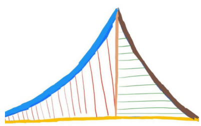
Random Variables through Simulation



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About this Book

This book introduces random variables through high school probability. All problems in the book are from NCERT mathematics textbooks from Class 9-12. A lot of college level concepts related to random variables are covered in the process. The content is sufficient for random variable simulations using Python/C. There is no copyright, so readers are free to print and share.

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CONTENTS

1 Definitions

1.1 NCERT

1.1.1 If a leap year is selected at random, what is the chance that it will contain 53 tuesdays?
Solution: The number of days in the leap year can be expressed as

$$366 = 52 \times 7 + 2 \tag{1.1.1.1}$$

The probability of one of the two remaining days being a Tuesday is $\frac{2}{7}$.

1.1.2 In a lottery, a person choses six different natural numbers at random from 1 to 20, and if these six numbers match with the six numbers already fixed by the lottery committee, he wins the prize. What is the probability of winning the prizes in the game? [Hint : order of the numbers is not important.]
Solution: The desired probability is given by

$$\frac{1}{{}^{20}C_6} = \frac{1}{38,760} = 0.0000258 \tag{1.1.2.1}$$

2 Boolean Logic

2.1 Formulae

2.1.1

$$A \cup B \triangleq A + B, A \cap B \triangleq AB. \tag{2.1.1.1}$$

2.1.2 Boolean Axioms: For $A \in \{0, 1\}$,

$$A + A' = 1 \tag{2.1.2.1}$$
$$AA' = 0 \tag{2.1.2.2}$$

2.1.3 De Morgan's Law

$$A'B' = (A + B)' \tag{2.1.3.1}$$

2.1.4 Axioms of Probability

a)

$$0 \le \Pr(A) \le 1 \tag{2.1.4.1}$$

b)

$$\Pr(1) = 1 \tag{2.1.4.2}$$

c) If $AB = 0$, i.e. A, B, are mutually exclusive,

$$\Pr(A + B) = \Pr(A) + \Pr(B). \tag{2.1.4.3}$$

2.1.5

$$\Pr(A + B) = \Pr(A) + \Pr(B) - \Pr(AB) \tag{2.1.5.1}$$

Proof.

$$A = A\left(B + B'\right) = AB + AB' \tag{2.1.5.2}$$

$$\implies \Pr\left(A\right) = \Pr\left(AB\right) + \Pr\left(AB'\right) \because \left(AB\right)\left(AB'\right) = 0, \tag{2.1.5.3}$$

from (2.1.4.3). Similarly,

$$A + B = A\left(B + B'\right) + B \tag{2.1.5.4}$$

$$= B\left(A + 1\right) + AB' \tag{2.1.5.5}$$

$$= B + AB' \tag{2.1.5.6}$$

$$\implies \Pr\left(A + B\right) = \Pr\left(B\right) + \Pr\left(AB'\right) \quad \because BAB' = 0 \tag{2.1.5.7}$$

From (2.1.5.3) and (2.1.5.7), we obtain (2.1.5.1). □

2.1.6 From (2.1.5.3) and (2.1.4.1),

$$\Pr\left(A\right) \geq \Pr\left(AB\right) \tag{2.1.6.1}$$

2.1.7 If A, B are independent,

$$\Pr\left(AB\right) = \Pr\left(A\right)\Pr\left(B\right) \tag{2.1.7.1}$$

2.1.8 Let $A + B = 1, AB = 0$. Then it is possible to define a real number X such that

$$X = 0 \implies A \text{ and } X = 1 \implies B \tag{2.1.8.1}$$

$$\text{or, } \Pr\left(A\right) = \Pr\left(X = 0\right), \Pr\left(B\right) = \Pr\left(X = 1\right) \tag{2.1.8.2}$$

$X \in \{0, 1\}$ is then defined to be a *random variable* with the *distribution*

$$p_X(n) = \begin{cases} \Pr\left(A\right) & X = 0, \\ \Pr\left(B\right) & X = 1. \end{cases} \tag{2.1.8.3}$$

Using (2.1.4.2),

$$\sum_n p_X(n) = 1. \tag{2.1.8.4}$$

2.2 NCERT

2.2.1 Which of the following cannot be the probability of an event ?

a) $\frac{2}{3}$ b) -1.5 c) 15% d) 0.7

Solution: We see that

$$\Pr\left(E\right) = -1.5 \tag{2.2.1.1}$$

violates (2.1.4.1). Hence, it cannot be a probability of any event.

2.2.2 If $P(E) = 0.05$, what is the probability of 'not E'?

Solution: From (2.1.4.2) and (2.1.4.3), the desired probability is

$$\Pr\left(E'\right) = 1 - \Pr\left(E\right) = 0.95 \tag{2.2.2.1}$$

2.2.3 Check whether the following probabilities $\Pr\left(A\right)$ and $\Pr\left(B\right)$ are consistently defined

a) $\Pr\left(A\right) = 0.5$, $\Pr\left(B\right) = 0.7$, $\Pr\left(A \cap B\right) = 0.6$

b) $\Pr(A) = 0.5$, $\Pr(B) = 0.7$, $\Pr(A \cup B) = 0.8$

Solution:

a)

$$\Pr(A) < \Pr(AB) = 0.6 \tag{2.2.3.1}$$

which violates (2.1.6.1). Inconsistent.

b) Given that

$$\Pr(A) = 0.5, \Pr(B) = 0.7, \Pr(A + B) = 0.8 \tag{2.2.3.2}$$

From (2.1.5.1), we get,

$$\Pr(AB) = 0.5 + 0.7 - 0.8 \tag{2.2.3.3}$$

$$= 0.4 \tag{2.2.3.4}$$

$\because$ no axioms are violated, the given probabilities are consistently defined

2.2.4 Given $\Pr(A) = \frac{3}{5}$ and $\Pr(B) = \frac{1}{5}$. Find $\Pr(A + B)$ if A and B are mutually exclusive events.

Solution: From (2.1.4.3),

$$\Pr(A + B) = \Pr(A) + \Pr(B) = \frac{4}{5} \tag{2.2.4.1}$$

2.2.5 If E and F are events such that $\Pr(E) = \frac{1}{4}$, $\Pr(F) = \frac{1}{2}$ and $\Pr(EF) = \frac{1}{8}$, find

a) $\Pr(E + F)$

b) $\Pr(E'F')$

Solution:

a)

$$\Pr(E + F) = \Pr(E) + \Pr(F) - \Pr(EF) = \frac{5}{8} \tag{2.2.5.1}$$

b) From (2.1.3.1),

$$(E'F') = (E + F)' \tag{2.2.5.2}$$

$$\implies \Pr(E'F') = \Pr((E + F)') \tag{2.2.5.3}$$

$$= 1 - \Pr(E + F) = \frac{3}{8} \tag{2.2.5.4}$$

upon substituting from (2.2.5.1).

2.2.6 Events E and F are such that P(not E or not F) = 0.25, state whether E and F are mutually exclusive.

Solution:

$$\Pr(E' + F') = \Pr((EF)') \tag{2.2.6.1}$$

$$= 1 - \Pr(EF) \tag{2.2.6.2}$$

$$\implies \Pr(EF) = 0.75 \tag{2.2.6.3}$$

$\because \Pr(EF) \neq 0, E$ and F are not mutually exclusive.

2.2.7 If A and B are two independent events with $\Pr(A) = \frac{3}{5}$ and $\Pr(B) = \frac{4}{9}$ then, $\Pr(A'B')$

a) $\frac{4}{15}$ b) $\frac{8}{45}$ c) $\frac{1}{3}$ d) $\frac{2}{9}$

Solution:

$$\Pr(A'B') = \Pr((A+B)') \tag{2.2.7.1}$$
$$= 1 - \Pr((A+B)) \tag{2.2.7.2}$$
$$= 1 - \Pr(A) - \Pr(B) + \Pr(A)\Pr(B) \tag{2.2.7.3}$$
$$= \frac{2}{9} \tag{2.2.7.4}$$

from (2.1.7.1) and (2.1.5.1).

2.2.8 A and B are events such that $\Pr(A) = 0.42, \Pr(B) = 0.48$ and $\Pr(A \text{ and } B) = 0.16$. Determine

a) Pr (not A)
b) Pr (not B)
c) Pr (A or B)

Solution: Solution:

a)

$$\Pr(A') = 1 - \Pr(A) = 0.58 \tag{2.2.8.1}$$

b)

$$\Pr(B') = 1 - \Pr(B) = 0.52 \tag{2.2.8.2}$$

c)

$$\Pr(\text{A+B}) = 0.42 + 0.48 - 0.16 = 0.74 \tag{2.2.8.3}$$

2.2.9 A and B are two events such that $\Pr(A) = 0.54$, $\Pr(B) = 0.69$ and $\Pr(AB) = 0.35$. Find

a) $\Pr(A+B)$ b) $\Pr(A'B')$ c) $\Pr(AB')$ d) $\Pr(BA')$

Solution:

a)

$$\Pr(A+B) = \Pr(A) + \Pr(B) - \Pr(AB) \tag{2.2.9.1}$$
$$= 0.88 \tag{2.2.9.2}$$

b) By De Morgan's Law,

$$A'B' = (A+B)' \tag{2.2.9.3}$$
$$\implies \Pr(A'B') = \Pr(A+B)' \tag{2.2.9.4}$$
$$= 1 - \Pr(A+B) \tag{2.2.9.5}$$
$$= 0.12 \tag{2.2.9.6}$$

c) From (2.1.5.3),

$$\Pr(A) = \Pr(AB) + \Pr(AB') \tag{2.2.9.7}$$
$$\implies \Pr(AB') = 0.19 \tag{2.2.9.8}$$

d) Similarly,

$$\Pr(BA') = \Pr(B) - \Pr(AB) = 0.34. \tag{2.2.9.9}$$

2.2.10 If $\Pr(A) = \frac{3}{5}$ and $\Pr(B) = \frac{1}{5}$ find $\Pr(A \cap B)$ if A and B are independent events.
Solution:

$$\Pr(AB) = \Pr(A)\Pr(B) = \frac{3}{25} \tag{2.2.10.1}$$

2.2.11 Let E and F be events with $\Pr(E) = \frac{3}{5}, \Pr(F) = \frac{3}{10}$ and $\Pr(EF) = \frac{1}{5}$. Are E and F independent?
Solution: From the given information,

$$\Pr(E)\Pr(F) = \frac{3}{5} \times \frac{9}{50}, \tag{2.2.11.1}$$
$$\Pr(EF) = \frac{1}{5} \tag{2.2.11.2}$$
$$\implies \Pr(EF) \neq P(E)P(F) \tag{2.2.11.3}$$

$\therefore$ E and F are not independent.

2.2.12 Given that the events A and B are such that $P(A) = \frac{1}{2}, P(A + B) = \frac{3}{5}$ and $P(B) = p$. Find p if they are

a) mutually exclusive
b) independent

Solution:

a)

$$\frac{3}{5} = \frac{1}{2} + p \tag{2.2.12.1}$$
$$\therefore p = \frac{1}{10} \tag{2.2.12.2}$$

b)

$$\frac{3}{5} = \frac{1}{2} + p - \frac{p}{2} \tag{2.2.12.3}$$
$$\therefore p = \frac{1}{5} \tag{2.2.12.4}$$

2.2.13 If A and B are two events such that $\Pr(A) = \frac{1}{4}, \Pr(B) = \frac{1}{2}$ and $\Pr(AB) = \frac{1}{8}$, find $\Pr$(not A and not B).

Solution:

$$\Pr(A+B) = \frac{1}{4} + \frac{1}{2} - \frac{1}{8} \tag{2.2.13.1}$$
$$= \frac{5}{8} \tag{2.2.13.2}$$

Hence,

$$\Pr(A'B') = 1 - \Pr((A+B)) = \frac{3}{8} \tag{2.2.13.3}$$

2.2.14 Events A and B are such that

$$\Pr(A) = \frac{1}{2}, \Pr(B) = \frac{7}{12}, \Pr(A' + B') = \frac{1}{4}. \tag{2.2.14.1}$$

State whether A and B are independent.

Solution:

$$\Pr(AB) = 1 - \Pr(A' + B') = \frac{3}{4}, \tag{2.2.14.2}$$
$$\Pr(A) \times \Pr(B) = \frac{7}{24} \tag{2.2.14.3}$$
$$\implies \Pr(AB) \neq \Pr(A)\Pr(B) \tag{2.2.14.4}$$

∴ A and B are not independent.

2.2.15 Two events A and B will be independent, if

a) A and B are mutually exclusive
b) P(not $A \cap$ not B) = $[1 - P(A)]\ [1 - P(B)]$
c) $P(A) = P(B)$
d) $P(A) + P(B) = 1$

Solution:

a) Let

$$\Pr(A) = \Pr(B) = \frac{1}{2} \implies \Pr(A) \times \Pr(B) = \frac{1}{4} \tag{2.2.15.1}$$
$$\text{or, } \Pr(AB) = 0 \neq \Pr(A) \times \Pr(B) \tag{2.2.15.2}$$

Hence A and B are not independent.

b)

$$\Pr(A'B') = [1 - \Pr(A)][1 - \Pr(B)] \tag{2.2.15.3}$$
$$\implies 1 - \Pr(A+B) = 1 - \Pr(A) - \Pr(B) + \Pr(A)\Pr(B) \tag{2.2.15.4}$$
$$\implies \Pr(AB) = \Pr(A)\Pr(B) \tag{2.2.15.5}$$

Thus, A and B are independent.

c) In 2.2.15a, $\Pr(A) = \Pr(B)$, but A and B are not independent.
d) In 2.2.15a, $\Pr(A) + \Pr(B)$=1, but A and B are not independent.

2.2.16 The probability that at least one of the two events A and B occurs is 0.6. If A and B occur simultaneously with probability 0.3, evaluate $\Pr(A') + \Pr(B')$.

Solution: Given:

$$\Pr(AB) = 0.3 \tag{2.2.16.1}$$
$$\Pr(A+B) = 0.6 \tag{2.2.16.2}$$
$$= \Pr(A) + \Pr(B) - \Pr(AB) \tag{2.2.16.3}$$
$$\implies 0.6 = \Pr(A) + \Pr(B) - 0.3 \tag{2.2.16.4}$$
$$\implies 0.9 = \Pr(A) + \Pr(B) \tag{2.2.16.5}$$

But

$$\Pr(A') = 1 - \Pr(A) \tag{2.2.16.6}$$
$$\Pr(B') = 1 - \Pr(B) \tag{2.2.16.7}$$
$$\therefore \Pr(A') + \Pr(B') = 2 - (\Pr(A) + \Pr(B)) \tag{2.2.16.8}$$
$$= 2 - 0.9 = 1.1 \tag{2.2.16.9}$$

2.2.17 Prove that

a) $\Pr(A) = \Pr(AB) + \Pr(AB')$
b) $\Pr(A+B) = \Pr(AB) + \Pr(AB') + \Pr(A'B)$

Solution:

a) See (2.1.5.3).
b) From (2.1.5.3) and (2.1.5.1),

$$\Pr(A) = \Pr(AB) + \Pr(AB') \tag{2.2.17.1}$$
$$\Pr(B) = \Pr(AB) + \Pr(A'B) \tag{2.2.17.2}$$
$$\Pr(A+B) = \Pr(A) + \Pr(B) - \Pr(AB) \tag{2.2.17.3}$$

yielding item 2.2.17b after addition.

2.2.18 A and B are events such that $\Pr(A) = 0.4$, $\Pr(B) = 0.3$ and $\Pr(A+B) = 0.5$. Find $\Pr(B'A)$.

Solution: Adding (2.2.17.1) and (2.2.17.3),

$$\Pr(A+B) = \Pr(B) + \Pr(AB') \tag{2.2.18.1}$$
$$\implies \Pr(AB') = \Pr(A+B) - \Pr(B) = 0.2 \tag{2.2.18.2}$$

State True or False.

2.2.19 If $\Pr(A) > 0$ and $\Pr(B) > 0$, then A and B can be mutually exclusive and independent.

Solution: Since $\Pr(A) > 0$ and $\Pr(B) > 0$,

$$\Pr(A)\Pr(B) > 0 \tag{2.2.19.1}$$

For $\Pr(A)$ and $\Pr(B)$ to be mutually exclusive and independent,

$$\Pr(AB) = 0 \tag{2.2.19.2}$$
$$\Pr(AB) = \Pr(A)\Pr(B) \tag{2.2.19.3}$$
$$\implies \Pr(A)\Pr(B) = 0 \tag{2.2.19.4}$$

which contradicts (2.2.19.1). Hence, the above statement is false.

2.2.20 If A and B are independent events, then A' and B' are also independent.

Solution: Given that

$$\Pr(AB) = \Pr(A)\Pr(B) \tag{2.2.20.1}$$

If A' and B' are independent,

$$\Pr(A'B') = \Pr(A+B)' = 1 - \Pr(A+B) \tag{2.2.20.2}$$
$$= 1 - \Pr(A) - \Pr(B) + \Pr(AB) \tag{2.2.20.3}$$
$$= 1 - \Pr(A) - \Pr(B) + \Pr(A)\Pr(B) \tag{2.2.20.4}$$
$$= [1 - \Pr(A)]\,[1 - \Pr(B)] \tag{2.2.20.5}$$
$$= \Pr(A')\Pr(B') \tag{2.2.20.6}$$

Hence, A' and B' are also independent. Therefore, the given statement is true.

2.2.21 If A and B are mutually exclusive events, $\Pr(A) = 0.35$ and $\Pr(B) = 0.45$ then find

a) $\Pr(A')$
b) $\Pr(B')$
c) $\Pr(A+B)$
d) $\Pr(AB)$
e) $\Pr(AB')$
f) $\Pr(A'B')$

Solution: See Table 2.2.21.1.

TABLE 2.2.21.1

Item	**Formula**	**Value**
$\Pr(A')$	$1 - \Pr(A)$	0.65
$\Pr(B')$	$1 - \Pr(B)$	0.55
$\Pr(A+B)$	$\Pr(A) + \Pr(B) - \Pr(AB)$	0.80
$\Pr(AB)$	$\because AB = 0$	0
$\Pr(AB')$	$\Pr(A) - \Pr(AB)$	0.35
$\Pr(A'B')$	$1 - \Pr(A+B)$	0.20

2.2.22 The accompanying Venn diagram shows three events, A, B, and C, and also the probabilities of the various intersections (for instance, $\Pr(AB) = 0.7$. Determine

a) $\Pr(A)$
b) $\Pr(BC')$
c) $\Pr(A+B)$
d) $\Pr(AB')$
e) $\Pr(BC)$
f) Probability that exactly one of the three occurs

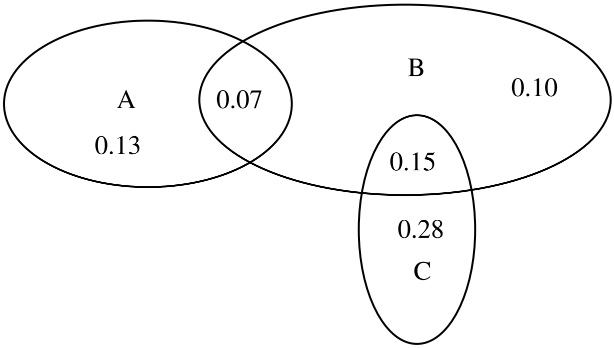


Fig. 2.2.22.1

Input		Output		
Probability	**Value**	**Probability**	**Formula**	**Value**
$\Pr(AB)$	0.07	$\Pr(A)$	$A = AB + AB'$	0.2
$\Pr(AB')$	0.13	$\Pr(BC')$	$BC' = AB + A'BC'$	0.17
$\Pr(BC)$	0.15	$\Pr(A + B)$	$\Pr(A) + \Pr(B) - \Pr(AB)$	0.45
$\Pr(BA'C')$	0.10	$\Pr(AB')$	Given	0.13
$\Pr(CB')$	0.28	$\Pr(BC)$	Given	0.15
$\Pr(AC)$	0	$\Pr(A'BC' + AB'C' + A'B'C)$	$\Pr(AB') + \Pr(CB') + \Pr(BA'C')$	0.51

TABLE 2.2.22.1

Solution: See Table 2.2.22.1. Fig. 2.2.22.1 is used to obtain the input probabilities.

a)

$$BC' = BC'(A + A') = BC'A + BC'A' \tag{2.2.22.1}$$

Also,

$$AB = AB(C + C') \tag{2.2.22.2}$$

$$= ABC + ABC' = ABC' \because AC = 0. \tag{2.2.22.3}$$

From (2.2.22.1) and (2.2.22.3),

$$BC' = AB + A'BC' \tag{2.2.22.4}$$

$$\implies \Pr(BC') = \Pr(AB) + \Pr(A'BC') \tag{2.2.22.5}$$

b) Also,

$$\Pr(B) = \Pr(BC) + \Pr(BC') = 0.17 + 0.15 = 0.32 \tag{2.2.22.6}$$

from Table 2.2.22.1. This is used to evaluate $\Pr(A + B)$.

c) $\because AC = 0$

$$\Pr(AB) = \Pr(AB'C') + \Pr(AB'C) = \Pr(AB'C') \tag{2.2.22.7}$$

$$\Pr(B'C) = \Pr(A'B'C) \tag{2.2.22.8}$$

2.2.23 The probability of happening of an event A is 0.5 and that of B is 0.3. If A and B are mutually exclusive events, then the probability of neither A nor B is _____.

Solution:

$$\because \Pr(AB) = 0, \tag{2.2.23.1}$$
$$\Pr((A+B)') = 1 - Pr(A+B) = 1 - Pr(A) - Pr(B) \tag{2.2.23.2}$$
$$= 0.2 \tag{2.2.23.3}$$

which is the desired probability.

2.2.24 If A and B are mutually exclusive events,then

a) $\Pr(A) \le \Pr(B')$
b) $\Pr(A) \ge \Pr(B')$
c) $\Pr(A) < \Pr(B')$
d) none of these

Solution:

$$\because \Pr(AB) = 0 \tag{2.2.24.1}$$
$$\Pr(A+B) \le 1 \implies \Pr(A) + \Pr(B) \le 1 \tag{2.2.24.2}$$
$$\implies \Pr(A) \le \Pr(B'). \tag{2.2.24.3}$$

where we have used the axiom of probability.

2.2.25 The probability of an occurrence of event A is .7 and that of the occurrence of event B is .3 and the probability of occurrence of both is .4. Is this statement true or false?

Solution:

$$\Pr(AB) > \Pr(B) \tag{2.2.25.1}$$

which violates (2.1.6.1). Hence, the given statement is false.

2.2.26 If $\Pr(A+B) = \Pr(AB)$ for any two events A and B , then

a) $\Pr(A) = \Pr(B)$ b) $\Pr(A) > \Pr(B)$ c) $\Pr(A) < \Pr(B)$ d) none of these

Solution:

$$\Pr(A) + \Pr(B) - \Pr(AB) = \Pr(A+B) \tag{2.2.26.1}$$
$$\implies \Pr(A) + \Pr(B) - \Pr(AB) = \Pr(AB) \tag{2.2.26.2}$$
$$\implies [\Pr(A) - \Pr(AB)] + [\Pr(B) - \Pr(AB)] = 0 \tag{2.2.26.3}$$

However, from (2.1.6.1),

$$\begin{aligned} \Pr(A) - \Pr(AB) \ge 0 \\ \Pr(B) - \Pr(AB) \ge 0 \end{aligned} \tag{2.2.26.4}$$

From (2.2.26.3) and (2.2.26.4),

$$\Pr(A) = \Pr(B) = \Pr(AB). \tag{2.2.26.5}$$

2.2.27 If A and B are such that $\Pr(A' \cup B') = \frac{2}{3}$ and $\Pr(A \cup B) = \frac{5}{9}$, then $\Pr(A') + \Pr(B') =$
Solution: Using De Morgan's law and axioms of probability,

$$\Pr\left((A+B)'\right) = \Pr\left(A'B'\right) \tag{2.2.27.1}$$

$$\Pr\left(A'+B'\right) = \Pr\left(A'\right) + \Pr\left(B'\right) - \Pr\left(A'B'\right) \tag{2.2.27.2}$$

Adding the above,

$$\Pr\left(A'\right) + \Pr\left(B'\right) = 1 + \Pr\left(A'+B'\right) - \Pr\left(A+B\right) = \frac{10}{9} \tag{2.2.27.3}$$

2.2.28 If A and B are independent, then $\Pr$ (exactly one of A, B occurs) $= \Pr(B)\Pr(A') + \Pr(A)\Pr(B')$.
Solution:

$$\because \Pr(AB) = \Pr(A)\Pr(B) \tag{2.2.28.1}$$

$$\Pr\left(AB'\right) = \Pr(A)\Pr\left(B'\right), \Pr\left(A'B\right) = \Pr\left(A'\right)\Pr(B) \tag{2.2.28.2}$$

$$\implies \Pr\left(A'B + AB'\right) = \Pr\left(A'B\right) + \Pr\left(A'B\right) \tag{2.2.28.3}$$

$$= \Pr(A)\Pr\left(B'\right) + \Pr\left(A'\right)\Pr(B). \tag{2.2.28.4}$$

2.2.29 Given two independent events A and B such that $P(A) = 0.3$, $P(B) = 0.6$. Find

a) $P(A \text{ and } B)$
b) $P(A \text{ and not } B)$
c) $P(A \text{ or } B)$
d) $P(\text{neither } A \text{ nor } B)$

2.2.30 The probability distribution of a discrete random variable X is given below in Table 2.2.30.1. The value of k is equal to

a) 8 b) 16 c) 32 d) 48

X	2	3	4	5
$p_X(n)$	$\frac{5}{k}$	$\frac{7}{k}$	$\frac{9}{k}$	$\frac{11}{k}$

TABLE 2.2.30.1

Solution: From (2.1.8.4),

$$\frac{5}{k} + \frac{7}{k} + \frac{9}{k} + \frac{11}{k} = 1 \tag{2.2.30.1}$$

$$\implies k = 32 \tag{2.2.30.2}$$

2.2.31 State True or False for the given statement: Two independent events are always mutually exclusive.
Solution: The given condition can be expressed as

$$\Pr(AB) = \Pr(A) \times \Pr(B) = 0 \tag{2.2.31.1}$$

$$\implies \Pr(A) = 0 \text{ or, } \Pr(B) = 0, \tag{2.2.31.2}$$

which is not always true.

2.2.32 If A and B' are independent events, then $\Pr(A'+B) = 1 - \Pr(A)\Pr(B')$.
Solution:

$$\Pr(A'+B) = \Pr((AB')') \tag{2.2.32.1}$$
$$= 1 - \Pr(AB') \tag{2.2.32.2}$$
$$= 1 - \Pr(A)\Pr(B'). \tag{2.2.32.3}$$

2.2.33 Let E_1 and E_2 be two independent events such that $\Pr(E_1) = p_1$ and $\Pr(E_2) = p_2$ Describe in words the events whose probabilities are

a) p_1p_2
b) $(1-p_1)p_2$
c) $1-(1-p_1)(1-p_2)$
d) $p_1+p_2-2p_1p_2$

Solution:

a)

$$p_1p_2 = \Pr(E_1)\Pr(E_2) \tag{2.2.33.1}$$
$$= \Pr(E_1E_2) \tag{2.2.33.2}$$

So, E_1 and E_2 occur simultaneously.

b)

$$(1-p_1)(p_2) = \Pr(E_1')\Pr(E_2) \tag{2.2.33.3}$$
$$= \Pr(E_1'E_2) \tag{2.2.33.4}$$

So E_1 does not occur but E_2 occurs.

c)

$$1-(1-p_1)(1-p_2) = 1 - \Pr(E_1')\Pr(E_2') \tag{2.2.33.5}$$
$$= 1 - \Pr(E_1'\ E_2') \tag{2.2.33.6}$$
$$= \Pr(E_1+E_2) \tag{2.2.33.7}$$

So, either E_1 or E_2 or both E_1 and E_2 occurs.

d)

$$p_1+p_2-2p_1p_2 = \Pr(E_1)+\Pr(E_2)-2\Pr(E_1)\Pr(E_2) \tag{2.2.33.8}$$
$$= \Pr(E_1)-\Pr(E_1)\Pr(E_2)+\Pr(E_2)-\Pr(E_1)\Pr(E_2) \tag{2.2.33.9}$$
$$= \Pr(E_1)(1-\Pr(E_2))+\Pr(E_2)(1-\Pr(E_1)) \tag{2.2.33.10}$$
$$= \Pr(E_1)\Pr(E_2')+\Pr(E_2)\Pr(E_1') \tag{2.2.33.11}$$
$$= \Pr(E_1E_2'+E_1'E_2) \tag{2.2.33.12}$$

So, either E_1 or E_2 occurs but not both.

2.2.34 Match the following in Table 2.2.34.1.

I	II
(a) if E_1 and E_2 are two mutually exclusive events	(i) $E_1 \cap E_2 = E_1$
(b) if E_1 and E_2 are mutually exclusive and exhaustive events	(ii) $(E_1 - E_2) \cup (E_1 \cap E_2) = E_1$
(c) if E_1 and E_2 have common outcomes, then	(iii) $E_1 \cap E_2 = \phi, E_1 \cup E_2 = S$
(d) if E_1 and E_2 are two events such that $E_1 \subset E_2$	(iv) $E_1 \cap E_2 = \phi$

TABLE 2.2.34.1

Solution:

a) If E_1 and E_2 are mutually exclusive events, then $E_1E_2 = \phi$.

b) If E_1 and E_2 are mutually exclusive and exhaustive events, then $E_1E_2 = \phi$ and $E_1 + E_2 = S$

c) If E_1 and E_2 have common outcomes, this means:

$$E_1E_2 \neq 0 \tag{2.2.34.1}$$

Let E_a be the outcomes that are present in E_1 and not in E_2. So,

$$E_a = E_1 - E_2 \tag{2.2.34.2}$$

Let E_b be the outcomes common between E_1 and E_2. So,

$$E_b = E_1E_2 \tag{2.2.34.3}$$

So, we can say that

$$E_1 = E_a + E_b \tag{2.2.34.4}$$

Refering to equation (2.2.34.2) and (2.2.34.3):

$$E_1 = (E_1 - E_2) + (E_1E_2) \tag{2.2.34.5}$$

d) If E_1 and E_2 are two events such that $E_1 \subset E_2$, then let E be subset of E_2 containing elements other than E_1. So,

$$E_1 + E = E_2 \text{ and } E_1E = E_2 \tag{2.2.34.6}$$

Refering to equation (2.2.34.6):

$$E_1E_2 = E_1(E_1 + E) \tag{2.2.34.7}$$
$$= (E_1E_1) + (E_1E) \tag{2.2.34.8}$$
$$= E_1 \tag{2.2.34.9}$$

Hence,

a) $\leftrightarrow (iv)$, b) $\leftrightarrow (iii)$, c) $\leftrightarrow (ii)$, d) $\leftrightarrow (i)$

2.2.35 If A and B are two candidates seeking admission in an engineering College. The probability that A is selected is 0.5 and the probability that both A and B are selected

is atmost 0.3. Is it possible that the probability of B getting selected is 0.7?
Solution:

$$\because \Pr(AB) \le 0.3 \tag{2.2.35.1}$$
$$\text{Let } \Pr(AB) = 0.1. \tag{2.2.35.2}$$

From (2.1.5.1)

$$\Pr(A+B) = 0.5 + 0.7 - 0.1 = 1.1 > 1, \tag{2.2.35.3}$$

which violates (2.1.4.1). Hence, it is not possible.

2.2.36 State whether the statement is True or False.
The probability that a person visiting a zoo will see the giraffe is 0.72, the probability that he will see the bears is 0.84 and the probability that he will see both is 0.52.
Solution: Let

$$\Pr(A) = 0.72, \ \Pr(B) = 0.84, \ \Pr(AB) = 0.52. \tag{2.2.36.1}$$

Using (2.1.5.1),

$$\Pr(A+B) = 0.72 + 0.84 - 0.52 = 1.04 \tag{2.2.36.2}$$

which violates (2.1.4.1). Hence, false.

2.2.37 Suppose an integer from 1 through 1000 is chosen at random, find the probability that the integer is a multiple of 2 or a multiple of 9.
Solution: See Table 2.2.37.1. From (2.1.5.1),

Event	**Description**	**Probability**
A	$n \mod 2 = 0$	$\Pr(A) = \frac{500}{1000}$
B	$n \mod 9 = 0$	$\Pr(B) = \frac{111}{1000}$
AB	$n \mod 18 = 0$	$\Pr(AB) = \frac{55}{1000}$

TABLE 2.2.37.1

$$\Pr(A+B) = \frac{500}{1000} + \frac{111}{1000} - \frac{55}{1000} = \frac{556}{1000} \tag{2.2.37.1}$$

2.2.38 If the probabilities for A to fail in an examination is 0.2 and that for B is 0.3, then the probability that either A or B fails is

a) > 0.5 b) 0.5 c) ≤ 0.5 d) 0

Solution:

$$\because \Pr(A) = 0.2, \Pr(B) = 0.3 \tag{2.2.38.1}$$
$$\Pr(AB) = 0.5 - \Pr(A+B) \tag{2.2.38.2}$$

from (2.1.5.1). Thus, from (2.1.4.1),

$$0.5 - \Pr(A+B) \ge 0 \tag{2.2.38.3}$$
$$\implies \Pr(A+B) \le 0.5 \tag{2.2.38.4}$$

2.2.39 It is given that in a group of 3 students, the probability of 2 students not having the same birthday is 0.992. What is the probability that the 2 students have the same birthday?

Solution: Let

$$\Pr(E) = 0.992. \tag{2.2.39.1}$$

Then,

$$\Pr(E') = 1 - \Pr(E) = 0.008 \tag{2.2.39.2}$$

2.2.40 In class XI of a school, 40% of the students study Mathematics and 30% study Biology. 10% of the class study both Mathematics and Biology. If a student is selected at random from the class,find the probability that he will be studying Mathematics or Biology

Solution:

$$\because \Pr(M) = 0.4, \Pr(B) = 0.3, \Pr(MB) = 0.1, \tag{2.2.40.1}$$

$$\Pr(M + B) = \Pr(M) + \Pr(B) - \Pr(MB) = 0.6 \tag{2.2.40.2}$$

using (2.1.5.1).

2.2.41 In an entrance test that is graded on the basis of two examinations, the probability of a randomly chosen student passing the first examination is 0.8 and the probability of passing the second examination is 0.7. The probability of passing atleast one of them is 0.95. What is the probability of passing both?

Solution:

$$\because \Pr(A) = 0.8,\ \Pr(B) = 0.7,\ \Pr(A + B) = 0.95, \tag{2.2.41.1}$$

$$\Pr(AB) = 0.55 \tag{2.2.41.2}$$

using (2.1.5.1).

2.2.42 In a class of 60 students, 30 opted for NCC, 32 opted for NSS and 24 opted for both NCC and NSS. If one of these students is selected at random, find the probability that

a) The student opted for NCC or NSS.
b) The student has opted neither NCC nor NSS.
c) The student has opted NSS but not NCC.

Solution: From the given data,

$$\Pr(A) = \frac{30}{60} = \frac{1}{2},\ \Pr(B) = \frac{32}{60} = \frac{8}{15},\ \Pr(AB) = \frac{24}{60} = \frac{2}{5}. \tag{2.2.42.1}$$

Thus, the desired probabilities are

a) $\Pr(A + B) = \frac{19}{30}$, from (2.1.5.1).
b) From (2.1.3.1) and the axioms of probability,

$$\Pr(A'B') = 1 - \Pr(A + B) = \frac{11}{30}. \tag{2.2.42.2}$$

c)

$$\Pr(A'B) = \Pr(B) - \Pr(AB) = \frac{2}{15} \tag{2.2.42.3}$$

from (4.1.2.1).

2.2.43 The probability of solving a specific problem independently by A and B are $\frac{1}{2}$ and $\frac{1}{3}$ respectively. If both try to solve the problem independently, find the probability that

a) the problem is solved
b) exactly one of them solves the problem

Solution:

$$\because \Pr(A) = \frac{1}{2}, \ \Pr(B) = \frac{1}{3}, \tag{2.2.43.1}$$

$$\Pr(AB) = \Pr(A)\Pr(B) = \frac{1}{6} \tag{2.2.43.2}$$

$\because A, B$ are independent.

a) From (2.1.5.1),

$$\Pr(A+B) = \frac{1}{2} + \frac{1}{3} - \frac{1}{6} = \frac{2}{3} \tag{2.2.43.3}$$

b)

$$\Pr(AB' + A'B) = \Pr(AB') + \Pr(A'B) \tag{2.2.43.4}$$

$$= \Pr(A)\Pr(B') + \Pr(A')\Pr(B) \tag{2.2.43.5}$$

$$= \Pr(A) + \Pr(B) - 2\Pr(A)\Pr(B) = \frac{1}{2} \tag{2.2.43.6}$$

2.2.44 One card is drawn at random from a well shuffled deck of 52 cards. In which of the following cases are the events E and F independent ?

a) E: 'the card drawn is spade'
   F: 'the card drawn is an ace'
b) E: 'the card drawn is black'
   F : 'the card drawn is a king'
c) E : 'the card drawn is a king or queen'
   F : 'the card drawn is a queen or jack'

Solution: See Table 2.2.44.1.

Item	$\Pr(E)$	$\Pr(F)$	$\Pr(EF)$	Independent
a)	$\frac{13}{52} = \frac{1}{4}$	$\frac{4}{52} = \frac{1}{13}$	$\frac{1}{52} = \Pr(E)\Pr(F)$	Yes
b)	$\frac{26}{52} = \frac{1}{2}$	$\frac{4}{52} = \frac{1}{13}$	$\frac{2}{52} = \frac{1}{26} = \Pr(E)\Pr(F)$	Yes
c)	$\frac{8}{52} = \frac{2}{13}$	$\frac{8}{52} = \frac{2}{13}$	$\frac{4}{52} = \frac{1}{13} \neq \Pr(E)\Pr(F)$	No

TABLE 2.2.44.1

2.2.45 A team of medical students doing their internship have to assist during surgeries at a city hospital. The probabilities of surgeries rated as very-complex, complex, routine, simple or very-simple are respectively, 0.15, 0.20, 0.31, 0.26, .08. Find the probability that a particular surgery will be rated

a) complex or very-complex
b) neither very-complex nor very simple
c) routine or complex
d) routine or simple

Solution: The given information is summarised in Table 2.2.45.2

a)

$$\Pr(E_1 + E_2) = \Pr(E_1) + \Pr(E_2) \qquad \because E_1E_2 = 0 \tag{2.2.45.1}$$
$$= 0.15 + 0.20 = 0.35 \tag{2.2.45.2}$$

b)

$$\Pr\left(E_1'E_5'\right) = \Pr\left((E_1 + E_5)'\right) \tag{2.2.45.3}$$
$$= 1 - \Pr(E_1 + E_5) \tag{2.2.45.4}$$
$$= 1 - [\Pr(E_1) + \Pr(E_5)] \qquad \because E_1E_5 = 0 \tag{2.2.45.5}$$
$$= 1 - [0.15 + 0.08] = 0.77 \tag{2.2.45.6}$$
$$\tag{2.2.45.7}$$

c)

$$\Pr(E_3 + E_2) = \Pr(E_3) + \Pr(E_2) \qquad \because E_3E_2 = 0 \tag{2.2.45.8}$$
$$= 0.31 + 0.20 = 0.51 \tag{2.2.45.9}$$

d)

$$\Pr(E_3 + E_4) = \Pr(E_3) + \Pr(E_4) \qquad \because E_3E_4 = 0 \tag{2.2.45.10}$$
$$= 0.31 + 0.26 = 0.57 \tag{2.2.45.11}$$

Variable	**Difficulty Levels**	**Probability**
E_1	Very-Complex	$\Pr(E_1)$=0.15
E_2	Complex	$\Pr(E_2)$=0.2
E_3	Routine	$\Pr(E_3)$=0.31
E_4	Simple	$\Pr(E_4)$=0.26
E_5	Very-Simple	$\Pr(E_5)$=0.08

TABLE 2.2.45.2

2.2.46 Without repetition of the numbers, four digit numbers are formed with the numbers 0,2,3,5. The probability of such a number divisible by 5 is

a) $\frac{1}{5}$ b) $\frac{4}{5}$ c) $\frac{1}{30}$ d) $\frac{5}{9}$

Solution: Let X denote the digit in the units place.

a) Number of four digit numbers possible are $3 \times 3 \times 2 \times 1 = 18$ because zero cannot be in the first place.
b) $n(X = 5) = 2 \times 2 \times 1 = 4$.
c) $n(X = 0) = 3 \times 2 \times 1 = 6$.

$$\therefore \Pr(X = 5) + \Pr(X = 0) = \frac{6+4}{18} = \frac{5}{9} \tag{2.2.46.1}$$

which is the desired probability.

2.2.47 Box A contains 25 slips of which 19 are marked Rs 1 and others are marked Rs 5 each. Box B contains 50 slips of which 45 are marked Rs 1 and others are marked Rs 13 each. Slips of both boxes are poured into a third box and reshuffled. A slip is drawn at random. What is the probability that it is marked other than Rs 1?

3 Bernoulli

3.1 Formulae

3.1.1 The Bernoulli distribution $X \in \{0, 1\}$ is defined as

$$X = \mathrm{Ber}(p). \tag{3.1.1.1}$$

with pmf

$$p_X(k) = \begin{cases} 1 - p & k = 0 \\ p & k = 1 \\ 0 & \text{otherwise} \end{cases} \tag{3.1.1.2}$$

3.1.2 The following code simulates 100 coin tosses

```
#Code by GVV Sharma
#November 18, 2020
#Released under GNU/GPL
#Given a Bernoulli probability and
#number of samples, the code generates the event data

import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import bernoulli

#100 samples
simlen=int(1e2)

#Probability of the event
prob = 0.5

#Generating sample date using Bernoulli r.v.
data_bern = bernoulli.rvs(size=simlen,p=prob)
#Calculating the number of favourable outcomes
err_ind = np.nonzero(data_bern == 1)
#calculating the probability
err_n = np.size(err_ind)/simlen

#Theory vs simulation
print(err_n,prob)
print(data_bern)
```

3.2 NCERT

3.2.1 A lot consists of 144 ball pens of which 20 are defective and the others are good. Navami will buy a pen if it is good, but will not buy if it is defective. The shopkeeper draws one pen at random and gives it to her. What is the probability that

a) She will buy it?
b) She will not buy it?

Solution: In this case, we have $X \sim \text{Ber}\left(\frac{67}{72}\right)$. Therefore, the desired probabilities are

a) $p_X(1) = \frac{67}{72}$
b) $p_X(0) = \frac{5}{72}$

3.2.2 A school has five houses A, B, C, D and E. A class has 23 students, 4 from house A, 8 from house B, 5 from house C, 2 from house D and rest from house E. A single student is selected at random to be the class monitor. The probability that the selected student is not from A, B and C is

Solution: See Table 3.2.2.1

House	A	B	C	D	E
Students	4	8	5	2	4

TABLE 3.2.2.1: Student distribution in each house

Define

$$X = \begin{cases} 0 & \text{A, B and C} \\ 1 & \text{Not from A, B and C} \end{cases} \tag{3.2.2.1}$$

Then, from Table 3.2.9.1,

$$X \sim \text{Ber}\left(\frac{6}{23}\right) \tag{3.2.2.2}$$

and the desired probability is

$$p_X(1) = \frac{6}{23} \tag{3.2.2.3}$$

3.2.3 A bag contains slips numbered from 1 to 100. If Phulan chooses a slip at random from the bag, it will either be an odd number or an even number. Since this situation has only two possible outcomes, so, the probability of each is $\frac{1}{2}$. Justify.

Solution: Let

$$X = \begin{cases} 1, & \text{if number is even} \\ 0, & \text{if number is odd} \end{cases} \tag{3.2.3.1}$$

Then

$$p_X(1) = \frac{50}{100} = \frac{1}{2} \tag{3.2.3.2}$$

$$p_X(0) = \frac{50}{100} = \frac{1}{2} \tag{3.2.3.3}$$

and $X \sim \text{Ber}\left(\frac{1}{2}\right)$.

3.2.4 A letter of English alphabets is chosen at random. Determine the probability that the letter is a consonant.

Solution: The desired probability is

$$p = \frac{21}{26} \tag{3.2.4.1}$$

3.2.5 A carton of 24 bulbs contain 6 defective bulbs. One bulb is drawn at random. What is the probability that the bulb is not defective? If the bulb selected is defective and it is not replaced and a second bulb is selected at random from the rest, what is the probability that the second bulb is defective?

Solution: Let

$$X_1 = \begin{cases} 1, & \text{if bulb is not defective} \\ 0, & \text{if bulb is defective} \end{cases} \tag{3.2.5.1}$$

Then the Bernoulli parameter

$$p_1 = 1 - \frac{6}{24} = \frac{3}{4} \tag{3.2.5.2}$$

which is the desired probability. In the second case,

$$1 - p_2 = \frac{6-1}{24-1} = \frac{5}{23} \tag{3.2.5.3}$$

which is the desired probability.

3.2.6 An integer is chosen between 0 and 100. What is the probability that it is

a) divisible by 7
b) not divisible by 7

Solution: Let X be a random variable such that

$$X = \begin{cases} 0 & n \not\equiv 0 \pmod 7 \\ 1 & n \equiv 0 \pmod 7 \end{cases} \tag{3.2.6.1}$$

Then,

a) $p = \frac{14}{99}$
b) $1 - p = \frac{85}{99}$

3.2.7 If the letters of the word **ALGORITHM** are arranged at random in a row what is the probability the letters GOR must remain together as a unit?

Solution: Let

$$X = \begin{cases} 1, & \text{if GOR remain together as a unit} \\ 0, & \text{otherwise} \end{cases} \tag{3.2.7.1}$$

Then

$$p = \frac{7!}{9!} = \frac{1}{72} \tag{3.2.7.2}$$

3.2.8 Six new employees, two of whom are married to each other, are to be assigned six desks that are lined up in a row. If the assignment of employees to desks is made randomly, what is the probability that the married couple will have nonadjacent desks?

Solution: Let X be a Random variable such that

RV	Values	Description
X	0	couple not sitting adjacent
	1	couple sitting adjacent

$$1 - p = \frac{5! \times 2}{6!} = \frac{1}{3} \tag{3.2.8.1}$$

$$\implies p = \frac{2}{3} \tag{3.2.8.2}$$

3.2.9 There are four men and six women on the city council. If one council member is selected for a committee at random, how likely is it that it is a woman?
Solution:

Parameter	**Value**	**Description**
X	0	Male
	1	Female

TABLE 3.2.9.1: Council distribution

$$X = \begin{cases} 0, & \text{if member is a man} \\ 1, & \text{if member is a woman} \end{cases} \tag{3.2.9.1}$$

From Table 3.2.9.1

$$p = \frac{6}{6+4} = \frac{3}{5} \tag{3.2.9.2}$$

3.2.10 A girl calculates that the probability of her winning the first prize in a lottery is 0.08. If 6000 tickets are sold,how many tickets has she bought?

a) 40 b) 240 c) 480 d) 750

Solution:

parameter	**value**	**description**
X	0	She didn't buy the ticket
	1	She bought the ticket
N	6000	Number of tickets sold

TABLE 3.2.10.1: Information table

See Table 3.2.10.1. The number of tickets bought is

$$Np = 0.08 \times 6000 = 480 \tag{3.2.10.1}$$

3.2.11 Three numbers are chosen from 1 to 20. Find the probability that they are not consecutive

a) $\frac{186}{190}$ b) $\frac{187}{190}$ c) $\frac{188}{190}$ d) $\frac{18}{^{20}C_3}$

Solution:

Random variable	Value	Description
X	0	The numbers are not consecutive
	1	The numbers are consecutive

TABLE 3.2.11.1: Random variable

See Table 3.2.11.1. The number of sets of three consecutive numbers from 1 to 20 is 18. Hence,

$$p = \frac{18}{^{20}C_3} \tag{3.2.11.1}$$

$$\implies 1 - p = 1 - \frac{18}{^{20}C_3} = \frac{187}{190} \tag{3.2.11.2}$$

3.2.12 Seven persons are to be seated in a row. What is the probability that two particular persons sit next to each other?
Solution:

RV	**Values**	**Description**
X	0	Not sitting next to each other
	1	Sitting next to each other

TABLE 3.2.12.1

See Table 3.2.12.1. The number of ways to arrange 7 people is 7! and the number of ways to arrange 7 people in which the two particular people are adjacent to each other is 6!×2 considering both of them as one unit and considering the arrangements within the unit. Thus,

$$p = \frac{6! \times 2}{7!} = \frac{2}{7} \tag{3.2.12.1}$$

3.2.13 A single letter is selected at random from the word 'PROBABILITY'. The probability that it is a vowel is ______.
Solution: Let X be an bernoulli rv defined as in Table 3.2.13.1. Then,

$$p = \frac{4}{11} \tag{3.2.13.1}$$

RV	Value	Description
X	0	Consonant
	1	Vowel

TABLE 3.2.13.1

3.2.14 A letter is chosen at random from the word 'ASSASSINATION'. Find the probability that letter is ______.

a) a vowel
b) a consonant

Solution: The number of vowels is 6 and consonants is 7. Therefore,

a) $p = \frac{6}{13}$
b) $1 - p = \frac{7}{13}$

3.2.15 A box contains 12 balls, out of which x are black. If one ball is drawn at random from the box, what is the probability that it will be a black ball? If 6 more black balls are put in the box, the probability of drawing a black ball is now double of what it was before. Find x.
Solution: From Table 3.2.15.1,

$$p_1 = \frac{x}{12}, p_2 = \frac{x+6}{18} \tag{3.2.15.1}$$

$$\because p_2 = 2p_1, \frac{x+6}{18} = 2\left(\frac{x}{12}\right) \tag{3.2.15.2}$$

$$\implies x = 3 \tag{3.2.15.3}$$

Random Variable	Sample space	Value	Event	Probability
X_1	12	0	not black	$\frac{12-x}{12}$
		1	choosing black ball	$\frac{x}{12}$
X_2	18	0	not black	$\frac{12-x}{18}$
		1	black	$\frac{x+6}{18}$

TABLE 3.2.15.1

3.2.16 Gopi buys a fish from a shop for his aquarium. The shopkeeper takes out one fish at random from a tank containing 5 male fish and 8 female fish . What is the probability that the fish taken out is a male fish?
Solution: For

$$X = \begin{cases} 1 & \text{male} \\ 0 & \text{female,} \end{cases} \tag{3.2.16.1}$$

$$p = \frac{5}{13} \tag{3.2.16.2}$$

3.2.17 A bag contains 3 red balls and 5 black balls. A ball is drawn at random from the bag. What is the probability that the ball drawn is

a) red ?
b) not red?

Solution: For

$$X = \begin{cases} 1 & \text{red} \\ 0 & \text{otherwise,} \end{cases} \tag{3.2.17.1}$$

a) $p = \frac{3}{8}$
b) $1 - p = \frac{5}{8}$

3.2.18 Someone is asked to take a number from 1 to 100. The probability that it is a prime

number is
Solution: See Table 3.2.18.1. Since there are 25 prime numbers in between 1 to 100,

$$p = \frac{25}{100} = \frac{1}{4} \tag{3.2.18.1}$$

RV	value	description
X	0	not prime
	1	prime

TABLE 3.2.18.1

4 Conditional Probability

4.1 Formulae

4.1.1

$$\Pr(A|B) = \frac{\Pr(AB)}{\Pr(B)} \tag{4.1.1.1}$$

If A and B are independent, from (4.1.1.1) and (2.1.7.1),

$$\Pr(A|B) = \frac{\Pr(A)\Pr(B)}{\Pr(B)} = \Pr(A) \tag{4.1.1.2}$$

4.1.2

$$\Pr(A'|B) = \frac{\Pr(A'B)}{\Pr(B)} = \frac{\Pr(B) - \Pr(AB)}{\Pr(B)} \tag{4.1.2.1}$$

4.1.3 Total probability

$$\Pr(A) = \sum_{i=1}^{2} \Pr(E_i)\Pr(A|E_i) \tag{4.1.3.1}$$

4.1.4 Bayes' Theorem

$$\Pr(E_1|A) = \frac{\Pr(E_1)\Pr(A|E_1)}{\sum_{i=1}^{2}\Pr(E_i)\Pr(A|E_i)} \tag{4.1.4.1}$$

4.1.5 Let $X, Y \in \{0, 1\}$ be two random variables. Then,

$$\Pr(Y = 1|X = 0) \triangleq p_{Y|X}(1|0) \tag{4.1.5.1}$$

4.1.6

$$p_{Y|X}(1|0) = \frac{p_{X,Y}(0,1)}{p_X(0)} = \frac{p_X(0) - p_{X,Y}(0,0)}{p_X(0)} \tag{4.1.6.1}$$

$$= 1 - \frac{p_{X,Y}(0,0)}{p_X(0)} = 1 - p_{Y|X}(0|0) \tag{4.1.6.2}$$

4.2 NCERT

4.2.1 Given that E and F are events such that $P(E) = 0.6, P(F) = 0.3$ and $P(EF) = 0.2$, find $P(E \mid F)$ and $P(F \mid E)$.

Solution: From (4.1.1.1)

$$\Pr(E|F) = \frac{\Pr(EF)}{\Pr(F)} = \frac{0.2}{0.3} = \frac{2}{3} \tag{4.2.1.1}$$

$$\Pr(F|E) = \frac{\Pr(EF)}{\Pr(E)} = \frac{0.2}{0.6} = \frac{1}{3} \tag{4.2.1.2}$$

4.2.2 Compute $\Pr(A|B)$, if $\Pr(B) = 0.5$ and $\Pr(AB) = 0.32$.

Solution:

$$\Pr(A|B) = \frac{0.32}{0.5} = 0.64 \tag{4.2.2.1}$$

4.2.3 If $\Pr(A) = 0.8$, $\Pr(B) = 0.5$ and $\Pr(B|A) = 0.4$, find

a) $\Pr(AB)$ b) $\Pr(A|B)$ c) $\Pr(A + B)$

Solution:

a)

$$\Pr(B|A) = \frac{\Pr(AB)}{\Pr(A)} \tag{4.2.3.1}$$

$$\implies 0.4 = \frac{\Pr(AB)}{0.8} \tag{4.2.3.2}$$

$$\text{or, } \Pr(AB) = 0.32 \tag{4.2.3.3}$$

b) Similarly,

$$\Pr(A|B) = \frac{\Pr(AB)}{\Pr(B)} = \frac{0.32}{0.5} = 0.64 \tag{4.2.3.4}$$

c)

$$\Pr(A + B) = \Pr(A) + \Pr(B) - \Pr(AB) \tag{4.2.3.5}$$

$$= 0.8 + 0.5 - 0.32 = 0.98 \tag{4.2.3.6}$$

4.2.4 If $\Pr(A) = \frac{6}{11}$, $\Pr(B) = \frac{5}{11}$ and $\Pr(A + B) = \frac{7}{11}$, find

a) $\Pr(AB)$ b) $\Pr(A \mid B)$ c) $\Pr(B \mid A)$

Solution:

a) From (2.1.5.1),

$$\Pr(AB) = \frac{6}{11} + \frac{5}{11} - \frac{7}{11} = \frac{4}{11} \tag{4.2.4.1}$$

b) From (4.2.4.1) and (4.1.1.1),

$$\Pr(A \mid B) = \frac{\frac{4}{11}}{\frac{5}{11}} = \frac{4}{5} \tag{4.2.4.2}$$

c) Similarly,

$$\Pr(B \mid A) = \frac{\frac{4}{11}}{\frac{6}{11}} = \frac{2}{3} \tag{4.2.4.3}$$

4.2.5 Mother, Father and Son line up at random for a family picture. Determine $\Pr(E \mid F)$ where E: Son on one end, F: Father in middle.

Solution: The total ways of arranging Father, Son, Mother in the family chart is 3! = 6. The probability that Father in middle is

$$\Pr(F) = \frac{2!}{3!} = \frac{1}{3} \tag{4.2.5.1}$$

The probability that Father in middle and Son is on one end is

$$\Pr(EF) = \frac{2!}{3!} = \frac{1}{3} \tag{4.2.5.2}$$

Thus,

$$\Pr(E \mid F) = \frac{\Pr(EF)}{\Pr(F)} = 1 \tag{4.2.5.3}$$

4.2.6 An instructor has a question bank consisting of 300 easy True / False questions, 200 difficult True / False questions, 500 easy multiple choice questions and 400 difficult multiple choice questions. If a question is selected at random from the question bank, what is the probability that it will be an easy question given that it is a multiple choice question?

Solution:

Variable	Event
$X = 0$	Easy question
$X = 1$	Difficult question
$Y = 0$	True/False question
$Y = 1$	Multiple choice question

TABLE 4.2.6.1

See Table 4.2.6.1. From the given information,

$$p_{XY}(0,0) = \frac{3}{14}, p_{XY}(0,1) = \frac{5}{14}, p_{XY}(1,0) = \frac{1}{7}, p_{XY}(1,1) = \frac{2}{7} \tag{4.2.6.1}$$

$$\implies p_Y(1) = \sum_{i=0}^{1} p_{XY}(1,i) = \frac{9}{14}. \tag{4.2.6.2}$$

$$\therefore p_{X|Y}(0|1) = \frac{p_{X,Y}(0,1)}{p_Y(1)} = \frac{\frac{5}{14}}{\frac{9}{14}} = \frac{5}{9} \tag{4.2.6.3}$$

4.2.7 If $\Pr(A) = \frac{1}{2}$, $\Pr(B) = 0$, then $\Pr(A \mid B)$ is

a) 0 b) $\frac{1}{2}$ c) not defined d) 1

4.2.8 If A and B are events such that

$$\Pr(A|B) = \Pr(B|A) \tag{4.2.8.1}$$

then

a) $A \subset B$ but $A \neq B$ b) $A = B$ c) $A \cap B = \phi$ d) $\Pr(A) = \Pr(B)$

Solution: Using Bayes' Rule,

$$\Pr(AB) = \Pr(A)\Pr(B|A) \tag{4.2.8.2}$$
$$= \Pr(B)\Pr(A|B) \tag{4.2.8.3}$$

Using (4.2.8.1) in (4.2.8.2) and (4.2.8.3),

$$\Pr(A) = \Pr(B) \tag{4.2.8.4}$$

We consider the options one by one.

a) If $A \subset B$ and $A \neq B$, then we can write $B = A + C$, where $AC = 0$ and $C \neq 0$. Thus,

$$\Pr(B) = \Pr(A + C) \tag{4.2.8.5}$$
$$= \Pr(A) + \Pr(C) - \Pr(AC) \tag{4.2.8.6}$$
$$= \Pr(A) + \Pr(C) > \Pr(A) \tag{4.2.8.7}$$

However, (4.2.8.7) contradicts (4.2.8.4).

b) We give a counterexample to show this is wrong. Consider A as the event that an even number shows on rolling a fair die and B as the event that a prime number shows on rolling a fair die. The joint pmf is shown in Table 4.2.8.1. Clearly,

$$\Pr(A|B) = \frac{\Pr(AB)}{\Pr(B)} = \frac{\frac{1}{6}}{\frac{1}{6} + \frac{1}{3}} = \frac{1}{2} \tag{4.2.8.8}$$
$$\Pr(B|A) = \frac{\Pr(AB)}{\Pr(A)} = \frac{\frac{1}{6}}{\frac{1}{6} + \frac{1}{3}} = \frac{1}{2} \tag{4.2.8.9}$$

c) The same example as before provides the required counterexample, as $\Pr(AB) = \frac{1}{6}$.

d) This is the correct answer, as discussed above.

	A	$\bar{A}$
B	$\frac{1}{6}$	$\frac{1}{3}$
$\bar{B}$	$\frac{1}{3}$	$\frac{1}{6}$

TABLE 4.2.8.1: Joint pmf for events A and B.

4.2.9 Consider the experiment of throwing a die, if a multiple of 3 comes up, throw the die again and if any other number comes, toss a coin. Find the conditional probability of the event 'the coin shows a tail', given that 'at least one die shows a 3'.

4.2.10 Two cards are drawn at random and without replacement from a pack of 52 playing cards. Find the probability that both the cards are black.

Solution: Let E_1 denote the event that the first card drawn is Black, E_2 denote the event that the second card drawn is Black. Then

$$\Pr(E_1) = \frac{26}{52}, \Pr(E_2 \mid E_1) = \frac{25}{51} \tag{4.2.10.1}$$

$$\implies \Pr(E_1E_2) = \Pr(E_1)\Pr(E_2 \mid E_1) = \frac{25}{102} \tag{4.2.10.2}$$

4.2.11 Let A and B be independent events with $P(A) = 0.3$ and $P(B) = 0.4$. Find

a) $P(A \cap B)$ b) $P(A \cup B)$ c) $P(A|B)$ d) $P(B|A)$

4.2.12 An urn contains 5 red and 5 black balls. A ball is drawn at random, its colour is noted and is returned to the urn. Moreover, 2 additional balls of the colour drawn are put in the urn and then a ball is drawn at random. What is the probability that the second ball is red?

Solution: The given information is summarized in Tables 4.2.12.2 and 4.2.12.4.

Variable	**Value**	**Colour**	**Description**
X	0	Red	1st draw
	1	Black	1st draw
Y	0	Red	2nd draw
	1	Black	2nd draw

TABLE 4.2.12.2

Probability	**Value**
$p_X(0)$	$\frac{5}{10}$
$p_X(1)$	$\frac{5}{10}$
$p_{Y\|X}(0\|0)$	$\frac{7}{12}$
$p_{Y\|X}(1\|0)$	$\frac{5}{12}$

TABLE 4.2.12.4

From (4.1.3.1), the required probability is given by

$$p_Y(0) = p_X(0)\, p_{Y|X}(0|0) + p_X(1)\, p_{Y|X}(0|1) \tag{4.2.12.1}$$

$$= \left(\frac{5}{10} \times \frac{7}{12}\right) + \left(\frac{5}{10} \times \frac{5}{12}\right) = \frac{1}{2} \tag{4.2.12.2}$$

4.2.13 A bag contains 4 red and 4 black balls, another bag contains 2 red and 6 black balls. One of the two bags is selected at random and a ball is drawn from the bag which is found to be red. Find the probability that the ball is drawn from the first bag.

4.2.14 Of the students in a college, it is known that 60% reside in hostel and 40% are day scholars (not residing in hostel). Previous year results report that 30% of all students who reside in hostel attain A grade and 20% of day scholars attain A grade in their annual examination. At the end of the year, one student is chosen at random from the college and he has an A grade, what is the probability that the student is a hostelier?

Solution: The given information is summarized in Table 4.2.14.2.

Variable	Value	Description
X	0	Hostel Student
	1	Day Scholar
Y	0	A grade
	1	No A grade

TABLE 4.2.14.2

From the given data,

$$p_X(0) = \frac{3}{5}, p_X(1) = \frac{2}{5}, p_Y(1|0) = \frac{3}{10}, p_Y(1|1) = \frac{1}{5} \tag{4.2.14.1}$$

The desired probability is

$$p_{X|Y}(0|1) = \frac{p_{Y|X}(1|0) \times p_X(0)}{\sum_{k=0}^{1} p_{Y|X}(1|k) \times p_X(k)} \tag{4.2.14.2}$$

$$= \frac{\frac{3}{10} \times \frac{3}{5}}{\frac{3}{10} \times \frac{3}{5} + \frac{1}{5} \times \frac{2}{5}} = \frac{9}{13} \tag{4.2.14.3}$$

4.2.15 In answering a question on a multiple choice test, a student either knows the answer or guesses. Let $\frac{3}{4}$ be the probability that he knows the answer and $\frac{1}{4}$ be the probability that he guesses. Assuming that a student who guesses at the answer will be correct with probability $\frac{1}{4}$, what is the probability that the student knows the answer given that he answered it correctly?
Solution: See Table 4.2.15.2

Variable	Value	Description
X	0	Guesses
	1	Knows
Y	0	Incorrect
	1	Correct

TABLE 4.2.15.2

From the given information,

$$p_{Y|X}(1|0) = \frac{1}{4}, p_{Y|X}(1|1) = 1, p_X(0) = \frac{1}{4}, p_X(1) = \frac{3}{4} \tag{4.2.15.1}$$

The desired probability is

$$p_{X|Y}(1|1) = \frac{p_{Y|X}(1|1)\, p_X(1)}{\sum_{i=0}^{1} p_{Y|X}(1|i)\, p_X(i)} \tag{4.2.15.2}$$

$$= \frac{\frac{3}{4}}{\frac{1}{4}\times\frac{1}{4} + 1\times\frac{3}{4}} = \frac{4}{5} \tag{4.2.15.3}$$

4.2.16 A laboratory blood test is 99% effective in detecting a certain disease when it is in fact, present. However, the test also yields a false positive result for 0.5% of the healthy person tested (i.e. if a healthy person is tested, then, with probability 0.005, the test will imply he has the disease). If 0.1 percent of the population actually has the disease, what is the probability that a person has the disease given that his test result is positive ?

Solution: See Table 4.2.16.2 for the given information.

Variable	Value	Description
X	0	Blood test negative
	1	Blood test positive
Y	0	No Disease
	1	Disease

TABLE 4.2.16.2

From the given information,

$$p_Y(0) = 1 - p_Y(1) = 1 - 0.001 = 0.999 \tag{4.2.16.1}$$

$$p_{X|Y}(1|1) = 0.99, p_{X|Y}(0|1) = 0.005 \tag{4.2.16.2}$$

$$\therefore p_{Y|X}(1|1) = \frac{p_Y(1)\, p_{X|Y}(1|1)}{\sum_{i=1}^{2} p_Y(i)\, p_{X|Y}(1|i)} \tag{4.2.16.3}$$

$$= \frac{0.001\times 0.99}{0.001\times 0.99 + 0.999\times 0.005} = \frac{22}{133} \tag{4.2.16.4}$$

4.2.17 There are three coins. One is a two headed coin (having head on both faces), another is a biased coin that comes up heads 75% of the time and third is an unbiased coin. One of the three coins is chosen at random and tossed, it shows heads, what is the probability that it was the two headed coin?

4.2.18 An insurance company insured 2000 scooter drivers, 4000 car drivers and 6000 truck drivers. The probability of an accidents are 0.01, 0.03 and 0.15 respectively. One of the insured persons meets with an accident. What is the probability that he is a scooter driver?

4.2.19 A factory has two machines A and B. Past record shows that machine A produced 60% of the items of output and machine B produced 40% of the items. Further, 2% of the items produced by machine A and 1% produced by machine B were defective. All the items are put into one stockpile and then one item is chosen at random from this and is found to be defective. What is the probability that it was produced by machine B?

4.2.20 Two groups are competing for the position on the Board of directors of a corporation. The probabilities that the first and the second groups will win are 0.6 and 0.4 respectively. Further, if the first group wins, the probability of introducing a new product is 0.7 and the corresponding prbability is 0.3 if the second group wins. Find the probability that the new product introduced was by the second group.

Solution: See Table 4.2.20.2.

Variable	Value	Description
X	1	Group 1 wins
	2	Group 2 wins
Y	0	New product introduced
	1	No new product introduced

TABLE 4.2.20.2

From the given information,

$$p_X(1) = 0.6, p_X(2) = 0.4, p_{Y|X}(1|1) = 0.7, p_{Y|X}(1|2) = 0.3 \tag{4.2.20.1}$$

$$\implies p_{X|Y}(2|1) = \frac{p_X(2)\, p_{Y|X}(1|1)}{\sum_{i=1}^{2} p_X(i)\, p_{Y|X}(1|i)} = \frac{2}{9} \tag{4.2.20.2}$$

upon substituting numerical values and simplifying.

4.2.21 Suppose a girl throws a die. If she gets a 5 or 6, she tosses a coin three times and notes the number of heads. If she gets 1, 2, 3 or 4, she tosses a coin once and notes whether a head or tail is obtained. If she obtained exactly one head, what is the probability that she threw 1, 2, 3 or 4 with the die?]

4.2.22 A manufacturer has three machine operators A, B and C. The first operator A produces 1% defective items, where as the other two operators B and C produce 5% and 7% defective items respectively. A is on the job for 50% of the time, B is on the job for 30% of the time and C is on the job for 20% of the time. A defective item is produced, what is the probability that it was produced by A?

4.2.23 A card from a pack of 52 cards is lost. From the remaining cards of the pack, two cards are drawn and are found to be both diamonds. Find the probability of the lost card being a diamond.

4.2.24 Probability that A speaks truth is $\frac{4}{5}$. A coin is tossed. A reports that a head appears. The probability that actually there was head is

a) $\frac{4}{5}$ b) $\frac{1}{2}$ c) $\frac{1}{5}$ d) $\frac{2}{5}$

Solution: See Table 4.2.24.2.

Variable	Value	Description
X	0	Tail
	1	Head
Y	0	A lies
	1	A tells the truth

TABLE 4.2.24.2

From the given information,

$$p_X(0) = p_X(1) = \frac{1}{2}, p_Y(1) = \frac{4}{5} \tag{4.2.24.1}$$

$$\therefore p_{Y|X}(1|1) = \frac{p_Y(1)\, p_{X|Y}(1|1)}{p_X(1)} \tag{4.2.24.2}$$

$$= \frac{\frac{4}{5} \times \frac{1}{2}}{\frac{1}{2}} = \frac{4}{5} \tag{4.2.24.3}$$

Here, we have used the fact that

$$p_{X|Y}(1|1) = p_X(1) = \frac{1}{2}. \tag{4.2.24.4}$$

since the appearance of a head upon a coin toss has nothing to do with A.

4.2.25 If A and B are two events such that $A \subset B$ and $\Pr(B) \neq 0$, then which of the following is correct ?

a) $\Pr(A|B) = \frac{\Pr(B)}{\Pr(A)}$
b) $\Pr(A|B) < \Pr(A)$
c) $\Pr(A|B) \geq \Pr(A)$
d) None of these

Solution:

a) If $A \subset B$ and $\Pr(B) \neq 0$ then

$$AB = A \tag{4.2.25.1}$$

$$\text{or, } P(A|B) = \frac{\Pr(AB)}{\Pr(B)} = \frac{\Pr(A)}{\Pr(B)} \tag{4.2.25.2}$$

c) Since

$$\Pr(B) \leq 1, \tag{4.2.25.3}$$

$$1 \leq \frac{1}{\Pr(B)} \tag{4.2.25.4}$$

$$\implies \Pr(A) \leq \frac{\Pr(A)}{\Pr(B)} \tag{4.2.25.5}$$

$$= \Pr(A|B) \tag{4.2.25.6}$$

from (4.2.25.2).

4.2.26 A and B are two events such that $\Pr(A) \neq 0$. Find $\Pr(B|A)$, if

a) A is a subset of B b) $A \cap B = \phi$

Solution: We use

$$\Pr(B \mid A) = \frac{\Pr(BA)}{\Pr(A)} \tag{4.2.26.1}$$

a) In this case,

$$BA = A \implies \Pr(BA) = \Pr(A) \tag{4.2.26.2}$$

From (4.2.26.1),

$$\Pr(B \mid A) = 1 \tag{4.2.26.3}$$

b) $A \cap B = \phi$. This implies

$$\Pr(BA) = 0 \tag{4.2.26.4}$$

From (4.2.26.1),

$$\Pr(B \mid A) = 0 \tag{4.2.26.5}$$

4.2.27 A couple has two children.

a) Find the probability that both children are males, if it is known that at least one of the children is male.
b) Find the probability that both children are females, if it is known that the elder child is a female.

4.2.28 Suppose that 5% of men and 0.25% of women have grey hair. A grey haired person is selected at random. What is the probability that this person being male? Assume that there are equal number of males and females.
Solution: See Table 4.2.28.1.

Variable	**Event**
$X = 0$	Men
$X = 1$	Women
$Y = 0$	Non-grey hair
$Y = 1$	grey hair

TABLE 4.2.28.1

From the given information,

$$p_X(0) = p_X(1) = \frac{1}{2} \tag{4.2.28.1}$$

$$p_{Y|X}(1|0) = \frac{5}{100} = \frac{1}{20} \tag{4.2.28.2}$$

$$p_{Y|X}(1|1) = \frac{0.25}{100} = \frac{1}{400} \tag{4.2.28.3}$$

Using (4.1.4.1)

$$p_{X|Y}(0|1) = \frac{p_{Y|X}(1|0)p_X(0)}{\sum_{i=0}^{1} p_{Y|X}(1|i)p_Y(i)} = \frac{\frac{1}{40}}{\frac{21}{800}} = \frac{20}{21} \tag{4.2.28.4}$$

4.2.29 Suppose we have four boxes A,B,C and D containing coloured marbles as given in Table 4.2.29.1.

Box	Marble colour		
	Red	White	Black
A	1	6	3
B	6	2	2
C	8	1	1
D	0	6	4

TABLE 4.2.29.1: Question Table

One of the boxes has been selected at random and a single marble is drawn from it. If the marble is red, what is the probability that it was drawn from

a) Box A ?

b) Box B ?

c) Box C ?

Solution: Let $X \in \{i\}_{i=0}^{2}$ represent the colour and $Y \in \{i\}_{i=0}^{3}$ represent the box. From the given information,

$$p_{X|Y}(0|i) = \begin{cases} \frac{1}{10} & i = 0 \\ \frac{6}{10}, & i = 1 \\ \frac{8}{10}, & i = 2 \\ 0 & i = 3 \end{cases}, \quad p_Y(i) = \frac{1}{4} \tag{4.2.29.1}$$

From (4.1.4.1)

$$p_{Y|X}(i|0) = \begin{cases} \frac{1}{15} & i = 0 \\ \frac{2}{5} & i = 1 \\ \frac{8}{15} & i = 2 \end{cases} \tag{4.2.29.2}$$

4.2.30 Bag I contains 3 red and 4 black balls and Bag II contains 4 red and 5 black balls. One ball is transferred from Bag I to Bag II and then a ball is drawn from Bag II. The ball so drawn is found to be red in colour. Find the probability that the transferred ball is black.

Solution: Let $X, Y \in \{0, 1\}, i \in \{1, 2\}$ represent the red and black balls in Bag 1 and 2 respectively. From the given information,

$$p_X(i) = \begin{cases} \frac{3}{7} & i = 0 \\ \frac{4}{7} & i = 1 \end{cases}, \quad p_Y(i) = \begin{cases} \frac{4}{9} & i = 0 \\ \frac{5}{9} & i = 1 \end{cases} \tag{4.2.30.1}$$

Also

$$p_{Y|X}(0|i) = \begin{cases} \frac{4}{10} & i = 0 \\ \frac{5}{10} & i = 1 \end{cases}, p_{Y|X}(1|i) = \begin{cases} \frac{5}{10} & i = 0 \\ \frac{6}{10} & i = 1 \end{cases} \tag{4.2.30.2}$$

$$\therefore p_{X|Y}(1|0) = \frac{p_{Y|X}(0|1)\, p_X(1)}{p_Y(0)} = \frac{\frac{1}{2} \times \frac{4}{7}}{\frac{5}{9}} = \frac{18}{35} \tag{4.2.30.3}$$

4.2.31 Bag I contains 3 black and 2 white balls, Bag II contains 2 black and 4 white balls. A bag and a ball is selected at random. Determine the probability of selecting a black ball.

Solution:

Random variable	Value	Definition
X	0	Bag 1
	1	Bag 2
Y	0	White ball
	1	Black ball

TABLE 4.2.31.1

From the given infrormation,

$$p_X(0) = p_X(1) = \frac{1}{2} \tag{4.2.31.1}$$

$$p_{Y|X}(1|0) = \frac{3}{5},\ p_{Y|X}(1|1) = \frac{1}{3} \tag{4.2.31.2}$$

From (4.1.3.1), the desired probability is

$$p_Y(1) = \sum_{i=0}^{1} p_{Y|X}(1|i)\, p_X(i) = \frac{7}{15} \tag{4.2.31.3}$$

4.2.32 While shuffling a pack of 52 playing cards, 2 cards are dropped. Find the probabilty that the missing cards are of different colours.

Solution: See Table 4.2.32.1.

Random Variable	Values	Description
X_1	0	First card is red
	1	First card is black
X_2	0	Second card is red
	1	Second card is black

TABLE 4.2.32.1

Since 26 out of 52 playing cards are red,

$$p_{X_1}(k) = \frac{26}{52} = \frac{1}{2} \quad \{k = 0, 1\} \tag{4.2.32.1}$$

Also,

$$p_{X_2|X_1}(i|j) = \begin{cases} \frac{25}{51} & i = j \\ \frac{26}{51} & i \neq j \end{cases} \tag{4.2.32.2}$$

The desired probability is

$$\Pr(X_1 \neq X_2) = \sum_{i=0}^{1} \Pr(X_1 \neq X_2 | X_1 = i) \, p_X(i) = \frac{26}{51} \tag{4.2.32.3}$$

4.2.33 A bag contains $(2n+1)$ coins. It is known that n of these coins have a head on both sides where as the rest of the coins are fair. A coin is picked up at random from the bag and is tossed. If the probability that the toss results in a head is $\frac{31}{42}$, determine the value of n.

4.2.34 An urn contains m white and n black balls. A ball is drawn at random and is put back into the urn along with k additional balls of the same colour as that of the ball drawn. A ball is again drawn at random. Show that the probability of drawing a white ball now does not depend on k.

4.2.35 If $\Pr(AB) = \frac{7}{10}$ and $\Pr(B) = \frac{17}{20}$, then $\Pr(A|B)$ equals

a) $\frac{14}{17}$ b) $\frac{17}{20}$ c) $\frac{7}{8}$ d) $\frac{1}{8}$

4.2.36 A letter is known to have come either from TATANAGAR or from CALCUTTA. On the envelope, just two consecutive letter TA are visible. What is the probability that the letter came from TATANAGAR.

4.2.37 A bag contain $(2n+1)$ coins. It is known that n of these coins have a head on both sides where as the rest of the coins are fair. A coin is picked up at random from the bag and is tossed. If the probability that the toss results in a head is $\frac{31}{42}$, determine the value of n.

4.2.38 By examining the chest X ray, the probability that TB is detected when a person is actually suffering is 0.99. The probability of an healthy person diagnosed to have TB is 0.001. In a certain city, 1 in 1000 people suffers from TB. A person is selected at random and is diagnosed to have TB. What is the probability that he actually has TB?

4.2.39 Compute $\Pr(A|B)$, if $\Pr(B) = 0.5$ and $\Pr(AB) = 0.32$.

4.2.40 Let $\Pr(A) = \frac{7}{13}$, $\Pr(B) = \frac{9}{13}$, $\Pr(AB) = \frac{4}{13}$. Then $\Pr(A'|B)$ is equal to

a) $\frac{6}{13}$ b) $\frac{4}{13}$ c) $\frac{4}{9}$ d) $\frac{5}{9}$

Solution: From (4.1.1.1),

$$\Pr(A'|B) = \frac{\Pr(A'B)}{\Pr(B)} \tag{4.2.40.1}$$

$$= \frac{\Pr(B) - \Pr(AB)}{\Pr(B)} = \frac{5}{9} \tag{4.2.40.2}$$

using (2.1.5.3).

4.2.41 If $\Pr(A) = \frac{2}{5}, \Pr(B) = \frac{3}{10}$ and $\Pr(AB) = \frac{1}{5}$, then $\Pr(A'|B')\Pr(B'|A')$ is equal to

a) $\frac{5}{6}$ b) $\frac{5}{7}$ c) $\frac{25}{42}$ d) 1

Solution: From (2.1.5.1),

$$\Pr(A+B) = \frac{1}{2}. \tag{4.2.41.1}$$

From (4.1.1.1),

$$\Pr(A'|B')\Pr(B'|A') = \frac{\Pr(A'B')}{\Pr(B')}.\frac{\Pr(A'B')}{\Pr(A')} \tag{4.2.41.2}$$

$$= \frac{(\Pr(A'B'))^2}{(1-\Pr(B))(1-\Pr(A))} \tag{4.2.41.3}$$

$$= \frac{(1-\Pr(A+B))^2}{(1-\Pr(B))(1-\Pr(A))} \tag{4.2.41.4}$$

$$= \frac{25}{42} \tag{4.2.41.5}$$

upon substituting numerical values.

4.2.42 A and B are two events such that $\Pr(A) = \frac{1}{2}$, $\Pr(B) = \frac{1}{3}$ and $\Pr(AB) = \frac{1}{4}$. Find

a) $\Pr(A|B)$ b) $\Pr(B|A)$ c) $\Pr(A'|B)$ d) $\Pr(A'|B')$

Solution:

a)

$$\Pr(A|B) = \frac{\Pr(AB)}{\Pr(B)} = \frac{3}{4} \tag{4.2.42.1}$$

b)

$$\Pr(B|A) = \frac{\Pr(AB)}{\Pr(A)} = \frac{1}{2} \tag{4.2.42.2}$$

c) From (4.1.2.1),

$$\Pr(A'|B) = \frac{\Pr(B)-\Pr(AB)}{\Pr(B)} = \frac{1}{4} \tag{4.2.42.3}$$

d)

$$\Pr(A'|B') = \frac{\Pr(A'B')}{\Pr(B')} \tag{4.2.42.4}$$

$$= \frac{\Pr(A+B)'}{\Pr(B')} = \frac{1-\Pr(A+B)}{1-\Pr(B)} \tag{4.2.42.5}$$

$$= \frac{5}{8} \tag{4.2.42.6}$$

using (2.1.5.1) in the numerator.

4.2.43 If $\Pr(A) = \frac{3}{10}, \Pr(B) = \frac{2}{5}$ and $\Pr(A+B) = \frac{3}{5}$, then $\Pr(B|A) + \Pr(A|B)$ equals

a) $\frac{1}{4}$ b) $\frac{1}{3}$ c) $\frac{5}{12}$ d) $\frac{7}{12}$

Solution:

$$\Pr(AB) = \Pr(A) + \Pr(B) - \Pr(A+B) \tag{4.2.43.1}$$
$$= \frac{1}{10} \tag{4.2.43.2}$$
$$\implies \Pr(B|A) + \Pr(A|B) = \frac{\Pr(AB)}{\Pr(A)} + \frac{\Pr(AB)}{\Pr(B)} = \frac{7}{12} \tag{4.2.43.3}$$

upon substituting numerical values.

4.2.44 Let A and B be two events such that $\Pr(A) = \frac{3}{8}$, $\Pr(B) = \frac{5}{8}$ and $\Pr(A+B) = \frac{3}{4}$. Then $\Pr(A|B)\Pr(A'|B)$ is equal to

a) $\frac{2}{5}$ b) $\frac{3}{8}$ c) $\frac{3}{20}$ d) $\frac{6}{25}$

Solution: From (2.1.5.1)

$$\Pr(AB) = \frac{1}{4} \tag{4.2.44.1}$$

Hence,

$$\Pr(A|B).\Pr(A'|B) = \frac{\Pr(AB)}{\Pr(B)} \times \frac{\Pr(B) - \Pr(AB)}{\Pr(B)} \tag{4.2.44.2}$$
$$= \frac{6}{25} \tag{4.2.44.3}$$

using (2.1.5.3) and substituting numerical values.

4.2.45 If $P(A) = 0.4$, $P(B) = 0.8$ and $P(B|A) = 0.6$, then $P(A \cup B)$ is equal to

a) 0.24 b) 0.3 c) 0.48 d) 0.96

Solution: From (4.1.1.1),

$$\Pr(AB) = \Pr(B|A)\Pr(A) = 0.24. \tag{4.2.45.1}$$

yielding

$$\Pr(A+B) = 0.96 \tag{4.2.45.2}$$

from (2.1.5.1).

4.2.46 If A and B are two events such that $\Pr(A) = \frac{1}{2}$ $\Pr(B) = \frac{1}{3}$, $\Pr(A|B) = \frac{1}{4}$, then $\Pr(A'B')$ equals

a) $\frac{1}{12}$ b) $\frac{3}{4}$ c) $\frac{1}{4}$ d) $\frac{3}{16}$

Solution: From (4.1.1.1),

$$\Pr(AB) = \Pr(A|B)\Pr(B) = \frac{1}{12} \tag{4.2.46.1}$$
$$\implies \Pr(A'B') = 1 - \Pr(A+B) = \frac{1}{4} \tag{4.2.46.2}$$

using (2.1.5.1) and substituting numerical values.

4.2.47 If A and B are such events that $\Pr(A) > 0$ and $\Pr(B) \neq 1$, then $\Pr(A'|B')$ is

a) $1 - \Pr(A|B)$ b) $1 - \Pr(A'|B)$ c) $\frac{1-\Pr(A+B)}{\Pr(B')}$ d) $\frac{\Pr(A')}{\Pr(B')}$

Solution:

$$\Pr(A'|B') = \frac{\Pr(A'B')}{\Pr(B')} = \frac{\Pr((A+B)')}{\Pr(B')} \tag{4.2.47.1}$$

$$= \frac{1 - \Pr(A+B)}{\Pr(B')} \tag{4.2.47.2}$$

4.2.48 Two events E and F are independent. If $\Pr(E) = 0.3$, $\Pr(E+F) = 0.5$, then $\Pr(E|F) - \Pr(F|E)$ equals

a) $\frac{2}{7}$ b) $\frac{3}{35}$ c) $\frac{1}{70}$ d) $\frac{1}{7}$

Solution:

$$\Pr(EF) = \Pr(E)\Pr(F) \tag{4.2.48.1}$$

$$\therefore \Pr(F) = \frac{\Pr(E+F) - \Pr(E)}{1 - \Pr(E)} = \frac{2}{7} \tag{4.2.48.2}$$

using (2.1.5.1) and simplifying. From (4.1.1.1),

$$\Pr(E|F) = \Pr(E), \ \Pr(F|E) = \Pr(F) \tag{4.2.48.3}$$

$$\implies \Pr(E|F) - \Pr(F|E) = \Pr(E) - \Pr(F) = \frac{1}{70} \tag{4.2.48.4}$$

4.2.49 If A and B are two events such that $\Pr(A|B) = p$, $\Pr(A) = p$, $\Pr(B) = \frac{1}{3}$ and $\Pr(A+B) = \frac{5}{9}$, then $p =$

Solution: From (4.1.1.1),

$$\Pr(AB) = \Pr(A|B)\Pr(B) = \frac{p}{3} \tag{4.2.49.1}$$

which, upon substituing in (2.1.5.1) and simplifying results in

$$p + \frac{1}{3} - \frac{p}{3} = \frac{5}{9} \tag{4.2.49.2}$$

$$\implies p = \frac{1}{3}. \tag{4.2.49.3}$$

4.2.50 If A and B are two events such that $\Pr(A) > 0$ and $\Pr(A) + \Pr(B) > 1$, then

$$\Pr(B|A) \geq 1 - \frac{\Pr(B')}{\Pr(A)} \tag{4.2.50.1}$$

Solution:

$$\Pr(B|A) = \frac{\Pr(AB)}{\Pr(A)} \tag{4.2.50.2}$$

$$= \frac{\Pr(A) + \Pr(B) - \Pr(A+B)}{\Pr(A)} \tag{4.2.50.3}$$

$$= \frac{\Pr(A) + 1 - \Pr(B') - \Pr(A+B)}{\Pr(A)} \tag{4.2.50.4}$$

$$= 1 - \frac{\Pr(B')}{\Pr(A)} + \frac{1 - \Pr(A+B)}{\Pr(A)} \tag{4.2.50.5}$$

From (2.1.4.1)

$$1 - \Pr(A+B) \ge 0 \tag{4.2.50.6}$$

Using this in (4.2.50.5) results in (4.2.50.1).

4.2.51 If

$$\Pr(B) = \frac{3}{5}, \Pr(A|B) = \frac{1}{2} \text{ and } \Pr(A+B) = \frac{4}{5}, \tag{4.2.51.1}$$

$$\text{then } \Pr(A+B') + \Pr(A'+B) = ? \tag{4.2.51.2}$$

Solution: From (4.1.1.1),

$$\Pr(AB) = \frac{3}{5} \times \frac{1}{2} = \frac{3}{10} \tag{4.2.51.3}$$

From (2.1.5.3),

$$\Pr(A'B) = \frac{3}{5} - \frac{3}{10} = \frac{3}{10} \tag{4.2.51.4}$$

From (2.1.5.1),

$$\Pr(A) = \frac{4}{5} + \frac{3}{10} - \frac{3}{5} = \frac{1}{2}. \tag{4.2.51.5}$$

Again, using (2.1.5.3),

$$\Pr(AB') = \frac{1}{2} - \frac{3}{10} = \frac{1}{5}. \tag{4.2.51.6}$$

Thus, using (2.1.3.1),

$$\Pr(A+B)' + \Pr(A'+B) = 1 - \frac{1}{5} + 1 - \frac{3}{10} = \frac{3}{2}. \tag{4.2.51.7}$$

4.2.52 If $\Pr(A|B) > \Pr(A)$, then which of the following is correct?

a) $\Pr(B|A) < \Pr(B)$

b) $\Pr(AB) < \Pr(A)\Pr(B)$

c) $\Pr(B|A) > \Pr(B)$

d) $\Pr(B|A) = \Pr(B)$

Solution:

$$\because \Pr(A|B) > \Pr(A), \ \frac{\Pr(AB)}{\Pr(B)} > \Pr(A) \tag{4.2.52.1}$$

$$\implies \Pr(AB) > \Pr(A)\Pr(B) \tag{4.2.52.2}$$

$$\text{or, } \frac{\Pr(AB)}{\Pr(A)} = \Pr(B|A) > \Pr(A) \tag{4.2.52.3}$$

4.2.53 Let A and B be independent events with $\Pr(A) = 0.3$ and $\Pr(B) = 0.4$. Find

a) $\Pr(AB)$ b) $\Pr(A + B)$ c) $\Pr(A|B)$ d) $\Pr(B|A)$

Solution:

a)

$$\Pr(AB) = 0.3 \times 0.4 = 0.12 \tag{4.2.53.1}$$

b)

$$\Pr(A + B) = 0.3 + 0.4 - 0.12 = 0.58 \tag{4.2.53.2}$$

c)

$$\Pr(A|B) = \Pr(A) = 0.3 \tag{4.2.53.3}$$

d)

$$\Pr(B|A) = \Pr(B) = 0.4 \tag{4.2.53.4}$$

4.2.54 Compute $\Pr(A|B)$, if $\Pr(B) = 0.5$ and $\Pr(AB) = 0.32$.

Solution:

$$\Pr(A|B) = \frac{0.32}{0.5} = 0.64 \tag{4.2.54.1}$$

4.2.55 If A and B are two events such that $P(A) \neq 0$ and $P(B \mid A) = 1$, then

a) $A \subset B$ b) $B \subset A$ c) $B = \phi$ d) $A = \phi$

Solution:

$$\Pr(B|A) = 1 \implies \Pr(BA) = \Pr(A) \tag{4.2.55.1}$$

yielding

$$BA = A, \text{ or, } A \subset B \tag{4.2.55.2}$$

4.2.56 You are given that A and B are two events such that $\Pr(B) = \frac{3}{5}, \Pr(A|B) = \frac{1}{2}$ and $\Pr(A + B) = \frac{4}{5}$, then $\Pr(A)$ equals ______.

Solution: From (4.1.1.1),

$$\Pr(AB) = \frac{3}{5} \times \frac{1}{2} = \frac{3}{10} \tag{4.2.56.1}$$

$$\implies \Pr(A) = \frac{4}{5} - \frac{3}{5} + \frac{3}{10} = \frac{1}{2} \tag{4.2.56.2}$$

from (2.1.5.1).

4.2.57 Three events A, B and C have probabilities $\frac{2}{5}$, $\frac{1}{3}$ and $\frac{1}{2}$ respetively. Given that $\Pr(AC) = \frac{1}{5}$ and $\Pr(BC) = \frac{1}{4}$, find the values of $\Pr(C|B)$ and $\Pr(A'C')$.

Solution:

a) From (4.1.1.1),

$$\Pr(C|B) = \frac{\frac{1}{4}}{\frac{1}{3}} = \frac{3}{4} \tag{4.2.57.1}$$

b)

$$\Pr(A'C') = 1 - \Pr(A + C) \tag{4.2.57.2}$$

$$= 1 - \left(\frac{2}{5} + \frac{1}{2} - \frac{1}{5}\right) = \frac{3}{10} \tag{4.2.57.3}$$

from (2.1.5.1).

4.2.58 If A and B are two events and A $\neq \phi$, B $\neq \phi$, then

a) $\Pr(A|B) = \Pr(A) . \Pr(B)$
b) $\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$
c) $\Pr(A|B) \Pr(B|A) = 1$
d) $\Pr(A|B) = \frac{\Pr(A)}{\Pr(B)}$

4.2.59 You are given that A and B are two events such that $\Pr(B) = \frac{3}{5}$, $\Pr(A|B) = \frac{1}{2}$, $\Pr(A + B) = \frac{4}{5}$ and $\Pr(A) = \frac{1}{2}$. $\Pr(B|A')$ is equal to ______.

