Inner Product

G V V Sharma*

CONTENTS

Abstract—**This mnual provides an introduction to inner product applications in school geometry based on the NCERT textbooks from Class 6-12.**

1 Angle

1.1. Find the angle between the two planes

$$\begin{pmatrix}2 & 1 & -2\end{pmatrix}\mathbf{x} = 5 \tag{1.1.1}$$

$$\begin{pmatrix}3 & -6 & -2\end{pmatrix}\mathbf{x} = 7. \tag{1.1.2}$$

Solution: The angle between two planes is the same as the angle between their normal vectors. This can be obtained from (1.8.6).

1.2. Find the angle between the two planes

$$\begin{pmatrix}2 & 2 & -2\end{pmatrix}\mathbf{x} = 5 \tag{1.2.1}$$

$$\begin{pmatrix}3 & -6 & 2\end{pmatrix}\mathbf{x} = 7. \tag{1.2.2}$$

Solution: See Problem (1.1).

1.3. Find the angle between the line

$$L: \quad \frac{x+1}{2} = \frac{y}{3} = \frac{z-3}{6} \tag{1.3.1}$$

and the plane

$$P: \quad \begin{pmatrix}10 & 2 & -11\end{pmatrix}\mathbf{x} = 3 \tag{1.3.2}$$

Solution: The angle between the direction vector of L and normal vector of P is

$$\cos\theta = \frac{\left|\begin{pmatrix}10 & 2 & -11\end{pmatrix}\begin{pmatrix}2\\3\\6\end{pmatrix}\right|}{\sqrt{225}\times\sqrt{49}} = \frac{8}{21} \tag{1.3.3}$$

Thus, the desired angle is $90° - \theta$.

*The author is with the Department of Electrical Engineering, Indian Institute of Technology, Hyderabad 502285 India e-mail: gadepall@iith.ac.in.

1.4. Find angles between the lines

$$\begin{pmatrix}\sqrt{3} & 1\end{pmatrix}\mathbf{x} = 1 \tag{1.4.1}$$

$$\begin{pmatrix}1 & \sqrt{3}\end{pmatrix}\mathbf{x} = 1 \tag{1.4.2}$$

Solution:

1.5. Find the angle between the vectors $\mathbf{a} = \begin{pmatrix}1\\1\\-1\end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix}1\\-1\\1\end{pmatrix}$.

Solution:

1.6. Find the angle between the pair of lines given by

$$\mathbf{x} = \begin{pmatrix}3\\2\\-4\end{pmatrix} + \lambda_1\begin{pmatrix}1\\2\\2\end{pmatrix} \tag{1.6.1}$$

$$\mathbf{x} = \begin{pmatrix}5\\-2\\0\end{pmatrix} + \lambda_2\begin{pmatrix}3\\2\\6\end{pmatrix} \tag{1.6.2}$$

Solution:

1.7. Find the angle between the pair of lines

$$\frac{x+3}{3} = \frac{y-1}{5} = \frac{z+3}{4}, \tag{1.7.1}$$

$$\frac{x+1}{1} = \frac{y-4}{1} = \frac{z-5}{2} \tag{1.7.2}$$

Solution:

1.8. Find the angle between two vectors $\mathbf{a}$ and $\mathbf{b}$ where

$$\|\mathbf{a}\| = 1, \|\mathbf{b}\| = 2, \mathbf{a}^T\mathbf{b} = 1. \tag{1.8.1}$$

Solution: In Fig. **??**, from the cosine formula,

$$\cos\theta = \frac{\|\mathbf{A}-\mathbf{B}\|^2 + \|\mathbf{B}-\mathbf{C}\|^2 - \|\mathbf{A}-\mathbf{C}\|^2}{2\|\mathbf{A}-\mathbf{B}\|\|\mathbf{B}-\mathbf{C}\|} \tag{1.8.2}$$

Letting $\mathbf{a} = \mathbf{A} - \mathbf{B}, \mathbf{b} = \mathbf{B} - \mathbf{C}$,

$$\cos\theta = \frac{\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - \|\mathbf{a}+\mathbf{b}\|^2}{2\|\mathbf{a}\|\|\mathbf{b}\|} \tag{1.8.3}$$

$$= \frac{\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - \left[\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\mathbf{a}^T\mathbf{b}\right]}{2\|\mathbf{a}\|\|\mathbf{b}\|} \tag{1.8.4}$$

$$\implies \cos\theta = \frac{\mathbf{a}^T\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} \tag{1.8.5}$$

Thus, the angle θ between two vectors is given by

$$\cos\theta = \frac{\mathbf{a}^T\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} \tag{1.8.6}$$

$$= \frac{1}{2} \tag{1.8.7}$$

$$\implies \theta = 60° \tag{1.8.8}$$

1.9. Find the angle between the lines

$$\begin{pmatrix}1 & -\sqrt{3}\end{pmatrix}\mathbf{x} = 5 \tag{1.9.1}$$

$$\begin{pmatrix}\sqrt{3} & -1\end{pmatrix}\mathbf{x} = -6. \tag{1.9.2}$$

Solution: The angle between the lines can also be expressed in terms of the normal vectors as

$$\cos\theta = \frac{\mathbf{n}_1\mathbf{n}_2}{\|\mathbf{n}_1\|\|\mathbf{n}_2\|} \tag{1.9.3}$$

$$= \frac{\sqrt{3}}{2} \implies \theta = 30° \tag{1.9.4}$$

1.10. Find the angle between the planes whose equations are $\begin{pmatrix}2 & 2 & -3\end{pmatrix}\mathbf{x} = 5$ and $\begin{pmatrix}3 & -3 & 5\end{pmatrix}\mathbf{x} = 3$
Solution:

1.11. Find the angle between the following pair of lines:

$$L_1: \quad \mathbf{x} = \begin{pmatrix}3\\1\\-2\end{pmatrix} + \lambda_1\begin{pmatrix}1\\-1\\-2\end{pmatrix} \tag{1.11.1}$$

$$L_2: \quad \mathbf{x} = \begin{pmatrix}2\\-1\\-56\end{pmatrix} + \lambda_2\begin{pmatrix}3\\-5\\-4\end{pmatrix} \tag{1.11.2}$$

Solution:

1.12. Find the angle between the following pair of lines:

$$L_1: \quad \mathbf{x} = \begin{pmatrix}2\\-5\\1\end{pmatrix} + \lambda_1\begin{pmatrix}3\\2\\6\end{pmatrix} \tag{1.12.1}$$

$$L_2: \quad \mathbf{x} = \begin{pmatrix}7\\-6\\0\end{pmatrix} + \lambda_2\begin{pmatrix}1\\2\\2\end{pmatrix} \tag{1.12.2}$$

Solution:

1.13. If the coordinates of the points $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}$ be $\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\7\end{pmatrix}, \begin{pmatrix}-4\\3\\-6\end{pmatrix}, \begin{pmatrix}2\\9\\2\end{pmatrix}$, then find the angle between the lines AB and CD.
Solution:

2 Orthogonality

2.1. Find the values of p so that the lines

$$\frac{1-x}{3} = \frac{7y-14}{2p} = \frac{z-3}{2}, \tag{2.1.1}$$

$$\frac{7-7x}{3p} = \frac{y-5}{1} = \frac{6-z}{5} \tag{2.1.2}$$

are at right angles.
Solution:

2.2. Show that the lines

$$\frac{x-5}{7} = \frac{y+2}{-5} = \frac{z}{1}, \tag{2.2.1}$$

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{3} \tag{2.2.2}$$

are perpendicular to each other.
Solution:

2.3. The line through the points $\begin{pmatrix}-2\\6\end{pmatrix}$ and $\begin{pmatrix}4\\8\end{pmatrix}$ is perpendicular to the line through the points $\begin{pmatrix}8\\12\end{pmatrix}$ and $\begin{pmatrix}x\\24\end{pmatrix}$. Find the value of x.
Solution:

2.4. Show that the line joining the origin to the point $\begin{pmatrix}2\\1\\1\end{pmatrix}$ is perpendicular to the line determined by the points $\begin{pmatrix}3\\5\\-1\end{pmatrix}, \begin{pmatrix}4\\3\\-1\end{pmatrix}$.

Solution:

2.5. Are the points

$$\mathbf{A} = \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 10 \\ 20 \\ 30 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 25 \\ -41 \\ 5 \end{pmatrix}, \tag{2.5.1}$$

the vertices of a right angled triangle?
Solution:

2.6. Show that the vectors $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix}, \begin{pmatrix} 3 \\ -4 \\ -4 \end{pmatrix}$ form the vertices of a right angled triangle.
Solution:

2.7. Show that the points

$$\mathbf{A} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 3 \\ -4 \\ -4 \end{pmatrix} \tag{2.7.1}$$

are the vertices of a right angled triangle.
Solution:

2.8. In $\triangle ABC$, $\mathbf{A} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$. Find $\angle B$.
Solution:

2.9. Without using the Pythagoras theorem, show that the points $\begin{pmatrix} 4 \\ 4 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -1 \end{pmatrix}$ are the vertices of a right angled triangle.
Solution:

2.10. Show that the points

$$\mathbf{A} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 3 \\ -4 \\ -4 \end{pmatrix} \tag{2.10.1}$$

are the vertices of a right angled triangle.
Solution: The following code plots Fig. **??**

```
codes/triangle/triangle_3d.py
```

From the figure, it appears that $\triangle ABC$ is right angled at $\mathbf{C}$. Since

$$(\mathbf{A} - \mathbf{C})^T (\mathbf{B} - \mathbf{C}) = 0 \tag{2.10.2}$$

it is proved that the triangle is indeed right angled.

2.11. Show that the points $\mathbf{A} = \begin{pmatrix} 1 \\ 7 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} -4 \\ 4 \end{pmatrix}$ are the vertices of a square.
Solution: By inspection,

$$\frac{\mathbf{A} + \mathbf{C}}{2} = \frac{\mathbf{B} + \mathbf{D}}{2} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} \tag{2.11.1}$$

Hence, the diagonals AC and BD bisect each other. Also,

$$(\mathbf{A} - \mathbf{C})^T (\mathbf{B} - \mathbf{D}) = 0 \tag{2.11.2}$$

$\implies AC \perp BD$. Hence $ABCD$ is a square.

2.12. Show that the points $\mathbf{A} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 4 \\ 7 \\ 6 \end{pmatrix}$. are the vertices of a parallelogram $ABCD$ but it is not a rectangle.
Solution: Since the direction vectors

$$\mathbf{A} - \mathbf{B} = \mathbf{D} - \mathbf{C} \tag{2.12.1}$$

$$\mathbf{A} - \mathbf{D} = \mathbf{B} - \mathbf{C} \tag{2.12.2}$$

$AB \parallel CD$ and $AD \parallel BC$. Hence $ABCD$ is a parallelogram. However,

$$(\mathbf{A} - \mathbf{B})^T (\mathbf{A} - \mathbf{D}) \neq 0 \tag{2.12.3}$$

Hence, it is not a rectangle. The following code plots Fig. **??**

```
codes/triangle/quad_3d.py
```

2.13. $ABCD$ is a rectangle formed by the points $\mathbf{A} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$. $\mathbf{P}, \mathbf{Q}, \mathbf{R}, \mathbf{S}$ are the mid points of AB, BC, CD, DA respectively. Is the quadrilateral $PQRS$ a

a) square?
b) rectangle?
c) rhombus?

Solution: