Linear Forms

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CONTENTS

***Abstract*—This manual provides a simple introduction to linear forms like lines and planes, based on the NCERT textbooks from Class 6-12.**

1 DEFINITIONS

1.1 Two Dimensions

1.1.1. The equation of a line is given by

$$\mathbf{n}^\top \mathbf{x} = c \tag{1.1.1.1}$$

where $\mathbf{n}$ is the normal vector of the line.

1.1.2. The equation of a line with normal vector $\mathbf{n}$ and passing through a point $\mathbf{A}$ is given by

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{A}) = 0 \tag{1.1.2.1}$$

1.1.3. The parametric equation of a line is given by

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m} \tag{1.1.3.1}$$

where $\mathbf{m}$ is the direction vector of the line and $\mathbf{A}$ is any point on the line.

1.1.4. The distance from a point $\mathbf{P}$ to the line in (1.1.1.1) is given by

$$d = \frac{\left|\mathbf{n}^\top \mathbf{P} - c\right|}{\left\|\mathbf{n}\right\|} \tag{1.1.4.1}$$

Solution: Without loss of generality, let $\mathbf{A}$ be the foot of the perpendicular from $\mathbf{P}$ to the line in (1.1.3.1). The equation of the normal to (1.1.1.1) can then be expressed as

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{n} \tag{1.1.4.2}$$

$$\implies \mathbf{P} - \mathbf{A} = \lambda \mathbf{n} \tag{1.1.4.3}$$

$\because$ $\mathbf{P}$ lies on (1.1.4.2). From the above, the desired distance can be expressed as

$$d = \left\|\mathbf{P} - \mathbf{A}\right\| = \left|\lambda\right| \left\|\mathbf{n}\right\| \tag{1.1.4.4}$$

From (1.1.4.3),

$$\mathbf{n}^\top (\mathbf{P} - \mathbf{A}) = \lambda \mathbf{n}^\top \mathbf{n} = \lambda \left\|\mathbf{n}\right\|^2 \tag{1.1.4.5}$$

$$\implies \left|\lambda\right| = \frac{\left|\mathbf{n}^\top (\mathbf{P} - \mathbf{A})\right|}{\left\|\mathbf{n}\right\|^2} \tag{1.1.4.6}$$

Substituting the above in (1.1.4.4) and using the fact that

$$\mathbf{n}^\top \mathbf{A} = c \tag{1.1.4.7}$$

from (1.1.1.1), yields (1.1.4.1).

1.1.5. The distance from the origin to the line in (1.1.1.1) is given by

$$d = \frac{\left|c\right|}{\left\|\mathbf{n}\right\|} \tag{1.1.5.1}$$

1.1.6. The distance between the parallel lines

$$\begin{aligned} \mathbf{n}^\top \mathbf{x} &= c_1 \\ \mathbf{n}^\top \mathbf{x} &= c_2 \end{aligned} \tag{1.1.6.1}$$

is given by

$$d = \frac{\left|c_1 - c_2\right|}{\left\|\mathbf{n}\right\|} \tag{1.1.6.2}$$

1.1.7. The equation of the line perpendicular to (1.1.1.1) and passing through the point $\mathbf{P}$ is given by

$$\mathbf{m}^\top (\mathbf{x} - \mathbf{P}) = 0 \tag{1.1.7.1}$$

1.1.8. The foot of the perpendicular from $\mathbf{P}$ to the line in (1.1.1.1) is given by

$$\begin{pmatrix} \mathbf{m} & \mathbf{n} \end{pmatrix}^\top \mathbf{x} = \begin{pmatrix} \mathbf{m}^\top \mathbf{P} \\ c \end{pmatrix} \tag{1.1.8.1}$$

Solution: From (1.1.1.1) and (1.1.2.1) the foot of the perpendicular satisfies the equations

$$\mathbf{n}^\top \mathbf{x} = c \tag{1.1.8.2}$$

$$\mathbf{m}^\top (\mathbf{x} - \mathbf{P}) = 0 \tag{1.1.8.3}$$

where $\mathbf{m}$ is the direction vector of the given line. Combining the above into a matrix equation results in (1.1.8.1).

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1.2 Three Dimensions

1.2.1. The equation of a line is given by (1.1.3.1)

1.2.2. The equation of a plane is given by (1.1.1.1)

1.2.3. The distance from the origin to the line in (1.1.1.1) is given by (1.1.5.1)

1.2.4. The equation of the line perpendicular to (1.1.1.1) and passing through the point $\mathbf{P}$ is given by

$$\mathbf{m}^\top (\mathbf{x} - \mathbf{P}) = 0 \tag{1.2.4.1}$$

1.2.5. The foot of the perpendicular from $\mathbf{P}$ to the line in (1.1.1.1) is given by

$$\begin{pmatrix} \mathbf{m} & \mathbf{n} \end{pmatrix}^\top \mathbf{x} = \begin{pmatrix} \mathbf{m}^\top \mathbf{P} \\ c \end{pmatrix} \tag{1.2.5.1}$$

1.2.6. The distance from a point $\mathbf{P}$ to the line in (1.1.3.1) is given by

$$d = \|\mathbf{A} - \mathbf{P}\|^2 - \frac{\left\{\mathbf{m}^\top (\mathbf{A} - \mathbf{P})\right\}^2}{\|\mathbf{m}\|^2} \tag{1.2.6.1}$$

Solution:

$$d(\lambda) = \|\mathbf{A} + \lambda\mathbf{m} - \mathbf{P}\| \tag{1.2.6.2}$$

$$\implies d^2(\lambda) = \|\mathbf{A} + \lambda\mathbf{m} - \mathbf{P}\|^2 \tag{1.2.6.3}$$

which can be simplified to obtain

$$d^2(\lambda) = \lambda^2 \|\mathbf{m}\|^2 + 2\lambda\mathbf{m}^\top (\mathbf{A} - \mathbf{P}) + \|\mathbf{A} - \mathbf{P}\|^2 \tag{1.2.6.4}$$

which is of the form

$$d^2(\lambda) = a\lambda^2 + 2b\lambda + c \tag{1.2.6.5}$$

$$= a\left\{\left(\lambda + \frac{b}{a}\right)^2 + \left[\frac{c}{a} - \left(\frac{b}{a}\right)^2\right]\right\} \tag{1.2.6.6}$$

with

$$a = \|\mathbf{m}\|^2, b = \mathbf{m}^\top (\mathbf{A} - \mathbf{P}), c = \|\mathbf{A} - \mathbf{P}\|^2 \tag{1.2.6.7}$$

which can be expressed as From the above, $d^2(\lambda)$ is smallest when upon substituting from (1.2.6.7)

$$\lambda + \frac{b}{2a} = 0 \implies \lambda = -\frac{b}{2a} \quad = -\frac{\mathbf{m}^\top (\mathbf{A} - \mathbf{P})}{\|\mathbf{m}\|^2} \tag{1.2.6.8}$$

and consequently,

$$d_{\min}(\lambda) = a\left(\frac{c}{a} - \left(\frac{b}{a}\right)^2\right) \tag{1.2.6.9}$$

$$= c - \frac{b^2}{a} \tag{1.2.6.10}$$

yielding (1.2.6.1) after substituting from (1.2.6.7).

1.2.7. The distance between the parallel planes (1.1.6.1) is given by (1.1.6.2).

1.2.8. The plane

$$\mathbf{n}^\top - \mathbf{x} = c \tag{1.2.8.1}$$

contains the line

$$\mathbf{x} = \mathbf{A} + \lambda\mathbf{m} \tag{1.2.8.2}$$

if

$$\mathbf{m}^\top \mathbf{n} = 0 \tag{1.2.8.3}$$

Solution: Any point on the line (1.2.8.2) should also satisfy (1.2.8.1). Hence,

$$\mathbf{n}^\top (\mathbf{A} + \lambda\mathbf{m}) = \mathbf{n}^\top \mathbf{A} = c \tag{1.2.8.4}$$

which can be simplified to obtain (1.2.8.3)

1.2.9. Let a plane pass through the points $\mathbf{A}, \mathbf{B}$ and be perpendicular to the plane

$$\mathbf{n}^\top \mathbf{x} = c \tag{1.2.9.1}$$

Then the equation of this plane is given by

$$\mathbf{p}^\top \mathbf{x} = 1 \tag{1.2.9.2}$$

where

$$\mathbf{p} = \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{n} \end{pmatrix}^{-\top} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \tag{1.2.9.3}$$

Solution: From the given information,

$$\mathbf{p}^\top \mathbf{A} = d \tag{1.2.9.4}$$

$$\mathbf{p}^\top \mathbf{B} = d \tag{1.2.9.5}$$

$$\mathbf{p}^\top \mathbf{n} = 0 \tag{1.2.9.6}$$

$\because$ the normal vectors to the two planes will also be perpendicular. The system of equations in (1.2.9.6) can be expressed as the matrix equation

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{n} \end{pmatrix}^\top \mathbf{p} = d \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \tag{1.2.9.7}$$

which yields (1.2.9.3) upon normalising with d.

1.2.10. The intersection of the line represented by (1.1.3.1) with the plane represented by (1.1.1.1) is given by

$$\mathbf{x} = \mathbf{A} + \frac{c - \mathbf{n}^\top \mathbf{A}}{\mathbf{n}^\top \mathbf{m}} \mathbf{m} \tag{1.2.10.1}$$

Solution: From (1.1.3.1) and (1.1.1.1),

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m} \tag{1.2.10.2}$$

$$\mathbf{n}^\top \mathbf{x} = c \tag{1.2.10.3}$$

$$\implies \mathbf{n}^\top \left(\mathbf{A} + \lambda \mathbf{m}\right) = c \tag{1.2.10.4}$$

which can be simplified to obtain

$$\mathbf{n}^\top \mathbf{A} + \lambda \mathbf{n}^\top \mathbf{m} = c \tag{1.2.10.5}$$

$$\implies \lambda = \frac{c - \mathbf{n}^\top \mathbf{A}}{\mathbf{n}^\top \mathbf{m}} \tag{1.2.10.6}$$

Substituting the above in (1.2.10.4) yields (1.2.10.1).

1.2.11. The foot of the perpendicular from the point $\mathbf{P}$ to the line represented by (1.1.3.1) is given by

$$\mathbf{x} = \mathbf{A} + \frac{\mathbf{m}^\top \left(\mathbf{P} - \mathbf{A}\right)}{\left\|\mathbf{m}\right\|^2} \mathbf{m} \tag{1.2.11.1}$$

Solution: Let the equation of the line be

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m} \tag{1.2.11.2}$$

The equation of the plane perpendicular to the given line passing through $\mathbf{P}$ is given by

$$\mathbf{m}^\top \left(\mathbf{x} - \mathbf{P}\right) = 0 \tag{1.2.11.3}$$

$$\implies \mathbf{m}^\top \mathbf{x} = \mathbf{m}^\top \mathbf{P} \tag{1.2.11.4}$$

The desired foot of the perpendicular is the intersection of (1.2.11.2) with (1.2.11.3) which can be obtained from (1.2.10.1) as (1.2.11.1)

1.2.12. The foot of the perpendicular from a point $\mathbf{P}$ to a plane is $\mathbf{Q}$. The equation of the plane is given by

$$\left(\mathbf{P} - \mathbf{Q}\right)^\top \left(\mathbf{x} - \mathbf{Q}\right) = 0 \tag{1.2.12.1}$$

Solution: The normal vector to the plane is given by

$$\mathbf{n} = \mathbf{P} - \mathbf{Q} \tag{1.2.12.2}$$

Hence, the equation of the plane is (1.2.12.1).

2 Examples

2.1.