Matrices

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Contents

***Abstract*—This manual provides a simple introduction to matrices and their properties based on the NCERT textbooks from Class 6-12.**

1 Definitions

1.1 Eigenvalues and Eigenvectors

1.1.1. The eigenvalue λ and the eigenvector $\mathbf{x}$ for a matrix $\mathbf{A}$ are defined as,

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \tag{1.1.1.1}$$

1.1.2. The eigenvalues are calculated by solving the equation

$$f(\lambda) = \left|\lambda\mathbf{I} - \mathbf{A}\right| = 0 \tag{1.1.2.1}$$

The above equation is known as the characteristic equation.

1.1.3. According to the Cayley-Hamilton theorem,

$$f(\lambda) = 0 \implies f(\mathbf{A}) = 0 \tag{1.1.3.1}$$

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1.1.4. The trace of a square matrix is defined to be the sum of the diagonal elements.

$$\operatorname{tr}(\mathbf{A}) = \sum_{i=1}^{N} a_{ii}. \tag{1.1.4.1}$$

where a_{ii} is the ith diagonal element of the matrix $\mathbf{A}$.

1.1.5. The trace of a matrix is equal to the sum of the eigenvalues

$$\operatorname{tr}(\mathbf{A}) = \sum_{i=1}^{N} \lambda_i \tag{1.1.5.1}$$

1.2 Determinants

1.2.1. Let

$$\mathbf{A} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}. \tag{1.2.1.1}$$

be a 3×3 matrix. Then,

$$\left|\mathbf{A}\right| = a_1 \begin{pmatrix} b_2 & c_2 \\ b_3 & c_3 \end{pmatrix} - a_2 \begin{pmatrix} b_1 & c_1 \\ b_3 & c_3 \end{pmatrix} + a_3 \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}. \tag{1.2.1.2}$$

1.2.2. Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of a matrix $\mathbf{A}$. Then, the product of the eigenvalues is equal to the determinant of $\mathbf{A}$.

$$\left|\mathbf{A}\right| = \prod_{i=1}^{n} \lambda_i \tag{1.2.2.1}$$

1.2.3.

$$\left|\mathbf{A}\mathbf{B}\right| = \left|\mathbf{A}\right|\left|\mathbf{B}\right| \tag{1.2.3.1}$$

1.2.4. If $\mathbf{A}$ be an $n \times n$ matrix,

$$\left|k\mathbf{A}\right| = k^n \left|\mathbf{A}\right| \tag{1.2.4.1}$$

2 Rank of a Matrix

2.1. The rank of a matrix is defined as the number of nonzero rows obtained after row reduction.

2.2. An $n \times n$ matrix is invertible if and only if its rank is n.

3 Inverse of a Matrix

3.1. For a 2×2 matrix

$$\mathbf{A} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}, \tag{3.1.1}$$

the inverse is given by

$$\mathbf{A}^{-1} = \frac{1}{\left|\mathbf{A}\right|} \begin{pmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{pmatrix}, \tag{3.1.2}$$

3.2. For higher order matrices, the inverse should be calculated using row operations.

4 Cayley-Hamilton Theorem

4.1. If

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{pmatrix}, \tag{4.1.1}$$

prove that

$$\mathbf{A}^3 - 6\mathbf{A}^2 + 7\mathbf{A} + 2\mathbf{I} = 0 \tag{4.1.2}$$

Solution: From (1.1.2.1), the characteristic equation is

$$\begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & 2-\lambda & 1 \\ 2 & 0 & 3-\lambda \end{vmatrix} = 0 \tag{4.1.3}$$

which can be expanded to obtain

$$(1-\lambda)(2-\lambda)(3-\lambda) + 2(-2(2-\lambda)) = 0 \tag{4.1.4}$$

yielding

$$\lambda^3 - 6\lambda^2 + 7\lambda + 2 = 0 \tag{4.1.5}$$

upon simplification. Using the Cayley-Hamilton theorem in (1.1.3.1), (4.1.2) is obtained

4.2. If

$$\mathbf{A} = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}, \tag{4.2.1}$$

show that

$$\mathbf{A}^2 - 5\mathbf{A} + 7\mathbf{I} = 0 \tag{4.2.2}$$

Solution: The characteristic equation is

$$\left|\mathbf{A} - \lambda\mathbf{I}\right| = 0 \tag{4.2.3}$$

$$\implies \begin{vmatrix} 3-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} = 0 \tag{4.2.4}$$

$$\implies (3-\lambda)(2-\lambda) + 1 = 0 \tag{4.2.5}$$

$$\text{or, } \lambda^2 - 5\lambda + 7 = 0 \tag{4.2.6}$$

Using the Cayley-Hamilton theorem, (4.2.2) is obtained.

4.3. If $\mathbf{A}$ is square matrix such that

$$\mathbf{A}^2 = \mathbf{A}, \tag{4.3.1}$$

then

$$g\left(\mathbf{A}\right) = \left(\mathbf{I} + \mathbf{A}\right)^3 - 7\mathbf{A} \tag{4.3.2}$$

is equal to

a) $\mathbf{A}$
b) $\mathbf{I} - \mathbf{A}$
c) $\mathbf{I}$
d) $3\mathbf{A}$

Solution: From (4.3.2),

$$g\left(\lambda\right) = \left(1 + \lambda\right)^3 - 7\lambda \tag{4.3.3}$$

Also,

$$\mathbf{A}^2 = \mathbf{A} \implies \mathbf{A}^2 - \mathbf{A} = 0 \tag{4.3.4}$$

Using the Cayley-Hamilton theorem, the eigenvalues satisfy the characteristic equation

$$f\left(\lambda\right) = \lambda^2 - \lambda = 0 \tag{4.3.5}$$

$\because f\left(\lambda\right)$ is of degree 2 and $g\left(\lambda\right)$ is of degree 3, (4.3.3) can be expressed as

$$g\left(\lambda\right) = f\left(\lambda\right) q\left(\lambda\right) + a\lambda + b \tag{4.3.6}$$

where a, b are real numbers and $q\left(\lambda\right)$ is some polynomial. Thus,

$$g\left(0\right) = b = 1 \tag{4.3.7}$$

$$g\left(1\right) = a + b = 1 \tag{4.3.8}$$

$$\implies a = 90, b = 1 \tag{4.3.9}$$

Thus,

$$g\left(\mathbf{A}\right) = f\left(\mathbf{A}\right) q\left(\mathbf{A}\right) + a\mathbf{A} + b\mathbf{I} \tag{4.3.10}$$

$$= \mathbf{I} \tag{4.3.11}$$

upon substituting from (4.3.5) and (4.3.9). Option c is the valid answer.

4.4. If

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{pmatrix} \tag{4.4.1}$$

then show that

$$\mathbf{A}^3 - 23\mathbf{A} - 40\mathbf{I} = 0 \tag{4.4.2}$$

Solution: The Characteristic equation is given by

$$|\mathbf{A}-\lambda\mathbf{I}| = 0 \quad (4.4.3)$$

$$\implies \begin{vmatrix} 1-\lambda & 2 & 3 \\ 3 & -2-\lambda & 1 \\ 4 & 2 & 1-\lambda \end{vmatrix} = 0 \quad (4.4.4)$$

which can be expressed as

$$\implies (1-\lambda)\left((-2-\lambda)(1-\lambda)-2\right) - 2\left(3(1-\lambda)-4\right)+3\left(6+4(2+\lambda)\right) = 0 \quad (4.4.5)$$

and simplified to obtain

$$\implies \lambda^3 - 23\lambda - 40 = 0. \quad (4.4.6)$$

Using the Cayley-Hamilton Theorem, (4.4.2) is obtained.

5 Matrix Polynomials

5.1. Find $\mathbf{A}^2 - 5\mathbf{A} + 6\mathbf{I}$,if $\mathbf{A} = \begin{pmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{pmatrix}$

Solution: Factorizing the corresponding polynomial,

$$f(x) = x^2 - 5x + 6 \quad (5.1.1)$$

$$= (x-3)(x-2) \quad (5.1.2)$$

$$\implies \mathbf{A}^2 - 5\mathbf{A} + 6\mathbf{I} = (\mathbf{A}-3\mathbf{I})(\mathbf{A}-5\mathbf{I}) \quad (5.1.3)$$

$$\mathbf{A}-3\mathbf{I} = \begin{pmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{pmatrix} + \begin{pmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix} \quad (5.1.4)$$

$$= \begin{pmatrix} -1 & 0 & 1 \\ 2 & -2 & 3 \\ 1 & -1 & -3 \end{pmatrix} \quad (5.1.5)$$

$$\mathbf{A}-5\mathbf{I} = \begin{pmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{pmatrix} + \begin{pmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{pmatrix} \quad (5.1.6)$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 2 & -1 & 3 \\ 1 & -1 & -2 \end{pmatrix} \quad (5.1.7)$$

Multiplying the above expressions,

$$\mathbf{A}^2 - 5\mathbf{A} + 6\mathbf{I} = \begin{pmatrix} -1 & 0 & 1 \\ 2 & -2 & 3 \\ 1 & -1 & -3 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 2 & -1 & 3 \\ 1 & -1 & -2 \end{pmatrix} = \begin{pmatrix} 1 & -1 & -3 \\ -1 & -1 & -10 \\ -5 & 4 & 4 \end{pmatrix} \quad (5.1.8)$$

6 Linear Equations and Rank

6.1. Two rails are represented by the equations

$$\begin{pmatrix} 1 & 2 \end{pmatrix}\mathbf{x} = 4 \text{ and} \quad \begin{pmatrix} 2 & 4 \end{pmatrix}\mathbf{x} = 12. \quad (6.1.1)$$

Will the rails cross each other?
Solution: The above equations can be expressed as the matrix equation

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 4 \\ 12 \end{pmatrix} \quad (6.1.2)$$

The augmented matrix for the above equation is row reduced as follows

$$\begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 12 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow \frac{R_2}{2}} \begin{pmatrix} 1 & 2 & 4 \\ 1 & 2 & 6 \end{pmatrix} \quad (6.1.3)$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 2 & 4 \\ 0 & 0 & 2 \end{pmatrix} \quad (6.1.4)$$

$\because$ row reduction of the 2×3 matrix

$$\begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 12 \end{pmatrix} \quad (6.1.5)$$

results in a matrix with 2 nonzero rows, its rank is 2. Similarly, the rank of the matrix

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \quad (6.1.6)$$

is 1, from 6.1.4.

$$\because rank \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \neq rank \begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 12 \end{pmatrix}, \quad (6.1.7)$$

(6.1.1) has no solution.

6.2. Check whether the pair of equations

$$\begin{pmatrix} 1 & 3 \end{pmatrix}\mathbf{x} = 6 \text{ and} \quad \begin{pmatrix} 2 & -3 \end{pmatrix}\mathbf{x} = 12 \quad (6.2.1)$$

is consistent.
Solution: The above equations can be expressed as the matrix equation

$$\begin{pmatrix}1 & 3\\ 2 & -3\end{pmatrix}\mathbf{x} = \begin{pmatrix}6\\ 12\end{pmatrix} \tag{6.2.2}$$

The augmented matrix for the above equation is row reduced as follows

$$\begin{pmatrix}1 & 3 & 6\\ 2 & -3 & 12\end{pmatrix} \xleftrightarrow{R_2 \leftarrow \frac{R_2-2R_1}{-9}} \tag{6.2.3}$$

$$\begin{pmatrix}1 & 3 & 6\\ 0 & 1 & 0\end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1-3R_2} \begin{pmatrix}1 & 0 & 6\\ 0 & 1 & 0\end{pmatrix} \tag{6.2.4}$$

$$\implies \mathbf{x} = \begin{pmatrix}6\\ 0\end{pmatrix} \tag{6.2.5}$$

which is the solution of 6.1.1.

6.3. Find whether the following pair of equations has no solution, unique solution or infinitely many solutions:

$$\begin{gathered}\begin{pmatrix}5 & -8\end{pmatrix}\mathbf{x} = -1 \text{ and}\\ \begin{pmatrix}3 & -\frac{24}{5}\end{pmatrix}\mathbf{x} = -\frac{3}{5}\end{gathered} \tag{6.3.1}$$

Solution: The above equations can be expressed as the matrix equation

$$\begin{pmatrix}5 & -8\\ 3 & -\frac{24}{5}\end{pmatrix}\mathbf{x} = -\begin{pmatrix}1\\ \frac{3}{5}\end{pmatrix} \tag{6.3.2}$$

The augmented matrix for the above equation is row reduced as follows

$$\begin{pmatrix}5 & -8 & -1\\ 3 & -\frac{24}{5} & -\frac{3}{5}\end{pmatrix} \xleftrightarrow{R_2 \leftarrow 5R_2} \begin{pmatrix}5 & -8 & 1\\ 15 & -24 & -3\end{pmatrix} \tag{6.3.3}$$

$$\xleftrightarrow{R_2 \leftarrow R_2-3R_1} \begin{pmatrix}5 & -8 & 1\\ 0 & 0 & 0\end{pmatrix} \tag{6.3.4}$$

$$\because rank\begin{pmatrix}5 & -8\\ 3 & -\frac{24}{5}\end{pmatrix} = rank\begin{pmatrix}5 & -8 & 1\\ 3 & -\frac{24}{5} & -\frac{3}{5}\end{pmatrix} \tag{6.3.5}$$

$$= 1 < dim\begin{pmatrix}5 & -8\\ 3 & -\frac{24}{5}\end{pmatrix} = 2, \tag{6.3.6}$$

(6.3.1) has infinitely many solutions.

7 Matrix Inverse

7.1. Using elementary transforamtions,find the inverse of $\begin{pmatrix}1 & -1\\ 2 & 3\end{pmatrix}$
Solution: Given that

$$\mathbf{A} = \begin{pmatrix}1 & -1\\ 2 & 3\end{pmatrix} \tag{7.1.1}$$

The augmented matrix $[\mathbf{A}|\mathbf{I}]$ is as given below:-

$$\left(\begin{array}{cc|cc}1 & -1 & 1 & 0\\ 2 & 3 & 0 & 1\end{array}\right) \tag{7.1.2}$$

We apply the elementary row operations on $[\mathbf{A}|\mathbf{I}]$ as follows :-

$$[\mathbf{A}|\mathbf{I}] = \left(\begin{array}{cc|cc}1 & -1 & 1 & 0\\ 2 & 3 & 0 & 1\end{array}\right) \tag{7.1.3}$$

$$\xleftrightarrow{R_2 \leftarrow R_2-2R_1} \left(\begin{array}{cc|cc}1 & -1 & 1 & 0\\ 0 & 5 & -2 & 1\end{array}\right) \tag{7.1.4}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{R_2}{5}} \left(\begin{array}{cc|cc}1 & -1 & 1 & 0\\ 0 & 1 & \frac{-2}{5} & \frac{1}{5}\end{array}\right) \tag{7.1.5}$$

$$\xleftrightarrow{R_2 \leftarrow R_1+R_2} \left(\begin{array}{cc|cc}1 & 0 & \frac{3}{5} & \frac{1}{5}\\ 0 & 1 & \frac{-2}{5} & \frac{1}{5}\end{array}\right) \tag{7.1.6}$$

By performing elementary transformations on augmented matrix$[\mathbf{A}|\mathbf{I}]$, we obtained the augmented matrix in the form $[\mathbf{I}|\mathbf{A}]$. Hence we can conclude that the matrix A is invertible and

$$\mathbf{A^{-1}} = \begin{pmatrix}\frac{3}{5} & \frac{1}{5}\\ \frac{-2}{5} & \frac{1}{5}\end{pmatrix} \tag{7.1.7}$$

7.2. Using elementary transforamtions, find the inverse of $\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}$
Solution: Given that

$$\mathbf{A} = \begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix} \tag{7.2.1}$$

The augmented matrix $[\mathbf{A}|\mathbf{I}]$ is as given below:-

$$\left(\begin{array}{cc|cc}2 & 1 & 1 & 0\\ 1 & 1 & 0 & 1\end{array}\right) \tag{7.2.2}$$

We apply the elementary row operations on $[\mathbf{A}|\mathbf{I}]$ as follows :-

$$[\mathbf{A}|\mathbf{I}] = \left(\begin{array}{cc|cc}2 & 1 & 1 & 0\\ 1 & 1 & 0 & 1\end{array}\right) \tag{7.2.3}$$

$$\xleftrightarrow{R_1 \leftarrow R_1-R_2} \left(\begin{array}{cc|cc}1 & 0 & 1 & -1\\ 1 & 1 & 0 & 1\end{array}\right) \tag{7.2.4}$$

$$\xleftrightarrow{R_2 \leftarrow R_2-R_1} \left(\begin{array}{cc|cc}1 & 0 & 1 & -1\\ 0 & 1 & -1 & 2\end{array}\right) \tag{7.2.5}$$

By performing elementary transformations on augmented matrix$[\mathbf{A}|\mathbf{I}]$, we obtained the augmented matrix in the form $[\mathbf{I}|\mathbf{A}]$. Hence we can conclude that the matrix A is invertible and inverse of the matrix is

$$\mathbf{A^{-1}} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \tag{7.2.6}$$

7.3. Obtain the inverse of the following matrix using elementary operations
A=$\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix}$.
Solution: Given that

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix}, \tag{7.3.1}$$

The augmented matrix $[\mathbf{A}|\mathbf{I}]$ is

$$\left(\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \tag{7.3.2}$$

Applying elementary row operations on $[\mathbf{A}|\mathbf{I}]$,

$$[\mathbf{A}|\mathbf{I}] = \left(\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \tag{7.3.3}$$

$$\xleftrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \tag{7.3.4}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - 3R_1} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -5 & -8 & 0 & -3 & 1 \end{array}\right) \tag{7.3.5}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - 2R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & -2 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -5 & -8 & 0 & -3 & 1 \end{array}\right) \tag{7.3.6}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 + 5R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & -2 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 5 & -3 & 1 \end{array}\right) \tag{7.3.7}$$

$$\xleftrightarrow{R_3 \leftarrow R_3/2} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & -2 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{array}\right) \tag{7.3.8}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{array}\right) \tag{7.3.9}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - 2R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & -4 & 3 & -1 \\ 0 & 0 & 1 & \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{array}\right) \tag{7.3.10}$$

By performing elementary transformations on augmented matrix$[\mathbf{A}|\mathbf{I}]$, we obtained the augmented matrix in the form $[\mathbf{I}|\mathbf{A}]$. Hence we can conclude that the matrix A is invertible and inverse of the matrix is

$$\mathbf{A^{-1}} = \begin{pmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{pmatrix} \tag{7.3.11}$$

7.4. Find P^{-1}, if it exists, given
P=$\begin{pmatrix} 10 & -2 \\ -5 & 1 \end{pmatrix}$.

Solution: Using row reduction,

$$\begin{pmatrix} 10 & -2 \\ -5 & 1 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 + \frac{R_1}{2}} \begin{pmatrix} 10 & -2 \\ 0 & 0 \end{pmatrix} \quad (7.4.1)$$

Since we obtain a zero row, $\mathbf{P}^{-1}$ does not exist.