Lines and Planes

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CONTENTS

***Abstract*—This manual provides an application of matrix algebra in coordinate geometry, based on the NCERT textbooks from Class 6-12.**

1 DISTANCE

1.1. Find the distance between the lines

$$L_1 : \quad \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} \tag{1.1.1}$$

$$L_2 : \quad \mathbf{x} = \begin{pmatrix} 3 \\ 3 \\ -5 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} \tag{1.1.2}$$

Solution: Both the lines have the same direction vector, so the lines are parallel. The following code plots

```
codes/line/line_dist_parallel.py
```

Fig. **??** From Fig. **??**, the distance is

$$\|\mathbf{A_2} - \mathbf{A_1}\| \sin\theta = \frac{\|\mathbf{m} \times (\mathbf{A}_2 - \mathbf{A}_1)\|}{\|\mathbf{m}\|} \tag{1.1.3}$$

where

$$\mathbf{A}_1 = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}, \mathbf{A}_2 = \begin{pmatrix} 3 \\ 3 \\ -5 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} \tag{1.1.4}$$

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1.2. Find the shortest distance between the lines

$$L_1 : \quad \mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \tag{1.2.1}$$

$$L_2 : \quad \mathbf{x} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} \tag{1.2.2}$$

Solution: In the given problem

$$\mathbf{A}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{m}_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \mathbf{A}_2 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \mathbf{m}_2 = \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix}. \tag{1.2.3}$$

The lines will intersect if

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} \tag{1.2.4}$$

$$\implies \lambda_1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} - \lambda_2 \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \tag{1.2.5}$$

$$\implies \begin{pmatrix} 2 & 3 \\ -1 & -5 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \tag{1.2.6}$$

Row reducing the augmented matrix,

$$\begin{pmatrix} 2 & 3 & 1 \\ -1 & -5 & 0 \\ 1 & 2 & -1 \end{pmatrix} \xleftrightarrow{R_3 \leftrightarrow R_1} \begin{pmatrix} 1 & 2 & -1 \\ -1 & -5 & 0 \\ 2 & 3 & 1 \end{pmatrix} \tag{1.2.7}$$

$$\xleftrightarrow[R_3 = 2R_1 - R_3]{R_2 = R_1 + R_2} \begin{pmatrix} 1 & 2 & -1 \\ 0 & -3 & -1 \\ 0 & 1 & -3 \end{pmatrix} \xleftrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -3 \\ 0 & -3 & -1 \end{pmatrix} \tag{1.2.8}$$

$$\xleftrightarrow{R_3 = 3R_2 + R_3} \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & -10 \end{pmatrix} \tag{1.2.9}$$

The above matrix has $rank = 3$. Hence, the lines do not intersect. Note that the lines are not parallel but they lie on parallel planes. Such lines are known as *skew* lines. The following code plots Fig. **??**

```
codes/line/line_dist_skew.py
```

The normal to both the lines (and corresponding planes) is

$$\mathbf{n} = \mathbf{m}_1 \times \mathbf{m}_2 \tag{1.2.10}$$

The equation of the second plane is then obtained as

$$\mathbf{n}^T\mathbf{x} = \mathbf{n}^T\mathbf{A}_2 \tag{1.2.11}$$

The distance from $\mathbf{A}_1$ to the above line is then obtained using (2.25.7) as

$$\frac{\left|\mathbf{n}^T\left(\mathbf{A}_2 - \mathbf{A}_1\right)\right|}{\left\|\mathbf{n}\right\|} = \frac{\left|\left(\mathbf{A}_2 - \mathbf{A}_1\right)^T\left(\mathbf{m}_1 \times \mathbf{m}_2\right)\right|}{\left\|\mathbf{m}_1 \times \mathbf{m}_2\right\|} \tag{1.2.12}$$

1.3. Find the distance of the plane

$$\begin{pmatrix} 2 & -3 & 4 \end{pmatrix}\mathbf{x} - 6 = 0 \tag{1.3.1}$$

from the origin.
Solution: From (2.25.7), the distance is obtained as

$$\frac{\left|c\right|}{\left\|\mathbf{n}\right\|} = \frac{6}{\sqrt{2^2+3^2+4^2}} \tag{1.3.2}$$

$$= \frac{6}{\sqrt{29}} \tag{1.3.3}$$

1.4. Show that the lines

$$\frac{x+3}{-3} = \frac{y-1}{1} = \frac{z-5}{5}, \tag{1.4.1}$$

$$\frac{x+1}{-1} = \frac{y-2}{2} = \frac{z-5}{5} \tag{1.4.2}$$

are coplanar.
Solution: Since the given lines have different direction vectors, they are not parallel. From Problem (1.2), the lines are coplanar if the distance between them is 0, i.e. they intersect. This is possible if

$$\left(\mathbf{A}_2 - \mathbf{A}_1\right)^T\left(\mathbf{m}_1 \times \mathbf{m}_2\right) = 0 \tag{1.4.3}$$

From the given information,

$$\mathbf{A}_2 - \mathbf{A}_1 = \begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix} - \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix} \tag{1.4.4}$$

$\mathbf{m}_1 \times \mathbf{m}_2$ is obtained by row reducing the matrix

$$\begin{pmatrix} -1 & 2 & 5 \\ -3 & 1 & 5 \end{pmatrix} \xleftrightarrow{R_2 = \frac{R_2 - 3R_1}{5}} \begin{pmatrix} -1 & 2 & 5 \\ 0 & 1 & 2 \end{pmatrix} \tag{1.4.5}$$

$$\xleftrightarrow{R_1 = -R_1 + 2R_2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix} \implies \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix} \times \begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \tag{1.4.6}$$

The LHS of (1.4.3) is

$$\begin{pmatrix} -2 & -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = 0 \tag{1.4.7}$$

which completes the proof. Alternatively, the lines are coplanar if

$$\begin{vmatrix} \mathbf{A}_1 - \mathbf{A}_2 & \mathbf{m}_1 & \mathbf{m}_2 \end{vmatrix} = 0 \tag{1.4.8}$$

1.5. Find the distance of a point $\begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix}$ from the plane

$$\begin{pmatrix} 6 & -3 & 2 \end{pmatrix}\mathbf{x} = 4 \tag{1.5.1}$$

Solution: Use (2.25.7).

