Matrix Analysis

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CONTENTS

***Abstract*—This manual provides an introduction to vectors and their properties, based on the question papers, year 2020, from Class 10 and 12, CBSE; JEE and JNTU.**

1 Class 10

1.1. Find the distance between the points $\begin{pmatrix} m \\ -n \end{pmatrix}$ and $\begin{pmatrix} -m \\ n \end{pmatrix}$

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Solution: Letting

$$\mathbf{A} = \begin{pmatrix} m \\ -n \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -m \\ n \end{pmatrix} \tag{1.1.1}$$

$$\mathbf{A} - \mathbf{B} = 2\begin{pmatrix} m \\ -n \end{pmatrix} \tag{1.1.2}$$

$$\implies \|\mathbf{A} - \mathbf{B}\| = 2\left\|\begin{pmatrix} m \\ -n \end{pmatrix}\right\| \tag{1.1.3}$$

$$= 2\sqrt{\begin{pmatrix} m & -n \end{pmatrix}\begin{pmatrix} m \\ -n \end{pmatrix}} \tag{1.1.4}$$

$$= 2\sqrt{m^2 + n^2} \tag{1.1.5}$$

1.2. Find a point on the $x-$axis which is equidistant from $\begin{pmatrix} -4 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 10 \\ 0 \end{pmatrix}$

Solution: Letting the given points be $\mathbf{A}, \mathbf{B}$.

a)

$$\because \mathbf{A} - \mathbf{B} = \begin{pmatrix} -4 \\ 0 \end{pmatrix} - \begin{pmatrix} 10 \\ 0 \end{pmatrix} \tag{1.2.1}$$

$$= \begin{pmatrix} -14 \\ 0 \end{pmatrix}, \tag{1.2.2}$$

$$\text{and } \frac{\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2}{2} = -42, \tag{1.2.3}$$

(5.1.8.4), can be expressed as

$$\begin{pmatrix} -14 & 0 \end{pmatrix}\mathbf{x} = 42 \tag{1.2.4}$$

$$\implies \begin{pmatrix} -14 & 0 \end{pmatrix}\begin{pmatrix} x \\ 0 \end{pmatrix} = -42 \tag{1.2.5}$$

$$\text{or, } x = 3 \tag{1.2.6}$$

Hence, the desired point is $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$.

b) In general, if $\mathbf{x}$ be the desired point,

$$\|\mathbf{x} - \mathbf{A}\| = \|\mathbf{x} - \mathbf{B}\| \tag{1.2.7}$$

$$\implies \|\mathbf{x} - \mathbf{A}\|^2 = \|\mathbf{x} - \mathbf{B}\|^2 \tag{1.2.8}$$

which can be expressed as

$$\|\mathbf{x}\|^2 + \|\mathbf{A}\|^2 - 2\mathbf{A}^\top \mathbf{x} = \|\mathbf{x}\|^2 + \|\mathbf{B}\|^2 - 2\mathbf{B}^\top \mathbf{x} \quad (1.2.9)$$

and simplified to obtain

$$(\mathbf{A} - \mathbf{B})^\top \mathbf{x} = \frac{\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2}{2} \quad (1.2.10)$$

Since $\mathbf{x}$ lies on the x-axis,

$$\mathbf{x} = x\mathbf{e}_1 \quad (1.2.11)$$

$$\implies x(\mathbf{A} - \mathbf{B})^\top \mathbf{e}_1 = \frac{\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2}{2} \quad (1.2.12)$$

$$\text{or, } x = \frac{\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2}{2(\mathbf{A} - \mathbf{B})^\top \mathbf{e}_1} \quad (1.2.13)$$

upon substituting from (1.2.10).

$$\because \mathbf{A} - \mathbf{B} = \begin{pmatrix} -4 \\ 0 \end{pmatrix} - \begin{pmatrix} 10 \\ 0 \end{pmatrix} \quad (1.2.14)$$

$$= \begin{pmatrix} -14 \\ 0 \end{pmatrix}, \quad (1.2.15)$$

$$\text{and } \frac{\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2}{2} = -42, \quad (1.2.16)$$

(5.1.8.4), can be expressed as

$$\begin{pmatrix} -14 & 0 \end{pmatrix} \mathbf{x} = 42 \quad (1.2.17)$$

$$\implies \begin{pmatrix} -14 & 0 \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} = -42 \quad (1.2.18)$$

$$\text{or, } x = 3 \quad (1.2.19)$$

Hence, the desired point is $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$.

1.3. Find the centre of a circle whose end points of a diameter are $\begin{pmatrix} -6 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ 4 \end{pmatrix}$.

Solution: Using section formula, from (5.1.15.1), the desired point is given by

$$\mathbf{O} = \frac{\mathbf{B} + \mathbf{A}}{2} \quad (1.3.1)$$

$$= \frac{1}{2}\left[\begin{pmatrix} -6 \\ 3 \end{pmatrix} + \begin{pmatrix} 6 \\ 4 \end{pmatrix}\right] \quad (1.3.2)$$

$$= \frac{1}{2}\begin{pmatrix} 0 \\ 7 \end{pmatrix} \quad (1.3.3)$$

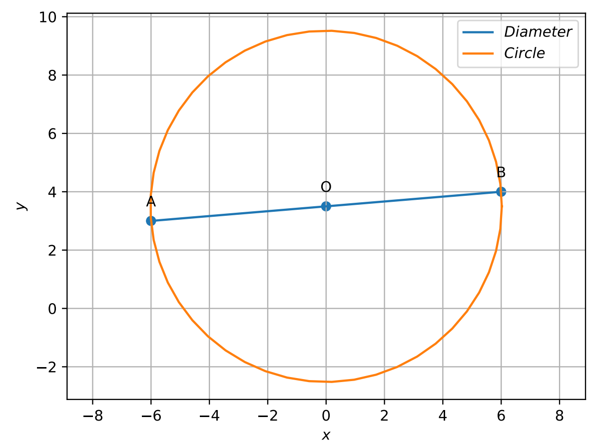

Fig. 1.3.

See Fig. 1.3

1.4. $AOBC$ is a rectangle whose three vertices are $A = \begin{pmatrix} 0 \\ -3 \end{pmatrix}, O = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, B = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$. Find the length of its diagonal.

Solution:

a) The fourth point is given by

$$OC = \mathbf{C} = \begin{pmatrix} 4 \\ -3 \end{pmatrix} \quad (1.4.1)$$

The length of the diagonal is

$$\|\mathbf{C}\| = \sqrt{\begin{pmatrix} 4 & -3 \end{pmatrix} \begin{pmatrix} 4 \\ -3 \end{pmatrix}} \quad (1.4.2)$$

$$= \sqrt{25} = 5 \quad (1.4.3)$$

b) In general, if three vertices of a rectangle are given, the adjacent sides need to be identified first to identify the opposite vertices. For example, in the given problem,

$$(\mathbf{A} - \mathbf{O})^\top (\mathbf{B} - \mathbf{O}) = 0 \quad (1.4.4)$$

$$\implies \angle AOB = 90° \quad (1.4.5)$$

and AB is the diagonal. This condition needs to be checked before proceeding to find the length of the diagonal.

1.5. Find the ratio in which the y-axis divides the line segment joining the points $\begin{pmatrix} 6 \\ -4 \end{pmatrix}, \begin{pmatrix} -2 \\ -7 \end{pmatrix}$. Also find the point of intersection.

Solution: Let the desired point on the y-axis be

$$\mathbf{P} = \begin{pmatrix} 0 \\ y \end{pmatrix} \tag{1.5.1}$$

a) Using section formula, from (5.1.15.1),

$$\mathbf{P} = \begin{pmatrix} 0 \\ y \end{pmatrix} = \frac{1}{k+1}\left[\begin{pmatrix} 6 \\ -4 \end{pmatrix} + k\begin{pmatrix} -2 \\ -7 \end{pmatrix}\right] \tag{1.5.2}$$

$$\implies 6 - 2k = 0 \text{ or, } k = 3 \tag{1.5.3}$$

Also,

$$y = \frac{-4-7k}{k+1} \tag{1.5.4}$$

$$= -\frac{25}{4} \tag{1.5.5}$$

Thus, the desired point is $-\frac{25}{4}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

b) In general, letting the given points be $\mathbf{A}, \mathbf{B}$,

$$\mathbf{P} = \frac{k\mathbf{B} + \mathbf{A}}{k+1} \tag{1.5.6}$$

Since the point lies on the y-axis,

$$\mathbf{e}_1^\top \mathbf{P} = 0 \tag{1.5.7}$$

$$\implies k\mathbf{e}_1^\top \mathbf{B} + \mathbf{e}_1^\top \mathbf{A} = 0 \tag{1.5.8}$$

$$\text{or, } k = -\frac{\mathbf{e}_1^\top \mathbf{A}}{\mathbf{e}_1^\top \mathbf{B}} \tag{1.5.9}$$

Substituting in (1.5.6) and simplifying,

$$\mathbf{P} = \frac{\left(\mathbf{e}_1^\top \mathbf{B}\right)\mathbf{A} - \left(\mathbf{e}_1^\top \mathbf{A}\right)\mathbf{B}}{\left(\mathbf{e}_1^\top \mathbf{B}\right) - \left(\mathbf{e}_1^\top \mathbf{A}\right)} \tag{1.5.10}$$

1.6. Show that the points $\begin{pmatrix} 7 \\ 10 \end{pmatrix}, \begin{pmatrix} -2 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$ are vertices of an isoscles right triangle.
Solution: Let the given points be $\mathbf{A}, \mathbf{B}, \mathbf{C}$ respectively.

a) Then, the direction vectors of AB, BC and CA are

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} 7 \\ 10 \end{pmatrix} - \begin{pmatrix} -2 \\ 5 \end{pmatrix} = \begin{pmatrix} 9 \\ 5 \end{pmatrix} \tag{1.6.1}$$

$$\mathbf{B} - \mathbf{C} = -\begin{pmatrix} -2 \\ 5 \end{pmatrix} - \begin{pmatrix} 3 \\ -4 \end{pmatrix} = \begin{pmatrix} -5 \\ 9 \end{pmatrix} \tag{1.6.2}$$

$$\mathbf{C} - \mathbf{A} = \begin{pmatrix} 3 \\ -4 \end{pmatrix} - \begin{pmatrix} 7 \\ 10 \end{pmatrix} = \begin{pmatrix} -4 \\ -14 \end{pmatrix} \tag{1.6.3}$$

From the above, we find that

$$(\mathbf{A} - \mathbf{B})^\top (\mathbf{B} - \mathbf{C}) = \begin{pmatrix} 9 & 5 \end{pmatrix}\begin{pmatrix} -5 \\ 9 \end{pmatrix} \tag{1.6.4}$$

$$= 0 \tag{1.6.5}$$

$$(\mathbf{B} - \mathbf{C})^\top (\mathbf{C} - \mathbf{A}) = \begin{pmatrix} -5 & 9 \end{pmatrix}\begin{pmatrix} -4 \\ -14 \end{pmatrix} \tag{1.6.6}$$

$$= -106 \tag{1.6.7}$$

$$(\mathbf{C} - \mathbf{A})^\top (\mathbf{A} - \mathbf{B}) = \begin{pmatrix} -4 & -14 \end{pmatrix}\begin{pmatrix} 9 \\ 5 \end{pmatrix} \tag{1.6.8}$$

$$= -106 \tag{1.6.9}$$

From the above equations, (5.1.9.1) and (5.1.10.1),

$$(\mathbf{A} - \mathbf{B}) \perp (\mathbf{B} - \mathbf{C}) \tag{1.6.10}$$

$$\angle BCA = \angle CAB \tag{1.6.11}$$

Thus, the triangle is right angled and isosceles.

b) In general, for an isosceles triangle, if $AB = BC$,

$$\angle BCA = \angle CAB, \tag{1.6.12}$$

$$\implies \frac{(\mathbf{C} - \mathbf{B})^\top (\mathbf{C} - \mathbf{A})}{\|\mathbf{C} - \mathbf{B}\| \|\mathbf{C} - \mathbf{A}\|} = \frac{(\mathbf{A} - \mathbf{C})^\top (\mathbf{A} - \mathbf{B})}{\|\mathbf{A} - \mathbf{C}\| \|\mathbf{A} - \mathbf{B}\|} \tag{1.6.13}$$

$$\text{or } (\mathbf{C} - \mathbf{B})^\top (\mathbf{C} - \mathbf{A}) = (\mathbf{A} - \mathbf{C})^\top (\mathbf{A} - \mathbf{B}) \tag{1.6.14}$$

The above condition needs to be checked for each pair of sides to determine whether the triangle is isosceles.

2 Class 12

2.1. Find the area of a triangle formed by vertices O, A and B, where

$$\mathbf{A} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix}, \tag{2.1.1}$$

Solution:

a) $\because$

$$\begin{pmatrix} 2 & -2 \\ 3 & 1 \end{pmatrix} = 8, \tag{2.1.2}$$

$$\begin{pmatrix} 3 & 1 \\ 1 & -3 \end{pmatrix} = -10, \tag{2.1.3}$$

$$\begin{pmatrix} 1 & -3 \\ 2 & -2 \end{pmatrix} = 4, \tag{2.1.4}$$

$$\mathbf{A} \times \mathbf{B} = \begin{pmatrix} 8 \\ -10 \\ 4 \end{pmatrix}, \tag{2.1.5}$$

and the desired area can be obtained from (5.2.2.1) and (6.2.1.1) as

$$\frac{1}{2} \left\| \begin{pmatrix} 8 \\ -10 \\ 4 \end{pmatrix} \right\| = 3\sqrt{5} \tag{2.1.6}$$

b) In general, the area of $\triangle OAB$ can be expressed as

$$\frac{1}{2} \left\| (\mathbf{A} - \mathbf{O}) \times (\mathbf{B} - \mathbf{O}) \right\| \tag{2.1.7}$$

2.2. Find the coordinates of the foot of the perpendicular drawn from the point $\begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$ on the y-axis.

Solution:

a) By definition of y-axis, the desired coordinates are

$$\begin{pmatrix} 0 \\ -3 \\ 0 \end{pmatrix} \tag{2.2.1}$$

Alternatively, the equation of the y-axis can be written as

$$\mathbf{x} = \lambda \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \tag{2.2.2}$$

The equation of the plane perpendicular to the y-axis and passing through the origin is

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \mathbf{x} = 0 \tag{2.2.3}$$

From (6.2.14.1), the desired point is given by

$$\mathbf{x} = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \tag{2.2.4}$$

$$= \begin{pmatrix} 0 \\ -3 \\ 0 \end{pmatrix} \tag{2.2.5}$$

b) In general, let $\mathbf{P}$ be any point and the line be

$$L : \quad \mathbf{x} = \mathbf{A} + \lambda \mathbf{m} \tag{2.2.6}$$

Then the plane perpendicular to L and passing through $\mathbf{P}$ has the equation

$$N : \quad \mathbf{m}^\top (\mathbf{x} - \mathbf{P}) = 0 \tag{2.2.7}$$

The intersection of N and P is the desired foot of the perpendicular. Thus, substituting (2.2.6) in (2.2.7),

$$\mathbf{m}^\top (\mathbf{A} + \lambda \mathbf{m} - \mathbf{P}) = 0 \tag{2.2.8}$$

which can be simplified to obtain

$$\lambda = \frac{\mathbf{m}^\top (\mathbf{P} - \mathbf{A})}{\left\| \mathbf{m} \right\|^2} \tag{2.2.9}$$

Substituting (2.2.9) in (2.2.6)

$$\mathbf{x} = \mathbf{A} + \frac{\mathbf{m}^\top (\mathbf{P} - \mathbf{A})}{\left\| \mathbf{m} \right\|^2} \mathbf{m} \tag{2.2.10}$$

2.3. Find the angle between the vectors $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

Solution:

a) $\because$

$$\begin{pmatrix} 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = -1, \left\| \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\| = \left\| \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\| = \sqrt{} \tag{2.3.1}$$

From (5.1.9.1), the angle between the given vectors is

$$\cos^{-1} \frac{-1}{\sqrt{2} \times \sqrt{2}} = \cos^{-1} \frac{-1}{2} \tag{2.3.2}$$

$$= \frac{2\pi}{3} \tag{2.3.3}$$

b) The angle between the vectors $\mathbf{m}_1, \mathbf{m}_2$ is given by

$$\cos\theta = \frac{\mathbf{m}_1^\top \mathbf{m}_2}{\|\mathbf{m}_1\| \|\mathbf{m}_2\|} \tag{2.3.4}$$

2.4. If A is a non-singular square matrix of order 3 such that $\mathbf{A}^2 = 3\mathbf{A}$, then find the value of $\left|\mathbf{A}\right|$
Solution:

$$\left|\mathbf{A}^2\right| = \left|3\mathbf{A}\right| \tag{2.4.1}$$

$$\implies \left|\mathbf{A}\right|^2 = 3^3 \left|\mathbf{A}\right| \tag{2.4.2}$$

from (5.4.4.1) yielding

$$\left|\mathbf{A}\right| = 27 \tag{2.4.3}$$

after simplification.

2.5. If $\|\mathbf{a}\| = 4$ and $-3 \le \lambda \le 2$ then find the range of values that $\|\lambda\mathbf{a}\|$ can satisfy.
Solution: From (5.1.6.3),

$$\|\lambda\mathbf{a}\| = |\lambda| \|\mathbf{A}\| \tag{2.5.1}$$

$$= 4|\lambda| \tag{2.5.2}$$

$\because$

$$0 \le |\lambda| \le 3, \tag{2.5.3}$$

$$0 \le 4|\lambda| \le 12 \tag{2.5.4}$$

2.6. If

$$\begin{vmatrix} 2x & -9 \\ -2 & x \end{vmatrix} = \begin{vmatrix} -4 & 8 \\ 1 & -2 \end{vmatrix} \tag{2.6.1}$$

then find the value of x.
Solution: Exapanding the above determinants,

$$2x^2 - 18 = 0 \tag{2.6.2}$$

$$\implies x = \pm 3 \tag{2.6.3}$$

2.7. Find the distance between parallel planes

$$2x + y - 2z - 6 = 0 \tag{2.7.1}$$

$$4x + 2y - 4z = 0 \tag{2.7.2}$$

Solution: The above planes have parameters

$$\mathbf{n} = \begin{pmatrix} 2 & 1 & -2 \end{pmatrix}, c_1 = 6, c_2 = 0 \tag{2.7.3}$$

Using (6.1.6.1), the distance is obtained as

$$d = \frac{|c_1 - c_2|}{\|\mathbf{n}\|} \tag{2.7.4}$$

$$= \frac{6}{3} = 2 \tag{2.7.5}$$

2.8. If

$$\mathbf{P} = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} \tag{2.8.1}$$

is the foot of the perpendicular from the origin to the plane, then find the equation of the plane.
Solution: Let the equation of the plane be

$$\mathbf{n}^\top \mathbf{x} = c \tag{2.8.2}$$

Since $\mathbf{P}$ is a point on the plane, it satisfies the above equation and

$$\mathbf{n}^\top \mathbf{P} = c \tag{2.8.3}$$

The normal vector to the plane is OP. Hence,

$$\mathbf{n} = \mathbf{P} \tag{2.8.4}$$

Substituting the above in (2.8.3),

$$\mathbf{P}^\top \mathbf{P} = c \tag{2.8.5}$$

and the desired equation of the plane is

$$\mathbf{P}^\top \mathbf{x} = \mathbf{P}^\top \mathbf{P} \tag{2.8.6}$$

$$\begin{pmatrix} 1 & 0 & -3 \end{pmatrix} \mathbf{x} = 10 \tag{2.8.7}$$

after substituting numerical values.

2.9. Find the coordinates of the point where the line $\frac{x-1}{3} = \frac{y+4}{7} = \frac{z+4}{2}$ cuts the xy-plane.
Solution: The given line can be expressed as

$$\mathbf{x} = \begin{pmatrix} 1 \\ -4 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 7 \\ 2 \end{pmatrix} \tag{2.9.1}$$

and the xy- plane is

$$\begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{2.9.2}$$

From (6.2.13.1),

$$\mathbf{x} = \mathbf{A} + \frac{c - \mathbf{n}^\top \mathbf{A}}{\mathbf{n}^\top \mathbf{m}} \mathbf{m} \tag{2.9.3}$$

$$= \begin{pmatrix} 1 \\ -4 \\ -4 \end{pmatrix} + \frac{0 - \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -4 \\ -4 \end{pmatrix}}{\begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 7 \\ 2 \end{pmatrix}} \begin{pmatrix} 3 \\ 7 \\ 2 \end{pmatrix} \tag{2.9.4}$$

$$= \begin{pmatrix} 1 \\ -4 \\ -4 \end{pmatrix} + 2 \begin{pmatrix} 3 \\ 7 \\ 2 \end{pmatrix} \tag{2.9.5}$$

$$= \begin{pmatrix} 7 \\ 10 \\ 0 \end{pmatrix} \tag{2.9.6}$$

2.10. Find a vector $\mathbf{r}$ equally inclined to the three axes and whose magnitude is $3\sqrt{3}$ units.
Solution: From (5.1.9.1),

$$\frac{\mathbf{e}_1^\top \mathbf{r}}{\|\mathbf{e}_1\| \|\mathbf{r}\|} = \frac{\mathbf{e}_2^\top \mathbf{r}}{\|\mathbf{e}_2\| \|\mathbf{r}\|} = \frac{\mathbf{e}_3^\top \mathbf{r}}{\|\mathbf{e}_3\| \|\mathbf{r}\|} = \cos\theta \tag{2.10.1}$$

which can be expressed as the system of equations

$$\mathbf{e}_1^\top \mathbf{r} = \|\mathbf{r}\| \cos\theta \tag{2.10.2}$$
$$\mathbf{e}_2^\top \mathbf{r} = \|\mathbf{r}\| \cos\theta \tag{2.10.3}$$
$$\mathbf{e}_3^\top \mathbf{r} = \|\mathbf{r}\| \cos\theta \tag{2.10.4}$$

which can be combined to obtain the matrix equation

$$\begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{pmatrix}^\top \mathbf{r} = \|\mathbf{r}\| \cos\theta \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{2.10.5}$$

$$\implies \frac{\mathbf{r}}{\|\mathbf{r}\|} = \cos\theta \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{2.10.6}$$

$$\because \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{pmatrix} = \mathbf{I} \tag{2.10.7}$$

From (2.10.6)

$$\left\| \cos\theta \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\| = 1 \tag{2.10.8}$$

$$\implies \cos\theta = \frac{1}{\sqrt{3}} \tag{2.10.9}$$

From (2.10.9) and (2.10.9),

$$\mathbf{r} = 3\sqrt{3} \times \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{2.10.10}$$

$$= 3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{2.10.11}$$

2.11. Find the angle between unit vectors $\mathbf{a}$ and $\mathbf{b}$ so that $\sqrt{3}$ **a - b** is also a unit vector.
Solution: From the given information,

$$\left\| \sqrt{3}\mathbf{a} - \mathbf{b} \right\| = 1 \tag{2.11.1}$$

$$\implies \left\| \sqrt{3}\mathbf{a} - \mathbf{b} \right\|^2 = 1 \tag{2.11.2}$$

which can be expressed as

$$\left(\sqrt{3}\mathbf{a} - \mathbf{b} \right)^\top \left(\sqrt{3}\mathbf{a} - \mathbf{b} \right) = 1 \tag{2.11.3}$$

$$\implies 3 \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\sqrt{3}\mathbf{a}^\top \mathbf{b} = 1 \tag{2.11.4}$$

and simplified to obtain

$$\mathbf{a}^\top \mathbf{b} = \frac{\sqrt{3}}{2} \tag{2.11.5}$$

Thus the desired angle is

$$\cos^{-1} \frac{\sqrt{3}}{2} = 30^\circ \tag{2.11.6}$$

2.12. If $\mathbf{A} = \begin{pmatrix} -3 & 2 \\ 1 & -1 \end{pmatrix}$ and $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, Find scalar k so that $\mathbf{A}^2 + \mathbf{I} = k\mathbf{A}$.
Solution: Using the Cayley-Hamilton theorem,

$$\lambda^2 - k\lambda + 1 = 0 \tag{2.12.1}$$

From (5.3.4.1),

$$k = \text{tr}\left(\mathbf{A}\right) = -3 - 1 = -4 \tag{2.12.2}$$

2.13. Show that the plane $x - 5y - 2z = 1$ contains the line $\frac{x-5}{3} = y = 2 - z$.
Solution: The plane and line can be expressed in vector form as

$$\begin{pmatrix} 1 & -5 & -2 \end{pmatrix} \mathbf{x} = 1 \tag{2.13.1}$$

$$\mathbf{x} = \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} \tag{2.13.2}$$

$\because$

$$\begin{pmatrix} 1 & -5 & -2 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} = 0 \tag{2.13.3}$$

from (6.2.9.3), the given plain contains the given line.

2.14. Find the equation of the plane passing through the points

$$\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} \tag{2.14.1}$$

and perpendicular to the plane $2x - y + z = 8$. Also find the distance of the plane thus obtained from the origin.

Solution: Let the equation of the desired plane be

$$\mathbf{n}^\top \mathbf{x} = 1 \tag{2.14.2}$$

$$\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} \tag{2.14.3}$$

From the given information,

$$\mathbf{n}^\top \mathbf{x} = 1 \tag{2.14.4}$$

$$\begin{aligned} &\begin{pmatrix} 1 & 0 & -2 \end{pmatrix} \mathbf{n} = 1 \\ \implies &\begin{pmatrix} 3 & -1 & 0 \end{pmatrix} \mathbf{n} = 1 \\ &\begin{pmatrix} 2 & -1 & 1 \end{pmatrix} \mathbf{n} = 0 \end{aligned} \tag{2.14.5}$$

From (2.14.5), we obtain the matrix equation

$$\begin{pmatrix} 1 & 0 & -2 \\ 3 & -1 & 0 \\ 2 & -1 & 1 \end{pmatrix} \mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \tag{2.14.6}$$

Forming the augmented matrix, and choosing the pivot,

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & 1 \\ 3 & -1 & 0 & 1 \\ 2 & -1 & 1 & 0 \end{array}\right) \tag{2.14.7}$$

$$\leftrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -2 & 1 \\ 0 & 1 & -6 & 2 \\ 0 & -1 & 5 & -2 \end{array}\right) \tag{2.14.8}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & 1 \\ 0 & 1 & -6 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right) \tag{2.14.9}$$

yielding

$$\mathbf{n} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \tag{2.14.10}$$

Thus, the equation of the desired plane is

$$\begin{pmatrix} 1 & 2 & 0 \end{pmatrix} \mathbf{n} = 1 \tag{2.14.11}$$

2.15. If $\mathbf{A} = \begin{pmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ 5 & -2 & 6 \end{pmatrix}$, Find $\mathbf{A}^{-1}$ and use it to solve the following system of the equations:

$$5x - y + 4z = 5 \tag{2.15.1}$$

$$2x + 3y + 5z = 2 \tag{2.15.2}$$

$$5x - 2y + 6z = -1 \tag{2.15.3}$$

Solution: Forming the augmented matrix and pivoting,

$$\left(\begin{array}{ccc|ccc} 5 & -1 & 4 & 1 & 0 & 0 \\ 2 & 3 & 5 & 0 & 1 & 0 \\ 5 & -2 & 6 & 0 & 0 & 1 \end{array}\right) \tag{2.15.4}$$

$$\leftrightarrow \left(\begin{array}{ccc|ccc} 5 & -1 & 4 & 1 & 0 & 0 \\ 0 & 17 & 17 & -2 & 5 & 0 \\ 0 & 1 & -2 & 1 & 0 & -1 \end{array}\right) \tag{2.15.5}$$

$$\leftrightarrow \left(\begin{array}{ccc|ccc} 17 & 0 & 17 & 3 & 1 & 0 \\ 0 & 17 & 17 & -2 & 5 & 0 \\ 0 & 0 & 51 & -19 & 5 & 17 \end{array}\right) \tag{2.15.6}$$

$$\xleftrightarrow[R_2 \leftarrow 3R_2 - R_3]{R_1 \leftarrow 3R_1 - R_3} \tag{2.15.7}$$

$$\left(\begin{array}{ccc|ccc} 51 & 0 & 0 & 28 & -2 & -17 \\ 0 & 51 & 0 & 13 & 10 & -17 \\ 0 & 0 & 51 & -19 & 5 & 17 \end{array}\right) \tag{2.15.8}$$

resulting in

$$\mathbf{A}^{-1} = \frac{1}{51} \begin{pmatrix} 28 & -2 & -17 \\ 13 & 10 & -17 \\ -19 & 5 & 17 \end{pmatrix} \tag{2.15.9}$$

2.16. If x, y, z are different and

$$\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0 \tag{2.16.1}$$

then using properties of determinants show that $1 + xyz = 0$.

Solution: The given determinant can be expressed as

$$\begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{vmatrix} \tag{2.16.2}$$

Since

$$\begin{vmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{vmatrix} = xyz \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \tag{2.16.3}$$

and

$$\begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}, \tag{2.16.4}$$

(2.16.2) can be expressed as

$$(1+xyz)\begin{vmatrix}1 & x & x^2\\1 & y & y^2\\1 & z & z^2\end{vmatrix}, \tag{2.16.5}$$

The above determinant can be simplified as

$$\begin{vmatrix}1 & x & x^2\\1 & y & y^2\\1 & z & z^2\end{vmatrix}, \tag{2.16.6}$$

$$\xleftrightarrow[R_2\leftarrow R_1-R_2]{R_3\leftarrow R_1-R_3}\begin{vmatrix}1 & x & x^2\\0 & x-y & x^2-y^2\\0 & x-z & x^2-z^2\end{vmatrix}, \tag{2.16.7}$$

$$=(x-y)(x-z)\begin{vmatrix}1 & x & x^2\\0 & 1 & x+y\\0 & 1 & x+z\end{vmatrix}, \tag{2.16.8}$$

$$=(x-y)(y-z)(z-x) \tag{2.16.9}$$

and (2.16.1) can be obtained from (2.16.5) as

$$(1+xyz)(x-y)(y-z)(z-x)=0 \tag{2.16.10}$$

Since

$$x\neq y\neq z,(1+xyz)=0 \tag{2.16.11}$$

2.17. Find the area of the triangle ABC, coordinates of whose vertices are

$$\mathbf{A}=\begin{pmatrix}2\\0\end{pmatrix},\mathbf{B}=\begin{pmatrix}4\\5\end{pmatrix} \text{ and } \mathbf{C}=\begin{pmatrix}6\\3\end{pmatrix}. \tag{2.17.1}$$

Solution: Since

$$\mathbf{A}-\mathbf{B}=\begin{pmatrix}-2\\-5\end{pmatrix}, \tag{2.17.2}$$

$$\mathbf{A}-\mathbf{C}=\begin{pmatrix}-4\\-3\end{pmatrix}, \tag{2.17.3}$$

the desired area is the magnitude of

$$\begin{vmatrix}2 & 4\\5 & 3\end{vmatrix} \tag{2.17.4}$$

Thus the desired area is 14 units.

