Inner Product

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CONTENTS

***Abstract*—This manual provides an introduction to inner product applications in school geometry based on the NCERT textbooks from Class 6-12.**

1 Angle

1.1. Find the angle between the vectors

$$\begin{pmatrix}1\\-2\\3\end{pmatrix}, \begin{pmatrix}3\\-2\\1\end{pmatrix} \tag{1.1.1}$$

Solution:

1.2. Find the angle between the force $\mathbf{F} = \begin{pmatrix}3\\4\\-5\end{pmatrix}$ and displacement $\mathbf{d} = \begin{pmatrix}5\\4\\3\end{pmatrix}$.

Solution:

1.3. Let $\|\mathbf{a}\| = 3, \|\mathbf{b}\| = 4, \|\mathbf{c}\| = 5$ such that each vector is perpendicular to the other two. Find $\|\mathbf{a}+\mathbf{b}+\mathbf{c}\|$.

Solution: Given that

$$\mathbf{a}^T\mathbf{b} = \mathbf{b}^T\mathbf{c} = \mathbf{c}^T\mathbf{a} = 0. \tag{1.3.1}$$

Then,

$$\begin{aligned}\|\mathbf{a}+\mathbf{b}+\mathbf{c}\|^2 &= \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 + \|\mathbf{c}\|^2 \\ &\quad + \mathbf{a}^T\mathbf{b} + \mathbf{b}^T\mathbf{c} + \mathbf{c}^T\mathbf{a}. \end{aligned} \tag{1.3.2}$$

which reduces to

$$\|\mathbf{a}+\mathbf{b}+\mathbf{c}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 + \|\mathbf{c}\|^2 \tag{1.3.3}$$

using (1.3.1)

1.4. Given

$$\mathbf{a}+\mathbf{b}+\mathbf{c} = \mathbf{0}, \tag{1.4.1}$$

evaluate

$$\mathbf{a}^T\mathbf{b} + \mathbf{b}^T\mathbf{c} + \mathbf{c}^T\mathbf{a}, \tag{1.4.2}$$

given that $\|\mathbf{a}\| = 3, \|\mathbf{b}\| = 4$ and $\|\mathbf{c}\| = 2$.

Solution: Multiplying (1.4.1) with $\mathbf{a}, \mathbf{b}, \mathbf{c}$,

$$\|\mathbf{a}\|^2 + \mathbf{a}^T\mathbf{b} + \mathbf{a}^T\mathbf{c} = 0 \tag{1.4.3}$$

$$\mathbf{a}^T\mathbf{b} + \|\mathbf{b}\|^2 + \mathbf{b}^T\mathbf{c} = 0 \tag{1.4.4}$$

$$+\mathbf{c}^T\mathbf{a} + \mathbf{b}^T\mathbf{c} + \|\mathbf{c}\|^2 = 0 \tag{1.4.5}$$

Adding all the above equations and rearranging,

$$\mathbf{a}^T\mathbf{b}+\mathbf{b}^T\mathbf{c}+\mathbf{c}^T\mathbf{a} = -\frac{\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 + \|\mathbf{c}\|^2}{2} \tag{1.4.6}$$

1.5. Find the angle between the x-axis and the line joining the points $\begin{pmatrix}3\\-1\end{pmatrix}$ and $\begin{pmatrix}4\\-2\end{pmatrix}$. **Solution:**

1.6. Find the angle between the two planes

$$\begin{pmatrix}2 & 1 & -2\end{pmatrix}\mathbf{x} = 5 \tag{1.6.1}$$

$$\begin{pmatrix}3 & -6 & -2\end{pmatrix}\mathbf{x} = 7. \tag{1.6.2}$$

Solution: The angle between two planes is the same as the angle between their normal vectors. This can be obtained from (1.13.6).

