

# Geometric Programming

## 1 12<sup>th</sup> Maths - Chapter 6

This is Problem-26 from Exercise 6.5

1. Show that semi-vertical angle of right circular cone of given surface area and maximum volume is  $\sin^{-1}\left(\frac{1}{3}\right)$ .

**Solution:** Let  $r, h, l$  be the radius, height and slant height of the right circular cone respectively. Let  $S$  be the given surface area and  $V$  be the volume of the cone. We have

$$l^2 = r^2 + h^2 \tag{1}$$

$$S = \pi r l + \pi r^2 \tag{2}$$

$$\implies l = \frac{S - \pi r^2}{\pi r} \tag{3}$$

$$V = \frac{1}{3} \pi r^2 h \tag{4}$$

The given problem can be formulated as

$$V = \max_{r, h} \frac{1}{3} \pi r^2 h \tag{5}$$

$$\text{s.t } \pi r \left( \sqrt{r^2 + h^2} \right) + \pi r^2 \leq S \tag{6}$$

(a) Theoretical proof:

$$(4) \implies V = \frac{1}{3}\pi r^2 \sqrt{l^2 - r^2} \quad (7)$$

$$V^2 = \frac{1}{9}\pi^2 r^4 (l^2 - r^2) \quad (8)$$

$$= \frac{1}{9}\pi^2 r^4 \left( \left( \frac{S - \pi r^2}{\pi r} \right)^2 - r^2 \right) \quad (9)$$

$$= \frac{1}{9}\pi^2 r^4 \left( \frac{(S - \pi r^2)^2 - \pi^2 r^4}{\pi^2 r^2} \right) \quad (10)$$

$$= \frac{1}{9}r^2 \left( (S - \pi r^2)^2 - \pi^2 r^4 \right) \quad (11)$$

$$= \frac{1}{9}r^2 (S^2 - 2\pi S r^2 + \pi^2 r^4 - \pi^2 r^4) \quad (12)$$

$$= \frac{1}{9} (S^2 r^2 - 2\pi S r^4) \quad (13)$$

Differentiating wrt  $r$ ,

$$2V \frac{dV}{dr} = \frac{S^2}{9} 2r - \frac{2\pi S}{9} 4r^3 \quad (14)$$

$$= \frac{2rS}{9} (S - 4\pi r^2) \quad (15)$$

For maximum volume,  $\frac{dV}{dr} = 0$

$$\implies \frac{2rS}{9} (S - 4\pi r^2) = 0 \quad (16)$$

$$\implies r = 0 \text{ or } S - 4\pi r^2 = 0 \quad (17)$$

Since  $r$  can't be equal to 0,

$$\implies S - 4\pi r^2 = 0 \quad (18)$$

$$\implies S = 4\pi r^2 \quad (19)$$

$$\implies r^2 = \frac{S}{4\pi} \quad (20)$$

$$\implies r^2 = \frac{\pi r l + \pi r^2}{4\pi} \quad (21)$$

$$\implies 4\pi r^2 = \pi r l + \pi r^2 \quad (22)$$

$$\implies 3\pi r^2 = \pi r l \quad (23)$$

$$\implies l = 3r \quad (24)$$

For  $V$  to be maximum,  $\frac{d^2V}{dr^2} < 0$

$$(14) \implies \frac{dV}{dr} = \frac{S}{3} \left[ \frac{S - 4\pi r^2}{\sqrt{S^2 - 2\pi S r^2}} \right] \quad (25)$$

$$\frac{d^2V}{dr^2} = \frac{S}{3} \left[ \frac{\sqrt{S^2 - 2\pi S r^2} (-8\pi r) + \frac{(S - 4\pi r^2)(4\pi S r)}{2\sqrt{S^2 - 2\pi S r^2}}}{S^2 - 2\pi S r^2} \right] \quad (26)$$

$$= \frac{S}{3} \left[ \frac{(S^2 - 2\pi S r^2) (-8\pi r) + (2\pi S r) (S - 4\pi r^2)}{(S^2 - 2\pi S r^2)^{\frac{3}{2}}} \right] \quad (27)$$

$$= \frac{S}{3} \left[ \frac{-8\pi r S^2 + 16\pi^2 S r^3 + 2\pi S^2 r - 8\pi^2 S r^3}{(S^2 - 2\pi S r^2)^{\frac{3}{2}}} \right] \quad (28)$$

$$= \frac{S}{3} \left[ \frac{8\pi^2 S r^3 - 6\pi r S^2}{(S^2 - 2\pi S r^2)^{\frac{3}{2}}} \right] \quad (29)$$

For maximum volume, substituting the value of  $S$  from (19) into

(29)

$$\Rightarrow \frac{d^2V}{dr^2} = \frac{4\pi r^2}{3} \left[ \frac{(8\pi^2 r^3) 4\pi r^2 - 6\pi r (16\pi^2 r^4)}{(16\pi^2 r^4 - 2\pi r^2 (4\pi r^2))^{\frac{3}{2}}} \right] \quad (30)$$

$$= \frac{4\pi r^2}{3} \left[ \frac{32\pi^3 r^5 - 96\pi^3 r^5}{(16\pi^2 r^4 - 8\pi^2 r^4)^{\frac{3}{2}}} \right] \quad (31)$$

$$= \frac{4\pi r^2}{3} \left[ \frac{-64\pi^3 r^5}{(8\pi^2 r^4)^{\frac{3}{2}}} \right] \quad (32)$$

$$= \frac{-8\sqrt{2}\pi r}{3} \quad (33)$$

$$\therefore \frac{d^2V}{dr^2} < 0 \quad (34)$$

Let  $\theta$  be the semi-vertical angle in Figure 1. Then,

$$\sin \theta = \frac{OA}{CA} = \frac{r}{l} \quad (35)$$

$$\sin \theta = \frac{r}{3r} \quad (36)$$

$$\Rightarrow \theta = \sin^{-1} \frac{1}{3} \quad (37)$$

- (b) Using Disciplined Geometric Programming (DGP) of cvxpy: Refer to equations (5) and (6) for formulation of optimization problem. Assume  $S$  to be 75.42857 sq.units. Solving this problem, yields following results:

$$r = 2.45 \text{ units} \quad (38)$$

$$l = 7.35 \text{ units} \quad (39)$$

$$h = 6.93 \text{ units} \quad (40)$$

$$\text{Optimal } V \approx 43.557 \text{ cu.units} \quad (41)$$

It can be seen from solution that  $l = 3r$  and semi-vertical angle is given as  $\sin^{-1} \left( \frac{1}{3} \right)$ . This is similar to what we proved theoretically.

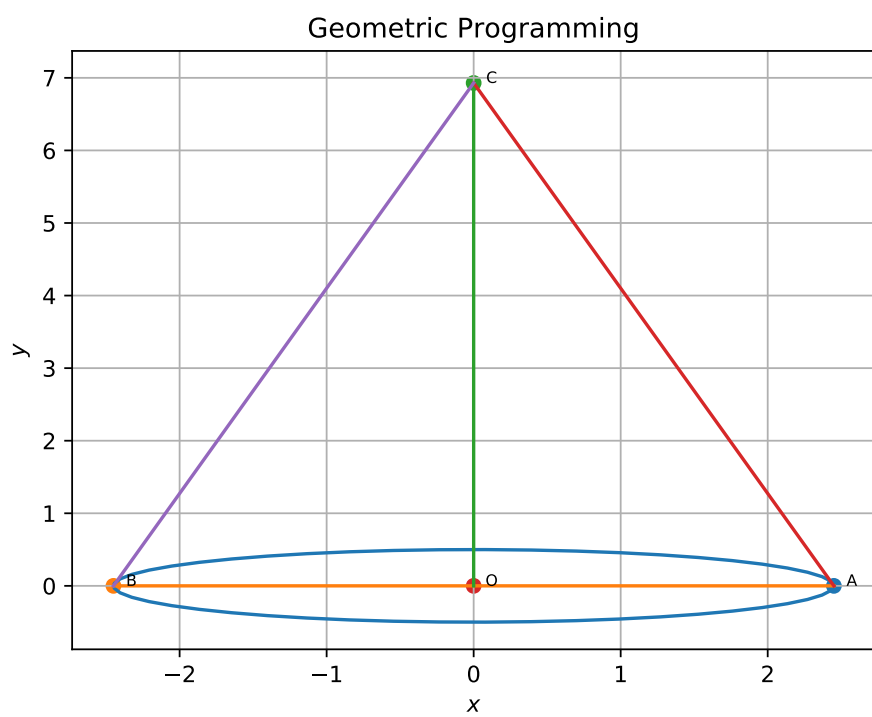


Figure 1