Optimization

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Contents

***Abstract*—This book provides some exercises related to coordinate geometry. The content and exercises are based on NCERT textbooks from Class 6-12.**

1 Examples

1.1. Find the maximum and minimum values, if any, of the following functions given by

a) $f(x) = (2x-1)^2 + 3$

Solution: Given Equation can be written as:

$$f(x) = (2x-1)^2 + 3 = 4x^2 - 4x + 4 \tag{1.1.1}$$

A function is said to be convex if following inequality is true:

$$\lambda f(x_1) + (1-\lambda) f(x_2) \geq f(\lambda x_1 + (1-\lambda)x_2) \tag{1.1.2}$$

and for $\lambda \in [0,1]$

$$\lambda(4x_1^2 - 4x_1 + 4) + (1-\lambda)((4x_2^2 - 4x_2 + 4) \geq 4(\lambda x_1 + (1-\lambda)x_2)^2 - 4(\lambda x_1 + (1-\lambda)x_2) + 4 \tag{1.1.3}$$

$$x_1^2(4\lambda - 4\lambda^2) + x_2^2(4\lambda - 4\lambda^2) - 2x_1x_2(4\lambda - 4\lambda^2) \tag{1.1.4}$$

$$4\lambda(1-\lambda)(x_1 - x_2)^2 \geq 0 \tag{1.1.5}$$

The above inequality is always true for all values from the domain. Hence the given function f(x) is convex.

Using gradient descent method,

$$x_n = x_{n-1} - \mu \frac{df(x)}{dx} \tag{1.1.6}$$

Here, μ is learning rate.

$$\frac{df(x)}{dx} = 8x - 4 \tag{1.1.7}$$

After substituting 1.1.7 in 1.1.6 we get:

$$x_n = x_{n-1} - \mu(8x_{n-1} - 4) \tag{1.1.8}$$

In equation (1.1.8), μ is a variable parameter known as step size. x_{n+1} is the next position. The minus sign refers to the minimization part of gradient descent. Assume, $\mu = 0.01$, $x_0 = 2$ and following the above method, we keep doing iterations until $x_{n+1} - x_n$ becomes less than the value of precision we have chosen.

$$x_n = 0.5 \tag{1.1.9}$$

The following python code computes the minimum value as plotted in Fig. 1.1. Hence, The minimum value of $f(x)$ at $x = 0.5$ is 3.

./codes/Assignment_4.py

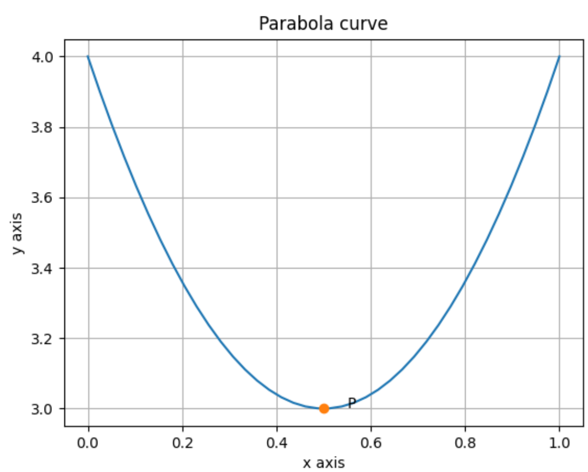

Fig. 1.1: Plot of the given polynomial

b)

$$f(x) = 9x^2 + 12x + 2 \tag{1.1.10}$$

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Solution: A function is said to be convex if following inequality is true:

$$\lambda f(x_1) + (1-\lambda)f(x_2) \ge f(\lambda x_1 + (1-\lambda)x_2) \tag{1.1.11}$$

and for $\lambda \in [0,1]$

$$\lambda(9x_1^2 + 12x_1 + 2) + (1-\lambda)((9x_2^2 + 12x_2 + 2) \ge 9(\lambda x_1 + (1-\lambda)x_2)^2 + 12(\lambda x_1 + (1-\lambda)x_2) + 2 \tag{1.1.12}$$

$$x_1^2(9\lambda - 9\lambda^2) + x_2^2(9\lambda - 9\lambda^2) - 2x_1x_2(9\lambda - 9\lambda^2) \tag{1.1.13}$$

$$(9\lambda - 9\lambda^2)(x_1^2 + x_2^2 - 2x_1x_2) \tag{1.1.14}$$

$$9\lambda(1-\lambda)(x_1 - x_2)^2 \ge 0 \tag{1.1.15}$$

Equation (1.1.15) holds true for all $\lambda \in (0,1)$. Hence the given function f(x) is convex. Fora general quadratic equation

$$f(x) = ax^2 + bx + c \tag{1.1.16}$$

The update equation for gradient descent to find minimum of a function is given by:

$$\lambda_{n+1} = \lambda_n - \mu f'(\lambda_n) \tag{1.1.17}$$

$$= \lambda_n - \mu(2a\lambda_n + b) \tag{1.1.18}$$

In equation (1.1.15) λ_0 is an initial guess and μ is a variable parameter,known as step size λ_{n+1} is the next position. The minus sign refers to the minimization part of gradient descent. Assume,

$$\lambda_0 = 1 \tag{1.1.19}$$

$$\mu = 0.001 \tag{1.1.20}$$

$$precision = 0.00000001 \tag{1.1.21}$$

$$\implies \lambda_1 = 1 - 0.001(2 \times 9 \times 1 + 12) \tag{1.1.22}$$

$$\implies \lambda_1 = 1 - 0.03 \tag{1.1.23}$$

$$= 0.97 \tag{1.1.24}$$

following the above method, we keep doing iterations until $\lambda_{n+1} - \lambda_n$ becomes less than the value of precision we have chosen. Results Using python, the results are:

i) The local minimum occurs at -0.666666130125316.
ii) The value of f(x) at minima is -1.9999999999974087

Figure 1 shows plot of parabola obtained from python code:

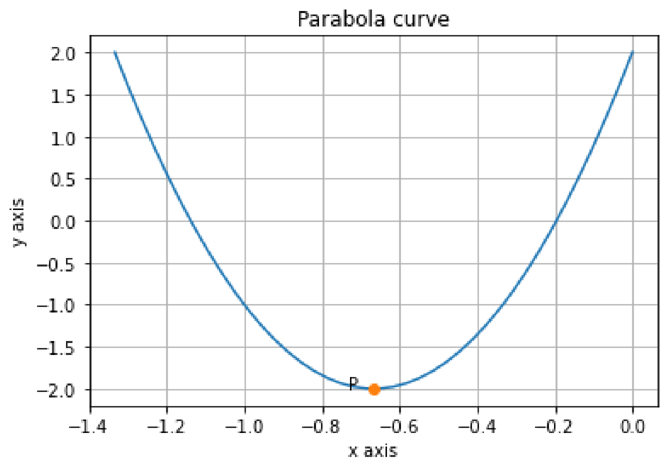

Fig. 1: Plot obtained from python code

c) $f(x) = -(x-1)^2 + 10$
d) $f(x) = x^2$.

1.2. Solve

$$\min_{\mathbf{x}} Z = \begin{pmatrix} 3 & 2 \end{pmatrix}\mathbf{x} \tag{1.2.1}$$

$$s.t. \quad \begin{pmatrix} -1 & -1 \\ 3 & 5 \end{pmatrix}\mathbf{x} \le \begin{pmatrix} -8 \\ 15 \end{pmatrix} \tag{1.2.2}$$

$$\mathbf{x} \ge \mathbf{0} \tag{1.2.3}$$

1.3. Solve

$$\min_{\mathbf{x}} Z = \begin{pmatrix} 200 & 500 \end{pmatrix}\mathbf{x} \tag{1.3.1}$$

$$s.t. \quad \begin{pmatrix} -1 & -2 \\ 3 & 4 \end{pmatrix}\mathbf{x} \le \begin{pmatrix} -10 \\ 24 \end{pmatrix} \tag{1.3.2}$$

$$\mathbf{x} \ge \mathbf{0} \tag{1.3.3}$$

1.4. Maximise Z=3x+4y
subject to the constraints : x+y≤4, x≥0, y≥ 0.

1.5. Minimise Z=-3x+4y
subject to x+2y≤8, 3x+2y≤12, x≥0, y≥0.

1.6. Maximise Z=5x+3y subject to 3x+5y≤15, 5x+2y≤10, x≥0, y≥0.

Solution:

$$Z - 5x - 3y = 0 \tag{1.6.1}$$
$$3x + 5y + s_1 = 15 \tag{1.6.2}$$
$$5x + 2y + s_2 = 10 \tag{1.6.3}$$

We will write the simplex tableau

$$\left(\begin{array}{ccccc} x & y & s_1 & s_2 & c \\ 3 & 5 & 1 & 0 & 15 \\ \boxed{5} & 2 & 0 & 1 & 10 \\ \hline -5 & -3 & 0 & 0 & 0 \end{array}\right) \tag{1.6.4}$$

Keeping the pivot element as 5, we will use gauss-jordan elimination.

$$\left(\begin{array}{ccccc} x & y & s_1 & s_2 & c \\ 0 & \boxed{\frac{19}{5}} & 1 & \frac{-3}{5} & 9 \\ 1 & \frac{2}{5} & 0 & \frac{1}{5} & 2 \\ \hline 0 & -1 & 0 & 1 & 10 \end{array}\right) \tag{1.6.5}$$

Keeping the pivot element as $\frac{19}{5}$, we will use gauss-jordan elimination.

$$\left(\begin{array}{ccccc} x & y & s_1 & s_2 & c \\ 0 & 1 & \frac{5}{19} & \frac{-3}{19} & \frac{45}{19} \\ 1 & 0 & \frac{-2}{19} & \frac{5}{19} & \frac{20}{19} \\ \hline 0 & 0 & \frac{5}{19} & \frac{16}{19} & \frac{235}{19} \end{array}\right) \tag{1.6.6}$$

In this tableau, there are no negative elements in the bottom row. We have therefore determined the optimal solution to be:

$$(x, y, s_1, s_2) = \left(\frac{20}{19}, \frac{45}{19}, 0, 0\right) \tag{1.6.7}$$
$$Z = 5x + 3y \tag{1.6.8}$$
$$Z = 5 \times \frac{20}{19} + 3 \times \frac{45}{19} \tag{1.6.9}$$
$$Z = \frac{235}{19} \tag{1.6.10}$$

The given problem can be expressed in general as matrix inequality as:

$$\max_{\mathbf{x}} \mathbf{c}^T \mathbf{x} \tag{1.6.11}$$
$$s.t. \quad \mathbf{A}\mathbf{x} \leq \mathbf{b}, \tag{1.6.12}$$
$$\mathbf{x} \geq \mathbf{0} \tag{1.6.13}$$
$$\mathbf{y} \geq \mathbf{0} \tag{1.6.14}$$

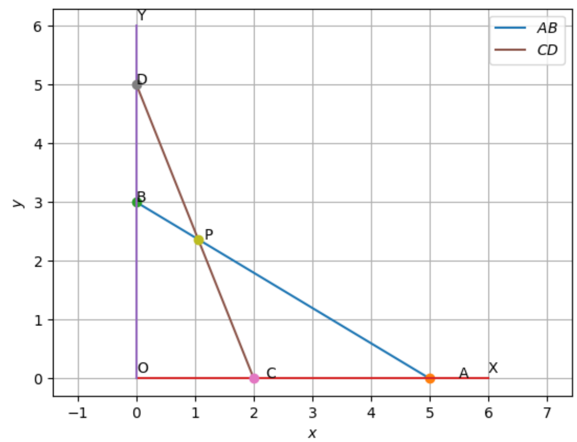

Fig. 1.6: optimal point through the intersection of various lines

where

$$\mathbf{c} = \begin{pmatrix} 5 \\ 3 \end{pmatrix} \tag{1.6.15}$$
$$\mathbf{A} = \begin{pmatrix} 3 & 5 \\ 5 & 2 \end{pmatrix} \tag{1.6.16}$$
$$\mathbf{b} = \begin{pmatrix} 15 \\ 10 \end{pmatrix} \tag{1.6.17}$$

and can be solved using *cvxpy*. Hence,

$$\mathbf{x} = \begin{pmatrix} 1.05263158 \\ 2.36842105 \end{pmatrix}, Z = 12.36842102 \tag{1.6.18}$$

1.7. Minimise Z=3x+5y such that x+3y≥3, x+y≥2, x,y≥0.

1.8. Maximise Z=3x+2y subject to x+2y≤10, 3x+y≤15, x,y≥0.
Solution:

$$Maximize : 3x_1 + 2x_2 \tag{1.8.1}$$
$$Subject - to : x_1 + 2x_2 \leq 10 \tag{1.8.2}$$
$$3x_1 + x_2 \leq 15 \tag{1.8.3}$$

The Problem is converted into canonical form by adding slack variables.Then Problem becomes,

$$Maximize : 3x_1 + 2x_2 + 0s_1 + 0s_2 \tag{1.8.4}$$
$$Constraints : x_1 + 2x_2 + s_1 = 10 \tag{1.8.5}$$
$$3x_1 + x_2 + s_2 = 15 \tag{1.8.6}$$

we write the Simplex tableau ,

$$\left(\begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & c \\ 1 & 2 & 1 & 0 & 10 \\ \boxed{3} & 1 & 0 & 1 & 15 \\ \hline -3 & -2 & 0 & 0 & 0 \end{array}\right) \tag{1.8.7}$$

Keeping the Pivot element as 3 and by using gauss-jordan Elimination we get

$$\left(\begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & c \\ 0 & \boxed{\frac{5}{3}} & 1 & \frac{-1}{3} & 5 \\ 1 & \frac{1}{3} & 0 & \frac{1}{3} & 5 \\ \hline 0 & -1 & 0 & 1 & 15 \end{array}\right) \tag{1.8.8}$$

Keeping the Pivot element as $\frac{5}{3}$ and by using gauss-jordan Elimination we get

$$\left(\begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & c \\ 0 & 1 & \frac{3}{5} & \frac{-1}{5} & 2 \\ 1 & 0 & \frac{-1}{5} & \frac{2}{5} & 3 \\ \hline 0 & 0 & \frac{3}{5} & \frac{4}{5} & 18 \end{array}\right) \tag{1.8.9}$$

In this tableau Since all indicators in last row are non-negative ,we found optimal solution to given problem. Therefore Optimal Solution will be:

$$(x_1, x_2) = (4, 3) \tag{1.8.10}$$
$$Z = 3x_1 + 2x_2 \tag{1.8.11}$$
$$Z = 3 \times 4 + 2 \times 3 \tag{1.8.12}$$
$$Z = 18 \tag{1.8.13}$$

This Problem can be represented in matrix form as follows,

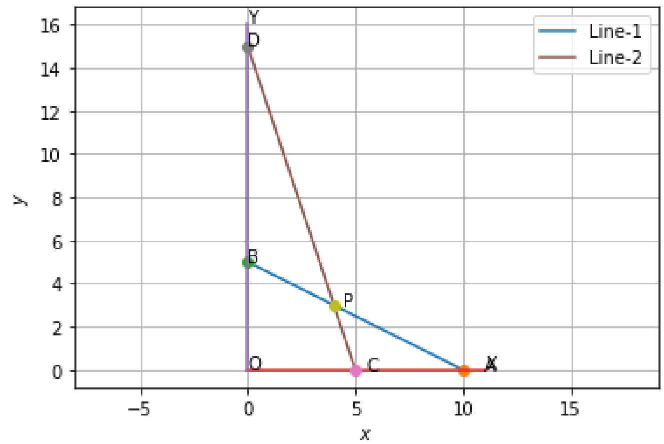

Fig. 1.8: Region OBPC is Valid region

$$\max_{\mathbf{x}} Z = \begin{pmatrix} 3 & 2 \end{pmatrix} \mathbf{x} \tag{1.8.14}$$
$$s.t. \quad \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} \mathbf{x} \leq \begin{pmatrix} 10 \\ 15 \end{pmatrix} \tag{1.8.15}$$
$$\mathbf{x} \geq \mathbf{0} \tag{1.8.16}$$
$$\mathbf{y} \geq \mathbf{0} \tag{1.8.17}$$

this is solved using cvxpy in python,we get

$$\mathbf{x} = \begin{pmatrix} 3.99999999 \\ 2.99999999 \end{pmatrix}, Z = 17.99999996 \tag{1.8.18}$$

1.9. Minimise Z=x+2y subject to 2x+y≥3, x+2y≥6, x,y≥0.
Show that the minimum of Z occurs at more than two points.
Solution: Here, to minimize the above given objective function, simplex method can be used.

The simplex algorithm for minimization problems works by converting the problem to a maximization problem, by forming an augmented matrix which includes given objective function in last row and constraints in above rows. It is transposed and then maximization is applied for the augmented matrix formed from those equations after transposing. Given objective function

$$f(x, y) = Z = x + 2y \tag{1.9.1}$$

subject to

$$2x + y \geq 3 \tag{1.9.2}$$
$$x + 2y \geq 6 \tag{1.9.3}$$
$$x, y \geq 0 \tag{1.9.4}$$

The matrix form of given objective function and constraints is given as below

$$\begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 6 \\ 1 & 2 & 0 \end{pmatrix} \tag{1.9.5}$$

After transposing the above matrix, we get

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 2 \\ 3 & 6 & 0 \end{pmatrix} \tag{1.9.6}$$

As per above transposed matrix, the constraints and objective function are as below.

$$f(x, y) = Z = 3x + 6y \tag{1.9.7}$$

subject to

$$2x + y \geq 1 \tag{1.9.8}$$
$$x + 2y \geq 2 \tag{1.9.9}$$
$$x, y \geq 0 \tag{1.9.10}$$

Apply Maximization now.
Adding slack variables s1, s2, the inequalities can be re-written as below.

$$f(x, y) = Z = 3x + 6y \Rightarrow Z - 3x - 6y = 0 \tag{1.9.11}$$
$$2x + y \geq 1 \Rightarrow 2x + y + s1 = 1 \tag{1.9.12}$$
$$x + 2y \geq 2 \Rightarrow x + 2y + s2 = 2 \tag{1.9.13}$$

The augmented matrix that can be formed using objective function and constraints is given as below.

$$\begin{pmatrix} 1 & -3 & -6 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 & 2 \end{pmatrix} \tag{1.9.14}$$

Here x and y are non-basic variables. s1, s2 are basic variables. Selecting x as entering variable.
Entering through x in row(1).
Reducing augmented matrix for eliminating x in row(1) through row(2) and row(3).

$$\begin{pmatrix} 1 & -3 & -6 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 & 2 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 + R_2 + R_3} \tag{1.9.15}$$

$$\xleftrightarrow{R_3 \leftarrow R_2 - 2R_3} \tag{1.9.16}$$

$$\begin{pmatrix} 1 & 0 & -3 & 1 & 1 & 3 \\ 0 & 2 & 1 & 1 & 0 & 1 \\ 0 & 0 & -3 & 1 & -2 & -3 \end{pmatrix} \tag{1.9.17}$$

eliminating x in row(1) through row(1).

$$\xleftrightarrow{R_2 \leftarrow R_2/2} \tag{1.9.18}$$

$$\begin{pmatrix} 1 & 0 & -3 & 1 & 1 & 3 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & -3 & 1 & -2 & -3 \end{pmatrix} \tag{1.9.19}$$

Here, the feasible solution may not be obtained since row(1) contains negative coefficients of non-basic variables. Hence the entering variable has to be changed to y(another non-basic variable). Entering through y in row(2) since ratio $\frac{1}{1}$ a positive number and minimum.
Reducing augmented matrix for eliminating y in row(1), row(2) through row(2) and row(3).

$$\begin{pmatrix} 1 & -3 & -6 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 & 2 \end{pmatrix} \tag{1.9.20}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + 3R_3} \tag{1.9.21}$$

$$\xleftrightarrow{R_2 \leftarrow 2R_3 - R_2} \tag{1.9.22}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 3 & 6 \\ 0 & 0 & 3 & -1 & 2 & 3 \\ 0 & 1 & 2 & 0 & 1 & 2 \end{pmatrix} \tag{1.9.23}$$

Here, the feasible solutions may be obtained as below and there may be multiple optimal solutions for Z since the row(1) contain zero coefficients of non-basic variables i.e. Z can take any any values of x and y. Hence, the objective function may have multiple minima points.

$$\Rightarrow (x, y) = (0, 0) \tag{1.9.24}$$
$$\Rightarrow Z = x + 2y \tag{1.9.25}$$
$$\Rightarrow Z = 0 + 0 = 0 \tag{1.9.26}$$

This Problem can be represented in matrix form as follows,

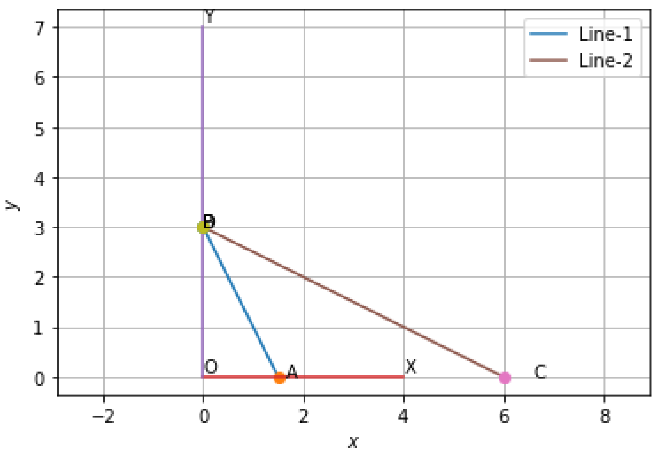

Fig. 1.9: Python plot .

$$\min_{\mathbf{x}} Z = \begin{pmatrix} 1 & 2 \end{pmatrix} \mathbf{x} \tag{1.9.27}$$

$$s.t. \quad \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \mathbf{x} \geq \begin{pmatrix} 3 \\ 6 \end{pmatrix} \tag{1.9.28}$$

$$\mathbf{x} \geq \mathbf{0} \tag{1.9.29}$$

$$\mathbf{y} \geq \mathbf{0} \tag{1.9.30}$$

The output as obtained through Python implementation:

$$\mathbf{x} = \begin{pmatrix} 6.89748716e-11 \\ 5.47282454e-10 \end{pmatrix}, Z = 1.16353978e-09 \tag{1.9.31}$$

1.10. Solve:

$$\max_{\{x\}} Z = \begin{pmatrix} 4 & 1 \end{pmatrix} \mathbf{x} \tag{1.10.1}$$

$$s.t \quad \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix} \leq \begin{pmatrix} 50 \\ 90 \end{pmatrix} \tag{1.10.2}$$

$$\mathbf{x} \geq 0 \tag{1.10.3}$$

Solution: Adding slack variables to the left side of (1.10.2) and (1.10.3),we get

$$Z - 4x - y = 0 \tag{1.10.4}$$

$$x + y + s_1 = 50 \tag{1.10.5}$$

$$3x + y + s_2 = 90 \tag{1.10.6}$$

Forming simplex tableau,

$$\left(\begin{array}{ccccc} x & y & s_1 & s_2 & b \\ \hline 1 & 1 & 1 & 0 & 50 \\ 3 & 1 & 0 & 1 & 90 \\ \hline -4 & -1 & 0 & 0 & 0 \end{array}\right) \tag{1.10.7}$$

-4 is the smallest entry in the bottom row. Therefore, we determine that x is the starting variable.
Also, the smallest positive ratio is 30, therefore, we chose s_2 as the departing variable.
Hence, keeping the pivot element as 3, we perform Gauss Jordan elimination,

$$\left(\begin{array}{ccccc} x & y & s_1 & s_2 & b \\ 1 & 1 & 1 & 0 & 50 \\ 1 & \frac{1}{3} & 0 & \frac{1}{3} & 30 \\ \hline -4 & -1 & 0 & 0 & 0 \end{array}\right) \tag{1.10.8}$$

$$\left(\begin{array}{ccccc} x & y & s_1 & s_2 & b \\ 0 & \frac{2}{3} & 1 & \frac{-1}{3} & 20 \\ 1 & \frac{1}{3} & 0 & \frac{1}{3} & 30 \\ \hline 0 & \frac{1}{3} & 0 & \frac{4}{3} & 120 \end{array}\right) \tag{1.10.9}$$

Note that x has replaced in the basis column s_2 and the improved solution

$$(x, y, s_1, s_2) = (30, 0, 20, 0) \tag{1.10.10}$$

maximizes Z to value

$$Z = 4(30) + 3(0) \tag{1.10.11}$$

$$Z = 120 \tag{1.10.12}$$

The given problem can be expressed in the form of matrix inequality as:

$$\max_{\{x\}} \mathbf{c}^T \mathbf{x} \tag{1.10.13}$$

$$s.t \quad \mathbf{A}\mathbf{x} \leq \mathbf{b} \tag{1.10.14}$$

$$\mathbf{x} \geq 0 \tag{1.10.15}$$

$$\tag{1.10.16}$$

where

$$\mathbf{c} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \tag{1.10.17}$$

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix} \tag{1.10.18}$$

$$\mathbf{b} = \begin{pmatrix} 50 \\ 90 \end{pmatrix} \tag{1.10.19}$$

can be solved using Python. The plot obtained from python is attached below:

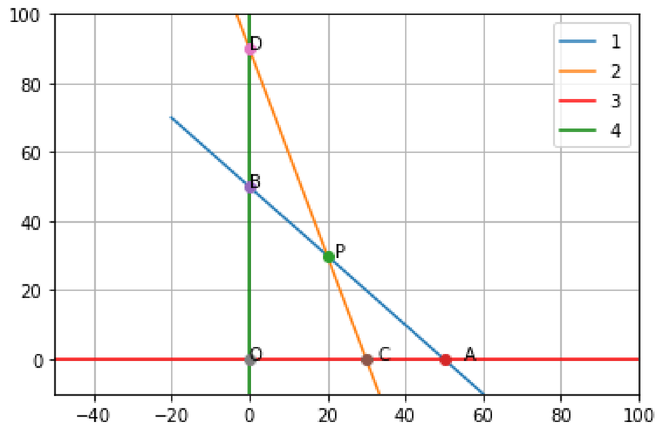

Fig. 1.10: Plot obtained from python code

1.11. Minimise and Maximise Z=5x+10y subject to x+2y≤120, x+y≥60, x-2y≥0, x,y≥0.
Solution: First we will plot these lines which are the constraints and the area enclosed by is the region we are interested in.

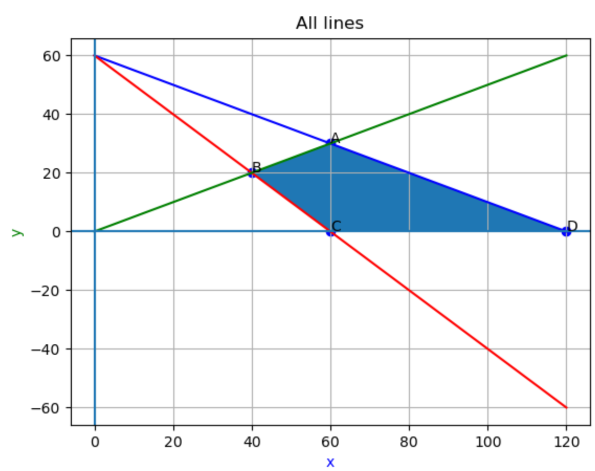

Fig. 1.11: optimal point through the intersection of various lines

The four points are the points which will maximize and minimize the function.These corner points are :

$$A = (60, 30)$$
$$B = (40, 20)$$
$$C = (60, 0)$$
$$D = (120, 0)$$

Value of Z at point

$$A = 5 \times 60 + 10 \times 30 = 600$$
$$B = 5 \times 40 + 10 \times 20 = 400$$
$$C = 5 \times 60 + 10 \times 0 = 300$$
$$D = 5 \times 120 + 10 \times 0 = 600$$

We can see that our function Z is maximum at points A and D that is (60,30) and (120,0)
and
Z is minimum at point C that is (60,0)
The given problem can be expressed in general as matrix inequality as:

$$\max_{\mathbf{x}} Z = \begin{pmatrix} 5 & 10 \end{pmatrix} \mathbf{x} \tag{1.11.1}$$

$$s.t. \quad \begin{pmatrix} 1 & 2 \\ -1 & -1 \\ -1 & 2 \end{pmatrix} \mathbf{x} \le \begin{pmatrix} 120 \\ -60 \\ 0 \end{pmatrix} \tag{1.11.2}$$

$$\mathbf{x} \ge \mathbf{0} \tag{1.11.3}$$

$$\mathbf{y} \ge \mathbf{0} \tag{1.11.4}$$

$$\max_{\mathbf{x}} \mathbf{c}^T \mathbf{x} \tag{1.11.5}$$

$$s.t. \quad \mathbf{A}\mathbf{x} \le \mathbf{b}, \tag{1.11.6}$$

$$\mathbf{x} \ge \mathbf{0} \tag{1.11.7}$$

$$\mathbf{y} \ge \mathbf{0} \tag{1.11.8}$$

where

$$\mathbf{c} = \begin{pmatrix} 5 \\ 10 \end{pmatrix} \tag{1.11.9}$$

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -1 & -1 \\ -1 & 2 \end{pmatrix} \tag{1.11.10}$$

$$\mathbf{b} = \begin{pmatrix} 120 \\ -60 \\ 0 \end{pmatrix} \tag{1.11.11}$$

1.12. Minimise and Maximise Z=x+2y subject to x+2y≥100, 2x-y≤0, 2x+y≤200; x,y≥0.
Solution: In order to obtain the maximum and minimum value we need to solve the system of inequalities by adding slack variables. The equations now become:

$$x + 2y - Z = 0 \tag{1.12.1}$$

$$x + 2y - S_1 = 100 \tag{1.12.2}$$

$$2x - y + S_2 = 0 \tag{1.12.3}$$

$$2x + y + S_3 = 200 \tag{1.12.4}$$

The simplex table can be formed as

$$\left(\begin{array}{cccccc} x & y & s_1 & s_2 & s_3 & b \\ \hline 1 & 2 & -1 & 0 & 0 & 50 \\ 2 & -1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 200 \\ \hline 1 & 2 & 0 & 0 & 0 & 0 \end{array}\right) \tag{1.12.5}$$

The pivot element is 2 as the minimum ratio 50 occurs for y as the entering variable. Now reducing the simplex matrix we get

$$\left(\begin{array}{cccccc} x & y & s_1 & s_2 & s_3 & b \\ \hline \frac{1}{2} & 1 & \frac{-1}{2} & 0 & 0 & 50 \\ \frac{5}{2} & 0 & \frac{-1}{2} & 1 & 0 & 50 \\ \frac{3}{2} & 0 & \frac{1}{2} & 0 & 1 & 150 \\ \hline 1 & 2 & 0 & 0 & 0 & 0 \end{array}\right) \tag{1.12.6}$$

This can be expressed in the form of matrix inequality for maximization and minimization respectively as:

$$\max_{\{x\}} \mathbf{c}^T\mathbf{x} \tag{1.12.7}$$

$$s.t \quad \mathbf{A}\mathbf{x} \le \mathbf{b}; \mathbf{x} \ge 0 \tag{1.12.8}$$

$$\min_{\{x\}} \mathbf{c}^T\mathbf{x} \tag{1.12.9}$$

$$s.t \quad \mathbf{A}\mathbf{x} \ge \mathbf{b}; \mathbf{x} \ge 0 \tag{1.12.10}$$

where

$$\mathbf{c} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{1.12.11}$$

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 2 & -1 \\ 2 & 1 \end{pmatrix} \tag{1.12.12}$$

$$\mathbf{b} = \begin{pmatrix} 100 \\ 0 \\ 200 \end{pmatrix} \tag{1.12.13}$$

Solving for Z by this reduction method we get

$$MaxZ = 400 \tag{1.12.14}$$

$$MinZ = 100 \tag{1.12.15}$$

This can be solved in Python which generates the result as shown in Fig 1.

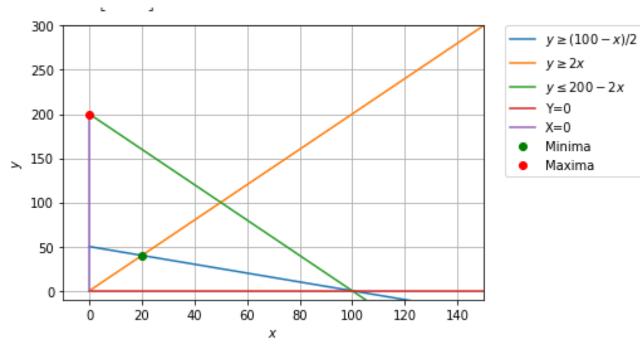

Fig. 1: Plot from python code with Maxima and Minima points

1.13. Reshma wishes to mix two types of food P and Q in such a way that the vitamin contents of the mixture contain at least 8 units of vitamin A and 11 units of vitamin B. Food P costs Rs 60/kg and Food Q costs Rs 80/kg. Food P contains 3 units/kg of Vitamin A and 5 units/kg of Vitamin B while food Q contains 4 units/kg of Vitamin A and 2 units/kg of vitamin B. Determine the minimum cost of the mixture.

Solution:

Food	Vitamin A	Vitamin B	Cost
P	3 units/kg	5 units/kg	60 Rs/kg
Q	4 units/kg	2 units/kg	80 Rs/kg
Requirement	8 units/kg	11 units/kg	

TABLE 1.13: Food Requirements

Let the mixture contain x kg of food P and y kg of food Q be y such that

$$x \ge 0 \tag{1.13.1}$$

$$y \ge 0 \tag{1.13.2}$$

According to the question,

$$3x + 4y \ge 8 \tag{1.13.3}$$

$$5x + 2y \ge 11 \tag{1.13.4}$$

$\therefore$ Our problem is

$$\min_{\mathbf{x}} Z = \begin{pmatrix} 60 & 80 \end{pmatrix} \mathbf{x} \tag{1.13.5}$$

$$s.t. \quad \begin{pmatrix} 3 & 4 \\ 5 & 2 \end{pmatrix} \mathbf{x} \le \begin{pmatrix} 8 \\ 11 \end{pmatrix} \tag{1.13.6}$$

$$\mathbf{x} \le \mathbf{0} \tag{1.13.7}$$

Lagrangian function is given by

$$\begin{aligned} & L(\mathbf{x}, \lambda) \\ & = \begin{pmatrix} 60 & 80 \end{pmatrix} \mathbf{x} + \left\{ \left[\begin{pmatrix} 3 & 4 \end{pmatrix} \mathbf{x} - 8 \right] \right. \\ & + \left[\begin{pmatrix} 5 & 2 \end{pmatrix} \mathbf{x} - 11 \right] \\ & \left. + \left[\begin{pmatrix} -1 & 0 \end{pmatrix} \mathbf{x} \right] + \left[\begin{pmatrix} 0 & -1 \end{pmatrix} \mathbf{x} \right] \right\} \lambda \end{aligned} \tag{1.13.8}$$

where,

$$\boldsymbol{\lambda} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} \tag{1.13.9}$$

Now,

$$\nabla L(\mathbf{x}, \boldsymbol{\lambda}) = \begin{pmatrix} 60 + \begin{pmatrix} 3 & 5 & -1 & 0 \end{pmatrix} \boldsymbol{\lambda} \\ 80 + \begin{pmatrix} 4 & 2 & 0 & -1 \end{pmatrix} \boldsymbol{\lambda} \\ \begin{pmatrix} 3 & 4 \end{pmatrix} \mathbf{x} - 8 \\ \begin{pmatrix} 5 & 2 \end{pmatrix} \mathbf{x} - 11 \\ \begin{pmatrix} -1 & 0 \end{pmatrix} \mathbf{x} \\ \begin{pmatrix} 0 & -1 \end{pmatrix} \mathbf{x} \end{pmatrix} \tag{1.13.10}$$

$\therefore$ Lagrangian matrix is given by

$$\begin{pmatrix} 0 & 0 & 3 & 5 & -1 & 0 \\ 0 & 0 & 4 & 2 & 0 & -1 \\ 3 & 4 & 0 & 0 & 0 & 0 \\ 5 & 2 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} -60 \\ -80 \\ 8 \\ 11 \\ 0 \\ 0 \end{pmatrix} \tag{1.13.11}$$

Considering λ_1, λ_2 as only active multiplier,

$$\begin{pmatrix} 0 & 0 & 3 & 5 \\ 0 & 0 & 4 & 2 \\ 3 & 4 & 0 & 0 \\ 5 & 2 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} -60 \\ -80 \\ 8 \\ 11 \end{pmatrix} \tag{1.13.12}$$

resulting in,

$$\begin{pmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 3 & 5 \\ 0 & 0 & 4 & 2 \\ 3 & 4 & 0 & 0 \\ 5 & 2 & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} -60 \\ -80 \\ 8 \\ 11 \end{pmatrix} \tag{1.13.13}$$

$$\implies \begin{pmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{-28}{196} & \frac{56}{196} \\ 0 & 0 & \frac{70}{196} & \frac{-42}{196} \\ \frac{-28}{196} & \frac{70}{196} & 0 & 0 \\ \frac{56}{196} & \frac{-42}{196} & 0 & 0 \end{pmatrix} \begin{pmatrix} -60 \\ -80 \\ 8 \\ 11 \end{pmatrix} \tag{1.13.14}$$

$$\implies \begin{pmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} 2 \\ \frac{1}{2} \\ -20 \\ 0 \end{pmatrix} \tag{1.13.15}$$

$\because \boldsymbol{\lambda} = \begin{pmatrix} -20 \\ 0 \end{pmatrix} \leq \mathbf{0}$

$\therefore$ Optimal solution is given by

$$\mathbf{x} = \begin{pmatrix} 2 \\ \frac{1}{2} \end{pmatrix} \tag{1.13.16}$$

$$Z = \begin{pmatrix} 60 & 80 \end{pmatrix} \mathbf{x} \tag{1.13.17}$$

$$= \begin{pmatrix} 60 & 80 \end{pmatrix} \begin{pmatrix} 2 \\ \frac{1}{2} \end{pmatrix} \tag{1.13.18}$$

$$= 160 \tag{1.13.19}$$

By using cvxpy in python,

$$\mathbf{x} = \begin{pmatrix} 2.11436237 \\ 0.41422822 \end{pmatrix} \tag{1.13.20}$$

$$Z = 159.99999999 \tag{1.13.21}$$

Fig. 1.13 verifies this result.

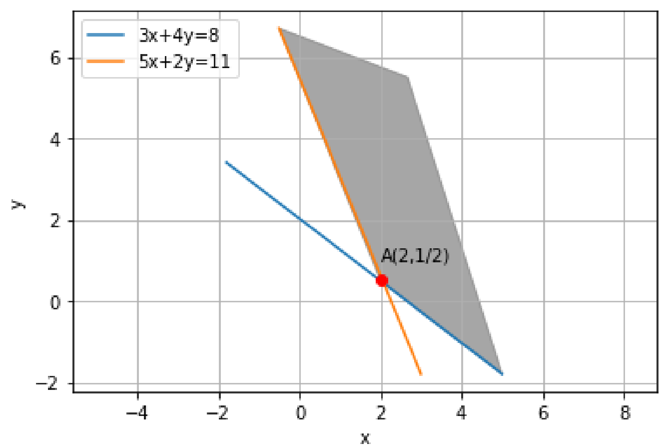

Fig. 1.13: Graphical Solution

1.14. One kind of cake requires 200g of flour and 25g of fat, and another kind of cake requires 100g of flour and 50g of fat. Find the maximum number of cakes which can be made from 5kg of flour and 1 kg of fat assuming that there is no shortage of the other ingredients used in making the cakes.
Solution:

kind of cake	No.of cakes	Flour	Fat
1st	x	200g	25g
2nd	y	100g	50g
Total	x+y	5 kg=5000g	1kg=1000g

TABLE 1.14: Ingredients used in making the cake is flour and fat

Let the 1st kind be x and the 2nd kind be y

such that

$$x \geq 0 \tag{1.14.1}$$
$$y \geq 0 \tag{1.14.2}$$

According to the question,

$$2x + y \leq 50 \tag{1.14.3}$$
$$x + 2y \leq 40 \tag{1.14.4}$$

$\therefore$ Our problem is

$$\max_{\mathbf{x}} Z = \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} \tag{1.14.5}$$
$$s.t. \quad \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \mathbf{x} \leq \begin{pmatrix} 50 \\ 40 \end{pmatrix} \tag{1.14.6}$$

Lagrangian function is given by

$$\begin{aligned} &L(\mathbf{x}, \boldsymbol{\lambda}) \\ &= \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} + \left\{ \left[\begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} - 50 \right] \right. \\ &+ \left[\begin{pmatrix} 1 & 2 \end{pmatrix} \mathbf{x} - 40 \right] \\ &+ \left. \left[\begin{pmatrix} -1 & 0 \end{pmatrix} \mathbf{x} \right] + \left[\begin{pmatrix} 0 & -1 \end{pmatrix} \mathbf{x} \right] \right\} \boldsymbol{\lambda} \end{aligned} \tag{1.14.7}$$

where,

$$\boldsymbol{\lambda} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \end{pmatrix} \tag{1.14.8}$$

Now,

$$\nabla L(\mathbf{x}, \boldsymbol{\lambda}) = \begin{pmatrix} 1 + \begin{pmatrix} 2 & 1 & -1 & 0 \end{pmatrix} \boldsymbol{\lambda} \\ 1 + \begin{pmatrix} 1 & 2 & 0 & -1 \end{pmatrix} \boldsymbol{\lambda} \\ \begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} - 50 \\ \begin{pmatrix} 1 & 2 \end{pmatrix} \mathbf{x} - 40 \\ \begin{pmatrix} -1 & 0 \end{pmatrix} \mathbf{x} \\ \begin{pmatrix} 0 & -1 \end{pmatrix} \mathbf{x} \end{pmatrix} \tag{1.14.9}$$

$\therefore$ Lagrangian matrix is given by

$$\begin{pmatrix} 0 & 0 & 2 & 1 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 & -1 \\ 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 50 \\ 4\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{1.14.10}$$

Considering λ_1, λ_2 as only active multiplier,

$$\begin{pmatrix} 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \\ 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 5\ 0 \\ 40 \end{pmatrix} \tag{1.14.11}$$

resulting in,

$$\begin{pmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \\ 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} -1 \\ -1 \\ 50 \\ 40 \end{pmatrix} \tag{1.14.12}$$

$$\implies \begin{pmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{2}{3} & \frac{-1}{3} \\ 0 & 0 & \frac{-1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{-1}{3} & 0 & 0 \\ \frac{-1}{3} & \frac{2}{3} & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \\ 50 \\ 40 \end{pmatrix} \tag{1.14.13}$$

$$\implies \begin{pmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} 20 \\ 10 \\ -0.3 \\ -0.3 \end{pmatrix} \tag{1.14.14}$$

$\because \boldsymbol{\lambda} = \begin{pmatrix} -0.3 \\ -0.3 \end{pmatrix} \succ \mathbf{0}$

$\therefore$ Optimal solution is given by

$$\mathbf{x} = \begin{pmatrix} 20 \\ 10 \end{pmatrix} \tag{1.14.15}$$
$$Z = \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} \tag{1.14.16}$$
$$= \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 20 \\ 10 \end{pmatrix} \tag{1.14.17}$$
$$= 60 \tag{1.14.18}$$

By using cvxpy in python ,

$$\mathbf{x} = \begin{pmatrix} 20 \\ 10 \end{pmatrix} \tag{1.14.19}$$
$$Z = 60 \tag{1.14.20}$$

Hence No.of cakes $\boxed{x = 20}$ 1st kind and .of cakes $\boxed{y = 10}$ 2nd kind should be used to maximum No. of cakes $\boxed{Z = 60}$. This is verified in Fig. 1.14.

1.15. A factory makes tennis rackets and cricket bats. A tennis racket takes 1.5 hours of machine time and 3 hours of craftman's time in its making while a cricket bat takes 3 hour of machine time and 1 hour of craftman's time. In a day, the factory has the availability of not more than 42 hours of machine time and 24 hours of

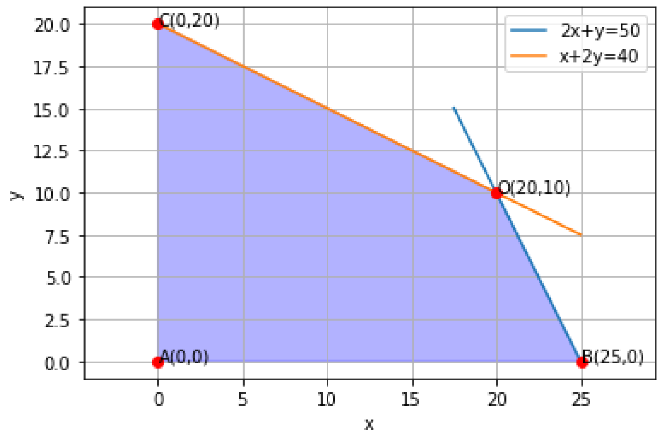

Fig. 1.14: Graphical Solution

craftsman's time.
(i) What number of rackets and bats must be made if the factory is to work at full capacity?
(ii)If the profit on a racket and on a bat is Rs 20 and Rs 10 respectively, find the maximum profit of the factory when it works at full capacity.
Solution:

item	Machine hours	Craftman's hours	profit
Tennis Racket	1.5	3	20
Cricket Bats	3	1	10
Maximum time Available	42	24	

TABLE 1.15: factory Requirements

Let the number of Tennis Rackets be x and the number of cricket bats be y such that

$$x \geq 0 \tag{1.15.1}$$

$$y \geq 0 \tag{1.15.2}$$

According to the question,

$$1.5x + 3y \leq 42 \tag{1.15.3}$$

$$\implies 3x + 6y \leq 84 \tag{1.15.4}$$

$$\implies x + 2y \leq 28 \tag{1.15.5}$$

and,

$$3x + y \leq 24 \tag{1.15.6}$$

$\therefore$ Our problem is

$$\max_{\mathbf{x}} Z = \begin{pmatrix} 20 & 10 \end{pmatrix} \mathbf{x} \tag{1.15.7}$$

$$s.t. \quad \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} \mathbf{x} \leq \begin{pmatrix} 28 \\ 24 \end{pmatrix} \tag{1.15.8}$$

Lagrangian function is given by

$$\begin{aligned} & L(\mathbf{x}, \boldsymbol{\lambda}) \\ &= \begin{pmatrix} 20 & 10 \end{pmatrix} \mathbf{x} + \left\{ \left[\begin{pmatrix} 1 & 2 \end{pmatrix} \mathbf{x} - 28 \right] \right. \\ &+ \left[\begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} - 24 \right] \\ &+ \left. \left[\begin{pmatrix} -1 & 0 \end{pmatrix} \mathbf{x} \right] + \left[\begin{pmatrix} 0 & -1 \end{pmatrix} \mathbf{x} \right] \right\} \boldsymbol{\lambda} \end{aligned} \tag{1.15.9}$$

where,

$$\boldsymbol{\lambda} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \end{pmatrix} \tag{1.15.10}$$

Now,

$$\nabla L(\mathbf{x}, \boldsymbol{\lambda}) = \begin{pmatrix} 20 + \begin{pmatrix} 1 & 3 & -1 & 0 \end{pmatrix} \boldsymbol{\lambda} \\ 10 + \begin{pmatrix} 2 & 1 & 0 & -1 \end{pmatrix} \boldsymbol{\lambda} \\ \begin{pmatrix} 1 & 2 \end{pmatrix} \mathbf{x} - 28 \\ \begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} - 24 \\ \begin{pmatrix} -1 & 0 \end{pmatrix} \mathbf{x} \\ \begin{pmatrix} 0 & -1 \end{pmatrix} \mathbf{x} \end{pmatrix} \tag{1.15.11}$$

$\therefore$ Lagrangian matrix is given by

$$\begin{pmatrix} 0 & 0 & 1 & 3 & -1 & 0 \\ 0 & 0 & 2 & 1 & 0 & -1 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} -20 \\ -10 \\ 28 \\ 24 \\ 0 \\ 0 \end{pmatrix} \tag{1.15.12}$$

Considering λ_1, λ_2 as only active multiplier,

$$\begin{pmatrix} 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & 1 \\ 1 & 2 & 0 & 0 \\ 3 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} -20 \\ -10 \\ 28 \\ 24 \end{pmatrix} \tag{1.15.13}$$

resulting in,

$$\begin{pmatrix}\mathbf{x}\\ \boldsymbol{\lambda}\end{pmatrix} = \begin{pmatrix}0 & 0 & 1 & 3\\ 0 & 0 & 2 & 1\\ 1 & 2 & 0 & 0\\ 3 & 1 & 0 & 0\end{pmatrix}^{-1}\begin{pmatrix}-20\\ -10\\ 28\\ 24\end{pmatrix} \tag{1.15.14}$$

$$\implies \begin{pmatrix}\mathbf{x}\\ \boldsymbol{\lambda}\end{pmatrix} = \begin{pmatrix}0 & 0 & \frac{-1}{5} & \frac{2}{5}\\ 0 & 0 & \frac{3}{5} & \frac{-1}{5}\\ \frac{-1}{5} & \frac{3}{5} & 0 & 0\\ \frac{2}{5} & \frac{-1}{5} & 0 & 0\end{pmatrix}\begin{pmatrix}-20\\ -10\\ 28\\ 24\end{pmatrix} \tag{1.15.15}$$

$$\implies \begin{pmatrix}\mathbf{x}\\ \boldsymbol{\lambda}\end{pmatrix} = \begin{pmatrix}4\\ 12\\ -2\\ -6\end{pmatrix} \tag{1.15.16}$$

$\because \boldsymbol{\lambda} = \begin{pmatrix}-2\\ -6\end{pmatrix} \succ \mathbf{0}$

$\therefore$ Optimal solution is given by

$$\mathbf{x} = \begin{pmatrix}4\\ 12\end{pmatrix} \tag{1.15.17}$$

$$Z = \begin{pmatrix}20 & 10\end{pmatrix}\mathbf{x} \tag{1.15.18}$$

$$= \begin{pmatrix}20 & 10\end{pmatrix}\begin{pmatrix}4\\ 12\end{pmatrix} \tag{1.15.19}$$

$$= 200 \tag{1.15.20}$$

By using cvxpy in python ,

$$\mathbf{x} = \begin{pmatrix}3.99999998\\ 12.0000000\end{pmatrix} \tag{1.15.21}$$

$$Z = 199.99999964 \tag{1.15.22}$$

Hence , $\boxed{x = 4}$ Tennis Rackets and $\boxed{y = 12}$ Cricket Bats should be used to maximum time Available profit $\boxed{Z = 200}$ as can be verified from Fig. 1.15.

a) 4 Tennis Rackets and 12 Cricket Bats must be made so that factory runs at full capacity.
b) Maximum profit is Rs 200, When 4 Tennis Bats and 12 Cricket Bats are produced.

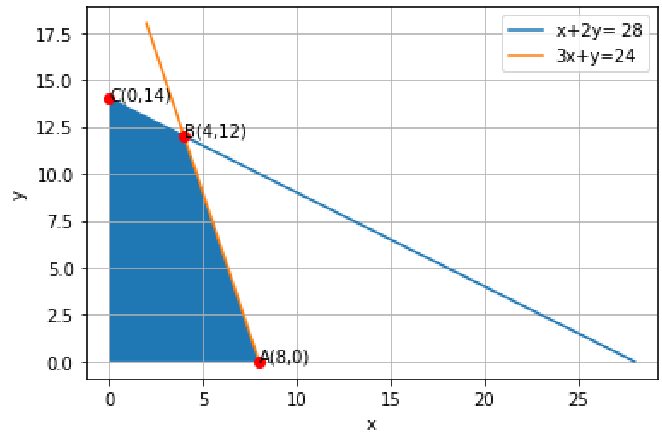

Fig. 1.15: Graphical Solution

1.16. A manufacturer produces nuts and bolts. It takes 1 hour of work on machine A and 3 hours on machine B to produce a package of nuts. It takes 3 hours on machine A and 1 hour on machine B to produce a package of bolts. He earns a profit of Rs17.50 per package on nuts and Rs 7.00 per package on bolts. How many packages of each should be produced each day so as to maximise his profit, if he operates his machines for at the most 12 hours a day?

Solution:

item	Machine A	Machine N	Earn profit
Nuts	1	3	17.5
Bolts	3	1	7
Hour's a day	12	12	

TABLE 1.16: Manufacturer produces nuts and bolts

Let the number of nuts be x and the number of bolts be y such that

$$x \geq 0 \tag{1.16.1}$$

$$y \geq 0 \tag{1.16.2}$$

According to the question,

$$x + 3y \leq 12 \tag{1.16.3}$$

$$3x + y \leq 12 \tag{1.16.4}$$

$\therefore$ Our problem is

$$\max_{\mathbf{x}} Z = \begin{pmatrix}17.5 & 7\end{pmatrix}\mathbf{x} \tag{1.16.5}$$

$$s.t. \quad \begin{pmatrix}1 & 3\\ 3 & 1\end{pmatrix}\mathbf{x} \preceq \begin{pmatrix}12\\ 12\end{pmatrix} \tag{1.16.6}$$

Lagrangian function is given by

$$\begin{aligned}&L(\mathbf{x}, \boldsymbol{\lambda})\\ &= \begin{pmatrix}17.5 & 7\end{pmatrix}\mathbf{x} + \Big\{\Big[\begin{pmatrix}1 & 3\end{pmatrix}\mathbf{x} - 12\Big]\\ &+ \Big[\begin{pmatrix}3 & 1\end{pmatrix}\mathbf{x} - 12\Big]\\ &+ \Big[\begin{pmatrix}-1 & 0\end{pmatrix}\mathbf{x}\Big] + \Big[\begin{pmatrix}0 & -1\end{pmatrix}\mathbf{x}\Big]\Big\}\boldsymbol{\lambda}\end{aligned} \tag{1.16.7}$$

where,

$$\boldsymbol{\lambda} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \end{pmatrix} \tag{1.16.8}$$

Now,

$$\nabla L(\mathbf{x}, \boldsymbol{\lambda}) = \begin{pmatrix} 17.5 + \begin{pmatrix} 1 & 3 & -1 & 0 \end{pmatrix} \boldsymbol{\lambda} \\ 7 + \begin{pmatrix} 3 & 1 & 0 & -1 \end{pmatrix} \boldsymbol{\lambda} \\ \begin{pmatrix} 1 & 3 \end{pmatrix} \mathbf{x} - 12 \\ \begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} - 12 \\ \begin{pmatrix} -1 & 0 \end{pmatrix} \mathbf{x} \\ \begin{pmatrix} 0 & -1 \end{pmatrix} \mathbf{x} \end{pmatrix} \tag{1.16.9}$$

$\therefore$ Lagrangian matrix is given by

$$\begin{pmatrix} 0 & 0 & 1 & 3 & -1 & 0 \\ 0 & 0 & 3 & 1 & 0 & -1 \\ 1 & 3 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} -17.5 \\ -7 \\ 12 \\ 12 \\ 0 \\ 0 \end{pmatrix} \tag{1.16.10}$$

Considering λ_1, λ_2 as only active multiplier,

$$\begin{pmatrix} 0 & 0 & 1 & 3 \\ 0 & 0 & 3 & 1 \\ 1 & 3 & 0 & 0 \\ 3 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} -17.5 \\ -7 \\ 12 \\ 12 \end{pmatrix} \tag{1.16.11}$$

resulting in,

$$\begin{pmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 3 \\ 0 & 0 & 3 & 1 \\ 1 & 3 & 0 & 0 \\ 3 & 1 & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} -17.5 \\ -7 \\ 12 \\ 12 \end{pmatrix} \tag{1.16.12}$$

$$\implies \begin{pmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{-1}{8} & \frac{3}{8} \\ 0 & 0 & \frac{3}{8} & \frac{-1}{8} \\ \frac{-1}{8} & \frac{3}{8} & 0 & 0 \\ \frac{3}{8} & \frac{-1}{8} & 0 & 0 \end{pmatrix} \begin{pmatrix} -17.5 \\ -7 \\ 12 \\ 12 \end{pmatrix} \tag{1.16.13}$$

$$\implies \begin{pmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ -0.5 \\ -5.7 \end{pmatrix} \tag{1.16.14}$$

$$\because \boldsymbol{\lambda} = \begin{pmatrix} -0.5 \\ -5.7 \end{pmatrix} \succ \mathbf{0}$$

$\therefore$ Optimal solution is given by

$$\mathbf{x} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} \tag{1.16.15}$$

$$Z = \begin{pmatrix} 17.5 & 7 \end{pmatrix} \mathbf{x} \tag{1.16.16}$$

$$= \begin{pmatrix} 17.5 & 7 \end{pmatrix} \begin{pmatrix} 3 \\ 3 \end{pmatrix} \tag{1.16.17}$$

$$= 73.5 \tag{1.16.18}$$

By using cvxpy in python ,

$$\mathbf{x} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} \tag{1.16.19}$$

$$Z = 73.49999997 \tag{1.16.20}$$

Hence, $\boxed{x = 3}$ Nuts and $\boxed{y = 3}$ Bolts should be used to maximum time Available profit $\boxed{Z = 73.5}$. This is verified in Fig. 1.16.

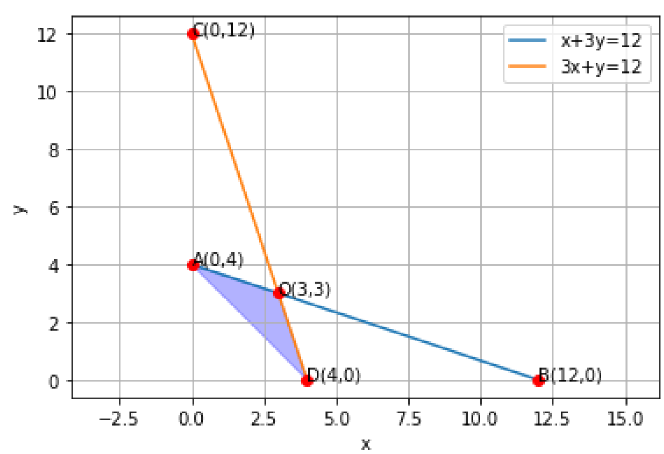

Fig. 1.16: Graphical Solution

1.17. A factory manufactures two types of screws, A and B. Each type of screw requires the use of two machines, an automatic and a hand operated. It takes 4 minutes on the automatic and 6 minutes on hand operated machines to manufacture a package of screws A, while it takes 6 minutes on automatic and 3 minutes on the hand operated machines to manufacture a package of screws B. Each machine is available for at the most 4 hours on any day. The manufacturer can sell a package of screws A at a profit of Rs 7 and screws B at a profit of Rs 10. Assuming that he can sell all the screws he manufactures, how many packages of each type should the factory owner produce in a day in order to maximise his profit? Determine the

maximum profit.
Solution:

Item	Number	Machine A	Machine B	Profit
Screw A	x	4 minutes	6 minutes	Rs 7
Screw B	y	6 minutes	3 minutes	Rs 10
Max Available Time		4hours =240minutes	4hours =240minutes	

TABLE 1.17: Screw Requirements

Let the number of packages of screw A be x and the number of packages of screw B be y such that

$$x \geq 0 \tag{1.17.1}$$

$$y \geq 0 \tag{1.17.2}$$

According to the question,

$$4x + 6y \leq 240 \tag{1.17.3}$$

$$\implies 2x + 3y \leq 120 \tag{1.17.4}$$

and,

$$6x + 3y \leq 240 \tag{1.17.5}$$

$$\implies 2x + y \leq 80 \tag{1.17.6}$$

$\therefore$ Our problem is

$$\max_{\mathbf{x}} Z = \begin{pmatrix} 7 & 10 \end{pmatrix} \mathbf{x} \tag{1.17.7}$$

$$s.t. \quad \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \mathbf{x} \leq \begin{pmatrix} 120 \\ 80 \end{pmatrix} \tag{1.17.8}$$

Lagrangian function is given by

$$\begin{aligned} &L(\mathbf{x}, \boldsymbol{\lambda}) \\ &= \begin{pmatrix} 7 & 10 \end{pmatrix} \mathbf{x} + \left\{ \left[\begin{pmatrix} 2 & 3 \end{pmatrix} \mathbf{x} + 120 \right] \right. \\ &+ \left[\begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} + 80 \right] \\ &+ \left. \left[\begin{pmatrix} -1 & 0 \end{pmatrix} \mathbf{x} \right] + \left[\begin{pmatrix} 0 & -1 \end{pmatrix} \mathbf{x} \right] \right\} \boldsymbol{\lambda} \end{aligned} \tag{1.17.9}$$

where,

$$\boldsymbol{\lambda} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \end{pmatrix} \tag{1.17.10}$$

Now,

$$\nabla L(\mathbf{x}, \boldsymbol{\lambda}) = \begin{pmatrix} 7 + \begin{pmatrix} 2 & 3 & -1 & 0 \end{pmatrix} \boldsymbol{\lambda} \\ 10 + \begin{pmatrix} 2 & 1 & 0 & -1 \end{pmatrix} \boldsymbol{\lambda} \\ \begin{pmatrix} 2 & 3 \end{pmatrix} \mathbf{x} + 120 \\ \begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} + 80 \\ \begin{pmatrix} -1 & 0 \end{pmatrix} \mathbf{x} \\ \begin{pmatrix} 0 & -1 \end{pmatrix} \mathbf{x} \end{pmatrix} \tag{1.17.11}$$

$\therefore$ Lagrangian matrix is given by

$$\begin{pmatrix} 0 & 0 & 2 & 3 & -1 & 0 \\ 0 & 0 & 2 & 1 & 0 & -1 \\ 2 & 3 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} -5 \\ -6 \\ 200 \\ 240 \\ 0 \\ 0 \end{pmatrix} \tag{1.17.12}$$

Considering λ_1, λ_2 as only active multiplier,

$$\begin{pmatrix} 0 & 0 & 2 & 3 \\ 0 & 0 & 2 & 1 \\ 2 & 3 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} -7 \\ -10 \\ 120 \\ 80 \end{pmatrix} \tag{1.17.13}$$

resulting in,

$$\begin{pmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 & 3 \\ 0 & 0 & 2 & 1 \\ 2 & 3 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} -7 \\ -10 \\ 120 \\ 80 \end{pmatrix} \tag{1.17.14}$$

$$\implies \begin{pmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{-1}{4} & \frac{3}{4} \\ 0 & 0 & \frac{1}{2} & \frac{-1}{2} \\ \frac{-1}{4} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{-1}{2} & 0 & 0 \end{pmatrix} \begin{pmatrix} -7 \\ -10 \\ 120 \\ 80 \end{pmatrix} \tag{1.17.15}$$

$$\implies \begin{pmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} 30 \\ 20 \\ \frac{-13}{4} \\ \frac{3}{2} \end{pmatrix} \tag{1.17.16}$$

$$\because \boldsymbol{\lambda} = \begin{pmatrix} \frac{-13}{4} \\ \frac{3}{2} \end{pmatrix} \succ \mathbf{0}$$

∴ Optimal solution is given by

$$\mathbf{x} = \begin{pmatrix} 30 \\ 20 \end{pmatrix} \tag{1.17.17}$$

$$Z = \begin{pmatrix} 7 & 10 \end{pmatrix} \mathbf{x} \tag{1.17.18}$$

$$= \begin{pmatrix} 7 & 10 \end{pmatrix} \begin{pmatrix} 30 \\ 20 \end{pmatrix} \tag{1.17.19}$$

$$= 410 \tag{1.17.20}$$

By using cvxpy in python ,

$$\mathbf{x} = \begin{pmatrix} 30.00000000 \\ 20.00000000 \end{pmatrix} \tag{1.17.21}$$

$$Z = 410.00000000 \tag{1.17.22}$$

Hence , $\boxed{x = 30}$ packages of screw A and $\boxed{y = 20}$ packages of screw B should be the factory owner produce in a day in order to maximise his profit is $\boxed{Z = 410}$. This is verified in Fig. 1.17.

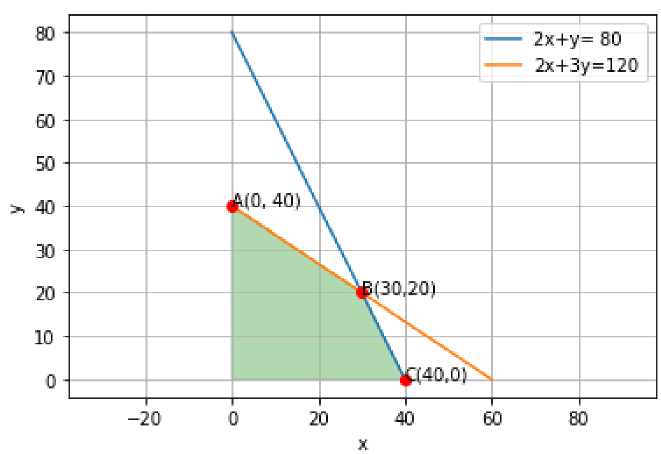

Fig. 1.17: graphical solution

1.18. A cottage industry manufactures pedestal lamps and wooden shades, each requiring the use of a grinding/cutting machine and a sprayer. It takes 2 hours on grinding/cutting machine and 3 hours on the sprayer to manufacture a pedestal lamp. It takes 1 hour on the grinding/cutting machine and 2 hours on the sprayer to manufacture a shade. On any day, the sprayer is available for at the most 20 hours and the grinding/cutting machine for at the most 12 hours. The profit from the sale of a lamp is Rs 5 and that from a shade is Rs 3. Assuming that the manufacturer can sell all the lamps and shades that he produces, how should he schedule his daily production in order to maximise his profit?

Solution:

- All the data can be tabularised as:

	Grinding machine	Sprayer	Profit
Pedestal lamps	2	3	5
Wooden shades	1	2	3
Max Hours	≤ 12	≤ 20	

TABLE 1.18: Time needed and Profit for each object

- Let the number of pieces of pedestal lamp manufactured be x and the number of pieces of wooden shades manufactured be y such that :

$$x \geq 0 \tag{1.18.1}$$

$$y \geq 0 \tag{1.18.2}$$

- From the data given we have:

$$2x + y \leq 12 \tag{1.18.3}$$

and,

$$3x + 2y \leq 20 \tag{1.18.4}$$

∴ The maximizing function is:

$$\max Z = \begin{pmatrix} 5 & 3 \end{pmatrix} \mathbf{x} \tag{1.18.5}$$

$$s.t. \quad \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \mathbf{x} \leq \begin{pmatrix} 12 \\ 20 \end{pmatrix} \tag{1.18.6}$$

$$-\mathbf{x} \leq \mathbf{0} \tag{1.18.7}$$

- The Lagrangian function can be given as:

$$\begin{aligned} &L(\mathbf{x}, \boldsymbol{\lambda}) \\ &= \begin{pmatrix} 5 & 3 \end{pmatrix} \mathbf{x} + \left\{ \left[\begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} - 12 \right] \right. \\ &+ \left[\begin{pmatrix} 3 & 2 \end{pmatrix} \mathbf{x} - 20 \right] \\ &+ \left. \left[\begin{pmatrix} -1 & 0 \end{pmatrix} \mathbf{x} \right] + \left[\begin{pmatrix} 0 & -1 \end{pmatrix} \mathbf{x} \right] \right\} \boldsymbol{\lambda} \end{aligned} \tag{1.18.8}$$

where,

$$\boldsymbol{\lambda} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} \tag{1.18.9}$$

- Now, we have

$$\nabla L(\mathbf{x},\lambda) = \begin{pmatrix} 5+\begin{pmatrix}2 & 3 & -1 & 0\end{pmatrix}\lambda \\ 3+\begin{pmatrix}1 & 2 & 0 & -1\end{pmatrix}\lambda \\ \begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x}-12 \\ \begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x}-20 \\ \begin{pmatrix}-1 & 0\end{pmatrix}\mathbf{x} \\ \begin{pmatrix}0 & -1\end{pmatrix}\mathbf{x} \end{pmatrix} \tag{1.18.10}$$

$\therefore$ The Lagrangian matrix is given by:-

$$\begin{pmatrix} 0 & 0 & 2 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 & -1 \\ 2 & 1 & 0 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix}\mathbf{x}\\ \lambda\end{pmatrix} = \begin{pmatrix}-5\\-3\\12\\20\\0\\0\end{pmatrix} \tag{1.18.11}$$

- Considering λ_1, λ_2 as only active multiplier,

$$\begin{pmatrix} 0 & 0 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 0 & 0 \end{pmatrix}\begin{pmatrix}\mathbf{x}\\ \lambda\end{pmatrix} = \begin{pmatrix}-5\\-3\\12\\20\end{pmatrix} \tag{1.18.12}$$

$$\implies \begin{pmatrix}\mathbf{x}\\ \lambda\end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 0 & 0 \end{pmatrix}^{-1}\begin{pmatrix}-5\\-3\\12\\20\end{pmatrix} \tag{1.18.13}$$

$$\implies \begin{pmatrix}\mathbf{x}\\ \lambda\end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 & -1 \\ 0 & 0 & -3 & 2 \\ 2 & -3 & 0 & 0 \\ -1 & 2 & 0 & 0 \end{pmatrix}\begin{pmatrix}-5\\-3\\12\\20\end{pmatrix} \tag{1.18.14}$$

$$\implies \begin{pmatrix}\mathbf{x}\\ \lambda\end{pmatrix} = \begin{pmatrix}4\\24\\-1\\-1\end{pmatrix} \tag{1.18.15}$$

$$\because \lambda = \begin{pmatrix}-1\\-1\end{pmatrix} < \mathbf{0}$$

- The Optimal solution is given by:

$$\mathbf{x} = \begin{pmatrix}4\\4\end{pmatrix} \tag{1.18.16}$$

$$Z = \begin{pmatrix}5 & 3\end{pmatrix}\mathbf{x} \tag{1.18.17}$$

$$Z = \begin{pmatrix}5 & 3\end{pmatrix}\begin{pmatrix}4\\4\end{pmatrix} \tag{1.18.18}$$

$$Z = \text{Rs}.32 \tag{1.18.19}$$

- So, to maximise profit
No. of pedestal lamps manufactured is $\boxed{x = 4}$ and
No. of wooden shades manufactured is $\boxed{y = 4}$. This is verified in Fig. 1.18
- The maximum daily profit is $\boxed{Z = \text{Rs}.32}$.

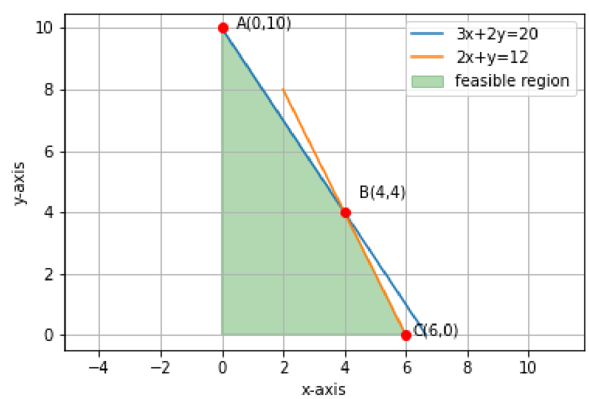

Fig. 1.18: Graphical Representataion

1.19. A company manufactures two types of novelty souvenirs made of plywood. Souvenirs of type A require 5 minutes each for cutting and 10 minutes each for assembling. Souvenirs of type B require 8 minutes each for cutting and 8 minutes each for assembling. There are 3 hours 20 minutes available for cutting and 4 hours for assembling. The profit is Rs 5 each for type A and Rs 6 each for type B souvenirs. How many souvenirs of each type should the company manufacture in order to maximise the profit?
Solution:

Item	Number	Cutting Time	Assembling Time	Profit
Type A	x	5 minutes	10 minutes	Rs 5
Type B	y	8 minutes	8 minutes	Rs 6
Max Available Time		3hours 20minutes =200minutes	4hours =240minutes	

TABLE 1.19: Plywood Requirements

Let the number of Souvenirs of type A be x and the number of Souvenirs of type B be y such that

$$x \geq 0 \tag{1.19.1}$$
$$y \geq 0 \tag{1.19.2}$$

According to the question,

$$5x + 8y \leq 200 \tag{1.19.3}$$

and,

$$10x + 8y \leq 240 \tag{1.19.4}$$

$\therefore$ Our problem is

$$\max_{\mathbf{x}} Z = \begin{pmatrix} 5 & 6 \end{pmatrix} \mathbf{x} \tag{1.19.5}$$
$$s.t. \quad \begin{pmatrix} 5 & 8 \\ 10 & 8 \end{pmatrix} \mathbf{x} \leq \begin{pmatrix} 200 \\ 240 \end{pmatrix} \tag{1.19.6}$$

Lagrangian function is given by

$$\begin{aligned} &L(\mathbf{x}, \boldsymbol{\lambda}) \\ &= \begin{pmatrix} 5 & 6 \end{pmatrix} \mathbf{x} + \left\{ \left[\begin{pmatrix} 5 & 8 \end{pmatrix} \mathbf{x} + 200 \right] \right. \\ &+ \left[\begin{pmatrix} 10 & 8 \end{pmatrix} \mathbf{x} + 240 \right] \\ &+ \left. \left[\begin{pmatrix} -1 & 0 \end{pmatrix} \mathbf{x} \right] + \left[\begin{pmatrix} 0 & -1 \end{pmatrix} \mathbf{x} \right] \right\} \boldsymbol{\lambda} \end{aligned} \tag{1.19.7}$$

where,

$$\boldsymbol{\lambda} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \end{pmatrix} \tag{1.19.8}$$

Now,

$$\nabla L(\mathbf{x}, \boldsymbol{\lambda}) = \begin{pmatrix} 5 + \begin{pmatrix} 5 & 8 & -1 & 0 \end{pmatrix} \boldsymbol{\lambda} \\ 6 + \begin{pmatrix} 10 & 8 & 0 & -1 \end{pmatrix} \boldsymbol{\lambda} \\ \begin{pmatrix} 5 & 8 \end{pmatrix} \mathbf{x} + 200 \\ \begin{pmatrix} 10 & 8 \end{pmatrix} \mathbf{x} + 240 \\ \begin{pmatrix} -1 & 0 \end{pmatrix} \mathbf{x} \\ \begin{pmatrix} 0 & -1 \end{pmatrix} \mathbf{x} \end{pmatrix} \tag{1.19.9}$$

$\therefore$ Lagrangian matrix is given by

$$\begin{pmatrix} 0 & 0 & 5 & 10 & -1 & 0 \\ 0 & 0 & 8 & 8 & 0 & -1 \\ 5 & 8 & 0 & 0 & 0 & 0 \\ 10 & 8 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} -5 \\ -6 \\ 200 \\ 240 \\ 0 \\ 0 \end{pmatrix} \tag{1.19.10}$$

Considering λ_1, λ_2 as only active multiplier,

$$\begin{pmatrix} 0 & 0 & 5 & 10 \\ 0 & 0 & 8 & 8 \\ 5 & 8 & 0 & 0 \\ 10 & 8 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} -5 \\ -6 \\ 200 \\ 240 \end{pmatrix} \tag{1.19.11}$$

resulting in,

$$\begin{pmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 5 & 10 \\ 0 & 0 & 8 & 8 \\ 5 & 8 & 0 & 0 \\ 10 & 8 & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} -5 \\ -6 \\ 200 \\ 240 \end{pmatrix} \tag{1.19.12}$$

$$\implies \begin{pmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{-8}{40} & \frac{8}{40} \\ 0 & 0 & \frac{10}{40} & \frac{-5}{40} \\ \frac{-3}{40} & \frac{8}{40} & 0 & 0 \\ \frac{10}{40} & \frac{-5}{40} & 0 & 0 \end{pmatrix} \begin{pmatrix} -5 \\ -6 \\ 200 \\ 240 \end{pmatrix} \tag{1.19.13}$$

$$\implies \begin{pmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} 8 \\ 20 \\ \frac{-1}{5} \\ \frac{-1}{2} \end{pmatrix} \tag{1.19.14}$$

$\because \boldsymbol{\lambda} = \begin{pmatrix} \frac{-1}{5} \\ \frac{-1}{2} \end{pmatrix} \succ \mathbf{0}$ $\therefore$ Optimal solution is given by

$$\mathbf{x} = \begin{pmatrix} 8 \\ 20 \end{pmatrix} \tag{1.19.15}$$
$$Z = \begin{pmatrix} 5 & 6 \end{pmatrix} \mathbf{x} \tag{1.19.16}$$
$$= \begin{pmatrix} 5 & 6 \end{pmatrix} \begin{pmatrix} 8 \\ 20 \end{pmatrix} \tag{1.19.17}$$
$$= 160 \tag{1.19.18}$$

By using cvxpy in python ,

$$\mathbf{x} = \begin{pmatrix} 8.00000000 \\ 20.00000000 \end{pmatrix} \tag{1.19.19}$$
$$Z = 160.00000000 \tag{1.19.20}$$

Hence , $\boxed{x = 8}$ Souvenirs of type A and $\boxed{y = 20}$ Souvenirs of type B should the company manufacture in order to maximise the profit is $\boxed{Z = 160}$ units. This is verified in Fig. 1.19

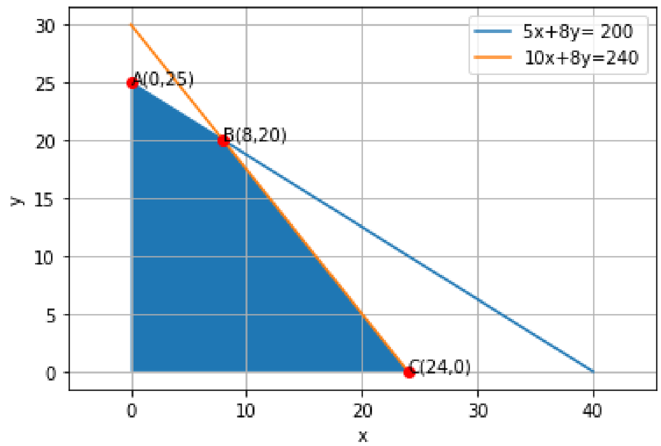

Fig. 1.19: Plywood Problem

1.20. A manufacturer makes two types of toys A and B. Three machines are needed for this purpose and the time (in minutes) required for each toy on the machines is given below:

	Machines		
Types of toys	I	II	III
A	12	18	6
B	6	0	9

TABLE 1.20: Toys table

Each machine is available for a maximum of 6 hours per day. If the profit on each toy of type A is Rs 7.50 and that on each toy of type B is Rs 5, show that 15 toys of type A and 30 of type B should be manufactured in a day to get maximum profit.

Solution:
Let the number of toys of type A be x and the number of toys of type B be y such that

$$x \geq 0 \tag{1.20.1}$$
$$y \geq 0 \tag{1.20.2}$$

According to the question,

$$12x + 6y \leq 360 \tag{1.20.3}$$
$$\implies 2x + y \leq 60 \tag{1.20.4}$$

and,

$$18x + 0y \leq 360 \tag{1.20.5}$$
$$\implies x \leq 20 \tag{1.20.6}$$

and,

$$6x + 9y \leq 360 \tag{1.20.7}$$
$$\implies 2x + 3y \leq 120 \tag{1.20.8}$$

$\therefore$ Our problem is

$$\max_{\mathbf{x}} Z = \begin{pmatrix} 7.5 & 5 \end{pmatrix} \mathbf{x} \tag{1.20.9}$$
$$s.t. \quad \begin{pmatrix} 2 & 1 \\ 1 & 0 \\ 2 & 3 \end{pmatrix} \mathbf{x} \preceq \begin{pmatrix} 60 \\ 20 \\ 120 \end{pmatrix} \tag{1.20.10}$$

Lagrangian function is given by

$$\begin{aligned} & L(\mathbf{x}, \boldsymbol{\lambda}) \\ & = \begin{pmatrix} 7.5 & 5 \end{pmatrix} \mathbf{x} + \left\{ \left[\begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} - 60 \right] \right. \\ & + \left[\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} - 20 \right] + \left[\begin{pmatrix} 2 & 3 \end{pmatrix} \mathbf{x} - 120 \right] \\ & + \left. \left[\begin{pmatrix} -1 & 0 \end{pmatrix} \mathbf{x} \right] + \left[\begin{pmatrix} 0 & -1 \end{pmatrix} \mathbf{x} \right] \right\} \boldsymbol{\lambda} \end{aligned} \tag{1.20.11}$$

where,

$$\boldsymbol{\lambda} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \end{pmatrix} \tag{1.20.12}$$

Now,

$$\nabla L(\mathbf{x}, \boldsymbol{\lambda}) = \begin{pmatrix} 7.5 + \begin{pmatrix} 2 & 1 & 2 & -1 & 0 \end{pmatrix} \boldsymbol{\lambda} \\ 5 + \begin{pmatrix} 1 & 0 & 3 & 0 & -1 \end{pmatrix} \boldsymbol{\lambda} \\ \begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} - 60 \\ \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} - 20 \\ \begin{pmatrix} 2 & 3 \end{pmatrix} \mathbf{x} - 120 \\ \begin{pmatrix} -1 & 0 \end{pmatrix} \mathbf{x} \\ \begin{pmatrix} 0 & -1 \end{pmatrix} \mathbf{x} \end{pmatrix} \tag{1.20.13}$$

$\therefore$ Lagrangian matrix is given by

$$\begin{pmatrix} 0 & 0 & 2 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 0 & 3 & 0 & -1 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \lambda \end{pmatrix} = \begin{pmatrix} -7.5 \\ -5 \\ 60 \\ 20 \\ 120 \\ 0 \\ 0 \end{pmatrix} \tag{1.20.14}$$

Considering λ_1, λ_2 as only active multiplier,

$$\begin{pmatrix} 0 & 0 & 2 & 2 \\ 0 & 0 & 1 & 3 \\ 2 & 1 & 0 & 0 \\ 2 & 3 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \lambda \end{pmatrix} = \begin{pmatrix} -7.5 \\ -5 \\ 60 \\ 120 \end{pmatrix} \tag{1.20.15}$$

resulting in,

$$\begin{pmatrix} \mathbf{x} \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 & 2 \\ 0 & 0 & 1 & 3 \\ 2 & 1 & 0 & 0 \\ 2 & 3 & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} -7.5 \\ -5 \\ 60 \\ 120 \end{pmatrix} \tag{1.20.16}$$

$$\implies \begin{pmatrix} \mathbf{x} \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{3}{4} & \frac{-1}{4} \\ 0 & 0 & \frac{-1}{2} & \frac{1}{2} \\ \frac{3}{4} & \frac{-1}{2} & 0 & 0 \\ \frac{-1}{4} & \frac{1}{2} & 0 & 0 \end{pmatrix} \begin{pmatrix} -7.5 \\ -5 \\ 60 \\ 120 \end{pmatrix} \tag{1.20.17}$$

$$\implies \begin{pmatrix} \mathbf{x} \\ \lambda \end{pmatrix} = \begin{pmatrix} 15 \\ 30 \\ -3.125 \\ -0.625 \end{pmatrix} \tag{1.20.18}$$

$\because \lambda = \begin{pmatrix} -3.125 \\ -0.625 \end{pmatrix} \succ \mathbf{0}$

$\therefore$ Optimal solution is given by

$$\mathbf{x} = \begin{pmatrix} 15 \\ 30 \end{pmatrix} \tag{1.20.19}$$

$$Z = \begin{pmatrix} 7.5 & 5 \end{pmatrix} \mathbf{x} \tag{1.20.20}$$

$$= \begin{pmatrix} 7.5 & 5 \end{pmatrix} \begin{pmatrix} 15 \\ 30 \end{pmatrix} \tag{1.20.21}$$

$$= 262.5 \tag{1.20.22}$$

By using cvxpy in python ,

$$\mathbf{x} = \begin{pmatrix} 14.99999998 \\ 29.99999996 \end{pmatrix} \tag{1.20.23}$$

$$Z = 262.49999967 \tag{1.20.24}$$

Hence, the manufacturer should manufacture $\boxed{x = 15}$ toys of type A and $\boxed{y = 30}$ toys of type B in a day to get maximum profit $\boxed{Z = Rs262.5}$. This can be verified in Fig. 1.20

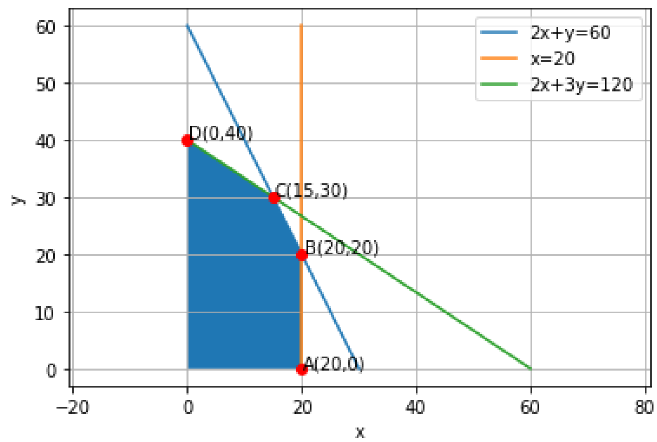

Fig. 1.20: Toy Problem

1.21. Two godowns A and B have grain capacity of 100 quintals and 50 quintals respectively.They supply to 3 ration shops, D, E and F whose requirements are 60, 50 and 40 quintals respectively. The cost of transportation per quintal from the godowns to the shops are given in the following table: How should the supplies

Transportation cost per quintal (in rupees)		
From/To	A	B
D	6	4
E	3	2
F	2.50	3

TABLE 1.9: Transportation table

be transported in order that the transportation cost is minimum? What is the minimum cost?

Solution: Let c_1 and c_2 be the column vectors of given matrix $\mathbf{A}$

$$c_1 = \begin{pmatrix} 4 \\ 6 \end{pmatrix} \tag{1.21.1}$$

$$c_2 = \begin{pmatrix} -3 \\ -2 \end{pmatrix} \tag{1.21.2}$$

We can express the matrix $\mathbf{A}$ as,

$$\mathbf{A} = \mathbf{QR} \tag{1.21.3}$$

Where, $\mathbf{Q}$ is an orthogonal matrix given as,

$$\mathbf{Q} = \begin{pmatrix} \mathbf{u_1} & \mathbf{u_2} \end{pmatrix} \quad (1.21.4)$$

and $\mathbf{R}$ is an upper triangular matrix given as,

$$\mathbf{R} = \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \quad (1.21.5)$$

Now, we can express α and β as,

$$c_1 = k_1 \mathbf{u_1} \quad (1.21.6)$$

$$c_2 = r_1 \mathbf{u_1} + k_2 \mathbf{u_2} \quad (1.21.7)$$

$$\text{where, } k_1 = \|c_1\| = \sqrt{4^2 + (6^2} = \sqrt{52} \quad (1.21.8)$$

Solving equation (1.21.6) for $\mathbf{u_1}$,

$$\mathbf{u_1} = \frac{c_1}{k_1} = \frac{1}{\sqrt{52}} \begin{pmatrix} 4 \\ 6 \end{pmatrix} \quad (1.21.9)$$

$$\text{Now, } r_1 = \frac{\mathbf{u_1}^T c_2}{\|\mathbf{u_1}\|^2} \quad (1.21.10)$$

$$\implies \frac{\frac{1}{\sqrt{52}} \begin{pmatrix} 4 & 6 \end{pmatrix} \begin{pmatrix} -3 \\ -2 \end{pmatrix}}{1} \quad (1.21.11)$$

$$\text{Hence, } r_1 = -\frac{-24}{\sqrt{52}} \quad (1.21.12)$$

$$\mathbf{u_2} = \frac{c_2 - r_1 \mathbf{u_1}}{\|c_2 - r_1 \mathbf{u_1}\|} \quad (1.21.13)$$

$$\implies \frac{\begin{pmatrix} -3 \\ -2 \end{pmatrix} - \left(\frac{-24}{\sqrt{52}}\right) \left(\frac{1}{\sqrt{52}} \begin{pmatrix} 4 \\ 6 \end{pmatrix}\right)}{\left\| \begin{pmatrix} -3 \\ -2 \end{pmatrix} - \left(-\frac{24}{\sqrt{52}} \frac{1}{\sqrt{52}} \begin{pmatrix} 4 \\ 6 \end{pmatrix}\right) \right\|} \quad (1.21.14)$$

$$\implies \mathbf{u_2} = \frac{1}{\sqrt{335}} \begin{pmatrix} -15 \\ 10 \end{pmatrix} \quad (1.21.15)$$

$$\text{Now, } k_2 = u_2^T c_2 \quad (1.21.16)$$

$$\implies \frac{1}{\sqrt{335}} \begin{pmatrix} -15 & 10 \end{pmatrix} \begin{pmatrix} -3 \\ -2 \end{pmatrix} \quad (1.21.17)$$

$$\implies k_2 = \frac{25}{\sqrt{335}} \quad (1.21.18)$$

Hence substituting the values of unknown parameter from equations (1.21.8), (1.21.18), (1.21.9), (1.21.15) and (1.21.12) to equation (1.21.4) and (1.21.5) we get,

$$\mathbf{Q} = \begin{pmatrix} \frac{4}{\sqrt{52}} & \frac{-15}{\sqrt{335}} \\ \frac{6}{\sqrt{52}} & \frac{10}{\sqrt{335}} \end{pmatrix} \quad (1.21.19)$$

$$\mathbf{R} = \begin{pmatrix} \sqrt{52} & \frac{-24}{\sqrt{52}} \\ 0 & \frac{25}{\sqrt{335}} \end{pmatrix} \quad (1.21.20)$$

1.22. A fruit grower can use two types of fertilizer in his garden, brand P and brand Q.The amounts(in kg) of nitrogen, phosphoric acid,potash and chlorine in a bag of each brand are given in the table.Tests indicate that garden needs atleast 240 kg of phosphoric acid,atleast 270 kg of potash and atmost 310 kg of chlorine. If the grower wants to minimise the amount of nitrogen added to garden, how many bags of each brand should be used?What is the minimum amount of nitrogen added in the ground?

	Brand P	**Brand Q**
Nitrogen	3	3.5
Phosphoric Acid	1	2
Potash	3	1.5
Chlorine	1.5	2

TABLE 1.10: kg per bag

Solution:

- All the data can be tabularised as:

	Brand P	**Brand Q**	Amounts Required
Nitrogen	3	3.5	?
Phosphoric Acid	1	2	≥ 240 kg
Potash	3	1.5	≥ 270 kg
Chlorine	1.5	2	≤ 310 kg

TABLE 1.11: Requirements of fertilizers

- Let the number of bags of Brand P be x &
- The number of bags of Brand Q be y such that :

$$x \geq 0 \quad (1.22.1)$$

$$y \geq 0 \quad (1.22.2)$$

- From the data given we have:

$$x + 2y \geq 240 \quad (1.22.3)$$

$$\implies -x - 2y \leq -240 \quad (1.22.4)$$

and,

$$3x + 1.5y \geq 270 \tag{1.22.5}$$
$$\implies -x - 0.5y \leq -90 \tag{1.22.6}$$

and,

$$1.5x + 2y \leq 310 \tag{1.22.7}$$
$$\tag{1.22.8}$$

$\therefore$ The minimizing function is:

$$\min_{\mathbf{x}} Z = \begin{pmatrix} 3 & 3.5 \end{pmatrix} \mathbf{x} \tag{1.22.9}$$
$$s.t. \quad \begin{pmatrix} -1 & -2 \\ -1 & -0.5 \\ 1.5 & 2 \end{pmatrix} \mathbf{x} \leq \begin{pmatrix} -240 \\ -90 \\ 310 \end{pmatrix} \tag{1.22.10}$$
$$-\mathbf{x} \leq \mathbf{0} \tag{1.22.11}$$

- The Lagrangian function can be given as:

$$\begin{aligned} &L(\mathbf{x}, \boldsymbol{\lambda}) \\ &= \begin{pmatrix} 3 & 3.5 \end{pmatrix} \mathbf{x} + \left\{ \left[\begin{pmatrix} -1 & -2 \end{pmatrix} \mathbf{x} + 240 \right] \right. \\ &+ \left[\begin{pmatrix} -1 & -0.5 \end{pmatrix} \mathbf{x} + 90 \right] + \left[\begin{pmatrix} 1.5 & 2 \end{pmatrix} \mathbf{x} - 310 \right] \\ &+ \left. \left[\begin{pmatrix} -1 & 0 \end{pmatrix} \mathbf{x} \right] + \left[\begin{pmatrix} 0 & -1 \end{pmatrix} \mathbf{x} \right] \right\} \boldsymbol{\lambda} \end{aligned} \tag{1.22.12}$$

where,

$$\boldsymbol{\lambda} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \end{pmatrix} \tag{1.22.13}$$

- Now, we have

$$\nabla L(\mathbf{x}, \lambda) = \begin{pmatrix} 3 + \begin{pmatrix} -1 & -1 & 1.5 & -1 & 0 \end{pmatrix} \boldsymbol{\lambda} \\ 3.5 + \begin{pmatrix} -2 & -0.5 & 2 & 0 & -1 \end{pmatrix} \boldsymbol{\lambda} \\ \begin{pmatrix} -1 & -2 \end{pmatrix} \mathbf{x} + 240 \\ \begin{pmatrix} -1 & -0.5 \end{pmatrix} \mathbf{x} + 90 \\ \begin{pmatrix} 1.5 & 2 \end{pmatrix} \mathbf{x} - 310 \\ \begin{pmatrix} -1 & 0 \end{pmatrix} \mathbf{x} \\ \begin{pmatrix} 0 & -1 \end{pmatrix} \mathbf{x} \end{pmatrix} \tag{1.22.14}$$

$\therefore$ The Lagrangian matrix is given by:-

$$\begin{pmatrix} 0 & 0 & -1 & -1 & 1.5 & -1 & 0 \\ 0 & 0 & -2 & -0.5 & 2 & 0 & -1 \\ -1 & -2 & 0 & 0 & 0 & 0 & 0 \\ -1 & -0.5 & 0 & 0 & 0 & 0 & 0 \\ 1.5 & 2 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} -3 \\ -3.5 \\ -240 \\ -90 \\ 310 \\ 0 \\ 0 \end{pmatrix} \tag{1.22.15}$$

- Considering λ_1, λ_2 as only active multiplier,

$$\begin{pmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & -2 & -0.5 \\ -1 & -2 & 0 & 0 \\ -1 & -0.5 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} -3 \\ -3.5 \\ -240 \\ -90 \end{pmatrix} \tag{1.22.16}$$

$$\implies \begin{pmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & -2 & -0.5 \\ -1 & -2 & 0 & 0 \\ -1 & -0.5 & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} -3 \\ -3.5 \\ -240 \\ -90 \end{pmatrix} \tag{1.22.17}$$

$$\implies \begin{pmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{1}{3} & \frac{4}{3} \\ 0 & 0 & \frac{-2}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{-2}{3} & 0 & 0 \\ \frac{-4}{3} & \frac{2}{3} & 0 & 0 \end{pmatrix} \begin{pmatrix} -3 \\ -3.5 \\ -240 \\ -90 \end{pmatrix} \tag{1.22.18}$$

$$\implies \begin{pmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} 40 \\ 100 \\ \frac{4}{3} \\ \frac{5}{3} \end{pmatrix} \tag{1.22.19}$$

$$\because \boldsymbol{\lambda} = \begin{pmatrix} \frac{4}{3} \\ \frac{5}{3} \end{pmatrix} \succ \mathbf{0}$$

- The Optimal solution is given by:

$$\mathbf{x} = \begin{pmatrix} 40 \\ 100 \end{pmatrix} \tag{1.22.20}$$
$$Z = \begin{pmatrix} 3 & 3.5 \end{pmatrix} \mathbf{x} \tag{1.22.21}$$
$$Z = \begin{pmatrix} 3 & 3.5 \end{pmatrix} \begin{pmatrix} 40 \\ 100 \end{pmatrix} \tag{1.22.22}$$
$$Z = 470 \text{ units} \tag{1.22.23}$$

- So, we get
Bags of brand **P** as $\boxed{x = 40}$ &
Bags of brand **Q** as $\boxed{y = 100}$ so as to minimise the amount of nitrogen added.
- The minimum amount of nitrogen required is $\boxed{Z = 470 \text{ units}}$. Fig. 1.22 verifies this result.

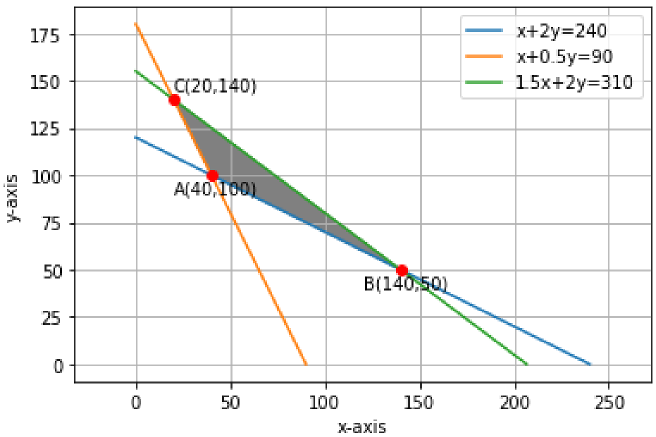

Fig. 1.22: Graphical Solution

1.23. A toy company manufactures two types of dolls, A and B. Market research and available resources have indicated that the combined production level should not exceed 1200 dolls per week and the demand for dolls of type B is at most half of that for dolls of type A. Further, the production level of dolls of type A can exceed three times the production of dolls of other type by at most 600 units. If the company makes profit of Rs 12 and Rs 16 per doll respectively on dolls A and B, how many of each should be produced weekly in order to maximise the profit? **Solution:**

Let x and y be number of dolls of type A and B respectively such that

$$x \geq 0 \tag{1.23.1}$$

$$y \geq 0 \tag{1.23.2}$$

According to the question,

$$x + y \leq 1200 \tag{1.23.3}$$

$$x - 3y \leq 600 \tag{1.23.4}$$

$$-x + 2y \leq 0 \tag{1.23.5}$$

$\therefore$

$$\max_{\mathbf{x}} Z = \begin{pmatrix} 12 & 16 \end{pmatrix} \mathbf{x} \tag{1.23.6}$$

$\therefore$ Our problem is

$$\max_{\mathbf{x}} Z = \begin{pmatrix} 12 & 16 \end{pmatrix} \mathbf{x} \tag{1.23.7}$$

$$s.t. \quad \begin{pmatrix} 1 & 1 \\ 1 & -3 \\ -1 & 2 \end{pmatrix} \mathbf{x} \leq \begin{pmatrix} 1200 \\ 600 \\ 0 \end{pmatrix} \tag{1.23.8}$$

Lagrangian function is given by

$$\begin{aligned} &L(\mathbf{x}, \boldsymbol{\lambda}) \\ &= \begin{pmatrix} 12 & 16 \end{pmatrix} \mathbf{x} + \left\{ \left[\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} - 1200 \right] \right. \\ &+ \left[\begin{pmatrix} 1 & -3 \end{pmatrix} \mathbf{x} - 600 \right] + \left[\begin{pmatrix} -1 & 2 \end{pmatrix} \mathbf{x} \right] \\ &+ \left. \left[\begin{pmatrix} -1 & 0 \end{pmatrix} \mathbf{x} \right] + \left[\begin{pmatrix} 0 & -1 \end{pmatrix} \mathbf{x} \right] \right\} \boldsymbol{\lambda} \end{aligned} \tag{1.23.9}$$

where,

$$\boldsymbol{\lambda} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \end{pmatrix} \tag{1.23.10}$$

Now,

$$\nabla L(\mathbf{x}, \boldsymbol{\lambda}) = \begin{pmatrix} 12 + \begin{pmatrix} 1 & 1 & 2 & -1 & 0 \end{pmatrix} \boldsymbol{\lambda} \\ 16 + \begin{pmatrix} 1 & -3 & -1 & 0 & -1 \end{pmatrix} \boldsymbol{\lambda} \\ \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} - 1200 \\ \begin{pmatrix} -1 & -3 \end{pmatrix} \mathbf{x} - 600 \\ \begin{pmatrix} -1 & 2 \end{pmatrix} \mathbf{x} \\ \begin{pmatrix} -1 & 0 \end{pmatrix} \mathbf{x} \\ \begin{pmatrix} 0 & -1 \end{pmatrix} \mathbf{x} \end{pmatrix} \tag{1.23.11}$$

$\therefore$ Lagrangian matrix is given by

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & -3 & -1 & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -3 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} -12 \\ -16 \\ 1200 \\ 600 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{1.23.12}$$

Considering λ_1, λ_2 as only active multiplier,

$$\begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ -1 & 2 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} -12 \\ -16 \\ 1200 \\ 0 \end{pmatrix} \tag{1.23.13}$$

resulting in,

$$\begin{pmatrix}\mathbf{x}\\ \lambda\end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 2\\ 0 & 0 & 1 & -1\\ 1 & 1 & 0 & 0\\ -1 & 2 & 0 & 0\end{pmatrix}^{-1}\begin{pmatrix}-12\\ -16\\ 1200\\ 0\end{pmatrix} \tag{1.23.14}$$

$$\implies \begin{pmatrix}\mathbf{x}\\ \lambda\end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{2}{3} & \frac{-1}{3}\\ 0 & 0 & \frac{1}{3} & \frac{1}{3}\\ \frac{1}{3} & \frac{2}{3} & 0 & 0\\ \frac{1}{3} & \frac{-1}{3} & 0 & 0\end{pmatrix}\begin{pmatrix}-12\\ -16\\ 1200\\ 0\end{pmatrix} \tag{1.23.15}$$

$$\implies \begin{pmatrix}\mathbf{x}\\ \lambda\end{pmatrix} = \begin{pmatrix}800\\ 400\\ \frac{-44}{3}\\ \frac{4}{3}\end{pmatrix} \tag{1.23.16}$$

$\because \lambda = \begin{pmatrix}\frac{-44}{3}\\ \frac{4}{3}\end{pmatrix} \succ \mathbf{0}$

$\therefore$ Optimal solution is given by

$$\mathbf{x} = \begin{pmatrix}800\\ 400\end{pmatrix} \tag{1.23.17}$$

$$Z = \begin{pmatrix}12 & 16\end{pmatrix}\mathbf{x} \tag{1.23.18}$$

$$= \begin{pmatrix}12 & 16\end{pmatrix}\begin{pmatrix}800\\ 400\end{pmatrix} \tag{1.23.19}$$

$$= 16000 \tag{1.23.20}$$

By using cvxpy in python ,

$$\mathbf{x} = \begin{pmatrix}800.00001236\\ 399.99998704\end{pmatrix} \tag{1.23.21}$$

$$Z = 15999.99994089 \tag{1.23.22}$$

Hence ,the profit will be maximum if company produces. Type A dolls = $\boxed{800}$, Type B dolls = $\boxed{400}$. This is verified in Fig. 1.23.

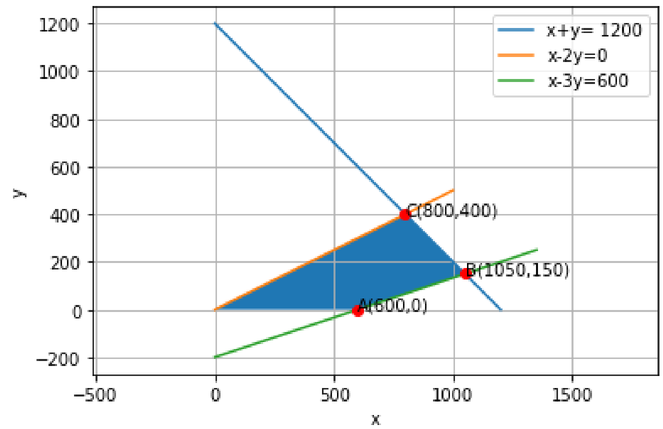

Fig. 1.23: TOY PROBLEM

1.24. **(Manufacturing problem)** A manufacturing company makes two models A and B of a product. Each piece of Model A requires 9 labour hours for fabricating and 1 labour hour for finishing. Each piece of Model B requires 12 labour hours for fabricating and 3 labour hours for finishing. For fabricating and finishing, the maximum labour hours available are 180 and 30 respectively. The company makes a profit of Rs 8000 on each piece of model A and Rs 12000 on each piece of Model B. How many pieces of Model A and Model B should be manufactured per week to realise a maximum profit? What is the maximum profit per week?

Solution:

- All the data can be tabularised as:

	Fabricating	Finishing	Profit
Model A	9	1	8000
Model B	12	3	12000
Max Hours	≤ 180	≤ 30	

TABLE 1.12: Labour Hours and Profit for each piece

- Let the number of pieces of model A manufactured be x and the number of pieces of model B manufactured be y such that :

$$x \ge 0 \tag{1.24.1}$$
$$y \ge 0 \tag{1.24.2}$$

- From the data given we have:

$$9x + 12y \le 180 \tag{1.24.3}$$
$$\implies 3x + 4y \le 60 \tag{1.24.4}$$

and,

$$x + 3y \le 30 \tag{1.24.5}$$

$\therefore$ The maximizing function is:

$$\max Z = \begin{pmatrix}8000 & 12000\end{pmatrix}\mathbf{x} \tag{1.24.6}$$

$$s.t. \quad \begin{pmatrix}3 & 4\\ 1 & 3\end{pmatrix}\mathbf{x} \preceq \begin{pmatrix}60\\ 30\end{pmatrix} \tag{1.24.7}$$

$$-\mathbf{x} \preceq \mathbf{0} \tag{1.24.8}$$

- The Lagrangian function can be given as:

$$L(\mathbf{x}, \boldsymbol{\lambda}) = \begin{pmatrix} 8000 & 12000 \end{pmatrix}\mathbf{x} + \left\{ \left[\begin{pmatrix} 3 & 4 \end{pmatrix}\mathbf{x} - 60\right] + \left[\begin{pmatrix} 1 & 3 \end{pmatrix}\mathbf{x} - 30\right] + \left[\begin{pmatrix} -1 & 0 \end{pmatrix}\mathbf{x}\right] + \left[\begin{pmatrix} 0 & -1 \end{pmatrix}\mathbf{x}\right] \right\} \boldsymbol{\lambda} \tag{1.24.9}$$

where,

$$\boldsymbol{\lambda} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} \tag{1.24.10}$$

- Now, we have

$$\nabla L(\mathbf{x}, \boldsymbol{\lambda}) = \begin{pmatrix} 8000 + \begin{pmatrix} 3 & 1 & -1 & 0 \end{pmatrix}\boldsymbol{\lambda} \\ 12000 + \begin{pmatrix} 4 & 3 & 0 & -1 \end{pmatrix}\boldsymbol{\lambda} \\ \begin{pmatrix} 3 & 4 \end{pmatrix}\mathbf{x} - 60 \\ \begin{pmatrix} 1 & 3 \end{pmatrix}\mathbf{x} - 30 \\ \begin{pmatrix} -1 & 0 \end{pmatrix}\mathbf{x} \\ \begin{pmatrix} 0 & -1 \end{pmatrix}\mathbf{x} \end{pmatrix} \tag{1.24.11}$$

$\therefore$ The Lagrangian matrix is given by:-

$$\begin{pmatrix} 0 & 0 & 3 & 1 & -1 & 0 \\ 0 & 0 & 4 & 3 & 0 & -1 \\ 3 & 4 & 0 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} -8000 \\ -12000 \\ 60 \\ 30 \\ 0 \\ 0 \end{pmatrix} \tag{1.24.12}$$

- Considering λ_1, λ_2 as only active multiplier,

$$\begin{pmatrix} 0 & 0 & 3 & 1 \\ 0 & 0 & 4 & 3 \\ 3 & 4 & 0 & 0 \\ 1 & 3 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} -8000 \\ -12000 \\ 60 \\ 30 \end{pmatrix} \tag{1.24.13}$$

$$\implies \begin{pmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 3 & 1 \\ 0 & 0 & 4 & 3 \\ 3 & 4 & 0 & 0 \\ 1 & 3 & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} -8000 \\ -12000 \\ 60 \\ 30 \end{pmatrix} \tag{1.24.14}$$

$$\implies \begin{pmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{3}{5} & \frac{-4}{5} \\ 0 & 0 & \frac{-1}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{-1}{5} & 0 & 0 \\ \frac{-4}{5} & \frac{3}{5} & 0 & 0 \end{pmatrix} \begin{pmatrix} -8000 \\ -12000 \\ 60 \\ 30 \end{pmatrix} \tag{1.24.15}$$

$$\implies \begin{pmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} 12 \\ 6 \\ -2400 \\ -800 \end{pmatrix} \tag{1.24.16}$$

$$\because \boldsymbol{\lambda} = \begin{pmatrix} -2400 \\ -800 \end{pmatrix} \prec \mathbf{0}$$

- The Optimal solution is given by:

$$\mathbf{x} = \begin{pmatrix} 12 \\ 6 \end{pmatrix} \tag{1.24.17}$$

$$Z = \begin{pmatrix} 8000 & 12000 \end{pmatrix}\mathbf{x} \tag{1.24.18}$$

$$Z = \begin{pmatrix} 8000 & 12000 \end{pmatrix}\begin{pmatrix} 12 \\ 6 \end{pmatrix} \tag{1.24.19}$$

$$Z = \text{Rs}168000 \tag{1.24.20}$$

- So, to maximise profit
  Pieces of model **A** manufactured is $\boxed{x = 12}$
  and
  Pieces of model **B** manufactured is $\boxed{y = 6}$.
- The maximum profit per week is $\boxed{Z = \text{Rs}168000}$.

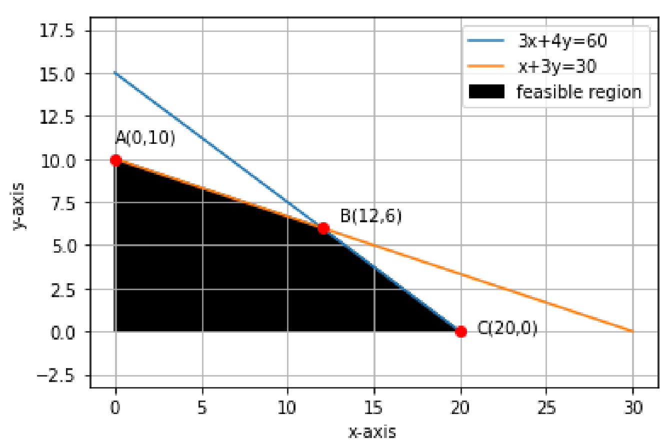

Fig. 1.24: Graphical Representataion

1.25. **(Diet problem)** A dietician has to develop a special diet using two foods P and Q. Each packet (containing 30 g) of food P contains 12 units of calcium, 4 units of iron, 6 units of cholesterol and 6 units of vitamin A. Each packet of the same quantity of food Q contains 3 units of calcium, 20 units of iron, 4 units of cholesterol and 3 units of vitamin A. The diet requires atleast 240 units of calcium, atleast 460 units of iron and at most 300 units of cholesterol. How many packets of each food should be used to minimise the amount of vitamin A in the diet? What is the minimum amount of vitamin A?

Solution: Let the number of packets of

Component	P	Q	Requirement
Calcium	12 units	3 units	≥ 240 units
Iron	4 units	20 units	≥ 460 units
Cholesterol	6 units	4 units	≤ 300 units
Vitamin A	6 units	3 units	

TABLE 1.13: Diet Requirements

food P be x and the number of packets of food Q be y such that

$$x \geq 0 \tag{1.25.1}$$

$$y \geq 0 \tag{1.25.2}$$

According to the question,

$$12x + 3y \geq 240 \tag{1.25.3}$$

$$\implies -4x - y \leq -80 \tag{1.25.4}$$

$$\tag{1.25.5}$$

and,

$$4x + 20y \geq 460 \tag{1.25.6}$$

$$\implies -x - 5y \leq -115 \tag{1.25.7}$$

and,

$$6x + 4y \leq 300 \tag{1.25.8}$$

$$\implies 3x + 2y \leq 150 \tag{1.25.9}$$

$\therefore$ Our problem is

$$\min_{\mathbf{x}} Z = \begin{pmatrix} 6 & 3 \end{pmatrix} \mathbf{x} \tag{1.25.10}$$

$$s.t. \quad \begin{pmatrix} -4 & -1 \\ -1 & -5 \\ 3 & 2 \end{pmatrix} \mathbf{x} \leq \begin{pmatrix} -80 \\ -115 \\ 150 \end{pmatrix} \tag{1.25.11}$$

$$-\mathbf{x} \leq \mathbf{0} \tag{1.25.12}$$

Lagrangian function is given by

$$\begin{aligned} &L(\mathbf{x}, \boldsymbol{\lambda}) \\ &= \begin{pmatrix} 6 & 3 \end{pmatrix} \mathbf{x} + \left\{ \left[\begin{pmatrix} -4 & -1 \end{pmatrix} \mathbf{x} + 80 \right] \right. \\ &+ \left[\begin{pmatrix} -1 & -5 \end{pmatrix} \mathbf{x} + 115 \right] + \left[\begin{pmatrix} 3 & 2 \end{pmatrix} \mathbf{x} - 150 \right] \\ &+ \left. \left[\begin{pmatrix} -1 & 0 \end{pmatrix} \mathbf{x} \right] + \left[\begin{pmatrix} 0 & -1 \end{pmatrix} \mathbf{x} \right] \right\} \boldsymbol{\lambda} \end{aligned} \tag{1.25.13}$$

where,

$$\boldsymbol{\lambda} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \end{pmatrix} \tag{1.25.14}$$

Now,

$$\nabla L(\mathbf{x}, \boldsymbol{\lambda}) = \begin{pmatrix} 6 + \begin{pmatrix} -4 & -1 & 3 & -1 & 0 \end{pmatrix} \boldsymbol{\lambda} \\ 3 + \begin{pmatrix} -1 & -5 & 2 & 0 & -1 \end{pmatrix} \boldsymbol{\lambda} \\ \begin{pmatrix} -4 & -1 \end{pmatrix} \mathbf{x} + 80 \\ \begin{pmatrix} -1 & -5 \end{pmatrix} \mathbf{x} + 115 \\ \begin{pmatrix} 3 & 2 \end{pmatrix} \mathbf{x} - 150 \\ \begin{pmatrix} -1 & 0 \end{pmatrix} \mathbf{x} \\ \begin{pmatrix} 0 & -1 \end{pmatrix} \mathbf{x} \end{pmatrix} \tag{1.25.15}$$

$\therefore$ Lagrangian matrix is given by

$$\begin{pmatrix} 0 & 0 & -4 & -1 & 3 & -1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 0 & -1 \\ -4 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -5 & 0 & 0 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} -6 \\ -3 \\ -80 \\ -115 \\ 150 \\ 0 \\ 0 \end{pmatrix} \tag{1.25.16}$$

Considering λ_1, λ_2 as only active multiplier,

$$\begin{pmatrix} 0 & 0 & -4 & -1 \\ 0 & 0 & -1 & -5 \\ -4 & -1 & 0 & 0 \\ -1 & -5 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} -6 \\ -3 \\ -80 \\ -115 \end{pmatrix} \quad (1.25.17)$$

resulting in,

$$\begin{pmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} 0 & 0 & -4 & -1 \\ 0 & 0 & -1 & -5 \\ -4 & -1 & 0 & 0 \\ -1 & -5 & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} -6 \\ -3 \\ -80 \\ -115 \end{pmatrix} \quad (1.25.18)$$

$$\implies \begin{pmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{-5}{19} & \frac{1}{19} \\ 0 & 0 & \frac{1}{19} & \frac{-4}{19} \\ \frac{-5}{19} & \frac{1}{19} & 0 & 0 \\ \frac{1}{19} & \frac{-4}{19} & 0 & 0 \end{pmatrix} \begin{pmatrix} -6 \\ -3 \\ -80 \\ -115 \end{pmatrix} \quad (1.25.19)$$

$$\implies \begin{pmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} 15 \\ 20 \\ \frac{27}{19} \\ \frac{6}{19} \end{pmatrix} \quad (1.25.20)$$

$\because \boldsymbol{\lambda} = \begin{pmatrix} \frac{27}{19} \\ \frac{6}{19} \end{pmatrix} \succ \mathbf{0}$

$\therefore$ Optimal solution is given by

$$\mathbf{x} = \begin{pmatrix} 15 \\ 20 \end{pmatrix} \quad (1.25.21)$$

$$Z = \begin{pmatrix} 6 & 3 \end{pmatrix} \mathbf{x} \quad (1.25.22)$$

$$= \begin{pmatrix} 6 & 3 \end{pmatrix} \begin{pmatrix} 15 \\ 20 \end{pmatrix} \quad (1.25.23)$$

$$= 150 \quad (1.25.24)$$

By using cvxpy in python ,

$$\mathbf{x} = \begin{pmatrix} 14.99999999 \\ 20.00000001 \end{pmatrix} \quad (1.25.25)$$

$$Z = 150.00000001 \quad (1.25.26)$$

Hence , $\boxed{x = 15}$ packets of food P and $\boxed{y = 20}$ packets of food Q should be used to minimise the amount of vitamin A in the diet and the minimum amount of vitamin A is $\boxed{Z = 150}$ units. This is verified in Fig. 1.25

2 Exercises

2.1. Find the absolute maximum and absoute minimum value of the following functions in the given intervals

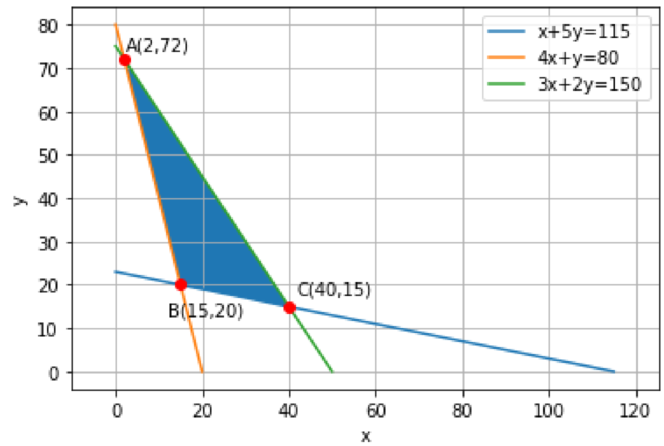

Fig. 1.25: Diet Problem

a) $f(x) = 4x - \frac{1}{2}x^2, x \in \left(-2, \frac{9}{2}\right)$

b) $f(x) = (x-1)^2 + 3, x \in (-3, 1)$

Solution:

a)

Lemma 2.1. *A function f(x) is said to be convex if following inequality is true for* $\lambda \in [0, 1]$:

$$\lambda f(x_1) + (1-\lambda) f(x_2) \geq f(\lambda x_1 + (1-\lambda) x_2) \quad (2.1.1)$$

Given :

$$f(x) = 4x - \frac{1}{2}x^2, x \in \left[-2, \frac{9}{2}\right] \quad (2.1.2)$$

Checking convexity of $f(x)$:

$$\lambda\left(4x_1 - \frac{1}{2}x_1^2\right) + (1-\lambda)\left(4x_2 - \frac{1}{2}x_2^2\right) \geq 4\left(\lambda x_1 + (1-\lambda)x_2\right) - \frac{1}{2}\left(\lambda x_1 + (1-\lambda)x_2\right)^2 \quad (2.1.3)$$

resulting in

$$x_1^2\left(\frac{\lambda^2-\lambda}{2}\right)+x_2^2\left(\frac{\lambda^2-\lambda}{2}\right)+2x_1x_2\left(\frac{\lambda-\lambda^2}{2}\right)\geq 0 \tag{2.1.4}$$

$$\implies \left(\frac{\lambda^2-\lambda}{2}\right)\left(x_1^2+x_2^2-2x_1x_2\right)\geq 0 \tag{2.1.5}$$

$$\implies -\frac{1}{2}\lambda(1-\lambda)(x_1-x_2)^2\geq 0 \tag{2.1.6}$$

$$\implies \frac{1}{2}\lambda(1-\lambda)(x_1-x_2)^2\leq 0 \tag{2.1.7}$$

Hence,using lemma 2.1, given $f(x)$ is a concave function .

i) For Maxima :
Using gradient ascent method,

$$x_{n+1} = x_n + \alpha\nabla f(x_n) \tag{2.1.8}$$

$$\implies x_{n+1} = x_n + \alpha(4-x_n) \tag{2.1.9}$$

Taking $x_0 = -2, \alpha = 0.001$ and precision= 0.00000001,values obtained using python are:

$$\boxed{\text{Maxima} = 7.999999999950196 \approx 8} \tag{2.1.10}$$

$$\boxed{\text{Maxima Point} = 3.9999900196756437 \approx 4} \tag{2.1.11}$$

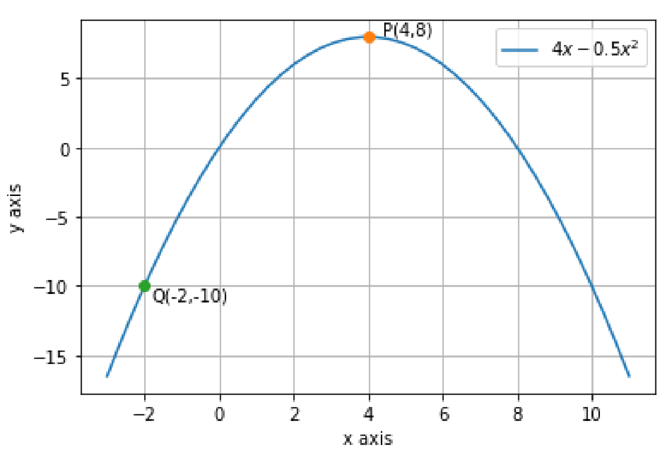

Fig. 2.1: $f(x) = 4x - 0.5x^2$

ii) For Minima :

x	$f(x)$
-2	-10
4	8
4.5	7.875

TABLE 2.1: Value of $f(x)$

Critical point is given by

$$\nabla f(x) = 0 \tag{2.1.12}$$

$$\implies x = 4 \tag{2.1.13}$$

and,end points are $x = -2$ and $x = 4.5$. Using table 2.1,

$$\boxed{\text{Minima} = -10} \tag{2.1.14}$$

$$\boxed{\text{Minima Point} = -2} \tag{2.1.15}$$

b)

Lemma 2.2. *A function f(x) is said to be convex if following inequality is true for* $\lambda \in [0,1]$:

$$\lambda f(x_1) + (1-\lambda)f(x_2) \geq f(\lambda x_1 + (1-\lambda)x_2) \tag{2.1.16}$$

Given :

$$(x-1)^2+3, x\in[-3,1], x\in[-3,1] \tag{2.1.17}$$

Checking convexity of $f(x)$:

$$\lambda\left((x_1-1)^2+3\right)+(1-\lambda)\left((x_2-1)^2+3\right)\geq ((\lambda x_1+(1-\lambda)x_2)-1)^2+3 \tag{2.1.18}$$

resulting in

$$x_1^2\left(\lambda-\lambda^2\right)+x_2^2\left(\lambda-\lambda^2\right)-2x_1x_2\left(\lambda-\lambda^2\right)\geq 0 \tag{2.1.19}$$

$$\implies \left(\lambda-\lambda^2\right)\left(x_1^2+x_2^2-2x_1x_2\right)\geq 0 \tag{2.1.20}$$

$$\implies \lambda(1-\lambda)(x_1-x_2)^2\geq 0 \tag{2.1.21}$$

Hence,using lemma 2.2, given $f(x)$ is a convex function .

i) For Maxima :
Using gradient ascent method,

$$x_{n+1} = x_n + \alpha\nabla f(x_n) \tag{2.1.22}$$

$$\implies x_{n+1} = x_n + \alpha(2x_n-2) \tag{2.1.23}$$

Taking $x_0 = -3, \alpha = 0.001$ and precision= 0.00000001,values obtained using python are:

$$\boxed{\text{Maxima} = 18.999999999940298 \approx 19} \tag{2.1.24}$$

$$\boxed{\text{Maxima Point} = -2.9999900126845568 \approx -3} \tag{2.1.25}$$

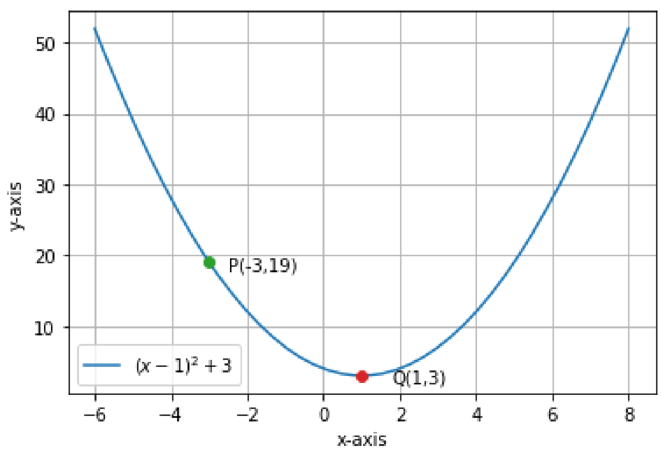

Fig. 2.1: $f(x) = (x-1)^2 + 3$

ii) For Minima :

x	$f(x)$
-3	19
1	3

TABLE 2.2: Value of $f(x)$

Critical point is given by

$$\nabla f(x) = 0 \tag{2.1.26}$$

$$\implies x = 1 \tag{2.1.27}$$

and,end points are $x = -3$ and $x = 1$. Using table 2.2,

$$\boxed{\text{Minima} = 3} \tag{2.1.28}$$

$$\boxed{\text{Minima Point} = 1} \tag{2.1.29}$$

2.2. Find the maximum profit that a company can make, if the profit function is given by

$$p(x) = 41 - 72x - 18x^2 \tag{2.2.1}$$

Solution:

Definition 1. *A function f(x) is said to be concave if following inequality is true for $\lambda \in [0, 1]$:*

$$\lambda f(x_1) + (1-\lambda) f(x_2) \le f(\lambda x_1 + (1-\lambda)x_2) \tag{2.2.2}$$

Lemma 2.3. *$p(x)$ is concave.*

Proof.

$$\because p(x) = 41 - 72x - 18x^2, \tag{2.2.3}$$

$$\begin{aligned} \lambda\left(41 - 72x_1 - 18{x_1}^2\right) \\ + (1-\lambda)\left(41 - 72x_2 - 18{x_2}^2\right) \\ \le 41 - 72(\lambda x_1 + (1-\lambda)x_2) \\ - 18\left(\lambda x_1 + (1-\lambda)x_2\right)^2 \end{aligned} \tag{2.2.4}$$

$$\implies 18\lambda(\lambda - 1)(x_1 - x_2)^2 \le 0 \tag{2.2.5}$$

$$\implies \lambda(\lambda - 1) \le 0 \tag{2.2.6}$$

$$\text{or, } 0 < \lambda < 1 \tag{2.2.7}$$

. Hence, $p(x)$ is concave. □

Gradient Ascent:

$$x_{n+1} = x_n + \alpha \nabla p(x_n) \tag{2.2.8}$$

$$\implies x_{n+1} = x_n + \alpha\left(-36x_n - 72\right) \tag{2.2.9}$$

Taking $x_0 = 2, \alpha = 0.001$ and precision= 0.00000001,values obtained using python are:

$$\boxed{\text{Maxima} = 112.99999999999876 \approx 113} \tag{2.2.10}$$

$$\boxed{\text{Maxima Point} = -1.9999997364868565 \approx -2} \tag{2.2.11}$$

We can verify this by the derivative test. Since $p(x)$ is a concave function it has a maxima.

$$\frac{dp(x)}{dx} = -36x - 72 \tag{2.2.12}$$

Critical point :

$$\frac{dp(x)}{dx} = 0 \tag{2.2.13}$$

$$-36x - 72 = 0 \tag{2.2.14}$$

$$x = -2 \tag{2.2.15}$$

is a critical point. $\because p(x)$ is a concave function, there will be a maxima at $x = -2$. Thus, the maxima is

$$p(-2) = 113 \tag{2.2.16}$$

This is verified in Fig. 2.2.

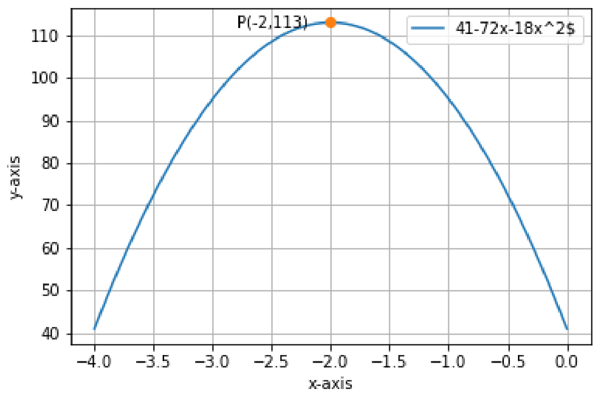

Fig. 2.2: $p(x) = 41 - 72x - 18x^2$

2.3. Find two positive numbers whose sum is 15 and the sum of whose squares is minimum.

2.4. Find two numbers whose sum is 24 and whose product is as large as possible.
Solution: Let x,y be two numbers Given

$$x + y = 24 \tag{2.4.1}$$

$$\implies y = 24 - x \tag{2.4.2}$$

For their product to be maximum

$$f(x) = xy = x(24 - x) = 24x - x^2 \tag{2.4.3}$$

Lemma 2.4. *A function f(x) is said to be concave if following inequality is true for* $\lambda \in [0, 1]$:

$$\lambda f(x_1) + (1 - \lambda) f(x_2) \leq f(\lambda x_1 + (1 - \lambda) x_2) \tag{2.4.4}$$

Checking convexity of $f(x)$:

$$\lambda(24x_1 - {x_1}^2) + (1 - \lambda)\left(24x_2 - {x_2}^2\right) \tag{2.4.5}$$

$$\leq 24(\lambda x_1 + (1 - \lambda)x_2) - (\lambda x_1 + (1 - \lambda)x_2)^2 \tag{2.4.6}$$

$$\lambda(\lambda - 1)(x_1 - x_2)^2 \leq 0 \tag{2.4.7}$$

$$\implies \lambda(\lambda - 1) \leq 0 \tag{2.4.8}$$

is true . $\implies$ The function is concave. Using gradient ascent method we can find its maxima,

$$x_{n+1} = x_n + \alpha \nabla f(x_n) \tag{2.4.9}$$

$$\implies x_{n+1} = x_n + \alpha(-2x_n + 24) \tag{2.4.10}$$

Taking $x_0 = 2, \alpha = 0.001$ and precision= 0.00000001,values obtained using python are:

$$\boxed{\text{Maxima} = 143.9999999999752 \approx 144} \tag{2.4.11}$$

$$\boxed{\text{Maxima Point} = 11.999995019260913 \approx 12} \tag{2.4.12}$$

We can verify this by the derivative test. Since $f(x)$ is a concave function it has a maxima.

$$\frac{df(x)}{dx} = -2x + 24 \tag{2.4.13}$$

Critical point :

$$\frac{df(x)}{dx} = 0 \tag{2.4.14}$$

$$-2x + 24 = 0 \tag{2.4.15}$$

$$x = 12 \tag{2.4.16}$$

is a critical point.And since $f(x)$ is a concave function there will be a maxima at $x = 12$. And the maxima is

$$f(12) = 144 \tag{2.4.17}$$

Fig. 2.4 provides a verification.

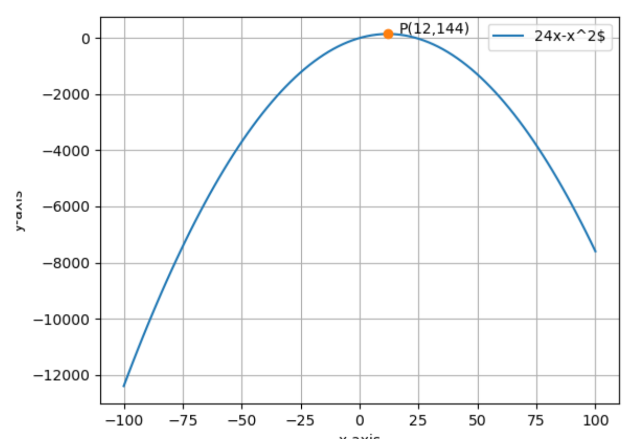

Fig. 2.4: $f(x) = 24x - x^2$

2.5. Find two positive numbers whose sum is 16 and the sum of whose cubes is minimum.

2.6. The sum of the perimeter of a circle and square is k, where k is some constant. Prove that the sum of their areas is least when the side of square is double the radius of the circle.

2.7. A window is in the form of a rectangle surmounted by a semicircular opening. The total perimeter of the window is 10 m. Find the dimensions of the window to admit maximum light through the whole opening.

2.8. Find the shortest distance of the point $\begin{pmatrix} 0 \\ c \end{pmatrix}$ from the parabola $y = x^2$, where $\frac{1}{2} \le c \le 5$.

2.9. Find the maximum area of an isosceles triangle inscribed in the ellipse

$$\mathbf{x}^T \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix} \mathbf{x} = a^2 b^2 \tag{2.9.1}$$

with its vertex at one end of the major axis.

2.10. Maximise Z=-x+2y, subject to the constraints: x≥3, x+y≥5, x+2y≥6, y≥0.

2.11. Maximise Z=x+y, subject to x-y≤-1,-x+y≤0, x,y≥0.

Solution: The given problem can be expressed in general as matrix inequality as:

$$\max_{\{x\}} \mathbf{c}^T \mathbf{x} \tag{2.11.1}$$

$$s.t \quad \mathbf{A}\mathbf{x} \le \mathbf{b} \tag{2.11.2}$$

$$\mathbf{x} \ge 0 \tag{2.11.3}$$

$$\mathbf{y} \ge 0 \tag{2.11.4}$$

where,

$$\mathbf{c} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.11.5}$$

$$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \tag{2.11.6}$$

$$\mathbf{b} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \tag{2.11.7}$$

Solving for Z by this reduction method we get

$$MaxZ = None \tag{2.11.8}$$

There is no optimal maximum solution for this. See Fig. 2.11.

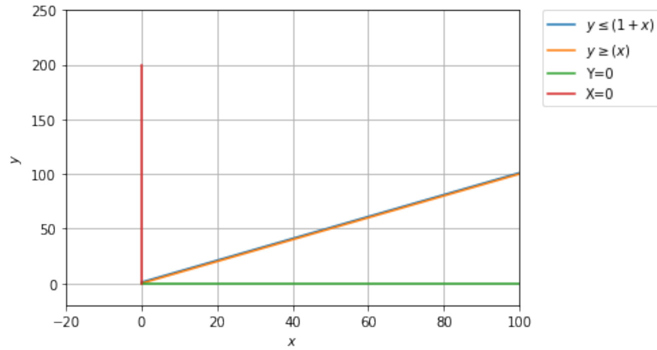

Fig. 2.11: Plot from python code

2.12. A merchant plans to sell two types of personal computers – a desktop model and a portable model that will cost Rs 25000 and Rs 40000 respectively. He estimates that the total monthly demand of computers will not exceed 250 units. Determine the number of units of each type of computers which the merchant should stock to get maximum profit if he does not want to invest more than Rs 70 lakhs and if his profit on the desktop model is Rs 4500 and on portable model is Rs 5000.

2.13. A diet is to contain at least 80 units of vitamin A and 100 units of minerals. Two foodsF_1 and F_2 are available. Food F_1 costs Rs 4 per unit food and F_2 costs Rs 6 per unit. One unit of food F_1 contains 3 units of vitamin A and 4 units of minerals. One unit of food F_2 contains 6 units of vitamin A and 3 units of minerals. Formulate this as a linear programming problem. Find the minimum cost for diet that consists of mixture of these two foods and also meets the minimal nutritional requirements.

2.14. There are two types of fertilisers F_1 and F_2.F_1 consists of 10% nitrogen and 6% phosphoric acid and F_2 consists of 5% nitrogen and 10% phosphoric acid. After testing the soil conditions, a farmer finds that she needs atleast 14 kg of nitrogen and 14 kg of phosphoric acid for her crop. If F_1 costs Rs 6/kg and F_2 costs Rs 5/kg, determine how much of each type of fertiliser should be used so that nutrient requirements are met at a minimum cost. What is the minimum cost?

2.15. The corner points of the feasible region determined by the following system of linear inequalities: 2x+y≤10, x+3y≤15, x,y≥0 are (0,0), (5,0),(3,4) and (0,5).Let Z=px+qy, where p,q>0.Condition on p and q so that the

maximum of Z occurs at both (3,4) and (0,5) is
(A) p = q
(B) p = 2q
(C) p = 3q
(D) q = 3p

2.16. Refer to Example 9. How many packets of each food should be used to maximise the amount of vitamin A in the diet? What is the maximum amount of vitamin A in the diet?

2.17. A farmer mixes two brands P and Q of cattle feed. Brand P, costing Rs 250 per bag, contains 3 units of nutritional element A, 2.5 units of element B and 2 units of element C. Brand Q costing Rs 200 per bag contains 1.5 units of nutritional element A, 11.25 units of element B, and 3 units of element C. The minimum requirements of nutrients A, B and C are 18 units, 45 units and 24 units respectively. Determine the number of bags of each brand which should be mixed in order to produce a mixture having a minimum cost per bag? What is the minimum cost of the mixture per bag?

2.18. A dietician wishes to mix together two kinds of food X and Y in such a way that the mixture contains at least 10 units of vitamin A, 12 units of vitamin B and 8 units of vitamin C. The vitamin contents of one kg food is given below:

Food	**Vitamin A**	**Vitamin B**	**VitaminC**
X	1	2	3
Y	2	2	1

One kg of food X costs Rs 16 and one kg of food Y costs Rs 20. Find the least cost of the mixture which will produce the required diet?

2.19. An aeroplane can carry a maximum of 200 passengers. A profit of Rs 1000 is made on each executive class ticket and a profit of Rs 600 is made on each economy class ticket. The airline reserves at least 20 seats for executive class. However, at least 4 times as many passengers prefer to travel by economy class than by the executive class. Determine how many tickets of each type must be sold in order to maximise the profit for the airline. What is the maximum profit?

2.20. An oil company has two depots A and B with capacities of 7000 L and 4000 L respectively. The company is to supply oil to three petrol pumps, D, E and F whose requirements are 4500L, 3000L and 3500L respectively. The distances (in km) between the depots and the petrol pumps is given in the following table:

Distance in (km.)		
From/To	**A**	**B**
D	7	3
E	6	4
F	3	2

Assuming that the transportation cost of 10 litres of oil is Re 1 per km, how should the delivery be scheduled in order that the transportation cost is minimum? What is the minimum cost?

2.21. Refer to Question 29. If the grower wants to maximise the amount of nitrogen added to the garden, how many bags of each brand should be added? What is the maximum amount of nitrogen added?

2.22. Find the shortest distance of the point $\begin{pmatrix} 0 \\ c \end{pmatrix}$ from the parabola $y = x^2$, where $\frac{1}{2} \le c \le 5$.

2.23. Find the maximum area of an isosceles triangle inscribed in the ellipse

$$\mathbf{x}^T \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix} \mathbf{x} = a^2 b^2 \tag{2.23.1}$$

with its vertex at one end of the major axis.