

# Probability

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*Abstract*—This book provides solved examples on Probability

## 1 AXIOMS

1.1. The probability that a given positive integer lying between 1 and 100 (both inclusive) is NOT divisible by 2,3 or 5 is ...

**Solution:** Table 1.1.1 summarizes the given information.

Event	Definition	Probability
A	$n \equiv 0 \pmod{2}$	$\frac{50}{100}$
B	$n \equiv 0 \pmod{3}$	$\frac{33}{100}$
C	$n \equiv 0 \pmod{5}$	$\frac{20}{100}$
AB	$n \equiv 0 \pmod{6}$	$\frac{16}{100}$
BC	$n \equiv 0 \pmod{15}$	$\frac{6}{100}$
AC	$n \equiv 0 \pmod{10}$	$\frac{10}{100}$
ABC	$n \equiv 0 \pmod{30}$	$\frac{3}{100}$

TABLE 1.1.1:  $1 \leq n \leq 100$

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$$\begin{aligned}
 \therefore \Pr(A + B + C) &= \Pr(A) + \Pr(B) + \Pr(C) \\
 &\quad - \Pr(AB) - \Pr(BC) \\
 &\quad - \Pr(AC) + \Pr(ABC) \quad (1.1.1)
 \end{aligned}$$

Substituting from Table 1.1.1 in (1.1.1),

$$\begin{aligned}\Pr(A + B + C) &= \frac{50}{100} + \frac{33}{100} + \frac{20}{100} \\ &\quad - \frac{16}{100} - \frac{6}{100} - \frac{10}{100} + \frac{3}{100} \\ &= \frac{74}{100} \quad (1.1.2)\end{aligned}$$

Thus, the required probability is

$$1 - \Pr(A + B + C) = \frac{26}{100} \quad (1.1.3)$$

1.2. P and Q are considering to apply for a job. The probability that P applies for the job is  $\frac{1}{4}$ , the probability that P applies for the job given that Q applies for the job is  $\frac{1}{2}$ , and the probability that Q applies for the job given that P applies for the job is  $\frac{1}{3}$ . Then the probability that P does not apply for the job given that Q does not apply for the job is

- a)  $\frac{4}{5}$       b)  $\frac{5}{6}$       c)  $\frac{7}{8}$       d)  $\frac{11}{12}$

**Solution:** The given information can be expressed as

$$\Pr(P) = \frac{1}{4} \quad (1.2.1)$$

$$\Pr(P|Q) = \frac{1}{2} = \frac{\Pr(PQ)}{\Pr(Q)} \quad (1.2.2)$$

$$\Pr(Q|P) = \frac{1}{3} = \frac{\Pr(PQ)}{\Pr(P)} \quad (1.2.3)$$

which yields

$$\Pr(PQ) = \frac{1}{3} \times \frac{1}{4} = \frac{1}{12} \quad (1.2.4)$$

$$\Pr(Q) = \frac{\frac{1}{12}}{\frac{1}{3}} = \frac{1}{6}$$

The objective is to find

$$\Pr(P'|Q') \quad (1.2.5)$$

(1.2.1) can be expressed as

$$\Pr(P'|Q') = \frac{\Pr(P'Q')}{\Pr(Q')} \quad (1.2.6)$$

$$= \frac{\Pr(1 - (P + Q)')}{\Pr(Q')} \quad (1.2.7)$$

$$= \frac{1 - \Pr(P) - \Pr(Q) + \Pr(PQ)}{1 - \Pr(Q)} \quad (1.2.8)$$

Substituting from (1.2.4) and (1.2.1) in (1.2.8),

$$\Pr(P'|Q') = \frac{4}{5} \quad (1.2.9)$$

1.3. An experiment consists of two papers. paper1 and paper2. The probability of failing in paper 1 is .3 and that in paper 2 is .2. Given that a student has failed in paper 2, the probability of failing in paper 1 is .6. The probability of student failing in both is

a) .5

b) .18

c) .12

d) .06

**Solution:** Table 1.3.1 summarises the given

	Description	Probability
0	failure	$\Pr(X = 0) = 0.3$
1	success	$\Pr(Y = 0) = 0.2$
X	Paper 1	$\Pr(X = 0 Y = 0) = 0.6$
Y	Paper 2	

TABLE 1.3.1: Description

information. The desired probability is

$$\Pr(X = 0, Y = 0) = \Pr(X = 0|Y = 0) \Pr(Y = 0) \quad (1.3.1)$$

$$= .12 \quad (1.3.2)$$

## 2 SUM OF RANDOM VARIABLES

2.1. Let  $X \in \{0, 1\}$  and  $Y \in \{0, 1\}$  be two independent binary random variables. If  $\Pr(X = 0) = p$  and  $\Pr(Y = 0) = q$ , then  $\Pr(X + Y \geq 1)$  is equal to

- a)  $pq + (1 - p)(1 - q)$
- b)  $pq$
- c)  $p(1 - q)$
- d)  $1 - pq$

**Solution:** From the given information,

$$p_X(n) = \begin{cases} p & n = 0 \\ 1 - p & n = 1 \end{cases} \quad (2.1.1)$$

$$p_Y(n) = \begin{cases} q & n = 0 \\ 1 - q & n = 1 \end{cases} \quad (2.1.2)$$

The characteristic functions of  $X$  and  $Y$  are

$$\phi_X(z) = E(z^{-X}) = p + (1 - p)z^{-1} \quad (2.1.3)$$

$$\phi_Y(z) = q + (1 - q)z^{-1} \quad (2.1.4)$$

and the CF of  $Z = X + Y$  is

$$\phi_{X+Y}(z) = E(z^{-(X+Y)}) \quad (2.1.5)$$

$$= \phi_X(z) \times \phi_Y(z) \quad (2.1.6)$$

$$= [p + (1 - p)z^{-1}] [q + (1 - q)z^{-1}] \quad (2.1.7)$$

$$\Rightarrow \phi_Z(z) = pq + (p + q - 2pq)z^{-1} + (1 - p)(1 - q)z^{-2} \quad (2.1.8)$$

yielding

$$p_Z(n) = \begin{cases} pq & n = 0 \\ p + q - 2pq & n = 1 \\ (1 - p)(1 - q) & n = 2 \end{cases} \quad (2.1.9)$$

Thus,

$$\Pr(X + Y \geq 1) = 1 - \Pr(Z < 1) = 1 - pq \quad (2.1.10)$$

### 3 BINOMIAL DISTRIBUTION

3.1. The probability that a part manufactured by a company will be defective is 0.05. If 15 such parts are selected randomly and inspected, the probability that atleast two parts will be defective is ...

**Solution:** The desired probability is

$$\Pr(X \geq 2) = 1 - \Pr(X < 2) \quad (3.1.1)$$

$$= 1 - \Pr(X = 0) - \Pr(X = 1) \quad (3.1.2)$$

$$= 1 - {}^{15}C_0 p^0 q^{15} - {}^{15}C_1 p^1 q^{14} \quad (3.1.3)$$

$$= 0.1709 \quad (3.1.4)$$

where

$$p = 0.05, q = 1 - p = 0.95 \quad (3.1.5)$$

and  $X$  is binomial with parameters  $(15, p)$ .

### 4 GEOMETRIC DISTRIBUTION

4.1. Suppose  $X$  has density

$$f(x|\theta) = \frac{1}{\theta} e^{-x/\theta}, x > 0 \quad (4.1.1)$$

Define

$$Y = k, \quad k \leq X < k + 1, \quad k = 0, 1, 2, \dots \quad (4.1.2)$$

Then the distribution of  $Y$  is

- a) Normal
- b) Binomial
- c) Poisson
- d) Geometric

**Solution:**

$$\Pr(Y = k) = \Pr(k \leq X < k + 1) \quad (4.1.3)$$

$$= \int_k^{k+1} f(x|\theta) dx \quad (4.1.4)$$

$$= \int_k^{k+1} \frac{1}{\theta} e^{-\frac{x}{\theta}} dx \quad (4.1.5)$$

$$= \left[ -e^{-\frac{x}{\theta}} \right]_k^{k+1} \quad (4.1.6)$$

$$= e^{-\frac{k}{\theta}} \left( 1 - e^{-\frac{1}{\theta}} \right) \quad (4.1.7)$$

$$\Rightarrow \Pr(Y = k) = (1 - p)^k p, k = 0, 1, 2, \dots \quad (4.1.8)$$

where

$$p = 1 - e^{-\frac{1}{\theta}} \quad (4.1.9)$$

Therefore, the distribution of  $Y$  is 4) Geometric.

## 5 TWO DIMENSIONS

5.1. Let  $c \in \mathbb{R}$  be a constant. Let  $X, Y$  be random variables with joint probability density function

$$f(x, y) = \begin{cases} cxy & 0 < x < y < 1, \\ 0 & \text{otherwise} \end{cases} \quad (5.1.1)$$

Which of the following statements are correct ?

- a)  $c = \frac{1}{8}$
- b)  $c = 8$
- c)  $X$  and  $Y$  are independent
- d)  $\Pr(X = Y) = 0$

**Solution:**

- a) False
- b) By definition,

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y) dx \quad (5.1.2)$$

$$= \int_0^y cxy dx \quad (5.1.3)$$

$$= cy \left( \frac{x^2}{2} \right) \Big|_0^y \quad (5.1.4)$$

$$= \frac{cy^3}{2} \quad (5.1.5)$$

$$\Rightarrow f_Y(y) = \begin{cases} \frac{cy^3}{2}, & 0 < y < 1 \\ 0 & \text{otherwise.} \end{cases} \quad (5.1.6)$$

$\therefore$  the area under the pdf is 1, from (5.1.6),

$$\Rightarrow \int_{-\infty}^{\infty} f_Y(y) dy = 1 \quad (5.1.7)$$

$$\Rightarrow \int_0^1 c \frac{y^3}{2} dy = 1 \quad (5.1.8)$$

$$\Rightarrow \frac{c}{8} = 1 \quad (5.1.9)$$

$$\text{or, } c = 8 \quad (5.1.10)$$

Also,

$$f_Y(y) = \begin{cases} 4y^3, & \text{if } 0 < y < 1 \\ 0, & \text{otherwise} \end{cases} \quad (5.1.11)$$

c)

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy \quad (5.1.12)$$

$$= \int_x^1 cxy dy \quad (5.1.13)$$

$$= cx \left( \frac{y^2}{2} \right) \Big|_x^1 \quad (5.1.14)$$

$$= cx \left( \frac{1 - x^2}{2} \right) \quad (5.1.15)$$

$$\Rightarrow f_X(x) = \begin{cases} 4x(1 - x^2), & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases} \quad (5.1.16)$$

From (5.1.16) and (5.1.11)

$$f_X(x) \times f_Y(y) = \begin{cases} 16xy^3(1 - x^2), & \text{if } 0 < x, y < 1 \\ 0 & \text{otherwise} \end{cases} \quad (5.1.17)$$

$$\neq f(x, y) \quad (5.1.18)$$

Hence,  $X$  and  $Y$  are not independent.

d)

$$F_X(x) = \int_{-\infty}^x f_X(x) dx \quad (5.1.19)$$

$$= \int_0^x 4x(1 - x^2) dx \quad (5.1.20)$$

$$= \int_0^x 4x - 4x^3 dx \quad (5.1.21)$$

$$= 2x^2 - 4x^4 \text{ for } 0 < x < 1 \quad (5.1.22)$$

yielding

$$F_X(x) = \begin{cases} 0 & x \leq 0 \\ 2x^2 - 4x^4 & 0 < x < 1 \\ 1 & x \geq 1 \end{cases} \quad (5.1.23)$$

From (5.1.23),

$$\begin{aligned} \Pr(Y - \epsilon < X < Y + \epsilon) &= F_X(Y + \epsilon) - F_X(Y - \epsilon) \\ &= 8\epsilon Y(1 - Y^2 - \epsilon^2) \end{aligned} \quad (5.1.24)$$

upon simplification. Letting

$$g(Y) = 8\epsilon Y(1 - Y^2 - \epsilon^2), \quad (5.1.25)$$

$$E[g(Y)] = \int_{-\infty}^{\infty} g(y)f_Y(y) dy \quad (5.1.26)$$

$$= \int_0^1 (4y^3)(8\epsilon y)(1 - y^2 - \epsilon^2) dy \quad (5.1.27)$$

$$\begin{aligned} \Rightarrow \Pr(Y - \epsilon < X < Y + \epsilon) \\ = 32\epsilon \left( \frac{2 - 7\epsilon^2}{35} \right) \end{aligned} \quad (5.1.28)$$

Now substituting  $\epsilon = 0$  in the above,

$$\Pr(X = Y) = 0 \quad (5.1.29)$$

5.2. Let  $X$  and  $Y$  be random variables having the joining probability density function

$$f_{XY}(x, y) = \begin{cases} \frac{1}{\sqrt{2\pi y}} e^{\frac{-1}{2y}(x-y)^2} & x \in (-\infty, \infty), \\ & y \in (0, 1) \\ 0 & \text{otherwise} \end{cases} \quad (5.2.1)$$

The covariance between the random variables  $X$  and  $Y$  is

**Solution:**

5.3. Let a random variable  $X$  follow exponential distribution with mean 2. Define  $Y = [X-2|X > 2]$ . The value of  $\Pr(Y \geq t)$  is ...

**Solution:** From the given information,

$$\Pr(Y \geq t) = \frac{\Pr(X - 2 \geq t, X > 2)}{\Pr(X > 2)} \quad (5.3.1)$$

$$= \frac{\Pr(X \geq t + 2, X > 2)}{\Pr(X > 2)} \quad (5.3.2)$$

$\therefore X$  has an exponential distribution with parameter  $\lambda = \frac{1}{2}$ ,

$$F_X(x) = \begin{cases} 1 - e^{-\lambda x}, & \text{if } 0 < x < \infty \\ 0, & \text{otherwise} \end{cases} \quad (5.3.3)$$

and

$$\Pr(X > 2) = 1 - F_X(2) = e^{-2\lambda} \quad (5.3.4)$$

Also,

$$\Pr(X \geq t + 2, X > 2) = \begin{cases} \Pr(X \geq t + 2) & t \geq 0 \\ \Pr(X > 2) & t < 0 \end{cases} \quad (5.3.5)$$

Substituting (5.3.5) in (5.3.2), using (5.3.4) and simplifying,

$$\Pr(Y \geq t) = \begin{cases} e^{-\frac{t}{2}} & t \geq 0 \\ 1 & t < 0 \end{cases} \quad (5.3.6)$$

## 6 MARKOV CHAIN

6.1. **Step 1.** Flip a coin twice.

**Step 2.** If the outcomes are (TAILS, HEADS) then output  $Y$  and stop.

**Step 3.** If the outcomes are either (HEADS, HEADS) or (HEADS, TAILS), then output  $N$  and stop.

**Step 4.** If the outcomes are (TAILS, TAILS), then go to Step 1.

The probability that the output of the experiment is  $Y$  is (upto two decimal places).....

**Solution:** The given problem can be repre-

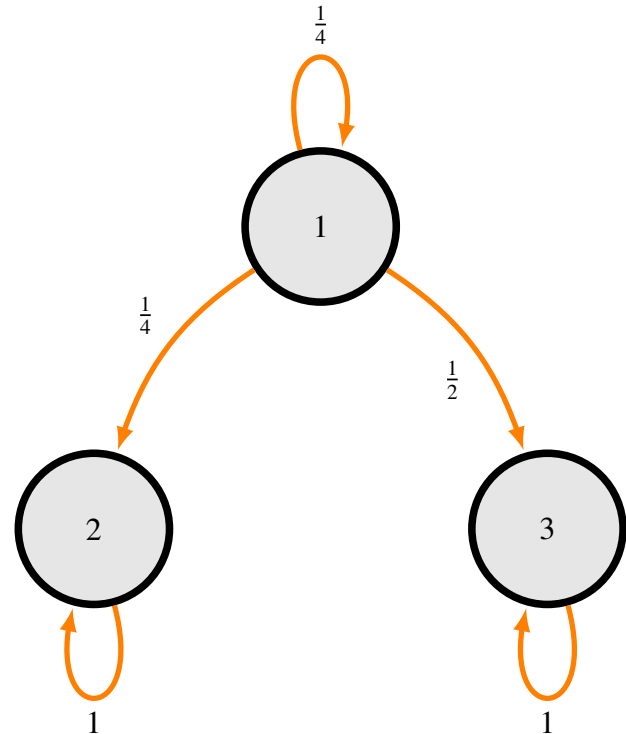


Fig. 6.1.1

sented using Table 6.1.1 and the Markov chain in Fig. 6.1.1. The state transition matrix for the

State	Description
1	$\{T, T\}$
2	$Y = \{T, H\}$
3	$N = \{\{H, H\}, \{H, T\}\}$

TABLE 6.1.1: States and their notations

Markov chain can be expressed as

$$P = \begin{matrix} & \begin{matrix} 2 & 3 & 1 \end{matrix} \\ \begin{matrix} 2 \\ 3 \\ 1 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0.25 & 0.5 & 0.25 \end{bmatrix} \end{matrix} \quad (6.1.1)$$

Clearly, the state 1 is transient, while 2, 3 are absorbing. Comparing (6.1.1) with the standard form of the state transition matrix

$$P = \begin{matrix} & \begin{matrix} A & N \end{matrix} \\ \begin{matrix} A \\ N \end{matrix} & \begin{bmatrix} I & O \\ R & Q \end{bmatrix} \end{matrix} \quad (6.1.2)$$

where, From (6.1.1) and (6.1.2),

TABLE 6.1.2: Notations and their meanings

Notation	Meaning
$A$	All absorbing states
$N$	All non-absorbing states
$I$	Identity matrix
$O$	Zero matrix
$R, Q$	Other submatrices

$$R = (0.25 \ 0.5), Q = (0.25) \quad (6.1.3)$$

The limiting matrix for absorbing Markov chain is

$$\bar{P} = \begin{pmatrix} I & O \\ FR & O \end{pmatrix} \quad (6.1.4)$$

where

$$F = (I - Q)^{-1} = (1 - 0.25)^{-1} = \frac{4}{3} \quad (6.1.5)$$

is called the fundamental matrix of  $P$ . Upon substituting from (6.1.3) in (6.1.5),

$$F = (1 - 0.25)^{-1} = \frac{4}{3} \quad (6.1.6)$$

and

$$FR = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \end{pmatrix} \quad (6.1.7)$$

which, upon substituting in (6.1.4) yields

$$\bar{P} = \begin{matrix} & \begin{matrix} 2 & 3 & 1 \end{matrix} \\ \begin{matrix} 2 \\ 3 \\ 1 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 \end{bmatrix} \end{matrix} \quad (6.1.8)$$

$$\therefore \bar{p}_{12} = \frac{1}{3} \quad (6.1.9)$$

## 7 CONVERGENCE

7.1. Let  $X_1, X_2, \dots$  be i.i.d.  $N(0, 1)$  random variables. Let

$$S_n = X_1^2 + X_2^2 + \dots + X_n^2 \forall n \geq 1. \quad (7.1.1)$$

Which of the following statements are correct?

a)

$$\frac{S_n - n}{\sqrt{2}} \sim N(0, 1) \quad \forall n \geq 1 \quad (7.1.2)$$

b)

$$\forall \epsilon > 0, \Pr\left(\left|\frac{S_n}{n} - 2\right| > \epsilon\right) \rightarrow 0, n \rightarrow \infty \quad (7.1.3)$$

c)  $\frac{S_n}{n} \rightarrow 1$  with probability 1

d)

$$\Pr(S_n \leq n + \sqrt{nx}) \rightarrow \Pr(Y \leq x) \forall x \in \mathbb{R}, Y \sim N(0, 2) \quad (7.1.4)$$