1.7. Find the angle between the two planes

$$\begin{pmatrix}2 & 2 & -2\end{pmatrix}\mathbf{x} = 5 \tag{1.7.1}$$

$$\begin{pmatrix}3 & -6 & 2\end{pmatrix}\mathbf{x} = 7. \tag{1.7.2}$$

Solution: See Problem (1.6).

1.8. Find the angle between the line

$$L: \quad \frac{x+1}{2} = \frac{y}{3} = \frac{z-3}{6} \tag{1.8.1}$$

and the plane

$$P: \quad \begin{pmatrix}10 & 2 & -11\end{pmatrix}\mathbf{x} = 3 \tag{1.8.2}$$

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Solution: The angle between the direction vector of L and normal vector of P is

$$\cos\theta = \frac{\left|\begin{pmatrix}10 & 2 & -11\end{pmatrix}\begin{pmatrix}2\\3\\6\end{pmatrix}\right|}{\sqrt{225}\times\sqrt{49}} = \frac{8}{21} \quad (1.8.3)$$

Thus, the desired angle is $90° - \theta$.

1.9. Find angles between the lines

$$\begin{pmatrix}\sqrt{3} & 1\end{pmatrix}\mathbf{x} = 1 \quad (1.9.1)$$
$$\begin{pmatrix}1 & \sqrt{3}\end{pmatrix}\mathbf{x} = 1 \quad (1.9.2)$$

Solution:

1.10. Find the angle between the vectors $\mathbf{a} = \begin{pmatrix}1\\1\\-1\end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix}1\\-1\\1\end{pmatrix}$.

Solution:

1.11. Find the angle between the pair of lines given by

$$\mathbf{x} = \begin{pmatrix}3\\2\\-4\end{pmatrix} + \lambda_1\begin{pmatrix}1\\2\\2\end{pmatrix} \quad (1.11.1)$$
$$\mathbf{x} = \begin{pmatrix}5\\-2\\0\end{pmatrix} + \lambda_2\begin{pmatrix}3\\2\\6\end{pmatrix} \quad (1.11.2)$$

Solution:

1.12. Find the angle between the pair of lines

$$\frac{x+3}{3} = \frac{y-1}{5} = \frac{z+3}{4}, \quad (1.12.1)$$
$$\frac{x+1}{1} = \frac{y-4}{1} = \frac{z-5}{2} \quad (1.12.2)$$

Solution:

1.13. Find the angle between two vectors $\mathbf{a}$ and $\mathbf{b}$ where

$$\|\mathbf{a}\| = 1, \|\mathbf{b}\| = 2, \mathbf{a}^T\mathbf{b} = 1. \quad (1.13.1)$$

Solution: In Fig. **??**, from the cosine formula,

$$\cos\theta = \frac{\|\mathbf{A}-\mathbf{B}\|^2 + \|\mathbf{B}-\mathbf{C}\|^2 - \|\mathbf{A}-\mathbf{C}\|^2}{2\|\mathbf{A}-\mathbf{B}\|\|\mathbf{B}-\mathbf{C}\|} \quad (1.13.2)$$

Letting $\mathbf{a} = \mathbf{A}-\mathbf{B}, \mathbf{b} = \mathbf{B}-\mathbf{C}$,

$$\cos\theta = \frac{\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - \|\mathbf{a}+\mathbf{b}\|^2}{2\|\mathbf{a}\|\|\mathbf{b}\|} \quad (1.13.3)$$
$$= \frac{\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - \left[\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\mathbf{a}^T\mathbf{b}\right]}{2\|\mathbf{a}\|\|\mathbf{b}\|} \quad (1.13.4)$$
$$\implies \cos\theta = \frac{\mathbf{a}^T\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} \quad (1.13.5)$$

Thus, the angle θ between two vectors is given by

$$\cos\theta = \frac{\mathbf{a}^T\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} \quad (1.13.6)$$
$$= \frac{1}{2} \quad (1.13.7)$$
$$\implies \theta = 60° \quad (1.13.8)$$

