Cross Product

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CONTENTS

***Abstract*—This manual provides an introduction to the cross product, based on the NCERT textbooks from Class 6-12.**

1 CROSS

1.1. Find the area of a parallelogram whose adjacent sides are determined by the vectors $\mathbf{a} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ -7 \\ 1 \end{pmatrix}$.
Solution:

1.2. The vertices of $\triangle ABC$ are $\mathbf{A} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$. A line is drawn to intersect sides AB and AC at D and E respectively, such that

$$\frac{AD}{AB} = \frac{AE}{AC} = \frac{1}{4} \tag{1.2.1}$$

Find

$$\frac{\text{area of } \triangle ADE}{\text{area of } \triangle ABC}. \tag{1.2.2}$$

Solution:

1.3. Find the scalar and vector products of the two vectors

$$\mathbf{a} = \begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} -2 \\ 1 \\ -3 \end{pmatrix} \tag{1.3.1}$$

Solution:

1.4. Find the torque of a force $\begin{pmatrix} 7 \\ 3 \\ -5 \end{pmatrix}$ about the origin. The force acts on a particle whose position vector is $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.
Solution:

1.5. Given

$$\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix}, \tag{1.5.1}$$

find $\|\mathbf{a} \times \mathbf{b}\|$.
Solution: Use (1.15.3).

1.6. Find area of the triangle with vertices at the point given in each of the following :
(i) $\begin{pmatrix} 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 6 & 0 \end{pmatrix}$, $\begin{pmatrix} 4 & 3 \end{pmatrix}$
(ii) $\begin{pmatrix} 2 & 7 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 \end{pmatrix}$, $\begin{pmatrix} 10 & 8 \end{pmatrix}$
(iii) $\begin{pmatrix} -2 & -3 \end{pmatrix}$, $\begin{pmatrix} 3 & 2 \end{pmatrix}$, $\begin{pmatrix} -1 & -8 \end{pmatrix}$
Solution:
a)

1.7. Find values of k if area of triangle is 4sq.units and vertices are
(i)) $\begin{pmatrix} k & 0 \end{pmatrix}$, $\begin{pmatrix} 4 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 2 \end{pmatrix}$
(ii) $\begin{pmatrix} -2 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 4 \end{pmatrix}$, $\begin{pmatrix} 0 & k \end{pmatrix}$

1.8. If the area of triangle is 35 sq.units with vertices $\begin{pmatrix} 2 & -6 \end{pmatrix}$, $\begin{pmatrix} 5 & 4 \end{pmatrix}$ and $\begin{pmatrix} k & 4 \end{pmatrix}$.then k is
a) 12
b) -2
c) -12,-2
d) 12,-2
Solution:

1.9. Find the area of a triangle having the points $\mathbf{A} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ as its vertices.
Solution:

1.10. Find the area of a triangle with vertices $\mathbf{A} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} 1 \\ 5 \\ 5 \end{pmatrix}$
Solution:

1.11. Find the area of the triangle whose vertices are

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a) $\begin{pmatrix}2\\3\end{pmatrix}, \begin{pmatrix}-1\\0\end{pmatrix}, \begin{pmatrix}2\\-4\end{pmatrix}$

b) $\begin{pmatrix}-5\\-1\end{pmatrix}, \begin{pmatrix}3\\-5\end{pmatrix}, \begin{pmatrix}5\\2\end{pmatrix}$

Solution:

1.12. Find the area of the triangle formed by joining the mid points of the sides of a triangle whose vertices are $\begin{pmatrix}0\\-1\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}0\\3\end{pmatrix}$.

Solution:

1.13. Verify that the median of $\triangle ABC$ with vertices $\mathbf{A} = \begin{pmatrix}4\\-6\end{pmatrix}, \mathbf{B} = \begin{pmatrix}3\\-2\end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix}5\\2\end{pmatrix}$ divides it into two triangles of equal areas.

Solution:

1.14. Find the area of a triangle whose vertices are $\mathbf{A} = \begin{pmatrix}1\\-1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-4\\6\end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix}-3\\-5\end{pmatrix}$.

Solution: Using Hero's formula, the following code computes the area of the triangle as 24.

```
codes/triangle/area_tri.py
```

1.15. Find the area of a triangle formed by the vertices $\mathbf{A} = \begin{pmatrix}5\\2\end{pmatrix}, \mathbf{B} = \begin{pmatrix}4\\7\end{pmatrix}, \mathbf{C} = \begin{pmatrix}7\\-4\end{pmatrix}$.

Solution: The area of $\triangle ABC$ is also obtained in terms of the *magnitude* of the determinant of the matrix $\mathbf{M}$ in (**??**) as

$$\frac{1}{2}\left|\mathbf{M}\right| \tag{1.15.1}$$

The computation is done in **area_tri.py**

1.16. Find the area of a triangle formed by the points $\mathbf{P} = \begin{pmatrix}-1.5\\3\end{pmatrix}, \mathbf{Q} = \begin{pmatrix}6\\-2\end{pmatrix}, \mathbf{R} = \begin{pmatrix}-3\\4\end{pmatrix}$.

Solution: Another formula for the area of $\triangle ABC$ is

$$\frac{1}{2}\begin{vmatrix}1 & 1 & 1\\ \mathbf{A} & \mathbf{B} & \mathbf{C}\end{vmatrix} \tag{1.16.1}$$

1.17. Find the area of a triangle having the points

$$\mathbf{A} = \begin{pmatrix}1\\1\\1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}1\\2\\3\end{pmatrix}, \mathbf{C} = \begin{pmatrix}2\\3\\1\end{pmatrix} \tag{1.17.1}$$

as its vertices.

Solution: The area of a triangle using the *vector product* is obtained as

$$\frac{1}{2}\left\|(\mathbf{B}-\mathbf{A}) \times (\mathbf{C}-\mathbf{A})\right\| \tag{1.17.2}$$

For any two vectors $\mathbf{a} = \begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}, \mathbf{b} = \begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$,

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix}0 & -a_3 & a_2\\ a_3 & 0 & -a_1\\ -a_2 & a_1 & 0\end{pmatrix}\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix} \tag{1.17.3}$$

The following code computes the area using the vector product.

```
codes/triangle/area_tri_vec.py
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1.18. Find the area of a parallelogram whose adjacent sides are given by the vectors $\begin{pmatrix}3\\1\\4\end{pmatrix}$ and $\begin{pmatrix}1\\-1\\1\end{pmatrix}$.

Solution: The area is given by

$$\frac{1}{2}\left\|\begin{pmatrix}3\\1\\4\end{pmatrix} \times \begin{pmatrix}1\\-1\\1\end{pmatrix}\right\| \tag{1.18.1}$$

1.19. Draw a quadrilateral in the Cartesian plane, whose vertices are $\begin{pmatrix}-4\\5\end{pmatrix}, \begin{pmatrix}0\\7\end{pmatrix}, \begin{pmatrix}5\\-5\end{pmatrix}$ and $\begin{pmatrix}-4\\-2\end{pmatrix}$. Also, find its area.

Solution:

1.20. Find the area of a rhombus if its vertices are

$$\mathbf{P} = \begin{pmatrix}3\\0\end{pmatrix}, \mathbf{Q} = \begin{pmatrix}4\\5\end{pmatrix}, \tag{1.20.1}$$

$$\mathbf{R} = \begin{pmatrix}-1\\4\end{pmatrix}, \mathbf{S} = \begin{pmatrix}-2\\-1\end{pmatrix} \tag{1.20.2}$$

taken in order.

Solution:

1.21. Find the area of the quadrilateral whose vertices, taken in order, are $\begin{pmatrix}-4\\2\end{pmatrix}, \begin{pmatrix}-3\\-5\end{pmatrix}, \begin{pmatrix}3\\-2\end{pmatrix}, \begin{pmatrix}2\\3\end{pmatrix}$.

Solution:

1.22. Find the area of a parallelogram whose adjacent sides are given by the vectors $\begin{pmatrix}3\\1\\4\end{pmatrix}$ and $\begin{pmatrix}1\\-1\\1\end{pmatrix}$.

Solution:

1.23. Find the area of a rectangle $ABCD$ with vertices $\mathbf{A} = \begin{pmatrix} -1 \\ \frac{1}{2} \\ 4 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ \frac{1}{2} \\ 4 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ 4 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} -1 \\ -\frac{1}{2} \\ 4 \end{pmatrix}$.

Solution:

1.24. The two adjacent sides of a parallelogram are $\begin{pmatrix} 2 \\ -4 \\ -5 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}$. Find the unit vector parallel to its diagonal. Also, find its area.

Solution:

1.25. If $\mathbf{A} = \begin{pmatrix} -5 \\ 7 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -4 \\ -5 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -1 \\ -6 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$, find the area of the quadrilateral $ABCD$.

Solution: The area of $ABCD$ is the sum of the areas of trianges ABD and CBD and is given by

$$\frac{1}{2}\left\|(\mathbf{A}-\mathbf{B})\times(\mathbf{A}-\mathbf{D})\right\| + \frac{1}{2}\left\|(\mathbf{C}-\mathbf{B})\times(\mathbf{C}-\mathbf{D})\right\| \quad (1.25.1)$$