1.6. Find the distance between the point $\mathbf{P} = \begin{pmatrix} 6 \\ 5 \\ 9 \end{pmatrix}$ and the plane determined by the points $\boldsymbol{A} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$, $\boldsymbol{B} = \begin{pmatrix} 5 \\ 2 \\ 4 \end{pmatrix}$ and $\boldsymbol{C} = \begin{pmatrix} -1 \\ -1 \\ 6 \end{pmatrix}$.
Solution: Find the equation of the plane using Problem 2.4. Find the distance using (2.25.7).

1.7. Find the distance of the point $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ from the line $\begin{pmatrix} 12 & -5 \end{pmatrix}\mathbf{x} = -82$.
Solution:

1.8. Find the points on the x-axis, whose distances from the line

$$\begin{pmatrix} 4 & 3 \end{pmatrix}\mathbf{x} = 12 \tag{1.8.1}$$

are 4 units.
Solution:

1.9. What are the points on the y-axis whose distance from the line

$$\begin{pmatrix} 4 & 3 \end{pmatrix}\mathbf{x} = 12 \tag{1.9.1}$$

4 units.
Solution:

1.10. Find the distance of the line

$$L_1: \quad \begin{pmatrix} 4 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{1.10.1}$$

from the point $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ measured along the line L_2 making an angle of $135°$ with the positive x-axis.
Solution: Let P be the point of intersection of L_1 and L_2. The direction vector of L_2 is

$$\mathbf{m} = \begin{pmatrix} 1 \\ \tan 135° \end{pmatrix} \tag{1.10.2}$$

Since $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ lies on L_2, the equation of L_2 is

$$\mathbf{x} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} + \lambda \mathbf{m} \tag{1.10.3}$$

$$\implies \mathbf{P} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} + \lambda \mathbf{m} \tag{1.10.4}$$

$$\text{or,} \quad \left\| \mathbf{P} - \begin{pmatrix} 4 \\ 1 \end{pmatrix} \right\| = d = |\lambda| \, \|\mathbf{m}\| \tag{1.10.5}$$

Since $\mathbf{P}$ lies on L_1, from (1.10.1),

$$\begin{pmatrix} 4 & 1 \end{pmatrix} \mathbf{P} = 0 \tag{1.10.6}$$

Substituting from the above in (1.10.3),

$$\begin{pmatrix} 4 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 4 & 1 \end{pmatrix} \mathbf{m} = 0 \tag{1.10.7}$$

$$\implies \lambda = \frac{\begin{pmatrix} 4 & 1 \end{pmatrix} \mathbf{m}}{17} \tag{1.10.8}$$

substituting $|\lambda|$ in (1.10.5) gives the desired answer.

1.11. Find the distance of the point $\begin{pmatrix} 3 \\ -5 \end{pmatrix}$ from the line

$$\begin{pmatrix} 3 & -4 \end{pmatrix} \mathbf{x} = 26 \tag{1.11.1}$$

Solution: Use (2.25.7).

1.12. Find the distance between the parallel lines

$$\begin{pmatrix} 15 & 8 \end{pmatrix} \mathbf{x} = 34 \tag{1.12.1}$$

$$\begin{pmatrix} 15 & 8 \end{pmatrix} \mathbf{x} = -31 \tag{1.12.2}$$

Solution:

1.13. Find the distance of the point $\begin{pmatrix} -1 \\ -5 \\ -10 \end{pmatrix}$ from the point of intersection of the line

$$\mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} \tag{1.13.1}$$

and the plane

$$\begin{pmatrix} 1 & -1 & 1 \end{pmatrix} \mathbf{x} = 5 \tag{1.13.2}$$

Solution:

1.14. A person standing at the junction of two straight paths represented by the equations

$$\begin{pmatrix} 2 & -3 \end{pmatrix} \mathbf{x} = 4 \tag{1.14.1}$$

$$\begin{pmatrix} 3 & 4 \end{pmatrix} \mathbf{x} = 5 \tag{1.14.2}$$

wants to reach the path whose equation is

$$\begin{pmatrix} 6 & -7 \end{pmatrix} \mathbf{x} = -8 \tag{1.14.3}$$

in the least time. Find the equation of the path that he should follow.
Solution:

1.15. In each of the following cases, determine the normal to the plane and the distance from the origin.

a) $\begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \mathbf{x} = 2$ c) $\begin{pmatrix} 0 & 5 & 0 \end{pmatrix} \mathbf{x} = -8$
b) $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \mathbf{x} = 1$ d) $\begin{pmatrix} 2 & 3 & -1 \end{pmatrix} \mathbf{x} = 5$

Solution:

1.16. Distance between the two planes

$$\begin{pmatrix} 2 & 3 & 4 \end{pmatrix} \mathbf{x} = 4 \tag{1.16.1}$$

$$\begin{pmatrix} 4 & 6 & 8 \end{pmatrix} \mathbf{x} = 12 \tag{1.16.2}$$

Solution:

a) 2 c) 8
b) 4 d) $\frac{2}{\sqrt{29}}$

1.17. Find the distance of the line

$$\begin{pmatrix} 4 & 7 \end{pmatrix} \mathbf{x} = -5 \tag{1.17.1}$$

from the point $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ along the line

$$\begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} = 0. \tag{1.17.2}$$

Solution:

2 Line Equation

2.1. Find equation of line joining
a) $\begin{pmatrix}1 & 2\end{pmatrix}$ and $\begin{pmatrix}3 & 6\end{pmatrix}$
b) $\begin{pmatrix}3 & 1\end{pmatrix}$ and $\begin{pmatrix}9 & 3\end{pmatrix}$.
Solution:
a)
b)

2.2. Find the equation of a plane which is at a distance of $\frac{6}{\sqrt{29}}$ from the origin and has normal vector $\mathbf{n} = \begin{pmatrix}2\\-3\\4\end{pmatrix}$.
Solution: From the previous problem, the desired equation is

$$\begin{pmatrix}2 & -3 & 4\end{pmatrix}\mathbf{x} - 6 = 0 \tag{2.2.1}$$

2.3. Find the equation of the plane which passes through the point $\mathbf{A} = \begin{pmatrix}5\\2\\-4\end{pmatrix}$ and perpendicular to the line with direction vector $\mathbf{n} = \begin{pmatrix}2\\3\\-1\end{pmatrix}$.
Solution: The normal vector to the plane is $\mathbf{n}$. Hence from (2.18.3), the equation of the plane is

$$\mathbf{n}^T(\mathbf{x} - \mathbf{A}) = 0 \tag{2.3.1}$$

$$\implies \begin{pmatrix}2\\3\\-1\end{pmatrix}\mathbf{x} = \begin{pmatrix}2 & 3 & -1\end{pmatrix}\begin{pmatrix}5\\2\\-4\end{pmatrix} \tag{2.3.2}$$

$$= 20 \tag{2.3.3}$$

2.4. Find the equation of the plane passing through $\boldsymbol{R} = \begin{pmatrix}2\\5\\-3\end{pmatrix}$, $\boldsymbol{S} = \begin{pmatrix}-2\\-3\\5\end{pmatrix}$ and $\boldsymbol{T} = \begin{pmatrix}5\\3\\-3\end{pmatrix}$.
Solution: If the equation of the plane be

$$\mathbf{n}^T\mathbf{x} = c, \tag{2.4.1}$$

$$\mathbf{n}^T\mathbf{R} = \mathbf{n}^T\mathbf{S} = \mathbf{n}^T\mathbf{T} = c, \tag{2.4.2}$$

$$\implies \begin{pmatrix}\mathbf{R}-\mathbf{S} & \mathbf{S}-\mathbf{T}\end{pmatrix}^T\mathbf{n} = 0 \tag{2.4.3}$$

after some algebra. Using row reduction on the above matrix,

$$\begin{pmatrix}4 & 8 & -8\\-7 & -6 & 8\end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{R_1}{4}} \begin{pmatrix}1 & 2 & -2\\-7 & -6 & 8\end{pmatrix} \tag{2.4.4}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 + 7R_1} \begin{pmatrix}1 & 2 & -2\\0 & 8 & -6\end{pmatrix} \xleftrightarrow{R_2 \leftarrow \frac{R_2}{2}} \begin{pmatrix}1 & 2 & -2\\0 & 4 & -3\end{pmatrix} \tag{2.4.5}$$

$$\xleftrightarrow{R_1 \leftarrow 2R_1 - R_2} \begin{pmatrix}2 & 0 & -1\\0 & 4 & -3\end{pmatrix} \tag{2.4.6}$$

Thus,

$$\mathbf{n} = \begin{pmatrix}\frac{1}{2}\\\frac{3}{4}\\1\end{pmatrix} = \begin{pmatrix}2\\3\\4\end{pmatrix} \text{ and} \tag{2.4.7}$$

$$c = \mathbf{n}^T\mathbf{T} = 7 \tag{2.4.8}$$

Thus, the equation of the plane is

$$\begin{pmatrix}2 & 3 & 4\end{pmatrix}\mathbf{n} = 7 \tag{2.4.9}$$

Alternatively, the normal vector to the plane can be obtained as

$$\mathbf{n} = (\mathbf{R} - \mathbf{S}) \times (\mathbf{S} - \mathbf{T}) \tag{2.4.10}$$

The equation of the plane is then obtained from (2.18.3) as

$$\mathbf{n}^T(\mathbf{x} - \mathbf{T}) = \left[(\mathbf{R} - \mathbf{S}) \times (\mathbf{S} - \mathbf{T})\right]^T(\mathbf{x} - \mathbf{T}) = 0 \tag{2.4.11}$$

2.5. Find the equation of the plane with intercepts 2, 3 and 4 on the x, y and z axis respectively.
Solution: From the given information, the plane passes through the points $\begin{pmatrix}2\\0\\0\end{pmatrix}$, $\begin{pmatrix}0\\3\\0\end{pmatrix}$ and $\begin{pmatrix}0\\0\\4\end{pmatrix}$ respectively. The equation can be obtained using Problem 2.4.

2.6. Find the equation of the plane passing through the intersection of the planes

$$\begin{pmatrix}1 & 1 & 1\end{pmatrix}\mathbf{x} = 6 \tag{2.6.1}$$

$$\begin{pmatrix}2 & 3 & 4\end{pmatrix}\mathbf{x} = -5 \tag{2.6.2}$$

and the point $\begin{pmatrix}1\\1\\1\end{pmatrix}$.

Solution: The intersection of the planes is obtained by row reducing the augmented matrix as

$$\begin{pmatrix}1 & 1 & 1 & 6\\2 & 3 & 4 & -5\end{pmatrix} \xleftrightarrow{R_2=R_2-2R_1} \begin{pmatrix}1 & 1 & 1 & 6\\0 & 1 & 2 & -17\end{pmatrix} \tag{2.6.3}$$

$$\xleftrightarrow{R_1=R_1-R_2} \begin{pmatrix}1 & 0 & -1 & 23\\0 & 1 & 2 & -17\end{pmatrix} \tag{2.6.4}$$

$$\implies \mathbf{x} = \begin{pmatrix}23\\-17\\0\end{pmatrix} + \lambda \begin{pmatrix}1\\-2\\1\end{pmatrix} \tag{2.6.5}$$

Thus, $\begin{pmatrix}23\\-17\\0\end{pmatrix}$ is another point on the plane. The normal vector to the plane is then obtained as The normal vector to the plane is then obtained as

$$\left(\begin{pmatrix}1\\1\\1\end{pmatrix} - \begin{pmatrix}23\\-17\\0\end{pmatrix}\right) \times \begin{pmatrix}1\\-2\\1\end{pmatrix} \tag{2.6.6}$$

which can be obtained by row reducing the matrix

$$\begin{pmatrix}1 & -2 & 1\\-22 & 18 & 1\end{pmatrix} \xleftrightarrow{R_2=R_2+22R_1} \begin{pmatrix}1 & -2 & 1\\0 & -26 & 23\end{pmatrix} \tag{2.6.7}$$

$$\xleftrightarrow{R_1=13R_1-R_2} \begin{pmatrix}13 & 0 & -10\\0 & -26 & 23\end{pmatrix} \tag{2.6.8}$$

$$\implies \mathbf{n} = \begin{pmatrix}\frac{10}{13}\\\frac{23}{26}\\1\end{pmatrix} = \begin{pmatrix}20\\23\\26\end{pmatrix} \tag{2.6.9}$$

Since the plane passes through $\begin{pmatrix}1\\1\\1\end{pmatrix}$, using (2.18.3),

$$\begin{pmatrix}20 & 23 & 26\end{pmatrix}\left(\mathbf{x} - \begin{pmatrix}1\\1\\1\end{pmatrix}\right) = 0 \tag{2.6.10}$$

$$\implies \begin{pmatrix}20 & 23 & 26\end{pmatrix}\mathbf{x} = 69 \tag{2.6.11}$$

Alternatively, the plane passing through the intersection of (2.6.1) and (2.6.2) has the form

$$\begin{pmatrix}1 & 1 & 1\end{pmatrix}\mathbf{x} + \lambda\begin{pmatrix}2 & 3 & 4\end{pmatrix}\mathbf{x} = 6 - 5\lambda \tag{2.6.12}$$

Substituting $\begin{pmatrix}1\\1\\1\end{pmatrix}$ in the above,

$$\begin{pmatrix}1 & 1 & 1\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix} + \lambda\begin{pmatrix}2 & 3 & 4\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix} = 6 - 5\lambda \tag{2.6.13}$$

$$\implies 3 + 9\lambda = 6 - 5\lambda \tag{2.6.14}$$

$$\implies \lambda = \frac{3}{14} \tag{2.6.15}$$

Substituting this value of λ in (2.6.12) yields the equation of the plane.

2.7. Find the equation of the plane that contains the point $\begin{pmatrix}1\\-1\\2\end{pmatrix}$ and is perpedicular to each of the planes

$$\begin{pmatrix}2 & 3 & -2\end{pmatrix}\mathbf{x} = 5 \tag{2.7.1}$$

$$\begin{pmatrix}1 & 2 & -3\end{pmatrix}\mathbf{x} = 8 \tag{2.7.2}$$

Solution: The normal vector to the desired plane is $\perp$ the normal vectors of both the given planes. Thus,

$$\mathbf{n} = \begin{pmatrix}2\\3\\-2\end{pmatrix} \times \begin{pmatrix}1\\2\\-3\end{pmatrix} \tag{2.7.3}$$

The equation of the plane is then obtained as

$$\mathbf{n}^T\left(\mathbf{x} - \mathbf{A}\right) = 0 \tag{2.7.4}$$

2.8. Find the equation of a line perpendicular to the line

$$\begin{pmatrix}1 & -7\end{pmatrix}\mathbf{x} = -5 \tag{2.8.1}$$

and having x intercept 3.

Solution:

2.9. Two lines passing through the point $\begin{pmatrix}2\\3\end{pmatrix}$ intersect each other at angle of $60°$. If the slope of one line is 2, find the equation of the other

line.
Solution:

2.10. Find the equation of the right bisector of the line segment joining the points $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$.
Solution:

2.11. Find the equations of the lines, which cut-off intercepts on the axes whose sum and product are 1 and -6 respectively.
Solution:

2.12. Find the equation of the line parallel to the y-axis drawn through the point of intersection of the lines

$$\begin{pmatrix} 1 & -7 \end{pmatrix} \mathbf{x} = -5 \tag{2.12.1}$$

$$\begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{2.12.2}$$

Solution:

2.13. Find the equation of the lines through the point $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ which make an angle of $45°$ with the line

$$\begin{pmatrix} 1 & -2 \end{pmatrix} \mathbf{x} = 3. \tag{2.13.1}$$

Solution:

2.14. Find the equation of the line passing through the point of intersection of the lines

$$\begin{pmatrix} 4 & 7 \end{pmatrix} \mathbf{x} = 3 \tag{2.14.1}$$

$$\begin{pmatrix} 2 & -3 \end{pmatrix} \mathbf{x} = -1 \tag{2.14.2}$$

that has equal intercepts on the axes.
Solution:

2.15. Two positions of time and distance are recorded as, when $T = 0, D = 2$ and when $T = 3, D = 8$. Using the concept of slope, find law of motion, i.e., how distance depends upon time.
Solution: The equation of the line joining the points $\mathbf{A} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 \\ 8 \end{pmatrix}$ is obtained as

$$\mathbf{x} = \mathbf{A} + \lambda \left(\mathbf{B} - \mathbf{A} \right) \tag{2.15.1}$$

$$\implies \begin{pmatrix} T \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} - \lambda \begin{pmatrix} -3 \\ -6 \end{pmatrix} \tag{2.15.2}$$

which can be expressed as

$$\begin{pmatrix} 2 & -1 \end{pmatrix} \begin{pmatrix} T \\ D \end{pmatrix} = \begin{pmatrix} 2 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \end{pmatrix} \tag{2.15.3}$$

$$\implies \begin{pmatrix} 2 & -1 \end{pmatrix} \begin{pmatrix} T \\ D \end{pmatrix} = -2 \tag{2.15.4}$$

$$\implies D = 2 + 2T \tag{2.15.5}$$

2.16. A line L is such that its segment between the lines is bisected at the point $\mathbf{P} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$. Obtain its equation.

$$L_1 : \quad \begin{pmatrix} 5 & -1 \end{pmatrix} \mathbf{x} = -4 \tag{2.16.1}$$

$$L_2 : \quad \begin{pmatrix} 3 & 4 \end{pmatrix} \mathbf{x} = 4 \tag{2.16.2}$$

Solution: Let

$$L : \quad \mathbf{x} = \mathbf{P} + \lambda \mathbf{m} \tag{2.16.3}$$

If L intersects L_1 and L_2 at $\mathbf{A}$ and $\mathbf{B}$ respectively,

$$\mathbf{A} = \mathbf{P} + \lambda \mathbf{m} \tag{2.16.4}$$

$$\mathbf{B} = \mathbf{P} - \lambda \mathbf{m} \tag{2.16.5}$$

since $\mathbf{P}$ bisects AB. Note that λ is a measure of the distance from P along the line L. From (2.16.1), (2.16.4) and (2.16.5),

$$\begin{pmatrix} 5 & -1 \end{pmatrix} \mathbf{A} = \begin{pmatrix} 5 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 5 & -1 \end{pmatrix} \mathbf{m} = -4 \tag{2.16.6}$$

$$\begin{pmatrix} 3 & 4 \end{pmatrix} \mathbf{B} = \begin{pmatrix} 3 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 5 \end{pmatrix} - \lambda \begin{pmatrix} 3 & 4 \end{pmatrix} \mathbf{m} = 4 \tag{2.16.7}$$

yielding

$$19 \begin{pmatrix} 5 & -1 \end{pmatrix} \mathbf{m} = -4 \begin{pmatrix} 3 & 4 \end{pmatrix} \mathbf{m} \tag{2.16.8}$$

$$\implies \begin{pmatrix} 107 & -3 \end{pmatrix} \mathbf{m} = 0 \tag{2.16.9}$$

$$\text{or, } \mathbf{n} = \begin{pmatrix} 107 \\ -3 \end{pmatrix} \tag{2.16.10}$$

after simplification. Thus, the equation of the line is

$$\mathbf{n}^T \left(\mathbf{x} - \mathbf{P} \right) = 0 \tag{2.16.11}$$

2.17. Find the equations of the lines parallel to the axes and passing through $\mathbf{A} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$.

Solution: The line parallel to the x-axis has direction vector $\mathbf{m} = \begin{pmatrix}1\\0\end{pmatrix}$. Hence,its equation is obtined as

$$\mathbf{x} = \begin{pmatrix}-2\\3\end{pmatrix} + \lambda_1 \begin{pmatrix}1\\0\end{pmatrix} \tag{2.17.1}$$

Similarly, the equation of the line parallel to the y-axis can be obtained as

$$\mathbf{x} = \begin{pmatrix}-2\\3\end{pmatrix} + \lambda_1 \begin{pmatrix}0\\1\end{pmatrix} \tag{2.17.2}$$

The following code plots Fig. **??**

```
codes/line/line_parallel_axes.py
```

2.18. Find the equation of the line through $\mathbf{A} = \begin{pmatrix}-2\\3\end{pmatrix}$ with slope –4.

Solution: The direction vector is $\mathbf{m} = \begin{pmatrix}1\\-4\end{pmatrix}$. Hence, the normal vector

$$\mathbf{n} = \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\mathbf{m} \tag{2.18.1}$$

$$= \begin{pmatrix}4\\1\end{pmatrix} \tag{2.18.2}$$

The equation of the line in terms of the normal vector is then obtained as

$$\mathbf{n}^T(\mathbf{x} - \mathbf{A}) = 0 \tag{2.18.3}$$

$$\implies \begin{pmatrix}4 & 1\end{pmatrix}\mathbf{x} = -5 \tag{2.18.4}$$

2.19. Write the equation of the line through the points $\begin{pmatrix}1\\-1\end{pmatrix}$ and $\begin{pmatrix}3\\5\end{pmatrix}$.
Solution: Use (2.17.1).

2.20. Write the equation of the lines for which $\tan\theta = \frac{1}{2}$, where θ is the inclination of the line and
a) y-intercept is $-\frac{3}{2}$
b) x-intercept is 4.

Solution: From the given information, $\tan\theta = \frac{1}{2} = m$.
a) y-intercept is $-\frac{3}{2} \implies$ the line cuts through the y-axis at $\begin{pmatrix}0\\-\frac{3}{2}\end{pmatrix}$.
b) x-intercept is 4 $\implies$ the line cuts through the x-axis at $\begin{pmatrix}4\\0\end{pmatrix}$.

Use the above information get the equations for the lines.

