Quadratic Forms

G V V Sharma*

Contents

***Abstract*—This book provides a computational approach to school geometry based on the NCERT textbooks from Class 6-12. Links to sample Python codes are available in the text.**

Download python codes using

```
svn co https://github.com/gadepall/school/trunk/
    ncert/computation/codes
```

1 Examples

1.1. Find the equation of a circle with centre $\begin{pmatrix}-3\\2\end{pmatrix}$ and radius 4.

Solution: From the given information, the desired equation is

$$\left\|\mathbf{x}-\begin{pmatrix}-3\\2\end{pmatrix}\right\|^2 = 4^2 \tag{1.1.1}$$

$$\implies \mathbf{x}^T\mathbf{x}+\begin{pmatrix}6 & -4\end{pmatrix}\mathbf{x}-3=0 \tag{1.1.2}$$

The python code for Fig. 1.1 is

```
solutions/1/codes/circle/circle1.py
```

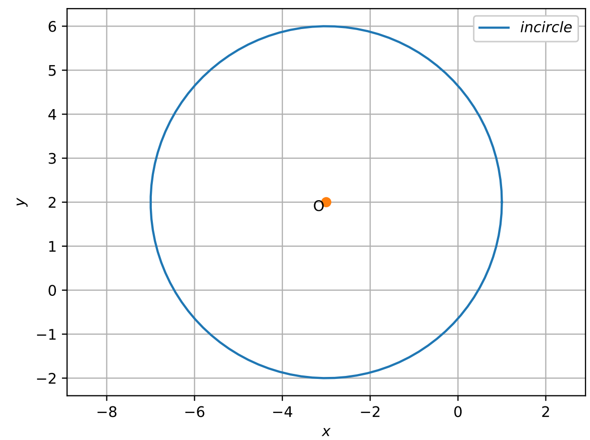

Fig. 1.1: Circle using python

1.2. Find the centre and radius of the circle

$$\mathbf{x}^T\mathbf{x}+\begin{pmatrix}8\\10\end{pmatrix}\mathbf{x}-8=0 \tag{1.2.1}$$

Solution:

The general equation of a circle is

$$\implies \mathbf{x}^T\mathbf{x}-2\mathbf{O}^T\mathbf{x}+\|\mathbf{O}\|^2-r^2=0 \tag{1.2.2}$$

Comparing equation (1.2.2) with the given circle equation:

$$\mathbf{O}=\begin{pmatrix}-4\\-5\end{pmatrix} \tag{1.2.3}$$

$$\|\mathbf{O}\|^2 = 41 \tag{1.2.4}$$

$$r^2 = 41+8 \tag{1.2.5}$$

$$\therefore r = 7 \tag{1.2.6}$$

The following Python code generates Fig. 1.2

```
solutions/2/codes/circle_exam.py
```

1.3. Find the equation of the circle which passes through the points $\begin{pmatrix}2\\-2\end{pmatrix}$ and $\begin{pmatrix}3\\4\end{pmatrix}$ and whose centre lies on the line

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x}=2. \tag{1.3.1}$$

Solution:

*The author is with the Department of Electrical Engineering, Indian Institute of Technology, Hyderabad 502285 India e-mail: gadepall@iith.ac.in.

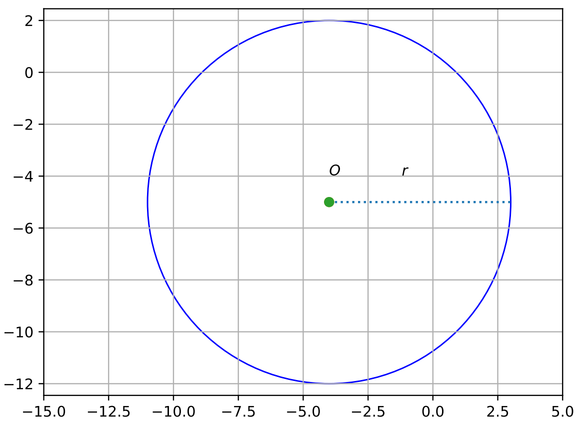

Fig. 1.2

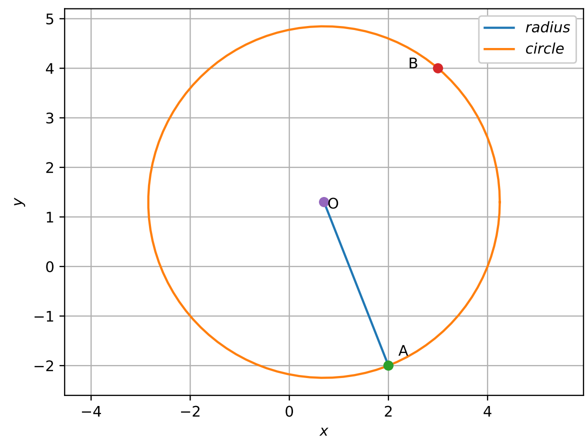

Fig. 1.4: Circle with centre at **O** and radius r

1.4. Let **O** be the centre and r be the radius. For

$$\mathbf{A} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \tag{1.4.1}$$

$$\implies \|\mathbf{A} - \mathbf{O}\| = \|\mathbf{B} - \mathbf{O}\| = r \tag{1.4.2}$$

$$\implies \|\mathbf{A} - \mathbf{O}\|^2 - \|\mathbf{B} - \mathbf{O}\|^2 = 0 \tag{1.4.3}$$

$$\implies (\mathbf{A} - \mathbf{B})^T \mathbf{O} = \frac{\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2}{2} \tag{1.4.4}$$

$$\text{or, } \begin{pmatrix} 1 & 6 \end{pmatrix} \mathbf{O} = \frac{17}{2} \tag{1.4.5}$$

Also centre O lies on the line in (1.3.1)

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{O} = 2 \tag{1.4.6}$$

(1.4.5) and (1.4.6) result in the matrix equation

$$\begin{pmatrix} 1 & 6 \\ 1 & 1 \end{pmatrix} \mathbf{O} = \begin{pmatrix} \frac{17}{2} \\ 2 \end{pmatrix} \tag{1.4.7}$$

The following code calculates centre and radius and plots figure 1.4

```
solutions/3/codes/circle1/circle1.py.py
```

1.5. Find the area enclosed by the circle $\|\mathbf{x}\| = a$
Solution: The area is $2\pi a^2$.

1.6. Find the area of the region in the first quadrant enclosed by the x-axis, the line $\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 0$, and the circle $\|\mathbf{x}\| = 1$.
Solution: The circle in Fig. 1.6 is generated using the following python code

```
solutions/6/codes/circle/example/circle.py
```

The angle that the line makes with the x-axis is given by

$$\cos\theta = \frac{\begin{pmatrix} 1 & -1 \end{pmatrix}^T \begin{pmatrix} 1 \\ 0 \end{pmatrix}}{\left\| \begin{pmatrix} 1 & -1 \end{pmatrix} \right\| \left\| \begin{pmatrix} 1 & -1 \end{pmatrix} \right\|} \tag{1.6.1}$$

$$= \frac{1}{\sqrt{2}} \tag{1.6.2}$$

$$\implies \theta = 45^\circ. \tag{1.6.3}$$

The area of the sector is then obtained as

$$\frac{\theta}{360^\circ}\pi r^2 = \frac{45^\circ}{360^\circ}\pi r^2 \tag{1.6.4}$$

$$= \frac{\pi}{8} \tag{1.6.5}$$

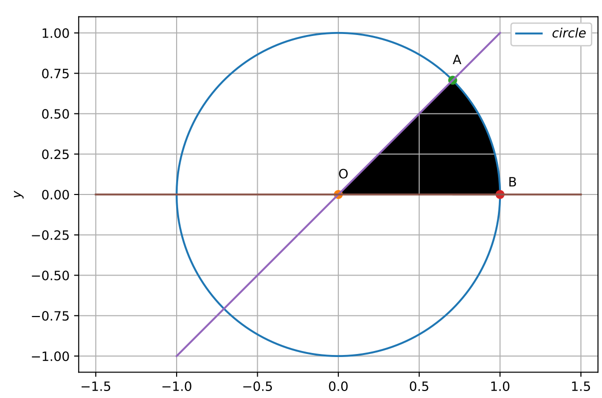

Fig. 1.6: Circle generated using python

1.7. Find the area of the region enclosed between the two circles: $\mathbf{x}^T\mathbf{x} = 4$ and $\left\|\mathbf{x} - \begin{pmatrix} 2 \\ 0 \end{pmatrix}\right\| = 2$.

1.8. Find the coordinates of a point $\mathbf{A}$, where AB is the diameter of a circle whose centre is $\begin{pmatrix} 2, -3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$.

Solution:

The input values for the question are given in the table (1.8) The $\mathbf{A}$ is at the end of diameter,

Input values	
Parameters	Values
$\mathbf{O}$	$\begin{pmatrix} 2 \\ -3 \end{pmatrix}$
$\mathbf{A}$	$\begin{pmatrix} 1 \\ 4 \end{pmatrix}$

TABLE 1.8: Input Values

so the centre($\mathbf{O}$) is the midpoint of $\mathbf{AB}$.

$$\mathbf{O} = \frac{\mathbf{A} + \mathbf{B}}{2} \tag{1.8.1}$$

$$\mathbf{A} = 2\mathbf{O} - \mathbf{B} \tag{1.8.2}$$

$$\therefore \mathbf{A} = \begin{pmatrix} 3 \\ -10 \end{pmatrix} \tag{1.8.3}$$

The python code for the figure (1.8) is

```
solutions/1/codes/circle/circle.py
```

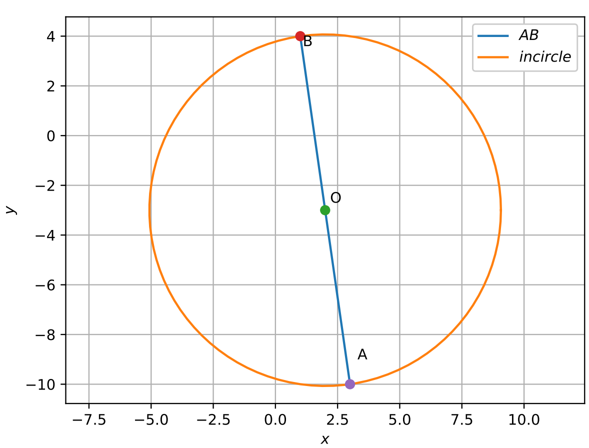

Fig. 1.8

1.9. Find the centre O of a circle passing through the points $\begin{pmatrix} 6 \\ -6 \end{pmatrix}$, $\begin{pmatrix} 3 \\ -7 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 3 \end{pmatrix}$. **Solution:** The general of a circle equation is:

$$\|\mathbf{x} - \mathbf{O}\| = r \tag{1.9.1}$$

Substituting the given coordinates

$$\left\|\begin{pmatrix} 6 \\ -6 \end{pmatrix} - \mathbf{O}\right\|^2 = r^2 \tag{1.9.2}$$

$$\left\|\begin{pmatrix} 3 \\ -7 \end{pmatrix} - \mathbf{O}\right\|^2 = r^2 \tag{1.9.3}$$

$$\left\|\begin{pmatrix} 3 \\ 3 \end{pmatrix} - \mathbf{O}\right\|^2 = r^2 \tag{1.9.4}$$

From (1.9.2), (1.9.3), (1.9.4):

$$\left\|\begin{pmatrix} 3 \\ -7 \end{pmatrix} - \mathbf{O}\right\|^2 - \left\|\begin{pmatrix} 6 \\ -6 \end{pmatrix} - \mathbf{O}\right\|^2 = 0 \tag{1.9.5}$$

$$\left\|\begin{pmatrix} 3 \\ 3 \end{pmatrix} - \mathbf{O}\right\|^2 - \left\|\begin{pmatrix} 6 \\ -6 \end{pmatrix} - \mathbf{O}\right\|^2 = 0 \tag{1.9.6}$$

Simplifying equations (1.9.5) and (1.9.6):

$$\begin{pmatrix} 3 & 1 \\ 1 & -3 \end{pmatrix}\mathbf{O} = \begin{pmatrix} 7 \\ 9 \end{pmatrix} \tag{1.9.7}$$

$$\begin{pmatrix} 3 & 1 & 7 \\ 1 & -3 & 9 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{R_1}{3}} \begin{pmatrix} 1 & \frac{1}{3} & \frac{7}{3} \\ 1 & -3 & 9 \end{pmatrix} \tag{1.9.8}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & \frac{1}{3} & \frac{7}{3} \\ 1 & -\frac{10}{3} & \frac{20}{3} \end{pmatrix} \tag{1.9.9}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{-3R_2}{10}} \begin{pmatrix} 1 & \frac{1}{3} & \frac{7}{3} \\ 1 & 1 & -2 \end{pmatrix} \tag{1.9.10}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - \frac{R_2}{3}} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \end{pmatrix} \tag{1.9.11}$$

$$\therefore \mathbf{O} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \tag{1.9.12}$$

The following Python code generates Fig. 1.9

```
solutions/2/codes/circle_ex/circumcircle.py
```

1.10. Sketch the circles with

a) centre $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ and radius 2

b) centre $\begin{pmatrix} -2 \\ 32 \end{pmatrix}$ and radius 4

c) centre $\begin{pmatrix} \frac{1}{2} \\ \frac{1}{4} \end{pmatrix}$ and radius $\frac{1}{12}$.

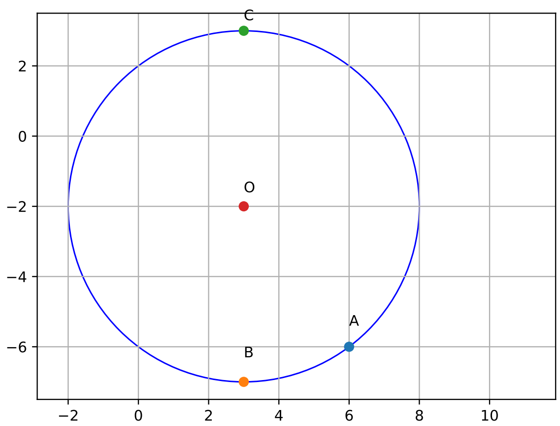

Fig. 1.9

d) centre $\begin{pmatrix}1\\1\end{pmatrix}$ and radius $\sqrt{2}$.

e) centre $\begin{pmatrix}-a\\-b\end{pmatrix}$ and radius $\sqrt{a^2-b^2}$.

Solution:

a) Let **O** be the centre , r be the radius of the circle.Any point **X** lying on the circle is at a distance r from **O**.
Therefore the equation of the circle is

$$\|\mathbf{X}-\mathbf{O}\| = r \tag{1.10.1}$$

b)

$$(a)\,\mathbf{O} = \begin{pmatrix}0\\2\end{pmatrix}, r = 2 \tag{1.10.2}$$

The following code sketches the circle (1.10.2) in figure 1.10 using the equation (1.10.1)

```
solutions/3/codes/circle2/circle2a.py
```

c)

$$(b)\,\mathbf{O} = \begin{pmatrix}-2\\32\end{pmatrix}, r = 4 \tag{1.10.3}$$

The following code sketches the circle (1.10.3) in figure 1.10 using the equation (1.10.1)

```
solutions/3/codes/circle2/circle2b.py
```

d)

$$(c)\,\mathbf{O} = \begin{pmatrix}\frac{1}{2}\\\frac{1}{4}\end{pmatrix}, r = \frac{1}{12} \tag{1.10.4}$$

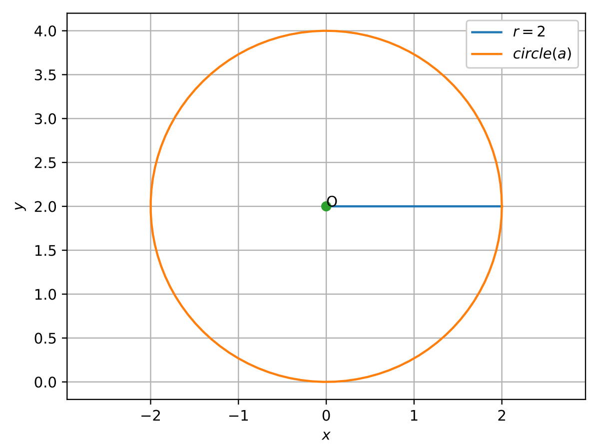

Fig. 1.10: Circle with centre at $\begin{pmatrix}0\\2\end{pmatrix}$ and radius 2

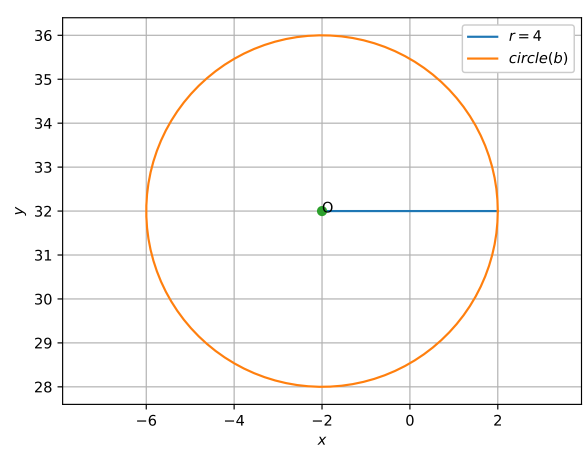

Fig. 1.10: Circle with centre at $\begin{pmatrix}-2\\32\end{pmatrix}$ and radius 4

The following code sketches the circle (1.10.4) in figure 1.10 using the equation (1.10.1)

```
solutions/3/codes/circle2/circle2c.py
```

e)

$$(d)\,\mathbf{O} = \begin{pmatrix}1\\1\end{pmatrix}, r = \sqrt{2} \tag{1.10.5}$$

The following code sketches the circle (1.10.5) in figure 1.10 using the equation (1.10.1)

```
solutions/3/codes/circle2/circle2d.py
```

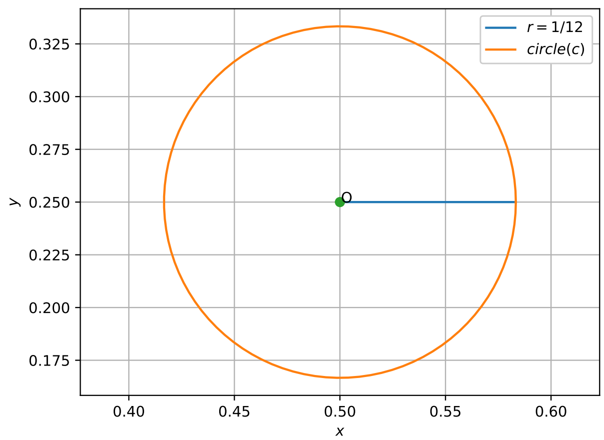

Fig. 1.10: Circle with centre at $\begin{pmatrix}\frac{1}{2}\\ \frac{1}{4}\end{pmatrix}$ and radius $\frac{1}{12}$

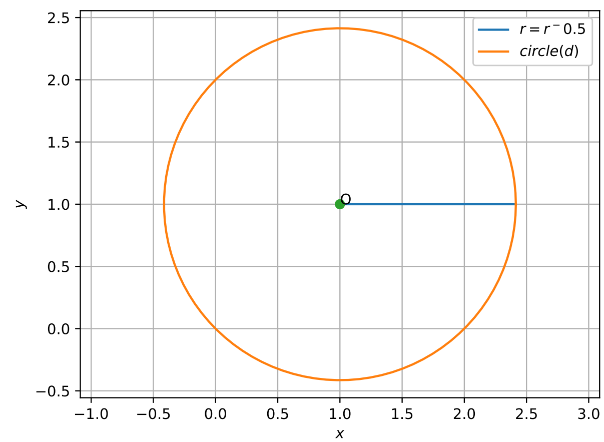

Fig. 1.10: Circle with centre at $\begin{pmatrix}1\\1\end{pmatrix}$ and radius $\sqrt{2}$

f)

$$(e)\,\mathbf{O} = \begin{pmatrix}-a\\-b\end{pmatrix}, r = \sqrt{a^2 - b^2} \tag{1.10.6}$$

The parameters used to sketch the circle are taken as

$$a = 5, b = 4 \implies \mathbf{O} = \begin{pmatrix}-5\\-4\end{pmatrix} \tag{1.10.7}$$

$$r = \sqrt{5^2 - 4^2} = 3 \tag{1.10.8}$$

The following code sketches the circle (1.10.8) in figure 1.10 using the equation (1.10.1)

```
solutions/3/codes/circle2/circle2e.py
```

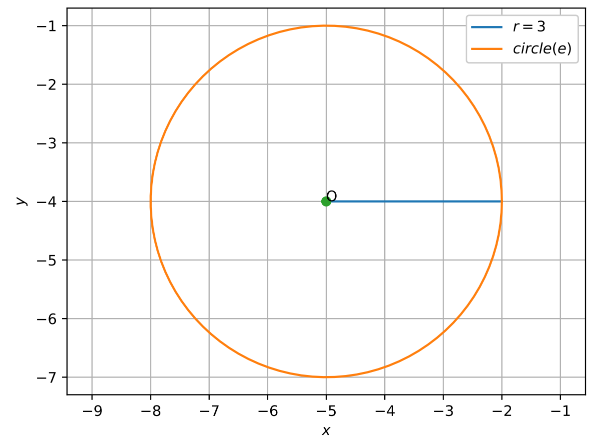

Fig. 1.10: Circle with centre at $\begin{pmatrix}-5\\-4\end{pmatrix}$ and radius 3

1.11. Does the point $\begin{pmatrix}-2.5\\3.5\end{pmatrix}$ lie inside, outside or on the circle $\mathbf{x}^T\mathbf{x} = 25$?

Solution: See Fig. 1.11. The general equation for the circle can be given as

$$\mathbf{x}^T\mathbf{x} + 2\mathbf{O}^T\mathbf{x} + \|O\|^2 - \mathbf{r}^2 = 0 \tag{1.11.1}$$

given equation of circle

$$\mathbf{x}^T\mathbf{x} - 25 = 0 \tag{1.11.2}$$

comparing bothe of equation we can find the value of r and value of $\mathbf{O}$

$$\mathbf{r} = 4 \tag{1.11.3}$$

$$\mathbf{O} = \begin{pmatrix}0\\0\end{pmatrix} \tag{1.11.4}$$

$$\implies \mathbf{B} - \mathbf{O} = \begin{pmatrix}-2.5\\3.5\end{pmatrix} \tag{1.11.5}$$

$$\implies \|\mathbf{B} - \mathbf{O}\|^2 = 18.5 < 25 \tag{1.11.6}$$

$$\text{or, } OB < r \tag{1.11.7}$$

Hence, $\mathbf{B}$ lies inside the circle.
The following code plots Fig. 1.11

```
solutions/4/codes/circle/circle2.py
```

1.12. Sketch the circles with equation

a) $\left\|\mathbf{x} - \begin{pmatrix}5\\-3\end{pmatrix}\right\|^2 = 36$

b) $\mathbf{x}^T\mathbf{x} - \begin{pmatrix}4\\8\end{pmatrix}\mathbf{x} - 45 = 0$

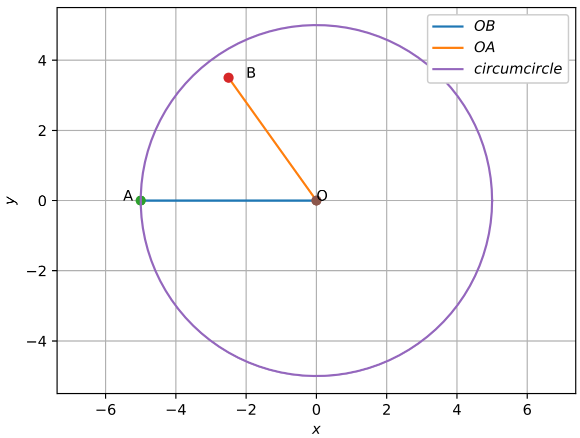

Fig. 1.11: circle

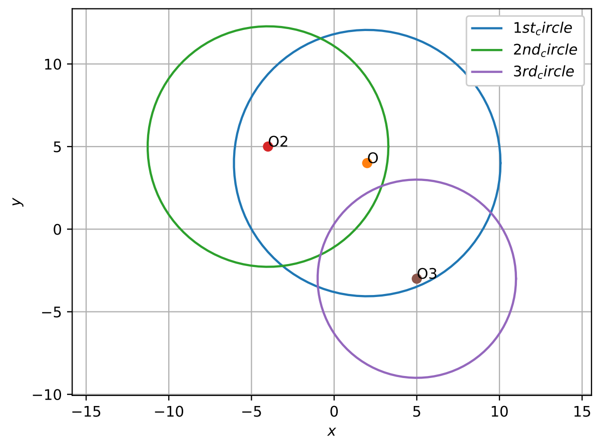

Fig. 1.12

c) $\mathbf{x}^T\mathbf{x} - \begin{pmatrix} 8 \\ -10 \end{pmatrix}\mathbf{x} - 12 = 0$

d) $2\mathbf{x}^T\mathbf{x} - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\mathbf{x} = 0$

Solution: The following python codes generate the required circle

```
./solutions/5/codes/circle/q18abc.py
./solutions/5/codes/circle/q18d.py
```

a)

$$\mathbf{x}^\mathbf{T}\mathbf{x} - \begin{pmatrix} 4 \\ 8 \end{pmatrix}\mathbf{x} - 45 = 0 \tag{1.12.1}$$

See Fig. 1.12.

$$\mathbf{O} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}, r = \sqrt{65} \tag{1.12.2}$$

b) See Fig. 1.12.

$$\mathbf{O} = \begin{pmatrix} -4 \\ 5 \end{pmatrix}, r = \sqrt{53} \tag{1.12.3}$$

$$\mathbf{x}^\mathbf{T}\mathbf{x} - \begin{pmatrix} 8 \\ -10 \end{pmatrix}\mathbf{x} - 12 = 0 \tag{1.12.4}$$

c) See Fig. 1.12.

$$\mathbf{O} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}, r = 6 \tag{1.12.5}$$

$$\left\| x - \begin{pmatrix} 5 \\ -3 \end{pmatrix} \right\| = 36 \tag{1.12.6}$$

d) See Fig. 1.12.

$$\mathbf{O} = \frac{1}{4}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, r = \frac{1}{4} \tag{1.12.7}$$

$$2\mathbf{x}^\mathbf{T}\mathbf{x} - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\mathbf{x} = 0 \tag{1.12.8}$$

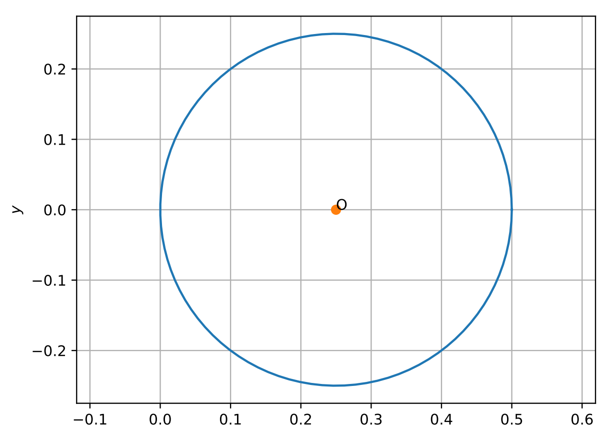

Fig. 1.12

1.13. Find the equation of the circle passing through the points $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ 5 \end{pmatrix}$ and whose centre is on the line $\begin{pmatrix} 4 & 1 \end{pmatrix}\mathbf{x} = 16$.
Solution: The vector form of general equation

of circle is,

$$\|\mathbf{x} - \mathbf{O}\|^2 = r^2 \quad (1.13.1)$$

$$\implies \mathbf{x}^T\mathbf{x} - 2\mathbf{O}^T\mathbf{x} + \|\mathbf{O}\|^2 - r^2 = 0 \quad (1.13.2)$$

whose centre is $\mathbf{O}$ and radius r. $\because$ $\mathbf{A} = \begin{pmatrix}4\\1\end{pmatrix}$ lies on the circle. Letting

$$F = \|\mathbf{O}\|^2 - r^2, \quad (1.13.3)$$

$$\begin{pmatrix}4 & 1\end{pmatrix}^T\begin{pmatrix}4\\1\end{pmatrix} - 2\mathbf{O}^T\begin{pmatrix}4\\1\end{pmatrix} + F = 0 \quad (1.13.4)$$

$$\implies 2\begin{pmatrix}4 & 1\end{pmatrix}\mathbf{O} - F = 17 \quad (1.13.5)$$

Similarly,

$$\begin{pmatrix}6 & 5\end{pmatrix}^T\begin{pmatrix}6\\5\end{pmatrix} - 2\mathbf{O}^T\begin{pmatrix}6\\5\end{pmatrix} + F = 0 \quad (1.13.6)$$

$$\implies 2\begin{pmatrix}6\\5\end{pmatrix}\mathbf{O} - F = 61 \quad (1.13.7)$$

Subtracting 1.13.5 from 1.13.7,

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{O} = 11 \quad (1.13.8)$$

Also, from the given information,

$$\begin{pmatrix}4 & 1\end{pmatrix}\mathbf{O} = 16 \quad (1.13.9)$$

From 1.13.9 and 1.13.8

$$\mathbf{O} = \begin{pmatrix}3\\4\end{pmatrix}, F = 15 \quad (1.13.10)$$

and the vector form of the circle is

$$\mathbf{x}^T\mathbf{x} - 2\begin{pmatrix}3 & 4\end{pmatrix}\mathbf{x} + 15 = 0 \quad (1.13.11)$$

The following code generates Fig. 1.13

solutions/6/codes/circle/exercise/circle.py

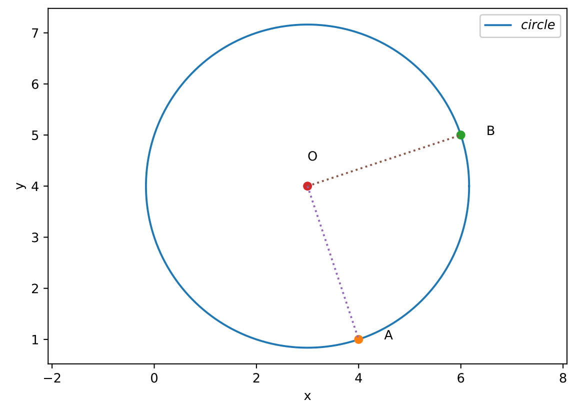

Fig. 1.13

1.14. Find the equation of the circle passing through the points $\mathbf{P} = \begin{pmatrix}2\\3\end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix}-1\\1\end{pmatrix}$ and whose centre is on the line $\begin{pmatrix}1 & -3\end{pmatrix}\mathbf{x} = 11$. **Solution:** Let $\mathbf{O}$ be the centre of the circle and r be the radius of the circle. Since centre lies on the given line

$$\begin{pmatrix}1 & -3\end{pmatrix}\mathbf{O} = 11 \quad (1.14.1)$$

Also

$$\|\mathbf{P} - \mathbf{O}\|^2 = \|\mathbf{Q} - \mathbf{O}\|^2 = r^2 \quad (1.14.2)$$

$$\implies \begin{pmatrix}6 & 4\end{pmatrix}\mathbf{O} = 11 \quad (1.14.3)$$

From (1.14.1) and (1.14.3),

$$\begin{pmatrix}1 & -3\\6 & 4\end{pmatrix}\mathbf{O} = \begin{pmatrix}11\\11\end{pmatrix} \quad (1.14.4)$$

$$\implies \mathbf{O} = \begin{pmatrix}\frac{7}{2}\\\frac{-5}{2}\end{pmatrix} \quad (1.14.5)$$

From $\mathbf{O}$ we get r= 5.7. This is verified in Fig. 1.14 by the following python code.

solutions/7/codes/circle/circle.py

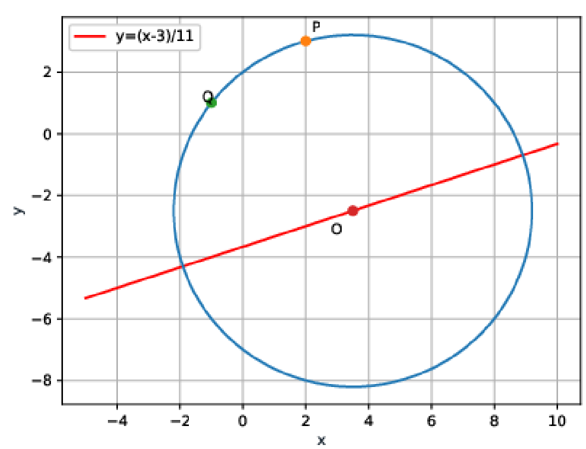

Fig. 1.14

1.15. Find the equation of the circle with radius 5 whose centre lies on x-axis and passes through the point $\begin{pmatrix}2\\3\end{pmatrix}$.

Solution: Let $\mathbf{O}$ be the centre and r be the radius of the given circle.

$\therefore$

$$\mathbf{O} = \begin{pmatrix}p\\0\end{pmatrix} \quad (1.15.1)$$
$$r = 5$$

The general equation of a circle is given by

$$\mathbf{x}^T\mathbf{x} - 2\mathbf{O}^T\mathbf{x} + \|\mathbf{O}\|^2 - r^2 = 0 \quad (1.15.2)$$

which, upon substitution from (1.15.1) yields

$$\mathbf{x}^T\mathbf{x} - 2\begin{pmatrix}p & 0\end{pmatrix}\mathbf{x} + p^2 - 25 = 0 \quad (1.15.3)$$

$\because$ Point $\mathbf{A}$=$\begin{pmatrix}2\\3\end{pmatrix}$ lies on the circle,

$$\begin{pmatrix}2 & 3\end{pmatrix}\begin{pmatrix}2\\3\end{pmatrix} - 2\begin{pmatrix}p & 0\end{pmatrix}\begin{pmatrix}2\\3\end{pmatrix} + p^2 - 25 = 0 \quad (1.15.4)$$
$$\implies p^2 - 4p - 12 = 0 \quad (1.15.5)$$
$$\implies p = 6 \text{ or } -2 \quad (1.15.6)$$

Hence,the possible equations of the circle are

$$\mathbf{x}^T\mathbf{x} - 2\begin{pmatrix}6 & 0\end{pmatrix}\mathbf{x} + 11 = 0 \quad (1.15.7)$$
$$\text{or}, \mathbf{x}^T\mathbf{x} - 2\begin{pmatrix}-2 & 0\end{pmatrix}\mathbf{x} - 21 = 0 \quad (1.15.8)$$

which are plotted in Fig. 1.15

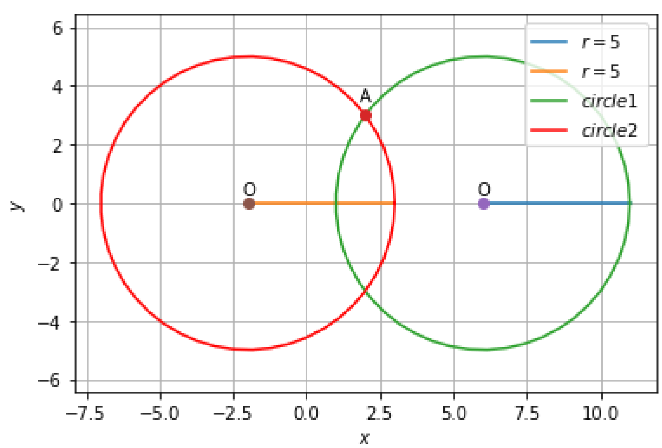

Fig. 1.15: Circles with centres (6,0) and (-2,0) respectively

1.16. Verify whether 2 and 0 are zeroes of the polynomial $x^2 - 2x$.

Solution: The given polynomial can be expressed as the parabola

$$\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\mathbf{x} + \begin{pmatrix}-2 & 0\end{pmatrix}\mathbf{x} + 0 = 0 \quad (1.16.1)$$

$$\because \begin{pmatrix}0 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\begin{pmatrix}0\\0\end{pmatrix} + \begin{pmatrix}-2 & 0\end{pmatrix}\begin{pmatrix}0\\0\end{pmatrix} = 0 \quad (1.16.2)$$

0 is a root.

$$\because \begin{pmatrix}2 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\begin{pmatrix}2\\0\end{pmatrix} + \begin{pmatrix}-2 & 0\end{pmatrix}\begin{pmatrix}2\\0\end{pmatrix} = 0 \quad (1.16.3)$$

2 is also a root. This is verified by plotting Fig. 1.16 through the following code.

```
solutions/2/codes/conics_example/conics.py
```

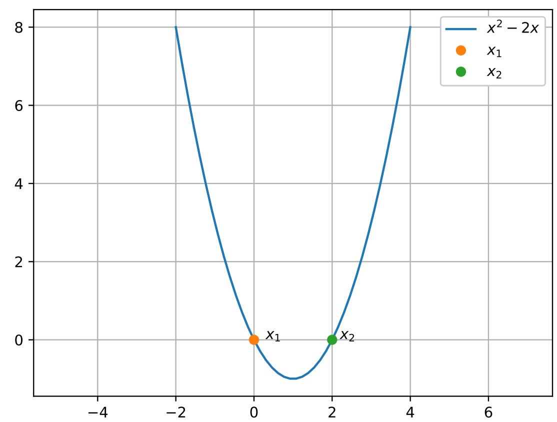

Fig. 1.16

1.17. Find $p(0)$, $p(1)$ and $p(2)$ for each of the following polynomials:

a) $p(y) = y^2$.
b) $p(x) = (x–1)(x + 1)$.

Solution:

a) To find p(0) we substitute 0 in place of the variable y in p(y). Similarly we find p(1) and p(2)

$$p(y) = y^2 \quad (1.17.1)$$
$$\implies p(0) = 0 \quad (1.17.2)$$
$$p(1) = 1 \quad (1.17.3)$$
$$p(2) = 4 \quad (1.17.4)$$

The following code sketches the graph of 1.17.1 in Fig. 1.17

```
solutions/3/codes/conic1/conic1a.py
```

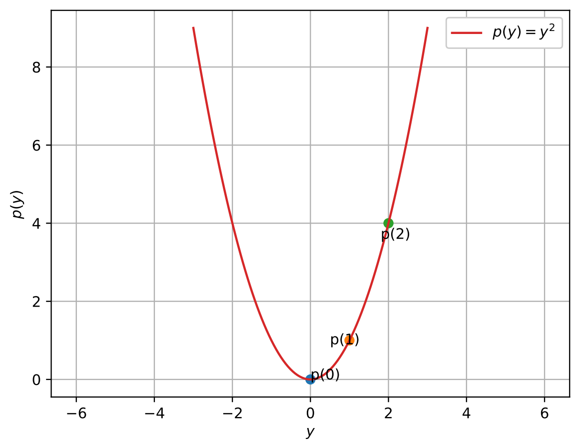

Fig. 1.17: Graph of p(y)

b) Similarly we find p(0), p(1) and p(2) of p(x) by replacing x

$$p(x) = (x-1)(x+1) \tag{1.17.5}$$
$$\implies p(0) = -1 \tag{1.17.6}$$
$$p(1) = 0 \tag{1.17.7}$$
$$p(2) = 3 \tag{1.17.8}$$

The following code sketches the graph of 1.17.5 in Fig. 1.17

```
solutions/3/codes/conic1/conic1b.py
```

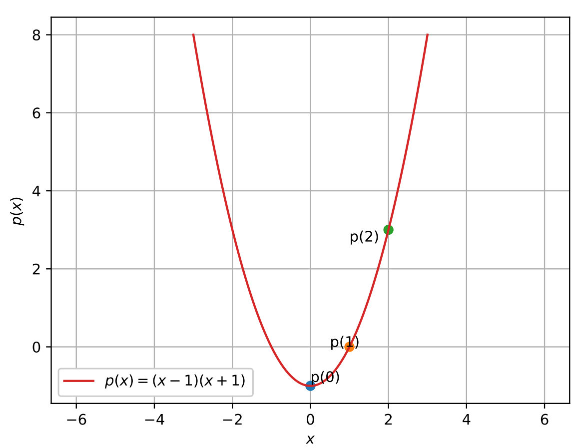

Fig. 1.17: Graph of p(x)

1.18. Find the roots of the quadratic equation $6x^2 - x - 2 = 0$.
Solution:
The vector form of

$$y = 6x^2 - x - 2 \tag{1.18.1}$$

is

$$\mathbf{x}^T \begin{pmatrix} 6 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -1 & -1 \end{pmatrix} \mathbf{x} - 2 = 0 \tag{1.18.2}$$

Thus,

$$y = 0 \implies 6x^2 - x - 2 = 0 \tag{1.18.3}$$
$$\left(x + \frac{1}{2}\right)\left(x - \frac{2}{3}\right) = 0 \tag{1.18.4}$$
$$x = \frac{-1}{2}, \frac{2}{3} \tag{1.18.5}$$

The following python code computes roots of the quadratic equation represented in Fig. 1.18.

```
./solutions/5/codes/conics/q19.py
```

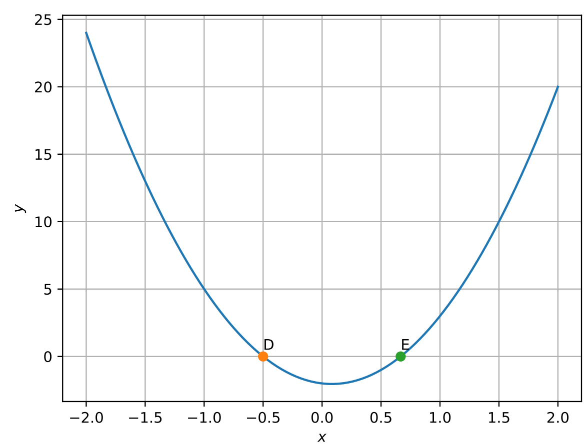

Fig. 1.18

1.19. Find the roots of the quadratic equation $3x^2 - 2\sqrt{6}x + 2 = 0$.
Solution: The vector form of the equation is

$$\mathbf{x}^T \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -2\sqrt{6} & 0 \end{pmatrix} \mathbf{x} + 2 = 0 \tag{1.19.1}$$

The values of $\mathbf{x}$ are found in the following python code

```
solutions/6/codes/conics/example/conics.py
```

$\mathbf{x} = \begin{pmatrix} 0.81649658 \\ 0 \end{pmatrix}, \begin{pmatrix} 0.81649658 \\ 0 \end{pmatrix}$
which can be verified from Fig. 1.19 generated by the following python code

```
codes/conics/example/conics.py
```

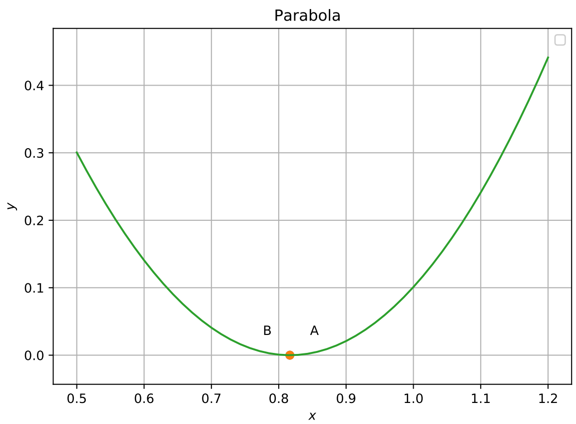

Fig. 1.19: Parabola

1.20. Verify whether the following are zeroes of the polynomial, indicated against them.

a) $p(x) = x^2 - 1, x = 1, -1$
b) $p(x) = (x+1)(x-2), x = -1, 2$
c) $p(x) = x^2, x = 0.$
d) $p(x) = 3x^2 - 1, x = -\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}.$

Solution: For a general polynomial equation of degree 2,
$p(x,y) \implies Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$
The vector form is

$$\mathbf{x}^T \begin{pmatrix} A & B/2 \\ B/2 & C \end{pmatrix} \mathbf{x} + \begin{pmatrix} D & E \end{pmatrix} \mathbf{x} + F = 0 \quad (1.20.1)$$

a)

$$y = x^2 - 1 \quad (1.20.2)$$

$$\implies \mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 & -1 \end{pmatrix} \mathbf{x} - 1 = 0 \quad (1.20.3)$$

For $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$,

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} - 1 = 0 \quad (1.20.4)$$

For $\mathbf{x} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$,

$$\begin{pmatrix} -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} - 1 = 0 \quad (1.20.5)$$

Hence +1,-1 are zeros, which can be verified from Fig. 1.20 The python code for Fig. 1.20 is

```
solutions/1/codes/conics/parab1.py
```

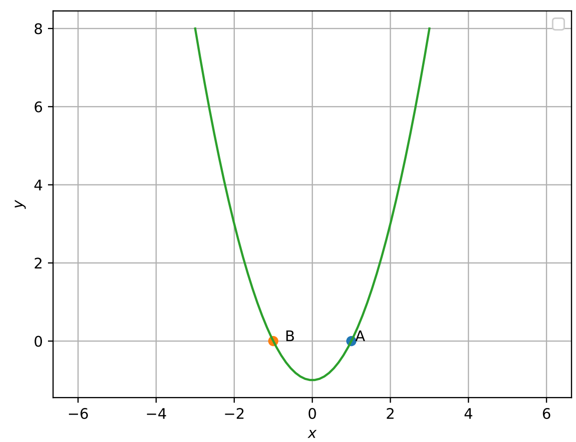

Fig. 1.20

b)

$$y = (x+1)(x-2) \quad (1.20.6)$$

$$\implies \mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -1 & -2 \end{pmatrix} \mathbf{x} - 2 = 0 \quad (1.20.7)$$

For $\mathbf{x} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$,

$$\begin{pmatrix} -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 & -2 \end{pmatrix} \mathbf{x} - 2 = 0 \quad (1.20.8)$$

Similarly, for For $\mathbf{x} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$,

$$\begin{pmatrix} 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 & -2 \end{pmatrix} \mathbf{x} - 2 = 0 \quad (1.20.9)$$

Hence -1,+2 are zeros, which can be verified from Fig. 1.20 The python code is

```
solutions/1/codes/conics/parab2.py
```

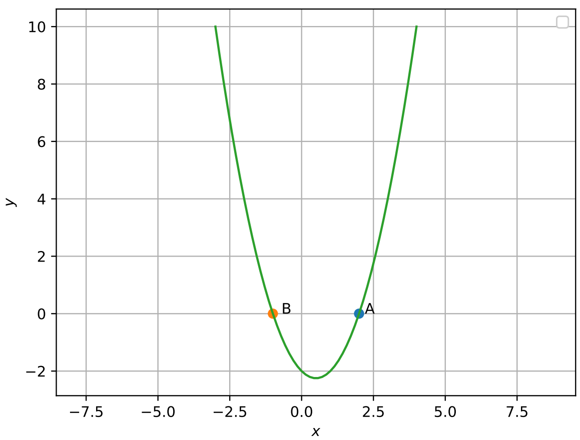

Fig. 1.20

c)

$$y = x^2 \tag{1.20.10}$$

$$\implies \mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} = 0 \tag{1.20.11}$$

For $\mathbf{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$,

$$\begin{pmatrix} 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0 \tag{1.20.12}$$

Hence 0 is the zero, which can be verified from the Fig. 1.20. The python code is

```
codes/conics/parab3.py
```

d)

$$y = 3x^2 - 1 \tag{1.20.13}$$

$$\implies \mathbf{x}^T \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 & -1 \end{pmatrix} \mathbf{x} - 1 = 0 \tag{1.20.14}$$

For $\mathbf{x} = \begin{pmatrix} -\frac{1}{\sqrt{3}} \\ 0 \end{pmatrix}$,

$$\begin{pmatrix} -\frac{1}{\sqrt{3}} & 0 \end{pmatrix}^T \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{3}} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{3}} \\ 0 \end{pmatrix} - 1 = 0 \tag{1.20.15}$$

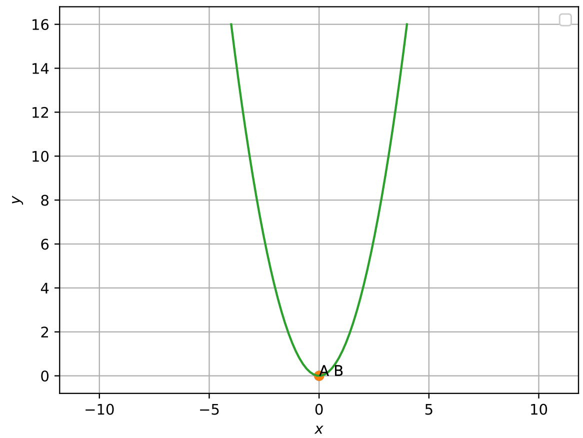

Fig. 1.20

For $\mathbf{x} = \begin{pmatrix} \frac{2}{\sqrt{3}} \\ 0 \end{pmatrix}$,

$$\begin{pmatrix} \frac{2}{\sqrt{3}} & 0 \end{pmatrix}^T \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{3}} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{3}} \\ 0 \end{pmatrix} - 1 \neq 0 \tag{1.20.16}$$

Hence $\frac{1}{\sqrt{3}}$ is a zero, but not $-\frac{2}{\sqrt{3}}$, which can be verified from Fig. 1.20 generated through the python code

```
solutions/1/codes/conics/parab4.py
```

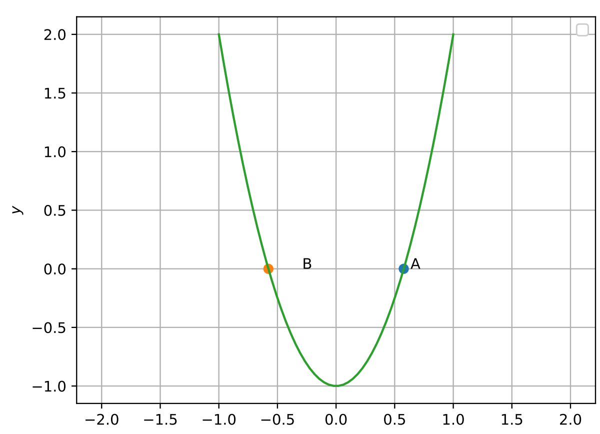

Fig. 1.20

1.21. Solve each of the following equations

a) $3x^2 - 4x + \frac{20}{3} = 0$
b) $x^2 - 2x + \frac{3}{2} = 0$
c) $27x^2 - 10x + 1 = 0$

d) $21x^2 - 28x + 10 = 0$

Solution:

a) To solve the equation - $3x^2 - 4x + \frac{20}{3} = 0$
The given equation can be represented as follows in the vector form

$$\mathbf{x}^T \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -4 & 0 \end{pmatrix} \mathbf{x} + \frac{20}{3} = 0 \quad (1.21.1)$$

$$\mathbf{x} = \begin{pmatrix} x \\ 0 \end{pmatrix} \quad (1.21.2)$$

$$3x^2 - 4x + \frac{20}{3} = 0 \quad (1.21.3)$$

$$\left(x - \begin{pmatrix} \frac{2}{3} \\ \frac{2\sqrt{14}}{3} \end{pmatrix}\right)\left(x - \begin{pmatrix} \frac{2}{3} \\ \frac{-2\sqrt{14}}{3} \end{pmatrix}\right) = 0 \quad (1.21.4)$$

$$x = \begin{pmatrix} \frac{2}{3} \\ \frac{2\sqrt{14}}{3} \end{pmatrix}, \begin{pmatrix} \frac{2}{3} \\ \frac{-2\sqrt{14}}{3} \end{pmatrix} \quad (1.21.5)$$

Figure 1.21 show that the equation does not intersect the x-axis hence there are no real roots.

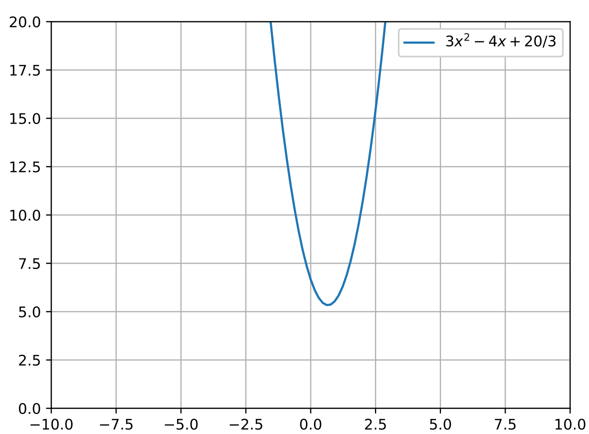

Fig. 1.21: $3x^2 - 4x + \frac{20}{3}$ generated using python

b) To solve the equation - $x^2 - 2x + \frac{3}{2} = 0$
The given equation can be represented as follows in the vector form

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -2 & 0 \end{pmatrix} \mathbf{x} + \frac{3}{2} = 0 \quad (1.21.6)$$

$$\mathbf{x} = \begin{pmatrix} x \\ 0 \end{pmatrix} \quad (1.21.7)$$

$$x^2 - 2x + \frac{3}{2} = 0 \quad (1.21.8)$$

$$\left(x - \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}\right)\left(x - \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}\right) = 0 \quad (1.21.9)$$

$$x = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}, \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} \quad (1.21.10)$$

Figure 1.21 show that the equation does not intersect the x-axis hence there are no real roots.

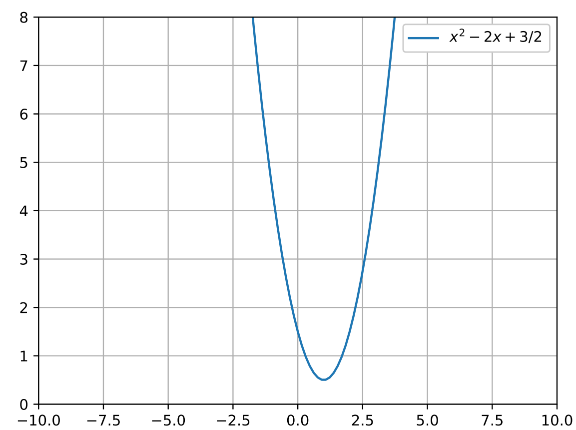

Fig. 1.21: $x^2 - 2x + \frac{3}{2}$ generated using python

c) To solve the equation - $27x^2 - 10x + 1 = 0$
The given equation can be represented as follows in the vector form

$$\mathbf{x}^T \begin{pmatrix} 27 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -10 & 0 \end{pmatrix} \mathbf{x} + 1 = 0 \quad (1.21.11)$$

$$\mathbf{x} = \begin{pmatrix} x \\ 0 \end{pmatrix} \quad (1.21.12)$$

$$27x^2 - 10x + 1 = 0 \quad (1.21.13)$$

$$\left(x - \begin{pmatrix} \frac{5}{27} \\ \frac{\sqrt{2}}{27} \end{pmatrix}\right)\left(x - \begin{pmatrix} \frac{5}{27} \\ \frac{-\sqrt{2}}{27} \end{pmatrix}\right) = 0 \quad (1.21.14)$$

$$x = \begin{pmatrix} \frac{5}{27} \\ \frac{\sqrt{2}}{27} \end{pmatrix}, \begin{pmatrix} \frac{5}{27} \\ \frac{-\sqrt{2}}{27} \end{pmatrix} \quad (1.21.15)$$

Figure 1.21 show that the equation does not intersect the x-axis hence there are no real roots.

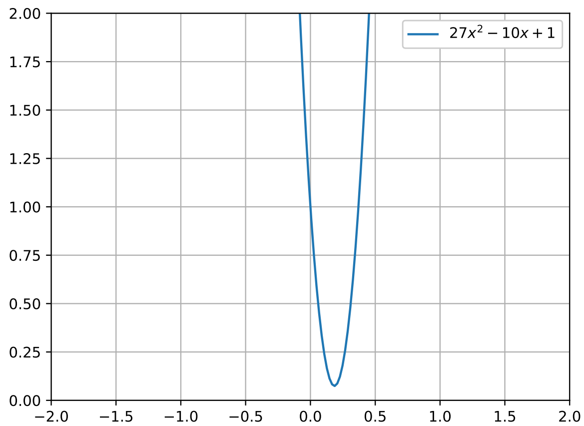

Fig. 1.21: $27x^2 - 10x + 1$ generated using python

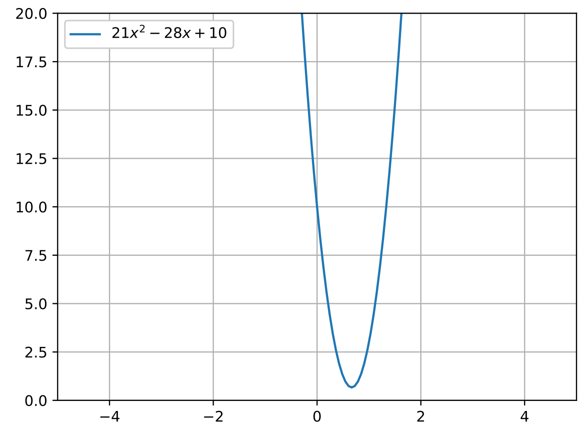

Fig. 1.21: $21x^2 - 28x + 10$ generated using python

d) To solve the equation - $21x^2 - 28x + 10 = 0$
The given equation can be represented as follows in the vector form

$$\mathbf{x}^T \begin{pmatrix} 21 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -28 & 0 \end{pmatrix} \mathbf{x} + 10 = 0 \tag{1.21.16}$$

$$\mathbf{x} = \begin{pmatrix} x \\ 0 \end{pmatrix} \tag{1.21.17}$$

$$21x^2 - 28x + 10 = 0 \tag{1.21.18}$$

$$\left(x - \left(\frac{\frac{2}{3}}{\frac{\sqrt{14}}{21}}\right)\right)\left(x - \left(\frac{\frac{2}{3}}{\frac{-\sqrt{14}}{21}}\right)\right) = 0 \tag{1.21.19}$$

$$x = \left(\frac{\frac{2}{3}}{\frac{\sqrt{14}}{21}}\right), \left(\frac{\frac{2}{3}}{\frac{-\sqrt{14}}{21}}\right) \tag{1.21.20}$$

Figure 1.21 show that the equation does not intersect the x-axis hence there are no real roots.
The following Python code generates Fig.1.21, 1.21, 1.21 and 1.21

```
solutions/2/codes/conics_ex/conics_ex.py
```

1.22. Factorise
a) $12x^2–7x + 1$
b) $6x^2 + 5x–6$
c) $2x^2 + 7x + 3$
d) $3x^2–x–4$

Solution:

a)

$$(a)\, 12x^2 - 7x + 1 \tag{1.22.1}$$

can be expressed as

$$\mathbf{x}^T \begin{pmatrix} 12 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -7 & 0 \end{pmatrix} \mathbf{x} + 1 = 0 \tag{1.22.2}$$

To find roots using 1.22.2, substitute

$$y = 0 \tag{1.22.3}$$

$$\implies 12x^2 - 7x + 1 = 0 \tag{1.22.4}$$

$$x = \frac{1}{3}, \frac{1}{4} \tag{1.22.5}$$

Hence $\left(x - \frac{1}{3}\right)$ and $\left(x - \frac{1}{4}\right)$ are the factors

$$\implies (3x - 1)(4x - 1) = 12x^2 - 7x + 1 \tag{1.22.6}$$

The following code sketches the graph of 1.22.1 in figure 1.22

```
solutions/3/codes/conic2/conic2a.py
```

b)

$$(b)\, 6x^2 + 5x - 6 \tag{1.22.7}$$

can be expressed as

$$\mathbf{x}^T \begin{pmatrix} 6 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 5 & 0 \end{pmatrix} \mathbf{x} - 6 = 0 \tag{1.22.8}$$

Substituting $y = 0$ in equation 1.22.8 to find roots,

$$\implies 6x^2 + 5x - 6 = 0 \tag{1.22.9}$$

$$x = \frac{-3}{2}, \frac{2}{3} \tag{1.22.10}$$

$$(2x + 3)(3x - 2) = 6x^2 + 5x - 6 \tag{1.22.11}$$

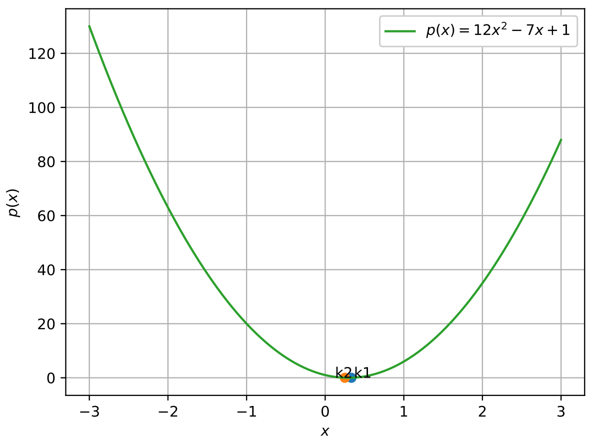

Fig. 1.22: Graph of $12x^2 - 7x + 1$

The following code sketches the graph of 1.22.7 in figure 1.22

```
solutions/3/codes/conic2/conic2b.py
```

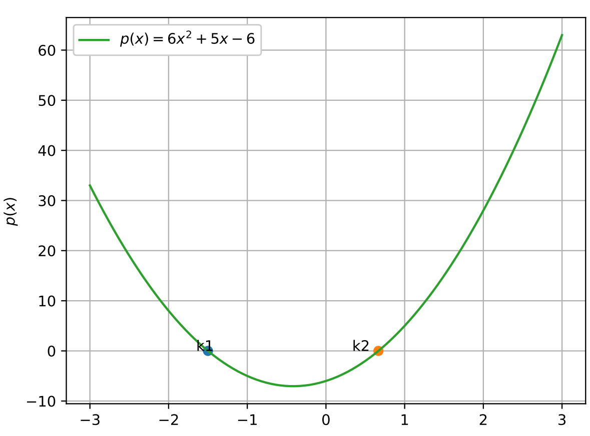

Fig. 1.22: Graph of $6x^2 + 5x - 6$

c)

$$(c)\, 2x^2 + 7x + 3 \tag{1.22.12}$$

can be expressed as

$$\mathbf{x}^T \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 7 & 0 \end{pmatrix} \mathbf{x} + 3 = 0 \tag{1.22.13}$$

Substituting $y = 0$ in equation 1.22.13,

$$\implies 2x^2 + 7x + 3 = 0 \tag{1.22.14}$$

$$x = \frac{-1}{2}, -3 \tag{1.22.15}$$

$$(2x + 1)(x + 3) = 2x^2 + 7x + 3 \tag{1.22.16}$$

The following code sketches the graph of 1.22.12 in figure 1.22

```
solutions/3/codes/conic2/conic2c.py
```

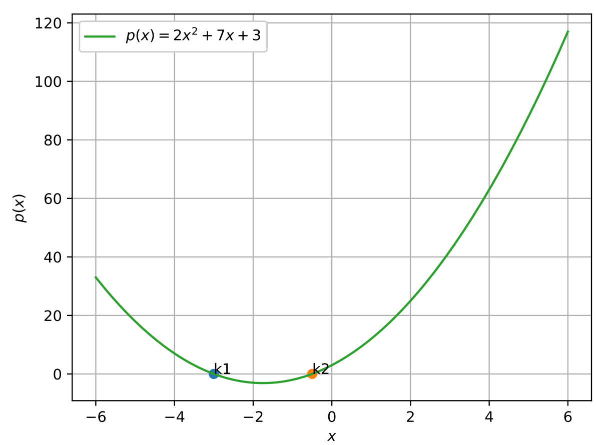

Fig. 1.22: Graph of $2x^2 + 7x + 3$

d)

$$(d)\, 3x^2 - x - 4 \tag{1.22.17}$$

can be expressed as

$$\mathbf{x}^T \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -1 & 0 \end{pmatrix} \mathbf{x} - 4 = 0 \tag{1.22.18}$$

Substituting $y = 0$ in equation 1.22.13,

$$\implies 3x^2 - x - 4 = 0 \tag{1.22.19}$$

$$x = \frac{4}{3}, -1 \tag{1.22.20}$$

$$(3x - 4)(x + 1) = 3x^2 - x - 4 \tag{1.22.21}$$

The following code sketches the graph of 1.22.17 in figure 1.22

```
solutions/3/codes/conic2/conic2d.py
```

1.23. Find the zeroes of the following quadratic polynomials and verify the relationship between the zeroes and the coefficients.

a) $x^2 – 2x – 8$

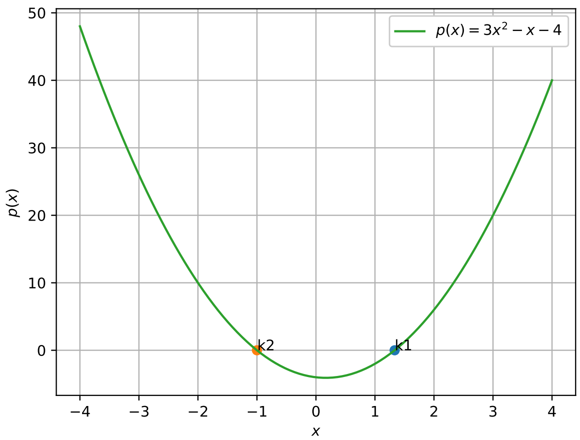

Fig. 1.22: Graph of $3x^2 - x - 4$

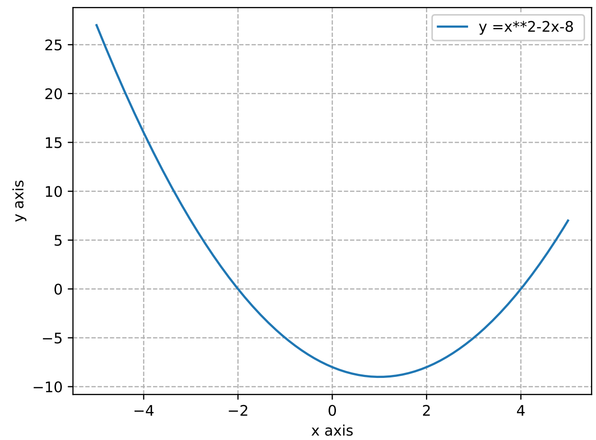

Fig. 1.23

b) $4u^2 + 8u$
c) $4s^2 – 4s + 1$
d) $t^2 – 15$
e) $6x^2 – 3 – 7x$
f) $3x^2 – x – 4$

Solution:

1. The vector equation for the conic is

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -2 & 0 \end{pmatrix} \mathbf{x} - 8 = 0 \tag{1.23.1}$$

$$x^2 - 2x - 8 = 0 \tag{1.23.2}$$

$$(x - 4)(x + 2) = 0 \tag{1.23.3}$$

$$\alpha = 4, \beta = -2 \tag{1.23.4}$$

quadratic equation can be represented as

$$ax^2 + bx + c = 0 \tag{1.23.5}$$

$$\alpha + \beta = -\frac{b}{a} = 2 \tag{1.23.6}$$

$$\alpha \times \beta = \frac{c}{a} = -8 \tag{1.23.7}$$

```
solutions/4/codes/conics/perabola2.py
```

2. The vector equation for the conic is

$$\mathbf{x}^T \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 8 & 0 \end{pmatrix} \mathbf{x} = 0 \tag{1.23.8}$$

$$4u^2 + 8u = 0 \tag{1.23.9}$$

$$(4u)(u + 2) = 0 \tag{1.23.10}$$

$$\alpha = 0, \beta = -2 \tag{1.23.11}$$

quadratic equation can be represented as

$$ax^2 + bx + c = 0 \tag{1.23.12}$$

$$\alpha + \beta = -\frac{b}{a} = -2 \tag{1.23.13}$$

$$\alpha \times \beta = \frac{c}{a} = 0 \tag{1.23.14}$$

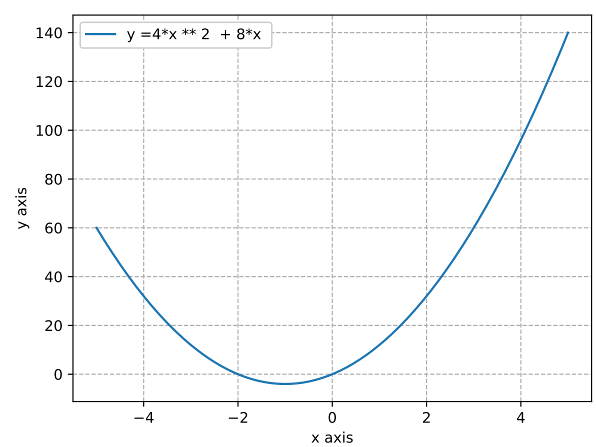

Fig. 1.23: equation 2

```
solutions/4/codes/conics/perabola2.py
```

3. The vector equation for the conic is

$$\mathbf{x}^T \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -4 & 0 \end{pmatrix} \mathbf{x} + 1 = 0 \quad (1.23.15)$$

$$4s^2 - 4s + 1 = 0 \quad (1.23.16)$$

$$(2s-1)(2s-1) = 0 \quad (1.23.17)$$

$$\alpha = \frac{1}{2}, \beta = -\frac{1}{2} \quad (1.23.18)$$

quadratic equation can be represented as

$$ax^2 + bx + c = 0 \quad (1.23.19)$$

$$\alpha + \beta = -\frac{b}{a} = 1 \quad (1.23.20)$$

$$\alpha \times \beta = \frac{c}{a} = \frac{1}{4} \quad (1.23.21)$$

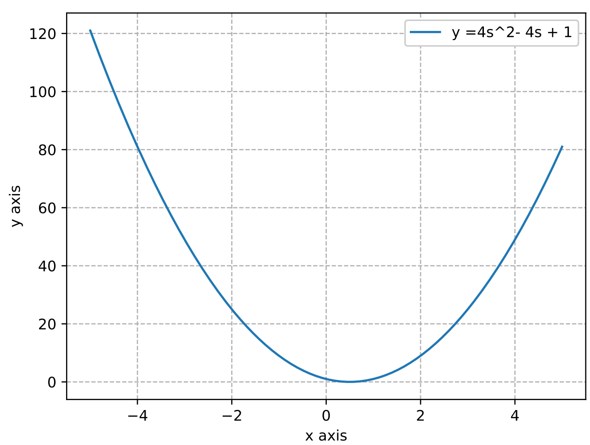

Fig. 1.23: equation 3

solutions/4/codes/conics/perabola3.py

4. The vector equation for the conic is

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 & 0 \end{pmatrix} \mathbf{x} - 15 = 0 \quad (1.23.22)$$

$$t^2 - 15 = 0 \quad (1.23.23)$$

$$\alpha = \sqrt{15}, \beta = -\sqrt{15} \quad (1.23.24)$$

quadratic equation can be represented as

$$ax^2 + bx + c = 0 \quad (1.23.25)$$

$$\alpha + \beta = -\frac{b}{a} = 0 \quad (1.23.26)$$

$$\alpha \times \beta = \frac{c}{a} = -15 \quad (1.23.27)$$

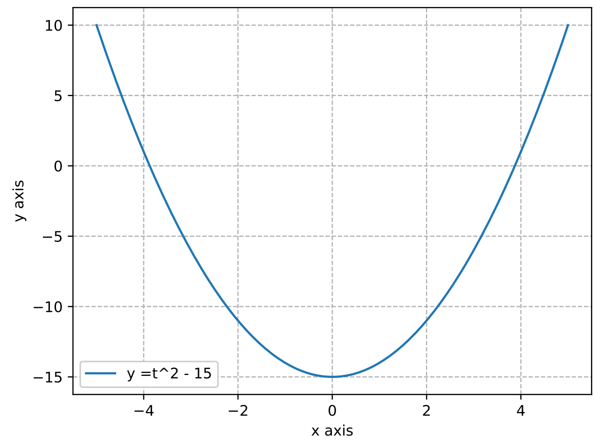

Fig. 1.23: equation 4

solutions/4/codes/conics/perabola4.py

5. The vector equation for the conic is

$$\mathbf{x}^T \begin{pmatrix} 6 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -7 & 0 \end{pmatrix} \mathbf{x} - 3 = 0 \quad (1.23.28)$$

$$6x^2 - 3 - 7x = 0 \quad (1.23.29)$$

$$(2x-3)(3x+1) = 0 \quad (1.23.30)$$

$$\alpha = \frac{3}{2}, \beta = -\frac{1}{3} \quad (1.23.31)$$

$$\alpha + \beta = -\frac{b}{a} = \frac{7}{6} \quad (1.23.32)$$

$$\alpha \times \beta = \frac{c}{a} = -\frac{1}{2} \quad (1.23.33)$$

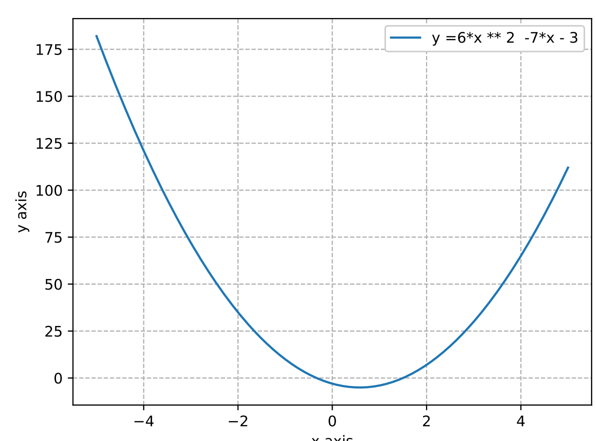

Fig. 1.23: equation 5

solutions/4/codes/conics/perabola5.py

6. The vector equation for the conic is

$$\mathbf{x}^T \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -1 & 0 \end{pmatrix} \mathbf{x} - 4 = 0 \quad (1.23.34)$$

$$3x^2 - 2x - 8 = 0 \quad (1.23.35)$$

$$(3x + 4)(x + 1) = 0 \quad (1.23.36)$$

$$\alpha = -1, \beta = -\frac{4}{3} \quad (1.23.37)$$

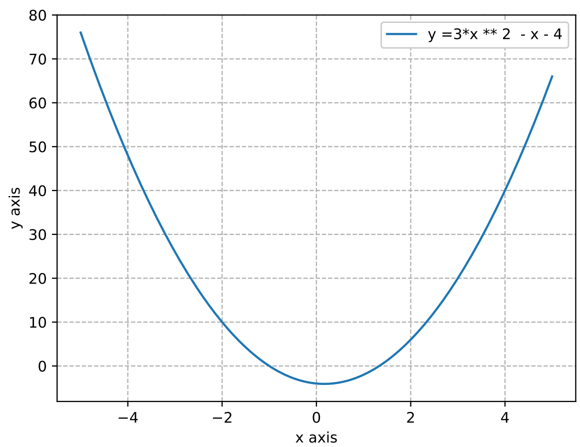

Fig. 1.23: equation 6

solutions/4/codes/conis/perabola6.py

quadratic equation can be represented as

$$ax^2 + bx + c = 0 \quad (1.23.38)$$

$$\alpha + \beta = -\frac{b}{a} = \frac{2}{3} \quad (1.23.39)$$

$$\alpha \times \beta = \frac{c}{a} = -\frac{8}{3} \quad (1.23.40)$$

1.24. Find a quadratic polynomial each with the given numbers as the sum and product of its zeroes respectively.

a) -1 , $\frac{1}{4}$
b) 1, 1
c) 0, $\sqrt{5}$
d) 4, 1
e) $\frac{1}{4}, \frac{1}{4}$
f) $\sqrt{2}, \frac{1}{3}$

Solution: The following python code computes roots of the quadratic equation obtained:

./solutions/5/codes/conics/q20a.py
./solutions/5/codes/conics/q20b.py
./solutions/5/codes/conics/q20c.py
./solutions/5/codes/conics/q20d.py
./solutions/5/codes/conics/q20e.py
./solutions/5/codes/conics/q20f.py

a) -1,$\frac{1}{4}$

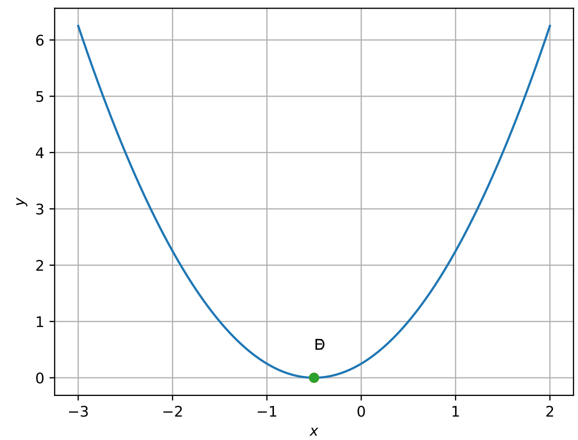

Fig. 1.24

For a general polynomial equation of degree 2,

$$p(x, y) = Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

The vector form is

$$\mathbf{x}^T \begin{pmatrix} A & \frac{B}{2} \\ \frac{B}{2} & C \end{pmatrix} \mathbf{x} + \begin{pmatrix} D & E \end{pmatrix} \mathbf{x} + F = 0 \quad (1.24.1)$$

Here, sum of zeroes = D = -1
Product of zeroes = F =$\frac{1}{4}$
Substituting the values in 1.24.1,

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} + \frac{1}{4} = 0 \quad (1.24.2)$$

$$\implies y = x^2 + x + \frac{1}{4} \quad (1.24.3)$$

The roots are -0.5 and -0.5 as represented in Fig. 1.24

b) 1,1
Here, sum of zeroes = D = 1
Product of zeroes = F =1

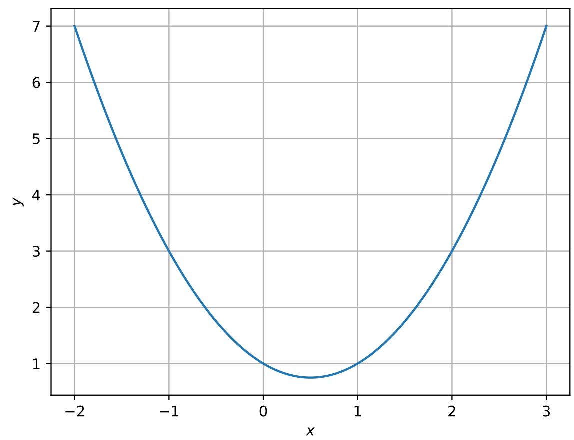

Fig. 1.24

Substituing the values in 1.24.1,

$$\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\mathbf{x}+\begin{pmatrix}-1 & -1\end{pmatrix}\mathbf{x}+1=0 \quad (1.24.4)$$

$$\implies y = x^2 - x + 1 \quad (1.24.5)$$

Since the curve doesn't meet the x-axis, real roots don't exist for this parabola as represented in Fig. 1.24

c) $0, \sqrt{5}$

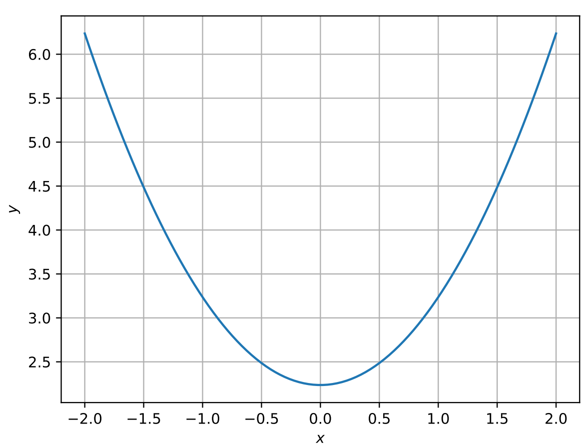

Fig. 1.24

Here, sum of zeroes = D = 0
Product of zeroes = F = $\sqrt{5}$
Substituing the values in 1.24.1,

$$\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\mathbf{x}+\begin{pmatrix}0 & -1\end{pmatrix}\mathbf{x}+\sqrt{5}=0 \quad (1.24.6)$$

$$\implies y = x^2 + \sqrt{5} \quad (1.24.7)$$

Since the curve doesn't meet the x-axis, real roots don't exist for this parabola as represented in Fig. 1.24

d) 4,1

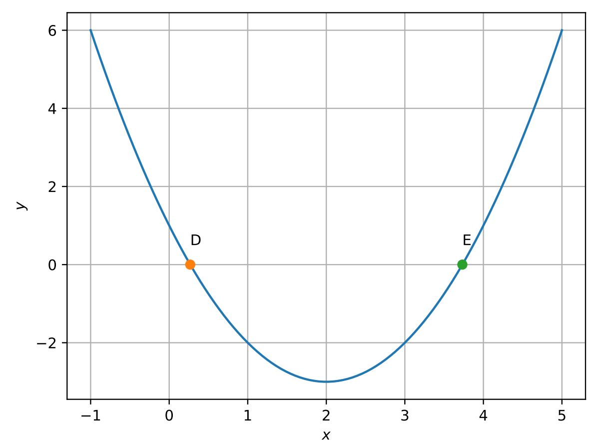

Fig. 1.24

Here, sum of zeroes = D = 4
Product of zeroes = F = 1
Substituing the values in 1.24.1,

$$\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\mathbf{x}+\begin{pmatrix}-4 & -1\end{pmatrix}\mathbf{x}+1=0 \quad (1.24.8)$$

$$\implies y = x^2 - 4x + 1 \quad (1.24.9)$$

The roots are 3.73 and 0.26 as represented in Fig. 1.24

e) $\frac{1}{4}, \frac{1}{4}$
Here, sum of zeroes = D = $\frac{1}{4}$
Product of zeroes = F = $\frac{1}{4}$
Substituing the values in 1.24.1,

$$\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\mathbf{x}+\begin{pmatrix}-\frac{1}{4} & -1\end{pmatrix}\mathbf{x}+\frac{1}{4}=0 \quad (1.24.10)$$

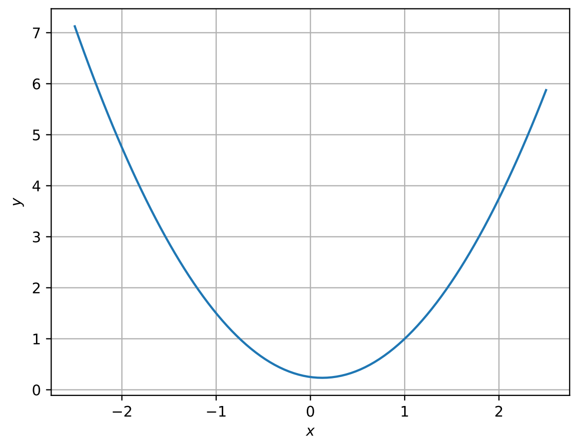

Fig. 1.24

$$\implies y = x^2 - \frac{1}{4}x + \frac{1}{4} \tag{1.24.11}$$

Since the curve doesn't meet the x-axis, real roots don't exist for this parabola as represented in Fig. 1.24

f) $\sqrt{2}, \frac{1}{3}$

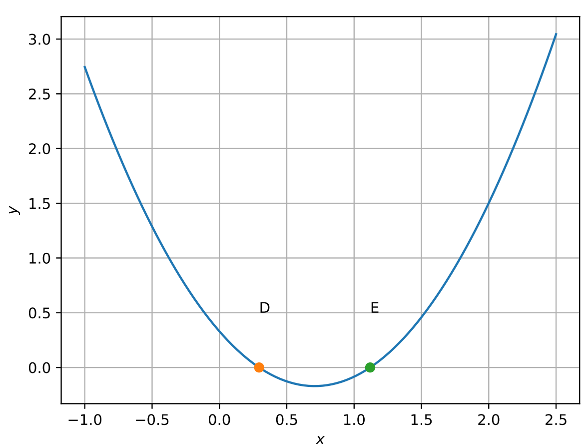

Fig. 1.24

Here, sum of zeroes = D = $\sqrt{2}$
Product of zeroes = F = $\frac{1}{3}$
Substituting the values in 1.24.1,

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -\sqrt{2} & -1 \end{pmatrix} \mathbf{x} + \frac{1}{3} = 0 \tag{1.24.12}$$

$$\implies y = x^2 - \sqrt{2}x + \frac{1}{3} \tag{1.24.13}$$

The roots are 1.11 and 0.29 as represented in Fig. 1.24

1.25. Find the roots of the following quadratic equations:

a) $x^2 – 3x – 10 = 0$
b) $2x^2 + x - 6 = 0$
c) $\sqrt{2}x^2 + 7x + 5\sqrt{2} = 0$
d) $2x^2 – x + \frac{1}{8} = 0$
e) $100x^2 – 20x + 1 = 0$

Solution:

a) $x^2 - 3x - 10 = 0$

The vector form from the equation is

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -3 & 0 \end{pmatrix} \mathbf{x} - 10 = 0 \tag{1.25.1}$$

The values of $\mathbf{x}$ are found in the following python code

```
solutions/6/codes/conics/exercise/conics_1.
    py
```

$$\mathbf{x} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \end{pmatrix}$$

which can be verified from the Fig.1.25. The following python code generates the fig.1.25

```
solutions/6/codes/conics/exercise/conics_1.
    py
```

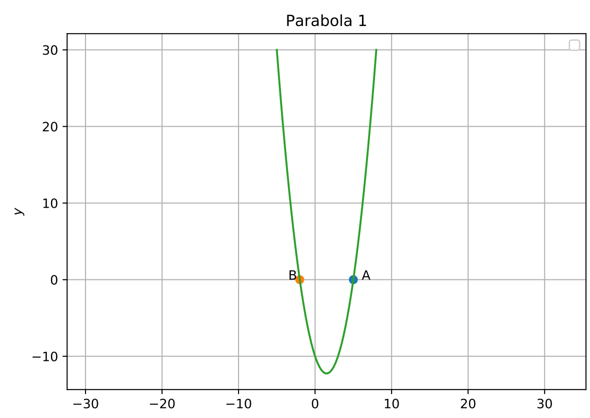

Fig. 1.25: Parabola 1

b) $2x^2 + x - 6 = 0$

The vector form from the equation is is

$$\mathbf{x}^T \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} - 6 = 0 \tag{1.25.2}$$

The values of $\mathbf{x}$ are found in the following python code

```
solutions/6/codes/conics/exercise/conics_2.
    py
```

$\mathbf{x} = \begin{pmatrix} 1.5 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \end{pmatrix}$
which can be verified from the Fig.1.25. The following python code generates the fig.1.25

```
solutions/6/codes/conics/exercise/conics_2.
    py
```

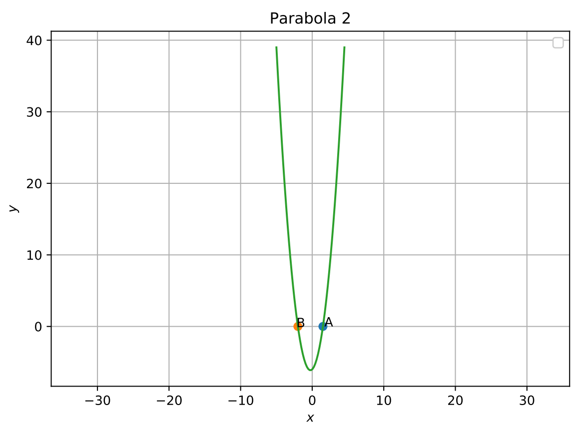

Fig. 1.25: Parabola 2

c) $\sqrt{2}x^2 + 7x + 5\sqrt{2} = 0$
The vector form from the equation is is

$$\mathbf{x}^T \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 7 & 0 \end{pmatrix} \mathbf{x} + 5\sqrt{2} = 0 \tag{1.25.3}$$

The values of $\mathbf{x}$ are found in the following python code

```
solutions/6/codes/conics/exercise/conics_3.
    py
```

$\mathbf{x} = \begin{pmatrix} -1.414 \\ 0 \end{pmatrix}, \begin{pmatrix} -3.535 \\ 0 \end{pmatrix}$ which can be verified from the Fig.1.25. The following python code generates the fig.1.25

```
solutions/6/codes/conics/exercise/conics_3.
    py
```

d) $2x^2 - x + \frac{1}{8} = 0$
The vector form from the equation is is

$$\mathbf{x}^T \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -1 & 0 \end{pmatrix} \mathbf{x} + \frac{1}{8} = 0 \tag{1.25.4}$$

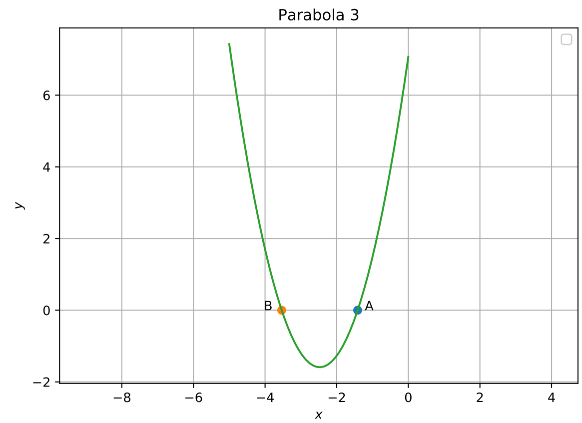

Fig. 1.25: Parabola 3

The values of $\mathbf{x}$ are found in the following python code

```
solutions/6/codes/conics/exercise/conics_4.
    py
```

$\mathbf{x} = \begin{pmatrix} 0.25 \\ 0 \end{pmatrix}, \begin{pmatrix} 0.25 \\ 0 \end{pmatrix}$
which can be verified from the Fig.1.25. The following python code generates the fig.1.25

```
solutions/6/codes/conics/exercise/conics_4.
    py
```

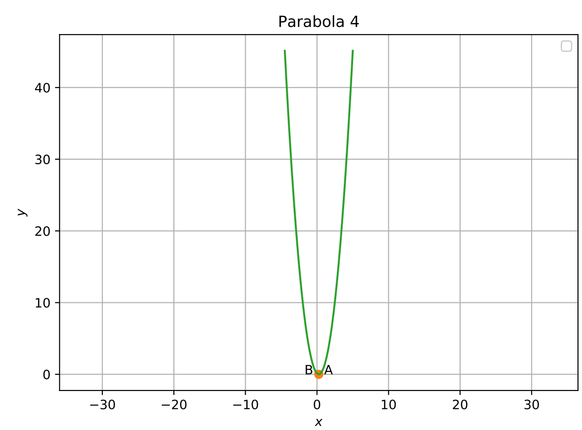

Fig. 1.25: Parabola 4

e) $100x^2 - 20x + 1 = 0$
The vector form from the equation is is

$$\mathbf{x}^T \begin{pmatrix} 100 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -20 & 0 \end{pmatrix} \mathbf{x} + 1 = 0 \tag{1.25.5}$$

The values of $\mathbf{x}$ are found in the following python code

```
solutions/6/codes/conics/exercise/conics_5.
    py
```

$\mathbf{x} = \begin{pmatrix} 0.1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0.1 \\ 0 \end{pmatrix}$ which can be verified from the Fig.1.25. The following python code generates the fig.1.25

```
solutions/6/codes/conics/exercise/conics_5.
    py
```

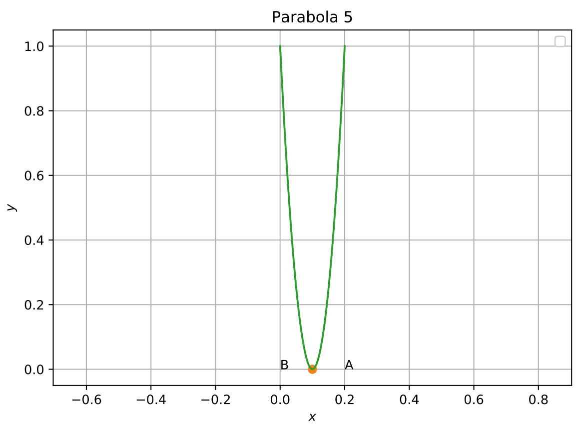

Fig. 1.25: Parabola 5

1.26. Find the roots of the following quadratic equations
a) $2x^2 - 7x + 3 = 0$
b) $2x^2 + x - 4 = 0$
c) $4x^2 + 4\sqrt{3}x + 3 = 0$
d) $2x^2 + x + 4 = 0$

Solution:

a) $2x^2 - 7x + 3 = 0$ can be expressed as

$$\mathbf{x}^T \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -7 & 0 \end{pmatrix} \mathbf{x} + 3 = 0 \quad (1.26.1)$$

If $\begin{pmatrix} k \\ 0 \end{pmatrix}$ satisfies (1.26.1) then k is the root of the equation (1.26.1).
From graph, the roots are the points where the quadratic equation cuts the x-axis. A quadratic equation can have a maximum of two distinct roots.

$$2k^2 - 7k + 3 = 0 \quad (1.26.2)$$
$$(k-3)(2k-1) = 0 \quad (1.26.3)$$

From the graph in 1.26, the roots are 3 and $\frac{1}{2}$. The python code can be downloaded from

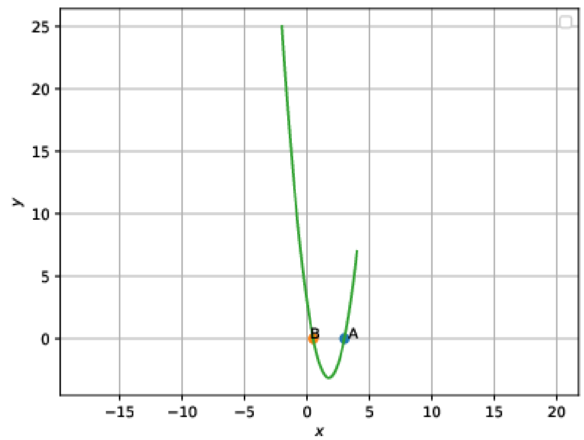

Fig. 1.26: Roots of $2x^2 - 7x + 3 = 0$

```
solutions/7/codes/conics/parabola1.py
```

b) $2x^2 + x - 4 = 0$ can be expressed as

$$\mathbf{x}^T \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} - 4 = 0 \quad (1.26.4)$$

From the 1.26, the roots are 1.186 and 1.686. The python code can be downloaded from

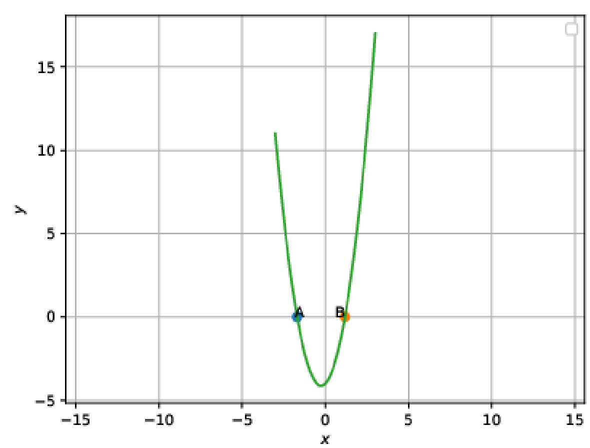

Fig. 1.26: Roots of $2x^2 + x - 4 = 0$

```
solutions/7/codes/conics/parabola2.py
```

c) $4x^2 + 4\sqrt{3}x + 3 = 0$ can be expressed as

$$\mathbf{x}^T \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 4\sqrt{3} & 0 \end{pmatrix} \mathbf{x} + 3 = 0 \quad (1.26.5)$$

From the graph in 1.26, the roots are real and equal. The root is $\frac{-\sqrt{3}}{2}$. The python code can

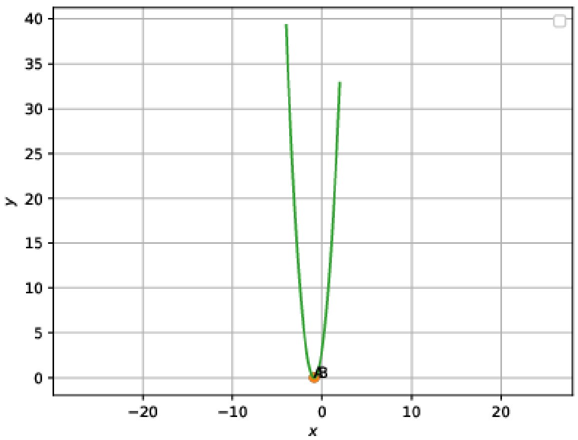

Fig. 1.26: Roots of $4x^2 + 4\sqrt{3}x + 3 = 0$

be downloaded from

```
solutions/7/codes/conics/parabola3.py
```

d) $2x^2 + x + 4 = 0$ can be expressed as

$$\mathbf{x}^T \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} + 4 = 0 \tag{1.26.6}$$

From the graph 1.26, the quadratic equation doesn't intersect x-axis. Thus it doesn't have real roots. It has ccomplex and conjugate roots. The python code can be downloaded

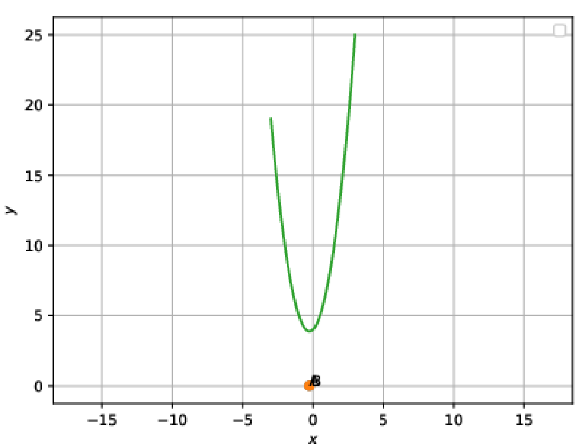

Fig. 1.26: Roots of $2x^2 + x + 4 = 0$

from

```
solutions/7/codes/conics/parabola4.py
```

1.27. Factorise $6x^2 + 17x + 5$.

Solution:

The given polynomial can be expressed as:

$$\mathbf{x}^T \begin{pmatrix} 6 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 17 & 0 \end{pmatrix} \mathbf{x} + 5 = 0 \tag{1.27.1}$$

Substituting y=0 in the above equation,

$$6x^2 + 17x + 5 = 0 \tag{1.27.2}$$

$$\implies x = \frac{-1}{3}, \frac{-5}{2} \tag{1.27.3}$$

$$\therefore (3x+1)(2x+5) = 6x^2 + 17x + 5 \tag{1.27.4}$$

which is verified in Fig. 1.27

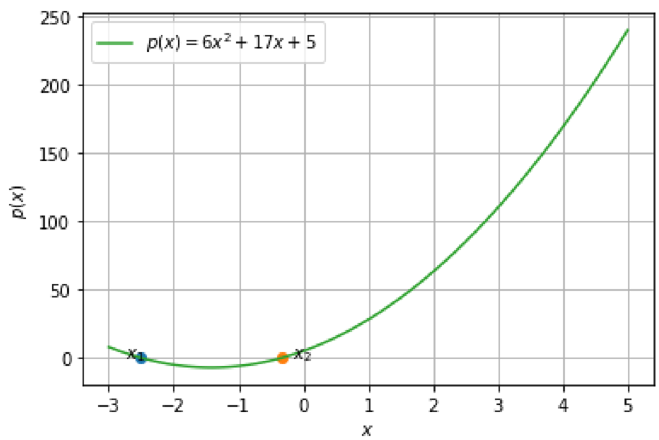

Fig. 1.27: Graph of $6x^2 + 17x + 5$.

1.28. Find the zeroes of the polynomial $x^2 - 3$ and verify the relationship between the zeroes and the coefficients.

Solution: The given equation can be expressed as a parabola with parameters

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 0 \\ \frac{-1}{2} \end{pmatrix}, f = -3 \tag{1.28.1}$$

Using eigenvalue decomposition,

$$\mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{1.28.2}$$

and

$$\begin{pmatrix} \mathbf{u}^T + \eta \mathbf{p_1}^T \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \eta \mathbf{p_1} - \mathbf{u} \end{pmatrix} \tag{1.28.3}$$

$$\implies \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{c} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} \tag{1.28.4}$$

$$\implies \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{c} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \tag{1.28.5}$$

$$\implies \mathbf{c} = \begin{pmatrix} 0 \\ -3 \end{pmatrix} \tag{1.28.6}$$

The zeros are at $\pm\sqrt{3}$, which can be verified in Fig. 1.28.

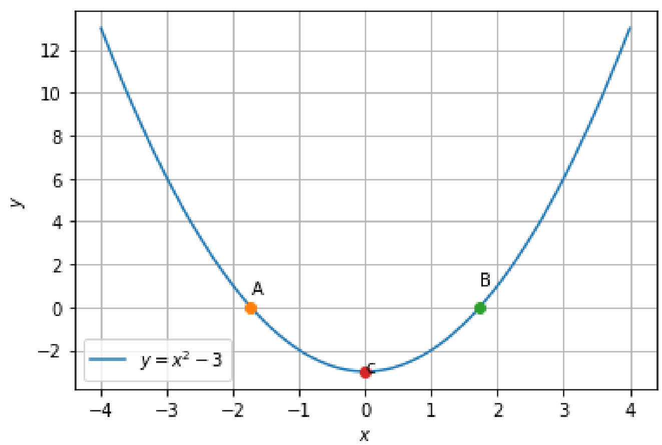

Fig. 1.28: $y = x^2 - 3$

1.29. Find the discriminant of the quadratic equation $2x^2 - 4x + 3 = 0$ hence find the nature of its roots.
Solution:
The given equation can be expressed as

$$\mathbf{x}^T \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -4 & 0 \end{pmatrix} \mathbf{x} + 3 = 0 \tag{1.29.1}$$

The discriminant is

$$D = b^2 - 4ac \tag{1.29.2}$$

$$= -8 \quad < 0 \tag{1.29.3}$$

Thus the equation has no real roots as can be seen from Fig. 1.29.

1.30. Find the discriminant of the quadratic equation $3x^2 - 2x + \frac{1}{3} = 0$ hence find the nature of its roots.
Solution:

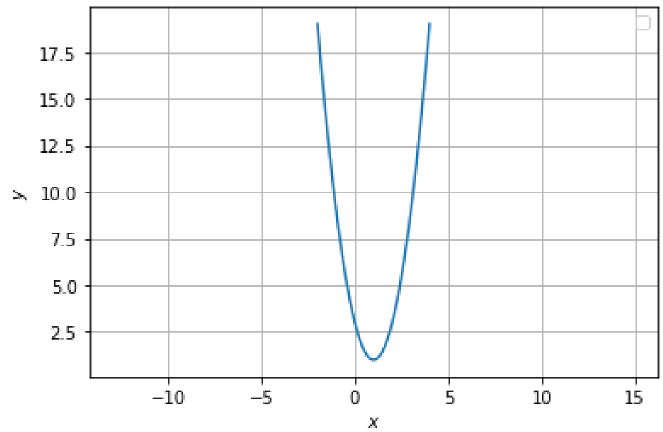

Fig. 1.29: Roots of $2x^2 - 4x + 3 = 0$

The given equation can be expressed as

$$\mathbf{x}^T \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} -1 \\ \frac{-1}{2} \end{pmatrix} \mathbf{x} + \frac{1}{3} = 0 \tag{1.30.1}$$

with

$$\mathbf{V} = \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} -1 \\ \frac{-1}{2} \end{pmatrix}, f = \frac{1}{3} \tag{1.30.2}$$

Using eigenvalue decomposition,

$$\mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}, \mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{1.30.3}$$

Thus,

$$\begin{pmatrix} \mathbf{u}^T + \eta \mathbf{p_1}^T \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \eta \mathbf{p_1} - \mathbf{u} \end{pmatrix} \tag{1.30.4}$$

$$\implies \begin{pmatrix} -1 & -1 \\ 3 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{c} = \begin{pmatrix} \frac{-1}{3} \\ 1 \\ 0 \end{pmatrix} \tag{1.30.5}$$

$$\implies \begin{pmatrix} -1 & -1 \\ 3 & 0 \end{pmatrix} \mathbf{c} = \begin{pmatrix} \frac{-1}{3} \\ 1 \end{pmatrix} \tag{1.30.6}$$

$$\implies \mathbf{c} = \begin{pmatrix} \frac{1}{3} \\ 0 \end{pmatrix} \tag{1.30.7}$$

$\because$

$$\mathbf{p_1}^T \mathbf{c} = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{3} \\ 0 \end{pmatrix} \tag{1.30.8}$$

$$= 0 \tag{1.30.9}$$

$$\mathbf{p_2}^T \mathbf{V} \mathbf{p_2} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.30.10}$$

$$= 3, \tag{1.30.11}$$

$\because$

$$(\mathbf{p_1}^T\mathbf{c})(\mathbf{p_2}^T\mathbf{V}\mathbf{p_2}) = 0 \tag{1.30.12}$$

Thus, the given equation has real and equal roots which can be verified from Fig. 1.30.

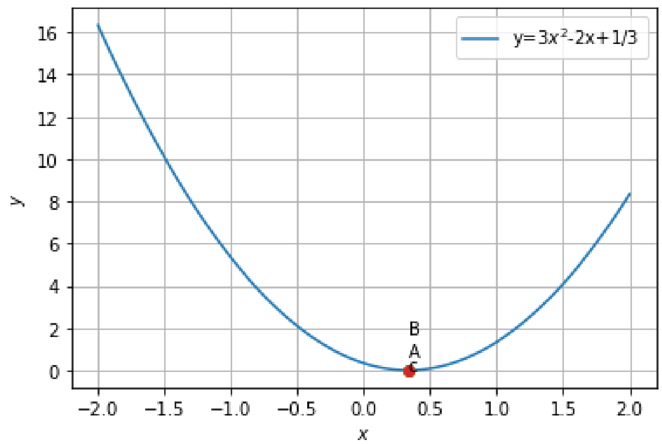

Fig. 1.30: Roots of $3x^2 - 2x + 1/3 = 0$

1.31. Find the equation of all lines having slope 2 and being tangent to the curve

$$y + \frac{2}{x-3} = 0 \tag{1.31.1}$$

Solution:
The equation of curve can be expressed as

$$y + \frac{2}{x-3} = 0 \tag{1.31.2}$$
$$xy - 3y + 2 = 0 \tag{1.31.3}$$

Comparing with the standard equation,

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{1.31.4}$$
$$\mathbf{V} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 0 \\ -\frac{3}{2} \end{pmatrix}, f = 2 \tag{1.31.5}$$
$$\mathbf{V}^{-1} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \tag{1.31.6}$$

$\because$

$$|\mathbf{V}| = \frac{-1}{4} \tag{1.31.7}$$
$$\implies |\mathbf{V}| < 0 \tag{1.31.8}$$

and (1.31.3) represents a hyperbola. The characteristic equation of $\mathbf{V}$ is

$$|\mathbf{V} - \lambda\mathbf{I}| = \begin{vmatrix} -\lambda & \frac{1}{2} \\ \frac{1}{2} & -\lambda \end{vmatrix} = 0 \tag{1.31.9}$$
$$\implies \lambda^2 - \frac{1}{4} = 0 \tag{1.31.10}$$

and the eigenvalues are

$$\lambda_1 = \frac{1}{2}, \lambda_2 = \frac{-1}{2} \tag{1.31.11}$$

Eigen vector $\mathbf{p}_1$ corressponding to λ_1 can be obtained as

$$(\mathbf{V} - \lambda_1\mathbf{I}) = \begin{pmatrix} \frac{-1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-1}{2} \end{pmatrix} \xleftrightarrow[R_1 \leftarrow -2R_1]{R_2 = R_1 + R_2} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \tag{1.31.12}$$
$$\implies \mathbf{p_1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.31.13}$$

Similarly,

$$\mathbf{p_2} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{1.31.14}$$

$\therefore$

$$\mathbf{P} = \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \tag{1.31.15}$$
$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{-1}{2} \end{pmatrix} \tag{1.31.16}$$

Now,

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \tag{1.31.17}$$
$$= -\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ \frac{-3}{2} \end{pmatrix} \tag{1.31.18}$$
$$= \begin{pmatrix} 3 \\ 0 \end{pmatrix} \tag{1.31.19}$$

$$\because \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f < 0, \tag{1.31.20}$$

the axes need to be swapped and

$$\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_2}} = 2 \tag{1.31.21}$$
$$\sqrt{\frac{f - \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}}{\lambda_1}} = 2 \tag{1.31.22}$$

$\therefore$ the equation of the standard hyperbola can be expressed as

$$\frac{y^2}{4} - \frac{x^2}{4} = 1 \tag{1.31.23}$$

The direction and normal vectors of tangent with slope 2 are

$$\mathbf{m} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \mathbf{n} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \tag{1.31.24}$$

From the conics table in the manual,

$$\kappa = \pm\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\mathbf{n}^T\mathbf{V}^{-1}\mathbf{n}}} = \pm\sqrt{\frac{1}{4}} = \pm\frac{1}{2} \tag{1.31.25}$$

and the desired points of contact are

$$\mathbf{q} = \mathbf{V}^{-1}(\kappa\mathbf{n} - \mathbf{u}) \tag{1.31.26}$$

$$\implies \mathbf{q}_1 = \begin{pmatrix} 4 \\ -2 \end{pmatrix}, \mathbf{q}_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \tag{1.31.27}$$

The equation of the tangents are given by

$$\mathbf{n}^T\mathbf{x} = \mathbf{n}^T\mathbf{q} \tag{1.31.28}$$

$$\implies \begin{pmatrix} -2 & 1 \end{pmatrix}\mathbf{x} = \begin{pmatrix} -2 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ -2 \end{pmatrix} = -10 \tag{1.31.29}$$

$$\text{and } \begin{pmatrix} -2 & 1 \end{pmatrix}\mathbf{x} = \begin{pmatrix} -2 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \end{pmatrix} = -2 \tag{1.31.30}$$

The above results are verified in Fig. 1.31.

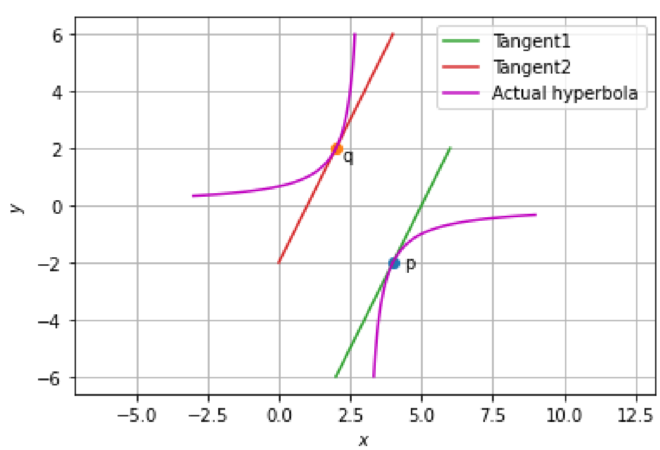

Fig. 1.31: The tangents to the curve with slope 2

1.32. Find the point at which the tangent to the curve $y = \sqrt{4x-3} - 1$ has its slope $\frac{2}{3}$.

Solution: Given curve,

$$y = \sqrt{4x-3} - 1 \tag{1.32.1}$$

$$\implies (y+1)^2 = 4x - 3 \tag{1.32.2}$$

$$\implies y^2 - 4x + 2y + 4 = 0 \tag{1.32.3}$$

which has the vector parameters

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} -2 & 1 \end{pmatrix}, \mathbf{f} = 4 \tag{1.32.4}$$

$$|\mathbf{V}| = 0 \tag{1.32.5}$$

$\therefore$ the given curve (1.32.1) is parabola. In standard form,

$$\mathbf{P} = \mathbf{I} \implies \mathbf{p_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.32.6}$$

Since the slope of the tangents is $\frac{2}{3}$, the direction and normal vectors are

$$\mathbf{m} = \begin{pmatrix} 1 \\ \frac{2}{3} \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \mathbf{n} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} \tag{1.32.7}$$

$$\text{and } \kappa = \frac{\mathbf{p_1^T u}}{\mathbf{p_1^T n}} = -1 \tag{1.32.8}$$

$\therefore$ Point of contact for the tangent is

$$\begin{pmatrix} \mathbf{u} + \kappa\mathbf{n^T} \\ \mathbf{V} \end{pmatrix}\mathbf{q} = \begin{pmatrix} -\mathbf{f} \\ \kappa\mathbf{n} - \mathbf{u} \end{pmatrix} \tag{1.32.9}$$

$$\implies \begin{pmatrix} -4 & 4 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{q} = \begin{pmatrix} -4 \\ 0 \\ 2 \end{pmatrix} \tag{1.32.10}$$

$$\implies \mathbf{q} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \tag{1.32.11}$$

which is verified in Fig. 1.32

1.33. Find the equation of the normal to the curve $x^2 = 4y$ which passes through the point $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Solution: The given curve can be expressed as

$$x^2 - 4y = 0 \tag{1.33.1}$$

$$\implies \mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}, \mathbf{f} = 0 \tag{1.33.2}$$

$\because$

$$|\mathbf{V}| = 0 \tag{1.33.3}$$

the given curve (1.33.1) represents a parabola.

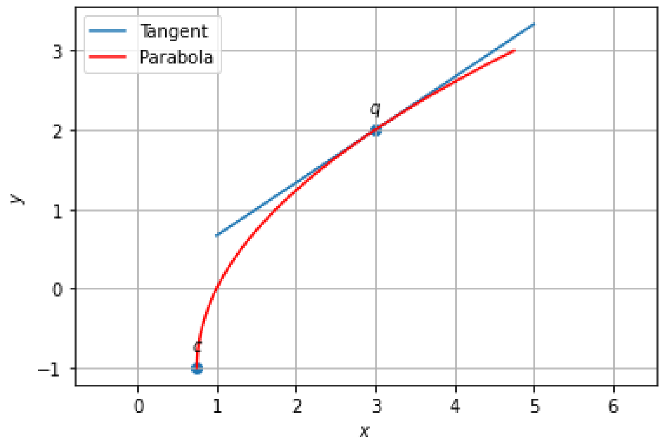

Fig. 1.32: Tangent to Parabola.

The eigenvalues are given by

$$\lambda_1 = 0, \lambda_2 = 1 \tag{1.33.4}$$

with corresponding eigenvectors

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} = 0 \implies \mathbf{p_1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.33.5}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{x} = 0 \implies \mathbf{p_2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.33.6}$$

To find the vertex of the parabola ,

$$\begin{pmatrix} \mathbf{u^T} + \kappa \mathbf{p_1^T} \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -\mathbf{f} \\ \kappa \mathbf{p_1} - \mathbf{u} \end{pmatrix} \tag{1.33.7}$$

$$\text{where, } \kappa = \mathbf{u^T p_1} = -2 \tag{1.33.8}$$

$$\implies \begin{pmatrix} 0 & -4 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{c} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{1.33.9}$$

from the above it can be observed that,

$$\mathbf{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1.33.10}$$

Now to evalute the direction vector m,

$$\mathbf{m}^T (\mathbf{Vq} + \mathbf{u}) = 0 \tag{1.33.11}$$

$$\implies \mathbf{m}^T \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ -2 \end{pmatrix} \right) = 0 \tag{1.33.12}$$

$$\implies \mathbf{m}^T \begin{pmatrix} 1 \\ -2 \end{pmatrix} = 0 \tag{1.33.13}$$

$$\implies \mathbf{m} = \begin{pmatrix} -2 \\ -1 \end{pmatrix} \tag{1.33.14}$$

The normal is obtained as

$$\mathbf{m}^T (\mathbf{x} - \mathbf{q}) = 0 \tag{1.33.15}$$

$$\begin{pmatrix} -2 & -1 \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right) = 0 \tag{1.33.16}$$

$$\begin{pmatrix} -2 & -1 \end{pmatrix} \mathbf{x} + 4 = 0 \tag{1.33.17}$$

The above results are verified in Fig. 1.33.

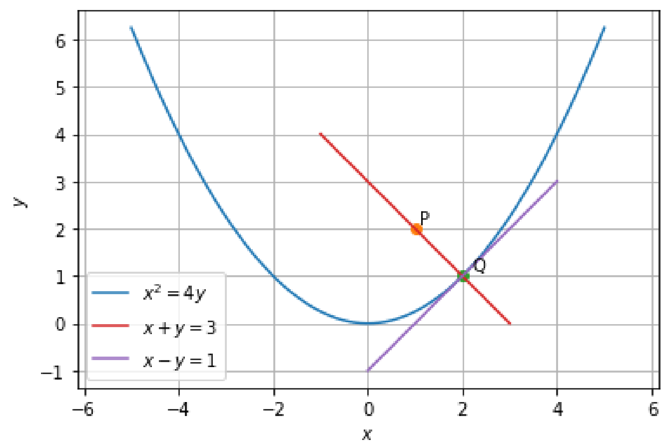

Fig. 1.33: Normal to Parabola.

1.34. Find the intervals in which the function

$$f(x) = x^2 - 4x + 6 \tag{1.34.1}$$

is
a) increasing
b) decreasing.

Solution:
Given equation can be written as

$$x^2 - 4x - y + 6 = 0 \tag{1.34.2}$$

$$\implies \mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 2 \\ \frac{-1}{2} \end{pmatrix}, f = 6 \tag{1.34.3}$$

Using eigen decomposition on $\mathbf{V}$,

$$\mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.34.4}$$

$$\mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{1.34.5}$$

The vertex of parabola $\mathbf{c}$ is given by

$$\begin{pmatrix} \mathbf{u}^T + \eta \mathbf{p_1}^T \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \eta \mathbf{p_1} - \mathbf{u} \end{pmatrix} \tag{1.34.6}$$

$$where, \eta = \mathbf{u}^T\mathbf{p_1} = \frac{-1}{2} \tag{1.34.7}$$

yielding

$$\begin{pmatrix} -2 & -1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}\mathbf{c} = \begin{pmatrix} -6 \\ 2 \\ 0 \end{pmatrix} \tag{1.34.8}$$

$$\implies \begin{pmatrix} -2 & -1 & -6 \\ 1 & 0 & 2 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{R_1}{-2}} \begin{pmatrix} 1 & \frac{1}{2} & 2 \\ 1 & 0 & 2 \end{pmatrix} \tag{1.34.9}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & \frac{1}{2} & 3 \\ 0 & \frac{-1}{2} & -1 \end{pmatrix} \tag{1.34.10}$$

$$\xleftrightarrow{R_2 \leftarrow (-2R_2)} \begin{pmatrix} 1 & \frac{1}{2} & 3 \\ 0 & 1 & 2 \end{pmatrix} \tag{1.34.11}$$

$$\implies \xleftrightarrow{R_1 \leftarrow R_1 - \frac{R_2}{2}} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{pmatrix} \tag{1.34.12}$$

From (1.34.12) it can be observed that,

$$\mathbf{c} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \tag{1.34.13}$$

$$\implies \mathbf{e_1}^T\mathbf{c} = 2 \quad \left(\because \mathbf{e_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) \tag{1.34.14}$$

$\therefore$ From -∞ to $\mathbf{e_1}^T\mathbf{c}$ the function is decreasing and from $\mathbf{e_1}^T\mathbf{c}$ to ∞ the function is increasing.

a) f is increasing in interval (2,∞)
b) f is decreasing in interval (-∞,2)

This is verified in Fig. 1.34.

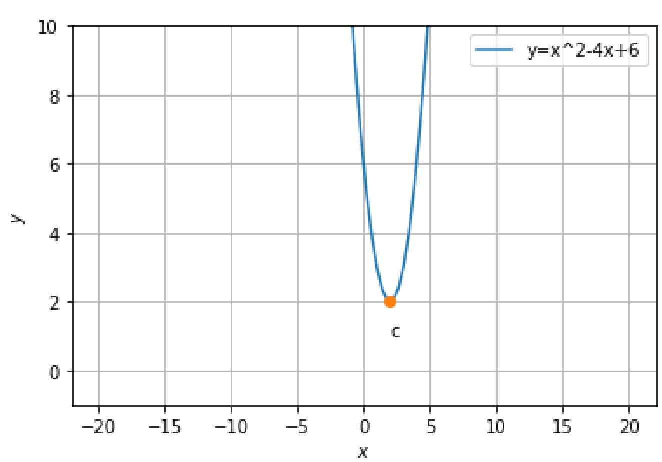

Fig. 1.34: Parabola

1.35. Find the equation of the parabola with focus $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and directrix $\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{x} = -2$.

1.36. Find the equation of the parabola with vertex at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and focus at $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$.

Solution:

Lemma 1.1. *The distance of a point* $\mathbf{P}$ *from a line* $L : \mathbf{n}^T\mathbf{x} = c$ *is given by:*

$$d = \frac{\left|c - \mathbf{P}^T\mathbf{n}\right|}{\|\mathbf{n}\|} \tag{1.36.1}$$

Definition 1. *The locus of* $\mathbf{P}$ *such that*

$$\frac{\|\mathbf{P} - \mathbf{F}\|}{d} = e \tag{1.36.2}$$

is known as a conic section. The line L *is known as the directrix and the point* $\mathbf{F}$ *is the focus.* e *is defined to be the eccentricity of the conic.*

a) *For* $e = 1$, *the conic is a parabola*
b) *For* $e < 1$, *the conic is an ellipse*
c) *For* $e > 1$, *the conic is a hyperbola*

Theorem 1.1. *The equation of a conic is given by any conicis given by:*

$$\mathbf{x}^T(t\mathbf{I} - \mathbf{n}\mathbf{n}^T)\mathbf{x} + 2(c\mathbf{n} - t\mathbf{F})^T\mathbf{x} + t\|\mathbf{F}\|^2 - c^2 = 0 \tag{1.36.3}$$

where

$$t = \frac{\|\mathbf{n}\|^2}{e^2} \tag{1.36.4}$$

Proof. Using Definition 1 and Lemma 1.1, for any point $\mathbf{x}$ on the conic,

$$\|\mathbf{x} - \mathbf{F}\|^2 = e^2\frac{(c - \mathbf{x}^T\mathbf{n})^2}{\|\mathbf{n}\|^2} \tag{1.36.5}$$

$$t(\mathbf{x} - \mathbf{F})^T(\mathbf{x} - \mathbf{F}) = (c - \mathbf{x}^T\mathbf{n})^2 \tag{1.36.6}$$

$$t(\mathbf{x}^T\mathbf{x} - 2\mathbf{F}^T\mathbf{x} + \|\mathbf{F}\|^2) = c^2 + (\mathbf{x}^T\mathbf{n})^2 - 2c\mathbf{x}^T\mathbf{n} \tag{1.36.7}$$

$$t\mathbf{x}^T\mathbf{x} - (\mathbf{x}^T\mathbf{n})^2 - 2t\mathbf{F}^T\mathbf{x} + 2c\mathbf{n}^T\mathbf{x} = c^2 - t\|\mathbf{F}\|^2 \tag{1.36.8}$$

$$t\mathbf{x}^T\mathbf{I}\mathbf{x} - \mathbf{x}^T\mathbf{n}\mathbf{n}^T\mathbf{x} + 2(c\mathbf{n} - t\mathbf{F})^T\mathbf{x} = c^2 - t\|\mathbf{F}\|^2 \tag{1.36.9}$$

$$\mathbf{x}^T(t\mathbf{I} - \mathbf{n}\mathbf{n}^T)\mathbf{x} + 2(c\mathbf{n} - t\mathbf{F})^T\mathbf{x} + t\|\mathbf{F}\|^2 - c^2 = 0 \tag{1.36.10}$$

$\square$

From the given information,

$$\mathbf{F} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \mathbf{n} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, c = -2, t = 1 \quad (1.36.11)$$

Hence, substituting from (1.36.11), the equation of the conic is

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} -4 & 0 \end{pmatrix} \mathbf{x} + 0 = 0 \quad (1.36.12)$$

Replacing $\mathbf{x}$ by $\begin{pmatrix} x \\ y \end{pmatrix}$ in (1.36.12) gives

$$y^2 = 8x \quad (1.36.13)$$

which is plotted in Fig. 1.36.

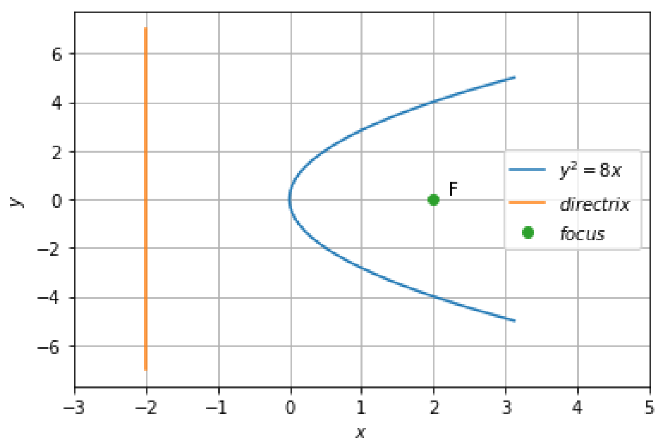

Fig. 1.36: Parabola $y^2 = 8x$

1.37. Find the equation of the ellipse, with major axis along the x-axis and passing through the points $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 4 \end{pmatrix}$.

Solution: Let

$$\mathbf{p} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}, \mathbf{q} = \begin{pmatrix} -1 \\ 4 \end{pmatrix} \quad (1.37.1)$$

In general, the equation of the ellipse passing through $\mathbf{p}, \mathbf{q}$ can be expressed as

$$(\mathbf{x} - \mathbf{c})^\top \mathbf{D} (\mathbf{x} - \mathbf{c}) = 1 \quad (1.37.2)$$

where the center $\mathbf{c} = \begin{pmatrix} \beta \\ 0 \end{pmatrix}$ and $\mathbf{D}$ is a diagonal matrix. $\because$ $\mathbf{p}, \mathbf{q}$ satisfy (1.37.2),

$$(\mathbf{p} - \mathbf{c})^\top \mathbf{D} (\mathbf{p} - \mathbf{c}) = 1, \quad (1.37.3)$$

$$(\mathbf{q} - \mathbf{c})^\top \mathbf{D} (\mathbf{q} - \mathbf{c}) = 1, \quad (1.37.4)$$

which can be simplified as

$$2 (\mathbf{p} - \mathbf{q})^\top \mathbf{D}\mathbf{c} = \mathbf{p}^\top \mathbf{D}\mathbf{p} - \mathbf{q}^\top \mathbf{D}\mathbf{q} \quad (1.37.5)$$

Using the identity,

$$(\mathbf{p} - \mathbf{q})^\top \mathbf{D}(\mathbf{p} + \mathbf{q}) = \mathbf{p}^\top \mathbf{D}\mathbf{p} - \mathbf{q}^\top \mathbf{D}\mathbf{q} \quad (1.37.6)$$

in the above,

$$2 (\mathbf{p} - \mathbf{q})^\top \mathbf{D}\mathbf{c} = (\mathbf{p} - \mathbf{q})^\top \mathbf{D}(\mathbf{p} + \mathbf{q})$$

$$\implies (\mathbf{p} - \mathbf{q})^\top \mathbf{D} (2\mathbf{c} - (\mathbf{p} + \mathbf{q})) = 0 \quad (1.37.7)$$

Thus, $\mathbf{c}$ can be expressed in parametric form as

$$\mathbf{c} = \frac{1}{2} \left[\mathbf{p} + \mathbf{q} + k\mathbf{D}^{-1}\mathbf{m} \right] \quad (1.37.8)$$

where

$$(\mathbf{p} - \mathbf{q})^\top \mathbf{m} = 0 \quad (1.37.9)$$

and k is a constant. Substituting numerical values in (1.37.9),

$$\mathbf{p} - \mathbf{q} = \begin{pmatrix} 5 \\ -1 \end{pmatrix} \implies \mathbf{m} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \quad (1.37.10)$$

Also,

$$\mathbf{p} + \mathbf{q} = \begin{pmatrix} 3 \\ 7 \end{pmatrix} \quad (1.37.11)$$

which, upon substitution in (1.37.8) yields

$$\begin{pmatrix} \beta \\ 0 \end{pmatrix} = \frac{1}{2} \left[\begin{pmatrix} 3 \\ 7 \end{pmatrix} + k \begin{pmatrix} \frac{1}{\lambda_1} & 0 \\ 0 & \frac{1}{\lambda_2} \end{pmatrix} \begin{pmatrix} 1 \\ 5 \end{pmatrix} \right] \quad (1.37.12)$$

From the given information, the $X-$axis is the major axis. Hence,

$$\frac{\lambda_2}{\lambda_1} > 1 \implies \frac{2\beta - 3}{\frac{-7}{5}} > 1 \quad (1.37.13)$$

$$\text{or, } \beta < 0.8 \quad (1.37.14)$$

The possible ellipses satisfying the above condition are plotted in Fig. 1.37.

1.38. Find the coordinates of the foci and the vertices, the eccentricity,the length of the latus rectum of the hyperbolas

a) $\mathbf{x}^T \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & -\frac{1}{16} \end{pmatrix} \mathbf{x} = 1$

b) $\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & -16 \end{pmatrix} \mathbf{x} = 16$

Solution:

a)

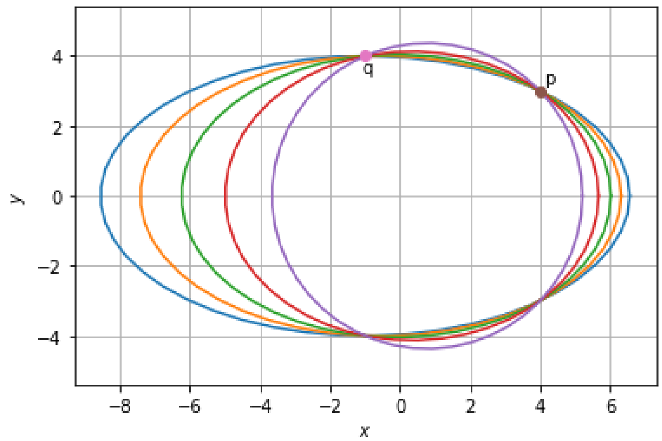

Fig. 1.37: Ellipses passing through the two points with X axis as major axis

Lemma 1.2. *The standard form of a conic is given by*

$$\frac{\mathbf{y}^\top D\mathbf{y}}{\mathbf{u}^\top\mathbf{V}^{-1}\mathbf{u}-f}=1 \tag{1.38.1}$$

Given

$$\mathbf{x}^\top\begin{pmatrix}\frac{1}{9} & 0\\ 0 & \frac{-1}{16}\end{pmatrix}\mathbf{x}=1 \tag{1.38.2}$$

we have,

$$\mathbf{V}=\begin{pmatrix}\frac{1}{9} & 0\\ 0 & \frac{-1}{16}\end{pmatrix} \tag{1.38.3}$$

$$\mathbf{u}^\top\mathbf{V}^{-1}\mathbf{u}-f=1 \tag{1.38.4}$$

$$\mathbf{c}=-\mathbf{V}^{-1}\mathbf{u}=\begin{pmatrix}0\\0\end{pmatrix} \tag{1.38.5}$$

$$\lambda_1=\frac{1}{9},\lambda_2=\frac{-1}{16} \tag{1.38.6}$$

Axes of hyperbola is given by

$$\sqrt{\frac{\mathbf{u}^\top\mathbf{V}^{-1}\mathbf{u}-f}{\lambda_1}}=4 \tag{1.38.7}$$

$$\sqrt{\frac{f-\mathbf{u}^\top\mathbf{V}^{-1}\mathbf{u}}{\lambda_2}}=3 \tag{1.38.8}$$

The vertices are given as

$$\pm\begin{pmatrix}4\\0\end{pmatrix} \tag{1.38.9}$$

Coordinates of foci are given by,

$$\mathbf{F}=\pm\left(\sqrt{\frac{(\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}-f)(\lambda_2-\lambda_1)}{\lambda_1\lambda_2}}\right)\mathbf{p_1} \tag{1.38.10}$$

where, $\mathbf{p_1}=\begin{pmatrix}1\\0\end{pmatrix}$ since the equation of hyperbola is in standard form. Substituting the values in (1.38.10) we have,

$$\mathbf{F}=\pm\begin{pmatrix}5\\0\end{pmatrix}. \tag{1.38.11}$$

Eccentricity of the hyperbola is given by,

$$e=\frac{\sqrt{\frac{(\mathbf{u}^\top\mathbf{V}^{-1}\mathbf{u})(\lambda_2-\lambda_1)}{\lambda_1\lambda_2}}}{\sqrt{\frac{\mathbf{u}^\top\mathbf{V}^{-1}\mathbf{u}-f}{\lambda_1}}} \tag{1.38.12}$$

substituting the values in (1.38.12),we have

$$e=\frac{5}{3}. \tag{1.38.13}$$

Length of the latus rectum is given by,

$$l=\frac{2\left(\sqrt{\frac{f-\mathbf{u}^\top\mathbf{V}^{-1}\mathbf{u}}{\lambda_2}}\right)^2}{\sqrt{\frac{\mathbf{u}^\top\mathbf{V}^{-1}\mathbf{u}-f}{\lambda_1}}} \tag{1.38.14}$$

substituting the values in (1.38.14),we have

$$l=\frac{32}{3} \tag{1.38.15}$$

Plot of the hyperbola:

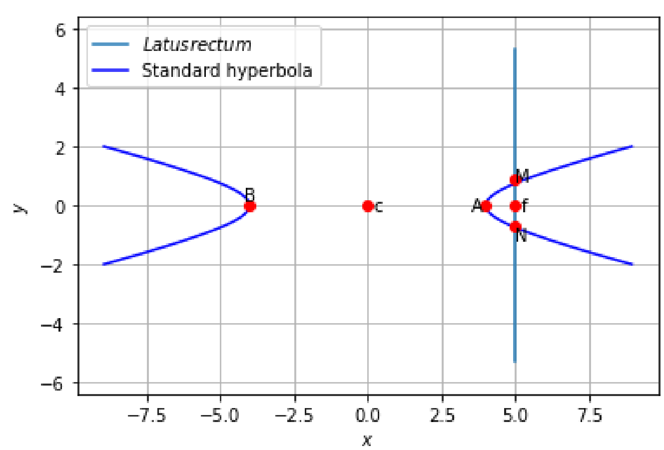

Fig. 1.38: Hyperbola

From the given equation,

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & -16 \end{pmatrix} \tag{1.38.16}$$

$$\mathbf{u}^\top \mathbf{V}^{-1} \mathbf{u} - f = 16 \tag{1.38.17}$$

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1.38.18}$$

$$\lambda_1 = 1, \lambda_2 = -16 \tag{1.38.19}$$

Axes of hyperbola are given by

$$\sqrt{\frac{\mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}} = 4 \tag{1.38.20}$$

$$\sqrt{\frac{f - \mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u}}{\lambda_2}} = 1 \tag{1.38.21}$$

and the vertices are

$$\pm \begin{pmatrix} 4 \\ 0 \end{pmatrix} \tag{1.38.22}$$

Coordinates of the foci are given by,

$$\mathbf{F} = \pm \left(\sqrt{\frac{(\mathbf{u}^T \mathbf{V}^{-1}\mathbf{u} - f)(\lambda_2 - \lambda_1)}{\lambda_1 \lambda_2}} \right) \mathbf{p_1} \tag{1.38.23}$$

where, $\mathbf{p_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ since the equation of hyperbola is in standard form. Substituting the values in (1.38.23) we have,

$$\mathbf{F} = \pm \begin{pmatrix} \sqrt{17} \\ 0 \end{pmatrix}. \tag{1.38.24}$$

Eccentricity of the hyperbola is given by,

$$e = \frac{\sqrt{\frac{(\mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u})(\lambda_2 - \lambda_1)}{\lambda_1 \lambda_2}}}{\sqrt{\frac{\mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}}} = \frac{\sqrt{17}}{4}. \tag{1.38.25}$$

upon substituting from (1.38.25). Length of the latus rectum is given by,

$$l = \frac{2\left(\sqrt{\frac{f - \mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u}}{\lambda_2}}\right)^2}{\sqrt{\frac{\mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}}} = \frac{1}{2} \tag{1.38.26}$$

upon substituting the values in (1.38.26). The above results are verified in Fig. 1.38.

1.39. Find the slope of the tangent to the curve $y = \frac{x-1}{x-2}, x \neq 2$ at $x = 10$.

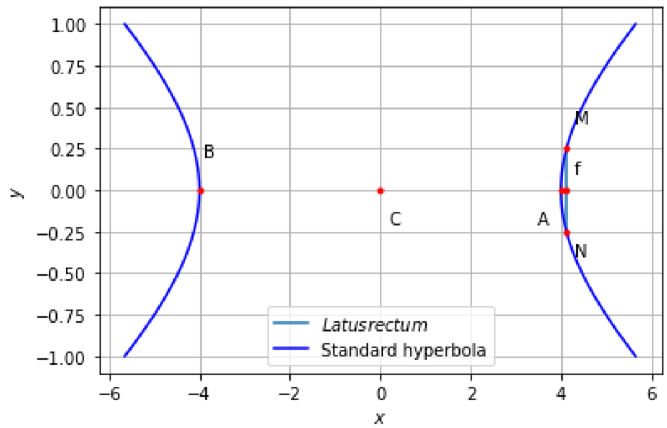

Fig. 1.38: Plot of standard hyperbola

Solution: The given curve, can be expressed as,

$$yx - 2y - x + 1 = 0 \tag{1.39.1}$$

yielding

$$\mathbf{V} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \mathbf{V}^{-1} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \mathbf{u} = \begin{pmatrix} -\frac{1}{2} \\ -1 \end{pmatrix}, f = 1 \tag{1.39.2}$$

$$\because |\mathbf{V}| < 0 \tag{1.39.3}$$

(1.39.1) is a hyperbola. Let the slope of tangent be r. Then the direction vector and normal vector of tangent to (1.39.1) are

$$\mathbf{m} = \begin{pmatrix} 1 \\ r \end{pmatrix}, \mathbf{n} = \begin{pmatrix} r \\ -1 \end{pmatrix} \tag{1.39.4}$$

$$\kappa = \pm \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1}\mathbf{u} - \mathbf{f}}{\mathbf{n}^T \mathbf{V}^{-1}\mathbf{n}}} \tag{1.39.5}$$

$$= \sqrt{\frac{-1}{4r}} \tag{1.39.6}$$

For hyperbola, the point of contact for the tangent is

$$\mathbf{q} = \mathbf{V}^{-1}(\kappa\mathbf{n} - \mathbf{u}) \tag{1.39.7}$$

$$\implies \mathbf{V}\mathbf{q} + \mathbf{u} = \kappa\mathbf{n} \tag{1.39.8}$$

$$\implies \begin{pmatrix} \frac{1}{16} \\ 4 \end{pmatrix} = \kappa\mathbf{n} \quad (\text{ From } \quad (1.39.2)) \tag{1.39.9}$$

$$\implies \begin{pmatrix} \frac{1}{16} \\ 4 \end{pmatrix} = \sqrt{\frac{-1}{4r}} \begin{pmatrix} r \\ -1 \end{pmatrix} \tag{1.39.10}$$

$$\implies \begin{pmatrix} \frac{1}{16} \\ 4 \end{pmatrix} = \begin{pmatrix} r\sqrt{\frac{-1}{4r}} \\ -\sqrt{\frac{-1}{4r}} \end{pmatrix} \tag{1.39.11}$$

$$\implies -\sqrt{\frac{-1}{4r}} = 4 \tag{1.39.12}$$

$$\implies r = -\frac{1}{64} \tag{1.39.13}$$

which is the desired slope. Fig. 1.39 verifies this result.

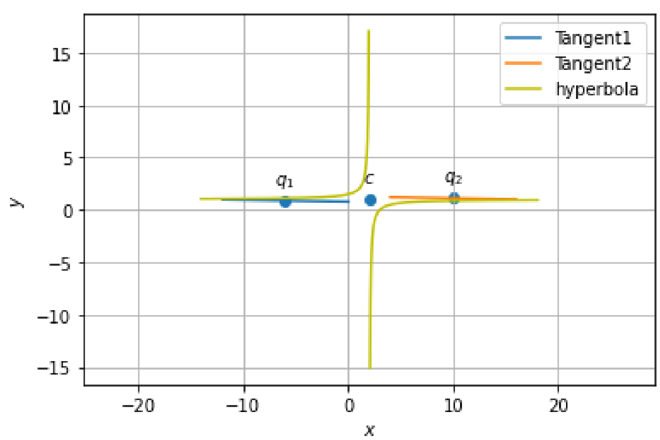

Fig. 1.39: Tangent to HYPERBOLA.

