

Linear Equations

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Abstract—This manual provides an introduction to linear equations based on the NCERT textbooks from Class 6-12.

1 LINEAR EQUATIONS

1.1. Solve the following

a)

$$\begin{aligned} \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} &= 5 \\ \begin{pmatrix} 2 & -3 \end{pmatrix} \mathbf{x} &= 4 \end{aligned} \quad (1.1.1)$$

b)

$$\begin{aligned} \begin{pmatrix} 3 & 4 \end{pmatrix} \mathbf{x} &= 10 \\ \begin{pmatrix} 2 & -2 \end{pmatrix} \mathbf{x} &= 2 \end{aligned} \quad (1.1.2)$$

c)

$$\begin{aligned} \begin{pmatrix} 3 & -5 \end{pmatrix} \mathbf{x} &= 4 \\ \begin{pmatrix} 9 & -2 \end{pmatrix} \mathbf{x} &= 7 \end{aligned} \quad (1.1.3)$$

$$\begin{aligned} \begin{pmatrix} \frac{1}{2} & \frac{2}{3} \end{pmatrix} \mathbf{x} &= -1 \\ \begin{pmatrix} 1 & -\frac{1}{3} \end{pmatrix} \mathbf{x} &= 3 \end{aligned} \quad (1.1.4)$$

Solution:

1.2. Solve the following pair of linear equations

$$\begin{aligned} \begin{pmatrix} 8 & 5 \end{pmatrix} \mathbf{x} &= 9 \\ \begin{pmatrix} 3 & 2 \end{pmatrix} \mathbf{x} &= 4 \end{aligned} \quad (1.2.1)$$

Solution:

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1.3. Solve the following pair of linear equations

$$\begin{aligned} \begin{pmatrix} 158 & -378 \end{pmatrix} \mathbf{x} &= -74 \\ \begin{pmatrix} -378 & 152 \end{pmatrix} \mathbf{x} &= -604 \end{aligned} \quad (1.3.1)$$

Solution:

1.4. Solve the following pair of equations

$$\begin{pmatrix} 7 & -15 \end{pmatrix} \mathbf{x} = 2 \quad (1.4.1)$$

$$\begin{pmatrix} 1 & 2 \end{pmatrix} \mathbf{x} = 3 \quad (1.4.2)$$

Solution: The above equations can be expressed as the matrix equation

$$\begin{pmatrix} 7 & -15 \\ 1 & 2 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad (1.4.3)$$

The augmented matrix for the above equation is row reduced as follows

$$\begin{pmatrix} 7 & -15 & 2 \\ 1 & 2 & 3 \end{pmatrix} \xrightarrow{R_2 \leftarrow 7R_2 - R_1} \begin{pmatrix} 7 & -15 & 2 \\ 0 & 29 & 19 \end{pmatrix} \quad (1.4.4)$$

$$\xrightarrow{R_1 \leftarrow \frac{15R_2 + 29R_1}{29}} \begin{pmatrix} 7 & 0 & 2 \\ 0 & 29 & 19 \end{pmatrix} \quad (1.4.5)$$

$$\Rightarrow \mathbf{x} = \begin{pmatrix} \frac{2}{7} \\ \frac{19}{29} \end{pmatrix} \quad (1.4.6)$$

The python code in Problem 2.10 can be used to plot Fig. ??, which shows that the lines are the same.

1.5. Find all possible solutions of

$$\begin{aligned} \begin{pmatrix} 2 & 3 \end{pmatrix} \mathbf{x} &= 8 \\ \begin{pmatrix} 4 & 6 \end{pmatrix} \mathbf{x} &= 7 \end{aligned} \quad (1.5.1)$$

Solution: The above equations can be expressed as the matrix equation

$$\begin{pmatrix} 2 & -3 \\ 4 & 6 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 8 \\ 7 \end{pmatrix} \quad (1.5.2)$$

The augmented matrix for the above equation is row reduced as follows

$$\begin{pmatrix} 2 & 3 & 8 \\ 4 & 6 & 7 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{pmatrix} 2 & -3 & 8 \\ 0 & 0 & -9 \end{pmatrix} \quad (1.5.3)$$

$$\Rightarrow \text{rank} \begin{pmatrix} 2 & -3 \\ 4 & 6 \end{pmatrix} \neq \begin{pmatrix} 2 & 3 & 8 \\ 4 & 6 & 7 \end{pmatrix}. \quad (1.5.4)$$

Hence, (1.5.1) has no solution. The python code in Problem 2.10 can be used to plot Fig. ??, which shows that the lines are parallel.

- 1.6. Which of the following pairs of linear equations are consistent/inconsistent? If consistent, obtain the solution:

a)

$$\begin{cases} (1 \ 1) \mathbf{x} = 5 \\ (2 \ 2) \mathbf{x} = 10 \end{cases} \quad (1.6.1)$$

b)

$$\begin{cases} (1 \ -1) \mathbf{x} = 8 \\ (3 \ -3) \mathbf{x} = 16 \end{cases} \quad (1.6.2)$$

c)

$$\begin{cases} (2 \ 1) \mathbf{x} = 6 \\ (4 \ -2) \mathbf{x} = 4 \end{cases} \quad (1.6.3)$$

d)

$$\begin{cases} (2 \ -2) \mathbf{x} = 2 \\ (4 \ -4) \mathbf{x} = 5 \end{cases} \quad (1.6.4)$$

Solution:

- 1.7. Find the intersection of the following lines

a)

$$\begin{cases} (1 \ 1) \mathbf{x} = 14 \\ (1 \ -1) \mathbf{x} = 4 \end{cases} \quad (1.7.1)$$

b)

$$\begin{cases} (1 \ -1) \mathbf{x} = 3 \\ \left(\frac{1}{3} \ \frac{1}{2}\right) \mathbf{x} = 6 \end{cases} \quad (1.7.2)$$

c)

$$\begin{cases} (3 \ -1) \mathbf{x} = 3 \\ (9 \ -3) \mathbf{x} = 9 \end{cases} \quad (1.7.3)$$

d)

$$\begin{cases} (0.2 \ 0.3) \mathbf{x} = 1.3 \\ (0.4 \ 0.5) \mathbf{x} = 2.3 \end{cases} \quad (1.7.4)$$

e)

$$\begin{cases} (\sqrt{2} \ \sqrt{3}) \mathbf{x} = 0 \\ (\sqrt{3} \ \sqrt{8}) \mathbf{x} = 0 \end{cases} \quad (1.7.5)$$

f)

$$\begin{cases} \left(\frac{3}{2} \ -\frac{5}{3}\right) \mathbf{x} = -2 \\ \left(\frac{1}{3} \ \frac{1}{2}\right) \mathbf{x} = \frac{13}{6} \end{cases} \quad (1.7.6)$$

Solution:

- 1.8. Draw the graphs of the equations

$$(1 \ -1) \mathbf{x} + 1 = 0 \quad (1.8.1)$$

$$(3 \ 2) \mathbf{x} - 12 = 0 \quad (1.8.2)$$

Determine the coordinates of the vertices of the triangle formed by these lines and the x-axis, and shade the triangular region.

Solution:

- 1.9. In a $\triangle ABC$, $\angle C = 3\angle B = 2(\angle A + \angle B)$. Find the three angles.

Solution:

- 1.10. Draw the graphs of the equations $5x - y = 5$ and $3x - y = 3$. Determine the co-ordinates of the vertices of the triangle formed by these lines and the y axis.

Solution:

- 1.11. $ABCD$ is a cyclic quadrilateral with

$$\angle A = 4y + 20 \quad (1.11.1)$$

$$\angle B = 3y - 5 \quad (1.11.2)$$

$$\angle C = -4x \quad (1.11.3)$$

$$\angle D = -7x + 5 \quad (1.11.4)$$

Find its angles.

Solution:

2 RANK

- 2.1. Show that points $A = (a \ b + c)$, $B = (b \ c + a)$, $C = (c \ a + b)$ are collinear.

Solution:

2.2. Find m if

$$\begin{aligned} \begin{pmatrix} 2 & 3 \end{pmatrix} \mathbf{x} &= 11 \\ \begin{pmatrix} 2 & -4 \end{pmatrix} \mathbf{x} &= -24 \\ \begin{pmatrix} m & -1 \end{pmatrix} \mathbf{x} &= -3 \end{aligned} \quad (2.2.1)$$

Solution:

2.3. For which values of a and b does the following pair of linear equations have an infinite number of solutions?

$$\begin{aligned} \begin{pmatrix} 2 & 3 \end{pmatrix} \mathbf{x} &= 7 \\ \begin{pmatrix} a-b & a+b \end{pmatrix} \mathbf{x} &= 3a+b-2 \end{aligned} \quad (2.3.1)$$

Solution:

2.4. For which value of k will the following pair of linear equations have no solution?

$$\begin{aligned} \begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} &= 1 \\ \begin{pmatrix} 2k-1 & k-1 \end{pmatrix} \mathbf{x} &= 2k+1 \end{aligned} \quad (2.4.1)$$

Solution:

2.5. If the lines

$$\begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = 3 \quad (2.5.1)$$

$$\begin{pmatrix} 5 & k \end{pmatrix} \mathbf{x} = 3 \quad (2.5.2)$$

$$\begin{pmatrix} 3 & -1 \end{pmatrix} \mathbf{x} = 2 \quad (2.5.3)$$

are concurrent, find the value of k .

Solution: If the lines are concurrent, the *augmented* matrix should have a 0 row upon row reduction. Hence,

$$\begin{pmatrix} 2 & 1 & 3 \\ 5 & k & 3 \\ 3 & -1 & 2 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 2 & 1 & 3 \\ 3 & -1 & 2 \\ 5 & k & 3 \end{pmatrix} \quad (2.5.4)$$

$$\xrightarrow[R_3 \leftarrow 2R_3 - 5R_1]{R_2 \leftarrow 2R_2 - 3R_1} \begin{pmatrix} 2 & 1 & 3 \\ 0 & -5 & -5 \\ 0 & 2k-5 & -9 \end{pmatrix} \quad (2.5.5)$$

$$\xrightarrow{R_2 \leftarrow -\frac{R_2}{5}} \begin{pmatrix} 2 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 2k-5 & -9 \end{pmatrix} \quad (2.5.6)$$

$$\xrightarrow{R_3 \leftarrow R_3 - (2k-5)R_2} \begin{pmatrix} 2 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -2k-4 \end{pmatrix} \quad (2.5.7)$$

$$\implies k = -2 \quad (2.5.8)$$

2.6. For which values of p does the pair of equations given below has unique solution?

$$\begin{aligned} \begin{pmatrix} 4 & p \end{pmatrix} \mathbf{x} &= -8 \\ \begin{pmatrix} 2 & 2 \end{pmatrix} \mathbf{x} &= -2 \end{aligned} \quad (2.6.1)$$

Solution: (2.6.1) has a unique solution

$$\iff \begin{vmatrix} 4 & p \\ 2 & 2 \end{vmatrix} \neq 0 \quad (2.6.2)$$

$$\text{or, } p \neq 4 \quad (2.6.3)$$

2.7. For what values of k will the following pair of linear equations have infinitely many solutions?

$$\begin{aligned} \begin{pmatrix} k & 3 \end{pmatrix} \mathbf{x} &= k-3 \\ \begin{pmatrix} 12 & k \end{pmatrix} \mathbf{x} &= k \end{aligned} \quad (2.7.1)$$

Solution: The first condition for (2.7.1) to have infinite solutions is

$$\begin{vmatrix} k & 3 \\ 12 & k \end{vmatrix} = 0 \quad (2.7.2)$$

$$\implies k^2 = 36, \text{ or, } k = \pm 6 \quad (2.7.3)$$

For $k = 6$, the augmented matrix for the above equation is row reduced as follows

$$\begin{pmatrix} 6 & 3 & 3 \\ 12 & 6 & 6 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{pmatrix} 6 & 3 & 3 \\ 0 & 0 & 0 \end{pmatrix} \quad (2.7.4)$$

indicating that (2.7.1) has infinite number of solutions. For $k = -6$, the augmented matrix is

$$\begin{pmatrix} 6 & 3 & -9 \\ 12 & 6 & -6 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{pmatrix} 6 & 3 & -9 \\ 0 & 0 & 12 \end{pmatrix} \quad (2.7.5)$$

indicating that (2.7.1) has no solution. Thus, (2.7.2) is a necessary condition but not sufficient.

2.8. Check whether the pair of equations

$$\begin{aligned} \begin{pmatrix} 1 & 3 \end{pmatrix} \mathbf{x} &= 6 \text{ and} \\ \begin{pmatrix} 2 & -3 \end{pmatrix} \mathbf{x} &= 12 \end{aligned} \quad (2.8.1)$$

is consistent.

Solution: The above equations can be expressed as the matrix equation

$$\begin{pmatrix} 1 & 3 \\ 2 & -3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 6 \\ 12 \end{pmatrix} \quad (2.8.2)$$

The augmented matrix for the above equation is row reduced as follows

$$\begin{pmatrix} 1 & 3 & 6 \\ 2 & -3 & 12 \end{pmatrix} \xrightarrow{R_2 \leftarrow \frac{R_2 - 2R_1}{-9}} \quad (2.8.3)$$

$$\begin{pmatrix} 1 & 3 & 6 \\ 0 & 1 & 0 \end{pmatrix} \xrightarrow{R_1 \leftarrow R_1 - 3R_2} \begin{pmatrix} 1 & 0 & 6 \\ 0 & 1 & 0 \end{pmatrix} \quad (2.8.4)$$

$$\implies \mathbf{x} = \begin{pmatrix} 6 \\ 0 \end{pmatrix} \quad (2.8.5)$$

which is the solution of 2.10.1. The python code in Problem 2.10 can be used to plot Fig. ??, which shows that the lines intersect.

2.9. Find whether the following pair of equations has no solution, unique solution or infinitely many solutions:

$$\begin{aligned} (5 \quad -8) \mathbf{x} &= -1 \text{ and} \\ (3 \quad -\frac{24}{5}) \mathbf{x} &= -\frac{3}{5} \end{aligned} \quad (2.9.1)$$

Solution: The above equations can be expressed as the matrix equation

$$\begin{pmatrix} 5 & -8 \\ 3 & -\frac{24}{5} \end{pmatrix} \mathbf{x} = -\begin{pmatrix} 1 \\ \frac{3}{5} \end{pmatrix} \quad (2.9.2)$$

The augmented matrix for the above equation is row reduced as follows

$$\begin{pmatrix} 5 & -8 & -1 \\ 3 & -\frac{24}{5} & -\frac{3}{5} \end{pmatrix} \xrightarrow{R_2 \leftarrow 5R_2} \begin{pmatrix} 5 & -8 & 1 \\ 15 & -24 & -3 \end{pmatrix} \quad (2.9.3)$$

$$\xrightarrow{R_2 \leftarrow R_2 - 3R_1} \begin{pmatrix} 5 & -8 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad (2.9.4)$$

$$\therefore \text{rank} \begin{pmatrix} 5 & -8 \\ 3 & -\frac{24}{5} \end{pmatrix} = \text{rank} \begin{pmatrix} 5 & -8 & 1 \\ 3 & -\frac{24}{5} & -\frac{3}{5} \end{pmatrix} \quad (2.9.5)$$

$$= 1 < \dim \begin{pmatrix} 5 & -8 \\ 3 & -\frac{24}{5} \end{pmatrix} = 2, \quad (2.9.6)$$

(2.9.1) has infinitely many solutions. The python code in Problem 2.10 can be used to plot Fig. ??, which shows that the lines are the same.

2.10. Two rails are represented by the equations

$$\begin{aligned} (1 \quad 2) \mathbf{x} &= 4 \text{ and} \\ (2 \quad 4) \mathbf{x} &= 12. \end{aligned} \quad (2.10.1)$$

Will the rails cross each other?

Solution: The above equations can be expressed as the matrix equation

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 4 \\ 12 \end{pmatrix} \quad (2.10.2)$$

The augmented matrix for the above equation is row reduced as follows

$$\begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 12 \end{pmatrix} \xrightarrow{R_2 \leftarrow \frac{R_2}{2}} \begin{pmatrix} 1 & 2 & 4 \\ 1 & 2 & 6 \end{pmatrix} \quad (2.10.3)$$

$$\xrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 2 & 4 \\ 0 & 0 & 2 \end{pmatrix} \quad (2.10.4)$$

\therefore row reduction of the 2×3 matrix

$$\begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 12 \end{pmatrix} \quad (2.10.5)$$

results in a matrix with 2 nonzero rows, its rank is 2. Similarly, the rank of the matrix

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \quad (2.10.6)$$

is 1, from 2.10.4.

$$\therefore \text{rank} \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \neq \text{rank} \begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 12 \end{pmatrix}, \quad (2.10.7)$$

(2.10.1) has no solution. The equivalent python code is

codes/line/line_check_sol.py

which plots Fig. ??, which shows that the rails are parallel.

2.11. Which of the following pairs of linear equations has a unique solution, no solution, or infinitely many solutions?

a)

$$\begin{aligned} (1 \quad -3) \mathbf{x} &= 3 \\ (3 \quad -9) \mathbf{x} &= 2 \end{aligned} \quad (2.11.1)$$

b)

$$\begin{aligned} (2 \quad 1) \mathbf{x} &= 5 \\ (3 \quad 2) \mathbf{x} &= 8 \end{aligned} \quad (2.11.2)$$

c)

$$\begin{aligned} (3 \quad -5) \mathbf{x} &= 20 \\ (6 \quad -10) \mathbf{x} &= 40 \end{aligned} \quad (2.11.3)$$

d)

$$\begin{aligned} \begin{pmatrix} 1 & -3 \end{pmatrix} \mathbf{x} &= 7 \\ \begin{pmatrix} 3 & -3 \end{pmatrix} \mathbf{x} &= 15 \end{aligned} \quad (2.11.4)$$

Solution:

2.12. Find out whether the lines representing the following pairs of linear equations intersect at a point, are parallel or coincident

a)

$$\begin{aligned} \begin{pmatrix} 5 & -4 \end{pmatrix} \mathbf{x} &= -8 \\ \begin{pmatrix} 7 & 6 \end{pmatrix} \mathbf{x} &= 9 \end{aligned} \quad (2.12.1)$$

b)

$$\begin{aligned} \begin{pmatrix} 9 & 3 \end{pmatrix} \mathbf{x} &= -12 \\ \begin{pmatrix} 18 & 6 \end{pmatrix} \mathbf{x} &= -24 \end{aligned} \quad (2.12.2)$$

c)

$$\begin{aligned} \begin{pmatrix} 6 & -3 \end{pmatrix} \mathbf{x} &= -10 \\ \begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} &= -9 \end{aligned} \quad (2.12.3)$$

Solution:

2.13. Find the value of p so that the three lines

$$\begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} = 2 \quad (2.13.1)$$

$$\begin{pmatrix} p & 2 \end{pmatrix} \mathbf{x} = 3 \quad (2.13.2)$$

$$\begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} = 3 \quad (2.13.3)$$

may intersect at one point. **Solution:**

2.14. Find a condition on \mathbf{x} such that the points

$\mathbf{x}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 7 \\ 0 \end{pmatrix}$ are collinear.

Solution: