

Normal to a Parabola

1 12th Maths - Chapter 6

This is Problem-23 from Exercise 6.6

1. Find the equation of the normal to the curve $x^2 = 4y$ and passing through the point $(1, 2)$.

Solution: The given equation of the curve can be written as

$$g(\mathbf{x}) = \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (1)$$

where

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (2)$$

$$\mathbf{u} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} \quad (3)$$

$$f = 0 \quad (4)$$

We are given that

$$\mathbf{h} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (5)$$

A point \mathbf{h} lies on a normal to the conic in (1) , if

$$\begin{aligned} (\mathbf{m}^T (\mathbf{V} \mathbf{h} + \mathbf{u}))^2 (\mathbf{n}^T \mathbf{V} \mathbf{n}) - 2 (\mathbf{m}^T \mathbf{V} \mathbf{n}) (\mathbf{m}^T (\mathbf{V} \mathbf{h} + \mathbf{u}) \mathbf{n}^T (\mathbf{V} \mathbf{h} + \mathbf{u})) \\ + g(\mathbf{h}) (\mathbf{m}^T \mathbf{V} \mathbf{n})^2 = 0 \end{aligned} \quad (6)$$

where \mathbf{m} is directional vector of the tangent(or normal vector of the normal) and \mathbf{n} is the normal vector of the tangent (or directional vector

of the normal). Assume

$$\mathbf{m} = \begin{pmatrix} 1 \\ m \end{pmatrix} \quad (7)$$

$$\mathbf{n} = \begin{pmatrix} 1 \\ -\frac{1}{m} \end{pmatrix} \quad (8)$$

Then

$$\mathbf{V}\mathbf{h} + \mathbf{u} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad (9)$$

$$\mathbf{m}^\top \mathbf{V}\mathbf{n} = (1 \quad m) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -\frac{1}{m} \end{pmatrix} = (1 \quad 0) \begin{pmatrix} 1 \\ -\frac{1}{m} \end{pmatrix} = 1 \quad (10)$$

$$\mathbf{n}^\top \mathbf{V}\mathbf{n} = (1 \quad -\frac{1}{m}) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -\frac{1}{m} \end{pmatrix} = (1 \quad 0) \begin{pmatrix} 1 \\ -\frac{1}{m} \end{pmatrix} = 1 \quad (11)$$

$$\mathbf{g}(\mathbf{h}) = (1 \quad 2) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 2(0 \quad -2) \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (12)$$

$$= (1 \quad 0) \begin{pmatrix} 1 \\ 2 \end{pmatrix} - 8 = -7 \quad (13)$$

$$(6) \implies \left((1 \quad m) \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right)^2 (1) - 2(1) \left((1 \quad m) \begin{pmatrix} 1 \\ -2 \end{pmatrix} (1 \quad -\frac{1}{m}) \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right) + (-7)(1)^2 = 0 \quad (14)$$

$$\implies (1 - 2m)^2 - 2(1 - 2m) \left(1 + \frac{2}{m} \right) - 7 = 0 \quad (15)$$

$$\implies 1 - 4m + 4m^2 - 2 \left(1 - 4 + \frac{2}{m} - 2m \right) - 7 = 0 \quad (16)$$

$$\implies 4m^2 - \frac{4}{m} = 0 \quad (17)$$

$$\implies 4m^3 = 4 \quad (18)$$

$$m = 1 \quad (19)$$

The equation of the normal is given by

$$\mathbf{m}^\top (\mathbf{x} - \mathbf{h}) = 0 \quad (20)$$

$$(1 \ 1) \left(\mathbf{x} - \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right) = 0 \quad (21)$$

$$(1 \ 1) (\mathbf{x}) = 3 \quad (22)$$

The relevant figure is shown in 1

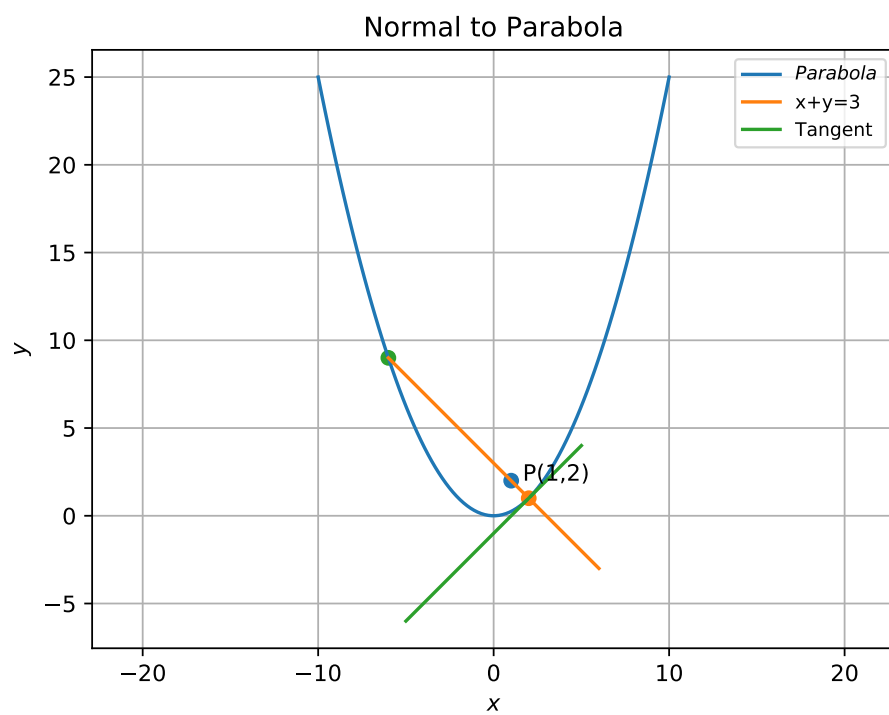


Figure 1