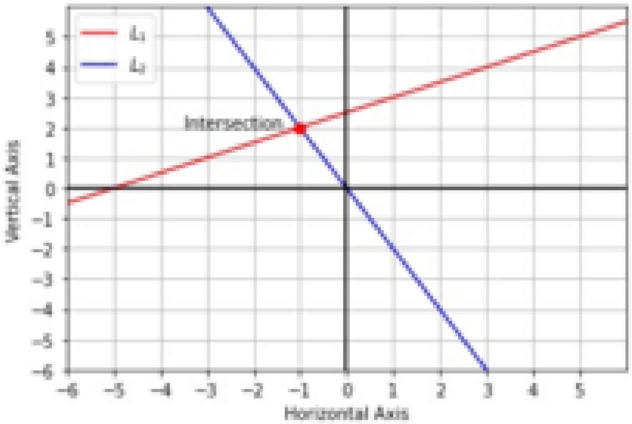

Fig. 1.101.1: Perpendicular Lines crossing

Let the intercepts be $\begin{pmatrix} a \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ b \end{pmatrix}$, respectively.
Given that: $\quad a + b = 1, \quad$ and $\quad ab = -6$
The quadratic equation whose roots are the x and y intercepts can be written as:

$$x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$$

$$\implies \quad x^2 - x - 6 = 0$$
$$\implies \quad x = (3, -2)$$

and corresponding y intercepts are $(-2, 3)$.
The line L_1 passes through $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ -2 \end{pmatrix}$.
Let direction vector of this line be $\mathbf{m}$.

$$\mathbf{m} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \end{pmatrix}$$

The normal vector, $\mathbf{n}$:

$$\mathbf{n} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{m} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

The equation of line in terms of normal vector and passing through a point A is

$$\mathbf{n}^{\mathbf{T}}(\mathbf{x} - A) = 0 \quad \implies \quad \mathbf{n}^{\mathbf{T}}\mathbf{x} = \mathbf{n}^{\mathbf{T}}A$$

$$\implies \mathbf{n}^{\mathbf{T}}\mathbf{x} = \begin{pmatrix} 2 & -3 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

$$\implies \begin{pmatrix} 2 & -3 \end{pmatrix} \mathbf{x} = 6$$

Similarly, the equation of second line L_2, with x and y intercepts $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$ and normal vector $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$ is

$$\begin{pmatrix} -3 & 2 \end{pmatrix} \mathbf{x} = 6$$

The equations of lines (1.102) and (1.102) can be represented collectively as

$$\begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 6 \\ 6 \end{pmatrix}$$

$x-$intercept	$y-$intercept	$\mathbf{n}$
$\begin{pmatrix} 3 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ -2 \end{pmatrix}$	$\begin{pmatrix} 2 \\ -3 \end{pmatrix}$
$\begin{pmatrix} -2 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 3 \end{pmatrix}$	$\begin{pmatrix} -3 \\ 2 \end{pmatrix}$

TABLE 1.102

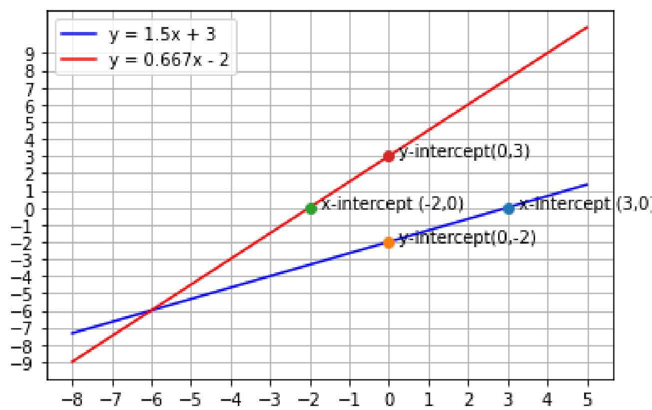

Fig. 1.102.1

1.103. What are the points on the y-axis whose distance from the line

$$\begin{pmatrix} 4 & 3 \end{pmatrix} \mathbf{x} = 12$$

4 units.
Solution: Here, direction vectors of the lines are $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$
Using the formula for the distance of a point P from a line

$$d = \frac{|\mathbf{n}^T P - c|}{\|\mathbf{n}\|}$$

normal vector n is given by,

$$\mathbf{n} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

Since the point lies on the y-axis. let

$$P = \begin{pmatrix} 0 \\ k \end{pmatrix}$$

If the equation of the line is :

$$\mathbf{n}^T x = c$$

$$\frac{|\mathbf{n}^T P - c|}{||\mathbf{n}||} = 4$$

$$\implies 3k - 12 = \pm 20$$

$$\implies k = \begin{pmatrix} 0 \\ -8 \end{pmatrix} and \begin{pmatrix} 0 \\ 32/3 \end{pmatrix}$$

therefore points on y-axis at distance of P from line are $\begin{pmatrix} 0 \\ -8 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 32/3 \end{pmatrix}$.

1.104. Find the equation of the line parallel to the y-axis drawn through the point of intersection of the lines

$$\begin{pmatrix} 1 & -7 \end{pmatrix} \mathbf{x} = -5$$

$$\begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} = 0$$

Solution: consider the equation of the system of lines

$$x - 7y = -5$$

$$3x + y = 0$$

consider the augmented matrix

$$\begin{pmatrix} 1 & -7 & -5 \\ 3 & 1 & 0 \end{pmatrix}$$

By applying row reduction reduction technique

$$\begin{pmatrix} 4 & -7 & -5 \\ 3 & 1 & 0 \end{pmatrix}$$

$$\xleftrightarrow[R_2 \leftarrow R_2/22]{R_2 \leftarrow R_2 - 3R_1} \begin{pmatrix} 1 & -7 & -5 \\ 0 & 1 & \frac{15}{22} \end{pmatrix}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + 7R_2} \begin{pmatrix} 1 & 0 & \frac{-5}{22} \\ 0 & 1 & \frac{15}{22} \end{pmatrix}$$

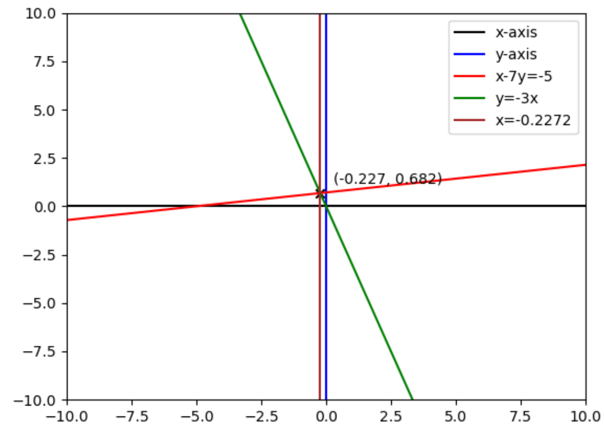

Fig. 1.104.1: graphical representation of systems of lines

The value of**A** is the point of intersection.

$$\mathbf{A} = \begin{pmatrix} \frac{-5}{22} \\ \frac{15}{22} \end{pmatrix}$$

Now the equation of line parallel to y-axis through the point of intersection.

$$\mathbf{n}^T (\mathbf{x} - \mathbf{A}) = 0$$

where**n** is the vector normal to the Y - axis and **A** is the point of intersection

$$\mathbf{n}^T \mathbf{x} = \mathbf{n}^T \mathbf{A}$$

$$\text{where } \mathbf{n}^T = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{-5}{22} \\ \frac{15}{22} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = -\frac{5}{22}$$

Shown in Fig. 1.104.1 is the equation of the line parallel to the Y-axis drawn through the point of intersection of the lines.

1.105. Find the equation of the lines through the point $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ which make an angle of 45° with the line

$$\begin{pmatrix} 1 & -2 \end{pmatrix} \mathbf{x} = 3.$$

Solution: On comparing $\begin{pmatrix} 1 & -2 \end{pmatrix} \mathbf{x} = 3$ with

Formulae 1 we get

$$\mathbf{n}_1^T = \begin{pmatrix} 1 & -2 \end{pmatrix}$$

Let the normal vector of the other line is

$$\mathbf{n}_2 = \begin{pmatrix} -m \\ 1 \end{pmatrix}$$

Angle between these two lines is 45°

$$\theta = 45^\circ \implies \cos 45^\circ = \frac{1}{\sqrt{2}}$$

Substituting $\mathbf{n}_1^T\mathbf{n}_2$ and $\cos\theta$ in Formulae 2 we get

$$\frac{1}{\sqrt{2}} = \frac{\begin{pmatrix} 1 & -2 \end{pmatrix} \times \begin{pmatrix} -m \\ 1 \end{pmatrix}}{\sqrt{5} \times \sqrt{m^2+1}}$$

$$3m^2 - 4m - 3 = 0$$

Solving this equation we get two roots

$$m_1 = 3$$
$$m_2 = -\frac{1}{3}$$

Equations of Line with Slope $m_1 = 3$ and passing through point $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ is given as

$$\mathbf{n}^T(\mathbf{x} - \mathbf{A}) = 0$$
$$\mathbf{n}^T = \begin{pmatrix} -3 & 1 \end{pmatrix}$$

Substituting Values we get

$$\begin{pmatrix} -3 & 1 \end{pmatrix}\left(\mathbf{x} - \begin{pmatrix} 3 \\ 2 \end{pmatrix}\right) = 0$$
$$\begin{pmatrix} -3 & 1 \end{pmatrix}\mathbf{x} = -7$$

Equations of Line with Slope $m_2 = -\frac{1}{3}$ and passing through point $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ is given as

$$\mathbf{n}^T(\mathbf{x} - \mathbf{A}) = 0$$
$$\mathbf{n}^T = \begin{pmatrix} -\frac{1}{3} & 1 \end{pmatrix}$$

Substituting Values we get

$$\begin{pmatrix} \frac{1}{3} & 1 \end{pmatrix}\left(\mathbf{x} - \begin{pmatrix} 3 \\ 2 \end{pmatrix}\right) = 0$$
$$\begin{pmatrix} \frac{1}{3} & 1 \end{pmatrix}\mathbf{x} = 3$$

the equation of lines through the point $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ which make an angle of 45°to the line

$$\begin{pmatrix} 1 & -2 \end{pmatrix}\mathbf{x} = 3$$

are

$$\begin{pmatrix} -3 & 1 \end{pmatrix}\mathbf{x} = -7$$
$$\begin{pmatrix} \frac{1}{3} & 1 \end{pmatrix}\mathbf{x} = 3$$

The Figure 1.105.1 shows the plot of all three lines

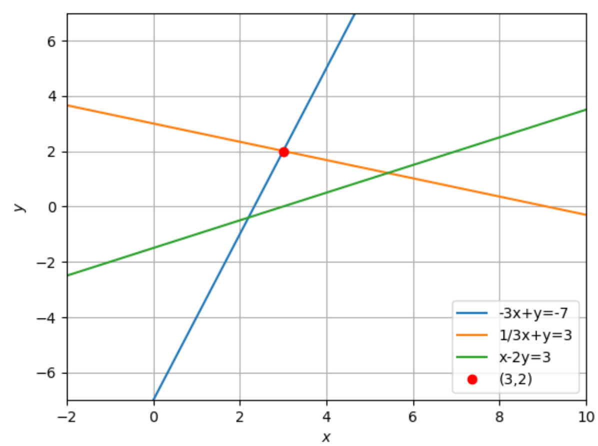

Fig. 1.105.1: Plotting these Equation

1.106. Find the equation of the line passing through the point of intersection of the lines

$$\begin{pmatrix} 4 & 7 \end{pmatrix}\mathbf{x} = 3$$
$$\begin{pmatrix} 2 & -3 \end{pmatrix}\mathbf{x} = -1$$

that has equal intercepts on the axes.
Solution: The above two line equations can be expressed as the matrix equation

$$\begin{pmatrix} 4 & 7 \\ 2 & -3 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

Constructing the augmented matrix

$$\begin{pmatrix} 4 & 7 & 3 \\ 2 & -3 & -1 \end{pmatrix}$$

Transforming the matrix into row-echelon form

$$\begin{pmatrix} 4 & 7 & 3 \\ 2 & -3 & -1 \end{pmatrix} \xleftrightarrow{R2 \leftarrow 2R2 - R1}$$

$$\begin{pmatrix} 4 & 7 & 3 \\ 0 & -13 & -5 \end{pmatrix} \xleftrightarrow{R2 \leftarrow -R2/13, R1 \leftarrow R1/4}$$

$$\begin{pmatrix} 1 & 7/4 & 3/4 \\ 0 & 1 & 5/13 \end{pmatrix} \xleftrightarrow{R1 \leftarrow R1 - 7/4R1/4}$$

$$\begin{pmatrix} 1 & 0 & 2/26 \\ 0 & 1 & 5/13 \end{pmatrix}$$

The solution for x can be written as

$$\mathbf{x} = \begin{pmatrix} 2/26 \\ 5/13 \end{pmatrix}$$

Thus, The point of intersection is at point (2/26 , 5/13) i.e. (0.07, 0.38)

Let the equation of the line be

$$\mathbf{n}^T \mathbf{x} = c \implies \mathbf{x}^T \mathbf{n} = c$$

Let the intercepts be a,b on the x and y axis respectively. Then,

$$\begin{pmatrix} a & 0 \end{pmatrix} \mathbf{n} = c$$

$$\begin{pmatrix} 0 & b \end{pmatrix} \mathbf{n} = c$$

resulting in the matrix equation

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mathbf{n} = c \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

As the intercepts are equal,Let a=b

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mathbf{n} = c \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\mathbf{n} = \frac{c}{a} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

As the line passes through point of intersection,We can use the equation (1.106) in equation (1.106) to find the value of c

$$c = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{2}{26} \\ \frac{2}{26} \end{pmatrix}$$

$$\implies c = \frac{2}{26} + \frac{2}{26} \implies c = \frac{6}{13}$$

So, the equation of line can be written as

$$\implies \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = \frac{6}{13}$$

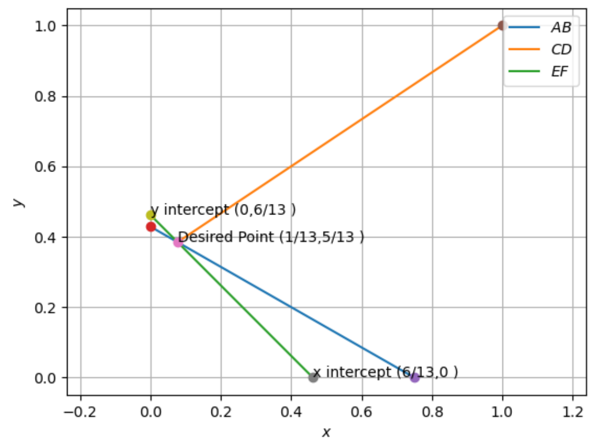

Fig. 1.106.1: The intercepts of the required line are equal

1.107. In what ratio is the line joining $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ 7 \end{pmatrix}$ divided by the line

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 4$$

Solution: The point $\mathbf{X}$ divides the line segment joining the two points $\mathbf{A}=\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix} 5 \\ 7 \end{pmatrix}$ in ratio k : 1. Then,

$$\mathbf{X} = \frac{(k\mathbf{B} + \mathbf{A})}{(k+1)}$$

From the equation (1.107)

$$(k+1)\mathbf{X} = k\mathbf{B} + \mathbf{A}$$

Let $\mathbf{n}=\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\implies (k+1)\,\mathbf{n}^T\mathbf{X} = \mathbf{n}^T\,(k\mathbf{B} + \mathbf{A})$$

$$\implies k\left(\mathbf{n}^T\mathbf{X} - \mathbf{n}^T\mathbf{B}\right) = \mathbf{n}^T\mathbf{A} - \mathbf{n}^T\mathbf{X}$$

$$\implies k = \frac{\mathbf{n}^T\mathbf{A} - \mathbf{n}^T\mathbf{X}}{\mathbf{n}^T\mathbf{X} - \mathbf{n}^T\mathbf{B}}$$

Hence on solving the equation (1.107) using

$$\mathbf{n}^T\mathbf{X} = 4$$

The line $\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x}=4$ divides the line joining points $\mathbf{A}=\begin{pmatrix}-1\\1\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}5\\7\end{pmatrix}$ in the ratio k=1/2 See Fig. 1.107.1

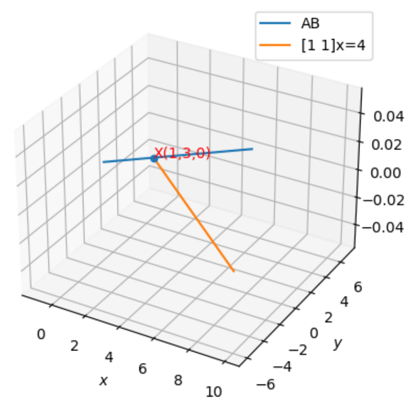

Fig. 1.107.1: Line as (1 1)**x**=4 intersecting the line joining points A and B

1.108. Find the distance of the line

$$\begin{pmatrix}4 & 7\end{pmatrix}\mathbf{x} = -5$$

from the point $\begin{pmatrix}1\\2\end{pmatrix}$ along the line

$$\begin{pmatrix}2 & -1\end{pmatrix}\mathbf{x} = 0.$$

Solution: We need to find the solution of equations

$$\begin{pmatrix}4 & 7\end{pmatrix}\mathbf{x} = -5$$
$$\begin{pmatrix}2 & -1\end{pmatrix}\mathbf{x} = 0$$

Transforming the matrix into row-echelon form

$$\begin{pmatrix}4 & 7 & -5\\2 & 1 & 0\end{pmatrix} \xleftrightarrow{R1\leftarrow\frac{1}{18}*(R1+7\times R2)} \begin{pmatrix}1 & 0 & -5/18\\2 & -1 & 0\end{pmatrix}$$

$$\begin{pmatrix}1 & 0 & -5/18\\2 & -1 & 0\end{pmatrix} \xleftrightarrow{R2\leftarrow-(R2-2\times R1)} \begin{pmatrix}1 & 0 & -5/18\\0 & 1 & -10/18\end{pmatrix}$$

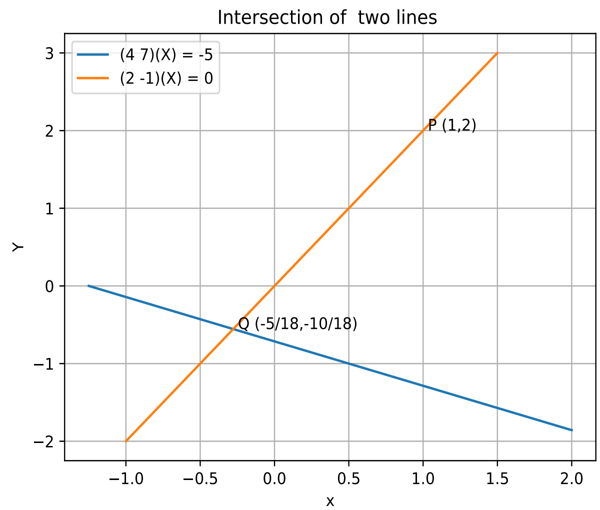

Fig. 1.108.1: Intersection of two lines

After solving this two equation we will get the point of intersection, which is intersection of these two lines segments. Thus, point of intersection is $\begin{pmatrix}-5/18\\-10/18\end{pmatrix}$. Now we have point of intersection

$$\mathbf{P} = \begin{pmatrix}-5/18\\-10/18\end{pmatrix}$$

and given point is

$$\mathbf{Q} = \begin{pmatrix}1\\2\end{pmatrix}$$

Now the distance between two points is given as :

$$\|\mathbf{P}-\mathbf{Q}\| = \left\|\begin{pmatrix}-5/18\\-10/18\end{pmatrix} - \begin{pmatrix}1\\2\end{pmatrix}\right\| = d = 2.85$$

1.109. Find the direction in which a straight line must be drawn through the point $\begin{pmatrix}-1\\2\end{pmatrix}$ so that its point of intersection with the line

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 4$$

may be at a distance of 3 units from this point. **Solution:** The given equation of the line in parametric form:

$$\mathbf{x} = \mathbf{A} + \lambda\mathbf{m}$$

where,

$$\mathbf{A} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\mathbf{m} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

If $\mathbf{x}$ be the point of intersection,

$$\|\mathbf{x} - \mathbf{B}\| = 3$$

$$\|\mathbf{A} + \lambda\mathbf{m} - \mathbf{B}\| = 3$$

$$(\mathbf{A} + \lambda\mathbf{m} - \mathbf{B})^T(\mathbf{A} + \lambda\mathbf{m} - \mathbf{B}) = 9$$

$$[(\mathbf{A} - \mathbf{B})^T\mathbf{m} = \mathbf{m}^T(\mathbf{A} - \mathbf{B})]$$

$$\|\mathbf{m}\|^2 \lambda^2 + \left[2(\mathbf{A} - \mathbf{B})^T\mathbf{m})\right]\lambda + \|\mathbf{A} - \mathbf{B}\|^2 = 9$$

$$2\lambda^2 + 10\lambda + 8 = 0$$

$$\lambda = -4, \lambda = -1$$

The point of intersection,

$$\boxed{\therefore \mathbf{x} = \begin{pmatrix} -1 \\ 5 \end{pmatrix} or \begin{pmatrix} 2 \\ 2 \end{pmatrix}}$$

The direction vector,

$$\mathbf{v} = \mathbf{B} - \mathbf{x}$$

$$\boxed{\mathbf{v} = \begin{pmatrix} 0 \\ -3 \end{pmatrix} or \begin{pmatrix} -3 \\ 0 \end{pmatrix}}$$

1.110. Find the image of the point $\begin{pmatrix} 3 \\ 8 \end{pmatrix}$ with respect to the line

$$\begin{pmatrix} 1 & 3 \end{pmatrix}\mathbf{x} = 7$$

assuming the line to be a plane mirror.
Solution: Let, given vector

$$\mathbf{P} = \begin{pmatrix} 3 \\ 8 \end{pmatrix} \tag{1.110.0}$$

Let, image point be $\mathbf{R}$. Let vector,

$$\mathbf{n} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \tag{1.110.0}$$

Let m be the directional vector along the line, $\begin{pmatrix} 1 & 3 \end{pmatrix}$ $\mathbf{x}$ =7 Hence m is,

$$\mathbf{m} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} \tag{1.110.0}$$

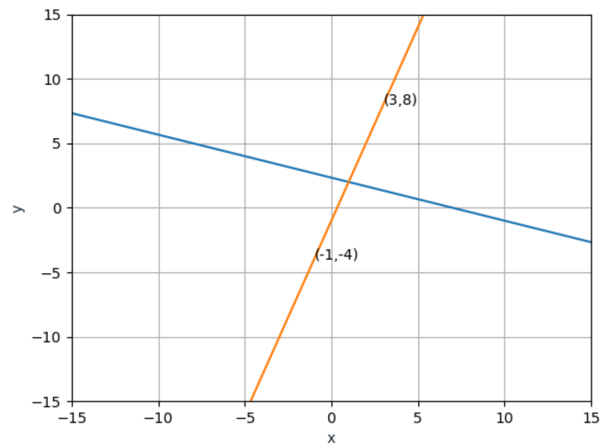

Fig. 1.110.1: Image of a point in 2D line

By property in Figure 1.110.1, the line PR bisects the mirror equation perpendicularly. Hence,

$$2\mathbf{Q} = \mathbf{P} + \mathbf{R} \tag{1.110.0}$$

Where, $\mathbf{Q}$ is the point on the line, $\begin{pmatrix} 1 & 3 \end{pmatrix}$ $\mathbf{x}$ =7. Hence the reflection vector $\mathbf{R}$ is given as,

$$\left(\frac{\mathbf{R}}{2}\right) = \left(\frac{\mathbf{m}\mathbf{m}^T - \mathbf{n}\mathbf{n}^T}{\mathbf{m}^T\mathbf{m} + \mathbf{n}^T\mathbf{n}}\right)\mathbf{P} + c\left(\frac{\mathbf{n}}{\|\mathbf{n}\|^2}\right) \tag{1.110.0}$$

$$\|\mathbf{n}\| = \sqrt{1^2 + 3^2} = \sqrt{10}$$

Substituting these values in equation (1.110) we get,

$$\mathbf{R} = \begin{pmatrix} -1 \\ -4 \end{pmatrix} \tag{1.110.0}$$

Hence, it is the required answer for image of $\mathbf{P}$ in line $\begin{pmatrix} 1 & 3 \end{pmatrix}$ $\mathbf{x}$ =7.

1.111. A ray of light passing through the point $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ reflects on the x-axis at point $\mathbf{A}$ and the reflected ray passes through the point $\begin{pmatrix} 5 \\ 3 \end{pmatrix}$. Find the coordinates of **A**.

Solution: Let point **P** be $\begin{pmatrix}1\\2\end{pmatrix}$ and point **Q** be $\begin{pmatrix}5\\3\end{pmatrix}$ Since, point **A** is on x-axis, its y-coordinate is zero. Assume

$$A = \begin{pmatrix}k\\0\end{pmatrix}$$

Incident vector

$$= \mathbf{P} - \mathbf{A}$$

Reflected vector

$$= \mathbf{Q} - \mathbf{A}$$

Vector along y-axis

$$\mathbf{e_1} = \begin{pmatrix}1\\0\end{pmatrix}$$

Vector along x-axis

$$\mathbf{e_2} = \begin{pmatrix}0\\1\end{pmatrix}$$

Angle between AP and the x axis = 180°- angle between AQ and the x axis,

$$\frac{(\mathbf{P}-\mathbf{A})^{\mathbf{T}}\mathbf{e_2}}{\|\mathbf{P}-\mathbf{A}\|} = \frac{(\mathbf{Q}-\mathbf{A})^{\mathbf{T}}\mathbf{e_2}}{\|\mathbf{Q}-\mathbf{A}\|}$$

$$\frac{\mathbf{P}^{\mathbf{T}}\mathbf{e_2}-\mathbf{A}^{\mathbf{T}}\mathbf{e_2}}{\|\mathbf{P}-\mathbf{A}\|} = \frac{\mathbf{Q}^{\mathbf{T}}\mathbf{e_2}-\mathbf{A}^{\mathbf{T}}\mathbf{e_2}}{\|\mathbf{Q}-\mathbf{A}\|}$$

$$\frac{\begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} - \begin{pmatrix}k & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}}{\left\|\begin{pmatrix}1-k\\2\end{pmatrix}\right\|} = \frac{\begin{pmatrix}5 & 3\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} - \begin{pmatrix}k & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}}{\left\|\begin{pmatrix}5-k\\3\end{pmatrix}\right\|}$$

$$\implies \frac{2}{\sqrt{(1-k)^2+(2)^2}} = \frac{3}{\sqrt{(5-k)^2+(3)^2}}$$

$$\implies 5k^2 + 22k - 91 = 0$$

Solving (1.111) we get: k=2.6, -7

Since, incident ray passes through $\begin{pmatrix}1\\2\end{pmatrix}$ and reflected ray passes through $\begin{pmatrix}5\\3\end{pmatrix}$,

k cannot be negative as reflection takes place in first quadrant.

$$k = 2.6$$

Figure plotted using python code:

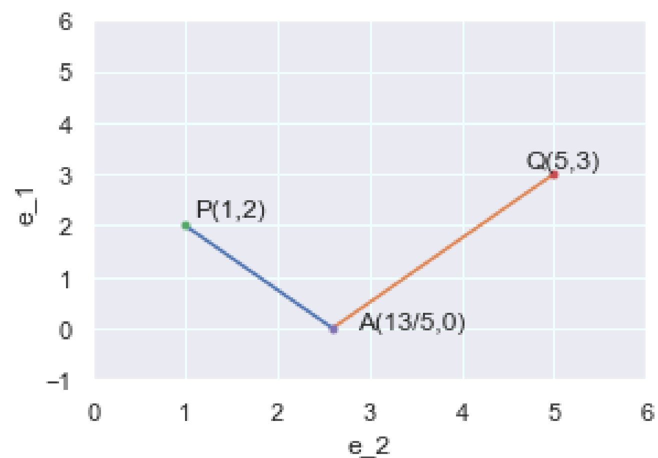

Fig. 1.111.1: Incident and reflected ray vectors plotted via Python code

1.112. Find the equation of a line which passes through the point $\begin{pmatrix}1\\2\\3\end{pmatrix}$ and is parallel to the vector $\begin{pmatrix}3\\2\\-2\end{pmatrix}$.

Solution:

$$\mathbf{a} = \begin{pmatrix}1\\2\\3\end{pmatrix}$$

$$\mathbf{b} = \begin{pmatrix}3\\2\\-2\end{pmatrix}$$

Equation of the desired line in vector form will be

$$\mathbf{r} = \begin{pmatrix}1\\2\\3\end{pmatrix} + k\begin{pmatrix}3\\2\\-2\end{pmatrix}$$

See Fig. 1.112.1

1.113. Find the equaion off the line that passes through $\begin{pmatrix}2\\-1\\4\end{pmatrix}$ and is in the direction $\begin{pmatrix}1\\2\\-1\end{pmatrix}$.

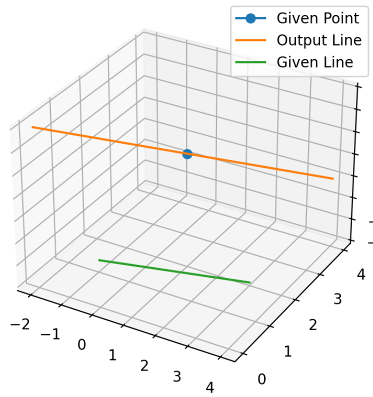

Fig. 1.112.1: Figure depicting provided as well as resultant data

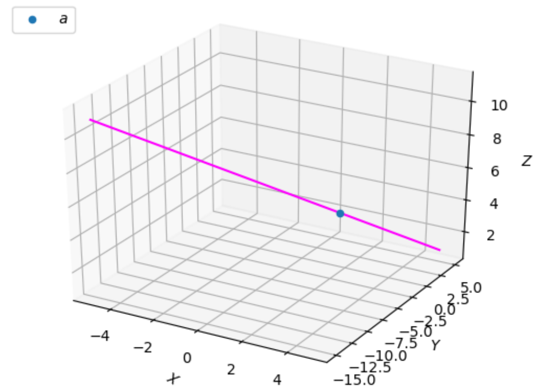

Fig. 1.113.1: The Straight line passing through a point at a particular direction

Solution:
As the line at the direction of (1 2 -1)^T is represented by $\mathbf{b}$, so $\mathbf{b}$ is the direction vector and the point is $\mathbf{a}$ = (2 -1 4)^T through which the line passes through.
From the problem statement, we got

$$\mathbf{a} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$$

So, let us consider another point P with $\mathbf{r}$ = (x y z)^T on the line that passes through $\mathbf{a}$.

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$$

Fig. 1.113.1

1.114. Find the equation of the line given by

$$\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2}.$$

Solution: Let,

$$\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2} = t$$

Equation of the line from the above (1.114) can be expressed as,

$$\implies \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3t+5 \\ 7t-4 \\ 2t+6 \end{pmatrix}$$

It can be further written as,

$$\implies \mathbf{x} = \begin{pmatrix} 5 \\ -4 \\ 6 \end{pmatrix} + t\begin{pmatrix} 3 \\ 7 \\ 2 \end{pmatrix}$$

where,

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Hence, equation 1.114 gives the equation of a line and for t=0, the line passes through the point $\begin{pmatrix} 5 \\ -4 \\ 6 \end{pmatrix}$
Plot of the line which passes through the point when t=0 is given below in Fig. 1.114.1

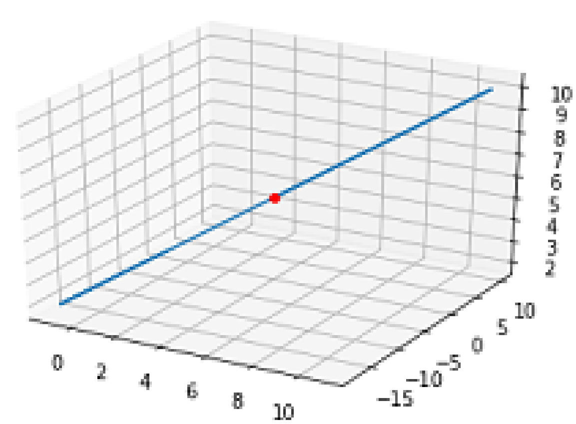

Fig. 1.114.1: Line passing through point (5,-4,6)

1.115. Find the equation of the line passing through the origin and the point $\begin{pmatrix} 5 \\ -2 \\ 3 \end{pmatrix}$.

Solution: Let the points be $\mathbf{O}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ which is the origin and $\mathbf{P}=\begin{pmatrix} 5 \\ -2 \\ 3 \end{pmatrix}$. The vector form of the line passing through $\mathbf{O}$ and $\mathbf{P}$, which is the line passing through the point $\mathbf{O}$ and along direction vector $\mathbf{A}$ is given by

$$\mathbf{r} = \mathbf{O} + k\mathbf{A}$$

$$\implies \mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + k\begin{pmatrix} 5 \\ -2 \\ 3 \end{pmatrix}$$

$$\implies \mathbf{r} = k\begin{pmatrix} 5 \\ -2 \\ 3 \end{pmatrix}$$

where k is a constant multiple. See Fig. 1.115.1

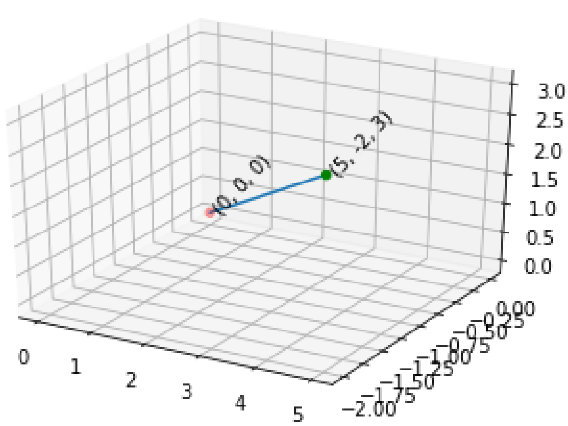

Fig. 1.115.1: Line passing through origin and point (5,-2,3)

1.116. Find the equation of the line passing through the points $\begin{pmatrix} 3 \\ -2 \\ -5 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \\ 6 \end{pmatrix}$.

Solution: Let ,

$$\mathbf{a} = \begin{pmatrix} 3 \\ -2 \\ -5 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 3 \\ -2 \\ 6 \end{pmatrix}$$

Direction vector $\mathbf{A}$ of the points a and b is given by,

$$\mathbf{A} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 0 \\ 0 \\ 11 \end{pmatrix}$$

Parametric equation is given by,

$$\mathbf{x} = \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix} + \lambda\begin{pmatrix} 0 \\ 0 \\ 11 \end{pmatrix}$$

See Fig. 1.116.1

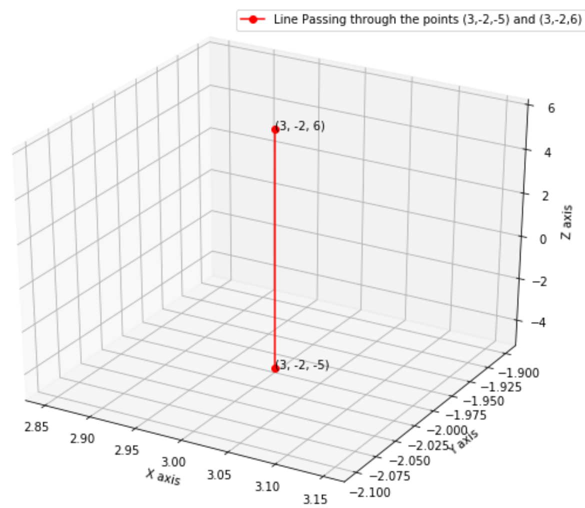

Fig. 1.116.1: Line passing through the points (3,-2,-5) and (3,-2,6)

1.117. Find the angle between the following pair of lines:

$$L_1 : \quad \mathbf{x} = \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} + \lambda_1\begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix}$$

$$L_2 : \quad \mathbf{x} = \begin{pmatrix} 7 \\ -6 \\ 0 \end{pmatrix} + \lambda_2\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

Solution:

a) The direction vectors of the lines are :

$$\mathbf{m_1} = \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix}$$

$$\mathbf{m_2} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

Thus, the angle θ between two vectors is given by

$$\cos\theta = \frac{\mathbf{m_1}^T\mathbf{m_2}}{\|\mathbf{m_1}\|\,\|\mathbf{m_2}\|}$$
$$= \frac{19}{3\times 7}$$
$$\implies \theta = 25.21^\circ$$

b) The direction vectors of the lines are:

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$$
$$\mathbf{m_2} = \begin{pmatrix} 3 \\ -5 \\ -4 \end{pmatrix}.$$

Thus, the angle θ between two vectors is given by

$$\cos\theta = \frac{\mathbf{m_1}^T\mathbf{m_2}}{\|\mathbf{m_1}\|\,\|\mathbf{m_2}\|}$$
$$= \frac{16}{\sqrt{6}\times\sqrt{50}}$$
$$\implies \theta = 22.52^\circ$$

1.118. A person standing at the junction of two straight paths represented by the equations

$$\begin{pmatrix} 2 & -3 \end{pmatrix}\mathbf{x} = 4$$
$$\begin{pmatrix} 3 & 4 \end{pmatrix}\mathbf{x} = 5$$

wants to reach the path whose equation is

$$\begin{pmatrix} 6 & -7 \end{pmatrix}\mathbf{x} = -8$$

in the least time. Find the equation of the path that he should follow.

Solution: Step1: we need to find the solution of equation:

$$\begin{pmatrix} 2 & -3 \end{pmatrix}\mathbf{x} = 4$$
$$\begin{pmatrix} 3 & 4 \end{pmatrix}\mathbf{x} = 5$$

$$\begin{pmatrix} 2 & -3 & 4 \\ 3 & 4 & 5 \end{pmatrix}$$

Transforming the matrix into row-echelon form

$$\begin{pmatrix} 2 & -3 & 4 \\ 3 & 4 & 5 \end{pmatrix} \xleftrightarrow{R1\leftarrow\frac{4}{17}*(R1+\frac{3}{4}R2)} \begin{pmatrix} 1 & 0 & 31/17 \\ 3 & 4 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 31/17 \\ 3 & 4 & 5 \end{pmatrix} \xleftrightarrow{R2\leftarrow\frac{1}{4}(R2-3*R1)} \begin{pmatrix} 1 & 0 & 31/17 \\ 0 & 1 & -2/17 \end{pmatrix}$$

After solving this two equation we will get the junction point, which is intersection of this line segments. Thus, Junction Point is (31/17 -2/17) To reach in the least time, he should follow the shortest path, i.e, perpendicular from the junction point to the line give by this equation:

$$\begin{pmatrix} 6 & -7 \end{pmatrix}\mathbf{x} = 8$$

normal vector to the given line is:

$$\mathbf{n} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 6 \\ -7 \end{pmatrix}$$
$$\mathbf{n} = \begin{pmatrix} 7 \\ 6 \end{pmatrix}$$

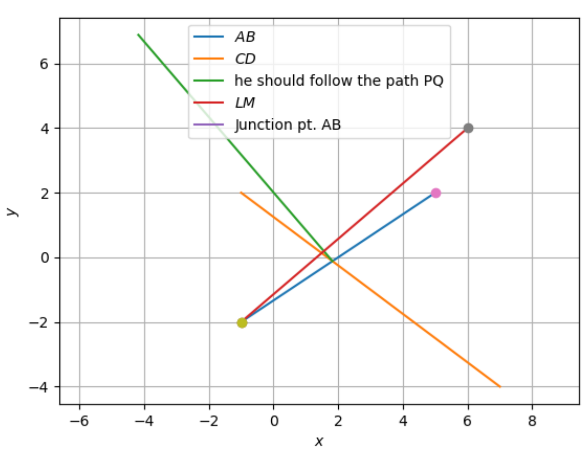

Fig. 1.118.1: The Required path is PQ.

The equation of the line in terms of normal vector passing through a given point is obtained as

$$\mathbf{n}^T(\mathbf{x}-\mathbf{A}) = 0$$
$$\mathbf{n}^T\mathbf{x} = \mathbf{n}^T\mathbf{A}$$

Hence, he should follow this path PQ:

$$\begin{pmatrix}7 & 6\end{pmatrix}\mathbf{x} = \frac{205}{17}$$

1.119. Find the values of p so that the lines

$$\frac{1-x}{3} = \frac{7y-14}{2p} = \frac{z-3}{2},$$
$$\frac{7-7x}{3p} = \frac{y-5}{1} = \frac{6-z}{5}$$

are at right angles.
Solution: Rewriting the given lines as

$$L_1 : \frac{x-1}{-3} = \frac{y-2}{\frac{2}{7}p} = \frac{z-3}{2}$$
$$L_2 : \frac{x-1}{\frac{-3}{7}p} = \frac{y-5}{1} = \frac{z-6}{-5}$$

Using the definition of a line in co-ordinate geometry, we see from the above two equations, the direction vectors $\mathbf{a}$ and $\mathbf{b}$ of the two lines are

$$\mathbf{a} = \begin{pmatrix}-3\\ \frac{2}{7}p\\ 2\end{pmatrix}$$
$$\mathbf{b} = \begin{pmatrix}\frac{-3}{7}p\\ 1\\ -5\end{pmatrix}$$

respectively.
In order for the two lines to be perpendicular, their dot product should be equal to 0 which gives,

$$\frac{\mathbf{a}^T\mathbf{b}}{\|\mathbf{a}\|\,\|\mathbf{b}\|} = 0 \tag{1.119.0}$$

Which in turn gives us,

$$\mathbf{a}^T\mathbf{b} = \frac{11}{7}p - 10$$
$$\implies \frac{11}{7}p - 10 = 0$$
$$\implies p = \frac{70}{11}$$
$$\implies p \approx 6.364$$

See Fig. 1.119.1

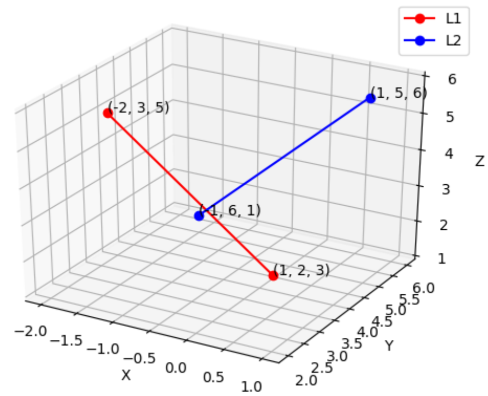

Fig. 1.119.1: The two lines plotted by substituting the value of p found

1.120. Show that the lines

$$\frac{x-5}{7} = \frac{y+2}{-5} = \frac{z}{1},$$
$$\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$$

are perpendicular to each other.
Solution: Let us consider a parameter t. Considering the first equation:

$$\frac{x-5}{7} = \frac{y+2}{-5} = \frac{z}{1} = t$$

Line equation of (1.120) can be written as,

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}7t+5\\5t-2\\t\end{pmatrix} = \begin{pmatrix}5\\2\\0\end{pmatrix} + t\begin{pmatrix}7\\-5\\1\end{pmatrix}$$

From (1.120), the direction vector is given by

$$\mathbf{d_1} = \begin{pmatrix}7\\-5\\1\end{pmatrix}$$

Similarly, let us consider second equation:

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{3} = t$$

Line equation of (1.120) can be written as,

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = t\begin{pmatrix}1\\2\\3\end{pmatrix}$$

From (1.120), the direction vector is given by

$$\mathbf{d_2} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Two lines are perpendicular to each other when the dot product of their direction vectors is 0.

Dot product of direction vectors $\mathbf{d_1}$ and $\mathbf{d_2}$ (from equation (1.120) and (1.120)) is given by:

$$\mathbf{d_1}^T\mathbf{d_2} = (7 \times 1) + (-5 \times 2) + (1 \times 3) = 0$$

$$\implies \boxed{\mathbf{d_1}^T\mathbf{d_2} = 0}$$

From (1.120), as the dot product of direction vectors of the lines is 0 ($\mathbf{d_1^T d_2} = 0$), we can say that the lines are perpendicular to each other.

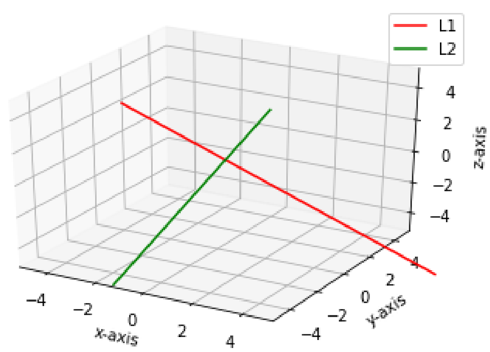

Fig. 1: Lines perpendicular to each other

1.121. Find the shortest distance between the lines

$$L_1: \quad \mathbf{x} = \begin{pmatrix} 1-t \\ t-2 \\ 3-2t \end{pmatrix}$$

$$L_2: \quad \mathbf{x} = \begin{pmatrix} s+1 \\ 2s-1 \\ -2s-1 \end{pmatrix}$$

Solution:

$$L_1 : \mathbf{x} = \begin{pmatrix} 1-t \\ t-2 \\ 3-2t \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$$

$$L_2 : \mathbf{x} = \begin{pmatrix} s+1 \\ 2s-1 \\ -2s-1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} + s\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$$

We have,

$$L_1 : \mathbf{x} = \mathbf{a_1} + t\mathbf{b_1}$$
$$L_2 : \mathbf{x} = \mathbf{a_2} + s\mathbf{b_2}$$

where, $\mathbf{a_i}$, $\mathbf{b_i}$ are positional and slope vectors of line L_i respectively.

As $\mathbf{b_1} \neq \lambda\mathbf{b_2}$, lines L_1 and L_2 are not parallel to each other.

Now, let us assume that L_1 and L_2 are intersecting at a point. Therefore,

$$\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} + s\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$$

$$s\begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix}$$

$$\begin{pmatrix} -1 & -1 \\ -2 & 1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix}$$

Using Gaussian elimination method:

$$E = E_{32}E_{31}E_{21} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ \frac{-1}{2} & 1 & \frac{3}{4} \end{pmatrix}$$

$$E\begin{pmatrix} -1 & -1 & : & 0 \\ -2 & 1 & : & 1 \\ 2 & -2 & : & -4 \end{pmatrix} = \begin{pmatrix} -1 & -1 & : & 0 \\ 0 & 3 & : & 1 \\ 0 & 0 & : & -2 \end{pmatrix}$$

From (1.121) it is clear that the system of linear equations are inconsistent. Therefore L_1 and L_2 are not intersecting at any point.
Hence our assumption was wrong, L_1, L_2 are **skew lines**.
Let d be the shortest distance between L_1, L_2 and $\mathbf{p_1}$, $\mathbf{p_2}$ be the positional vectors of its end points.

For d to be the shortest, we know that,

$$\mathbf{b_1}^T(\mathbf{p_2} - \mathbf{p_1}) = 0$$
$$\mathbf{b_2}^T(\mathbf{p_2} - \mathbf{p_1}) = 0$$
$$\mathbf{b_1}^T\left((\mathbf{a_2} - \mathbf{a_1}) + \begin{pmatrix}\mathbf{b_2} & \mathbf{b_1}\end{pmatrix}\begin{pmatrix}s \\ -t\end{pmatrix}\right) = 0$$
$$\mathbf{b_2}^T\left((\mathbf{a_2} - \mathbf{a_1}) + \begin{pmatrix}\mathbf{b_2} & \mathbf{b_1}\end{pmatrix}\begin{pmatrix}s \\ -t\end{pmatrix}\right) = 0$$
$$\mathbf{B} = \begin{pmatrix}\mathbf{b_2} & \mathbf{b_1}\end{pmatrix}, \mathbf{B^T} = \begin{pmatrix}\mathbf{b_2}^T \\ \mathbf{b_1}^T\end{pmatrix}$$

By combining equation (1.121) and (1.121) and writing in terms of $\mathbf{B}$ and $\mathbf{B}^T$ using (1.121) we get:

$$\mathbf{B}^T\mathbf{B}\begin{pmatrix}s \\ -t\end{pmatrix} = \mathbf{B}^T(\mathbf{a_1} - \mathbf{a_2})$$

By putting the values of $\mathbf{a_1}, \mathbf{a_2}, \mathbf{b_1}$ and $\mathbf{b_2}$ in equation (1.121) we get:

$$\begin{pmatrix}5 & 6 \\ 9 & 5\end{pmatrix}\begin{pmatrix}s \\ -t\end{pmatrix} = \begin{pmatrix}-9 \\ -10\end{pmatrix}$$

Solving equation (1.121) we get:

$$s = \frac{-15}{29}, t = \frac{31}{29}$$

By putting the values of t and s in equation (1.121) and (1.121) respectively we get:

$$\mathbf{p_1} = \begin{pmatrix}\frac{-17}{250} \\ \frac{-93}{100} \\ \frac{43}{50}\end{pmatrix}, \mathbf{p_2} = \begin{pmatrix}\frac{12}{25} \\ -2 \\ \frac{17}{500}\end{pmatrix}$$

Hence the shortest distance d between the two skew lines is :

$$d = \|\mathbf{p_2} - \mathbf{p_1}\| = 1.4855$$

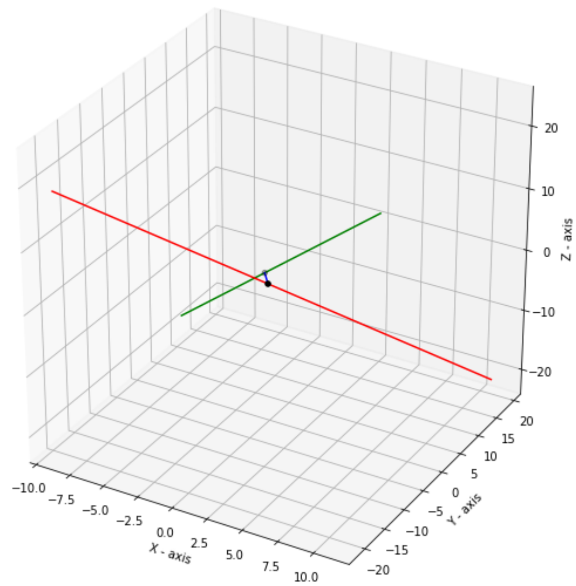

Fig. 1: 3-D plot for the skew lines and the shortest distance between them.

1.122. In each of the following cases, determine the normal to the plane and the distance from the origin.

a) $\begin{pmatrix}0 & 0 & 1\end{pmatrix}\mathbf{x} = 2$ c) $\begin{pmatrix}0 & 5 & 0\end{pmatrix}\mathbf{x} = -8$

b) $\begin{pmatrix}1 & 1 & 1\end{pmatrix}\mathbf{x} = 1$ d) $\begin{pmatrix}2 & 3 & -1\end{pmatrix}\mathbf{x} = 5$

Solution:

(a) $\begin{pmatrix}0 & 0 & 1\end{pmatrix}\mathbf{x} = 2$

$$\mathbf{n} = \begin{pmatrix}0 \\ 0 \\ 1\end{pmatrix}$$
$$c = 2$$

shortest distance from origin =

$$\frac{|2|}{\sqrt{0^2 + 0^2 + 1^2}} = 2$$

(b) $\begin{pmatrix}1 & 1 & 1\end{pmatrix}\mathbf{x} = 1$

$$\mathbf{n} = \begin{pmatrix}1 \\ 1 \\ 1\end{pmatrix}$$
$$c = 1$$

shortest distance from origin =

$$\frac{|1|}{\sqrt{1^2 + 1^2 + 1^1}} = \frac{1}{\sqrt{3}}$$

(c) $\begin{pmatrix}0 & 5 & 0\end{pmatrix}\mathbf{x} = -8$

$$\mathbf{n} = \begin{pmatrix}0 \\ 5 \\ 0\end{pmatrix}$$
$$c = -8$$

shortest distance from origin =

$$\frac{|-8|}{\sqrt{0^2+5^2+0^2}} = \frac{8}{5}$$

(d) $\begin{pmatrix} 2 & 3 & -1 \end{pmatrix} \mathbf{x} = 5$

$$\mathbf{n} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$$

$$c = 5$$

shortest distance from origin =

$$\frac{|5|}{\sqrt{2^2+3^2+(-1)^2}} = \frac{5}{\sqrt{14}}$$

1.123. Find the coordinates of the point where the line through $\begin{pmatrix} 3 \\ -4 \\ -5 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$ crosses the plane

$$\begin{pmatrix} 2 & 1 & 1 \end{pmatrix} \mathbf{x} = 7$$

Solution: We know that vector equation of line passing through two points , say A and B is

$$\mathbf{x} = \mathbf{A} + \lambda \left(\mathbf{B} - \mathbf{A} \right)$$

We also know that equation of a plane is

$$\mathbf{n}^T \mathbf{x} = c$$

Substituting (1.123) in (1.123) as line passes through the plane we can get the point of contact.
Let us first findout the equation of line passing through two given points using (1.123)

$$\mathbf{x} = \begin{pmatrix} 3 \\ -4 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} 2-3 \\ -3+4 \\ 1+5 \end{pmatrix}$$

$$\mathbf{x} = \begin{pmatrix} 3 \\ -4 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 6 \end{pmatrix}$$

Now let us construct the equation of plane from the given data. Using the values we can construct

$$\mathbf{n} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

Now using (1.123) , (1.123) in (1.123)

$$\begin{pmatrix} 2 & 1 & 1 \end{pmatrix} \left(\begin{pmatrix} 3 \\ -4 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 6 \end{pmatrix} \right) = 7$$

solving (1.123) we get

$$6 - 4 - 5 - 2\lambda + \lambda + 6\lambda = 7$$
$$5\lambda = 10$$
$$\lambda = 2$$

Now substituting the value of λ in (1.123) we get the point of contact of line on plane

$$\mathbf{x} = \begin{pmatrix} 1 \\ -2 \\ 7 \end{pmatrix}$$

1.124. Find the distance of the point $\begin{pmatrix} -1 \\ -5 \\ -10 \end{pmatrix}$ from the point of intersection of the line

$$\mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}$$

and the plane

$$\begin{pmatrix} 1 & -1 & 1 \end{pmatrix} \mathbf{x} = 5$$

Solution:
We know that equation of the line passing through given a point and a plane

$$\mathbf{r} = \mathbf{a} + \lambda \mathbf{m} \tag{1.124.0}$$

Also we can find direction vector from the Cartesian form of equation

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} \tag{1.124.0}$$

This can be expressed as

$$\mathbf{x} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \lambda \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{1.124.0}$$

where $\mathbf{a} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ is a point on given line and

$\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is the direction vector.

Distance between the point and point of intersection.

$$D = \sqrt{(x2 - x1)^2 + (y2 - y1)^2 + (z2 - z1)^2} \tag{1.124.0}$$

Writing given equation (1.0.1) in vector form as

$$\mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} \tag{1.124.0}$$

substitute (3.0.1) in (1.0.2) to find the value of λ

$$\begin{pmatrix} 1 & -1 & 1 \end{pmatrix} \left\{ \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} \right\} = 5 \tag{1.124.0}$$

by multiplying the row vector with the first column vector

$$1(2) - 1(-1) + 1(2) = 5 \tag{1.124.0}$$

by multiplying the row vector with the coefficient column vector of lambda

$$1(3\lambda) - 1(4\lambda) + 1(2\lambda) = \lambda \tag{1.124.0}$$

we get as

$$\lambda = 0 \tag{1.124.0}$$

The line intersects the plane at

$$\mathbf{x}_0 = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \tag{1.124.0}$$

Finally the distance between the point $\mathbf{P} = \begin{pmatrix} -1 \\ -5 \\ -10 \end{pmatrix}$ and intersection point $\mathbf{x}_0 = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ is

$$\|x_0 - P\| = \sqrt{(2+1)^2 + (-1+5)^2 + (2+10)^2} \tag{1.124.0}$$

$$\|x_0 - P\| = 13 \tag{1.124.0}$$

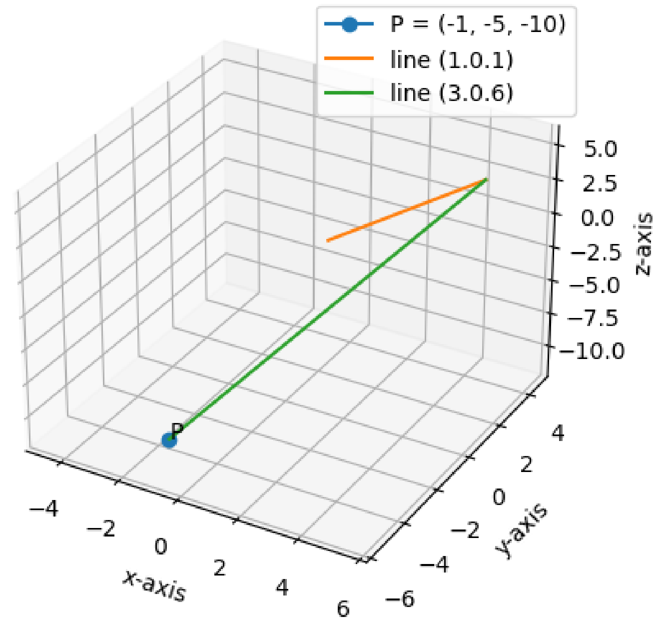

Fig. 1: Equation of line passing through point x_0 and intersection to line (1.0.1)

1.125. Find the vector equation of the line passing through the point $\begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}$ and perpendicular to the two lines

$$\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7},$$
$$\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}$$

Solution: The line passes through $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}$

$$\mathbf{x} = \begin{pmatrix} 8 \\ -19 \\ 10 \end{pmatrix} + \lambda_1 \begin{pmatrix} 3 \\ -16 \\ 7 \end{pmatrix}$$
$$\mathbf{x} = \begin{pmatrix} 15 \\ 29 \\ 5 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ 8 \\ -5 \end{pmatrix}$$

Let $\mathbf{n}$ be the normal vector to both lines. If $\mathbf{m_1}$ and $\mathbf{m_2}$ are the direction vectors of the lines,then

$$\mathbf{m_1}^T \mathbf{n} = 0$$
$$\mathbf{m_2}^T \mathbf{n} = 0$$

Let the matrix $\mathbf{M}$ be

$$\mathbf{M} = \begin{pmatrix} \mathbf{m_1}^T \\ \mathbf{m_2}^T \end{pmatrix}$$

$$\mathbf{m_1} = \begin{pmatrix} 3 \\ -16 \\ 7 \end{pmatrix}, \mathbf{m_2} = \begin{pmatrix} 3 \\ 8 \\ -5 \end{pmatrix}, \mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$$

$$\mathbf{Mn} = 0$$

The matrix form is

$$\begin{pmatrix} 3 & -16 & 7 \\ 3 & 8 & -5 \end{pmatrix}$$

$$\begin{pmatrix} 3 & -16 & 7 \\ 3 & 8 & -5 \end{pmatrix} \xleftrightarrow{R_2=R_1-R_2} \begin{pmatrix} 3 & -16 & 7 \\ 0 & -24 & 12 \end{pmatrix}$$

$$\begin{pmatrix} 3 & -16 & 7 \\ 0 & -24 & 12 \end{pmatrix} \xleftrightarrow{R_2=\frac{R_2}{2}} \begin{pmatrix} 3 & -16 & 7 \\ 0 & -2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 3 & -16 & 7 \\ 0 & -2 & 1 \end{pmatrix} \xleftrightarrow{R_1=R_1-8R_2} \begin{pmatrix} 3 & 0 & -1 \\ 0 & -2 & 1 \end{pmatrix}$$

We have 2 equations and 3 unknowns , we will have parametric solution

$$n_1 = \frac{k}{3}$$
$$n_2 = \frac{k}{2}$$
$$n_3 = k$$
$$\mathbf{n} = \frac{k}{6}\begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}$$

The equation of required line is

$$\begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}$$

See Fig. 1

1.126. Distance between the two planes

$$\begin{pmatrix} 2 & 3 & 4 \end{pmatrix} \mathbf{x} = 4$$
$$\begin{pmatrix} 4 & 6 & 8 \end{pmatrix} \mathbf{x} = 12$$

Solution: So, the distance between the given

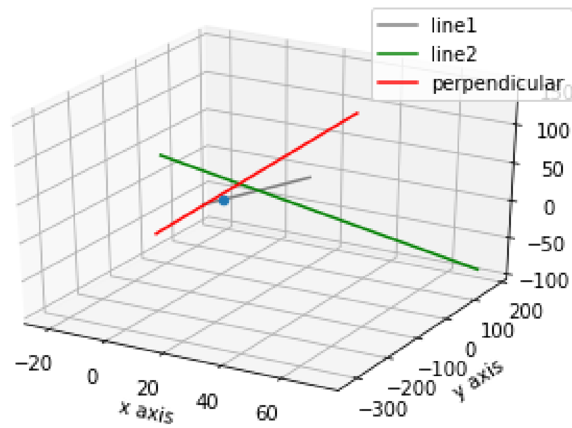

Fig. 1: Perpendicular Line

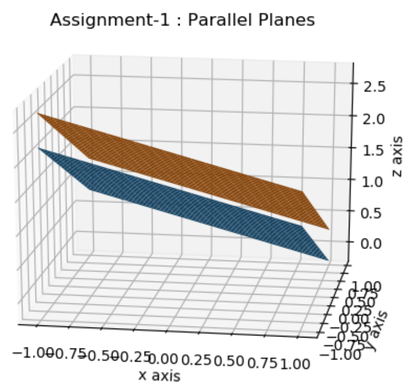

Fig. 1: Example of Two parallel planes

planes is:

$$\frac{|4-6|}{\sqrt{2^2+3^2+4^2}} = \frac{2}{\sqrt{29}}$$

See Fig. 1

a) 2
b) 4
c) 8
d) $\frac{2}{\sqrt{29}}$

1.127. If $\mathbf{O}$ be the origin and the coordinates of $\mathbf{P}$ be $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, then find the equation of the plane passing through $\mathbf{P}$ and perpendicular to OP.

Solution: The normal vector to the plane is

$$\mathbf{n} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Thus, the equation of the plane is given by

$$\mathbf{n}^T(\mathbf{x}-\mathbf{P}) = 0$$

$$\implies \begin{pmatrix}1 & 2 & 3\end{pmatrix}\mathbf{x} = \begin{pmatrix}1 & 2 & 3\end{pmatrix}\begin{pmatrix}1\\2\\3\end{pmatrix}$$

$$\implies \begin{pmatrix}1 & 2 & 3\end{pmatrix}\mathbf{x} = 14$$

and is plotted in Fig. 1.

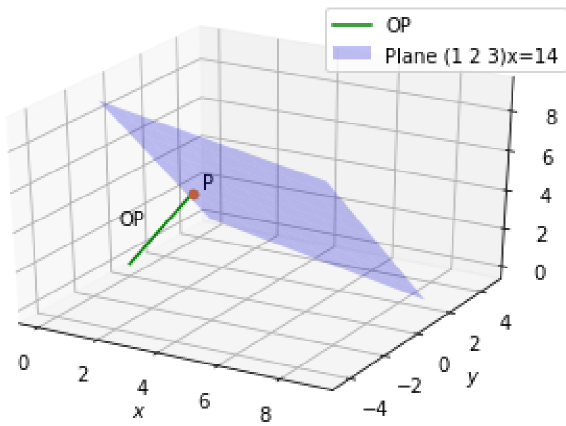

Fig. 1: Plot of the plane

1.128. If the point $\begin{pmatrix}3\\4\end{pmatrix}$ lies on the graph of the equation $3y = ax + 7$, find the value of a

Solution:

The given equation can be expressed as

$$\implies \begin{pmatrix}-a & 3\end{pmatrix}\mathbf{x} = 7$$

$\because$ the given point $\mathbf{P}$=$\begin{pmatrix}3\\4\end{pmatrix}$ satisfies the above equation,

$$\begin{pmatrix}-a & 3\end{pmatrix}\begin{pmatrix}3\\4\end{pmatrix} = 7$$

$$\implies -3a + 12 = 7$$

$$\implies a = \frac{5}{3}$$

Hence,the equation can be written as

$$\begin{pmatrix}\frac{-5}{3} & 3\end{pmatrix}\mathbf{x} = 7$$

and is plotted in Fig. 1.

1.129. Given the linear equation $\begin{pmatrix}2 & 3\end{pmatrix}\mathbf{x}$–$8 = 0$, write another linear equation in two variables such that the geometrical representation of the pair so formed is:

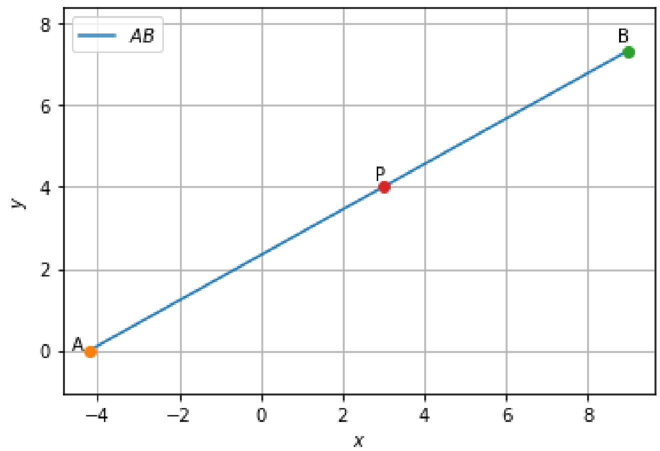

Fig. 1: Line AB

a) intersecting lines
b) parallel lines
c) coincident lines

Solution: Consider the lines

a)

$$\begin{pmatrix}2 & 3\end{pmatrix}\mathbf{x} = 8$$

$$\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x} = 4$$

The above equations can be expressed as the matrix equation

$$\begin{pmatrix}2 & 3\\3 & 2\end{pmatrix}\mathbf{x} = \begin{pmatrix}8\\4\end{pmatrix}$$

The augmented matrix for the above equation is row reduced as follows

$$\begin{pmatrix}2 & 3 & 8\\3 & 2 & 4\end{pmatrix} \xleftrightarrow{R_1\leftarrow\frac{R_1}{2}} \begin{pmatrix}1 & \frac{3}{2} & 4\\3 & 2 & 4\end{pmatrix}$$

$$\xleftrightarrow{R_2\leftarrow R_2-3R_1} \begin{pmatrix}1 & \frac{3}{2} & 4\\0 & \frac{-5}{2} & -8\end{pmatrix}$$

$$\xleftrightarrow{R_2\leftarrow\frac{2R_2}{-5}} \begin{pmatrix}1 & \frac{3}{2} & 4\\0 & 1 & \frac{16}{5}\end{pmatrix}$$

$$\xleftrightarrow{R_1\leftarrow R_1-\frac{3R_2}{2}} \begin{pmatrix}1 & 0 & \frac{-4}{5}\\0 & 1 & \frac{16}{5}\end{pmatrix}$$

$\because$ row reduction of the 2×3 matrix

$$\begin{pmatrix}2 & 3 & 8\\3 & 2 & 4\end{pmatrix}$$

results in a matrix with 2 nonzero rows, its rank is 2. Similarly, the rank of the matrix

$$\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$$

is 2.

$$\because Rank\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} = Rank\begin{pmatrix} 2 & 3 & 8 \\ 3 & 2 & 4 \end{pmatrix}$$
$$= dim\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$$
$$= 2$$

$\therefore$ the lines in (1.129a) intersect as can be seen from Fig. 1.

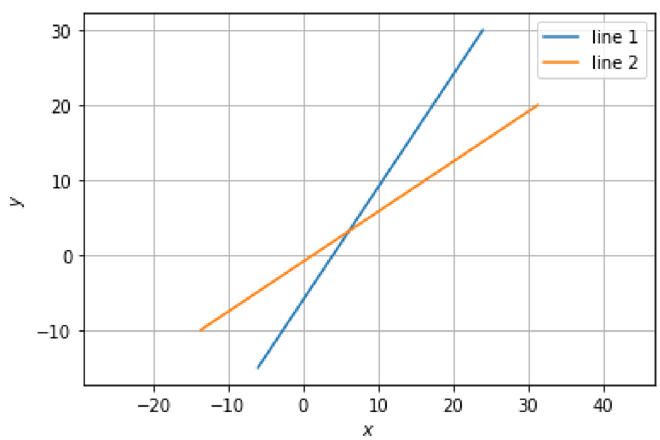

Fig. 1: INTERSECTING LINES.

b)

$$\begin{pmatrix} 2 & 3 \end{pmatrix}\mathbf{x} = 8$$
$$\begin{pmatrix} 4 & 6 \end{pmatrix}\mathbf{x} = 16$$

The above equations can be expressed as the matrix equation

$$\begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 8 \\ 16 \end{pmatrix}$$

The augmented matrix for the above equation is row reduced as follows

$$\begin{pmatrix} 2 & 3 & 8 \\ 4 & 6 & 16 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{R_1}{2}} \begin{pmatrix} 1 & \frac{3}{2} & 4 \\ 4 & 6 & 16 \end{pmatrix}$$
$$\xleftrightarrow{R_2 \leftarrow R_2 - 4R_1} \begin{pmatrix} 1 & \frac{3}{2} & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

$\because$ row reduction of the 2×3 matrix

$$\begin{pmatrix} 2 & 3 & 8 \\ 4 & 6 & 16 \end{pmatrix}$$

results in a matrix with 1 nonzero rows, its rank is 1. Similarly, the rank of the matrix

$$\begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}$$

is also 1.

$$\because Rank\begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix} = Rank\begin{pmatrix} 2 & 3 & 8 \\ 4 & 6 & 16 \end{pmatrix} = 1$$
$$= dim\begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix} = 1$$

$\therefore$ the lines in (1.129b) coincide as can be seen from Fig. 2.

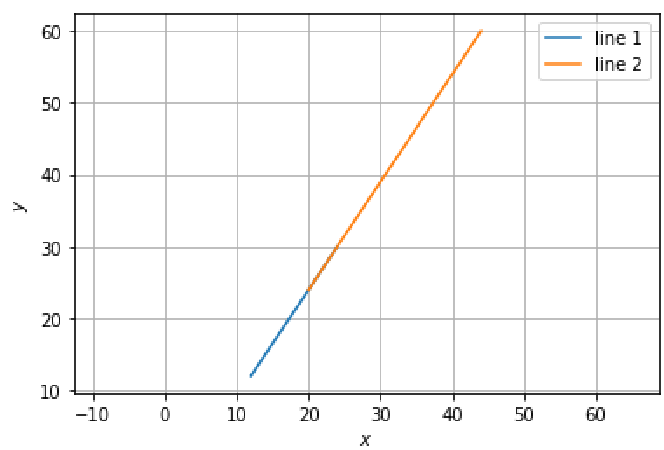

Fig. 2: COINCIDENT LINES

c)

$$\begin{pmatrix} 2 & 3 \end{pmatrix}\mathbf{x} = 8$$
$$\begin{pmatrix} 2 & 3 \end{pmatrix}\mathbf{x} = 4$$

The above equations can be expressed as the matrix equation

$$\begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 8 \\ 4 \end{pmatrix}$$

The augmented matrix for the above equa-

tion is row reduced as follows

$$\begin{pmatrix}2 & 3 & 8\\2 & 3 & 4\end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{R_1}{2}} \begin{pmatrix}1 & \frac{3}{2} & 4\\2 & 3 & 4\end{pmatrix}$$
$$\xleftrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{pmatrix}1 & \frac{3}{2} & 4\\0 & 0 & -4\end{pmatrix}$$

$\because$ row reduction of the 2×3 matrix

$$\begin{pmatrix}2 & 3 & 8\\2 & 3 & 4\end{pmatrix}$$

results in a matrix with 2 nonzero rows, its rank is 2. Similarly, the rank of the matrix

$$\begin{pmatrix}2 & 3\\2 & 3\end{pmatrix}$$

is 1.

$$\because Rank\begin{pmatrix}2 & 3\\2 & 3\end{pmatrix} \neq Rank\begin{pmatrix}2 & 3 & 8\\2 & 3 & 4\end{pmatrix}$$

$\therefore$ the lines in (1.129c) are parallel as can be seen from Fig. 3.

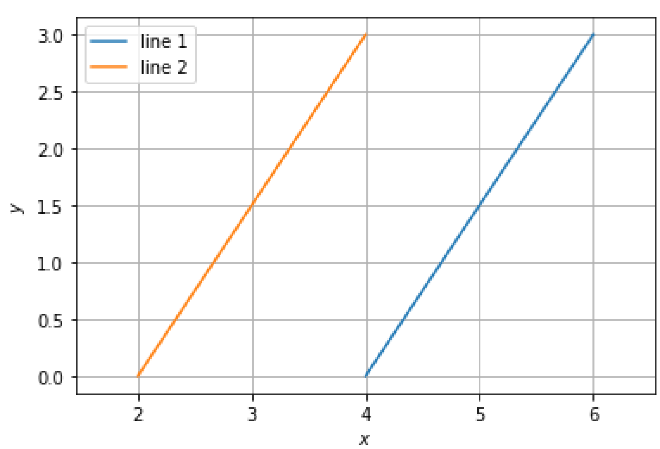

Fig. 3: PARALLEL LINES

1.130. Find the equation of the line passing through $\begin{pmatrix}-3\\5\end{pmatrix}$ and perpendicular to the line through the points $\begin{pmatrix}2\\5\end{pmatrix}$ and $\begin{pmatrix}-3\\6\end{pmatrix}$. **Solution:** From the given information, for

$$\mathbf{A} = \begin{pmatrix}2\\5\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-3\\6\end{pmatrix}, \mathbf{m} \quad = \mathbf{A} - \mathbf{B}$$
$$= \begin{pmatrix}5\\-1\end{pmatrix}$$
$$\implies \mathbf{n} = \begin{pmatrix}5\\-1\end{pmatrix}$$

Let

$$\mathbf{P} = \begin{pmatrix}-3\\5\end{pmatrix}$$

The equation of the desired line is then obtained as

$$\mathbf{n}^T (\mathbf{x} - \mathbf{P}) = 0$$
$$\implies \begin{pmatrix}5 & -1\end{pmatrix}\mathbf{x} = -20$$

and plotted in Fig. 1

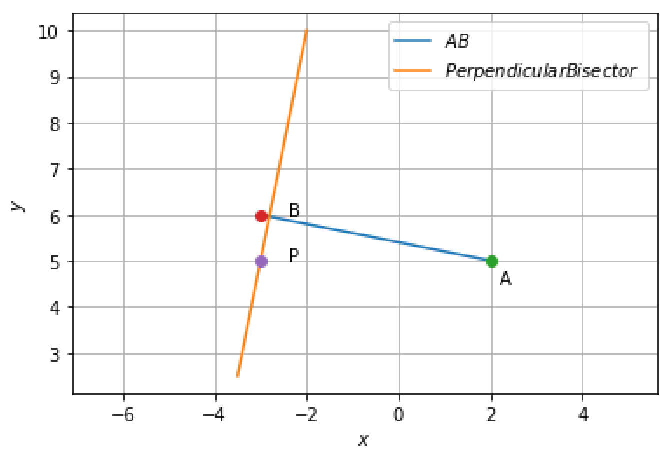

Fig. 1: Perpendicular Bisector

1.131. Find the distance between the parallel lines

$$\begin{pmatrix}15 & 8\end{pmatrix}\mathbf{x} = 34$$
$$\begin{pmatrix}15 & 8\end{pmatrix}\mathbf{x} = -31$$

Solution:
The distance between the two parallel lines is

$$d = \frac{|c_2 - c_1|}{\|\mathbf{n}\|}$$

By substituting the given values

$$\mathbf{n} = \begin{pmatrix}15\\8\end{pmatrix}, c_1 = 34, c_2 = -31$$

we get

$$d = \frac{65}{17}$$

1.132. Find the alue of p so that the three lines

$$\begin{pmatrix}3 & 1\end{pmatrix}\mathbf{x} = 2$$
$$\begin{pmatrix}p & 2\end{pmatrix}\mathbf{x} = 3$$
$$\begin{pmatrix}2 & -1\end{pmatrix}\mathbf{x} = 3$$

may intersect at one point. **Solution:** The given system of equations can be expressed in matrix form as

$$\begin{pmatrix}3 & 1\\ p & 2\\ 2 & -1\end{pmatrix}\mathbf{x} = \begin{pmatrix}2\\3\\3\end{pmatrix}$$

Assuming the system of equations are consistent , lets reduce the augmented matrix to find the value of p

$$\begin{pmatrix}3 & 1 & 2\\ p & 2 & 3\\ 2 & -1 & 3\end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{pmatrix}3 & 1 & 2\\ P-6 & 0 & -1\\ 2 & -1 & 3\end{pmatrix}$$

$$\xleftrightarrow{R_3 \leftarrow 3R_3 - 2R_1} \begin{pmatrix}3 & 1 & 2\\ p-6 & 0 & -1\\ 0 & -5 & 5\end{pmatrix}$$

$$\xleftrightarrow{R_3 \leftarrow -\frac{R_3}{5}} \begin{pmatrix}3 & 1 & 2\\ p-6 & 0 & -1\\ 0 & 1 & -1\end{pmatrix}$$

$$\begin{pmatrix}3 & 1 & 2\\ p-6 & 0 & -1\\ 0 & 1 & -1\end{pmatrix}$$

Since the system of equations are assumed consistent,

$$p - 6 = -1$$
$$\implies p = 5$$

Thus, the system of equations is given by

$$\begin{pmatrix}3 & 1\end{pmatrix}\mathbf{x} = 2$$
$$\begin{pmatrix}5 & 2\end{pmatrix}\mathbf{x} = 3$$
$$\begin{pmatrix}2 & -1\end{pmatrix}\mathbf{x} = 3$$

and plotted in Fig. 1.

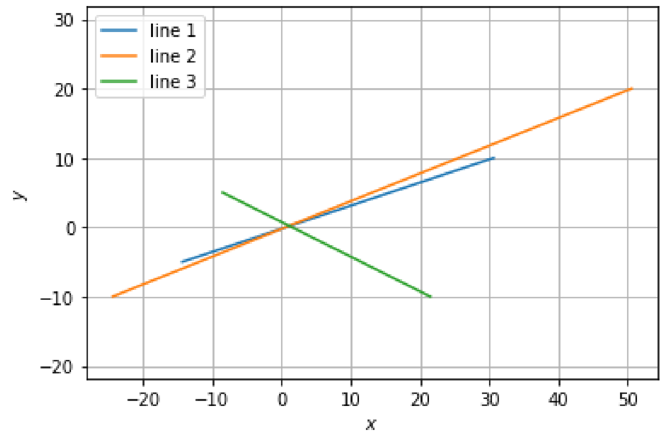

Fig. 1: INTERSECTING LINES.

1.133. Find the shortest distance between the lines

$$L_1: \quad \mathbf{x} = \begin{pmatrix}1\\2\\1\end{pmatrix} + \lambda_1 \begin{pmatrix}1\\-1\\1\end{pmatrix}$$

$$L_2: \quad \mathbf{x} = \begin{pmatrix}2\\-1\\-1\end{pmatrix} + \lambda_2 \begin{pmatrix}2\\1\\2\end{pmatrix}$$

Solution: The lines will intersect if

$$\begin{pmatrix}1\\2\\1\end{pmatrix} + \lambda_1 \begin{pmatrix}1\\-1\\1\end{pmatrix} = \begin{pmatrix}2\\-1\\-1\end{pmatrix} + \lambda_2 \begin{pmatrix}2\\1\\2\end{pmatrix}$$

$$\implies \begin{pmatrix}1 & 2\\ -1 & 1\\ 1 & 2\end{pmatrix}\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix} = \begin{pmatrix}1\\-3\\-2\end{pmatrix}$$

The augmented matrix for the above equation is row reduced form

$$\begin{pmatrix}1 & 2 & 1\\ -1 & 1 & -3\\ 1 & 2 & 2\end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 + R_1} \begin{pmatrix}1 & 2 & 1\\ 0 & 3 & -2\\ 1 & 2 & 2\end{pmatrix}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - R_1} \begin{pmatrix}1 & 2 & 1\\ 0 & 3 & -2\\ 0 & 0 & 1\end{pmatrix}$$

$\therefore$ The above matrix has rank=3. Hence the lines do not interest. Since they are not parallel, they are skew lines as can be seen in Fig. 1. $\therefore$

the distance between given two lines are

$$\frac{\left|\mathbf{n}^T(\mathbf{A_2}-\mathbf{A_1})\right|}{\|\mathbf{n}\|} = \frac{\left|(\mathbf{A_2}-\mathbf{A_1})^T(\mathbf{m_1}\times\mathbf{m_2})\right|}{\|\mathbf{m_1}\times\mathbf{m_2}\|} = 4.5$$

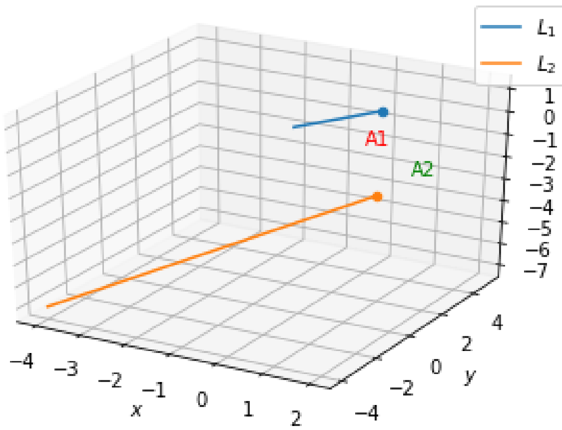

Fig. 1: Skew Lines

1.134. Find the equation of the planes

a) that passes through the point $\begin{pmatrix}1\\0\\-2\end{pmatrix}$ and the normal to the plane is $\begin{pmatrix}1\\1\\-1\end{pmatrix}$.

Solution: The equation of the plane is given by

$$\mathbf{n}^T(\mathbf{x}-\mathbf{A}) = 0$$

$$\implies \begin{pmatrix}1 & 1 & -1\end{pmatrix}\mathbf{x} = \begin{pmatrix}1 & 1 & -1\end{pmatrix}\begin{pmatrix}1\\0\\-2\end{pmatrix}$$

$$\text{or, } \begin{pmatrix}1 & 1 & -1\end{pmatrix}\mathbf{x} = 3$$

and plotted in Fig. 1.

b) that passes through the point $\begin{pmatrix}1\\4\\6\end{pmatrix}$ and the normal vetor the plane is $\begin{pmatrix}1\\-2\\1\end{pmatrix}$. **Solution:** From the given information, the equation of

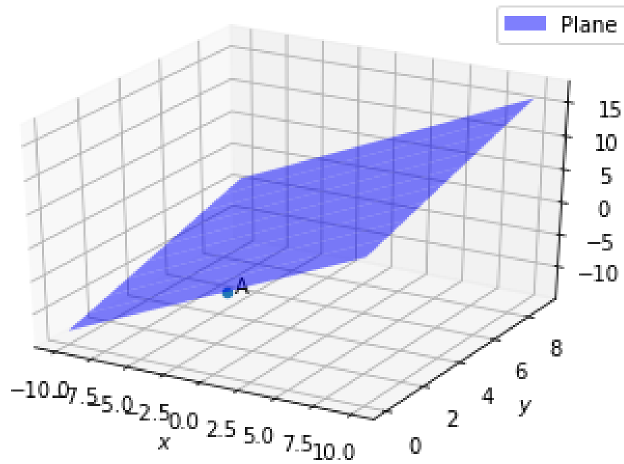

Fig. 1: Plot of the plane

the plane is

$$\mathbf{n}^T(\mathbf{x}-\mathbf{A}) = 0$$

$$\begin{pmatrix}1 & -2 & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}1 & -2 & 1\end{pmatrix}\begin{pmatrix}1\\4\\6\end{pmatrix}$$

$$\begin{pmatrix}1 & -2 & 1\end{pmatrix}\mathbf{x} = -1$$

The corresponding plot is available in Fig. 2.

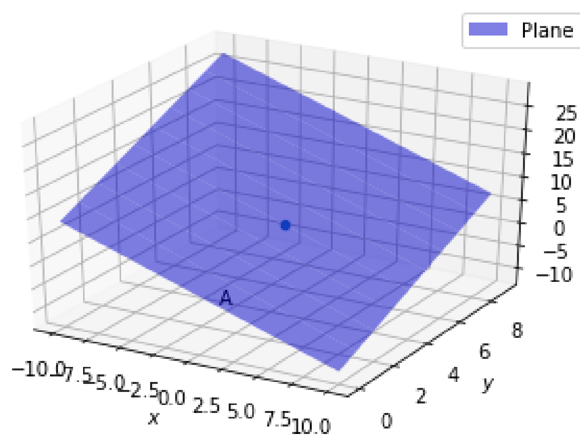

Fig. 2: Plot of the plane

1.135. Find the equation of the plane through the intersection of the planes $\begin{pmatrix}3 & -1 & 2\end{pmatrix}\mathbf{x} = 4$ and $\begin{pmatrix}1 & 1 & 1\end{pmatrix}\mathbf{x} = -2$ and the pont $\begin{pmatrix}2\\2\\1\end{pmatrix}$.

Solution:

From the given information,

$$\mathbf{n_1} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}, \mathbf{n_2} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$c_1 = 4, c_2 = -2$$

The intersection of the planes is given by

$$\mathbf{n_1^T x} + \lambda \mathbf{n_2^T x} = c_1 + \lambda c_2$$

$$\implies (\mathbf{n_1^T} + \lambda \mathbf{n_2^T})\mathbf{x} = c_1 + \lambda c_2$$

yielding

$$\lambda = \frac{(c_1 - \mathbf{An_1^T})}{(\mathbf{An_2^T} - c_2)} = \frac{-2}{7}$$

$\therefore$ By substituting the numerical values in (1.135), the desired equation of the plane is

$$\begin{pmatrix} 19 & -9 & 12 \end{pmatrix} \mathbf{x} = 32$$

which is plotted in Fig. 1

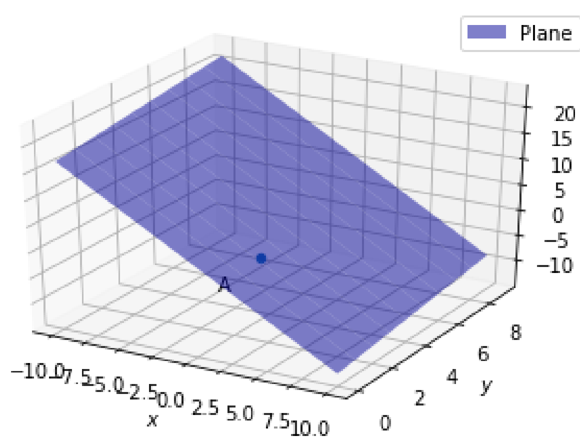

Fig. 1: Plot of the plane

1.136. Find the angle between the planes whose equations are $\begin{pmatrix} 2 & 2 & -3 \end{pmatrix}\mathbf{x} = 5$ and $\begin{pmatrix} 3 & -3 & 5 \end{pmatrix}\mathbf{x} = 3$

Solution: From the given info, The normal vectors of the given planes are

$$\mathbf{n}_1 = \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix}, \mathbf{n}_2 = \begin{pmatrix} 3 \\ -3 \\ 5 \end{pmatrix}, \qquad (1.136.0)$$

Let θ be angle between vectors $\mathbf{n}_1$,$\mathbf{n}_2$. Then,

$$\theta = \cos^{-1}\left(\frac{\mathbf{n}_1^T \mathbf{n}_2}{\|\mathbf{n_1}\| \, \|\mathbf{n}_2\|}\right) = \cos^{-1}\left(-\frac{1}{2}\right)$$

$$\implies \theta = 120^\circ$$

Fig. 1 shows the two planes.

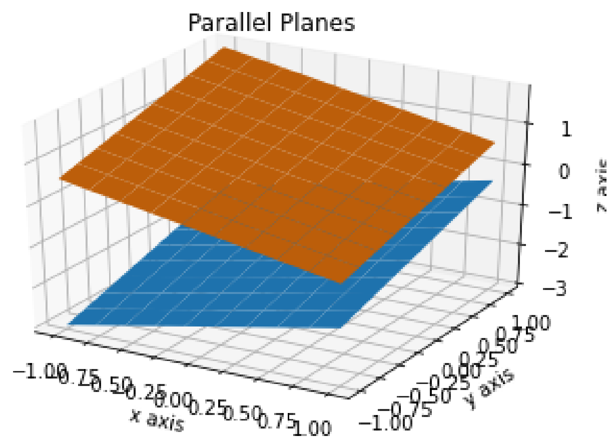

Fig. 1: Parallel planes

1.137. Which of the following pairs of linear equations has a unique solution, no solution, or infinitely many solutions?

a)

$$\begin{pmatrix} 1 & -3 \end{pmatrix}\mathbf{x} = 3$$
$$\begin{pmatrix} 3 & -9 \end{pmatrix}\mathbf{x} = 2$$

b)

$$\begin{pmatrix} 2 & 1 \end{pmatrix}\mathbf{x} = 5$$
$$\begin{pmatrix} 3 & 2 \end{pmatrix}\mathbf{x} = 8$$

c)

$$\begin{pmatrix} 3 & -5 \end{pmatrix}\mathbf{x} = 20$$
$$\begin{pmatrix} 6 & -10 \end{pmatrix}\mathbf{x} = 40$$

d)

$$\begin{pmatrix} 1 & -3 \end{pmatrix}\mathbf{x} = 7$$
$$\begin{pmatrix} 3 & -3 \end{pmatrix}\mathbf{x} = 15$$

Solution:

a) The given equations can be expressed as the matrix equation

$$\begin{pmatrix} 1 & -3 \\ 3 & -9 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

The augmented matrix for the above equation is row reduced as follows

$$\begin{pmatrix} 1 & -3 & 3 \\ 3 & -9 & 2 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - 3R_1} \begin{pmatrix} 1 & -3 & 3 \\ 0 & 0 & -7 \end{pmatrix}$$

$\therefore$ the lines in (1.137a) are parallel as can be seen in Fig. 1.

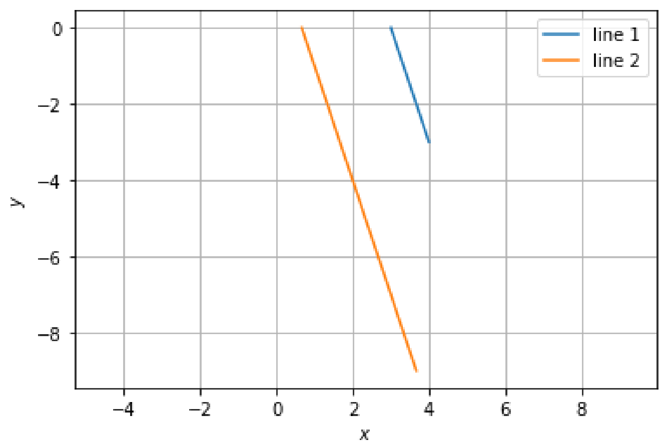

Fig. 1: PARALLEL LINES.

b) The given equations can be expressed as the matrix equation

$$\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 5 \\ 8 \end{pmatrix}$$

The augmented matrix for the above equation is row reduced as follows

$$\begin{pmatrix} 2 & 1 & 5 \\ 3 & 2 & 8 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow 2R_2 - 3R_1} \begin{pmatrix} 2 & 1 & 5 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 & 5 \\ 0 & 1 & 1 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 - R_2} \begin{pmatrix} 2 & 0 & 4 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 & 4 \\ 0 & 1 & 1 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{R_1}{2}} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}$$

Thus,

$$\mathbf{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

is the point of intersection of the lines in (1.137b) as verified in Fig. 2.

c) The given equations can be expressed as the

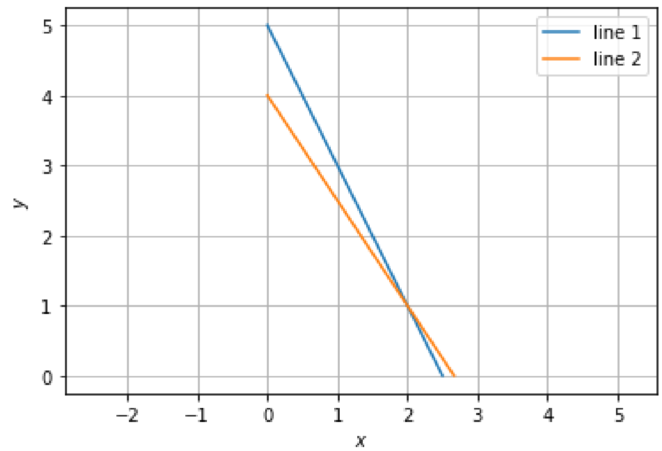

Fig. 2: INTERSECTING LINES.

matrix equation

$$\begin{pmatrix} 3 & -5 \\ 6 & -10 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 20 \\ 40 \end{pmatrix}$$

The augmented matrix for the above equation is row reduced as follows

$$\begin{pmatrix} 3 & -5 & 20 \\ 6 & -10 & 40 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{pmatrix} 3 & -5 & 20 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus 1.137c has infinitely many solutions and Fig. 3 shows that the lines are the same.

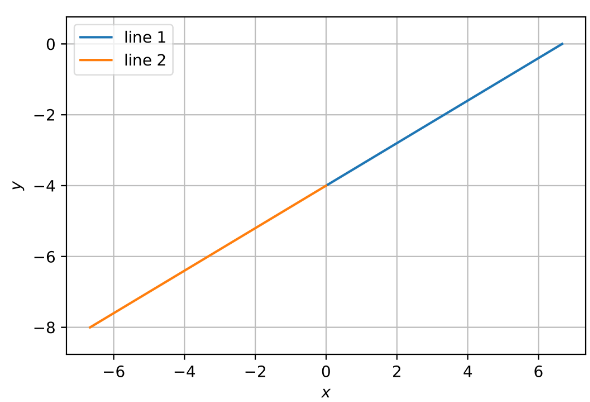

Fig. 3: Lines coincide:infinitely many solutions

d) The above equations can be expressed as the matrix equation

$$\begin{pmatrix} 1 & -3 \\ 3 & -3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 7 \\ 15 \end{pmatrix}$$

The augmented matrix for the above equation is row reduced as follows

$$\begin{pmatrix} 1 & -3 & 7 \\ 3 & -3 & 15 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & -3 & 7 \\ 2 & 0 & 8 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -3 & 7 \\ 2 & 0 & 8 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{pmatrix} 1 & -3 & 7 \\ 0 & 6 & -6 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -3 & 7 \\ 0 & 6 & -6 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 + \frac{R_2}{2}} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 6 & -6 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 4 \\ 0 & 6 & -6 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow \frac{R_2}{6}} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -1 \end{pmatrix}$$

$$\implies \mathbf{x} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

is a solution of 1.137d which is verified through Fig. 4

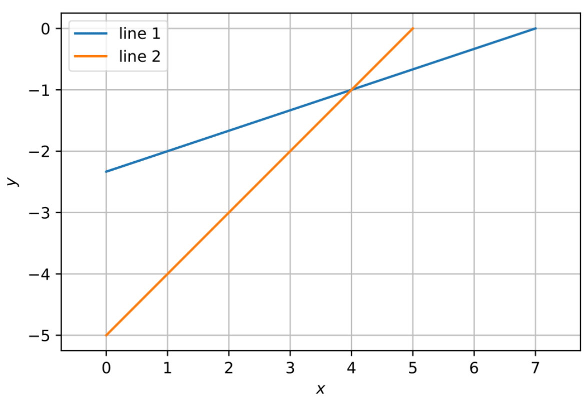

Fig. 4: Lines intersecting only at one point:unique solution

1.138. Which of the following pairs of linear equations are consistent/inconsistent? If consistent, obtain the solution:

a)

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 5$$
$$\begin{pmatrix} 2 & 2 \end{pmatrix} \mathbf{x} = 10$$

b)

$$\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 8$$
$$\begin{pmatrix} 3 & -3 \end{pmatrix} \mathbf{x} = 16$$

c)

$$\begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = 6$$
$$\begin{pmatrix} 4 & -2 \end{pmatrix} \mathbf{x} = 4$$

d)

$$\begin{pmatrix} 2 & -2 \end{pmatrix} \mathbf{x} = 2$$
$$\begin{pmatrix} 4 & -4 \end{pmatrix} \mathbf{x} = 5$$

Solution:

a)

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 5$$
$$\begin{pmatrix} 2 & 2 \end{pmatrix} \mathbf{x} = 10$$

The above equations can be expressed as the matrix equation

$$\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 5 \\ 10 \end{pmatrix}$$

The augmented matrix for the above equation is row reduced as follows

$$\begin{pmatrix} 1 & 1 & 5 \\ 2 & 2 & 10 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{pmatrix} 1 & 1 & 5 \\ 0 & 0 & 0 \end{pmatrix}$$

$\because$ row reduction of the 2×3 matrix

$$\begin{pmatrix} 1 & 1 & 5 \\ 0 & 0 & 0 \end{pmatrix}$$

results in a matrix with 1 nonzero row, its rank is 1. Similarly, the rank of the matrix

$$\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$$

is also 1.

$$\begin{aligned} \because Rank \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} &= Rank \begin{pmatrix} 1 & 1 & 5 \\ 2 & 2 & 10 \end{pmatrix} \\ &= dim \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \\ &= 1 \end{aligned}$$

$\therefore$ the lines in (1.138a) have infinitely many solutions and concide as seen in Fig. 1. The given lines are consistent.

b)

$$\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 8$$
$$\begin{pmatrix} 3 & -3 \end{pmatrix} \mathbf{x} = 16$$

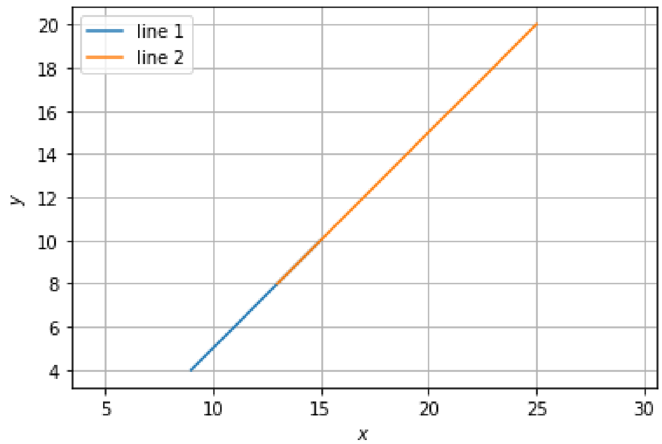

Fig. 1: SAME LINES

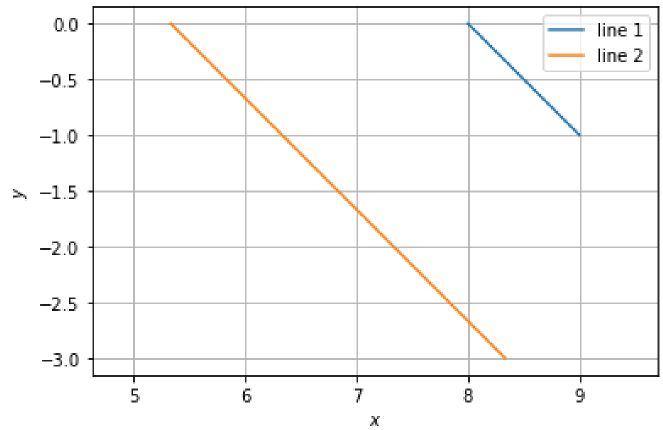

Fig. 2: Parallel lines

The above equations can be expressed as the matrix equation

$$\begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 8 \\ 16 \end{pmatrix}$$

The augmented matrix for the above equation is row reduced as follows

$$\begin{pmatrix} 1 & -1 & 8 \\ 3 & -3 & 16 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - 3R_1} \begin{pmatrix} 1 & -1 & 8 \\ 0 & 0 & -8 \end{pmatrix}$$

$\because$ row reduction of the 2×3 matrix

$$\begin{pmatrix} 1 & -1 & 8 \\ 3 & -3 & 16 \end{pmatrix}$$

results in a matrix with 2 nonzero rows, its rank is 2. Similarly, the rank of the matrix

$$\begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix}$$

is also 1.

$$\because Rank \begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix} \neq Rank \begin{pmatrix} 1 & -1 & 8 \\ 3 & -3 & 16 \end{pmatrix}$$

$\therefore$ Given lines (1.138b) have no solution and are parallel as can be seen in Fig. 2

c)

$$\begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = 6$$
$$\begin{pmatrix} 4 & -2 \end{pmatrix} \mathbf{x} = 4$$

The above equations can be expressed as the matrix equation

$$\begin{pmatrix} 2 & 1 \\ 4 & -2 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$$

The augmented matrix for the above equation is row reduced as follows

$$\begin{pmatrix} 2 & 1 & 6 \\ 4 & -2 & 4 \end{pmatrix} \xleftrightarrow{R_2 \rightarrow \frac{R_1}{2} - \frac{R_2}{4}} \begin{pmatrix} 2 & 1 & 6 \\ 0 & 1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 & 6 \\ 0 & 1 & 2 \end{pmatrix} \xleftrightarrow{R_1 \rightarrow R_1 - R_2} \begin{pmatrix} 2 & 0 & 4 \\ 0 & 1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 & 4 \\ 0 & 1 & 2 \end{pmatrix} \xleftrightarrow{R_1 \rightarrow \frac{R_1}{2}} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{pmatrix}$$

$\because$ row reduction of the 2×3 matrix

$$\begin{pmatrix} 2 & 1 & 6 \\ 4 & -2 & 4 \end{pmatrix}$$

results in a matrix with 2 nonzero row, its rank is 2. Similarly, the rank of the matrix

$$\begin{pmatrix} 2 & 1 \\ 4 & -2 \end{pmatrix}$$

is also 2.

$$\because Rank \begin{pmatrix} 2 & 1 \\ 4 & -2 \end{pmatrix} = Rank \begin{pmatrix} 2 & 1 & 6 \\ 4 & -2 & 4 \end{pmatrix} = 2$$
$$= dim \begin{pmatrix} 2 & 1 \\ 4 & -2 \end{pmatrix} = 2$$

$\therefore$ the lines in (1.138c) intersect as can be seen from Fig. 3

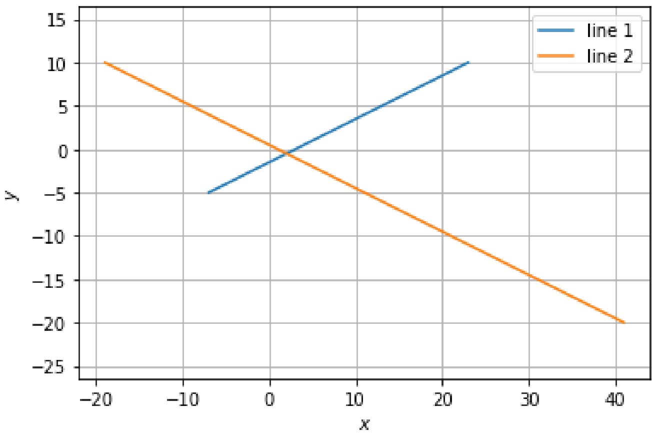

Fig. 3: INTERSECTING LINES

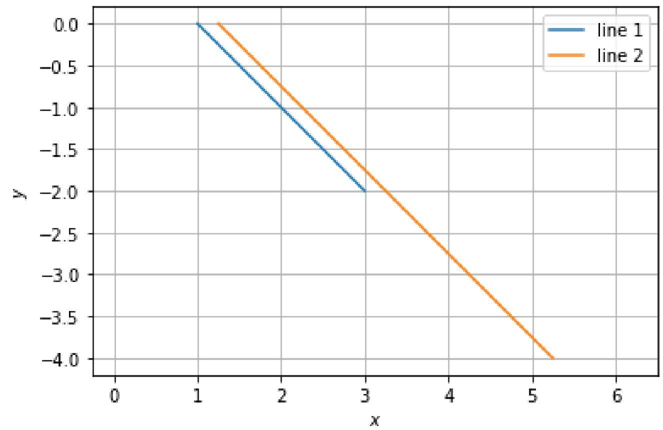

Fig. 4: PARALLEL LINES

d)

$$\begin{pmatrix} 2 & -2 \end{pmatrix} \mathbf{x} = 2$$
$$\begin{pmatrix} 4 & -4 \end{pmatrix} \mathbf{x} = 5$$

The above equations can be expressed as the matrix equation

$$\begin{pmatrix} 2 & -2 \\ 4 & -4 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

The augmented matrix for the above equation is row reduced as follows

$$\begin{pmatrix} 2 & -2 & 2 \\ 4 & -4 & 5 \end{pmatrix} \xleftrightarrow{R_2 \rightarrow R_2 - 2R_1} \begin{pmatrix} 2 & -2 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

$\because$ row reduction of the 2×3 matrix

$$\begin{pmatrix} 2 & -2 & 2 \\ 4 & -4 & 5 \end{pmatrix}$$

results in a matrix with 2 nonzero rows, its rank is 2. Similarly, the rank of the matrix

$$\begin{pmatrix} 2 & -2 \\ 4 & -4 \end{pmatrix}$$

is also 1.

$$\because Rank \begin{pmatrix} 2 & -2 \\ 4 & -4 \end{pmatrix} \neq Rank \begin{pmatrix} 2 & -2 & 2 \\ 4 & -4 & 5 \end{pmatrix} = 1$$
$$< dim \begin{pmatrix} 2 & -2 \\ 4 & -4 \end{pmatrix} = 2$$

$\therefore$ the lines in (1.138d) are parallel as can be seen in Fig. 4

1.139. Find out whether the lines representing the following pairs of linear equations intersect at a point, are parallel or coincident

a)

$$\begin{pmatrix} 5 & -4 \end{pmatrix} \mathbf{x} = -8$$
$$\begin{pmatrix} 7 & 6 \end{pmatrix} \mathbf{x} = 9$$

b)

$$\begin{pmatrix} 9 & 3 \end{pmatrix} \mathbf{x} = -12$$
$$\begin{pmatrix} 18 & 6 \end{pmatrix} \mathbf{x} = -24$$

c)

$$\begin{pmatrix} 6 & -3 \end{pmatrix} \mathbf{x} = -10$$
$$\begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} = -9$$

Solution:

a)

$$\begin{pmatrix} 5 & -4 \end{pmatrix} \mathbf{x} = -8$$
$$\begin{pmatrix} 7 & 6 \end{pmatrix} \mathbf{x} = 9$$

The above equations can be expressed as the matrix equation

$$\begin{pmatrix} 5 & -4 \\ 7 & 6 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -8 \\ 9 \end{pmatrix}$$

The augmented matrix for the above equa-

tion is row reduced as follows

$$\begin{pmatrix}5 & -4 & -8\\7 & 6 & 9\end{pmatrix} \xleftrightarrow{R_1\leftarrow\ 7\frac{R_1}{5}} \begin{pmatrix}7 & \frac{-28}{5} & \frac{-56}{5}\\7 & 6 & 9\end{pmatrix}$$

$$\xleftrightarrow{R_2\leftarrow R_2-R_1} \begin{pmatrix}7 & \frac{-28}{5} & \frac{-56}{5}\\0 & \frac{58}{5} & \frac{101}{5}\end{pmatrix}$$

$$\xleftrightarrow{R_1\leftarrow 5\frac{R_1}{7}} \begin{pmatrix}5 & -4 & -8\\0 & \frac{58}{5} & \frac{101}{5}\end{pmatrix}$$

$$\xleftrightarrow{R_2\leftarrow 5R_2} \begin{pmatrix}5 & -4 & -8\\0 & 58 & 101\end{pmatrix}$$

$\because$ row reduction of the 2×3 matrix

$$\begin{pmatrix}5 & -4 & -8\\7 & 6 & 9\end{pmatrix}$$

results in a matrix with 2 nonzero rows, its rank is 2. Similarly, the rank of the matrix

$$\begin{pmatrix}5 & -4\\7 & 6\end{pmatrix}$$

is also 2.

$$\because Rank\begin{pmatrix}5 & -4\\7 & 6\end{pmatrix} = Rank\begin{pmatrix}5 & -4 & -8\\7 & 6 & 9\end{pmatrix}$$
$$= dim\begin{pmatrix}5 & -4\\7 & 6\end{pmatrix}$$
$$= 2$$

$\therefore$ the lines given in (1.139a) intersect and are plotted in Fig. 1.

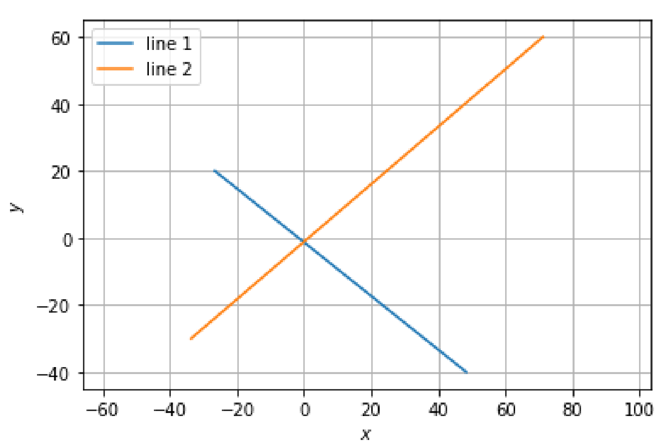

Fig. 1: INTERSECTING LINES.

b)

$$\begin{pmatrix}9 & 3\end{pmatrix}\mathbf{x} = -12$$
$$\begin{pmatrix}18 & 6\end{pmatrix}\mathbf{x} = -24$$

The above equations can be expressed as the matrix equation

$$\begin{pmatrix}9 & 3\\18 & 6\end{pmatrix}\mathbf{x} = \begin{pmatrix}-12\\-24\end{pmatrix}$$

The augmented matrix for the above equation is row reduced as follows

$$\begin{pmatrix}9 & 3 & -12\\18 & 6 & -24\end{pmatrix} \xleftrightarrow{R_1\leftarrow\ 7\frac{R_1}{5}} \begin{pmatrix}7 & \frac{-28}{5} & \frac{-56}{5}\\7 & 6 & 9\end{pmatrix}$$

$$\xleftrightarrow{R_1\leftarrow\frac{18R_1}{9}} \begin{pmatrix}18 & 6 & -24\\18 & 6 & -24\end{pmatrix}$$

$$\xleftrightarrow{R_2\leftarrow R_2-R_1} \begin{pmatrix}18 & 6 & -24\\0 & 0 & 0\end{pmatrix}$$

$\because$ row reduction of the 2×3 matrix

$$\begin{pmatrix}9 & 3 & -12\\18 & 6 & -24\end{pmatrix}$$

results in a matrix with 1 nonzero rows, its rank is 1. Similarly, the rank of the matrix

$$\begin{pmatrix}9 & 3\\18 & 6\end{pmatrix}$$

is also 1.

$$\because Rank\begin{pmatrix}9 & 3\\18 & 6\end{pmatrix} = Rank\begin{pmatrix}9 & 3 & -12\\18 & 6 & -24\end{pmatrix} = 1$$
$$< dim\begin{pmatrix}9 & 3\\18 & 6\end{pmatrix} = 2$$

$\therefore$ the lines (1.139b) coincide and are plotted in Fig. 2.

c)

$$\begin{pmatrix}6 & -3\end{pmatrix}\mathbf{x} = -10$$
$$\begin{pmatrix}2 & -1\end{pmatrix}\mathbf{x} = -9$$

The above equations can be expressed as the

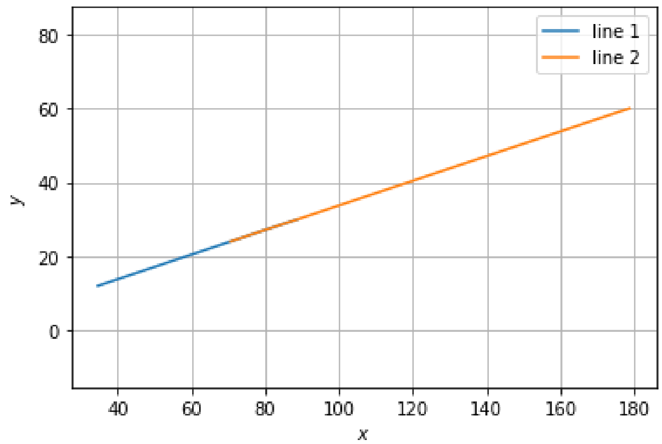

Fig. 2: SAME LINES

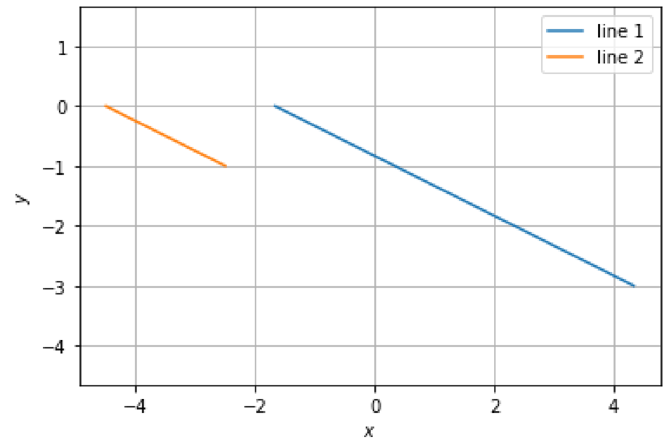

Fig. 3: Parallel lines

matrix equation

$$\begin{pmatrix} 6 & -3 \\ 2 & -1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -10 \\ -9 \end{pmatrix}$$

The augmented matrix for the above equation is row reduced as follows

$$\begin{pmatrix} 6 & -3 & -10 \\ 2 & -1 & -9 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{2R_1}{6}} \begin{pmatrix} 2 & -1 & \frac{-10}{3} \\ 2 & -1 & -9 \end{pmatrix}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 2 & -1 & \frac{-10}{3} \\ 0 & 0 & \frac{-17}{3} \end{pmatrix}$$

$$\xleftrightarrow{R_1 \leftarrow 3R_1} \begin{pmatrix} 6 & -3 & -10 \\ 0 & 0 & \frac{-17}{3} \end{pmatrix}$$

$\because$ row reduction of the 2×3 matrix

$$\begin{pmatrix} 6 & -3 & -10 \\ 2 & -1 & -9 \end{pmatrix}$$

results in a matrix with 2 nonzero rows, its rank is 2. Similarly, the rank of the matrix

$$\begin{pmatrix} 6 & -3 \\ 2 & -1 \end{pmatrix}$$

is 1.

$$\because Rank \begin{pmatrix} 6 & -3 \\ 2 & -1 \end{pmatrix} \neq Rank \begin{pmatrix} 6 & -3 & -10 \\ 2 & -1 & -9 \end{pmatrix}$$

$\therefore$ the lines in (1.139c) are parallel and plotted in Fig. 3.

1.140. Find the intersection of the following lines

a)

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 14$$
$$\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 4$$

b)

$$\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 3$$
$$\begin{pmatrix} \frac{1}{3} & \frac{1}{2} \end{pmatrix} \mathbf{x} = 6$$

c)

$$\begin{pmatrix} 3 & -1 \end{pmatrix} \mathbf{x} = 3$$
$$\begin{pmatrix} 9 & -3 \end{pmatrix} \mathbf{x} = 9$$

d)

$$\begin{pmatrix} 0.2 & 0.3 \end{pmatrix} \mathbf{x} = 1.3$$
$$\begin{pmatrix} 0.4 & 0.5 \end{pmatrix} \mathbf{x} = 2.3$$

e)

$$\begin{pmatrix} \sqrt{2} & \sqrt{3} \end{pmatrix} \mathbf{x} = 0$$
$$\begin{pmatrix} \sqrt{3} & \sqrt{8} \end{pmatrix} \mathbf{x} = 0$$

f)

$$\begin{pmatrix} \frac{3}{2} & -\frac{5}{3} \end{pmatrix} \mathbf{x} = -2$$
$$\begin{pmatrix} \frac{1}{3} & \frac{1}{2} \end{pmatrix} \mathbf{x} = \frac{13}{6}$$

Solution:

a)

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 14$$
$$\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 4$$

The above equations can be expressed as the matrix equation

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 14 \\ 4 \end{pmatrix}$$

The augmented matrix for the above equation is row reduced as follows

$$\begin{pmatrix} 1 & 1 & 14 \\ 1 & -1 & 4 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 1 & 14 \\ 0 & -2 & -10 \end{pmatrix}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - R_2/-2} \begin{pmatrix} 1 & 0 & 9 \\ 0 & -2 & -10 \end{pmatrix}$$

$$\xleftrightarrow{R_2 \leftarrow R_2/-2} \begin{pmatrix} 1 & 0 & 9 \\ 0 & 1 & 5 \end{pmatrix}$$

As left part is converted into identity matrix the intersection vector is $\begin{pmatrix} 9 \\ 5 \end{pmatrix}$ which is plotted in Fig. 1

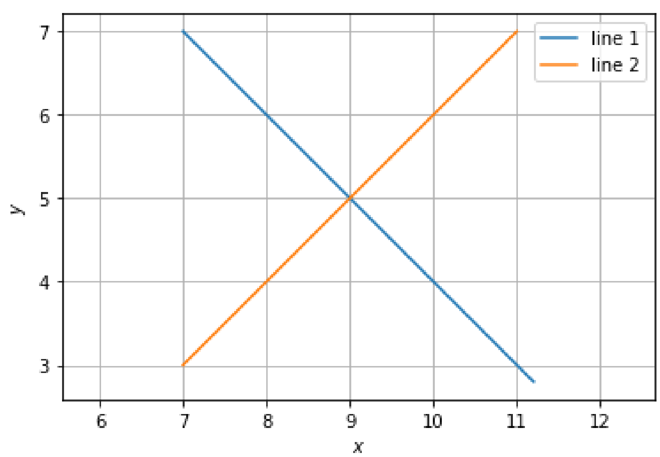

Fig. 1: INTERSECTING LINES

b)

$$\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 3$$
$$\begin{pmatrix} \frac{1}{3} & \frac{1}{2} \end{pmatrix} \mathbf{x} = 6$$

The above equations can be expressed as the matrix equation

$$\begin{pmatrix} 1 & -1 \\ \frac{1}{3} & \frac{1}{2} \end{pmatrix} \mathbf{x} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

The augmented matrix for the above equation is row reduced as follows

$$\begin{pmatrix} 1 & -1 & 3 \\ \frac{1}{3} & \frac{1}{2} & 6 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - R_1/3} \begin{pmatrix} 1 & -1 & 3 \\ 0 & \frac{5}{6} & 5 \end{pmatrix}$$

$$\xleftrightarrow{R_2 \leftarrow R_2/5} \begin{pmatrix} 1 & -1 & 3 \\ 0 & \frac{1}{6} & 1 \end{pmatrix}$$

$$\xleftrightarrow{R_2 \leftarrow 6R_2} \begin{pmatrix} 1 & -1 & 3 \\ 0 & 1 & 6 \end{pmatrix}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 + R_1} \begin{pmatrix} 1 & 0 & 9 \\ 0 & 1 & 6 \end{pmatrix}$$

As left part is converted into identity matrix the intersection vector is $\begin{pmatrix} 9 \\ 6 \end{pmatrix}$ which is plotted in Fig. 2.

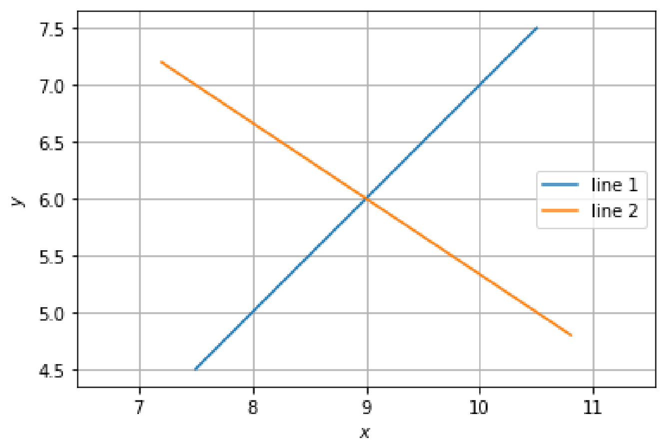

Fig. 2: INTERSECTING LINES

c)

$$\begin{pmatrix} 3 & -1 \end{pmatrix} \mathbf{x} = 3$$
$$\begin{pmatrix} 9 & -3 \end{pmatrix} \mathbf{x} = 9$$

The above equations can be expressed as the matrix equation

$$\begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 3 \\ 9 \end{pmatrix}$$

Now we converted these matrix equation in augmented matrix form using row reduction

$$\begin{pmatrix} 3 & -1 & 3 \\ 9 & -3 & 9 \end{pmatrix} \xleftrightarrow{R_2 \rightarrow \frac{R_2}{3}} \begin{pmatrix} 3 & -1 & 3 \\ 3 & -1 & 3 \end{pmatrix}$$

Since the rows are linearly dependent, the given set of equations has infinite solutions and the lines are coincident as can be seen from Fig. 3.

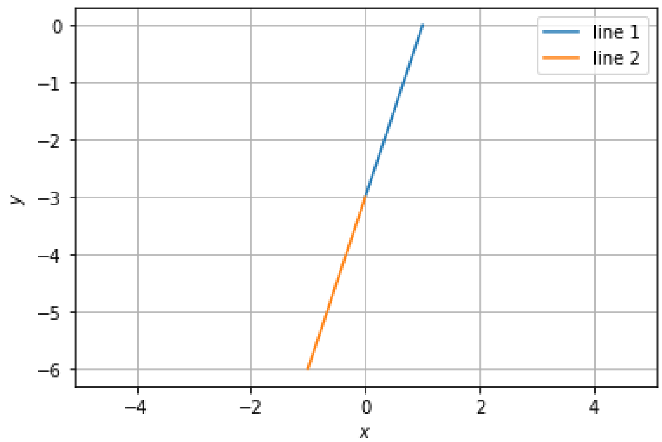

Fig. 3: SAME-LINES

d)

$$\begin{pmatrix} 0.2 & 0.3 \end{pmatrix} \mathbf{x} = 1.3$$
$$\begin{pmatrix} 0.4 & 0.5 \end{pmatrix} \mathbf{x} = 2.3$$

The above equations can be expressed as the matrix equation

$$\begin{pmatrix} 0.2 & 0.3 \\ 0.4 & 0.5 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1.3 \\ 1.4 \end{pmatrix}$$

Now we converted these matrix equation in augmented matrix form using row reduction

$$\begin{pmatrix} 0.2 & 0.3 & 1.3 \\ 0.4 & 0.5 & 1.4 \end{pmatrix} \xleftrightarrow{R_2 \to R_2 - 2R_1} \begin{pmatrix} 0.2 & 0.3 & 1.3 \\ 0 & -0.1 & -0.3 \end{pmatrix}$$

$$\xleftrightarrow{R_2 \to \frac{R_2}{-0.1}} \begin{pmatrix} 0.2 & 0.3 & 1.3 \\ 0 & 1 & 3 \end{pmatrix}$$

$$\xleftrightarrow{R_1 \to R_1 - 0.3R_2} \begin{pmatrix} 0.2 & 0 & 0.4 \\ 0 & 1 & 3 \end{pmatrix}$$

$$\xleftrightarrow{R_1 \to \frac{R_1}{0.2}} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix}$$

Thus, the point of intersection is $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$, as can be seen from Fig. 4

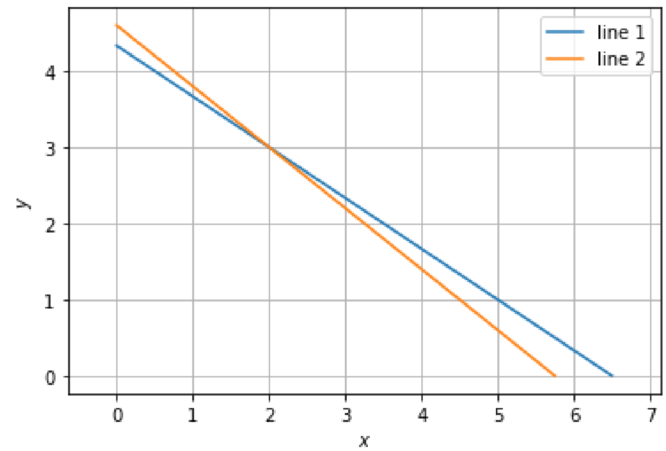

Fig. 4: INTERSECTING-LINES

e)

$$\begin{pmatrix} \sqrt{2} & \sqrt{3} \end{pmatrix} \mathbf{x} = 0$$
$$\begin{pmatrix} \sqrt{3} & \sqrt{8} \end{pmatrix} \mathbf{x} = 0$$

The above equations can be expressed as the matrix equation

$$\begin{pmatrix} \sqrt{2} & \sqrt{3} \\ \sqrt{3} & \sqrt{8} \end{pmatrix} \mathbf{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The augmented matrix for the above equation is row reduced as follows

$$\begin{pmatrix} \sqrt{2} & \sqrt{3} & 0 \\ \sqrt{3} & \sqrt{8} & 0 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - \frac{\sqrt{3}}{\sqrt{2}} R_1} \begin{pmatrix} \sqrt{2} & \sqrt{3} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

As left part is converted into identity matrix the intersection vector is $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ as can be seen in Fig. 5

f)

$$\begin{pmatrix} \frac{3}{2} & -\frac{5}{3} \end{pmatrix} \mathbf{x} = -2$$
$$\begin{pmatrix} \frac{1}{3} & \frac{1}{2} \end{pmatrix} \mathbf{x} = \frac{13}{6}$$

The above equations can be expressed as the matrix equation

$$\begin{pmatrix} \frac{3}{2} & -\frac{5}{3} \\ \frac{1}{3} & \frac{1}{2} \end{pmatrix} \mathbf{x} = \begin{pmatrix} -2 \\ \frac{13}{6} \end{pmatrix}$$

The augmented matrix for the above equa-

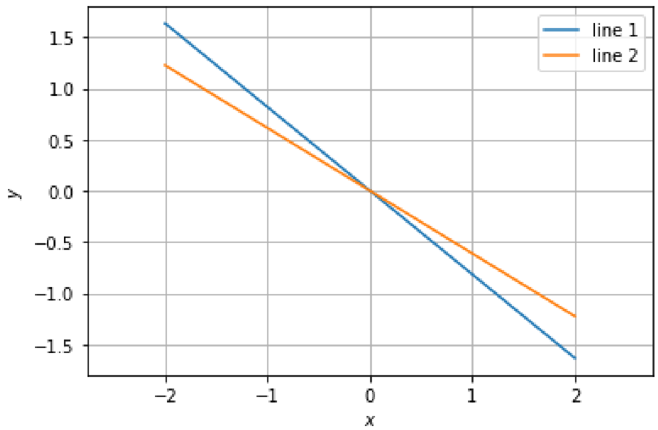

Fig. 5: intersecting lines

tion is row reduced as follows

$$\begin{pmatrix} \frac{3}{2} & -\frac{5}{3} & -2 \\ \frac{1}{3} & \frac{1}{2} & \frac{13}{6} \end{pmatrix} \xleftrightarrow{R_1 \leftarrow 6R_1}, \xleftrightarrow{R_2 \leftarrow 6R_2} \begin{pmatrix} 9 & -10 & -12 \\ 2 & 3 & 13 \end{pmatrix}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - 4R_2} \begin{pmatrix} 1 & -22 & -64 \\ 2 & 3 & 13 \end{pmatrix}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{pmatrix} 1 & -22 & -64 \\ 0 & 47 & 141 \end{pmatrix}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 / 47} \begin{pmatrix} 1 & -22 & -64 \\ 0 & 1 & 3 \end{pmatrix}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + 22R_2} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix}$$

As left part is converted into identity matrix the intersection vector is $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ as can be seen in Fig. 6

1.141. Find the angle between the following pair of lines:

$$L_1 : \quad \mathbf{x} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$$

$$L_2 : \quad \mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ -56 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -5 \\ -4 \end{pmatrix}$$

Solution: From (1.141) and (1.141), we get the

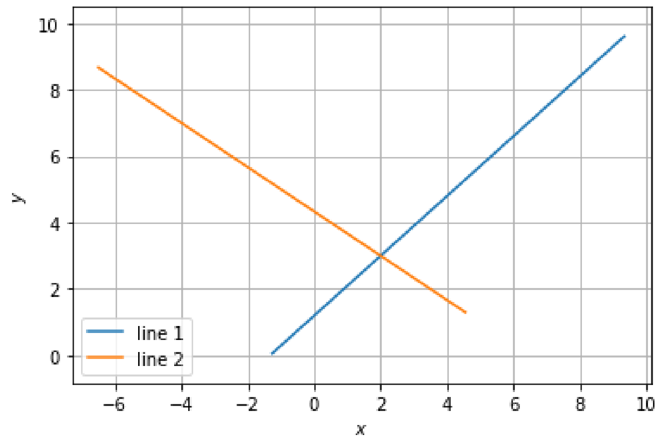

Fig. 6: intersecting lines

directional vector of L_1 and L_2 as

$$\mathbf{a} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 3 \\ -5 \\ -4 \end{pmatrix}$$

Angle between the pair of lines is calculated by using cosine formula

$$\cos\theta = \frac{\mathbf{a}^T \mathbf{b}}{\|\mathbf{a}\| \, \|\mathbf{b}\|}$$

$\because$

$$\mathbf{a}^T \mathbf{b} = \begin{pmatrix} 1 & -1 & -2 \end{pmatrix} \begin{pmatrix} 3 \\ -5 \\ -4 \end{pmatrix} = 3 + 5 + 8 = 16$$

$$\|\mathbf{a}\| = \sqrt{1^2 + (-1)^2 + (-2)^2} = \sqrt{6}$$

$$\|\mathbf{b}\| = \sqrt{(3)^2 + (-5)^2 + (-4)^2} = \sqrt{50},$$

$$\cos\theta = \frac{16}{\sqrt{6}\sqrt{50}}$$

$$\implies \theta = \cos^{-1} \frac{16}{\sqrt{6}\sqrt{50}} = 22.5178253587°$$