Vector Properties

G V V Sharma*

CONTENTS

***Abstract*—This book provides a computational approach to school geometry based on the NCERT textbooks from Class 6-12. Links to sample Python codes are available in the text.**

1 Direction Vector

1.1. Find a unit vector in the direction of $\mathbf{A}+\mathbf{B}$, where

$$\mathbf{A}=\begin{pmatrix}2\\2\\-5\end{pmatrix}, \mathbf{B}=\begin{pmatrix}2\\1\\3\end{pmatrix}. \tag{1.1.1}$$

Solution:

1.2. If $\mathbf{a}=\begin{pmatrix}1\\1\\1\end{pmatrix}, \mathbf{b}=\begin{pmatrix}2\\-1\\3\end{pmatrix}, \mathbf{c}=\begin{pmatrix}1\\-2\\1\end{pmatrix}$, find a unit vector parallel to the vector $2\mathbf{a}-\mathbf{b}+3\mathbf{c}$. **Solution:**

1.3. Find a vector of magnitude 5 units, and parallel to the resultant of the vectors $\mathbf{a}=\begin{pmatrix}2\\3\\-1\end{pmatrix}, \mathbf{b}=\begin{pmatrix}1\\-2\\1\end{pmatrix}$, **Solution:**

1.4. Show that the unit direction vector inclined equally to the coordinate axes is $\begin{pmatrix}\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\end{pmatrix}$. **Solution:**

1.5. Find a unit vector that makes an angle of $90°, 135°$ and $45°$ with the positive x, y and z axis respectively. **Solution:**

1.6. Show that the line through the points $\begin{pmatrix}4\\7\\8\end{pmatrix}$, $\begin{pmatrix}2\\3\\4\end{pmatrix}$ is parallel to the line through the points $\begin{pmatrix}-1\\-2\\1\end{pmatrix}, \begin{pmatrix}1\\2\\5\end{pmatrix}$. **Solution:**

1.7. Find a vector $\mathbf{x}$ in the direction of $\begin{pmatrix}1\\-2\end{pmatrix}$ such that $\|\mathbf{x}\|=7$. **Solution:** Let $\mathbf{x}=k\begin{pmatrix}1\\-2\end{pmatrix}$. Then

$$\|\mathbf{x}\|=|k|\left\|\begin{pmatrix}1\\-2\end{pmatrix}\right\|=7 \tag{1.7.1}$$

$$\implies |k|=\frac{7}{\sqrt{5}} \tag{1.7.2}$$

$$\text{or, } \mathbf{x}=\frac{7}{\sqrt{5}}\begin{pmatrix}1\\-2\end{pmatrix} \tag{1.7.3}$$

1.8. Find the direction vector of PQ, where

$$\mathbf{P}=\begin{pmatrix}2\\3\\0\end{pmatrix}, \mathbf{Q}=\begin{pmatrix}-1\\-2\\-4\end{pmatrix} \tag{1.8.1}$$

Solution: The direction vector of PQ is

$$\mathbf{P}-\mathbf{Q}=\begin{pmatrix}3\\5\\4\end{pmatrix}, \tag{1.8.2}$$

1.9. Find the slope of a line, which passes through the origin, and the mid-point of the line segment joining the points $\mathbf{P}=\begin{pmatrix}0\\-4\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}8\\0\end{pmatrix}$. **Solution:**

1.10. The slope of a line is double of the slope of another line. If the tangent of the angle between them is $\frac{1}{3}$, find the slopes of the lines. **Solution:**

1.11. Find a unit vector that makes an angle of $90°, 60°$ and $30°$ with the positive x, y and z

*The author is with the Department of Electrical Engineering, Indian Institute of Technology, Hyderabad 502285 India e-mail: gadepall@iith.ac.in.

axis respectively.

Solution: The direction vector is

$$\mathbf{x} = \begin{pmatrix} \cos 90^\circ \\ \cos 60^\circ \\ \cos 30^\circ \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} \tag{1.11.1}$$

$\because \|\mathbf{x}\| = 1$, it is the desired unit vector.

1.12. Find the direction vectors and slopes of the lines passing through the points

a) $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 4 \end{pmatrix}$.

b) $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 7 \\ -2 \end{pmatrix}$.

c) $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

d) Making an inclination of 60° with the positive direction of the x-axis.

Solution:

a) If the direction vector is

$$\begin{pmatrix} 1 \\ m \end{pmatrix}, \tag{1.12.1}$$

the slope is m. Thus, the direction vector is

$$\begin{pmatrix} -1 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} -4 \\ 6 \end{pmatrix} = -\frac{1}{4}\begin{pmatrix} -4 \\ 6 \end{pmatrix} \tag{1.12.2}$$

$$= \begin{pmatrix} 1 \\ -\frac{3}{2} \end{pmatrix} \implies m = -\frac{3}{2} \tag{1.12.3}$$

b) The direction vector is

$$\begin{pmatrix} 7 \\ -2 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \tag{1.12.4}$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \implies m = 0 \tag{1.12.5}$$

c) The direction vector is

$$\begin{pmatrix} 3 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \end{pmatrix} \tag{1.12.6}$$

$$= \begin{pmatrix} 1 \\ \infty \end{pmatrix} \implies m = \infty \tag{1.12.7}$$

d) The slope is $m = \tan 60^\circ = \sqrt{3}$ and the direction vector is

$$\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} \tag{1.12.8}$$

1.13. If the angle between two lines is $\frac{\pi}{4}$ and the slope of one of the lines is $\frac{1}{4}$ find the slope of the other line.

Solution: The angle θ between two lines is given by

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2} \tag{1.13.1}$$

$$\implies 1 = \frac{m_1 - \frac{1}{4}}{1 + \frac{m_1}{4}} \tag{1.13.2}$$

$$\text{or } m_1 = \frac{5}{3} \tag{1.13.3}$$

1.14. If the points $\mathbf{A} = \begin{pmatrix} 6 \\ 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 8 \\ 2 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 9 \\ 4 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} p \\ 3 \end{pmatrix}$ are the vertices of a parallelogram, taken in order, find the value of p.

Solution: In the parallelogram $ABCD$, AC and BD bisect each other. This can be used to find p.

1.15. Without using distance formula, show that points $\begin{pmatrix} -2 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 4 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$ are the vertices of a parallelogram.

Solution:

1.16. The two opposite vertices of a square are $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$. Find the coordinates of the other two vertices.

Solution:

1.17. Find the direction vectors of the sides of a triangle with vertices $\mathbf{A} = \begin{pmatrix} 3 \\ 5 \\ -4 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} -5 \\ -5 \\ -2 \end{pmatrix}$

Solution:

1.18. Find a unit vector in the direction of

$$\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}. \tag{1.18.1}$$

Solution:

1.19. Find a unit vector in the direction of $\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$.

Solution:

1.20. Find a unit vector in the direction of the line passing through $\begin{pmatrix}-2\\4\\-5\end{pmatrix}$ and $\begin{pmatrix}1\\2\\3\end{pmatrix}$.
Solution:

2 Norm

2.1. Find a point on the x-axis, which is equidistant from the points $\begin{pmatrix}7\\6\end{pmatrix}$ and $\begin{pmatrix}3\\4\end{pmatrix}$.
Solution:

2.2. Write down a unit vector in the xy-plane, makeing an angle of 30° with the positive direction of the x-axis.
Solution:

2.3. Find the value of x for which $x\begin{pmatrix}1\\1\\1\end{pmatrix}$ is a unit vector.
Solution:

2.4. If

$$\mathbf{a} = \begin{pmatrix}1\\2\end{pmatrix}, \mathbf{b} = \begin{pmatrix}2\\1\end{pmatrix}, \tag{2.4.1}$$

verify if

a) $\|\mathbf{a}\| = \|\mathbf{b}\|$
b) $\mathbf{a} = \mathbf{b}$

Solution:

a) $\|\mathbf{a}\| = \|\mathbf{b}\|, \mathbf{a} \neq \mathbf{b}$.

2.5. Find a unit vector in the direction of $\begin{pmatrix}2\\3\\1\end{pmatrix}$.
Solution: The unit vector is given by

$$\frac{\begin{pmatrix}2\\3\\1\end{pmatrix}}{\left\|\begin{pmatrix}2\\3\\1\end{pmatrix}\right\|} = \frac{1}{\sqrt{14}}\begin{pmatrix}2\\3\\1\end{pmatrix} \tag{2.5.1}$$

2.6. Find the distance between the points

$$\mathbf{P} = \begin{pmatrix}1\\-3\\4\end{pmatrix}, \mathbf{Q} = \begin{pmatrix}-4\\1\\2\end{pmatrix} \tag{2.6.1}$$

