# OPTIMIZATION

## Through High School Math

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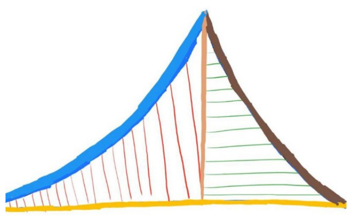

# Contents

# Introduction

This book introduces optimization through high school math.

# Chapter 1

# Convex Functons

# Chapter 2

# Gradient Descent

2.1 Find the maximum and minimum values of

(a)

$$f(x) = 9x^2 + 12x + 2 \tag{2.1.1}$$

**Solution:** From Fig. 2.1.1, the given function has a minimum. Since

$$f'(x) = 18x + 12, \tag{2.1.2}$$

the minimum value is calculated using the gradient descent method as

$$x_{n+1} = x_n - \alpha \nabla f(x_n) \tag{2.1.3}$$

$$= x_n - \alpha \left(18x_n + 12\right) \tag{2.1.4}$$

Choosing

i. $\alpha = 0.001$

ii. precision $= 0.00000001$

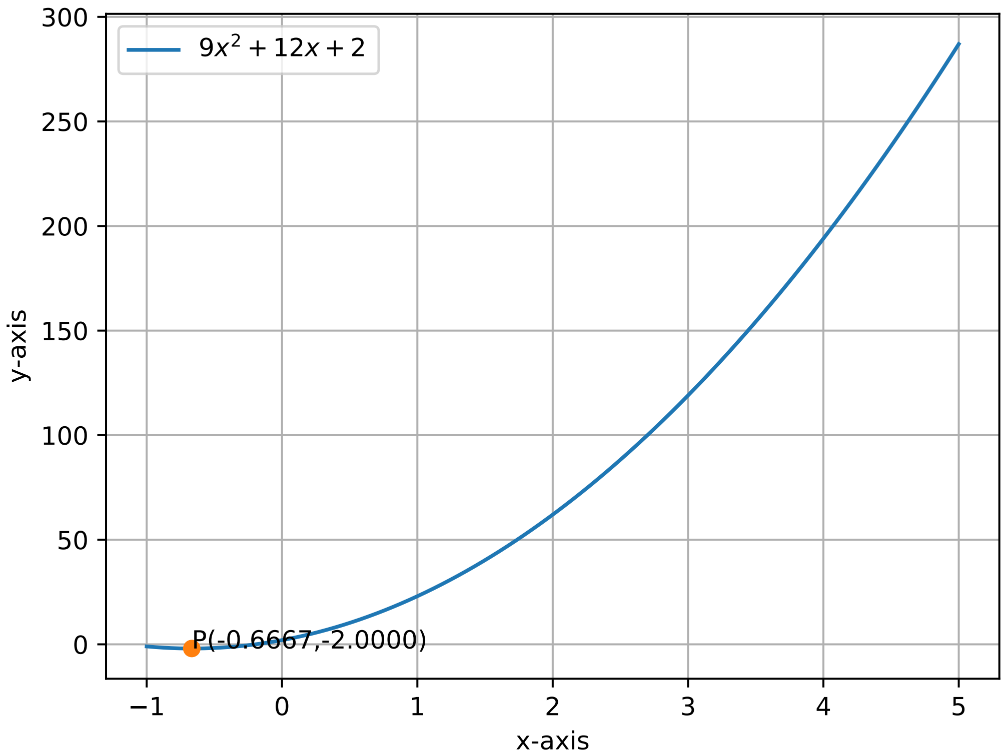


Figure 2.1.1:

iii. n = 100000000

$$x_{min} = \frac{-2}{3}, f_{min} = -2 \tag{2.1.5}$$

2.2 Find the maximum profit that a company can make if the profit function is given by

$$f(x) = 41 - 72x + 18x^2 \tag{2.2.1}$$

**Solution:** Considering

$$\lambda \left(41 - 72x_1 - 18x_1^2\right) + (1-\lambda)\left(41 - 72x_2 - 18x_2^2\right) \geq \tag{2.2.2}$$

$$41 - 72\left(\lambda x_1 + (1-\lambda)x_2\right) - 18\left(\lambda x_1 + (1-\lambda)x_2\right)^2, \tag{2.2.3}$$

we obtain

$$18x_1^2\left(\lambda^2 - \lambda\right) + 18x_2^2\left(\lambda^2 - \lambda\right) + 36x_1x_2\left(\lambda^2 - \lambda\right) \geq 0 \tag{2.2.4}$$

$$x_1^2\left(\lambda^2 - \lambda\right) + x_2^2\left(\lambda^2 - \lambda\right) + 2x_1x_2\left(\lambda^2 - \lambda\right) \geq 0 \tag{2.2.5}$$

$$-\lambda\left(1-\lambda\right)\left(x_1 - x_2\right)^2 \geq 0 \tag{2.2.6}$$

$$\implies \lambda\left(1-\lambda\right)\left(x_1 - x_2\right)^2 \leq 0 \tag{2.2.7}$$

which is false for all $\lambda \in (0,1)$. Hence the given function $f(x)$ is concave. Using the gradient ascent method,

$$x_n = x_{n-1} + \mu\frac{df(x)}{dx} \tag{2.2.8}$$

Since

$$\frac{df(x)}{dx} = -72 - 36x, \tag{2.2.9}$$

substituting (2.2.9) in (2.2.8),

$$x_n = x_{n-1} + \mu(-72 - 36x_{n-1}) \tag{2.2.10}$$

Choosing

$$x_0 = 1, \alpha = 0.001, precision = 0.00000001, \tag{2.2.11}$$

$$f_{max} \approx 113, x_{max} \approx -2.0, \tag{2.2.12}$$

which is verified in Fig. 2.2.1.

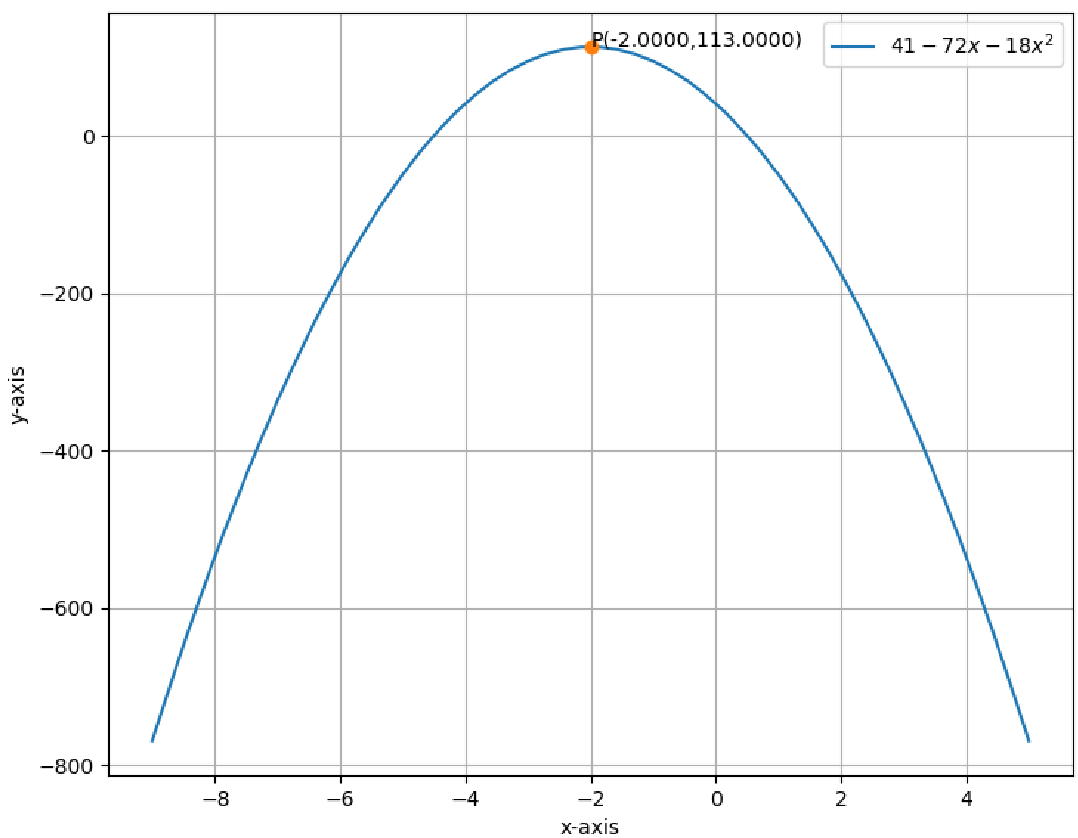


Figure 2.2.1:

2.3 Find both the maximum value and the minimum value of

$$f(x) = 3x^4 - 8x^3 + 12x^2 - 48x + 25 = 0 \qquad x \in (0, 3) \tag{2.3.1}$$

**Solution:**

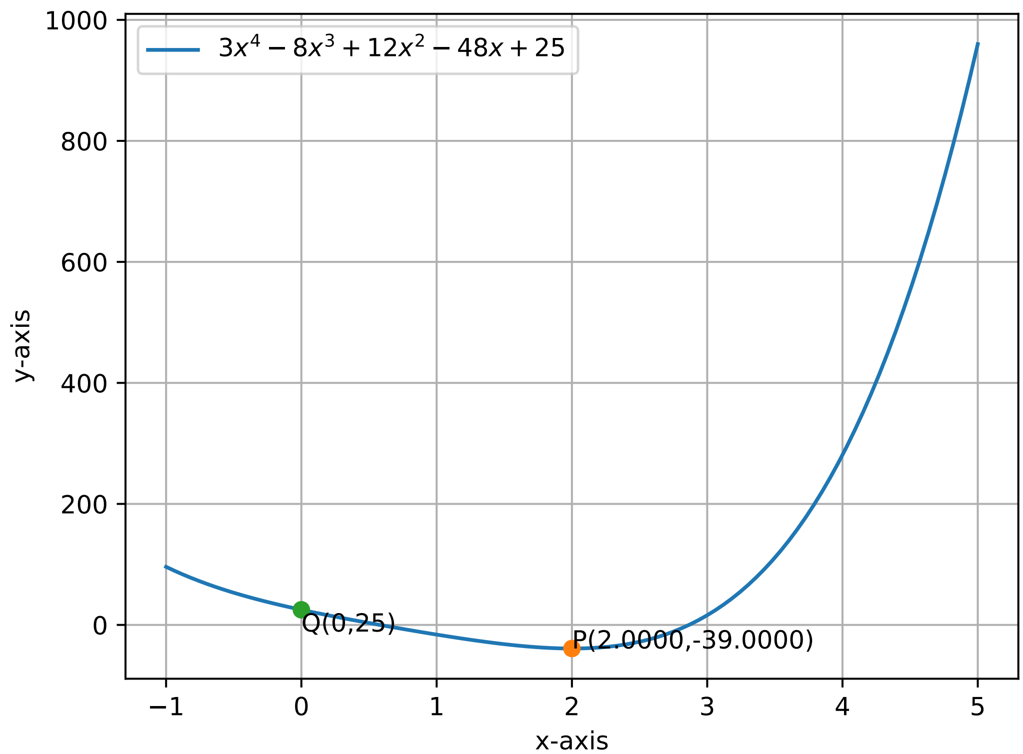


Figure 2.3.1:

$$\frac{df(x)}{dx} = 12x^3 - 24x^2 + 24x - 48 \tag{2.3.2}$$

The minimum can be found using

$$x_{n+1} = x_n - \alpha \frac{df(x)}{dx} \tag{2.3.3}$$

$$= x_n - \alpha(12x_n^3 - 24x_n^2 + 24x_n - 48) \tag{2.3.4}$$

where

(a) $\alpha = 0.001$

(b) $x_{n+1}$ is current value

(c) $x_n$ is previous value

(d) precession $= 0.00000001$

(e) maximum iterations $= 100000000$

as

$$f_{min} = -39 \tag{2.3.5}$$

$$x_{min} = 2 \tag{2.3.6}$$

2.4 At what points in the interval $(0,2\pi)$ does the function $\sin 2x$ attain its maximum value.

**Solution:** Since

$$f(x) = \sin 2x, \tag{2.4.1}$$

$$f'(x) = 2\cos 2x \tag{2.4.2}$$

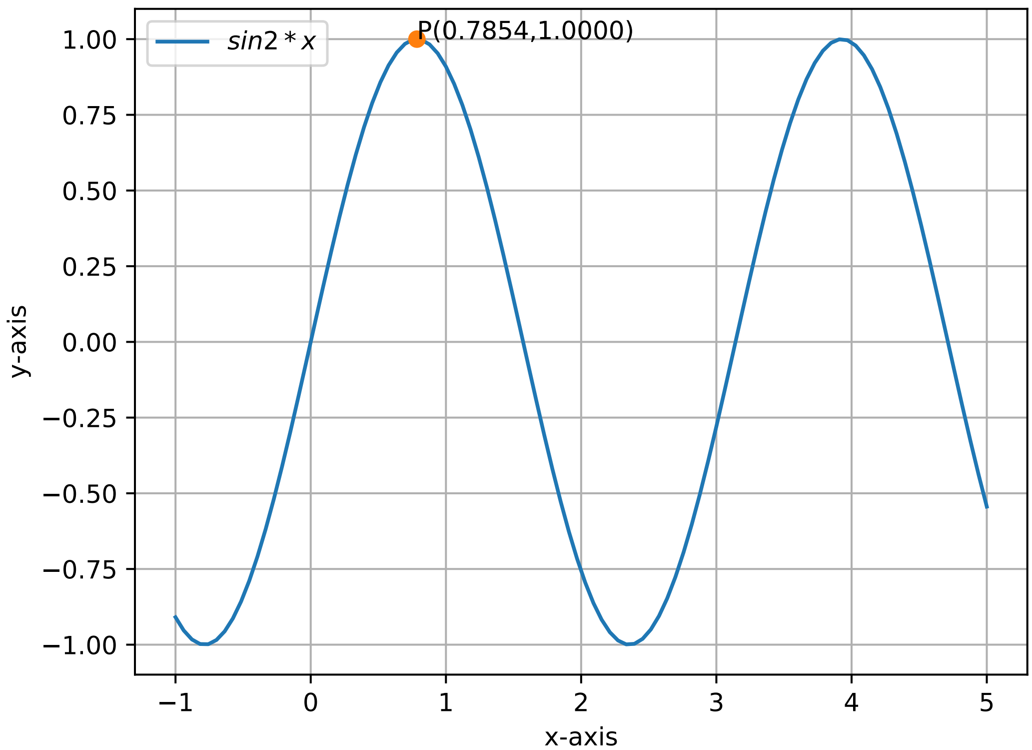


Figure 2.4.1:

Using gradient ascent,

$$x_{n+1} = x_n + \alpha \nabla f(x_n) \tag{2.4.3}$$

$$= x_n + \alpha(2\cos 2x) \tag{2.4.4}$$

Choosing

$$x_0 = 0.5, \alpha = 0.001, precision = 0.00000001, \tag{2.4.5}$$

$$f_{max} = 1.0000, x_{max} = 0.7854. \tag{2.4.6}$$

2.5 Find the maximum value of $2x^3–24x + 107$ in the interval [1, 3]. Find the maximum value of the same function in [–3, –1].

**Solution:** Using gradient ascent method,

$$x_n = x_{n-1} + \mu\frac{df(x)}{dx} \tag{2.5.1}$$

where

$$\frac{df(x)}{dx} = 6x^2 - 24 \tag{2.5.2}$$

yielding

$$x_n = x_{n-1} + \mu(6x^2 - 24_{n-1}) \tag{2.5.3}$$

Choosing

$$x_0 = 1, \mu = 0.001 \text{ and precision} = 0.00000001, \tag{2.5.4}$$

$$f_{max} \approx 139, x_{max} \approx -2.0 \tag{2.5.5}$$

2.6 It is given that at x=1, the function $x^4 - 62x^2 + ax + 9$ attains its maximum value,

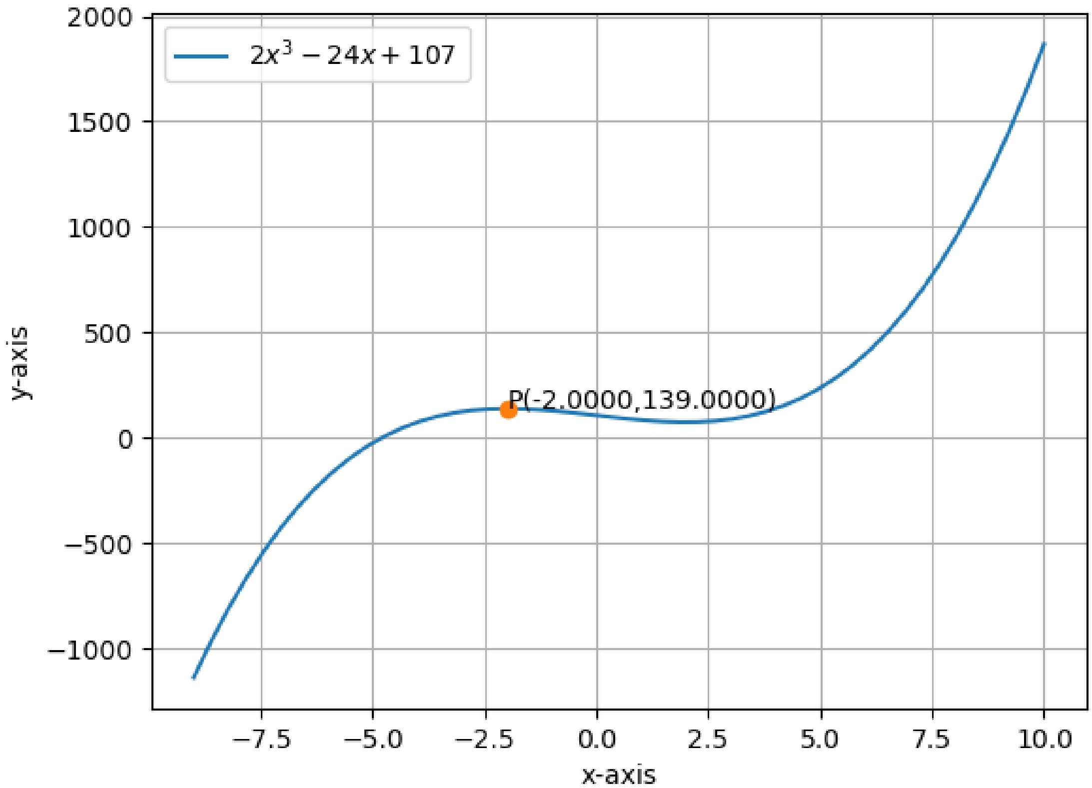


Figure 2.5.1:

on the interval [0,2]. Find the value of a.

**Solution:** Differentiating the given function,

$$\nabla f(x) = 4x^3 - 124x + a \tag{2.6.1}$$

Since $f$ attains its maximum value on the interval [0,2] at $x = 1$,

$$\nabla f(1) = 0 \implies a = 120 \tag{2.6.2}$$

Using gradient descent,

$$x_{n+1} = x_n + \alpha \nabla f(x_n) \tag{2.6.3}$$

$$= x_n + \alpha \left(4x_n^3 - 124x_n + 120\right) \tag{2.6.4}$$

and choosing

$$x_0 = 0.5, \alpha = 0.001 \text{ and precision} = 0.00000001, \tag{2.6.5}$$

$$f_{max} = 68, x_{max} = 1 \tag{2.6.6}$$

2.7 Find the absolute maximum and minimum values of the function $f$ given by

$$f(x) = \cos^2 x + \sin x, \quad x \in [0, \pi] \tag{2.7.1}$$

**Solution:** The derivative of the given function is

$$\nabla f(x) = \cos x - 2 \sin x \cos x \tag{2.7.2}$$

The maxima is calculated by

$$x_{n+1} = x_n + \alpha \nabla f(x_n) \tag{2.7.3}$$

$$= x_n + \alpha \left(cosx_n - 2sinx_n cosx_n\right) \tag{2.7.4}$$

where

(a) $x_0 = 0.5$

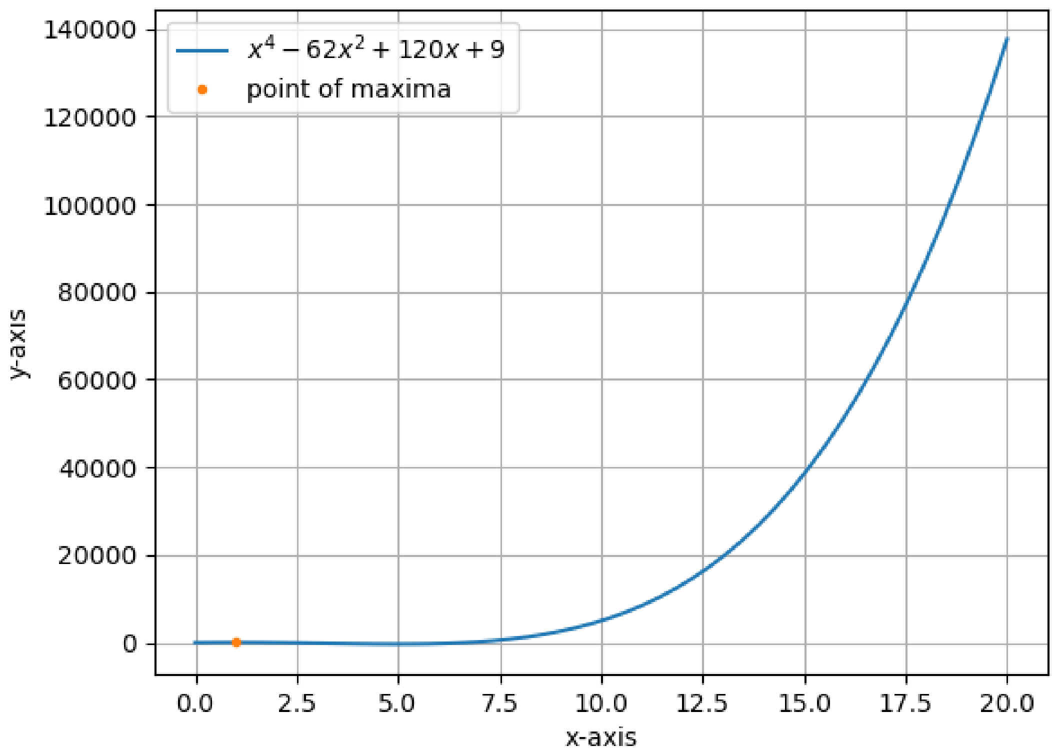


Figure 2.6.1:

(b) $\alpha = 0.001$

(c) precision $= 0.00000001$

yielding

$$f_{max} = 1.25, x_{max} = 0.52. \tag{2.7.5}$$

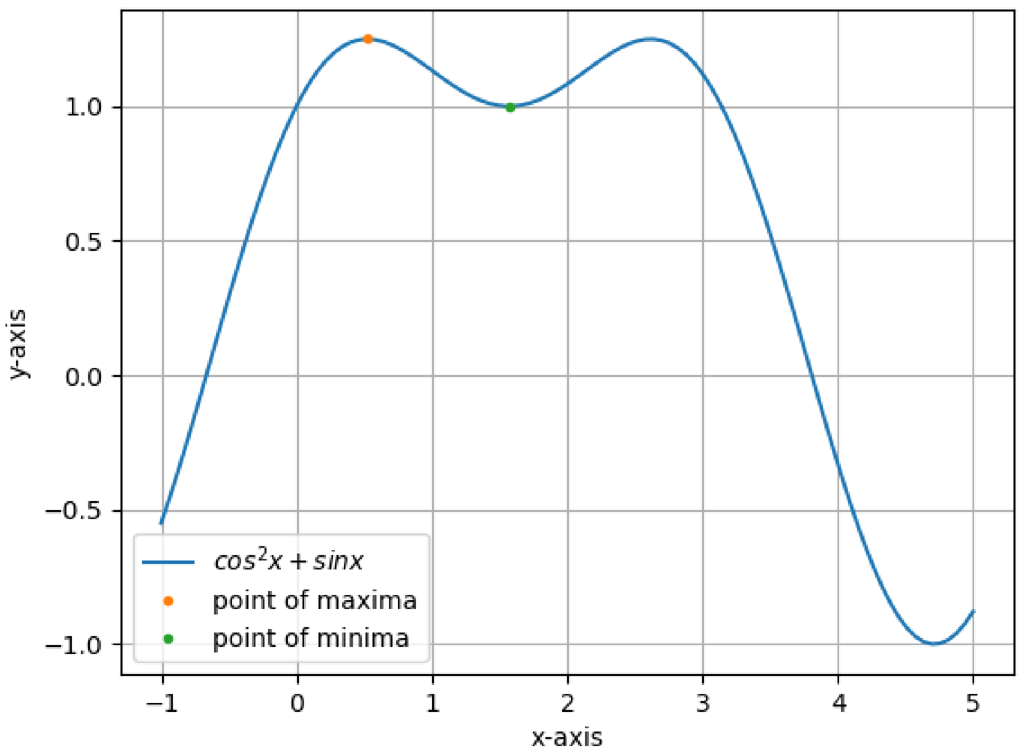


Figure 2.7.1:

The minima is found by

$$x_{n+1} = x_n - \alpha \nabla f(x_n) \tag{2.7.6}$$

$$= x_n - \alpha \left( cosx_n - 2sinx_n cosx_n \right) \tag{2.7.7}$$

# Chapter 3

# Geometric Programming

3.1 Find the maximum area of an isosceles triangle inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with its vertex at one end of the major axis.

**Solution:**

3.2 A tank with rectangular base and rectangular sides, open at the top is to be constructed so that its depth is 2 m and volume is $8m^3$. If building of tank costs Rs 70 per sq metres for the base and Rs 45 per square metre for sides. What is the cost of least expensive tank?

**Solution:**

Let l,b and h be the length, width and height of a tank. The volume of tank is given by,

$$V = lbh \tag{3.2.1}$$

$$h = 2 \tag{3.2.2}$$

Cost of Building

$$R_b = 70/m^2 \tag{3.2.3}$$

$$R_s = 45/m^2 \tag{3.2.4}$$

The given problem can then be formulated as

$$S = \min_{l,b} R_b(lb) + R_s(4(l+b)) \tag{3.2.5}$$

$$\text{s.t} \quad lb = 4 \tag{3.2.6}$$

which is a disciplined geometric programming (DGP) problem that can be solved using cvxpy. DGP is a subset of log-log-convex program (LLCP).

Chapter 4

# Linear Programming

## 4.1. Introduction

4.1.1

4.1.2 Minimise

$$Z = -3x + 4y \tag{4.1.2.1}$$

such that

$$x + 2y < 8, \tag{4.1.2.2}$$

$$3x + 2y < 12, \tag{4.1.2.3}$$

$$x > 0, y > 0 \tag{4.1.2.4}$$

**Solution:** The given problem can be formulated as

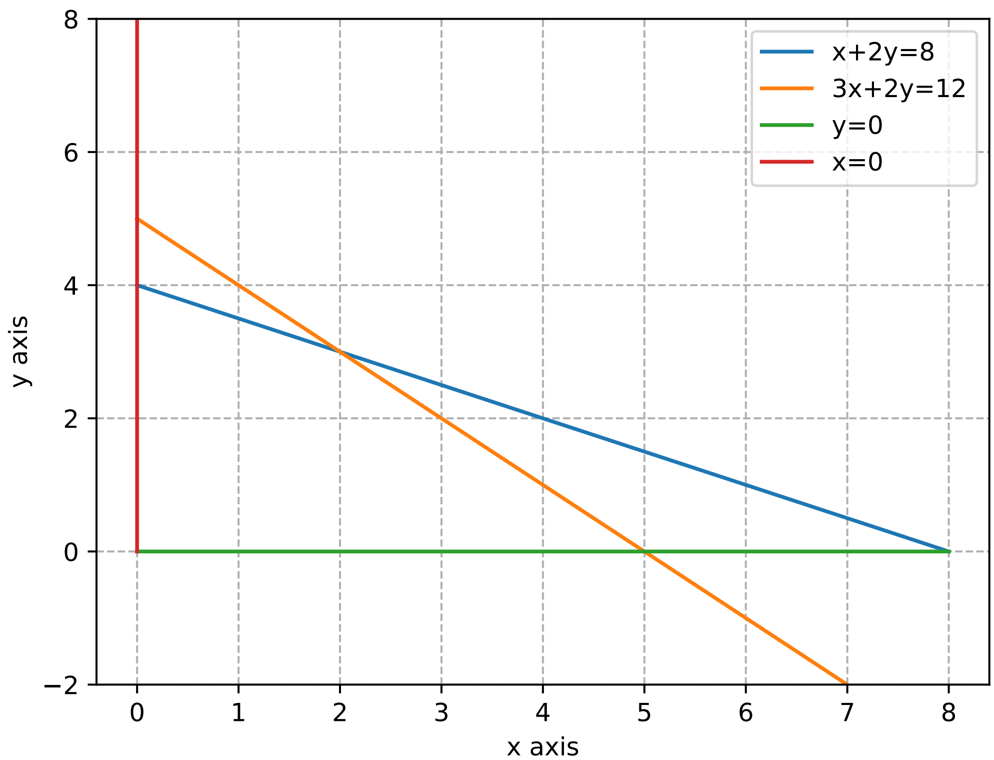


Figure 4.1.2.1:

$$\min_{\mathbf{x}} Z = \begin{pmatrix} -3 & 4 \end{pmatrix} \mathbf{x} \tag{4.1.2.5}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} \succeq \begin{pmatrix} 8 \\ 12 \\ 0 \\ 0 \end{pmatrix} \tag{4.1.2.6}$$

Solving above equations using cvxpy, we get

$$\min_{\mathbf{x}} Z = -12 \tag{4.1.2.7}$$

$$\mathbf{x} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \tag{4.1.2.8}$$

4.1.3 Maximize $Z = 5x + 3y$ such that

$$3x + 5y \leq 15, \tag{4.1.3.1}$$

$$5x + 2y \leq 10, \tag{4.1.3.2}$$

$$x \geq 0, y \geq 0. \tag{4.1.3.3}$$

**Solution:** The given problem can be expressed as

$$Z = \begin{pmatrix} 5 & 3 \end{pmatrix} \mathbf{x} \tag{4.1.3.4}$$

$$s.t. \quad \begin{pmatrix} 3 & 5 \\ 5 & 2 \end{pmatrix} \mathbf{x} \leq \begin{pmatrix} 15 \\ 10 \end{pmatrix} \tag{4.1.3.5}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{4.1.3.6}$$

$$\tag{4.1.3.7}$$

Using cvxpy, the solution is

$$\mathbf{x} = \begin{pmatrix} \frac{20}{19} \\ \frac{45}{19} \end{pmatrix}, Z_{max} = \frac{235}{19} \tag{4.1.3.8}$$

4.1.4 Minimize $Z = 3x + 5y$ such that

$$x + 3y \geq 3 \tag{4.1.4.1}$$

$$x + y \geq 2 \tag{4.1.4.2}$$

$$x \geq 0, y \geq 0. \tag{4.1.4.3}$$

**Solution:** The given problem can be expressed as

$$Z = \min_{\mathbf{x}} \begin{pmatrix} 3 & 5 \end{pmatrix} \mathbf{x} \tag{4.1.4.4}$$

$$\begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} \mathbf{x} \succeq \begin{pmatrix} 3 \\ 2 \end{pmatrix} \tag{4.1.4.5}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} \succeq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{4.1.4.6}$$

Solving using cvxpy, we get,

$$\mathbf{x} = \begin{pmatrix} \frac{3}{2} \\ \frac{1}{2} \end{pmatrix}, Z_{min} = 7 \tag{4.1.4.7}$$

4.1.5

4.1.6 Minimize Z=x+2y subject to

$$2x + 3y \geq 3 \tag{4.1.6.1}$$

$$x + 2y \geq 6 \tag{4.1.6.2}$$

$$x, y \geq 0. \tag{4.1.6.3}$$

**Solution:** The optimization problem can be defined as

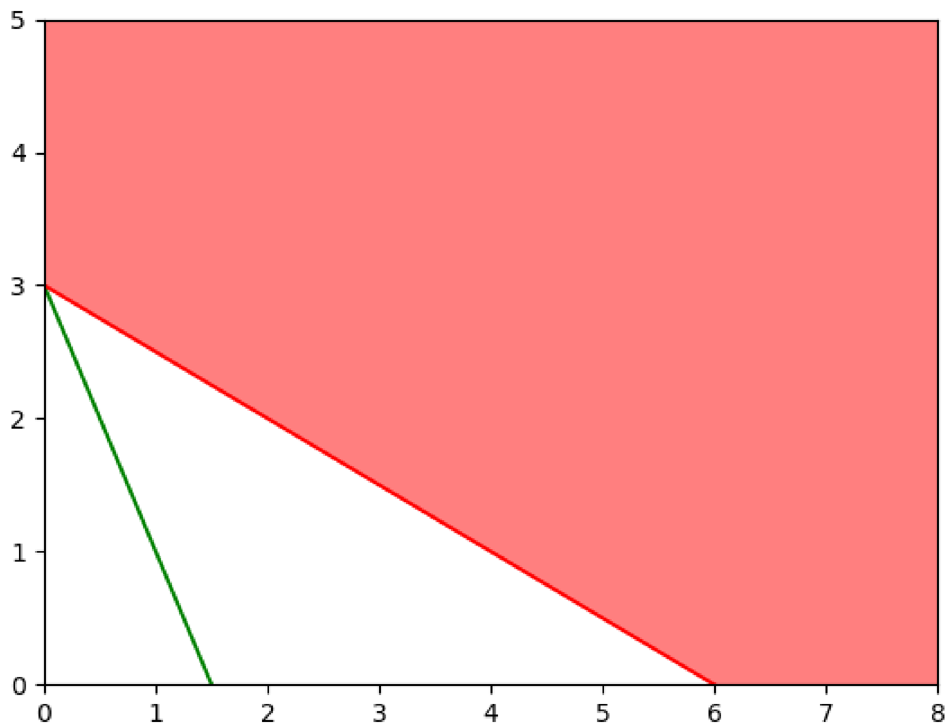


Figure 4.1.6.1:

$$P = \min_{\mathbf{x}} \begin{pmatrix} 1 & 2 \end{pmatrix} \mathbf{x} \tag{4.1.6.4}$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \mathbf{x} \succeq \begin{pmatrix} 3 \\ 6 \end{pmatrix} \tag{4.1.6.5}$$

$$x, y \geq \mathbf{0} \tag{4.1.6.6}$$

From Fig. 4.1.6.1, the feasible region vertices are

$$\begin{pmatrix} 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 6 \\ 0 \end{pmatrix} \tag{4.1.6.7}$$

yielding

$$\begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \end{pmatrix} = 6 \tag{4.1.6.8}$$

$$\begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 6 \\ 0 \end{pmatrix} = 6 \tag{4.1.6.9}$$

$$\tag{4.1.6.10}$$

Thus, the minimum value of Z is 6.

### 4.1.7 Minimise and Maximise

$$Z = 5x + 10y \tag{4.1.7.1}$$

subject to

$$x + 2y \leq 120 \tag{4.1.7.2}$$

$$x + y \geq 60 \tag{4.1.7.3}$$

$$x - 2y \geq 0 \tag{4.1.7.4}$$

$$x \geq 0, y \geq 0 \tag{4.1.7.5}$$

**Solution:** The given problem can be formulated as

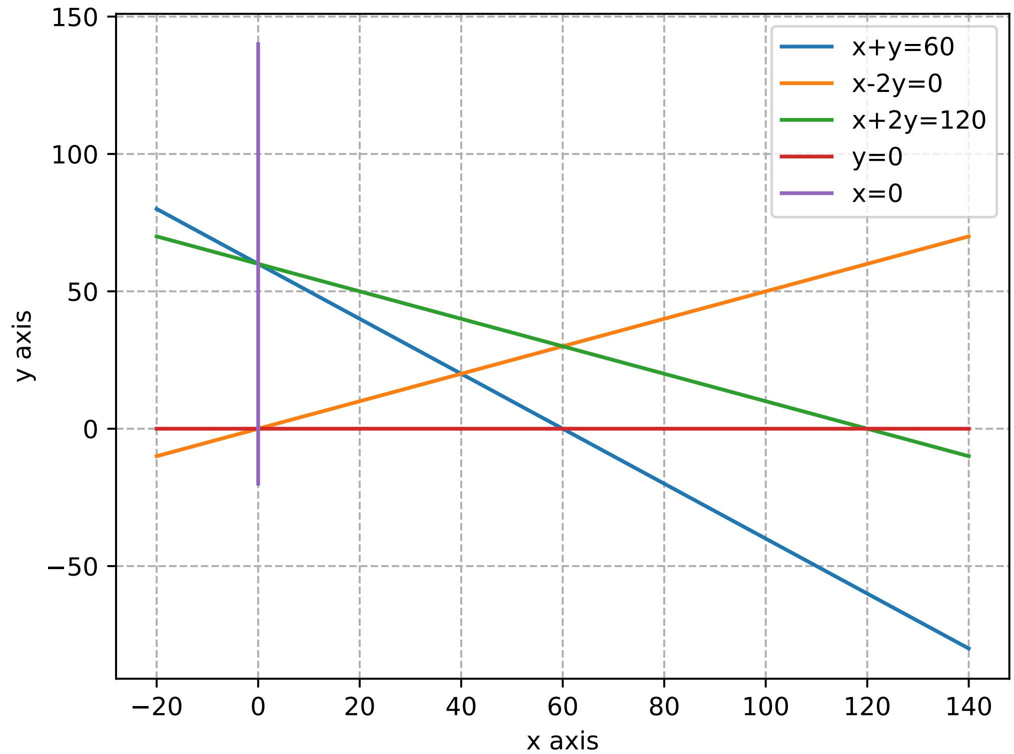


Figure 4.1.7.1:

$$\min_{\mathbf{x}} \mathbf{Z} = \begin{pmatrix} 5 & 10 \end{pmatrix} \mathbf{x} \tag{4.1.7.6}$$

$$\max_{\mathbf{x}} \mathbf{Z} = \begin{pmatrix} 5 & 10 \end{pmatrix} \mathbf{x} \tag{4.1.7.7}$$

$$s.t. \quad \begin{pmatrix} -1 & -2 \\ 1 & 1 \\ 1 & -2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} \succeq \begin{pmatrix} -120 \\ 60 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{4.1.7.8}$$

$$\tag{4.1.7.9}$$

Solving above equations using cvxpy,

$$\min_{\mathbf{x}} Z = 300, \mathbf{x} = \begin{pmatrix} 60 \\ 0 \end{pmatrix} \tag{4.1.7.10}$$

$$\max_{\mathbf{x}} Z = 600, \mathbf{x} = \begin{pmatrix} 60 \\ 30 \end{pmatrix} \tag{4.1.7.11}$$

4.1.8

4.1.9 Maximise

$$Z = -x + 2y \tag{4.1.9.1}$$

subject to the constraints

$$x + y \geq 5 \tag{4.1.9.2}$$

$$x + 2y \geq 6 \tag{4.1.9.3}$$

$$x \geq 3, y \geq 0. \tag{4.1.9.4}$$

**Solution:** The given problem can be expressed as

$$z = \max_{\mathbf{x}} \begin{pmatrix} -1 & 2 \end{pmatrix} \mathbf{x} \tag{4.1.9.5}$$

$$s.t. \quad \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 5 \\ 6 \\ 3 \\ 0 \end{pmatrix} \tag{4.1.9.6}$$

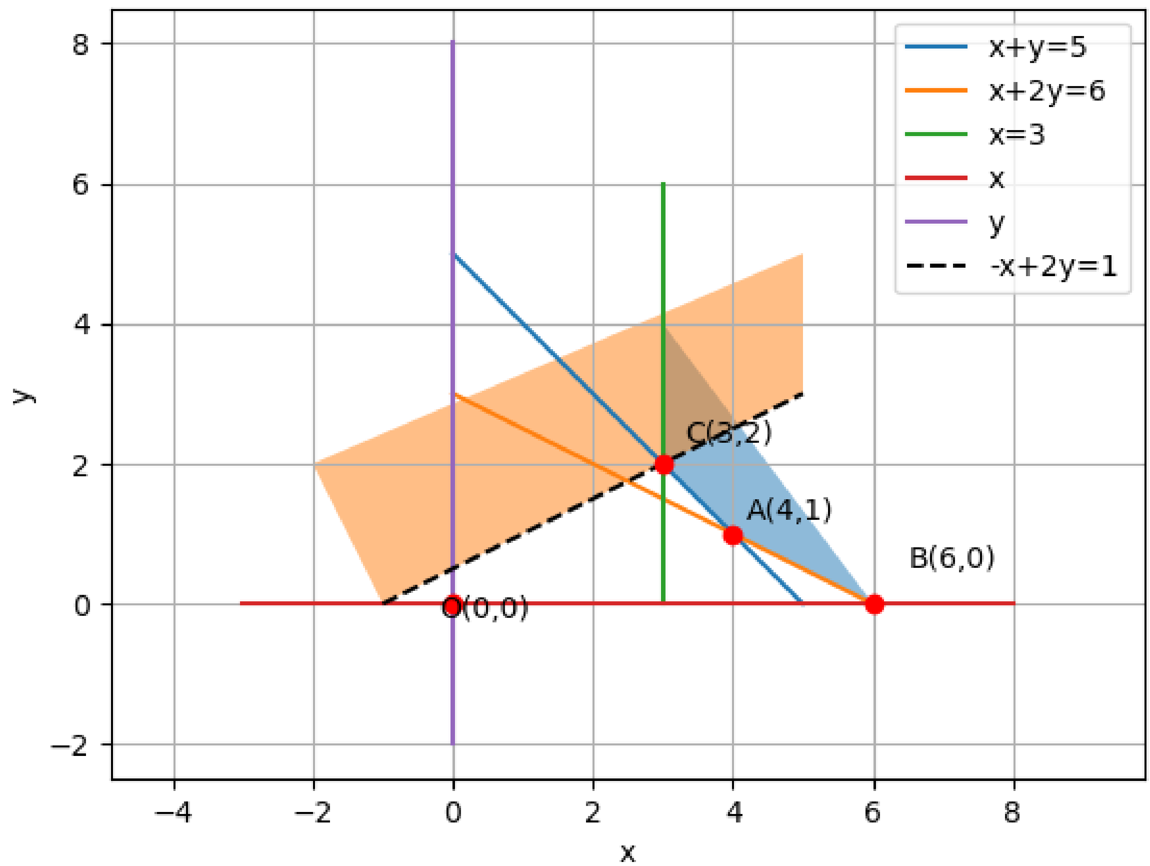


Figure 4.1.9.1:

By providing the objective function and constraints to cvxpy, the optimal value gives infinity as result and the problem is unbounded. This is verified from Fig. 4.1.9.1.

4.1.10 Maximize

$$Z = x + y \tag{4.1.10.1}$$

subject to

$$x - y \leq -1 \tag{4.1.10.2}$$

$$-x + y \leq 0 \tag{4.1.10.3}$$

$$x, y \geq 0 \tag{4.1.10.4}$$

**Solution:** From Fig. 4.1.10.1, the given problem has no optimal solution. This is

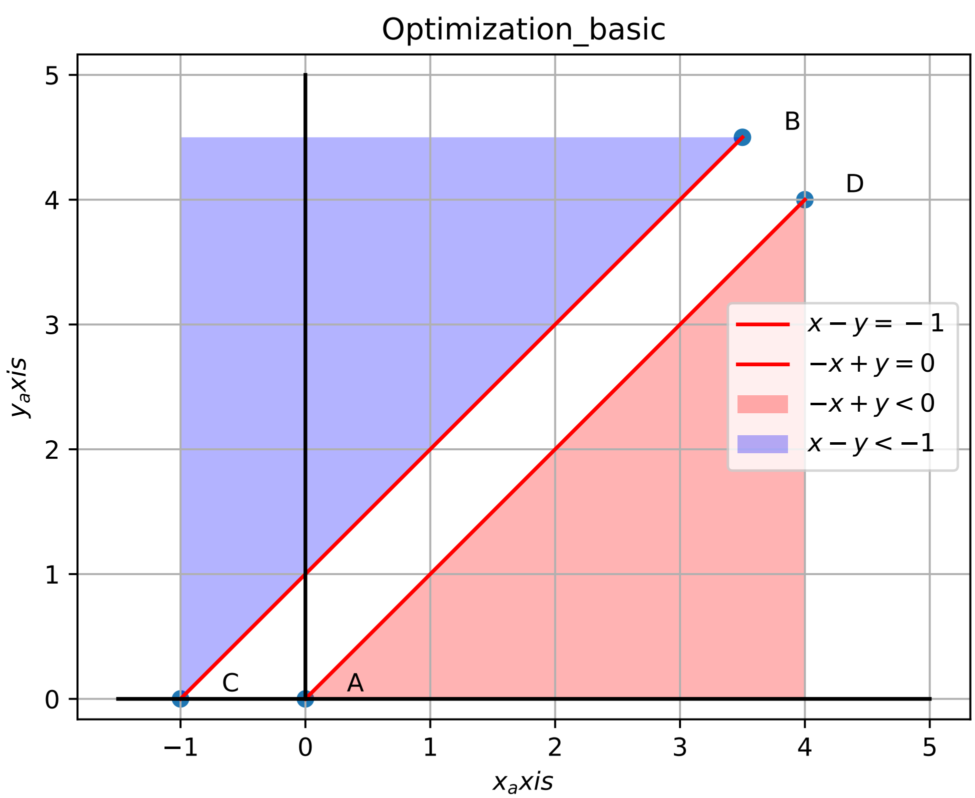


Figure 4.1.10.1:

verified from cvxpy by considering the following optimization problem.

$$z = \max_{x} \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} \tag{4.1.10.5}$$

$$s.t. \quad \begin{pmatrix} 1 & -1 \\ -1 & 0 \\ 0 & -1 \\ -1 & 0 \end{pmatrix} x \preceq \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{4.1.10.6}$$

## 4.2. Applications

4.2.1 Reshma wishes to mix two types of food P and Q in such a way that the vitamin contents of the mixture contain at least 8 units of vitamin A and 11 units of vitamin B. Food P costs Rs 60/kg and Food Q costs Rs 80/kg. Food P contains 3 units/kg of Vitamin A and 5 units / kg of Vitamin B while food Q contains 4 units/kg of Vitamin A and 2 units/kg of vitamin B. Determine the minimum cost of the mixture.

**Solution:** Let the mixture contain $x$ kg of food and $y$ kg of food. The given information can be compiled in a table as and which can be expressed in vector form as

| | Vitamin A(units/kg) | Vitamin B(units/kg) | Cost(Rs/kg) |
|---|---|---|---|
| Food P | 3 | 5 | 60 |
| Food Q | 4 | 2 | 80 |
| Requirement(units/kg) | 8 | 11 | |

Table 4.2.1.1:

$$P = \min_{\mathbf{x}} \begin{pmatrix} 60 & 80 \end{pmatrix} \mathbf{x} \tag{4.2.1.1}$$

$$\begin{pmatrix} 3 & 4 \\ 5 & 2 \end{pmatrix} \mathbf{x} \succeq \begin{pmatrix} 8 \\ 11 \end{pmatrix} \tag{4.2.1.2}$$

$$\mathbf{x} \succeq \mathbf{0} \tag{4.2.1.3}$$

Solving using cvxpy, we get

$$P_{min} = 159.99999999 \tag{4.2.1.4}$$

$$\mathbf{x} = \begin{pmatrix} 2.11436236 \\ 0.41422823 \end{pmatrix} \tag{4.2.1.5}$$

4.2.2 One kind of cake requires 200g of flour and 25g of fat, and another kind of cake requires 100g of flour and 50g of fat. Find the maximum number of cakes which can be made from 5kg of flour and 1 kg of fat assuming that there is no shortage of the other ingredients used in making the cakes.

**Solution:** Let $x, y$ be the number of cakes of first kind and second kind that can be made from the given amount of floor and fat respectively. From the given information,

| **Kind of cake** | **No. of cakes** | **Flour (in gm)** | **Fat (in gm)** |
|---|---|---|---|
| $Cake_1$ | x | 200 | 25 |
| $Cake_2$ | y | 100 | 50 |

Table 4.2.2.1:

$$200x + 100y \leq 5000 \tag{4.2.2.1}$$

$$100x + 50y \leq 1000 \tag{4.2.2.2}$$

Let P be the maximum number of cakes that can be made from the given amount of flour and fat. The problem can be formulated as

$$P = \max_{\mathbf{x}} \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} \tag{4.2.2.3}$$

$$\begin{pmatrix} 200 & 100 \\ 100 & 50 \end{pmatrix} \mathbf{x} \leq \begin{pmatrix} 5000 \\ 1000 \end{pmatrix} \tag{4.2.2.4}$$

$$\mathbf{x} \geq \mathbf{0} \tag{4.2.2.5}$$

Solving the above equations using cvxpy, we get

$$P_{max} = 30, \mathbf{x} = \begin{pmatrix} 20 \\ 10 \end{pmatrix} \tag{4.2.2.6}$$

4.2.3 A factory makes tennis rackets and cricket bats. A tennis racket takes 1.5 hours of machine time and 3 hours of craftman's time in its making while a cricket bat takes 3 hour of machine time and 1 hour of craftman's time. In a day, the factory has the availability of not more than 42 hours of machine time and 24 hours of craftsman's time.

(a) What number of rackets and bats must be made if the factory is to work at full capacity?

(b) If the profit on a racket and on a bat is Rs 20 and Rs 10 respectively, find the maximum profit of the factory when it works at full capacity.

**Solution:** The given information is summarized in Table 4.2.3.1. From the given

| Item | Number | Machine hours | Craftman's hours | Profit |
|---|---|---|---|---|
| Tennis Rackets | x | 1.5 | 3 | $Rs.20$ |
| Cricket Bats | y | 3 | 1 | $Rs.10$ |
| Maximum time available | | 42 | 24 | |

Table 4.2.3.1:

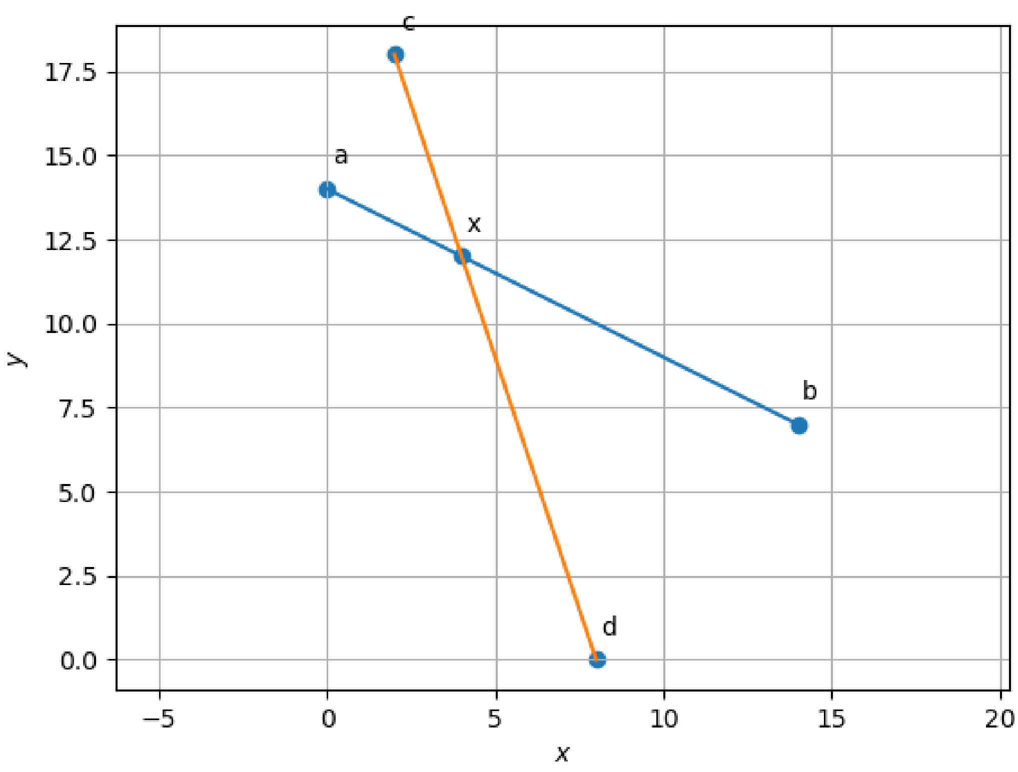


Figure 4.2.3.1:

information, the optimization problem can be expressed as

$$Z = \max_{\mathbf{x}} \begin{pmatrix} 20 & 10 \end{pmatrix} \mathbf{x} \tag{4.2.3.1}$$

$$s.t. \quad \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} \mathbf{x} \preceq \begin{pmatrix} 28 \\ 24 \end{pmatrix} \tag{4.2.3.2}$$

$$\mathbf{x} \succeq \mathbf{0} \tag{4.2.3.3}$$

From Fig. 4.2.3.1, the values at the corner points are obtained in Table 4.2.3.2.

| **Corner points** | **Value of Z** |
|---|---|
| (0,14) | 140 |
| (4,12) | 200 |
| (8,0) | 160 |

Table 4.2.3.2:

At full capacity,

$$\mathbf{x} = \begin{pmatrix} 4 \\ 12 \end{pmatrix}, \tag{4.2.3.4}$$

$$\implies Z = \begin{pmatrix} 20 & 10 \end{pmatrix} \begin{pmatrix} 4 \\ 12 \end{pmatrix} \tag{4.2.3.5}$$

$$= 200 \tag{4.2.3.6}$$

This is verified using cvxpy.

4.2.4 A manufacturer produces nuts and bolts. It takes 1 hour of work on machine A and 3 hours on machine B to produce a package of nuts. It takes 3 hours on machine A and 1 hour on machine B to produce a package of bolts. He earns a profit of Rs17.50 per package on nuts and Rs 7.00 per package on bolts.

How many packages of each should be produced each day so as to maximise his profit, if he operates his machines for at the most 12 hours a day?.

**Solution:** Table 4.2.4.1 summarizes the given information. The optimization problem

| Symbol | Name | Machine A | Machine B | Profit |
|---|---|---|---|---|
| **x** | nuts | 1x | 3x | 17.5x |
| **y** | bolts | 3y | 1y | 7y |
| **Sum** | x+y | x+3y | 3x+y | 17.5x+7y |
| **t** | time | 12 h | 12 h | |

Table 4.2.4.1:

is formulated as

$$z = \begin{pmatrix} 17.5 & 7 \end{pmatrix} \mathbf{x} \tag{4.2.4.1}$$

$$\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \mathbf{x} \preceq \begin{pmatrix} 12 \\ 12 \end{pmatrix} \tag{4.2.4.2}$$

Fig. 4.2.4.1 represents the given constraints from which, the corner points of the feasible region intersected by above two lines are

$$\mathbf{0}, \mathbf{A} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \text{ and } \mathbf{C} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} \tag{4.2.4.3}$$

The values of $Z$ at these points are listed in Table 4.2.4.2. Thus, the maximum profit

| Corner Point | z=17.5x+7y | Remarks |
|---|---|---|
| O=$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ | 0 | |
| A=$\begin{pmatrix} 0 \\ 4 \end{pmatrix}$ | 28 | |
| C=$\begin{pmatrix} 3 \\ 3 \end{pmatrix}$ | 73.5 | Maximum |
| B=$\begin{pmatrix} 4 \\ 0 \end{pmatrix}$ | 70 | |

Table 4.2.4.2:

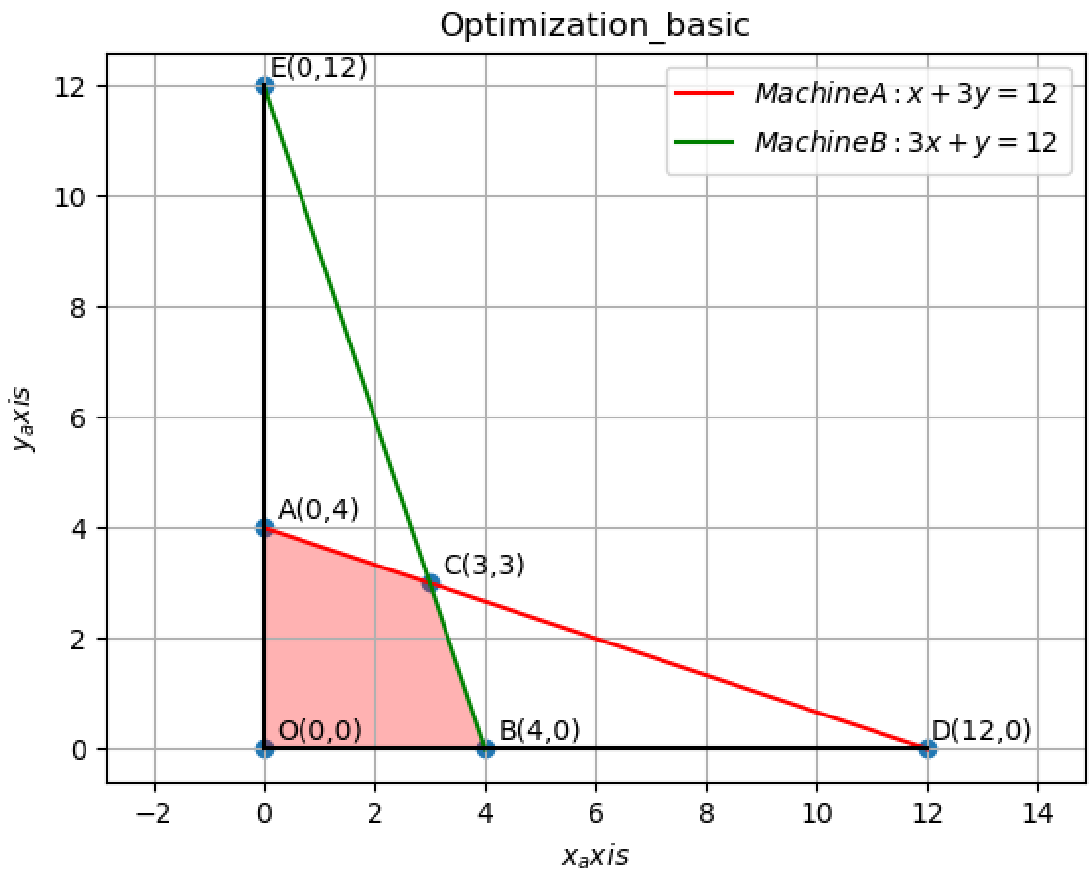


Figure 4.2.4.1:

is

$$z = 73.50, \mathbf{x} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}. \tag{4.2.4.4}$$

4.2.5 A factory manufactures two types of screws , A and B. Each type screw requires the use of two machines , an automatic and a hand operated. It takes 4 minutes on the automatic and 6 minutes on hand operated machines to manufacture a package of screws A,while it takes 6 minutes on automatic and 3 minutes on the hand operated machines to manufacture a package of screws B. Each machine is available for at the most 4hrs on any day. The manufacturer can sell a package of screws A at a profit of Rs. 7 and screws B at a profit of Rs. 10. Assuming that he can sell all the screws he manufactures, how many packages of each type should the factory owner produce in a day in order to maximise his profit? Determine the maximum profit.

**Solution:** The given information is summarized in Table 4.2.5.1 resulting in the

| **Item** | **Number** | **Machine A** | **Machine B** | **Profit** |
|---|---|---|---|---|
| Screw A | x | 4(min) | 6(min) | 7 |
| SCREW B | y | 6(min) | 3(min) | 10 |
| Max.Time available | | 4 hrs | 4 hrs | |

Table 4.2.5.1:

following optimization problem.

$$z = \max_{\mathbf{x}} \begin{pmatrix} 7 & 10 \end{pmatrix} \begin{pmatrix} x \end{pmatrix} \tag{4.2.5.1}$$

$$s.t. \quad \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \preceq \begin{pmatrix} 120 \\ 80 \end{pmatrix} \tag{4.2.5.2}$$

$$\mathbf{x} \succeq \mathbf{0} \tag{4.2.5.3}$$

The maximum profit is obtained as

$$\begin{pmatrix} 2 & 3 & 120 \\ 2 & 1 & 80 \end{pmatrix} \longrightarrow R_2 - R_1 \begin{pmatrix} 2 & 3 & 120 \\ 0 & -2 & -40 \end{pmatrix} \longrightarrow R_2 * \frac{-1}{2} \tag{4.2.5.4}$$

$$\begin{pmatrix} 2 & 3 & 120 \\ 0 & 1 & 20 \end{pmatrix} \longrightarrow R_1 - 3R_2 \begin{pmatrix} 2 & 0 & 60 \\ 0 & 1 & 20 \end{pmatrix} \longrightarrow \frac{R_1}{2} \begin{pmatrix} 1 & 0 & 30 \\ 0 & 1 & 20 \end{pmatrix} \tag{4.2.5.5}$$

yielding the maximum profit

$$z = 410 \tag{4.2.5.6}$$

at

$$\mathbf{x} = \begin{pmatrix} 30 \\ 20 \end{pmatrix}. \tag{4.2.5.7}$$

This is verified by Table 4.2.5.2, where the corner points are obtained from Fig. 4.2.5.1.

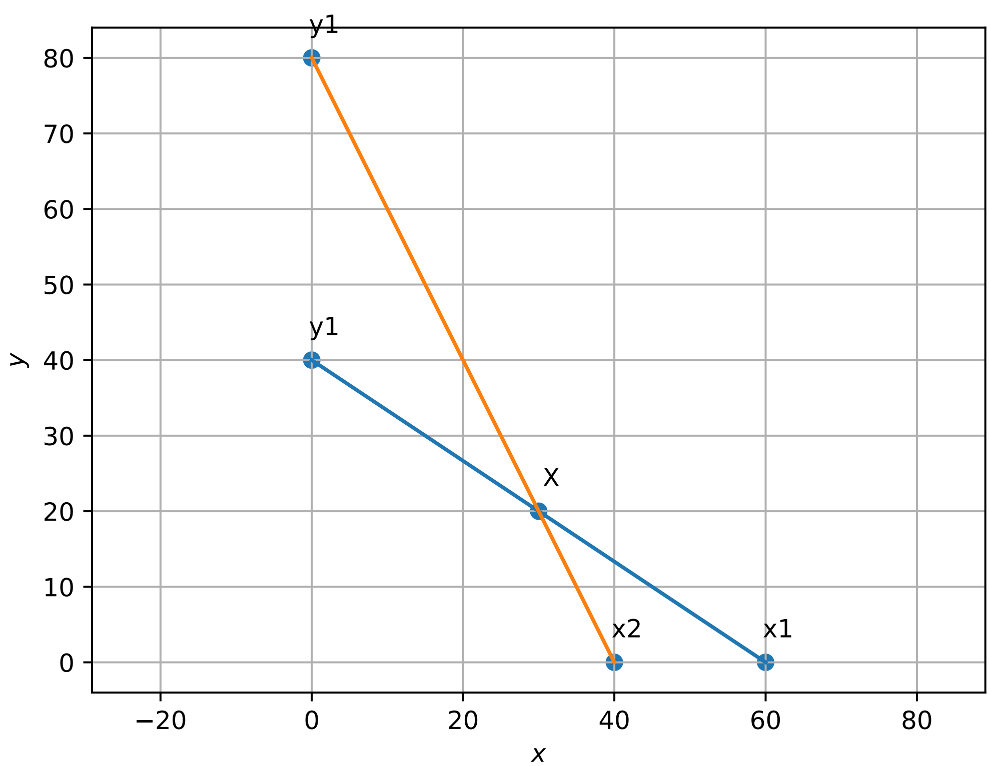


Figure 4.2.5.1:

| Corner points | Value of Z |
|---|---|
| (0,40) | 400 |
| (30,20) | 410 |
| (40,0) | 280 |

Table 4.2.5.2:

4.2.6

4.2.7

4.2.8 A merchant plans to sell two types of personal computers, a desktop model and a portable model that will cost Rs 25000 and Rs 40000 respectively. He estimates that the total monthly demand of computers will not exceed 250 units. Determine the number of units of each type of computers which the merchant should stock to get maximum profit if he does not want to invest more than Rs 70 lakhs and if his profit on the desktop model is Rs 4500 and on portable model is Rs 5000.

**Solution:** Table 4.2.8.1 summarizes the given information. The optimization problem

| Item | Number | Cost | Profit |
|---|---|---|---|
| Desktop | x | 25000 | 4500 |
| Portable Computers | y | 40000 | 5000 |
| Max Investment | | 7000000 | |

Table 4.2.8.1:

can then be summarized as

$$Z = \begin{pmatrix} 4500 & 5000 \end{pmatrix} \mathbf{x} \tag{4.2.8.1}$$

$$s.t. \quad \begin{pmatrix} 1 & 1 \\ 5 & 8 \end{pmatrix} \mathbf{x} \preceq \begin{pmatrix} 250 \\ 1400 \end{pmatrix} \tag{4.2.8.2}$$

$$\mathbf{x} \succeq \mathbf{0} \tag{4.2.8.3}$$

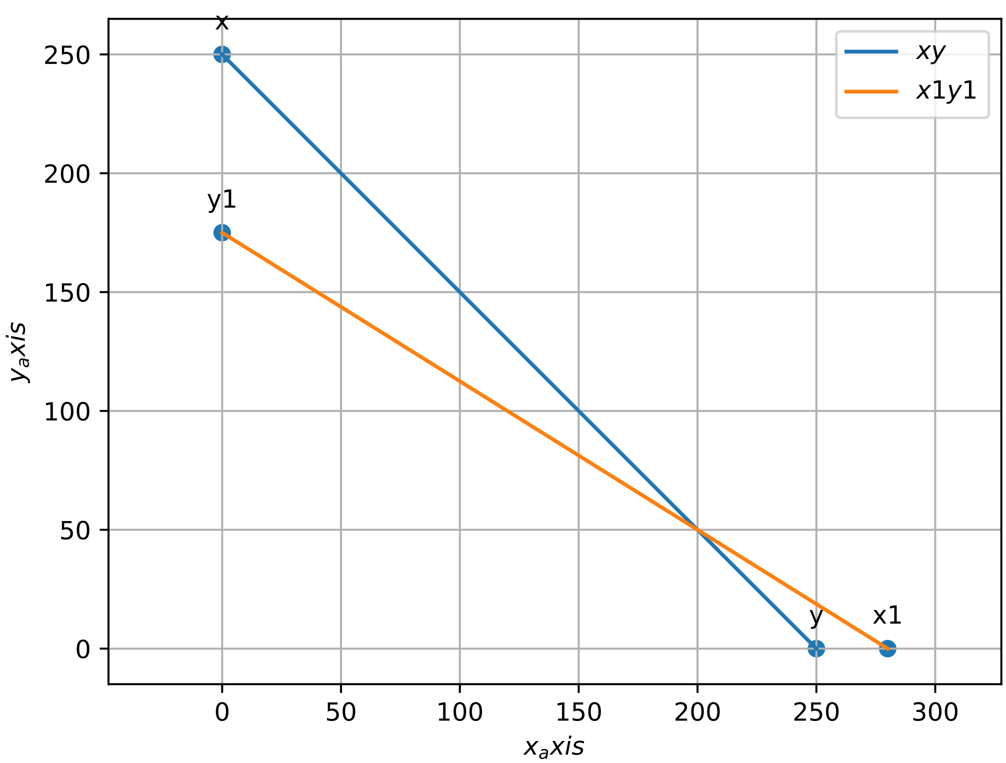


Figure 4.2.8.1:

From Fig. 4.2.8.1, the corner points are listed in Table 4.2.8.2 yielding the solution

$$Z = 1150000, \mathbf{x} = \begin{pmatrix} 200 \\ 50 \end{pmatrix} \tag{4.2.8.4}$$

| **Corner points** | **Value of Z** |
|---|---|
| (250,0) | 112500 |
| (200,50) | 1150000 |
| (0,175) | 875000 |

Table 4.2.8.2:

4.2.9

4.2.10 There are two types of fertilisers $F_1$ and $F_2$. $F_1$ consists of 10% Nitrogen and 6% Phosphoric acidand $F_2$ consists of 5% Nitrogen and 10% Phosphoric acid. After testing the soil conditions, a farmer finds that she needs atleast 14 kg of nitrogen and 14 kg of phosphoric acid for her crop. If $F_1$ costs Rs 6/kg and $F_2$ costs Rs 5/kg, determine how much of each type of fertiliser should be used so that nutrient requirements are met at a minimum cost. What is the minimum cost?

**Solution:** The optimization problem can be framed from Table 4.2.10.1 as

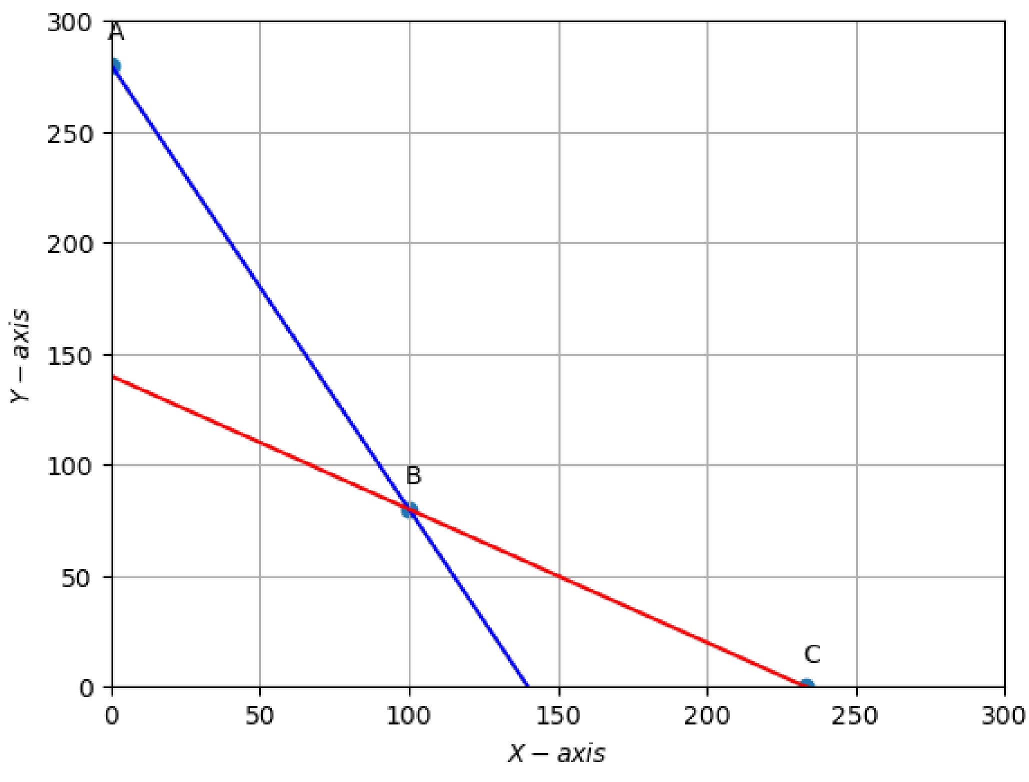


Figure 4.2.10.1:

| **Fertiliser** | **Nitrogen** | **Phosphoric Acid** |
|---|---|---|
| $F_1$ | 10% | 6% |
| $F_2$ | 5% | 10% |
| Total | 14 kg | 14 kg |

Table 4.2.10.1:

$$Z = \min_{\mathbf{x}} \begin{pmatrix} 6 & 5 \end{pmatrix} \mathbf{x} \tag{4.2.10.1}$$

$$\begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix} \mathbf{x} \preceq \begin{pmatrix} 280 \\ 700 \end{pmatrix} \tag{4.2.10.2}$$

$$\mathbf{x} \succeq \mathbf{0} \tag{4.2.10.3}$$

yielding

$$Z_{min} = Rs.1000, \mathbf{x} = \begin{pmatrix} 100 \\ 80 \end{pmatrix} \tag{4.2.10.4}$$

## 4.3. Miscellenaeous

4.3.1

4.3.2

4.3.3 A dietician wishes to mix together two kinds of food X and Y in such away that the mixture contains atleast 10 units of vitamin A ,12 units of vitamin B and 8 units of vitamin C .The vitamin contents of one kg food is given below One kg of food X costs

| **Food** | **Vitamin A** | **Vitamin B** | **Vitamin C** |
|---|---|---|---|
| **X** | 1 | 2 | 3 |
| **Y** | 2 | 2 | 1 |

Table 4.3.3.1:

Rs 16 and one kg of food Y costs Rs 20. Find the least cost of the mixture which will produce the required diet.

**Solution:**

4.3.4 A manufacturer makes two types of toys A and B.Three machines are needed for this purpose and the time(in minutes)required for each toy on the machines is given below

| **Types of Toys** | **I** | **II** | **III** |
|---|---|---|---|
| A | 12 | 18 | 6 |
| B | 6 | 0 | 9 |

Each machine is available for a maximum 6 hours per day. If the profit on each toy of type A is Rs.7.50 and that on each toy of type B is Rs.5, show that 15 toys of type A and 30 type B should be manufactured in a day to get maximum profit.

**Solution:** The given information can be framed as the optimization problem

$$Z = \max_{\mathbf{x}} \begin{pmatrix} 7.50 & 5 \end{pmatrix} \mathbf{x} \tag{4.3.4.1}$$

$$\begin{pmatrix} 2 & 1 \\ 3 & 0 \\ 2 & 3 \end{pmatrix} \mathbf{x} \preceq \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \tag{4.3.4.2}$$

$$\mathbf{x} \succeq \mathbf{0} \tag{4.3.4.3}$$

Solving the above equations using cvxpy, we obtain

$$Z_{max} = Rs.262.50, \mathbf{x} = \begin{pmatrix} 15 \\ 30 \end{pmatrix} \tag{4.3.4.4}$$

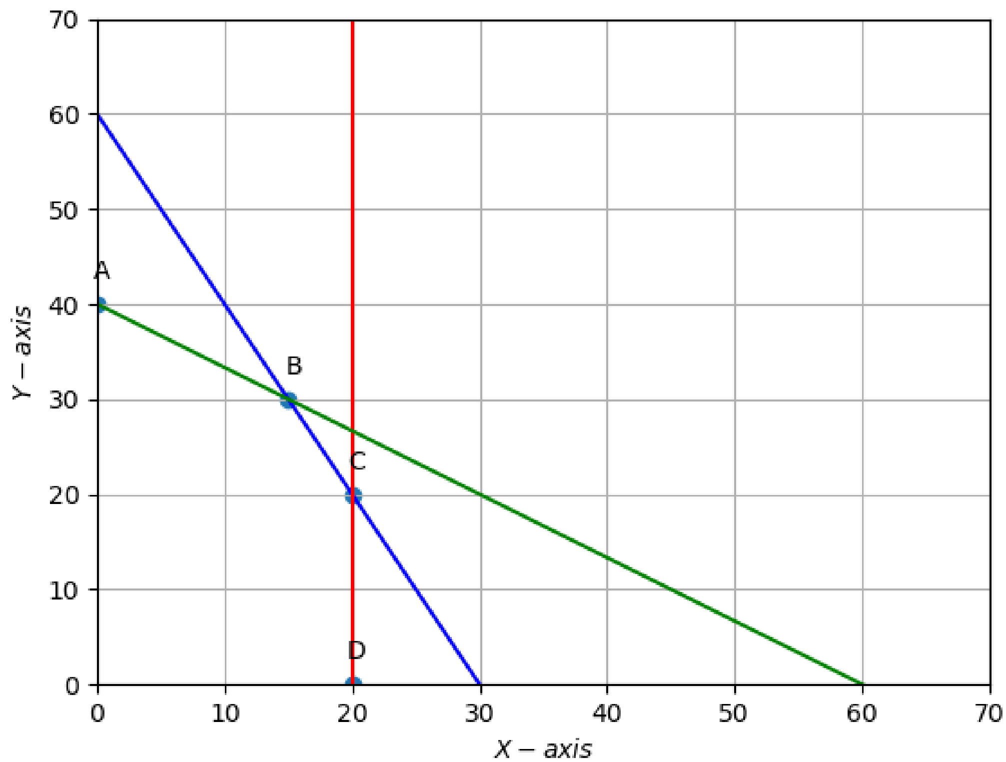


Figure 4.3.4.1:

4.3.5 An aeroplane can carry a maximum of 200 passengers.A profit of Rs.1000 is made on each executive class ticket and a profit of Rs.600 is made on each economy class ticket. The airline reserves at least 20 seats for executive class. However, at least 4 times as many passengers prefer to travel by economy class than by the executive class. Determine how many tickets of each type must be sold in order to maximise the profit for the airline. What is the maximum profit?

**Solution:** Let $P$ be the maximum number of tickets of each type must be sold in order to maximise the profit for the airline . The problem can be formulated as

$$P = \max_{\mathbf{x}} \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} \tag{4.3.5.1}$$

$$s.t. \quad \begin{pmatrix} -1 & -1 \\ 1 & 0 \\ -4 & 1 \end{pmatrix} \mathbf{x} \succeq \begin{pmatrix} 200 \\ 20 \\ 0 \end{pmatrix} \tag{4.3.5.2}$$

yielding

$$P_{max} = 136000, \mathbf{x} = \begin{pmatrix} 40 \\ 160 \end{pmatrix} \tag{4.3.5.3}$$

4.3.6 Two godowns A and B have grain capacity of 100 quintals and 50 quintals respectively. They supply to 3 ration shops, D, E and F whose requirements are 60, 50 and 40 quintals respectively. The cost of transportation per quintal from the godowns to the shops are given in the following table How should the supplies be

| From/to | A | B |
|---|---|---|
| D | 6 | 4 |
| E | 3 | 2 |
| F | 2.5 | 3 |

transported in order that the transportation cost is minimum? What is the minimum cost?

**Solution:** Let's assume that

(a) A supplies $x$ quintals grain to ration shop D.

(b) A supplies $y$ quintals grain to ration shop E.

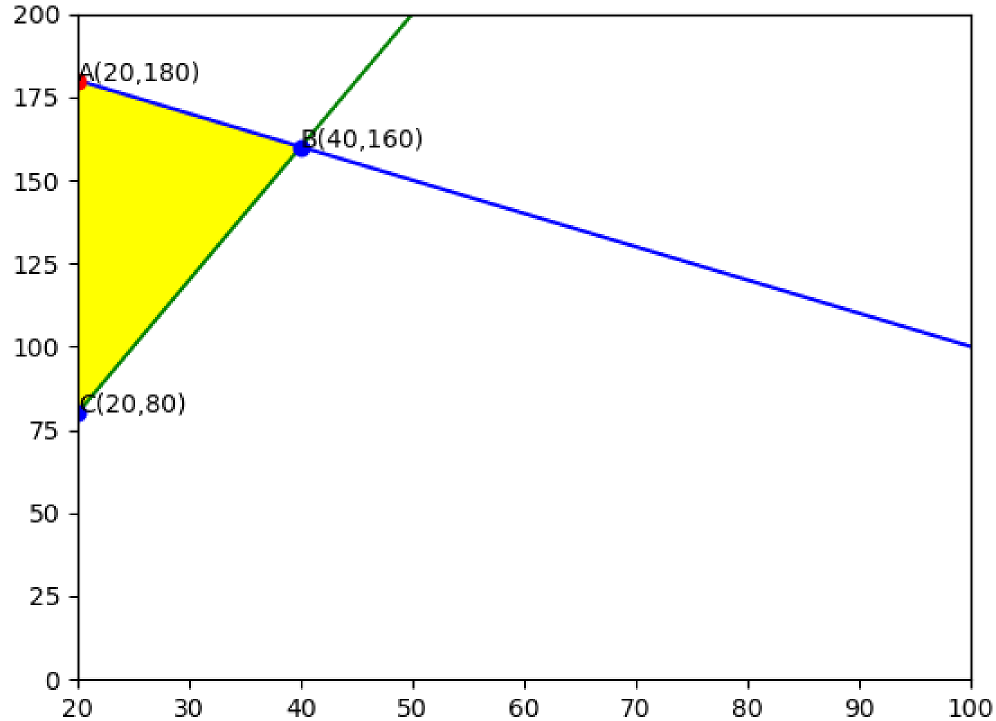


Figure 4.3.5.1:

(c) A will supply remaining grains 100-$x$-$y$ quintals to F.

(d) B will supply 60-$x$ quintals grain to ration shop D.

(e) B will supply 50-$y$ quintals grain to ration shop E.

(f) B will supply $x$+$y$-60 quintals grain to ration shop F.

Total transportation cost is given by :

$$P = 2.5x + 1.5y + 410 \tag{4.3.6.1}$$

Now, Since godown A can supply maximum 60 quintals to ration shop D and 50 quintals to ration shop E and have maximum 100 quintals capacity to supply.

Also, if godown A supplies all 40 quintals to ration shop F, then remaining 60 quintals will be supplied to ration shop D and E and $x$ and $y$ is amount of grains. It can never be negative. This leads to the following conditions

$$x + y \leq 100 \tag{4.3.6.2}$$

$$x \leq 60 \tag{4.3.6.3}$$

$$y \leq 50 \tag{4.3.6.4}$$

$$-x - y \leq -60 \tag{4.3.6.5}$$

$$x \geq 0 \tag{4.3.6.6}$$

$$y \geq 0 \tag{4.3.6.7}$$

The optimization problem can then be expressed as

$$P = \max_{\mathbf{x}} \begin{pmatrix} 2.5 & 1.5 \end{pmatrix} \mathbf{x} + 410 \tag{4.3.6.8}$$

$$s.t. \quad \begin{pmatrix} 1 & 1 \\ -1 & -1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{x} \preceq \begin{pmatrix} 100 \\ -60 \\ -60 \\ -50 \end{pmatrix} \tag{4.3.6.9}$$

yielding

$$P = 510, \mathbf{x} = \begin{pmatrix} 10 \\ 50 \end{pmatrix} \tag{4.3.6.10}$$

Hence,

(a) The minimum transportation cost is : 510 /-

(b) A supplies 10 quintals grain to ration shop D.

(c) A supplies 50 quintals grain to ration shop E.

(d) A supplies 40 quintals grain to ration shop F.

(e) A supplies 50 quintals grain to ration shop D.

(f) A supplies 0 quintals grain to ration shop E.

(g) A supplies 0 quintals grain to ration shop F.

4.3.7

4.3.8 A fruit grower can use two types of fertilizer in his garden, brand P and brand Q.The amounts (in kg) of nitrogen, phosphoric acid, potash, and chlorine in a bag of each brand are given in the table. Tests indicate that the garden needs at least 240 kg of phosphoric acid, at least 270 kg of potash and at most 310 kg of chlorine. If the grower wants to minimise the amount of nitrogen added to the garden, how many bags of each brand should be used? What is the minimum amount of nitrogen added in the garden?

**Solution:** The given information is summarized in Table 4.3.8.1.

| Kg per bag | Brand P | Brand Q |
|---|---|---|
| Nitrogen | 3 | 3.5 |
| Phosphoric acid | 1 | 2 |
| Potash | 3 | 1.5 |
| Chlorine | 1.5 | 2 |

Table 4.3.8.1:

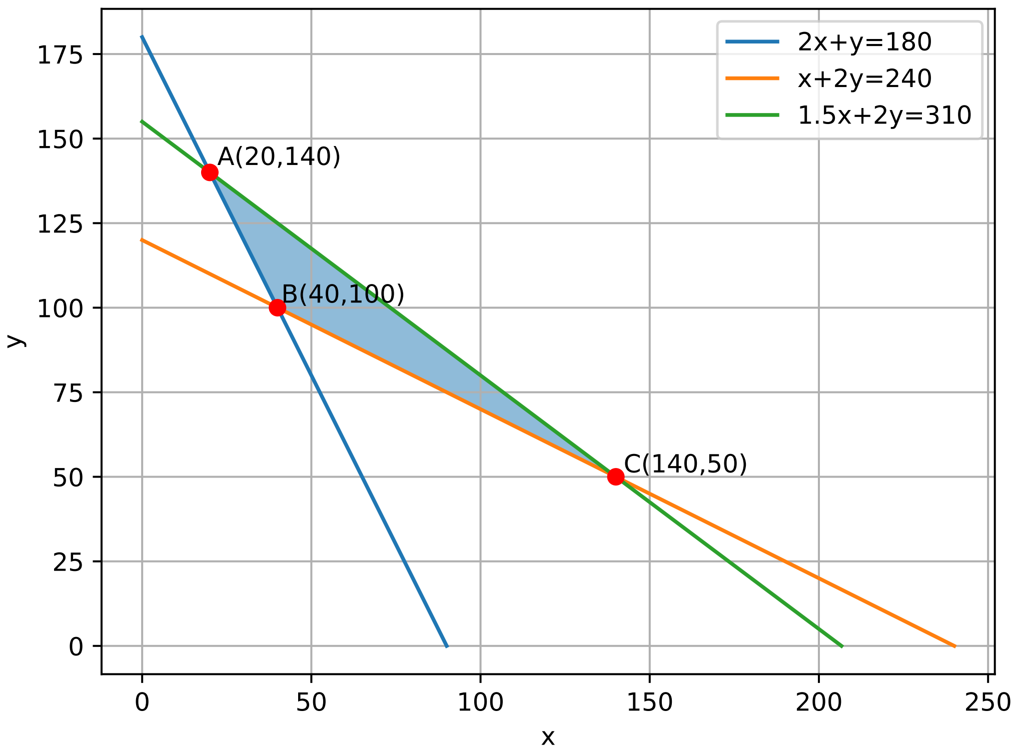


Figure 4.3.8.1:

The given problem can be expressed as

$$P = \min_{\mathbf{x}} \begin{pmatrix} 3 & 3.5 \end{pmatrix} \mathbf{x} \tag{4.3.8.1}$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \\ -1.5 & -2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 180 \\ 240 \\ -310 \\ 0 \\ 0 \end{pmatrix} \tag{4.3.8.2}$$

$$\mathbf{x} \succeq \mathbf{0} \tag{4.3.8.3}$$

yielding

$$P_{min} = 470, \mathbf{x} = \begin{pmatrix} 40 \\ 100 \end{pmatrix} \tag{4.3.8.4}$$

This can be verified using Fig. 4.3.8.1.

## 4.3.9

## 4.3.10

# Chapter 5

# Quadratic Programming

# Chapter 6

# Semi-Definite Programming

6.1 Find the normal to the curve $2y + x^2 = 3$ passing through (2,2).

**Solution:** The parameters of the given conic are

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, f = -3 \tag{6.1.1}$$

If $\mathbf{x}$ be the point of contact on the conic, the optimization problem can be formulated as

$$\mathbf{q} = \min_{\mathbf{x}} \left\|\mathbf{x} - \mathbf{p}\right\|^2 \tag{6.1.2}$$

$$s.t. \quad \mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{6.1.3}$$

where

$$\mathbf{p} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \tag{6.1.4}$$

Since

$$\|\mathbf{x}-\mathbf{p}\|^2 = \|\mathbf{x}\|^2 - 2\mathbf{p}^\top\mathbf{x} + \|\mathbf{p}\|^2 \tag{6.1.5}$$

$$= \mathbf{y}^\top\mathbf{C}\mathbf{y} \tag{6.1.6}$$

where

$$\mathbf{C} = \begin{pmatrix} \mathbf{I} & -\mathbf{p} \\ -\mathbf{p}^\top & \|\mathbf{p}\|^2 \end{pmatrix} \; \mathbf{y} = \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} \tag{6.1.7}$$

and (6.1.3) can be expressed as

$$\mathbf{y}^\top\mathbf{A}\mathbf{y} = 0, \tag{6.1.8}$$

where

$$\mathbf{A} = \begin{pmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^\top & f \end{pmatrix}, \tag{6.1.9}$$

Using SDR (Semi Definite Relaxation), (6.1.2) can be expressed as

$$\min_{\mathbf{X}} tr\left(\mathbf{C}\mathbf{X}\right) \tag{6.1.10}$$

$$s.t. \quad tr\left(\mathbf{A}\mathbf{X}\right) = 0, \tag{6.1.11}$$

$$\mathbf{X} \succeq \mathbf{0} \tag{6.1.12}$$

yielding

$$\mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{6.1.13}$$

Thus, the equation of the normal is

$$\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 0 \tag{6.1.14}$$

6.2 Find the normal to the curve $x^2 = 4y$ passing through (1,2). The parameters of the given conic are

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}, f = 0 \tag{6.2.1}$$

If $\mathbf{x}$ be the point of contact on the conic, the optimization problem can be formulated as

$$\mathbf{q} = \min_{\mathbf{x}} \|\mathbf{x} - \mathbf{p}\|^2 \tag{6.2.2}$$

$$s.t. \quad \mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{6.2.3}$$

where

$$\mathbf{p} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{6.2.4}$$

Since

$$\|\mathbf{x} - \mathbf{p}\|^2 = \|\mathbf{x}\|^2 - 2\mathbf{p}^\top\mathbf{x} + \|\mathbf{p}\|^2 \tag{6.2.5}$$

$$= \mathbf{y}^\top\mathbf{C}\mathbf{y} \tag{6.2.6}$$

where

$$\mathbf{C} = \begin{pmatrix} \mathbf{I} & -\mathbf{p} \\ -\mathbf{p}^\top & \|\mathbf{p}\|^2 \end{pmatrix} \; \mathbf{y} = \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} \tag{6.2.7}$$

and (6.2.3) can be expressed as

$$\mathbf{y}^\top\mathbf{A}\mathbf{y} = 0, \tag{6.2.8}$$

where

$$\mathbf{A} = \begin{pmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^\top & f \end{pmatrix}, \tag{6.2.9}$$

Using SDR (Semi Definite Relaxation), (6.2.2) can be expressed as

$$\min_{\mathbf{X}} tr\left(\mathbf{C}\mathbf{X}\right) \tag{6.2.10}$$

$$s.t. \quad tr\left(\mathbf{A}\mathbf{X}\right) = 0, \tag{6.2.11}$$

$$\mathbf{X} \succeq \mathbf{0} \tag{6.2.12}$$

yielding

$$\mathbf{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}. \tag{6.2.13}$$

Thus, the equation of the normal is

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 3 \tag{6.2.14}$$

# Appendix A

# Manual

## A.1. Convex Functions

A single variable function $f$ is said to be convex if

$$f\left[\lambda x + (1-\lambda)\, y\right] \leq \lambda f\left(x\right) + (1-\lambda)\, f\left(y\right), \tag{2.1}$$

for $\quad 0 < \lambda < 1$.

A.1.1 Download and execute the following python script. Is $\ln x$ convex or concave?

```
wget https://raw.githubusercontent.com/gadepall/optimization/master/
```

A.1.2 Modify the above python script as follows to plot the parabola $f(x) = x^2$. Is it convex or concave?

```
wget https://raw.githubusercontent.com/gadepall/optimization/master/
```

A.1.3 Execute the following script to obtain Fig. A.1.3.1. Comment.

```
wget https://raw.githubusercontent.com/gadepall/optimization/master/
```

A.1.4 Modify the script in the previous problem for $f(x) = x^2$. What can you conclude?

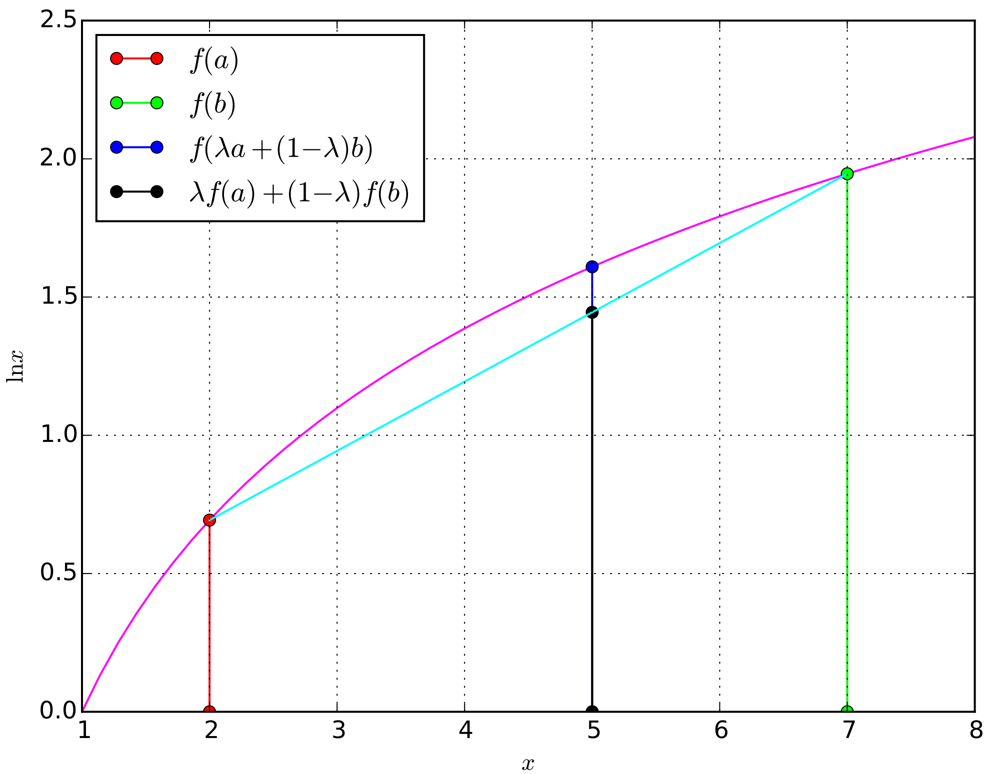


Figure A.1.1.1: $\ln x$ versus $x$

.

A.1.5 Let

$$f(\mathbf{x}) = x_1 x_2, \quad \mathbf{x} \in \mathbf{R}^2 \tag{A.1.5.1}$$

Sketch $f(\mathbf{x})$ and deduce whether it is convex.

A.1.6 Show that

$$f(\mathbf{x}) = \mathbf{x}^T \mathbf{V} \mathbf{x} \tag{A.1.6.1}$$

and find $\mathbf{V}$.

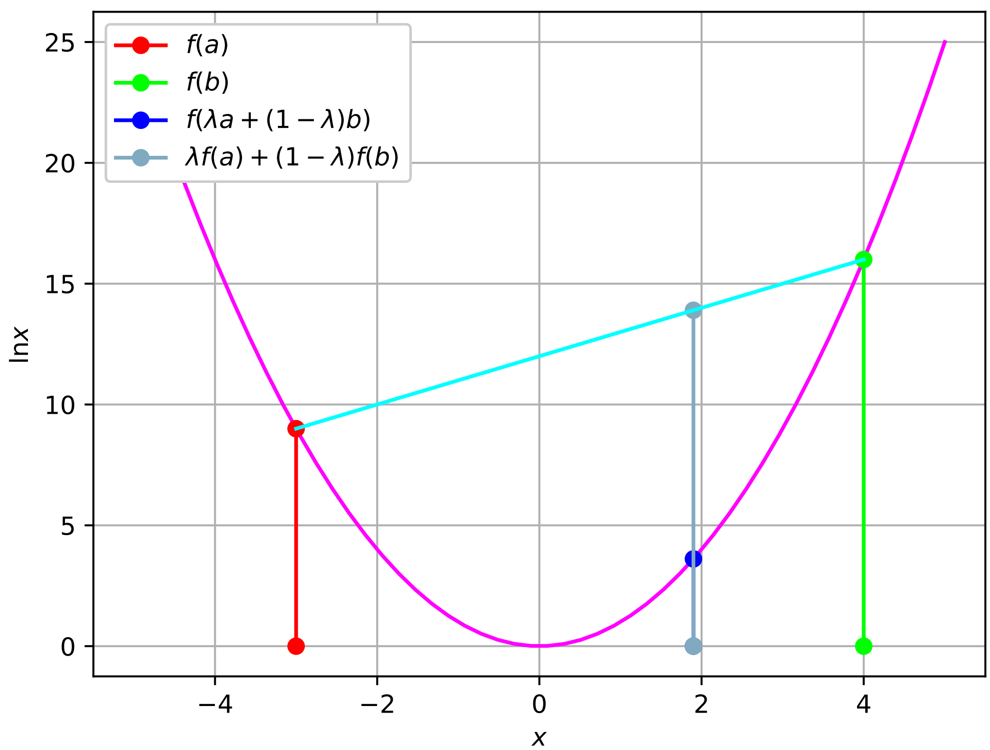


Figure A.1.2.1: $x^2$ versus $x$

.

A.1.7 Show that

$$\frac{1}{2}\nabla^2 f(\mathbf{x}) = \mathbf{V} \tag{A.1.7.1}$$

A.1.8 Use (2.1) to examine the convexity of $f(\mathbf{x})$.

A.1.9 How can you deduce the convexity of $f(\mathbf{x})$ using the eigenvalues of $\mathbf{V}$?

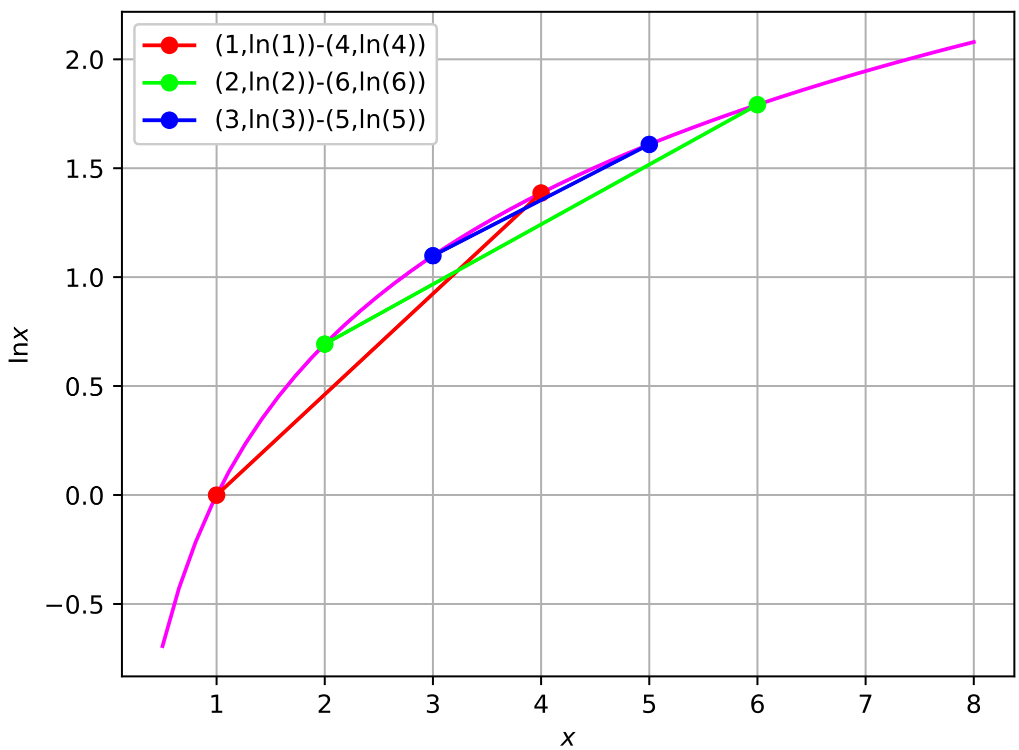


Figure A.1.3.1: Segments are below the curve

.

# A.2. Gradient Descent Method

Consider the problem of finding the square root of a number $c$. This can be expressed as the equation

$$x^2 - c = 0 \tag{9.1}$$

A.2.1 Sketch the function for different values of $c$

$$f(x) = x^3 - 3xc \tag{A.2.1.1}$$

and comment upon its convexity.

A.2.2 Show that (9.1) results from

$$\min_x f(x) = x^3 - 3xc \tag{A.2.2.1}$$

A.2.3 Find a numerical solution for (9.1).

**Solution:** A numerical solution for (9.1) is obtained as

$$x_{n+1} = x_n - \mu f'(x) \tag{A.2.3.1}$$

$$= x_n - \mu\left(3x_n^2 - 3c\right) \tag{A.2.3.2}$$

where $x_0$ is an inital guess.

A.2.4 Write a program to implement (A.2.3.2).

**Solution:** Download and execute

```
wget
https://raw.githubusercontent.com/gadepall/optimization/master/manu
```

# A.3. Convex Optimization

## A.3.1. Lagrange Multipliers

A.3.1 Find

$$\min_{\mathbf{x}} f(\mathbf{x}) = \left\| \mathbf{x} - \begin{pmatrix} 8 \\ 6 \end{pmatrix} \right\|^2 = r^2 \tag{A.3.1.1}$$

$$\text{s.t.} \quad g(\mathbf{x}) = \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} - 9 = 0 \tag{A.3.1.2}$$

by plotting the circles $f(\mathbf{x})$ for different values of $r$ along with the line $g(\mathbf{x})$.

**Solution:** The following code plots Fig. A.3.1.1

```
wget https://raw.githubusercontent.com/gadepall/optimization/master/manual/cod
```

A.3.2 Show that

$$\min r = \frac{5}{\sqrt{2}} \tag{A.3.2.1}$$

A.3.3 Show that

$$\nabla g(\mathbf{x}) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{A.3.3.1}$$

where

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \end{pmatrix} \tag{A.3.3.2}$$

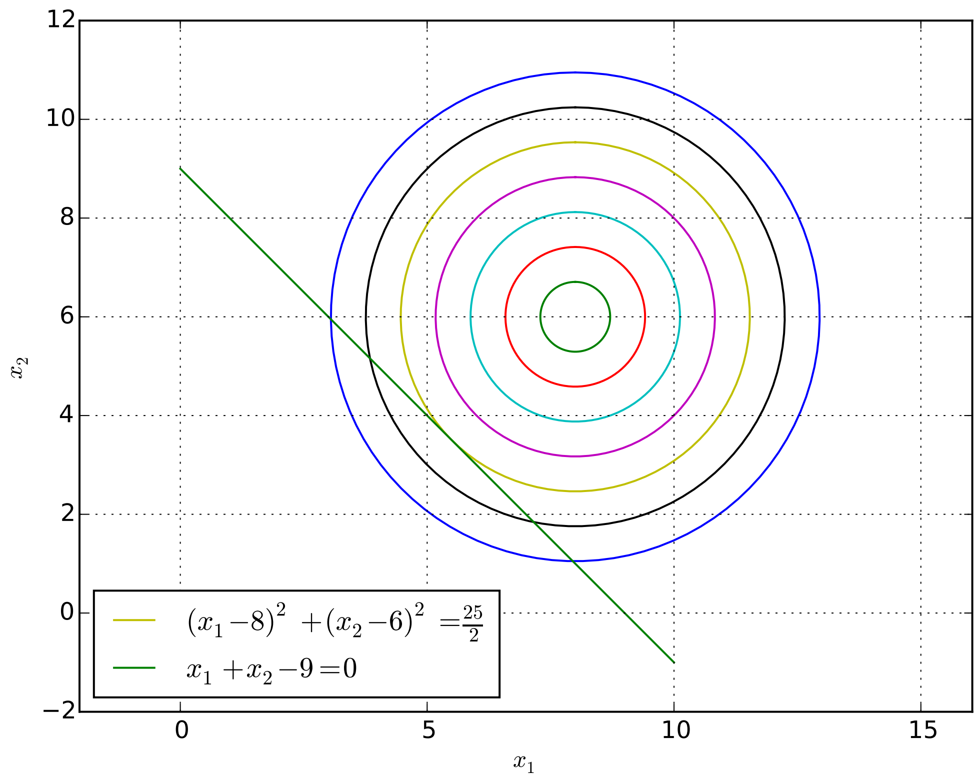


Figure A.3.1.1: Finding $\min_{\mathbf{x}} f(\mathbf{x})$

.

A.3.4 Show that

$$\nabla f(\mathbf{x}) = 2\left\{\mathbf{x} - \begin{pmatrix} 8 \\ 6 \end{pmatrix}\right\} \tag{A.3.4.1}$$

A.3.5 From Fig. A.3.1.1, show that

$$\nabla f(\mathbf{p}) = \lambda \nabla g(\mathbf{p}), \tag{A.3.5.1}$$

where $\mathbf{p}$ is the point of contact.

A.3.6 Use (A.3.5.1) and $\mathbf{g}(\mathbf{p}) = 0$ from (A.3.1.2) to obtain $\mathbf{p}$.

A.3.7 Define

$$L(\mathbf{x}, \lambda) = f(\mathbf{x}) - \lambda g(\mathbf{x}) \tag{A.3.7.1}$$

and show that $\mathbf{p}$ can also be obtained by solving the equations

$$\nabla L(\mathbf{x}, \lambda) = 0. \tag{A.3.7.2}$$

What is the sign of $\lambda$? $L$ is known as the Lagrangian and the above technique is known as the Method of Lagrange Multipliers.

**Solution:**

```
wget https://raw.githubusercontent.com/gadepall/optimization/master/manual/cod
```

## A.3.2. Inequality Constraints

A.3.8 Modify the code in problem A.3.1 to find a graphical solution for minimising

$$f(\mathbf{x}) \tag{A.3.8.1}$$

with constraint

$$g(\mathbf{x}) \geq 0 \tag{A.3.8.2}$$

**Solution:** This problem reduces to finding the radius of the smallest circle in the shaded area in Fig. A.3.8.1 . It is clear that this radius is 0.

```
wget https://raw.githubusercontent.com/gadepall/optimization/master/
```

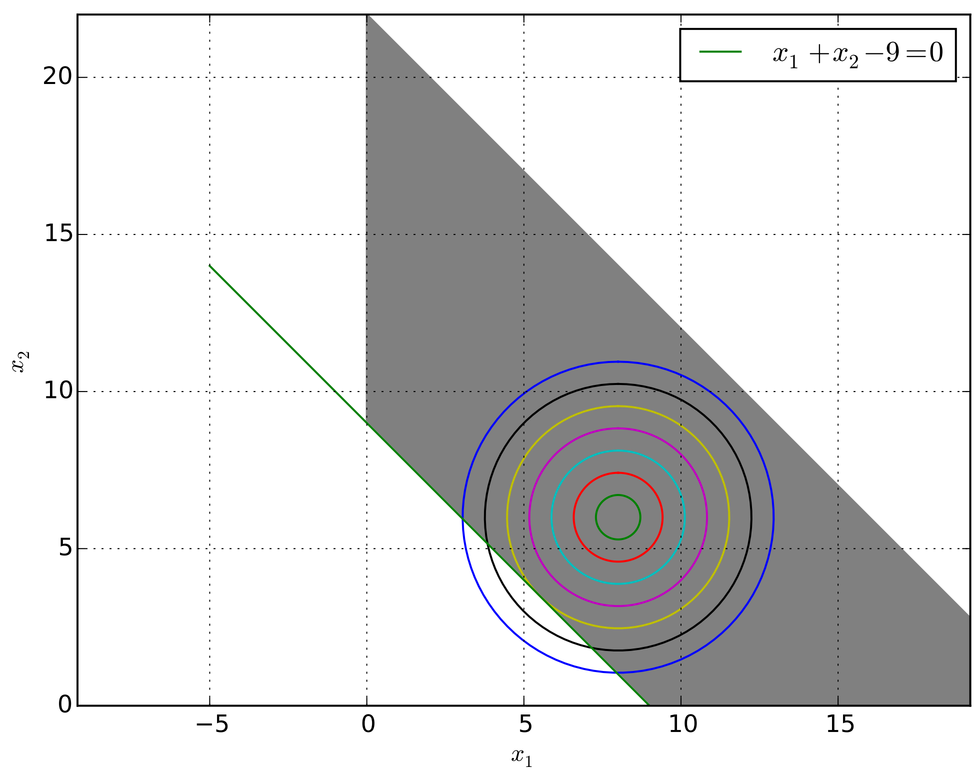


Figure A.3.8.1: Smallest circle in the shaded region is a point.

A.3.9 Now use the method of Lagrange multipliers to solve problem A.3.8 and compare with the graphical solution. Comment.

**Solution:** Using the method of Lagrange multipliers, the solution is the same as the one obtained in problem A.3.8, which is different from the graphical solution. This means that the Lagrange multipliers method cannot be applied blindly.

A.3.10 Repeat problem A.3.9 by keeping $\lambda = 0$. Comment.

**Solution:** Keeping $\lambda = 0$ results in $\mathbf{x} = \begin{pmatrix} 8 \\ 6 \end{pmatrix}$, which is the correct solution. The minimum value of $f(\mathbf{x})$ without any constraints lies in the region $g(\mathbf{x}) = 0$. In this case, $\lambda = 0$.

A.3.11 Find a graphical solution for minimising

$$f(\mathbf{x}) \tag{A.3.11.1}$$

with constraint

$$g(\mathbf{x}) \leq 0 \tag{A.3.11.2}$$

Summarize your observations.

**Solution:** In Fig. A.3.11.1, the shaded region represents the constraint. Thus, the solution is the same as the one in problem A.3.8. This implies that the method of Lagrange multipliers can be used to solve the optimization problem with this inequality constraint as well. Table A.3.11.1 summarizes the conditions for this based on the observations so far.

```
wget https://raw.githubusercontent.com/gadepall/optimization/master/manual/cod
```

Table A.3.11.1: Summary of conditions.

<table>
<tr><th>Cost</th><th>Constraint</th><th>$\lambda$</th></tr>
<tr><td rowspan="3">$f(\mathbf{x})$</td><td>$g(\mathbf{x}) = 0$</td><td>$< 0$</td></tr>
<tr><td>$g(\mathbf{x}) \geq 0$</td><td>0</td></tr>
<tr><td>$g(\mathbf{x}) \leq 0$</td><td>$< 0$</td></tr>
</table>

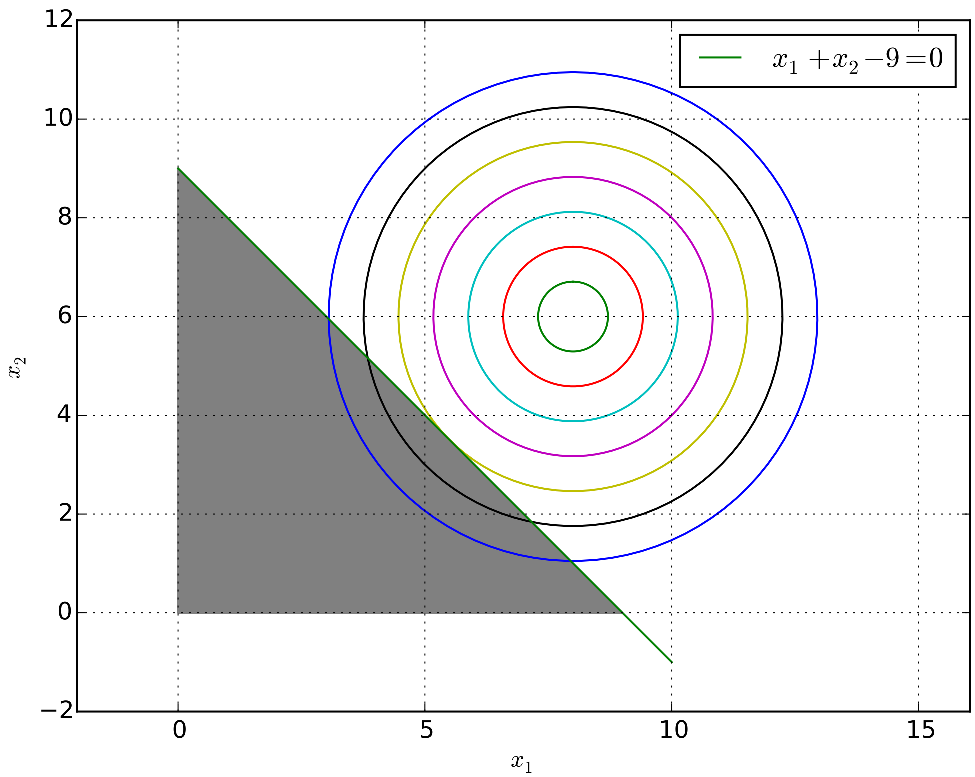


Figure A.3.11.1: Finding $\min_{\mathbf{x}} f(\mathbf{x})$.

A.3.12 Find a graphical solution for

$$\min_{\mathbf{x}} f(\mathbf{x}) = \left\| \mathbf{x} - \begin{pmatrix} 8 \\ 6 \end{pmatrix} \right\|^2 \tag{A.3.12.1}$$

with constraint

$$g(\mathbf{x}) = \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} - 18 = 0 \tag{A.3.12.2}$$

**Solution:**

```
wget https://raw.githubusercontent.com/gadepall/optimization/master/manual/code
```

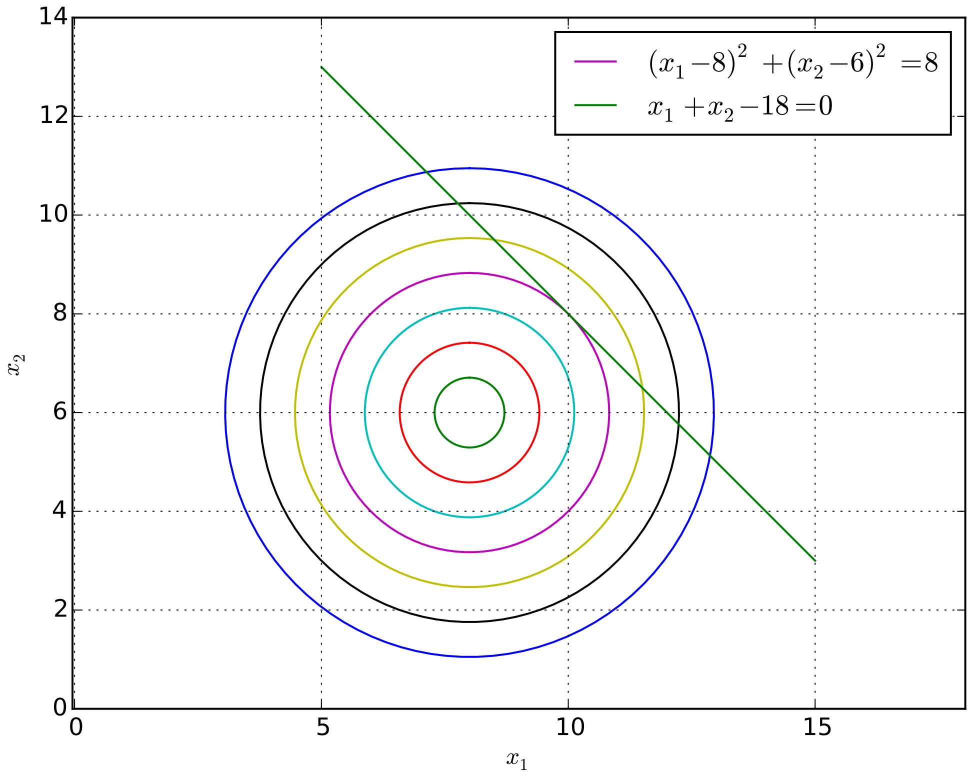


Figure A.3.12.1: Finding $\min_{\mathbf{x}} f(\mathbf{x})$.

A.3.13 Repeat problem A.3.12 using the method of Lagrange mutipliers. What is the sign of $\lambda$?

**Solution:** Using the following python script, $\lambda$ is positive and the minimum value of $f$ is 8.

```
wget https://raw.githubusercontent.com/gadepall/optimization/master/manual/code
```

A.3.14 Solve

$$\min_{\mathbf{x}} f(\mathbf{x}) \tag{A.3.14.1}$$

with constraint

$$g(\mathbf{x}) \geq 0 \tag{A.3.14.2}$$

**Solution:** Since the unconstrained solution is outside the region $g(\mathbf{x}) \geq 0$, the solution is the same as the one in problem A.3.12.

A.3.15 Based on the problems so far, generalise the Lagrange multipliers method for

$$\min_{\mathbf{x}} f(\mathbf{x}), \quad g(\mathbf{x}) \geq 0 \tag{A.3.15.1}$$

**Solution:** Considering $L(\mathbf{x}, \lambda) = f(\mathbf{x}) - \lambda g(\mathbf{x})$, for $g(\mathbf{x}) = \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} - 18 \geq 0$ we found $\lambda > 0$ and for $g(\mathbf{x}) = \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} - 9 \leq 0, \lambda < 0$. A single condition can be obtained by framing the optimization problem as

$$\min_{\mathbf{x}} f(\mathbf{x}), \quad g(\mathbf{x}) \leq 0 \tag{A.3.15.2}$$

with the Lagrangian

$$L(\mathbf{x}, \lambda) = f(\mathbf{x}) + \lambda g(\mathbf{x}), \tag{A.3.15.3}$$

provided

$$\nabla L(\mathbf{x}, \lambda) = 0 \Rightarrow \lambda > 0 \tag{A.3.15.4}$$

else, $\lambda = 0$.

## A.3.3. Karush Kuhn-Tucker Conditions

A.3.16 Solve

$$\min_{\mathbf{x}} f(\mathbf{x}) = \mathbf{x}^T \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{x} \tag{A.3.16.1}$$

with constraints

$$g_1(\mathbf{x}) = \begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} - 8 = 0 \tag{A.3.16.2}$$

$$g_2(\mathbf{x}) = 15 - \begin{pmatrix} 2 & 4 \end{pmatrix} \mathbf{x} \geq 0 \tag{A.3.16.3}$$

**Solution:** Considering the Lagrangian

$$\nabla L(\mathbf{x}, \lambda, \mu) = 0 \tag{A.3.16.4}$$

resulting in the matrix equation

$$\Rightarrow \begin{pmatrix} 8 & 0 & 3 & 2 \\ 0 & 4 & 1 & 4 \\ 3 & 1 & 0 & 0 \\ 2 & 4 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 8 \\ 15 \end{pmatrix} \tag{A.3.16.5}$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} 1.7 \\ 2.9 \\ -3.12 \\ -2.12 \end{pmatrix} \tag{A.3.16.6}$$

using the following python script. The (incorrect) graphical solution is available in Fig. A.3.16.1

```
wget https://raw.githubusercontent.com/gadepall/optimization/master/
```

Note that $\mu < 0$, contradicting the necessary condition in (A.3.15.4).

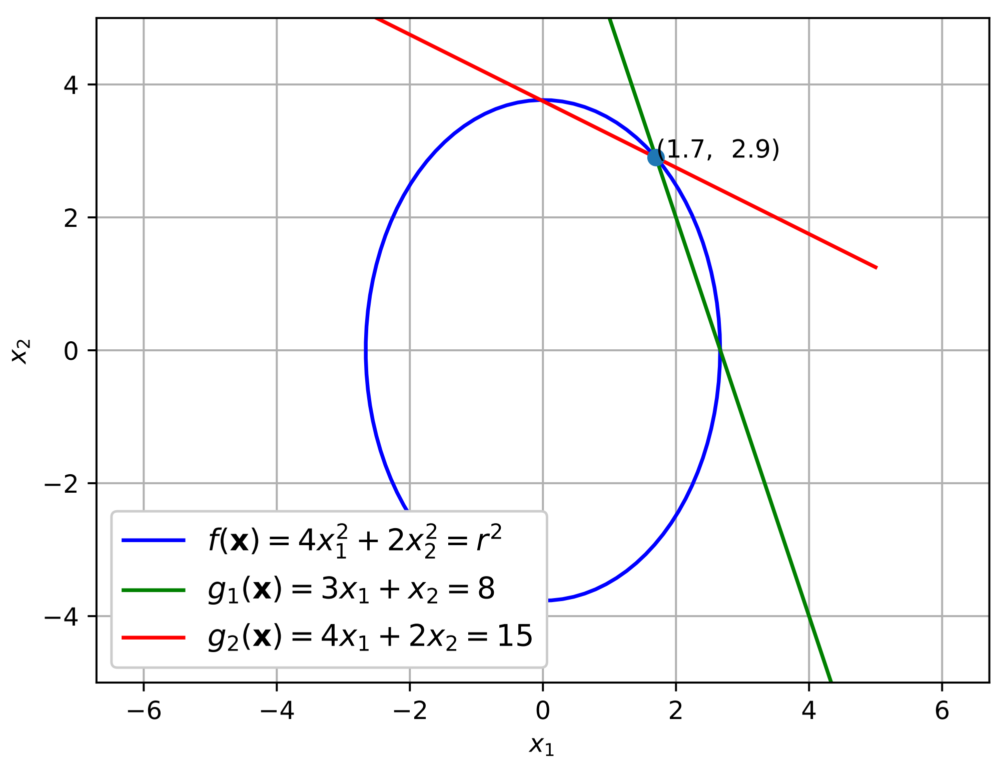


Figure A.3.16.1: Incorrect solution is at intersection of all curves $r = 5.33$

A.3.17 Obtain the correct solution to the previous problem by considering $\mu = 0$.

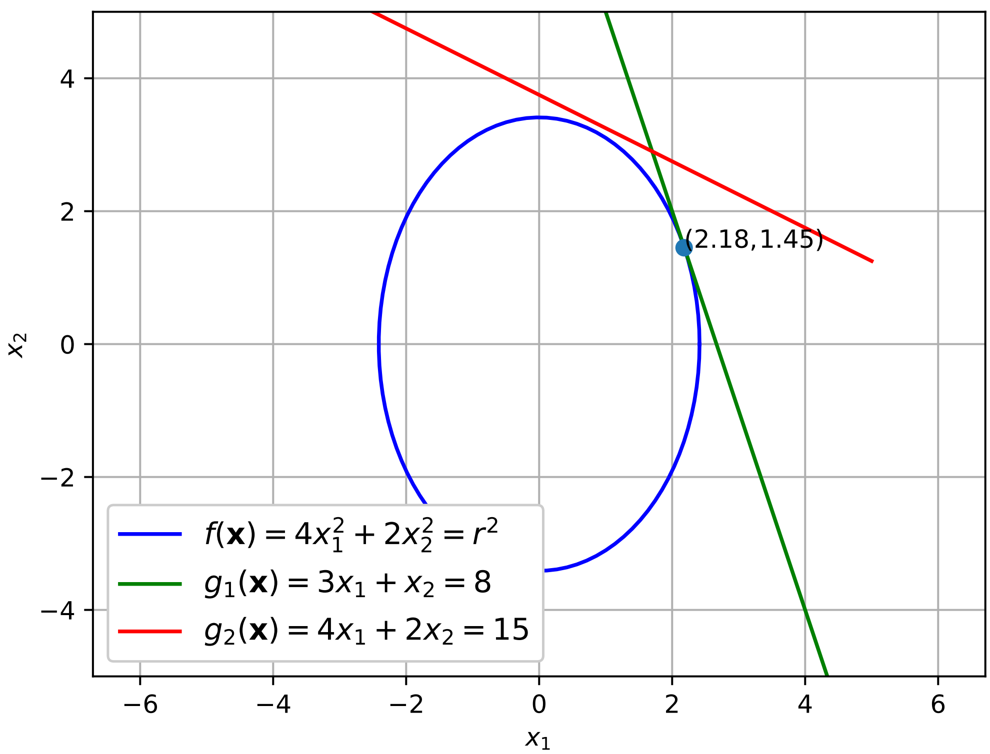


Figure A.3.17.1: Optimal solution is where $g_1(x)$ touches the curve $r = 4.82$

A.3.18 Solve

$$\min_{\mathbf{x}} f(\mathbf{x}) \tag{A.3.18.1}$$

with constraints

$$g_1(\mathbf{x}) = 0 \tag{A.3.18.2}$$

$$g_2(\mathbf{x}) \leq 0 \tag{A.3.18.3}$$

A.3.19 Based on whatever you have done so far, list the steps that you would use in general for solving a convex optimization problem like (A.3.16.1) using Lagrange Multipliers. These are called Karush-Kuhn-Tucker(KKT) conditions.

**Solution:** For a problem defined by

$$\mathbf{x}^* = \min_{\mathbf{x}} f(\mathbf{x}) \tag{A.3.19.1}$$

$$\text{subject to } h_i(\mathbf{x}) = 0, \forall i = 1, .., m \tag{A.3.19.2}$$

$$\text{subject to } g_i(\mathbf{x}) \leq 0, \forall i = 1, .., n \tag{A.3.19.3}$$

the optimal solution is obtained through

$$\mathbf{x}^* = \min_{\mathbf{x}} L(\mathbf{x}, \lambda, \mu) \tag{A.3.19.4}$$

$$= \min_{\mathbf{x}} f(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i h_i(\mathbf{x}) + \sum_{i=1}^{n} \mu_i g_i(\mathbf{x}), \tag{A.3.19.5}$$

using the KKT conditions

$$\Rightarrow \nabla_{\mathbf{x}} f(\mathbf{x}) + \sum_{i=1}^{m} \nabla_{\mathbf{x}} \lambda_i h_i(\mathbf{x}) + \sum_{i=1}^{n} \mu_i \nabla_{\mathbf{x}} g_i(\mathbf{x}) = 0 \tag{A.3.19.6}$$

$$\text{subject to } \mu_i g_i(\mathbf{x}) = 0, \forall i = 1, .., n \tag{A.3.19.7}$$

$$\text{and } \mu_i \geq 0, \forall i = 1, .., n \tag{A.3.19.8}$$

A.3.20 Maxmimize

$$f(\mathbf{x}) = \sqrt{x_1 x_2} \tag{A.3.20.1}$$

with the constraints

$$x_1^2 + x_2^2 \leq 5 \tag{A.3.20.2}$$

$$x_1 \geq 0, x_2 \geq 0 \tag{A.3.20.3}$$

A.3.21 Solve

$$\min_{\mathbf{x}} \quad x_1 + x_2 \tag{A.3.21.1}$$

with the constraints

$$x_1^2 - x_1 + x_2^2 \leq 0 \tag{A.3.21.2}$$

where $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

**Solution:**

Graphical solution:

```
wget https://raw.githubusercontent.com/gadepall/optimization/master/manual/code
```

# A.4. Semi-definite Programming

A.4.1 The problem

$$\min_{\mathbf{X}} x_{11} + x_{12} \tag{A.4.1.1}$$

with constraints

$$x_{11} + x_{22} = 1 \tag{A.4.1.2}$$

$$\mathbf{X} \succeq 0 \quad (\succeq \text{ means positive definite}) \tag{A.4.1.3}$$

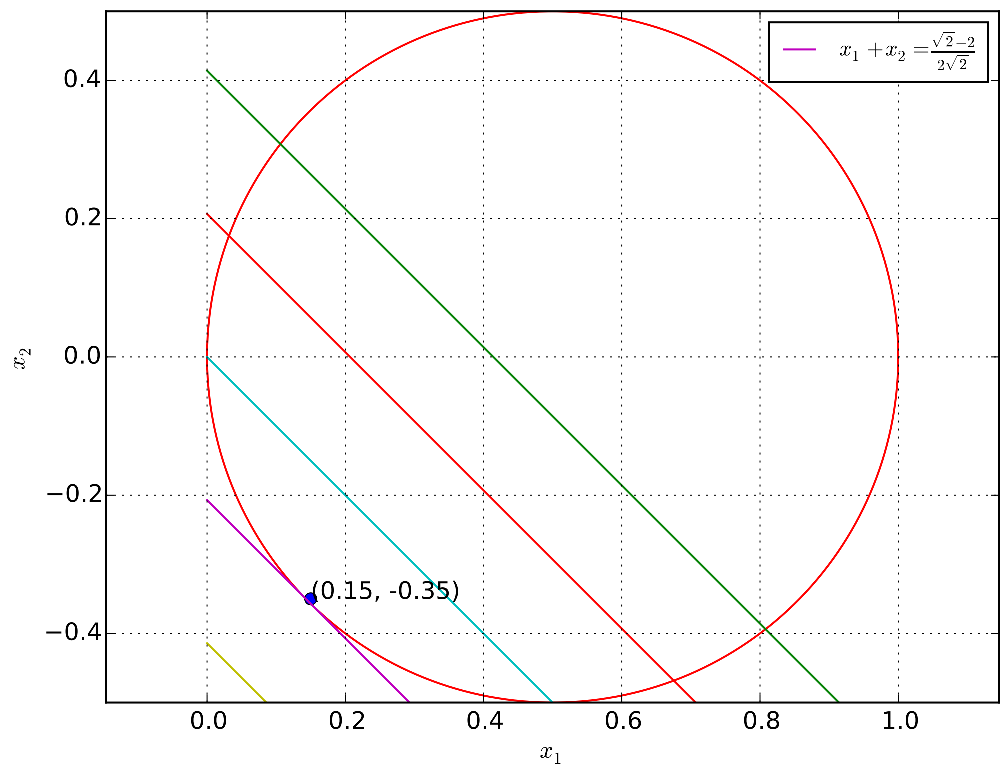


Figure A.3.21.1: Optimal solution is the lower tangent to the circle

where

$$\mathbf{X} = \begin{pmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{pmatrix} \tag{A.4.1.4}$$

is known as a semi-definite program. Find a numerical solution to this problem. Compare with the solution in problem A.3.21.

**Solution:** The cvxopt solver needs to be used in order to find a numerical solution.

For this, the given problem has to be reformulated as

$$\min_{\mathbf{x}} \begin{pmatrix} 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{12} \\ x_{22} \end{pmatrix} \quad \text{s.t} \tag{A.4.1.5}$$

$$\begin{pmatrix} 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{12} \\ x_{22} \end{pmatrix} = 1 \tag{A.4.1.6}$$

$$x_{11} \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} + x_{12} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} + x_{22} \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \preceq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{A.4.1.7}$$

The following script provides the solution to this problem.

```
wget https://raw.githubusercontent.com/gadepall/optimization/master/manual/cod
```

A.4.2 Frame Problem A.4.1 in terms of matrices.

**Solution:** It is easy to verify that

$$x_{11} + x_{12} = \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{X}^T \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{A.4.2.1}$$

and

$$x_{11} + x_{22} = \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{X} & \mathbf{0} \\ \mathbf{0} & \mathbf{X} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{A.4.2.2}$$

Thus, Problem A.4.1 can be expressed as

$$\begin{gathered} \min_{\mathbf{X}} \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{X}^T \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad s.t \\ \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{X} & \mathbf{0} \\ \mathbf{0} & \mathbf{X} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = 1, \\ \mathbf{X} \succeq 0 \end{gathered} \tag{A.4.2.3}$$

A.4.3 Solve (A.4.2.3) using cvxpy.

**Solution:**

```
wget https://raw.githubusercontent.com/gadepall/optimization/master/
```

A.4.4 Minimize

$$-x_{11} - 2x_{12} - 5x_{22} \tag{A.4.4.1}$$

subject to

$$2x_{11} + 3x_{12} + x_{22} = 7 \tag{A.4.4.2}$$

$$x_{11} + x_{12} \geq 1 \tag{A.4.4.3}$$

$$x_{11}, x_{12}, x_{22} \geq 0 \tag{A.4.4.4}$$

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{pmatrix} \succeq 0 \tag{A.4.4.5}$$

using cvxpy.

A.4.5 Repeat the above exercise by converting the problem into a convex optimization problem in two variables and using graphical plots.

A.4.6 Solve the above problem using the KKT conditions. Comment.

# A.5. Linear Programming

A.5.1 Graphically obtain a solution to the following

$$\max_{\mathbf{x}} 6x_1 + 5x_2 \tag{A.5.1.1}$$

with constraints

$$x_1 + x_2 \leq 5 \tag{A.5.1.2}$$

$$3x_1 + 2x_2 \leq 12 \tag{A.5.1.3}$$

$$\text{where } x_1, x_2 \geq 0 \tag{A.5.1.4}$$

**Solution:** The following program plots the solution in Fig. A.5.1.1

```
wget https://raw.githubusercontent.com/gadepall/optimization/master/
```

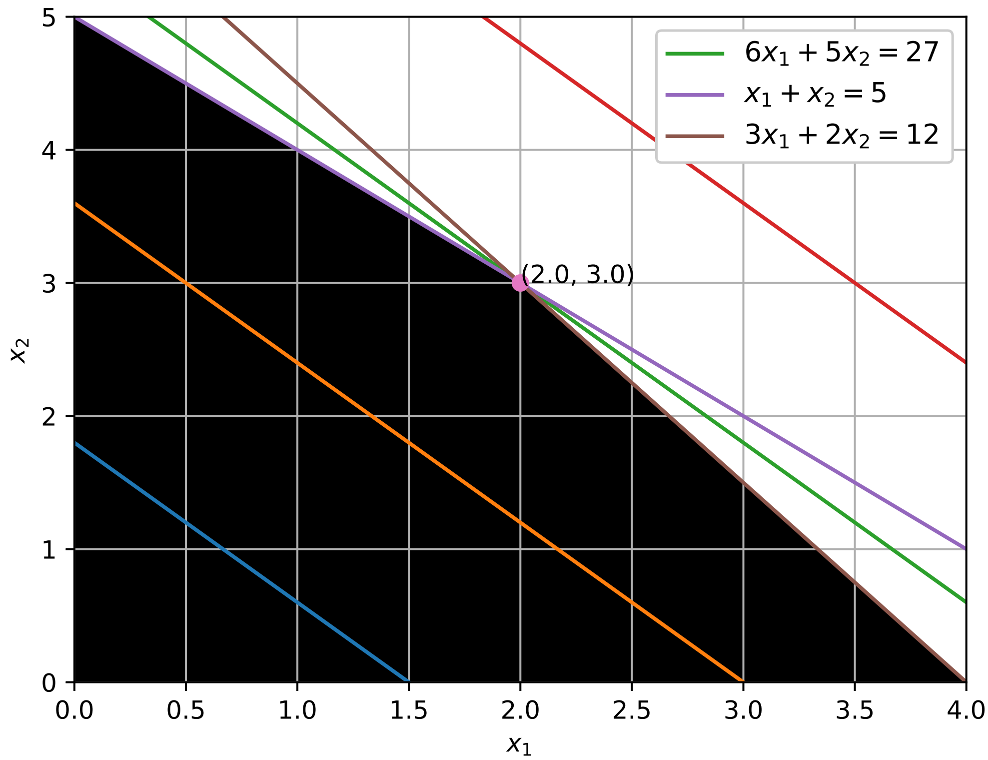


Figure A.5.1.1: The cost function intersects with the two constraints at $\mathbf{x} = (2, 3)$.

A.5.2 Now use cvxpy to obtain a solution to problem A.5.1.

**Solution:** The given problem is expressed as follows

$$\min_{\mathbf{x}} \mathbf{c}^T \mathbf{x} \quad s.t. \tag{A.5.2.1}$$

$$\mathbf{A}\mathbf{x} \preceq \mathbf{b} \tag{A.5.2.2}$$

where

$$\mathbf{c} = \begin{pmatrix} -6 \\ -5 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 1 & 1 \\ 3 & 2 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 5 \\ 12 \\ 0 \\ 0 \end{pmatrix} \tag{A.5.2.3}$$

The desired solution is then obtained using the following program.

```
wget https://raw.githubusercontent.com/gadepall/optimization/master/manual/code
```

A.5.3 Verify your solution to the above problem using the method of Lagrange multipliers.

A.5.4 Maximise $5x_1 + 3x_2$ w.r.t the constraints

$$x_1 + x_2 \le 2$$
$$5x_1 + 2x_2 \le 10$$
$$3x_1 + 8x_2 \le 12$$
$$\text{where } x_1, x_2 \ge 0$$

# A.6. Convex Polygon

A.6.1 Show that $\mathbf{D}$ lies inside $\triangle ABC$ iff

$$\mathbf{D} = \lambda_1 \mathbf{A} + \lambda_2 \mathbf{B} + \lambda_3 \mathbf{C} \tag{A.6.1.1}$$

such that

$$0 \le \lambda_1, \lambda_2, \lambda_3 \le 1, \tag{A.6.1.2}$$

$$0 \le \lambda_1 + \lambda_2 + \lambda_3 \le 1, \tag{A.6.1.3}$$

A.6.2 Prove that the point $\begin{pmatrix} 4 \\ 4 \end{pmatrix}$ lies outside the triangle whose sides are the lines

$$\begin{pmatrix} 3 & 4 \end{pmatrix} \mathbf{x} = 24 \tag{A.6.2.1}$$

$$\begin{pmatrix} 5 & -3 \end{pmatrix} \mathbf{x} = 15 \tag{A.6.2.2}$$

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{A.6.2.3}$$

## A.7. Complex Numbers: Optimization

A.7.1 Consider the optimization problem

$$\max_{z} \frac{1}{|z-1|} \tag{A.7.1.1}$$

$$s.t. \quad |z - 2 + \jmath| \ge \sqrt{5} \tag{A.7.1.2}$$

Show that it can be reframed as

$$\min_{\mathbf{x}} \left\|\mathbf{x} - \mathbf{c}_1\right\|^2 \tag{A.7.1.3}$$

$$s.t. \quad \left\|\mathbf{x} - \mathbf{c}_2\right\|^2 \ge 5 \tag{A.7.1.4}$$

where

$$z = \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \mathbf{c}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{c}_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \tag{A.7.1.5}$$

A.7.2 Explain the optimization problem with a figure.

**Solution:** Fig. A.7.2.1 explains (A.7.1.3) where $z_0$ is the set of points comprising of the intersection of the smallest circle $\Gamma$ : with the largest circle $\Omega : r_2 \geq \sqrt{5}$ with radii $r_1$ and $r_2 \geq \sqrt{5}$ respectively.

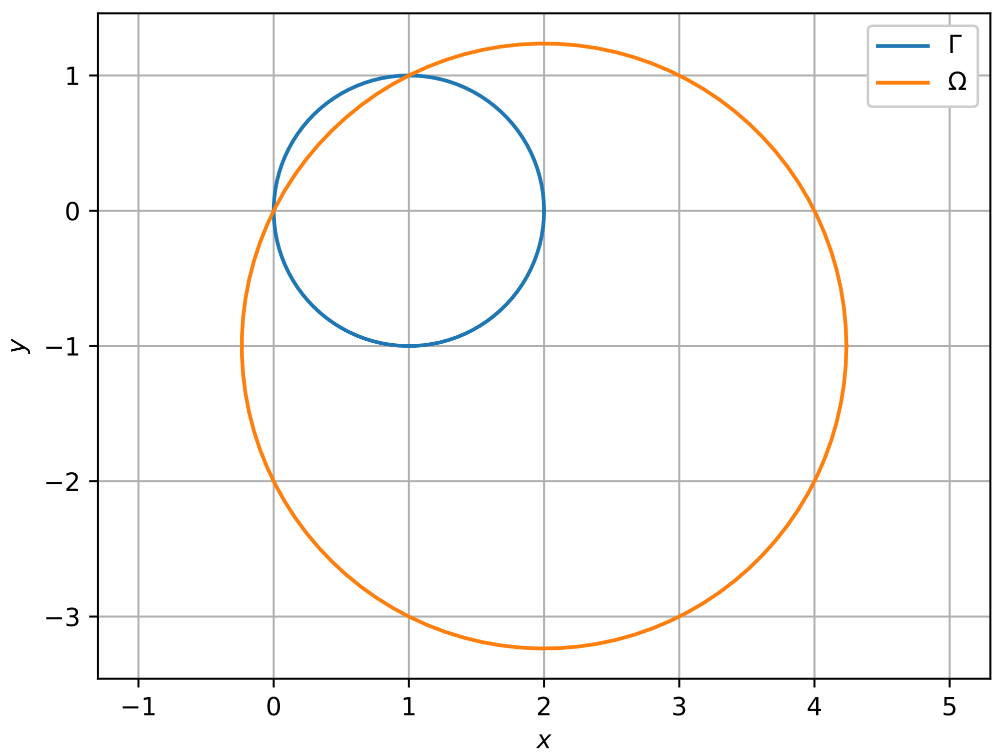


Figure A.7.2.1:

A.7.3 Obtain the Lagrangian.

**Solution:** The Lagrangian is

$$L(\mathbf{x}, \lambda) = \|\mathbf{x} - \mathbf{c}_1\|^2 - \lambda \left\{ \|\mathbf{x} - \mathbf{c}_2\|^2 - r_2^2 \right\} \tag{A.7.3.1}$$

A.7.4 Use the KKT conditions to obtain the minima.

**Solution:** From the KKT conditions,

$$\frac{\partial L(\mathbf{x}, \lambda)}{\partial \mathbf{x}} = 0 \tag{A.7.4.1}$$

$$\implies \mathbf{x} - \mathbf{c}_1 - \lambda(\mathbf{x} - \mathbf{c}_2) = 0 \tag{A.7.4.2}$$

$$\implies \mathbf{x} = \frac{\mathbf{c}_1 - \lambda \mathbf{c}_2}{1 - \lambda} \tag{A.7.4.3}$$

and

$$\frac{\partial L(\mathbf{x}, \lambda)}{\partial \lambda} = 0 \tag{A.7.4.4}$$

$$\implies \|\mathbf{x} - \mathbf{c}_2\|^2 - r_2^2 = 0 \tag{A.7.4.5}$$

Substituting from (A.7.4.3) in (A.7.4.5),

$$\left\| \frac{\mathbf{c}_1 - \lambda \mathbf{c}_2}{1 - \lambda} - \mathbf{c}_2 \right\|^2 - r_2^2 = 0 \tag{A.7.4.6}$$

$$\implies \lambda = 1 \pm \frac{\|\mathbf{c}_1 - \mathbf{c}_2\|}{r_2} \tag{A.7.4.7}$$

$$= 1 \pm \sqrt{\frac{2}{5}} \tag{A.7.4.8}$$

Fig. A.7.5.1 plots $\Gamma$ for

$$\lambda = 1 - \sqrt{\frac{2}{5}} \tag{A.7.4.9}$$

A.7.5 If the maximum value is obtained at $z_0$, find the principal argument of

$$\frac{4 - z_0 - \bar{z}_0}{z_0 - \bar{z}_0 + 2\jmath} \tag{A.7.5.1}$$

**Solution:** From (A.7.4.3),

$$\mathbf{x}_0 = \frac{\mathbf{c}_1 - \lambda\mathbf{c}_2}{1 - \lambda} \tag{A.7.5.2}$$

$$\implies z_0 = \frac{1}{1-\lambda}\left(1 - 2\lambda + \jmath\lambda\right) \tag{A.7.5.3}$$

$$\text{or,}\ \arg\frac{4 - z_0 - \bar{z}_0}{z_0 - \bar{z}_0 + 2\jmath} = \frac{2 - \Re\left\{z_0\right\}}{\jmath\left(\Im\left\{z_0\right\} + 1\right)} \tag{A.7.5.4}$$

$$= \frac{2\left(1-\lambda\right) - \left(1 - 2\lambda\right)}{\jmath} \tag{A.7.5.5}$$

$$= -\jmath \tag{A.7.5.6}$$

Thus, the principal argument is $-\frac{\pi}{2}$.

A.7.6 Show that the set

$$D = \left\{\mathbf{x} : \|\mathbf{x} - \mathbf{C}_2\| \geq r_2\right\}, r_2 > 0 \tag{A.7.6.1}$$

is nonconvex.

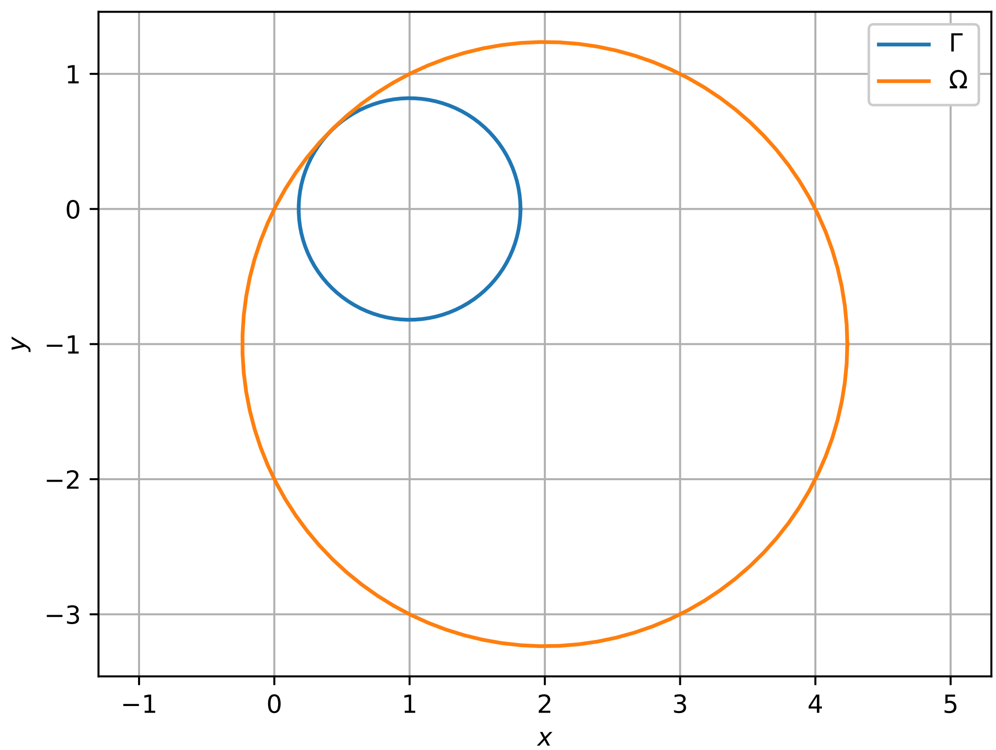


Figure A.7.5.1:

**Solution:** Let $\mathbf{x}_1 \in D$ and

$$\mathbf{x}_2 = 2\mathbf{C}_2 - \mathbf{x}_1 \tag{A.7.6.2}$$

Then

$$\|\mathbf{x}_2 - \mathbf{C}_2\| = \|\mathbf{C}_2 - \mathbf{x}_1\| \geq r_2 \tag{A.7.6.3}$$

$$\implies \mathbf{x}_2 \in D. \tag{A.7.6.4}$$

Suppose

$$\mathbf{x} = \theta \mathbf{x}_1 + (1 - \theta)\, \mathbf{x}_2 \tag{A.7.6.5}$$

For $\theta = \frac{1}{2}$,

$$\mathbf{x} = \mathbf{C}_2 \tag{A.7.6.6}$$

$$\implies \|\mathbf{x} - \mathbf{C}_2\| = 0, \tag{A.7.6.7}$$

$$\text{or, } \mathbf{x} \notin D \tag{A.7.6.8}$$

Thus, by definition, $D$ is not a convex set.