Points and Vectors

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CONTENTS

***Abstract*—This book provides a computational approach to school geometry based on the NCERT textbooks from Class 6-12. Links to sample Python codes are available in the text.**

Download python codes using

```
svn co https://github.com/gadepall/school/trunk/
    ncert/computation/codes
```

1 DEFINITIONS

1.1 2×1 *vectors*

1.1.1. Let

$$\mathbf{A} \equiv \overrightarrow{A} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \tag{1.1.1.1}$$

$$\equiv a_1 \overrightarrow{i} + a_2 \overrightarrow{j}, \tag{1.1.1.2}$$

$$\mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \tag{1.1.1.3}$$

be 2×1 vectors. Then, the determinant of the 2×2 matrix

$$\mathbf{M} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} \tag{1.1.1.4}$$

is defined as

$$\left|\mathbf{M}\right| = \left|\mathbf{A} \quad \mathbf{B}\right| \tag{1.1.1.5}$$

$$= \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1 \tag{1.1.1.6}$$

1.1.2. The area of the triangle with vertices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ is given by the absolute value of

$$\frac{1}{2} \left|\mathbf{A} - \mathbf{B} \quad \mathbf{A} - \mathbf{C}\right| \tag{1.1.2.1}$$

1.1.3. The transpose of $\mathbf{A}$ is defined as

$$\mathbf{A}^\top = \begin{pmatrix} a_1 & a_2 \end{pmatrix} \tag{1.1.3.1}$$

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1.1.4. The *inner product* or *dot product* is defined as

$$\mathbf{A}^\top \mathbf{B} \equiv \mathbf{A} \cdot \mathbf{B} \tag{1.1.4.1}$$

$$= \begin{pmatrix} a_1 & a_2 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = a_1 b_1 + a_2 b_2 \tag{1.1.4.2}$$

1.1.5. *norm* of $\mathbf{A}$ is defined as

$$\|A\| \equiv \left|\overrightarrow{A}\right| \tag{1.1.5.1}$$

$$= \sqrt{\mathbf{A}^\top \mathbf{A}} = \sqrt{a_1^2 + a_2^2} \tag{1.1.5.2}$$

Thus,

$$\|\lambda \mathbf{A}\| \equiv \left|\lambda \overrightarrow{A}\right| \tag{1.1.5.3}$$

$$= |\lambda| \, \|\mathbf{A}\| \tag{1.1.5.4}$$

1.1.6. The angle between two vectors is given by

$$\theta = \cos^{-1} \frac{\mathbf{A}^\top \mathbf{B}}{\|A\| \, \|B\|} \tag{1.1.6.1}$$

1.2 3×1 *vectors*

1.2.1. Let

$$\mathbf{A} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \equiv a_1 \overrightarrow{i} + a_2 \overrightarrow{j} + a_3 \overrightarrow{j}, \tag{1.2.1.1}$$

$$\mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \tag{1.2.1.2}$$

and

$$\mathbf{A}_{ij} = \begin{pmatrix} a_i \\ a_j \end{pmatrix}, \mathbf{B}_{ij} \quad = \begin{pmatrix} b_i \\ b_j \end{pmatrix}, \tag{1.2.1.3}$$

1.2.2. The *cross product* or *vector product* of $\mathbf{A}, \mathbf{B}$ is defined as

$$\mathbf{A} \times \mathbf{B} = \begin{pmatrix} \mathbf{A}_{23} \times \mathbf{B}_{23} \\ \mathbf{A}_{31} \times \mathbf{B}_{31} \\ \mathbf{A}_{12} \times \mathbf{B}_{12} \end{pmatrix} \tag{1.2.2.1}$$

1.2.3. Verify that

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \tag{1.2.3.1}$$

1.2.4. The area of a triangle with vertices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ is given by

$$\frac{1}{2} \|(\mathbf{A} - \mathbf{B}) \times (\mathbf{A} - \mathbf{C})\| \tag{1.2.4.1}$$

2 Area of a Triangle

2.1. Find the area of a triangle whose vertices are $\mathbf{A} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -4 \\ 6 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} -3 \\ -5 \end{pmatrix}$.
Solution:

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \begin{pmatrix} -4 \\ 6 \end{pmatrix} = \begin{pmatrix} 5 \\ -7 \end{pmatrix} \quad (2.1.1)$$

$$\mathbf{A} - \mathbf{C} = \begin{pmatrix} -1 \\ 11 \end{pmatrix} \quad (2.1.2)$$

Hence, the desired area is

$$\frac{1}{2}\begin{vmatrix} 5 & -1 \\ -7 & 11 \end{vmatrix} = \frac{1}{2}(55 - 7) = 24 \quad (2.1.3)$$

2.2. Find the area of a triangle formed by the vertices $\mathbf{A} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 7 \\ -4 \end{pmatrix}$.
Solution:

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} 1 \\ -5 \end{pmatrix} \quad (2.2.1)$$

$$\mathbf{A} - \mathbf{C} = \begin{pmatrix} -2 \\ 6 \end{pmatrix} \quad (2.2.2)$$

Hence, the desired area is

$$\frac{1}{2}\begin{vmatrix} 1 & -2 \\ -5 & 6 \end{vmatrix} = \frac{1}{2}(6 - 10) = 2 \quad (2.2.3)$$

after taking the absolute value.

2.3. Find the area of a triangle formed by the points $\mathbf{P} = \begin{pmatrix} -1.5 \\ 3 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} 6 \\ -2 \end{pmatrix}, \mathbf{R} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$.
Solution:

$$\mathbf{P} - \mathbf{Q} = \begin{pmatrix} -7.5 \\ 5 \end{pmatrix} \quad (2.3.1)$$

$$\mathbf{P} - \mathbf{R} = \begin{pmatrix} 1.5 \\ -1 \end{pmatrix} \quad (2.3.2)$$

Hence, the desired area is the absolute value of

$$\frac{1}{2}\begin{vmatrix} -7.5 & 1.5 \\ 5 & -1 \end{vmatrix} = \frac{1}{2}(7.5 - 7.5) = 0 \quad (2.3.3)$$

This means that the points are on a straight line.

$$\mathbf{P} - \mathbf{Q} = \begin{pmatrix} -7.5 \\ 5 \end{pmatrix} \quad (2.3.4)$$

$$\mathbf{P} - \mathbf{R} = \begin{pmatrix} 1.5 \\ -1 \end{pmatrix} \quad (2.3.5)$$

Hence, the desired area is the absolute value of

$$\frac{1}{2}\begin{vmatrix} -7.5 & 1.5 \\ 5 & -1 \end{vmatrix} = \frac{1}{2}(7.5 - 7.5) = 0 \quad (2.3.6)$$

This means that the points are on a straight line.

2.4. Find the area of a triangle having the points

$$\mathbf{A} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \quad (2.4.1)$$

as its vertices.
Solution:

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} 0 \\ -1 \\ -2 \end{pmatrix}, \mathbf{A} - \mathbf{C} = \begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix} \quad (2.4.2)$$

and

$$\begin{vmatrix} -1 & -2 \\ -2 & 0 \end{vmatrix} = -4 \quad (2.4.3)$$

$$\begin{vmatrix} -2 & 0 \\ 0 & -1 \end{vmatrix} = 2 \quad (2.4.4)$$

$$\begin{vmatrix} 0 & -1 \\ -1 & -2 \end{vmatrix} = -1 \quad (2.4.5)$$

$$(2.4.6)$$

From (**??**),

$$\frac{1}{2}(\mathbf{A} - \mathbf{B}) \times (\mathbf{A} - \mathbf{C}) = \frac{1}{2}\begin{pmatrix} -4 \\ 2 \\ -1 \end{pmatrix} \quad (2.4.7)$$

and from (**??**), the area of the triangle is

$$\frac{1}{2}\left\|\begin{pmatrix} -4 \\ 2 \\ -1 \end{pmatrix}\right\| = \frac{1}{2}\sqrt{4^2 + 2^2 + 1^2} = \frac{1}{2}\sqrt{21} \quad (2.4.8)$$

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} 0 \\ -1 \\ -2 \end{pmatrix}, \mathbf{A} - \mathbf{C} = \begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix} \quad (2.4.9)$$

and

$$\begin{vmatrix} -1 & -2 \\ -2 & 0 \end{vmatrix} = -4 \tag{2.4.10}$$

$$\begin{vmatrix} -2 & 0 \\ 0 & -1 \end{vmatrix} = 2 \tag{2.4.11}$$

$$\begin{vmatrix} 0 & -1 \\ -1 & -2 \end{vmatrix} = -1 \tag{2.4.12}$$

$$\tag{2.4.13}$$

From (**??**),

$$\frac{1}{2}(\mathbf{A}-\mathbf{B})\times(\mathbf{A}-\mathbf{C}) = \frac{1}{2}\begin{pmatrix} -4 \\ 2 \\ -1 \end{pmatrix} \tag{2.4.14}$$

and from (**??**), the area of the triangle is

$$\frac{1}{2}\left\|\begin{pmatrix} -4 \\ 2 \\ -1 \end{pmatrix}\right\| = \frac{1}{2}\sqrt{4^2+2^2+1^2} = \frac{1}{2}\sqrt{21} \tag{2.4.15}$$

3 Angle Between Vectors

3.1. Find the angle between the vectors

$$\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \tag{3.1.1}$$

Solution: Let

$$\mathbf{a} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \tag{3.1.2}$$

Angle between the vectors is given by,

$$\theta = \cos^{-1}\left(\frac{\mathbf{a}^T\mathbf{b}}{\|\mathbf{a}\|\,\|\mathbf{b}\|}\right) \tag{3.1.3}$$

$$\|\mathbf{a}\| = \sqrt{1^2+(-2)^2+3^2} = \sqrt{14} \tag{3.1.4}$$

$$\|\mathbf{b}\| = \sqrt{3^2+(-2)^2+1^2} = \sqrt{14} \tag{3.1.5}$$

$$\mathbf{a}^T\mathbf{b} = (1)(3)+(-2)(-2)+(3)(1) = 10 \tag{3.1.6}$$

$$\theta = \cos^{-1}\left(\frac{10}{(\sqrt{14})(\sqrt{14})}\right) \tag{3.1.7}$$

$$= \cos^{-1}\left(\frac{10}{14}\right) \tag{3.1.8}$$

$$\tag{3.1.9}$$