Matrix Applications

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CONTENTS

***Abstract*—This manual shows how to balance chemical equations using matrices, based on exercises from the NCERT textbooks from Class 6-12.**

Download python codes using

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svn co https://github.com/gadepall/school/trunk/
    ncert/computation/codes
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1 Balancing Chemical Equations

1.1. Balance the following chemical equation.

$$HNO_3 + Ca(OH)_2 \to Ca(NO_3)_2 + H_2O \tag{1.1.1}$$

Solution: Let the balanced version of (1.1.1) be

$$\begin{aligned} x_1HNO_3 + x_2Ca(OH)_2 \\ \to x_3Ca(NO_3)_2 + x_4H_2O \end{aligned} \tag{1.1.2}$$

which results in the following equations:

$$(x_1 + 2x_2 - 2x_4)H = 0 \tag{1.1.3}$$

$$(x_1 - 2x_3)N = 0 \tag{1.1.4}$$

$$(3x_1 + 2x_2 - 6x_3 - x_4)O = 0 \tag{1.1.5}$$

$$(x_2 - x_3)Ca = 0 \tag{1.1.6}$$

which can be expressed as

$$x_1 + 2x_2 + 0x_3 - 2x_4 = 0 \tag{1.1.7}$$

$$x_1 + 0x_2 - 2x_3 + 0.x_4 = 0 \tag{1.1.8}$$

$$3x_1 + 2x_2 - 6x_3 - x_4 = 0 \tag{1.1.9}$$

$$0x_1 + x_2 - x_3 + 0.x_4 = 0 \tag{1.1.10}$$

resulting in the matrix equation

$$\begin{pmatrix} 1 & 2 & 0 & -2 \\ 1 & 0 & -2 & 0 \\ 3 & 2 & -6 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix} \mathbf{x} = \mathbf{0} \tag{1.1.11}$$

where,

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \tag{1.1.12}$$

(1.1.11) can be reduced as follows

$$\begin{pmatrix} 1 & 2 & 0 & -2 \\ 1 & 0 & -2 & 0 \\ 3 & 2 & -6 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix} \tag{1.1.13}$$

$$\xleftrightarrow[R_3 \leftarrow \frac{R_3}{3} - R_1]{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 2 & 0 & -2 \\ 0 & -2 & -2 & 2 \\ 0 & -\frac{4}{3} & -2 & \frac{5}{3} \\ 0 & 1 & -1 & 0 \end{pmatrix} \tag{1.1.14}$$

$$\xleftrightarrow{R_2 \leftarrow -\frac{R_2}{2}} \begin{pmatrix} 1 & 2 & 0 & -2 \\ 0 & 1 & 1 & -1 \\ 0 & -\frac{4}{3} & -2 & \frac{5}{3} \\ 0 & 1 & -1 & 0 \end{pmatrix} \tag{1.1.15}$$

$$\xleftrightarrow[R_4 \leftarrow R_4 - R_2]{R_3 \leftarrow R_3 + \frac{4}{3}R_2} \begin{pmatrix} 1 & 2 & 0 & -2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -\frac{2}{3} & \frac{1}{3} \\ 0 & 0 & -2 & 1 \end{pmatrix} \tag{1.1.16}$$

$$\xleftrightarrow[R_3 \leftarrow -\frac{3}{2}R_3]{R_1 \leftarrow R_1 - 2R_2} \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & -2 & 1 \end{pmatrix} \tag{1.1.17}$$

$$\xleftrightarrow{R_4 \leftarrow R_4 + 2R_3} \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1.1.18}$$

$$\xleftrightarrow[R_2 \leftarrow R_2 - R_3]{R_1 \leftarrow R_1 + 2R_3} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1.1.19}$$

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Thus,

$$x_1 = x_4, x_2 = \frac{1}{2}x_4, x_3 = \frac{1}{2}x_4 \tag{1.1.20}$$

$$\implies \quad \mathbf{x} = x_4 \begin{pmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 2 \end{pmatrix} \tag{1.1.21}$$

by substituting $x_4 = 2$

Hence, (1.1.2) finally becomes

$$2HNO_3 + Ca(OH)_2 \to Ca(NO_3)_2 + 2H_2O \tag{1.1.22}$$

1.2. Balance the following chemical equation.

$$\text{Zinc + Silver nitrate} \to \text{Zinc nitrate + Silver} \tag{1.2.1}$$

Solution:
1.2.1 can be written as

$$Zn + AgNO_3 \to Ag + Zn(NO_3)_2 \tag{1.2.2}$$

Suppose the balanced form of the equation is

$$x_1Zn + x_2AgNO_3 \to x_3Ag + x_4Zn(NO_3)_2 \tag{1.2.3}$$

which results in the following equations:

$$(x_1 - 2x_4)Zn = 0 \tag{1.2.4}$$
$$(x_2 - x_3)Ag = 0 \tag{1.2.5}$$
$$(x_3 - 2x_4)N = 0 \tag{1.2.6}$$
$$(3x_3 - 6x_4)O = 0 \tag{1.2.7}$$

which can be expressed as

$$x_1 + 0x_2 + 0x_3 - x_4 = 0 \tag{1.2.8}$$
$$0x_1 + x_2 - x_3 + 0x_4 = 0 \tag{1.2.9}$$
$$0x_1 + 0x_2 + x_3 - 2x_4 = 0 \tag{1.2.10}$$
$$0x_1 + 0x_2 + 3x_3 - 6x_4 = 0 \tag{1.2.11}$$

resulting in the matrix equation

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 3 & -6 \end{pmatrix} \mathbf{x} = \mathbf{0} \tag{1.2.12}$$

where,

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \tag{1.2.13}$$

(1.2.12) can be reduced as

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 3 & -6 \end{pmatrix} \xleftrightarrow{R_4 \leftarrow R_4 - 3R_3} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1.2.14}$$

Thus,

$$x_1 = x_4, x_2 = 2x_4, x_3 = 2x_4 \tag{1.2.15}$$

$$\implies \quad \mathbf{x} = \begin{pmatrix} x_4 \\ 2x_4 \\ 2x_4 \\ x_4 \end{pmatrix} = x_4 \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix} \tag{1.2.16}$$

by substituting $x_4 = 1$, we get

$$\implies \quad \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix} \tag{1.2.17}$$

Hence, (1.2.3) finally becomes

$$Zn + 2AgNO_3 \to 2Ag + Zn(NO_3)_2 \tag{1.2.18}$$

1.3. Write the balanced chemical equations for the following reaction :

$$BaCl_2 + K_2SO_4 \to BaSO_4 + KCl \tag{1.3.1}$$

Solution: We know that the number of atoms of each element remains the same, before and after a chemical reaction.
Equation (1.3.1) can be written as

$$x_1BaCl_2 + x_2K_2SO_4 \to x_3BaSO_4 + x_4KCl \tag{1.3.2}$$

Element wise contribution in forming the respective chemical compound can be written in the form of equation as

$$Ba : x_1 + 0x_2 - x_3 - 0x_4 = 0 \tag{1.3.3}$$
$$Cl : 2x_1 + 0x_2 - 0x_3 - 1x_4 = 0 \tag{1.3.4}$$
$$K : 0x_1 + 2x_2 - 0x_3 - 1x_4 = 0 \tag{1.3.5}$$
$$S : 0x_1 + 1x_2 - 1x_3 - 0x_4 = 0 \tag{1.3.6}$$
$$O : 0x_1 + 4x_2 - 4x_3 - 0x_4 = 0 \tag{1.3.7}$$

In matrix form this can be written as

$$A\mathbf{x} = 0 \tag{1.3.8}$$

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 2 & 0 & 0 & -1 \\ 0 & 2 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 4 & -4 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{1.3.9}$$

Using Gaussian Elimination method

$$\xleftrightarrow{R_2 \leftrightarrow R_5} \left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 4 & -4 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 2 & 0 & 0 & -1 & 0 \end{array}\right) \tag{1.3.10}$$

$$\xleftrightarrow{R_5 \leftarrow 2R_1 - R_5} \left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 4 & -4 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 \end{array}\right) \tag{1.3.11}$$

$$\xleftrightarrow[R_4 \leftarrow 4R_4 - R_2]{R_3 \leftarrow 2R_3 - R_2} \left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 4 & -4 & 0 & 0 \\ 0 & 0 & 4 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 \end{array}\right) \tag{1.3.12}$$

$$\xleftrightarrow{R_5 \leftrightarrow R_5} \left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 4 & -4 & 0 & 0 \\ 0 & 0 & 4 & -2 & 0 \\ 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \tag{1.3.13}$$

$$\xleftrightarrow{R_4 \leftarrow 2R_4 - R_3} \left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 4 & -4 & 0 & 0 \\ 0 & 0 & 4 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \tag{1.3.14}$$

Clearly the system is linearly dependent. Therefore by fixing the value of $x_4 = 2$, one of the possible vectors $\mathbf{x}$ is

$$\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \end{pmatrix} \tag{1.3.15}$$

Hence by putting the values of x_1, x_2, x_3, x_4 in equation (1.3.1) we get our balanced chemical equation as

$$BaCl_2 + K_2SO_4 \rightarrow BaSO_4 + 2KCl \tag{1.3.16}$$

1.4. Balance the following chemical equation.

$$Fe + H_2O \rightarrow Fe_3O_4 + H_2 \tag{1.4.1}$$

Solution:
Let the balanced version of (1.4.1) be

$$x_1Fe + x_2H_2O \rightarrow x_3Fe_3O_4 + x_4H_2 \tag{1.4.2}$$

which results in the following equations

$$\begin{aligned} (x_1 - 3x_3)\, Fe &= 0 \\ (2x_2 - 2x_4)\, H &= 0 \\ (x_2 - 4x_3)\, O &= 0 \end{aligned} \tag{1.4.3}$$

which can be expressed as

$$\begin{aligned} x_1 + 0.x_2 - 3x_3 + 0.x_4 &= 0 \\ 0.x_1 + 2x_2 + 0.x_3 - 2x_4 &= 0 \\ 0.x_1 + x_2 - 4x_3 + 0.x_4 &= 0 \end{aligned} \tag{1.4.4}$$

resulting in the matrix equation

$$\begin{pmatrix} 1 & 0 & -3 & 0 \\ 0 & 2 & 0 & -2 \\ 0 & 1 & -4 & 0 \end{pmatrix} \mathbf{x} = \mathbf{0} \tag{1.4.5}$$

where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \tag{1.4.6}$$

(1.4.5) can be row reduced as follows

$$\begin{pmatrix}1 & 0 & -3 & 0\\0 & 2 & 0 & -2\\0 & 1 & -4 & 0\end{pmatrix} \xleftrightarrow{R_2\leftarrow\frac{R_2}{2}} \begin{pmatrix}1 & 0 & -3 & 0\\0 & 1 & 0 & -1\\0 & 1 & -4 & 0\end{pmatrix} \tag{1.4.7}$$

$$\xleftrightarrow{R_3\leftarrow R_3-R_2} \begin{pmatrix}1 & 0 & -3 & 0\\0 & 1 & 0 & -1\\0 & 0 & -4 & 1\end{pmatrix} \tag{1.4.8}$$

$$\xleftrightarrow{R_1\leftarrow 4R_1-3R_3} \begin{pmatrix}4 & 0 & 0 & -3\\0 & 1 & 0 & -1\\0 & 0 & -4 & 1\end{pmatrix} \tag{1.4.9}$$

$$\xleftrightarrow[R_3\leftarrow-\frac{1}{4}R_3]{R_1\leftarrow\frac{1}{4}} \begin{pmatrix}1 & 0 & 0 & -\frac{3}{4}\\0 & 1 & 0 & -1\\0 & 0 & 1 & -\frac{1}{4}\end{pmatrix} \tag{1.4.10}$$

Thus,

$$x_1 = \frac{3}{4}x_4, x_2 = x_4, x_3 = \frac{1}{4}x_4 \tag{1.4.11}$$

$$\tag{1.4.12}$$

$$\implies \mathbf{x} = x_4\begin{pmatrix}\frac{3}{4}\\1\\\frac{1}{4}\\1\end{pmatrix} = \begin{pmatrix}3\\4\\1\\4\end{pmatrix} \tag{1.4.13}$$

upon substituting $x_4 = 4$. (1.4.2) then becomes

$$3Fe + 4H_2O \to Fe_3O_4 + 4H_2 \tag{1.4.14}$$

1.5. Balance the following chemical equation.

$$NaOH + H_2SO_4 \to Na_2SO_4 + H_2O \tag{1.5.1}$$

Solution: Let the balanced version of (1.5.1) be

$$x_1NaOH+x_2H_2SO_4 \to x_3Na_2SO_4+x_4H_2O \tag{1.5.2}$$

which results in the following equations:

$$(x_1 - 2x_3)Na = 0 \tag{1.5.3}$$
$$(x_1 + 4x_2 - 4x_3 - x_4)O = 0 \tag{1.5.4}$$
$$(x_1 + 2x_2 - 2x_4)H = 0 \tag{1.5.5}$$
$$(x_2 - x_3)S = 0 \tag{1.5.6}$$

which can be expressed as

$$x_1 + 0x_2 - 2x_3 + 0x_4 = 0 \tag{1.5.7}$$
$$x_1 + 4x_2 - 4x_3 - x_4 = 0 \tag{1.5.8}$$
$$x_1 + 2x_2 + 0x_3 - 2x_4 = 0 \tag{1.5.9}$$
$$0x_1 + x_2 - x_3 + 0x_4 = 0 \tag{1.5.10}$$

resulting in the matrix equation

$$\begin{pmatrix}1 & 0 & -2 & 0\\1 & 4 & -4 & -1\\1 & 2 & 0 & -2\\0 & 1 & -1 & 0\end{pmatrix}\mathbf{x} = \mathbf{0} \tag{1.5.11}$$

where,

$$\mathbf{x} = \begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} \tag{1.5.12}$$

(1.5.11) can be reduced as

$$\begin{pmatrix}1 & 0 & -2 & 0\\1 & 4 & -4 & -1\\1 & 2 & 0 & -2\\0 & 1 & -1 & 0\end{pmatrix} \xleftrightarrow[R_3\leftarrow R_3-R_1]{R_2\leftarrow R_2-R_1} \begin{pmatrix}1 & 0 & -2 & 0\\0 & 4 & -2 & -1\\0 & 2 & 2 & -2\\0 & 1 & -1 & 0\end{pmatrix} \tag{1.5.13}$$

$$\xleftrightarrow{R_2\leftarrow\frac{R_2}{4}} \begin{pmatrix}1 & 0 & -2 & 0\\0 & 1 & -\frac{1}{2} & -\frac{1}{4}\\0 & 2 & 2 & -2\\0 & 1 & -1 & 0\end{pmatrix} \tag{1.5.14}$$

$$\xleftrightarrow[R_4\leftarrow R_4-R_2]{R_3\leftarrow R_3-2R_2} \begin{pmatrix}1 & 0 & -2 & 0\\0 & 1 & -\frac{1}{2} & -\frac{1}{4}\\0 & 0 & 3 & -\frac{3}{2}\\0 & 0 & -\frac{1}{2} & \frac{1}{4}\end{pmatrix} \tag{1.5.15}$$

$$\xleftrightarrow{R_3\leftarrow\frac{R_3}{3}} \begin{pmatrix}1 & 0 & -2 & 0\\0 & 1 & -\frac{1}{2} & -\frac{1}{4}\\0 & 0 & 1 & -\frac{1}{2}\\0 & 0 & -\frac{1}{2} & \frac{1}{4}\end{pmatrix} \tag{1.5.16}$$

$$\xleftrightarrow[R_4\leftarrow R_4+\frac{R_3}{2}]{R_2\leftarrow R_2+\frac{R_3}{2}} \begin{pmatrix}1 & 0 & -2 & 0\\0 & 1 & 0 & -\frac{1}{2}\\0 & 0 & 1 & -\frac{1}{2}\\0 & 0 & 0 & 0\end{pmatrix} \tag{1.5.17}$$

$$\xleftrightarrow{R_1\leftarrow R_1+2R_3} \begin{pmatrix}1 & 0 & 0 & -1\\0 & 1 & 0 & -\frac{1}{2}\\0 & 0 & 1 & -\frac{1}{2}\\0 & 0 & 0 & 0\end{pmatrix} \tag{1.5.18}$$

Thus,

$$x_1 = x_4, x_2 = \frac{1}{2}x_4, x_3 = \frac{1}{2}x_4 \tag{1.5.19}$$

$$\implies \quad \mathbf{x} = x_4 \begin{pmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix} \tag{1.5.20}$$

by substituting $x_4 = 2$

$$\mathbf{x} = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 2 \end{pmatrix} \tag{1.5.21}$$

Hence, (1.5.2) finally becomes

$$2NaOH + H_2SO_4 \to Na_2SO_4 + 2H_2O \tag{1.5.22}$$

2 Miscellaneous

2.1. A trust fund has ₹30,000 that must be invested in two different types of bonds. The first bond pays 5% interest per year, and the second bond pays 7% interest per year. Using matrix multiplication, determine how to divide ₹ 30,000 among the two types of bonds if the trust fund must obtain an annual total interest of

a) ₹1800
b) ₹2000

Solution: Let

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{2.1.1}$$

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \tag{2.1.2}$$

so that ₹30000 is divided into two parts x_1 and x_2 in part $\mathbf{a}$), and into two part y_1 and y_2 in part $\mathbf{b}$). Then x_1, x_2, y_1, y_2 satisfy the following equations

$$x_1 + x_2 = 30000 \tag{2.1.3}$$

$$0.05x_1 + 0.07x_2 = 1800 \tag{2.1.4}$$

$$y_1 + y_2 = 30000 \tag{2.1.5}$$

$$0.05y_1 + 0.07y_2 = 2000 \tag{2.1.6}$$

which can be expressed as

$$\mathbf{A}\mathbf{x} = \mathbf{c}_1 = \begin{pmatrix} 30000 \\ 1800 \end{pmatrix} \tag{2.1.7}$$

$$\mathbf{A}\mathbf{y} = \mathbf{c}_2 = \begin{pmatrix} 30000 \\ 2000 \end{pmatrix} \tag{2.1.8}$$

and combined to obtain

$$\mathbf{A}\mathbf{X} = \mathbf{C} \tag{2.1.9}$$

where

$$\mathbf{X} = \begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix}, \tag{2.1.10}$$

$$\mathbf{C} = \begin{pmatrix} \mathbf{c}_1 & \mathbf{c}_2 \end{pmatrix} \tag{2.1.11}$$

Substituting numerical values,

$$\begin{pmatrix} 1 & 1 \\ 0.05 & 0.07 \end{pmatrix} \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} = \begin{pmatrix} 30000 & 30000 \\ 1800 & 2000 \end{pmatrix} \tag{2.1.12}$$

with the augmented matrix followed by row reduction

$$\begin{aligned} &\xleftrightarrow{R_2=R_2-0.05R_1} \left(\begin{array}{cc|cc} 1 & 1 & 30000 & 30000 \\ 0 & 0.02 & 300 & 500 \end{array}\right) \\ &\xleftrightarrow{R_2=50R_2} \left(\begin{array}{cc|cc} 1 & 1 & 30000 & 30000 \\ 0 & 1 & 15000 & 25000 \end{array}\right) \\ &\xleftrightarrow{R_1=R_1-R_2} \left(\begin{array}{cc|cc} 1 & 0 & 115000 & 5000 \\ 0 & 1 & 15000 & 25000 \end{array}\right) \end{aligned} \tag{2.1.13}$$

Thus, the desired division is

$$\mathbf{a} = \begin{pmatrix} 15000 \\ 15000 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 5000 \\ 25000 \end{pmatrix} \tag{2.1.14}$$

2.2. The bookshop of a particular school has 10 dozen chemistry books, 8 dozen physics books, 10 dozen economics books. Their selling prices are ₹80, ₹60 and ₹40 each respectively. Find the total amount the bookshop will receive from selling all the books using matrix algebra.
Solution: