MATHEMATICS

Time: 4 hours

Instructions:

- **Calculators (in any form) and protractors are not allowed.**
- **Rulers and compasses are allowed.**
- **All questions carry equal marks. Maximum marks: 102.**
- **No marks will be awarded for stating an answer without justification.**
- **Answer all the questions.**
- **PLEASE READ THE INSTRUCTIONS ON THE ANSWER BOOKLET VERY CAREFULLY BEFORE ANSWERING THE QUESTIONS.**

1. Suppose $r \geq 2$ is an integer, and let $m_1, n_1, m_2, n_2, \cdots, m_r, n_r$ be $2r$ integers such that
$$|m_i n_j - m_j n_i| = 1$$
for any two integers i and j satisfying $1 \leq i < j \leq r$. Determine the maximum possible value of r.

2. Find all pairs of integers (a, b) so that each of the two cubic polynomials
$$x^3 + ax + b \text{ and } x^3 + bx + a$$
has all the roots to be integers.

3. Betal marks 2021 points on the plane such that no three are collinear, and draws all possible line segments joining these. He then chooses any 1011 of these line segments, and marks their midpoints. Finally, he chooses a line segment whose midpoint is not marked yet, and challenges Vikram to construct its midpoint using **only** a straightedge. Can Vikram always complete this challenge?

   Note: A straightedge is an infinitely long ruler without markings, which can only be used to draw the line joining any two given distinct points.

4. A Magician and a Detective play a game. The Magician lays down cards numbered from 1 to 52 face-down on a table. On each move, the Detective can point to two cards and inquire if the numbers on them are consecutive. The Magician replies truthfully. After a finite number of moves the Detective points to two cards. She wins if the numbers on these two cards are consecutive, and loses otherwise.

   Prove that the Detective can guarantee a win if and only if she is allowed to ask at least 50 questions.

5. In a convex quadrilateral $ABCD$, $\angle ABD = 30^\circ$, $\angle BCA = 75^\circ$, $\angle ACD = 25^\circ$ and $CD = CB$. Extend CB to meet the circumcircle of triangle DAC at E. Prove that $CE = BD$.

6. Let $\mathbb{R}[x]$ be the set of all polynomials with real coefficients, and let $\deg P$ denote the degree of a nonzero polynomial P. Find all functions $f : \mathbb{R}[x] \to \mathbb{R}[x]$ satisfying the following conditions:

   - f maps the zero polynomial to itself,
   - for any non-zero polynomial $P \in \mathbb{R}[x]$, $\deg f(P) \leq 1 + \deg P$, and
   - for any two polynomials $P, Q \in \mathbb{R}[x]$, the polynomials $P - f(Q)$ and $Q - f(P)$ have the same set of real roots.

———00000———