Matrices

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CONTENTS

***Abstract*—This book provides a computational approach to school geometry based on the NCERT textbooks from Class 6-12. Links to sample Python codes are available in the text.**

Download python codes using

```
svn co https://github.com/gadepall/school/trunk/
    ncert/computation/codes
```

1 Definitions

1.1 Eigenvalues and Eigenvectors

1.1.1. The eigenvalue λ and the eigenvector $\mathbf{x}$ for a matrix $\mathbf{A}$ are defined as,

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \tag{1.1.1.1}$$

1.1.2. The eigenvalues are calculated by solving the equation

$$f(\lambda) = \left|\lambda\mathbf{I} - \mathbf{A}\right| = 0 \tag{1.1.2.1}$$

The above equation is known as the characteristic equation.

1.1.3. According to the Cayley-Hamilton theorem,

$$f(\lambda) = 0 \implies f(\mathbf{A}) = 0 \tag{1.1.3.1}$$

1.1.4. The trace of a square matrix is defined to be the sum of the diagonal elements.

$$\operatorname{tr}(\mathbf{A}) = \sum_{i=1}^{N} a_{ii}. \tag{1.1.4.1}$$

where a_{ii} is the ith diagonal element of the matrix $\mathbf{A}$.

1.1.5. The trace of a matrix is equal to the sum of the eigenvalues

$$\operatorname{tr}(\mathbf{A}) = \sum_{i=1}^{N} \lambda_i \tag{1.1.5.1}$$

1.2 Determinants

1.2.1. Let

$$\mathbf{A} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}. \tag{1.2.1.1}$$

be a 3×3 matrix. Then,

$$\left|\mathbf{A}\right| = a_1 \begin{pmatrix} b_2 & c_2 \\ b_3 & c_3 \end{pmatrix} - a_2 \begin{pmatrix} b_1 & c_1 \\ b_3 & c_3 \end{pmatrix} + a_3 \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}. \tag{1.2.1.2}$$

1.2.2. Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of a matrix $\mathbf{A}$. Then, the product of the eigenvalues is equal to the determinant of $\mathbf{A}$.

$$\left|\mathbf{A}\right| = \prod_{i=1}^{n} \lambda_i \tag{1.2.2.1}$$

1.2.3.

$$\left|\mathbf{AB}\right| = \left|\mathbf{A}\right|\left|\mathbf{B}\right| \tag{1.2.3.1}$$

1.2.4. If $\mathbf{A}$ be an $n \times n$ matrix,

$$\left|k\mathbf{A}\right| = k^n \left|\mathbf{A}\right| \tag{1.2.4.1}$$

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1.3 Inverse of a Matrix

1.3.1. For a 2×2 matrix

$$\mathbf{A} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}, \quad (1.3.1.1)$$

the inverse is given by

$$\mathbf{A}^{-1} = \frac{1}{\left|\mathbf{A}\right|} \begin{pmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{pmatrix}, \quad (1.3.1.2)$$

1.3.2. For higher order matrices, the inverse should be calculated using row operations.

2 Cayley-Hamilton Theorem

2.1. If

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{pmatrix}, \quad (2.1.1)$$

prove that

$$\mathbf{A}^3 - 6\mathbf{A}^2 + 7\mathbf{A} + 2\mathbf{I} = 0 \quad (2.1.2)$$

Solution: From (1.1.2.1), the characteristic equation is

$$\begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & 2-\lambda & 1 \\ 2 & 0 & 3-\lambda \end{vmatrix} = 0 \quad (2.1.3)$$

which can be expanded to obtain

$$(1-\lambda)(2-\lambda)(3-\lambda) + 2(-2(2-\lambda)) = 0 \quad (2.1.4)$$

yielding

$$\lambda^3 - 6\lambda^2 + 7\lambda + 2 = 0 \quad (2.1.5)$$

upon simplification. Using the Cayley-Hamilton theorem in (1.1.3.1), (2.1.2) is obtained

2.2. If

$$\mathbf{A} = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}, \quad (2.2.1)$$

show that

$$\mathbf{A}^2 - 5\mathbf{A} + 7\mathbf{I} = 0 \quad (2.2.2)$$

Solution: The characteristic equation is

$$\left|\mathbf{A} - \lambda\mathbf{I}\right| = 0 \quad (2.2.3)$$

$$\implies \begin{vmatrix} 3-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} = 0 \quad (2.2.4)$$

$$\implies (3-\lambda)(2-\lambda) + 1 = 0 \quad (2.2.5)$$

$$\text{or, } \lambda^2 - 5\lambda + 7 = 0 \quad (2.2.6)$$

Using the Cayley-Hamilton theorem, (2.2.2) is obtained.

2.3. If $\mathbf{A}$ is square matrix such that

$$\mathbf{A}^2 = \mathbf{A}, \quad (2.3.1)$$

then

$$g(\mathbf{A}) = (\mathbf{I} + \mathbf{A})^3 - 7\mathbf{A} \quad (2.3.2)$$

is equal to

a) $\mathbf{A}$
b) $\mathbf{I} - \mathbf{A}$
c) $\mathbf{I}$
d) $3\mathbf{A}$

Solution: From (2.3.2),

$$g(\lambda) = (1+\lambda)^3 - 7\lambda \quad (2.3.3)$$

Also,

$$\mathbf{A}^2 = \mathbf{A} \implies \mathbf{A}^2 - \mathbf{A} = 0 \quad (2.3.4)$$

Using the Cayley-Hamilton theorem, the eigenvalues satisfy the characteristic equation

$$f(\lambda) = \lambda^2 - \lambda = 0 \quad (2.3.5)$$

$\because f(\lambda)$ is of degree 2 and $g(\lambda)$ is of degree 3, (2.3.3) can be expressed as

$$g(\lambda) = f(\lambda)\, q(\lambda) + a\lambda + b \quad (2.3.6)$$

where a, b are real numbers and $q(\lambda)$ is some polynomial. Thus,

$$g(0) = b = 1 \quad (2.3.7)$$

$$g(1) = a + b = 1 \quad (2.3.8)$$

$$\implies a = 90, b = 1 \quad (2.3.9)$$

Thus,

$$g(\mathbf{A}) = f(\mathbf{A})\, q(\mathbf{A}) + a\mathbf{A} + b\mathbf{I} \quad (2.3.10)$$

$$= \mathbf{I} \quad (2.3.11)$$

upon substituting from (2.3.5) and (2.3.9). Option c is the valid answer.

2.4. If

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{pmatrix} \quad (2.4.1)$$

then show that

$$\mathbf{A}^3 - 23\mathbf{A} - 40\mathbf{I} = 0 \quad (2.4.2)$$

Solution: The Characteristic equation is given by

$$|\mathbf{A} - \lambda\mathbf{I}| = 0 \quad (2.4.3)$$

$$\implies \begin{vmatrix} 1-\lambda & 2 & 3 \\ 3 & -2-\lambda & 1 \\ 4 & 2 & 1-\lambda \end{vmatrix} = 0 \quad (2.4.4)$$

which can be expressed as

$$\implies (1-\lambda)\left((-2-\lambda)(1-\lambda)-2\right) - 2\left(3(1-\lambda)-4\right) + 3\left(6+4(2+\lambda)\right) = 0 \quad (2.4.5)$$

and simplified to obtain

$$\implies \lambda^3 - 23\lambda - 40 = 0. \quad (2.4.6)$$

Using the Cayley-Hamilton Theorem, (2.4.2) is obtained.

3 Matrix Polynomials

3.1. Find $\mathbf{A}^2 - 5\mathbf{A} + 6\mathbf{I}$,if $\mathbf{A} = \begin{pmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{pmatrix}$

Solution: Factorizing the corresponding polynomial,

$$f(x) = x^2 - 5x + 6 \quad (3.1.1)$$

$$= (x-3)(x-2) \quad (3.1.2)$$

$$\implies \mathbf{A}^2 - 5\mathbf{A} + 6\mathbf{I} = (\mathbf{A} - 3\mathbf{I})(\mathbf{A} - 5\mathbf{I}) \quad (3.1.3)$$

$$\mathbf{A} - 3\mathbf{I} = \begin{pmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{pmatrix} + \begin{pmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix} \quad (3.1.4)$$

$$= \begin{pmatrix} -1 & 0 & 1 \\ 2 & -2 & 3 \\ 1 & -1 & -3 \end{pmatrix} \quad (3.1.5)$$

$$\mathbf{A} - 5\mathbf{I} = \begin{pmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{pmatrix} + \begin{pmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{pmatrix} \quad (3.1.6)$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 2 & -1 & 3 \\ 1 & -1 & -2 \end{pmatrix} \quad (3.1.7)$$

Multiplying the above expressions,

$$\mathbf{A}^2 - 5\mathbf{A} + 6\mathbf{I} = \begin{pmatrix} -1 & 0 & 1 \\ 2 & -2 & 3 \\ 1 & -1 & -3 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 2 & -1 & 3 \\ 1 & -1 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 & -3 \\ -1 & -1 & -10 \\ -5 & 4 & 4 \end{pmatrix} \quad (3.1.8)$$

4 Matrix Inverse

4.1. Using elementary transforamtions,find the inverse of $\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$

Solution: Given that

$$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} \quad (4.1.1)$$

The augmented matrix $[\mathbf{A}|\mathbf{I}]$ is as given below:-

$$\left(\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right) \quad (4.1.2)$$

We apply the elementary row operations on $[\mathbf{A}|\mathbf{I}]$ as follows :-

$$[\mathbf{A}|\mathbf{I}] = \left(\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right) \quad (4.1.3)$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - 2R_1} \left(\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & 5 & -2 & 1 \end{array}\right) \quad (4.1.4)$$

$$\xleftrightarrow{R_2 \leftarrow \frac{R_2}{5}} \left(\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & 1 & \frac{-2}{5} & \frac{1}{5} \end{array}\right) \quad (4.1.5)$$

$$\xleftrightarrow{R_2 \leftarrow R_1 + R_2} \left(\begin{array}{cc|cc} 1 & 0 & \frac{3}{5} & \frac{1}{5} \\ 0 & 1 & \frac{-2}{5} & \frac{1}{5} \end{array}\right) \quad (4.1.6)$$

By performing elementary transformations on augmented matrix$[\mathbf{A}|\mathbf{I}]$, we obtained the augmented matrix in the form $[\mathbf{I}|\mathbf{A}]$. Hence we can conclude that the matrix A is invertible and

$$\mathbf{A^{-1}} = \begin{pmatrix} \frac{3}{5} & \frac{1}{5} \\ \frac{-2}{5} & \frac{1}{5} \end{pmatrix} \quad (4.1.7)$$

4.2. Using elementary transforamtions, find the inverse of $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

Solution: Given that

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad (4.2.1)$$

The augmented matrix $[\mathbf{A}|\mathbf{I}]$ is as given below:-

$$\left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right) \tag{4.2.2}$$

We apply the elementary row operations on $[\mathbf{A}|\mathbf{I}]$ as follows :-

$$[\mathbf{A}|\mathbf{I}] = \left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right) \tag{4.2.3}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - R_2} \left(\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 1 & 1 & 0 & 1 \end{array}\right) \tag{4.2.4}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - R_1} \left(\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 2 \end{array}\right) \tag{4.2.5}$$

By performing elementary transformations on augmented matrix$[\mathbf{A}|\mathbf{I}]$, we obtained the augmented matrix in the form $[\mathbf{I}|\mathbf{A}]$. Hence we can conclude that the matrix A is invertible and inverse of the matrix is

$$\mathbf{A}^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \tag{4.2.6}$$

4.3. Obtain the inverse of the following matrix using elementary operations
A=$\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix}$.
Solution: Given that

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix}, \tag{4.3.1}$$

The augmented matrix $[\mathbf{A}|\mathbf{I}]$ is

$$\left(\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \tag{4.3.2}$$

Applying elementary row operations on $[\mathbf{A}|\mathbf{I}]$,

$$[\mathbf{A}|\mathbf{I}] = \left(\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \tag{4.3.3}$$

$$\xleftrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \tag{4.3.4}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - 3R_1} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -5 & -8 & 0 & -3 & 1 \end{array}\right) \tag{4.3.5}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - 2R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & -2 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -5 & -8 & 0 & -3 & 1 \end{array}\right) \tag{4.3.6}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 + 5R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & -2 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 5 & -3 & 1 \end{array}\right) \tag{4.3.7}$$

$$\xleftrightarrow{R_3 \leftarrow R_3/2} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & -2 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{array}\right) \tag{4.3.8}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{array}\right) \tag{4.3.9}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - 2R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & -4 & 3 & -1 \\ 0 & 0 & 1 & \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{array}\right) \tag{4.3.10}$$

By performing elementary transformations on augmented matrix$[\mathbf{A}|\mathbf{I}]$, we obtained the augmented matrix in the form $[\mathbf{I}|\mathbf{A}]$. Hence we can conclude that the matrix A is invertible and inverse of the matrix is

$$\mathbf{A}^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{pmatrix} \tag{4.3.11}$$

4.4. Find P^{-1}, if it exists, given
P=$\begin{pmatrix} 10 & -2 \\ -5 & 1 \end{pmatrix}$.

Solution: Using row reduction,

$$\begin{pmatrix} 10 & -2 \\ -5 & 1 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 + \frac{R_1}{2}} \begin{pmatrix} 10 & -2 \\ 0 & 0 \end{pmatrix} \quad (4.4.1)$$

Since we obtain a zero row, $\mathbf{P}^{-1}$ does not exist.