Solution:
The distance is given by $\|\mathbf{P} - \mathbf{Q}\|$

2.7. Find $\|\mathbf{a} - \mathbf{b}\|$, if

$$\|\mathbf{a}\| = 2, \|\mathbf{b}\| = 3, \mathbf{a}^T\mathbf{b} = 4. \tag{2.7.1}$$

Solution:

2.8. If $\mathbf{a}$ is a unit vector and

$$(\mathbf{x} - \mathbf{a})(\mathbf{x} + \mathbf{a}) = 8, \tag{2.8.1}$$

then find $\mathbf{x}$.
Solution:

$$(\mathbf{x} - \mathbf{a})(\mathbf{x} + \mathbf{a}) = \|\mathbf{x}\|^2 - \|\mathbf{a}\|^2 \tag{2.8.2}$$

$$\implies \|\mathbf{x}\|^2 = 9 \text{ or, } \|\mathbf{x}\| = 3. \tag{2.8.3}$$

2.9. Find the point on the x-axis which is equidistant from

$$\begin{pmatrix}2\\-5\end{pmatrix}, \begin{pmatrix}-2\\9\end{pmatrix}, \tag{2.9.1}$$

Solution:

2.10. Find the values of y for which the distance between the points

$$\mathbf{P} = \begin{pmatrix}2\\-3\end{pmatrix}, \mathbf{Q} = \begin{pmatrix}10\\y\end{pmatrix} \tag{2.10.1}$$

is 10 units. **Solution:**

2.11. A town B is located 36km east and 15 km north of the town A. How would you find the distance from town A to town B without actually measuring it?
Solution:

2.12. Find the unit normal vector of the plane

$$\begin{pmatrix}6 & -3 & -2\end{pmatrix}\mathbf{x} = 1. \tag{2.12.1}$$

Solution: The normal vector is

$$\mathbf{n} = \begin{pmatrix}6 & -3 & -2\end{pmatrix} \tag{2.12.2}$$

$$\because \|\mathbf{n}\| = 7, \tag{2.12.3}$$

the unit normal vector is

$$\frac{\mathbf{n}}{\|\mathbf{n}\|} = \frac{1}{7}\begin{pmatrix}6 & -3 & -2\end{pmatrix} \tag{2.12.4}$$

2.13. Find the condition for $\mathbf{x} = \begin{pmatrix}x_1\\x_2\end{pmatrix}$ to be equidistant from the points $\begin{pmatrix}7\\1\end{pmatrix}, \begin{pmatrix}3\\5\end{pmatrix}$.
Solution: From the given information,

$$\left\|\mathbf{x} - \begin{pmatrix}7\\1\end{pmatrix}\right\|^2 = \left\|\mathbf{x} - \begin{pmatrix}3\\5\end{pmatrix}\right\|^2 \tag{2.13.1}$$

$$\implies \|\mathbf{x}\|^2 + \left\|\begin{pmatrix}7\\1\end{pmatrix}\right\|^2 - 2\begin{pmatrix}7 & 1\end{pmatrix}\mathbf{x}$$
$$= \|\mathbf{x}\|^2 + \left\|\begin{pmatrix}3\\5\end{pmatrix}\right\|^2 - 2\begin{pmatrix}3 & 5\end{pmatrix}\mathbf{x} \quad (2.13.2)$$

which can be simplified to obtain

$$\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = 2 \quad (2.13.3)$$

which is the desired condition. The following code plots Fig. **??**clearly showing that the above equation is the perpendicular bisector of AB.

```
codes/line/line_perp_bisect.py
```

2.14. Find a point on the y-axis which is equidistant from the points $\mathbf{A} = \begin{pmatrix}6\\5\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-4\\3\end{pmatrix}$.
Solution:

2.15. Find the equation of set of points $\mathbf{P}$ such that

$$PA^2 + PB^2 = 2k^2, \quad (2.15.1)$$

$$\mathbf{A} = \begin{pmatrix}3\\4\\5\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-1\\3\\-7\end{pmatrix}, \quad (2.15.2)$$

respectively. **Solution:**

2.16. Find the equation of the set of points $\mathbf{P}$ such that its distances from the points $\mathbf{A} = \begin{pmatrix}3\\4\\-5\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-2\\1\\4\end{pmatrix}$ are equal.
Solution:

3 Section

3.1. If

$$\mathbf{P} = 3\mathbf{a} - 2\mathbf{b} \quad (3.1.1)$$
$$\mathbf{Q} = \mathbf{a} + \mathbf{b} \quad (3.1.2)$$

find $\mathbf{R}$, which divides PQ in the ratio $2:1$

a) internally,
b) externally.

Solution:

3.2. Draw a line segement of length 7.6 cm and divide it in the ratio $5:8$.
Solution: Let the end points of the line be

$$\mathbf{A} = \begin{pmatrix}0\\0\end{pmatrix}, \mathbf{B} = \begin{pmatrix}7.6\\0\end{pmatrix} \quad (3.2.1)$$

Using section formula, the point $\mathbf{C}$

$$\mathbf{C} = \frac{k\mathbf{B} + \mathbf{A}}{k+1} \quad (3.2.2)$$

If $\mathbf{C}$ divides AB in the ratio

$$m = \frac{5}{8}, \quad (3.2.3)$$

then,

$$\frac{\|\mathbf{C} - \mathbf{A}\|^2}{\|\mathbf{B} - \mathbf{C}\|^2} = m^2 \quad (3.2.4)$$
$$\implies \frac{\frac{k^2\|\mathbf{B}-\mathbf{A}\|^2}{(k+1)^2}}{\frac{\|\mathbf{B}-\mathbf{A}\|^2}{(k+1)^2}} = m^2 \quad (3.2.5)$$
$$\implies k = m \quad (3.2.6)$$

upon substituting from (3.2.4) and simplifying. (3.2.2) is known as the section formula. The following code plots Fig. **??**

```
codes/line/draw_section.py
```

3.3. Find the coordinates of the point which divides the line segment joining the points $\begin{pmatrix}4\\-3\end{pmatrix}$ and $\begin{pmatrix}8\\5\end{pmatrix}$ in the ratio $3:1$ internally.
Solution: Using (3.2.2), the desired point is

$$\mathbf{P} = \frac{3\begin{pmatrix}4\\-3\end{pmatrix} + \begin{pmatrix}8\\5\end{pmatrix}}{4} \quad (3.3.1)$$

3.4. In what ratio does the point $\begin{pmatrix}-4\\6\end{pmatrix}$ divide the line segment joining the points

$$\mathbf{A} = \begin{pmatrix}-6\\10\end{pmatrix}, \mathbf{B} = \begin{pmatrix}3\\-8\end{pmatrix} \quad (3.4.1)$$

Solution: Use (3.2.2).

3.5. Find the coordinates of the points of trisection of the line segement joining the points

$$\mathbf{A} = \begin{pmatrix}2\\-2\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-7\\4\end{pmatrix} \quad (3.5.1)$$

Solution: Using (3.2.2), the coordinates are

$$\mathbf{P} = \frac{2\mathbf{A} + \mathbf{B}}{3} \quad (3.5.2)$$
$$\mathbf{Q} = \frac{\mathbf{A} + 2\mathbf{B}}{3} \quad (3.5.3)$$

3.6. Find the ratio in which the y-axis divides the line segment joining the points $\begin{pmatrix} 5 \\ -6 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -4 \end{pmatrix}$.
Solution: Let the corresponding point on the y-axis be $\begin{pmatrix} 0 \\ y \end{pmatrix}$. If the ratio be $k : 1$, using (3.2.2), the coordinates are

$$\begin{pmatrix} 0 \\ y \end{pmatrix} = k \begin{pmatrix} 5 \\ -6 \end{pmatrix} + \begin{pmatrix} -1 \\ -4 \end{pmatrix} \tag{3.6.1}$$

$$\implies 0 = 5k - 1 \implies k = \frac{1}{5} \tag{3.6.2}$$

3.7. Find the coordinates of the point which divides the join of

$$\begin{pmatrix} -1 \\ 7 \end{pmatrix}, \begin{pmatrix} 4 \\ -3 \end{pmatrix} \tag{3.7.1}$$

in the ratio $2 : 3$.
Solution:

3.8. Find the coordinates of the points of trisection of the line segment joining $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ -3 \end{pmatrix}$.
Solution:

3.9. Find the ratio in which the line segment joining the points $\begin{pmatrix} -3 \\ 10 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ -8 \end{pmatrix}$ is divided by $\begin{pmatrix} -1 \\ 6 \end{pmatrix}$.
Solution:

3.10. Find the ratio in whcih the line segment joining $\mathbf{A} = \begin{pmatrix} 1 \\ -5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -4 \\ 5 \end{pmatrix}$ is divided by the x-axis. Also find the coordinates of the point of division.
Solution:

3.11. If $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 4 \\ y \end{pmatrix}$, $\begin{pmatrix} x \\ 6 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$ are the vertices of a parallelogram taken in order, find x and y.
Solution:

3.12. If $\mathbf{A} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$ respectively, find the coordinates of $\mathbf{P}$ such that $AP = \frac{3}{7}AB$ and $\mathbf{P}$ lies on the line segment AB.
Solution:

3.13. Find the coordinates of the points which divide the line segment joining $\mathbf{A} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 \\ 8 \end{pmatrix}$ into four equal parts.
Solution:

4 Projection

4.1. Find the projection of the vector

$$\begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix} \tag{4.1.1}$$

on the vector

$$\begin{pmatrix} 7 \\ -1 \\ 8 \end{pmatrix} \tag{4.1.2}$$

Solution:

4.2. Find the projection of the vector

$$\mathbf{a} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} \tag{4.2.1}$$

on the vector

$$\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}. \tag{4.2.2}$$

Solution: The projection of $\mathbf{a}$ on $\mathbf{b}$ is shown in Fig. **??**. It has magnitude $\|\mathbf{a}\| \cos\theta$ and is in the direction of $\mathbf{b}$. Thus, the projection is defined as

$$(\|\mathbf{a}\| \cos\theta) \frac{\mathbf{b}}{\|\mathbf{b}\|} = \frac{\left(\mathbf{a}^T\mathbf{b}\right) \|\mathbf{a}\|}{\|\mathbf{b}\|}\mathbf{b} \tag{4.2.3}$$

5 Applications

5.1. A bullet fired at an angle of 30° with the horizontal hits the ground 3.0 km away. By adjusting its angle of projection, can one hope to hit a target 5.0 km away ? Assume the muzzle speed to be fixed, and neglect air resistance.

5.2. A fighter plane flying horizontally at an altitude of 1.5 km with speed 720 km/h passes directly overhead an anti-aircraft gun. At what angle from the vertical should the gun be fired for the shell with muzzle speed 600 ms^{-1} to hit the plane ? At what minimum altitude should the pilot fly the plane to avoid being hit ? (Take g = $10ms^{-2}$).

5.3. Consider the collision depicted in Fig. **??** to be between two billiard balls with equal masses

$m_1 = m_2$. The first ball is called the cue while the second ball is called the target. The billiard player wants to 'sink' the target ball in a corner pocket, which is at an angle $\theta_2 = 37°$. Assume that the collosion is elastic and that friction and rotational motion are not important. Obtain θ_1.

5.4. Rain is falling vertically with a speed of 35 ms^{-1} . Winds starts blowing after sometime with a speed of 12 ms^{-1} in east to west direction. In which direction should a boy waiting at a bus stop hold his umbrella ?
Solution:

5.5. A motorboat is racing towards north at 25 km/h and the water current in that region is 10 km/h in the direction of 60° east of south. Find the resultant velocity of the boat.
Solution:

5.6. Rain is falling vertically with a speed of 35 ms^{-1} . A woman rides a bicycle with a speed of 12 ms^{-1} in east to west direction. What is the direction in which she should hold her umbrella ?
Solution:

5.7. A hiker stands on the edge of a cliff 490 m above the ground and throws a stone horizontally with an initial speed of 15 ms^{-1} . Neglecting air resistance, find the time taken by the stone to reach the ground, and the speed with which it hits the ground. (Take g = 9.8 ms^{-2}).
Solution:

5.8. Rain is falling vertically with a speed of 30 ms^{-1}. A woman rides a bicycle with a speed of 10 ms^{-1} in the north to south direction. What is the direction in which she should hold her umbrella?
Solution:

5.9. A man can swim with a speed of 4.0 km/h in still water. How long does he take to cross a river 1.0 km wide if the river flows steadily at 3.0 km/h and he makes his strokes normal to the river current? How far down the river does he go when he reaches the other bank ?
Solution:

5.10. In a harbour, wind is blowing at the speed of 72 km/h and the flag on the mast of a boat anchored in the harbour flutters along the N-E direction. If the boat starts moving at a speed of 51 km/h to the north, what is the direction of the flag on the mast of the boat ?

Solution: