Matrix Inversion

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***Abstract*—This manual explains matrix inversion by solving problems from NCERT textbooks from Class 6-12.**

1 DEFINITIONS

1.1. For a 2×2 matrix

$$\mathbf{A} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}, \tag{1.1.1}$$

the inverse is given by

$$\mathbf{A}^{-1} = \frac{1}{\left|\mathbf{A}\right|} \begin{pmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{pmatrix}, \tag{1.1.2}$$

1.2. Using elementary transformations,find the inverse of $\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$

Solution:
Given that

$$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} \tag{1.2.1}$$

The augmented matrix $[\mathbf{A}|\mathbf{I}]$ is as given below

$$\left(\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right) \tag{1.2.2}$$

We now apply the elementary row operations on $[\mathbf{A}|\mathbf{I}]$ by first reducing $\mathbf{A}$ to an upper triangular matrix as follows

$$[\mathbf{A}|\mathbf{I}] = \left(\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right) \tag{1.2.3}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - 2R_1} \left(\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & 5 & -2 & 1 \end{array}\right) \tag{1.2.4}$$

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Now we try to transform the left matrix above to the identity matrix using row operations

$$\xleftrightarrow{R_2 \leftarrow \frac{R_2}{5}} \left(\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & 1 & \frac{-2}{5} & \frac{1}{5} \end{array}\right) \tag{1.2.5}$$

$$\xleftrightarrow{R_2 \leftarrow R_1 + R_2} \left(\begin{array}{cc|cc} 1 & 0 & \frac{3}{5} & \frac{1}{5} \\ 0 & 1 & \frac{-2}{5} & \frac{1}{5} \end{array}\right) \tag{1.2.6}$$

By performing elementary transformations on augmented matrix$[\mathbf{A}|\mathbf{I}]$, we obtained the augmented matrix in the form $[\mathbf{I}|\mathbf{A}]$. Hence we can conclude that the matrix A is invertible and the right matrix above is the desired inverse.

$$\mathbf{A^{-1}} = \begin{pmatrix} \frac{3}{5} & \frac{1}{5} \\ \frac{-2}{5} & \frac{1}{5} \end{pmatrix} \tag{1.2.7}$$

1.3. Obtain the inverse of the following matrix using elementary operations

A=$\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix}$.

Solution:
Given that

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix}, \tag{1.3.1}$$

The augmented matrix $[\mathbf{A}|\mathbf{I}]$ is

$$\left(\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \tag{1.3.2}$$

Since there is 0 in the first row above, the first row can be interchanged with the second so that the first entry of the first row becomes nonzero.

$$[\mathbf{A}|\mathbf{I}] = \left(\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \tag{1.3.3}$$

$$\xleftrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \tag{1.3.4}$$

Continuing with row operations to obtain a lower triangular matrix,

$$\xleftrightarrow{R_3\leftarrow R_3-3R_1}\left(\begin{array}{ccc|ccc}1&2&3&0&1&0\\0&1&2&1&0&0\\0&-5&-8&0&-3&1\end{array}\right) \tag{1.3.5}$$

$$\xleftrightarrow{R_1\leftarrow R_1-2R_2}\left(\begin{array}{ccc|ccc}1&0&-1&-2&1&0\\0&1&2&1&0&0\\0&-5&-8&0&-3&1\end{array}\right) \tag{1.3.6}$$

$$\xleftrightarrow{R_3\leftarrow R_3+5R_2}\left(\begin{array}{ccc|ccc}1&0&-1&-2&1&0\\0&1&2&1&0&0\\0&0&2&5&-3&1\end{array}\right) \tag{1.3.7}$$

Further row operations are performed to obtain the identity matrix.

$$\xleftrightarrow{R_3\leftarrow R_3/2}\left(\begin{array}{ccc|ccc}1&0&-1&-2&1&0\\0&1&2&1&0&0\\0&0&1&\frac{5}{2}&\frac{-3}{2}&\frac{1}{2}\end{array}\right) \tag{1.3.8}$$

$$\xleftrightarrow{R_1\leftarrow R_1+R_3}\left(\begin{array}{ccc|ccc}1&0&0&\frac{1}{2}&\frac{-1}{2}&\frac{1}{2}\\0&1&2&1&0&0\\0&0&1&\frac{5}{2}&\frac{-3}{2}&\frac{1}{2}\end{array}\right) \tag{1.3.9}$$

$$\xleftrightarrow{R_2\leftarrow R_2-2R_3}\left(\begin{array}{ccc|ccc}1&0&0&\frac{1}{2}&\frac{-1}{2}&\frac{1}{2}\\0&1&0&-4&3&-1\\0&0&1&\frac{5}{2}&\frac{-3}{2}&\frac{1}{2}\end{array}\right) \tag{1.3.10}$$

By performing elementary transformations on augmented matrix$[\mathbf{A}|\mathbf{I}]$, we obtained the augmented matrix in the form $[\mathbf{I}|\mathbf{A}]$. Hence we can conclude that the matrix A is invertible and inverse of the matrix is

$$\mathbf{A^{-1}} = \begin{pmatrix}\frac{1}{2}&\frac{-1}{2}&\frac{1}{2}\\-4&3&-1\\\frac{5}{2}&\frac{-3}{2}&\frac{1}{2}\end{pmatrix} \tag{1.3.11}$$

1.4. For higher order matrices, the inverse should be calculated using row operations.

2 Examples

2.1. Using elementary transformations, find the inverse of $\begin{pmatrix}2&1\\1&1\end{pmatrix}$
Solution:
Given that

$$\mathbf{A} = \begin{pmatrix}2&1\\1&1\end{pmatrix} \tag{2.1.1}$$

The augmented matrix $[\mathbf{A}|\mathbf{I}]$ is as given below:-

$$\left(\begin{array}{cc|cc}2&1&1&0\\1&1&0&1\end{array}\right) \tag{2.1.2}$$

We apply the elementary row operations on $[\mathbf{A}|\mathbf{I}]$ as follows :-

$$[\mathbf{A}|\mathbf{I}] = \left(\begin{array}{cc|cc}2&1&1&0\\1&1&0&1\end{array}\right) \tag{2.1.3}$$

$$\xleftrightarrow{R_1\leftarrow R_1-R_2}\left(\begin{array}{cc|cc}1&0&1&-1\\1&1&0&1\end{array}\right) \tag{2.1.4}$$

$$\xleftrightarrow{R_2\leftarrow R_2-R_1}\left(\begin{array}{cc|cc}1&0&1&-1\\0&1&-1&2\end{array}\right) \tag{2.1.5}$$

By performing elementary transformations on augmented matrix$[\mathbf{A}|\mathbf{I}]$, we obtained the augmented matrix in the form $[\mathbf{I}|\mathbf{A}]$. Hence we can conclude that the matrix A is invertible and inverse of the matrix is

$$\mathbf{A^{-1}} = \begin{pmatrix}1&-1\\-1&2\end{pmatrix} \tag{2.1.6}$$

2.2. Find P^{-1}, if it exists, given
P=$\begin{pmatrix}10&-2\\-5&1\end{pmatrix}$.
Solution:
Using row reduction,

$$\begin{pmatrix}10&-2\\-5&1\end{pmatrix}\xleftrightarrow{R_2\leftarrow R_2+\frac{R_1}{2}}\begin{pmatrix}10&-2\\0&0\end{pmatrix} \tag{2.2.1}$$

Since we obtain a zero row, $\mathbf{P}^{-1}$ does not exist.
Using elementary transformations,find the inverse of each of the matrices, if it exists

