Vector Properties

G V V Sharma*

Contents

***Abstract*—This book provides a computational approach to school geometry based on the NCERT textbooks from Class 6-12. Links to sample Python codes are available in the text.**

1 Direction Vector

1.1. Show that the line through the points $\begin{pmatrix}4\\7\\8\end{pmatrix}$, $\begin{pmatrix}2\\3\\4\end{pmatrix}$ is parallel to the line through the points $\begin{pmatrix}-1\\-2\\1\end{pmatrix}$, $\begin{pmatrix}1\\2\\5\end{pmatrix}$.

Solution: The direction vector of the line joining $\begin{pmatrix}4\\7\\8\end{pmatrix}$, $\begin{pmatrix}2\\3\\4\end{pmatrix}$ is

$$\mathbf{m}_1 = \begin{pmatrix}4\\7\\8\end{pmatrix} - \begin{pmatrix}2\\3\\4\end{pmatrix} \tag{1.1.1}$$

$$= \begin{pmatrix}2\\4\\4\end{pmatrix} \tag{1.1.2}$$

Similarly, the direction vector of the line joining $\begin{pmatrix}-1\\-2\\1\end{pmatrix}$, $\begin{pmatrix}1\\2\\5\end{pmatrix}$ is

$$\mathbf{m}_2 = \begin{pmatrix}-1\\-2\\1\end{pmatrix} - \begin{pmatrix}1\\2\\5\end{pmatrix} \tag{1.1.3}$$

$$= \begin{pmatrix}-2\\-4\\-4\end{pmatrix} = -\mathbf{m}_1 \tag{1.1.4}$$

By definition, from (1.1.4), the lines with direction vectors $\mathbf{m}_1, \mathbf{m}_2$ are parallel.

1.2. Find the slope of a line, which passes through the origin, and the mid-point of the line segment joining the points $\mathbf{P} = \begin{pmatrix}0\\-4\end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix}8\\0\end{pmatrix}$.

Solution: The mid-point of the line segment joining the given points is

$$\mathbf{Q} = \frac{\mathbf{P}+\mathbf{B}}{2} \tag{1.2.1}$$

The direction vector of OQ, where $\mathbf{O}$ is the origin, is

$$\mathbf{m} = \mathbf{Q} - \mathbf{O} = \mathbf{Q} \tag{1.2.2}$$

Substituting numerical values in (1.2.1)

$$\mathbf{Q} = \frac{1}{2}\left[\begin{pmatrix}0\\-4\end{pmatrix} + \begin{pmatrix}8\\0\end{pmatrix}\right] \tag{1.2.3}$$

$$= \begin{pmatrix}4\\-2\end{pmatrix} \tag{1.2.4}$$

which can be simplified to express

$$\mathbf{m} = \begin{pmatrix}1\\-\frac{1}{2}\end{pmatrix} \tag{1.2.5}$$

1.3. Find the direction vector of PQ, where

$$\mathbf{P} = \begin{pmatrix}2\\3\\0\end{pmatrix}, \mathbf{Q} = \begin{pmatrix}-1\\-2\\-4\end{pmatrix} \tag{1.3.1}$$

*The author is with the Department of Electrical Engineering, Indian Institute of Technology, Hyderabad 502285 India e-mail: gadepall@iith.ac.in.

Solution: The direction vector of PQ is

$$\mathbf{P}-\mathbf{Q}=\begin{pmatrix}3\\5\\4\end{pmatrix}, \tag{1.3.2}$$

1.4. Find the direction vectors and slopes of the lines passing through the points

a) $\begin{pmatrix}3\\-2\end{pmatrix}$ and $\begin{pmatrix}-1\\4\end{pmatrix}$.

b) $\begin{pmatrix}3\\-2\end{pmatrix}$ and $\begin{pmatrix}7\\-2\end{pmatrix}$.

c) $\begin{pmatrix}3\\-2\end{pmatrix}$ and $\begin{pmatrix}3\\4\end{pmatrix}$.

d) Making an inclination of $60°$ with the positive direction of the x-axis.

Solution:

a) If the direction vector is

$$\begin{pmatrix}1\\m\end{pmatrix}, \tag{1.4.1}$$

the slope is m. Thus, the direction vector is

$$\begin{pmatrix}-1\\4\end{pmatrix}-\begin{pmatrix}3\\-2\end{pmatrix}=\begin{pmatrix}-4\\6\end{pmatrix}\equiv-\frac{1}{4}\begin{pmatrix}-4\\6\end{pmatrix} \tag{1.4.2}$$

$$=\begin{pmatrix}1\\-\frac{3}{2}\end{pmatrix}\implies m=-\frac{3}{2} \tag{1.4.3}$$

b) The direction vector is

$$\begin{pmatrix}7\\-2\end{pmatrix}-\begin{pmatrix}3\\-2\end{pmatrix}=\begin{pmatrix}4\\0\end{pmatrix} \tag{1.4.4}$$

$$=\begin{pmatrix}1\\0\end{pmatrix}\implies m=0 \tag{1.4.5}$$

c) The direction vector is

$$\begin{pmatrix}3\\4\end{pmatrix}-\begin{pmatrix}3\\-2\end{pmatrix}=\begin{pmatrix}0\\6\end{pmatrix} \tag{1.4.6}$$

$$=\begin{pmatrix}1\\\infty\end{pmatrix}\implies m=\infty \tag{1.4.7}$$

d) The slope is $m=\tan 60°=\sqrt{3}$ and the direction vector is

$$\begin{pmatrix}1\\\sqrt{3}\end{pmatrix} \tag{1.4.8}$$

1.5. Without using distance formula, show that points $\begin{pmatrix}-2\\-1\end{pmatrix}$, $\begin{pmatrix}4\\0\end{pmatrix}$, $\begin{pmatrix}3\\3\end{pmatrix}$ and $\begin{pmatrix}-3\\2\end{pmatrix}$ are the vertices of a parallelogram.

Solution: Since

$$\because \mathbf{A}-\mathbf{B}=\mathbf{D}-\mathbf{C} \tag{1.5.1}$$

$$\mathbf{A}-\mathbf{D}=\mathbf{B}-\mathbf{C}, \tag{1.5.2}$$

$AB \parallel CD$ and $AD \parallel BC$. Hence, $ABCD$ is a $\parallel$gm.

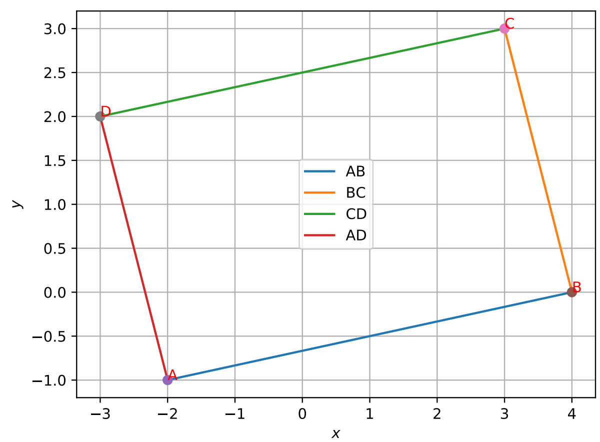

Fig. 1.5.

1.6. The two opposite vertices of a square are $\begin{pmatrix}-1\\2\end{pmatrix}$, $\begin{pmatrix}3\\2\end{pmatrix}$. Find the coordinates of the other two vertices.

Solution: See Fig. **??**.

a) From inspection we see that the opposite vertices forms a diagonal which is parallel to x-axis. Then the diagonal formed by other two vertices is parallel to y-axis(i.e. their x coordinates are equal). Let $\mathbf{A}=\begin{pmatrix}-1\\2\end{pmatrix}$ and $\mathbf{C}=\begin{pmatrix}3\\2\end{pmatrix}$.

b) Diagonals bisect each other at $90°$. Let $\mathbf{B}$ and $\mathbf{D}$ be other two vertices.

c) Using the property that diagonals bisect each other at $90°$, we can obtain other vertices by rotating diagonal AC by $90°$about $\mathbf{E}$ in clockwise or anticlockwise direction.

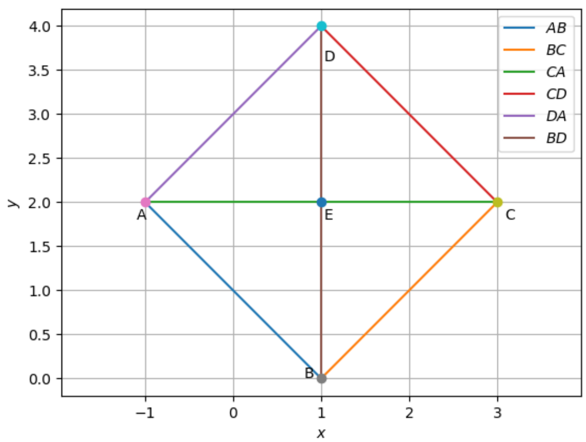

Fig. 1.6. Square ABCD

d) The rotation matrix for a rotation of angle 90°about origin in anticlockwise direction is given by

$$\begin{pmatrix} \cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{1.6.1}$$

The $\mathbf{E}$ is given by

$$\mathbf{E} = \frac{\mathbf{A}+\mathbf{C}}{2} \tag{1.6.2}$$

$$= \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{1.6.3}$$

e) To make the rotation we need to shift the $\mathbf{E}$ to origin. So the change in other vectors are

$$\mathbf{A} - \mathbf{E} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \tag{1.6.4}$$

$$\mathbf{C} - \mathbf{E} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \tag{1.6.5}$$

The required matrix now is $\begin{pmatrix} -2 & 2 \\ 0 & 0 \end{pmatrix}$. Multiplying this with rotation matrix

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -2 & 2 \\ 0 & 0 \end{pmatrix} \tag{1.6.6}$$

$$= \begin{pmatrix} 0 & 0 \\ -2 & 2 \end{pmatrix} \tag{1.6.7}$$

Now we obtained the coordinates as $\begin{pmatrix} 0 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$. To obtain the final coordinates we will add $\mathbf{E}$ to shift to the actual position.

$$\mathbf{B} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{1.6.8}$$

$$\mathbf{D} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{1.6.9}$$

Thus

$$\mathbf{B} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.6.10}$$

$$\mathbf{D} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \tag{1.6.11}$$

f) The python code for the figure can be downloaded from

```
solutions/7/codes/quad/quad.py
```

1.7. Find the direction vectors of the sides of a triangle with vertices $\mathbf{A} = \begin{pmatrix} 3 \\ 5 \\ -4 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} -5 \\ -5 \\ -2 \end{pmatrix}$

Solution:

1.8. Find the direction vector of the line, which makes an angle of 30° with the y-axis measured anticlockwise.

1.9. Find the direction vectors and and y-intercepts of the following lines

a) $\begin{pmatrix} 1 & 7 \end{pmatrix} \mathbf{x} = 0.$
b) $\begin{pmatrix} 6 & 3 \end{pmatrix} \mathbf{x} = 5.$
c) $\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 0.$

2 Norm

2.1. Find a unit vector in the direction of $\mathbf{A} + \mathbf{B}$, where

$$\mathbf{A} = \begin{pmatrix} 2 \\ 2 \\ -5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}. \tag{2.1.1}$$

Solution: Let

$$\mathbf{C} = \mathbf{A} + \mathbf{B} \tag{2.1.2}$$

$$\tag{2.1.3}$$

$$\therefore \mathbf{C} = \begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix} \tag{2.1.4}$$

$$\text{and } \|\mathbf{C}\| = \sqrt{(4)^2 + (3)^2 + (-2)^2} \tag{2.1.5}$$

$$= \sqrt{29} \tag{2.1.6}$$

Thus, the unit vector in the direction of $\mathbf{C}$ is

$$\frac{\mathbf{C}}{\|\mathbf{C}\|} = \frac{1}{\sqrt{29}} \begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix} \tag{2.1.7}$$

2.2. If $\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$, find a unit vector parallel to the vector $2\mathbf{a} - \mathbf{b} + 3\mathbf{c}$.
Solution:

$$\mathbf{d} = \mathbf{2a} - \mathbf{b} + \mathbf{3c} \tag{2.2.1}$$

$$= 2\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + 3\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \tag{2.2.2}$$

$$= \begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix} \tag{2.2.3}$$

Hence,

$$\|\mathbf{d}\| = \sqrt{3^2 + (-3)^2 + 2^2} = \sqrt{22} \tag{2.2.4}$$

$$\implies \frac{\mathbf{d}}{\|\mathbf{d}\|} = \frac{1}{\sqrt{22}} \begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix} \tag{2.2.5}$$

is the unit vector parallel to given vector.

2.3. Find a vector of magnitude 5 units, and parallel to the resultant of the vectors $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$.
Solution: The desired vector can be expressed as

$$\mathbf{R} = k\left(\mathbf{a} + \mathbf{b}\right) \tag{2.3.1}$$

$$\implies \|\mathbf{R}\| = |k| \, \|\mathbf{a} + \mathbf{b}\| = 5 \tag{2.3.2}$$

$\because$

$$\mathbf{a} + \mathbf{b} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \tag{2.3.3}$$

$$= \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \tag{2.3.4}$$

$$\|\mathbf{a} + \mathbf{b}\| = \sqrt{3^2 + 1^2 + 0^2} \tag{2.3.5}$$

$$= \sqrt{10} \tag{2.3.6}$$

Using the above result in (2.3.2),

$$k\sqrt{10} = 5 \implies k = \frac{5}{\sqrt{10}} \tag{2.3.7}$$

Substituting the above in (2.3.1),

$$\mathbf{R} = \frac{5}{\sqrt{10}} \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \tag{2.3.8}$$

2.4. Find a unit vector in the direction of

$$\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}. \tag{2.4.1}$$

Solution:

2.5. Find a unit vector in the direction of $\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$.
Solution:

2.6. Find a unit vector in the direction of the line passing through $\begin{pmatrix} -2 \\ 4 \\ -5 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.
Solution:

2.7. Find a unit vector that makes an angle of $90°, 60°$ and $30°$ with the positive x, y and z axis respectively.
Solution: The direction vector is

$$\mathbf{x} = \begin{pmatrix} \cos 90° \\ \cos 60° \\ \cos 30° \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} \tag{2.7.1}$$

$\because \|\mathbf{x}\| = 1$, it is the desired unit vector.

2.8. Find a vector $\mathbf{x}$ in the direction of $\begin{pmatrix}1\\-2\end{pmatrix}$ such that $\|\mathbf{x}\| = 7$.

Solution: Let $\mathbf{x} = k\begin{pmatrix}1\\-2\end{pmatrix}$. Then

$$\|\mathbf{x}\| = |k|\left\|\begin{pmatrix}1\\-2\end{pmatrix}\right\| = 7 \tag{2.8.1}$$

$$\implies |k| = \frac{7}{\sqrt{5}} \tag{2.8.2}$$

$$\text{or, } \mathbf{x} = \frac{7}{\sqrt{5}}\begin{pmatrix}1\\-2\end{pmatrix} \tag{2.8.3}$$

2.9. Find a point on the x-axis, which is equidistant from the points $\begin{pmatrix}7\\6\end{pmatrix}$ and $\begin{pmatrix}3\\4\end{pmatrix}$.

Solution:

2.10. Write down a unit vector in the xy-plane, makeing an angle of $30°$ with the positive direction of the x-axis.

Solution:

2.11. Find the value of x for which $x\begin{pmatrix}1\\1\\1\end{pmatrix}$ is a unit vector.

Solution:

2.12. If

$$\mathbf{a} = \begin{pmatrix}1\\2\end{pmatrix}, \mathbf{b} = \begin{pmatrix}2\\1\end{pmatrix}, \tag{2.12.1}$$

verify if

a) $\|\mathbf{a}\| = \|\mathbf{b}\|$
b) $\mathbf{a} = \mathbf{b}$

Solution:

a) $\|\mathbf{a}\| = \|\mathbf{b}\|, \mathbf{a} \neq \mathbf{b}$.

2.13. Find a unit vector in the direction of $\begin{pmatrix}2\\3\\1\end{pmatrix}$.

Solution: The unit vector is given by

$$\frac{\begin{pmatrix}2\\3\\1\end{pmatrix}}{\left\|\begin{pmatrix}2\\3\\1\end{pmatrix}\right\|} = \frac{1}{\sqrt{14}}\begin{pmatrix}2\\3\\1\end{pmatrix} \tag{2.13.1}$$

2.14. Find the distance between the points

$$\mathbf{P} = \begin{pmatrix}1\\-3\\4\end{pmatrix}, \mathbf{Q} = \begin{pmatrix}-4\\1\\2\end{pmatrix} \tag{2.14.1}$$

Solution:
The distance is given by $\|\mathbf{P} - \mathbf{Q}\|$

2.15. Find $\|\mathbf{a} - \mathbf{b}\|$, if

$$\|\mathbf{a}\| = 2, \|\mathbf{b}\| = 3, \mathbf{a}^T\mathbf{b} = 4. \tag{2.15.1}$$

Solution:

2.16. If $\mathbf{a}$ is a unit vector and

$$(\mathbf{x} - \mathbf{a})(\mathbf{x} + \mathbf{a}) = 8, \tag{2.16.1}$$

then find $\mathbf{x}$.

Solution:

$$(\mathbf{x} - \mathbf{a})(\mathbf{x} + \mathbf{a}) = \|\mathbf{x}\|^2 - \|\mathbf{a}\|^2 \tag{2.16.2}$$

$$\implies \|\mathbf{x}\|^2 = 9 \text{ or, } \|\mathbf{x}\| = 3. \tag{2.16.3}$$

2.17. Find the point on the x-axis which is equidistant from

$$\begin{pmatrix}2\\-5\end{pmatrix}, \begin{pmatrix}-2\\9\end{pmatrix}, \tag{2.17.1}$$

Solution:

2.18. Find the values of y for which the distance between the points

$$\mathbf{P} = \begin{pmatrix}2\\-3\end{pmatrix}, \mathbf{Q} = \begin{pmatrix}10\\y\end{pmatrix} \tag{2.18.1}$$

is 10 units. **Solution:**

2.19. A town B is located 36km east and 15 km north of the town A. How would you find the distance from town A to town B without actually measuring it?

Solution:

2.20. Find the unit normal vector of the plane

$$\begin{pmatrix}6 & -3 & -2\end{pmatrix}\mathbf{x} = 1. \tag{2.20.1}$$

Solution: The normal vector is

$$\mathbf{n} = \begin{pmatrix}6 & -3 & -2\end{pmatrix} \tag{2.20.2}$$

$$\because \|\mathbf{n}\| = 7, \tag{2.20.3}$$

the unit normal vector is

$$\frac{\mathbf{n}}{\|\mathbf{n}\|} = \frac{1}{7}\begin{pmatrix}6 & -3 & -2\end{pmatrix} \tag{2.20.4}$$

2.21. Find the condition for $\mathbf{x} = \begin{pmatrix}x_1\\x_2\end{pmatrix}$ to be equidistant from the points $\begin{pmatrix}7\\1\end{pmatrix}, \begin{pmatrix}3\\5\end{pmatrix}$.

Solution: From the given information,

$$\left\|\mathbf{x} - \begin{pmatrix}7\\1\end{pmatrix}\right\|^2 = \left\|\mathbf{x} - \begin{pmatrix}3\\5\end{pmatrix}\right\|^2 \tag{2.21.1}$$

$$\implies \|\mathbf{x}\|^2 + \left\|\begin{pmatrix}7\\1\end{pmatrix}\right\|^2 - 2\begin{pmatrix}7 & 1\end{pmatrix}\mathbf{x}$$
$$= \|\mathbf{x}\|^2 + \left\|\begin{pmatrix}3\\5\end{pmatrix}\right\|^2 - 2\begin{pmatrix}3 & 5\end{pmatrix}\mathbf{x} \quad (2.21.2)$$

which can be simplified to obtain

$$\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = 2 \quad (2.21.3)$$

which is the desired condition. The following code plots Fig. **??**clearly showing that the above equation is the perpendicular bisector of AB.

```
codes/line/line_perp_bisect.py
```

2.22. Find a point on the y-axis which is equidistant from the points $\mathbf{A} = \begin{pmatrix}6\\5\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-4\\3\end{pmatrix}$.
Solution:

2.23. Find the equation of set of points $\mathbf{P}$ such that

$$PA^2 + PB^2 = 2k^2, \quad (2.23.1)$$

$$\mathbf{A} = \begin{pmatrix}3\\4\\5\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-1\\3\\-7\end{pmatrix}, \quad (2.23.2)$$

respectively. **Solution:**

2.24. Find the equation of the set of points $\mathbf{P}$ such that its distances from the points $\mathbf{A} = \begin{pmatrix}3\\4\\-5\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-2\\1\\4\end{pmatrix}$ are equal.
Solution:

3 Section

3.1. If the points $\mathbf{A} = \begin{pmatrix}6\\1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}8\\2\end{pmatrix}, \mathbf{C} = \begin{pmatrix}9\\4\end{pmatrix}, \mathbf{D} = \begin{pmatrix}p\\3\end{pmatrix}$ are the vertices of a parallelogram, taken in order, find the value of p.
Solution: In the parallelogram $ABCD$, AC and BD bisect each other. This can be used to find p.

3.2. Find $\mathbf{R}$ which divides the line joining the points

$$\mathbf{P} = 2\mathbf{a} + \mathbf{b} \quad (3.2.1)$$
$$\mathbf{Q} = \mathbf{a} - \mathbf{b} \quad (3.2.2)$$

externally in the ratio $1 : 2$.

3.3. If

$$\mathbf{P} = 3\mathbf{a} - 2\mathbf{b} \quad (3.3.1)$$
$$\mathbf{Q} = \mathbf{a} + \mathbf{b} \quad (3.3.2)$$

find $\mathbf{R}$, which divides PQ in the ratio $2 : 1$

a) internally,
b) externally.

Solution:

3.4. Draw a line segement of length 7.6 cm and divide it in the ratio $5 : 8$.
Solution: Let the end points of the line be

$$\mathbf{A} = \begin{pmatrix}0\\0\end{pmatrix}, \mathbf{B} = \begin{pmatrix}7.6\\0\end{pmatrix} \quad (3.4.1)$$

Using section formula, the point $\mathbf{C}$

$$\mathbf{C} = \frac{k\mathbf{B} + \mathbf{A}}{k+1} \quad (3.4.2)$$

If $\mathbf{C}$ divides AB in the ratio

$$m = \frac{5}{8}, \quad (3.4.3)$$

then,

$$\frac{\|\mathbf{C} - \mathbf{A}\|^2}{\|\mathbf{B} - \mathbf{C}\|^2} = m^2 \quad (3.4.4)$$
$$\implies \frac{\frac{k^2\|\mathbf{B}-\mathbf{A}\|^2}{(k+1)^2}}{\frac{\|\mathbf{B}-\mathbf{A}\|^2}{(k+1)^2}} = m^2 \quad (3.4.5)$$
$$\implies k = m \quad (3.4.6)$$

upon substituting from (3.4.4) and simplifying. (3.4.2) is known as the section formula. The following code plots Fig. **??**

```
codes/line/draw_section.py
```

3.5. Find the coordinates of the point which divides the line segment joining the points $\begin{pmatrix}4\\-3\end{pmatrix}$ and $\begin{pmatrix}8\\5\end{pmatrix}$ in the ratio $3 : 1$ internally.
Solution: Using (3.4.2), the desired point is

$$\mathbf{P} = \frac{3\begin{pmatrix}4\\-3\end{pmatrix} + \begin{pmatrix}8\\5\end{pmatrix}}{4} \quad (3.5.1)$$

3.6. In what ratio does the point $\begin{pmatrix}-4\\6\end{pmatrix}$ divide the line segment joining the points

$$\mathbf{A} = \begin{pmatrix}-6\\10\end{pmatrix}, \mathbf{B} = \begin{pmatrix}3\\-8\end{pmatrix} \quad (3.6.1)$$

Solution: Use (3.4.2).

3.7. Find the coordinates of the points of trisection of the line segement joining the points

$$\mathbf{A} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -7 \\ 4 \end{pmatrix} \tag{3.7.1}$$

Solution: Using (3.4.2), the coordinates are

$$\mathbf{P} = \frac{2\mathbf{A} + \mathbf{B}}{3} \tag{3.7.2}$$

$$\mathbf{Q} = \frac{\mathbf{A} + 2\mathbf{B}}{3} \tag{3.7.3}$$

3.8. Find the ratio in which the y-axis divides the line segment joining the points $\begin{pmatrix} 5 \\ -6 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -4 \end{pmatrix}$.

Solution: Let the corresponding point on the y-axis be $\begin{pmatrix} 0 \\ y \end{pmatrix}$. If the ratio be $k : 1$, using (3.4.2), the coordinates are

$$\begin{pmatrix} 0 \\ y \end{pmatrix} = k \begin{pmatrix} 5 \\ -6 \end{pmatrix} + \begin{pmatrix} -1 \\ -4 \end{pmatrix} \tag{3.8.1}$$

$$\implies 0 = 5k - 1 \implies k = \frac{1}{5} \tag{3.8.2}$$

3.9. Find the coordinates of the point which divides the join of

$$\begin{pmatrix} -1 \\ 7 \end{pmatrix}, \begin{pmatrix} 4 \\ -3 \end{pmatrix} \tag{3.9.1}$$

in the ratio $2 : 3$.

Solution:

3.10. Find the coordinates of the points of trisection of the line segment joining $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ -3 \end{pmatrix}$.

Solution:

3.11. Find the ratio in which the line segment joining the points $\begin{pmatrix} -3 \\ 10 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ -8 \end{pmatrix}$ is divided by $\begin{pmatrix} -1 \\ 6 \end{pmatrix}$.

Solution:

3.12. Find the ratio in whcih the line segment joining $\mathbf{A} = \begin{pmatrix} 1 \\ -5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -4 \\ 5 \end{pmatrix}$ is divided by the x-axis. Also find the coordinates of the point of division.

Solution:

3.13. If $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 4 \\ y \end{pmatrix}$, $\begin{pmatrix} x \\ 6 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$ are the vertices of a parallelogram taken in order, find x and y.

Solution:

3.14. If $\mathbf{A} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$ respectively, find the coordinates of $\mathbf{P}$ such that $AP = \frac{3}{7}AB$ and $\mathbf{P}$ lies on the line segment AB.

Solution:

3.15. Find the coordinates of the points which divide the line segment joining $\mathbf{A} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 \\ 8 \end{pmatrix}$ into four equal parts.

Solution:

4 Projection

4.1. Find the projection of the vector

$$\begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix} \tag{4.1.1}$$

on the vector

$$\begin{pmatrix} 7 \\ -1 \\ 8 \end{pmatrix} \tag{4.1.2}$$

Solution:

4.2. Find the projection of the vector

$$\mathbf{a} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} \tag{4.2.1}$$

on the vector

$$\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}. \tag{4.2.2}$$

Solution: The projection of $\mathbf{a}$ on $\mathbf{b}$ is shown in Fig. **??**. It has magnitude $\|\mathbf{a}\| \cos\theta$ and is in the direction of $\mathbf{b}$. Thus, the projection is defined as

$$\left(\|\mathbf{a}\| \cos\theta\right) \frac{\mathbf{b}}{\|\mathbf{b}\|} = \frac{\left(\mathbf{a}^T\mathbf{b}\right) \|\mathbf{a}\|}{\|\mathbf{b}\|}\mathbf{b} \tag{4.2.3}$$

5 Applications

5.1. Find the perpendicular distances of the following lines from the origin and angle between the perpendicular and the positive x-axis.

a) $\begin{pmatrix}1 & -\sqrt{3}\end{pmatrix}\mathbf{x} = -8.$

b) $\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} = 2.$

c) $\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = 4.$

5.2. Find $\|\mathbf{a}\|$ and $\|\mathbf{b}\|$ if

$$(\mathbf{a}+\mathbf{b})^T(\mathbf{a}-\mathbf{b}) = 8 \quad (5.2.1)$$

$$\|\mathbf{a}\| = 8\|\mathbf{b}\| \quad (5.2.2)$$

5.3. Evaluate the product

$$(3\mathbf{a}-5\mathbf{b})^T(2\mathbf{a}+7\mathbf{b}) \quad (5.3.1)$$

5.4. Find $\|\mathbf{a}\|$ and $\|\mathbf{b}\|$, if

$$\|\mathbf{a}\| = \|\mathbf{b}\|, \quad (5.4.1)$$

$$\mathbf{a}^T\mathbf{b} = \frac{1}{2} \quad (5.4.2)$$

and the angle between $\mathbf{a}$ and $\mathbf{b}$ is $60°$.

5.5. Show that

$$(\|\mathbf{a}\|\mathbf{b}+\|\mathbf{b}\|\mathbf{a}) \perp (\|\mathbf{a}\|\mathbf{b}-\|\mathbf{b}\|\mathbf{a}) \quad (5.5.1)$$

$$(5.5.2)$$

5.6. If $\mathbf{a}^T\mathbf{a} = 0$ and $\mathbf{ab} = 0$, what can be concluded about the vector $\mathbf{b}$?

5.7. If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are unit vectors such that

$$\mathbf{a}+\mathbf{b}+\mathbf{c} = 0, \quad (5.7.1)$$

find the value of

$$\mathbf{a}^T\mathbf{b}+\mathbf{b}^T\mathbf{c}+\mathbf{c}^T\mathbf{a}. \quad (5.7.2)$$

5.8. If $\mathbf{a} \neq \mathbf{0}, \lambda \neq 0$, then $\|\lambda\mathbf{a}\| = 1$ if

a) $\lambda = 1$

b) $\lambda = -1$

c) $\|\mathbf{a}\| = |\lambda|$

d) $\|\mathbf{a}\| = \frac{1}{|\lambda|}$

5.9. If a unit vector $\mathbf{a}$ makes angles $\frac{\pi}{3}$ with the x-axis and $\frac{\pi}{4}$ with the y-axis and an acute angle θ with the z-axis, find θ and $\mathbf{a}$.

5.10. Show that

$$(\mathbf{a}-\mathbf{b})\times(\mathbf{a}+\mathbf{b}) = 2(\mathbf{a}\times\mathbf{b}) \quad (5.10.1)$$

5.11. If $\mathbf{a}^T\mathbf{b} = 0$ and $\mathbf{a}\times\mathbf{b} = 0$, what can you conclude about $\mathbf{a}$ and $\mathbf{b}$?

5.12. Find $\mathbf{x}$ if $\mathbf{a}$ is a unit vector such that

$$(\mathbf{x}-\mathbf{a})^T(\mathbf{x}+\mathbf{a}) = 12. \quad (5.12.1)$$

5.13. If $\|\mathbf{a}\| = 3, \|\mathbf{b}\| = \frac{\sqrt{2}}{3}$, then $\mathbf{a}\times\mathbf{b}$ is a unit vector if the angle between $\mathbf{a}$ and $\mathbf{b}$ is

a) $\frac{\pi}{6}$ c) $\frac{\pi}{3}$

b) $\frac{\pi}{4}$ d) $\frac{\pi}{2}$

5.14. Prove that

$$(\mathbf{a}+\mathbf{b})^T(\mathbf{a}+\mathbf{b}) = \|\mathbf{a}\|^2+\|\mathbf{b}\|^2 \quad (5.14.1)$$

$$\iff \mathbf{a}\perp\mathbf{b}. \quad (5.14.2)$$

5.15. If θ is the angle between two vectors $\mathbf{a}$ and $\mathbf{b}$, then $\mathbf{a}^T\mathbf{b} \geq$ only when

a) $0 < \theta < \frac{\pi}{2}$ c) $0 < \theta < \pi$

b) $0 \leq \theta \leq \frac{\pi}{2}$ d) $0 \leq \theta \leq \pi$

5.16. Let $\mathbf{a}$ and $\mathbf{b}$ be two unit vectors and θ be the angle between them. Then $\mathbf{a}+\mathbf{b}$ is a unit vector if

a) $\theta = \frac{\pi}{4}$ c) $\theta = \frac{\pi}{2}$

b) $\theta = \frac{\pi}{3}$ d) $\theta = \frac{2\pi}{3}$

5.17. If θ is the angle between any two vectors $\mathbf{a}$ and $\mathbf{b}$, then $\left\|\mathbf{a}^T\mathbf{b}\right\| = \|\mathbf{a}\times\mathbf{b}\|$ when θ is equal to

a) 0 c) $\frac{\pi}{2}$

b) $\frac{\pi}{4}$ d) π.

5.18. A bullet fired at an angle of $30°$ with the horizontal hits the ground 3.0 km away. By adjusting its angle of projection, can one hope to hit a target 5.0 km away ? Assume the muzzle speed to be fixed, and neglect air resistance.

5.19. A fighter plane flying horizontally at an altitude of 1.5 km with speed 720 km/h passes directly overhead an anti-aircraft gun. At what angle from the vertical should the gun be fired for the shell with muzzle speed 600 ms^{-1} to hit the plane ? At what minimum altitude should the pilot fly the plane to avoid being hit ? (Take g = 10ms^{-2}).

5.20. Consider the collision depicted in Fig. **??** to be between two billiard balls with equal masses $m_1 = m_2$. The first ball is called the cue while the second ball is called the target. The billiard player wants to 'sink' the target ball in a corner pocket, which is at an angle $\theta_2 = 37°$. Assume that the collosion is elastic and that friction and rotational motion are not important. Obtain θ_1.

5.21. Rain is falling vertically with a speed of 35 ms^{-1} . Winds starts blowing after sometime with a speed of 12 ms^{-1} in east to west direction. In which direction should a boy waiting at a bus stop hold his umbrella ?

Solution:

5.22. A motorboat is racing towards north at 25 km/h and the water current in that region is 10 km/h in the direction of 60° east of south. Find the resultant velocity of the boat.

Solution:

5.23. Rain is falling vertically with a speed of 35 ms^{-1} . A woman rides a bicycle with a speed of 12 ms^{-1} in east to west direction. What is the direction in which she should hold her umbrella ?

Solution:

5.24. A hiker stands on the edge of a cliff 490 m above the ground and throws a stone horizontally with an initial speed of 15 ms^{-1} . Neglecting air resistance, find the time taken by the stone to reach the ground, and the speed with which it hits the ground. (Take g = 9.8 ms^{-2}).

Solution:

5.25. Rain is falling vertically with a speed of 30 ms^{-1}. A woman rides a bicycle with a speed of 10 ms^{-1} in the north to south direction. What is the direction in which she should hold her umbrella?

Solution:

5.26. A man can swim with a speed of 4.0 km/h in still water. How long does he take to cross a river 1.0 km wide if the river flows steadily at 3.0 km/h and he makes his strokes normal to the river current? How far down the river does he go when he reaches the other bank ?

Solution:

5.27. In a harbour, wind is blowing at the speed of 72 km/h and the flag on the mast of a boat anchored in the harbour flutters along the N-E direction. If the boat starts moving at a speed of 51 km/h to the north, what is the direction of the flag on the mast of the boat ?

Solution:

5.28. Find the intercepts cut off by the plane $\begin{pmatrix}2 & 1 & 1\end{pmatrix}\mathbf{x} = 5$.

6 Locus

6.1. The sum of the perpendicular distances of a variable point $\mathbf{P}$ from the lines

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 0 \tag{6.1.1}$$

$$\begin{pmatrix}3 & -2\end{pmatrix}\mathbf{x} = -7 \tag{6.1.2}$$

is always 10. Show that $\mathbf{P}$ must move on a line.