Problems in Linear Algebra

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1 Introduction

1.1 Points

1. Find the distance between

$$\mathbf{P} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} 3 \\ -5 \end{pmatrix} \tag{1.1.1}$$

Solution: Two point are $P(x_1, y_1)$ and $Q(x_2, y_2)$. The distance between both points is d.

$$\mathbf{Z} = \mathbf{P} - \mathbf{Q} \tag{1.1.1}$$

Then the distance between P and Q is given

by:

$$d = \|\mathbf{Z}\| \tag{1.1.1}$$

$$d = \|\mathbf{P} - \mathbf{Q}\| \tag{1.1.1}$$

So, the distance between given points P and Q is:

$$d = \sqrt{(-2-3)^2 + (4-(-5))^2} \tag{1.1.1}$$

$$d = \sqrt{25+81} \tag{1.1.1}$$

$$d = \sqrt{106} \tag{1.1.1}$$

So, the distance between P(-2,4) and Q(3,-5) is :

$$d = \sqrt{106} \tag{1.1.1}$$

2. Find the length of PQ for

a) $\mathbf{P} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$;

b) $\mathbf{P} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$;

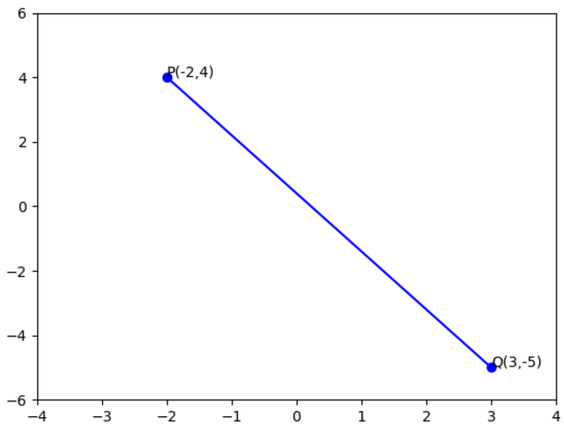

Fig. 1.1.1: Line between two points

c) $\mathbf{P} = \begin{pmatrix} a \\ b \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} -b \\ a \end{pmatrix}$.

Solution:

a) The distance between **P** and **Q** is given by:

$$d = \|\mathbf{P} - \mathbf{Q}\| \tag{1.1.2}$$

$$= \sqrt{(-1-2)^2 + (1+1)^2} \tag{1.1.2}$$

$$= \sqrt{9+4} = 3.6055 \tag{1.1.2}$$

b)

$$\mathbf{R} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} - \begin{pmatrix} -2 \\ 2 \end{pmatrix} \tag{1.1.2}$$

$$= \begin{pmatrix} 6 \\ 1 \end{pmatrix} \tag{1.1.2}$$

The desired distance between **P** and **Q** is

$$d = \|\mathbf{P} - \mathbf{Q}\| \tag{1.1.2}$$

From (1.1.2) and (1.1.2)

$$d = \|\mathbf{R}\| \tag{1.1.2}$$

$$= \sqrt{37} \tag{1.1.2}$$

3. Using direction vectors, show that $\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 7 \end{pmatrix}, \begin{pmatrix} 5 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ are the vertices of a parallelogram. **Solution:** Two lines are parallel if their respective directional vectors are in the same ratio.

Let the points be denoted by:

$$\mathbf{A} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{1.1.3}$$

$$\mathbf{B} = \begin{pmatrix} 5 \\ 4 \end{pmatrix} \tag{1.1.3}$$

$$\mathbf{C} = \begin{pmatrix} 4 \\ 7 \end{pmatrix} \tag{1.1.3}$$

$$\mathbf{D} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \tag{1.1.3}$$

The directional vector of **AB** is

$$\begin{pmatrix} 2-5 \\ 1-4 \end{pmatrix} = \begin{pmatrix} -3 \\ -3 \end{pmatrix} \tag{1.1.3}$$

The directional vector of **BC** is

$$\begin{pmatrix} 5-4 \\ 4-7 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix} \tag{1.1.3}$$

The directional vector of **CD** is

$$\begin{pmatrix} 4-1 \\ 7-4 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} \tag{1.1.3}$$

The directional vector of **AD** is

$$\begin{pmatrix} 2-1 \\ 1-4 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix} \tag{1.1.3}$$

The directional vector of **AC** is

$$\begin{pmatrix} 2-4 \\ 1-7 \end{pmatrix} = \begin{pmatrix} -2 \\ -6 \end{pmatrix} \tag{1.1.3}$$

Since the directional vectors of **AB** and **CD** are in the same ratio, so **AB** and **CD** are parallel and also opposite to each other.
Similarly, the directional vectors of **BC** and **AD** are in the same ratio,hence **BC** and **AD** are parallel and opposite.
Since the two pairs of opposite sides are parallel, the given points are the vertices of the parallelogram.

Moreover the sum of the directional vectors of **AB** and **BC**

$$\begin{pmatrix} -3 \\ -3 \end{pmatrix} + \begin{pmatrix} 1 \\ -3 \end{pmatrix} = \begin{pmatrix} -3+1 \\ -3-3 \end{pmatrix} = \begin{pmatrix} -2 \\ -6 \end{pmatrix}$$

Thus **AB** + **BC**= **AC**, which satisfy parallelogram law of vector addition i.e vector sum of two adjacent side of a parallelogram is the diagonal vector of the parallelogram. See Fig.

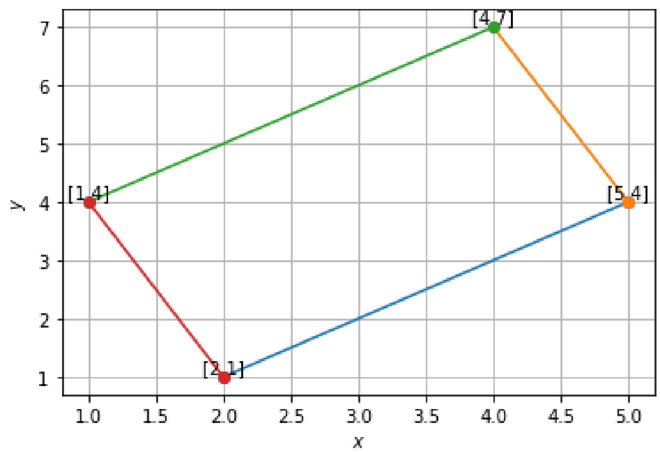

Fig. 1.1.3: This is the 2D diagram of the parallelogram with the given vertices

1.1.3

4. Using Baudhayana's theorem, show that the points $\begin{pmatrix}-3\\-4\end{pmatrix}$, $\begin{pmatrix}2\\6\end{pmatrix}$ and $\begin{pmatrix}-6\\10\end{pmatrix}$ are the vertices of a right-angled traingle. Repeat using orthogonality. **Solution:** Say there exists two points

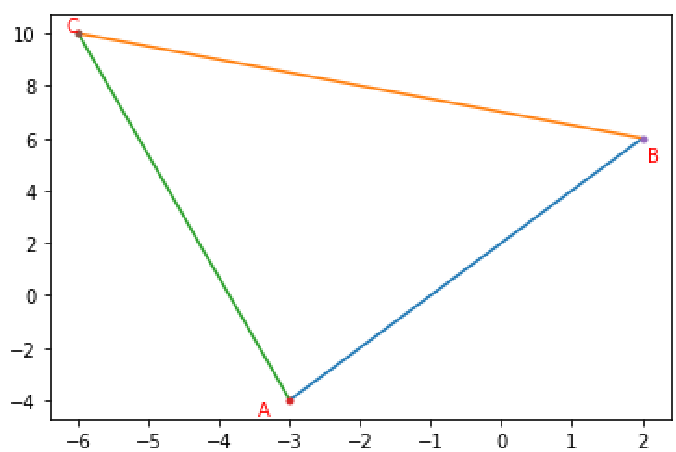

Fig. 1.1.4: Right Angled Triangle

$\mathbf{P}(x_1, y_1)$ and $\mathbf{Q}(x_2, y_2)$. The distance between the points is:

$$\mathbf{Z = P - Q} \tag{1.1.4}$$

Distance between **P** and **Q** is given by

$$\|\mathbf{Z}\| = \|\mathbf{P\text{-}Q}\| \tag{1.1.4}$$

Let $\mathbf{P} = (-3, -4)$, $\mathbf{Q} = (2, 6)$ and $\mathbf{R} = (-6, 10)$.

a) Distance between **P** and **Q** is

$$\|\mathbf{P\text{-}Q}\| = \sqrt{(-3-2)^2 + (-4-6)^2} = \sqrt{125} \tag{1.1.4}$$

b) Distance between **Q** and **R** is

$$\|\mathbf{Q\text{-}R}\| = \sqrt{(2-(-6))^2 + (6-10)^2} = \sqrt{80} \tag{1.1.4}$$

c) Distance between **P** and **R** is

$$\|\mathbf{P\text{-}R}\| = \sqrt{(-3-(-6))^2 + (-4-10)^2} = \sqrt{205} \tag{1.1.4}$$

Here, the largest distance is $\sqrt{205}$. To be vertices of a right angled triangle, we should have

$$\|\mathbf{P\text{-}Q}\|^2 + \|\mathbf{Q\text{-}R}\|^2 = \|\mathbf{R\text{-}P}\|^2 \tag{1.1.4}$$

$$(\sqrt{205})^2 = (\sqrt{125})^2 + (\sqrt{80})^2 \tag{1.1.4}$$

$$205 = 205 \tag{1.1.4}$$

So, the condition is satisfied. So, using Baudhayana's theorem, it is proved that 3 points given are vertices of a right angled triangle. Now, for orthogonality,

$$(\mathbf{P}-\mathbf{Q})^T(\mathbf{Q}-\mathbf{R}) = 0 \tag{1.1.4}$$

We have

a)

$$\mathbf{P}-\mathbf{Q} = (2-(-3), 6-(-4)) \tag{1.1.4}$$

$$\mathbf{P}-\mathbf{Q} = \begin{pmatrix}5\\10\end{pmatrix} \tag{1.1.4}$$

b)

$$\mathbf{Q}-\mathbf{R} = (2-(-6), 6-10) \tag{1.1.4}$$

$$\mathbf{Q}-\mathbf{R} = \begin{pmatrix}8\\-4\end{pmatrix} \tag{1.1.4}$$

c)

$$\mathbf{P}-\mathbf{R} = (-3-(-6), -4-10) \tag{1.1.4}$$

$$\mathbf{P}-\mathbf{R} = \begin{pmatrix}3\\-14\end{pmatrix} \tag{1.1.4}$$

For orthogonality, product of transpose of one point and other must be 0. Here, checking for

$$\begin{pmatrix}5\\10\end{pmatrix}^T\begin{pmatrix}8\\-4\end{pmatrix}=\begin{pmatrix}5\\10\end{pmatrix}^T\begin{pmatrix}8\\-4\end{pmatrix}=0 \tag{1.1.4}$$

Hence, using orthogonality, it is proved that the points form a right angled triangle.
Figure 1.1.4 Right anlged triangle where A=P and B=Q and C=R

5. Plot the points $\begin{pmatrix}0\\2\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}4\\4\end{pmatrix}$ and $\begin{pmatrix}3\\5\end{pmatrix}$ and prove that they are the vertices of a rectangle.
6. Show that $\mathbf{B}=\begin{pmatrix}-2\\-2\end{pmatrix},\mathbf{A}=\begin{pmatrix}-1\\2\end{pmatrix}$ and $\mathbf{C}=\begin{pmatrix}3\\1\end{pmatrix}$ are the vertices of an isosceles triangle.
Solution:
Define a matrix $\mathbf{M}$ such that,

$$\mathbf{M}=\begin{pmatrix}\mathbf{B}-\mathbf{A} & \mathbf{C}-\mathbf{A}\end{pmatrix}^T \tag{1.1.6}$$

$$\mathbf{M}=\begin{pmatrix}-1 & -4\\4 & -1\end{pmatrix} \tag{1.1.6}$$

Using matrix transformation,

$$\mathbf{M}=\begin{pmatrix}-1 & -4\\4 & -1\end{pmatrix}\xleftrightarrow{R_1\leftarrow -R_1-\frac{R_2}{4}}\begin{pmatrix}0 & \frac{17}{4}\\4 & -1\end{pmatrix} \tag{1.1.6}$$

$$\implies rank(\mathbf{M})=2 \tag{1.1.6}$$

Since the rank of matrix $\mathbf{M}$ is 2, the points form a triangle.