1.14. Find the angle between the lines

$$\begin{pmatrix}1 & -\sqrt{3}\end{pmatrix}\mathbf{x} = 5 \quad (1.14.1)$$
$$\begin{pmatrix}\sqrt{3} & -1\end{pmatrix}\mathbf{x} = -6. \quad (1.14.2)$$

Solution: The angle between the lines can also be expressed in terms of the normal vectors as

$$\cos\theta = \frac{\mathbf{n}_1\mathbf{n}_2}{\|\mathbf{n}_1\|\|\mathbf{n}_2\|} \quad (1.14.3)$$
$$= \frac{\sqrt{3}}{2} \implies \theta = 30° \quad (1.14.4)$$

1.15. Find the angle between the planes whose equations are $\begin{pmatrix}2 & 2 & -3\end{pmatrix}\mathbf{x} = 5$ and $\begin{pmatrix}3 & -3 & 5\end{pmatrix}\mathbf{x} = 3$

Solution:

1.16. Find the angle between the following pair of lines:

$$L_1: \quad \mathbf{x} = \begin{pmatrix}3\\1\\-2\end{pmatrix} + \lambda_1\begin{pmatrix}1\\-1\\-2\end{pmatrix} \quad (1.16.1)$$
$$L_2: \quad \mathbf{x} = \begin{pmatrix}2\\-1\\-56\end{pmatrix} + \lambda_2\begin{pmatrix}3\\-5\\-4\end{pmatrix} \quad (1.16.2)$$

Solution:

1.17. Find the angle between the following pair of lines:

$$L_1: \quad \mathbf{x} = \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} \tag{1.17.1}$$

$$L_2: \quad \mathbf{x} = \begin{pmatrix} 7 \\ -6 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \tag{1.17.2}$$

Solution:

1.18. If the coordinates of the points $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}$ be $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 7 \end{pmatrix}, \begin{pmatrix} -4 \\ 3 \\ -6 \end{pmatrix}, \begin{pmatrix} 2 \\ 9 \\ 2 \end{pmatrix}$, then find the angle between the lines AB and CD.
Solution:

2 Orthogonality

2.1. Check whether

$$\begin{pmatrix} 5 \\ -2 \end{pmatrix}, \begin{pmatrix} 6 \\ 4 \end{pmatrix}, \begin{pmatrix} 7 \\ -2 \end{pmatrix} \tag{2.1.1}$$

are the vertices of an isosceles triangle.
Solution:

2.2. Show that each of the given three vectors is a unit vector

$$\frac{1}{7}\begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}, \frac{1}{7}\begin{pmatrix} 3 \\ -6 \\ 2 \end{pmatrix}, \frac{1}{7}\begin{pmatrix} 6 \\ 2 \\ -3 \end{pmatrix}. \tag{2.2.1}$$

Also, show that they are mutually perpendicular to each other.
Solution:

2.3. For

$$\mathbf{a} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \tag{2.3.1}$$

$(\mathbf{a} + k\mathbf{b}) \perp \mathbf{c}$. Find λ. **Solution:**

2.4. Find $\mathbf{a} \times \mathbf{b}$ if

$$\mathbf{a} = \begin{pmatrix} 1 \\ -7 \\ 7 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}. \tag{2.4.1}$$

Solution:

2.5. The scalar product of $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ with a unit vector along the sum of the vectors $\begin{pmatrix} 2 \\ 4 \\ -5 \end{pmatrix}$ and $\begin{pmatrix} \lambda \\ 2 \\ 3 \end{pmatrix}$ is unity. Find the value of λ.

2.6. The value of

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}^T \left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}^T \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}^T \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right) \tag{2.6.1}$$

is

a) 0
b) -1
c) 1
d) 3

Solution:

2.7. Show that the lines with direction vectors $\begin{pmatrix} 12 \\ -3 \\ -4 \end{pmatrix}, \begin{pmatrix} 4 \\ 12 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -4 \\ 12 \end{pmatrix}$ are mutually perpendicular.

2.8. Show that the line through the points $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix}$ is perpendicular to the line through the points $\begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \\ 6 \end{pmatrix}$.
Solution:

2.9. If $\mathbf{a} = \begin{pmatrix} 5 \\ -1 \\ -3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix}$, then show that the vectors $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$ are perpendicular.
Solution:

2.10. Find the values of p so that the lines

$$\frac{1-x}{3} = \frac{7y-14}{2p} = \frac{z-3}{2}, \tag{2.10.1}$$

$$\frac{7-7x}{3p} = \frac{y-5}{1} = \frac{6-z}{5} \tag{2.10.2}$$

are at right angles.
Solution:

2.11. Show that the lines

$$\frac{x-5}{7} = \frac{y+2}{-5} = \frac{z}{1}, \tag{2.11.1}$$

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{3} \tag{2.11.2}$$

are perpendicular to each other.
Solution:

2.12. The line through the points $\begin{pmatrix}-2\\6\end{pmatrix}$ and $\begin{pmatrix}4\\8\end{pmatrix}$ is perpendicular to the line through the points $\begin{pmatrix}8\\12\end{pmatrix}$ and $\begin{pmatrix}x\\24\end{pmatrix}$. Find the value of x.
Solution:

2.13. Show that the line joining the origin to the point $\begin{pmatrix}2\\1\\1\end{pmatrix}$ is perpendicular to the line determined by the points $\begin{pmatrix}3\\5\\-1\end{pmatrix}, \begin{pmatrix}4\\3\\-1\end{pmatrix}$.
Solution:

2.14. Are the points

$$\mathbf{A} = \begin{pmatrix}3\\6\\9\end{pmatrix}, \mathbf{B} = \begin{pmatrix}10\\20\\30\end{pmatrix}, \mathbf{C} = \begin{pmatrix}25\\-41\\5\end{pmatrix}, \tag{2.14.1}$$

the vertices of a right angled triangle?
Solution:

2.15. Show that the vectors $\begin{pmatrix}2\\-1\\1\end{pmatrix}, \begin{pmatrix}1\\-3\\-5\end{pmatrix}, \begin{pmatrix}3\\-4\\-4\end{pmatrix}$ form the vertices of a right angled triangle.
Solution:

2.16. Show that the points

$$\mathbf{A} = \begin{pmatrix}2\\-1\\1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}1\\-3\\-5\end{pmatrix}, \mathbf{C} = \begin{pmatrix}3\\-4\\-4\end{pmatrix} \tag{2.16.1}$$

are the vertices of a right angled triangle.
Solution:

2.17. In $\triangle ABC$, $\mathbf{A} = \begin{pmatrix}1\\2\\3\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-1\\0\\0\end{pmatrix}, \mathbf{C} = \begin{pmatrix}0\\1\\2\end{pmatrix}$. Find $\angle B$.
Solution:

2.18. Without using the Pythagoras theorem, show that the points $\begin{pmatrix}4\\4\end{pmatrix}$, $\begin{pmatrix}3\\5\end{pmatrix}$ and $\begin{pmatrix}-1\\-1\end{pmatrix}$ are the vertices of a right angled triangle.
Solution:

2.19. Show that the points

$$\mathbf{A} = \begin{pmatrix}2\\-1\\1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}1\\-3\\-5\end{pmatrix}, \mathbf{C} = \begin{pmatrix}3\\-4\\-4\end{pmatrix} \tag{2.19.1}$$

are the vertices of a right angled triangle.
Solution: The following code plots Fig. **??**

```
codes/triangle/triangle_3d.py
```

From the figure, it appears that $\triangle ABC$ is right angled at $\mathbf{C}$. Since

$$(\mathbf{A}-\mathbf{C})^T(\mathbf{B}-\mathbf{C}) = 0 \tag{2.19.2}$$

it is proved that the triangle is indeed right angled.

2.20. Show that the points $\mathbf{A} = \begin{pmatrix}1\\7\end{pmatrix}, \mathbf{B} = \begin{pmatrix}4\\2\end{pmatrix}, \mathbf{C} = \begin{pmatrix}-1\\-1\end{pmatrix}, \mathbf{D} = \begin{pmatrix}-4\\4\end{pmatrix}$ are the vertices of a square.
Solution: By inspection,

$$\frac{\mathbf{A}+\mathbf{C}}{2} = \frac{\mathbf{B}+\mathbf{D}}{2} = \begin{pmatrix}0\\3\end{pmatrix} \tag{2.20.1}$$

Hence, the diagonals AC and BD bisect each other. Also,

$$(\mathbf{A}-\mathbf{C})^T(\mathbf{B}-\mathbf{D}) = 0 \tag{2.20.2}$$

$\implies AC \perp BD$. Hence $ABCD$ is a square.

2.21. Show that the points $\mathbf{A} = \begin{pmatrix}1\\2\\3\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-1\\-2\\-1\end{pmatrix}, \mathbf{C} = \begin{pmatrix}2\\3\\2\end{pmatrix}, \mathbf{D} = \begin{pmatrix}4\\7\\6\end{pmatrix}$. are the vertices of a parallelogram $ABCD$ but it is not a rectangle.
Solution: Since the direction vectors

$$\mathbf{A}-\mathbf{B} = \mathbf{D}-\mathbf{C} \tag{2.21.1}$$

$$\mathbf{A}-\mathbf{D} = \mathbf{B}-\mathbf{C} \tag{2.21.2}$$

$AB \parallel CD$ and $AD \parallel BC$. Hence $ABCD$ is a parallelogram. However,

$$(\mathbf{A}-\mathbf{B})^T(\mathbf{A}-\mathbf{D}) \neq 0 \tag{2.21.3}$$

Hence, it is not a rectangle. The following code plots Fig. **??**

```
codes/triangle/quad_3d.py
```

2.22. $ABCD$ is a rectangle formed by the points $\mathbf{A} = \begin{pmatrix}-1\\-1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-1\\4\end{pmatrix}, \mathbf{C} = \begin{pmatrix}5\\4\end{pmatrix}, \mathbf{D} = \begin{pmatrix}5\\-1\end{pmatrix}$. $\mathbf{P}, \mathbf{Q}, \mathbf{R}, \mathbf{S}$ are the mid points of

AB, BC, CD, DA respectively. Is the quadrilateral $PQRS$ a

a) square?
b) rectangle?
c) rhombus?

Solution:

3 Application

3.1. Let $\boldsymbol{\alpha} = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}, \boldsymbol{\beta} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$. Find $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2$ such that $\boldsymbol{\beta} = \boldsymbol{\beta}_1 + \boldsymbol{\beta}_2, \boldsymbol{\beta}_1 \parallel \boldsymbol{\alpha}$ and $\boldsymbol{\beta}_2 \perp \boldsymbol{\alpha}$.
Solution: Let $\beta_1 = k\alpha$. Then,

$$\boldsymbol{\beta} = k\boldsymbol{\alpha} + \boldsymbol{\beta}_2 \tag{3.1.1}$$

$$\implies k = \frac{\boldsymbol{\alpha}^T\boldsymbol{\beta}}{\left\|\boldsymbol{\alpha}\right\|^2} \tag{3.1.2}$$

and

$$\boldsymbol{\beta}_2 = \boldsymbol{\beta} - k\boldsymbol{\alpha} \tag{3.1.3}$$

This process is known as *Gram-Schmidth orthogonalization*.

3.2. A body constrained to move along the z-axis of a coordinate system is subject to a constant force

$$\mathbf{F} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} \tag{3.2.1}$$

What is the work done by this force in moving the body a distance of 4 m along the z-axis ?
Solution: