Probability

G V V Sharma*

Contents

*The author is with the Department of Electrical Engineering, Indian Institute of Technology, Hyderabad 502285 India e-mail: gadepall@iith.ac.in.

***Abstract*—This book provides solved examples on Probability**

1 Axioms

1.1. The probability that a given positive integer lying between 1 and 100 (both inclusive) is NOT divisible by 2,3 or 5 is ...
Solution: Table 1.1.1 summarizes the given information.

Event	Definition	Probability
A	$n \equiv 0 \pmod 2$	$\frac{50}{100}$
B	$n \equiv 0 \pmod 3$	$\frac{33}{100}$
C	$n \equiv 0 \pmod 5$	$\frac{20}{100}$
AB	$n \equiv 0 \pmod 6$	$\frac{16}{100}$
BC	$n \equiv 0 \pmod{15}$	$\frac{6}{100}$
AC	$n \equiv 0 \pmod{10}$	$\frac{10}{100}$
ABC	$n \equiv 0 \pmod{30}$	$\frac{3}{100}$

TABLE 1.1.1: $1 \le n \le 100$

$$\because \Pr(A+B+C) = \Pr(A) + \Pr(B) + \Pr(C) - \Pr(AB) - \Pr(BC) - \Pr(AC) + \Pr(ABC) \quad (1.1.1)$$

Substituting from Table 1.1.1 in (1.1.1),

$$\Pr(A+B+C) = \frac{50}{100} + \frac{33}{100} + \frac{20}{100} - \frac{16}{100} - \frac{6}{100} - \frac{10}{100} + \frac{3}{100} \tag{1.1.2}$$
$$= \frac{74}{100}$$

Thus, the required probability is

$$1 - \Pr(A+B+C) = \frac{26}{100} \tag{1.1.3}$$

1.2. P and Q are considering to apply for a job. The probability that P applies for the job is $\frac{1}{4}$, the probability that P applies for the job given that Q applies for the job is $\frac{1}{2}$, and the probability that Q applies for the job given that P applies for the job is $\frac{1}{3}$. Then the probability that P does not apply for the job given that Q does not apply for the job is

a) $\frac{4}{5}$ b) $\frac{5}{6}$ c) $\frac{7}{8}$ d) $\frac{11}{12}$

Solution: The given information can be expressed as

$$\Pr(P) = \frac{1}{4} \tag{1.2.1}$$
$$\Pr(P|Q) = \frac{1}{2} = \frac{\Pr(PQ)}{\Pr(Q)} \tag{1.2.2}$$
$$\Pr(Q|P) = \frac{1}{3} = \frac{\Pr(PQ)}{\Pr(P)} \tag{1.2.3}$$

which yields

$$\Pr(PQ) = \frac{1}{3} \times \frac{1}{4} = \frac{1}{12}, \quad \Pr(Q) = \frac{\frac{1}{12}}{\frac{1}{2}} = \frac{1}{6} \tag{1.2.4}$$

The objective is to find

$$\Pr(P'|Q') \tag{1.2.5}$$

(1.2.1) can be expressed as

$$\Pr(P'|Q') = \frac{\Pr(P'Q')}{\Pr(Q')} \tag{1.2.6}$$
$$= \frac{\Pr(1-(P+Q)')}{\Pr(Q')} \tag{1.2.7}$$
$$= \frac{1 - \Pr(P) - \Pr(Q) + \Pr(PQ)}{1 - \Pr(Q)} \tag{1.2.8}$$

Substituting from (1.2.4) and (1.2.1) in (1.2.8),

$$\Pr(P'|Q') = \frac{4}{5} \tag{1.2.9}$$

1.3. An experiment consists of two papers.paper1 and paper2.The probability of failing in paper 1 is .3 and that in paper 2 is .2.Given that a student has failed in paper 2,the probability of failing in paper 1 is .6.The probability of student failing in both is

a) .5

b) .18

c) .12

d) .06

Solution: Table 1.3.1 summarises the given information.

	Description	Probability
0	failure	$\Pr(X=0) = 0.3$
1	success	$\Pr(Y=0) = 0.2$
X	Paper 1	$\Pr(X=0\|Y=0) = 0.6$
Y	Paper 2	

TABLE 1.3.1: Description

The desired probability is

$$\Pr(X=0, Y=0) = \Pr(X=0|Y=0)\Pr(Y=0) \tag{1.3.1}$$
$$= .12 \tag{1.3.2}$$

2 Geometric Distribution

2.1. Suppose X has density

$$f(x|\theta) = \frac{1}{\theta}e^{-x/\theta}, x > 0 \tag{2.1.1}$$

Define

$$Y = k, \quad k \le X < k+1, \quad k = 0,1,2\dots \tag{2.1.2}$$

Then the distribution of Y is

a) Normal
b) Binomial
c) Poisson
d) Geometric

Solution:

$$\Pr(Y=k) = \Pr(k \le X < k+1) \tag{2.1.3}$$
$$= \int_k^{k+1} f(x|\theta)\,dx \tag{2.1.4}$$
$$= \int_k^{k+1} \frac{1}{\theta} e^{-\frac{x}{\theta}} dx \tag{2.1.5}$$
$$= \left[-e^{-\frac{x}{\theta}}\right]_k^{k+1} \tag{2.1.6}$$
$$= e^{-\frac{k}{\theta}}\left(1 - e^{-\frac{1}{\theta}}\right) \tag{2.1.7}$$
$$\implies \Pr(Y=k) = (1-p)^k\,p\,k = 0,1,2\dots \tag{2.1.8}$$

where

$$p = 1 - e^{-\frac{1}{\theta}} \tag{2.1.9}$$

Therefore, the distribution of Y is 4) Geometric.

3 Two Dimensions

3.1. Let $c \in \mathbb{R}$ be a constant. Let X, Y be random variables with joint probability density function

$$f(x,y) = \begin{cases} cxy & 0 < x < y < 1, \\ 0 & \text{otherwise} \end{cases} \tag{3.1.1}$$

Which of the following statements are correct ?

a) $c = \frac{1}{8}$
b) $c = 8$
c) X and Y are independent
d) $\Pr(X = Y) = 0$

Solution:

a) False
b) By definition,

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx \tag{3.1.2}$$
$$= \int_0^y cxy\,dx \tag{3.1.3}$$
$$= cy\left(\frac{x^2}{2}\right)\Bigg|_0^y \tag{3.1.4}$$
$$= \frac{cy^3}{2} \tag{3.1.5}$$

$$\implies f_Y(y) = \begin{cases} \frac{cy^3}{2}, & 0 < y < 1 \\ 0 & \text{otherwise.} \end{cases} \tag{3.1.6}$$

$\because$ the area under the pdf is 1, from (3.1.6),

$$\implies \int_{-\infty}^{\infty} f_Y(y)\,dy = 1 \tag{3.1.7}$$
$$\implies \int_0^1 c\frac{y^3}{2} = 1 \tag{3.1.8}$$
$$\implies \frac{c}{8} = 1 \tag{3.1.9}$$
$$\text{or, } c = 8 \tag{3.1.10}$$

Also,

$$f_Y(y) = \begin{cases} 4y^3 & \text{, if } 0 < y < 1 \\ 0 & \text{, otherwise} \end{cases} \tag{3.1.11}$$

c)

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy \tag{3.1.12}$$
$$= \int_x^1 cxy\,dy \tag{3.1.13}$$
$$= cx\left(\frac{y^2}{2}\right)\Bigg|_x^1 \tag{3.1.14}$$
$$= cx\left(\frac{1-x^2}{2}\right) \tag{3.1.15}$$

$$\implies f_X(x) = \begin{cases} 4x\left(1-x^2\right), & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases} \tag{3.1.16}$$

From (3.1.16) and (3.1.11)

$$f_X(x) \times f_Y(y) = \begin{cases} 16xy^3\left(1-x^2\right) & \text{, if } 0 < x, y < 1 \\ 0 & \text{, otherwise} \end{cases} \tag{3.1.17}$$

$$\neq f(x,y) \tag{3.1.18}$$

Hence, X and Y are not independent.

d)

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx \tag{3.1.19}$$

$$= \int_0^x 4x(1-x^2)\,dx \tag{3.1.20}$$

$$= \int_0^x 4x - 4x^3\,dx \tag{3.1.21}$$

$$= 2x^2 - 4x^4 \text{ for } 0 < x < 1 \tag{3.1.22}$$

yielding

$$F_X(x) = \begin{cases} 0 & x \le 0 \\ 2x^2 - 4x^4 & 0 < x < 1 \\ 1 & x \ge 1 \end{cases} \tag{3.1.23}$$

From (3.1.23),

$$\Pr(Y-\epsilon < X < Y+\epsilon)$$
$$= F_X(Y+\epsilon) - F_X(Y-\epsilon)$$
$$= 8\epsilon Y\left(1 - Y^2 - \epsilon^2\right) \tag{3.1.24}$$

upon simplification. Letting

$$g(Y) = 8\epsilon Y(1 - Y^2 - \epsilon^2), \tag{3.1.25}$$

$$E[g(Y)] = \int_{-\infty}^{\infty} g(y) f_Y(y)\,dy \tag{3.1.26}$$

$$= \int_0^1 (4y^3)(8\epsilon y)(1 - y^2 - \epsilon^2)\,dy \tag{3.1.27}$$

$$\implies \Pr(Y-\epsilon < X < Y+\epsilon)$$
$$= 32\epsilon\left(\frac{2-7\epsilon^2}{35}\right) \tag{3.1.28}$$

Now substituting $\epsilon = 0$ in the above,

$$\Pr(X = Y) = 0 \tag{3.1.29}$$

3.2. Let X and Y be random variables having the joining probability density function

$$f_{XY}(x,y) = \begin{cases} \frac{1}{\sqrt{2\pi y}} e^{\frac{-1}{2y}(x-y)^2} & x \in (-\infty,\infty), \\ & y \in (0,1) \\ 0 & \text{otherwise} \end{cases} \tag{3.2.1}$$

The covariance between the random variables X and Y is

Solution:

3.3. Let a random variable X follow exponential distribution with mean 2. Define $Y = [X-2|X > 2]$. The value of $\Pr(Y \ge t)$ is ...

Solution: From the given information,

$$\Pr(Y \ge t) = \frac{\Pr(X-2 \ge t, X > 2)}{\Pr(X > 2)} \tag{3.3.1}$$

$$= \frac{\Pr(X \ge t+2, X > 2)}{\Pr(X > 2)} \tag{3.3.2}$$

$\because$ X has an exponential distribution with parameter $\lambda = \frac{1}{2}$,

$$F_X(x) = \begin{cases} 1 - e^{-\lambda x}, & \text{if } 0 < x < \infty \\ 0, & \text{otherwise} \end{cases} \tag{3.3.3}$$

and

$$\Pr(X > 2) = 1 - F_X(2) = e^{-2\lambda} \tag{3.3.4}$$

Also,

$$\Pr(X \ge t+2, X > 2) = \begin{cases} \Pr(X \ge t+2) & t \ge 0 \\ \Pr(X > 2) & t < 0 \end{cases} \tag{3.3.5}$$

Substituting (3.3.5) in (3.3.2), using (3.3.4) and simplifying,

$$\Pr(Y \ge t) = \begin{cases} e^{-\frac{t}{2}} & t \ge 0 \\ 1 & t < 0 \end{cases} \tag{3.3.6}$$

4 Markov Chain

4.1. **Step 1.** Flip a coin twice.
Step 2. If the outcomes are (TAILS, HEADS) then output Y and stop.
Step 3. If the outcomes are either (HEADS, HEADS) or (HEADS, TAILS), then output N and stop.
Step 4. If the outcomes are (TAILS, TAILS), then go to Step 1.
The probability that the output of the experiment is Y is (upto two decimal places)......

Solution: The given problem can be repre-

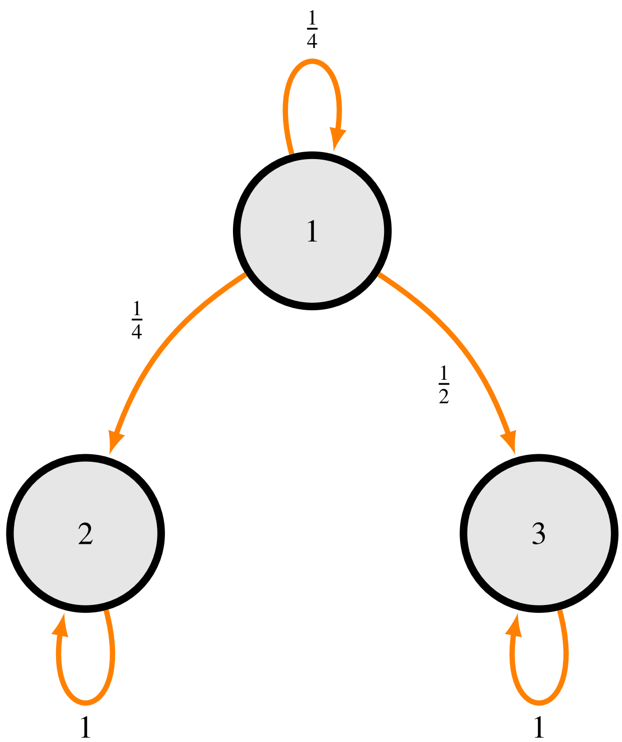

Fig. 4.1.1

sented using Table 4.1.1 and the Markov chain in Fig. 4.1.1. The state transition matrix for the

State	Description
1	$\{T, T\}$
2	$Y = \{T, H\}$
3	$N = \{\{H, H\}, \{H, T\}\}$

TABLE 4.1.1: States and their notations

Markov chain can be expressed as

$$P = \begin{array}{c c} & \begin{array}{ccc} 2 & 3 & 1 \end{array} \\ \begin{array}{c} 2 \\ 3 \\ 1 \end{array} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0.25 & 0.5 & 0.25 \end{bmatrix} \end{array} \tag{4.1.1}$$

Clearly, the state 1 is transient, while 2, 3 are absorbing. Comparing (4.1.1) with the standard form of the state transition matrix

$$P = \begin{array}{c c} & \begin{array}{cc} A & N \end{array} \\ \begin{array}{c} A \\ N \end{array} & \begin{bmatrix} I & O \\ R & Q \end{bmatrix} \end{array} \tag{4.1.2}$$

TABLE 4.1.2: Notations and their meanings

Notation	Meaning
A	All absorbing states
N	All non-absorbing states
I	Identity matrix
O	Zero matrix
R, Q	Other submatices

where, From (4.1.1) and (4.1.2),

$$R = \begin{pmatrix} 0.25 & 0.5 \end{pmatrix}, Q = \begin{pmatrix} 0.25 \end{pmatrix} \tag{4.1.3}$$

The limiting matrix for absorbing Markov chain is

$$\bar{P} = \begin{pmatrix} I & O \\ FR & O \end{pmatrix} \tag{4.1.4}$$

where

$$F = (I - Q)^{-1} = (1 - 0.25)^{-1} = \frac{4}{3} \tag{4.1.5}$$

is called the fundamental matrix of P. Upon substituting from (4.1.3) in (4.1.5),

$$F = (1 - 0.25)^{-1} = \frac{4}{3} \tag{4.1.6}$$

and

$$FR = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \end{pmatrix} \tag{4.1.7}$$

which, upon substituting in (4.1.4) yields

$$\bar{P} = \begin{array}{c c} & \begin{array}{ccc} 2 & 3 & 1 \end{array} \\ \begin{array}{c} 2 \\ 3 \\ 1 \end{array} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 \end{bmatrix} \end{array} \tag{4.1.8}$$

$$\therefore \bar{p}_{12} = \frac{1}{3} \tag{4.1.9}$$