Matrix Equations and the Cayley-Hamilton Theorem

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Contents

Abstract—**This manual provides some applications of the Cayley-Hamilton theorem, based on exercises from the NCERT textbooks from Class 6-12.**

1 Eigenvalues and Eigenvectors

1.1. The eigenvalue λ and the eigenvector $\mathbf{x}$ for a matrix $\mathbf{A}$ are defined as,

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \tag{1.1.1}$$

1.2. The eigenvalues are calculated by solving the equation

$$f(\lambda) = \left|\lambda\mathbf{I} - \mathbf{A}\right| = 0 \tag{1.2.1}$$

The above equation is known as the characteristic equation.

1.3. According to the Cayley-Hamilton theorem,

$$f(\lambda) = 0 \implies f(\mathbf{A}) = 0 \tag{1.3.1}$$

1.4. The trace of a square matrix is defined to be the sum of the diagonal elements.

$$\operatorname{tr}(\mathbf{A}) = \sum_{i=1}^{N} a_{ii}. \tag{1.4.1}$$

where a_{ii} is the ith diagonal element of the matrix $\mathbf{A}$.

1.5. The trace of a matrix is equal to the sum of the eigenvalues

$$\operatorname{tr}(\mathbf{A}) = \sum_{i=1}^{N} \lambda_i \tag{1.5.1}$$

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2 Examples

2.1. If

$$\mathbf{A} = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}, \tag{2.1.1}$$

show that

$$\mathbf{A}^2 - 5\mathbf{A} + 7\mathbf{I} = 0 \tag{2.1.2}$$

Solution:
The characteristic equation is

$$\left|\mathbf{A} - \lambda\mathbf{I}\right| = 0 \tag{2.1.3}$$

$$\implies \begin{vmatrix} 3-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} = 0 \tag{2.1.4}$$

$$\implies (3-\lambda)(2-\lambda) + 1 = 0 \tag{2.1.5}$$

$$\text{or, } \lambda^2 - 5\lambda + 7 = 0 \tag{2.1.6}$$

Using the Cayley-Hamilton theorem, (2.1.2) is obtained.

2.2. If $\mathbf{A} = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}$ is such that $\mathbf{A}^2 = \mathbf{I}$,then

a) $1 + \alpha^2 + \beta\gamma = 0$
b) $1 - \alpha^2 + \beta\gamma = 0$
c) $1 - \alpha^2 - \beta\gamma = 0$
d) $1 + \alpha^2 - \beta\gamma = 0$

Solution: If

$$\mathbf{A} = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}, \ \mathbf{A}^2 = I \tag{2.2.1}$$

The characteristic equation is

$$|\mathbf{A} - \lambda\mathbf{I}| = 0 \tag{2.2.2}$$

$$\implies \begin{vmatrix} \alpha-\lambda & \beta \\ \gamma & -\alpha-\lambda \end{vmatrix} = 0 \tag{2.2.3}$$

$$\implies (\alpha-\lambda)(-\alpha-\lambda) - \gamma\beta = 0 \tag{2.2.4}$$

$$\implies \lambda^2 - \alpha^2 - \gamma\beta = 0 \tag{2.2.5}$$

By the Cayley-Hamilton theorem, every square matrix satisfies its own characteristic equation. Hence,on substituting from (2.2.1) in (2.2.5)

$$\mathbf{A^2} - \alpha^2\mathbf{I} - \gamma\beta\mathbf{I} = 0. \tag{2.2.6}$$

$$\implies \mathbf{I}\left(1 - \alpha^2 - \gamma\beta\right) = 0 \tag{2.2.7}$$

Hence, (c) is the correct answer

2.3. If $\mathbf{A}$ is square matrix such that

$$\mathbf{A}^2 = \mathbf{A}, \tag{2.3.1}$$

then

$$g(\mathbf{A}) = (\mathbf{I} + \mathbf{A})^3 - 7\mathbf{A} \tag{2.3.2}$$

is equal to

a) $\mathbf{A}$
b) $\mathbf{I} - \mathbf{A}$
c) $\mathbf{I}$
d) $3\mathbf{A}$

Solution:
From (2.3.2),

$$g(\lambda) = (1 + \lambda)^3 - 7\lambda \tag{2.3.3}$$

Also,

$$\mathbf{A}^2 = \mathbf{A} \implies \mathbf{A}^2 - \mathbf{A} = 0 \tag{2.3.4}$$

Using the Cayley-Hamilton theorem, the eigenvalues satisfy the characteristic equation

$$f(\lambda) = \lambda^2 - \lambda = 0 \tag{2.3.5}$$

$\because f(\lambda)$ is of degree 2 and $g(\lambda)$ is of degree 3, (2.3.3) can be expressed as

$$g(\lambda) = f(\lambda)\, q(\lambda) + a\lambda + b \tag{2.3.6}$$

where a, b are real numbers and $q(\lambda)$ is some polynomial. Thus,

$$g(0) = b = 1 \tag{2.3.7}$$
$$g(1) = a + b = 1 \tag{2.3.8}$$
$$\implies a = 90, b = 1 \tag{2.3.9}$$

Thus,

$$g(\mathbf{A}) = f(\mathbf{A})\, q(\mathbf{A}) + a\mathbf{A} + b\mathbf{I} \tag{2.3.10}$$
$$= \mathbf{I} \tag{2.3.11}$$

upon substituting from (2.3.5) and (2.3.9). Option c is the valid answer.

2.4. If

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{pmatrix}, \tag{2.4.1}$$

prove that

$$\mathbf{A}^3 - 6\mathbf{A}^2 + 7\mathbf{A} + 2\mathbf{I} = 0 \tag{2.4.2}$$

Solution:
From (1.2.1), the characteristic equation is

$$\begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & 2-\lambda & 1 \\ 2 & 0 & 3-\lambda \end{vmatrix} = 0 \tag{2.4.3}$$

which can be expanded to obtain

$$(1-\lambda)(2-\lambda)(3-\lambda) + 2(-2(2-\lambda)) = 0 \tag{2.4.4}$$

yielding

$$\lambda^3 - 6\lambda^2 + 7\lambda + 2 = 0 \tag{2.4.5}$$

upon simplification. Using the Cayley-Hamilton theorem in (1.3.1), (2.4.2) is obtained

2.5. If

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{pmatrix} \tag{2.5.1}$$

then show that

$$\mathbf{A}^3 - 23\mathbf{A} - 40\mathbf{I} = 0 \tag{2.5.2}$$

Solution:
The Characteristic equation is given by

$$|\mathbf{A} - \lambda\mathbf{I}| = 0 \tag{2.5.3}$$
$$\implies \begin{vmatrix} 1-\lambda & 2 & 3 \\ 3 & -2-\lambda & 1 \\ 4 & 2 & 1-\lambda \end{vmatrix} = 0 \tag{2.5.4}$$

which can be expressed as

$$\implies (1-\lambda)\left((-2-\lambda)(1-\lambda) - 2\right) \\ - 2\left(3(1-\lambda) - 4\right) + 3\left(6 + 4(2+\lambda)\right) = 0 \tag{2.5.5}$$

and simplified to obtain

$$\implies \lambda^3 - 23\lambda - 40 = 0. \tag{2.5.6}$$

Using the Cayley-Hamilton Theorem, (2.5.2) is obtained.