1.40. Find the equation of all lines having slope 2 which are tangents to the curve $\frac{1}{x-3}, x \neq 3$.
Solution:
Given curve

$$y = \frac{1}{x-3}, x \neq 3 \tag{1.40.1}$$

$$\implies xy - 3y - 1 = 0 \tag{1.40.2}$$

$\therefore$

$$\mathbf{V} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{1.40.3}$$

$$\mathbf{u} = \frac{-3}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.40.4}$$

$$f = -1 \tag{1.40.5}$$

$\because$

$$|\mathbf{V}| = \frac{-1}{4} \tag{1.40.6}$$

$$\implies |\mathbf{V}| < 0 \tag{1.40.7}$$

$\therefore$ (1.40.1) represents a hyperbola . Now,the characteristic equation of $\mathbf{V}$ is

$$|\mathbf{V} - \lambda\mathbf{I}| = \begin{vmatrix} -\lambda & \frac{1}{2} \\ \frac{1}{2} & -\lambda \end{vmatrix} = 0 \tag{1.40.8}$$

$$\implies \lambda^2 - \frac{1}{4} = 0 \tag{1.40.9}$$

$\therefore$ Eigen values are

$$\lambda_1 = \frac{1}{2}, \lambda_2 = \frac{-1}{2} \tag{1.40.10}$$

Eigen vector $\mathbf{p}$ is

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \tag{1.40.11}$$

$$\implies (\mathbf{V} - \lambda\mathbf{I})\mathbf{p} = 0 \tag{1.40.12}$$

Eigen vector $\mathbf{p}_1$ corressponding to λ_1 can be obtained as

$$(\mathbf{V} - \lambda_1\mathbf{I}) = \begin{pmatrix} \frac{-1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-1}{2} \end{pmatrix} \xleftrightarrow[R_1 \leftarrow -2R_1]{R_2 = R_1 + R_2} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \tag{1.40.13}$$

$$\implies \mathbf{p_1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.40.14}$$

Similarly,

$$\mathbf{p_2} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{1.40.15}$$

$\therefore$

$$\mathbf{P} = \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \tag{1.40.16}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{-1}{2} \end{pmatrix} \tag{1.40.17}$$

Now,

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \quad (1.40.18)$$

$$= -\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ \frac{-3}{2} \end{pmatrix} \quad (1.40.19)$$

$$= \begin{pmatrix} 3 \\ 0 \end{pmatrix} \quad (1.40.20)$$

and

$$\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}} = \sqrt{2} \quad (1.40.21)$$

$$\sqrt{\frac{f - \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}}{\lambda_2}} = \sqrt{2} \quad (1.40.22)$$

$\therefore$ Equation of standard hyperbola can be expressed as

$$\frac{x^2}{2} - \frac{y^2}{2} = 1 \quad (1.40.23)$$

Now,direction vector of tangent with slope = 2 is

$$\mathbf{m} = \begin{pmatrix} 1 \\ m \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (1.40.24)$$

and,normal vector of same tangent is

$$\mathbf{m}^T\mathbf{n} = 0 \quad (1.40.25)$$

$$\implies \mathbf{n} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad (1.40.26)$$

Now,

$$\kappa = \pm\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\mathbf{n}^T\mathbf{V}^{-1}\mathbf{n}}} \quad (1.40.27)$$

$$= \pm\sqrt{\frac{1}{-8}} \quad (1.40.28)$$

$\therefore$ Real value of κ does not exist and hence points of contacts of tangent $\mathbf{q_1}, \mathbf{q_2}$ also does not exist.
Hence,there exists no tangent to the curve having slope = 2. The hyperbola is plotted in Fig. 1.40.

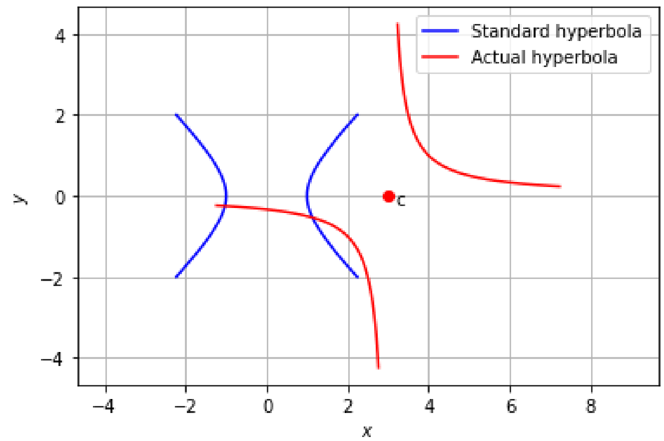

Fig. 1.40: Standard and actual hyperbola

1.41. Find points on the curve $\mathbf{x}^T\begin{pmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{16} \end{pmatrix}\mathbf{x} = 1$ at which tangents are
a) parallel to x-axis
b) parallel to y-axis.

Solution:

Given curve,

$$\mathbf{x}^T\begin{pmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{16} \end{pmatrix}\mathbf{x} = 1 \quad (1.41.1)$$

where,

$$\mathbf{V} = \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{16} \end{pmatrix}, \mathbf{V}^{-1} = \begin{pmatrix} 9 & 0 \\ 0 & 16 \end{pmatrix} \mathbf{u} = 0, f = -1 \quad (1.41.2)$$

$$\because |\mathbf{V}| > 0 \quad (1.41.3)$$

given curve (1.41.1) is ellipse. For an ellipse, the point of contact for the tangent is

$$\mathbf{q} = \mathbf{V}^{-1}(\kappa\mathbf{n} - \mathbf{u}) \quad (1.41.4)$$

$$= \mathbf{V}^{-1}\kappa\mathbf{n} \qquad (\because \mathbf{u} = 0). \quad (1.41.5)$$

where,

$$\kappa = \pm\sqrt{\frac{\mathbf{u}^\mathbf{T}\mathbf{V}^{-1}\mathbf{u} - f}{\mathbf{n}^\mathbf{T}\mathbf{V}^{-1}\mathbf{n}}} \quad (1.41.6)$$

$$= \pm\sqrt{\frac{-f}{\mathbf{n}^\mathbf{T}\mathbf{V}^{-1}\mathbf{n}}} \qquad (\because \mathbf{u} = 0) \quad (1.41.7)$$

a) For the tangents parallel to x-axis, then direction and normal vectors are,

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{n_1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \kappa_1 = \pm\sqrt{\frac{-f}{\mathbf{n_1^T}\mathbf{V}^{-1}\mathbf{n_1}}} \quad (1.41.8)$$

$$= \pm\frac{1}{4} \quad (1.41.9)$$

$\therefore$ By Substituting $\kappa_1, \mathbf{n_1}, \mathbf{V}^{-1}$ in (1.41.5)

$$\mathbf{q} = \mathbf{V}^{-1}\kappa_1\mathbf{n_1} \tag{1.41.10}$$

$$= \begin{pmatrix} 0 \\ \pm 4 \end{pmatrix} \tag{1.41.11}$$

b) For the tangents parallel to the y-axis, the direction and normal vectors are

$$\mathbf{m_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \mathbf{n_2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \kappa_2 = \pm\sqrt{\frac{-f}{\mathbf{n_2^T}\mathbf{V}^{-1}\mathbf{n_2}}} \tag{1.41.12}$$

$$= \pm\frac{1}{3} \tag{1.41.13}$$

$\therefore$ substituting $\kappa_2, \mathbf{n_2}, \mathbf{V}^{-1}$ in (1.41.5)

$$\mathbf{q} = \mathbf{V}^{-1}\kappa_2\mathbf{n_2} \tag{1.41.14}$$

$$= \begin{pmatrix} 0 \\ \pm 3 \end{pmatrix} \tag{1.41.15}$$

The above results are verified in Fig. 1.41

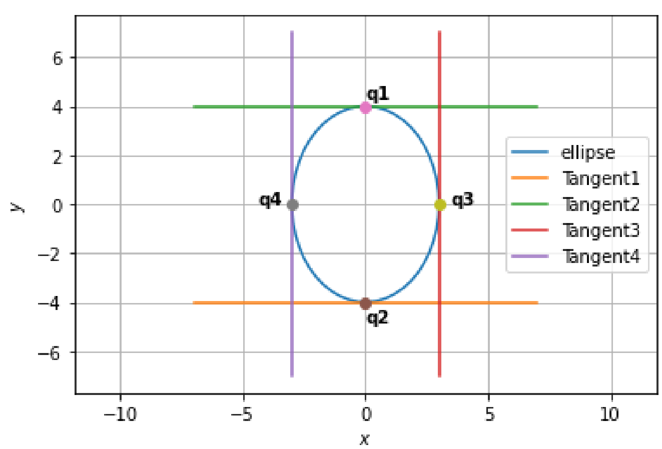

Fig. 1.41: Tangents to ELLIPSE.

1.42. Find the equation of the tangent to the curve $y = \sqrt{3x-2}$ which is parallel to the line $\begin{pmatrix} 4 & -2 \end{pmatrix}\mathbf{x} + 5 = 0$. **Solution:** The given equation can be expressed as

$$\mathbf{x}^\top \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} -3 & 0 \end{pmatrix}\mathbf{x} + 2 = 0 \tag{1.42.1}$$

and

$$\therefore \mathbf{u} = \begin{pmatrix} \frac{-3}{2} \\ 0 \end{pmatrix} \tag{1.42.2}$$

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.42.3}$$

$$\implies \left|\mathbf{V}\right| = 0 \tag{1.42.4}$$

Thus the curve is a parabola with eigenvalues

$$\lambda = 0, 1 \tag{1.42.5}$$

The corresponding eigenvectors are

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x} = 0 \implies \mathbf{p_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.42.6}$$

$$\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}\mathbf{x} = \mathbf{x} \implies \mathbf{p_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.42.7}$$

and

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^\top \tag{1.42.8}$$

where

$$\mathbf{P} = \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.42.9}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.42.10}$$

Since the given parallel line equation is

$$\begin{pmatrix} 4 & -2 \end{pmatrix}\mathbf{x} + 5 = 0, \tag{1.42.11}$$

the normal vector $\mathbf{n}$ and the direction vector $\mathbf{m}$ of the tangent to the parabola are given by

$$\mathbf{n} = \begin{pmatrix} 4 \\ -2 \end{pmatrix} \tag{1.42.12}$$

$$\mathbf{m} = \begin{pmatrix} -2 \\ -4 \end{pmatrix} \tag{1.42.13}$$

The point of contact for the parabola is given by,

$$\begin{pmatrix} \mathbf{u} + \kappa\mathbf{n}^\top \\ \mathbf{V} \end{pmatrix}\mathbf{q} = \begin{pmatrix} -\mathbf{f} \\ \kappa\mathbf{n} - \mathbf{u} \end{pmatrix} \tag{1.42.14}$$

$$\implies \kappa = \frac{\mathbf{p_1}^\top\mathbf{u}}{\mathbf{p_1}^\top\mathbf{n}} = \frac{\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{-3}{2} \\ 0 \end{pmatrix}}{\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 4 \\ 2 \end{pmatrix}} = \frac{-3}{8} \tag{1.42.15}$$

and

$$\begin{pmatrix} -3 & \frac{3}{4} \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{q} = \begin{pmatrix} -2 \\ 0 \\ \frac{3}{4} \end{pmatrix} \tag{1.42.16}$$

yielding

$$\mathbf{q} = \begin{pmatrix} \frac{41}{48} \\ \frac{3}{4} \end{pmatrix} \tag{1.42.17}$$

The desired equation of the line is

$$\mathbf{n}^\top(\mathbf{x} - \mathbf{q}) = 0 \tag{1.42.18}$$

$$\implies \begin{pmatrix} 4 & -2 \end{pmatrix} \mathbf{x} = \frac{23}{12} \tag{1.42.19}$$

which is verified in Fig. 1.42.

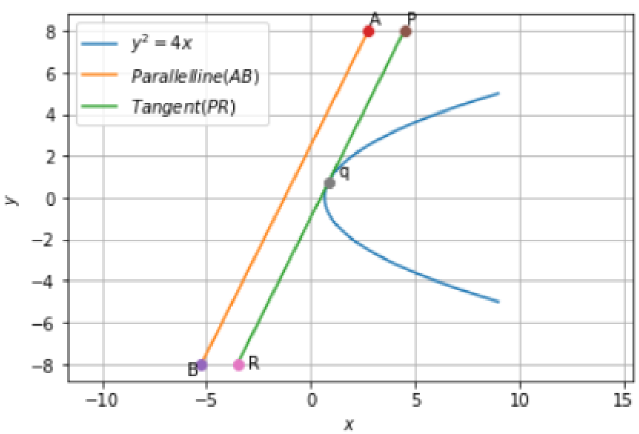

Fig. 1.42: Plot of curve and the lines

1.43. AOBA is the part of the ellipse $\mathbf{x}^T \begin{pmatrix} 9 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = 36$ in the first quadrant such that $OA = 2$ and $OB = 6$. Find the area between the arc AB and the chord AB.

Solution: Given ellipse is

$$\mathbf{x}^T \begin{pmatrix} 9 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = 36 \tag{1.43.1}$$

On comparing it with standard form

$$\mathbf{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1.43.2}$$

$$\mathbf{D} = \begin{pmatrix} 9 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.43.3}$$

$$\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f = 36 \tag{1.43.4}$$

$$\lambda_1 = 9 \tag{1.43.5}$$

$$\lambda_2 = 1 \tag{1.43.6}$$

$\therefore$ Semi major and minor axes of ellipse are

$$a = \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_2}} = \sqrt{\frac{36}{1}} = 6 \tag{1.43.7}$$

$$b = \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}} = \sqrt{\frac{36}{9}} = 2 \tag{1.43.8}$$

$\therefore$ Equation of ellipse can be written as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{1.43.9}$$

$$\implies \frac{x^2}{4} + \frac{y^2}{36} = 1 \tag{1.43.10}$$

Now,area of ellipse is given by

$$A = \pi ab \tag{1.43.11}$$

$$\implies A = 12\pi \tag{1.43.12}$$

$\therefore$ Area of a quadrant of ellipse is given by

$$A_1 = A/4 = 3\pi \tag{1.43.13}$$

Now,from Fig. 1.43 , $AOBA$ is a right angled triangle whose area is given by

$$A_2 = \frac{1}{2} ab = 6 \tag{1.43.14}$$

$\therefore$ Area between arc AB and chord AB is given by

$$A_3 = A_1 - A_2 = 3\pi - 6 \tag{1.43.15}$$

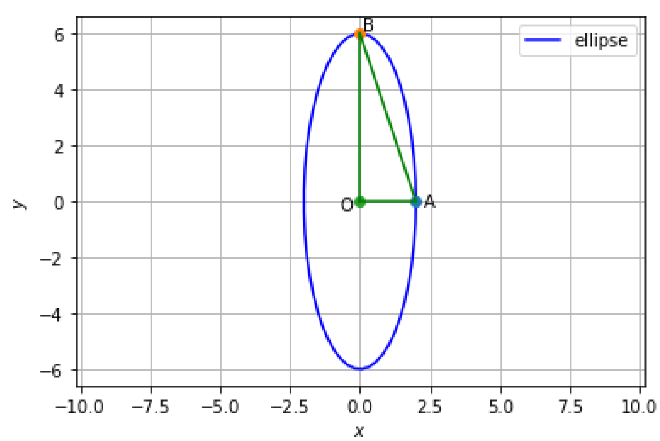

Fig. 1.43: Ellipse $\frac{x^2}{4} + \frac{y^2}{36} = 1$

1.44. Find the area enclosed by the parabola $4y = 3x^2$ and the line $\begin{pmatrix} -3 & 2 \end{pmatrix} \mathbf{x} = 12$.

Solution:

Lemma 1.3. *The points of intersection of* ***Line***

$L : \mathbf{x} = \mathbf{q} + \mu\mathbf{m}$ *with **parabola**,are given by:*

$$\mathbf{x}_i = \mathbf{q} + \mu_i\mathbf{m} \tag{1.44.1}$$

where,

$$\mu_i = \frac{1}{\mathbf{m}^T\mathbf{V}\mathbf{m}}\Big(-\mathbf{m}^T(\mathbf{V}\mathbf{q}+\mathbf{u}) \pm \sqrt{[\mathbf{m}^T(\mathbf{V}\mathbf{q}+\mathbf{u})]^2 - (\mathbf{q}^T\mathbf{V}\mathbf{q} + 2\mathbf{u}^T\mathbf{q} + f)(\mathbf{m}^T\mathbf{V}\mathbf{m})}\Big) \tag{1.44.2}$$

The matrix parameters of the parabola are

$$\mathbf{V} = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \mathbf{u} = \begin{pmatrix}0\\ -\frac{2}{3}\end{pmatrix}, f = 0 \tag{1.44.3}$$

with eigen parameters

$$\lambda_1 = 0, \lambda_2 = 1 \tag{1.44.4}$$

$$\mathbf{p_1} = \begin{pmatrix}0\\1\end{pmatrix}, \mathbf{p_2} = \begin{pmatrix}1\\0\end{pmatrix} \tag{1.44.5}$$

The vertex of the parabola can be expressed as

$$\begin{pmatrix}\mathbf{u^T} + \eta\mathbf{p_1^T}\\ \mathbf{V}\end{pmatrix}\mathbf{c} = \begin{pmatrix}-\mathbf{f}\\ \eta\mathbf{p_1} - \mathbf{u}\end{pmatrix} \tag{1.44.6}$$

$$\text{where, } \eta = \mathbf{u^T}\mathbf{p_1} = \frac{-2}{3} \tag{1.44.7}$$

$$\implies \begin{pmatrix}0 & \frac{-4}{3}\\ 1 & 0\\ 0 & 0\end{pmatrix}\mathbf{c} = \begin{pmatrix}0\\0\\0\end{pmatrix} \tag{1.44.8}$$

$$\text{or, } \mathbf{c} = \begin{pmatrix}0\\0\end{pmatrix} \tag{1.44.9}$$

From (1.44.2),

$$\mu_i = \frac{1}{4}(10) \pm 6 \tag{1.44.10}$$

$$\implies \mu_1 = 1, \mu_2 = 4 \tag{1.44.11}$$

The given line is

$$\begin{pmatrix}-3 & 2\end{pmatrix}\mathbf{x} = 12 \tag{1.44.12}$$

In parametric form, the given line can be written as:

$$L : \mathbf{x} = \mathbf{q} + \mu\mathbf{m} \tag{1.44.13}$$

$$\implies \mathbf{x} = \begin{pmatrix}-4\\0\end{pmatrix} + \mu\begin{pmatrix}2\\3\end{pmatrix} \tag{1.44.14}$$

Substitutuing μ_1 and μ_2 in (1.44.14),the points of intersection

$$\mathbf{K} = \begin{pmatrix}-2\\3\end{pmatrix}, \mathbf{L} = \begin{pmatrix}4\\12\end{pmatrix} \tag{1.44.15}$$

a) Thus, from Fig. 1.44 the area enclosed by parabola and line can be given as

$$A = \text{Area under line} - \text{Area under parabola} \tag{1.44.16}$$

$$A = Ar(KLMNK) - Ar(KCLMCNK) \tag{1.44.17}$$

$$A = A_1 - A_2 \tag{1.44.18}$$

b) Area under the line 2y=3x+12 i.e, A_1-

$$A_1 = \int_{-2}^{4} y dx \tag{1.44.19}$$

$$A_1 = \frac{1}{2}\int_{-2}^{4}(3x+12)\,dx \tag{1.44.20}$$

$$A_1 = \frac{3}{2}\int_{-2}^{4} x + \frac{1}{2}\int_{-2}^{4} 12dx \tag{1.44.21}$$

$$A_1 = \frac{3}{4}\left(4^2 - 2^2\right) + \frac{12}{2}(4+2) \tag{1.44.22}$$

$$A_1 = \frac{3}{4}(12) + \frac{12}{2}(6) \tag{1.44.23}$$

$$A_1 = 9 + 36 \tag{1.44.24}$$

$$A_1 = 45 \text{ units} \tag{1.44.25}$$

c) Area under the parabola that is A_2-

$$A_2 = \int_{-2}^{4} y dx \tag{1.44.26}$$

$$A_2 = \int_{-2}^{4} \frac{3}{4}x^2 dx \tag{1.44.27}$$

$$A_2 = \frac{3}{4}\int_{-2}^{4} x^2 dx \tag{1.44.28}$$

$$A_2 = \frac{3}{4\times 3}\left(4^3 - (-2)^3\right) \tag{1.44.29}$$

$$A_2 = \frac{1}{4}(64+8) \tag{1.44.30}$$

$$A_2 = \frac{72}{4} \tag{1.44.31}$$

$$A_2 = 18 \text{ units} \tag{1.44.32}$$

d) Putting (1.44.25) and (1.44.32) in (1.44.18) we get required area A as:

$$A = A_1 - A_2 \tag{1.44.33}$$
$$A = 45 - 18 \tag{1.44.34}$$
$$A = 27 \text{ units} \tag{1.44.35}$$

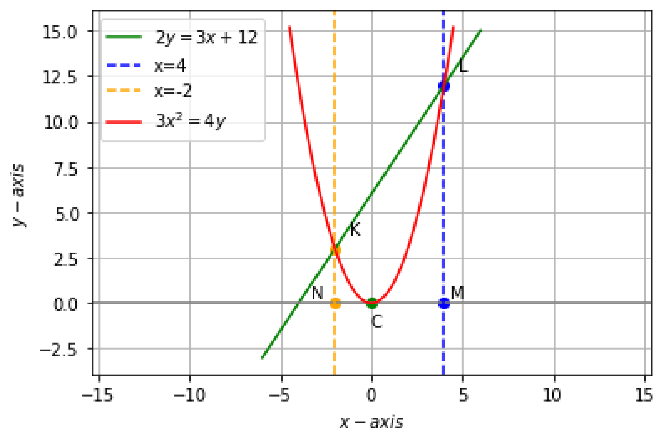

Fig. 1.44: Plot of the parabola and line

1.45. Find the area of the region enclosed between the two circles: $\mathbf{x}^T\mathbf{x} = 4$ and $\left\|\mathbf{x} - \begin{pmatrix}2\\0\end{pmatrix}\right\| = 2$.

Solution: General equation of circle is

$$\mathbf{x}^T\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{1.45.1}$$

Taking equation of the first circle to be,

$$\|\mathbf{x}\|^2 + 2\mathbf{u}_1^T\mathbf{x} + f_1 = 0 \tag{1.45.2}$$
$$\mathbf{x}^T\mathbf{x} - 4 = 0 \tag{1.45.3}$$
$$\mathbf{u_1} = \begin{pmatrix}0\\0\end{pmatrix} \tag{1.45.4}$$
$$f_1 = -4 \tag{1.45.5}$$
$$\mathbf{O_1} = \begin{pmatrix}0\\0\end{pmatrix} \tag{1.45.6}$$

Taking equation of the second circle to be,

$$\left\|\mathbf{x} - \begin{pmatrix}2\\0\end{pmatrix}\right\|^2 = 2^2 \tag{1.45.7}$$
$$\mathbf{x}^T\mathbf{x} + 2\mathbf{u_2}^T\mathbf{x} = 0 \tag{1.45.8}$$
$$\mathbf{u_2} = \begin{pmatrix}-2\\0\end{pmatrix} \tag{1.45.9}$$
$$f_2 = 0 \tag{1.45.10}$$
$$\mathbf{O_2} = \begin{pmatrix}2\\0\end{pmatrix} \tag{1.45.11}$$

Now,Subtracting equation (1.45.8) from (1.45.3) We get,

$$\mathbf{x}^T\mathbf{x} - 2\mathbf{u_2}^T\mathbf{x} + f_1 - \mathbf{x}^T\mathbf{x} = 0 \tag{1.45.12}$$
$$2\mathbf{u}^T\mathbf{x} = -4 \tag{1.45.13}$$
$$\begin{pmatrix}-4 & 0\end{pmatrix}\mathbf{x} = -4 \tag{1.45.14}$$

Which can be written as:-

$$\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = 1 \tag{1.45.15}$$
$$\mathbf{x} = \begin{pmatrix}1\\0\end{pmatrix} + \lambda\begin{pmatrix}0\\1\end{pmatrix} \tag{1.45.16}$$
$$\mathbf{x} = \mathbf{q} + \lambda\mathbf{m} \tag{1.45.17}$$
$$\mathbf{q} = \begin{pmatrix}1\\0\end{pmatrix} \tag{1.45.18}$$
$$\mathbf{m} = \begin{pmatrix}0\\1\end{pmatrix} \tag{1.45.19}$$

Substituting (1.45.17) in (1.45.2)

$$\|\mathbf{x}\|^2 + 2\mathbf{u}_1^T\mathbf{x} + f_1 = 0 \tag{1.45.20}$$
$$\|\mathbf{q} + \lambda\mathbf{m}\|^2 + f_1 = 0 \tag{1.45.21}$$
$$(\mathbf{q} + \lambda\mathbf{m})^T(\mathbf{q} + \lambda\mathbf{m}) + f_1 = 0 \tag{1.45.22}$$
$$\mathbf{q}^T(\mathbf{q} + \lambda\mathbf{m}) + \lambda\mathbf{m}^T(\mathbf{q} + \lambda\mathbf{m}) + f_1 = 0 \tag{1.45.23}$$
$$\|\mathbf{q}\|^2 + \lambda\mathbf{q}^T\mathbf{m} + \lambda\mathbf{m}^T\mathbf{q} + \lambda^2\|\mathbf{m}\|^2 + f_1 = 0 \tag{1.45.24}$$
$$\|\mathbf{q}\|^2 + 2\lambda\mathbf{q}^T\mathbf{m} + \lambda^2\|\mathbf{m}\|^2 + f_1 = 0 \tag{1.45.25}$$
$$\lambda(\lambda\|\mathbf{m}\|^2 + 2\mathbf{q}^T\mathbf{m}) = -f_1 - \|\mathbf{q}\|^2 \tag{1.45.26}$$
$$\lambda^2\|\mathbf{m}\|^2 = -f_1 - \|\mathbf{q}\|^2 \tag{1.45.27}$$
$$\lambda^2 = \frac{-f_1 - \|\mathbf{q}\|^2}{\|\mathbf{m}\|^2} \tag{1.45.28}$$
$$\lambda^2 = 3 \tag{1.45.29}$$
$$\lambda = +\sqrt{3}, -\sqrt{3} \tag{1.45.30}$$

Substituting the value of λ in(1.45.17)

$$\mathbf{x} = \mathbf{q} + \lambda \mathbf{m} \tag{1.45.31}$$

$$\mathbf{A} = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} \tag{1.45.32}$$

$$\mathbf{B} = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} \tag{1.45.33}$$

Now finding the direction vector $\mathbf{m}_{O_1A}$, $\mathbf{m}_{O_1B}$,$\mathbf{m}_{O_2A}$ and $\mathbf{m}_{O_2B}$.

$$\mathbf{m}_{O_1A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} = \begin{pmatrix} -1 \\ -\sqrt{3} \end{pmatrix} \tag{1.45.34}$$

$$\mathbf{m}_{O_1B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} = \begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix} \tag{1.45.35}$$

$$\mathbf{m}_{O_2A} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} \tag{1.45.36}$$

$$\mathbf{m}_{O_2B} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} \tag{1.45.37}$$

Now finding the angle $\angle O_1AB$.

$$\mathbf{m}_{O_1A}^T \mathbf{m}_{O_1B} = \left\|\mathbf{m}_{O_1A}\right\| \left\|\mathbf{m}_{O_1B}\right\| \cos\theta_1 \tag{1.45.38}$$

$$\frac{\mathbf{m}_{O_1A}^T \mathbf{m}_{O_1B}}{\left\|\mathbf{m}_{O_1A}\right\| \left\|\mathbf{m}_{O_1B}\right\|} = \cos\theta_1 \tag{1.45.39}$$

$$\frac{-2}{4} = \cos\theta_1 \tag{1.45.40}$$

$$\frac{-1}{2} = \cos\theta_1 \tag{1.45.41}$$

$$\theta_1 = 120^\circ \tag{1.45.42}$$

Now finding the angle $\angle O_2AB$.

$$\mathbf{m}_{O_2A}^T \mathbf{m}_{O_2B} = \left\|\mathbf{m}_{O_2A}\right\| \left\|\mathbf{m}_{O_2B}\right\| \cos\theta_2 \tag{1.45.43}$$

$$\frac{\mathbf{m}_{O_2A}^T \mathbf{m}_{O_2B}}{\left\|\mathbf{m}_{O_2A}\right\| \left\|\mathbf{m}_{O_2B}\right\|} = \cos\theta_2 \tag{1.45.44}$$

$$\frac{-2}{4} = \cos\theta_2 \tag{1.45.45}$$

$$\frac{-1}{2} = \cos\theta_2 \tag{1.45.46}$$

$$\theta_2 = 120^\circ \tag{1.45.47}$$

Finding area of $\mathbf{O_1AB}$ and $\mathbf{O_2AB}$.

$$A_{O_1AB} = \frac{\theta_1}{360} r^2 - \frac{1}{2} 2\sqrt{3} \tag{1.45.48}$$

$$= \frac{120}{360} 4\pi - \frac{1}{2} 2\sqrt{3} \tag{1.45.49}$$

$$A_{O_2AB} = \frac{\pi\theta_2}{360} r^2 - \frac{1}{2} 2\sqrt{3} \tag{1.45.50}$$

$$= \frac{120}{360} 4\pi - \frac{1}{2} 2\sqrt{3} \tag{1.45.51}$$

Area of $\mathbf{O_1AO_2B}$

$$A_{O_1AO_2B} = \frac{120}{360} 4\pi - \frac{1}{2} 2\sqrt{3} + \frac{120}{360} 4\pi - \frac{1}{2} 2\sqrt{3} \tag{1.45.52}$$

$$= \frac{8\pi}{3} - 2\sqrt{3} \tag{1.45.53}$$

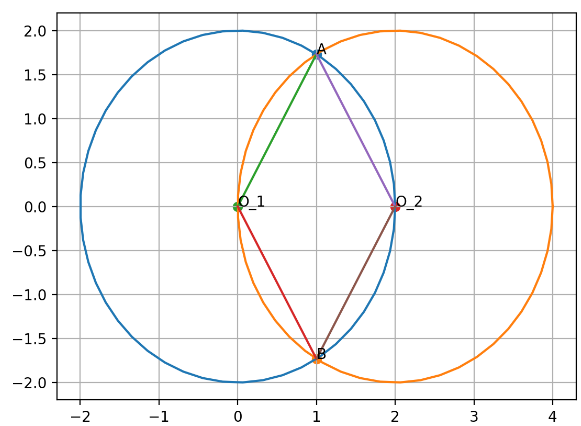

Fig. 1.45: Figure depicting intersection points of circle

1.46. Find the equation of the circle with radius 5 whose centre lies on x-axis and passes through the point $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

Solution:

Equation of the circle with radius r and centre(h,k) is given by,

$$x^T x + 2u^T x + f = 0 \tag{1.46.1}$$

where,

$$f = \mathbf{u}^T \mathbf{u} - r^2 \tag{1.46.2}$$

The radius and centre are respectively given by,

$$r = 5 \tag{1.46.3}$$
$$\mathbf{c} = -u = k\mathbf{e} \tag{1.46.4}$$

Where ,

$$\mathbf{e} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.46.5}$$
$$\mathbf{x_1} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \tag{1.46.6}$$

From the given data , we modify equation 1.46.1 as,

$$\mathbf{x_1}^T \mathbf{x_1} + 2\begin{pmatrix} -k & 0 \end{pmatrix}\begin{pmatrix} -k \\ 0 \end{pmatrix} + f = 0 \tag{1.46.7}$$
$$\|\mathbf{x_1}\|^2 + 2\left(k^2\right) + f = 0 \tag{1.46.8}$$
$$2k^2 + f = -\|\mathbf{x_1}\|^2 \tag{1.46.9}$$

Subsituting $\mathbf{u}$ in equation 1.46.2 , we get ,

$$f = \begin{pmatrix} -k & 0 \end{pmatrix}\begin{pmatrix} -k \\ 0 \end{pmatrix} - r^2 \tag{1.46.10}$$
$$f = \left(k^2\right) - r^2 \tag{1.46.11}$$
$$k^2 - f = r^2 \tag{1.46.12}$$

From equations 1.46.9 and 1.46.12,

$$\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} k^2 \\ f \end{pmatrix} = \begin{pmatrix} -\|\mathbf{x_1}\|^2 \\ r^2 \end{pmatrix} \tag{1.46.13}$$

Here ,$\|\mathbf{x_1}\|$ is given by ,

$$\|\mathbf{x_1}\| = \sqrt{2^2 + 3^2} \tag{1.46.14}$$
$$\|\mathbf{x_1}\| = \sqrt{13} \tag{1.46.15}$$

Subsituting equation 1.46.6,1.46.3 in equation 1.46.13 we get ,

$$\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} k^2 \\ f \end{pmatrix} = \begin{pmatrix} -13 \\ 25 \end{pmatrix} \tag{1.46.16}$$

The augumented matrix of 1.46.16 is given by ,

$$\left(\begin{array}{cc|c} 2 & 1 & -13 \\ 1 & -1 & 25 \end{array}\right) \tag{1.46.17}$$

By using row reduction technique, we get ,

$$\left(\begin{array}{cc|c} 2 & 1 & -13 \\ 1 & -1 & 25 \end{array}\right) \xleftrightarrow{R_2 \leftrightarrow R_1} \left(\begin{array}{cc|c} 1 & -1 & 25 \\ 2 & 1 & -13 \end{array}\right) \tag{1.46.18}$$

$$\left(\begin{array}{cc|c} 1 & -1 & 25 \\ 2 & 1 & -13 \end{array}\right) \xleftrightarrow{R_2 = R_2 - 2R_1} \left(\begin{array}{cc|c} 1 & -1 & 25 \\ 0 & 3 & -63 \end{array}\right) \tag{1.46.19}$$

$$\left(\begin{array}{cc|c} 1 & -1 & 25 \\ 0 & 3 & -63 \end{array}\right) \xleftrightarrow{R_2 = \frac{R_2}{3}} \left(\begin{array}{cc|c} 1 & -1 & 25 \\ 0 & 1 & -21 \end{array}\right) \tag{1.46.20}$$

$$\left(\begin{array}{cc|c} 1 & -1 & 25 \\ 0 & 1 & -21 \end{array}\right) \xleftrightarrow{R_1 = R_1 + R_2} \left(\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -21 \end{array}\right) \tag{1.46.21}$$

Equation 1.46.16 can we rewritten as ,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} k^2 \\ f \end{pmatrix} = \begin{pmatrix} 4 \\ -21 \end{pmatrix} \tag{1.46.22}$$

Expanding the above equation 1.46.22 we get ,

$$k^2 = 4 \tag{1.46.23}$$
$$k = \pm 2 \tag{1.46.24}$$
$$f = -21 \tag{1.46.25}$$

To get the centre substitute equation 1.46.24 in equation 1.46.4 To verify the above results we plot the circle with centre $\mathbf{c}$ as $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$,

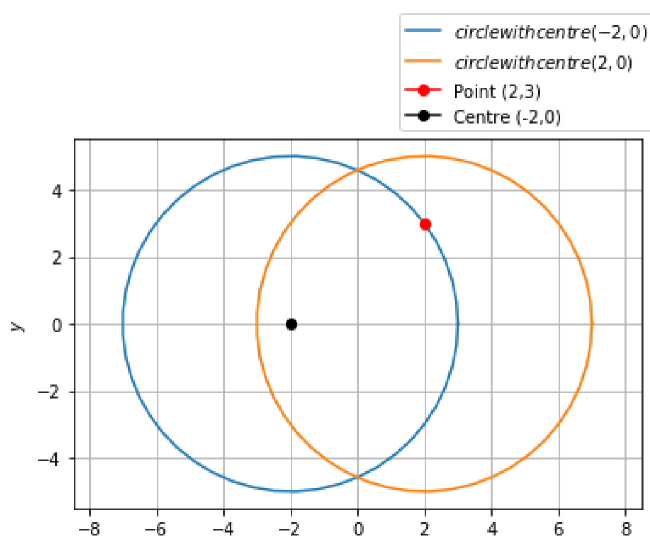

Fig. 1.46: Circle of radius 5 centre lies on x-axis and passing through the point(2,3)

qFrom the above figure 1.46 it is clear that circle with centre $\mathbf{c}=\begin{pmatrix}-2\\0\end{pmatrix}$ passes through the point $\mathbf{x_1}$
Desired equation of circle is given by ,

$$c = \begin{pmatrix}-2\\0\end{pmatrix} \tag{1.46.26}$$
$$f = -21 \tag{1.46.27}$$

1.47. Find the equation of a circle with centre $\begin{pmatrix}2\\2\end{pmatrix}$ and passes through the point $\begin{pmatrix}4\\5\end{pmatrix}$.
Solution: he general equation of a circle is

$$\mathbf{x}^T\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{1.47.1}$$
$$\text{If } r \text{ is radius, } f = \mathbf{u}^T\mathbf{u} - r^2 \tag{1.47.2}$$
$$\text{center } \mathbf{c} = -\mathbf{u} \tag{1.47.3}$$

Given centre is $\begin{pmatrix}2\\2\end{pmatrix}$

$$\implies \mathbf{c} = \begin{pmatrix}2\\2\end{pmatrix} \tag{1.47.4}$$
$$\implies \mathbf{u} = \begin{pmatrix}-2\\-2\end{pmatrix} \tag{1.47.5}$$

Equation (1.47.1) becomes

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}-4 & -4\end{pmatrix}\mathbf{x} + f = 0 \tag{1.47.6}$$

This passes through point $\begin{pmatrix}4\\5\end{pmatrix}$

Substituting $\mathbf{x} = \begin{pmatrix}4\\5\end{pmatrix}$ in (1.47.6)

$$\begin{pmatrix}4 & 5\end{pmatrix}\begin{pmatrix}4\\5\end{pmatrix} + \begin{pmatrix}-4 & -4\end{pmatrix}\begin{pmatrix}4\\5\end{pmatrix} + f = 0 \tag{1.47.7}$$
$$\implies f = -5 \tag{1.47.8}$$

Also, radius can be determined as follows

$$f = \mathbf{u}^T\mathbf{u} - r^2 \tag{1.47.9}$$
$$\implies -5 = \begin{pmatrix}-2 & -2\end{pmatrix}\begin{pmatrix}-2\\-2\end{pmatrix} - r^2 \tag{1.47.10}$$
$$\implies -5 = 8 - r^2 \tag{1.47.11}$$
$$\implies r = \sqrt{13} \tag{1.47.12}$$

The equation of required circle is

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}-4 & -4\end{pmatrix}\mathbf{x} - 5 = 0 \tag{1.47.13}$$

See Fig. 1.47

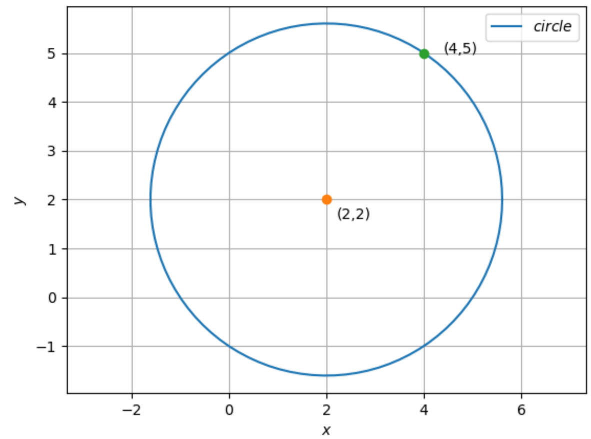

Fig. 1.47: plot showing the circle

1.48. Find the points on the curve $\mathbf{x}^T\mathbf{x} - 2\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} - 3 = 0$ at which the tangents are parallel to the x-axis.
Solution: General equation of circle is

$$\mathbf{x}^T\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{1.48.1}$$

The centre and the radius can be obtained as,

$$\mathbf{u} = \begin{pmatrix}-1\\0\end{pmatrix} \tag{1.48.2}$$
$$f = -3 \tag{1.48.3}$$
$$\mathbf{c} = -\mathbf{u} = \begin{pmatrix}1\\0\end{pmatrix} \tag{1.48.4}$$
$$r = \sqrt{\|\mathbf{u}\|^2 - f} = 2 \tag{1.48.5}$$

$\because$ The tangents are parallel to the x-axis, their direction and normal vectors, $\mathbf{m}$ and $\mathbf{n}$ are respectively,

$$\mathbf{m} = \begin{pmatrix}1\\0\end{pmatrix} \tag{1.48.6}$$
$$\mathbf{n} = \begin{pmatrix}0\\1\end{pmatrix} \tag{1.48.7}$$

For a circle, given the normal vector $\mathbf{n}$, the tangent points of contact to circle given by equation (1.48.1) are given by

$$\mathbf{q_i} = (\kappa_i\mathbf{n} - \mathbf{u}), i = 1, 2 \tag{1.48.8}$$

where

$$\kappa_i = \pm\sqrt{\frac{\mathbf{u^T u} - f}{\mathbf{n^T n}}} \tag{1.48.9}$$

$$\kappa = \pm\sqrt{\frac{\begin{pmatrix}-1 & 0\end{pmatrix}\begin{pmatrix}-1\\0\end{pmatrix} - (-3)}{\begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}}} \tag{1.48.10}$$

$$\implies \kappa = \pm\sqrt{\frac{4}{1}} \tag{1.48.11}$$

$$\implies \kappa = \pm 2 \tag{1.48.12}$$

and from (1.48.8) , the point of contact $\mathbf{q_i}$ are,

$$\mathbf{q_1} = 2\begin{pmatrix}0\\1\end{pmatrix} - \begin{pmatrix}-1\\0\end{pmatrix} \tag{1.48.13}$$

$$= \begin{pmatrix}1\\2\end{pmatrix} \tag{1.48.14}$$

$$\mathbf{q_2} = -2\begin{pmatrix}0\\1\end{pmatrix} - \begin{pmatrix}-1\\0\end{pmatrix} \tag{1.48.15}$$

$$= \begin{pmatrix}1\\-2\end{pmatrix} \tag{1.48.16}$$

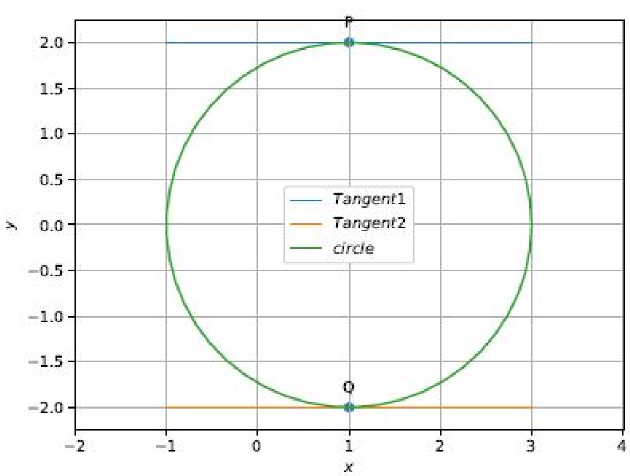

Fig. 1.48: Figure depicting tangents of circle parallel to x-axis

1.49. Find the area of the region in the first quadrant enclosed by x-axis, line $\begin{pmatrix}1 & -\sqrt{3}\end{pmatrix}\mathbf{x} = 0$ and the circle $\mathbf{x}^T\mathbf{x} = 4$.

Solution: The equation of a circle can be expressed as,

$$\mathbf{x}^T\mathbf{x} - 2\mathbf{c}^T\mathbf{x} + f = 0 \tag{1.49.1}$$

where $\mathbf{c}$ is the center.
Comparing equation (1.49.1) with the circle equation given,

$$\mathbf{x}^T\mathbf{x} = 4 \tag{1.49.2}$$

$$\implies \mathbf{c} = \begin{pmatrix}0\\0\end{pmatrix} \quad f = -4 \tag{1.49.3}$$

$$r = \sqrt{\mathbf{c}^T\mathbf{c} - f} = \sqrt{4} \tag{1.49.4}$$

$$\implies \boxed{r = 2} \tag{1.49.5}$$

From equation (1.49.5), the point at which circle touches x-axis is $\begin{pmatrix}2\\0\end{pmatrix}$.

