Determinants

G V V Sharma*

CONTENTS

***Abstract*—This manual provides a simple introduction to determinants, based on exercises from the NCERT textbooks from Class 6-12.**

1 Minor and Cofactor

1.1. **Write Minors and Coafactors of the elements of following determinants:**(i) $\begin{vmatrix} 2 & -4 \\ 0 & 3 \end{vmatrix}$

(ii) $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$

1.2. (i) $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

(ii) $\begin{vmatrix} 1 & 0 & 4 \\ 3 & 5 & -1 \\ 0 & 1 & 2 \end{vmatrix}$

1.3. Using Cofactors of elements of second row,evaluate $\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$.

1.4. Using Cofactors of elements of third column ,evaluate $\Delta = \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}$.

*The author is with the Department of Electrical Engineering, Indian Institute of Technology, Hyderabad 502285 India e-mail: gadepall@iith.ac.in.

1.5. If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ and A_{ij} is Cofactors of a_{ij} then value of Δ is given by

a) $a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33}$
b) $a_{11}A_{11} + a_{12}A_{21} + a_{13}A_{31}$
c) $a_{21}A_{11} + a_{22}A_{12} + a_{23}A_{13}$
d) $a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$

2 Adjoint

2.1. **Find adjoint of each of the matrices** $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

2.2. $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$ Verify A(adjA)=(adjA)A=$|A|\, I$

2.3. $\begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}$

2.4. $\begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}$

2.5. $\begin{bmatrix} 2 & -2 \\ 4 & 3 \end{bmatrix}$

2.6. $\begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix}$

2.7. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$

2.8. $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{bmatrix}$

2.9. $\begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{bmatrix}$

2.10. $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$

2.11. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{bmatrix}$

3 Properties

3.1. If$\mathbf{A} = \begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix}$,then show that $|2\mathbf{A}| = 4\,|\mathbf{A}|$

Solution:

3.2. If $\mathbf{A} = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{vmatrix}$, then show that $|3\mathbf{A}| = 27\,|\mathbf{A}|$

Solution: Choose the correct answer in Exercises 23 and 24.

3.3. Let A be a square matrix of order 3X3, then $|kA|$ is equal to

a) $k\,|A|$
b) $k^2\,|A|$
c) $k^3\,|A|$
d) $3k\,|A|$

3.4. Which of the following is correct

a) Determinant is a square matrix.
b) Determinant is a number associated to a matrix.
c) Determinant is a number associated to a square matrix.
d) None of these.

3.5. Let A= $\begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$ and B= $\begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}$. Verify that $(AB)^{-1} = B^{-1}A^{-1}$

3.6. Let A be a nonsingular square matrix of order 3X3 .Then $|adjA|$ is equal to

a) $|A|$
b) $|A|^2$
c) $|A|^3$
d) $3\,|A|$

3.7. If A is an invertible matrix of order 2, then det(A^{-1}) is equal to

a) det(A)
b) $\frac{1}{det(A)}$
c) 1
d) 0

3.8. If

$$A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$ and

B=$\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$, find $(AB)^{-1}$

3.9. Let A= $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$. Verify that

(i) $[adjA]^{-1} = adj(A)^{-1}$
(ii) $(A^{-1})^{-1} = A$

4 Cramer's Rule

4.1. The cost of 4 kg onion, 3 kg wheat and 2 kg rice is ₹60. The cost of 2 kg onion,4 kg wheat and 6 kg rice is ₹90.The cost of 6kg onion 2kg wheat and 3kg rice is ₹70.Find the cost of each item per kg by matrix mathod. Solve the system linear equations,using matrix method.

4.2. $5x + 2y = 4$

$7x + 3y = 5$

Solution:

4.3. $2x - y = -2$

$3x + 4y = 3$

Solution:

4.4. $4x - 3y = 3$

$3x - 5y = 7$

Solution:

4.5. $5x + 2y = 3$

$3x + 2y = 5$

Solution:

4.6. 2x+y+z = 1
x-2y-z = $\frac{3}{2}$
3y- 5z = 9

4.7. x-y+z = 4
2x+y-3z = 0
x+y+z = 2

4.8. 2x+3y+3z = 5
x-2y+z = -4
3x-y-2z = 3

4.9. x-y+2z = 7
3x+4y-5z = -5
2x-y+3z = 12

4.10. If A=$\begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$, find A^{-1}. Using A^{-1} solve the system of equations
2x-3y+5z = 11,
3x+2y-4z = -5,
x+y-2z =-3.

5 Algebra

5.1. (i) $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$ (ii) $\begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix}$

Solution:

5.2. Find the values of x,If

(i) $\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$ (ii) $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} x & 3 \\ 2x & 5 \end{vmatrix}$

Solution:

5.3. If $\begin{vmatrix} x & 2 \\ 18 & x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix}$, then x is equal to

a) 6
b) ± 6
c) -6
d) 0

5.4. $\begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix} = 0$

Solution:

5.5. $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$

Solution:

5.6. $\begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix} = 0$

Solution:

5.7. $\begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix} = 0$

Solution:

5.8. $\begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix} = 2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$

Solution:

5.9. $\begin{vmatrix} -a^2 & ab & ab \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix} = 4a^2b^2c^2$

Solution: By Using properties of determinants, show that;

5.10. (i) $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$=(a-b)(b-c)(c-a)

(ii) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$=(a-b)(b-c)(c-a)(a+b+c)

Solution:

5.11. $\begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix}$=(x-y)(y-z)(z-x)(xy+yz+zx)

5.12. (i) $\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} = (5x+4)(4-x)^2$

Solution: (ii) $\begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & xy+k \end{vmatrix} = k^2(3y+k)$

Solution:

5.13. $\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix} = 1+a^2+b^2+c^2$

Solution:

5.14. Prove that the determinant $\begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$ is independent of θ

Solution:

5.15. Without expanding the determinant, prove that $\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$.

Solution:

5.16. Evaluate $\begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{vmatrix}$.

Solution:

5.17. If a,b and c are real numbers, and $\Delta = \begin{vmatrix} b+c & c=a & a=b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0$, Show that either a+b+c=0 or a=b=c.

Solution:

5.18. Solve the equation $\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0, a \neq 0$

Solution:

5.19. Prove that $\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2$

Solution:

5.20. Evaluate $\begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$

Solution:

5.21. Evaluate $\begin{vmatrix} 1 & x & y \\ 1 & x+y & y \\ 1 & x & x+y \end{vmatrix}$ **Solution:** Using properties of determinants ,prove that:

5.22. $\begin{vmatrix} \alpha & \alpha^2 & \beta+\gamma \\ \beta & \beta^2 & \gamma+\alpha \\ \gamma & \gamma^2 & \alpha+\beta \end{vmatrix} = (\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)(\alpha+\beta+\gamma)$

Solution:

5.23. $\begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix} = (1+pxyz)(x-y)(y-z)(z-x)$, where p is any scalar.

Solution:

5.24. $\begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix}$=3(a+b+c)(ab+bc+ca)

Solution:

6 Arithmetic

6.1. Find $\begin{vmatrix} 2 & 4 \\ -5 & -1 \end{vmatrix}$

Solution:

6.2. Evaluate the determinants

a) $\begin{vmatrix} 3 & -1 & -2 \\ 0 & 0 & -1 \\ 3 & -5 & 0 \end{vmatrix}$

b) $\begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix}$

Solution:

c) $\begin{vmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{vmatrix}$

d) $\begin{vmatrix} 2 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{vmatrix}$

6.3. If A=$\begin{vmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & -9 \end{vmatrix}$, find $|A|$

Solution: Examine the consistency of the system of given Equations.

6.4. $x + 3y = 5$

$2x + 6y = 8$

Solution:

6.5. x+y+z=1
2x+3y+2z=2
ax+ay+2az=4

Solution:

6.6. 3x-y-2z=2
2y-z=-1
3x-5y=3

Solution:

6.7. 5x-y+4z=5
2x+3y+5z=2
5x-2y+6z=-1

Solution: