11.10.3.10

Lokesh Surana

Class 11, Chapter 10, Exercise 3.10

Q. The line through the points $(h, 3)$ and $(4, 1)$ intersects the line $7x - 9y - 19 = 0$ at right angle. Find the value of h.

Solution: Let the point $\mathbf{P}$ be the foot of the perpendicular on the line $7x-9y-19 = 0$ from point $\begin{pmatrix} 22/9 \\ 3 \end{pmatrix}$ (Let's say point $\mathbf{O}$). The optimization problem can be expressed as

$$\min_{\mathbf{x}} \|\mathbf{x} - \mathbf{O}\|^2 \tag{1}$$

$$\text{s.t.} \quad g(\mathbf{x}) = \mathbf{n}^T \mathbf{x} - c = 0 \tag{2}$$

where

$$\mathbf{n} = \begin{pmatrix} 7 \\ -9 \end{pmatrix}, \ \mathbf{O} = \begin{pmatrix} 22/9 \\ 3 \end{pmatrix}, \ c = 19 \tag{3}$$

The line equation can be expressed as

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m} \tag{4}$$

where

$$\mathbf{m} = \begin{pmatrix} 9 \\ 7 \end{pmatrix}, \ \mathbf{A} = \begin{pmatrix} \frac{19}{7} \\ 0 \end{pmatrix} \tag{5}$$

Using the parameric form, Substituting (4) in (1), the optimization problem becomes

$$\min_{\lambda} \|\lambda \mathbf{m} + (\mathbf{A} - \mathbf{O})\|^2 \tag{6}$$

$$\implies \min_{\lambda} f(\lambda) = \lambda^2 \|\mathbf{m}\|^2 + 2\lambda (\mathbf{A} - \mathbf{O})^\top \mathbf{m} + \|\mathbf{A} - \mathbf{O}\|^2 \tag{7}$$

yielding

$$f(\lambda) = 130\lambda^2 - \frac{260}{7}\lambda + \frac{\sqrt{36010}}{63} \tag{8}$$

$$f'(\lambda) = 260\lambda - \frac{260}{7} \tag{9}$$

Define

$$H(\mathbf{x}, \lambda) = f(\mathbf{x}) - \lambda g(\mathbf{x}) \tag{10}$$

and we find that

$$\nabla f(\mathbf{x}) = 2(\mathbf{x} - \mathbf{O}) \tag{11}$$

$$\nabla g(\mathbf{x}) = \mathbf{n} \tag{12}$$

We have to find $\lambda \in \mathbb{R}$ such that

$$\nabla H(\mathbf{x}, \lambda) = 0 \tag{13}$$

$$\implies 2(\mathbf{x} - \mathbf{O}) - \lambda \mathbf{n} = 0 \tag{14}$$

$$\implies \mathbf{x} = \frac{\lambda}{2}\mathbf{n} + \mathbf{O} \tag{15}$$

Substituting (15) in (2)

$$\mathbf{n}^\top \left(\frac{\lambda}{2}\mathbf{n} + \mathbf{O} \right) - c = 0 \tag{16}$$

$$\implies \lambda = \frac{2(c - \mathbf{n}^\top \mathbf{O})}{\|\mathbf{n}\|^2} \tag{17}$$

Substituting the value of λ in (14),

$$\mathbf{x}_{min} = \mathbf{P} = \mathbf{O} + \frac{\mathbf{n}(c - \mathbf{n}^\top \mathbf{O})}{\|\mathbf{n}\|^2} \tag{18}$$

$$= \begin{pmatrix} \frac{22}{9} \\ 3 \end{pmatrix} + \frac{\begin{pmatrix} 7 \\ -9 \end{pmatrix} \left(19 - \begin{pmatrix} 7 & -9 \end{pmatrix} \begin{pmatrix} 22/9 \\ 3 \end{pmatrix} \right)}{130} \tag{19}$$

$$= \begin{pmatrix} \frac{22}{9} \\ 3 \end{pmatrix} + \begin{pmatrix} \frac{14}{9} \\ -2 \end{pmatrix} \tag{20}$$

$$= \begin{pmatrix} 4 \\ 1 \end{pmatrix} \tag{21}$$

This validates that $\mathbf{P} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ is foot of perpendicular from $\mathbf{O} = \begin{pmatrix} \frac{22}{9} \\ 3 \end{pmatrix}$ on the line $7x - 9y - 19 = 0$.

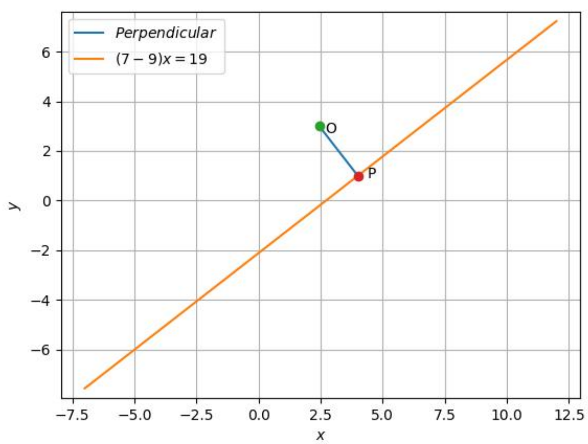

Fig. 1: lines