Linear Forms

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Contents

***Abstract*—This book provides a computational approach to school geometry based on the NCERT textbooks from Class 6-12. Links to sample Python codes are available in the text.**

Download python codes using

```
svn co https://github.com/gadepall/school/trunk/
    ncert/computation/codes
```

1 Exercises

1.1. Check which of the following are solutions of the equation

$$\begin{pmatrix}1 & -2\end{pmatrix}\mathbf{x} = 4 \tag{1.1.1}$$

a) $\begin{pmatrix}0\\2\end{pmatrix}$

b) $\begin{pmatrix}2\\0\end{pmatrix}$

c) $\begin{pmatrix}4\\0\end{pmatrix}$

d) $\begin{pmatrix}\sqrt{2}\\4\sqrt{2}\end{pmatrix}$

e) $\begin{pmatrix}1\\1\end{pmatrix}$

Solution:

Let

$$\mathbf{A} = \begin{pmatrix}0\\2\end{pmatrix} \tag{1.1.2}$$

$$\mathbf{B} = \begin{pmatrix}4\\0\end{pmatrix} \tag{1.1.3}$$

$$\mathbf{C} = \begin{pmatrix}2\\0\end{pmatrix} \tag{1.1.4}$$

$$\mathbf{D} = \begin{pmatrix}\sqrt{2}\\4\sqrt{2}\end{pmatrix} \tag{1.1.5}$$

$$\mathbf{E} = \begin{pmatrix}1\\1\end{pmatrix} \tag{1.1.6}$$

If

$$\boxed{\mathbf{y} = \begin{pmatrix}1 & -2\end{pmatrix}\mathbf{x}}, \tag{1.1.7}$$

Substituting (1.1.2)in (1.1.7),

$$\mathbf{x} = \mathbf{A} = \begin{pmatrix}0\\2\end{pmatrix} \text{in(1.1.7)} \tag{1.1.8}$$

$$\mathbf{y} = \begin{pmatrix}1 & -2\end{pmatrix}\begin{pmatrix}0\\2\end{pmatrix} \tag{1.1.9}$$

$$\boxed{\mathbf{y} = -4} \tag{1.1.10}$$

Substitute (1.1.3)in (1.1.7)

$$\mathbf{x} = \mathbf{B} = \begin{pmatrix}4\\0\end{pmatrix} \text{in(1.1.7)} \tag{1.1.11}$$

$$\mathbf{y} = \begin{pmatrix}1 & -2\end{pmatrix}\begin{pmatrix}4\\0\end{pmatrix} \tag{1.1.12}$$

$$\boxed{\mathbf{y} = 4} \tag{1.1.13}$$

Substituting (1.1.4)in (1.1.7)

$$\mathbf{x} = \mathbf{C} = \begin{pmatrix}2\\0\end{pmatrix} \text{in(1.1.7)} \tag{1.1.14}$$

$$\mathbf{y} = \begin{pmatrix}1 & -2\end{pmatrix}\begin{pmatrix}2\\0\end{pmatrix} \tag{1.1.15}$$

$$\boxed{\mathbf{y} = 2} \tag{1.1.16}$$

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Substituting (1.1.5)in (1.1.7)

$$\mathbf{x} = \mathbf{D} = \begin{pmatrix} \sqrt{2} \\ 4\sqrt{2} \end{pmatrix} \text{in}(1.1.7) \tag{1.1.17}$$

$$\mathbf{y} = \begin{pmatrix} 1 & -2 \end{pmatrix} \begin{pmatrix} \sqrt{2} \\ 4\sqrt{2} \end{pmatrix} \tag{1.1.18}$$

$$\boxed{\mathbf{y} = -7\sqrt{2}} \tag{1.1.19}$$

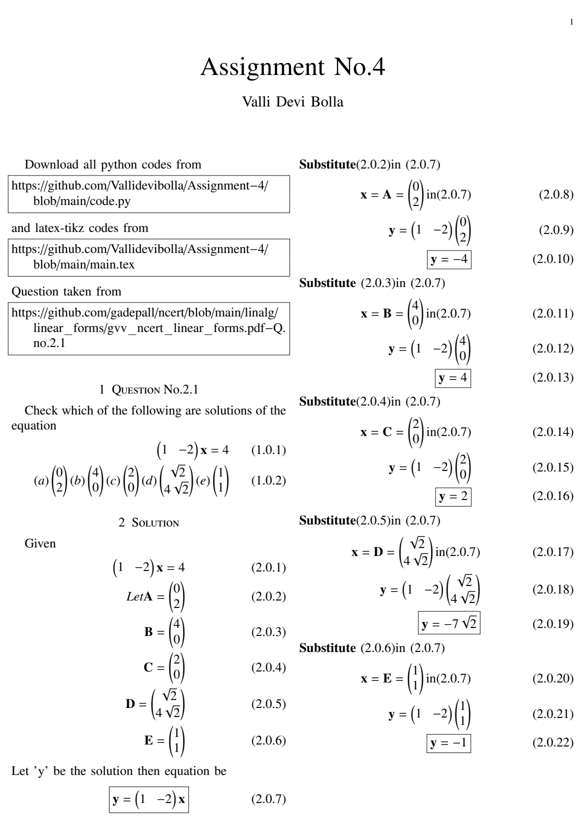
Substituting (1.1.6)in (1.1.7)

$$\mathbf{x} = \mathbf{E} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{in}(1.1.7) \tag{1.1.20}$$

$$\mathbf{y} = \begin{pmatrix} 1 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.1.21}$$

$$\boxed{\mathbf{y} = -1} \tag{1.1.22}$$

Thus, **B** is the desired solution as can be seen from Fig. 1.1.



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Assignment No.4

Valli Devi Bolla

Download all python codes from

https://github.com/Vallidevibolla/Assignment−4/blob/main/code.py

and latex-tikz codes from

https://github.com/Vallidevibolla/Assignment−4/blob/main/main.tex

Question taken from

https://github.com/gadepall/ncert/blob/main/linalg/linear_forms/gvv_ncert_linear_forms.pdf−Q.no.2.1

1 Question No.2.1

Check which of the following are solutions of the equation

$$\begin{pmatrix} 1 & -2 \end{pmatrix} \mathbf{x} = 4 \tag{1.0.1}$$

$$(a)\begin{pmatrix} 0 \\ 2 \end{pmatrix} (b)\begin{pmatrix} 4 \\ 0 \end{pmatrix} (c)\begin{pmatrix} 2 \\ 0 \end{pmatrix} (d)\begin{pmatrix} \sqrt{2} \\ 4\sqrt{2} \end{pmatrix} (e)\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.0.2}$$

2 Solution

Given

$$\begin{pmatrix} 1 & -2 \end{pmatrix} \mathbf{x} = 4 \tag{2.0.1}$$

$$Let\mathbf{A} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \tag{2.0.2}$$

$$\mathbf{B} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \tag{2.0.3}$$

$$\mathbf{C} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \tag{2.0.4}$$

$$\mathbf{D} = \begin{pmatrix} \sqrt{2} \\ 4\sqrt{2} \end{pmatrix} \tag{2.0.5}$$

$$\mathbf{E} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.0.6}$$

Let 'y' be the solution then equation be

$$\boxed{\mathbf{y} = \begin{pmatrix} 1 & -2 \end{pmatrix} \mathbf{x}} \tag{2.0.7}$$

Substitute(2.0.2)in (2.0.7)

$$\mathbf{x} = \mathbf{A} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \text{in}(2.0.7) \tag{2.0.8}$$

$$\mathbf{y} = \begin{pmatrix} 1 & -2 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \end{pmatrix} \tag{2.0.9}$$

$$\boxed{\mathbf{y} = -4} \tag{2.0.10}$$

Substitute (2.0.3)in (2.0.7)

$$\mathbf{x} = \mathbf{B} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \text{in}(2.0.7) \tag{2.0.11}$$

$$\mathbf{y} = \begin{pmatrix} 1 & -2 \end{pmatrix} \begin{pmatrix} 4 \\ 0 \end{pmatrix} \tag{2.0.12}$$

$$\boxed{\mathbf{y} = 4} \tag{2.0.13}$$

Substitute(2.0.4)in (2.0.7)

$$\mathbf{x} = \mathbf{C} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \text{in}(2.0.7) \tag{2.0.14}$$

$$\mathbf{y} = \begin{pmatrix} 1 & -2 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix} \tag{2.0.15}$$

$$\boxed{\mathbf{y} = 2} \tag{2.0.16}$$

Substitute(2.0.5)in (2.0.7)

$$\mathbf{x} = \mathbf{D} = \begin{pmatrix} \sqrt{2} \\ 4\sqrt{2} \end{pmatrix} \text{in}(2.0.7) \tag{2.0.17}$$

$$\mathbf{y} = \begin{pmatrix} 1 & -2 \end{pmatrix} \begin{pmatrix} \sqrt{2} \\ 4\sqrt{2} \end{pmatrix} \tag{2.0.18}$$

$$\boxed{\mathbf{y} = -7\sqrt{2}} \tag{2.0.19}$$

Substitute (2.0.6)in (2.0.7)

$$\mathbf{x} = \mathbf{E} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{in}(2.0.7) \tag{2.0.20}$$

$$\mathbf{y} = \begin{pmatrix} 1 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.0.21}$$

$$\boxed{\mathbf{y} = -1} \tag{2.0.22}$$

Fig. 1.1: Solution

1.2. Find the value of k, if $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is a solution of the equation

Solution:

We have

$$\begin{pmatrix} 2 & 3 \end{pmatrix} \mathbf{x} = k \tag{1.2.1}$$

where,

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \tag{1.2.2}$$

Let

$$\mathbf{A} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{1.2.3}$$

be the solution of (1.2.1). Substituting **A** in (1.2.1), we get,

$$k = 7 \tag{1.2.4}$$

See Fig. 1.2.

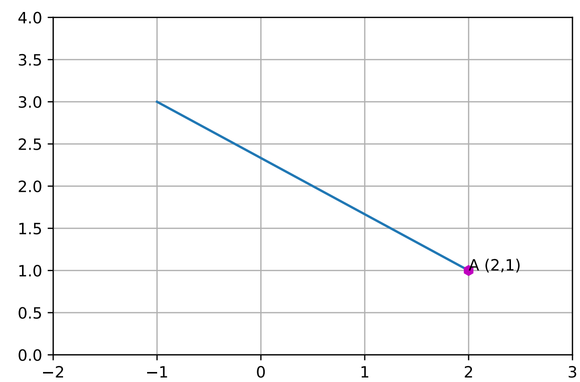

Fig. 1.2

$$\begin{pmatrix} 2 & 3 \end{pmatrix} \mathbf{x} = k \tag{1.2.5}$$

1.3. Draw the graphs of the following equations

a) $\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 4$
b) $\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 2$
c) $\begin{pmatrix} 3 & -1 \end{pmatrix} \mathbf{x} = 0$
d) $\begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = 3$
e) $\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 0$
f) $\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 0$
g) $\begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} = 0$
h) $\begin{pmatrix} 7 & -3 \end{pmatrix} \mathbf{x} = 2$
i) $\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 0$
j) $\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = -2$
k) $\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 2$

l) $\begin{pmatrix}1 & 2\end{pmatrix}\mathbf{x} = 6$

Solution:

a)

b) Substituting $\mathbf{x} = \begin{pmatrix}x\\0\end{pmatrix}$,

$$\begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}x\\0\end{pmatrix} = 2 \tag{1.3.1}$$

$$\implies x = 2 \tag{1.3.2}$$

Also, substituting $\mathbf{x} = \begin{pmatrix}0\\y\end{pmatrix}$,

$$\begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}0\\y\end{pmatrix} = 2 \tag{1.3.3}$$

$$\implies y = -2 \tag{1.3.4}$$

Thus,

$$\mathbf{P} = \begin{pmatrix}2\\0\end{pmatrix}, \mathbf{Q} = \begin{pmatrix}0\\-2\end{pmatrix} \tag{1.3.5}$$

c) Since the constant $c = 0$, the line passes through the origin Substituting $\mathbf{x} = \begin{pmatrix}1\\y\end{pmatrix}$ in the equation,

$$\mathbf{A} = \begin{pmatrix}0\\0\end{pmatrix}, \mathbf{B} = \begin{pmatrix}1\\1\end{pmatrix} \tag{1.3.6}$$

See Fig. 1.3.

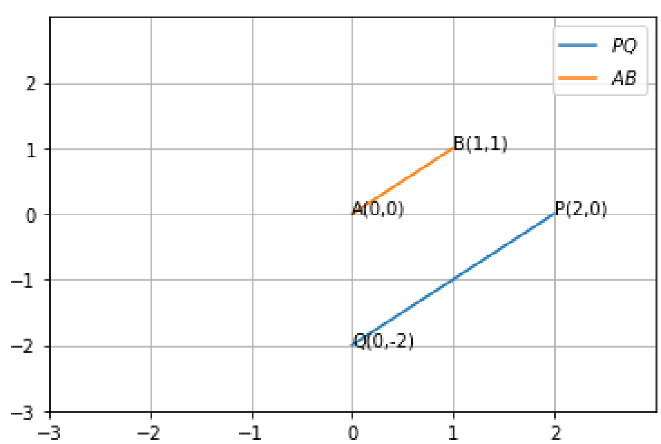

Fig. 1.3: Graphs of Equations (a) and (b)

1.4. Give the equations of two lines passing through $\begin{pmatrix}2\\14\end{pmatrix}$. How many more such lines are there, and why?

1.5. Find out whether the following pair of linear equations are consistent, or inconsistent.

a)

$$\begin{aligned}\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x} &= 5\\ \begin{pmatrix}2 & -3\end{pmatrix}\mathbf{x} &= 7\end{aligned} \tag{1.5.1}$$

b)

$$\begin{aligned}\begin{pmatrix}2 & -3\end{pmatrix}\mathbf{x} &= 8\\ \begin{pmatrix}4 & -6\end{pmatrix}\mathbf{x} &= 9\end{aligned} \tag{1.5.2}$$

c)

$$\begin{aligned}\begin{pmatrix}\frac{3}{2} & \frac{5}{3}\end{pmatrix}\mathbf{x} &= 7\\ \begin{pmatrix}9 & -10\end{pmatrix}\mathbf{x} &= 14\end{aligned} \tag{1.5.3}$$

d)

$$\begin{aligned}\begin{pmatrix}5 & -3\end{pmatrix}\mathbf{x} &= 11\\ \begin{pmatrix}-10 & 6\end{pmatrix}\mathbf{x} &= -22\end{aligned} \tag{1.5.4}$$

e)

$$\begin{aligned}\begin{pmatrix}\frac{4}{3} & 2\end{pmatrix}\mathbf{x} &= 8\\ \begin{pmatrix}2 & 3\end{pmatrix}\mathbf{x} &= 12\end{aligned} \tag{1.5.5}$$

Solution:

a) The above equations can be expressed as the matrix equation

$$\begin{pmatrix}3 & 2\\2 & -3\end{pmatrix}\mathbf{x} = \begin{pmatrix}5\\7\end{pmatrix} \tag{1.5.6}$$

The augmented matrix for the above equation is row reduced as follows

$$\begin{pmatrix}3 & 2 & 5\\2 & -3 & 7\end{pmatrix} \xleftrightarrow{R_1\leftarrow\frac{1}{3}R_1} \begin{pmatrix}1 & \frac{2}{3} & \frac{5}{3}\\2 & -3 & 7\end{pmatrix} \tag{1.5.7}$$

$$\xleftrightarrow{R_2\leftarrow-2R_1+R_2} \begin{pmatrix}1 & \frac{2}{3} & \frac{5}{3}\\0 & \frac{-13}{3} & \frac{11}{3}\end{pmatrix} \tag{1.5.8}$$

$$\xleftrightarrow{R_2\leftarrow\frac{-3}{13}R_2} \begin{pmatrix}1 & \frac{2}{3} & \frac{5}{3}\\0 & 1 & \frac{-11}{13}\end{pmatrix} \tag{1.5.9}$$

$$\xleftrightarrow{R_1\leftarrow\frac{-2}{3}R_2+R_1} \begin{pmatrix}1 & 0 & \frac{29}{13}\\0 & 1 & \frac{-11}{13}\end{pmatrix} \tag{1.5.10}$$

resulting in

$$\begin{pmatrix}3 & 2 & 5\\2 & -3 & 7\end{pmatrix} \tag{1.5.11}$$

with 2 nonzero rows. So it's rank is 2. The rank of the following matrix is also 2.

$$\begin{pmatrix} 3 & 2 \\ 2 & -3 \end{pmatrix} \tag{1.5.12}$$

$\therefore$ lines are consistent and give a unique solution.

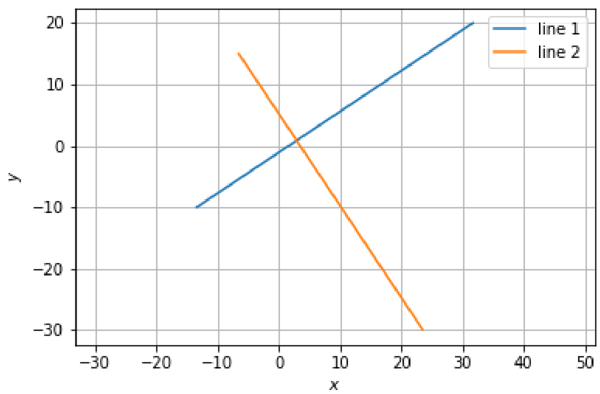

Fig. 1.5: Graphical solution

This is verified in Fig. 1.5.

b)

c) The given equations can be expressed as the matrix equation

$$\begin{pmatrix} \frac{3}{2} & \frac{5}{3} \\ 9 & -10 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 7 \\ 14 \end{pmatrix} \tag{1.5.13}$$

The augmented matrix is row reduced as

$$\begin{pmatrix} \frac{3}{2} & \frac{5}{3} & 7 \\ 9 & -10 & 14 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{2}{3} R_1} \begin{pmatrix} 1 & \frac{10}{9} & \frac{14}{3} \\ 9 & -10 & 14 \end{pmatrix} \tag{1.5.14}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - 9R_1} \begin{pmatrix} 1 & \frac{10}{9} & \frac{14}{3} \\ 0 & -20 & -28 \end{pmatrix} \tag{1.5.15}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{-1}{20} R_2} \begin{pmatrix} 1 & \frac{10}{9} & \frac{14}{3} \\ 0 & 1 & \frac{7}{5} \end{pmatrix} \tag{1.5.16}$$

$$\xleftrightarrow{R_1 \leftarrow \frac{-10}{9} R_2 + R_1} \begin{pmatrix} 1 & 0 & \frac{28}{9} \\ 0 & 1 & \frac{7}{5} \end{pmatrix} \tag{1.5.17}$$

$\therefore$ the given system is consistent as can be verified from Fig. 1.5

1.6. Find the slope of the line, which makes an angle of 30° of y-axis measured anticlockwise.

1.7. Write the equations for the x and y axes.

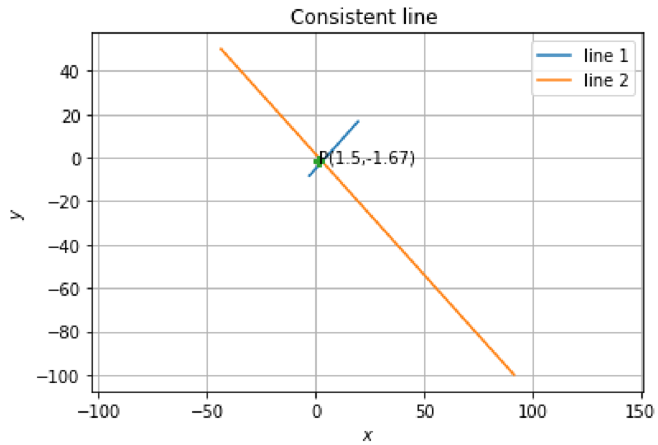

Fig. 1.5: Graphical solution

1.8. Find the equation of the line satisfying the following conditions

a) passing through the point $\begin{pmatrix} -4 \\ 3 \end{pmatrix}$ with slope $\frac{1}{2}$.

b) passing through the point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ with slope m.

c) passing through the point $\begin{pmatrix} 2 \\ 2\sqrt{3} \end{pmatrix}$ and inclined with the x-axis at an angle of 75°.

d) Intersecting the x-axis at a distance of 3 units to the let of the origin with slope -2.

e) intersecting the y-axis at a distance of 2 units above the origin and making an angle of 30° with the positive direction of the x-axis.

f) passing through the points $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -4 \end{pmatrix}$.

g) perpendicular distance from the origin is 5 and the angle made by the perpendicular with the positive x-axis is 30°.

Solution:

a) i) Given point $\mathbf{P} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$ and slope $m = \frac{1}{2}$.

The direction vector is $\mathbf{m} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Hence, the normal vector

$$\mathbf{n} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{m} \tag{1.8.1}$$

$$= \begin{pmatrix} -1 \\ 2 \end{pmatrix} \tag{1.8.2}$$

The equation of the line in terms of the normal vector is then obtained as

$$\mathbf{n}^T(\mathbf{x}-\mathbf{A}) = 0 \tag{1.8.3}$$

$$\implies \begin{pmatrix}-1 & 2\end{pmatrix}\mathbf{x} = 10 \tag{1.8.4}$$

See Fig. 1.8

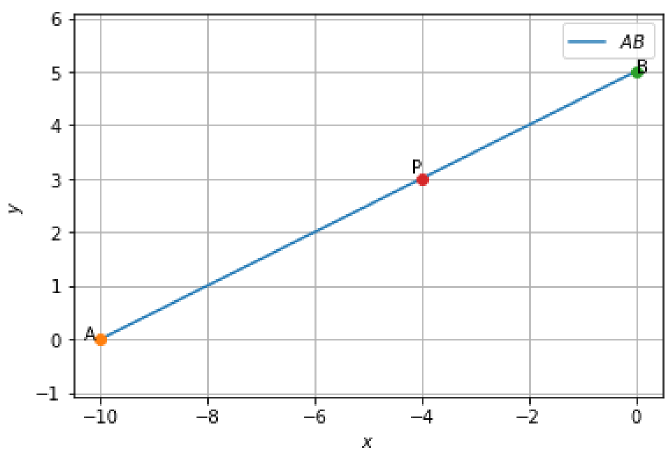

Fig. 1.8: Plot of Line AB (Part-1)

ii) Given point $\mathbf{P} = \begin{pmatrix}2\\2\sqrt{3}\end{pmatrix}$.From the given information we have, $\tan 75° = m = \frac{\sqrt{3}+1}{\sqrt{3}-1}$. The direction vector is $\mathbf{m} = \begin{pmatrix}1\\\tan 75°\end{pmatrix}$. Hence, the normal vector

$$\mathbf{n} = \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\mathbf{m} \tag{1.8.5}$$

$$= \begin{pmatrix}-\tan 75°\\1\end{pmatrix} \tag{1.8.6}$$

The equation of the line in terms of the normal vector is then obtained as

$$\mathbf{n}^T(\mathbf{x}-\mathbf{A}) = 0 \tag{1.8.7}$$

$$\implies \begin{pmatrix}-\sqrt{3}+1 & \sqrt{3}-1\end{pmatrix}\mathbf{x} = -4\left(\sqrt{3}-1\right) \tag{1.8.8}$$

See Fig. 1.8

iii)

iv) Given point $\mathbf{A} = \begin{pmatrix}-3\\0\end{pmatrix}$ and slope $m = -2$. The direction vector is $\mathbf{m} = \begin{pmatrix}1\\-2\end{pmatrix}$. Hence,

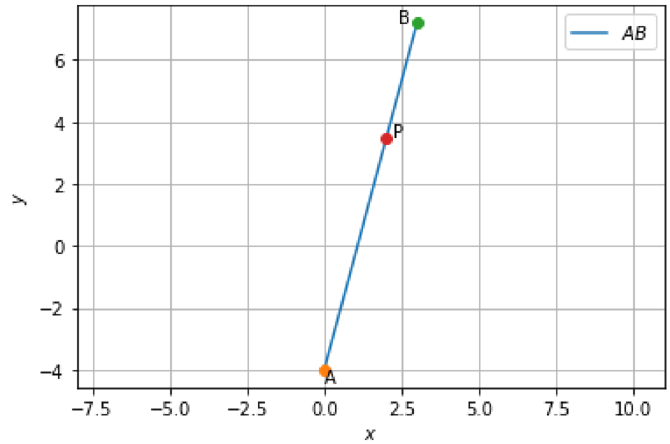

Fig. 1.8: Plot of Line AB (Part-2)

the normal vector

$$\mathbf{n} = \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\mathbf{m} \tag{1.8.9}$$

$$= \begin{pmatrix}2\\1\end{pmatrix} \tag{1.8.10}$$

The equation of the line in terms of the normal vector is then obtained as

$$\mathbf{n}^T(\mathbf{x}-\mathbf{A}) = 0 \tag{1.8.11}$$

$$\implies \begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} = -6 \tag{1.8.12}$$

and plotted in Fig. 1.8.

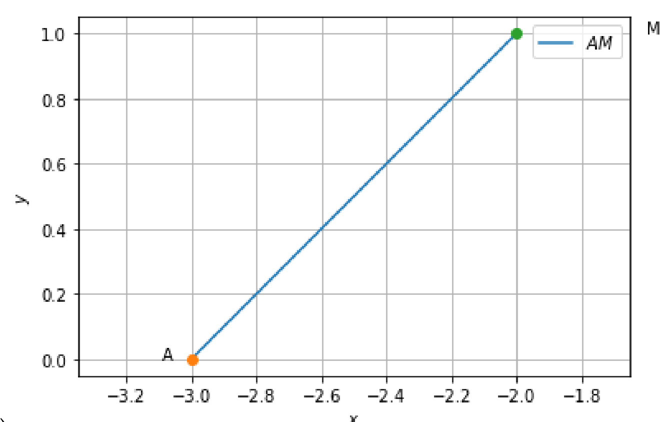

Fig. 1.8: Plot of Line AB (Part-1)

v) Given point $\mathbf{A} = \begin{pmatrix}0\\2\end{pmatrix}$.From the given information we have, $\tan 30° = m = \frac{1}{\sqrt{3}}$. The direction vector is $\mathbf{m} = \begin{pmatrix}1\\\tan 30°\end{pmatrix}$.

Hence, the normal vector

$$\mathbf{n} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{m} \tag{1.8.13}$$

$$= \begin{pmatrix} -\tan 30^\circ \\ 1 \end{pmatrix} \tag{1.8.14}$$

The equation of the line in terms of the normal vector is then obtained as

$$\mathbf{n}^T (\mathbf{x} - \mathbf{A}) = 0 \tag{1.8.15}$$

$$\implies \begin{pmatrix} -1 & \sqrt{3} \end{pmatrix} \mathbf{x} = 2\sqrt{3} \tag{1.8.16}$$

and plotted in Fig. 1.8.

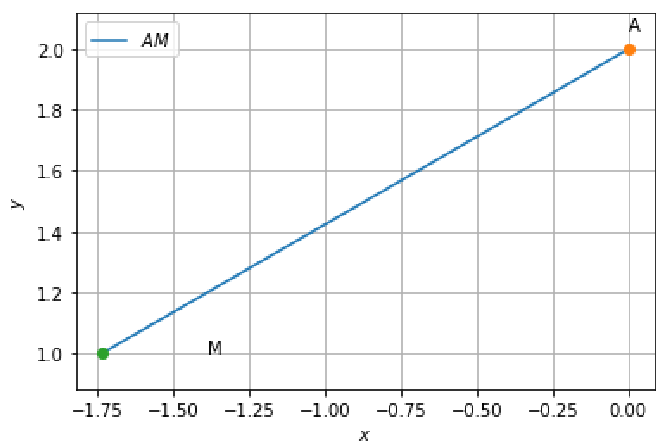

Fig. 1.8: Plot of Line AB (Part-2)

1.9. Find the direction vectors and and y-intercepts of the following lines
   a) $\begin{pmatrix} 1 & 7 \end{pmatrix} \mathbf{x} = 0.$
   b) $\begin{pmatrix} 6 & 3 \end{pmatrix} \mathbf{x} = 5.$
   c) $\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 0.$
1.10. Find the perpendicular distances of the following lines from the origin and angle between the perpendicular and the positive x-axis.
   a) $\begin{pmatrix} 1 & -\sqrt{3} \end{pmatrix} \mathbf{x} = -8.$
   b) $\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 2.$
   c) $\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 4.$
1.11. Find the equation of the line parallel to the line

$$\begin{pmatrix} 3 & -4 \end{pmatrix} \mathbf{x} = -2 \tag{1.11.1}$$

and passing through the point $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$.
1.12. The hypotenuse of a right angled triangle has its ends at the points $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -4 \\ 1 \end{pmatrix}$. Find an equation of the legs of the triangle.
1.13. If the lines

$$\begin{pmatrix} -3 & 1 \end{pmatrix} \mathbf{x} = 1 \tag{1.13.1}$$

$$\begin{pmatrix} -1 & 2 \end{pmatrix} \mathbf{x} = 3 \tag{1.13.2}$$

are equally inclined to the line

$$\begin{pmatrix} -m & 1 \end{pmatrix} \mathbf{x} = 4, \tag{1.13.3}$$

find the value of m.
1.14. The sum of the perpendicular distances of a variable point $\mathbf{P}$ from the lines

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{1.14.1}$$

$$\begin{pmatrix} 3 & -2 \end{pmatrix} \mathbf{x} = -7 \tag{1.14.2}$$

is always 10. Show that $\mathbf{P}$ must move on a line.
1.15. Find the equation of the line which is equidistant from parallel lines

$$\begin{pmatrix} 9 & 7 \end{pmatrix} \mathbf{x} = 7 \tag{1.15.1}$$

$$\begin{pmatrix} 3 & 2 \end{pmatrix} \mathbf{x} = -6. \tag{1.15.2}$$

Solution:
In general, we can obtain the following lemma:

Lemma 1.1. *Given the two following parallel lines:*

$$a\mathbf{n}^T \mathbf{x} - c_1 = 0 \tag{1.15.3}$$

$$b\mathbf{n}^T \mathbf{x} - c_2 = 0 \tag{1.15.4}$$

The line equidistant from both parallel lines would be given by:

$$\mathbf{n}^T \mathbf{x} - \frac{1}{2}\left(\frac{c_1}{a} + \frac{c_2}{b}\right) = 0 \tag{1.15.5}$$

Proof. The distance between a point $\mathbf{A}$ and a line $L = \mathbf{n}^T \mathbf{x} - c$ is given by:

$$\|\mathbf{P} - \mathbf{A}\| = \frac{\left|\mathbf{n}^T \mathbf{A} - c\right|}{\|\mathbf{n}\|} \tag{1.15.6}$$

where $\mathbf{P}$ is the foot of perpendicular from $\mathbf{A}$ onto L.
Consider a point $\mathbf{x}$ equidistant from both par-

allel lines, then:

$$\frac{\left|a\mathbf{n}^T\mathbf{x}-c_1\right|}{\|a\mathbf{n}\|}=\frac{\left|b\mathbf{n}^T\mathbf{x}-c_2\right|}{\|b\mathbf{n}\|} \tag{1.15.7}$$

$$\frac{\left|a\mathbf{n}^T\mathbf{x}-c_1\right|}{|a|}=\frac{\left|b\mathbf{n}^T\mathbf{x}-c_2\right|}{|b|} \tag{1.15.8}$$

$$\left|ab\mathbf{n}^T\mathbf{x}-bc_1\right|=\left|ab\mathbf{n}^T\mathbf{x}-ac_2\right| \tag{1.15.9}$$

$$2ab\mathbf{n}^T\mathbf{x}-bc_1-ac_2=0 \tag{1.15.10}$$

$$\mathbf{n}^T\mathbf{x}-\frac{1}{2}\left(\frac{c_1}{a}+\frac{c_2}{b}\right)=0 \tag{1.15.11}$$

□

The two given parallel lines can be written as:

$$3\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x}-7=0 \tag{1.15.12}$$

$$\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x}+6=0 \tag{1.15.13}$$

On comparing the equations with (1.15.4),

$$\mathbf{n}=\begin{pmatrix}3 & 2\end{pmatrix} \tag{1.15.14}$$

$$a=3 \tag{1.15.15}$$

$$b=1 \tag{1.15.16}$$

$$c_1=7 \tag{1.15.17}$$

$$c_2=-6 \tag{1.15.18}$$

On substituting these values into (1.15.11),

$$\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x}-\frac{1}{2}\left(\frac{7}{3}-6\right)=0 \tag{1.15.19}$$

$$\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x}-\frac{11}{6}=0 \tag{1.15.20}$$

See Fig. 1.15.

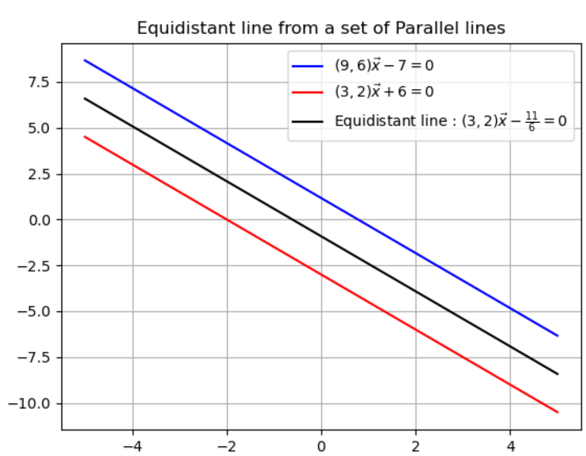

Fig. 1.15: The equidistant line

1.16. Determine the ratio in which the line

$$\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x}-4=0 \tag{1.16.1}$$

divides the line segment joining the points $\mathbf{A}=\begin{pmatrix}2\\-2\end{pmatrix}, \mathbf{B}=\begin{pmatrix}3\\7\end{pmatrix}$.

1.17. A line perpendicular to the line segment joining the points $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}2\\3\end{pmatrix}$ divides it in the ratio $1:n$. Find the equation of the line.

Solution: Let $\mathbf{M}$ be the point that divides the two points $\mathbf{A}=\begin{pmatrix}1\\0\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}2\\3\end{pmatrix}$ in ratio $1:n$.

$$\mathbf{M}=\frac{n\mathbf{A}+\mathbf{B}}{n+1}=\frac{n\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}2\\3\end{pmatrix}}{n+1} \tag{1.17.1}$$

$$\implies \mathbf{M}=\frac{1}{n+1}\begin{pmatrix}n+2\\3\end{pmatrix}$$

The direction vector of line AB is

$$\begin{pmatrix}1\\0\end{pmatrix}-\begin{pmatrix}2\\3\end{pmatrix}=\begin{pmatrix}-1\\-3\end{pmatrix} \tag{1.17.2}$$

The direction vector of line AB is normal vector of perpendicular line. Then

$$\mathbf{n}=\begin{pmatrix}-1\\-3\end{pmatrix} \tag{1.17.3}$$

The equation of line in terms of normal vector is then obtained as

$$\mathbf{n}^T(\mathbf{x}-\mathbf{M})=0 \tag{1.17.4}$$

$$\implies \begin{pmatrix}-1 & -3\end{pmatrix}\left(\mathbf{x}-\frac{1}{n+1}\begin{pmatrix}n+2\\3\end{pmatrix}\right)=0 \tag{1.17.5}$$

$$\therefore \begin{pmatrix}-1 & -3\end{pmatrix}\mathbf{x}=\frac{-n-11}{n+1} \tag{1.17.6}$$

We got equation of the line perpendicular to line segment joining points $\mathbf{A}$ and $\mathbf{B}$ and dividing them in the ratio $1:n$.

For plotting let us take $n=2$, Then the perpendicular line equation will be as,

$$\begin{pmatrix}-1 & -3\end{pmatrix}\mathbf{x}=\frac{-13}{3} \tag{1.17.7}$$

See Fig. 1.17

1.18. Find the equation of a line that cuts off equal intercepts on the coordinate axes and passes

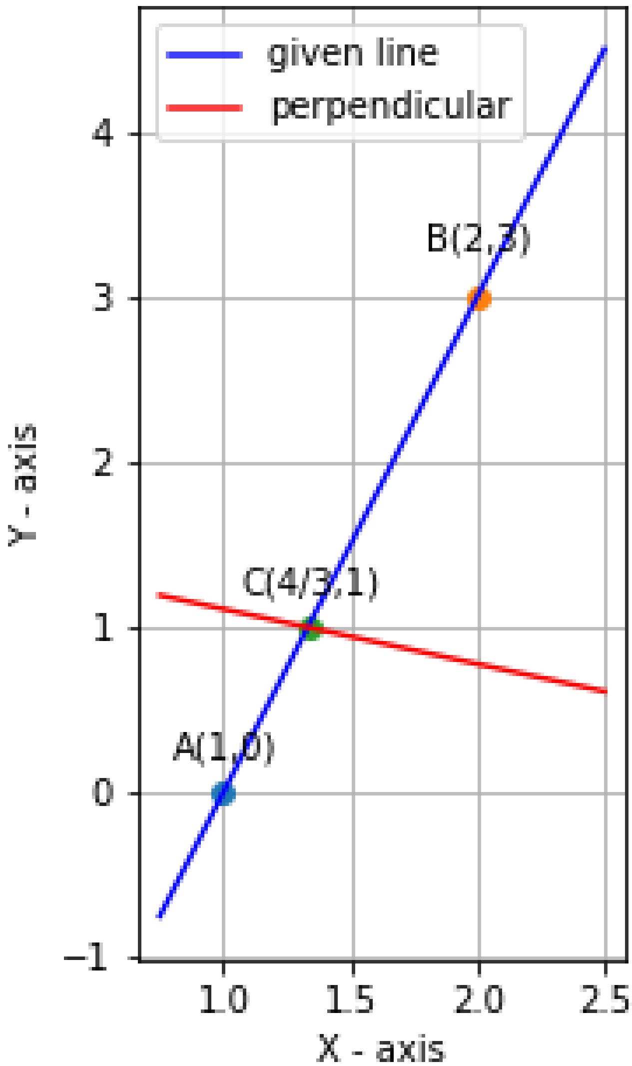

Fig. 1.17: graphical interpretation

through the point $\begin{pmatrix}2\\3\end{pmatrix}$.

Solution:
The general equation of a line can be written as :

$$\mathbf{n}^T\mathbf{x} = c \tag{1.18.1}$$

where $\mathbf{n}$ is the normal to the line. The standard basis vectors in 2D plane are given by:

$$\mathbf{e_1} = \begin{pmatrix}1\\0\end{pmatrix} \tag{1.18.2}$$

$$\mathbf{e_2} = \begin{pmatrix}0\\1\end{pmatrix} \tag{1.18.3}$$

Let the line (1.18.1) cut the x and y co-ordinate axes at $\mathbf{A}$ and $\mathbf{B}$ respectively. They can be written as

$$\mathbf{A} = \frac{c\mathbf{e_1}}{\mathbf{n}^T\mathbf{e_1}} \tag{1.18.4}$$

$$\mathbf{B} = \frac{c\mathbf{e_2}}{\mathbf{n}^T\mathbf{e_2}} \tag{1.18.5}$$

It is given that the line cuts off equal intercepts on the co-ordinate axes. Hence from (1.18.4) and (1.18.5) we have

$$\mathbf{n}^T\mathbf{e_1} = \mathbf{n}^T\mathbf{e_2} \tag{1.18.6}$$

which is equivalent to

$$\mathbf{n}^T(\mathbf{e_1} - \mathbf{e_2}) = 0 \tag{1.18.7}$$

$$\mathbf{n}^T\begin{pmatrix}1\\-1\end{pmatrix} = 0 \tag{1.18.8}$$

$$\implies \mathbf{n} = \begin{pmatrix}1\\1\end{pmatrix} \tag{1.18.9}$$

Hence from (1.18.1) and (1.18.9) , the equation of the line is given by:

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = c \tag{1.18.10}$$

It is given that $\begin{pmatrix}2\\3\end{pmatrix}$ lies on the line. Hence from (1.18.10) we have:

$$c = \begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}2\\3\end{pmatrix} = 5 \tag{1.18.11}$$

Therefore the equation of the line is

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 5 \tag{1.18.12}$$

This is illustrated in Fig. 1.18.

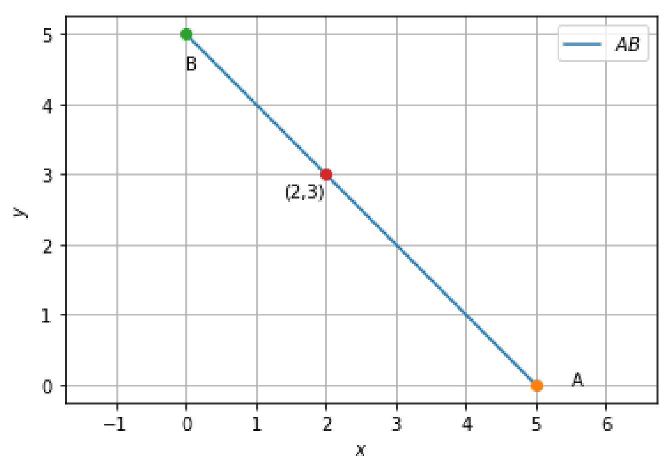

Fig. 1.18: Line **AB** making equal intercepts on co-ordinate axes

1.19. Find the equation of the line passing through the point $\begin{pmatrix}2\\2\end{pmatrix}$ and cutting off intercepts on the axes whose sum is 9.

1.20. Find the equation of the line through the point $\begin{pmatrix}0\\2\end{pmatrix}$ making an angle $\frac{2\pi}{3}$ with the positive x-axis. Also, find the equation of the line parallel to it and crossing the y-axis at a distance of 2 units below the origin.

Solution: The direction vector of the line is $\begin{pmatrix}1\\-\sqrt{3}\end{pmatrix}$. The normal vector **n**

$$\mathbf{n} = \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\begin{pmatrix}1\\-\sqrt{3}\end{pmatrix} \tag{1.20.1}$$

$$= \begin{pmatrix}\sqrt{3}\\1\end{pmatrix} \tag{1.20.2}$$

Let **P** be $\begin{pmatrix}0\\2\end{pmatrix}$. The equation of line in terms of normal vector

$$\mathbf{n}^T(\mathbf{x}-\mathbf{P}) = 0 \tag{1.20.3}$$

$$\implies \begin{pmatrix}\sqrt{3} & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}\sqrt{3} & 1\end{pmatrix}\mathbf{P} \tag{1.20.4}$$

$$\implies \begin{pmatrix}\sqrt{3} & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}\sqrt{3} & 1\end{pmatrix}\begin{pmatrix}0\\2\end{pmatrix} \tag{1.20.5}$$

$$\implies \begin{pmatrix}\sqrt{3} & 1\end{pmatrix}\mathbf{x} = 2 \tag{1.20.6}$$

The point which crosses the y-axis at a distance of 2 units below the origin

$$\mathbf{Q} = \begin{pmatrix}0\\-2\end{pmatrix} \tag{1.20.7}$$

The equation of line which passes through **Q**

$$\mathbf{n}^T(\mathbf{x}-\mathbf{Q}) = 0 \tag{1.20.8}$$

$$\implies \begin{pmatrix}\sqrt{3} & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}\sqrt{3} & 1\end{pmatrix}\mathbf{Q} \tag{1.20.9}$$

$$\implies \begin{pmatrix}\sqrt{3} & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}\sqrt{3} & 1\end{pmatrix}\begin{pmatrix}0\\-2\end{pmatrix} \tag{1.20.10}$$

$$\implies \begin{pmatrix}\sqrt{3} & 1\end{pmatrix}\mathbf{x} = -2 \tag{1.20.11}$$

See Fig. 1.20.

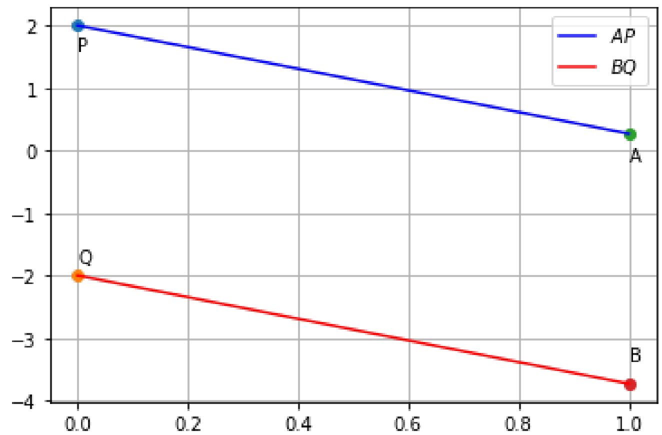

Fig. 1.20: Plot of the given points and lines

1.21. The perpendicular from the origin to a line meets it at a point $\begin{pmatrix}-2\\9\end{pmatrix}$, find the equation of the line.

Solution: Let the equation of line be

$$\mathbf{n}^{\top}(\mathbf{x}-\mathbf{P}) = 0 \tag{1.21.1}$$

So the perpendicular from the origin meets the line at $\mathbf{P} = \begin{pmatrix}-2\\9\end{pmatrix}$.Since,

$$\mathbf{n} = \mathbf{P}-\mathbf{O} \tag{1.21.2}$$

$$= \begin{pmatrix}-2-0\\9-0\end{pmatrix} \tag{1.21.3}$$

$$= \begin{pmatrix}-2\\9\end{pmatrix} \tag{1.21.4}$$

is the normal vector where **O** is the origin then is the direction vector, Hence the equation of line is given by

$$\begin{pmatrix}-2 & 9\end{pmatrix}\left(\mathbf{x}-\begin{pmatrix}-2\\9\end{pmatrix}\right) = 0 \tag{1.21.5}$$

$$\implies \begin{pmatrix}-2 & 9\end{pmatrix}\mathbf{x} = 85 \tag{1.21.6}$$

See Fig. 1.21

1.22. Find the shortest distance between the lines

$$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}, \tag{1.22.1}$$

$$\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1} \tag{1.22.2}$$

Solution:

a) In the given problem

$$\mathbf{A}_1 = \begin{pmatrix}-1\\-1\\-1\end{pmatrix}, \mathbf{m}_1 = \begin{pmatrix}7\\-6\\1\end{pmatrix}, \mathbf{A}_2 = \begin{pmatrix}3\\5\\7\end{pmatrix}, \mathbf{m}_2 = \begin{pmatrix}1\\-2\\1\end{pmatrix}. \tag{1.22.3}$$

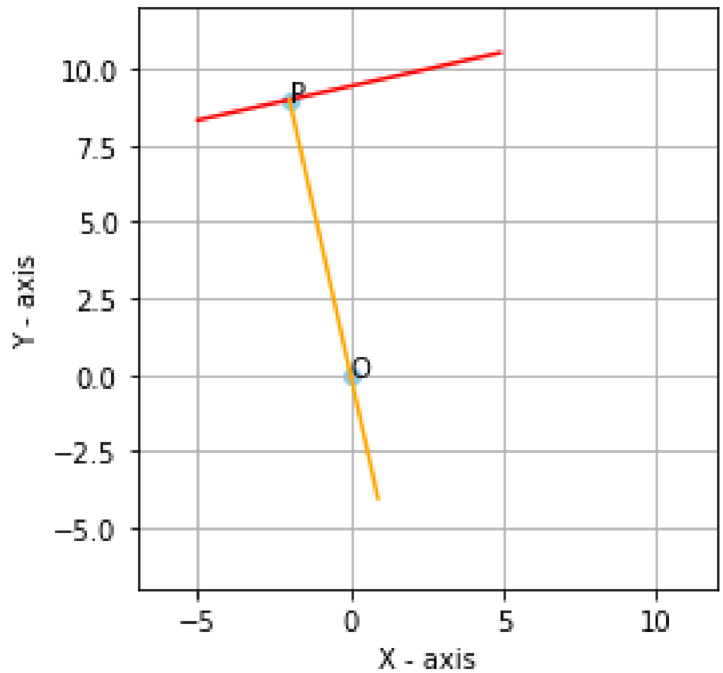

Fig. 1.21: graph

The lines will intersect if

$$\begin{pmatrix}-1\\-1\\-1\end{pmatrix}+\lambda_1\begin{pmatrix}7\\-6\\1\end{pmatrix}=\begin{pmatrix}3\\5\\7\end{pmatrix}+\lambda_2\begin{pmatrix}1\\-2\\1\end{pmatrix} \tag{1.22.4}$$

$$\implies \lambda_1\begin{pmatrix}7\\-6\\1\end{pmatrix}-\lambda_2\begin{pmatrix}1\\-2\\1\end{pmatrix}=\begin{pmatrix}3\\5\\7\end{pmatrix}-\begin{pmatrix}-1\\-1\\-1\end{pmatrix} \tag{1.22.5}$$

$$\implies \begin{pmatrix}7 & 1\\-6 & -2\\1 & 1\end{pmatrix}\begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix}=\begin{pmatrix}4\\6\\8\end{pmatrix} \tag{1.22.6}$$

Row reducing the augmented matrix,

$$\begin{pmatrix}7 & 1 & 4\\-6 & -2 & 6\\1 & 1 & 8\end{pmatrix}\xleftrightarrow{R_3\leftrightarrow R_1}\begin{pmatrix}1 & 1 & 8\\-6 & -2 & 6\\7 & 1 & 4\end{pmatrix} \tag{1.22.7}$$

$$\xleftrightarrow[R_3=-7R_1+R_3]{R_2=6R_1+R_2}\begin{pmatrix}1 & 1 & 8\\0 & 4 & 54\\0 & -6 & -52\end{pmatrix}\xleftrightarrow{R_2=\frac{R_2}{4}}\begin{pmatrix}1 & 1 & 8\\0 & 1 & \frac{27}{2}\\0 & -6 & -52\end{pmatrix} \tag{1.22.8}$$

$$\xleftrightarrow{R_3=6R_2+R_3}\begin{pmatrix}1 & 1 & 8\\0 & 1 & \frac{27}{2}\\0 & 0 & 29\end{pmatrix} \tag{1.22.9}$$

The above matrix has *rank* = 3. Hence, the lines do not intersect. Note that the lines are not parallel but they lie on parallel planes. Such lines are known as *skew* lines as can be seen in Fig 1.22.
$\therefore$ The distance between given two lines is (using equation (1.22.6))

$$\left\|\begin{pmatrix}7 & 1\\-6 & -2\\1 & 1\end{pmatrix}\begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix}-\begin{pmatrix}4\\6\\8\end{pmatrix}\right\| \tag{1.22.10}$$

We know that, the minimizer of $\|\mathbf{Ax}-\mathbf{B}\|$ is given by the solution to the normal equations $\mathbf{A}^T\mathbf{Ax}=\mathbf{A}^T\mathbf{B}$. Since

$$\begin{pmatrix}7 & 1\\-6 & -2\\1 & 1\end{pmatrix}^T\begin{pmatrix}7 & 1\\-6 & -2\\1 & 1\end{pmatrix}=\begin{pmatrix}86 & 20\\20 & 6\end{pmatrix} \tag{1.22.11}$$

$$\begin{pmatrix}7 & 1\\-6 & -2\\1 & 1\end{pmatrix}^T\begin{pmatrix}4\\6\\8\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \tag{1.22.12}$$

the normal equations give us the following system of equations

$$\begin{pmatrix}86 & 20\\20 & 6\end{pmatrix}\begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \tag{1.22.13}$$

whose solution is $\begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$. The minimum distance between this two lines is, thus,

$$\left\|\begin{pmatrix}7 & 1\\-6 & -2\\1 & 1\end{pmatrix}\begin{pmatrix}0\\0\end{pmatrix}-\begin{pmatrix}4\\6\\8\end{pmatrix}\right\|=\left\|\begin{pmatrix}4\\6\\8\end{pmatrix}\right\| \tag{1.22.14}$$

$$=\sqrt{116} \tag{1.22.15}$$

1.23. Find the shortest distance between the lines

$$L_1:\quad \mathbf{x}=\begin{pmatrix}1\\2\\3\end{pmatrix}+\lambda_1\begin{pmatrix}1\\-3\\2\end{pmatrix} \tag{1.23.1}$$

$$L_2:\quad \mathbf{x}=\begin{pmatrix}4\\5\\6\end{pmatrix}+\lambda_2\begin{pmatrix}2\\3\\1\end{pmatrix} \tag{1.23.2}$$

Solution: We have,

$$L_1:\ \mathbf{x}=\mathbf{a_1}+\lambda_1\mathbf{b_1} \tag{1.23.3}$$

$$L_2:\ \mathbf{x}=\mathbf{a_2}+\lambda_2\mathbf{b_2} \tag{1.23.4}$$

where $\mathbf{a_i},\mathbf{b_i}$ are positional vector, slope vector

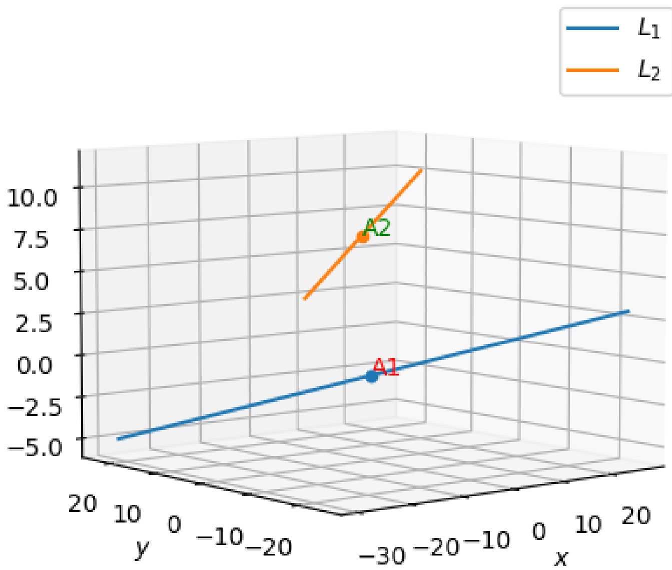

Fig. 1.22: plot of lines

of line L_i respectively.

$$\mathbf{b_1} \neq k\mathbf{b_2}, \tag{1.23.5}$$

the lines are not parallel to each other. If L_1 and L_2 intersect at a point,

$$\begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda_1\begin{pmatrix}1\\-3\\2\end{pmatrix} = \begin{pmatrix}4\\5\\6\end{pmatrix} + \lambda_2\begin{pmatrix}2\\3\\1\end{pmatrix} \tag{1.23.6}$$

$$\implies \begin{pmatrix}1 & -2\\-3 & -3\\2 & -1\end{pmatrix}\begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix} = \begin{pmatrix}3\\3\\3\end{pmatrix} \tag{1.23.7}$$

Rown reducing the augmented matrix,

$$\begin{pmatrix}1 & -2 & 3\\-3 & -3 & 3\\2 & -1 & 3\end{pmatrix} \xleftrightarrow{R_2\leftarrow R_2+3R_1} \begin{pmatrix}1 & -2 & 3\\0 & 3 & 9\\2 & -1 & 3\end{pmatrix}$$
$$\xleftrightarrow{R_3\leftarrow R_3-2R_1} \begin{pmatrix}1 & -2 & 3\\0 & 3 & 9\\0 & 3 & -3\end{pmatrix} \xleftrightarrow[R_3\leftarrow R_3-3R_2]{R_2\leftarrow \frac{R_2}{3}} \begin{pmatrix}1 & -2 & 3\\0 & 1 & 9\\0 & 0 & -3\end{pmatrix} \tag{1.23.8}$$

$\therefore$ the rank of the matrix = 3. Hence the lines do not intersect and L_1 and L_2 are skew lines. Let d be the shortest distance and $\mathbf{p_1}, \mathbf{p_2}$ be positional vectors of its end points. Then,

$$\mathbf{b_1}^\top(\mathbf{p_2} - \mathbf{p_1}) = 0 \tag{1.23.9}$$
$$\mathbf{b_2}^\top(\mathbf{p_2} - \mathbf{p_1}) = 0 \tag{1.23.10}$$
$$\mathbf{b_1}^\top((\mathbf{a}_2 - \mathbf{a}_1)) + \begin{pmatrix}\mathbf{b_2} & \mathbf{b}_1\end{pmatrix}\begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix} = 0 \tag{1.23.11}$$
$$\mathbf{b_2}^\top((\mathbf{a}_2 - \mathbf{a}_1)) + \begin{pmatrix}\mathbf{b_2} & \mathbf{b}_1\end{pmatrix}\begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix} = 0 \tag{1.23.12}$$

Let

$$\mathbf{B} = \begin{pmatrix}\mathbf{b_2} & \mathbf{b}_1\end{pmatrix} \qquad \mathbf{B}^\top = \begin{pmatrix}\mathbf{b_2}^\top\\\mathbf{b_1}^\top\end{pmatrix} \tag{1.23.13}$$

By combining equations (1.23.11) and (1.23.12) and writing in terms of $\mathbf{B}$ and $\mathbf{B}^\top$ using (1.23.13), we get

$$\mathbf{B}^\top\mathbf{B}\begin{pmatrix}\lambda_2\\-\lambda_1\end{pmatrix} = \mathbf{B}^\top(\mathbf{a}_1 - \mathbf{a}_2) \tag{1.23.14}$$

By putting the values of a_1, a_2, b_1, b_2 in (1.23.14), we get

$$\begin{pmatrix}14 & -5\\-5 & 14\end{pmatrix}\begin{pmatrix}\lambda_2\\-\lambda_1\end{pmatrix} = \begin{pmatrix}-18\\0\end{pmatrix} \tag{1.23.15}$$

Solving (1.23.15), we get

$$\begin{pmatrix}\lambda_2\\-\lambda_1\end{pmatrix} = \begin{pmatrix}-1.4736\\-0.5263\end{pmatrix} \tag{1.23.16}$$

Substituting the value of λ_1 and λ_2 in (1.23.3) and (1.23.4), we get

$$\mathbf{p_1} = \begin{pmatrix}1.5263\\0.4210\\4.0526\end{pmatrix} \qquad \mathbf{p_2} = \begin{pmatrix}1.0526\\0.5789\\4.5263\end{pmatrix} \tag{1.23.17}$$

Hence, the shortest distance between these two skew lines is

$$d = \|\mathbf{p_2} - \mathbf{p_1}\| = 0.6882 \tag{1.23.18}$$

1.24. Find the equation of the planes that passes through three points

a) $\begin{pmatrix}1\\1\\-1\end{pmatrix}, \begin{pmatrix}6\\4\\-5\end{pmatrix}, \begin{pmatrix}-4\\-2\\3\end{pmatrix}$

b) $\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}-2\\2\\-1\end{pmatrix}$.

Solution:

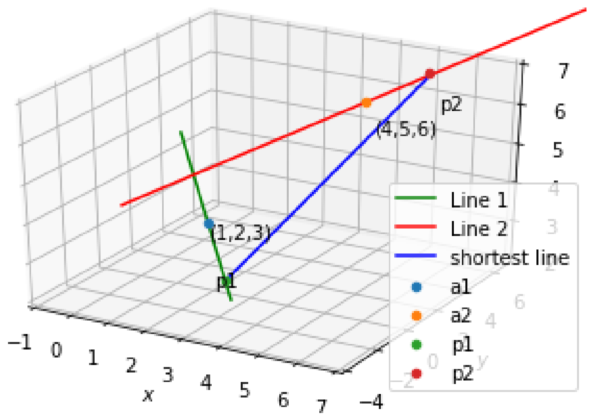

Fig. 1.23: Plot of skew lines

a) If the equation of the plane is given by

$$\mathbf{n}^T\mathbf{x} = 1, \tag{1.24.1}$$

$$\begin{pmatrix} 1 & 1 & -1 \\ 6 & 4 & -5 \\ -4 & -2 & 3 \end{pmatrix}\mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{1.24.2}$$

Row reducing the augmented matrix,

$$\begin{pmatrix} 1 & 0 & -0.5 & -1.5 \\ 0 & 1 & -0.5 & 2.5 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1.24.3}$$

which yields the equation of the line

$$\mathbf{x} = \begin{pmatrix} 1 \\ 6 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix} \tag{1.24.4}$$

and is plotted in Fig. 1.24.

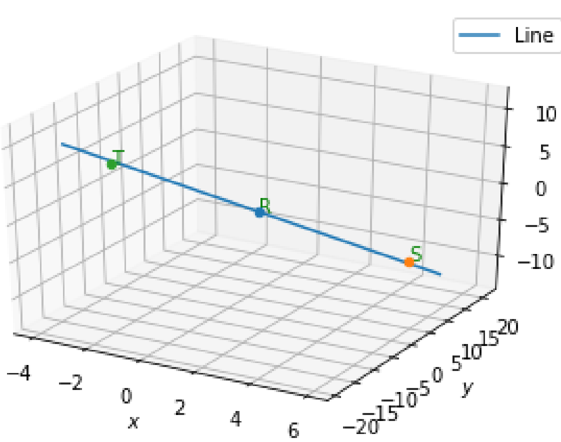

Fig. 1.24: plot of the line

1.25. Find the intercepts cut off by the plane $\begin{pmatrix} 2 & 1 & 1 \end{pmatrix}\mathbf{x} = 5$.

1.26. Find the equaion of the plane with intercept 3 on the y-axis and parallel to ZOX plane.

Solution:
Since plane cuts an intercept of 3 units on y-axis, point $\mathbf{C} = \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}$ lies on the plane.
Also, as the plane is parallel to the ZOX plane, both must have same normal vector. So,

$$\mathbf{n} = \mathbf{n}_{\text{ZOX}} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \tag{1.26.1}$$

If $\mathbf{x}$ is a general point on the plane, then the equation of plane is given by

$$\mathbf{n}^\top(\mathbf{x} - \mathbf{C}) = 0 \tag{1.26.2}$$

$$\implies \mathbf{n}^\top\mathbf{x} = \mathbf{n}^\top\mathbf{C} \tag{1.26.3}$$

$$\implies \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} \tag{1.26.4}$$

$$\implies \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}\mathbf{x} = 3 \tag{1.26.5}$$

See Fig. 1.26 for a verification.

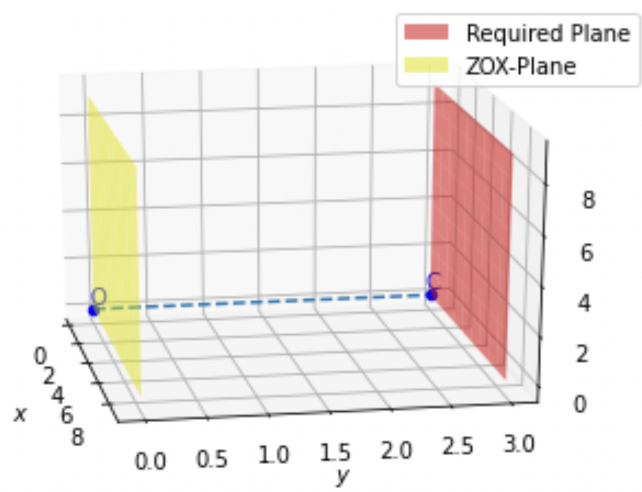

Fig. 1.26: 3D plot

1.27. Find the equation of the plane passing through the intersection of the planes $\begin{pmatrix} 2 & 2 & -3 \end{pmatrix}\mathbf{x} = 7$ and $\begin{pmatrix} 2 & 5 & 3 \end{pmatrix}\mathbf{x} = 9$ and the pont $\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$.

Solution:
Lemma 1.2. *The equation of a plane passing*

through the intersection of two planes and given point will be
Let

$$P_1: \quad \mathbf{n_1}^T\mathbf{x} = c_1 \tag{1.27.1}$$
$$P_2: \quad \mathbf{n_2}^T\mathbf{x} = c_2 \tag{1.27.2}$$

Then,

$$P: \quad \mathbf{n^T x} = c \tag{1.27.3}$$

where

$$\mathbf{n} = \mathbf{n_1} + \left(\frac{c_1 - \mathbf{n_1}^T\mathbf{A}}{\mathbf{n_2}^T\mathbf{A} - c_2}\right)\mathbf{n_2} \tag{1.27.4}$$
$$c = c_1 + \left(\frac{c_1 - \mathbf{A}\mathbf{n_1}^T}{\mathbf{A}\mathbf{n_2}^T - c_2}\right)c_2 \tag{1.27.5}$$

Proof. P has the equation

$$\mathbf{n_1}^T\mathbf{x} + \lambda\left(\mathbf{n_2}^T\mathbf{x}\right) = c_1 + \lambda(c_2) \tag{1.27.6}$$
$$\implies (\mathbf{n_1} + \lambda\mathbf{n_2})^T\mathbf{x} = c_1 + \lambda(c_2) \tag{1.27.7}$$

Then

$$\mathbf{n} = \mathbf{n_1} + \lambda\mathbf{n_2} \tag{1.27.8}$$
$$c = c_1 + \lambda c_2 \tag{1.27.9}$$

Given that plane P passes through point $\mathbf{A}$ then

$$(\mathbf{n_1} + \lambda\mathbf{n_2})^T\mathbf{A} = c_1 + \lambda(c_2) \tag{1.27.10}$$
$$\implies \lambda = \frac{c_1 - \mathbf{n_1}^T\mathbf{A}}{\mathbf{n_2}^T\mathbf{A} - c_2} \tag{1.27.11}$$

Substituting λ in (1.27.8) yields 1.27.4. □

For the given problem

$$\mathbf{n_1} = \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix} \tag{1.27.12}$$
$$\mathbf{n_2} = \begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix} \tag{1.27.13}$$
$$c_1 = 7 \tag{1.27.14}$$
$$c_2 = 9 \tag{1.27.15}$$

By substituting the given values,

$$\lambda = \frac{10}{9} \tag{1.27.16}$$
$$\mathbf{n} = \begin{pmatrix} \frac{38}{9} \\ \frac{68}{9} \\ \frac{1}{3} \end{pmatrix} \tag{1.27.17}$$
$$c = 17 \tag{1.27.18}$$

So the equation of plane P is given by

$$\begin{pmatrix} \frac{38}{9} & \frac{68}{9} & \frac{1}{3} \end{pmatrix}\mathbf{x} = 17 \tag{1.27.19}$$

See Fig. 1.27.

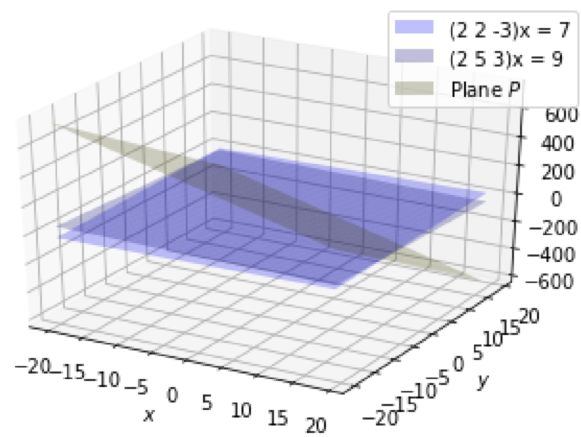

Fig. 1.27: Plane P passing through intersection of P_1 and P_2 and through a point $\mathbf{A}$

1.28. Find the equation of the plane through the intersection of the planes $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}\mathbf{x} = 1$ and $\begin{pmatrix} 2 & 3 & 4 \end{pmatrix}\mathbf{x} = 5$ which is perpendicular to the plane $\begin{pmatrix} 1 & -1 & 1 \end{pmatrix}\mathbf{x} = 0$.
Solution:

Lemma 1.3. *The equation of a plane passing through intersection of planes*

$$\mathbf{n_1}^T\mathbf{x} = c_1 \tag{1.28.1}$$
$$\mathbf{n_2}^T\mathbf{x} = c_2 \tag{1.28.2}$$

and perpendicular to plane

$$\mathbf{n_3}^T\mathbf{x} = c_3 \tag{1.28.3}$$

is given by

$$\mathbf{n_4}^T\mathbf{x} = c_4 \tag{1.28.4}$$

where

$$\mathbf{n_4} = \mathbf{n_1} - \left(\frac{\mathbf{n_3^T n_1}}{\mathbf{n_3^T n_2}}\right)\mathbf{n_2} \tag{1.28.5}$$

$$c_4 = c_1 - \left(\frac{\mathbf{n_3^T n_1}}{\mathbf{n_3^T n_2}}\right)c_2 \tag{1.28.6}$$

Proof. Let P be the plane that passes through intersection 2 given planes. Then P has the form

$$\mathbf{n_1}^T\mathbf{x} + \lambda\left(\mathbf{n_2}^T\mathbf{x}\right) = c_1 + \lambda\left(c_2\right) \tag{1.28.7}$$

$$\implies \left(\mathbf{n_1} + \lambda\mathbf{n_2}\right)^T\mathbf{x} = c_1 + \lambda\left(c_2\right) \tag{1.28.8}$$

Normal vector to plane P is,

$$\mathbf{n_4} = \mathbf{n_1} + \lambda\mathbf{n_2} \tag{1.28.9}$$

As P is perpendicular to the third plane i.e. angle between normal vectors is 90°,

$$\cos\left(90°\right) = 0 = \frac{\mathbf{n_3}^T\mathbf{n_4}}{\|\mathbf{n_3}\|\,\|\mathbf{n_4}\|} \tag{1.28.10}$$

$$\implies \mathbf{n_3}^T\mathbf{n_4} = 0 \tag{1.28.11}$$

$$\implies \mathbf{n_3}^T\left(\mathbf{n_1} + \lambda\mathbf{n_2}\right) = 0 \tag{1.28.12}$$

$$\implies \lambda = \frac{-\mathbf{n_3^T n_1}}{\mathbf{n_3^T n_2}} \tag{1.28.13}$$

Therefore equation of plane P is,

$$\left(\mathbf{n_1} - \left(\frac{\mathbf{n_3^T n_1}}{\mathbf{n_3^T n_2}}\right)\mathbf{n_2}\right)^T\mathbf{x} = c_1 - \left(\frac{\mathbf{n_3^T n_1}}{\mathbf{n_3^T n_2}}\right)c_2 \tag{1.28.14}$$

□

For the given problem,

$$\mathbf{n_1} = \begin{pmatrix}1\\1\\1\end{pmatrix} \tag{1.28.15}$$

$$c_1 = 1, \mathbf{n_2} = \begin{pmatrix}2\\3\\4\end{pmatrix}, c_2 = 5 \tag{1.28.16}$$

$$\mathbf{n_3} = \begin{pmatrix}1\\-1\\1\end{pmatrix} \tag{1.28.17}$$

$$c_3 = 0 \tag{1.28.18}$$

Solving the above we get,

$$\lambda = \frac{-1}{3}, \mathbf{n_4} = \begin{pmatrix}\frac{1}{3}\\0\\\frac{-1}{3}\end{pmatrix}, c_4 = \frac{-2}{3} \tag{1.28.19}$$

We have equation of the plane as,

$$\begin{pmatrix}\frac{1}{3} & 0 & \frac{-1}{3}\end{pmatrix}\mathbf{x} = \frac{-2}{3} \tag{1.28.20}$$

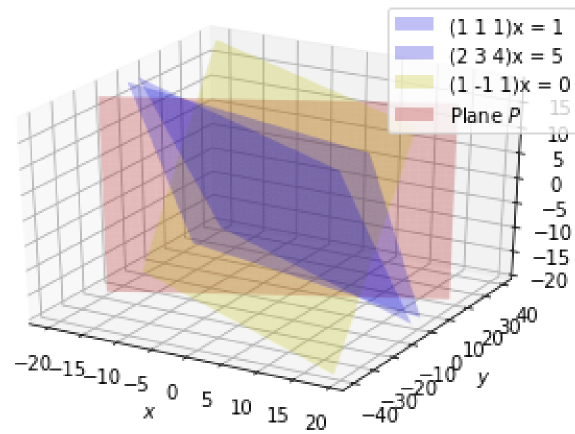

Fig. 1.28: Plane P passing through intersection of P_1 and P_2 and perpendicular to P_3

1.29. In the following cases, determine whether the given planes are parallel or perpendicular, and in case they are neither, find the angles between them.

a) $\begin{pmatrix}7 & 5 & 6\end{pmatrix}\mathbf{x} = -30$ and $\begin{pmatrix}3 & -1 & -10\end{pmatrix}\mathbf{x} = -4$
b) $\begin{pmatrix}2 & 1 & 3\end{pmatrix}\mathbf{x} = 2$ and $\begin{pmatrix}1 & -2 & 5\end{pmatrix}\mathbf{x} = 0$
c) $\begin{pmatrix}2 & -2 & 4\end{pmatrix}\mathbf{x} = -5$ and $\begin{pmatrix}3 & -3 & 6\end{pmatrix}\mathbf{x} = 1$
d) $\begin{pmatrix}2 & -1 & 3\end{pmatrix}\mathbf{x} = 1$ and $\begin{pmatrix}2 & -1 & 3\end{pmatrix}\mathbf{x} = -3$
e) $\begin{pmatrix}4 & 8 & 1\end{pmatrix}\mathbf{x} = 8$ and $\begin{pmatrix}0 & 1 & 1\end{pmatrix}\mathbf{x} = 4$

Solution:

a) The normal vectors of the planes are

$$\mathbf{n}_1 = \begin{pmatrix}2\\1\\3\end{pmatrix}, \mathbf{n}_2 = \begin{pmatrix}1\\-2\\5\end{pmatrix}, \tag{1.29.1}$$

respectively. Consequently, the angle between the planes is

$$\cos\theta = \frac{\mathbf{n_1^\top n_2}}{\|\mathbf{n_1}\|\,\|\mathbf{n_2}\|} \tag{1.29.2}$$

$$== \sqrt{\frac{15}{28}} \tag{1.29.3}$$

$\therefore$ the planes are neither parallel nor perpendicular, as can be verfied from Fig. 1.29.

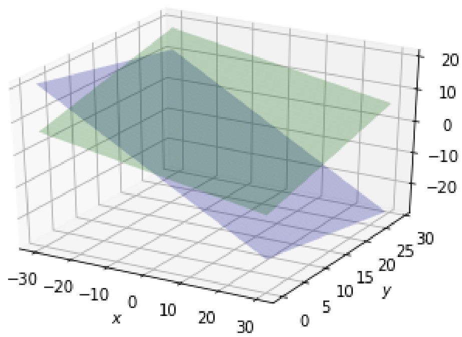

Fig. 1.29: Planes P_1 and P_2

b) From the given information,

$$\mathbf{n}_1 = \begin{pmatrix} 2 \\ -2 \\ 4 \end{pmatrix}, \mathbf{n}_2 = \begin{pmatrix} 3 \\ -3 \\ 6 \end{pmatrix}, \tag{1.29.4}$$

and

$$\theta = \cos^{-1}\left(\frac{\mathbf{n}_1^T \mathbf{n}_2}{\|\mathbf{n}_1\| \, \|\mathbf{n}_2\|}\right) \tag{1.29.5}$$

$$= 0^\circ \tag{1.29.6}$$

Hence, the given planes are parallel, as can be seen from Fig. 1.29

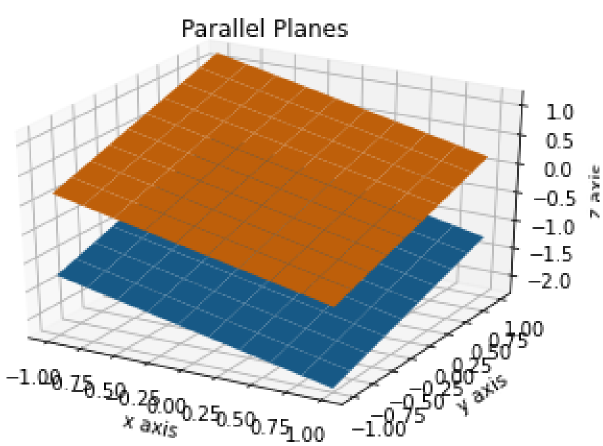

Fig. 1.29: Parallel planes

c) For the given planes, $\mathbf{n}_1 = \mathbf{n}_2$. Hence, the given planes are parallel. This is verified in Fig. 1.29.

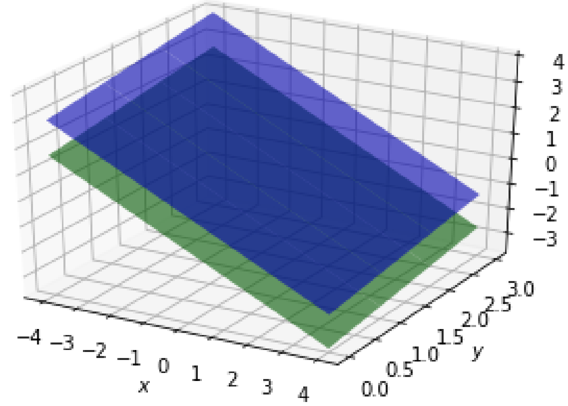

Fig. 1.29: PARALLEL PLANES

1.30. In the following cases, find the distance of each of the given points from the corresponding plane.

Item	Point	Plane
a)	$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 3 & -4 & 12 \end{pmatrix} \boldsymbol{x} = 3$
b)	$\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 2 & -1 & 2 \end{pmatrix} \boldsymbol{x} = -3$
c)	$\begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & -2 \end{pmatrix} \boldsymbol{x} = 9$
d)	$\begin{pmatrix} -6 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 2 & -3 & 6 \end{pmatrix} \boldsymbol{x} = 2$

TABLE 1.30

1.31.

1.32.

1.33. If the lines

$$\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}, \tag{1.33.1}$$

$$\frac{x-3}{3k} = \frac{y-1}{1} = \frac{z-6}{-5}, \tag{1.33.2}$$

find the value of k.

1.34. Find the equation of the line passing through

$\begin{pmatrix}1\\2\\3\end{pmatrix}$ and perpendicular to the plane

$$\begin{pmatrix}1 & 2 & -5\end{pmatrix}\mathbf{x} = -9 \tag{1.34.1}$$

Solution:
Let $\mathbf{p} = \begin{pmatrix}1\\2\\3\end{pmatrix}$ be a point on the line L. Direction vector of the line perpendicular to the given plane is

$$\begin{pmatrix}1\\2\\-5\end{pmatrix} \tag{1.34.2}$$

Thus, the equation of required line is

$$L: \quad \mathbf{x} = \mathbf{p} + \lambda\mathbf{a} \tag{1.34.3}$$

$$= \begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\-5\end{pmatrix} \tag{1.34.4}$$

See Fig. 1.34.

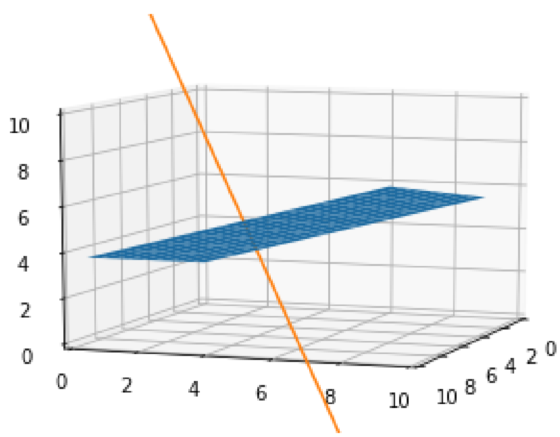

Fig. 1.34: Plot of plane and the line

1.35. Find the shortest distance between the lines

$$\mathbf{x} = \begin{pmatrix}6\\2\\2\end{pmatrix} + \lambda_1\begin{pmatrix}1\\-2\\2\end{pmatrix} \text{ and} \tag{1.35.1}$$

$$\mathbf{x} = \begin{pmatrix}-4\\0\\-1\end{pmatrix} + \lambda_2\begin{pmatrix}3\\-2\\-2\end{pmatrix} \tag{1.35.2}$$

1.36. Find the coordinates of the point where the line through $\begin{pmatrix}5\\1\\6\end{pmatrix}$ and $\begin{pmatrix}3\\4\\1\end{pmatrix}$ crosses the YZ-plane.

Solution: The equation of the line is

$$\mathbf{x} = \mathbf{A} + \lambda(\mathbf{B} - \mathbf{A}) \tag{1.36.1}$$

The equation of the plane can be represented as

$$\mathbf{n}^T\mathbf{x} = c \tag{1.36.2}$$

The point of intersection of the line and the plane satisfies the plane equation and is given by

$$c = \mathbf{n}^T(\mathbf{x}) \tag{1.36.3}$$

$$= \mathbf{n}^T(\mathbf{A} + \lambda(\mathbf{B} - \mathbf{A})) \tag{1.36.4}$$

Thus,

$$\lambda = \frac{c - \mathbf{n}^T\mathbf{A}}{\mathbf{n}^T(\mathbf{B} - \mathbf{A})} \tag{1.36.5}$$

The point of intersection is then given by

$$\mathbf{x} = \mathbf{A} + \left(\frac{c - \mathbf{n}^T\mathbf{A}}{\mathbf{n}^T(\mathbf{B} - \mathbf{A})}\right)(\mathbf{B} - \mathbf{A}) \tag{1.36.6}$$

For the given problem,

$$\mathbf{A} = \begin{pmatrix}5\\1\\6\end{pmatrix} \tag{1.36.7}$$

$$\mathbf{B} = \begin{pmatrix}2\\-3\\5\end{pmatrix} \tag{1.36.8}$$

$$\mathbf{n} = \begin{pmatrix}1\\0\\0\end{pmatrix} \tag{1.36.9}$$

$$c = 0 \tag{1.36.10}$$

Solving the above we get

$$\lambda = \frac{-5}{2} \tag{1.36.11}$$

Substituting the value of λ we have the point of contact as

$$\mathbf{x} = \begin{pmatrix}5\\1\\6\end{pmatrix} - \frac{5}{2}\begin{pmatrix}2\\-3\\5\end{pmatrix} = \frac{1}{2}\begin{pmatrix}0\\17\\-13\end{pmatrix} \tag{1.36.12}$$

See fig. 1.36 .

1.37. Find the coordinates of the point where the line through $\begin{pmatrix}5\\1\\6\end{pmatrix}$ and $\begin{pmatrix}3\\4\\1\end{pmatrix}$ crosses the ZX-plane.

Solution: The equation of line joining A and

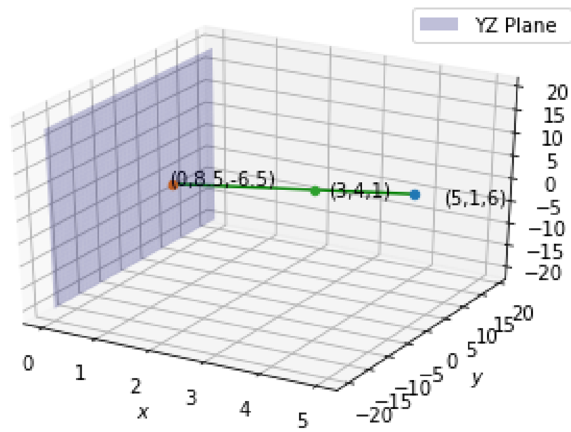

Fig. 1.36: Line and point of intersection

B is given by

$$\mathbf{r} = \mathbf{A} + \lambda(\mathbf{B} - \mathbf{A}) \tag{1.37.1}$$

General equation of plane is given by

$$\mathbf{n}^T\mathbf{r} = c \tag{1.37.2}$$

where $\mathbf{n}$ is normal vector to the plane

$$\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \tag{1.37.3}$$

Lemma 1.4. *The point of intersection of line*

$$\mathbf{r} = \mathbf{A} + \lambda(\mathbf{B} - \mathbf{A}) \tag{1.37.4}$$

and plane

$$\mathbf{n}^T\mathbf{r} = c \tag{1.37.5}$$

is given by

$$\mathbf{r_0} = \mathbf{A} + \left(\frac{c - \mathbf{n}^T\mathbf{A}}{\mathbf{n}^T(\mathbf{B} - \mathbf{A})}\right)(\mathbf{B} - \mathbf{A}) \tag{1.37.6}$$

Proof. Let r_0 be the point of intersection of the line and the plane then the point lies on both line and plane so,

$$\mathbf{r_0} = \mathbf{A} + \lambda(\mathbf{B} - \mathbf{A}) \tag{1.37.7}$$

As r_0 also lies on plane

$$\mathbf{n}^T\mathbf{r_0} = c \tag{1.37.8}$$

$$\implies \mathbf{n}^T(\mathbf{A} + \lambda(\mathbf{B} - \mathbf{A})) = c \tag{1.37.9}$$

$$\implies \lambda\mathbf{n}^T(\mathbf{B} - \mathbf{A}) = c - \mathbf{n}^T\mathbf{A} \tag{1.37.10}$$

$$\implies \lambda = \left(\frac{c - \mathbf{n}^T\mathbf{A}}{\mathbf{n}^T(\mathbf{B} - \mathbf{A})}\right) \tag{1.37.11}$$

Therefore,

$$\mathbf{r_0} = \mathbf{A} + \left(\frac{c - \mathbf{n}^T\mathbf{A}}{\mathbf{n}^T(\mathbf{B} - \mathbf{A})}\right)(\mathbf{B} - \mathbf{A}) \tag{1.37.12}$$

□

Given,

$$\mathbf{A} = \begin{pmatrix} 5 \\ 1 \\ 6 \end{pmatrix} \tag{1.37.13}$$

$$\mathbf{B} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} \tag{1.37.14}$$

Equation of line joining A and B is given by

$$\mathbf{r} = \mathbf{A} + \lambda(\mathbf{B} - \mathbf{A}) \tag{1.37.15}$$

$$\tag{1.37.16}$$

Equation of ZX-plane is given by

$$y = 0 \tag{1.37.17}$$

$$\implies \mathbf{n} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, c = 0 \tag{1.37.18}$$

Therefore coordinates of the intersection point are

$$\mathbf{r_0} = \mathbf{A} + \left(\frac{c - \mathbf{n}^T\mathbf{A}}{\mathbf{n}^T(\mathbf{B} - \mathbf{A})}\right)(\mathbf{B} - \mathbf{A}) \tag{1.37.19}$$

substituting all the vectors gives,

$$\mathbf{r_0} = \frac{1}{3}\begin{pmatrix} 17 \\ 0 \\ 23 \end{pmatrix} \tag{1.37.20}$$

1.38. Find the equation of the plane passing through the point $\begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$ and perpendicular to each of the planes

$$\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}\mathbf{x} = 5 \tag{1.38.1}$$

$$\begin{pmatrix} 3 & 3 & 1 \end{pmatrix}\mathbf{x} = 0 \tag{1.38.2}$$

1.39. If the points $\begin{pmatrix} 1 \\ 1 \\ p \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$ be equidistant from the plane

$$\begin{pmatrix} 3 & 4 & -12 \end{pmatrix}\mathbf{x} = -13, \tag{1.39.1}$$

then find the value of p.

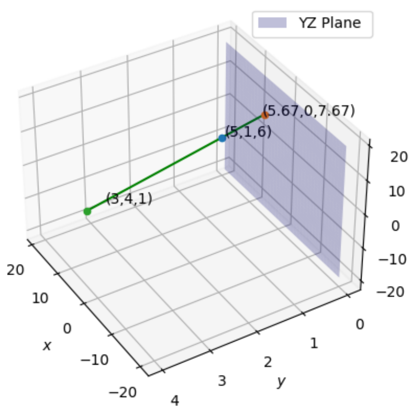

Fig. 1.37: Line and point of intersection

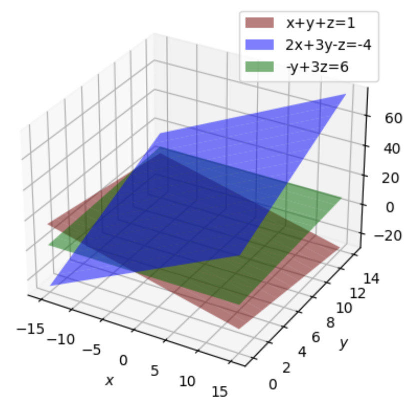

Fig. 1.40: Plot of the planes

1.40. Find the equation of the plane passing through the line of intersection of the planes

$$\begin{pmatrix}1 & 1 & 1\end{pmatrix}\mathbf{x} = 1 \text{ and} \tag{1.40.1}$$

$$\begin{pmatrix}2 & 3 & -1\end{pmatrix}\mathbf{x} = -4 \tag{1.40.2}$$

and parallel to the x-axis.

Solution: The equations of planes are

$$p_1 : \mathbf{n_1}^T x = 1 \tag{1.40.3}$$

$$p_2 : \mathbf{n_2}^T x = -4 \tag{1.40.4}$$

where

$$\mathbf{n_1} = \begin{pmatrix}1 & 1 & 1\end{pmatrix}^T \tag{1.40.5}$$

$$\mathbf{n_2} = \begin{pmatrix}2 & 3 & -1\end{pmatrix}^T \tag{1.40.6}$$

The equation of the desired plane is given by

$$(\mathbf{n_2} + \lambda\mathbf{n_1})^T\mathbf{x} = d_2 + \lambda d_1 \tag{1.40.7}$$

Since the plane is parallel to x-axis,

$$\mathbf{e_1}^T(\mathbf{n_2} + \lambda\mathbf{n_1}) = 0 \tag{1.40.8}$$

$$\implies \lambda = -2 \tag{1.40.9}$$

The required plane is given by

$$(\mathbf{n_2} - 2\mathbf{n_1})^T\mathbf{x} = d_2 - 2d_1 \tag{1.40.10}$$

$$\begin{pmatrix}0 & 1 & -3\end{pmatrix}\mathbf{x} = -6 \tag{1.40.11}$$

A plot for the planes is given in Fig. 1.40.

1.41. Find the equation of the plane which contains the line of intersection of the planes

$$\begin{pmatrix}1 & 2 & 3\end{pmatrix}\mathbf{x} = 4 \tag{1.41.1}$$

$$\begin{pmatrix}2 & 1 & -1\end{pmatrix}\mathbf{x} = -5 \tag{1.41.2}$$

and which is perpendicular to the plane

$$\begin{pmatrix}5 & 3 & -6\end{pmatrix}\mathbf{x} = -8 \tag{1.41.3}$$

1.42. Find the vector equation of the line passing through $\begin{pmatrix}1\\2\\3\end{pmatrix}$ and parallel to the planes

$$\begin{pmatrix}1 & -1 & 2\end{pmatrix}\mathbf{x} = 5 \tag{1.42.1}$$

$$\begin{pmatrix}3 & 1 & 1\end{pmatrix}\mathbf{x} = 6 \tag{1.42.2}$$

Solution:

1.43. The planes

$$\begin{pmatrix}2 & -1 & 4\end{pmatrix}\mathbf{x} = 5 \tag{1.43.1}$$

$$\begin{pmatrix}5 & -\frac{5}{2} & 10\end{pmatrix}\mathbf{x} = 6 \tag{1.43.2}$$

are

a) Perpendicular
b) Parallel
c) intersect y-axis
d) passes through $\begin{pmatrix}0\\0\\\frac{5}{4}\end{pmatrix}$

1.44. Find the maximum and minimum values, if any of the following functions given by

a) $f(x) = |x+2| - 1$
b) $f(x) = -|x+1| + 3$
c) $h(x) = x + 1, x \in (-1, 1)$.

1.45. Using integration find the area of region bounded by the triangle whose vertices are $\begin{pmatrix}1\\0\end{pmatrix}$, $\begin{pmatrix}2\\2\end{pmatrix}$ and $\begin{pmatrix}3\\1\end{pmatrix}$.

1.46. Using integration find the area of region bounded by the triangle whose vertices are (–1, 0), (1, 3) and (3, 2).

1.47. Using integration find the area of the triangular region whose sides have the equations $\begin{pmatrix}2 & -1\end{pmatrix}\mathbf{x} = -1$, $\begin{pmatrix}3 & -1\end{pmatrix}\mathbf{x} = -1$ and x = 4.

1.48. Find the area of the region bounded by the line $\begin{pmatrix}3 & -1\end{pmatrix}\mathbf{x} = -2$, the x-axis and the ordinates $x = -1, x = 1$.

1.49. Find the area bounded by the curve $|x|+|y| = 1$.

1.50. Using the method of integration find the area of $\triangle ABC$, whose vertices are $\mathbf{A} = \begin{pmatrix}2\\0\end{pmatrix}, \mathbf{B} = \begin{pmatrix}4\\5\end{pmatrix}, \mathbf{C} = \begin{pmatrix}6\\3\end{pmatrix}$.

1.51. Using integration find the area of the triangular region whose sides have the equations $\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} = 4$, $\begin{pmatrix}3 & -2\end{pmatrix}\mathbf{x} = 6$ and $\begin{pmatrix}1 & -3\end{pmatrix}\mathbf{x} = -5$.

1.52. The two equal sides of an isosceles triangle with fixed base b are decreasing at the rate of 3 cm per second. How fast is the area decreasing when the two equal sides are equal to the base ?

1.53. A tank with rectangular base and rectangular sides, open at the top is to be constructed so that its depth is 2 m and volume is 8 m^3 . If building of tank costs ₹70 per sq metres for the base and Rs 45 per square metre for sides. What is the cost of least expensive tank?

1.54. A point on the hypotenuse of a triangle is at distance a and b from the sides of the triangle. Show that the minimum length of the hypotenuse is

$$\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}} \tag{1.54.1}$$

1.55. Prove that the function $f(x) = 5x–3$ is continuous at $x = 0, at x = –3$ and at $x = 5$.

1.56. Examine the following functions for continuity.

a) $f(x) = x - 5$

b) $f(x) = |x - 1|$

1.57. Is the function defined by

$$f(x) = \begin{cases} x, & x \le 1, \\ 5, & x > 1 \end{cases} \tag{1.57.1}$$

continuous at $x = 0$? At $x = 1$? At $x = 2$?

1.58. Find all points of discontinuity of f, where f is defined by

a) $f(x) = \begin{cases} 2x + 3, & x \le 2, \\ 2x - 3, & x > 2 \end{cases}$

b) $f(x) = \begin{cases} |x| + 3, & x \le -3, \\ -2x, & -3 < x < 3 \\ 6x + 2, & x \ge 2 \end{cases}$

c) $f(x) = \begin{cases} \frac{|x|}{x}, & x \ne 0, \\ 0, & x = 0, \end{cases}$

d) $f(x) = \begin{cases} \frac{x}{|x|}, & x < 0, \\ -1, & x \ge 0, \end{cases}$

1.59. Is the function defined by

$$f(x) = \begin{cases} x + 5, & x \le 1, \\ x - 5, & x > 1 \end{cases} \tag{1.59.1}$$

a continuous function?

1.60. Discuss the continuity of the function f, where f is defined by

a) $f(x) = \begin{cases} 3, & 0 \le x \le 1, \\ 4, & 0 < x \le 3, \\ 5, & 3 \le x \le 10, \end{cases}$

b) $f(x) = \begin{cases} 2x, & x < 0, \\ 0, & 0 \le x \le 1 \\ 4x, & x > 1 \end{cases}$

c) $f(x) = \begin{cases} -2, & x < -1, \\ 2x, & -1 \le x \le 1 \\ 2, & x > 1 \end{cases}$

1.61. Find the relationship between a and b so that the function defined by

$$f(x) = \begin{cases} ax + 1, & x \le 3, \\ bx + 3, & x > 3 \end{cases} \tag{1.61.1}$$

is continuous at $x = 3$

1.62. Prove that the function $f(x) = x$ is continiuous at every real number.

1.63. Is $f(x) = |x|$ a continuous function?

1.64. Discuss the continuity of the function f defined by

$$f(x) = \begin{cases} x+2 & x \le 1 \\ x-2 & x > 1 \end{cases} \quad (1.64.1)$$

1.65. Show that the function defined by $g(x) = x–[x]$ is discontinuous at all integral points. Here $[x]$ denotes the greatest integer less than or equal to x.

1.66. For what value of k is the following function continuous at the given point.

$$f(x) = \begin{cases} kx+1, & x \le 5, \\ 3x-5, & x > 5, \end{cases} \quad x = 5 \quad (1.66.1)$$

1.67. Prove that the function f given by

$$f(x) = |x-1|, x \in \mathbf{R} \quad (1.67.1)$$

is not differentiable at $x = 1$.

1.68. Prove that the greatest integer function defined by

$$f(x) = |x|, 0 < x < 3 \quad (1.68.1)$$

is not differentiable at $x = 1$ and $x = 2$.

1.69. Examine if Rolle's theorem is applicable to the following functions

a) $f(x) = [x], x \in [5, 9]$.

b) $f(x) = [x], x \in [-2, 2]$.

Can you say some thing about the converse of Rolle's theorem from this example?

1.70. Examine the applicability of the mean value theorem for all functions in Problem 1.69a.

1.71. Find $\lim_{x\to 5} x + 10$

1.72. Find $\lim_{x\to 2} 3x$

1.73. Find $\lim_{x\to 0} f(x)$ where

$$f(x) = \begin{cases} 1 & x \le 0 \\ 2 & x > 0 \end{cases} \quad (1.73.1)$$

1.74. Find $\lim_{x\to 0} f(x)$ where

$$f(x) = \begin{cases} x-2 & x < 0 \\ 0 & x = 0 \\ x+2 & x > 0 \end{cases} \quad (1.74.1)$$

1.75. Evaluate the following limits

a) $\lim_{x\to 3} x + 3$

b) $\lim_{x\to \pi} \left(x - \frac{22}{7}\right)$

1.76. Find $\lim_{x\to 0} f(x)$ where

$$f(x) = \begin{cases} \frac{|x|}{x} & x \ne 0 \\ 0, & x = 0 \end{cases} \quad (1.76.1)$$

1.77. Find $\lim_{x\to 0} f(x)$ where

$$f(x) = \begin{cases} \frac{x}{|x|} & x \ne 0 \\ 0, & x = 0 \end{cases} \quad (1.77.1)$$

1.78. Find $\lim_{x\to 5} |x| - 5$.

1.79. Suppose

$$f(x) = \begin{cases} a+bx & x \ne 1 \\ 4, & x = 1 \\ b-ax & x > 1 \end{cases} \quad (1.79.1)$$

and if $\lim_{x\to 1} f(x) = f(1)$, what are the possible values of a and b?

1.80. If

$$f(x) = \begin{cases} |x|+1 & x < 0 \\ 0, & x = 0 \\ |x|-1 & x > 0 \end{cases} \quad (1.80.1)$$

for what value(s) of a does $\lim_{x\to a} f(x)$ exists?

1.81. Find the derivative of x at $x = 1$.

1.82. Find the derivative of $99x$ at $x = 100$.

1.83. Find the derivative of the following functions:

a) $-x$

b) $x + a$

1.84. Integrate the following as limit of sums:

(i) $\int_a^b x\,dx$

(ii) $\int_0^5 (x+1)\,dx$

(iii) $\int_{-1}^1 (x+1)\,dx$

(iv) $\int_{-5}^5 |x+2|\,dx$

(v) $\int_2^8 |x-5|\,dx$

(vi) $\int_0^4 |x-1|\,dx$

(vii) $\int_1^4 [|x-1| + |x-2| + |x-3|]\,dx$

1.85. Form the differential equation representing the following family of curves

$$\begin{pmatrix} \frac{1}{a} & \frac{1}{b} \end{pmatrix} \mathbf{x} = 1 \quad (1.85.1)$$

1.86. Find θ and p if

$$\begin{pmatrix} \sqrt{3} & 1 \end{pmatrix} \mathbf{x} = -2 \quad (1.86.1)$$

is equivalent to

$$\begin{pmatrix} \cos\theta & \sin\theta \end{pmatrix} \mathbf{x} = p \quad (1.86.2)$$

1.87. Find the equation of the line which passes through the point $\begin{pmatrix}-2\\4\\-5\end{pmatrix}$ and parallel to the line given by

$$\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}. \tag{1.87.1}$$

1.88. Find the angle between the following pair of lines

a)

$$\frac{x-2}{2} = \frac{y-1}{5} = \frac{z+3}{-3}, \tag{1.88.1}$$

$$\frac{x+2}{-1} = \frac{y-4}{8} = \frac{z-5}{4} \tag{1.88.2}$$

b)

$$\frac{x}{2} = \frac{y}{2} = \frac{z}{1}, \tag{1.88.3}$$

$$\frac{x-5}{4} = \frac{y-2}{1} = \frac{z-3}{8} \tag{1.88.4}$$

Solution:

a)

$$\frac{x-2}{2} = \frac{y-1}{5} = \frac{z+3}{-3} \tag{1.88.5}$$

$$\frac{x+2}{-1} = \frac{y-4}{8} = \frac{z-5}{4} \tag{1.88.6}$$

b)

$$\frac{x}{2} = \frac{y}{2} = \frac{z}{1} \tag{1.88.7}$$

$$\frac{x-5}{4} = \frac{y-4}{1} = \frac{z-3}{8} \tag{1.88.8}$$

2 Solution(Linear forms 2.88)

a) The direction vectors **a** and **b** of the two lines are

$$\mathbf{a} = \begin{pmatrix}2\\5\\-3\end{pmatrix} \tag{2.88.1}$$

$$\mathbf{b} = \begin{pmatrix}-1\\8\\4\end{pmatrix} \tag{2.88.2}$$

Let θ be the angle between the vectors,

$$\cos\theta = \frac{\mathbf{a}^\top\mathbf{b}}{\|\mathbf{a}\|\,\|\mathbf{b}\|} \tag{2.88.3}$$

$$\mathbf{a}^\top\mathbf{b} = \begin{pmatrix}2 & 5 & -3\end{pmatrix}\begin{pmatrix}-1\\8\\4\end{pmatrix} \tag{2.88.4}$$

$$= 26 \tag{2.88.5}$$

$$\|\mathbf{a}\| = \sqrt{38} \tag{2.88.6}$$

$$\|\mathbf{b}\| = 9 \tag{2.88.7}$$

$$\implies \cos\theta = \frac{26}{9\sqrt{38}} \tag{2.88.8}$$

$$\theta = \cos^{-1}\left(\frac{26}{9\sqrt{38}}\right) \tag{2.88.9}$$

$$= 62.053^\circ \tag{2.88.10}$$

b) The direction vectors **a** and **b** of the two lines are

$$\mathbf{c} = \begin{pmatrix}2\\2\\1\end{pmatrix} \tag{2.88.11}$$

$$\mathbf{d} = \begin{pmatrix}4\\1\\8\end{pmatrix} \tag{2.88.12}$$

Let θ be the angle between the vectors,

$$\cos\theta = \frac{\mathbf{c}^\top\mathbf{d}}{\|\mathbf{c}\|\,\|\mathbf{d}\|} \tag{2.88.13}$$

$$\mathbf{c}^\top\mathbf{d} = \begin{pmatrix}2 & 2 & 1\end{pmatrix}\begin{pmatrix}4\\1\\8\end{pmatrix} \tag{2.88.14}$$

$$= 18 \tag{2.88.15}$$

$$\|\mathbf{c}\| = 3 \tag{2.88.16}$$

$$\|\mathbf{d}\| = 9 \tag{2.88.17}$$

$$\implies \cos\theta = \frac{18}{9\times 3} \tag{2.88.18}$$

$$\theta = \cos^{-1}\left(\frac{2}{3}\right) \tag{2.88.19}$$

$$= 48.189^\circ \tag{2.88.20}$$

2.89. Find the equation of a plane which is at a distance of 7 units from the origin and normal to $\begin{pmatrix}3\\5\\-6\end{pmatrix}$.

2.90. For the following planes, find the coordinates of the foot of the perpendicular drawn from the origin

a) $\begin{pmatrix}2 & 3 & 4\end{pmatrix}\mathbf{x} = 12$ c) $\begin{pmatrix}1 & 1 & 1\end{pmatrix}\mathbf{x} = 1$
b) $\begin{pmatrix}3 & 4 & -6\end{pmatrix}\mathbf{x} = 0$ d) $\begin{pmatrix}0 & 5 & 0\end{pmatrix}\mathbf{x} = -8$

2.91. Solve the following pair of linear equations

a)

$$\begin{aligned}\begin{pmatrix}p & q\end{pmatrix}\mathbf{x} &= p - q \\ \begin{pmatrix}q & -p\end{pmatrix}\mathbf{x} &= p + q\end{aligned} \tag{2.91.1}$$

b)

$$\begin{aligned}\begin{pmatrix}a & b\end{pmatrix}\mathbf{x} &= c \\ \begin{pmatrix}b & a\end{pmatrix}\mathbf{x} &= 1 + c\end{aligned} \tag{2.91.2}$$

c)

$$\begin{aligned}\begin{pmatrix}\frac{1}{a} & -\frac{1}{b}\end{pmatrix}\mathbf{x} &= 0 \\ \begin{pmatrix}a & b\end{pmatrix}\mathbf{x} &= a^2 + b^2\end{aligned} \tag{2.91.3}$$

2.92. Solve the following pair of equations

$$\begin{aligned}\begin{pmatrix}a - b & a + b\end{pmatrix}\mathbf{x} &= a^2 - 2ab - b^2 \\ \begin{pmatrix}a + b & a + b\end{pmatrix}\mathbf{x} &= a^2 + b^2\end{aligned} \tag{2.92.1}$$

2.93. In $\triangle ABC$, Show that the centroid

$$\mathbf{O} = \frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} \tag{2.93.1}$$

2.94. The base of an equilateral triangle with side $2a$ lies along the y-axis such that the mid-point of the base is at the origin. Find vertices of the triangle.

2.95. Find the distance between $\mathbf{P} = \begin{pmatrix}x_1 y_1\end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix}x_2 y_2\end{pmatrix}$ when

a) PQ is parallel to the y-axis.
b) PQ is parallel to the x-axis.

2.96. If three points $\begin{pmatrix}h 0\end{pmatrix}$, $\begin{pmatrix}a b\end{pmatrix}$ and $\begin{pmatrix}0 k\end{pmatrix}$ lie on a line, show that $\frac{a}{h} + \frac{b}{k} = 1$.

2.97. $\mathbf{P} = \begin{pmatrix}a b\end{pmatrix}$ is the mid-point of a line segment between axes. Show that equation of the line is

$$\begin{pmatrix}\frac{1}{a} & \frac{1}{b}\end{pmatrix}\mathbf{x} = 2 \tag{2.97.1}$$

2.98. Point $\mathbf{R} = \begin{pmatrix}h k\end{pmatrix}$ divides a line segment between the axes in the ratio 1: 2. Find equation of the line.

2.99. Show that two lines

$$\begin{pmatrix}a_1 & b_1\end{pmatrix}\mathbf{x} + c_1 = 0 \tag{2.99.1}$$

$$\begin{pmatrix}a_2 & b_2\end{pmatrix}\mathbf{x} + c_2 = 0 \tag{2.99.2}$$

are

a) parallel if $\frac{a_1}{b_1} = \frac{a_2}{b_2}$ and
b) perpendicular if $a_1 a_2 - b_1 b_2 = 0$.

2.100. Find the distance between the parallel lines

$$l\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = -p \tag{2.100.1}$$

$$l\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = r \tag{2.100.2}$$

2.101. Find th equation of the line through the point $\mathbf{x}_1$ and parallel to the line

$$\begin{pmatrix}A & B\end{pmatrix}\mathbf{x} = -C \tag{2.101.1}$$

2.102. If p and q are the lengths of perpendiculars from the origin to the lines

$$\begin{pmatrix}\cos\theta & \sin\theta\end{pmatrix}\mathbf{x} = k\cos 2\theta \tag{2.102.1}$$

$$\begin{pmatrix}\sec\theta & \operatorname{cosec}\theta\end{pmatrix}\mathbf{x} = k \tag{2.102.2}$$

respectively, prove that $p^2 + 4q^2 = k^2$.

2.103. If p is the length of the perpendicular from the origin to the line whose intercepts on the axes are a and b, then show that

$$\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}. \tag{2.103.1}$$

2.104. Show that the area of the triangle formed by the lines

$$\begin{pmatrix}-m_1 & 1\end{pmatrix}\mathbf{x} = c_1 \tag{2.104.1}$$

$$\begin{pmatrix}-m_2 & 1\end{pmatrix}\mathbf{x} = c_2 \tag{2.104.2}$$

$$\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = 0 \tag{2.104.3}$$

is $\frac{(c_1 - c_2)^2}{2|m_1 - m_2|}$.

2.105. Find the values of k for which the line

$$\begin{pmatrix}k - 3 & -\left(4 - k^2\right)\end{pmatrix}\mathbf{x} + k^2 - 7k + 6 = 0 \tag{2.105.1}$$

is

a) parallel to the x-axis
b) parallel to the y-axis
c) passing through the origin.

2.106. Find the perpendicular distance from the origin to the line joining the points $\begin{pmatrix}\cos\theta \sin\theta\end{pmatrix}$ and $\begin{pmatrix}\cos\phi \sin\phi\end{pmatrix}$.

2.107. Find the area of the triangle formed by the lines

$$\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 0 \tag{2.107.1}$$
$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{2.107.2}$$
$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = k \tag{2.107.3}$$

2.108. If three lines whose equations are

$$\begin{pmatrix} -m_1 & 1 \end{pmatrix} \mathbf{x} = c_1 \tag{2.108.1}$$
$$\begin{pmatrix} -m_2 & 1 \end{pmatrix} \mathbf{x} = c_2 \tag{2.108.2}$$
$$\begin{pmatrix} -m_3 & 1 \end{pmatrix} \mathbf{x} = c_3 \tag{2.108.3}$$

are concurrent, show that

$$m_1 (c_2 - c_3) + m_2 (c_3 - c_1) + m_3 (c_1 - c_2) = 0 \tag{2.108.4}$$

2.109. Find the equation of the line passing through the origin and making an angle θ with the line

$$\begin{pmatrix} -m & 1 \end{pmatrix} \mathbf{x} = c \tag{2.109.1}$$

Solution:

2.110. Prove that the product of the lengths of the perpendiculars drawn from the points $\begin{pmatrix} \sqrt{a^2 - b^2} 0 \end{pmatrix}$ and $\begin{pmatrix} \sqrt{a^2 - b^2} 0 \end{pmatrix}$ to the line

$$\begin{pmatrix} \frac{\cos\theta}{a} & \frac{\sin\theta}{b} \end{pmatrix} \mathbf{x} = 1 \tag{2.110.1}$$

is b^2.

2.111. If $\begin{pmatrix} l_1 m_1 n_1 \end{pmatrix}$ and $\begin{pmatrix} l_2 m_2 n_2 \end{pmatrix}$ are the unit direction vectors of two mutually perpendicular lines, the shown that the unit direction vector of the line perpendicular to both of these is $\begin{pmatrix} m_1 n_2 - m_2 n_1 n_1 l_2 - n_2 l_1 l_1 m_2 - l_2 m_1 \end{pmatrix}$.

2.112. A line makes angles $\alpha, \beta, \gamma, \delta$ with the diagonals of a cube, prove that

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma + \cos^2 \delta = \frac{4}{3}. \tag{2.112.1}$$

2.113. Show that the lines

$$\frac{x - a + d}{\alpha - \delta} = \frac{y - a}{\alpha} = \frac{z - a - d}{\alpha + \delta}, \tag{2.113.1}$$
$$\frac{x - b + c}{\beta - \gamma} = \frac{y - b}{\beta} = \frac{z - b - c}{\beta + \gamma} \tag{2.113.2}$$

are coplanar.

2.114. Find $\mathbf{R}$ which divides the line joining the points

$$\mathbf{P} = 2\mathbf{a} + \mathbf{b} \tag{2.114.1}$$
$$\mathbf{Q} = \mathbf{a} - \mathbf{b} \tag{2.114.2}$$

externally in the ratio 1 : 2.

2.115. Find $\|\mathbf{a}\|$ and $\|\mathbf{b}\|$ if

$$(\mathbf{a} + \mathbf{b})^T (\mathbf{a} - \mathbf{b}) = 8 \tag{2.115.1}$$
$$\|\mathbf{a}\| = 8 \|\mathbf{b}\| \tag{2.115.2}$$

2.116. Evaluate the product

$$(3\mathbf{a} - 5\mathbf{b})^T (2\mathbf{a} + 7\mathbf{b}) \tag{2.116.1}$$

2.117. Find $\|\mathbf{a}\|$ and $\|\mathbf{b}\|$, if

$$\|\mathbf{a}\| = \|\mathbf{b}\|, \tag{2.117.1}$$
$$\mathbf{a}^T \mathbf{b} = \frac{1}{2} \tag{2.117.2}$$

and the angle between $\mathbf{a}$ and $\mathbf{b}$ is 60°.

2.118. Show that

$$(\|\mathbf{a}\| \mathbf{b} + \|\mathbf{b}\| \mathbf{a}) \perp (\|\mathbf{a}\| \mathbf{b} - \|\mathbf{b}\| \mathbf{a}) \tag{2.118.1}$$
$$\tag{2.118.2}$$

2.119. If $\mathbf{a}^T \mathbf{a} = 0$ and $\mathbf{ab} = 0$, what can be concluded about the vector $\mathbf{b}$?

2.120. If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are unit vectors such that

$$\mathbf{a} + \mathbf{b} + \mathbf{c} = 0, \tag{2.120.1}$$

find the value of

$$\mathbf{a}^T \mathbf{b} + \mathbf{b}^T \mathbf{c} + \mathbf{c}^T \mathbf{a}. \tag{2.120.2}$$

2.121. If $\mathbf{a} \neq \mathbf{0}, \lambda \neq 0$, then $\|\lambda \mathbf{a}\| = 1$ if

a) $\lambda = 1$
b) $\lambda = -1$
c) $\|\mathbf{a}\| = |\lambda|$
d) $\|\mathbf{a}\| = \frac{1}{|\lambda|}$

2.122. If a unit vector $\mathbf{a}$ makes angles $\frac{\pi}{3}$ with the x-axis and $\frac{\pi}{4}$ with the y-axis and an acute angle θ with the z-axis, find θ and $\mathbf{a}$.

2.123. Show that

$$(\mathbf{a} - \mathbf{b}) \times (\mathbf{a} + \mathbf{b}) = 2 (\mathbf{a} \times \mathbf{b}) \tag{2.123.1}$$

2.124. If $\mathbf{a}^T \mathbf{b} = 0$ and $\mathbf{a} \times \mathbf{b} = 0$, what can you conclude about $\mathbf{a}$ and $\mathbf{b}$?

2.125. Find $\mathbf{x}$ if $\mathbf{a}$ is a unit vector such that

$$(\mathbf{x} - \mathbf{a})^T (\mathbf{x} + \mathbf{a}) = 12. \tag{2.125.1}$$

2.126. If $\|\mathbf{a}\| = 3, \|\mathbf{b}\| = \frac{\sqrt{2}}{3}$, then $\mathbf{a} \times \mathbf{b}$ is a unit vector if the angle between $\mathbf{a}$ and $\mathbf{b}$ is

a) $\frac{\pi}{6}$
b) $\frac{\pi}{4}$
c) $\frac{\pi}{3}$
d) $\frac{\pi}{2}$

2.127. Prove that

$$\left(\mathbf{a}+\mathbf{b}\right)^T\left(\mathbf{a}+\mathbf{b}\right) = \left\|\mathbf{a}\right\|^2 + \left\|\mathbf{b}\right\|^2 \tag{2.127.1}$$
$$\iff \mathbf{a} \perp \mathbf{b}. \tag{2.127.2}$$

2.128. If θ is the angle between two vectors $\mathbf{a}$ and $\mathbf{b}$, then $\mathbf{a}^T\mathbf{b} \geq$ only when

a) $0 < \theta < \frac{\pi}{2}$
b) $0 \leq \theta \leq \frac{\pi}{2}$
c) $0 < \theta < \pi$
d) $0 \leq \theta \leq \pi$

2.129. Let $\mathbf{a}$ and $\mathbf{b}$ be two unit vectors and θ be the angle between them. Then $\mathbf{a}+\mathbf{b}$ is a unit vector if

a) $\theta = \frac{\pi}{4}$
b) $\theta = \frac{\pi}{3}$
c) $\theta = \frac{\pi}{2}$
d) $\theta = \frac{2\pi}{3}$

2.130. If θ is the angle between any two vectors $\mathbf{a}$ and $\mathbf{b}$, then $\left\|\mathbf{a}^T\mathbf{b}\right\| = \left\|\mathbf{a}\times\mathbf{b}\right\|$ when θ is equal to

a) 0
b) $\frac{\pi}{4}$
c) $\frac{\pi}{2}$
d) π.

2.131. Find the angle between the lines whose direction vectors are $\begin{pmatrix} abc \end{pmatrix}$ and $\begin{pmatrix} b-cc-aa-b \end{pmatrix}$.

2.132. Find the equation of a line parallel to the x-axis and passing through the origin.

2.133. Find the equation of a plane passing through $\begin{pmatrix} abc \end{pmatrix}$ and parallel to the plane

$$\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}\mathbf{x}x = 2 \tag{2.133.1}$$

2.134. Prove that if a plane has the intercepts a, b, c and is at a distance of p units from the origin, then,

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{p^2} \tag{2.134.1}$$

2.135. In an experiment, a solution of hydrochloric acid is to be kept between 30° and 35° Celsius. What is the range of temperature in degree Fahrenheit if conversion formula is given by C = $\frac{5}{9}(F-32)$, where C and F represent temperature in degree Celsius and degree Fahrenheit, respectively.

2.136. A manufacturer has 600 litres of a 12% solution of acid. How many litres of a 30% acid solution must be added to it so that acid content in the resulting mixture will be more than 15% but less than 18%?

2.137. Ravi obtained 70 and 75 marks in first two unit test. Find the minimum marks he should get in the third test to have an average of at least 60 marks.

2.138. To receive Grade A in a course, one must obtain an average of 90 marks or more in five examinations (each of 100 marks). If Sunita's marks in first four examinations are 87, 92, 94 and 95, find minimum marks that Sunita must obtain in fifth examination to get grade 'A' in the course.

2.139. Find all pairs of consecutive odd positive integers both of which are smaller than 10 such that their sum is more than 11.

2.140. Find all pairs of consecutive even positive integers, both of which are larger than 5 such that their sum is less than 23.

2.141. A man wants to cut three lengths from a single piece of board of length 91cm.The second length is to be 3cm longer than the shortest and the third length is to be twice as long as the shortest. What are the possible lengths of the shortest board if the third piece is to be at least 5cm longer than the second?

2.142. A solution is to be kept between 68°F and 77°F. What is the range in temperature in degree Celsius(C) if the Celsius / Fahrenheit(F) conversion formula is given by F= $\frac{9}{5}$ C+32?

2.143. A solution of 8% boric acid is to be diluted by adding a 2% boric acid solution to it. The resulting mixture is to be more than 4% but less than 6% boric acid. If we have 640 litres of the 8% solution, how many litres of the 2% solution will have to be added?

2.144. How many litres of water will have to be added to 1125 litres of the 45% solution of acid so that the resulting mixture will contain more than 25% but less than 30% acid content?

2.145. IQ of a person is given by the formula IQ = $\frac{MA}{CA} \times 100$, where MA is mental age and CA is chronological age.If $80 \leq$ IQ ≤ 140 for a group of 12 years old children, find the range of their mental age.