Linear Forms

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CONTENTS

***Abstract*—This manual provides a simple introduction to linear forms like lines and planes, based on the NCERT textbooks from Class 6-12.**

1 Definitions

1.1 Two Dimensions

1.1.1. The equation of a line is given by

$$\mathbf{n}^\top \mathbf{x} = c \tag{1.1.1.1}$$

where $\mathbf{n}$ is the normal vector of the line.

1.1.2. The parametric equation of a line is given by

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m} \tag{1.1.2.1}$$

where $\mathbf{m}$ is the direction vector of the line and $\mathbf{A}$ is any point on the line.

1.1.3. The distance from a point $\mathbf{P}$ to the line in (1.1.1.1) is given by

$$d = \frac{\left|\mathbf{n}^\top \mathbf{P} - c\right|}{\left\|\mathbf{n}\right\|} \tag{1.1.3.1}$$

1.1.4. The distance from the origin to the line in (1.1.1.1) is given by

$$d = \frac{\left|c\right|}{\left\|\mathbf{n}\right\|} \tag{1.1.4.1}$$

1.1.5. The distance between the parallel lines

$$\mathbf{n}^\top \mathbf{x} = c_1 \mathbf{n}^\top \mathbf{x} = c_2 \tag{1.1.5.1}$$

is given by

$$d = \frac{\left|c_1 - c_2\right|}{\left\|\mathbf{n}\right\|} \tag{1.1.5.2}$$

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1.1.6. The equation of the line perpendicular to (1.1.1.1) and passing through the point $\mathbf{P}$ is given by

$$\mathbf{m}^\top \left(\mathbf{x} - \mathbf{P}\right) = 0 \tag{1.1.6.1}$$

1.1.7. The foot of the perpendicular from $\mathbf{P}$ to the line in (1.1.1.1) is given by

$$\begin{pmatrix} \mathbf{m} & \mathbf{n} \end{pmatrix}^\top \mathbf{x} = \begin{pmatrix} \mathbf{m}^\top \mathbf{P} \\ c \end{pmatrix} \tag{1.1.7.1}$$

1.2 Three Dimensions

1.2.1. The equation of a line is given by (1.1.2.1)

1.2.2. The equation of a plane is given by (1.1.1.1)

1.2.3. The distance from the origin to the line in (1.1.1.1) is given by (1.1.4.1)

1.2.4. The equation of the line perpendicular to (1.1.1.1) and passing through the point $\mathbf{P}$ is given by

$$\mathbf{m}^\top \left(\mathbf{x} - \mathbf{P}\right) = 0 \tag{1.2.4.1}$$

1.2.5. The foot of the perpendicular from $\mathbf{P}$ to the line in (1.1.1.1) is given by

$$\begin{pmatrix} \mathbf{m} & \mathbf{n} \end{pmatrix}^\top \mathbf{x} = \begin{pmatrix} \mathbf{m}^\top \mathbf{P} \\ c \end{pmatrix} \tag{1.2.5.1}$$

1.2.6. The distance from a point $\mathbf{P}$ to the line in (1.1.2.1) is given by

$$d = \left\|\mathbf{A} - \mathbf{P} - \frac{\mathbf{m}^\top \left(\mathbf{A} - \mathbf{P}\right)}{\left\|\mathbf{m}\right\|^2}\mathbf{m}\right\| \tag{1.2.6.1}$$

Solution:

$$d\left(\lambda\right) = \left\|\mathbf{A} + \lambda\mathbf{m} - \mathbf{P}\right\| \tag{1.2.6.2}$$

$$\implies d^2\left(\lambda\right) = \left\|\mathbf{A} + \lambda\mathbf{m} - \mathbf{P}\right\|^2 \tag{1.2.6.3}$$

which can be simplified to obtain

$$d^2\left(\lambda\right) = \lambda^2 \left\|\mathbf{m}\right\|^2 + 2\lambda \mathbf{m}^\top \left(\mathbf{A} - \mathbf{P}\right) + \left\|\mathbf{A} - \mathbf{P}\right\|^2 \tag{1.2.6.4}$$

Now, $d(\lambda)$ is smallest when the above quadratic equation has a single root, i.e.

$$\lambda = -\frac{\mathbf{m}^\top (\mathbf{A} - \mathbf{P})}{\|\mathbf{m}\|^2} \tag{1.2.6.5}$$

From (1.2.6.2) and (1.2.6.5),

1.2.7. The distance between the parallel planes (1.1.5.1) is given by (1.1.5.2).