Linear Equations

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CONTENTS

***Abstract*—This manual provides an introduction to linear equations based on the NCERT textbooks from Class 6-12.**

1 Linear Equations

1.1. Solve the following

a)

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 5$$
$$\begin{pmatrix}2 & -3\end{pmatrix}\mathbf{x} = 4 \tag{1.1.1}$$

b)

$$\begin{pmatrix}3 & 4\end{pmatrix}\mathbf{x} = 10$$
$$\begin{pmatrix}2 & -2\end{pmatrix}\mathbf{x} = 2 \tag{1.1.2}$$

c)

$$\begin{pmatrix}3 & -5\end{pmatrix}\mathbf{x} = 4$$
$$\begin{pmatrix}9 & -2\end{pmatrix}\mathbf{x} = 7 \tag{1.1.3}$$

$$\begin{pmatrix}\frac{1}{2} & \frac{2}{3}\end{pmatrix}\mathbf{x} = -1$$
$$\begin{pmatrix}1 & -\frac{1}{3}\end{pmatrix}\mathbf{x} = 3 \tag{1.1.4}$$

Solution:

1.2. Solve the following pair of linear equations

$$\begin{pmatrix}8 & 5\end{pmatrix}\mathbf{x} = 9$$
$$\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x} = 4 \tag{1.2.1}$$

Solution:

1.3. Solve the following pair of linear equations

$$\begin{pmatrix}158 & -378\end{pmatrix}\mathbf{x} = -74$$
$$\begin{pmatrix}-378 & 152\end{pmatrix}\mathbf{x} = -604 \tag{1.3.1}$$

Solution:

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1.4. Solve the following pair of equations

$$\begin{pmatrix}7 & -15\end{pmatrix}\mathbf{x} = 2 \tag{1.4.1}$$
$$\begin{pmatrix}1 & 2\end{pmatrix}\mathbf{x} = 3 \tag{1.4.2}$$

Solution: The above equations can be expressed as the matrix equation

$$\begin{pmatrix}7 & -15\\ 1 & 2\end{pmatrix}\mathbf{x} = \begin{pmatrix}2\\3\end{pmatrix} \tag{1.4.3}$$

The augmented matrix for the above equation is row reduced as follows

$$\begin{pmatrix}7 & -15 & 2\\ 1 & 2 & 3\end{pmatrix} \xleftrightarrow{R_2 \leftarrow 7R_2 - R_1} \begin{pmatrix}7 & -15 & 2\\ 0 & 29 & 19\end{pmatrix} \tag{1.4.4}$$

$$\xleftrightarrow{R_1 \leftarrow \frac{15R_2 29 + 29R_1}{29}} \begin{pmatrix}7 & 0 & 2\\ 0 & 29 & 19\end{pmatrix} \tag{1.4.5}$$

$$\implies \mathbf{x} = \begin{pmatrix}\frac{2}{7}\\ \frac{19}{29}\end{pmatrix} \tag{1.4.6}$$

The python code in Problem 2.25 can be used to plot Fig. **??**, which shows that the lines are the same.

1.5. Find all possibe solutions of

$$\begin{pmatrix}2 & 3\end{pmatrix}\mathbf{x} = 8$$
$$\begin{pmatrix}4 & 6\end{pmatrix}\mathbf{x} = 7 \tag{1.5.1}$$

Solution: The above equations can be expressed as the matrix equation

$$\begin{pmatrix}2 & -3\\ 4 & 6\end{pmatrix}\mathbf{x} = \begin{pmatrix}8\\7\end{pmatrix} \tag{1.5.2}$$

The augmented matrix for the above equation is row reduced as follows

$$\begin{pmatrix}2 & 3 & 8\\ 4 & 6 & 7\end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{pmatrix}2 & -3 & 8\\ 0 & 0 & -9\end{pmatrix} \tag{1.5.3}$$

$$\implies rank\begin{pmatrix}2 & -3\\ 4 & 6\end{pmatrix} \neq \begin{pmatrix}2 & 3 & 8\\ 4 & 6 & 7\end{pmatrix}. \tag{1.5.4}$$

Hence, (1.5.1) has no solution. The python code in Problem 2.25 can be used to plot Fig. **??**, which shows that the lines are parallel.

1.6. Which of the following pairs of linear equations are consistent/inconsistent? If consistent, obtain the solution:

a)

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 5 \\ \begin{pmatrix}2 & 2\end{pmatrix}\mathbf{x} = 10 \tag{1.6.1}$$

b)

$$\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = 8 \\ \begin{pmatrix}3 & -3\end{pmatrix}\mathbf{x} = 16 \tag{1.6.2}$$

c)

$$\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} = 6 \\ \begin{pmatrix}4 & -2\end{pmatrix}\mathbf{x} = 4 \tag{1.6.3}$$

d)

$$\begin{pmatrix}2 & -2\end{pmatrix}\mathbf{x} = 2 \\ \begin{pmatrix}4 & -4\end{pmatrix}\mathbf{x} = 5 \tag{1.6.4}$$

Solution:

1.7. Find the intersection of the following lines

a)

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 14 \\ \begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = 4 \tag{1.7.1}$$

b)

$$\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = 3 \\ \begin{pmatrix}\frac{1}{3} & \frac{1}{2}\end{pmatrix}\mathbf{x} = 6 \tag{1.7.2}$$

c)

$$\begin{pmatrix}3 & -1\end{pmatrix}\mathbf{x} = 3 \\ \begin{pmatrix}9 & -3\end{pmatrix}\mathbf{x} = 9 \tag{1.7.3}$$

d)

$$\begin{pmatrix}0.2 & 0.3\end{pmatrix}\mathbf{x} = 1.3 \\ \begin{pmatrix}0.4 & 0.5\end{pmatrix}\mathbf{x} = 2.3 \tag{1.7.4}$$

e)

$$\begin{pmatrix}\sqrt{2} & \sqrt{3}\end{pmatrix}\mathbf{x} = 0 \\ \begin{pmatrix}\sqrt{3} & \sqrt{8}\end{pmatrix}\mathbf{x} = 0 \tag{1.7.5}$$

f)

$$\begin{pmatrix}\frac{3}{2} & -\frac{5}{3}\end{pmatrix}\mathbf{x} = -2 \\ \begin{pmatrix}\frac{1}{3} & \frac{1}{2}\end{pmatrix}\mathbf{x} = \frac{13}{6} \tag{1.7.6}$$

Solution:

1.8. Draw the graphs of the equations

$$\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} + 1 = 0 \tag{1.8.1}$$

$$\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x} - 12 = 0 \tag{1.8.2}$$

Determine the coordinates of the vertices of the triangle formed by these lines and the x-axis, and shade the triangular region.

Solution:

1.9. In a $\triangle ABC, \angle C = 3\angle B = 2(\angle A + \angle B)$. Find the three angles.

Solution:

1.10. Draw the graphs of the equations $5x–y = 5$ and $3x–y = 3$. Determine the co-ordinates of the vertices of the triangle formed by these lines and the y axis.

Solution:

1.11. $ABCD$ is a cyclic quadrilateral with

$$\angle A = 4y + 20 \tag{1.11.1}$$

$$\angle B = 3y - 5 \tag{1.11.2}$$

$$\angle C = -4x \tag{1.11.3}$$

$$\angle D = -7x + 5 \tag{1.11.4}$$

Find its angles.

Solution:

2 Rank

2.1. Verify if $\mathbf{A} = \begin{pmatrix}3\\1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}6\\4\end{pmatrix}, \mathbf{C} = \begin{pmatrix}8\\6\end{pmatrix}$ are points on a line.

2.2. Determine if the points

$$\begin{pmatrix}1\\5\end{pmatrix}, \begin{pmatrix}2\\3\end{pmatrix}, \begin{pmatrix}-2\\-11\end{pmatrix} \tag{2.2.1}$$

are collinear.

Solution:

2.3. By using the concept of equation of a line, prove that the three points $\begin{pmatrix}3\\0\end{pmatrix}, \begin{pmatrix}-2\\-2\end{pmatrix}$ and $\begin{pmatrix}8\\2\end{pmatrix}$ are collinear.

Solution:

2.4. Find the value of x for which the points $\begin{pmatrix} x \\ -1 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 5 \end{pmatrix}$ are collinear.
Solution:

2.5. In each of the following, find the value of k for which the points are collinear
a) $\begin{pmatrix} 7 \\ -2 \end{pmatrix}, \begin{pmatrix} 5 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ k \end{pmatrix}$
b) $\begin{pmatrix} 8 \\ 1 \end{pmatrix}, \begin{pmatrix} k \\ -4 \end{pmatrix}, \begin{pmatrix} 2 \\ -5 \end{pmatrix}$
Solution:

2.6. Show that the points $\mathbf{A} = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 \\ 6 \\ 3 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 3 \\ 10 \\ -1 \end{pmatrix}$ are collinear.
Solution:

2.7. Show that the points $\mathbf{A} = \begin{pmatrix} 1 \\ -2 \\ -8 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 5 \\ 0 \\ -2 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 11 \\ 3 \\ 7 \end{pmatrix}$ are collinear, and find the ratio in which $\mathbf{B}$ divides AC.
Solution:

2.8. Show that $\mathbf{A} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 5 \\ 8 \\ 7 \end{pmatrix}$ are collinear.
Solution:

2.9. Show that the points $\mathbf{A} = \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 7 \\ 0 \\ -1 \end{pmatrix}$ are collinear.
Solution: Forming the matrix in (**??**)

$$\mathbf{M} = \begin{pmatrix} 3 & -1 & -2 \\ 9 & -3 & -6 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - 3R_1} \begin{pmatrix} 3 & -1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.9.1}$$

$\implies rank(\mathbf{M}) = 1$. The following code plots Fig. **??** showing that the points are collinear.

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codes/line/draw_lines_3d.py
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2.10. If $\mathbf{A} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 3 \\ 2 \\ -3 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 1 \\ -6 \\ -1 \end{pmatrix}$, show that $\mathbf{A} - \mathbf{B}$ and $\mathbf{C} - \mathbf{D}$ are collinear.
Solution:

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} -1 \\ -4 \\ 1 \end{pmatrix} \tag{2.10.1}$$

$$\mathbf{C} - \mathbf{D} = \begin{pmatrix} 2 \\ 8 \\ -2 \end{pmatrix} \tag{2.10.2}$$

$$\because -2\left(\mathbf{A} - \mathbf{B}\right) = \mathbf{C} - \mathbf{D}, \tag{2.10.3}$$

$\mathbf{A} - \mathbf{B}$ and $\mathbf{C} - \mathbf{D}$ are collinear.

2.11. Find the value of k if the points $\mathbf{A} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 4 \\ k \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 6 \\ -3 \end{pmatrix}$ are collinear.
Solution: Forming the matrix

$$\mathbf{M} = \begin{pmatrix} \mathbf{B} - \mathbf{A} & \mathbf{B} - \mathbf{A} \end{pmatrix}^T = \begin{pmatrix} 2 & k-3 \\ 4 & -6 \end{pmatrix} \tag{2.11.1}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{R_2}{2}} \begin{pmatrix} 2 & k-3 \\ 2 & -3 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 2 & k-3 \\ 0 & -k \end{pmatrix} \tag{2.11.2}$$

$$\implies rank(\mathbf{M}) = 1 \iff R_2 = \mathbf{0}, \text{or } k = 0 \tag{2.11.3}$$

2.12. Find a unit vector perpendicular to each of the vectors $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$, where

$$\mathbf{a} = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}. \tag{2.12.1}$$

Solution:

2.13. Let $\mathbf{a} = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 3 \\ -2 \\ 7 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$. Find a vector $\mathbf{d}$ such that $\mathbf{d} \perp \mathbf{a}, \mathbf{d} \perp \mathbf{b}$ and $\mathbf{d}^T\mathbf{c} = 15$.
Solution:

2.14. Find a unit vector perpendicular to each of the vectors $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$, where

$$\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}. \tag{2.14.1}$$

Solution: If $\mathbf{x}$ is the desired vector,

$$(\mathbf{a}+\mathbf{b})^T\mathbf{x} = 0 \tag{2.14.2}$$
$$(\mathbf{a}-\mathbf{b})^T\mathbf{x} = 0 \tag{2.14.3}$$

resulting in the matrix equation

$$\begin{pmatrix}2 & 3 & 4\\ 0 & -1 & -2\end{pmatrix}\mathbf{x} = 0 \tag{2.14.4}$$

Performing row operations,

$$\begin{pmatrix}2 & 3 & 4\\ 0 & -1 & -2\end{pmatrix} \xleftrightarrow[R_2\leftarrow -R_2]{R_1\leftarrow R_1+3R_2} \begin{pmatrix}2 & 0 & -2\\ 0 & -1 & -2\end{pmatrix} \tag{2.14.5}$$

$$\xleftrightarrow{R_1\leftarrow \frac{R_1}{2}} \begin{pmatrix}1 & 0 & -1\\ 0 & 1 & 2\end{pmatrix} \implies \begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} = x_3\begin{pmatrix}1\\ -2\\ 1\end{pmatrix} \tag{2.14.6}$$

The desired unit vector is then obtained as

$$\mathbf{x} = \frac{\begin{pmatrix}1\\ -2\\ 1\end{pmatrix}}{\left\|\begin{pmatrix}1\\ -2\\ 1\end{pmatrix}\right\|} = \frac{1}{\sqrt{6}}\begin{pmatrix}1\\ -2\\ 1\end{pmatrix} \tag{2.14.7}$$

2.15. Show that $\mathbf{A} = \begin{pmatrix}-2\\ 3\\ 5\end{pmatrix}, \mathbf{B} = \begin{pmatrix}1\\ 2\\ 3\end{pmatrix}, \mathbf{C} = \begin{pmatrix}7\\ 0\\ -1\end{pmatrix}$, are collinear.
Solution: See Problem 2.9.

2.16. Show that points A=$\begin{pmatrix}a & b+c\end{pmatrix}$, B=$\begin{pmatrix}b & c+a\end{pmatrix}$, C=$\begin{pmatrix}c & a+b\end{pmatrix}$ are collinear.
Solution:

2.17. Find m if

$$\begin{aligned}\begin{pmatrix}2 & 3\end{pmatrix}\mathbf{x} &= 11\\ \begin{pmatrix}2 & -4\end{pmatrix}\mathbf{x} &= -24\\ \begin{pmatrix}m & -1\end{pmatrix}\mathbf{x} &= -3\end{aligned} \tag{2.17.1}$$

Solution:

2.18. For which values of a and b does the following pair of linear equations have an infinite number of solutions?

$$\begin{aligned}\begin{pmatrix}2 & 3\end{pmatrix}\mathbf{x} &= 7\\ \begin{pmatrix}a-b & a+b\end{pmatrix}\mathbf{x} &= 3a+b-2\end{aligned} \tag{2.18.1}$$

Solution:

2.19. For which value of k will the following pair of linear equations have no solution?

$$\begin{aligned}\begin{pmatrix}3 & 1\end{pmatrix}\mathbf{x} &= 1\\ \begin{pmatrix}2k-1 & k-1\end{pmatrix}\mathbf{x} &= 2k+1\end{aligned} \tag{2.19.1}$$

Solution:

2.20. If the lines

$$\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} = 3 \tag{2.20.1}$$
$$\begin{pmatrix}5 & k\end{pmatrix}\mathbf{x} = 3 \tag{2.20.2}$$
$$\begin{pmatrix}3 & -1\end{pmatrix}\mathbf{x} = 2 \tag{2.20.3}$$

are concurrent, find the value of k.
Solution: If the lines are concurrent, the *augmented* matrix should have a 0 row upon row reduction. Hence,

$$\begin{pmatrix}2 & 1 & 3\\ 5 & k & 3\\ 3 & -1 & 2\end{pmatrix} \xleftrightarrow{R_2\leftrightarrow R_3} \begin{pmatrix}2 & 1 & 3\\ 3 & -1 & 2\\ 5 & k & 3\end{pmatrix} \tag{2.20.4}$$

$$\xleftrightarrow[R_3\leftarrow 2R_3-5R_1]{R_2\leftrightarrow 2R_2-3R_1} \begin{pmatrix}2 & 1 & 3\\ 0 & -5 & -5\\ 0 & 2k-5 & -9\end{pmatrix} \tag{2.20.5}$$

$$\xleftrightarrow{R_2\leftarrow -\frac{R_2}{5}} \begin{pmatrix}2 & 1 & 3\\ 0 & 1 & 1\\ 0 & 2k-5 & -9\end{pmatrix} \tag{2.20.6}$$

$$\xleftrightarrow{R_3\leftarrow R_3-(2k-5)R_2} \begin{pmatrix}2 & 1 & 3\\ 0 & 1 & 1\\ 0 & 0 & -2k-4\end{pmatrix} \tag{2.20.7}$$

$$\implies k = -2 \tag{2.20.8}$$

2.21. For which values of p does the pair of equations given below has unique solution?

$$\begin{aligned}\begin{pmatrix}4 & p\end{pmatrix}\mathbf{x} &= -8\\ \begin{pmatrix}2 & 2\end{pmatrix}\mathbf{x} &= -2\end{aligned} \tag{2.21.1}$$

Solution: (2.21.1) has a unique solution

$$\iff \begin{vmatrix}4 & p\\ 2 & 2\end{vmatrix} \neq 0 \tag{2.21.2}$$
$$\text{or, } p \neq 4 \tag{2.21.3}$$

2.22. For what values of k will the following pair of linear equations have infinitely many solutions?

$$\begin{aligned}\begin{pmatrix}k & 3\end{pmatrix}\mathbf{x} &= k-3\\ \begin{pmatrix}12 & k\end{pmatrix}\mathbf{x} &= k\end{aligned} \tag{2.22.1}$$

Solution: The first condition for (2.22.1) to have infinite solutions is

$$\begin{vmatrix} k & 3 \\ 12 & k \end{vmatrix} = 0 \tag{2.22.2}$$

$$\implies k^2 = 36, \text{or}, \ k = \pm 6 \tag{2.22.3}$$

For $k = 6$, the augmented matrix for the above equation is row reduced as follows

$$\begin{pmatrix} 6 & 3 & 3 \\ 12 & 6 & 6 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{pmatrix} 6 & 3 & 3 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.22.4}$$

indicating that (2.22.1) has infinite number of solutions. For $k = -6$, the augmented matrix is

$$\begin{pmatrix} 6 & 3 & -9 \\ 12 & 6 & -6 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{pmatrix} 6 & 3 & -9 \\ 0 & 0 & 12 \end{pmatrix} \tag{2.22.5}$$

indicating that (2.22.1) has no solution Thus, (2.22.2) is a necessary condition but not sufficient.

2.23. Check whether the pair of equations

$$\begin{pmatrix} 1 & 3 \end{pmatrix} \mathbf{x} = 6 \text{ and} \\ \begin{pmatrix} 2 & -3 \end{pmatrix} \mathbf{x} = 12 \tag{2.23.1}$$

is consistent.

Solution: The above equations can be expressed as the matrix equation

$$\begin{pmatrix} 1 & 3 \\ 2 & -3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 6 \\ 12 \end{pmatrix} \tag{2.23.2}$$

The augmented matrix for the above equation is row reduced as follows

$$\begin{pmatrix} 1 & 3 & 6 \\ 2 & -3 & 12 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow \frac{R_2 - 2R_1}{-9}} \tag{2.23.3}$$

$$\begin{pmatrix} 1 & 3 & 6 \\ 0 & 1 & 0 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 - 3R_2} \begin{pmatrix} 1 & 0 & 6 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.23.4}$$

$$\implies \mathbf{x} = \begin{pmatrix} 6 \\ 0 \end{pmatrix} \tag{2.23.5}$$

which is the solution of 2.25.1. The python code in Problem 2.25 can be used to plot Fig. **??**, which shows that the lines intersect.

2.24. Find whether the following pair of equations has no solution, unique solution or infinitely many solutions:

$$\begin{pmatrix} 5 & -8 \end{pmatrix} \mathbf{x} = -1 \text{ and} \\ \begin{pmatrix} 3 & -\frac{24}{5} \end{pmatrix} \mathbf{x} = -\frac{3}{5} \tag{2.24.1}$$

Solution: The above equations can be expressed as the matrix equation

$$\begin{pmatrix} 5 & -8 \\ 3 & -\frac{24}{5} \end{pmatrix} \mathbf{x} = -\begin{pmatrix} 1 \\ \frac{3}{5} \end{pmatrix} \tag{2.24.2}$$

The augmented matrix for the above equation is row reduced as follows

$$\begin{pmatrix} 5 & -8 & -1 \\ 3 & -\frac{24}{5} & -\frac{3}{5} \end{pmatrix} \xleftrightarrow{R_2 \leftarrow 5R_2} \begin{pmatrix} 5 & -8 & 1 \\ 15 & -24 & -3 \end{pmatrix} \tag{2.24.3}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - 3R_1} \begin{pmatrix} 5 & -8 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.24.4}$$

$$\because rank \begin{pmatrix} 5 & -8 \\ 3 & -\frac{24}{5} \end{pmatrix} = rank \begin{pmatrix} 5 & -8 & 1 \\ 3 & -\frac{24}{5} & -\frac{3}{5} \end{pmatrix} \tag{2.24.5}$$

$$= 1 < dim \begin{pmatrix} 5 & -8 \\ 3 & -\frac{24}{5} \end{pmatrix} = 2, \tag{2.24.6}$$

(2.24.1) has infinitely many solutions. The python code in Problem 2.25 can be used to plot Fig. **??**, which shows that the lines are the same.

2.25. Two rails are represented by the equations

$$\begin{pmatrix} 1 & 2 \end{pmatrix} \mathbf{x} = 4 \text{ and} \\ \begin{pmatrix} 2 & 4 \end{pmatrix} \mathbf{x} = 12. \tag{2.25.1}$$

Will the rails cross each other?

Solution: The above equations can be expressed as the matrix equation

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 4 \\ 12 \end{pmatrix} \tag{2.25.2}$$

The augmented matrix for the above equation is row reduced as follows

$$\begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 12 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow \frac{R_2}{2}} \begin{pmatrix} 1 & 2 & 4 \\ 1 & 2 & 6 \end{pmatrix} \tag{2.25.3}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 2 & 4 \\ 0 & 0 & 2 \end{pmatrix} \tag{2.25.4}$$

$\because$ row reduction of the 2×3 matrix

$$\begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 12 \end{pmatrix} \tag{2.25.5}$$

results in a matrix with 2 nonzero rows, its rank is 2. Similarly, the rank of the matrix

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \tag{2.25.6}$$

is 1, from **??**.

$$\because rank \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \neq rank \begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 12 \end{pmatrix}, \tag{2.25.7}$$

(2.25.1) has no solution. The equivalent python code is

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codes/line/line_check_sol.py
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which plots Fig. **??**, which shows that the rails are parallel.

2.26. Which of the following pairs of linear equations has a unique solution, no solution, or infinitely many solutions?

a)

$$\begin{aligned} \begin{pmatrix} 1 & -3 \end{pmatrix} \mathbf{x} = 3 \\ \begin{pmatrix} 3 & -9 \end{pmatrix} \mathbf{x} = 2 \end{aligned} \tag{2.26.1}$$

b)

$$\begin{aligned} \begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = 5 \\ \begin{pmatrix} 3 & 2 \end{pmatrix} \mathbf{x} = 8 \end{aligned} \tag{2.26.2}$$

c)

$$\begin{aligned} \begin{pmatrix} 3 & -5 \end{pmatrix} \mathbf{x} = 20 \\ \begin{pmatrix} 6 & -10 \end{pmatrix} \mathbf{x} = 40 \end{aligned} \tag{2.26.3}$$

d)

$$\begin{aligned} \begin{pmatrix} 1 & -3 \end{pmatrix} \mathbf{x} = 7 \\ \begin{pmatrix} 3 & -3 \end{pmatrix} \mathbf{x} = 15 \end{aligned} \tag{2.26.4}$$

Solution:

2.27. Find out whether the lines representing the following pairs of linear equations intersect at a point, are parallel or coincident

a)

$$\begin{aligned} \begin{pmatrix} 5 & -4 \end{pmatrix} \mathbf{x} = -8 \\ \begin{pmatrix} 7 & 6 \end{pmatrix} \mathbf{x} = 9 \end{aligned} \tag{2.27.1}$$

b)

$$\begin{aligned} \begin{pmatrix} 9 & 3 \end{pmatrix} \mathbf{x} = -12 \\ \begin{pmatrix} 18 & 6 \end{pmatrix} \mathbf{x} = -24 \end{aligned} \tag{2.27.2}$$

c)

$$\begin{aligned} \begin{pmatrix} 6 & -3 \end{pmatrix} \mathbf{x} = -10 \\ \begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} = -9 \end{aligned} \tag{2.27.3}$$

Solution:

2.28. Find the value of p so that the three lines

$$\begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} = 2 \tag{2.28.1}$$

$$\begin{pmatrix} p & 2 \end{pmatrix} \mathbf{x} = 3 \tag{2.28.2}$$

$$\begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} = 3 \tag{2.28.3}$$

may intersect at one point. **Solution:**

2.29. Find a condition on $\mathbf{x}$ such that the points $\mathbf{x}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 7 \\ 0 \end{pmatrix}$ are collinear.

Solution: