DIGITAL COMMUNICATION

Through Simulations

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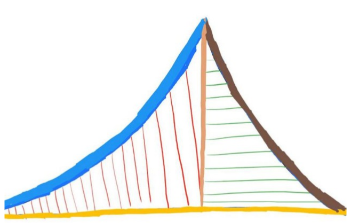

Contents

Introduction

This book introduces digital communication through probability.

Chapter 1

Axioms

1.1. Examples

1.1.1 Why is tossing a coin considered to be a fair way of deciding which team should get the ball at the beginning of a football game?

Solution: Let X be a random variable which takes the values 0 and 1.

$$X = \begin{cases} 1, & \text{if coin toss results in Head} \\ 0, & \text{if coin toss results in Tail} \end{cases} \tag{1.1}$$

From law of total probability,

$$\Pr(X = 0) + \Pr(X = 1) = 1 \tag{1.2}$$

Since there is only one head,

$$\Pr(X = 1) = \frac{1}{2} \tag{1.3}$$

Similarly,

$$\Pr(X=0) = 1 - \Pr(X=1) = \frac{1}{2} \tag{1.4}$$

Thus,

$$\Pr(X=0) = \Pr(X=1) \tag{1.5}$$

which is why tossing the coin is a fair way to decide.

1.1.2 Which of the following cannot be the probability of an event ?

(a) $\frac{2}{3}$

(b) -1.5

(c) 15%

(d) 0.7

Solution: From the axioms of probability,

$$0 \leq \Pr(E) \leq 1 \tag{1.6}$$

(a) $\Pr(E) = \frac{2}{3}$

$$\because 0 \leq \frac{2}{3} \leq 1 \tag{1.7}$$

from (1.6), it can be probability of an event.

(b) $\Pr(E) = -1.5$

$$\because -1.5 < 0 \tag{1.8}$$

from (1.6), it cannot be a probability of any event.

(c)

$$\Pr(E) = \frac{15}{100} \tag{1.9}$$

$$\because 0 \leq \frac{15}{100} \leq 1, \tag{1.10}$$

from (1.6), it can be probability of an event.

(d) $\Pr(E) = 0.7$

$$\because 0 \leq 0.7 \leq 1 \tag{1.11}$$

from (1.6), it can be a probability of an event.

1.1.3 If P(E) = 0.05, what is the probability of 'not E'?

Solution: The desired probability is

$$\Pr\left(E'\right) = 1 - \Pr(E) = 0.95 \tag{1.12}$$

1.1.4 A bag contains lemon flavoured candies only. Malini takes out one candy without looking into the bag. What is the probability that she takes out

(a) an orange flavoured candy?

(b) a lemon flavoured candy?

Solution:

$$\Pr(O) = 0 \tag{1.13}$$

$$\Pr(L) = 1 \tag{1.14}$$

1.1.5 It is given that in a group of 3 students, the probability of 2 students not having the same birthday is 0.992. What is the probability that the 2 students have the same birthday?

Solution: Let E be the event that no 2 students in a group of 3 share a birthday. Then

$$\Pr(E) = 0.992 \implies \Pr\left(E'\right) = 1 - \Pr(E) = 0.008 \tag{1.15}$$

1.1.6 A bag contains 3 red balls and 5 black balls. A ball is drawn at random from the bag. What is the probability that the ball drawn is

(i) red ?

(ii) not red?

Solution: Let

$$X = \begin{cases} 1 & \text{if drawn ball is red} \\ 0 & \text{otherwise.} \end{cases} \tag{1.16}$$

(i) Probability that the drawn ball is red

$$\Pr(X = 1) = \frac{3}{8} \tag{1.17}$$

$$\tag{1.18}$$

(ii) Probability that the drawn ball is not red

$$\Pr(X = 0) = 1 - \frac{3}{8} = \frac{5}{8} \tag{1.19}$$

1.1.7 Gopi buys a fish from a shop for his aquarium. The shopkeeper takes out one fish at random from a tank containing 5 male fish and 8 female fish . What is the probability that the fish taken out is a male fish?

Solution:

Let

$$X = \begin{cases} 1, & \text{if the chosen fish is male} \\ 0, & \text{if the chosen fish is female} \end{cases} \tag{1.20}$$

Then

$$\Pr(X = 1) = \frac{5}{13} \tag{1.21}$$

1.1.8 A box contains 12 balls, out of which x are black. If one ball is drawn at random from the box, what is the probability that it will be a black ball? If 6 more black balls are put in the box, the probability of drawing a black ball is now double of what it was before. Find x.

Solution: From Table 1.1,

Random Variable	Sample space	Value	Event	Probability
X_1	12	0	not choosing black ball	12-x/12
		1	choosing black ball	x/12
X_2	18	0	not choosing black ball	12-x/18
		1	choosing black ball	x+6/18

Table 1.1:

$$\Pr\left(X_1 = 1\right) = \frac{x}{12} \tag{1.22}$$

Since

$$\Pr\left(X_2 = 1\right) = 2\Pr\left(X_1 = 1\right), \tag{1.23}$$

$$\frac{x+6}{18} = 2\left(\frac{x}{12}\right) \tag{1.24}$$

$$\implies x = 3 \tag{1.25}$$

1.1.9 A letter is chosen at random from the word ‘ASSASSINATION’. Find the probability that letter is

(a) a vowel

(b) a consonant

. **Solution:** The number of vowels is 6 and consonants is 7.

(a)

$$\Pr\left(X\right) = \frac{6}{13} \tag{1.26}$$

(b)

$$\Pr(Y) = \frac{7}{13} \tag{1.27}$$

1.1.10 In a lottery, a person choses six different natural numbers at random from 1 to 20, and if these six numbers match with the six numbers already fixed by the lottery committee, he wins the prize. What is the probability of winning the prizes in the game? [Hint : order of the numbers is not important.]

Solution: The desired probability is given by

$$\frac{1}{{}^{20}C_6} = \frac{1}{38,760} = 0.0000258 \tag{1.28}$$

1.1.11 Check whether the following probabilities $\Pr(A)$ and $\Pr(B)$ are consistently defined

(a) $\Pr(A) = 0.5$, $\Pr(B) = 0.7$, $\Pr(A \cap B) = 0.6$

(b) $\Pr(A) = 0.5$, $\Pr(B) = 0.7$, $\Pr(A \cup B) = 0.8$

Solution: To check whether the given probabilities are consistently defined, we check whether the following property holds correctly with the probability axioms

$$\Pr(A + B) = \Pr(A) + \Pr(B) - \Pr(AB) \tag{1.29}$$

(a) Given that

$$\Pr(A) = 0.5,\ \Pr(B) = 0.7,\ \Pr(AB) = 0.6 \tag{1.30}$$

From (1.29),

$$\Pr(A+B) = 0.5 + 0.7 - 0.6 = 0.6 \tag{1.31}$$

From (1.31) we have

$$0 \le \Pr(A+B) \le 1 \tag{1.32}$$

Hence the given probabilities are consistently defined.

(b) Given that

$$\Pr(A) = 0.5,\ \Pr(B) = 0.7,\ \Pr(A+B) = 0.8 \tag{1.33}$$

From (1.29) we get,

$$\Pr(AB) = 0.5 + 0.7 - 0.8 \tag{1.34}$$

$$= 0.4 \tag{1.35}$$

From (1.35) we have

$$0 \le \Pr(AB) \le 1 \tag{1.36}$$

Hence the given probabilities are consistently defined

1.1.12 Given $\Pr(A) = \frac{3}{5}$ and $\Pr(B) = \frac{1}{5}$. Find $\Pr(A+B)$ if A and B are mutually exclusive events.

Solution: Since $AB = 0$,

$$\Pr(A+B) = \Pr(A) + \Pr(B) = \frac{3}{5} + \frac{1}{5} = \frac{4}{5} \tag{1.37}$$

1.1.13 If E and F are events such that $\Pr(E) = \frac{1}{4}$, $\Pr(F) = \frac{1}{2}$ and $\Pr(EF) = \frac{1}{8}$, find

(a) $\Pr(E+F)$

(b) $\Pr(E'F')$

Solution:

(a)

$$\Pr(E+F) = \Pr(E) + \Pr(F) - \Pr(EF) = \frac{1}{4} + \frac{1}{2} - \frac{1}{8} = \frac{5}{8} \tag{1.38}$$

$$\tag{1.39}$$

(b)

$$(E'F') = (E+F)' \implies \Pr\left(E'F'\right) = \Pr\left((E+F)'\right) = 1 - \Pr(E+F) \tag{1.40}$$

$$\implies \Pr\left(E'F'\right) = 1 - \frac{5}{8} = \frac{3}{8} \tag{1.41}$$

1.1.14 Events E and F are such that $\Pr(E' + F') = 0.25$, state whether E and F are mutually exclusive.

Solution:

$$\Pr\left(E' + F'\right) = \Pr\left(EF\right)' = 1 - \Pr\left(EF\right) = 0.25 \tag{1.42}$$

$$\implies \Pr\left(EF\right) = 0.75 \tag{1.43}$$

$$\because \Pr\left(EF\right) \neq 0 \tag{1.44}$$

E and F are not mutually exclusive events.

1.1.15 A and B are events such that $\Pr\left(A\right) = 0.42, \Pr\left(B\right) = 0.48$ and $\Pr\left(\text{A and B}\right) = 0.16$. Determine

(a) $\Pr\left(\text{not A}\right)$

(b) $\Pr\left(\text{not B}\right)$

(c) $\Pr\left(\text{A or B}\right)$

Solution: Solution:

(a) $\Pr\left(\text{not A}\right)$

$$\Pr\left(A'\right) = 1 - \Pr\left(A\right) \tag{1.45}$$

$$= 1 - 0.42 \tag{1.46}$$

$$= 0.58 \tag{1.47}$$

(b) $\Pr\left(\text{not B}\right)$

$$\Pr\left(B'\right) = 1 - \Pr\left(B\right) \tag{1.48}$$

$$= 1 - 0.48 \tag{1.49}$$

$$= 0.52 \tag{1.50}$$

(c) Pr (A or B)

$$\Pr\left(\text{A+B}\right) = \Pr\left(A\right) + \Pr\left(B\right) - \Pr\left(AB\right) \tag{1.51}$$

$$= 0.42 + 0.48 - 0.16 \tag{1.52}$$

$$= 0.74 \tag{1.53}$$

1.1.16 If A and B are two independent events with $\Pr\left(A\right) = \frac{3}{5}$ and $\Pr\left(B\right) = \frac{4}{9}$ then, $\Pr\left(A'B'\right)$

(a) $\frac{4}{15}$

(b) $\frac{8}{45}$

(c) $\frac{1}{3}$

(d) $\frac{2}{9}$

Solution: Using de morgan's law and axioms of probability,

$$\Pr\left(A'B'\right) = \Pr\left((A+B)'\right) \tag{1.54}$$

$$\Pr\left(A+B\right) = \Pr\left(A\right) - \Pr\left(B\right) - \Pr\left(AB\right) \tag{1.55}$$

Also, As A and B are independent,

$$\Pr\left(AB\right) = \Pr\left(A\right)\Pr\left(B\right) \tag{1.56}$$

Now,using the equations of (1.54), (1.55) and (1.56)

$$\Pr\left(A'B'\right) = \Pr\left((A+B)'\right) \tag{1.57}$$
$$= 1 - \Pr\left((A+B)\right) \tag{1.58}$$
$$= 1 - \Pr(A) - \Pr(B) - \Pr(A)\Pr(B) \tag{1.59}$$
$$= 1 - \left(\frac{3}{5}\right) - \left(\frac{4}{9}\right) - \left(\frac{3}{5}\right)\left(\frac{4}{9}\right) \tag{1.60}$$
$$= \frac{2}{9} \tag{1.61}$$

so option (4) is correct

1.1.17 Let A and B be two events such that $\Pr(A) = \frac{3}{8}$, $\Pr(B) = \frac{5}{8}$ and $\Pr(A+B) = \frac{3}{4}$. Then $\Pr(A|\, B) . \Pr(A'|\, B)$ is equal to

(a) $\frac{2}{5}$

(b) $\frac{3}{8}$

(c) $\frac{3}{20}$

(d) $\frac{6}{25}$

Solution: Given

$$\Pr(A) = \frac{3}{8} \tag{1.62}$$
$$\Pr(B) = \frac{5}{8} \tag{1.63}$$
$$\Pr(A+B) = \frac{3}{4} \tag{1.64}$$

As we know

$$\Pr(A+B) = \Pr(A) + \Pr(B) - \Pr(AB) \tag{1.65}$$

$$\implies \Pr(AB) = \Pr(A) + \Pr(B) - \Pr(A+B) \tag{1.66}$$

$$\implies \Pr(AB) = \frac{1}{4} \tag{1.67}$$

Now,

$$\Pr(A|\,B) = \frac{\Pr(AB)}{\Pr(B)} \tag{1.68}$$

$$\Pr\left(A'|\,B\right) = \frac{\Pr(A'B)}{\Pr(B)} \tag{1.69}$$

$$= \frac{\Pr(B) - \Pr(AB)}{\Pr(B))} \tag{1.70}$$

From (1.68) and (1.70)

$$\Pr(A|\,B) \,.\, \Pr\left(A'|\,B\right) = \frac{\Pr(AB)}{\Pr(B)} \times \frac{\Pr(B) - \Pr(AB)}{\Pr(B)} \tag{1.71}$$

$$= \frac{\left(\frac{1}{4}\right)}{\frac{5}{8}} \times \frac{\left(\frac{5}{8} - \frac{1}{4}\right)}{\frac{5}{8}} \tag{1.72}$$

$$= \frac{6}{25} \tag{1.73}$$

1.1.18 In class XI of a school 40% of the students study Mathematics and 30% study Biology. 10% of the class study both Mathematics and Biology. If a student is selected at random from the class,find the probability that he will be studying Mathematics or Biology

Solution: The given information is summarised in Table 1.3. Thus,

Random Variable	**Subject**	**Probability**
M	Mathematics	$\Pr(M)$=0.4
B	Biology	$\Pr(B)$=0.3
M, B	Both	$\Pr(MB)$=0.10

Table 1.3:

$$\Pr(M+B) = \Pr(M) + \Pr(B) - \Pr(M, B) \tag{1.74}$$

$$= 0.6 \tag{1.75}$$

1.1.19 In an entrance test that is graded on the basis of two examinations, the probability of a randomly chosen student passing the first examination is 0.8 and the probability of passing the second examination is 0.7. The probability of passing atleast one of them is 0.95. What is the probability of passing both?

Solution: Let

A : Probability of random student passing the first exam

B : Probability of random student passing the second exam

Given

$$\Pr(A) = 0.8, \Pr(B) = 0.7, \Pr(A+B) = 0.95. \tag{1.76}$$

$$\because \Pr(A+B) = \Pr(A) + \Pr(B) - \Pr(AB), \tag{1.77}$$

$$\implies \Pr(AB) = \Pr(A) + \Pr(B) - \Pr(A+B) \tag{1.78}$$

$$= 0.55 \tag{1.79}$$

1.1.20 In a class of 60 students, 30 opted for NCC, 32 opted for NSS and 24 opted for both NCC and NSS. If one of these students is selected at random, find the probability that

(a) The student opted for NCC or NSS.

(b) The student has opted neither NCC nor NSS.

(c) The student has opted NSS but not NCC.

Solution: Define random variables X and Y as shown in Tables 1.4 and 1.5. From

$X = 0$	Student does not opt for NCC.
$X = 1$	Student opts for NCC.

Table 1.4: Definition of X.

$Y = 0$	Student does not opt for NSS.
$Y = 1$	Student opts for NSS.

Table 1.5: Definition of Y.

the given data

$$\Pr(X = 1) = \frac{30}{60} = \frac{1}{2} \tag{1.80}$$

$$\Pr(Y = 1) = \frac{32}{60} = \frac{8}{15} \tag{1.81}$$

$$\Pr(X = 1,\ Y = 1) = \frac{24}{60} = \frac{2}{5} \tag{1.82}$$

Thus, we write

$$\Pr(X=1,\ Y=0) = \Pr(X=1) - \Pr(X=1,\ Y=1) = \frac{1}{10} \tag{1.83}$$

$$\Pr(X=0,\ Y=1) = \Pr(Y=1) - \Pr(X=1,\ Y=1) = \frac{2}{15} \tag{1.84}$$

$$\Pr(X=0,\ Y=0) = \Pr(Y=0) - \Pr(X=1,\ Y=0) \tag{1.85}$$

$$= 1 - \Pr(Y=1) - \Pr(X=1,\ Y=0) \tag{1.86}$$

$$= 1 - \frac{8}{15} - \frac{1}{10} = \frac{11}{30} \tag{1.87}$$

and form the joint pmf as in Table 1.6.

	$X=0$	$X=1$
$Y=0$	$\frac{11}{30}$	$\frac{1}{10}$
$Y=1$	$\frac{2}{15}$	$\frac{2}{5}$

Table 1.6: Joint pmf of X and Y.

(a) From Table 1.6,

$$\Pr(X+Y \geq 1) = 1 - \Pr(X+Y=0) \tag{1.88}$$

$$= \frac{19}{30} \tag{1.89}$$

(b) From Table 1.6,

$$\Pr(X=0,\ Y=0) = \frac{11}{30} \tag{1.90}$$

(c) From Table 1.6,

$$\Pr(X=0,\ Y=1) = \frac{2}{15} \tag{1.91}$$

1.1.21 A die has two faces each with number '1', three faces each with number '2' and one face with number '3'. If die is rolled once, determine

(a) $\Pr(2)$

(b) $\Pr(1 \text{ or } 3)$

(c) $\Pr(\text{not } 3)$

Solution: The given information is summarized in the following table 1.7

RV	**Description**	**Probability**
$X = 1$	Die rolls to 1	$\frac{1}{3}$
$X = 2$	Die rolls to 2	$\frac{1}{2}$
$X = 3$	Die rolls to 3	$\frac{1}{6}$

Table 1.7: Random variable X

(a)

$$\Pr(X = 2) = \frac{1}{2} \tag{1.92}$$

(b) Since

$$X = 1 \text{ or } X = 3 \equiv X \in \{1, 3\} \tag{1.93}$$

$$X = 1 \text{ and } X = 3 \equiv X = \phi \tag{1.94}$$

$$\Pr(X \in \{1, 3\}) = \Pr(X = 1) + \Pr(X = 3) - \Pr(X = \phi) \tag{1.95}$$

$$= \frac{1}{3} + \frac{1}{6} \tag{1.96}$$

$$= \frac{1}{2} \tag{1.97}$$

(c)

$$\Pr(X \neq 3) = 1 - \Pr(X = 3) \tag{1.98}$$
$$= 1 - \frac{1}{6} \tag{1.99}$$
$$= \frac{5}{6} \tag{1.100}$$

1.1.22 A and B are two events such that $\Pr(A) = 0.54$, $\Pr(B) = 0.69$ and $\Pr(AB) = 0.35$. Find

(a) $\Pr(A + B)$

(b) $\Pr(A'B')$

(c) $\Pr(AB')$

(d) $\Pr(BA')$

Solution:

(a)

$$\Pr(A + B) = \Pr(A) + \Pr(B) - \Pr(AB) \tag{1.101}$$
$$= 0.54 + 0.69 - 0.35 = 0.88 \tag{1.102}$$

(b) By De Morgan's Law,

$$A'B' = (A + B)' \tag{1.103}$$
$$\implies \Pr\left(A'B'\right) = \Pr(A + B)' \tag{1.104}$$
$$= 1 - \Pr(A + B) \tag{1.105}$$
$$= 1 - 0.88 = 0.12 \tag{1.106}$$

(c) We know that,

$$B + B' = 1BB' = 0 \tag{1.107}$$

$$\implies A = A(B + B') = AB + AB' \tag{1.108}$$

$$\implies \Pr(A) = \Pr(AB) + \Pr\left(AB'\right) - \Pr\left(ABB'\right) \tag{1.109}$$

$$= \Pr(AB) + \Pr\left(AB'\right) \tag{1.110}$$

$$\implies \Pr\left(AB'\right) = \Pr(A) - \Pr(AB) \tag{1.111}$$

$$= 0.54 - 0.35 = 0.19 \tag{1.112}$$

(d) From (1.111),

$$\Pr\left(BA'\right) = \Pr(B) - \Pr(AB) = 0.69 - 0.35 = 0.34. \tag{1.113}$$

1.1.23 If $\Pr(A) = \frac{3}{5}$ and $\Pr(B) = \frac{1}{5}$ find $\Pr(A \cap B)$ if A and B are independent events.

Solution: Since the events A, B are independent, we have

$$\Pr(AB) = \Pr(A)\Pr(B) = \frac{3}{25} \tag{1.114}$$

1.1.24 A die marked 1, 2, 3 in red and 4, 5, 6 in green is tossed. Let A be the event,‘the number is even,’ and B be the event, ‘the number is red’. Are A and B independent?

Solution: Let

$$X = \begin{cases} 0, & \text{if number is odd} \\ 1, & \text{if number is even} \end{cases} \tag{1.115}$$

$$Y = \begin{cases} 0, & \text{if number is green} \\ 1, & \text{if number is red} \end{cases} \tag{1.116}$$

From the given information,

$$\Pr(X = 1) = \frac{3}{6} = \frac{1}{2}, \Pr(Y = 1) = \frac{3}{6} = \frac{1}{2} \tag{1.117}$$

$$\Pr(X = 1, Y = 1) = \frac{1}{6} \tag{1.118}$$

Now,

$$\Pr(X = 1) \times \Pr(Y = 1) = \frac{1}{4} \tag{1.119}$$

$$\implies \Pr(X = 1, Y = 1) \neq \Pr(X = 1) \times \Pr(Y = 1) \tag{1.120}$$

Hence, A and B are not independent.

1.1.25 Let E and F be events with $\Pr(E) = \frac{3}{5}, \Pr(F) = \frac{3}{10}$ and $\Pr(EF) = \frac{1}{5}$. Are E and F independent?

Solution: From the given information,

$$\Pr(E)\Pr(F) = \frac{3}{5} \times \frac{9}{50} \tag{1.121}$$

$$\Pr(EF) = \frac{1}{50} \tag{1.122}$$

$$\implies \Pr(EF) \neq P(E)P(F) \tag{1.123}$$

$\therefore E$ and F are not independent events.

1.1.26 Given that the events A and B are such that $P(A) = \frac{1}{2}, P(A+B) = \frac{3}{5}$ and $P(B) = p$. Find p if they are

(a) mutually exclusive

(b) independent

Solution: Solution:

(a) In this case

$$\Pr(A+B) = \Pr(A) + \Pr(B) \tag{1.124}$$
$$\implies \frac{3}{5} = \frac{1}{2} + p \tag{1.125}$$
$$\therefore p = \frac{1}{10} \tag{1.126}$$

(b) Given A and B are independent events, then,

$$\Pr(A+B) = \Pr(A) + \Pr(B) - \Pr(AB) \tag{1.127}$$
$$\implies \Pr(A+B) = \Pr(A) + \Pr(B) - \Pr(A)\Pr(B) \tag{1.128}$$
$$\implies \frac{3}{5} = \frac{1}{2} + p - \frac{p}{2} \tag{1.129}$$
$$\therefore p = \frac{1}{5} \tag{1.130}$$

1.1.27 Let and B be independent events with $\Pr(A) = 0.3$ and $\Pr(B) = 0.4$. Find

(a) $\Pr(AB)$

(b) $\Pr(A+B)$

(c) $\Pr(A|B)$

(d) $\Pr(B|A)$

Solution:

(a)

$$\Pr(AB) = \Pr(A) \times \Pr(B) = 0.3 \times 0.4 = 0.12 \tag{1.131}$$

(b)

$$\Pr(A+B) = \Pr(A) + \Pr(B) - \Pr(AB) = 0.3 + 0.4 - 0.12 = 0.58 \tag{1.132}$$

(c)

$$\Pr(A|B) = \frac{\Pr(AB)}{\Pr(B)} = \frac{\Pr(A) \times \Pr(B)}{\Pr(B)} = \Pr(A) = 0.3 \tag{1.133}$$

(d)

$$\Pr(B|A) = \frac{\Pr(AB)}{\Pr(A)} = \frac{\Pr(B) \times \Pr(A)}{\Pr(A)} = \Pr(B) = 0.4 \tag{1.134}$$

1.1.28 If A and B are two events such that $\Pr(A) = \frac{1}{4}, \Pr(B) = \frac{1}{2}$ and $\Pr(AB) = \frac{1}{8}$, find $\Pr(\text{not A and not B})$.

Solution: Since,

$$A'B' = (A+B)' \tag{1.135}$$

$$\implies \Pr\left(A'B'\right) = \Pr\left((A+B)'\right) \tag{1.136}$$

$$= 1 - \Pr((A+B)) \tag{1.137}$$

we also know that,

$$\Pr(A+B) = \Pr(A) + \Pr(B) - \Pr(AB) \tag{1.138}$$
$$= \frac{1}{4} + \frac{1}{2} - \frac{1}{8} \tag{1.139}$$
$$= \frac{5}{8} \tag{1.140}$$

Hence, by substituting in (1.136) we get

$$\Pr\left(A'B'\right) = 1 - \frac{5}{8} \tag{1.141}$$
$$= \frac{3}{8} \tag{1.142}$$

1.1.29 Events A and B are such that

$$\Pr(A) = \frac{1}{2}, \Pr(B) = \frac{7}{12} \text{and} \Pr\left(A' + B'\right) = \frac{1}{4}. \tag{1.143}$$

State whether A and B are independent.

Solution:

$$\Pr(AB) = 1 - \Pr\left(A' + B'\right) = 1 - \frac{1}{4} = \frac{3}{4} \tag{1.144}$$
$$\Pr(A) \times \Pr(B) = \frac{1}{2} \times \frac{7}{12} = \frac{7}{24} \tag{1.145}$$
$$\implies \Pr(AB) \neq \Pr(A)\Pr(B) \tag{1.146}$$

$\therefore$ A and B are not independent.

1.1.30 A fair coin and an unbiased die are tossed. Let A be the event 'head appears on the coin' and B be the event '3 on the die'. Check whether A and B are independent events or not.

1.1.31 A die marked 1, 2, 3 in red and 4, 5, 6 in green is tossed. Let A be the event, 'the number is even, ' and B be the event, 'the number is red'. Are A and B independent?

1.1.32 Let E and F be events with $P(E) = \frac{3}{5}$, $P(F) = \frac{3}{10}$ and $P(E \cap F) = \frac{1}{5}$. Are E and F independent?

1.1.33 Given that the events A and B are such that $P(A) = \frac{1}{2}$, $P(A \cup B) = \frac{3}{5}$ and $P(B) = p$. Find p if they are

(a) mutually exclusive

(b) independent

1.1.34 If A and B are two events such that $P(A) = \frac{1}{4}$, $P(B) = \frac{1}{2}$ and $P(A \cap B) = \frac{1}{8}$, find $P(\text{not } A \text{ and not } B)$

1.1.35 Events A and B are such that $P(A) = \frac{1}{2}$, $P(B) = \frac{7}{12}$ and $P(\text{not } A \text{ or not } B) = \frac{1}{4}$. State whether A and B are independent?

1.1.36 Given two independent events A and B such that $P(A) = 0.3$, $P(B) = 0.6$. Find

(a) $P(A \text{ and } B)$

(b) $P(A \text{ and not } B)$

(c) $P(A \text{ or } B)$

(d) $P(\text{neither } A \text{ nor } B)$

1.1.37 Probability of solving specific problem independently by A and B are $\frac{1}{2}$ and $\frac{1}{3}$ respectively. If both try to solve the problem independently, find the probability that

(a) the problem is solved

(b) exactly one of them solves the problem

Solution: Given that $\Pr(A) = \frac{1}{2}$ and $\Pr(B) = \frac{1}{3}$ A, B are independent so

$$\Pr(AB) = \Pr(A)\Pr(B) = \frac{1}{6} \tag{1.147}$$

(a) The probability of the problem being solved is

$$\Pr(A+B) = \Pr(A) + \Pr(B) - \Pr(AB) \tag{1.148}$$

$$= \Pr(A) + \Pr(B) - \Pr(A)\Pr(B) \tag{1.149}$$

$$= \frac{1}{2} + \frac{1}{3} - \frac{1}{6} = \frac{2}{3} \tag{1.150}$$

(b) Probalbility that exactly one person solves problem is

$$\Pr\left(AB'\right) + \Pr\left(A'B\right) = \Pr(A)\Pr\left(B'\right) + \Pr\left(A'\right)\Pr(B) \tag{1.151}$$

$$= \Pr(A) + \Pr(B) - 2\Pr(A)\Pr(B) = \frac{1}{2} \tag{1.152}$$

1.1.38 One card is drawn at random from a well shuffled deck of 52 cards. In which of the following cases are the events E and F independent ?

(a) E: 'the card drawn is spade'

F: 'the card drawn is an ace'

(b) E: 'the card drawn is black'

F : 'the card drawn is a king'

(c) E : 'the card drawn is a king or queen'

F : 'the card drawn is a queen or jack'

Solution:

(i) E denotes the event that the card drawn is spade

$$\Pr(E) = \frac{13}{52} = \frac{1}{4} \tag{1.153}$$

F denotes the event that card drawn is ace

$$\Pr(F) = \frac{4}{52} = \frac{1}{13} \tag{1.154}$$

$$\Pr(EF) = \frac{1}{52} \tag{1.155}$$

$$\Pr(E)\Pr(F) = \frac{1}{4} \times \frac{1}{13} = \frac{1}{52} \tag{1.156}$$

$$\therefore \ \Pr(EF) = \Pr(E)\Pr(F) \tag{1.157}$$

and the events are independent.

(ii) E denotes the event that the card drawn is black

$$\Pr(E) = \frac{26}{52} = \frac{1}{2} \tag{1.158}$$

F denotes the event that card drawn is a king

$$\Pr(F) = \frac{4}{52} = \frac{1}{13} \tag{1.159}$$

$$\Pr(EF) = \frac{2}{52} = \frac{1}{26} \tag{1.160}$$

$$\Pr(E)\Pr(F) = \frac{1}{2} \times \frac{1}{13} = \frac{1}{26} \tag{1.161}$$

$$\therefore \ \Pr(EF) = \Pr(E)\Pr(F) \tag{1.162}$$

and E and F are independent events.

(iii) E denotes the event that the card drawn is king or queen

$$\Pr(E) = \frac{8}{52} = \frac{2}{13} \tag{1.163}$$

F denotes the event that card drawn is a queen or jack

$$\Pr(F) = \frac{8}{52} = \frac{2}{13} \tag{1.164}$$

$$\Pr(EF) = \frac{4}{52} = \frac{1}{13} \tag{1.165}$$

$$\Pr(E)\Pr(F) = \frac{2}{13} \times \frac{2}{13} = \frac{4}{169} \tag{1.166}$$

$$\therefore\ \Pr(EF) \neq \Pr(E)\Pr(F) \tag{1.167}$$

and E and F are not independent events.

Choose the correct answer in the following exercises

1.1.39 The probability of obtaining an even prime number on each die, when a pair of dice is rolled is

(a) 0

(b) $\frac{1}{3}$

(c) $\frac{1}{12}$

(d) $\frac{1}{36}$

1.1.40 Two events A and B will be independent, if

(a) A and B are mutually exclusive

(b) $\mathrm{P}(\text{not } A \cap \text{not } B) = [1 - P(A)]\,[1 - P(B)]$

(c) $P(A) = P(B)$

(d) $P(A) + P(B) = 1$

Solution:

(a) When tossing a coin, the event of getting a head and tail are mutually exclusive and let them be denoted by A and B respectively.

$$\Pr(A) = \Pr(B) = \frac{1}{2} \implies \Pr(A) \times \Pr(B) = \frac{1}{4} \tag{1.168}$$

$$\text{or,}\ \Pr(AB) = 0 \neq \Pr(A) \times \Pr(B) \tag{1.169}$$

Hence A and B are not independent.

(b)

$$\Pr(A'B') = [1 - \Pr(A)][1 - \Pr(B)] \tag{1.170}$$

$$= 1 - \Pr(A) - \Pr(B) + \Pr(A)\Pr(B) \tag{1.171}$$

$$\implies 1 - \Pr(A + B) = 1 - \Pr(A) - \Pr(B) + \Pr(A)\Pr(B) \tag{1.172}$$

$$\implies \Pr(AB) = \Pr(A)\Pr(B) \tag{1.173}$$

which implies that A and B are independent.

(c) For the same counter example given for option 1.1.40a, $\Pr(A) = \Pr(B)$, but A and B are not independent events.

(d) For the same counter example given for option 1.1.40a, $\Pr(A) + \Pr(B) = 1$, but A and B are not independent events.

1.1.41 If $P(A) = \frac{3}{5}$ and $P(B) = \frac{1}{5}$, find $P(A \cap B)$ if A and B are independent events.

1.1.42 Compute $\Pr(A|B)$, if $\Pr(B) = 0.5$ and $\Pr(AB) = 0.32$.

Solution:

$$\Pr(A|B) = \frac{\Pr(AB)}{\Pr(B)} = \frac{0.32}{0.5} = 0.64 \tag{1.174}$$

1.1.43 A fair die is rolled. Consider events $E = 1, 3, 5,\ F = 2, 3$ and $G = 2, 3, 4, 5$. Find

(a) $\Pr(E \mid F)$ and $\Pr(F \mid E)$

(b) $\Pr(E \mid G)$ and $\Pr(G \mid E)$

(c) $\Pr(E \cup F \mid G)$ and $\Pr(E \cap F \mid G)$

Solution: See Table 1.8.

E = {1,3,5}	Pr(E) = $\frac{1}{2}$
F = {2,3}	Pr(F) = $\frac{1}{3}$
G = {2,3,4,5}	Pr(G) = $\frac{2}{3}$
EF = {3}	Pr(EF) = $\frac{1}{6}$
FG = {2,3}	Pr(FG) = $\frac{1}{3}$
EG = {3,5}	Pr(EG) = $\frac{1}{3}$
EFG = {3}	Pr(EFG) = $\frac{1}{6}$

Table 1.8: From given data

(a)

$$\Pr(E|F) = \frac{\Pr(EF)}{\Pr(F)} = \frac{1/6}{1/3} = 1/2 \tag{1.175}$$

(b)

$$\Pr(F|E) = \frac{\Pr(EF)}{\Pr(E)} = \frac{1/6}{1/2} = 1/3 \tag{1.176}$$

(c)

$$\Pr(E|G) = \frac{\Pr(EG)}{\Pr(G)} = \frac{1/3}{2/3} = 1/2 \tag{1.177}$$

(d)

$$\Pr(G|E) = \frac{\Pr(EG)}{\Pr(E)} = \frac{1/3}{1/2} = 2/3 \tag{1.178}$$

(e) Since

$$\Pr\left((E+F)G\right) = \Pr\left((EG)+(FG)\right) = \Pr(EG)+\Pr(FG)-\Pr(EFG), \tag{1.179}$$

$$= \frac{1}{3}+\frac{1}{3}-\frac{1}{6} = \frac{1}{2} \tag{1.180}$$

$$\implies \Pr((E+F)|G) = \frac{\Pr((E+F)G)}{\Pr(G)} = \frac{1/2}{2/3} = \frac{3}{4} \tag{1.181}$$

(f)

$$\Pr((EF)|G) = \frac{\Pr(EFG)}{\Pr(G)} = \frac{1/6}{2/3} = \frac{1}{4} \tag{1.182}$$

1.1.44 State which of the following are not the probability distributions of a random variable. Give reasons for your answer.

Table 1.9:

(a)

X	0	1	2
P(X)	0.4	0.4	0.2

Table 1.10:

(b)

X	0	1	2	3	4
P(X)	0.1	0.5	0.2	-0.1	0.3

Table 1.11:

(c)

Y	-1	0	1
P(Y)	0.6	0.1	0.2

Solution: From the axioms of probability,

$$0 < \Pr\left(X = i\right) < 1, i = 1, 2, 3...n. \tag{1.183}$$

$$\sum_{i=1}^{n} \Pr\left(X = i\right) = 1, i = 1, 2, 3...n. \tag{1.184}$$

(a)

$$\sum_{i=0}^{2} \Pr\left(X = i\right) = 0.4 + 0.4 + 0.2 = 1 \tag{1.185}$$

satisfies both (1.184) and (1.183), so it is a probability distribution.

(b)

$$\sum_{i=0}^{4} \Pr\left(X = i\right) = 0.1 + 0.5 + 0.2 - 0.1 + 0.3 = 1 \tag{1.186}$$

Satisfies (1.184) but does not satisfy (1.183) as $P(3) < 0$. Hence NOT a probability distribution.

(c)

$$\sum_{i=-1}^{1} \Pr\left(X = i\right) = 0.6 + 0.1 + 0.2 = 0.9 \tag{1.187}$$

Table 1.12:

(d)

X	0	1	2	3	4
P(Z)	0.3	0.2	0.4	0.1	0.05

(1.184) not satisfied, so it is NOT a probability distribution.

(d)

$$\sum_{i=0}^{4} \Pr(X = i) = 0.3 + 0.2 + 0.4 + 0.1 + 0.05 = 1.05 \tag{1.188}$$

(1.184) not satisfied, so it is NOT a probability distribution.

1.1.45 If a leap year is selected at random, what is the chance that it will contain 53 tuesdays?

Solution: The number of days in the leap year can be expressed as

$$366 = 52 \times 7 + 2 \tag{1.189}$$

The probability of one of the two remaining days being a Tuesday is $\frac{2}{7}$.

1.1.46 Assume that the chances of a patient having a heart attack is 40%. It is also assumed that a meditation and yoga course reduce the risk of heart attack by 30% and prescription of certain drug reduces its chances by 25%. At a time a patient can choose any one of the two options with equal probabilities. It is given that after going through one of the two options the patient selected at random suffers a heart attack. Find the probability that the patient followed a course of meditation and yoga?

Solution: The given information is summarised in Table From the given information,

$$\Pr(A|E_1) = \Pr(A)(1 - (0.30)) = 0.40 \times 0.70 = 0.28 \tag{1.190}$$

A:	Person with heat attack	$\Pr(A)$=0.40
E_1:	Person treated with meditation and yoga	$\Pr(E_1)$=0.50
E_2:	Person treated with drug	$\Pr(E_2)$=0.50

Table 1.13: Given Information

and

$$\Pr(A|E_2) = \Pr(A)(1 - (0.25)) = 0.40 \times 0.75 = 0.30 \tag{1.191}$$

From (1.190) and (1.191),

$$\Pr(E_1|A) = \frac{\Pr(E_1)\Pr(A|E_1)}{\sum_{i=1}^{2}\Pr(E_i)\Pr(A|E_i)} = \frac{\frac{1}{2}\times 0.28}{\frac{1}{2}\times 0.28 + \frac{1}{2}\times 0.30} = \frac{14}{29} \tag{1.192}$$

1.1.47 An electronic assembley consists of two subsystems,say A and B.From previous testing procedures, the following probabilities are assumed to be known

$$\Pr(\text{A fails}) = 0.20 \tag{1.193}$$

$$\Pr(\text{B alone fails}) = 0.15 \tag{1.194}$$

$$\Pr(\text{A and B fails}) = 0.15 \tag{1.195}$$

Evaluate the following probabilities

(a) $\Pr(\text{A fails given B has failed})$

(b) $\Pr(\text{A fails alone})$

Solution: From the given information,

$$\Pr\left(A'\right) = 0.20 \tag{1.196}$$

$$\Pr\left(AB'\right) = 0.15 \tag{1.197}$$

$$\Pr\left(A'B'\right) = 0.15 \tag{1.198}$$

(a)

$$\Pr\left(A'|B'\right) = \frac{\Pr\left(A'B'\right)}{\Pr\left(B'\right)} \tag{1.199}$$

Since

$$B'(1) = B'(A + A') = B'A + B'A' \tag{1.200}$$

$$\Pr\left(B'\right) = \Pr\left(AB'\right) + \Pr\left(A'B'\right) \because ((B'A)(B'A')) = 0 \tag{1.201}$$

$$= 0.15 + 0.15 = 0.30 \tag{1.202}$$

Thus,

$$\Pr\left(A'|B'\right) = 0.15/0.30 == 0.50 \tag{1.203}$$

(b) Similiarly,

$$\Pr\left(A'\right) = \Pr\left(BA'\right) + \Pr\left(A'B'\right) \tag{1.204}$$

$$\implies \Pr\left(BA'\right) = \Pr\left(A'\right) - \Pr\left(A'B'\right) = 0.20 - 0.15 = 0.05 \tag{1.205}$$

1.1.48 If A and B are two events such that $P(A) \neq 0$ and $P(B \mid A) = 1$, then

(a) $A \subset B$

(b) $B \subset A$

(c) $B = \phi$

(d) $A = \phi$

Solution:

$$\Pr(B|A) = 1 \implies \frac{\Pr(BA)}{\Pr(A)} = 1 \tag{1.206}$$

$$\implies \Pr(BA) = \Pr(A) \tag{1.207}$$

yielding

$$BA = A, \text{ or, } A \subset B \tag{1.208}$$

1.1.49 If $\Pr(A \mid B) > \Pr(A)$, then which of the following is correct

(a) $\Pr(B \mid A) < \Pr((B))$

(b) $\Pr(AB) < \Pr(A)\Pr(B)$

(c) $\Pr(B \mid A) > \Pr(B)$

(d) $\Pr(B \mid A) = \Pr(B)$

Solution:

$$\Pr(A \mid B) > \Pr(A) \implies \frac{\Pr(AB)}{\Pr(B)} > \Pr(A) \tag{1.209}$$

$$\implies \Pr(AB) > \Pr(A)\Pr(B) \implies \frac{\Pr(AB)}{\Pr(A)} > \Pr(B) \tag{1.210}$$

Since

$$\Pr(B \mid A) = \frac{\Pr(AB)}{\Pr(A)}, \Pr(B \mid A) > \Pr(B) \tag{1.211}$$

Hence, option 1.1.49c is correct.

1.1.50 If A and B are any two events such that $\Pr(A)$+$\Pr(B)$-$\Pr(AB)$=$\Pr(A)$, then choose the correct option

(a) $\Pr(B|A) = 1$

(b) $\Pr(A|B) = 1$

(c) $\Pr(B|A) = 0$

(d) $\Pr(A|B) = 0$

Solution: From the given information,

$$\Pr(A) + \Pr(B) - \Pr(AB) = \Pr(A) \tag{1.212}$$

$$\implies \Pr(B) = \Pr(AB) \tag{1.213}$$

Hence,

(a)

$$\Pr(B|A) = \frac{\Pr(AB)}{\Pr(A)} = \frac{\Pr(B)}{\Pr(A)} \tag{1.214}$$

(b)

$$\Pr(A|B) = \frac{\Pr(AB)}{\Pr(B)} = \frac{\Pr(B)}{\Pr(B)} = 1 \tag{1.215}$$

Hence option 1.1.50b is correct.

1.1.51 In a hostel, 60% of the students read Hindi newspaper, 40% read English newspaper and 20% read both Hindi and English newspapers. A student is selected at random.

(a) Find the probability that she reads neither Hindi nor English newspapers.

(b) If she reads Hindi newspaper, find the probability that she reads English newspaper.

(c) If she reads English newspaper, find the probability that she reads Hindi newspaper.

Solution: Given,

Random Variables	Events
A	Hindi
B	English

(a)

$$\Pr\left(A'B'\right) = \Pr\left((A+B)'\right) \tag{1.216}$$

$$= 1 - \Pr(A+B) \tag{1.217}$$

$$= 1 - (\Pr(A) + \Pr(B) - \Pr(AB)) \tag{1.218}$$

$$= 1 - \left(\frac{6}{10} + \frac{4}{10} - \frac{2}{10}\right) \tag{1.219}$$

$$= \frac{2}{10} \tag{1.220}$$

(b)

$$\Pr(B|A) = \frac{\Pr(BA)}{\Pr(A)} \tag{1.221}$$

$$= \frac{\frac{2}{10}}{\frac{6}{10}} \tag{1.222}$$

$$= \frac{1}{3} \tag{1.223}$$

(c)

$$\Pr(A|B) = \frac{\Pr(AB)}{\Pr(B)} \tag{1.224}$$

$$= \frac{\frac{2}{10}}{\frac{4}{10}} \tag{1.225}$$

$$= \frac{1}{2} \tag{1.226}$$

1.1.52 Two Coins are tossed once, where

(i) E : Tail appears on one coin, F : one coin shows head

(ii) E : no tail appears, F : no head appears Determine $\Pr(E \mid F)$. **Solution:** The random variables X_1 and X_2 are shown in Table 1.14.

$X_1 = 0$	First coin shows Tail.
$X_1 = 1$	First coin shows Head.
$X_2 = 0$	Second coin shows Tail.
$X_2 = 1$	Second coin shows Head.

Table 1.14: Definition of X_1 and X_2.

Since the coins are fair.

$$P_{X_1 X_2}(k, m) = \frac{1}{4} \tag{1.227}$$

In (1.227), $k, m \in \{0, 1\}$. So, total four different k,m combinations.

(a) E: Here one coin is tail and other is head. We are required to find $\Pr(X_1 + X_2 = 1)$. Thus, from (1.227).

$$\Pr(E) = \Pr(X_1 + X_2 = 1) \tag{1.228}$$
$$= \Pr(X_1 = 0,\ X_2 = 1) + \Pr(X_1 = 1,\ X_2 = 0) \tag{1.229}$$
$$= \frac{1}{2} \tag{1.230}$$

F: Here one coin is head and other is tail. We are required to find $\Pr(X_1 + X_2 = 1)$. Thus, from (1.227).

$$\Pr(F) = \Pr(X_1 + X_2 = 1) \tag{1.231}$$
$$= \Pr(X_1 = 0,\ X_2 = 1) + \Pr(X_1 = 1,\ X_2 = 0) \tag{1.232}$$
$$= \frac{1}{2} \tag{1.233}$$

EF: Here one coin is head and other is tail. We are required to find $\Pr(X_1 + X_2 = 1)$.

Thus, from (1.227).

$$\Pr(EF) = \Pr(X_1 + X_2 = 1) \tag{1.234}$$

$$= \Pr(X_1 = 0,\ X_2 = 1) + \Pr(X_1 = 1,\ X_2 = 0) \tag{1.235}$$

$$= \frac{1}{2} \tag{1.236}$$

$$\Pr(E|F) = \frac{\Pr(EF)}{\Pr(F)} \tag{1.237}$$

$$= \frac{\frac{1}{2}}{\frac{1}{2}} \tag{1.238}$$

$$= 1 \tag{1.239}$$

(b) E: no tail appears.We are required to find $\Pr(X_1 \neq 0, X_2 \neq 0)$. Thus, from (1.227).

$$\Pr(E) = \Pr(X_1 \neq 0, X_2 \neq 0) \tag{1.240}$$

$$= \Pr(X_1 = 1,\ X_2 = 1) \tag{1.241}$$

$$= \frac{1}{4} \tag{1.242}$$

F: no head appears.We are required to find $\Pr(X_1 \neq 1, X_2 \neq 1)$. Thus, from (1.227).

$$\Pr(F) = \Pr(X_1 \neq 1, X_2 \neq 1) \tag{1.243}$$

$$= \Pr(X_1 = 0,\ X_2 = 0) \tag{1.244}$$

$$= \frac{1}{4} \tag{1.245}$$

EF: coins should show neither head nor tail. From Table 1.14, we have coins showing head or tail. So, this is an impossible event

$$\Pr(EF) = 0 \tag{1.246}$$

$$\Pr(E|F) = \frac{\Pr(EF)}{\Pr(F)} \tag{1.247}$$

$$= \frac{0}{\frac{1}{4}} \tag{1.248}$$

$$= 0 \tag{1.249}$$

1.1.53 If the probabilities for A to fail in an examination is 0.2 and that for B is 0.3, then the probability that either A or B fails is

(a) > 0.5

(b) 0.5

(c) ≤ 0.5

(d) 0

Solution: Given,

$$\Pr(A) = 0.2 \tag{1.250}$$

$$\Pr(B) = 0.3 \tag{1.251}$$

$$\Pr(A + B) = \Pr(A) + \Pr(B) - \Pr(AB) \tag{1.252}$$

$$= 0.2 + 0.3 - \Pr(AB) \tag{1.253}$$

$$= 0.5 - \Pr(AB) \tag{1.254}$$

By the first axiom of Probability

$$0 \le \Pr(AB) \tag{1.255}$$

$$0.5 - \Pr(AB) \le 0.5 \tag{1.256}$$

$$\Pr(A+B) \le 0.5 \tag{1.257}$$

1.1.54 The probability that at least one of the two events A and B occurs is 0.6. If A and B occur simultaneously with probability 0.3, evaluate $\Pr(A') + \Pr(B')$.

Solution: Given:

$$\Pr(AB) = 0.3 \tag{1.258}$$

$$\Pr(A+B) = 0.6 \tag{1.259}$$

$$= \Pr(A) + \Pr(B) - \Pr(AB) \tag{1.260}$$

$$\implies 0.6 = \Pr(A) + \Pr(B) - 0.3 \tag{1.261}$$

$$\implies 0.9 = \Pr(A) + \Pr(B) \tag{1.262}$$

But

$$\Pr\left(A'\right) = 1 - \Pr(A) \tag{1.263}$$

$$\Pr\left(B'\right) = 1 - \Pr(B) \tag{1.264}$$

$$\therefore \Pr\left(A'\right) + \Pr\left(B'\right) = 2 - (\Pr(A) + \Pr(B)) \tag{1.265}$$

$$= 2 - 0.9 \tag{1.266}$$

$$= 1.1 \tag{1.267}$$

1.1.55 Three events A, B and C have probabilities $\frac{2}{5}$,$\frac{1}{3}$ and $\frac{1}{2}$ respetively.Given that $\Pr(AC) =$

$\frac{1}{5}$ and $\Pr(BC) = \frac{1}{4}$,find the values of $\Pr(C|B)$ and $\Pr(A'C')$

Solution:

(a) $\Pr(C|B)$

$$= \frac{\Pr(BC)}{\Pr(B)} \tag{1.268}$$

$$= \frac{\frac{1}{4}}{\frac{1}{3}} \tag{1.269}$$

$$= \frac{3}{4} \tag{1.270}$$

(b) $\Pr(A'C')$

$$= \Pr\left((A+C)'\right) \tag{1.271}$$

$$= 1 - \Pr(A+C) \tag{1.272}$$

$$= 1 - (\Pr(A) + \Pr(C) - \Pr(AC)) \tag{1.273}$$

$$= 1 - \left(\frac{2}{5} + \frac{1}{2} - \frac{1}{5}\right) \tag{1.274}$$

$$= 1 - \frac{7}{10} \tag{1.275}$$

$$= \frac{3}{10} \tag{1.276}$$

1.1.56 Prove that

(a) $\Pr(A) = \Pr(AB) + \Pr(AB')$

(b) $\Pr(A+B) = \Pr(AB) + \Pr(AB') + \Pr(A'B)$

Solution:

(a) consider RHS:

$$A = A(B + B') \tag{1.277}$$

$$\Pr(A) = \Pr\big(A(B + B')\big) \tag{1.278}$$

$$= \Pr\big(AB + AB'\big) \tag{1.279}$$

$$= \Pr(AB) + \Pr\big(AB'\big) - \Pr\big((AB)(AB')\big) \tag{1.280}$$

$$= \Pr(AB) + \Pr\big(AB'\big) - \Pr\big(ABB'\big) \tag{1.281}$$

$$= \Pr(AB) + \Pr\big(AB'\big) \tag{1.282}$$

(b) consider RHS:

$$A + B = A(B + B') + B(A + A') \tag{1.283}$$

$$\Pr(A + B) = \Pr\big(A(B + B') + B(A + A')\big) \tag{1.284}$$

$$= \Pr\big(AB + AB + AB' + BA'\big) \tag{1.285}$$

$$= \Pr\big(AB + AB' + BA'\big) \tag{1.286}$$

But,

$$AB(AB') = 0 \tag{1.287}$$

$$AB(A'B) = 0 \tag{1.288}$$

$$AB'(A'B) = 0 \tag{1.289}$$

$\implies AB, AB', A'B$ are mutually exclusive as their pairwise product is zero.

$$\Pr(A+B) = \Pr(AB) + \Pr\left(AB'\right) + \Pr\left(A'B\right) - \Pr\left(AB(AB')\right)$$
$$- \Pr\left(AB'(A'B)\right) - \Pr\left(A'B(AB)\right) + \Pr\left((AB)(AB')(A'B)\right) \quad (1.290)$$

From (1.287), (1.288) and (1.289), we get:

$$= \Pr(AB) + \Pr\left(AB'\right) + \Pr\left(A'B\right) - 0 - 0 - 0 + 0 \quad (1.291)$$
$$= \Pr(AB) + \Pr\left(AB'\right) + \Pr\left(A'B\right) \quad (1.292)$$

1.1.57 A and B are events such that $\Pr(A) = 0.4$ and $\Pr(B) = 0.3$ and $\Pr(A+B) = 0.5$ .Then $\Pr(B'A)$ is equal to

Solution: By axioms,

$$A = A(B+B') \quad (1.293)$$
$$A = AB + AB' \quad (1.294)$$
$$\implies \Pr(A) = \Pr(AB) + \Pr\left(AB'\right) \quad (1.295)$$
$$\implies \Pr\left(AB'\right) = \Pr(A) - \Pr(AB) \quad (1.296)$$

we also know that,

$$\Pr(AB) = \Pr(A) + \Pr(B) - \Pr(A+B) \quad (1.297)$$
$$= 0.4 + 0.3 - 0.5 \quad (1.298)$$
$$= 0.2 \quad (1.299)$$

Hence,

$$\Pr\left(AB'\right) = 0.4 - 0.2 \tag{1.300}$$
$$= 0.2 \tag{1.301}$$

1.1.58 You are given that A and B are two events such that $\Pr(B) = \frac{3}{5}, \Pr(A|B) = \frac{1}{2}$ and $\Pr(A+B) = \frac{4}{5}$,then $\Pr(A)$ equals

Solution:

$$\Pr(A|B) = \frac{1}{2} \tag{1.302}$$
$$\frac{\Pr(AB)}{\Pr(B)} = \frac{1}{2} \tag{1.303}$$
$$\Pr(AB) = \frac{\Pr(B)}{2} \tag{1.304}$$
$$= \frac{3}{10} \tag{1.305}$$

$$\Pr(A+B) = \frac{4}{5} \tag{1.306}$$
$$\Pr(A) + \Pr(B) - \Pr(AB) = \frac{4}{5} \tag{1.307}$$
$$\Pr(A) = \frac{4}{5} - \Pr(B) + \Pr(AB) \tag{1.308}$$

Substitute $\Pr(AB)$ from above

$$\Pr(A) = \frac{4}{5} - \frac{3}{5} + \frac{3}{10} \tag{1.309}$$
$$= \frac{1}{2} \tag{1.310}$$

1.1.59 If $P(A) = 0.4$, $P(B) = 0.8$ and $P(B|A) = 0.6$, then $P(A \cup B)$ is equal to

(a) 0.24

(b) 0.3

(c) 0.48

(d) 0.96

Solution: Given,

$$\Pr(A) = 0.4 \tag{1.311}$$

$$\Pr(B) = 0.8 \tag{1.312}$$

$$\Pr(B|A) = 0.6 \tag{1.313}$$

We know that,

$$\Pr(B|A) = \frac{\Pr(AB)}{\Pr(A)} \tag{1.314}$$

Therefore,

$$\Pr(AB) = \Pr(B|A)\Pr(A) \tag{1.315}$$

$$= (0.6)(0.4) \tag{1.316}$$

$$= 0.24 \tag{1.317}$$

$$\Pr(A+B) = \Pr(A) + \Pr(B) - \Pr(AB) \tag{1.318}$$

$$= 0.4 + 0.8 - 0.24 \tag{1.319}$$

$$= 0.96 \tag{1.320}$$

Hence, option (D) 0.96 is the correct option.

1.1.60 If A and B aret two events such that $\Pr(A) = \frac{1}{2}$ $\Pr(B) = \frac{1}{3}$, $\Pr(A|B) = \frac{1}{4}$, Then

$\Pr(A'B')$ equals

(a) $\frac{1}{12}$

(b) $\frac{3}{4}$

(c) $\frac{1}{4}$

(d) $\frac{3}{16}$

Solution: from De Morgan's Law

$$\Pr\left((A+B)'\right) = \Pr\left(A'B'\right) \tag{1.321}$$

$$1 - \Pr(A+B) = \Pr\left(A'B'\right) \tag{1.322}$$

So,

$$\implies \Pr(A+B) = \Pr(A) + \Pr(B) - \Pr(AB) \tag{1.323}$$

Finding $\Pr(AB)$ by,

$$\Pr(A|B) = \frac{\Pr(AB)}{\Pr(B)} \tag{1.324}$$

$$\tag{1.325}$$

by substituting the values $\Pr(B) = \frac{1}{3}$ and $\Pr(A|B) = \frac{1}{4}$

$$\implies \Pr(AB) = \frac{1}{12} \tag{1.326}$$

$$\Pr(A+B) = \frac{1}{2} + \frac{1}{3} - \frac{1}{12} \tag{1.327}$$

$$= \frac{3}{4} \tag{1.328}$$

Then,

$$\Pr\left(A'B'\right) = 1 - \Pr(A+B) \tag{1.329}$$
$$= 1 - \frac{3}{4} \tag{1.330}$$
$$= \frac{1}{4} \tag{1.331}$$

1.1.61 If A and B are such events that $\Pr(A) > 0$ and $\Pr(B) \neq 1$, then $\Pr(A'|B')$ is

(a) $1 - \Pr(A|B)$

(b) $1 - \Pr(A'|B)$

(c) $\frac{1-\Pr(A+B)}{\Pr(B')}$

(d) $\frac{\Pr(A')}{\Pr(B')}$

Solution:

$$\Pr\left(A'|B'\right) = \frac{\Pr(A'B')}{\Pr(B')} \tag{1.332}$$
$$= \frac{\Pr((A+B)')}{\Pr(B')} \tag{1.333}$$

We know $\Pr(A') = 1 - \Pr(A)$

$$\Pr\left(A'|B'\right) = \frac{1 - \Pr(A+B)}{\Pr(B')} \tag{1.334}$$

1.1.62 Two events E and F are independent. If $\Pr(E) = 0.3$, $\Pr(E+F) = 0.5$, then $\Pr(E|F) - \Pr(F|E)$ equals

(a) $\frac{2}{7}$

(b) $\frac{3}{35}$

(c) $\frac{1}{70}$

(d) $\frac{1}{7}$

Solution: As E and F are independent:

$$\Pr(EF) = \Pr(E)\Pr(F) \tag{1.335}$$
$$\text{But } \Pr(E+F) = \Pr(E) + \Pr(F) - \Pr(EF) \tag{1.336}$$
$$\therefore \Pr(E+F) = \Pr(E) + \Pr(F) - \Pr(E)\Pr(F) \tag{1.337}$$
$$0.5 = 0.3 + \Pr(F) - 0.3\Pr(F) \tag{1.338}$$
$$\Pr(F) = \frac{2}{7} \tag{1.339}$$

As E and F are independent:

$$\Pr(E|F) = \Pr(E) \tag{1.340}$$
$$\Pr(F|E) = \Pr(F) \tag{1.341}$$
$$\therefore \Pr(E|F) - \Pr(F|E) = \Pr(E) - \Pr(F) \tag{1.342}$$
$$= \frac{3}{10} - \frac{2}{7} \tag{1.343}$$
$$= \frac{1}{70} \tag{1.344}$$

1.1.63 You are given that A and B are two events such that $\Pr(B)=\frac{3}{5}$ $\Pr(A \mid B)=\frac{1}{2}$ $\Pr(A+B)=\frac{4}{5}$ and $\Pr(A)=\frac{1}{2}$ $\Pr(B \mid A')$ is equal to

Solution: Let E be the event for getting exactly one of A,B occurs.

If A and B are independent events

$$\Pr(AB) = \Pr(A)\Pr(B) \tag{1.345}$$

$$\Pr(B) = \Pr\left(B\left(A + A'\right)\right) \tag{1.346}$$

$$= \Pr\left(BA + BA'\right) \tag{1.347}$$

$$= \Pr(BA) + \Pr\left(BA'\right) + \Pr\left((BA)\left(BA'\right)\right) \tag{1.348}$$

$$= \Pr(BA) + \Pr\left(BA'\right) + \Pr\left((BB)\left(AA'\right)\right) \tag{1.349}$$

$$= \Pr(BA) + \Pr\left(BA'\right) \tag{1.350}$$

$$\implies \Pr\left(BA'\right) = \Pr(B) - \Pr(BA) \tag{1.351}$$

$$= \Pr(B) - \Pr(A)\Pr(B) \tag{1.352}$$

$$= \Pr(B)(1 - \Pr(A)) \tag{1.353}$$

$$= \Pr(B)\Pr\left(A'\right) \tag{1.354}$$

$$\Pr\left(A'B\right) = \Pr\left(A'\right)\Pr(B) \tag{1.355}$$

$$\Pr\left(AB'\right) = \Pr(A)\Pr\left(B'\right) \tag{1.356}$$

$$\Pr(E) = \Pr\left(A'B + AB'\right) \tag{1.357}$$

$$= \Pr\left(A'B\right) + \Pr\left(AB'\right) - \Pr\left(A'BAB'\right) \tag{1.358}$$

$$= \Pr\left(A'\right)\Pr(B) + \Pr(A)\Pr\left(B'\right) - 0 \quad \left(AA' = 0\right) \tag{1.359}$$

$$= \Pr\left(A'\right)\Pr(B) + \Pr(A)\Pr\left(B'\right) \tag{1.360}$$

$\therefore$ The statement is true

1.1.64 If A and B' are independent events, then $\Pr(A' + B) = 1 - \Pr(A)\Pr(B')$.

Solution: Given A and B' are independent events

$$\Pr\left(AB'\right) = \Pr\left(A\right)\Pr\left(B'\right) \tag{1.361}$$

Consider

$$\Pr\left(A' + B\right) = \Pr\left(\left(AB'\right)'\right) \tag{1.362}$$

$$= 1 - \Pr\left(AB'\right) \tag{1.363}$$

From (1.361)

$$\Pr\left(A' + B\right) = 1 - \Pr\left(A\right)\Pr\left(B'\right) \tag{1.364}$$

Given statement is true

1.1.65 State True or False for the statement.

If $\Pr\left(A\right) > 0$ and $\Pr\left(B\right) > 0$. Then A and B can be mutually exclusive and independent.

Solution: Since $\Pr\left(A\right) > 0$ and $\Pr\left(B\right) > 0$, then,

$$\Pr\left(A\right)\Pr\left(B\right) > 0 \tag{1.365}$$

But, for $\Pr\left(A\right)$ and $\Pr\left(B\right)$ to be mutually exclusive and independent events

$$\Pr\left(AB\right) = 0 \tag{1.366}$$

$$\Pr\left(AB\right) = \Pr\left(A\right)\Pr\left(B\right) \tag{1.367}$$

$$\implies \Pr\left(A\right)\Pr\left(B\right) = 0 \tag{1.368}$$

Thus, (1.368) contradicts our initial assumption (1.365) Hence, The above statement is false

1.1.66 If A and B are independent events, then A' and B' are also independent.

Solution: Given that A and B are independent events.

$$\implies \Pr(AB) = \Pr(A)\Pr(B) \tag{1.369}$$

We know that

$$\Pr\left(A'\right) = 1 - \Pr(A) \tag{1.370}$$

$$\Pr\left(B'\right) = 1 - \Pr(B) \tag{1.371}$$

Demorgan's law states that:

$$\Pr(AB)' = \Pr\left(A' + B'\right) \tag{1.372}$$

$$\Pr(A+B)' = \Pr\left(A'B'\right) \tag{1.373}$$

For A' and B', using the above properties we get:

$$\Pr\left(A'B'\right) = \Pr\left(A'\right) + \Pr\left(B'\right) - \Pr\left(A' + B'\right) \tag{1.374}$$
$$= 1 - \Pr(A) + 1 - \Pr(B) - \Pr(AB)' \tag{1.375}$$
$$= 2 - \Pr(A) - \Pr(B) - 1 + \Pr(AB) \tag{1.376}$$
$$= 1 - \Pr(A) - \Pr(B) + \Pr(A)\Pr(B) \tag{1.377}$$
$$= (1 - \Pr(A))(1 - \Pr(B)) \tag{1.378}$$
$$= (\Pr\left(A'\right))(\Pr\left(B'\right)) \tag{1.379}$$

Hence, A' and B' are also independent vectors.

Therefore, the given statement is true.

1.1.67 If A and B are two events such that $\Pr(A|B) = p, \Pr(A) = p, \Pr(B) = \frac{1}{3}$ and $\Pr(A+B) = \frac{5}{9}$,then $p =$

Solution:

$$\Pr(A|B) = p \tag{1.380}$$
$$\frac{\Pr(AB)}{\Pr(B)} = p \tag{1.381}$$
$$\Pr(AB) = p\Pr(B) \tag{1.382}$$
$$= \frac{p}{3} \tag{1.383}$$

$$\Pr(A+B) = \frac{5}{9} \tag{1.384}$$

$$\Pr(A) + \Pr(B) - \Pr(AB) = \frac{5}{9} \tag{1.385}$$

$$p + \frac{1}{3} - \frac{p}{3} = \frac{5}{9} \tag{1.386}$$

$$\frac{2p}{3} = \frac{2}{9} \tag{1.387}$$

$$\implies p = \frac{1}{3} \tag{1.388}$$

1.1.68 If A and B are two events such that $\Pr(A) > 0$ and $\Pr(A) + \Pr(B) > 1$, then

$$\Pr(B|A) \geq 1 - \frac{\Pr(B')}{\Pr(A)}$$

Solution:

$$\Pr(B|A) = \frac{\Pr(AB)}{\Pr(A)} \tag{1.389}$$

$$= \frac{\Pr(A) + \Pr(B) - \Pr(A+B)}{\Pr(A)} \tag{1.390}$$

For any event E, $\Pr(E) \leq 1$

$$\therefore \Pr(A+B) \leq 1 \tag{1.391}$$

$$-\Pr(A+B) \geq -1 \tag{1.392}$$

$$\frac{\Pr(A) + \Pr(B) - \Pr(A+B)}{\Pr(A)} \geq \frac{\Pr(A) + \Pr(B) - 1}{\Pr(A)} \tag{1.393}$$

$$\Pr(B|A) \geq \frac{\Pr(A) - (1 - \Pr(B))}{\Pr(A)} \tag{1.394}$$

$$\Pr(B|A) \geq \frac{\Pr(A) - \Pr(B')}{\Pr(A)} \tag{1.395}$$

$$\Pr(B|A) \geq 1 - \frac{\Pr(B')}{\Pr(A)} \tag{1.396}$$

$\therefore$ Proved.

1.1.69 If

$$\Pr(B) = \frac{3}{5}, \Pr(A|B) = \frac{1}{2} \text{and} \Pr(A+B) = \frac{4}{5}, \text{then} \Pr(A+B)' + \Pr\left(A' + B\right) \tag{1.397}$$

Solution:

$$\Pr(AB) = \Pr(A|B)\Pr(B) \tag{1.398}$$
$$= \frac{1}{2} \times \frac{3}{5} \tag{1.399}$$
$$= \frac{3}{10} \tag{1.400}$$
$$= \Pr(A)\Pr(B) \tag{1.401}$$

Hence, A and B are independent of each other.

We know that

$$\Pr\left(A'B\right) = \Pr(B) - \Pr(AB) \tag{1.402}$$
$$= \frac{3}{5} - \frac{3}{10} \tag{1.403}$$
$$= \frac{3}{10} \tag{1.404}$$

Calculating the probability of A

$$\begin{aligned}
\Pr(A) &= \Pr(A+B) - \Pr(B) + \Pr(AB) && (1.405)\\
&= \frac{4}{5} - \frac{3}{5} + \frac{3}{10} && (1.406)\\
&= \frac{5}{10} && (1.407)\\
&= \frac{1}{2} && (1.408)
\end{aligned}$$

Complement of A is given by

$$\begin{aligned}
\Pr\left(A'\right) &= 1 - \Pr(A) && (1.409)\\
&= 1 - \frac{1}{2} && (1.410)\\
&= \frac{1}{2} && (1.411)
\end{aligned}$$

$$\begin{aligned}
\Pr(A+B)' &= 1 - \Pr(A+B) && (1.412)\\
&= 1 - \frac{4}{5} && (1.413)\\
&= \frac{1}{5} && (1.414)
\end{aligned}$$

$$\begin{aligned}
\Pr\left(A'+B\right) &= \Pr\left(A'\right) + \Pr(B) - \Pr\left(A'B\right) && (1.415)\\
&= \frac{1}{2} + \frac{3}{5} - \frac{3}{10} && (1.416)\\
&= \frac{8}{10} && (1.417)\\
&= \frac{4}{5} && (1.418)
\end{aligned}$$

Therefore, the required probability is

$$\Pr(A+B)' + \Pr\left(A'+B\right) = \frac{1}{5} + \frac{4}{5} \tag{1.419}$$
$$= \frac{5}{5} \tag{1.420}$$
$$= 1 \tag{1.421}$$

1.1.70 Let A and B be two events. If

$$\Pr(A|B) = \Pr(A), \tag{1.422}$$

then A is of B.

Solution: Given that

$$\Pr(A|B) = \Pr(A) \tag{1.423}$$
$$\implies \frac{\Pr(AB)}{\Pr(B)} = \Pr(A) \tag{1.424}$$
$$\implies \Pr(AB) = \Pr(A)\Pr(B) \tag{1.425}$$

Hence, we can conclude that A is independent of B.

1.1.71 If $\Pr(A|B) > \Pr(A)$, then which of the following is correct :

(A) $\Pr(B|A) < \Pr(B)$

(B) $\Pr(AB) < \Pr(A)\Pr(B)$

(C) $\Pr(B|A) > \Pr(B)$

(D) $\Pr(B|A) = \Pr(B)$

Solution: We know:

$$\Pr(A|B) > \Pr(A) \tag{1.426}$$

$$\implies \frac{\Pr(AB)}{\Pr(B)} > \Pr(A) \tag{1.427}$$

$$\implies \Pr(AB) > \Pr(A)\Pr(B) \tag{1.428}$$

(A) To find, $\Pr(B|A)$

$$\Pr(B|A) = \frac{\Pr(AB)}{\Pr(A)} \tag{1.429}$$

Dividing $\Pr(A)$ on both sides of equation ((1.426))

$$\implies \frac{\Pr(AB)}{\Pr(A)} > \frac{\Pr(A)\Pr(B)}{\Pr(A)} \tag{1.430}$$

$$\implies \Pr(B|A) > \Pr(B) \tag{1.431}$$

But given $\Pr(B|A) < \Pr(B)$ so option (A) is incorrect

(B) from equation (1.428) we have

$$\Pr(AB) > \Pr(A)\Pr(B) \tag{1.432}$$

Therefore, option (B) is incorrect

(C) from equation (1.428) we have

$$\Pr(B|A) > \Pr(B) \tag{1.433}$$

which matches the given option

Therefore, option (C) is correct

(D) from equation (1.431) we have

$$\Pr(B|A) > \Pr(B) \tag{1.434}$$

but given $\Pr(B|A) = \Pr(B)$

Therefore, option (D) is incorrect

1.2. Exercises

1.2.1 A team of medical students doing their internship have to assist during surgeries at a city hospital. The probabilities of surgeries rated as very-complex, complex, routine, simple or very-simple are respectively, 0.15, 0.20, 0.31, 0.26, .08. Find the probabilities that a particular surgery will be rated

(a) complex or very-complex

(b) neither very-complex nor very simple

(c) routine or complex

(d) routine or simple

Solution: The given information is summarised in Table 1.16

Random Variables	Difficulty Levels	Probability
E_1	Very-Complex	$\Pr(E_1)$=0.15
E_2	Complex	$\Pr(E_2)$=0.2
E_3	Routine	$\Pr(E_3)$=0.31
E_4	Simple	$\Pr(E_4)$=0.26
E_5	Very-Simple	$\Pr(E_5)$=0.08

Table 1.16:

(a)

$$\Pr(E_1 + E_2) = \Pr(E_1) + \Pr(E_2) \qquad \because E_1 E_2 = 0 \tag{1.435}$$

$$= 0.15 + 0.20 = 0.35 \tag{1.436}$$

(b)

$$\Pr\left(E_1' E_5'\right) = \Pr\left((E_1 + E_5)'\right) \tag{1.437}$$

$$= 1 - \Pr(E_1 + E_5) \tag{1.438}$$

$$= 1 - \ [\Pr(E_1) + \Pr(E_5)] \qquad \because E_1 E_5 = 0 \tag{1.439}$$

$$= 1 - \ [0.15 + 0.08] = 0.77 \tag{1.440}$$

$$\tag{1.441}$$

(c)

$$\Pr(E_3 + E_2) = \Pr(E_3) + \Pr(E_2) \qquad \because E_3E_2 = 0 \tag{1.442}$$

$$= 0.31 + 0.20 = 0.51 \tag{1.443}$$

(d) To find the probabilities that a particular surgery will be rated routine or simple:

$$\Pr(E_3 + E_4) = \Pr(E_3) + \Pr(E_4) \qquad \because E_3E_4 = 0 \tag{1.444}$$

$$= 0.31 + 0.26 = 0.57 \tag{1.445}$$

$$\tag{1.446}$$

1.2.2 If A and B are mutually exclusive events, $\Pr(A) = 0.35$ and $\Pr(B) = 0.45$ then find

(a) $\Pr(A')$

(b) $\Pr(B')$

(c) $\Pr(A + B)$

(d) $\Pr(AB)$

(e) $\Pr(AB')$

(f) $\Pr(A'B')$

Solution: Given that A and B are mutually exclusive events.

$$AB = 0 \tag{1.447}$$

(a)

$$\Pr\left(A'\right) = 1 - \Pr(A) \tag{1.448}$$

$$= 1 - 0.35 \tag{1.449}$$

$$= 0.65 \tag{1.450}$$

(b)

$$\Pr\left(B'\right) = 1 - \Pr(B) \tag{1.451}$$

$$= 1 - 0.45 \tag{1.452}$$

$$= 0.55 \tag{1.453}$$

(c) From (1.447).

$$\Pr(A + B) = \Pr(A) + \Pr(B) - \Pr(AB) \tag{1.454}$$

$$= 0.35 + 0.45 - 0 \tag{1.455}$$

$$= 0.80 \tag{1.456}$$

(d) From (1.447)

$$\Pr(AB) = 0 \tag{1.457}$$

(e)

$$\Pr(A) = \Pr(A(1)) \tag{1.458}$$
$$= \Pr\left(A(B+B')\right) \tag{1.459}$$
$$= \Pr\left(AB+AB'\right) \tag{1.460}$$
$$= \Pr(AB) + \Pr\left(AB'\right) - \Pr\left(AB(AB')\right) \tag{1.461}$$
$$= \Pr(AB) + \Pr\left(AB'\right) - \Pr\left(AA(BB')\right) \tag{1.462}$$
$$= \Pr(AB) + \Pr\left(AB'\right) \tag{1.463}$$

$BB' = 0$. So, $\Pr(A(BB')) = 0$

$$\Pr\left(AB'\right) = \Pr(A) - \Pr(AB) \tag{1.464}$$
$$= 0.35 - 0 \tag{1.465}$$
$$= 0.35 \tag{1.466}$$

(f)

$$\Pr\left(A'B'\right) = \Pr\left((A+B)'\right) \tag{1.467}$$
$$= 1 - \Pr(A+B) \tag{1.468}$$
$$= 1 - 0.80 \tag{1.469}$$
$$= 0.20 \tag{1.470}$$

1.2.3 The accompanying Venn diagram shows three events, A, B, and C, and also the probabilities of the various intersections (for instance, $\Pr(AB) = 0.7$. Determine

(a) $\Pr(A)$

(b) $\Pr(BC')$

(c) $\Pr(A+B)$

(d) $\Pr(AB')$

(e) $\Pr(BC)$

(f) Probability of exactly one of the three occurs

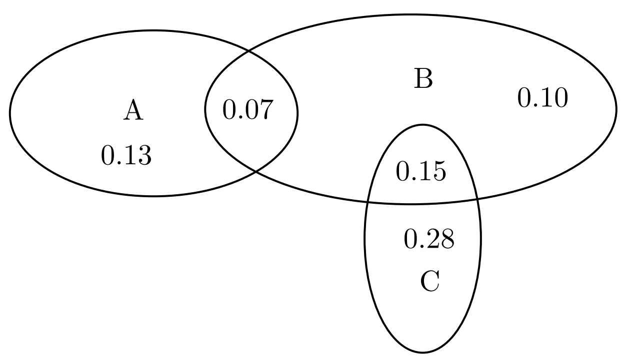

Figure 1.1: generated by Latextikz

Solution: Given:

From Fig. 1.1

$$\Pr(AB) = 0.07 \tag{1.471}$$

$$\Pr\left(AB'\right) = 0.13 \tag{1.472}$$

$$\Pr(BC) = 0.15 \tag{1.473}$$

$$\Pr\left(BA'C'\right) = 0.10 \tag{1.474}$$

$$\Pr\left(CB'\right) = 0.28 \tag{1.475}$$

(a)

$$\Pr(A) = 0.13 + 0.07 \tag{1.476}$$

$$= 0.2 \tag{1.477}$$

(b)

$$\Pr\left(BC'\right) = 0.07 + 0.10 + 0.15 - 0.15 \tag{1.478}$$

$$= 0.17 \tag{1.479}$$

(c)

$$\Pr(A + B) = \Pr(A) + \Pr(B) - \Pr(AB) \tag{1.480}$$

$$= 0.20 + (0.07 + 0.10 + 0.15) - 0.07 \tag{1.481}$$

$$= 0.45 \tag{1.482}$$

(d)

$$\Pr\left(AB'\right) = 0.20 - 0.07 \tag{1.483}$$

$$= 0.13 \tag{1.484}$$

(e)

$$\Pr(BC) = 0.15 \tag{1.485}$$

(f)

$$\Pr(\text{AB'}) + \Pr\left(CB'\right) + \Pr\left(BA'C'\right) = 0.13 + 0.10 + 0.28 \tag{1.486}$$

$$= 0.51 \tag{1.487}$$

1.2.4 One urn contains two black balls (labelled B1 and B2) and one white ball. A second urn contains one black ball and two white balls (labelled W1 and W2). Suppose the following experiment is performed. One of the two urns is chosen at random. Next a ball is randomly chosen from the urn. Then a second ball is chosen at random from the same urn without replacing the first ball.

(a) What is the probability that two black balls are chosen?

(b) What is the probability that two balls of opposite colour are chosen?

Solution: Let X be a Bernoulli random variable

$$X = \begin{cases} 0, & \text{Urn 1} \\ 1, & \text{Urn 2} \end{cases} \tag{1.488}$$

Since both events are equally likely

$$\Pr(X = 0) = \Pr(X = 1) \tag{1.489}$$

$$= \frac{1}{2} \tag{1.490}$$

Let Y_i be a random variable to denote the turn

$$Y_i = \begin{cases} 0, & \text{Black ball} \\ 1, & \text{White ball} \end{cases} \tag{1.491}$$

Y_1 denotes the first ball and Y_2 denotes the second ball.

Y_1	Y_2	**Description**
0	0	Both Black
1	1	Both White
0	1	Black,White
1	0	White,Black

Table 1.17: Random variables for each ball

(a)

$$E = (Y_1 + Y_2)' \tag{1.492}$$

$$= Y_1'Y_2' \tag{1.493}$$

Required Probability:

$$\Pr\left(Y_1'Y_2'\right) = \Pr\left(Y_1'Y_2'|X'\right)\Pr\left(X'\right) \tag{1.494}$$

$$= \frac{2}{3} \times \frac{1}{2} \times \frac{1}{2} \tag{1.495}$$

$$= 1/6 \tag{1.496}$$

Therefore,

$$\Pr(E) = \frac{1}{6} \tag{1.497}$$

(b)

$$E = Y_1 Y_2' + Y_1' Y_2 \tag{1.498}$$

Required Probability :

$$\Pr\left(Y_1 Y_2' + Y_1' Y_2\right) = \Pr\left(Y_1 Y_2'\right) + \Pr\left(Y_1' Y_2\right) \tag{1.499}$$

$$= \left(\frac{1}{3} \times \frac{1}{2} + \frac{2}{3} \times \frac{1}{2} \times \frac{1}{2}\right) \times 2 \tag{1.500}$$

$$= \frac{2}{3} \tag{1.501}$$

Therefore,

$$\Pr(E) = \frac{2}{3} \tag{1.502}$$

1.2.5 Events E and F are such that P(not E or not F) = 0.25, State whether E and F are mutually exclusive.

Solution: Given that:

$$\Pr\left(E' + F'\right) = \frac{1}{4} \tag{1.503}$$

From De-Morgan's Law, We can state that

$$\Pr\left(E' + F'\right) = \Pr(EF)' \tag{1.504}$$

From (1.503) and (1.504), We get

$$\Pr(EF)' = \frac{1}{4} \tag{1.505}$$

$$\implies 1 - \Pr(EF) = \frac{1}{4} \tag{1.506}$$

$$\implies \Pr(EF) = \frac{3}{4} \tag{1.507}$$

We can say that,

$$\because \Pr(EF) \neq 0 \tag{1.508}$$

E and F are not mutually exclusive events.

1.2.6 Suppose an integer from 1 through 1000 is chosen at random, find the probability that the integer is a multiple of 2 or a multiple of 9.

Solution:

Declare a random variable X.

$p_X(k)$ = Probability that the chosen number is a multiple of k.

Parameter	**Value**	**Description**
X=2	$P_X(2) = \frac{500}{1000}$	n mod 2 = 0, n is divisible by 2
X=9	$P_X(9) = \frac{111}{1000}$	n mod 9 = 0, n is divisible by 9
X=18	$P_X(18) = \frac{55}{1000}$	n mod 18 = 0, n is divisible by 2 & 9

Table 1.18: Random Variables

$$p_X(k) = \begin{cases} \frac{500}{1000} & k = 2 \\ \frac{111}{1000} & k = 9 \\ \frac{55}{1000} & k = 18 \end{cases} \tag{1.509}$$

$$\Pr\left((X = 2) + (X = 9)\right) = p_X(2) + p_X(9) - p_X(18) \tag{1.510}$$

$$= \frac{500}{1000} + \frac{111}{1000} - \frac{55}{1000} \tag{1.511}$$

$$= \frac{556}{1000} \tag{1.512}$$

$$= 0.556 \tag{1.513}$$

1.2.7 The probability that at least one of the events A and B occurs is 0.6. If A and B occur simultaneously with probability 0.2, then $\Pr(A') + \Pr(B')$is

(a) 0.4

(b) 0.8

(c) 1.2

(d) 1.6

Solution:

Given,

$$\Pr(A+B) = 0.6 \tag{1.514}$$

$$\Pr(AB) = 0.2 \tag{1.515}$$

$$\Pr(A+B) = \Pr(A) + \Pr(B) - \Pr(AB) \tag{1.516}$$

$$0.6 = \Pr(A) + \Pr(B) - 0.2 \tag{1.517}$$

$$\implies \Pr(A) + \Pr(B) = 0.8 \tag{1.518}$$

$$1 - \Pr\left(A'\right) + 1 - \Pr\left(B'\right) = 0.8 \tag{1.519}$$

$$\therefore \Pr\left(A'\right) + \Pr\left(B'\right) = 1.2 \tag{1.520}$$

1.2.8 State whether the statement is True or False.

The probability that a person visiting a zoo will see the giraffe is 0.72, the probability that he will see the bears is 0.84 and the probability that he will see both is 0.52.

Solution:

Variable	Event
A	Event of seeing the giraffe
B	Event of seeing the bears

Table 1.19: Events

Given,

$$\Pr(A) = 0.72 \tag{1.521}$$
$$\Pr(B) = 0.84 \tag{1.522}$$
$$\Pr(AB) = 0.52 \tag{1.523}$$

$$\Pr(A+B) = \Pr(A) + \Pr(B) - \Pr(AB) \tag{1.524}$$
$$= 0.72 + 0.84 - 0.52 \tag{1.525}$$
$$= 1.04 \tag{1.526}$$

which does not satisfy the first axiom of probability. Hence,it is a false statement.

1.2.9 The probability of happening of an event A is 0.5 and that of B is 0.3. If A and B are mutually exclusive events, then the probability of neither A nor B is _____.

Solution: Given:

$$Pr(A) = 0.5 \tag{1.527}$$

$$Pr(B) = 0.3 \tag{1.528}$$

As A and B are mutually exclusive,

$$Pr(AB) = 0 \tag{1.529}$$

Probability of atleast one of A and B happening is given by:

$$Pr(A + B) = Pr(A) + Pr(B) - Pr(AB) \tag{1.530}$$

$$= 0.5 + 0.3 - 0 \tag{1.531}$$

$$= 0.8 \tag{1.532}$$

$\therefore$ probability of neither A nor B happening is:

$$Pr((A + B)') = 1 - Pr(A + B) \tag{1.533}$$

$$= 1 - 0.8 \tag{1.534}$$

$$= 0.2 \tag{1.535}$$

1.2.10 If A and B are mutually exclusive events,then

(a) $\Pr(A) \leq \Pr(B')$

(b) $\Pr(A) \geq \Pr(B')$

(c) $\Pr(A) < \Pr(B')$

(d) none of these

Solution: Given, A and B are mutually exclusive events

So,

$$\Pr(AB) = 0 \tag{1.536}$$

Using axioms of probability we can write:

$$\Pr(A+B) \leq 1 \tag{1.537}$$

$$\Pr(A) + \Pr(B) - \Pr(AB) \leq 1 \tag{1.538}$$

$$\Pr(A) \leq 1 - \Pr(B) \tag{1.539}$$

$$\Pr(A) \leq \Pr\left(B'\right) \tag{1.540}$$

$\therefore$ Option (1) is correct.

1.2.11 State whether the statement is True or False. The probabilities that a typist will make 0, 1, 2, 3, 4, 5 or more mistakes in typing a report are, respectively, 0.12, 0.25, 0.36, 0.14, 0.08, 0.11.

Solution:

$$p_X(k) = \begin{cases} 0.12 & k = 0 \\ 0.25 & k = 1 \\ 0.36 & k = 2 \\ 0.14 & k = 3 \\ 0.08 & k = 4 \\ 0.11 & k \geq 5 \end{cases} \tag{1.541}$$

Since

$$\sum_{i=0}^{5} p_X(k) = 1 \tag{1.542}$$

We will use the above property to determine the validity of the statement.

$$\sum_{i=0}^{5} p_X(k) = 1.06 \tag{1.543}$$

$$> 1 \tag{1.544}$$

Hence the given statement is false.

1.2.12 If A and B are two candidates seeking admission in an engineering College. The probability that A is selected is 0.5 and the probability that both A and B are selected is atmost 0.3. Is it possible that the probability of B getting selected is 0.7?

Solution:

Let, $\Pr(A)$ = Probability that A is selected

$\Pr(B)$ = Probability that B is selected

Given that,

$$\Pr(AB) \leq 0.3 \tag{1.545}$$

$$\Pr(A) = 0.5 \tag{1.546}$$

We know that,

$$\Pr(A+B) \leq 1 \tag{1.547}$$

$$\Pr(A+B) = \Pr(A) + \Pr(B) - \Pr(AB) \tag{1.548}$$

$$\implies \Pr(AB) = \Pr(A) + \Pr(B) - \Pr(A+B) \tag{1.549}$$

Therefore,

$$\implies \Pr(A) + \Pr(B) - \Pr(A+B) \leq 0.3 \tag{1.550}$$

$$\implies \Pr(A) + \Pr(B) \leq 0.3 + \Pr(A+B) \tag{1.551}$$

$$\implies \Pr(A) + \Pr(B) \leq 0.3 + 1 \tag{1.552}$$

$$\implies 0.5 + \Pr(B) \leq 1.3 \tag{1.553}$$

$$\implies \Pr(B) \leq 0.8 \tag{1.554}$$

$\therefore$ It is possible that the probability of B getting selected is 0.7.

1.2.13 Prove if the given statement is true or false - The probability of intersection of two events A and B is always less than or equal to those favourable to the event A. **Solution:** We have to prove that:

$$\Pr(AB) \leq \Pr(A) \tag{1.555}$$

We know that,

$$BB' = 0 \tag{1.556}$$

$$B + B' = 1 \tag{1.557}$$

$$\implies \Pr(A) = \Pr\left(A\left(B + B'\right)\right) \tag{1.558}$$

$$= \Pr\left(AB + AB'\right) \tag{1.559}$$

By using inclusion-exclusion principle,

$$\Pr(A) = \Pr(AB) + \Pr\left(AB'\right) - \Pr\left((AB)\left(AB'\right)\right) \tag{1.560}$$

$$= \Pr(AB) + \Pr\left(AB'\right) - \Pr\left((AA)\left(BB'\right)\right) \tag{1.561}$$

$$= \Pr(AB) + \Pr\left(AB'\right) \tag{1.562}$$

We know that the value of probability ranges from 0 to 1.

$$0 \le \Pr\left(AB'\right) \le 1 \tag{1.563}$$

$$\text{Adding } \Pr(AB) \text{ both sides}$$

$$\Pr(AB) \le \Pr(AB) + \Pr\left(AB'\right) \tag{1.564}$$

Substituting value from equation (1.562)

$$\implies \Pr(AB) \le \Pr(A) \tag{1.565}$$

Hence, the given statement is true.

1.2.14 The probability of an occurrence of event A is .7 and that of the occurrence of event B is .3 and the probability of occurrence of both is .4.Is this statement true or false?

Solution: Given,

$$\Pr(A) = 0.7 \tag{1.566}$$

$$\Pr(B) = 0.3 \tag{1.567}$$

$$\Pr(AB) = 0.4 \tag{1.568}$$

Consider,

$$\Pr(A|B) = \frac{\Pr(AB)}{\Pr(B)} \tag{1.569}$$

Now, we know that

$$0 \leq \frac{\Pr(AB)}{\Pr(B)} \leq 1 \tag{1.570}$$

Since, probabilities are always between 0 and 1.

$$0 \leq \frac{0.4}{0.3} \leq 1 \tag{1.571}$$

$$0 \leq 0.4 \leq 0.3 \tag{1.572}$$

But given that $0.4 > 0.3$

$\therefore \Pr(AB) \neq 0.4$

The given statement is false.

1.2.15 If $\Pr(A+B) = \Pr(AB)$ for any two events A and B , then

A) $\Pr(A)$=$\Pr(B)$

B) $\Pr(A) > \Pr(B)$

C) $\Pr(A) < \Pr(B)$

D) none of these

Solution:

$$\Pr(A) + \Pr(B) - \Pr(AB) = \Pr(A+B) \tag{1.573}$$

$$\implies \Pr(A) + \Pr(B) - \Pr(AB) = \Pr(AB) \tag{1.574}$$

$$\implies [\Pr(A) - \Pr(AB)] + [\Pr(B) - \Pr(AB)] = 0 \tag{1.575}$$

But,

$$\Pr(A|B) = \frac{\Pr(AB)}{\Pr(B)} \tag{1.576}$$

Also,

$$0 \le \Pr(A|B) \le 1 \tag{1.577}$$

$$\implies 0 \le \frac{\Pr(AB)}{\Pr(B)} \le 1 \tag{1.578}$$

$$\implies 0 \le \Pr(AB) \le \Pr(B) \tag{1.579}$$

Similarly,

$$0 \le \Pr(AB) \le \Pr(A) \tag{1.580}$$

$$\therefore \Pr(A) - \Pr(AB) \ge 0 \tag{1.581}$$

$$\Pr(B) - \Pr(AB) \ge 0 \tag{1.582}$$

$$\implies \Pr(A) - \Pr(AB) = 0 \tag{1.583}$$

$$\implies \Pr(A) = \Pr(AB) \tag{1.584}$$

Also,

$$\Pr(B) - \Pr(AB) = 0 \tag{1.585}$$

$$\implies \Pr(B) = \Pr(AB) \tag{1.586}$$

From (1.584) and (1.586), it can be said that

$$\Pr(A) = \Pr(B) \tag{1.587}$$

1.2.16 Let E_1 and E_2 be two independent events such that $\Pr(E_1) = p_1$ and $\Pr(E_2) = p_2$ Describe in words the events whose probabilities are:

(a) p_1p_2

(b) $(1-p_1)p_2$

(c) $1-(1-p_1)(1-p_2)$

(d) $p_1+p_2-2p_1p_2$

Solution:

$\Pr(E1) = p_1 \quad \Pr(E_2) = p_2$

(a)

$$p_1p_2 = \Pr(E_1)\Pr(E_2) \tag{1.588}$$

$$= \Pr(E_1E_2) \tag{1.589}$$

So, E_1 and E_2 occur simultaneously.

(b)

$$(1 - p_1)(p_2) = \Pr\left(E_1'\right)\Pr\left(E_2\right) \tag{1.590}$$

$$= \Pr\left(E_1' E_2\right) \tag{1.591}$$

So E_1 does not occur but E_2 occurs.

(c)

$$1 - (1 - p_1)(1 - p_2) = 1 - \Pr\left(E_1'\right)\Pr\left(E_2'\right) \tag{1.592}$$

$$= 1 - \Pr\left(E_1'\ E_2'\right) \tag{1.593}$$

$$= \Pr\left(E_1 + E_2\right) \tag{1.594}$$

So, either E_1 or E_2 or both E_1 and E_2 occurs.

(d)

$$p_1 + p_2 - 2p_1p_2 = \Pr\left(E_1\right) + \Pr\left(E_2\right) - 2\Pr\left(E_1\right)\Pr\left(E_2\right) \tag{1.595}$$

$$= \Pr\left(E_1\right) - \Pr\left(E_1\right)\Pr\left(E_2\right) + \Pr\left(E_2\right) - \Pr\left(E_1\right)\Pr\left(E_2\right) \tag{1.596}$$

$$= \Pr\left(E_1\right)\left(1 - \Pr\left(E_2\right)\right) + \Pr\left(E_2\right)\left(1 - \Pr\left(E_1\right)\right) \tag{1.597}$$

$$= \Pr\left(E_1\right)\Pr\left(E_2'\right) + \Pr\left(E_2\right)\Pr\left(E_1'\right) \tag{1.598}$$

$$= \Pr\left(E_1E_2' + E_1'E_2\right) \tag{1.599}$$

Since, $\Pr\left(E_1\right) + \Pr\left(E_1'\right) = 1$

So, either E_1 or E_2 occurs but not both

1.2.17 Suppose that 6% of the people with blood group O are left handed and 10% of those with other blood groups are left handed 30% of the people have blood group O. If a left handed person is selected at random, what is the probability that he/she will have blood group O? **Solution:** Let us consider two random variables A and B. We are given that,

RV	Value	Description	Representation
A	0	Any other Blood Group	A'
	1	Blood Group O	A
B	0	Right Handed Person	B'
	1	Left Handed Person	B

$$\Pr(A) = 0.3 \tag{1.600}$$

$$\Pr(B|A) = 0.06 \tag{1.601}$$

$$\Pr\left(B|A'\right) = 0.1 \tag{1.602}$$

So, we can write that,

$$\Pr(A) = 0.3 \tag{1.603}$$

$$\therefore \Pr\left(A'\right) = 1 - \Pr(A) \tag{1.604}$$

$$= 1 - 0.3 \tag{1.605}$$

$$= 0.7 \tag{1.606}$$

$$\Pr(B|\,A) = 0.06 \tag{1.607}$$

$$\implies \frac{\Pr(BA)}{\Pr(A)} = 0.06 \tag{1.608}$$

$$\implies \Pr(BA) = 0.06 \Pr(A) \tag{1.609}$$

$$= 0.018 \tag{1.610}$$

$$\Pr\left(B|\,A'\right) = 0.1 \tag{1.611}$$

$$\implies \frac{\Pr(BA')}{\Pr(A')} = 0.1 \tag{1.612}$$

$$\implies \Pr\left(BA'\right) = 0.1 \Pr\left(A'\right) \tag{1.613}$$

$$= 0.07 \tag{1.614}$$

Hence,

$$\Pr(BA) = 0.018 \tag{1.615}$$

$$\Pr\left(BA'\right) = 0.07 \tag{1.616}$$

We know that,

$$A + A' = 1 \tag{1.617}$$

$$AA' = 0 \tag{1.618}$$

We can write $\Pr(B)$ as:

$$\Pr(B) = \Pr\left(B\left(A + A'\right)\right) \tag{1.619}$$

$$= \Pr\left(BA + BA'\right) \tag{1.620}$$

By inclusion-exclusion principle,

$$\Pr(B) = \Pr(BA) + \Pr\left(BA'\right) + \Pr\left((BA)\left(BA'\right)\right) \tag{1.621}$$

$$= \Pr(BA) + \Pr\left(BA'\right) + \Pr\left((BB)\left(AA'\right)\right) \tag{1.622}$$

$$= \Pr(BA) + \Pr\left(BA'\right) \tag{1.623}$$

By substituting values from equation (1.615) and (1.616),

$$\Pr(B) = 0.018 + 0.07 \tag{1.624}$$

$$= 0.088 \tag{1.625}$$

So, $\Pr(A|B)$ can be written as,

$$\Pr(A|B) = \frac{\Pr(BA)}{\Pr(B)} \tag{1.626}$$

$$= \frac{0.018}{0.088} \tag{1.627}$$

$$= \frac{9}{44} \tag{1.628}$$

Hence, if a left handed is selected at random, the probability of the person having blood group O is $\frac{9}{44}$.

1.2.18 Match the following:

(a) if E_1 and E_2 are the two mutually exclusive events	(i) $E_1 \cap E_2 = E_1$
(b) if E_1 and E_2 are the mutually exclusive and exhaustive events	(ii) $(E_1 - E_2) \cup (E_1 \cap E_2) = E_1$
(c) if E_1 and E_2 have common outcomes, then	(iii) $E_1 \cap E_2 = \phi, E_1 \cup E_2 = S$
(d) if E_1 and E_2 are two events such that $E_1 \subset E_2$	(iv) $E_1 \cap E_2 = \phi$

Table 1.2.18:

Solution:

(a) If E_1 and E_2 are mutually exclusive events, then $E_1E_2 = \phi$.

(b) If E_1 and E_2 are mutually exclusive and exhaustive events, then $E_1E_2 = \phi$ and $E_1 + E_2 = S$

(c) If E_1 and E_2 have common outcomes, this means:

$$E_1E_2 \neq 0 \tag{1.629}$$

Let E_a be the outcomes that are present in E_1 and not in E_2. So,

$$E_a = E_1 - E_2 \tag{1.630}$$

Let E_b be the outcomes common between E_1 and E_2. So,

$$E_b = E_1E_2 \tag{1.631}$$

So, we can say that

$$E_1 = E_a + E_b \tag{1.632}$$

Refering to equation (1.630) and (1.631):

$$E_1 = (E_1 - E_2) + (E_1E_2) \tag{1.633}$$

(d) If E_1 and E_2 are two events such that $E_1 \subset E_2$, then let E be subset of E_2 containing elements other than E_1. So,

$$E_1 + E = E_2 \text{ and } E_1E = E_2 \tag{1.634}$$

Refering to equation (1.634):

$$E_1E_2 = E_1(E_1 + E) \tag{1.635}$$

$$= (E_1E_1) + (E_1E) \tag{1.636}$$

$$= E_1 \tag{1.637}$$

Hence,

(a) $\leftrightarrow (iv)$,

(b) $\leftrightarrow (iii)$,

(c) $\leftrightarrow (ii)$,

(d) $\leftrightarrow (i)$

1.2.19 If $\Pr(A) = \frac{2}{5}$,$\Pr(B) = \frac{3}{10}$ and $\Pr(AB) = \frac{1}{5}$, then $\Pr(A'|B').\Pr(B'|A')$ is equal to

(A) $\frac{5}{6}$

(B) $\frac{5}{7}$

(C) $\frac{25}{42}$

(D) 1

Solution: Using the following equation:

$$\Pr(A|B) = \frac{\Pr(AB)}{\Pr(B)} \tag{1.638}$$

$$\Pr\left(A'|B'\right).\Pr\left(B'|A'\right) = \frac{\Pr(A'B')}{\Pr(B')}.\frac{\Pr(A'B')}{\Pr(A')} \tag{1.639}$$

$$= \frac{(\Pr(A'B'))^2}{(1-\Pr(B))(1-\Pr(A))} \tag{1.640}$$

$$= \frac{(1-\Pr(A+B))^2}{(1-\Pr(B))(1-\Pr(A))} \tag{1.641}$$

$$= \frac{\left\{1-\left(\frac{2}{5}+\frac{3}{10}-\frac{1}{5}\right)\right\}^2}{\left(1-\frac{3}{10}\right)\left(1-\frac{2}{5}\right)} \tag{1.642}$$

$$= \frac{\left(\frac{1}{2}\right)^2}{\left(\frac{7}{10}\right)\left(\frac{3}{5}\right)} \tag{1.643}$$

$$= \frac{25}{42} \tag{1.644}$$

1.2.20 A and B are two events such that $\Pr(A) = \frac{1}{2}$, $\Pr(B) = \frac{1}{3}$ and $\Pr(AB) = \frac{1}{4}$.

Find:

i $\Pr(A|B)$

ii $\Pr(B|A)$

iii $\Pr(A'|B)$

iv $\Pr(A'|B')$

Solution: :

Given, $\Pr(A) = \frac{1}{2}$, $\Pr(B) = \frac{1}{3}$ and $\Pr(AB) = \frac{1}{4}$. Then,

$$\begin{aligned} \Pr\left(A'\right) &= 1 - \Pr(A) = \frac{1}{2} \\ \Pr\left(B'\right) &= 1 - \Pr(B) = \frac{2}{3} \\ \Pr(A+B) &= \Pr(A) + \Pr(B) - \Pr(AB) \\ &= \frac{7}{12} \end{aligned} \tag{1.645}$$

(a)

$$\Pr(A|B) = \frac{\Pr(AB)}{\Pr(B)} = \frac{3}{4} \tag{1.646}$$

(b)

$$\Pr(B|A) = \frac{\Pr(AB)}{\Pr(A)} = \frac{1}{2} \tag{1.647}$$

(c)

$$\Pr\left(A'|B\right) = \frac{\Pr(A'B)}{\Pr(B)}$$

We have,

$$B = AB + A'B$$

Applying probabilities on both sides,

$$\begin{aligned}
\Pr(B) &= \Pr(AB) + \Pr\left(A'B\right) \\
\Pr\left(A'B\right) &= \Pr(B) - \Pr(AB) \\
&= \frac{1}{12} \\
\therefore \Pr\left(A'|B\right) &= \frac{1}{4}
\end{aligned} \tag{1.648}$$

(d)

$$\begin{aligned}
\Pr\left(A'|B'\right) &= \frac{\Pr(A'B')}{\Pr(B')} \\
\because \Pr\left(A'B'\right) &= \Pr(A+B)' \\
&= 1 - \Pr(A+B)
\end{aligned} \tag{1.649}$$

using the conclusions from (1.645)

$$\begin{aligned}
\Pr\left(A'B'\right) &= \frac{5}{12} \\
\therefore \Pr\left(A'|B'\right) &= \frac{5}{8}
\end{aligned} \tag{1.650}$$

1.2.21 Let $\Pr(A) = \frac{7}{13}$, $\Pr(B) = \frac{9}{13}$, $\Pr(AB) = \frac{4}{13}$. Then $\Pr(A'|B)$ is equal to

(a) $\frac{6}{13}$

(b) $\frac{4}{13}$

(c) $\frac{4}{9}$

(d) $\frac{5}{9}$

Solution: We are given that:

$$\Pr(A) = \frac{7}{13} \tag{1.651}$$
$$\Pr(B) = \frac{9}{13} \tag{1.652}$$
$$\Pr(AB) = \frac{4}{13} \tag{1.653}$$

We know that:

$$A + A' = 1 \tag{1.654}$$
$$AA' = 0 \tag{1.655}$$

Hence, we can say that:

$$\Pr(B) = \Pr\left(B\left(A + A'\right)\right) \tag{1.656}$$
$$= \Pr\left(AB + A'B\right) \tag{1.657}$$

By inclusion-exclusion principle,

$$\Pr(B) = \Pr(AB) + \Pr\left(A'B\right) + \Pr\left((AB)\left(A'B\right)\right) \tag{1.658}$$
$$= \Pr(AB) + \Pr\left(A'B\right) + \Pr\left((BB)\left(AA'\right)\right) \tag{1.659}$$
$$= \Pr(AB) + \Pr\left(A'B\right) \tag{1.660}$$

We get that,

$$\Pr\left(A'B\right) = \Pr(B) - \Pr(AB) \tag{1.661}$$
$$= \frac{9}{13} - \frac{4}{13} \tag{1.662}$$
$$= \frac{5}{13} \tag{1.663}$$

Then, $\Pr(A'|\, B)$ is:

$$\Pr\left(A'\middle|\, B\right) = \frac{\Pr(A'B)}{\Pr(B)} \tag{1.664}$$
$$= \frac{\frac{5}{13}}{\frac{9}{13}} \tag{1.665}$$
$$= \frac{5}{9} \tag{1.666}$$

Hence, option (d) is correct.

1.2.22 If $\Pr(A) = \frac{3}{10}, \Pr(B) = \frac{2}{5}$ and $\Pr(A+B) = \frac{3}{5}$, then $\Pr(B|\, A) + \Pr(A|\, B)$ equals

(a) $\frac{1}{4}$

(b) $\frac{1}{3}$

(c) $\frac{5}{12}$

(d) $\frac{7}{12}$

Solution:

$$\Pr(AB) = \Pr(A) + \Pr(B) - \Pr(A+B) \tag{1.667}$$
$$= \frac{3}{10} + \frac{2}{5} - \frac{3}{5} \tag{1.668}$$
$$= \frac{1}{10} \tag{1.669}$$

$$\Pr(B|\, A) = \frac{\Pr(AB)}{\Pr(A)} \tag{1.670}$$
$$= \frac{\frac{1}{10}}{\frac{3}{10}} \tag{1.671}$$
$$= \frac{1}{3} \tag{1.672}$$

$$\Pr(A|\, B) = \frac{\Pr(AB)}{\Pr(B)} \tag{1.673}$$
$$= \frac{\frac{1}{10}}{\frac{2}{5}} \tag{1.674}$$
$$= \frac{1}{4} \tag{1.675}$$

$$\Pr(B|\, A) + \Pr(A|\, B) = \frac{1}{3} + \frac{1}{4} \tag{1.676}$$
$$= \frac{7}{12} \tag{1.677}$$

1.2.23 The probability distribution of a discrete random variable X is given below. The value of k is equal to:

X	2	3	4	5
P(X)	$\frac{5}{k}$	$\frac{7}{k}$	$\frac{9}{k}$	$\frac{11}{k}$

(a) 8

(b) 16

(c) 32

(d) 48

Solution: We know that the sum of probabilities for all the values of random variable is equal to 1. Hence,

$$\sum_{i=2}^{5} P(X_i) = 1 \tag{1.678}$$

$$\implies \frac{5}{k} + \frac{7}{k} + \frac{9}{k} + \frac{11}{k} = 1 \tag{1.679}$$

$$\frac{5+7+9+11}{k} = 1 \tag{1.680}$$

$$\frac{32}{k} = 1 \tag{1.681}$$

$$k = 32 \tag{1.682}$$

Hence, option (c) is correct.

1.2.24 If A and B are such that

$\Pr(A' \cup B') = \frac{2}{3}$ and $\Pr(A \cup B) = \frac{5}{9}$

then $\Pr(A') + \Pr(B') =$

Solution: Using de morgan's law and axioms of probability.

$$\Pr(A'B') = \Pr((A+B)') \tag{1.683}$$

$$\Pr(A'+B') = \Pr(A') + \Pr(B') - \Pr(A'B') \tag{1.684}$$

We have,

$$\Pr(A'B') = \Pr((A+B)') \tag{1.685}$$

$$\Pr(A'B') = 1 - \Pr(A+B) \tag{1.686}$$

$$= 1 - \frac{5}{9} \tag{1.687}$$

$$= \frac{4}{9} \tag{1.688}$$

$$\Pr(A'+B') = \Pr(A') + \Pr(B') - \Pr(A'B') \tag{1.689}$$

$$\Pr(A') + \Pr(B') = \Pr(A'+B') + \Pr(A'B') \tag{1.690}$$

$$= \frac{2}{3} + \frac{4}{9} \tag{1.691}$$

$$= \frac{10}{9} \tag{1.692}$$

1.2.25 state True or False for the given statement: Two independent events are always mutually exclusive.

Solution:

Two events A and B are said to be independent if the occurrence of one does not affect the probability of the probability of the occurrence of the other. for these events to be independent

$$\Pr(AB) = \Pr(A) \times \Pr(B) \tag{1.693}$$

$$0 = \Pr(A) \times \Pr(B) \tag{1.694}$$

Example: There are two Events A and B for a pack of cards.

Event A: Drawing a red card (hearts or diamonds).

Event B: Drawing a face card (jack, queen, or king).

Now,

$$\Pr(AB) = 0 \tag{1.695}$$

$$\Pr(A) = \frac{26}{52} \tag{1.696}$$

$$\Pr(B) = \frac{12}{52} \tag{1.697}$$

$$\Pr(A) \times \Pr(B) = \frac{3}{26} \neq \Pr(AB) \tag{1.698}$$

$P(AB) > 0$, this demonstrates that Events A and B are not mutually exclusive.

1.2.26 If A and B are independent, then

$\Pr(exactlyoneof A, Boccurs)$ =$\Pr(B)$$\Pr(A')$+$\Pr(A)$$\Pr(B')$

Solution: Let E be the event for getting exactly one of A,B occurs.

If A and B are independent events

$$\Pr(AB) = \Pr(A)\Pr(B) \tag{1.699}$$

$$\Pr(B) = \Pr\left(B\left(A + A'\right)\right) \tag{1.700}$$
$$= \Pr\left(BA + BA'\right) \tag{1.701}$$
$$= \Pr(BA) + \Pr\left(BA'\right) + \Pr\left((BA)\left(BA'\right)\right) \tag{1.702}$$
$$= \Pr(BA) + \Pr\left(BA'\right) + \Pr\left((BB)\left(AA'\right)\right) \tag{1.703}$$
$$= \Pr(BA) + \Pr\left(BA'\right) \tag{1.704}$$
$$\implies \Pr\left(BA'\right) = \Pr(B) - \Pr(BA) \tag{1.705}$$
$$= \Pr(B) - \Pr(A)\Pr(B) \tag{1.706}$$
$$= \Pr(B)(1 - \Pr(A)) \tag{1.707}$$
$$= \Pr(B)\Pr\left(A'\right) \tag{1.708}$$

$$\Pr\left(A'B\right) = \Pr\left(A'\right)\Pr(B) \tag{1.709}$$
$$\Pr\left(AB'\right) = \Pr(A)\Pr\left(B'\right) \tag{1.710}$$

$$\Pr(E) = \Pr\left(A'B + AB'\right) \tag{1.711}$$
$$= \Pr\left(A'B\right) + \Pr\left(AB'\right) - \Pr\left(A'BAB'\right) \tag{1.712}$$
$$= \Pr\left(A'\right)\Pr(B) + \Pr(A)\Pr\left(B'\right) - 0 \quad \left(AA' = 0\right) \tag{1.713}$$
$$= \Pr\left(A'\right)\Pr(B) + \Pr(A)\Pr\left(B'\right) \tag{1.714}$$

$\therefore$ The statement is true

1.2.27 If A and B are two events and A $\neq \phi$, B $\neq \phi$, then

(a) $\Pr(A|B) = \Pr(A) . \Pr(B)$

(b) $\Pr(A|B) = \frac{\Pr(A\cap B)}{\Pr(B)}$

(c) $\Pr(A|B) . \Pr(B|A) = 1$

(d) $\Pr(A|B) = \frac{\Pr(A)}{\Pr(B)}$

Solution:

Let us take an example. The following table describes events when a die is rolled.

Event	Description	Probability
A	Even number shows up	$\frac{1}{2}$
B	Perfect number shows up	$\frac{1}{3}$
AB	Both Event A and B happen	$\frac{1}{6}$

(a) As $\Pr(A|B)$ represents the probability of occurence of A given that B has occured. Hence,

$$\Pr(A|B) = \frac{1/6}{1/3} \tag{1.715}$$

$$= \frac{1}{2} \tag{1.716}$$

$$\Pr(A) . \Pr(B) = \frac{1}{2} \times \frac{1}{3} \tag{1.717}$$

$$= \frac{1}{6} \tag{1.718}$$

As RHS of (1.716) and (1.718) are not equal, option 1 is not correct.

(b)

$$\frac{\Pr(AB)}{\Pr(B)} = \frac{\frac{1}{6}}{\frac{1}{3}} \tag{1.719}$$

$$= \frac{1}{2} \tag{1.720}$$

As (1.716) and (1.720) are equal, option 2 is correct.

$$\Pr(A|B) = \frac{\Pr(AB)}{\Pr(B)} \tag{1.721}$$

Equation (1.721) is one of the axioms of probability.

(c) From (1.721),

$$\Pr(A|B) . \Pr(B|A) = \frac{\Pr(AB)}{\Pr(B)} \times \frac{\Pr(AB)}{\Pr(A)} \tag{1.722}$$

$$= \frac{\Pr(AB)^2}{\Pr(A) . \Pr(B)} \tag{1.723}$$

$$\neq 1 \tag{1.724}$$

Hence, option 3 is incorrect. Let us verify it using the example.

$$\frac{\Pr(AB)^2}{\Pr(A) . \Pr(B)} = \frac{\frac{1}{6}^2}{\frac{1}{2} \times \frac{1}{3}} \tag{1.725}$$

$$= \frac{1}{6} \tag{1.726}$$

Hence, option 3 is incorrect.

(d)

$$\frac{\Pr(A)}{\Pr(B)} = \frac{\frac{1}{2}}{\frac{1}{3}} \tag{1.727}$$

$$= \frac{3}{2} \tag{1.728}$$

As (1.716) and (1.728) are not equal, option 4 is incorrect.

Chapter 2

Random Variables

2.1. Examples

2.1.1 One card is drawn from a well-shuffled deck of 52 cards. Find the probability of getting

(a) A king of red colour

(b) A face card

(c) A red face card

(d) The jack of hearts

(e) A spade

(f) The queen of diamonds

Solution: See Table 2.1. Consider 3 random variables X, Y and Z, which represent the Colour, Class and Value of each card respectively. The pmfs are

$$p_X(i) = \frac{1}{2} \; \forall \; i \in [0, 1] \tag{2.1}$$

$$p_Y(i) = \frac{1}{4} \; \forall \; i \in [1, 4] \tag{2.2}$$

$$p_Z(i) = \frac{1}{13} \; \forall \; i \in [1, 13] \tag{2.3}$$

Event	Value of X	Value of Y	Value of Z
Draw Red King	1	N/A	3
Draw Face Card	N/A	N/A	1,2 or 3
Draw Red Face Card	1	N/A	1,2 or 3
Draw Hearts Jack	N/A	3	1
Draw Spade	N/A	4	N/A
Draw Diamonds Queen	N/A	1	2

Table 2.1: Values of X,Y,Z for each event

and

$$\begin{aligned} F_Z(z) = \Pr(Z \leq z) &= \sum_{i=1}^{z} \Pr(Z = i) \\ &= z \times \Pr(Z = 1) = \frac{z}{13} \end{aligned} \tag{2.4}$$

Also, the random variable pairs X,Z and Y,Z are independent.

(a) Probability of drawing a King of Red colour

$$\begin{aligned} \Pr(X = 1, Z = 3) &= \Pr(X = 1) \times \Pr(Z = 3) \\ &= \frac{1}{2} \times \frac{1}{13} = \frac{1}{26} \end{aligned} \tag{2.5}$$

(b) Probability of drawing a Face Card

$$\Pr(1 \leq Z \leq 3) = F_Z(3) = \frac{3}{13} \tag{2.6}$$

(c) Probability of drawing a Red Face Card

$$\Pr(X = 1, 1 \leq Z \leq 3) = \Pr(X = 1) \times F_Z(3) \tag{2.7}$$

$$= \frac{1}{2} \times \frac{3}{13} = \frac{3}{26} \tag{2.8}$$

(d) Probability of drawing the Jack of Hearts

$$\Pr(Y = 3, Z = 1) = \Pr(Y = 3) \times \Pr(Z = 1) \tag{2.9}$$

$$= \frac{1}{4} \times \frac{1}{13} = \frac{1}{52} \tag{2.10}$$

(e) Probability of drawing a Spade

$$\Pr(Y = 4) = \frac{1}{4} \tag{2.11}$$

(f) Probability of drawing the Queen of Diamonds:

$$\Pr(Y = 1, Z = 2) = \Pr(Y = 1) \times \Pr(Z = 2) \tag{2.12}$$

$$= \frac{1}{4} \times \frac{1}{13} = \frac{1}{52} \tag{2.13}$$

2.1.2 Five cards—the ten, jack, queen, king and ace of diamonds, are well-shuffled with their face downwards. One card is then picked up at random.

(a) What is the probability that the card is the queen?

(b) If the queen is drawn and put aside, what is the probability that the second card picked up is (a) an ace? (b) a queen?

Solution: See Table 2.3.

EVENT	DESCRIPTION
E	Event of picking a card.
S	Sample space of picking a card.
Q	Event of the card picked be Queen.
A	Event of the card picked be Ace.

Table 2.3:

(a)

$$\Pr(Q) = \frac{1}{5} \tag{2.14}$$

(b) i.

$$\Pr(A) = \frac{1}{4} \tag{2.15}$$

ii.

$$\Pr(Q) = 0 \tag{2.16}$$

2.1.3 A bag contains 5 red balls and some blue balls. If the probability of drawing a blue ball is double that if a red ball, determine the number of blue balls in the bag.

Solution: The probability of drawing a red ball is

$$\Pr(R) = \frac{5}{5+x} \tag{2.17}$$

The probability of drawing a blue ball is

$$\Pr(B) = \frac{10}{5+x} \tag{2.18}$$

Thus,

$$\left(\frac{5}{5+x}\right)+2\left(\frac{5}{5+x}\right)=1 \tag{2.19}$$

$$\Longrightarrow \frac{15}{5+x}=1 \tag{2.20}$$

$$\text{or, } x=10 \tag{2.21}$$

2.1.4 A card is selected from a pack of 52 cards.

(a) How many points are there in the sample space?

(b) Calculate the probability that the card is an ace of spades.

(c) Calculate the probability that the card is (i) an ace and (ii) black card.

Solution: See Table 2.4

Table 2.4: Random Variable and probability Table

Random variable	value of R.V	Probability
\$X\$	1,2	26/52
\$Y\$	1,2,3,4	13/52
\$Z\$	1 \le Z \le 13	1/13

(a)

$$\Pr(Y=1,Z=1)=\Pr(Y=1)\Pr(Z=1)=\left(\frac{1}{4}\right)\left(\frac{1}{13}\right)=\frac{1}{52} \tag{2.22}$$

(b) The probability when the card choosen is ,

i. an ace $(Z=1)$

$$\Pr(Z=1)=\frac{1}{13}. \tag{2.23}$$

ii. black card $(X = 1)$

$$\Pr(X = 1) = \frac{1}{2}. \tag{2.24}$$

2.1.5 Four cards are drawn from a well-shuffled deck of 52 cards. What is the probability of obtaining 3 diamonds and one spade.

Solution: The given information is summarised in Table 2.6. yielding

RV	**Values**	**Description**
X	{0,1,2,3}	Cards drawn randomly
Y	{0,1}	0:diamond ,1:spade
X,Y	{00,10,20,31}	3 diamonds and one spade out of 13 each

Table 2.6: Random variables(RV) X,Y and X,Y

$$\Pr(00, 10, 20, 31) = \frac{{}^{13}C_3 \times {}^{13}C_1}{{}^{52}C_4} \tag{2.25}$$

$$= \frac{286}{20285} \tag{2.26}$$

2.1.6 In a certain lottery 10,000 tickets are sold and ten equal prizes are awarded. What is the probability of not getting a prize if you buy (a) one ticket (b) two tickets (c) 10 tickets ?

Solution: The given information is summarised in Table 2.8 The total number of possible outcomes is ${}^{N}C_n$ and the total number of favourable outcomes is ${}^{q}C_n$ yielding the desired probability

$$\Pr(n) = \frac{{}^{q}C_n}{{}^{N}C_n} \tag{2.27}$$

Substituting numerical values,

Variable	Value	Description
N	10000	Total number of tickets sold
k	10	Total number of prizes awarded
n	{0,1,2,...,N}	Number of tickets purchased
$\Pr(n)$		probability of not wining a prize
q	N-k	number of tickets with no prize

Table 2.8:

(a) For one ticket,

$$\Pr(1) = \frac{^{9990}C_1}{^{10000}C_1} = 0.9990 \tag{2.28}$$

(b) For two tickets,

$$\Pr(2) = \frac{^{9990}C_2}{^{10000}C_2} = 0.9980 \tag{2.29}$$

(c) For 10 tickets

$$\Pr(3) = \frac{^{9990}C_{10}}{^{10000}C_{10}} = 0.9901 \tag{2.30}$$

2.1.7 Out of 100 students, two sections of 40 and 60 are formed. If you and your friend are among the 100 students, what is the probability that

(a) you both enter the same section?

(b) you both enter the different sections?

Solution: Table 2.10 summarises the given information.

RV	Values	Description
X	{0,1}	0: sectioin1, 1: section2
Y	{0,1}	0: student1, 1: student2
XY	{001,101}	Students enter same section

(a) When both enter the same section, the probability is

$$\Pr(001, 101) = \frac{{}^{40}C_2}{{}^{100}C_2} + \frac{{}^{60}C_2}{{}^{100}C_2} = \frac{156}{990} + \frac{354}{990} = 0.51 \tag{2.31}$$

(b) When both enter different sections, the desired probability is

$$\Pr(00, 01, 10, 11) = 1 - 0.51 = 0.49 \tag{2.32}$$

2.1.8 The number lock of a suitcase has 4 wheels each labelled with ten digits i.e. from 0 to 9.The lock opens with a sequence of four digits with no repeats.What is the probability of a person getting the right sequence to open the suitcase.

Solution: Let

$$X_i = \begin{cases} 1, & \text{correct number choosen in } i^{th} \text{ wheel} \\ 0, & \text{otherwise} \end{cases} \tag{2.33}$$

and since repetition is not allowed, sample space for every next wheel will reduce by 1 unit. Therefore,

$$p_{X_i}(1) = \frac{1}{11 - i} \tag{2.34}$$

$$p_{X_i}(0) = 1 - \frac{1}{11 - i} \tag{2.35}$$

$$= \frac{10 - i}{11 - i} \tag{2.36}$$

Therefore, the desired probability is

$$\Pr(E) = \prod_{i=1}^{4} p_{X_i}(1) \tag{2.37}$$

$$= \frac{1}{10} \times \frac{1}{9} \times \frac{1}{8} \times \frac{1}{7} \tag{2.38}$$

$$= \frac{1}{5040} \tag{2.39}$$

2.1.9 Two cards are drawn at random and without replacement from a pack of 52 playing cards. Find the probability that both the cards are black.

Solution: Table 2.12 summarizes the various events Given that the cards are drawn

RV	Values	Description
X	{0,1}	number of cards drawn 2
Y	{0,1}	0: black card, 1: red card
XY	{00,10}	card drawn is black

Table 2.12:

at random without replacement. Without replacement means only one card is random at a time and is excluded from the total while next card is drawn at random. Thus, the probability that both the cards are black is,

$$\Pr(00, 10) = \frac{^{26}C_1}{^{52}C_1} \times \frac{^{25}C_1}{^{51}C_1} = \frac{1}{2} \times \frac{25}{51} = 0.24 \tag{2.40}$$

2.1.10 A box of oranges is inspected by examining three randomly selected oranges drawn without replacement. If all the three oranges are good, the box is approved for sale, otherwise, it is rejected. Find the probability that a box containing 15 oranges out of which 12 are good and 3 are bad ones will be approved for sale.

2.1.11 Two balls are drawn at random with replacement from a box containing 10 black and 8 red balls. Find the probability that

(a) both balls are red.

(b) first ball is black and second is red.

(c) one of them is black and other is red.

2.1.12 In a hostel, 60% of the students read Hindi newspaper, 40% read English newspaper and 20% read both Hindi and English newspapers. A student is selected at random.

(a) Find the probability that she reads neither Hindi nor English newspapers.

(b) If she reads Hindi newspaper, find the probability that she reads English newspaper.

(c) If she reads English newspaper, find the probability that she reads Hindi newspaper.

2.1.13 The probability of obtaining an even prime number on each die, when a pair of dice is rolled is

(a) 0

(b) $\frac{1}{3}$

(c) $\frac{1}{12}$

(d) $\frac{1}{36}$

Solution: Let X and Y be two random variables representing outcomes on both the die, See Table 2.13. Since both die rolls are independent,

$$\Pr\left(X = 2, Y = 2\right) = \Pr\left(X = 2\right)\Pr\left(Y = 2\right) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36} \tag{2.41}$$

$\Pr(X=2)$	The probability of occurence of 2 on die roll 1.
$\Pr(Y=2)$	The probability of occurence of 2 on die roll 2.
$\Pr(X=2, Y=2)$	The probability of occurence of 2 on both the die.

Table 2.13:

2.1.14 A bag contains 4 red and 4 black balls, another bag contains 2 red and 6 black balls. One of the two bags is selected at random and a ball is drawn from the bag which is found to be red. Find the probability that the ball is drawn from the first bag.

Solution: See Table 2.14 Given,

Table 2.14: Random Variable Declaration

Random Variable	Value of the random variable	Event
B	0	selecting first bag
	1	selecting second bag
R	0	choosing white ball from the bag
	1	choosing red ball from the bag

$$\Pr(R=1|B=0) = \frac{4}{8} = \frac{1}{2} \tag{2.42}$$

$$\Pr(R=1|B=1) = \frac{2}{8} = \frac{1}{4} \tag{2.43}$$

$$\Pr(B=0) = \frac{1}{2} \tag{2.44}$$

$$\Pr(B=1) = \frac{1}{2} \tag{2.45}$$

$$\Pr\left(B=0|R=1\right)=\frac{\Pr\left(R=1|B=0\right)\Pr\left(B=0\right)}{\Pr\left(R=1|B=0\right)\Pr\left(B=0\right)+\Pr\left(R=1|B=1\right)\Pr\left(B=1\right)} \tag{2.46}$$

$$=\frac{\frac{1}{2}\times\frac{1}{2}}{\frac{1}{2}\times\frac{1}{2}+\frac{1}{4}\times\frac{1}{2}} \tag{2.47}$$

$$=\frac{\frac{1}{4}}{\frac{1}{4}+\frac{1}{8}}=\frac{2}{3} \tag{2.48}$$

2.1.15 Cards with numbers 2 to 101 are placed in a box. A card is selected at random.Find the probability that the card has

(i) an even number

(ii) a square number

Solution:

(i) an even number

$$X=\begin{cases}1, & \text{if number is even}\\ 0, & \text{otherwise}\end{cases} \tag{2.49}$$

Then

$$p_X\left(1\right)=\frac{50}{100} \tag{2.50}$$

$$=\frac{1}{2} \tag{2.51}$$

(ii) a square number

$$Y = \begin{cases} 1, & \text{if square number} \\ 0, & \text{otherwise} \end{cases} \tag{2.52}$$

Then

$$p_Y(1) = \frac{9}{100} \tag{2.53}$$

2.1.16 The king, queen and jack of clubs are removed from a deck of 52 playing cards and then well shuffled. Now one card is drawn at random from the remaining cards. Determine the probability that the card is

(i) a club

(ii) 10 of hearts

Solution: Total number of cards left

$$52 - 3 = 49 \tag{2.54}$$

Random variable	Sample space	Value	Event	Probability
X_1	49	1	the card is a club	$\frac{10}{49}$
X_2	49	1	the card is a 10 of hearts	$\frac{1}{49}$

Table 2.15: Distribution

2.1.17 A team of medical students doing their internship have to assist during surgeries at a city hospital. The probabilities of surgeries rated as very complex, complex, routine, simple or very simple are respectively, 0.15, 0.20, 0.31, 0.26, .08. Find the probabilities that a particular surgery will be rated

(a) complex or very complex;

(b) neither very complex nor very simple;

(c) routine or complex

(d) routine or simple

Solution:

$$X = \begin{cases} 1, & \text{very complex} \\ 2, & \text{complex} \\ 3, & \text{routine} \\ 4, & \text{simple} \\ 5, & \text{very simple} \end{cases} \tag{2.55}$$

(a) complex or very complex

$$= p_X(1) + p_X(2) \tag{2.56}$$

$$= 0.15 + 0.20 \tag{2.57}$$

$$= 0.35 \tag{2.58}$$

(b) neither very complex nor very simple

$$= 1 - (p_X(1) + p_X(5)) \tag{2.59}$$

$$= 1 - (0.15 + 0.08) \tag{2.60}$$

$$= 0.77 \tag{2.61}$$

(c) routine or complex

$$= p_X(2) + p_X(3) \tag{2.62}$$

$$= 0.20 + 0.31 \tag{2.63}$$

$$= 0.51 \tag{2.64}$$

(d) routine or simple

$$= p_X(3) + p_X(4) \tag{2.65}$$

$$= 0.31 + 0.26 \tag{2.66}$$

$$= 0.57 \tag{2.67}$$

2.1.18 A card is selected from a pack of 52 cards.

(a) How many points are there in the sample space?

(b) Calculate the probability that the card is an ace of spades.

(c) Calculate the probability that the card is (i) an ace and (ii) black card.

Solution:

Table 2.1.18: Random Variables and Probability Table

Random Variable	Value of R.V.	Description
X	1, 2, 3, 4	Type of the card
Y	1, 2, 3, ..., 13	Number of the card
Z	1, 2	Colour of the card

The probabilities are as follows:

$$p_X(k) = \frac{1}{4}, \quad k \in [1,4] \tag{2.68}$$
$$p_Y(k) = \frac{1}{13}, \quad k \in [1,13] \tag{2.69}$$
$$p_Z(k) = \frac{1}{2}, \quad k \in [1,2] \tag{2.70}$$

(a) The sample space consists of all possible outcomes when selecting a card. Therefore, the sample space contains 52 points.

(b)

$$p_{XY}(1,1) = p_X(1)p_Y(1) \tag{2.71}$$
$$= \left(\frac{1}{4}\right)\left(\frac{1}{13}\right) = \frac{1}{52} \tag{2.72}$$

(c) The probability when the card choosen is ,

(i) An ace

$$p_Y(1) = \frac{1}{13} \tag{2.73}$$

(ii) Black card (Z=1)

$$p_Z(1) = \frac{1}{2} \tag{2.74}$$

2.1.19 The probability that a non leap year selected at random will contain 53 sundays.

Solution: A non-leap year has 365 days, and a week has 7 days. Using the modulo operator, we can calculate the number of weeks and the remaining days as follows:

$$\text{no. of remaining days} = 365 \pmod 7 = 1. \tag{2.75}$$

$$\text{no. of weeks} = \frac{365 - 1}{7} = \frac{364}{7} = 52. \tag{2.76}$$

Therefore, a non-leap year has 52 weeks and 1 day in total.

$$\implies 52 \text{ sundays} \tag{2.77}$$

Let X denote the day of a week.

$$p_X(k) = \frac{1}{7} \quad \{1 \le k \le 7\} \tag{2.78}$$

Hence probability of the extra day being a sunday is

$$p_X(1) = \frac{1}{7} \tag{2.79}$$

Table 2.1.19: Representation of X

Parameters	Values	Description
X	1	Sunday
	2	Monday
	3	Tuesday
	4	Wednesday
	5	Thursday
	6	Friday
	7	Saturday

2.1.20 One of the four persons John, Rita, Aslam or Gurpreet will be promoted next month. Consequently the sample space consists of four elementary outcomes S = John promoted, Rita promoted, Aslam promoted, Gurpreet promoted You are told that the chances of John's promotion is same as that of Gurpreet, Rita's chances of promotion are twice as likely as Johns. Aslam's chances are four times that of John.

(a) Determine

i. P (John promoted)

ii. P (Rita promoted)

iii. P (Aslam promoted)

iv. P (Gurpreet promoted)

(b) If A = John promoted or Gurpreet promoted, find P (A).

Solution: Let X be a random variable such that

Table 2.1.20: Random variable declaration.

RV	Value	Description
X	0	Promotion of John
	1	Promotion of Rita
	2	Promotion of Aslam
	3	Promotion of Gurpreet

Given that,

$$p_X(1) = 2p_X(0) \tag{2.80}$$

$$p_X(2) = 4p_X(0) \tag{2.81}$$

$$p_X(3) = p_X(0) \tag{2.82}$$

Also,

$$\sum_{i=0}^{3} p_X(i) = 1 \tag{2.83}$$

Hence, we get PMF as follows

(a)

$$p_X(k) = \begin{cases} \frac{1}{8} & k = 0 \\ \frac{1}{4} & k = 2 \\ \frac{1}{2} & k = 2 \\ \frac{1}{8} & k = 3 \end{cases} \tag{2.84}$$

(b) $p_X(0) + p_X(3) = \frac{1}{4}$

2.1.21 A card is drawn from a deck of 52 cards. Find the probability of getting a king or a heart or a red card.

Solution:

parameters	values	decription
X	1	red card
	0	not a red card
Y	1	king
	0	not a king

Table 2.1.21: Random variable description

Hearts is a subset of red cards,hence X covers it.Then the probabilities are

$$p_{XY}(k,m) = \begin{cases} \frac{1}{26} & k=1, m=1 \\ \frac{12}{13} & k=1, m=0 \\ \frac{1}{26} & k=0, m=1 \\ \frac{6}{13} & k=0, m=0 \end{cases} \tag{2.85}$$

The desired probability is

$$= p_{XY}(11) + p_{XY}(10) + p_{XY}(01) \tag{2.86}$$

$$= 1 - p_{XY}(00) \tag{2.87}$$

$$= 1 - \frac{6}{13} \tag{2.88}$$

$$= \frac{7}{13} \tag{2.89}$$

2.1.22 The probability that a student will pass his examination is 0.73, the probability of the student getting a compartment is 0.13, and the probability that the student will either pass or get compartment is 0.96. State True or False.

Solution:

RV	Values	Descripotion
X	0	Not pasiing exam
	1	Passing exam
	2	getting compartment

Table 2.1.22: Random variable declaration.

Given,

$$p_X(1) = 0.73 \tag{2.90}$$

$$p_X(2) = 0.13 \tag{2.91}$$

Also,

$$p_X(1) + p_X(2) = 0.86 \tag{2.92}$$

Hence, The statement is False

2.1.23 A card is selected from a pack of 52 cards

(a) How many points are there in the sample space?

(b) Calculate the probability that the cards is an ace of spades.

(c) Calculate the probability that the card is (i) an ace (ii)black card.

Solution: S is a sample space of given cards ,

Let the random variables(r.v), where X,Y and Z are uniformly distributed r.v's.

Table 2.1.23: Random Variable and probability Table

Random independent variable	value of R.V
X(denotes colour)	1,2
Y(denotes type of card)	1,2,3,4
Z(denotes value of card chosen)	$1 \leq \mathrm{Z} \leq 13$

$$\text{PMF for Z is:} \tag{2.93}$$

$$p_Z(k) = \frac{1}{13} \qquad \text{k} = 1, 2, 3, ..., 13 \tag{2.94}$$

$$\text{Similarly the PMF for Y and X are:} \tag{2.95}$$

$$p_Y(m) = \frac{1}{4} \qquad \text{m} = 1, 2, 3 \tag{2.96}$$

$$p_X(n) = \frac{1}{2} \qquad \text{n} = 1, 2 \tag{2.97}$$

$$\tag{2.98}$$

(a) The number of sample space points is 52

(b)

$$p_{ZY}(1,1) = p_Z(1)p_Y(1) \qquad \text{as } ZY = 0 \tag{2.99}$$

$$= \left(\frac{1}{4}\right)\left(\frac{1}{13}\right) \tag{2.100}$$

$$= \frac{1}{52} \tag{2.101}$$

(c) The probability when the card choosen is ,

(i) an ace

$$p_Z(1) = \frac{1}{13}. \tag{2.102}$$

(ii) black card

$$p_X(1) = \frac{1}{2}. \tag{2.103}$$

2.1.24 In a non-leap year, the probability of having 53 tuesdays or 53 wednesdays is

Solution: A non-leap year has a total of 365 days, and a week has 7 days.

So it can be expressed as

$$365\text{days} = 52 \times 7 + 1\text{day} \tag{2.104}$$

$\implies$ 52 tuesdays or wednesdays

Random variable X denotes the days of a week

$$p_X(k) = \frac{1}{7}; \quad (1 < k < 7) \tag{2.105}$$

So the probability of extra day being tuesday or wednesday is

$$p_X(3) + p_X(4) = \frac{1}{7} + \frac{1}{7} = \frac{2}{7} \tag{2.106}$$

2.1.25 There are 1000 sealed envelopes in a box, 10 of them contain a cash prize of Rs 100 each, 100 of them contain a cash prize of Rs 50 each and 200 of them contain a cash prize of Rs 10 each and rest do not contain any cash prize. If they are well shuffled and an envelope is picked up out, what is the probability that it contains no cash prize?

Solution:

RV	Value	Description
X	0	Rs. 0
	1	Rs. 10
	2	Rs. 50
	3	Rs. 100

Table 2.1.25: Random variable declaration.

PMF is

$$p_X(k) = \begin{cases} \frac{690}{1000} & k = 0 \\ \frac{200}{1000} & k = 1 \\ \frac{100}{1000} & k = 2 \\ \frac{10}{1000} & k = 3 \end{cases} \tag{2.107}$$

Hence,

$$p_X(0) = \frac{690}{1000} \tag{2.108}$$
$$= \frac{69}{100} \tag{2.109}$$

2.1.26 A die is thrown and a card is selected at random from a deck of 52 playing cards. The probability of getting an even number on the die and a spade card.

Solution: We know that 52 playing cards contain 13 spades, 13 clubs, 13 diamonds and 13 hearts.

Let X_1 and X_2 be two random variables denoting the number on dice and card selected respectively:

Table 2.1.26: Description of random variables

Parameters	Values	Description
X_1	0	Even number
	1	Odd number
X_2	0	Spade
	1	Club
	2	Diamond
	3	Heart

Probability of the number on the dice:

$$p_{X_1}(k) = \frac{1}{2} \quad \{k = 0, 1\} \tag{2.110}$$

Probability of selecting a card:

$$p_{X_2}(k) = \frac{13}{52} = \frac{1}{4} \quad \{k = 0, 1\} \tag{2.111}$$

Therefore, the probability of getting an even number on the die and a spade card

$$= pr(X_1 = 0.X_2 = 0) \tag{2.112}$$

$$= p_{X_1}(0) \times p_{X_2}(0) \tag{2.113}$$

$$= \frac{1}{2} \times \frac{1}{4} \tag{2.114}$$

$$= \frac{1}{8} \tag{2.115}$$

2.1.27 If 4-digit numbers greater than 5,000 are randomly formed from the digits 0, 1, 3, 5, and 7, what is the probability of forming a number divisible by 5 when:

(a) The digits are repeated?

(b) The repetition of digits is not allowed?

Solution: Solution: Let X be a random variable such that:

$$X = \begin{cases} 0 & n \not\equiv 0 \pmod 5 \\ 1 & n \equiv 0 \pmod 5 \end{cases} \tag{2.116}$$

Let N be a 4 digit number $X_1X_2X_3X_4$ where X_1, X_2, X_3, X_4 are digits of the number N.

Digit	**Position**
X_1	$Thousands's\ Digit$
X_2	$Hundred's\ Digit$
X_3	$Ten's\ Digit$
X_4	$One's\ Digit$

Table 2.1.27: Listing variables

Let's solve each part separately.

(i) Repetition of digits

Let number of favourable outcomes be N(A) and total outcomes be N(T).

For N > 5000,

Digit	**Favourable**
X_1	$5, 7$
X_2, X_3, X_4	$0, 1, 3, 5, 7$

Table 2.1.27: Conditions for N greater than 5000

We must also exclude the case of 5000. Hence,

$$N(T) = (2 \times 5 \times 5 \times 5) - 1 \tag{2.117}$$

$$\implies N(T) = 249 \tag{2.118}$$

Digit	**Favourable**
X_1	$5, 7$
X_2, X_3	$0, 1, 3, 5, 7$
X_4	$0, 5$

Table 2.1.27: Conditions for N greater than 5000 and divisible by 5

Here also we must exclude the case of 5000.

$$N(A) = (2 \times 5 \times 5 \times 2) - 1 \tag{2.119}$$

$$\implies N(A) = 99 \tag{2.120}$$

With this information we can find the required answer,

$$\Pr(X = 1) = \frac{N(A)}{N(T)} \tag{2.121}$$

$$\implies \Pr(X = 1) = \frac{33}{83} \tag{2.122}$$

(ii) No Repetition of Digits

Let number of favourable outcomes be N(B) and total outcomes be N(T).

For N $>$ 5000,

H Hence,

Digit	**Favourable**
X_1	$5, 7$
X_2, X_3, X_4	$0, 1, 3, 5, 7$

Table 2.1.27: Conditions for N greater than 5000

$$N(T) = (2 \times 4 \times 3 \times 2) \tag{2.123}$$

$$\implies N(T) = 48 \tag{2.124}$$

For N $>$ 5000 and also divisble by 5:

$$X_4 = \begin{cases} 0 & X_1 = 5 \\ 5, 0 & X_1 = 7 \end{cases} \tag{2.125}$$

Hence,

$$N(B) = (1 \times 3 \times 2 \times 1) + (1 \times 3 \times 2 \times 2) \tag{2.126}$$

$$\implies N(B) = 18 \tag{2.127}$$

With this information we can find the required answer,

$$\Pr(X = 1) = \frac{N(B)}{N(T)} \tag{2.128}$$

$$\implies \Pr(X = 1) = \frac{3}{8} \tag{2.129}$$

2.1.28 Consider the probability space $(\Omega, \mathcal{G}, P)$ where $\Omega = [0, 2]$ and $\mathcal{G} = \{\phi, \Omega, [0, 1], (1, 2]\}$. Let X and Y be two functions on Ω defined as

$$X(\omega) = \begin{cases} 1 & \text{if } \omega \in [0, 1] \\ 2 & \text{if } \omega \in (1, 2] \end{cases}$$

and

$$Y(\omega) = \begin{cases} 2 & \text{if } \omega \in [0, 1.5] \\ 3 & \text{if } \omega \in (1.5, 2]. \end{cases}$$

Then which one of the following statements is true?

(A) X is a random variable with respect to $\mathcal{G}$, but Y is not a random variable with respect to $\mathcal{G}$.

(B) Y is a random variable with respect to $\mathcal{G}$, but X is not a random variable with respect to $\mathcal{G}$.

(C) Neither X nor Y is a random variable with respect to $\mathcal{G}$.

(D) Both X and Y are random variables with respect to $\mathcal{G}$.

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Solution:

(a) For X to be a random variable with respect to $\mathcal{G}$:

$$X^{-1}(X(\omega)) \in \mathcal{G} \qquad \forall X(\omega) \in \mathcal{T}_1 \tag{2.130}$$

where $\mathcal{T}_1$ is the range of $X(\omega)$.

If $X(\omega) = 1:$

$$X^{-1}(X(\omega)) = X^{-1}(1) \tag{2.131}$$

$$= [0, 1] \tag{2.132}$$

$$\in \mathcal{G} \tag{2.133}$$

If $X(\omega) = 2:$

$$X^{-1}(X(\omega)) = X^{-1}(2) \tag{2.134}$$

$$= (1, 2] \tag{2.135}$$

$$\in \mathcal{G} \tag{2.136}$$

$\therefore X$ is a random variable with respect to $\mathcal{G}$.

(b) For Y to be a random variable with respect to $\mathcal{G}$:

$$Y^{-1}(Y(\omega)) \in \mathcal{G} \qquad \forall Y(\omega) \in \mathcal{T}_2 \tag{2.137}$$

where $\mathcal{T}_2$ is the range of $Y(\omega)$.

If $Y(\omega) = 2:$

$$Y^{-1}(Y(\omega)) = Y^{-1}(2) \tag{2.138}$$

$$= [0, 1.5] \tag{2.139}$$

$$\notin \mathcal{G} \tag{2.140}$$

$\therefore Y$ is not a random variable with respect to $\mathcal{G}$.

2.2. Exercises

2.2.1 A die is loaded in such a way that each odd number is twice as likely to occur as each even number. Find $P(G)$, where G is the event that a number greater than 3 occurs on a single roll of the die.

Solution: See Table 2.29 for the input parameters. Then,

Parameter	Value	Description
X	{1,2,3,4,5,6}	Number obtained on the die

Table 2.29: Parameters and their Description

$$p_X(k) = \begin{cases} 2p, & if\ k = 2m-1 \\ p, & if\ k = 2m \end{cases} \tag{2.141}$$

Since $1 \leq X \leq 6$,

$$\sum_{i=1}^{6} \Pr(X = i) = 1 \tag{2.142}$$

$$\implies 6p + 3p = 1 \tag{2.143}$$

$$\implies p = \frac{1}{9} \tag{2.144}$$

The CDF

$$F_X(k) = \begin{cases} \frac{3k+1}{18}, & if\ k = 2m-1 \\ \frac{k}{6}, & if\ k = 2m \end{cases} \tag{2.145}$$

Thus,

$$\Pr(G) = \Pr(X > 3) = F_X(6) - F_X(3) \tag{2.146}$$

$$= 1 - \frac{3(3)+1}{18} = \frac{4}{9} \tag{2.147}$$

2.2.2 All the jacks, queens and kings are removed from a deck of 52 playing cards. The remaining cards are well shuffled and then one card is drawn at random. Giving ace a value 1 similar value for other cards, find the probability that the card has a value

(a) 7

(b) greater than 7

(c) less than 7

Solution: Number of cards left after removing all jacks, queens and kings(=N)

$$= 52 - 4 \times 3 \tag{2.148}$$

$$= 40 \tag{2.149}$$

Parameter	Value	Description
X	1-10	Represents the value of the card picked

Finding pmf:

$$p_X(k) = \Pr(X = k) \ \forall \ 1 \le k \le 10 \tag{2.150}$$

$$= \frac{4 \times 1}{40} \tag{2.151}$$

$$= \frac{1}{10} \tag{2.152}$$

$$\therefore p_X(k) = \begin{cases} \frac{1}{10} & 1 \le k \le 10 \\ 0 & \text{otherwise} \end{cases} \tag{2.153}$$

CDF for the following pmf is:

$$F_X(k) = \sum_{m=0}^{k} p_X(m) \;\forall\; 1 \leq k \leq 10 \tag{2.154}$$
$$= k \times \frac{1}{10} \tag{2.155}$$
$$= \frac{k}{10} \tag{2.156}$$
$$\therefore F_X(k) = \begin{cases} 0 & k \leq 0 \\ \frac{k}{10} & 1 \leq k \leq 10 \\ 1 & k > 10 \end{cases} \tag{2.157}$$

(a) Probability that card has value equal to 7:

$$= p_X(7) \tag{2.158}$$
$$= 1 \times \frac{1}{10} = \frac{1}{10} \tag{2.159}$$

(b) Probability that card has value greater than 7

$$= F_X(10) - F_X(7) \tag{2.160}$$
$$= 1 - \frac{7}{10} = \frac{3}{10} \tag{2.161}$$

(c) Probability that card has value less than 7

$$= F_X(6) \tag{2.162}$$

$$= \frac{6}{10} \tag{2.163}$$

2.2.3 A Lot consists of 48 mobile phones of which 42 are good, 3 have only minor defects and 3 have major defects.Varnika will buy a phone if it is good but the trader will only buy a mobile if it has no major defects. One phone is selected at random from the lot. What is the probability that it is

(a) acceptable to Varnika?

(b) acceptable to the trader?

Solution: Then

Random variable	values	Events
X	0	The Mobile is good
	1	The Mobile has major defects
	2	The Mobile has minor defects

(a)

$$\Pr\left(X_1 = 0\right) = \frac{42}{48} \tag{2.164}$$

$$= \frac{7}{8} \tag{2.165}$$

(b)

$$1 - \Pr(X_1 = 1) = 1 - \frac{3}{48} \tag{2.166}$$

$$= \frac{15}{16} \tag{2.167}$$

2.2.4 A student says that if you throw a die, it will show up 1 or not 1. Therefore, the probability of getting 1 and the probability of getting 'not 1' each is equal to $\frac{1}{2}$. Is this correct? Give reasons.

Solution:

Solution:

Let

$$p_X(k) = \begin{cases} \frac{1}{6}, & 1 \le X \le 6 \\ 0, & \text{otherwise} \end{cases} \tag{2.168}$$

$$\Pr(X \neq 1) = 1 - \Pr(X = 1) \tag{2.169}$$

$$= 1 - p_X(1) \tag{2.170}$$

$$= 1 - \frac{1}{6} \tag{2.171}$$

$$= \frac{5}{6} \tag{2.172}$$

$$\implies \Pr(X \neq 1) \neq \Pr(X = 1) \tag{2.173}$$

Since, $\Pr(X = 1)$ and $\Pr(X \neq 1)$ are not equal.

$\therefore$ The given statement is not true.

2.2.5 Four candidates A, B, C, D have ap- plied for the assignment to coach a school cricket team. If A is twice as likely to be selected as B, and B and C are given about the same chance of being selected, while C is twice as likely to be selected as D, what are the probabilities that

(a) C will be selected?

(b) A will not be selected?

Solution: Let X be a random variable

$$X = \begin{cases} 0 & \text{A is selected} \\ 1 & \text{B is selected} \\ 2 & \text{C is selected} \\ 3 & \text{D is selected} \end{cases} \tag{2.174}$$

Given,

$$p_X(0) = 2p_X(1) \tag{2.175}$$

$$p_X(1) = p_X(2) \tag{2.176}$$

$$p_X(2) = 2p_X(3) \tag{2.177}$$

Using axioms of probability:

$$\sum_{k=0}^{3} p_X(k) = 1 \tag{2.178}$$

which gives

$$p_X(k) = \begin{cases} \frac{4}{9} & k = 0 \\ \frac{2}{9} & k = 1 \\ \frac{2}{9} & k = 2 \\ \frac{1}{9} & k = 3 \end{cases} \tag{2.179}$$

(a) For C getting selected:

$$\implies p_X(2) = \frac{2}{9} \tag{2.180}$$

(b) For A not getting selected:

$$= 1 - p_X(1) \tag{2.181}$$

$$= 1 - \frac{4}{9} \tag{2.182}$$

$$= \frac{5}{9} \tag{2.183}$$

2.2.6 A bag contain 24 balls of which x balls are red, $2x$ are white and $3x$ are blue. A ball is selected at random, What is the probability that it is

a) not red ?

b) white ?

Solution: Total number of balls :

$$x + 2x + 3x = 24 \tag{2.184}$$

$$\implies 6x = 24 \tag{2.185}$$

$$\implies x = 4 \tag{2.186}$$

Let X be a Random variable such that

Parameter	**Value**	**Description**
X	0	Red ball
	1	White ball
	2	Blue ball

a) Required probability:

$$= 1 - \Pr(X = 0) \tag{2.187}$$

$$= 1 - \frac{x}{24} \tag{2.188}$$

$$= 1 - \frac{1}{6} \tag{2.189}$$

$$= \frac{5}{6} \tag{2.190}$$

b) Required probability:

$$\Pr(X = 1) = \frac{2x}{24} \tag{2.191}$$

$$= \frac{1}{3} \tag{2.192}$$

If the letters of the word ASSASSINATION are arranged at random. Find the Probability that

(a) Four $S's$ come consecutively in the word

(b) Two $I's$ and two $N's$ come together

(c) All $A's$ are not coming together

(d) No two $A's$ are coming together

Solution: Number of letters in word 'ASSASSINATION' = 13

Letter's are $3A's, 4S's, 2I's, 2N's, 1T's$ and $1O's$

$$\text{Total ways of arranging letters} = \frac{13!}{3!4!2!2!} \tag{2.193}$$

]

Random variable	values	Events
X	0	All $S's$ are together
	1	All $S's$ are not together
Y	0	$2I's$ and $2N's$ are together
	1	$2I's$ and $2N's$ are not together
Z	0	All $A's$ together
	1	Only $2A's$ are together
	2	No $2A's$ are together

(a)

$$p_X(0) = \frac{\frac{10!}{3!2!2!}}{\frac{13!}{3!4!2!2!}} \tag{2.194}$$

$$= \frac{2}{143} \tag{2.195}$$

(b)

$$p_Y(0) = \frac{\frac{10!4!}{3!4!2!2!}}{\frac{13!}{3!4!2!2!}} \tag{2.196}$$

$$= \frac{2}{143} \tag{2.197}$$

(c)

$$p_Z(0) = \frac{\frac{11!}{4!2!2!}}{\frac{13!}{3!4!2!2!}} \tag{2.198}$$

$$= \frac{1}{26} \tag{2.199}$$

Probability of all $A's$ not coming together:

$$p_Z(1) + p_Z(2) = 1 - p_Z(0) \tag{2.200}$$

$$= 1 - \frac{1}{26} \tag{2.201}$$

$$= \frac{25}{26} \tag{2.202}$$

(d)

$$p_Z(2) = \frac{\frac{11!10!}{3!8!4!2!2!}}{\frac{13!}{3!4!2!2!}} \tag{2.203}$$

$$= \frac{15}{26} \tag{2.204}$$

Chapter 3

Conditional Probability

3.1. Examples

3.1 Given that E and F are events such that $P(E) = 0.6, P(F) = 0.3$ and $P(EF) = 0.2$, find $P(E \mid F)$ and $P(F \mid E)$.

Solution:

$$\Pr(E|F) = \frac{\Pr(EF)}{\Pr(F)} = \frac{0.2}{0.3} = \frac{2}{3} \tag{3.1}$$

$$\Pr(F|E) = \frac{\Pr(EF)}{\Pr(E)} = \frac{0.2}{0.6} = \frac{1}{3} \tag{3.2}$$

3.2 Compute $\Pr(A|B)$, if $\Pr(B) = 0.5$ and $\Pr(AB) = 0.32$.

3.3 Probability that A speaks truth is $\frac{4}{5}$. A coin is tossed. A reports that a head appears. The probability that actually there was head is

(a) $\frac{4}{5}$

(b) $\frac{1}{2}$

(c) $\frac{1}{5}$

(d) $\frac{2}{5}$

3.4 Compute $\Pr(A|B)$, if $\Pr(B) = 0.5$ and $\Pr(AB) = 0.32$.

Solution: By using property of conditional probability we have,

$$\Pr(A|B) = \frac{\Pr(AB)}{\Pr B} = \frac{0.32}{0.5} = 0.64 \tag{3.3}$$

3.5 If $\Pr(A) = 0.8$, $\Pr(B) = 0.5$ and $\Pr(B|A = 0.4)$, find

(a) $\Pr(AB)$

(b) $\Pr(A|B)$

(c) $\Pr(A+B)$

Solution:

3.6 Mother, Father and Son line up at random for a family picture. Determine $\Pr(E \mid F)$ where E: Son on one end, F: Father in middle

Solution: The total ways of arranging Father, Son, Mother in the family chart is 3! = 6. The probability that Father in middle is

$$\Pr(F) = \frac{2!}{3!} = \frac{1}{3} \tag{3.4}$$

The probability that Father in middle and Son is on one end is

$$\Pr(EF) = \frac{2!}{3!} = \frac{1}{3} \tag{3.5}$$

Thus,

$$\Pr(E \mid F) = \frac{\Pr(EF)}{\Pr(F)} = 1 \tag{3.6}$$

3.7 Assume that each born child is equally likely to be a boy or a girl. If a family has two children, what is the conditional probability that both are girls given that

(a) The youngest is a girl

(b) At least one is a girl

Solution:

Variable	Description	Probability
$X_i = 1$	ith born child is a boy	$\Pr(X_i = 1) = 0.50$
$X_i = 0$	ith born child is a girl	$\Pr(X_i = 0) = 0.50$

Table 3.1: Random variable definitions.

(a)

$$\Pr\left((X_1 + X_2)'|X_2'\right) = \frac{\Pr\left((X_1'X_2')X_2'\right)}{\Pr\left(X_2'\right)} \tag{3.7}$$

$$= \frac{\Pr\left(X_1'\right)\Pr\left(X_2'\right)}{\Pr\left(X_2'\right)} \tag{3.8}$$

$$= \Pr\left(X_1'\right) = \frac{1}{2} \tag{3.9}$$

(b)

$$\Pr\left((X_1 + X_2)'|(X_1X_2)'\right) = \frac{\Pr\left((X_1'X_2')(X_1' + X_2')\right)}{1 - \Pr\left(X_1X_2\right)} \tag{3.10}$$

$$= \frac{\Pr\left(X_1'X_2'\right)}{1 - \Pr\left(X_1X_2\right)} \tag{3.11}$$

$$= \frac{\Pr\left(X_1'\right)\Pr\left(X_2'\right)}{1 - \Pr\left(X_1\right)\Pr\left(X_2\right)} \tag{3.12}$$

$$= \frac{\frac{1}{2} \times \frac{1}{2}}{1 - (\frac{1}{2} \times \frac{1}{2})} = \frac{1}{3} \tag{3.13}$$

3.8 An instructor has a question bank consisting of 300 easy True / False questions, 200 difficult True / False questions, 500 easy multiple choice questions and 400 difficult multiple choice questions. If a question is selected at random from the question bank, what is the probability that it will be an easy question given that it is a multiple choice question?

Solution: From the law of total probability,

Variable	**Event**
$X = 0$	Easy question
$X = 1$	Difficult question
$Y = 0$	True/False question
$Y = 1$	Multiple choice question

Table 3.2:

$$p_X(0) + p_X(1) = 1 \tag{3.14}$$

$$p_Y(0) + p_Y(1) = 1 \tag{3.15}$$

From Table 3.2,

$$p_X(0) = p_{X,Y}(0,0) + p_{X,Y}(0,1) \tag{3.16}$$

$$= \frac{300 + 500}{300 + 200 + 500 + 400} = \frac{4}{7} \tag{3.17}$$

$$p_Y(0) = p_{X,Y}(0,0) + p_{X,Y}(1,0) \tag{3.18}$$

$$= \frac{300 + 200}{300 + 200 + 500 + 400} = \frac{5}{14} \tag{3.19}$$

From (3.14), (3.15) and (3.19),

$$p_X(1) = 1 - p_X(0) = \frac{3}{7} \tag{3.20}$$

$$p_Y(1) = 1 - p_Y(0) = \frac{9}{14} \tag{3.21}$$

$$p_{X,Y}(0,0) = \frac{300}{1400} = \frac{3}{14} \tag{3.22}$$

$$p_{X,Y}(1,1) = \frac{400}{1400} = \frac{2}{7} \tag{3.23}$$

From (3.16), (3.18), (3.19) and (3.23),

$$p_{X,Y}(0,1) = p_X(0) - p_{X,Y}(0,0) = \frac{5}{14} \tag{3.24}$$

$$p_{X,Y}(1,0) = p_Y(0) - p_{X,Y}(0,0) = \frac{1}{7} \tag{3.25}$$

By definition,

$$p_{X|Y}(x|y) = \frac{p_{X,Y}(x,y)}{p_Y(y)} \tag{3.26}$$

From (3.22) and (3.25),

$$p_{X|Y}(0|1) = \frac{p_{X,Y}(0,1)}{p_Y(1)} \tag{3.27}$$

$$= \frac{\frac{5}{14}}{\frac{9}{14}} = \frac{5}{9} \tag{3.28}$$

3.9 If $\Pr(A) = \frac{1}{2}$, $\Pr(B) = 0$, then $\Pr(A \mid B)$ is

(a) 0

(b) $\frac{1}{2}$

(c) not defined

(d) 1

Since

$$\Pr(A \mid B) = \frac{\Pr(AB)}{\Pr(B)}, \tag{3.29}$$

$$\Pr(A \mid B) \text{ is not defined} \tag{3.30}$$

3.10 If A and B are events such that

$$\Pr(A|B) = \Pr(B|A) \tag{3.31}$$

then

(a) $A \subset B$ but $A \neq B$

(b) $A = B$

(c) $A \cap B = \phi$

(d) $\Pr(A) = \Pr(B)$

Solution: Using Bayes' Rule,

$$\Pr(AB) = \Pr(A)\Pr(B|A) \tag{3.32}$$

$$= \Pr(B)\Pr(A|B) \tag{3.33}$$

Using (3.31) in (3.32) and (3.33),

$$\Pr(A) = \Pr(B) \tag{3.34}$$

We consider the options one by one.

(a) If $A \subset B$ and $A \neq B$, then we can write $B = A + C$, where $AC = 0$ and $C \neq 0$. Thus,

$$\Pr(B) = \Pr(A + C) \tag{3.35}$$

$$= \Pr(A) + \Pr(C) - \Pr(AC) \tag{3.36}$$

$$= \Pr(A) + \Pr(C) > \Pr(A) \tag{3.37}$$

However, (3.37) contradicts (3.34).

(b) We give a counterexample to show this is wrong. Consider A as the event that an even number shows on rolling a fair die and B as the event that a prime number shows on rolling a fair die. The joint pmf is shown in Table 3.3. Clearly,

$$\Pr(A|B) = \frac{\Pr(AB)}{\Pr(B)} = \frac{\frac{1}{6}}{\frac{1}{6} + \frac{1}{3}} = \frac{1}{2} \tag{3.38}$$

$$\Pr(B|A) = \frac{\Pr(AB)}{\Pr(A)} = \frac{\frac{1}{6}}{\frac{1}{6} + \frac{1}{3}} = \frac{1}{2} \tag{3.39}$$

(c) The same example as before provides the required counterexample, as $\Pr(AB) = \frac{1}{6}$.

(d) This is the correct answer, as discussed above.

	A	$\bar{A}$
B	$\frac{1}{6}$	$\frac{1}{3}$
$\bar{B}$	$\frac{1}{3}$	$\frac{1}{6}$

Table 3.3: Joint pmf for events A and B.

3.11 Consider the experiment of throwing a die, if a multiple of 3 comes up, throw the die again and if any other number comes, toss a coin. Find the conditional probability of the event 'the coin shows a tail', given that 'at least one die shows a 3'.

Solution: Let X denote the die roll for the first trial. The pmf of X is

$$\Pr(X = k) = \begin{cases} \frac{1}{6} & 1 \le i \le 6 \\ 0 & \text{otherwise} \end{cases} \tag{3.40}$$

Let Y be the random variable denoting the outcome of the coin toss in the second trial. The pmf of Y is

$$\Pr(Y = k) = \begin{cases} \frac{1}{2} & 0 \le i \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{3.41}$$

We are required to find $\Pr(Y = 1|X = 3)$. However, from the given data,

$$\Pr(Y = 1,\ X = k) = \begin{cases} \frac{1}{12} & k \in \{1, 2, 4, 5\} \\ 0 & \text{otherwise} \end{cases} \tag{3.42}$$

Therefore, from (3.42),

$$\Pr(Y = 1|X = 3) = \frac{\Pr(X = 3,\ Y = 1)}{\Pr(X = 3)} = 0 \tag{3.43}$$

3.12 Two cards are drawn at random and without replacement from a pack of 52 playing cards. Find the probability that both the cards are black.

Solution: Let E_1 denote the event that the first card drawn is Black, E_2 denote the event that the second card drawn is Black. Then

$$\Pr(E_1) = \frac{26}{52},\ \Pr(E_2 \mid E_1) = \frac{25}{51} \tag{3.44}$$

$$\implies \Pr(E_1 E_2) = \Pr(E_1)\Pr(E_2 \mid E_1) = \frac{25}{102} \tag{3.45}$$

3.13 Let A and B be independent events with $P(A) = 0.3$ and $P(B) = 0.4$. Find

(a) $P(A \cap B)$

(b) $P(A \cup B)$

(c) $P(A|B)$

(d) $P(B|A)$

3.14 An urn contains 5 red and 5 black balls. A ball is drawn at random, its colour is noted and is returned to the urn. Moreover, 2 additional balls of the colour drawn are put in the urn and then a ball is drawn at random. What is the probability that the second ball is red?

Solution: The given information is summarized in Tables 3.4 and 3.5.

RV	**Values**	**Description**
X	$\{0, 1\}$	1st draw - 0: Red, 1: Black
Y	$\{0, 1\}$	2nd draw - 0: Red, 1: Black

Table 3.4: Random variables X,Y

The required probability is given by

$$\Pr(Y = 0) = \Pr(X = 0)\Pr(Y = 0 \mid X = 0) + \Pr(X = 1)\Pr(Y = 0 \mid X = 1) \tag{3.46}$$

$$= \left(\frac{5}{10} \times \frac{7}{12}\right) + \left(\frac{5}{10} \times \frac{5}{12}\right) = \frac{1}{2} \tag{3.47}$$

Event	Probability
$\Pr(X=0)$	$\frac{5}{10}$
$\Pr(X=1)$	$\frac{5}{10}$
$\Pr(Y=1 \mid X=0)$	$\frac{7}{12}$
$\Pr(Y=1 \mid X=1)$	$\frac{5}{12}$

Table 3.5: Probabilities

3.15 A bag contains 4 red and 4 black balls, another bag contains 2 red and 6 black balls. One of the two bags is selected at random and a ball is drawn from the bag which is found to be red. Find the probability that the ball is drawn from the first bag.

3.16 Of the students in a college, it is known that 60% reside in hostel and 40% are day scholars (not residing in hostel). Previous year results report that 30% of all students who reside in hostel attain A grade and 20% of day scholars attain A grade in their annual examination. At the end of the year, one student is chosen at random from the college and he has an A grade, what is the probability that the student is a hostlier?

Solution:

Let

$$X = \begin{cases} 0, & \text{if student is resides in hostel} \\ 1, & \text{if student is a day scholar} \end{cases} \tag{3.48}$$

$$Y = \begin{cases} 0, & \text{if student does not attain A grade} \\ 1, & \text{if student attains A grade} \end{cases} \tag{3.49}$$

From the given data,

$$\Pr(X=0) = \frac{3}{5} \tag{3.50}$$

$$\Pr(X=1) = \frac{2}{5} \tag{3.51}$$

$$\Pr(Y=1 \mid X=0) = \frac{3}{10} \tag{3.52}$$

$$\Pr(Y=1 \mid X=1) = \frac{1}{5} \tag{3.53}$$

The desired probability is

$$\Pr(X=0 \mid Y=1) = \frac{\Pr(Y=1 \mid X=0) \times \Pr(X=0)}{\sum_{k=0}^{1} \Pr(Y=1 \mid X=k) \times \Pr(X=k)} \tag{3.54}$$

$$= \frac{\frac{3}{10} \times \frac{3}{5}}{\frac{3}{10} \times \frac{3}{5} + \frac{1}{5} \times \frac{2}{5}} = \frac{9}{13} \tag{3.55}$$

3.17 In answering a question on a multiple choice test, a student either knows the answer or guesses. Let $\frac{3}{4}$ be the probability that he knows the answer and $\frac{1}{4}$ be the probability that he guesses. Assuming that a student who guesses at the answer will be correct with probability $\frac{1}{4}$. What is the probability that the student knows the answer given that he answered it correctly?

Solution: See Tables 3.17 and 3.17

Random Variable	Description
$X = 0$	Student guesses the answer
$X = 1$	Student knows the answer
$Y = 0$	Answer is incorrect
$Y = 1$	Answer is correct

Table 3.6: Random Variable and their description

Pr(Event)	Value
Pr(Y=1 | X=0)	0.25
Pr(Y=1 | X=1)	1
Pr(X=0)	0.25
Pr(X=1)	0.75

Table 3.7: Probability of events

The probability that the student knows the answer and he answered it correctly is

$$\Pr(X = 1|Y = 1) = \frac{\Pr(Y = 1|X = 1)\Pr(X = 1)}{\sum_{i=0}^{i=1} Pr(Y = 1|X = i)Pr(X = i)} \tag{3.56}$$

$$= \frac{0.75}{0.25 \times 0.25 + 1 \times 0.75} = 0.92308 \tag{3.57}$$

3.18 A laboratory blood test is 99% effective in detecting a certain disease when it is in fact, present. However, the test also yields a false positive result for 0.5% of the healthy person tested (i.e. if a healthy person is tested, then, with probability 0.005, the test will imply he has the disease). If 0.1 percent of the population actually has the disease, what is the probability that a person has the disease given that his test result is positive ?

Solution: See Table 3.18 for the given information.

$$\Pr(E_2) = 1 - \Pr(E_1) = 1 - 0.001 = 0.999 \tag{3.58}$$

A:	Person with positive blood test	$\Pr(A)$
E_1:	Person suffering from a disease	$\Pr(E_1)$=0.001
E_2:	Person not suffering from a disease	$\Pr(E_2)$=0.999
$A\|E_1$:	Event of positive blood test when person suffers from disease	$\Pr(A\|E_1)$=0.99
$A\|E_2$:	Event of positive blood test when person not suffers from disease	$\Pr(A\|E_2)$=0.005

Table 3.8: Given Information

$$\Pr(E_1|A) = \frac{\Pr(E_1)\Pr(A|E_1)}{\sum_{i=1}^{2}\Pr(E_i)\Pr(A|E_i)} \tag{3.59}$$

$$= \frac{0.001 \times 0.99}{0.001 \times 0.99 + 0.999 \times 0.005} = \frac{22}{133} \tag{3.60}$$

3.19 There are three coins. One is a two headed coin (having head on both faces), another is a biased coin that comes up heads 75% of the time and third is an unbiased coin. One of the three coins is chosen at random and tossed, it shows heads, what is the probability that it was the two headed coin ?

Solution: Define the random variable X as in Table 3.9. Clearly, the pmf of X is

$X = 1$	Two-headed coin is selected.
$X = 2$	75% biased coin is selected.
$X = 3$	Fair coin is selected.

Table 3.9: Definition of X.

$$\Pr(X = k) = \begin{cases} \frac{1}{3} & 1 \le k \le 3 \\ 0 & \text{otherwise} \end{cases} \tag{3.61}$$

Let the random variables Y_1, Y_2 and Y_3 (one for each coin) be defined as

$$Y_1 \sim \text{Ber}(1) \tag{3.62}$$

$$Y_2 \sim \text{Ber}\left(\frac{3}{4}\right) \tag{3.63}$$

$$Y_3 \sim \text{Ber}\left(\frac{1}{2}\right) \tag{3.64}$$

Define Y as

$$Y \triangleq \sum_{i=1}^{3} \mathbf{1}_i (X) \, Y_i \tag{3.65}$$

where $\mathbf{1}$ denotes the indicator random variable, defined as

$$\mathbf{1}_i (X) = \begin{cases} 1 & \text{if } X = i \\ 0 & \text{otherwise} \end{cases} \tag{3.66}$$

We are required to find $\Pr(X = 1 | Y = 1)$. However, from Bayes' Rule,

$$\Pr(X = 1, Y = 1) = \Pr(X = 1) \Pr(Y = 1 | X = 1) \tag{3.67}$$

$$= \Pr(Y = 1) \Pr(X = 1 | Y = 1) \tag{3.68}$$

Note from (3.65) that

$$X = 1 \implies Y = Y_1 \tag{3.69}$$

and also,

$$\Pr(Y = 1) = \sum_{i=1}^{3} \Pr(X = i) \Pr(Y_i = 1) \tag{3.70}$$

$$= \frac{1}{3} \left(1 + \frac{3}{4} + \frac{1}{2} \right) = \frac{3}{4} \tag{3.71}$$

Thus, from (3.67), (3.68) and (3.71), we see that

$$\Pr(X = 1|Y = 1) = \frac{\Pr(X = 1)\Pr(Y_1 = 1)}{\Pr(Y = 1)} \tag{3.72}$$

$$= \frac{\frac{1}{3}}{\frac{3}{4}} = \frac{4}{9} \tag{3.73}$$

3.20 An insurance company insured 2000 scooter drivers, 4000 car drivers and 6000 truck drivers. The probability of an accidents are 0.01, 0.03 and 0.15 respectively. One of the insured persons meets with an accident. What is the probability that he is a scooter driver?

3.21 A factory has two machines A and B. Past record shows that machine A produced 60% of the items of output and machine B produced 40% of the items. Further, 2% of the items produced by machine A and 1% produced by machine B were defective. All the items are put into one stockpile and then one item is chosen at random from this and is found to be defective. What is the probability that it was produced by machine B?

3.22 . Two groups are competing for the position on the Board of directors of a corporation. The probabilities that the first and the second groups will win are 0.6 and 0.4 respectively. Further, if the first group wins, the probability of introducing a new product is 0.7 and the corresponding prbability is 0.3 if the second group wins. Find the probability that the new product introduced was by the second group.

Solution: The given information is listed in Tables 3.11 and 3.13

RV	Values	Description
X	{1,2}	1:Group1 ,2:Group2
Y	{0,1}	0:New product not introduced ,1:New product introduced

Table 3.11: Random variables(RV) X,Y

Event	Probability	Description
$\Pr(X = 1)$	0.6	First group winning
$\Pr(X = 2)$	0.4	Second group winning
$\Pr(Y = 1 \mid X = 1)$	0.7	Introducing 1 if 1 wins
$\Pr(Y = 1 \mid X = 2)$	0.3	Introducing 1 if 2 wins

Table 3.13: Probabilities

$$\Pr(X = 2 \mid Y = 1) = \frac{\Pr(2)\Pr(1 \mid 2)}{\Pr(1)\Pr(1 \mid 1) + \Pr(2)\Pr(1 \mid 2)} = \frac{2}{9} \tag{3.74}$$

3.23 Suppose a girl throws a die. If she gets a 5 or 6, she tosses a coin three times and notes the number of heads. If she gets 1, 2, 3 or 4, she tosses a coin once and notes whether a head or tail is obtained. If she obtained exactly one head, what is the probability that she threw 1, 2, 3 or 4 with the die?]

3.24 A manufacturer has three machine operators A, B and C. The first operator A produces 1% defective items, where as the other two operators B and C produce 5% and 7% defective items respectively. A is on the job for 50% of the time, B is on the job for 30% of the time and C is on the job for 20% of the time. A defective item is produced, what is the probability that it was produced by A?

3.25 A card from a pack of 52 cards is lost. From the remaining cards of the pack, two cards are drawn and are found to be both diamonds. Find the probability of the lost card being a diamond.

3.26 Probability that A speaks truth is $\frac{4}{5}$. A coin is tossed. A reports that a head appears. The probability that actually there was head is

(a) $\frac{4}{5}$

(b) $\frac{1}{2}$

(c) $\frac{1}{5}$

(d) $\frac{2}{5}$

Solution: Consider the random variables A, X as described in the table 3.14.

RV	Values	Description
A	$\{0, 1\}$	1: A speaks truth, 0: A lies
X	$\{0, 1\}$	1: Heads, 0: Tails

Table 3.14: Random variables A, X

The given information about probabilities is listed in table 3.15.

Event	Probability
$\Pr(A = 1)$	$\frac{4}{5}$
$\Pr(X = 1)$	$\frac{1}{2}$
$\Pr(X = 1 \mid A = 1)$	$\frac{1}{2}$

Table 3.15: Probabilities

The required probability is given by

$$\Pr(A = 1 \mid X = 1) = \frac{\Pr(A = 1)\Pr(X = 1 \mid A = 1)}{\Pr(X = 1)} \tag{3.75}$$

$$= \frac{\frac{4}{5} \times \frac{1}{2}}{\frac{1}{2}} \tag{3.76}$$

$$= \frac{4}{5} \tag{3.77}$$

3.27 If A and B are two events such that $A \subset B$ and $\Pr(B) \neq 0$, then which of the following is correct ?

(a) $\Pr(A \mid B) = \frac{\Pr(B)}{\Pr(A)}$

(b) $\Pr(A \mid B) < \Pr(A)$

(c) $\Pr(A \mid B) \geq \Pr(A)$

(d) None of these

Solution: if $A \subset B$ and $\Pr(B) \neq 0$ then

$$AB = A \tag{3.78}$$

$$\text{or, } P(A|B) = \frac{\Pr(AB)}{\Pr(B)} = \frac{\Pr(A)}{\Pr(B)} \tag{3.79}$$

we know that

$$\Pr(B) \leq 1 \tag{3.80}$$

$$\implies 1 \leq \frac{1}{\Pr(B)} \tag{3.81}$$

Multiplying both sides with $\Pr(A)$,

$$\Pr(A) \leq \frac{\Pr(A)}{\Pr(B)} \tag{3.82}$$

$$= \Pr(A \mid B) \tag{3.83}$$

from (3.79).

3.28 A and B are two events such that $\Pr(A) \neq 0$. Find $\Pr(B \mid A)$, if

(a) A is a subset of B

(b) $A \cap B = \phi$

Solution: We use

$$\Pr(B \mid A) = \frac{\Pr(BA)}{\Pr(A)} \tag{3.84}$$

(a) In this case,

$$BA = A \implies \Pr(BA) = \Pr(A) \tag{3.85}$$

From (3.84),

$$\Pr(B \mid A) = 1 \tag{3.86}$$

(b) $A \cap B = \phi$. This implies

$$\Pr(BA) = 0 \tag{3.87}$$

From (3.84),

$$\Pr(B \mid A) = 0 \tag{3.88}$$

3.29 A couple has two children.

(a) Find the probability that both children are males, if it is known that at least one of the children is male.

(b) Find the probability that both children are females, if it is known that the elder child is a female.

Solution: Consider the random variables X, Y, which denotes the first child, second child gender respectively as described in table 3.16.

RV	**Values**	**Description**
X	$\{0, 1\}$	0: Male , 1: Female
Y	$\{0, 1\}$	0: Male, 1: Female

Table 3.16: Random variables X

The probabilities for the random variables X, Y is listed in table 3.17.

Event	**Probability**
$\Pr(X = 0)$	$\frac{1}{2}$
$\Pr(X = 1)$	$\frac{1}{2}$
$\Pr(Y = 0)$	$\frac{1}{2}$
$\Pr(Y = 1)$	$\frac{1}{2}$
$\Pr(X + Y = 0)$	$\frac{1}{4}$
$\Pr(X + Y = 2)$	$\frac{1}{4}$
$\Pr(XY = 0)$	$\frac{3}{4}$

Table 3.17: Probabilities

The probability $\Pr(XY = 0)$ is given by

$$= \Pr(X = 0) + \Pr(Y = 0) - \Pr(X + Y = 0) \tag{3.89}$$

$$= \frac{1}{2} + \frac{1}{2} - \frac{1}{4} \tag{3.90}$$

$$= \frac{3}{4} \tag{3.91}$$

(a) The event of both children being Male is when $X + Y = 0$. The event of atleast one of the children being Male is when $XY = 0$.

$$\{X + Y = 0\} \cap \{XY = 0\} \equiv \{X + Y = 0\} \tag{3.92}$$

The required probability is given by,

$$\Pr(X + Y = 0 \mid XY = 0) \tag{3.93}$$

$$= \frac{\Pr(X + Y = 0)}{\Pr(XY = 0)} \tag{3.94}$$

$$= \frac{1}{3} \tag{3.95}$$

(b) The event of both children being Female is when $X + Y = 2$. The event of elder child being Female is when $X = 1$.

$$\{X + Y = 2\} \cap \{X = 1\} \equiv \{X + Y = 2\} \tag{3.96}$$

The required probability is given by,

$$\Pr(X + Y = 2 \mid X = 1) \tag{3.97}$$

$$= \frac{\Pr(X + Y = 2)}{\Pr(X = 1)} \tag{3.98}$$

$$= \frac{1}{2} \tag{3.99}$$

3.30 Suppose that 5% of men and 0.25% of women have grey hair. A grey haired person is selected at random. What is the probability that this person being male? Assume that there are equal number of males and females.

Solution: See Table 3.18. It is given that,

Variable	**Event**
$X = 0$	Men
$X = 1$	Women
$Y = 0$	Non-grey hair
$Y = 1$	grey hair

Table 3.18:

$$p_X(0) = p_X(1) = \frac{1}{2} \tag{3.100}$$

$$p_{Y|X}(1|0) = \frac{5}{100} = \frac{1}{20} \tag{3.101}$$

$$p_{Y|X}(1|1) = \frac{0.25}{100} = \frac{1}{400} \tag{3.102}$$

From the law of total probability,

$$p_Y(1) = p_{Y|X}(1|0) \times p_X(0) + p_{Y|X}(1|1) \times p_X(1) \tag{3.103}$$

$$= \frac{1}{20} \times \frac{1}{2} + \frac{1}{400} \times \frac{1}{2} = \frac{21}{800} \tag{3.104}$$

$$\therefore\ p_Y(1) = \frac{21}{800} \tag{3.105}$$

Thus,

$$p_{X|Y}(x|y) = \frac{p_{X,Y}(x,y)}{p_Y(y)},\ p_{X,Y}(x,y) = p_{Y|X}(y|x)p_X(x) \tag{3.106}$$

yielding

$$p_{X,Y}(0,1) = p_{Y|X}(1|0) \times p_X(0) \tag{3.107}$$
$$= \frac{1}{20} \times \frac{1}{2} = \frac{1}{40} \tag{3.108}$$
$$\tag{3.109}$$

resulting in

$$p_{X|Y}(0|1) = \frac{p_{X,Y}(0,1)}{p_Y(1)} = \frac{\frac{1}{40}}{\frac{21}{800}} = \frac{20}{21} \tag{3.110}$$

3.31 Suppose we have four boxes A,B,C and D containing coloured marbles as given in Table 3.19. One of the boxes has been selected at random and a single marble is

Box	Marble colour		
	Red	White	Black
A	1	6	3
B	6	2	2
C	8	1	1
D	0	6	4

Table 3.19: Question Table

drawn from it. If the marble is red, what is the probability that it was drawn from

1) Box A ?

2) Box B ?

3) Box C ?

Solution: See Table Table 3.20. Here,

Events	Definition
E	drawn marble is red
E_1	selected box is A
E_2	selected box is B
E_3	selected box is C
E_4	selected box is D

Table 3.20: Events Table

$$\Pr\left(E|E_1\right) = \frac{1}{10}, \Pr\left(E|E_2\right) = \frac{6}{10}, \Pr\left(E|E_3\right) = \frac{8}{10}, \Pr\left(E|E_4\right) = \frac{0}{10} \tag{3.111}$$

$$\Pr\left(E_i\right) = \frac{1}{4}\ \forall 1 \le i \le 4 \tag{3.112}$$

(a)

$$\Pr\left(E_1|E\right) = \frac{\Pr\left(E|E_1\right)\Pr\left(E_1\right)}{\sum_{i=1}^{i=4}(\Pr\left(E|E_i\right)\Pr\left(E_i\right))} \tag{3.113}$$

$$= \frac{\frac{1}{10} \times \frac{1}{4}}{\frac{1}{10} \times \frac{1}{4} + \frac{6}{10} \times \frac{1}{4} + \frac{8}{10} \times \frac{1}{4} + \frac{0}{10} \times \frac{1}{4}} = \frac{1}{15} \tag{3.114}$$

(b)

$$\Pr\left(E_2|E\right) = \frac{\Pr\left(E|E_2\right)\Pr\left(E_2\right)}{\sum_{i=1}^{i=4}(\Pr\left(E|E_i\right)\Pr\left(E_i\right))} \tag{3.115}$$

$$= \frac{\frac{6}{10} \times \frac{1}{4}}{\frac{1}{10} \times \frac{1}{4} + \frac{6}{10} \times \frac{1}{4} + \frac{8}{10} \times \frac{1}{4} + \frac{0}{10} \times \frac{1}{4}} = \frac{2}{5} \tag{3.116}$$

(c)

$$\Pr\left(E_3|E\right) = \frac{\Pr\left(E|E_3\right)\Pr\left(E_3\right)}{\sum_{i=1}^{i=4}(\Pr\left(E|E_i\right)\Pr\left(E_i\right))} \tag{3.117}$$

$$= \frac{\frac{8}{10} \times \frac{1}{4}}{\frac{1}{10} \times \frac{1}{4} + \frac{6}{10} \times \frac{1}{4} + \frac{8}{10} \times \frac{1}{4} + \frac{0}{10} \times \frac{1}{4}} = \frac{8}{15} \tag{3.118}$$

3.32 Bag I contains 3 red and 4 black balls and Bag II contains 4 red and 5 black balls.

One ball is transferred from Bag I to Bag II and then a ball is drawn from Bag II. The ball so drawn is found to be red in colour. Find the probability that the transferred ball is black.

Solution: Let

$$X = \begin{cases} 1, & \text{if ball is being drawn from Bag I} \\ 2, & \text{if ball is being drawn from Bag I} \end{cases} \tag{3.119}$$

$$Y = \begin{cases} 1, & \text{if ball drawn is Red} \\ 2, & \text{if ball drawn is Black} \end{cases} \tag{3.120}$$

Then

$$\Pr\left(X = 1, Y = 1\right) = \frac{3}{7} \Pr\left(X = 1, Y = 2\right) = \frac{4}{7} \tag{3.121}$$

$$\implies \Pr\left(X = 2, Y = 1\right) = \Pr\left(X = 1, Y = 1\right) \times \frac{5}{10} + \Pr\left(X = 1, Y = 2\right) \times \frac{4}{10} = \frac{15}{70} + \frac{16}{70} = \frac{3}{7} \tag{3.122}$$

Consequently,

$$\Pr\left(X = 2, Y = 2\right) = 1 - \Pr\left(X = 2, Y = 1\right) = \frac{39}{70} \tag{3.123}$$

See Thus, the desired probability is

Red ball from Bag I:	$\Pr\left(X = 1, Y = 1\right)=\frac{3}{7}$
Black ball from Bag I:	$\Pr\left(X = 1, Y = 2\right)=\frac{4}{7}$
Red ball from Bag II:	$\Pr\left(X = 2, Y = 1\right)=\frac{31}{70}$
Black ball from Bag II:	$\Pr\left(X = 2, Y = 2\right)=\frac{39}{70}$

(3.124)

Table 3.21: Final probabilities of the events.

$$\Pr\left(X=1, Y=2 | X=2, Y=2\right)$$
$$= \frac{\Pr\left(X=2, Y=1 | X=1, Y=1\right)}{\sum_{i=1}^{i=2} \Pr\left(X=2, Y=1 | X=1, Y=i\right) \Pr\left(X=1, Y=i\right)}$$
$$= \frac{\frac{4}{10} \times \frac{4}{7}}{\frac{4}{10} \times \frac{4}{7} + \frac{5}{10} \times \frac{3}{7}} = \frac{16}{31} \quad (3.125)$$

3.33 Box A contains 25 slips of which 19 are marked Rs 1 and others are marked Rs 5 each. Box B contains 50 slips of which 45 are marked Rs 1 and others are marked Rs 13 each. Slips of both boxes are poured into a third box and reshuffled. A slip is drawn at random. What is the probability that it is marked other than Rs 1?

Solution:

Random variable	Value	Definition
X	0	Slips of Rs 1
	1	Slips of Rs 5
	2	Slips of Rs 13
Y	0	Box A
	1	Box B

Table 3.22: Distribution

The PMF of Y (the box that the slip came from) is given as.

$$p_Y\left(k\right) = \begin{cases} \frac{1}{3} & \text{if k=0} \\ \\ \frac{2}{3} & \text{if k=1} \end{cases} \quad (3.126)$$

Conditional Probabilty,

$$\begin{cases} \Pr\left(Y = 0 | X = 0\right) = \frac{19}{25} \\ \Pr\left(Y = 0 | X = 1\right) = \frac{6}{25} \\ \Pr\left(Y = 1 | X = 0\right) = \frac{45}{50} \\ \Pr\left(Y = 1 | X = 2\right) = \frac{5}{50} \end{cases} \tag{3.127}$$

The desired probability is the probability that a slip drawn at random is marked other than Rs 1,

$$= 1 - p_X\left(0\right) \tag{3.128}$$

$$= p_X(1) + p_X(2) \tag{3.129}$$

Using Bayes theorem,

$$= p_Y\left(0\right) \times \Pr\left(Y = 0 | X = 1\right) + p_Y\left(1\right) \times \Pr\left(Y = 1 | X = 2\right) \tag{3.130}$$

$$= \frac{1}{3} \times \frac{6}{25} + \frac{2}{3} \times \frac{5}{50} \tag{3.131}$$

$$= \frac{11}{75} \tag{3.132}$$

3.34 Bag I contains 3 black and 2 white balls, Bag II contains 2 black and 4 white balls. A bag and a ball is selected at random. Determine the probability of selecting a black ball.

Solution:

Random variable	Value	Definition
X	0	Bag 1
	1	Bag 2
Y	0	White ball
	1	Black ball

Table 3.34: Distribution

Probablity of chosing Bag

$$\Pr(X = 0) = \frac{1}{2} \tag{3.133}$$

$$\Pr(X = 1) = \frac{1}{2} \tag{3.134}$$

conditional probablity

$$\Pr(Y = 1|X = 0) = \frac{3}{5} \tag{3.135}$$

$$\Pr(Y = 1|X = 1) = \frac{1}{3} \tag{3.136}$$

Probablity of black balls

$$\begin{aligned}\Pr(Y=1) &= \Pr(Y=1|X=0)\Pr(X=0)\\ &\quad+\Pr(Y=1|X=1)\Pr(X=1)\\ \Pr(Y=1) &= \frac{3}{10}+\frac{1}{6}\\ &= \frac{7}{15}\end{aligned} \tag{3.137}$$

3.35 While shuffling a pack of 52 playing cards, 2 cards are dropped. Find the probabilty that the missing cards to be of different colours.

Solution: We know that the 52 playing cards contain 26 red cards and 26 black cards. Let X_1 and X_2 be two random variables denoting the colour of first card and second card respectively:

$$X_i = \begin{cases} 0, & \text{red card} \\ 1, & \text{black card} \end{cases} \qquad \{i=1,2\} \tag{3.138}$$

Probability of choosing the first card:

$$p_{X_1}(k) = \frac{26}{52} \quad \{k=0,1\} \tag{3.139}$$

$$= \frac{1}{2} \tag{3.140}$$

Probability that second card choosen is red after selecting the first card as red

$$= p(X_2=0|X_1=0) = \frac{25}{51}\times\frac{26}{52}\div\frac{26}{52} = \frac{25}{51} \tag{3.141}$$

Probability that second card choosen is black after selecting the first card as red

$$= p(X_2 = 1|X_1 = 0) = \frac{26}{51} \times \frac{26}{52} \div \frac{26}{52} = \frac{26}{51} \tag{3.142}$$

Probability that second card choosen is red after selecting the first card as black

$$= p(X_2 = 0|X_1 = 1) = \frac{26}{51} \times \frac{26}{52} \div \frac{26}{52} = \frac{26}{51} \tag{3.143}$$

Probability that second card choosen is black after selecting the first card as black

$$= p(X_2 = 1|X_1 = 1) = \frac{25}{51} \times \frac{26}{52} \div \frac{26}{52} = \frac{25}{51} \tag{3.144}$$

Probabilty that both cards have different colour:

$$\begin{aligned}
pr(X_1 \neq X_2) &= p_{X_1}(0)\, p(X_2 = 1|X_1 = 0) + p_{X_1}(1)\, p(X_2 = 0|X_1 = 1) && (3.145)\\
&= \frac{26}{52} \times \frac{26}{51} + \frac{26}{52} \times \frac{26}{51} && (3.146)\\
&= 2 \times \frac{26}{52} \times \frac{13}{51} && (3.147)\\
&= \frac{26}{51} && (3.148)
\end{aligned}$$

Table 3.35: Description of random variables

Random Variable	Values	Description
X_1	0	First card is red
	1	First card is black
X_2	0	Second card is red
	1	Second card is black

3.36 A bag contains $(2n+1)$ coins. It is known that n of these coins have a head on both sides where as the rest of the coins are fair. A coin is picked up at random from the bag and is tossed. If the probability that the toss results in a head is $\frac{31}{42}$, determine the value of n.

Solution:

RV	Values	Description
X	0	Getting unfair coin
	1	Getting fair coin
Y	0	Getting Head
	1	Getting Tail

Table 3.36: Random variable declaration.

PMF is

$$p_X(k) = \begin{cases} \frac{n}{2n+1} & k = 0 \\ \frac{n+1}{2n+1} & k = 1 \end{cases} \tag{3.149}$$

Conditional probability,

$$\Pr(Y = 0|X = 0) = 1 \tag{3.150}$$

$$\Pr(Y = 0|X = 1) = \frac{1}{2} \tag{3.151}$$

Given that,

$$p_Y(0) = p_X(0)\Pr(Y = 0|X = 0) + p_X(1)\Pr(Y = 0|X = 1) \tag{3.152}$$

$$\implies \frac{31}{42} = \frac{n}{2n+1} + \frac{1}{2} \times \frac{n+1}{2n+1} \tag{3.153}$$

$$\implies n = 10 \tag{3.154}$$

3.37 An urn contains m white and n black balls. A ball is drawn at random and is put back into the urn along with k additional balls of the same colour as that of the ball drawn. A ball is again drawn at random. Show that the probability of drawing a white ball now does not depend on k.

Solution:

parameters	values	decription
X	0	first ball
	1	second ball
Y	0	black ball
	1	white ball

Table 3.37: Random variable description

The probabilities are

$$\begin{cases} p_{XY}(00) = \frac{n}{m+n} \\ p_{XY}(01) = \frac{m}{m+n} \\ \Pr(X=1, Y=1 | X=0, Y=0) = \frac{m}{m+n+k} \\ \Pr(X=1, Y=1 | X=0, Y=1) = \frac{m+k}{m+n+k} \\ \Pr(X=1, Y=0 | X=0, Y=0) = \frac{n+k}{m+n+k} \\ \Pr(X=1, Y=0 | X=0, Y=1) = \frac{n}{m+n+k} \end{cases} \tag{3.155}$$

The desired probability is

$$= p_{XY}(11) \tag{3.156}$$

Using total probability theorem,

$$= p_{XY}(00)\Pr(X=1,Y=1|X=0,Y=0) + p_{XY}(01)\Pr(X=1,Y=1|X=0,Y=1) \tag{3.157}$$

$$= \left(\frac{n}{m+n}\right)\left(\frac{m}{m+n+k}\right) + \left(\frac{m}{m+n}\right)\left(\frac{m+k}{m+n+k}\right) \tag{3.158}$$

$$= \frac{mn + m(m+k)}{(m+n)(m+n+k)} \tag{3.159}$$

$$= \frac{m(m+n+k)}{(m+n)(m+n+k)} \tag{3.160}$$

$$= \frac{m}{m+n} \tag{3.161}$$

which is independent of k.

3.38 If $\Pr(AB) = \frac{7}{10}$ and $\Pr(B) = \frac{17}{20}$, then $\Pr(A|B)$ equals

(a) $\frac{14}{17}$

(b) $\frac{17}{20}$

(c) $\frac{7}{8}$

(d) $\frac{1}{8}$

Solution: We know

$$\Pr(A|B) = \frac{\Pr(AB)}{\Pr(B)} \tag{3.162}$$

$$= \frac{\frac{7}{10}}{\frac{17}{20}} \tag{3.163}$$

$$= \frac{14}{17} \tag{3.164}$$

3.39 A letter is known to have come either from TATANAGAR or from CALCUTTA. On the envelope, just two consecutive letter TA are visible. What is the probability that the letter came from TATANAGAR.

Solution: We know that the 52 playing cards contain 26 red cards and 26 black cards. Let X_1 and X_2 be two random variables denoting the city and TA respectively:

$$X_1 = \begin{cases} 0, & \text{TATANAGAR} \\ 1, & \text{CALCUTTA} \end{cases} \tag{3.165}$$

$$X_2 = \begin{cases} 0, & \text{Choosing TA from TATANAGAR} \\ 1, & \text{Choosing TA from CALCUTTA} \end{cases} \tag{3.166}$$

Probability of choosing the city:

$$p_{X_1}(k) = \frac{1}{2} \quad \{k = 0, 1\} \tag{3.167}$$

Probability of choosing consecutive letters TA from TATANAGAR:

$$= pr(X_2 = 0|X_1 = 0) = \frac{1}{2} \times \frac{2}{8} \div \frac{1}{2} = \frac{1}{4} \tag{3.168}$$

Probability of choosing consecutive letters TA from CALCUTTA:

$$= pr(X_2 = 1|X_1 = 1) = \frac{1}{2} \times \frac{1}{7} \div \frac{1}{2} = \frac{1}{7} \tag{3.169}$$

By using bayes theorem, we get the probability of getting two consecutive letters TA from TATANAGAR:

$$= \{p_{X_1}(0)\, pr(X_2 = 0|X_1 = 0)\} \div \{p_{X_1}(0)\, pr(X_2 = 0|X_1 = 0) + p_{X_2}(1)\, pr(X_2 = 1|X_1 = 1)\} \tag{3.170}$$

$$= \frac{1}{2} \times \frac{1}{4} \div \left\{ \frac{1}{2} \times \frac{1}{4} + \frac{1}{2} \times \frac{1}{7} \right\} \tag{3.171}$$

$$= \frac{1}{8} \div \frac{22}{112} \tag{3.172}$$

$$= \frac{7}{11} \tag{3.173}$$

Table 3.39: Description of random variables

Random Variable	Values	Description
X_1	0	TATANAGAR
	1	CALCUTTA
X_2	0	TA is choosen from TATANAGAR
	1	TA is choosen from CALCUTTA

3.40 A bag contain (2n+1) coins. It is known that n of these coins have a head on both sides where as the rest of the coins are fair. A coin is picked up at random from the bag and is tossed. If the probability that the toss results in a head is $\frac{31}{42}$, determine the value of n.

Solution:

parameter	**value**	**description**
Random Variable X	0	Fair coin is selected
	1	Unfair coin is selected
Random Variable Y	0	Head comes in the toss
	1	Tail comes in the toss

Table 3.40: Tosses of coins

$$p_X(0) = \frac{n+1}{2n+1} \tag{3.174}$$

$$p_X(1) = \frac{n}{2n+1} \tag{3.175}$$

$$\Pr(Y = 0|X = 0) = \frac{1}{2} \tag{3.176}$$

$$\Pr(Y = 0|X = 1) = 1 \tag{3.177}$$

Hence, the probability of head is

$$p_X(0)\Pr(Y = 0|X = 0) + p_X(1)\Pr(Y = 0|X = 1) = \frac{n+1}{2n+1} \times \frac{1}{2} + \frac{n}{2n+1} \times 1 \tag{3.178}$$

$$= \frac{3n+1}{2(2n+1)} \tag{3.179}$$

Now,

$$\frac{3n+1}{2(2n+1)} = \frac{31}{42} \tag{3.180}$$

$$\implies \frac{3n+1}{2n+1} = \frac{31}{21} \tag{3.181}$$

$$\implies 63n + 21 = 62n + 31 \tag{3.182}$$

$$\implies n = 10 \tag{3.183}$$

3.41 By examining the chest X ray, the probability that TB is detected when a person is actually suffering is 0.99. The probability of an healthy person diagnosed to have TB is 0.001. In a certain city, 1 in 1000 people suffers from TB. A person is selected at random and is diagnosed to have TB. What is the probability that he actually has TB?

Solution: Let us define two random variables A and B, where A is probability of a person being healthy and B is the probability of a person getting diagnosed with TB.

Random Variable	**Value**	**Description**
A	0	Person not having TB
	1	Person having TB
B	0	not diagnosed with TB
	1	diagnosed with TB

The following information is given to us:

$$\Pr(A=1) = 0.001 \tag{3.184}$$

$$\Pr(A=0) = 0.999 \tag{3.185}$$

$$\Pr(B=1|A=1) = 0.99 \tag{3.186}$$

$$\Pr(B=1|A=0) = 0.001 \tag{3.187}$$

To find the probability that the person actually has TB given that they were diagnosed with TB, we can use Bayes' theorem.

$$\Pr(A=1|B=1) = \frac{\Pr(B=1|A=1)\cdot\Pr(A=1)}{\Pr(B=1)} \tag{3.188}$$

We can calculate $\Pr(B=1)$, the probability of being diagnosed with TB:

$$\Pr(B=1) = \Pr(B=1|A=1)\cdot\Pr(A=1) + \Pr(B=1|A=0)\cdot\Pr(A=0) \quad (3.189)$$
$$= (0.99\cdot 0.001) + (0.001\cdot 0.999) \quad (3.190)$$
$$= 0.001989 \quad (3.191)$$

Now, we can use Bayes' theorem to calculate $\Pr(A=1|B=1)$, the probability that the person actually has TB given the diagnosis:

$$\Pr(A=1|B=1) = \frac{\Pr(B=1|A=1)\cdot\Pr(A=1)}{\Pr(B=1)} \quad (3.192)$$
$$= \frac{0.99\cdot 0.001}{0.001989} \quad (3.193)$$
$$= \frac{0.00099}{0.001989} \quad (3.194)$$
$$= 0.4987 \quad (3.195)$$

So, the probability that the person actually has TB given the diagnosis is approximately 0.4987.

3.2. Exercises

1. A child's game has 8 triangles of which 3 are blue and rest are red, and 10 squares of which 6 are blue and rest are red. One piece is lost at random. Find the probability that it is

   (a) triangle

(b) square

(c) square of blue colour

(d) triangle of red colour

Solution:

Table 3.29: Random Variables

Variable	Value	Description
X	1	Triangle
	0	Square
Y	1	Blue coloured
	0	Red coloured

$$p_X(X) = \begin{cases} \frac{10}{18}, & if X = 0 \\ \frac{8}{18}, & if X = 1 \end{cases} \tag{3.196}$$

$$\Pr(Y = 0|X = 1) = \frac{5}{8} \tag{3.197}$$

$$\Pr(Y = 1|X = 1) = \frac{3}{8} \tag{3.198}$$

$$\Pr(Y = 0|X = 0) = \frac{4}{10} \tag{3.199}$$

$$\Pr(Y = 1|X = 0) = \frac{6}{10} \tag{3.200}$$

(a) $p_X(1) = \frac{8}{18}$

(b) $p_X(0) = \frac{10}{18}$

(c) $p_{XY}(0, 1) = \Pr(Y = 1|X = 0)\, p_X(0) = \frac{6}{18}$

(d) $p_{XY}(1, 0) = \Pr(Y = 0|X = 1)\, p_X(1) = \frac{5}{18}$

2. At a fete, cards bearing numbers 1 to 1000, one number on a card, are put in a box. Each player selects one card at random and that card is not replaced. If the

selected card has a perfect square greater than 500, the player wins a prize. What is the probability that

(a) the first player wins a prize

(b) the second player wins a prize, if the first has won?

Solution: Let

Variable	Value	Description
X_i	0	i^{th} player doesn't win a prize
	1	i^{th} player wins a prize

If n^2 is the value of the chosen number that is greater than 500 and also a perfect square, then

$$n^2 \in (500, 1000] \tag{3.201}$$

$$\implies n \in (22.36, 31.62] \tag{3.202}$$

n can take 9 integer values in the above interval.

$$Pr\left(X_1 = 1\right) = \frac{9}{1000} \tag{3.203}$$

$$= 0.009 \tag{3.204}$$

If first player gets a number greater than 500 which is a perfect square then the second player can get a number from the remaining 8 numbers in the above interval to win a prize.

(a) Probability that first player wins a prize

$$= Pr\,(X_1 = 1) \tag{3.205}$$

$$= 0.009 \tag{3.206}$$

(b) Probability that second player wins given that the first player has won prize

$$= \Pr((X_2 = 1)|(X_1 = 1)) \tag{3.207}$$

$$= 0.008 \tag{3.208}$$

3. Without repetition of the numbers, four digit numbers are formed with the numbers 0,2,3,5.The probability of such a number divisible by 5 is

   (a) $\frac{1}{5}$

   (b) $\frac{4}{5}$

   (c) $\frac{1}{30}$

   (d) $\frac{5}{9}$

   Solution: Number of four digit numbers possible are $3 \times 3 \times 2 \times 1 = 18$ because zero cannot be in the first place.

   As number of four digit numbers with fourth digit being 0 is $3 \times 2 \times 1 \times 1 = 6$

$$p\,(Y = 0, X = 4) = \frac{3 \times 2 \times 1 \times 1}{3 \times 3 \times 2 \times 1 \times 1} \tag{3.209}$$

$$= \frac{1}{3} \tag{3.210}$$

$$\tag{3.211}$$

Random Variable	Values	Description
X	1	first digit
	2	second digit
	3	third digit
	4	fourth digit
Y	0	0 as digit
	1	5 as digit

Table 3.30: Table 1

As number of four digit numbers with fourth digit being 5 and second digit being 0 is $2 \times 1 \times 1 \times 1 = 2$

$$p\left(Y = 1, X = 4 | Y = 0, X = 2\right) = \frac{2 \times 1 \times 1 \times 1}{3 \times 3 \times 2 \times 1} \tag{3.212}$$

$$= \frac{1}{9} \tag{3.213}$$

$$\tag{3.214}$$

As number of four digit numbers with fourth digit being 5 and third digit being 0 is $2 \times 1 \times 1 \times 1 = 2$

$$p\left(Y = 1, X = 4 | Y = 0, X = 3\right) = \frac{2 \times 1 \times 1 \times 1}{3 \times 3 \times 2 \times 1} \tag{3.215}$$

$$= \frac{1}{9} \tag{3.216}$$

$$\tag{3.217}$$

Probability of forming four digit number divisible by 5, without repetition,

$$p = p(Y = 0, X = 4) + p(Y = 1, X = 4|Y = 0, X = 2) + p(Y = 1, X = 4|Y = 0, X = 3) \tag{3.218}$$

$$= \frac{5}{9} \tag{3.219}$$

Hence, option (D)$\frac{5}{9}$ is the correct option.

4. Four cards are successively drawn without from a deck of 52 playing cards. What is the probability that all the four cards are kings?

Solution:

So we have to find the probability that all four cards are kings,

RV	Value	Description
X_i	1	first card is king
	2	second card is king
	3	third card is king
	4	fourth card is king

Table 3.31: RV description table

$$\Pr(\text{all four cards are kings}) = \Pr(X_1 X_2 X_3 X_4) \tag{3.220}$$

$$= \Pr(X_1) \times \Pr(X_2|X_1) \times \Pr(X_3|X_1 X_2) \times \Pr(X_4|X_1 X_2 X_3) \tag{3.221}$$

$$= \frac{4}{52} \times \frac{3}{51} \times \frac{2}{50} \times \frac{1}{49} \tag{3.222}$$

$$= \frac{1}{270725} \tag{3.223}$$

5. Suppose you have two coins which appear identical in your pocket. You know that

one is fair and one is 2-headed. If you take one out, toss it and get a head, what is the probability that it was a fair coin?

Table 3.32: Random Variables

Variable	Value	Description
X	1	Fair coin
	0	2-headed coin
Y	1	if output is heads
	0	if output is tails

Given,

$$\Pr(X = 1) = \frac{1}{2}, \tag{3.224}$$
$$\Pr(X = 0) = \frac{1}{2}, \tag{3.225}$$
$$\Pr(Y = 1 \mid X = 1) = \frac{1}{2}, \tag{3.226}$$
$$\Pr(Y = 1 \mid X = 0) = 1, \tag{3.227}$$

The probability of the coin being fair when the output comes as heads is

$$\Pr(X = 1 \mid Y = 1) = \frac{\Pr(Y = 1 \mid X = 1) \times \Pr(X = 1)}{\sum_{k=0}^{1} \Pr(Y = 1 \mid X = k) \times \Pr(X = k)} \tag{3.228}$$
$$= \frac{\frac{1}{2} \times \frac{1}{2}}{1 \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2}} \tag{3.229}$$
$$= \frac{1}{3} \tag{3.230}$$

6. A box has 5 blue and 4 red balls. One ball is drawn at random and not replaced. Its colour is also not noted. Then another ball is drawn at random. What is the probability of second ball being blue? **Solution:**

Let X and Y denote the random variables for the first and second draw respectively as follows:

RV	Values	Description
X	$\{0,1\}$	1st draw :- 0: blue, 1: red
Y	$\{0,1\}$	2nd draw :- 0: blue, 1: red

Table 3.33: Random Variables

The probabilities are given as:

$$\Pr(X=0) = \frac{5}{9} \tag{3.231}$$
$$\Pr(X=1) = \frac{4}{9} \tag{3.232}$$

$$\Pr(Y=0|X=0) = \frac{\Pr(Y=0, X=0)}{\Pr(X=0)} = \frac{1}{2} \tag{3.233}$$
$$\Pr(Y=0|X=1) = \frac{\Pr(Y=0, X=1)}{\Pr(X=1)} = \frac{5}{8} \tag{3.234}$$

The probability of the second ball beign drawn being blue is given as:

$$\Pr(Y=0) = \Pr(X=0)\Pr(Y=0|X=0) + \Pr(X=1)\Pr(Y=0|X=1) \tag{3.235}$$
$$= \frac{5}{9} \times \frac{1}{2} + \frac{4}{9} \times \frac{5}{8} \tag{3.236}$$
$$= \frac{5}{9} \tag{3.237}$$

7. A bag contains 4 white and 5 black balls. Another bag contains 9 white and 7 black balls. A ball is transferred from the first bag to the second and then a ball is drawn at random from the second bag. Find the probability that the ball drawn is white.

Solution: Let X be the random variables that denotes which ball is picked and transferred from the first bag to the second.

$$X = \begin{cases} 0, & \text{White} \\ 1, & \text{Black} \end{cases} \tag{3.238}$$

The pmf of X is:

$$p_X(k) = \begin{cases} \frac{4}{9}, & k = 0 \\ \frac{5}{9}, & k = 1 \end{cases} \tag{3.239}$$

Let Y be a random variable that denotes which ball is drawn from the second bag after a ball from first bag is transferred to it.

$$Y = \begin{cases} 0, & \text{White} \\ 1, & \text{Black} \end{cases} \tag{3.240}$$

$$p_Y(0) = \Pr(Y = 0, X = 0) + \Pr(Y = 0, X = 1) \tag{3.241}$$

$$= \Pr(Y = 0| X = 0)\Pr(X = 0) + \Pr(Y = 0| X = 1)\Pr(X = 1) \tag{3.242}$$

$$= \left(\frac{10}{17}\right)\left(\frac{4}{9}\right) + \left(\frac{9}{17}\right)\left(\frac{5}{9}\right) \tag{3.243}$$

$$= \frac{5}{9} \tag{3.244}$$

parameter	value	description
X	$\{0, 1\}$	Denotes which ball is drawn from the first bag
Y	$\{0, 1\}$	Denotes which ball is drawn from the second bag

Table 3.34: Random Variables

8. An item is manufactured by three machines A, B and C. Out of the total number of items manufactured during a specified period, 50% are manufactured on A, 30% on B and 20% on C, 2% of the items produced on A and 2% of items produced on B are defective, and 3% of these products produced on C are defective. All the items are stored at one godown. One item is drawn at random and is found to be defective. What is the probability that is was manufactured on machine A? **Solution:**

Parameter	Values	Description
X	0	not defective
	1	defective
Y	1	manufactured on A
	2	manufactured on B
	3	manufactured on C

Table 3.35: Table 1

Given,

$$\Pr(Y=1) = \frac{50}{100} \tag{3.245}$$

$$= 0.5 \tag{3.246}$$

$$\Pr(Y=2) = \frac{30}{100} \tag{3.247}$$

$$= 0.3 \tag{3.248}$$

$$\Pr(Y=3) = \frac{20}{100} \tag{3.249}$$

$$= 0.2 \tag{3.250}$$

$$\Pr(X=1|Y=1) = \frac{2}{100} \tag{3.251}$$

$$= 0.02 \tag{3.252}$$

$$\Pr(X=1|Y=2) = \frac{2}{100} \tag{3.253}$$

$$= 0.02 \tag{3.254}$$

$$\Pr(X=1|Y=3) = \frac{3}{100} \tag{3.255}$$

$$= 0.03 \tag{3.256}$$

We need to find $\Pr(Y=1|X=1)$,

$$\Pr(Y=1|X=1) = \frac{\Pr(Y=1)\Pr(X=1|Y=1)}{\Pr(Y=1)\Pr(X=1|Y=1)+\Pr(Y=2)\Pr(X=1|Y=2)+\Pr(Y=3)\Pr(X=1|Y=} \tag{3.257}$$

$$= \frac{0.5\times 0.02}{0.5\times 0.02+0.3\times 0.02+0.2\times 0.03} \tag{3.258}$$

$$= \frac{5}{11} \tag{3.259}$$

$$\tag{3.260}$$

9. Two natural numbers r, s are drawn one at a time, without replacement from the set S = 1,2,3, ...,n. Find P$[r \leq p | s \leq p]$

Solution: There are two conditions,

(a) Case 1: s is chosen first

Let X, Y and p be random variables as defined in Table 3.36,

RV	Value	Description
X	$\{1, 2, 3, \ldots, n\}$	First number (s)
Y	$\{1, 2, 3, \ldots, n\} \setminus \{s\}$	Second number (r)
p	$\mathbb{Z}$	number to be compared

Table 3.36: Random variable X declaration

We need to find the value of

$$\Pr(Y \leq p \,|\, X \leq p) = \frac{\Pr(Y \leq p,\, X \leq p)}{\Pr(X \leq p)} \tag{3.261}$$

There are 3 cases for the value of p;

i. if $p < 1$: This case is never possible as $X, Y \geq 1$

ii. if $1 \leq p \leq n$: Then we can say that,

$$\Pr(Y \leq p,\, X \leq p) = \frac{p(p-1)}{n(n-1)}, \tag{3.262}$$

$$\Pr(X \leq p) = \frac{p}{n} \tag{3.263}$$

From (3.262) and (3.263):

$$\Pr(Y \leq p | X \leq p) = \frac{\Pr(Y \leq p,\, X \leq p)}{\Pr(X \leq p)} \tag{3.264}$$

$$= \frac{\frac{p(p-1)}{n(n-1)}}{\frac{p}{n}} = \frac{p-1}{n-1} \tag{3.265}$$

iii. if $p > n$: Then we can say that,

$$\Pr(Y \leq p,\, X \leq p) = 1, \tag{3.266}$$

$$\Pr(X \leq p) = 1 \tag{3.267}$$

From (3.266) and (3.267):

$$\Pr(Y \leq p | X \leq p) = \frac{\Pr(Y \leq p,\, X \leq p)}{\Pr(X \leq p)} \tag{3.268}$$

$$= 1 \tag{3.269}$$

(b) Case 2: r is chosen first

Let X, Y and p be random variables as defined in Table 3.37,

RV	Value	Description
X	$\{1, 2, 3, \ldots, n\}$	First number (r)
Y	$\{1, 2, 3, \ldots, n\} \setminus \{s\}$	Second number (s)
p	$\mathbb{Z}$	number to be compared

Table 3.37: Random variable X declaration

We need to find the value of

$$\Pr(X \leq p | Y \leq p) = \frac{\Pr(X \leq p,\, Y \leq p)}{\Pr(Y \leq p)} \tag{3.270}$$

There are 3 cases for the value of p;

i. if $p < 1$: This case is never possible as $X, Y \geq 1$

ii. if $1 \leq p \leq n$: Then we can say that,

$$\Pr(X \leq p, Y \leq p) = \frac{p(p-1)}{n(n-1)}, \tag{3.271}$$

$$\Pr(Y \leq p) = \frac{p-1}{n-1} \tag{3.272}$$

From (3.271) and (3.272):

$$\Pr(X \leq p | Y \leq p) = \frac{\Pr(X \leq p, Y \leq p)}{\Pr(Y \leq p)} \tag{3.273}$$

$$= \frac{\frac{p(p-1)}{n(n-1)}}{\frac{p}{n}} = \frac{p}{n} \tag{3.274}$$

iii. if $p > n$: Then we can say that,

$$\Pr(X \leq p, Y \leq p) = 1, \tag{3.275}$$

$$\Pr(Y \leq p) = 1 \tag{3.276}$$

From (3.275) and (3.276):

$$\Pr(X \leq p | Y \leq p) = \frac{\Pr(X \leq p, Y \leq p)}{\Pr(Y \leq p)} \tag{3.277}$$

$$= 1 \tag{3.278}$$

10. There are two bags, one which contains 3 black balls and 4 white balls while the other contains 4 black balls and 3 white balls. A die is thrown. If it shows up 1 or 3, a ball is taken from the first bag; but it shown up any other number, a ball is taken from the second bag. Find the probability of choosing a black ball.

Solution:

So we already know,

RV	Value	Description
X	0	first bag is selected
	1	second bag is selected
Y	0	black ball is drawn
	1	white ball is drawn
Z	0	1 or 3 is shown up
	1	another number is shown up

Table 3.38: RV description table

$$\Pr(X=0) = \Pr(Z=0) \tag{3.279}$$
$$= \frac{1}{3} \tag{3.280}$$
$$\Pr(X=1) = \Pr(Z=1) \tag{3.281}$$
$$= \frac{2}{3} \tag{3.282}$$
$$\Pr(Y=0|\,X=0) = \frac{3}{7} \tag{3.283}$$
$$\Pr(Y=0|\,X=1) = \frac{4}{7} \tag{3.284}$$

So the required probability will be:

$$\Pr(\text{getting a black ball}) = \Pr(X=0) \times \Pr(Y=0|\,X=0) + \Pr(X=1) \times \Pr(Y=0|\,X=1) \tag{3.285}$$
$$= \frac{1}{3} \times \frac{3}{7} + \frac{2}{3} \times \frac{4}{7} \tag{3.286}$$
$$= \frac{7}{21} \tag{3.287}$$

Hence, the probability of getting a black ball is $\frac{7}{21}$.

11. A shopkeeper sells three types of flower seeds A1, A2 and A3. They are sold as a mixture where the proportions are 4:4:2 respectively. The germination rates of the three types of seeds are 0.45, 0.60 and 0.35. Calculate the probability:

(a) of a randomly chosen seed to germinate

(b) that it will not germinate given that the seed is of type A_3,

(c) that it is of the type A_2 given that a randomly chosen seed does not germinate.

Solution: Let the two random variables be X and Y for denoting type of seed and germination status.

Variable	Description	Value
X	Type of seed 1:A_1, 2:A_2, 3:A_3	$\{1, 2, 3\}$
Y	0:germinates, 1:not germinate	$\{0, 1\}$

$$p_X(X) = \begin{cases} \frac{4}{10}, & if X = 1 \\ \frac{4}{10}, & if X = 2 \\ \frac{2}{10}, & if X = 3 \end{cases} \tag{3.288}$$

$$\Pr(Y = 0|X = 1) = 0.45 \tag{3.289}$$

$$\Pr(Y = 0|X = 2) = 0.60 \tag{3.290}$$

$$\Pr(Y = 0|X = 3) = 0.35 \tag{3.291}$$

$$\tag{3.292}$$

(a)

$$p_Y(0) = \Pr(Y=0|X=1)\Pr(X=1) + \Pr(Y=0|X=2)\Pr(X=2) + \Pr(Y=0|X=3)\Pr(X=3) \tag{3.293}$$

$$= \frac{49}{100} \tag{3.294}$$

$$= 0.49 \tag{3.295}$$

(b)

$$\Pr(Y=1|X=2) = 1 - \Pr(Y=0|X=2) \tag{3.296}$$

$$= 1 - 0.35 \tag{3.297}$$

$$= 0.65 \tag{3.298}$$

(c)

$$\Pr(X=2|Y=1) = \frac{\Pr(Y=1|X=2)\Pr(X=2)}{\Pr(Y=1)} \tag{3.299}$$

$$= \frac{(1-\Pr(Y=0|X=2))\Pr(X=2)}{1-\Pr(Y=0)} \tag{3.300}$$

$$= \frac{16}{51} \tag{3.301}$$

Hence all the three probabilities have been found.

12. Three bags contains a no of red and white balls as follows: B_1: 3 red balls,B_2:2 red balls and 1 white ball,B_3:3 white balls The probability that bag i will be chosen and a ball is selected is i/6,i=1,2,3. what is the probability that (i) a red ball will be selected? (ii) a white ball will be selected?

Solution:

object	RV	values	description
bag	X	1	bag-1 is selected
		2	bag-2 is selected
		3	bag-3 isselected
ball	Y	0	white ball is selected
		1	red ball is selected

Table 3.39: random variables of objects

$$\Pr(X=i) = \begin{cases} \frac{1}{6}, \text{when i=1} \\ \frac{2}{6}, \text{when i=2} \\ \frac{3}{6}, \text{when i=3} \end{cases} \tag{3.302}$$

we know that that the conditional probability is defined as

$\Pr(A|B) = \frac{\Pr(A,B)}{\Pr(B)}$

(a) The probability that a red ball will be selected is:

$$\Pr(Y=1) = \Pr(Y=1, X=1) + \Pr(Y=1, X=2) + \Pr(Y=1, X=3) \tag{3.303}$$

$$= \Pr(X=1) \times \Pr(Y=1|X=1) + \Pr(X=2) \times \Pr(Y=1|X=2) + \Pr(X \tag{3.304}$$

$$= \frac{1}{6} \times \frac{3}{3} + \frac{2}{6} \times \frac{2}{3} + \frac{3}{6} \times 0 \tag{3.305}$$

$$= \frac{7}{18} \tag{3.306}$$

(b) The probability that a white ball will be selected is:

$$\Pr(Y=0) = \Pr(Y=0, X=1) + \Pr(Y=0, X=2) + \Pr(Y=0, X=3) \quad (3.307)$$

$$= \Pr(X=1) \times \Pr(Y=0|X=1) + \Pr(X=2) \times \Pr(Y=0|X=2) + \Pr(X=3) \times \Pr(Y = \quad (3.308)$$

$$= \frac{1}{6} \times 0 + \frac{2}{6} \times \frac{1}{3} + \frac{3}{6} \times \frac{3}{3} \quad (3.309)$$

$$= \frac{11}{18} \quad (3.310)$$

13. Refer to Question 41 above. If a white ball is selected, what is the probability that it came from

(a) Bag 2

(b) Bag 3

Solution: Referring to the above question,

Parameter	Values	Description
X	0	red balls
	1	white balls
Y	1	Bag 1
	2	Bag 2
	3	Bag 3

Table 3.40: Table 1

(a) $\Pr(Y = 2|X = 1)$

$$\Pr(Y=2|X=1) = \frac{\Pr(Y=2)\cdot\Pr(X=1|Y=2)}{\Pr(Y=1)\cdot\Pr(X=1|Y=1)+\Pr(Y=2)\cdot\Pr(X=1|Y=2)+\Pr(} \tag{3.311}$$

$$= \frac{\frac{2}{6}\cdot\frac{1}{3}}{\frac{1}{6}\cdot 0+\frac{2}{6}\cdot\frac{1}{3}+\frac{3}{6}\cdot 1} \tag{3.312}$$

$$= \frac{\frac{2}{18}}{\frac{2}{18}+\frac{3}{6}} \tag{3.313}$$

$$= \frac{2}{11} \tag{3.314}$$

(b) $\Pr(Y=3|X=1)$

$$\Pr(Y=3|X=1) = \frac{\Pr(Y=3)\cdot\Pr(X=1|Y=3)}{\Pr(Y=1)\cdot\Pr(X=1|Y=1)+\Pr(Y=2)\cdot\Pr(X=1|Y=2)+\Pr(} \tag{3.315}$$

$$= \frac{\frac{3}{6}\cdot 1}{\frac{1}{6}\cdot 0+\frac{2}{6}\cdot\frac{1}{3}+\frac{3}{6}\cdot 1} \tag{3.316}$$

$$= \frac{\frac{3}{6}}{\frac{2}{18}+\frac{3}{6}} \tag{3.317}$$

$$= \frac{9}{11} \tag{3.318}$$

14. If P(A) $= \frac{4}{5}$ and P(AB) $= \frac{7}{10}$, then $P(B|A)$ is equal to

Solution: The conditional probabilty of B given A is defined as :

$$P(B|A) = \frac{P(BA)}{P(A)} \tag{3.319}$$

Hence the required probability is:

$$P(B|A) = \frac{\left(\frac{7}{10}\right)}{\left(\frac{4}{5}\right)} = \left(\frac{7}{8}\right) \tag{3.320}$$

15. A flashlight has 8 batteries out of which 3 are dead. If two batteries are selected without replacement and tested, find the probability that both are dead.

    Solution: Let the random variable be X_i for denoting the battery being dead.

Variable	Description	Value
X_i	ith battery being dead $\forall i = 1, 2$	$\{0, 1\}$
n	no of batteries selected	2

Table 3.41: definition of random variables

$$\Pr(X_1 = 0) = \frac{3}{8} \tag{3.321}$$

$$\Pr(X_2 = 0|X_1 = 0) = \frac{\Pr(X_2 = 0)\Pr(X_1 = 0)}{\Pr(X_1 = 0)} \tag{3.322}$$

$$\Pr(X_2 = 0)\Pr(X_1 = 0) = \Pr(X_1 = 0)\Pr(X_2 = 0|X_1 = 0) \tag{3.323}$$

$$= \frac{3}{8} \times \frac{2}{7} \tag{3.324}$$

$$= \frac{3}{28} \tag{3.325}$$

16. Assume that in a family, each child is equally likely to be a boy or a girl. A family with three children is chosen at random. The probability that the eldest child is a girl given that the family has at least one girl is

    (a) $\frac{1}{2}$

    (b) $\frac{1}{3}$

    (c) $\frac{2}{3}$

(d) $\frac{4}{7}$

Solution:

Let X_0, X_1, X_2 be the random variables which denotes the three children, where X_0 is the eldest child and X_2 is the youngest child.

RV	Value	Description
X_i	0	child is boy
	1	child is girl

Table 3.42: RV description table

so the required probability is,

$$\Pr\left(X_0 = 1 \middle| X_0 + X_1 + X_2 \geq 1\right) = \frac{\Pr\left(X_0 = 1, X_0 + X_1 + X_2 \geq 1\right)}{\Pr\left(X_0 + X_1 + X_2 \geq 1\right)} \tag{3.326}$$

$$= \frac{\Pr\left(X_0 = 1\right) \times \Pr\left(X_1 + X_2 \geq 0\right)}{\Pr\left(X_0 + X_1 + X_2 \geq 1\right)} \tag{3.327}$$

$$= \frac{\frac{1}{2} \times \sum_{k=0}^{2} {}^{2}C_k \times \frac{1}{2}^k \times \frac{1}{2}^{2-k}}{\sum_{k=1}^{3} {}^{3}C_k \times \frac{1}{2}^k \times \frac{1}{2}^{3-k}} \tag{3.328}$$

$$= \frac{\frac{1}{2} \times 1}{\frac{3}{8} + \frac{3}{8} + \frac{1}{8}} \tag{3.329}$$

$$= \frac{4}{7} \tag{3.330}$$

Therefore, the probability that the eldest child is a girl given that the family has atleast one girl is $\frac{4}{7}$

17. In a college, 30% students fail in physics, 25% fail in mathematics and 10% fail in both. One student is chosen at random. The probability that she fails in physics if she has failed in mathematics is

(a) $\frac{1}{10}$

(b) $\frac{2}{5}$

(c) $\frac{9}{20}$

(d) $\frac{1}{3}$

Random Variable	Value	Description
X	0	Subject being mathematics
	1	Subject being physics
Y	0	Fail
	1	Pass

Table 3.43: Table 1

$$\Pr\left(X = 1, Y = 0\right) = 0.3 \tag{3.331}$$

$$\Pr\left(X = 0, Y = 0\right) = 0.25 \tag{3.332}$$

$$\Pr\left(Y = 0\right) = 0.1 \tag{3.333}$$

$$\Pr\left(X = 1, Y = 0 | X = 0, Y = 0\right) = \frac{\Pr\left(Y = 0\right)}{\Pr\left(X = 0, Y = 0\right)} \tag{3.334}$$

$$= \frac{0.1}{0.3} \tag{3.335}$$

$$= \frac{1}{3} \tag{3.336}$$

Hence, option (d) $\frac{1}{3}$ is the correct option.

Chapter 4

Discrete Distributions

4.1. Bernoulli

4.1.1 A lot consists of 144 ball pens of which 20 are defective and the others are good. Nuri will buy a pen if it is good, but will not buy if it is defective. The shopkeeper draws one pen at random and gives it to her. What is the probability that

(a) She will buy it?

(b) She will not buy it?

Solution: We can model this situation using the random variable $X \sim \mathrm{Ber}(p)$, where p is the probability of success, *i.e.* the pen is purchased. From the given data,

$$1 - p = \frac{20}{144} \implies p = \frac{67}{72} \tag{4.1}$$

(a) Probability that the pen is purchased is

$$\Pr(X = 1) = p = \frac{67}{72} \tag{4.2}$$

(b) Probability that the pen is not purchased is

$$\Pr(X=0) = 1 - p = \frac{5}{72} \tag{4.3}$$

4.1.2 A school has five houses A, B, C, D and E. A class has 23 students, 4 from house A, 8 from house B, 5 from house C, 2 from house D and rest from house E. A single student is selected at random to be the class monitor. The probability that the selected student is not from A, B and C is

Solution: Total no of students=23

House	A	B	C	D	E
Students	4	8	5	2	4

Table 4.1: Student distribution in each house

$$X = \begin{cases} 0, & \text{if student is from A,B and C} \\ 1, & \text{if student is not from A,B and C} \end{cases} \tag{4.4}$$

With reference to Table 4.2

$$p_X(0) = \Pr(A) + \Pr(B) + \Pr(C) \tag{4.5}$$
$$= \frac{4}{23} + \frac{8}{23} + \frac{5}{23} \tag{4.6}$$
$$= \frac{17}{23} \tag{4.7}$$
$$p_X(1) = 1 - p_X(0) \tag{4.8}$$
$$= 1 - \frac{17}{23} \tag{4.9}$$
$$= \frac{6}{23} \tag{4.10}$$

The desired probability is:

$$p_X(1) = \frac{6}{23} \tag{4.11}$$

4.1.3 A bag contains slips numbered from 1 to 100. If Fatima chooses a slip at random from the bag, it will either be an odd number or an even number. Since this situation has only two possible outcomes, so, the probability of each is $\frac{1}{2}$.Justify.

Solution: Let

$$X = \begin{cases} 1, & \text{if number is even} \\ 0, & \text{if number is odd} \end{cases} \tag{4.12}$$

Then

$$p_X(1) = \frac{50}{100} \tag{4.13}$$
$$= \frac{1}{2} \tag{4.14}$$

Similarly

$$p_X(0) = \frac{50}{100} \tag{4.15}$$
$$= \frac{1}{2} \tag{4.16}$$

4.1.4 A letter of English alphabets is chosen at random. Determine the probability that the letter is a consonant.

Solution: Let

$$X = \begin{cases} 1, & \text{if alphabet is consonant} \\ 0, & \text{if alphabet is vowel} \end{cases} \tag{4.17}$$

Then

$$p_X(0) = \frac{5}{26} \tag{4.18}$$

$$p_X(1) = 1 - p_X(0) \tag{4.19}$$

$$= 1 - \frac{5}{26} \tag{4.20}$$

$$= \frac{21}{26} \tag{4.21}$$

4.1.5 A carton of 24 bulbs contain 6 defective bulbs. One bulbs is drawn at random. What is the probability that the bulb is not defective? If the bulb selected is defective and it is not replaced and a second bulb is selected at random from the rest, what is the probability that the second bulb is defective?

Solution: Let

$$X = \begin{cases} 1, & \text{if bulb is not defective} \\ 0, & \text{if bulb is defective} \end{cases} \tag{4.22}$$

Then

$$p_X(0) = \frac{6}{24} \tag{4.23}$$
$$= \frac{1}{4} \tag{4.24}$$
$$p_X(1) = 1 - p_X(0) \tag{4.25}$$
$$= 1 - \frac{1}{4} \tag{4.26}$$
$$= \frac{3}{4} \tag{4.27}$$

When a defective bulb is selected and not replaced

$$Y = \begin{cases} 1, & \text{if bulb is not defective} \\ 0, & \text{if bulb is defective} \end{cases} \tag{4.28}$$

Then

$$p_Y(0) = \frac{6-1}{24-1} \tag{4.29}$$
$$= \frac{5}{23} \tag{4.30}$$

4.1.6 An integer is chosen between 0 and 100. What is the probability that it is

(a) divisible by 7

(b) not divisible by 7

Solution: Let X be a random variable such that

$$X = \begin{cases} 0 & n \not\equiv 0 \pmod 7 \\ 1 & n \equiv 0 \pmod 7 \end{cases} \tag{4.31}$$

$$\text{Hence, } \Pr(X = 1) = \frac{14}{99} \tag{4.32}$$

$$\Pr(X = 0) = 1 - \Pr(X = 1) \tag{4.33}$$

$$= \frac{85}{99} \tag{4.34}$$

4.1.7 In an examination, 20 questions of true-false type are asked. Suppose a student tosses a fair coin to determine his answer to each question. If the coin falls heads, he answer true; if it falls tails, he answer false. Find the probability that he answers at least 12 questions correctly.

Solution:

$$X = \begin{cases} 1 & \text{if } True, \\ 0 & \text{if } False, \end{cases}$$

$$X \sim \text{Ber}(p) \tag{4.35}$$

Suppose $X_i, 1 \leq i \leq n$ represent each of the n draws. Define Y as

$$Y = \sum_{i=1}^{n} X_i \tag{4.36}$$

Then, since the X_i are iid, the pmf of Y is given by

$$Y \sim \mathrm{Bin}\left(n, p\right) \tag{4.37}$$

The cdf of Y is given by

$$F_Y\left(k\right) = \Pr\left(Y \leq k\right) \tag{4.38}$$

$$= \begin{cases} 0 & k < 0 \\ \sum_{i=1}^{k} \begin{pmatrix} n \\ \\ i \end{pmatrix} p^i \left(1-p\right)^{n-i} & 1 \leq k \leq n \\ 1 & k \geq n \end{cases} \tag{4.39}$$

In this case,

$$p = \frac{1}{2}, \; n = 20 \tag{4.40}$$

(a) We require $\Pr\left(Y \geq 12\right)$. Since $n = 20$,

$$\Pr\left(Y \geq 12\right) = 1 - \Pr\left(Y < 12\right) \tag{4.41}$$

$$= F_Y\left(20\right) - F_Y\left(11\right) \tag{4.42}$$

$$= \sum_{k=12}^{20} p_Y(k) \tag{4.43}$$

$$= \sum_{k=12}^{20} \begin{pmatrix} n \\ \\ k \end{pmatrix} p^k \left(1-p\right)^{n-k} \tag{4.44}$$

$$= 0.2517 \tag{4.45}$$

4.1.8 If the letters of the word **ALGORITHM** are arranged at random in a row what is the probability the letters GOR must remain together as a unit?

Solution: Let

$$X = \begin{cases} 1, & \text{if GOR remain together as a unit} \\ 0, & \text{otherwise} \end{cases} \tag{4.46}$$

Then

$$p_X(1) = \frac{7!}{9!} \tag{4.47}$$

$$= \frac{1}{72} \tag{4.48}$$

4.1.9 Six new employees, two of whom are married to each other, are to be assigned six desks that are lined up in a row. If the assignment of employees to desks is made randomly, what is the probability that the married couple will have nonadjacent desks?

Solution: Let X be a Random variable such that

RV	Values	Description
X	0	couple not sitting adjacent
	1	couple sitting adjacent

$$p_X(1) = \frac{5! \times 2}{6!} \tag{4.49}$$

$$= \frac{1}{3} \tag{4.50}$$

$$p_X(0) = 1 - p_X(1) \tag{4.51}$$

$$= \frac{2}{3} \tag{4.52}$$

4.1.10 There are four men and six women on the city council. If one council member is selected for a committee at random,how likely is it that it is a woman?

Solution:

Parameter	**Value**	**Description**
X	0	Male
	1	Female

Table 4.2: Council distribution

$$X = \begin{cases} 0, & \text{if member is a man} \\ 1, & \text{if member is a woman} \end{cases} \tag{4.53}$$

With reference to Table 4.2

$$p_X(1) = \frac{6}{6+4} = \frac{6}{10} \tag{4.54}$$

The desired probability is:

$$p_X(1) = \frac{6}{10} \tag{4.55}$$

4.1.11 A girl calculates that the probability of her winning the first prize in a lottery is 0.08. If 6000 tickets are sold,how many tickets has she bought?

(A) 40 (B)240 (C)480 (D)750

Solution:

parameter	value	description
Random Variable X	0	She didn't buy the ticket
	1	She bought the ticket
N	6000	Number of tickets sold

Table 4.3: Information table

$$\Pr\left(X = 1\right) = 0.08 \tag{4.56}$$

$$\implies \frac{n(X = 1)}{N} = 0.08 \tag{4.57}$$

$$\implies n(X = 1) = 0.08 \times 6000 \tag{4.58}$$

$$n(X = 1) = 480 \tag{4.59}$$

4.1.12 Three numbers are chosen from 1 to 20. Find the probability that they are not consecutive

(a) $\frac{186}{190}$

(b) $\frac{187}{190}$

(c) $\frac{188}{190}$

(d) $\frac{18}{^{20}C_3}$

Solution: The number of set of three consecutive numbers from 1 to 20 is 18 ie,

Random variable	values	Events
X	0	The numbers are consecutive
	1	The numbers are not consecutive

Table 4.4: Random variable

$$\Pr(X=0) = \frac{18}{{}^{20}C_3} \tag{4.60}$$

Then

$$\Pr(X=1) = 1 - \Pr(X=0) \tag{4.61}$$

$$= 1 - \frac{18}{{}^{20}C_3} \tag{4.62}$$

$$= \frac{187}{190} \tag{4.63}$$

4.1.13 Seven persons are to be seated in a row. What is the probabiity that two particular persons sit next to each other?

Solution: Let X be a random variable as defined in the following table. The number

RV	**Values**	**Description**
X	0	Not sitting next to each other
	1	Sitting next to each other

Table 4.5: Random Variables

of ways to arrange 7 people is 7! and the number of ways to arrange 7 people in which the two particular people are adjacent to each other is $6! \times 2$ considering both of them as one unit and considering the arrangements within the unit. Thus,

$$p_X(1) = \frac{6! \times 2}{7!} \tag{4.64}$$

$$= \frac{2}{7} \tag{4.65}$$

$$p_X(0) = 1 - p_X(1) \tag{4.66}$$

$$= \frac{5}{7} \tag{4.67}$$

4.1.14 A single letter is selected at random from the word 'PROBABILITY'. The probability that it is a vowel is

Solution: Let X be an bernoulli rv defined as in Table 4.6,

RV	Value	Description
X	0	Selection of non-vowels
	1	Selection of vowels

Table 4.6: Random variable X declaration.

Where, The probabilities are as follows:

$$p_X(k) = \begin{cases} \frac{7}{11} & k = 0 \\ \frac{4}{11} & k = 1 \end{cases} \tag{4.68}$$

From (4.68), The probability that the selected letter is a vowel is given by:

$$p_X(1) = \frac{4}{11} \tag{4.69}$$

Therefore, the probability that the selected letter is a vowel is $\frac{4}{11}$.

4.1.15 The probability of getting a bad egg in a lot of 400 is 0.035. The number of bad eggs in the lot is

Solution: Let

$$\Pr(X = k) = \begin{cases} 1 - p & k = 0 \\ p & k = 1 \end{cases} \tag{4.70}$$

Bad eggs

$$= Pr(X = 1) \times n \tag{4.71}$$

$$= 14 \tag{4.72}$$

Table 4.7: Table

Parameters	Values	Description
X	0	good egg
	1	bad egg
n	400	Total number of eggs
p	0.035	Probability of getting bad egg

4.1.16 someone is asked to take a number from 1 to 100.The probability that it is a prime number is

Solution:

RV	value	description
X	0	not prime number
	1	prime number

Table 4.9: random variable

We know that there are 25 prime numbers in between 1 to 100.

hence,

$$\Pr(X = 1) = \frac{25}{100} \tag{4.73}$$

$$= \frac{1}{4} \tag{4.74}$$

4.2. Multinomial

4.2.1 A box contains 5 red marbles, 8 white marbles and 4 green marbles.One marble is taken out of the box at random. What is the probability that the marble taken out will be

Solution:

(a) red ?

(b) white ?

(c) not green?

Solution: Let

$$\mathrm{N} = \mathrm{R} + \mathrm{W} + \mathrm{G} \tag{4.75}$$

$$\mathrm{n} = \mathrm{r} + \mathrm{w} + \mathrm{g} \tag{4.76}$$

where R,B,G and r, b, g represent the number of red, white and green marbles respectively within N and n. Then

$$\Pr(r, w, g) = \frac{{}^{R}C_{r}\,{}^{W}C_{w}\,{}^{G}C_{g}}{{}^{R+W+G}C_{r+w+g}} \tag{4.77}$$

(a) Probability that the marble taken out is red

$$\Pr(1, 0, 0) = \frac{{}^{5}C_{1}\,{}^{8}C_{0}\,{}^{4}C_{0}}{{}^{17}C_{1}} = \frac{5}{17} \approx 0.2941 \tag{4.78}$$

(b) Probability that the marble taken out is white

$$\Pr(0, 1, 0) = \frac{{}^{5}C_{0}\,{}^{8}C_{1}\,{}^{4}C_{0}}{{}^{17}C_{1}} = \frac{8}{17} \approx 0.4706 \tag{4.79}$$

(c) Probability that the marble taken out is not green

$$1 - \Pr(0,0,1) = 1 - \frac{{}^{5}C_0 {}^{8}C_0 {}^{4}C_1}{{}^{17}C_1} = 1 - \frac{4}{17} = \frac{13}{17} \approx 0.7647 \tag{4.80}$$

4.2.2 A box contains 3 orange balls, 3 green balls and 2 blue balls. Three balls are drawn at random from the box without replacement. The probability of drawing 2 green balls and one blue ball is

(a) $\frac{3}{28}$

(b) $\frac{2}{21}$

(c) $\frac{1}{28}$

(d) $\frac{167}{168}$

Solution:

Let,

$$N = O + G + B \tag{4.81}$$

$$n = o + g + b \tag{4.82}$$

where O,G,B and o,g,b represents the number of Orange, Green and Blue balls respectively within N, n. Then

$$\Pr(o,g,b) = \frac{{}^{O}C_o {}^{G}C_g {}^{B}C_b}{{}^{O+G+B}C_{o+g+b}} \tag{4.83}$$

So,Probability of 2 Green and 1 blue ball,

$$\Pr(0,2,1) = \frac{{}^3C_0 {}^3C_2 {}^2C_1}{{}^8C_3} \tag{4.84}$$

$$= \frac{3}{28} \tag{4.85}$$

$\therefore$ Option (1) is correct.

4.2.3 A game of chance consists of spinning an arrow which comes to rest pointing at one of the numbers 1, 2, 3, 4, 5, 6, 7, 8 (see Fig. 4.1), and these are equally likely outcomes. What is the probability that it will point at:

(a) 8?

(b) an odd number?

(c) a number greater than 2?

(d) a number less than 9?

Solution: Let X be a random variable defined as the value given by the pointer. Then,

$$\Pr(X = i) = \frac{1}{8} \quad 1 \le i \le 8 \tag{4.86}$$

$$F_X(i) = \Pr(X \le i) \tag{4.87}$$

$$= \begin{cases} 0, & i \le 0 \\ \frac{i}{8} & 1 \le i \le 8 \\ 1, & i \ge 9 \end{cases} \tag{4.88}$$

which are plotted in Fig. 4.2 and Fig. 4.3 respectively.

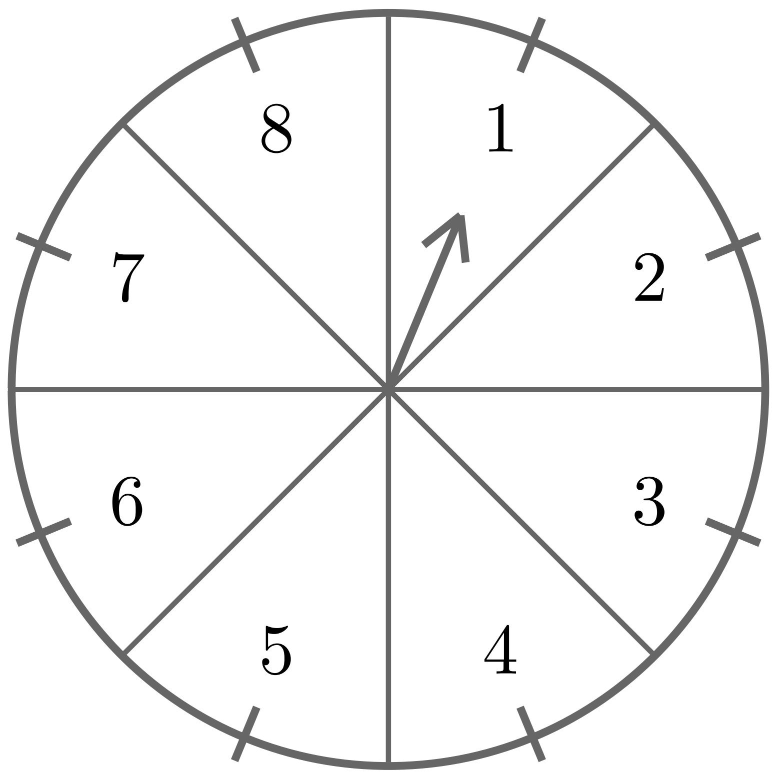

Figure 4.1: Spinner

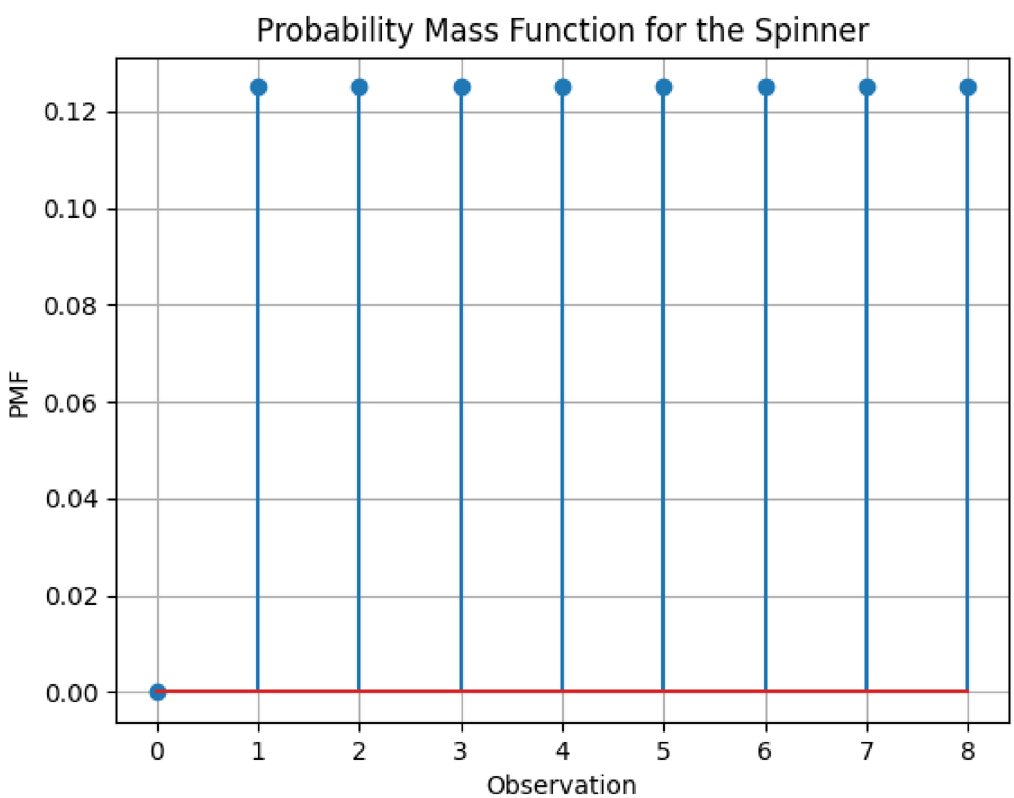

Figure 4.2: Plot of Probability Mass Function

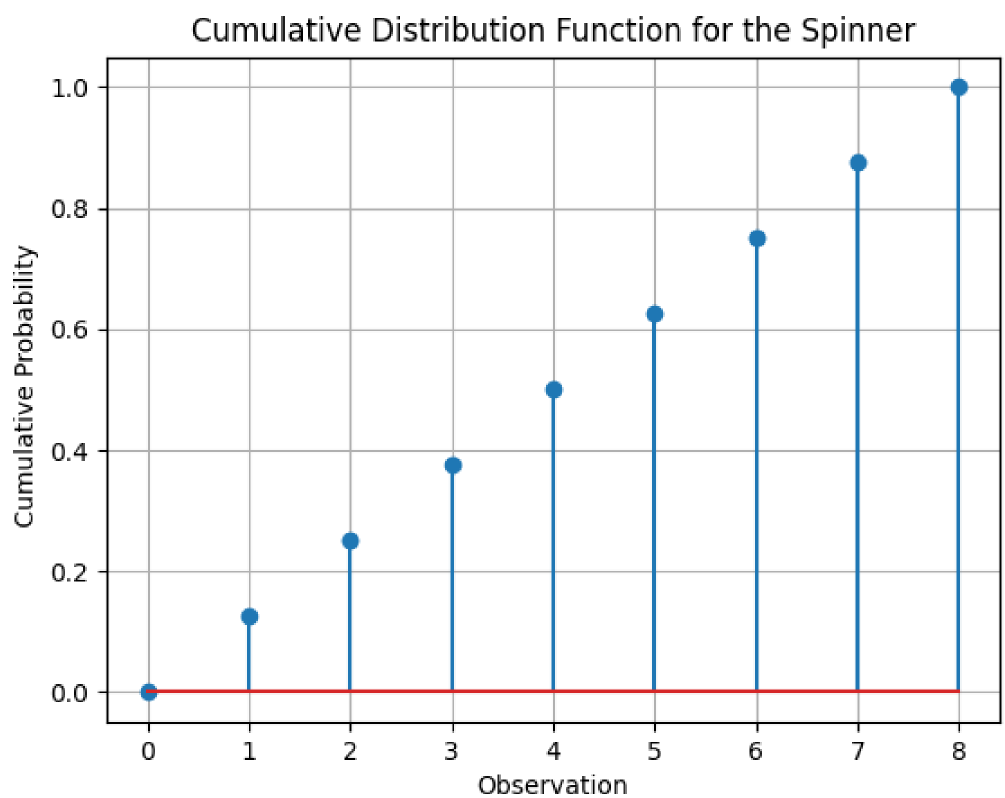

Figure 4.3: Plot of Cumulative Distribution Function

(a)

$$\Pr(X = 8) = \frac{1}{8} = 0.125 \tag{4.89}$$

(b) For i being odd,

$$\Pr(X = \{1, 3, 5, 7\}) = \frac{4}{8} = 0.5 \tag{4.90}$$

(c)

$$\Pr(X > 2) = 1 - \Pr(X \leq 2) \tag{4.91}$$

$$= 1 - (F_X(2) - F_X(0)) \tag{4.92}$$

$$= \frac{6}{8} \tag{4.93}$$

(d)

$$\Pr(1 \leq X < 9) = F_X(8) - F_X(0) = 1 \tag{4.94}$$

4.2.4 A box contains 10 red marbles, 20 blue marbles and 30 green marbles. 5 marbles are drawn from the box, what is the probability that

(a) all will be blue?

(b) atleast one will be green?

Solution: See (E.1.2). In this question,

$$N = 60,\ R = 10,\ B = 20,\ G = 30,\ n = 5 \tag{4.95}$$

(a) From (E.1.2),

$$\Pr(0,5,0) = \frac{{}^{20}C_5}{{}^{60}C_5} \tag{4.96}$$

(b) Since

$$\Pr(r,b,0) = \frac{{}^{R}C_r \, {}^{B}C_b}{{}^{R+B+G}C_{r+b}} \tag{4.97}$$

The probability that at least one marble is green is given by

$$1 - \sum_{r+b=n} \Pr(r,b,0) = 1 - \sum_{r+b=n} \frac{{}^{R}C_r \, {}^{B}C_b}{{}^{R+B+G}C_{r+b}} = 1 - \frac{{}^{R+B}C_n}{{}^{R+B+G}C_n} \tag{4.98}$$

from (E.2.1). Substituting numerical values, the desired probability is

$$1 - \frac{{}^{30}C_5}{{}^{60}C_5} \tag{4.99}$$

4.2.5 A box of oranges is inspected by examining three randomly selected oranges drawn without replacement. If all the three oranges are good, the box is approved for sale, otherwise, it is rejected. Find the probability that a box containing 15 oranges out of which 12 are good and 3 are bad ones will be approved for sale.

Solution: Choosing

$$R = 12, B = 3, G = 0, n = 3, r = 3, b = 0, g = 0 \tag{4.100}$$

in (E.1.2) the desired probability is

$$\Pr(3,0,0) = \frac{{}^{12}C_3}{{}^{15}C_3} = \frac{44}{91} \tag{4.101}$$

4.2.6 A bag contain 5 red and 3 blue balls. If 3 balls are drawn at random without replacement, the probability of getting exactly one red ball is

(a) $\frac{45}{196}$

(b) $\frac{135}{392}$

(c) $\frac{15}{56}$

(d) $\frac{15}{29}$

Solution:

Let,

$$N = R + B \tag{4.102}$$

$$n = r + b \tag{4.103}$$

where R,B and r,b represents the number of Red and Blue balls respectively within N, n. Then

$$\Pr(r, b) = \frac{{}^{R}C_{r}\,{}^{B}C_{b}}{{}^{R+B}C_{r+b}} \tag{4.104}$$

So,Probability of getting exactly 1 Red ball,

$$\Pr(1, 2) = \frac{{}^{5}C_{1}\,{}^{3}C_{2}}{{}^{8}C_{3}} \tag{4.105}$$

$$= \frac{15}{56} \tag{4.106}$$

$\therefore$ Option (3) is correct.

4.2.7 A bag contains 5 red and 3 blue balls. If 3 balls are drawn at random without replacement the probability that exactly two of the three balls were red, the first ball being red is

See (E.1.2). In this question,

As the first ball drawn is red,

$$N = 7, R = 4, B = 3, G = 0, r = 1, b = 1, g = 0 \tag{4.107}$$

The desired probability is,

$$\Pr(1,1,0) = \frac{{}^4C_1 {}^3C_1}{{}^7C_2} \tag{4.108}$$

$$= \frac{4}{7} \tag{4.109}$$

4.3. Uniform

4.3.1 A die is thrown, find the probability of following events:

(a) A prime number will appear

(b) A number greater than or equal to 3 will appear

(c) A number less than or equal to one will appear

(d) A number more than 6 will appear

(e) A number less than 6 will appear

Solution: The CDF of the random variable X representing the roll of a dice, is available in (C.3.3.1).

(a) The set of possible prime numbers in a die roll contains 2,3,5

$$\Pr\left(X \in \{2,3,5\}\right) = p_X\left(2\right) + p_X\left(3\right) + p_X\left(5\right) \tag{4.110}$$
$$= \frac{1}{2} \tag{4.111}$$

(b) The probability that a number greater than or equal to 3 will appear is given by

$$\Pr\left(X \geq 3\right) = 1 - \Pr\left(X \leq 2\right) \tag{4.112}$$
$$= 1 - F_X\left(2\right) \tag{4.113}$$
$$= \frac{2}{3} \tag{4.114}$$

(c) The probability that a number less than or equal to 1 will appear is given by

$$\Pr\left(X \leq 1\right) = F_X\left(1\right) \tag{4.115}$$
$$= \frac{1}{6} \tag{4.116}$$

(d) The probability that a number greater than 6 will appear is given by

$$\Pr(X > 6) = 1 - \Pr(X \leq 6) \tag{4.117}$$

$$= 1 - F_X(6) \tag{4.118}$$

$$= 0 \tag{4.119}$$

(e) The probability that a number less than 6 will appear is given by

$$\Pr(X < 6) = \Pr(X \leq 5) \tag{4.120}$$

$$= F_X(5) \tag{4.121}$$

$$= \frac{5}{6} \tag{4.122}$$

4.3.2 All the jacks, queens and kings are removed from a deck of 52 playing cards. The remaining cards are well shuffled and then one card is drawn at random. Giving ace a value 1 similar value for other cards, find the probability that the card has a value

(a) 7

(b) greater than 7

(c) less than 7

Solution: Number of cards left after removing all jacks, queens and kings(=N)

$$= 52 - 4 \times 3 \tag{4.123}$$

$$= 40 \tag{4.124}$$

Parameter	**Value**	**Description**
X	1-10	Represents the value of the card picked

Finding pmf:

$$p_X(k) = \Pr(X = k) \; \forall \; 1 \leq k \leq 10 \tag{4.125}$$
$$= \frac{4 \times 1}{40} \tag{4.126}$$
$$= \frac{1}{10} \tag{4.127}$$
$$\therefore p_X(k) = \begin{cases} \frac{1}{10} & 1 \leq k \leq 10 \\ 0 & \text{otherwise} \end{cases} \tag{4.128}$$

CDF for the following pmf is:

$$F_X(k) = \sum_{m=0}^{k} p_X(m) \; \forall \; 1 \leq k \leq 10 \tag{4.129}$$
$$= k \times \frac{1}{10} \tag{4.130}$$
$$= \frac{k}{10} \tag{4.131}$$
$$\therefore F_X(k) = \begin{cases} 0 & k \leq 0 \\ \frac{k}{10} & 1 \leq k \leq 10 \\ 1 & k > 10 \end{cases} \tag{4.132}$$

(a) Probability that card has value equal to 7:

$$= p_X(7) \tag{4.133}$$
$$= 1 \times \frac{1}{10} = \frac{1}{10} \tag{4.134}$$

(b) Probability that card has value greater than 7

$$= F_X(10) - F_X(7) \tag{4.135}$$

$$= 1 - \frac{7}{10} = \frac{3}{10} \tag{4.136}$$

(c) Probability that card has value less than 7

$$= F_X(6) \tag{4.137}$$

$$= \frac{6}{10} \tag{4.138}$$

4.3.3 Two dice are numbered 1,2,3,4,5,6 and 1,1,2,2,3,3 respectively. They are thrown and the sum of then numbers on them is noted. Find the probability of getting each sum from 2 to 9 seperately **Solution:** The Z-transform of a die is defined as

$$M_X(z) = z^{-X} = \sum_{k=-\infty}^{\infty} p_X(k) z^{-k} \tag{4.139}$$

The Z-transform of the first die X_1 is given by

$$M_{X_1}(z) = \frac{1}{6} \sum_{n=1}^{6} z^{-n} = \frac{z^{-1}(1 - z^{-6})}{6(1 - z^{-1})}, |z| > 1 \tag{4.140}$$

The Z-transform of the second die X_2 is given by

$$M_{X_2}(z) = \frac{1}{3} \sum_{n=1}^{3} z^{-n} = \frac{z^{-1}(1 - z^{-3})}{3(1 - z^{-1})}, |z| > 1 \tag{4.141}$$

The Z-transform of X is given as:

$$M_X(z) = \frac{z^{-1}(1-z^{-6})}{6(1-z^{-1})} \times \frac{z^{-1}(1-z^{-3})}{3(1-z^{-1})} \tag{4.142}$$

$$M_X(z) = \frac{1}{18}\left[\frac{z^{-2}(1-z^{-3}-z^{-6}-z^{-9})}{(1-z^{-1})^2}\right] \tag{4.143}$$

We also know that,

$$p_X(n-k) \overset{Z}{\longleftrightarrow} M_X(z)z^{-k}; \tag{4.144}$$

$$nu(n) \overset{Z}{\longleftrightarrow} \frac{z^{-1}}{(1-z^{-1})^2} \tag{4.145}$$

Hence, after some algebra, it can be shown that,

$$\frac{1}{18}[n-1u(n-1)-n-4u(n-4)$$
$$-(n-7)u(n-7)-(n-10)u(n-10)]$$
$$\overset{Z}{\longleftrightarrow}$$
$$\frac{1}{18}\left[\frac{z^{-2}1-z^{-3}-z^{-6}-z^{-9}}{(1-z^{-1})^2}\right] \tag{4.146}$$

where,

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} \tag{4.147}$$

hence,

$$p_X(n) = \frac{1}{18}[n - 1u(n-1) - n - 4u(n-4) -$$
$$(n-7)u(n-7) - (n-10)u(n-10)] \quad (4.148)$$

$$p_X(n) = \begin{cases} 0 & n \leq 1 \\ \frac{n-1}{18} & 2 \leq n \leq 4 \\ \frac{1}{6} & 5 \leq n \leq 7 \\ \frac{10-n}{18} & 8 \leq n \leq 9 \\ 0 & n \geq 10 \end{cases} \quad (4.149)$$

hence, the probabilities are,

$$p_X(n) = \begin{cases} \frac{1}{18} & n = 2 \\ \frac{1}{9} & n = 3 \\ \frac{1}{6} & n = 4 \\ \frac{1}{6} & n = 5 \\ \frac{1}{6} & n = 6 \\ \frac{1}{6} & n = 7 \\ \frac{1}{9} & n = 8 \\ \frac{1}{18} & n = 9 \end{cases} \quad (4.150)$$

The experiment of rolling the dice was simulated using Python for 10000 samples.

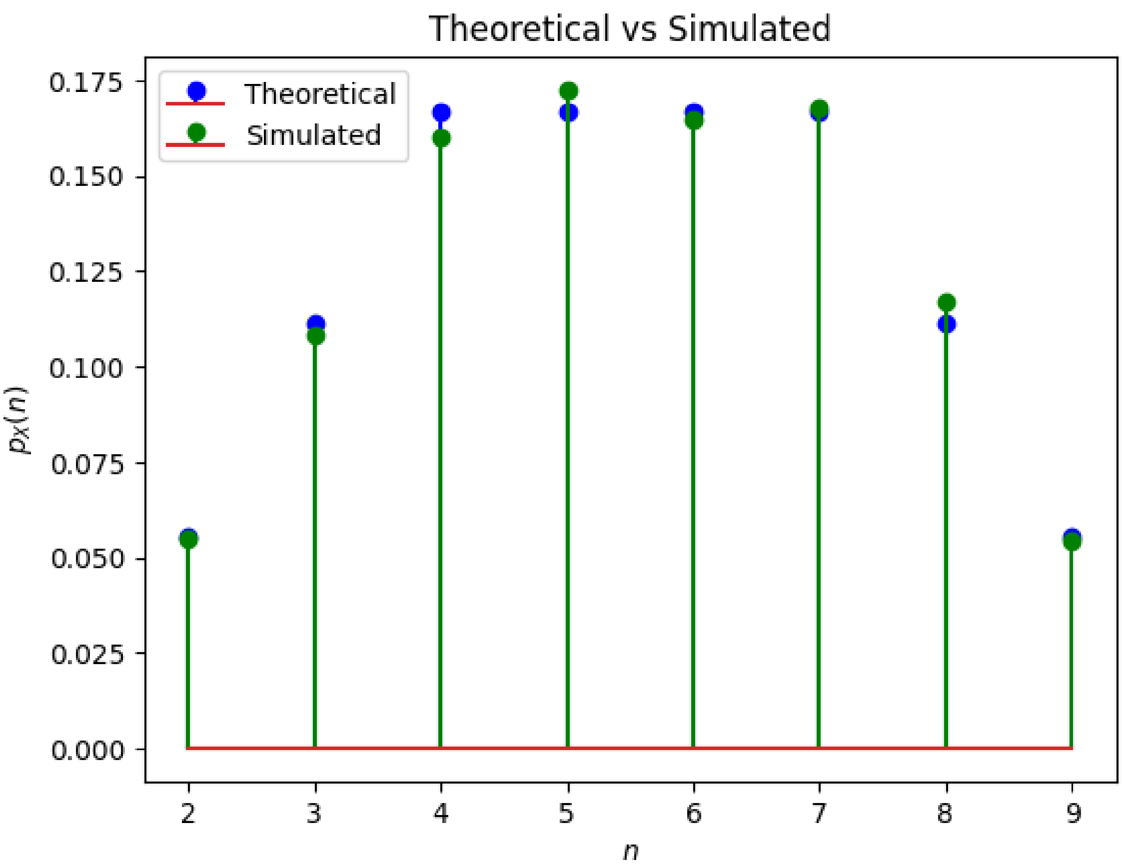

Figure 4.4: Plot of $p_X(n)$.Simulations are close to the analysis.

4.4. Binomial

4.4.1 A die is thrown twice. What is the probability that

(a) 5 will not come up either time?

(b) 5 will come up at least once?

Solution: 4.11 From Table 4.11, the PMF of X is

Parameters	**Value**	**Description**
n	2	Number of trials in an Experiment
p	1/6	Probability of Success
q	5/6	Probability of Failure

Table 4.11:

$$\Pr(X = k) = {}^{n}C_k p^k q^{n-k} \tag{4.151}$$

$$= {}^{2}C_k \left(\frac{1}{6}\right)^k \left(\frac{5}{6}\right)^{2-k} \qquad \forall k = 0, 1, 2 \tag{4.152}$$

and the CDF is

$$F_X(k) = \Pr(X \le k) = \sum_{i=0}^{k} {}^{n}C_i p^i q^{n-i} \tag{4.153}$$

(a)

$$\Pr(X = 0) = {}^{2}C_0 \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^2 = \frac{25}{36} \tag{4.154}$$

(b)

$$\Pr(X \ge 1) = 1 - \Pr(X \le 0) = 1 - F_X(0) \tag{4.155}$$

$$= 1 - \frac{25}{36} = \frac{11}{36} \tag{4.156}$$

4.4.2 Three coins are tossed once. Find the probability of getting

(a) 3 heads

(b) 2 heads

(c) atleast 2 heads

(d) atmost 2 heads

(e) no head

(f) 3 tails

(g) exactly two tails

(h) no tail

(i) atmost two tails

Solution: Let the random variable X denote one single coin toss, where obtaining a head is considered a success. Then,

$$X \sim \text{Ber}(p) \tag{4.157}$$

Suppose $X_i, 1 \leq i \leq n$ represent each of the n tosses. Define Y as

$$Y = \sum_{i=1}^{n} X_i \tag{4.158}$$

Then, since the X_i are iid, the pmf of Y is given by

$$Y \sim \text{Bin}(n, p) \tag{4.159}$$

The cdf of Y is given by

$$F_Y(k) = \Pr(Y \le k) \tag{4.160}$$

$$= \begin{cases} 0 & k < 0 \\ \sum_{i=1}^{k} \begin{pmatrix} n \\ i \end{pmatrix} p^i (1-p)^{n-i} & 1 \le k \le n \\ 1 & k \ge n \end{cases} \tag{4.161}$$

In this case,

$$p = \frac{1}{2}, \ n = 3 \tag{4.162}$$

(a) We require $\Pr(Y = 3)$. Thus, from (4.159),

$$\Pr(Y = 3) = \begin{pmatrix} n \\ 3 \end{pmatrix} p^3 (1-p)^{n-3} \tag{4.163}$$

$$= \frac{1}{8} \tag{4.164}$$

(b) We require $\Pr(Y = 2)$. Thus, from (4.159),

$$\Pr(Y = 2) = \begin{pmatrix} n \\ 2 \end{pmatrix} p^2 (1-p)^{n-2} \tag{4.165}$$

$$= \frac{3}{8} \tag{4.166}$$

(c) We require $\Pr(Y \geq 2)$. Since $n = 3$ in (4.161),

$$\Pr(Y \geq 2) = 1 - \Pr(Y < 2) \tag{4.167}$$

$$= F_Y(3) - F_Y(1) \tag{4.168}$$

$$= \sum_{k=2}^{3} \binom{n}{k} p^k (1-p)^{n-k} \tag{4.169}$$

$$= \frac{1}{2} \tag{4.170}$$

(d) We require $\Pr(Y \leq 2)$. Thus, from (4.161),

$$\Pr(Y \leq 2) = \sum_{k=0}^{2} \binom{n}{k} p^k (1-p)^{n-k} \tag{4.171}$$

$$= \frac{7}{8} \tag{4.172}$$

(e) Let, X_i be the sequence of independent Bernoulli random varibles

$$X_i = \begin{cases} 1, & \text{Heads} \\ 0, & \text{Tails} \end{cases} \tag{4.173}$$

and X be summation of all sequences

$$X = \sum_{0}^{n} X_i \tag{4.174}$$

which means,

$$p_X(k) = \begin{cases} \frac{1}{2} & = p \qquad k = 1 \\ \frac{1}{2} & = q \qquad k = 0 \end{cases} \tag{4.175}$$

For number of trials be n and the pmf of getting k heads is given by:

$$p_X(k) = {}^nC_k (p)^k (q)^{n-k} \tag{4.176}$$

$$= {}^nC_k (0.5)^k (0.5)^{n-k} \tag{4.177}$$

Using the above equation, for $n = 3$ and $k = 0$:

$$p_X(0) = {}^3C_0 (0.5)^3 (0.5)^0 \tag{4.178}$$

$$= \left(\frac{3!}{(3-0)!\,(0)!}\right)(0.5)^3 \tag{4.179}$$

$$= \left(\frac{3!}{3!}\right)\left(\frac{1}{8}\right) \tag{4.180}$$

$$= \frac{1}{8} \tag{4.181}$$

Hence, the given statement is wrong$\left(\because \frac{1}{8} \neq \frac{1}{4}\right)$

(f) We require $\Pr(Y = 1)$ (since only one head is obtained). Thus, from (4.159),

$$\Pr(Y = 1) = \begin{pmatrix} n \\ 1 \end{pmatrix} p^1 (1-p)^{n-1} \tag{4.182}$$

$$= \frac{3}{8} \tag{4.183}$$

(g) We require $\Pr(Y = 3) = \frac{1}{8}$ from (4.164).

(h) We require $\Pr(Y \geq 1)$ (since at least one head is obtained). Thus, from above solved questions,

$$\Pr(Y \geq 1) = 1 - \Pr(Y < 1) \tag{4.184}$$

$$= 1 - F_Y(0) \tag{4.185}$$

$$= 1 - \Pr(Y = 0) \tag{4.186}$$

$$= \frac{7}{8} \tag{4.187}$$

4.4.3 In a game, the entry fee is Rs 5.The game consists of a tossing a coin 3 times.If one or two heads show, Sweta gets her entry fee back.If she throws 3 heads, she receives double the entry fees.Otherwise she will lose. For tossing a coin three times, find the probability that she

(a) loses the entry fee.

(b) gets double entry fee.

(c) just gets her entry fee.

Solution: Let, X_i be the random variable that represent the a single toss resulting in head and Y be the random variable that represent the total tosses.

Parameter	value	description
X_i	1	first toss
	2	second toss
	3	third toss
n	3	number of tosses
p,q	$\frac{1}{2}$	toss result in heads/tails
Y	$\sum_{i=0}^{3} X_i$	three tosses

The pmf of Y is,

$$\Pr(Y = k) = {}^{n}C_k\,(p)^k\,(q)^{n-k} \qquad ,0 \le k \le n \tag{4.188}$$

From above equations (4.188) we can say, Probability that she loss the fees (0 heads),

$$p_Y(0) = {}^{3}C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^{3-0} \tag{4.189}$$

$$= \frac{1}{8} \tag{4.190}$$

$$= 0.125 \tag{4.191}$$

Probability that she gets double entry fees(3 heads),

$$p_Y(3) = {}^{3}C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^{3-3} \tag{4.192}$$

$$= \frac{1}{8} \tag{4.193}$$

$$= 0.125 \tag{4.194}$$

Probability that she just gets the entry fees(1 heads + 2 heads),

$$p_Y(1) + p_Y(2) = {}^{3}C_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^{3-1} + {}^{3}C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{3-2} \tag{4.195}$$

$$= \frac{3}{8} + \frac{3}{8} \tag{4.196}$$

$$= 0.750 \tag{4.197}$$

4.4.4 A game consists of tossing a one rupee coin 3 times and noting its outcome each time. Hanif wins if all the tosses give the same result i.e., three heads or three tails, and loses otherwise. Calculate the probability that Hanif will lose the game

Solution: A coin toss can have only two outcomes which are:

(a) Heads

(b) Tails

Both of these outcomes are equally likely. Let us consider random variable X for the number of heads in the experiment.

Parameter	Value	Description
X	$\{0, 1, 2, 3\}$	Number of heads
n	3	Number of trails
p	0.5	Probability of success
q	0.5	Probability of Failure

The PMF of X is

$$\Pr(X = k) = {}^{n}C_k p^k q^{n-k} \tag{4.198}$$

$$= {}^{3}C_k \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{3-k} \qquad \forall k = 0, 1, 2, 3 \tag{4.199}$$

$$= {}^{3}C_k \left(\frac{1}{2}\right)^3 \qquad \forall k = 0, 1, 2, 3 \tag{4.200}$$

The Cumulative Distribution Function (CDF) of X is given by the probability that X is less than or equal to a given value k, for k = 0, 1, 2, 3. The CDF can be expressed as:

$$F_X(k) = \Pr(X \le k) \tag{4.201}$$
$$= \sum_{i=0}^{k} {}^nC_i p^i q^{n-i} \tag{4.202}$$
$$= \sum_{i=0}^{k} {}^3C_i \left(\frac{1}{2}\right)^3 \tag{4.203}$$

Hanif will lose the game if the value of X is 1 or 2. Hence, we need to find $\Pr(X = 1) + \Pr(X = 2)$

$$\Pr(X = 1) + \Pr(X = 2) = {}^3C_1 \left(\frac{1}{2}\right)^3 + {}^3C_2 \left(\frac{1}{2}\right)^3 \tag{4.204}$$
$$= 3\left(\frac{1}{8}\right) + 3\left(\frac{1}{8}\right) \tag{4.205}$$
$$= 3\left(\frac{1}{4}\right) \tag{4.206}$$
$$= 0.75 \tag{4.207}$$

Hence, the probability of hanif losing the game is 0.75

4.4.5 A coin is tossed three times, where. Determine $\Pr(E \mid F)$ where

(a) E : head on third toss, F : heads on first two tosses

(b) E : at least two heads, F : at most two heads

(c) E : at most two tails, F : at least one tail

Solution: Consider the random variables X_1, X_2, X_3, X, which denotes the first, second, third toss and number of heads in the 3 tosses respectively as described in table 4.12.

RV	Values	Description
X	$\{0,1,2,3\}$	Number of heads in 3 tosses
X_1	$\{0,1\}$	0: Heads , 1: Tails
X_2	$\{0,1\}$	0: Heads , 1: Tails
X_3	$\{0,1\}$	0: Heads , 1: Tails

Table 4.12: Random variables X_1, X_2, X_3, X

The random variable X follows binomial distribution

$$X = X_1 + X_2 + X_3 \tag{4.208}$$

The PMF of the random variable X is given by,

$$P_X\left(n\right) = {}^{N}C_n\, p^n \left(1-p\right)^{N-n} \tag{4.209}$$

Here we have

$$N = 3,\, p = \frac{1}{2} \tag{4.210}$$

The CDF of the random variable X is given by,

$$F_X\left(n\right) = \Pr\left(X \le n\right) = \sum_{i=0}^{n} {}^{N}C_i\, p^i \left(1-p\right)^{N-i} \tag{4.211}$$

(a) The events E, F can be described by the RV as

$$E : X_3 = 0 \tag{4.212}$$

$$F : X_1 + X_2 = 0 \tag{4.213}$$

Y is another random variable which represents the number of heads in first two tosses.

$$Y = X_1 + X_2 \tag{4.214}$$

The PMF of the random variable Y is given by,

$$P_Y(n) = {}^N C_n \, p^n (1-p)^{N-n} \tag{4.215}$$

Here we have

$$N = 2, \, p = \frac{1}{2} \tag{4.216}$$

The event EF can be expressed as,

$$X_3 = 0 \cap X_1 + X_2 = 0 \tag{4.217}$$

$$\triangleq X_1 + X_2 + X_3 = 0 \tag{4.218}$$

$$\implies X = 0 \tag{4.219}$$

The required probability is given by,

$$\Pr(X_3 = 0 \mid Y = 0) \tag{4.220}$$

$$= \frac{\Pr(X = 0)}{\Pr(Y = 0)} \tag{4.221}$$

$$= \frac{1}{2} \tag{4.222}$$

(b) The events E, F, F' can be described by the RV as

$$E : X \leq 1 \tag{4.223}$$

$$F : X \geq 1 \tag{4.224}$$

$$F' : X = 0 \tag{4.225}$$

The required probability is given by,

$$= \frac{\Pr(EF)}{1 - \Pr(F')} \tag{4.226}$$

The event EF can be expressed as,

$$X \leq 1 \cap X \geq 1 \tag{4.227}$$

$$\implies X = 1 \tag{4.228}$$

Hence, the required probability is given by,

$$= \frac{\Pr(X = 1)}{1 - \Pr(X = 0)} \tag{4.229}$$

$$= \frac{\frac{3}{8}}{1 - \frac{1}{8}} \tag{4.230}$$

$$= \frac{3}{7} \tag{4.231}$$

(c) For the events E, F, their complements are E' : all 3 tails, F' : zero tails. The events E', F' can be described by the RV as

$$E' : X = 3 \tag{4.232}$$

$$F' : X = 0 \tag{4.233}$$

By using property of conditional probability we have,

$$\Pr(E \mid F) = \frac{\Pr(EF)}{\Pr(F)} \tag{4.234}$$

$$= \frac{1 - \Pr(E' + F')}{\Pr(F)} \tag{4.235}$$

The required probability is given by,

$$= \frac{1 - \Pr(X = 0 + X = 3)}{1 - \Pr(X = 0)} \tag{4.236}$$

$$= \frac{1 - (\Pr(X = 0) + \Pr(X = 3) - \Pr(\phi))}{1 - \Pr(X = 0)} \tag{4.237}$$

$$= \frac{1 - \left(\frac{1}{8} + \frac{1}{8} - 0\right)}{1 - \frac{1}{8}} \tag{4.238}$$

$$= \frac{6}{7} \tag{4.239}$$

4.4.6 A die is tossed thrice. Find the probability of getting an odd number at least once.

Solution: The parameters for the equivalent binomial distribution is

$$p = \frac{1}{2},\ n = 3 \tag{4.240}$$

The CDF is given by

$$F_X(k) = \sum_{0}^{k} {}^nC_k p^k (1-p)^{(n-k)} \tag{4.241}$$

and the required probability is

$$\Pr(1 \leq X \leq 3) = F_X(3) - F_X(0) = \frac{7}{8} \tag{4.242}$$

4.4.7 Find the probability distribution of

(a) number of heads in two tosses of a coin.

(b) number of tails in the simultaneous tosses of three coins.

(c) number of heads in four tosses of a coin.

Solution: Table 4.14 summarises the given information.

Variable	**Value**	**Description**
n	$\{2,3,4\}$	Number of trials in 2,3,4 tosses of a coin
p	$\frac{1}{2}$	Probability of getting a head
q	$1-p$	Probability of not getting a head
X_1	$\{0,1,2\}$	Number of heads in 2 tosses of a coin
X_2	$\{0,1,2,3\}$	Number of tails in 3 tosses of a coin
X_3	$\{0,1,2,3,4\}$	Number of heads in 4 tosses of a coin

Table 4.14: Variable Description

(a) Number of heads in two tosses of a coin.

$$p_{X_1}(k) = {}^nC_k p^k q^{n-k}, 0 \leq k \leq 2, n = 2 \tag{4.243}$$

(b) Number of tails in the simultaneous tosses of three coins.

$$p_{X_2}(k) = {}^nC_k p^k q^{n-k}, 0 \leq k \leq 3, n = 3 \tag{4.244}$$

(c) Number of heads in four tosses of a coin.

$$p_{X_3}(k) = {}^nC_k p^k q^{n-k}, 0 \leq k \leq 4, n = 4 \tag{4.245}$$

4.4.8 Find the probability distribution of the number of successes in two tosses of a die, where a success is defined as

(a) number greater than 4

(b) six appears on at least one die

Solution: Let X be a random variable denoting the outcome of a die toss.

(a)

$$\Pr(X > 4) = 1 - F_X(3) = \frac{1}{3} \tag{4.246}$$

Let Y be the random variable denoting number of successes. Then,

$$Y \sim \text{Bin}(n, p) \tag{4.247}$$

where

$$n = 2, p = \frac{1}{3}. \tag{4.248}$$

Thus,

$$\therefore \Pr(Y = i) = {}^2C_i (1-p)^{2-i} p^i \tag{4.249}$$

and the desired distribution is

$$p_Y(k) = \begin{cases} \frac{4}{9}, & k = 0 \\ \frac{4}{9}, & k = 1 \\ \frac{1}{9}, & k = 2 \\ 0, & \text{otherwise} \end{cases} \tag{4.250}$$

(b) In this case, the binomial distribution has parameters

$$n = 2, p = \frac{1}{6} \tag{4.251}$$

yielding

$$p_Z(k) = \begin{cases} \frac{25}{36}, & k = 0 \\ \frac{11}{36}, & k = 1 \\ 0, & \text{otherwise} \end{cases} \tag{4.252}$$

4.4.9 There are 5% defective items in a large bulk of items. What is the probability that a sample of 10 items will include not more than one defective item?

Solution:

Solution:

The cdf of Y is given by

$$F_Y(n) = \Pr(Y \le n) \tag{4.253}$$

$$= \sum_{k=0}^{n} {}^{10}C_k p^k (1-p)^{10-k} \tag{4.254}$$

Parameter	Values	Description
n	10	Number of items
k	0,1	Number of defective items
p	0.05	Probability of being defective
X	1 if defective 0 if not defective	Bernoulli Random Variable
Y	$\sum_{i=1}^{n} X_i$	Binomial Random Variable

Table 4.15: Definition of X and parameters.

We require $\Pr(Y \leq 1)$. Since $n = 1$,

$$F_Y(1) = \Pr(Y \leq 1) \tag{4.255}$$

$$= \sum_{k=0}^{1} {}^{10}C_k (0.05)^k (0.95)^{10-k} \tag{4.256}$$

$$= 0.9138 \tag{4.257}$$

4.4.10 Five cards are drawn successively with replacement from a well-shuffled deck of 52 cards. What is the probability that

(a) all the five cards are spades?

(b) only 3 cards are spades?

(c) none is a spade?

Solution: A deck has 52 cards among which 13 are spades. Since we are replacing drawn cards, the probability of getting spade on any draw is

$$p = \frac{13}{52} = \frac{1}{4} \tag{4.258}$$

This is a binomial distribution where getting a card of spades is considered success.

The pmf is given by

$$\Pr(X = r) = {}^nC_r p^r (1-p)^{n-r}, \quad p = \frac{1}{4}, n = 5 \tag{4.259}$$

The desired probabilities are then obtained as

(i)

$$\Pr(X = 5) = {}^5C_5 \left(\frac{1}{4}\right)^5 \left(\frac{3}{4}\right)^0 \tag{4.260}$$

$$= \frac{1}{1024} \approx 0.00098 \tag{4.261}$$

(ii)

$$\Pr(X = 3) = {}^5C_5 \left(\frac{1}{4}\right)^3 \left(\frac{3}{4}\right)^2 \tag{4.262}$$

$$= \frac{45}{512} \approx 0.08789 \tag{4.263}$$

(iii)

$$\Pr(X = 0) = {}^5C_0 \left(\frac{1}{4}\right)^0 \left(\frac{3}{4}\right)^5 \tag{4.264}$$

$$= \frac{243}{1024} \approx 0.23730 \tag{4.265}$$

4.4.11 The probability that a bulb produced by a factory will fuse after 150 days of use is 0.05. Find the probability that out of 5 such bulbs

(a) none

(b) not more than one

(c) more than one

(d) at least one

will fuse after 150 days of use.

Solution: The binomial distribution parameters are

$$n = 5,\, p = 0.05,\, q = 1 - p = 0.95. \tag{4.266}$$

The pmf and CDF are

$$p_X(i) = {}^5C_i p^i q^{5-i} \tag{4.267}$$

$$F_X(i) = \Pr(X \le i) = \sum_{r=0}^{i} {}^5C_r p^r q^{5-r} \tag{4.268}$$

(a) Probability that none of the 5 bulbs fuses is

$$\Pr(X = 0) = F_X(0) = 0.95^5 \tag{4.269}$$

(b) Probability that not more than one bulb fuses is

$$\Pr(X \le 1) = F_X(1) = 0.9774075 \tag{4.270}$$

(c) Probability that more than one bulb will fuse will be

$$\Pr(1 < X \le 5) = F_X(5) - F_X(1) = 0.0225925 \tag{4.271}$$

(d) Probability that at least one bulb is fused is

$$\Pr(1 \le X \le 5) = F_X(5) - F_X(0) = 1 - (0.95)^5 \tag{4.272}$$

4.4.12 A bag consists of 10 balls each marked with one of the digits 0 to 9. If 4 balls are drawn successively with replacement from the bag, what is the probability that none is marked with the digit 0?

Solution: Let, p_X be the sequence of independent Bernoulli random varibles.

$$X = \begin{cases} 0, & \text{non-zero marked ball} \\ 1, & \text{zero marked ball} \end{cases} \tag{4.273}$$

which means

$$p_X(0) = \frac{9}{10} \tag{4.274}$$

$$p_X(1) = \frac{1}{10} \tag{4.275}$$

Let, the total number of trials be n and Z be the random variable that represents the number of balls marked zero in n trials which is given by:

$$p_X(Z = k) = {}^nC_k p^{n-k} q^k \tag{4.276}$$

where,

$$Z = X_1 + X_2 + ... + X_n \tag{4.277}$$

For only non-zero marked balls in 4 trials,

$$p_X(Z = 0) = {}^4C_0 \left(\frac{9}{10}\right)^{4-0} \left(\frac{1}{10}\right)^{0} \tag{4.278}$$

$$= (1) \left(\frac{9}{10}\right)^{4} (1) \tag{4.279}$$

$$= \left(\frac{9}{10}\right)^{4} \tag{4.280}$$

$$= 0.6561 \tag{4.281}$$

4.4.13 How many times must a man toss a fair coin so that the probability of having at least one head is more than 90%?

Solution: Let, X_i be the sequence of independent Bernoulli random varibles.

$$\implies X = \sum_{0}^{n} X_i \tag{4.282}$$

$$X_i = \begin{cases} 1, & \text{Heads} \\ 0, & \text{Tails} \end{cases} \tag{4.283}$$

which means

$$p_X(k) = \begin{cases} 0.5 = p & k = 0 \\ 0.5 = q & k = 1 \end{cases} \tag{4.284}$$

Let, the total number of trials be n and the pmf of getting k heads is given by:

$$p_X(k) = \Pr(X = k) \tag{4.285}$$

$$= {}^nC_k \,(p)^k \,(q)^{n-k} \tag{4.286}$$

$$= {}^nC_k \,(0.5)^k \,(0.5)^{n-k} \tag{4.287}$$

The cdf for the following pmf:

$$F_X(k) = \sum_{i=0}^{k} p_X(i) \tag{4.288}$$

$$= \sum_{i=0}^{k} {}^nC_i \,(0.5)^{n-i} \,(0.5)^i \tag{4.289}$$

Then the probability of getting atleast 1 heads is:

$$\Pr(X \geq 1) > 0.9 \tag{4.290}$$

$$\implies 1 - p_X(0) > 0.9 \tag{4.291}$$

$$(2)^n > 10 \tag{4.292}$$

Taking ln both side we get:

$$n > \log_2(10) \tag{4.293}$$

$$\implies n > 3.32 \tag{4.294}$$

As we know, n can be a positive integer value.

So,

$$\implies n = 4 \tag{4.295}$$

4.4.14 In an examination, 20 questions of true-false type are asked. Suppose a student tosses a fair coin to determine his answer to each question. If the coin falls heads, he answer true; if it falls tails, he answer false. Find the probability that he answers at least 12 questions correctly.

Solution: Let X denote the number of correct answers out of 20 questions. Then X is binomial with

$$n = 20,\ p = \frac{1}{2},\ q = 1 - p = \frac{1}{2} \tag{4.296}$$

The desired probability is then given by

$$\Pr\left(X >= 12\right) = 1 - F_X(11) = 0.2517 \tag{4.297}$$

4.4.15 Find the probability of getting 5 twice in 7 throws of a dice.

Solution: The Binomial r.v. parameters are

$$\therefore\ q = 1 - p = 1 - \frac{1}{6} = \frac{5}{6}, n = 7 \tag{4.298}$$

with pmf

$$\therefore \Pr(X = k) = {}^{n}C_k \times q^{n-k} \times p^k = {}^{7}C_k \times \left(\frac{5}{6}\right)^{(7-k)} \times \left(\frac{1}{6}\right)^{k} \tag{4.299}$$

The desired probability is

$$\Pr(X = 2) = {}^{7}C_2 \times \left(\frac{5}{6}\right)^{(7-2)} \times \left(\frac{1}{6}\right)^{2} = \left(\frac{7}{12}\right) \times \left(\frac{5}{6}\right)^{5} \tag{4.300}$$

4.4.16 On a multiple choice examination with three possible answers for each of the five questions, what is the probability that a candidate would get four or more correct answers just by guessing?

Solution: See

Parameter	Value	Description
X	$bin(n,p)$	no of correct answers that candidate gets by guessing
n	5	total no of questions
p	$\frac{1}{3}$	probability of getting correct answer by guessing

Table 4.16: The binomial random variable,it's parameters and their values

The pmf and CDF are

$$p_X(k) = {}^{5}C_k \times \left(\frac{2}{3}\right)^{5} \times \frac{1}{2^k} \tag{4.301}$$

$$F_X(k) = \left(\frac{2}{3}\right)^{5} \times \left(\sum_{i=0}^{k} {}^{5}C_i \times \frac{1}{2^i}\right) \tag{4.302}$$

The desired probability is

$$\Pr(X \geq 4) = 1 - F_X(3) = \frac{11}{243} \tag{4.303}$$

4.4.17 Find the probability of throwing at most 2 sixes in 6 throws of a single die.

Solution: The binomial distribution parameters are

$$n = 6,\ p = \frac{1}{6},\ q = 1 - p = \frac{5}{6} \tag{4.304}$$

with pmf

$$Pr\left(X = r\right) = {}^{n}C_{r}\left(p\right)^{r}\left(q\right)^{n-r} \tag{4.305}$$

and CDF

$$F_X(r) = \sum_{i=0}^{r} {}^{n}C_{i} p^{i} q^{n-i} \tag{4.306}$$

The desired probability is

$$F_X\left(2\right) = \frac{21875}{23328} \tag{4.307}$$

4.4.18 Suppose that 90 % of people are right-handed. What is the probability that atmost 6 of a random sample of 10 people are right-handed.

Solution:

Solution: :

Table 4.17: Table-1

Parameters	values	Description
	1	if right-handed
X_i	0	if not right-handed
n	10	Total people
k	6	People right-handed
p	0.9	probability of being right-handed

$$X \sim \mathrm{Ber}\left(p\right) \tag{4.308}$$

Suppose $X_i, 1 \leq i \leq n$ represent each of the n draws. Define Y as

$$Y = \sum_{i=1}^{n} X_i \tag{4.309}$$

Then, since the X_i are iid, the pmf of Y is given by

$$Y \sim \mathrm{Bin}\left(n, p\right) \tag{4.310}$$

The cdf of Y is given by

$$F_Y\left(k\right) = \Pr\left(Y \leq k\right) \tag{4.311}$$

$$= \begin{cases} 0 & k < 0 \\ \sum_{i=1}^{k} \begin{pmatrix} n \\ i \end{pmatrix} p^i \left(1-p\right)^{n-i} & 1 \leq k \leq n \\ 1 & k \geq n \end{cases} \tag{4.312}$$

(a) We require $\Pr\left(Y \leq 6\right)$. Since $n = 10$,

$$\Pr\left(Y \leq 6\right) = F_Y(6) \tag{4.313}$$

$$= 0.01279 \tag{4.314}$$

4.4.19 An urn contains 25 balls of which 10 balls bear a mark 'X' and the remaining 15 bear a mark ‘Y’. A ball is drawn at random from the urn, its mark is noted down and it is replaced. If 6 balls are drawn in this way, find the probability that

(a) all will bear ‘X’ mark.

(b) not more than 2 will bear ‘Y’ mark.

(c) at least one ball will bear ‘Y’ mark.

(d) the number of balls with ‘X’ mark and ‘Y’ mark will be equal.

Solution: The given information is listed in Table 4.18 The pmf and CDF are given

Variables	Definition	values
N	Balls in the urn	25
N_X	Balls marked with X	10
N_Y	Balls marked with Y	15
n	No. Of trials	6
k	No. Of balls marked X in n trials	
p	$\Pr(X)$	0.4
q	$\Pr(Y)$ = 1 - $\Pr(X)$	0.6

Table 4.18: Given Information

by

$$\Pr\left(Z = i\right) = {}^{6}C_{i} p^{i} q^{6-i} \tag{4.315}$$

$$\therefore F_Z(i) = \sum_{r=0}^{i} {}^{6}C_{r} p^{r} q^{6-r} \tag{4.316}$$

(a)

$$\Pr\left(Z = 6\right) = {}^{6}C_{6}\left(0.4\right)^{6}\left(0.6\right)^{0} = 0.004096 \tag{4.317}$$

(b)

$$\Pr(Z \geq 4) = 1 - F_Z(3) = 0.1792 \tag{4.318}$$

(c)

$$\Pr(Z < 6) = F_Z(5) = 0.995904 \tag{4.319}$$

(d)

$$\Pr(Z = 3) = {}^6C_3\,(0.4)^3\,(0.6)^3 = 0.13824 \tag{4.320}$$

4.4.20 An urn contains 5 red and 2 black balls. Two balls are randomly drawn. Let X represent the number of black balls. What are the possible values of X? Is X a random variable ?

4.4.21 Find the probability distribution of

(a) number of heads in two tosses of a coin.

(b) number of tails in the simultaneous tosses of three coins.

(c) number of heads in four tosses of a coin.

4.4.22 Find the probability distribution of the number of successes in two tosses of a die, where a success is defined as

(a) number greater than 4

(b) six appears on at least one die

4.4.23 From a lot of 30 bulbs which include 6 defectives, a sample of 4 bulbs is drawn at random with replacement. Find the probability distribution of the number of defective bulbs.

4.4.24 A coin is biased so that the head is 3 times as likely to occur as tail. If the coin is tossed twice, find the probability distribution of number of tails.

4.4.25 A coin is tossed twice, what is the probability that atleast one tail occurs?

Solution: By using binomial distribution, the desired probability is given by

$$\Pr(Y \geq 1) = \sum_{k=1}^{2} \binom{n}{k} p^k (1-p)^{n-k} = \frac{3}{4} \tag{4.321}$$

upon substituting $p = \frac{1}{2}$.

4.4.26 From a lot of 30 bulbs which include 6 defectives, a sample of 4 bulbs is drawn at random with replacement. Find the probability distribution of the number of defective bulbs.

4.4.27 Suppose X is a binomial distribution $B\left(6, \frac{1}{2}\right)$. Show that $X = 3$ is the most likely outcome. (Hint : $P(X = 3)$ is the maximum among all $P(x_i), x_i = 0, 1, 2, 3, 4, 5, 6$)

Solution: Given that, X is a binomial distribution with parameters

$$n = 6 \quad p = \frac{1}{2} \tag{4.322}$$

the probability of getting exactly k successes in n trials is given by

$$p_X(k) = {}^nC_k p^k (1-p)^{n-k} \tag{4.323}$$

From equations in (4.322) , The pmf simplifies as,

$$p_X(k) = {}^nC_k \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{n-k} \tag{4.324}$$

$$= {}^nC_k \left(\frac{1}{2}\right)^6 \tag{4.325}$$

We know that ${}^nC_k$ can be written as,

$${}^nC_k = \frac{n!}{(n-k)!k!} \tag{4.326}$$

If pmf is the greatest, then ${}^nC_k$ is the maximum for $k \in [0, n]$, Therefore It can be said that,

$${}^nC_k \geq {}^nC_{k-1} \quad \text{and} \tag{4.327}$$

$${}^nC_k \geq {}^nC_{k+1} \tag{4.328}$$

From (4.326) and (4.327), we can state that

$$\frac{n!}{(n-k)!k!} \geq \frac{n!}{(n-k+1)!(k-1)!} \tag{4.329}$$

$$\implies \frac{n!}{(n-k)!k!} \geq \frac{n!}{(n-k)!k!} \frac{k}{n-k+1} \tag{4.330}$$

$$\implies 1 \geq \frac{k}{n-k+1} \tag{4.331}$$

$$\therefore k \leq \frac{n+1}{2} \tag{4.332}$$

From (4.326) and (4.328), we can state that

$$\frac{n!}{(n-k)!k!} \geq \frac{n!}{(n-k-1)!(k+1)!} \tag{4.333}$$

$$\implies \frac{n!}{(n-k)!k!} \geq \frac{n!}{(n-k)!k!}\frac{n-k}{k+1} \tag{4.334}$$

$$\implies 1 \geq \frac{n-k}{k+1} \tag{4.335}$$

$$\therefore k \geq \frac{n-1}{2} \tag{4.336}$$

From (4.332) and (4.336), we can state that

$$\frac{n-1}{2} \leq k \leq \frac{n+1}{2} \tag{4.337}$$

We know that, $k \in \mathbb{W}$ and $k \in [0, n]$ and from (4.337),

$$k = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even} \\ \frac{n+1}{2} \quad \text{or} \quad \frac{n-1}{2}, & \text{if } n \text{ is odd} \end{cases} \tag{4.338}$$

As,

$$n = 6 \tag{4.339}$$

$$\implies k = \frac{n}{2} = 3 \tag{4.340}$$

Hence proved that,

$$X = 3 \tag{4.341}$$

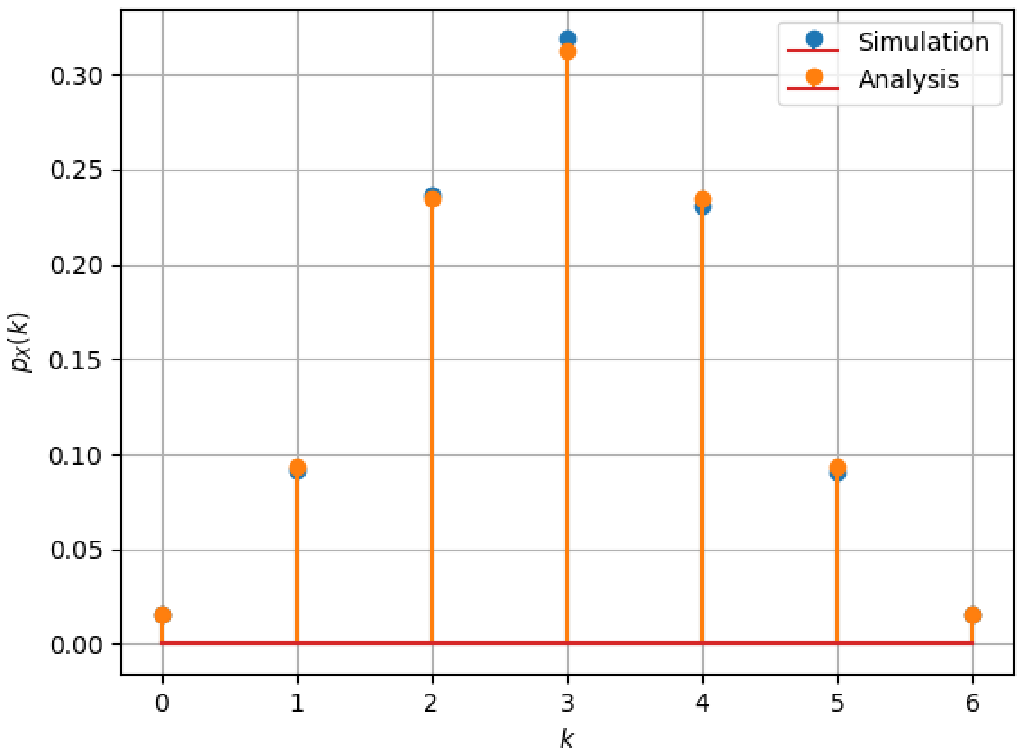

Figure 4.5: Figure compares the therotical and simulation output

is the most likely outcome;

$$p_X(3) = {}^6C_3 \left(\frac{1}{2}\right)^6 \tag{4.342}$$

$$= \frac{5}{16} \tag{4.343}$$

4.4.28 A fair coin is tossed four times, and a person win Re 1 for each head and lose Rs 1.5 for each tail that turns up.

From the sample space calculate how many different amounts of money you can have after four tosses and the probability of having each of these amounts.

Solution: According to the Question:

Variable	Description	Value
n	Number of tosses	4
A	Amount gained/lost	A
p	Profit when it is heads	Re 1
q	Loss when it is tails	Rs 1.5
X	Number of heads in n tosses	X
Y	Number of tails in n tosses	Y

$$X + Y = n \tag{4.344}$$

The amount of money the person will have after n tosses is:

$$A = (X \times 1) - (Y \times 1.5) \tag{4.345}$$

$$= (X \times 1) - ((n - X) \times 1.5) \tag{4.346}$$

$$= (2.5X) - (1.5n) \tag{4.347}$$

For the given question the value of n = 4

$$A = (2.5X) - 6 \tag{4.348}$$

The probability of getting a profit/loss is:

$$p_X(k) = {}^4C_k(0.5)^k(0.5)^{4-k} = {}^4C_k(0.5)^4 \tag{4.349}$$

Let $F_X(k)$ denote the cumulative distribution function of X:

$$F_X(k) = p(X \leq k) = \sum_{i=0}^{k} {}^4C_i \left(\frac{1}{2}\right)^4 \tag{4.350}$$

Let $F_A(k)$ denote the cumulative distribution function of A:

$$F_A(k) = p(A \leq k) \tag{4.351}$$

$$= p(2.5X - 6 \leq k) \tag{4.352}$$

$$= p\left(X \leq \frac{k+6}{2.5}\right) \tag{4.353}$$

$$= F_X\left(\frac{k+6}{2.5}\right) \tag{4.354}$$

$$\tag{4.355}$$

By (4.350)

$$= \sum_{i=0}^{\frac{k+6}{2.5}} {}^4C_i \left(\frac{1}{2}\right)^4 \tag{4.356}$$

$$p_A(k) = \begin{cases} {}^4C_{\frac{k+6}{2.5}} \left(\frac{1}{2}\right)^4, & \frac{k+6}{2.5} \in I \text{ and } 0 \leq \frac{k+6}{2.5} \leq 4 \\ 0, & \text{otherwise} \end{cases} \tag{4.357}$$

Now, for 4 tosses as given in the question the different amount of money and its

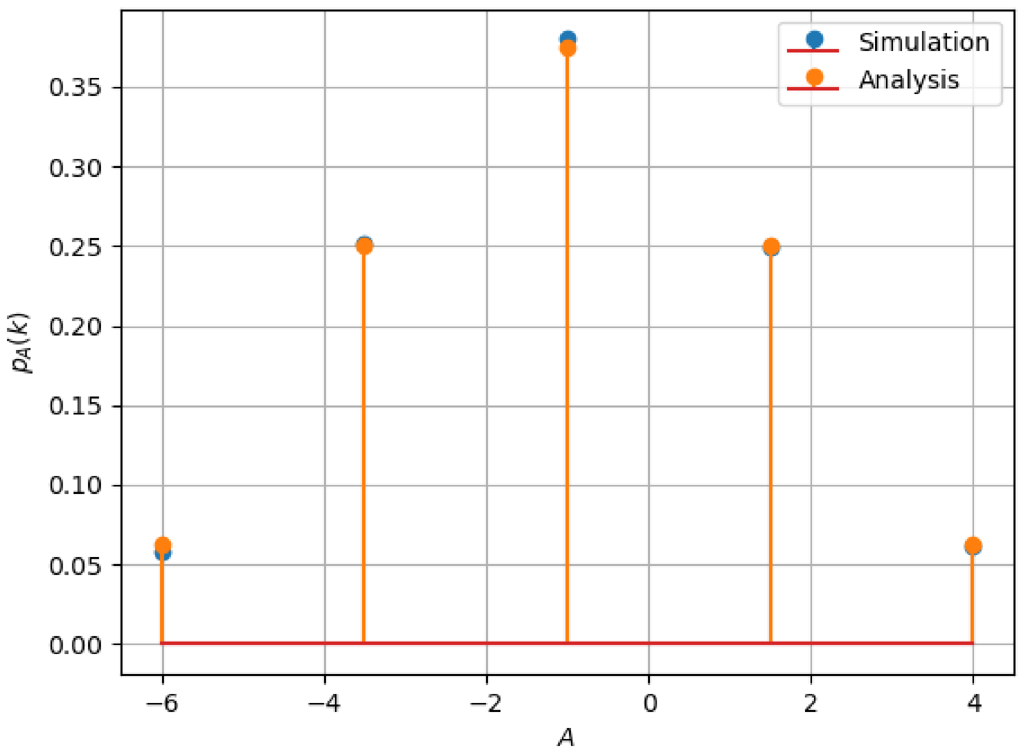

Figure 4.6: PMF of A

$$\text{probability} = \left(4, \tfrac{1}{16}\right), \left(1.5, \tfrac{1}{4}\right), \left(-1, \tfrac{3}{8}\right), \left(-3.5, \tfrac{1}{4}\right), \left(-6, \tfrac{1}{16}\right)$$

4.4.29 It is known that 10 % of certain articles manufactured are defective. What is the probability that in a random sample space of 12 such articles,9 are defective?

Solution: From Table 4.20,

Parameters	**Value**	**Description**
n	12	Number of Articles
p	0.1	Probability of Defective Articles
q	0.9	Probability of Non-Defective Articles

Table 4.20:

Variable	Description	Value
n	Number of tosses	2
X_i	Result of ith coin	$X_i, i = 1, 2$
X	No of heads	$X_1 + X_2$

$$X = \sum_{i=1}^{2} X_i \tag{4.361}$$

$$X = X_1 + X_2 \tag{4.362}$$

$$X \leq 2 \tag{4.363}$$

The probability of getting a head is:

$$p_X(k) = {}^2C_k (0.5)^k (0.5)^{2-k} \tag{4.364}$$

$$= {}^2C_k (0.5)^2 \qquad \forall k = 0, 1, 2 \tag{4.365}$$

The above equation gives the PMF of getting k heads on 2 coint tosses. Let $F_X(k)$ denote the cumulative distribution function of X:

$$F_X(k) = p(X \leq k) \tag{4.366}$$

$$= \sum_{i=0}^{k} {}^2C_i \left(\frac{1}{2}\right)^2 \tag{4.367}$$

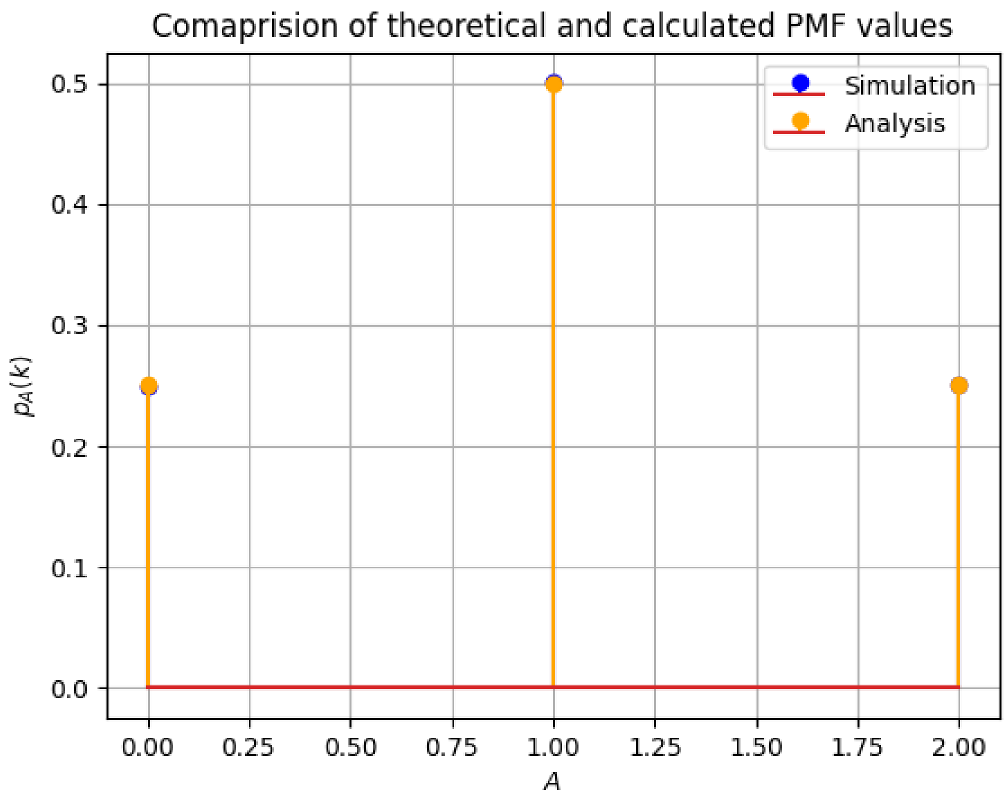

Figure 4.7: PMF of X

Let $F_X(k)$ denote the cumulative distribution function of X:

$$F_X(k) = p(X \leq k) \tag{4.368}$$

$$F_X(1) = \sum_{i=0}^{1} {}^2C_i \left(\frac{1}{2}\right)^2 \tag{4.369}$$

$$= \frac{3}{4} \tag{4.370}$$

4.4.31 An experiment succeeds twice as often as it fails. Find the probability that in the next

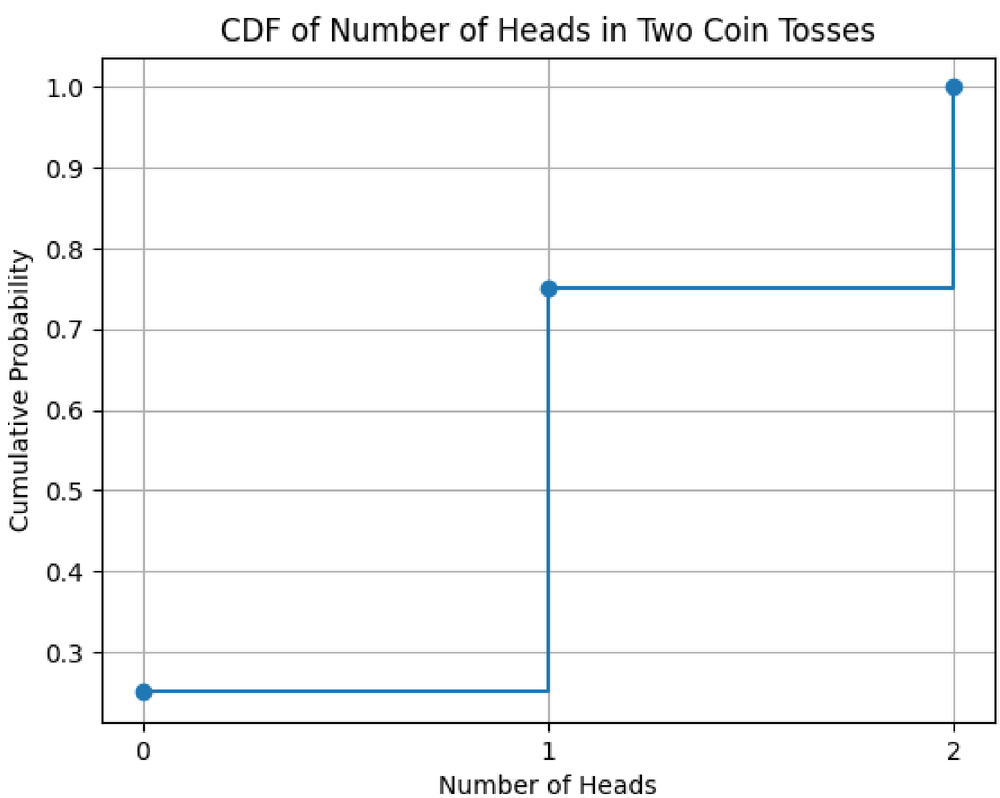

Figure 4.8: CDF of X

six trials, there will be atleast 4 successes. **Solution:** Let p be the probability for the experiment to succeed and q for the failure.

Here, it is given that probability of success is twice that of the failure, so

$$\begin{aligned} p &= 2q \\ q &= \frac{1}{3} \\ p &= \frac{2}{3} \end{aligned} \tag{4.371}$$

Now, let's consider a single trial as a bernuolli random variable $X_i = 1$ represents success and $X_i = 0$ represents failure. Therefore we have,

Table 4.21: random variables

X_i	1	success
	0	failure

$$P_X(X_i) = \begin{cases} \frac{2}{3}, & \text{when } X_i{=}1 \\ \frac{1}{3}, & \text{when } X_i{=}0 \end{cases} \tag{4.372}$$

Since we have n=6 trials, the random variable X representing the number of successes in 6 trials follows a binomial distribution. The cumulative distribution function (CDF) of X is given by

$$F_X(k) = P_X(X \leq k) = \Sigma_{k=0}^{n} {}^nC_k q^{n-k} p^k \tag{4.373}$$

We need to find the probability for the experiment to succeed to atleast 4 times i.e.

Table 4.22: parameters for CDF

parameter	value
n	6
p	$\frac{2}{3}$
q	$\frac{1}{3}$
k	0,1,2,...,6

$\Pr(X \geq 4)$. Using equation 4.373 we get,

$$\begin{aligned}\Pr(X \geq 4) &= 1 - P_X(X \leq 3) \\ &= 1 - F_X(3) \qquad (4.374) \\ &= 1 - \frac{233}{3^6} \approxeq 0.680\end{aligned}$$

Therefore the probability that in the next six trials, there will be atleast 4 successes is 0.680.

4.4.32 A die is thrown 5 times. Find the probability that an odd number will come up exactly three times. **Solution:**

Parameter	Values	Description
n	5	Number of throws
k	3	Number being odd numbers
p	$\frac{3}{6} = \frac{1}{2}$	Probability of being odd number
X	1 if odd 0 if even	Bernoulli Random Variable
Y	$\sum_{i=1}^{n} X_i$	Binomial Random Variable

Table 4.23: Table 1

$$p_Y(k) = \Pr(Y = k) \tag{4.375}$$
$$= {}^nC_k p^k (1-p)^{n-k}, (1 \le k \le n) \tag{4.376}$$

We require $\Pr(Y = 3)$. Since $n = 5$,

$$\Pr(Y = 3) = p_Y(3) \tag{4.377}$$
$$= {}^nC_k p^k (1-p)^{n-k} \tag{4.378}$$
$$= \frac{5}{16} \tag{4.379}$$

4.4.33 A coin is tossed 3 times. List the possible outcomes. Find the probability of getting (i) all heads (ii) at least 2 heads

Solution: As the coin is tossed 3 times we will get 8 different outcomes.Let us define a random variable X, where getting heads is success.

$$X_i \sim \mathrm{Ber}(p) \tag{4.380}$$

The cdf of X is given by

$$F_X(k) = \Pr(X \le k) \tag{4.381}$$
$$= \begin{cases} 0 & k < 0 \\ \sum_{i=1}^{k} \begin{pmatrix} n \\ k \end{pmatrix} p^i (1-p)^{n-i} & 1 \le k \le n \\ 1 & k \ge n \end{cases} \tag{4.382}$$

Parameter	Value	Description
	0	0 heads
	1	1 head
X	2	2 heads
	3	3 heads
n	3	no. of tosses
p	$\frac{1}{2}$	probability of heads

(a) To get all heads: To get all heads Z should be equal to 3. So we need

$$\Pr(Z = 3) = \begin{pmatrix} n \\ 3 \end{pmatrix} p^3 (1-p)^{n-3} \tag{4.383}$$

$$= \frac{1}{8} \tag{4.384}$$

(b) To get atleast 2 heads: To get atleast two heads the value of $Z \geq 2$.

$$\Pr(Z \geq 2) = 1 - \Pr(Z < 2) \tag{4.385}$$

$$= F_Z(3) - F_Z(1) \tag{4.386}$$

$$= \sum_{k=2}^{3} \begin{pmatrix} n \\ k \end{pmatrix} p^k (1-p)^{n-k} \tag{4.387}$$

$$= \frac{1}{2} \tag{4.388}$$

4.4.34 Ten coins are tossed. What is the probability of getting atleast 8 heads?

Solution: Let the event of getting a head on one coin toss be H. Then

$$\Pr(H) = \frac{1}{2} \tag{4.389}$$

Variable	Description	Value
n	Number of tosses	10
X_i	Result of ith coin	$X_i, i = \{1, 2, ..., 10\}$
X	No of heads	$\sum_{i=1}^{10} X_i$

$$X = \sum_{i=1}^{10} X_i \tag{4.390}$$

$$X \leq 10 \tag{4.391}$$

The probability of getting a head is:

$$p_X(k) = {}^{10}C_k (0.5)^k (0.5)^{10-k} \tag{4.392}$$

$$= {}^{10}C_k (0.5)^{10} \qquad \forall k = 0, 1, 2, ..., 10 \tag{4.393}$$

The above equation gives the PMF of getting k heads on 10 coint tosses. Let $F_X(k)$ denote the cumulative distribution function of X:

$$F_X(k) = p(X \leq k) \tag{4.394}$$

$$= \sum_{i=0}^{k} {}^{10}C_i \left(\frac{1}{2}\right)^{10} \tag{4.395}$$

Let $F_X(k)$ denote the cumulative distribution function of X:

$$F_X(k) = p(X \leq k) \tag{4.396}$$

$$\implies F_X(10) - F_X(7) = \sum_{i=8}^{10} {}^{10}C_i \left(\frac{1}{2}\right)^{10} \tag{4.397}$$

$$= \frac{7}{128} \tag{4.398}$$

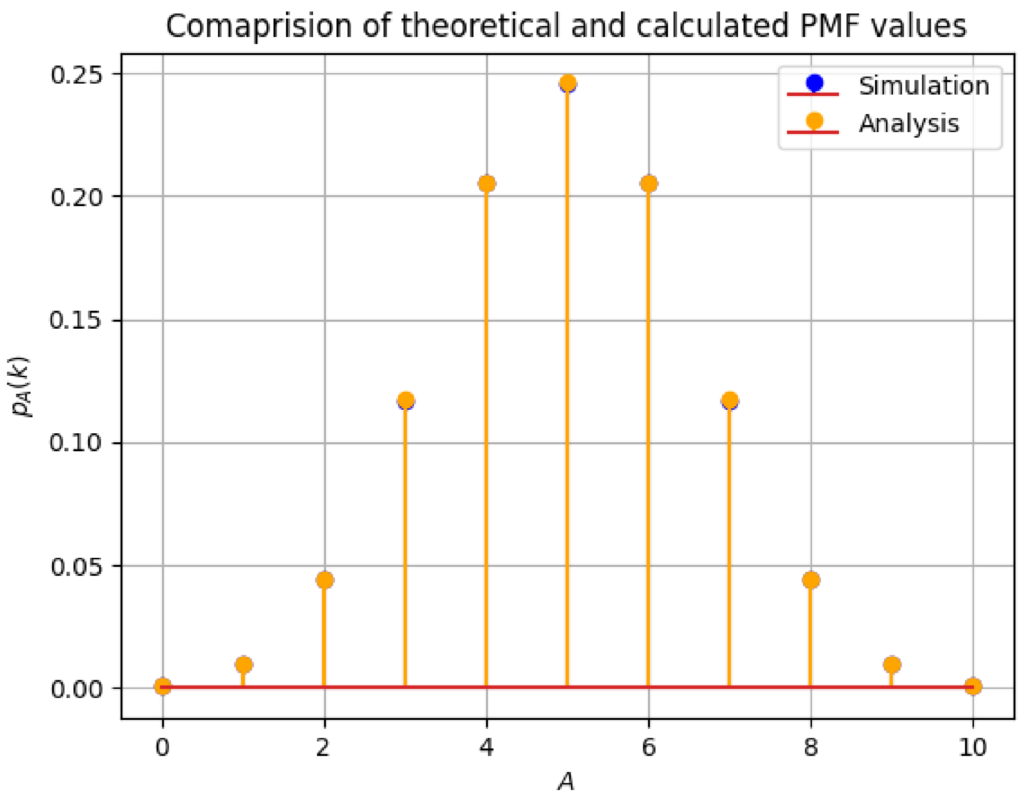

Figure 4.9: PMF of X

4.4.35 A factory produces bulbs. The probability that any one bulb is defective is $\frac{1}{50}$ and they are packed in boxes of 10. From a single box, find the probability that

(a) none of the bulb is defective

(b) exactly two bulbs are defective

(c) more than 8 bulbs are working properly

Solution:

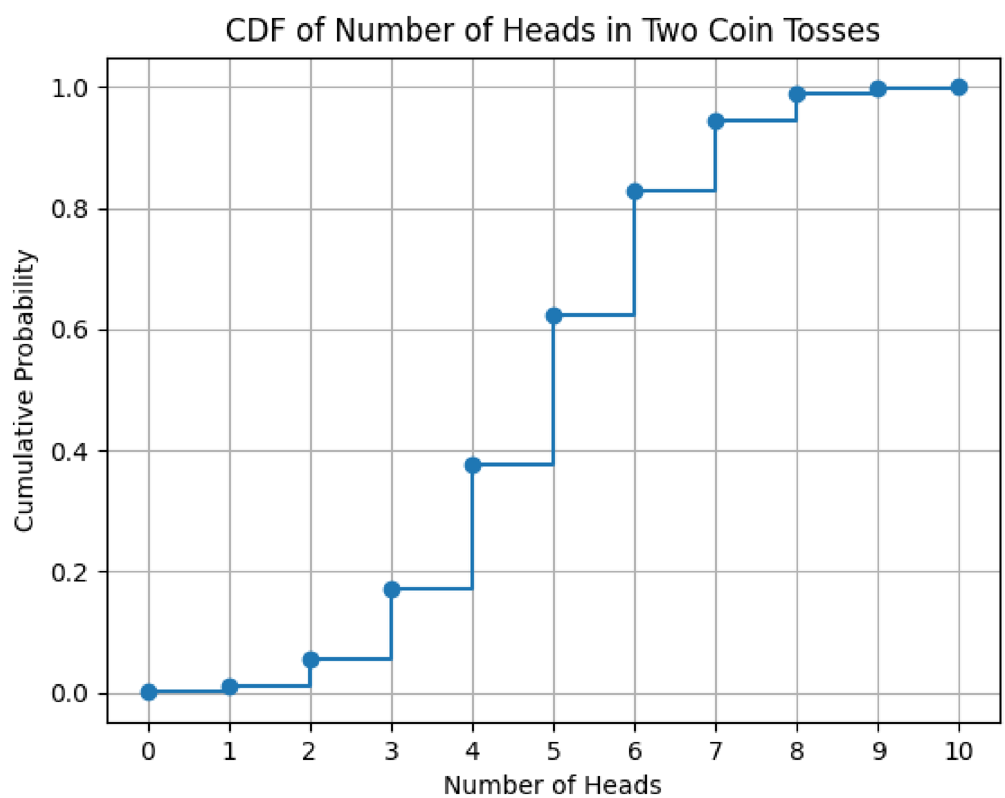

Figure 4.10: CDF of X

Parameter	Values	Description
n	10	Number of boxes
p	$\frac{1}{50}$	Probability of being defective
X	1 if defective 0 if not defective	Bernoulli Random Variable
Y	$\sum_{i=1}^{n} X_i$	Binomial Random Variable

Table 4.24: Table 1

$$p_Y(k) = \Pr(Y = k) \tag{4.399}$$

$$= {}^{n}C_k p^k (1-p)^{n-k}, (1 \le k \le n) \tag{4.400}$$

(a) none of the bulb is defective

We require $\Pr(Y = 0)$.

$$\Pr(Y = 0) = p_Y(0) \tag{4.401}$$

$$= {}^{n}C_k p^k (1-p)^{n-k} \tag{4.402}$$

$$= \left(\frac{49}{50}\right)^{10} \tag{4.403}$$

(b) exactly two bulbs are defective

We require $\Pr(Y = 2)$.

$$\Pr(Y = 2) = p_Y(2) \tag{4.404}$$

$$= {}^{n}C_k p^k (1-p)^{n-k} \tag{4.405}$$

$$= 45\left(\frac{1}{50}\right)^{2}\left(\frac{49}{50}\right)^{8} \tag{4.406}$$

(c) more than 8 bulbs are working properly

$$\implies \text{Atmost 1 is defective} \tag{4.407}$$

Since,

$$F_Y(k) = \Pr(Y \le k) \tag{4.408}$$

$$= \sum_{i=0}^{k} {}^nC_i p^i (1-p)^{n-i} \qquad 0 \le k \le n \tag{4.409}$$

$$\tag{4.410}$$

We require $\Pr(Y \le 1)$.

$$\Pr Y \le 1 = \sum_{i=0}^{1} {}^nC_i p^i (1-p)^{n-i} \tag{4.411}$$

$$= {}^{10}C_0 \left(\frac{1}{50}\right)^0 \left(\frac{49}{50}\right)^{10} + {}^{10}C_1 \left(\frac{1}{50}\right)^1 \left(\frac{49}{50}\right)^9 \tag{4.412}$$

$$= \frac{59 \cdot 49^9}{50^{10}} \tag{4.413}$$

4.4.36 A lot of 100 watches is known to have 10 defective watches. If 8 watches are selected (one by one with replacement) at random, what is the probability that there will be at least one defective watch?

Solution:

Parameter	**Value**	**Description**
X_i	0,1	0-Defective watch, 1-Good watch and X_i denotes i^{th} selection
Y	$\sum_{i=1}^{8} X_i$	Represents number of defective watches selected in 8 selections

pmf of X_i:

$$p_{X_i}(k) = \begin{cases} \frac{1}{10}, & k = 0 \\ \\ \frac{9}{10}, & k = 1 \end{cases} \quad \forall \quad 1 \le i \le 8 \tag{4.414}$$

pmf of Y is given by:

$$p_Y(k) = \Pr(Y = k) = \binom{8}{k} \times (0.1)^k \times (0.9)^{8-k} \quad \forall \quad 0 \le k \le 8 \tag{4.415}$$

$$\tag{4.416}$$

CDF of Y:

$$F_Y(k) = \sum_{i=0}^{k} \binom{8}{i} \times (0.1)^i \times (0.9)^{8-i} \quad \forall \quad 0 \le k \le 8 \tag{4.417}$$

$$\tag{4.418}$$

$\therefore$ probability of choosing atleast one defective watch in 8 selections

$$= \sum_{k=1}^{8} \Pr(Y = k) \tag{4.419}$$

$$= F_Y(8) - F_Y(0) \tag{4.420}$$

$$= 0.569533 \tag{4.421}$$

4.4.37 The Probability of a man hitting target is 0.25.He shoots 7 times. What is the probability of his hitting atleast twice?

Solution: Let the probability of hitting the targets correctly be p

-

Table 4.25: Random Variable and probability Table

Random independent variable	value of R.V	Description
n	7	Total no. of trials
X	$0 \leq \text{X} \leq 7$	no. of times he hits the target

$$= \frac{1}{4} = 0.25 \tag{4.422}$$

CDF of binomial distribution is:

$$F_X(k) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \qquad \text{k} = 0, 1, 2, ..7 \tag{4.423}$$

$$\tag{4.424}$$

Probability of Hitting the target atleast twice is

$$\Pr(X \geq 2) = 1 - \Pr(X \leq 1) \tag{4.425}$$

$$= F_X(7) - F_X(1) \tag{4.426}$$

$$= 1 - \left\{ \sum_{k=0}^{1} \binom{7}{k} \left(\frac{1}{4}\right)^k \left(\frac{3}{4}\right)^{7-k} \right\} \tag{4.427}$$

$$= 0.55505 \tag{4.428}$$

Hence, the probability of hitting the target atleast twice is 0.55505

4.4.38 Explain why the experiment of tossing a coin three times is said to have binomial distribution

Solution:

let X be the event of tossing coin and bernoulli distribution is

$$\Pr(X = k) = \begin{cases} q = 1 - p & k = 0 \\ p & k = 1 \\ 0 & otherwise \end{cases} \tag{4.429}$$

$$\tag{4.430}$$

Then the Z transform of X is

$$M_X(z) = E\left[z^{-X}\right] = \sum_{k=-\infty}^{\infty} \Pr(X = k) z^{-k} \tag{4.431}$$

$$= qz^0 + pz^{-1} \tag{4.432}$$

$$= q + pz^{-1} \tag{4.433}$$

Then for n trials,

$$M_X(z) = \left(pz^{-1} + q\right)^n \tag{4.434}$$

$$= \sum_{k=0}^{n} {}^nC_k \left(pz^{-1}\right)^k q^{n-k} \tag{4.435}$$

$$= \sum_{k=0}^{n} {}^nC_k (p)^k (1 - p)^{n-k} z^{-k} \tag{4.436}$$

By comparing Coefficients of z^{-k},

$$p_X(k) = {}^nC_k p^k (1 - p)^{n-k} \qquad 0 \le k \le n \tag{4.437}$$

which is a binomial distribution

$\therefore$ Tossing 3 coins also has a binomial distribution

4.4.39 A person buys a lottery ticket in 50 lotteries in each of which his chance of winning a prize is $\frac{1}{100}$. What is the probability that he will win a prize

(a) atleast once

(b) exactly once

(c) atleast twice ?

Solution: Let X be number of winning prizes in 50 lotteries. The trials are Bernoulli trials.X has binomial distribution with n = 50 and p = $\frac{1}{100}$

Table 4.26: parameters for CDF

parameter	value
n	50
p	$\frac{1}{100}$
q	$\frac{99}{100}$

$$q = 1 - p = 1 - \frac{1}{100} \tag{4.438}$$

$$q = \frac{99}{100} \tag{4.439}$$

$$p_X(k) = \Pr(X = k) \tag{4.440}$$

$$p_X(k) = {}^nC_k q^{n-k} p^k \tag{4.441}$$

$$= {}^{50}C_k \left(\frac{99}{100}\right)^{50-k} \left(\frac{1}{100}\right)^k \tag{4.442}$$

The Cdf for the following pmf :

$$F_X(k) = \sum_{i=0}^{k} {}^5C_i \left(\frac{99}{100}\right)^{50-i} \left(\frac{1}{100}\right)^i \tag{4.443}$$

(a)

$$\Pr(X \geq 1) = 1 - \Pr(X < 1) \tag{4.444}$$
$$= 1 - F_X(0) \tag{4.445}$$
$$= 1 - {}^{50}C_0 \left(\frac{99}{100}\right)^{50} \tag{4.446}$$
$$= 0.394 \tag{4.447}$$

(b)

$$\Pr(X = 1) = {}^{50}C_1 \left(\frac{99}{100}\right)^{49} \left(\frac{1}{100}\right)^{1} \tag{4.448}$$
$$= 0.3055 \tag{4.449}$$

(c)

$$\Pr(X \geq 2) = 1 - \Pr(X < 2) \tag{4.450}$$
$$= 1 - F_X(1) \tag{4.451}$$
$$= \left(1 - \frac{99}{100}\right)^{50} - \frac{1}{2}\left(\frac{99}{100}\right)^{49} \tag{4.452}$$
$$= 0.0894 \tag{4.453}$$

4.4.40 The probability that a person is not a swimmer is 0.3. The probability that out of 5 persons 4 are swimmers is

(a) ${}^{5}C_4\,(0.7)^4\,(0.3)$

(b) ${}^{5}C_1\,(0.7)\,(0.3)^4$

(c) ${}^{5}C_4\,(0.7)\,(0.3)^4$

(d) $(0.7)^4(0.3)$

Solution: Let,X_i denote the sequence of independent bernoulli random variables

$$X_i = \begin{cases} 1, & \text{if person is swimmer} \\ 0, & \text{otherwise} \end{cases} \tag{4.454}$$

which means

$$p_X(k) = \begin{cases} 0.7 = p, & k = 1 \\ 0.3 = q, & k = 0 \end{cases} \tag{4.455}$$

and Y be summation of all such sequences

$$Y = \sum_{i=0}^{n} X_i \tag{4.456}$$

The pmf of having k swimmers out of n swimmers is given by:

$$p_Y(k) = {}^nC_k p^k q^{n-k} \tag{4.457}$$

$$= {}^nC_k (0.7)^k (0.3)^{n-k} \tag{4.458}$$

for $n = 5$ and $k = 4$:

$$p_Y(4) = {}^5C_4 (0.7)^4 (0.3) \tag{4.459}$$

4.4.41 Suppose a random variable X follows binomial distribution with parameters n and p, where $0 < p < 1$. If $\Pr(X = r)/\Pr(X = n - r)$ is independent of n and r, then p equals,

(a) $\frac{1}{2}$

(b) $\frac{1}{3}$

(c) $\frac{1}{5}$

(d) $\frac{1}{7}$

Solution:

In a binomial distribution, The PMF of random variable X,

$$\Pr\left(X = r\right) = {}^{n}C_{r} p^{r} \left(1 - p\right)^{n-r} \tag{4.460}$$

Therefore,

$$\frac{\Pr\left(X = r\right)}{\Pr\left(X = n - r\right)} = \frac{{}^{n}C_{r} p^{r} \left(1 - p\right)^{n-r}}{{}^{n}C_{n-r} p^{n-r} \left(1 - p\right)^{r}} \tag{4.461}$$

$$= \frac{p^{r} \left(1 - p\right)^{n-r}}{p^{n-r} \left(1 - p\right)^{r}} \tag{4.462}$$

since

$${}^{n}C_{r} = {}^{n}C_{n-r} \tag{4.463}$$

$$\implies \frac{\Pr\left(X = r\right)}{\Pr\left(X = n - r\right)} = \left(\frac{1 - p}{p}\right)^{n-2r} \tag{4.464}$$

Since it is independent of n and r,

$$\frac{1 - p}{p} = 1 \tag{4.465}$$

$$\implies p = \frac{1}{2} \tag{4.466}$$

4.4.42 The probability of guessing correctly at least 8 out of 10 answers on a true-false type examination is

Solution:

Parameter	Value	Description
n	10	number of questions
p	1/2	correct answer
q	1/2	wrong answer

Table 4.27: Random variable declaration.

Let X be a random variable which denotes the number of correct answers

The PMF of X is

$$\Pr\left(X=k\right) = {}^{n}C_{k}p^{k}q^{n-k} \qquad \forall k = 0,1,2...10 \tag{4.467}$$

And the CDF is given by

$$F_X(k) = \Pr\left(X \le k\right) \tag{4.468}$$

$$= \sum_{i=0}^{k} {}^{n}C_{i}p^{i}q^{n-i} \tag{4.469}$$

$$\Pr\left(X \ge 8\right) = 1 - \Pr\left(X \le 7\right) \tag{4.470}$$

$$= 1 - F_X(7) \tag{4.471}$$

$$= 1 - \sum_{i=0}^{2} {}^{10}C_{i}\left(\frac{1}{2}\right)^{i}\left(\frac{1}{2}\right)^{10-i} \tag{4.472}$$

$$= \frac{7}{128} \tag{4.473}$$

4.4.43 In a family having three children, there may be no girl, one girl, two girls, or three girls. So the probability of each is 1/4. Is this correct? Justify your answer.

Solution:

NO, it is not correct.

Parameter	Values	Description
n	3	Number of children
k	0,1,2,3	Number of girls
p	0.5	Probability of girl
X	1 if girl 0 if boy	Bernoulli Random Variable
Y	$\sum_{i=1}^{n} X_i$	Binomial Random Variable

Table 4.28: variables

The cdf of Y is given by

$$F_Y(n) = \Pr(Y \leq n) \tag{4.474}$$

$$= \sum_{k=0}^{n} {}^{3}C_k p^k (1-p)^{3-k} \tag{4.475}$$

(a)

$$F_Y(0) = \Pr(Y \leq 0) \tag{4.476}$$

$$= \sum_{k=0}^{0} {}^{3}C_k (0.5)^k (0.5)^{3-k} \tag{4.477}$$

$$= 0.125 \tag{4.478}$$

$$\tag{4.479}$$

(b)

$$F_Y\,(1) = \Pr(Y \leq 1) \tag{4.480}$$
$$= \sum_{k=0}^{1} {}^3C_k (0.5)^k \, (0.5)^{3-k} \tag{4.481}$$
$$= 0.375 \tag{4.482}$$
$$\tag{4.483}$$

(c)

$$F_Y\,(2) = \Pr(Y \leq 2) \tag{4.484}$$
$$= \sum_{k=0}^{2} {}^3C_k (0.5)^k \, (0.5)^{3-k} \tag{4.485}$$
$$= 0.375 \tag{4.486}$$
$$\tag{4.487}$$

(d)

$$F_Y\,(3) = \Pr(Y \leq 3) \tag{4.488}$$
$$= \sum_{k=0}^{3} {}^3C_k (0.5)^k \, (0.5)^{3-k} \tag{4.489}$$
$$= 0.125 \tag{4.490}$$

$\therefore$ Hence Proved that the probability is not 1/4 for each of them.

4.4.44 Two cards are drawn from a well shuffled deck of 52 playing cards with replacement. The probability, that both cards are queens, is

A $\frac{1}{13} \times \frac{1}{13}$

B $\frac{1}{13} + \frac{1}{13}$

C $\frac{1}{13} \times \frac{1}{17}$

D $\frac{1}{13} \times \frac{4}{51}$

Solution: Let p be the probability of selecting queens from the deck of 52 cards.There are 4 queens in a standard deck.So,

$$p = \frac{1}{13} \tag{4.491}$$

We can consider each draw as a Bernoulli trial with success defined as drawing a queen. Then, we can use the binomial probability formula:

$$\Pr(X = 2) = {}^{n}C_k (1 - p)^{n-k} p^k \tag{4.492}$$

where,$\Pr(X = 2)$ is the probability of getting k number of queens in trails The probability for the trails to succeed to 2 times is $\Pr(X = 2)$. Using (4.492) we get,

Table 4.29: parameters for CDF

parameter	value
n	2
p	$\frac{1}{13}$
k	0,1,2

$$\begin{aligned} \Pr(X = 2) &= {}^{2}C_2 \left(1 - \frac{1}{13}\right)^{2-2} \left(\frac{1}{13}\right)^2 \\ &= \frac{1}{13} \times \frac{1}{13} \end{aligned} \tag{4.493}$$

$\therefore$ the probability, that both cards are queens, is $\frac{1}{13} \times \frac{1}{13}$.

4.4.45 Eight coins are tossed together. The probability of getting exactly 3 heads is

(a) $\frac{1}{256}$

(b) $\frac{7}{32}$

(c) $\frac{5}{32}$

(d) $\frac{3}{32}$

Solution: The probability or getting a head is

-

Table 4.30: Random Variable and probability Table

Random independent variable	value of R.V	Description
n	8	Total no. of coin toss
X	0 ≤ X ≤ 8	no. of heads

$$p = \frac{1}{2} \tag{4.494}$$

The pmf for the binomial distribution is

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k} \qquad \text{k} = 0, 1, 2, ..8 \tag{4.495}$$

$$\tag{4.496}$$

For exactly 3 heads

$$p_X(3) = \begin{pmatrix} 8 \\ \\ 3 \end{pmatrix} p^3 (1-p)^5 \tag{4.497}$$

$$= 56 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^5 \tag{4.498}$$

$$= \frac{7}{32} \tag{4.499}$$

Hence, the probability of getting 3 heads is 0.21875

4.4.46 If X follows binomial distribution with parameters $n = 5$, p and

$$p_X(2) = 9p_X(3) \tag{4.500}$$

then p is ? **Solution:**

Given, X follows binomial distribution and pmf of X is:

$$p_X(k) = {}^nC_k p^k (1-p)^{n-k} \tag{4.501}$$

We have, $n = 5$ and p as parameters.

So,

$$p_X(2) = {}^5C_2 p^2 (1-p)^3 \tag{4.502}$$

$$p_X(3) = {}^5C_3 p^3 (1-p)^2 \tag{4.503}$$

As, $p_X(2) = 9p_X(3)$

$$^5C_2p^2(1-p)^3 = 9\left[^5C_3p^3(1-p)^2\right] \tag{4.504}$$

$$(1-p) = 9p \tag{4.505}$$

$$\implies p = \frac{1}{10} \tag{4.506}$$

4.4.47 A box has 100 pens of which 10 are defective. What is the probability that out of a sample of 5 pens drawn one by one with replacement at most one is defective?

(a) $\left(\frac{9}{10}\right)^5$

(b) $\frac{1}{2}\left(\frac{9}{10}\right)^4$

(c) $\frac{1}{2}\left(\frac{9}{10}\right)^5$

(d) $\frac{1}{2}\left(\frac{9}{10}\right)^4 + \left(\frac{9}{10}\right)^5$

Solution: Let X be a random variable such that

Variable	Description	Value
X	Defective pens drawn in sample of 5	$\{0,1,2,3,4,5\}$
p	Probability of drawing defective pens	$\frac{1}{10}$

$$p_X(k) = {}^5C_k\left(\frac{1}{10}\right)^k\left(\frac{9}{10}\right)^{5-k} \tag{4.507}$$

Now,

$$F_X(k) = \sum_{i=0}^{k} p_X(i) \tag{4.508}$$

$$\implies F_X(1) = \sum_{i=0}^{1} p_X(i) \tag{4.509}$$

$$= p_X(0) + p_X(1) \tag{4.510}$$

$$= {}^5C_0 \left(\frac{1}{10}\right)^0 \left(\frac{9}{10}\right)^5 + {}^5C_1 \left(\frac{1}{10}\right)^1 \left(\frac{9}{10}\right)^4 \tag{4.511}$$

$$= \left(\frac{9}{10}\right)^5 + \frac{1}{2}\left(\frac{9}{10}\right)^4 \tag{4.512}$$

4.4.48 If X follows a binomial distribution with parameters n = 5, p and $p_X(2) = 9p_X(3)$ then p is?

Solution:

$$p_X(2) = {}^5C_2 p^2 (1-p)^{5-2} \tag{4.513}$$
$$= \frac{5!}{2!3!} p^2 (1-p)^3 \tag{4.514}$$
$$= 10p^2(1-p)^3 \tag{4.515}$$
$$p_X(3) = {}^5C_3 p^3 (1-p)^{5-3} \tag{4.516}$$
$$= \frac{5!}{3!2!} p^3 (1-p)^2 \tag{4.517}$$
$$= 10p^3(1-p)^2 \tag{4.518}$$
$$9p_X(3) = 9 \times 10p^3(1-p)^2 \tag{4.519}$$
$$= 90p^3(1-p)^2 \tag{4.520}$$
$$\text{Given that } p_X(2) = 9p_X(3) \tag{4.521}$$
$$\implies 10p^2(1-p)^3 = 90p^3(1-p)^2 \tag{4.522}$$
$$\implies (1-p) = 9p \tag{4.523}$$
$$\implies 10p = 1 \tag{4.524}$$
$$\implies p = \frac{1}{10} \tag{4.525}$$

4.4.49 A die is thrown again and again until three sixes are obtained. Find the probability of obtaining third six on sixth throw of a die.

Solution: Let E the event be getting third six on sixth throw

Binomial pmf given by,

$\Pr(X = k) = {}^nC_k p^k (1-p)^{n-k}$

probability of getting two sixes in first five throws,

Table 4.31: parameters for PMF

parameter	value
n	5
p	$\frac{1}{6}$
k	2
$1-p$	$\frac{5}{6}$

$$\Pr(k=2) = {}^5C_2\left(\frac{1}{6}\right)^2\left(1-\frac{1}{6}\right)^{5-2} \tag{4.526}$$

$$= {}^5C_2\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^3 \tag{4.527}$$

$$= \frac{10\times 5^3}{6^5} \tag{4.528}$$

$$= \frac{1250}{7776} \tag{4.529}$$

Now,

$$\Pr(E) = p.\Pr(k=2) \tag{4.530}$$

$$= \frac{1}{6}\times\frac{1250}{7776} \tag{4.531}$$

$$= 0.026 \tag{4.532}$$

Hence, probability of getting third six on sixth throw of a die is 0.026

4.4.50 A fair coin is tossed four times and a person win Re 1 for each head and lose Re 1.5 for each tail that turns up.from the sample space calculate how many different amounts of money you can have after four tosses and the probability of having each of these amounts.

Solution: Let us define a random variable X, where getting heads is success.

parameters	values	descrption
X	0,1,2,3,4	no of heads
n	4	times event occured
p	0.5	prob of getting head
q	0.5	prob of getting tail

Table 4.32: Random variable X declaration

$$p_X(k) = \begin{cases} 0 & k < 0 \\ {}^nC_k \left(p\right)^k \left(q\right)^{n-k} & 0 \leq k \leq n \end{cases} \tag{4.533}$$

let Y be a random variable with "y" denoting amounts possible, whose value is given by:

$$y = k(1) + (4-k)(-1.5) \tag{4.534}$$

k	y(profit in Rs)
0	-6
1	-3.5
2	-1
3	1.5
4	4

Table 4.33: amounts possible

now using PMF the probability of different amounts is:

$$p_Y(y) = \begin{cases} 0 & y < -6 \\ {}^nC_{\frac{y+6}{2.5}} \left(p\right)^{\frac{y+6}{2.5}} \left(q\right)^{n-\frac{y+6}{2.5}} & -6 \leq y \leq 4 \\ 0 & y > 4 \end{cases} \tag{4.535}$$

simulation steps:

(a) define a function called binomial probability

(b) set the parameters for the binomial distribution with n=4,$p = \frac{1}{2}, q = \frac{1}{2}$

(c) define a list called money changes and this list contains all possible amounts ranging from -6 to 4.

(d) create a empty list called probabilities to store probabilities of each amount.

(e) calculated the probabilities using loops and stored them in the above list.

(f) create a plot

4.5. Triangular

4.5.1 Two dice, one blue and one grey, are thrown at the same time. The event defined by the sum of the two numbers appearing on the top of the dice can have 11 possible outcomes 2, 3, 4, 5, 6, 6, 8, 9, 10, 11 and 12. A student argues that each of these outcomes has a probability $\frac{1}{11}$. Do you agree with this argument? Justify your answer.

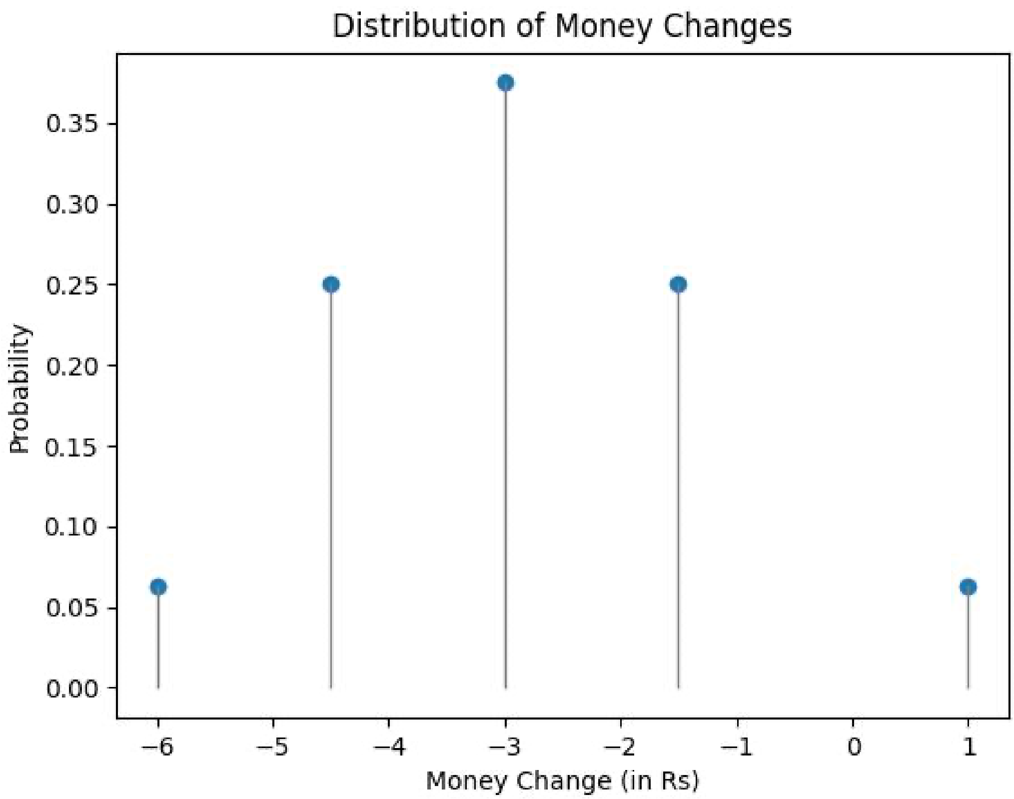

Figure 4.11: distribution of Y

4.6. Miscellaneous

4.6.1 The random variable X has a probability distribution $\Pr(X)$ of the following form, where k is some number

$$\Pr(X) = \begin{cases} k, & x = 0 \\ 2k, & x = 1 \\ 3k, & x = 2 \\ 0, & \text{otherwise} \end{cases} \tag{4.536}$$

(a) Determine the value of k

(b) Find $\Pr(X < 2)$,$\Pr(X \leq 2)$,$\Pr(X \geq 2)$

Solution:

(a) Using the axioms of probability,

$$k + 2k + 3k = 1 \implies k = \frac{1}{6} \tag{4.537}$$

(b) The CDF is given by

$$F_X(k) = \begin{cases} 0, & x < 0 \\ \frac{1}{6}, & 0 \leq x < 1 \\ \frac{1}{2}, & 1 \leq x < 2 \\ 1, & x \geq 2 \end{cases} \tag{4.538}$$

Thus,

i.

$$\Pr\left(X<2\right)=F(1)=\frac{1}{2} \tag{4.539}$$

ii.

$$\Pr\left(X\leq 2\right)=F(2)=1 \tag{4.540}$$

iii.

$$\Pr\left(X\geq 2\right)=1-\Pr\left(X<2\right)=1-F(1)=\frac{1}{2} \tag{4.541}$$

4.6.2 State which of the following are not the probability distributions of a random variable. Give reasons for your answer

X	0	1	2
P(X)	0.4	0.4	0.2

i

X	0	1	2	3	4
P(X)	0.1	0.5	0.2	-0.1	0.3

ii

Y	-1	0	1
P(Y)	0.6	0.1	0.2

iii

Z	3	2	1	0	-1
P(Z)	0.3	0.2	0.4	0.1	-0.05

iv

4.6.3 A random variable X has the following probability distribution

Determine

X	0	1	2	3	4	5	6	7
P(X)	0	K	2K	2K	3K	K^2	$2K^2$	$7K^2 + K$

i k

ii P$(X < 3)$

iii P$(X > 6)$

iv P$(0 < X < 3)$

4.6.4 The random variable X has a probability distribution P(X) of the following form, where k is some number :

$$P(x) = \begin{cases} k, & \text{if } x = 0 \\ 2k, & \text{if } x = 1 \\ 3k, & \text{if } x = 2 \\ 0, & otherwise \end{cases}$$

i Determine the value of k.

ii Find P $(X < 2)$, P $(X \leq 2)$, P$(X \geq 2)$

4.6.5 A game consists of spinning an arrow which comes to rest pointing at one of the

regions (1, 2 or 3) (Fig. 13.1). Are the outcomes 1, 2 and 3 equally likely to occur?

Give reasons.

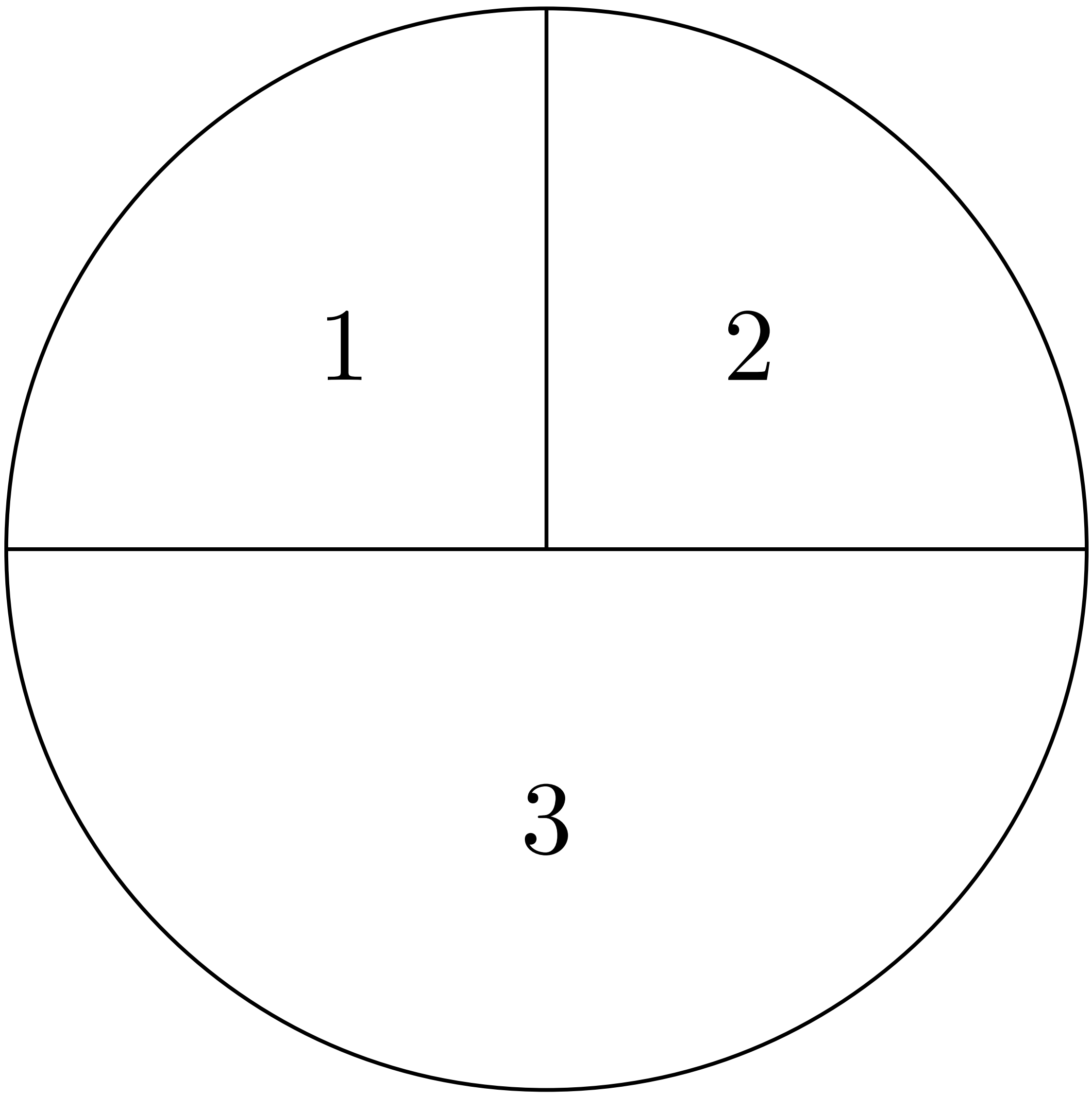

Figure 4.12: Fig.13.1

Solution:

$$X \in \{1, 2, 3\}$$

From the figure given,

$$p_X(k) = \begin{cases} \frac{90^\circ}{360^\circ} = 0.25 & k = 1 \\ \frac{90^\circ}{360^\circ} = 0.25 & k = 2 \\ \frac{180^\circ}{360^\circ} = 0.5 & k = 3 \end{cases} \tag{4.542}$$

$p_X(k)$ are not equal for all k. Therefore, the events are not equally likely.

4.6.6 Apoorv throws two dice once and computes the product of the numbers appearing on the dice. Peehu throws one die and squares the number that appears on it. Who has the better chance of getting the number 36? Why?

Solution: Let the random variables be defined as:

Random Variable	**Values**	**Description**
X	$1 \leq X \leq 6$	Apoorv's First Dice Roll
Y	$1 \leq Y \leq 6$	Apoorv's Second Dice Roll
Z	$1 \leq Z \leq 6$	Peehu's Dice Roll

i **Product:** Assuming all dice rolls and equally likely,:

$$p_X(k) = \begin{cases} \frac{1}{6} & \text{if } k \in \{1, 2, 3, 4, 5, 6\} \\ 0 & \text{otherwise} \end{cases} \tag{4.543}$$

$$p_Y(k) = \begin{cases} \frac{1}{6} & \text{if } k \in \{1, 2, 3, 4, 5, 6\} \\ 0 & \text{otherwise} \end{cases} \tag{4.544}$$

The probability mass function for Apoorv is:

$$
\begin{aligned}
p_{XY}(k) &= \Pr(XY = k) && (4.545)\\
&= \Pr\left(X = \frac{k}{Y}\right) && (4.546)\\
&= E\left(p_X\left(\frac{k}{Y}\right)\right) && (4.547)\\
&= \sum_{i=1}^{6} p_X\left(\frac{k}{i}\right) p_Y(i) && (4.548)\\
&= \frac{1}{6}\sum_{i=1}^{6} p_X\left(\frac{k}{i}\right) && (4.549)\\
&= \frac{1}{6}\sum_{i=1}^{6} \frac{[k \bmod i = 0]}{6}\left[\frac{k}{i} \le 6\right] && (4.550)\\
&= \frac{1}{36}\sum_{i=1}^{6} [k \bmod i = 0]\left[\frac{k}{i} \le 6\right] && (4.551)
\end{aligned}
$$

Thus, the probability of Apoorv rolling a 36 is:

$$
\begin{aligned}
p_{XY}(36) &= \frac{1}{36}\sum_{i=1}^{6} [36 \bmod i = 0]\left[\frac{36}{i} \le 6\right] && (4.552)\\
&= \frac{1}{36}(0+0+0+0+0+1) && (4.553)\\
&= \frac{1}{36} && (4.554)
\end{aligned}
$$

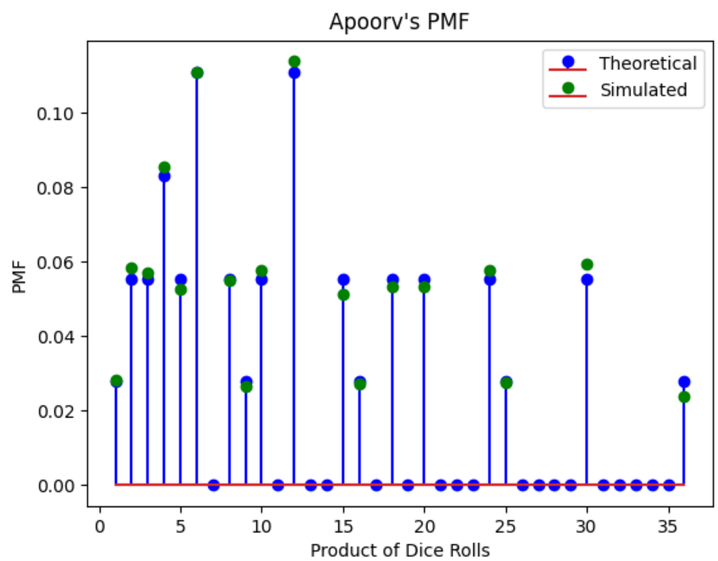

Figure 4.13: Sketch of Probability Mass Function for Product obtained by taking a sample of random variables

The cumulative distribution function for Apoorv is:

$$F_{XY}(k) = \Pr\left(XY \le k\right) \tag{4.555}$$

$$= \frac{1}{6}\sum_{j=1}^{k}\sum_{i=1}^{6} p_X\left(\frac{j}{i}\right) \tag{4.556}$$

$$= \frac{1}{36}\sum_{j=1}^{k}\sum_{i=1}^{6} [j \bmod i = 0]\left[\frac{j}{i} \le 6\right] \tag{4.557}$$

ii **Square:** The probability mass function for Peehu is:

$$p_{Z^2}(k) = \begin{cases} \frac{1}{6} & \text{if } k \in \{1, 4, 9, 16, 25, 36\} \\ 0 & \text{otherwise} \end{cases} \tag{4.558}$$

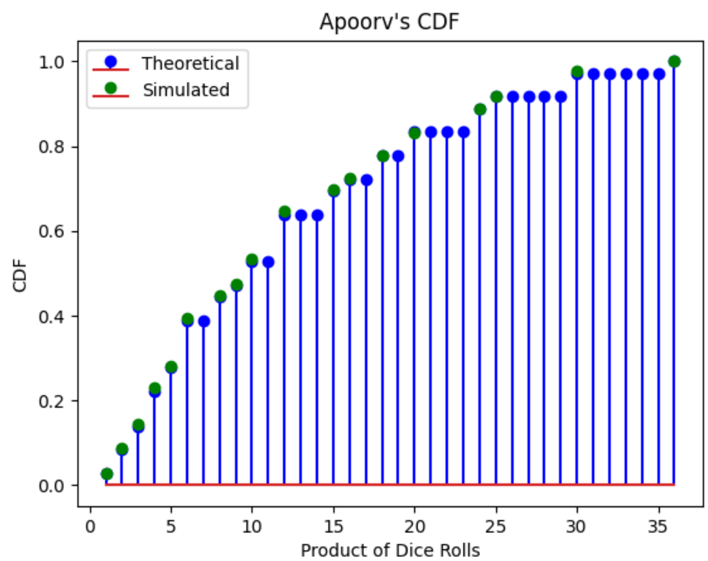

Figure 4.14: Sketch of Cumulative Distribution Function for product obtained by taking a sample of random variables

Thus, the probability of Peehu rolling a 36 is

$$p_{Z^2}(36) = \frac{1}{6} \tag{4.559}$$

The cumulative distribution function for Peehu is:

$$F_{Z^2}(k) = \Pr\left(Z^2 \leq k\right) \tag{4.560}$$

$$= \begin{cases} 0 & \text{if } k \leq 0 \\ \frac{\lfloor \sqrt{k} \rfloor}{6} & \text{if } k \in \{1, 2, ..., 35\} \\ 1 & \text{if } k \geq 36 \end{cases} \tag{4.561}$$

From (4.554) and (4.559), $p_{Z^2}(36) > p_{XY}(36)$. Therefore, Peehu has a better chance

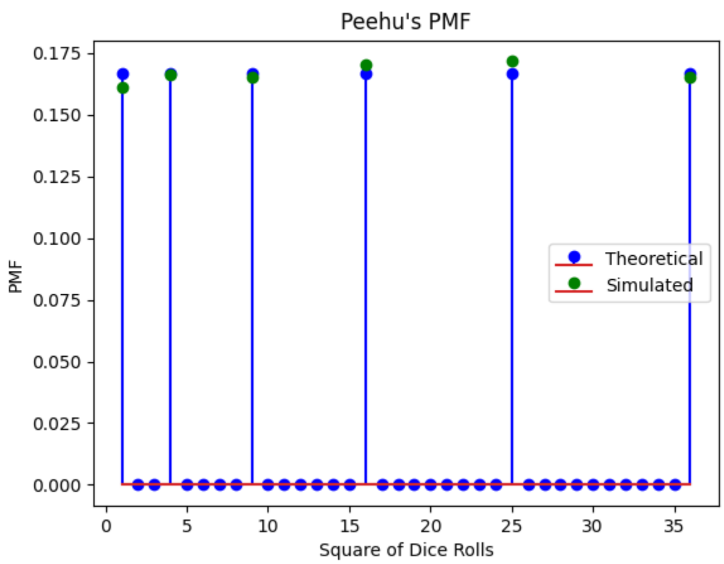

Figure 4.15: Sketch of Probability Mass Function for square obtained by taking a sample of random variables

of getting the number 36 than Apoorv.

4.6.7 6 boys and 6 girls sit in a row at random. The probability that all the girls sit together is

i $\frac{1}{432}$

ii $\frac{12}{431}$

iii $\frac{1}{132}$

iv none of the above

Solution: The number of ways in which n people can sit in a row

$$= n! \tag{4.562}$$

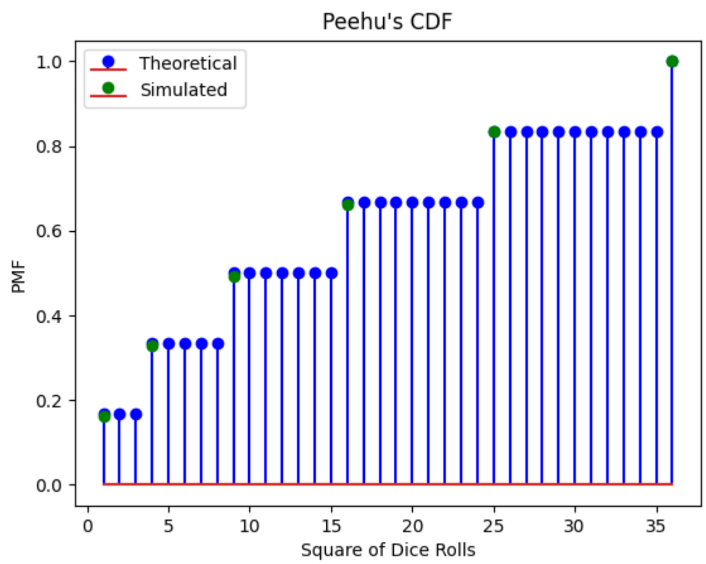

Figure 4.16: Sketch of Cumulative Distribution Function for square obtained by taking a sample of random variables

$\therefore$ for 6 boys and 6 girls, total number of arrangements

$$= 12! \tag{4.563}$$

Parameter	Value	Description
X	1-12	Represents the number of selected people sitting together

Finding pmf:

$$p_X(k) = Pr(X = k) \tag{4.564}$$
$$= \frac{(12 - k + 1)! \times k!}{12!} \tag{4.565}$$

$\therefore$ probability of 6 girls sitting together

$$= p_X(6) \tag{4.566}$$
$$= \frac{7! \times 6!}{12!} \tag{4.567}$$
$$= \frac{1}{132} \tag{4.568}$$

$\therefore$ option 3 is correct.

4.6.8 A card is selected from a deck of 52 cards. The probability of its being a red face card is

Solution: :

$$p_X(k) = \begin{cases} \frac{12}{52} & k = \{1, 2, 3...12\} \\ \frac{40}{52} & k = \{13, 14, 15....52\} \\ 0 & otherwise \end{cases} \tag{4.569}$$

$$p_Y(k) = \begin{cases} \frac{26}{52} & k = \{1, 2, 3, ...26\} \\ \frac{26}{52} & k = \{27, 28, 29....52\} \\ 0 & otherwise \end{cases} \tag{4.570}$$

Parameter	Value	Description
X	0	Not a face card
	1	A face card
Y	0	A black card
	1	A red card

Table 4.39: Table with random varaible outcomes

$$Z = XY \tag{4.571}$$

$$p_Z(k) = p\left(XY = k\right) \tag{4.572}$$

$$= \Pr\left(X = \frac{k}{Y}\right) \tag{4.573}$$

$$= E\left(p_X\left(\frac{k}{Y}\right)\right) \tag{4.574}$$

$$= \sum_{i=1}^{52} p_X\left(\frac{k}{i}\right) p_Y\left(i\right) \tag{4.575}$$

$$= \frac{1}{2}\sum_{i=1}^{52} p_X\left(\frac{k}{i}\right) \tag{4.576}$$

$$Z = 1 \quad X = 1, Y = 1 \tag{4.577}$$

$$\implies p_Z(1) = \frac{1}{2}\{p_X\left(1\right) + 0 + 0...\} \tag{4.578}$$

$$= \frac{1}{2}\left(\frac{12}{52}\right) \tag{4.579}$$

$$= \frac{3}{26} \tag{4.580}$$

4.6.9 A die is loaded in such a way that each odd number is twice as likely to occur as each even number. Find P(G), where G is the event that a number greater than 3 occurs on a single roll of the die.

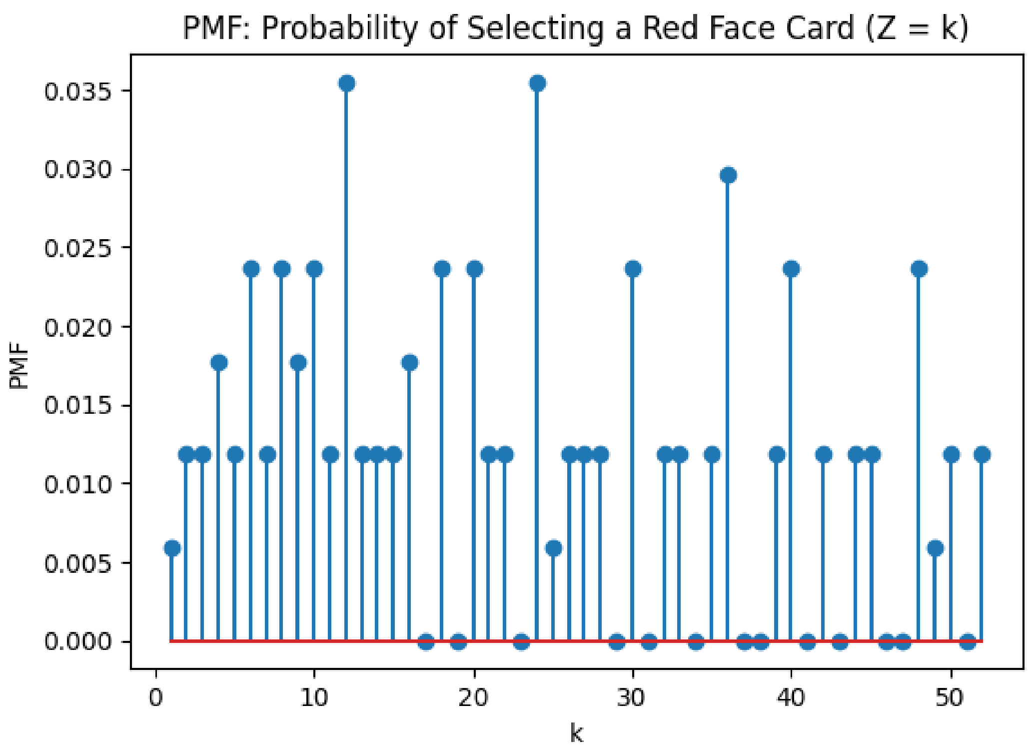

Figure 4.17: Plot of Probability Mass Function

Solution: Let the random variable be X and p is the probability that rolled number is odd.

The PMF of X in terms of p is given by:

$$p_X(k) = \begin{cases} 2p & \text{if } k \text{ is odd} \\ p & \text{if } k \text{ is even} \end{cases}$$

To find Probability that rolled number is odd: We know,

$$\sum_{k=1}^{6} p_X(k) = 1 \tag{4.581}$$

$$\Rightarrow \quad p + \frac{p}{2} + p + \frac{p}{2} + p + \frac{p}{2} = 1 \tag{4.582}$$

$$\Rightarrow \quad \frac{9p}{2} = 1 \tag{4.583}$$

$$\Rightarrow \quad p = \frac{2}{9} \tag{4.584}$$

$$p_X(k) = \begin{cases} \frac{2}{9} & \text{if } k \text{ is odd} \\ \frac{1}{9} & \text{if } k \text{ is even} \end{cases} \tag{4.585}$$

The cdf of X is given by

$$F_X(k) = \Pr(X \leq k) \tag{4.586}$$

The cdf of X in terms of k is given by:

$$F_X(k) = \begin{cases} \frac{k+1}{9} & \text{if } k \text{ is odd} \\ \frac{k}{6} & \text{if } k \text{ is even} \end{cases} \tag{4.587}$$

i We require $\Pr(X > 3)$.

$$\Pr(X > 3) = 1 - \Pr(X \leq 3) \tag{4.588}$$
$$= 1 - F_X(3) \tag{4.589}$$
$$= 1 - \frac{5}{9} \tag{4.590}$$
$$= \frac{4}{9} \tag{4.591}$$
$$= 0.44 \tag{4.592}$$

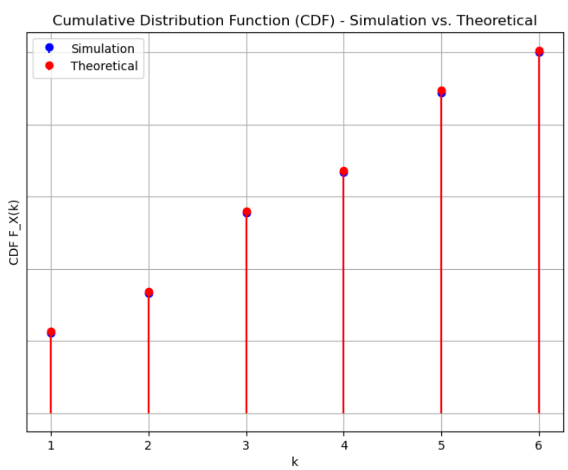

Figure 4.18: CDF plot - Simulation and theoretical.

4.6.10 Determine the probability p,for each of following events.

i An odd number appears in a single roll of dice.

ii Atleast one head appears in two tosses of fair coin.

iii A king,9 of hearts or 3 of spades appears in drawing a single card from a well shuffled deck of 52 cards.

iv The sum of 6 appears in single toss of a pair of fair dice.

Solution:

i Let the random variable X be defined as:

Random Variable	**Values**	**Description**
X	$1 \leq X \leq 6$	Number appeared on a roll

$$p_X(k) = \begin{cases} \frac{1}{6} & \text{if } k \in \{1,2,3,4,5,6\} \\ 0 & \text{otherwise} \end{cases} \tag{4.593}$$

Let E be event occuring odd number on single roll. Since, the dice rolls are mutually exclusive. From (4.593).

$$\Pr(E) = p_X(1) + p_X(3) + p_X(5) \tag{4.594}$$
$$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} \tag{4.595}$$
$$= \frac{1}{2} \tag{4.596}$$

ii Let 1 be Head and 0 be Tail. Consider random variable X_i where $i \in \{1,2\}$ as

Random Variable	**Values**	**Description**
X_i	$\{0,1\}$	Result appeared on "i"th coin

Let $Y = X_1 + X_2$ be a binomial distribution with parameters

$$n = 2 \qquad p = \frac{1}{2} \tag{4.597}$$

Probabilty of getting k Head in $n = 2$ coins is

$$p_Y(k) = {}^2C_k p^k (1-p)^{2-k} \tag{4.598}$$

Now, Let F be event of getting alteast one head.

$$\begin{aligned}
\Pr(F) &= p_Y(1) + p_Y(2) &(4.599)\\
&= {}^2C_1 \left(\frac{1}{2}\right)^1 \left(1 - \frac{1}{2}\right)^{2-1} + {}^2C_2 \left(\frac{1}{2}\right)^2 \left(1 - \frac{1}{2}\right)^{2-2} &(4.600)\\
&= \frac{3}{4} &(4.601)
\end{aligned}$$

iii Let the random variables X and Y be defined as:

Random Variable	**Values**	**Description**
X	$1 \leq X \leq 4$	Shape of Card
Y	$1 \leq Y \leq 13$	Number on Card

Let For $X \in \{1, 2, 3, 4\}$ represents Diamonds, Clubs, Hearts, Spades respectively.

$$p_X(k) = \begin{cases} \frac{1}{4} & \text{if } 1 \leq k \leq 4 \\ 0 & \text{otherwise} \end{cases} \tag{4.602}$$

$$p_Y(k) = \begin{cases} \frac{1}{13} & \text{if } 1 \leq k \leq 13 \\ 0 & \text{otherwise} \end{cases} \tag{4.603}$$

$$p_{XY}(k, m) = \begin{cases} \frac{1}{52} & \text{if } 1 \leq k \leq 4 \text{ and } 1 \leq m \leq 13 \\ 0 & \text{otherwise} \end{cases} \tag{4.604}$$

Let $Y = 13$ represent king Card. So, Let G be event to get 4 kings, 9 of hearts,3 of spades. From (4.603) and (4.604).

$$\Pr(G) = p_Y(13) + p_{XY}(3,9) + p_{XY}(4,3) \tag{4.605}$$
$$= \frac{1}{13} + \frac{1}{52} + \frac{1}{52} \tag{4.606}$$
$$= \frac{3}{26} \tag{4.607}$$

iv Let random variables X_i where $i \in \{1, 2\}$ be defined as

Random Variable	**Values**	**Description**
X_i	$1 \leq X_i \leq 6$	Number appeared on "i"th dice

Let $Y = X_1 + X_2$. Then

$$p_Y(k) = \begin{cases} \frac{k-1}{36} & \text{if } 1 \leq k \leq 7 \\ \frac{13-k}{36} & \text{if } 8 \leq k \leq 13 \\ 0 & \text{otherwise} \end{cases} \tag{4.608}$$

Consider an H for which sum of both dice is six. Since the dice are independent.

$$\Pr(H) = p_Y(6) \tag{4.609}$$
$$= \frac{5}{36} \tag{4.610}$$

4.6.11 Determine the probability p, for each of the following events.

(a) An odd number appears in a single toss of a fair die.

(b) At least one head appears in two tosses of a fair coin.

(c) A king, 9 of hearts, or 3 of spades appears in drawing a single card from a well shuffled ordinary deck of 52 cards.

(d) The sum of 6 appears in a single toss of a pair of fair dice.

Solution:

parameter	**value**	**description**
Random Variable X	0	Odd number appears in the throw (1,3,5)
	1	Even number appears in the throw (2,4,6)

Table 4.40: Single toss of Die

(a)

$$p = p_X(0) = \frac{1}{2} \tag{4.611}$$

(b) Let the random variable X denote one single coin toss where obtaining a head is considered as sucess.Then,

$$X \sim \text{Ber}\left(p\right) \tag{4.612}$$

Suppose $X_i, 1 \leq i \leq n$ represent each of the n draws. Define Y as

$$Y = \sum_{i=1}^{n} X_i \tag{4.613}$$

Then, since the X_i are iid, the pmf of Y is given by

$$Y \sim \text{Bin}\left(n, p\right) \tag{4.614}$$

The cdf of Y is given by

$$F_Y(k) = \Pr(Y \leq k) \tag{4.615}$$

$$= \begin{cases} 0 & k < 0 \\ \sum_{i=1}^{k} \begin{pmatrix} n \\ i \end{pmatrix} p^i (1-p)^{n-i} & 1 \leq k \leq n \\ 1 & k \geq n \end{cases} \tag{4.616}$$

In this case,

$$p = \frac{1}{2}, \; n = 2 \tag{4.617}$$

We require $\Pr(Y \geq 1)$. Since $n = 2$,

$$\Pr(Y \geq 1) = 1 - \Pr(Y < 1) \tag{4.618}$$

$$= F_Y(2) - F_Y(0) \tag{4.619}$$

$$= \sum_{k=1}^{2} p_Y(k) \tag{4.620}$$

$$= \sum_{k=1}^{2} \begin{pmatrix} n \\ k \end{pmatrix} p^k (1-p)^{n-k} \tag{4.621}$$

$$= 0.75 \tag{4.622}$$

parameter	value	description
Random Variable X	$1 \leq X \leq 13$	Different cards in deck
Random Variable Y	1	Hearts of cards
	2	Spades of cards
	3	Clubs of cards
	4	Diamonds of cards

Table 4.41: Deck of cards

(c)

$$p_X(13) = \frac{1}{13} \tag{4.623}$$

$$p_{XY}(9,1) = \frac{1}{52} \tag{4.624}$$

$$p_{XY}(3,2) = \frac{1}{52} \tag{4.625}$$

$$p = p_X(13) + p_{XY}(9,1) + p_{XY}(3,2) \tag{4.626}$$

$$p = \frac{3}{26} \tag{4.627}$$

random variables	description
X	number appearing on first dice
Y	number appearing on second dice
Z	Sum of numbers appearing on both dice

Table 4.42: Sum of two dices

(d) We know that

$$p_Z(n) = \begin{cases} 0 & n \leq 1 \\ \frac{n-1}{36} & 2 \leq n \leq 6 \\ \frac{13-n}{36} & 7 \leq n \leq 12 \\ 0 & n \geq 13 \end{cases} \tag{4.628}$$

Hence, the probability ,

$$p = p_Z(6) = \frac{5}{36} \tag{4.629}$$

4.6.12 The probability distribution of a random variable X is given below:

X	0	1	2	3
$P(X)$	k	$\frac{k}{2}$	$\frac{k}{4}$	$\frac{k}{8}$

i Determine the value of k.

ii Determine $P(X \leq 2)$ and $P(X > 2)$.

iii Find $P(X \leq 2) + P(X > 2)$.

Solution:

i The cumulative distribution function of X is,

$$F_X(k) = \Pr(X \leq k) \tag{4.630}$$

$$= \sum_{i=0}^{k} p_X(i) \tag{4.631}$$

As X can only take values to 3, we can say that,

$$F_X(k \geq 3) = 1 \tag{4.632}$$

$$\sum_{i=0}^{k} p_X(k) = 1 \tag{4.633}$$

$$\implies p_X(0) + p_X(1) + p_X(2) + p_X(3) = 1 \tag{4.634}$$

$$\implies k + \frac{k}{2} + \frac{k}{4} + \frac{k}{8} = 1 \tag{4.635}$$

$$\implies \frac{15k}{8} = 1 \tag{4.636}$$

$$\implies k = \frac{8}{15} \tag{4.637}$$

Hence, the value of k is $\frac{8}{15}$. This makes the data given in the question as follows,

$$p_X(k) = \begin{cases} \frac{8}{15} & \text{if } k = 0 \\ \frac{4}{15} & \text{if } k = 1 \\ \frac{2}{15} & \text{if } k = 2 \\ \frac{1}{15} & \text{if } k = 3 \\ 0 & \text{Otherwise} \end{cases} \tag{4.638}$$

Using the value of k we can also find CDF of X,

$$F_X(k) = \begin{cases} 0 & \text{if } k < 0 \\ \frac{8}{15} & \text{if } k = 0 \\ \frac{4}{5} & \text{if } k = 1 \\ \frac{14}{15} & \text{if } k = 2 \\ 1 & \text{if } k \geq 3 \end{cases} \tag{4.639}$$

ii

$$\Pr(X \leq 2) = F_X(2) \tag{4.640}$$
$$= \frac{14}{15} \tag{4.641}$$

$$\Pr(X > 2) = 1 - F_X(2) \tag{4.642}$$
$$= 1 - \frac{14}{15} \tag{4.643}$$
$$= \frac{1}{15} \tag{4.644}$$

iii

$$\Pr(X \leq 2) + \Pr(X > 2) = F_X(2) + 1 - F_X(2) \tag{4.645}$$
$$= 1 \tag{4.646}$$

4.6.13 One ticket is drawn at random from a bag containing tickets numbered 1 to 40.The probability that the selected ticket has a number which is a multiple of 5 is (a) $\frac{1}{5}$ (b)

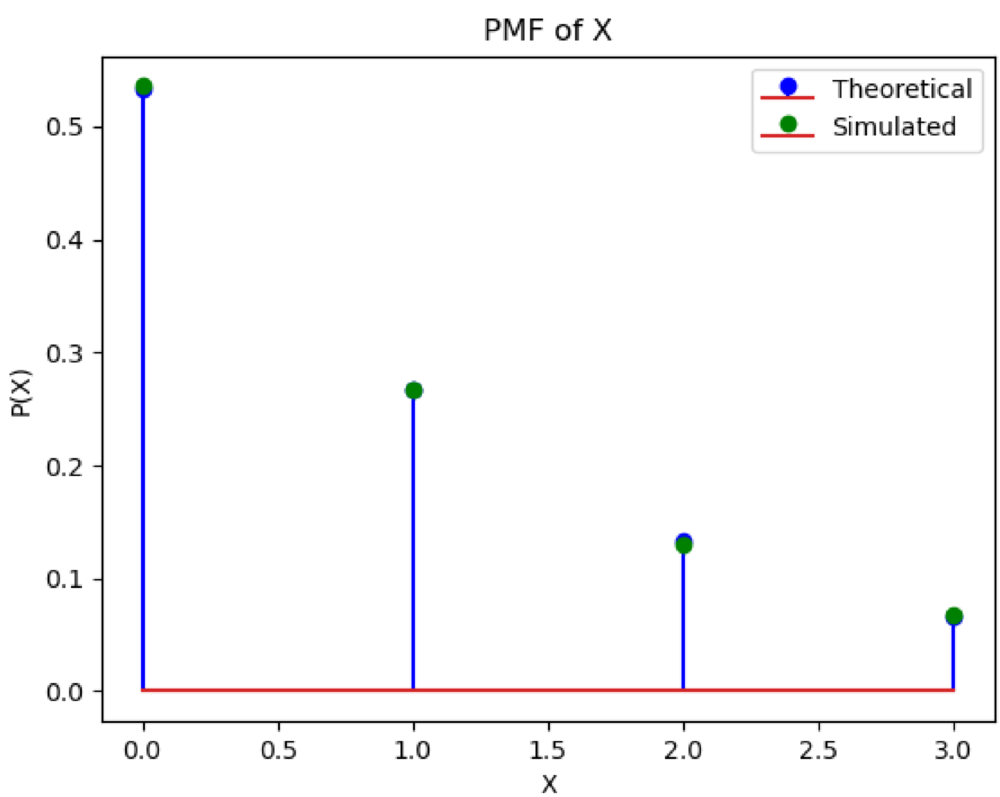

Figure 4.19: Generated using (4.6.12a)

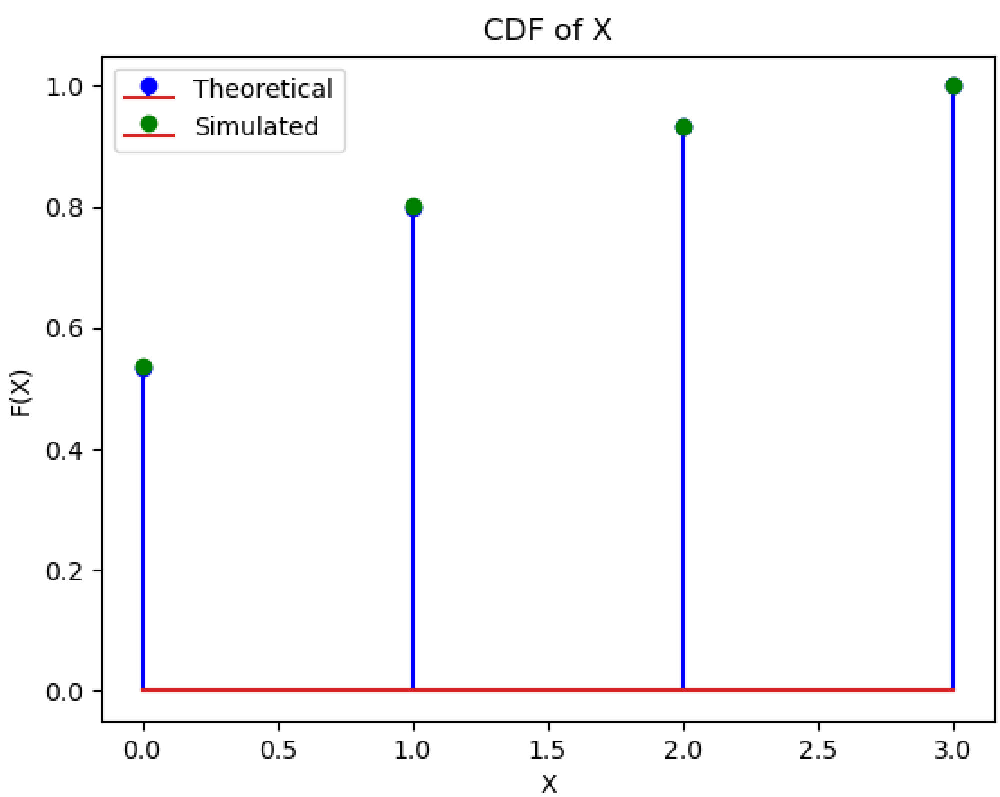

Figure 4.20: Generated using (4.639)

$\frac{3}{5}$ (c) $\frac{4}{5}$ (d) $\frac{1}{3}$

Solution:

$$P_r(k \text{ is divisible by } 5) = \frac{8}{40} \tag{4.647}$$

$$= \frac{1}{5} \tag{4.648}$$

4.6.14 Three persons, A, B and C, fire at a target in turn, starting with A. Their probability of hitting the target are 0.4, 0.3 and 0.2 respectively. The probability of two hits is

i 0.024

ii 0.188

iii 0.336

iv 0.452

Solution:

Let X, Y and Z be random variables with definition given as under:

Random Variable	Description
X	A hitting the target
Y	B hitting the target
Z	C hitting the target

Table 4.43: Definition of Random variables

We want to find the probability of two hits, which corresponds to $S = X + Y + Z$ being equal to 2

Bernouli probability	Value	Description
p_1	Probability of A hitting the target	0.4
p_2	Probability of B hitting the target	0.3
p_3	Probability of C hitting the target	0.2

Table 4.44: Probabilities

PMF of S using z-transform:

Applying the z-transform on both the sides

$$M_S(z) = M_{X+Y+Z}(z) \tag{4.649}$$

Using the expectation operator:

$$E\left[z^{-S}\right] = M_X(z) \cdot M_Y(z) \cdot M_Z(z) \tag{4.650}$$

For a Bernoulli random variable X with parameter p, the Z-transform is given by:

$$M_X(z) = (1 - p_i) + zp_i \tag{4.651}$$

For the random variables

$$M_X(z) = (1 - 0.4) + 0.4z = 0.6 + 0.4z \tag{4.652}$$

$$M_Y(z) = (1 - 0.3) + 0.3z = 0.7 + 0.3z \tag{4.653}$$

$$M_Z(z) = (1 - 0.2) + 0.2z = 0.8 + 0.2z \tag{4.654}$$

Now, we can find the z-transform of S, denoted as $M_S(z)$, which is the product of the Z-transforms of X,Y and Z since they are independent

Coefficient of the z^2 term in $M_S(z)$ is corresponds to the probability of two hits.

$$\begin{aligned} M_S(z) &= M_X(z)M_Y(z)M_Z(z) && (4.655)\\ &= \prod_{k=1}^{3}(1-p_i)+zp_i && (4.656)\\ &= (0.6+0.4z)(0.7+0.3z)(0.8+0.2z) && (4.657)\\ &= 0.188z^2+1.208z+.. && (4.658)\end{aligned}$$

$$\therefore \text{The probability of two hits} = 0.188 \quad (4.659)$$

item If two events are independent, then

i they must be mutually exclusive

ii the sum of their probabilities must be equal to 1

iii (A) and (B) both are correct

iv None of the above is correct

Solution: Let X,Y be bernoulli random variables as defined in Table 4.45, Lets us

RV	Value	Description
X	$\{0,1\}$	Event A
Y	$\{0,1\}$	Event B

Table 4.45: Random variable X declaration

consider the pmf's as follows:

$$p_X(k) = \begin{cases} x & \text{if } k = 1 \\ 1 - x & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.660}$$

$$p_Y(k) = \begin{cases} y & \text{if } k = 1 \\ 1 - y & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.661}$$

Two events A and B are independent if and only if:

$$\Pr(X = 1, Y = 1) = p_X(1)p_Y(1) = xy \tag{4.662}$$

i Two events A and B are mutually-exclusive if and only if:

$$\Pr(X = 1, Y = 1) = 0 \tag{4.663}$$

from (4.662) and (4.663), We can say that X and Y are not mutually exclusive as

$$\Pr(X = 1, Y = 1) \neq 0 \tag{4.664}$$

Hence Option A is false.

ii The sum of the probabilities of two events A and B can be represented by:

$$\Pr(X = 1) + \Pr(Y = 1) = x + y \tag{4.665}$$

Here, It is not always true that $x + y$ should be 1 rather

$$0 \leq x + y \leq 2 \tag{4.666}$$

Hence Option B is false.

iii Option C cant be true as Both A and B are false.

iv So option D none of these is the most appropriate answer

For example: Let event X and Y be indepedent and defined as follows: Then the pmf's

RV	Value	Description
X	$\{0, 1\}$	Getting Head in a coin toss
Y	$\{0, 1\}$	Getting 4 by a dice roll

Table 4.46: Random variable X declaration

as follows:

$$p_X(k) = \begin{cases} \frac{1}{2} & \text{if } k = 1 \\ \frac{1}{2} & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.667}$$

$$p_Y(k) = \begin{cases} \frac{1}{6} & \text{if } k = 1 \\ \frac{5}{6} & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.668}$$

Then

i

$$\Pr(X = 1\,, Y = 1) = p_X(1)p_Y(1) \tag{4.669}$$
$$= \frac{1}{12} \neq 0 \tag{4.670}$$

From (4.670), we can say that X and Y are not mutually exclusive and

ii

$$\Pr(X = 1) + \Pr(Y = 1) = \frac{2}{3} \tag{4.671}$$

From (4.671), the sum of their probabilities may not be equal to 1

Therefore, Option A and B are wrong and Option D is correct

4.6.15 Three letters are dictated to three persons and an envelope is addressed to each of them, the letters are inserted into the envelopes at random so that each envelope contains exactly one letter. Find the probability that at least one letter in its proper envelope.

Solution:

Parameter	Values	Description
X	0,1,2,3	No.of envelopes with correct letters.

Table 4.47: Definition of X and parameters.

$$p_X(k) = \begin{cases} \frac{2}{6} & \text{k} = 0 \\ \frac{3}{6} & \text{k} = 1 \\ 0 & \text{k} = 2 \\ \frac{1}{6} & \text{k} = 3 \end{cases} \tag{4.672}$$

$$F_X(k) = \Pr(X \leq k) \tag{4.673}$$

$$= \sum_{k=0}^{k} p_X(k) \tag{4.674}$$

$$\implies F_X(k) = \begin{cases} \frac{1}{3} & \text{k} = 0 \\ \frac{5}{6} & \text{k} = 1 \\ 0 & \text{k} = 2 \\ 1 & \text{k} = 3 \end{cases} \tag{4.675}$$

$$\Pr(X \geq k) = 1 - F_X(k-1) \tag{4.676}$$

$$\implies \Pr(X \geq 1) = 1 - F_X(0) \tag{4.677}$$

$$= 1 - \frac{1}{3} \tag{4.678}$$

$$= \frac{2}{3} \tag{4.679}$$

$\therefore$ The probability that at least one letter in its proper envelope is 0.667.

Chapter 5

Moments

5.1 Find the mean number of heads in three tosses of a fair coin.

Solution: Substituting $n = 3, p = \frac{1}{2}$ in (C.2.3.1), the mean is $\frac{3}{2}$.

5.2 Two dice are thrown simultaneously. If X denotes the number of sixes, find the expectation of X.

5.3 Two numbers are selected at random (without replacement) from the first six positive integers. Let X denote the larger of the two numbers obtained. Find E(X).

5.4 Let X denote the sum of the numbers obtained when two fair dice are rolled. Find the variance and standard deviation of X.

5.5 A class has 15 students whose ages are 14, 17, 15, 14, 21, 17, 19, 20, 16, 18, 20, 17, 16, 19 and 20 years. One student is selected in such a manner that each has the same chance of being chosen and the age X of the selected student is recorded. What is the probability distribution of the random variable X? Find mean, variance and standard deviation of X.

5.6 In a meeting, 70A member is selected at random and we take X = 0 if he opposed, and X = 1 if he is in favour. Find E(X) and Var (X).

5.7 In a dice game, a player pays a stake of Re 1 for each throw of a die. She receives Rs 5 if the die shows a 3, Rs 2 if the die shows a 1 or 6, and nothing otherwise. What is the player's expected profit per throw over a long series of throws?.

Solution: Let the random variable X denote the net profit on the roll of a die.

RV	**Values**	**Description**
X	-1	If the number 2 or 4 or 5 are rolled
	1	If the number 1 or 6 are rolled
	4	If the number 3 is rolled

Table 5.1: Net gain

Thus, the probability distribution function of X is:

$$p_X(k) = \begin{cases} \frac{3}{6} & \text{if } k = -1 \\ \frac{2}{6} & \text{if } k = 1 \\ \frac{1}{6} & \text{if } k = 4 \end{cases} \tag{5.1}$$

The expected profit per roll over a long series of throws, in Rupees, is

$$E(X) = \sum_k k p_X(k) \tag{5.2}$$

$$= \frac{3}{6}(-1) + \frac{2}{6}(1) + \frac{1}{6}(4) \tag{5.3}$$

$$= 0.5 \tag{5.4}$$

5.8 A die is thrown three times. Let X be 'the number of twos seen'. Find the expectation of X.

Solution: Let the random variables be:

RV	Values	Description
X	$\{0, 1, 2, 3\}$	The number of twos rolled in three dice rolls
X_1	$\{0, 1\}$	The number of twos rolled on first die
X_2	$\{0, 1\}$	The number of twos rolled on second die
X_3	$\{0, 1\}$	The number of twos rolled on third die

Table 5.2: Random Variables

For a single die roll, since the probability of rolling a two is $\frac{1}{6}$ the probability distribution function of X_i is:

$$p_{X_i}(k) = \begin{cases} \frac{1}{6} & \text{if } k = 1 \\ \frac{5}{6} & \text{if } k = 0 \end{cases} \tag{5.5}$$

Thus,

$$\begin{aligned} E(X_1) = E(X_2) = E(X_3) &= \sum_{k=0}^{1} k p_{X_i}(k) && (5.6) \\ &= \frac{5}{6}(0) + \frac{1}{6}(1) && (5.7) \\ &= \frac{1}{6} && (5.8) \end{aligned}$$

But, as all three dice rolls are independent, and expectation is linear:

$$\begin{aligned} X &= X_1 + X_2 + X_3 && (5.9) \\ \therefore E(X) &= E(X_1 + X_2 + X_3) && (5.10) \\ &= E(X_1) + E(X_2) + E(X_3) && (5.11) \\ &= \frac{1}{2} && (5.12) \end{aligned}$$

Choose the correct answer in each of the following:

5.9 The mean of the numbers obtained on throwing a die having written 1 on three faces, 2 on two faces and 5 on one face is

(a) 1

(b) 2

(c) 5

(d) $\frac{8}{3}$

5.10 Suppose that two cards are drawn at random from a deck of cards. Let X be the number of aces obtained. Then the value of E(X) is

(a) $\frac{37}{221}$

(b) $\frac{5}{13}$

(c) $\frac{1}{13}$

(d) $\frac{2}{13}$

5.11 Suppose 10000 tickets are sold in a lottery each for Re. 1. First prize is of Rs 3000 and the second prize is of Rs 2000. There are three third prizes of Rs. 500 each. If you buy one ticket, what is your expectation. **Solution:** Let X be a random variable such that it denotes winning amount

RV	Value	Description
X	0	Winning no amount
	500	Winning Rs 500
	2000	Winning Rs 2000
	3000	Winning Rs 3000

Table 5.3: Random variable declaration.

$$p_X(k) = \begin{cases} \frac{9995}{10000} & k = 0 \\ \frac{3}{10000} & k = 500 \\ \frac{1}{10000} & k = 2000 \\ \frac{1}{10000} & k = 3000 \end{cases} \tag{5.13}$$

Expectation is defined as:

$$E(X) = \sum k p_X(k) \tag{5.14}$$

$$= 0p_X(0) + 500p_X(500) + 2000p_X(2000) + 3000p_X(3000) \tag{5.15}$$

$$= 0 + \frac{3}{20} + \frac{1}{5} + \frac{3}{10} \tag{5.16}$$

$$= 0.65 \tag{5.17}$$

5.12 Consider the probability distribution of a random variable X:

X	0	1	2	3	4
P(X)	0.1	0.25	0.3	0.2	0.15

Calculate

(i) $\text{var}(X)$

(ii) $\text{var}\left(\frac{X}{2}\right)$

Solution:

(i) var(X)

$$= E\left[X - E\left(X\right)\right]^2 \tag{5.18}$$

$$= E\left[X^2 + \left[E\left(X\right)\right]^2 - 2XE\left(X\right)\right] \tag{5.19}$$

$$= E\left(X^2\right) + \left[E\left(X\right)\right]^2 - 2\left[E\left(X\right)\right]^2 \tag{5.20}$$

$$= E\left(X^2\right) - \left[E\left(X\right)\right]^2 \tag{5.21}$$

where

$$E\left(X\right) = \sum_{k=0}^{4} k p_X\left(k\right) \tag{5.22}$$

$$= 2.05 \tag{5.23}$$

and

$$E\left(X^2\right) = \sum_{k=0}^{4} k^2 p_X\left(k\right) \tag{5.24}$$

$$= 5.65 \tag{5.25}$$

Then

$$\operatorname{var}\left(X\right) = 5.65 - \left(2.05\right)^2 \tag{5.26}$$

$$= 1.4475 \tag{5.27}$$

(ii) $\operatorname{var}\left(\frac{X}{2}\right)$

$$= E\left(\frac{X^2}{4}\right) - \left[E\left(\frac{X}{2}\right)\right]^2 \tag{5.28}$$

$$= \frac{1}{4}\left[E\left(X^2\right) - \left[E\left(X\right)\right]^2\right] \tag{5.29}$$

$$= \frac{\operatorname{var}\left(X\right)}{4} \tag{5.30}$$

$$= \frac{1.4475}{4} \tag{5.31}$$

$$= 0.361875 \tag{5.32}$$

5.13 The random variable X can take only the values $0, 1, 2$. Given that $\Pr\left(X = 0\right) = \Pr\left(X = 1\right) = p$ and that $E\left(X^2\right) = E\left(X\right)$, find the value of p.

Solution: Given that X is a random variable such that

$$X = \{0, 1, 2\} \tag{5.33}$$

$$\Pr\left(X = k\right) = p_X(k) \tag{5.34}$$

$$p_X(0) = p_X(1) = p \tag{5.35}$$

Then,

$$p_X(0) + p_X(1) + p_X(2) = 1 \tag{5.36}$$

$$\implies p + p + p_X(2) = 1 \tag{5.37}$$

$$\implies p_X(2) = 1 - 2p \tag{5.38}$$

Expectation is defined as:

$$E(X) = \sum_{k=0}^{2} kp_X(k) \tag{5.39}$$

$$= 0p_X(0) + 1p_X(1) + 2p_X(2) \tag{5.40}$$

$$= 2 - 3p \tag{5.41}$$

And

$$E\left(X^2\right) = \sum_{k=0}^{2} k^2 p_X(k) \tag{5.42}$$

$$= 0p_X(0) + 1p_X(1) + 4p_X(2) \tag{5.43}$$

$$= 4 - 7p \tag{5.44}$$

Given,

$$E(X) = E\left(X^2\right) \tag{5.45}$$

using (5.41) and (5.44)

$$\implies 2 - 3p = 4 - 7p \tag{5.46}$$

$$\implies p = \frac{1}{2} \tag{5.47}$$

5.14 Find the variance of distribution.

X	0	1	2	3	4	5
$P(X)$	$\frac{1}{6}$	$\frac{5}{18}$	$\frac{2}{9}$	$\frac{1}{6}$	$\frac{1}{9}$	$\frac{1}{18}$

Solution: Calculating $E(X)$.

$$E(X) = \sum_{k=0}^{5} k p_X(k) \tag{5.48}$$
$$= 0\left(\frac{1}{6}\right) + 1\left(\frac{5}{18}\right) + 2\left(\frac{2}{9}\right) + 3\left(\frac{1}{6}\right) + 4\left(\frac{1}{9}\right) + 5\left(\frac{1}{18}\right) \tag{5.49}$$
$$= \frac{35}{18} \tag{5.50}$$

Calculating $E(X^2)$

$$E(X^2) = \sum_{k=0}^{5} k^2 p_X(k) \tag{5.51}$$
$$= 0^2\left(\frac{1}{6}\right) + 1^2\left(\frac{5}{18}\right) + 2^2\left(\frac{2}{9}\right) + 3^2\left(\frac{1}{6}\right) + 4^2\left(\frac{1}{9}\right) + 5^2\left(\frac{1}{18}\right) \tag{5.52}$$
$$= \frac{105}{18} \tag{5.53}$$

From (5.50) and (5.53).

$$\sigma^2 = E(X^2) - [E(X)]^2 \tag{5.54}$$
$$= \frac{105}{18} - \left(\frac{35}{18}\right)^2 \tag{5.55}$$
$$= \frac{665}{324} \tag{5.56}$$

5.15 Two cards are drawn successively without replacement from a well shuffled deck of cards. Find the mean and standard variation of random variable X where X is the number of aces. **Solution:** Let X be a random variable such that

RV	Value	Description
X	0	Drawing no ace
	1	Drawing only 1 ace
	2	Drawing both aces

Table 5.4: Random variable declaration

$$
\begin{aligned}
p_X(0) &= \frac{48}{52} \times \frac{47}{51} && (5.57)\\
&= \frac{188}{221} && (5.58)\\
p_X(1) &= \frac{4}{52} \times \frac{48}{51} + \frac{48}{52} \times \frac{4}{51} && (5.59)\\
&= \frac{32}{221} && (5.60)\\
p_X(2) &= \frac{4}{52} \times \frac{3}{51} && (5.61)\\
&= \frac{1}{221} && (5.62)
\end{aligned}
$$

Now,

$$
\begin{aligned}
E\left(X\right) &= \sum_{k=0}^{2} k p_X(k) && (5.63)\\
&= 0p_X(0) + 1p_X(1) + 2p_X(2) && (5.64)\\
&= \frac{34}{221} && (5.65)\\
&= \frac{2}{13} && (5.66)
\end{aligned}
$$

And

$$E\left(X^{2}\right)=\sum_{k=0}^{2}k^{2}p_X(k) \tag{5.67}$$

$$=0p_X(0)+1p_X(1)+4p_X(2) \tag{5.68}$$

$$=\frac{36}{221} \tag{5.69}$$

Now,

$$Var\left(X\right)=E\left(X^{2}\right)-\left(E\left(X\right)\right)^{2} \tag{5.70}$$

Using (5.66) and (5.69)

$$Var\left(X\right)=\frac{36}{221}-\left(\frac{2}{13}\right)^{2} \tag{5.71}$$

$$=\frac{400}{2873} \tag{5.72}$$

$$\implies\sqrt{Var\left(X\right)}=\sqrt{\left(\frac{400}{2873}\right)} \tag{5.73}$$

$$\approx 0.373 \tag{5.74}$$

5.16 The probability distribution of a discrete random variable X is given as under:

X	1	2	4	2A	3A	5A
$\Pr\left(X\right)$	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{3}{25}$	$\frac{1}{10}$	$\frac{1}{25}$	$\frac{1}{25}$

Calculate:

(a) The value of A if E(X) = 2.94

(b) Variance of X.

(a) Since,

$$E(X) = \sum k p_X(k) \tag{5.75}$$

$$2.94 = \frac{1}{2} + \frac{2}{5} + \frac{12}{25} + \frac{2A}{10} + \frac{3A}{25} + \frac{5A}{25} \tag{5.76}$$

$$\implies A = \frac{78}{26} = 3 \tag{5.77}$$

(b) We know that,

$$Var(X) = E\left(k^2\right) - [E(X)]^2 \tag{5.78}$$

$$= \sum k^2 p_X(k) - [E(X)]^2 \tag{5.79}$$

$$= \frac{1}{2} + \frac{4}{5} + \frac{48}{25} + \frac{4A^2}{10} + \frac{9A^2}{25} + \frac{25A^2}{25} - [2.94]^2 \tag{5.80}$$

$$= \frac{161 + 88A^2}{50} - [2.94]^2 \tag{5.81}$$

$$= \frac{953}{50} - [2.94]^2 \tag{5.82}$$

$$= 10.4164 \tag{5.83}$$

5.17 The probability distribution of a random variable X is given as under:

$$p_X(x) = \begin{cases} kx^2 & \text{for } x = 1, 2, 3 \\ 2kx & \text{for } x = 4, 5, 6 \\ 0 & \text{otherwise} \end{cases}$$

where k is a constant. Calculate:

(a) $E(X)$

(b) $E\left(3X^2\right)$

(c) $\Pr(X \geq 4)$

Solution: From the axiom of total probability,

$$\sum_{i=1}^{6} p_X(i) = 1 \tag{5.84}$$

$$\implies \sum_{i=1}^{3} ki^2 + \sum_{i=4}^{6} 2ki = 1 \tag{5.85}$$

$$\implies k + 4k + 9k + 8k + 10k + 12k = 1 \tag{5.86}$$

$$\implies k = \frac{1}{44} \tag{5.87}$$

Thus, the probability distribution of X is

$$p_X(x) = \begin{cases} \frac{x^2}{44} & \text{for } x = 1, 2, 3 \\ \frac{2x}{44} & \text{for } x = 4, 5, 6 \\ 0 & \text{otherwise} \end{cases}$$

(a) Calculating $E(X)$:

$$E(X) = \sum_{i=1}^{6} i p_X(i) \tag{5.88}$$

$$= 1\left(\frac{1}{44}\right) + 2\left(\frac{4}{44}\right) + 3\left(\frac{9}{44}\right) + 4\left(\frac{8}{44}\right) + 5\left(\frac{10}{44}\right) + 6\left(\frac{12}{44}\right) \tag{5.89}$$

$$= \frac{95}{22} \tag{5.90}$$

$$= 4.32 \tag{5.91}$$

(b) Calculating $E\left(3X^2\right)$:

$$
\begin{aligned}
E\left(3X^2\right) &= 3E\left(X^2\right) && (5.92)\\
&= 3\sum_{i=1}^{6} i^2 p_X(i) && (5.93)\\
&= 3\left(1\left(\frac{1}{44}\right)+4\left(\frac{4}{44}\right)+9\left(\frac{9}{44}\right)+16\left(\frac{8}{44}\right)+25\left(\frac{10}{44}\right)+36\left(\frac{12}{44}\right)\right) && (5.94)\\
&= \frac{2724}{44} && (5.95)\\
&= 61.91 && (5.96)
\end{aligned}
$$

(c) Firstly, calculating the CDF:

$$
\begin{aligned}
F_X(x) &= \sum_{i=1}^{x} p_X(i) && (5.97)\\
&= \begin{cases} \sum_{i=1}^{x} \frac{i^2}{44} & \text{if } x \leq 3 \\ \sum_{i=1}^{3} \frac{i^2}{44} + \sum_{i=4}^{x} \frac{2i}{44} & \text{if } x \geq 4 \end{cases} && (5.98)\\
&= \begin{cases} \frac{x(x+1)(2x+1)}{6\times 44} & \text{if } x \leq 3 \\ \frac{14}{44} + \frac{x(x+1)}{44} - \frac{3\times 4}{44} & \text{if } x \geq 4 \end{cases} && (5.99)\\
&= \begin{cases} \frac{x(x+1)(2x+1)}{264} & \text{if } x \leq 3 \\ \frac{x(x+1)+2}{44} & \text{if } x \geq 4 \end{cases} && (5.100)
\end{aligned}
$$

Calculating $\Pr(X \geq 4)$:

$$\Pr(X \geq 4) = 1 - \Pr(X \leq 3) \tag{5.101}$$
$$= 1 - F_X(3) \tag{5.102}$$
$$= 1 - \frac{3 \times 4 \times 7}{264} \tag{5.103}$$
$$= \frac{15}{22} \tag{5.104}$$
$$= 0.68 \tag{5.105}$$

5.18 Two probability distributions of the discrete random variable X and Y are given below.

Table 5.5: Table-1

X	0	1	2	3
$P(X)$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

Table 5.6: Table-2

Y	0	1	2	3
$P(Y)$	$\frac{1}{5}$	$\frac{3}{10}$	$\frac{2}{5}$	$\frac{1}{10}$

Prove that $E\left(Y^2\right) = 2E(X)$

Solution:

The probability distribution function of X is:

$$p_Y(k) = \begin{cases} \frac{1}{5} & k = 0 \\ \\ \frac{3}{10} & k = 1 \\ \\ \frac{2}{5} & k = 2 \\ \\ \frac{1}{10} & k = 3 \end{cases} \tag{5.106}$$

$$\begin{aligned} E\left(Y^2\right) &= \sum_k (k)^2 \times p_Y(k) & (5.107) \\ &= 0 \times \frac{1}{5} + 1^2 \times \frac{3}{10} + (2)^2 \times \frac{2}{5} + (3)^2 \times \frac{1}{10} & (5.108) \\ &= \frac{14}{5} & (5.109) \end{aligned}$$

The probability distribution function of Y is:

$$p_Y(k) = \begin{cases} \frac{1}{5} & k = 0 \\ \\ \frac{2}{5} & k = 1 \\ \\ \frac{1}{5} & k = 2 \\ \\ \frac{1}{5} & k = 3 \end{cases} \tag{5.110}$$

$$\begin{aligned} E(X) &= \sum_k k \times p_X(k) & (5.111) \\ &= 0 \times \frac{1}{5} + 1 \times \frac{2}{5} + 2 \times \frac{1}{5} + 3 \times \frac{1}{5} & (5.112) \\ &= \frac{7}{5} & (5.113) \end{aligned}$$

From above;

$$\frac{14}{5} = 2 \times \frac{7}{5} \tag{5.114}$$

$$\therefore E\left(Y^2\right) = 2E\left(X\right) \tag{5.115}$$

$$\text{Hence proved} \tag{5.116}$$

5.19 For the following probability distribution determine standard deviation of the random variable X.

X	2	3	4
$P(X)$	0.2	0.5	0.3

Solution: Given, X be the random variable and $p_X(k)$ is the probability distribution. Variance of X is given by

$$\sigma_X^2 = E\left[X - E\left(X\right)\right]^2 \tag{5.117}$$

$$= E\left(X^2\right) - \left[E\left(X\right)\right]^2 \tag{5.118}$$

Now,

$$E\left(X^2\right) = \sum_{k=2}^{4} k^2 p_X(k) \tag{5.119}$$

$$= 10.1 \tag{5.120}$$

Similarly,

$$[E(X)]^2 = \left[\sum_{k=2}^{4} k p_X(k)\right]^2 \tag{5.121}$$

$$= 9.61 \tag{5.122}$$

Now putting the values in (5.118)

$$\sigma_X^2 = 10.1 - 9.61 \tag{5.123}$$

$$\implies \sigma_X = 0.7 \tag{5.124}$$

$\therefore$ Standard deviation is 0.7

5.20 Let X be a discrete random variable whose probability distribution is defined as follows:

$$\Pr(X = x) = \begin{cases} k(x+1) & , x = 1, 2, 3, 4 \\ 2kx & , x = 5, 6, 7 \\ 0 & , otherwise \end{cases}$$

where k is a constant. Calculate

(i) the value of k

(ii) E(X)

(iii) Standard deviation of X

Solution:

(i) the value of k:

Using the third axiom of probability,

$$\sum_{i=1}^{n} p_X(i) = 1 \tag{5.125}$$

We get,

$$2k + 3k + 4k + 5k + 10k + 12k + 14k = 1 \tag{5.126}$$

$$50k = 1 \tag{5.127}$$

$$k = 0.02 \tag{5.128}$$

(ii) E(X):

$$E\left(X\right) = \sum_{i=1}^{n} x_i p_X(i) \tag{5.129}$$

$$E\left(X\right) = 2k + 6k + 12k + 20k + 50k + 72k + 98k \tag{5.130}$$

$$= 260k \tag{5.131}$$

$$= 260 \times 0.02 \tag{5.132}$$

$$= 5.2 \tag{5.133}$$

(iii) Standard deviation of X:

$$Var\,(X) = E(X^2) - [E(X)]^2 \tag{5.134}$$

$$= \sum_{i=1}^{n} x_i^2 p_X(i) - \left[\sum_{i=1}^{n} x_i p_X(i)\right]^2 \tag{5.135}$$

$$= 1498k - (5.2)^2 \tag{5.136}$$

$$= 1498 \times 0.02 - 27.04 \tag{5.137}$$

$$= 2.92 \tag{5.138}$$

Standard deviation of X is

$$X = \sqrt{Var\,(X)} \tag{5.139}$$

$$= \sqrt{2.92} \tag{5.140}$$

$$= 1.7 \tag{5.141}$$

5.21 The probability distribution of a discrete random variable X is given as under:

X	1	2	4	$2A$	$3A$	$5A$
p(X)	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{3}{25}$	$\frac{1}{10}$	$\frac{1}{25}$	$\frac{1}{25}$

Calculate:

a) The value of A if $E(X)$ =2.94

b) Variance of X.

Solution: We know,

$$E(X) = \sum_{k=X} k p_X(k) \tag{5.142}$$

$$\implies 2.94 = \frac{1}{2} + \frac{2}{5} + \frac{12}{25} + \frac{2A}{10} + \frac{3A}{25} + \frac{5A}{25} \tag{5.143}$$

$$\implies 2.94 = \frac{25 + 20 + 24 + 10A + 6A + 10A}{50} \tag{5.144}$$

$$\implies 2.94 = \frac{69 + 26A}{50} \tag{5.145}$$

$$\implies 147 = 69 + 26A \tag{5.146}$$

$$\implies A = 3 \tag{5.147}$$

Now , we know

$$Var(X) = E(X^2) - [E(X)]^2 \tag{5.148}$$

$$= \sum_{K=X} k^2 p_X(k) - [E(X)]^2 \tag{5.149}$$

$$= \frac{1}{2} + \frac{4}{5} + \frac{48}{25} + \frac{4A^2}{10} + \frac{9A^2}{25} + \frac{25A^2}{25} - [E(X)]^2 \tag{5.150}$$

$$= \frac{161 + 88A^2}{50} - [E(X)]^2 \tag{5.151}$$

$$\because A = 3 \;\; E(X) = 2.94 \tag{5.152}$$

$$= \frac{953}{50} - [2.94]^2 \tag{5.153}$$

$$= 19.06 - 8.6436 = 10.4164 \tag{5.154}$$

5.22 A discrete random variable X has the probability distribution given as below:

X	0.5	1	1.5	2
P(X)	k	k^2	$2k^2$	k

(a) Find the value of k

(b) Determine the mean of the distribution

Solution:

(a) We know,

$$\sum_i p_X(i) = 1 \tag{5.155}$$

$$k + k^2 + 2k^2 + k = 1 \tag{5.156}$$

$$\implies 3k^2 + 2k - 1 = 0 \tag{5.157}$$

$$\implies (3k - 1)(k + 1) = 0 \tag{5.158}$$

$$\implies k = \frac{1}{3} \text{ or } k = -1 \tag{5.159}$$

$$\text{As probability cannot be negative,} \tag{5.160}$$

$$k = \frac{1}{3} \tag{5.161}$$

(b)

$$\text{Mean of the distribution}(\mu) = E(X) = \sum_i i p_X(i) \tag{5.162}$$

$$= 0.5(k) + 1(k^2) + 1.5(2k^2) + 2k \tag{5.163}$$

$$= 4k^2 + 2.5k \tag{5.164}$$

$$= (4)\frac{1}{9} + (2.5)\frac{1}{3} \tag{5.165}$$

$$= \frac{23}{18} \tag{5.166}$$

5.23 There are 5 cards numbered 1 to 5, one number on one card. Two cards are drawn at random without replacement. Let X denote the sum of the numbers on two cards drawn. Find the mean and variance of X. **Solution:**

parameters	description
A	number on the first card
B	number on the second card

$$p_A(i) = \begin{cases} \frac{1}{5}, & 1 \leq i \leq 5 \\ 0, & \text{otherwise} \end{cases} \tag{5.167}$$

$$p_{AB}(i,k) = \Pr\left(B = k \mid A = i\right) p_A(i) \tag{5.168}$$

$$= \begin{cases} \frac{1}{20}, & 1 \leq k \leq 5, k \neq i \\ 0, & \text{otherwise} \end{cases} \tag{5.169}$$

$$p_B(k) = \sum_{i=-\infty}^{\infty} p_{AB}(i,k) \tag{5.170}$$

$$= \sum_{i=1}^{5} p_{AB}(i,k) \tag{5.171}$$

Using the result from (5.169)

$$\implies p_B(k) = \begin{cases} \frac{1}{5}, & 1 \leq k \leq 5 \\ 0, & \text{otherwise} \end{cases} \tag{5.172}$$

Since $p_A(k) = p_B(k)$, A and B are identical.

$$X = A + B \tag{5.173}$$

$$\mathbb{E}[X] = \mathbb{E}[A + B] \tag{5.174}$$

$$= \mathbb{E}[A] + \mathbb{E}[B] \tag{5.175}$$

$$= \mathbb{E}[A] + \mathbb{E}[A] \tag{5.176}$$

$$= 2\mathbb{E}[A] \tag{5.177}$$

$$= 2\sum_{i=1}^{5} i p_A(i) \tag{5.178}$$

$$= 6 \tag{5.179}$$

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 \tag{5.180}$$

$$= \mathbb{E}[(A + B)^2] - 6^2 \tag{5.181}$$

$$= \mathbb{E}[A^2 + B^2 + 2AB] - 36 \tag{5.182}$$

$$= \mathbb{E}[A^2] + \mathbb{E}[B^2] + 2\mathbb{E}[AB] - 36 \tag{5.183}$$

$$= \mathbb{E}[A^2] + \mathbb{E}[A^2] + 2\sum_{i=1}^{5}\sum_{k=1}^{5} ik p_{AB}(i, k) - 36 \tag{5.184}$$

Using the result from (5.169)

$$\implies \mathrm{Var}(X) = 2\sum_{i=1}^{5} i^2 p_A(i) + \frac{2}{20}(170) - 36 \tag{5.185}$$

$$= 3 \tag{5.186}$$

5.24 For the following probability distribution:

X	-4	-3	-2	-1	0
P(X)	0.1	0.2	0.3	0.2	0.2

$E(X)$ is equal to:

(a) 0

(b) -1

(c) -2

(d) -1.8

Solution: $E(X)$ is given by

$$E(X) = \sum_{k=-4}^{0} k p_X(k) \tag{5.187}$$

$$= -4p_X(-4) - 3p_X(-3) - 2p_X(-2) - p_X(-1) \tag{5.188}$$

$$= -4(0.1) - 3(0.2) - 2(0.3) - 1(0.2) \tag{5.189}$$

$$= -1.8 \tag{5.190}$$

5.25 If A, B and C are three independent events such that $\Pr(A) = \Pr(B) = \Pr(C) = p$, then P(At least two of A, B, C occur) $= 3p^2 - 2p^3$

Solution:

RV	Value	Description
X	0	none of the events occur
	1	only one of the events occur
	2	any two of the events occur
	3	all the three events occur

Table 5.10: Random variable declaration

$$
\begin{aligned}
\Pr\left(X=0\right) &= \Pr\left(A'B'C'\right) && (5.191)\\
&= \Pr\left(A'\right)\Pr\left(B'\right)\Pr\left(C'\right) && (5.192)\\
&= (1-p)^3 && (5.193)\\
\Pr\left(X=1\right) &= \Pr\left(AB'C'\right)+\Pr\left(BC'A'\right)+\Pr\left(CA'B'\right) && (5.194)\\
&= \Pr\left(A\right)\Pr\left(B'\right)\Pr\left(C'\right)+\Pr\left(B\right)\Pr\left(C'\right)\Pr\left(A'\right)\\
&\quad+\Pr\left(C\right)\Pr\left(A'\right)\Pr\left(B'\right) && (5.195)\\
&= 3p(1-p)^2 && (5.196)\\
\Pr\left(X=2\right) &= \Pr\left(ABC'\right)+\Pr\left(BCA'\right)+\Pr\left(CAB'\right) && (5.197)\\
&= \Pr\left(A\right)\Pr\left(B\right)\Pr\left(C'\right)+\Pr\left(B\right)\Pr\left(C\right)\Pr\left(A'\right)\\
&\quad+\Pr\left(C\right)\Pr\left(A\right)\Pr\left(B'\right) && (5.198)\\
&= 3p^2(1-p) && (5.199)\\
\Pr\left(X=3\right) &= \Pr\left(ABC\right) && (5.200)\\
&= \Pr\left(A\right)\Pr\left(B\right)\Pr\left(C\right) && (5.201)\\
&= p^3 && (5.202)
\end{aligned}
$$

$$
\begin{aligned}
\Pr\left(X\geq 2\right) &= \Pr\left(X=2\right)+\Pr\left(X=3\right) && (5.203)\\
&= 3p^2(1-p)+p^3 && (5.204)\\
&= 3p^2-2p^3 && (5.205)
\end{aligned}
$$

5.26 For the following probability distribution

E(X^2)is equal to

(A)3 (B)5 (C)7 (D)10

X	1	2	3	4
P(X)	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{3}{10}$	$\frac{2}{5}$

Table 5.11: Probability Distribution

Solution: As we know

$$E\left(X^2\right) = \sum_{k=1}^{4} k^2 p_X(k) \tag{5.206}$$

$$= (1)^2 \times \frac{1}{10} + (2)^2 \times \frac{1}{5} + (3)^2 \times \frac{3}{10} + (4)^2 \times \frac{2}{5} \tag{5.207}$$

$$= \frac{1}{10} + \frac{4}{5} + \frac{27}{10} + \frac{32}{5} \tag{5.208}$$

$$= 10 \tag{5.209}$$

$$\therefore E\left(X^2\right) = 10 \tag{5.210}$$

5.27 A die is tossed twice. A ‘success’ is getting an even number on a toss. Find the variance of the number of successes.

Solution:

Parameter	**Value**	**Description**
X_i	0,1	0-Not a success, 1-Success and it represents outcome of i^{th} throw
X	0,1,2	X = $X_1 + X_2$, denoting number of outcomes in two throws

pmf of X_i is

$$p_{X_i}(k) = \begin{cases} \frac{1}{2}, & k = 0 \\ \frac{1}{2}, & k = 1 \end{cases} \quad \forall \quad 1 \le i \le 2 \tag{5.211}$$

Mean value of X_i is

$$\mu_{X_i} = E[X_i], \quad i = 0, 1 \tag{5.212}$$
$$= \frac{1}{2} \tag{5.213}$$

Variance of X_i is

$$\sigma^2_{X_i} = E[(X_i - \mu_{X_i})^2], \quad i = 0, 1 \tag{5.214}$$
$$= \frac{1}{4} \tag{5.215}$$

Variance of getting successes in two throws of a die is

$$\sigma^2_X = \sigma^2_{X_1} + \sigma^2_{X_2} \tag{5.216}$$
$$= \frac{1}{2} \tag{5.217}$$

5.28 A biased die is such that $\Pr(4) = \frac{1}{10}$ and other scores being equally likely. The die is tossed twice. If X is the 'number of fours seen', find the variance of the random variable X.

Solution: Since, X = number of fours seen on tossing a die twice, X=$\{0, 1, 2\}$

Number of trials, $n = 2$

Probability of seeing 4, $p = \frac{1}{10}$

Parameter	**Value**	**Description**
X	$\{0, 1, 2\}$	Random Variable
n	2	Number of trials
p	$\frac{1}{10}$	Probability of success

Table 5.12: Random Variables

$$Var(X) = np(1-p) \tag{5.218}$$
$$= (2)\left(\frac{1}{10}\right)\left(1-\frac{1}{10}\right) \tag{5.219}$$
$$= (2)\frac{1}{10}\frac{9}{10} \tag{5.220}$$
$$= \frac{18}{100} \tag{5.221}$$

$\therefore Var(X) = 0.18$

5.29 One million random numbers are generated from a statistically stationary process with a Gaussian distribution with mean zero and standard deviation σ_0.

The σ_0 is estimated by randomly drawing out 10,000 number of samples (x_n). The estimates $\hat{\sigma}_1^2$ and $\hat{\sigma}_2^2$ are computed in the following two ways:

$$\hat{\sigma}_1^2 = \frac{1}{10000}\sum_{n=1}^{10000} x_n^2$$

$$\hat{\sigma}_2^2 = \frac{1}{9999}\sum_{n=1}^{10000} x_n^2$$

Which of the following statements is true?

(a) $E(\hat{\sigma}_2^2) = \sigma_0^2$

(b) $E(\hat{\sigma}_2) = \sigma_0$

(c) $E(\hat{\sigma}_1^2) = \sigma_0^2$

(d) $E(\hat{\sigma}_1) = E(\hat{\sigma}_2)$

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Solution:

$$\hat{\sigma}_1^2 = \frac{1}{10000}\sum_{n=1}^{10000} x_n^2 \tag{5.222}$$

$$\hat{\sigma}_2^2 = \frac{1}{9999}\sum_{n=1}^{10000} x_n^2 \tag{5.223}$$

We know,

$$\mathrm{Var}(x_n) = E(x_n^2) - [E(x_n)]^2 \tag{5.224}$$

$$= E(x_n^2) - 0 \tag{5.225}$$

$$= E(x_n^2) \tag{5.226}$$

$$= \sigma_0^2 \tag{5.227}$$

$$\implies E(x_n^2) = \sigma_0^2 \tag{5.228}$$

$$\mathrm{E}[\hat{\sigma}_1^2] = \frac{1}{10000}\sum_{n=1}^{10000} \mathrm{E}[x_n^2] \tag{5.229}$$

$$\mathrm{E}[\hat{\sigma}_2^2] = \frac{1}{9999}\sum_{n=1}^{10000} \mathrm{E}[x_n^2] \tag{5.230}$$

Using Equation (5.228), we get:

$$\mathrm{E}[\hat{\sigma}_1^2] = \sigma_0^2 \tag{5.231}$$

$$\mathrm{E}[\hat{\sigma}_2^2] = \frac{10000}{9999}\sigma_0^2 \tag{5.232}$$

Hence, the correct answer is (5.29c)

5.30 If X is the number of tails in three tosses of coin, determine the standard deviation of

X.

Solution: Let number of tails obtained be defined by random variable

$$X = 0, 1, 2, 3 \tag{5.233}$$

$$n = \text{Number of trails} \tag{5.234}$$

$$p = \text{Probability of getting tails in a toss} \tag{5.235}$$

$$= \frac{1}{2} \tag{5.236}$$

$$q = \text{Probability of not getting tails} \tag{5.237}$$

$$= (1 - p) \tag{5.238}$$

$$= \frac{1}{2} \tag{5.239}$$

$$\sigma_{X^2}^2 = E\left(X - E\left(X\right)\right)^2 \tag{5.240}$$

$$= E\left(X^2 - 2XE\left(X\right) + E\left(X\right)^2\right) \tag{5.241}$$

$$= E\left(X^2\right) - 2E\left(X\right).E\left(X\right) + E\left(X\right)^2 \tag{5.242}$$

$$= E\left(X^2\right) - E\left(X\right)^2 \tag{5.243}$$

$$E[X^n] = \frac{d^n M\left(z^{-1}\right)}{dz^n} \tag{5.244}$$

$$M\left(z^{-1}\right) = \sum_{-\infty}^{\infty} P_X\left(k\right) z^k \tag{5.245}$$

$$E[X] = \frac{d\left(q + pz\right)^n}{dz}|_{z=1} \tag{5.246}$$

$$= np\left(q + pz\right)^{n-1}|_{z=1} \tag{5.247}$$

$$= np\left(p + q\right)^{n-1} \tag{5.248}$$

$$= np \tag{5.249}$$

$$E[X^2] = \frac{d^2 M\left(z^{-1}\right)}{dz^2}|_{z=1} \tag{5.250}$$

$$\frac{d\left(M_X\left(z^{-1}\right)\right)}{dz} = \sum_{-\infty}^{\infty} kP_X\left(k\right) z^{k-1} \tag{5.251}$$

$$\frac{d\left(q + pz\right)^n}{dz} = \sum_{-\infty}^{\infty} kP_X\left(k\right) z^{k-1} \tag{5.252}$$

Multiplying z on both sides, we get

$$znp\left(q+pz\right)^{n-1}=\sum_{-\infty}^{\infty}kP_{X}\left(k\right)z^{k} \tag{5.255}$$

Differentiating both sides,

$$zn\left(n-1\right)p^{2}\left(q+pz\right)^{n-2}+np\left(q+pz\right)^{n-1}=\sum_{-\infty}^{\infty}k^{2}P_{X}\left(k\right)z^{k-1} \tag{5.256}$$

$$n^{2}p^{2}-np^{2}+np=\sum_{-\infty}^{\infty}k^{2}P_{X}\left(k\right) \tag{5.257}$$

$$E[X^{2}]=n^{2}p^{2}-np^{2}+np \tag{5.258}$$

$$\sigma_{X^{2}}^{2}=n^{2}p^{2}-np^{2}+np-\left(np\right)^{2} \tag{5.259}$$

$$\sigma_{X^{2}}^{2}=np-np^{2} \tag{5.260}$$

$$=np\left(1-p\right) \tag{5.261}$$

$$=npq \tag{5.262}$$

$$\tag{5.263}$$

Standard deviation

$$=\sigma_{X^{2}} \tag{5.264}$$

$$=\sqrt{npq} \tag{5.265}$$

$$=\sqrt{\frac{3}{4}} \tag{5.266}$$

5.31 In a game, a man wins a rupee for a six and loses a rupee for any other number when a fair die is thrown. The man decided to throw a die thrice but to quit as and when he gets a six. Find the expected value of the amount he wins / loses.

Solution: Random variables defined as

Random Variable	Values	Description
X	$\{0, 1, 2, 3\}$	6 appeared roll

k be roll on which 6 appeared.Let m_k be money recieved till 6 appeared.

$$p_X(k) = \begin{cases} \left(\frac{5}{6}\right)^{k-1} \frac{1}{6} & \text{if } k \in \{1, 2, 3\} \\ \left(\frac{5}{6}\right)^3 & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases} \tag{5.267}$$

$$m_k = \begin{cases} (-1)(k-1) + 1 & \text{if } k \in \{1, 2, 3\} \\ -3 & \text{if } k = 0 \end{cases} \tag{5.268}$$

Calculating the expected value

$$\text{Expected value} = \sum_{k=0}^{3} m_k p_X(k) \tag{5.269}$$

$$= \left(1 \times \frac{1}{6}\right) + \left(0 \times \frac{5}{36}\right) + \left(-1 \times \frac{25}{216}\right)$$
$$+ \left(-3 \times \frac{125}{216}\right) \tag{5.270}$$

$$= \frac{1}{6} - 0 + \left(-\frac{25}{216}\right) - \frac{375}{216} \tag{5.271}$$

$$= \frac{36}{216} - 0 + \left(-\frac{25}{216}\right) - \frac{375}{216} \tag{5.272}$$

$$= \frac{36 - 0 - 25 - 375}{216} \tag{5.273}$$

$$= -\frac{364}{216} \tag{5.274}$$

$$\approx -1.685 \tag{5.275}$$

Chapter 6

Random Algebra

6.1. Examples

6.1 Given that a fair coin is marked 1 on one face and 6 on the other and a fair die are tossed. Find the probability that the sum turns up to be 3 and 12.

Solution: Let the random variables X, Y denote the toss of a coin and roll of a dice respectively. Since,

$$M_X(z) = \frac{z^{-1}}{2}\left(1 + z^{-5}\right), \tag{6.1}$$

$$M_Y(z) = \frac{z^{-1}\left(1 - z^{-6}\right)}{6\left(1 - z^{-1}\right)}, \tag{6.2}$$

$$M_Z(z) = \left[\frac{z^{-1} + z^{-6}}{2}\right]\left[\frac{z^{-1}\left(1 - z^{-6}\right)}{6\left(1 - z^{-1}\right)}\right] \tag{6.3}$$

yielding

$$p_Z(n) = \begin{cases} \frac{1}{12} & 2 \leqslant n < 7, \\ \frac{1}{6} & n = 7, \\ \frac{1}{12} & 8 \leqslant n < 13 \end{cases} \tag{6.4}$$

See Fig. 6.1. Thus,

$$\Pr(Z = 3) = \frac{1}{12}, \Pr(Z = 12) = \frac{1}{12} \tag{6.5}$$

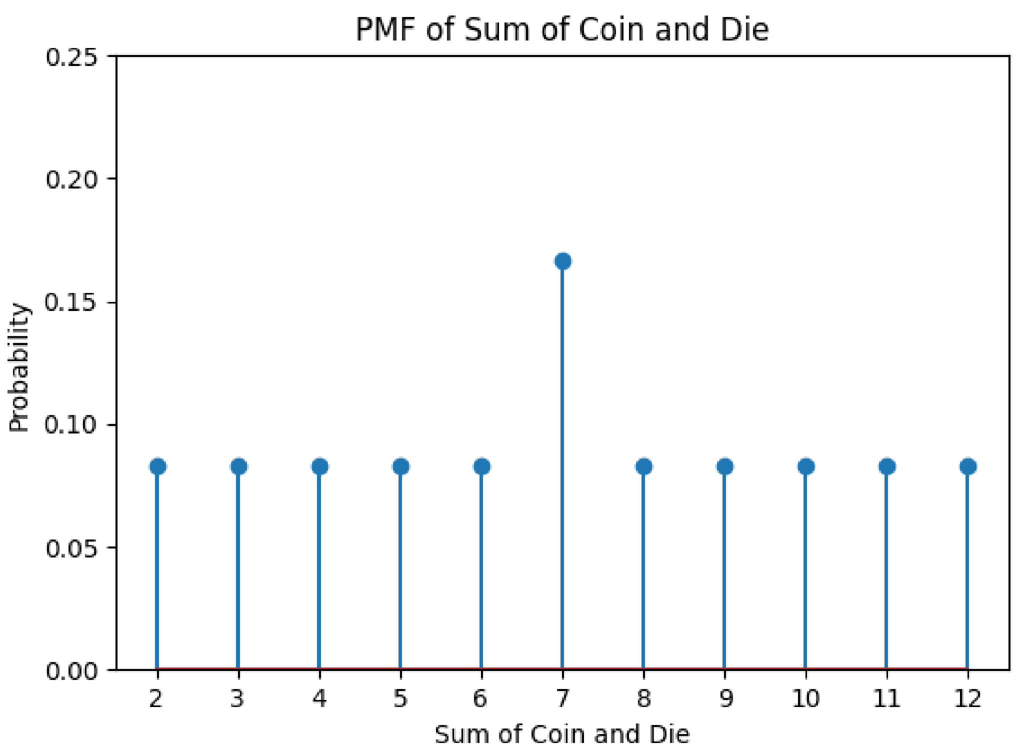


Figure 6.1: pmf of the sum when coin and die are rolled simultaneously

6.2 A fair coin is tossed four times, and a person win Re 1 for each head and lose Rs 1.50 for each tail that turns up. From the sample space calculate how many different amounts of money you can have after four tosses and the probability of having each of these amounts.

Solution: The input parameters are listed in Table 6.1. The amount of money the

Variable	Description	Value
n	Number of tosses	4
A	Amount gained/lost	A
p	Profit when it is heads	Re 1
q	Loss when it is tails	Rs 1.5
X	Number of heads in n tosses	X

Table 6.1: Given Information

person will have after n tosses is

$$Y = (X \times 1) - ((n - X) \times 1.50) = 2.5X - 1.5n \tag{6.6}$$

The pmf and CDF of X are

$$p_X(k) = {}^nC_k(0.5)^k(0.5)^{n-k} = {}^nC_k(0.5)^n \tag{6.7}$$

$$F_X(k) = \Pr(X \le k) = \sum_{i=0}^{i=k} {}^nC_i \left(\frac{1}{2}\right)^n \tag{6.8}$$

The CDF of Y is

$$F_Y(k) = \Pr(A \le k) \tag{6.9}$$

$$= \Pr(2.5X - 1.5n \le k) \tag{6.10}$$

$$= \Pr\left(X \le \frac{k + 1.5n}{2.5}\right) \tag{6.11}$$

$$= F_X\left(\frac{k + 1.5n}{2.5}\right) = \sum_{i=0}^{i=\lfloor \frac{k+1.5n}{2.5} \rfloor} {}^nC_i \left(\frac{1}{2}\right)^n \tag{6.12}$$

from (6.8). Consequently,

$$p_Y(k) = \begin{cases} {}^nC_{\frac{k+1.5n}{2.5}} \left(\frac{1}{2}\right)^n, & \frac{k+1.5n}{2.5} \in I \text{ and } 0 \leq \frac{k+1.5n}{2.5} \leq n \\ 0, & \text{otherwise} \end{cases} \tag{6.13}$$

See Figs. 6.2, 6.3 and 6.4. for the distribution of Y.

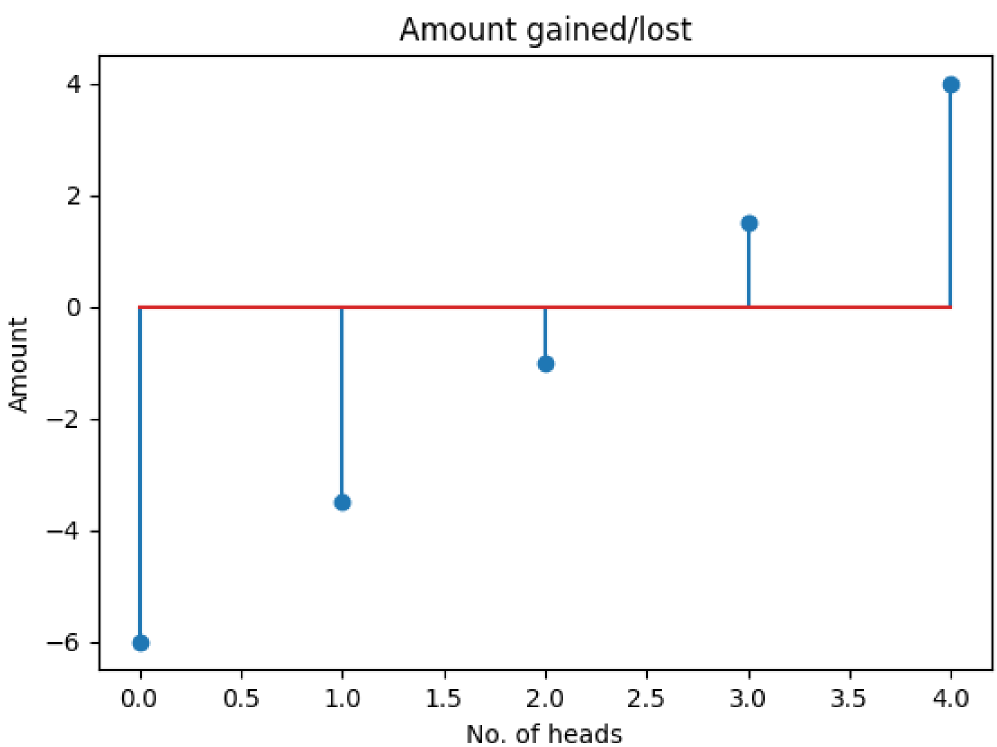


Figure 6.2: Plot of amount gained/lost

6.3 A black and a red dice are rolled.

(a) Find the conditional probability of obtaining a sum greater than 9, given that the black die resulted in a 5.

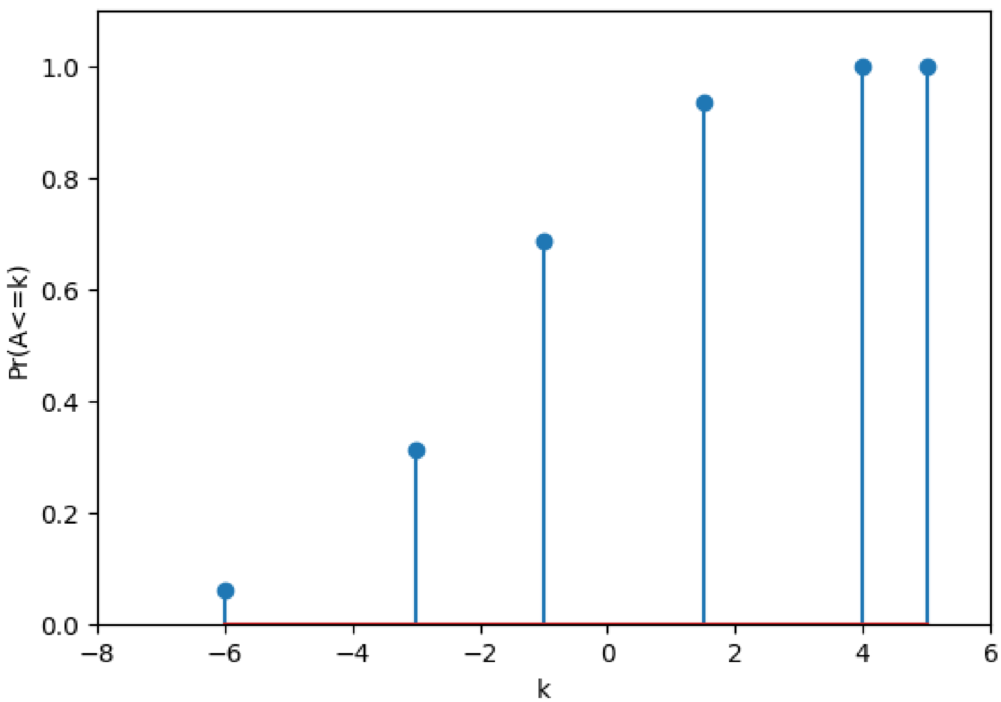


Figure 6.3: CDF of A

(b) Find the conditional probability of obtaining the sum 8, given that the red die resulted in a number less than 4.

Solution: Let X and Y be i.i.d, denoting the number which comes up on black and red die respectively.

$$F_X(i) = F_Y(i) = \begin{cases} 0 & i < 1 \\ \frac{i}{6} & 0 < i \leq 6 \\ 1 & i > 6 \end{cases} \tag{6.14}$$

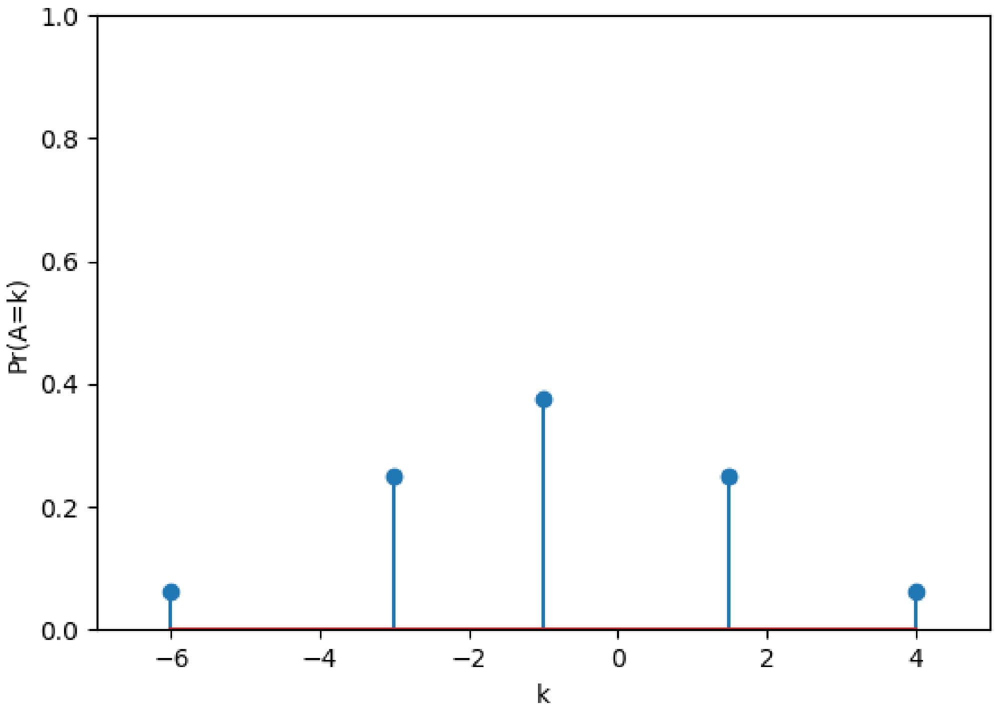


Figure 6.4: PMF of A

Since X and Y are independent random variables.

$$\Pr(X = k, Y = r) = \Pr(X = k)\Pr(Y = r) \tag{6.15}$$

$$\implies p_{X,Y}(k, r) = \frac{1}{36} \tag{6.16}$$

(a)

$$\Pr(X+Y>9|X=5) = \frac{\Pr(X+Y>9, X=5)}{\Pr(X=5)} \tag{6.17}$$

$$= \frac{\Pr(Y>4, X=5)}{\Pr(X=5)} = \frac{\Pr(6 \geq Y > 4)\Pr(X=5)}{\Pr(X=5)} \tag{6.18}$$

$$= 1 - F_Y(4) = \frac{1}{3} \tag{6.19}$$

(b) From Fig. 6.5,

$$X+Y=8, Y<4 = (2,6)+(5,3) \tag{6.20}$$

Thus,

$$\Pr(X+Y=8, Y<4) = p_{X,Y}(2,6) + p_{X,Y}(5,3) = \frac{2}{36} \tag{6.21}$$

$$\Longrightarrow \Pr(X+Y=8|Y<4) = \frac{\Pr(X+Y=8, Y<4)}{\Pr(Y<4)} \tag{6.22}$$

$$= \frac{\frac{2}{36}}{\frac{1}{2}} = \frac{1}{9} \tag{6.23}$$

6.4 Given that 2 numbers appearing on throwing two dice are different. Find the probability of the event ‘ the sum of numbers on the dice is 4’ .

Solution: See Tables 6.2 and 6.3.

Random Variable	Description
X	Number which comes up on Die1
Y	Number which comes up on Die2

Table 6.2: Random Variables for Die Rolls

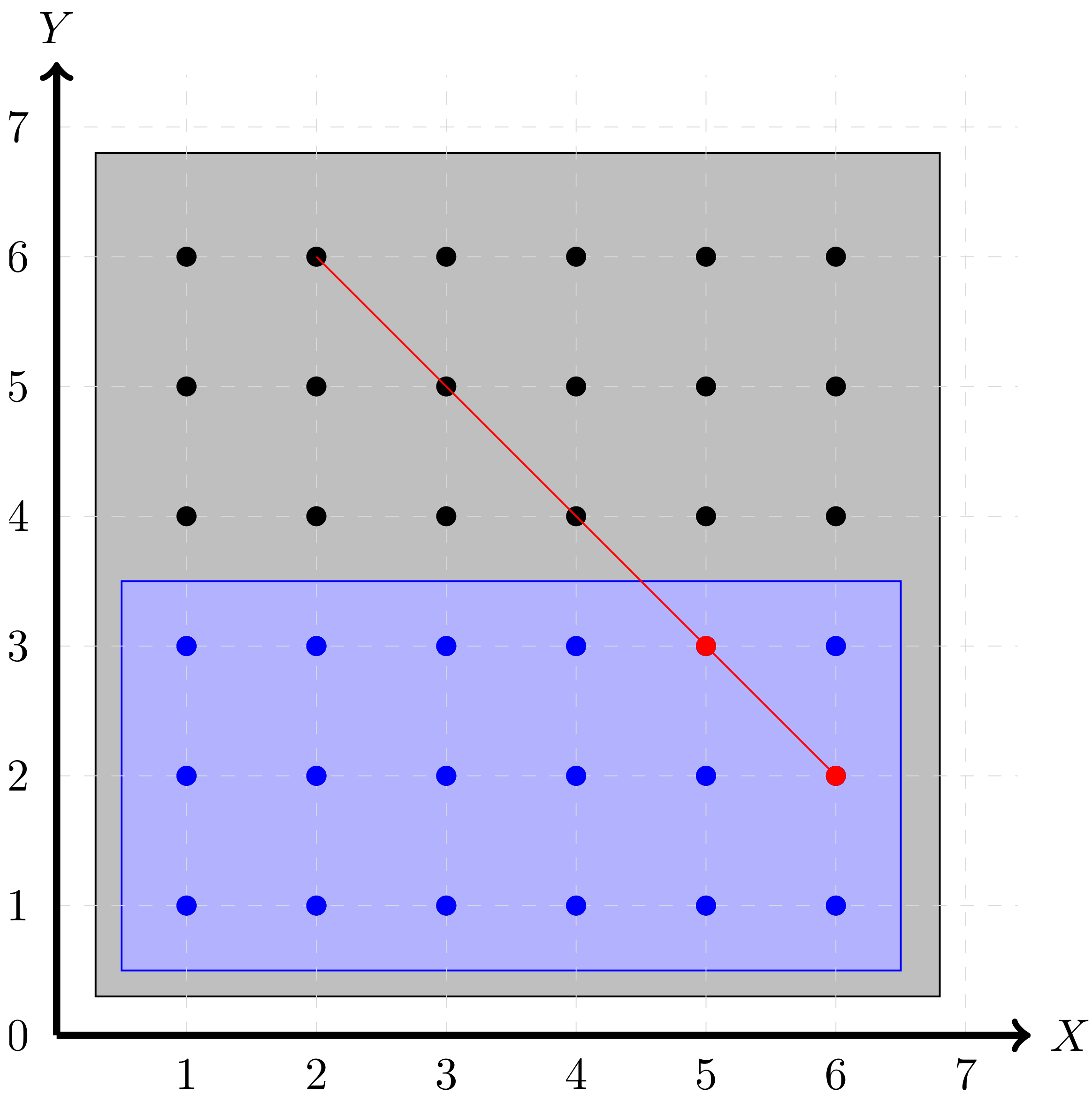


Figure 6.5: $X + Y = 8 | Y < 4$

$$\Pr\left(X + Y = 4 | X \neq Y\right) = \frac{\Pr\left(X + Y = 4, X \neq Y\right)}{\Pr\left(X \neq Y\right)} \tag{6.24}$$

Event	Description
A	$X+Y=4$
B	$X\neq Y$

Table 6.3: Events A and B

$$\Pr\left(X\neq Y\right) = 1-\Pr\left(X=Y\right) \tag{6.25}$$

$$= 1-\frac{6}{36}=\frac{5}{6} \tag{6.26}$$

$$\Pr\left(AB\right)=\Pr\left(A\right)-\Pr\left(AB'\right) \tag{6.27}$$

If

$$X=Y,\ X+Y=4 \tag{6.28}$$

$$\implies X=Y=2 \tag{6.29}$$

$$\therefore \Pr\left(AB'\right)=\Pr\left(X+Y=4, X=Y\right)=\frac{1}{36} \tag{6.30}$$

Since,

$$\Pr\left(X+Y=n\right)=\begin{cases}0 & n<1\\ \frac{n-1}{36} & 2\leq n\leq 7\\ \frac{13-n}{36} & 7<n\leq 12\\ 0 & n>12\end{cases}, \tag{6.31}$$

$$\Pr\left(A\right)=\Pr\left(X+Y=4\right)=\frac{4-1}{36}=\frac{1}{12} \tag{6.32}$$

and from (6.30)

$$\Pr(AB) = \Pr(X+Y=4, X \neq Y) = \frac{1}{12} - \frac{1}{36} \tag{6.33}$$

$$= \frac{1}{18} \tag{6.34}$$

Consequently, from (6.24) and (6.26)

$$\Pr(X+Y=4|X \neq Y) = \frac{\left(\frac{1}{18}\right)}{\left(\frac{5}{6}\right)} \tag{6.35}$$

$$\implies \Pr(X+Y=4|X \neq Y) = \frac{1}{15} \tag{6.36}$$

6.5 If each element of a 2×2 determinant is either zero or one. What is the probability that the value of the determinant is positive ? (Assume that the individual entries of the determinant are chosen independently each value being assumed with probability $\frac{1}{2}$)

Solution: Let the matrix be

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{6.37}$$

(a) The desired probability can be expressed as

$$\Pr(ad - bc > 0) = \Pr\left(a > \frac{bc}{d}\right) = 1 - \Pr\left(a \leq \frac{bc}{d}\right) \tag{6.38}$$

$$= 1 - F_A\left(\frac{bc}{d}\right) \tag{6.39}$$

(b) Since

$$F_A(x) = \begin{cases} 0 & x = 0, \\ \frac{1}{2} & 0 \leqslant x < 1, \\ 1 & 1 \leqslant x < \infty \end{cases}, \tag{6.40}$$

$$E_d\left(F_A\left(\frac{bc}{d}\right)\right) = \frac{1}{2}F_A(bc) + \frac{1}{2}F_A(\infty) = \frac{1}{2}F_A(bc) + \frac{1}{2} \tag{6.41}$$

(c) and

$$E_b\left(\frac{1}{2}F_A(bc) + \frac{1}{2}\right) = \frac{1}{2}E_b(F_A(bc)) + \frac{1}{2} = \frac{1}{2}\left(\frac{1}{2}F_A(0) + \frac{1}{2}F_A(c)\right) \tag{6.42}$$

$$= \frac{1}{2} + \frac{1}{8} + \frac{1}{4}F_A(c) = \frac{5}{8} + \frac{1}{4}F_A(c) \tag{6.43}$$

(d) yielding

$$E_c\left(\frac{5}{8} + \frac{1}{4}F_A(c)\right) = \frac{5}{8} + \frac{1}{4}E_c(F_A(c)) \tag{6.44}$$

$$= \frac{5}{8} + \frac{1}{4}\left(\frac{1}{2}F_A(0) + \frac{1}{2}F_A(1)\right) \tag{6.45}$$

$$= \frac{5}{8} + \frac{1}{4}\left(\frac{1}{4} + \frac{1}{2}\right) = \frac{13}{16} \tag{6.46}$$

(e) Thus, the required probability is

$$\Pr\left(a > \frac{bc}{d}\right) = 1 - E_b, c, d\left(F_A\left(\frac{bc}{d}\right)\right) = 1 - \frac{13}{16} = \frac{3}{16} \tag{6.47}$$

6.6 Two customers Shyam and Ekta are visiting a particular shop in the same week(Tuesday to Saturday). Each is equally likely to visit the shop on any day as on another day. What is the probability that both will visit the shop on (i) the same day? (ii) consecutive days? (iii) different days?

Solution: Let X and Y be the random variables that denote the day on which Shyam and Ekta visit the shop respectively.

$$X, Y = \begin{cases} 0, & \text{Tuesday} \\ 1, & \text{Wednesday} \\ 2, & \text{Thursday} \\ 3, & \text{Friday} \\ 4, & \text{Saturday} \end{cases} \tag{6.48}$$

The pmf of X is:

$$p_X(k) = \begin{cases} 0, & k < 0 \\ \frac{1}{5}, & 0 \leq k \leq 4 \\ 0 & k > 4 \end{cases} \tag{6.49}$$

Let Z be a random variable such that,

$$Z = X - Y \tag{6.50}$$

$$p_Z(k) = P(k = Z) \tag{6.51}$$

$$= P(X - Y = k) \tag{6.52}$$

$$= P(X = k + Y) \tag{6.53}$$

$$= E\left[p_X(k + Y)\right] \tag{6.54}$$

$$= \sum_{m=0}^{4} p_X(k + m)\, p_Y(m) \tag{6.55}$$

$$= \begin{pmatrix} \frac{1}{5} & 0 & 0 & 0 & 0 \\ \frac{1}{5} & \frac{1}{5} & 0 & 0 & 0 \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & 0 & 0 \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & 0 \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ 0 & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ 0 & 0 & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ 0 & 0 & 0 & \frac{1}{5} & \frac{1}{5} \\ 0 & 0 & 0 & 0 & \frac{1}{5} \end{pmatrix} \begin{pmatrix} \frac{1}{5} \\ \frac{1}{5} \\ \frac{1}{5} \\ \frac{1}{5} \\ \frac{1}{5} \end{pmatrix} \tag{6.56}$$

$$= \begin{pmatrix} \frac{1}{25} \\ \frac{2}{25} \\ \frac{3}{25} \\ \frac{4}{25} \\ \frac{1}{5} \\ \frac{4}{25} \\ \frac{3}{25} \\ \frac{2}{25} \\ \frac{1}{25} \end{pmatrix} \tag{6.57}$$

The pmf of Z is:

$$p_Z(k) = \begin{cases} \frac{1}{25}, & k \in \{-4, 4\} \\ \frac{2}{25}, & k \in \{-3, 3\} \\ \frac{3}{25}, & k \in \{-2, 2\} \\ \frac{4}{25}, & k \in \{-1, 1\} \\ \frac{1}{5}, & k = 0 \end{cases} \tag{6.58}$$

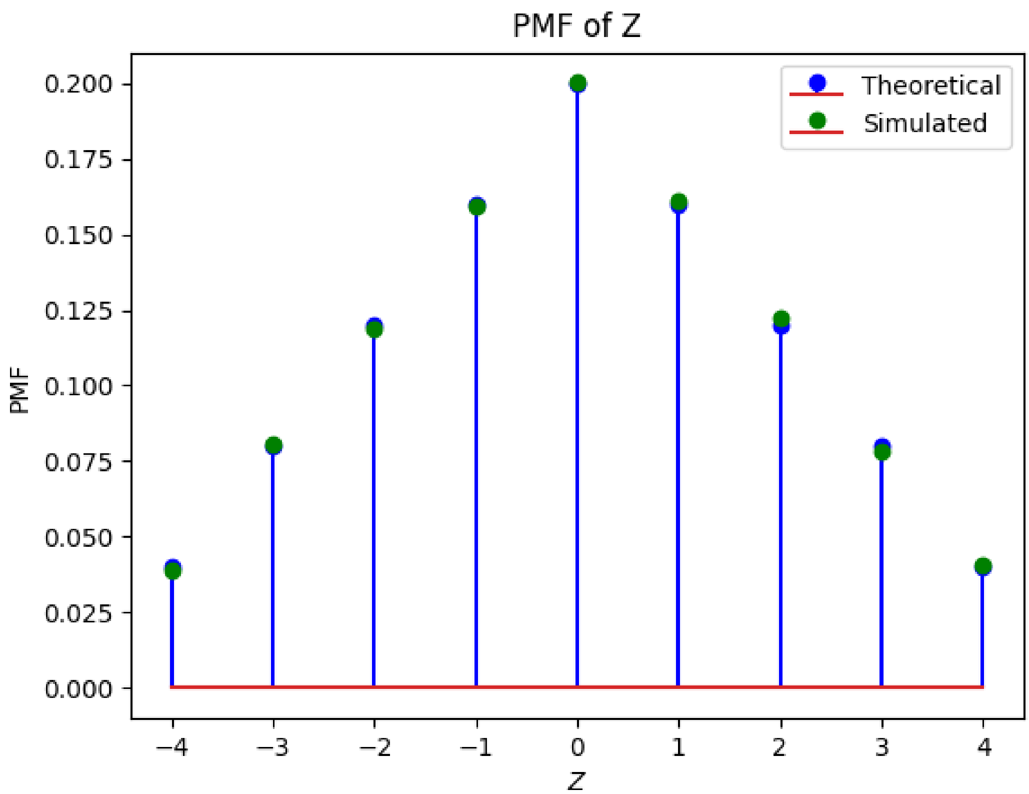


Figure 6.6: PMF of Z

The cumulative distributive function for $|Z|$ is:

$$
\begin{aligned}
F_{|Z|}(k) &= \Pr\left(|Z| \leq k\right) && (6.59)\\
&= \sum_{i=-k}^{k} p_Z\left(i\right) && (6.60)\\
&= \begin{cases} 0, & k < 0 \\ \frac{1}{5}, & k = 0 \\ \frac{13}{25}, & k = 1 \\ \frac{19}{25}, & k = 2 \\ \frac{23}{}\ & k = 3 \end{cases} && (6.61)
\end{aligned}
$$

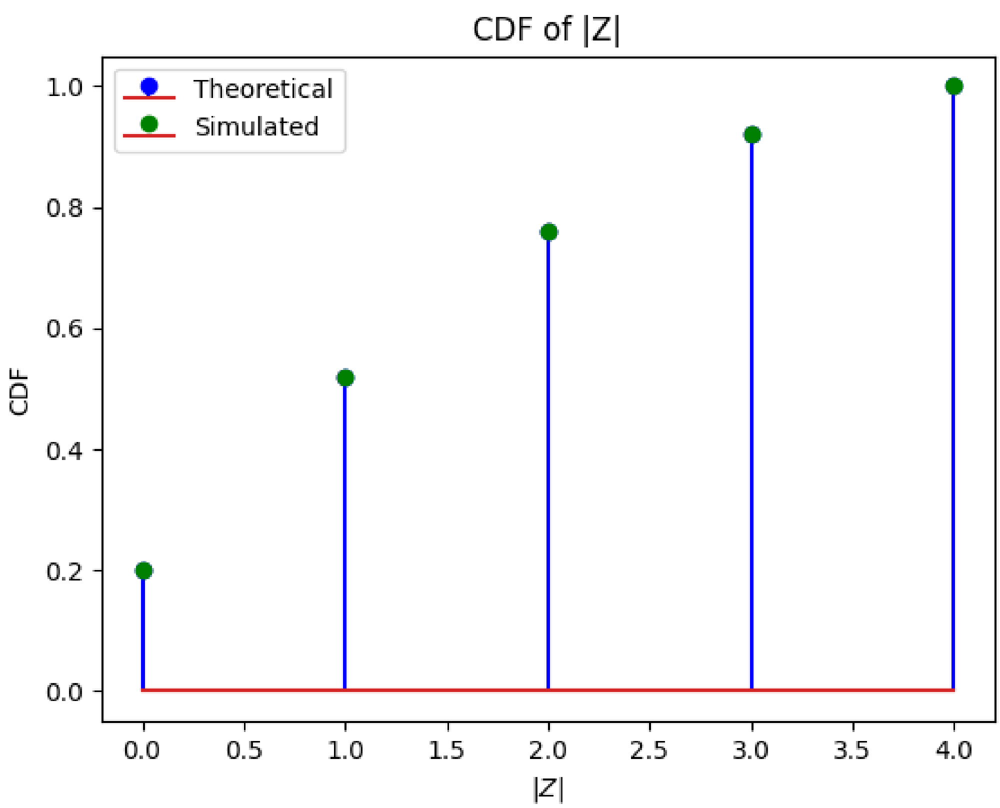


Figure 6.7: CDF of $|Z|$

The pmf of $|Z|$ is:

$$p_{|Z|}(k) = F_{|Z|}(k) - F_{|Z|}(k-1) \tag{6.62}$$

$$= \begin{cases} \frac{1}{5}, & k = 0 \\ \frac{8}{25}, & k = 1 \\ \frac{6}{25}, & k = 2 \\ \frac{4}{25}, & k = 3 \\ \frac{2}{25}, & k = 4 \end{cases} \tag{6.63}$$

(i) the same day

$$\implies X = Y \tag{6.64}$$

$$|Z| = 0 \tag{6.65}$$

$$p_{|Z|}(0) = \frac{1}{5} \tag{6.66}$$

(ii) consecutive days

$$\implies |X - Y| = 1 \tag{6.67}$$

$$|Z| = 1 \tag{6.68}$$

$$p_{|Z|}(1) = \frac{8}{25} \tag{6.69}$$

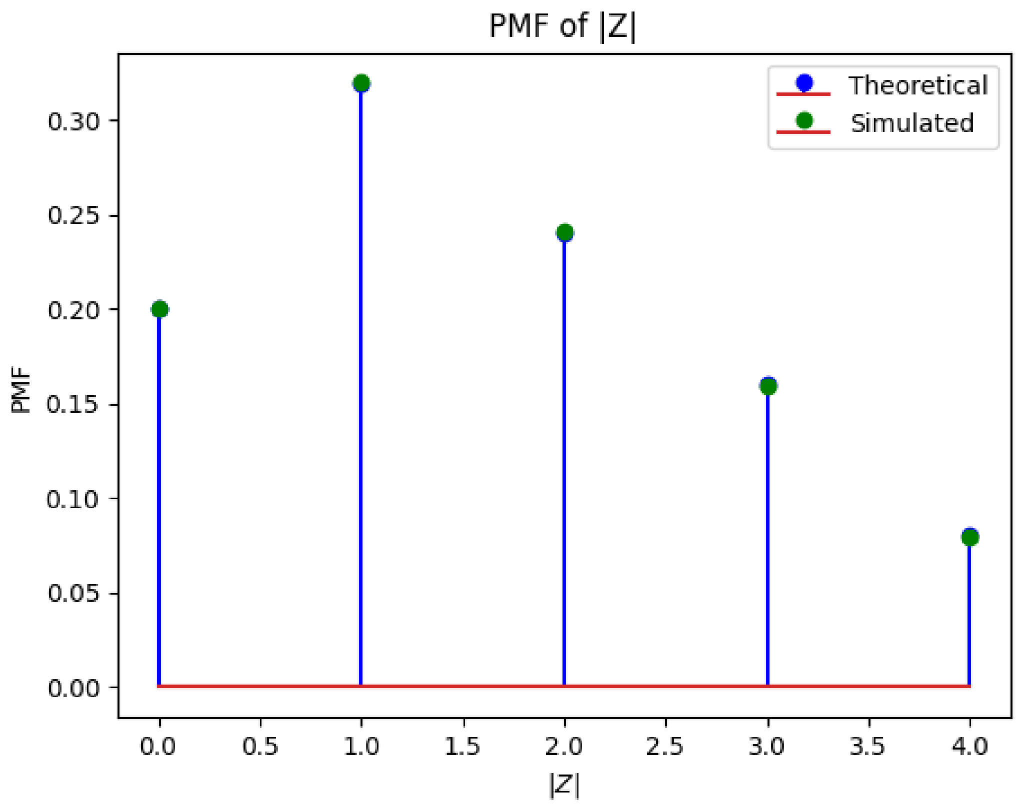


Figure 6.8: PMF of $|Z|$

(iii) different days

$$\implies X \neq Y \tag{6.70}$$

$$|X - Y| \neq 0 \tag{6.71}$$

$$|Z| \neq 0 \tag{6.72}$$

$$p_{|Z|}(k \neq 0) = 1 - p_{|Z|}(0) \tag{6.73}$$

$$= 1 - \frac{1}{5} \tag{6.74}$$

$$= \frac{4}{5} \tag{6.75}$$

parameter	**value**	**description**
X	$\{0, 1, 2, 3, 4\}$	Denotes the day on which Shyam visits the shop
Y	$\{0, 1, 2, 3, 4\}$	Denotes the day on which Ekta visits the shop
Z	$\{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$	$X - Y$

Table 6.4: Random Variables

6.7 Let X represent the difference between the number of heads and the number of tails obtained when a coin is tossed 6 times. What are the possible values of X? Also find the Probability distribution of X.

Solution:

It is given that the coin is tossed 6 times.

Let H be a random variable which denotes the number of heads,

$$H = \{0, 1, 2, 3, 4, 5, 6\} \tag{6.76}$$

Let T be a random variable which denotes the number of tails,

$$T = 6 - H \tag{6.77}$$

$$= \{6, 5, 4, 3, 2, 1, 0\} \tag{6.78}$$

Let X be a random variable which denotes the absolute value of the difference between the number of heads and number of tails,

$$X = |H - T| \tag{6.79}$$

$$= |H - (6 - H)| \tag{6.80}$$

$$= |2H - 6| \tag{6.81}$$

Therefore, X can take values from the set {0,2,4,6}.

Now we will find the probability distribution of X using the CDF approach,

The CDF of H is given by:

$$F_H(k) = \Pr(H \le k) \tag{6.82}$$

$$= \sum_{i=0}^{k} \Pr(H = i) \tag{6.83}$$

$$= \sum_{i=0}^{k} \frac{{}^6C_i}{2^6} \tag{6.84}$$

First we will do the analysis for k = 2,4 and 6

So the CDF of X will be:

$$\begin{aligned}
F_X(k) &= \Pr(X \leq k) && (6.85)\\
&= \Pr(|2H-6| \leq k) && (6.86)\\
&= \Pr(-k \leq 2H-6 \leq k) && (6.87)\\
&= \Pr\left(\frac{6-k}{2} \leq H \leq \frac{6+k}{2}\right) && (6.88)\\
&= F_H\left(3+\frac{k}{2}\right) - F_H\left((3-\frac{k}{2})-1\right) && (6.89)\\
&= F_H\left(3+\frac{k}{2}\right) - F_H\left(2-\frac{k}{2}\right) && (6.90)
\end{aligned}$$

So the probability distribution of X would be given by:

$$\begin{aligned}
p_X(k) &= F_X(k) - F_X(k-1) && (6.91)\\
&= F_H\left(3+\frac{k}{2}\right) - F_H\left(2-\frac{k}{2}\right) - F_H\left(2.5+\frac{k}{2}\right) + F_H\left(3.5-\frac{k}{2}\right) && (6.92)\\
&= \left(\sum_{i=0}^{3+\frac{k}{2}} \frac{{}^6C_i}{2^6} - \sum_{i=0}^{2.5+\frac{k}{2}} \frac{{}^6C_i}{2^6}\right) + \left(\sum_{i=0}^{3.5-\frac{k}{2}} \frac{{}^6C_i}{2^6} - \sum_{i=0}^{2-\frac{k}{2}} \frac{{}^6C_i}{2^6}\right) && (6.93)\\
&= \frac{{}^6C_k}{2^6} + \frac{{}^6C_{6-k}}{2^6} && (6.94)\\
&= \frac{{}^6C_k}{2^5} \quad (k = 2, 4, 6) && (6.95)
\end{aligned}$$

Now we will do the analysis for k = 0,

CDF of X for will be

$$F_X(k) = F_H\left(\left[3+\frac{k}{2}\right]\right) - F_H\left(\left[3-\frac{k}{2}\right]\right) \qquad (6.96)$$

So the probability distribution of X would be given by:

$$p_X(k) = F_X(k) - F_X(k-1) \tag{6.97}$$

$$= F_H\left(\left[3+\frac{k}{2}\right]\right) - F_H\left(\left[3-\frac{k}{2}\right]\right) - F_H\left(\left[2.5+\frac{k}{2}\right]\right) + F_H\left(\left[3.5-\frac{k}{2}\right]\right) \tag{6.98}$$

$$= F_H\left(3+\left[\frac{k}{2}\right]\right) - F_H\left(3-\left[\frac{k}{2}\right]\right) - F_H\left(2+\left[\frac{k}{2}\right]\right) + F_H\left(3-\left[\frac{k}{2}\right]\right) \tag{6.99}$$

$$= F_H\left(3+\left[\frac{k}{2}\right]\right) - F_H\left(2+\left[\frac{k}{2}\right]\right) \tag{6.100}$$

$$= \sum_{i=0}^{3+\left[\frac{k}{2}\right]} \frac{{}^6C_i}{2^6} - \sum_{i=0}^{2+\left[\frac{k}{2}\right]} \frac{{}^6C_i}{2^6} \quad (k=0) \tag{6.101}$$

The plot of probability distribution of X is shown below:

6.8 A black and a red dice are rolled.

(a) find the conditional probability of obtaining a sum greater than 9,given that the black dice resulted in a 5.

(b) find the conditional probability of obtaining the sum 8,given that the red die resulted in a number less than 4.

Solution:

RV	description
X_1	Black die
X_2	Red die

Table 6.5: random variables of X_1 and X_2

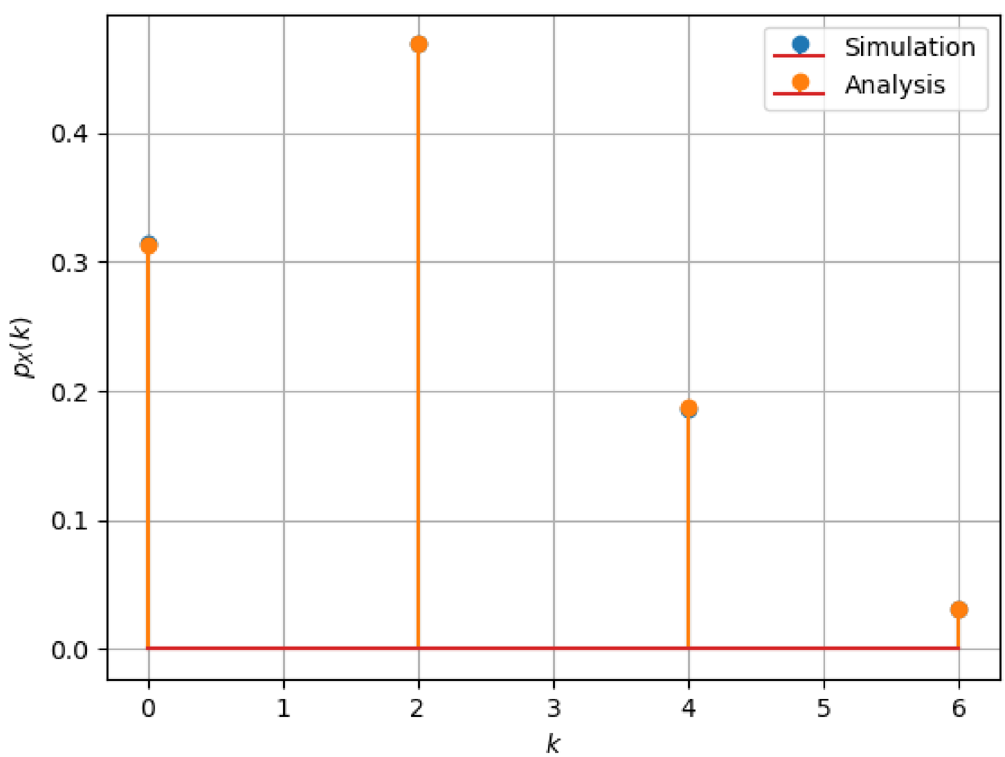


Figure 6.9: PMF of X

PMF of the random variables is:

$$p_{X_j}(i) = \begin{cases} \frac{1}{6} & j = 1, 2; 1 \leq i \leq 6 \end{cases} \tag{6.102}$$

CDF of the random variables is:

$$F_{X_j}(i) = \begin{cases} \frac{i}{6} & j = 1, 2; 1 \leq i \leq 6 \end{cases} \tag{6.103}$$

(a)

$$\begin{aligned}
\Pr(X_1 + X_2 > 9 \mid X_1 = 5) &= \Pr(X_2 > 4 \mid X_1 = 5) &(6.104)\\
&= \frac{\Pr(X_2 > 4, X_1 = 5)}{\Pr(X_1 = 5)} &(6.105)\\
&= \Pr(X_2 > 4) &(6.106)\\
&= 1 - \Pr(X_2 \leq 4) &(6.107)\\
&= 1 - F_{X_2}(4) &(6.108)\\
&= 1 - \frac{4}{6} &(6.109)\\
&= 1 - \frac{2}{3} &(6.110)\\
&= \frac{1}{3} &(6.111)
\end{aligned}$$

(b)

$$\Pr\left(X_1 + X_2 = 8 | X_2 < 4\right) = \frac{\Pr\left(X_1 + X_2 = 8, X_2 < 4\right)}{\Pr\left(X_2 < 4\right)} \tag{6.112}$$

$$= \frac{\sum_{i=1}^{3} p_{X_1}\left(8 - i\right) p_{X_2}\left(i\right)}{\Pr\left(X_2 < 4\right)} \tag{6.113}$$

$$= \frac{p_{X_1}\left(5\right) p_{X_2}\left(3\right) + p_{X_1}\left(6\right) p_{X_2}\left(2\right)}{\Pr\left(X_2 < 4\right)} \tag{6.114}$$

$$= \frac{\frac{1}{36} + \frac{1}{36}}{\frac{3}{6}} \tag{6.115}$$

$$= \frac{1}{9} \tag{6.116}$$

6.9 A fair die is thrown two times. Let A and B be the events, 'same number each time', and a 'a total score is 10 or more', respectively. Determine whether or not A and B are independent.

Solution:

Random Variable	Description
X	Number appearing on throwing the first time
Y	Number appearing on throwing the second time.

Table 6.6: Random Variables

The probability mass functions(PMFs) of X and Y are given by,

$$\Pr\left(X = i\right) = \frac{1}{6},\ i = 1, 2, 3, 4, 5, 6 \tag{6.117}$$

$$\Pr\left(Y = i\right) = \frac{1}{6},\ i = 1, 2, 3, 4, 5, 6 \tag{6.118}$$

Event A occurs if and only if $X = Y$

$$\Pr(A) = \Pr(X = Y) \tag{6.119}$$

Let n be the differnce of two numbers appearing on the dice.

$$\Pr(X - Y = n) = \begin{cases} \frac{6-|n|}{36}, -5 \leq n \leq 5 \\ 0, otherwise \end{cases} \tag{6.120}$$

In this case, n is 0 as $X = Y$

$$\Pr(A) = \frac{6}{36} \tag{6.121}$$

Event B occurs if and only if $X + Y \geq 10$

$$\Pr(B) = \Pr(X + Y \geq 10) \tag{6.122}$$

Let n be the sum of two numbers appearing on the dice,

$$\Pr(X + Y = n) = \begin{cases} 0, n < 1 \\ \frac{n-1}{36}, 2 \leq n \leq 7 \\ \frac{13-n}{36}, 7 \leq n \leq 12 \\ 0, n > 12 \end{cases} \tag{6.123}$$

In this case, n can be $10, 11$ or 12

$$\Pr(B) = \frac{3}{36} + \frac{2}{36} + \frac{1}{36} \tag{6.124}$$
$$= \frac{6}{36} \tag{6.125}$$

For two events A and B to be independent,

$$\Pr(A)\Pr(B) = \Pr(AB) \tag{6.126}$$

$$AB = ((5,5),(6,6)) \tag{6.127}$$
$$\Pr(AB) = \Pr(X = Y, X + Y \geq 10) \tag{6.128}$$
$$= \frac{1}{6} \times \frac{1}{6} + \frac{1}{6} \times \frac{1}{6} \tag{6.129}$$
$$= \frac{2}{36} \tag{6.130}$$

$$\Pr(A)\Pr(B) = \frac{6}{36} \times \frac{6}{36} \tag{6.131}$$
$$= \frac{1}{36} \tag{6.132}$$

Thus,

$$\Pr(A)\Pr(B) \neq \Pr(AB) \tag{6.133}$$
$$\frac{1}{36} \neq \frac{2}{36} \tag{6.134}$$

Hence A and B are not independent events.

6.10 Two dice are thrown together. Find the probability that the product of the numbers on the top of the dice is less than 9

Solution: Let the X and Y be random variables denoting the roll of first dice snd second dice respectively. Assuming both dice rolls and equally likely:

$$p_X(k) = \begin{cases} \frac{1}{6} & \text{if } k \in \{1,2,3,4,5,6\} \\ 0 & \text{otherwise} \end{cases} \tag{6.135}$$

$$p_Y(k) = \begin{cases} \frac{1}{6} & \text{if } k \in \{1,2,3,4,5,6\} \\ 0 & \text{otherwise} \end{cases} \tag{6.136}$$

The probability mass function is:

$$\begin{aligned}
p_{XY}(k) &= \Pr\left(XY = k\right) && (6.137)\\
&= \Pr\left(X = \frac{k}{Y}\right) && (6.138)\\
&= E\left(p_X\left(\frac{k}{Y}\right)\right) && (6.139)\\
&= \sum_{i=1}^{6} p_X\left(\frac{k}{i}\right) p_Y(i) && (6.140)\\
&= \frac{1}{6}\sum_{i=1}^{6} p_X\left(\frac{k}{i}\right) && (6.141)\\
&= \frac{1}{6}\sum_{i=1}^{6} \frac{[k \bmod i = 0]}{6}\left[\frac{k}{i} \le 6\right] && (6.142)\\
&= \frac{1}{36}\sum_{i=1}^{6} [k \bmod i = 0]\left[\frac{k}{i} \le 6\right] && (6.143)
\end{aligned}$$

Thus, the probability of getting product less than 9 is:

$$\Pr(XY < 9) = \sum_{i=1}^{8} \left(\frac{1}{36} \sum_{i=1}^{6} [j \bmod i = 0] \left[\frac{j}{i} \le 6 \right] \right) \tag{6.144}$$
$$= \frac{1}{36} (1 + 2 + 2 + 3 + 2 + 4 + 2) \tag{6.145}$$
$$= \frac{16}{36} \tag{6.146}$$
$$= \frac{4}{9} \tag{6.147}$$

Table 6.7: Table

Parameters	Values	Description
X	$1 \le X \le 6$	First dice roll
Y	$1 \le Y \le 6$	Second dice roll

6.11 Two dice are thrown together. Find the probability that the product of the numbers on the top of the dice is 6 , 7 , 12

Solution: Let X and Y denote the random variables for the roll of first dice and second dice respectively. Assuming both dice rolls and equally likely,:

$$p_X(k) = \begin{cases} \frac{1}{6} & \text{if } k \in \{1,2,3,4,5,6\} \\ 0 & \text{otherwise} \end{cases} \tag{6.148}$$

$$p_Y(k) = \begin{cases} \frac{1}{6} & \text{if } k \in \{1,2,3,4,5,6\} \\ 0 & \text{otherwise} \end{cases} \tag{6.149}$$

The probability mass function is:

$$p_{XY}(k) = \Pr(XY = k) \tag{6.150}$$

$$= \Pr\left(X = \frac{k}{Y}\right) \tag{6.151}$$

$$= E\left(p_X\left(\frac{k}{Y}\right)\right) \tag{6.152}$$

$$= \sum_{i=1}^{6} p_X\left(\frac{k}{i}\right) p_Y(i) \tag{6.153}$$

$$= \frac{1}{6}\sum_{i=1}^{6} p_X\left(\frac{k}{i}\right) \tag{6.154}$$

$$= \frac{1}{6}\sum_{i=1}^{6} \frac{[k \bmod i = 0]}{6}\left[\frac{k}{i} \le 6\right] \tag{6.155}$$

$$= \frac{1}{36}\sum_{i=1}^{6} [k \bmod i = 0]\left[\frac{k}{i} \le 6\right] \tag{6.156}$$

Thus, the probability of getting product 6 is:

$$\Pr(XY = 6) = \left(\frac{1}{36}\sum_{i=1}^{6} [6 \bmod i = 0]\left[\frac{6}{i} \le 6\right]\right) \tag{6.157}$$

$$= \frac{1}{36}(1 + 1 + 1 + 1 + 0 + 0) \tag{6.158}$$

$$= \frac{4}{36} \tag{6.159}$$

$$= \frac{1}{9} \tag{6.160}$$

Probability of getting product 7 is:

$$\Pr\left(XY=7\right)=\left(\frac{1}{36}\sum_{i=1}^{6}\left[7 \bmod i=0\right]\left[\frac{7}{i}\leq 6\right]\right) \tag{6.161}$$
$$=\frac{1}{36}\left(0+0+0+0+0+0\right) \tag{6.162}$$
$$=\frac{0}{36} \tag{6.163}$$
$$=0 \tag{6.164}$$

Probability of getting product 12 is:

$$\Pr\left(XY=12\right)=\left(\frac{1}{36}\sum_{i=1}^{6}\left[12 \bmod i=0\right]\left[\frac{12}{i}\leq 6\right]\right) \tag{6.165}$$
$$=\frac{1}{36}\left(0+1+1+1+0+1\right) \tag{6.166}$$
$$=\frac{4}{36} \tag{6.167}$$
$$=\frac{1}{9} \tag{6.168}$$

Table 6.8: Table

Variable	Values	Description
X	$1 \leq X \leq 6$	First Dice Roll
Y	$1 \leq Y \leq 6$	Second Dice Roll

6.2. Exercises

6.2.1 Two dice are thrown together. Find the probability that the product of the numbers on the top of the dice is

(a) 6

(b) 12

(c) 7

Solution:

Let X, Y be the outcome of the two dice and

$$Z = XY \tag{6.169}$$

$$p_X(k) = \begin{cases} 0 & k < 1 \\ \frac{1}{6} & 1 \leq k \leq 6 \\ 0 & k > 6 \end{cases} \tag{6.170}$$

$$F_X(k) = \begin{cases} 0 & k < 1 \\ \frac{k}{6} & 1 \leq k \leq 6 \\ 1 & k > 6 \end{cases} \tag{6.171}$$

$$F_Z(n) = E\left[F_Y\left(\frac{n}{k}\middle|k\right)\right] = \frac{1}{6}\sum_{k=1}^{6} F_Y\left(\frac{n}{k}\middle|k\right) \tag{6.172}$$

and

$$F\left(\frac{n}{k}\right) = \begin{cases} 1 & k < \frac{n}{6} \\ \frac{\left[\frac{n}{k}\right]}{6} & k \geq \frac{n}{6} \cap \frac{n}{k} \notin I \\ \frac{\left(\frac{n}{k}\right)}{6} & k \geq \frac{n}{6} \cap \frac{n}{k} \in I \end{cases} \tag{6.173}$$

Thus,

$$F_Z(n) = \frac{1}{6}\left[\sum_{k=1}^{\left[\frac{n}{6}\right]} F_Y\left(\frac{n}{k}\right) + \sum_{k=\left[\frac{n}{6}\right]+1}^{6} F_Y\left(\frac{n}{k}\right)\right] \tag{6.174}$$

$$= \frac{1}{6}\left[\sum_{k=1}^{\left[\frac{n}{6}\right]} 1 + \sum_{k=\left[\frac{n}{6}\right]+1}^{6} F_Y\left(\frac{n}{k}\right)\right] \tag{6.175}$$

$$= \frac{1}{6}\left[\left[\frac{n}{6}\right] + \sum_{k=\left[\frac{n}{6}\right]+1}^{6} F_Y\left(\frac{n}{k}\right)\right] \tag{6.176}$$

and

$$p_Z(n) = F_Z(n) - F_Z(n-1) \tag{6.177}$$

See Fig. 6.10 and Fig. 6.11 for the plots.

From (6.176),

(a)

$$p_Z(6) = F_Z(6) - F_Z(5) = \frac{1}{9} \tag{6.178}$$

(b)

$$p_Z(12) = F_Z(12) - F_Z(11) = \frac{1}{9} \tag{6.179}$$

(c)

$$p_Z(7) = 0 \tag{6.180}$$

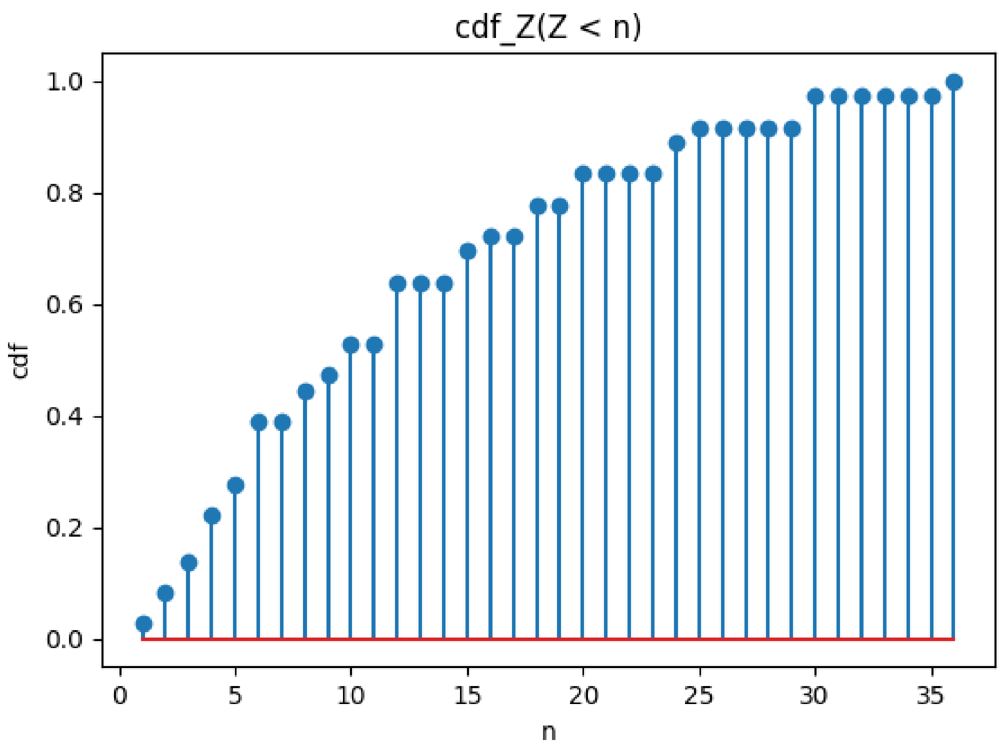


Figure 6.10: Plot of cummulative Distribution function

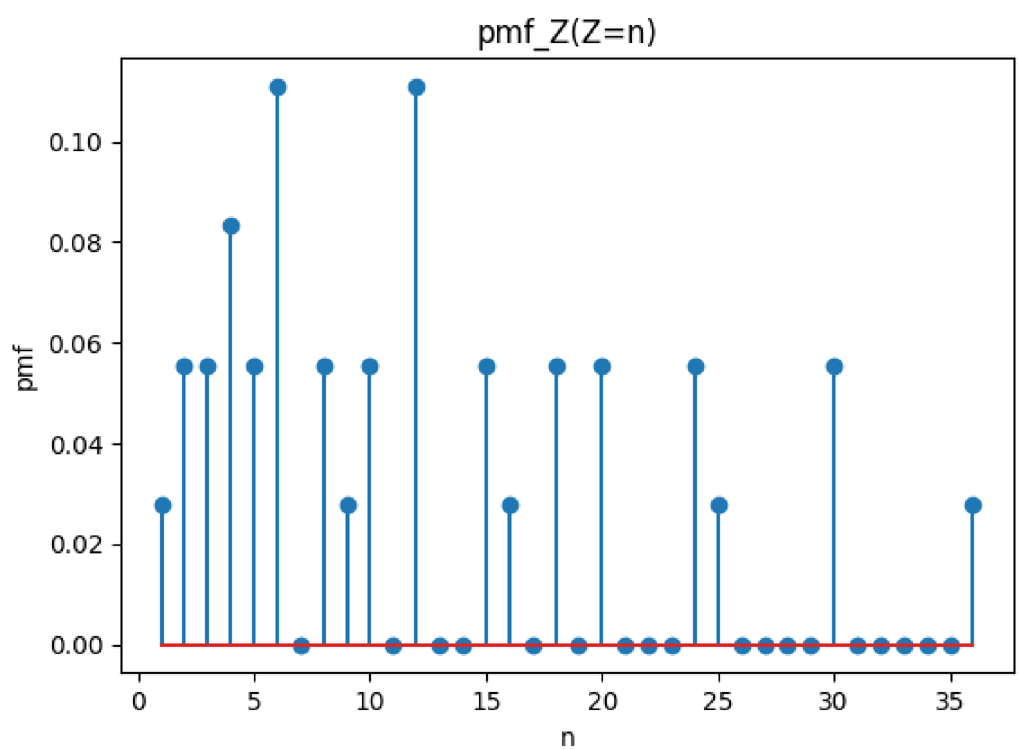


Figure 6.11: Plot of Probability Mass Function

6.2.2 Two dice are thrown simultaneously. What is the probability that the sum of the numbers appearing on the dice is

(a) 7?

(b) a prime number?

(c) 1?

Solution: Let X and Y represent number appearing on two dice. Let Z be the sum of the numbers appearing on two dice.

$$Z = X + Y$$

random variables	description
X	number appearing on first dice
Y	number appearing on second dice
Z	sum of numbers appearing on both dice

Table 6.9: Random Variables for die rolls

We know,

$$\Pr(Z = n) = \begin{cases} 0 & n \leq 1 \\ \frac{n-1}{36} & 2 \leq x \leq 7 \\ \frac{13-n}{36} & 7 < n \leq 12 \\ 0 & n > 12 \end{cases} \tag{6.181}$$

Then,

(a) The sum of numbers appearing on the dice is 7. Then from (6.181),

$$\Pr(Z=7) = \frac{7-1}{36} \tag{6.182}$$
$$= \frac{1}{6} \tag{6.183}$$

(b) The sum of numbers appearing on dice is a prime number. From (6.181),

$$\Pr(Z \text{ is a prime number}) = \sum_{p\in\{2,3,5,7,11\}} \Pr(Z=p) \tag{6.184}$$
$$= \frac{1}{36} + \frac{2}{36} + \frac{4}{36} + \frac{6}{36} + \frac{2}{36} \tag{6.185}$$
$$= \frac{5}{12} \tag{6.186}$$

(c) From (6.181), the probability of the sum of numbers appearing on the dice is 1 is,

$$\Pr(Z=1) = 0 \tag{6.187}$$

6.2.3 Two dice are thrown at the same time. Find the probability of getting

(i) same number on both dice.

(ii) different numbers on both dice.

Solution: Let the random variables:

parameters	**value**	**description**
X	$1 \le X \le 6$	outcome of the first die
Y	$1 \le Y \le 6$	outcome of the second die

Consider a random variable W:

$$W = X - Y \tag{6.188}$$

W can take values ranging from {-5 to 5}.

$$p_X(k) = \begin{cases} \frac{1}{6}, & 1 \le k \le 6 \\ 0, & \text{otherwise} \end{cases} \tag{6.189}$$

$$p_X(k) = p_Y(k) \tag{6.190}$$

PMF of W using z-transform:

applying the z-transform on both the sides

$$z\{W\} = z\{X - Y\} \tag{6.191}$$

$$M_W(z) = M_{X-Y}(z) \tag{6.192}$$

Using the expectation operator:

$$E[z^{-W}] = E[z^{-X+Y}] \tag{6.193}$$

$$= E[z^{-X}] \cdot E[z^{Y}] \tag{6.194}$$

$$= \left(\sum_{i=1}^{6} p_X(i) \cdot z^{-i} \right) \cdot \left(\sum_{j=1}^{6} p_Y(j) \cdot z^{j} \right) \tag{6.195}$$

Extracting the PMF by considering the defenition of z-transform

$$
\begin{aligned}
M_W(z) &= p_W(0) + p_W(1)z + ... + p_W(k)z^k + ... && (6.196)\\
&= \frac{1}{36}\left(z^{-1} + ... + z^{-6}\right)\cdot\left(z^1 + ... + z^6\right) && (6.197)\\
&= \frac{1}{36}(z^{-5} + 2z^{-4} + 3z^{-3} + 4z^{-2} + 5z^{-1} + 6\\
&\quad + 5z^1 + 4z^2 + 3z^3 + 2z^4 + z^5) && (6.198)
\end{aligned}
$$

defined for all the values of $-5 \le k \le 5$

$$
\begin{aligned}
p_W(k) &= \frac{1}{k!}\left(\frac{d^{|k|}}{dz^{|k|}} M_W(z)\right)_{z=0} && (6.199)\\
&= \frac{1}{36k!}\left(\frac{d^{|k|}}{dz^{|k|}}\left(z^{-5} + ... + z^5\right)\right)_{z=0} && (6.200)\\
&= \frac{1}{36k!}\left(\frac{d^{|k|}}{dz^{|k|}}\left(\sum_{m=-5}^{5}\left(6 - |m|\right) z^m\right)\right)_{z=0} && (6.201)\\
&= \frac{1}{36k!}\left(6 - |k|\right)\frac{d^{|k|}}{dz^{|k|}} z^k && (6.202)\\
&= \frac{1}{36k!}\left(6 - |k|\right) k! && (6.203)\\
&= \frac{1}{36}\left(6 - |k|\right) && (6.204)
\end{aligned}
$$

(i) Finding the probability for $W = 0$

From the result (6.204)

$$p_W(0) = \frac{1}{36}(6 - |0|) \tag{6.205}$$

$$\implies \Pr(W = 0) = \frac{1}{36}(6) \tag{6.206}$$

$$= \frac{1}{6} \tag{6.207}$$

(ii) Finding the probability for $W \neq 0$

$$\Pr(W \neq 0) = 1 - \Pr(W = 0) \tag{6.208}$$

$$= 1 - \frac{1}{6} \tag{6.209}$$

$$= \frac{5}{6} \tag{6.210}$$

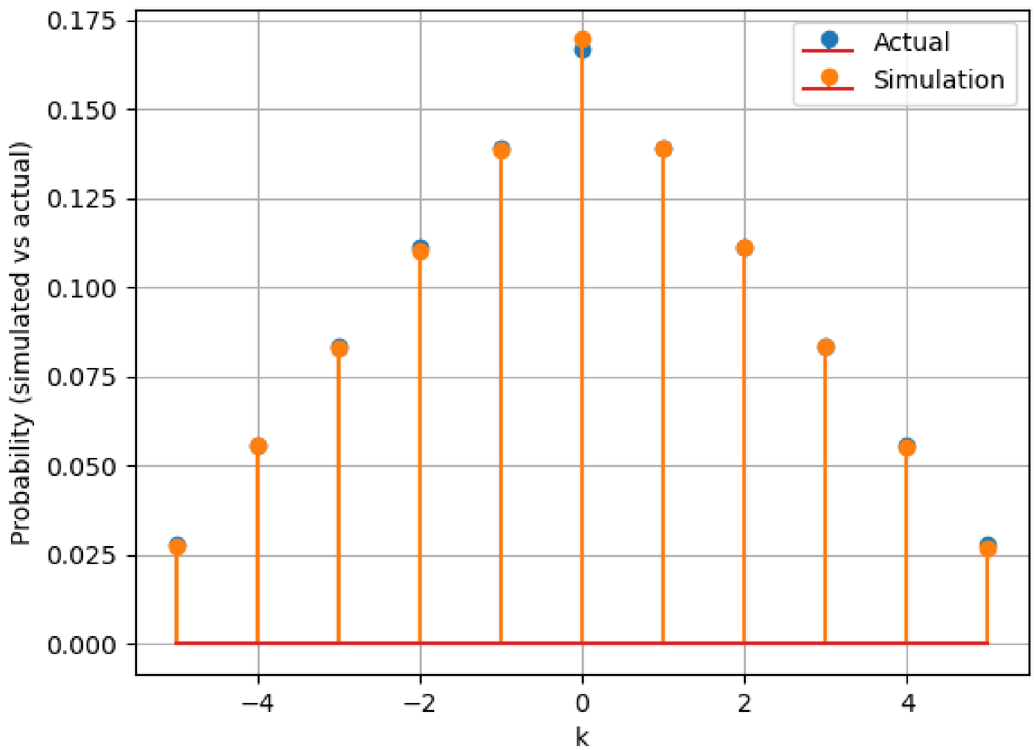


Figure 6.12: PMF analysis of W ($p_W(k)$)

6.2.4 A die has its face marked 0,1,1,1,6,6. Two such dice are thrown together and their score is recorded.

(a) How many different scores are possible ?

(b) What is the probability of getting a total 7 ?

Solution: Let the random variables be defined as:

Random Variable	Values	Description
X	X = {0,1,6}	First Dice Roll
Y	Y = {0,1,6}	Second Dice Roll

(a) **Possible outcomes:** The following data can be interpreted from the data given in the question,

$$p_X(k) = \begin{cases} \frac{1}{6} & \text{if } k = 0 \\ \frac{1}{2} & \text{if } k = 1 \\ \frac{1}{3} & \text{if } k = 6 \\ 0 & \text{Otherwise} \end{cases} \tag{6.211}$$

$$p_Y(k) = \begin{cases} \frac{1}{6} & \text{if } k = 0 \\ \frac{1}{2} & \text{if } k = 1 \\ \frac{1}{3} & \text{if } k = 6 \\ 0 & \text{Otherwise} \end{cases} \tag{6.212}$$

$$\tag{6.213}$$

The Z-transform of a die is defined as

$$M_X(z) = z^{-X} = \sum_{k=-\infty}^{\infty} p_X(k) z^{-k} \tag{6.214}$$

The Z-transform of the first die X_1 is given by

$$M_{X_1}(z) = \frac{1}{6}z^0 + \frac{1}{2}z^{-1} + \frac{1}{3}z^{-6} \tag{6.215}$$

$$= \frac{z^0 + 3z^{-1} + 2z^{-6}}{6} \tag{6.216}$$

$$\tag{6.217}$$

The Z-transform of the second die X_2 is given by

$$M_{X_2}(z) = \frac{1}{6}z^0 + \frac{1}{2}z^{-1} + \frac{1}{3}z^{-6} \tag{6.218}$$
$$= \frac{z^0 + 3z^{-1} + 2z^{-6}}{6} \tag{6.219}$$
$$\tag{6.220}$$

$$M_{X_1+X_2}(z) = \frac{z^0 + 3z^{-1} + 2z^{-6}}{6} \times \frac{z^0 + 3z^{-1} + 2z^{-6}}{6} \tag{6.221}$$
$$M_{X_1+X_2}(z) = \frac{\left(1 + 3z^{-1} + 2z^{-6}\right)^2 \left(1 - z^{-1}\right)^2}{36\left(1 - z^{-1}\right)^2} \tag{6.222}$$

We also know that,

$$p_{X_1+X_2}(n-k) \overset{Z}{\longleftrightarrow} M_{X_1+X_2}(z)z^{-k}; \tag{6.223}$$
$$nu(n) \overset{Z}{\longleftrightarrow} \frac{z^{-1}}{(1 - z^{-1})^2} \tag{6.224}$$

Hence, after some algebra, it can be shown that,

$$\begin{aligned}
&\frac{1}{36}[(n+1)u(n+1) + 4nu(n) - 2n - 1u(n-1) \\
&\quad -12(n-2)u(n-2) + 9(n-3)u(n-3) + 4(n-5)u(n-5) \\
&\quad -20(n-7)u(n-7) + 12(n-8)u(n-8) \\
&\quad +4(n-11)u(n-11) - 8(n-12)u(n-12) \\
&\quad +4(n-13)u(n-13)] \overset{Z}{\longleftrightarrow} \\
&\quad \frac{\left(1 + 3z^{-1} + 2z^{-6}\right)^2 \left(1 - z^{-1}\right)^2}{36\left(1 - z^{-1}\right)^2}
\end{aligned} \tag{6.225}$$

The probability function for the sum of the numbers appearing on both the dice,

$$p_X(n) = \begin{cases} \frac{1}{36} & n = 0 \\ \frac{1}{6} & n = 1 \\ \frac{1}{4} & n = 2 \\ \frac{1}{9} & n = 6, 12 \\ \frac{1}{3} & n = 7 \\ 0 & else \end{cases} \tag{6.226}$$

The possible outcomes: 0,1,2,6,7&12

(b) **Probability of getting a 7 :** From (6.226), the probability of getting the sum of 2 numbers appearing on the dice to be 7, would be $\frac{1}{3}$.

6.2.5 For a loaded die, the probabilities of outcomes are given as under: $\Pr(1) = \Pr(2) = 0.2, \Pr(3) = \Pr(5) = \Pr(6) = 0.1$ and $\Pr(4) = 0.3$ The die is thrown two times. Let A and B be the events, 'same number each time', and 'a total score is 10 or more' ,respectively. Determine whether or not A and B are independent. Let X, Y and Z be random variables with definition given as under:

X	Number appearing on dice the first time	
Y	Number appearing on dice the second time	
Z	Sum of the numbers appearing on the dice	X + Y
W	Difference of the numbers appearing on the dice	X - Y

Table 6.10: Definition of Random variables.

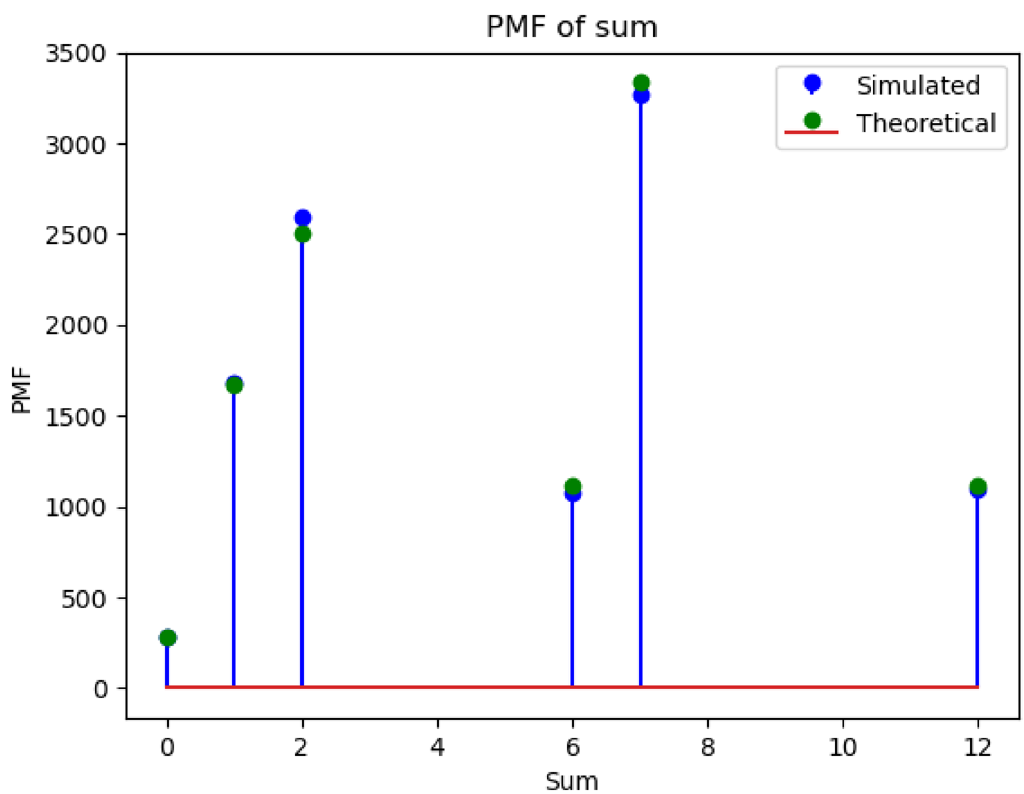


Figure 6.13: Sketch of Probability Mass Function for Sum generated using (6.225)

$$p_X(k) = \begin{cases} 0.2, & k = 1, 2 \\ 0.1, & k = 3, 5, 6 \\ 0.3, & k = 4 \end{cases} \tag{6.227}$$

$$p_X(k) = p_Y(k) \tag{6.228}$$

PMF of W using z-transform: applying the z-transform on both the sides

$$M_W(z) = M_{X-Y}(z) \tag{6.229}$$

Using the expectation operator:

$$E\left[z^{-W}\right] = E\left[z^{-X+Y}\right] \tag{6.230}$$

$$= M_X(z) \cdot M_Y(z^{-1}) \tag{6.231}$$

Extracting the PMF by considering the definition of z-transform

$$M_W(z) = p_W(0) + p_W(1)z^{-1} + p_W(1)z^{-2} + ... + p_W(k)z^k + ... \tag{6.232}$$

$$= 0.01(2z^{-5} + 4z^{-4} + 9z^{-3} + 12z^{-2} + 13z^{-1} + 20$$

$$+ 13z^1 + 12z^2 + 9z^3 + 4z^4 + 2z^5) \tag{6.233}$$

defined for all the values of $-5 \leq k \leq 5$ Now, Z can take values ranging from $\{2$ to $12\}$. PMF of Z using z-transform: applying the z-transform on both the sides

$$M_Z(z) = M_{X+Y}(z) \tag{6.234}$$

$$E\left[z^{-Z}\right] = M_X(z) \cdot M_Y(z) \tag{6.235}$$

Extracting the PMF by considering the defenition of z-transform

$$M_W(z) = (0.1z^{-6} + 0.1z^{-5} + 0.3z^{-4} + 0.1z^{-3} + 0.2z^{-2} + 0.2z^{-1})^2 \tag{6.236}$$

defined for all the values of $2 \leq k \leq 12$

For event A, Finding the probability for $W = 0$

$$p_W(0) = 0.2 \tag{6.237}$$

For event B, Finding the probability for $Z \geq 10$

$$p_Z(10) = 0.07 \tag{6.238}$$

$$p_Z(11) = 0.02 \tag{6.239}$$

$$p_Z(12) = 0.01 \tag{6.240}$$

Hence,

$$\Pr(B) = \Pr(Z = 10) + \Pr(Z = 11) + \Pr(Z = 12) \tag{6.241}$$

$$= 0.1 \tag{6.242}$$

Now, A and B will be independent if,

$$\Pr(AB) = \Pr(A)\Pr(B) \tag{6.243}$$

$$AB = ((5,5),(6,6)) \tag{6.244}$$

$$\Pr(AB) = 0.1 \times 0.1 + 0.1 \times 0.1 \tag{6.245}$$

$$= 0.02 \tag{6.246}$$

$$= \Pr(A)\Pr(B) \tag{6.247}$$

Hence, events A and B are independent.

6.2.6 Three dice are thrown at the sametime. Find the probability of getting three two's, if

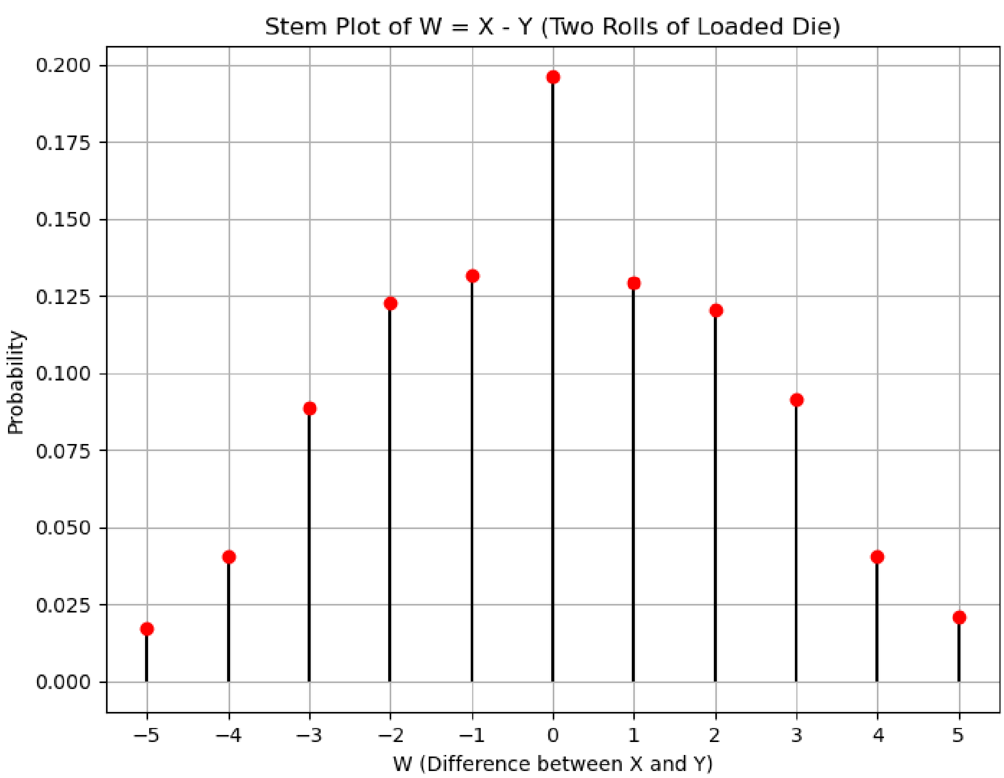


Figure 6.14: Stem plot for P(W)

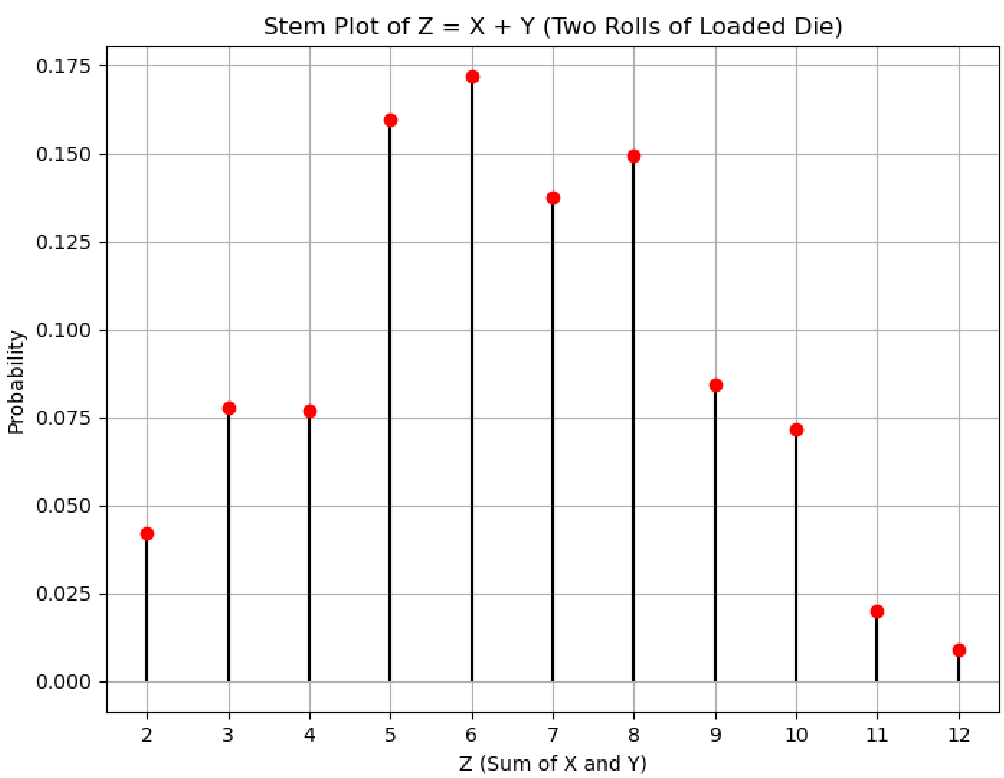


Figure 6.15: Stem plot for P(Z)

it is known that the sum of the numbers on the dice was six.

Solution: Let X_1, X_2, X_3 be Bernoulli Random Variables as defined in Table 6.11,

RV	Value	Description
X_i	0,1,2,3,4,5,6	Outcome of i^{th} die

Table 6.11: Random variable X_i declaration $\forall i \in 1, 2, 3$.

We need to find the value of

$$\Pr\left(X_1 = 2, X_2 = 2, X_3 = 2 | X_1 + X_2 + X_3 = 6\right) \tag{6.248}$$

The pmf's of the random variables are as follows:

$$p_{X_1}(k) = \begin{cases} \frac{1}{6} & x \in \{1, 2, 3, 4, 5, 6\} \\ 0 & \text{otherwise} \end{cases} \tag{6.249}$$

Now, If

$$X = X_1 + X_2 + X_3 \tag{6.250}$$

Then

$$M_X(z) = \prod_{i=1}^{3} M_{X_i}(z) \tag{6.251}$$

$$= \prod_{i=1}^{3} \left(\sum_{k=1}^{6} \frac{z^{-k}}{6} \right) \tag{6.252}$$

$$= \frac{z^{-3}}{216} + \frac{3z^{-4}}{216} + \frac{6z^{-5}}{216} + \frac{10z^{-6}}{216} + \dots \tag{6.253}$$

From (6.249), When all the three die's roll 2, then thier sum will be 6. So,

$$\Pr\left(X_1 = X_2 = X_3 = 2,\ X_1 + X_2 + X_3 = 6\right) \tag{6.254}$$

$$= p_{X_1}(2)p_{X_2}(2)p_{X_3}(2) = \left(\frac{1}{6}\right)^3 = \frac{1}{216} \tag{6.255}$$

From (6.252), the probability that the sum of all three die's is 6 is,

$$\Pr\left(X_1 + X_2 + X_3 = 6\right) = \frac{10}{216} \tag{6.256}$$

From (6.254) and (6.256), the probability of getting three two's, if it is known that the sum of the numbers on the dice was six will be

$$\Pr\left(X_1 = 2, X_2 = 2, X_3 = 2 | X_1 + X_2 + X_3 = 6\right) \tag{6.257}$$

$$= \frac{\Pr\left(X_1 = X_2 = X_3 = 2,\ X_1 + X_2 + X_3 = 6\right)}{\Pr\left(X_1 + X_2 + X_3 = 6\right)} \tag{6.258}$$

$$= \frac{\frac{1}{216}}{\frac{10}{216}} = \frac{1}{10} \tag{6.259}$$

6.2.7 Two dice are tossed. Find whether the following two events A and B are independent:

A = {(x,y) : x+y=11} B = {(x,y): x $\neq$ 5}

where (x,y) denotes a typical sample point.

Solution: We know that

random variables	**description**
X	number appearing on first dice
Y	number appearing on second dice
Z	Sum of numbers appearing on both dice

Table 6.12: Two dice roll

$$p_Z(n) = \begin{cases} 0 & n \le 1 \\ \frac{n-1}{36} & 2 \le n \le 6 \\ \frac{13-n}{36} & 7 \le n \le 12 \\ 0 & n \ge 13 \end{cases} \tag{6.260}$$

$$\Pr(A) = p_Z(11) \tag{6.261}$$
$$= \frac{1}{18} \tag{6.262}$$
$$\Pr(B) = 1 - p_X(5) \tag{6.263}$$
$$= 1 - \frac{1}{6} \tag{6.264}$$
$$= \frac{5}{6} \tag{6.265}$$
$$\Pr(AB) = p_{XY}(6,5) \tag{6.266}$$
$$= p_X(6) \times p_Y(5) \tag{6.267}$$
$$= \frac{1}{36} \tag{6.268}$$
$$\Pr(A) \times \Pr(B) = \frac{5}{108} \tag{6.269}$$

$\because \Pr(AB) \neq \Pr(A) \times \Pr(B)$

$\therefore$ A and B are not independent events

6.2.8 Find the probability distribution of the maximum of the two scores obtained when a die is thrown twice. Determine also the mean of the distribution.

Solution: Let us define two random variables X and Y which represent the scores of the two dices which are rolled and random variable Z which represents the maximum of the two scores.

Random variables	value
X	$1 \leq X \leq 6$
Y	$1 \leq Y \leq 6$

$$Z = \max(X, Y) = \begin{cases} X & \text{if } X > Y \\ Y & \text{if } Y \geq X \end{cases} \tag{6.270}$$

$$F_Z(z) = \Pr\left(\{\max(X, Y) \leq z\}\right) \tag{6.271}$$

$$= \Pr\left(\{(X \leq z, X > Y) \cup (Y \leq z, X \leq Y)\}\right) \tag{6.272}$$

$$= \Pr\left(\{X \leq z, X > Y\}\right) + \Pr\left(\{Y \leq z, X \leq Y\}\right) \tag{6.273}$$

Since $\{X > Y\}$ and $\{X \leq Y\}$ are mutually exclusive sets that form a partition.

$$F_Z(z) = \Pr\left(\{X \leq z, Y \leq z\}\right) = F_{XY}(z, z) \tag{6.274}$$

if X,Y are independent, then

$$F_Z(z) = F_X(z) \cdot F_Y(z) \tag{6.275}$$

Finding $F_X(z)$ and $F_Y(z)$ for some random z.

$$F_X(z) = \begin{cases} 0, & \text{if } z < 1 \\ \frac{z}{6} & 1 \leq z \leq 6 \\ 1, & \text{if } z > 6 \end{cases} \tag{6.276}$$

$$F_Y(z) = \begin{cases} 0, & \text{if } z < 1 \\ \frac{z}{6}, & \text{if } 1 \leq z \leq 6 \\ 1, & \text{if } z > 6 \end{cases} \tag{6.277}$$

Finding $F_Z(z)$ for some random z.

$$F_Z(z) = \begin{cases} 0, & \text{if } z < 1 \\ \frac{z^2}{6}, & 1 \leq z \leq 6 \\ 1, & \text{if } z > 6 \end{cases} \tag{6.278}$$

$$p_Z(z) = F_Z(z) - F_Z(z-1) = \frac{2z-1}{36} \tag{6.279}$$

$$p_Z(z) = \begin{cases} \frac{2z-1}{36} & \text{if } 1 \leq z \leq 6 \\ 0 & \text{otherwise} \end{cases} \tag{6.280}$$

The mean of the distribution is given by:

$$\mu = \sum_{z=1}^{6} z \cdot p_Z(z) \tag{6.281}$$

$$= \sum_{z=1}^{6} z \cdot \frac{2z-1}{36} \tag{6.282}$$

$$= \sum_{z=1}^{6} \frac{2z^2 - z}{36} \tag{6.283}$$

$$= \sum_{z=1}^{6} \frac{2z^2}{36} - \sum_{z=1}^{6} \frac{z}{36} \tag{6.284}$$

$$= \frac{161}{36} \tag{6.285}$$

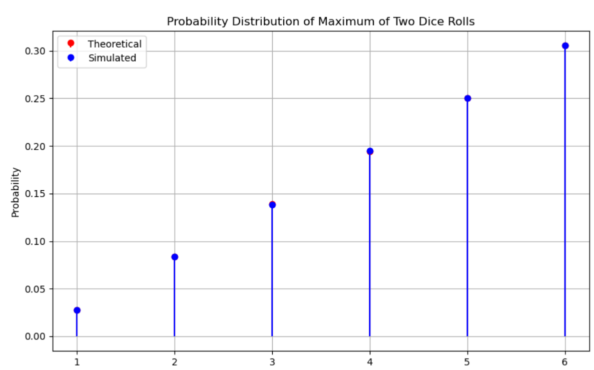


Figure 6.16: Probabilities - Simulation and theoretical.

6.2.9 Two dice are thrown. If it is known that sum of the numbers on the dice was less than 6,the probability of getting a sum 3, is

A) $\frac{1}{18}$

B) $\frac{5}{18}$

C) $\frac{1}{5}$

D) $\frac{2}{5}$

Solution: Let random variables such that

parameters	**value**	**description**
X	$1 \leq X \leq 6$	outcome of the first die
Y	$1 \leq Y \leq 6$	outcome of the second die

Consider a random variable W such that

$$W = X + Y; \tag{6.286}$$

W can take values from $\{2 \text{ to } 12\}$,

$$p_X(k) = \begin{cases} \frac{1}{6}, & 1 \leq k \leq 6 \\ 0, & \text{otherwise} \end{cases} \tag{6.287}$$

$$p_X(k) = p_Y(k) \tag{6.288}$$

Since X and Y are independent events,

$$M_W(z) = P_X(z)P_Y(z) \tag{6.289}$$

$$M_W(z) = \frac{1}{36}\left(z^{-1} + ... + z^{-6}\right) \cdot \left(z^{-1} + ... + z^{-6}\right) \tag{6.290}$$

$$= p_W(2) + p_W(3)z + ... + p_W(k)z^k + ... \tag{6.291}$$

$$= \frac{1}{36}(z^{-2} + 2z^{-3} + 3z^{-4} + 4z^{-5} + 5z^{-6} + 6z^{-7}$$

$$+ 5z^{-8} + 4z^{-9} + 3z^{-10} + 2z^{-11} + z^{-12}) \tag{6.292}$$

From (6.292) ,

$$\Pr(W = 3) = \frac{2}{36} \tag{6.293}$$

$$\Pr(W < 6) = \Pr(W = 2) + \Pr(W = 3) + \Pr(W = 4) + \Pr(W = 5) \tag{6.294}$$

$$= \frac{1}{36} + \frac{2}{36} + \frac{3}{36} + \frac{4}{36} \tag{6.295}$$

$$= \frac{10}{36} \tag{6.296}$$

We know,

$$\Pr(W = 3|W < 6) = \frac{\Pr((W = 3)(W < 6))}{\Pr(W < 6)} \tag{6.297}$$

$$= \frac{\frac{2}{36}}{\frac{10}{36}} \tag{6.298}$$

$$= \frac{2}{10} \tag{6.299}$$

$$= \frac{1}{5} \tag{6.300}$$

6.2.10 Two dice are thrown at the same time. Determine the probabiity that the difference of the numbers on the two dice is 2.

Solution: Let X, Y and Z be random variables with definition given as under:

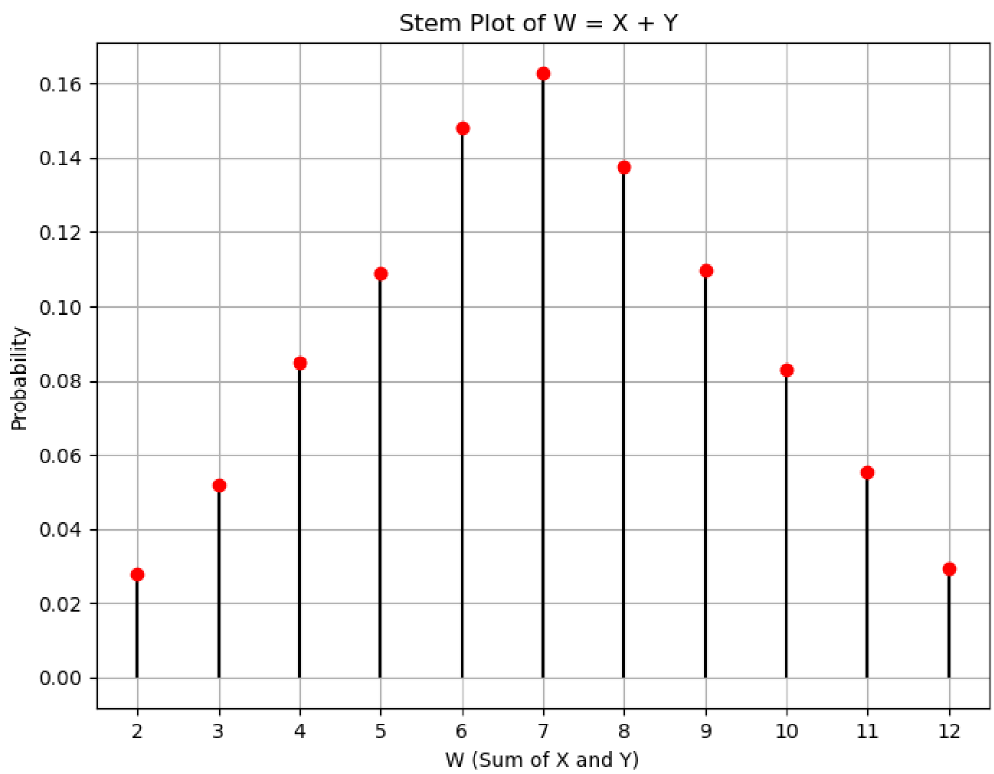


Figure 6.17: Stem plot for P(Z)

X	Number appearing on the first dice	
Y	Number appearing on the second dice	
Z	Difference of the numbers appearing on the dice	X - Y

Table 6.13: Definition of Random variables.

$$p_X(k) = \frac{1}{6} \tag{6.301}$$

$$p_Y(k) = p_X(k) \tag{6.302}$$

PMF of W using z-transform: applying the z-transform on both the sides

$$\begin{aligned}
M_Z(z) &= M_{X-Y}(z) && (6.303)\\
&= M_X(z)\cdot M_Y(z^{-1}) && (6.304)\\
&= \Sigma p_X(k)z^{-k}\cdot\Sigma p_Y(k)z^{k} && (6.305)\\
&= \frac{1}{6}(z^{-1}+z^{-2}+z^{-3}+z^{-4}+z^{-5}+z^{-6})\cdot\frac{1}{6}(z+z^2\\
&\quad +z^3+z^4+z^5+z^6) && (6.306)\\
&= \frac{1}{36}(z^{-5}+2z^{-4}+3z^{-3}+4z^{-2}+5z^{-1}+6z^0\\
&\quad +5z^1+4z^2+3z^3+2z^4+z^5) && (6.307)
\end{aligned}$$

Now, we also know that,

$$M_Z(z) = \Sigma p_Z(k)z^{-k} \tag{6.308}$$

Therefore,

$$\begin{aligned}
\Sigma p_Z(k)z^{-k} &= \frac{1}{36}(z^{-5}+2z^{-4}+3z^{-3}+4z^{-2}+5z^{-1}+6z^0\\
&\quad +5z^1+4z^2+3z^3+2z^4+z^5) && (6.309)
\end{aligned}$$

From (6.309), we can extract the PMF of Z.

Now, let E be the event that the difference of the numbers on the two dice is 2. Then,

$$p(E) = p_Z(2)+p_Z(-2) \tag{6.310}$$

$$\tag{6.311}$$

From (6.309), we find that,

$$p_Z(2) = \frac{1}{9} \tag{6.312}$$

$$p_Z(-2) = \frac{1}{9} \tag{6.313}$$

Therefore,

$$p(E) = \frac{2}{9} \tag{6.314}$$

Shown below is a graph visualising a simulation of the given question, with the 2 dice being rolled 10,000 times and comparing it to the theoretical probability

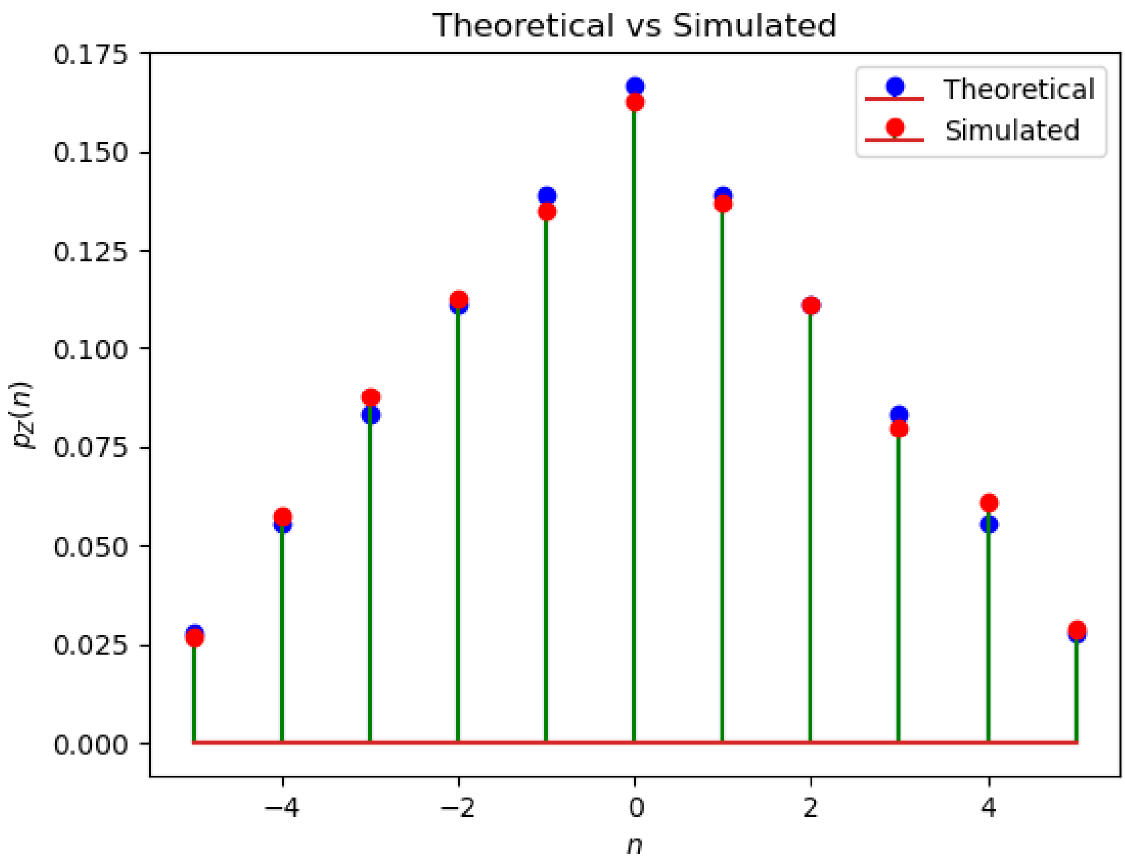


Figure 6.18: Stem plot for P(Z)

Chapter 7

Markov Chain

7.1 Consider the experiment of throwing a die.

- If a multiple of 3 comes up, throw the die again
- If any other number comes, toss a coin.

Find the conditional probability of the event 'the coin shows a tail', given that 'at least one die shows a 3'.

(a) See Table 7.1 and Fig. 7.1.

i	**State** (e_i)
0	$Y = 3$ OR $Y = 6$
1	$\sum (Y = k) \, ; k \in \{1, 2, 4, 5\}$
2	Obtaining heads from coin toss
3	Obtaining tails from coin toss

Table 7.1: States in Markov Chain

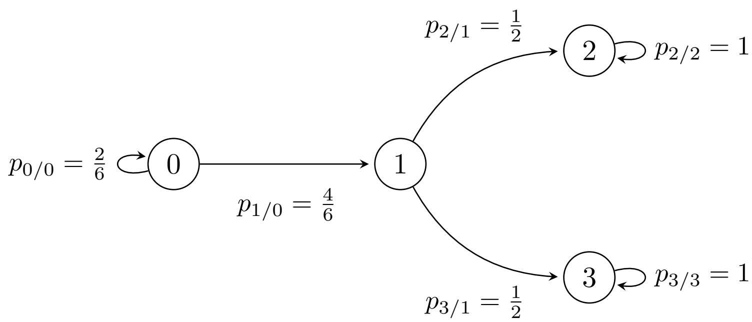


Figure 7.1: Graph of Markov Chain

(b) The state vector is,

$$\mathbf{Q}_n = \begin{pmatrix} p_0^{(n)} \\ p_1^{(n)} \\ p_2^{(n)} \\ p_3^{(n)} \end{pmatrix} \tag{7.1}$$

The probabilities after one step in time are

$$p_0^{(n+1)} = \frac{2}{6} \times p_0^{(n)} \tag{7.2}$$

$$p_1^{(n+1)} = \frac{4}{6} \times p_0^{(n)} \tag{7.3}$$

$$p_2^{(n+1)} = \frac{1}{2} \times p_1^{(n)} + 1 \times p_2^{(n)} \tag{7.4}$$

$$p_3^{(n+1)} = \frac{1}{2} \times p_1^{(n)} + 1 \times p_3^{(n)} \tag{7.5}$$

(c) The previous equations can be summarized as

$$\mathbf{Q}_{n+1} = \mathbf{P}\mathbf{Q}_n \tag{7.6}$$

Where $\mathbf{P}$ is the transition probability matrix. Its elements are values of $p_{i|j}$

$$\mathbf{P} = \begin{pmatrix} 2/6 & 0 & 0 & 0 \\ 4/6 & 0 & 0 & 0 \\ 0 & 1/2 & 1 & 0 \\ 0 & 1/2 & 0 & 1 \end{pmatrix} \tag{7.7}$$

(d) The given condition is that '3 occurs at least once'. Let the first occurrence of 3 be the initial state $\mathbf{Q}_0$.

$$\mathbf{Q}_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{7.8}$$

Using (7.6), further states can be generated.

$$\mathbf{Q}_1 = \mathbf{P}\mathbf{Q}_0 = \begin{pmatrix} \frac{2}{6} \\ \frac{4}{6} \\ 0 \\ 0 \end{pmatrix} \tag{7.9}$$

$$\mathbf{Q}_2 = \mathbf{P}\mathbf{Q}_1 = \mathbf{P}^2\mathbf{Q}_0 = \begin{pmatrix} \frac{4}{9} \\ \frac{8}{9} \\ \frac{5}{24} \\ \frac{5}{12} \end{pmatrix} \tag{7.10}$$

$$\vdots \tag{7.11}$$

$$\mathbf{Q}_n = \mathbf{P}^n\mathbf{Q}_0 \tag{7.12}$$

(e) Now to find the eigen values,

$$\left|\mathbf{P} - \lambda\mathbf{I}\right| = 0 \tag{7.13}$$

$$\implies \begin{pmatrix} 2/6-\lambda & 0 & 0 & 0 \\ 4/6 & -\lambda & 0 & 0 \\ 0 & 1/2 & 1-\lambda & 0 \\ 0 & 1/2 & 0 & 1-\lambda \end{pmatrix} = 0 \tag{7.14}$$

$$\implies \lambda\left(\frac{2}{6} - \lambda\right)\left(1-\lambda^2\right) = 0 \tag{7.15}$$

$$\text{or, } \lambda = \frac{2}{6}, 0, 1, 1 \tag{7.16}$$

The corresponding eigenvectors are

i. $\lambda = \frac{2}{6}$

$$\mathbf{X} = \begin{pmatrix} \frac{-2}{3} \\ \frac{-4}{3} \\ 1 \\ 1 \end{pmatrix} \tag{7.17}$$

ii. $\lambda = 0$

$$\mathbf{X} = \begin{pmatrix} 0 \\ -2 \\ 1 \\ 1 \end{pmatrix} \tag{7.18}$$

iii. $\lambda = 1$

$$\mathbf{X} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{7.19}$$

$$\tag{7.20}$$

resulting in the eigenvector matrix

$$\mathbf{S} = \begin{pmatrix} -2/3 & 0 & 0 & 0 \\ -4/3 & -2 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \tag{7.21}$$

Thus,

$$\mathbf{P} = \mathbf{S}\mathbf{D}\mathbf{S}^{-1} \tag{7.22}$$

Where $\mathbf{D}$ is eigenvalue matrix

$$\mathbf{D} = \begin{pmatrix} 1/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{7.23}$$

(f)

$$\mathbf{P}^n = (\mathbf{S}\mathbf{D}\mathbf{S}^{-1})(\mathbf{S}\mathbf{D}\mathbf{S}^{-1})\dots(\mathbf{S}\mathbf{D}\mathbf{S}^{-1}) \tag{7.24}$$

$$\implies = \mathbf{S}\mathbf{D}^n\mathbf{S}^{-1} \tag{7.25}$$

$$\implies \lim_{n\to\infty} \mathbf{P}^n = \lim_{n\to\infty} \mathbf{S}\mathbf{D}^n\mathbf{S}^{-1} \tag{7.26}$$

From (7.6),

$$\mathbf{Q}_n = \mathbf{P}^n\mathbf{Q}_0 \tag{7.27}$$

and Now,

$$\lim_{\mathbf{n}\to\infty} \mathbf{D^n} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{7.28}$$

$$\implies \lim_{\mathbf{n}\to\infty} \mathbf{SD^nS^{-1}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathbf{P^n} \tag{7.29}$$

$$\implies \mathbf{Q}_n = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mathbf{Q}_0 \tag{7.30}$$

$$\implies \mathbf{Q}_n = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{7.31}$$

(g) Probability of the coin showing tails, given that at least one die shows a 3,

$$\lim_{n\to\infty} p_3^{(n)} = 0 \tag{7.32}$$

7.2 A and B throw a pair of dice alternately. A wins the game if he gets a total of 6 and B wins if she gets a total of 7. It A starts the game, find the probability of winning the game by A in third throw of the pair of dice.

Solution:

Let state defined be

State	description
X_0	A rolls dice
X_1	B rolls dice
X_2	A wins
X_3	B wins
X_4	game stops

Table 7.2: States

Markov chain

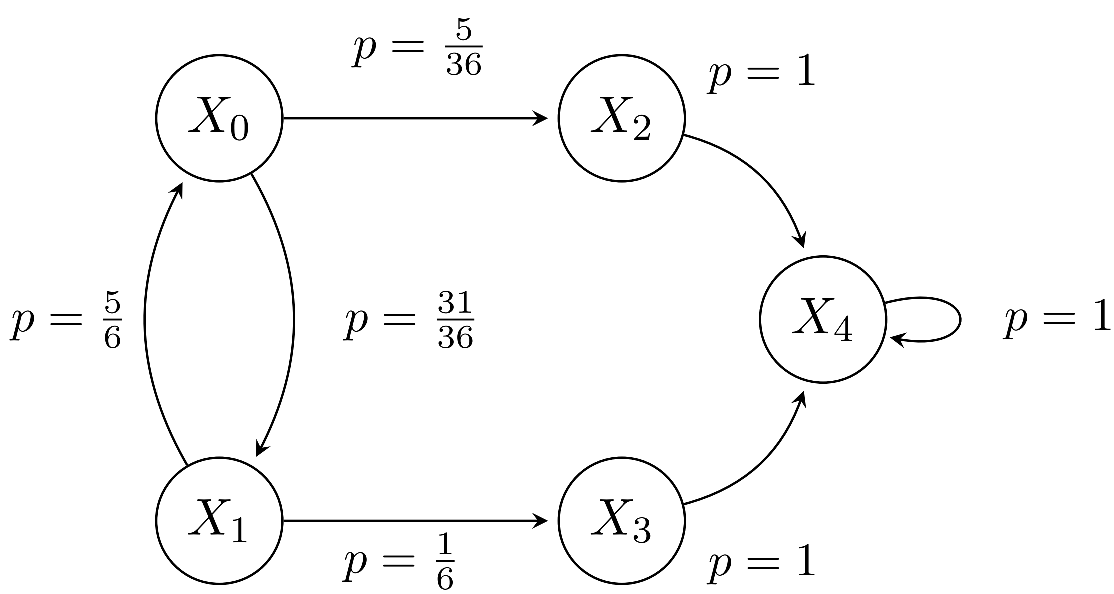


Figure 7.2: State diagram generated using LatexTikZ

Initial state vector

$$\mathbf{S}_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{7.33}$$

Transition matrix

$$\mathbf{P} = \begin{pmatrix} 0 & \frac{5}{6} & 0 & 0 & 0 \\ \frac{31}{36} & 0 & 0 & 0 & 0 \\ \frac{5}{36} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{6} & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix} \tag{7.34}$$

Probablity of A winning in third throw given A starts first

$$\mathbf{S}_1 = \mathbf{P}\mathbf{S_0} \tag{7.35}$$

$$\mathbf{S_2} = \mathbf{P}\mathbf{S_1} \tag{7.36}$$

$$\mathbf{S_3} = \mathbf{P}\mathbf{S_2} \tag{7.37}$$

$$= \mathbf{P^3}\mathbf{S_0} \tag{7.38}$$

$$= \begin{pmatrix} \frac{775}{1296} & 0 & 0 & 0 & 0 \\ \frac{4805}{7776} & 0 & 0 & 0 & 0 \\ \frac{775}{7776} & 0 & 0 & 0 & 0 \\ 0 & \frac{155}{1296} & 0 & 0 & 0 \\ \frac{61}{216} & \frac{61}{216} & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{7.39}$$

$$\mathbf{S_3} = \begin{pmatrix} 0.597 \\ 0.617 \\ 0.099 \\ 0 \\ 0.282 \end{pmatrix} \tag{7.40}$$

$$\mathbf{S_3}[2] = 0.099 \tag{7.41}$$

7.3 A state transition diagram with states A, B, and C, and transition probabilities p_1, p_2,...., p_7 is shown in the figure (e.g., p_1 denotes the probability of transition from state A to B). For this state diagram, select the statement(s) which is/are universally true

(a) $p_2 + p_3 = p_5 + p_6$

(b) $p_1 + p_3 = p_4 + p_6$

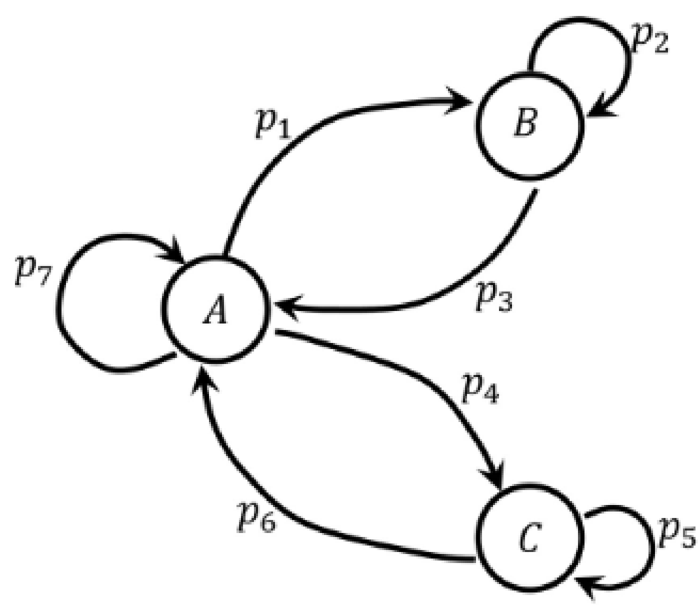


Figure 7.3: Figure1

(c) $p_1 + p_4 + p_7 = 1$

(d) $p_2 + p_5 + p_7 = 1$

Solution:

From, the given state diagram we can make a Transition matrix as:

$$M = \begin{pmatrix} p_7 & p_1 & p_4 \\ p_3 & p_2 & 0 \\ p_6 & 0 & p_5 \end{pmatrix} \tag{7.42}$$

And a valid transition matrix for a Markov Chain, the sum of rows should be 1 which gives:

$$p_7 + p_1 + p_4 = 1 \tag{7.43}$$

$$p_3 + p_2 = 1 \tag{7.44}$$

$$p_6 + p_5 = 1 \tag{7.45}$$

From (7.44) and (7.45) we have:

$$p_3 + p_2 = p_6 + p_5 \tag{7.46}$$

$\therefore$ Option (7.3a) and (7.3c) are correct.

Chapter 8

Continuous Distributions

8.1. Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

8.1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .

Solution: Execute the following C program.

```
#include <stdio.h>
#include <stdlib.h>
#include <math.h>
#include "coeffs.h"

int main(void) //main function begins
{

//Uniform random numbers
uniform("uni.dat", 1000000);

//Gaussian random numbers
```

```
gaussian("gau.dat", 1000000);


//Mean of uniform
//printf("%lf",mean("uni.dat"));
return 0;
}
```

```
//Function declaration
void uniform(char *str, int len);
void gaussian(char *str, int len);
double mean(char *str);
//End function declaration




//Defining the function for generating uniform random numbers
void uniform(char *str, int len)
{
int i;
FILE *fp;

fp = fopen(str,"w");
//Generate numbers
for (i = 0; i < len; i++)
{
```

```
fprintf(fp,"%lf\n",(double)rand()/RAND_MAX);
}
fclose(fp);


}
//End function for generating uniform random numbers


//Defining the function for calculating the mean of random numbers
double mean(char *str)
{
int i=0,c;
FILE *fp;
double x, temp=0.0;


fp = fopen(str,"r");
//get numbers from file
while(fscanf(fp,"%lf",&x)!=EOF)
{
//Count numbers in file
i=i+1;
//Add all numbers in file
temp = temp+x;
}
fclose(fp);
temp = temp/(i-1);
```

```
return temp;


}
//End function for calculating the mean of random numbers


//Defining the function for generating Gaussian random numbers
void gaussian(char *str, int len)
{
int i,j;
double temp;
FILE *fp;


fp = fopen(str,"w");
//Generate numbers
for (i = 0; i < len; i++)
{
temp = 0;
for (j = 0; j < 12; j++)
{
temp += (double)rand()/RAND_MAX;
}
temp-=6;
fprintf(fp,"%lf\n",temp);
}
fclose(fp);
```

```
}
//End function for generating Gaussian random numbers
```

8.1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr\left(U \le x\right) \tag{8.1}$$

Solution: The following code plots Fig. 8.1

```
#Importing numpy, scipy, mpmath and pyplot
import numpy as np
import matplotlib.pyplot as plt

#if using termux
import subprocess
import shlex
#end if



x = np.linspace(-4,4,30)#points on the x axis
simlen = int(1e6) #number of samples
err = [] #declaring probability list
#randvar = np.random.normal(0,1,simlen)
```

```
randvar = np.loadtxt('uni.dat',dtype='double')
#randvar = np.loadtxt('gau.dat',dtype='double')
for i in range(0,30):
	err_ind = np.nonzero(randvar < x[i]) #checking probability condition
	err_n = np.size(err_ind) #computing the probability
	err.append(err_n/simlen) #storing the probability values in a list



plt.plot(x.T,err)#plotting the CDF
plt.grid() #creating the grid
plt.xlabel('$x$')
plt.ylabel('$F_X(x)$')


#if using termux
plt.savefig('../figs/uni_cdf.pdf')
plt.savefig('../figs/uni_cdf.eps')
subprocess.run(shlex.split("termux-open ../figs/uni_cdf.pdf"))
#if using termux
#plt.savefig('../figs/gauss_cdf.pdf')
#plt.savefig('../figs/gauss_cdf.eps')
#subprocess.run(shlex.split("termux-open ../figs/gauss_cdf.pdf"))
#else
#plt.show() #opening the plot window
```

8.1.3 Find a theoretical expression for $F_U(x)$.

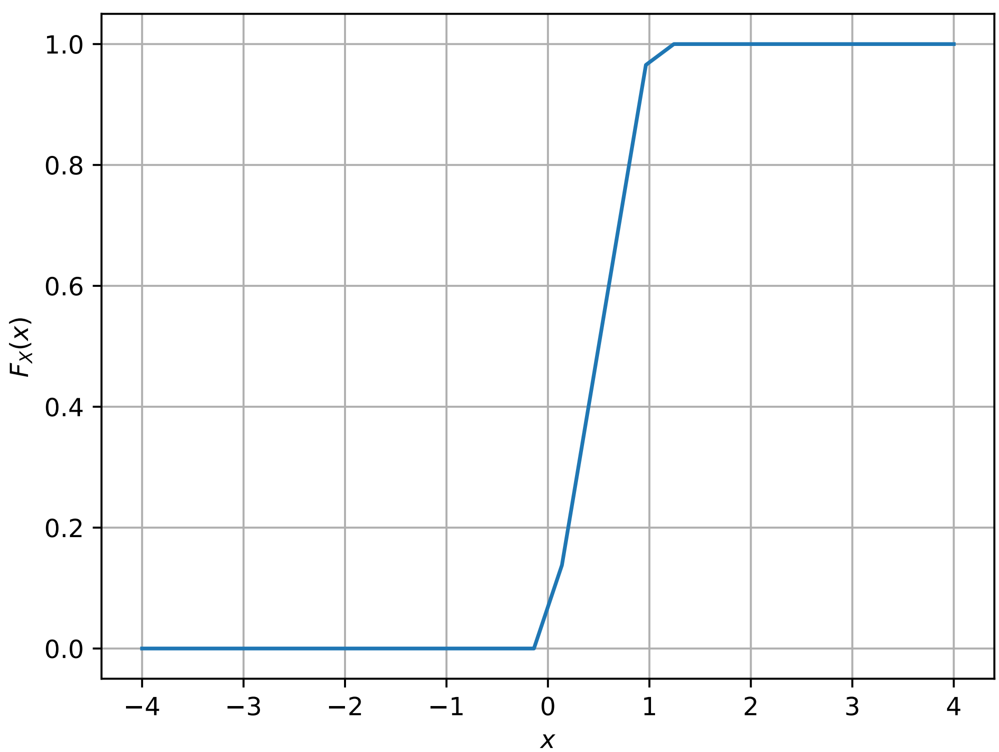


Figure 8.1: The CDF of U

8.1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{8.2}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{8.3}$$

Write a C program to find the mean and variance of U.

8.1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{8.4}$$

8.2. Central Limit Theorem

8.2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{8.5}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

8.2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?

Solution: The CDF of X is plotted in Fig. 8.2

8.2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat.

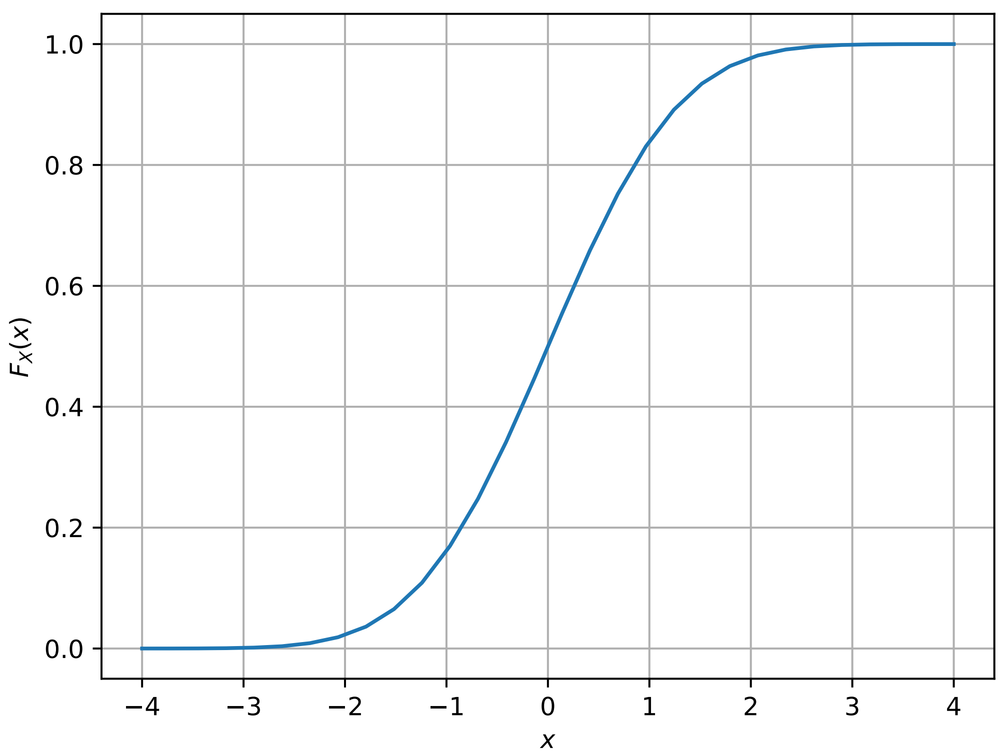


Figure 8.2: The CDF of X

The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{8.6}$$

What properties does the PDF have?

Solution: The PDF of X is plotted in Fig. 8.3 using the code below

```
#Importing numpy, scipy, mpmath and pyplot
import numpy as np
import mpmath as mp
import scipy
import matplotlib.pyplot as plt

#if using termux
import subprocess
import shlex
#end if



maxrange=50
maxlim=6.0
x = np.linspace(-maxlim,maxlim,maxrange)#points on the x axis
simlen = int(1e6) #number of samples
err = [] #declaring probability list
pdf = [] #declaring pdf list
h = 2*maxlim/(maxrange-1);
#randvar = np.random.normal(0,1,simlen)
```

```
#randvar = np.loadtxt('uni.dat',dtype='double')
randvar = np.loadtxt('gau.dat',dtype='double')

for i in range(0,maxrange):
        err_ind = np.nonzero(randvar < x[i]) #checking probability condition
        err_n = np.size(err_ind) #computing the probability
        err.append(err_n/simlen) #storing the probability values in a list



for i in range(0,maxrange-1):
        test = (err[i+1]-err[i])/(x[i+1]-x[i])
        pdf.append(test) #storing the pdf values in a list

def gauss_pdf(x):
        return 1/mp.sqrt(2*np.pi)*np.exp(-x**2/2.0)

vec_gauss_pdf = scipy.vectorize(gauss_pdf)

plt.plot(x[0:(maxrange-1)].T,pdf,'o')
plt.plot(x,vec_gauss_pdf(x))#plotting the CDF
plt.grid() #creating the grid
plt.xlabel('$x_i$')
plt.ylabel('$p_X(x_i)$')
plt.legend(["Numerical","Theory"])
```

```
#if using termux
#plt.savefig('../figs/uni_pdf.pdf')
#plt.savefig('../figs/uni_pdf.eps')
#subprocess.run(shlex.split("termux-open ../figs/uni_pdf.pdf"))
#if using termux
plt.savefig('../figs/gauss_pdf.pdf')
plt.savefig('../figs/gauss_pdf.eps')
subprocess.run(shlex.split("termux-open ../figs/gauss_pdf.pdf"))
#else
#plt.show() #opening the plot window
```

8.2.4 Find the mean and variance of X by writing a C program.

8.2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{8.7}$$

repeat the above exercise theoretically.

8.3. Gaussian

8.3.1 Suppose that 90 % of people are right-handed. What is the probability that atmost 6 of a random sample of 10 people are right-handed.

Solution:

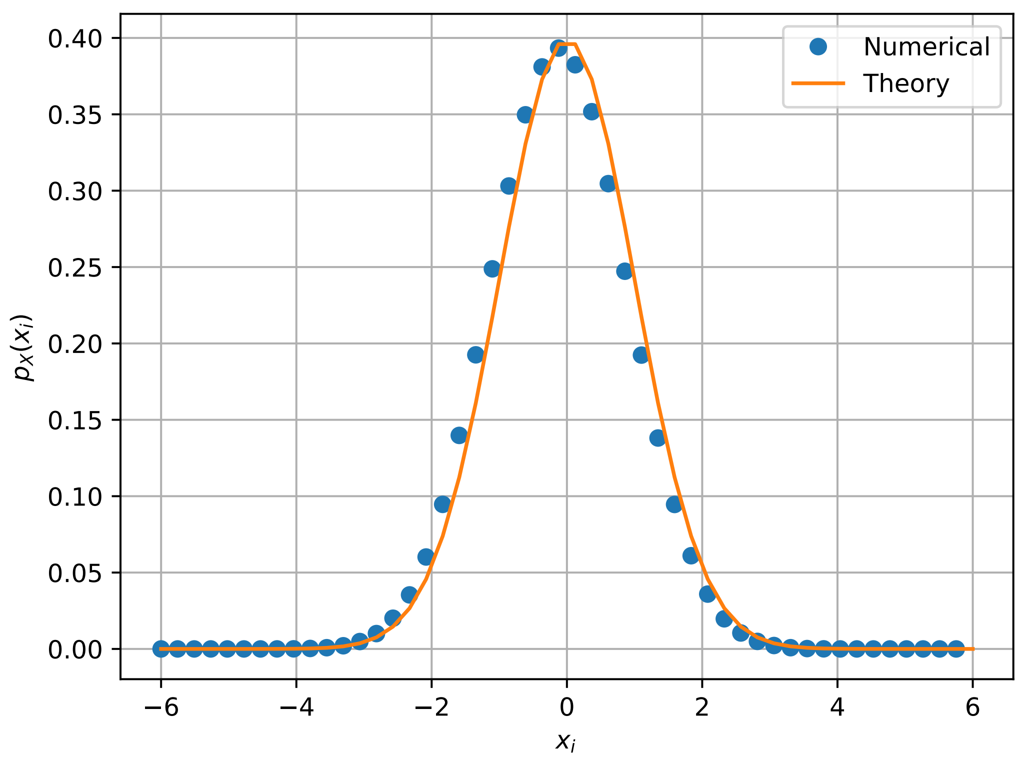


Figure 8.3: The PDF of X

Let,

$$Z = \frac{X - \mu}{\sigma}, \ \mu = np, \ \sigma^2 = npq \tag{8.8}$$

For large n,

$$Z \sim \mathcal{N}(0, 1) \tag{8.9}$$

$$\implies \Pr(Z < -2.63) = \Pr(-Z > 2.63) \equiv \Pr(Z > -2.63) = Q(2.63) \tag{8.10}$$

8.3.2 There are 5% defective items in a large bulk of items. What is the probability that a sample of 10 items will include not more than one defective item?

Solution: Gaussian Distribution

Parameter	Values	Description
n	10	Number of articles
p	0.05	Probability of being defective
Y	$0 \le Y \le 10$	Number of defective elements
$\mu = np$	0.5	mean
$\sigma = \sqrt{np(1-p)}$	0.475	standard deviation

(a) Central limit theorm:

$$Y \sim \mathcal{N}\left(\mu, \frac{\sigma}{\sqrt{n}}\right) \tag{8.11}$$

$$\tag{8.12}$$

Due to continuity correction $\Pr(X = x)$ can be approximated using gaussian

distribution as

$$p_Y(x) \approx \Pr(x - 0.05 < Y < x + 0.05) \tag{8.13}$$

$$\approx \Pr(Y < x + 0.05) - \Pr(Y < x - 0.05) \tag{8.14}$$

$$\approx F_Y(x + 0.05) - F_Y(x - 0.05) \tag{8.15}$$

Now, the CDF of Y can be found by;

$$F_Y(y) = \Pr(Y \le y) \tag{8.16}$$

$$= p_Y(y) \tag{8.17}$$

We also know that;

$$Q(x) = \Pr(X > x), x > 0, X \sim N(0,1) \tag{8.18}$$

$$Q(-x) = \Pr(X > -x), x < 0, X \sim N(0,1) \tag{8.19}$$

$$= 1 - Q(x) \tag{8.20}$$

Hence, the CDF is given as:

$$F_Y(y) = \begin{cases} 1 - Q\left(\frac{y-\mu}{\sigma}\right), & \text{if } y > \mu \\ 1 - Q\left(\frac{y-\mu}{\sigma}\right) = Q\left(\frac{\mu-y}{\sigma}\right), & \text{if } y < \mu \end{cases} \tag{8.21}$$

Now, we get:

$$
\begin{aligned}
F_Y(1) &= p_Y(1.05) && (8.22)\\
&= 1 - Q\left(\frac{1.05 - 0.5}{\sqrt{0.05}}\right) && (8.23)\\
&= 1 - Q\left(\frac{0.55}{0.2236}\right) && (8.24)\\
&= 1 - Q(2.4596) && (8.25)\\
&= 0.99304 && (8.26)
\end{aligned}
$$

(b) Binomial Distribution:

$$n = 10; p = \frac{1}{20} \tag{8.27}$$

Pmf of X for $0 \leq k \leq 10$ is

$$p_X(k) = {}^{n}C_k p^k (1-p)^{n-k} \tag{8.28}$$

Then the probability is given as:

$$p_X(0) + p_X(1) = {}^{10}C_0 \left(\frac{1}{20}\right)^0 \left(1 - \frac{1}{20}\right)^{10} + {}^{10}C_1 \left(\frac{1}{20}\right)^1 \left(1 - \frac{1}{20}\right)^9 \tag{8.29}$$

Hence we get;

$$p_X(0) + p_X(1) = 29\left(\frac{19^9}{20^{10}}\right) = 0.91386 \tag{8.30}$$

Hence we can say probability calculated through central limit theorem is very close to the one calculated through binomial distribution.

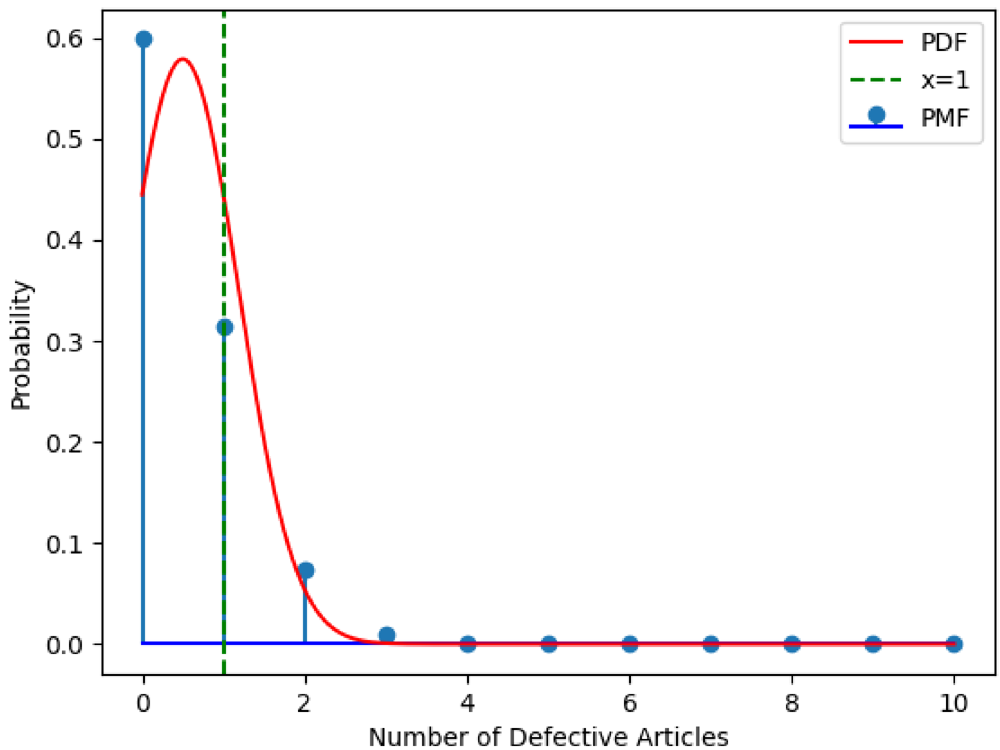


Figure 8.4: Binomial vs Gaussian

8.3.3 Five cards are drawn successively with replacement from a well-shuffled deck of 52 cards. What is the probability that

(a) all the five cards are spades?

(b) only 3 cards are spades?

(c) none is a spade?

Solution:

let Y be a gaussian Random variable

Parameter	Value	Description
X	{0,1,2,3,4,5}	Number of spade cards drawn
n	5	Number of cards drawn
p	0.25	Drawing a spade card
q	0.75	Drawing any other card
$\mu = np$	1.25	Mean of Binomial distribution
$\sigma^2 = npq$	0.9375	Varience of Binomial distribution

Table 8.1: Random variable and Parameter

$$Y \sim N(\mu, \sigma) \tag{8.31}$$

$$\sim N(1.25, 0.9375) \tag{8.32}$$

Due to continuity correction $\Pr(X = x)$ can be approximated using gaussian distribution as

$$p_Y(x) \approx \Pr(x - 0.5 < Y < x + 0.5) \tag{8.33}$$

$$\approx \Pr(Y < x + 0.5) - \Pr(Y < x - 0.5) \tag{8.34}$$

$$\approx F_Y(x + 0.5) - F_Y(x - 0.5) \tag{8.35}$$

CDF of Y is defined as:

$$F_Y(x) = \Pr(Y < x) \tag{8.36}$$

$$= \Pr\left(\frac{Y-\mu}{\sigma} < \frac{x-\mu}{\sigma}\right) \tag{8.37}$$

$$\implies \frac{Y-\mu}{\sigma} \sim N(0,1) \tag{8.38}$$

$$= 1 - \Pr\left(\frac{Y-\mu}{\sigma} > \frac{x-\mu}{\sigma}\right) \tag{8.39}$$

$$= \begin{cases} 1 - Q\left(\frac{x-\mu}{\sigma}\right) & x \geq \mu \\ Q\left(\frac{\mu-x}{\sigma}\right) & x < \mu \end{cases} \tag{8.40}$$

Then probability in terms of Q funtion is

$$\implies p_Y(x) \approx Q\left(\frac{(x-0.5)-\mu}{\sigma}\right) - Q\left(\frac{(x+0.5)-\mu}{\sigma}\right) \tag{8.41}$$

(a) The Gaussian approximation for $\Pr(X = 5)$ is

$$p_Y(5) \approx Q\left(\frac{4.5-1.25}{0.9375}\right) - Q\left(\frac{5.5-1.25}{0.9375}\right) \tag{8.42}$$

$$\approx Q(3.356) - Q(4.389) \tag{8.43}$$

$$\approx 0.0003888 \tag{8.44}$$

(b) The Gaussian approximation for $\Pr(X = 3)$ is

$$p_Y(3) \approx Q\left(\frac{2.5-1.25}{0.9375}\right) - Q\left(\frac{3.5-1.25}{0.9375}\right) \tag{8.45}$$

$$\approx Q(1.2909) - Q(2.3237) \tag{8.46}$$

$$\approx 0.08828 \tag{8.47}$$

(c) The Gaussian approximation for $\Pr(X = 0)$ is

$$p_Y(0) \approx Q\left(\frac{-0.5 - 1.25}{0.9375}\right) - Q\left(\frac{0.5 - 1.25}{0.9375}\right) \tag{8.48}$$

$$\approx (1 - Q(1.8073)) - (1 - Q(0.7745)) \tag{8.49}$$

$$= Q(0.7745) - Q(1.8073) \tag{8.50}$$

$$\approx 0.1839 \tag{8.51}$$

Comparison			
Number of spade cards	Binomial distribution	Gaussian approximation	Error (%)
5	0.0009765625	0.00038880	60.18688
3	0.087890625	0.088279	0.4430
0	0.2373046875	0.18390	22.5046

Table 8.2: Comparison between the approximation

8.3.4 In an examination, 20 questions of true-false type are asked. Suppose a student tosses a fair coin to determine his answer to each question. If the coin falls heads, he answer true; if it falls tails, he answer false. Find the probability that he answers at least 12 questions correctly.

Solution: Gaussian:

Table 8.3: Variables

Variable	Value	Description
n	20	Number of questions
p	0.5	probability of question being correct
$\mu = np$	10	mean of distribution
$\sigma = \sqrt{npq}$	$\sqrt{5}$	variance of distribution
X	$0 \leq X \leq 20$	Number of correct questions

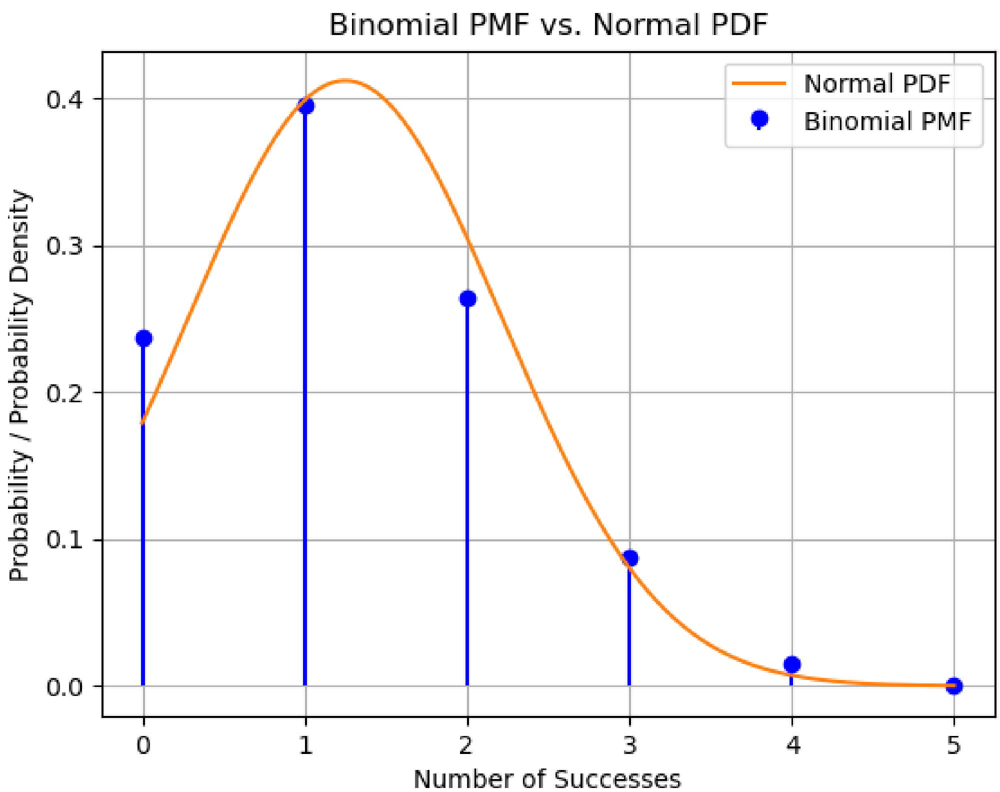


Figure 8.5: Binomial and gaussian distribution

$$Y \sim \mathcal{N}\left(\mu, \sigma^2\right) \tag{8.52}$$

CDF of Y is defined as:

$$F_Y(x) = \Pr(Y \le x) \tag{8.53}$$

$$= \Pr\left(\frac{Y-\mu}{\sigma} \le \frac{X-\mu}{\sigma}\right) \tag{8.54}$$

$$\frac{Y-\mu}{\sigma} \sim \mathcal{N}(0,1) \tag{8.55}$$

$$= 1 - \Pr\left(\frac{Y-\mu}{\sigma} > \frac{X-\mu}{\sigma}\right) \tag{8.56}$$

$$= 1 - Q\left(\frac{X-\mu}{\sigma}\right) \tag{8.57}$$

(a) Without correction:

$$\Pr(Y > 11) = 1 - \Pr(Y \le 11) \tag{8.58}$$

$$= 1 - F_Y(11) \tag{8.59}$$

$$\implies \Pr(Y > 11) = Q\left(\frac{X-\mu}{\sigma}\right) \tag{8.60}$$

$$= Q(0.894) \tag{8.61}$$

$$\Pr(Y > 11) = 0.1855 \tag{8.62}$$

(b) With a 0.5 correction:

$$\Pr(Y > 11) = Q\left(\frac{X-\mu+0.5}{\sigma}\right) \tag{8.63}$$

$$= Q(0.67) \tag{8.64}$$

$$\implies \Pr(Y > 11) = 0.2511 \tag{8.65}$$

Binomial:

$$\Pr(X \geq 12) = 1 - \Pr(X < 12) \tag{8.66}$$

$$= \sum_{k=12}^{20} {}^nC_k p^k (1-p)^{n-k} \tag{8.67}$$

$$= 0.2517 \tag{8.68}$$

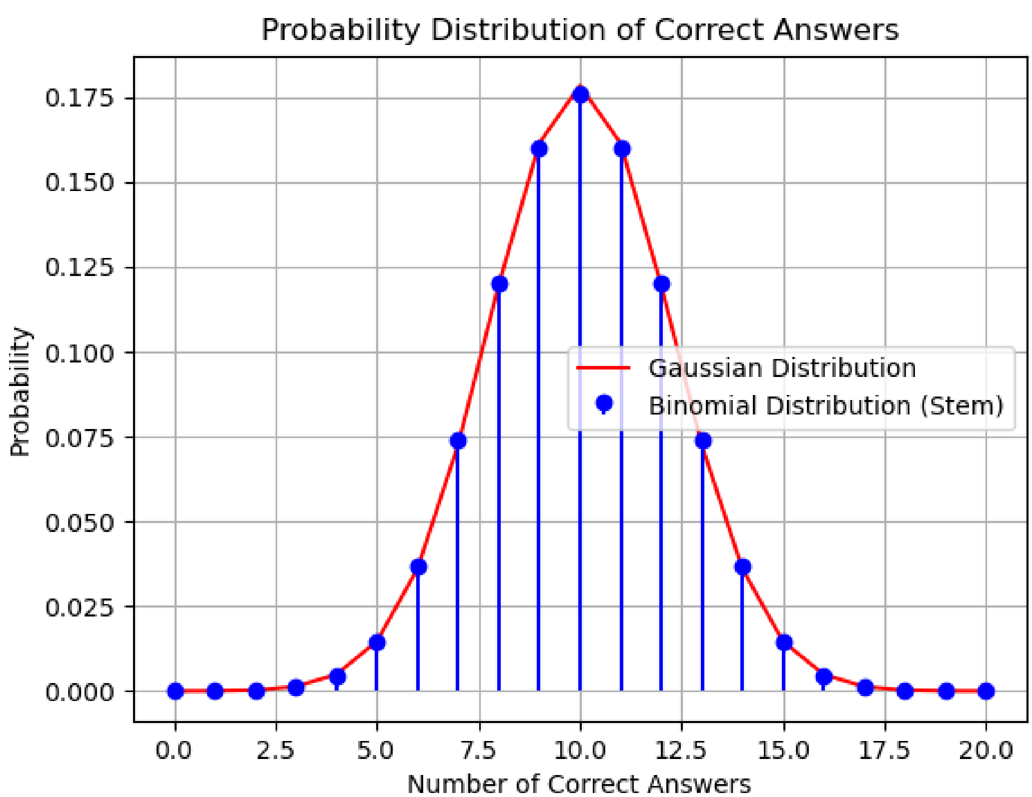


Figure 8.6: Binomial vs Gaussian

8.3.5 It is known that 10 % of certain articles manufactured are defective. What is the probability that in a random sample space of 12 such articles,9 are defective?

Solution: Let X be random variable defined as

Random Variable	Values	Description
X	$1 \leq X \leq 12$	Number of defective in 12 articles

X has a binomial distribution with parameters

$$n = 12 \qquad p = \frac{10}{100} = \frac{1}{10} \tag{8.69}$$

Pmf of X for $1 \leq k \leq 12$ is

$$p_X(k) = {}^nC_k p^k (1-p)^{n-k} \tag{8.70}$$

Probability that a random sample space of 12 such articles,9 are defective is

$$p_X(9) = {}^{12}C_9 \left(\frac{1}{10}\right)^9 \left(1 - \frac{1}{10}\right)^{12-9} \tag{8.71}$$

$$= \frac{12!}{9!3!} \left(\frac{1}{10}\right)^9 \left(\frac{9}{10}\right)^3 \tag{8.72}$$

$$= 220 \left(\frac{1}{10^9}\right) \left(\frac{9^3}{10^3}\right) \tag{8.73}$$

$$= 22 \left(\frac{9^3}{10^{11}}\right) \tag{8.74}$$

$$= 1.603773(10^{-7}) \tag{8.75}$$

Let Y be goussian variable

$$\mu = np \tag{8.76}$$

$$= \frac{6}{5} \tag{8.77}$$

$$\sigma^2 = np(1-p) \tag{8.78}$$
$$= \frac{27}{25} \tag{8.79}$$

Using Normal distribution at X=9.

$$Z = \frac{X-\mu}{\sigma} \tag{8.80}$$
$$= \frac{9-\frac{6}{5}}{\sqrt{\frac{27}{25}}} \tag{8.81}$$
$$= 7.50555 \tag{8.82}$$

For pdf calculation

$$f_Y(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \tag{8.83}$$

From the plot, pmf is close to normal distribution pdf.

$$p_Y(9) = p_Z(7.5055) \tag{8.84}$$
$$= 1.6109(10^{-7}) \tag{8.85}$$

From (8.75) and (8.85),

$$p_X(9) \approx p_Y(9) \tag{8.86}$$

8.3.6 The probability that a student is not a swimmer is $\frac{1}{5}$ .Then the probability that out of five students, four are swimmers

(a) ${}^5C_4 \left(\frac{4}{5}\right)^4 \frac{1}{5}$

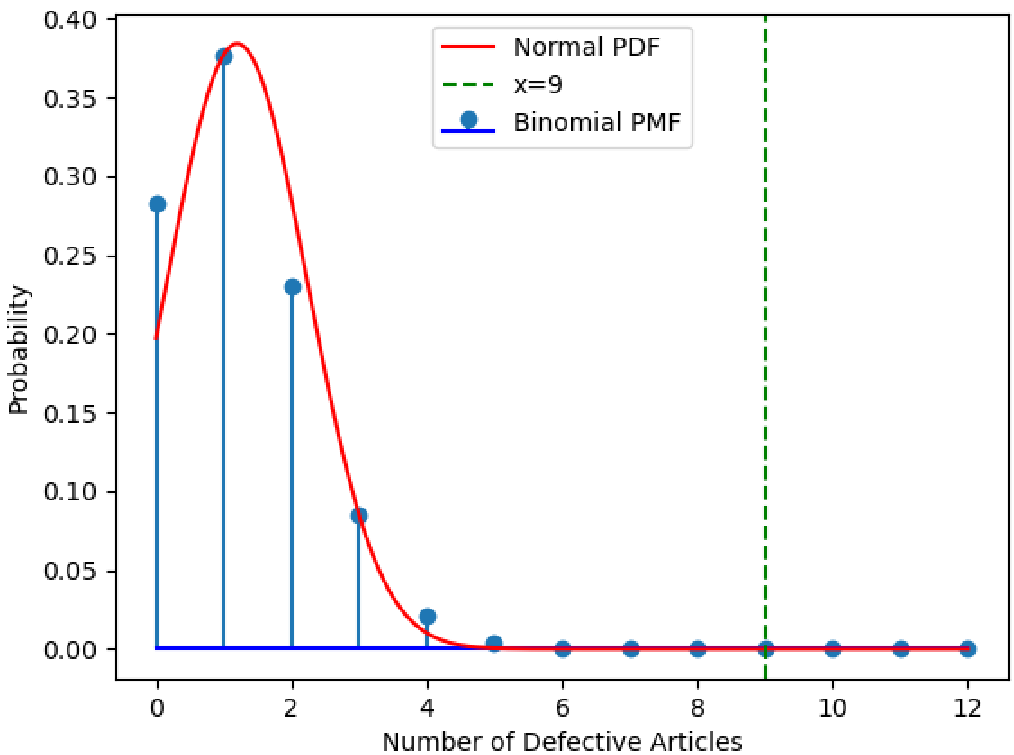


Figure 8.7: Binomial pmf vs Gaussian pdf

(b) $\left(\frac{4}{5}\right)^4 \frac{1}{5}$

(c) ${}^5C_1 \frac{1}{5} \left(\frac{4}{5}\right)^4$

(d) None of these

Solution: The X is the random variable,We require pmf at $X = 4$,

Parameter	Value	Description
n	5	number of students
q	$\frac{1}{5}$	not a swimmer
p	$\frac{4}{5}$	swimmer
k	4	number of swimmers
X	$0 \leq X \leq 5$	X swimmer out of 5
Y	$0 \leq Y \leq 5$	Gaussian variable
μ	$np = 4$	mean
σ^2	$npq = \frac{4}{5}$	variance

Table 8.4: Given Information

$$p_X(4) = {}^5C_4 \left(\frac{4}{5}\right)^4 \left(\frac{1}{5}\right)^{5-4} = 0.4096 \tag{8.87}$$

$$X \approx Y \sim \mathcal{N}(\mu, \sigma^2) \tag{8.88}$$

Using central limit theorem, we can use the gaussian distribution function:

$$p_Y(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad (x \in Y) \tag{8.89}$$

Now, using Normal distribution at Y=4

$$p_Y(4) = \frac{1}{\sqrt{2\pi\left(\frac{4}{5}\right)}} e^{-\frac{(4-4)^2}{2\left(\frac{4}{5}\right)}} \tag{8.90}$$

$$= \frac{1}{\sqrt{2\pi\left(\frac{4}{5}\right)}} e^0 \tag{8.91}$$

$$= 0.4463 \tag{8.92}$$

From the plot also the pmf is close to normal distribution pdf. Hence, $p_Y(4) \approx p_X(4)$

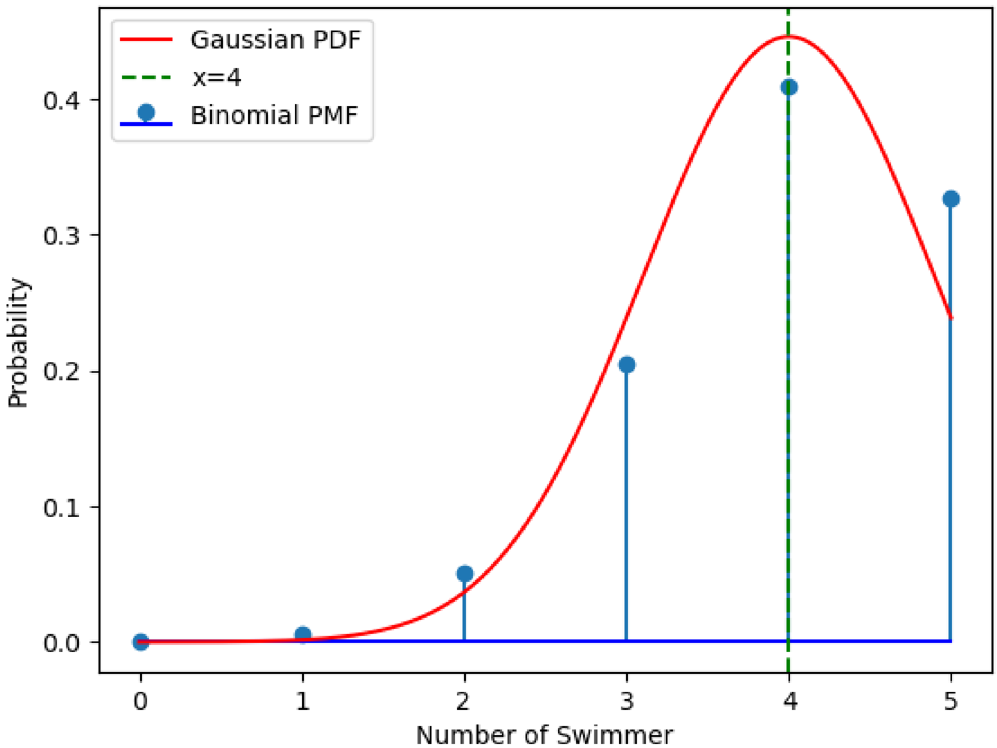


Figure 8.8: Binomial pmf vs Gaussian pdf

so, option (8.3.6c) is correct

8.3.7 There are 5% defective items in a large bulk of items. What is the probability that a sample of 10 items will include not more than one defective item?

Solution:

Parameter	Values	Description
n	10	Number of items
p	0.05	Probability of being defective
q	0.95	Probability of not being defective
$\mu = np$	0.5	Mean
$\sigma^2 = npq$	0.475	Variance

Table 8.5: Definition of parameters and their values

(a) Binomial: The cdf using binomial is given by

$$F_Y(n) = \Pr(Y \leq n) \tag{8.93}$$

$$= \sum_{k=0}^{n} {}^{10}C_k p^k (1-p)^{10-k} \tag{8.94}$$

We require $\Pr(Y \leq 1)$. Since $n = 1$,

$$F_Y(1) = \Pr(Y \leq 1) \tag{8.95}$$

$$= \sum_{k=0}^{1} {}^{10}C_k (0.05)^k (0.95)^{10-k} \tag{8.96}$$

$$= 0.9138 \tag{8.97}$$

(b) Gaussian: $Y \sim \mathcal{N}\left(\mu, \sigma^2\right)$

To obtain cdf,

$$\Pr(Y \leq 1) = F_Y(1) \tag{8.98}$$

$$F_Y(x) = \Pr(Y \leq x) \tag{8.99}$$

$$= \Pr(Y - \mu \leq x - \mu) \tag{8.100}$$

$$= \Pr(\frac{Y - \mu}{\sigma} \leq \frac{x - \mu}{\sigma}) \tag{8.101}$$

$$= 1 - \Pr(\frac{Y - \mu}{\sigma} > \frac{x - \mu}{\sigma}) \tag{8.102}$$

We know that,

$$\frac{Y - \mu}{\sigma} \sim \mathcal{N}(0, 1) \tag{8.103}$$

$$\Pr(X > x) = Q(x) \tag{8.104}$$

Hence,

$$F_Y(x) = 1 - Q\left(\frac{x - \mu}{\sigma}\right), \text{if } x > \mu \tag{8.105}$$

$$= Q\left(\frac{\mu - x}{\sigma}\right), \text{if } x < \mu \tag{8.106}$$

$$\implies F_Y(1) = 1 - Q\left(\frac{0.5}{\sqrt{0.475}}\right) \tag{8.107}$$

$$= 0.766 \tag{8.108}$$

With correction of 0.5,

$$\Pr(Y \leq 1.5) = F_Y(1.5) \tag{8.109}$$

$$F_Y(1.5) = 1 - Q\left(\frac{1}{\sqrt{0.475}}\right) \tag{8.110}$$

$$= 0.927 \tag{8.111}$$

From (8.97) and (8.111)

$$\Pr(Y \leq 1) \approx F_Y(1.5) \tag{8.112}$$

Binomial	Gaussian (without correction)	Gaussian (with correction)
0.9138	0.766	0.927

Table 8.6: Probability obtained using different methods

8.3.8 Five cards are drawn successively with replacement from a well-shuffled deck of 52 cards. What is the probability that

(a) all the five cards are spades?

(b) only 3 cards are spades?

(c) none is a spade?

Solution: Let us define:

(i) Gaussian Distribution

The gaussian distribution function is defined as:

$$p_Y(x) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad (x \in Y) \tag{8.113}$$

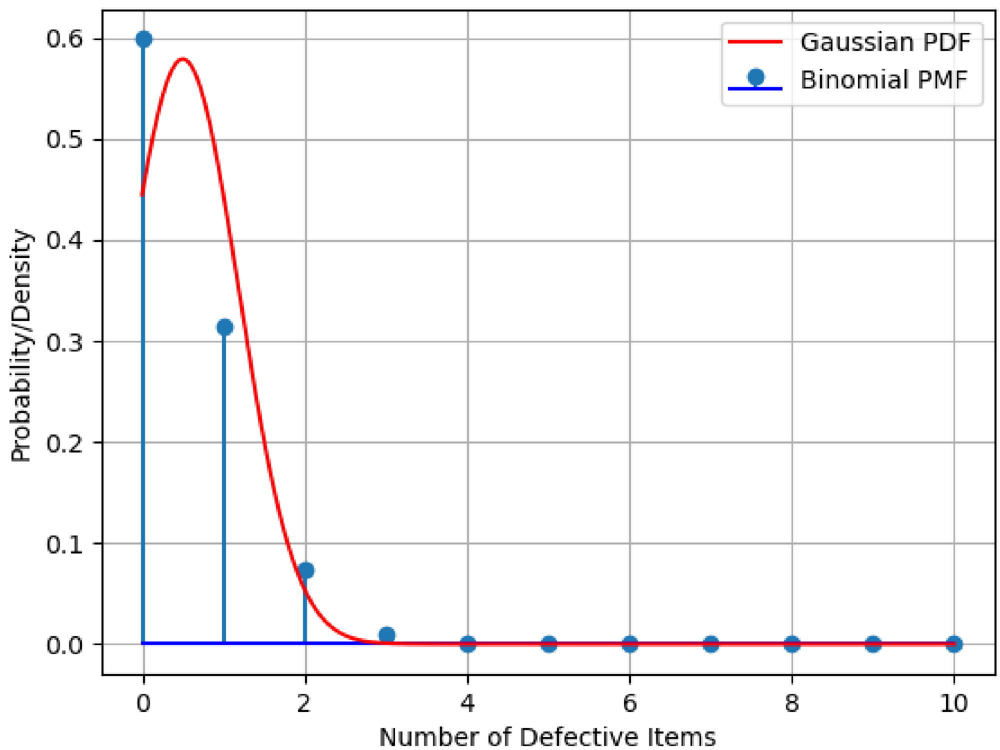


Figure 8.9: Binomial pmf vs Gaussian pdf

Parameter	Value	Description
n	5	number of cards drawn
p	$\frac{1}{4}$	drawing a spade card
q	$\frac{3}{4}$	drawing any other card
$\mu = np$	$\frac{5}{4}$	mean of the distribution
$\sigma^2 = npq$	$\frac{15}{16}$	variance of the distribution
Y	{0,1,2,3,4,5}	Number of spade cards drawn

If we consider all cards to be spades,

$$Y = 5p_Y(5) \qquad = \frac{1}{\sqrt{2\pi\left(\frac{15}{16}\right)}} e^{-\frac{\left(5-\frac{5}{4}\right)^2}{2\left(\frac{15}{16}\right)}} \tag{8.114}$$

$$= \frac{1}{\sqrt{2\pi\left(\frac{15}{16}\right)}} e^{-\frac{15}{2}} \tag{8.115}$$

If we consider 3 cards to be spades,

$$Y = 3p_Y(3) \qquad = \frac{1}{\sqrt{2\pi\left(\frac{15}{16}\right)}}e^{-\frac{\left(3-\frac{5}{4}\right)^2}{2\left(\frac{15}{16}\right)}} \tag{8.117}$$

$$= \frac{1}{\sqrt{2\pi\left(\frac{15}{16}\right)}}e^{-\frac{49}{30}} \tag{8.118}$$

$$= 0.044 \tag{8.119}$$

If we consider 0 cards to be spades,

$$Y = 0p_Y(0) \qquad = \frac{1}{\sqrt{2\pi\left(\frac{15}{16}\right)}}e^{-\frac{\left(0-\frac{5}{4}\right)^2}{2\left(\frac{15}{16}\right)}} \tag{8.120}$$

$$= \frac{1}{\sqrt{2\pi\left(\frac{15}{16}\right)}}e^{-\frac{5}{6}} \tag{8.121}$$

$$= 0.0978 \tag{8.122}$$

(ii) Solving using Q function

Consider a gaussian random variable Z,

$$Z \sim N(\mu, \sigma) \tag{8.123}$$

$$\sim N\left(\frac{5}{4}, \frac{\sqrt{15}}{4}\right) \tag{8.124}$$

Due to continuity correction $\Pr(Y = x)$ can be approximated using gaussian

distribution as

$$p_Z(x) \approx \Pr(x - 0.5 < Z < x + 0.5) \tag{8.125}$$

$$\approx \Pr(Z < x + 0.5) - \Pr(Z < x - 0.5) \tag{8.126}$$

$$\approx F_Z(x + 0.5) - F_Z(x - 0.5) \tag{8.127}$$

CDF of Z is defined as:

$$F_Z(x) = \Pr(Z < x) \tag{8.128}$$

$$= \Pr\left(\frac{Z - \mu}{\sigma} < \frac{x - \mu}{\sigma}\right) \tag{8.129}$$

$$\implies \frac{Z - \mu}{\sigma} \sim N(0, 1) \tag{8.130}$$

$$= 1 - \Pr\left(\frac{Z - \mu}{\sigma} > \frac{x - \mu}{\sigma}\right) \tag{8.131}$$

$$= \begin{cases} 1 - Q\left(\frac{x-\mu}{\sigma}\right) & x \geq \mu \\ Q\left(\frac{\mu-x}{\sigma}\right) & x < \mu \end{cases} \tag{8.132}$$

Then probability in terms of Q funtion is

$$\implies p_Z(x) \approx Q\left(\frac{(x - 0.5) - \mu}{\sigma}\right) - Q\left(\frac{(x + 0.5) - \mu}{\sigma}\right) \tag{8.133}$$

The Gaussian approximation for $\Pr(Y = 5)$ is

$$p_Z(5) \approx Q\left(\frac{4.5 - 1.25}{0.9375}\right) - Q\left(\frac{5.5 - 1.25}{0.9375}\right) \tag{8.134}$$

$$\approx Q(3.356) - Q(4.389) \tag{8.135}$$

$$\approx 0.0003888 \tag{8.136}$$

The Gaussian approximation for $\Pr(Y = 3)$ is

$$p_Y(3) \approx Q\left(\frac{2.5 - 1.25}{0.9375}\right) - Q\left(\frac{3.5 - 1.25}{0.9375}\right) \tag{8.137}$$

$$\approx Q(1.2909) - Q(2.3237) \tag{8.138}$$

$$\approx 0.08828 \tag{8.139}$$

The Gaussian approximation for $\Pr(Y = 0)$ is

$$p_Z(0) \approx Q\left(\frac{-0.5 - 1.25}{0.9375}\right) - Q\left(\frac{0.5 - 1.25}{0.9375}\right) \tag{8.140}$$

$$\approx (1 - Q(1.8073)) - (1 - Q(0.7745)) \tag{8.141}$$

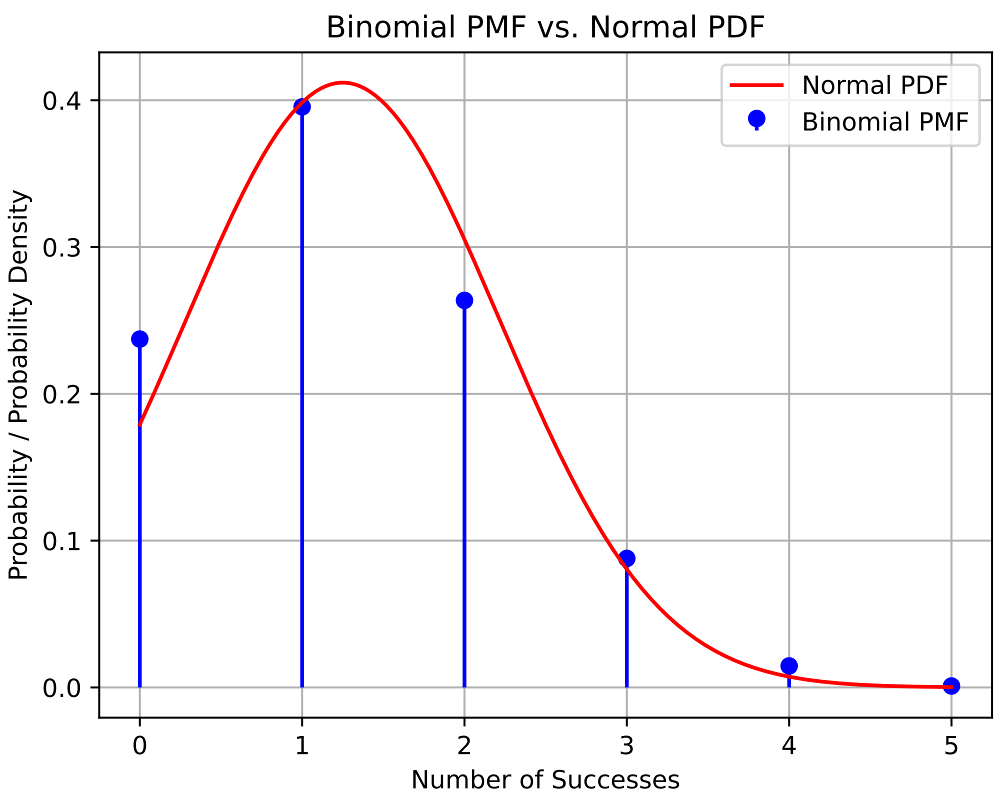
$$= Q(0.7745) - Q(1.8073) \tag{8.142}$$

$$\approx 0.1839 \tag{8.143}$$

(iii) Gaussian vs Binomial vs Q-function Comparison

Y	Gaussian	Q-function	Binomial
0	0.0978	0.1839	0.2373
3	0.044	0.08828	0.08789
5	0.0001245	0.0003888	0.00098

(iv) Binomial vs Gaussian Graph



8.3.9 The probability that a bulb produced by a factory will fuse after 150 days of use is 0.05. Find the probability that out of 5 such bulbs

(a) none

(b) not more than one

(c) more than one

(d) at least one

will fuse after 150 days of use.

Solution:

Guassian :

let Y be a gaussian Random variable

Parameter	Value	Description
X	0,1,2,3,4,5	No. Of bulbs fused
n	5	Total no. Of bulbs
p	0.05	bulb fusing
q	0.95	not fusing
$\mu = np$	0.25	Mean of Binomial Distribution
$\sigma^2 = npq$	0.2375	Varience of binomial Distribution

Table 8.7: Random variable and Parameter

$$Y \sim N(\mu, \sigma) \tag{8.144}$$

$$\sim N(1.25, 0.9375) \tag{8.145}$$

Due to continuity correction $\Pr(X = x)$ can be approximated using gaussian distribution as

$$p_Y(x) \approx \Pr(x - 0.5 < Y < x + 0.5) \tag{8.146}$$

$$\approx \Pr(Y < x + 0.5) - \Pr(Y < x - 0.5) \tag{8.147}$$

$$\approx F_Y(x + 0.5) - F_Y(x - 0.5) \tag{8.148}$$

CDF of Y is defined as:

$$F_Y(x) = \Pr(Y < x) \tag{8.149}$$

$$= \Pr\left(\frac{Y-\mu}{\sigma} < \frac{x-\mu}{\sigma}\right) \tag{8.150}$$

$$\implies \frac{Y-\mu}{\sigma} \sim N(0,1) \tag{8.151}$$

$$= 1 - \Pr\left(\frac{Y-\mu}{\sigma} > \frac{x-\mu}{\sigma}\right) \tag{8.152}$$

$$= \begin{cases} 1 - Q\left(\frac{x-\mu}{\sigma}\right) & x \geq \mu \\ Q\left(\frac{\mu-x}{\sigma}\right) & x < \mu \end{cases} \tag{8.153}$$

Then probability in terms of Q funtion is

$$\implies p_Y(x) \approx Q\left(\frac{(x-0.5)-\mu}{\sigma}\right) - Q\left(\frac{(x+0.5)-\mu}{\sigma}\right) \tag{8.154}$$

Binomial :

$$\Pr(X = k) = {}^nC_k p^k (1-p)^{n-k} \tag{8.155}$$

$$= {}^5C_k (0.05)^k (0.95)^{5-k} \tag{8.156}$$

CDF of X

$$F_X(k) = \Pr(X \leq k) \tag{8.157}$$

$$= \sum_{i=0}^{k} {}^{10}C_i (0.05)^i (0.95)^{5-i} \tag{8.158}$$

The solution

The graph

$\Pr(X=x)$	in term of Q	Numercal value	Binomial solution
$\Pr(X=0)$	$Q(1.5389) - Q(0.512)$	0.6960	0.773
$\Pr(X \leq 1)$	$Q(1.5896)$	0.9948	0.9774075
$1 - \Pr(X=0)$	$1 - (Q(1.5389) - Q(0.512))$	0.304	0.227
$1 - \Pr(X \leq 1)$	$1 - Q(1.5896)$	0.006	0.0226

Table 8.8: Random variable and Parameter

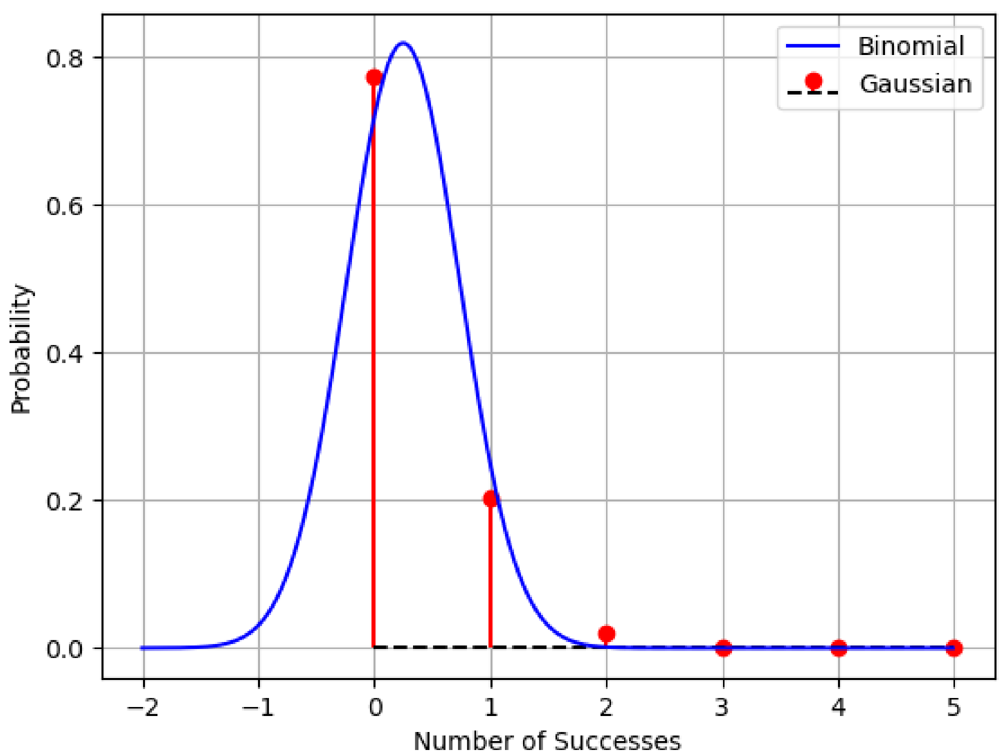


Figure 8.10: Binomial vs guassian

8.3.10 A bag consists of 10 balls each marked with one of the digits 0 to 9. If 4 balls are drawn successively with replacement from the bag, what is the probability that none is marked with the digit 0?

Solution:

(a) Gaussian PDF

Parameter	**Values**	**Description**
n	4	Number of balls drawn
p	0.1	Probability that the ball drawn is marked zero
$\mu = np$	0.4	Mean of distribution
$\sigma^2 = np(1-p)$	0.36	Variance of distribution
Y	0,1,2,3,4	Number of balls drawn which are zero

$$Y \sim \mathcal{N}\left(\mu, \sigma^2\right) \tag{8.159}$$

The gaussian distribution function is defined as:

$$p_Y(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \tag{8.160}$$

The probability that none of the balls drawn is marked with zero is given by:

$$p_Y(0) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(-\mu)^2}{2\sigma^2}} \tag{8.161}$$

$$= 0.532 \tag{8.162}$$

(b) CDF Approximation

$$\Pr(Y \le 0) = F_Y(0) \tag{8.163}$$

CDF of Y is:

$$F_Y(x) = \Pr(Y \le x) \tag{8.164}$$

$$= \Pr(Y - \mu \le x - \mu) \tag{8.165}$$

$$= \Pr\left(\frac{Y-\mu}{\sigma} \le \frac{x-\mu}{\sigma}\right) \tag{8.166}$$

Since,

$$\frac{Y-\mu}{\sigma} \sim \mathcal{N}(0,1) \tag{8.167}$$

Q function is defined

$$Q(x) = \Pr(Y > x)\,\forall x \in Y \sim \mathcal{N}(0,1) \tag{8.168}$$

$$F_Y(x) = 1 - \Pr\left(\frac{Y-\mu}{\sigma} > \frac{x-\mu}{\sigma}\right) \tag{8.169}$$

$$= \begin{cases} 1 - Q\left(\frac{x-\mu}{\sigma}\right), & x > \mu \\ Q\left(\frac{\mu-x}{\sigma}\right), & x < \mu \end{cases} \tag{8.170}$$

$$F_Y(0) = Q\left(\frac{0.4-0}{0.6}\right) \tag{8.171}$$

$$= Q\left(\frac{2}{3}\right) \tag{8.172}$$

$$= 0.252 \tag{8.173}$$

(c) Binomial PMF Let X be a random variable which denotes the number of balls drawn that are marked with zero,

$$p_X(k) = {}^nC_k p^k (1-p)^{n-k} \tag{8.174}$$

$$p_X(0) = {}^4C_0 (0.1)^0 (0.9)^4 \tag{8.175}$$

$$= 0.6561 \tag{8.176}$$

Y	Gaussian PDF	CDF Approximation	Binomial PMF
0	0.532	0.252	0.6561

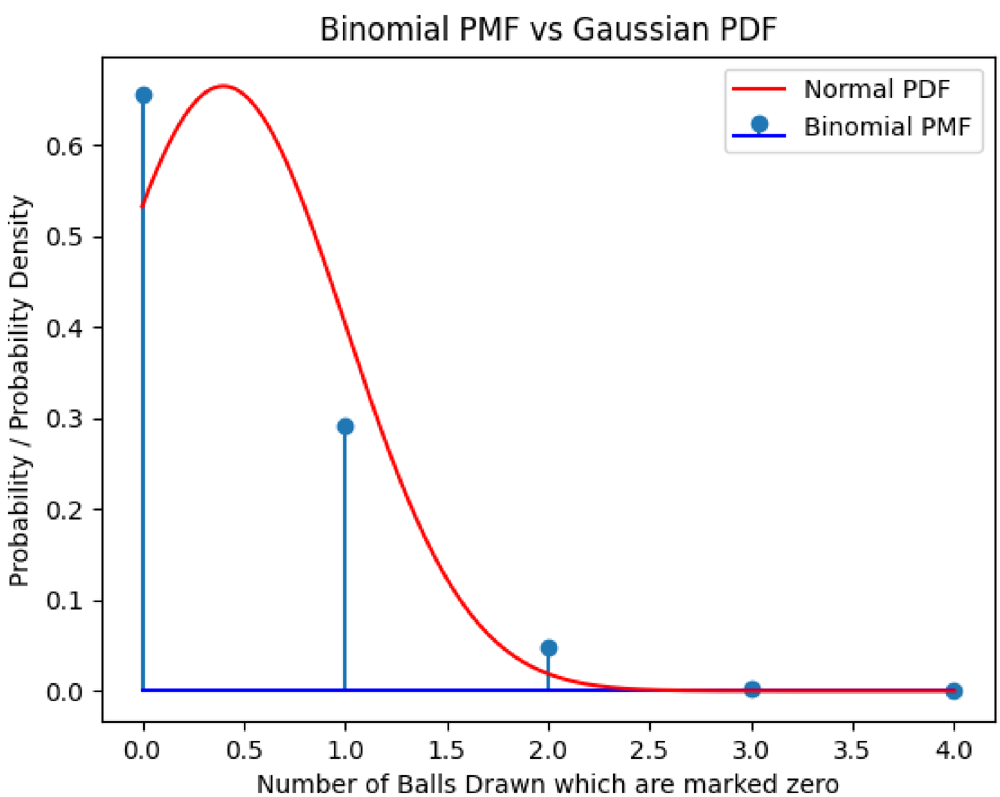


Figure 8.11: Binomial PMF vs Gaussian PDF

8.3.11 How many times must a man toss a fair coin so that the probability of having at least one head is more than 90%?

Solution:

Parameter	**Value**	**Description**
n	n	number of coin tosses
p	$\frac{1}{2}$	getting a head on a coin toss
q	$\frac{1}{2}$	getting a tail on a coin toss
$\mu = np$	$\frac{n}{2}$	mean of the distribution
$\sigma^2 = npq$	$\frac{n}{4}$	variance of the distribution
Y	≥ 1	Number of heads

(a) Gaussian:

$$Y \sim \mathcal{N}\left(\mu, \sigma^2\right) \tag{8.177}$$

The CDF of Y:

$$\begin{aligned} F_Y(y) &= 1 - \Pr\left(Y > y\right) && (8.178) \\ &= 1 - \Pr\left(\frac{Y-\mu}{\sigma} > \frac{y-\mu}{\sigma}\right) && (8.179) \end{aligned}$$

But,

$$\begin{aligned} \frac{Y-\mu}{\sigma} &\sim \mathcal{N}\left(0, 1\right) && (8.180) \\ \implies F_Y(y) &= 1 - Q\left(\frac{y-\mu}{\sigma}\right) && (8.181) \end{aligned}$$

i. Without correction

$$\Pr\left(Y \geq 1\right) = 1 - F_Y(1) \tag{8.182}$$

From the result (8.181)

$$Q\left(\frac{2-n}{\sqrt{n}}\right) > 0.9 \tag{8.183}$$

$$\frac{2-n}{\sqrt{n}} < Q^{-1}(0.9) \tag{8.184}$$

$$\frac{2-n}{\sqrt{n}} < -1.28 \tag{8.185}$$

Squaring on both the sides

$$\implies (n-2)^2 > (1.28\sqrt{n})^2 \tag{8.186}$$

$$\implies n^2 - 5.6384n + 4 > 0 \tag{8.187}$$

$$\implies n > 4.86, n < 0.8 \tag{8.188}$$

$$\implies n = 5 \tag{8.189}$$

ii. With correction: 0.5 as correction term

$$\Pr(Y > 0.5) = 1 - F_Y(0.5) \tag{8.190}$$

From the result (8.181)

$$Q\left(\frac{1-n}{\sqrt{n}}\right) > 0.9 \tag{8.191}$$

$$\frac{1-n}{\sqrt{n}} < Q^{-1}(0.9) \tag{8.192}$$

$$\frac{1-n}{\sqrt{n}} < -1.28 \tag{8.193}$$

Squaring on both the sides

$$\implies (n-1)^2 > (1.28\sqrt{n})^2 \tag{8.194}$$

$$\implies n^2 - 3.6384n + 1 > 0 \tag{8.195}$$

$$\implies n > 3.38,\, n < 0.29 \tag{8.196}$$

$$\implies n = 4 \tag{8.197}$$

(b) Binomial:

$$X \sim \text{Bin}(n, p) \tag{8.198}$$

$$\Pr\left(X \geq 1\right) > 0.9 \tag{8.199}$$

$$\implies n = 4 \tag{8.200}$$

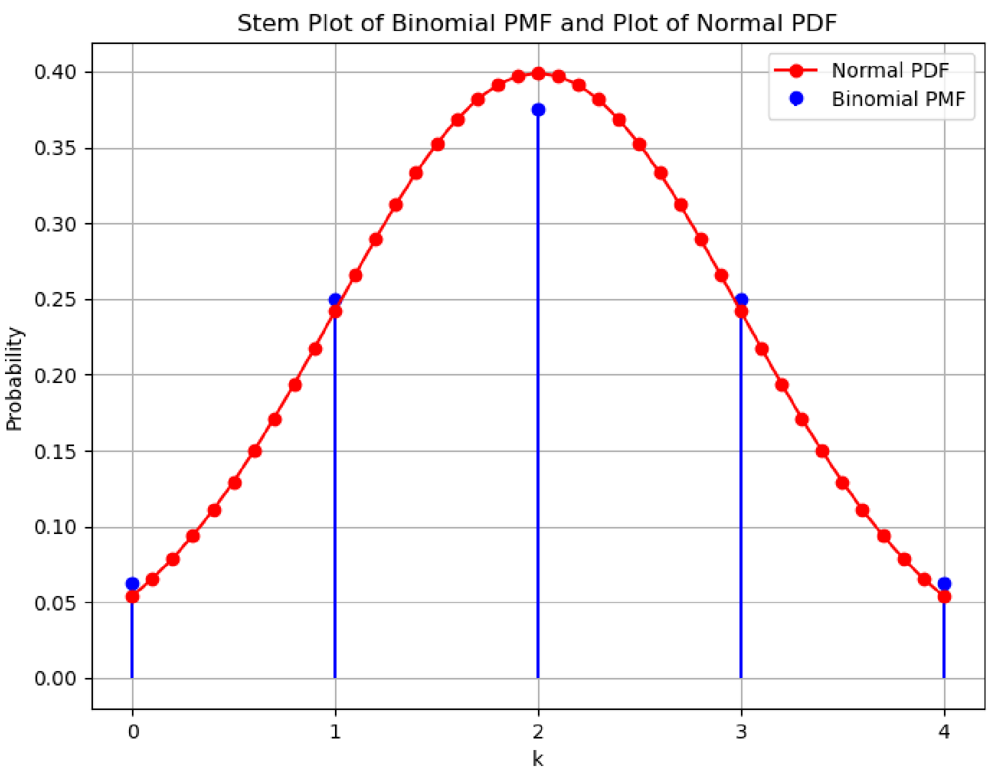
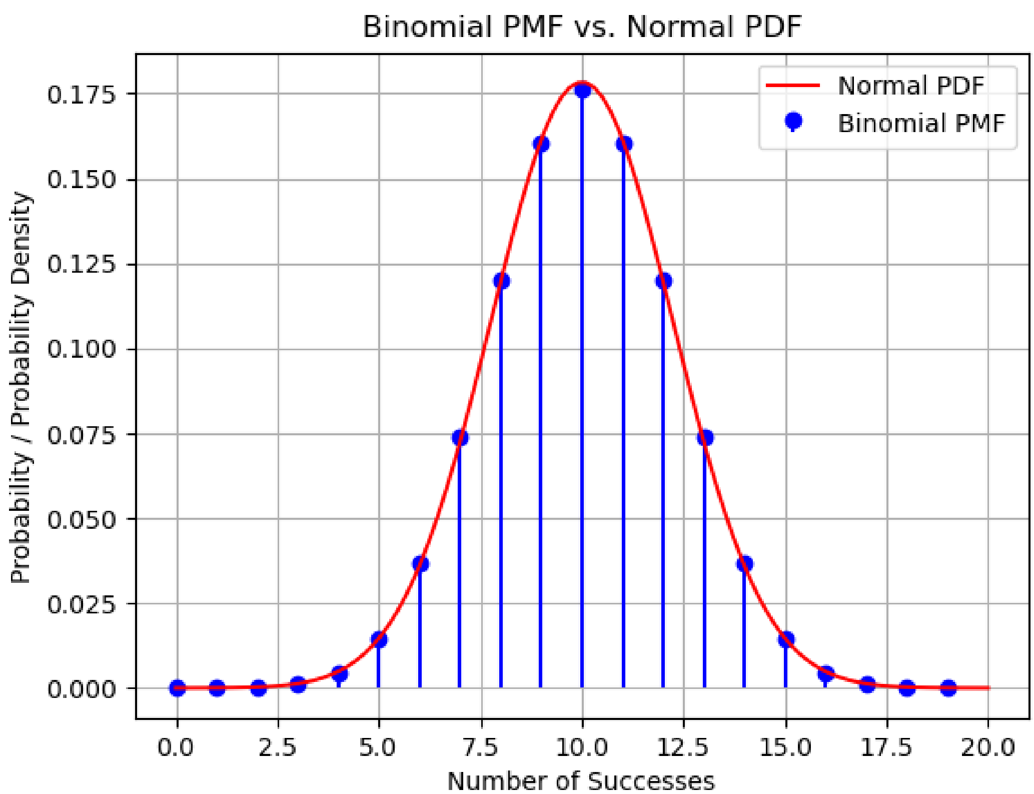


Figure 8.12: Binomial PMF of X vs Normal PDF of Y

8.3.12 In an examination, 20 questions of true-false type are asked. Suppose a student tosses a fair coin to determine his answer to each question. If the coin falls heads, he answer true; if it falls tails, he answer false. Find the probability that he answers at least 12 questions correctly.

Solution: Gaussian vs Binomial

Let us define:



Parameter	**Value**	**Description**
n	20	number of Questions
p	0.5	probability of answering correct
q	0.5	probability of answering wrong
$\mu = np$	10	mean of the distribution
$\sigma^2 = npq$	5	variance of the distribution
Y	0,1,2,3,...,20	Number of correct answers

(a) Gaussian:

$$Y \sim \mathcal{N}\left(\mu, \sigma^2\right) \tag{8.201}$$

The CDF of Y:

$$F_Y(y) = 1 - \Pr(Y > y) \tag{8.202}$$

$$= 1 - \Pr\left(\frac{Y-\mu}{\sigma} > \frac{y-\mu}{\sigma}\right) \tag{8.203}$$

But,

$$\frac{Y-\mu}{\sigma} \sim \mathcal{N}(0,1) \tag{8.204}$$

$$\tag{8.205}$$

the Q-function is defined as:

$$Q(x) = \Pr(Y > x)\ \forall x \in Y \sim \mathcal{N}(0,1) \tag{8.206}$$

therefore the cdf will be:

$$F_Y(y) = \begin{cases} 1 - Q\left(\frac{y-\mu}{\sigma}\right), & y > \mu \\ Q\left(\frac{\mu-y}{\sigma}\right), & y < \mu \end{cases} \tag{8.207}$$

The probability of getting atleast 12 answers correct:

considering 0.5 as coorection term:

$$\Pr\left(Y > 12.5\right) = 1 - F_Y\left(12.5\right) \tag{8.208}$$

$$= Q\left(\frac{12.5-\mu}{\sigma}\right) \tag{8.209}$$

$$= Q\left(\frac{2.5}{\sqrt{5}}\right) \tag{8.210}$$

$$= Q\left(1.118\right) \tag{8.211}$$

$$= 0.13178 \tag{8.212}$$

Questions answered correctly	**Binomial**	**Gaussian**
Atleast 12	0.2517	0.13178

8.3.13 Find the probability of getting 5 twice in 7 throws of a dice.

Solution:

Parameter	Value	Description
X	{0,1,2,3,4,5,6,7}	Number of 5 appearing on dice
n	7	Number of cards drawn
p	$\frac{1}{6}$	getting 5
q	$\frac{5}{6}$	getting any other number
$\mu = np$	$\frac{7}{6}$	Mean of Binomial distribution
$\sigma^2 = npq$	$\frac{35}{36}$	Varience of Binomial distribution

Table 8.9: Random variable and Parameter

(a) Binomial Distribution :

The X is the random variable, the pmf of X is given by

$$p_X(k) = {}^nC_k p^k (1-p)^{n-k} \tag{8.213}$$

We require $\Pr(X = 2)$. Since $n = 7$,

$$p_X(2) = 0.234 \tag{8.214}$$

(b) Gaussian Distribution

Let Y be gaussian variable. Using central limit theorem, we can use the gaussian distribution function:

$$p_Y(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad (x \in Y) \tag{8.215}$$

Using Normal distribution at X=2,

$$p_Y(2) = \frac{1}{\sqrt{2\pi\left(\frac{35}{36}\right)}} e^{-\frac{\left(2-\frac{7}{6}\right)^2}{2\left(\frac{35}{36}\right)}} \tag{8.216}$$

$$= \frac{1}{\sqrt{2\pi\left(\frac{35}{36}\right)}} e^{-\frac{5}{14}} \tag{8.217}$$

$$= 0.283 \tag{8.218}$$

(c) Using Q function:

let Y be a gaussian Random variable

$$Y \sim N(\mu, \sigma) \tag{8.219}$$

$$\sim N(1.166, 0.972) \tag{8.220}$$

Due to continuity correction $\Pr(X = x)$ can be approximated using gaussian

distribution as

$$p_Y(x) \approx \Pr(x - 0.5 < Y < x + 0.5) \tag{8.221}$$

$$\approx \Pr(Y < x + 0.5) - \Pr(Y < x - 0.5) \tag{8.222}$$

$$\approx F_Y(x + 0.5) - F_Y(x - 0.5) \tag{8.223}$$

CDF of Y is defined as:

$$F_Y(x) = \Pr(Y < x) \tag{8.224}$$

$$= \Pr\left(\frac{Y - \mu}{\sigma} < \frac{x - \mu}{\sigma}\right) \tag{8.225}$$

$$\implies \frac{Y - \mu}{\sigma} \sim N(0, 1) \tag{8.226}$$

$$= 1 - \Pr\left(\frac{Y - \mu}{\sigma} > \frac{x - \mu}{\sigma}\right) \tag{8.227}$$

$$= \begin{cases} 1 - Q\left(\frac{x-\mu}{\sigma}\right) & x \geq \mu \\ Q\left(\frac{\mu-x}{\sigma}\right) & x < \mu \end{cases} \tag{8.228}$$

Then probability in terms of Q funtion is

$$\implies p_Y(x) \approx Q\left(\frac{(x - 0.5) - \mu}{\sigma}\right) - Q\left(\frac{(x + 0.5) - \mu}{\sigma}\right) \tag{8.229}$$

The Gaussian approximation for $\Pr(X = 2)$ is

$$p_Y(2) \approx Q\left(\frac{1.5 - 1.166}{0.972}\right) - Q\left(\frac{2.5 - 1.166}{0.972}\right) \tag{8.230}$$

$$\approx Q(0.343) - Q(1.371) \tag{8.231}$$

$$\approx 0.282 \tag{8.232}$$

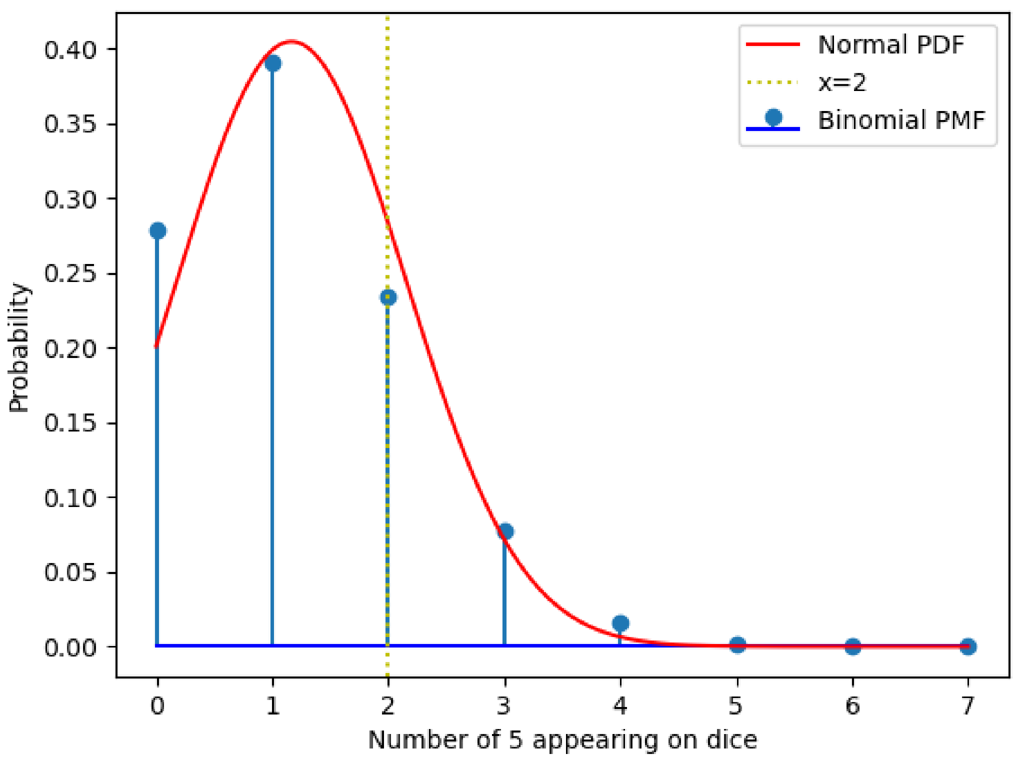


Figure 8.13: Binomial and gaussian distribution

8.3.14 On a multiple choice examination with three possible answers for each of the five questions, what is the probability that a candidate would get four or more correct answers just by guessing?

Solution:

Gaussian:

$$Y \sim \mathcal{N}\left(\mu, \sigma^2\right) \tag{8.233}$$

Table 8.10: Variables

Variable	Value	Description
n	5	Number of questions
p	$\frac{1}{3}$	probability of question being correct
$\mu = np$	$\frac{5}{3}$	mean of distribution
$\sigma = \sqrt{npq}$	$\sqrt{\frac{10}{9}}$	variance of distribution
X	$0 \le X \le 5$	Number of correct questions

CDF of Y is defined as:

$$F_Y(x) = \Pr(Y \le x) \tag{8.234}$$

$$= \Pr\left(\frac{Y-\mu}{\sigma} \le \frac{X-\mu}{\sigma}\right) \tag{8.235}$$

$$\frac{Y-\mu}{\sigma} \sim \mathcal{N}(0,1) \tag{8.236}$$

$$= 1 - \Pr\left(\frac{Y-\mu}{\sigma} > \frac{X-\mu}{\sigma}\right) \tag{8.237}$$

$$\tag{8.238}$$

therefore the cdf will be:

$$F_Y(y) = \begin{cases} 1 - Q\left(\frac{y-\mu}{\sigma}\right), & y \ge \mu \\ Q\left(\frac{\mu-y}{\sigma}\right), & y \le \mu \end{cases} \tag{8.239}$$

(a) Without correction:

$$\Pr(Y \geq 4) = 1 - \Pr(Y \leq 4) \tag{8.240}$$

$$= 1 - F_Y(4) \tag{8.241}$$

$$\implies \Pr(Y \geq 4) = Q\left(\frac{X - \mu}{\sigma}\right) \tag{8.242}$$

$$= Q(2.22286) \tag{8.243}$$

$$\Pr(Y \geq 4) = 0.013113 \tag{8.244}$$

(b) With a 0.5 correction:

$$\Pr(Y \geq 4) = Q\left(\frac{X - \mu + 0.5}{\sigma}\right) \tag{8.245}$$

$$= Q(1.74604) \tag{8.246}$$

$$\implies \Pr(Y \geq 4) = 0.040402 \tag{8.247}$$

Binomial:

$$p_X(k) = {}^nC_k p^k (1-p)^{n-k} \tag{8.248}$$

$$\tag{8.249}$$

Probablity that 4 or more are correct

$$\implies P\left(X \geq 4\right) = \sum_{k=4}^{5} {}^{5}C_k \left(\frac{1}{3}\right)^k \left(\frac{2}{3}\right)^{5-k} \tag{8.250}$$

$$= \frac{11}{243} \tag{8.251}$$

$$= 0.04526 \tag{8.252}$$

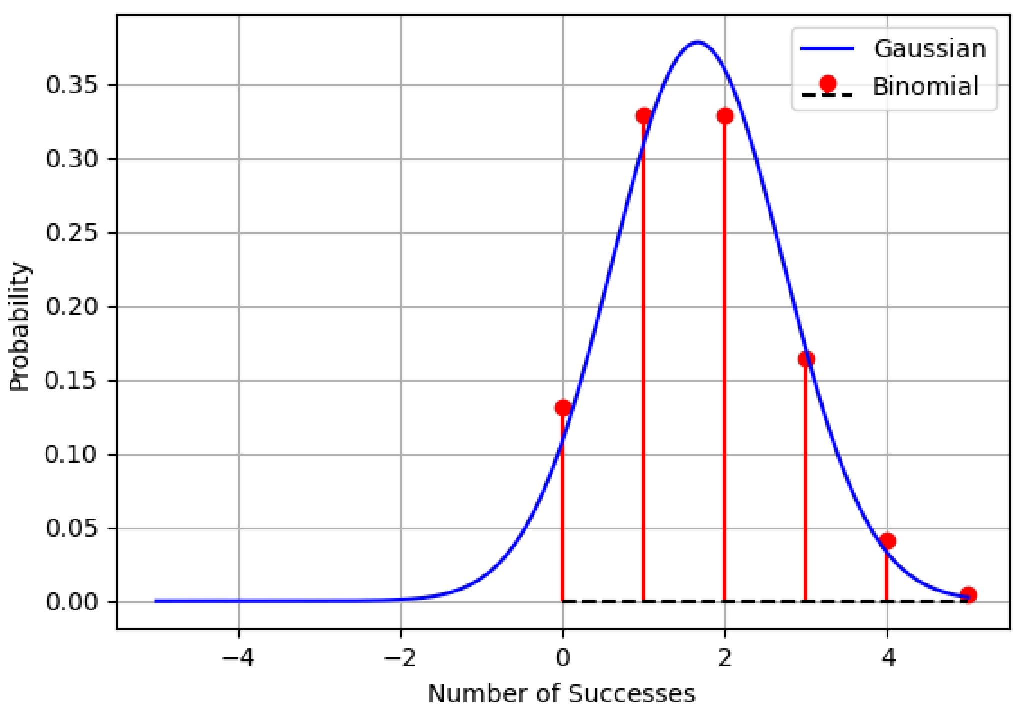


Figure 8.14: Binomial vs guassian

8.3.15 Find the probability of throwing at most 2 sixes in 6 throws of a single die.

Solution:

8.3.16 Suppose that 90 % of people are right-handed. What is the probability that atmost 6 of a random sample of 10 people are right-handed.

Solution: Given that 90% of the people are right-handed.

Table 8.11: Description of random variables

Parameters	Values	Description
n	10	Sample space
p	0.9	Probability that the person is right-handed
Y	$0 \leq Y \leq 10$	Number of people thatare right-handed
$\mu = np$	9	Mean
$\sigma = \sqrt{np(1-p)}$	0.9	Standard deviation

Gaussian Distribution

Central limit theorm:

$$Y \sim \mathcal{N}\left(\mu, \frac{\sigma}{\sqrt{n}}\right) \tag{8.253}$$

$$\tag{8.254}$$

CDF of Y is

$$F_Y(y) = \Pr(Y \leq y) \tag{8.255}$$

We know that

$$Q(x) = \Pr(X > x), x > 0, X \sim N(0,1) \tag{8.256}$$

$$Q(-x) = \Pr(X > -x), x < 0, X \sim N(0,1) \tag{8.257}$$

$$= 1 - Q(x) \tag{8.258}$$

Hence,

CDF :

$$F_Y(y) = \begin{cases} 1 - Q\left(\frac{y-\mu}{\sigma}\right), & \text{if } y > \mu \\ 1 - Q\left(\frac{y-\mu}{\sigma}\right) = Q\left(\frac{\mu-y}{\sigma}\right), & \text{if } y < \mu \end{cases} \tag{8.259}$$

With a 0.9 correction:

$$F_Y(6) = \Pr(Y < 6.9) \tag{8.260}$$

$$= 1 - Q\left(\frac{6.9 - 9}{\sqrt{0.9}}\right) \tag{8.261}$$

$$= Q\left(\frac{2.1}{0.9487}\right) \tag{8.262}$$

$$= Q(2.21) \tag{8.263}$$

$$= 0.013553 \tag{8.264}$$

Without correction:

$$F_Y(6) = \Pr(Y \le 6) \tag{8.265}$$

$$= 1 - Q\left(\frac{6-9}{\sqrt{0.9}}\right) \tag{8.266}$$

$$= Q\left(\frac{3}{0.9487}\right) \tag{8.267}$$

$$= Q(3.1622) \tag{8.268}$$

$$= 0.000783 \tag{8.269}$$

Table 8.12: Comparision

Number of people(RH)	Binomial	Gaussian	Gaussian(C)	Error(%)	Error(C)(%)
Atmost 6	0.012795	0.000783	0.013553	-93.88	55.92

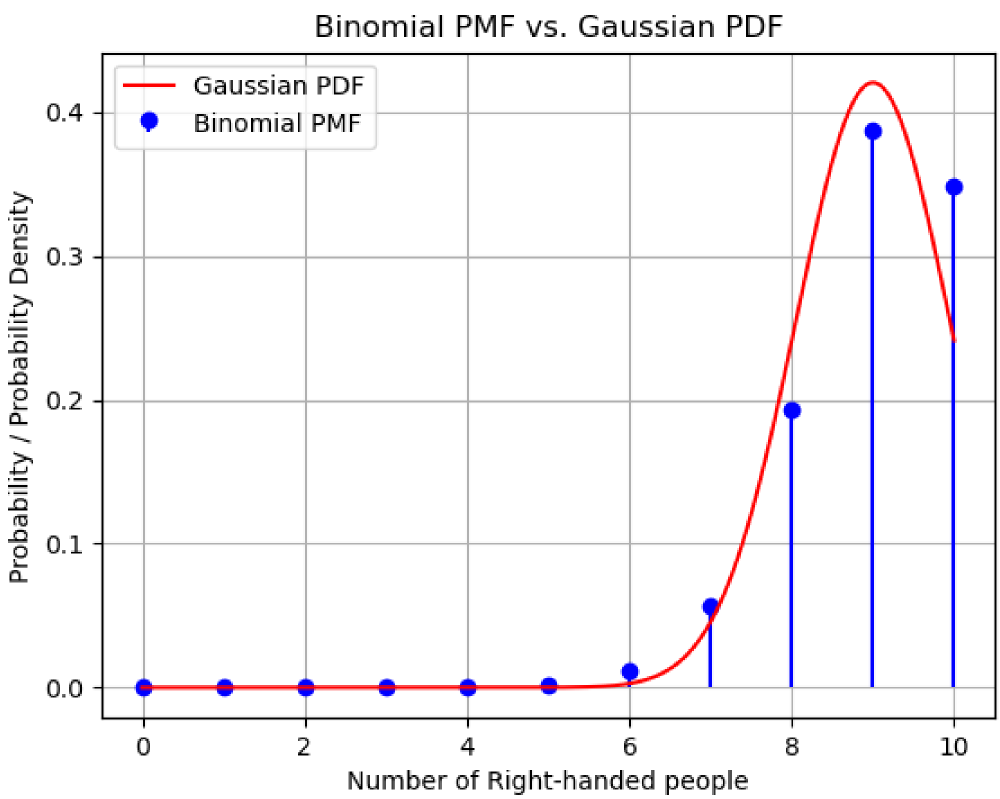


Figure 8.15: Binomial vs Gaussian

8.3.17 An urn contains 25 balls of which 10 balls bear a mark 'X' and the remaining 15 bear a mark 'Y'. A ball is drawn at random from the urn, its mark is noted down and it is replaced. If 6 balls are drawn in this way, find the probability that

(a) all will bear 'X' mark.

(b) not more than 2 will bear 'Y' mark.

(c) at least one ball will bear 'Y' mark.

(d) the number of balls with 'X' mark and 'Y' mark will be equal.

Solution:

Parameter	Values	Description
n	6	Number of draws
p	0.4	Probability that ball bears X mark
q	0.6	Probability that ball bears Y mark
$\mu = np$	2.4	mean of the distribution
$\sigma = npq$	1.2	variance of the distribution
X		Number of cards bear mark X
Y		Number of cards bear mark Y

Table 8.13: Definition of parameters

using Gaussian

$$Y \sim \mathcal{N}\left(\mu, \sigma^2\right) \tag{8.270}$$

The CDF of Y:

$$F_Y(y) = 1 - \Pr\left(Y > y\right) \tag{8.271}$$

$$= 1 - \Pr\left(\frac{Y-\mu}{\sigma} > \frac{y-\mu}{\sigma}\right) \tag{8.272}$$

But,

$$\frac{Y-\mu}{\sigma} \sim \mathcal{N}\left(2.4, 1.44\right) \tag{8.273}$$

$$\tag{8.274}$$

the Q-function is defined as:

$$Q\left(x\right) = \Pr\left(Y > x\right) \forall x \in Y \sim \mathcal{N}(2.4, 1.44) \tag{8.275}$$

therefore the cdf will be:

$$F_Y(y) = \begin{cases} 1 - Q\left(\frac{y-\mu}{\sigma}\right), & y > \mu \\ Q\left(\frac{\mu-y}{\sigma}\right), & y < \mu \end{cases} \tag{8.276}$$

(a) all will bear X mark.

using Gaussian

considering 0.5 as the correction term,

$$\Pr(X > 5.5) = 1 - F_X(5.5) \tag{8.277}$$

$$= Q\left(\frac{5.5-\mu}{\sigma}\right) \tag{8.278}$$

$$= Q\left(\frac{3.1}{1.2}\right) \tag{8.279}$$

$$= Q(2.583) \tag{8.280}$$

$$= 0.00489 \tag{8.281}$$

(b) not more than 2 will bear Y mark.

using Gaussian

considering 0.5 as the correction term,

$$\Pr(Y < 2.5) = 1 - Q\left(\frac{2.5 - \mu}{\sigma}\right) \tag{8.282}$$

$$= 1 - Q\left(\frac{-1.1}{1.2}\right) \tag{8.283}$$

$$= 1 - Q(-0.9166) \tag{8.284}$$

$$= Q(0.9166) \tag{8.285}$$

$$= 0.1796 \tag{8.286}$$

(c) at least one ball will bear Y mark.

using Gaussian

considering 0.5 as the correction term,

$$\Pr(Y < 0.5) = 1 - Q\left(\frac{0.5 - \mu}{\sigma}\right) \tag{8.287}$$

$$= 1 - Q\left(\frac{-1.1}{1.2}\right) \tag{8.288}$$

$$= 1 - Q(-2.588) \tag{8.289}$$

$$= 1 - 0.0048 \tag{8.290}$$

$$= 0.9952 \tag{8.291}$$

(d) the number of balls with X mark and Y mark will be equal.

using Gaussian

Calculate the mean(μ_D) and variance(σ_D) of the difference variance D,

$$\mu_D = -1.2 \tag{8.292}$$

$$\sigma_D = 2.4 \tag{8.293}$$

$$\Pr\left(D=0\right) = Q\left(\frac{0-\mu_D}{\sigma_D}\right) \tag{8.294}$$

$$= Q\left(\frac{1.2}{1.697}\right) \tag{8.295}$$

$$= Q\left(0.71\right) \tag{8.296}$$

$$= 0.2388 \tag{8.297}$$

Gaussian vs Binomial Table

Question	Gaussian	Binomial
all will bear X mark	0.00489	0.00409
not more than 2 will bear Y mark	0.1796	0.1792
at least one ball will bear Y mark	0.9952	0.9959
the number of balls with X mark and Y mark will be equal	0.2388	0.2764

Table 8.14: Definition of parameters

Gaussian vs Binomial graph

8.3.18 From a lot of 30 bulbs which include 6 defectives, a sample of 4 bulbs is drawn at random with replacement. Find the probability distribution of the number of defective bulbs.

Solution:

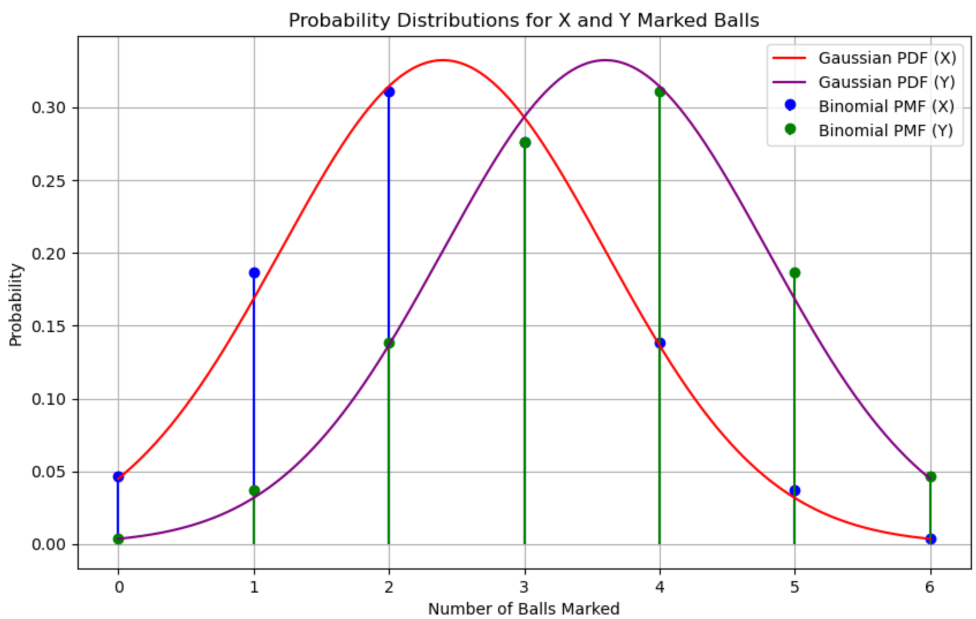


Figure 8.16: pmf of binomial and pdf of Gaussian of X and Y marked balls

Parameter	Value	Description
X	{0,1,2,3,4}	Number of defective bulbs taken
n	4	Number of bulbs taken
p	0.2	Taking a defective bulb
q	0.8	Taking a non defective bulb
$\mu = np$	0.8	Mean of Binomial distribution
$\sigma^2 = npq$	0.64	Varience of Binomial distribution

Table 8.15: Parameter description

let Y be a gaussian Random variable

$$Y \sim N(np, npq) \tag{8.298}$$

$$\sim N(0.8, 0.64) \tag{8.299}$$

Due to continuity correction $\Pr(X = x)$ can be approximated using gaussian distribution as

$$p_Y(x) \approx \Pr(x - 0.5 < Y < x + 0.5) \tag{8.300}$$

$$\approx \Pr(Y < x + 0.5) - \Pr(Y < x - 0.5) \tag{8.301}$$

$$\approx F_Y(x + 0.5) - F_Y(x - 0.5) \tag{8.302}$$

CDF of Y is defined as:

$$F_Y(x) = \Pr(Y < x) \tag{8.303}$$

$$= \Pr\left(\frac{Y - \mu}{\sigma} < \frac{x - \mu}{\sigma}\right) \tag{8.304}$$

$$\implies \frac{Y - \mu}{\sigma} \sim N(0, 1) \tag{8.305}$$

$$= 1 - \Pr\left(\frac{Y - \mu}{\sigma} > \frac{x - \mu}{\sigma}\right) \tag{8.306}$$

$$= \begin{cases} 1 - Q\left(\frac{x-\mu}{\sigma}\right) & x \geq \mu \\ Q\left(\frac{\mu-x}{\sigma}\right) & x < \mu \end{cases} \tag{8.307}$$

Number of defective bulbs	Binomial distribution	Gaussian approximation	Error
0	0.4096	0.3017	26.342773437
1	0.4096	0.4555	11.206054688
2	0.1536	0.1739	13.216145833
3	0.0256	0.0164	35.9375
4	0.0016	0.00036	77.5

Table 8.16: Comparing the gaussian approximation with binomial

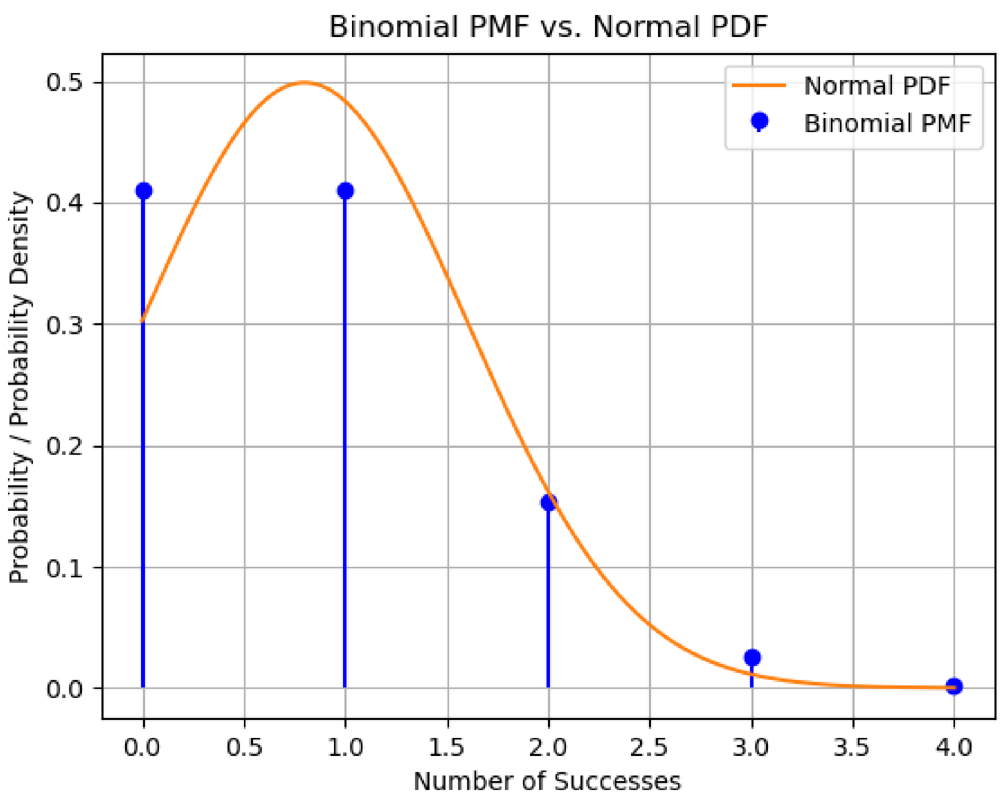


Figure 8.17: Binomial and gaussian distribution

8.3.19 Suppose X is a binomial distribution $B\left(6, \frac{1}{2}\right)$. Show that $X = 3$ is the most likely outcome. (Hint : $P(X = 3)$ is the maximum among all $P(x_i), x_i = 0, 1, 2, 3, 4, 5, 6$)

Solution:

RV	Values	Description
X	$\{0, 1, 2, 3, 4, 5, 6\}$	Outcomes of the binomial distribution
Y	$[-\infty, \infty]$	Outcomes of the Gaussian distribution

Table 8.17: Random Variables

(a) **Binomial:**

$$X \sim Bin\left(6, \frac{1}{2}\right) \tag{8.308}$$

We know that, for $k \in \mathbb{W}$ and $k \in [0, n]$, the maximum of ${}^nC_k$ occurs at

$$k = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even} \\ \frac{n+1}{2} \quad \text{or} \quad \frac{n-1}{2}, & \text{if } n \text{ is odd} \end{cases} \tag{8.309}$$

As,

$$n = 6 \tag{8.310}$$

$$\implies k = \frac{n}{2} = 3 \tag{8.311}$$

$\therefore X = 3$ is the most likely outcome.

$$p_X(k) = {}^6C_k \left(\frac{1}{2}\right)^6 \tag{8.312}$$

$$p_X(3) = {}^6C_3 \left(\frac{1}{2}\right)^6 \tag{8.313}$$

$$= \frac{5}{16} \tag{8.314}$$

(b) **Gaussian:** The binomial distribution $X \sim Bin\left(6, \frac{1}{2}\right)$ can be approximated as a Gaussian distribution $Y \sim \mathcal{N}\left(\mu, \sigma^2\right)$ using the Mean μ and Standard Deviation

σ parameters.

$$\mu = np = 6 \times \frac{1}{2} = 3 \tag{8.315}$$

$$\sigma^2 = npq = 6 \times \frac{1}{2} \times \frac{1}{2} = \frac{3}{2} \tag{8.316}$$

Thus, the Gaussian (normal) approximation is:

$$Y \sim \mathcal{N}\left(3, \frac{3}{2}\right) \tag{8.317}$$

$$\implies p_Y(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \tag{8.318}$$

$$= \frac{1}{\sqrt{3\pi}} e^{-\frac{(x-3)^2}{3}} \tag{8.319}$$

The most likely outcome is the mean of the Gaussian distribution. Thus, $Y = 3$ is the most likely outcome, as seen in the following plot.

Comparing the values numerically:

(a) Binomial

$$p_X(0) = p_X(6) = \frac{1}{64} = 0.015625 \tag{8.320}$$

$$p_X(1) = p_X(5) = \frac{6}{64} = 0.09375 \tag{8.321}$$

$$p_X(2) = p_X(4) = \frac{15}{64} = 0.234375 \tag{8.322}$$

$$p_X(3) = \frac{20}{64} = 0.3125 \tag{8.323}$$

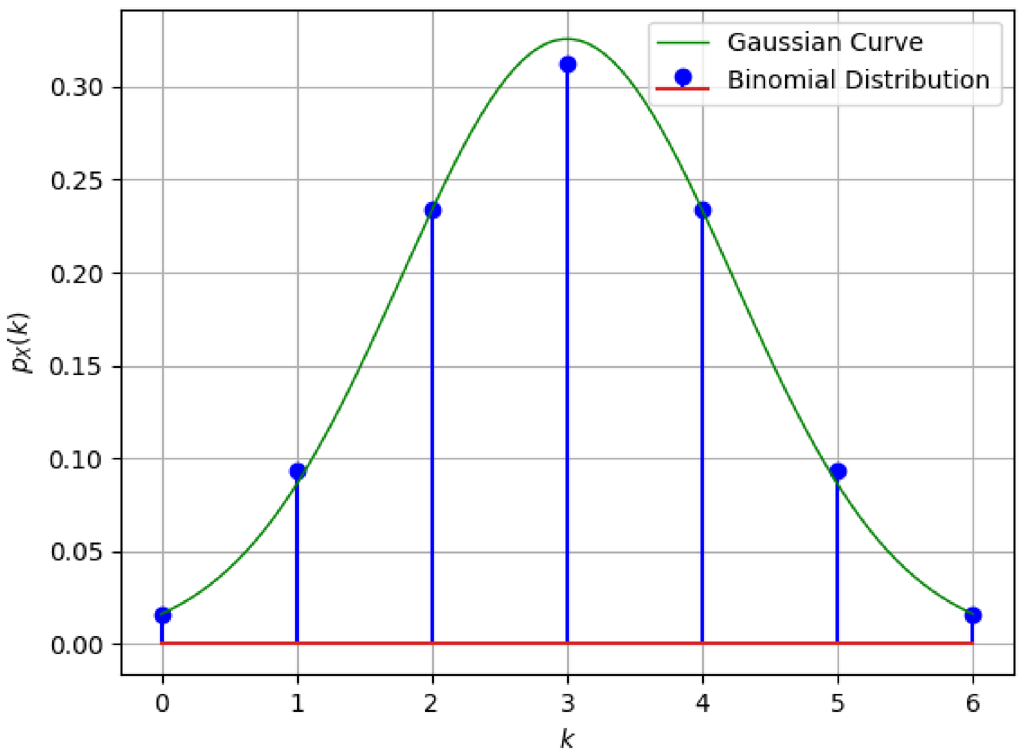


Figure 8.18: Binomial Distribution and Gaussian Approximation

(b) Gaussian

$$p_Y(0) = p_Y(6) = 0.01621739 \tag{8.324}$$

$$p_Y(1) = p_Y(5) = 0.08586282 \tag{8.325}$$

$$p_Y(2) = p_Y(4) = 0.23339933 \tag{8.326}$$

$$p_Y(3) = 0.32573501 \tag{8.327}$$

8.3.20 A fair coin is tossed four times, and a person win Re 1 for each head and lose Rs 1.5

for each tail that turns up.

From the sample space calculate how many different amounts of money you can have after four tosses and the probability of having each of these amounts. **Solution:**

8.3.21 It is known that 10 % of certain articles manufactured are defective. What is the probability that in a random sample space of 12 such articles,9 are defective?

Solution:

Parameter	Values	Description
n	12	Number of articles
k	9	Number of defective articles
p	0.1	Probability of being defective
X	$1 \leq X \leq 12$	X defective elements out of 12
Y	$1 \leq Y \leq 12$	gaussian variable
$\mu = np$	1.2	mean
$\sigma = \sqrt{np(1-p)}$	1.039	standard deviation

Table 8.18: Table 1

(a) Binomial Distribution :

The X is the random variable, the pmf of X is given by

$$p_X(k) = {}^nC_k p^k (1-p)^{n-k} \tag{8.328}$$

We require $\Pr(X = 9)$. Since $n = 12$,

$$p_X(9) = 1.60379(10^{-7}) \tag{8.329}$$

(b) Gaussian Distribution

Let Y be gaussian variable. Using central limit theorem, we can use the gaussian distribution function:

$$p_Y\left(x\right) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad \left(x \in Y\right) \tag{8.330}$$

Using Normal distribution at X=9.

$$p_Y\left(9\right) = \frac{1}{\sqrt{2\pi\left(\frac{27}{25}\right)}}e^{-\frac{\left(x-\frac{6}{5}\right)^2}{2\left(\frac{27}{25}\right)}} \tag{8.331}$$

$$= \frac{1}{\sqrt{2\pi\left(\frac{27}{25}\right)}}e^{-\frac{169}{3}} \tag{8.332}$$

$$= 3.89010(10^{-9}) \tag{8.333}$$

(c) <u>using Q function:</u>

$$Y \sim \mathcal{N}\left(\mu, \sigma^2\right) \tag{8.334}$$

The CDF of Y:

$$F_Y\left(y\right) = \begin{cases} 1 - Q\left(\frac{y-\mu}{\sigma}\right), & y > \mu \\ Q\left(\frac{\mu-y}{\sigma}\right), & y < \mu \end{cases} \tag{8.335}$$

But,

$$\frac{Y-\mu}{\sigma} \sim \mathcal{N}\left(0,1\right) \tag{8.336}$$

$$\implies F_Y(y) = 1 - Q\left(\frac{y-\mu}{\sigma}\right) \tag{8.337}$$

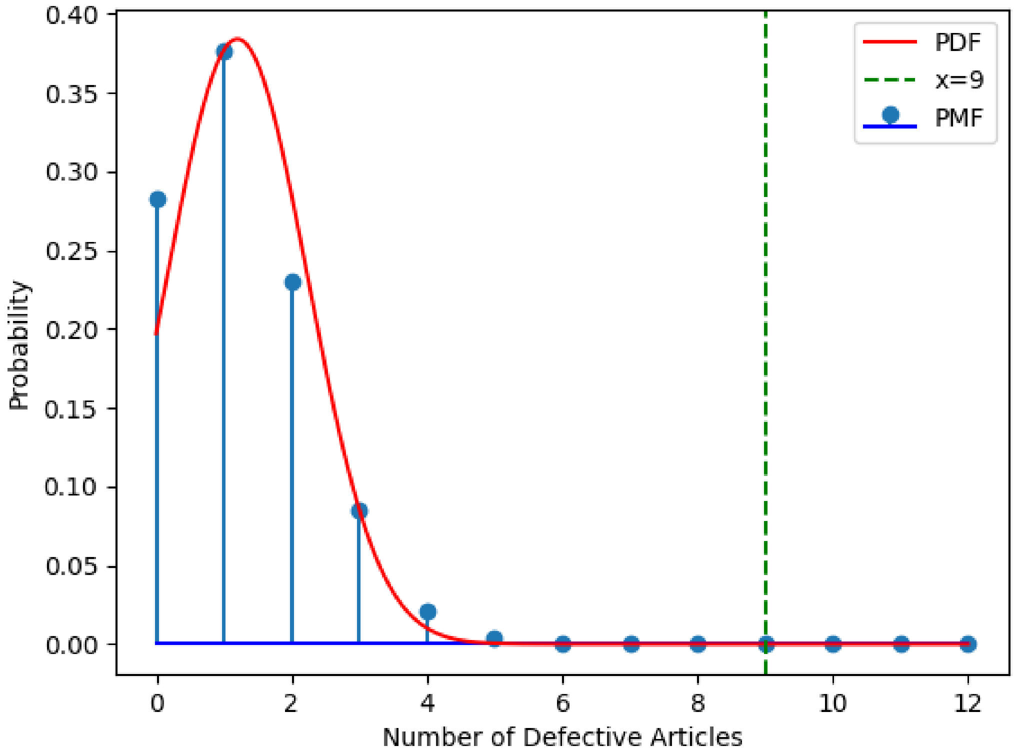


Figure 8.19: Binomial-PMF and Gaussian-PDFof X

to include correction of 0.5,

$$p_Y\left(8.5 < Y < 9.5\right) = F_Y(9.5) - F_Y(8.5) \tag{8.338}$$

$$= Q\left(\frac{8.5-\mu}{\sigma}\right) - Q\left(\frac{9.5-\mu}{\sigma}\right) \tag{8.339}$$

$$= Q\left(7.02\right) - Q\left(7.98\right) \tag{8.340}$$

$$= 1.2798(10^{-12}) \tag{8.341}$$

8.3.22 An experiment succeeds twice as often as it fails. Find the probability that in the next six trials, there will be atleast 4 successes. **Solution:**

8.3.23 A die is thrown 5 times. Find the probability that an odd number will come up exactly three times. **Solution:**

Parameter	Values	Description
n	5	Number of throws
k	3	Number of favourable outcomes
p	0.5	Probability of getting odd number
X	$1 \leq X \leq 5$	X favourable out of 5 total outcomes
Y	$1 \leq Y \leq 5$	gaussian variable
$\mu = np$	2.5	mean
$\sigma = \sqrt{np(1-p)}$	1.118	standard deviation

(a) Binomial Distribution :

The X is the random variable, the pmf of X is given by

$$p_X(k) = {}^nC_k p^k (1-p)^{n-k} \tag{8.342}$$

We require $\Pr(X = 3)$. Since $n = 5$,

$$p_X(3) = 0.3125 \tag{8.343}$$

(b) Gaussian Distribution

Let Y be gaussian variable. Using central limit theorem, we can use the gaussian distribution function:

$$p_Y(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \tag{8.344}$$

(8.345)

Using Normal distribution at X=3.

$$p_Y(3) = \frac{1}{\sqrt{2\pi\left(\frac{5}{4}\right)}} e^{-\frac{\left(x-\frac{5}{2}\right)^2}{2\left(\frac{5}{4}\right)}} \tag{8.346}$$

$$= \frac{1}{\sqrt{2\pi\left(\frac{5}{4}\right)}} e^{-\frac{1}{10}} \tag{8.347}$$

$$= 0.3228684517 \tag{8.348}$$

(c) <u>using Q function:</u>

$$Y \sim \mathcal{N}\left(\mu, \sigma^2\right) \tag{8.349}$$

The CDF of Y:

$$F_Y(y) = \begin{cases} 1 - Q\left(\frac{y-\mu}{\sigma}\right), & y > \mu \\ Q\left(\frac{\mu-y}{\sigma}\right), & y < \mu \end{cases} \tag{8.350}$$

But,

$$\frac{Y-\mu}{\sigma} \sim \mathcal{N}(0,1) \tag{8.351}$$

$$\implies F_Y(y) = 1 - Q\left(\frac{y-\mu}{\sigma}\right) \tag{8.352}$$

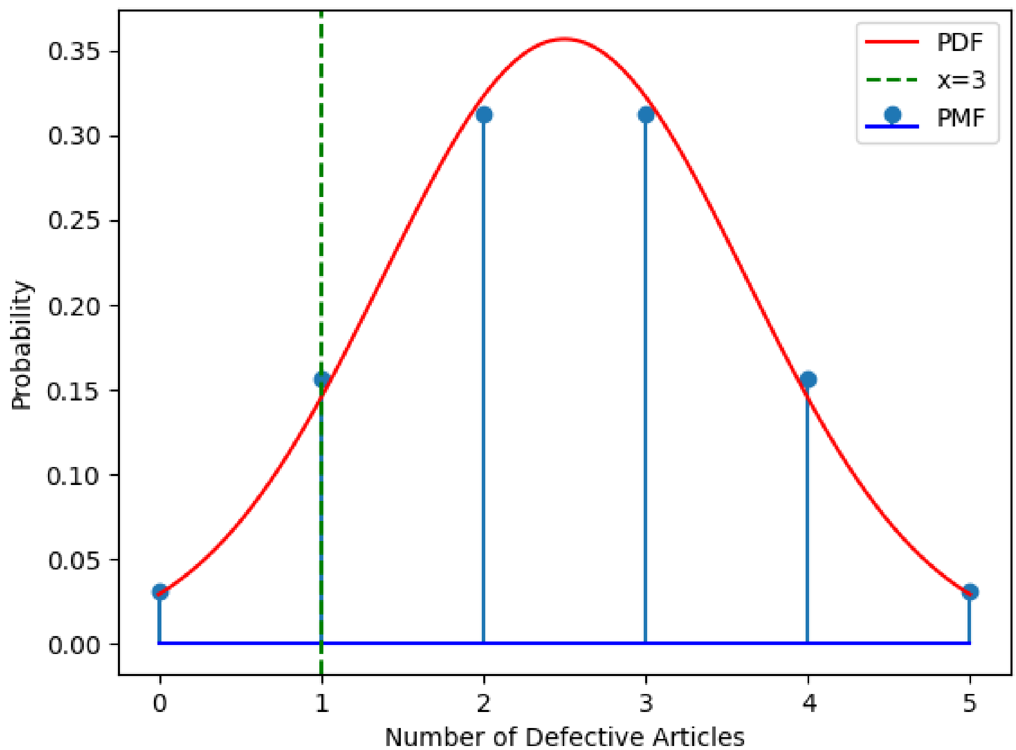


Figure 8.20: Binomial-PMF and Gaussian-PDFof X

to include correction of 0.5,

$$p_Y(2.5 < Y < 3.5) = F_Y(2.5) - F_Y(3.5) \tag{8.353}$$

$$= Q\left(\frac{2.5-\mu}{\sigma}\right) - Q\left(\frac{3.5-\mu}{\sigma}\right) \tag{8.354}$$

$$= Q(0) - Q(0.8944) \tag{8.355}$$

$$= 0.314446 \tag{8.356}$$

8.3.24 Ten coins are tossed. What is the probability of getting atleast 8 heads?

Solution:

Parameter	**Value**	**Description**
n	10	number of tosses
p	$\frac{1}{2}$	Probability for Heads
q	$\frac{1}{2}$	Probability for Tails
$\mu = np$	5	mean of the distribution
$\sigma^2 = npq$	2.5	variance of the distribution
X	$0 \leq X \leq 10$	Number of heads

Gaussian Distribution

$$X \approx Y \sim \mathcal{N}(5, 2.5) \tag{8.357}$$

(a) With a 0.5 correction:

$$\Pr(Y \geq 8) = Q\left(\frac{7.5-\mu}{\sigma}\right) \tag{8.358}$$

$$\implies \Pr(Y \geq 8) = Q\left(\sqrt{2.5}\right) = Q(1.5811) \tag{8.359}$$

$$\implies \Pr(Y \geq 8) = 0.0569276 \tag{8.360}$$

(b) Without correction:

$$\Pr(Y \geq 8) = Q\left(\frac{8-\mu}{\sigma}\right) \tag{8.361}$$

$$\implies \Pr(Y \geq 8) = Q\left(\frac{3}{\sqrt{2.5}}\right) = Q(1.8973) \tag{8.362}$$

$$\implies \Pr(Y \geq 8) = 0.0288898 \tag{8.363}$$

Binomial Distribution

$$\Pr(X \geq 8) = \sum_{k=8}^{10} \binom{n}{k} p^k (1-p)^{n-k} \tag{8.364}$$

$$= 0.0546875 \tag{8.365}$$

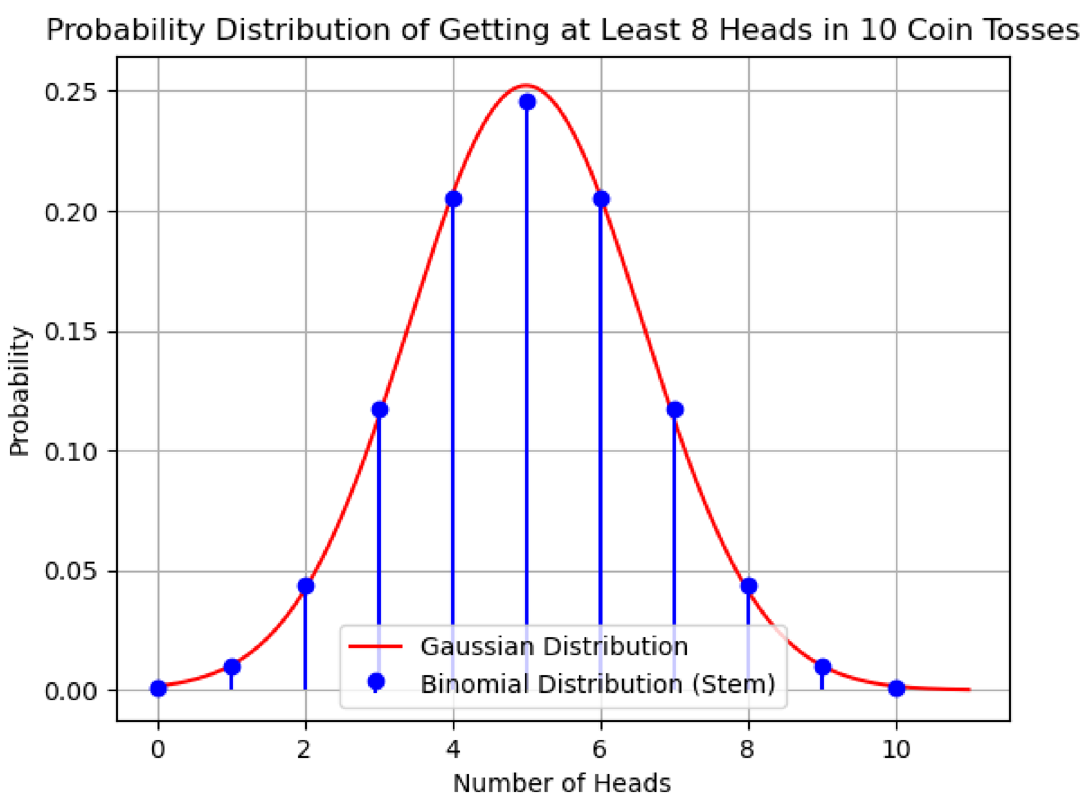


Figure 8.21: Binomial vs Guassian

8.3.25 A factory produces bulbs. The probability that any one bulb is defective is $\frac{1}{50}$ and they are packed in boxes of 10. From a single box, find the probability that

(a) none of the bulb is defective

(b) exactly two bulbs are defective

(c) more than 8 bulbs are working properly

Solution:

parameter	**value**	**description**
n	10	Number of bulbs in the bag
p	$\frac{1}{50}$	Bulb chosen is defective
q	$\frac{49}{50}$	Bulb chosen is proper
$\mu = np$	$\frac{1}{5}$	Mean of the distribution
$\sigma^2 = npq$	$\frac{49}{250}$	Variance of the distribution

Table 8.19: Gaussian Info Table

(i) Gaussian Distribution

Let Y is the Gaussian obtained by approximating binomial with parameters n,p then By Central limit theroem,

$$Y \sim \mathcal{N}(np, npq) \tag{8.366}$$

CDF of Y is:

$$F_Y(x) = \Pr(Y \leq x) \tag{8.367}$$

$$= \Pr(Y - \mu \leq x - \mu) \tag{8.368}$$

$$= \Pr\left(\frac{Y-\mu}{\sigma} \leq \frac{x-\mu}{\sigma}\right) \tag{8.369}$$

Since,

$$\frac{Y-\mu}{\sigma} \sim \mathcal{N}(0,1) \tag{8.370}$$

Q function is defined

$$Q(x) = \Pr(Y > x) \ \forall x \in Y \sim \mathcal{N}(0,1) \tag{8.371}$$

From (8.369) and (8.371),

$$F_Y(x) = 1 - \Pr\left(\frac{Y-\mu}{\sigma} > \frac{x-\mu}{\sigma}\right) \tag{8.372}$$

$$= \begin{cases} 1 - Q\left(\frac{x-\mu}{\sigma}\right), & x > \mu \\ Q\left(\frac{\mu-x}{\sigma}\right), & x < \mu \end{cases} \tag{8.373}$$

(a) If we consider no bulb is defective, we need to find

$$\Pr(Y=0) = \Pr(Y \leq 1) - \Pr(Y \leq 0) \tag{8.374}$$

$$= F_Y(1) - F_Y(0) \tag{8.375}$$

From (8.373) and Table 8.19,

$$F_Y(0) = Q\left(\frac{0.2-0}{0.196}\right) \tag{8.376}$$

$$= Q\,(1) \tag{8.377}$$

$$= 0.1587 \tag{8.378}$$

$$F_Y(1) = 1 - Q\left(\frac{1-0.2}{0.44}\right) \tag{8.379}$$

$$= 1 - Q\,(1.8) \tag{8.380}$$

$$= 0.964 \tag{8.381}$$

$$\Pr(Y=0) = F_Y(1) - F_Y(0) \tag{8.382}$$

$$= 0.964 - 0.1587 \tag{8.383}$$

$$= 0.8053 \tag{8.384}$$

(b) If we consider exactly 2 bulbs to be defective, the we need to find

$$\Pr(Y=2) = \Pr(Y \leq 2) - \Pr(Y \leq 1) \tag{8.385}$$

$$= F_Y(2) - F_Y(1) \tag{8.386}$$

From (8.373) and Table 8.19,

$$F_Y(2) = 1 - Q\left(\frac{2-0.2}{0.44}\right) \tag{8.387}$$

$$= 1 - Q(4) \tag{8.388}$$

$$= 0.999 \tag{8.389}$$

$$F_Y(1) = 1 - Q\left(\frac{1-0.2}{0.44}\right) \tag{8.390}$$

$$= 1 - Q(1.8) \tag{8.391}$$

$$= 0.964 \tag{8.392}$$

$$\Pr(Y = 2) = F_Y(2) - F_Y(1) \tag{8.393}$$

$$= 0.999 - 0.964 \tag{8.394}$$

$$= 0.036 \tag{8.395}$$

(c) If more than 8 bulbs are working properly then either 1 bulb is defective or no bulb is defetive we need to find

$$\Pr(Y \leq 1) = F_Y(1) \tag{8.396}$$

From (8.373) and Table 8.19,

$$F_Y(1) = 1 - Q\left(\frac{1-0.2}{0.44}\right) \tag{8.397}$$

$$= 1 - Q(1.8) \tag{8.398}$$

$$= 0.964 \tag{8.399}$$

(ii) Binomial Distribution

Lets define a random variable X which represents the number of defective bulbs.

$$X = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \tag{8.400}$$

The pmf is given by

$$P_X(r) = {}^nC_r p^r (1-p)^{n-r} \tag{8.401}$$

(a) If we consider there is no defective bulb,

$$P_X(0) = 0.817 \tag{8.402}$$

(b) If we consider there are 2 defective bulbs,

$$P_X(2) = 0.0153 \tag{8.403}$$

(c) If we consider there are more than 8 proper bulbs,

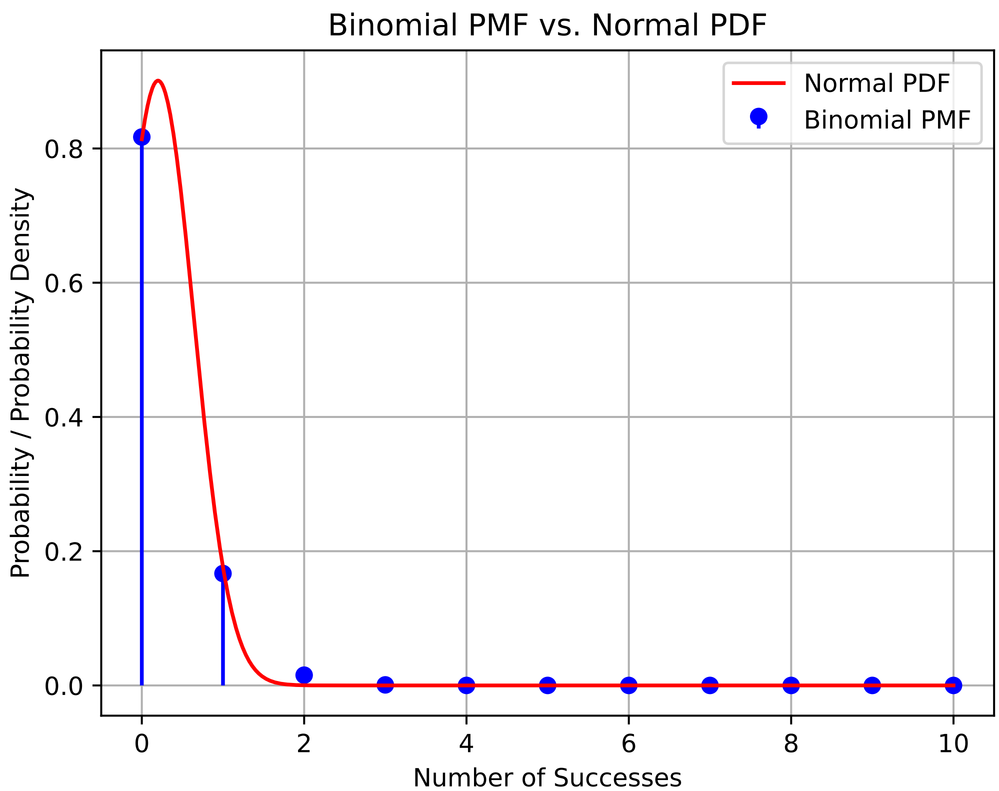
$$P_X(0) + P_X(1) = 0.817 + 0.1667 \tag{8.404}$$

$$= 0.9837 \tag{8.405}$$

Number of defective bulbs	**Binomial distribution**	**Gaussian approximation**
0	0.817	0.8053
2	0.0153	0.036
≤ 1	0.9837	0.964

Table 8.20: Comparing Binomial distribution and Gaussian approximation

(iii) Binomial vs Gaussian Graph



8.3.26 A lot of 100 watches is known to have 10 defective watches. If 8 watches are selected (one by one with replacement) at random, what is the probability that there will be at least one defective watch? **Solution:**

parameter	**value**	**description**
n	8	Number of watches selected
p	$\frac{1}{10}$	Chosen watch is defective
q	$\frac{9}{10}$	Chosen watch is non-defective
$\mu = np$	$\frac{8}{10}$	Mean of the distribution
$\sigma^2 = npq$	$\frac{72}{100}$	Variance of the distribution

Table 8.21: Gaussian Info Table

(i) Gaussian Distribution

Let Y is the Gaussian obtained by approximating binomial with parameters n,p then By Central limit theroem,

$$Y \sim \mathcal{N}(np, npq) \tag{8.406}$$

CDF of Y is:

$$F_Y(x) = \Pr\left(Y \leq x\right) \tag{8.407}$$

$$= \Pr\left(Y - \mu \leq x - \mu\right) \tag{8.408}$$

$$= \Pr\left(\frac{Y-\mu}{\sigma} \leq \frac{x-\mu}{\sigma}\right) \tag{8.409}$$

Since,

$$\frac{Y-\mu}{\sigma} \sim \mathcal{N}(0,1) \tag{8.410}$$

Q function is defined

$$Q\left(x\right) = \Pr\left(Y > x\right) \forall x \in Y \sim \mathcal{N}(0,1) \tag{8.411}$$

$$F_Y(x) = 1 - \Pr\left(\frac{Y-\mu}{\sigma} > \frac{x-\mu}{\sigma}\right) \tag{8.412}$$

$$= \begin{cases} 1 - Q\left(\frac{x-\mu}{\sigma}\right), & x > \mu \\ Q\left(\frac{\mu-x}{\sigma}\right), & x < \mu \end{cases} \tag{8.413}$$

(a) For atleast one watch to be defective, we need to find

$$1 - \Pr(Y = 0) \tag{8.414}$$

$$\Pr(Y = 0) = \Pr(Y \le 1) \tag{8.415}$$
$$= F_Y(1) \tag{8.416}$$

$$F_Y(1) = 1 - Q\left(\frac{1 - 0.8}{0.848}\right) \tag{8.417}$$
$$= 1 - Q(0.235) \tag{8.418}$$
$$= 0.58 \tag{8.419}$$
$$\Pr(Y = 0) = F_Y(1) \tag{8.420}$$
$$= 0.58 \tag{8.421}$$

(ii) Binomial Distribution

Lets define a random variable X which represents the number of defective bulbs.

$$X = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \tag{8.422}$$

The pmf is given by

$$P_X(r) = {}^nC_r p^r (1 - p)^{n-r} \tag{8.423}$$

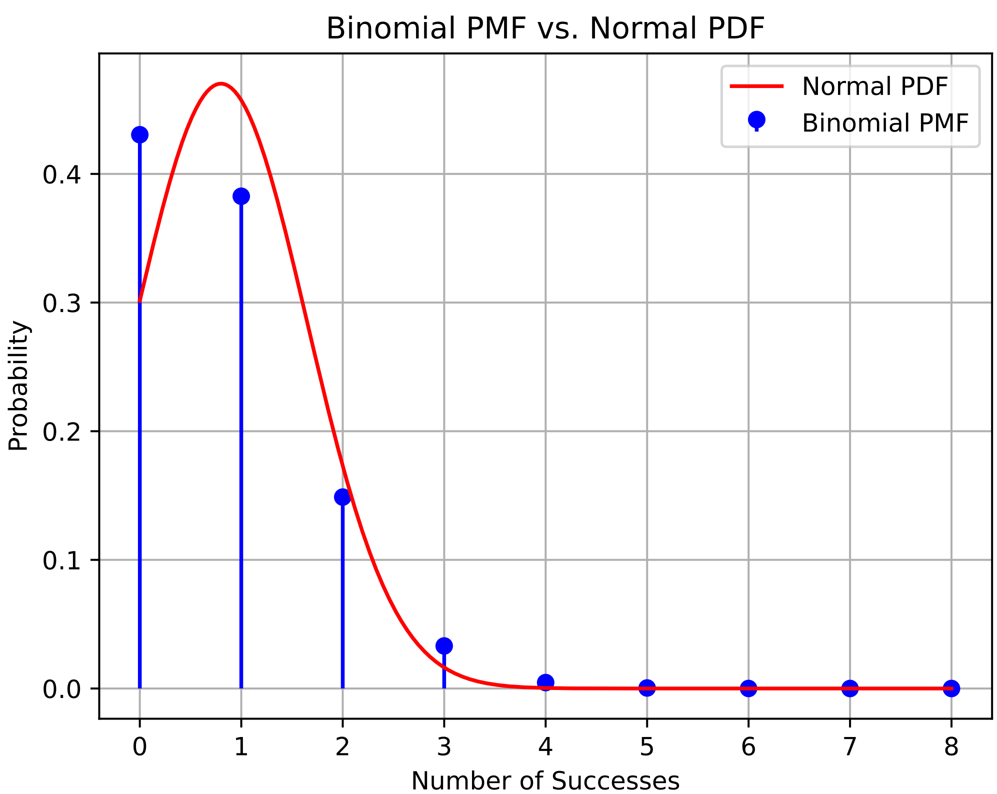
If we consider atleast one watch to be defective, we need,

$$1 - P_X(0) \tag{8.424}$$

$$P_X(0) = 0.430 \tag{8.425}$$

$$1 - P_X(0) = 0.569 \tag{8.426}$$

(iii) Binomial vs Gaussian Graph



8.3.27 The Probability of a man hitting target is 0.25.He shoots 7 times. What is the probability of his hitting atleast twice?

Solution: :

Table 8.22: Variables

Variable	Value	Description
n	7	Number of trails
p	0.25	The probability of man hitting the target
q	0.75	The probability of man not hitting the target
$\mu = np$	1.75	mean of distribution
$\sigma = \sqrt{npq}$	1.145	variance of distribution
X	$X \geq 2$	Number of times man hits the target

From gaussian,

$$Y \sim \mathcal{N}\left(\mu, \sigma^2\right) \tag{8.427}$$

CDF of Y is defined as:

$$F_Y\left(X\right) = \Pr\left(Y < X\right) \tag{8.428}$$

$$= \Pr\left(\frac{Y-\mu}{\sigma} \leq \frac{X-\mu}{\sigma}\right) \tag{8.429}$$

$$\implies \frac{Y-\mu}{\sigma} \sim N\left(0,1\right) \tag{8.430}$$

$$= 1 - \Pr\left(\frac{Y-\mu}{\sigma} > \frac{X-\mu}{\sigma}\right) \tag{8.431}$$

$$= \begin{cases} 1 - Q\left(\frac{X-\mu}{\sigma}\right) & X \geq \mu \\ Q\left(\frac{\mu-X}{\sigma}\right) & X < \mu \end{cases} \tag{8.432}$$

Hence, the probability of hitting target atleast twice using gaussian distribution is:

without correction:

$$\Pr(Y \geq 2) = 1 - \Pr(Y < 2) \tag{8.433}$$

$$= 1 - F_Y(2) \tag{8.434}$$

$$\implies \Pr(Y \geq 2) = Q\left(\frac{X - \mu}{\sigma}\right) \tag{8.435}$$

$$= Q(0.218) \tag{8.436}$$

$$\Pr(Y \geq 2) = 0.4137 \tag{8.437}$$

with correction:

$$\Pr(Y \geq 2) = Q\left(\frac{X - 0.5 - \mu}{\sigma}\right) \tag{8.438}$$

$$= Q(-0.218) \tag{8.439}$$

$$\Pr(Y \geq 2) = 0.5862 \tag{8.440}$$

Hence, the probability of hitting target atleast twice using binomial distribution is:

$$\Pr(X \geq 2) = 1 - \Pr(X < 2) \tag{8.441}$$

$$= 1 - \sum_{k=0}^{1} {}^{n}C_k p^k (1-p)^{n-k} \tag{8.442}$$

$$= 0.55 \tag{8.443}$$

8.3.28 A person buys a lottery ticket in 50 lotteries in each of which his chance of winning a prize is $\frac{1}{100}$. What is the probability that he will win a prize

(a) atleast once

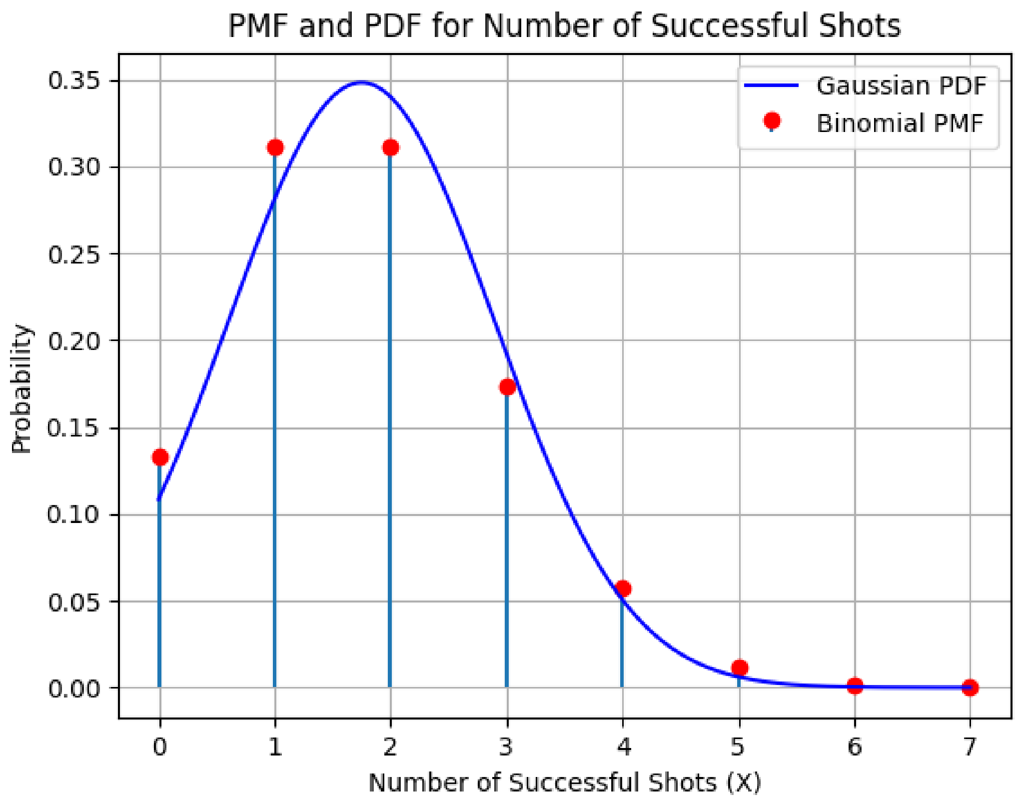


Figure 8.22: gaussian and binomial

(b) exactly once

(c) atleast twice ?

Solution: Let us define:

(a) **using Gaussian**

$$Y \sim \mathcal{N}\left(\mu, \sigma^2\right) \tag{8.444}$$

Parameter	Value	Description
n	50	number of lotteries
p	0.01	probability of winning a prize
q	0.99	probability of not winning
$\mu = np$	0.5	mean of the distribution
$\sigma^2 = npq$	0.495	variance of the distribution
Y	0,1,2,3,...,50	Number of successes

The CDF of Y:

$$F_Y(y) = 1 - \Pr(Y > y) \tag{8.445}$$

$$= 1 - \Pr\left(\frac{Y-\mu}{\sigma} > \frac{y-\mu}{\sigma}\right) \tag{8.446}$$

But,

$$\frac{Y-\mu}{\sigma} \sim \mathcal{N}(0,1) \tag{8.447}$$

$$\tag{8.448}$$

the Q-function is defined as:

$$Q(x) = \Pr(Y > x)\ \forall x \in Y \sim \mathcal{N}(0,1) \tag{8.449}$$

therefore the cdf will be:

$$F_Y(y) = \begin{cases} 1 - Q\left(\frac{y-\mu}{\sigma}\right), & y > \mu \\ Q\left(\frac{\mu-y}{\sigma}\right), & y < \mu \end{cases} \tag{8.450}$$

The probabbility of winning the prize atleast once is given by:

Considering 0.3 as the correction term,

$$\Pr(Y > 0.7) = 1 - F_Y(0.7) \tag{8.451}$$

$$= Q\left(\frac{0.7 - \mu}{\sigma}\right) \quad from(8.450) \tag{8.452}$$

$$= Q(0.2842) \tag{8.453}$$

$$= 0.3881 \tag{8.454}$$

(b) **using Gaussian**

the gaussian distribution function is defined as:

$$p_Y(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \tag{8.455}$$

the probability of the person winning the prize exactly once is given by:

$$p_Y(1) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(1-\mu)^2}{2\sigma^2}} \tag{8.456}$$

$$= 0.44 \tag{8.457}$$

(c) **using Gaussian**

the probability of the person winning the prize atleast twice is given by:

considering 0.5 as the correction term,

$$\Pr(Y > 1.5) = 1 - F_Y(1.5) \tag{8.458}$$
$$= Q\left(\frac{1.5 - \mu}{\sigma}\right) \quad from(8.450) \tag{8.459}$$
$$= Q\left(\frac{0.1}{\sqrt{0.495}}\right) \tag{8.460}$$
$$= Q(1.42) \tag{8.461}$$
$$= 0.0776 \tag{8.462}$$

Gaussian vs Binomial Table

Y	Gaussian	Binomial
atleast one	0.3881	0.395
exactly one	0.441	0.305
atleast two	0.0776	0.089

Gaussian vs Binomial graph

8.3.29 The probability that a person is not a swimmer is 0.3. The probability that out of 5 persons 4 are swimmers is

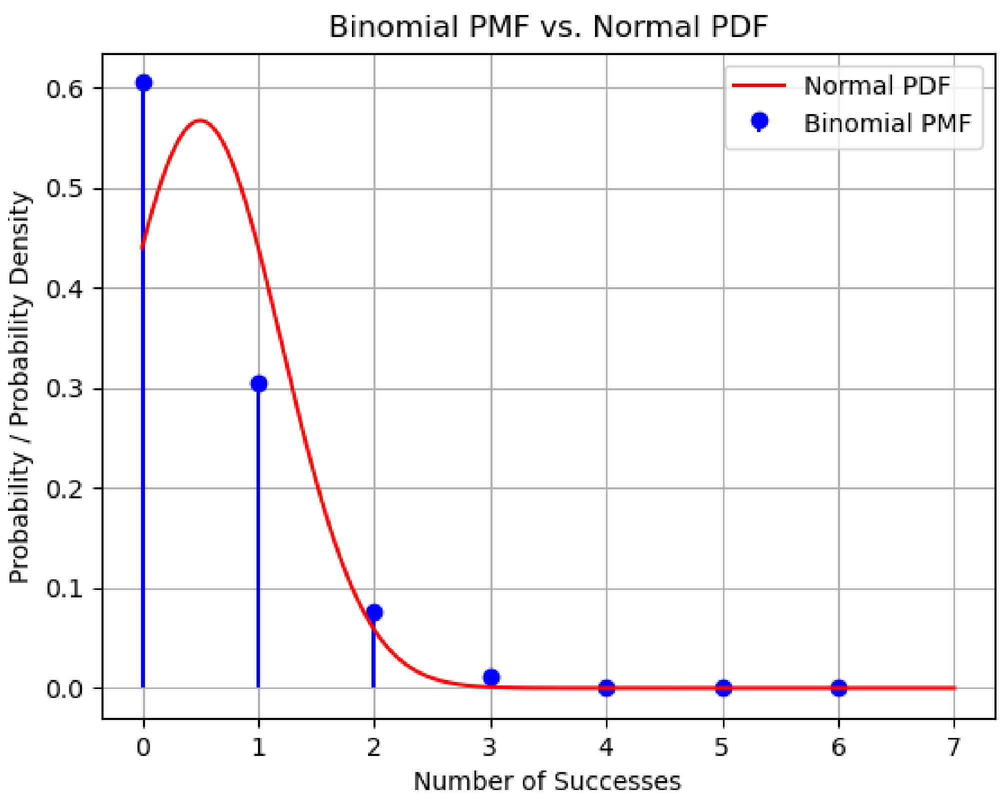
(a) ${}^5C_4 (0.7)^4 (0.3)$

(b) ${}^5C_1 (0.7) (0.3)^4$

(c) ${}^5C_4 (0.7) (0.3)^4$

(d) $(0.7)^4 (0.3)$

Solution: Let Y be the gaussian random variable,



Parameter	Values	Description
n	5	Number of draws
p	0.3	Probability that person is not a swimmer
q	0.7	Probability that person is a swimmer
$\mu = np$	3.5	Mean
$\sigma^2 = npq$	1.05	Variance

$$Y \sim \mathcal{N}\left(\mu, \sigma^2\right) \tag{8.463}$$

$$\sim \mathcal{N}\left(3.5, 1.05\right) \tag{8.464}$$

Due to continuity correction $\Pr\left(X = x\right)$ can be approximated using gaussian distribution as

$$p_Y(x) \approx \Pr(x - 0.5 < Y < x + 0.5) \tag{8.465}$$

$$\approx \Pr(Y < x + 0.5) - \Pr(Y < x - 0.5) \tag{8.466}$$

$$\approx F_Y(x + 0.5) - F_Y(x - 0.5) \tag{8.467}$$

$$\tag{8.468}$$

then CDF of Y is:

$$F_Y(x) = \Pr(Y < x) \tag{8.469}$$

$$= \Pr\left(\frac{Y - \mu}{\sigma} < \frac{x - \mu}{\sigma}\right) \tag{8.470}$$

$$\implies \frac{Y - \mu}{\sigma} \sim N(0, 1) \tag{8.471}$$

$$= 1 - \Pr\left(\frac{Y - \mu}{\sigma} > \frac{x - \mu}{\sigma}\right) \tag{8.472}$$

$$= \begin{cases} 1 - Q\left(\frac{x-\mu}{\sigma}\right) & x \geq \mu \\ Q\left(\frac{\mu-x}{\sigma}\right) & x < \mu \end{cases} \tag{8.473}$$

Hence, probability that out of 5 persons 4 are swimmers using gaussian approximation is

$$\Pr(Y = 4) = \Pr(3.5 < Y < 4.5) \tag{8.474}$$

$$= 0.335 \tag{8.475}$$

Probability that out of 5 persons 4 are swimmers using bernoulli distribution is

$$\Pr(Y = 4) = p_Y(4) \tag{8.476}$$

$$= {}^nC_k p^k (1-p)^{n-k} \tag{8.477}$$

$$= 0.360 \tag{8.478}$$

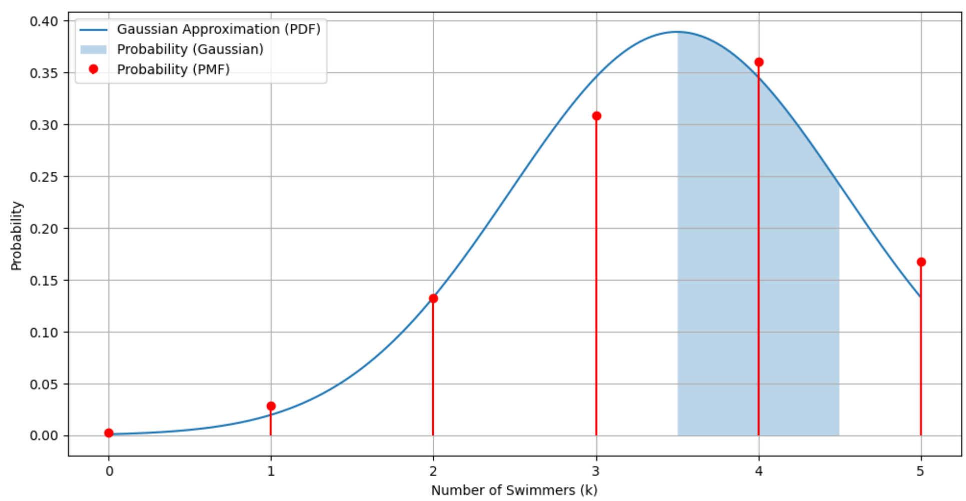


Figure 8.23: PDF vs Gaussian

8.3.30 The probability of guessing correctly at least 8 out of 10 answers on a true-false type examination is

Solution: Defining variables:

Parameter	Value	Description
n	10	Number of questions
p	0.5	probability of guessing correctly
$\mu = np$	5	mean of the distribution
$\sigma^2 = np(1-p)$	2.5	variance of the distribution
Y	0-10	denotes number of questions guessed correctly

(a) **Binomial distribution:** the probability of getting exactly 8 correct answers is

$$= \binom{10}{8} \times 0.5^8 \times 0.5^2 \tag{8.479}$$

$$= 0.043946 \tag{8.480}$$

(b) **Gaussian Distribution:**

The gaussian distribution for Y is

$$p_Y(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} \tag{8.481}$$

For getting exactly 8 correct answers

$$Y = 8 \tag{8.482}$$

Substituting in equation (8.498), probability for getting exactly 8 correct answers is

$$p_Y(8) = \frac{1}{\sqrt{2\pi \times 2.5}} e^{\frac{-(8-5)^2}{2\times 2.5}} \tag{8.483}$$

$$= 0.05204 \tag{8.484}$$

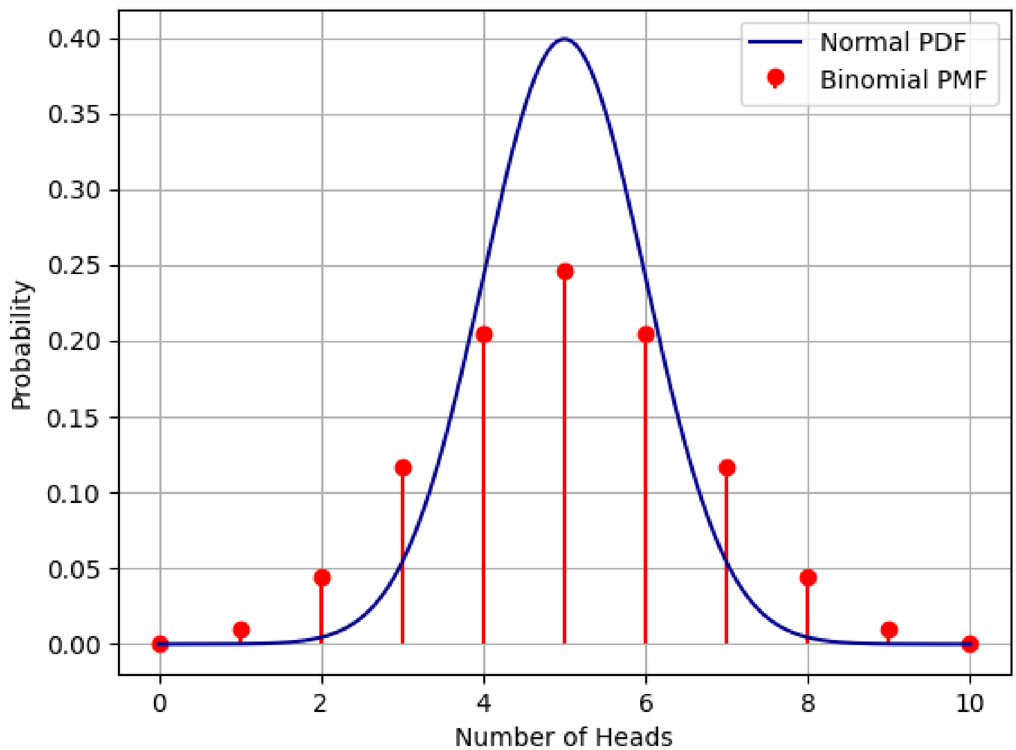


Figure 8.24: Binomial distribution vs Gaussian distribution

(c) **Using Q function:** Defining a gaussian random variable Z such that

$$Z \sim \mathcal{N}\left(\mu, \sigma^2\right) \tag{8.485}$$

Due to continuity correction, $\Pr(Z = x)$ can be approximated as

$$p_Z(x) \approx \Pr(x - 0.5 \le Z < x + 0.5) \tag{8.486}$$

$$\approx \Pr(Z < x + 0.5) - \Pr(Z < x - 0.5) \tag{8.487}$$

$$\approx F_Z(x + 0.5) - F_Z(x - 0.5) \tag{8.488}$$

CDF of Z is defined as -

$$F_Z(x) = \Pr(Z < x) \tag{8.489}$$

$$= \Pr\left(\frac{Z - \mu}{\sigma} < \frac{x - \mu}{\sigma}\right) \tag{8.490}$$

As

$$\frac{Z - \mu}{\sigma} \sim \mathcal{N}(0, 1) \tag{8.491}$$

$$\implies F_Z(x) = 1 - \Pr\left(\frac{Z - \mu}{\sigma} > \frac{x - \mu}{\sigma}\right) \tag{8.492}$$

$$= \begin{cases} 1 - Q(\frac{x-\mu}{\sigma}) & x \ge \mu \\ Q(\frac{\mu-x}{\sigma}) & x < \mu \end{cases} \tag{8.493}$$

$\therefore$ Gaussian approximation for $\Pr(Z = 8)$ is

$$p_Z(8) = 1 - Q(1.63273) \tag{8.494}$$

$$= 0.051263 \tag{8.495}$$

8.3.31 Eight coins are tossed together. The probability of getting exactly 3 heads is

(a) $\frac{1}{256}$

(b) $\frac{7}{32}$

(c) $\frac{5}{32}$

(d) $\frac{3}{32}$

Solution: Defining variables:

Parameter	**Value**	**Description**
n	8	Number of coins tossed
p	0.5	probability of getting heads
$\mu = np$	4	mean of the distribution
$\sigma^2 = np(1-p)$	2	variance of the distribution
Y	0-8	denotes number of heads obtained

(a) **Binomial distribution:** the probability of getting exactly 3 heads is

$$= \binom{8}{3} \times 0.5^3 \times 0.5^5 \tag{8.496}$$

$$= 0.21875 \tag{8.497}$$

$\therefore$ option 2 is correct.

(b) **Gaussian Distribution:**

The gaussian distribution for Y is

$$p_Y(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} \tag{8.498}$$

For getting 3 exactly heads

$$Y = 3 \tag{8.499}$$

Substituting in equation (8.498), probability for getting exactly 3 heads is

$$Y = 3 \tag{8.500}$$

$$p_Y(3) = \frac{1}{\sqrt{2\pi \times 2}} e^{\frac{-(3-4)^2}{2\times 2}} \tag{8.501}$$

$$= 0.35206 \tag{8.502}$$

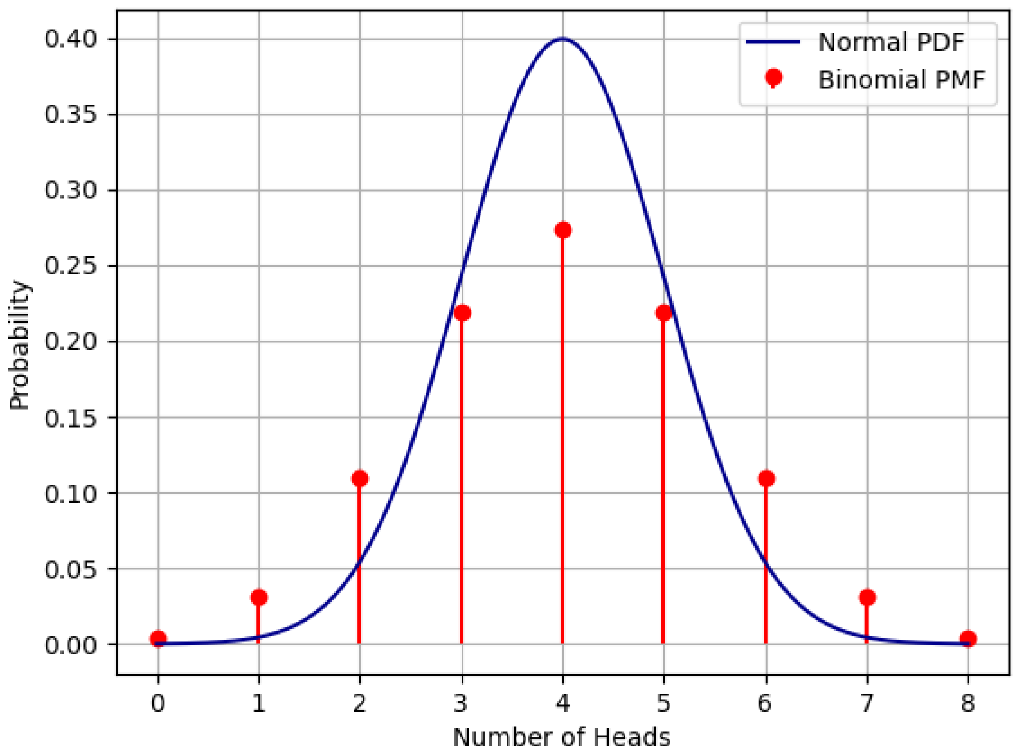


Figure 8.25: Binomial distribution vs Gaussian distribution

(c) **Using Q function:** Defining a gaussian random variable Z such that

$$Z \sim \mathcal{N}\left(\mu, \sigma^2\right) \tag{8.503}$$

Due to continuity correction, $\Pr(Z = x)$ can be approximated as

$$p_Z(x) \approx \Pr(x - 0.5 \leq Z < x + 0.5) \tag{8.504}$$

$$\approx \Pr(Z < x + 0.5) - \Pr(Z < x - 0.5) \tag{8.505}$$

$$\approx F_Z(x + 0.5) - F_Z(x - 0.5) \tag{8.506}$$

CDF of Z is defined as

$$F_Z(x) = \Pr(Z < x) \tag{8.507}$$

$$= \Pr\left(\frac{Z - \mu}{\sigma} < \frac{x - \mu}{\sigma}\right) \tag{8.508}$$

As

$$\frac{Z - \mu}{\sigma} \sim \mathcal{N}(0, 1) \tag{8.509}$$

$$\implies F_Z(x) = 1 - \Pr\left(\frac{Z - \mu}{\sigma} > \frac{x - \mu}{\sigma}\right) \tag{8.510}$$

$$= \begin{cases} 1 - Q(\frac{x-\mu}{\sigma}) & x \geq \mu \\ Q(\frac{\mu-x}{\sigma}) & x < \mu \end{cases} \tag{8.511}$$

Probability in terms of Q function is

$$p_Z(x) \approx Q\left(\frac{(x - 0.5) - \mu}{\sigma}\right) - Q\left(\frac{(x + 0.5) - \mu}{\sigma}\right) \tag{8.512}$$

$\therefore$ Gaussian approximation for $\Pr(Z = 3)$ is

$$p_Z(3) \approx Q(0.3536) - Q(1.0608) \tag{8.513}$$

$$= 0.2174 \tag{8.514}$$

(d) **Comparing all three techniques:**

Event	**Binomial**	**Gaussian**	**Q function**
Getting exactly 3 heads	0.21875	0.35206	0.2174

8.3.32 If X follows binomial distribution with parameters $n = 5$, p and

$$p_X(2) = 9p_X(3) \tag{8.515}$$

then p is ?

$$\mu = np \tag{8.516}$$

$$= 5p \tag{8.517}$$

$$\sigma^2 = np(1-p) \tag{8.518}$$

$$= 5p(1-p) \tag{8.519}$$

$$Y \sim N(\mu, \sigma) \tag{8.520}$$

Using the condition $p_Y(2) = 9p_Y(3)$, we get:

$$e^{-\frac{1}{2}(\frac{2-\mu}{\sigma})^2} = 9e^{-\frac{1}{2}\left(\frac{3-\mu}{\sigma}\right)^2} \tag{8.521}$$

$$\implies e^{-\frac{1}{2}\left(\frac{2-5p}{\sqrt{5p(1-p)}}\right)^2} = 9e^{-\frac{1}{2}\left(\frac{3-5p}{\sqrt{5p(1-p)}}\right)^2} \tag{8.522}$$

$$\implies e^{-\frac{1}{2}\left(\frac{2-5p}{\sqrt{5p(1-p)}}\right)^2} = 9e^{-\frac{1}{2}\left(\frac{3-5p}{\sqrt{5p(1-p)}}\right)^2} \tag{8.523}$$

$$\implies e^{-\frac{1}{2}\left(\frac{(2-5p)^2-(3-5p)^2}{(\sqrt{5p(1-p)})^2}\right)} = 9 \tag{8.524}$$

Taking the natural logarithm of both sides, we have:

$$\implies -\frac{1}{2}\left(\frac{(2-5p)^2-(3-5p)^2}{5p(1-p)}\right) = \ln(9) \tag{8.525}$$

$$\implies 4 + 25p^2 - 20p - 9 - 25p^2 + 30p = -10p(1-p)\ln(9) \tag{8.526}$$

$$\implies 10p - 5 = -10p(1-p)\ln(9) \tag{8.527}$$

$$\implies 1 - 2p = (2p - 2p^2)\ln(9) \tag{8.528}$$

$$\implies 2p^2\ln(9) - 2p\ln(9) - 2p + 1 = 0 \tag{8.529}$$

$$p = \frac{2\ln(9) + 2 \pm \sqrt{(-2\ln(9)-2)^2 - 4(2\ln(9))(1)}}{2(2\ln(9))} \tag{8.530}$$

$$= \frac{2\ln(9) + 2 \pm \sqrt{4(\ln(9))^2 + 4}}{4\ln(9)} \tag{8.531}$$

$$= 0.178211588 \tag{8.532}$$

8.3.33 A box has 100 pens of which 10 are defective. What is the probability that out of a sample of 5 pens drawn one by one with replacement at most one is defective?

(a) $\left(\frac{9}{10}\right)^5$

(b) $\frac{1}{2}\left(\frac{9}{10}\right)^4$

(c) $\frac{1}{2}\left(\frac{9}{10}\right)^5$

(d) $\frac{1}{2}\left(\frac{9}{10}\right)^4 + \left(\frac{9}{10}\right)^5$

Solution:

Parameter	Values	Description
n	5	Number of defective pens
p	0.1	probability of drawing a defective pen
μ	0.5	np
σ	0.671	$\sqrt{np(1-p)}$
X		Defective pens

Using Binomial

Given,

Probability of drawing a defective pen $= \frac{1}{10}$

Probability of drawing a non-defective pen $= \frac{9}{10}$

Let,

Probability of drawing atmost one pen out of 5 defective with replacement $= \Pr(X \leq 1)$

$$\Pr(X \leq 1) = p_X(0) + p_X(1) \tag{8.533}$$

$$\implies \Pr(X \leq 1) = \binom{5}{0}\left(\frac{9}{10}\right)^5 + \binom{5}{1}\left(\frac{9}{10}\right)^4\left(\frac{1}{10}\right) \tag{8.534}$$

$$= \left(\frac{9}{10}\right)^5 + 5\left(\frac{9}{10}\right)^4\left(\frac{1}{10}\right) \tag{8.535}$$

$$= \left(\frac{9}{10}\right)^5 + \frac{1}{2}\left(\frac{9}{10}\right)^4 \tag{8.536}$$

$$= 0.91854 \tag{8.537}$$

Gaussian

$$Y \sim \mathcal{N}\left(\mu, \sigma^2\right) \tag{8.538}$$

CDF of Y is

$$F_Y\left(y\right) = \Pr\left(Y \leq y\right) \tag{8.539}$$

We know that

$$Q\left(x\right) = \Pr\left(X > x\right), x > 0, X \sim N(0,1) \tag{8.540}$$

$$Q\left(-x\right) = \Pr\left(X > -x\right), x < 0, X \sim N(0,1) \tag{8.541}$$

$$= 1 - Q\left(x\right) \tag{8.542}$$

Hence,

CDF :

$$F_Y\left(y\right) = \begin{cases} 1 - Q\left(\frac{y-\mu}{\sigma}\right), & \text{if } y > \mu \\ 1 - Q\left(\frac{y-\mu}{\sigma}\right) = Q\left(\frac{\mu-y}{\sigma}\right), & \text{if } y < \mu \end{cases} \tag{8.543}$$

$$
\begin{aligned}
F_Y(1) &= \Pr(Y \leq 1) && (8.544)\\
&= 1 - Q\left(\frac{1-0.5}{\sqrt{0.671}}\right) && (8.545)\\
&= 1 - Q\left(\frac{0.5}{0.819}\right) && (8.546)\\
&= 1 - Q(0.6104) && (8.547)\\
&= 0.729198876 && (8.548)
\end{aligned}
$$

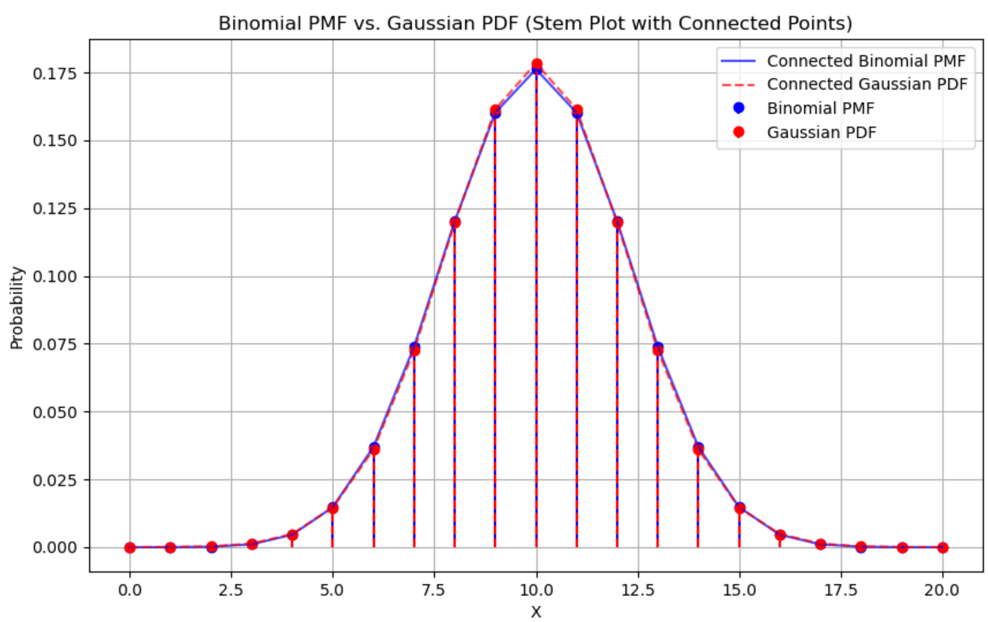


Figure 8.26: pmf of binomial and pdf of Gaussian of X and Y marked balls

8.4. From Uniform to Other

8.4.1 Generate samples of

$$V = -2\ln(1-U) \tag{8.549}$$

and plot its CDF.

8.4.2 Find a theoretical expression for $F_V(x)$.

Appendix A

Axioms

A.1. Definitions

A.1.1

$$0 \le \Pr(A) \le 1 \tag{A.1.1.1}$$

A.1.2 If $AB = 0$,

$$\Pr(A + B) = \Pr(A) + \Pr(B). \tag{A.1.2.1}$$

A.1.3 If A, B are independent,

$$\Pr(AB) = \Pr(A)\Pr(B) \tag{A.1.3.1}$$

A.1.4

$$\Pr(A|B) = \frac{\Pr(AB)}{\Pr(B)} \tag{A.1.4.1}$$

A.2. Boolean Logic

A.2.1

$$A + A' = 1 \tag{A.2.1.1}$$

A.2.2

$$A'B' = (A + B)' \tag{A.2.2.1}$$

A.2.3

$$A + B = A\left(B + B'\right) + B \tag{A.2.3.1}$$

$$= B\left(A + 1\right) + AB' \tag{A.2.3.2}$$

$$= B + AB' \tag{A.2.3.3}$$

A.2.4

$$A = A\left(B + B'\right) = AB + AB' \tag{A.2.4.1}$$

and

$$\left(AB\right)\left(AB'\right) = 0, \because BB' = 0 \tag{A.2.4.2}$$

Hence, AB and AB' are mutually exclusive.

A.2.5 Let A, B and C be three events.

Let X be the event that exactly one of A, B and C occurs.

Let Y be the event that at least one of A, B or C occur.

Let Z be the event that at least two of A, B or C occur.

$$Y = A + B + C \tag{A.2.5.1}$$

Similarly,

$$Z = AB + BC + CA \tag{A.2.5.2}$$

And,

$$X = (AB'C' + A'BC' + A'B'C) \tag{A.2.5.3}$$

A.3. Properties

A.3.1

$$\Pr\left(A'\right) = 1 - \Pr\left(A\right). \tag{A.3.1.1}$$

A.3.2

$$\Pr\left(A'B'\right) = \Pr\left(\left(A+B\right)'\right) \tag{A.3.2.1}$$

$$= 1 - \Pr\left(A+B\right) \tag{A.3.2.2}$$

A.3.3

$$\Pr\left(A+B\right) = \Pr\left(B + AB'\right) \tag{A.3.3.1}$$

$$= \Pr\left(B\right) + \Pr\left(AB'\right) \tag{A.3.3.2}$$

$$\because B\left(AB'\right) = 0 \tag{A.3.3.3}$$

A.3.4

$$\Pr\left(A\right) = \Pr\left(AB\right) + \Pr\left(AB'\right) \tag{A.3.4.1}$$

$$\implies \Pr\left(AB'\right) = \Pr\left(A\right) - \Pr\left(AB\right) \tag{A.3.4.2}$$

A.3.5 Substituting (A.3.4.2) in (A.3.3.2),

$$\Pr\left(A+B\right) = \Pr\left(A\right) + \Pr\left(B\right) - \Pr\left(AB\right) \tag{A.3.5.1}$$

Appendix B

Z-transform

B.1 The Z-transform of X is defined as

$$M_X(z) = E\left[z^{-X}\right] = \sum_{k=-\infty}^{\infty} p_X(k) z^{-k} \tag{B.1.1}$$

B.2 If X_1 and X_2 are independent, the MGF of

$$X = X_1 + X_2 \tag{B.2.1}$$

is given by

$$M_X(z) = P_{X_1}(z) P_{X_2}(z) \tag{B.2.2}$$

The above property follows from Fourier analysis and is fundamental to signal processing.

B.3 Let X_i be independent. For

$$X = X_1 + X_2 + \ldots + X_n, \tag{B.3.1}$$

$$M_X(z) = \prod_{i=1}^{n} M_{X_i}(z) \tag{B.3.2}$$

B.4 The nth moment of X can be expressed as

$$E\left[X^n\right] = \frac{d^n M_X(z^{-1})}{dz^n}\Big|_{z=1} \tag{B.4.1}$$

Appendix C

Distributions

C.1. Bernoulli

C.1.1. The pmf of a Bernoulli distribution is defined as

$$p_X(k) = \begin{cases} q = 1 - p & k = 0 \\ p & k = 1 \\ 0 & \text{otherwise} \end{cases} \tag{C.1.1.1}$$

C.1.2. For a Bernoulli random variable X with success probability p,

$$M_X(z) = q + pz^{-1} \tag{C.1.2.1}$$

C.1.3. The mean of the Bernoulli distribution is

$$E(X) = p \tag{C.1.3.1}$$

C.1.4. The following code simulates 100 coin tosses

```
#Code by GVV Sharma
```

```
#November 18, 2020
#Released under GNU/GPL
#Given a Bernoulli probability and
#number of samples, the code generates the event data

import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import bernoulli

#100 samples
simlen=int(1e2)

#Probability of the event
prob = 0.5

#Generating sample date using Bernoulli r.v.
data_bern = bernoulli.rvs(size=simlen,p=prob)
#Calculating the number of favourable outcomes
err_ind = np.nonzero(data_bern == 1)
#calculating the probability
err_n = np.size(err_ind)/simlen

#Theory vs simulation
print(err_n,prob)
print(data_bern)
```

C.2. Binomial Distribution

C.2.1. The Binomial distribution is defined as

$$X = X_1 + X_2 + ... + X_n, \tag{C.2.1.1}$$

Where X_i are i.i.d bernoulli.

C.2.2. For a Binomial random variable X with parameters n, p,

$$M_X(z) = \left(q + pz^{-1}\right)^n \tag{C.2.2.1}$$

C.2.3. The mean for the Binomial r.v. is

$$E[X] = np \tag{C.2.3.1}$$

Solution: From (B.4.1) and (C.2.2.1),

$$E[X] = \frac{d(q+pz)^n}{dz}|_{z=1} \tag{C.2.3.2}$$

$$= np(q+pz)^{n-1}|_{z=1} \tag{C.2.3.3}$$

$$= np(q+p)^{n-1} \tag{C.2.3.4}$$

yielding (C.2.3.1)

$$\because p + q = 1 \tag{C.2.3.5}$$

C.2.4. In a hurdle race, a player has to cross 10 hurdles. The probability that he will clear each hurdle is $\frac{5}{6}$. What is the probability that he will knock down fewer than 2 hurdles?

Solution: See the following code

```
#Code by GVV Sharma
#November 20,2020
#Released under GNU/GPL
#To find the probability of an event using the binomial distribution

import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import bernoulli
from scipy.stats import norm
from scipy.stats import binom




#Simlen
simlen=1000

#Number of hurdles
n = 10

#Probability of clearing a hurdle
p = 1-5/6

#Mean
```

```
mu = p

#Variance
sigma = np.sqrt(p*(1-p))

#Theoretical probability of knocking down fewer than 2 hurdles
k = 1
print(binom.cdf(k, n, p),3*(5/6)**10)

#Using the Gaussian approximation for the binomial pdf
print(1/(sigma*np.sqrt(n))*(norm.pdf((k-n*mu)/(sigma*np.sqrt(n)))+norm.pdf((k
    -1-n*mu)/(sigma*np.sqrt(n)))))



#Simulating the probability using the binomial random variable
data_binom = binom.rvs(n,p,size=simlen) #Simulating the event of jumping 10
    hurdles
err_ind = np.nonzero(data_binom <=k) #checking probability condition
err_n = np.size(err_ind) #computing the probability
print(err_n/simlen)
#print(data_binom)



#Simulating the probability using the bernoulli random variable
data_bern_mat = bernoulli.rvs(p,size=(n,simlen))
```

```
data_binom=np.sum(data_bern_mat, axis=0)
#print(data_bern_mat)
#print(data_binom)
err_ind = np.nonzero(data_binom <=k) #checking probability condition
err_n = np.size(err_ind) #computing the probability
print(err_n/simlen)
```

C.3. Uniform Distribution

C.3.1. Let $X \in \{1, 2, 3, 4, 5, 6\}$ be the random variables representing the outcome for a die. Assuming the die to be fair, the probability mass function (pmf) is expressed as

$$p_X(n) = \begin{cases} \frac{1}{6} & 1 \le n \le 6 \\ 0 & otherwise \end{cases} \tag{C.3.1.1}$$

C.3.2. The Z-transform of X is given by

$$P_X(z) = \frac{1}{6}\sum_{n=1}^{6} z^{-n} = \frac{z^{-1}\left(1 - z^{-6}\right)}{6\left(1 - z^{-1}\right)}, \quad |z| > 1 \tag{C.3.2.1}$$

upon summing up the geometric progression.

C.3.3. From (C.3.2.1), the CDF of X is given by

$$F_X(n) = \Pr(X \le n) = \begin{cases} 0 & n < 1 \\ \frac{n}{6} & 1 \le n \le 6 \\ 1 & \text{otherwise} \end{cases} \tag{C.3.3.1}$$

and plotted in Fig. C.3.3.1.

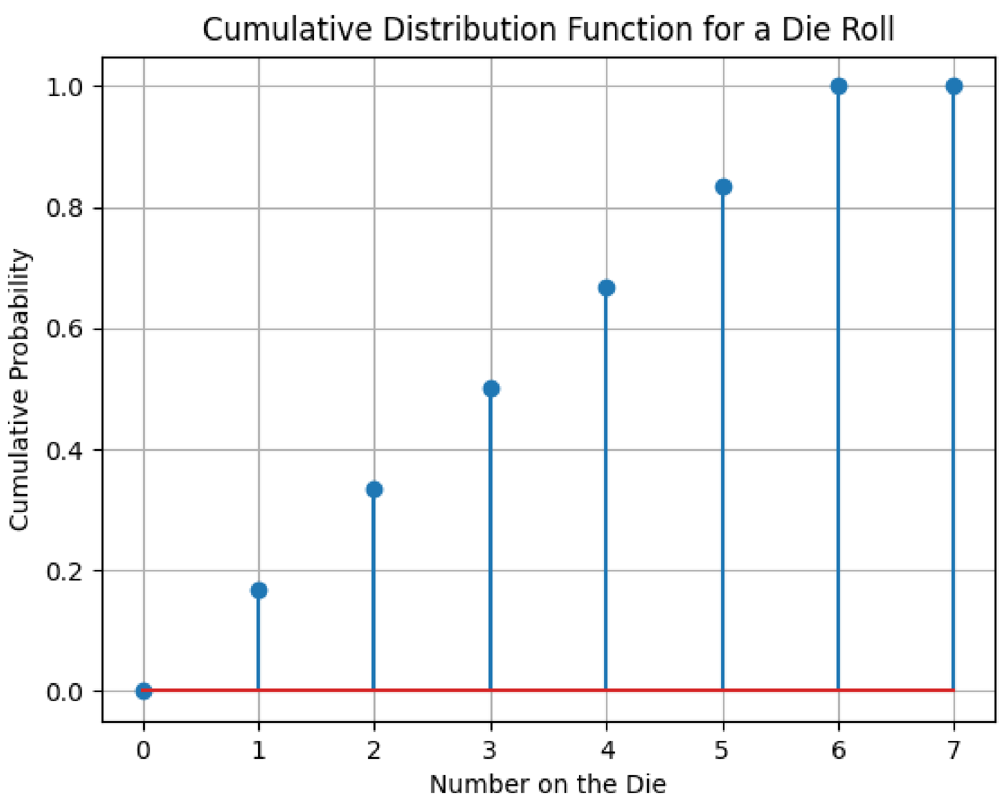


Figure C.3.3.1: CDF

C.4. Triangular Distribution

C.4.1. The desired outcome is

$$X = X_1 + X_2, \tag{C.4.1.1}$$
$$\implies X \in \{1, 2, \dots, 12\} \tag{C.4.1.2}$$

The objective is to show that

$$p_X(n) \neq \frac{1}{11} \tag{C.4.1.3}$$

C.4.2. <u>Convolution:</u> From (C.4.1.1),

$$p_X(n) = \Pr(X_1 + X_2 = n) = \Pr(X_1 = n - X_2) \tag{C.4.2.1}$$
$$= \sum_k \Pr(X_1 = n - k | X_2 = k) \, p_{X_2}(k) \tag{C.4.2.2}$$

after unconditioning. $\because$ X_1 and X_2 are independent,

$$\Pr(X_1 = n - k | X_2 = k)$$
$$= \Pr(X_1 = n - k) = p_{X_1}(n - k) \tag{C.4.2.3}$$

From (C.4.2.2) and (C.4.2.3),

$$p_X(n) = \sum_k p_{X_1}(n - k) p_{X_2}(k) = p_{X_1}(n) * p_{X_2}(n) \tag{C.4.2.4}$$

where $*$ denotes the convolution operation. Substituting from (C.3.1.1) in (C.4.2.4),

$$p_X(n) = \frac{1}{6}\sum_{k=1}^{6} p_{X_1}(n-k) = \frac{1}{6}\sum_{k=n-6}^{n-1} p_{X_1}(k) \tag{C.4.2.5}$$

$$\because p_{X_1}(k) = 0, \quad k \le 1, k \ge 6. \tag{C.4.2.6}$$

From (C.4.2.5),

$$p_X(n) = \begin{cases} 0 & n < 1 \\ \frac{1}{6}\sum_{k=1}^{n-1} p_{X_1}(k) & 1 \le n-1 \le 6 \\ \frac{1}{6}\sum_{k=n-6}^{6} p_{X_1}(k) & 1 < n-6 \le 6 \\ 0 & n > 12 \end{cases} \tag{C.4.2.7}$$

Substituting from (C.3.1.1) in (C.4.2.7),

$$p_X(n) = \begin{cases} 0 & n < 1 \\ \frac{n-1}{36} & 2 \le n \le 7 \\ \frac{13-n}{36} & 7 < n \le 12 \\ 0 & n > 12 \end{cases} \tag{C.4.2.8}$$

satisfying (C.4.1.3).

C.4.3. <u>The Z-transform:</u> From (C.3.2.1) and (B.2.2),

$$P_X(z) = \left\{ \frac{z^{-1}\left(1 - z^{-6}\right)}{6\left(1 - z^{-1}\right)} \right\}^2 \tag{C.4.3.1}$$

$$= \frac{1}{36} \frac{z^{-2}\left(1 - 2z^{-6} + z^{-12}\right)}{\left(1 - z^{-1}\right)^2} \tag{C.4.3.2}$$

Using the fact that

$$p_X(n-k) \overset{\mathcal{Z}}{\longleftrightarrow} P_X(z) z^{-k}, \tag{C.4.3.3}$$

$$nu(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}}{\left(1 - z^{-1}\right)^2} \tag{C.4.3.4}$$

after some algebra, it can be shown that

$$\frac{1}{36}\left[(n-1)\,u(n-1) - 2\,(n-7)\,u(n-7) + (n-13)\,u(n-13)\right] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{36} \frac{z^{-2}\left(1 - 2z^{-6} + z^{-12}\right)}{\left(1 - z^{-1}\right)^2} \tag{C.4.3.5}$$

where

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} \tag{C.4.3.6}$$

From (B.1.1), (C.4.3.2) and (C.4.3.5)

$$p_X(n) = \frac{1}{36}\left[(n-1)\,u(n-1) - 2\,(n-7)\,u(n-7) + (n-13)\,u(n-13)\right] \tag{C.4.3.7}$$

which is the same as (C.4.2.8). Note that (C.4.2.8) can be obtained from (C.4.3.5) using contour integration as well.

C.4.4. The experiment of rolling the dice was simulated using Python for 10000 samples. These were generated using Python libraries for uniform distribution. The frequencies for each outcome were then used to compute the resulting pmf, which is plotted in Figure C.4.4.1. The theoretical pmf obtained in (C.4.2.8) is plotted for comparison.

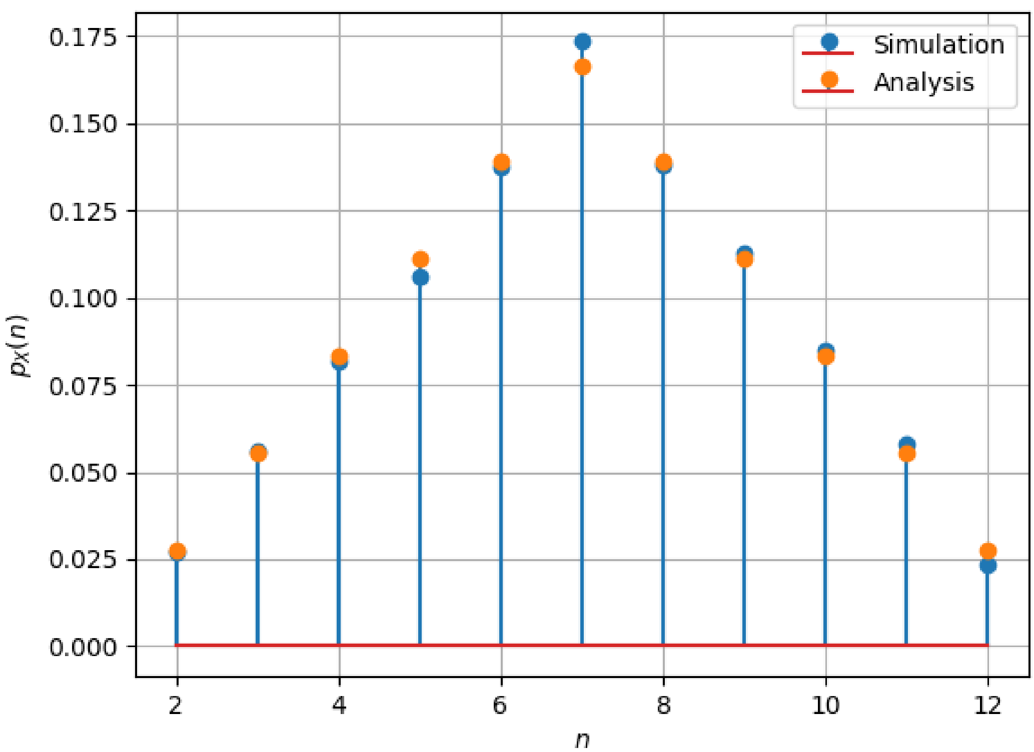


Figure C.4.4.1: Plot of $p_X(n)$. Simulations are close to the analysis.

C.4.5. The python code is available below

```
import numpy as np
import matplotlib.pyplot as plt
#If using termux
import subprocess
import shlex
#end if

#Sample size
simlen = 10000
#Possible outcomes
n = range(2,13)
# Generate X1 and X2
y = np.random.randint(1,7, size=(2, simlen))

#Generate X
X = np.sum(y, axis = 0)
#Find the frequency of each outcome
unique, counts = np.unique(X, return_counts=True)
#Simulated probability
psim = counts/simlen
#Theoretical probability
n1 = range(2,8)
n2 = range(8,13)
panal1 = (n1 -np.ones((1,6)))
panal2 = (13*np.ones((1,5))-n2)
```

```
panal = np.concatenate((panal1,panal2),axis=None)/36


#Plotting
plt.stem(n,psim, markerfmt='o', use_line_collection=True, label='Simulation')
plt.stem(n,panal, markerfmt='o',use_line_collection=True, label='Analysis')
plt.xlabel('$n$')
plt.ylabel('$p_{X}(n)$')
plt.legend()
plt.grid()# minor


#If using termux
plt.savefig('figs/pmf.pdf')
plt.savefig('figs/pmf.png')
subprocess.run(shlex.split("termux-open figs/pmf.pdf"))
#else
#plt.show()
```

Appendix D

Central Limit Theorem

D.1. Binomial

D.1 Let

$$X = \mathrm{Bin}(n, p). \tag{D.1.1}$$

The mean and variance are then given by

$$\mu = np,\ \sigma^2 = npq. \tag{D.1.2}$$

D.2 For large values of $n, k, n-k$, by Stirling's Approximation.

$$\begin{aligned} n! &\approx n^n e^{-n}\sqrt{2\pi n} \\ k! &\approx k^k e^{-k}\sqrt{2\pi k} \\ (n-k)! &\approx (n-k)^{(n-k)}\, e^{-(n-k)}\sqrt{2\pi\,(n-k)} \end{aligned} \tag{D.2.1}$$

D.3 Then,

$$p_X(k) = \frac{n!}{k!(n-k)!} p^k q^{n-k} \approx \frac{n^n e^{-n}\sqrt{2\pi n}}{k^k e^{-k}\sqrt{2\pi k}(n-k)^{n-k} e^{-(n-k)}\sqrt{2\pi(n-k)}} p^k q^{n-k} \tag{D.3.1}$$

$$= \left(\frac{np}{k}\right)^k \left(\frac{nq}{n-k}\right)^{n-k} \sqrt{\frac{n}{2\pi k(n-k)}} \tag{D.3.2}$$

from (D.2.1)

D.4 For

$$\delta \ll np, nq, \tag{D.4.1}$$

and

$$k = np + \delta, \tag{D.4.2}$$

$$n - k = nq - \delta \tag{D.4.3}$$

$$\implies \begin{aligned} \frac{k}{np} &= 1 + \frac{\delta}{np} \\ \frac{n-k}{nq} &= 1 - \frac{\delta}{nq} \end{aligned} \tag{D.4.4}$$

D.5 Taking logarithms in (D.3.2),

$$\ln\left[p_X(k)\right] = -k \ln\left(\frac{np}{k}\right) - (n-k)\left(\frac{nq}{n-k}\right) + \frac{1}{2}\ln\left(\frac{n}{2\pi k(n-k)}\right) \tag{D.5.1}$$

$$\begin{aligned} &= -\left(np+\delta\right)\ln\left(1+\frac{\delta}{np}\right) - \left(nq-\delta\right)\left(1-\frac{\delta}{nq}\right) \\ &\quad + \frac{1}{2}\ln\left(\frac{n}{2\pi\left(np+\delta\right)\left(nq-\delta\right)}\right) \end{aligned} \tag{D.5.2}$$

upon substituting from (D.4.4). From (D.4.1),

$$\frac{1}{2}\ln\left(\frac{n}{2\pi\left(np+\delta\right)\left(nq-\delta\right)}\right)\approx\frac{1}{2}\ln\left(\frac{1}{2\pi npq}\right)\tag{D.5.3}$$

From (D.4.1) and the approximation

$$\ln\left(1+x\right)\approx x-\frac{x^2}{2},\tag{D.5.4}$$

the first sum in (D.5.2) can be expressed as,

$$\begin{aligned}-\left(np+\delta\right)\ln\left(1+\frac{\delta}{np}\right)-\left(nq-\delta\right)\left(1-\frac{\delta}{nq}\right)\\ \approx-\left(np+\delta\right)\left(\frac{\delta}{np}-\frac{\delta^2}{2n^2p^2}\right)-\left(nq-\delta\right)\left(-\frac{\delta}{nq}-\frac{\delta^2}{2n^2q^2}\right)\\ =-\delta\left[1+\frac{\delta}{2np}-1+\frac{\delta}{2nq}\right]=-\frac{\delta^2}{2npq}\end{aligned}\tag{D.5.5}$$

Substituting from (D.5.5) and (D.5.5) in (D.5.2),

$$\ln\left[p_X(k)\right]\approx\frac{1}{2}\ln\left(\frac{1}{2\pi npq}\right)-\frac{\delta^2}{2npq}\tag{D.5.6}$$

$$\implies p_X(k)=\sqrt{\frac{1}{2\pi npq}}e^{-\frac{(k-np)^2}{2npq}}=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(k-\mu)^2}{2\sigma^2}}\tag{D.5.7}$$

D.6 A comparison of Binomial and Gaussian pmf/pdf is provided in Figs. D.6.1 and D.6.2.

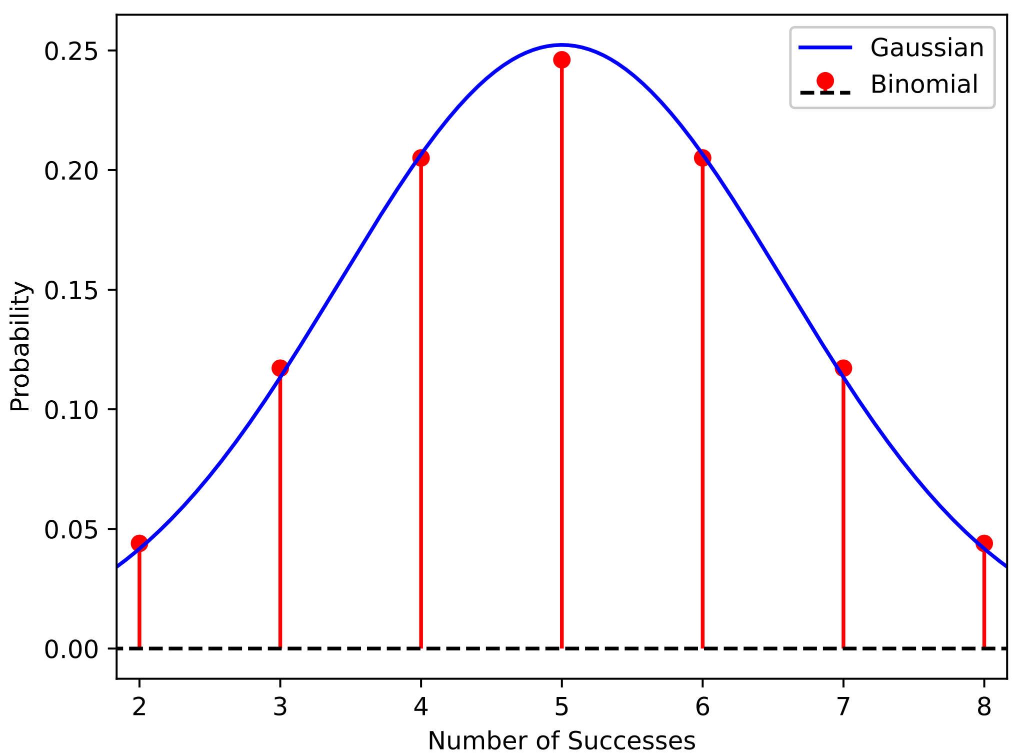


Figure D.6.1: 10 trials

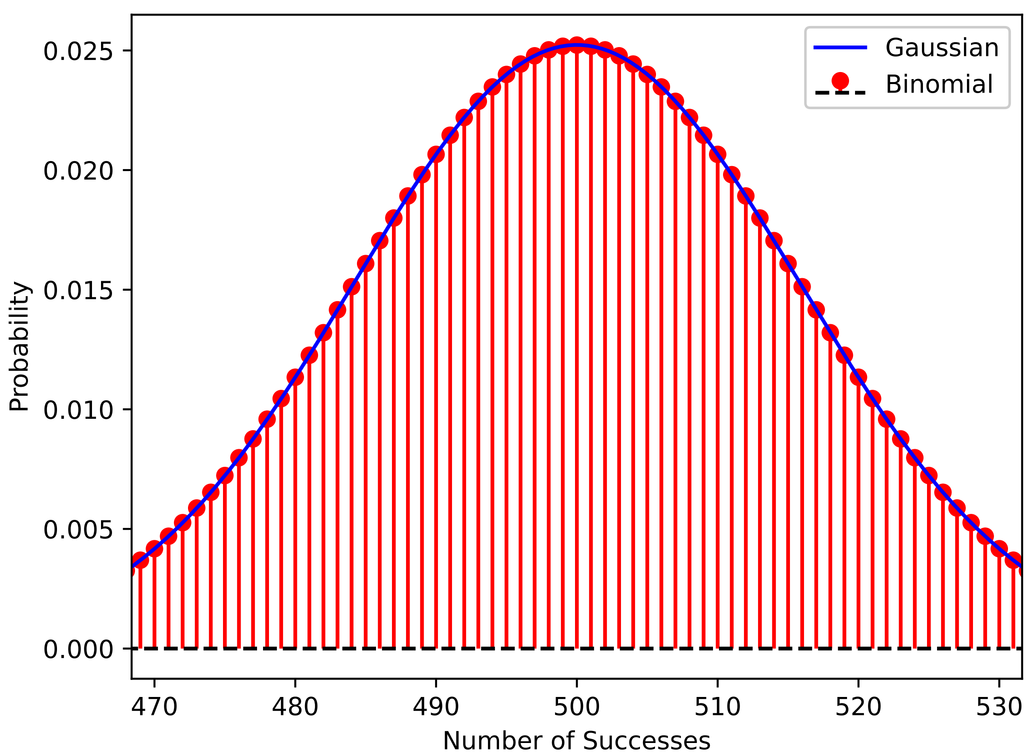


Figure D.6.2: 1000 trails

Appendix E

Identities

E.1 Let

$$N = R + B + G, n = r + b + g \tag{E.1.1}$$

where R, B, G and r, b, g represent the number of red, blue and green marbles respectively within N and n. Then

$$\Pr(r, b, g) = \frac{{}^{R}C_{r}\,{}^{B}C_{b}\,{}^{G}C_{g}}{{}^{R+B+G}C_{r+b+g}} \tag{E.1.2}$$

Solution: The number of ways of choosing n marbles from N is

$${}^{N}C_{n} \tag{E.1.3}$$

The number of ways of choosing r,b,g marbles is

$${}^{R}C_{r}\,{}^{B}C_{b}\,{}^{G}C_{g} \tag{E.1.4}$$

Using the definition of probability, we obtain (E.1.2).

E.2

$$ {}^{R+B}C_n = \sum_{k=0}^{R} \sum_{m=n-k}^{B} {}^{R}C_k \, {}^{B}C_m \tag{E.2.1} $$

Solution: Since

$$ (x+1)^R = \sum_{k=0}^{R} {}^{R}C_k \, x^k, \tag{E.2.2} $$

$$ (x+1)^R \, (x+1)^B = \sum_{k=0}^{R} \sum_{m=0}^{B} {}^{R}C_k \, {}^{B}C_m \, x^{k+m} \tag{E.2.3} $$

$$ \Longrightarrow \; (x+1)^{R+B} = \sum_{k=0}^{R} \sum_{m=n-k}^{B} {}^{R}C_k \, {}^{B}C_m \, x^n + \sum_{k=0}^{R} \sum_{m \neq n-k}^{B} {}^{R}C_k \, {}^{B}C_m \, x^{k+m} \tag{E.2.4} $$

(E.2.5)

yielding (E.2.1) upon comparing the coefficients of x^n on both sides.

Appendix F

Random Number Genration Using Shift Registers

F.0.1. Components

Component	**Value**	**Quantity**
Breadboard		1
Seven Segment Display	Common Anode	1
Decoder	7447	1
Flip Flop	7474	2
X-OR GATE	7486	1
555 IC		1
Resistor	$1K\Omega$	1
Resistor	$1M\Omega$	1
Capacitor	100nF	1
Capacitor	10nF	1
Jumper Wires		20

Table 2.1:

F.0.1.1. Generate the CLOCK signal using the 555 timer circuit as shown in the figure F.0.1.1

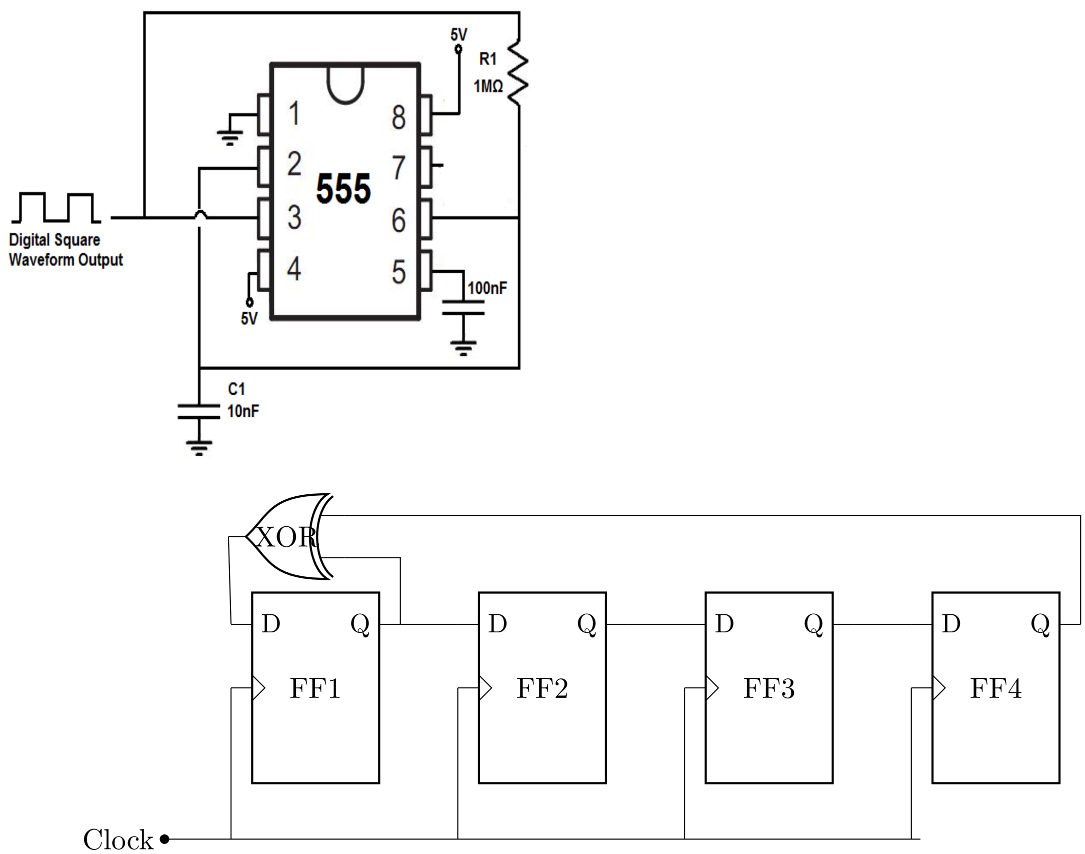


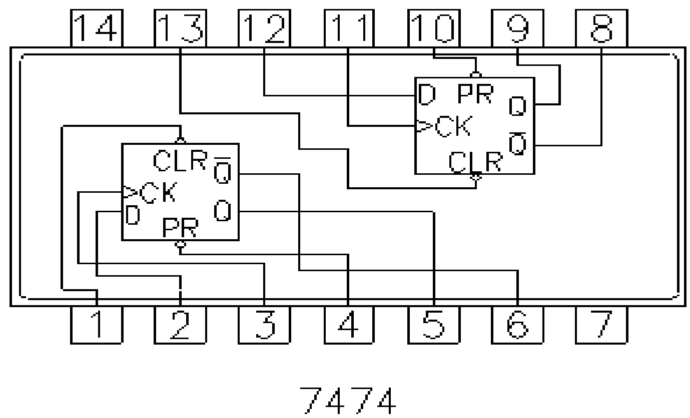
Figure F.0.1.3.1: Circuit Connections

F.0.1.2. Connect the CLOCK output of 555 timer circuit to CLOCK signal of D-Flip flops, change the resistor value to 1MΩ

F.0.1.3. Now make the cicuit for shift registers uisng 4 D-Flip flops (by using two 7474 IC's) and one X-OR gate (7486 IC) as shown in figure F.0.1.3.1. Pin out for 7474 IC is shown in figure F.0.1.4.1

F.0.1.4. Connect the output of each D-flip flop to Decoder IC (7447 IC), The pin out of 7447 IC is shown in figure F.0.1.4.1

F.0.1.5. As per the pinout of IC 7474 [2,12] pins of both IC's need to connected to the [7,1,2,6]



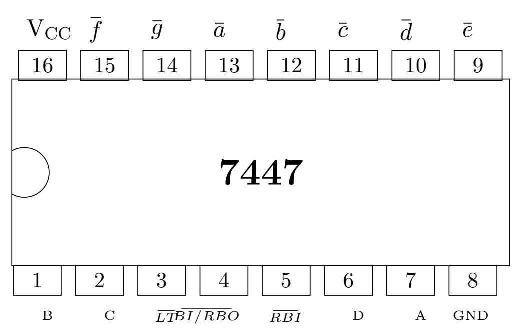


Figure F.0.1.4.1:

of decoder IC respectively

F.0.1.6. Make connections between the seven segment display in FigF.0.1.6.1 and the 7447 IC in Fig.F.0.1.6.1 as shown in Table 2.1

7447	$\bar{a}$	$\bar{b}$	$\bar{c}$	$\bar{d}$	$\bar{e}$	$\bar{f}$	$\bar{g}$
Display	a	b	c	d	e	f	g

Table F.0.1.6.1:

F.0.1.7. Additionally make conections like Vcc and GNG to every IC as per the respective IC

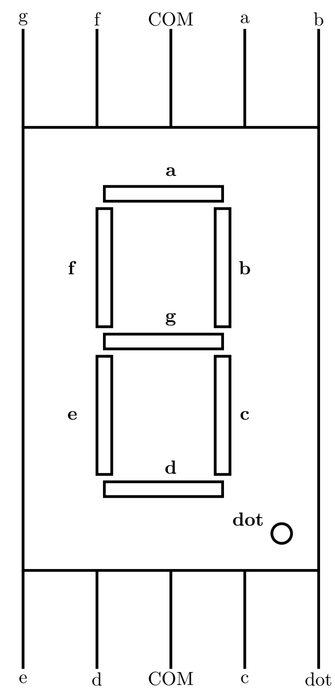


Figure F.0.1.6.1:

pinout for IC's 7474,7447,7486.