2.18. A cottage industry manufactures pedestal lamps and wooden shades. Both the products require machine time as well as craftsman time in the making. The number of hours required for producing 1 unit of each and the corresponding profit is given in the following table.

Item	Machine Time	Craftsman Time	Profit(in INR)
Pedestal Lamp	1.5 hours	3 hours	30
Wooden shades	3 hours	1 hour	20

TABLE 2.18

In a day, the factory has availability of not more than 42 hours of machine time and 24 hours of craftsman time. Assuming that all items manufactured are sold, how should the manufacturer schedule his daily production in order to maximise the profit? Formulate it as an LPP and solve it graphically.

Solution: Let x be the number of lamps and y be the number of wooden shades produced. From the given information, the problem can be formulated as

$$P=\max_{x,y} 30x+20y \tag{2.18.1}$$

$$1.5x+3y\leq 42 \tag{2.18.2}$$

$$3x+y\leq 24 \tag{2.18.3}$$

which can be expressed in vector form as

$$P=\max_{\mathbf{x}}\begin{pmatrix}30 & 20\end{pmatrix}\mathbf{x} \tag{2.18.4}$$

$$\begin{pmatrix}1 & 2\\3 & 1\end{pmatrix}\mathbf{x}\preceq\begin{pmatrix}28\\24\end{pmatrix} \tag{2.18.5}$$

The feasible region is a quadrilateral with vertices

$$\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}8\\0\end{pmatrix},\begin{pmatrix}0\\14\end{pmatrix},\begin{pmatrix}4\\12\end{pmatrix} \tag{2.18.6}$$

with respective profit

$$\begin{pmatrix}30 & 20\end{pmatrix}\begin{pmatrix}0\\0\end{pmatrix}=0 \tag{2.18.7}$$

$$\begin{pmatrix}30 & 20\end{pmatrix}\begin{pmatrix}8\\0\end{pmatrix}=240 \tag{2.18.8}$$

$$\begin{pmatrix}30 & 20\end{pmatrix}\begin{pmatrix}0\\14\end{pmatrix}=280 \tag{2.18.9}$$

$$\begin{pmatrix}30 & 20\end{pmatrix}\begin{pmatrix}4\\12\end{pmatrix}=360 \tag{2.18.10}$$

Thus, the manufacturer should produce 4 pedestal lamps and 12 wooden shades daily.

2.19. The corner points of the feasible region of an LPP are

$$\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}0\\8\end{pmatrix},\begin{pmatrix}2\\7\end{pmatrix},\begin{pmatrix}5\\4\end{pmatrix} \text{ and } \begin{pmatrix}6\\0\end{pmatrix}. \tag{2.19.1}$$

Find the point at which the maximum profit $P = 3x + 2y$ occurs.
Solution: The profit can be expressed as

$$P = \begin{pmatrix} 3 & 2 \end{pmatrix} \mathbf{x} \tag{2.19.2}$$

and the respective values at each of the above points are given by

$$\begin{pmatrix} 3 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0, \tag{2.19.3}$$

$$\begin{pmatrix} 3 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 8 \end{pmatrix} = 16 \tag{2.19.4}$$

$$\begin{pmatrix} 3 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 7 \end{pmatrix} = 20 \tag{2.19.5}$$

$$\begin{pmatrix} 3 & 2 \end{pmatrix} \begin{pmatrix} 5 \\ 4 \end{pmatrix} = 23 \tag{2.19.6}$$

$$\begin{pmatrix} 3 & 2 \end{pmatrix} \begin{pmatrix} 6 \\ 0 \end{pmatrix} = 18 \tag{2.19.7}$$

Hence, the maximum profit is $P = 23$ which occurs at $\begin{pmatrix} 5 \\ 4 \end{pmatrix}$

3 JEE

3.1. Let α be a root of the equation

$$x^2 + x + 1 = 0 \tag{3.1.1}$$

and the matrix

$$\mathbf{A} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha^4 \end{pmatrix} \tag{3.1.2}$$

Find $\mathbf{A}^{31}$.
Solution: Since

$$\mathbf{A}^2 = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha^4 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha^4 \end{pmatrix} \tag{3.1.3}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \mathbf{E}, \tag{3.1.4}$$

where $\mathbf{E}$ is an elementary matrix that interchanges the 2nd and 3rd row,

$$\mathbf{E}^2 = \mathbf{I}. \tag{3.1.5}$$

Thus, from (3.1.4) and (3.1.5),

$$\mathbf{A}^{31} = \mathbf{A} \left(\mathbf{A}^2\right)^{15} = \mathbf{A} \left(\mathbf{E}^2\right)^{15} \tag{3.1.6}$$

$$= \mathbf{A} \left(\mathbf{I}\right)^{15} = \mathbf{A} \tag{3.1.7}$$

3.2. A vector

$$\mathbf{a} = \begin{pmatrix} \alpha \\ 2 \\ \beta \end{pmatrix} \tag{3.2.1}$$

lies in the plane of the vectors

$$\mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} \tag{3.2.2}$$

If $\mathbf{a}$ bisects the angle between $\mathbf{b}, \mathbf{c}$, find $\mathbf{a}$.
Solution: From the given information,

$$\frac{\mathbf{a}^\top \mathbf{b}}{\|\mathbf{a}\| \|\mathbf{b}\|} = \frac{\mathbf{a}^\top \mathbf{c}}{\|\mathbf{a}\| \|\mathbf{c}\|} \tag{3.2.3}$$

$$\implies \frac{\mathbf{a}^\top \mathbf{b}}{\|\mathbf{b}\|} = \frac{\mathbf{a}^\top \mathbf{c}}{\|\mathbf{c}\|} \tag{3.2.4}$$

Since

$$\|\mathbf{b}\| = \sqrt{2}, \|\mathbf{c}\| = 3\sqrt{2} \tag{3.2.5}$$

from (3.2.4),

$$3\mathbf{a}^\top \mathbf{b} = \mathbf{a}^\top \mathbf{c} \tag{3.2.6}$$

$$\implies \mathbf{a}^\top \left(3\mathbf{b} - \mathbf{c}\right) = 0 \tag{3.2.7}$$

$$\text{or, } \begin{pmatrix} 1 & 2 & -2 \end{pmatrix} \mathbf{a} = 0 \tag{3.2.8}$$

Also, since $\mathbf{a}, \mathbf{b}, \mathbf{c}$ lie on the same plane,

$$\mathbf{a}^\top \left(\mathbf{b} \times \mathbf{c}\right) = 0 \tag{3.2.9}$$

$$\implies \begin{pmatrix} 2 & -2 & -1 \end{pmatrix} \mathbf{a} = 0 \tag{3.2.10}$$

From (3.2.8) and (3.2.10)

$$\begin{pmatrix} 1 & 2 & -2 \\ 2 & -2 & -1 \end{pmatrix} \mathbf{a} = 0 \tag{3.2.11}$$

Row reducing the above coefficient matrix,

$$\begin{pmatrix} 1 & 2 & -2 \\ 2 & -2 & -1 \end{pmatrix} \tag{3.2.12}$$

$$\text{(pivoting)} \begin{pmatrix} 1 & 2 & -2 \\ 0 & 2 & -1 \end{pmatrix} \tag{3.2.13}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - R_2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & -1 \end{pmatrix} \tag{3.2.14}$$

Thus,

$$\mathbf{a} = k \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \tag{3.2.15}$$

where k is some constant. Comparing the above with (3.2.1),

$$k = 2 \implies \mathbf{a} = \begin{pmatrix} 4 \\ 2 \\ 4 \end{pmatrix} \tag{3.2.16}$$

3.3. If the system of linear equations

$$2x + 2ay + az = 0 \tag{3.3.1}$$
$$2x + 3by + bz = 0 \tag{3.3.2}$$
$$2x + 4cy + cz = 0 \tag{3.3.3}$$

where $a, b, c \in \mathbb{R}$ are nonzero and distinct; has a nonzero solution, find the relation between a, b, c.

Solution: The given system of equations can be expressed as the matrix equation

$$\begin{pmatrix} 2 & 2a & a \\ 2 & 3b & b \\ 2 & 4c & c \end{pmatrix} \mathbf{x} = \mathbf{0} \tag{3.3.4}$$

Row reducing the coefficient matrix in, (3.3.5) yields (3.3.7). For the given system to have a nontrivial solution, the rank of the coefficient matrix should be 2. Hence, from (3.3.7),

$$-bc - ab + 2ac = 0 \tag{3.3.8}$$
$$\implies ab + bc = 2ac \tag{3.3.9}$$
$$\text{or, } \frac{1}{a} + \frac{1}{c} = \frac{2}{b} \tag{3.3.10}$$

Thus, a, b, c are in H.P.

3.4. Let P be a plane passing through the points

$$\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix} \tag{3.4.1}$$

and

$$\mathbf{R} = \begin{pmatrix} 2 \\ 1 \\ 6 \end{pmatrix} \tag{3.4.2}$$

be any point. Find the image of $\mathbf{R}$ in the plane $\mathbf{P}$.

Solution: From (6.2.16.1), the normal vector of the plane is given by

$$\begin{pmatrix} 2 & 1 & 0 \\ 4 & 1 & 1 \\ 5 & 0 & 1 \end{pmatrix} \mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{3.4.3}$$

which can be solved by row reducing the augmented matrix as follows

$$\begin{pmatrix} 2 & 1 & 0 & 1 \\ 4 & 1 & 1 & 1 \\ 5 & 0 & 1 & 1 \end{pmatrix} \tag{3.4.4}$$

$$\xleftrightarrow{\text{pivoting}} \begin{pmatrix} 2 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & -5 & 2 & -3 \end{pmatrix} \tag{3.4.5}$$

$$\xleftrightarrow{\text{pivoting}} \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 3 & -2 \end{pmatrix} \xleftrightarrow{\text{pivoting}} \begin{pmatrix} 6 & 0 & 0 & 2 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 3 & -2 \end{pmatrix} \tag{3.4.6}$$

yielding

$$\mathbf{n} = \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \tag{3.4.7}$$

$$\begin{pmatrix} 2 & 2a & a \\ 2 & 3b & b \\ 2 & 4c & c \end{pmatrix} \xleftrightarrow[R_2 \leftarrow R_2 - R_1]{R_3 \leftarrow R_3 - R_1} \begin{pmatrix} 2 & 2a & a \\ 0 & \text{3b-2a} & b - a \\ 0 & 4c - 2a & c - a \end{pmatrix} \tag{3.3.5}$$

$$\text{pivoting} \begin{pmatrix} 2(3b - 2a) & 0 & a(3b - 2a) - 2a(b - a) \\ 0 & \text{3b-2a} & b - a \\ 0 & 0 & (3b - 2a)(c - a) - (b - a)4c - 2a \end{pmatrix} \tag{3.3.6}$$

$$= \begin{pmatrix} 2(3b - 2a) & 0 & ab \\ 0 & 3b - 2a & b - a \\ 0 & 0 & -bc - ab + 2ac \end{pmatrix} \tag{3.3.7}$$

Thus, the equation of the desired plane can be expressed as

$$\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \mathbf{x} = 3 \tag{3.4.8}$$

From (6.2.11.2), the desired image is then obtained as

$$\mathbf{Q} = \begin{pmatrix} 2 \\ 1 \\ 6 \end{pmatrix} + 2\frac{3 - \begin{pmatrix} 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 6 \end{pmatrix}}{6}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \tag{3.4.9}$$

$$= \begin{pmatrix} 6 \\ 5 \\ -2 \end{pmatrix} \tag{3.4.10}$$

3.5. Let

$$\mathbf{A} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} \frac{3}{2} \\ 6 \end{pmatrix} \tag{3.5.1}$$

be the vertices of a triangle. If $\mathbf{P}$ is a point inside the triangle ABC such that the triangles APC, APB and BPC have equal areas, then find the length of the line segment PQ, where

$$\mathbf{Q} = \begin{pmatrix} -\frac{7}{6} \\ -\frac{1}{3} \end{pmatrix} \tag{3.5.2}$$

Solution: The point $\mathbf{P}$ is the median of the given triangle. Hence,

$$\mathbf{P} = \frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} \tag{3.5.3}$$

$$= \begin{pmatrix} \frac{17}{6} \\ \frac{8}{3} \end{pmatrix} \tag{3.5.4}$$

Thus,

$$\mathbf{P} - \mathbf{Q} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \tag{3.5.5}$$

$$\implies PQ = 5 \tag{3.5.6}$$

3.6. The line

$$y = mx + 4 \tag{3.6.1}$$

is tangent to both the parabolas

$$y^2 = 4x \tag{3.6.2}$$

$$x^2 = 2by \tag{3.6.3}$$

Find the value of b.

Solution: The given equations can be expressed as

$$\begin{pmatrix} m & -1 \end{pmatrix} \mathbf{x} = -4 \tag{3.6.4}$$

$$\mathbf{x}^\top \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - 4\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 0 \tag{3.6.5}$$

$$\mathbf{x}^\top \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} - 2\begin{pmatrix} 0 & b \end{pmatrix} \mathbf{x} = 0 \tag{3.6.6}$$

and the respective conic parameters for (3.6.5), (3.6.6) can be expressed as

$$\mathbf{n} = \begin{pmatrix} m \\ -1 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} 1 \\ m \end{pmatrix} \tag{3.6.7}$$

$$\mathbf{V}_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u}_1 = -2\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{p}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.6.8}$$

$$\mathbf{V}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u}_2 = -\begin{pmatrix} 0 \\ b \end{pmatrix}, \mathbf{p}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.6.9}$$

where $\mathbf{p}_1, \mathbf{p}_2$ are the eigenvectors corresponding to the 0 eigenvalue for $\mathbf{V}_1, \mathbf{V}_2$ respectively. From (7.3.10.1) and (7.3.10.2) the expression for the point of contact is given by

$$\begin{pmatrix} \mathbf{u}^\top + \kappa\mathbf{n}^\top \\ \mathbf{V} \end{pmatrix} \mathbf{q} = \begin{pmatrix} -f \\ \kappa\mathbf{n} - \mathbf{u} \end{pmatrix} \tag{3.6.10}$$

$$\text{where } \kappa = \frac{\mathbf{p}^T\mathbf{u}}{\mathbf{p}^T\mathbf{n}}, \quad \mathbf{V}\mathbf{p} = 0 \tag{3.6.11}$$

a) Let $\mathbf{q}_1$ be the point of contact for (3.6.5). Then, from (7.3.10.2),

$$\kappa_1 = \frac{\mathbf{p}_1^T\mathbf{u}_1}{\mathbf{p}_1^T\mathbf{n}} \tag{3.6.12}$$

$$= -2\frac{\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}}{\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} m \\ -1 \end{pmatrix}} \tag{3.6.13}$$

$$= -\frac{2}{m} \tag{3.6.14}$$

Substituting the above in (7.3.10.1),

$$\begin{pmatrix} -2\begin{pmatrix}1 & 0\end{pmatrix} - \frac{2}{m}\begin{pmatrix}m & -1\end{pmatrix} \\ \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\end{pmatrix}\mathbf{q}_1 = \begin{pmatrix}0\\ -\frac{2}{m}\begin{pmatrix}m\\-1\end{pmatrix}+2\begin{pmatrix}1\\0\end{pmatrix}\end{pmatrix}$$

$$\implies \begin{pmatrix}-4 & \frac{2}{m}\\ 0 & 0\\ 0 & 1\end{pmatrix}\mathbf{q}_1 = \begin{pmatrix}0\\0\\\frac{2}{m}\end{pmatrix} \tag{3.6.15}$$

which can be expressed as

$$\begin{pmatrix}-4 & \frac{2}{m}\\ 0 & 1\end{pmatrix}\mathbf{q}_1 = \begin{pmatrix}0\\\frac{2}{m}\end{pmatrix} \tag{3.6.16}$$

$$\implies \mathbf{q}_1 = \begin{pmatrix}\frac{1}{m^2}\\ \\ \frac{2}{m}\end{pmatrix} \tag{3.6.17}$$

b) Similarly,

$$\kappa_2 = \frac{\mathbf{p}_2^T\mathbf{u}_2}{\mathbf{p}_2^T\mathbf{n}} \tag{3.6.18}$$

$$= -\frac{\begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}0\\b\end{pmatrix}}{\begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}m\\-1\end{pmatrix}} \tag{3.6.19}$$

$$= b \tag{3.6.20}$$

Substituting the above in (7.3.10.1),

$$\begin{pmatrix} -\begin{pmatrix}0 & b\end{pmatrix} + b\begin{pmatrix}m & -1\end{pmatrix} \\ \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\end{pmatrix}\mathbf{q}_2 = \begin{pmatrix}0\\ b\begin{pmatrix}m\\-1\end{pmatrix}+\begin{pmatrix}0\\b\end{pmatrix}\end{pmatrix}$$

$$\implies \begin{pmatrix}bm & -2b\\ 1 & 0\\ 0 & 0\end{pmatrix}\mathbf{q}_2 = \begin{pmatrix}0\\bm\\0\end{pmatrix} \tag{3.6.21}$$

which can be expressed as

$$\begin{pmatrix}m & -2\\ 1 & 0\end{pmatrix}\mathbf{q}_1 = \begin{pmatrix}0\\bm\end{pmatrix} \tag{3.6.22}$$

$$\implies \mathbf{q}_2 = \begin{pmatrix}bm\\ \\ \frac{bm^2}{2}\end{pmatrix} \tag{3.6.23}$$

c) Since $\mathbf{q}_1, \mathbf{q}_2$ lie on (3.6.4),

$$\begin{pmatrix}m & -1\end{pmatrix}\mathbf{q}_1 = -4 \tag{3.6.24}$$

$$\begin{pmatrix}m & -1\end{pmatrix}\mathbf{q}_2 = -4 \tag{3.6.25}$$

which can be expressed as

$$\begin{pmatrix}bm & \frac{bm^2}{2}\\ \frac{1}{m^2} & \frac{2}{m}\end{pmatrix}\begin{pmatrix}m\\-1\end{pmatrix} = -4\begin{pmatrix}1\\1\end{pmatrix} \tag{3.6.26}$$

yielding

$$m = \frac{1}{4}, b = -128 \tag{3.6.27}$$

3.7. If the distance between the foci of an ellipse is 6 and the distance between its directrices is 12, then find the length of its latus rectum.
Solution:

3.8. Find the area of the region, enclosed by the circle

$$\mathbf{x}^\top\mathbf{x} = 2 \tag{3.8.1}$$

which is not common to the region bounded by the parabola

$$\mathbf{x}^\top\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\mathbf{x} - \begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = 0 \tag{3.8.2}$$

and the straight line

$$\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = 0. \tag{3.8.3}$$

4 JNTU

4.1. Find the value of k such that the rank of the matrix

$$\begin{pmatrix}1 & 2 & 3\\2 & k & 7\\3 & 6 & 10\end{pmatrix} \tag{4.1.1}$$

is 2.
Solution: Using pivoting,

$$\begin{pmatrix}1 & 2 & 3\\2 & k & 7\\3 & 6 & 10\end{pmatrix} \tag{4.1.2}$$

$$\text{(pivoting)}\ \begin{pmatrix}1 & 2 & 3\\0 & k-4 & 1\\0 & 0 & 1\end{pmatrix} \tag{4.1.3}$$

Thus, the rank of the matrix will be 2 if $k = 4$.

4.2. If a square matrix $\mathbf{A}$ has an eigenvalue λ, then what is the eigenvalue of the matrix $k\mathbf{A}$, where $k \neq 0$ is a scalar.

Solution: By definition of the eigenvalue,

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \tag{4.2.1}$$

$$\implies k\left(\mathbf{A}\mathbf{x}\right) = k\left(\lambda\mathbf{x}\right) \tag{4.2.2}$$

$$\text{or}\left(k\mathbf{A}\right)\mathbf{x} = \left(k\lambda\right)\mathbf{x} \tag{4.2.3}$$

Thus,$k\lambda$ is the eigenvalue of $k\mathbf{A}$.

4.3. If a matrix

$$\mathbf{A} = \begin{pmatrix} -1 & 0 & 0 \\ 2 & -3 & 0 \\ 1 & 4 & 2 \end{pmatrix} \tag{4.3.1}$$

then what are the eigenvalues of $\mathbf{A}^2$? **Solution:** By definition of the eigenvalue,

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \tag{4.3.2}$$

$$\implies \mathbf{A}\left(\mathbf{A}\mathbf{x}\right) = \mathbf{A}\left(\lambda\mathbf{x}\right) = \lambda\left(\mathbf{A}\mathbf{x}\right) \tag{4.3.3}$$

$$\text{or}\mathbf{A}^2\mathbf{x} = \lambda^2\mathbf{x} \tag{4.3.4}$$

Thus,λ^2 is the eigenvalue of $\mathbf{A}^2$. Since $\mathbf{A}$ is a lower triangular matrix, its eigenvalues are the digonal elements. Thus,

$$\lambda = -1, -3, 2 \tag{4.3.5}$$

$$\implies \lambda^2 = 1, 9, 4 \tag{4.3.6}$$

4.4. For what values of λ and μ do the system of equations

$$x + y + z = 6 \tag{4.4.1}$$

$$x + 2y + 3z = 10 \tag{4.4.2}$$

$$x + 2y + \lambda z = \mu \tag{4.4.3}$$

have

a) no solution
b) unique solution
c) more than one solution

Solution: The augmented matrix for the given system can be expressed as

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right) \tag{4.4.4}$$

$$\leftrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda - 1 & \mu - 6 \end{array}\right) \tag{4.4.5}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda - 3 & \mu - 10 \end{array}\right) \tag{4.4.6}$$

From the above, the system has

a) no solution if

$$\lambda - 3 = 0, \mu - 10 \neq 0 \tag{4.4.7}$$

$$\implies \lambda = 3, \mu \neq 10 \tag{4.4.8}$$

b) a unique solution if

$$\lambda - 3 \neq 0 \implies \lambda \neq -3 \tag{4.4.9}$$

c) more than one solution if

$$\lambda - 3 = 0, \mu - 10 = 0 \tag{4.4.10}$$

$$\implies \lambda = 3, \mu = 10 \tag{4.4.11}$$

4.5. Find the value of k for which the system of equations

$$\left(k + 1\right)x + 8y = 4k \tag{4.5.1}$$

$$kx + \left(k + 3\right)y = 3k - 1 \tag{4.5.2}$$

has infinitely many solutions.

Solution: Forming the augmented matrix

$$\left(\begin{array}{cc|c} \text{k+1} & 8 & 4k \\ k & k + 3 & 3k - 1 \end{array}\right) \tag{4.5.3}$$

(4.5.4)

$$\leftrightarrow \left(\begin{array}{cc|c} k + 1 & 1 & 4k \\ 0 & \left(k + 1\right)\left(k + 3\right) - 8k & \left(k + 1\right)\left(3k - 1\right) - 4k^2 \end{array}\right) \tag{4.5.5}$$

Thus, for the given system to have infinite solutions, the above matrix should have a 0 row. Hence,

$$\left(k + 1\right)\left(k + 3\right) - 8k = 0 \tag{4.5.6}$$

$$\left(k + 1\right)\left(3k - 1\right) - 4k^2 = 0 \tag{4.5.7}$$

yielding

$$k^2 - 4k + 3 = 0 \tag{4.5.8}$$
$$k^2 - 2k + 1 = 0 \tag{4.5.9}$$

From the above, it is obvious that $k = 1$ is the only value for which the given system has infinite solutions.

4.6. Verify Cayley-Hamilton Theorem for the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 \\ -1 & -1 & 2 \\ 1 & 2 & 1 \end{pmatrix} \tag{4.6.1}$$

and obtain $\mathbf{A}^{-1}$ and $\mathbf{A}^3$.
Solution: The characteristic equation for the given matrix is obtained a

$$\left|\lambda \mathbf{I} - \mathbf{A}\right| = \begin{vmatrix} \lambda - 1 & 2 & 0 \\ -1 & \lambda + 1 & 2 \\ 1 & 2 & \lambda - 1 \end{vmatrix} = 0 \tag{4.6.2}$$

Doing row operations the above equation can be expressed as

$$\xleftrightarrow{R_2 \leftarrow R_2 + R_3} \begin{vmatrix} \lambda - 1 & 2 & 0 \\ 0 & \lambda + 3 & \lambda + 1 \\ 1 & 2 & \lambda - 1 \end{vmatrix} = 0 \tag{4.6.3}$$

Expanding the above determinant,

$$(\lambda - 1) \begin{vmatrix} \lambda + 3 & \lambda + 1 \\ 2 & \lambda - 1 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ \lambda + 3 & \lambda + 1 \end{vmatrix} = 0 \tag{4.6.4}$$

yielding

$$\lambda^3 - 5\lambda^2 + \lambda + 7 = 0 \tag{4.6.5}$$

Thus, using the Cayley-Hamilton theorem,

$$\mathbf{A}^3 - 5\mathbf{A}^2 + \mathbf{A} + 7\mathbf{I} = 0 \tag{4.6.6}$$

from which,

$$\mathbf{A}^3 = 5\mathbf{A}^2 - \mathbf{A} - 7\mathbf{I} \tag{4.6.7}$$
$$\mathbf{A}^{-1} = \frac{-\mathbf{A}^2 + 5\mathbf{A} - \mathbf{I}}{7} \tag{4.6.8}$$

4.7. Find the LU decomposition of

$$\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 4 & -1 \end{pmatrix} \tag{4.7.1}$$

Solution: Performing row operations,

$$\begin{pmatrix} 1 & 3 \\ 4 & -1 \end{pmatrix} \tag{4.7.2}$$
$$\leftrightarrow \begin{pmatrix} 1 & 3 \\ 0 & -13 \end{pmatrix} = \mathbf{U} \tag{4.7.3}$$

By organizing the pivot columns in the above matrices,

$$\mathbf{L} = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} \tag{4.7.4}$$

It can be easily verified that

$$\mathbf{LU} = \mathbf{A} \tag{4.7.5}$$

4.8. Factorize the matrix

$$\begin{pmatrix} 2 & -3 & 1 \\ 3 & 4 & 2 \\ 2 & -3 & 4 \end{pmatrix} \tag{4.8.1}$$

by the LU decomposition method.
Solution: Row reducing the given matrix,

$$\begin{pmatrix} 2 & -3 & 1 \\ 3 & 4 & 2 \\ 2 & -3 & 4 \end{pmatrix} \leftrightarrow \begin{pmatrix} 2 & -3 & 1 \\ 0 & 17 & 1 \\ 0 & 0 & 6 \end{pmatrix} = \mathbf{U} \tag{4.8.2}$$

Thus,

$$\mathbf{L} = \frac{1}{2} \begin{pmatrix} 2 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \tag{4.8.3}$$

4.9. Reduce the quadratic form

$$3x^2 + 3y^2 + 3z^2 - 2yz + 2zx + 2xy \tag{4.9.1}$$

to its canonical form.
Solution: The given quadratic form can be expressed as

$$\mathbf{x}^\top \mathbf{V} \mathbf{x} \tag{4.9.2}$$

where

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \mathbf{V} = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{pmatrix} \tag{4.9.3}$$

The eigenvalues of $\mathbf{V}$ can be obtained by solving

$$\left|\mathbf{A} - \lambda \mathbf{I}\right| = 0 \tag{4.9.4}$$

which can be expressed as

$$\begin{pmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{pmatrix} = 0 \tag{4.9.5}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - R_2} (\lambda - 4) \begin{pmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & -1 \\ 0 & 1 & -1 \end{pmatrix} = 0 \tag{4.9.6}$$

$$\xleftrightarrow{C_2 \leftarrow C_2 + C_3} (\lambda - 4) \begin{pmatrix} 3-\lambda & 2 & 1 \\ 1 & 2-\lambda & -1 \\ 0 & 0 & -1 \end{pmatrix} = 0 \tag{4.9.7}$$

which can be simplified to obtain

$$(\lambda - 1)(\lambda - 4)^2 = 0 \tag{4.9.8}$$

Thus, the eigenvalues of $\mathbf{V}$ are $1, 4$. The eigenvector corresponding to 4 is given by

$$(\mathbf{A} - \lambda \mathbf{I})\,\mathbf{x} = 0 \tag{4.9.9}$$

$$\implies \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{pmatrix} \mathbf{x} = 0 \tag{4.9.10}$$

Row reducing the above matrix,

$$\begin{pmatrix} 1 & -1 & -1 \end{pmatrix} \mathbf{x} = 0 \tag{4.9.11}$$

resulting in

$$\mathbf{x} = y \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \tag{4.9.12}$$

It is easy to verify that

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \tag{4.9.13}$$

are the eigenvectors corresponding to the eigenvalue 4. The eigenvector corresponding to the eigenvalue 1 can be obtained from (4.9.9) by row reducing the matrix as

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} \tag{4.9.14}$$

$$\xleftrightarrow[R_2 \leftarrow R_1 - 2R_2]{R_3 \leftarrow R_1 - 2R_3} \begin{pmatrix} 2 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \tag{4.9.15}$$

$$\xleftrightarrow[R_1 \leftarrow R_1 + R_2]{} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} \tag{4.9.16}$$

Thus, the desired eigenvector is

$$\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \tag{4.9.17}$$

Stacking the the eigenvectors into a matrix

$$\mathbf{P} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \tag{4.9.18}$$

$$\mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} \tag{4.9.19}$$

Thus, the desired canonical form is

$$\mathbf{y}^\top \mathbf{D} \mathbf{y} \tag{4.9.20}$$

5 Definitions

5.1 2×1 *vectors*

5.1.1. Let

$$\mathbf{A} \equiv \overrightarrow{A} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \tag{5.1.1.1}$$

$$\equiv a_1 \overrightarrow{i} + a_2 \overrightarrow{j}, \tag{5.1.1.2}$$

$$\mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \tag{5.1.1.3}$$

be 2×1 vectors. Then, the determinant of the 2×2 matrix

$$\mathbf{M} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} \tag{5.1.1.4}$$

is defined as

$$\left|\mathbf{M}\right| = \left|\mathbf{A} \quad \mathbf{B}\right| \tag{5.1.1.5}$$

$$= \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1 \tag{5.1.1.6}$$

5.1.2. The value of the cross product of two vectors is given by (5.1.1.5).

5.1.3. The area of the triangle with vertices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ is given by the absolute value of

$$\frac{1}{2} \left|\mathbf{A} - \mathbf{B} \quad \mathbf{A} - \mathbf{C}\right| \tag{5.1.3.1}$$

5.1.4. The transpose of $\mathbf{A}$ is defined as

$$\mathbf{A}^\top = \begin{pmatrix} a_1 & a_2 \end{pmatrix} \tag{5.1.4.1}$$

5.1.5. The *inner product* or *dot product* is defined as

$$\mathbf{A}^{\top}\mathbf{B} \equiv \mathbf{A}\cdot\mathbf{B} \tag{5.1.5.1}$$

$$= \begin{pmatrix} a_1 & a_2 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = a_1b_1 + a_2b_2 \tag{5.1.5.2}$$

5.1.6. *norm* of $\mathbf{A}$ is defined as

$$\|A\| \equiv \left|\overrightarrow{A}\right| \tag{5.1.6.1}$$

$$= \sqrt{\mathbf{A}^{\top}\mathbf{A}} = \sqrt{a_1^2 + a_2^2} \tag{5.1.6.2}$$

Thus,

$$\|\lambda\mathbf{A}\| \equiv \left|\lambda\overrightarrow{A}\right| \tag{5.1.6.3}$$

$$= |\lambda|\,\|\mathbf{A}\| \tag{5.1.6.4}$$

5.1.7. The distance betwen the points $\mathbf{A}$ and $\mathbf{B}$ is given by

$$\|\mathbf{A}-\mathbf{B}\| \tag{5.1.7.1}$$

5.1.8. Let $\mathbf{x}$ be equidistant from the points $\mathbf{A}$ and $\mathbf{B}$. Then

$$\|\mathbf{x}-\mathbf{A}\| = \|\mathbf{A}-\mathbf{B}\| \tag{5.1.8.1}$$

$$\implies \|\mathbf{x}-\mathbf{A}\|^2 = \|\mathbf{x}-\mathbf{B}\|^2 \tag{5.1.8.2}$$

which can be expressed as

$$\left(\mathbf{x}-\mathbf{A}\right)^{\top}\left(\mathbf{x}-\mathbf{A}\right) = \left(\mathbf{x}-\mathbf{B}\right)^{\top}\left(\mathbf{x}-\mathbf{B}\right)$$

$$\implies \|\mathbf{x}\|^2 - 2\mathbf{x}^{\top}\mathbf{A} + \|\mathbf{A}\|^2$$

$$= \|\mathbf{x}\|^2 - 2\mathbf{x}^{\top}\mathbf{B} + \|\mathbf{B}\|^2 \tag{5.1.8.3}$$

which can be simplified to obtain

$$\left(\mathbf{A}-\mathbf{B}\right)^{\top}\mathbf{x} = \frac{\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2}{2} \tag{5.1.8.4}$$

5.1.9. The angle between two vectors is given by

$$\theta = \cos^{-1}\frac{\mathbf{A}^{\top}\mathbf{B}}{\|A\|\,\|B\|} \tag{5.1.9.1}$$

5.1.10. If two vectors are orthogonal (perpendicular),

$$\mathbf{A}^{\top}\mathbf{B} = 0 \tag{5.1.10.1}$$

5.1.11. The *direction vector* of the line joining two points $\mathbf{A}, \mathbf{B}$ is given by

$$\mathbf{m} = \mathbf{A}-\mathbf{B} \tag{5.1.11.1}$$

5.1.12. The unit vector in the direction of $\mathbf{m}$ is defined as

$$\frac{\mathbf{m}}{\|\mathbf{m}\|} \tag{5.1.12.1}$$

5.1.13. If the direction vector of a line is expressed as

$$\mathbf{m} = \begin{pmatrix} 1 \\ m \end{pmatrix}, \tag{5.1.13.1}$$

the m is defined to be the slope of the line.

5.1.14. The *normal vector* to $\mathbf{m}$ is defined by

$$\mathbf{m}^{\top}\mathbf{n} = 0 \tag{5.1.14.1}$$

5.1.15. The point $\mathbf{P}$ that divides the line segment AB in the ratio $k:1$ is given by

$$\mathbf{P} = \frac{k\mathbf{B}+\mathbf{A}}{k+1} \tag{5.1.15.1}$$

5.1.16. The standard basis vectors are defined as

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{5.1.16.1}$$

5.2 3×1 *vectors*

5.2.1. Let

$$\mathbf{A} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \equiv a_1\overrightarrow{i} + a_2\overrightarrow{j} + a_3\overrightarrow{j}, \tag{5.2.1.1}$$

$$\mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \tag{5.2.1.2}$$

and

$$\mathbf{A}_{ij} = \begin{pmatrix} a_i \\ a_j \end{pmatrix}, \mathbf{B}_{ij} = \begin{pmatrix} b_i \\ b_j \end{pmatrix}, \tag{5.2.1.3}$$

5.2.2. The *cross product* or *vector product* of $\mathbf{A}, \mathbf{B}$ is defined as

$$\mathbf{A}\times\mathbf{B} = \begin{pmatrix} \begin{vmatrix} \mathbf{A}_{23} & \mathbf{B}_{23} \end{vmatrix} \\ \begin{vmatrix} \mathbf{A}_{31} & \mathbf{B}_{31} \end{vmatrix} \\ \begin{vmatrix} \mathbf{A}_{12} & \mathbf{B}_{12} \end{vmatrix} \end{pmatrix} \tag{5.2.2.1}$$

5.2.3. Verify that

$$\mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A} \tag{5.2.3.1}$$

5.3 Eigenvalues and Eigenvectors

5.3.1. The eigenvalue λ and the eigenvector $\mathbf{x}$ for a matrix $\mathbf{A}$ are defined as,

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \tag{5.3.1.1}$$

5.3.2. The eigenvalues are calculated by solving the equation

$$f(\lambda) = \left|\lambda\mathbf{I} - \mathbf{A}\right| = 0 \tag{5.3.2.1}$$

The above equation is known as the characteristic equation.

5.3.3. According to the Cayley-Hamilton theorem,

$$f(\lambda) = 0 \implies f(\mathbf{A}) = 0 \tag{5.3.3.1}$$

5.3.4. The trace of a square matrix is defined to be the sum of the diagonal elements.

$$\text{tr}(\mathbf{A}) = \sum_{i=1}^{N} a_{ii}. \tag{5.3.4.1}$$

where a_{ii} is the ith diagonal element of the matrix $\mathbf{A}$.

5.3.5. The trace of a matrix is equal to the sum of the eigenvalues

$$\text{tr}(\mathbf{A}) = \sum_{i=1}^{N} \lambda_i \tag{5.3.5.1}$$

5.4 Determinants

5.4.1. Let

$$\mathbf{A} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}. \tag{5.4.1.1}$$

be a 3×3 matrix. Then,

$$\left|\mathbf{A}\right| = a_1 \begin{pmatrix} b_2 & c_2 \\ b_3 & c_3 \end{pmatrix} - a_2 \begin{pmatrix} b_1 & c_1 \\ b_3 & c_3 \end{pmatrix} + a_3 \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}. \tag{5.4.1.2}$$

5.4.2. Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of a matrix $\mathbf{A}$. Then, the product of the eigenvalues is equal to the determinant of $\mathbf{A}$.

$$\left|\mathbf{A}\right| = \prod_{i=1}^{n} \lambda_i \tag{5.4.2.1}$$

5.4.3.

$$\left|\mathbf{A}\mathbf{B}\right| = \left|\mathbf{A}\right|\left|\mathbf{B}\right| \tag{5.4.3.1}$$

5.4.4. If $\mathbf{A}$ be an $n \times n$ matrix,

$$\left|k\mathbf{A}\right| = k^n \left|\mathbf{A}\right| \tag{5.4.4.1}$$

5.5 Rank of a Matrix

5.5.1. The rank of a matrix is defined as the number of linearly independent rows. This is also known as the row rank.

5.5.2. Row rank = Column rank.

5.5.3. The rank of a matrix is obtained as the number of nonzero rows obtained after row reduction.

5.5.4. An $n \times n$ matrix is invertible if and only if its rank is n.

5.6 Inverse of a Matrix

5.6.1. For a 2×2 matrix

$$\mathbf{A} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}, \tag{5.6.1.1}$$

the inverse is given by

$$\mathbf{A}^{-1} = \frac{1}{\left|\mathbf{A}\right|} \begin{pmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{pmatrix}, \tag{5.6.1.2}$$

5.6.2. For higher order matrices, the inverse should be calculated using row operations.

6 Linear Forms

6.1 Two Dimensions

6.1.1. The equation of a line is given by

$$\mathbf{n}^\top \mathbf{x} = c \tag{6.1.1.1}$$

where $\mathbf{n}$ is the normal vector of the line.

6.1.2. The equation of a line with normal vector $\mathbf{n}$ and passing through a point $\mathbf{A}$ is given by

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{A}) = 0 \tag{6.1.2.1}$$

6.1.3. The parametric equation of a line is given by

$$\mathbf{x} = \mathbf{A} + \lambda\mathbf{m} \tag{6.1.3.1}$$

where $\mathbf{m}$ is the direction vector of the line and $\mathbf{A}$ is any point on the line.

6.1.4. The distance from a point $\mathbf{P}$ to the line in (6.1.1.1) is given by

$$d = \frac{\left|\mathbf{n}^\top \mathbf{P} - c\right|}{\|\mathbf{n}\|} \tag{6.1.4.1}$$

Solution: Without loss of generality, let $\mathbf{A}$ be the foot of the perpendicular from $\mathbf{P}$ to the line in (6.1.3.1). The equation of the normal to (6.1.1.1) can then be expressed as

$$\mathbf{x} = \mathbf{A} + \lambda\mathbf{n} \tag{6.1.4.2}$$

$$\implies \mathbf{P} - \mathbf{A} = \lambda\mathbf{n} \tag{6.1.4.3}$$

$\because$ $\mathbf{P}$ lies on (6.1.4.2). From the above, the desired distance can be expressed as

$$d = \|\mathbf{P}-\mathbf{A}\| = |\lambda| \|\mathbf{n}\| \tag{6.1.4.4}$$

From (6.1.4.3),

$$\mathbf{n}^\top(\mathbf{P}-\mathbf{A}) = \lambda\mathbf{n}^\top\mathbf{n} = \lambda\|\mathbf{n}\|^2 \tag{6.1.4.5}$$

$$\implies |\lambda| = \frac{\left|\mathbf{n}^\top(\mathbf{P}-\mathbf{A})\right|}{\|\mathbf{n}\|^2} \tag{6.1.4.6}$$

Substituting the above in (6.1.4.4) and using the fact that

$$\mathbf{n}^\top\mathbf{A} = c \tag{6.1.4.7}$$

from (6.1.1.1), yields (6.1.4.1).

6.1.5. The distance from the origin to the line in (6.1.1.1) is given by

$$d = \frac{|c|}{\|\mathbf{n}\|} \tag{6.1.5.1}$$

6.1.6. The distance between the parallel lines

$$\begin{aligned}\mathbf{n}^\top\mathbf{x} &= c_1\\ \mathbf{n}^\top\mathbf{x} &= c_2\end{aligned} \tag{6.1.6.1}$$

is given by

$$d = \frac{|c_1-c_2|}{\|\mathbf{n}\|} \tag{6.1.6.2}$$

6.1.7. The equation of the line perpendicular to (6.1.1.1) and passing through the point $\mathbf{P}$ is given by

$$\mathbf{m}^\top(\mathbf{x}-\mathbf{P}) = 0 \tag{6.1.7.1}$$

6.1.8. The foot of the perpendicular from $\mathbf{P}$ to the line in (6.1.1.1) is given by

$$\begin{pmatrix}\mathbf{m} & \mathbf{n}\end{pmatrix}^\top \mathbf{x} = \begin{pmatrix}\mathbf{m}^\top\mathbf{P}\\ c\end{pmatrix} \tag{6.1.8.1}$$

Solution: From (6.1.1.1) and (6.1.2.1) the foot of the perpendicular satisfies the equations

$$\mathbf{n}^\top\mathbf{x} = c \tag{6.1.8.2}$$

$$\mathbf{m}^\top(\mathbf{x}-\mathbf{P}) = 0 \tag{6.1.8.3}$$

where $\mathbf{m}$ is the direction vector of the given line. Combining the above into a matrix equation results in (6.1.8.1).

6.2 Three Dimensions

6.2.1. The area of a triangle with vertices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ is given by

$$\frac{1}{2}\|(\mathbf{A}-\mathbf{B})\times(\mathbf{A}-\mathbf{C})\| \tag{6.2.1.1}$$

6.2.2. Points $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are on a line if

$$\operatorname{rank}\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\\ \mathbf{C}\end{pmatrix} = 1 \tag{6.2.2.1}$$

6.2.3. Points $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ form a paralelogram if

$$\operatorname{rank}\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\\ \mathbf{C}\\ \mathbf{D}\end{pmatrix} = 1, \operatorname{rank}\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\\ \mathbf{C}\end{pmatrix} = 2 \tag{6.2.3.1}$$

6.2.4. The equation of a line is given by (6.1.3.1)

6.2.5. The equation of a plane is given by (6.1.1.1)

6.2.6. The distance from the origin to the line in (6.1.1.1) is given by (6.1.5.1)

6.2.7. The distance from a point $\mathbf{P}$ to the line in (6.1.3.1) is given by

$$d = \|\mathbf{A}-\mathbf{P}\|^2 - \frac{\left\{\mathbf{m}^\top(\mathbf{A}-\mathbf{P})\right\}^2}{\|\mathbf{m}\|^2} \tag{6.2.7.1}$$

Solution:

$$d(\lambda) = \|\mathbf{A}+\lambda\mathbf{m}-\mathbf{P}\| \tag{6.2.7.2}$$

$$\implies d^2(\lambda) = \|\mathbf{A}+\lambda\mathbf{m}-\mathbf{P}\|^2 \tag{6.2.7.3}$$

which can be simplified to obtain

$$\begin{aligned}d^2(\lambda) = \lambda^2\|\mathbf{m}\|^2 + 2\lambda\mathbf{m}^\top(\mathbf{A}-\mathbf{P})\\ + \|\mathbf{A}-\mathbf{P}\|^2\end{aligned} \tag{6.2.7.4}$$

which is of the form

$$d^2(\lambda) = a\lambda^2 + 2b\lambda + c \tag{6.2.7.5}$$

$$= a\left\{\left(\lambda + \frac{b}{a}\right)^2 + \left[\frac{c}{a} - \left(\frac{b}{a}\right)^2\right]\right\} \tag{6.2.7.6}$$

with

$$a = \|\mathbf{m}\|^2, b = \mathbf{m}^\top(\mathbf{A}-\mathbf{P}), c = \|\mathbf{A}-\mathbf{P}\|^2 \tag{6.2.7.7}$$

which can be expressed as From the above, $d^2(\lambda)$ is smallest when upon substituting from (6.2.7.7)

$$\lambda + \frac{b}{2a} = 0 \implies \lambda = -\frac{b}{2a} \tag{6.2.7.8}$$

$$= -\frac{\mathbf{m}^\top(\mathbf{A}-\mathbf{P})}{\|\mathbf{m}\|^2} \tag{6.2.7.9}$$

and consequently,

$$d_{\min}(\lambda) = a\left(\frac{c}{a} - \left(\frac{b}{a}\right)^2\right) \tag{6.2.7.10}$$

$$= c - \frac{b^2}{a} \tag{6.2.7.11}$$

yielding (6.2.7.1) after substituting from (6.2.7.7).

6.2.8. The distance between the parallel planes (6.1.6.1) is given by (6.1.6.2).

6.2.9. The plane

$$\mathbf{n}^\top\mathbf{x} = c \tag{6.2.9.1}$$

contains the line

$$\mathbf{x} = \mathbf{A} + \lambda\mathbf{m} \tag{6.2.9.2}$$

if

$$\mathbf{m}^\top\mathbf{n} = 0 \tag{6.2.9.3}$$

Solution: Any point on the line (6.2.9.2) should also satisfy (6.2.9.1). Hence,

$$\mathbf{n}^\top(\mathbf{A}+\lambda\mathbf{m}) = \mathbf{n}^\top\mathbf{A} = c \tag{6.2.9.4}$$

which can be simplified to obtain (6.2.9.3)

6.2.10. The foot of the perpendicular from a point $\mathbf{P}$ to the plane

$$\mathbf{n}^\top\mathbf{x} = c \tag{6.2.10.1}$$

is given by

Solution: The equation of the line perpendicular to the given plane and passing through $\mathbf{P}$ is

$$\mathbf{x} = \mathbf{P} + \lambda\mathbf{n} \tag{6.2.10.2}$$

From (6.2.13.1), the intersection of the above line with the given plane is

$$\mathbf{x} = \mathbf{P} + \frac{c - \mathbf{n}^\top\mathbf{P}}{\|\mathbf{n}\|^2}\mathbf{n} \tag{6.2.10.3}$$

6.2.11. The image of a point $\mathbf{P}$ with respect to the plane

$$\mathbf{n}^\top\mathbf{x} = c \tag{6.2.11.1}$$

is given by

$$\mathbf{R} = \mathbf{P} + 2\frac{c - \mathbf{n}^\top\mathbf{P}}{\|\mathbf{n}\|^2} \tag{6.2.11.2}$$

Solution: Let $\mathbf{R}$ be the desired image. Then, subtituting the expression for the foot of the perpendicular from $\mathbf{P}$ to the given plane using (6.2.10.3),

$$\frac{\mathbf{P}+\mathbf{R}}{2} = \mathbf{P} + \frac{c - \mathbf{n}^\top\mathbf{P}}{\|\mathbf{n}\|^2} \tag{6.2.11.3}$$

6.2.12. Let a plane pass through the points $\mathbf{A}, \mathbf{B}$ and be perpendicular to the plane

$$\mathbf{n}^\top\mathbf{x} = c \tag{6.2.12.1}$$

Then the equation of this plane is given by

$$\mathbf{p}^\top\mathbf{x} = 1 \tag{6.2.12.2}$$

where

$$\mathbf{p} = \begin{pmatrix}\mathbf{A} & \mathbf{B} & \mathbf{n}\end{pmatrix}^{-\top}\begin{pmatrix}1\\1\\0\end{pmatrix} \tag{6.2.12.3}$$

Solution: From the given information,

$$\mathbf{p}^\top\mathbf{A} = d \tag{6.2.12.4}$$

$$\mathbf{p}^\top\mathbf{B} = d \tag{6.2.12.5}$$

$$\mathbf{p}^\top\mathbf{n} = 0 \tag{6.2.12.6}$$

$\because$ the normal vectors to the two planes will also be perpendicular. The system of equations in (6.2.12.6) can be expressed as the matrix equation

$$\begin{pmatrix}\mathbf{A} & \mathbf{B} & \mathbf{n}\end{pmatrix}^\top\mathbf{p} = d\begin{pmatrix}1\\1\\0\end{pmatrix} \tag{6.2.12.7}$$

which yields (6.2.12.3) upon normalising with d.

6.2.13. The intersection of the line represented by (6.1.3.1) with the plane represented by (6.1.1.1) is given by

$$\mathbf{x} = \mathbf{A} + \frac{c - \mathbf{n}^\top\mathbf{A}}{\mathbf{n}^\top\mathbf{m}}\mathbf{m} \tag{6.2.13.1}$$

Solution: From (6.1.3.1) and (6.1.1.1),

$$\mathbf{x} = \mathbf{A} + \lambda\mathbf{m} \tag{6.2.13.2}$$
$$\mathbf{n}^\top\mathbf{x} = c \tag{6.2.13.3}$$
$$\implies \mathbf{n}^\top\left(\mathbf{A} + \lambda\mathbf{m}\right) = c \tag{6.2.13.4}$$

which can be simplified to obtain

$$\mathbf{n}^\top\mathbf{A} + \lambda\mathbf{n}^\top\mathbf{m} = c \tag{6.2.13.5}$$
$$\implies \lambda = \frac{c - \mathbf{n}^\top\mathbf{A}}{\mathbf{n}^\top\mathbf{m}} \tag{6.2.13.6}$$

Substituting the above in (6.2.13.4) yields (6.2.13.1).

6.2.14. The foot of the perpendicular from the point $\mathbf{P}$ to the line represented by (6.1.3.1) is given by

$$\mathbf{x} = \mathbf{A} + \frac{\mathbf{m}^\top\left(\mathbf{P} - \mathbf{A}\right)}{\left\|\mathbf{m}\right\|^2}\mathbf{m} \tag{6.2.14.1}$$

Solution: Let the equation of the line be

$$\mathbf{x} = \mathbf{A} + \lambda\mathbf{m} \tag{6.2.14.2}$$

The equation of the plane perpendicular to the given line passing through $\mathbf{P}$ is given by

$$\mathbf{m}^\top\left(\mathbf{x} - \mathbf{P}\right) = 0 \tag{6.2.14.3}$$
$$\implies \mathbf{m}^\top\mathbf{x} = \mathbf{m}^\top\mathbf{P} \tag{6.2.14.4}$$

The desired foot of the perpendicular is the intersection of (6.2.14.2) with (6.2.14.3) which can be obtained from (6.2.13.1) as (6.2.14.1)

6.2.15. The foot of the perpendicular from a point $\mathbf{P}$ to a plane is $\mathbf{Q}$. The equation of the plane is given by

$$\left(\mathbf{P} - \mathbf{Q}\right)^\top\left(\mathbf{x} - \mathbf{Q}\right) = 0 \tag{6.2.15.1}$$

Solution: The normal vector to the plane is given by

$$\mathbf{n} = \mathbf{P} - \mathbf{Q} \tag{6.2.15.2}$$

Hence, the equation of the plane is (6.2.15.1).

6.2.16. Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be points on a plane. The equation of the plane is then given by

$$\begin{pmatrix}\mathbf{A} & \mathbf{B} & \mathbf{C}\end{pmatrix}^\top\mathbf{n} = \begin{pmatrix}1\\1\\1\end{pmatrix} \tag{6.2.16.1}$$

Solution: Let the equation of the plane be

$$\mathbf{n}^\top\mathbf{x} = 1 \tag{6.2.16.2}$$

Then

$$\mathbf{n}^\top\mathbf{A} = 1 \tag{6.2.16.3}$$
$$\mathbf{n}^\top\mathbf{B} = 1 \tag{6.2.16.4}$$
$$\mathbf{n}^\top\mathbf{C} = 1 \tag{6.2.16.5}$$

which can be combined to obtain (6.2.16.1).

6.2.17. (Affine Transformation) Let $\mathbf{A}, \mathbf{C}$, be opposite sides of a square. The other two points can be obtained as

$$\mathbf{B} = \frac{\left\|\mathbf{A} - \mathbf{C}\right\|}{\sqrt{2}}\mathbf{P}\mathbf{e}_1 + \mathbf{A} \tag{6.2.17.1}$$
$$\mathbf{D} = \frac{\left\|\mathbf{A} - \mathbf{C}\right\|}{\sqrt{2}}\mathbf{P}\mathbf{e}_2 + \mathbf{A} \tag{6.2.17.2}$$

where

$$\mathbf{P} = \begin{pmatrix}\cos\left(\theta - \frac{\pi}{4}\right) & \sin\left(\theta - \frac{\pi}{4}\right)\\ \sin\left(\theta - \frac{\pi}{4}\right) & \cos\left(\theta - \frac{\pi}{4}\right)\end{pmatrix} \tag{6.2.17.3}$$

and

$$\cos\theta = \frac{\left(\mathbf{C} - \mathbf{A}\right)^\top\mathbf{e}_1}{\left\|\mathbf{A} - \mathbf{C}\right\|\left\|\mathbf{e}_1\right\|} \tag{6.2.17.4}$$

7 Quadratic Forms

7.1 Conic Sections

7.1.1. Let $\mathbf{P}$ be a point such that the ratio of its distance from a fixed point $\mathbf{F}$ and the distance (d) from a fixed line $L : \mathbf{n}^\top\mathbf{x} = c$ is constant, given by

$$\frac{\left\|\mathbf{P} - \mathbf{F}\right\|}{d} = e \tag{7.1.1.1}$$

The locus of $\mathbf{P}$ such is known as a conic section. The line L is known as the directrix and the point $\mathbf{F}$ is the focus. e is defined to be the eccentricity of the conic.

a) For $e = 1$, the conic is a parabola
b) For $e < 1$, the conic is an ellipse
c) For $e > 1$, the conic is a hyperbola

7.1.2. The equation of a conic with directrix $\mathbf{n}^\top\mathbf{x} = c$, eccentricity e and focus $\mathbf{F}$ is given by

$$\mathbf{x}^\top\mathbf{V}\mathbf{x} + 2\mathbf{u}^\top\mathbf{x} + f = 0 \tag{7.1.2.1}$$

where

$$\mathbf{V} = \left\|\mathbf{n}\right\|^2\mathbf{I} - e^2\mathbf{n}\mathbf{n}^\top, \tag{7.1.2.2}$$
$$\mathbf{u} = ce^2\mathbf{n} - \left\|\mathbf{n}\right\|^2\mathbf{F}, \tag{7.1.2.3}$$
$$f = \left\|\mathbf{n}\right\|^2\left\|\mathbf{F}\right\|^2 - c^2e^2 \tag{7.1.2.4}$$

Solution: From (7.1.1.1) and (6.1.4.1), for any point $\mathbf{x}$ on the conic,

$$\left\|\mathbf{x}-\mathbf{F}\right\|^2 = e^2\frac{\left(\mathbf{n}^\top\mathbf{x}-c\right)^2}{\left\|\mathbf{n}\right\|^2} \tag{7.1.2.5}$$

$$\implies \left\|\mathbf{n}\right\|^2\left(\mathbf{x}-\mathbf{F}\right)^\top\left(\mathbf{x}-\mathbf{F}\right) = e^2\left(\mathbf{n}^\top\mathbf{x}-c\right)^2 \tag{7.1.2.6}$$

yielding

$$\begin{aligned}\left\|\mathbf{n}\right\|^2&\left(\mathbf{x}^\top\mathbf{x}-2\mathbf{F}^\top\mathbf{x}+\left\|\mathbf{F}\right\|^2\right)\\ &= e^2\left(c^2+\left(\mathbf{n}^\top\mathbf{x}\right)^2-2c\mathbf{n}^\top\mathbf{x}\right)\\ &= e^2\left(c^2+\left(\mathbf{x}^\top\mathbf{n}\mathbf{n}^\top\mathbf{x}\right)-2c\mathbf{n}^\top\mathbf{x}\right)\end{aligned} \tag{7.1.2.7}$$

which can be expressed as (7.1.2.1) after simplification.

7.1.3. (7.1.2.1) represents
   a) a parabola for $\left|\mathbf{V}\right| = 0$,
   b) ellipse for $\left|\mathbf{V}\right| > 0$ and
   c) hyperbola for $\left|\mathbf{V}\right| < 0$.

In general (7.1.2.1) represents a conic if and only if

$$\begin{vmatrix}\mathbf{V} & \mathbf{u}\\ \mathbf{u}^\top & f\end{vmatrix} \neq 0 \tag{7.1.3.1}$$

else, it represents a pair of straight lines.

7.2 Conic Parameters

7.2.1. The conic in (7.1.2.1) can be expressed in standard form (centre/vertex at the origin, major axis - x axis) as

$$\mathbf{y}^\top\mathbf{D}\mathbf{y} = \mathbf{u}^\top\mathbf{V}^{-1}\mathbf{u}-f \quad |V| \neq 0 \tag{7.2.1.1}$$

$$\mathbf{y}^\top\mathbf{D}\mathbf{y} = -2\eta\mathbf{e}_1^\top\mathbf{y} \quad |V| = 0 \tag{7.2.1.2}$$

where

$$\mathbf{P}^\top\mathbf{V}\mathbf{P} = \mathbf{D}. \quad \text{(Eigenvalue Decomposition)} \tag{7.2.1.3}$$

$$\mathbf{D} = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}, \tag{7.2.1.4}$$

$$\mathbf{P} = \begin{pmatrix}\mathbf{p}_1 & \mathbf{p}_2\end{pmatrix}, \quad \mathbf{P}^\top = \mathbf{P}^{-1}, \tag{7.2.1.5}$$

$$\eta = \mathbf{u}^\top\mathbf{p}_1 \tag{7.2.1.6}$$

$$\mathbf{e}_1 = \begin{pmatrix}1\\0\end{pmatrix} \tag{7.2.1.7}$$

Solution: Using

$$\mathbf{x} = \mathbf{P}\mathbf{y}+\mathbf{c} \quad \text{(Affine Transformation)} \tag{7.2.1.8}$$

(7.1.2.1) can be expressed as

$$\begin{aligned}\left(\mathbf{P}\mathbf{y}+\mathbf{c}\right)^T\mathbf{V}\left(\mathbf{P}\mathbf{y}+\mathbf{c}\right)+2\mathbf{u}^T\left(\mathbf{P}\mathbf{y}+\mathbf{c}\right)+f\\ = 0,\end{aligned} \tag{7.2.1.9}$$

yielding

$$\begin{aligned}\mathbf{y}^T\mathbf{P}^T\mathbf{V}\mathbf{P}\mathbf{y}+2\left(\mathbf{V}\mathbf{c}+\mathbf{u}\right)^T\mathbf{P}\mathbf{y}\\ +\mathbf{c}^T\mathbf{V}\mathbf{c}+2\mathbf{u}^T\mathbf{c}+f = 0\end{aligned} \tag{7.2.1.10}$$

From (7.2.1.10) and (7.2.1.3),

$$\begin{aligned}\mathbf{y}^T\mathbf{D}\mathbf{y}+2\left(\mathbf{V}\mathbf{c}+\mathbf{u}\right)^T\mathbf{P}\mathbf{y}\\ +\mathbf{c}^T\left(\mathbf{V}\mathbf{c}+\mathbf{u}\right)+\mathbf{u}^T\mathbf{c}+f = 0\end{aligned} \tag{7.2.1.11}$$

When $\mathbf{V}^{-1}$ exists,

$$\mathbf{V}\mathbf{c}+\mathbf{u} = \mathbf{0}, \quad \text{or}, \mathbf{c} = -\mathbf{V}^{-1}\mathbf{u}, \tag{7.2.1.12}$$

and substituting (7.2.1.12) in (7.2.1.11) yields (7.2.1.1). When $|\mathbf{V}| = 0, \lambda_1 = 0$ and

$$\mathbf{V}\mathbf{p}_1 = 0, \mathbf{V}\mathbf{p}_2 = \lambda_2\mathbf{p}_2. \tag{7.2.1.13}$$

where $\mathbf{p}_1, \mathbf{p}_2$ are the eigenvectors of $\mathbf{V}$ such that (7.2.1.3)

$$\mathbf{P} = \begin{pmatrix}\mathbf{p}_1 & \mathbf{p}_2\end{pmatrix}, \tag{7.2.1.14}$$

Substituting (7.2.1.14) in (7.2.1.11),

$$\begin{aligned}&\mathbf{y}^T\mathbf{D}\mathbf{y}+2\left(\mathbf{c}^T\mathbf{V}+\mathbf{u}^T\right)\begin{pmatrix}\mathbf{p}_1 & \mathbf{p}_2\end{pmatrix}\mathbf{y}\\ &\quad+\mathbf{c}^T\left(\mathbf{V}\mathbf{c}+\mathbf{u}\right)+\mathbf{u}^T\mathbf{c}+f = 0\\ &\implies \mathbf{y}^T\mathbf{D}\mathbf{y}\\ &+2\left(\left(\mathbf{c}^T\mathbf{V}+\mathbf{u}^T\right)\mathbf{p}_1 \; \left(\mathbf{c}^T\mathbf{V}+\mathbf{u}^T\right)\mathbf{p}_2\right)\mathbf{y}\\ &\quad+\mathbf{c}^T\left(\mathbf{V}\mathbf{c}+\mathbf{u}\right)+\mathbf{u}^T\mathbf{c}+f = 0\\ &\implies \mathbf{y}^T\mathbf{D}\mathbf{y}\\ &+2\left(\mathbf{u}^T\mathbf{p}_1 \quad \left(\lambda_2\mathbf{c}^T+\mathbf{u}^T\right)\mathbf{p}_2\right)\mathbf{y}\\ &+\mathbf{c}^T\left(\mathbf{V}\mathbf{c}+\mathbf{u}\right)+\mathbf{u}^T\mathbf{c}+f = 0 \text{ from (7.2.1.13)}\end{aligned}$$

$$\begin{aligned}\implies \lambda_2y_2^2+2\left(\mathbf{u}^T\mathbf{p}_1\right)y_1+2y_2\left(\lambda_2\mathbf{c}+\mathbf{u}\right)^T\mathbf{p}_2\\ +\mathbf{c}^T\left(\mathbf{V}\mathbf{c}+\mathbf{u}\right)+\mathbf{u}^T\mathbf{c}+f = 0\end{aligned}$$

which is the equation of a parabola. Thus, (7.2.1.15) can be expressed as (7.2.1.2) by choosing

$$\eta = \mathbf{u}^T \mathbf{p}_1 \tag{7.2.1.15}$$

and $\mathbf{c}$ in (7.2.1.11) such that

$$\mathbf{P}^T \left(\mathbf{V}\mathbf{c} + \mathbf{u}\right) = \eta \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{7.2.1.16}$$

$$\mathbf{c}^T \left(\mathbf{V}\mathbf{c} + \mathbf{u}\right) + \mathbf{u}^T\mathbf{c} + f = 0 \tag{7.2.1.17}$$

Multiplying (7.2.1.16) by $\mathbf{P}$ yields

$$\left(\mathbf{V}\mathbf{c} + \mathbf{u}\right) = \eta\mathbf{p}_1, \tag{7.2.1.18}$$

which, upon substituting in (7.2.1.17) results in

$$\eta\mathbf{c}^T\mathbf{p}_1 + \mathbf{u}^T\mathbf{c} + f = 0 \tag{7.2.1.19}$$

(7.2.1.18) and (7.2.1.19) can be clubbed together to obtain (7.2.2.2).

7.2.2. The centre/vertex of the conic is given by

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \qquad |V| \neq 0 \tag{7.2.2.1}$$

$$\begin{pmatrix} \mathbf{u}^\top + \eta\mathbf{p}_1^\top \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \eta\mathbf{p}_1 - \mathbf{u} \end{pmatrix} \qquad |V| = 0 \tag{7.2.2.2}$$

Solution: From (7.2.1.8),

$$\mathbf{y} = \mathbf{P}^\top \left(\mathbf{x} - \mathbf{c}\right) \tag{7.2.2.3}$$

For the standard conic, $\mathbf{y} = \mathbf{0}$ is the centre/vertex and in (7.2.2.3),

$$\mathbf{y} = \mathbf{0} \implies \mathbf{x} = \mathbf{c} \tag{7.2.2.4}$$

7.2.3. The focal length of the parabola in (7.2.1.2) is given by

$$\left|\frac{2\eta}{\lambda_2}\right| \tag{7.2.3.1}$$

where λ_2 is the nonzero eigenvalue of $\mathbf{V}$ and η is defined in (7.2.1.6).

7.2.4. For $\left|V\right| \neq 0$, the lengths of the semi-major and semi-minor axes of the conic in (7.1.2.1) are given by

$$\sqrt{\frac{\mathbf{u}^\top\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}}, \sqrt{\frac{\mathbf{u}^\top\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_2}}. \qquad \text{(ellipse)} \tag{7.2.4.1}$$

$$\sqrt{\frac{\mathbf{u}^\top\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}}, \sqrt{\frac{f - \mathbf{u}^\top\mathbf{V}^{-1}\mathbf{u}}{\lambda_2}}, \qquad \text{(hyperbola)} \tag{7.2.4.2}$$

Solution: For

$$|\mathbf{V}| > 0, \quad \text{or, } \lambda_1 > 0, \lambda_2 > 0 \tag{7.2.4.3}$$

and (7.2.1.1) becomes

$$\lambda_1 y_1^2 + \lambda_2 y_2^2 = \mathbf{u}^\top\mathbf{V}^{-1}\mathbf{u} - f \tag{7.2.4.4}$$

yielding (7.2.4.1). Similarly, (7.2.4.2) can be obtained for

$$|\mathbf{V}| < 0, \quad \text{or, } \lambda_1 > 0, \lambda_2 < 0 \tag{7.2.4.5}$$

7.2.5. The equation of the minor and major axes are respectively given by

$$\mathbf{p}_i^\top \left(\mathbf{x} - \mathbf{c}\right) = 0, i = 1, 2 \tag{7.2.5.1}$$

7.2.6. The eccentricity, directrices and foci of (7.1.2.1) are given by (7.2.6.1) - (7.2.6.4) **Solution:** From (7.1.2.2),

$$\begin{aligned} \mathbf{V}^\top\mathbf{V} &= \left(\left\|\mathbf{n}\right\|^2 \mathbf{I} - e^2\mathbf{n}\mathbf{n}^\top\right)^\top \\ &\quad \left(\left\|\mathbf{n}\right\|^2 \mathbf{I} - e^2\mathbf{n}\mathbf{n}^\top\right) \\ \implies \mathbf{V}^2 &= \left\|\mathbf{n}\right\|^4 \mathbf{I} + e^4\mathbf{n}\mathbf{n}^\top\mathbf{n}\mathbf{n}^\top \\ &\quad - 2e^2 \left\|\mathbf{n}\right\|^2 \mathbf{n}\mathbf{n}^\top \\ &= \left\|\mathbf{n}\right\|^4 \mathbf{I} + e^4 \left\|\mathbf{n}\right\|^2 \mathbf{n}\mathbf{n}^\top - 2e^2 \left\|\mathbf{n}\right\|^2 \mathbf{n}\mathbf{n}^\top \\ &= \left\|\mathbf{n}\right\|^4 \mathbf{I} + e^2 \left(e^2 - 2\right) \left\|\mathbf{n}\right\|^2 \mathbf{n}\mathbf{n}^\top \\ &= \left\|\mathbf{n}\right\|^4 \mathbf{I} + \left(e^2 - 2\right) \left\|\mathbf{n}\right\|^2 \left(\left\|\mathbf{n}\right\|^2 \mathbf{I} - \mathbf{V}\right) \end{aligned} \tag{7.2.6.5}$$

which can be expressed as

$$\mathbf{V}^2 + \left(e^2 - 2\right) \left\|\mathbf{n}\right\|^2 \mathbf{V} - \left(e^2 - 1\right) \left\|\mathbf{n}\right\|^4 \mathbf{I} = 0 \tag{7.2.6.6}$$

Using the Cayley-Hamilton theorem, (7.2.6.6) results in the characteristic equation,

$$\lambda^2 - \left(2 - e^2\right) \left\|\mathbf{n}\right\|^2 \lambda + \left(1 - e^2\right) \left\|\mathbf{n}\right\|^4 = 0 \tag{7.2.6.7}$$

which can be expressed as

$$\left(\frac{\lambda}{\left\|\mathbf{n}\right\|^2}\right)^2 - \left(2 - e^2\right)\left(\frac{\lambda}{\left\|\mathbf{n}\right\|^2}\right) + \left(1 - e^2\right) = 0 \tag{7.2.6.8}$$

$$\implies \frac{\lambda}{\left\|\mathbf{n}\right\|^2} = 1 - e^2, 1 \tag{7.2.6.9}$$

$$\text{or, } \lambda_2 = \left\|\mathbf{n}\right\|^2, \lambda_1 = \left(1 - e^2\right)\lambda_2 \tag{7.2.6.10}$$

From (7.2.6.10), the eccentricity of (7.1.2.1) is given by (7.2.6.1). Multiplying both sides of (7.1.2.2) by $\mathbf{n}$,

$$\mathbf{V}\mathbf{n} = \|\mathbf{n}\|^2 \mathbf{n} - e^2 \mathbf{n}\mathbf{n}^\top \mathbf{n} \tag{7.2.6.11}$$
$$= \|\mathbf{n}\|^2 \left(1 - e^2\right) \mathbf{n} \tag{7.2.6.12}$$
$$= \lambda_1 \mathbf{n} \tag{7.2.6.13}$$

from (7.2.6.10) Thus, λ_1 is the corresponding eigenvalue for $\mathbf{n}$. From (7.2.1.5), (7.2.6.10) and (7.2.6.13),

$$\mathbf{n} = \|\mathbf{n}\| \mathbf{p}_1 = \sqrt{\lambda_2}\mathbf{p}_1 \tag{7.2.6.14}$$

From (7.1.2.3) and (7.2.6.10),

$$\mathbf{F} = \frac{ce^2\mathbf{n} - \mathbf{u}}{\lambda_2} \tag{7.2.6.15}$$
$$\implies \|\mathbf{F}\|^2 = \frac{\left(ce^2\mathbf{n} - \mathbf{u}\right)^\top \left(ce^2\mathbf{n} - \mathbf{u}\right)}{\lambda_2^2} \tag{7.2.6.16}$$
$$\implies \lambda_2^2 \|\mathbf{F}\|^2 = c^2e^4\lambda_2 - 2ce^2\mathbf{u}^\top\mathbf{n} + \|\mathbf{u}\|^2 \tag{7.2.6.17}$$

Also, (7.1.2.4) can be expressed as

$$\lambda_2 \|\mathbf{F}\|^2 = f + c^2e^2 \tag{7.2.6.18}$$

From (7.2.6.17) and (7.2.6.18),

$$c^2e^4\lambda_2 - 2ce^2\mathbf{u}^\top\mathbf{n} + \|\mathbf{u}\|^2 = \lambda_2 \left(f + c^2e^2\right) \tag{7.2.6.19}$$

$$\implies \lambda_2 e^2 \left(e^2 - 1\right) c^2 - 2ce^2\mathbf{u}^\top\mathbf{n} + \|\mathbf{u}\|^2 - \lambda_2 f = 0 \tag{7.2.6.20}$$

yielding (7.2.6.4).

7.3 Tangent and Normal

7.3.1. The points of intersection of the line

$$L: \quad \mathbf{x} = \mathbf{q} + \mu\mathbf{m} \quad \mu \in \mathbb{R} \tag{7.3.1.1}$$

with the conic section in (7.1.2.1) are given by

$$\mathbf{x}_i = \mathbf{q} + \mu_i\mathbf{m} \tag{7.3.1.2}$$

where

$$\mu_i = \frac{1}{\mathbf{m}^T\mathbf{V}\mathbf{m}} \left(-\mathbf{m}^T \left(\mathbf{V}\mathbf{q} + \mathbf{u}\right) \pm \sqrt{\left[\mathbf{m}^T \left(\mathbf{V}\mathbf{q} + \mathbf{u}\right)\right]^2 - \left(\mathbf{q}^T\mathbf{V}\mathbf{q} + 2\mathbf{u}^T\mathbf{q} + f\right)\left(\mathbf{m}^T\mathbf{V}\mathbf{m}\right)}\right) \tag{7.3.1.3}$$

Solution: Substituting (7.3.1.1) in (7.1.2.1),

$$\left(\mathbf{q} + \mu\mathbf{m}\right)^T \mathbf{V} \left(\mathbf{q} + \mu\mathbf{m}\right) \tag{7.3.1.4}$$
$$+2\mathbf{u}^T \left(\mathbf{q} + \mu\mathbf{m}\right) + f = 0 \tag{7.3.1.5}$$
$$\implies \mu^2\mathbf{m}^T\mathbf{V}\mathbf{m} + 2\mu\mathbf{m}^T \left(\mathbf{V}\mathbf{q} + \mathbf{u}\right) \tag{7.3.1.6}$$
$$+\mathbf{q}^T\mathbf{V}\mathbf{q} + 2\mathbf{u}^T\mathbf{q} + f = 0 \tag{7.3.1.7}$$

Solving the above quadratic in (7.3.1.7) yields (7.3.1.3).

7.3.2. (*Latus Rectum*) The latus rectum of a conic section is the chord that passes through the focus, is perpendicular to the major axis and has both endpoints on the curve.

7.3.3. The latus rectum is parallel to the directrix.

7.3.4. The equation of the latus rectum is given by

$$\mathbf{n}^\top \left(\mathbf{x} - \mathbf{F}\right) = 0 \tag{7.3.4.1}$$
$$\text{or, } \mathbf{x} = \mathbf{F} + \mu\mathbf{m} \tag{7.3.4.2}$$

$$e = \sqrt{1 - \frac{\lambda_1}{\lambda_2}} \tag{7.2.6.1}$$
$$\mathbf{n} = \sqrt{\lambda_2}\mathbf{p}_1, \tag{7.2.6.2}$$
$$c = \begin{cases} \frac{e\mathbf{u}^\top\mathbf{n} \pm \sqrt{e^2\left(\mathbf{u}^\top\mathbf{n}\right)^2 - \lambda_2\left(e^2-1\right)\left(\|\mathbf{u}\|^2 - \lambda_2 f\right)}}{\lambda_2 e\left(e^2-1\right)} & e \neq 1 \\ \frac{\|\mathbf{u}\|^2 - \lambda_2 f}{2e^2\mathbf{u}^\top\mathbf{n}} & e = 1 \end{cases} \tag{7.2.6.3}$$
$$\mathbf{F} = \frac{ce^2\mathbf{n} - \mathbf{u}}{\lambda_2} \tag{7.2.6.4}$$

where $\mathbf{F}$ is the focus and $\mathbf{m}$ is the normal to the directrix, i.e.

$$\mathbf{m}^\top \mathbf{n} = 0 \tag{7.3.4.3}$$

7.3.5. The affine transform preserves the norm. This implies that the length of any chord of a conic is invariant to translation and/or rotation. **Solution:** Let From (7.2.1.8),

$$\mathbf{x}_i = \mathbf{P}\mathbf{y}_i + \mathbf{c} \tag{7.3.5.1}$$

be any two points on the conic. Then the distance between the points is given by

$$\|\mathbf{x}_1 - \mathbf{x}_2\| = \|\mathbf{P}\mathbf{y}_1 - \mathbf{y}_2\| \tag{7.3.5.2}$$

$$\implies \|\mathbf{x}_1 - \mathbf{x}_2\|^2 = (\mathbf{y}_1 - \mathbf{y}_2)^\top \mathbf{P}^\top \mathbf{P} (\mathbf{y}_1 - \mathbf{y}_2) \tag{7.3.5.3}$$

$$= \|\mathbf{y}_1 - \mathbf{y}_2\|^2 \tag{7.3.5.4}$$

since

$$\mathbf{P}^\top \mathbf{P} = \mathbf{I} \tag{7.3.5.5}$$

7.3.6. The length of the latus rectum is given by **Solution:**
From (7.3.1.2) and (7.3.1.3), substituting $\mathbf{q} = \mathbf{F}$, the end points of the latus rectum on the conic section can be obtained. Thus, the distance between these points is given by

$$\|\mathbf{x}_1 - \mathbf{x}_2\| = |\mu_1 - \mu_2| \, \|\mathbf{m}\| \tag{7.3.6.1}$$

which can be used to obtain the length of the latus rectum as

$$\frac{2\sqrt{\left[\mathbf{m}^T\left(\mathbf{V}\mathbf{F}+\mathbf{u}\right)\right]^2 - \left(\mathbf{F}^T\mathbf{V}\mathbf{F} + 2\mathbf{u}^T\mathbf{F} + f\right)\left(\mathbf{m}^T\mathbf{V}\mathbf{m}\right)}}{\mathbf{m}^T\mathbf{V}\mathbf{m}} \|\mathbf{m}\| \tag{7.3.6.2}$$

a) From (7.3.5.4), we may consider the standard ellipse/hyperbola given by (7.2.1.1) as

$$\mathbf{y}^\top \mathbf{D}\mathbf{y} = -f, \mathbf{V} = \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$
$$f = -1, \mathbf{u} = 0, \mathbf{p}_1 = \mathbf{e}_1, \mathbf{p}_2 = \mathbf{e}_2 \tag{7.3.6.3}$$

for computing the length of the latus rectum in (7.3.6.9). Note that $\mathbf{p}_1, \mathbf{p}_2$ are the eigenvectors and $\mathbf{e}_1, \mathbf{e}_2$ are the standard basis vectors. Substituting from (7.3.6.3) in (7.2.6.2), the parameters of the directrix are obtained as

$$\mathbf{n} = \sqrt{\lambda_2}\mathbf{e}_1, \tag{7.3.6.4}$$
$$c = \pm\frac{1}{e\sqrt{e^2 - 1}} \tag{7.3.6.5}$$

and the focus is

$$e = \sqrt{1 - \frac{\lambda_1}{\lambda_2}} \tag{7.3.6.6}$$
$$\mathbf{F} = \frac{e}{\sqrt{\lambda_2}\sqrt{e^2 - 1}}\mathbf{e}_1 \tag{7.3.6.7}$$

From (7.3.6.4),

$$\mathbf{m} = \mathbf{e}_1 \tag{7.3.6.8}$$

Substituting the above in (7.3.6.9) along with (7.3.6.6) and (7.3.6.7),
the length of the latus rectum for an ellipse and hyperbola is obtained from (7.3.6.9) as
.

$$\frac{2\sqrt{\left[\mathbf{m}^T\left(\mathbf{V}\mathbf{F}+\mathbf{u}\right)\right]^2 - \left(\mathbf{F}^T\mathbf{V}\mathbf{F} + 2\mathbf{u}^T\mathbf{F} + f\right)\left(\mathbf{m}^T\mathbf{V}\mathbf{m}\right)}}{\mathbf{m}^T\mathbf{V}\mathbf{m}} \|\mathbf{m}\| \tag{7.3.6.9}$$

Solution: For simplicity, we consider the standard ellipse given by

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} = 1, \tag{7.3.6.10}$$

$$\text{where } \mathbf{V} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \mathbf{u} = 0, f = -1, \tag{7.3.6.11}$$

From (7.3.6.11) and (7.2.6.4), the distance between the foci can be expressed as

$$, \|\mathbf{F}_1 - \mathbf{F}_2\| = e^2 \left|\frac{c_1 - c_2}{\lambda_2}\right| \|\mathbf{n}\| \tag{7.3.6.12}$$

where c_1, c_2 are the scalar parameters of the directrices. The distances between the directrices is given by

$$\frac{|c_1 - c_2|}{\|\mathbf{n}\|} \tag{7.3.6.13}$$

Thus, substituting the above in (7.3.6.12),

$$, \frac{\|\mathbf{F}_1 - \mathbf{F}_2\|}{|c_1 - c_2|} \|n\| = e^2 \frac{\|\mathbf{n}\|^2}{|\lambda_2|} \tag{7.3.6.14}$$
$$= \frac{6}{12} = \frac{1}{2}, \tag{7.3.6.15}$$

based on the given information. For the standard ellipse,

$$\mathbf{p}_1 = \mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{7.3.6.16}$$

$$\implies \mathbf{n} = \sqrt{\lambda_2}\mathbf{e}_1, \tag{7.3.6.17}$$

$$\text{or, } \mathbf{m} = \sqrt{\lambda_2}\mathbf{e}_2, \tag{7.3.6.18}$$

Hence, substituting in (7.3.6.14) and using (7.2.6.2)

$$e^2 \frac{\|\mathbf{n}\|^2}{\lambda_2} = \frac{1}{2} \tag{7.3.6.19}$$

$$\implies e^2 = \frac{1}{2} \tag{7.3.6.20}$$

Substituting $\mathbf{u} = 0$ in (7.2.6.2) and (7.2.6.4) and simplifying using (7.2.6.1),

$$c = \pm \frac{1}{e}\sqrt{\frac{\lambda_1}{\lambda_2}} \tag{7.3.6.21}$$

$$\mathbf{F} = \pm \frac{e}{\sqrt{\lambda_1}}\mathbf{p}_1 \tag{7.3.6.22}$$

For the standard ellipse, $\mathbf{m}$, orthogonal to $\mathbf{n}$ is also an eigenvector, such that

$$\mathbf{V}\mathbf{m} = \lambda_1\mathbf{m} \tag{7.3.6.23}$$

$$\implies \mathbf{m} = \sqrt{\lambda_1}\mathbf{e}_2 \tag{7.3.6.24}$$

Thus,

$$\mathbf{m}^\top\mathbf{V}\mathbf{m} = {\lambda_1}^2 \tag{7.3.6.25}$$

$$\mathbf{m}^\top\mathbf{V}\mathbf{F} = 0 \tag{7.3.6.26}$$

$$\mathbf{F}^\top\mathbf{V}\mathbf{F} = e^2, \tag{7.3.6.27}$$

From (7.3.6.9), the length of the latus rectum is given by

$$\frac{2\sqrt{\left[\mathbf{m}^T\left(\mathbf{V}\mathbf{F}+\mathbf{u}\right)\right]^2 - \left(\mathbf{F}^T\mathbf{V}\mathbf{F} + 2\mathbf{u}^T\mathbf{F} + f\right)\left(\mathbf{m}^T\mathbf{V}\mathbf{m}\right)}}{\mathbf{m}^T\mathbf{V}\mathbf{m}} \|\mathbf{m}\| \tag{7.3.6.28}$$

Substituting (7.3.6.27) in the above, the desired length can be expressed as,

7.3.7. If L in (7.3.1.1) touches (7.1.2.1) at exactly one point $\mathbf{q}$,

$$\mathbf{m}^T\left(\mathbf{V}\mathbf{q}+\mathbf{u}\right) = 0 \tag{7.3.7.1}$$

Solution: In this case, (7.3.1.7) has exactly one root. Hence, in (7.3.1.3)

$$\left[\mathbf{m}^T\left(\mathbf{V}\mathbf{q}+\mathbf{u}\right)\right]^2 - \left(\mathbf{m}^T\mathbf{V}\mathbf{m}\right) \left(\mathbf{q}^T\mathbf{V}\mathbf{q} + 2\mathbf{u}^T\mathbf{q} + f\right) = 0 \tag{7.3.7.2}$$

$\because$ $\mathbf{q}$ is the point of contact, $\mathbf{q}$ satisfies (7.1.2.1) and

$$\mathbf{q}^T\mathbf{V}\mathbf{q} + 2\mathbf{u}^T\mathbf{q} + f = 0 \tag{7.3.7.3}$$

Substituting (7.3.7.3) in (7.3.7.2) and simplifying, we obtain (7.3.7.1).

7.3.8. Given the point of contact $\mathbf{q}$, the equation of a tangent to (7.1.2.1) is

$$\left(\mathbf{V}\mathbf{q}+\mathbf{u}\right)^T\mathbf{x} + \mathbf{u}^T\mathbf{q} + f = 0 \tag{7.3.8.1}$$

Solution: The normal vector is obtained from (7.3.7.1) and (6.1.1.1) as

$$\mathbf{n} = \mathbf{V}\mathbf{q}+\mathbf{u} \tag{7.3.8.2}$$

From (7.3.8.2) and (6.1.2.1), the equation of the tangent is

$$\left(\mathbf{V}\mathbf{q}+\mathbf{u}\right)^T\left(\mathbf{x}-\mathbf{q}\right) = 0 \tag{7.3.8.3}$$

$$\implies \left(\mathbf{V}\mathbf{q}+\mathbf{u}\right)^T\mathbf{x} - \mathbf{q}^T\mathbf{V}\mathbf{q} - \mathbf{u}^T\mathbf{q} = 0 \tag{7.3.8.4}$$

which, upon substituting from (7.3.7.3) and simplifying yields (7.3.1.1).

7.3.9. If $\mathbf{V}^{-1}$ exists, given the normal vector $\mathbf{n}$, the tangent points of contact to (7.1.2.1) are given by

$$\mathbf{q}_i = \mathbf{V}^{-1}\left(\kappa_i\mathbf{n} - \mathbf{u}\right), i = 1, 2$$
$$\text{where } \kappa_i = \pm\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\mathbf{n}^T\mathbf{V}^{-1}\mathbf{n}}} \tag{7.3.9.1}$$

Solution: From (7.3.8.2),

$$\mathbf{q} = \mathbf{V}^{-1}\left(\kappa\mathbf{n} - \mathbf{u}\right), \quad \kappa \in \mathbb{R} \tag{7.3.9.2}$$

Substituting (7.3.9.2) in (7.3.7.3),

$$\left(\kappa\mathbf{n} - \mathbf{u}\right)^T\mathbf{V}^{-1}\left(\kappa\mathbf{n} - \mathbf{u}\right) + 2\mathbf{u}^T\mathbf{V}^{-1}\left(\kappa\mathbf{n} - \mathbf{u}\right) + f = 0 \tag{7.3.9.3}$$

$$\implies \kappa^2\mathbf{n}^T\mathbf{V}^{-1}\mathbf{n} - \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} + f = 0 \tag{7.3.9.4}$$

$$\text{or, } \kappa = \pm\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\mathbf{n}^T\mathbf{V}^{-1}\mathbf{n}}} \tag{7.3.9.5}$$

Substituting (7.3.9.5) in (7.3.9.2) yields (7.3.9.1).

7.3.10. If $\mathbf{V}$ is not invertible, given the normal vector $\mathbf{n}$, the point of contact to (7.1.2.1) is given by the matrix equation

$$\begin{pmatrix} \mathbf{u}^\top + \kappa\mathbf{n}^\top \\ \mathbf{V} \end{pmatrix} \mathbf{q} = \begin{pmatrix} -f \\ \kappa\mathbf{n} - \mathbf{u} \end{pmatrix} \tag{7.3.10.1}$$

$$\text{where } \kappa = \frac{\mathbf{p}_1^T\mathbf{u}}{\mathbf{p}_1^T\mathbf{n}}, \quad \mathbf{V}\mathbf{p}_1 = 0 \tag{7.3.10.2}$$

Solution: If $\mathbf{V}$ is non-invertible, it has a zero eigenvalue. If the corresponding eigenvector is $\mathbf{p}_1$, then,

$$\mathbf{V}\mathbf{p}_1 = 0 \tag{7.3.10.3}$$

From (7.3.8.2),

$$\kappa\mathbf{n} = \mathbf{V}\mathbf{q} + \mathbf{u}, \quad \kappa \in \mathbb{R} \tag{7.3.10.4}$$

$$\implies \kappa\mathbf{p}_1^T\mathbf{n} = \mathbf{p}_1^T\mathbf{V}\mathbf{q} + \mathbf{p}_1^T\mathbf{u} \tag{7.3.10.5}$$

$$\text{or, } \kappa\mathbf{p}_1^T\mathbf{n} = \mathbf{p}_1^T\mathbf{u}, \quad \because \mathbf{p}_1^T\mathbf{V} = 0, \tag{7.3.10.6}$$

$$(\text{ from } (7.3.10.3)) \tag{7.3.10.7}$$

yielding κ in (7.3.10.2). From (7.3.10.4),

$$\kappa\mathbf{q}^T\mathbf{n} = \mathbf{q}^T\mathbf{V}\mathbf{q} + \mathbf{q}^T\mathbf{u} \tag{7.3.10.8}$$

$$\implies \kappa\mathbf{q}^T\mathbf{n} = -f - \mathbf{q}^T\mathbf{u} \quad \text{from (7.3.7.3)}, \tag{7.3.10.9}$$

$$\text{or, } (\kappa\mathbf{n} + \mathbf{u})^\top \mathbf{q} = -f \tag{7.3.10.10}$$

(7.3.10.4) can be expressed as

$$\mathbf{V}\mathbf{q} = \kappa\mathbf{n} - \mathbf{u}. \tag{7.3.10.11}$$

(7.3.10.10) and (7.3.10.11) clubbed together result in (7.3.10.1).

7.3.11. When (7.1.2.1) is a hyperbola, its *asymptotes* are defined as the pair of intersecting straight lines

$$\mathbf{x}^\top\mathbf{V}\mathbf{x} + 2\mathbf{u}^\top\mathbf{x} + \mathbf{u}^\top\mathbf{V}^{-1}\mathbf{u} = 0, \quad |\mathbf{V}| < 0 \tag{7.3.11.1}$$

7.3.12. (7.3.11.1) can be expressed as the lines

$$\left(\sqrt{|\lambda_1|} \quad \pm\sqrt{|\lambda_2|}\right) \mathbf{P}^\top (\mathbf{x} - \mathbf{c}) = 0 \tag{7.3.12.1}$$

Solution: Reducing (7.3.11.1) to standard form using the *affine transformation* yields

$$\lambda_1 y_1^2 - (-\lambda_2)\, y_1^2 = 0 \tag{7.3.12.2}$$

From (7.3.11.1), the equation of the asymptotes for (7.3.12.2) is

$$\left(\sqrt{|\lambda_1|} \quad \pm\sqrt{|\lambda_2|}\right) \mathbf{y} = 0 \tag{7.3.12.3}$$

from which (7.3.12.1) is obtained using (7.2.1.8).

7.3.13. The angle between the asymptotes is then given by using the inner product

$$\cos\theta = \frac{|\lambda_1| - |\lambda_2|}{|\lambda_1| + |\lambda_2|} \tag{7.3.13.1}$$

7.3.14. The normal vectors of the lines in (7.3.12.1) are

$$\begin{aligned} \mathbf{n}_1 &= \mathbf{P}\begin{pmatrix} \sqrt{|\lambda_1|} \\ \sqrt{|\lambda_2|} \end{pmatrix} \\ \mathbf{n}_2 &= \mathbf{P}\begin{pmatrix} \sqrt{|\lambda_1|} \\ -\sqrt{|\lambda_2|} \end{pmatrix} \end{aligned} \tag{7.3.14.1}$$

The angle between the asymptotes is given by

$$\cos\theta = \frac{\mathbf{n_1}^\top\mathbf{n_2}}{\|\mathbf{n_1}\|\,\|\mathbf{n_2}\|} \tag{7.3.14.2}$$

The orthogonal matrix $\mathbf{P}$ preserves the norm, i.e.

$$\|\mathbf{n_1}\| = \left\|\mathbf{P}\begin{pmatrix} \sqrt{|\lambda_1|} \\ \sqrt{|\lambda_2|} \end{pmatrix}\right\| \tag{7.3.14.3}$$

$$= \left\|\begin{pmatrix} \sqrt{|\lambda_1|} \\ \sqrt{|\lambda_2|} \end{pmatrix}\right\| = \sqrt{|\lambda_1| + |\lambda_2|} = \|\mathbf{n_2}\| \tag{7.3.14.4}$$

It is easy to verify that

$$\mathbf{n_1}^\top\mathbf{n_2} = |\lambda_1| - |\lambda_2| \tag{7.3.14.5}$$

Thus, the angle between the asymptotes is obtained from (7.3.14.2) as (7.3.13.1).

7.3.15. Another hyperbola with the same asymptotes as (7.3.12.1) can be obtained from (7.1.2.1) and (7.3.11.1) as

$$\mathbf{x}^\top\mathbf{V}\mathbf{x} + 2\mathbf{u}^\top\mathbf{x} + 2\mathbf{u}^\top\mathbf{V}^{-1}\mathbf{u} - f = 0 \tag{7.3.15.1}$$