Vector Properties

G V V Sharma*

CONTENTS

***Abstract*—This book provides a computational approach to school geometry based on the NCERT textbooks from Class 6-12. Links to sample Python codes are available in the text.**

1 Direction Vector

1.1. If $\mathbf{a} = \begin{pmatrix}1\\1\\1\end{pmatrix}, \mathbf{b} = \begin{pmatrix}2\\-1\\3\end{pmatrix}, \mathbf{c} = \begin{pmatrix}1\\-2\\1\end{pmatrix}$, find a unit vector parallel to the vector $2\mathbf{a} - \mathbf{b} + 3\mathbf{c}$.
Solution:

1.2. Find a vector of magnitude 5 units, and parallel to the resultant of the vectors $\mathbf{a} = \begin{pmatrix}2\\3\\-1\end{pmatrix}, \mathbf{b} = \begin{pmatrix}1\\-2\\1\end{pmatrix}$,
Solution:

1.3. Show that the unit direction vector inclined equally to the coordinate axes is $\begin{pmatrix}\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\end{pmatrix}$.
Solution:

1.4. Find a unit vector that makes an angle of $90°, 135°$ and $45°$ with the positive x, y and z axis respectively. **Solution:**

1.5. Show that the line through the points $\begin{pmatrix}4\\7\\8\end{pmatrix}$, $\begin{pmatrix}2\\3\\4\end{pmatrix}$ is parallel to the line through the points $\begin{pmatrix}-1\\-2\\1\end{pmatrix}, \begin{pmatrix}1\\2\\5\end{pmatrix}$.
Solution:

*The author is with the Department of Electrical Engineering, Indian Institute of Technology, Hyderabad 502285 India e-mail: gadepall@iith.ac.in. All content in this manual is released under GNU GPL. Free and open source.

1.6. Find a vector $\mathbf{x}$ in the direction of $\begin{pmatrix}1\\-2\end{pmatrix}$ such that $\|\mathbf{x}\| = 7$. **Solution:** Let $\mathbf{x} = k\begin{pmatrix}1\\-2\end{pmatrix}$. Then

$$\|\mathbf{x}\| = |k|\left\|\begin{pmatrix}1\\-2\end{pmatrix}\right\| = 7 \tag{1.6.1}$$

$$\implies |k| = \frac{7}{\sqrt{5}} \tag{1.6.2}$$

$$\text{or, } \mathbf{x} = \frac{7}{\sqrt{5}}\begin{pmatrix}1\\-2\end{pmatrix} \tag{1.6.3}$$

1.7. Find the direction vector of PQ, where

$$\mathbf{P} = \begin{pmatrix}2\\3\\0\end{pmatrix}, \mathbf{Q} = \begin{pmatrix}-1\\-2\\-4\end{pmatrix} \tag{1.7.1}$$

Solution: The direction vector of PQ is

$$\mathbf{P} - \mathbf{Q} = \begin{pmatrix}3\\5\\4\end{pmatrix}, \tag{1.7.2}$$

1.8. Find the slope of a line, which passes through the origin, and the mid-point of the line segment joining the points $\mathbf{P} = \begin{pmatrix}0\\-4\end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix}8\\0\end{pmatrix}$. **Solution:**

1.9. The slope of a line is double of the slope of another line. If the tangent of the angle between them is $\frac{1}{3}$, find the slopes of the lines. **Solution:**

1.10. Find a unit vector that makes an angle of $90°, 60°$ and $30°$ with the positive x, y and z axis respectively.
Solution: The direction vector is

$$\mathbf{x} = \begin{pmatrix}\cos 90°\\\cos 60°\\\cos 30°\end{pmatrix} = \begin{pmatrix}0\\\frac{1}{2}\\\frac{\sqrt{3}}{2}\end{pmatrix} \tag{1.10.1}$$

$\because \|\mathbf{x}\| = 1$, it is the desired unit vector.

1.11. Find the direction vectors and slopes of the lines passing through the points

a) $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 4 \end{pmatrix}$.
b) $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 7 \\ -2 \end{pmatrix}$.
c) $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$.
d) Making an inclination of $60°$ with the positive direction of the x-axis.

Solution:

a) If the direction vector is

$$\begin{pmatrix} 1 \\ m \end{pmatrix}, \tag{1.11.1}$$

the slope is m. Thus, the direction vector is

$$\begin{pmatrix} -1 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} -4 \\ 6 \end{pmatrix} = -\frac{1}{4}\begin{pmatrix} -4 \\ 6 \end{pmatrix} \tag{1.11.2}$$

$$= \begin{pmatrix} 1 \\ -\frac{3}{2} \end{pmatrix} \implies m = -\frac{3}{2} \tag{1.11.3}$$

b) The direction vector is

$$\begin{pmatrix} 7 \\ -2 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \tag{1.11.4}$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \implies m = 0 \tag{1.11.5}$$

c) The direction vector is

$$\begin{pmatrix} 3 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \end{pmatrix} \tag{1.11.6}$$

$$= \begin{pmatrix} 1 \\ \infty \end{pmatrix} \implies m = \infty \tag{1.11.7}$$

d) The slope is $m = \tan 60° = \sqrt{3}$ and the direction vector is

$$\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} \tag{1.11.8}$$

1.12. If the angle between two lines is $\frac{\pi}{4}$ and the slope of one of the lines is $\frac{1}{4}$ find the slope of the other line.

Solution: The angle θ between two lines is given by

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2} \tag{1.12.1}$$

$$\implies 1 = \frac{m_1 - \frac{1}{4}}{1 + \frac{m_1}{4}} \tag{1.12.2}$$

$$\text{or } m_1 = \frac{5}{3} \tag{1.12.3}$$

1.13. If the points $\mathbf{A} = \begin{pmatrix} 6 \\ 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 8 \\ 2 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 9 \\ 4 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} p \\ 3 \end{pmatrix}$ are the vertices of a parallelogram, taken in order, find the value of p.

Solution: In the parallelogram $ABCD$, AC and BD bisect each other. This can be used to find p.

1.14. Without using distance formula, show that points $\begin{pmatrix} -2 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 4 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$ are the vertices of a parallelogram.

Solution:

1.15. The two opposite vertices of a square are $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$. Find the coordinates of the other two vertices.

Solution:

1.16. Find the direction vectors of the sides of a triangle with vertices $\mathbf{A} = \begin{pmatrix} 3 \\ 5 \\ -4 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} -5 \\ -5 \\ -2 \end{pmatrix}$

Solution:

1.17. Find a unit vector in the direction of

$$\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}. \tag{1.17.1}$$

Solution:

1.18. Find a unit vector in the direction of $\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$.

Solution:

1.19. Find a unit vector in the direction of the line passing through $\begin{pmatrix} -2 \\ 4 \\ -5 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

Solution:

2 Norm

2.1. Find a point on the x-axis, which is equidistant from the points $\begin{pmatrix}7\\6\end{pmatrix}$ and $\begin{pmatrix}3\\4\end{pmatrix}$.
Solution:

2.2. Write down a unit vector in the xy-plane, makeing an angle of $30°$ with the positive direction of the x-axis.
Solution:

2.3. Find the value of x for which $x\begin{pmatrix}1\\1\\1\end{pmatrix}$ is a unit vector.
Solution:

2.4. If

$$\mathbf{a} = \begin{pmatrix}1\\2\end{pmatrix}, \mathbf{b} = \begin{pmatrix}2\\1\end{pmatrix}, \tag{2.4.1}$$

verify if
a) $\|\mathbf{a}\| = \|\mathbf{b}\|$
b) $\mathbf{a} = \mathbf{b}$
Solution:
a) $\|\mathbf{a}\| = \|\mathbf{b}\|, \mathbf{a} \neq \mathbf{b}$.

2.5. Find a unit vector in the direction of $\begin{pmatrix}2\\3\\1\end{pmatrix}$.
Solution: The unit vector is given by

$$\frac{\begin{pmatrix}2\\3\\1\end{pmatrix}}{\left\|\begin{pmatrix}2\\3\\1\end{pmatrix}\right\|} = \frac{1}{\sqrt{14}}\begin{pmatrix}2\\3\\1\end{pmatrix} \tag{2.5.1}$$

2.6. Find the distance between the points

$$\mathbf{P} = \begin{pmatrix}1\\-3\\4\end{pmatrix}, \mathbf{Q} = \begin{pmatrix}-4\\1\\2\end{pmatrix} \tag{2.6.1}$$

Solution:
The distance is given by $\|\mathbf{P} - \mathbf{Q}\|$

2.7. Find $\|\mathbf{a} - \mathbf{b}\|$, if

$$\|\mathbf{a}\| = 2, \|\mathbf{b}\| = 3, \mathbf{a}^T\mathbf{b} = 4. \tag{2.7.1}$$

Solution:

2.8. If $\mathbf{a}$ is a unit vector and

$$(\mathbf{x} - \mathbf{a})(\mathbf{x} + \mathbf{a}) = 8, \tag{2.8.1}$$

then find $\mathbf{x}$.
Solution:

$$(\mathbf{x} - \mathbf{a})(\mathbf{x} + \mathbf{a}) = \|\mathbf{x}\|^2 - \|\mathbf{a}\|^2 \tag{2.8.2}$$

$$\implies \|\mathbf{x}\|^2 = 9 \text{ or, } \|\mathbf{x}\| = 3. \tag{2.8.3}$$

2.9. Find the point on the x-axis which is equidistant from

$$\begin{pmatrix}2\\-5\end{pmatrix}, \begin{pmatrix}-2\\9\end{pmatrix}, \tag{2.9.1}$$

Solution:

2.10. Find the values of y for which the distance between the points

$$\mathbf{P} = \begin{pmatrix}2\\-3\end{pmatrix}, \mathbf{Q} = \begin{pmatrix}10\\y\end{pmatrix} \tag{2.10.1}$$

is 10 units. **Solution:**

2.11. A town B is located 36km east and 15 km north of the town A. How would you find the distance from town A to town B without actually measuring it?
Solution:

2.12. Find the unit normal vector of the plane

$$\begin{pmatrix}6 & -3 & -2\end{pmatrix}\mathbf{x} = 1. \tag{2.12.1}$$

Solution: The normal vector is

$$\mathbf{n} = \begin{pmatrix}6 & -3 & -2\end{pmatrix} \tag{2.12.2}$$

$$\because \|\mathbf{n}\| = 7, \tag{2.12.3}$$

the unit normal vector is

$$\frac{\mathbf{n}}{\|\mathbf{n}\|} = \frac{1}{7}\begin{pmatrix}6 & -3 & -2\end{pmatrix} \tag{2.12.4}$$

2.13. Find the condition for $\mathbf{x} = \begin{pmatrix}x_1\\x_2\end{pmatrix}$ to be equidistant from the points $\begin{pmatrix}7\\1\end{pmatrix}, \begin{pmatrix}3\\5\end{pmatrix}$.
Solution: From the given information,

$$\left\|\mathbf{x} - \begin{pmatrix}7\\1\end{pmatrix}\right\|^2 = \left\|\mathbf{x} - \begin{pmatrix}3\\5\end{pmatrix}\right\|^2 \tag{2.13.1}$$

$$\implies \|\mathbf{x}\|^2 + \left\|\begin{pmatrix}7\\1\end{pmatrix}\right\|^2 - 2\begin{pmatrix}7 & 1\end{pmatrix}\mathbf{x}$$

$$= \|\mathbf{x}\|^2 + \left\|\begin{pmatrix}3\\5\end{pmatrix}\right\|^2 - 2\begin{pmatrix}3 & 5\end{pmatrix}\mathbf{x} \tag{2.13.2}$$

which can be simplified to obtain

$$\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = 2 \tag{2.13.3}$$

which is the desired condition. The following code plots Fig. **??**clearly showing that the above equation is the perpendicular bisector of AB.

```
codes/line/line_perp_bisect.py
```

2.14. Find a point on the y-axis which is equidistant from the points $\mathbf{A} = \begin{pmatrix}6\\5\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-4\\3\end{pmatrix}$.
Solution:

2.15. Find the equation of set of points $\mathbf{P}$ such that

$$PA^2 + PB^2 = 2k^2, \tag{2.15.1}$$

$$\mathbf{A} = \begin{pmatrix}3\\4\\5\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-1\\3\\-7\end{pmatrix}, \tag{2.15.2}$$

respectively. **Solution:**

2.16. Find the equation of the set of points $\mathbf{P}$ such that its distances from the points $\mathbf{A} = \begin{pmatrix}3\\4\\-5\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-2\\1\\4\end{pmatrix}$ are equal.
Solution:

3 Section

3.1. Draw a line segement of length 7.6 cm and divide it in the ratio $5 : 8$.
Solution: Let the end points of the line be

$$\mathbf{A} = \begin{pmatrix}0\\0\end{pmatrix}, \mathbf{B} = \begin{pmatrix}7.6\\0\end{pmatrix} \tag{3.1.1}$$

Using section formula, the point $\mathbf{C}$

$$\mathbf{C} = \frac{k\mathbf{B} + \mathbf{A}}{k+1} \tag{3.1.2}$$

If $\mathbf{C}$ divides AB in the ratio

$$m = \frac{5}{8}, \tag{3.1.3}$$

then,

$$\frac{\|\mathbf{C} - \mathbf{A}\|^2}{\|\mathbf{B} - \mathbf{C}\|^2} = m^2 \tag{3.1.4}$$

$$\implies \frac{\frac{k^2\|\mathbf{B}-\mathbf{A}\|^2}{(k+1)^2}}{\frac{\|\mathbf{B}-\mathbf{A}\|^2}{(k+1)^2}} = m^2 \tag{3.1.5}$$

$$\implies k = m \tag{3.1.6}$$

upon substituting from (3.1.4) and simplifying. (3.1.2) is known as the section formula. The following code plots Fig. **??**

```
codes/line/draw_section.py
```

3.2. Find the coordinates of the point which divides the line segment joining the points $\begin{pmatrix}4\\-3\end{pmatrix}$ and $\begin{pmatrix}8\\5\end{pmatrix}$ in the ratio $3 : 1$ internally.
Solution: Using (3.1.2), the desired point is

$$\mathbf{P} = \frac{3\begin{pmatrix}4\\-3\end{pmatrix} + \begin{pmatrix}8\\5\end{pmatrix}}{4} \tag{3.2.1}$$

3.3. In what ratio does the point $\begin{pmatrix}-4\\6\end{pmatrix}$ divide the line segment joining the points

$$\mathbf{A} = \begin{pmatrix}-6\\10\end{pmatrix}, \mathbf{B} = \begin{pmatrix}3\\-8\end{pmatrix} \tag{3.3.1}$$

Solution: Use (3.1.2).

3.4. Find the coordinates of the points of trisection of the line segement joining the points

$$\mathbf{A} = \begin{pmatrix}2\\-2\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-7\\4\end{pmatrix} \tag{3.4.1}$$

Solution: Using (3.1.2), the coordinates are

$$\mathbf{P} = \frac{2\mathbf{A} + \mathbf{B}}{3} \tag{3.4.2}$$

$$\mathbf{Q} = \frac{\mathbf{A} + 2\mathbf{B}}{3} \tag{3.4.3}$$

3.5. Find the ratio in which the y-axis divides the line segment joining the points $\begin{pmatrix}5\\-6\end{pmatrix}$ and $\begin{pmatrix}-1\\-4\end{pmatrix}$.
Solution: Let the corresponding point on the y-axis be $\begin{pmatrix}0\\y\end{pmatrix}$. If the ratio be $k : 1$, using (3.1.2), the coordinates are

$$\begin{pmatrix}0\\y\end{pmatrix} = k\begin{pmatrix}5\\-6\end{pmatrix} + \begin{pmatrix}-1\\-4\end{pmatrix} \tag{3.5.1}$$

$$\implies 0 = 5k - 1 \implies k = \frac{1}{5} \tag{3.5.2}$$

4 Projection

4.1. Find the projection of the vector

$$\begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix} \tag{4.1.1}$$

on the vector

$$\begin{pmatrix} 7 \\ -1 \\ 8 \end{pmatrix} \tag{4.1.2}$$

Solution:

4.2. Find the projection of the vector

$$\mathbf{a} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} \tag{4.2.1}$$

on the vector

$$\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}. \tag{4.2.2}$$

Solution: The projection of $\mathbf{a}$ on $\mathbf{b}$ is shown in Fig. **??**. It has magnitude $\|\mathbf{a}\| \cos\theta$ and is in the direction of $\mathbf{b}$. Thus, the projection is defined as

$$\left(\|\mathbf{a}\| \cos\theta\right) \frac{\mathbf{b}}{\|\mathbf{b}\|} = \frac{\left(\mathbf{a}^T\mathbf{b}\right) \|\mathbf{a}\|}{\|\mathbf{b}\|}\mathbf{b} \tag{4.2.3}$$