4.2.60 A fair die is rolled. Consider events $E = 1, 3, 5$, $F = 2, 3$ and $G = 2, 3, 4, 5$. Find

a) $\Pr(E \mid F)$ and $\Pr(F \mid E)$
b) $\Pr(E \mid G)$ and $\Pr(G \mid E)$
c) $\Pr(E \cup F \mid G)$ and $\Pr(E \cap F \mid G)$

Solution: See Table 4.2.60.1.

E = {1,3,5}	Pr(E) = $\frac{1}{2}$
F = {2,3}	Pr(F) = $\frac{1}{3}$
G = {2,3,4,5}	Pr(G) = $\frac{2}{3}$
EF = {3}	Pr(EF) = $\frac{1}{6}$
FG = {2,3}	Pr(FG) = $\frac{1}{3}$
EG = {3,5}	Pr(EG) = $\frac{1}{3}$
EFG = {3}	Pr(EFG) = $\frac{1}{6}$

TABLE 4.2.60.1

a)

$$\Pr(E|F) = \frac{\Pr(EF)}{\Pr(F)} = \frac{1/6}{1/3} = 1/2 \tag{4.2.60.1}$$

b)

$$\Pr(F|E) = \frac{\Pr(EF)}{\Pr(E)} = \frac{1/6}{1/2} = 1/3 \tag{4.2.60.2}$$

c)

$$\Pr(E|G) = \frac{\Pr(EG)}{\Pr(G)} = \frac{1/3}{2/3} = 1/2 \tag{4.2.60.3}$$

d)

$$\Pr(G|E) = \frac{\Pr(EG)}{\Pr(E)} = \frac{1/3}{1/2} = 2/3 \tag{4.2.60.4}$$

e)

$$\because \Pr((E+F)G) = \Pr(EG+FG) = \Pr(EG) + \Pr(FG) - \Pr(EFG),$$

$$= \frac{1}{3} + \frac{1}{3} - \frac{1}{6} = \frac{1}{2} \tag{4.2.60.5}$$

$$\Pr((E+F)|G) = \frac{\Pr((E+F)G)}{\Pr(G)} = \frac{1/2}{2/3} = \frac{3}{4} \tag{4.2.60.6}$$

f)

$$\Pr(EF|G) = \frac{\Pr(EFG)}{\Pr(G)} = \frac{1/6}{2/3} = \frac{1}{4} \tag{4.2.60.7}$$

4.2.61 An electronic assembly consists of two subsystems, say A and B. From previous testing procedures, the following probabilities are assumed to be known

$$\Pr(A \text{ fails}) = 0.20 \tag{4.2.61.1}$$

$$\Pr(B \text{ alone fails}) = 0.15 \tag{4.2.61.2}$$

$$\Pr(A \text{ and } B \text{ fails}) = 0.15 \tag{4.2.61.3}$$

Evaluate the following probabilities

a) Pr(A fails given B has failed) b) Pr(A fails alone)

Solution: From the given information,

$$\Pr(A') = 0.20,\ \Pr(AB') = 0.15,\ \Pr(A'B') = 0.15 \tag{4.2.61.4}$$

a)

$$\Pr(A'|B') = \frac{\Pr(A'B')}{\Pr(B')} \tag{4.2.61.5}$$

From (2.1.5.3),

$$\Pr(B') = 0.15 + 0.15 = 0.30 \tag{4.2.61.6}$$

$$\Pr(A'|B') = \frac{0.15}{0.30} = 0.50 \tag{4.2.61.7}$$

b) Similiarly, from (2.1.5.3),

$$\Pr(BA') = \Pr(A') - \Pr(A'B') = 0.20 - 0.15 = 0.05 \tag{4.2.61.8}$$

4.2.62 In a hostel, 60% of the students read Hindi newspaper, 40% read English newspaper and 20% read both Hindi and English newspapers. A student is selected at random.

a) Find the probability that she reads neither Hindi nor English newspapers.
b) If she reads Hindi newspaper, find the probability that she reads English newspaper.
c) If she reads English newspaper, find the probability that she reads Hindi newspaper.

Solution: From the given information,

$$\Pr(A) = \frac{6}{10}, \; \Pr(B) = \frac{4}{10}, \; \Pr(AB) = \frac{2}{10} \tag{4.2.62.1}$$

a)

$$\Pr(A'B') = \Pr((A+B)') \tag{4.2.62.2}$$

$$= 1 - \Pr(A+B) \tag{4.2.62.3}$$

$$= 1 - (\Pr(A) + \Pr(B) - \Pr(AB)) \tag{4.2.62.4}$$

$$= 1 - \left(\frac{6}{10} + \frac{4}{10} - \frac{2}{10}\right) = \frac{2}{10} \tag{4.2.62.5}$$

b)

$$\Pr(B|A) = \frac{\Pr(AB)}{\Pr(A)} = \frac{\frac{2}{10}}{\frac{6}{10}} = \frac{1}{3} \tag{4.2.62.6}$$

c)

$$\Pr(A|B) = \frac{\Pr(AB)}{\Pr(B)} = \frac{\frac{2}{10}}{\frac{4}{10}} = \frac{1}{2} \tag{4.2.62.7}$$

4.2.63 Assume that the chances of a patient having a heart attack is 40%. It is also assumed that a meditation and yoga course reduce the risk of heart attack by 30% and prescription of certain drug reduces its chances by 25%. At a time a patient can choose any one of the two options with equal probabilities. It is given that after going through one of the two options the patient selected at random suffers a heart attack. Find the probability that the patient followed a course of meditation and yoga.

Solution: The given information is summarised in Table 4.2.63.2. The given

Variable	Value	Description
X	0	Heart attack
	1	No heart attack
Y	0	Drugs
	1	Meditation and Yoga

TABLE 4.2.63.2

probabilities are

$$p_X(0) = 0.4, p_Y(0) = p_Y(1) = 0.5 \tag{4.2.63.1}$$

$$p_{X|Y}(0|1) = p_X(1 - 0.30) = 0.28 \tag{4.2.63.2}$$

$$p_{X|Y}(0|0) = p_X(1 - 0.25) = 0.30 \tag{4.2.63.3}$$

From (4.1.4.1),

$$p_{Y|X}(1|0) = \frac{p_{X|Y}(0|1)p_Y(1)}{\sum_{i=0}^{2} p_{X|Y}(0|i)p_Y(i)} = \frac{\frac{1}{2} \times 0.28}{\frac{1}{2} \times 0.28 + \frac{1}{2} \times 0.30} = \frac{14}{29} \tag{4.2.63.4}$$

which is the desired probability.

4.2.64 Suppose that 6% of the people with blood group O are left handed and 10% of those with other blood groups are left handed. 30% of the people have blood group O. If a left handed person is selected at random, what is the probability that he/she will have blood group O?

Solution: Let A represent blood group O and B represent left handedness. From the given information,

$$\Pr(A) = 0.3, \ \Pr(B|A) = 0.06, \ \Pr(B|A') = 0.1. \tag{4.2.64.1}$$

Using (4.1.4.1),

$$\Pr(A|B) = \frac{\Pr(A)\Pr(B|A)}{\Pr(A)\Pr(B|A) + \Pr(A')\Pr(B|A')} = \frac{9}{44} \tag{4.2.64.2}$$

upon substituting numerical values.

4.2.65 At a fete, cards bearing numbers 1 to 1000, one number on a card, are put in a box. Each player selects one card at random and that card is not replaced. If the selected card has a perfect square greater than 500, the player wins a prize. What is the probability that

a) the first player wins a prize
b) the second player wins a prize, if the first has won?

Solution: If n^2 is the value of the chosen number that is greater than 500 and also a perfect square, then

$$n^2 \in (500, 1000] \tag{4.2.65.1}$$
$$\implies n \in (22.36, 31.62] \tag{4.2.65.2}$$

n can take 9 integer values in the above interval. If A, B represent the first and second player winning a prize respectively,

a)

$$\Pr(A) = \frac{9}{1000} \tag{4.2.65.3}$$

b) Given that the first player has won, the second player has only 8 numbers left to choose. Hence,

$$\Pr(B|A) = \frac{8}{1000} \tag{4.2.65.4}$$

4.2.66 Four cards are successively drawn without replaccement from a deck of 52 playing cards. What is the probability that all the four cards are kings?

Solution: Let $X_i, i = 1, 2, 3, 4$ denote a king in the ith draw. Then,

$$\Pr(X_1) = \frac{4}{52},\ \Pr(X_2|X_1) = \frac{3}{51},\ \Pr(X_3|X_2X_1) = \frac{2}{50},\ \Pr(X_4|X_1X_2X_3) = \frac{1}{49}$$
$$\implies \Pr(X_1X_2X_3X_4) = \frac{4}{52} \times \frac{3}{51} \times \frac{2}{50} \times \frac{1}{49} = \frac{1}{270725} \tag{4.2.66.1}$$

which is the desired probability.

4.2.67 Two natural numbers r, s are drawn one at a time, without replacement from the set $S = 1, 2, 3, \dots, n$. Find $P[r \le p|s \le p]$.

Solution: There are two conditions,

a) s is chosen first:

$$\Pr(r \le p \mid s \le p) = \frac{\Pr(r \le p,\ s \le p)}{\Pr(s \le p)} \tag{4.2.67.1}$$

i) $p < 1$: This case is never possible as $s, r \ge 1$

ii) $1 \le p \le n$: Then we can say that,

$$\Pr(r \le p,\ s \le p) = \frac{p(p-1)}{n(n-1)}, \tag{4.2.67.2}$$
$$\Pr(s \le p) = \frac{p}{n} \tag{4.2.67.3}$$

From (4.2.67.2) and (4.2.67.3):

$$\Pr(r \le p|s \le p) = \frac{\Pr(r \le p,\ s \le p)}{\Pr(s \le p)} \tag{4.2.67.4}$$
$$= \frac{\frac{p(p-1)}{n(n-1)}}{\frac{p}{n}} = \frac{p-1}{n-1} \tag{4.2.67.5}$$

iii) $p > n$:

$$\Pr(r \le p,\ s \le p) = 1, \tag{4.2.67.6}$$
$$\Pr(s \le p) = 1 \tag{4.2.67.7}$$

From (4.2.67.6) and (4.2.67.7):

$$\Pr(r \le p | s \le p) = \frac{\Pr(r \le p,\ s \le p)}{\Pr(s \le p)} \tag{4.2.67.8}$$
$$= 1 \tag{4.2.67.9}$$

b) r is chosen first:

$$\Pr(r \le p | s \le p) = \frac{\Pr(r \le p,\ s \le p)}{\Pr(s \le p)} \tag{4.2.67.10}$$

i) $p < 1$: This case is never possible as $r, s \ge 1$

ii) $1 \le p \le n$:

$$\Pr(r \le p,\ s \le p) = \frac{p(p-1)}{n(n-1)}, \tag{4.2.67.11}$$
$$\Pr(s \le p) = \frac{p-1}{n-1} \tag{4.2.67.12}$$

From (4.2.67.11) and (4.2.67.12):

$$\Pr(r \le p | s \le p) = \frac{\Pr(r \le p,\ s \le p)}{\Pr(s \le p)} \tag{4.2.67.13}$$
$$= \frac{\frac{p(p-1)}{n(n-1)}}{\frac{p}{n}} = \frac{p}{n} \tag{4.2.67.14}$$

iii) $p > n$:

$$\Pr(r \le p,\ s \le p) = 1, \tag{4.2.67.15}$$
$$\Pr(s \le p) = 1 \tag{4.2.67.16}$$

From (4.2.67.15) and (4.2.67.16):

$$\Pr(r \le p | s \le p) = \frac{\Pr(r \le p,\ s \le p)}{\Pr(s \le p)} \tag{4.2.67.17}$$
$$= 1 \tag{4.2.67.18}$$

4.2.68 Three bags contain a number of red and white balls as follows: B_1: 3 red balls, B_2: 2 red balls and 1 white ball, B_3: 3 white balls. The probability that bag i will be chosen and a ball is selected is $i/6, i = 1, 2, 3$. what is the probability that

a) a red ball will be selected?
b) a white ball will be selected?

Solution: The r.vs are listed in Table 4.2.68.2. From the given information,

RV	Value	Description
X	1	Bag selection
	2	
	3	
Y	0	white ball
	1	red ball

TABLE 4.2.68.2: Random variable description

$$p_X(i) = \begin{cases} \frac{1}{6} & i=1 \\ \frac{2}{6} & i=2 \\ \frac{3}{6} & i=3 \end{cases}, \quad p_{Y|X}(1|i) = \begin{cases} 1 & i=1 \\ \frac{2}{3} & i=2 \\ 0 & i=3 \end{cases}, \quad p_{Y|X}(0|i) = \begin{cases} 0 & i=1 \\ \frac{1}{3} & i=2 \\ 1 & i=3 \end{cases} \tag{4.2.68.1}$$

a) The probability that a red ball will be selected is

$$p_Y(1) = \sum_{i=1}^{3} p_{Y|X}(1|i)\, p_X(i) \tag{4.2.68.2}$$

$$= \frac{1}{6} \times \frac{3}{3} + \frac{2}{6} \times \frac{2}{3} + \frac{3}{6} \times 0 = \frac{7}{18} \tag{4.2.68.3}$$

from (4.2.68.1).

b) The probability that a white ball will be selected is

$$p_Y(0) = \sum_{i=1}^{3} p_{Y|X}(0|i)\, p_X(i) \tag{4.2.68.4}$$

$$= \frac{1}{6} \times 0 + \frac{2}{6} \times \frac{1}{3} + \frac{3}{6} \times \frac{3}{3} = \frac{11}{18} \tag{4.2.68.5}$$

from (4.2.68.1).

4.2.69 Refer to Problem 4.2.68. If a white ball is selected, what is the probability that it came from

a) B_2
b) B_3

Solution:

a) The desired probability is

$$p_{X|Y}(2|0) = \frac{p_{Y|X}(0|2)\, p_X(2)}{p_Y(0)} \tag{4.2.69.1}$$

$$= \frac{\frac{1}{3} \times \frac{1}{3}}{\frac{11}{18}} = \frac{2}{11} \tag{4.2.69.2}$$

b) Similarly,

$$p_{X|Y}(3|0) = \frac{p_{Y|X}(0|3)\,p_X(3)}{p_Y(0)} \tag{4.2.69.3}$$

$$= \frac{\frac{1}{2}}{\frac{11}{18}} = \frac{9}{11} \tag{4.2.69.4}$$

4.2.70 If P(A) = $\frac{4}{5}$ and P(AB) = $\frac{7}{10}$, then $P(B|A)$ is equal to
Solution: From (4.1.1.1), the required probability is

$$\Pr(B|A) = \frac{\left(\frac{7}{10}\right)}{\left(\frac{4}{5}\right)} = \frac{7}{8} \tag{4.2.70.1}$$

4.2.71 A flashlight has 8 batteries out of which 3 are dead. If two batteries are selected without replacement and tested, find the probability that both are dead.
Solution: Let $X_i \in \{0, 1\}, i \in 1, 2$ represent the ith battery, 0 denoting the battery being dead. From the given information,

$$p_{X_1}(0) = \frac{3}{8}, p_{X_2|X_1}(0|0) = \frac{2}{7}, \tag{4.2.71.1}$$

$$\implies p_{X_1,X_2}(0,0) = p_{X_1}(0)\,p_{X_2|X_1}(0|0) = \frac{3}{28} \tag{4.2.71.2}$$

from (4.1.1.1).

4.2.72 In a college, 30% students fail in physics, 25% fail in mathematics and 10% fail in both. One student is chosen at random. The probability that she fails in physics if she has failed in mathematics is

a) $\frac{1}{10}$ b) $\frac{2}{5}$ c) $\frac{9}{20}$ d) $\frac{1}{3}$

Solution: From the given information,

$$\Pr(P) = 0.3,\ \Pr(M) = 0.1,\ \Pr(PM) = 0.25 \tag{4.2.72.1}$$

$$\implies \Pr(P|M) = \frac{\Pr(PM)}{\Pr(M)} = \frac{0.1}{0.25} = \frac{2}{5} \tag{4.2.72.2}$$

5 Uniform Distribution

5.1 Formulae

5.1.1. Let $X \in \{1,2,3,4,5,6\}$ be the random variables representing the outcome for a die. Assuming the die to be fair, the probability mass function (pmf) is expressed as

$$p_X(n) = \begin{cases} \frac{1}{6} & 1 \le n \le 6 \\ 0 & \text{otherwise} \end{cases} \tag{5.1.1.1}$$

5.1.2. The CDF of X is given by

$$F_X(n) = \Pr(X \le n) = \sum_{k=1}^{n} p_X(k) = \begin{cases} 0 & n < 1 \\ \frac{n}{6} & 1 \le n \le 6 \\ 1 & \text{otherwise} \end{cases} \tag{5.1.2.1}$$

and plotted in Fig. 5.1.2.1.

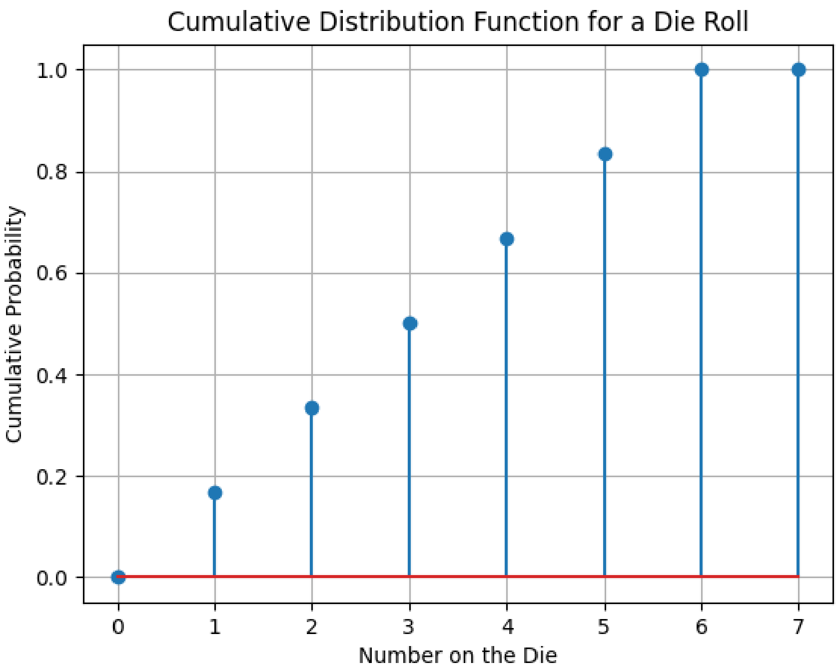


Fig. 5.1.2.1: CDF

5.2 NCERT

5.1 A die is thrown, find the probability of following events:

a) A prime number will appear

b) A number greater than or equal to 3 will appear
c) A number less than or equal to one will appear
d) A number more than 6 will appear
e) A number less than 6 will appear

Solution: The CDF of the random variable X representing the roll of a dice, is available in (5.1.2.1).

a) The set of possible prime numbers in a die roll contains 2,3,5

$$\Pr(X \in \{2,3,5\}) = p_X(2) + p_X(3) + p_X(5) \tag{5.1.1}$$
$$= \frac{1}{2} \tag{5.1.2}$$

b) The probability that a number greater than or equal to 3 will appear is given by

$$\Pr(X \geq 3) = 1 - \Pr(X \leq 2) \tag{5.1.3}$$
$$= 1 - F_X(2) \tag{5.1.4}$$
$$= \frac{2}{3} \tag{5.1.5}$$

c) The probability that a number less than or equal to 1 will appear is given by

$$\Pr(X \leq 1) = F_X(1) \tag{5.1.6}$$
$$= \frac{1}{6} \tag{5.1.7}$$

d) The probability that a number greater than 6 will appear is given by

$$\Pr(X > 6) = 1 - \Pr(X \leq 6) \tag{5.1.8}$$
$$= 1 - F_X(6) \tag{5.1.9}$$
$$= 0 \tag{5.1.10}$$

e) The probability that a number less than 6 will appear is given by

$$\Pr(X < 6) = \Pr(X \leq 5) \tag{5.1.11}$$
$$= F_X(5) \tag{5.1.12}$$
$$= \frac{5}{6} \tag{5.1.13}$$

5.2 All the jacks, queens and kings are removed from a deck of 52 playing cards. The remaining cards are well shuffled and then one card is drawn at random. Giving ace a value 1 similar value for other cards, find the probability that the card has a value

a) 7
b) greater than 7
c) less than 7

Solution: Number of cards left after removing all jacks, queens and kings

$$N = 52 - 4 \times 3 = 40 \tag{5.2.1}$$

Let $1 \leq X \leq 10$ be the value of the card picked. Then,

$$p_X(k) = \Pr(X = k) \ \forall \ 1 \leq k \leq 10 \tag{5.2.2}$$

$$= \frac{4 \times 1}{40} \tag{5.2.3}$$

$$= \frac{1}{10} \tag{5.2.4}$$

$$\therefore p_X(k) = \begin{cases} \frac{1}{10} & 1 \leq k \leq 10 \\ 0 & \text{otherwise} \end{cases} \tag{5.2.5}$$

and

$$F_X(k) = \sum_{m=0}^{k} p_X(m) \quad 1 \leq k \leq 10 \tag{5.2.6}$$

$$= \frac{k}{10} \tag{5.2.7}$$

$$\therefore F_X(k) = \begin{cases} 0 & k \leq 0 \\ \frac{k}{10} & 1 \leq k \leq 10 \\ 1 & k > 10 \end{cases} \tag{5.2.8}$$

a) Probability that card has value equal to 7 is

$$p_X(7) = \frac{1}{10} \tag{5.2.9}$$

b) Probability that card has value greater than 7 is

$$1 - F_X(7) = 1 - \frac{7}{10} \tag{5.2.10}$$

$$= \frac{3}{10} \tag{5.2.11}$$

c) Probability that card has value less than 7 is

$$F_X(6) = \frac{6}{10} \tag{5.2.12}$$

5.3 A game of chance consists of spinning an arrow which comes to rest pointing at one of the numbers 1, 2, 3, 4, 5, 6, 7, 8 (see Fig. 5.3.1), and these are equally likely outcomes. What is the probability that it will point at

a) 8?

b) an odd number?

c) a number greater than 2?

d) a number less than 9?

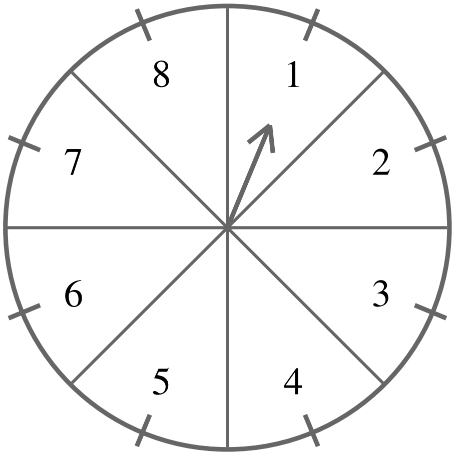


Fig. 5.3.1: Spinner

Solution: Let X be a random variable defined as the value given by the pointer. Then,

$$\Pr(X = i) = \frac{1}{8} \quad 1 \le i \le 8 \tag{5.3.1}$$

$$F_X(i) = \Pr(X \le i) \tag{5.3.2}$$

$$= \begin{cases} 0, & i \le 0 \\ \frac{i}{8} & 1 \le i \le 8 \\ 1, & i \ge 9 \end{cases} \tag{5.3.3}$$

which are plotted in Fig. 5.3.2 and Fig. 5.3.3 respectively.

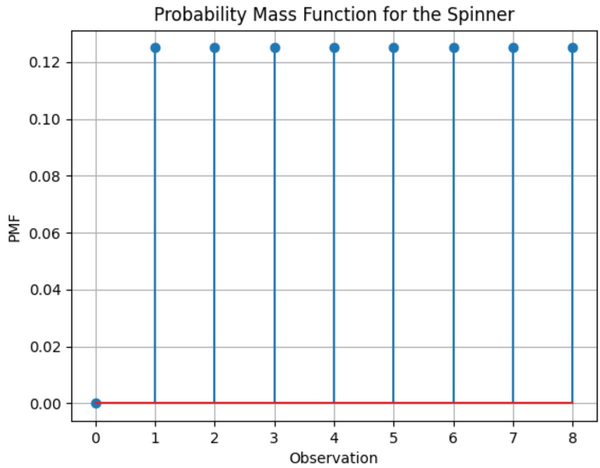


Fig. 5.3.2: Plot of Probability Mass Function

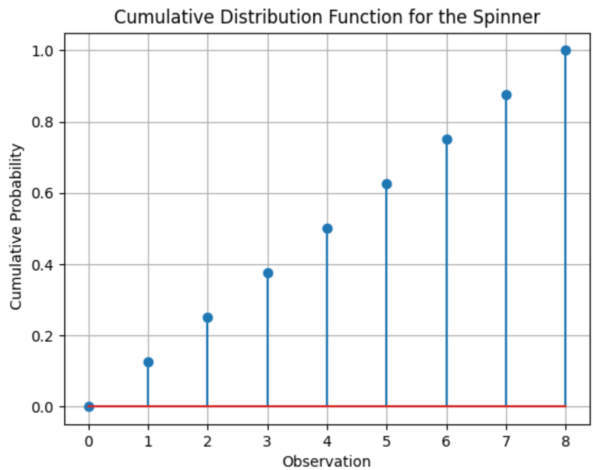


Fig. 5.3.3: Plot of Cumulative Distribution Function

a)

$$\Pr(X = 8) = \frac{1}{8} = 0.125 \tag{5.3.4}$$

b) For i being odd,

$$\Pr(X = \{1, 3, 5, 7\}) = \frac{4}{8} = 0.5 \tag{5.3.5}$$

c)

$$\Pr(X > 2) = 1 - \Pr(X \leq 2) \tag{5.3.6}$$

$$= 1 - (F_X(2) - F_X(0)) \tag{5.3.7}$$

$$= \frac{6}{8} \tag{5.3.8}$$

d)

$$\Pr(1 \leq X < 9) = F_X(8) - F_X(0) = 1 \tag{5.3.9}$$

6 Sum of Random Variables

6.1 Formulae

6.1.1 Consider the rv

$$X = X_1 + X_2, \tag{6.1.1.1}$$

where X_1 and X_2 are independent uniform rvs with pmf given in (5.1.1.1).

6.1.2 *Convolution:* From (6.1.1.1),

$$p_X(n) = \Pr(X_1 + X_2 = n) = \Pr(X_1 = n - X_2) \tag{6.1.2.1}$$

$$= \sum_k \Pr(X_1 = n - k|X_2 = k)\, p_{X_2}(k) \tag{6.1.2.2}$$

after unconditioning. $\because$ X_1 and X_2 are independent,

$$\Pr(X_1 = n - k|X_2 = k) = \Pr(X_1 = n - k) = p_{X_1}(n - k) \tag{6.1.2.3}$$

From (6.1.2.2) and (6.1.2.3),

$$p_X(n) = \sum_k p_{X_1}(n - k) p_{X_2}(k) = p_{X_1}(n) * p_{X_2}(n) \tag{6.1.2.4}$$

where $*$ denotes the convolution operation.

6.1.3 (Triangular PMF:) Substituting from (5.1.1.1) in (6.1.2.4),

$$p_X(n) = \frac{1}{6}\sum_{k=1}^{6} p_{X_1}(n - k) = \frac{1}{6}\sum_{k=n-6}^{n-1} p_{X_1}(k) \tag{6.1.3.1}$$

$$\because p_{X_1}(k) = 0, \quad k \le 1, k \ge 6. \tag{6.1.3.2}$$

From (6.1.3.1),

$$p_X(n) = \begin{cases} 0 & n < 1 \\ \frac{1}{6}\sum_{k=1}^{n-1} p_{X_1}(k) & 1 \le n - 1 \le 6 \\ \frac{1}{6}\sum_{k=n-6}^{6} p_{X_1}(k) & 1 < n - 6 \le 6 \\ 0 & n > 12 \end{cases} \tag{6.1.3.3}$$

Substituting from (5.1.1.1) in (6.1.3.3),

$$p_X(n) = \begin{cases} 0 & n < 1 \\ \frac{n-1}{36} & 2 \le n \le 7 \\ \frac{13-n}{36} & 7 < n \le 12 \\ 0 & n > 12 \end{cases} \tag{6.1.3.4}$$

6.1.4 The experiment of rolling the dice was simulated using Python for 10000 samples. These were generated using Python libraries for uniform distribution. The frequencies for each outcome were then used to compute the resulting pmf, which is plotted in Figure 6.1.4.1. The theoretical pmf obtained in (6.1.3.4) is plotted for comparison.

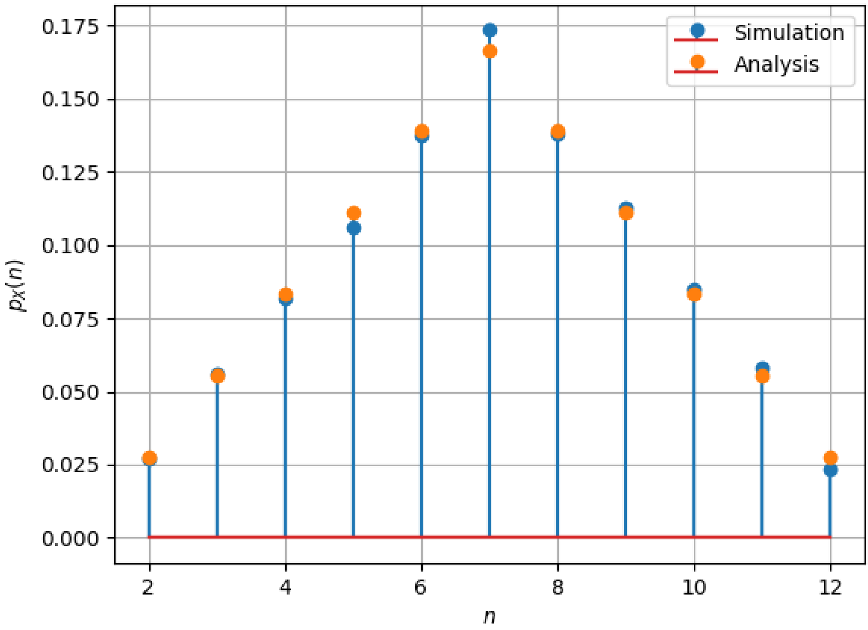


Fig. 6.1.4.1: Plot of $p_X(n)$. Simulations are close to the analysis.

6.1.5 The python code is available below

```
import numpy as np
import matplotlib.pyplot as plt
#If using termux
import subprocess
import shlex
#end if

#Sample size
simlen = 10000
#Possible outcomes
n = range(2,13)
# Generate X1 and X2
y = np.random.randint(1,7, size=(2, simlen))

#Generate X
X = np.sum(y, axis = 0)
#Find the frequency of each outcome
unique, counts = np.unique(X, return_counts=True)
```

```
#Simulated probability
psim = counts/simlen
#Theoretical probability
n1 = range(2,8)
n2 = range(8,13)
panal1 = (n1 -np.ones((1,6)))
panal2 = (13*np.ones((1,5))-n2)
panal = np.concatenate((panal1,panal2),axis=None)/36

#Plotting
plt.stem(n,psim, markerfmt='o', use_line_collection=True, label='Simulation')
plt.stem(n,panal, markerfmt='o',use_line_collection=True, label='Analysis')
plt.xlabel('$n$')
plt.ylabel('$p_{X}(n)$')
plt.legend()
plt.grid()# minor

#If using termux
plt.savefig('figs/pmf.pdf')
plt.savefig('figs/pmf.png')
subprocess.run(shlex.split("termux-open figs/pmf.pdf"))
#else
#plt.show()
```

6.1.6 The Z-transform of X is defined as

$$M_X(z) = E\left[z^{-X}\right] = \sum_{k=-\infty}^{\infty} p_X(k) z^{-k} \tag{6.1.6.1}$$

6.1.7 If X_1 and X_2 are independent, the Z-transform of

$$X = X_1 + X_2 \tag{6.1.7.1}$$

is given by

$$M_X(z) = M_{X_1}(z) M_{X_2}(z) \tag{6.1.7.2}$$

The above property follows from Fourier analysis and is fundamental to signal processing.

6.1.8 For (5.1.1.1), the Z-transform of X_1 is given by

$$M_{X_1}(z) = \frac{1}{6}\sum_{n=1}^{6} z^{-n} = \frac{z^{-1}\left(1 - z^{-6}\right)}{6\left(1 - z^{-1}\right)}, \quad |z| > 1 \tag{6.1.8.1}$$

upon summing up the geometric progression.

6.1.9 From (6.1.8.1) and (6.1.7.2),

$$M_X(z) = \left\{ \frac{z^{-1}\left(1 - z^{-6}\right)}{6\left(1 - z^{-1}\right)} \right\}^2 \tag{6.1.9.1}$$

$$= \frac{1}{36} \frac{z^{-2}\left(1 - 2z^{-6} + z^{-12}\right)}{\left(1 - z^{-1}\right)^2} \tag{6.1.9.2}$$

Using the fact that

$$p_X(n-k) \overset{\mathcal{Z}}{\longleftrightarrow} P_X(z)z^{-k}, \qquad nu(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}}{\left(1 - z^{-1}\right)^2} \tag{6.1.9.3}$$

after some algebra, it can be shown that

$$\frac{1}{36}\left[(n-1)\,u(n-1) - 2\,(n-7)\,u(n-7) + (n-13)\,u(n-13)\right] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{36} \frac{z^{-2}\left(1 - 2z^{-6} + z^{-12}\right)}{\left(1 - z^{-1}\right)^2} \tag{6.1.9.4}$$

where

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} \tag{6.1.9.5}$$

From (6.1.6.1), (6.1.9.2) and (6.1.9.4)

$$p_X(n) = \frac{1}{36}\left[(n-1)\,u(n-1) \;\; -2\,(n-7)\,u(n-7) + (n-13)\,u(n-13)\right] \tag{6.1.9.6}$$

which is the same as (6.1.3.4). Note that (6.1.3.4) can be obtained from (6.1.9.4) using contour integration as well.

6.2 NCERT

6.1 Two dice, one blue and one grey, are thrown at the same time. The event defined by the sum of the two numbers appearing on the top of the dice can have 11 possible outcomes 2, 3, 4, 5, 6, 6, 8, 9, 10, 11 and 12. A student argues that each of these outcomes has a probability $\frac{1}{11}$. Do you agree with this argument? Justify your answer.
Solution: From (6.1.3.4) and Fig. 6.1.4.1, it is obvious that

$$p_X(n) \neq \frac{1}{11}. \tag{6.1.1}$$

6.2 Two dice are numbered 1,2,3,4,5,6 and 1,1,2,2,3,3 respectively. They are thrown and the sum of then numbers on them is noted. Find the probability of getting each sum from 2 to 9 seperately
Solution: The Z-transform of the first die X_1 is given by (6.1.8.1). The pmf of the

second die is

$$p_{X_2}(n) = \begin{cases} \frac{1}{3} & 1 \le n \le 3 \\ 0 & \text{otherwise} \end{cases} \tag{6.2.1}$$

yielding

$$M_{X_2}(z) = \frac{1}{3}\sum_{n=1}^{3} z^{-n} = \frac{z^{-1}(1-z^{-3})}{3(1-z^{-1})}, |z| > 1 \tag{6.2.2}$$

upon substituting in (6.1.6.1). From (6.1.7.2), The Z-transform of X is given as

$$M_X(z) = \frac{z^{-1}(1-z^{-6})}{6(1-z^{-1})} \times \frac{z^{-1}(1-z^{-3})}{3(1-z^{-1})} \tag{6.2.3}$$

$$= \frac{1}{18}\left[\frac{z^{-2}\left((1-z^{-3}-z^{-6}-z^{-9})\right)}{(1-z^{-1})^2}\right] \tag{6.2.4}$$

Using (6.1.9.3), after some algebra, it can be shown that,

$$\frac{1}{18}[n-1u(n-1)-n-4u(n-4)-(n-7)u(n-7)-(n-10)u(n-10)]$$
$$\overset{Z}{\longleftrightarrow}$$
$$\frac{1}{18}\left[\frac{z^{-2}1-z^{-3}-z^{-6}-z^{-9}}{(1-z^{-1})^2}\right] \tag{6.2.5}$$

Hence,

$$p_X(n) = \begin{cases} 0 & n \le 1 \\ \frac{n-1}{18} & 2 \le n \le 4 \\ \frac{1}{6} & 5 \le n \le 7 \\ \frac{10-n}{18} & 8 \le n \le 9 \\ 0 & n \ge 10 \end{cases} \tag{6.2.6}$$

See Fig. 6.2.1. The experiment of rolling the dice was simulated using Python for 10000 samples.

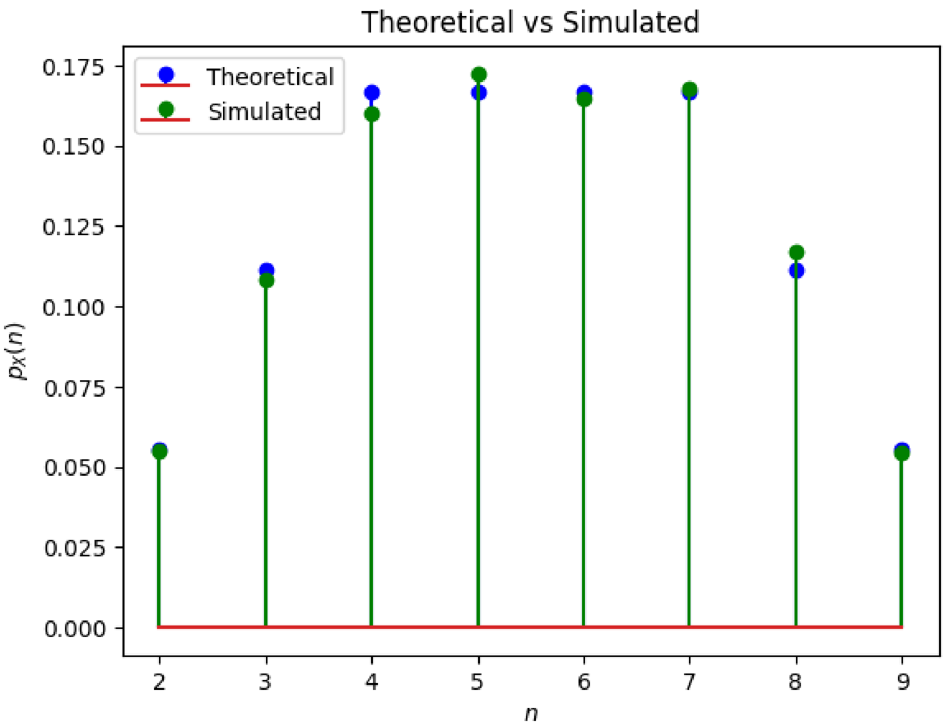


Fig. 6.2.1: Plot of $p_X(n)$.Simulations are close to the analysis.

6.3 A die is tossed thrice. Find the probability of getting an odd number at least once.

7 Binomial

7.1 Formulae

7.1.1. The Binomial distribution is defined as

$$X = X_1 + X_2 + \cdots + X_n, \tag{7.1.1.1}$$

Where X_i are i.i.d bernoulli.

7.1.2. For $X_i \sim \text{Ber}(p)$,

$$M_{X_i}(z) = q + pz^{-1}, \quad q = 1 - p \tag{7.1.2.1}$$

upon substituting from (3.1.1.2) in (6.1.6.1).

7.1.3. For $X \sim \text{Binom}(n, p)$,

$$M_X(z) = \left(q + pz^{-1}\right)^n \tag{7.1.3.1}$$

using (7.1.1.1) and (6.1.7.2).

7.1.4. Expanding (7.1.3.1),

$$M_X(z) = \sum_{k=0}^{n} {}^nC_k (1-p)^{n-k} p^k z^{-k} \tag{7.1.4.1}$$

$$\implies p_X(k) = \begin{cases} {}^nC_k (1-p)^{n-k} p^k & 0 \le k \le n \\ 0 & \text{otherwise} \end{cases} \tag{7.1.4.2}$$

upon comparing with (6.1.6.1).

7.1.5. The CDF of X is

$$\Pr(X \le k) = F_X(k) = \sum_{r=0}^{k} (1-p)^{n-r} p^r 0 \le k \le n \tag{7.1.5.1}$$

7.1.6. In a hurdle race, a player has to cross 10 hurdles. The probability that he will clear each hurdle is $\frac{5}{6}$. What is the probability that he will knock down fewer than 2 hurdles?

Solution: See the following code

```
#Code by GVV Sharma
#November 20,2020
#Released under GNU/GPL
#To find the probability of an event using the binomial distribution

import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import bernoulli
from scipy.stats import norm
from scipy.stats import binom
```

```
#Simlen
simlen=1000

#Number of hurdles
n = 10

#Probability of clearing a hurdle
p = 1-5/6

#Mean
mu = p

#Variance
sigma = np.sqrt(p*(1-p))

#Theoretical probability of knocking down fewer than 2 hurdles
k = 1
print(binom.cdf(k, n, p),3*(5/6)**10)

#Using the Gaussian approximation for the binomial pdf
print(1/(sigma*np.sqrt(n))*(norm.pdf((k-n*mu)/(sigma*np.sqrt(n)))+norm.pdf((k
    -1-n*mu)/(sigma*np.sqrt(n)))))


#Simulating the probability using the binomial random variable
data_binom = binom.rvs(n,p,size=simlen) #Simulating the event of jumping 10
    hurdles
err_ind = np.nonzero(data_binom <=k) #checking probability condition
err_n = np.size(err_ind) #computing the probability
print(err_n/simlen)
#print(data_binom)


#Simulating the probability using the bernoulli random variable
data_bern_mat = bernoulli.rvs(p,size=(n,simlen))
data_binom=np.sum(data_bern_mat, axis=0)
#print(data_bern_mat)
#print(data_binom)
err_ind = np.nonzero(data_binom <=k) #checking probability condition
err_n = np.size(err_ind) #computing the probability
print(err_n/simlen)
```

7.2 NCERT

7.2.1 A die is tossed thrice. Find the probability of getting an odd number at least once.
Solution:

7.2.2 A die is thrown twice. What is the probability that

a) 5 will not come up either time?
b) 5 will come up at least once?

Solution: Since $X \sim \text{Binom}\left(\frac{1}{2}\right)$, the desired probabilites are

a)

$$\Pr(X = 0) = {}^{2}C_0 \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^2 = \frac{25}{36} \tag{7.2.2.1}$$

b)

$$\Pr(X \geq 1) = 1 - \Pr(X \leq 0) = 1 - F_X(0) \tag{7.2.2.2}$$

$$= 1 - \frac{25}{36} = \frac{11}{36} \tag{7.2.2.3}$$

using (7.1.5.1).

7.2.3 Three coins are tossed once. Find the probability of getting

a) 3 heads
b) 2 heads
c) atleast 2 heads
d) atmost 2 heads
e) no head
f) 3 tails
g) exactly two tails
h) no tail
i) atmost two tails

Solution: The rv representing the given event is

$$Y \sim \text{Binom}\left(3, \frac{1}{2}\right) \tag{7.2.3.1}$$

a)

$$p_Y(3) = \frac{1}{8} \tag{7.2.3.2}$$

b)

$$p_Y(2) = \frac{3}{8} \tag{7.2.3.3}$$

c)

$$\Pr(Y \geq 2) = 1 - \Pr(Y < 2) \tag{7.2.3.4}$$

$$= F_Y(3) - F_Y(1) \tag{7.2.3.5}$$

$$= \frac{1}{2} \tag{7.2.3.6}$$

d)

$$\Pr(Y \le 2) = \sum_{k=0}^{2} \binom{n}{k} p^k (1-p)^{n-k} \tag{7.2.3.7}$$
$$= \frac{7}{8} \tag{7.2.3.8}$$

e)

$$p_X(0) = {}^3C_0 (0.5)^3 (0.5)^0 \tag{7.2.3.9}$$
$$= \frac{1}{8} \tag{7.2.3.10}$$

f)

$$p_Y(1) = \binom{n}{1} p^1 (1-p)^{n-1} \tag{7.2.3.11}$$
$$= \frac{3}{8} \tag{7.2.3.12}$$

g) $p_Y(3) = \frac{1}{8}$ from (7.2.3.2).

h)

$$\Pr(Y \ge 1) = 1 - \Pr(Y < 1) \tag{7.2.3.13}$$
$$= 1 - F_Y(0) \tag{7.2.3.14}$$
$$= \frac{7}{8} \tag{7.2.3.15}$$

7.2.4 A game consists of tossing a one rupee coin 3 times and noting its outcome each time. Hans wins if all the tosses give the same result i.e., three heads or three tails, and loses otherwise. Calculate the probability that Hans will lose the game.

Solution: Since $X \sim \mathrm{B}\left(3, \frac{1}{2}\right)$, the desired probability is

$$\Pr(X = 1) + \Pr(X = 2) = {}^3C_1 \left(\frac{1}{2}\right)^3 + {}^3C_2 \left(\frac{1}{2}\right)^3 \tag{7.2.4.1}$$
$$= \left(\frac{3}{4}\right) \tag{7.2.4.2}$$

7.2.5 A coin is tossed three times. Determine $\Pr(E|F)$ where

a) E : head on third toss, F : heads on first two tosses

b) E : at least two heads, F : at most two heads

c) E : at most two tails, F : at least one tail

Solution: If $X_i \sim \mathrm{B}\left(\frac{1}{2}\right), i = 1, 2, 3$ with 0 denoting head,

a) The events E, F can be described as

$$E : X_3 = 0 \tag{7.2.5.1}$$
$$F : X_1 + X_2 = 0 \tag{7.2.5.2}$$

Therefore,

$$\Pr(E|F) = \frac{\Pr(EF)}{\Pr(F)} = \frac{\Pr(X_3 = 0, X_1 + X_2 = 0)}{\Pr(X_1 + X_2 = 0)} \tag{7.2.5.3}$$

$$= \frac{\Pr(X_1 + X_2 + X_3 = 0)}{\Pr(X_1 + X_2 = 0)} = \frac{{}^3C_0\left(\frac{1}{2}\right)^3}{{}^2C_0\left(\frac{1}{2}\right)^2} = \frac{1}{2} \tag{7.2.5.4}$$

b) The desired probability can be expressed as

$$\frac{\Pr(EF)}{1 - \Pr(F')} = \frac{\Pr(X = 1)}{1 - \Pr(X = 0)} \tag{7.2.5.5}$$

$$= \frac{\frac{3}{8}}{1 - \frac{1}{8}} = \frac{3}{7} \tag{7.2.5.6}$$

where

$$X = X_1 + X_2 + X_3. \tag{7.2.5.7}$$

c) In this case, the desired probability is

$$\frac{\Pr(X \le 2, X \ge 1)}{\Pr(X \ge 1)} = \frac{\Pr(1 \le X \le 2)}{\Pr(X \ge 1)} = \frac{6}{7} \tag{7.2.5.8}$$

7.2.6 Find the probability distribution of

a) number of heads in two tosses of a coin.
b) number of tails in the simultaneous tosses of three coins.
c) number of heads in four tosses of a coin.

Solution: The desired probabilities are

a) $X \sim \mathrm{B}\left(2, \frac{1}{2}\right)$

$$p_X(2) = {}^2C_2 \frac{1}{2}^2 = \frac{1}{4} \tag{7.2.6.1}$$

b) $X \sim \mathrm{B}\left(3, \frac{1}{2}\right)$

$$p_X(3) = {}^2C_0 \frac{1}{2}^3 = \frac{1}{8} \tag{7.2.6.2}$$

c) $X \sim \mathrm{B}\left(4, \frac{1}{2}\right)$

$$p_X(4) = {}^4C_4 \frac{1}{2}^4 = \frac{1}{16} \tag{7.2.6.3}$$

7.2.7 Find the probability distribution of the number of successes in two tosses of a die, where a success is defined as

a) number greater than 4
b) six appears on at least one die

Solution:

a) From (5.1.2.1),

$$\Pr(X > 4) = 1 - F_X(3) = \frac{1}{3} \tag{7.2.7.1}$$

The distribution is then given by $Y \sim \mathrm{B}\left(2, \frac{1}{3}\right)$

b) In this case, from (5.1.1.1), $p_X(6) = \frac{1}{6}$ yielding the distribution $Y \sim \mathrm{B}\left(2, \frac{1}{6}\right)$

7.2.8 There are 5% defective items in a large bulk of items. What is the probability that a sample of 10 items will include not more than one defective item?
Solution: Since $X \sim \mathrm{B}(10, 0.5)$, the desired probability is $F_X(1)$.

7.2.9 Five cards are drawn successively with replacement from a well-shuffled deck of 52 cards. What is the probability that

a) all the five cards are spades?
b) only 3 cards are spades?
c) none is a spade?

Solution: The probability of getting spade on any draw is

$$p = \frac{13}{52} = \frac{1}{4} \tag{7.2.9.1}$$

The given distribution is $X \sim \mathrm{B}\left(5, \frac{1}{4}\right)$. The desired probabilities are

a) $p_X(5)$
b) $p_X(3)$
c) $p_X(0)$

7.2.10 The probability that a bulb produced by a factory will fuse after 150 days of use is 0.05. Find the probability that out of 5 such bulbs

a) none
b) not more than one
c) more than one
d) at least one

will fuse after 150 days of use.
Solution: The given distribution can be expressed as $X \sim \mathrm{B}(5, 0.05)$. The desired probabilities are

a) $p_X(0)$
b) $F_X(1)$
c) $\Pr(X > 1) = 1 - F_X(2)$
d) $\Pr(X \geq 1) = 1 - F_X(0)$

7.2.11 A bag consists of 10 balls each marked with one of the digits 0 to 9. If 4 balls are drawn successively with replacement from the bag, what is the probability that none is marked with the digit 0?
Solution: The probability that a ball is marked 0 is the Bernoulli parameter

$$p = \frac{1}{10} \tag{7.2.11.1}$$

In the given problem, the relevant Binomial distribution is $X \sim \mathrm{B}\left(4, \frac{1}{10}\right)$. The desired probability is then given by

$$1 - p_X(4) \tag{7.2.11.2}$$

7.2.12 How many times must a man toss a fair coin so that the probability of having at least one head is more than 90%?

Solution: If n be the number of coin tosses, the distribution is $X \sim \mathrm{B}\left(n, \frac{1}{2}\right)$, where X represents the number of heads. The given probability is

$$\Pr(X \geq 1) > 0.9 \tag{7.2.12.1}$$
$$\implies 1 - p_X(0) > 0.9 \tag{7.2.12.2}$$
$$\text{or, } (2)^n > 10 \tag{7.2.12.3}$$
$$\therefore n > \log_2(10) \tag{7.2.12.4}$$
$$\implies n > 3.32 \implies n = 4, \tag{7.2.12.5}$$

Since n is a positive integer.

7.2.13 In an examination, 20 questions of true-false type are asked. Suppose a student tosses a fair coin to determine his answer to each question. If the coin falls heads, he answer true; if it falls tails, he answer false. Find the probability that he answers at least 12 questions correctly.

Solution: Here, $X \sim \mathrm{B}\left(20, \frac{1}{2}\right)$. Therefore, the desired probability is given by

$$\Pr(X >= 12) = 1 - F_X(11) = 0.2517 \tag{7.2.13.1}$$

7.2.14 Find the probability of getting 5 twice in 7 throws of a dice.

Solution: The Binomial distribution here is $X \sim \mathrm{B}\left(7, \frac{1}{6}\right)$. $\therefore$ the desired probability is $p_X(2)$.

7.2.15 On a multiple choice examination with three possible answers for each of the five questions, what is the probability that a candidate would get four or more correct answers just by guessing?

Solution: The relevant distribution here is $X \sim \mathrm{B}\left(5, \frac{1}{3}\right)$ The desired probability is

$$\Pr(X \geq 4) = 1 - F_X(3) = \frac{11}{243}. \tag{7.2.15.1}$$

7.2.16 Find the probability of throwing at most 2 sixes in 6 throws of a single die.

Solution: The given distribution is $X \sim \mathrm{B}\left(6, \frac{1}{6}\right)$ and the desired probability is

$$F_X(2) = \frac{21875}{23328} \tag{7.2.16.1}$$

7.2.17 Suppose that 90 % of people are right-handed. What is the probability that atmost 6 of a random sample of 10 people are right-handed.

Solution: The given distribution is $X \sim \mathrm{B}\left(10, \frac{9}{10}\right)$ and the desired probability is $F_X(6)$.

7.2.18 An urn contains 25 balls of which 10 balls bear a mark 'X' and the remaining 15 bear a mark 'Y'. A ball is drawn at random from the urn, its mark is noted down and it is replaced. If 6 balls are drawn in this way, find the probability that

a) all will bear 'X' mark.
b) not more than 2 will bear 'Y' mark.
c) at least one ball will bear 'Y' mark.
d) the number of balls with 'X' mark and 'Y' mark will be equal.

Solution: The given distribution is $X \sim \mathrm{B}\left(6, \frac{2}{5}\right)$. The desired probabilities are

a) $p_X(6)$
b) $\Pr(X \geq 4) = 1 - F_X(3)$
c) $\Pr(X < 6) = F_X(5)$
d) $p_X(3)$

7.2.19 An urn contains 5 red and 2 black balls. Two balls are randomly drawn. Let X represent the number of black balls. What are the possible values of X? Is X a random variable ?

7.2.20 From a lot of 30 bulbs which include 6 defectives, a sample of 4 bulbs is drawn at random with replacement. Find the probability distribution of the number of defective bulbs.

7.2.21 A coin is biased so that the head is 3 times as likely to occur as tail. If the coin is tossed twice, find the probability distribution of number of tails.

7.2.22 A coin is tossed twice, what is the probability that atleast one tail occurs?
Solution: Here, $X \sim \mathrm{B}\left(2, \frac{1}{2}\right)$, where X denotes the number of heads. The desired probability is $\Pr(X \geq 1) = 1 - p_X(0) = \frac{3}{4}$

7.2.23 Suppose X is a binomial distribution $B\left(6, \frac{1}{2}\right)$. Show that $X = 3$ is the most likely outcome. (Hint : $P(X = 3)$ is the maximum among all $P(x_i), x_i = 0, 1, 2, 3, 4, 5, 6$)
Solution: From the given information,

$$p_X(k) = {}^nC_k p^k (1-p)^{n-k}, \quad n = 6, p = \frac{1}{2}. \tag{7.2.23.1}$$

yielding

$$p_X(k) = {}^nC_k \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{n-k} \tag{7.2.23.2}$$

$$= {}^nC_k \left(\frac{1}{2}\right)^n \tag{7.2.23.3}$$

upon substituting for p. For $p_X(k)$ to be maximum,

$${}^nC_k \geq {}^nC_{k-1} \quad \text{and} \tag{7.2.23.4}$$

$${}^nC_k \geq {}^nC_{k+1} \tag{7.2.23.5}$$

$$\because {}^nC_k = \frac{n!}{(n-k)!k!}, \tag{7.2.23.6}$$

from (7.2.23.6) and (7.2.23.4),

$$\frac{n!}{(n-k)!k!} \geq \frac{n!}{(n-k+1)!(k-1)!} \tag{7.2.23.7}$$

$$\implies \frac{n!}{(n-k)!k!} \geq \frac{n!}{(n-k)!k!}\frac{k}{n-k+1} \tag{7.2.23.8}$$

$$\implies 1 \geq \frac{k}{n-k+1} \tag{7.2.23.9}$$

$$\therefore k \leq \frac{n+1}{2} \tag{7.2.23.10}$$

From (7.2.23.6) and (7.2.23.5),

$$\frac{n!}{(n-k)!k!} \geq \frac{n!}{(n-k-1)!(k+1)!} \tag{7.2.23.11}$$

$$\implies \frac{n!}{(n-k)!k!} \geq \frac{n!}{(n-k)!k!}\frac{n-k}{k+1} \tag{7.2.23.12}$$

$$\implies 1 \geq \frac{n-k}{k+1} \tag{7.2.23.13}$$

$$\therefore k \geq \frac{n-1}{2} \tag{7.2.23.14}$$

Thus, from (7.2.23.10) and (7.2.23.14),

$$\frac{n-1}{2} \leq k \leq \frac{n+1}{2} \tag{7.2.23.15}$$

$$\implies k = \begin{cases} \frac{n}{2}, & n \text{ even} \\ \frac{n+1}{2} \text{ or } \frac{n-1}{2}, & n \text{ odd} \end{cases} \tag{7.2.23.16}$$

Since

$$n = 6, k = \frac{n}{2} = 3 \tag{7.2.23.17}$$

See Fig. 7.2.23.1.

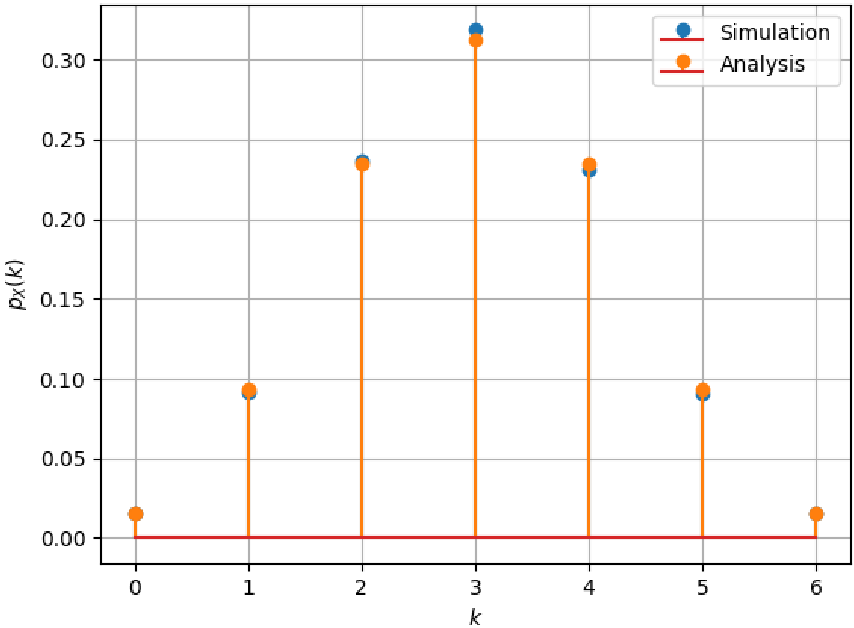


Fig. 7.2.23.1

7.2.24 A fair coin is tossed four times, and a person win Re 1 for each head and lose Rs 1.5 for each tail that turns up. From the sample space calculate how many different amounts of money you can have after four tosses and the probability of having each of these amounts.

Solution: If $X \sim \mathrm{B}\left(4, \frac{1}{2}\right)$ denotes a head, the amount of money the person will have after n tosses is

$$A = (X \times 1) - ((n - X) \times 1.5) \tag{7.2.24.1}$$

$$= 2.5X - 1.5n \tag{7.2.24.2}$$

Therefore,

$$p_A(k) = \Pr(A = k) = \Pr(2.5X - 1.5n = k) \tag{7.2.24.3}$$

$$= \Pr\left(X = \frac{k + 1.5n}{2.5}\right) = p_X\left(\frac{k + 6}{2.5}\right) \tag{7.2.24.4}$$

See Fig. 7.2.24.1.

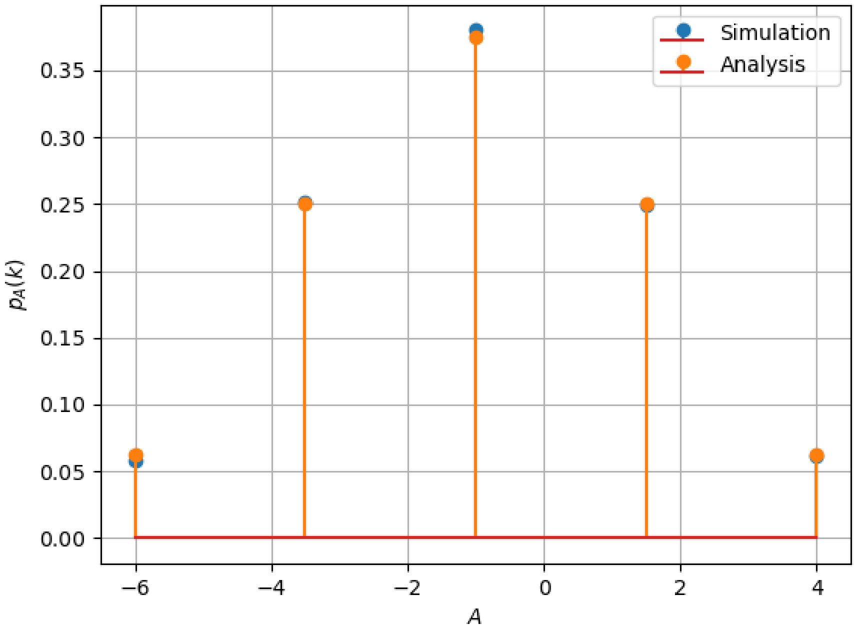


Fig. 7.2.24.1: PMF of A

7.2.25 It is known that 10 % of certain articles manufactured are defective. What is the probability that in a random sample space of 12 such articles,9 are defective?
Solution: In this case, $X \sim \text{B}\left(12, \frac{1}{10}\right)$, where X represents a defective item. The desired probability is $p_X(9)$.

7.2.26 A coin is tossed two times. Find the probability of getting at most one head.
Solution: In this case, the distribution is $X \sim \text{B}\left(2, \frac{1}{2}\right)$, where X represents a head. The desired probability is $F_X(1) = \frac{3}{4}$

7.2.27 An experiment succeeds twice as often as it fails. Find the probability that in the next six trials, there will be atleast 4 successes.
Solution: The success distribution is given by $X \sim \text{B}\left(6, \frac{2}{3}\right)$. Thus, the desired probability is $\Pr\left(X \geq 4\right) = 1 - F_X(3)$

7.2.28 A die is thrown 5 times. Find the probability that an odd number will come up exactly three times.
Solution: The odd number proability is $p = \frac{1}{2}$. $\therefore X \sim \text{B}\left(5, \frac{1}{2}\right)$. Thus, the desired probability is $p_Y\left(3\right)$.

7.2.29 A coin is tossed 3 times. List the possible outcomes. Find the probability of getting (i) all heads (ii) at least 2 heads
Solution: In this case, $X \sim \text{B}\left(3, \frac{1}{2}\right)$.

a) To get all heads, the probability is $p_X\left(3\right)$.

b) To get atleast 2 heads, the desired probability is $\Pr(Z \geq 2) = F_Z(1)$

7.2.30 Ten coins are tossed. What is the probability of getting atleast 8 heads?

Solution: Here, $X \sim \mathrm{B}\left(10, \frac{1}{2}\right)$. The desired probability is

$$\Pr(X \geq 8) = F_X(10) - F_X(7) = \frac{7}{128} \tag{7.2.30.1}$$

7.2.31 A factory produces bulbs. The probability that any one bulb is defective is $\frac{1}{50}$ and they are packed in boxes of 10. From a single box, find the probability that

a) none of the bulb is defective
b) exactly two bulbs are defective
c) more than 8 bulbs are working properly

Solution: The defective bulb distribution is $X \sim \mathrm{B}\left(10, \frac{1}{50}\right)$. The desired probabilities are

a) $p_X(0)$
b) $p_X(2)$
c) $\Pr(X \leq 1) = F_X(1)$

7.2.32 A lot of 100 watches is known to have 10 defective watches. If 8 watches are selected (one by one with replacement) at random, what is the probability that there will be at least one defective watch?

Solution: The defective distribution is $X \sim \mathrm{B}\left(8, \frac{10}{100}\right)$. The desired probability is $F_Y(8) - F_Y(0)$.

8 Moments

8.1 Formulae

8.1.1 The mean of an rv is defined as

$$\mu = E(X) = \sum_k k p_X(k) \tag{8.1.1.1}$$

8.1.2 The variance of an rv is defined as

$$\sigma^2 = E(X-\mu)^2 = \sum_k (k-\mu)^2 \, p_X(k) \tag{8.1.2.1}$$

8.1.3 In general, the nth moment is defined as

$$E(X)^n = \sum_k k^n p_X(k) \tag{8.1.3.1}$$

8.1.4

$$E(X)^2 = \sigma^2 + \mu^2 \tag{8.1.4.1}$$

Solution: (8.1.2.1) can be expressed as

$$E[X - E(X)]^2 = E\left[X^2 + [E(X)]^2 - 2XE(X)\right] \tag{8.1.4.2}$$

$$= E\left(X^2\right) + [E(X)]^2 - 2[E(X)]^2 \tag{8.1.4.3}$$

$$= E\left(X^2\right) - [E(X)]^2 \tag{8.1.4.4}$$

yielding (8.1.4.1).

8.1.5

$$\text{var}(aX) = a^2 \text{var}(X). \tag{8.1.5.1}$$

Solution:

$$\text{var}(aX) = E\left(a^2X^2\right) - [E(aX)]^2 \tag{8.1.5.2}$$

$$= a^2\left[E\left(X^2\right) - [E(X)]^2\right] \tag{8.1.5.3}$$

$$\tag{8.1.5.4}$$

yielding (8.1.5.1).

8.1.6 For $X \sim \text{B}(n, p)$,

$$\mu = np, \sigma^2 = np(1-p) \tag{8.1.6.1}$$

8.2 NCERT

8.2.1 Find the mean number of heads in three tosses of a fair coin.

Solution: The given distribution is $\text{B}\left(3, \frac{1}{2}\right)$. Substituting $n = 3, p = \frac{1}{2}$ in (8.1.6.1), $\mu = \frac{3}{2}$.

8.2.2 Two dice are thrown simultaneously. If X denotes the number of sixes, find the expectation of X.

8.2.3 Two numbers are selected at random (without replacement) from the first six positive integers. Let X denote the larger of the two numbers obtained. Find $E(X)$.

8.2.4 Let X denote the sum of the numbers obtained when two fair dice are rolled. Find the variance and standard deviation of X.

8.2.5 A class has 15 students whose ages are 14, 17, 15, 14, 21, 17, 19, 20, 16, 18, 20, 17, 16, 19 and 20 years. One student is selected in such a manner that each has the same chance of being chosen and the age X of the selected student is recorded. What is the probability distribution of the random variable X? Find mean, variance and standard deviation of X.

8.2.6 In a meeting, 70% of the members favour and 30% oppose a certain proposal. A member is selected at random and we take $X = 0$ if he opposed, and $X = 1$ if he is in favour. Find $E(X)$ and $Var(X)$.

8.2.7 In a dice game, a player pays a stake of Re 1 for each throw of a die. She receives Rs 5 if the die shows a 3, Rs 2 if the die shows a 1 or 6, and nothing otherwise. What is the player's expected profit per throw over a long series of throws?.

Solution: Let the random variable X denote the net profit on the roll of a die.

RV	Values	Event
X	-1	$\{2, 4, 5\}$
	1	$\{1, 6\}$
	4	$\{3\}$

TABLE 8.2.7.1: Net gain

Thus, the pmf of X is

$$p_X(k) = \begin{cases} \frac{3}{6} & k = -1 \\ \frac{2}{6} & k = 1 \\ \frac{1}{6} & k = 4 \end{cases} \tag{8.2.7.1}$$

The expected profit is

$$E(X) = \sum_k k p_X(k) \tag{8.2.7.2}$$

$$= \frac{3}{6}(-1) + \frac{2}{6}(1) + \frac{1}{6}(4) \tag{8.2.7.3}$$

$$= \frac{1}{2} \tag{8.2.7.4}$$

8.2.8 A die is thrown three times. Let X be 'the number of twos seen'. Find the expectation of X.

Solution: The probability of a two appearing in one throw of the die is

$$p = \frac{1}{6} \tag{8.2.8.1}$$

Therefore, $X \sim B\left(3, \frac{1}{6}\right)$. From (8.1.6.1),

$$E(X) = 3 \times \frac{1}{6} = \frac{1}{2} \tag{8.2.8.2}$$

Choose the correct answer in each of the following:

8.2.9 The mean of the numbers obtained on throwing a die having written 1 on three faces, 2 on two faces and 5 on one face is

a) 1 b) 2 c) 5 d) $\frac{8}{3}$

8.2.10 Suppose that two cards are drawn at random from a deck of cards. Let X be the number of aces obtained. Then the value of E(X) is

a) $\frac{37}{221}$ b) $\frac{5}{13}$ c) $\frac{1}{13}$ d) $\frac{2}{13}$

8.2.11 Suppose 10000 tickets are sold in a lottery each for Re. 1. First prize is of Rs 3000 and the second prize is of Rs 2000. There are three third prizes of Rs. 500 each. If you buy one ticket, what is your expectation ?

Solution: The given information is summarized in Table 8.2.11.1.

RV	Value	Description
X	0	Winning no amount
	500	Winning Rs 500
	2000	Winning Rs 2000
	3000	Winning Rs 3000

TABLE 8.2.11.1: Random variable declaration.

The corresponding pmf is

$$p_X(k) = \begin{cases} \frac{9995}{10000} & k = 0 \\ \frac{3}{10000} & k = 500 \\ \frac{1}{10000} & k = 2000 \\ \frac{1}{10000} & k = 3000 \end{cases} \tag{8.2.11.1}$$

From (8.1.1.1),

$$E(X) = 0p_X(0) + 500p_X(500) + 2000p_X(2000) + 3000p_X(3000) \tag{8.2.11.2}$$

$$= 0 + \frac{3}{20} + \frac{1}{5} + \frac{3}{10} \tag{8.2.11.3}$$

$$= 0.65 \tag{8.2.11.4}$$

8.2.12 Consider the probability distribution of a random variable X given in Table 8.2.12.1. Calculate

a) $\text{var}(X)$

b) $\text{var}\left(\frac{X}{2}\right)$

X	0	1	2	3	4
$P(X)$	0.1	0.25	0.3	0.2	0.15

TABLE 8.2.12.1

Solution:

a) From (8.1.1.1) and (8.1.3.1)

$$E(X) = \sum_{k=0}^{4} k p_X(k) = 2.05 \tag{8.2.12.1}$$

and

$$E\left(X^2\right) = \sum_{k=0}^{4} k^2 p_X(k) = 5.65 \tag{8.2.12.2}$$

Substituting from the above in (8.1.4.1),

$$\text{var}(X) = 5.65 - (2.05)^2 = 1.4475 \tag{8.2.12.3}$$

b) From (8.1.5.1),

$$\text{var}\left(\frac{X}{2}\right) = \frac{\text{var}(X)}{4} = 0.361875 \tag{8.2.12.4}$$

8.2.13 The random variable X can take only the values $0, 1, 2$. Given that $\Pr(X = 0) = \Pr(X = 1) = p$ and that $E\left(X^2\right) = E(X)$, find the value of p.

Solution: From the given information,

$$p_X(0) = p_X(1) = p, p_X(2) = 1 - 2p. \tag{8.2.13.1}$$

Then,

$$E(X) = \sum_{k=0}^{2} k p_X(k) \tag{8.2.13.2}$$

$$= 0p_X(0) + 1p_X(1) + 2p_X(2) \tag{8.2.13.3}$$

$$= 2 - 3p \tag{8.2.13.4}$$

and

$$E\left(X^2\right) = \sum_{k=0}^{2} k^2 p_X(k) \tag{8.2.13.5}$$

$$= 0p_X(0) + 1p_X(1) + 4p_X(2) \tag{8.2.13.6}$$

$$= 4 - 7p \tag{8.2.13.7}$$

Since

$$E(X) = E\left(X^2\right), \tag{8.2.13.8}$$

from (8.2.13.4) and (8.2.13.7)

$$2 - 3p = 4 - 7p \tag{8.2.13.9}$$

$$\implies p = \frac{1}{2} \tag{8.2.13.10}$$

8.2.14 Find the variance of distribution in Table 8.2.14.1

X	0	1	2	3	4	5
$P(X)$	$\frac{1}{6}$	$\frac{5}{18}$	$\frac{2}{9}$	$\frac{1}{6}$	$\frac{1}{9}$	$\frac{1}{18}$

TABLE 8.2.14.1

Solution:

$$E(X) = \sum_{k=0}^{5} k p_X(k) \tag{8.2.14.1}$$

$$= 0\left(\frac{1}{6}\right) + 1\left(\frac{5}{18}\right) + 2\left(\frac{2}{9}\right) + 3\left(\frac{1}{6}\right) + 4\left(\frac{1}{9}\right) + 5\left(\frac{1}{18}\right) \tag{8.2.14.2}$$

$$= \frac{35}{18} \tag{8.2.14.3}$$

$$E(X^2) = \sum_{k=0}^{5} k^2 p_X(k) \tag{8.2.14.4}$$

$$= 0^2\left(\frac{1}{6}\right) + 1^2\left(\frac{5}{18}\right) + 2^2\left(\frac{2}{9}\right) + 3^2\left(\frac{1}{6}\right) + 4^2\left(\frac{1}{9}\right) + 5^2\left(\frac{1}{18}\right) \tag{8.2.14.5}$$

$$= \frac{105}{18} \tag{8.2.14.6}$$

From (8.2.14.3) and (8.2.14.6).

$$\sigma^2 = E(X^2) - [E(X)]^2 \tag{8.2.14.7}$$

$$= \frac{105}{18} - \left(\frac{35}{18}\right)^2 \tag{8.2.14.8}$$

$$= \frac{665}{324} \tag{8.2.14.9}$$

8.2.15 Two cards are drawn successively without replacement from a well shuffled deck of cards. Find the mean and standard variation of random variable X where X is the number of aces.

Solution: See Table 8.2.15.1

RV	Value	Description
X	0	Drawing no ace
	1	Drawing only 1 ace
	2	Drawing both aces

TABLE 8.2.15.1

The pmf of X is given by

$$p_X(0) = \frac{48}{52} \times \frac{47}{51} = \frac{188}{221} \tag{8.2.15.1}$$

$$p_X(1) = \frac{4}{52} \times \frac{48}{51} + \frac{48}{52} \times \frac{4}{51} = \frac{32}{221} \tag{8.2.15.2}$$

$$p_X(2) = \frac{4}{52} \times \frac{3}{51} = \frac{1}{221} \tag{8.2.15.3}$$

Therefore,

$$\mu = E(X) = \sum_{k=0}^{2} k p_X(k) \tag{8.2.15.4}$$

$$= 0p_X(0) + 1p_X(1) + 2p_X(2) \tag{8.2.15.5}$$

$$= \frac{34}{221} = \frac{2}{13} \tag{8.2.15.6}$$

and

$$E\left(X^2\right) = \sum_{k=0}^{2} k^2 p_X(k) \tag{8.2.15.7}$$

$$= 0p_X(0) + 1p_X(1) + 4p_X(2) \tag{8.2.15.8}$$

$$= \frac{36}{221} \tag{8.2.15.9}$$

Now,

$$Var(X) = E\left(X^2\right) - (E(X))^2 \tag{8.2.15.10}$$

Using (8.2.15.6) and (8.2.15.9)

$$\sigma^2 = \frac{36}{221} - \left(\frac{2}{13}\right)^2 = \frac{400}{2873} \tag{8.2.15.11}$$

$$\implies \sigma = \sqrt{\frac{400}{2873}} \approx 0.373 \tag{8.2.15.12}$$

8.2.16 The probability distribution of a discrete random variable X is given in Table 8.2.16.1.

X	1	2	4	2A	3A	5A
$\Pr(X)$	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{3}{25}$	$\frac{1}{10}$	$\frac{1}{25}$	$\frac{1}{25}$

TABLE 8.2.16.1

Calculate

a) The value of A if $E(X) = 2.94$.

b) Variance of X.

Solution:

a) Since,

$$E(X) = \sum k p_X(k) \tag{8.2.16.1}$$
$$2.94 = \frac{1}{2} + \frac{2}{5} + \frac{12}{25} + \frac{2A}{10} + \frac{3A}{25} + \frac{5A}{25} \tag{8.2.16.2}$$
$$\implies A = \frac{78}{26} = 3 \tag{8.2.16.3}$$

b)

$$\sigma^2 = E\left(X^2\right) - [E(X)]^2 \tag{8.2.16.4}$$
$$= 10.4164 \tag{8.2.16.5}$$

8.2.17 The probability distribution of a random variable X is given as under

$$p_X(x) = \begin{cases} kx^2 & \text{for } x = 1, 2, 3 \\ 2kx & \text{for } x = 4, 5, 6 \\ 0 & \text{otherwise} \end{cases}$$

where k is a constant. Calculate

a) $E(X)$
b) $E\left(3X^2\right)$
c) $\Pr(X \geq 4)$

Solution: From the axiom of total probability,

$$\sum_{i=1}^{6} p_X(i) = 1 \tag{8.2.17.1}$$
$$\implies \sum_{i=1}^{3} ki^2 + \sum_{i=4}^{6} 2ki = 1 \tag{8.2.17.2}$$
$$\implies k + 4k + 9k + 8k + 10k + 12k = 1 \tag{8.2.17.3}$$
$$\implies k = \frac{1}{44} \tag{8.2.17.4}$$

Therefore,

$$p_X(x) = \begin{cases} \frac{x^2}{44} & \text{for } x = 1, 2, 3 \\ \frac{2x}{44} & \text{for } x = 4, 5, 6 \\ 0 & \text{otherwise} \end{cases}$$

a)

$$E(X) = \sum_{i=1}^{6} i p_X(i) \tag{8.2.17.5}$$
$$= \frac{95}{22} \tag{8.2.17.6}$$
$$\tag{8.2.17.7}$$

b)

$$E\left(3X^2\right) = 3E\left(X^2\right) \tag{8.2.17.8}$$
$$= 3\sum_{i=1}^{6} i^2 p_X(i) \tag{8.2.17.9}$$
$$= \frac{2724}{44} \tag{8.2.17.10}$$
$$\tag{8.2.17.11}$$

c) The CDF

$$F_X(x) = \sum_{i=1}^{x} p_X(i) \tag{8.2.17.12}$$
$$= \begin{cases} \sum_{i=1}^{x} \frac{i^2}{44} & \text{if } x \le 3 \\ \sum_{i=1}^{3} \frac{i^2}{44} + \sum_{i=4}^{x} \frac{2i}{44} & \text{if } x \ge 4 \end{cases} \tag{8.2.17.13}$$
$$= \begin{cases} \frac{x(x+1)(2x+1)}{6\times 44} & \text{if } x \le 3 \\ \frac{14}{44} + \frac{x(x+1)}{44} - \frac{3\times 4}{44} & \text{if } x \ge 4 \end{cases} \tag{8.2.17.14}$$
$$= \begin{cases} \frac{x(x+1)(2x+1)}{264} & \text{if } x \le 3 \\ \frac{x(x+1)+2}{44} & \text{if } x \ge 4 \end{cases} \tag{8.2.17.15}$$

Therefore,

$$\Pr(X \ge 4) = 1 - \Pr(X \le 3) \tag{8.2.17.16}$$
$$= 1 - F_X(3) \tag{8.2.17.17}$$
$$= 1 - \frac{3 \times 4 \times 7}{264} \tag{8.2.17.18}$$
$$= \frac{15}{22} \tag{8.2.17.19}$$

8.2.18 Two probability distributions of the discrete random variable X and Y are given below.

TABLE 8.2.18.1

X	0	1	2	3
$P(X)$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

TABLE 8.2.18.2

Y	0	1	2	3
$P(Y)$	$\frac{1}{5}$	$\frac{3}{10}$	$\frac{2}{5}$	$\frac{1}{10}$

Prove that $E\left(Y^2\right) = 2E(X)$

Solution: From the given information,

$$E\left(Y^2\right) = \sum_k (k)^2 \times p_Y(k) = \frac{14}{5} \tag{8.2.18.1}$$
$$E(X) = \sum_k k \times p_X(k) = \frac{7}{5} \tag{8.2.18.2}$$

which verifies the given condition.

8.2.19 For the following probability distribution in Table 8.2.19.2, determine the standard deviation of the random variable X.

X	2	3	4
$P(X)$	0.2	0.5	0.3

TABLE 8.2.19.2

Solution:

$$E\left(X^2\right) = \sum_{k=2}^{4} k^2 p_X(k) = 10.1 \tag{8.2.19.1}$$

$$[E(X)]^2 = \left[\sum_{k=2}^{4} k p_X(k)\right]^2 = 9.61 \tag{8.2.19.2}$$

Therefore,

$$\sigma_X^2 = E\left(X^2\right) - [E(X)]^2 = 10.1 - 9.61 \tag{8.2.19.3}$$

$$\implies \sigma_X = 0.7 \tag{8.2.19.4}$$

8.2.20 Let X be a discrete random variable whose probability distribution is defined as follows:

$$\Pr(X = x) = \begin{cases} k(x+1) & x = 1, 2, 3, 4, \\ 2kx & x = 5, 6, 7, \\ 0 & otherwise \end{cases}$$

where k is a constant. Calculate

a) the value of k

b) $E(X)$

c) Standard deviation of X

Solution:

a)

$$\sum_{i=1}^{n} p_X(i) = 1 \tag{8.2.20.1}$$

$$\implies 50k = 1 \tag{8.2.20.2}$$

$$\text{or, } k = 0.02 \tag{8.2.20.3}$$

b)

$$E(X) = 260k \tag{8.2.20.4}$$

$$= 5.2 \tag{8.2.20.5}$$

c)

$$\sigma_X^2 = 2.92 \tag{8.2.20.6}$$

$$\implies \sigma_X = 1.7 \tag{8.2.20.7}$$

8.2.21 A discrete random variable X has the probability distribution given in Table 8.2.21.1.

X	0.5	1	1.5	2
P(X)	k	k^2	$2k^2$	k

TABLE 8.2.21.1

a) Find the value of k
b) Determine the mean of the distribution.

Solution:

a)

$$\sum_i p_X(i) = 1 \tag{8.2.21.1}$$

$$\implies k + k^2 + 2k^2 + k = 1 \tag{8.2.21.2}$$

$$\implies 3k^2 + 2k - 1 = 0 \tag{8.2.21.3}$$

$$\implies (3k-1)(k+1) = 0 \tag{8.2.21.4}$$

$$\implies k = \frac{1}{3} \text{ or } k = -1 \tag{8.2.21.5}$$

(8.2.21.6)

Therefore, $k = \frac{1}{3}$, since probability cannot be negative.

b)

$$\mu = E(X) = \sum_i i p_X(i) \tag{8.2.21.7}$$

$$= 4k^2 + 2.5k = \frac{23}{18} \tag{8.2.21.8}$$

8.2.22 There are 5 cards numbered 1 to 5, one number on one card. Two cards are drawn at random without replacement. Let X denote the sum of the numbers on two cards drawn. Find the mean and variance of X.

8.2.23 For the following probability distribution:

X	−4	−3	−2	−1	0
P(X)	0.1	0.2	0.3	0.2	0.2

$E\,(X)$ is equal to:

a) 0
b) −1
c) −2

d) -1.8

8.2.24 For the following probability distribution

X	1	2	3	4
P(X)	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{3}{10}$	$\frac{2}{5}$

TABLE 8.2.24.1: Probability Distribution

E(X^2)is equal to

(A)3 (B)5 (C)7 (D)10

8.2.25 A die is tossed twice. A 'success' is getting an even number on a toss. Find the variance of the number of successes.

Solution:

Parameter	Value	Description
X_i	0,1	0-Not a success, 1-Success and it represents outcome of i^{th} throw
X	0,1,2	X = $X_1 + X_2$, denoting number of outcomes in two throws

pmf of X_i is

$$p_{X_i}(k) = \begin{cases} \frac{1}{2}, & k = 0 \\ \frac{1}{2}, & k = 1 \end{cases} \quad \forall \quad 1 \le i \le 2 \tag{8.2.25.1}$$

Mean value of X_i is

$$\mu_{X_i} = E[X_i], \quad i = 0, 1 \tag{8.2.25.2}$$

$$= \frac{1}{2} \tag{8.2.25.3}$$

Variance of X_i is

$$\sigma^2_{X_i} = E[(X_i - \mu_{X_i})^2], \quad i = 0, 1 \tag{8.2.25.4}$$

$$= \frac{1}{4} \tag{8.2.25.5}$$

Variance of getting successes in two throws of a die is

$$\sigma^2_X = \sigma^2_{X_1} + \sigma^2_{X_2} \tag{8.2.25.6}$$

$$= \frac{1}{2} \tag{8.2.25.7}$$

8.2.26 A biased die is such that $\Pr(4) = \frac{1}{10}$ and other scores being equally likely. The die is tossed twice. If X is the 'number of fours seen', find the variance of the random variable X.

8.2.27 If X is the number of tails in three tosses of coin, determine the standard deviation of X.

8.2.28 In a game, a man wins a rupee for a six and loses a rupee for any other number when a fair die is thrown. The man decided to throw a die thrice but to quit as and when he gets a six. Find the expected value of the amount he wins / loses.

8.2.29 Find the mean of $\mathrm{B}\left(4, \frac{1}{3}\right)$.

8.2.30 Two cards are drawn simultaneously (or successively without replacement) from a well shuffled pack of 52 cards. Find the mean, variance and standard deviation of the number of kings.

8.2.31 Find the variance of the number obtained on a throw of an unbiased die

8.2.32 Let a pair of dice be thrown and the random variable X be the sum of the numbers that appear on the two dice. Find the mean or expectation of X.

9 Gaussian

9.1 Central Limit Theorem

9.1.1 The pdf of a Gaussian rv with mean μ and variance σ^2, defined as $Y \sim \mathcal{N}\left(\mu, \sigma^2\right)$ is given by

$$p_Y(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad x \in (-\infty, \infty). \tag{9.1.1.1}$$

9.1.2 Let

$$X \sim \mathrm{B}(n, p) \tag{9.1.2.1}$$

The mean and variance are then given by

$$\mu = np, \sigma^2 = npq. \tag{9.1.2.2}$$

9.1.3 For large n,

$$Z = \frac{X - \mu}{\sigma} \xleftrightarrow{d} \mathcal{N}(0, 1) \tag{9.1.3.1}$$

which implies that Z converges in distribution to the standard Gaussian.
Solution: See Appendix A. A comparison of the Binomial and Gaussian pmf/pdf is provided in Figs. 9.1.3.1 and 9.1.3.2.

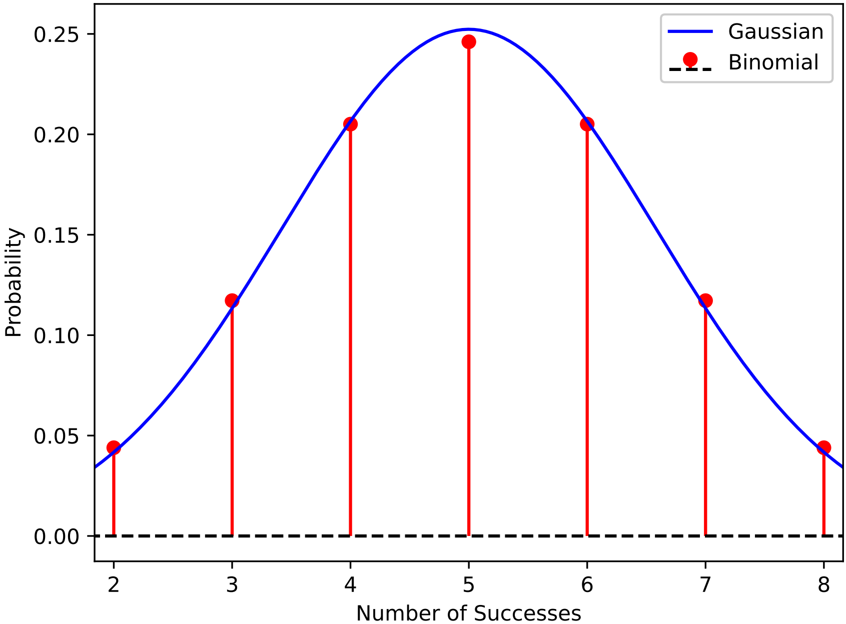


Fig. 9.1.3.1: 10 trials

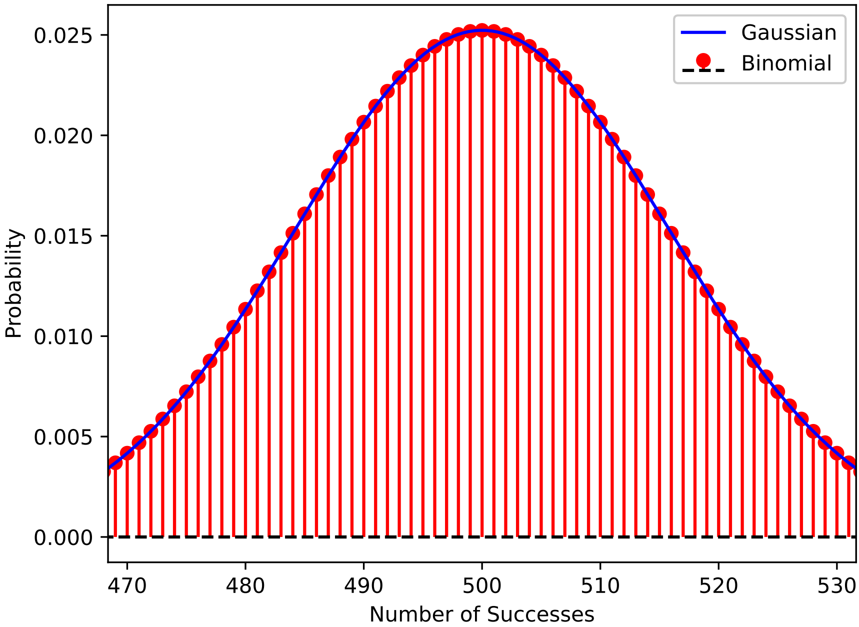


Fig. 9.1.3.2: 1000 trials

9.1.4 The CDF of Z is

$$
\begin{aligned}
F_Z(x) &= \Pr(Z \leq x) && (9.1.4.1)\\
&= \Phi_Z(x) && (9.1.4.2)
\end{aligned}
$$

9.1.5 The Q-function is defined as

$$
\begin{aligned}
Q(x) &= \Pr(Z > x), \qquad x > 0,\\
Q(-x) &= \Pr(Z > -x), \qquad x < 0,\\
&= 1 - Q(x).
\end{aligned}
\tag{9.1.5.1}
$$

9.1.6

$$
\Phi_Z(x) = \begin{cases} 1 - Q(x), & x > 0,\\ Q(-x), & x < 0. \end{cases} \tag{9.1.6.1}
$$

9.1.7

$$
F_Y(x) = \begin{cases} 1 - Q\left(\frac{x-\mu}{\sigma}\right), & x > \mu\\ Q\left(\frac{\mu-x}{\sigma}\right), & x < \mu \end{cases} \tag{9.1.7.1}
$$

9.2.1 Suppose that 90 % of people are right-handed. What is the probability that atmost 6 of a random sample of 10 people are right-handed.
Solution: From the given information, $X \sim B\left(10, \frac{9}{10}\right)$. From (9.1.2.2),

$$\mu = 10 \times \frac{9}{10} = 9, \tag{9.2.1.1}$$

$$\sigma^2 = 10 \times \frac{9}{10} \times \frac{1}{10} = \frac{9}{10} \tag{9.2.1.2}$$

The desired probability is

$$\Pr(X \le 6) \approx F_Y(6) = Q\left(\frac{9-6}{3}\sqrt{10}\right) \tag{9.2.1.3}$$

$$= Q\left(\sqrt{10}\right) \tag{9.2.1.4}$$

from (9.1.7.1).

9.2.2 There are 5% defective items in a large bulk of items. What is the probability that a sample of 10 items will include not more than one defective item?
Solution: Gaussian Distribution

Parameter	Values	Description
n	10	Number of articles
p	0.05	Probability of being defective
Y	$0 \le Y \le 10$	Number of defective elements
$\mu = np$	0.5	mean
$\sigma = \sqrt{np(1-p)}$	0.475	standard deviation

a) Central limit theorm:

$$Y \sim \mathcal{N}\left(\mu, \frac{\sigma}{\sqrt{n}}\right) \tag{9.2.2.1}$$

$$\tag{9.2.2.2}$$

Due to continuity correction $\Pr(X = x)$ can be approximated using gaussian distribution as

$$p_Y(x) \approx \Pr(x - 0.05 < Y < x + 0.05) \tag{9.2.2.3}$$

$$\approx \Pr(Y < x + 0.05) - \Pr(Y < x - 0.05) \tag{9.2.2.4}$$

$$\approx F_Y(x + 0.05) - F_Y(x - 0.05) \tag{9.2.2.5}$$

Now, we get:

$$F_Y(1) = p_Y(1.05) \tag{9.2.2.6}$$

$$= 1 - Q\left(\frac{1.05 - 0.5}{\sqrt{0.05}}\right) \tag{9.2.2.7}$$

$$= 1 - Q\left(\frac{0.55}{0.2236}\right) \tag{9.2.2.8}$$

$$= 1 - Q(2.4596) \tag{9.2.2.9}$$

$$= 0.99304 \tag{9.2.2.10}$$

b) Binomial Distribution:

$$n = 10; p = \frac{1}{20} \tag{9.2.2.11}$$

Pmf of X for $0 \le k \le 10$ is

$$p_X(k) = {}^{n}C_k p^k (1-p)^{n-k} \tag{9.2.2.12}$$

Then the probability is given as:

$$p_X(0) + p_X(1) = {}^{10}C_0 \left(\frac{1}{20}\right)^0 \left(1 - \frac{1}{20}\right)^{10} + {}^{10}C_1 \left(\frac{1}{20}\right)^1 \left(1 - \frac{1}{20}\right)^9 \tag{9.2.2.13}$$

Hence we get;

$$p_X(0) + p_X(1) = 29\left(\frac{19^9}{20^{10}}\right) = 0.91386 \tag{9.2.2.14}$$

Hence we can say probability calculated through central limit theorem is very close to the one calculated through binomial distribution.

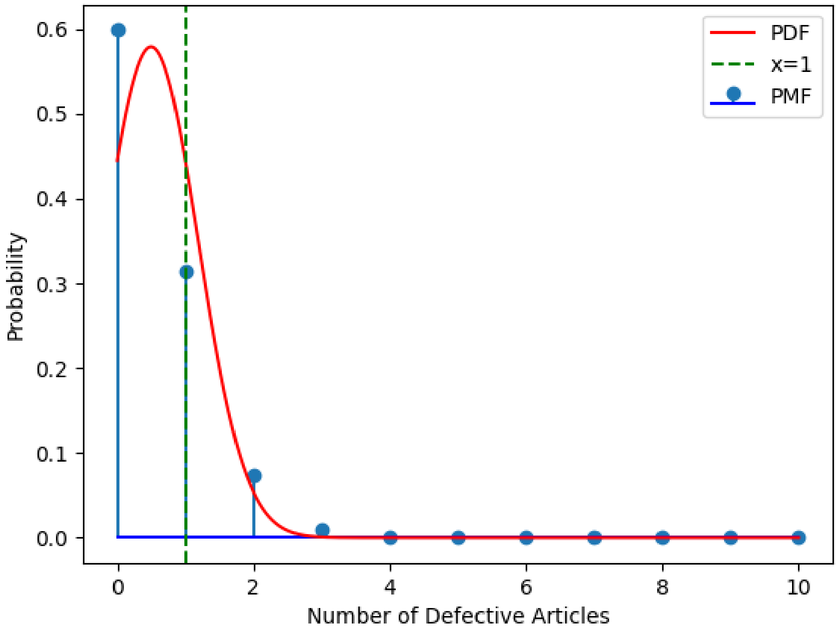


Fig. 9.2.2.1: Binomial vs Gaussian

9.2.3 Five cards are drawn successively with replacement from a well-shuffled deck of 52 cards. What is the probability that

a) all the five cards are spades?
b) only 3 cards are spades?
c) none is a spade?

Solution:
let Y be a gaussian Random variable

Parameter	Value	Description
X	{0,1,2,3,4,5}	Number of spade cards drawn
n	5	Number of cards drawn
p	0.25	Drawing a spade card
q	0.75	Drawing any other card
$\mu = np$	1.25	Mean of Binomial distribution
$\sigma^2 = npq$	0.9375	Varience of Binomial distribution

TABLE 9.2.3.1: Random variable and Parameter

$$Y \sim N(\mu, \sigma) \tag{9.2.3.1}$$
$$\sim N(1.25, 0.9375) \tag{9.2.3.2}$$

Due to continuity correction $\Pr(X = x)$ can be approximated using gaussian distribution as

$$p_Y(x) \approx \Pr(x - 0.5 < Y < x + 0.5) \tag{9.2.3.3}$$
$$\approx \Pr(Y < x + 0.5) - \Pr(Y < x - 0.5) \tag{9.2.3.4}$$
$$\approx F_Y(x + 0.5) - F_Y(x - 0.5) \tag{9.2.3.5}$$

CDF of Y is defined as:

$$F_Y(x) = \Pr(Y < x) \tag{9.2.3.6}$$
$$= \Pr\left(\frac{Y-\mu}{\sigma} < \frac{x-\mu}{\sigma}\right) \tag{9.2.3.7}$$
$$\implies \frac{Y-\mu}{\sigma} \sim N(0, 1) \tag{9.2.3.8}$$
$$= 1 - \Pr\left(\frac{Y-\mu}{\sigma} > \frac{x-\mu}{\sigma}\right) \tag{9.2.3.9}$$
$$= \begin{cases} 1 - Q\left(\frac{x-\mu}{\sigma}\right) & x \geq \mu \\ Q\left(\frac{\mu-x}{\sigma}\right) & x < \mu \end{cases} \tag{9.2.3.10}$$

Then probability in terms of Q funtion is

$$\implies p_Y(x) \approx Q\left(\frac{(x-0.5)-\mu}{\sigma}\right) - Q\left(\frac{(x+0.5)-\mu}{\sigma}\right) \tag{9.2.3.11}$$

a) The Gaussian approximation for $\Pr(X = 5)$ is

$$p_Y(5) \approx Q\left(\frac{4.5 - 1.25}{0.9375}\right) - Q\left(\frac{5.5 - 1.25}{0.9375}\right) \tag{9.2.3.12}$$
$$\approx Q(3.356) - Q(4.389) \tag{9.2.3.13}$$
$$\approx 0.0003888 \tag{9.2.3.14}$$

b) The Gaussian approximation for $\Pr(X = 3)$ is

$$p_Y(3) \approx Q\left(\frac{2.5 - 1.25}{0.9375}\right) - Q\left(\frac{3.5 - 1.25}{0.9375}\right) \tag{9.2.3.15}$$
$$\approx Q(1.2909) - Q(2.3237) \tag{9.2.3.16}$$
$$\approx 0.08828 \tag{9.2.3.17}$$

c) The Gaussian approximation for $\Pr(X = 0)$ is

$$p_Y(0) \approx Q\left(\frac{-0.5 - 1.25}{0.9375}\right) - Q\left(\frac{0.5 - 1.25}{0.9375}\right) \tag{9.2.3.18}$$

$$\approx (1 - Q(1.8073)) - (1 - Q(0.7745)) \tag{9.2.3.19}$$

$$= Q(0.7745) - Q(1.8073) \tag{9.2.3.20}$$

$$\approx 0.1839 \tag{9.2.3.21}$$

Comparison			
Number of spade cards	Binomial distribution	Gaussian approximation	Error (%)
5	0.0009765625	0.00038880	60.18688
3	0.087890625	0.088279	0.4430
0	0.2373046875	0.18390	22.5046

TABLE 9.2.3.2: Comparison between the approximation

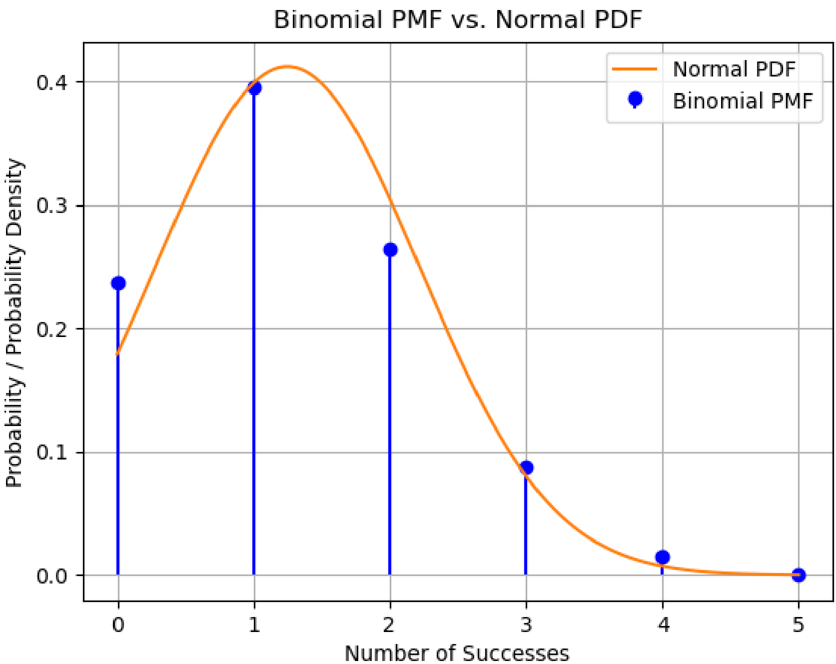


Fig. 9.2.3.1: Binomial and gaussian distribution

9.2.4 In an examination, 20 questions of true-false type are asked. Suppose a student tosses a fair coin to determine his answer to each question. If the coin falls heads, he answer true; if it falls tails, he answer false. Find the probability that he answers at least 12 questions correctly.

Solution: Gaussian:

TABLE 9.2.4.1: Variables

Variable	Value	Description
n	20	Number of questions
p	0.5	probability of question being correct
$\mu = np$	10	mean of distribution
$\sigma = \sqrt{npq}$	$\sqrt{5}$	variance of distribution
X	$0 \le X \le 20$	Number of correct questions

$$Y \sim \mathcal{N}\left(\mu, \sigma^2\right) \tag{9.2.4.1}$$

CDF of Y is defined as:

$$F_Y(x) = \Pr(Y \le x) \tag{9.2.4.2}$$

$$= \Pr\left(\frac{Y-\mu}{\sigma} \le \frac{X-\mu}{\sigma}\right) \tag{9.2.4.3}$$

$$\frac{Y-\mu}{\sigma} \sim \mathcal{N}(0,1) \tag{9.2.4.4}$$

$$= 1 - \Pr\left(\frac{Y-\mu}{\sigma} > \frac{X-\mu}{\sigma}\right) \tag{9.2.4.5}$$

$$= 1 - Q\left(\frac{X-\mu}{\sigma}\right) \tag{9.2.4.6}$$

a) Without correction:

$$\Pr(Y > 11) = 1 - \Pr(Y \le 11) \tag{9.2.4.7}$$

$$= 1 - F_Y(11) \tag{9.2.4.8}$$

$$\implies \Pr(Y > 11) = Q\left(\frac{X-\mu}{\sigma}\right) \tag{9.2.4.9}$$

$$= Q(0.894) \tag{9.2.4.10}$$

$$\Pr(Y > 11) = 0.1855 \tag{9.2.4.11}$$

b) With a 0.5 correction:

$$\Pr(Y > 11) = Q\left(\frac{X-\mu+0.5}{\sigma}\right) \tag{9.2.4.12}$$

$$= Q(0.67) \tag{9.2.4.13}$$

$$\implies \Pr(Y > 11) = 0.2511 \tag{9.2.4.14}$$

Binomial:

$$\Pr(X \geq 12) = 1 - \Pr(X < 12) \tag{9.2.4.15}$$
$$= \sum_{k=12}^{20} {}^{n}C_k p^k (1-p)^{n-k} \tag{9.2.4.16}$$
$$= 0.2517 \tag{9.2.4.17}$$

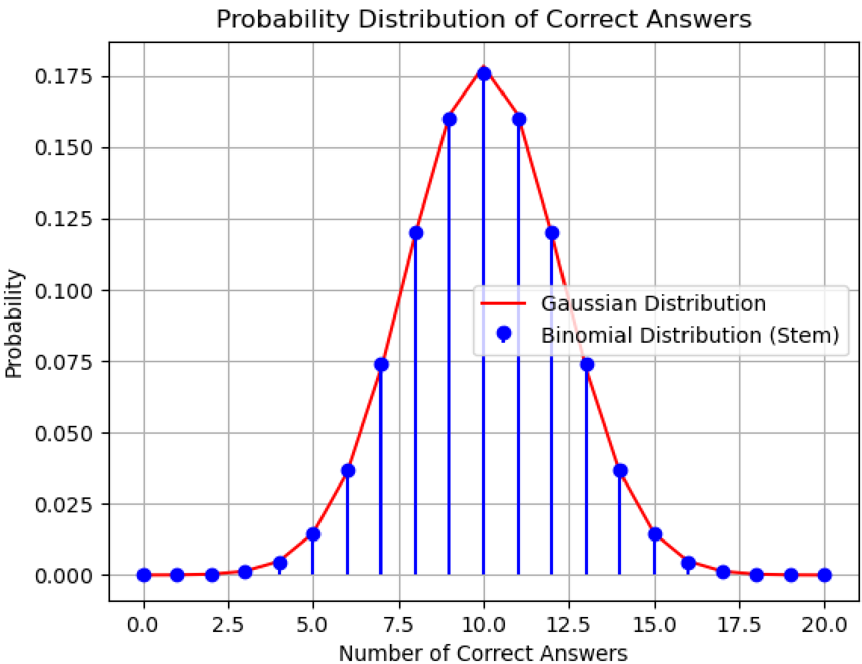


Fig. 9.2.4.1: Binomial vs Gaussian

9.2.5 It is known that 10 % of certain articles manufactured are defective. What is the probability that in a random sample space of 12 such articles,9 are defective?
Solution: Let X be random variable defined as

Random Variable	Values	Description
X	$1 \leq X \leq 12$	Number of defective in 12 articles

X has a binomial distribution with parameters

$$n = 12 \qquad p = \frac{10}{100} = \frac{1}{10} \tag{9.2.5.1}$$

Pmf of X for $1 \leq k \leq 12$ is

$$p_X(k) = {}^nC_k p^k (1-p)^{n-k} \tag{9.2.5.2}$$

Probability that a random sample space of 12 such articles,9 are defective is

$$p_X(9) = {}^{12}C_9 \left(\frac{1}{10}\right)^9 \left(1 - \frac{1}{10}\right)^{12-9} \tag{9.2.5.3}$$

$$= \frac{12!}{9!3!} \left(\frac{1}{10}\right)^9 \left(\frac{9}{10}\right)^3 \tag{9.2.5.4}$$

$$= 220 \left(\frac{1}{10^9}\right) \left(\frac{9^3}{10^3}\right) \tag{9.2.5.5}$$

$$= 22 \left(\frac{9^3}{10^{11}}\right) \tag{9.2.5.6}$$

$$= 1.603773(10^{-7}) \tag{9.2.5.7}$$

Let Y be goussian variable

$$\mu = np \tag{9.2.5.8}$$

$$= \frac{6}{5} \tag{9.2.5.9}$$

$$\sigma^2 = np(1-p) \tag{9.2.5.10}$$

$$= \frac{27}{25} \tag{9.2.5.11}$$

Using Normal distribution at X=9.

$$Z = \frac{X - \mu}{\sigma} \tag{9.2.5.12}$$

$$= \frac{9 - \frac{6}{5}}{\sqrt{\frac{27}{25}}} \tag{9.2.5.13}$$

$$= 7.50555 \tag{9.2.5.14}$$

For pdf calculation

$$f_Y(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \tag{9.2.5.15}$$

From the plot, pmf is close to normal distribution pdf.

$$p_Y(9) = p_Z(7.5055) \tag{9.2.5.16}$$

$$= 1.6109(10^{-7}) \tag{9.2.5.17}$$

From (9.2.5.7) and (9.2.5.17),

$$p_X(9) \approx p_Y(9) \tag{9.2.5.18}$$

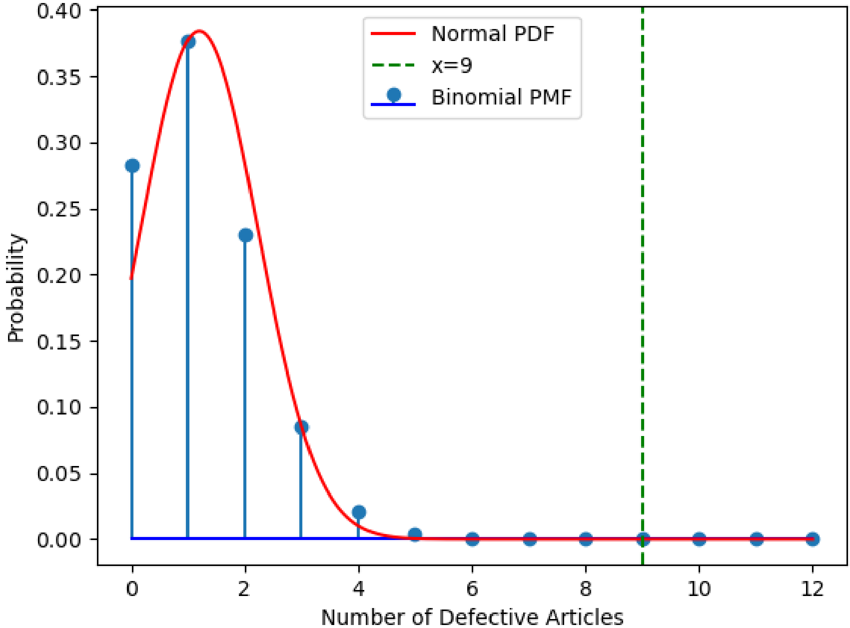


Fig. 9.2.5.1: Binomial pmf vs Gaussian pdf

9.2.6 The probability that a student is not a swimmer is $\frac{1}{5}$ .Then the probability that out of five students, four are swimmers

a) ${}^5C_4\left(\frac{4}{5}\right)^4\frac{1}{5}$

b) $\left(\frac{4}{5}\right)^4\frac{1}{5}$

c) ${}^5C_1\frac{1}{5}\left(\frac{4}{5}\right)^4$

d) None of these

Solution: The X is the random variable,We require pmf at $X = 4$,

$$p_X(4) = {}^5C_4\left(\frac{4}{5}\right)^4\left(\frac{1}{5}\right)^{5-4} = 0.4096 \tag{9.2.6.1}$$

$$X \approx Y \sim \mathcal{N}(\mu, \sigma^2) \tag{9.2.6.2}$$

Parameter	**Value**	**Description**
n	5	number of students
q	$\frac{1}{5}$	not a swimmer
p	$\frac{4}{5}$	swimmer
k	4	number of swimmers
X	$0 \leq X \leq 5$	X swimmer out of 5
Y	$0 \leq Y \leq 5$	Gaussian variable
μ	$np = 4$	mean
σ^2	$npq = \frac{4}{5}$	variance

TABLE 9.2.6.1: Given Information

Using central limit theorem, we can use the gaussian distribution function:

$$p_Y(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad (x \in Y) \tag{9.2.6.3}$$

Now, using Normal distribution at Y=4

$$p_Y(4) = \frac{1}{\sqrt{2\pi\left(\frac{4}{5}\right)}} e^{-\frac{(4-4)^2}{2\left(\frac{4}{5}\right)}} \tag{9.2.6.4}$$

$$= \frac{1}{\sqrt{2\pi\left(\frac{4}{5}\right)}} e^{0} \tag{9.2.6.5}$$

$$= 0.4463 \tag{9.2.6.6}$$

From the plot also the pmf is close to normal distribution pdf. Hence, $p_Y(4) \approx p_X(4)$ so, option (9.2.6c) is correct

9.2.7 There are 5% defective items in a large bulk of items. What is the probability that a sample of 10 items will include not more than one defective item?

Solution:

Parameter	Values	Description
n	10	Number of items
p	0.05	Probability of being defective
q	0.95	Probability of not being defective
$\mu = np$	0.5	Mean
$\sigma^2 = npq$	0.475	Variance

TABLE 9.2.7.1: Definition of parameters and their values

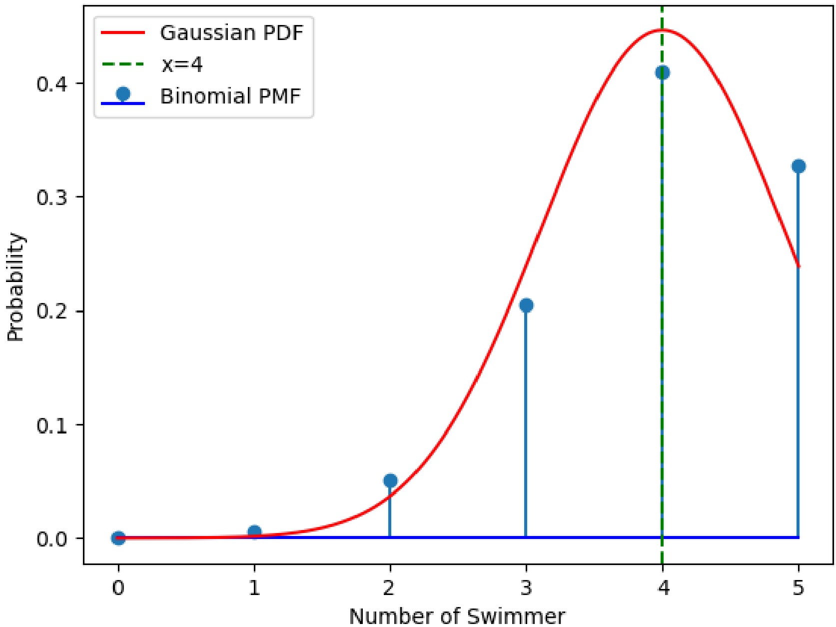


Fig. 9.2.6.1: Binomial pmf vs Gaussian pdf

a) Binomial: The cdf using binomial is given by

$$F_Y(n) = \Pr(Y \le n) \tag{9.2.7.1}$$

$$= \sum_{k=0}^{n} {}^{10}C_k p^k (1-p)^{10-k} \tag{9.2.7.2}$$

We require $\Pr(Y \le 1)$. Since $n = 1$,

$$F_Y(1) = \Pr(Y \le 1) \tag{9.2.7.3}$$

$$= \sum_{k=0}^{1} {}^{10}C_k (0.05)^k (0.95)^{10-k} \tag{9.2.7.4}$$

$$= 0.9138 \tag{9.2.7.5}$$

b) Gaussian: $Y \sim \mathcal{N}\left(\mu, \sigma^2\right)$

To obtain cdf,

$$\Pr(Y \le 1) = F_Y(1) \tag{9.2.7.6}$$
$$F_Y(x) = \Pr(Y \le x) \tag{9.2.7.7}$$
$$= \Pr(Y - \mu \le x - \mu) \tag{9.2.7.8}$$
$$= \Pr(\frac{Y-\mu}{\sigma} \le \frac{x-\mu}{\sigma}) \tag{9.2.7.9}$$
$$= 1 - \Pr(\frac{Y-\mu}{\sigma} > \frac{x-\mu}{\sigma}) \tag{9.2.7.10}$$

We know that,

$$\frac{Y-\mu}{\sigma} \sim \mathcal{N}(0,1) \tag{9.2.7.11}$$
$$\Pr(X > x) = Q(x) \tag{9.2.7.12}$$

Hence,

$$F_Y(x) = 1 - Q\left(\frac{x-\mu}{\sigma}\right), \text{if } x > \mu \tag{9.2.7.13}$$
$$= Q\left(\frac{\mu-x}{\sigma}\right), \text{if } x < \mu \tag{9.2.7.14}$$
$$\implies F_Y(1) = 1 - Q\left(\frac{0.5}{\sqrt{0.475}}\right) \tag{9.2.7.15}$$
$$= 0.766 \tag{9.2.7.16}$$

With correction of 0.5,

$$\Pr(Y \le 1.5) = F_Y(1.5) \tag{9.2.7.17}$$
$$F_Y(1.5) = 1 - Q\left(\frac{1}{\sqrt{0.475}}\right) \tag{9.2.7.18}$$
$$= 0.927 \tag{9.2.7.19}$$

From (9.2.7.5) and (9.2.7.19)

$$\Pr(Y \le 1) \approx F_Y(1.5) \tag{9.2.7.20}$$

Binomial	Gaussian (without correction)	Gaussian (with correction)
0.9138	0.766	0.927

TABLE 9.2.7.2: Probability obtained using different methods

9.2.8 Five cards are drawn successively with replacement from a well-shuffled deck of 52 cards. What is the probability that
  a) all the five cards are spades?
  b) only 3 cards are spades?
  c) none is a spade?

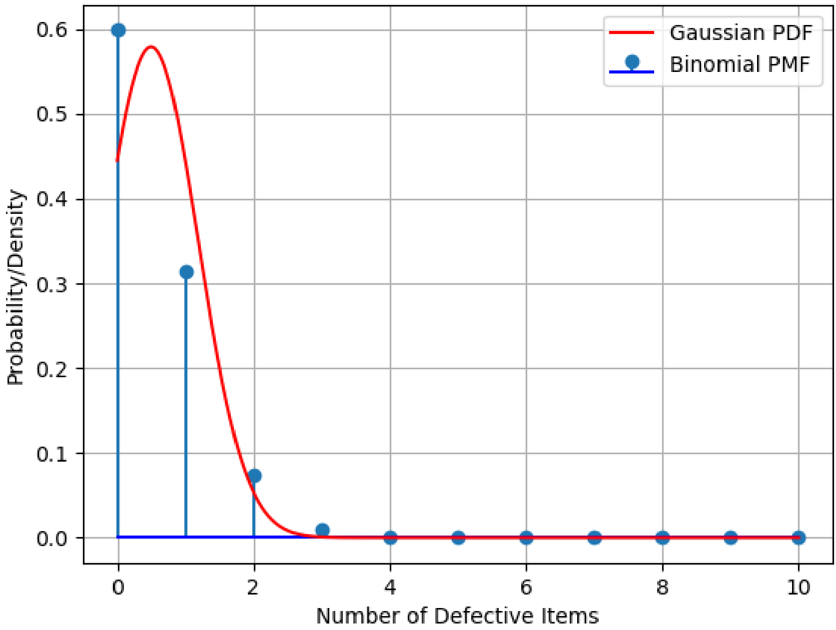


Fig. 9.2.7.1: Binomial pmf vs Gaussian pdf

Solution: Let us define:

Parameter	**Value**	**Description**
n	5	number of cards drawn
p	$\frac{1}{4}$	drawing a spade card
q	$\frac{3}{4}$	drawing any other card
$\mu = np$	$\frac{5}{4}$	mean of the distribution
$\sigma^2 = npq$	$\frac{15}{16}$	variance of the distribution
Y	{0,1,2,3,4,5}	Number of spade cards drawn

(i) Gaussian Distribution
The gaussian distribution function is defined as:

$$p_Y(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad (x \in Y) \tag{9.2.8.1}$$

If we consider all cards to be spades,

$$Y = 5p_Y(5) \qquad = \frac{1}{\sqrt{2\pi\left(\frac{15}{16}\right)}}e^{-\frac{\left(5-\frac{5}{4}\right)^2}{2\left(\frac{15}{16}\right)}} \tag{9.2.8.2}$$

$$= \frac{1}{\sqrt{2\pi\left(\frac{15}{16}\right)}}e^{-\frac{15}{2}} \tag{9.2.8.3}$$

$$= 0.0001245 \tag{9.2.8.4}$$

If we consider 3 cards to be spades,

$$Y = 3p_Y(3) \qquad = \frac{1}{\sqrt{2\pi\left(\frac{15}{16}\right)}}e^{-\frac{\left(3-\frac{5}{4}\right)^2}{2\left(\frac{15}{16}\right)}} \tag{9.2.8.5}$$

$$= \frac{1}{\sqrt{2\pi\left(\frac{15}{16}\right)}}e^{-\frac{49}{30}} \tag{9.2.8.6}$$

$$= 0.044 \tag{9.2.8.7}$$

If we consider 0 cards to be spades,

$$Y = 0p_Y(0) \qquad = \frac{1}{\sqrt{2\pi\left(\frac{15}{16}\right)}}e^{-\frac{\left(0-\frac{5}{4}\right)^2}{2\left(\frac{15}{16}\right)}} \tag{9.2.8.8}$$

$$= \frac{1}{\sqrt{2\pi\left(\frac{15}{16}\right)}}e^{-\frac{5}{6}} \tag{9.2.8.9}$$

$$= 0.0978 \tag{9.2.8.10}$$

(ii) Solving using Q function
Consider a gaussian random variable Z,

$$Z \sim N(\mu, \sigma) \tag{9.2.8.11}$$

$$\sim N\left(\frac{5}{4}, \frac{\sqrt{15}}{4}\right) \tag{9.2.8.12}$$

Due to continuity correction $\Pr(Y = x)$ can be approximated using gaussian distribution as

$$p_Z(x) \approx \Pr(x - 0.5 < Z < x + 0.5) \tag{9.2.8.13}$$

$$\approx \Pr(Z < x + 0.5) - \Pr(Z < x - 0.5) \tag{9.2.8.14}$$

$$\approx F_Z(x + 0.5) - F_Z(x - 0.5) \tag{9.2.8.15}$$

CDF of Z is defined as:

$$F_Z(x) = \Pr(Z < x) \tag{9.2.8.16}$$

$$= \Pr\left(\frac{Z-\mu}{\sigma} < \frac{x-\mu}{\sigma}\right) \tag{9.2.8.17}$$

$$\implies \frac{Z-\mu}{\sigma} \sim N(0,1) \tag{9.2.8.18}$$

$$= 1 - \Pr\left(\frac{Z-\mu}{\sigma} > \frac{x-\mu}{\sigma}\right) \tag{9.2.8.19}$$

$$= \begin{cases} 1 - Q\left(\frac{x-\mu}{\sigma}\right) & x \geq \mu \\ Q\left(\frac{\mu-x}{\sigma}\right) & x < \mu \end{cases} \tag{9.2.8.20}$$

Then probability in terms of Q funtion is

$$\implies p_Z(x) \approx Q\left(\frac{(x-0.5)-\mu}{\sigma}\right) - Q\left(\frac{(x+0.5)-\mu}{\sigma}\right) \tag{9.2.8.21}$$

The Gaussian approximation for $\Pr(Y = 5)$ is

$$p_Z(5) \approx Q\left(\frac{4.5-1.25}{0.9375}\right) - Q\left(\frac{5.5-1.25}{0.9375}\right) \tag{9.2.8.22}$$

$$\approx Q(3.356) - Q(4.389) \tag{9.2.8.23}$$

$$\approx 0.0003888 \tag{9.2.8.24}$$

The Gaussian approximation for $\Pr(Y = 3)$ is

$$p_Y(3) \approx Q\left(\frac{2.5-1.25}{0.9375}\right) - Q\left(\frac{3.5-1.25}{0.9375}\right) \tag{9.2.8.25}$$

$$\approx Q(1.2909) - Q(2.3237) \tag{9.2.8.26}$$

$$\approx 0.08828 \tag{9.2.8.27}$$

The Gaussian approximation for $\Pr(Y = 0)$ is

$$p_Z(0) \approx Q\left(\frac{-0.5-1.25}{0.9375}\right) - Q\left(\frac{0.5-1.25}{0.9375}\right) \tag{9.2.8.28}$$

$$\approx (1 - Q(1.8073)) - (1 - Q(0.7745)) \tag{9.2.8.29}$$

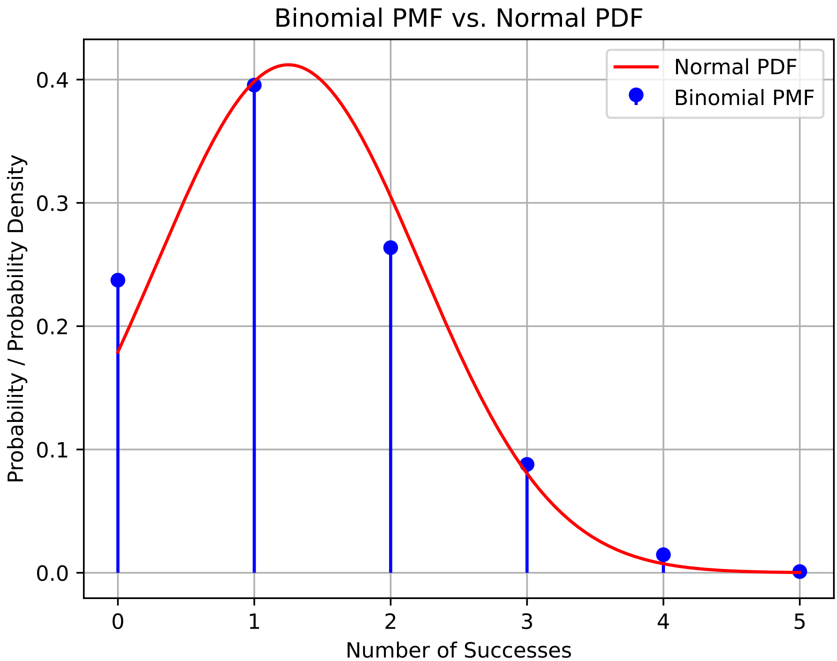
$$= Q(0.7745) - Q(1.8073) \tag{9.2.8.30}$$

$$\approx 0.1839 \tag{9.2.8.31}$$

(iii) Gaussian vs Binomial vs Q-function Comparison

Y	Gaussian	Q-function	Binomial
0	0.0978	0.1839	0.2373
3	0.044	0.08828	0.08789
5	0.0001245	0.0003888	0.00098

(iv) Binomial vs Gaussian Graph



9.2.9 The probability that a bulb produced by a factory will fuse after 150 days of use is 0.05. Find the probability that out of 5 such bulbs

a) none
b) not more than one
c) more than one
d) at least one

will fuse after 150 days of use.

Solution:

Guassian :

let Y be a gaussian Random variable

$$Y \sim N(\mu, \sigma) \tag{9.2.9.1}$$
$$\sim N(1.25, 0.9375) \tag{9.2.9.2}$$

Due to continuity correction $\Pr(X = x)$ can be approximated using gaussian distribution as

$$p_Y(x) \approx \Pr(x - 0.5 < Y < x + 0.5) \tag{9.2.9.3}$$
$$\approx \Pr(Y < x + 0.5) - \Pr(Y < x - 0.5) \tag{9.2.9.4}$$
$$\approx F_Y(x + 0.5) - F_Y(x - 0.5) \tag{9.2.9.5}$$

Parameter	Value	Description
X	0,1,2,3,4,5	No. Of bulbs fused
n	5	Total no. Of bulbs
p	0.05	bulb fusing
q	0.95	not fusing
$\mu = np$	0.25	Mean of Binomial Distribution
$\sigma^2 = npq$	0.2375	Varience of binomial Distribution

TABLE 9.2.9.1: Random variable and Parameter

CDF of Y is defined as:

$$F_Y(x) = \Pr(Y < x) \tag{9.2.9.6}$$

$$= \Pr\left(\frac{Y-\mu}{\sigma} < \frac{x-\mu}{\sigma}\right) \tag{9.2.9.7}$$

$$\implies \frac{Y-\mu}{\sigma} \sim N(0,1) \tag{9.2.9.8}$$

$$= 1 - \Pr\left(\frac{Y-\mu}{\sigma} > \frac{x-\mu}{\sigma}\right) \tag{9.2.9.9}$$

$$= \begin{cases} 1 - Q\left(\frac{x-\mu}{\sigma}\right) & x \geq \mu \\ Q\left(\frac{\mu-x}{\sigma}\right) & x < \mu \end{cases} \tag{9.2.9.10}$$

Then probability in terms of Q funtion is

$$\implies p_Y(x) \approx Q\left(\frac{(x-0.5)-\mu}{\sigma}\right) - Q\left(\frac{(x+0.5)-\mu}{\sigma}\right) \tag{9.2.9.11}$$

Binomial :

$$\Pr(X = k) = {}^nC_k p^k (1-p)^{n-k} \tag{9.2.9.12}$$

$$= {}^5C_k (0.05)^k (0.95)^{5-k} \tag{9.2.9.13}$$

CDF of X

$$F_X(k) = \Pr(X \leq k) \tag{9.2.9.14}$$

$$= \sum_{i=0}^{k} {}^{10}C_i (0.05)^i (0.95)^{5-i} \tag{9.2.9.15}$$

The solution
The graph

$\Pr(X = x)$	in term of Q	Numercal value	Binomial solution
$\Pr(X = 0)$	$Q(1.5389) - Q(0.512)$	0.6960	0.773
$\Pr(X \leq 1)$	$Q(1.5896)$	0.9948	0.9774075
$1 - \Pr(X = 0)$	$1 - (Q(1.5389) - Q(0.512))$	0.304	0.227
$1 - \Pr(X \leq 1)$	$1 - Q(1.5896)$	0.006	0.0226

TABLE 9.2.9.2: Random variable and Parameter

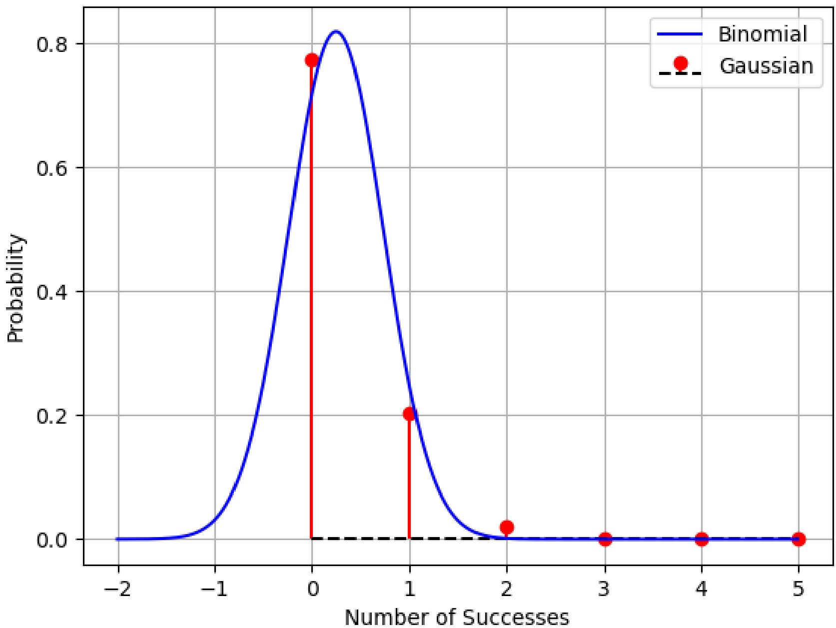


Fig. 9.2.9.1: Binomial vs guassian

9.2.10 A bag consists of 10 balls each marked with one of the digits 0 to 9. If 4 balls are drawn successively with replacement from the bag, what is the probability that none is marked with the digit 0?

Solution:

(a) Gaussian PDF

$$Y \sim \mathcal{N}\left(\mu, \sigma^2\right) \tag{9.2.10.1}$$

Parameter	Values	Description
n	4	Number of balls drawn
p	0.1	Probability that the ball drawn is marked zero
$\mu = np$	0.4	Mean of distribution
$\sigma^2 = np(1-p)$	0.36	Variance of distribution
Y	0,1,2,3,4	Number of balls drawn which are zero

The gaussian distribution function is defined as:

$$p_Y(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \tag{9.2.10.2}$$

The probability that none of the balls drawn is marked with zero is given by:

$$p_Y(0) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(-\mu)^2}{2\sigma^2}} \tag{9.2.10.3}$$

$$= 0.532 \tag{9.2.10.4}$$

(b) CDF Approximation

$$\Pr(Y \le 0) = F_Y(0) \tag{9.2.10.5}$$

CDF of Y is:

$$F_Y(x) = \Pr(Y \le x) \tag{9.2.10.6}$$

$$= \Pr(Y - \mu \le x - \mu) \tag{9.2.10.7}$$

$$= \Pr\left(\frac{Y-\mu}{\sigma} \le \frac{x-\mu}{\sigma}\right) \tag{9.2.10.8}$$

Since,

$$\frac{Y-\mu}{\sigma} \sim \mathcal{N}(0,1) \tag{9.2.10.9}$$

Q function is defined

$$Q(x) = \Pr(Y > x)\, \forall x \in Y \sim \mathcal{N}(0,1) \tag{9.2.10.10}$$

$$F_Y(x) = 1 - \Pr\left(\frac{Y-\mu}{\sigma} > \frac{x-\mu}{\sigma}\right) \tag{9.2.10.11}$$

$$= \begin{cases} 1 - Q\left(\frac{x-\mu}{\sigma}\right), & x > \mu \\ Q\left(\frac{\mu-x}{\sigma}\right), & x < \mu \end{cases} \tag{9.2.10.12}$$

$$F_Y(0) = Q\left(\frac{0.4-0}{0.6}\right) \tag{9.2.10.13}$$

$$= Q\left(\frac{2}{3}\right) \tag{9.2.10.14}$$

$$= 0.252 \tag{9.2.10.15}$$

(c) Binomial PMF Let X be a random variable which denotes the number of balls drawn that are marked with zero,

$$p_X(k) = {}^{n}C_k p^k (1-p)^{n-k} \tag{9.2.10.16}$$

$$p_X(0) = {}^{4}C_0 (0.1)^0 (0.9)^4 \tag{9.2.10.17}$$

$$= 0.6561 \tag{9.2.10.18}$$

Y	Gaussian PDF	CDF Approximation	Binomial PMF
0	0.532	0.252	0.6561

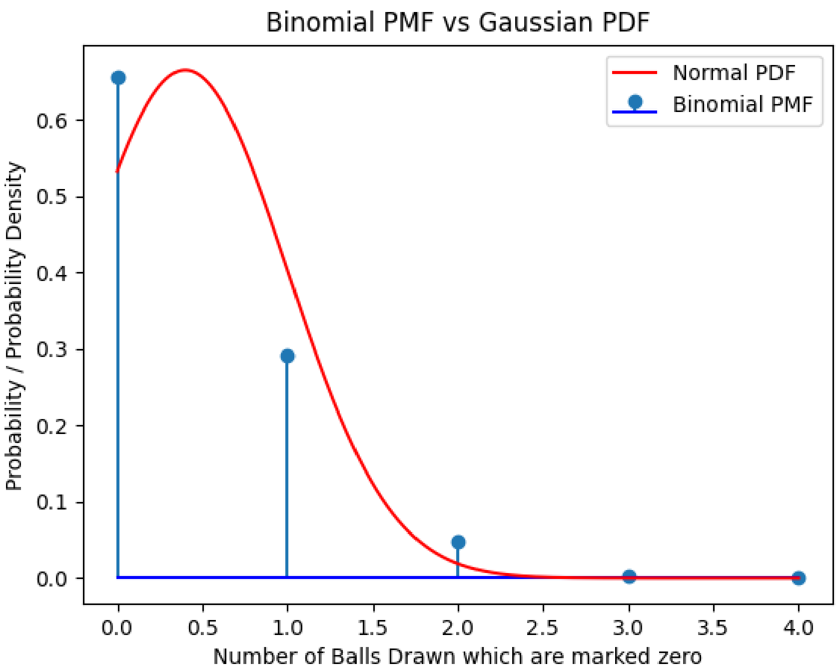


Fig. 9.2.10.1: Binomial PMF vs Gaussian PDF

9.2.11 How many times must a man toss a fair coin so that the probability of having at least one head is more than 90%?

Solution:

a) Gaussian:

$$Y \sim \mathcal{N}\left(\mu, \sigma^2\right) \tag{9.2.11.1}$$

Parameter	Value	Description
n	n	number of coin tosses
p	$\frac{1}{2}$	getting a head on a coin toss
q	$\frac{1}{2}$	getting a tail on a coin toss
$\mu = np$	$\frac{n}{2}$	mean of the distribution
$\sigma^2 = npq$	$\frac{n}{4}$	variance of the distribution
Y	≥ 1	Number of heads

The CDF of Y:

$$F_Y(y) = 1 - \Pr(Y > y) \tag{9.2.11.2}$$

$$= 1 - \Pr\left(\frac{Y-\mu}{\sigma} > \frac{y-\mu}{\sigma}\right) \tag{9.2.11.3}$$

But,

$$\frac{Y-\mu}{\sigma} \sim \mathcal{N}(0,1) \tag{9.2.11.4}$$

$$\implies F_Y(y) = 1 - Q\left(\frac{y-\mu}{\sigma}\right) \tag{9.2.11.5}$$

i) Without correction

$$\Pr(Y \geq 1) = 1 - F_Y(1) \tag{9.2.11.6}$$

From the result (9.2.11.5)

$$Q\left(\frac{2-n}{\sqrt{n}}\right) > 0.9 \tag{9.2.11.7}$$

$$\frac{2-n}{\sqrt{n}} < Q^{-1}(0.9) \tag{9.2.11.8}$$

$$\frac{2-n}{\sqrt{n}} < -1.28 \tag{9.2.11.9}$$

Squaring on both the sides

$$\implies (n-2)^2 > (1.28\sqrt{n})^2 \tag{9.2.11.10}$$

$$\implies n^2 - 5.6384n + 4 > 0 \tag{9.2.11.11}$$

$$\implies n > 4.86, n < 0.8 \tag{9.2.11.12}$$

$$\implies n = 5 \tag{9.2.11.13}$$

ii) With correction: 0.5 as correction term

$$\Pr(Y > 0.5) = 1 - F_Y(0.5) \tag{9.2.11.14}$$

From the result (9.2.11.5)

$$Q\left(\frac{1-n}{\sqrt{n}}\right) > 0.9 \tag{9.2.11.15}$$

$$\frac{1-n}{\sqrt{n}} < Q^{-1}(0.9) \tag{9.2.11.16}$$

$$\frac{1-n}{\sqrt{n}} < -1.28 \tag{9.2.11.17}$$

Squaring on both the sides

$$\implies (n-1)^2 > (1.28\sqrt{n})^2 \tag{9.2.11.18}$$

$$\implies n^2 - 3.6384n + 1 > 0 \tag{9.2.11.19}$$

$$\implies n > 3.38, n < 0.29 \tag{9.2.11.20}$$

$$\implies n = 4 \tag{9.2.11.21}$$

b) Binomial:

$$X \sim \text{Bin}(n, p) \tag{9.2.11.22}$$

$$\Pr(X \geq 1) > 0.9 \tag{9.2.11.23}$$

$$\implies n = 4 \tag{9.2.11.24}$$

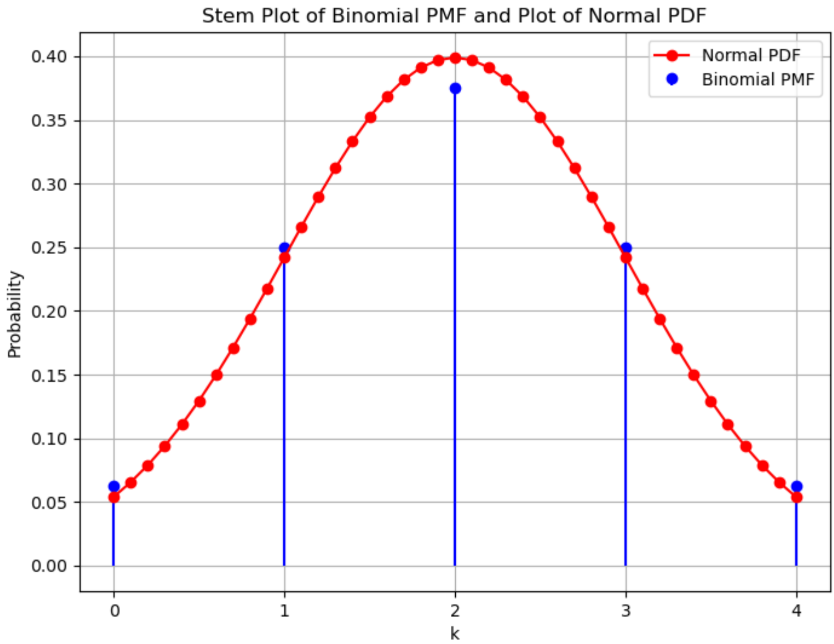


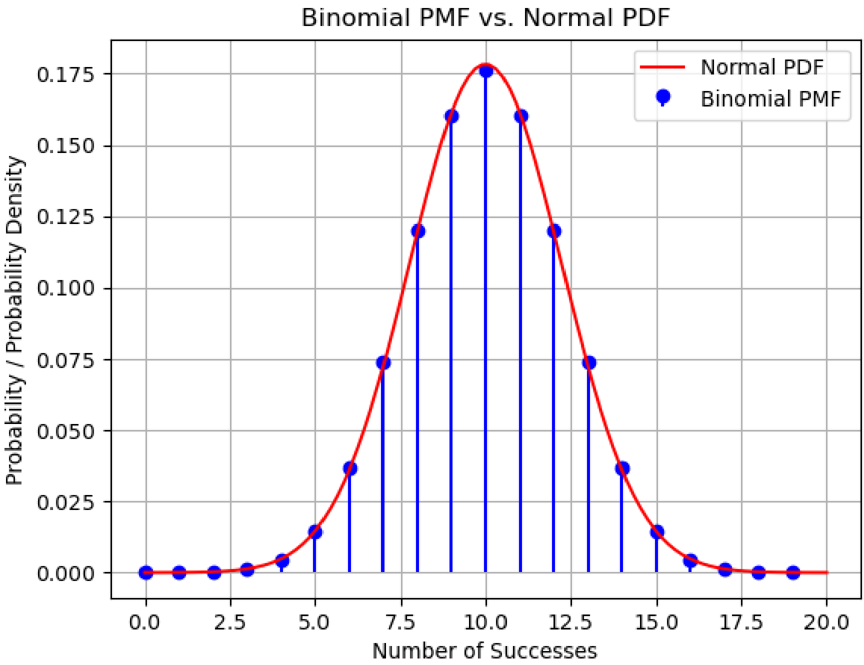
Fig. 9.2.11.1: Binomial PMF of X vs Normal PDF of Y

9.2.12 In an examination, 20 questions of true-false type are asked. Suppose a student tosses a fair coin to determine his answer to each question. If the coin falls heads, he answer true; if it falls tails, he answer false. Find the probability that he answers at least 12 questions correctly.

Solution: Gaussian vs Binomial

Let us define:

Parameter	**Value**	**Description**
n	20	number of Questions
p	0.5	probability of answering correct
q	0.5	probability of answering wrong
$\mu = np$	10	mean of the distribution
$\sigma^2 = npq$	5	variance of the distribution
Y	0,1,2,3,...,20	Number of correct answers



a) Gaussian:

$$Y \sim \mathcal{N}\left(\mu, \sigma^2\right) \tag{9.2.12.1}$$

The CDF of Y:

$$F_Y(y) = 1 - \Pr\left(Y > y\right) \tag{9.2.12.2}$$

$$= 1 - \Pr\left(\frac{Y-\mu}{\sigma} > \frac{y-\mu}{\sigma}\right) \tag{9.2.12.3}$$

But,

$$\frac{Y-\mu}{\sigma} \sim \mathcal{N}\left(0, 1\right) \tag{9.2.12.4}$$

$$\tag{9.2.12.5}$$

the Q-function is defined as:

$$Q\left(x\right) = \Pr\left(Y > x\right) \; \forall x \in Y \sim \mathcal{N}(0, 1) \tag{9.2.12.6}$$

therefore the cdf will be:

$$F_Y\left(y\right) = \begin{cases} 1 - Q\left(\frac{y-\mu}{\sigma}\right), & y > \mu \\ Q\left(\frac{\mu-y}{\sigma}\right), & y < \mu \end{cases} \tag{9.2.12.7}$$

The probability of getting atleast 12 answers correct:
considering 0.5 as coorection term:

$$\Pr(Y > 12.5) = 1 - F_Y(12.5) \tag{9.2.12.8}$$
$$= Q\left(\frac{12.5 - \mu}{\sigma}\right) \tag{9.2.12.9}$$
$$= Q\left(\frac{2.5}{\sqrt{5}}\right) \tag{9.2.12.10}$$
$$= Q(1.118) \tag{9.2.12.11}$$
$$= 0.13178 \tag{9.2.12.12}$$

Questions answered correctly	**Binomial**	**Gaussian**
Atleast 12	0.2517	0.13178

9.2.13 Find the probability of getting 5 twice in 7 throws of a dice.
Solution:

Parameter	Value	Description
X	{0,1,2,3,4,5,6,7}	Number of 5 appearing on dice
n	7	Number of cards drawn
p	$\frac{1}{6}$	getting 5
q	$\frac{5}{6}$	getting any other number
$\mu = np$	$\frac{7}{6}$	Mean of Binomial distribution
$\sigma^2 = npq$	$\frac{35}{36}$	Varience of Binomial distribution

TABLE 9.2.13.1: Random variable and Parameter

a) *Binomial Distribution :*
The X is the random variable, the pmf of X is given by

$$p_X(k) = {}^nC_k p^k (1-p)^{n-k} \tag{9.2.13.1}$$

We require $\Pr(X = 2)$. Since $n = 7$,

$$p_X(2) = 0.234 \tag{9.2.13.2}$$

b) *Gaussian Distribution*
Let Y be gaussian variable. Using central limit theorem, we can use the gaussian distribution function:

$$p_Y(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad (x \in Y) \tag{9.2.13.3}$$

Using Normal distribution at X=2,

$$p_Y(2) = \frac{1}{\sqrt{2\pi\left(\frac{35}{36}\right)}} e^{-\frac{\left(2-\frac{7}{6}\right)^2}{2\left(\frac{35}{36}\right)}} \tag{9.2.13.4}$$

$$= \frac{1}{\sqrt{2\pi\left(\frac{35}{36}\right)}} e^{-\frac{5}{14}} \tag{9.2.13.5}$$

$$= 0.283 \tag{9.2.13.6}$$

c) *Using Q function:*
let Y be a gaussian Random variable

$$Y \sim N(\mu, \sigma) \tag{9.2.13.7}$$

$$\sim N(1.166, 0.972) \tag{9.2.13.8}$$

Due to continuity correction $\Pr(X = x)$ can be approximated using gaussian distribution as

$$p_Y(x) \approx \Pr(x - 0.5 < Y < x + 0.5) \tag{9.2.13.9}$$

$$\approx \Pr(Y < x + 0.5) - \Pr(Y < x - 0.5) \tag{9.2.13.10}$$

$$\approx F_Y(x + 0.5) - F_Y(x - 0.5) \tag{9.2.13.11}$$

CDF of Y is defined as:

$$F_Y(x) = \Pr(Y < x) \tag{9.2.13.12}$$

$$= \Pr\left(\frac{Y-\mu}{\sigma} < \frac{x-\mu}{\sigma}\right) \tag{9.2.13.13}$$

$$\implies \frac{Y-\mu}{\sigma} \sim N(0, 1) \tag{9.2.13.14}$$

$$= 1 - \Pr\left(\frac{Y-\mu}{\sigma} > \frac{x-\mu}{\sigma}\right) \tag{9.2.13.15}$$

$$= \begin{cases} 1 - Q\left(\frac{x-\mu}{\sigma}\right) & x \geq \mu \\ Q\left(\frac{\mu-x}{\sigma}\right) & x < \mu \end{cases} \tag{9.2.13.16}$$

Then probability in terms of Q funtion is

$$\implies p_Y(x) \approx Q\left(\frac{(x-0.5)-\mu}{\sigma}\right) - Q\left(\frac{(x+0.5)-\mu}{\sigma}\right) \tag{9.2.13.17}$$

The Gaussian approximation for $\Pr(X = 2)$ is

$$p_Y(2) \approx Q\left(\frac{1.5 - 1.166}{0.972}\right) - Q\left(\frac{2.5 - 1.166}{0.972}\right) \tag{9.2.13.18}$$

$$\approx Q(0.343) - Q(1.371) \tag{9.2.13.19}$$

$$\approx 0.282 \tag{9.2.13.20}$$

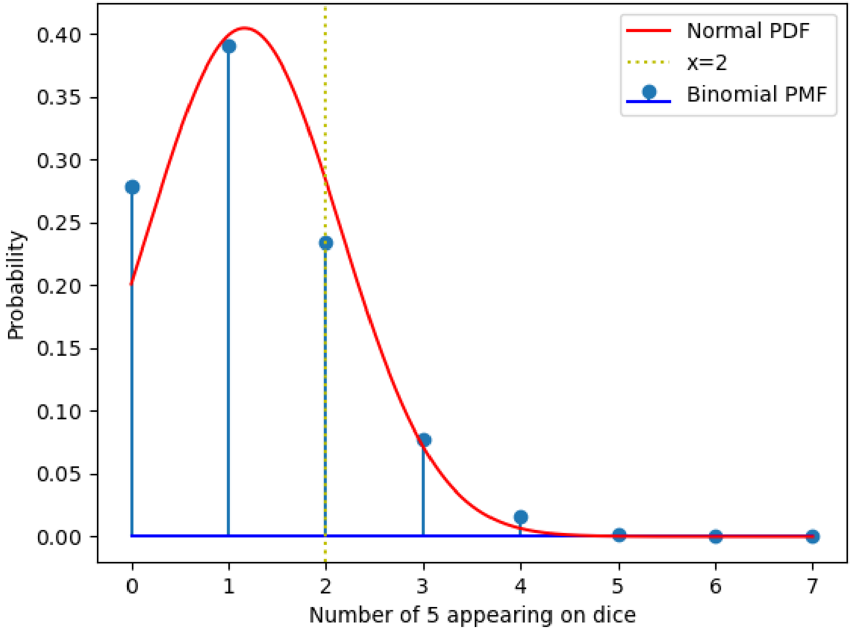


Fig. 9.2.13.1: Binomial and gaussian distribution

9.2.14 On a multiple choice examination with three possible answers for each of the five questions, what is the probability that a candidate would get four or more correct answers just by guessing?

Solution:

Gaussian:

TABLE 9.2.14.1: Variables

Variable	Value	Description
n	5	Number of questions
p	$\frac{1}{3}$	probability of question being correct
$\mu = np$	$\frac{5}{3}$	mean of distribution
$\sigma = \sqrt{npq}$	$\sqrt{\frac{10}{9}}$	variance of distribution
X	$0 \leq X \leq 5$	Number of correct questions

$$Y \sim \mathcal{N}\left(\mu, \sigma^2\right) \tag{9.2.14.1}$$

CDF of Y is defined as:

$$F_Y(x) = \Pr(Y \le x) \tag{9.2.14.2}$$
$$= \Pr\left(\frac{Y-\mu}{\sigma} \le \frac{X-\mu}{\sigma}\right) \tag{9.2.14.3}$$
$$\frac{Y-\mu}{\sigma} \sim \mathcal{N}(0,1) \tag{9.2.14.4}$$
$$= 1 - \Pr\left(\frac{Y-\mu}{\sigma} > \frac{X-\mu}{\sigma}\right) \tag{9.2.14.5}$$
$$\tag{9.2.14.6}$$

therefore the cdf will be:

$$F_Y(y) = \begin{cases} 1 - Q\left(\frac{y-\mu}{\sigma}\right), & y \ge \mu \\ Q\left(\frac{\mu-y}{\sigma}\right), & y \le \mu \end{cases} \tag{9.2.14.7}$$

a) Without correction:

$$\Pr(Y \ge 4) = 1 - \Pr(Y \le 4) \tag{9.2.14.8}$$
$$= 1 - F_Y(4) \tag{9.2.14.9}$$
$$\implies \Pr(Y \ge 4) = Q\left(\frac{X-\mu}{\sigma}\right) \tag{9.2.14.10}$$
$$= Q(2.22286) \tag{9.2.14.11}$$
$$\Pr(Y \ge 4) = 0.013113 \tag{9.2.14.12}$$

b) With a 0.5 correction:

$$\Pr(Y \ge 4) = Q\left(\frac{X-\mu+0.5}{\sigma}\right) \tag{9.2.14.13}$$
$$= Q(1.74604) \tag{9.2.14.14}$$
$$\implies \Pr(Y \ge 4) = 0.040402 \tag{9.2.14.15}$$

Binomial:

$$p_X(k) = {}^nC_k p^k (1-p)^{n-k} \tag{9.2.14.16}$$
$$\tag{9.2.14.17}$$

Probablity that 4 or more are correct

$$\implies P(X \ge 4) = \sum_{k=4}^{5} {}^5C_k \left(\frac{1}{3}\right)^k \left(\frac{2}{3}\right)^{5-k} \tag{9.2.14.18}$$
$$= \frac{11}{243} \tag{9.2.14.19}$$
$$= 0.04526 \tag{9.2.14.20}$$

9.2.15 Find the probability of throwing at most 2 sixes in 6 throws of a single die.

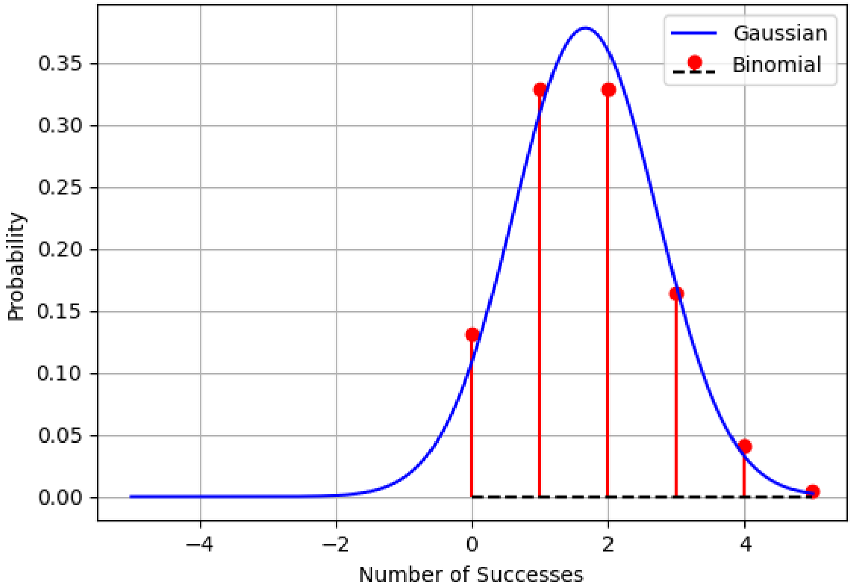


Fig. 9.2.14.1: Binomial vs guassian

Solution:

9.2.16 Suppose that 90 % of people are right-handed. What is the probability that atmost 6 of a random sample of 10 people are right-handed.

Solution: Given that 90% of the people are right-handed.

TABLE 9.2.16.1: Description of random variables

Parameters	Values	Description
n	10	Sample space
p	0.9	Probability that the person is right-handed
Y	$0 \leq Y \leq 10$	Number of people thatare right-handed
$\mu = np$	9	Mean
$\sigma = \sqrt{np(1-p)}$	0.9	Standard deviation

Gaussian Distribution

Central limit theorm:

$$Y \sim \mathcal{N}\left(\mu, \frac{\sigma}{\sqrt{n}}\right) \tag{9.2.16.1}$$

$$\tag{9.2.16.2}$$

CDF of Y is

$$F_Y(y) = \Pr(Y \leq y) \tag{9.2.16.3}$$

We know that

$$Q(x) = \Pr(X > x), x > 0, X \sim N(0, 1) \tag{9.2.16.4}$$
$$Q(-x) = \Pr(X > -x), x < 0, X \sim N(0, 1) \tag{9.2.16.5}$$
$$= 1 - Q(x) \tag{9.2.16.6}$$

Hence,
CDF :

$$F_Y(y) = \begin{cases} 1 - Q\left(\frac{y-\mu}{\sigma}\right), & \text{if } y > \mu \\ 1 - Q\left(\frac{y-\mu}{\sigma}\right) = Q\left(\frac{\mu-y}{\sigma}\right), & \text{if } y < \mu \end{cases} \tag{9.2.16.7}$$

With a 0.9 correction:

$$F_Y(6) = \Pr(Y < 6.9) \tag{9.2.16.8}$$
$$= 1 - Q\left(\frac{6.9 - 9}{\sqrt{0.9}}\right) \tag{9.2.16.9}$$
$$= Q\left(\frac{2.1}{0.9487}\right) \tag{9.2.16.10}$$
$$= Q(2.21) \tag{9.2.16.11}$$
$$= 0.013553 \tag{9.2.16.12}$$

Without correction:

$$F_Y(6) = \Pr(Y \leq 6) \tag{9.2.16.13}$$
$$= 1 - Q\left(\frac{6 - 9}{\sqrt{0.9}}\right) \tag{9.2.16.14}$$
$$= Q\left(\frac{3}{0.9487}\right) \tag{9.2.16.15}$$
$$= Q(3.1622) \tag{9.2.16.16}$$
$$= 0.000783 \tag{9.2.16.17}$$

TABLE 9.2.16.2: Comparision

Number of people(RH)	Binomial	Gaussian	Gaussian(C)	Error(%)	Error(C)(%)
Atmost 6	0.012795	0.000783	0.013553	-93.88	55.92

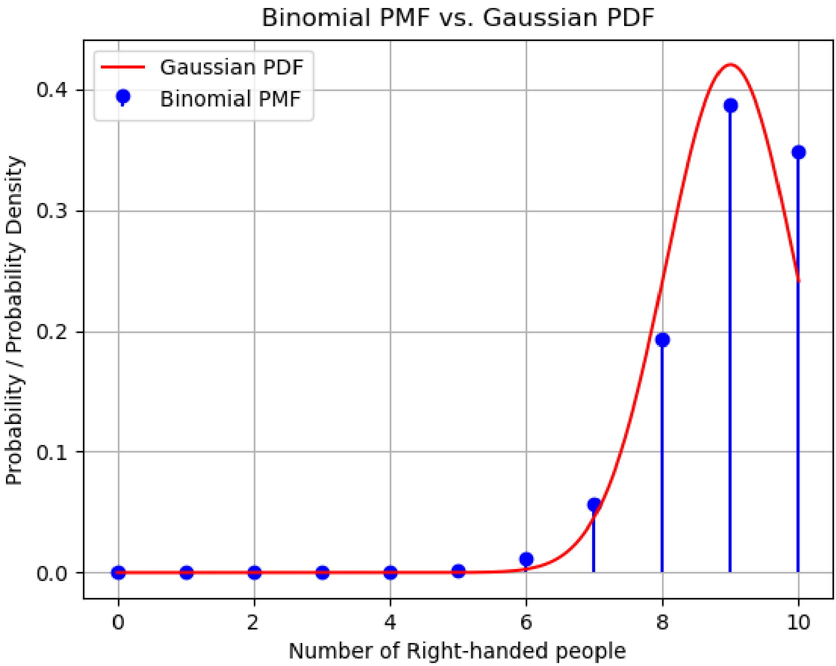


Fig. 9.2.16.1: Binomial vs Gaussian

9.2.17 An urn contains 25 balls of which 10 balls bear a mark 'X' and the remaining 15 bear a mark 'Y'. A ball is drawn at random from the urn, its mark is noted down and it is replaced. If 6 balls are drawn in this way, find the probability that

a) all will bear 'X' mark.
b) not more than 2 will bear 'Y' mark.
c) at least one ball will bear 'Y' mark.
d) the number of balls with 'X' mark and 'Y' mark will be equal.

Solution:

Parameter	Values	Description
n	6	Number of draws
p	0.4	Probability that ball bears X mark
q	0.6	Probability that ball bears Y mark
$\mu = np$	2.4	mean of the distribution
$\sigma = npq$	1.2	variance of the distribution
X		Number of cards bear mark X
Y		Number of cards bear mark Y

TABLE 9.2.17.1: Definition of parameters

using Gaussian

$$Y \sim \mathcal{N}\left(\mu, \sigma^2\right) \tag{9.2.17.1}$$

The CDF of Y:

$$F_Y(y) = 1 - \Pr\left(Y > y\right) \tag{9.2.17.2}$$

$$= 1 - \Pr\left(\frac{Y - \mu}{\sigma} > \frac{y - \mu}{\sigma}\right) \tag{9.2.17.3}$$

But,

$$\frac{Y - \mu}{\sigma} \sim \mathcal{N}\left(2.4, 1.44\right) \tag{9.2.17.4}$$

$$\tag{9.2.17.5}$$

the Q-function is defined as:

$$Q\left(x\right) = \Pr\left(Y > x\right) \, \forall x \in Y \sim \mathcal{N}(2.4, 1.44) \tag{9.2.17.6}$$

therefore the cdf will be:

$$F_Y\left(y\right) = \begin{cases} 1 - Q\left(\frac{y-\mu}{\sigma}\right), & y > \mu \\ Q\left(\frac{\mu-y}{\sigma}\right), & y < \mu \end{cases} \tag{9.2.17.7}$$

(a) all will bear X mark.
using Gaussian
considering 0.5 as the correction term,

$$\Pr\left(X > 5.5\right) = 1 - F_X\left(5.5\right) \tag{9.2.17.8}$$

$$= Q\left(\frac{5.5 - \mu}{\sigma}\right) \tag{9.2.17.9}$$

$$= Q\left(\frac{3.1}{1.2}\right) \tag{9.2.17.10}$$

$$= Q\left(2.583\right) \tag{9.2.17.11}$$

$$= 0.00489 \tag{9.2.17.12}$$

(b) not more than 2 will bear Y mark.
using Gaussian
considering 0.5 as the correction term,

$$\Pr(Y < 2.5) = 1 - Q\left(\frac{2.5 - \mu}{\sigma}\right) \quad (9.2.17.13)$$

$$= 1 - Q\left(\frac{-1.1}{1.2}\right) \quad (9.2.17.14)$$

$$= 1 - Q(-0.9166) \quad (9.2.17.15)$$

$$= Q(0.9166) \quad (9.2.17.16)$$

$$= 0.1796 \quad (9.2.17.17)$$

(c) at least one ball will bear Y mark.
using Gaussian
considering 0.5 as the correction term,

$$\Pr(Y < 0.5) = 1 - Q\left(\frac{0.5 - \mu}{\sigma}\right) \quad (9.2.17.18)$$

$$= 1 - Q\left(\frac{-1.1}{1.2}\right) \quad (9.2.17.19)$$

$$= 1 - Q(-2.588) \quad (9.2.17.20)$$

$$= 1 - 0.0048 \quad (9.2.17.21)$$

$$= 0.9952 \quad (9.2.17.22)$$

(d) the number of balls with X mark and Y mark will be equal.
using Gaussian
Calculate the mean(μ_D) and variance(σ_D) of the difference variance D,

$$\mu_D = -1.2 \quad (9.2.17.23)$$

$$\sigma_D = 2.4 \quad (9.2.17.24)$$

$$\Pr(D = 0) = Q\left(\frac{0 - \mu_D}{\sigma_D}\right) \quad (9.2.17.25)$$

$$= Q\left(\frac{1.2}{1.697}\right) \quad (9.2.17.26)$$

$$= Q(0.71) \quad (9.2.17.27)$$

$$= 0.2388 \quad (9.2.17.28)$$

Gaussian vs Binomial Table
Gaussian vs Binomial graph

9.2.18 From a lot of 30 bulbs which include 6 defectives, a sample of 4 bulbs is drawn at random with replacement. Find the probability distribution of the number of defective bulbs.
Solution:

Question	Gaussian	Binomial
all will bear X mark	0.00489	0.00409
not more than 2 will bear Y mark	0.1796	0.1792
at least one ball will bear Y mark	0.9952	0.9959
the number of balls with X mark and Y mark will be equal	0.2388	0.2764

TABLE 9.2.17.2: Definition of parameters

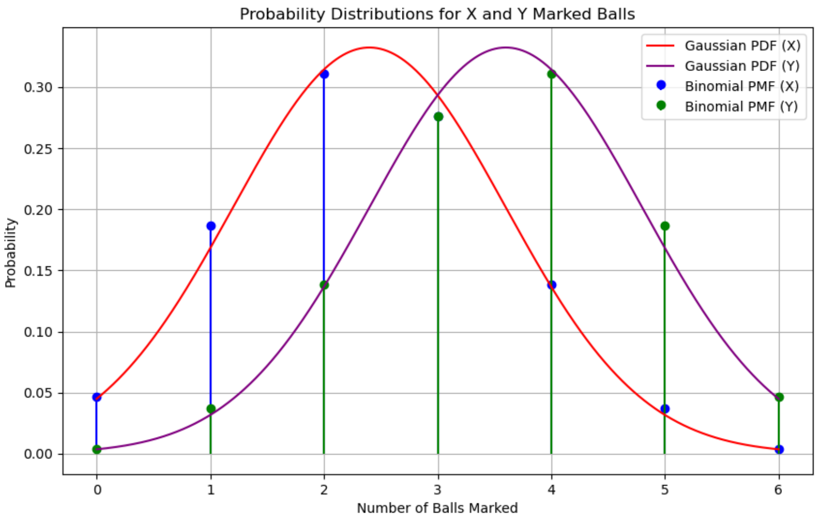


Fig. 9.2.17.1: pmf of binomial and pdf of Gaussian of X and Y marked balls

Parameter	Value	Description
X	{0,1,2,3,4}	Number of defective bulbs taken
n	4	Number of bulbs taken
p	0.2	Taking a defective bulb
q	0.8	Taking a non defective bulb
$\mu = np$	0.8	Mean of Binomial distribution
$\sigma^2 = npq$	0.64	Varience of Binomial distribution

TABLE 9.2.18.1: Parameter description

let Y be a gaussian Random variable

$$Y \sim N(np, npq) \tag{9.2.18.1}$$

$$\sim N(0.8, 0.64) \tag{9.2.18.2}$$

Due to continuity correction $\Pr(X = x)$ can be approximated using gaussian distribution as

$$p_Y(x) \approx \Pr(x - 0.5 < Y < x + 0.5) \tag{9.2.18.3}$$
$$\approx \Pr(Y < x + 0.5) - \Pr(Y < x - 0.5) \tag{9.2.18.4}$$
$$\approx F_Y(x + 0.5) - F_Y(x - 0.5) \tag{9.2.18.5}$$

CDF of Y is defined as:

$$F_Y(x) = \Pr(Y < x) \tag{9.2.18.6}$$
$$= \Pr\left(\frac{Y-\mu}{\sigma} < \frac{x-\mu}{\sigma}\right) \tag{9.2.18.7}$$
$$\implies \frac{Y-\mu}{\sigma} \sim N(0,1) \tag{9.2.18.8}$$
$$= 1 - \Pr\left(\frac{Y-\mu}{\sigma} > \frac{x-\mu}{\sigma}\right) \tag{9.2.18.9}$$
$$= \begin{cases} 1 - Q\left(\frac{x-\mu}{\sigma}\right) & x \geq \mu \\ Q\left(\frac{\mu-x}{\sigma}\right) & x < \mu \end{cases} \tag{9.2.18.10}$$

Number of defective bulbs	Binomial distribution	Gaussian approximation	Error
0	0.4096	0.3017	26.342773437
1	0.4096	0.4555	11.206054688
2	0.1536	0.1739	13.216145833
3	0.0256	0.0164	35.9375
4	0.0016	0.00036	77.5

TABLE 9.2.18.2: Comparing the gaussian approximation with binomial

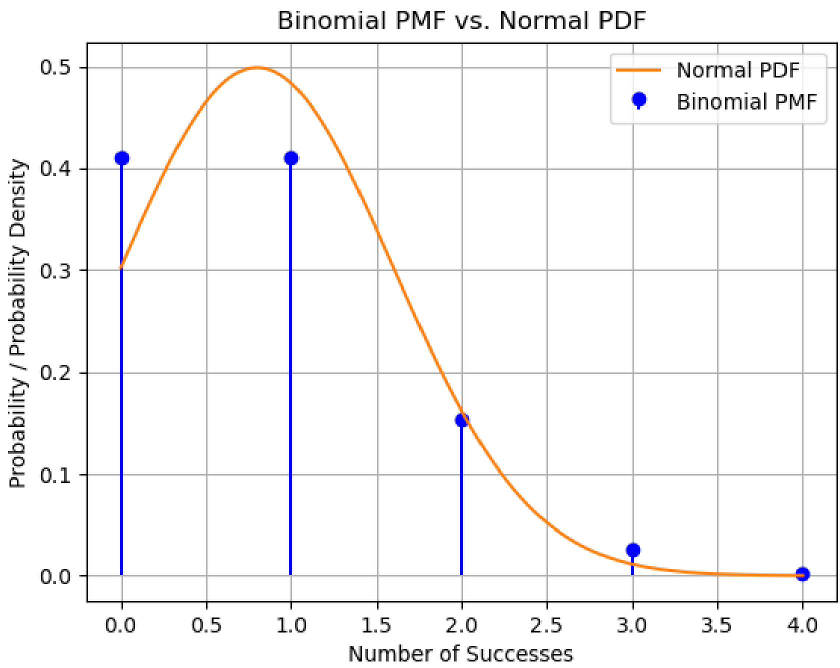


Fig. 9.2.18.1: Binomial and gaussian distribution

9.2.19 Suppose X is a binomial distribution $B\left(6, \frac{1}{2}\right)$. Show that $X = 3$ is the most likely outcome. (Hint : $P(X = 3)$ is the maximum among all $P(x_i), x_i = 0, 1, 2, 3, 4, 5, 6$)
Solution:

RV	Values	Description
X	$\{0, 1, 2, 3, 4, 5, 6\}$	Outcomes of the binomial distribution
Y	$[-\infty, \infty]$	Outcomes of the Gaussian distribution

TABLE 9.2.19.1: Random Variables

a) **Binomial:**

$$X \sim Bin\left(6, \frac{1}{2}\right) \tag{9.2.19.1}$$

We know that, for $k \in \mathbb{W}$ and $k \in [0, n]$, the maximum of ${}^nC_k$ occurs at

$$k = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even} \\ \frac{n+1}{2} \quad \text{or} \quad \frac{n-1}{2}, & \text{if } n \text{ is odd} \end{cases} \tag{9.2.19.2}$$

As,

$$n = 6 \tag{9.2.19.3}$$
$$\implies k = \frac{n}{2} = 3 \tag{9.2.19.4}$$

$\therefore X = 3$ is the most likely outcome.

$$p_X(k) = {}^6C_k \left(\frac{1}{2}\right)^6 \tag{9.2.19.5}$$
$$p_X(3) = {}^6C_3 \left(\frac{1}{2}\right)^6 \tag{9.2.19.6}$$
$$= \frac{5}{16} \tag{9.2.19.7}$$

b) **Gaussian:** The binomial distribution $X \sim Bin\left(6, \frac{1}{2}\right)$ can be approximated as a Gaussian distribution $Y \sim \mathcal{N}\left(\mu, \sigma^2\right)$ using the Mean μ and Standard Deviation σ parameters.

$$\mu = np = 6 \times \frac{1}{2} = 3 \tag{9.2.19.8}$$
$$\sigma^2 = npq = 6 \times \frac{1}{2} \times \frac{1}{2} = \frac{3}{2} \tag{9.2.19.9}$$

Thus, the Gaussian (normal) approximation is:

$$Y \sim \mathcal{N}\left(3, \frac{3}{2}\right) \tag{9.2.19.10}$$
$$\implies p_Y(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \tag{9.2.19.11}$$
$$= \frac{1}{\sqrt{3\pi}} e^{-\frac{(x-3)^2}{3}} \tag{9.2.19.12}$$

The most likely outcome is the mean of the Gaussian distribution. Thus, $Y = 3$ is the most likely outcome, as seen in the following plot.

Comparing the values numerically:

a) Binomial

$$p_X(0) = p_X(6) = \frac{1}{64} = 0.015625 \tag{9.2.19.13}$$
$$p_X(1) = p_X(5) = \frac{6}{64} = 0.09375 \tag{9.2.19.14}$$
$$p_X(2) = p_X(4) = \frac{15}{64} = 0.234375 \tag{9.2.19.15}$$
$$p_X(3) = \frac{20}{64} = 0.3125 \tag{9.2.19.16}$$

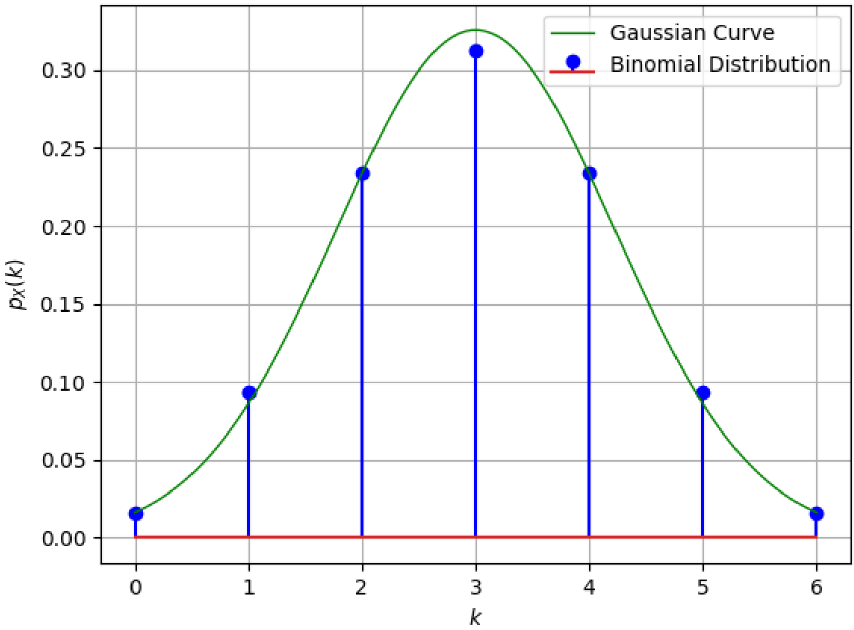


Fig. 9.2.19.1: Binomial Distribution and Gaussian Approximation

b) Gaussian

$$p_Y(0) = p_Y(6) = 0.01621739 \tag{9.2.19.17}$$

$$p_Y(1) = p_Y(5) = 0.08586282 \tag{9.2.19.18}$$

$$p_Y(2) = p_Y(4) = 0.23339933 \tag{9.2.19.19}$$

$$p_Y(3) = 0.32573501 \tag{9.2.19.20}$$

9.2.20 A fair coin is tossed four times, and a person win Re 1 for each head and lose Rs 1.5 for each tail that turns up.
From the sample space calculate how many different amounts of money you can have after four tosses and the probability of having each of these amounts. **Solution:**

9.2.21 It is known that 10 % of certain articles manufactured are defective. What is the probability that in a random sample space of 12 such articles,9 are defective?
Solution:

a) *Binomial Distribution :*
The X is the random variable, the pmf of X is given by

$$p_X(k) = {}^{n}C_k p^k (1-p)^{n-k} \tag{9.2.21.1}$$

Parameter	Values	Description
n	12	Number of articles
k	9	Number of defective articles
p	0.1	Probability of being defective
X	$1 \leq X \leq 12$	X defective elements out of 12
Y	$1 \leq Y \leq 12$	gaussian variable
$\mu = np$	1.2	mean
$\sigma = \sqrt{np(1-p)}$	1.039	standard deviation

TABLE 9.2.21.1: Table 1

We require $\Pr(X = 9)$. Since $n = 12$,

$$p_X(9) = 1.60379(10^{-7}) \tag{9.2.21.2}$$

b) *Gaussian Distribution*
Let Y be gaussian variable. Using central limit theorem, we can use the gaussian distribution function:

$$p_Y(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad (x \in Y) \tag{9.2.21.3}$$

Using Normal distribution at X=9.

$$p_Y(9) = \frac{1}{\sqrt{2\pi\left(\frac{27}{25}\right)}} e^{-\frac{\left(x-\frac{6}{5}\right)^2}{2\left(\frac{27}{25}\right)}} \tag{9.2.21.4}$$

$$= \frac{1}{\sqrt{2\pi\left(\frac{27}{25}\right)}} e^{-\frac{169}{3}} \tag{9.2.21.5}$$

$$= 3.89010(10^{-9}) \tag{9.2.21.6}$$

c) *using Q function:*

$$Y \sim \mathcal{N}\left(\mu, \sigma^2\right) \tag{9.2.21.7}$$

The CDF of Y:

$$F_Y(y) = \begin{cases} 1 - Q\left(\frac{y-\mu}{\sigma}\right), & y > \mu \\ Q\left(\frac{\mu-y}{\sigma}\right), & y < \mu \end{cases} \tag{9.2.21.8}$$

But,

$$\frac{Y-\mu}{\sigma} \sim \mathcal{N}(0, 1) \tag{9.2.21.9}$$

$$\implies F_Y(y) = 1 - Q\left(\frac{y-\mu}{\sigma}\right) \tag{9.2.21.10}$$

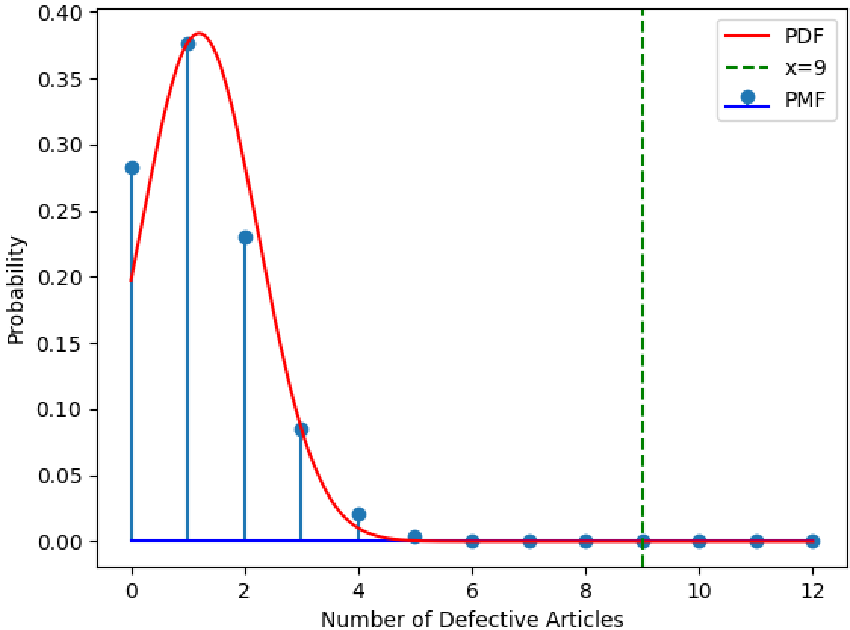


Fig. 9.2.21.1: Binomial-PMF and Gaussian-PDFof X

to include correction of 0.5,

$$p_Y(8.5 < Y < 9.5) = F_Y(9.5) - F_Y(8.5) \tag{9.2.21.11}$$

$$= Q\left(\frac{8.5-\mu}{\sigma}\right) - Q\left(\frac{9.5-\mu}{\sigma}\right) \tag{9.2.21.12}$$

$$= Q(7.02) - Q(7.98) \tag{9.2.21.13}$$

$$= 1.2798(10^{-12}) \tag{9.2.21.14}$$

9.2.22 An experiment succeeds twice as often as it fails. Find the probability that in the next six trials, there will be atleast 4 successes. **Solution:**

9.2.23 A die is thrown 5 times. Find the probability that an odd number will come up exactly three times. **Solution:**

a) *Binomial Distribution :*

The X is the random variable, the pmf of X is given by

$$p_X(k) = {}^nC_k p^k (1-p)^{n-k} \tag{9.2.23.1}$$

Parameter	Values	Description
n	5	Number of throws
k	3	Number of favourable outcomes
p	0.5	Probability of getting odd number
X	$1 \leq X \leq 5$	X favourable out of 5 total outcomes
Y	$1 \leq Y \leq 5$	gaussian variable
$\mu = np$	2.5	mean
$\sigma = \sqrt{np(1-p)}$	1.118	standard deviation

We require $\Pr(X = 3)$. Since $n = 5$,

$$p_X(3) = 0.3125 \tag{9.2.23.2}$$

b) *Gaussian Distribution*
Let Y be gaussian variable. Using central limit theorem, we can use the gaussian distribution function:

$$p_Y(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \tag{9.2.23.3}$$

$$\tag{9.2.23.4}$$

Using Normal distribution at X=3.

$$p_Y(3) = \frac{1}{\sqrt{2\pi\left(\frac{5}{4}\right)}} e^{-\frac{\left(x-\frac{5}{2}\right)^2}{2\left(\frac{5}{4}\right)}} \tag{9.2.23.5}$$

$$= \frac{1}{\sqrt{2\pi\left(\frac{5}{4}\right)}} e^{-\frac{1}{10}} \tag{9.2.23.6}$$

$$= 0.3228684517 \tag{9.2.23.7}$$

c) *using Q function:*

$$Y \sim \mathcal{N}\left(\mu, \sigma^2\right) \tag{9.2.23.8}$$

The CDF of Y:

$$F_Y(y) = \begin{cases} 1 - Q\left(\frac{y-\mu}{\sigma}\right), & y > \mu \\ Q\left(\frac{\mu-y}{\sigma}\right), & y < \mu \end{cases} \tag{9.2.23.9}$$

But,

$$\frac{Y-\mu}{\sigma} \sim \mathcal{N}(0, 1) \tag{9.2.23.10}$$

$$\implies F_Y(y) = 1 - Q\left(\frac{y-\mu}{\sigma}\right) \tag{9.2.23.11}$$

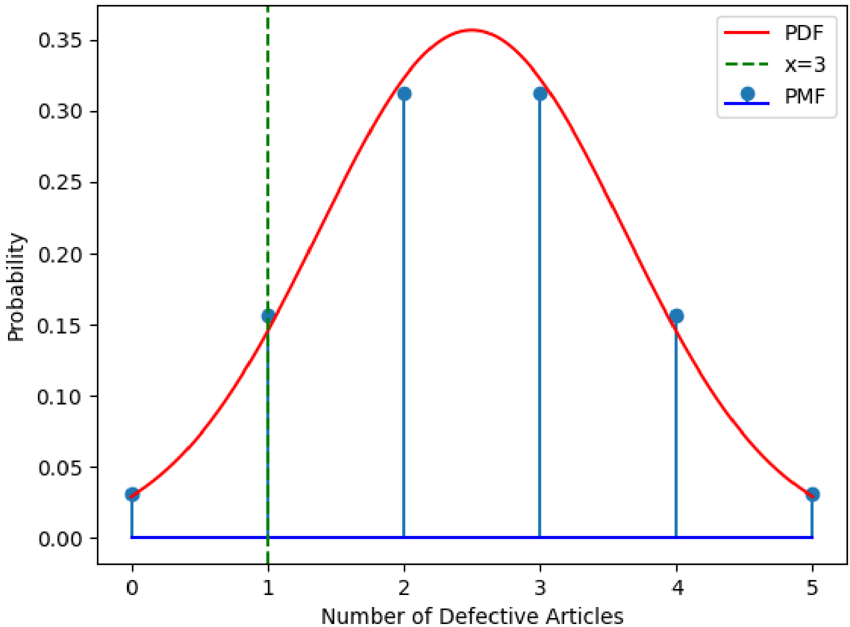


Fig. 9.2.23.1: Binomial-PMF and Gaussian-PDFof X

to include correction of 0.5,

$$p_Y(2.5 < Y < 3.5) = F_Y(2.5) - F_Y(3.5) \tag{9.2.23.12}$$

$$= Q\left(\frac{2.5-\mu}{\sigma}\right) - Q\left(\frac{3.5-\mu}{\sigma}\right) \tag{9.2.23.13}$$

$$= Q(0) - Q(0.8944) \tag{9.2.23.14}$$

$$= 0.314446 \tag{9.2.23.15}$$

9.2.24 Ten coins are tossed. What is the probability of getting atleast 8 heads?
Solution:
Gaussian Distribution

$$X \approx Y \sim \mathcal{N}(5, 2.5) \tag{9.2.24.1}$$

Parameter	Value	Description
n	10	number of tosses
p	$\frac{1}{2}$	Probability for Heads
q	$\frac{1}{2}$	Probability for Tails
$\mu = np$	5	mean of the distribution
$\sigma^2 = npq$	2.5	variance of the distribution
X	$0 \le X \le 10$	Number of heads

a) With a 0.5 correction:

$$\Pr(Y \geq 8) = Q\left(\frac{7.5 - \mu}{\sigma}\right) \tag{9.2.24.2}$$

$$\implies \Pr(Y \geq 8) = Q\left(\sqrt{2.5}\right) = Q(1.5811) \tag{9.2.24.3}$$

$$\implies \Pr(Y \geq 8) = 0.0569276 \tag{9.2.24.4}$$

b) Without correction:

$$\Pr(Y \geq 8) = Q\left(\frac{8 - \mu}{\sigma}\right) \tag{9.2.24.5}$$

$$\implies \Pr(Y \geq 8) = Q\left(\frac{3}{\sqrt{2.5}}\right) = Q(1.8973) \tag{9.2.24.6}$$

$$\implies \Pr(Y \geq 8) = 0.0288898 \tag{9.2.24.7}$$

Binomial Distribution

$$\Pr(X \geq 8) = \sum_{k=8}^{10} \binom{n}{k} p^k (1-p)^{n-k} \tag{9.2.24.8}$$

$$= 0.0546875 \tag{9.2.24.9}$$

9.2.25 A factory produces bulbs. The probability that any one bulb is defective is $\frac{1}{50}$ and they are packed in boxes of 10. From a single box, find the probability that

a) none of the bulb is defective
b) exactly two bulbs are defective
c) more than 8 bulbs are working properly

Solution:

(i) Gaussian Distribution
Let Y is the Gaussian obtained by approximating binomial with parameters n,p then By Central limit theroem,

$$Y \sim \mathcal{N}(np, npq) \tag{9.2.25.1}$$

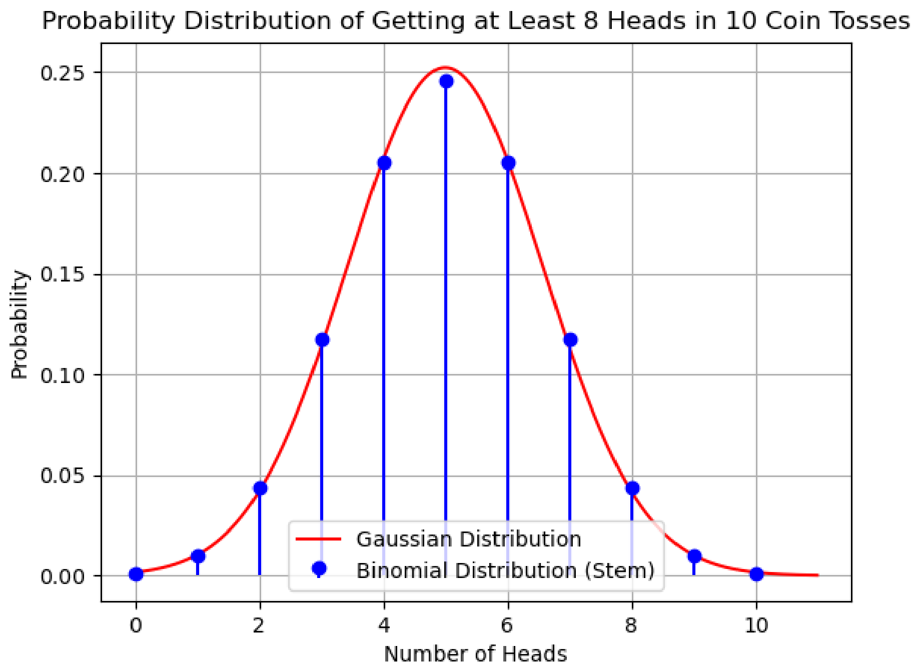


Fig. 9.2.24.1: Binomial vs Guassian

parameter	**value**	**description**
n	10	Number of bulbs in the bag
p	$\frac{1}{50}$	Bulb chosen is defective
q	$\frac{49}{50}$	Bulb chosen is proper
$\mu = np$	$\frac{1}{5}$	Mean of the distribution
$\sigma^2 = npq$	$\frac{49}{250}$	Variance of the distribution

TABLE 9.2.25.1: Gaussian Info Table

CDF of Y is:

$$F_Y(x) = \Pr(Y \leq x) \tag{9.2.25.2}$$

$$= \Pr(Y - \mu \leq x - \mu) \tag{9.2.25.3}$$

$$= \Pr\left(\frac{Y - \mu}{\sigma} \leq \frac{x - \mu}{\sigma}\right) \tag{9.2.25.4}$$

Since,

$$\frac{Y-\mu}{\sigma} \sim \mathcal{N}(0,1) \tag{9.2.25.5}$$

Q function is defined

$$Q(x) = \Pr(Y > x)\ \forall x \in Y \sim \mathcal{N}(0,1) \tag{9.2.25.6}$$

From (9.2.25.4) and (9.2.25.6),

$$F_Y(x) = 1 - \Pr\left(\frac{Y-\mu}{\sigma} > \frac{x-\mu}{\sigma}\right) \tag{9.2.25.7}$$

$$= \begin{cases} 1 - Q\left(\frac{x-\mu}{\sigma}\right), & x > \mu \\ Q\left(\frac{\mu-x}{\sigma}\right), & x < \mu \end{cases} \tag{9.2.25.8}$$

(a) If we consider no bulb is defective, we need to find

$$\Pr(Y = 0) = \Pr(Y \leq 1) - \Pr(Y \leq 0) \tag{9.2.25.9}$$

$$= F_Y(1) - F_Y(0) \tag{9.2.25.10}$$

From (9.2.25.8) and Table 9.2.25.1,

$$F_Y(0) = Q\left(\frac{0.2-0}{0.196}\right) \tag{9.2.25.11}$$

$$= Q(1) \tag{9.2.25.12}$$

$$= 0.1587 \tag{9.2.25.13}$$

$$F_Y(1) = 1 - Q\left(\frac{1-0.2}{0.44}\right) \tag{9.2.25.14}$$

$$= 1 - Q(1.8) \tag{9.2.25.15}$$

$$= 0.964 \tag{9.2.25.16}$$

$$\Pr(Y = 0) = F_Y(1) - F_Y(0) \tag{9.2.25.17}$$

$$= 0.964 - 0.1587 \tag{9.2.25.18}$$

$$= 0.8053 \tag{9.2.25.19}$$

(b) If we consider exactly 2 bulbs to be defective, the we need to find

$$\Pr(Y = 2) = \Pr(Y \leq 2) - \Pr(Y \leq 1) \tag{9.2.25.20}$$

$$= F_Y(2) - F_Y(1) \tag{9.2.25.21}$$

From (9.2.25.8) and Table 9.2.25.1,

$$F_Y(2) = 1 - Q\left(\frac{2-0.2}{0.44}\right) \tag{9.2.25.22}$$
$$= 1 - Q(4) \tag{9.2.25.23}$$
$$= 0.999 \tag{9.2.25.24}$$
$$F_Y(1) = 1 - Q\left(\frac{1-0.2}{0.44}\right) \tag{9.2.25.25}$$
$$= 1 - Q(1.8) \tag{9.2.25.26}$$
$$= 0.964 \tag{9.2.25.27}$$
$$\Pr(Y = 2) = F_Y(2) - F_Y(1) \tag{9.2.25.28}$$
$$= 0.999 - 0.964 \tag{9.2.25.29}$$
$$= 0.036 \tag{9.2.25.30}$$

(c) If more than 8 bulbs are working properly then either 1 bulb is defective or no bulb is defetive we need to find

$$\Pr(Y \leq 1) = F_Y(1) \tag{9.2.25.31}$$

From (9.2.25.8) and Table 9.2.25.1,

$$F_Y(1) = 1 - Q\left(\frac{1-0.2}{0.44}\right) \tag{9.2.25.32}$$
$$= 1 - Q(1.8) \tag{9.2.25.33}$$
$$= 0.964 \tag{9.2.25.34}$$

(ii) Binomial Distribution
Lets define a random variable X which represents the number of defective bulbs.

$$X = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \tag{9.2.25.35}$$

The pmf is given by

$$P_X(r) = {}^nC_r p^r (1-p)^{n-r} \tag{9.2.25.36}$$

(a) If we consider there is no defective bulb,

$$P_X(0) = 0.817 \tag{9.2.25.37}$$

(b) If we consider there are 2 defective bulbs,

$$P_X(2) = 0.0153 \tag{9.2.25.38}$$

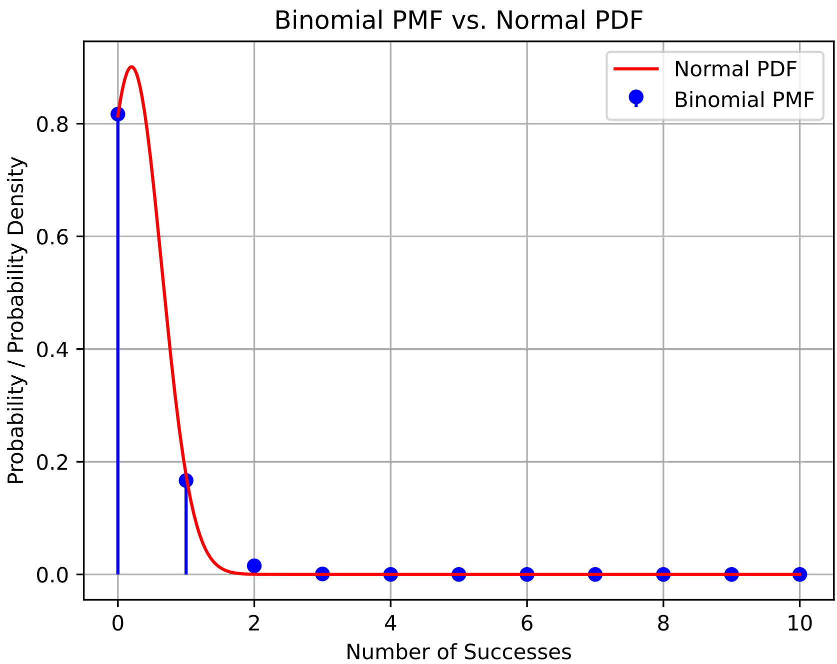
(c) If we consider there are more than 8 proper bulbs,

$$P_X(0) + P_X(1) = 0.817 + 0.1667 \tag{9.2.25.39}$$
$$= 0.9837 \tag{9.2.25.40}$$

(iii) Binomial vs Gaussian Graph

Number of defective bulbs	Binomial distribution	Gaussian
0	0.817	0.8053
2	0.0153	0.036
≤ 1	0.9837	0.964

TABLE 9.2.25.2: Comparing Binomial distribution and Gaussian approximation



9.2.26 A lot of 100 watches is known to have 10 defective watches. If 8 watches are selected (one by one with replacement) at random, what is the probability that there will be at least one defective watch? **Solution:**

parameter	value	description
n	8	Number of watches selected
p	$\frac{1}{10}$	Chosen watch is defective
q	$\frac{9}{10}$	Chosen watch is non-defective
$\mu = np$	$\frac{8}{10}$	Mean of the distribution
$\sigma^2 = npq$	$\frac{72}{100}$	Variance of the distribution

TABLE 9.2.26.1: Gaussian Info Table

(i) Gaussian Distribution
Let Y is the Gaussian obtained by approximating binomial with parameters n,p then By Central limit theroem,

$$Y \sim \mathcal{N}(np, npq) \tag{9.2.26.1}$$

CDF of Y is:

$$F_Y(x) = \Pr(Y \leq x) \tag{9.2.26.2}$$
$$= \Pr(Y - \mu \leq x - \mu) \tag{9.2.26.3}$$
$$= \Pr\left(\frac{Y-\mu}{\sigma} \leq \frac{x-\mu}{\sigma}\right) \tag{9.2.26.4}$$

Since,

$$\frac{Y-\mu}{\sigma} \sim \mathcal{N}(0,1) \tag{9.2.26.5}$$

Q function is defined

$$Q(x) = \Pr(Y > x)\ \forall x \in Y \sim \mathcal{N}(0,1) \tag{9.2.26.6}$$

$$F_Y(x) = 1 - \Pr\left(\frac{Y-\mu}{\sigma} > \frac{x-\mu}{\sigma}\right) \tag{9.2.26.7}$$
$$= \begin{cases} 1 - Q\left(\frac{x-\mu}{\sigma}\right), & x > \mu \\ Q\left(\frac{\mu-x}{\sigma}\right), & x < \mu \end{cases} \tag{9.2.26.8}$$

(a) For atleast one watch to be defective, we need to find

$$1 - \Pr(Y = 0) \tag{9.2.26.9}$$

$$\Pr(Y = 0) = \Pr(Y \leq 1) \tag{9.2.26.10}$$
$$= F_Y(1) \tag{9.2.26.11}$$

$$F_Y(1) = 1 - Q\left(\frac{1-0.8}{0.848}\right) \tag{9.2.26.12}$$
$$= 1 - Q(0.235) \tag{9.2.26.13}$$
$$= 0.58 \tag{9.2.26.14}$$
$$\Pr(Y = 0) = F_Y(1) \tag{9.2.26.15}$$
$$= 0.58 \tag{9.2.26.16}$$

(ii) Binomial Distribution
Lets define a random variable X which represents the number of defective bulbs.

$$X = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \tag{9.2.26.17}$$

The pmf is given by

$$P_X(r) = {}^nC_r p^r (1-p)^{n-r} \tag{9.2.26.18}$$

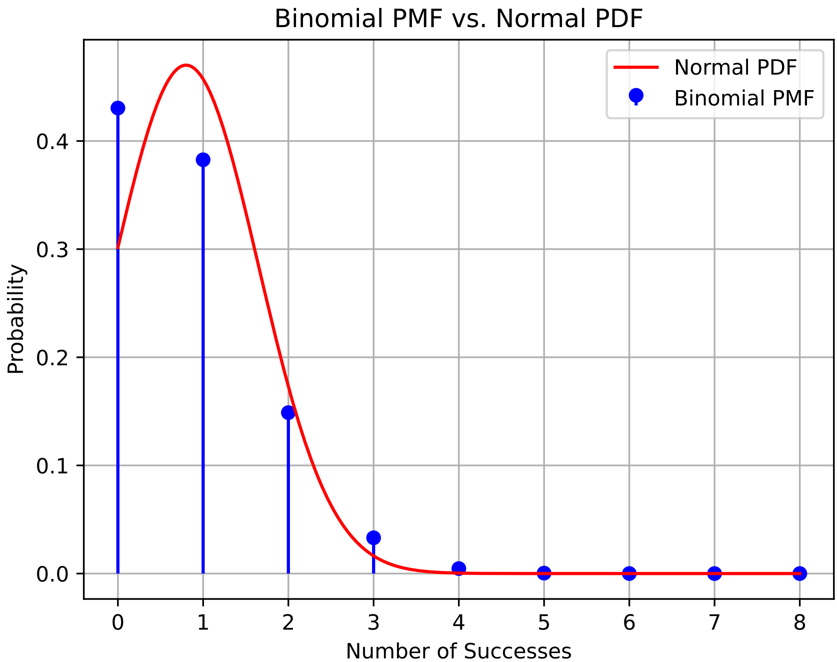
If we consider atleast one watch to be defective, we need,

$$1 - P_X(0) \tag{9.2.26.19}$$

$$P_X(0) = 0.430 \tag{9.2.26.20}$$

$$1 - P_X(0) = 0.569 \tag{9.2.26.21}$$

(iii) Binomial vs Gaussian Graph



9.2.27 The Probability of a man hitting target is 0.25.He shoots 7 times. What is the probability of his hitting atleast twice?

Solution: :

TABLE 9.2.27.1: Variables

Variable	Value	Description
n	7	Number of trails
p	0.25	The probability of man hitting the target
q	0.75	The probability of man not hitting the target
$\mu = np$	1.75	mean of distribution
$\sigma = \sqrt{npq}$	1.145	variance of distribution
X	$X \geq 2$	Number of times man hits the target

From gaussian,

$$Y \sim \mathcal{N}\left(\mu, \sigma^2\right) \tag{9.2.27.1}$$

CDF of Y is defined as:

$$F_Y(X) = \Pr(Y < X) \tag{9.2.27.2}$$

$$= \Pr\left(\frac{Y-\mu}{\sigma} \le \frac{X-\mu}{\sigma}\right) \tag{9.2.27.3}$$

$$\implies \frac{Y-\mu}{\sigma} \sim N(0,1) \tag{9.2.27.4}$$

$$= 1 - \Pr\left(\frac{Y-\mu}{\sigma} > \frac{X-\mu}{\sigma}\right) \tag{9.2.27.5}$$

$$= \begin{cases} 1 - Q\left(\frac{X-\mu}{\sigma}\right) & X \ge \mu \\ Q\left(\frac{\mu-X}{\sigma}\right) & X < \mu \end{cases} \tag{9.2.27.6}$$

Hence, the probability of hitting target atleast twice using gaussian distribution is: without correction:

$$\Pr(Y \ge 2) = 1 - \Pr(Y < 2) \tag{9.2.27.7}$$

$$= 1 - F_Y(2) \tag{9.2.27.8}$$

$$\implies \Pr(Y \ge 2) = Q\left(\frac{X-\mu}{\sigma}\right) \tag{9.2.27.9}$$

$$= Q(0.218) \tag{9.2.27.10}$$

$$\Pr(Y \ge 2) = 0.4137 \tag{9.2.27.11}$$

with correction:

$$\Pr(Y \ge 2) = Q\left(\frac{X - 0.5 - \mu}{\sigma}\right) \tag{9.2.27.12}$$

$$= Q(-0.218) \tag{9.2.27.13}$$

$$\Pr(Y \ge 2) = 0.5862 \tag{9.2.27.14}$$

Hence, the probability of hitting target atleast twice using binomial distribution is:

$$\Pr(X \ge 2) = 1 - \Pr(X < 2) \tag{9.2.27.15}$$

$$= 1 - \sum_{k=0}^{1} {}^{n}C_k p^k (1-p)^{n-k} \tag{9.2.27.16}$$

$$= 0.55 \tag{9.2.27.17}$$

9.2.28 A person buys a lottery ticket in 50 lotteries in each of which his chance of winning a prize is $\frac{1}{100}$. What is the probability that he will win a prize

(a) atleast once
(b) exactly once
(c) atleast twice ?

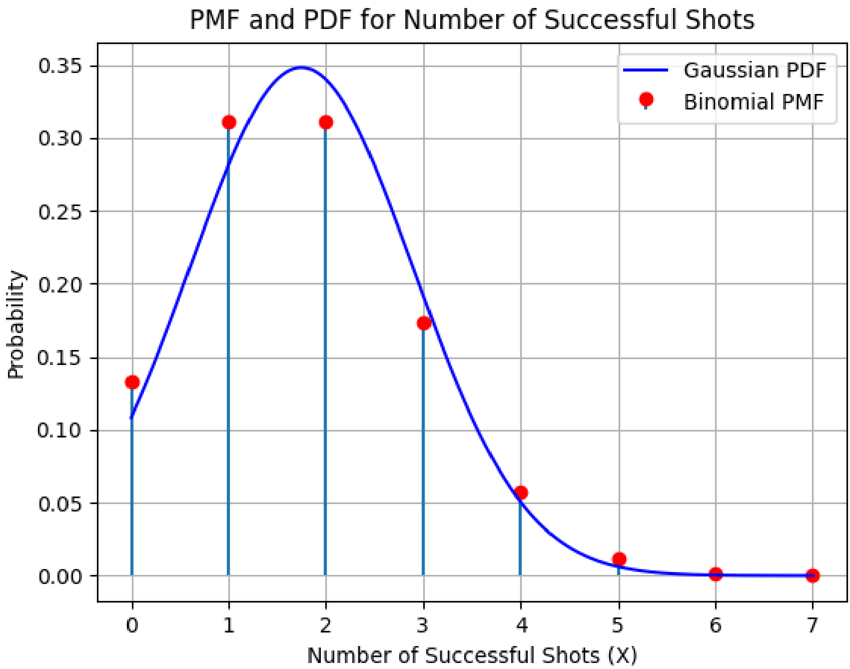


Fig. 9.2.27.1: gaussian and binomial

Solution: Let us define:

Parameter	Value	Description
n	50	number of lotteries
p	0.01	probability of winning a prize
q	0.99	probability of not winning
$\mu = np$	0.5	mean of the distribution
$\sigma^2 = npq$	0.495	variance of the distribution
Y	0,1,2,3,...,50	Number of successes

(a) **using Gaussian**

$$Y \sim \mathcal{N}\left(\mu, \sigma^2\right) \tag{9.2.28.1}$$

The CDF of Y:

$$F_Y(y) = 1 - \Pr(Y > y) \tag{9.2.28.2}$$

$$= 1 - \Pr\left(\frac{Y-\mu}{\sigma} > \frac{y-\mu}{\sigma}\right) \tag{9.2.28.3}$$

But,

$$\frac{Y-\mu}{\sigma} \sim \mathcal{N}(0,1) \tag{9.2.28.4}$$

$$\tag{9.2.28.5}$$

the Q-function is defined as:

$$Q(x) = \Pr(Y > x)\ \forall x \in Y \sim \mathcal{N}(0,1) \tag{9.2.28.6}$$

therefore the cdf will be:

$$F_Y(y) = \begin{cases} 1 - Q\left(\frac{y-\mu}{\sigma}\right), & y > \mu \\ Q\left(\frac{\mu-y}{\sigma}\right), & y < \mu \end{cases} \tag{9.2.28.7}$$

The probabbility of winning the prize atleast once is given by:
Considering 0.3 as the correction term,

$$\Pr(Y > 0.7) = 1 - F_Y(0.7) \tag{9.2.28.8}$$

$$= Q\left(\frac{0.7-\mu}{\sigma}\right) \quad from(9.2.28.7) \tag{9.2.28.9}$$

$$= Q(0.2842) \tag{9.2.28.10}$$

$$= 0.3881 \tag{9.2.28.11}$$

(b) **using Gaussian**
the gaussian distribution function is defined as:

$$p_Y(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \tag{9.2.28.12}$$

the probability of the person winning the prize exactly once is given by:

$$p_Y(1) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(1-\mu)^2}{2\sigma^2}} \tag{9.2.28.13}$$

$$= 0.44 \tag{9.2.28.14}$$

(c) **using Gaussian**
the probability of the person winning the prize atleast twice is given by:
considering 0.5 as the correction term,

$$\Pr(Y > 1.5) = 1 - F_Y(1.5) \tag{9.2.28.15}$$

$$= Q\left(\frac{1.5-\mu}{\sigma}\right) \quad from(9.2.28.7) \tag{9.2.28.16}$$

$$= Q\left(\frac{0.1}{\sqrt{0.495}}\right) \tag{9.2.28.17}$$

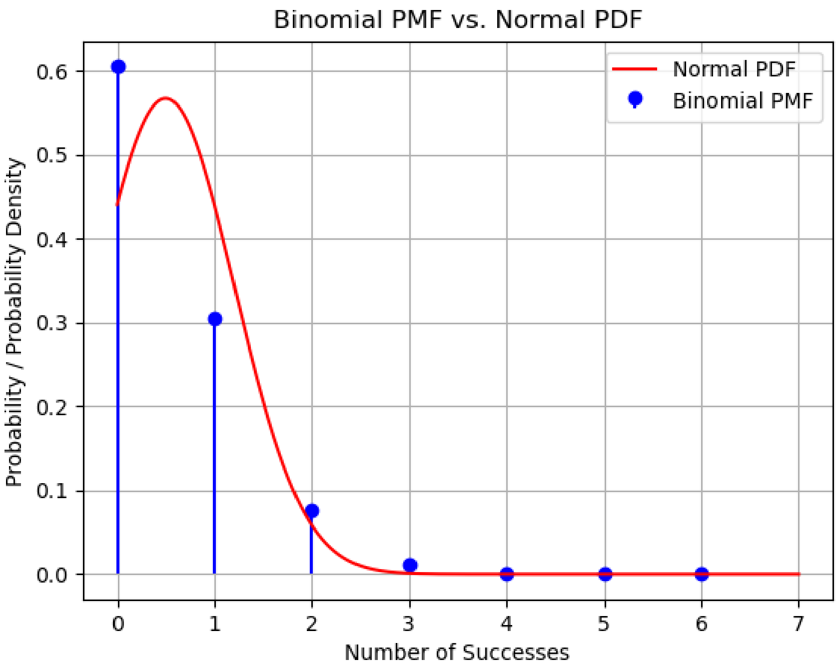
$$= Q(1.42) \tag{9.2.28.18}$$

$$= 0.0776 \tag{9.2.28.19}$$

Gaussian vs Binomial Table

Y	Gaussian	Binomial
atleast one	0.3881	0.395
exactly one	0.441	0.305
atleast two	0.0776	0.089

Gaussian vs Binomial graph



9.2.29 The probability that a person is not a swimmer is 0.3. The probability that out of 5 persons 4 are swimmers is

a) ${}^{5}C_{4}\,(0.7)^{4}\,(0.3)$
b) ${}^{5}C_{1}\,(0.7)\,(0.3)^{4}$
c) ${}^{5}C_{4}\,(0.7)\,(0.3)^{4}$
d) $(0.7)^{4}\,(0.3)$

Solution: Let Y be the gaussian random variable,

$$Y \sim \mathcal{N}\left(\mu, \sigma^2\right) \tag{9.2.29.1}$$

$$\sim \mathcal{N}\,(3.5, 1.05) \tag{9.2.29.2}$$

Due to continuity correction $\Pr\left(X = x\right)$ can be approximated using gaussian distri-

Parameter	Values	Description
n	5	Number of draws
p	0.3	Probability that person is not a swimmer
q	0.7	Probability that person is a swimmer
$\mu = np$	3.5	Mean
$\sigma^2 = npq$	1.05	Variance

bution as

$$p_Y(x) \approx \Pr(x - 0.5 < Y < x + 0.5) \tag{9.2.29.3}$$
$$\approx \Pr(Y < x + 0.5) - \Pr(Y < x - 0.5) \tag{9.2.29.4}$$
$$\approx F_Y(x + 0.5) - F_Y(x - 0.5) \tag{9.2.29.5}$$
$$\tag{9.2.29.6}$$

then CDF of Y is:

$$F_Y(x) = \Pr(Y < x) \tag{9.2.29.7}$$
$$= \Pr\left(\frac{Y-\mu}{\sigma} < \frac{x-\mu}{\sigma}\right) \tag{9.2.29.8}$$
$$\implies \frac{Y-\mu}{\sigma} \sim N(0,1) \tag{9.2.29.9}$$
$$= 1 - \Pr\left(\frac{Y-\mu}{\sigma} > \frac{x-\mu}{\sigma}\right) \tag{9.2.29.10}$$
$$= \begin{cases} 1 - Q\left(\frac{x-\mu}{\sigma}\right) & x \geq \mu \\ Q\left(\frac{\mu-x}{\sigma}\right) & x < \mu \end{cases} \tag{9.2.29.11}$$

Hence, probability that out of 5 persons 4 are swimmers using gaussian approximation is

$$\Pr(Y = 4) = \Pr(3.5 < Y < 4.5) \tag{9.2.29.12}$$
$$= 0.335 \tag{9.2.29.13}$$

Probability that out of 5 persons 4 are swimmers using bernoulli distribution is

$$\Pr(Y = 4) = p_Y(4) \tag{9.2.29.14}$$
$$= {}^nC_k p^k (1-p)^{n-k} \tag{9.2.29.15}$$
$$= 0.360 \tag{9.2.29.16}$$

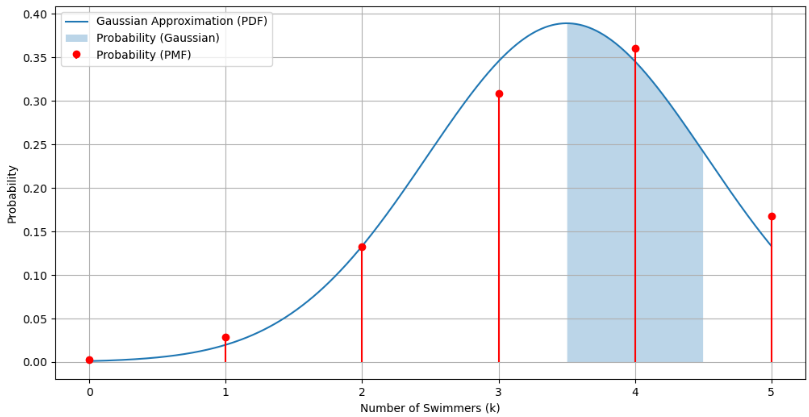


Fig. 9.2.29.1: PDF vs Gaussian

9.2.30 The probability of guessing correctly at least 8 out of 10 answers on a true-false type examination is

Solution: Defining variables:

Parameter	Value	Description
n	10	Number of questions
p	0.5	probability of guessing correctly
$\mu = np$	5	mean of the distribution
$\sigma^2 = np(1-p)$	2.5	variance of the distribution
Y	0-10	denotes number of questions guessed correctly

a) **Binomial distribution:** the probability of getting exactly 8 correct answers is

$$= \binom{10}{8} \times 0.5^8 \times 0.5^2 \tag{9.2.30.1}$$

$$= 0.043946 \tag{9.2.30.2}$$

b) **Gaussian Distribution:**

The gaussian distribution for Y is

$$p_Y(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} \tag{9.2.30.3}$$

For getting exactly 8 correct answers

$$Y = 8 \tag{9.2.30.4}$$

Substituting in equation (9.2.31.3), probability for getting exactly 8 correct answers is

$$p_Y(8) = \frac{1}{\sqrt{2\pi \times 2.5}} e^{\frac{-(8-5)^2}{2\times 2.5}} \tag{9.2.30.5}$$
$$= 0.05204 \tag{9.2.30.6}$$

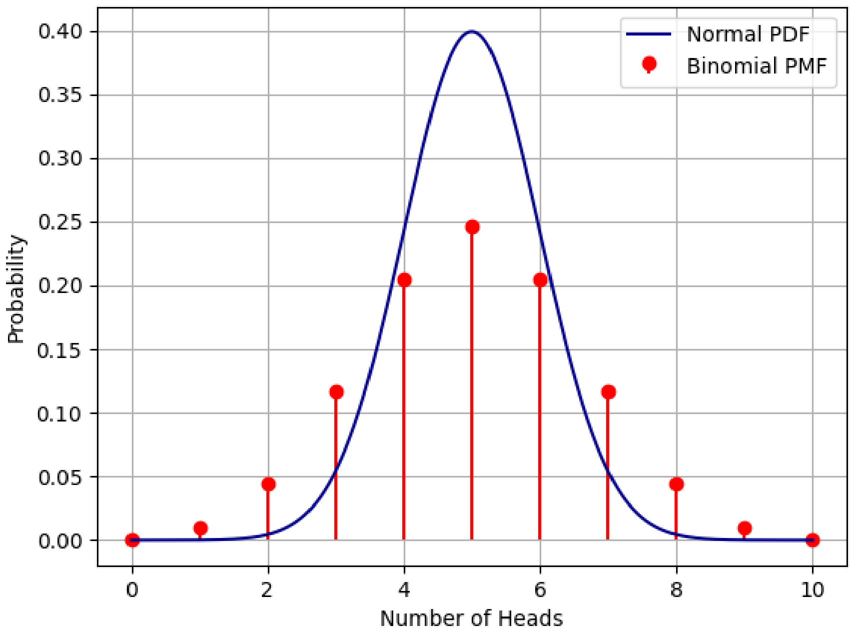


Fig. 9.2.30.1: Binomial distribution vs Gaussian distribution

c) **Using Q function:** Defining a gaussian random variable Z such that

$$Z \sim \mathcal{N}\left(\mu, \sigma^2\right) \tag{9.2.30.7}$$

Due to continuity correction, $\Pr(Z = x)$ can be approximated as

$$p_Z(x) \approx \Pr(x - 0.5 \le Z < x + 0.5) \tag{9.2.30.8}$$
$$\approx \Pr(Z < x + 0.5) - \Pr(Z < x - 0.5) \tag{9.2.30.9}$$
$$\approx F_Z(x + 0.5) - F_Z(x - 0.5) \tag{9.2.30.10}$$

CDF of Z is defined as -

$$F_Z(x) = \Pr(Z < x) \tag{9.2.30.11}$$
$$= \Pr\left(\frac{Z-\mu}{\sigma} < \frac{x-\mu}{\sigma}\right) \tag{9.2.30.12}$$

As

$$\frac{Z-\mu}{\sigma} \sim \mathcal{N}(0,1) \tag{9.2.30.13}$$

$$\implies F_Z(x) = 1 - \Pr\left(\frac{Z-\mu}{\sigma} > \frac{x-\mu}{\sigma}\right) \tag{9.2.30.14}$$

$$= \begin{cases} 1 - Q(\frac{x-\mu}{\sigma}) & x \geq \mu \\ Q(\frac{\mu-x}{\sigma}) & x < \mu \end{cases} \tag{9.2.30.15}$$

$\therefore$ Gaussian approximation for $\Pr(Z=8)$ is

$$p_Z(8) = 1 - Q(1.63273) \tag{9.2.30.16}$$

$$= 0.051263 \tag{9.2.30.17}$$

9.2.31 Eight coins are tossed together. The probability of getting exactly 3 heads is

a) $\frac{1}{256}$

b) $\frac{7}{32}$

c) $\frac{5}{32}$

d) $\frac{3}{32}$

Solution: Defining variables:

Parameter	Value	Description
n	8	Number of coins tossed
p	0.5	probability of getting heads
$\mu = np$	4	mean of the distribution
$\sigma^2 = np(1-p)$	2	variance of the distribution
Y	0-8	denotes number of heads obtained

a) **Binomial distribution:** the probability of getting exactly 3 heads is

$$= \binom{8}{3} \times 0.5^3 \times 0.5^5 \tag{9.2.31.1}$$

$$= 0.21875 \tag{9.2.31.2}$$

$\therefore$ option 2 is correct.

b) **Gaussian Distribution:**
The gaussian distribution for Y is

$$p_Y(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} \tag{9.2.31.3}$$

For getting 3 exactly heads

$$Y = 3 \tag{9.2.31.4}$$

Substituting in equation (9.2.31.3), probability for getting exactly 3 heads is

$$Y = 3 \tag{9.2.31.5}$$

$$p_Y(3) = \frac{1}{\sqrt{2\pi \times 2}} e^{\frac{-(3-4)^2}{2\times 2}} \tag{9.2.31.6}$$

$$= 0.35206 \tag{9.2.31.7}$$

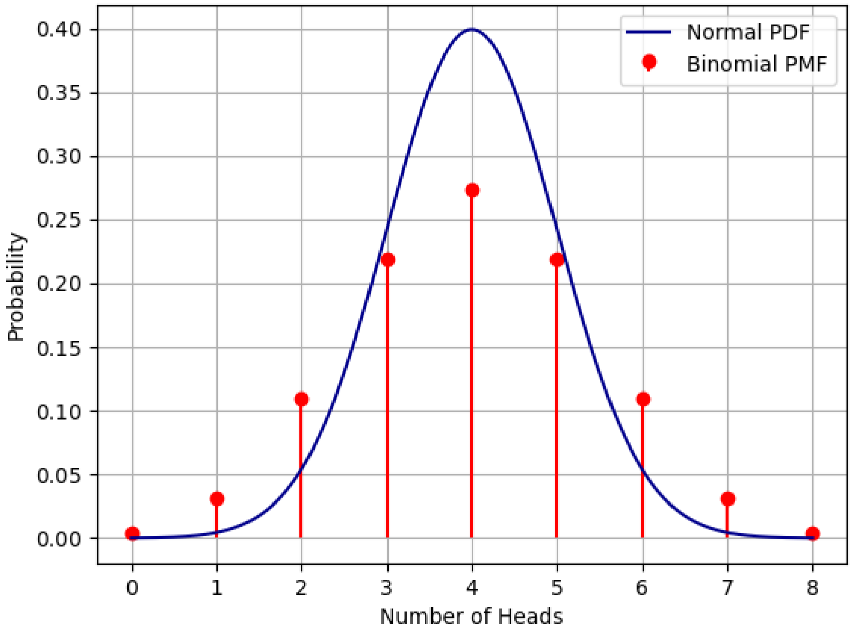


Fig. 9.2.31.1: Binomial distribution vs Gaussian distribution

c) **Using Q function:** Defining a gaussian random variable Z such that

$$Z \sim \mathcal{N}\left(\mu, \sigma^2\right) \tag{9.2.31.8}$$

Due to continuity correction, $\Pr(Z = x)$ can be approximated as

$$p_Z(x) \approx \Pr(x - 0.5 \le Z < x + 0.5) \tag{9.2.31.9}$$

$$\approx \Pr(Z < x + 0.5) - \Pr(Z < x - 0.5) \tag{9.2.31.10}$$

$$\approx F_Z(x + 0.5) - F_Z(x - 0.5) \tag{9.2.31.11}$$

CDF of Z is defined as

$$F_Z(x) = \Pr(Z < x) \tag{9.2.31.12}$$

$$= \Pr\left(\frac{Z-\mu}{\sigma} < \frac{x-\mu}{\sigma}\right) \tag{9.2.31.13}$$

As

$$\frac{Z-\mu}{\sigma} \sim \mathcal{N}(0,1) \tag{9.2.31.14}$$

$$\implies F_Z(x) = 1 - \Pr\left(\frac{Z-\mu}{\sigma} > \frac{x-\mu}{\sigma}\right) \tag{9.2.31.15}$$

$$= \begin{cases} 1 - Q(\frac{x-\mu}{\sigma}) & x \geq \mu \\ Q(\frac{\mu-x}{\sigma}) & x < \mu \end{cases} \tag{9.2.31.16}$$

Probability in terms of Q function is

$$p_Z(x) \approx Q\left(\frac{(x-0.5)-\mu}{\sigma}\right) - Q\left(\frac{(x+0.5)-\mu}{\sigma}\right) \tag{9.2.31.17}$$

$\therefore$ Gaussian approximation for $\Pr(Z = 3)$ is

$$p_Z(3) \approx Q(0.3536) - Q(1.0608) \tag{9.2.31.18}$$

$$= 0.2174 \tag{9.2.31.19}$$

d) **Comparing all three techniques:**

Event	**Binomial**	**Gaussian**	**Q function**
Getting exactly 3 heads	0.21875	0.35206	0.2174

9.2.32 If X follows binomial distribution with parameters $n = 5$, p and

$$p_X(2) = 9p_X(3) \tag{9.2.32.1}$$

then p is ?

$$\mu = np \tag{9.2.32.2}$$

$$= 5p \tag{9.2.32.3}$$

$$\sigma^2 = np(1-p) \tag{9.2.32.4}$$

$$= 5p(1-p) \tag{9.2.32.5}$$

$$Y \sim N(\mu, \sigma) \tag{9.2.32.6}$$

Using the condition $p_Y(2) = 9p_Y(3)$, we get:

$$e^{-\frac{1}{2}(\frac{2-\mu}{\sigma})^2} = 9e^{-\frac{1}{2}\left(\frac{3-\mu}{\sigma}\right)^2} \tag{9.2.32.7}$$

$$\implies e^{-\frac{1}{2}\left(\frac{2-5p}{\sqrt{5p(1-p)}}\right)^2} = 9e^{-\frac{1}{2}\left(\frac{3-5p}{\sqrt{5p(1-p)}}\right)^2} \tag{9.2.32.8}$$

$$\implies e^{-\frac{1}{2}\left(\frac{2-5p}{\sqrt{5p(1-p)}}\right)^2} = 9e^{-\frac{1}{2}\left(\frac{3-5p}{\sqrt{5p(1-p)}}\right)^2} \tag{9.2.32.9}$$

$$\implies e^{-\frac{1}{2}\left(\frac{(2-5p)^2-(3-5p)^2}{(\sqrt{5p(1-p)})^2}\right)} = 9 \tag{9.2.32.10}$$

Taking the natural logarithm of both sides, we have:

$$\implies -\frac{1}{2}\left(\frac{(2-5p)^2-(3-5p)^2}{5p(1-p)}\right) = \ln(9) \tag{9.2.32.11}$$

$$\implies 4 + 25p^2 - 20p - 9 - 25p^2 + 30p = -10p(1-p)\ln(9) \tag{9.2.32.12}$$

$$\implies 10p - 5 = -10p(1-p)\ln(9) \tag{9.2.32.13}$$

$$\implies 1 - 2p = (2p - 2p^2)\ln(9) \tag{9.2.32.14}$$

$$\implies 2p^2\ln(9) - 2p\ln(9) - 2p + 1 = 0 \tag{9.2.32.15}$$

$$p = \frac{2\ln(9) + 2 \pm \sqrt{(-2\ln(9) - 2)^2 - 4(2\ln(9))(1)}}{2(2\ln(9))} \tag{9.2.32.16}$$

$$= \frac{2\ln(9) + 2 \pm \sqrt{4(\ln(9))^2 + 4}}{4\ln(9)} \tag{9.2.32.17}$$

$$= 0.178211588 \tag{9.2.32.18}$$

9.2.33 A box has 100 pens of which 10 are defective. What is the probability that out of a sample of 5 pens drawn one by one with replacement at most one is defective?

(a) $\left(\frac{9}{10}\right)^5$

(b) $\frac{1}{2}\left(\frac{9}{10}\right)^4$

(c) $\frac{1}{2}\left(\frac{9}{10}\right)^5$

(d) $\frac{1}{2}\left(\frac{9}{10}\right)^4 + \left(\frac{9}{10}\right)^5$

Solution:

Parameter	Values	Description
n	5	Number of defective pens
p	0.1	probability of drawing a defective pen
μ	0.5	np
σ	0.671	$\sqrt{np(1-p)}$
X		Defective pens

Using Binomial

Given,

Probability of drawing a defective pen = $\frac{1}{10}$
Probability of drawing a non-defective pen = $\frac{9}{10}$
Let,
Probability of drawing atmost one pen out of 5 defective with replacement = $\Pr(X \leq 1)$

$$\Pr(X \leq 1) = p_X(0) + p_X(1) \tag{9.2.33.1}$$

$$\implies \Pr(X \leq 1) = \binom{5}{0}\left(\frac{9}{10}\right)^5 + \binom{5}{1}\left(\frac{9}{10}\right)^4\left(\frac{1}{10}\right) \tag{9.2.33.2}$$

$$= \left(\frac{9}{10}\right)^5 + 5\left(\frac{9}{10}\right)^4\left(\frac{1}{10}\right) \tag{9.2.33.3}$$

$$= \left(\frac{9}{10}\right)^5 + \frac{1}{2}\left(\frac{9}{10}\right)^4 \tag{9.2.33.4}$$

$$= 0.91854 \tag{9.2.33.5}$$

Gaussian

$$Y \sim \mathcal{N}\left(\mu, \sigma^2\right) \tag{9.2.33.6}$$

CDF of Y is

$$F_Y(y) = \Pr(Y \leq y) \tag{9.2.33.7}$$

We know that

$$Q(x) = \Pr(X > x), x > 0, X \sim N(0, 1) \tag{9.2.33.8}$$

$$Q(-x) = \Pr(X > -x), x < 0, X \sim N(0, 1) \tag{9.2.33.9}$$

$$= 1 - Q(x) \tag{9.2.33.10}$$

Hence,
CDF :

$$F_Y(y) = \begin{cases} 1 - Q\left(\frac{y-\mu}{\sigma}\right), & \text{if } y > \mu \\ 1 - Q\left(\frac{y-\mu}{\sigma}\right) = Q\left(\frac{\mu-y}{\sigma}\right), & \text{if } y < \mu \end{cases} \tag{9.2.33.11}$$

$$F_Y(1) = \Pr(Y \leq 1) \tag{9.2.33.12}$$

$$= 1 - Q\left(\frac{1 - 0.5}{\sqrt{0.671}}\right) \tag{9.2.33.13}$$

$$= 1 - Q\left(\frac{0.5}{0.819}\right) \tag{9.2.33.14}$$

$$= 1 - Q(0.6104) \tag{9.2.33.15}$$

$$= 0.729198876 \tag{9.2.33.16}$$

9.2.34 There are 5% defective items in a large bulk of items. What is the probability that

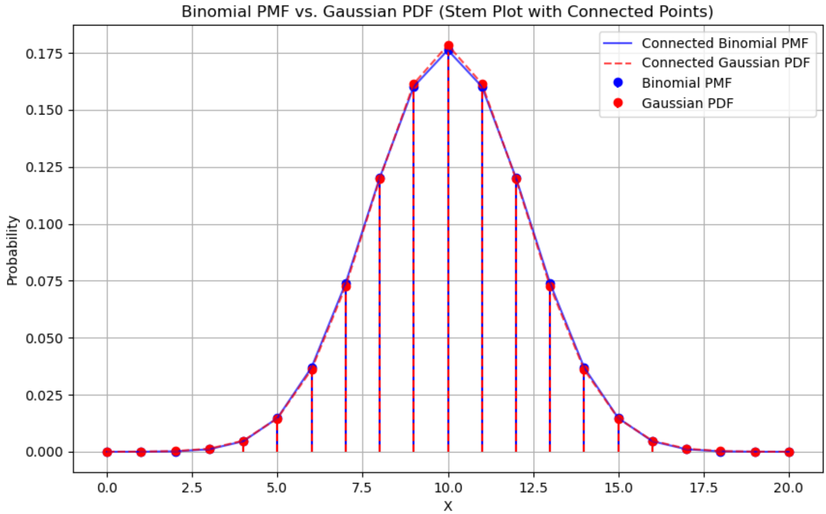


Fig. 9.2.33.1: pmf of binomial and pdf of Gaussian of X and Y marked balls

a sample of 10 items will include not more than one defective item?

Solution:

9.2.35 Five cards are drawn successively with replacement from a well-shuffled deck of 52 cards. What is the probability that

a) all the five cards are spades?
b) only 3 cards are spades?
c) none is a spade?

Solution:

9.2.36 In an examination, 20 questions of true-false type are asked. Suppose a student tosses a fair coin to determine his answer to each question. If the coin falls heads, he answer true; if it falls tails, he answer false. Find the probability that he answers at least 12 questions correctly.

Solution:

9.2.37 It is known that 10 % of certain articles manufactured are defective. What is the probability that in a random sample space of 12 such articles,9 are defective?

Solution:

9.2.38 The probability that a student is not a swimmer is $\frac{1}{5}$ .Then the probability that out of five students, four are swimmers

a) ${}^5C_4\left(\frac{4}{5}\right)^4\frac{1}{5}$

b) $\left(\frac{4}{5}\right)^4\frac{1}{5}$

c) ${}^5C_1\frac{1}{5}\left(\frac{4}{5}\right)^4$

d) None of these

Solution: See Table 9.2.38.1. The pmf of X is

Parameter	Value	Description
n	5	number of students
q	$\frac{1}{5}$	probability for not a swimmer
p	$\frac{4}{5}$	probability for a swimmer
k	4	number of swimmers

TABLE 9.2.38.1: Given Information

$$p_X(k) = {}^nC_k p^k q^{n-k} \tag{9.2.38.1}$$

and the desired probability is

$$p_X(4) = {}^5C_4\left(\frac{4}{5}\right)^4\left(\frac{1}{5}\right)^{5-4} \tag{9.2.38.2}$$

Hence, option 9.2.38a is correct.

9.2.39 Suppose that 90 % of people are right-handed. What is the probability that atmost 6 of a random sample of 10 people are right-handed.

Solution: From the given information, $X \sim \text{B}\left(10, \frac{9}{10}\right)$. From (9.1.2.2),

$$\mu = 10 \times \frac{9}{10} = 9, \tag{9.2.39.1}$$

$$\sigma^2 = 10 \times \frac{9}{10} \times \frac{1}{10} = \frac{9}{10} \tag{9.2.39.2}$$

The desired probability is

$$\Pr(X \leq 6) \approx F_Y(6) = Q\left(\frac{9-6}{3}\sqrt{10}\right) \tag{9.2.39.3}$$

$$= Q\left(\sqrt{10}\right) \tag{9.2.39.4}$$

from (9.1.7.1).

9.2.40 A die is thrown again and again until three sixes are obtained. Find the probability of obtaining third six on sixth throw of a die.

Solution: Let E the event be getting third six on sixth throw

Binomial pmf given by,

$\Pr(X = k) = {}^nC_k p^k (1-p)^{n-k}$

probability of getting two sixes in first five throws,

TABLE 9.2.40.1: parameters for PMF

parameter	value
n	5
p	$\frac{1}{6}$
k	2
$1-p$	$\frac{5}{6}$

$$\Pr(k=2) = {}^5C_2\left(\frac{1}{6}\right)^2\left(1-\frac{1}{6}\right)^{5-2} \tag{9.2.40.1}$$
$$= {}^5C_2\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^3 \tag{9.2.40.2}$$
$$= \frac{10\times 5^3}{6^5} \tag{9.2.40.3}$$
$$= \frac{1250}{7776} \tag{9.2.40.4}$$

Now,

$$\Pr(E) = p.\Pr(k=2) \tag{9.2.40.5}$$
$$= \frac{1}{6}\times\frac{1250}{7776} \tag{9.2.40.6}$$
$$= 0.026 \tag{9.2.40.7}$$

Hence, probability of getting third six on sixth throw of a die is 0.026

10 Random Variables

10.1 NCERT

10.1.1 Assume that in a family, each child is equally likely to be a boy or a girl. A family with three children is chosen at random. The probability that the eldest child is a girl given that the family has at least one girl is

a) $\frac{1}{2}$
b) $\frac{1}{3}$
c) $\frac{2}{3}$
d) $\frac{4}{7}$

Solution:
Let X_0, X_1, X_2 be the random variables which denotes the three children, where X_0 is the eldest child and X_2 is the youngest child.

RV	Value	Description
X_i	0	child is boy
	1	child is girl

TABLE 10.1.1.1: RV description table

so the required probability is,

$$\Pr(X_0 = 1 | X_0 + X_1 + X_2 \geq 1) = \frac{\Pr(X_0 = 1, X_0 + X_1 + X_2 \geq 1)}{\Pr(X_0 + X_1 + X_2 \geq 1)} \tag{10.1.1.1}$$

$$= \frac{\Pr(X_0 = 1) \times \Pr(X_1 + X_2 \geq 0)}{\Pr(X_0 + X_1 + X_2 \geq 1)} \tag{10.1.1.2}$$

$$= \frac{\frac{1}{2} \times \sum_{k=0}^{2} {}^2C_k \times \frac{1}{2}^k \times \frac{1}{2}^{2-k}}{\sum_{k=1}^{3} {}^3C_k \times \frac{1}{2}^k \times \frac{1}{2}^{3-k}} \tag{10.1.1.3}$$

$$= \frac{\frac{1}{2} \times 1}{\frac{3}{8} + \frac{3}{8} + \frac{1}{8}} \tag{10.1.1.4}$$

$$= \frac{4}{7} \tag{10.1.1.5}$$

Therefore, the probability that the eldest child is a girl given that the family has atleast one girl is $\frac{4}{7}$

10.1.2 State whether the statement is True or False. The probabilities that a typist will make 0, 1, 2, 3, 4, 5 or more mistakes in typing a report are, respectively, 0.12, 0.25, 0.36, 0.14, 0.08, 0.11.

Solution: From the given information, we obtain the distribution

$$p_X(k) = \begin{cases} 0.12 & k = 0 \\ 0.25 & k = 1 \\ 0.36 & k = 2 \\ 0.14 & k = 3 \\ 0.08 & k = 4 \\ 0.11 & k \geq 5 \end{cases} \tag{10.1.2.1}$$

Since

$$\sum_{i=0}^{5} p_X(k) = 1.06 > 1 \tag{10.1.2.2}$$

violates (2.1.4.1), the given statement is false.

10.1.3 State which of the following are not the probability distributions of a random variable. Give reasons for your answer.

a)

X	0	1	2
$P(X)$	0.4	0.4	0.2

c)

Y	-1	0	1
$P(Y)$	0.6	0.1	0.2

b)

X	0	1	2	3	4
$P(X)$	0.1	0.5	0.2	-0.1	0.3

d)

X	0	1	2	3	4
$P(Z)$	0.3	0.2	0.4	0.1	0.05

Solution:

a) The given distribution satisfies (2.1.4.1) and (2.1.8.4), so it is a valid probability distribution.

b)

$$p_X(3) = -0.1 < 0 \tag{10.1.3.1}$$

which violates (2.1.4.1). Hence, not a probability distribution.

c)

$$\sum_{k=-1}^{1} p_X(k) = 0.9 < 1 \tag{10.1.3.2}$$

which violates (2.1.8.4). So, not a probability distribution.

d)

$$\sum_{k=0}^{4} p_X(k) = 1.05 > 1 \tag{10.1.3.3}$$

which violates (2.1.8.4). So, not a probability distribution.

10.1.4 A die has two faces each with number ‘1’, three faces each with number ‘2’ and one face with number ‘3’. If die is rolled once, determine

a) Pr (2)
b) Pr (1 or 3)
c) Pr (not 3)

Solution: The given information is summarized in the following table 10.1.4.1

RV	**Description**	**Probability**
$X = 1$	Die rolls to 1	$\frac{1}{3}$
$X = 2$	Die rolls to 2	$\frac{1}{2}$
$X = 3$	Die rolls to 3	$\frac{1}{6}$

TABLE 10.1.4.1: Random variable X

a)

$$\Pr(X = 2) = \frac{1}{2} \tag{10.1.4.1}$$

b) Since

$$X = 1 \text{ or } X = 3 \equiv X \in \{1, 3\} \tag{10.1.4.2}$$

$$X = 1 \text{ and } X = 3 \equiv X = \phi \tag{10.1.4.3}$$

$$\Pr(X \in \{1,3\}) = \Pr(X = 1) + \Pr(X = 3) - \Pr(X = \phi) \tag{10.1.4.4}$$
$$= \frac{1}{3} + \frac{1}{6} \tag{10.1.4.5}$$
$$= \frac{1}{2} \tag{10.1.4.6}$$

c)

$$\Pr(X \neq 3) = 1 - \Pr(X = 3) \tag{10.1.4.7}$$
$$= 1 - \frac{1}{6} \tag{10.1.4.8}$$
$$= \frac{5}{6} \tag{10.1.4.9}$$

10.1.5 A die marked 1, 2, 3 in red and 4, 5, 6 in green is tossed. Let A be the event,'the number is even,' and B be the event, 'the number is red'. Are A and B independent?
Solution: Let

$$X = \begin{cases} 0, & \text{if number is odd} \\ 1, & \text{if number is even} \end{cases} \tag{10.1.5.1}$$
$$Y = \begin{cases} 0, & \text{if number is green} \\ 1, & \text{if number is red} \end{cases} \tag{10.1.5.2}$$

From the given information,

$$\Pr(X = 1) = \frac{3}{6} = \frac{1}{2}, \Pr(Y = 1) = \frac{3}{6} = \frac{1}{2} \tag{10.1.5.3}$$
$$\Pr(X = 1, Y = 1) = \frac{1}{6} \tag{10.1.5.4}$$

Now,

$$\Pr(X = 1) \times \Pr(Y = 1) = \frac{1}{4} \tag{10.1.5.5}$$
$$\implies \Pr(X = 1, Y = 1) \neq \Pr(X = 1) \times \Pr(Y = 1) \tag{10.1.5.6}$$

Hence, A and B are not independent.

10.1.6 A fair coin and an unbiased die are tossed. Let A be the event 'head appears on the coin' and B be the event '3 on the die'. Check whether A and B are independent events or not.

10.1.7 A child's game has 8 triangles of which 3 are blue and rest are red, and 10 squares of which 6 are blue and rest are red. One piece is lost at random. Find the probability that it is a

a) triangle
b) square
c) square of blue colour
d) triangle of red colour

Solution: The random variables in the problem are summarized in Table 10.1.7.1.

From the given information,

$$p_X(k) = \begin{cases} \frac{10}{18} & k = 0 \\ \frac{8}{18} & k = 1 \end{cases} \tag{10.1.7.1}$$

$$\Pr(Y = 0|X = 1) = \frac{5}{8} \tag{10.1.7.2}$$

$$\Pr(Y = 1|X = 1) = \frac{3}{8} \tag{10.1.7.3}$$

$$\Pr(Y = 0|X = 0) = \frac{4}{10} \tag{10.1.7.4}$$

$$\Pr(Y = 1|X = 0) = \frac{6}{10} \tag{10.1.7.5}$$

Consequently,

a) $p_X(1) = \frac{8}{18}$

b) $p_X(0) = \frac{10}{18}$

c) $p_{XY}(0, 1) = \Pr(Y = 1|X = 0)\, p_X(0) = \frac{6}{18}$

d) $p_{XY}(1, 0) = \Pr(Y = 0|X = 1)\, p_X(1) = \frac{5}{18}$

TABLE 10.1.7.1

Variable	Value	Description
X	1	Triangle
	0	Square
Y	1	Blue
	0	Red

10.1.8 Suppose you have two coins which appear identical in your pocket. You know that one is fair and one is 2-headed. If you take one out, toss it and get a head, what is the probability that it was a fair coin?

See Table 10.1.8.1. Given,

$$\Pr(X = 1) = \frac{1}{2},\ \Pr(X = 0) = \frac{1}{2},\ \Pr(Y = 1 \mid X = 1) = \frac{1}{2}, \tag{10.1.8.1}$$

$$\Pr(Y = 1 \mid X = 0) = 1. \tag{10.1.8.2}$$

Hence, the desired probability is

$$\Pr(X = 1 \mid Y = 1) = \frac{\Pr(Y = 1 \mid X = 1) \times \Pr(X = 1)}{\sum_{k=0}^{1} \Pr(Y = 1 \mid X = k) \times \Pr(X = k)} \tag{10.1.8.3}$$

$$= \frac{\frac{1}{2} \times \frac{1}{2}}{1 \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2}} = \frac{1}{3} \tag{10.1.8.4}$$

TABLE 10.1.8.1: Random Variables

Variable	Value	Description
X	1	Fair coin
	0	2-headed coin
Y	1	heads
	0	tails

10.1.9 A box has 5 blue and 4 red balls. One ball is drawn at random and not replaced. Its colour is also not noted. Then another ball is drawn at random. What is the probability of second ball being blue?

Solution: See Table 10.1.9.1. From the given information,

$$\Pr(X=0) = \frac{5}{9}, \Pr(X=1) = \frac{4}{9} \tag{10.1.9.1}$$

$$\Pr(Y=0|X=0) = \frac{1}{2}, \Pr(Y=0|X=1) = \frac{5}{8} \tag{10.1.9.2}$$

The desired probability is

$$\Pr(Y=0) = \Pr(X=0)\Pr(Y=0|X=0) + \Pr(X=1)\Pr(Y=0|X=1) \tag{10.1.9.3}$$

$$= \frac{5}{9} \times \frac{1}{2} + \frac{4}{9} \times \frac{5}{8} = \frac{5}{9} \tag{10.1.9.4}$$

TABLE 10.1.9.1

Variable	Value	Description
X	1	Red in first draw
	0	Blue in first draw
Y	1	Red in second draw
	0	Blue in second draw

10.1.10 A bag contains 4 white and 5 black balls. Another bag contains 9 white and 7 black balls. A ball is transferred from the first bag to the second and then a ball is drawn at random from the second bag. Find the probability that the ball drawn is white.

10.1.11 An item is manufactured by three machines A, B and C. Out of the total number of items manufactured during a specified period, 50% are manufactured on A, 30% on B and 20% on C, 2% of the items produced on A and 2% of items produced on B are defective, and 3% of these products produced on C are defective. All the items are stored at one godown. One item is drawn at random and is found to be defective. What is the probability that is was manufactured on machine A?

Solution: See Table 10.1.11.1.

Parameter	Values	Description
X	0	not defective
	1	defective
Y	1	manufactured on A
	2	manufactured on B
	3	manufactured on C

TABLE 10.1.11.1

Given that,

$$\Pr(Y = 1) = \frac{50}{100} = 0.5 \tag{10.1.11.1}$$
$$\Pr(Y = 2) = \frac{30}{100} = 0.3 \tag{10.1.11.2}$$
$$\Pr(Y = 3) = \frac{20}{100} = 0.2 \tag{10.1.11.3}$$
$$\Pr(X = 1|Y = 1) = \frac{2}{100} = 0.02 \tag{10.1.11.4}$$
$$\Pr(X = 1|Y = 2) = \frac{2}{100} = 0.02 \tag{10.1.11.5}$$
$$\Pr(X = 1|Y = 3) = \frac{3}{100} = 0.03 \tag{10.1.11.6}$$

Thus,

$$\Pr(Y = 1|X = 1) = \frac{\Pr(Y = 1)\Pr(X = 1|Y = 1)}{\sum_i \Pr(Y = i)\Pr(X = 1|Y = i)} \tag{10.1.11.7}$$
$$= \frac{0.5 \times 0.02}{0.5 \times 0.02 + 0.3 \times 0.02 + 0.2 \times 0.03} = \frac{5}{11} \tag{10.1.11.8}$$

10.1.12 There are two bags, one which contains 3 black balls and 4 white balls while the other contains 4 black balls and 3 white balls. A die is thrown. If it shows up 1 or 3, a ball is taken from the first bag; but it shown up any other number, a ball is taken from the second bag. Find the probability of choosing a black ball.
Solution: See Table 10.1.12.1. From the given information,

$$\Pr(X = 0) = \Pr(Z = 0) = \frac{1}{3} \tag{10.1.12.1}$$
$$\Pr(X = 1) = \Pr(Z = 1) = \frac{2}{3} \tag{10.1.12.2}$$
$$\Pr(Y = 0|X = 0) = \frac{3}{7} \tag{10.1.12.3}$$
$$\Pr(Y = 0|X = 1) = \frac{4}{7} \tag{10.1.12.4}$$

Hence, the desierd probability is

$$\Pr(Y = 0) = \Pr(X = 0) \times \Pr(Y = 0|X = 0) + \Pr(X = 1) \times \Pr(Y = 0|X = 1) \tag{10.1.12.5}$$
$$= \frac{1}{3} \times \frac{3}{7} + \frac{2}{3} \times \frac{4}{7} = \frac{7}{21} \tag{10.1.12.6}$$

10.1.13 A shopkeeper sells three types of flower seeds A_1, A_2 and A_3. They are sold as a mixture where the proportions are 4:4:2 respectively. The germination rates of the three types of seeds are 0.45, 0.60 and 0.35. Calculate the probability

a) of a randomly chosen seed to germinate
b) that it will not germinate given that the seed is of type A_3,

RV	Value	Description
X	0	first bag is selected
	1	second bag is selected
Y	0	black ball is drawn
	1	white ball is drawn
Z	0	1 or 3 is shown up
	1	another number is shown up

TABLE 10.1.12.1

c) that it is of the type A_2 given that a randomly chosen seed does not germinate.

Solution: See Table 10.1.13.1. From the given information,

$$p_X(k) = \begin{cases} \frac{4}{10} & k = 1 \\ \frac{4}{10} & k = 2 \\ \frac{2}{10} & k = 3 \end{cases} \tag{10.1.13.1}$$

$$p_{Y|X}(0|1) = 0.45 \tag{10.1.13.2}$$

$$p_{Y|X}(0|2) = 0.60 \tag{10.1.13.3}$$

$$p_{Y|X}(0|3) = 0.35 \tag{10.1.13.4}$$

using the definition in (4.1.5.1).

a)

$$p_Y(0) = \sum_{k=0}^{3} p_{Y|X}(0|k)\, p_X(k) = \frac{49}{100} \tag{10.1.13.5}$$

Also,

$$p_Y(1) = 1 - p_Y(0) = \frac{51}{100} \tag{10.1.13.6}$$

b) From (4.1.6.2),

$$p_{Y|X}(1|2) = 1 - p_{Y|X}(0|2) = 1 - 0.35 = 0.65 \tag{10.1.13.7}$$

c)

$$p_{X|Y}(2|1) = \frac{p_{Y|X}(1|2)\, p_X(2)}{p_Y(1)} = \frac{16}{51} \tag{10.1.13.8}$$

upon substituting from (10.1.13.7) and (10.1.13.6).

10.1 One card is drawn from a well-shuffled deck of 52 cards. Find the probability of getting

a) A king of red colour
b) A face card
c) A red face card

Variable	Description	Value
X	A_1	1
	A_2	2
	A_3	3
Y	germinate	0
	not germinate	1

TABLE 10.1.13.1

d) The jack of hearts
e) A spade
f) The queen of diamonds

Solution:

10.2 Five cards—the ten, jack, queen, king and ace of diamonds, are well-shuffled with their face downwards. One card is then picked up at random.

a) What is the probability that the card is the queen?
b) If the queen is drawn and put aside, what is the probability that the second card picked up is (a) an ace? (b) a queen?

Solution:

10.3 A bag contains 5 red balls and some blue balls. If the probability of drawing a blue ball is double that if a red ball, determine the number of blue balls in the bag.

Solution:

10.4 A card is selected from a pack of 52 cards.

(a) How many points are there in the sample space?
(b) Calculate the probability that the card is an ace of spades.
(c) Calculate the probability that the card is (i) an ace and (ii) black card.

Solution:

10.5 Four cards are drawn from a well-shuffled deck of 52 cards. What is the probability of obtaining 3 diamonds and one spade.

Solution:

10.6 In a certain lottery 10,000 tickets are sold and ten equal prizes are awarded. What is the probability of not getting a prize if you buy (a) one ticket (b) two tickets (c) 10 tickets ?

Solution:

10.7 Out of 100 students, two sections of 40 and 60 are formed. If you and your friend are among the 100 students, what is the probability that

a) you both enter the same section?
b) you both enter the different sections?

Solution:

10.8 The number lock of a suitcase has 4 wheels each labelled with ten digits i.e. from 0 to 9.The lock opens with a sequence of four digits with no repeats.What is the probability of a person getting the right sequence to open the suitcase.

Solution:

10.9 Two cards are drawn at random and without replacement from a pack of 52 playing cards. Find the probability that both the cards are black.

Solution:

10.10 A box of oranges is inspected by examining three randomly selected oranges drawn without replacement. If all the three oranges are good, the box is approved for sale, otherwise, it is rejected. Find the probability that a box containing 15 oranges out of which 12 are good and 3 are bad ones will be approved for sale.

10.11 Two balls are drawn at random with replacement from a box containing 10 black and 8 red balls. Find the probability that

a) both balls are red.
b) first ball is black and second is red.
c) one of them is black and other is red.

10.12 In a hostel, 60% of the students read Hindi newspaper, 40% read English newspaper and 20% read both Hindi and English newspapers. A student is selected at random.

a) Find the probability that she reads neither Hindi nor English newspapers.
b) If she reads Hindi newspaper, find the probability that she reads English newspaper.
c) If she reads English newspaper, find the probability that she reads Hindi newspaper.

10.13 The probability of obtaining an even prime number on each die, when a pair of dice is rolled is

a) 0
b) $\frac{1}{3}$
c) $\frac{1}{12}$
d) $\frac{1}{36}$

Solution:

10.14 A bag contains 4 red and 4 black balls, another bag contains 2 red and 6 black balls. One of the two bags is selected at random and a ball is drawn from the bag which is found to be red. Find the probability that the ball is drawn from the first bag.

Solution:

10.15 Cards with numbers 2 to 101 are placed in a box. A card is selected at random.Find the probability that the card has

(i) an even number
(ii) a square number

Solution:

10.16 The king, queen and jack of clubs are removed from a deck of 52 playing cards and then well shuffled. Now one card is drawn at random from the remaining cards. Determine the probability that the card is

(i) a club
(ii) 10 of hearts

Solution:

10.17 A team of medical students doing their internship have to assist during surgeries at a city hospital. The probabilities of surgeries rated as very complex, complex, routine, simple or very simple are respectively, 0.15, 0.20, 0.31, 0.26, .08. Find the probabilities that a particular surgery will be rated

a) complex or very complex;
b) neither very complex nor very simple;
c) routine or complex
d) routine or simple

Solution:

10.18 A card is selected from a pack of 52 cards.

(a) How many points are there in the sample space?
(b) Calculate the probability that the card is an ace of spades.
(c) Calculate the probability that the card is (i) an ace and (ii) black card.

Solution:

10.19 The probability that a non leap year selected at random will contain 53 sundays.

Solution:

10.20 One of the four persons John, Rita, Aslam or Gurpreet will be promoted next month. Consequently the sample space consists of four elementary outcomes S = John promoted, Rita promoted, Aslam promoted, Gurpreet promoted You are told that the chances of John's promotion is same as that of Gurpreet, Rita's chances of promotion are twice as likely as Johns. Aslam's chances are four times that of John.

a) Determine
   i) P (John promoted)
   ii) P (Rita promoted)
   iii) P (Aslam promoted)
   iv) P (Gurpreet promoted)
b) If A = John promoted or Gurpreet promoted, find P (A).

Solution:

10.21 A card is drawn from a deck of 52 cards. Find the probability of getting a king or a heart or a red card.

Solution:

10.22 The probability that a student will pass his examination is 0.73, the probability of the student getting a compartment is 0.13, and the probability that the student will either pass or get compartment is 0.96. State True or False.

Solution:

10.23 A card is selected from a pack of 52 cards

(a) How many points are there in the sample space?
(b) Calculate the probability that the cards is an ace of spades.
(c) Calculate the probability that the card is (i) an ace (ii)black card.

10.24 In a non-leap year, the probability of having 53 tuesdays or 53 wednesdays is

Solution:

10.25 There are 1000 sealed envelopes in a box, 10 of them contain a cash prize of Rs

100 each, 100 of them contain a cash prize of Rs 50 each and 200 of them contain a cash prize of Rs 10 each and rest do not contain any cash prize. If they are well shuffled and an envelope is picked up out, what is the probability that it contains no cash prize?

Solution:

10.26 A die is thrown and a card is selected at random from a deck of 52 playing cards. The probability of getting an even number on the die and a spade card.

Solution:

10.27 If 4-digit numbers greater than 5,000 are randomly formed from the digits 0, 1, 3, 5, and 7, what is the probability of forming a number divisible by 5 when:

a) The digits are repeated?
b) The repetition of digits is not allowed?

Solution:

10.28 Consider the probability space $(\Omega, \mathcal{G}, P)$ where $\Omega = [0, 2]$ and $\mathcal{G} = \{\phi, \Omega, [0, 1], (1, 2]\}$. Let X and Y be two functions on Ω defined as

$$X(\omega) = \begin{cases} 1 & \text{if } \omega \in [0, 1] \\ 2 & \text{if } \omega \in (1, 2] \end{cases}$$

and

$$Y(\omega) = \begin{cases} 2 & \text{if } \omega \in [0, 1.5] \\ 3 & \text{if } \omega \in (1.5, 2]. \end{cases}$$

Then which one of the following statements is true?

(A) X is a random variable with respect to $\mathcal{G}$, but Y is not a random variable with respect to $\mathcal{G}$.
(B) Y is a random variable with respect to $\mathcal{G}$, but X is not a random variable with respect to $\mathcal{G}$.
(C) Neither X nor Y is a random variable with respect to $\mathcal{G}$.
(D) Both X and Y are random variables with respect to $\mathcal{G}$.

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Solution:

10.29 A die is loaded in such a way that each odd number is twice as likely to occur as each even number. Find $P(G)$, where G is the event that a number greater than 3 occurs on a single roll of the die.

Solution:

10.30 All the jacks, queens and kings are removed from a deck of 52 playing cards. The remaining cards are well shuffled and then one card is drawn at random. Giving ace a value 1 similar value for other cards, find the probability that the card has a value

a) 7
b) greater than 7
c) less than 7

10.31 A Lot consists of 48 mobile phones of which 42 are good, 3 have only minor defects and 3 have major defects.Varnika will buy a phone if it is good but the trader will only buy a mobile if it has no major defects. One phone is selected at random from the lot. What is the probability that it is

a) acceptable to Varnika?
b) acceptable to the trader?

Solution:

10.32 A student says that if you throw a die, it will show up 1 or not 1. Therefore, the probability of getting 1 and the probability of getting 'not 1' each is equal to $\frac{1}{2}$. Is this correct? Give reasons.

Solution:

10.33 Four candidates A, B, C, D have ap- plied for the assignment to coach a school cricket team. If A is twice as likely to be selected as B, and B and C are given about the same chance of being selected, while C is twice as likely to be selected as D, what are the probabilities that

a) C will be selected?
b) A will not be selected?

10.34 A bag contain 24 balls of which x balls are red, $2x$ are white and $3x$ are blue. A ball is selected at random, What is the probability that it is

a) not red ?
b) white ?

If the letters of the word ASSASSINATION are arranged at random. Find the Probability that

(a) Four $S's$ come consecutively in the word
(b) Two $I's$ and two $N's$ come together
(c) All $A's$ are not coming together
(d) No two $A's$ are coming together

10.35 One urn contains two black balls (labelled B1 and B2) and one white ball. A second urn contains one black ball and two white balls (labelled W1 and W2). Suppose the following experiment is performed. One of the two urns is chosen at random. Next a ball is randomly chosen from the urn. Then a second ball is chosen at random from the same urn without replacing the first ball.

a) What is the probability that two black balls are chosen?
b) What is the probability that two balls of opposite colour are chosen?

Solution:

11 Multinomial

11.1 Formulae

11.1.1 Let

$$N = R + B + G, n = r + b + g \tag{11.1.1.1}$$

where R, B, G and r, b, g represent the number of red, blue and green marbles respectively within N and n. Then

$$p_{R,G,B}(r, b, g) = \frac{{}^{R}C_r\,{}^{B}C_b\,{}^{G}C_g}{{}^{R+B+G}C_{r+b+g}} \tag{11.1.1.2}$$

Solution: The number of ways of choosing n marbles from N is

$$^{N}C_n = {}^{R+B+G}C_{r+b+g} \tag{11.1.1.3}$$

The number of ways of choosing r, b, g marbles is

$$^{R}C_r\,{}^{B}C_b\,{}^{G}C_g \tag{11.1.1.4}$$

Using the definition of probability, we obtain (11.1.1.2).

11.1.2

$$^{R+B}C_n = \sum_{k=0}^{R}\sum_{m=n-k}^{B} {}^{R}C_k\,{}^{B}C_m \tag{11.1.2.1}$$

Solution: Since

$$(x+1)^R = \sum_{k=0}^{R} {}^{R}C_k\, x^k, \tag{11.1.2.2}$$

$$(x+1)^R (x+1)^B = \sum_{k=0}^{R}\sum_{m=0}^{B} {}^{R}C_k\,{}^{B}C_m\, x^{k+m} \tag{11.1.2.3}$$

$$\implies (x+1)^{R+B} = \sum_{k=0}^{R}\sum_{m=n-k}^{B} {}^{R}C_k\,{}^{B}C_m\, x^n + \sum_{k=0}^{R}\sum_{m\neq n-k}^{B} {}^{R}C_k\,{}^{B}C_m\, x^{k+m} \tag{11.1.2.4}$$

$$\tag{11.1.2.5}$$

yielding (11.1.2.1) upon comparing the coefficients of x^n on both sides.

11.2 NCERT

11.2.1 A box contains 5 red marbles, 8 white marbles and 4 green marbles. One marble is taken out of the box at random. What is the probability that the marble taken out will be

a) red ? b) white ? c) not green?

Solution: From (11.1.1.2),

a) Probability that the marble taken out is red

$$p_{R,W,G}(1, 0, 0) = \frac{{}^{5}C_1\,{}^{8}C_0\,{}^{4}C_0}{{}^{17}C_1} = \frac{5}{17} \tag{11.2.1.1}$$

b) Probability that the marble taken out is white

$$p_{R,W,G}(0, 1, 0) = \frac{{}^{5}C_0\,{}^{8}C_1\,{}^{4}C_0}{{}^{17}C_1} = \frac{8}{17} \tag{11.2.1.2}$$

c) Probability that the marble taken out is not green

$$1 - p_{R,W,G}(0,0,1) = 1 - \frac{{}^{5}C_0 {}^{8}C_0 {}^{4}C_1}{{}^{17}C_1} = 1 - \frac{4}{17} = \frac{13}{17} \tag{11.2.1.3}$$

11.2.2 A box contains 10 red marbles, 20 blue marbles and 30 green marbles. 5 marbles are drawn from the box, what is the probability that

a) all will be blue?
b) atleast one will be green?

Solution: See (11.1.1.2). In this question,

$$N = 60,\ R = 10,\ B = 20,\ G = 30,\ n = 5 \tag{11.2.2.1}$$

a) From (11.1.1.2),

$$p_{R,B,G}(0,5,0) = \frac{{}^{20}C_5}{{}^{60}C_5} \tag{11.2.2.2}$$

b) Since

$$p_{R,B,G}(r,b,0) = \frac{{}^{R}C_r {}^{B}C_b}{{}^{R+B+G}C_{r+b}} \tag{11.2.2.3}$$

The probability that at least one marble is green is given by

$$1 - \sum_{r+b=n} p_{R,B,G}(r,b,0) = 1 - \sum_{r+b=n} \frac{{}^{R}C_r {}^{B}C_b}{{}^{R+B+G}C_{r+b}} = 1 - \frac{{}^{R+B}C_n}{{}^{R+B+G}C_n} \tag{11.2.2.4}$$

from (11.1.2.1). Substituting numerical values, the desired probability is

$$1 - \frac{{}^{30}C_5}{{}^{60}C_5} \tag{11.2.2.5}$$

11.2.3 A box of oranges is inspected by examining three randomly selected oranges drawn without replacement. If all the three oranges are good, the box is approved for sale, otherwise, it is rejected. Find the probability that a box containing 15 oranges out of which 12 are good and 3 are bad ones will be approved for sale.

Solution: Choosing

$$R = 12, B = 3, G = 0, n = 3, r = 3, b = 0, g = 0 \tag{11.2.3.1}$$

in (11.1.1.2) the desired probability is

$$p_{R,B,G}(3,0,0) = \frac{{}^{12}C_3}{{}^{15}C_3} = \frac{44}{91} \tag{11.2.3.2}$$

11.2.4 A box contains 3 orange balls, 3 green balls and 2 blue balls. Three balls are drawn at random from the box without replacement. The probability of drawing 2 green balls and one blue ball is

a) $\frac{3}{28}$ b) $\frac{2}{21}$ c) $\frac{1}{28}$ d) $\frac{167}{168}$

Solution: The desired probability is

$$p_{O,G,B}(0,2,1) = \frac{{}^{3}C_{0}{}^{3}C_{2}{}^{2}C_{1}}{{}^{8}C_{3}} = \frac{3}{28} \tag{11.2.4.1}$$

11.2.5 A bag contain 5 red and 3 blue balls. If 3 balls are drawn at random without replacement, the probability of getting exactly one red ball is

a) $\frac{45}{196}$ b) $\frac{135}{392}$ c) $\frac{15}{56}$ d) $\frac{15}{29}$

Solution: The desired probability is

$$p_{R,B}(1,2) = \frac{{}^{5}C_{1}{}^{3}C_{2}}{{}^{8}C_{3}} = \frac{15}{56} \tag{11.2.5.1}$$

11.2.6 A bag contains 5 red and 3 blue balls. If 3 balls are drawn at random without replacement the probability that exactly two of the three balls were red, the first ball being red is
Solution: See (11.1.1.2). In this question,
As the first ball drawn is red,

$$N = 7, R = 4, B = 3, G = 0, r = 1, b = 1, g = 0 \tag{11.2.6.1}$$

The desired probability is,

$$p_{R,B}(1,1) = \frac{{}^{4}C_{1}{}^{3}C_{1}}{{}^{7}C_{2}} = \frac{4}{7} \tag{11.2.6.2}$$

12 Miscellaneous

12.1 The random variable X has a probability distribution $\Pr(X)$ of the following form, where k is some number

$$\Pr(X) = \begin{cases} k, & x = 0 \\ 2k, & x = 1 \\ 3k, & x = 2 \\ 0, & \text{otherwise} \end{cases} \tag{12.1.1}$$

a) Determine the value of k
b) Find $\Pr(X < 2)$,$\Pr(X \leq 2)$,$\Pr(X \geq 2)$

Solution:

12.2 State which of the following are not the probability distributions of a random variable. Give reasons for your answer

i

ii

iii
iv

12.3 A random variable X has the following probability distribution

Determine

i k
ii P($X < 3$)
iii P($X > 6$)
iv P($0 < X < 3$)

12.4 The random variable X has a probability distribution P(X) of the following form, where k is some number :

$$P(x) = \begin{cases} k, & \text{if } x = 0 \\ 2k, & \text{if } x = 1 \\ 3k, & \text{if } x = 2 \\ 0, & \textit{otherwise} \end{cases}$$

i Determine the value of k.
ii Find P ($X < 2$), P ($X \leq 2$), P($X \geq 2$)

12.5 A game consists of spinning an arrow which comes to rest pointing at one of the regions (1, 2 or 3) (Fig. 13.1). Are the outcomes 1, 2 and 3 equally likely to occur? Give reasons.

Solution:

12.6 Apoorv throws two dice once and computes the product of the numbers appearing on the dice. Peehu throws one die and squares the number that appears on it. Who has the better chance of getting the number 36? Why?

Solution:

12.7 6 boys and 6 girls sit in a row at random. The probability that all the girls sit together is

i $\frac{1}{432}$
ii $\frac{12}{431}$
iii $\frac{1}{132}$
iv none of the above

12.8 A card is selected from a deck of 52 cards. The probability of its being a red face card is

12.9 A die is loaded in such a way that each odd number is twice as likely to occur as each even number. Find P(G), where G is the event that a number greater than 3 occurs on a single roll of the die.

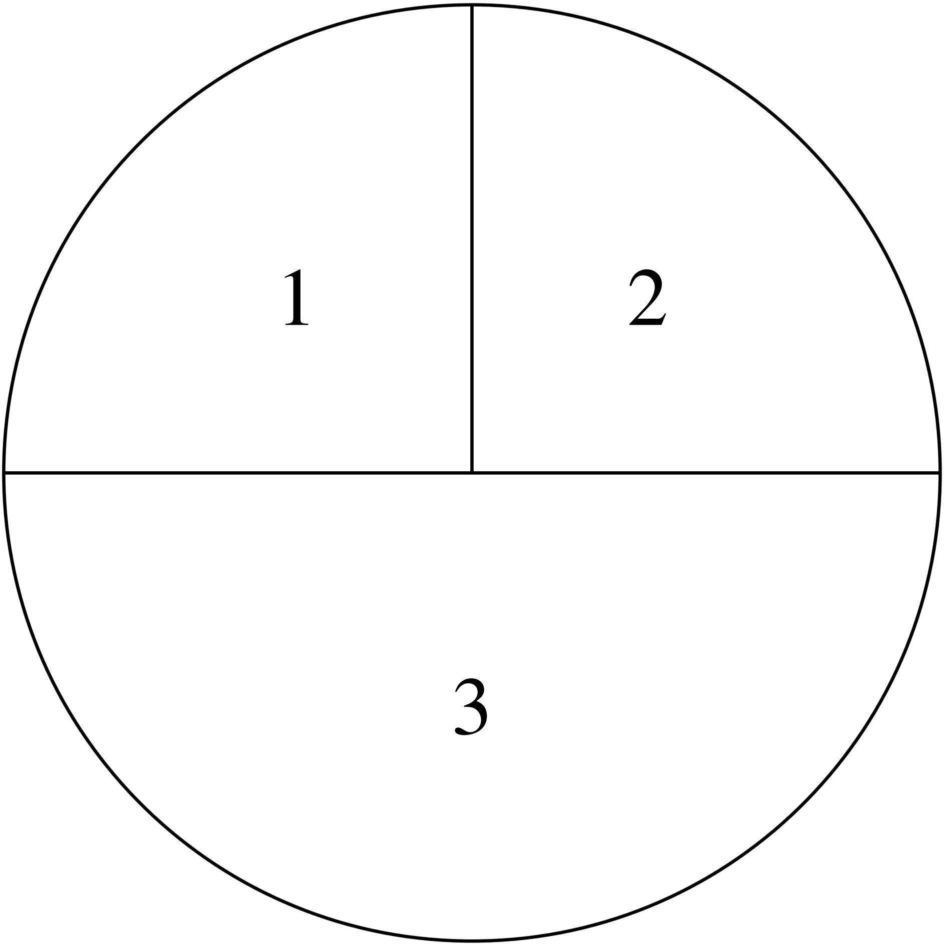

Fig. 12.5.1: Fig.13.1

12.10 Determine the probability p,for each of following events.

i An odd number appears in a single roll of dice.

ii Atleast one head appears in two tosses of fair coin.

iii A king,9 of hearts or 3 of spades appears in drawing a single card from a well shuffled deck of 52 cards.

iv The sum of 6 appears in single toss of a pair of fair dice.

12.11 Determine the probability p, for each of the following events.

(a) An odd number appears in a single toss of a fair die.

(b) At least one head appears in two tosses of a fair coin.

(c) A king, 9 of hearts, or 3 of spades appears in drawing a single card from a well shuffled ordinary deck of 52 cards.

(d) The sum of 6 appears in a single toss of a pair of fair dice.

12.12 The probability distribution of a random variable X is given below:

X	0	1	2	3
$P(X)$	k	$\frac{k}{2}$	$\frac{k}{4}$	$\frac{k}{8}$

i Determine the value of k.
ii Determine $P(X \leq 2)$ and $P(X > 2)$.
iii Find $P(X \leq 2) + P(X > 2)$.

Solution:

12.13

12.14 Three persons, A, B and C, fire at a target in turn, starting with A. Their probability of hitting the target are 0.4, 0.3 and 0.2 respectively. The probability of two hits is

i 0.024
ii 0.188
iii 0.336
iv 0.452

Solution:

item If two events are independent, then

i they must be mutually exclusive
ii the sum of their probabilities must be equal to 1
iii (A) and (B) both are correct
iv None of the above is correct

12.15 Three letters are dictated to three persons and an envelope is addressed to each of them, the letters are inserted into the envelopes at random so that each envelope contains exactly one letter. Find the probability that at least one letter in its proper envelope.

12.16 If A, B and C are three independent events such that $\Pr(A) = \Pr(B) = \Pr(C) = p$, then P(At least two of A, B, C occur) $= 3p^2 - 2p^3$

13 Random Algebra

13.1 Examples

13.1.1 Given that a fair coin is marked 1 on one face and 6 on the other and a fair die are tossed. Find the probability that the sum turns up to be 3 and 12.

Solution:

13.1.2 A fair coin is tossed four times, and a person win Re 1 for each head and lose Rs 1.50 for each tail that turns up. From the sample space calculate how many different amounts of money you can have after four tosses and the probability of having each of these amounts.

Solution:

13.1.3 A black and a red dice are rolled.

a) Find the conditional probability of obtaining a sum greater than 9, given that the black die resulted in a 5.
b) Find the conditional probability of obtaining the sum 8, given that the red die resulted in a number less than 4.

Solution:

13.1.4 Given that 2 numbers appearing on throwing two dice are different. Find the probability of the event ‘ the sum of numbers on the dice is 4’ .

Solution:

13.1.5 If each element of a 2×2 determinant is either zero or one. What is the probability that the value of the determinant is positive ? (Assume that the individual entries of the determinant are chosen independently each value being assumed with probability $\frac{1}{2}$)

Solution:

13.1.6 Two customers Shyam and Ekta are visiting a particular shop in the same week(Tuesday to Saturday). Each is equally likely to visit the shop on any day as on another day. What is the probability that both will visit the shop on (i) the same day? (ii) consecutive days? (iii) different days?

13.1.7 Let X represent the difference between the number of heads and the number of tails obtained when a coin is tossed 6 times. What are the possible values of X? Also find the Probability distribution of X.

13.1.8 A black and a red dice are rolled.

(a) find the conditional probability of obtaining a sum greater than 9,given that the black dice resulted in a 5.

(b) find the conditional probability of obtaining the sum 8,given that the red die resulted in a number less than 4.

Solution:

13.1.9 A fair die is thrown two times. Let A and B be the events, ’same number each time’, and a ’a total score is 10 or more’, respectively. Determine whether or not A and B are independent.

Solution:

13.1.10 Two dice are thrown together. Find the probability that the product of the numbers on the top of the dice is less than 9

Solution:

13.1.11 Two dice are thrown together. Find the probability that the product of the numbers on the top of the dice is 6 , 7 , 12

Solution:

13.2 Exercises

13.2.1 Two dice are thrown together. Find the probability that the product of the numbers on the top of the dice is

a) 6
b) 12
c) 7

Solution:

13.2.2 Two dice are thrown simultaneously. What is the probability that the sum of the numbers appearing on the dice is

a) 7?
b) a prime number?
c) 1?

Solution:

13.2.3 Two dice are thrown at the same time. Find the probability of getting

(i) same number on both dice.
(ii) different numbers on both dice.

13.2.4 A die has its face marked 0,1,1,1,6,6. Two such dice are thrown together and their score is recorded.

a) How many different scores are possible ?
b) What is the probability of getting a total 7 ?

13.2.5 For a loaded die, the probabilities of outcomes are given as under: Pr(1) = Pr(2) = 0.2, Pr(3) = Pr(5) = Pr(6) = 0.1 and Pr(4) = 0.3 The die is thrown two times. Let A and B be the events, 'same number each time', and 'a total score is 10 or more' ,respectively. Determine whether or not A and B are independent.

13.2.6 Three dice are thrown at the sametime. Find the probability of getting three two's, if it is known that the sum of the numbers on the dice was six.

13.2.7 Two dice are tossed. Find whether the following two events A and B are independent: $A = \{(x,y) : x+y=11\}$ $B = \{(x,y): x \neq 5\}$
where (x,y) denotes a typical sample point.

13.2.8 Find the probability distribution of the maximum of the two scores obtained when a die is thrown twice. Determine also the mean of the distribution.

13.2.9 Two dice are thrown. If it is known that sum of the numbers on the dice was less than 6,the probability of getting a sum 3, is

A) $\frac{1}{18}$

B) $\frac{5}{18}$

C) $\frac{1}{5}$

D) $\frac{2}{5}$

13.2.10 Two dice are thrown at the same time. Determine the probabiity that the difference of the numbers on the two dice is 2.

14.0.1 Consider the experiment of throwing a die.

- If a multiple of 3 comes up, throw the die again
- If any other number comes, toss a coin.

Find the conditional probability of the event 'the coin shows a tail', given that 'at least one die shows a 3'.

14.0.2 A and B throw a pair of dice alternately. A wins the game if he gets a total of 6 and B wins if she gets a total of 7. It A starts the game, find the probability of winning the game by A in third throw of the pair of dice.

14.0.3 A state transition diagram with states A, B, and C, and transition probabilities p_1, p_2,...., p_7 is shown in the figure (e.g., p_1 denotes the probability of transition from state A to B). For this state diagram, select the statement(s) which is/are universally true

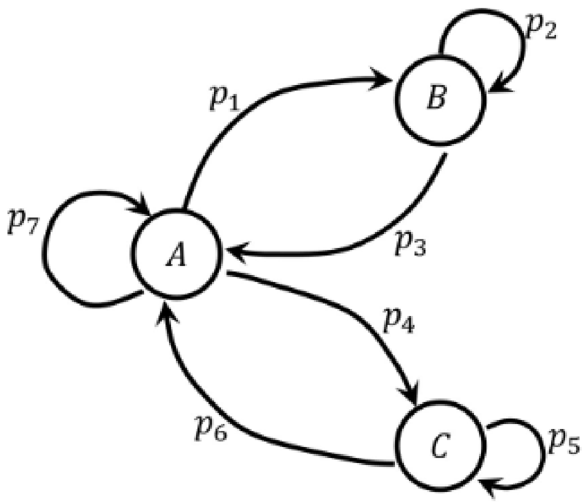

Fig. 14.0.3.1: Figure1

a) $p_2 + p_3 = p_5 + p_6$
b) $p_1 + p_3 = p_4 + p_6$
c) $p_1 + p_4 + p_7 = 1$
d) $p_2 + p_5 + p_7 = 1$

Appendix A
Central Limit Theorem

A.1 For large values of $n, k, n-k$, by Stirling's Approximation.

$$\begin{aligned} n! &\approx n^n e^{-n}\sqrt{2\pi n} \\ k! &\approx k^k e^{-k}\sqrt{2\pi k} \\ (n-k)! &\approx (n-k)^{(n-k)}\, e^{-(n-k)}\sqrt{2\pi\,(n-k)} \end{aligned} \tag{A.1.1}$$

A.2 Then,

$$p_X(k) = \frac{n!}{k!(n-k)!}p^k q^{n-k} \approx \frac{n^n e^{-n}\sqrt{2\pi n}}{k^k e^{-k}\sqrt{2\pi k}(n-k)^{n-k}e^{-(n-k)}\sqrt{2\pi(n-k)}}p^k q^{n-k} \tag{A.2.1}$$

$$= \left(\frac{np}{k}\right)^k \left(\frac{nq}{n-k}\right)^{n-k} \sqrt{\frac{n}{2\pi k(n-k)}} \tag{A.2.2}$$

from (A.1.1)

A.3 For

$$\delta \ll np, nq, \tag{A.3.1}$$

and

$$k = np + \delta, \tag{A.3.2}$$

$$n - k = nq - \delta \tag{A.3.3}$$

$$\implies \begin{aligned} \frac{k}{np} &= 1 + \frac{\delta}{np} \\ \frac{n-k}{nq} &= 1 - \frac{\delta}{nq} \end{aligned} \tag{A.3.4}$$

A.4 Taking logarithms in (A.2.2),

$$\ln\left[p_X(k)\right] = -k\ln\left(\frac{np}{k}\right) - (n-k)\left(\frac{nq}{n-k}\right) + \frac{1}{2}\ln\left(\frac{n}{2\pi k(n-k)}\right) \tag{A.4.1}$$

$$\begin{aligned} &= -\,(np+\delta)\ln\left(1+\frac{\delta}{np}\right) - (nq-\delta)\left(1-\frac{\delta}{nq}\right) \\ &\quad + \frac{1}{2}\ln\left(\frac{n}{2\pi\,(np+\delta)\,(nq-\delta)}\right) \end{aligned} \tag{A.4.2}$$

upon substituting from (A.3.4). From (A.3.1),

$$\frac{1}{2}\ln\left(\frac{n}{2\pi\,(np+\delta)\,(nq-\delta)}\right) \approx \frac{1}{2}\ln\left(\frac{1}{2\pi npq}\right) \tag{A.4.3}$$

From (A.3.1) and the approximation

$$\ln(1+x) \approx x - \frac{x^2}{2}, \tag{A.4.4}$$

the first sum in (A.4.2) can be expressed as,

$$
\begin{aligned}
-(np+\delta)\ln\left(1+\frac{\delta}{np}\right)&-(nq-\delta)\left(1-\frac{\delta}{nq}\right)\\
&\approx -(np+\delta)\left(\frac{\delta}{np}-\frac{\delta^2}{2n^2p^2}\right)-(nq-\delta)\left(-\frac{\delta}{nq}-\frac{\delta^2}{2n^2q^2}\right)\\
&= -\delta\left[1+\frac{\delta}{2np}-1+\frac{\delta}{2nq}\right]=-\frac{\delta^2}{2npq} \qquad \text{(A.4.5)}
\end{aligned}
$$

Substituting from (A.4.5) and (A.4.5) in (A.4.2),

$$
\ln\left[p_X(k)\right]\approx\frac{1}{2}\ln\left(\frac{1}{2\pi npq}\right)-\frac{\delta^2}{2npq} \qquad \text{(A.4.6)}
$$

$$
\implies p_X(k)=\sqrt{\frac{1}{2\pi npq}}e^{-\frac{(k-np)^2}{2npq}}=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(k-\mu)^2}{2\sigma^2}} \qquad \text{(A.4.7)}
$$