2.3. $\begin{pmatrix}2 & -3 & 3\\ 2 & 2 & 3\\ 3 & -2 & 2\end{pmatrix}$

Solution: Forming the augmented matrix,

$$\left(\begin{array}{ccc|ccc}2 & -3 & 3 & 1 & 0 & 0\\ 2 & 2 & 3 & 0 & 1 & 0\\ 3 & -2 & 2 & 0 & 0 & 1\end{array}\right) \tag{2.3.1}$$

$$\xleftrightarrow{C_2 \leftarrow C_2 + C_1} \left(\begin{array}{ccc|ccc}2 & 0 & 3 & 1 & 0 & 0\\ 2 & 5 & 3 & 0 & 1 & 0\\ 3 & 0 & 2 & 0 & 1 & 1\end{array}\right) \tag{2.3.2}$$

$$\xleftrightarrow{C_1 \leftarrow C_3 - C_1} \left(\begin{array}{ccc|ccc}1 & 0 & 3 & -1 & 0 & 0\\ 1 & 5 & 3 & 0 & 1 & 0\\ -1 & 0 & 2 & 1 & 1 & 1\end{array}\right) \tag{2.3.3}$$

$$\xleftrightarrow{C_3 \leftarrow C_3 - 3C_1} \left(\begin{array}{ccc|ccc}1 & 0 & 0 & -1 & 0 & 3\\ 1 & 5 & 0 & 0 & 1 & 0\\ -1 & 0 & 5 & 1 & 1 & -2\end{array}\right) \tag{2.3.4}$$

$$\xleftrightarrow{C_3 \leftarrow \frac{1}{5}C_3} \left(\begin{array}{ccc|ccc}1 & 0 & 0 & -1 & 0 & 3/5\\ 0 & 5 & 0 & 0 & 1 & 0\\ -1 & 0 & 1 & 1 & 1 & -2/5\end{array}\right) \tag{2.3.5}$$

$$\xleftrightarrow{C_2 \leftarrow \frac{1}{5}C_2} \left(\begin{array}{ccc|ccc}1 & 0 & 0 & -1 & 0 & 3/5\\ 0 & 1 & 0 & 0 & 1/5 & 0\\ -1 & 0 & 1 & 1 & 1/5 & -2/5\end{array}\right) \tag{2.3.6}$$

yielding

$$\xleftrightarrow{C_1 \leftarrow C_1 - C_2} \left(\begin{array}{ccc|ccc}1 & 0 & 0 & -1 & 0 & 3/5\\ 0 & 1 & 0 & -1/5 & 1/5 & 0\\ -1 & 0 & 1 & 4/5 & 1/5 & -2/5\end{array}\right) \tag{2.3.7}$$

$$\xleftrightarrow{C_1 \leftarrow C_1 + C_3} \left(\begin{array}{ccc|ccc}1 & 0 & 0 & -2/5 & 0 & 3/5\\ 0 & 1 & 0 & -1/5 & 1/5 & 0\\ 0 & 0 & 1 & 2/5 & 1/5 & -2/5\end{array}\right) \tag{2.3.8}$$

Thus the desired inverse is

$$\mathbf{A}^{-1} = \begin{pmatrix}-2/5 & 0 & 3/5\\ -1/5 & 1/5 & 0\\ 2/5 & 1/5 & -2/5\end{pmatrix} \tag{2.3.9}$$

2.4. $\begin{pmatrix}1 & 3 & -2\\ -3 & 0 & -5\\ 2 & 5 & 0\end{pmatrix}$

Solution: Let

$$\mathbf{A} = \begin{pmatrix}1 & 3 & -2\\ -3 & 0 & -5\\ 2 & 5 & 0\end{pmatrix} \tag{2.4.1}$$

The augmented matrix can then be represented as

$$\left(\begin{array}{ccc|ccc}1 & 3 & -2 & 1 & 0 & 0\\ -3 & 0 & -5 & 0 & 1 & 0\\ 2 & 5 & 0 & 0 & 0 & 1\end{array}\right) \tag{2.4.2}$$

Applying elementary transformations,

$$\xleftrightarrow[R_3 \leftarrow R_3 - 2R_1]{R_2 \leftarrow R_2 + 3R_1} \left(\begin{array}{ccc|ccc}1 & 3 & -2 & 1 & 0 & 0\\ 0 & 9 & -11 & 3 & 1 & 0\\ 0 & -1 & 4 & -2 & 0 & 1\end{array}\right) \tag{2.4.3}$$

$$\xleftrightarrow[R_1 \leftarrow R_1 - 3R_2]{R_2 \leftrightarrow -R_3} \left(\begin{array}{ccc|ccc}1 & 0 & 10 & -5 & 0 & 3\\ 0 & 1 & -4 & 2 & 0 & -1\\ 0 & 9 & -11 & 3 & 1 & 0\end{array}\right) \tag{2.4.4}$$

$$\xleftrightarrow[R_3 \leftarrow \frac{R_3}{25}]{R_3 \leftarrow R_3 - 4R_2} \left(\begin{array}{ccc|ccc}1 & 0 & 10 & -5 & 0 & 3\\ 0 & 1 & -4 & 2 & 0 & -1\\ 0 & 0 & 1 & \frac{-3}{5} & \frac{1}{25} & \frac{9}{25}\end{array}\right) \tag{2.4.5}$$

$$\xleftrightarrow[R_2 \leftarrow R_2 + 4R_3]{R_1 \leftarrow R_1 - 10R_3} \left(\begin{array}{ccc|ccc}1 & 0 & 0 & 1 & \frac{-2}{5} & \frac{-3}{5}\\ 0 & 1 & 0 & \frac{-2}{5} & \frac{4}{25} & \frac{11}{25}\\ 0 & 0 & 1 & \frac{-3}{5} & \frac{1}{25} & \frac{9}{25}\end{array}\right) \tag{2.4.6}$$

Thus

$$\mathbf{A^{-1}} = \begin{pmatrix}1 & \frac{-2}{5} & \frac{-3}{5}\\ \frac{-2}{5} & \frac{4}{25} & \frac{11}{25}\\ \frac{-3}{5} & \frac{1}{25} & \frac{9}{25}\end{pmatrix} \tag{2.4.7}$$

2.5. $\begin{pmatrix}2 & 0 & -1\\ 5 & 1 & 0\\ 0 & 1 & 3\end{pmatrix}$

Solution: The augmented matrix $[\mathbf{A}|\mathbf{I}]$ is as given below:-

$$\left(\begin{array}{ccc|ccc}2 & 0 & -1 & 1 & 0 & 0\\ 5 & 1 & 0 & 0 & 1 & 0\\ 0 & 1 & 3 & 0 & 0 & 1\end{array}\right) \tag{2.5.1}$$

We apply the elementary row operations on $[\mathbf{A}|\mathbf{I}]$ as follows :-

$$[\mathbf{A}|\mathbf{I}] = \left(\begin{array}{ccc|ccc} 2 & 0 & -1 & 1 & 0 & 0 \\ 5 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array}\right) \tag{2.5.2}$$

$$\xrightarrow{R_2 \leftarrow 2R_2 - 5R_1} \left(\begin{array}{ccc|ccc} 2 & 0 & -1 & 1 & 0 & 0 \\ 0 & 2 & 5 & -5 & 2 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array}\right) \tag{2.5.3}$$

$$\xrightarrow{R_3 \leftarrow 2R_3 - R_2} \left(\begin{array}{ccc|ccc} 2 & 0 & -1 & 1 & 0 & 0 \\ 0 & 2 & 5 & -5 & 2 & 0 \\ 0 & 0 & 1 & 5 & -2 & 2 \end{array}\right) \tag{2.5.4}$$

$$\xleftrightarrow[R_2 \leftarrow \frac{R_2}{2}]{R_1 \leftarrow \frac{R_1}{2}} \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{-1}{2} & \frac{-1}{2} & 0 & 0 \\ 0 & 1 & \frac{5}{2} & \frac{-5}{2} & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & 2 \end{array}\right) \tag{2.5.5}$$

$$\xleftrightarrow[R_1 \leftarrow R_1 + \frac{R_3}{2}]{R_2 \leftarrow R_2 - \frac{5}{2}R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -1 & 1 \\ 0 & 1 & 0 & -15 & 6 & -5 \\ 0 & 0 & 1 & 5 & -2 & 2 \end{array}\right) \tag{2.5.6}$$

By performing elementary transformations on augmented matrix $[\mathbf{A}|\mathbf{I}]$, we obtained the augmented matrix in the form $[\mathbf{I}|\mathbf{B}]$. Hence we can conclude that the matrix A is invertible and inverse of the matrix is:-

$$\mathbf{A}^{-1} = \begin{pmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix} \tag{2.5.7}$$

2.6. $\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$

2.7. $\begin{pmatrix} 2 & 3 \\ 3 & -4 \end{pmatrix}$

2.8. $\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$

2.9. $\begin{pmatrix} 4 & 3 \\ 5 & -2 \end{pmatrix}$

2.10.

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} \tag{2.10.1}$$

2.11.

$$\begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix} \tag{2.11.1}$$

2.12.

$$\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \tag{2.12.1}$$

2.13.

$$\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 3 & -5 \end{pmatrix} \tag{2.13.1}$$

2.14.

$$\mathbf{A} = \begin{pmatrix} 1 & -7 \\ 3 & 1 \end{pmatrix} \tag{2.14.1}$$

2.15. $\mathbf{A} = \begin{pmatrix} 3 & -2 \\ 4 & -2 \end{pmatrix}$,

2.16. $\mathbf{A} = \begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix}$.

2.17.

$$A = \begin{pmatrix} 3 & -4 \\ -4 & 3 \end{pmatrix}$$

2.18.

$$\mathbf{A} = \begin{pmatrix} 2 & -6 \\ 1 & -2 \end{pmatrix} \tag{2.18.1}$$

2.19.

$$\mathbf{A} = \begin{pmatrix} 4 & 7 \\ 3 & 5 \end{pmatrix} \tag{2.19.1}$$

2.20.

$$\mathbf{A} = \begin{pmatrix} 4 & -3 \\ 6 & -2 \end{pmatrix} \tag{2.20.1}$$

2.21.

$$\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \tag{2.21.1}$$

2.22. ,

$$\mathbf{A} = \begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix} \tag{2.22.1}$$

2.23. $\begin{pmatrix} 6 & 1 \\ -8 & 2 \end{pmatrix}$

2.24. $\begin{pmatrix} 3 & 1 \\ -4 & 1 \end{pmatrix}$

2.25. $\begin{pmatrix} 55 & -60 \\ -60 & 20 \end{pmatrix}$

2.26.

$$\mathbf{V} = \begin{pmatrix} 6 & \frac{17}{2} \\ \frac{17}{2} & 12 \end{pmatrix} \tag{2.26.1}$$

2.27.

$$\mathbf{A} = \begin{pmatrix} 7 & 3 \\ 2 & 4 \end{pmatrix} \tag{2.27.1}$$

2.28.

$$\mathbf{V} = \begin{pmatrix} 14 & -2 \\ -2 & 11 \end{pmatrix} \tag{2.28.1}$$

2.29. $\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$

2.30.

$$\mathbf{V} = \begin{pmatrix} 12 & \frac{7}{2} \\ \frac{7}{2} & -10 \end{pmatrix} \tag{2.30.1}$$

2.31.

$$\mathbf{V} = \begin{pmatrix} 12 & -5 \\ -5 & 2 \end{pmatrix} \tag{2.31.1}$$

2.32.

$$\mathbf{V} = \begin{pmatrix} 19 & 12 \\ 12 & 1 \end{pmatrix} \tag{2.32.1}$$