11.10.3.10

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CLASS 11, CHAPTER 10, EXERCISE 3.10

Q. The line through the points $(h, 3)$ and $(4, 1)$ intersects the line $7x - 9y - 19 = 0$ at right angle. Find the value of h.

Solution: Let the point **P** be the foot of the perpendicular on the line $7x-9y-19 = 0$ from point $\begin{pmatrix} 22/9 \\ 3 \end{pmatrix}$ (Let's say point **O**). The optimization problem can be expressed as

$$\min_{\mathbf{x}} \|\mathbf{x} - \mathbf{O}\|^2 \tag{1}$$

$$\text{s.t.} \quad \mathbf{n}^\top \mathbf{x} = c \tag{2}$$

where

$$\mathbf{n} = \begin{pmatrix} 7 \\ -9 \end{pmatrix}, \ c = 19 \tag{3}$$

The line equation can be expressed as

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m} \tag{4}$$

where

$$\mathbf{m} = \begin{pmatrix} 9 \\ 7 \end{pmatrix}, \ \mathbf{A} = \begin{pmatrix} \frac{19}{7} \\ 0 \end{pmatrix} \tag{5}$$

Using the parameric form, Substituting (4) in (1), the optimization problem becomes

$$\min_{\lambda} \|\lambda \mathbf{m} + (\mathbf{A} - \mathbf{O})\|^2 \tag{6}$$

$$\implies \min_{\lambda} f(\lambda) = \lambda^2 \|\mathbf{m}\|^2 + 2\lambda (\mathbf{A} - \mathbf{O})^\top \mathbf{m} + \|\mathbf{A} - \mathbf{O}\|^2 \tag{7}$$

yielding

$$f(\lambda) = 130\lambda^2 - \frac{260}{7}\lambda + \frac{\sqrt{36010}}{63} \tag{8}$$

$$f'(\lambda) = 260\lambda - \frac{260}{7} \tag{9}$$

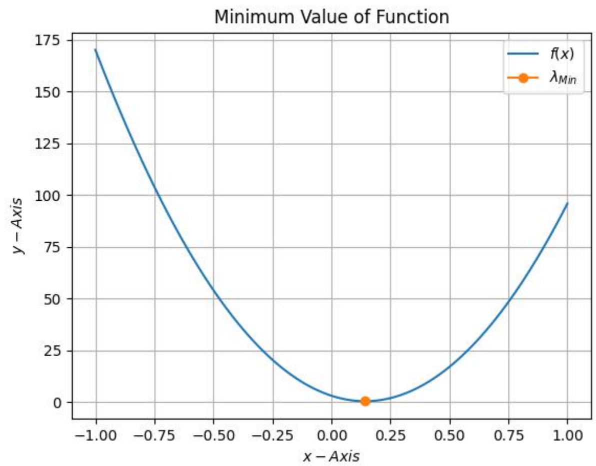

Fig. 1: Plot of objective function

Computing λ_{min} using Gradient Descent method:

$$\lambda_{n+1} = \lambda_n - \alpha f'(\lambda_n) \tag{10}$$

$$\lambda_{n+1} = \lambda_n (1 - 260\alpha) - \frac{260}{7}\alpha \tag{11}$$

Taking the one-sided Z-transform on both sides of (11),

$$z\Lambda(z) = (1 - 260\alpha)\Lambda(z) + \frac{260\alpha}{7(1 - z^{-1})} \tag{12}$$

$$\Lambda(z) = \frac{260\alpha z^{-1}}{7(1 - z^{-1})(1 - (1 - 260\alpha)z^{-1})} \tag{13}$$

$$= \frac{1}{7}\left(\frac{1}{1 - z^{-1}} - \frac{1}{1 - (1 - 260\alpha)z^{-1}}\right) \tag{14}$$

$$= \frac{1}{7}\sum_{k=0}^{\infty}\left(1 - (1 - 260\alpha)^k\right)z^{-k} \tag{15}$$

From (15), the ROC is

$$|z| > \max\{1, |1 - 260\alpha|\} \tag{16}$$

$$\implies -1 < |1 - 260\alpha| < 1 \tag{17}$$

$$\implies 0 < \alpha < \frac{1}{130} \tag{18}$$

Thus, if α satisfies (18), then from (15),

$$\lim_{n\to\infty} \lambda_n = \frac{1}{7} \tag{19}$$

Choosing

1) α = 0.001
2) precision = 0.0000001
3) n = 10000000
4) λ_0 = -5

$$\lambda_{min} = \frac{1}{7} \tag{20}$$

Substituting the values of $\mathbf{A}$, $\mathbf{m}$ and λ_{min} in equation (4)

$$\mathbf{x}_{min} = \mathbf{P} = \begin{pmatrix} \frac{19}{7} \\ 0 \end{pmatrix} + \frac{1}{7}\begin{pmatrix} 9 \\ 7 \end{pmatrix} \tag{21}$$

$$= \begin{pmatrix} 4 \\ 1 \end{pmatrix} \tag{22}$$

This validates that $\mathbf{P} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ is foot of perpendicular from $\mathbf{O} = \begin{pmatrix} \frac{22}{9} \\ 3 \end{pmatrix}$ on the line $7x - 9y - 19 = 0$.

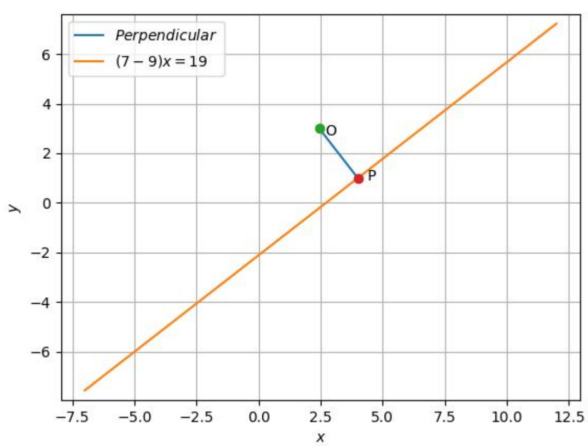

Fig. 2: lines