

Matrices

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Abstract—This book provides a computational approach to school geometry based on the NCERT textbooks from Class 6-12. Links to sample Python codes are available in the text.

Download python codes using

svn co <https://github.com/gadepall/school/trunk/ncert/computation/codes>

1 DEFINITIONS

1.1 Eigenvalues and Eigenvectors

1.1.1. The eigenvalue λ and the eigenvector \mathbf{x} for a matrix \mathbf{A} are defined as,

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \quad (1.1.1.1)$$

1.1.2. The eigenvalues are calculated by solving the equation

$$f(\lambda) = |\lambda\mathbf{I} - \mathbf{A}| = 0 \quad (1.1.2.1)$$

The above equation is known as the characteristic equation.

1.1.3. According to the Cayley-Hamilton theorem,

$$f(\lambda) = 0 \implies f(\mathbf{A}) = 0 \quad (1.1.3.1)$$

1.2 Determinants

1.2.1. Let

$$\mathbf{A} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}. \quad (1.2.1.1)$$

be a 3×3 matrix. Then,

$$|\mathbf{A}| = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_1 \\ a_2 & b_2 \end{vmatrix}. \quad (1.2.1.2)$$

1.2.2. Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of a matrix \mathbf{A} . Then, the product of the eigenvalues is equal to the determinant of \mathbf{A} .

$$|\mathbf{A}| = \prod_{i=1}^n \lambda_i \quad (1.2.2.1)$$

1.2.3.

$$|\mathbf{AB}| = |\mathbf{A}| |\mathbf{B}| \quad (1.2.3.1)$$

1.2.4. If \mathbf{A} be an $n \times n$ matrix,

$$|k\mathbf{A}| = k^n |\mathbf{A}| \quad (1.2.4.1)$$

2 CAYLEY-HAMILTON THEOREM

2.1. If

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{pmatrix}, \quad (2.1.1)$$

prove that

$$\mathbf{A}^3 - 6\mathbf{A}^2 + 7\mathbf{A} + 2\mathbf{I} = 0 \quad (2.1.2)$$

Solution: From (1.1.2.1), the characteristic equation is

$$\begin{vmatrix} 1 - \lambda & 0 & 2 \\ 0 & 2 - \lambda & 1 \\ 2 & 0 & 3 - \lambda \end{vmatrix} = 0 \quad (2.1.3)$$

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which can be expanded to obtain

$$(1 - \lambda)(2 - \lambda)(3 - \lambda) + 2(-2(2 - \lambda)) = 0 \quad (2.1.4)$$

yielding

$$\lambda^3 - 6\lambda^2 + 7\lambda + 2 = 0 \quad (2.1.5)$$

upon simplification. Using the Cayley-Hamilton theorem in (1.1.3.1), (2.1.2) is obtained

2.2. If

$$\mathbf{A} = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}, \quad (2.2.1)$$

show that

$$\mathbf{A}^2 - 5\mathbf{A} + 7\mathbf{I} = 0 \quad (2.2.2)$$

Solution: The characteristic equation is

$$|\mathbf{A} - \lambda\mathbf{I}| = 0 \quad (2.2.3)$$

$$\Rightarrow \begin{vmatrix} 3 - \lambda & 1 \\ -1 & 2 - \lambda \end{vmatrix} = 0 \quad (2.2.4)$$

$$\Rightarrow (3 - \lambda)(2 - \lambda) + 1 = 0 \quad (2.2.5)$$

$$\text{or, } \lambda^2 - 5\lambda + 7 = 0 \quad (2.2.6)$$

Using the Cayley-Hamilton theorem, (2.2.2) is obtained.

2.3. If \mathbf{A} is square matrix such that

$$\mathbf{A}^2 = \mathbf{A}, \quad (2.3.1)$$

then

$$g(\mathbf{A}) = (\mathbf{I} + \mathbf{A})^3 - 7\mathbf{A} \quad (2.3.2)$$

is equal to

- a) \mathbf{A}
- b) $\mathbf{I} - \mathbf{A}$
- c) \mathbf{I}
- d) $3\mathbf{A}$

Solution: From (2.3.2),

$$g(\lambda) = (1 + \lambda)^3 - 7\lambda \quad (2.3.3)$$

Also,

$$\mathbf{A}^2 = \mathbf{A} \Rightarrow \mathbf{A}^2 - \mathbf{A} = 0 \quad (2.3.4)$$

Using the Cayley-Hamilton theorem, the eigenvalues satisfy the characteristic equation

$$f(\lambda) = \lambda^2 - \lambda = 0 \quad (2.3.5)$$

$\therefore f(\lambda)$ is of degree 2 and $g(\lambda)$ is of degree 3, (2.3.3) can be expressed as

$$g(\lambda) = f(\lambda)q(\lambda) + a\lambda + b \quad (2.3.6)$$

where a, b are real numbers and $q(\lambda)$ is some polynomial. Thus,

$$g(0) = b = 1 \quad (2.3.7)$$

$$g(1) = a + b = 1 \quad (2.3.8)$$

$$\Rightarrow a = 90, b = 1 \quad (2.3.9)$$

Thus,

$$g(\mathbf{A}) = f(\mathbf{A})q(\mathbf{A}) + a\mathbf{A} + b\mathbf{I} \quad (2.3.10)$$

$$= \mathbf{I} \quad (2.3.11)$$

upon substituting from (2.3.5) and (2.3.9). Option c is the valid answer.

2.4. If

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{pmatrix} \quad (2.4.1)$$

then show that

$$\mathbf{A}^3 - 23\mathbf{A} - 40\mathbf{I} = 0 \quad (2.4.2)$$

Solution: The Characteristic equation is given by

$$|\mathbf{A} - \lambda\mathbf{I}| = 0 \quad (2.4.3)$$

$$\Rightarrow \begin{vmatrix} 1 - \lambda & 2 & 3 \\ 3 & -2 - \lambda & 1 \\ 4 & 2 & 1 - \lambda \end{vmatrix} = 0 \quad (2.4.4)$$

which can be expressed as

$$\begin{aligned} &\Rightarrow (1 - \lambda)((-2 - \lambda)(1 - \lambda) - 2) \\ &- 2(3(1 - \lambda) - 4) + 3(6 + 4(2 + \lambda)) = 0 \end{aligned} \quad (2.4.5)$$

and simplified to obtain

$$\Rightarrow \lambda^3 - 23\lambda - 40 = 0. \quad (2.4.6)$$

Using the Cayley-Hamilton Theorem, (2.4.2) is obtained.

3 MATRIX POLYNOMIALS

3.1. Find $\mathbf{A}^2 - 5\mathbf{A} + 6\mathbf{I}$, if $\mathbf{A} = \begin{pmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{pmatrix}$

Solution: Factorizing the corresponding polynomial,

$$f(x) = x^2 - 5x + 6 \quad (3.1.1)$$

$$= (x - 3)(x - 2) \quad (3.1.2)$$

$$\implies \mathbf{A}^2 - 5\mathbf{A} + 6\mathbf{I} = (\mathbf{A} - 3\mathbf{I})(\mathbf{A} - 2\mathbf{I}) \quad (3.1.3)$$

$$\mathbf{A} - 3\mathbf{I} = \begin{pmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{pmatrix} + \begin{pmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix} \quad (3.1.4)$$

$$= \begin{pmatrix} -1 & 0 & 1 \\ 2 & -2 & 3 \\ 1 & -1 & -3 \end{pmatrix} \quad (3.1.5)$$

$$\mathbf{A} - 2\mathbf{I} = \begin{pmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{pmatrix} + \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix} \quad (3.1.6)$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 2 & -1 & 3 \\ 1 & -1 & -2 \end{pmatrix} \quad (3.1.7)$$

Multiplying the above expressions,

$$\begin{aligned} \mathbf{A}^2 - 5\mathbf{A} + 6\mathbf{I} &= \\ &= \begin{pmatrix} -1 & 0 & 1 \\ 2 & -2 & 3 \\ 1 & -1 & -3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 2 & -1 & 3 \\ 1 & -1 & -2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -1 & -3 \\ -1 & -1 & -10 \\ -5 & 4 & 4 \end{pmatrix} \quad (3.1.8) \end{aligned}$$