Determinants

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Contents

Abstract—**This manual provides a simple introduction to determinants, based on exercises from the NCERT textbooks from Class 6-12.**

1 Determinants

1.1. Let

$$\mathbf{A} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}. \tag{1.1.1}$$

be a 3×3 matrix. Then,

$$\left|\mathbf{A}\right| = a_1 \begin{pmatrix} b_2 & c_2 \\ b_3 & c_3 \end{pmatrix} - a_2 \begin{pmatrix} b_1 & c_1 \\ b_3 & c_3 \end{pmatrix} + a_3 \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}. \tag{1.1.2}$$

1.2. Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of a matrix $\mathbf{A}$. Then, the product of the eigenvalues is equal to the determinant of $\mathbf{A}$.

$$\left|\mathbf{A}\right| = \prod_{i=1}^{n} \lambda_i \tag{1.2.1}$$

1.3.

$$\left|\mathbf{AB}\right| = \left|\mathbf{A}\right| \left|\mathbf{B}\right| \tag{1.3.1}$$

1.4. If $\mathbf{A}$ be an $n \times n$ matrix,

$$\left|k\mathbf{A}\right| = k^n \left|\mathbf{A}\right| \tag{1.4.1}$$

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2 Examples

2.1. Find $\begin{vmatrix} 2 & 4 \\ -5 & -1 \end{vmatrix}$

Solution:

2.2. (i) $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$ (ii) $\begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix}$

Solution:

2.3. If$\mathbf{A} = \begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix}$,then show that $|2\mathbf{A}| = 4\,|\mathbf{A}|$

Solution:

2.4. If $\mathbf{A} = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{vmatrix}$, then show that $|3\mathbf{A}| = 27\,|\mathbf{A}|$

Solution:

2.5. Evaluate the determinants

a) $\begin{vmatrix} 3 & -1 & -2 \\ 0 & 0 & -1 \\ 3 & -5 & 0 \end{vmatrix}$

b) $\begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix}$

Solution:

c) $\begin{vmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{vmatrix}$

d) $\begin{vmatrix} 2 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{vmatrix}$

2.6. If A=$\begin{vmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & -9 \end{vmatrix}$, find $|A|$

Solution:

2.7. Find the values of x,If
(i)$\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix}$ =$\begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$ (ii)$\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix}$ =$\begin{vmatrix} x & 3 \\ 2x & 5 \end{vmatrix}$

Solution:

2.8. If $\begin{vmatrix} x & 2 \\ 18 & x \end{vmatrix}$ =$\begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix}$, then x is equal to

a) 6
b) ± 6
c) -6
d) 0

2.9. $\begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix} = 0$

Solution:

2.10. $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$

Solution:

2.11. $\begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix} = 0$

Solution:

2.12. $\begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix} = 0$

Solution:

2.13. $\begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix}$=2$\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$

Solution:

2.14. $\begin{vmatrix} -a^2 & ab & ab \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix}$=$4a^2b^2c^2$

Solution: By Using properties of determinants, in Exercises 16 to 22,Show that;

2.15. (i)$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$=(a-b)(b-c)(c-a)

(ii) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$=(a-b)(b-c)(c-a)(a+b+c)

Solution:

2.16. $\begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix}$=(x-y)(y-z)(z-x)(xy+yz+zx)

2.17. (i) $\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$=$(5x+4)(4-x)^2$

Solution: (ii) $\begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & xy+k \end{vmatrix}$=$k^2(3y+k)$

Solution:

2.18. $\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix}$=$1+a^2+b^2+c^2$

Solution: Choose the correct answer in Exercises 23 and 24.

2.19. Let A be a square matrix of order 3X3, then $|kA|$ is equal to

a) $k|A|$
b) $k^2|A|$
c) $k^3|A|$
d) $3k|A|$

2.20. Which of the following is correct

a) Determinant is a square matrix.
b) Determinant is a number associated to a matrix.
c) Determinant is a number associated to a square matrix.
d) None of these.

2.21. Find area of the triangle with vertices at the point given in each of the following :

(i) $\begin{pmatrix} 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 6 & 0 \end{pmatrix}$, $\begin{pmatrix} 4 & 3 \end{pmatrix}$
(ii) $\begin{pmatrix} 2 & 7 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 \end{pmatrix}$, $\begin{pmatrix} 10 & 8 \end{pmatrix}$
(iii) $\begin{pmatrix} -2 & -3 \end{pmatrix}$, $\begin{pmatrix} 3 & 2 \end{pmatrix}$, $\begin{pmatrix} -1 & -8 \end{pmatrix}$

Solution:

a)

2.22. Show that points A=$\begin{pmatrix} a & b+c \end{pmatrix}$, B=$\begin{pmatrix} b & c+a \end{pmatrix}$, C=$\begin{pmatrix} c & a+b \end{pmatrix}$ are collinear.

Solution:

2.23. Find values of k if area of triangle is 4sq.units and vertices are

(i)) $\begin{pmatrix} k & 0 \end{pmatrix}$, $\begin{pmatrix} 4 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 2 \end{pmatrix}$
(ii) $\begin{pmatrix} -2 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 4 \end{pmatrix}$, $\begin{pmatrix} 0 & k \end{pmatrix}$

2.24. Find equation of line joining

a) $\begin{pmatrix} 1 & 2 \end{pmatrix}$ and $\begin{pmatrix} 3 & 6 \end{pmatrix}$
b) $\begin{pmatrix} 3 & 1 \end{pmatrix}$ and $\begin{pmatrix} 9 & 3 \end{pmatrix}$.

Solution:

a)
b)

2.25. If the area of triangle is 35 sq.units with vertices $\begin{pmatrix} 2 & -6 \end{pmatrix}$, $\begin{pmatrix} 5 & 4 \end{pmatrix}$ and $\begin{pmatrix} k & 4 \end{pmatrix}$.then k is

a) 12
b) -2
c) -12,-2
d) 12,-2

Solution:

Write Minors and Coafactors of the elements of following determinants:

2.26. (i) $\begin{vmatrix} 2 & -4 \\ 0 & 3 \end{vmatrix}$

(ii) $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$

2.27. (i) $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

(ii) $\begin{vmatrix} 1 & 0 & 4 \\ 3 & 5 & -1 \\ 0 & 1 & 2 \end{vmatrix}$

2.28. Using Cofactors of elements of second row,evaluate $\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$.

2.29. Using Cofactors of elements of third column ,evaluate $\Delta = \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}$.

2.30. If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ and A_{ij} is Cofactors of a_{ij} then value of Δ is given by

a) $a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33}$
b) $a_{11}A_{11} + a_{12}A_{21} + a_{13}A_{31}$
c) $a_{21}A_{11} + a_{22}A_{12} + a_{23}A_{13}$
d) $a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$

Find adjoint of each of the matrices

2.31. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

2.32. $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$ Verify A(adjA)=(adjA)A=$|A|\ I$

2.33. $\begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}$

2.34. $\begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}$

2.35. $\begin{bmatrix} 2 & -2 \\ 4 & 3 \end{bmatrix}$

2.36. $\begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix}$

2.37. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$

2.38. $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{bmatrix}$

2.39. $\begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{bmatrix}$

2.40. $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$

2.41. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{bmatrix}$

2.42. Let A= $\begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$ and B= $\begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}$. Verify that $(AB)^{-1} = B^{-1}A^{-1}$

2.43. Let A = $\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I = O$. Hence find A^{-1} **Solution:**

2.44. For the matrix A= $\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, find the numbers a and b such that $A^2 + aA + bI = O$.
Solution:

2.45. For the matrix A= $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{pmatrix}$.Show that

$$A^3 - 6A^2 + 5A + 11I = 0 \qquad (2.45.1)$$

and hence find A^{-1}. **Solution:**

2.46. Let A be a nonsingular square matrix of order 3X3 .Then $|adjA|$ is equal to

a) $|A|$
b) $|A|^2$
c) $|A|^3$
d) $3\,|A|$

2.47. If A is an invertible matrix of order 2, then det(A^{-1}) is equal to

a) det(A)
b) $\frac{1}{det(A)}$
c) 1
d) 0

Examine the consistency of the system of given Equations.

2.48. $x + 3y = 5$

$2x + 6y = 8$

Solution:

2.49. x+y+z=1
2x+3y+2z=2
ax+ay+2az=4

Solution:

2.50. 3x-y-2z=2
2y-z=-1
3x-5y=3

Solution:

2.51. 5x-y+4z=5
2x+3y+5z=2
5x-2y+6z=-1

Solution: Solve the system linear equations,using matrix method.

2.52. $5x+2y=4$
$7x+3y=5$
Solution:

2.53. $2x-y=-2$
$3x+4y=3$
Solution:

2.54. $4x-3y=3$
$3x-5y=7$
Solution:

2.55. $5x+2y=3$
$3x+2y=5$
Solution:

2.56. 2x+y+z = 1
x-2y-z = $\frac{3}{2}$
3y- 5z = 9

2.57. x-y+z = 4
2x+y-3z = 0
x+y+z = 2

2.58. 2x+3y+3z = 5
x-2y+z = -4
3x-y-2z = 3

2.59. x-y+2z = 7
3x+4y-5z = -5
2x-y+3z = 12

2.60. If A=$\begin{bmatrix}2 & -3 & 5\\ 3 & 2 & -4\\ 1 & 1 & -2\end{bmatrix}$, find A^{-1}. Using A^{-1} solve the system of equations
2x-3y+5z = 11,
3x+2y-4z = -5,
x+y-2z =-3.

2.61. The cost of 4 kg onion, 3 kg wheat and 2 kg rice is ₹60. The cost of 2 kg onion,4 kg wheat and 6 kg rice is ₹90.The cost of 6kg onion 2kg wheat and 3kg rice is ₹70.Find the cost of each item per kg by matrix mathod.

2.62. Prove that the determinant $\begin{vmatrix}x & \sin\theta & \cos\theta\\ -\sin\theta & -x & 1\\ \cos\theta & 1 & x\end{vmatrix}$ is independent of θ
Solution:

2.63. Without expanding the determinant, prove that $\begin{vmatrix}a & a^2 & bc\\ b & b^2 & ca\\ c & c^2 & ab\end{vmatrix} = \begin{vmatrix}1 & a^2 & a^3\\ 1 & b^2 & b^3\\ 1 & c^2 & c^3\end{vmatrix}$.
Solution:

2.64. Evaluate $\begin{vmatrix}\cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin\alpha\\ -\sin\beta & \cos\beta & 0\\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha\end{vmatrix}$.
Solution:

2.65. If a,b and c are real numbers, and $\Delta = \begin{vmatrix}b+c & c=a & a=b\\ c+a & a+b & b+c\\ a+b & b+c & c+a\end{vmatrix} = 0$, Show that either a+b+c=0 or a=b=c.
Solution:

2.66. Solve the equation
$\begin{vmatrix}x+a & x & x\\ x & x+a & x\\ x & x & x+a\end{vmatrix} = 0, a \neq 0$
Solution:

2.67. Prove that
$\begin{vmatrix}a^2 & bc & ac+c^2\\ a^2+ab & b^2 & ac\\ ab & b^2+bc & c^2\end{vmatrix} = 4a^2b^2c^2$
Solution:

2.68. If
$A^{-1} = \begin{bmatrix}3 & -1 & 1\\ -15 & 6 & -5\\ 5 & -2 & 2\end{bmatrix}$ and B=$\begin{bmatrix}1 & 2 & -2\\ -1 & 3 & 0\\ 0 & -2 & 1\end{bmatrix}$, find $(AB)^{-1}$

2.69. Let A= $\begin{bmatrix}1 & 2 & 1\\ 2 & 3 & 1\\ 1 & 1 & 5\end{bmatrix}$. Verify that
(i) $[adjA]^{-1} = adj(A)^{-1}$
(ii) $(A^{-1})^{-1} = A$

2.70. Evaluate $\begin{vmatrix}x & y & x+y\\ y & x+y & x\\ x+y & x & y\end{vmatrix}$
Solution:

2.71. Evaluate $\begin{vmatrix}1 & x & y\\ 1 & x+y & y\\ 1 & x & x+y\end{vmatrix}$ **Solution:** Using properties of determinants ,prove that:

2.72. $\begin{vmatrix}\alpha & \alpha^2 & \beta+\gamma\\ \beta & \beta^2 & \gamma+\alpha\\ \gamma & \gamma^2 & \alpha+\beta\end{vmatrix} = (\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)(\alpha+\beta+\gamma)$
Solution:

2.73. $\begin{vmatrix}x & x^2 & 1+px^3\\ y & y^2 & 1+py^3\\ z & z^2 & 1+pz^3\end{vmatrix} = (1+pxyz)(x-y)(y-z)(z-x)$, where p is any scalar.

Solution:

2.74. $\begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix}$=3(a+b+c)(ab+bc+ca)

Solution: