Probability

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Contents

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***Abstract*—This book provides solved examples on Probability**

1 Axioms

1.1. The probability that a given positive integer lying between 1 and 100 (both inclusive) is NOT divisible by 2,3 or 5 is …

Solution: Table 1.1.1 summarizes the given information.

Event	Definition	Probability
A	$n \equiv 0 \pmod 2$	$\frac{50}{100}$
B	$n \equiv 0 \pmod 3$	$\frac{33}{100}$
C	$n \equiv 0 \pmod 5$	$\frac{20}{100}$
AB	$n \equiv 0 \pmod 6$	$\frac{16}{100}$
BC	$n \equiv 0 \pmod{15}$	$\frac{6}{100}$
AC	$n \equiv 0 \pmod{10}$	$\frac{10}{100}$
ABC	$n \equiv 0 \pmod{30}$	$\frac{3}{100}$

TABLE 1.1.1: $1 \le n \le 100$

$$\because \Pr(A+B+C) = \Pr(A) + \Pr(B) + \Pr(C) - \Pr(AB) - \Pr(BC) - \Pr(AC) + \Pr(ABC) \quad (1.1.1)$$

Substituting from Table 1.1.1 in (1.1.1),

$$\Pr(A+B+C) = \frac{50}{100} + \frac{33}{100} + \frac{20}{100} - \frac{16}{100} - \frac{6}{100} - \frac{10}{100} + \frac{3}{100}$$
$$= \frac{74}{100} \quad (1.1.2)$$

Thus, the required probability is

$$1 - \Pr(A+B+C) = \frac{26}{100} \quad (1.1.3)$$

1.2. P and Q are considering to apply for a job. The probability that P applies for the job is $\frac{1}{4}$, the probability that P applies for the job given that Q applies for the job is $\frac{1}{2}$, and the probability that Q applies for the job given that P applies for the job is $\frac{1}{3}$. Then the probability that P does not apply for the job given that Q does not apply for the job is

a) $\frac{4}{5}$ b) $\frac{5}{6}$ c) $\frac{7}{8}$ d) $\frac{11}{12}$

Solution: The given information can be expressed as

$$\Pr(P) = \frac{1}{4} \quad (1.2.1)$$
$$\Pr(P|Q) = \frac{1}{2} = \frac{\Pr(PQ)}{\Pr(Q)} \quad (1.2.2)$$
$$\Pr(Q|P) = \frac{1}{3} = \frac{\Pr(PQ)}{\Pr(P)} \quad (1.2.3)$$

which yields

$$\Pr(PQ) = \frac{1}{3} \times \frac{1}{4} = \frac{1}{12}$$
$$\Pr(Q) = \frac{\frac{1}{12}}{\frac{1}{2}} = \frac{1}{6} \quad (1.2.4)$$

The objective is to find

$$\Pr(P'|Q') \quad (1.2.5)$$

(1.2.1) can be expressed as

$$\Pr(P'|Q') = \frac{\Pr(P'Q')}{\Pr(Q')} \quad (1.2.6)$$
$$= \frac{\Pr(1-(P+Q)')}{\Pr(Q')} \quad (1.2.7)$$
$$= \frac{1-\Pr(P)-\Pr(Q)+\Pr(PQ)}{1-\Pr(Q)} \quad (1.2.8)$$

Substituting from (1.2.4) and (1.2.1) in (1.2.8),

$$\Pr(P'|Q') = \frac{4}{5} \quad (1.2.9)$$

2 Elementary Probability

2.1. An experiment consists of two papers.paper1 and paper2.The probability of failing in paper 1 is .3 and that in paper 2 is .2.Given that a student has failed in paper 2,the probability of failing in paper 1 is .6.The probability of student failing in both is

a) .5

b) .18

c) .12

d) .06

Solution: Table 2.1.1 summarises the given

	Description	Probability
0	failure	$\Pr(X=0) = 0.3$
1	success	$\Pr(Y=0) = 0.2$
X	Paper 1	$\Pr(X=0\|Y=0) = 0.6$
Y	Paper 2	

TABLE 2.1.1: Description

information. The desired probability is

$$\Pr(X=0, Y=0) = \Pr(X=0|Y=0)\Pr(Y=0) \quad (2.1.1)$$
$$= .12 \quad (2.1.2)$$

2.2. An urn contains 5 red balls and 5 black balls.In the first draw, one ball is picked at random and discarded without noticing its colour.The probability to get a red ball in the second draw is

a) $\frac{1}{2}$ b) $\frac{4}{9}$ c) $\frac{5}{9}$ d) $\frac{6}{9}$

Solution: Let $X_i \in \{0, 1\}$ represent the i^{th} draw where 1 denotes a red ball being drawn.

	$X_1 = 0$	$X_1 = 1$
$X_2 = 0$	4/18	5/18
$X_2 = 1$	5/18	4/18

TABLE 2.2.1: The probabilities of all possible cases when two balls are drawn one by one from the urn.

From Table 2.2.1,

$$\Pr(X_2 = 1) = \Pr(X_2 = 1, X_1 = 0) + \Pr(X_2 = 1, X_1 = 1) \quad (2.2.1)$$
$$= \frac{5}{18} + \frac{4}{18} \quad (2.2.2)$$
$$= \frac{1}{2} \quad (2.2.3)$$

The required option is (A).

3 Independent Random Variables

3.1. Let $X \in \{0, 1\}$ and $Y \in \{0, 1\}$ be two independent binary random variables. If $\Pr(X = 0) =$ p and $\Pr(Y = 0) =$ q, then $\Pr(X + Y \geqslant 1))$ is equal to

a) $pq + (1 - p)(1 - q)$
b) pq
c) $p(1 - q)$
d) $1 - pq$

Solution: From the given information,

$$p_X(n) = \begin{cases} p & n = 0 \\ 1 - p & n = 1 \end{cases} \quad (3.1.1)$$
$$p_Y(n) = \begin{cases} q & n = 0 \\ 1 - q & n = 1 \end{cases} \quad (3.1.2)$$

The characteristic functions of X and Y are

$$\phi_X(z) = \mathrm{E}\left(z^{-X}\right) = p + (1 - p)z^{-1} \quad (3.1.3)$$
$$\phi_Y(z) = q + (1 - q)z^{-1} \quad (3.1.4)$$

and the CF of $Z = X + Y$ is

$$\phi_{X+Y}(z) = \mathrm{E}\left(z^{-(X+Y)}\right) \quad (3.1.5)$$
$$= \phi_X(z) \times \phi_Y(z) \quad (3.1.6)$$
$$= \left[p + (1 - p)z^{-1}\right]\left[q + (1 - q)z^{-1}\right] \quad (3.1.7)$$

$$\implies \phi_Z(z) = pq + (p + q - 2pq)z^{-1} + (1 - p)(1 - q)z^{-2} \quad (3.1.8)$$

yielding

$$p_Z(n) = \begin{cases} pq & n = 0 \\ p + q - 2pq & n = 1 \\ (1 - p)(1 - q) & n = 2 \end{cases} \quad (3.1.9)$$

Thus,

$$\Pr(X + Y \geq 1) = 1 - \Pr(Z < 1) = 1 - pq \quad (3.1.10)$$

3.2. Two independent random variables X and Y are uniformly distributed in the interval $[-1, 1]$. The probability that $\max(X, Y)$ is less than $\frac{1}{2}$ is

a) $\frac{3}{4}$ b) $\frac{9}{16}$ c) $\frac{1}{4}$ d) $\frac{2}{3}$

Solution:

4 Binomial Distribution

4.1. The probability that a part manufactured by a company will be defective is 0.05. If 15 such parts are selected randomly and inspected,the probability that atleast two parts will be defective is . . .

Solution: The desired probabilty is

$$\Pr(X \geq 2) = 1 - \Pr(X < 2) \quad (4.1.1)$$
$$= 1 - \Pr(X = 0) - \Pr(X = 1) \quad (4.1.2)$$
$$= 1 - {}^{15}C_0 p^0 q^{15} - {}^{15}C_1 p^1 q^{14} \quad (4.1.3)$$
$$= 0.1709 \quad (4.1.4)$$

where

$$p = 0.0.5, q = 1 - p = 0.95 \quad (4.1.5)$$

and X is binomial with parameters $(15, p)$.

5 Gaussian Distribution

5.1. Let U and V be two independent zero mean Gaussian random variables of variances $\frac{1}{4}$ and $\frac{1}{9}$ respectively. The probability $\Pr(3V \ge 2U)$ is . . .

Solution: From the given information,

$$U = \mathcal{N}\left(0, \frac{1}{4}\right) V \qquad = \mathcal{N}\left(0, \frac{1}{9}\right) \tag{5.1.1}$$

Let $Y = 3V - 2U$. Then,

$$E(Y) = 3E(V) - 2E(U) = 0 \tag{5.1.2}$$
$$\text{var}(Y) = 3^2\text{var}(V) + 2^2\text{var}(U) = 2 \tag{5.1.3}$$

$$\therefore Y = \mathcal{N}(0, 2) \tag{5.1.4}$$

Thus,

$$\Pr(3V \ge 2U) = \Pr(3V - 2U \ge 0) \tag{5.1.5}$$
$$= \Pr(Y \ge 0) = \frac{1}{2} \tag{5.1.6}$$

$\because$ Y is symmetric about the origin.

6 Geometric Distribution

6.1. Suppose X has density

$$f(x|\theta) = \frac{1}{\theta}e^{-x/\theta}, x > 0 \tag{6.1.1}$$

Define

$$Y = k, \quad k \le X < k + 1, \quad k = 0, 1, 2 \dots \tag{6.1.2}$$

Then the distribution of Y is

a) Normal
b) Binomial
c) Poisson
d) Geometric

Solution:

$$\Pr(Y = k) = \Pr(k \le X < k + 1) \tag{6.1.3}$$
$$= \int_k^{k+1} f(x|\theta)\, dx \tag{6.1.4}$$
$$= \int_k^{k+1} \frac{1}{\theta}e^{-\frac{x}{\theta}} dx \tag{6.1.5}$$
$$= \left[-e^{-\frac{x}{\theta}}\right]_k^{k+1} \tag{6.1.6}$$
$$= e^{-\frac{k}{\theta}}\left(1 - e^{-\frac{1}{\theta}}\right) \tag{6.1.7}$$
$$\implies \Pr(Y = k) = (1 - p)^k\, p\; k = 0, 1, 2 \dots \tag{6.1.8}$$

where

$$p = 1 - e^{-\frac{1}{\theta}} \tag{6.1.9}$$

Therefore, the distribution of Y is 4) Geometric.

7 Two Dimensions

7.1. Let $c \in \mathbb{R}$ be a constant. Let X, Y be random variables with joint probability density function

$$f(x, y) = \begin{cases} cxy & 0 < x < y < 1, \\ 0 & \text{otherwise} \end{cases} \tag{7.1.1}$$

Which of the following statements are correct ?

a) $c = \frac{1}{8}$
b) $c = 8$
c) X and Y are independent
d) $\Pr(X = Y) = 0$

Solution:

a) False
b) By definition,

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\, dx \tag{7.1.2}$$
$$= \int_0^y cxy\, dx \tag{7.1.3}$$
$$= cy\left(\frac{x^2}{2}\right)\Bigg|_0^y \tag{7.1.4}$$
$$= \frac{cy^3}{2} \tag{7.1.5}$$

$$\implies f_Y(y) = \begin{cases} \frac{cy^3}{2}, & 0 < y < 1 \\ 0 & \text{otherwise.} \end{cases} \tag{7.1.6}$$

$\because$ the area under the pdf is 1, from (7.1.6),

$$\implies \int_{-\infty}^{\infty} f_Y(y)\, dy = 1 \tag{7.1.7}$$
$$\implies \int_0^1 c\frac{y^3}{2} = 1 \tag{7.1.8}$$
$$\implies \frac{c}{8} = 1 \tag{7.1.9}$$
$$\text{or, } c = 8 \tag{7.1.10}$$

Also,

$$f_Y(y) = \begin{cases} 4y^3 & \text{, if } 0 < y < 1 \\ 0 & \text{, otherwise} \end{cases} \tag{7.1.11}$$

c)

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy \tag{7.1.12}$$
$$= \int_x^1 cxy\,dy \tag{7.1.13}$$
$$= cx\left(\frac{y^2}{2}\right)\Big|_x^1 \tag{7.1.14}$$
$$= cx\left(\frac{1-x^2}{2}\right) \tag{7.1.15}$$

$$\implies f_X(x) = \begin{cases} 4x\left(1-x^2\right), & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases} \tag{7.1.16}$$

From (7.1.16) and (7.1.11)

$$f_X(x) \times f_Y(y) = \begin{cases} 16xy^3\left(1-x^2\right) & \text{, if } 0 < x, y < 1 \\ 0 & \text{, otherwise} \end{cases} \tag{7.1.17}$$
$$\neq f(x,y) \tag{7.1.18}$$

Hence, X and Y are not independent.

d)

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx \tag{7.1.19}$$
$$= \int_0^x 4x(1-x^2)\,dx \tag{7.1.20}$$
$$= \int_0^x 4x - 4x^3\,dx \tag{7.1.21}$$
$$= 2x^2 - 4x^4 \text{ for } 0 < x < 1 \tag{7.1.22}$$

yielding

$$F_X(x) = \begin{cases} 0 & x \le 0 \\ 2x^2 - 4x^4 & 0 < x < 1 \\ 1 & x \ge 1 \end{cases} \tag{7.1.23}$$

From (7.1.23),

$$\Pr(Y - \epsilon < X < Y + \epsilon) = F_X(Y+\epsilon) - F_X(Y-\epsilon) = 8\epsilon Y\left(1 - Y^2 - \epsilon^2\right) \tag{7.1.24}$$

upon simplification. Letting

$$g(Y) = 8\epsilon Y(1 - Y^2 - \epsilon^2), \tag{7.1.25}$$
$$E[g(Y)] = \int_{-\infty}^{\infty} g(y) f_Y(y)\,dy \tag{7.1.26}$$
$$= \int_0^1 (4y^3)(8\epsilon y)(1 - y^2 - \epsilon^2)\,dy \tag{7.1.27}$$

$$\implies \Pr(Y - \epsilon < X < Y + \epsilon) = 32\epsilon\left(\frac{2 - 7\epsilon^2}{35}\right) \tag{7.1.28}$$

Now substituting $\epsilon = 0$ in the above,

$$\Pr(X = Y) = 0 \tag{7.1.29}$$

7.2. Let X and Y be random variables having the joining probability density function

$$f_{XY}(x,y) = \begin{cases} \frac{1}{\sqrt{2\pi y}} e^{\frac{-1}{2y}(x-y)^2} & x \in (-\infty, \infty), \\ & y \in (0,1) \\ 0 & \text{otherwise} \end{cases} \tag{7.2.1}$$

The covariance between the random variables X and Y is

Solution:

7.3. Let a random variable X follow exponential distribution with mean 2. Define $Y = [X-2|X > 2]$. The value of $\Pr(Y \ge t)$ is . . .

Solution: From the given information,

$$\Pr(Y \ge t) = \frac{\Pr(X - 2 \ge t, X > 2)}{\Pr(X > 2)} \tag{7.3.1}$$
$$= \frac{\Pr(X \ge t + 2, X > 2)}{\Pr(X > 2)} \tag{7.3.2}$$

$\because$ X has an exponential distribution with parameter $\lambda = \frac{1}{2}$,

$$F_X(x) = \begin{cases} 1 - e^{-\lambda x}, & \text{if } 0 < x < \infty \\ 0, & \text{otherwise} \end{cases} \tag{7.3.3}$$

and

$$\Pr(X > 2) = 1 - F_X(2) = e^{-2\lambda} \tag{7.3.4}$$

Also,

$$\Pr\left(X \geq t+2, X > 2\right) = \begin{cases} \Pr\left(X \geq t+2\right) & t \geq 0 \\ \Pr\left(X > 2\right) & t < 0 \end{cases} \tag{7.3.5}$$

Substituting (7.3.5) in (7.3.2), using (7.3.4) and simplifying,

$$\Pr\left(Y \geq t\right) = \begin{cases} e^{-\frac{t}{2}} & t \geq 0 \\ 1 & t < 0 \end{cases} \tag{7.3.6}$$

8 Markov Chain

8.1. **Step 1.** Flip a coin twice.
Step 2. If the outcomes are (TAILS, HEADS) then output Y and stop.
Step 3. If the outcomes are either (HEADS, HEADS) or (HEADS, TAILS), then output N and stop.
Step 4. If the outcomes are (TAILS, TAILS), then go to Step 1.
The probability that the output of the experiment is Y is (upto two decimal places)......
Solution: The given problem can be represented using Table 8.1.1 and the Markov chain in Fig. 8.1.1. The state transition matrix for the

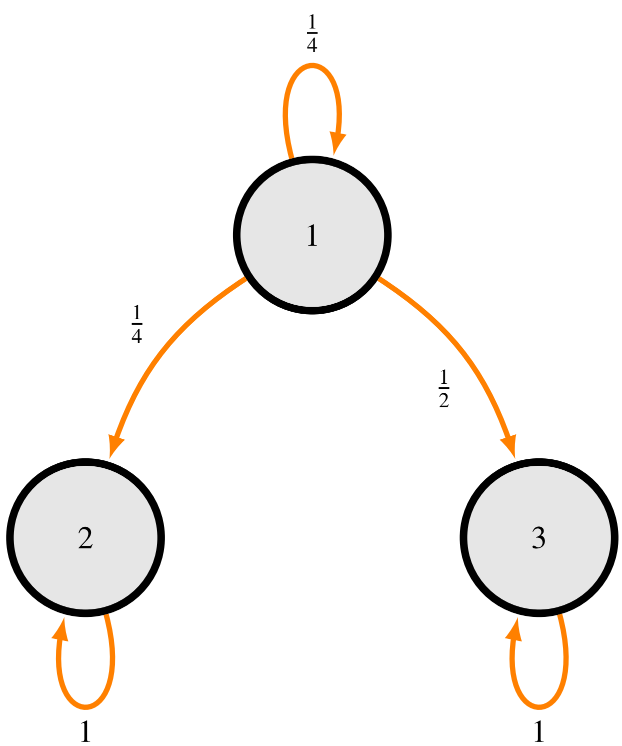

Fig. 8.1.1

State	Description
1	$\{T, T\}$
2	$Y = \{T, H\}$
3	$N = \{\{H, H\}, \{H, T\}\}$

TABLE 8.1.1: States and their notations

Markov chain can be expressed as

$$P = \begin{array}{c c} & \begin{array}{ccc} 2 & 3 & 1 \end{array} \\ \begin{array}{c} 2 \\ 3 \\ 1 \end{array} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0.25 & 0.5 & 0.25 \end{bmatrix} \end{array} \tag{8.1.1}$$

Clearly, the state 1 is transient, while $2, 3$ are absorbing. Comparing (8.1.1) with the standard form of the state transition matrix

$$P = \begin{array}{c c} & \begin{array}{cc} A & N \end{array} \\ \begin{array}{c} A \\ N \end{array} & \begin{bmatrix} I & O \\ R & Q \end{bmatrix} \end{array} \tag{8.1.2}$$

where, From (8.1.1) and (8.1.2),

TABLE 8.1.2: Notations and their meanings

Notation	Meaning
A	All absorbing states
N	All non-absorbing states
I	Identity matrix
O	Zero matrix
R, Q	Other submatices

$$R = \begin{pmatrix} 0.25 & 0.5 \end{pmatrix}, Q = \begin{pmatrix} 0.25 \end{pmatrix} \tag{8.1.3}$$

The limiting matrix for absorbing Markov chain is

$$\bar{P} = \begin{pmatrix} I & O \\ FR & O \end{pmatrix} \tag{8.1.4}$$

where

$$F = (I - Q)^{-1} = (1 - 0.25)^{-1} = \frac{4}{3} \tag{8.1.5}$$

is called the fundamental matrix of P. Upon substituting from (8.1.3) in (8.1.5),

$$F = (1 - 0.25)^{-1} = \frac{4}{3} \tag{8.1.6}$$

and

$$FR = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \end{pmatrix} \tag{8.1.7}$$

which, upon substituting in (8.1.4) yields

$$\bar{P} = \begin{array}{c} \\ 2 \\ 3 \\ 1 \end{array} \begin{array}{c} \begin{array}{ccc} 2 & 3 & 1 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 \end{bmatrix} \end{array} \tag{8.1.8}$$

$$\therefore \bar{p}_{12} = \frac{1}{3} \tag{8.1.9}$$

9 Convergence

9.1. Let $X_1, X_2, \dots$ be i.i.d. $N(0,1)$ random variables. Let

$$S_n = X_1^2 + X_2^2 + \cdots + X_n^2 \forall n \geq 1. \tag{9.1.1}$$

Which of the following statements are correct?

a)

$$\frac{S_n - n}{\sqrt{2}} \sim N(0,1) \quad \forall n \geq 1 \tag{9.1.2}$$

b)

$$\forall \epsilon > 0, \Pr\left(\left| \frac{S_n}{n} - 2 \right| > \epsilon \right) \to 0, n \to \infty \tag{9.1.3}$$

c) $\frac{S_n}{n} \to 1$ with probability 1

d)

$$\Pr\left(S_n \leq n + \sqrt{n}x \right) \to \Pr\left(Y \leq x\right) \forall x \in \mathbb{R}, Y \sim N\left(0,2\right) \tag{9.1.4}$$