Quadratic Programming Assignment

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Abstract—**This document contains the solution to Question 27 of Exercise 5 in Chapter 6 of the class 12 NCERT textbook.**

1) The point on the curve

$$x^2 = 2y \tag{1}$$

which is nearest to the point $\mathbf{P} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$ is

a) $\begin{pmatrix} 2\sqrt{2} \\ 4 \end{pmatrix}$
b) $\begin{pmatrix} 2\sqrt{2} \\ 0 \end{pmatrix}$
c) $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
d) $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$

Solution: We need to find

$$\min_{\mathbf{x}} g(\mathbf{x}) = \left\|\mathbf{x} - \mathbf{P}\right\|^2 \tag{2}$$
$$\text{s.t. } h(\mathbf{x}) = \mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} = 0 \tag{3}$$

where

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ \mathbf{u} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \tag{4}$$

First, we show that (3) is nonconvex. Indeed, suppose $\mathbf{x_1}$ and $\mathbf{x_2}$ satisfy $h(\mathbf{x}) = 0$. Then,

$$\mathbf{x_1}^\top \mathbf{V}\mathbf{x_1} + 2\mathbf{u}^\top \mathbf{x_1} + f = 0 \tag{5}$$
$$\mathbf{x_2}^\top \mathbf{V}\mathbf{x_2} + 2\mathbf{u}^\top \mathbf{x_2} + f = 0 \tag{6}$$

Then, for any $0 \le \lambda \le 1$, substituting

$$\mathbf{x}_\lambda \leftarrow \lambda\mathbf{x_1} + (1-\lambda)\mathbf{x_2} \tag{7}$$

into (3), we get

$$h(\mathbf{x}_\lambda) = \lambda(\lambda-1)(\mathbf{x_1}-\mathbf{x_2})^\top \mathbf{V}(\mathbf{x_1}-\mathbf{x_2}) + f \neq 0 \tag{8}$$

Hence, the optimization problem is nonconvex. The constraints throw an error when *cvxpy* is used, as shown in the erroneous Python code `codes/parab_cvx.py`.

We use the method of Lagrange multipliers instead to find the optima. Here, we need to find $\lambda \in \mathbb{R}$ such that there exists a $\mathbf{x}$ satisfying

$$\nabla g(\mathbf{x}) = \lambda \nabla h(\mathbf{x}) \tag{9}$$
$$\implies 2(\mathbf{x} - \mathbf{P}) = 2\lambda(\mathbf{A}\mathbf{x} + \mathbf{u}) \tag{10}$$
$$\implies (\mathbf{I} - \lambda\mathbf{A})\mathbf{x} = \lambda\mathbf{u} + \mathbf{P} \tag{11}$$
$$\implies \begin{pmatrix} 1-\lambda & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 0 \\ 5-\lambda \end{pmatrix} \tag{12}$$

From (12), we have two cases:

a) $\lambda \neq 1$. In this case, we form the augmented matrix

$$\begin{pmatrix} 1-\lambda & 0 & 0 \\ 0 & 1 & 5-\lambda \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{R_1}{1-\lambda}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 5-\lambda \end{pmatrix} \tag{13}$$

and get that

$$\mathbf{x_m} = \begin{pmatrix} 0 \\ 5-\lambda \end{pmatrix} \tag{14}$$

Substituting in (3) gives $\lambda = 5$. Thus, $\mathbf{x_m} = \mathbf{0}$.

b) $\lambda = 1$. In this case, (12) becomes

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} \tag{15}$$
$$\implies \mathbf{e_2}^\top \mathbf{x} = 4 \tag{16}$$

Substituting (16) into (3) becomes

$$\left(\mathbf{e_1}^\top \mathbf{x}\right)^2 = 8 \tag{17}$$
$$\implies \mathbf{e_1}^\top \mathbf{x} = \pm 2\sqrt{2} \tag{18}$$

Using (18) and (16),

$$\mathbf{x_m} = \begin{pmatrix} \pm 2\sqrt{2} \\ 4 \end{pmatrix} \tag{19}$$

Using these values of $\mathbf{x_m}$, the distances are

$$\left\|\begin{pmatrix} 0 \\ 5 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\| = 5 \tag{20}$$
$$\left\|\begin{pmatrix} 0 \\ 5 \end{pmatrix} - \begin{pmatrix} \pm 2\sqrt{2} \\ 4 \end{pmatrix}\right\| = 3 \tag{21}$$

Thus, the correct answer is **a)**.
We demonstrate both of the minima in (21) is obtained using constrained gradient descent in Fig. 1, plotted using the Python code `codes/grad_pits.py`.

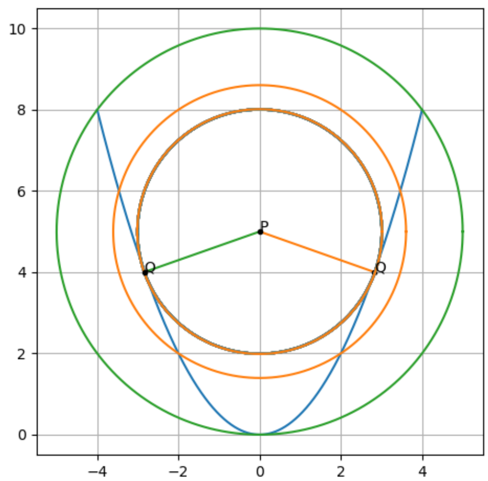

Fig. 1: Gradient descent for a nonconvex optimization problem.