

Convex Optimization

1 12th Maths - Chapter 6

This is Problem-1(i) from Exercise 6.5

1. Determine whether the function $f(x) = (2x - 1)^2 + 3$ is convex or not.

Solution: A single variable function f is said to be convex if

$$f[\lambda x_1 + (1 - \lambda)x_2] \leq \lambda f(x_1) + (1 - \lambda)f(x_2), \quad (1)$$

for $0 < \lambda < 1$ and $x_1, x_2 \in \mathbb{R}$.

For a generic quadratic function $ax^2 + bx + c$, let us determine the sufficient condition for it to be convex. Let

$$f(x) = ax^2 + bx + c \quad (2)$$

Substituting LHS of inequality from (1) in (2)

$$\begin{aligned} f[\lambda x_1 + (1 - \lambda)x_2] &= f[x_2 + \lambda(x_1 - x_2)] \\ \implies a[x_2 + \lambda(x_1 - x_2)]^2 &+ b[x_2 + \lambda(x_1 - x_2)] + c \\ \implies ax_2^2 + a\lambda^2 x_1^2 + a\lambda^2 x_2^2 - 2a\lambda^2 x_1 x_2 \\ &+ 2a\lambda x_1 x_2 - 2a\lambda x_2^2 + bx_2 + b\lambda x_1 - b\lambda x_2 + c \end{aligned} \quad (3)$$

Substituting RHS of inequality from (1) in (2)

$$\begin{aligned} \lambda f(x_1) + (1 - \lambda)f(x_2) &= a\lambda x_1^2 + b\lambda x_1 + \lambda c \\ &+ (1 - \lambda)(ax_2^2 + bx_2 + c) \\ \implies a\lambda x_1^2 + b\lambda x_1 + a\lambda^2 x_2^2 + bx_2 + c - a\lambda x_2^2 - b\lambda x_2 \end{aligned} \quad (4)$$

Combining (3) and (4) with inequality and simplifying

$$a\lambda^2 x_1^2 + a\lambda^2 x_2^2 - 2a\lambda^2 x_1 x_2 + 2a\lambda x_1 x_2 - 2a\lambda x_2^2 \leq a\lambda x_1^2 - a\lambda x_2^2 \quad (5)$$

$$\begin{aligned}
&\implies a\lambda^2x_1^2 + a\lambda^2x_2^2 - 2a\lambda^2x_1x_2 + 2a\lambda x_1x_2 - a\lambda x_2^2 - a\lambda x_1^2 \leq 0 \\
&\implies x_1^2(a\lambda^2 - a\lambda) + x_2^2(a\lambda^2 - a\lambda) - 2x_1x_2(a\lambda^2 - a\lambda) \leq 0 \\
&\implies (a\lambda^2 - a\lambda)(x_1 - x_2)^2 \leq 0 \\
&\implies a\lambda(1 - \lambda)(x_1 - x_2)^2 \geq 0 \quad (6)
\end{aligned}$$

For the inequality in (6) to be true,

$$a \geq 0 \because \lambda, 1 - \lambda \geq 0, (x_1 - x_2)^2 \geq 0 \quad (7)$$

However, $a \neq 0$, since it is a quadratic function. Hence $a > 0$, for $f(x)$ to be convex.

The given function is

$$f(x) = (2x - 1)^2 + 3 \quad (8)$$

$$= 4x^2 + 4x + 4 \quad (9)$$

$$\therefore a = 4, > 0 \quad (10)$$

Hence, the function in equation (8) is convex.

The figure is as shown in Fig1

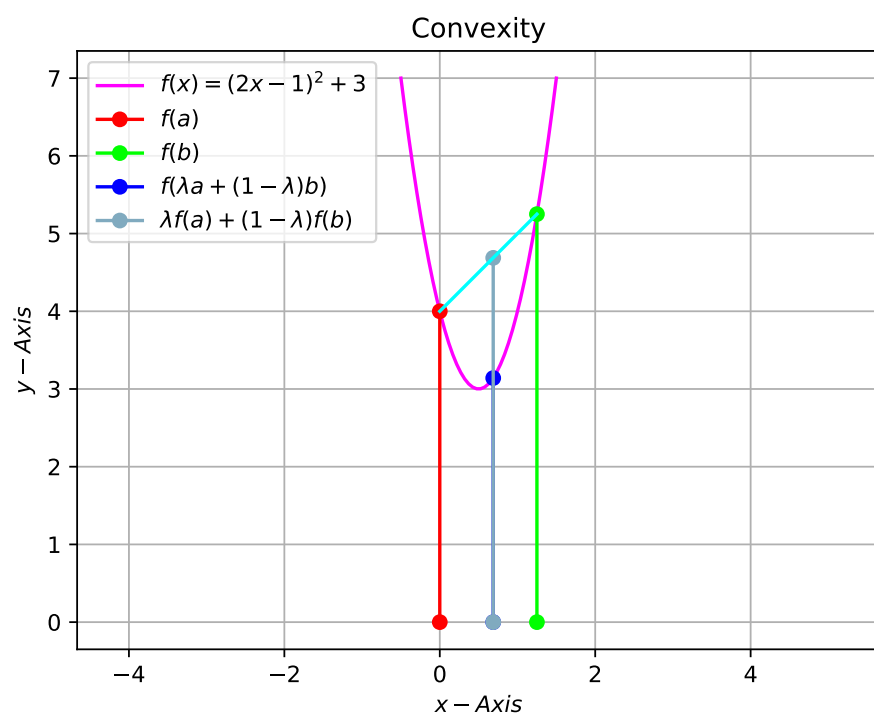


Figure 1