2.21. Find the equation of a line through the point $\begin{pmatrix}5\\2\\-4\end{pmatrix}$ and parallel to the vector $\begin{pmatrix}3\\2\\-8\end{pmatrix}$.
Solution: The equation of the line is

$$\mathbf{x} = \begin{pmatrix}5 & 2\\-4 & \end{pmatrix} + \lambda \begin{pmatrix}3\\2\\-8\end{pmatrix} \tag{2.21.1}$$

2.22. Find the equation of a line passing through the points $\begin{pmatrix}-1\\0\\2\end{pmatrix}$ and $\begin{pmatrix}3\\4\\6\end{pmatrix}$.
Solution: Using (2.15.1), the desired equation of the line is

$$\mathbf{x} = \begin{pmatrix}-1 & 0\\2 & \end{pmatrix} + \lambda \begin{pmatrix}4\\4\\4\end{pmatrix} \tag{2.22.1}$$

$$= \begin{pmatrix}-1 & 0\\2 & \end{pmatrix} + \lambda \begin{pmatrix}1\\1\\1\end{pmatrix} \tag{2.22.2}$$

2.23. If

$$\frac{x+3}{2} = \frac{y-5}{4} = \frac{z+6}{2} = \lambda \tag{2.23.1}$$

find the equation of the line.
Solution: The line can be expressed from (2.23.1) as

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-3+2\lambda\\5+4\lambda\\-6+2\lambda\end{pmatrix} \tag{2.23.2}$$

$$\implies \mathbf{x} = \begin{pmatrix}-3\\5\\-6\end{pmatrix} + \lambda \begin{pmatrix}2\\4\\2\end{pmatrix} \tag{2.23.3}$$

$$\implies \mathbf{x} = \begin{pmatrix}-3\\5\\-6\end{pmatrix} + \lambda \begin{pmatrix}1\\2\\1\end{pmatrix} \tag{2.23.4}$$

2.24. Find the equation of the line, which makes intercepts -3 and 2 on the x and y axes respectively.
Solution:

2.25. Find the equation of the line whose perpendicular distance from the origin is 4 units and the angle which the normal makes with the positive direction of x-axis is $15°$.

Solution: In Fig. **??**, the foot of the perpendicular P is the intersection of the lines L and M. Thus,

$$\mathbf{n}^T\mathbf{P} = c \quad (2.25.1)$$

$$\mathbf{P} = \mathbf{A} + \lambda\mathbf{n} \quad (2.25.2)$$

$$\text{or, } \mathbf{n}^T\mathbf{P} = \mathbf{n}^T\mathbf{A} + \lambda\left\|\mathbf{n}\right\|^2 = c \quad (2.25.3)$$

$$\implies -\lambda = \frac{\mathbf{n}^T\mathbf{A} - c}{\left\|\mathbf{n}\right\|^2} \quad (2.25.4)$$

Also, the distance between $\mathbf{A}$ and L is obtained from

$$\mathbf{P} = \mathbf{A} + \lambda\mathbf{n} \quad (2.25.5)$$

$$\implies \left\|\mathbf{P} - \mathbf{A}\right\| = |\lambda|\left\|\mathbf{n}\right\| \quad (2.25.6)$$

From (2.25.4) and (2.25.6)

$$\left\|\mathbf{P} - \mathbf{A}\right\| = \frac{\left|\mathbf{n}^T\mathbf{A} - c\right|}{\left\|\mathbf{n}\right\|} \quad (2.25.7)$$

$$\mathbf{n} = \begin{pmatrix} 1 \\ \tan 15^\circ \end{pmatrix} \quad (2.25.8)$$

$\because \mathbf{A} = \mathbf{0}$,

$$4 = \frac{|c|}{\left\|\mathbf{n}\right\|} \implies c = \pm 4\sqrt{1 + \tan^2 15^\circ} \quad (2.25.9)$$

$$= \pm 4\sec 15^\circ \quad (2.25.10)$$

where

$$\sec\theta = \frac{1}{\cos\theta} \quad (2.25.11)$$

This follows from the fact that

$$\cos^2\theta + \sin^2\theta = 1 \quad (2.25.12)$$

$$\implies 1 + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \quad (2.25.13)$$

It is easy to verify that

$$\frac{\sin\theta}{\cos\theta} = \tan\theta \quad (2.25.14)$$

$$\implies 1 + \tan^2\theta = \sec^2\theta \quad (2.25.15)$$

Thus, the equation of the line is

$$\begin{pmatrix} 1 & \tan 15^\circ \end{pmatrix}\mathbf{c} = \pm 4\sec 15^\circ \quad (2.25.16)$$

2.26. The Farenheit temperature F and absolute temperature K satisfy a linear equation. Given $K = 273$ when $F = 32$ and that $K = 373$ when $F = 212$, express K in terms of F and find the value of F, when $K = 0$.
Solution: Let

$$\mathbf{x} = \begin{pmatrix} F & K \end{pmatrix} \quad (2.26.1)$$

Since the relation between F, K is linear, $\begin{pmatrix} 273 \\ 32 \end{pmatrix}, \begin{pmatrix} 373 \\ 21 \end{pmatrix}$ are on a line. The corresponding equation is obtained from (2.18.3) and (2.18.1) as

$$\begin{pmatrix} 11 & -100 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 11 & -100 \end{pmatrix}\begin{pmatrix} 273 \\ 32 \end{pmatrix} \quad (2.26.2)$$

$$\implies \begin{pmatrix} 11 & -100 \end{pmatrix}\mathbf{x} = -197 \quad (2.26.3)$$

If $\begin{pmatrix} F \\ 0 \end{pmatrix}$ is a point on the line,

$$\begin{pmatrix} 11 & -100 \end{pmatrix}\begin{pmatrix} F \\ 0 \end{pmatrix} = -197 \implies F = -\frac{197}{11} \quad (2.26.4)$$

2.27. Equation of a line is

$$\begin{pmatrix} 3 & -4 \end{pmatrix}\mathbf{x} + 10 = 0. \quad (2.27.1)$$

Find its
a) slope,
b) x - and y-intercepts.

Solution: From the given information,

$$\mathbf{n} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}, \quad (2.27.2)$$

$$\mathbf{m} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}, \quad (2.27.3)$$

a) $m = \frac{3}{4}$
b) x-intercept is $-\frac{10}{3}$ and y-intercept is $\frac{10}{4} = \frac{5}{2}$.

2.28. Find the equation of a line perpendicular to the line

$$\begin{pmatrix} 1 & -2 \end{pmatrix}\mathbf{x} = 3 \quad (2.28.1)$$

and passes through the point $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$.
Solution: The normal vector of the perpendicular line is

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad (2.28.2)$$

Thus, the desired equation of the line is

$$\begin{pmatrix} 2 & 1 \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right) = 0 \quad (2.28.3)$$

$$\implies \begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = 0 \quad (2.28.4)$$

2.29. Find the equation of a line which passes through the point $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and is parallel to the vector $\begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}$.
Solution:

2.30. Find the equaion off the line that passes through $\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$ and is in the direction $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$.
Solution:

2.31. Find the equation of the line given by

$$\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2}. \quad (2.31.1)$$

Solution:

2.32. Find the equation of the line passing through the origin and the point $\begin{pmatrix} 5 \\ -2 \\ 3 \end{pmatrix}$.
Solution:

2.33. Find the equation of the line passing through the points $\begin{pmatrix} 3 \\ -2 \\ -5 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \\ 6 \end{pmatrix}$.
Solution:

2.34. Find the vector equation of the line passing through the point $\begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}$ and perpendicular to the two lines

$$\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7}, \quad (2.34.1)$$

$$\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5} \quad (2.34.2)$$

Solution:

2.35. If $\mathbf{O}$ be the origin and the coordinates of $\mathbf{P}$ be $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, then find the equation of the plane passing through $\mathbf{P}$ and perpendicular to OP.
Solution:

2.36. Find the equation of the planes

a) that passes through the point $\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$ and the normal to the plane is $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$.
Solution:

b) that passes through the point $\begin{pmatrix} 1 \\ 4 \\ 6 \end{pmatrix}$ and the normal vector the plane is $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$. **Solution:**

2.37. Find the equation of the plane through the intersection of the planes $\begin{pmatrix} 3 & -1 & 2 \end{pmatrix} \mathbf{x} = 4$ and $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \mathbf{x} = -2$ and the pont $\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$.
Solution:

2.38. Find the equation of the line passing through $\begin{pmatrix} -3 \\ 5 \end{pmatrix}$ and perpendicular to the line through the points $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 6 \end{pmatrix}$. **Solution:**

3 Properties

3.1. Find the ratio in which the line segment joining the points $\begin{pmatrix} 4 \\ 8 \\ 10 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ 10 \\ -8 \end{pmatrix}$ is divided by the YZ-plane.
Solution:

3.2. Find the coordinates of the foot of the perpendicular drawn from the origin to the plane

$$\begin{pmatrix} 2 & -3 & 4 \end{pmatrix} \mathbf{x} - 6 = 0 \quad (3.2.1)$$

Solution: The normal vector is

$$\mathbf{n} = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} \quad (3.2.2)$$

Hence, the foot of the perpendicular from the origin is $\lambda \mathbf{n}$. Substituting in (3.2.1),

$$\lambda \left\| \mathbf{n} \right\|^2 = 6 \implies \lambda = \frac{6}{\left\| \mathbf{n} \right\|^2} = \frac{6}{29} \quad (3.2.3)$$

Thus, the foot of the perpendicular is

$$\frac{6}{29} \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} \quad (3.2.4)$$

3.3. Find the coordinates of the point where the line through the points $\mathbf{A} = \begin{pmatrix}3\\4\\1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}5\\1\\6\end{pmatrix}$ crosses the XY plane.
Solution: The equation of the line is

$$\mathbf{x} = \mathbf{A} + \lambda\left(\mathbf{B} - \mathbf{A}\right) \tag{3.3.1}$$

$$= \begin{pmatrix}3\\4\\1\end{pmatrix} + \lambda\begin{pmatrix}2\\-3\\5\end{pmatrix} \tag{3.3.2}$$

The line crosses the XY plane for $x_3 = 0 \implies \lambda = -\frac{1}{5}$. Thus, the desired point is

$$\begin{pmatrix}3\\4\\1\end{pmatrix} - \frac{1}{5}\begin{pmatrix}2\\-3\\5\end{pmatrix} = \frac{1}{5}\begin{pmatrix}13\\23\\0\end{pmatrix} \tag{3.3.3}$$

3.4. The line through the points $\begin{pmatrix}h\\3\end{pmatrix}$ and $\begin{pmatrix}4\\1\end{pmatrix}$ intersects the line

$$\begin{pmatrix}7 & -9\end{pmatrix}\mathbf{x} = 19 \tag{3.4.1}$$

at right angle. Find the value of h.
Solution:

3.5. Find the coordinates of the foot of the perpendicular from the point $\begin{pmatrix}-1\\3\end{pmatrix}$ to the line

$$\begin{pmatrix}3 & -4\end{pmatrix}\mathbf{x} = 16. \tag{3.5.1}$$

Solution:

3.6. The perpendicular from the origin to the line

$$\begin{pmatrix}-m & 1\end{pmatrix}\mathbf{x} = c \tag{3.6.1}$$

meets it at the point $\begin{pmatrix}-1\\2\end{pmatrix}$. Find the values of m and c.
Solution:

3.7. In what ratio is the line joining $\begin{pmatrix}-1\\1\end{pmatrix}$ and $\begin{pmatrix}5\\7\end{pmatrix}$ divided by the line

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 4 \tag{3.7.1}$$

Solution:

3.8. Find the direction in which a straight line must be drawn through the point $\begin{pmatrix}-1\\2\end{pmatrix}$ so that its point of intersection with the line

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 4 \tag{3.8.1}$$

may be at a distance of 3 units from this point.
Solution:

3.9. Show that the path of a moving point such that its distances from two lines

$$\begin{pmatrix}3 & -2\end{pmatrix}\mathbf{x} = 5 \tag{3.9.1}$$

$$\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x} = 5 \tag{3.9.2}$$

are equal is a straight line.
Solution: Using (2.25.7) the point $\mathbf{x}$ satisfies

$$\frac{\left|\begin{pmatrix}3 & -2\end{pmatrix}\mathbf{x} - 5\right|}{\left\|\begin{pmatrix}3\\-2\end{pmatrix}\right\|} = \frac{\left|\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x} - 5\right|}{\left\|\begin{pmatrix}3\\2\end{pmatrix}\right\|} \tag{3.9.3}$$

$$\implies \left|\begin{pmatrix}3 & -2\end{pmatrix}\mathbf{x} - 5\right| = \left|\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x} - 5\right| \tag{3.9.4}$$

resulting in

$$\begin{pmatrix}3 & -2\end{pmatrix}\mathbf{x} - 5 = \pm\left(\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x} - 5\right) \tag{3.9.5}$$

leading to the possible lines

$$L_1 : \quad \begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} = 0 \tag{3.9.6}$$

$$L_2 : \quad \begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = \frac{5}{3} \tag{3.9.7}$$

3.10. Find the coordinates of the point where the line through $\begin{pmatrix}3\\-4\\-5\end{pmatrix}$ and $\begin{pmatrix}2\\-3\\1\end{pmatrix}$ crosses the plane

$$\begin{pmatrix}2 & 1 & 1\end{pmatrix}\mathbf{x} = 7 \tag{3.10.1}$$

Solution:

4 Least Squares

4.1. Find the shortest distance between the lines

$$L_1 : \quad \mathbf{x} = \begin{pmatrix}1-t\\t-2\\3-2t\end{pmatrix} \tag{4.1.1}$$

$$L_2 : \quad \mathbf{x} = \begin{pmatrix}s+1\\2s-1\\-2s-1\end{pmatrix} \tag{4.1.2}$$

Solution:

4.2. Find the shortest distance between the lines

$$L_1 : \quad \mathbf{x} = \begin{pmatrix}1\\2\\1\end{pmatrix} + \lambda_1\begin{pmatrix}1\\-1\\1\end{pmatrix} \tag{4.2.1}$$

$$L_2 : \quad \mathbf{x} = \begin{pmatrix}2\\-1\\-1\end{pmatrix} + \lambda_2\begin{pmatrix}2\\1\\2\end{pmatrix} \tag{4.2.2}$$

Solution:

5 Reflection

5.1. Assuming that straight lines work as a plane mirror for a point, find the image of the point $\begin{pmatrix}1\\2\end{pmatrix}$ in the line

$$\begin{pmatrix}1 & -3\end{pmatrix}\mathbf{x} = -4. \tag{5.1.1}$$

Solution: Since $\mathbf{R}$ is the reflection of $\mathbf{P}$ and $\mathbf{Q}$ lies on L, $\mathbf{Q}$ bisects PR. This leads to the following equations

$$2\mathbf{Q} = \mathbf{P} + \mathbf{R} \tag{5.1.2}$$
$$\mathbf{n}^T\mathbf{Q} = c \tag{5.1.3}$$
$$\mathbf{m}^T\mathbf{R} = \mathbf{m}^T\mathbf{P} \tag{5.1.4}$$

where $\mathbf{m}$ is the direction vector of L. From (5.1.2) and (5.1.3),

$$\mathbf{n}^T\mathbf{R} = 2c - \mathbf{n}^T\mathbf{P} \tag{5.1.5}$$

From (5.1.5) and (5.1.4),

$$\begin{pmatrix}\mathbf{m} & \mathbf{n}\end{pmatrix}^T\mathbf{R} = \begin{pmatrix}\mathbf{m} & -\mathbf{n}\end{pmatrix}^T\mathbf{P} + \begin{pmatrix}0\\2c\end{pmatrix} \tag{5.1.6}$$

Letting

$$\mathbf{V} = \begin{pmatrix}\mathbf{m} & \mathbf{n}\end{pmatrix} \tag{5.1.7}$$

with the condition that $\mathbf{m}, \mathbf{n}$ are orthonormal, i.e.

$$\mathbf{V}^T\mathbf{V} = \mathbf{I} \tag{5.1.8}$$

Noting that

$$\begin{pmatrix}\mathbf{m} & -\mathbf{n}\end{pmatrix} = \begin{pmatrix}\mathbf{m} & \mathbf{n}\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}, \tag{5.1.9}$$

(5.1.6) can be expressed as

$$\mathbf{V}^T\mathbf{R} = \left[\mathbf{V}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\right]^T\mathbf{P} + \begin{pmatrix}0\\2c\end{pmatrix} \tag{5.1.10}$$

$$\implies \mathbf{R} = \left[\mathbf{V}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\mathbf{V}^{-1}\right]^T\mathbf{P} + \mathbf{V}\begin{pmatrix}0\\2c\end{pmatrix} \tag{5.1.11}$$

$$= \mathbf{V}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\mathbf{V}^T\mathbf{P} + 2c\mathbf{n} \tag{5.1.12}$$

It can be verified that the reflection is also given by

$$\frac{\mathbf{R}}{2} = \frac{\mathbf{m}\mathbf{m}^T - \mathbf{n}\mathbf{n}^T}{\mathbf{m}^T\mathbf{m} + \mathbf{n}^T\mathbf{n}}\mathbf{P} + c\frac{\mathbf{n}}{\|\mathbf{n}\|^2} \tag{5.1.13}$$

The following code plots Fig. **??** while computing the reflection

```
codes/line/line_reflect.py
```

5.2. Find the image of the point $\begin{pmatrix}3\\8\end{pmatrix}$ with respect to the line

$$\begin{pmatrix}1 & 3\end{pmatrix}\mathbf{x} = 7 \tag{5.2.1}$$

assuming the line to be a plane mirror.
Solution:

5.3. A ray of light passing through the point $\begin{pmatrix}1\\2\end{pmatrix}$ reflects on the x-axis at point $\mathbf{A}$ and the reflected ray passes through the point $\begin{pmatrix}5\\3\end{pmatrix}$. Find the coordinates of $\mathbf{A}$.
Solution: