Matrix Inversion

G V V Sharma*

CONTENTS

Abstract—**This manual explains matrix inversion by solving problems from NCERT textbooks from Class 6-12.**

1 EXAMPLES

1.1. Using elementary transformations, find the inverse of $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

Solution:
Given that

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \tag{1.1.1}$$

The augmented matrix $[\mathbf{A}|\mathbf{I}]$ is as given below:-

$$\left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right) \tag{1.1.2}$$

We apply the elementary row operations on $[\mathbf{A}|\mathbf{I}]$ as follows :-

$$[\mathbf{A}|\mathbf{I}] = \left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right) \tag{1.1.3}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - R_2} \left(\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 1 & 1 & 0 & 1 \end{array}\right) \tag{1.1.4}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - R_1} \left(\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 2 \end{array}\right) \tag{1.1.5}$$

By performing elementary transformations on augmented matrix$[\mathbf{A}|\mathbf{I}]$, we obtained the augmented matrix in the form $[\mathbf{I}|\mathbf{A}]$. Hence we can conclude that the matrix A is invertible and inverse of the matrix is

$$\mathbf{A^{-1}} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \tag{1.1.6}$$

*The author is with the Department of Electrical Engineering, Indian Institute of Technology, Hyderabad 502285 India e-mail: gadepall@iith.ac.in.

1.2. Find P^{-1}, if it exists, given $\mathrm{P}=\begin{pmatrix} 10 & -2 \\ -5 & 1 \end{pmatrix}$.

Solution:
Using row reduction,

$$\begin{pmatrix} 10 & -2 \\ -5 & 1 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 + \frac{R_1}{2}} \begin{pmatrix} 10 & -2 \\ 0 & 0 \end{pmatrix} \tag{1.2.1}$$

Since we obtain a zero row, $\mathbf{P}^{-1}$ does not exist.
Using elementary transformations,find the inverse of each of the matrices, if it exists

1.3. $\begin{pmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{pmatrix}$

Solution: Forming the augmented matrix,

$$\left(\begin{array}{ccc|ccc} 2 & -3 & 3 & 1 & 0 & 0 \\ 2 & 2 & 3 & 0 & 1 & 0 \\ 3 & -2 & 2 & 0 & 0 & 1 \end{array}\right) \tag{1.3.1}$$

$$\xleftrightarrow{C_2 \leftarrow C_2 + C_1} \left(\begin{array}{ccc|ccc} 2 & 0 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 3 & 0 & 2 & 0 & 1 & 1 \end{array}\right) \tag{1.3.2}$$

$$\xleftrightarrow{C_1 \leftarrow C_3 - C_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & -1 & 0 & 0 \\ 1 & 5 & 3 & 0 & 1 & 0 \\ -1 & 0 & 2 & 1 & 1 & 1 \end{array}\right) \tag{1.3.3}$$

$$\xleftrightarrow{C_3 \leftarrow C_3 - 3C_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 3 \\ 1 & 5 & 0 & 0 & 1 & 0 \\ -1 & 0 & 5 & 1 & 1 & -2 \end{array}\right) \tag{1.3.4}$$

$$\xleftrightarrow{C_3 \leftarrow \frac{1}{5} C_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 3/5 \\ 0 & 5 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 1 & 1 & -2/5 \end{array}\right) \tag{1.3.5}$$

$$\xleftrightarrow{C_2 \leftarrow \frac{1}{5} C_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 3/5 \\ 0 & 1 & 0 & 0 & 1/5 & 0 \\ -1 & 0 & 1 & 1 & 1/5 & -2/5 \end{array}\right) \tag{1.3.6}$$

yielding

$$\xleftrightarrow{C_1 \leftarrow C_1 - C_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 3/5 \\ 0 & 1 & 0 & -1/5 & 1/5 & 0 \\ -1 & 0 & 1 & 4/5 & 1/5 & -2/5 \end{array}\right) \tag{1.3.7}$$

$$\xleftrightarrow{C_1 \leftarrow C_1 + C_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2/5 & 0 & 3/5 \\ 0 & 1 & 0 & -1/5 & 1/5 & 0 \\ 0 & 0 & 1 & 2/5 & 1/5 & -2/5 \end{array}\right) \tag{1.3.8}$$

Thus the desired inverse is

$$\mathbf{A}^{-1} = \begin{pmatrix} -2/5 & 0 & 3/5 \\ -1/5 & 1/5 & 0 \\ 2/5 & 1/5 & -2/5 \end{pmatrix} \tag{1.3.9}$$

1.4. $\begin{pmatrix} 1 & 3 & -2 \\ -3 & 0 & -5 \\ 2 & 5 & 0 \end{pmatrix}$

Solution: Let

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & -2 \\ -3 & 0 & -5 \\ 2 & 5 & 0 \end{pmatrix} \tag{1.4.1}$$

The augmented matrix can then be represented as

$$\left(\begin{array}{ccc|ccc} 1 & 3 & -2 & 1 & 0 & 0 \\ -3 & 0 & -5 & 0 & 1 & 0 \\ 2 & 5 & 0 & 0 & 0 & 1 \end{array}\right) \tag{1.4.2}$$

Applying elementary transformations,

$$\xleftrightarrow[R_3 \leftarrow R_3 - 2R_1]{R_2 \leftarrow R_2 + 3R_1} \left(\begin{array}{ccc|ccc} 1 & 3 & -2 & 1 & 0 & 0 \\ 0 & 9 & -11 & 3 & 1 & 0 \\ 0 & -1 & 4 & -2 & 0 & 1 \end{array}\right) \tag{1.4.3}$$

$$\xleftrightarrow[R_1 \leftarrow R_1 - 3R_2]{R_2 \leftrightarrow -R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 10 & -5 & 0 & 3 \\ 0 & 1 & -4 & 2 & 0 & -1 \\ 0 & 9 & -11 & 3 & 1 & 0 \end{array}\right) \tag{1.4.4}$$

$$\xleftrightarrow[R_3 \leftarrow \frac{R_3}{25}]{R_3 \leftarrow R_3 - 4R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 10 & -5 & 0 & 3 \\ 0 & 1 & -4 & 2 & 0 & -1 \\ 0 & 0 & 1 & \frac{-3}{5} & \frac{1}{25} & \frac{9}{25} \end{array}\right) \tag{1.4.5}$$

$$\xleftrightarrow[R_2 \leftarrow R_2 + 4R_3]{R_1 \leftarrow R_1 - 10R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & \frac{-2}{5} & \frac{-3}{5} \\ 0 & 1 & 0 & \frac{-2}{5} & \frac{4}{25} & \frac{11}{25} \\ 0 & 0 & 1 & \frac{-3}{5} & \frac{1}{25} & \frac{9}{25} \end{array}\right) \tag{1.4.6}$$

Thus

$$\mathbf{A^{-1}} = \begin{pmatrix} 1 & \frac{-2}{5} & \frac{-3}{5} \\ \frac{-2}{5} & \frac{4}{25} & \frac{11}{25} \\ \frac{-3}{5} & \frac{1}{25} & \frac{9}{25} \end{pmatrix} \tag{1.4.7}$$

1.5. $\begin{pmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix}$

Solution: The augmented matrix $[\mathbf{A}|\mathbf{I}]$ is as given below:-

$$\left(\begin{array}{ccc|ccc} 2 & 0 & -1 & 1 & 0 & 0 \\ 5 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array}\right) \tag{1.5.1}$$

We apply the elementary row operations on $[\mathbf{A}|\mathbf{I}]$ as follows :-

$$[\mathbf{A}|\mathbf{I}] = \left(\begin{array}{ccc|ccc} 2 & 0 & -1 & 1 & 0 & 0 \\ 5 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array}\right) \tag{1.5.2}$$

$$\xrightarrow{R_2 \leftarrow 2R_2 - 5R_1} \left(\begin{array}{ccc|ccc} 2 & 0 & -1 & 1 & 0 & 0 \\ 0 & 2 & 5 & -5 & 2 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array}\right) \tag{1.5.3}$$

$$\xrightarrow{R_3 \leftarrow 2R_3 - R_2} \left(\begin{array}{ccc|ccc} 2 & 0 & -1 & 1 & 0 & 0 \\ 0 & 2 & 5 & -5 & 2 & 0 \\ 0 & 0 & 1 & 5 & -2 & 2 \end{array}\right) \tag{1.5.4}$$

$$\xleftrightarrow[R_2 \leftarrow \frac{R_2}{2}]{R_1 \leftarrow \frac{R_1}{2}} \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{-1}{2} & \frac{-1}{2} & 0 & 0 \\ 0 & 1 & \frac{5}{2} & \frac{-5}{2} & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & 2 \end{array}\right) \tag{1.5.5}$$

$$\xleftrightarrow[R_1 \leftarrow R_1 + \frac{R_3}{2}]{R_2 \leftarrow R_2 - \frac{5}{2}R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -1 & 1 \\ 0 & 1 & 0 & -15 & 6 & -5 \\ 0 & 0 & 1 & 5 & -2 & 2 \end{array}\right) \tag{1.5.6}$$

By performing elementary transormations on augmented matrix $[\mathbf{A}|\mathbf{I}]$, we obtained the augmented matrix in the form $[\mathbf{I}|\mathbf{B}]$. Hence we can conclude that the matrix A is invertible and inverse of the matrix is:-

$$\mathbf{A}^{-1} = \begin{pmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix} \tag{1.5.7}$$

1.6. $\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$

1.7. $\begin{pmatrix} 2 & 3 \\ 3 & -4 \end{pmatrix}$

1.8. $\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$

1.9. $\begin{pmatrix} 4 & 3 \\ 5 & -2 \end{pmatrix}$

1.10.

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} \tag{1.10.1}$$

1.11.

$$\begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix} \tag{1.11.1}$$

1.12.

$$\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \tag{1.12.1}$$

1.13.

$$\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 3 & -5 \end{pmatrix} \tag{1.13.1}$$

1.14.

$$\mathbf{A} = \begin{pmatrix} 1 & -7 \\ 3 & 1 \end{pmatrix} \tag{1.14.1}$$

1.15. $\mathbf{A} = \begin{pmatrix} 3 & -2 \\ 4 & -2 \end{pmatrix}$,

1.16. $\mathbf{A} = \begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix}$.

1.17.

$$A = \begin{pmatrix} 3 & -4 \\ -4 & 3 \end{pmatrix}$$

1.18.

$$\mathbf{A} = \begin{pmatrix} 2 & -6 \\ 1 & -2 \end{pmatrix} \tag{1.18.1}$$

1.19.

$$\mathbf{A} = \begin{pmatrix} 4 & 7 \\ 3 & 5 \end{pmatrix} \tag{1.19.1}$$

1.20.

$$\mathbf{A} = \begin{pmatrix} 4 & -3 \\ 6 & -2 \end{pmatrix} \tag{1.20.1}$$

1.21.

$$\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \tag{1.21.1}$$

1.22. ,

$$\mathbf{A} = \begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix} \tag{1.22.1}$$

1.23. $\begin{pmatrix} 6 & 1 \\ -8 & 2 \end{pmatrix}$

1.24. $\begin{pmatrix} 3 & 1 \\ -4 & 1 \end{pmatrix}$

1.25. $\begin{pmatrix} 55 & -60 \\ -60 & 20 \end{pmatrix}$

1.26.

$$\mathbf{V} = \begin{pmatrix} 6 & \frac{17}{2} \\ \frac{17}{2} & 12 \end{pmatrix} \tag{1.26.1}$$

1.27.

$$\mathbf{A} = \begin{pmatrix} 7 & 3 \\ 2 & 4 \end{pmatrix} \tag{1.27.1}$$

1.28.

$$\mathbf{V} = \begin{pmatrix} 14 & -2 \\ -2 & 11 \end{pmatrix} \tag{1.28.1}$$

1.29. $\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$

1.30.

$$\mathbf{V} = \begin{pmatrix} 12 & \frac{7}{2} \\ \frac{7}{2} & -10 \end{pmatrix} \tag{1.30.1}$$

1.31.

$$\mathbf{V} = \begin{pmatrix} 12 & -5 \\ -5 & 2 \end{pmatrix} \tag{1.31.1}$$

1.32.

$$\mathbf{V} = \begin{pmatrix} 19 & 12 \\ 12 & 1 \end{pmatrix} \tag{1.32.1}$$