# Series

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## About this Book

This book introduces progressions, binomial theorem, limits and sequences. All problems in the book are from NCERT mathematics textbooks from Class 9-12. Exercises are from CBSE, JEE and Olympiad exam papers.



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Github: https://github.com/gadepall/algebra

Contents

# 1 Arithmetic Progression

## 1.1 Formulae

1.1.1 Find the sum

$$S = 1 + 2 + \cdots + 10 \tag{1.1.1.1}$$

**Solution:** Reversing the sum in (1.1.1.1) as

$$S = 10 + 9 + \cdots + 1 \tag{1.1.1.2}$$

and adding (1.1.1.1) and (1.1.1.2),

$$2S = 11 + 11 + \cdots + 11 \quad 10 \text{ times} \tag{1.1.1.3}$$

$$\implies S = \frac{11 \times 10}{2} = 55 \tag{1.1.1.4}$$

1.1.2 The sum of the first $n$ natural numbers is

$$\sum_{k=1}^{n} k = 1 + 2 + \cdots + n \tag{1.1.2.1}$$

$$= \frac{n\,(n+1)}{2} \tag{1.1.2.2}$$

1.1.3 The $n^{th}$ term of an arithmetic progression (AP) is

$$x(n) = x(0) + nd, \quad n = 0, 1, \ldots \tag{1.1.3.1}$$

1.1.4

$$y(n) = \sum_{k=0}^{n} x(k) = \frac{n+1}{2}\left(x(0) + \frac{dn}{2}\right) \tag{1.1.4.1}$$

**Solution:** From (1.1.3.1) and (1.1.4.1),

$$\sum_{k=0}^{n} x(k) = \sum_{k=0}^{n} (x(0) + kd) \tag{1.1.4.2}$$

$$= \sum_{k=0}^{n} x(0) + d \sum_{k=1}^{n} k = (n+1)\,x(0) + d\frac{n\,(n+1)}{2} \tag{1.1.4.3}$$

upon substituting from (1.1.2.2), yielding (1.1.4.2) upon simplification.

## 1.2 NCERT

1.2.1 Determine the AP whose 3rd term is 5 and the 7th term is 9.
**Solution:**

$$x(0) + n_1 d = x(2) \tag{1.2.1.1}$$

$$x(0) + n_2 d = x(6) \tag{1.2.1.2}$$

$$\implies \begin{pmatrix} 1 & n_1 \\ 1 & n_2 \end{pmatrix} \begin{pmatrix} x(0) \\ d \end{pmatrix} \begin{pmatrix} x(2) \\ x(6) \end{pmatrix} \tag{1.2.1.3}$$

Substituting numerical values yields

$$\begin{pmatrix} 1 & 2 \\ 1 & 6 \end{pmatrix} \begin{pmatrix} x(0) \\ d \end{pmatrix} = \begin{pmatrix} 5 \\ 9 \end{pmatrix} \tag{1.2.1.4}$$

$$\implies \begin{pmatrix} x(0) \\ d \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \tag{1.2.1.5}$$

1.2.2 For the AP

$$\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, \dots$$

write the first term $x(0)$ and the common difference $d$.
**Solution:**

$$x(0) = \frac{3}{2}, d = \frac{1}{2} - \frac{3}{2} = -1. \tag{1.2.2.1}$$

1.2.3 Which of the following list of numbers form an AP? If they form an AP, write the next two terms

a) $4, 10, 16, 22, \dots$
b) $-2, 2, -2, 2, -2, \dots$
c) $1, -1, -3, -5, \dots$
d) $1, 1, 1, 2, 2, 2, 3, 3, 3, \dots$

a) AP

$$x(0) = 4, d = 6 \tag{1.2.3.1}$$

$$\implies x(n) = 4 + 6n \tag{1.2.3.2}$$

$$\text{or, } x(4) = 28, x(5) = 34. \tag{1.2.3.3}$$

b) Not an AP.

$$x(1) - x(0) = 4 \tag{1.2.3.4}$$

$$x(2) - x(1) = -4 \tag{1.2.3.5}$$

1.2.4 Find the 10th term of the AP : $2, 7, 12, \dots$
1.2.5 Which term of the AP : $21, 18, 15, \dots$ is - 81? Also, is any term 0? Give reason for your answer.
**Solution:**

$$x(0) = 21, d = -6 \tag{1.2.5.1}$$

$$\implies x(n) = 21 - 6n \tag{1.2.5.2}$$

$$\text{or, } -81 = 21 - 6n \implies n = 17 \tag{1.2.5.3}$$

If

$$x(n) = 0, n = \frac{21}{6} \tag{1.2.5.4}$$

using (1.2.5.2) which is impossible.
1.2.6 Check whether 301 is a term of the list of numbers $5, 11, 17, 23, \dots$
1.2.7 How many two-digit numbers are divisible by 3?

1.2.8 Find the 11th term from the last term (towards the first term) of the AP : $10, 7, 4, \ldots, -62$.

**Solution:** Reversing the AP,

$$x(0) = -62, d = 3, \tag{1.2.8.1}$$

$$\implies x(10) = -62 + 10 \times 3 = -32 \tag{1.2.8.2}$$

upon substituting in (1.1.3.1).

1.2.9 A sum of ₹1000 is invested at 8% simple interest per year. Calculate the interest at the end of each year. Do these interests form an AP? If so, find the interest at the end of 30 years making use of this fact.

1.2.10 In a flower bed, there are 23 rose plants in the first row, 21 in the second, 19 in the third, and so on. There are 5 rose plants in the last row. How many rows are there in the flower bed?

1.2.11 Find the sum of the first 22 terms of the AP : $8, 3, -2, \ldots$

**Solution:**

$$x(0) = 8, d = -5, n = 21 \tag{1.2.11.1}$$

$$\implies y(21) = -979 \tag{1.2.11.2}$$

upon substituting in (1.1.4.1).

1.2.12 If the sum of the first 14 terms of an AP is 1050 and its first term is 10, find the 20th term.

**Solution:**

$$y(13) = 1050, x(0) = 10, n = 13 \tag{1.2.12.1}$$

$$\implies 1050 = 14\left(10 + \frac{13d}{2}\right) \tag{1.2.12.2}$$

$$\text{or, } d = 10 \tag{1.2.12.3}$$

$$\therefore x(19) = 10 + 19d = 200 \tag{1.2.12.4}$$

1.2.13 How many terms of the AP : $24, 21, 18, \ldots$ must be taken so that their sum is 78?

1.2.14 Find the sum of

a) the first 1000 positive integers.
b) the first $n$ positive integers.

1.2.15 Find the sum of first 24 terms of the list of numbers whose $n^{th}$ term is given by $x(n) = 3 + 2n$

**Solution:**

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} 3 + \sum_{k=1}^{n} 2k \tag{1.2.15.1}$$

$$= 3n + 2\frac{n(n+1)}{2} = n(n+4) \tag{1.2.15.2}$$

$$= 672 \tag{1.2.15.3}$$

upon substituting numerical values.

1.2.16 A manufacturer of TV sets produced 600 sets in the third year and 700 sets in the seventh year. Assuming that the production increases uniformly by a fixed number every year, find

a) the production in the 1st year
b) the production in the 10th year
c) the total production in first 7 years.

1.2.17 In which of the following situations, does the list of numbers involved make an arithmetic progression, and why?

a) The taxi fare after each km when the fare is ₹15 for the first km and ₹8 for each additional km.
b) The amount of air present in a cylinder when a vacuum pump removes $\frac{1}{4}$ of the air remaining in the cylinder at a time.
c) The cost of digging a well after every metre of digging, when it costs ₹150 for the first metre and rises by ₹50 for each subsequent metre.
d) The amount of money in the account every year, when ₹10000 is deposited at compound interest at 8 % per annum.

1.2.18 Write first four terms of the AP, when the first term $a$ and the common difference $d$ are given as follows

a) $x(0) = 10, d = 10$
b) $x(0) = 4, d = -3$
c) $x(0) = -2, d = 0$
d) $x(0) = -1, d = \frac{1}{2}$
e) $x(0) = -1.25, d = -0.25$

1.2.19 For the following APs, write the first term and the common difference

a) $3, 1, -1, -3, \dots$
b) $-5, -1, 3, 7, \dots$
c) $\frac{1}{3}, \frac{5}{3}, \frac{9}{3}, \frac{13}{3}, \dots$
d) $0.6, 1.7, 2.8, 3.9, \dots$

1.2.20 Which of the following are APs? If they form an AP, find the common difference $d$ and write three more terms.

a) $2, 4, 8, 16, \dots$
b) $2, \frac{5}{2}, 3, \frac{7}{2}, \dots$
c) $-1.2, -3.2, -5.2, -7.2, \dots$
d) $-10, -6, -2, 2, \dots$
e) $3, 3 + \sqrt{2}, 3 + 2\sqrt{2}, 3 + 3\sqrt{2}, \dots$
f) $0.2, 0.22, 0.222, 0.2222, \dots$
g) $0, -4, -8, -12, \dots$
h) $-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \dots$
i) $1, 3, 9, 27, \dots$
j) $a, 2a, 3a, 4a, \dots$
k) $a, a^2, a^3, a^4, \dots$
l) $\sqrt{2}, \sqrt{8}, \sqrt{18}, \sqrt{32}, \dots$
m) $\sqrt{3}, \sqrt{6}, \sqrt{9}, \sqrt{12}, \dots$
n) $1^2, 3^2, 5^2, 7^2, \dots$
o) $1^2, 5^2, 7^2, 73, \dots$

1.2.21 Fill in the blanks in Table 1.2.21, given that $a$ is the first term, $d$ the common difference and $x(n)$ the $n^{th}$ term of the AP.

| | $x(0)$ | $d$ | $n$ | $x(n)$ |
|---|---|---|---|---|
| (i) | 7 | 3 | 8 | $\ldots$ |
| (ii) | -18 | $\ldots$ | 10 | 0 |
| (iii) | $\ldots$ | -3 | 18 | -5 |
| (iv) | -18.9 | 2.5 | $\ldots$ | 3.6 |
| (v) | 3.5 | 0 | 105 | $\ldots$ |

TABLE 1.2.21

1.2.22 Choose the correct choice in the following and justify

a) $30^{th}$ term of the AP: $10, 7, 4, \ldots$ is

i) 97 ii) 77 iii) -77 iv) -87

b) $11^{th}$ term of the AP: $-3, -\frac{1}{2}, 2, \ldots$ is

i) 28 ii) 22 iii) -38 iv) $-48\frac{1}{2}$

c) In the following APs, find the missing terms in the blanks

i) $2, \ldots, 26$

ii) $\ldots, 13, \ldots, 3$

iii) $5, \ldots, \ldots, 9\frac{1}{2}$

iv) $-4, \ldots, \ldots, \ldots, \ldots, 6$

v) $\ldots, 38, \ldots, \ldots, \ldots, -22$

1.2.23 Which term of the AP : $3, 8, 13, 18, \ldots$ is 78?

1.2.24 Find the number of terms in each of the following APs:

a) $7, 13, 19, \ldots, 205$.

b) $18, 15\frac{1}{2}, 13, \ldots, -47$

1.2.25 Check whether -150 is a term of the AP : $11, 8, 5, 2 \ldots$

1.2.26 Find the 31st term of an AP whose 11th term is 38 and the 16th term is 73.

1.2.27 An AP consists of 50 terms of which 3rd term is 12 and the last term is 106. Find the 29th term.

1.2.28 If the 3rd and the 9th terms of an AP are 4 and -8 respectively, which term of this AP is zero?

1.2.29 The 17th term of an AP exceeds its 10th term by 7. Find the common difference.

1.2.30 Which term of the AP : $3, 15, 27, 39, \ldots$ will be 132 more than its 54th term?

1.2.31 How many three-digit numbers are divisible by 7?

1.2.32 How many multiples of 4 lie between 10 and 250?

1.2.33 For what value of $n$, are the $n^{th}$ terms of two APs: $63, 65, 67, \ldots$ and $3, 10, 17, \ldots$ equal?

1.2.34 Determine the AP whose third term is 16 and the 7th term exceeds the 5th term by 12.

1.2.35 Find the 20th term from the last term of the AP : $3, 8, 13, \ldots, 253$.

1.2.36 The sum of the 4th and 8th terms of an AP is 24 and the sum of the 6th and 10th terms is 44. Find the first three terms of the AP.

1.2.37 Subba Rao started work in 1995 at an annual salary of ₹5000 and received an increment of ₹200 each year. In which year did his income reach ₹7000?

1.2.38 Ramkali saved ₹5 in the first week of a year and then increased her weekly savings by ₹1.75. If in the $n^{th}$ week, her weekly savings become ₹20.75, find $n$.

1.2.39 Find the sum of the following APs

a) $2, 7, 12, \ldots,$ to 10 terms.

b) $-37, -33, -29, \ldots,$ to 12 terms.

c) $0.6, 1.7, 2.8, \ldots,$ to 100 terms.

d) $\frac{1}{15}, \frac{1}{12}, \frac{1}{10}, \ldots$ to 11 terms.

1.2.40 Find the sums given below

a) $7 + 10\frac{1}{2} + 14 + \cdots + 84$

b) $34 + 32 + 30 + \cdots + 10$

c) $-5 + (-8) + (-11) + \cdots + (-230)$

1.2.41 In an A.P

a) given $x(0) = 5, d = 3, x(n) = 50$, find $n$ and $y(n)$.

b) given $x(0) = 7, x(13) = 35$, find $d$ and $y(13)$.

c) given $x(12) = 37, d = 3$, find $x(0)$ and $y(12)$.

d) given $x(3) = 15, y(10) = 125$, find $d$ and $x(10)$.

e) given $d = 5, y(9) = 75$, find $x(0)$ and $x(9)$.

f) given $x(0) = 2, d = 8, y(n) = 90$, find $n$ and $x(n)$.

g) given $x(0) = 8, x(n) = 62, y(n) = 210$, find $n$ and $d$.

h) given $x(n) = 4, d = 2, y(n) = -14$, find $n$ and $x(0)$.

i) given $x(0) = 3, n = 8, S = 192$, find $d$.

j) given $l = 28, S = 144$, and there are total 9 terms. Find $x(0)$.

1.2.42 How many terms of the AP : $9, 17, 25, \ldots$ must be taken to give a sum of 636?

1.2.43 The first term of an AP is 5, the last term is 45 and the sum is 400. Find the number of terms and the common difference.

1.2.44 The first and the last terms of an AP are 17 and 350 respectively. If the common difference is 9, how many terms are there and what is their sum?

1.2.45 Find the sum of first 22 terms of an AP in which $d = 7$ and $22^{nd}$ term is 149.

1.2.46 Find the sum of first 51 terms of an AP whose second and third terms are 14 and 18 respectively.

1.2.47 If the sum of first 7 terms of an AP is 49 and that of 17 terms is 289, find the sum of first $n$ terms.

1.2.48 Show that $x(0), x(1), x(2), \ldots, x(n), \ldots$ form an AP where $x(n)$ is defined as below

a) $x(n) = 3 + 4n$

b) $x(n) = 9 - 5n$

Also find the sum of the first 15 terms in each case.

1.2.49 Find the sum of the first 40 positive integers divisible by 6.

1.2.50 Find the sum of the first 15 multiples of 8.

1.2.51 Find the sum of the odd numbers between 0 and 50.

1.2.52 A contract on construction job specifies a penalty for delay of completion beyond a certain date as follows: ₹200 for the first day, ₹250 for the second day, ₹300 for the third day, etc., the penalty for each succeeding day being ₹50 more than for the preceding day. How much money the contractor has to pay as penalty, if he has delayed the work by 30 days?

1.2.53 A sum of ₹700 is to be used to give seven cash prizes to students of a school for their overall academic performance. If each prize is ₹20 less than its preceding prize, find the value of each of the prizes.

1.2.54 In a school, students thought of planting trees in and around the school to reduce air pollution. It was decided that the number of trees, that each section of each class will plant, will be the same as the class, in which they are studying, e.g., a section of Class I will plant 1 tree, a section of Class II will plant 2 trees and so on till Class XII. There are three sections of each class. How many trees will be planted by the students?

1.2.55 A spiral is made up of successive semicircles, with centres alternately at A and B, starting with centre at A, of radii $0.5cm, 1.0cm, 1.5cm, 2.0cm, \ldots$ as shown in Fig. 1.2.1 What is the total length of such a spiral made up of thirteen consecutive 22 semicircles? (Take $\pi = \frac{22}{7}$)

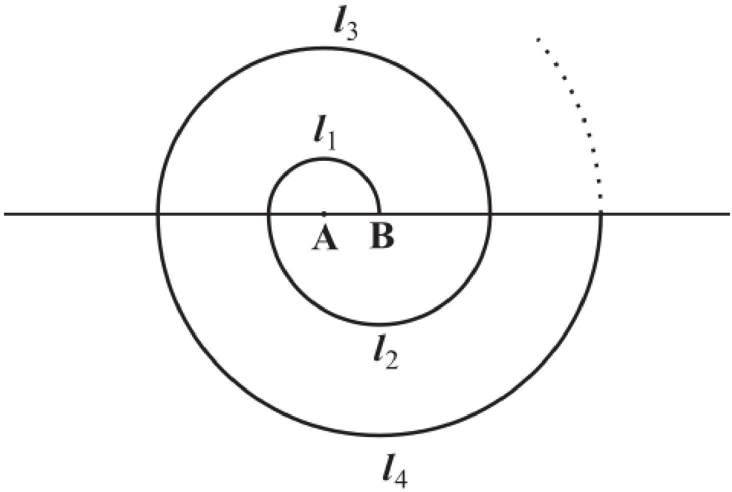


Fig. 1.2.1

*Hint:* Length of successive semicircles is $l_1, l_2, l_3, l_4, \ldots$ with centres at A, B, A, B, $\ldots$, respectively.

1.2.56 200 logs are stacked in the following manner: 20 logs in the bottom row, 19 in the next row, 18 in the row next to it and so on (see Fig 1.2.2 ). In how many rows are the 200 logs placed and how many logs are in the top row?

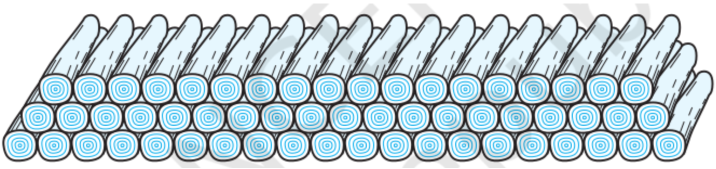

Fig. 1.2.2

1.2.57 In a potato race, a bucket is placed at the starting point, which is $5m$ from the first potato, and the other potatoes are placed $3m$ apart in a straight line. There are ten potatoes in the line as shown in Fig. 1.2.3.


Fig. 1.2.3

A competitor starts from the bucket, picks up the nearest potato, runs back with it, drops it in the bucket, runs back to pick up the next potato, runs to the bucket to drop it in, and she continues in the same way until all the potatoes are in the bucket. What is the total distance the competitor has to run? [*Hint:* To pick up the first potato and the second potato, the total distance (in metres) run by a competitor is $2 \times 5 + 2 \times (5 + 3)$].

1.2.58 Which term of the AP : $121, 117, 113, \ldots$ is its first negative term? [*Hint:* Find $n$ for $x(n) < 0$]

1.2.59 The sum of the third and the seventh terms of an AP is 6 and their product is 8. Find the sum of first sixteen terms of the AP.

1.2.60 The houses of a row are numbered consecutively from 1 to 49. Show that there is a value of $x$ such that the sum of the numbers of the houses preceding the house numbered $x$ is equal to the sum of the numbers of the houses following it. Find this value of $x$.[*Hint:* $y(x-1) = y(49) - y(x)$]

1.2.61 A small terrace at a football ground comprises of 15 steps each of which is $50m$ long and built of solid concrete. Each step has rise of $\frac{1}{4}m$ and a tread of $\frac{1}{2}m$ (see Fig. 1.2.4). Calculate the total volume of concrete required to build the terrace. [*Hint:* Volume of concrete required to build the first step $= \frac{1}{4} \times \frac{1}{2} \times 50m^3$]

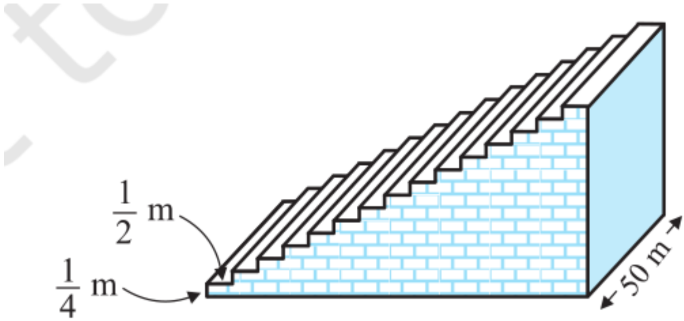


Fig. 1.2.4

1.2.62 Find the sum to $n$ terms of the following series

a) $x(n) = 2n + 5$ b) $x(n) = \frac{n-3}{4}$ c) $x(n) = \frac{2n-3}{6}$ d) $x(n) = 4n - 3$

1.2.63 Find the sum of all natural numbers lying between 100 and 1000, which are multiples of 5.

1.2.64 In an AP, the first term is 2 and the sum of the first five terms is one-fourth of the next five terms. Show that $20^{th}$ term is -112.

1.2.65 How many terms of the AP $-6, -\frac{11}{2}, -5, \ldots$ are needed to give the sum -25?

1.2.66 If the sum of a certain number of terms of the AP: $25, 22, 19, \ldots$ is 116, find the last term.

1.2.67 Insert five numbers between 8 and 26 such that the resulting sequence is an AP

1.2.68 A man starts repaying a loan as first instalment of ₹100. If he increases the instalment by Rs 5 every month, what amount he will pay in the $30^{th}$ instalment?

1.2.69 The difference between any two consecutive interior angles of a polygon is 5°. If the smallest angle is 120°, find the number of the sides of the polygon.

1.2.70 If the sum of three numbers in AP is 24 and their product is 440, find the numbers.

1.2.71 Find the sum of all numbers between 200 and 400 which are divisible by 7.

1.2.72 Find the sum of integers from 1 to 100 that are divisible by 2 or 5.

1.2.73 The sum of the first four terms of an AP is 56. The sum of the last four terms is 112. If its first term is 11, then find the number of terms.

1.2.74 The sum of $n$ terms of two arithmetic progressions are in the ratio $(3n + 8) : (7n + 15)$. Find the ratio of their $12^{th}$ terms.

1.2.75 The income of a person is ₹3,00,000 in the first year and he receives an increase of ₹10,000 to his income per year for the next 19 years. Find the total amount he received in 20 years.

1.2.76 Insert 6 numbers between 3 and 24 such that the resulting sequence is an AP.

1.2.77 Two APs have the same common difference. The difference between their 100th terms is 100, what is the difference between their 1000th terms?

1.2.78 Find the sum of odd integers from 1 to 2001.

1.2.79 Find the sum of all two digit numbers which when divided by 4, yield 1 as remainder.

1.2.80 A farmer buys a used tractor for Rs 12000. He pays Rs 6000 cash and agrees to pay the balance in annual instalments of Rs 500 plus 12% interest on the unpaid amount. How much will the tractor cost him?

1.2.81 Shyam Anand buys a scooter for Rs 22000. He pays Rs 4000 cash and agrees to pay the balance in annual instalment of Rs 1000 plus 10% interest on the unpaid amount. How much will the scooter cost him?

1.2.82 A man deposited Rs 10000 in a bank at the rate of 5% simple interest annually. Find the amount in the $15^{th}$ year since he deposited the amount and also calculate the total amount after 20 years.

### *1.3 CBSE*

1.3.1 In an AP, if $d = 2, n = 5$ and $a_n = 0$, then value of $a$ is (10, 2011)

a) 10 b) 5 c) -8 d) 8

1.3.2 Find whether -150 is a term of the AP: $17, 12, 7, 2, \ldots$? (10, 2011)

1.3.3 Find the value of the middle term of the following AP: $-6, -2, 2, \ldots, 58$. (10, 2011)

1.3.4 Determine the AP whose fourth term is 18 and the difference of the ninth term from the fifteenth term is 30. (10, 2011)

1.3.5 Find how many two-digit numbers are divisible by 6. How many multiples of 4 lie between 10 and 250 ? Also find their sum. (10, 2011)

1.3.6 The ratio of the $11^{\text{th}}$ term to $17^{\text{th}}$ term of an AP is 3 : 4. Find the ratio of $5^{\text{th}}$ to $21^{\text{th}}$ of the same AP. Also, find the ratio of the sum of first 5 terms to that of first 21 terms (10, 2023)

1.3.7 250 logs are stacked in the following manner: 22 logs in the bottom row, 21 in the next row, 20 in the row next to it and so on. In how many rows are the 250 logs placed and how many logs are there in top row ? (10, 2023)

1.3.8 If $-\frac{5}{7}$, $a$, 2 are consecutive terms in an Arthimetic Progression, then the value of $a$ is (10, 2022)

a) $\frac{9}{7}$ b) $\frac{9}{14}$ c) $\frac{19}{7}$ d) $\frac{19}{14}$

1.3.9 Find the sum of first 16 terms of an Arithmetic Progression whose $4^{\text{th}}$ and $9^{\text{th}}$ terms are $-15$ and $-30$ respectively. (10, 2022)

1.3.10 If the sum of first 14 terms of an Arithmetic Progression is 1050 and its fourth term is 40, find its $20^{\text{th}}$ term. (10, 2022)

1.3.11 Find the sum of the first twelve 2-digit numbers which are multiples of 6. (10, 2022)

1.3.12 In an AP, if $a_2 = 26$ and $a_{15} = -26$, then write the AP. (10, 2022)

1.3.13 In Mathematics, relations can be expressed in various ways. The matchstick patterns are based on linear relations. Different strategies can be used to calculate the number of matchsticks used in different Fig. 1.3.1 One such pattern is shown below. Observe the pattern and answer the following questions using Arithmetic Progression (10, 2022)

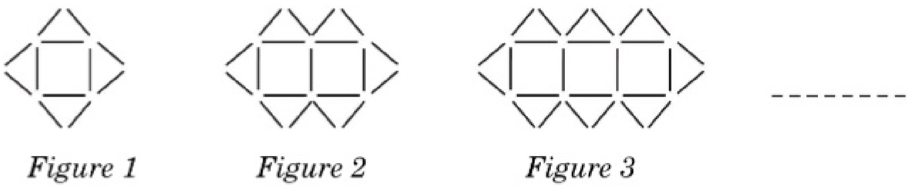


Fig. 1.3.1

a) Write the AP for the number of triangles used in the Fig. 1.3.1. Also, write the nth term of this AP.
b) Which figure has 61 matchsticks ?

1.3.14 In an AP if the sum of third and seventh term is zero, find its $5^{\text{th}}$ term. (10, 2022)

1.3.15 Determine the AP whose third term is 5 and seventh term is 9. (10, 2022)

1.3.16 Find the sum of the first 20 terms of an AP whose $n^{\text{th}}$ term is given as $a_n = 5 - 2n$ (10, 2022)

1.3.17 Find the common difference $d$ of an AP whose first term is 10 and the sum of the first 14 terms is 1505. (10, 2022)

1.3.18 For what value of $n$, are the $n^{\text{th}}$ terms of the APs: $9, 7, 5, \ldots$ and $15, 12, 9, \ldots$ the same? (10, 2022)

1.3.19 Write the common difference of the AP: $\frac{1}{5}, \frac{4}{5}, \frac{7}{5}, \frac{10}{5}, \cdots$ (10, 2021)

1.3.20 Find the $8^{th}$ term of the AP whose first term is $-2$ and common difference is 3. (10, 2021)

1.3.21 Roshini being a plant lover decides to start a nursery. She bought few plants with pots. She placed the pots in such a way that the number of pots in the first row is 2, in the second is 5, in the third row is 8 and so on as shown in Fig. 1.3.2.

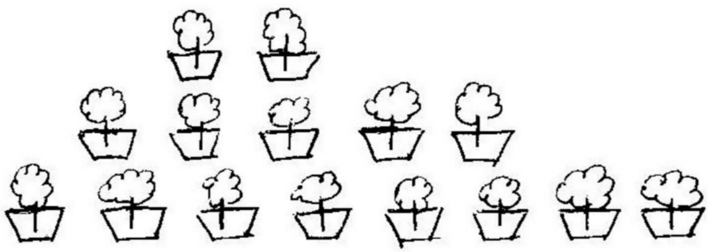

Fig. 1.3.2

Based on the above, answer the following questions (10, 2021)

a) How many pots were placed in the $7^{th}$ row ?

i) 20 ii) 23 iii) 77 iv) 29

b) If Roshini wants to place 100 pots in total, then total number of rows formed in the arrangement will be ?

i) 8 ii) 9 iii) 10 iv) 12

c) How many pots are placed in the last row ?

i) 20 ii) 23 iii) 26 iv) 29

d) If Roshini has sufficient space for 12 rows, then how many total number of pots are placed by her wih the same arrangement ?

i) 222 ii) 155 iii) 187 iv) 313

1.3.22 The sum of the first 4 terms of an AP is zero and its $4^{th}$ term is 2. Find the AP. (10, 2021)

1.3.23 If the sum of the first $n$ terms of an AP is given by $S_n = 4n - n^2$, then find its $n^{th}$ term. Hence, find the $25^{th}$ term and the sum if the first 25 terms of this AP. (10, 2021)

1.3.24 Find the mean of first 10 composite numbers. (10, 2021)

1.3.25 If $S_n$ denotes the sum of first $n$ terms of an AP, prove that $S_{12} = 3(S_8 - S_4)$. (10, 2021)

1.3.26 After how many decimal places will the decimal expansion of the rational number $\frac{14587}{1250}$ terminate ? (10, 2021)

1.3.27 If the $6^{th}$ and $14^{th}$ terms of an AP are 29 and 69 respectively, then find the $10^{th}$ term of the AP. (10, 2021)

1.3.28 If the first three consecutive terms of an AP are $3y - 1, 3y + 5$ and $5y + 1$ find the value of $y$. (10, 2021)

1.3.29 Which of the following is not an AP? (10, 2020)

a) $-1.2, 0.8, 2.8, \ldots$

b) $3, 3 + \sqrt{2}, 3 + 2\sqrt{2}, 3 + 3\sqrt{2}, \ldots$

c) $\frac{4}{3}, \frac{7}{3}, \frac{9}{3}, \frac{12}{3}, \ldots$

d) $\frac{-1}{5}, \frac{-2}{5}, \frac{-3}{5}, \ldots$

1.3.30 Find the sum of the first 100 natural numbers. (10, 2020)

1.3.31 Find the sum (10, 2020)

$$(-5) + (-8) + (-11) + \cdots + (-230)$$

1.3.32 Find the number of terms in the AP $18, 15\frac{1}{2}, 13, \ldots, 47$. (10, 2019)

1.3.33 Determine the AP whose third term is 16 and $7^{th}$ term exceeds the $5^{th}$ term by 12. (10, 2019)

1.3.34 Find the value of $x$, when in the AP given below (10, 2019)

$$2 + 6 + 10 + \cdots + x = 1800.$$

1.3.35 Which term of the AP $-4, -1, 2, \ldots$ is 101? (10, 2019)

1.3.36 In an AP, the first term is $-4$, the last term is 29 and the sum of all its terms is 150. Find its common difference. (10, 2019)

1.3.37 Find the $21^{st}$ term of the AP $-4\frac{1}{2}, -3, -1\frac{1}{2}, \ldots$ (10, 2019)

1.3.38 Find the common difference of the AP (10, 2019)

$$\frac{1}{a}, \frac{3-a}{3a}, \frac{3-2a}{3a}, \ldots (a \neq 0)$$

1.3.39 Which term of the Arithmetic Progression $-7, -12, -17, -22, \ldots$ will be $-82$? Is $-100$ any term of the AP? Give reason for your answer. (10, 2019)

1.3.40 How many terms of the Arithmetic Progression $45, 39, 33, \ldots$ must be taken so that their sum is 180 ? Explain the double answer. (10, 2019)

1.3.41 Find after how many places of decimal the decimal form of the number $\frac{27}{2^3 5^4 3^2}$ will terminate. (10, 2019)

1.3.42 Find the sum of first 10 multiples of 6

(10, 2019)

1.3.43 If $m$ times the $m^{th}$ term of an Arithmetic Progression is equal to $n$ times its $n^{th}$ term and $m \neq n$, show that the $(m+n)^{th}$ term of the A.P is zero (10, 2019)

1.3.44 The sum of the first three numbers in an Arithmetic Progression is 18. If the product of the first and the third term is 5 times the common difference, find the three numbers.

1.3.45 Find the sum of all the two digit numbers which leave the remainder 2 when divided by 5. (10, 2019)

1.3.46 If in an AP, $a = 15$, $d = -3$ and $a_n = 0$, then find the value of $n$. (10, 2019)

1.3.47 If $S_n$, the sum of the first $n$ terms of an AP is given by $S_n = 2n^2 + n$, then find its $n^{th}$ term. (10, 2019)

1.3.48 If the $17^{th}$ term of an AP exceeds its $10^{th}$ term by 7, find the common difference.

(10, 2019)

1.3.49 If the sum of the first $p$ terms of an AP is $q$ and the sum of the first $q$ terms is $p$, then show that the sum of the first $(p+q)$ terms is $\{-(p+q)\}$. (10, 2019)

1.3.50 Write the common difference of the AP $\sqrt{3}$, $\sqrt{12}$, $\sqrt{27}$, $\sqrt{48}$ , ... (10, 2019)

1.3.51 In an AP, the $n^{th}$ term is $\frac{1}{m}$ and the $m^{th}$ term is $\frac{1}{n}$. Find (10, 2019)

a) $(mn)^{th}$ term,

b) sum of first $(mn)$ terms.

1.3.52 The first term of an AP is 3, the last term is 83 and the sum of all its terms is 903. Find the number of terms and the common difference of the AP. (10, 2019)

1.3.53 If the sum of first $n$ terms of an AP is $n^2$, then find its $10^{th}$term. (10, 2019)

1.3.54 Which term of the AP: $3, 15, 27, 39, \ldots$ will be 120 more than its 21st term?

(10, 2019)

1.3.55 If $S_n$, the sum of first $n$ terms of an AP is given by $S_n = 3n^2 - 4n$, find the $n^{th}$ term.

(10, 2019)

1.3.56 If the sum of first four terms of an AP is 40 and that of first 14 terms is 280. Find the sum of its first $n$ terms. (10, 2019)

1.3.57 In an AP , if the common difference $d = -4$, and the seventh term $a_7$ is 4, then find the first term. (10, 2018)

1.3.58 The sum of four consecuive numbers in an AP is 32 and the ratio of the product of the first and the last term to the product of two middle term is 7 : 15. Find the numbers. (10, 2018)

1.3.59 Find the sum of 8 multiples of 3. (10, 2018)

1.3.60 In an AP , if the common difference $d = -4$, and the seventh term $a_7$ is 4, then find the first term. (10, 2018)

1.3.61 The sum of four consecuive numbers in an AP is 32 and the ratio of the product of the first and the last term to the product of two middle term is 7 : 15. Find the numbers. (10, 2018)

1.3.62 Find the sum of 8 multiples of 3. (10, 2018)

1.3.63 The $5^{th}$ and $15^{th}$ terms of an AP are 13 and $-17$ respectively. Find the sum of first 21 terms of the AP. (10, 2018)

1.3.64 The sum of the first $n$ terms of an AP is $5n^2 + 3n$. If its $m^{th}$ terms is 168, find the value of $m$. Also find the $20^{th}$ term of the AP. (10, 2018)

1.3.65 The $4^{th}$ and the last terms of an AP are 11 and 89 respectively. If there are 30 terms in the AP, find the A.P and its $23^{rd}$ term. (10, 2018)

1.3.66 Write the $m^{th}$ term of the AP

$$\frac{1}{k}, \frac{1+k}{k}, \frac{1+2k}{k}, \ldots$$

(10, 2018)

1.3.67 Which term of the AP: $8, 14, 20, 26, \ldots$ will be 72 more than its $41^{st}$ term ? (10, 2017)

1.3.68 If the $10^{th}$ term of an AP is 52 and the $17^{th}$ term is 20 more than the $13^{th}$ term, find the AP (10, 2017)

1.3.69 If the ratio of the sum of the first n terms of two A.Ps is $(7n + 1) : (4n + 27)$, then find the ratio of their $9^{th}$ terms. (10, 2017)

1.3.70 For what value of $n$, are the $n^{th}$ terms of two APs $63, 65, 67, \ldots$ and $3, 10, 17, \ldots$ equal? (10, 2017)

1.3.71 How many terms of an AP: $9, 17, 254, \ldots$ must be taken to give a sum of 636? (10, 2017)

1.3.72 What is the common difference of an A.P in which $a_{21} - a_7 = 84$ ? (10, 2017)

1.3.73 Which term of the progression $20, 19\frac{1}{4}, 18\frac{1}{2}, 17\frac{3}{4}, \ldots$ is the first negative term ? (10, 2017)

1.3.74 The first term of an AP is 5, the last term is 45 and the sum of all its terms is 400. Find the number of terms and the common difference of the AP. (10, 2017)

1.3.75 For what value of $k$ will $k + 9, 2k - 1$ and $2k + 7$ are the consecutive terms of an AP ? (10, 2016)

1.3.76 The $4^{th}$ term of an AP is zero. Prove that the $25^{th}$ term of the AP is three times its $11^{th}$ term. (10, 2016)

1.3.77 If the ratio of the sum of first $n$ terms of two APs is $(7n + 1) : (4n + 27)$, find the ratio of their $m^{th}$ terms. (10, 2016)

1.3.78 The sums of first $n$ terms of three arithmetic progressions are $S_1, S_2$ and $S_3$ respectively. The first term of each AP is 1 and their common differences are $1, 2$ and 3 respectively. Prove that $S_1 + S_3 = S_2$.
(10, 2016)

1.3.79 The digits of a positive number of three digits are in AP and their sum is 15. The number obtained by reversing the digits is 594 less than the original number. Find the number. (10, 2016)

1.3.80 The houses in a row are numbered consecutively from 1 to 49. Show that there exists a value of $X$ such that sum of numbers of houses preceeding the house numbered $X$ is equal to sum of the numbers of houses following $X$. (10, 2016)

1.3.81 In an AP, if $S_5 + S_7 = 167$ and $S_10 = 235$, then find the AP, where $S_n$ denotes the sum of its first n terms. (10, 2015)

1.3.82 The $14^{th}$ term of an AP is twice its $8^{th}$ term. If its $6^{th}$ term is – 8, then find the sum of its first 20 terms. (10, 2015)

1.3.83 Find the $60^{th}$ term of the AP $8, 10, 12, \ldots,$ if it has a total of 60 terms and hence find the sum of its last 10 terms. (10, 2015)

1.3.84 The $16^{th}$ term of an AP is five times its third term. If its $10^{th}$ term is 41, then find the sum of its first fifteen terms. (10, 2015)

1.3.85 An arithmetic progression $5, 12, 19, \ldots$ has 50 terms. Find its last term. Hence find the sum of its last 15 terms. (10, 2015)

1.3.86 The $13^{th}$ term of an AP is four times its $3^{rd}$ term. If its fifth term is 16, then find the sum of its first ten terms. (10, 2015)

1.3.87 If the $n^{th}$ term of an AP is $(2n + 1)$, then sum of its first three terms is (10, 2012)

a) $6n + 3$ b) 15 c) 12 d) 21

1.3.88 The next term of AP: $\sqrt{18}, \sqrt{50}, \sqrt{98}, \ldots$ is (10, 2012)

a) $\sqrt{146}$ b) $\sqrt{128}$ c) $\sqrt{162}$ d) $\sqrt{200}$

1.3.89 Find the common difference of an AP whose first term is 5 and the sum of its first four terms is half the sum of the next four terms. (10, 2012)

1.3.90 The $17th$ term of an AP is 5 more than twice its $8^{th}$ term. If the $11th$ term of the AP is 43, then find the $n^{th}$ term. (10, 2012)

1.3.91 Sum of the first 14 terms of an AP is 1505 and its first term is 10. Find its $25^{th}$ term. (10, 2012)

1.3.92 In an AP , the first term is 12 and the common difference is 6. If the last term of the AP is 252, find its middle term. (10, 2012)

1.3.93 If 4 times the fourth term of an AP is equal to 18 times its $18^{th}$ term, then find its $22^{th}$ term. (10, 2012)

1.3.94 The sum of $4^{th}$ and $8^{th}$ term terms of an AP is 24 and the sum of its $6^{th}$ and $10^{th}$ terms is 44. Find the sum of first ten terms of the AP (10, 2012)

1.3.95 In an AP, if $d = 2, n = 5$ and $a_n = 0$, then value of $a$ is (10, 2011)

a) 10 b) 5 c) -8 d) 8

1.3.96 Find whether -150 is a term of the AP: $17, 12, 7, 2, \ldots$? (10, 2011)

1.3.97 Find the value of the middle term of the following AP: $-6, -2, 2, \ldots, 58$. (10, 2011)

1.3.98 Determine the AP whose fourth term is 18 and the difference of the ninth term from the fifteenth term is 30. (10, 2011)

1.3.99 Assertion ($A$): $a, b, c$ are in AP if and if only if $2b = a + c$. Reason($R$): The sum of first $n$ natural numbers is $n^2$. (10, 2023)

a) Both Assertion ($A$) and Reason ($R$) are true and Reason ($R$) is the correct explanation of Assertion ($A$).
b) Both Assertion ($A$) and Reason ($R$) are true and Reason ($R$) is not the correct explanation of Assertion ($A$).
c) Assertion ($A$) is true but Reason ($R$) is false.
d) Assertion ($A$) is false but Reason ($R$) is true.

1.3.100 How many terms are there in AP whsoe first and fifth term are $-14$ and 2, respectively and the last term is 62. (10, 2023)

1.3.101 Which term of the AP: $65, 61, 57, 53, \ldots$ is the first negative term? (10, 2023)

1.3.102 Three bells ring at intervals of 6, 12 and 18 minutes. If all the three bells rang at 6 a.m., when will they ring together again ? (10, 2023)

1.3.103 How many terms of the arithmetic progression $45, 39, 33, \ldots$ must be taken so that their sum is 180? Explain the double answer. (10, 2023)

1.3.104 For what value of $k$ will $(k+9), (2k-1)$ and $(2k+7)$ be consecutive terms of an AP? (10, 2016)

1.3.105 The sums of the first $n$ terms of three arithmetic progressions are $S_0, S_2$ and $S_3$ respectively. The first term of each AP is 1 and their common differences are $1, 2$ and 3 respectively. Prove that $S_1 + S_3 = 2S_2$. (10, 2016)

1.3.106 The $5^{\text{th}}$ term of an Arithmetic Progression (AP) is 26 and the 10th term is 51. Determine the $15^{\text{th}}$ term of the AP. (10, 2006)

1.3.107 Find the sum of all the natural numbers less than 100 which are divisible by 6. (10, 2006)

### *1.4 JEE*

1.4.1 The sum of integers from 1 to 100 that are divisible by 2 or 5 is ______. (1984)

1.4.2 A pack contains $n$ cards numbered from 1 to $n$, two consecutive numbered cards are removed from the pack and then the sum of the numbers on the remaining cards is 1224. If the smaller of the numbers on the removed cards is $k$, then $k-20 =$ ______. (2013)

1.4.3 Suppose all the numbers of an (AP) are natural numbers. If the ratio of the sum of the first seven terms to the sum of the first eleven terms is 6 : 11 and the seventh term lies between 130 and 140, then the common difference of the AP is ______. (2015)

1.4.4 The sides of a right angled triangle are in AP . If the triangle has area 24, what is the length of its smallest side? (2018)

1.4.5 Let $X$ be the set consisting of the first 2018 terms of the AP 1, 6, 11, . . . and $Y$ be the set consisting of the first 2018 terms of the AP 9, 16, 23, . . . Then, the number of elements in the set $X \cup Y$ is ______. (2018)

1.4.6 Let AP $(a; d)$ denote the set of all the terms of an infinite AP with the first term $a$ and the common difference $d > 0$. If

$$AP(1; 3) \cap AP(2; 5) \cap AP(3; 7) = AP(a; d)$$

then $a + d$ equals ______. (2019)

1.4.7 A person is to count 4500 currency notes. Let $a_n$ denote the number of notes he counts in the $n^{th}$ minute. If $a_1 = a_2 = \cdots = a_{10} = 150$ and $a_{10}, a_{11}, \ldots$ are in an AP with common difference $-2$, then the time taken by him to count all notes is (2010)

a) 34 minutes b) 125 minutes c) 135 minutes d) 24 minutes

1.4.8 A man saves ₹200 in each of the first three months of his service. In each of the subsequent months his saving increases by ₹40 more than the saving of immediately previous month. His total saving from the start of service will be ₹11040 after (2011)

a) 19 months b) 20 months c) 21 months d) 18 months

1.4.9 Let $a_1, a_2, \ldots, a_{30}$ be an AP. $S = \sum_{i=1}^{30} a_i$ and $T = \sum_{i=2}^{15} a_{(2i-1)}$. If $a_5 = 27$ and $S - 2T = 75$, then $a_{10}$ is equal to (2019)

a) 52 b) 57 c) 47 d) 42

1.4.10 Let the sum of the first $n$ terms of a non-constant AP: $a_1, a_2, a_3, \ldots$ be $50n + \frac{n(n-7)}{2}A$, where $A$ is a constant. If $d$ is the common difference of this AP then ordered pair $(d,\ a_{50})$ is equal to (2019)

a) $(50,\ 50 + 46A)$ b) $(50,\ 50 + 45A)$ c) $(A,\ 50 + 45A)$ d) $(A,\ 50 + 46A)$

1.4.11 Let $a_1, a_2, \ldots a_{10}$ be in AP and $h_1, h_2, \ldots h_{10}$ be in HP. If $a_1 = h_1 = 2$ and $a_{10} = h_{10} = 3$, then $a_4 h_7$ is (1992)

a) 2 b) 3 c) 5 d) 6

1.4.12 The harmonic mean of the roots of the equation

$$(5 + \sqrt{2})x^2 - (4 + \sqrt{5})x + 8 + 2\sqrt{5} = 0$$

is (1999)

a) 2 b) 4 c) 6 d) 8

1.4.13 If the sum of the first $2n$ terms of the AP: $2, 5, 8, \ldots,$ is equal to the sum of the first $n$ terms of the AP: $57, 59, 61, \ldots,$ then $n$ equals (2001)

a) 10 b) 12 c) 11 d) 13

1.4.14 The interior angles of a polygon are in AP . The smallest angle is $120°$ and the common difference is $5°$. Find the number of sides of the polygon. (1980)

1.4.15 Five numbers are in AP, whose sum is 25 and product is 2520. If one of these five numbers is $-\frac{1}{2}$, then the greatest number among them is (2020)

a) $\frac{21}{2}$ b) 27 c) 16 d) 7

1.4.16 Let $l_1, l_2, \ldots, l_{100}$ be consecutive terms of an AP with common difference $d_1$, and let $w_1, w_2, \ldots, w_{100}$ be consecutive terms of another AP with common difference $d_2$, where $d_1 d_2 = 10$. For each $i = 1, 2, \ldots, 100$, let $R_i$ be a rectangle with length $L_i$, width $W_i$ and area $A_i$. If $A_{51} - A_{50} = 1000$, then the value of $A_{100} - A_{90}$ is ______. (2022)

1.4.17 Let $7\underbrace{555\cdots5}_{r\text{ times}}7$ denote the $(r+2)$-digit number where the first and the last digits are 7, and the remaining $r$ digits are 5. Consider the sum

$$S = 77 + 757 + 7557 + \cdots + 7\underbrace{555\cdots5}_{98\text{ times}}7.$$

If $S = \frac{7\overbrace{555\cdots5}^{99\text{ times}}7+m}{n}$ where $m$ and $n$ are natural numbers less than 3000, then the value of $m+n$ is ______. (2023)

1.4.18 If the sum of the first 40 terms of the series: 3 + 4 + 8 + 9 + 13 + 14 + 18 + 19 + …is $102m$, then $m$ is equal to (2024)

a) 20 b) 5 c) 10 d) 25

1.4.19 If the $10^{\text{th}}$ term of an AP is $\frac{1}{20}$ and its $20^{\text{th}}$ term is $\frac{1}{10}$, then the sum of its first 200 terms is (2020)

a) $50\frac{1}{4}$ b) $100\frac{1}{2}$ c) 50 d) 100

# 2 Geometric Progression

## 2.1 Formulae

2.1.1 The $n^{th}$ term of a GP is given by

$$x(n) = x(0)r^n, n = 0, 1, 2, \ldots \tag{2.1.1.1}$$

where $r$ is defined to be the *common ratio*.

2.1.2 Find the sum

$$S = 1 + 2 + 2^2 + 2^3 + 2^4 \tag{2.1.2.1}$$

**Solution:** Mulitpying the sum in (2.1.2.1) as

$$2S = 2 + 2^2 + 2^3 + 2^4 + 2^5 \tag{2.1.2.2}$$

and subtracting (2.1.2.1) from (2.1.2.2),

$$S = 2^5 - 1 \tag{2.1.2.3}$$

2.1.3 The sum of a GP is

$$y(n) = \sum_{k=0}^{n-1} x(k) = x(0)\left(\frac{1-r^n}{1-r}\right) \tag{2.1.3.1}$$

2.1.4 For an infinite GP,

$$\lim_{r\to\infty} y(n) = \left(\frac{x(0)}{1-r}\right), \quad |r| < 1 \tag{2.1.4.1}$$

## *2.2 NCERT*

2.2.1. Find the $20^{th}$ and $n^{th}$ terms of the GP: $\frac{5}{2}, \frac{5}{4}, \frac{5}{8}, \ldots,$
**Solution:**

$$x_0 = \frac{5}{2}, r = \frac{1}{2} \tag{2.2.1.1}$$

$$\implies x_{19} = \frac{5}{2}\left(\frac{1}{2}\right)^{19} = \frac{5}{2^{20}} \tag{2.2.1.2}$$

$$x_{n-1} = \frac{5}{2^n} \tag{2.2.1.3}$$

using (2.1.1.1).

2.2.2. The $4^{th}$ term of a GP is square of its second term, and the first term is -3. Determine its $7^{th}$ term.
**Solution:** From the given information,

$$x_3 = x_1^2, x_0 = -3 \tag{2.2.2.1}$$

$$\implies x_0 r^3 = x_0^2 r^2 \tag{2.2.2.2}$$

$$\text{or, } r = x_0 \tag{2.2.2.3}$$

$$\therefore x_6 = x_0^7 = (-3)^7 \tag{2.2.2.4}$$

2.2.3. Which term of the following sequences
a) 2, 2 $\sqrt{2}$, 4,. . . , is 128 ?
b) $\sqrt{3}, 3, 3\sqrt{3}, \ldots,$ is 729?
c) $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots,$ is $\frac{1}{19683}$?
**Solution:**
a)

$$x_0 = 2, r = \sqrt{2} \tag{2.2.3.1}$$

$$x_n = x_0 r^n = 128 \tag{2.2.3.2}$$

$$\implies 2\left(\sqrt{2}\right)^n = 128 \tag{2.2.3.3}$$

$$\text{or, } n = 2\left(\frac{\log 128}{\log 2} - 1\right) = 12 \tag{2.2.3.4}$$

2.2.4. For what values of $x$, the numbers $-\frac{2}{7}, x, -\frac{7}{2}$ are in GP?

Find the sum to indicated number of terms in each of the geometric progressions.

2.2.5. 0.15, 0.015, 0.0015,..., 20 terms.

2.2.6. $\sqrt{7}$, $\sqrt{21}$, $\sqrt[3]{7}, \ldots, n$ terms.

2.2.7. $1, -a, a^2, -a^2, a^3, \ldots, n$ terms $(a \neq -1)$.

2.2.8. $x^3, x^5, x^7, \ldots, n$ terms $(x \neq \pm 1)$.

2.2.9. Evaluate

$$\sum_{k=1}^{11} \left(2 + 3^k\right).$$

**Solution:** The summation can be expressed as

$$\sum_{k=1}^{11} 2 + \sum_{k=1}^{11} 3^k = 2 \times 11 + \sum_{k=0}^{10} 3^{k+1} \tag{2.2.9.1}$$

$$= 22 + 3\sum_{k=0}^{10} 3^k = 22 + 3\left(\frac{3^{11} - 1}{2}\right) \tag{2.2.9.2}$$

2.2.10. The sum of first three terms of a GP is $\frac{39}{10}$ and their product is 1. Find the common ratio and the terms.

**Solution:**

$$x_0 \left(1 + r + r^2\right) = \frac{39}{10} \tag{2.2.10.1}$$

$$x_0^3 r^3 = 1 \implies x_0 r = 1 \tag{2.2.10.2}$$

$$\therefore 10\left(1 + r + r^2\right) = 39r \tag{2.2.10.3}$$

yielding the quadratic

$$10r^2 - 29r + 10 = 0 \tag{2.2.10.4}$$

$$\implies (5r - 2)(2r - 5) = 0 \tag{2.2.10.5}$$

$$\text{or, } r = \frac{5}{2}, \frac{2}{5} \tag{2.2.10.6}$$

2.2.11. How many terms of GP $3, 3^2, 3^3, \ldots,$ are needed to give the sum 120?

2.2.12. The sum of first three terms of a GP is 16 and the sum of the next three terms is 128. Determine the first term, the common ratio and the sum to $n$ terms of the GP

2.2.13. Given a GP with $a = 729$ and $7^{th}$ term 64, determine $y_6$.

2.2.14. The number of bacteria in a certain culture doubles every hour. If there were 30 bacteria present in the culture originally, how many bacteria will be present at the end of $2^{nd}$ hour, $4^{th}$ hour and $n^{th}$ hour ?

2.2.15. What will Rs 500 amount to in 10 years after its deposit in a bank which pays annual interest rate of 10% compounded annually?

2.2.16. If AM and GM of roots of a quadratic equation are 8 and 5, respectively, then obtain the quadratic equation.

**Solution:** If the roots are $a, b$,

$$\frac{a+b}{2} = 8, \implies a+b = 16, \tag{2.2.16.1}$$

$$\sqrt{ab} = 5 \implies ab = 25 \tag{2.2.16.2}$$

$$\therefore x^2 - 16x + 25 = 0 \tag{2.2.16.3}$$

is the desired quadratic equation.

2.2.17. The sum of some terms of GP is 315 whose first term and the common ratio are 5 and 2, respectively. Find the last term and the number of terms.

2.2.18. The first term of a GP is 1. The sum of the third term and fifth term is 90. Find the common ratio of the GP.

2.2.19. The sum of three numbers in GP is 56. If we subtract 1, 7, 21 from these numbers in that order, we obtain an arithmetic progression. Find the numbers.

2.2.20. A person writes a letter to four of his friends. He asks each one of them to copy the letter and mail to four different persons with instruction that they move the chain similarly. Assuming that the chain is not broken and that it costs 50 paise to mail one letter. Find the amount spent on the postage when $8^{th}$ set of letter is mailed.

2.2.21. A manufacturer reckons that the value of a machine, which costs him Rs. 15625, will depreciate each year by 20%. Find the estimated value at the end of 5 years.

2.2.22. 150 workers were engaged to finish a job in a certain number of days. 4 workers dropped out on second day, 4 more workers dropped out on third day and so on.It took 8 more days to finish the work. Find the number of days in which the work was completed.

2.2.23. Find the $10^{th}$ and $n^{th}$ and terms of the GP: $5, 25, 125, \ldots$

2.2.24. Which term of the GP: $2, 8, 32, \ldots$ upto $n$ terms is 131072.

2.2.25. In a GP the $3^{rd}$ term is 24 and the $6^{th}$ term is 192. Find the $10^{th}$ term.

2.2.26. Find the sum of the first $n$ terms and the sum of the first 5 terms of the series

$$1 + \frac{2}{3} + \frac{4}{9} + \ldots$$

2.2.27. How many terms of the GP:

$$3 + \frac{3}{2} + \frac{3}{4} + \ldots$$

are needed to give the sum $\frac{3069}{512}$.

2.2.28. The sum of the first 3 terms of a GP is $\frac{13}{12}$ and their product is -1. Find the common ratio and the terms.

2.2.29. A person has 2 parents, 4 grandparents, 8 great grand parents and so on. Find the number of his ancestors during the ten generations preceeding his own.

2.2.30. Insert 3 numbers between 1 and 256 so that the resulting sequence is a GP.

2.2.31. If the AM and GM of two positive numbers $a$ and $b$ are 10 and 8 respectively, find the numbers.

2.2.32. Find the $12^{th}$ term of a GP whose $8^{th}$ term is 192 and the common ratio is 2.

2.2.33. Find a GP for which sum of the first two terms is -4 and the fifth term is 4 times the third term.

2.2.34. Find four numbers forming a geometric progression in which the third term is greater than the first term by 9, and the second term is greater than the $4^{th}$ by 18.

## 2.3 JEE

2.3.1 Fifth term of a GP is 2, then the product of its 9 terms is (2002)

a) 256 b) 512 c) 1024 d) none of these

2.3.2 The first two terms of a geometric progression add up to 12. The sum of the third and the fourth terms is 48. If the terms of the geometric progression are alternately positive and negative, then the first term is (2008)

a) $-4$ b) $-12$ c) 12 d) 4

2.3.3 The third term of a geometric progression is 4. The product of five terms is (1982)

a) $4^3$ b) $4^5$ c) $4^4$ d) none of these

2.3.4 Consider an infinite geometric series with first term $a$ and common ratio $r$. If its sum is 4 and the second term is $3/4$, then (2000)

a) $a = 4/7, r = 3/7$ b) $a = 2, r = 3/8$ c) $a = 3/2, r = 1/2$ d) $a = 3, r = 1/4$

2.3.5 The harmonic mean of two numbers is 4. Their arithmetic mean $A$ and the geometric mean $G$ satisfy the relation $2A + G^2 = 27$. Find the two numbers. (1979)

2.3.6 Does there exist a geometric progression containing 27, 8 and 12 as three of its terms? If it exits, how many such progressions are possible? (1982)

2.3.7 Three positive numbers form an increasing GP. If the middle term in this GP is doubled, the new numbers are in AP. Then the common ratio of the GP is (2014)

a) $2 - \sqrt{3}$ b) $2 + \sqrt{3}$ c) $\sqrt{2} + \sqrt{3}$ d) $3 + \sqrt{2}$

2.3.8 An infinite GP has first term $x$ and sum 5 then $x$ belongs to (2004)

a) $x < -10$ b) $-10 < x < 0$ c) $0 < x < 10$ d) $x > 10$

2.3.9 Find three numbers $a, b, c$ between 2 and 18 such that

a) their sum is 25
b) the numbers $2, a, b$ are consecutive terms of an AP
and
c) the numbers $b, c, 18$ are consecutive terms of a GP (1983)

# 3 Z Transform

## 3.1 Formulae

3.1.1 The $Z$-transform of $x(n)$ is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{3.1.1.1}$$

3.1.2 The unit step function is defined as

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{3.1.2.1}$$

3.1.3 For a Geometric progression

$$x(n) = r^n u(n), \tag{3.1.3.1}$$

$$\implies X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=0}^{\infty} r^n z^{-n} \tag{3.1.3.2}$$

$$= \sum_{n=0}^{\infty} \left(rz^{-1}\right)^n \tag{3.1.3.3}$$

$$= \frac{1}{1 - rz^{-1}}, \quad |z| > |r| \tag{3.1.3.4}$$

from (2.1.4.1).

3.1.4 Substituting $r = 1$ in (3.1.3.4),

$$u(n) \overset{\mathcal{Z}}{\longleftrightarrow} U(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{3.1.4.1}$$

3.1.5 From (3.1.6.1) and (3.1.4.1),

$$nu(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}}{(1 - z^{-1})^2}, \quad |z| > 1 \tag{3.1.5.1}$$

3.1.6 From (3.1.6.1)

$$nx(n) \overset{\mathcal{Z}}{\longleftrightarrow} -zX'(z) \tag{3.1.6.1}$$

yielding

$$nu(n) \xleftrightarrow{\mathcal{Z}} \frac{z^{-1}}{\left(1-z^{-1}\right)^2}, |z| > 1 \tag{3.1.6.2}$$

$$n^2u(n) \xleftrightarrow{\mathcal{Z}} \frac{z^{-1}\left(z^{-1}+1\right)}{\left(1-z^{-1}\right)^3}, |z| > 1 \tag{3.1.6.3}$$

$$n^3u(n) \xleftrightarrow{\mathcal{Z}} \frac{z^{-1}\left(1+4z^{-1}+z^{-2}\right)}{\left(1-z^{-1}\right)^4}, |z| > 1 \tag{3.1.6.4}$$

$$n^4u(n) \xleftrightarrow{\mathcal{Z}} \frac{z^{-1}\left(1+11z^{-1}+11z^{-2}+z^{-3}\right)}{\left(1-z^{-1}\right)^5}, |z| > 1 \tag{3.1.6.5}$$

3.1.7 The convolution sum is defined as

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \tag{3.1.7.1}$$

3.1.8

$$x(n-k) \xleftrightarrow{\mathcal{Z}} z^{-k}X(z) \tag{3.1.8.1}$$

3.1.9 Using (3.1.8.1):

$$nu(n-1) \xleftrightarrow{\mathcal{Z}} z\frac{2z^{-2}}{\left(1-z^{-1}\right)^2} \tag{3.1.9.1}$$

Now ,

$$\frac{(n-1)}{2}u(n-2) \xleftrightarrow{\mathcal{Z}} \frac{z^{-2}}{\left(1-z^{-1}\right)^2} \tag{3.1.9.2}$$

$$\frac{(n-1)(n-2)}{6}u(n-3) \xleftrightarrow{\mathcal{Z}} \frac{z^{-3}}{\left(1-z^{-1}\right)^3} \tag{3.1.9.3}$$

$$\vdots$$

$$\frac{(n-1)(n-2)\dots(n-k+1)}{k!}u(n-k) \xleftrightarrow{\mathcal{Z}} \frac{z^{-k}}{\left(1-z^{-1}\right)^k} \tag{3.1.9.4}$$

3.1.10

$$\delta(n) \xleftrightarrow{\mathcal{Z}} 1 \tag{3.1.10.1}$$

$$\delta(n+k) \xleftrightarrow{\mathcal{Z}} z^k, \forall k \in \mathbb{R} \tag{3.1.10.2}$$

3.1.11 If

$$y(n) = x(n) * h(n), \tag{3.1.11.1}$$

$$Y(z) = X(z)H(z) \tag{3.1.11.2}$$

3.1.12 For $x(n) = 0, n < 0$,

$$\sum_{0}^{n} x(k) = x(n) * u(n) \tag{3.1.12.1}$$

**Solution:**

$$\sum_{k=-\infty}^{\infty} u(k)\, x(n-k) = \sum_{k=0}^{\infty} x(n-k) \tag{3.1.12.2}$$

$$= \sum_{k=-\infty}^{n} x(k) \tag{3.1.12.3}$$

yielding (3.1.12.1).

## 3.2 NCERT

3.2.1 Find the sum to $n$ terms of the series $n^2 + 2^n$

**Solution:** Let

$$x(n) = n^2 u(n) \tag{3.2.1.1}$$

$$\therefore X(z) = \frac{z^{-1}\left(1 + z^{-1}\right)}{\left(1 - z^{-1}\right)^3} \tag{3.2.1.2}$$

from (3.1.6.3). From (3.1.12.1) and (3.1.11.2),

$$\sum_{k=0}^{n} x(k) \stackrel{\mathcal{Z}}{\longleftrightarrow} X(z)U(z) \tag{3.2.1.3}$$

$$\sum_{k=1}^{n} n^2 \stackrel{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}\left(1 + z^{-1}\right)}{\left(1 - z^{-1}\right)^4} \tag{3.2.1.4}$$

From (3.1.9.4)

$$\frac{(n-1)(n-2)(n-3)}{3!} u(n-4) \stackrel{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-4}}{\left(1 - z^{-1}\right)^4} \tag{3.2.1.5}$$

$$\implies \sum_{k=1}^{n} n^2 = \frac{n(n+2)(n+1)}{3!} u(n-1) + \frac{n(n+1)(n-1)}{3!} u(n-2) \tag{3.2.1.6}$$

$$= \left\{ \tfrac{n(n+1)(2n+1)}{3!} u(n-1) \right. \tag{3.2.1.7}$$

after some algebra.

3.2.2 Find the sum to $n$ terms of the series: $5 + 11 + 19 + 29 + 41 + \ldots$

**Solution:** Let

$$y(n) = 5 + 11 + 19 + 29 + 41 + \cdots + x(n) \tag{3.2.2.1}$$

$$\implies y(n) = \qquad 5 + 11 + 19 + 29 + 41 + \cdots + x(n) \tag{3.2.2.2}$$

Substracting,

$$0 = 5 + 6 + 8 + 10 + \cdots - x(n) \tag{3.2.2.3}$$
$$\implies x(n) = 5 + 6 + 8 + 10 + \dots \tag{3.2.2.4}$$
$$= 2 + 4 + 6 + 8 + 10 + \cdots - 1 \tag{3.2.2.5}$$
$$= [(n+2)(n+3) - 1]\, u(n) \tag{3.2.2.6}$$

From (3.1.9.4)

$$\frac{(n+2)(n+3)}{(k-1)!} u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{\left(1 - z^{-1}\right)^3} \tag{3.2.2.7}$$
$$\implies X(z) = \frac{1}{\left(1 - z^{-1}\right)^3} - \frac{1}{1 - z^{-1}} \tag{3.2.2.8}$$

Therefore,

$$y(n) = \sum_{k=0}^{n-1} x(n) = x(n) * u(n-1) \tag{3.2.2.9}$$
$$\implies Y(z) = X(z)U(z)z^{-1} \tag{3.2.2.10}$$
$$= \left[\frac{1}{\left(1 - z^{-1}\right)^3} - \frac{1}{1 - z^{-1}}\right]\left(\frac{z^{-1}}{1 - z^{-1}}\right) \tag{3.2.2.11}$$
$$= \frac{z^{-1}}{\left(1 - z^{-1}\right)^4} - \frac{z^{-1}}{\left(1 - z^{-1}\right)^2} \tag{3.2.2.12}$$

Hence,

$$y(n) = \frac{n(n+1)(n+2)}{4!} u(n-1) - \frac{n(n-1)}{3!} u(n-2) \tag{3.2.2.13}$$

3.2.3 Find the sum to $n$ terms of the series whose $n^{th}$ term is $n(n+3)$.

3.2.4 Find the sum to $n$ terms of each of the series

a) $n(n+2)$
b) $n\left(\frac{n^2+5}{4}\right)$
c) $1 \times 2 + 2 \times 3 + 3 \times 4 + 4 \times 5 + \dots$
d) $1 \times 2 \times 3 + 2 \times 3 \times 4 + 3 \times 4 \times 5 + \dots$
e) $3 \times 1^2 + 5 \times 2^2 + 7 \times 3^2 + \dots$
f) $\frac{1}{1\times2} + \frac{1}{2\times3} + \frac{1}{3\times4} + \dots$
g) $5^2 + 6^2 + 7^2 + \cdots + 20^2$
h) $3 \times 8 + 6 \times 11 + 9 \times 14 + \dots$
i) $1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + \dots$
j) $n(n+1)(n+4)$.
k) $(2n-1)^2$
l) $(n-1)(2-n)(3+n)$

3.2.5 Find the sum to $n$ terms of the following series

a) $a_n = (-1)^{n-1} 5^{n+1}$
b) $a_n = (-1)^{n-1} n^3$
c) $a_n = \frac{n^2}{2^n}$

3.2.6 Obtain the closed form expression for the following

a) $a_1 = 1, a_n = a_{n-1} + 2 \quad \forall n > 1$
b) $a_1 = 3, a_n = 3a_{n-1} + 2 \quad \forall n > 1$
c) $a_1 = a_2 = 2, a_n = a_{n-1} - 1, n > 2$
d) The Pingala sequence, defined by

$$a_n = a_{n-1} + a_{n-2}, \quad n > 2, a_1 = a_2 = 1$$

3.2.7 Find the sum of the following series up to $n$ terms

$$\frac{1^3}{1} + \frac{1^3 + 2^2}{1+3} + \frac{1^3 + 2^3 + 3^3}{1+3+5} + \dots$$

3.2.8 Show that

$$\frac{1 \times 2^2 + 2 \times 3^2 + \dots + n \times (n+1)^2}{1^2 \times 2 + 2^2 \times 3 + \dots + n^2 \times (n+1)} = \frac{3n+5}{3n+1}.$$

3.2.9 Find the $20^{th}$ term of the series: $2 \times 4 + 4 \times 6 + 6 \times 8 + \dots$

3.2.10 Find the sum of the first $n$ terms of the series: 3 + 7 + 13 + 21 + 31 + ...,

3.2.11 If $S_1, S_2, S_3$ are the sum of first $n$ natural numbers, their squares and their cubes, respectively, show that

$$9S_2^2 = S_3 (1 + 8S_1).$$

3.2.12 Find the sum to $n$ terms of the sequence, $8, 88, 888, 8888, \dots$.

3.2.13 Find the sum of the products of the corresponding terms of the sequences 2, 4, 8, 16, 32, and 128, 32, 8, 2, $\frac{1}{2}$.

3.2.14 Find the sum of the following series up to $n$ terms

a) 5 + 55 +555 + ...,
b) .6 + .66 + .666+...

3.2.15 Find the sum of the sequence $7, 77, 777, \dots$ to $n$ terms.

### *3.3 JEE*

3.3.1 The sum of the first $n$ terms of the series $1^2 + 2.2^2 + 3^2 + 2.4^2 + 5^2 + 2.6^2 + \dots$ is $n(n+1)^2/2$, when $n$ is even. When $n$ is odd, the sum is... (1988)

3.3.2 For any odd integer $n \geq 1$, $n^3 - (n-1)^3 + (-1)^{n-1} 1^3 = \dots$ (1996)

3.3.3 Let $S_k, k = 1, 2, \dots, 100$, denote the sum of the infinite geometric series whose first term is $\frac{k-1}{k!}$ and the common ratio is $\frac{1}{k}$. Then the value of

$$\frac{100^2}{100!} + \sum_{k=1}^{100} \left| \left(k^2 - 3k + 1\right) S_k \right|$$

is ______. (2010)

3.3.4 let $a_1, a_2, a_3, \dots, a_{11}$ be real numbers satisfying $a_1 = 15, 27 - 2a_2 > 0$ and $a_k = 2a_{k-1} - a_{k-2}$ for $k = 3, 4 \dots 11$. If

$$\frac{a_1^2 + a_2^2 + \dots + a_{11}^2}{11} = 90$$

then the value of

$$\frac{a_1 + a_2 + \dots + a_{11}}{11}$$

is equal to ______. (2010)

3.3.5 The value of $2^{\frac{1}{4}} \cdot 4^{\frac{1}{8}} \cdot 8^{\frac{1}{16}} \dots \infty$ is (2002)

a) 1
b) 2
c) $\frac{3}{2}$
d) 4

3.3.6 Sum of infinite number of terms of a GP is 20 and sum of their squares is 100. The common ratio of GP is (2002)

a) 5
b) $\frac{3}{5}$
c) $\frac{8}{5}$
d) $\frac{1}{5}$

3.3.7 $1^3 - 2^3 + 3^3 - 4^3 + ... + 9^3 =$ (2002)

a) 425
b) −425
c) 475
d) −475

3.3.8 The sum of the first $n$ terms of the series $1^2 + 2 \cdot 2^2 + 3^2 + 2 \cdot 4^2 + 5^2 + 2 \cdot 6^2 + \cdots$ is $\frac{n(n+1)^2}{2}$ when $n$ is even. When $n$ is odd the sum is (2004)

a) $\left[\frac{n(n+1)}{2}\right]^2$
b) $\frac{n^2(n+1)}{2}$
c) $\frac{n(n+1)^2}{4}$
d) $\frac{3n(n+1)}{2}$

3.3.9 If $x = \sum_{n=0}^{\infty} a^n$, $y = \sum_{n=0}^{\infty} b^n$, $z = \sum_{n=0}^{\infty} c^n$ where $a, b, c$ are in AP and $|a| < 1, |b| < 1, |c| < 1$ then $x, y, z$ are in (2005)

a) GP
b) AP
c) Arithmetic - Geometric Progression
d) HP

3.3.10 The sum of first 9 terms of the series: $\frac{1^3}{1} + \frac{1^3+2^3}{1+3} + \frac{1^3+2^3+3^3}{1+3+5} + \dots$ (2015)

a) 142
b) 192
c) 71
d) 96

3.3.11 The sum to infinite term of the series $1 + \frac{2}{3} + \frac{6}{3^2} + \frac{10}{3^3} + \frac{14}{3^4} + \dots$ is (2009)

a) 3
b) 4
c) 6
d) 2

3.3.12 Statement-1 : The sum of the series

$$1 + (1 + 2 + 4) + (4 + 6 + 9) + (9 + 12 + 16) + \cdots + (361 + 380 + 400)$$

is 8000.

Statement-2 :

$$\sum_{k=1}^{n} \left(k^3 - (k-1)^3\right) = n^3,$$

for any natural number $n$. (2012)

a) Statement-1 is false, Statement-2 is true.

b) Statement-1 is true; Statement-2 is true; Statement-2 is a correct explanation for Statement-1

c) Statement-1 is true; Statement-2 is true; Statement-2 is not a correct explanation for Statement-1

d) Statement-1 is true; Statement-2 is false.

3.3.13 The sum of first 20 terms of the sequence $0.7, 0.77, 0.777, \ldots$ is (2013)

a) $\frac{7}{81}\left(179 - 10^{-20}\right)$ b) $\frac{7}{9}\left(99 - 10^{-20}\right)$ c) $\frac{7}{81}\left(179 + 10^{-20}\right)$ d) $\frac{7}{9}\left(99 + 10^{-20}\right)$

3.3.14 If $(10)^9 + 2(11)^1(10)^8 + 3(11)^2(10)^7 + \cdots + 10(11)^9 = k(10)^9$, then $k$ is equal to (2014)

a) 100 b) 110 c) $\frac{121}{10}$ d) $\frac{441}{100}$

3.3.15 If the sum of the first ten terms of the series $\left(1\frac{3}{5}\right)^2 + \left(2\frac{2}{5}\right)^2 + \left(3\frac{1}{5}\right)^2 + 4^2 + \left(4\frac{4}{5}\right)^2 + \ldots,$ is $\frac{16}{5}m$, then $m$ is equal to (2016)

a) 100 b) 99 c) 102 d) 101

3.3.16 If, for positive integer $n$, the quadratic equation,

$$x(x+1) + (x+1)(x+2) + \cdots + (x + \overline{n-1})(x+n) = 10n$$

has two consecutive integral solutions, then $n$ is equal to (2017)

a) 11 b) 12 c) 9 d) 10

3.3.17 Let $a_1, a_2, a_3, \ldots, a_{49}$ be an AP such that $\sum_{k=0}^{12} a_{4k+1} = 416$ and $a_9 + a_{43} = 66$. If $a_1^2 + a_2^2 + \cdots + a_{17}^2 = 140m$, then $m$ is equal to (2018)

a) 68 b) 34 c) 33 d) 66

3.3.18 Let $A$ be the sum of the frst 20 terms and $B$ be the sum of the first 40 terms of the series $1^2 + 2 \cdot 2^2 + 3^2 + 2 \cdot 4^2 + 5^2 + 2 \cdot 6^2 + \ldots$. If $B - 2A = 100\lambda$, then $\lambda$ can be (2018)

a) 248 b) 464 c) 496 d) 232

3.3.19 Let

$$S_n = \sum_{k=1}^{4n} (-1)^{\frac{k(k+1)}{2}} k^2.$$

Then $S_n$ can take value (2013)

a) 1056 b) 1088 c) 1120 d) 1332

3.3.20 Let $\alpha$ and $\beta$ be the roots of $x^2 - x - 1 = 0$, with $\alpha > \beta$. For all positive integers $n$, define

$$a_n = \frac{\alpha_n - \beta_n}{\alpha - \beta}, n \geq 2b_1 = 1 \text{ and } b_n = a_{n-1} + a_{n+1}, n \geq 1$$

Then which of the following options is/are correct? (2019)

a) $\sum_{n=1}^{\infty} \frac{a_n}{10^n} = \frac{10}{89}$
b) $B_n = a^n + b^n \; \forall \; n \geq 1$
c) $a_1 + a_2 + a_3 + \cdots + a_n = a_{n+2} - 1 \forall n \geq 1$
d) $\sum_{n=1}^{\infty} \frac{b_n}{10^n} = \frac{8}{89}$

3.3.21 The rational number, which equals the number 2.357 with recurring decimal is (1983)

a) $\frac{2355}{1001}$ b) $\frac{2379}{997}$ c) $\frac{2355}{999}$ d) none of these

3.3.22 Sum of the first $n$ terms of the series

$$\frac{1}{2} + \frac{3}{4} + \frac{7}{8} + \frac{15}{16} + \ldots$$

is equal to (1998)

a) $2^n - n - 1$ b) $1 - 2^{-n}$ c) $n + 2^{-n} - 1$ d) $2^n + 1$

3.3.23 If $a_n = \frac{3}{4} - \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + \ldots (-1)^{n-1} \left(\frac{3}{4}\right)^n$ and $b_n = 1 - a_n$, then find the least natural number $n_0$ such that $b_n > a_n \forall n \geq n_0$. (2006)

3.3.24 In a geometric progression consisting of positive terms, each term equals the sum of the next two terms. Then the common ratio of its progression equals (2007)

a) $\sqrt{5}$ b) $\frac{1}{2}\left(\sqrt{5} - 1\right)$ c) $\frac{1}{2}\left(1 - \sqrt{5}\right)$ d) $\frac{1}{2}\sqrt{5}$

3.3.25 If $S_1, S_2, S_3, \ldots, S_n$ are the sums of infinite geometric series whose first terms are $1, 2, 3, \ldots, n$ and whose common ratios are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n+1}$ respectively, then find the values of $S_1^2 + S_2^2 + S_3^2 + \cdots + S_{2n-1}^2$ (1991)

3.3.26 Let $a_1, a_2, a_3, \ldots$ be an arithmetic progression with $a_1 = 7$ and common difference 8. Let $T_1, T_2, T_3, \ldots$ be such that $T_1 = 3$ and $T_{n+1} - T_n = a_n$ for $n \geq 1$. Then, which of the following is/are TRUE? (2022)

a) $T_{20} = 1604$
b) $\sum_{K=1}^{20} T_k = 10510$
c) $T_{30} = 3454$
d) $\sum_{K=1}^{30} T_k = 357610$

3.3.27 If the sum of first $n$ terms of an AP is $cn^2$, then the sum of squares of these $n$ terms is (2009)

a) $\frac{n(4n^2-1)c^2}{6}$ b) $\frac{n(4n^2+1)c^2}{3}$ c) $\frac{n(4n^2-1)c^2}{3}$ d) $\frac{n(4n^2+1)c^2}{6}$

# 4 Miscellaneous

## 4.1 NCERT

4.1.1 In an AP, If $p^{th}$ term is $\frac{1}{q}, q^{th}$ term is $\frac{1}{p}$, prove that the sum of first $pq$ terms is $\frac{1}{2}(pq+1)$, where $p \neq q$.

4.1.2 Sum of the first $p, q$ and $r$ terms of an AP are $a, b$ and $c$, respectively. Prove that

$$\frac{a}{p}(q-r)+\frac{b}{q}(r-p)+\frac{c}{r}(p-q)=0$$

4.1.3 Find the sum to $n$ terms of the AP, whose $k^{th}$ term is $5k+1$.

4.1.4 If the sum of $n$ terms of an AP is $pn+qn^2$, where $p$ and $q$ are constants, find the common difference.

4.1.5 The sums of $n$ terms of two arithmetic progressions are in the ratio $5n+4 : 9n+6$. Find the ratio of their $18^{th}$ terms.

4.1.6 If the sum of first $p$ terms of an AP is equal to the sum of the first $q$ terms, then find the sum of the first $p+q$ terms.

4.1.7 The ratio of the sums of $m$ and $n$ terms of an AP is $m^2 : n^2$. Show that the ratio of $m^{th}$ and $n^{th}$ term is $(2m-1) : (2n-1)$.

4.1.8 If the sum of $n$ terms of an AP is $3n^2+5n$ and its $m^{th}$ term is 164, find the value of $m$.

4.1.9 Show that the sum of $(m+n)^{th}$ and $(m-n)^{th}$ terms of an AP is equal to twice the $m^{th}$ term.

4.1.10 Let the sum of $n, 2n, 3n$ terms of an AP be $y(1), y(2)$ and $y(3)$, respectively, show that

$$y(3)=3(y(2)-y(1))$$

4.1.11 The $p^{th}, q^{th}$ and $r^{th}$ terms of an AP are $a, b, c$, respectively. Show that

$$(q-r)a+(r-p)b+(p-q)c=0.$$

4.1.12 If

$$a\left(\frac{1}{b}+\frac{1}{c}\right), b\left(\frac{1}{c}+\frac{1}{a}\right), c\left(\frac{1}{a}+\frac{1}{b}\right)$$

are in AP, prove that $a, b, c$ are in AP.

4.1.13 In an AP if the $m^{th}$ is $n$ and the $n^{th}$ term is $m$, where $m \neq n$, find the $p^{th}$ term.

4.1.14 If the sum of $n$ terms of an AP is

$$nP+\frac{1}{2}n(n-1)Q,$$

where $P$ and $Q$ are constants, find the common difference.

4.1.15 If $\frac{a^n+b^n}{a^{n-1}+b^{n-1}}$ is the AM between $a$ and $b$, then find the value of $n$.

4.1.16 If the sum of the first $n$ terms of an AP is $4n-n^2$, what is the first term (that is $y(1)$ )? What is the sum of first two terms? What is the second term? Similarly, find the 3rd, the 10th and the $n$th terms.

4.1.17 Between 1 and 31, $m$ numbers have been inserted in such a way that the resulting sequence is an AP and the ratio of $7^{th}$ and $(m-1)^{th}$ numbers is 5 : 9. Find the value of $m$.

4.1.18 The $5^{th}$, $8^{th}$ and $11^{th}$ terms of a GP are $p, q$ and $s$, respectively. Show that $q^2 = ps$.

4.1.19 If the $4^{th}$, $10^{th}$ and $16^{th}$ terms of a GP are $x, y$ and $z$ respectively, prove that $x, y, z$ are in GP.

4.1.20 Show that the products of the corresponding terms of the sequences $a, ar, ar^2, \ldots, ar^{n-1}$ and $A, AR, AR^2, \ldots, AR^{n-1}$ form a GP and find the common ratio.

4.1.21 If the $p^{th}, q^{th}$ and $r^{th}$ terms of a GP are $a, b$ and $c$, respectively. Prove that

$$a^{q-r} b^{r-p} c^{p-q} = 1.$$

4.1.22 If the first and the $n^{th}$ term of a GP are $a$ and $b$, respectively, and if $P$ is the product of $n$ terms, prove that $P^2 = (ab)^n$.

4.1.23 If $a, b, c$ and $d$ are in GP show that $\left(a^2 + b^2 + c^2\right)\left(b^2 + c^2 + d^2\right) = (ab + bc + cd)^2$.

4.1.24 If $f$ is a function satisfying

$$f(x+y) = f(x) f(y) \, \forall x, y \in N$$

such that

$$f(1) = 3 \text{ and } \sum_{x=1}^{n} f(x) = 120,$$

find the value of $n$.

**Solution:** Since

$$f(2) = f(1) f(1) = [f(1)]^2, \tag{4.1.24.1}$$

it is easy to verify that

$$f(k) = [f(1)]^k \tag{4.1.24.2}$$

$$\therefore \sum_{k=1}^{n} f(k) = \sum_{k=1}^{n} 3^k \tag{4.1.24.3}$$

$$\frac{3(3^n - 1)}{2} = 120, \tag{4.1.24.4}$$

$$\text{or, } n = \frac{\log 81}{\log 3} = 4 \tag{4.1.24.5}$$

4.1.25 A GP consists of an even number of terms. If the sum of all the terms is 5 times the sum of terms occupying odd places, then find its common ratio.

4.1.26 The sum of the first four terms of an AP is 56. The sum of the last four terms is 112. If its first term is 11, then find the number of terms.

4.1.27 If

$$\frac{a+bx}{a-bx} = \frac{b+cx}{b-cx} = \frac{c+dx}{c-dx} \, (x \neq 0),$$

then show that $a, b, c$ and $d$ are in GP.

4.1.28 Let $S$ be the sum, $P$ the product and $R$ the sum of reciprocals of $n$ terms in a GP. Prove that $P^2 R^n = S^n$.

4.1.29 The $p^{th}, q^{th}$ and $r^{th}$ terms of an AP are $a, b, c$, respectively. Show that

$$(q-r)\,a+(r-p)\,b+(p-q)\,c=0.$$

4.1.30 If

$$a\left(\frac{1}{b}+\frac{1}{c}\right), b\left(\frac{1}{c}+\frac{1}{a}\right), c\left(\frac{1}{a}+\frac{1}{b}\right)$$

are in AP, prove that $a, b, c$ are in AP

4.1.31 If $a, b, c, d$ are in GP prove that

$$(a^n+b^n)\,,(b^n+c^n)\,,(c^n+d^n)$$

are in GP.

4.1.32 If $a$ and $b$ are the roots of

$$x^2-3x+p=0$$

and $c, d$ are roots of

$$x^2-12x+q=0,$$

where $a, b, c, d$ form a GP, prove that

$$(q+p):(q-p)=17:15.$$

4.1.33 The ratio of the AM and GM of two positive numbers $a$ and $b$, is $m:n$. Show that

$$a:b=\left(m+\sqrt{m^2-n^2}\right):\left(m-\sqrt{m^2-n^2}\right).$$

4.1.34 If $a, b, c$ are in AP; $b, c, d$ are in GP and

$$\frac{1}{c}, \frac{1}{d}, \frac{1}{e}$$

are in AP prove that $a, c, e$ are in GP.

4.1.35 Show that the ratio of the sum of first $n$ terms of a GP to the sum of terms from $(n+1)^{th}$ to $(2n)^{th}$ term is $\frac{1}{r^n}$.

4.1.36 Insert two numbers between 3 and 81 so that the resulting sequence is GP

4.1.37 Find the value of $n$ so that

$$\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$$

may be the geometric mean between $a$ and $b$.

4.1.38 The sum of two numbers is 6 times their geometric mean, show that numbers are in the ratio

$$\left(3+2\sqrt{2}\right):\left(3-2\sqrt{2}\right).$$

4.1.39 If $A$ and $G$ be AM and GM, respectively between two positive numbers, prove that the numbers are

$$A\pm\sqrt{(A+G)\,(A-G)}.$$

4.1.40 If the $p^{th}, q^{th}, r^{th}$ and $s^{th}$ terms of an AP are in GP, then show that $(p-q)\,,(q-r)\,,(r-s)$ are also in GP.

4.1.41 If $a, b, c$ are in GP and $a^{\frac{1}{x}}b^{\frac{1}{y}}c^{\frac{1}{z}}$, prove that $x, y, z$ are in AP.

4.1.42 If $a, b, c, d, p$ are different real numbers such that

$$\left(a^2 + b^2 + c^2\right) p^2 - 2\,(ab + bc + cd)\,p + \left(b^2 + c^2 + d^2\right) \leq 0,$$

then show that $a, b, c$ and $d$ are in GP.

4.1.43 If $p, q, r$ are in GP and the equations

$$px^2 + qx + r = 0, \quad dx^2 + 2ex + f = 0$$

have a common root, then show that

$$\frac{d}{p}, \frac{e}{q}, \frac{f}{r}$$

are in AP.

## *4.2 JEE*

4.2.1 Let $a_1, a_2, a_3 \dots a_{100}$ be an AP with $a_1 = 3$ and

$$S_p = \sum_{i=1}^{p} a_i, 1 \leq p \leq 100.$$

For any integer $n$ with $1 \leq n \leq 20$, let $m = 5n$. If $\frac{S_m}{S_n}$ does not depend on $n$, then $a_2$ is ______. (2011)

4.2.2 Let $p$ and $q$ be the roots of the equation

$$x^2 - 2x + A = 0$$

and $r$ and $s$ be the roots of the equation $x^2 - 18x + B = 0$. If $p < q < r < s$ are in AP , then find $A$ and $B$. (1977)

4.2.3 If $1, \log_9\left(3^{1-x} + 2\right), \log_3\left(4 \cdot 3^x - 1\right)$ are in AP then $x$ equals (2002)

a) $\log_3 4$ b) $1 - \log_3 4$ c) $1 - \log_4 3$ d) $\log_4 3$

4.2.4 Let $T_r$ be the $r^{th}$ term of an AP whose first term is a and common difference is $d$. If for some positive integers $m, n, m \neq n, T_m = \frac{1}{n}$ and $T_n = \frac{1}{m}$, then $a - d$ equals (2004)

a) $\frac{1}{m} + \frac{1}{n}$ b) 1 c) $\frac{1}{mn}$ d) 0

4.2.5 Let $a_1, a_2, a_3 \dots$ be terms of an AP. If $\frac{a_1 + a_2 + \dots a_p}{a_1 + a_2 + \dots a_q} = \frac{p^2}{q^2}, p \neq q$, then $\frac{a_6}{a_{21}}$ equals (2006)

a) $\frac{41}{11}$ b) $\frac{7}{2}$ c) $\frac{2}{7}$ d) $\frac{11}{41}$

4.2.6 If $a_1, a_2, \dots, a_n$ are in HP, then the expression $a_1a_2 + a_2a_3 + \dots + a_{n-1}a_n$ is equal to (2006)

a) $n(a_1 - a_n)$ b) $(n-1)(a_1 - a_n)$ c) $na_1a_n$ d) $(n-1)a_1a_n$

4.2.7 Let $a, b, c \in R$. If $f(x) = ax^2 + bx + c$ is such that $a + b + c = 3$ and

$$f(x+y) = f(x) + f(y) \forall x, y \epsilon R,$$

then $\sum_{n=1}^{10} f(n)$ is equal to (2017)

a) 255 b) 330 c) 165 d) 190

4.2.8 Let $T_r$ be the $r^{th}$ term of an AP, for $r = 1, 2, 3, \ldots$ If for some positive integers $m, n$ we have $T_m = \frac{1}{n}$ and $T_n = \frac{1}{m}$ , then $T_{mn}$ equals (1998)

a) $\frac{1}{mn}$ b) $\frac{1}{m} + \frac{1}{m}$ c) 1 d) 0

4.2.9 Let $a_1, a_2, a_3, \ldots$ be in harmonic progression with $a_1 = 5$ and $a_{20} = 25$. The least positive integer $n$ for which $a_n < 0$ is (2012)

a) 22 b) 23 c) 24 d) 25

4.2.10 Let $b_i > 1$ for $i = 1, 2, \ldots, 101$. Suppose $\log_e b_1, \log_e b_2, \ldots, \log_e b_{101}$ are in AP with the common difference $\log_e 2$. Suppose $a_1, a_2, \ldots, a_{101}$ are in AP such that $a_1 = b_1$ and $a_{51} = b_{51}$. If $t = b_1 + b_2 + \cdots + b_{51}$ and $s = a_1 + a_2 + \cdots + a_{53}$, then ( 2016)

a) $s > t$ and $a_{101} > b_{101}$
b) $s > t$ and $a_{101} < b_{101}$
c) $s < t$ and $a_{101} > b_{101}$
d) $s < t$ and $a_{101} < b_{101}$

Let $V_r$ denote the sum of first $r$ terms of an AP whose first term is $r$ and the common difference is $(2r - 1)$. Let $T_r = V_{r+1} - V_r - 2$ and $Q_r = T_{r+1} - T_r$ for $r = 1, 2, \ldots$

4.2.11 The sum $V_1 + V_2 + \cdots + V_n$ is (2007)

a) $\frac{1}{12}n(n+1)\left(3n^2 - n + 1\right)$
b) $\frac{1}{12}n(n+1)\left(3n^2 + n + 2\right)$
c) $\frac{1}{2}n\left(2n^2 - n + 1\right)$
d) $\frac{1}{3}\left(2n^3 - 2n + 3\right)$

4.2.12 $T_r$ is always (2007)

a) an odd number
b) an even number
c) a prime number
d) composite number

4.2.13 Which one of the following is a correct statement? (2007)

a) $Q_1, Q_2, Q_3, \ldots$ are in AP with common difference 5
b) $Q_1, Q_2, Q_3, \ldots$ are in AP with common difference 6
c) $Q_1, Q_2, Q_3, \ldots$ are in AP with common difference 11
d) $Q_1 = Q_2 = Q_3 = \ldots$

4.2.14 If $\log_3 2, \log_3 2^x - 3$ and $\log_3\left(2^x - \frac{7}{2}\right)$ are in AP , determine the value of $x$. (1990)

4.2.15 Let $p$ be the first of $n$ arithmetic means between two numbers and $q$ the first of $n$ harmonic means between the same numbers . Show that $q$ does not lie between $p$ and $\left(\frac{n+1}{n-1}\right)^2 p$ (1991)

4.2.16 The real numbers $x_1, x_2, x_3$ satisfying the equation $x^3 - x^2 + \beta x + \gamma = 0$ are in AP. Find the intervals in which $\beta$ and $\gamma$ lie. (1996)

4.2.17 The fourth power of the common difference of an AP with integer entries is added to the product of any four consecutive terms of it. Prove that the resulting sum is the square of an integer. (2000)

4.2.18 Let $p, q$ amd $r$ be nonzero real numbers that are the $10^{th}, 100^{th}$, and $1000^{th}$ terms of a harmonic progression, respectively. Consider the following system of linear equations (2022)

$$\begin{aligned} x + y + z &= 1 \\ 10x + 100y + 1000z &= 0 \\ qrx + pry + pqz &= 0 \end{aligned}$$

| | |
|---|---|
| (I) If $\frac{q}{r} = 10$, then the system of linear equations has | (A) $x = 0, y = \frac{10}{9}, z = -\frac{1}{9}$ as a solution |
| (II) If $\frac{p}{r} \neq 100$, then the system of linear equations has | (B) $x = \frac{10}{9}, y = -\frac{1}{9}, z = 0$ as a solution |
| (III) If $\frac{p}{q} \neq 10$, then the system of linear equations has | (C) infinitely many solutions |
| (IV) If $\frac{p}{q} = 10$, then the system of linear equations has | (D) no solution |

a) $(I) \to (T); (II) \to (C); (III) \to (D); (IV) \to (T)$
b) $(I) \to (B); (II) \to (D); (III) \to (D); (IV) \to (C)$
c) $(I) \to (B); (II) \to (C); (III) \to (A); (IV) \to (C)$
d) $(I) \to (T); (II) \to (D); (III) \to (A); (IV) \to (T)$

4.2.19 Suppose four positive numbers $a_1, a_2, a_3, a_4$ are in GP. Let $b_1 = a_1, b_2 = b_1 + a_2, b_3 = b_2 + a_3$ and $b_4 = b_3 + a_4$.
STATEMENT-1 : The numbers $b_1, b_2, b_3, b_4$ are neither in AP nor in GP and
STATEMENT-2 : The numbers $b_1, b_2, b_3, b4$ are in H.P. (2008)

a) STATEMENT-1 is True, STATEMENT-2 is True; STATEMENT-2 is the correct explanation for STATEMENT-1
b) STATEMENT-1 is True, STATEMENT-2 is True;
STATEMENT-2 is NOT a correct explanation for STATEMENT-1
c) STATEMENT-1 is True, STATEMENT-2 is False
d) STATEMENT-1 is False, STATEMENT-2 is True

4.2.20 Let the harmonic mean and geometric mean of two positive numbers be the ratio 4 : 5. Then the two numbers are in ratio ... (1992)

4.2.21 Let $a, b, c$ be positive integers such that $\frac{b}{a}$ is an integer. If $a$,$b$,$c$ are in geometric progression and the arithmetic mean of $a, b, c$ is $b + 2$, then the value of $\frac{a^2+a-14}{a+1}$ is ______. (2014)

4.2.22 $l, m, n$ are the $p^{th}, q^{th}$ and $r^{th}$ term of a GP all positive, then $\begin{vmatrix} \log l & p & 1 \\ \log m & q & 1 \\ \log n & r & 1 \end{vmatrix}$ equals (2002)

a) 1 b) 2 c) 1 d) 0

4.2.23 If $m$ is the AM of two distinct real numbers $l$ and $n$ $(l, n > 1)$ and $G_1$, $G_2$ and $G_3$ are three geometric means between $l$ and $n$, then $G_1^4 + 2G_2^4 + G_3^4$ equals (2015)

a) $4lmn^2$ b) $4l^2m^2n^2$ c) $4l^2mn$ d) $4lm^2n$

4.2.24 If the $2^{nd}$, $5^{th}$ and $9^{th}$ terms of a non-constant AP are in GP, then the common ratio of this GP is (2016)

a) 1 b) $\frac{7}{4}$ c) $\frac{8}{5}$ d) $\frac{4}{3}$

Let $A_1, G_1, H_1$ denote the arithmetic, geometric and harmonic means, respectively, of two distinct positive numbers. For $n \geq 2$, Let $A_{n-1}$ and $H_{n-1}$ have arithmetic, geometric and harmonic means as $A_n, G_n, H_n$ respectively. (2007)

a) Which one of the following statements is correct ?
   i) $G_1 > G_2 > G_3 > \dots$
   ii) $G_1 < G_2 < G_3 < \dots$
   iii) $G_1 = G_2 = G_3 = \dots$
   iv) $G_1 < G_3 < G_5 < \dots$ and $G_2 > G_4 > G_6 > \dots$

b) Which one of the following statements is correct ?
   i) $A_1 > A_2 > A_3 > \dots$
   ii) $A_1 < A_2 < A_3 < \dots$
   iii) $A_1 > A_3 > A_5 > \dots$ and $A_2 < A_4 < A_6 < \dots$
   iv) $A_1 < A_3 < A_5 < \dots$ and $A_2 > A_4 > A_6 > \dots$

c) Which one of the following statements is correct ?
   i) $H_1 > H_2 > H_3 > \dots$
   ii) $H_1 < H_2 < H_3 < \dots$
   iii) $H_1 > H_3 > H_5 > \dots$ and $H_2 < H_4 < H_6 < \dots$
   iv) $H_1 < H_3 < H_5 < \dots$ and $H_2 > H_4 > H_6 > \dots$

4.2.25 Let $a_1, a_2, \dots, a_n$ be positive real numbers in geometric progression. For each $n$, let $A_n, G_n, H_n$ be respectively, the arithmetic mean, geometric mean and harmonic mean of $a_1, a_2, \dots, a_n$. Find an expression for the geometric mean of $G_1, G_2, \dots, G_n$ in terms of $A_1, A_2, \dots, A_n, H_1, H_2, \dots, H_n$. (2001)

4.2.26 Let $a, b$ be positive real numbers. If $a, A_1, A_2, b$ are in arithmetic progression, $a, G_1, G_2, b$ are in geometric progression and $a, H_1, H_2, b$ are in harmonic progression, show that

$$\frac{G_1G_2}{H_1H_2} = \frac{A_1 + A_2}{H_1 + H_2} = \frac{(2a + b)(a + 2b)}{9ab}$$

(2002)

4.2.27 If $a, b, c$ are in AP, $a^2, b^2, c^2$ are in HP, then prove that either $a = b = c$ or $a, b, -\frac{c}{2}$ form a GP. (2003)

4.2.28 If $a, b$ and $c$ be three distinct real numbers in GP and $a + b + c = xb$, then $x$ cannot be (2019)

a) -2 b) 4 c) -3 d) 2

4.2.29 If the first and the $(2n - 1)^{th}$ terms of an AP, a GP and an HP are equal and their $n^{th}$ terms are $a, b$ and $c$ respectively, then (1988)

a) $a = b = c$ b) $a \geq b \geq c$ c) $a + b = c$ d) $ac - b^2 = 0$

4.2.30 If $x > 1, y > 1, z > 1$ are in GP, then $\frac{1}{1+\ln x}, \frac{1}{1+\ln y}, \frac{1}{1+\ln z}$ are in (1998)

a) AP b) HP c) GP d) None of these

4.2.31 If $x$, $y$ and $z$ are $p^{th}, q^{th}$ and $r^{th}$ terms respectively of an AP and also of a GP, then $x^{y-z}y^{z-x}z^{x-y}$ is equal to (1982)

a) $xyz$ b) 0 c) 1 d) none of these

4.2.32 If $\log_e(a + c), \log_e(a - c), \log_e(a - 2b + c)$ are in AP, then (1994)

a) $a, b, c$ are in AP
b) $a^2, b^2, c^2$ are in AP
c) $a, b, c$ are in GP
d) $a, b, c$ are in HP

4.2.33 Let $\alpha$, $\beta$ be the roots of $x^2 - x + p = 0$ and $\gamma$, $\delta$ be the roots of $x^2 - 4x + q = 0$. If $\alpha$, $\beta$, $\gamma$, $\delta$ are in GP, then the integral values of $p$ and $q$ respectively are (2001)

a) $-2, -32$ b) $-2, 3$ c) $-6, 3$ d) $6, -32$

4.2.34 Let the positive numbers $a, b, c, d$ be in AP. Then $abc, abd, acd, bcd$ are (2001)

a) NOT in AP/GP/H.P
b) in AP
c) in GP
d) in HP

4.2.35 Suppose $a, b, c$ are in AP and $a^2, b^2, c^2$ are in GP if $a < b < c$ and $a + b + c = 3/2$, then the value of $a$ is (2002)

a) $\frac{1}{2\sqrt{2}}$ b) $\frac{1}{2\sqrt{3}}$ c) $\frac{1}{2} - \frac{1}{\sqrt{3}}$ d) $\frac{1}{2} - \frac{1}{\sqrt{2}}$

4.2.36 In the quadratic equation $ax^2 + bx + c = 0, \triangle = b^2 - 4ac$ and $\alpha+\beta$, $\alpha^2+\beta^2$, $\alpha^3+\beta^3$ are in GP where $\alpha$, $\beta$ are roots of $ax^2 + bx + c = 0$, then (2005)

a) $\triangle \neq 0$ b) $b\triangle = 0$ c) $c\triangle = 0$ d) $\triangle = 0$

4.2.37 Let $a, b, c, d$ be real numbers in GP. If $u, v, w$ satisfy the system of equations (1999)

$$\begin{aligned} u + 2v + 3w &= 6 \\ 4u + 5v + 6w &= 12 \\ 6u + 9v &= 4 \end{aligned}$$

then show that the roots of the equations

$$\left(\frac{1}{u} + \frac{1}{v} + \frac{1}{w}\right)x^2 + \left[(b-c)^2 + (c-a)^2 + (d-b)^2\right]x + u + v + w = 0$$

and

$$20x^2 + 10\,(a-d)^2\,x - 9 = 0$$

are reciprocals of each other.

4.2.38 For any three positive real numbers $a, b$ and $c$,

$$9(25a^2 + b^2) + 25(c^2 - 3ac) = 15b(3a + c).$$

Then (2017)

a) $a, b$ and $c$ are in AP
c) $b, c$ and $a$ are in AP
b) $b, c$ and $a$ are in GP
d) $a, b$ and $c$ are in GP

4.2.39 Let $a_1, a_2, a_3, \ldots$ be a sequence of positive integers in arithmetic progression with common difference 2. Also, let $b_1, b_2, b_3, \ldots$ be a sequence of positive integers in geometric progression with common ratio 2. If $a_1 = b_1 = c$, then the number of all possible values of $c$ for which the equality

$$2(a_1 + a_2 + \cdots + a_n) = b_1 + b_2 + \cdots + b_n$$

holds for some positive integer $n$, is ______. (2020)

# 5 Binomial Theorem

## *5.1 NCERT*

Expand

5.1.1 $\left(x^2 + \frac{3}{4}\right)^4, x \neq 0$5.1.3 $\left(\frac{2}{x} - \frac{x}{2}\right)^5$ 5.1.5 $\left(\frac{x}{3} + \frac{1}{x}\right)^5$ 5.1.6 $\left(x + \frac{1}{x}\right)^6$

5.1.2 $(1-2x)^5$ 5.1.4 $(2x-3)^6$

Using binomial theorem, evaluate each of the following

5.1.7 $(96)^3$ 5.1.9 $(101)^4$ 5.1.11 $(98)^5$.

5.1.8 $(102)^5$ 5.1.10 $(99)^5$

Find the coefficient of

5.1.12 $x^5$ in $(x+3)^8$

5.1.13 $a^5b^7$ in $(a-2b)^{12}$.

5.1.14 $x^6y^3$ in $(x+2y)^9$.

Write the general term in the expansion of

5.1.15 $\left(x^2-y\right)^6$

5.1.16 $\left(x^2-yx\right)^{12}$, $x \neq 0$.

Find the middle terms in the expansions of

5.1.17 $\left(3-\frac{x^3}{6}\right)^7$

5.1.18 $\left(\frac{x}{3}+9y\right)^{10}$

5.1.19 Which is larger $(1.01)^{1000000}$ or 10,000?

5.1.20 Using binomial theorem, prove that $6^n-5n$ always leaves remainder 1 when divided by 25.

5.1.21 Find $a$ if the $17^{th}$ and $18^{th}$ terms of expansiom $(2+a)^{50}$ are equal.

5.1.22 Show that the middle term in the expansion of $(1+x)^{2n}$ is $\frac{1.3.5...(2n-1)}{n!}2nx^n$, where $n$ is a positive integer.

5.1.23 The second, third and fourth terms in the binomial expansion $(x+a)^n$ are 240, 720 and 1080 respectively. Find $x, a$ and $n$.

5.1.24 The coefficients of three consecutive terms in the expansion of $(1+a)^n$ are in the ratio 1:7:42. Find $n$.

5.1.25 Find the term independent of $x$ in the expansion of $\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^6$.

5.1.26 If the coefficients of $a^{r-1}, a^r$ and $a^{r+1}$ in the expansion of $(1+a)^n$ are in arithmetic progression, prove that $n^2-n(4r+1)+4r^2-2=0$.

5.1.27 Show that the coefficient of the middle term in the expansion of $(1+x)^{2n}$ is equal to the sum of the coefficients of two middle terms in the expansion of $(1+x)^{2n-1}$.

5.1.28 Find the coefficient of $a^4$ in the product $(1+2a)^4(2-a)^5$ using binomial theorem.

5.1.29 Find thr $r^{th}$ term from the end in the expansion of $(x+a)^n$.

5.1.30 Find the term independent of $x$ in the expansion of $\left(\sqrt[3]{x}+\frac{1}{2\sqrt[3]{x}}\right)^{18}$, $x>0$.

5.1.31 The sum of the coefficients of the first three terms in the expansion of $\left(x-\frac{3}{x^2}\right)^m$, $x \neq 0$, $m$ being a natural number, is 559. Find the term of the expansion containing $x^3$.

5.1.32 If the coefficients of $(r-5)^{th}$ and $(2r-1)^{th}$ terms in the expansion of $(1+x)^{34}$ are equal, find $r$.

5.1.33 Using binomial theorem, indicate which number is larger $(1.1)^{10000}$ or 1000.

5.1.34 Find $(a+b)^4 - (a-b)^4$. Hence evaluate $\left(\sqrt{3}+\sqrt{2}\right)^4$ - $\left(\sqrt{3}-\sqrt{2}\right)^4$.

5.1.35 Find $(x+1)^6 + (x-1)^6$. Hence or otherwise evaluate $\left(\sqrt{2}+1\right)^6 + \left(\sqrt{2}-1\right)^6$.

5.1.36 Show that $9^{n+1} - 8n - 9$ is divisible by 64, whenever $n$ is a positive integer.

5.1.37 Prove that $\sum\limits_{r=0}^{n} 3^r \, {}^nC_r = 4^n$.

5.1.38 Find the $4^{th}$ term in the expansion of $(x-2y)^{12}$.

5.1.39 Find the $13^{th}$ term in the expansion of $\left(9x - \frac{1}{3\sqrt{x}}\right)^{18}, x \neq 0$.

5.1.40 In the expansion of $(1+a)^{m+n}$, prove that coefficients of $a^m$ and $a^n$ are equal.

5.1.41 The coefficients of the $(r-1)^{th}$, $r^{th}$ and $(r+1)^{th}$ terms in the expansion of $(x+1)^n$ are in the ratio 1 : 3: 5. Find $n$ and $r$.

5.1.42 Prove that the coefficient of $x^n$ in the expansion of $(1+x)^{2n}$ is twice the coefficient of $x^n$ in the expansion of $(1+x)^{2n-1}$.

5.1.43 Find the positive value of $m$ for which the coefficient of $x^2$ in the expansion $(1+x)^m$ is 6.

5.1.44 Find $a, b$ and $n$ in the expansion of $(a+b)^n$ if the first three terms of the expansion are 729, 7290 and 30375 respectively.

5.1.45 Find the coefficient of $x^5$ in the product $(1+2x)^6(1-x)^7$ using binomial theorem.

5.1.46 Find $a$ if the coefficients of $x^2$ and $x^3$ in the expansion of $(3+ax)^9$ are equal.

5.1.47 If $a$ and $b$ are distinct integers, prove that $a-b$ is a factor of $a^n - b^n$, whenever $n$ is a positive integer.

5.1.48 Evaluate $\left(\sqrt{3}+\sqrt{2}\right)^6$ - $\left(\sqrt{3}-\sqrt{2}\right)^6$

5.1.49 Find the value of $\left(a^2+\sqrt{a^2-1}\right)^4 + \left(a^2-\sqrt{a^2-1}\right)^4$.

5.1.50 Find an approximation of $(0.99)^5$ using the first three terms of its expansion.

5.1.51 Find $n$, if the ratio of the fifth term from the beginning to the fifth term from the end in the expansion of $\left(\sqrt[4]{2}+\frac{1}{\sqrt[4]{3}}\right)^n$ is $\sqrt{6}:1$

5.1.52 Expand using binomial theorem $\left(1+\frac{x}{2}-\frac{2}{x}\right)^4$, $x \neq 0$.

5.1.53 Find the expansion of $\left(3x^2-2ax+3a^2\right)^3$ using binomial theorem.

## 5.2 JEE

5.2.1 For $0 < \phi < \frac{\pi}{2}$, if

$$x = \sum_{n=0}^{\infty} \cos^{2n}\phi,\ y = \sum_{n=0}^{\infty} \sin^{2n}\phi,\ z = \sum_{n=0}^{\infty} \cos^{2n}\phi \sin^{2n}\phi$$

then (1993)

a) $xyz = xz + y$ c) $xyz = x + y + z$
b) $xyz = xy + z$ d) $xyz + yz + x$

5.2.2 Let $n$ be a odd integer. If

$$\sin n\theta = \sum_{r=0}^{n} b_r \sin^r \theta,$$

for every value of $\theta$, then (1998)

a) $b_0 = 1, b_1 = 3$ c) $b_0 = -1, b_1 = 3$
b) $b_0 = 0, b_1 = n$ d) $b_0 = 0, b_1 = n^2 - 3n + 3$

5.2.3 Find the sum of the series (1985)

$$\sum_{r=0}^{n}(-1)^r\binom{n}{r}\left(\frac{1}{2^r} + \frac{3^r}{2^{2r}} + \frac{7^r}{2^{3r}} + \frac{15^r}{2^{4r}} \cdots \text{up to } m \text{ terms}\right)$$

5.2.4 The larger of $99^{50} + 100^{50}$ and $101^{50}$ is ... (1982)
5.2.5 The sum of the coefficients of the polynomial $\left(1 + x - 3x^2\right)^{2163}$ is ... (1982)
5.2.6 If $(1 + ax)^n = 1 + 8x + 24x^2 + \ldots$ then $a = \ldots$ and $n = \ldots$ (1983)
5.2.7 Let $n$ be a positive integer. If the coefficients of 2nd, 3rd and 4th terms in the expansion of $(1 + x)^n$ are in AP, then the value of $n$ is ... (1994)
5.2.8 The sum of the rational terms in the expansion of $\left(\sqrt{2} + 3^{\frac{1}{5}}\right)^{10}$ is ... (1997)
5.2.9 The coefficients of three consecutive terms of $(1 + x)^{n+5}$ are in the ratio 5:10:14. Then $n =$ (2013)
5.2.10 Let $m$ be the smallest positive integer such that the coefficient of $x^2$ in the expansion of $(1 + x)^2 + (1 + x)^3 + ... + (1 + x)^{49} + (1 + x)^{50} + (1 + mx)^{50}$ is $(3n + 1)\,{}^{51}C_3$ for some positive integer $n$. Then the value of $n$ is (2016)
5.2.11 Let

$$X = \left({}^{10}C_1{}^2\right) + 2\left({}^{10}C_2{}^2\right) + 3\left({}^{10}C_3{}^2\right) + \cdots + 10\left({}^{10}C_{10}{}^2\right),$$

where ${}^{10}C_r, r \in \{1, 2, \ldots, 10\}$ denote binomial coefficients. Then, the value of $\frac{X}{1430}$ is ______ (2018)
5.2.12 Suppose

$$\begin{vmatrix} \sum\limits_{k=0}^{n} k & \sum\limits_{k=0}^{n} k^2\,{}^nC_k \\ \sum\limits_{k=0}^{n} {}^nC_k k & \sum\limits_{k=0}^{n} {}^nC_k 3^k \end{vmatrix} = 0$$

holds for some positive integer $n$. The $\sum\limits_{k=0}^{n} \frac{{}^nC_k}{k+1}$ equals ______. (2019)
5.2.13 The coefficients of $x^p$ and $x^q$ in the expansion of $(1 + x)^{p+q}$ are (2002)

a) equal
b) equal with opposite signs
c) reciprocals of each other
d) none of these

5.2.14 If the sum of coefficients in the expansion of $(a+b)^n$ is 4096, then the greatest coefficient in the expansion is (2002)

a) 1594
b) 792
c) 924
d) 2924

5.2.15 $r$ and $n$ are positive integers, $r>1, n>2$ and coefficient of $(r+2)^{th}$ term and $(3r)^{th}$ term in the expansion of $(1+x)^{2n}$ are equal, then $n$ equals
(2002)

a) $3r$
b) $3r+1$
c) $2r$
d) $2r+1$

5.2.16 The number of integral terms in the expansion of $\left(\sqrt{3}+\sqrt[8]{5}\right)^{256}$ is (2003)

a) 35
b) 32
c) 33
d) 34

4

5.2.17 The positive integer just greater than $(1+0.0001)^{10000}$ is (2002)

a) 4
b) 5
c) 2
d) 3

5.2.18 If $x$ is positive, the first negative term in the expansion of $(1+x)^{\frac{27}{5}}$ is (2003)

a) $6^{th}$ term
b) $7^{th}$ term
c) $5^{th}$ term
d) $8^{th}$ term

5.2.19 If $C_r$ stands for ${}^nC_r$, then the sum of the series $\frac{2\left(\frac{n}{2}!\right)\left(\frac{n}{2}!\right)}{n!}\left[C_0^2-2C_1^2+3C_2^2-\cdots+(-1)^n(n+1)C_n^2\right]$, where $n$ is an even positive integer is equal to (1992)

a) 0
b) $(-1)^{\frac{n}{2}}(n+1)$
c) $(-1)^{\frac{n}{2}}(n+2)$
d) $(-1)^n n$
e) none of these

5.2.20 If $a_n=\sum_{r=0}^{n}\frac{1}{{}^nC_r}$, then $\sum_{r=0}^{n}\frac{r}{{}^nC_r}$ equals (1998)

a) $(n-1)a_n$
b) $na_n$
c) $\frac{1}{2}na_n$
d) None of the above

5.2.21 Given positive integers $r>1, n>2$ and that the coefficient of the $(3r)^{th}$ terms in the binomial expansion of $(1+x)^{2n}$ are equal. Then (1983)

a) $n = 2r$ b) $n = 2r + 1$ c) $n = 3r$ d) none of these

5.2.22 The coefficient of $x^4$ in $\left(\frac{x}{2} - \frac{3}{x^2}\right)^{10}$ is (1983)

a) $\frac{405}{256}$ b) $\frac{504}{259}$ c) $\frac{450}{263}$ d) none of these

5.2.23 The expression $\left(x + \left(x^3 - 1\right)^{\frac{1}{2}}\right)^5 + \left(x - \left(x^3 - 1\right)^{\frac{1}{2}}\right)^5$ is a polynomial of degree (1992)

a) 5 b) 6 c) 7 d) 8

5.2.24 If in the expansion of $(1 + x)^m (1 - x))^n$, the coefficients of $x$ and $x^2$ are 3 and $-6$ respectively, then $m$ is (1999)

a) 6 b) 9 c) 12 d) 24

5.2.25 For $2 \le r \le n$, ${}^nC_r + 2\ {}^nC_{r-1} + {}^nC_{r-2} =$ (2000)

a) ${}^{n+1}C_{r-1}$
c) $2\ {}^{n+2}C_r$
b) $2\ {}^{n+1}C_{r+1}$
d) ${}^{n+2}C_r$

5.2.26 In the binomial expansion of $(a - b)^n$, $n \ge 5$, the sum of the $5^{th}$ and $6^{th}$ terms is zero. Then $a/b$ equals (2001)

a) $\frac{n-5}{6}$ b) $\frac{n-4}{5}$ c) $\frac{5}{n-4}$ d) $\frac{6}{n-5}$

5.2.27 The sum $\sum_{i=0}^{9} {}^{10}C_i {}^{20}C_{m-i}$, (where ${}^pC_q = 0$ if $p < q$) is maximum when $m$ is (2002)

a) 5 b) 10 c) 15 d) 20

5.2.28 Coefficient of $t^{24}$ in $\left(1 + t^2\right)^{12} \left(1 + t^{12}\right)\left(1 + t^{24}\right)$ is (2003)

a) ${}^{12}C_6$+3 b) ${}^{12}C_6$+1 c) ${}^{12}C_6$ d) ${}^{12}C_6$+2

5.2.29 If

$$ {}^{n-1}C_r = \left(k^2 - 3\right) {}^nC_{r+1} $$

then ($k \in$ ) (2004)

a) $(-8, -2]$ b) $[2, \infty)$ c) $\left[-\sqrt{3}, \sqrt{3}\right]$ d) $(\sqrt{3}, 2]$

5.2.30 The value of ${}^{30}C_0 {}^{30}C_{10}$-${}^{30}C_1 {}^{30}C_{11}$+${}^{30}C_2 {}^{30}C_{12} \ldots {}^{30}C_{20} {}^{30}C_{30}$ is (2005)

a) $^{30}C_{10}$ b) $^{30}C_{15}$ c) $^{60}C_{30}$ d) $^{31}C_{10}$

5.2.31 For $r = 0, 1 \ldots, 10$, let $A_r, B_r$ and $C_r$ denote, respectively, the coefficients of $x^r$ in the expansions of $(1+x)^{10}$,$(1+x)^{20}$and $(1+x)^{30}$. Then $\sum_{r=1}^{10} A_r\left(B_{10}B_r - C_{10}A_r\right)$ is equal to (2010)

a) $B_{10} - C_{10}$ b) $A_{10}\left(B_{10}^2 C_{10} A_{10}\right)$ c) 0 d) $C_{10} - B_{10}$

5.2.32 Coefficient of $x^{11}$ in the expansion of $\left(1+x^2\right)^4\left(1+x^3\right)^7\left(1+x^4\right)^{12}$ is (2014)

a) 1051
b) 1106
c) 1113
d) 1120

5.2.33 Given that (1979)

$$C_1 + 2C_2x + 3C_3x^2 + \cdots + 2nC_{2n}x^{2n-1} = 2n(1+x)^{2n-1}$$

where $C_r = \frac{(2n)!}{r!(2n-r)!}, r = 0, 1, 2, \ldots, 2n$. Prove that

$$C_1^2 - 2C_2^2 + 3C_3^2 - \cdots - 2nC_{2n}^2 = (-1)^n nC_n.$$

5.2.34 Prove that $7^{2n} + \left(2^{3n-2}\right)\left(3^{n-1}\right)$ is divisible by 25 for any natural number $n$. (1982)

5.2.35 If $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots + C_nx^n$ then show that the sum of products of $C_i$'s taken two at a time, represented by $\sum_{0\le i<j\le n}\sum C_iC_j$ is equal to $2^{2n-1}$-$\frac{(2n)!}{2(n!)^2}$ (1983)

5.2.36 If $p$ be a natural number then prove that $p^{n+1} + (p+1)^{2n-1}$ is divisible by $p^2 + p + 1$ for every positive integer $n$. (1984)

5.2.37 Given $s_n = 1 + q + q^2 + \ldots. + q^n$; $S_n = 1 + \frac{q+1}{2} + \left(\frac{q+1}{2}\right)^2 + \ldots.. + \left(\frac{q+1}{2}\right)^n, q \neq 1$ . Prove that

$$^{n+1}C_1 + {}^{n+1}C_2 s_1 + {}^{n+1}C_3 s_2 + \ldots.. + {}^{n+1}C_n s_n = 2^n S_n$$

(1984)

5.2.38 Let $R = \left(5\sqrt{5} + 11\right)^{2n}$ and $f = R - [R]$, where [] denotes the greatest integer function.Prove that $Rf = 4^{2n+4}$ (1988)

5.2.39 Prove that (1989)

$$C_0 - 2^2C_1 + 3^2C_2 - \ldots\ldots\ldots.. + (-1)^n(n+1)^2C_n = 0$$

,$n > 2$ , where $C_r = {}^nC_r$

5.2.40 Prove that $\frac{n^7}{7} + \frac{n^5}{5} + \frac{2n^3}{3} - \frac{n}{105}$ is an integer for every positive integer $n$. (1990)

5.2.41 If $\sum_{r=0}^{2n} a_r(x-2)^r = \sum_{r=0}^{2n} b_r(x-3)^r$ and $a_k = 1$ for all $k \ge n$ then show that $b_n = {}^{2n+1}C_{n+1}$ (1992)

5.2.42 If $a_n = \sqrt{7 + \sqrt{7 + \sqrt{7 + \ldots}}}$ having $n$ radical signs, then by methods of mathematical induction, which is true? (2002)

a) $a_n > 7 \ \forall \ n \geq 1$
b) $a_n < 7 \ \forall \ n \geq 1$
c) $a_n < 4 \ \forall \ n \geq 1$
d) $a_n < 3 \ \forall \ n \geq 1$

5.2.43 If $a_n = \sqrt{7+\sqrt{7+\sqrt{7+\ldots}}}$ having $n$ radical signs, then by methods of mathematical induction, which is true? (2002)

a) $a_n > 7 \ \forall \ n \geq 1$
b) $a_n < 7 \ \forall \ n \geq 1$
c) $a_n < 4 \ \forall \ n \geq 1$
d) $a_n < 3 \ \forall \ n \geq 1$

5.2.44 Use mathematical Induction to prove : If $n$ is any odd positive integer , then $n\left(n^2-1\right)$ is divisible by 24. (1983)

5.2.45 Use method of mathematical Induction $2.7^n + 3.5^n - 5$ is divisible by 24 for all $n > 0$ (1985)

5.2.46 Prove by mathematical induction that -$\frac{(2n)!}{2^{2n}(n!)^2} \leq \frac{1}{(3n+1)^{\frac{1}{2}}}$ for all postive Integers $n$. (1987)

5.2.47 Using mathematical induction,prove that

$$^mC_0\,^nC_k + \,^mC_1\,^nC_{k-1} + \ldots.. + \,^mC_k\,^nC_0 = \,^{m+k}C_k$$

(1989)

5.2.48 Using induction or otherwise , prove that for any non-negative integers $m, n, r$ and $k$,

$$\sum_{r=0}^{k}(n-m)\frac{(r+m)!}{m!} = \frac{(r+k+1)!}{k!}\left[\frac{n}{r+1} - \frac{k}{r+2}\right]$$

(1991)

5.2.49 Let $p \leq 3$ be an integer and $\alpha,\beta$ be the roots of $x^2-(p+1)x+1 = 0$ using mathematical induction show that $\alpha^n + \beta^n$ (1992)

(i) is an integer and
(ii) is not divisible by $p$

# 6 Others

## 6.1 JEE

6.1.1 The sum of series $\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots$ upto infinity is (2007)

a) $e^{-\frac{1}{2}}$
b) $e^{+\frac{1}{2}}$
c) $e^{-2}$
d) $e^{-1}$

6.1.2 For a positive integer $n$, let $a_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{(2^n)-1}$. Then (1999)

a) $a100) \leq 100$ b) $a(100) > 100$ c) $a(200) \leq 100$ d) $a(200) > 100$

6.1.3 Let

$$S_n = \sum_{k=1}^{n}\frac{n}{n^2+kn+k^2} \text{ and } T_n = \sum_{k=0}^{n-1}\frac{n}{n^2+kn+k^2}$$

for $n = 1, 2, 3, \ldots$ Then, (2008)

a) $S_n < \frac{\pi}{3\sqrt{3}}$ b) $S_n > \frac{\pi}{3\sqrt{3}}$ c) $T_n < \frac{\pi}{3\sqrt{3}}$ d) $T_n > \frac{\pi}{3\sqrt{3}}$