Optimization Assignment

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***Abstract*—This document contains the solution to Question 4 of Exercise 2 in Chapter 10 of the class 11 NCERT textbook.**

1) Find the coordinates of the foot of perpendicular from the point

$$\mathbf{P} = \begin{pmatrix} -1 \\ 3 \end{pmatrix} \tag{1}$$

to the line

$$\begin{pmatrix} 3 & -4 \end{pmatrix} \mathbf{x} = 16 \tag{2}$$

Solution: We rewrite the problem as

$$\min_{\mathbf{x}} h(\mathbf{x}) \triangleq \|\mathbf{x} - \mathbf{P}\|^2 \tag{3}$$

$$\text{s.t. } g(\mathbf{x}) \triangleq \mathbf{n}^\top \mathbf{x} - c = 0 \tag{4}$$

where

$$\mathbf{P} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}, \ \mathbf{n} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}, \ c = 16 \tag{5}$$

Define

$$C(\mathbf{x}, \lambda) = h(\mathbf{x}) - \lambda g(\mathbf{x}) \tag{6}$$

and note that

$$\nabla h(\mathbf{x}) = 2(\mathbf{x} - \mathbf{P}) \tag{7}$$

$$\nabla g(\mathbf{x}) = \mathbf{n} \tag{8}$$

We are required to find $\lambda \in \mathbb{R}$ such that

$$\nabla C(\mathbf{x}, \lambda) = 0 \tag{9}$$

$$\implies 2(\mathbf{x} - \mathbf{P}) - \lambda \mathbf{n} = 0 \tag{10}$$

However, $\mathbf{x}$ lies on the line (2). Thus, from (10),

$$\mathbf{n}^\top \left(\frac{\lambda}{2} \mathbf{n} + \mathbf{P} \right) - c = 0 \tag{11}$$

$$\implies \lambda = \frac{2(c - \mathbf{n}^\top \mathbf{P})}{\|\mathbf{n}\|^2} \tag{12}$$

Substituting (12) in (10), the optimal point is given by

$$\mathbf{Q} = \mathbf{P} + \frac{\lambda}{2} \mathbf{n} \tag{13}$$

$$= \mathbf{P} - \frac{\mathbf{n}^\top \mathbf{P} - c}{\|\mathbf{n}\|^2} \mathbf{n} \tag{14}$$

Substituting from (5),

$$\lambda = \frac{62}{25}, \ \mathbf{Q} = \frac{1}{25} \begin{pmatrix} 68 \\ -49 \end{pmatrix} \tag{15}$$

To find $\mathbf{Q}$ graphically, we use constrained gradient descent, with learning rate $\alpha = 0.01$. The results are shown in Fig. 1, plotted using the Python code `codes/gd_lagrange.py`.

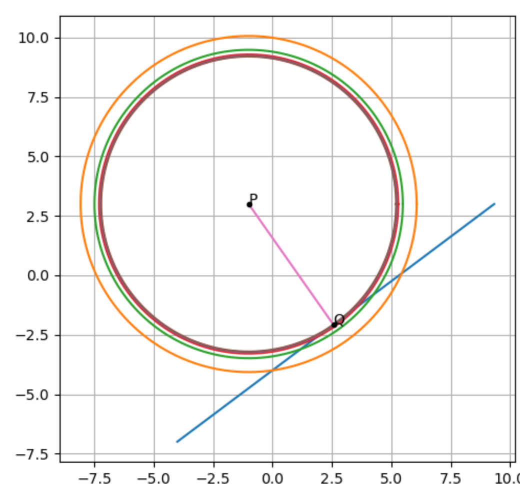

Fig. 1: Constrained gradient descent to find optimal $\mathbf{Q}$.

1 Appendix

1.1 Constrained Gradient Descent

Constrained gradient descent is a method of optimizing the cost function subject to some constraints, represented as follows.

$$\max_{\mathbf{x}} f(\mathbf{x}) \tag{16}$$

$$\text{s.t. } g(\mathbf{x}) = 0 \tag{17}$$

Unlike the unconstrained version, one cannot move in the negative direction of the gradient vector of $f(\mathbf{x})$. However, we must move along the constraint in (17).

The algorithm terminates when the gradient vector of f is parallel to the normal vector of g at that point. Mathematically, at an optimum $\mathbf{x_o}$,

$$\nabla f(\mathbf{x_o}) = \lambda \nabla g(\mathbf{x_o}) \tag{18}$$

where $\lambda \in \mathbb{R}\backslash\{0\}$. Observe that (18) may be rewritten as

$$\nabla C(\mathbf{x}, \lambda) = \nabla (f(\mathbf{x}) - \lambda g(\mathbf{x})) = 0 \tag{19}$$

which is analogous to the method of Lagrangian multipliers.