

# Random Variables through Simulation



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## ABOUT THIS BOOK

This book introduces random variables through high school probability. All problems in the book are from NCERT mathematics textbooks from Class 9-12. A lot of college level concepts related to random variables are covered in the process. The content is sufficient for random variable simulations using Python/C. There is no copyright, so readers are free to print and share.

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## 1 DEFINITIONS

### 1.1 NCERT

1.1.1 If a leap year is selected at random, what is the chance that it will contain 53 tuesdays?

**Solution:** The number of days in the leap year can be expressed as

$$366 = 52 \times 7 + 2 \quad (1.1.1.1)$$

The probability of one of the two remaining days being a Tuesday is  $\frac{2}{7}$ .

1.1.2 In a lottery, a person chooses six different natural numbers at random from 1 to 20, and if these six numbers match with the six numbers already fixed by the lottery committee, he wins the prize. What is the probability of winning the prizes in the game? [Hint : order of the numbers is not important.]

**Solution:** The desired probability is given by

$$\frac{1}{20C_6} = \frac{1}{38,760} = 0.0000258 \quad (1.1.2.1)$$

## 2 BOOLEAN LOGIC

### 2.1 Formulae

#### 2.1.1

$$A \cup B \triangleq A + B, A \cap B \triangleq AB. \quad (2.1.1.1)$$

2.1.2 Boolean Axioms: For  $A \in \{0, 1\}$ ,

$$A + A' = 1 \quad (2.1.2.1)$$

$$AA' = 0 \quad (2.1.2.2)$$

2.1.3 De Morgan's Law

$$A'B' = (A + B)' \quad (2.1.3.1)$$

2.1.4 Axioms of Probability

a)

$$0 \leq \Pr(A) \leq 1 \quad (2.1.4.1)$$

b)

$$\Pr(1) = 1 \quad (2.1.4.2)$$

c) If  $AB = 0$ , i.e.  $A, B$ , are mutually exclusive,

$$\Pr(A + B) = \Pr(A) + \Pr(B). \quad (2.1.4.3)$$

#### 2.1.5

$$\Pr(A + B) = \Pr(A) + \Pr(B) - \Pr(AB) \quad (2.1.5.1)$$

*Proof.*

$$A = A(B + B') = AB + AB' \quad (2.1.5.2)$$

$$\implies \Pr(A) = \Pr(AB) + \Pr(AB') \because (AB)(AB') = 0, \quad (2.1.5.3)$$

from (2.1.4.3). Similarly,

$$A + B = A(B + B') + B \quad (2.1.5.4)$$

$$= B(A + 1) + AB' \quad (2.1.5.5)$$

$$= B + AB' \quad (2.1.5.6)$$

$$\implies \Pr(A + B) = \Pr(B) + \Pr(AB') \quad \therefore BAB' = 0 \quad (2.1.5.7)$$

From (2.1.5.3) and (2.1.5.7), we obtain (2.1.5.1).  $\square$

2.1.6 From (2.1.5.3) and (2.1.4.1),

$$\Pr(A) \geq \Pr(AB) \quad (2.1.6.1)$$

2.1.7 If  $A, B$  are independent,

$$\Pr(AB) = \Pr(A)\Pr(B) \quad (2.1.7.1)$$

2.1.8 Let  $A + B = 1, AB = 0$ . Then it is possible to define a real number  $X$  such that

$$X = 0 \implies A \text{ and } X = 1 \implies B \quad (2.1.8.1)$$

$$\text{or, } \Pr(A) = \Pr(X = 0), \Pr(B) = \Pr(X = 1) \quad (2.1.8.2)$$

$X \in \{0, 1\}$  is then defined to be a *random variable* with the *distribution*

$$p_X(n) = \begin{cases} \Pr(A) & X = 0, \\ \Pr(B) & X = 1. \end{cases} \quad (2.1.8.3)$$

Using (2.1.4.2),

$$\sum_n p_X(n) = 1. \quad (2.1.8.4)$$

## 2.2 NCERT

2.2.1 Which of the following cannot be the probability of an event ?

- a)  $\frac{2}{3}$       b)  $-1.5$       c)  $15\%$       d)  $0.7$

**Solution:** We see that

$$\Pr(E) = -1.5 \quad (2.2.1.1)$$

violates (2.1.4.1). Hence, it cannot be a probability of any event.

2.2.2 If  $P(E) = 0.05$ , what is the probability of ‘not  $E$ ’?

**Solution:** From (2.1.4.2) and (2.1.4.3), the desired probability is

$$\Pr(E') = 1 - \Pr(E) = 0.95 \quad (2.2.2.1)$$

2.2.3 Check whether the following probabilities  $\Pr(A)$  and  $\Pr(B)$  are consistently defined

- a)  $\Pr(A) = 0.5, \Pr(B) = 0.7, \Pr(A \cap B) = 0.6$

- b)  $\Pr(A) = 0.5$ ,  $\Pr(B) = 0.7$ ,  $\Pr(A \cup B) = 0.8$

**Solution:**

a)

$$\Pr(A) < \Pr(AB) = 0.6 \quad (2.2.3.1)$$

which violates (2.1.6.1). Inconsistent.

- b) Given that

$$\Pr(A) = 0.5, \Pr(B) = 0.7, \Pr(A + B) = 0.8 \quad (2.2.3.2)$$

From (2.1.5.1), we get,

$$\Pr(AB) = 0.5 + 0.7 - 0.8 \quad (2.2.3.3)$$

$$= 0.4 \quad (2.2.3.4)$$

$\therefore$  no axioms are violated, the given probabilities are consistently defined

- 2.2.4 Given  $\Pr(A) = \frac{3}{5}$  and  $\Pr(B) = \frac{1}{5}$ . Find  $\Pr(A + B)$  if  $A$  and  $B$  are mutually exclusive events.

**Solution:** From (2.1.4.3),

$$\Pr(A + B) = \Pr(A) + \Pr(B) = \frac{4}{5} \quad (2.2.4.1)$$

- 2.2.5 If  $E$  and  $F$  are events such that  $\Pr(E) = \frac{1}{4}$ ,  $\Pr(F) = \frac{1}{2}$  and  $\Pr(EF) = \frac{1}{8}$ , find

- a)  $\Pr(E + F)$   
b)  $\Pr(E'F')$

**Solution:**

a)

$$\Pr(E + F) = \Pr(E) + \Pr(F) - \Pr(EF) = \frac{5}{8} \quad (2.2.5.1)$$

- b) From (2.1.3.1),

$$(E'F') = (E + F)' \quad (2.2.5.2)$$

$$\implies \Pr(E'F') = \Pr((E + F)') \quad (2.2.5.3)$$

$$= 1 - \Pr(E + F) = \frac{3}{8} \quad (2.2.5.4)$$

upon substituting from (2.2.5.1).

- 2.2.6 Events  $E$  and  $F$  are such that  $P(\text{not } E \text{ or not } F) = 0.25$ , state whether  $E$  and  $F$  are mutually exclusive.

**Solution:**

$$\Pr(E' + F') = \Pr((EF)') \quad (2.2.6.1)$$

$$= 1 - \Pr(EF) \quad (2.2.6.2)$$

$$\implies \Pr(EF) = 0.75 \quad (2.2.6.3)$$

$\therefore \Pr(EF) \neq 0$ ,  $E$  and  $F$  are not mutually exclusive.

2.2.7 If A and B are two independent events with  $\Pr(A) = \frac{3}{5}$  and  $\Pr(B) = \frac{4}{9}$  then,  $\Pr(A'B')$

a)  $\frac{4}{15}$

b)  $\frac{8}{45}$

c)  $\frac{1}{3}$

d)  $\frac{2}{9}$

**Solution:**

$$\Pr(A'B') = \Pr((A+B)') \quad (2.2.7.1)$$

$$= 1 - \Pr((A+B)) \quad (2.2.7.2)$$

$$= 1 - \Pr(A) - \Pr(B) + \Pr(A)\Pr(B) \quad (2.2.7.3)$$

$$= \frac{2}{9} \quad (2.2.7.4)$$

from (2.1.7.1) and (2.1.5.1).

2.2.8 A and B are events such that  $\Pr(A) = 0.42$ ,  $\Pr(B) = 0.48$  and  $\Pr(A \text{ and } B) = 0.16$ .

Determine

a)  $\Pr(\text{not } A)$

b)  $\Pr(\text{not } B)$

c)  $\Pr(A \text{ or } B)$

**Solution:** **Solution:**

a)

$$\Pr(A') = 1 - \Pr(A) = 0.58 \quad (2.2.8.1)$$

b)

$$\Pr(B') = 1 - \Pr(B) = 0.52 \quad (2.2.8.2)$$

c)

$$\Pr(A+B) = 0.42 + 0.48 - 0.16 = 0.74 \quad (2.2.8.3)$$

2.2.9 A and B are two events such that  $\Pr(A) = 0.54$ ,  $\Pr(B) = 0.69$  and  $\Pr(AB) = 0.35$ .

Find

a)  $\Pr(A+B)$

b)  $\Pr(A'B')$

c)  $\Pr(AB')$

d)  $\Pr(BA')$

**Solution:**

a)

$$\Pr(A+B) = \Pr(A) + \Pr(B) - \Pr(AB) \quad (2.2.9.1)$$

$$= 0.88 \quad (2.2.9.2)$$

b) By De Morgan's Law,

$$A'B' = (A+B)' \quad (2.2.9.3)$$

$$\implies \Pr(A'B') = \Pr(A+B)' \quad (2.2.9.4)$$

$$= 1 - \Pr(A+B) \quad (2.2.9.5)$$

$$= 0.12 \quad (2.2.9.6)$$

c) From (2.1.5.3),

$$\Pr(A) = \Pr(AB) + \Pr(AB') \quad (2.2.9.7)$$

$$\implies \Pr(AB') = 0.19 \quad (2.2.9.8)$$

d) Similarly,

$$\Pr(BA') = \Pr(B) - \Pr(AB) = 0.34. \quad (2.2.9.9)$$

2.2.10 If  $\Pr(A) = \frac{3}{5}$  and  $\Pr(B) = \frac{1}{5}$  find  $\Pr(A \cap B)$  if A and B are independent events.

**Solution:**

$$\Pr(AB) = \Pr(A)\Pr(B) = \frac{3}{25} \quad (2.2.10.1)$$

2.2.11 Let E and F be events with  $\Pr(E) = \frac{3}{5}$ ,  $\Pr(F) = \frac{3}{10}$  and  $\Pr(EF) = \frac{1}{5}$ . Are E and F independent?

**Solution:** From the given information,

$$\Pr(E)\Pr(F) = \frac{3}{5} \times \frac{9}{50}, \quad (2.2.11.1)$$

$$\Pr(EF) = \frac{1}{5} \quad (2.2.11.2)$$

$$\implies \Pr(EF) \neq P(E)P(F) \quad (2.2.11.3)$$

$\therefore E$  and  $F$  are not independent.

2.2.12 Given that the events A and B are such that  $P(A) = \frac{1}{2}$ ,  $P(A + B) = \frac{3}{5}$  and  $P(B) = p$ . Find  $p$  if they are

- a) mutually exclusive
- b) independent

**Solution:**

a)

$$\frac{3}{5} = \frac{1}{2} + p \quad (2.2.12.1)$$

$$\therefore p = \frac{1}{10} \quad (2.2.12.2)$$

b)

$$\frac{3}{5} = \frac{1}{2} + p - \frac{p}{2} \quad (2.2.12.3)$$

$$\therefore p = \frac{1}{5} \quad (2.2.12.4)$$

2.2.13 If A and B are two events such that  $\Pr(A) = \frac{1}{4}$ ,  $\Pr(B) = \frac{1}{2}$  and  $\Pr(AB) = \frac{1}{8}$ , find  $\Pr(\text{not } A \text{ and not } B)$ .

**Solution:**

$$\Pr(A + B) = \frac{1}{4} + \frac{1}{2} - \frac{1}{8} \quad (2.2.13.1)$$

$$= \frac{5}{8} \quad (2.2.13.2)$$

Hence,

$$\Pr(A'B') = 1 - \Pr((A + B)) = \frac{3}{8} \quad (2.2.13.3)$$

2.2.14 Events A and B are such that

$$\Pr(A) = \frac{1}{2}, \Pr(B) = \frac{7}{12}, \Pr(A' + B') = \frac{1}{4}. \quad (2.2.14.1)$$

State whether A and B are independent.

**Solution:**

$$\Pr(AB) = 1 - \Pr(A' + B') = \frac{3}{4}, \quad (2.2.14.2)$$

$$\Pr(A) \times \Pr(B) = \frac{7}{24} \quad (2.2.14.3)$$

$$\implies \Pr(AB) \neq \Pr(A)\Pr(B) \quad (2.2.14.4)$$

$\therefore$  A and B are not independent.

2.2.15 Two events A and B will be independent, if

- a) A and B are mutually exclusive
- b)  $P(\text{not } A \cap \text{not } B) = [1 - P(A)][1 - P(B)]$
- c)  $P(A) = P(B)$
- d)  $P(A) + P(B) = 1$

**Solution:**

- a) Let

$$\Pr(A) = \Pr(B) = \frac{1}{2} \implies \Pr(A) \times \Pr(B) = \frac{1}{4} \quad (2.2.15.1)$$

$$\text{or, } \Pr(AB) = 0 \neq \Pr(A) \times \Pr(B) \quad (2.2.15.2)$$

Hence A and B are not independent.

- b)

$$\Pr(A'B') = [1 - \Pr(A)][1 - \Pr(B)] \quad (2.2.15.3)$$

$$\implies 1 - \Pr(A + B) = 1 - \Pr(A) - \Pr(B) + \Pr(A)\Pr(B) \quad (2.2.15.4)$$

$$\implies \Pr(AB) = \Pr(A)\Pr(B) \quad (2.2.15.5)$$

Thus, A and B are independent.

- c) In 2.2.15a,  $\Pr(A) = \Pr(B)$ , but A and B are not independent.
- d) In 2.2.15a,  $\Pr(A) + \Pr(B) = 1$ , but A and B are not independent.

2.2.16 The probability that at least one of the two events A and B occurs is 0.6. If A and B occur simultaneously with probability 0.3, evaluate  $\Pr(A') + \Pr(B')$ .

**Solution:** Given:

$$\Pr(AB) = 0.3 \quad (2.2.16.1)$$

$$\Pr(A + B) = 0.6 \quad (2.2.16.2)$$

$$= \Pr(A) + \Pr(B) - \Pr(AB) \quad (2.2.16.3)$$

$$\implies 0.6 = \Pr(A) + \Pr(B) - 0.3 \quad (2.2.16.4)$$

$$\implies 0.9 = \Pr(A) + \Pr(B) \quad (2.2.16.5)$$

But

$$\Pr(A') = 1 - \Pr(A) \quad (2.2.16.6)$$

$$\Pr(B') = 1 - \Pr(B) \quad (2.2.16.7)$$

$$\therefore \Pr(A') + \Pr(B') = 2 - (\Pr(A) + \Pr(B)) \quad (2.2.16.8)$$

$$= 2 - 0.9 = 1.1 \quad (2.2.16.9)$$

2.2.17 Prove that

- a)  $\Pr(A) = \Pr(AB) + \Pr(AB')$
- b)  $\Pr(A + B) = \Pr(AB) + \Pr(AB') + \Pr(A'B)$

**Solution:**

- a) See (2.1.5.3).
- b) From (2.1.5.3) and (2.1.5.1),

$$\Pr(A) = \Pr(AB) + \Pr(AB') \quad (2.2.17.1)$$

$$\Pr(B) = \Pr(AB) + \Pr(A'B) \quad (2.2.17.2)$$

$$\Pr(A + B) = \Pr(A) + \Pr(B) - \Pr(AB) \quad (2.2.17.3)$$

yielding item 2.2.17b after addition.

2.2.18 A and B are events such that  $\Pr(A) = 0.4$ ,  $\Pr(B) = 0.3$  and  $\Pr(A + B) = 0.5$ . Find  $\Pr(B'A)$ .

**Solution:** Adding (2.2.17.1) and (2.2.17.3),

$$\Pr(A + B) = \Pr(B) + \Pr(AB') \quad (2.2.18.1)$$

$$\implies \Pr(AB') = \Pr(A + B) - \Pr(B) = 0.2 \quad (2.2.18.2)$$

State True or False.

2.2.19 If  $\Pr(A) > 0$  and  $\Pr(B) > 0$ , then A and B can be mutually exclusive and independent.

**Solution:** Since  $\Pr(A) > 0$  and  $\Pr(B) > 0$ ,

$$\Pr(A)\Pr(B) > 0 \quad (2.2.19.1)$$

For  $\Pr(A)$  and  $\Pr(B)$  to be mutually exclusive and independent,

$$\Pr(AB) = 0 \quad (2.2.19.2)$$

$$\Pr(AB) = \Pr(A)\Pr(B) \quad (2.2.19.3)$$

$$\implies \Pr(A)\Pr(B) = 0 \quad (2.2.19.4)$$

which contradicts (2.2.19.1). Hence, the above statement is false.

2.2.20 If  $A$  and  $B$  are independent events, then  $A'$  and  $B'$  are also independent.

**Solution:** Given that

$$\Pr(AB) = \Pr(A)\Pr(B) \quad (2.2.20.1)$$

If  $A'$  and  $B'$  are independent,

$$\Pr(A'B') = \Pr(A+B)' = 1 - \Pr(A+B) \quad (2.2.20.2)$$

$$= 1 - \Pr(A) - \Pr(B) + \Pr(AB) \quad (2.2.20.3)$$

$$= 1 - \Pr(A) - \Pr(B) + \Pr(A)\Pr(B) \quad (2.2.20.4)$$

$$= [1 - \Pr(A)][1 - \Pr(B)] \quad (2.2.20.5)$$

$$= \Pr(A')\Pr(B') \quad (2.2.20.6)$$

Hence,  $A'$  and  $B'$  are also independent. Therefore, the given statement is true.

2.2.21 If  $A$  and  $B$  are mutually exclusive events,  $\Pr(A) = 0.35$  and  $\Pr(B) = 0.45$  then find

- |              |               |                |
|--------------|---------------|----------------|
| a) $\Pr(A')$ | c) $\Pr(A+B)$ | e) $\Pr(AB')$  |
| b) $\Pr(B')$ | d) $\Pr(AB)$  | f) $\Pr(A'B')$ |

**Solution:** See Table 2.2.21.1.

TABLE 2.2.21.1

| Item        | Formula                     | Value |
|-------------|-----------------------------|-------|
| $\Pr(A')$   | $1 - \Pr(A)$                | 0.65  |
| $\Pr(B')$   | $1 - \Pr(B)$                | 0.55  |
| $\Pr(A+B)$  | $\Pr(A) + \Pr(B) - \Pr(AB)$ | 0.80  |
| $\Pr(AB)$   | $\because AB = 0$           | 0     |
| $\Pr(AB')$  | $\Pr(A) - \Pr(AB)$          | 0.35  |
| $\Pr(A'B')$ | $1 - \Pr(A+B)$              | 0.20  |

2.2.22 The accompanying Venn diagram shows three events,  $A$ ,  $B$ , and  $C$ , and also the probabilities of the various intersections (for instance,  $\Pr(AB) = 0.7$ ). Determine

- |               |               |   |
|---------------|---------------|---|
| a) $\Pr(A)$   | c) $\Pr(A+B)$ | e) $\Pr(BC)$  |
| b) $\Pr(BC')$ | d) $\Pr(AB')$ | f) Probability that exactly one of the three occurs |

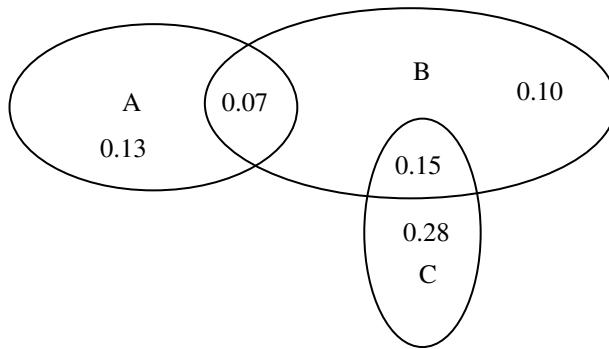


Fig. 2.2.22.1

| Input        |       | Output                       |                                    |       |
|--------------|-------|------------------------------|------------------------------------|-------|
| Probability  | Value | Probability                  | Formula                            | Value |
| $\Pr(AB)$    | 0.07  | $\Pr(A)$                     | $A = AB + AB'$                     | 0.2   |
| $\Pr(AB')$   | 0.13  | $\Pr(BC')$                   | $BC' = AB + A'BC'$                 | 0.17  |
| $\Pr(BC)$    | 0.15  | $\Pr(A + B)$                 | $\Pr(A) + \Pr(B) - \Pr(AB)$        | 0.45  |
| $\Pr(BA'C')$ | 0.10  | $\Pr(AB')$                   | Given                              | 0.13  |
| $\Pr(CB')$   | 0.28  | $\Pr(BC)$                    | Given                              | 0.15  |
| $\Pr(AC)$    | 0     | $\Pr(A'BC' + AB'C' + A'B'C)$ | $\Pr(AB') + \Pr(CB') + \Pr(BA'C')$ | 0.51  |

TABLE 2.2.22.1

**Solution:** See Table 2.2.22.1. Fig. 2.2.22.1 is used to obtain the input probabilities.  
a)

$$BC' = BC' (A + A') = BC'A + BC'A' \quad (2.2.22.1)$$

Also,

$$AB = AB(C + C') \quad (2.2.22.2)$$

$$= ABC + ABC' = ABC' \because AC = 0. \quad (2.2.22.3)$$

From (2.2.22.1) and (2.2.22.3),

$$BC' = AB + A'BC' \quad (2.2.22.4)$$

$$\implies \Pr(BC') = \Pr(AB) + \Pr(A'BC') \quad (2.2.22.5)$$

b) Also,

$$\Pr(B) = \Pr(BC) + \Pr(BC') = 0.17 + 0.15 = 0.32 \quad (2.2.22.6)$$

from Table 2.2.22.1. This is used to evaluate  $\Pr(A + B)$ .

c)  $\because AC = 0$

$$\Pr(AB) = \Pr(AB'C') + \Pr(AB'C) = \Pr(AB'C') \quad (2.2.22.7)$$

$$\Pr(B'C) = \Pr(A'B'C) \quad (2.2.22.8)$$

2.2.23 The probability of happening of an event  $A$  is 0.5 and that of  $B$  is 0.3. If  $A$  and  $B$  are mutually exclusive events, then the probability of neither  $A$  nor  $B$  is \_\_\_\_.

**Solution:**

$$\therefore \Pr(AB) = 0, \quad (2.2.23.1)$$

$$\Pr((A+B)') = 1 - \Pr(A+B) = 1 - \Pr(A) - \Pr(B) \quad (2.2.23.2)$$

$$= 0.2 \quad (2.2.23.3)$$

which is the desired probability.

2.2.24 If  $A$  and  $B$  are mutually exclusive events, then

- a)  $\Pr(A) \leq \Pr(B')$   
 b)  $\Pr(A) \geq \Pr(B')$

- c)  $\Pr(A) < \Pr(B')$   
 d) none of these

**Solution:**

$$\therefore \Pr(AB) = 0 \quad (2.2.24.1)$$

$$\Pr(A+B) \leq 1 \implies \Pr(A) + \Pr(B) \leq 1 \quad (2.2.24.2)$$

$$\implies \Pr(A) \leq \Pr(B'). \quad (2.2.24.3)$$

where we have used the axiom of probability.

2.2.25 The probability of an occurrence of event  $A$  is .7 and that of the occurrence of event  $B$  is .3 and the probability of occurrence of both is .4. Is this statement true or false?

**Solution:**

$$\Pr(AB) > \Pr(B) \quad (2.2.25.1)$$

which violates (2.1.6.1). Hence, the given statement is false.

2.2.26 If  $\Pr(A+B) = \Pr(AB)$  for any two events  $A$  and  $B$ , then

- a)  $\Pr(A) = \Pr(B)$     b)  $\Pr(A) > \Pr(B)$     c)  $\Pr(A) < \Pr(B)$     d) none of these

**Solution:**

$$\Pr(A) + \Pr(B) - \Pr(AB) = \Pr(A+B) \quad (2.2.26.1)$$

$$\implies \Pr(A) + \Pr(B) - \Pr(AB) = \Pr(AB) \quad (2.2.26.2)$$

$$\implies [\Pr(A) - \Pr(AB)] + [\Pr(B) - \Pr(AB)] = 0 \quad (2.2.26.3)$$

However, from (2.1.6.1),

$$\Pr(A) - \Pr(AB) \geq 0 \quad (2.2.26.4)$$

$$\Pr(B) - \Pr(AB) \geq 0$$

From (2.2.26.3) and (2.2.26.4),

$$\Pr(A) = \Pr(B) = \Pr(AB). \quad (2.2.26.5)$$

- 2.2.27** If  $A$  and  $B$  are such that  $\Pr(A' \cup B') = \frac{2}{3}$  and  $\Pr(A \cup B) = \frac{5}{9}$ , then  $\Pr(A') + \Pr(B') =$   
**Solution:** Using De Morgan's law and axioms of probability,

$$\Pr((A + B)') = \Pr(A'B') \quad (2.2.27.1)$$

$$\Pr(A' + B') = \Pr(A') + \Pr(B') - \Pr(A'B') \quad (2.2.27.2)$$

Adding the above,

$$\Pr(A') + \Pr(B') = 1 + \Pr(A' + B') - \Pr(A + B) = \frac{10}{9} \quad (2.2.27.3)$$

- 2.2.28 If  $A$  and  $B$  are independent, then  $\Pr(\text{exactly one of } A, B \text{ occurs}) = \Pr(B)\Pr(A') + \Pr(A)\Pr(B')$ .

**Solution:**

$$\therefore \Pr(AB) = \Pr(A)\Pr(B) \quad (2.2.28.1)$$

$$\Pr(AB') = \Pr(A)\Pr(B'), \Pr(A'B) = \Pr(A')\Pr(B) \quad (2.2.28.2)$$

$$\Rightarrow \Pr(A'B + AB') = \Pr(A'B) + \Pr(AB') \quad (2.2.28.3)$$

$$= \Pr(A) \Pr(B') + \Pr(A') \Pr(B). \quad (2.2.28.4)$$

- 2.2.29 Given two independent events A and B such that  $P(A) = 0.3$ ,  $P(B) = 0.6$ . Find

- a)  $P(A \text{ and } B)$       c)  $P(A \text{ or } B)$   
b)  $P(A \text{ and not } B)$       d)  $P(\text{neither } A \text{ nor } B)$

- 2.2.30 The probability distribution of a discrete random variable  $X$  is given below in Table 2.2.30.1. The value of  $k$  is equal to



| $X$      | 2             | 3             | 4             | 5              |
|----------|---------------|---------------|---------------|----------------|
| $p_X(n)$ | $\frac{5}{k}$ | $\frac{7}{k}$ | $\frac{9}{k}$ | $\frac{11}{k}$ |

TABLE 2.2.30.1

**Solution:** From (2.1.8.4),

$$\frac{5}{k} + \frac{7}{k} + \frac{9}{k} + \frac{11}{k} = 1 \quad (2.2.30.1)$$

$$\implies k = 32 \quad (2.2.30.2)$$

- 2.2.31 State True or False for the given statement: Two independent events are always mutually exclusive.

**Solution:** The given condition can be expressed as

$$\Pr(AB) = \Pr(A) \times \Pr(B) = 0 \quad (2.2.31.1)$$

$$\implies \Pr(A) = 0 \text{ or, } \Pr(B) = 0, \quad (2.2.31.2)$$

which is not always true.

2.2.32 If  $A$  and  $B'$  are independent events, then  $\Pr(A' + B) = 1 - \Pr(A)\Pr(B')$ .

**Solution:**

$$\Pr(A' + B) = \Pr((AB')') \quad (2.2.32.1)$$

$$= 1 - \Pr(AB') \quad (2.2.32.2)$$

$$= 1 - \Pr(A)\Pr(B'). \quad (2.2.32.3)$$

2.2.33 Let  $E_1$  and  $E_2$  be two independent events such that  $\Pr(E_1) = p_1$  and  $\Pr(E_2) = p_2$ . Describe in words the events whose probabilities are

a)  $p_1p_2$   
b)  $(1 - p_1)p_2$

c)  $1 - (1 - p_1)(1 - p_2)$   
d)  $p_1 + p_2 - 2p_1p_2$

**Solution:**

a)

$$p_1p_2 = \Pr(E_1)\Pr(E_2) \quad (2.2.33.1)$$

$$= \Pr(E_1E_2) \quad (2.2.33.2)$$

So,  $E_1$  and  $E_2$  occur simultaneously.

b)

$$(1 - p_1)(p_2) = \Pr(E'_1)\Pr(E_2) \quad (2.2.33.3)$$

$$= \Pr(E'_1E_2) \quad (2.2.33.4)$$

So  $E_1$  does not occur but  $E_2$  occurs.

c)

$$1 - (1 - p_1)(1 - p_2) = 1 - \Pr(E'_1)\Pr(E'_2) \quad (2.2.33.5)$$

$$= 1 - \Pr(E'_1 E'_2) \quad (2.2.33.6)$$

$$= \Pr(E_1 + E_2) \quad (2.2.33.7)$$

So, either  $E_1$  or  $E_2$  or both  $E_1$  and  $E_2$  occurs.

d)

$$p_1 + p_2 - 2p_1p_2 = \Pr(E_1) + \Pr(E_2) - 2\Pr(E_1)\Pr(E_2) \quad (2.2.33.8)$$

$$= \Pr(E_1) - \Pr(E_1)\Pr(E_2) + \Pr(E_2) - \Pr(E_1)\Pr(E_2) \quad (2.2.33.9)$$

$$= \Pr(E_1)(1 - \Pr(E_2)) + \Pr(E_2)(1 - \Pr(E_1)) \quad (2.2.33.10)$$

$$= \Pr(E_1)\Pr(E'_2) + \Pr(E_2)\Pr(E'_1) \quad (2.2.33.11)$$

$$= \Pr(E_1E'_2 + E'_1E_2) \quad (2.2.33.12)$$

So, either  $E_1$  or  $E_2$  occurs but not both.

2.2.34 Match the following in Table 2.2.34.1.

| I   | II   |
|---|--|
| (a) if $E_1$ and $E_2$ are two mutually exclusive events            | (i) $E_1 \cap E_2 = E_1$                           |
| (b) if $E_1$ and $E_2$ are mutually exclusive and exhaustive events | (ii) $(E_1 - E_2) \cup (E_1 \cap E_2) = E_1$       |
| (c) if $E_1$ and $E_2$ have common outcomes, then                   | (iii) $E_1 \cap E_2 = \emptyset, E_1 \cup E_2 = S$ |
| (d) if $E_1$ and $E_2$ are two events such that $E_1 \subset E_2$   | (iv) $E_1 \cap E_2 = \emptyset$                    |

TABLE 2.2.34.1

**Solution:**

- a) If  $E_1$  and  $E_2$  are mutually exclusive events, then  $E_1 E_2 = \emptyset$ .
- b) If  $E_1$  and  $E_2$  are mutually exclusive and exhaustive events, then  $E_1 E_2 = \emptyset$  and  $E_1 + E_2 = S$
- c) If  $E_1$  and  $E_2$  have common outcomes, this means:

$$E_1 E_2 \neq 0 \quad (2.2.34.1)$$

Let  $E_a$  be the outcomes that are present in  $E_1$  and not in  $E_2$ . So,

$$E_a = E_1 - E_2 \quad (2.2.34.2)$$

Let  $E_b$  be the outcomes common between  $E_1$  and  $E_2$ . So,

$$E_b = E_1 E_2 \quad (2.2.34.3)$$

So, we can say that

$$E_1 = E_a + E_b \quad (2.2.34.4)$$

Referring to equation (2.2.34.2) and (2.2.34.3):

$$E_1 = (E_1 - E_2) + (E_1 E_2) \quad (2.2.34.5)$$

- d) If  $E_1$  and  $E_2$  are two events such that  $E_1 \subset E_2$ , then let  $E$  be subset of  $E_2$  containing elements other than  $E_1$ . So,

$$E_1 + E = E_2 \text{ and } E_1 E = E_2 \quad (2.2.34.6)$$

Referring to equation (2.2.34.6):

$$E_1 E_2 = E_1(E_1 + E) \quad (2.2.34.7)$$

$$= (E_1 E_1) + (E_1 E) \quad (2.2.34.8)$$

$$= E_1 \quad (2.2.34.9)$$

Hence,

- a)  $\leftrightarrow$  (iv),      b)  $\leftrightarrow$  (iii),      c)  $\leftrightarrow$  (ii),      d)  $\leftrightarrow$  (i)

- 2.2.35 If  $A$  and  $B$  are two candidates seeking admission in an engineering College. The probability that  $A$  is selected is 0.5 and the probability that both  $A$  and  $B$  are selected

is atmost 0.3. Is it possible that the probability of  $B$  getting selected is 0.7?

**Solution:**

$$\therefore \Pr(AB) \leq 0.3 \quad (2.2.35.1)$$

$$\text{Let } \Pr(AB) = 0.1. \quad (2.2.35.2)$$

From (2.1.5.1)

$$\Pr(A + B) = 0.5 + 0.7 - 0.1 = 1.1 > 1, \quad (2.2.35.3)$$

which violates (2.1.4.1). Hence, it is not possible.

2.2.36 State whether the statement is True or False.

The probability that a person visiting a zoo will see the giraffe is 0.72, the probability that he will see the bears is 0.84 and the probability that he will see both is 0.52.

**Solution:** Let

$$\Pr(A) = 0.72, \Pr(B) = 0.84, \Pr(AB) = 0.52. \quad (2.2.36.1)$$

Using (2.1.5.1),

$$\Pr(A + B) = 0.72 + 0.84 - 0.52 = 1.04 \quad (2.2.36.2)$$

which violates (2.1.4.1). Hence, false.

2.2.37 Suppose an integer from 1 through 1000 is chosen at random, find the probability that the integer is a multiple of 2 or a multiple of 9.

**Solution:** See Table 2.2.37.1. From (2.1.5.1),

| Event | Description      | Probability                 |
|-------|------------------|-----------------------------|
| A     | $n \bmod 2 = 0$  | $\Pr(A) = \frac{500}{1000}$ |
| B     | $n \bmod 9 = 0$  | $\Pr(B) = \frac{111}{1000}$ |
| AB    | $n \bmod 18 = 0$ | $\Pr(AB) = \frac{55}{1000}$ |

TABLE 2.2.37.1

$$\Pr(A + B) = \frac{500}{1000} + \frac{111}{1000} - \frac{55}{1000} = \frac{556}{1000} \quad (2.2.37.1)$$

2.2.38 If the probabilities for  $A$  to fail in an examination is 0.2 and that for  $B$  is 0.3, then the probability that either  $A$  or  $B$  fails is

- a)  $> 0.5$       b) 0.5      c)  $\leq 0.5$       d) 0

**Solution:**

$$\therefore \Pr(A) = 0.2, \Pr(B) = 0.3 \quad (2.2.38.1)$$

$$\Pr(AB) = 0.5 - \Pr(A + B) \quad (2.2.38.2)$$

from (2.1.5.1). Thus, from (2.1.4.1),

$$0.5 - \Pr(A + B) \geq 0 \quad (2.2.38.3)$$

$$\implies \Pr(A + B) \leq 0.5 \quad (2.2.38.4)$$

- 2.2.39 It is given that in a group of 3 students, the probability of 2 students not having the same birthday is 0.992. What is the probability that the 2 students have the same birthday?

**Solution:** Let

$$\Pr(E) = 0.992. \quad (2.2.39.1)$$

Then,

$$\Pr(E') = 1 - \Pr(E) = 0.008 \quad (2.2.39.2)$$

- 2.2.40 In class XI of a school, 40% of the students study Mathematics and 30% study Biology. 10% of the class study both Mathematics and Biology. If a student is selected at random from the class, find the probability that he will be studying Mathematics or Biology

**Solution:**

$$\because \Pr(M) = 0.4, \Pr(B) = 0.3, \Pr(MB) = 0.1, \quad (2.2.40.1)$$

$$\Pr(M + B) = \Pr(M) + \Pr(B) - \Pr(MB) = 0.6 \quad (2.2.40.2)$$

using (2.1.5.1).

- 2.2.41 In an entrance test that is graded on the basis of two examinations, the probability of a randomly chosen student passing the first examination is 0.8 and the probability of passing the second examination is 0.7. The probability of passing atleast one of them is 0.95. What is the probability of passing both?

**Solution:**

$$\because \Pr(A) = 0.8, \Pr(B) = 0.7, \Pr(A + B) = 0.95, \quad (2.2.41.1)$$

$$\Pr(AB) = 0.55 \quad (2.2.41.2)$$

using (2.1.5.1).

- 2.2.42 In a class of 60 students, 30 opted for NCC, 32 opted for NSS and 24 opted for both NCC and NSS. If one of these students is selected at random, find the probability that

- The student opted for NCC or NSS.
- The student has opted neither NCC nor NSS.
- The student has opted NSS but not NCC.

**Solution:** From the given data,

$$\Pr(A) = \frac{30}{60} = \frac{1}{2}, \Pr(B) = \frac{32}{60} = \frac{8}{15}, \Pr(AB) = \frac{24}{60} = \frac{2}{5}. \quad (2.2.42.1)$$

Thus, the desired probabilities are

- $\Pr(A + B) = \frac{19}{30}$ , from (2.1.5.1).
- From (2.1.3.1) and the axioms of probability,

$$\Pr(A'B') = 1 - \Pr(A + B) = \frac{11}{30}. \quad (2.2.42.2)$$

c)

$$\Pr(A'B) = \Pr(B) - \Pr(AB) = \frac{2}{15} \quad (2.2.42.3)$$

from (4.1.2.1).

- 2.2.43 The probability of solving a specific problem independently by A and B are  $\frac{1}{2}$  and  $\frac{1}{3}$  respectively. If both try to solve the problem independently, find the probability that

- a) the problem is solved
- b) exactly one of them solves the problem

**Solution:**

$$\because \Pr(A) = \frac{1}{2}, \Pr(B) = \frac{1}{3}, \quad (2.2.43.1)$$

$$\Pr(AB) = \Pr(A)\Pr(B) = \frac{1}{6} \quad (2.2.43.2)$$

$\therefore A, B$  are independent.

- a) From (2.1.5.1),

$$\Pr(A + B) = \frac{1}{2} + \frac{1}{3} - \frac{1}{6} = \frac{2}{3} \quad (2.2.43.3)$$

b)

$$\Pr(AB' + A'B) = \Pr(AB') + \Pr(A'B) \quad (2.2.43.4)$$

$$= \Pr(A)\Pr(B') + \Pr(A')\Pr(B) \quad (2.2.43.5)$$

$$= \Pr(A) + \Pr(B) - 2\Pr(A)\Pr(B) = \frac{1}{2} \quad (2.2.43.6)$$

- 2.2.44 One card is drawn at random from a well shuffled deck of 52 cards. In which of the following cases are the events E and F independent?

- a) E: 'the card drawn is spade'  
F: 'the card drawn is an ace'
- b) E: 'the card drawn is black'  
F: 'the card drawn is a king'
- c) E: 'the card drawn is a king or queen'  
F: 'the card drawn is a queen or jack'

**Solution:** See Table 2.2.44.1.

| Item | $\Pr(E)$                      | $\Pr(F)$                      | $\Pr(EF)$                                       | Independent |
|------|-------------------------------|-------------------------------|---|-------------|
| a)   | $\frac{13}{52} = \frac{1}{4}$ | $\frac{4}{52} = \frac{1}{13}$ | $\frac{1}{52} = \Pr(E)\Pr(F)$                   | Yes         |
| b)   | $\frac{26}{52} = \frac{1}{2}$ | $\frac{4}{52} = \frac{1}{13}$ | $\frac{2}{52} = \frac{1}{26} = \Pr(E)\Pr(F)$    | Yes         |
| c)   | $\frac{8}{52} = \frac{2}{13}$ | $\frac{8}{52} = \frac{2}{13}$ | $\frac{4}{52} = \frac{1}{13} \neq \Pr(E)\Pr(F)$ | No          |

TABLE 2.2.44.1

- 2.2.45 A team of medical students doing their internship have to assist during surgeries

at a city hospital. The probabilities of surgeries rated as very-complex, complex, routine, simple or very-simple are respectively, 0.15, 0.20, 0.31, 0.26, .08. Find the probability that a particular surgery will be rated

- a) complex or very-complex                            c) routine or complex
- b) neither very-complex nor very simple      d) routine or simple

**Solution:** The given information is summarised in Table 2.2.45.2

a)

$$\Pr(E_1 + E_2) = \Pr(E_1) + \Pr(E_2) \quad \because E_1 E_2 = 0 \quad (2.2.45.1)$$

$$= 0.15 + 0.20 = 0.35 \quad (2.2.45.2)$$

b)

$$\Pr(E'_1 E'_5) = \Pr((E_1 + E_5)') \quad (2.2.45.3)$$

$$= 1 - \Pr(E_1 + E_5) \quad (2.2.45.4)$$

$$= 1 - [\Pr(E_1) + \Pr(E_5)] \quad \because E_1 E_5 = 0 \quad (2.2.45.5)$$

$$= 1 - [0.15 + 0.08] = 0.77 \quad (2.2.45.6)$$

$$(2.2.45.7)$$

c)

$$\Pr(E_3 + E_2) = \Pr(E_3) + \Pr(E_2) \quad \because E_3 E_2 = 0 \quad (2.2.45.8)$$

$$= 0.31 + 0.20 = 0.51 \quad (2.2.45.9)$$

d)

$$\Pr(E_3 + E_4) = \Pr(E_3) + \Pr(E_4) \quad \because E_3 E_4 = 0 \quad (2.2.45.10)$$

$$= 0.31 + 0.26 = 0.57 \quad (2.2.45.11)$$

| Variable | Difficulty Levels | Probability     |
|----------|-------------------|-----------------|
| $E_1$    | Very-Complex      | $\Pr(E_1)=0.15$ |
| $E_2$    | Complex           | $\Pr(E_2)=0.2$  |
| $E_3$    | Routine           | $\Pr(E_3)=0.31$ |
| $E_4$    | Simple            | $\Pr(E_4)=0.26$ |
| $E_5$    | Very-Simple       | $\Pr(E_5)=0.08$ |

TABLE 2.2.45.2

2.2.46 Without repetition of the numbers, four digit numbers are formed with the numbers 0,2,3,5. The probability of such a number divisible by 5 is

a)  $\frac{1}{5}$

b)  $\frac{4}{5}$

c)  $\frac{1}{30}$

d)  $\frac{5}{9}$

**Solution:** Let  $X$  denote the digit in the units place.

- a) Number of four digit numbers possible are  $3 \times 3 \times 2 \times 1 = 18$  because zero cannot be in the first place.
- b)  $n(X = 5) = 2 \times 2 \times 1 = 4$ .
- c)  $n(X = 0) = 3 \times 2 \times 1 = 6$ .

$$\therefore \Pr(X = 5) + \Pr(X = 0) = \frac{6+4}{18} = \frac{5}{9} \quad (2.2.46.1)$$

which is the desired probability.

- 2.2.47 Box A contains 25 slips of which 19 are marked Rs 1 and others are marked Rs 5 each. Box B contains 50 slips of which 45 are marked Rs 1 and others are marked Rs 13 each. Slips of both boxes are poured into a third box and reshuffled. A slip is drawn at random. What is the probability that it is marked other than Rs 1?

### 3 BERNOULLI

#### 3.1 Formulae

3.1.1 The Bernoulli distribution  $X \in \{0, 1\}$  is defined as

$$X = \text{Ber}(p). \quad (3.1.1.1)$$

with pmf

$$p_X(k) = \begin{cases} 1 - p & k = 0 \\ p & k = 1 \\ 0 & \text{otherwise} \end{cases} \quad (3.1.1.2)$$

3.1.2 The following code simulates 100 coin tosses

```
#Code by GVV Sharma
#November 18, 2020
#Released under GNU/GPL
#Given a Bernoulli probability and
#number of samples, the code generates the event data

import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import bernoulli

#100 samples
simlen=int(1e2)

#Probability of the event
prob = 0.5

#Generating sample date using Bernoulli r.v.
data_bern = bernoulli.rvs(size=simlen,p=prob)
#Calculating the number of favourable outcomes
err_ind = np.nonzero(data_bern == 1)
#calculating the probability
err_n = np.size(err_ind)/simlen

#Theory vs simulation
print(err_n,prob)
print(data_bern)
```

#### 3.2 NCERT

3.2.1 A lot consists of 144 ball pens of which 20 are defective and the others are good. Navami will buy a pen if it is good, but will not buy if it is defective. The shopkeeper draws one pen at random and gives it to her. What is the probability that

- a) She will buy it?  
 b) She will not buy it?

**Solution:** In this case, we have  $X \sim \text{Ber}\left(\frac{67}{72}\right)$ . Therefore, the desired probabilities are

a)  $p_X(1) = \frac{67}{72}$

b)  $p_X(0) = \frac{5}{72}$

- 3.2.2 A school has five houses A, B, C, D and E. A class has 23 students, 4 from house A, 8 from house B, 5 from house C, 2 from house D and rest from house E. A single student is selected at random to be the class monitor. The probability that the selected student is not from A, B and C is

**Solution:** See Table 3.2.2.1

| House    | A | B | C | D | E |
|----------|---|---|---|---|---|
| Students | 4 | 8 | 5 | 2 | 4 |

TABLE 3.2.2.1: Student distribution in each house

Define

$$X = \begin{cases} 0 & \text{A, B and C} \\ 1 & \text{Not from A, B and C} \end{cases} \quad (3.2.2.1)$$

Then, from Table 3.2.9.1,

$$X \sim \text{Ber}\left(\frac{6}{23}\right) \quad (3.2.2.2)$$

and the desired probability is

$$p_X(1) = \frac{6}{23} \quad (3.2.2.3)$$

- 3.2.3 A bag contains slips numbered from 1 to 100. If Phulan chooses a slip at random from the bag, it will either be an odd number or an even number. Since this situation has only two possible outcomes, so, the probability of each is  $\frac{1}{2}$ . Justify.

**Solution:** Let

$$X = \begin{cases} 1, & \text{if number is even} \\ 0, & \text{if number is odd} \end{cases} \quad (3.2.3.1)$$

Then

$$p_X(1) = \frac{50}{100} = \frac{1}{2} \quad (3.2.3.2)$$

$$p_X(0) = \frac{50}{100} = \frac{1}{2} \quad (3.2.3.3)$$

and  $X \sim \text{Ber}\left(\frac{1}{2}\right)$ .

- 3.2.4 A letter of English alphabets is chosen at random. Determine the probability that the letter is a consonant.

**Solution:** The desired probability is

$$p = \frac{21}{26} \quad (3.2.4.1)$$

- 3.2.5 A carton of 24 bulbs contain 6 defective bulbs. One bulb is drawn at random. What is the probability that the bulb is not defective? If the bulb selected is defective and it is not replaced and a second bulb is selected at random from the rest, what is the probability that the second bulb is defective?

**Solution:** Let

$$X_1 = \begin{cases} 1, & \text{if bulb is not defective} \\ 0, & \text{if bulb is defective} \end{cases} \quad (3.2.5.1)$$

Then the Bernoulli parameter

$$p_1 = 1 - \frac{6}{24} = \frac{3}{4} \quad (3.2.5.2)$$

which is the desired probability. In the second case,

$$1 - p_2 = \frac{6 - 1}{24 - 1} = \frac{5}{23} \quad (3.2.5.3)$$

which is the desired probability.

- 3.2.6 An integer is chosen between 0 and 100. What is the probability that it is

- a) divisible by 7
- b) not divisible by 7

**Solution:** Let  $X$  be a random variable such that

$$X = \begin{cases} 0 & n \not\equiv 0 \pmod{7} \\ 1 & n \equiv 0 \pmod{7} \end{cases} \quad (3.2.6.1)$$

Then,

- a)  $p = \frac{14}{99}$
- b)  $1 - p = \frac{85}{99}$

- 3.2.7 If the letters of the word **ALGORITHM** are arranged at random in a row what is the probability the letters GOR must remain together as a unit?

**Solution:** Let

$$X = \begin{cases} 1, & \text{if GOR remain together as a unit} \\ 0, & \text{otherwise} \end{cases} \quad (3.2.7.1)$$

Then

$$p = \frac{7!}{9!} = \frac{1}{72} \quad (3.2.7.2)$$

- 3.2.8 Six new employees, two of whom are married to each other, are to be assigned six desks that are lined up in a row. If the assignment of employees to desks is made randomly, what is the probability that the married couple will have nonadjacent desks?

**Solution:** Let  $X$  be a Random variable such that

| RV  | Values | Description                 |
|-----|--------|-----------------------------|
| $X$ | 0      | couple not sitting adjacent |
|     | 1      | couple sitting adjacent     |

$$1 - p = \frac{5! \times 2}{6!} = \frac{1}{3} \quad (3.2.8.1)$$

$$\implies p = \frac{2}{3} \quad (3.2.8.2)$$

3.2.9 There are four men and six women on the city council. If one council member is selected for a committee at random, how likely is it that it is a woman?

**Solution:**

| Parameter | Value | Description |
|-----------|-------|-------------|
| $X$       | 0     | Male        |
|           | 1     | Female      |

TABLE 3.2.9.1: Council distribution

$$X = \begin{cases} 0, & \text{if member is a man} \\ 1, & \text{if member is a woman} \end{cases} \quad (3.2.9.1)$$

From Table 3.2.9.1

$$p = \frac{6}{6+4} = \frac{3}{5} \quad (3.2.9.2)$$

3.2.10 A girl calculates that the probability of her winning the first prize in a lottery is 0.08. If 6000 tickets are sold, how many tickets has she bought?

- a) 40                    b) 240                    c) 480                    d) 750

**Solution:**

| parameter | value | description               |
|-----------|-------|---------------------------|
| $X$       | 0     | She didn't buy the ticket |
|           | 1     | She bought the ticket     |
| $N$       | 6000  | Number of tickets sold    |

TABLE 3.2.10.1: Information table

See Table 3.2.10.1. The number of tickets bought is

$$Np = 0.08 \times 6000 = 480 \quad (3.2.10.1)$$

3.2.11 Three numbers are chosen from 1 to 20. Find the probability that they are not consecutive

a)  $\frac{186}{190}$

b)  $\frac{187}{190}$

c)  $\frac{188}{190}$

d)  $\frac{18}{20C_3}$

**Solution:**

| Random variable | Value | Description                     |
|-----------------|-------|---------------------------------|
| X               | 0     | The numbers are not consecutive |
|                 | 1     | The numbers are consecutive     |

TABLE 3.2.11.1: Random variable

See Table 3.2.11.1. The number of sets of three consecutive numbers from 1 to 20 is 18. Hence,

$$p = \frac{18}{20C_3} \quad (3.2.11.1)$$

$$\implies 1 - p = 1 - \frac{18}{20C_3} = \frac{187}{190} \quad (3.2.11.2)$$

- 3.2.12 Seven persons are to be seated in a row. What is the probability that two particular persons sit next to each other?

**Solution:**

| RV | Values | Description                    |
|----|--------|--------------------------------|
| X  | 0      | Not sitting next to each other |
|    | 1      | Sitting next to each other     |

TABLE 3.2.12.1

See Table 3.2.12.1. The number of ways to arrange 7 people is  $7!$  and the number of ways to arrange 7 people in which the two particular people are adjacent to each other is  $6! \times 2$  considering both of them as one unit and considering the arrangements within the unit. Thus,

$$p = \frac{6! \times 2}{7!} = \frac{2}{7} \quad (3.2.12.1)$$

- 3.2.13 A single letter is selected at random from the word ‘PROBABILITY’. The probability that it is a vowel is \_\_\_\_\_.

**Solution:** Let  $X$  be an bernoulli rv defined as in Table 3.2.13.1. Then,

$$p = \frac{4}{11} \quad (3.2.13.1)$$

| RV | Value | Description |
|----|-------|-------------|
| X  | 0     | Consonant   |
|    | 1     | Vowel       |

TABLE 3.2.13.1

- 3.2.14 A letter is chosen at random from the word ‘ASSASSINATION’. Find the probability that letter is \_\_\_\_\_.

- a) a vowel  
b) a consonant

**Solution:** The number of vowels is 6 and consonants is 7. Therefore,

a)  $p = \frac{6}{13}$

b)  $1 - p = \frac{7}{13}$

- 3.2.15 A box contains 12 balls, out of which  $x$  are black. If one ball is drawn at random from the box, what is the probability that it will be a black ball? If 6 more black balls are put in the box, the probability of drawing a black ball is now double of what it was before. Find  $x$ .

**Solution:** From Table 3.2.15.1,

$$p_1 = \frac{x}{12}, p_2 = \frac{x+6}{18} \quad (3.2.15.1)$$

$$\therefore p_2 = 2p_1, \frac{x+6}{18} = 2\left(\frac{x}{12}\right) \quad (3.2.15.2)$$

$$\implies x = 3 \quad (3.2.15.3)$$

| Random Variable | Sample space | Value | Event               | Probability       |
|-----------------|--------------|-------|---------------------|-------------------|
| $X_1$           | 12           | 0     | not black           | $\frac{12-x}{12}$ |
|                 |              | 1     | choosing black ball | $\frac{x}{12}$    |
| $X_2$           | 18           | 0     | not black           | $\frac{12-x}{18}$ |
|                 |              | 1     | black               | $\frac{x+6}{18}$  |

TABLE 3.2.15.1

- 3.2.16 Gopi buys a fish from a shop for his aquarium. The shopkeeper takes out one fish at random from a tank containing 5 male fish and 8 female fish . What is the probability that the fish taken out is a male fish?

**Solution:** For

$$X = \begin{cases} 1 & \text{male} \\ 0 & \text{female,} \end{cases} \quad (3.2.16.1)$$

$$p = \frac{5}{13} \quad (3.2.16.2)$$

- 3.2.17 A bag contains 3 red balls and 5 black balls. A ball is drawn at random from the bag. What is the probability that the ball drawn is

- a) red ?  
b) not red?

**Solution:** For

$$X = \begin{cases} 1 & \text{red} \\ 0 & \text{otherwise,} \end{cases} \quad (3.2.17.1)$$

- a)  $p = \frac{3}{8}$   
b)  $1 - p = \frac{5}{8}$

- 3.2.18 Someone is asked to take a number from 1 to 100. The probability that it is a prime

number is

**Solution:** See Table 3.2.18.1. Since there are 25 prime numbers in between 1 to 100,

$$p = \frac{25}{100} = \frac{1}{4} \quad (3.2.18.1)$$

| RV | value | description |
|----|-------|-------------|
| X  | 0     | not prime   |
|    | 1     | prime       |

TABLE 3.2.18.1

## 4 CONDITIONAL PROBABILITY

### 4.1 Formulae

#### 4.1.1

$$\Pr(A|B) = \frac{\Pr(AB)}{\Pr(B)} \quad (4.1.1.1)$$

If  $A$  and  $B$  are independent, from (4.1.1.1) and (2.1.7.1),

$$\Pr(A|B) = \frac{\Pr(A)\Pr(B)}{\Pr(B)} = \Pr(A) \quad (4.1.1.2)$$

#### 4.1.2

$$\Pr(A'|B) = \frac{\Pr(A'B)}{\Pr(B)} = \frac{\Pr(B) - \Pr(AB)}{\Pr(B)} \quad (4.1.2.1)$$

#### 4.1.3 Total probability

$$\Pr(A) = \sum_{i=1}^2 \Pr(E_i) \Pr(A|E_i) \quad (4.1.3.1)$$

#### 4.1.4 Bayes' Theorem

$$\Pr(E_1|A) = \frac{\Pr(E_1)\Pr(A|E_1)}{\sum_{i=1}^2 \Pr(E_i)\Pr(A|E_i)} \quad (4.1.4.1)$$

4.1.5 Let  $X, Y \in \{0, 1\}$  be two random variables. Then,

$$\Pr(Y = 1|X = 0) \triangleq p_{Y|X}(1|0) \quad (4.1.5.1)$$

#### 4.1.6

$$p_{Y|X}(1|0) = \frac{p_{X,Y}(0, 1)}{p_X(0)} = \frac{p_X(0) - p_{X,Y}(0, 0)}{p_X(0)} \quad (4.1.6.1)$$

$$= 1 - \frac{p_{X,Y}(0, 0)}{p_X(0)} = 1 - p_{Y|X}(0|0) \quad (4.1.6.2)$$

## 4.2 NCERT

4.2.1 Given that  $E$  and  $F$  are events such that  $P(E) = 0.6$ ,  $P(F) = 0.3$  and  $P(EF) = 0.2$ , find  $P(E|F)$  and  $P(F|E)$ .

**Solution:** From (4.1.1.1)

$$\Pr(E|F) = \frac{\Pr(EF)}{\Pr(F)} = \frac{0.2}{0.3} = \frac{2}{3} \quad (4.2.1.1)$$

$$\Pr(F|E) = \frac{\Pr(EF)}{\Pr(E)} = \frac{0.2}{0.6} = \frac{1}{3} \quad (4.2.1.2)$$

4.2.2 Compute  $\Pr(A|B)$ , if  $\Pr(B) = 0.5$  and  $\Pr(AB) = 0.32$ .

**Solution:**

$$\Pr(A|B) = \frac{0.32}{0.5} = 0.64 \quad (4.2.2.1)$$

4.2.3 If  $\Pr(A) = 0.8$ ,  $\Pr(B) = 0.5$  and  $\Pr(B|A) = 0.4$ , find

a)  $\Pr(AB)$

b)  $\Pr(A|B)$

c)  $\Pr(A + B)$

**Solution:**

a)

$$\Pr(B|A) = \frac{\Pr(AB)}{\Pr(A)} \quad (4.2.3.1)$$

$$\implies 0.4 = \frac{\Pr(AB)}{0.8} \quad (4.2.3.2)$$

$$\text{or, } \Pr(AB) = 0.32 \quad (4.2.3.3)$$

b) Similarly,

$$\Pr(A|B) = \frac{\Pr(AB)}{\Pr(B)} = \frac{0.32}{0.5} = 0.64 \quad (4.2.3.4)$$

c)

$$\Pr(A + B) = \Pr(A) + \Pr(B) - \Pr(AB) \quad (4.2.3.5)$$

$$= 0.8 + 0.5 - 0.32 = 0.98 \quad (4.2.3.6)$$

4.2.4 If  $\Pr(A) = \frac{6}{11}$ ,  $\Pr(B) = \frac{5}{11}$  and  $\Pr(A + B) = \frac{7}{11}$ , find

a)  $\Pr(AB)$

b)  $\Pr(A | B)$

c)  $\Pr(B | A)$

**Solution:**

a) From (2.1.5.1),

$$\Pr(AB) = \frac{6}{11} + \frac{5}{11} - \frac{7}{11} = \frac{4}{11} \quad (4.2.4.1)$$

b) From (4.2.4.1) and (4.1.1.1),

$$\Pr(A | B) = \frac{\frac{4}{11}}{\frac{5}{11}} = \frac{4}{5} \quad (4.2.4.2)$$

c) Similarly,

$$\Pr(B | A) = \frac{\frac{4}{11}}{\frac{6}{11}} = \frac{2}{3} \quad (4.2.4.3)$$

4.2.5 Mother, Father and Son line up at random for a family picture. Determine  $\Pr(E | F)$  where E: Son on one end, F: Father in middle.

**Solution:** The total ways of arranging Father, Son, Mother in the family chart is  $3! = 6$ . The probability that Father in middle is

$$\Pr(F) = \frac{2!}{3!} = \frac{1}{3} \quad (4.2.5.1)$$

The probability that Father in middle and Son is on one end is

$$\Pr(EF) = \frac{2!}{3!} = \frac{1}{3} \quad (4.2.5.2)$$

Thus,

$$\Pr(E | F) = \frac{\Pr(EF)}{\Pr(F)} = 1 \quad (4.2.5.3)$$

4.2.6 An instructor has a question bank consisting of 300 easy True / False questions, 200 difficult True / False questions, 500 easy multiple choice questions and 400 difficult multiple choice questions. If a question is selected at random from the question bank, what is the probability that it will be an easy question given that it is a multiple choice question?

**Solution:**

| Variable | Event                    |
|----------|--------------------------|
| $X = 0$  | Easy question            |
| $X = 1$  | Difficult question       |
| $Y = 0$  | True/False question      |
| $Y = 1$  | Multiple choice question |

TABLE 4.2.6.1

See Table 4.2.6.1. From the given information,

$$p_{XY}(0,0) = \frac{3}{14}, p_{XY}(0,1) = \frac{5}{14}, p_{XY}(1,0) = \frac{1}{7}, p_{XY}(1,1) = \frac{2}{7} \quad (4.2.6.1)$$

$$\implies p_Y(1) = \sum_{i=0}^1 p_{XY}(1, i) = \frac{9}{14}. \quad (4.2.6.2)$$

$$\therefore p_{X|Y}(0|1) = \frac{p_{X,Y}(0,1)}{p_Y(1)} = \frac{\frac{5}{14}}{\frac{9}{14}} = \frac{5}{9} \quad (4.2.6.3)$$

4.2.7 If  $\Pr(A) = \frac{1}{2}$ ,  $\Pr(B) = 0$ , then  $\Pr(A | B)$  is



4.2.8 If  $A$  and  $B$  are events such that

$$\Pr(A|B) = \Pr(B|A) \quad (4.2.8.1)$$

then

**Solution:** Using Bayes' Rule,

$$\Pr(AB) = \Pr(A)\Pr(B|A) \quad (4.2.8.2)$$

$$= \Pr(B)\Pr(A|B) \quad (4.2.8.3)$$

Using (4.2.8.1) in (4.2.8.2) and (4.2.8.3),

$$\Pr(A) = \Pr(B) \quad (4.2.8.4)$$

We consider the options one by one.

- a) If  $A \subset B$  and  $A \neq B$ , then we can write  $B = A + C$ , where  $AC = 0$  and  $C \neq 0$ . Thus,

$$\Pr(B) = \Pr(A + C) \quad (4.2.8.5)$$

$$= \Pr(A) + \Pr(C) - \Pr(AC) \quad (4.2.8.6)$$

$$= \Pr(A) + \Pr(C) > \Pr(A) \quad (4.2.8.7)$$

However, (4.2.8.7) contradicts (4.2.8.4).

- b) We give a counterexample to show this is wrong. Consider  $A$  as the event that an even number shows on rolling a fair die and  $B$  as the event that a prime number shows on rolling a fair die. The joint pmf is shown in Table 4.2.8.1. Clearly,

$$\Pr(A|B) = \frac{\Pr(AB)}{\Pr(B)} = \frac{\frac{1}{6}}{\frac{1}{6} + \frac{1}{3}} = \frac{1}{2} \quad (4.2.8.8)$$

$$\Pr(B|A) = \frac{\Pr(AB)}{\Pr(A)} = \frac{\frac{1}{6}}{\frac{1}{6} + \frac{1}{3}} = \frac{1}{2} \quad (4.2.8.9)$$

- c) The same example as before provides the required counterexample, as  $\Pr(AB) = \frac{1}{6}$ .  
d) This is the correct answer, as discussed above.

|           | $A$           | $\bar{A}$     |
|-----------|---------------|---------------|
| $B$       | $\frac{1}{6}$ | $\frac{1}{3}$ |
| $\bar{B}$ | $\frac{1}{3}$ | $\frac{1}{6}$ |

TABLE 4.2.8.1: Joint pmf for events  $A$  and  $B$ .

- 4.2.9 Consider the experiment of throwing a die, if a multiple of 3 comes up, throw the die again and if any other number comes, toss a coin. Find the conditional probability of the event ‘the coin shows a tail’, given that ‘at least one die shows a 3’.  
4.2.10 Two cards are drawn at random and without replacement from a pack of 52 playing cards. Find the probability that both the cards are black.  
**Solution:** Let  $E_1$  denote the event that the first card drawn is Black,  $E_2$  denote the

event that the second card drawn is Black. Then

$$\Pr(E_1) = \frac{26}{52}, \Pr(E_2 | E_1) = \frac{25}{51} \quad (4.2.10.1)$$

$$\implies \Pr(E_1 E_2) = \Pr(E_1) \Pr(E_2 | E_1) = \frac{25}{102} \quad (4.2.10.2)$$

4.2.11 Let  $A$  and  $B$  be independent events with  $P(A) = 0.3$  and  $P(B) = 0.4$ . Find

- a)  $P(A \cap B)$       b)  $P(A \cup B)$       c)  $P(A|B)$       d)  $P(B|A)$

4.2.12 An urn contains 5 red and 5 black balls. A ball is drawn at random, its colour is noted and is returned to the urn. Moreover, 2 additional balls of the colour drawn are put in the urn and then a ball is drawn at random. What is the probability that the second ball is red?

**Solution:** The given information is summarized in Tables 4.2.12.2 and 4.2.12.4.

| Variable | Value | Colour | Description |
|----------|-------|--------|-------------|
| X        | 0     | Red    | 1st draw    |
|          | 1     | Black  | 1st draw    |
| Y        | 0     | Red    | 2nd draw    |
|          | 1     | Black  | 2nd draw    |

TABLE 4.2.12.2

| Probability    | Value          |
|----------------|----------------|
| $p_X(0)$       | $\frac{5}{10}$ |
| $p_X(1)$       | $\frac{5}{10}$ |
| $p_{Y X}(0 0)$ | $\frac{7}{12}$ |
| $p_{Y X}(1 0)$ | $\frac{5}{12}$ |

TABLE 4.2.12.4

From (4.1.3.1), the required probability is given by

$$p_Y(0) = p_X(0) p_{Y|X}(0|0) + p_X(1) p_{Y|X}(0|1) \quad (4.2.12.1)$$

$$= \left( \frac{5}{10} \times \frac{7}{12} \right) + \left( \frac{5}{10} \times \frac{5}{12} \right) = \frac{1}{2} \quad (4.2.12.2)$$

4.2.13 A bag contains 4 red and 4 black balls, another bag contains 2 red and 6 black balls. One of the two bags is selected at random and a ball is drawn from the bag which is found to be red. Find the probability that the ball is drawn from the first bag.

4.2.14 Of the students in a college, it is known that 60% reside in hostel and 40% are day scholars (not residing in hostel). Previous year results report that 30% of all students who reside in hostel attain A grade and 20% of day scholars attain A grade in their annual examination. At the end of the year, one student is chosen at random from the

college and he has an A grade, what is the probability that the student is a hostelier?

**Solution:** The given information is summarized in Table 4.2.14.2.

| Variable | Value | Description    |
|----------|-------|----------------|
| X        | 0     | Hostel Student |
|          | 1     | Day Scholar    |
| Y        | 0     | A grade        |
|          | 1     | No A grade     |

TABLE 4.2.14.2

From the given data,

$$p_X(0) = \frac{3}{5}, p_X(1) = \frac{2}{5}, p_Y(1|0) = \frac{3}{10}, p_Y(1|1) = \frac{1}{5} \quad (4.2.14.1)$$

The desired probability is

$$p_{X|Y}(0|1) = \frac{p_{Y|X}(1|0) \times p_X(0)}{\sum_{k=0}^1 p_{Y|X}(1|k) \times p_X(k)} \quad (4.2.14.2)$$

$$= \frac{\frac{3}{10} \times \frac{3}{5}}{\frac{3}{10} \times \frac{3}{5} + \frac{1}{5} \times \frac{2}{5}} = \frac{9}{13} \quad (4.2.14.3)$$

- 4.2.15 In answering a question on a multiple choice test, a student either knows the answer or guesses. Let  $\frac{3}{4}$  be the probability that he knows the answer and  $\frac{1}{4}$  be the probability that he guesses. Assuming that a student who guesses at the answer will be correct with probability  $\frac{1}{4}$ , what is the probability that the student knows the answer given that he answered it correctly?

**Solution:** See Table 4.2.15.2

| Variable | Value | Description |
|----------|-------|-------------|
| X        | 0     | Guesses     |
|          | 1     | Knows       |
| Y        | 0     | Incorrect   |
|          | 1     | Correct     |

TABLE 4.2.15.2

From the given information,

$$p_{Y|X}(1|0) = \frac{1}{4}, p_{Y|X}(1|1) = 1, p_X(0) = \frac{1}{4}, p_X(1) = \frac{3}{4} \quad (4.2.15.1)$$

The desired probability is

$$p_{X|Y}(1|1) = \frac{p_{Y|X}(1|1) p_X(1)}{\sum_{i=0}^1 p_{Y|X}(1|i) p_X(i)} \quad (4.2.15.2)$$

$$= \frac{\frac{3}{4}}{\frac{1}{4} \times \frac{1}{4} + 1 \times \frac{3}{4}} = \frac{4}{5} \quad (4.2.15.3)$$

- 4.2.16 A laboratory blood test is 99% effective in detecting a certain disease when it is in fact, present. However, the test also yields a false positive result for 0.5% of the healthy person tested (i.e. if a healthy person is tested, then, with probability 0.005, the test will imply he has the disease). If 0.1 percent of the population actually has the disease, what is the probability that a person has the disease given that his test result is positive ?

**Solution:** See Table 4.2.16.2 for the given information.

| Variable | Value | Description         |
|----------|-------|---------------------|
| X        | 0     | Blood test negative |
|          | 1     | Blood test positive |
| Y        | 0     | No Disease          |
|          | 1     | Disease             |

TABLE 4.2.16.2

From the given information,

$$p_Y(0) = 1 - p_Y(1) = 1 - 0.001 = 0.999 \quad (4.2.16.1)$$

$$p_{X|Y}(1|1) = 0.99, p_{X|Y}(0|1) = 0.005 \quad (4.2.16.2)$$

$$\therefore p_{Y|X}(1|1) = \frac{p_Y(1) p_{X|Y}(1|1)}{\sum_{i=1}^2 p_Y(i) p_{X|Y}(1|i)} \quad (4.2.16.3)$$

$$= \frac{0.001 \times 0.99}{0.001 \times 0.99 + 0.999 \times 0.005} = \frac{22}{133} \quad (4.2.16.4)$$

- 4.2.17 There are three coins. One is a two headed coin (having head on both faces), another is a biased coin that comes up heads 75% of the time and third is an unbiased coin. One of the three coins is chosen at random and tossed, it shows heads, what is the probability that it was the two headed coin?

- 4.2.18 An insurance company insured 2000 scooter drivers, 4000 car drivers and 6000 truck drivers. The probability of an accidents are 0.01, 0.03 and 0.15 respectively. One of the insured persons meets with an accident. What is the probability that he is a scooter driver?

- 4.2.19 A factory has two machines A and B. Past record shows that machine A produced 60% of the items of output and machine B produced 40% of the items. Further, 2% of the items produced by machine A and 1% produced by machine B were defective. All the items are put into one stockpile and then one item is chosen at random from this and is found to be defective. What is the probability that it was produced by

machine B?

- 4.2.20 Two groups are competing for the position on the Board of directors of a corporation. The probabilities that the first and the second groups will win are 0.6 and 0.4 respectively. Further, if the first group wins, the probability of introducing a new product is 0.7 and the corresponding probability is 0.3 if the second group wins. Find the probability that the new product introduced was by the second group.

**Solution:** See Table 4.2.20.2.

| Variable | Value | Description               |
|----------|-------|---------------------------|
| X        | 1     | Group 1 wins              |
|          | 2     | Group 2 wins              |
| Y        | 0     | New product introduced    |
|          | 1     | No new product introduced |

TABLE 4.2.20.2

From the given information,

$$p_X(1) = 0.6, p_X(2) = 0.4, p_{Y|X}(1|1) = 0.7, p_{Y|X}(1|2) = 0.3 \quad (4.2.20.1)$$

$$\Rightarrow p_{X|Y}(2|1) = \frac{p_X(2) p_{Y|X}(1|1)}{\sum_{i=1}^2 p_X(i) p_{Y|X}(1|i)} = \frac{2}{9} \quad (4.2.20.2)$$

upon substituting numerical values and simplifying.

- 4.2.21 Suppose a girl throws a die. If she gets a 5 or 6, she tosses a coin three times and notes the number of heads. If she gets 1, 2, 3 or 4, she tosses a coin once and notes whether a head or tail is obtained. If she obtained exactly one head, what is the probability that she threw 1, 2, 3 or 4 with the die?]
- 4.2.22 A manufacturer has three machine operators A, B and C. The first operator A produces 1% defective items, where as the other two operators B and C produce 5% and 7% defective items respectively. A is on the job for 50% of the time, B is on the job for 30% of the time and C is on the job for 20% of the time. A defective item is produced, what is the probability that it was produced by A?
- 4.2.23 A card from a pack of 52 cards is lost. From the remaining cards of the pack, two cards are drawn and are found to be both diamonds. Find the probability of the lost card being a diamond.
- 4.2.24 Probability that A speaks truth is  $\frac{4}{5}$ . A coin is tossed. A reports that a head appears. The probability that actually there was head is

a)  $\frac{4}{5}$

b)  $\frac{1}{2}$

c)  $\frac{1}{5}$

d)  $\frac{2}{5}$

**Solution:** See Table 4.2.24.2.

| Variable | Value | Description       |
|----------|-------|-------------------|
| X        | 0     | Tail              |
|          | 1     | Head              |
| Y        | 0     | A lies            |
|          | 1     | A tells the truth |

TABLE 4.2.24.2

From the given information,

$$p_X(0) = p_X(1) = \frac{1}{2}, p_Y(1) = \frac{4}{5} \quad (4.2.24.1)$$

$$\therefore p_{Y|X}(1|1) = \frac{p_Y(1)p_{X|Y}(1|1)}{p_X(1)} \quad (4.2.24.2)$$

$$= \frac{\frac{4}{5} \times \frac{1}{2}}{\frac{1}{2}} = \frac{4}{5} \quad (4.2.24.3)$$

Here, we have used the fact that

$$p_{X|Y}(1|1) = p_X(1) = \frac{1}{2}. \quad (4.2.24.4)$$

since the appearance of a head upon a coin toss has nothing to do with A.

- 4.2.25 If A and B are two events such that  $A \subset B$  and  $\Pr(B) \neq 0$ , then which of the following is correct ?

- a)  $\Pr(A|B) = \frac{\Pr(B)}{\Pr(A)}$   
 b)  $\Pr(A|B) < \Pr(A)$   
 c)  $\Pr(A|B) \geq \Pr(A)$   
 d) None of these

### Solution:

- a) If  $A \subset B$  and  $\Pr(B) \neq 0$  then

$$AB = A \quad (4.2.25.1)$$

$$\text{or, } P(A|B) = \frac{\Pr(AB)}{\Pr(B)} = \frac{\Pr(A)}{\Pr(B)} \quad (4.2.25.2)$$

- c) Since

$$\Pr(B) \leq 1, \quad (4.2.25.3)$$

$$1 \leq \frac{1}{\Pr(B)} \quad (4.2.25.4)$$

$$\implies \Pr(A) \leq \frac{\Pr(A)}{\Pr(B)} \quad (4.2.25.5)$$

$$= \Pr(A|B) \quad (4.2.25.6)$$

from (4.2.25.2).

- 4.2.26 A and B are two events such that  $\Pr(A) \neq 0$ . Find  $\Pr(B|A)$ , if

a)  $A$  is a subset of  $B$

b)  $A \cap B = \emptyset$

**Solution:** We use

$$\Pr(B | A) = \frac{\Pr(BA)}{\Pr(A)} \quad (4.2.26.1)$$

a) In this case,

$$BA = A \implies \Pr(BA) = \Pr(A) \quad (4.2.26.2)$$

From (4.2.26.1),

$$\Pr(B | A) = 1 \quad (4.2.26.3)$$

b)  $A \cap B = \emptyset$ . This implies

$$\Pr(BA) = 0 \quad (4.2.26.4)$$

From (4.2.26.1),

$$\Pr(B | A) = 0 \quad (4.2.26.5)$$

4.2.27 A couple has two children.

- a) Find the probability that both children are males, if it is known that at least one of the children is male.
- b) Find the probability that both children are females, if it is known that the elder child is a female.

4.2.28 Suppose that 5% of men and 0.25% of women have grey hair. A grey haired person is selected at random. What is the probability that this person being male? Assume that there are equal number of males and females.

**Solution:** See Table 4.2.28.1.

| Variable | Event         |
|----------|---------------|
| $X = 0$  | Men           |
| $X = 1$  | Women         |
| $Y = 0$  | Non-grey hair |
| $Y = 1$  | grey hair     |

TABLE 4.2.28.1

From the given information,

$$p_X(0) = p_X(1) = \frac{1}{2} \quad (4.2.28.1)$$

$$p_{Y|X}(1|0) = \frac{5}{100} = \frac{1}{20} \quad (4.2.28.2)$$

$$p_{Y|X}(1|1) = \frac{0.25}{100} = \frac{1}{400} \quad (4.2.28.3)$$

Using (4.1.4.1)

$$p_{X|Y}(0|1) = \frac{p_{Y|X}(1|0)p_X(0)}{\sum_{i=0}^1 p_{Y|X}(1|i)p_Y(i)} = \frac{\frac{1}{40}}{\frac{21}{800}} = \frac{20}{21} \quad (4.2.28.4)$$

4.2.29 Suppose we have four boxes A,B,C and D containing coloured marbles as given in Table 4.2.29.1.

| Box | Marble colour |       |       |
|-----|---------------|-------|-------|
|     | Red           | White | Black |
| A   | 1             | 6     | 3     |
| B   | 6             | 2     | 2     |
| C   | 8             | 1     | 1     |
| D   | 0             | 6     | 4     |

TABLE 4.2.29.1: Question Table

One of the boxes has been selected at random and a single marble is drawn from it. If the marble is red, what is the probability that it was drawn from

a) Box A ?

b) Box B ?

c) Box C ?

**Solution:** Let  $X \in \{i\}_{i=0}^2$  represent the colour and  $Y \in \{i\}_{i=0}^3$  represent the box. From the given information,

$$p_{X|Y}(0|i) = \begin{cases} \frac{1}{10}, & i = 0 \\ \frac{6}{10}, & i = 1 \\ \frac{8}{10}, & i = 2 \\ 0 & i = 3 \end{cases}, p_Y(i) = \frac{1}{4} \quad (4.2.29.1)$$

From (4.1.4.1)

$$p_{Y|X}(i|0) = \begin{cases} \frac{1}{15} & i = 0 \\ \frac{2}{5} & i = 1 \\ \frac{8}{15} & i = 2 \end{cases} \quad (4.2.29.2)$$

4.2.30 Bag I contains 3 red and 4 black balls and Bag II contains 4 red and 5 black balls. One ball is transferred from Bag I to Bag II and then a ball is drawn from Bag II. The ball so drawn is found to be red in colour. Find the probability that the transferred ball is black.

**Solution:** Let  $X, Y \in \{0, 1\}, i \in \{1, 2\}$  represent the red and black balls in Bag 1 and 2 respectively. From the given information,

$$p_X(i) = \begin{cases} \frac{3}{7} & i = 0 \\ \frac{4}{7} & i = 1 \end{cases}, p_Y(i) = \begin{cases} \frac{4}{9} & i = 0 \\ \frac{5}{9} & i = 1 \end{cases} \quad (4.2.30.1)$$

Also

$$p_{Y|X}(0|i) = \begin{cases} \frac{4}{10} & i = 0 \\ \frac{5}{10} & i = 1 \end{cases}, p_{Y|X}(1|i) = \begin{cases} \frac{5}{10} & i = 0 \\ \frac{6}{10} & i = 1 \end{cases} \quad (4.2.30.2)$$

$$\therefore p_{X|Y}(1|0) = \frac{p_{Y|X}(0|1)p_X(1)}{p_Y(0)} = \frac{\frac{1}{2} \times \frac{4}{7}}{\frac{5}{9}} = \frac{18}{35} \quad (4.2.30.3)$$

- 4.2.31 Bag I contains 3 black and 2 white balls, Bag II contains 2 black and 4 white balls. A bag and a ball is selected at random. Determine the probability of selecting a black ball.

**Solution:**

| Random variable | Value | Definition |
|-----------------|-------|------------|
| X               | 0     | Bag 1      |
|                 | 1     | Bag 2      |
| Y               | 0     | White ball |
|                 | 1     | Black ball |

TABLE 4.2.31.1

From the given information,

$$p_X(0) = p_X(1) = \frac{1}{2} \quad (4.2.31.1)$$

$$p_{Y|X}(1|0) = \frac{3}{5}, p_{Y|X}(1|1) = \frac{1}{3} \quad (4.2.31.2)$$

From (4.1.3.1), the desired probability is

$$p_Y(1) = \sum_{i=0}^1 p_{Y|X}(1|i)p_X(i) = \frac{7}{15} \quad (4.2.31.3)$$

- 4.2.32 While shuffling a pack of 52 playing cards, 2 cards are dropped. Find the probability that the missing cards are of different colours.

**Solution:** See Table 4.2.32.1.

| Random Variable | Values | Description          |
|-----------------|--------|----------------------|
| X <sub>1</sub>  | 0      | First card is red    |
|                 | 1      | First card is black  |
| X <sub>2</sub>  | 0      | Second card is red   |
|                 | 1      | Second card is black |

TABLE 4.2.32.1

Since 26 out of 52 playing cards are red,

$$p_{X_1}(k) = \frac{26}{52} = \frac{1}{2} \quad \{k = 0, 1\} \quad (4.2.32.1)$$

Also,

$$p_{X_2|X_1}(i|j) = \begin{cases} \frac{25}{51} & i = j \\ \frac{26}{51} & i \neq j \end{cases} \quad (4.2.32.2)$$

The desired probability is

$$\Pr(X_1 \neq X_2) = \sum_{i=0}^1 \Pr(X_1 \neq X_2 | X_1 = i) p_X(i) = \frac{26}{51} \quad (4.2.32.3)$$

- 4.2.33 A bag contains  $(2n + 1)$  coins. It is known that  $n$  of these coins have a head on both sides whereas the rest of the coins are fair. A coin is picked up at random from the bag and is tossed. If the probability that the toss results in a head is  $\frac{31}{42}$ , determine the value of  $n$ .
- 4.2.34 An urn contains  $m$  white and  $n$  black balls. A ball is drawn at random and is put back into the urn along with  $k$  additional balls of the same colour as that of the ball drawn. A ball is again drawn at random. Show that the probability of drawing a white ball now does not depend on  $k$ .
- 4.2.35 If  $\Pr(AB) = \frac{7}{10}$  and  $\Pr(B) = \frac{17}{20}$ , then  $\Pr(A|B)$  equals

a)  $\frac{14}{17}$

b)  $\frac{17}{20}$

c)  $\frac{7}{8}$

d)  $\frac{1}{8}$

- 4.2.36 A letter is known to have come either from TATANAGAR or from CALCUTTA. On the envelope, just two consecutive letters TA are visible. What is the probability that the letter came from TATANAGAR.
- 4.2.37 A bag contains  $(2n + 1)$  coins. It is known that  $n$  of these coins have a head on both sides whereas the rest of the coins are fair. A coin is picked up at random from the bag and is tossed. If the probability that the toss results in a head is  $\frac{31}{42}$ , determine the value of  $n$ .
- 4.2.38 By examining the chest X-ray, the probability that TB is detected when a person is actually suffering is 0.99. The probability of a healthy person diagnosed to have TB is 0.001. In a certain city, 1 in 1000 people suffers from TB. A person is selected at random and is diagnosed to have TB. What is the probability that he actually has TB?
- 4.2.39 Compute  $\Pr(A|B)$ , if  $\Pr(B) = 0.5$  and  $\Pr(AB) = 0.32$ .
- 4.2.40 Let  $\Pr(A) = \frac{7}{13}$ ,  $\Pr(B) = \frac{9}{13}$ ,  $\Pr(AB) = \frac{4}{13}$ . Then  $\Pr(A'|B)$  is equal to

a)  $\frac{6}{13}$

b)  $\frac{4}{13}$

c)  $\frac{4}{9}$

d)  $\frac{5}{9}$

**Solution:** From (4.1.1.1),

$$\Pr(A'|B) = \frac{\Pr(A'B)}{\Pr(B)} \quad (4.2.40.1)$$

$$= \frac{\Pr(B) - \Pr(AB)}{\Pr(B)} = \frac{5}{9} \quad (4.2.40.2)$$

using (2.1.5.3).

- 4.2.41 If  $\Pr(A) = \frac{2}{5}$ ,  $\Pr(B) = \frac{3}{10}$  and  $\Pr(AB) = \frac{1}{5}$ , then  $\Pr(A'|B')\Pr(B'|A')$  is equal to

a)  $\frac{5}{6}$

b)  $\frac{5}{7}$

c)  $\frac{25}{42}$

d) 1

**Solution:** From (2.1.5.1),

$$\Pr(A + B) = \frac{1}{2}. \quad (4.2.41.1)$$

From (4.1.1.1),

$$\Pr(A'|B')\Pr(B'|A') = \frac{\Pr(A'B')}{\Pr(B')}\cdot\frac{\Pr(A'B')}{\Pr(A')} \quad (4.2.41.2)$$

$$= \frac{(\Pr(A'B'))^2}{(1 - \Pr(B))(1 - \Pr(A))} \quad (4.2.41.3)$$

$$= \frac{(1 - \Pr(A + B))^2}{(1 - \Pr(B))(1 - \Pr(A))} \quad (4.2.41.4)$$

$$= \frac{25}{42} \quad (4.2.41.5)$$

upon substituting numerical values.

4.2.42  $A$  and  $B$  are two events such that  $\Pr(A) = \frac{1}{2}$ ,  $\Pr(B) = \frac{1}{3}$  and  $\Pr(AB) = \frac{1}{4}$ . Find

- a)  $\Pr(A|B)$       b)  $\Pr(B|A)$       c)  $\Pr(A'|B)$       d)  $\Pr(A'|B')$

**Solution:**

a)

$$\Pr(A|B) = \frac{\Pr(AB)}{\Pr(B)} = \frac{3}{4} \quad (4.2.42.1)$$

b)

$$\Pr(B|A) = \frac{\Pr(AB)}{\Pr(A)} = \frac{1}{2} \quad (4.2.42.2)$$

c) From (4.1.2.1),

$$\Pr(A'|B) = \frac{\Pr(B) - \Pr(AB)}{\Pr(B)} = \frac{1}{4} \quad (4.2.42.3)$$

d)

$$\Pr(A'|B') = \frac{\Pr(A'B')}{\Pr(B')} \quad (4.2.42.4)$$

$$= \frac{\Pr(A + B)'}{\Pr(B')} = \frac{1 - \Pr(A + B)}{1 - \Pr(B)} \quad (4.2.42.5)$$

$$= \frac{5}{8} \quad (4.2.42.6)$$

using (2.1.5.1) in the numerator.

4.2.43 If  $\Pr(A) = \frac{3}{10}$ ,  $\Pr(B) = \frac{2}{5}$  and  $\Pr(A + B) = \frac{3}{5}$ , then  $\Pr(B|A) + \Pr(A|B)$  equals

a)  $\frac{1}{4}$

b)  $\frac{1}{3}$

c)  $\frac{5}{12}$

d)  $\frac{7}{12}$

**Solution:**

$$\Pr(AB) = \Pr(A) + \Pr(B) - \Pr(A + B) \quad (4.2.43.1)$$

$$= \frac{1}{10} \quad (4.2.43.2)$$

$$\implies \Pr(B|A) + \Pr(A|B) = \frac{\Pr(AB)}{\Pr(A)} + \frac{\Pr(AB)}{\Pr(B)} = \frac{7}{12} \quad (4.2.43.3)$$

upon substituting numerical values.

- 4.2.44 Let  $A$  and  $B$  be two events such that  $\Pr(A) = \frac{3}{8}$ ,  $\Pr(B) = \frac{5}{8}$  and  $\Pr(A + B) = \frac{3}{4}$ . Then  $\Pr(A|B)\Pr(A'|B)$  is equal to

a)  $\frac{2}{5}$

b)  $\frac{3}{8}$

c)  $\frac{3}{20}$

d)  $\frac{6}{25}$

**Solution:** From (2.1.5.1)

$$\Pr(AB) = \frac{1}{4} \quad (4.2.44.1)$$

Hence,

$$\Pr(A|B) \cdot \Pr(A'|B) = \frac{\Pr(AB)}{\Pr(B)} \times \frac{\Pr(B) - \Pr(AB)}{\Pr(B)} \quad (4.2.44.2)$$

$$= \frac{6}{25} \quad (4.2.44.3)$$

using (2.1.5.3) and substituting numerical values.

- 4.2.45 If  $P(A) = 0.4$ ,  $P(B) = 0.8$  and  $P(B|A) = 0.6$ , then  $P(A \cup B)$  is equal to

a) 0.24

b) 0.3

c) 0.48

d) 0.96

**Solution:** From (4.1.1.1),

$$\Pr(AB) = \Pr(B|A)\Pr(A) = 0.24. \quad (4.2.45.1)$$

yielding

$$\Pr(A + B) = 0.96 \quad (4.2.45.2)$$

from (2.1.5.1).

- 4.2.46 If  $A$  and  $B$  are two events such that  $\Pr(A) = \frac{1}{2}$ ,  $\Pr(B) = \frac{1}{3}$ ,  $\Pr(A|B) = \frac{1}{4}$ , then  $\Pr(A'B')$  equals

a)  $\frac{1}{12}$

b)  $\frac{3}{4}$

c)  $\frac{1}{4}$

d)  $\frac{3}{16}$

**Solution:** From (4.1.1.1),

$$\Pr(AB) = \Pr(A|B)\Pr(B) = \frac{1}{12} \quad (4.2.46.1)$$

$$\implies \Pr(A'B') = 1 - \Pr(A + B) = \frac{1}{4} \quad (4.2.46.2)$$

using (2.1.5.1) and substituting numerical values.

4.2.47 If  $A$  and  $B$  are such events that  $\Pr(A) > 0$  and  $\Pr(B) \neq 1$ , then  $\Pr(A'|B')$  is

- a)  $1 - \Pr(A|B)$       b)  $1 - \Pr(A'|B)$       c)  $\frac{1 - \Pr(A+B)}{\Pr(B')}$       d)  $\frac{\Pr(A')}{\Pr(B')}$

**Solution:**

$$\Pr(A'|B') = \frac{\Pr(A'B')}{\Pr(B')} = \frac{\Pr((A+B)')}{\Pr(B')} \quad (4.2.47.1)$$

$$= \frac{1 - \Pr(A+B)}{\Pr(B')} \quad (4.2.47.2)$$

4.2.48 Two events  $E$  and  $F$  are independent. If  $\Pr(E) = 0.3$ ,  $\Pr(E+F) = 0.5$ , then  $\Pr(E|F) - \Pr(F|E)$  equals

- a)  $\frac{2}{7}$       b)  $\frac{3}{35}$       c)  $\frac{1}{70}$       d)  $\frac{1}{7}$

**Solution:**

$$\Pr(EF) = \Pr(E)\Pr(F) \quad (4.2.48.1)$$

$$\therefore \Pr(F) = \frac{\Pr(E+F) - \Pr(E)}{1 - \Pr(E)} = \frac{2}{7} \quad (4.2.48.2)$$

using (2.1.5.1) and simplifying. From (4.1.1.1),

$$\Pr(E|F) = \Pr(E), \quad \Pr(F|E) = \Pr(F) \quad (4.2.48.3)$$

$$\implies \Pr(E|F) - \Pr(F|E) = \Pr(E) - \Pr(F) = \frac{1}{70} \quad (4.2.48.4)$$

4.2.49 If  $A$  and  $B$  are two events such that  $\Pr(A|B) = p$ ,  $\Pr(A) = p$ ,  $\Pr(B) = \frac{1}{3}$  and

$\Pr(A+B) = \frac{5}{9}$ , then  $p =$

**Solution:** From (4.1.1.1),

$$\Pr(AB) = \Pr(A|B)\Pr(B) = \frac{p}{3} \quad (4.2.49.1)$$

which, upon substituting in (2.1.5.1) and simplifying results in

$$p + \frac{1}{3} - \frac{p}{3} = \frac{5}{9} \quad (4.2.49.2)$$

$$\implies p = \frac{1}{3}. \quad (4.2.49.3)$$

4.2.50 If  $A$  and  $B$  are two events such that  $\Pr(A) > 0$  and  $\Pr(A) + \Pr(B) > 1$ , then

$$\Pr(B|A) \geq 1 - \frac{\Pr(B')}{\Pr(A)} \quad (4.2.50.1)$$

**Solution:**

$$\Pr(B|A) = \frac{\Pr(AB)}{\Pr(A)} \quad (4.2.50.2)$$

$$= \frac{\Pr(A) + \Pr(B) - \Pr(A+B)}{\Pr(A)} \quad (4.2.50.3)$$

$$= \frac{\Pr(A) + 1 - \Pr(B') - \Pr(A+B)}{\Pr(A)} \quad (4.2.50.4)$$

$$= 1 - \frac{\Pr(B')}{\Pr(A)} + \frac{1 - \Pr(A+B)}{\Pr(A)} \quad (4.2.50.5)$$

From (2.1.4.1)

$$1 - \Pr(A+B) \geq 0 \quad (4.2.50.6)$$

Using this in (4.2.50.5) results in (4.2.50.1).

4.2.51 If

$$\Pr(B) = \frac{3}{5}, \Pr(A|B) = \frac{1}{2} \text{ and } \Pr(A+B) = \frac{4}{5}, \quad (4.2.51.1)$$

$$\text{then } \Pr(A+B') + \Pr(A'+B) = ? \quad (4.2.51.2)$$

**Solution:** From (4.1.1.1),

$$\Pr(AB) = \frac{3}{5} \times \frac{1}{2} = \frac{3}{10} \quad (4.2.51.3)$$

From (2.1.5.3),

$$\Pr(A'B) = \frac{3}{5} - \frac{3}{10} = \frac{3}{10} \quad (4.2.51.4)$$

From (2.1.5.1),

$$\Pr(A) = \frac{4}{5} + \frac{3}{10} - \frac{3}{5} = \frac{1}{2}. \quad (4.2.51.5)$$

Again, using (2.1.5.3),

$$\Pr(AB') = \frac{1}{2} - \frac{3}{10} = \frac{1}{5}. \quad (4.2.51.6)$$

Thus, using (2.1.3.1),

$$\Pr(A+B)' + \Pr(A'+B) = 1 - \frac{1}{5} + 1 - \frac{3}{10} = \frac{3}{2}. \quad (4.2.51.7)$$

4.2.52 If  $\Pr(A|B) > \Pr(A)$ , then which of the following is correct?

- a)  $\Pr(B|A) < \Pr(B)$
- b)  $\Pr(AB) < \Pr(A)\Pr(B)$
- c)  $\Pr(B|A) > \Pr(B)$
- d)  $\Pr(B|A) = \Pr(B)$

**Solution:**

$$\because \Pr(A|B) > \Pr(A), \frac{\Pr(AB)}{\Pr(B)} > \Pr(A) \quad (4.2.52.1)$$

$$\implies \Pr(AB) > \Pr(A)\Pr(B) \quad (4.2.52.2)$$

$$\text{or, } \frac{\Pr(AB)}{\Pr(A)} = \Pr(B|A) > \Pr(A) \quad (4.2.52.3)$$

4.2.53 Let  $A$  and  $B$  be independent events with  $\Pr(A) = 0.3$  and  $\Pr(B) = 0.4$ . Find

- a)  $\Pr(AB)$       b)  $\Pr(A + B)$       c)  $\Pr(A|B)$       d)  $\Pr(B|A)$

**Solution:**

a)

$$\Pr(AB) = 0.3 \times 0.4 = 0.12 \quad (4.2.53.1)$$

b)

$$\Pr(A + B) = 0.3 + 0.4 - 0.12 = 0.58 \quad (4.2.53.2)$$

c)

$$\Pr(A|B) = \Pr(A) = 0.3 \quad (4.2.53.3)$$

d)

$$\Pr(B|A) = \Pr(B) = 0.4 \quad (4.2.53.4)$$

4.2.54 Compute  $\Pr(A|B)$ , if  $\Pr(B) = 0.5$  and  $\Pr(AB) = 0.32$ .

**Solution:**

$$\Pr(A|B) = \frac{0.32}{0.5} = 0.64 \quad (4.2.54.1)$$

4.2.55 If  $A$  and  $B$  are two events such that  $P(A) \neq 0$  and  $P(B|A) = 1$ , then

- a)  $A \subset B$       b)  $B \subset A$       c)  $B = \emptyset$       d)  $A = \emptyset$

**Solution:**

$$\Pr(B|A) = 1 \implies \Pr(BA) = \Pr(A) \quad (4.2.55.1)$$

yielding

$$BA = A, \text{ or, } A \subset B \quad (4.2.55.2)$$

4.2.56 You are given that  $A$  and  $B$  are two events such that  $\Pr(B) = \frac{3}{5}$ ,  $\Pr(A|B) = \frac{1}{2}$  and  $\Pr(A + B) = \frac{4}{5}$ , then  $\Pr(A)$  equals \_\_\_\_\_.

**Solution:** From (4.1.1.1),

$$\Pr(AB) = \frac{3}{5} \times \frac{1}{2} = \frac{3}{10} \quad (4.2.56.1)$$

$$\implies \Pr(A) = \frac{4}{5} - \frac{3}{5} + \frac{3}{10} = \frac{1}{2} \quad (4.2.56.2)$$

from (2.1.5.1).

- 4.2.57 Three events  $A, B$  and  $C$  have probabilities  $\frac{2}{5}$ ,  $\frac{1}{3}$  and  $\frac{1}{2}$  respectively. Given that  $\Pr(AC) = \frac{1}{5}$  and  $\Pr(BC) = \frac{1}{4}$ , find the values of  $\Pr(C|B)$  and  $\Pr(A'C')$ .

**Solution:**

- a) From (4.1.1.1),

$$\Pr(C|B) = \frac{\frac{1}{4}}{\frac{1}{3}} = \frac{3}{4} \quad (4.2.57.1)$$

b)

$$\Pr(A'C') = 1 - \Pr(A + C) \quad (4.2.57.2)$$

$$= 1 - \left( \frac{2}{5} + \frac{1}{2} - \frac{1}{5} \right) = \frac{3}{10} \quad (4.2.57.3)$$

from (2.1.5.1).

- 4.2.58 If  $A$  and  $B$  are two events and  $A \neq \phi, B \neq \phi$ , then

a)  $\Pr(A|B) = \Pr(A) \cdot \Pr(B)$   
b)  $\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$

c)  $\Pr(A|B) \Pr(B|A) = 1$   
d)  $\Pr(A|B) = \frac{\Pr(A)}{\Pr(B)}$

- 4.2.59 You are given that  $A$  and  $B$  are two events such that  $\Pr(B) = \frac{3}{5}$ ,  $\Pr(A|B) = \frac{1}{2}$ ,  $\Pr(A + B) = \frac{4}{5}$  and  $\Pr(A) = \frac{1}{2}$ .  $\Pr(B|A')$  is equal to \_\_\_\_\_.

- 4.2.60 A fair die is rolled. Consider events  $E = \{1, 3, 5\}$ ,  $F = \{2, 3\}$  and  $G = \{2, 3, 4, 5\}$ . Find

- a)  $\Pr(E | F)$  and  $\Pr(F | E)$   
b)  $\Pr(E | G)$  and  $\Pr(G | E)$   
c)  $\Pr(E \cup F | G)$  and  $\Pr(E \cap F | G)$

**Solution:** See Table 4.2.60.1.

|                      |                          |
|----------------------|--------------------------|
| $E = \{1, 3, 5\}$    | $\Pr(E) = \frac{1}{2}$   |
| $F = \{2, 3\}$       | $\Pr(F) = \frac{1}{3}$   |
| $G = \{2, 3, 4, 5\}$ | $\Pr(G) = \frac{2}{3}$   |
| $EF = \{3\}$         | $\Pr(EF) = \frac{1}{6}$  |
| $FG = \{2, 3\}$      | $\Pr(FG) = \frac{1}{3}$  |
| $EG = \{3, 5\}$      | $\Pr(EG) = \frac{1}{3}$  |
| $EFG = \{3\}$        | $\Pr(EFG) = \frac{1}{6}$ |

TABLE 4.2.60.1

a)

$$\Pr(E|F) = \frac{\Pr(EF)}{\Pr(F)} = \frac{1/6}{1/3} = 1/2 \quad (4.2.60.1)$$

b)

$$\Pr(F|E) = \frac{\Pr(EF)}{\Pr(E)} = \frac{1/6}{1/2} = 1/3 \quad (4.2.60.2)$$

c)

$$\Pr(E|G) = \frac{\Pr(EG)}{\Pr(G)} = \frac{1/3}{2/3} = 1/2 \quad (4.2.60.3)$$

d)

$$\Pr(G|E) = \frac{\Pr(EG)}{\Pr(E)} = \frac{1/3}{1/2} = 2/3 \quad (4.2.60.4)$$

e)

$$\begin{aligned} \because \Pr((E+F)G) &= \Pr(EG + FG) = \Pr(EG) + \Pr(FG) - \Pr(EFG), \\ &= \frac{1}{3} + \frac{1}{3} - \frac{1}{6} = \frac{1}{2} \end{aligned} \quad (4.2.60.5)$$

$$\Pr((E+F)|G) = \frac{\Pr((E+F)G)}{\Pr(G)} = \frac{1/2}{2/3} = \frac{3}{4} \quad (4.2.60.6)$$

f)

$$\Pr(EF|G) = \frac{\Pr(EFG)}{\Pr(G)} = \frac{1/6}{2/3} = \frac{1}{4} \quad (4.2.60.7)$$

4.2.61 An electronic assembly consists of two subsystems, say  $A$  and  $B$ . From previous testing procedures, the following probabilities are assumed to be known

$$\Pr(A \text{ fails}) = 0.20 \quad (4.2.61.1)$$

$$\Pr(B \text{ alone fails}) = 0.15 \quad (4.2.61.2)$$

$$\Pr(A \text{ and } B \text{ fails}) = 0.15 \quad (4.2.61.3)$$

Evaluate the following probabilities

a)  $\Pr(A \text{ fails given } B \text{ has failed})$

b)  $\Pr(A \text{ fails alone})$

**Solution:** From the given information,

$$\Pr(A') = 0.20, \quad \Pr(AB') = 0.15, \quad \Pr(A'B') = 0.15 \quad (4.2.61.4)$$

a)

$$\Pr(A'|B') = \frac{\Pr(A'B')}{\Pr(B')} \quad (4.2.61.5)$$

From (2.1.5.3),

$$\Pr(B') = 0.15 + 0.15 = 0.30 \quad (4.2.61.6)$$

$$\Pr(A'|B') = \frac{0.15}{0.30} = 0.50 \quad (4.2.61.7)$$

b) Similarly, from (2.1.5.3),

$$\Pr(BA') = \Pr(A') - \Pr(A'B') = 0.20 - 0.15 = 0.05 \quad (4.2.61.8)$$

- 4.2.62 In a hostel, 60% of the students read Hindi newspaper, 40% read English newspaper and 20% read both Hindi and English newspapers. A student is selected at random.

- a) Find the probability that she reads neither Hindi nor English newspapers.
- b) If she reads Hindi newspaper, find the probability that she reads English newspaper.
- c) If she reads English newspaper, find the probability that she reads Hindi newspaper.

**Solution:** From the given information,

$$\Pr(A) = \frac{6}{10}, \quad \Pr(B) = \frac{4}{10}, \quad \Pr(AB) = \frac{2}{10} \quad (4.2.62.1)$$

a)

$$\Pr(A'B') = \Pr((A+B)') \quad (4.2.62.2)$$

$$= 1 - \Pr(A+B) \quad (4.2.62.3)$$

$$= 1 - (\Pr(A) + \Pr(B) - \Pr(AB)) \quad (4.2.62.4)$$

$$= 1 - \left( \frac{6}{10} + \frac{4}{10} - \frac{2}{10} \right) = \frac{2}{10} \quad (4.2.62.5)$$

b)

$$\Pr(B|A) = \frac{\Pr(AB)}{\Pr(A)} = \frac{\frac{2}{10}}{\frac{6}{10}} = \frac{1}{3} \quad (4.2.62.6)$$

c)

$$\Pr(A|B) = \frac{\Pr(AB)}{\Pr(B)} = \frac{\frac{2}{10}}{\frac{4}{10}} = \frac{1}{2} \quad (4.2.62.7)$$

- 4.2.63 Assume that the chances of a patient having a heart attack is 40%. It is also assumed that a meditation and yoga course reduce the risk of heart attack by 30% and prescription of certain drug reduces its chances by 25%. At a time a patient can choose any one of the two options with equal probabilities. It is given that after going through one of the two options the patient selected at random suffers a heart attack. Find the probability that the patient followed a course of meditation and yoga.

**Solution:** The given information is summarised in Table 4.2.63.2. The given

| Variable | Value | Description         |
|----------|-------|---------------------|
| X        | 0     | Heart attack        |
|          | 1     | No heart attack     |
| Y        | 0     | Drugs               |
|          | 1     | Meditation and Yoga |

TABLE 4.2.63.2

probabilities are

$$p_X(0) = 0.4, p_Y(0) = p_Y(1) = 0.5 \quad (4.2.63.1)$$

$$p_{X|Y}(0|1) = p_X(1 - 0.30) = 0.28 \quad (4.2.63.2)$$

$$p_{X|Y}(0|0) = p_X(1 - 0.25) = 0.30 \quad (4.2.63.3)$$

From (4.1.4.1),

$$p_{Y|X}(1|0) = \frac{p_{X|Y}(0|1)p_Y(1)}{\sum_{i=0}^2 p_{X|Y}(0|i)p_Y(i)} = \frac{\frac{1}{2} \times 0.28}{\frac{1}{2} \times 0.28 + \frac{1}{2} \times 0.30} = \frac{14}{29} \quad (4.2.63.4)$$

which is the desired probability.

- 4.2.64 Suppose that 6% of the people with blood group O are left handed and 10% of those with other blood groups are left handed. 30% of the people have blood group O. If a left handed person is selected at random, what is the probability that he/she will have blood group O?

**Solution:** Let A represent blood group O and B represent left handedness. From the given information,

$$\Pr(A) = 0.3, \Pr(B|A) = 0.06, \Pr(B|A') = 0.1. \quad (4.2.64.1)$$

Using (4.1.4.1),

$$\Pr(A|B) = \frac{\Pr(A)\Pr(B|A)}{\Pr(A)\Pr(B|A) + \Pr(A')\Pr(B|A')} = \frac{9}{44} \quad (4.2.64.2)$$

upon substituting numerical values.

- 4.2.65 At a fete, cards bearing numbers 1 to 1000, one number on a card, are put in a box. Each player selects one card at random and that card is not replaced. If the selected card has a perfect square greater than 500, the player wins a prize. What is the probability that

- the first player wins a prize
- the second player wins a prize, if the first has won?

**Solution:** If  $n^2$  is the value of the chosen number that is greater than 500 and also

a perfect square, then

$$n^2 \in (500, 1000] \quad (4.2.65.1)$$

$$\implies n \in (22.36, 31.62] \quad (4.2.65.2)$$

$n$  can take 9 integer values in the above interval. If  $A, B$  represent the first and second player winning a prize respectively,

a)

$$\Pr(A) = \frac{9}{1000} \quad (4.2.65.3)$$

b) Given that the first player has won, the second player has only 8 numbers left to choose. Hence,

$$\Pr(B|A) = \frac{8}{1000} \quad (4.2.65.4)$$

4.2.66 Four cards are successively drawn without replacement from a deck of 52 playing cards. What is the probability that all the four cards are kings?

**Solution:** Let  $X_i, i = 1, 2, 3, 4$  denote a king in the  $i$ th draw. Then,

$$\begin{aligned} \Pr(X_1) &= \frac{4}{52}, \quad \Pr(X_2|X_1) = \frac{3}{51}, \quad \Pr(X_3|X_2X_1) = \frac{2}{50}, \quad \Pr(X_4|X_1X_2X_3) = \frac{1}{49} \\ \implies \Pr(X_1X_2X_3X_4) &= \frac{4}{52} \times \frac{3}{51} \times \frac{2}{50} \times \frac{1}{49} = \frac{1}{270725} \end{aligned} \quad (4.2.66.1)$$

which is the desired probability.

4.2.67 Two natural numbers  $r, s$  are drawn one at a time, without replacement from the set  $S = 1, 2, 3, \dots, n$ . Find  $P[r \leq p | s \leq p]$ .

**Solution:** There are two conditions,

a)  $s$  is chosen first:

$$\Pr(r \leq p | s \leq p) = \frac{\Pr(r \leq p, s \leq p)}{\Pr(s \leq p)} \quad (4.2.67.1)$$

i)  $p < 1$ : This case is never possible as  $s, r \geq 1$

ii)  $1 \leq p \leq n$ : Then we can say that,

$$\Pr(r \leq p, s \leq p) = \frac{p(p-1)}{n(n-1)}, \quad (4.2.67.2)$$

$$\Pr(s \leq p) = \frac{p}{n} \quad (4.2.67.3)$$

From (4.2.67.2) and (4.2.67.3):

$$\Pr(r \leq p | s \leq p) = \frac{\Pr(r \leq p, s \leq p)}{\Pr(s \leq p)} \quad (4.2.67.4)$$

$$= \frac{\frac{p(p-1)}{n(n-1)}}{\frac{p}{n}} = \frac{p-1}{n-1} \quad (4.2.67.5)$$

iii)  $p > n$ :

$$\Pr(r \leq p, s \leq p) = 1, \quad (4.2.67.6)$$

$$\Pr(s \leq p) = 1 \quad (4.2.67.7)$$

From (4.2.67.6) and (4.2.67.7):

$$\Pr(r \leq p | s \leq p) = \frac{\Pr(r \leq p, s \leq p)}{\Pr(s \leq p)} \quad (4.2.67.8)$$

$$= 1 \quad (4.2.67.9)$$

b)  $r$  is chosen first:

$$\Pr(r \leq p | s \leq p) = \frac{\Pr(r \leq p, s \leq p)}{\Pr(s \leq p)} \quad (4.2.67.10)$$

i)  $p < 1$ : This case is never possible as  $r, s \geq 1$

ii)  $1 \leq p \leq n$ :

$$\Pr(r \leq p, s \leq p) = \frac{p(p-1)}{n(n-1)}, \quad (4.2.67.11)$$

$$\Pr(s \leq p) = \frac{p-1}{n-1} \quad (4.2.67.12)$$

From (4.2.67.11) and (4.2.67.12):

$$\Pr(r \leq p | s \leq p) = \frac{\Pr(r \leq p, s \leq p)}{\Pr(s \leq p)} \quad (4.2.67.13)$$

$$= \frac{\frac{p(p-1)}{n(n-1)}}{\frac{p}{n}} = \frac{p}{n} \quad (4.2.67.14)$$

iii)  $p > n$ :

$$\Pr(r \leq p, s \leq p) = 1, \quad (4.2.67.15)$$

$$\Pr(s \leq p) = 1 \quad (4.2.67.16)$$

From (4.2.67.15) and (4.2.67.16):

$$\Pr(r \leq p | s \leq p) = \frac{\Pr(r \leq p, s \leq p)}{\Pr(s \leq p)} \quad (4.2.67.17)$$

$$= 1 \quad (4.2.67.18)$$

4.2.68 Three bags contain a number of red and white balls as follows:  $B_1$ : 3 red balls,  $B_2$ : 2 red balls and 1 white ball,  $B_3$ : 3 white balls. The probability that bag  $i$  will be chosen and a ball is selected is  $i/6, i = 1, 2, 3$ . what is the probability that

a) a red ball will be selected?

b) a white ball will be selected?

**Solution:** The r.vs are listed in Table 4.2.68.2. From the given information,

| RV | Value | Description   |
|----|-------|---------------|
| X  | 1     | Bag selection |
|    | 2     |               |
|    | 3     |               |
| Y  | 0     | white ball    |
|    | 1     | red ball      |

TABLE 4.2.68.2: Random variable description

$$p_X(i) = \begin{cases} \frac{1}{6} & i = 1 \\ \frac{2}{6} & i = 2 \\ \frac{3}{6} & i = 3 \end{cases}, \quad p_{Y|X}(1|i) = \begin{cases} 1 & i = 1 \\ \frac{2}{3} & i = 2 \\ 0 & i = 3 \end{cases}, \quad p_{Y|X}(0|i) = \begin{cases} 0 & i = 1 \\ \frac{1}{3} & i = 2 \\ 1 & i = 3 \end{cases} \quad (4.2.68.1)$$

a) The probability that a red ball will be selected is

$$p_Y(1) = \sum_{i=1}^3 p_{Y|X}(1|i) p_X(i) \quad (4.2.68.2)$$

$$= \frac{1}{6} \times \frac{3}{3} + \frac{2}{6} \times \frac{2}{3} + \frac{3}{6} \times 0 = \frac{7}{18} \quad (4.2.68.3)$$

from (4.2.68.1).

b) The probability that a white ball will be selected is

$$p_Y(0) = \sum_{i=1}^3 p_{Y|X}(0|i) p_X(i) \quad (4.2.68.4)$$

$$= \frac{1}{6} \times 0 + \frac{2}{6} \times \frac{1}{3} + \frac{3}{6} \times \frac{3}{3} = \frac{11}{18} \quad (4.2.68.5)$$

from (4.2.68.1).

4.2.69 Refer to Problem 4.2.68. If a white ball is selected, what is the probability that it came from

- a)  $B_2$
- b)  $B_3$

### Solution:

a) The desired probability is

$$p_{X|Y}(2|0) = \frac{p_{Y|X}(0|2) p_X(2)}{p_Y(0)} \quad (4.2.69.1)$$

$$= \frac{\frac{1}{3} \times \frac{1}{3}}{\frac{11}{18}} = \frac{2}{11} \quad (4.2.69.2)$$

b) Similarly,

$$p_{X|Y}(3|0) = \frac{p_{Y|X}(0|3)p_X(3)}{p_Y(0)} \quad (4.2.69.3)$$

$$= \frac{\frac{1}{2}}{\frac{11}{18}} = \frac{9}{11} \quad (4.2.69.4)$$

4.2.70 If  $P(A) = \frac{4}{5}$  and  $P(AB) = \frac{7}{10}$ , then  $P(B|A)$  is equal to

**Solution:** From (4.1.1.1), the required probability is

$$\Pr(B|A) = \frac{\left(\frac{7}{10}\right)}{\left(\frac{4}{5}\right)} = \frac{7}{8} \quad (4.2.70.1)$$

4.2.71 A flashlight has 8 batteries out of which 3 are dead. If two batteries are selected without replacement and tested, find the probability that both are dead.

**Solution:** Let  $X_i \in \{0, 1\}, i \in 1, 2$  represent the  $i$ th battery, 0 denoting the battery being dead. From the given information,

$$p_{X_1}(0) = \frac{3}{8}, p_{X_2|X_1}(0|0) = \frac{2}{7}, \quad (4.2.71.1)$$

$$\implies p_{X_1, X_2}(0, 0) = p_{X_1}(0)p_{X_2|X_1}(0|0) = \frac{3}{28} \quad (4.2.71.2)$$

from (4.1.1.1).

4.2.72 In a college, 30% students fail in physics, 25% fail in mathematics and 10% fail in both. One student is chosen at random. The probability that she fails in physics if she has failed in mathematics is

- a)  $\frac{1}{10}$       b)  $\frac{2}{5}$       c)  $\frac{9}{20}$       d)  $\frac{1}{3}$

**Solution:** From the given information,

$$\Pr(P) = 0.3, \Pr(M) = 0.1, \Pr(PM) = 0.25 \quad (4.2.72.1)$$

$$\implies \Pr(P|M) = \frac{\Pr(PM)}{\Pr(M)} = \frac{0.1}{0.25} = \frac{2}{5} \quad (4.2.72.2)$$

## 5 UNIFORM DISTRIBUTION

### 5.1 Formulae

- 5.1.1. Let  $X \in \{1, 2, 3, 4, 5, 6\}$  be the random variables representing the outcome for a die. Assuming the die to be fair, the probability mass function (pmf) is expressed as

$$p_X(n) = \begin{cases} \frac{1}{6} & 1 \leq n \leq 6 \\ 0 & \text{otherwise} \end{cases} \quad (5.1.1.1)$$

- 5.1.2. The CDF of  $X$  is given by

$$F_X(n) = \Pr(X \leq n) = \sum_{k=1}^n p_X(k) = \begin{cases} 0 & n < 1 \\ \frac{n}{6} & 1 \leq n \leq 6 \\ 1 & \text{otherwise} \end{cases} \quad (5.1.2.1)$$

and plotted in Fig. 5.1.2.1.

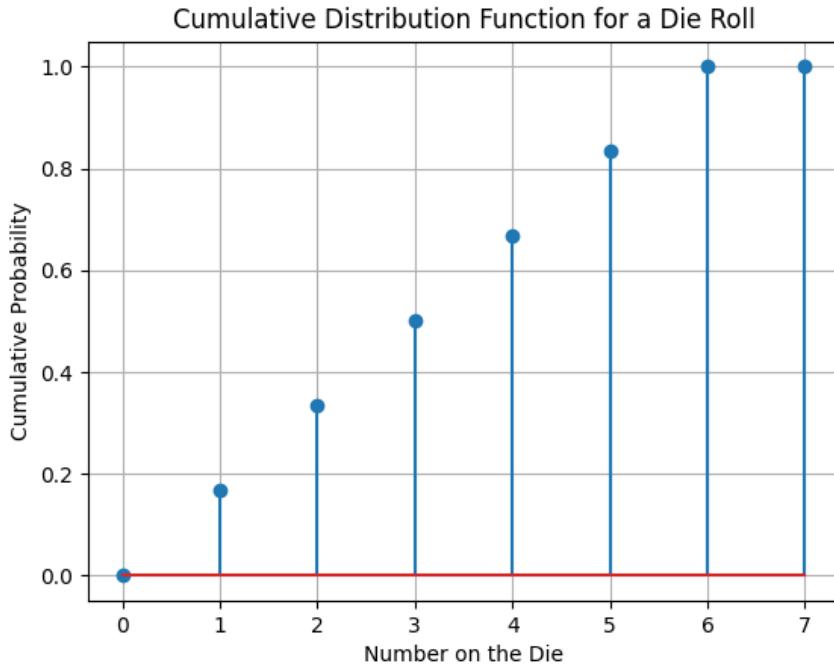


Fig. 5.1.2.1: CDF

### 5.2 NCERT

- 5.1 A die is thrown, find the probability of following events:
- A prime number will appear

- b) A number greater than or equal to 3 will appear
- c) A number less than or equal to one will appear
- d) A number more than 6 will appear
- e) A number less than 6 will appear

**Solution:** The CDF of the random variable  $X$  representing the roll of a dice, is available in (5.1.2.1).

- a) The set of possible prime numbers in a die roll contains 2,3,5

$$\Pr(X \in \{2, 3, 5\}) = p_X(2) + p_X(3) + p_X(5) \quad (5.1.1)$$

$$= \frac{1}{2} \quad (5.1.2)$$

- b) The probability that a number greater than or equal to 3 will appear is given by

$$\Pr(X \geq 3) = 1 - \Pr(X \leq 2) \quad (5.1.3)$$

$$= 1 - F_X(2) \quad (5.1.4)$$

$$= \frac{2}{3} \quad (5.1.5)$$

- c) The probability that a number less than or equal to 1 will appear is given by

$$\Pr(X \leq 1) = F_X(1) \quad (5.1.6)$$

$$= \frac{1}{6} \quad (5.1.7)$$

- d) The probability that a number greater than 6 will appear is given by

$$\Pr(X > 6) = 1 - \Pr(X \leq 6) \quad (5.1.8)$$

$$= 1 - F_X(6) \quad (5.1.9)$$

$$= 0 \quad (5.1.10)$$

- e) The probability that a number less than 6 will appear is given by

$$\Pr(X < 6) = \Pr(X \leq 5) \quad (5.1.11)$$

$$= F_X(5) \quad (5.1.12)$$

$$= \frac{5}{6} \quad (5.1.13)$$

- 5.2 All the jacks, queens and kings are removed from a deck of 52 playing cards. The remaining cards are well shuffled and then one card is drawn at random. Giving ace a value 1 similar value for other cards, find the probability that the card has a value
- a) 7
  - b) greater than 7
  - c) less than 7

**Solution:** Number of cards left after removing all jacks, queens and kings

$$N = 52 - 4 \times 3 = 40 \quad (5.2.1)$$

Let  $1 \leq X \leq 10$  be the value of the card picked. Then,

$$p_X(k) = \Pr(X = k) \quad \forall 1 \leq k \leq 10 \quad (5.2.2)$$

$$= \frac{4 \times 1}{40} \quad (5.2.3)$$

$$= \frac{1}{10} \quad (5.2.4)$$

$$\therefore p_X(k) = \begin{cases} \frac{1}{10} & 1 \leq k \leq 10 \\ 0 & \text{otherwise} \end{cases} \quad (5.2.5)$$

and

$$F_X(k) = \sum_{m=0}^k p_X(m) \quad 1 \leq k \leq 10 \quad (5.2.6)$$

$$= \frac{k}{10} \quad (5.2.7)$$

$$\therefore F_X(k) = \begin{cases} 0 & k \leq 0 \\ \frac{k}{10} & 1 \leq k \leq 10 \\ 1 & k > 10 \end{cases} \quad (5.2.8)$$

a) Probability that card has value equal to 7 is

$$p_X(7) = \frac{1}{10} \quad (5.2.9)$$

b) Probability that card has value greater than 7 is

$$1 - F_X(7) = 1 - \frac{7}{10} \quad (5.2.10)$$

$$= \frac{3}{10} \quad (5.2.11)$$

c) Probability that card has value less than 7 is

$$F_X(6) = \frac{6}{10} \quad (5.2.12)$$

5.3 A game of chance consists of spinning an arrow which comes to rest pointing at one of the numbers 1, 2, 3, 4, 5, 6, 7, 8 (see Fig. 5.3.1), and these are equally likely outcomes. What is the probability that it will point at

- a) 8?
- b) an odd number?
- c) a number greater than 2?
- d) a number less than 9?



Fig. 5.3.1: Spinner

**Solution:** Let  $X$  be a random variable defined as the value given by the pointer. Then,

$$\Pr(X = i) = \frac{1}{8} \quad 1 \leq i \leq 8 \quad (5.3.1)$$

$$F_X(i) = \Pr(X \leq i) \quad (5.3.2)$$

$$= \begin{cases} 0, & i \leq 0 \\ \frac{i}{8}, & 1 \leq i \leq 8 \\ 1, & i \geq 9 \end{cases} \quad (5.3.3)$$

which are plotted in Fig. 5.3.2 and Fig. 5.3.3 respectively.



Fig. 5.3.2: Plot of Probability Mass Function

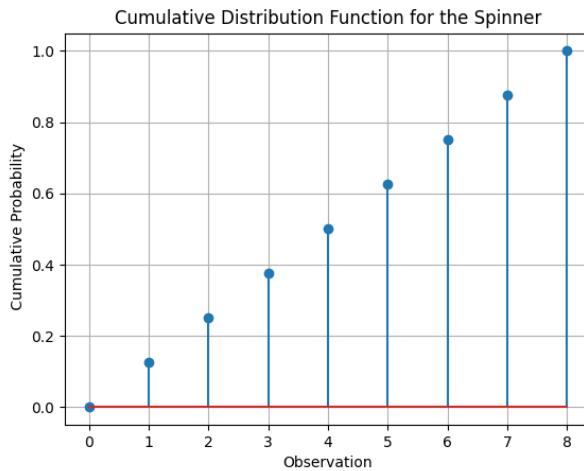


Fig. 5.3.3: Plot of Cumulative Distribution Function

a)

$$\Pr(X = 8) = \frac{1}{8} = 0.125 \quad (5.3.4)$$

b) For  $i$  being odd,

$$\Pr(X = \{1, 3, 5, 7\}) = \frac{4}{8} = 0.5 \quad (5.3.5)$$

c)

$$\Pr(X > 2) = 1 - \Pr(X \leq 2) \quad (5.3.6)$$

$$= 1 - (F_X(2) - F_X(0)) \quad (5.3.7)$$

$$= \frac{6}{8} \quad (5.3.8)$$

d)

$$\Pr(1 \leq X < 9) = F_X(8) - F_X(0) = 1 \quad (5.3.9)$$

## 6 SUM OF RANDOM VARIABLES

### 6.1 Formulae

6.1.1 Consider the rv

$$X = X_1 + X_2, \quad (6.1.1.1)$$

where  $X_1$  and  $X_2$  are independent uniform rvs with pmf given in (5.1.1.1).

6.1.2 Convolution: From (6.1.1.1),

$$p_X(n) = \Pr(X_1 + X_2 = n) = \Pr(X_1 = n - X_2) \quad (6.1.2.1)$$

$$= \sum_k \Pr(X_1 = n - k | X_2 = k) p_{X_2}(k) \quad (6.1.2.2)$$

after unconditioning.  $\because X_1$  and  $X_2$  are independent,

$$\Pr(X_1 = n - k | X_2 = k) = \Pr(X_1 = n - k) = p_{X_1}(n - k) \quad (6.1.2.3)$$

From (6.1.2.2) and (6.1.2.3),

$$p_X(n) = \sum_k p_{X_1}(n - k) p_{X_2}(k) = p_{X_1}(n) * p_{X_2}(n) \quad (6.1.2.4)$$

where  $*$  denotes the convolution operation.

6.1.3 (Triangular PMF:) Substituting from (5.1.1.1) in (6.1.2.4),

$$p_X(n) = \frac{1}{6} \sum_{k=1}^6 p_{X_1}(n - k) = \frac{1}{6} \sum_{k=n-6}^{n-1} p_{X_1}(k) \quad (6.1.3.1)$$

$$\therefore p_{X_1}(k) = 0, \quad k \leq 1, k \geq 6. \quad (6.1.3.2)$$

From (6.1.3.1),

$$p_X(n) = \begin{cases} 0 & n < 1 \\ \frac{1}{6} \sum_{k=1}^{n-1} p_{X_1}(k) & 1 \leq n - 1 \leq 6 \\ \frac{1}{6} \sum_{k=n-6}^6 p_{X_1}(k) & 1 < n - 6 \leq 6 \\ 0 & n > 12 \end{cases} \quad (6.1.3.3)$$

Substituting from (5.1.1.1) in (6.1.3.3),

$$p_X(n) = \begin{cases} 0 & n < 1 \\ \frac{n-1}{36} & 2 \leq n \leq 7 \\ \frac{13-n}{36} & 7 < n \leq 12 \\ 0 & n > 12 \end{cases} \quad (6.1.3.4)$$

6.1.4 The experiment of rolling the dice was simulated using Python for 10000 samples. These were generated using Python libraries for uniform distribution. The frequencies for each outcome were then used to compute the resulting pmf, which is plotted in Figure 6.1.4.1. The theoretical pmf obtained in (6.1.3.4) is plotted for comparison.

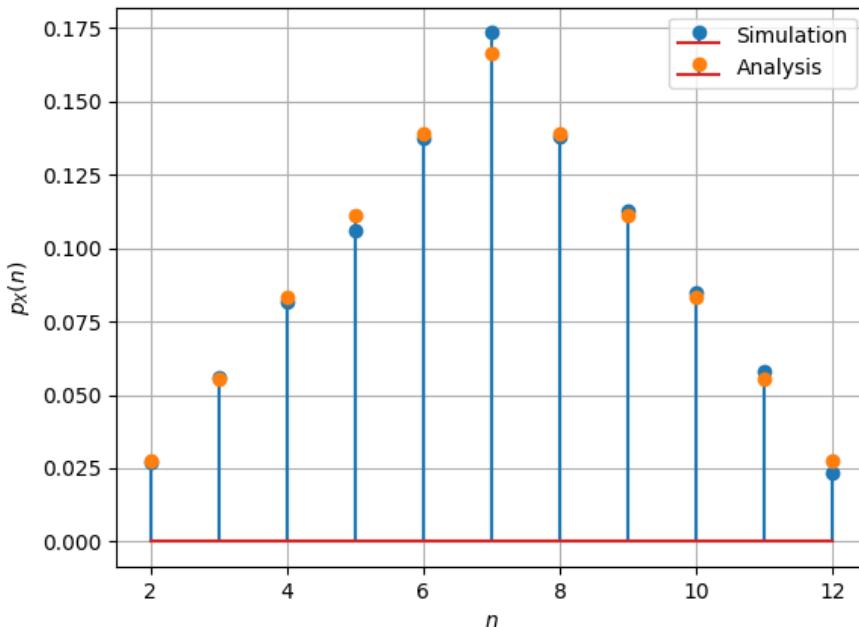


Fig. 6.1.4.1: Plot of  $p_X(n)$ . Simulations are close to the analysis.

6.1.5 The python code is available below

```

import numpy as np
import matplotlib.pyplot as plt
#If using termux
import subprocess
import shlex
#end if

#Sample size
simlen = 10000
#Possible outcomes
n = range(2,13)
# Generate X1 and X2
y = np.random.randint(1,7, size=(2, simlen))

#Generate X
X = np.sum(y, axis = 0)
#Find the frequency of each outcome
unique, counts = np.unique(X, return_counts=True)

```

```

#Simulated probability
psim = counts/simlen
#Theoretical probability
n1 = range(2,8)
n2 = range(8,13)
panal1 = (n1 - np.ones((1,6)))
panal2 = (13*np.ones((1,5))-n2)
panal = np.concatenate((panal1,panal2),axis=None)/36

#Plotting
plt.stem(n,psim, markerfmt='o', use_line_collection=True, label='Simulation')
plt.stem(n,panal, markerfmt='o',use_line_collection=True, label='Analysis')
plt.xlabel('n$')
plt.ylabel('$p_{\{X\}(n)$')
plt.legend()
plt.grid()# minor

#If using termux
plt.savefig('figs/pmf.pdf')
plt.savefig('figs/pmf.png')
subprocess.run(shlex.split("termux-open figs/pmf.pdf"))
#else
#plt.show()

```

### 6.1.6 The Z-transform of $X$ is defined as

$$M_X(z) = E[z^{-X}] = \sum_{k=-\infty}^{\infty} p_X(k)z^{-k} \quad (6.1.6.1)$$

### 6.1.7 If $X_1$ and $X_2$ are independent, the Z-transform of

$$X = X_1 + X_2 \quad (6.1.7.1)$$

is given by

$$M_X(z) = M_{X_1}(z)M_{X_2}(z) \quad (6.1.7.2)$$

The above property follows from Fourier analysis and is fundamental to signal processing.

### 6.1.8 For (5.1.1.1), the Z-transform of $X_1$ is given by

$$M_{X_1}(z) = \frac{1}{6} \sum_{n=1}^6 z^{-n} = \frac{z^{-1} (1 - z^{-6})}{6 (1 - z^{-1})}, \quad |z| > 1 \quad (6.1.8.1)$$

upon summing up the geometric progression.

### 6.1.9 From (6.1.8.1) and (6.1.7.2),

$$M_X(z) = \left\{ \frac{z^{-1} (1 - z^{-6})}{6(1 - z^{-1})} \right\}^2 \quad (6.1.9.1)$$

$$= \frac{1}{36} \frac{z^{-2} (1 - 2z^{-6} + z^{-12})}{(1 - z^{-1})^2} \quad (6.1.9.2)$$

Using the fact that

$$\begin{aligned} p_X(n-k) &\xrightarrow{\mathcal{Z}} P_X(z)z^{-k}, \\ nu(n) &\xrightarrow{\mathcal{Z}} \frac{z^{-1}}{(1-z^{-1})^2} \end{aligned} \quad (6.1.9.3)$$

after some algebra, it can be shown that

$$\begin{aligned} \frac{1}{36} [(n-1)u(n-1) - 2(n-7)u(n-7) + (n-13)u(n-13)] \\ \xrightarrow{\mathcal{Z}} \frac{1}{36} \frac{z^{-2} (1 - 2z^{-6} + z^{-12})}{(1 - z^{-1})^2} \end{aligned} \quad (6.1.9.4)$$

where

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} \quad (6.1.9.5)$$

From (6.1.6.1), (6.1.9.2) and (6.1.9.4)

$$p_X(n) = \frac{1}{36} [(n-1)u(n-1) - 2(n-7)u(n-7) + (n-13)u(n-13)] \quad (6.1.9.6)$$

which is the same as (6.1.3.4). Note that (6.1.3.4) can be obtained from (6.1.9.4) using contour integration as well.

## 6.2 NCERT

- 6.1 Two dice, one blue and one grey, are thrown at the same time. The event defined by the sum of the two numbers appearing on the top of the dice can have 11 possible outcomes 2, 3, 4, 5, 6, 6, 8, 9, 10, 11 and 12. A student argues that each of these outcomes has a probability  $\frac{1}{11}$ . Do you agree with this argument? Justify your answer.
- Solution:** From (6.1.3.4) and Fig. 6.1.4.1, it is obvious that

$$p_X(n) \neq \frac{1}{11}. \quad (6.1.1)$$

- 6.2 Two dice are numbered 1,2,3,4,5,6 and 1,1,2,2,3,3 respectively. They are thrown and the sum of then numbers on them is noted. Find the probability of getting each sum from 2 to 9 seperately

**Solution:** The Z-transform of the first die  $X_1$  is given by (6.1.8.1). The pmf of the

second die is

$$p_{X_2}(n) = \begin{cases} \frac{1}{3} & 1 \leq n \leq 3 \\ 0 & \text{otherwise} \end{cases} \quad (6.2.1)$$

yielding

$$M_{X_2}(z) = \frac{1}{3} \sum_{n=1}^3 z^{-n} = \frac{z^{-1}(1-z^{-3})}{3(1-z^{-1})}, |z| > 1 \quad (6.2.2)$$

upon substituting in (6.1.6.1). From (6.1.7.2), The Z-transform of X is given as

$$M_X(z) = \frac{z^{-1}(1-z^{-6})}{6(1-z^{-1})} \times \frac{z^{-1}(1-z^{-3})}{3(1-z^{-1})} \quad (6.2.3)$$

$$= \frac{1}{18} \left[ \frac{z^{-2}((1-z^{-3}-z^{-6}-z^{-9}))}{(1-z^{-1})^2} \right] \quad (6.2.4)$$

Using (6.1.9.3), after some algebra, it can be shown that,

$$\begin{aligned} \frac{1}{18}[n-1u(n-1)-n-4u(n-4)-(n-7)u(n-7)-(n-10)u(n-10)] \\ \xleftrightarrow{Z} \\ \frac{1}{18} \left[ \frac{z^{-2}1-z^{-3}-z^{-6}-z^{-9}}{(1-z^{-1})^2} \right] \end{aligned} \quad (6.2.5)$$

Hence,

$$p_X(n) = \begin{cases} 0 & n \leq 1 \\ \frac{n-1}{18} & 2 \leq n \leq 4 \\ \frac{1}{6} & 5 \leq n \leq 7 \\ \frac{10-n}{18} & 8 \leq n \leq 9 \\ 0 & n \geq 10 \end{cases} \quad (6.2.6)$$

See Fig. 6.2.1. The experiment of rolling the dice was simulated using Python for 10000 samples.

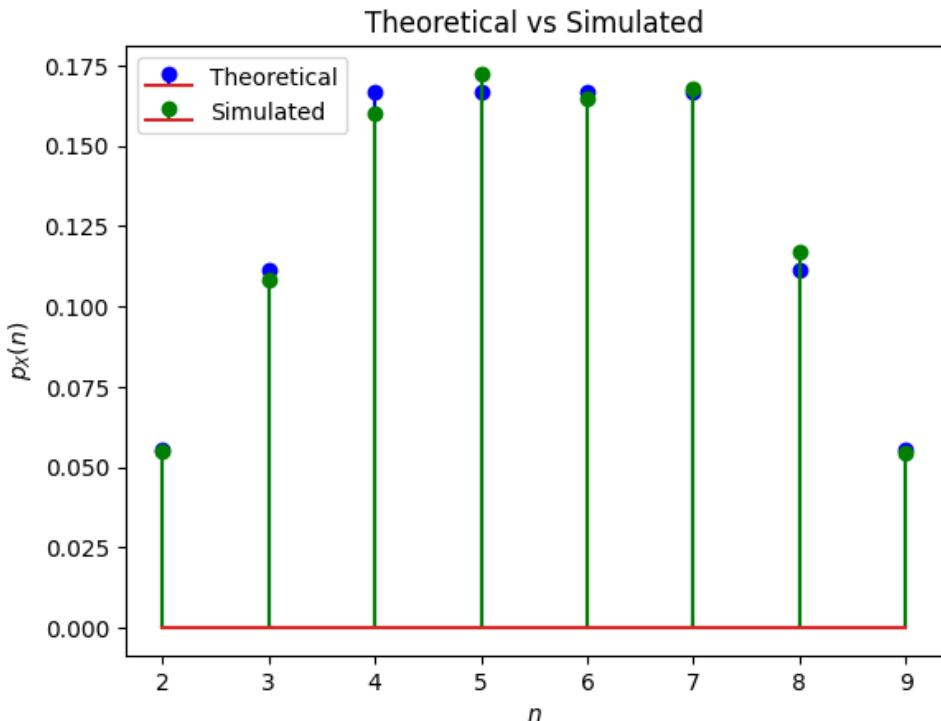


Fig. 6.2.1: Plot of  $p_X(n)$ . Simulations are close to the analysis.

## 7 BINOMIAL

### 7.1 Formulae

7.1.1. The Binomial distribution is defined as

$$X = X_1 + X_2 + \cdots + X_n, \quad (7.1.1.1)$$

Where  $X_i$  are i.i.d bernoulli.

7.1.2. For  $X_i \sim \text{Ber}(p)$ ,

$$M_{X_i}(z) = q + pz^{-1}, \quad q = 1 - p \quad (7.1.2.1)$$

upon substituting from (3.1.1.2) in (6.1.6.1).

7.1.3. For  $X \sim \text{Binom}(n, p)$ ,

$$M_X(z) = (q + pz^{-1})^n \quad (7.1.3.1)$$

using (7.1.1.1) and (6.1.7.2).

7.1.4. Expanding (7.1.3.1),

$$M_X(z) = \sum_{k=0}^n (1-p)^{n-k} p^k z^{-k} \quad (7.1.4.1)$$

$$\implies p_X(k) = \begin{cases} (1-p)^{n-k} p^k & 0 \leq k \leq n \\ 0 & \text{otherwise} \end{cases} \quad (7.1.4.2)$$

upon comparing with (6.1.6.1).

7.1.5. In a hurdle race, a player has to cross 10 hurdles. The probability that he will clear each hurdle is  $\frac{5}{6}$ . What is the probability that he will knock down fewer than 2 hurdles?

**Solution:** See the following code

```
#Code by GVV Sharma
#November 20,2020
#Released under GNU/GPL
#To find the probability of an event using the binomial distribution

import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import bernoulli
from scipy.stats import norm
from scipy.stats import binom

#Simlen
simlen=1000
```

```

#Number of hurdles
n = 10

#Probability of clearing a hurdle
p = 1-5/6

#Mean
mu = p

#Variance
sigma = np.sqrt(p*(1-p))

#Theoretical probability of knocking down fewer than 2 hurdles
k = 1
print(binom.cdf(k, n, p),3*(5/6)**10)

#Using the Gaussian approximation for the binomial pdf
print(1/(sigma*np.sqrt(n))*(norm.pdf((k-n*mu)/(sigma*np.sqrt(n)))+norm.pdf((k
-1-n*mu)/(sigma*np.sqrt(n)))))

#Simulating the probability using the binomial random variable
data_binom = binom.rvs(n,p,size=simlen) #Simulating the event of jumping 10
hurdles
err_ind = np.nonzero(data_binom <=k) #checking probability condition
err_n = np.size(err_ind) #computing the probability
print(err_n/simlen)
#print(data_binom)

#Simulating the probability using the bernoulli random variable
data_bern_mat = bernoulli.rvs(p,size=(n,simlen))
data_binom=np.sum(data_bern_mat, axis=0)
#print(data_bern_mat)
#print(data_binom)
err_ind = np.nonzero(data_binom <=k) #checking probability condition
err_n = np.size(err_ind) #computing the probability
print(err_n/simlen)

```

## 7.2 NCERT

### 8 GAUSSIAN

#### 8.1 NCERT

## 9 RANDOM VARIABLES

### 9.1 NCERT

9.1.1 Assume that in a family, each child is equally likely to be a boy or a girl. A family with three children is chosen at random. The probability that the eldest child is a girl given that the family has at least one girl is

- a)  $\frac{1}{2}$
- b)  $\frac{1}{3}$
- c)  $\frac{3}{4}$
- d)  $\frac{4}{7}$

**Solution:**

Let  $X_0, X_1, X_2$  be the random variables which denotes the three children, where  $X_0$  is the eldest child and  $X_2$  is the youngest child.

| RV    | Value | Description   |
|-------|-------|---------------|
| $X_i$ | 0     | child is boy  |
|       | 1     | child is girl |

TABLE 9.1.1.1: RV description table

so the required probability is,

$$\Pr(X_0 = 1 | X_0 + X_1 + X_2 \geq 1) = \frac{\Pr(X_0 = 1, X_0 + X_1 + X_2 \geq 1)}{\Pr(X_0 + X_1 + X_2 \geq 1)} \quad (9.1.1.1)$$

$$= \frac{\Pr(X_0 = 1) \times \Pr(X_1 + X_2 \geq 0)}{\Pr(X_0 + X_1 + X_2 \geq 1)} \quad (9.1.1.2)$$

$$= \frac{\frac{1}{2} \times \sum_{k=0}^2 {}^2C_k \times \left(\frac{1}{2}\right)^k \times \left(\frac{1}{2}\right)^{2-k}}{\sum_{k=1}^3 {}^3C_k \times \left(\frac{1}{2}\right)^k \times \left(\frac{1}{2}\right)^{3-k}} \quad (9.1.1.3)$$

$$= \frac{\frac{1}{2} \times 1}{\frac{3}{8} + \frac{3}{8} + \frac{1}{8}} \quad (9.1.1.4)$$

$$= \frac{4}{7} \quad (9.1.1.5)$$

Therefore, the probability that the eldest child is a girl given that the family has atleast one girl is  $\frac{4}{7}$

9.1.2 State whether the statement is True or False. The probabilities that a typist will make 0, 1, 2, 3, 4, 5 or more mistakes in typing a report are, respectively, 0.12, 0.25, 0.36, 0.14, 0.08, 0.11.

**Solution:** From the given information, we obtain the distribution

$$p_X(k) = \begin{cases} 0.12 & k = 0 \\ 0.25 & k = 1 \\ 0.36 & k = 2 \\ 0.14 & k = 3 \\ 0.08 & k = 4 \\ 0.11 & k \geq 5 \end{cases} \quad (9.1.2.1)$$

Since

$$\sum_{i=0}^5 p_X(k) = 1.06 > 1 \quad (9.1.2.2)$$

violates (2.1.4.1), the given statement is false.

- 9.1.3 State which of the following are not the probability distributions of a random variable. Give reasons for your answer.

|      |   |     |     |   |   |      |     |     |     |
|------|---|-----|-----|---|---|------|-----|-----|-----|
| a)   | <table border="1" style="display: inline-table; vertical-align: middle;"> <tr> <td>X</td><td>0</td><td>1</td><td>2</td></tr> <tr> <td>P(X)</td><td>0.4</td><td>0.4</td><td>0.2</td></tr> </table> | X   | 0   | 1 | 2 | P(X) | 0.4 | 0.4 | 0.2 |
| X    | 0   | 1   | 2   |   |   |      |     |     |     |
| P(X) | 0.4   | 0.4 | 0.2 |   |   |      |     |     |     |

|      |  |     |     |   |   |      |     |     |     |
|------|--|-----|-----|---|---|------|-----|-----|-----|
| c)   | <table border="1" style="display: inline-table; vertical-align: middle;"> <tr> <td>Y</td><td>-1</td><td>0</td><td>1</td></tr> <tr> <td>P(Y)</td><td>0.6</td><td>0.1</td><td>0.2</td></tr> </table> | Y   | -1  | 0 | 1 | P(Y) | 0.6 | 0.1 | 0.2 |
| Y    | -1   | 0   | 1   |   |   |      |     |     |     |
| P(Y) | 0.6  | 0.1 | 0.2 |   |   |      |     |     |     |

|      |  |     |     |      |     |   |   |      |     |     |     |      |     |
|------|--|-----|-----|------|-----|---|---|------|-----|-----|-----|------|-----|
| b)   | <table border="1" style="display: inline-table; vertical-align: middle;"> <tr> <td>X</td><td>0</td><td>1</td><td>2</td><td>3</td><td>4</td></tr> <tr> <td>P(X)</td><td>0.1</td><td>0.5</td><td>0.2</td><td>-0.1</td><td>0.3</td></tr> </table> | X   | 0   | 1    | 2   | 3 | 4 | P(X) | 0.1 | 0.5 | 0.2 | -0.1 | 0.3 |
| X    | 0  | 1   | 2   | 3    | 4   |   |   |      |     |     |     |      |     |
| P(X) | 0.1  | 0.5 | 0.2 | -0.1 | 0.3 |   |   |      |     |     |     |      |     |

|      |  |     |     |     |      |   |   |      |     |     |     |     |      |
|------|--|-----|-----|-----|------|---|---|------|-----|-----|-----|-----|------|
| d)   | <table border="1" style="display: inline-table; vertical-align: middle;"> <tr> <td>X</td><td>0</td><td>1</td><td>2</td><td>3</td><td>4</td></tr> <tr> <td>P(Z)</td><td>0.3</td><td>0.2</td><td>0.4</td><td>0.1</td><td>0.05</td></tr> </table> | X   | 0   | 1   | 2    | 3 | 4 | P(Z) | 0.3 | 0.2 | 0.4 | 0.1 | 0.05 |
| X    | 0  | 1   | 2   | 3   | 4    |   |   |      |     |     |     |     |      |
| P(Z) | 0.3  | 0.2 | 0.4 | 0.1 | 0.05 |   |   |      |     |     |     |     |      |

**Solution:**

- a) The given distribution satisfies (2.1.4.1) and (2.1.8.4), so it is a valid probability distribution.  
 b)

$$p_X(3) = -0.1 < 0 \quad (9.1.3.1)$$

which violates (2.1.4.1). Hence, not a probability distribution.  
 c)

$$\sum_{k=-1}^1 p_X(k) = 0.9 < 1 \quad (9.1.3.2)$$

which violates (2.1.8.4). So, not a probability distribution.  
 d)

$$\sum_{k=0}^4 p_X(k) = 1.05 > 1 \quad (9.1.3.3)$$

which violates (2.1.8.4). So, not a probability distribution.  
 9.1.4 A die has two faces each with number '1', three faces each with number '2' and one face with number '3'. If die is rolled once, determine  
 a)  $\Pr(2)$   
 b)  $\Pr(1 \text{ or } 3)$

c)  $\Pr(\text{not } 3)$

**Solution:** The given information is summarized in the following table 9.1.4.1

| RV      | Description    | Probability   |
|---------|----------------|---------------|
| $X = 1$ | Die rolls to 1 | $\frac{1}{3}$ |
| $X = 2$ | Die rolls to 2 | $\frac{1}{2}$ |
| $X = 3$ | Die rolls to 3 | $\frac{1}{6}$ |

TABLE 9.1.4.1: Random variable X

a)

$$\Pr(X = 2) = \frac{1}{2} \quad (9.1.4.1)$$

b) Since

$$X = 1 \text{ or } X = 3 \equiv X \in \{1, 3\} \quad (9.1.4.2)$$

$$X = 1 \text{ and } X = 3 \equiv X = \emptyset \quad (9.1.4.3)$$

$$\Pr(X \in \{1, 3\}) = \Pr(X = 1) + \Pr(X = 3) - \Pr(X = \emptyset) \quad (9.1.4.4)$$

$$= \frac{1}{3} + \frac{1}{6} \quad (9.1.4.5)$$

$$= \frac{1}{2} \quad (9.1.4.6)$$

c)

$$\Pr(X \neq 3) = 1 - \Pr(X = 3) \quad (9.1.4.7)$$

$$= 1 - \frac{1}{6} \quad (9.1.4.8)$$

$$= \frac{5}{6} \quad (9.1.4.9)$$

9.1.5 A die marked 1, 2, 3 in red and 4, 5, 6 in green is tossed. Let A be the event, ‘the number is even,’ and B be the event, ‘the number is red’. Are A and B independent?

**Solution:** Let

$$X = \begin{cases} 0, & \text{if number is odd} \\ 1, & \text{if number is even} \end{cases} \quad (9.1.5.1)$$

$$Y = \begin{cases} 0, & \text{if number is green} \\ 1, & \text{if number is red} \end{cases} \quad (9.1.5.2)$$

From the given information,

$$\Pr(X = 1) = \frac{3}{6} = \frac{1}{2}, \Pr(Y = 1) = \frac{3}{6} = \frac{1}{2} \quad (9.1.5.3)$$

$$\Pr(X = 1, Y = 1) = \frac{1}{6} \quad (9.1.5.4)$$

Now,

$$\Pr(X = 1) \times \Pr(Y = 1) = \frac{1}{4} \quad (9.1.5.5)$$

$$\implies \Pr(X = 1, Y = 1) \neq \Pr(X = 1) \times \Pr(Y = 1) \quad (9.1.5.6)$$

Hence, A and B are not independent.

9.1.6 A fair coin and an unbiased die are tossed. Let A be the event 'head appears on the coin' and B be the event '3 on the die'. Check whether A and B are independent events or not.

9.1.7 A child's game has 8 triangles of which 3 are blue and rest are red, and 10 squares of which 6 are blue and rest are red. One piece is lost at random. Find the probability that it is a

- a) triangle
- b) square
- c) square of blue colour
- d) triangle of red colour

**Solution:** The random variables in the problem are summarized in Table 9.1.7.1. From the given information,

$$p_X(k) = \begin{cases} \frac{10}{18} & k = 0 \\ \frac{8}{18} & k = 1 \end{cases} \quad (9.1.7.1)$$

$$\Pr(Y = 0|X = 1) = \frac{5}{8} \quad (9.1.7.2)$$

$$\Pr(Y = 1|X = 1) = \frac{3}{8} \quad (9.1.7.3)$$

$$\Pr(Y = 0|X = 0) = \frac{4}{10} \quad (9.1.7.4)$$

$$\Pr(Y = 1|X = 0) = \frac{6}{10} \quad (9.1.7.5)$$

Consequently,

a)  $p_X(1) = \frac{8}{18}$

b)  $p_X(0) = \frac{10}{18}$

c)  $p_{XY}(0, 1) = \Pr(Y = 1|X = 0) p_X(0) = \frac{6}{18}$

d)  $p_{XY}(1, 0) = \Pr(Y = 0|X = 1) p_X(1) = \frac{5}{18}$

TABLE 9.1.7.1

| Variable | Value | Description |
|----------|-------|-------------|
| X        | 1     | Triangle    |
|          | 0     | Square      |
| Y        | 1     | Blue        |
|          | 0     | Red         |

- 9.1.8 Suppose you have two coins which appear identical in your pocket. You know that one is fair and one is 2-headed. If you take one out, toss it and get a head, what is the probability that it was a fair coin?

See Table 9.1.8.1. Given,

$$\Pr(X = 1) = \frac{1}{2}, \quad \Pr(X = 0) = \frac{1}{2}, \quad \Pr(Y = 1 | X = 1) = \frac{1}{2}, \quad (9.1.8.1)$$

$$\Pr(Y = 1 | X = 0) = 1. \quad (9.1.8.2)$$

Hence, the desired probability is

$$\Pr(X = 1 | Y = 1) = \frac{\Pr(Y = 1 | X = 1) \times \Pr(X = 1)}{\sum_{k=0}^1 \Pr(Y = 1 | X = k) \times \Pr(X = k)} \quad (9.1.8.3)$$

$$= \frac{\frac{1}{2} \times \frac{1}{2}}{1 \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2}} = \frac{1}{3} \quad (9.1.8.4)$$

TABLE 9.1.8.1: Random Variables

| Variable | Value | Description   |
|----------|-------|---------------|
| X        | 1     | Fair coin     |
|          | 0     | 2-headed coin |
| Y        | 1     | heads         |
|          | 0     | tails         |

- 9.1.9 A box has 5 blue and 4 red balls. One ball is drawn at random and not replaced. Its colour is also not noted. Then another ball is drawn at random. What is the probability of second ball being blue?

**Solution:** See Table 9.1.9.1. From the given information,

$$\Pr(X = 0) = \frac{5}{9}, \quad \Pr(X = 1) = \frac{4}{9} \quad (9.1.9.1)$$

$$\Pr(Y = 0 | X = 0) = \frac{1}{2}, \quad \Pr(Y = 0 | X = 1) = \frac{5}{8} \quad (9.1.9.2)$$

The desired probability is

$$\Pr(Y = 0) = \Pr(X = 0) \Pr(Y = 0 | X = 0) + \Pr(X = 1) \Pr(Y = 0 | X = 1) \quad (9.1.9.3)$$

$$= \frac{5}{9} \times \frac{1}{2} + \frac{4}{9} \times \frac{5}{8} = \frac{5}{9} \quad (9.1.9.4)$$

TABLE 9.1.9.1

| Variable | Value | Description         |
|----------|-------|---------------------|
| X        | 1     | Red in first draw   |
|          | 0     | Blue in first draw  |
| Y        | 1     | Red in second draw  |
|          | 0     | Blue in second draw |

- 9.1.10 A bag contains 4 white and 5 black balls. Another bag contains 9 white and 7 black balls. A ball is transferred from the first bag to the second and then a ball is drawn

at random from the second bag. Find the probability that the ball drawn is white.

- 9.1.11 An item is manufactured by three machines A, B and C. Out of the total number of items manufactured during a specified period, 50% are manufactured on A, 30% on B and 20% on C, 2% of the items produced on A and 2% of items produced on B are defective, and 3% of these products produced on C are defective. All the items are stored at one godown. One item is drawn at random and is found to be defective. What is the probability that it was manufactured on machine A?

**Solution:** See Table 9.1.11.1.

| Parameter | Values | Description       |
|-----------|--------|-------------------|
| $X$       | 0      | not defective     |
|           | 1      | defective         |
| $Y$       | 1      | manufactured on A |
|           | 2      | manufactured on B |
|           | 3      | manufactured on C |

TABLE 9.1.11.1

Given that,

$$\Pr(Y = 1) = \frac{50}{100} = 0.5 \quad (9.1.11.1)$$

$$\Pr(Y = 2) = \frac{30}{100} = 0.3 \quad (9.1.11.2)$$

$$\Pr(Y = 3) = \frac{20}{100} = 0.2 \quad (9.1.11.3)$$

$$\Pr(X = 1|Y = 1) = \frac{2}{100} = 0.02 \quad (9.1.11.4)$$

$$\Pr(X = 1|Y = 2) = \frac{2}{100} = 0.02 \quad (9.1.11.5)$$

$$\Pr(X = 1|Y = 3) = \frac{3}{100} = 0.03 \quad (9.1.11.6)$$

Thus,

$$\Pr(Y = 1|X = 1) = \frac{\Pr(Y = 1)\Pr(X = 1|Y = 1)}{\sum_i \Pr(Y = i)\Pr(X = 1|Y = i)} \quad (9.1.11.7)$$

$$= \frac{0.5 \times 0.02}{0.5 \times 0.02 + 0.3 \times 0.02 + 0.2 \times 0.03} = \frac{5}{11} \quad (9.1.11.8)$$

- 9.1.12 There are two bags, one which contains 3 black balls and 4 white balls while the other contains 4 black balls and 3 white balls. A die is thrown. If it shows up 1 or 3, a ball is taken from the first bag; but if it shows up any other number, a ball is taken from the second bag. Find the probability of choosing a black ball.

**Solution:** See Table 9.1.12.1. From the given information,

$$\Pr(X = 0) = \Pr(Z = 0) = \frac{1}{3} \quad (9.1.12.1)$$

$$\Pr(X = 1) = \Pr(Z = 1) = \frac{2}{3} \quad (9.1.12.2)$$

$$\Pr(Y = 0|X = 0) = \frac{3}{7} \quad (9.1.12.3)$$

$$\Pr(Y = 0|X = 1) = \frac{4}{7} \quad (9.1.12.4)$$

Hence, the desired probability is

$$\begin{aligned} \Pr(Y = 0) &= \Pr(X = 0) \times \Pr(Y = 0|X = 0) + \Pr(X = 1) \times \Pr(Y = 0|X = 1) \\ &\quad (9.1.12.5) \end{aligned}$$

$$= \frac{1}{3} \times \frac{3}{7} + \frac{2}{3} \times \frac{4}{7} = \frac{7}{21} \quad (9.1.12.6)$$

| RV | Value | Description                |
|----|-------|----------------------------|
| X  | 0     | first bag is selected      |
|    | 1     | second bag is selected     |
| Y  | 0     | black ball is drawn        |
|    | 1     | white ball is drawn        |
| Z  | 0     | 1 or 3 is shown up         |
|    | 1     | another number is shown up |

TABLE 9.1.12.1

9.1.13 A shopkeeper sells three types of flower seeds  $A_1, A_2$  and  $A_3$ . They are sold as a mixture where the proportions are 4:4:2 respectively. The germination rates of the three types of seeds are 0.45, 0.60 and 0.35. Calculate the probability

- a) of a randomly chosen seed to germinate
- b) that it will not germinate given that the seed is of type  $A_3$ ,
- c) that it is of the type  $A_2$  given that a randomly chosen seed does not germinate.

**Solution:** See Table 9.1.13.1. From the given information,

$$p_X(k) = \begin{cases} \frac{4}{10} & k = 1 \\ \frac{4}{10} & k = 2 \\ \frac{2}{10} & k = 3 \end{cases} \quad (9.1.13.1)$$

$$p_{Y|X}(0|1) = 0.45 \quad (9.1.13.2)$$

$$p_{Y|X}(0|2) = 0.60 \quad (9.1.13.3)$$

$$p_{Y|X}(0|3) = 0.35 \quad (9.1.13.4)$$

using the definition in (4.1.5.1).

a)

$$p_Y(0) = \sum_{k=0}^3 p_{Y|X}(0|k) p_X(k) = \frac{49}{100} \quad (9.1.13.5)$$

Also,

$$p_Y(1) = 1 - p_Y(0) = \frac{51}{100} \quad (9.1.13.6)$$

b) From (4.1.6.2),

$$p_{Y|X}(1|2) = 1 - p_{Y|X}(0|2) = 1 - 0.35 = 0.65 \quad (9.1.13.7)$$

c)

$$p_{X|Y}(2|1) = \frac{p_{Y|X}(1|2) p_X(2)}{p_Y(1)} = \frac{16}{51} \quad (9.1.13.8)$$

upon substituting from (9.1.13.7) and (9.1.13.6).

| Variable | Description   | Value |
|----------|---------------|-------|
| $X$      | $A_1$         | 1     |
|          | $A_2$         | 2     |
|          | $A_3$         | 3     |
| $Y$      | germinate     | 0     |
|          | not germinate | 1     |

TABLE 9.1.13.1

- 9.1 One card is drawn from a well-shuffled deck of 52 cards. Find the probability of getting
- A king of red colour
  - A face card
  - A red face card
  - The jack of hearts
  - A spade
  - The queen of diamonds

**Solution:**

- 9.2 Five cards—the ten, jack, queen, king and ace of diamonds, are well-shuffled with their face downwards. One card is then picked up at random.
- What is the probability that the card is the queen?
  - If the queen is drawn and put aside, what is the probability that the second card picked up is (a) an ace? (b) a queen?

**Solution:**

- 9.3 A bag contains 5 red balls and some blue balls. If the probability of drawing a blue ball is double that if a red ball, determine the number of blue balls in the bag.

**Solution:**

9.4 A card is selected from a pack of 52 cards.

- (a) How many points are there in the sample space?
- (b) Calculate the probability that the card is an ace of spades.
- (c) Calculate the probability that the card is (i) an ace and (ii) black card.

**Solution:**

9.5 Four cards are drawn from a well-shuffled deck of 52 cards. What is the probability of obtaining 3 diamonds and one spade.

**Solution:**

9.6 In a certain lottery 10,000 tickets are sold and ten equal prizes are awarded. What is the probability of not getting a prize if you buy (a) one ticket (b) two tickets (c) 10 tickets ?

**Solution:**

9.7 Out of 100 students, two sections of 40 and 60 are formed. If you and your friend are among the 100 students, what is the probability that

- a) you both enter the same section?
- b) you both enter the different sections?

**Solution:**

9.8 The number lock of a suitcase has 4 wheels each labelled with ten digits i.e. from 0 to 9. The lock opens with a sequence of four digits with no repeats. What is the probability of a person getting the right sequence to open the suitcase.

**Solution:**

9.9 Two cards are drawn at random and without replacement from a pack of 52 playing cards. Find the probability that both the cards are black.

**Solution:**

9.10 A box of oranges is inspected by examining three randomly selected oranges drawn without replacement. If all the three oranges are good, the box is approved for sale, otherwise, it is rejected. Find the probability that a box containing 15 oranges out of which 12 are good and 3 are bad ones will be approved for sale.

9.11 Two balls are drawn at random with replacement from a box containing 10 black and 8 red balls. Find the probability that

- a) both balls are red.
- b) first ball is black and second is red.
- c) one of them is black and other is red.

9.12 In a hostel, 60% of the students read Hindi newspaper, 40% read English newspaper and 20% read both Hindi and English newspapers. A student is selected at random.

- a) Find the probability that she reads neither Hindi nor English newspapers.
- b) If she reads Hindi newspaper, find the probability that she reads English newspaper.
- c) If she reads English newspaper, find the probability that she reads Hindi newspaper.

9.13 The probability of obtaining an even prime number on each die, when a pair of dice is rolled is

- a) 0

- b)  $\frac{1}{3}$
- c)  $\frac{1}{12}$
- d)  $\frac{1}{36}$

**Solution:**

- 9.14 A bag contains 4 red and 4 black balls, another bag contains 2 red and 6 black balls. One of the two bags is selected at random and a ball is drawn from the bag which is found to be red. Find the probability that the ball is drawn from the first bag.

**Solution:**

- 9.15 Cards with numbers 2 to 101 are placed in a box. A card is selected at random. Find the probability that the card has

- (i) an even number
- (ii) a square number

**Solution:**

- 9.16 The king, queen and jack of clubs are removed from a deck of 52 playing cards and then well shuffled. Now one card is drawn at random from the remaining cards. Determine the probability that the card is

- (i) a club
- (ii) 10 of hearts

**Solution:**

- 9.17 A team of medical students doing their internship have to assist during surgeries at a city hospital. The probabilities of surgeries rated as very complex, complex, routine, simple or very simple are respectively, 0.15, 0.20, 0.31, 0.26, .08. Find the probabilities that a particular surgery will be rated

- a) complex or very complex;
- b) neither very complex nor very simple;
- c) routine or complex
- d) routine or simple

**Solution:**

- 9.18 A card is selected from a pack of 52 cards.

- (a) How many points are there in the sample space?
- (b) Calculate the probability that the card is an ace of spades.
- (c) Calculate the probability that the card is (i) an ace and (ii) black card.

**Solution:**

- 9.19 The probability that a non leap year selected at random will contain 53 sundays.

**Solution:**

- 9.20 One of the four persons John, Rita, Aslam or Gurpreet will be promoted next month. Consequently the sample space consists of four elementary outcomes  $S = \text{John promoted, Rita promoted, Aslam promoted, Gurpreet promoted}$ . You are told that the chances of John's promotion is same as that of Gurpreet, Rita's chances of promotion are twice as likely as Johns. Aslam's chances are four times that of John.

- a) Determine
  - i)  $P(\text{John promoted})$
  - ii)  $P(\text{Rita promoted})$

- iii)  $P(\text{Aslam promoted})$   
 iv)  $P(\text{Gurpreet promoted})$
- b) If  $A = \text{John promoted or Gurpreet promoted}$ , find  $P(A)$ .

**Solution:**

- 9.21 A card is drawn from a deck of 52 cards. Find the probability of getting a king or a heart or a red card.

**Solution:**

- 9.22 The probability that a student will pass his examination is 0.73, the probability of the student getting a compartment is 0.13, and the probability that the student will either pass or get compartment is 0.96. State True or False.

**Solution:**

- 9.23 A card is selected from a pack of 52 cards

- (a) How many points are there in the sample space?
- (b) Calculate the probability that the card is an ace of spades.
- (c) Calculate the probability that the card is (i) an ace (ii)black card.

- 9.24 In a non-leap year, the probability of having 53 tuesdays or 53 wednesdays is

**Solution:**

- 9.25 There are 1000 sealed envelopes in a box, 10 of them contain a cash prize of Rs 100 each, 100 of them contain a cash prize of Rs 50 each and 200 of them contain a cash prize of Rs 10 each and rest do not contain any cash prize. If they are well shuffled and an envelope is picked up out, what is the probability that it contains no cash prize?

**Solution:**

- 9.26 A die is thrown and a card is selected at random from a deck of 52 playing cards. The probability of getting an even number on the die and a spade card.

**Solution:**

- 9.27 If 4-digit numbers greater than 5,000 are randomly formed from the digits 0, 1, 3, 5, and 7, what is the probability of forming a number divisible by 5 when:

- a) The digits are repeated?
- b) The repetition of digits is not allowed?

**Solution:**

- 9.28 Consider the probability space  $(\Omega, \mathcal{G}, P)$  where  $\Omega = [0, 2]$  and  $\mathcal{G} = \{\phi, \Omega, [0, 1], (1, 2]\}$ . Let  $X$  and  $Y$  be two functions on  $\Omega$  defined as

$$X(\omega) = \begin{cases} 1 & \text{if } \omega \in [0, 1] \\ 2 & \text{if } \omega \in (1, 2] \end{cases}$$

and

$$Y(\omega) = \begin{cases} 2 & \text{if } \omega \in [0, 1.5] \\ 3 & \text{if } \omega \in (1.5, 2]. \end{cases}$$

Then which one of the following statements is true?

- (A)  $X$  is a random variable with respect to  $\mathcal{G}$ , but  $Y$  is not a random variable with respect to  $\mathcal{G}$ .
- (B)  $Y$  is a random variable with respect to  $\mathcal{G}$ , but  $X$  is not a random variable with respect to  $\mathcal{G}$ .
- (C) Neither  $X$  nor  $Y$  is a random variable with respect to  $\mathcal{G}$ .
- (D) Both  $X$  and  $Y$  are random variables with respect to  $\mathcal{G}$ .

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**Solution:**

- 9.29 A die is loaded in such a way that each odd number is twice as likely to occur as each even number. Find  $P(G)$ , where  $G$  is the event that a number greater than 3 occurs on a single roll of the die.

**Solution:**

- 9.30 All the jacks, queens and kings are removed from a deck of 52 playing cards. The remaining cards are well shuffled and then one card is drawn at random. Giving ace a value 1 similar value for other cards, find the probability that the card has a value

- a) 7
- b) greater than 7
- c) less than 7

- 9.31 A Lot consists of 48 mobile phones of which 42 are good, 3 have only minor defects and 3 have major defects. Varnika will buy a phone if it is good but the trader will only buy a mobile if it has no major defects. One phone is selected at random from the lot. What is the probability that it is

- a) acceptable to Varnika?
- b) acceptable to the trader?

**Solution:**

- 9.32 A student says that if you throw a die, it will show up 1 or not 1. Therefore, the probability of getting 1 and the probability of getting 'not 1' each is equal to  $\frac{1}{2}$ . Is this correct? Give reasons.

**Solution:**

- 9.33 Four candidates A, B, C, D have applied for the assignment to coach a school cricket team. If A is twice as likely to be selected as B, and B and C are given about the same chance of being selected, while C is twice as likely to be selected as D, what are the probabilities that

- a) C will be selected?
- b) A will not be selected?

- 9.34 A bag contains 24 balls of which  $x$  balls are red,  $2x$  are white and  $3x$  are blue. A ball is selected at random. What is the probability that it is

- a) not red ?
- b) white ?

If the letters of the word ASSASSINATION are arranged at random. Find the Probability that

- (a) Four S's come consecutively in the word
- (b) Two I's and two N's come together

- (c) All  $A'$ s are not coming together  
 (d) No two  $A'$ s are coming together
- 9.35 One urn contains two black balls (labelled B1 and B2) and one white ball. A second urn contains one black ball and two white balls (labelled W1 and W2). Suppose the following experiment is performed. One of the two urns is chosen at random. Next a ball is randomly chosen from the urn. Then a second ball is chosen at random from the same urn without replacing the first ball.

- a) What is the probability that two black balls are chosen?  
 b) What is the probability that two balls of opposite colour are chosen?

**Solution:**

## 10 MULTINOMIAL

### 10.1 Formulae

10.1.1 Let

$$N = R + B + G, n = r + b + g \quad (10.1.1.1)$$

where  $R, B, G$  and  $r, b, g$  represent the number of red, blue and green marbles respectively within  $N$  and  $n$ . Then

$${}^R C_r {}^B C_b {}^G C_g = \frac{{}^R C_r {}^B C_b {}^G C_g}{R+B+G} \quad (10.1.1.2)$$

**Solution:** The number of ways of choosing  $n$  marbles from  $N$  is

$${}^N C_n = {}^{R+B+G} C_{r+b+g} \quad (10.1.1.3)$$

The number of ways of choosing  $r, b, g$  marbles is

$${}^R C_r {}^B C_b {}^G C_g \quad (10.1.1.4)$$

Using the definition of probability, we obtain (10.1.1.2).

10.1.2

$${}^{R+B} C_n = \sum_{k=0}^R \sum_{m=n-k}^B {}^R C_k {}^B C_m \quad (10.1.2.1)$$

**Solution:** Since

$$(x+1)^R = \sum_{k=0}^R {}^R C_k x^k, \quad (10.1.2.2)$$

$$(x+1)^R (x+1)^B = \sum_{k=0}^R \sum_{m=0}^B {}^R C_k {}^B C_m x^{k+m} \quad (10.1.2.3)$$

$$\Rightarrow (x+1)^{R+B} = \sum_{k=0}^R \sum_{m=n-k}^B {}^R C_k {}^B C_m x^n + \sum_{k=0}^R \sum_{m \neq n-k}^B {}^R C_k {}^B C_m x^{k+m} \quad (10.1.2.4)$$

$$= {}^N C_n \quad (10.1.2.5)$$

yielding (10.1.2.1) upon comparing the coefficients of  $x^n$  on both sides.

## 10.2 NCERT

10.2.1 A box contains 5 red marbles, 8 white marbles and 4 green marbles. One marble is taken out of the box at random. What is the probability that the marble taken out will be

- a) red ?
- b) white ?
- c) not green?

**Solution:** From (10.1.1.2),

- a) Probability that the marble taken out is red

$$p_{R,W,G}(1,0,0) = \frac{^5C_1 \cdot ^8C_0 \cdot ^4C_0}{^{17}C_1} = \frac{5}{17} \quad (10.2.1.1)$$

- b) Probability that the marble taken out is white

$$p_{R,W,G}(0,1,0) = \frac{^5C_0 \cdot ^8C_1 \cdot ^4C_0}{^{17}C_1} = \frac{8}{17} \quad (10.2.1.2)$$

- c) Probability that the marble taken out is not green

$$1 - p_{R,W,G}(0,0,1) = 1 - \frac{^5C_0 \cdot ^8C_0 \cdot ^4C_1}{^{17}C_1} = 1 - \frac{4}{17} = \frac{13}{17} \quad (10.2.1.3)$$

10.2.2 A box contains 10 red marbles, 20 blue marbles and 30 green marbles. 5 marbles are drawn from the box, what is the probability that

- a) all will be blue?
- b) atleast one will be green?

**Solution:** See (10.1.1.2). In this question,

$$N = 60, R = 10, B = 20, G = 30, n = 5 \quad (10.2.2.1)$$

- a) From (10.1.1.2),

$$p_{R,B,G}(0,5,0) = \frac{^{20}C_5}{^{60}C_5} \quad (10.2.2.2)$$

- b) Since

$$p_{R,B,G}(r,b,0) = \frac{^R C_r \cdot ^B C_b}{^{R+B+G} C_{r+b}} \quad (10.2.2.3)$$

The probability that at least one marble is green is given by

$$1 - \sum_{r+b=n} p_{R,B,G}(r,b,0) = 1 - \sum_{r+b=n} \frac{^R C_r \cdot ^B C_b}{^{R+B+G} C_{r+b}} = 1 - \frac{^{R+B} C_n}{^{R+B+G} C_n} \quad (10.2.2.4)$$

from (10.1.2.1). Substituting numerical values, the desired probability is

$$1 - \frac{^{30}C_5}{^{60}C_5} \quad (10.2.2.5)$$

- 10.2.3 A box of oranges is inspected by examining three randomly selected oranges drawn without replacement. If all the three oranges are good, the box is approved for sale, otherwise, it is rejected. Find the probability that a box containing 15 oranges out of which 12 are good and 3 are bad ones will be approved for sale.

**Solution:** Choosing

$$R = 12, B = 3, G = 0, n = 3, r = 3, b = 0, g = 0 \quad (10.2.3.1)$$

in (10.1.1.2) the desired probability is

$$p_{R,B,G}(3,0,0) = \frac{^{12}C_3}{^{15}C_3} = \frac{44}{91} \quad (10.2.3.2)$$

- 10.2.4 A box contains 3 orange balls, 3 green balls and 2 blue balls. Three balls are drawn at random from the box without replacement. The probability of drawing 2 green balls and one blue ball is

a)  $\frac{3}{28}$

b)  $\frac{2}{21}$

c)  $\frac{1}{28}$

d)  $\frac{167}{168}$

**Solution:** The desired probability is

$$p_{O,G,B}(0,2,1) = \frac{^3C_0 ^3C_2 ^2C_1}{^8C_3} = \frac{3}{28} \quad (10.2.4.1)$$

- 10.2.5 A bag contain 5 red and 3 blue balls. If 3 balls are drawn at random without replacement, the probability of getting exactly one red ball is

a)  $\frac{45}{196}$

b)  $\frac{135}{392}$

c)  $\frac{15}{56}$

d)  $\frac{15}{29}$

**Solution:** The desired probability is

$$p_{R,B}(1,2) = \frac{^5C_1 ^3C_2}{^8C_3} = \frac{15}{56} \quad (10.2.5.1)$$

- 10.2.6 A bag contains 5 red and 3 blue balls. If 3 balls are drawn at random without replacement the probability that exactly two of the three balls were red, the first ball being red is

**Solution:** See (10.1.1.2). In this question,

As the first ball drawn is red,

$$N = 7, R = 4, B = 3, G = 0, r = 1, b = 1, g = 0 \quad (10.2.6.1)$$

The desired probability is,

$$p_{R,B}(1,1) = \frac{^4C_1 ^3C_1}{^7C_2} = \frac{4}{7} \quad (10.2.6.2)$$

## 11 MISCELLANEOUS

- 11.1 The random variable  $X$  has a probability distribution  $\Pr(X)$  of the following form, where  $k$  is some number

$$\Pr(X) = \begin{cases} k, & x = 0 \\ 2k, & x = 1 \\ 3k, & x = 2 \\ 0, & \text{otherwise} \end{cases} \quad (11.1.1)$$

- a) Determine the value of  $k$
- b) Find  $\Pr(X < 2), \Pr(X \leq 2), \Pr(X \geq 2)$

**Solution:**

- 11.2 State which of the following are not the probability distributions of a random variable. Give reasons for your answer

i

ii

iii

iv

- 11.3 A random variable  $X$  has the following probability distribution

Determine

- i  $k$
- ii  $P(X < 3)$
- iii  $P(X > 6)$
- iv  $P(0 < X < 3)$

- 11.4 The random variable  $X$  has a probability distribution  $P(X)$  of the following form, where  $k$  is some number :

$$P(x) = \begin{cases} k, & \text{if } x = 0 \\ 2k, & \text{if } x = 1 \\ 3k, & \text{if } x = 2 \\ 0, & \text{otherwise} \end{cases}$$

- i Determine the value of  $k$ .
- ii Find  $P(X < 2), P(X \leq 2), P(X \geq 2)$

- 11.5 A game consists of spinning an arrow which comes to rest pointing at one of the regions (1, 2 or 3) (Fig. 13.1). Are the outcomes 1, 2 and 3 equally likely to occur?

Give reasons.

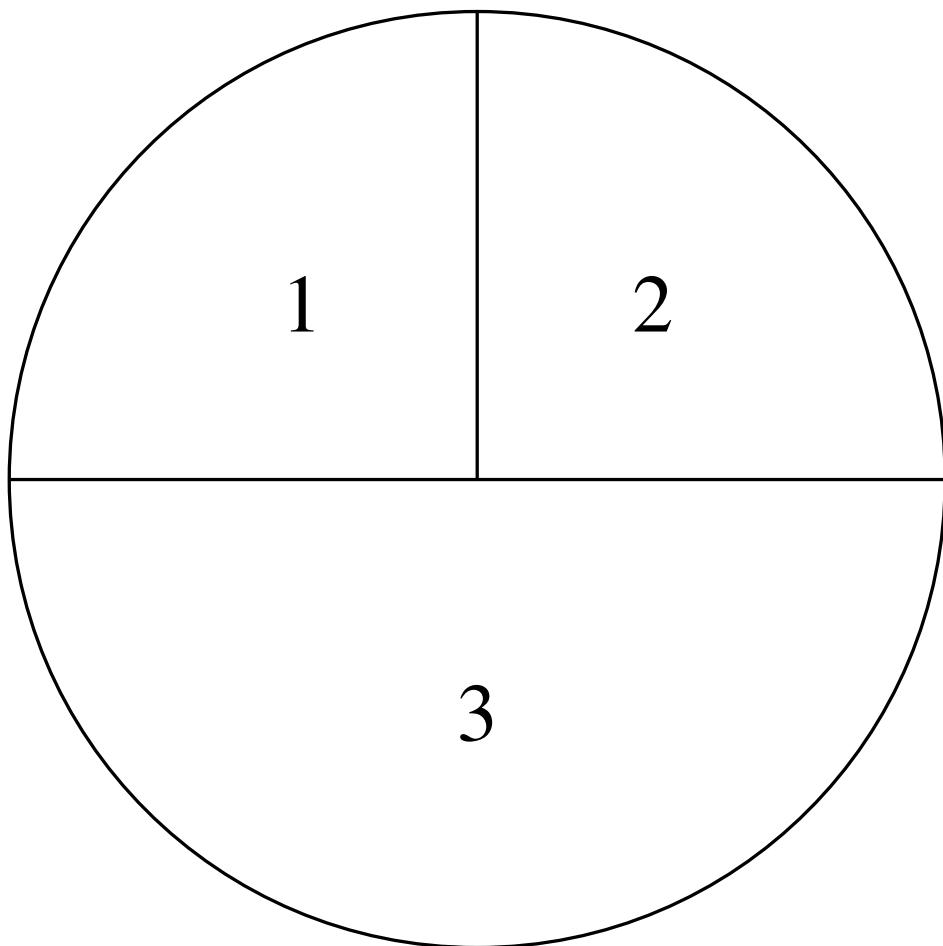


Fig. 11.5.1: Fig.13.1

**Solution:**

- 11.6 Apoorv throws two dice once and computes the product of the numbers appearing on the dice. Peehu throws one die and squares the number that appears on it. Who has the better chance of getting the number 36? Why?

**Solution:**

- 11.7 6 boys and 6 girls sit in a row at random. The probability that all the girls sit together is

- i  $\frac{1}{432}$
- ii  $\frac{12}{431}$
- iii  $\frac{1}{132}$

- iv none of the above
- 11.8 A card is selected from a deck of 52 cards. The probability of its being a red face card is
- 11.9 A die is loaded in such a way that each odd number is twice as likely to occur as each even number. Find  $P(G)$ , where  $G$  is the event that a number greater than 3 occurs on a single roll of the die.
- 11.10 Determine the probability  $p$ , for each of following events.
- An odd number appears in a single roll of dice.
  - Atleast one head appears in two tosses of fair coin.
  - A king, 9 of hearts or 3 of spades appears in drawing a single card from a well shuffled deck of 52 cards.
  - The sum of 6 appears in single toss of a pair of fair dice.
- 11.11 Determine the probability  $p$ , for each of the following events.
- An odd number appears in a single toss of a fair die.
  - At least one head appears in two tosses of a fair coin.
  - A king, 9 of hearts, or 3 of spades appears in drawing a single card from a well shuffled ordinary deck of 52 cards.
  - The sum of 6 appears in a single toss of a pair of fair dice.
- 11.12 The probability distribution of a random variable  $X$  is given below:

|        |     |               |               |               |
|--------|-----|---------------|---------------|---------------|
| $X$    | 0   | 1             | 2             | 3             |
| $P(X)$ | $k$ | $\frac{k}{2}$ | $\frac{k}{4}$ | $\frac{k}{8}$ |

- Determine the value of  $k$ .
- Determine  $P(X \leq 2)$  and  $P(X > 2)$ .
- Find  $P(X \leq 2) + P(X > 2)$ .

**Solution:**

- 11.13
- 11.14 Three persons, A, B and C, fire at a target in turn, starting with A. Their probability of hitting the target are 0.4, 0.3 and 0.2 respectively. The probability of two hits is
- 0.024
  - 0.188
  - 0.336
  - 0.452
- Solution:**
- item If two events are independent, then
- they must be mutually exclusive
  - the sum of their probabilities must be equal to 1
  - (A) and (B) both are correct
  - None of the above is correct
- 11.15 Three letters are dictated to three persons and an envelope is addressed to each of them, the letters are inserted into the envelopes at random so that each envelope

contains exactly one letter. Find the probability that at least one letter in its proper envelope.

## 12 MOMENTS

- 12.0.1 Find the mean number of heads in three tosses of a fair coin.

**Solution:**

- 12.0.2 Two dice are thrown simultaneously. If  $X$  denotes the number of sixes, find the expectation of  $X$ .

- 12.0.3 Two numbers are selected at random (without replacement) from the first six positive integers. Let  $X$  denote the larger of the two numbers obtained. Find  $E(X)$ .

- 12.0.4 Let  $X$  denote the sum of the numbers obtained when two fair dice are rolled. Find the variance and standard deviation of  $X$ .

- 12.0.5 A class has 15 students whose ages are 14, 17, 15, 14, 21, 17, 19, 20, 16, 18, 20, 17, 16, 19 and 20 years. One student is selected in such a manner that each has the same chance of being chosen and the age  $X$  of the selected student is recorded. What is the probability distribution of the random variable  $X$ ? Find mean, variance and standard deviation of  $X$ .

- 12.0.6 In a meeting, 70A member is selected at random and we take  $X = 0$  if he opposed, and  $X = 1$  if he is in favour. Find  $E(X)$  and  $\text{Var}(X)$ .

- 12.0.7 In a dice game, a player pays a stake of Re 1 for each throw of a die. She receives Rs 5 if the die shows a 3, Rs 2 if the die shows a 1 or 6, and nothing otherwise. What is the player's expected profit per throw over a long series of throws?

**Solution:**

- 12.0.8 A die is thrown three times. Let  $X$  be 'the number of twos seen'. Find the expectation of  $X$ .

**Solution:**

Choose the correct answer in each of the following:

- 12.0.9 The mean of the numbers obtained on throwing a die having written 1 on three faces, 2 on two faces and 5 on one face is

- a) 1
- b) 2
- c) 5
- d)  $\frac{8}{3}$

- 12.0.10 Suppose that two cards are drawn at random from a deck of cards. Let  $X$  be the number of aces obtained. Then the value of  $E(X)$  is

- a)  $\frac{37}{221}$
- b)  $\frac{3}{13}$
- c)  $\frac{1}{13}$
- d)  $\frac{2}{13}$

- 12.0.11 Suppose 10000 tickets are sold in a lottery each for Re. 1. First prize is of Rs 3000 and the second prize is of Rs 2000. There are three third prizes of Rs. 500 each. If you buy one ticket, what is your expectation.

12.0.12 Consider the probability distribution of a random variable X:

Calculate

- (i)  $\text{var}(X)$
- (ii)  $\text{var}\left(\frac{X}{2}\right)$

**Solution:**

12.0.13 The random variable X can take only the values 0, 1, 2. Given that  $\Pr(X = 0) = \Pr(X = 1) = p$  and that  $E(X^2) = E(X)$ , find the value of p.

12.0.14 Find the variance of distribution.

12.0.15 Two cards are drawn successively without replacement from a well shuffled deck of cards. Find the mean and standard variation of random variable X where X is the number of aces.

12.0.16 The probability distribution of a discrete random variable X is given as under:

|          |               |               |                |                |                |                |
|----------|---------------|---------------|----------------|----------------|----------------|----------------|
| X        | 1             | 2             | 4              | 2A             | 3A             | 5A             |
| $\Pr(X)$ | $\frac{1}{2}$ | $\frac{1}{5}$ | $\frac{3}{25}$ | $\frac{1}{10}$ | $\frac{1}{25}$ | $\frac{1}{25}$ |

Calculate:

- a) The value of A if  $E(X) = 2.94$
- b) Variance of X.

12.0.17 The probability distribution of a random variable X is given as under:

$$p_X(x) = \begin{cases} kx^2 & \text{for } x = 1, 2, 3 \\ 2kx & \text{for } x = 4, 5, 6 \\ 0 & \text{otherwise} \end{cases}$$

where k is a constant. Calculate:

- a)  $E(X)$
- b)  $E(3X^2)$
- c)  $\Pr(X \geq 4)$

12.0.18 Two probability distributions of the discrete random variable X and Y are given below.

TABLE 12.0.18.1: Table-1

|        |               |               |               |               |
|--------|---------------|---------------|---------------|---------------|
| X      | 0             | 1             | 2             | 3             |
| $P(X)$ | $\frac{1}{5}$ | $\frac{2}{5}$ | $\frac{1}{5}$ | $\frac{1}{5}$ |

TABLE 12.0.18.2: Table-2

| $Y$    | 0             | 1              | 2             | 3              |
|--------|---------------|----------------|---------------|----------------|
| $P(Y)$ | $\frac{1}{5}$ | $\frac{3}{10}$ | $\frac{2}{5}$ | $\frac{1}{10}$ |

Prove that  $E(Y^2) = 2E(X)$

**Solution:**

- 12.0.19 For the following probability distribution determine standard deviation of the random variable  $X$ .

- 12.0.20 Let  $X$  be a discrete random variable whose probability distribution is defined as follows:

$$\Pr(X = x) = \begin{cases} k(x+1) & , x = 1, 2, 3, 4 \\ 2kx & , x = 5, 6, 7 \\ 0 & , \text{otherwise} \end{cases}$$

where  $k$  is a constant. Calculate

- (i) the value of  $k$
- (ii)  $E(X)$
- (iii) Standard deviation of  $X$

**Solution:**

- 12.0.21 The probability distribution of a discrete random variable  $X$  is given as under:

Calculate:

- a) The value of  $A$  if  $E(X) = 2.94$
- b) Variance of  $X$ .

- 12.0.22 A discrete random variable  $X$  has the probability distribution given as below:

| $X$    | 0.5 | 1     | 1.5    | 2   |
|--------|-----|-------|--------|-----|
| $P(X)$ | $k$ | $k^2$ | $2k^2$ | $k$ |

- a) Find the value of  $k$
- b) Determine the mean of the distribution

**Solution:**

- 12.0.23 There are 5 cards numbered 1 to 5, one number on one card. Two cards are drawn at random without replacement. Let  $X$  denote the sum of the numbers on two cards drawn. Find the mean and variance of  $X$ .

- 12.0.24 For the following probability distribution:

$E(X)$  is equal to:

- a) 0
- b) -1
- c) -2
- d) -1.8

12.0.25 If  $A$ ,  $B$  and  $C$  are three independent events such that  $\Pr(A) = \Pr(B) = \Pr(C) = p$ , then  $P(\text{At least two of } A, B, C \text{ occur}) = 3p^2 - 2p^3$

12.0.26 For the following probability distribution

TABLE 12.0.26.1: Probability Distribution

$E(X^2)$  is equal to

- (A) 3
- (B) 5
- (C) 7
- (D) 10

12.0.27 A die is tossed twice. A ‘success’ is getting an even number on a toss. Find the variance of the number of successes.

**Solution:**

| Parameter | Value | Description  |
|-----------|-------|--|
| $X_i$     | 0,1   | 0-Not a success, 1-Success and it represents outcome of $i^{th}$ throw |
| $X$       | 0,1,2 | $X = X_1 + X_2$ , denoting number of outcomes in two throws            |

pmf of  $X_i$  is

$$p_{X_i}(k) = \begin{cases} \frac{1}{2}, & k = 0 \\ \frac{1}{2}, & k = 1 \end{cases} \quad \forall \quad 1 \leq i \leq 2 \quad (12.0.27.1)$$

Mean value of  $X_i$  is

$$\mu_{X_i} = E[X_i], \quad i = 0, 1 \quad (12.0.27.2)$$

$$= \frac{1}{2} \quad (12.0.27.3)$$

Variance of  $X_i$  is

$$\sigma_{X_i}^2 = E[(X_i - \mu_{X_i})^2], \quad i = 0, 1 \quad (12.0.27.4)$$

$$= \frac{1}{4} \quad (12.0.27.5)$$

Variance of getting successes in two throws of a die is

$$\sigma_X^2 = \sigma_{X_1}^2 + \sigma_{X_2}^2 \quad (12.0.27.6)$$

$$= \frac{1}{2} \quad (12.0.27.7)$$

12.0.28 A biased die is such that  $\Pr(4) = \frac{1}{10}$  and other scores being equally likely. The die is tossed twice. If  $X$  is the ‘number of fours seen’, find the variance of the random variable  $X$ .

12.0.29 One million random numbers are generated from a statistically stationary process with a Gaussian distribution with mean zero and standard deviation  $\sigma_0$ .

The  $\sigma_0$  is estimated by randomly drawing out 10,000 number of samples ( $x_n$ ). The estimates  $\hat{\sigma}_1^2$  and  $\hat{\sigma}_2^2$  are computed in the following two ways:

$$\hat{\sigma}_1^2 = \frac{1}{10000} \sum_{n=1}^{10000} x_n^2$$

$$\hat{\sigma}_2^2 = \frac{1}{9999} \sum_{n=1}^{10000} x_n^2$$

Which of the following statements is true?

- a)  $E(\hat{\sigma}_2^2) = \sigma_0^2$
- b)  $E(\hat{\sigma}_2) = \sigma_0$
- c)  $E(\hat{\sigma}_1^2) = \sigma_0^2$
- d)  $E(\hat{\sigma}_1) = E(\hat{\sigma}_2)$

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**Solution:**

12.0.30 If X is the number of tails in three tosses of coin, determine the standard deviation of X.

12.0.31 In a game, a man wins a rupee for a six and loses a rupee for any other number when a fair die is thrown. The man decided to throw a die thrice but to quit as and when he gets a six. Find the expected value of the amount he wins / loses.

## 13 RANDOM ALGEBRA

### 13.1 Examples

13.1.1 Given that a fair coin is marked 1 on one face and 6 on the other and a fair die are tossed. Find the probability that the sum turns up to be 3 and 12.

**Solution:**

13.1.2 A fair coin is tossed four times, and a person win Re 1 for each head and lose Rs 1.50 for each tail that turns up. From the sample space calculate how many different amounts of money you can have after four tosses and the probability of having each of these amounts.

**Solution:**

13.1.3 A black and a red dice are rolled.

- a) Find the conditional probability of obtaining a sum greater than 9, given that the black die resulted in a 5.
- b) Find the conditional probability of obtaining the sum 8, given that the red die resulted in a number less than 4.

**Solution:**

- 13.1.4 Given that 2 numbers appearing on throwing two dice are different. Find the probability of the event ‘the sum of numbers on the dice is 4’.

**Solution:**

- 13.1.5 If each element of a  $2 \times 2$  determinant is either zero or one. What is the probability that the value of the determinant is positive ? (Assume that the individual entries of the determinant are chosen independently each value being assumed with probability  $\frac{1}{2}$ )

**Solution:**

- 13.1.6 Two customers Shyam and Ekta are visiting a particular shop in the same week(Tuesday to Saturday). Each is equally likely to visit the shop on any day as on another day. What is the probability that both will visit the shop on (i) the same day? (ii) consecutive days? (iii) different days?

- 13.1.7 Let  $X$  represent the difference between the number of heads and the number of tails obtained when a coin is tossed 6 times. What are the possible values of  $X$ ? Also find the Probability distribution of  $X$ .

- 13.1.8 A black and a red dice are rolled.

(a) find the conditional probability of obtaining a sum greater than 9,given that the black dice resulted in a 5.

(b) find the conditional probability of obtaining the sum 8,given that the red die resulted in a number less than 4.

**Solution:**

- 13.1.9 A fair die is thrown two times. Let A and B be the events, 'same number each time', and a 'a total score is 10 or more', respectively. Determine whether or not A and B are independent.

**Solution:**

- 13.1.10 Two dice are thrown together. Find the probability that the product of the numbers on the top of the dice is less than 9

**Solution:**

- 13.1.11 Two dice are thrown together. Find the probability that the product of the numbers on the top of the dice is 6 , 7 , 12

**Solution:**

## 13.2 Exercises

- 13.2.1 Two dice are thrown together. Find the probability that the product of the numbers on the top of the dice is

- a) 6
- b) 12
- c) 7

**Solution:**

- 13.2.2 Two dice are thrown simultaneously. What is the probability that the sum of the numbers appearing on the dice is

- a) 7?
- b) a prime number?
- c) 1?

**Solution:**

- 13.2.3 Two dice are thrown at the same time. Find the probability of getting
- (i) same number on both dice.
  - (ii) different numbers on both dice.
- 13.2.4 A die has its face marked 0,1,1,1,6,6. Two such dice are thrown together and their score is recorded.
- a) How many different scores are possible ?
  - b) What is the probability of getting a total 7 ?
- 13.2.5 For a loaded die, the probabilities of outcomes are given as under:  $\Pr(1) = \Pr(2) = 0.2$ ,  $\Pr(3) = \Pr(5) = \Pr(6) = 0.1$  and  $\Pr(4) = 0.3$ . The die is thrown two times. Let A and B be the events, 'same number each time', and 'a total score is 10 or more' respectively. Determine whether or not A and B are independent.
- 13.2.6 Three dice are thrown at the sametime. Find the probability of getting three two's, if it is known that the sum of the numbers on the dice was six.
- 13.2.7 Two dice are tossed. Find whether the following two events A and B are independent:  
 $A = \{(x,y) : x+y=11\}$   $B = \{(x,y) : x \neq 5\}$   
where  $(x,y)$  denotes a typical sample point.
- 13.2.8 Find the probability distribution of the maximum of the two scores obtained when a die is thrown twice. Determine also the mean of the distribution.
- 13.2.9 Two dice are thrown. If it is known that sum of the numbers on the dice was less than 6, the probability of getting a sum 3, is
- A)  $\frac{1}{18}$
  - B)  $\frac{5}{18}$
  - C)  $\frac{1}{5}$
  - D)  $\frac{2}{5}$
- 13.2.10 Two dice are thrown at the same time. Determine the probabiity that the difference of the numbers on the two dice is 2.

## 14 MARKOV CHAIN

- 14.0.1 Consider the experiment of throwing a die.
- If a multiple of 3 comes up, throw the die again
  - If any other number comes, toss a coin.
- Find the conditional probability of the event 'the coin shows a tail', given that 'at least one die shows a 3'.

14.0.2 A and B throw a pair of dice alternately. A wins the game if he gets a total of 6 and B wins if she gets a total of 7. If A starts the game, find the probability of winning the game by A in third throw of the pair of dice.

14.0.3 A state transition diagram with states  $A$ ,  $B$ , and  $C$ , and transition probabilities  $p_1, p_2, \dots, p_7$  is shown in the figure (e.g.,  $p_1$  denotes the probability of transition from state  $A$  to  $B$ ). For this state diagram, select the statement(s) which is/are universally true

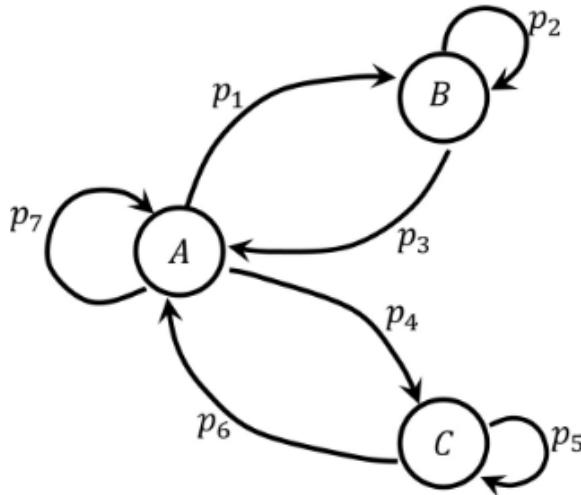


Fig. 14.0.3.1: Figure1

- a)  $p_2 + p_3 = p_5 + p_6$
- b)  $p_1 + p_3 = p_4 + p_6$
- c)  $p_1 + p_4 + p_7 = 1$
- d)  $p_2 + p_5 + p_7 = 1$

## APPENDIX

A.1 The mean of the Bernoulli distribution is

$$E(X) = p \quad (\text{A.1.1})$$

A.2 Let  $X_i$  be independent. For

$$X = X_1 + X_2 + \dots + X_n, \quad (\text{A.2.1})$$

$$M_X(z) = \prod_{i=1}^n M_{X_i}(z) \quad (\text{A.2.2})$$

A.3 The  $n$ th moment of  $X$  can be expressed as

$$E [X^n] = \frac{d^n M_X(z^{-1})}{dz^n} \Big|_{z=1} \quad (\text{A.3.1})$$

A.4 The mean for the Binomial r.v. is

$$E [X] = np \quad (\text{A.4.1})$$

**Solution:** From (A.3.1) and (7.1.3.1),

$$E [X] = \frac{d(q + pz)^n}{dz} \Big|_{z=1} \quad (\text{A.4.2})$$

$$= np(q + pz)^{n-1} \Big|_{z=1} \quad (\text{A.4.3})$$

$$= np(q + p)^{n-1} \quad (\text{A.4.4})$$

yielding (A.4.1)

$$\therefore p + q = 1 \quad (\text{A.4.5})$$