The direction vector of x-axis is $\begin{pmatrix}1\\0\end{pmatrix}$.
The direction vector of the given line $\begin{pmatrix}1 & -\sqrt{3}\end{pmatrix}\mathbf{x} = 0$ is $\begin{pmatrix}\sqrt{3}\\1\end{pmatrix}$.
The angle that the line makes with the x-axis is given by,

$$\cos\theta = \frac{\begin{pmatrix}\sqrt{3} & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}}{\left\|\begin{pmatrix}\sqrt{3} & 1\end{pmatrix}\right\|\left\|\begin{pmatrix}1 & 0\end{pmatrix}\right\|} = \frac{\sqrt{3}}{2} \tag{1.49.6}$$

$$\implies \boxed{\theta = 30^\circ} \tag{1.49.7}$$

Using equation (1.49.5) and (1.49.7),the area of the sector is obtained as,

$$\implies \boxed{\frac{\theta}{360^\circ}\pi r^2 = \frac{30^\circ}{360^\circ}\pi(2)^2 = \frac{\pi}{3}} \tag{1.49.8}$$

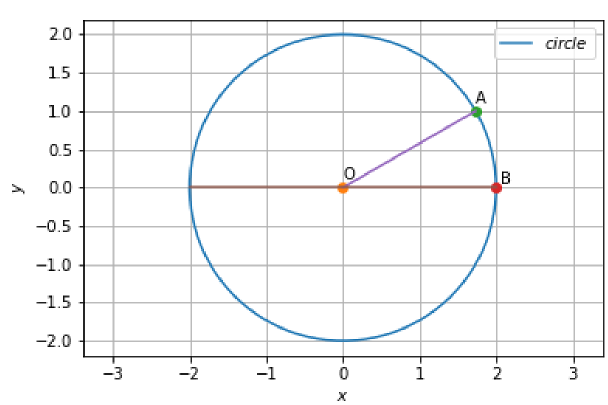

Fig. 1.49: Region enclosed by x-axis, line and circle

To find points **A** and **B**,

The parametric form of x-axis is,

$$\mathbf{B} = \mathbf{q} + \lambda \mathbf{m} \tag{1.49.9}$$
$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.49.10}$$

From the intersection of circle and line, the value of λ can be found by,

$$\lambda^2 = \frac{-f_1 - \|\mathbf{q}\|^2}{\|\mathbf{m}\|^2} \tag{1.49.11}$$
$$= \frac{4-0}{1} = 4 \tag{1.49.12}$$
$$\implies \lambda = \pm 2 \tag{1.49.13}$$

Sub equation (1.49.13) in (1.49.10),

$$\mathbf{B} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \tag{1.49.14}$$

As given in question as first quadrant,

$$\implies \boxed{\mathbf{B} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}} \tag{1.49.15}$$

Similarly, to find point $\mathbf{A}$, The parametric form of line is,

$$\mathbf{A} = \mathbf{q} + \lambda \mathbf{m} \tag{1.49.16}$$
$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} \tag{1.49.17}$$
$$\lambda^2 = \frac{-f_1 - \|\mathbf{q}\|^2}{\|\mathbf{m}\|^2} \tag{1.49.18}$$
$$= \frac{4-0}{4} = 1 \tag{1.49.19}$$
$$\implies \lambda = \pm 1 \tag{1.49.20}$$
$$\mathbf{A} = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} \quad \mathbf{A} = \begin{pmatrix} -\sqrt{3} \\ -1 \end{pmatrix} \tag{1.49.21}$$
$$\implies \boxed{\mathbf{A} = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}} \tag{1.49.22}$$

1.50. Find the area bounded by curves $\left\| \mathbf{x} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\| = 1$ and $\|\mathbf{x}\| = 1$

Solution:

General equation of circle is $\mathbf{x}^T\mathbf{x} + 2\mathbf{u}^T\mathbf{x}+$ f = 0

Taking equation of the first curve to be,

$$\left\| \mathbf{x} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\|^2 = 1^2 \tag{1.50.1}$$
$$\mathbf{x}^T\mathbf{x} + 2\mathbf{u_1}^T\mathbf{x} = 0 \tag{1.50.2}$$
$$\mathbf{u_1} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \tag{1.50.3}$$
$$f_1 = 0 \tag{1.50.4}$$
$$\mathbf{O_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.50.5}$$

Taking equation of the second curve to be,

$$\|\mathbf{x}\|^2 + 2\mathbf{u}_2^T\mathbf{x} + f_2 = 0 \tag{1.50.6}$$
$$\mathbf{x}^T\mathbf{x} - 1 = 0 \tag{1.50.7}$$
$$\mathbf{u_2} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1.50.8}$$
$$f_2 = -1 \tag{1.50.9}$$
$$\mathbf{O_2} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1.50.10}$$

Now, subtracting equation (1.50.2) from (1.50.7) We get,

$$\mathbf{x}^T\mathbf{x} + 2\mathbf{u_1}^T\mathbf{x} - \mathbf{x}^T\mathbf{x} - f_2 = 0 \tag{1.50.11}$$
$$2\mathbf{u}^T\mathbf{x} = -1 \tag{1.50.12}$$
$$\begin{pmatrix} -2 & 0 \end{pmatrix}\mathbf{x} = -1 \tag{1.50.13}$$

which can be written as:-

$$\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{x} = 1/2 \tag{1.50.14}$$
$$\mathbf{x} = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.50.15}$$
$$\mathbf{x} = \mathbf{q} + \lambda \mathbf{m} \tag{1.50.16}$$
$$\mathbf{q} = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} \tag{1.50.17}$$
$$\mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.50.18}$$

Substituting (1.50.16) in (1.50.6)

$$\|\mathbf{x}\|^2 + 2\mathbf{u}_2^T\mathbf{x} + f_2 = 0 \quad (1.50.19)$$

$$\|\mathbf{q} + \lambda\mathbf{m}\|^2 + f_2 = 0 \quad (1.50.20)$$

$$(\mathbf{q} + \lambda\mathbf{m})^T(\mathbf{q} + \lambda\mathbf{m}) + f_2 = 0 \quad (1.50.21)$$

$$\mathbf{q}^T(\mathbf{q} + \lambda\mathbf{m}) + \lambda\mathbf{m}^T(\mathbf{q} + \lambda\mathbf{m}) + f_2 = 0 \quad (1.50.22)$$

$$\|\mathbf{q}\|^2 + \lambda\mathbf{q}^T\mathbf{m} + \lambda\mathbf{m}^T\mathbf{q} + \lambda^2\|\mathbf{m}\|^2 + f_2 = 0 \quad (1.50.23)$$

$$\|\mathbf{q}\|^2 + 2\lambda\mathbf{q}^T\mathbf{m} + \lambda^2\|\mathbf{m}\|^2 + f_2 = 0 \quad (1.50.24)$$

Taking λ as common :

$$\lambda(\lambda\|\mathbf{m}\|^2 + 2\mathbf{q}^T\mathbf{m}) = -f_2 - \|\mathbf{q}\|^2 \quad (1.50.25)$$

$$\lambda^2\|\mathbf{m}\|^2 = -f_2 - \|\mathbf{q}\|^2 \quad (1.50.26)$$

$$\lambda^2 = \frac{-f_2 - \|\mathbf{q}\|^2}{\|\mathbf{m}\|^2} \quad (1.50.27)$$

$$\lambda^2 = \frac{3}{4} \quad (1.50.28)$$

$$\lambda = +\sqrt{\frac{3}{4}}, -\sqrt{\frac{3}{4}} \quad (1.50.29)$$

$$\lambda = +\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2} \quad (1.50.30)$$

Substituting the value of λ in (1.50.16)

$$\mathbf{x} = \mathbf{q} + \lambda\mathbf{m} \quad (1.50.31)$$

$$\mathbf{A} = \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} \quad (1.50.32)$$

$$\mathbf{B} = \begin{pmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix} \quad (1.50.33)$$

Now finding the direction vector $\mathbf{m}_{O_1A}$, $\mathbf{m}_{O_1B}$,$\mathbf{m}_{O_2A}$ and $\mathbf{m}_{O_2B}$.

$$\mathbf{m}_{O_1A} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix} \quad (1.50.34)$$

$$\mathbf{m}_{O_1B} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} \quad (1.50.35)$$

$$\mathbf{m}_{O_2A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix} \quad (1.50.36)$$

$$\mathbf{m}_{O_2B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} \quad (1.50.37)$$

Now finding the angle $\angle O_1AB$.

$$\mathbf{m}_{O_1A}^T\mathbf{m}_{O_1B} = \left\|\mathbf{m}_{O_1A}\right\|\left\|\mathbf{m}_{O_1B}\right\|\cos\theta_1 \quad (1.50.38)$$

$$\frac{\mathbf{m}_{O_1A}^T\mathbf{m}_{O_1B}}{\left\|\mathbf{m}_{O_1A}\right\|\left\|\mathbf{m}_{O_1B}\right\|} = \cos\theta_1 \quad (1.50.39)$$

$$\frac{-2}{4} = \cos\theta_1 \quad (1.50.40)$$

$$\frac{-1}{2} = \cos\theta_1 \quad (1.50.41)$$

$$\theta_1 = 120° \quad (1.50.42)$$

Now finding the angle $\angle O_2AB$.

$$\mathbf{m}_{O_2A}^T\mathbf{m}_{O_2B} = \left\|\mathbf{m}_{O_2A}\right\|\left\|\mathbf{m}_{O_2B}\right\|\cos\theta_2 \quad (1.50.43)$$

$$\frac{\mathbf{m}_{O_2A}^T\mathbf{m}_{O_2B}}{\left\|\mathbf{m}_{O_2A}\right\|\left\|\mathbf{m}_{O_2B}\right\|} = \cos\theta_2 \quad (1.50.44)$$

$$\frac{-2}{4} = \cos\theta_2 \quad (1.50.45)$$

$$\frac{-1}{2} = \cos\theta_2 \quad (1.50.46)$$

$$\theta_2 = 120° \quad (1.50.47)$$

Finding area of $\mathbf{O_1AB}$ and $\mathbf{O_2AB}$.

$$A_{O_1AB} = \frac{\pi\theta_1}{360}r^2 - \frac{1}{2}2\sqrt{3} \quad (1.50.48)$$

$$= \frac{120}{360}\pi - \frac{1}{2}2\sqrt{3} \quad (1.50.49)$$

$$A_{O_2AB} = \frac{\pi\theta_2}{360}r^2 - \frac{1}{2}2\sqrt{3} \quad (1.50.50)$$

$$= \frac{120}{360}\pi - \frac{1}{2}2\sqrt{3} \quad (1.50.51)$$

Area of $\mathbf{O_1AO_2B}$

$$A_{O_1AO_2B} = \frac{120}{360}\pi - \frac{1}{2}2\sqrt{3} + \frac{120}{360}\pi - \frac{1}{2}2\sqrt{3} \quad (1.50.52)$$

$$= \frac{2\pi}{3} - 2\sqrt{3} \quad (1.50.53)$$

1.51. Find the smaller area enclosed by the circle $\mathbf{x}^T\mathbf{x} = 4$ and the line $\begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} = 2$.
Solution:
Find the smaller area enclosed by the circle $\mathbf{x}\mathbf{x}^T = 4$ and the line $\begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} = 2$. General

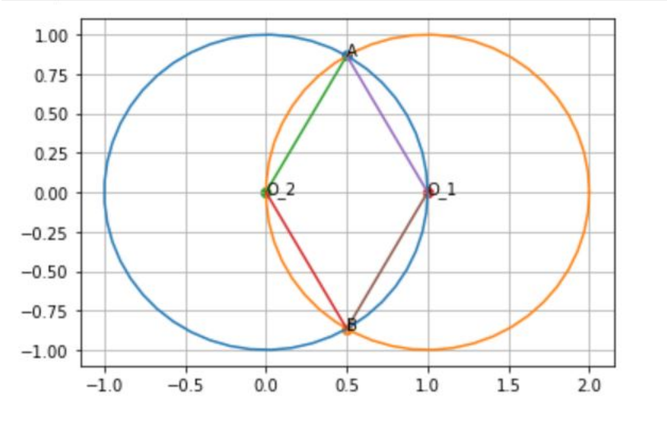

Fig. 1.50: Figure depicting intersection points of circle

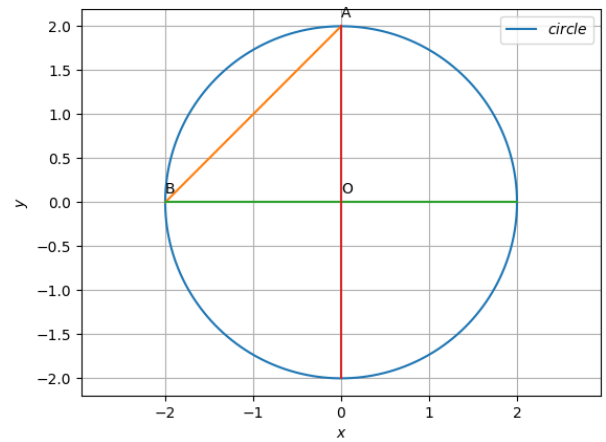

Fig. 1.51: Smaller area enclosed by line and circle

equation of circle is

$$\mathbf{x}^T\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{1.51.1}$$

$$\|\mathbf{x}\|^2 + 2\mathbf{u}_1^T\mathbf{x} + f_1 = 0 \tag{1.51.2}$$

$$\mathbf{x}^T\mathbf{x} - 4 = 0 \tag{1.51.3}$$

$$\mathbf{u_1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1.51.4}$$

$$f_1 = -4 \tag{1.51.5}$$

$$\mathbf{O_1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1.51.6}$$

$$r = \sqrt{\mathbf{c}^T\mathbf{c} - f} = \sqrt{4} \tag{1.51.7}$$

$$\implies \boxed{r = 2} \tag{1.51.8}$$

From equation (1.51.8), the point at which circle touches x-axis is $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$.
The direction vector of the given line $\begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} = 2$ is $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

To find point **A** and **B**, The parametric form of line is,

$$\mathbf{A} = \mathbf{q} + \lambda\mathbf{m} \tag{1.51.9}$$

$$= \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{1.51.10}$$

$$\lambda^2 = \frac{-f_1 - \|\mathbf{q}\|^2}{\|\mathbf{m}\|^2} \tag{1.51.11}$$

$$= \frac{4-2}{2} = 1 \tag{1.51.12}$$

$$\implies \lambda = \pm 1 \tag{1.51.13}$$

$$\mathbf{A} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \tag{1.51.14}$$

$$\mathbf{O} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1.51.15}$$

$$(\mathbf{A} - \mathbf{O}) = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \tag{1.51.16}$$

$$(\mathbf{B} - \mathbf{O}) = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \tag{1.51.17}$$

Inner product of $(\mathbf{A} - \mathbf{O})$ and $(\mathbf{B} - \mathbf{O})$ is given as:

$$(\mathbf{A} - \mathbf{O})^T(\mathbf{B} - \mathbf{O}) = 0 \tag{1.51.18}$$

Therefore, $(\mathbf{A} - \mathbf{O}) \perp (\mathbf{B} - \mathbf{O})$
Smaller area enclosed by circle and line **AB** is:
Area = (Area of circle in 2nd Quadrant) - (Area of right triangle formed by line AB, X and Y

axis)

$$Area = \frac{\pi\theta_1}{360}r^2 - \frac{1}{2} \times 2 \times 2 \tag{1.51.19}$$
$$= \frac{90}{360}\pi \times 2^2 - 2 \tag{1.51.20}$$
$$= \pi - 2 \tag{1.51.21}$$

Hence, the smaller area enclosed by the circle $\mathbf{x}\mathbf{x}^T = 4$ and the line $\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 2$ is $(\pi - 2)$

1.52. Find the slope of the tangent to the curve $y = \frac{x-1}{x-2}, x \neq 2$ at $x = 10$.
Solution:

$$y = \frac{x-1}{x-2} \tag{1.52.1}$$

Equation (1.52.1) can be expressed as

$$y(x-2) = x - 1 \tag{1.52.2}$$
$$yx - 2y - x + 1 = 0 \tag{1.52.3}$$

From above we can say,

$$\mathbf{V} = \frac{1}{2}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} \tag{1.52.4}$$
$$\mathbf{u} = \begin{pmatrix}-\frac{1}{2} & -1\end{pmatrix} \tag{1.52.5}$$
$$f = 1 \tag{1.52.6}$$

Now,

$$\because |V| = \begin{vmatrix}0 & \frac{1}{2}\\ \frac{1}{2} & 0\end{vmatrix} < 0, \tag{1.52.7}$$

(1.52.1) is the equation of a hyperbola. To verify that this we will find the the characteristic equation of $\mathbf{V}$.

$$\left|\lambda\mathbf{I} - \mathbf{V}\right| = \begin{vmatrix}\lambda & \frac{1}{2}\\ \frac{1}{2} & \lambda\end{vmatrix} = 0 \tag{1.52.8}$$
$$\implies \lambda^2 - 2\lambda + \frac{3}{4} = 0 \tag{1.52.9}$$

The eigenvalues are the roots of (1.52.9) given by

$$\lambda_1 = \frac{1}{2}, \lambda_2 = -\frac{1}{2} \tag{1.52.10}$$

The eigenvector $\mathbf{p}$ is defined as

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \tag{1.52.11}$$
$$\implies (\lambda\mathbf{I} - \mathbf{V})\,\mathbf{p} = 0 \tag{1.52.12}$$

where λ is the eigenvalue. For $\lambda_1 = \frac{1}{2}$,

$$(\lambda_1\mathbf{I} - \mathbf{V}) = \begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix} \xleftrightarrow[R_1 \leftarrow 2R_1]{R_2 \leftarrow R_2 - R_1} \begin{pmatrix}1 & 1\\0 & 0\end{pmatrix} \tag{1.52.13}$$
$$\implies \mathbf{p}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} \tag{1.52.14}$$

Now,λ is the eigenvalue. For $\lambda_2 = -\frac{1}{2}$,

$$(\lambda_2\mathbf{I} - \mathbf{V}) = \begin{pmatrix}-\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & -\frac{1}{2}\end{pmatrix} \xleftrightarrow[R_1 \leftarrow 2R_1]{R_2 \leftarrow R_2 + R_1} \begin{pmatrix}-1 & 1\\0 & 0\end{pmatrix} \tag{1.52.15}$$
$$\implies \mathbf{p}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \tag{1.52.16}$$

From Equations,

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1} = \mathbf{P}\mathbf{D}\mathbf{P}^T \quad \because \mathbf{P}^{-1} = \mathbf{P}^T \tag{1.52.17}$$
$$\text{or, } \mathbf{D} = \mathbf{P}^T\mathbf{V}\mathbf{P} \tag{1.52.18}$$

We can say that

$$\mathbf{P} = \begin{pmatrix}\mathbf{p}_1 & \mathbf{p}_2\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\1 & 1\end{pmatrix} \tag{1.52.19}$$
$$\mathbf{D} = \begin{pmatrix}\lambda_1 & 0\\0 & \lambda_2\end{pmatrix} = \begin{pmatrix}\frac{1}{2} & 0\\0 & -\frac{1}{2}\end{pmatrix} \tag{1.52.20}$$

$\because \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f > 0$,there isn't a need to swap axes. In hyperbola,

$$\mathbf{c} = -\mathbf{V}^-1\mathbf{u} \tag{1.52.21}$$
$$axes = \begin{cases}\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}}\\ \sqrt{\frac{f - \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}}{\lambda_2}}\end{cases} \tag{1.52.22}$$

From above equations we can say that,

$$\mathbf{c} = \begin{pmatrix}-2\\-1\end{pmatrix} \tag{1.52.23}$$
$$\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}} = \sqrt{2} \tag{1.52.24}$$
$$\sqrt{\frac{f - \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}}{\lambda_2}} = \sqrt{2} \tag{1.52.25}$$

with the standard hyperbola equation becoming

$$\frac{x^2}{2} - \frac{y^2}{2} = 1, \tag{1.52.26}$$

Let us assume slope to be l,now finding the direction vector and normal vector of the tangent with slope l.

$$\mathbf{m} = \begin{pmatrix} 1 \\ l \end{pmatrix} \tag{1.52.27}$$

$$\mathbf{n} = \begin{pmatrix} l \\ -1 \end{pmatrix} \tag{1.52.28}$$

Now considering the equations to find point of contact

$$\mathbf{q} = \mathbf{V}^{-1}(\kappa\mathbf{n} - \mathbf{u}) \tag{1.52.29}$$

$$\kappa = \pm\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\mathbf{n}^T\mathbf{V}^{-1}\mathbf{n}}} \tag{1.52.30}$$

By using (1.52.30)

$$\kappa = \sqrt{-\frac{1}{4l}} \tag{1.52.31}$$

Now substituting this κ in (1.52.29)

$$\mathbf{q} = \begin{pmatrix} -2\sqrt{-\frac{1}{4l}} + 2 \\ 2\sqrt{\frac{-l}{4}} + 1 \end{pmatrix} \tag{1.52.32}$$

We know that x=10.

$$-2\sqrt{-\frac{1}{4l}} + 2 = 10 \tag{1.52.33}$$

$$-2\sqrt{-\frac{1}{4l}} = 8 \tag{1.52.34}$$

$$\sqrt{-\frac{1}{4l}} = 4 \tag{1.52.35}$$

$$-\frac{1}{4l} = 16 \tag{1.52.36}$$

$$l = -\frac{1}{64} \tag{1.52.37}$$

The slope of the tangent to the curve y=$\frac{x-1}{x-2}$, x≠2 at x=10 is $\frac{1}{64}$. So,from the above we can say that κ=4,-4 and from equation (1.52.27) and (1.52.28) direction and normal vectors will come out to be

$$\mathbf{m} = \begin{pmatrix} 1 \\ -\frac{1}{64} \end{pmatrix} \tag{1.52.38}$$

$$\mathbf{n} = \begin{pmatrix} -\frac{1}{64} \\ -1 \end{pmatrix} \tag{1.52.39}$$

Now using equation (1.52.29)

$$\mathbf{q}_1 = \mathbf{V}^{-1}(\kappa_1\mathbf{n} - \mathbf{u}) \tag{1.52.40}$$

$$\mathbf{q}_1 = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}\left(-4\begin{pmatrix} -\frac{1}{64} \\ -1 \end{pmatrix} - \begin{pmatrix} -\frac{1}{2} \\ -1 \end{pmatrix}\right) \tag{1.52.41}$$

$$\mathbf{q}_1 = \begin{pmatrix} 10 \\ \frac{9}{8} \end{pmatrix} \tag{1.52.42}$$

$$\mathbf{q}_2 = \mathbf{V}^{-1}(\kappa_2\mathbf{n} - \mathbf{u}) \tag{1.52.43}$$

$$\mathbf{q}_2 = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}\left(4\begin{pmatrix} -\frac{1}{64} \\ -1 \end{pmatrix} - \begin{pmatrix} -\frac{1}{2} \\ -1 \end{pmatrix}\right) \tag{1.52.44}$$

$$\mathbf{q}_2 = \begin{pmatrix} -6 \\ \frac{7}{8} \end{pmatrix} \tag{1.52.45}$$

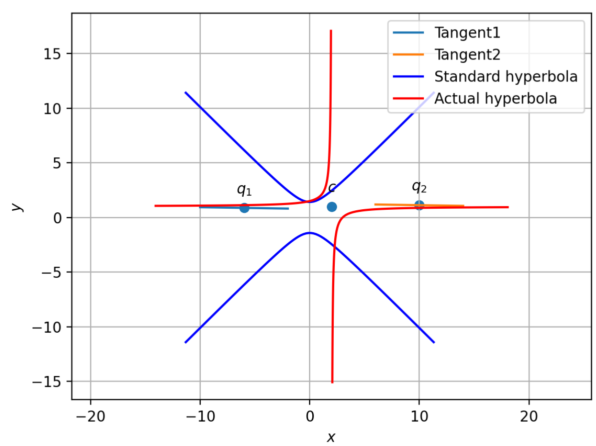

Fig. 1.52: Tangent 2 shows the tangent

1.53. Find a point on the curve $y = (x–2)^2$ at which the tangent is parallel to the chord joining the points $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 4 \end{pmatrix}$.

Solution: $y = (x-2)^2$ can be written as,

$$x^2 - 4x - y + 4 = 0 \tag{1.53.1}$$

From (1.53.1),

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}; \mathbf{u} = \begin{pmatrix} -2 \\ -\frac{1}{2} \end{pmatrix}; f = 4 \tag{1.53.2}$$

$$\left|V\right| = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0 \tag{1.53.3}$$

(1.53.3) implies that the curve is a parabola. Now, finding the eigen values corresponding to the $\mathbf{V}$,

$$\left|V - \lambda I\right| = 0$$

$$\begin{vmatrix} 1-\lambda & 0 \\ 0 & -\lambda \end{vmatrix} = 0$$

$$\implies \lambda = 0, 1 \tag{1.53.4}$$

Calculating the eigenvectors corresponding to $\lambda = 0, 1$ respectively,

$$\mathbf{V}\mathbf{x} = \lambda\mathbf{x}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\mathbf{x} = 0; \implies \mathbf{p}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.53.5}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}\mathbf{x} = 0; \implies \mathbf{p}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.53.6}$$

By Eigen decomposition on $\mathbf{V}$,

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T$$

$$where, \mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{1.53.7}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.53.8}$$

To find the vertex of the parabola,

$$\begin{pmatrix} \mathbf{u}^T + \eta\mathbf{p}_1^T \\ \mathbf{V} \end{pmatrix}\mathbf{c} = \begin{pmatrix} -f \\ \eta\mathbf{p}_1 - \mathbf{u} \end{pmatrix} \tag{1.53.9}$$

$$where, \eta = \mathbf{u}^T\mathbf{p}_1 = -\frac{1}{2} \tag{1.53.10}$$

Substituting values from (1.53.2), (1.53.5) and (1.53.10) in (1.53.9),

$$\begin{pmatrix} -2 & -1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}\mathbf{c} = \begin{pmatrix} -4 \\ 2 \\ 0 \end{pmatrix} \tag{1.53.11}$$

Removing last row and representing (1.53.11) as augmented matrix and then converting the matrix to echelon form,

$$\begin{pmatrix} -2 & -1 & -4 \\ 1 & 0 & 2 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{R_1}{-2}} \begin{pmatrix} 1 & \frac{1}{2} & 2 \\ 1 & 0 & 2 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - R_1}$$

$$\begin{pmatrix} 1 & \frac{1}{2} & 2 \\ 0 & -\frac{1}{2} & 0 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow (-2R_2)} \begin{pmatrix} 1 & \frac{1}{2} & 2 \\ 0 & 1 & 0 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 - \frac{R_2}{2}}$$

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix} \tag{1.53.12}$$

From (1.53.12) it can be observed that,

$$\mathbf{c} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \tag{1.53.13}$$

Direction vector of the chord joining A(4,4) and B(2,0) can be calculated as,

$$\mathbf{m} = \mathbf{A} - \mathbf{B} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\implies \mathbf{m} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{1.53.14}$$

We know that,

$$\mathbf{m}^T\mathbf{n} = 0; \implies \mathbf{n} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \tag{1.53.15}$$

To find the point of contact $\mathbf{q}$, which is intersection point for normal of the chord AB and also tangent of the curve,

$$\begin{pmatrix} \mathbf{u}^T + \kappa\mathbf{n}^T \\ \mathbf{V} \end{pmatrix}\mathbf{q} = \begin{pmatrix} -f \\ \kappa\mathbf{n} - \mathbf{u} \end{pmatrix} \tag{1.53.16}$$

$$where, \kappa = \frac{\mathbf{p_1}^T\mathbf{u}}{\mathbf{p_1}^T\mathbf{n}} = \frac{1}{2} \tag{1.53.17}$$

Substituting the values from (1.53.2),(1.53.15) and (1.53.17) in (1.53.16),

$$\begin{pmatrix} -1 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}\mathbf{q} = \begin{pmatrix} -4 \\ 3 \\ 0 \end{pmatrix} \tag{1.53.18}$$

Removing last row and representing (1.53.18) as augmented matrix and then converting the matrix to echelon form,

$$\begin{pmatrix} -1 & -1 & -4 \\ 1 & 0 & 3 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow (-R_1)} \begin{pmatrix} 1 & 1 & 4 \\ 1 & 0 & 3 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - R_1}$$

$$\begin{pmatrix} 1 & 1 & 4 \\ 0 & -1 & -1 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow (-R_2)} \begin{pmatrix} 1 & 1 & 4 \\ 0 & 1 & 1 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 - R_2}$$

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \end{pmatrix} \tag{1.53.19}$$

From (1.53.19), it can be observed,

$$\mathbf{q} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \tag{1.53.20}$$

which is the required point of contact

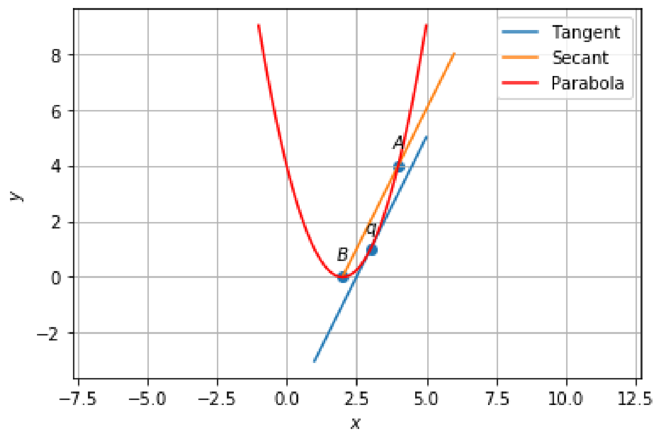

Fig. 1.53: Parabola with AB as chord, a tangent parallel to the chord

1.54. Find the equation of all lines having slope – 1 that are tangents to the curve $\frac{1}{x-1}, x \neq 1$

Solution: The given curve

$$y = \frac{1}{x-1} \tag{1.54.1}$$

can be expressed as

$$xy - y - 1 = 0 \tag{1.54.2}$$

Hence, we have

$$\mathbf{V} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \mathbf{u} = \frac{1}{2}\begin{pmatrix} 0 \\ -1 \end{pmatrix}, f = -1 \tag{1.54.3}$$

Since $\left|\mathbf{V}\right| < 0$, the equation (1.54.2) represents hyperbola. To find the values of λ_1 and λ_2, consider the characteristic equation,

$$\left|\lambda\mathbf{I} - \mathbf{V}\right| = 0 \tag{1.54.4}$$

$$\implies \left|\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}\right| = 0 \tag{1.54.5}$$

$$\implies \begin{vmatrix} \lambda & \frac{-1}{2} \\ \frac{-1}{2} & \lambda \end{vmatrix} = 0 \tag{1.54.6}$$

$$\implies \lambda_1 = \frac{1}{2}, \lambda_2 = \frac{-1}{2} \tag{1.54.7}$$

In addition, given the slope -1, the direction and normal vectors are given by

$$\mathbf{m} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{1.54.8}$$

$$\mathbf{n} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.54.9}$$

The parameters of hyperbola are as follows:

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \tag{1.54.10}$$

$$= -\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix} \tag{1.54.11}$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.54.12}$$

$$axes = \begin{cases} \sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}} = \sqrt{2} \\ \sqrt{\frac{f - \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}}{\lambda_2}} = \sqrt{2} \end{cases} \tag{1.54.13}$$

which represents the standard hyperbola equation,

$$\frac{x^2}{2} - \frac{x^2}{2} = 1 \tag{1.54.14}$$

The points of contact are given by

$$K = \pm\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\mathbf{n}^T\mathbf{V}^{-1}\mathbf{n}}} = \pm\frac{1}{2} \tag{1.54.15}$$

$$\mathbf{q} = \mathbf{V}^{-1}(k\mathbf{n} - \mathbf{u}) \tag{1.54.16}$$

$$\mathbf{q_1} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}\left[\frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ \frac{-1}{2} \end{pmatrix}\right] \tag{1.54.17}$$

$$= \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{1.54.18}$$

$$\mathbf{q_2} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}\left[\frac{-1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ \frac{-1}{2} \end{pmatrix}\right] \tag{1.54.19}$$

$$= \begin{pmatrix} 0 \\ -1 \end{pmatrix} \tag{1.54.20}$$

$\therefore$ The tangents are given by

$$\begin{pmatrix} 1 & 1 \end{pmatrix}\left(\mathbf{x} - \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right) = 0 \tag{1.54.21}$$

$$\begin{pmatrix} 1 & 1 \end{pmatrix}\left(\mathbf{x} - \begin{pmatrix} 0 \\ -1 \end{pmatrix}\right) = 0 \tag{1.54.22}$$

The desired equations of all lines having slope -1 that are tangents to the curve $\frac{1}{x-1}, x \neq 1$ are given by

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 3 \tag{1.54.23}$$

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = -1 \tag{1.54.24}$$

The above results are verified in the following figure.

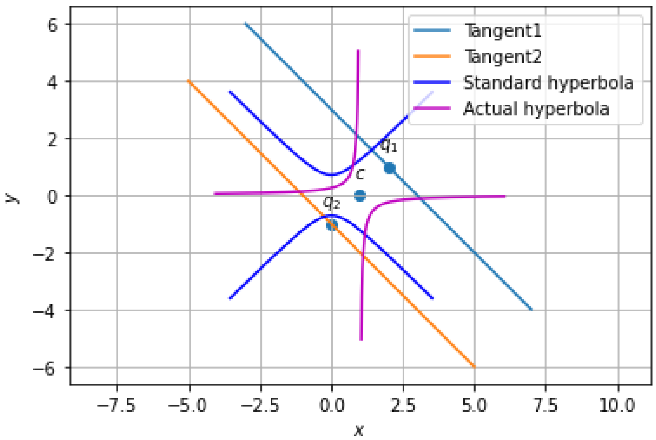

Fig. 1.54: The standard and actual hyperbola.

1.55. Find the equation of all lines having slope -2 which are tangents to the curve $\frac{1}{x-3}, x \neq 3$.
Solution: Given the curve,

$$y = \frac{1}{x-3} \tag{1.55.1}$$

$$\implies xy - 3y - 1 = 0 \tag{1.55.2}$$

From (1.55.2) we get,

$$\mathbf{V} = \frac{1}{2}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}, \mathbf{u} = \frac{-3}{2}\begin{pmatrix}0\\1\end{pmatrix}, f = -1 \tag{1.55.3}$$

Now,

$$\because \left|V\right| = \begin{vmatrix}0 & \frac{1}{2}\\ \frac{1}{2} & 0\end{vmatrix} = \frac{-1}{2} < 0 \tag{1.55.4}$$

(1.55.1) is equation of hyperbola. Now,

$$\left|\lambda\mathbf{I} - \mathbf{V}\right| = \begin{vmatrix}\lambda & \frac{-1}{2}\\ \frac{-1}{2} & \lambda\end{vmatrix} = 0 \tag{1.55.5}$$

$$\implies \lambda^2 - \frac{1}{4} = 0 \tag{1.55.6}$$

Thus the eigen values are,

$$\lambda_1 = \frac{1}{2}, \lambda_2 = \frac{-1}{2} \tag{1.55.7}$$

The eigen vector $\mathbf{p}$ is given by,

$$(\lambda\mathbf{I} - \mathbf{V})\mathbf{p} = 0 \tag{1.55.8}$$

For $\lambda_1 = \frac{1}{2}$,

$$(\lambda_1\mathbf{I} - \mathbf{V}) = \begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix} \xleftrightarrow[R_1 \leftarrow 2R_1]{R_2 \leftarrow R_2 + R_1} \begin{pmatrix}1 & -1\\0 & 0\end{pmatrix} \tag{1.55.9}$$

$$\implies \mathbf{p_1} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \tag{1.55.10}$$

Similarly for λ_2,

$$(\lambda_2\mathbf{I} - \mathbf{V}) = \begin{pmatrix}\frac{-1}{2} & \frac{-1}{2}\\ \frac{-1}{2} & \frac{-1}{2}\end{pmatrix} \xleftrightarrow[R_1 \leftarrow 2R_1]{R_2 \leftarrow R_{-}R_1} \begin{pmatrix}1 & 1\\0 & 0\end{pmatrix} \tag{1.55.11}$$

$$\implies \mathbf{p_2} = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\end{pmatrix} \tag{1.55.12}$$

Now,

$$\mathbf{P} = \begin{pmatrix}\mathbf{p_1} & \mathbf{p_2}\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\1 & 1\end{pmatrix} \tag{1.55.13}$$

$$\mathbf{D} = \begin{pmatrix}\frac{1}{2} & 0\\0 & \frac{-1}{2}\end{pmatrix} \tag{1.55.14}$$

$$\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f = 1 \tag{1.55.15}$$

$\because \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f = 1 > 0$, there is no need to swap the axes. The hyperbola parameters are,

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} = 3\begin{pmatrix}1\\0\end{pmatrix} \tag{1.55.16}$$

$$\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}} = \sqrt{2} \tag{1.55.17}$$

$$\sqrt{\frac{f - \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}}{\lambda_1}} = \sqrt{2} \tag{1.55.18}$$

with the standard hyperbola becoming,

$$\frac{x^2}{2} - \frac{y^2}{2} = 1 \tag{1.55.19}$$

The direction and normal vectors of the tangent with slope -2 are given as,

$$\mathbf{m} = \begin{pmatrix}1\\-2\end{pmatrix}, \mathbf{n} = \begin{pmatrix}2\\1\end{pmatrix} \tag{1.55.20}$$

Now considering the equations to find the point of contact,

$$\mathbf{q} = \mathbf{V}^{-1}(k\mathbf{n} - \mathbf{u}) \tag{1.55.21}$$

$$k = \pm\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\mathbf{n}^T\mathbf{V}^{-1}\mathbf{n}}} \tag{1.55.22}$$

Thus,

$$\mathbf{n}^T\mathbf{V}^{-1}\mathbf{n} = 8 \tag{1.55.23}$$

$$k = \pm\frac{1}{2\sqrt{2}} \tag{1.55.24}$$

$$\mathbf{q_1} = \begin{pmatrix} \frac{1+3\sqrt{2}}{\sqrt{2}} \\ \sqrt{2} \end{pmatrix} \tag{1.55.25}$$

$$\mathbf{q_2} = \begin{pmatrix} \frac{-1+3\sqrt{2}}{\sqrt{2}} \\ -\sqrt{2} \end{pmatrix} \tag{1.55.26}$$

The desired tangents are,

$$\begin{pmatrix} 2 & 1 \end{pmatrix}\left\{\mathbf{x} - \begin{pmatrix} \frac{1+3\sqrt{2}}{\sqrt{2}} \\ \sqrt{2} \end{pmatrix}\right\} = 0 \tag{1.55.27}$$

$$\implies \begin{pmatrix} 2 & 1 \end{pmatrix}\mathbf{x} = 6 + 2\sqrt{2} \tag{1.55.28}$$

$$\begin{pmatrix} 2 & 1 \end{pmatrix}\left\{\mathbf{x} - \begin{pmatrix} \frac{-1+3\sqrt{2}}{\sqrt{2}} \\ -\sqrt{2} \end{pmatrix}\right\} = 0 \tag{1.55.29}$$

$$\implies \begin{pmatrix} 2 & 1 \end{pmatrix}\mathbf{x} = 6 - 2\sqrt{2} \tag{1.55.30}$$

Below figure corresponds to the tangents on the hyperbola, represented by (1.55.28) and (1.55.30) each having slope of -2.

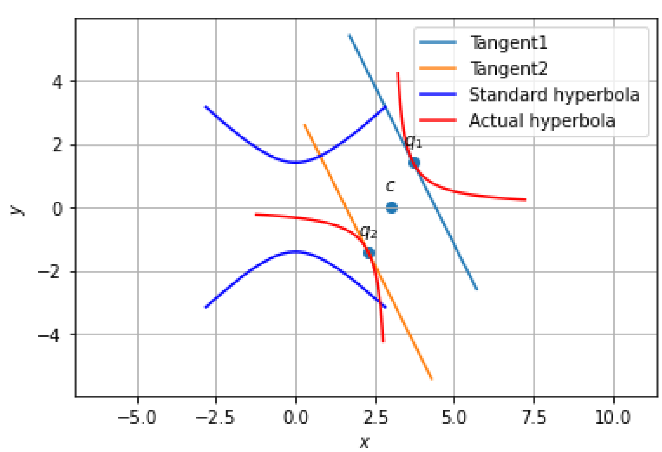

Fig. 1.55: Tangents to the hyperbola

1.56. Find points on the curve $\mathbf{x}^T\begin{pmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{16} \end{pmatrix}\mathbf{x} = 1$ at which tangents are

a) parallel to x-axis

b) parallel to y-axis.

Solution:

General equation of conics is

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{1.56.1}$$

Comparing with the equation given,

$$\mathbf{V} = \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{16} \end{pmatrix} \tag{1.56.2}$$

$$\mathbf{u} = \mathbf{0} \tag{1.56.3}$$

$$f = -1 \tag{1.56.4}$$

$$\left|\mathbf{v}\right| = \left|\begin{pmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{16} \end{pmatrix}\right| > 0 \tag{1.56.5}$$

$\because$ $|\mathbf{V}| > 0$, the given equation is of ellipse.

a)The tangents are parallel to the x-axis, hence, their direction and normal vectors, $\mathbf{m_1}$ and $\mathbf{n_1}$ are respectively,

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.56.6}$$

$$\mathbf{n_1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.56.7}$$

For an ellipse, given the normal vector $\mathbf{n}$, the tangent points of contact to the ellipse are given by

$$\mathbf{q} = \mathbf{V}^{-1}(\kappa\mathbf{n} - \mathbf{u}) = \mathbf{V}^{-1}\kappa\mathbf{n} \tag{1.56.8}$$

where

$$\kappa = \pm\sqrt{\frac{\mathbf{u^T}\mathbf{V}^{-1}\mathbf{u} - f}{\mathbf{n^T}\mathbf{V}^{-1}\mathbf{n}}} \tag{1.56.9}$$

$$= \pm\sqrt{\frac{-f}{\mathbf{n^T}\mathbf{V}^{-1}\mathbf{n}}} \tag{1.56.10}$$

$$\mathbf{V}^{-1} = \begin{pmatrix} 9 & 0 \\ 0 & 16 \end{pmatrix} \tag{1.56.11}$$

$$\kappa_1 = \pm\sqrt{\frac{-(-1)}{\begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} 9 & 0 \\ 0 & 16 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}}} \tag{1.56.12}$$

$$\implies \kappa_1 = \pm\sqrt{\frac{1}{16}} \tag{1.56.13}$$

$$\implies \kappa_1 = \pm\frac{1}{4} \tag{1.56.14}$$

From (1.56.8) , the point of contact $\mathbf{q_i}$ are,

$$\mathbf{q_1} = \begin{pmatrix} 9 & 0 \\ 0 & 16 \end{pmatrix} \frac{1}{4} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (1.56.15)$$

$$= \begin{pmatrix} 9 & 0 \\ 0 & 16 \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{4} \end{pmatrix} \quad (1.56.16)$$

$$= \begin{pmatrix} 0 \\ 4 \end{pmatrix} \quad (1.56.17)$$

$$\mathbf{q_2} = \begin{pmatrix} 9 & 0 \\ 0 & 16 \end{pmatrix} \left(-\frac{1}{4}\right) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (1.56.18)$$

$$= \begin{pmatrix} 9 & 0 \\ 0 & 16 \end{pmatrix} \begin{pmatrix} 0 \\ -\frac{1}{4} \end{pmatrix} \quad (1.56.19)$$

$$= \begin{pmatrix} 0 \\ -4 \end{pmatrix} \quad (1.56.20)$$

b) The tangents are parallel to the y-axis, hence, their direction and normal vectors, $\mathbf{m_2}$ and $\mathbf{n_2}$ are respectively,

$$\mathbf{m_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (1.56.21)$$

$$\mathbf{n_2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.56.22)$$

Using equation (1.56.9), the values of κ for this case are

$$\kappa_2 = \pm \sqrt{\frac{-(-1)}{\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 9 & 0 \\ 0 & 16 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}}} \quad (1.56.23)$$

$$\implies \kappa_2 = \pm \sqrt{\frac{1}{9}} \quad (1.56.24)$$

$$\implies \kappa_2 = \pm \frac{1}{3} \quad (1.56.25)$$

and from (1.56.8) , the point of contact $\mathbf{q_i}$ are,

$$\mathbf{q_3} = \begin{pmatrix} 9 & 0 \\ 0 & 16 \end{pmatrix} \frac{1}{3} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.56.26)$$

$$= \begin{pmatrix} 9 & 0 \\ 0 & 16 \end{pmatrix} \begin{pmatrix} \frac{1}{3} \\ 0 \end{pmatrix} \quad (1.56.27)$$

$$= \begin{pmatrix} 3 \\ 0 \end{pmatrix} \quad (1.56.28)$$

$$\mathbf{q_4} = \begin{pmatrix} 9 & 0 \\ 0 & 16 \end{pmatrix} \left(-\frac{1}{3}\right) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.56.29)$$

$$= \begin{pmatrix} 9 & 0 \\ 0 & 16 \end{pmatrix} \begin{pmatrix} -\frac{1}{3} \\ 0 \end{pmatrix} \quad (1.56.30)$$

$$= \begin{pmatrix} -3 \\ 0 \end{pmatrix} \quad (1.56.31)$$

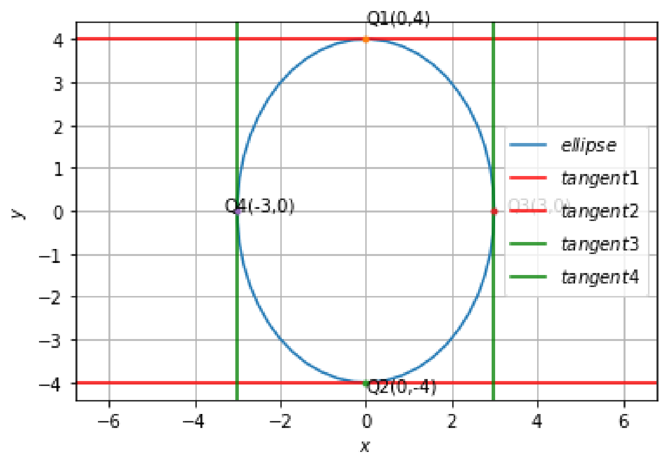

Fig. 1.56: Figure depicting point of contact of tangents of ellipse parallel to x-axis and y-axis

1.57. Find the equation of the tangent line to the curve $y = x^2 - 2x + 7$

a) parallel to the line $\begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} = -9$

b) perpendicular to the line $\begin{pmatrix} -15 & 5 \end{pmatrix} \mathbf{x} = 13$.

Solution:

Given equation

$$y = x^2 - 2x + 7 \quad (1.57.1)$$

The equation (1.57.1) can be written as,

$$x^2 - 2x - y + 7 = 0 \quad (1.57.2)$$

Comparing it with standard equation,

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \quad (1.57.3)$$

$\therefore$ a = 1, b = 0, c = 0, d = -1, e = $\frac{-1}{2}$, f = 7.

$$\therefore \mathbf{V} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{1.57.4}$$

$$\therefore \mathbf{u} = \begin{pmatrix} d \\ e \end{pmatrix} = \begin{pmatrix} -1 \\ \frac{-1}{2} \end{pmatrix} \tag{1.57.5}$$

$$\text{Now,} \left|V\right| = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0 \tag{1.57.6}$$

$\implies$ that the curve is a parabola. Now, finding the eigen values corresponding to the $\mathbf{V}$,

$$\left|\mathbf{V} - \lambda\mathbf{I}\right| = 0 \tag{1.57.7}$$

$$\begin{vmatrix} 1-\lambda & 0 \\ 0 & -\lambda \end{vmatrix} = 0 \tag{1.57.8}$$

$$\implies \lambda = 0, 1. \tag{1.57.9}$$

Calculating the eigenvectors corresponding to $\lambda = 0, 1$ respectively,

$$\mathbf{V}\mathbf{x} = \lambda\mathbf{x} \tag{1.57.10}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\mathbf{x} = 0 \implies \mathbf{p_1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.57.11}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}\mathbf{x} = \mathbf{x} \implies \mathbf{p_2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.57.12}$$

Now by eigen decomposition on $\mathbf{V}$,

$$\mathbf{V} = \mathbf{PDP^T} \tag{1.57.13}$$

$$\text{where,}\mathbf{P} = \begin{pmatrix} \mathbf{p_1}\mathbf{p_2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{1.57.14}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.57.15}$$

Hence equation (1.57.5) becomes,

$$\mathbf{V} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{1.57.16}$$

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{1.57.17}$$

a) The given parallel line equation is

$$\begin{pmatrix} 2 & -1 \end{pmatrix}\mathbf{x} = -9 \tag{1.57.18}$$

$$\implies 2x - y + 9 = 0 \tag{1.57.19}$$

Now the tangent to parabola is parallel to the line equation (1.57.19), the general straight line equation is of the form

$$ax + by + c = 0 \tag{1.57.20}$$

The normal vector ($\mathbf{n}$) and direction ($\mathbf{m}$) are given by,

$$\mathbf{n} = \begin{pmatrix} a \\ b \end{pmatrix} \tag{1.57.21}$$

$$\mathbf{m} = \begin{pmatrix} b \\ -a \end{pmatrix} \tag{1.57.22}$$

Comparing (1.57.19), (1.57.13), (1.57.21), the direction vectors ($\mathbf{m}$) and normal ($\mathbf{n}$) vectors are,

$$\mathbf{m} = \begin{pmatrix} -1 \\ -2 \end{pmatrix} \tag{1.57.23}$$

$$\mathbf{n} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \tag{1.57.24}$$

Now, the equation for the point of contact for the parabola is given as,

$$\begin{pmatrix} \mathbf{u}^T + \kappa\mathbf{n}^T \\ \mathbf{V} \end{pmatrix}\mathbf{q} = \begin{pmatrix} -f \\ \kappa\mathbf{n} - \mathbf{u} \end{pmatrix} \tag{1.57.25}$$

$$\text{where, } \kappa = \frac{\mathbf{p_1}^T\mathbf{u}}{\mathbf{p_1}^T\mathbf{n}} = \frac{1}{2} \tag{1.57.26}$$

Hence substituting the values of (1.57.5), (1.57.24), (1.57.13), (1.57.26) in equation (1.57.25) we get,

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}\mathbf{q} = \begin{pmatrix} -7 \\ 2 \\ 0 \end{pmatrix} \tag{1.57.27}$$

Solving for $\mathbf{q}$ by removing the zero row and representing (1.57.27) as augmented matrix and then converting the matrix to echelon form,

$$\implies \begin{pmatrix} 0 & -1 & -7 \\ 1 & 0 & 2 \end{pmatrix} \xleftrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & -7 \end{pmatrix} \tag{1.57.28}$$

$$\xleftrightarrow{R_2 \leftarrow -R_2} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 7 \end{pmatrix} \tag{1.57.29}$$

Hence from equation (1.57.29) it can be concluded that the point of contact is,

$$\mathbf{q} = \begin{pmatrix} 2 \\ 7 \end{pmatrix} \tag{1.57.30}$$

Now $\mathbf{q}$ is a point on the tangent. Hence, the equation of the line can be expressed as

$$\mathbf{n}^T \mathbf{x} = c \tag{1.57.31}$$

where c is,

$$c = \mathbf{n}^T \mathbf{q} = \begin{pmatrix} 2 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 7 \end{pmatrix} = -3 \tag{1.57.32}$$

Hence equation of tangent to the curve (1.57.1) parallel to (1.57.19) is given by substituting the value of c and $\mathbf{n}$ from equation (1.57.32) and (1.57.24) respectively to the equation (1.57.31),

$$\implies \begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} = -3 \tag{1.57.33}$$

Figure 1.57 verifies that the $\begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} = -3$ is a tangent to parabola $y = x^2 - 2x + 7$

b) The given perpendicular line equation is

$$\begin{pmatrix} -15 & 5 \end{pmatrix} \mathbf{x} = 13 \tag{1.57.34}$$

$$\implies -15x + 5y - 13 = 0 \tag{1.57.35}$$

Now the tangent to parabola is perpendicular to the line equation (1.57.35), the general straight line equation is of the form

$$ax + by + c = 0 \tag{1.57.36}$$

Therefore, if we find the line that is parallel to the line (1.57.35), it will be parallel to the tangent itself. For the given line the normal vector ($\mathbf{n}$) and direction ($\mathbf{m}$) are given by,

$$\mathbf{n} = \begin{pmatrix} a \\ b \end{pmatrix} \tag{1.57.37}$$

$$\mathbf{m} = \begin{pmatrix} b \\ -a \end{pmatrix} \tag{1.57.38}$$

Comparing (1.57.35), (1.57.37), (1.57.38), the direction vectors ($\mathbf{m}$) and normal ($\mathbf{n}$) vectors are,

$$\mathbf{m} = \begin{pmatrix} 5 \\ 15 \end{pmatrix} \tag{1.57.39}$$

$$\mathbf{n} = \begin{pmatrix} -15 \\ 5 \end{pmatrix} \tag{1.57.40}$$

The parallel line for this vector will have the normal vector ($\mathbf{n_1}$) and direction ($\mathbf{m_1}$) are given by

$$\mathbf{m_1} = \begin{pmatrix} 15 \\ -5 \end{pmatrix} \tag{1.57.41}$$

$$\mathbf{n_1} = \begin{pmatrix} 5 \\ 15 \end{pmatrix} \tag{1.57.42}$$

Now, the equation for the point of contact for the parabola is given as,

$$\begin{pmatrix} \mathbf{u}^T + \kappa \mathbf{n_1}^T \\ \mathbf{V} \end{pmatrix} \mathbf{q} = \begin{pmatrix} -f \\ \kappa \mathbf{n_1} - \mathbf{u} \end{pmatrix} \tag{1.57.43}$$

$$\text{where, } \kappa = \frac{\mathbf{p_1}^T \mathbf{u}}{\mathbf{p_1}^T \mathbf{n_1}} = \frac{-1}{30} \tag{1.57.44}$$

Hence substituting the values of (1.57.5), (1.57.42), (1.57.13), (1.57.44) in equation (1.57.43) we get,

$$\begin{pmatrix} \frac{-7}{6} & -1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{q} = \begin{pmatrix} -7 \\ \frac{5}{6} \\ 0 \end{pmatrix} \tag{1.57.45}$$

Solving for $\mathbf{q}$ by removing the zero row and representing (1.57.45) as augmented matrix and then converting the matrix to echelon form,

$$\implies \begin{pmatrix} \frac{-7}{6} & -1 & -7 \\ 1 & 0 & \frac{5}{6} \end{pmatrix} \xleftrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 0 & \frac{5}{6} \\ \frac{-7}{6} & -1 & -7 \end{pmatrix} \tag{1.57.46}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - \left(\frac{7}{6}\right) R_1} \begin{pmatrix} 1 & 0 & \frac{5}{6} \\ 0 & -1 & \frac{-217}{36} \end{pmatrix} \tag{1.57.47}$$

$$\xleftrightarrow{R_2 \leftarrow -R_2} \begin{pmatrix} 1 & 0 & \frac{5}{6} \\ 0 & 1 & \frac{217}{36} \end{pmatrix} \tag{1.57.48}$$

Hence from equation (1.57.48) it can be concluded that the point of contact is,

$$\mathbf{q} = \begin{pmatrix} \frac{5}{6} \\ \frac{217}{36} \end{pmatrix} \tag{1.57.49}$$

Now $\mathbf{q}$ is a point on the tangent. Hence, the

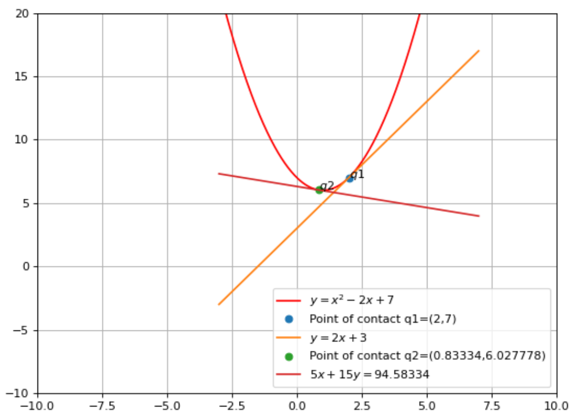

Fig. 1.57: Tangents to parabola $y = x^2 - 2x + 7$

equation of the line can be expressed as

$$\mathbf{n_1}^T\mathbf{x} = c \tag{1.57.50}$$

where c is,

$$c = \mathbf{n_1}^T\mathbf{q} = \begin{pmatrix} 5 & 15 \end{pmatrix}\begin{pmatrix} \frac{5}{6} \\ \frac{217}{36} \end{pmatrix} = \frac{3405}{36} \tag{1.57.51}$$

Hence equation of tangent to the curve (1.57.1) parallel to (1.57.35) is given by substituting the value of c and $\mathbf{n_1}$ from equation (1.57.51) and (1.57.42) respectively to the equation (1.57.50),

$$\implies \begin{pmatrix} 5 & 15 \end{pmatrix}\mathbf{x} = \frac{3405}{36} \tag{1.57.52}$$

Figure 1.57 verifies that the $\begin{pmatrix} 5 & 15 \end{pmatrix}\mathbf{x} = \frac{3405}{36}$ is a tangent to parabola $y = x^2 - 2x + 7$

1.58. Find the point at which the line $\begin{pmatrix} -1 & 1 \end{pmatrix}\mathbf{x} = 1$ is a tangent to the curve $y^2 = 4x$.

Solution: Comparing $y^2 = 4x$ to standard equation,

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{1.58.1}$$

$\therefore$ a = b = e = 0, d =-2, c = 1, f = 0.

$$\therefore \mathbf{V} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.58.2}$$

$$\therefore \mathbf{u} = \begin{pmatrix} d \\ e \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \tag{1.58.3}$$

$$\text{Now, } \left|V\right| = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0 \tag{1.58.4}$$

$\implies$ That the curve is a parabola.

$$\text{Since } \mathbf{Vp_1} = 0 \tag{1.58.5}$$

$$\therefore \mathbf{p_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.58.6}$$

Since the slope of the line is 1 The direction vector $\mathbf{m}$ is as follows:

$$\mathbf{m} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.58.7}$$

$$\text{Since } \mathbf{m}^T\mathbf{n} = 0 \tag{1.58.8}$$

$$\therefore \mathbf{n} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{1.58.9}$$

Now, the equation for the point of contact for the parabola is given as,

$$\begin{pmatrix} \mathbf{u}^T + \kappa\mathbf{n}^T \\ \mathbf{V} \end{pmatrix}\mathbf{q} = \begin{pmatrix} -f \\ \kappa\mathbf{n} - \mathbf{u} \end{pmatrix} \tag{1.58.10}$$

$$\text{where, } \kappa = \frac{\mathbf{p_1}^T\mathbf{u}}{\mathbf{p_1}^T\mathbf{n}} = -2 \tag{1.58.11}$$

By substituting the values ,we get:

$$\begin{pmatrix} -4 & 2 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{q} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \tag{1.58.12}$$

Solving for $\mathbf{q}$ by removing the zero row and representing (1.58.12) as augmented matrix and then converting the matrix to echelon form,

$$\implies \begin{pmatrix} -4 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \left(\frac{-R_1}{4}\right)} \begin{pmatrix} 1 & \frac{-1}{2} & 0 \\ 0 & 1 & 2 \end{pmatrix} \tag{1.58.13}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + \frac{1}{2}R_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} \tag{1.58.14}$$

Threrefore the point at which the line $\begin{pmatrix} -1 & 1 \end{pmatrix}\mathbf{x} = 1$ is a tangent to the curve $y^2 = 4x$ is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

1.59. Find the normal at the point $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ on the curve $2y + x^2 = 3$

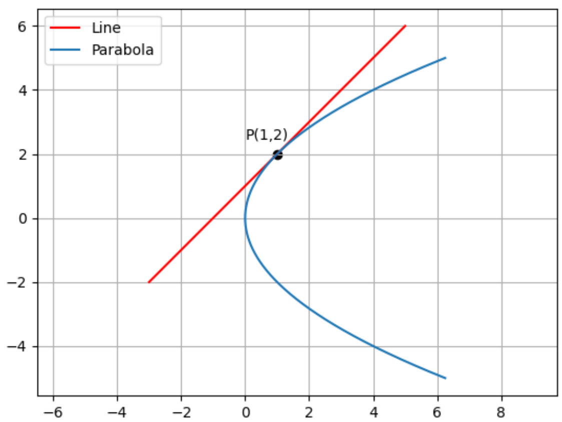

Fig. 1.58: Figure depicting the point at which the line is tangent to the parabola

Solution: Given,

$$x^2 + 2y - 3 = 0 \tag{1.59.1}$$

From (1.59.1),

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{1.59.2}$$

$$\mathbf{u} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.59.3}$$

$$f = -3 \tag{1.59.4}$$

From (1.59.2),

$$\left|V\right| = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0 \tag{1.59.5}$$

Now (1.59.5) implies that the curve is a parabola. We can find the Eigen values corresponding to the $\mathbf{V}$,

$$\left|V - \lambda I\right| = 0$$

$$\begin{vmatrix} 1-\lambda & 0 \\ 0 & -\lambda \end{vmatrix} = 0$$

$$\implies \lambda = 0, 1 \tag{1.59.6}$$

Calculating the Eigen Vectors corresponding to $\lambda = 0, 1$ respectively,

$$\mathbf{V}\mathbf{x} = \lambda\mathbf{x}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\mathbf{x} = 0; \implies \mathbf{p}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.59.7}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}\mathbf{x} = 0; \implies \mathbf{p}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.59.8}$$

By Eigen decomposition on $\mathbf{V}$,

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T$$

$$where, \mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{1.59.9}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.59.10}$$

To find the vertex of the parabola,

$$\begin{pmatrix} \mathbf{u}^T + \eta\mathbf{p}_1^T \\ \mathbf{V} \end{pmatrix}\mathbf{c} = \begin{pmatrix} -f \\ \eta\mathbf{p}_1 - \mathbf{u} \end{pmatrix} \tag{1.59.11}$$

$$where, \eta = \mathbf{u}^T\mathbf{p}_1 = 1 \tag{1.59.12}$$

Substituting values from (1.59.2), (1.59.7) and (1.59.12) in (1.59.11),

$$\begin{pmatrix} 0 & 2 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}\mathbf{c} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} \tag{1.59.13}$$

Removing last row and representing (1.59.13) as augmented matrix and then converting the matrix to echelon form,

$$\begin{pmatrix} 0 & 2 & 3 \\ 1 & 0 & 0 \end{pmatrix} \xleftrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & -3 \end{pmatrix} \tag{1.59.14}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & -3 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow -\frac{R_2}{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} \end{pmatrix} \tag{1.59.15}$$

From (1.59.15) it can be observed that,

$$\mathbf{c} = \begin{pmatrix} 0 \\ \frac{3}{2} \end{pmatrix} \tag{1.59.16}$$

Now to evaluate the direction vector $\mathbf{m}$,

$$\mathbf{m}^T(\mathbf{V}\mathbf{q} + \mathbf{u}) = 0 \tag{1.59.17}$$

$$\mathbf{m}^T\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = 0 \tag{1.59.18}$$

$$\mathbf{m}^T\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = 0 \tag{1.59.19}$$

$$\mathbf{m}^T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0 \tag{1.59.20}$$

$$\implies \mathbf{m} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{1.59.21}$$

Now to obtain the equation of normal using,

$$\mathbf{m}^T(\mathbf{x}-\mathbf{q}) = 0 \tag{1.59.22}$$

$$\begin{pmatrix}1 & -1\end{pmatrix}\left(\mathbf{x}-\begin{pmatrix}1\\1\end{pmatrix}\right) = 0 \tag{1.59.23}$$

$$\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = 0 \tag{1.59.24}$$

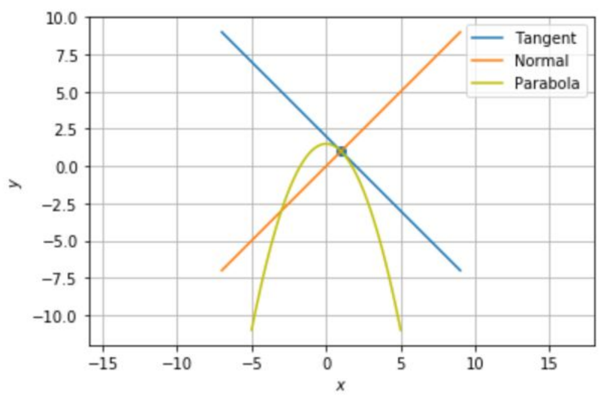

Fig. 1.59: Parabola showing tangent perpendicular to the normal

2 Exercises

2.1. Find the equation of the circle passing through $\begin{pmatrix}0\\0\end{pmatrix}$ and making intercepts a and b on the coordinate axes.
Solution: Let

$$\mathbf{A} = \begin{pmatrix}a\\0\end{pmatrix}, \mathbf{B} = \begin{pmatrix}b\\0\end{pmatrix}, \mathbf{C} = \begin{pmatrix}0\\0\end{pmatrix} \tag{2.1.1}$$

Since the circle passes through $\begin{pmatrix}0\\0\end{pmatrix}$, the equation of given circle is,

$$\mathbf{x}^\top\mathbf{x} + 2\mathbf{u}^\top\mathbf{x} = 0 \tag{2.1.2}$$

The general equation of circle is,

$$\mathbf{x}^\top\mathbf{x} + 2\mathbf{u}^\top\mathbf{x} + f = 0 \tag{2.1.3}$$

Substituting $\mathbf{A}, \mathbf{B}$ in (2.1.2),

$$\mathbf{A}^\top\mathbf{A} + 2\mathbf{u}^\top\mathbf{A} = 0 \tag{2.1.4}$$

$$\mathbf{B}^\top\mathbf{B} + 2\mathbf{u}^\top\mathbf{B} = 0 \tag{2.1.5}$$

Simplifying (2.1.4) and (2.1.5)

$$\begin{pmatrix}a & 0\\0 & b\end{pmatrix}\mathbf{u}^\top = -\frac{1}{2}\begin{pmatrix}a^2\\b^2\end{pmatrix} \tag{2.1.6}$$

$$\implies \begin{pmatrix}a & 0 & -a^2/2\\0 & b & -b^2/2\end{pmatrix} \tag{2.1.7}$$

$$\xleftrightarrow[R_1\leftarrow R_1/a]{R_2\leftarrow R_2/b} \begin{pmatrix}1 & 0 & -a/2\\0 & 1 & -b/2\end{pmatrix} \tag{2.1.8}$$

$$\implies \mathbf{u} = -\frac{1}{2}\begin{pmatrix}a\\b\end{pmatrix} \tag{2.1.9}$$

$$\text{or, } \mathbf{x}^\top\mathbf{x} - \begin{pmatrix}a\\b\end{pmatrix}\mathbf{x} = 0 \tag{2.1.10}$$

upon substituting in (2.1.2).

2.2. Find the equation of a circle with centre $\begin{pmatrix}2\\2\end{pmatrix}$ and passes through the point $\begin{pmatrix}4\\5\end{pmatrix}$.
Solution: From the given information, the centre

$$\mathbf{c} = \begin{pmatrix}2\\2\end{pmatrix} \implies \mathbf{u} = -\mathbf{c} = \begin{pmatrix}-2\\-2\end{pmatrix} \tag{2.2.1}$$

Using the general equation of a circle and substituting $\mathbf{x} = \begin{pmatrix}4\\5\end{pmatrix}$,

$$\begin{aligned}\mathbf{x}^\top\mathbf{x} + 2\mathbf{u}^\top\mathbf{x} + f &= 0\\ \implies \begin{pmatrix}4 & 5\end{pmatrix}\begin{pmatrix}4\\5\end{pmatrix} + \begin{pmatrix}-4 & -4\end{pmatrix}\begin{pmatrix}4\\5\end{pmatrix} + f &= 0\\ \implies f + \begin{pmatrix}41\end{pmatrix} + \begin{pmatrix}-36\end{pmatrix} &= 0\\ \text{or, } f &= -5\end{aligned} \tag{2.2.2}$$

Hence , the equation of the circle is,

$$\mathbf{x}^\top\mathbf{x} + \begin{pmatrix}-4 & -4\end{pmatrix}\mathbf{x} - 5 = 0 \tag{2.2.3}$$

The radius of the circle is then given by

$$r = \sqrt{\mathbf{u}^\top\mathbf{u} - f} \implies r = \sqrt{13} \tag{2.2.4}$$

The above results are verified in Fig. 2.2

2.3. Find the locus of all the unit vectors in the xy-plane.

2.4. Find the points on the curve $\mathbf{x}^T\mathbf{x} - 2\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} - 3 = 0$ at which the tangents are parallel to the x-axis.
Solution: The given curve,

$$\mathbf{x}^T\mathbf{x} - 2\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} - 3 = 0 \tag{2.4.1}$$

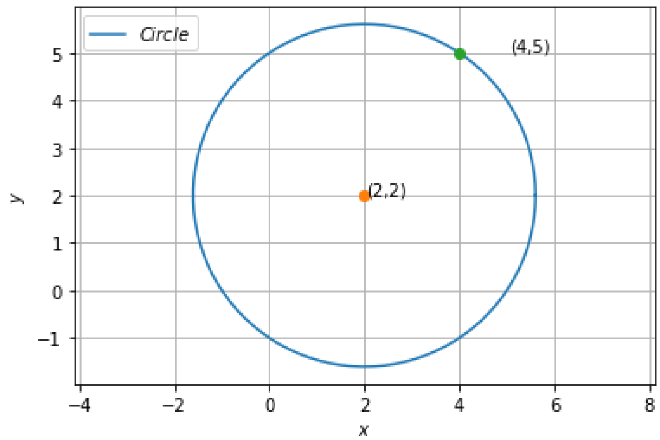

Fig. 2.2: Plot of the required circle

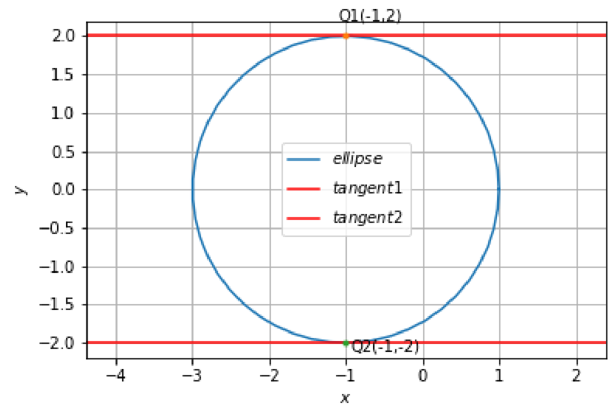

Fig. 2.4: Ellipse with tangents parallel to x axis at points $\mathbf{q} = \begin{pmatrix} -1 \\ \pm 2 \end{pmatrix}$

has parameters

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{V}^{-1}, \mathbf{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, f = 0 \tag{2.4.2}$$

Hence, the given curve is a circle. The point of contact for the tangent is

$$\mathbf{q} = \mathbf{V}^{-1}(\kappa \mathbf{n} - \mathbf{u}) \tag{2.4.3}$$

where,

$$\kappa = \pm \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\mathbf{n}^T \mathbf{V}^{-1} \mathbf{n}}} \tag{2.4.4}$$

For the tangents parallel to x-axis, the direction and normal vectors are

$$\mathbf{m} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{n} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.4.5}$$

$$\kappa = \pm \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\mathbf{n}^T \mathbf{V}^{-1} \mathbf{n}}} \tag{2.4.6}$$

$$= \pm 2 \tag{2.4.7}$$

By substituting $\kappa, \mathbf{n}, \mathbf{V}^{-1}$ in (2.4.3)

$$\mathbf{q} = \mathbf{V}^{-1}(\kappa \mathbf{n} - \mathbf{u}) \tag{2.4.8}$$

$$= \begin{pmatrix} -1 \\ \pm 2 \end{pmatrix} \tag{2.4.9}$$

2.5. Find the area of the region in the first quadrant enclosed by x-axis, line $\begin{pmatrix} 1 & -\sqrt{3} \end{pmatrix} \mathbf{x} = 0$ and the circle $\mathbf{x}^T \mathbf{x} = 4$.

2.6. Find the area lying in the first quadrant and bounded by the circle $\mathbf{x}^T \mathbf{x} = 4$ and the lines $x = 0$ and $x = 2$.

Solution: The general equation of a circle is

$$\mathbf{x}\mathbf{x}^\top - 2\mathbf{O}^\top \mathbf{x} + \|O\|^2 - r^2 = 0 \tag{2.6.1}$$

Given equation of the circle is

$$\mathbf{x}\mathbf{x}^\top = 4 \tag{2.6.2}$$

Comparing (2.6.2) with (2.6.1), we get

$$\mathbf{O} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.6.3}$$

$$r = 2 \tag{2.6.4}$$

Given lines are

$$L_1 : \mathbf{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.6.5}$$

$$L_2 : \mathbf{x} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.6.6}$$

where α and β are real numbers. We know the points of intersection of the line

$$L : \mathbf{x} = \mathbf{q} + \mu \mathbf{m} \tag{2.6.7}$$

with the circle in (2.6.2) is given by

$$\mathbf{x_i} = \mathbf{q} + \mu_i \mathbf{m} \tag{2.6.8}$$

where

$$\mu_i = \frac{1}{\mathbf{m}^\top \mathbf{I} \mathbf{m}} \Big(-\mathbf{m}^\top (\mathbf{I}\mathbf{q} + \mathbf{O}) \pm \sqrt{\left[-\mathbf{m}^\top (\mathbf{I}\mathbf{q} + \mathbf{O}) \right]^2 - \left(\mathbf{q}^\top \mathbf{I} \mathbf{q} + \mathbf{O}^\top \mathbf{q} - r^2 \right) \left(\mathbf{m}^\top \mathbf{I} \mathbf{m} \right)} \Big) \tag{2.6.9}$$

Solving for α and β, we get

$$\alpha = \pm 2 \qquad \beta = 0 \tag{2.6.10}$$

Points of intersection of line L_1 are $\begin{pmatrix}0\\2\end{pmatrix}$ and $\begin{pmatrix}0\\-2\end{pmatrix}$ and line L_2 is $\begin{pmatrix}0\\0\end{pmatrix}$.
The angle made by lines L_1 and L_2 with the x axis i.e $\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} = 0$ is

$$\cos\theta = \frac{\begin{pmatrix}1\\0\end{pmatrix}^{\top}\begin{pmatrix}0 & 1\end{pmatrix}}{\left\|\begin{pmatrix}1 & 0\end{pmatrix}\right\|\left\|\begin{pmatrix}0 & 1\end{pmatrix}\right\|} \tag{2.6.11}$$

$$= 0 \tag{2.6.12}$$

$$\implies \theta = 90^\circ \tag{2.6.13}$$

The area of sector thus obtained is

$$\frac{\theta^\circ}{360^\circ}\pi r^2 = \frac{90^\circ}{360^\circ}\pi r^2 \tag{2.6.14}$$

$$= \frac{\pi}{4}2^2 \tag{2.6.15}$$

$$= \pi \tag{2.6.16}$$

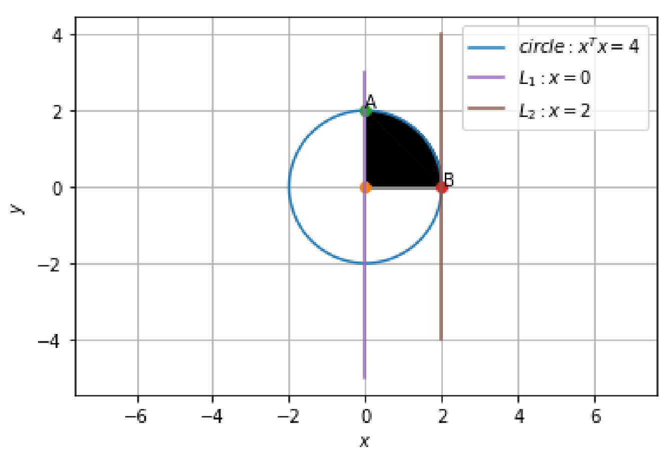

Fig. 2.6: Plotting the region bounded by circle and lines in first quadrant

2.7. Find the area of the circle $4\mathbf{x}^T\mathbf{x} = 9$.

2.8. Find the area bounded by curves $\left\|\mathbf{x} - \begin{pmatrix}1\\0\end{pmatrix}\right\| = 1$ and $\|\mathbf{x}\| = 1$

2.9. Find the smaller area enclosed by the circle $\mathbf{x}^T\mathbf{x} = 4$ and the line $\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 2$.

2.10. If $(x-a)^2+(y-b)^2 = c^2$, for some $c > 0$, prove that

$$\frac{(1+y_2)^{\frac{3}{2}}}{y_2} \tag{2.10.1}$$

is a constant independent of a and b.

2.11. Form the differential equation of the family of circles touching the y-axis at origin.

2.12. Form the differential equation of the family of circles having centre on y-axis and radius 3 units.

2.13. Form the differntial equation of the fmaily of circles touching the x-axis at the origin.

2.14. Form the differential equation of the family of circles in the second quadrant and touching the coordinate axes.

2.15. Factorise y^2–$5y + 6$.

2.16. Find the zeroes of the quadratic polynomial $x^2+7x+10$ and verify the relationship between the zeroes and the coefficients.

2.17. Find a quadratic polynomial, the sum and product of whose zeroes are – 3 and 2, respectively.
Solution: A general polynomial equation $p(x, y)$ of degree 2 is given by :

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \tag{2.17.1}$$

The vector equation of $p(x, y)$ is given by

$$\mathbf{x}^{\mathbf{T}}\begin{pmatrix}A & \frac{B}{2}\\ \frac{B}{2} & C\end{pmatrix}\mathbf{x} + \begin{pmatrix}D & E\end{pmatrix}\mathbf{x} + F = 0 \tag{2.17.2}$$

As the polynomial we have to find is a quadratic polynomial we have :

$$B = 0, C = 0, E = 0 \tag{2.17.3}$$

If we take $A = 1$, we have :

$$\text{Sum of zeroes} = -D = -3 \implies D = 3 \tag{2.17.4}$$

$$\text{Product of zeroes} = F = 2 \tag{2.17.5}$$

Substituting the values in (2.17.2), the required quadratic polynomial is given by :

$$\mathbf{x}^{\mathbf{T}}\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\mathbf{x} + \begin{pmatrix}3 & 0\end{pmatrix}\mathbf{x} + 2 = 0 \tag{2.17.6}$$

See Fig. 2.17

2.18. Find the roots of the equation $5x^2$–$6x$–$2 = 0$.

2.19. Find the roots of $4x^2 + 3x + 5 = 0$.
Solution: The given equation can be written

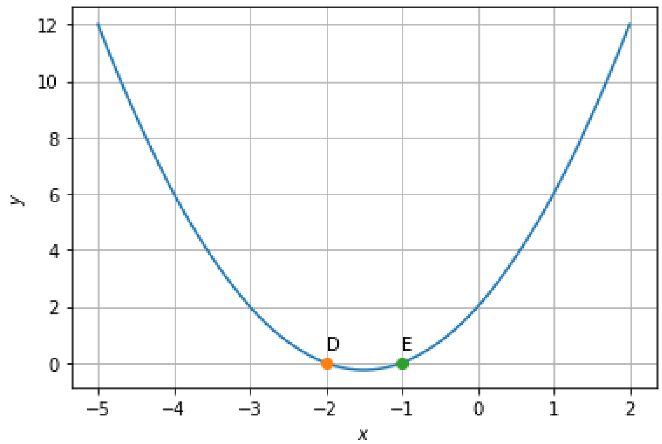

Fig. 2.17: Quadratic polynomial with zeroes -1 and -2

as,

$$\mathbf{x}^T \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 3 & 0 \end{pmatrix} \mathbf{x} + 5 = 0 \tag{2.19.1}$$

where,

$$\mathbf{x} = \begin{pmatrix} x \\ 0 \end{pmatrix} \tag{2.19.2}$$

Substituting (2.19.2) in (2.19.1),

$$\begin{pmatrix} x & 0 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} + \begin{pmatrix} 3 & 0 \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} + 5 = 0 \tag{2.19.3}$$

$$\implies 4x^2 + 3x + 5 = 0 \tag{2.19.4}$$

$$\implies \left(2x + \frac{3}{4}\right)^2 = -\frac{71}{16} \tag{2.19.5}$$

The square of a real number is always non-negative. In (2.19.5), we can say that $2x + \frac{3}{4}$ is not a real number. So, the roots are not real. From the figure, we can see that the function does not cross the x-axis, so, the quadratic equation has no real roots. Obtaining the affine transformation,

$$\mathbf{V} = \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} \tag{2.19.6}$$

$$\mathbf{u} = \begin{pmatrix} \frac{3}{2} \\ \frac{-3}{2} \end{pmatrix} \tag{2.19.7}$$

$$f = 5 \tag{2.19.8}$$

The equation used for affine transformation

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c} \tag{2.19.9}$$

The eigenvalues of V are

$$\lambda_1 = 4 \tag{2.19.10}$$

$$\lambda_2 = 0 \tag{2.19.11}$$

$$\mathbf{D} = \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} \tag{2.19.12}$$

The eigenvectors of V are

$$\mathbf{p}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.19.13}$$

$$\mathbf{p}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.19.14}$$

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix} \tag{2.19.15}$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.19.16}$$

Since $|\mathbf{V}| = 0$,

$$\begin{pmatrix} \mathbf{u}^T + \eta \mathbf{p}_1^T \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \eta \mathbf{p}_1 - \mathbf{u} \end{pmatrix} \tag{2.19.17}$$

$$\eta = \mathbf{u}^\top \mathbf{p}_1 \tag{2.19.18}$$

$$\implies \eta = -\frac{3}{2} \tag{2.19.19}$$

$$\implies \begin{pmatrix} \frac{3}{2} & -3 \\ 4 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -5 \\ -\frac{3}{2} \\ 0 \end{pmatrix} \tag{2.19.20}$$

$$\implies \mathbf{c} = \begin{pmatrix} \frac{-3}{8} \\ \frac{71}{48} \end{pmatrix} \tag{2.19.21}$$

The quadratic equation will not have real roots if

$$\left(\mathbf{p_1}^T \mathbf{c}\right)\left(\mathbf{p_2}^T \mathbf{V} \mathbf{p_2}\right) > 0 \tag{2.19.22}$$

Substituting the values in LHS,

$$\left(\mathbf{p_1}^T \mathbf{c}\right)\left(\mathbf{p_2}^T \mathbf{V} \mathbf{p_2}\right) = \left(\frac{71}{48}\right)(4) \tag{2.19.23}$$

$$= \frac{71}{12} \tag{2.19.24}$$

Since the value is positive, the quadratic equation has no real roots. Finding the roots of the equation. The equation of line is,

$$L : \mathbf{x} = \mathbf{q} + \mu \mathbf{m}, \mu \in \mathbb{R} \tag{2.19.25}$$

The line L is the x-axis,

$$\mathbf{q} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.19.26}$$

$$\mathbf{m} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.19.27}$$

The line L intersects the conic and to find μ,

$$\mu = \frac{-\mathbf{m}^T(\mathbf{V}\mathbf{q}+\mathbf{u})}{\mathbf{m}^T\mathbf{V}\mathbf{m}} \pm \frac{\sqrt{\left(\mathbf{m}^T(\mathbf{V}\mathbf{q}+\mathbf{u})\right)^2 - \left(\mathbf{q}^T\mathbf{V}\mathbf{q}+2\mathbf{u}^T\mathbf{q}+f\right)\left(\mathbf{m}^T\mathbf{V}\mathbf{m}\right)}}{\mathbf{m}^T\mathbf{V}\mathbf{m}} \tag{2.19.28}$$

$$\mu = \frac{-3}{8} \pm \frac{\sqrt{-71}}{8} \tag{2.19.29}$$

$$= \begin{pmatrix} \frac{-3}{8} \\ \frac{\sqrt{71}}{8} \end{pmatrix}, \begin{pmatrix} \frac{-3}{8} \\ -\frac{\sqrt{71}}{8} \end{pmatrix} \tag{2.19.30}$$

So, the roots of the equation are $\begin{pmatrix} \frac{-3}{8} \\ \frac{\sqrt{71}}{8} \end{pmatrix}, \begin{pmatrix} \frac{-3}{8} \\ -\frac{\sqrt{71}}{8} \end{pmatrix}$.

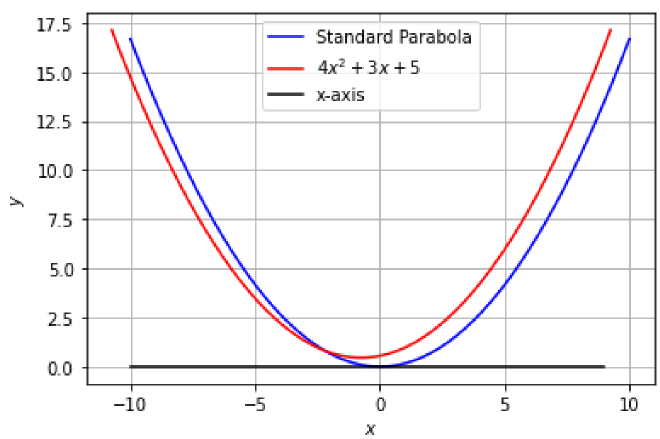

Fig. 2.19: Plot of the function

2.20. Find the roots of the following quadratic equations, if they exist.

a)

$$3x^2 - 5x + 2 = 0 \tag{2.20.1}$$

b)

$$x^2 + 4x + 5 = 0 \tag{2.20.2}$$

c) $2x^2 - 2\sqrt{2}x + 1 = 0$

Solution:

a) From the given info,

$$\mathbf{x}^T \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} \frac{-5}{2} \\ \frac{-1}{2} \end{pmatrix} \mathbf{x} + 2 = 0 \tag{2.20.3}$$

$$\implies \mathbf{V} = \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} \frac{-5}{2} \\ \frac{-1}{2} \end{pmatrix}, f = 2 \tag{2.20.4}$$

Using eigenvalue decomposition,

$$\mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}, \mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.20.5}$$

Also,

$$\begin{pmatrix} \mathbf{u}^T + \eta\mathbf{p_1}^T \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \eta\mathbf{p_1} - \mathbf{u} \end{pmatrix} \tag{2.20.6}$$

$$\implies \begin{pmatrix} \frac{-5}{2} & -1 \\ 3 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -2 \\ \frac{5}{2} \\ 0 \end{pmatrix} \tag{2.20.7}$$

$$\implies \begin{pmatrix} \frac{-5}{2} & -1 \\ 3 & 0 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -2 \\ \frac{5}{2} \end{pmatrix} \tag{2.20.8}$$

$$\implies \mathbf{c} = \begin{pmatrix} \frac{5}{6} \\ \frac{-1}{12} \end{pmatrix} \tag{2.20.9}$$

$\because$

$$\mathbf{p_1}^T\mathbf{c} = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{5}{6} \\ \frac{-1}{12} \end{pmatrix} \tag{2.20.10}$$

$$= \frac{-1}{12} \tag{2.20.11}$$

and,

$$\mathbf{p_2}^T\mathbf{V}\mathbf{p_2} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.20.12}$$

$$= 3 \tag{2.20.13}$$

$$(\mathbf{p_1}^T\mathbf{c})(\mathbf{p_2}^T\mathbf{V}\mathbf{p_2}) = \frac{-1}{4} < 0 \tag{2.20.14}$$

Hence,the given equation has real roots which is verified in Fig. 2.20.

b)

$$y = x^2 + 4x + 5 \tag{2.20.15}$$

Here,

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 2 \\ \frac{-1}{2} \end{pmatrix}, f = 5 \tag{2.20.16}$$

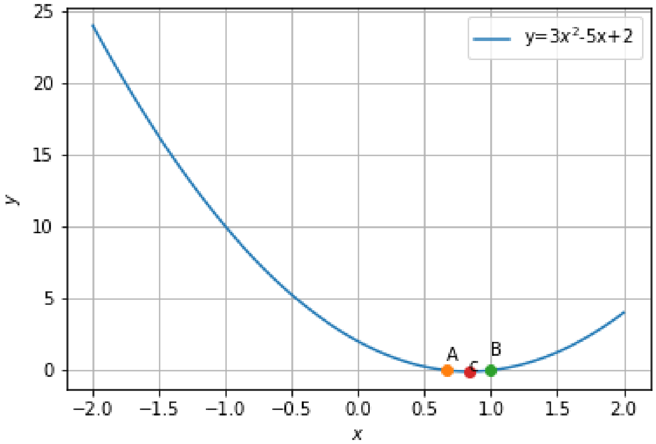

Fig. 2.20: $y = 3x^2 - 5x + 2$

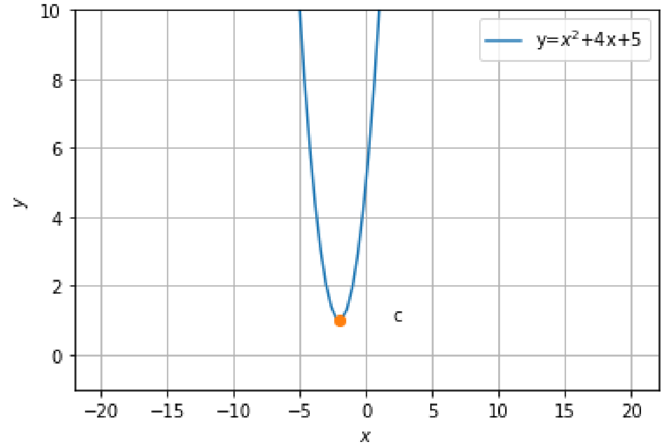

Fig. 2.20: $y = x^2 + 4x + 5$

Using eigenvalue decomposition,

$$\mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.20.17}$$

Now,

$$\begin{pmatrix} \mathbf{u}^T + \eta \mathbf{p_1}^T \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \eta \mathbf{p_1} - \mathbf{u} \end{pmatrix} \tag{2.20.18}$$

$\therefore$

$$\begin{pmatrix} 2 & -1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -5 \\ 2 \\ 0 \end{pmatrix} \tag{2.20.19}$$

$$\implies \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -5 \\ 2 \end{pmatrix} \tag{2.20.20}$$

$$\implies \mathbf{c} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{2.20.21}$$

$\because$

$$\mathbf{p_1}^T \mathbf{c} = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{2.20.22}$$

$$= 1 \tag{2.20.23}$$

and

$$\mathbf{p_2}^T \mathbf{V} \mathbf{p_2} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.20.24}$$

$$= 1 \tag{2.20.25}$$

$$(\mathbf{p_1}^T \mathbf{c})(\mathbf{p_2}^T \mathbf{V} \mathbf{p_2}) = (1)(1) = 1 > 0 \tag{2.20.26}$$

Hence,the given equation does not have real roots which is verified in Fig. 2.20

2.21. Solve $x^2 + 2 = 0$.

Solution: The given equation can be represented as follows in the vector form

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 & 0 \end{pmatrix} \mathbf{x} + 2 = 0 \tag{2.21.1}$$

where,

$$\mathbf{x} = \begin{pmatrix} x \\ 0 \end{pmatrix} \tag{2.21.2}$$

$$x^2 + 2 = 0 \tag{2.21.3}$$

$$\implies x^2 = -2 \tag{2.21.4}$$

Thus, the equation has no real roots as can be seen from Fig. 2.21

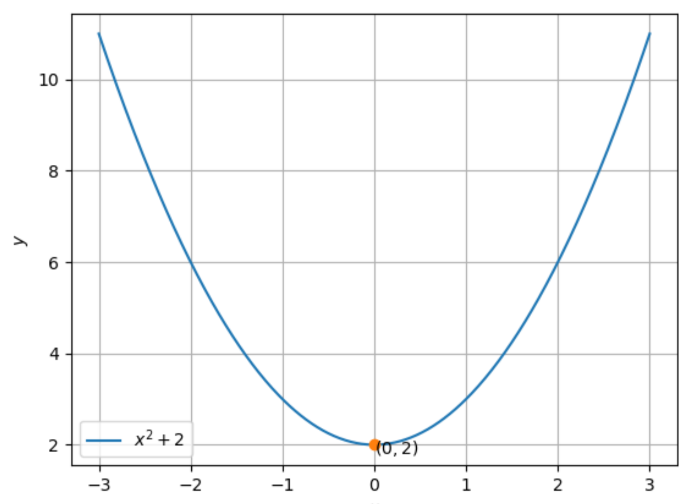

Fig. 2.21: $x^2 + 2$ generated using python

2.22. Solve $x^2 + x + 1 = 0$.

2.23. Solve $\sqrt{5}x^2 + x + \sqrt{5} = 0$.

Solution: Since

$$\sqrt{5}x^2 + x + \sqrt{5} = 0 \tag{2.23.1}$$

$$\implies \left(x + \frac{1}{2\sqrt{5}}\right)^2 + \frac{19}{20} = 0, \tag{2.23.2}$$

the equation has no real roots.
Consider $y = \sqrt{5}x^2 + x + \sqrt{5}$, which can be written in the vector form as

$$\mathbf{x}^T \begin{pmatrix} \sqrt{5} & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + 2\begin{pmatrix} \frac{1}{2} & \frac{-1}{2} \end{pmatrix} \mathbf{x} + \sqrt{5} = 0 \tag{2.23.3}$$

$$\mathbf{V} = \begin{pmatrix} \sqrt{5} & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} \frac{1}{2} \\ \frac{-1}{2} \end{pmatrix}, f = \sqrt{5} \tag{2.23.4}$$

For obtaining the affine transformation, we use:

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c} \tag{2.23.5}$$

The corresponding eigenvalues of $\mathbf{V}$ are:

$$\lambda_1 = 0, \lambda_2 = \sqrt{5} \tag{2.23.6}$$

$$\mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{5} \end{pmatrix} \tag{2.23.7}$$

The corresponding eigenvectors are:

$$\mathbf{p}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \mathbf{p}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.23.8}$$

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.23.9}$$

Since $|\mathbf{V}| = 0$, the given curve is a parabola and is plotted in Fig. 2.23 using

$$\begin{pmatrix} \mathbf{u}^\top + \eta \mathbf{p}_1^\top \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \eta \mathbf{p}_1 - \mathbf{u} \end{pmatrix} \tag{2.23.10}$$

$$\eta = \mathbf{u}^\top \mathbf{p}_1 \tag{2.23.11}$$

$$\implies \eta = \frac{-1}{2} \tag{2.23.12}$$

$$\implies \begin{pmatrix} \frac{1}{2} & -1 \\ \sqrt{5} & 0 \\ 0 & 0 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -\sqrt{5} \\ \frac{-1}{2} \\ 0 \end{pmatrix} \tag{2.23.13}$$

$$\implies \mathbf{c} = \begin{pmatrix} \frac{-1}{2\sqrt{5}} \\ \frac{19}{4\sqrt{5}} \end{pmatrix} \tag{2.23.14}$$

As we can see from the graph, the curve does not intersect the x-axis anywhere, and hence, has no real roots.

2.24. Find the coordinates of the focus, axis, the equation of the directrix and latus rectum of the parabola $y^2 = 8x$.

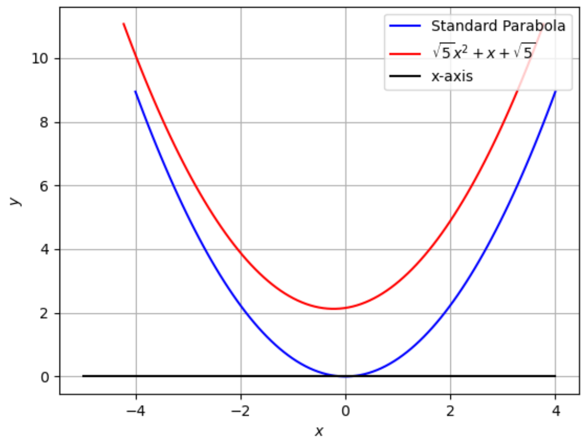

Fig. 2.23: Graph of $y = \sqrt{5}x^2 + x + \sqrt{5}$

Solution:
Given equation of the parabola is:

$$y^2 = 8x \tag{2.24.1}$$

$$y^2 - 8x = 0 \tag{2.24.2}$$

$$y^2 + 2(-4) = 0 \tag{2.24.3}$$

comparing it with standard equation

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{2.24.4}$$

$\therefore$ a = b = e = 0, d = -4, c = 1, f = 0

$$\mathbf{V} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.24.5}$$

$$\mathbf{u} = \begin{pmatrix} -4 \\ 0 \end{pmatrix} \tag{2.24.6}$$

Lemma 2.1. *The equation of a parabola is:*

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \tag{2.24.7}$$

Then its vertex can be calculated as :

$$\begin{pmatrix} \mathbf{u}^T + \eta \mathbf{p_1}^T \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -\mathbf{f} \\ \eta \mathbf{p_1} - \mathbf{u} \end{pmatrix} \tag{2.24.8}$$

$$\text{where, } \eta = \mathbf{u^T p_1} \tag{2.24.9}$$

Equation of the parabola can be written as

$$\implies \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + 2\begin{pmatrix} -4 & 0 \end{pmatrix} \mathbf{x} + 0 = 0 \tag{2.24.10}$$

We can find the eigen values corresponding to the $\mathbf{V}$,

$$|\mathbf{V}-\lambda\mathbf{I}| = \left|\begin{pmatrix}0-\lambda & 0\\ 0 & 1-\lambda\end{pmatrix}\right| \tag{2.24.11}$$

$$(-\lambda)(1-\lambda) = 0 \tag{2.24.12}$$

$\therefore$ Eigen values are $\lambda_1 = 0, \lambda_2 = 1$
Calculating the eigen vectors corresponding to $\lambda_1 = 0, \lambda_2 = 1$ respectively

$$\mathbf{V}\mathbf{x} = \lambda\mathbf{x} \tag{2.24.13}$$

$$\implies \mathbf{p_1} = \begin{pmatrix}1\\0\end{pmatrix} \tag{2.24.14}$$

$$\implies \mathbf{p_2} = \begin{pmatrix}0\\1\end{pmatrix} \tag{2.24.15}$$

The vertex of the parabola can be given as

$$\begin{pmatrix}\mathbf{u}^T + \eta\mathbf{p_1}^T\\ \mathbf{V}\end{pmatrix}\mathbf{c} = \begin{pmatrix}-\mathbf{f}\\ \eta\mathbf{p_1}-\mathbf{u}\end{pmatrix} \tag{2.24.16}$$

$$\text{where, } \eta = \mathbf{u^T}\mathbf{p_1} = \begin{pmatrix}-4 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} \tag{2.24.17}$$

$$\begin{pmatrix}-8 & 1\\ 0 & 0\\ 0 & 1\end{pmatrix}\mathbf{c} = \begin{pmatrix}0\\0\\0\end{pmatrix} \tag{2.24.18}$$

$$\implies \mathbf{c} = \begin{pmatrix}0\\0\end{pmatrix} \tag{2.24.19}$$

Since $\lambda_2 > \lambda_1$
Hence, the axis using $\mathbf{p_2}$ is given by

$$\mathbf{p_2}^T\left(\mathbf{x}-\mathbf{c}\right) = 0 \tag{2.24.20}$$

$$\begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = 0 \tag{2.24.21}$$

$$\implies y = 0 \tag{2.24.22}$$

$$\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} = 0 \tag{2.24.23}$$

Theorem 2.1. *The eccentricity, directrices and foci of parabola are given by*
Eccentricity,

$$e = \sqrt{1-\frac{\lambda_1}{\lambda_2}} \tag{2.24.24}$$

$$\mathbf{n} = \sqrt{\lambda_2}\mathbf{p}_1 \tag{2.24.25}$$

$$c = \frac{\|\mathbf{u}\|^2 - \lambda_2 f}{2e^2\mathbf{u}^\top\mathbf{n}} \tag{2.24.26}$$

Focus,

$$\mathbf{F} = \frac{ce^2\mathbf{n}-\mathbf{u}}{\lambda_2} \tag{2.24.27}$$

Directrix,

$$\mathbf{n}^T\mathbf{x} = c \tag{2.24.28}$$

From Equation 2.24.24, eccentricity can be calculated as

$$e = \sqrt{1-\frac{\lambda_1}{\lambda_2}} \tag{2.24.29}$$

$$e = \sqrt{1-\frac{0}{1}} = \sqrt{1} \tag{2.24.30}$$

$$\implies e = 1 \tag{2.24.31}$$

Substituting λ_2 and $\mathbf{p_1}$ values in Equation 2.24.25

$$\mathbf{n} = \sqrt{\lambda_2}\mathbf{p}_1 \tag{2.24.32}$$

$$\mathbf{n} = \sqrt{1}\begin{pmatrix}1\\0\end{pmatrix} \tag{2.24.33}$$

$$\implies \mathbf{n} = \begin{pmatrix}1\\0\end{pmatrix} \tag{2.24.34}$$

Equation 2.24.26 can be calculated as

$$c = \frac{\|\mathbf{u}\|^2 - \lambda_2 f}{2e^2\mathbf{u}^\top\mathbf{n}} \tag{2.24.35}$$

$$c = \frac{16 - 1\times 0}{2\times 1^2\times\begin{pmatrix}-4 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}} \tag{2.24.36}$$

$$= \frac{16}{-8} \tag{2.24.37}$$

$$\implies c = -2 \tag{2.24.38}$$

Focus of the parabolic equation can be calculated from the equation 2.24.27

$$\mathbf{F} = \frac{ce^2\mathbf{n}-\mathbf{u}}{\lambda_2} \tag{2.24.39}$$

$$\mathbf{F} = \frac{-2 \times 1 \times \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} -4 \\ 0 \end{pmatrix}}{1} \tag{2.24.40}$$

$$\implies \mathbf{F} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \tag{2.24.41}$$

Directrix of the parabolic equation can be calculated from equation 2.24.28

$$\mathbf{n}^T \mathbf{x} = c \tag{2.24.42}$$

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = -2 \tag{2.24.43}$$

Latus rectum is the line which is parallel to the directrix, passes through the focus and four times of the focal length. Since, the focal length of the parabola is 2.
∴ Latus rectum is 8
Equation of the latus rectum is

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 2 \tag{2.24.44}$$

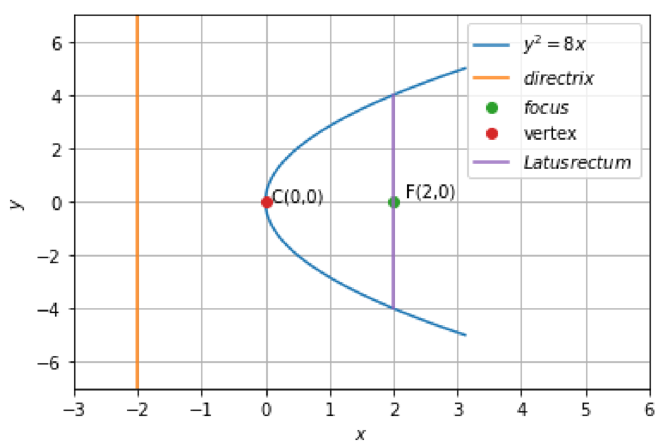

Fig. 2.35: Parabola

2.25. Find the equation of the parabola which is symmetric about the y-axis, and passes through the point $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$.

2.26. Find the coordinates of the foci, the vertices, the length of major axis, the minor axis, the eccentricity and the latus rectum of the ellipse

$$\mathbf{x}^T \begin{pmatrix} \frac{1}{25} & 0 \\ 0 & \frac{1}{9} \end{pmatrix} \mathbf{x} = 1 \tag{2.26.1}$$

2.27. Find the coordinates of the foci, the vertices, the lengths of major and minor axes and the eccentricity of the ellipse

$$\mathbf{x}^T \begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix} \mathbf{x} = 36 \tag{2.27.1}$$

Solution: Given ellipse is

$$\mathbf{x}^T \begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix} \mathbf{x} = 36 \tag{2.27.2}$$

On comparing it with standard form we have,

$$\mathbf{V} = \begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix}, \mathbf{u} = 0, f = -36 \tag{2.27.3}$$

$$\implies \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f = 36 \tag{2.27.4}$$

$$\implies \mathbf{c} = -\mathbf{V}^{-1} \mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.27.5}$$

$$\tag{2.27.6}$$

The eigen vector decomposition of

$$\mathbf{V} = \begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix} \tag{2.27.7}$$

is given by

$$\mathbf{D} = \begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix} \implies \lambda_1 = 9, \lambda_2 = 4 \tag{2.27.8}$$

$$\mathbf{P} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \implies \mathbf{p}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{p}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.27.9}$$

Since

$$\lambda_1 > \lambda_2 \tag{2.27.10}$$

Eccentricity of the ellipse is,

$$e = \sqrt{1 - \frac{\lambda_2}{\lambda_1}} = \frac{\sqrt{5}}{3} \tag{2.27.11}$$

Semi major and minor axes of ellipse are,

$$a = \sqrt{\frac{\mathbf{u}^\top \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_2}} = 3 \tag{2.27.12}$$

$$b = \sqrt{\frac{\mathbf{u}^\top \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}} = 2 \tag{2.27.13}$$

The co-ordinates of vertices are,

$$\pm \begin{pmatrix} 0 \\ 3 \end{pmatrix} \tag{2.27.14}$$

The co-ordinates of foci are given by,

$$\mathbf{F} = \frac{ce^2 \mathbf{n} - \mathbf{u}}{\lambda_1} \tag{2.27.15}$$

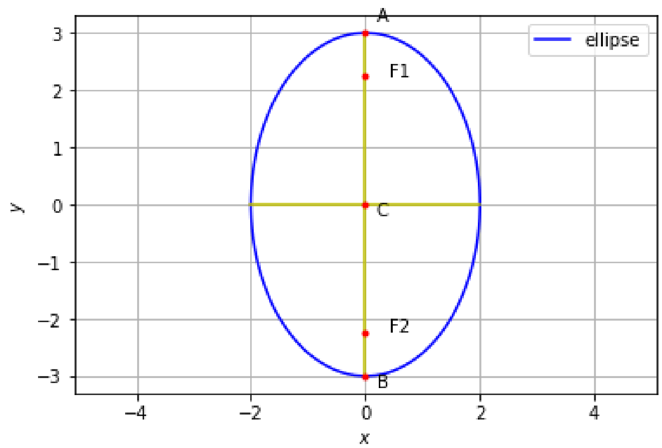

Fig. 2.36: Plot of the ellipse

Where,

$$\mathbf{n} = \sqrt{\lambda_1}\mathbf{p}_2 \tag{2.27.16}$$

$$c = \frac{e\mathbf{u}^\top \mathbf{n} \pm \sqrt{e^2 \left(\mathbf{u}^\top \mathbf{n}\right)^2 - \lambda_2 \left(e^2 - 1\right)\left(\|\mathbf{u}\|^2 - \lambda_2 f\right)}}{\lambda_2 e \left(e^2 - 1\right)} \tag{2.27.17}$$

Substituting we have,

$$\mathbf{n} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} \tag{2.27.18}$$

$$c = \pm \frac{27}{\sqrt{5}} \tag{2.27.19}$$

$$\mathbf{F} = \pm \begin{pmatrix} 0 \\ \sqrt{5} \end{pmatrix}. \tag{2.27.20}$$

See Fig. 2.36.

2.28. Find the equation of the ellipse whose vertices are $\begin{pmatrix} \pm 13 \\ 0 \end{pmatrix}$ and foci are $\begin{pmatrix} \pm 5 \\ 0 \end{pmatrix}$.

2.29. Find the equation of the ellipse, whose length of the major axis is 20 and foci are $\begin{pmatrix} 0 \\ \pm 5 \end{pmatrix}$

2.30. Find the equation of the hyperbola with vertices $\begin{pmatrix} 0 \\ \pm\frac{\sqrt{11}}{2} \end{pmatrix}$, foci $\begin{pmatrix} 0 \\ \pm 3 \end{pmatrix}$

2.31. Find the equation of the hyperbola with foci $\begin{pmatrix} 0 \\ \pm 12 \end{pmatrix}$ and length of latus rectum 36.

Solution:

Theorem 2.2. *The equation of a conic with directrix $\mathbf{n}^\top \mathbf{x} = c$, eccentricity e and focus $\mathbf{F}$ is given by*

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{2.31.1}$$

where

$$\mathbf{V} = \|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{n}\mathbf{n}^\top, \tag{2.31.2}$$

$$\mathbf{u} = ce^2 \mathbf{n} - \|\mathbf{n}\|^2 \mathbf{F}, \tag{2.31.3}$$

$$f = \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2 e^2 \tag{2.31.4}$$

For $|\mathbf{V}| > 0$, the equation represents an ellipse, while for $|\mathbf{V}| < 0$, the equation represents a hyperbola.

Theorem 2.3. *For $|\mathbf{V}| \neq 0$ the equations of minor and major axes of the conic in* (2.31.1) *are given by*

$$\mathbf{p_i}^\top (\mathbf{x} - \mathbf{c}) = 0, i = 1, 2 \tag{2.31.5}$$

Theorem 2.4. *The eccentricity of the conic in* (2.31.1) *is given by*

$$e = \sqrt{1 - \frac{\lambda_1}{\lambda_2}} \tag{2.31.6}$$

Definition 2 (Latus rectum)**.** *The latus rectum of a conic section is the chord (line segment) that passes through the focus, is perpendicular to the major axis and has both endpoints on the curve.*

Theorem 2.5. *The equation latus rectum of the conic in* (2.31.1) *is given by*

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{F}) = 0 \tag{2.31.7}$$

Theorem 2.6. *For $|\mathbf{V}| \neq 0$, the lengths of semi-major and semi-minor axes of the conic in* (2.31.1) *are*

$$\sqrt{\frac{\mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}}, \sqrt{\left|\frac{f - \mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u}}{\lambda_2}\right|} \tag{2.31.8}$$

Theorem 2.7. *For $|\mathbf{V}| \neq 0$, the length of latus rectum (LLR) of the conic in* (2.31.1) *is given by*

$$LLR = \frac{2\left|\frac{f - \mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u}}{\lambda_2}\right|}{\sqrt{\frac{\mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}}} \tag{2.31.9}$$

Proof. Using (2.31.8), we can write

$$\mathbf{F} = \mathbf{c} \pm \left(\sqrt{\frac{(\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f)(\lambda_2 - \lambda_1)}{\lambda_1\lambda_2}} \right) \mathbf{p_1} \tag{2.31.10}$$

Also, we know, for $|\mathbf{V}| \neq 0$,

$$\mathbf{n} = \sqrt{\lambda_2}\mathbf{p_1} \tag{2.31.11}$$

$$\implies \mathbf{p_1}^\top (\mathbf{x} - \mathbf{F}) = 0 \tag{2.31.12}$$

is the equation of latus rectum. Solving this with (2.31.1) and simplifying, we get

$$\mathbf{H} = \mathbf{F} \pm \sqrt{\lambda_1 \left(\frac{\mathbf{u}^\top\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_2^2} \right)} \mathbf{p_2} \tag{2.31.13}$$

as the points of intersection. Hence, the length of line segment is

$$LLR = \left\| 2\sqrt{\lambda_1 \left(\frac{\mathbf{u}^\top\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_2^2} \right)} \mathbf{p_2} \right\| = \frac{2\left| \frac{f - \mathbf{u}^\top\mathbf{V}^{-1}\mathbf{u}}{\lambda_2} \right|}{\sqrt{\frac{\mathbf{u}^\top\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}}} \tag{2.31.14}$$

□

Given, length of latus rectum is 36 and focii are $\begin{pmatrix} 0 \\ \pm 12 \end{pmatrix}$. Let us consider $\begin{pmatrix} 0 \\ 12 \end{pmatrix}$ for solving the problem.

$$\mathbf{F} = \begin{pmatrix} 0 \\ 12 \end{pmatrix} \implies \|\mathbf{F}\| = 12 \tag{2.31.15}$$

Let $\mathbf{u}^\top\mathbf{V}^{-1}\mathbf{u} - f = \alpha$. From (2.31.8),(2.31.6),(2.31.9)

$$\sqrt{\frac{\alpha}{\lambda_1}}\sqrt{1 - \frac{\lambda_1}{\lambda_2}} = 12 \tag{2.31.16}$$

$$\frac{2\left(\frac{-\alpha}{\lambda_2}\right)}{\sqrt{\frac{\alpha}{\lambda_1}}} = 36 \tag{2.31.17}$$

Dividing (2.31.16) by (2.31.17) gives

$$\frac{\lambda_1}{\lambda_2} = -3 \tag{2.31.18}$$

$$\implies e = 2 \tag{2.31.19}$$

$$\implies \sqrt{\frac{\alpha}{\lambda_1}} = 6 \tag{2.31.20}$$

The associated directrix is perpendicular to the y-axis and passes through the point

$$\begin{pmatrix} 0 \\ \sqrt{\frac{\alpha}{e^2\lambda_1}} \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} \tag{2.31.21}$$

Hence, its equation is

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} 0 \\ 3 \end{pmatrix} \right) = 0 \tag{2.31.22}$$

$$\implies \begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 3 \tag{2.31.23}$$

Comparing it with $\mathbf{n}^\top\mathbf{x} = c$

$$\mathbf{n} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, c = 3 \implies \|\mathbf{n}\| = 1 \tag{2.31.24}$$

Calculating $\mathbf{V}, \mathbf{u}$ and f,

$$\mathbf{V} = 1^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - 2^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} \tag{2.31.25}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix} \tag{2.31.26}$$

$$\mathbf{u} = 3(2^2) \begin{pmatrix} 0 \\ 1 \end{pmatrix} - 1^2 \begin{pmatrix} 0 \\ 12 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.31.27}$$

$$f = 1^2(12^2) - 3^2(2^2) = 108 \tag{2.31.28}$$

Hence, the required equation is

$$\mathbf{x}^\top \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix} \mathbf{x} + 108 = 0 \tag{2.31.29}$$

Also, from (2.31.7), the equations of latus rectum is

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} 0 \\ 12 \end{pmatrix} \right) = 0 \tag{2.31.30}$$

$$\implies \begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 12 \tag{2.31.31}$$

Similarly, the equations of directrix and latus rectum associated with $\begin{pmatrix} 0 \\ -12 \end{pmatrix}$ are given by

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = -3 \tag{2.31.32}$$

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = -12 \tag{2.31.33}$$

2.32. Find the roots of the following equations:

a) $x + \frac{1}{x} = 3, x \neq= 0$

b) $\frac{1}{x} + \frac{1}{x-2} = 3, x \neq= 0, 2$

Solution:

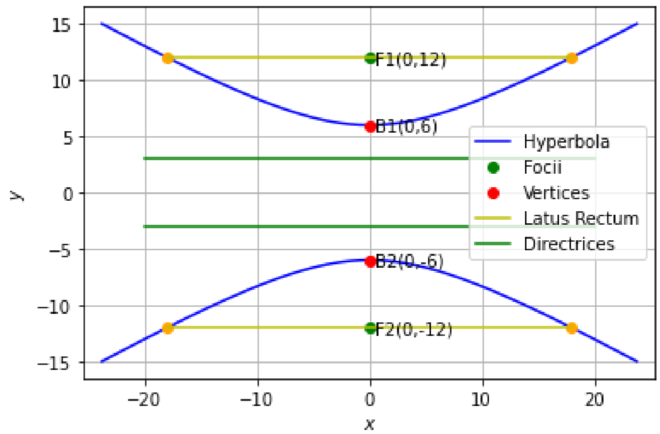

Fig. 2.1: Hyperbola

a) The given equation can be writen as:

$$x^2 + 1 = 3x \implies x^2 - \tag{2.32.1}$$

$$\text{or } \mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -3 & -1 \end{pmatrix} \mathbf{x} + 1 = 0 \tag{2.32.2}$$

$$\because \mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} \frac{-3}{2} \\ \frac{-1}{2} \end{pmatrix}, f = 1, \tag{2.32.3}$$

and

$$\mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.32.4}$$

$$\begin{pmatrix} \mathbf{u}^T + \eta \mathbf{p_1}^T \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \eta \mathbf{p_1} - \mathbf{u} \end{pmatrix} \tag{2.32.5}$$

$$\implies \begin{pmatrix} \frac{-3}{2} & -1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -1 \\ \frac{-3}{2} \\ 0 \end{pmatrix} \tag{2.32.6}$$

$$\implies \begin{pmatrix} -\frac{-3}{2} & -1 \\ 1 & 0 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -1 \\ \frac{3}{2} \end{pmatrix} \tag{2.32.7}$$

$$\implies \mathbf{c} = \begin{pmatrix} \frac{3}{2} \\ \frac{-5}{4} \end{pmatrix} \tag{2.32.8}$$

Now,

$$\mathbf{p_1}^T \mathbf{c} = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{3}{2} \\ \frac{-5}{4} \end{pmatrix} \tag{2.32.9}$$

$$= \frac{-5}{4} \tag{2.32.10}$$

and,

$$\mathbf{p_2}^T \mathbf{V} \mathbf{p_2} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.32.11}$$

$$= 1 \tag{2.32.12}$$

$\because$

$$(\mathbf{p_1}^T \mathbf{c})(\mathbf{p_2}^T \mathbf{V} \mathbf{p_2}) = \frac{-5}{4} < 0, \tag{2.32.13}$$

the given equation has real and distinct roots given by

$$\mathbf{x} = \begin{pmatrix} \pm \frac{-2\mathbf{u}^T \mathbf{e_1} \pm \sqrt{(2\mathbf{u}^T \mathbf{e_1})^2 - (4\mathbf{e_1}^T \mathbf{V} \mathbf{e_1} \mathbf{f})^2}}{2\mathbf{e_1}^T \mathbf{V} \mathbf{e_1}} \\ 0 \end{pmatrix} \tag{2.32.14}$$

$$= \begin{pmatrix} 2.6180 \\ 0 \end{pmatrix}, \begin{pmatrix} 0.38196 \\ 0 \end{pmatrix} \tag{2.32.15}$$

This is verified in Fig. 2.2.

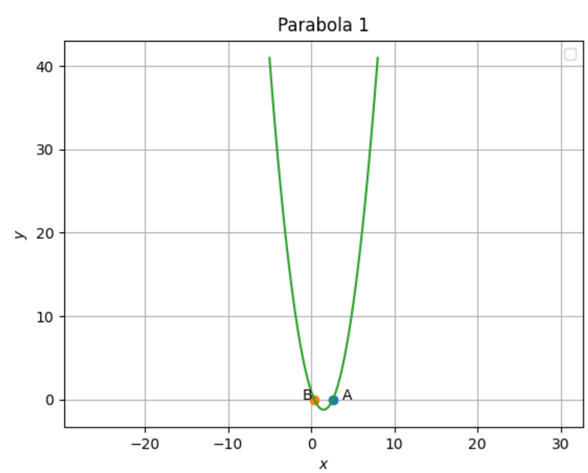

Fig. 2.2: curve

2.33. Find points on the curve $\mathbf{x}^T \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{25} \end{pmatrix} \mathbf{x} = 1$ at which the tangents are
  a) parallel to x-axis
  b) parallel to y-axis

2.34. Find the area enclosed by the ellipse $\mathbf{x}^T \begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{b^2} \end{pmatrix} \mathbf{x} = 1$

2.35. Find the area of the region bounded by the curve $y = x^2$ and the line $y = 4$.

2.36. Find the area bounded by the ellipse $\mathbf{x}^T \begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{b^2} \end{pmatrix} \mathbf{x} = 1$ and $x = ae$, where, $b^2 = a^2(1-e^2)$ and $e < 1$.

2.37. Prove that the curves $y^2 = 4x$ and $x^2 = 4y$

divide the area of the square bounded by $x = 0, x = 4, y = 4$ and $y = 0$ into three equal parts.

2.38. Find the area of the region

$$\left\{(x, y) = 0 \le y \le x^2 + 1, 0 \le y \le x + 1, 0 \le x \le 2\right\} \tag{2.38.1}$$

2.39. Examine whether the function f given by $f(x) = x^2$ is continuous at $x = 0$.

2.40. Discuss the continuity of the function f defined by

$$f(x) = \begin{cases} x & x \ge 0 \\ x^2 & x < 0 \end{cases} \tag{2.40.1}$$

2.41. Verify Rolle's theorem for the function $y = x^2 + 2, a = -2$ and $b = 2$.

2.42. Verify Mean Value Theorem for the function $f(x) = x^2$ in the interval $[2, -4]$.

2.43. Find the derivative of $f(x) = x^2$.

2.44. Find the derivative of $x^2 - 2$ at $x = 10$.

2.45. Find the derivative of $(x - 1)(x - 2)$.

2.46. Find

$$\int_0^2 \left(x^2 + 1\right) dx \tag{2.46.1}$$

as a limit of a sum.

2.47. Evaluate the following integral:

$$\int_2^3 x^2\, dx \tag{2.47.1}$$

2.48. Form the differntial equation representing the family of ellipses having foci on x-axis and cenre at the origin.

2.49. Form the differntial equation representing the family of parabolas having vertex at origin and axis along positive direction of x-axis.

2.50. Form a differntial equation representing the following family of curves

$$y^2 = a\left(b^2 - x^2\right) \tag{2.50.1}$$

2.51. A cricket ball is thrown at a speed of 28 ms^{-1} in a direction 30° above the horizontal. Calculate

a) the maximum height,
b) the time taken by the ball to return to the same level, and
c) the distance from the thrower to the point where the ball returns to the same level.

2.52. Find the roots of the equation $2x^2–5x + 3 = 0$.

2.53. Find the value of the following polynomial at the indicated value of variables

$$p(x) = 5x^2–3x + 7 \text{ at } x = 1. \tag{2.53.1}$$

2.54. Find the nature of the roots of the following quadratic equations. If the real roots exist, find them:

a) $2x^2 - 3x + 5 = 0$
b) $2x^2 - 6x + 3 = 0$
c) $3x^2 - 4\sqrt{3}x + 4 = 0$

2.55. Solve each of the following equations

a) $x^2 + 3 = 0$
b) $2x^2 + x + 1 = 0$
c) $x^2 + 3x + 9 = 0$
d) $-x^2 + x - 2 = 0$
e) $x^2 + 3x + 5 = 0$
f) $x^2 - 3x + 2 = 0$
g) $\sqrt{2}x^2 + x + \sqrt{2} = 0$
h) $\sqrt{3}x^2 - \sqrt{2}x + 3\sqrt{3} = 0$
i) $x^2 + x + \frac{1}{\sqrt{2}} = 0$
j) $x^2 + \frac{x}{\sqrt{2}} + 1 = 0$

2.56. In each of the following exercises, find the coordinates of the focus, axis of the parabola, the equation of the directrix and the length of the latus rectum

a) $y^2 = 12x$
b) $x^2 = 6y$
c) $y^2 = -8x$
d) $x^2 = -16y$
e) $y^2 = 10x$
f) $x^2 = -9y$

2.57. In each of the following exercises, find the equation of the parabola that satisfies the following conditions:

a) Focus $\begin{pmatrix} 6 \\ 0 \end{pmatrix}$, directrix $\begin{pmatrix} 1 & 0 \end{pmatrix} = -6$.
b) Focus $\begin{pmatrix} 0 \\ -3 \end{pmatrix}$, directrix $\begin{pmatrix} 0 & 1 \end{pmatrix} = 3$.
c) Focus $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$, vertex $\begin{pmatrix} 0 & 0 \end{pmatrix}$.
d) Focus $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$, vertex $\begin{pmatrix} 0 & 0 \end{pmatrix}$.
e) vertex $\begin{pmatrix} 0 & 0 \end{pmatrix}$ passing through $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ and axis is along the x-axis
f) vertex $\begin{pmatrix} 0 & 0 \end{pmatrix}$ passing through $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$ and symmetric with respect to the y-axis.

2.58. In each of the exercises, find the coordinates of the foci, the vertices, the length of major axis, the minor axis, the eccentricity and the length of the latus rectum of the ellipse.

a) $\mathbf{x}^T \begin{pmatrix} \frac{1}{36} & 0 \\ 0 & \frac{1}{16} \end{pmatrix} \mathbf{x} = 1$
b) $\mathbf{x}^T \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{25} \end{pmatrix} \mathbf{x} = 1$
c) $\mathbf{x}^T \begin{pmatrix} \frac{1}{16} & 0 \\ 0 & \frac{1}{9} \end{pmatrix} \mathbf{x} = 1$
d) $\mathbf{x}^T \begin{pmatrix} \frac{1}{25} & 0 \\ 0 & \frac{1}{100} \end{pmatrix} \mathbf{x} = 1$
e) $\mathbf{x}^T \begin{pmatrix} \frac{1}{49} & 0 \\ 0 & \frac{1}{36} \end{pmatrix} \mathbf{x} = 1$
f) $\mathbf{x}^T \begin{pmatrix} \frac{1}{100} & 0 \\ 0 & \frac{1}{16} \end{pmatrix} \mathbf{x} = 1$
g) $\mathbf{x}^T \begin{pmatrix} 36 & 0 \\ 0 & 4 \end{pmatrix} \mathbf{x} = 144$
h) $\mathbf{x}^T \begin{pmatrix} 16 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = 16$
i) $\mathbf{x}^T \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix} \mathbf{x} = 36$

2.59. In each of the following, find the equation for the ellipse that satisfies the given conditions:

a) Vertices $\begin{pmatrix} \pm 5 \\ 0 \end{pmatrix}$, foci $\begin{pmatrix} \pm 4 \\ 0 \end{pmatrix}$
b) Vertices $\begin{pmatrix} 0 \\ \pm 13 \end{pmatrix}$, foci $\begin{pmatrix} 0 \\ \pm 5 \end{pmatrix}$
c) Vertices $\begin{pmatrix} \pm 6 \\ 0 \end{pmatrix}$, foci $\begin{pmatrix} \pm 4 \\ 0 \end{pmatrix}$
d) Ends of major axis $\begin{pmatrix} \pm 3 \\ 0 \end{pmatrix}$, ends of minor axis $\begin{pmatrix} 0 \\ \pm 2 \end{pmatrix}$
e) Ends of major axis $\begin{pmatrix} 0 \\ \pm 5 \end{pmatrix}$, ends of minor axis $\begin{pmatrix} \pm 1 \\ 0 \end{pmatrix}$
f) Length of major axis 26, foci $\begin{pmatrix} \pm 5 \\ 0 \end{pmatrix}$
g) Length of minor axis 16, foci $\begin{pmatrix} 0 \\ \pm 6 \end{pmatrix}$.
h) Foci $\begin{pmatrix} \pm 3 \\ 0 \end{pmatrix}$, a = 4
i) b = 3, c = 4, centre at the origin; foci on the x axis.
j) Centre at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, major axis on the y-axis and passes through the points $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 6 \end{pmatrix}$.
k) Major axis on the x-axis and passes through the points $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ 2 \end{pmatrix}$.

Solution:

a) Given,

$$\mathbf{p} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}, \mathbf{q} = \begin{pmatrix} 6 \\ 2 \end{pmatrix} \tag{2.59.1}$$

are the points on the ellipse. The general form of the conic is given by

$$\mathbf{x}^\top \mathbf{D} \mathbf{x} = 1, \quad \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \lambda_1, \lambda_2 > 0 \tag{2.59.2}$$

The points $\mathbf{p}$ and $\mathbf{q}$ satisfy (2.59.2), and thus we have

$$\mathbf{p}^\top \mathbf{D} \mathbf{p} = 1, \tag{2.59.3}$$

$$\mathbf{q}^\top \mathbf{D} \mathbf{q} = 1 \tag{2.59.4}$$

which can be further expressed as,

$$\begin{aligned} \mathbf{p}^\top \mathbf{P} \mathbf{d} &= 1, \\ \mathbf{q}^\top \mathbf{Q} \mathbf{d} &= 1 \end{aligned} \tag{2.59.5}$$

where,

$$\mathbf{d} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}, \mathbf{P} = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix}. \tag{2.59.6}$$

(2.59.5) can then be expressed as,

$$\begin{pmatrix} \mathbf{p}^\top \mathbf{P} \\ \mathbf{q}^\top \mathbf{Q} \end{pmatrix} \mathbf{d} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.59.7}$$

$$\begin{pmatrix} 16 & 9 \\ 36 & 4 \end{pmatrix} \mathbf{d} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.59.8}$$

The augmented matrix is

$$\begin{pmatrix} 16 & 9 & 1 \\ 36 & 4 & 1 \end{pmatrix} \tag{2.59.9}$$

and we perform row reduction,

$$\begin{pmatrix}16 & 9 & 1\\ 1 & 16 & 1\end{pmatrix} \xleftrightarrow{R_1 \to \frac{R_1}{16}} \begin{pmatrix}1 & \frac{9}{16} & \frac{1}{16}\\ 36 & 4 & 1\end{pmatrix} \quad (2.59.10)$$

$$\xleftrightarrow{R_2 \to R_2 - 36R_1} \begin{pmatrix}1 & \frac{9}{16} & \frac{1}{16}\\ 0 & \frac{-65}{4} & \frac{-5}{4}\end{pmatrix} \quad (2.59.11)$$

$$\xleftrightarrow{R_2 \to \frac{-4}{65} R_2} \begin{pmatrix}1 & \frac{9}{16} & \frac{1}{16}\\ 0 & 1 & \frac{1}{13}\end{pmatrix} \quad (2.59.12)$$

$$\xleftrightarrow{R_1 \to R_1 - \frac{9}{16} R_2} \begin{pmatrix}1 & 0 & \frac{1}{52}\\ 0 & 1 & \frac{1}{13}\end{pmatrix} \quad (2.59.13)$$

$$\implies \mathbf{d} = \begin{pmatrix}\frac{1}{52}\\ \frac{1}{13}\end{pmatrix}. \quad (2.59.14)$$

Thus we have,

$$\mathbf{D} = \begin{pmatrix}\frac{1}{52} & 0\\ 0 & \frac{1}{13}\end{pmatrix} \quad (2.59.15)$$

Hence equation of ellipse is given by,

$$\mathbf{x}^\top \begin{pmatrix}\frac{1}{52} & 0\\ 0 & \frac{1}{13}\end{pmatrix} \mathbf{x} = 1 \quad (2.59.16)$$

The plot of the ellipse is given below The

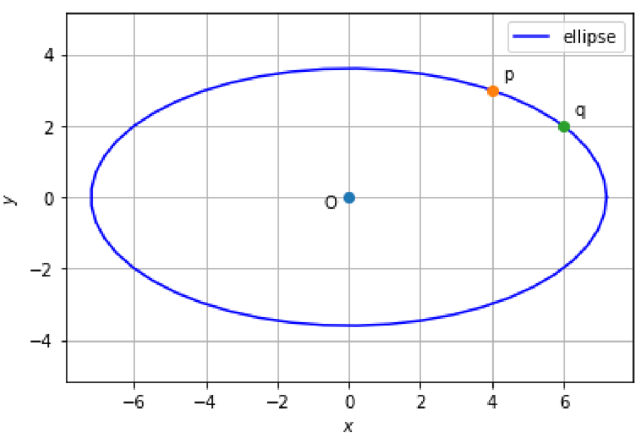

Fig. 2.3: Plot of standard ellipse

center and axes of the ellipse is given as

$$\mathbf{c} = \mathbf{0}; \frac{1}{\sqrt{\lambda_1}} = \sqrt{52}, \frac{1}{\sqrt{\lambda_2}} = \sqrt{13}. \quad (2.59.17)$$

Now let us consider the case when the ellipse is not in the standard form then we have the center to be $\mathbf{c} = \begin{pmatrix}\beta\\ 0\end{pmatrix}$.The equation is given by:

$$(\mathbf{x} - \mathbf{c})^\top \mathbf{D}(\mathbf{x} - \mathbf{c}) = 1 \quad (2.59.18)$$

where $\mathbf{D}$ is a diagonal matrix. $\because$ $\mathbf{p}, \mathbf{q}$ satisfy (2.59.18), we have

$$(\mathbf{p} - \mathbf{c})^\top \mathbf{D}(\mathbf{p} - \mathbf{c}) = 1, \quad (2.59.19)$$
$$(\mathbf{q} - \mathbf{c})^\top \mathbf{D}(\mathbf{q} - \mathbf{c}) = 1, \quad (2.59.20)$$

which can be simplified as

$$2(\mathbf{p} - \mathbf{q})^\top \mathbf{D}\mathbf{c} = \mathbf{p}^\top \mathbf{D}\mathbf{p} - \mathbf{q}^\top \mathbf{D}\mathbf{q} \quad (2.59.21)$$

Using the identity,

$$(\mathbf{p}^\top - \mathbf{q}^\top)\mathbf{D}(\mathbf{p} + \mathbf{q}) = \mathbf{p}^\top \mathbf{D}\mathbf{p} - \mathbf{q}^\top \mathbf{D}\mathbf{q} \quad (2.59.22)$$

in the equation (2.59.21)

$$2(\mathbf{p} - \mathbf{q})^\top \mathbf{D}\mathbf{c} = (\mathbf{p} - \mathbf{q})^\top \mathbf{D}(\mathbf{p} + \mathbf{q}) \quad (2.59.23)$$

$$\implies (\mathbf{p} - \mathbf{q})^\top \mathbf{D}(2\mathbf{c} - (\mathbf{p} + \mathbf{q})) \quad (2.59.24)$$

Thus $\mathbf{c}$ can be expressed in parametric form as

$$\mathbf{c} = \frac{1}{2}((\mathbf{p} + \mathbf{q}) + k\mathbf{D}^{-1}\mathbf{m}) \quad (2.59.25)$$

where,

$$(\mathbf{p} - \mathbf{q})^\top \mathbf{m} = 0 \quad (2.59.26)$$

and k is a constant.Substituting numerical values in (2.59.26),

$$\mathbf{p} - \mathbf{q} = \begin{pmatrix}-2\\ 1\end{pmatrix} \implies \mathbf{m} = \begin{pmatrix}-1\\ -2\end{pmatrix} \quad (2.59.27)$$

and

$$\mathbf{p} + \mathbf{q} = \begin{pmatrix}10\\ 5\end{pmatrix} \quad (2.59.28)$$

substituting in (2.59.25),we get

$$\begin{pmatrix}\beta\\ 0\end{pmatrix} = \frac{1}{2}\left(\begin{pmatrix}10\\ 5\end{pmatrix} + k\begin{pmatrix}\frac{1}{\lambda_1} & 0\\ 0 & \frac{1}{\lambda_2}\end{pmatrix}\begin{pmatrix}-1\\ -2\end{pmatrix}\right) \quad (2.59.29)$$

From the given information,the X-axis is the major axis.Hence,

$$\frac{\lambda_2}{\lambda_1} > 1 \implies \frac{20 - 4\beta}{5} > 1 \quad (2.59.30)$$
$$\beta < 3.75 \quad (2.59.31)$$

The possible ellipse satisfying the above conditions are plotted below

2.60. In each of the exercises, find the coordinates of the foci, the vertices, the length of major axis, the minor axis, the eccentricity and the length

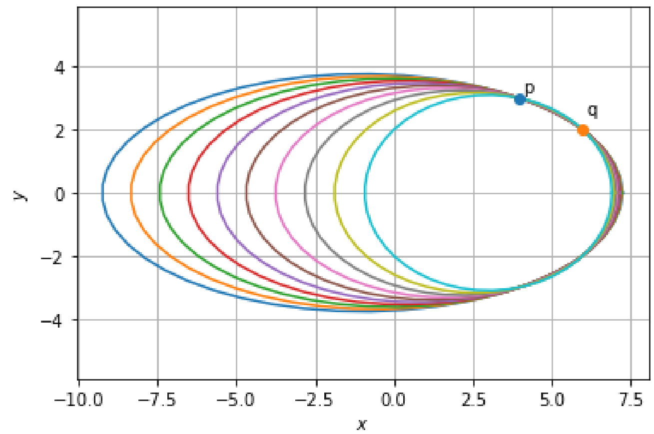

Fig. 2.4: Ellipses passing through the two points with X axis as major axis

of the latus rectum of the ellipse.

a) $\mathbf{x}^T\begin{pmatrix}\frac{1}{16} & 0\\ 0 & -\frac{1}{9}\end{pmatrix}\mathbf{x} = 1$

b) $\mathbf{x}^T\begin{pmatrix}\frac{1}{9} & 0\\ 0 & -\frac{1}{27}\end{pmatrix}\mathbf{x} = 1$

c) $\mathbf{x}^T\begin{pmatrix}9 & 0\\ 0 & -4\end{pmatrix}\mathbf{x} = 36$

d) $\mathbf{x}^T\begin{pmatrix}16 & 0\\ 0 & -9\end{pmatrix}\mathbf{x} = 576$

e) $\mathbf{x}^T\begin{pmatrix}5 & 0\\ 0 & -9\end{pmatrix}\mathbf{x} = 36$

f) $\mathbf{x}^T\begin{pmatrix}49 & 0\\ 0 & -16\end{pmatrix}\mathbf{x} = 784$

2.61. In each of the following, find the equation for the ellipse that satisfies the given conditions:

a) Vertices $\begin{pmatrix}\pm 2\\ 0\end{pmatrix}$, foci $\begin{pmatrix}\pm 3\\ 0\end{pmatrix}$

b) Vertices $\begin{pmatrix}0\\ \pm 5\end{pmatrix}$, foci $\begin{pmatrix}0\\ \pm 8\end{pmatrix}$

c) Vertices $\begin{pmatrix}0\\ \pm 3\end{pmatrix}$, foci $\begin{pmatrix}0\\ \pm 5\end{pmatrix}$

d) Transverse axis length 8, foci $\begin{pmatrix}\pm 5\\ 0\end{pmatrix}$.

e) Conjugate axis length 24, foci $\begin{pmatrix}0\\ \pm 13\end{pmatrix}$.

f) Latus rectum length 8, foci $\begin{pmatrix}\pm 3\sqrt{5}\\ 0\end{pmatrix}$.

g) Latus rectum lenght 12, foci $\begin{pmatrix}\pm 4\\ 0\end{pmatrix}$.

h) Ends of major axis $\begin{pmatrix}0\\ \pm 5\end{pmatrix}$, ends of minor axis $\begin{pmatrix}\pm 1\\ 0\end{pmatrix}$

i) Vertices $\begin{pmatrix}\pm 7\\ 0\end{pmatrix}$, $e = \frac{4}{3}$

j) Foci $\begin{pmatrix}0\\ \pm\sqrt{10}\end{pmatrix}$, passing through $\begin{pmatrix}2\\ 3\end{pmatrix}$.

2.62. Find a point on the curve $y = (x–2)^2$ at which the tangent is parallel to the chord joining the points $\begin{pmatrix}2\\ 0\end{pmatrix}$ and $\begin{pmatrix}4\\ 4\end{pmatrix}$.

2.63. Find the equation of all lines having slope -1 that are tangents to the curve $\frac{1}{x-1}, x \neq 1$

2.64. Find the equations of the tangent and normal to the given curves at the indicated points: $y = x^2$ at $\begin{pmatrix}0\\ 0\end{pmatrix}$.

2.65. Find the equation of the tangent line to the curve $y = x^2 - 2x + 7$

a) parallel to the line $\begin{pmatrix}2 & -1\end{pmatrix}\mathbf{x} = -9$

b) perpendicular to the line $\begin{pmatrix}-15 & 5\end{pmatrix}\mathbf{x} = 13$.

2.66. Find the point at which the line $\begin{pmatrix}-1 & 1\end{pmatrix}\mathbf{x} = 1$ is a tangent to the curve $y^2 = 4x$.

2.67. The line $\begin{pmatrix}-m & 1\end{pmatrix}\mathbf{x} = 1$ is a tangent to the curve $y^2 = 4x$. Find the value of m.

2.68. Find the normal at the point $\begin{pmatrix}1\\ 1\end{pmatrix}$ on the curve $2y + x^2 = 3$

Solution: The given curve can be expressed as

$$x^2 + 2y - 3 = 0 \tag{2.68.1}$$

$$\implies \mathbf{V} = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \ \mathbf{u} = \begin{pmatrix}0\\ 1\end{pmatrix}, \ \mathbf{f} = -3 \tag{2.68.2}$$

Since $|\mathbf{V}| = 0$, the given curve represents a parabola. The eigenvalues are given by

$$\lambda_1 = 0, \ \lambda_2 = 1 \tag{2.68.3}$$

with corresponding eigenvectors

$$\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\mathbf{x} = 0 \implies \mathbf{p}_1 = \begin{pmatrix}0\\ 1\end{pmatrix} \tag{2.68.4}$$

$$\begin{pmatrix}0 & 0\\ 0 & -1\end{pmatrix}\mathbf{x} = 0 \implies \mathbf{p}_2 = \begin{pmatrix}1\\ 0\end{pmatrix} \tag{2.68.5}$$

To find the vertex of the parabola,

$$\begin{pmatrix}\mathbf{u}^\top + \kappa\mathbf{p}_1^\top\\ \mathbf{V}\end{pmatrix}\mathbf{c} = \begin{pmatrix}-\mathbf{f}\\ \kappa\mathbf{p}_1 - \mathbf{u}\end{pmatrix} \tag{2.68.6}$$

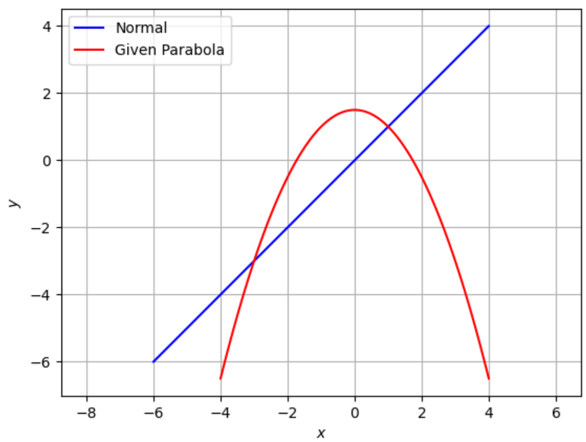

Fig. 2.5: Plot of the normal

where, $\kappa = \mathbf{u}^\top \mathbf{p}_1 = 1$

$$\implies \begin{pmatrix} 0 & 2 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{c} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} \tag{2.68.7}$$

$$\implies \mathbf{c} = \begin{pmatrix} 0 \\ 1.5 \end{pmatrix} \tag{2.68.8}$$

Now to evaluate the direction vector $\mathbf{m}$,

$$\mathbf{m}^\top (\mathbf{V}\mathbf{q} + \mathbf{u}) = 0 \tag{2.68.9}$$

$$\implies \mathbf{m}^\top \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = 0 \tag{2.68.10}$$

$$\implies \mathbf{m}^\top \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0 \tag{2.68.11}$$

$$\implies \mathbf{m} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{2.68.12}$$

The normal is obtained as

$$\mathbf{m}^\top (\mathbf{x} - \mathbf{q}) = 0 \tag{2.68.13}$$

$$\implies \begin{pmatrix} -1 & 1 \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) = 0 \tag{2.68.14}$$

$$\implies \begin{pmatrix} -1 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{2.68.15}$$

The above results are verified in Fig. 2.5.

2.69. Find the normal to the curve $x^2 = 4y$ passing through $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

2.70. Find the area of the region bounded by the curve $y^2 = x$ and the lines $x = 1, x = 4$ and the x-axis in the first quadrant.

Solution:

The matrix parameters for (**??**) are

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} \frac{-1}{2} \\ 0 \end{pmatrix}, \mathbf{f} = 0. \tag{2.70.1}$$

Thus, the given curve is a parabola. $\because$ $\mathbf{V}$ is diagonal and in standard form,

$$\mathbf{V}\mathbf{p} = \mathbf{0} \tag{2.70.2}$$

$$\implies \mathbf{p} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.70.3}$$

with eigen parameters

$$\lambda_1 = 0, \lambda_2 = 1 \tag{2.70.4}$$

The line $x = 1$ can be expressed as

$$\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mathbf{y} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.70.5}$$

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mathbf{y} \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.70.6}$$

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 1 \tag{2.70.7}$$

Similarly, for $x = 4$ we get

$$\begin{pmatrix} 4 & 0 \end{pmatrix} \mathbf{x} = 16 \tag{2.70.8}$$

$$\implies \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 4 \tag{2.70.9}$$

The direction vector and normal vectors are

$$\mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \mathbf{n} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{2.70.10}$$

The equation of parabola is

$$\mathbf{q}^T \mathbf{V} \mathbf{q} + 2\mathbf{u}^T \mathbf{q} + f = 0 \tag{2.70.11}$$

The vertex of conic section in (2.70.11) is given by $\mathbf{c}$ using

$$\begin{pmatrix} \mathbf{u^T} + \mathbf{n}\mathbf{p}^T \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \mathbf{n}\mathbf{p} - \mathbf{u} \end{pmatrix} \tag{2.70.12}$$

$$\begin{pmatrix} \frac{-1}{4} & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{c} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{2.70.13}$$

$$\implies \begin{pmatrix} \frac{-1}{4} & 0 \\ 0 & 1 \end{pmatrix} \mathbf{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.70.14}$$

$$\text{or, } \mathbf{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.70.15}$$

$$\kappa = \frac{\mathbf{p}_1^T \mathbf{u}}{\mathbf{p}_1^T \mathbf{n}}, \tag{2.70.16}$$

From (2.70.16), (2.70.10) and (2.70.3),

$$\kappa = \frac{-1}{2} \tag{2.70.17}$$

which, upon substitution in (2.70.18)

$$\begin{pmatrix} \mathbf{u} + \kappa \mathbf{n}^T \\ \mathbf{V} \end{pmatrix} \mathbf{q} = \begin{pmatrix} -f \\ \kappa \mathbf{n} - \mathbf{u} \end{pmatrix} \tag{2.70.18}$$

and simplification yields the matrix equation

$$\begin{pmatrix} \frac{-1}{2} & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{q} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{2.70.19}$$

$$\implies \begin{pmatrix} \frac{-1}{2} & 0 \\ 0 & 1 \end{pmatrix} \mathbf{q} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.70.20}$$

$$\text{or, } \mathbf{q} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.70.21}$$

Secant: The points of intersection of the line

$$L: \quad \mathbf{x} = \mathbf{q} + \mu \mathbf{m} \quad \mu \in \mathbb{R} \tag{2.70.22}$$

$$\mathbf{x}_i = \mathbf{q} + \mu_i \mathbf{m} \tag{2.70.23}$$

where

$$\mu_i = \frac{1}{\mathbf{m}^T \mathbf{V} \mathbf{m}} \Big(-\mathbf{m}^T (\mathbf{V}\mathbf{q} + \mathbf{u}) \pm \sqrt{[\mathbf{m}^T (\mathbf{V}\mathbf{q} + \mathbf{u})]^2 - (\mathbf{q}^T \mathbf{V} \mathbf{q} + 2\mathbf{u}^T \mathbf{q} + f)(\mathbf{m}^T \mathbf{V} \mathbf{m})} \Big) \tag{2.70.24}$$

$\because$ $\mathbf{q}$ is the point of contact, $\mathbf{q}$ satisfies parabola equation

Given the point of contact $\mathbf{q}$, the equation of a tangent is

$$(\mathbf{V}\mathbf{q} + \mathbf{u})^T \mathbf{x} + \mathbf{u}^T \mathbf{q} + f = 0 \tag{2.70.25}$$

From (2.70.24) we get $\mu_1 = 1, \mu_2 = -1$

The lines (2.70.7), (2.70.8) can be written in parametric form in (2.70.23) we get

$$\mathbf{x}_i = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mu_i \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.70.26}$$

Substituting μ_1, μ_2 value in (2.70.26) we get

$$\mathbf{x}_i = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.70.27}$$

$$\implies \mathbf{K_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.70.28}$$

$$\mathbf{x}_i = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + -1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.70.29}$$

$$\implies \mathbf{L_1} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{2.70.30}$$

For $x = 4$,

$$\mathbf{x}_i = \begin{pmatrix} 4 \\ 0 \end{pmatrix} + \mu_i \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.70.31}$$

$$\mathbf{x}_i = \begin{pmatrix} 4 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.70.32}$$

$$\implies \mathbf{K_2} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} \tag{2.70.33}$$

$$\mathbf{x}_i = \begin{pmatrix} 4 \\ 0 \end{pmatrix} + -2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.70.34}$$

$$\implies \mathbf{L_2} = \begin{pmatrix} 4 \\ -2 \end{pmatrix} \tag{2.70.35}$$

The area enclosed by parabola and line on x-axis can be given as A = Area under line - Area under curve

$$\implies \boxed{\mathbf{A} = \mathbf{A_1} - \mathbf{A_3}} \tag{2.70.36}$$

Performing integration,

In Fig. 2.6 , the area under the lines (2.70.7), (2.70.8) is given by

$$\mathbf{A_1} = \frac{2}{3}, \mathbf{A_2} = \frac{16}{3} \tag{2.70.37}$$

Putting these values in (2.70.36) we get

$$\implies \boxed{\mathbf{A} = \frac{14}{3}} \tag{2.70.38}$$

2.71. Find the area of the region bounded by $y^2 = 9x, x = 2, x = 4$ and the x-axis in the first quadrant.

2.72. Find the area of the region bounded by $x^2 = 4y, y = 2, y = 4$ and the y-axis in the first quadrant.

2.73. Find the area of the region bounded by the

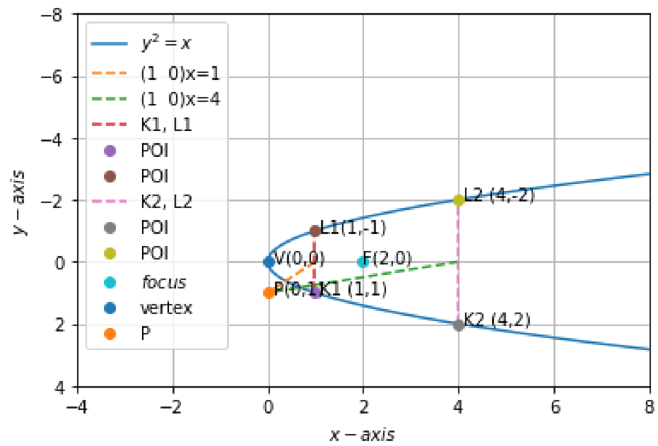

Fig. 2.6: Parabola $\mathbf{y^2} = \mathbf{x}$

ellipse $\mathbf{x}^T \begin{pmatrix} \frac{1}{16} & 0 \\ 0 & \frac{1}{9} \end{pmatrix} \mathbf{x} = 1$

2.74. Find the area of the region bounded by the ellipse $\mathbf{x}^T \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{9} \end{pmatrix} \mathbf{x} = 1$

2.75. The area between $x = y^2$ and $x = 4$ is divided into two equal parts by the line $x = a$, find the value of a.

2.76. Find the area of the region bounded by the parabola $y = x^2$ and $y = |x|$.

2.77. Find the area bounded by the curve $x^2 = 4y$ and the line $\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = -2$.

Solution:

Lemma 2.2. *The points of intersection of* **Line** $L : \mathbf{x} = \mathbf{q} + \mu \mathbf{m}$ *with the conic*

$$\mathbf{x}^{\mathbf{T}} \mathbf{V} \mathbf{x} + 2\mathbf{u}^{\mathbf{T}} \mathbf{x} + f = 0 \tag{2.77.1}$$

are given by:

$$\mathbf{x}_i = \mathbf{q} + \mu_i \mathbf{m} \tag{2.77.2}$$

where,

$$\mu_i = \frac{1}{\mathbf{m}^T \mathbf{V} \mathbf{m}} \left(-\mathbf{m}^T (\mathbf{V}\mathbf{q} + \mathbf{u}) \pm \sqrt{[\mathbf{m}^T (\mathbf{V}\mathbf{q} + \mathbf{u})]^2 - (\mathbf{q}^T \mathbf{V} \mathbf{q} + 2\mathbf{u}^T \mathbf{q} + f)(\mathbf{m}^T \mathbf{V} \mathbf{m})} \right) \tag{2.77.3}$$

For the given conic, the matrix parameters are

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}, f = 0 \tag{2.77.4}$$

with eigen parameters

$$\lambda_1 = 0, \lambda_2 = 1 \tag{2.77.5}$$

$$\mathbf{p_1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \mathbf{p_2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.77.6}$$

The vertex of the parabola can be expressed as

$$\begin{pmatrix} \mathbf{u^T} + \eta \mathbf{p_1^T} \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -\mathbf{f} \\ \eta \mathbf{p_1} - \mathbf{u} \end{pmatrix} \tag{2.77.7}$$

where,

$$\eta = \mathbf{u^T p_1} = -2 \tag{2.77.8}$$

$$\implies \begin{pmatrix} 0 & -2 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{c} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{2.77.9}$$

or

$$\mathbf{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.77.10}$$

From (2.77.3),

$$\mu_1 = 4 + 2\sqrt{3}, \mu_2 = 4 - 2\sqrt{3} \tag{2.77.11}$$

The given line is

$$\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = -2 \tag{2.77.12}$$

In parametric form, the given line can be written as:

$$L : \mathbf{x} = \mathbf{q} + \mu \mathbf{m} \tag{2.77.13}$$

$$\implies \mathbf{x} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.77.14}$$

Substituting μ_1 and μ_2 in (2.77.14),the points of intersection

$$\mathbf{K} = \begin{pmatrix} 2 + 2\sqrt{3} \\ 4 + 2\sqrt{3} \end{pmatrix}, \mathbf{L} = \begin{pmatrix} 2 - 2\sqrt{3} \\ 4 - 2\sqrt{3} \end{pmatrix} \tag{2.77.15}$$

a) Thus, from Fig. 2.7 the area enclosed by parabola and line can be given as

$$A = \text{Area under line} - \text{Area under parabola} \tag{2.77.16}$$

$$A = Ar(KLMNK) - Ar(KCLMCNK) \tag{2.77.17}$$

$$A = A_1 - A_2 \tag{2.77.18}$$

b) Area under the line y=x+2 i.e, A_1-

$$A_1 = \int_{2-2\sqrt{3}}^{2+2\sqrt{3}} y dx \tag{2.77.19}$$
$$= \int_{2-2\sqrt{3}}^{2+2\sqrt{3}} (x+2)\, dx \tag{2.77.20}$$
$$= \int_{2-2\sqrt{3}}^{2+2\sqrt{3}} x dx + \int_{2-2\sqrt{3}}^{2+2\sqrt{3}} 2dx \tag{2.77.21}$$
$$= \frac{1}{2}\left((2+2\sqrt{3})^2 - (2-2\sqrt{3})^2\right) + 2\left(4\sqrt{3}\right) \tag{2.77.22}$$
$$= 16\sqrt{3} \text{ units} \tag{2.77.23}$$

c) Area under the parabola that is A_2-

$$A_2 = \int_{2-2\sqrt{3}}^{2+2\sqrt{3}} y dx \tag{2.77.24}$$
$$= \int_{2-2\sqrt{3}}^{2+2\sqrt{3}} \frac{1}{4} x^2 dx \tag{2.77.25}$$
$$= \frac{1}{12} \int_{2-2\sqrt{3}}^{2+2\sqrt{3}} x^3 dx \tag{2.77.26}$$
$$= \frac{1}{12}\left((2+2\sqrt{3})^3 - \left(2-2\sqrt{3}\right)^3\right) \tag{2.77.27}$$
$$= 8\sqrt{3} \text{ units} \tag{2.77.28}$$

d) Putting (2.77.23) and (2.77.28) in (2.77.18) we get required area A as:

$$A = A_1 - A_2 \tag{2.77.29}$$
$$A = 8\sqrt{3} \text{ units} \tag{2.77.30}$$

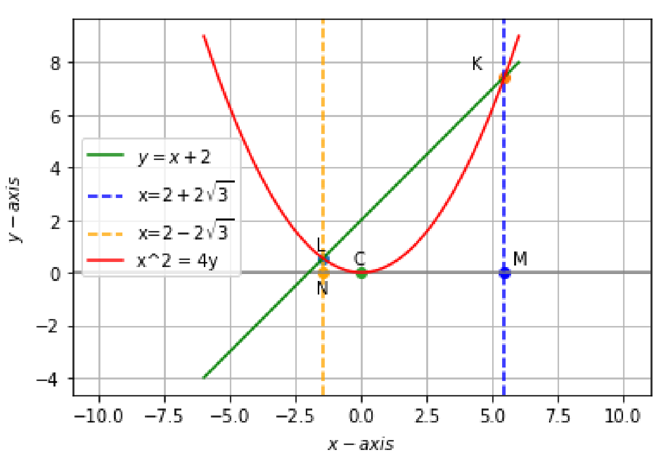

Fig. 2.7: Plot of the parabola and line

2.78. Find the area of the region bounded by the curve $y^2 = 4x$ and the line $x = 3$.
2.79. Find the area of the region bounded by the curve $y^2 = x$, y-axis and the line $y = 3$.
2.80. Find the area of the region bounded by the two parabolas $y = x^2, y^2 = x$.
2.81. Find the area lying above x-axis and included between the circle $\mathbf{x}^T\mathbf{x} - 8\begin{pmatrix}1 & 0\end{pmatrix} = 0$ and inside of the parabola $y^2 = 4x$.
2.82. Find the area lying between the curves $y^2 = 4x$ and $y = 2x$.
2.83. Find the area of the region bounded by the curves $y = x^2 + 2, y = x, x = 0$ and $x = 3$.
2.84. Find the area under $y = x^2, x = 1, x = 2$ and x-axis.
2.85. Find the area between $y = x^2$ and $y = x$.
2.86. Find the area of the region lying in the first quadrant and bounded by $y = 4x^2, x = 0, y = 1$ and $y = 4$.
2.87. Find the area of the smaller region bounded by the ellipse $\mathbf{x}^T\begin{pmatrix}\frac{1}{9} & 0\\ 0 & \frac{1}{4}\end{pmatrix}\mathbf{x} = 1$ and the line $\begin{pmatrix}\frac{1}{a} & \frac{1}{b}\end{pmatrix}\mathbf{x} = 1$
2.88. Find the area of the region enclosed by the parabola $x^2 = y$, the line $\begin{pmatrix}-1 & 1\end{pmatrix}\mathbf{x} = 2$ and the x-axis.
2.89. Find the area bounded by the curves

$$\left\{(x,y) : y > x^2, y = |x|\right\} \tag{2.89.1}$$

2.90. Find the area of the region

$$\left\{(x,y) : y^2 \le 4x, 4\mathbf{x}^T\mathbf{x} = 9\right\} \tag{2.90.1}$$

2.91. Find the area of the circle $\mathbf{x}^T\mathbf{x} = 16$ exterior to the parabola $y^2 = 6$.
2.92. Find the intervals in which the function given by

$$f(x) = 2x^2 - 3x \tag{2.92.1}$$

is

a) increasing
b) decreasing.

2.93. Find the intervals in which the following functions are strictly increasing or decreasing

a) $x^2 + 2x - 5$
b) $10 - 6x - 2x^2$
c) $6 - 9x - x^2$

2.94. Prove that the function f given by $f(x) = x^2 - x + 1$ is neither strictly increasing nor decreasing on $(1,-1)$.

2.95. Examine the continuity of the function $f(x) = 2x^2-1$ at $x = 3$.

2.96. Find all points of discontinuity of f, where $f$is defined by

$$f(x) = \begin{cases} x+1, & x \geq 1, \\ x^2+1, & x < 1, \end{cases} \tag{2.96.1}$$

2.97. For what value of $\lambda$is the function defined by

$$f(x) = \begin{cases} \lambda\left(x^2-2x\right), & x \leq 0, \\ 4x+1, & x > 0 \end{cases} \tag{2.97.1}$$

continuous at $x = 0$? What about continuity at $x = 1$?

2.98. For what value of k is the following function continuous at the given point.

$$f(x) = \begin{cases} kx^2, & x \leq 2, \\ 3, & x > 2, \end{cases} \quad x = 2 \tag{2.98.1}$$

2.99. Find $\frac{dy}{dx}$ in the following

$$x^2 + xy + y^2 = 100 \tag{2.99.1}$$

2.100. Verify Rolle's theorem for the function . $f(x) = x^2+2x-8, x \in [-4,2]$

2.101. Examine if Rolle's theorem is applicable to the following function $f(x) = x^2-1, x \in [1,2]$. Can you say some thing about the converse of Rolle's theorem from this example?

2.102. Examine the applicability of the mean value theorem for the function in Problem 2.100.

2.103. Find $\lim_{x\to 1} \pi r^2$.

2.104. Find $\lim_{x\to 0} f(x)$ where

$$f(x) = \begin{cases} x^2-1 & x \leq 1 \\ -x^2-1, & x > 1 \end{cases} \tag{2.104.1}$$

2.105. For some constans a and b, find the derivative of

$$(x-a)(x-b) \tag{2.105.1}$$

2.106. Integrate the following as limit of sums:

(i) $\int_2^3 x^2\,dx$

(ii) $\int_1^4 \left(x^2-x\right)\,dx$

2.107. Form the differential equation of the family of parabolas having vertex at origin and axis along positive y-axis.

2.108. Form the differential equation of the family of ellipses having foci on y-axis and centre at origin.

2.109. Form the differential equation of the family of hyperbolas having foci on x-axis and centre at origin.

2.110. The ceiling of a long hall is 25 m high. What is the maximum horizontal distance that a ball thrown with a speed of 40 ms^{-1} can go without hitting the ceiling of the hall ?

2.111. A cricketer can throw a ball to a maximum horizontal distance of 100 m. How much high above the ground can the cricketer throw the same ball ?

2.112. Find the normal to the curve $x^2 = 4y$ passing through $\begin{pmatrix}1\\2\end{pmatrix}$.

2.113. Find the area of the region bounded by the curve $y^2 = x$ and the lines $x = 1, x = 4$ and the x-axis in the first quadrant.

2.114. Find the area of the region bounded by $y^2 = 9x, x = 2, x = 4$ and the x-axis in the first quadrant.

2.115. Find the area of the region bounded by $x^2 = 4y, y = 2, y = 4$ and the y-axis in the first quadrant.

2.116. Find the area of the region bounded by the ellipse $\mathbf{x}^T\begin{pmatrix}\frac{1}{16} & 0\\0 & \frac{1}{9}\end{pmatrix}\mathbf{x} = 1$

2.117. Find the area of the region bounded by the ellipse $\mathbf{x}^T\begin{pmatrix}\frac{1}{4} & 0\\0 & \frac{1}{9}\end{pmatrix}\mathbf{x} = 1$

2.118. The area between $x = y^2$ and $x = 4$ is divided into two equal parts by the line $x = a$, find the value of a.

2.119. Find the area of the region bounded by the parabola $y = x^2$ and $y = |x|$.

2.120. Find the area bounded by the curve $x^2 = 4y$ and the line $\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = -2$.

2.121. Find the area of the region bounded by the curve $y^2 = 4x$ and the line $x = 3$.

2.122. Find the area of the region bounded by the curve $y^2 = x$, y-axis and the line $y = 3$.

2.123. Find the area of the region bounded by the two parabolas $y = x^2, y^2 = x$.

2.124. Find the area lying above x-axis and included between the circle $\mathbf{x}^T\mathbf{x}-8\begin{pmatrix}1 & 0\end{pmatrix} = 0$ and inside of the parabola $y^2 = 4x$.

2.125. AOBA is the part of the ellipse $\mathbf{x}^T\begin{pmatrix}9 & 0\\0 & 1\end{pmatrix}\mathbf{x} = 36$ in the first quadrant such that $OA = 2$ and

$OB = 6$. Find the area between the arc AB and the chord AB.

2.126. Find the area lying between the curves $y^2 = 4x$ and $y = 2x$.

2.127. Find the area of the region bounded by the curves $y = x^2 + 2, y = x, x = 0$ and $x = 3$.

2.128. Find the area under $y = x^2, x = 1, x = 2$ and x-axis.

2.129. Find the area between $y = x^2$ and $y = x$.

2.130. Find the area of the region lying in the first quadrant and bounded by $y = 4x^2, x = 0, y = 1$ and $y = 4$.

2.131. Find the area enclosed by the parabola $4y = 3x^2$ and the line $\begin{pmatrix}-3 & 2\end{pmatrix}\mathbf{x} = 12$.

2.132. Find the area of the smaller region bounded by the ellipse $\mathbf{x}^T\begin{pmatrix}\frac{1}{9} & 0\\ 0 & \frac{1}{4}\end{pmatrix}\mathbf{x} = 1$ and the line $\begin{pmatrix}\frac{1}{a} & \frac{1}{b}\end{pmatrix}\mathbf{x} = 1$

2.133. Find the area of the region enclosed by the parabola $x^2 = y$, the line $\begin{pmatrix}-1 & 1\end{pmatrix}\mathbf{x} = 2$ and the x-axis.

2.134. Find the area bounded by the curves

$$\left\{(x, y) : y > x^2, y = |x|\right\} \tag{2.134.1}$$

2.135. Find the area of the region

$$\left\{(x, y) : y^2 \le 4x, 4\mathbf{x}^T\mathbf{x} = 9\right\} \tag{2.135.1}$$

2.136. Find the area of the circle $\mathbf{x}^T\mathbf{x} = 16$ exterior to the parabola $y^2 = 6$.

2.137. Find the equation of the circle passing through $\begin{pmatrix}0\\0\end{pmatrix}$ and making intercepts a and b on the coordinate axes.

2.138. Find the locus of all the unit vectors in the xy-plane.

2.139. Find the area lying in the first quadrant and bounded by the circle $\mathbf{x}^T\mathbf{x} = 4$ and the lines $x = 0$ and $x = 2$.

2.140. Find the area of the circle $4\mathbf{x}^T\mathbf{x} = 9$.

2.141. Find the equation of the tangent to the curve,

$$y = \sqrt{3x - 2} \tag{2.141.1}$$

which is parallel to the line,

$$\begin{pmatrix}4 & 2\end{pmatrix}\mathbf{x} + 5 = 0 \tag{2.141.2}$$

2.142. Find the equations of the tangent and normal to the given curves at the indicated points: $y = x^2$ at $\begin{pmatrix}0\\0\end{pmatrix}$.

2.143. The line

$$\begin{pmatrix}-m & 1\end{pmatrix}\mathbf{x} = 1 \tag{2.143.1}$$

is a tangent to the curve $y^2 = 4x$. Find the value of m.

3 Appendices

3.1 Roots of a quadratic equation

Lemma 3.1. *Consider the parabola defined as*

$$y = ax^2 + bx + c \tag{143.1}$$

$$\text{or, } \mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{143.2}$$

where

$$\mathbf{V} = \begin{pmatrix}a & 0\\ 0 & 0\end{pmatrix} \tag{143.3}$$

$$\mathbf{u} = \begin{pmatrix}\frac{b}{2}\\ \frac{-1}{2}\end{pmatrix} \tag{143.4}$$

$$f = c \tag{143.5}$$

$$\eta = \mathbf{u}^T\mathbf{p_1} = \frac{-1}{2} \tag{143.6}$$

with eigenvalue decomposition

$$\mathbf{D} = \begin{pmatrix}0 & 0\\ 0 & a\end{pmatrix} \tag{143.7}$$

$$\mathbf{P} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \tag{143.8}$$

Then the quadratic equation in (143.1) *will not have real roots if*

$$\boxed{(\mathbf{p_1}^T\mathbf{c})(\mathbf{p_2}^T\mathbf{V}\mathbf{p_2}) > 0} \tag{143.9}$$

Proof. For the quadratic equation to not have any real roots, the y coordinate should always be either positive or negative . Express this in terms of the matrix/vector parameters of the parabola. Now,for y coordinate to be always positive,two conditions need to be satisfied:

1) y-coordinate of vertex $\mathbf{c}$ of parabola needs to be always positive.
2) Coefficient a of x^2 needs to be always positive.

$\therefore$ For condition 1,

$$\begin{pmatrix}\mathbf{u}^T + \eta\mathbf{p_1}^T\\ \mathbf{V}\end{pmatrix}\mathbf{c} = \begin{pmatrix}-f\\ \eta\mathbf{p_1} - \mathbf{u}\end{pmatrix} \tag{2.1}$$

$$\mathbf{p_1}^T\mathbf{c} > 0 \tag{2.2}$$

and for condition 2,

$$\mathbf{p_2}^T\mathbf{V}\mathbf{p_2} > 0 \tag{2.3}$$

Also, for y coordinate to be always negative, two conditions need to be satisfied:

1) y-coordinate of vertex $\mathbf{c}$ of parabola needs to be always negative.
2) Coefficient a of x^2 needs to be always negative.

$\therefore$ for condition 1,

$$\begin{pmatrix} \mathbf{u}^T + \eta \mathbf{p_1}^T \\ -\mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \eta \mathbf{p_1} - \mathbf{u} \end{pmatrix} \tag{2.1}$$

$$\mathbf{p_1}^T \mathbf{c} < 0 \tag{2.2}$$

and for condition 2,

$$\mathbf{p_2}^T \mathbf{V} \mathbf{p_2} < 0 \tag{2.3}$$

All the above can be clubbed together to obtain (143.9). □

3.2 Examples

1)

$$y = 21x^2 - 28x + 10 \tag{1.1}$$

Here,

$$\mathbf{V} = \begin{pmatrix} 21 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = -\begin{pmatrix} 14 \\ \frac{1}{2} \end{pmatrix}, f = 10 \tag{1.2}$$

Using eigenvalue decomposition,

$$\mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & 21 \end{pmatrix}, \mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{1.3}$$

$\therefore$Vertex $\mathbf{c}$ is given by

$$\begin{pmatrix} -14 & -1 \\ 21 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -10 \\ 14 \\ 0 \end{pmatrix} \tag{1.4}$$

$$\implies \begin{pmatrix} -14 & -1 \\ 21 & 0 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -10 \\ 14 \end{pmatrix} \tag{1.5}$$

$$\implies \mathbf{c} = \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \end{pmatrix} \tag{1.6}$$

Now,

$$\mathbf{p_1}^T \mathbf{c} = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \end{pmatrix} \tag{1.7}$$

$$= \frac{2}{3} \tag{1.8}$$

and,

$$\mathbf{p_2}^T \mathbf{V} \mathbf{p_2} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 21 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.9}$$

$$= 21 \tag{1.10}$$

$\because$

$$(\mathbf{p_1}^T \mathbf{c})(\mathbf{p_2}^T \mathbf{V} \mathbf{p_2}) = (\frac{2}{3})(21) = \frac{42}{3} > 0 \tag{1.11}$$

Hence,the given equation does not have any real roots.

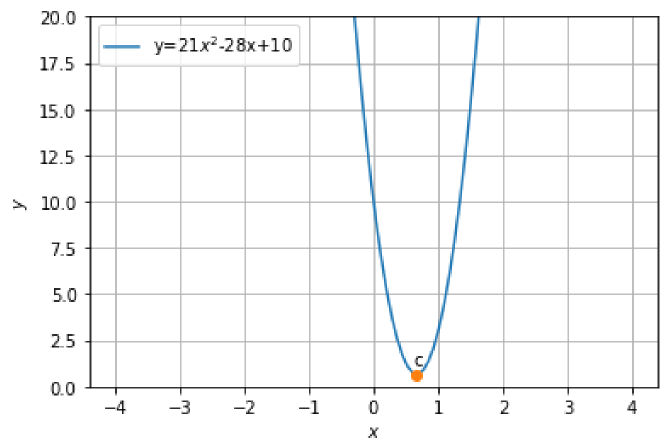

Fig. 3.1: $y = 21x^2 - 28x + 10$

2)

$$y = 6x^2 - x - 2 \tag{2.1}$$

Here,

$$\mathbf{V} = \begin{pmatrix} 6 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = -\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, f = -2 \tag{2.2}$$

Using eigenvalue decomposition,

$$\mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & 6 \end{pmatrix}, \mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.3}$$

$\therefore$Vertex $\mathbf{c}$ is given by

$$\begin{pmatrix} \frac{-1}{2} & -1 \\ 6 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{c} = \begin{pmatrix} 2 \\ \frac{1}{2} \\ 0 \end{pmatrix} \tag{2.4}$$

$$\implies \begin{pmatrix} \frac{-1}{2} & -1 \\ 6 & 0 \end{pmatrix} \mathbf{c} = \begin{pmatrix} 2 \\ \frac{1}{2} \end{pmatrix} \tag{2.5}$$

$$\implies \mathbf{c} = \begin{pmatrix} \frac{1}{12} \\ \frac{-49}{24} \end{pmatrix} \tag{2.6}$$

Now,

$$\mathbf{p_1}^T \mathbf{c} = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{12} \\ \frac{-49}{24} \end{pmatrix} \tag{2.7}$$

$$= \frac{-49}{24} \tag{2.8}$$

and,

$$\mathbf{p_2}^T\mathbf{V}\mathbf{p_2} = \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}6 & 0\\0 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} \tag{2.9}$$
$$= 6 \tag{2.10}$$

$\because$

$$(\mathbf{p_1}^T\mathbf{c})(\mathbf{p_2}^T\mathbf{V}\mathbf{p_2}) = (\frac{-49}{24})(6) = \frac{-49}{4} < 0 \tag{2.11}$$

Hence,the given equation has real roots.

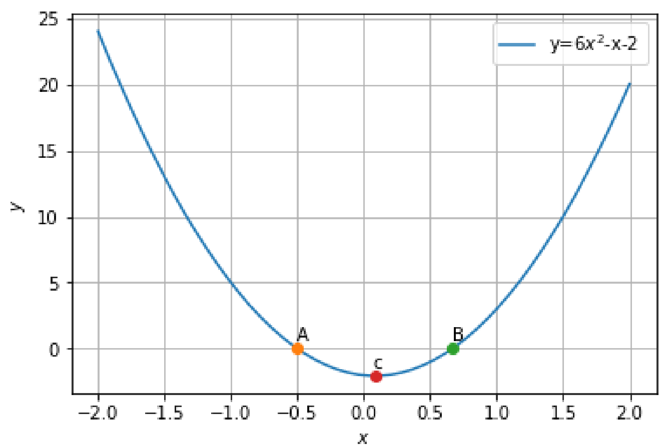

Fig. 3.2: $y = 6x^2 - x - 2$

3)

$$y = -x^2 - 4x - 5 \tag{3.1}$$

Here,

$$\mathbf{V} = \begin{pmatrix}-1 & 0\\0 & 0\end{pmatrix}, \mathbf{u} = -\begin{pmatrix}2\\\frac{1}{2}\end{pmatrix}, f = -5 \tag{3.2}$$

Using eigenvalue decomposition,

$$\mathbf{D} = \begin{pmatrix}0 & 0\\0 & -1\end{pmatrix}, \mathbf{P} = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} \tag{3.3}$$

$\therefore$Vertex $\mathbf{c}$ is given by

$$\begin{pmatrix}-2 & -1\\1 & 0\\0 & 0\end{pmatrix}\mathbf{c} = \begin{pmatrix}5\\2\\0\end{pmatrix} \tag{3.4}$$
$$\implies \begin{pmatrix}-2 & -1\\1 & 0\end{pmatrix}\mathbf{c} = \begin{pmatrix}5\\2\end{pmatrix} \tag{3.5}$$
$$\implies \mathbf{c} = \begin{pmatrix}2\\-1\end{pmatrix} \tag{3.6}$$

Now,

$$\mathbf{p_1}^T\mathbf{c} = \begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}2\\-1\end{pmatrix} \tag{3.7}$$
$$= -1 \tag{3.8}$$

and,

$$\mathbf{p_2}^T\mathbf{V}\mathbf{p_2} = \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}-1 & 0\\0 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} \tag{3.9}$$
$$= -1 \tag{3.10}$$

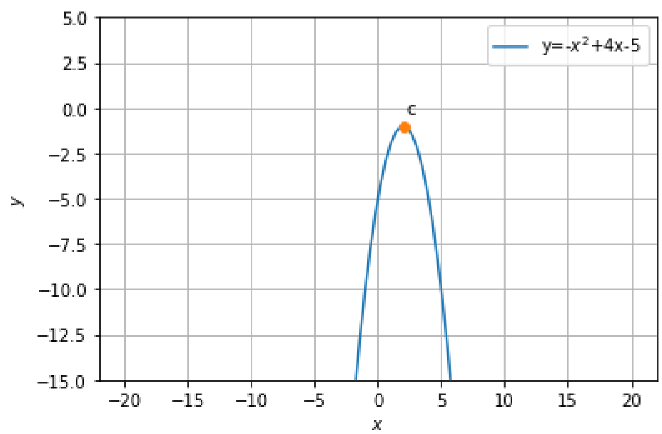

Fig. 3.3: $y = -x^2 + 4x - 5$

$\because$

$$(\mathbf{p_1}^T\mathbf{c})(\mathbf{p_2}^T\mathbf{V}\mathbf{p_2}) = (-1)(-1) = 1 > 0 \tag{3.11}$$

Hence,the given equation does not have real roots.

4)

$$y = -x^2 - 4x + 9 \tag{4.1}$$

Here,

$$\mathbf{V} = \begin{pmatrix}-1 & 0\\0 & 0\end{pmatrix}, \mathbf{u} = -\begin{pmatrix}2\\\frac{1}{2}\end{pmatrix}, f = 9 \tag{4.2}$$

Using eigenvalue decomposition,

$$\mathbf{D} = \begin{pmatrix}0 & 0\\0 & -1\end{pmatrix}, \mathbf{P} = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} \tag{4.3}$$

$\therefore$Vertex $\mathbf{c}$ is given by

$$\begin{pmatrix} -2 & -1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -9 \\ 2 \\ 0 \end{pmatrix} \tag{4.4}$$

$$\implies \begin{pmatrix} -2 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -9 \\ 2 \end{pmatrix} \tag{4.5}$$

$$\implies \mathbf{c} = \begin{pmatrix} 2 \\ 13 \end{pmatrix} \tag{4.6}$$

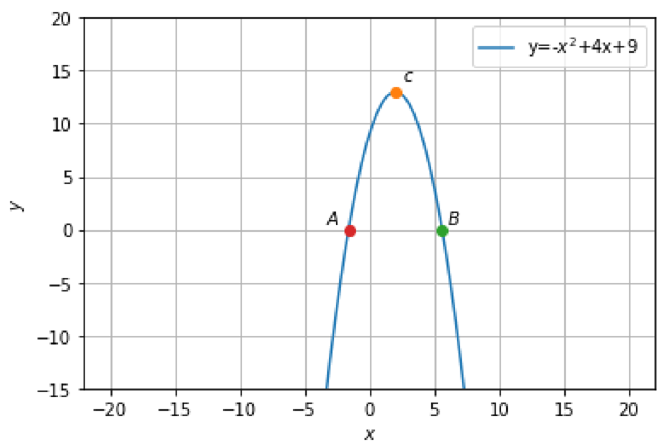

Fig. 3.4: $y = -x^2 + 4x + 9$

Now,

$$\mathbf{p_1}^T\mathbf{c} = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 13 \end{pmatrix} \tag{4.7}$$

$$= 13 \tag{4.8}$$

and,

$$\mathbf{p_2}^T\mathbf{V}\mathbf{p_2} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{4.9}$$

$$= -1 \tag{4.10}$$

$\because$

$$(\mathbf{p_1}^T\mathbf{c})(\mathbf{p_2}^T\mathbf{V}\mathbf{p_2}) = (13)(-1) = -13 < 0 \tag{4.11}$$

Hence,the given equation has real roots.

3.3 Axes of a conic

Lemma 3.2. *Let $\mathbf{p}_1, \mathbf{p}_2$ be the orthogonal eigenvectors of the symmetric matrix $\mathbf{V}$, defined by*

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{4.1}$$

The axes of the conics are given by

$$\mathbf{p}_1^T(\mathbf{x} - \mathbf{c}) = 0 \tag{4.2}$$

$$\mathbf{p}_2^T(\mathbf{x} - \mathbf{c}) = 0 \tag{4.3}$$

where,$\mathbf{c}$ is the vertex of conic.

Proof. According to the principal axis theorem,

1) Each eigen vector of $\mathbf{V}$ is parallel to either the major axis or minor axis.
2) Axes pass through the vertex $\mathbf{c}$ of the conic.

□

Examples:

1) Parabola

$$y^2 - 4x + 2y + 4 = 0 \tag{1.1}$$

Here,

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.2}$$

$$\mathbf{u} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \tag{1.3}$$

$$f = 4 \tag{1.4}$$

Now,

$$\begin{pmatrix} -4 & 1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -4 \\ 0 \\ -1 \end{pmatrix} \tag{1.5}$$

$$\implies \mathbf{c} = \begin{pmatrix} \frac{3}{4} \\ -1 \end{pmatrix} \tag{1.6}$$

So,

$$(\mathbf{x} - \mathbf{c}) = \begin{pmatrix} x - \frac{3}{4} \\ y + 1 \end{pmatrix} \tag{1.7}$$

Now,

$$\left|\mathbf{V} - \lambda\mathbf{I}\right| = 0 \tag{1.8}$$

$$\implies \begin{vmatrix} -\lambda & 0 \\ 0 & 1 - \lambda \end{vmatrix} = 0 \tag{1.9}$$

$$\implies \lambda_1 = 0, \lambda_2 = 1 \tag{1.10}$$

For $\lambda_1 = 0$,

$$\mathbf{V} - \lambda_1\mathbf{I} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.11}$$

$$\implies \mathbf{p_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.12}$$

Similarly for $\lambda_2 = 1$,

$$\mathbf{p_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.13}$$

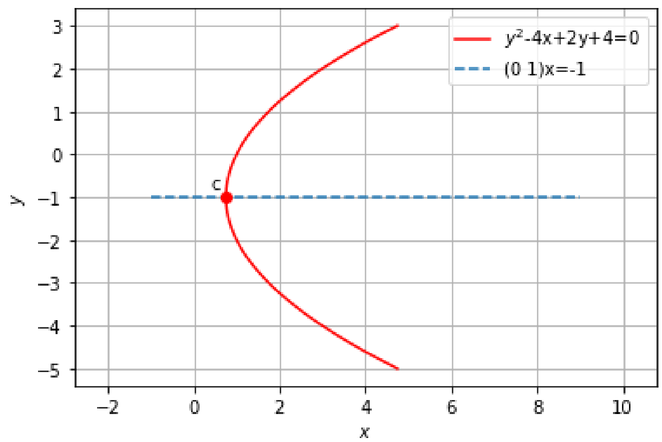

Fig. 3.5: y^2-4x+2y+4=0

$\because \lambda_2 > \lambda_1$
Hence,the axis using $\mathbf{p_2}$ is given by

$$\mathbf{p_2}^T(\mathbf{x} - \mathbf{c}) = 0 \tag{1.14}$$

$$\implies \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} x - \frac{3}{4} \\ y + 1 \end{pmatrix} = 0 \tag{1.15}$$

$$\implies y + 1 = 0 \tag{1.16}$$

$$\implies \boxed{\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = -1} \tag{1.17}$$

2) Parabola

$$y^2 = 8x \tag{2.1}$$

$$\implies y^2 - 8x = 0 \tag{2.2}$$

Here,

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.3}$$

$$\mathbf{u} = \begin{pmatrix} -4 \\ 0 \end{pmatrix} \tag{2.4}$$

$$f = 0 \tag{2.5}$$

Now,

$$\begin{pmatrix} -8 & 1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{c} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{2.6}$$

$$\implies \mathbf{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.7}$$

So,

$$(\mathbf{x} - \mathbf{c}) = \begin{pmatrix} x \\ y \end{pmatrix} \tag{2.8}$$

Now,

$$\left|\mathbf{V} - \lambda \mathbf{I}\right| = 0 \tag{2.9}$$

$$\implies \begin{vmatrix} -\lambda & 0 \\ 0 & 1 - \lambda \end{vmatrix} = 0 \tag{2.10}$$

$$\implies \lambda_1 = 0, \lambda_2 = 1 \tag{2.11}$$

For $\lambda_1 = 0$,

$$\mathbf{V} - \lambda_1 \mathbf{I} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.12}$$

$$\implies \mathbf{p_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.13}$$

Similarly for $\lambda_2 = 1$,

$$\mathbf{p_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.14}$$

$\because \lambda_2 > \lambda_1$
Hence,the axis using $\mathbf{p_2}$ is given by

$$\mathbf{p_2}^T(\mathbf{x} - \mathbf{c}) = 0 \tag{2.15}$$

$$\implies \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \tag{2.16}$$

$$\implies y = 0 \tag{2.17}$$

$$\implies \boxed{\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 0} \tag{2.18}$$

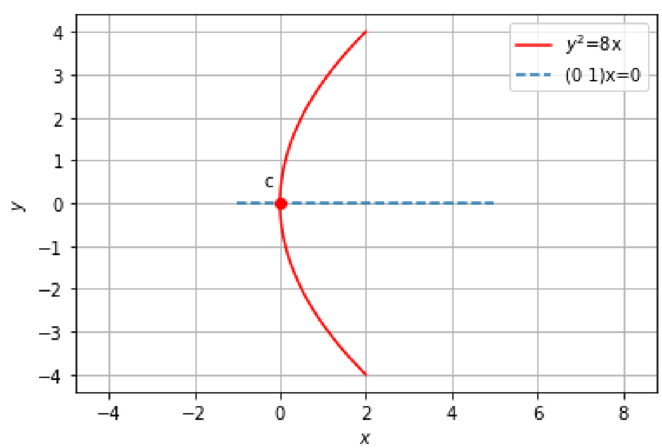

Fig. 3.6: y^2=8x

3) Ellipse

$$x^2 + xy + y^2 = 100 \tag{3.1}$$

Here,

$$\mathbf{V} = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} \tag{3.2}$$

$$\mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.3}$$

$$f = -100 \tag{3.4}$$

Now,

$$\mathbf{c} = \mathbf{V}^{-1}\mathbf{u} \tag{3.5}$$

$$\implies \mathbf{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.6}$$

So,

$$(\mathbf{x} - \mathbf{c}) = \begin{pmatrix} x \\ y \end{pmatrix} \tag{3.7}$$

Now,

$$\left|\mathbf{V} - \lambda\mathbf{I}\right| = 0 \tag{3.8}$$

$$\implies \begin{vmatrix} 1-\lambda & \frac{1}{2} \\ \frac{1}{2} & 1-\lambda \end{vmatrix} = 0 \tag{3.9}$$

$$\implies \lambda^2 - 2\lambda + \frac{3}{4} = 0 \tag{3.10}$$

$$\implies \lambda_1 = \frac{1}{2}, \lambda_2 = \frac{3}{2} \tag{3.11}$$

For $\lambda_1 = \frac{1}{2}$,

$$\mathbf{V} - \lambda_1\mathbf{I} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \tag{3.12}$$

$$\implies \mathbf{p_1} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{3.13}$$

Similarly for $\lambda_2 = \frac{3}{2}$,

$$\mathbf{p_2} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{3.14}$$

$\because \lambda_2 > \lambda_1$
Hence,the major axis using $\mathbf{p_2}$ is given by

$$\mathbf{p_2}^T(\mathbf{x} - \mathbf{c}) = 0 \tag{3.15}$$

$$\implies \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \tag{3.16}$$

$$\implies x + y = 0 \tag{3.17}$$

$$\implies \boxed{\begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} = 0} \tag{3.18}$$

and the minor axis using $\mathbf{p_1}$ is given by

$$\mathbf{p_1}^T(\mathbf{x} - \mathbf{c}) = 0 \tag{3.19}$$

$$\implies \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \tag{3.20}$$

$$\implies -x + y = 0 \tag{3.21}$$

$$\implies \boxed{\begin{pmatrix} -1 & 1 \end{pmatrix}\mathbf{x} = 0} \tag{3.22}$$

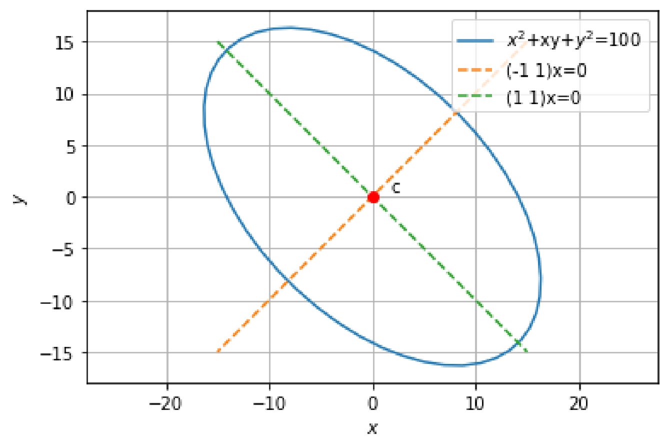

Fig. 3.7: x^2+xy+y^2=100

4) Hyperbola

$$xy - 3y + 2 = 0 \tag{4.1}$$

Here,

$$\mathbf{V} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{4.2}$$

$$\mathbf{u} = \frac{-3}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{4.3}$$

$$f = 2 \tag{4.4}$$

Now,

$$\mathbf{c} = \mathbf{V}^{-1}\mathbf{u} \tag{4.5}$$

$$\implies \mathbf{c} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \tag{4.6}$$

So,

$$(\mathbf{x} - \mathbf{c}) = \begin{pmatrix} x-3 \\ y \end{pmatrix} \tag{4.7}$$

Now,

$$\left|\mathbf{V} - \lambda \mathbf{I}\right| = 0 \tag{4.8}$$

$$\implies \begin{vmatrix} -\lambda & \frac{1}{2} \\ \frac{1}{2} & -\lambda \end{vmatrix} = 0 \tag{4.9}$$

$$\implies \lambda^2 - \frac{1}{4} = 0 \tag{4.10}$$

$$\implies \lambda_1 = \frac{-1}{2}, \lambda_2 = \frac{1}{2} \tag{4.11}$$

For $\lambda_1 = \frac{-1}{2}$,

$$\mathbf{V} - \lambda_1 \mathbf{I} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \tag{4.12}$$

$$\implies \mathbf{p_1} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{4.13}$$

Similarly for $\lambda_2 = \frac{1}{2}$,

$$\mathbf{p_2} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{4.14}$$

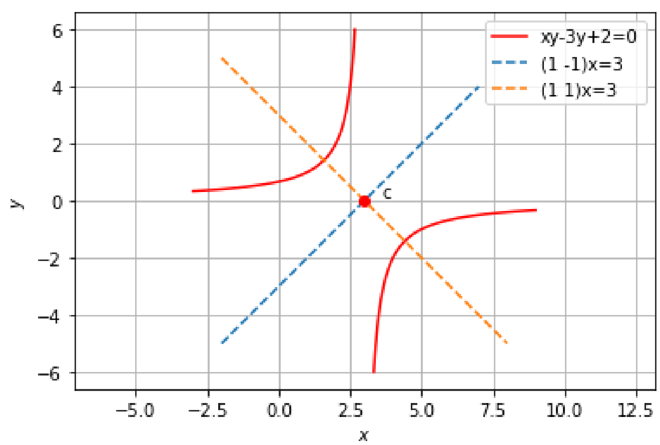

Fig. 3.8: xy-3y+2=0

$\because \lambda_2 > \lambda_1$
Hence,the major axis using $\mathbf{p_2}$ is given by

$$\mathbf{p_2}^T \left(\mathbf{x} - \mathbf{c}\right) = 0 \tag{4.15}$$

$$\implies \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} x - 3 \\ y \end{pmatrix} = 0 \tag{4.16}$$

$$\implies x + y = 3 \tag{4.17}$$

$$\implies \boxed{\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 3} \tag{4.18}$$

and the minor axis using $\mathbf{p_1}$ is given by

$$\mathbf{p_1}^T \left(\mathbf{x} - \mathbf{c}\right) = 0 \tag{4.19}$$

$$\implies \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} x - 3 \\ y \end{pmatrix} = 0 \tag{4.20}$$

$$\implies x - y = 3 \tag{4.21}$$

$$\implies \boxed{\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 3} \tag{4.22}$$