$$AB^2=(\mathbf{A}-\mathbf{B})^T(\mathbf{A}-\mathbf{B}) \tag{1.1.6}$$
$$=17 \tag{1.1.6}$$
$$BC^2=(\mathbf{B}-\mathbf{C})^T(\mathbf{B}-\mathbf{C}) \tag{1.1.6}$$
$$=34 \tag{1.1.6}$$
$$CA^2=(\mathbf{C}-\mathbf{A})^T(\mathbf{C}-\mathbf{A}) \tag{1.1.6}$$
$$=17 \tag{1.1.6}$$
$$\implies AB=AC \tag{1.1.6}$$

Hence, the triangle is isosceles. See Fig. 1.1.6

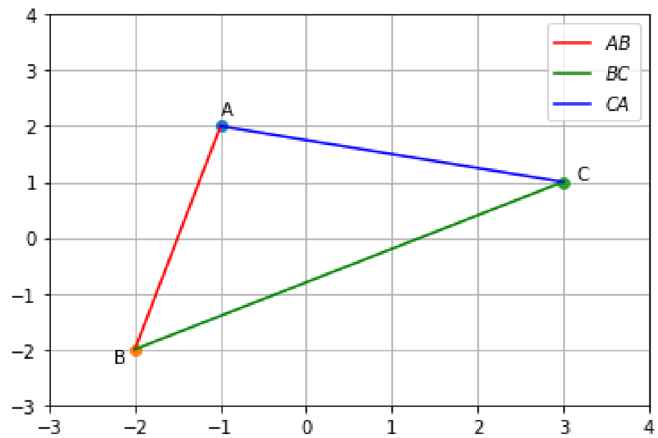

Fig. 1.1.6: Plot of the given points

7. In the last question, find the distance of the vertex $\mathbf{A}$ of the triangle from the middle point of the base BC.
Solution:

From the given information,

$$a^2=(\mathbf{B}-\mathbf{C})^T(\mathbf{B}-\mathbf{C}) \tag{1.1.7}$$
$$=34 \tag{1.1.7}$$
$$c^2=(\mathbf{A}-\mathbf{B})^T(\mathbf{A}-\mathbf{B}) \tag{1.1.7}$$
$$=17 \tag{1.1.7}$$
$$b^2=(\mathbf{C}-\mathbf{A})^T(\mathbf{C}-\mathbf{A}) \tag{1.1.7}$$
$$=17 \tag{1.1.7}$$
$$\implies AB=AC \tag{1.1.7}$$

Thus, the required distance is AD where

$$\mathbf{D}=\frac{\mathbf{B}+\mathbf{C}}{2}=\begin{pmatrix}\frac{-1}{2}\\\frac{-1}{2}\end{pmatrix} \tag{1.1.7}$$

and

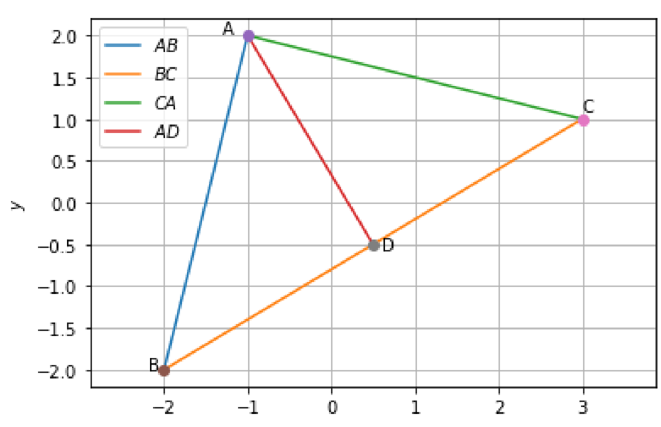

Fig. 1.1.7: plot

$$AD = \|\mathbf{A} - \mathbf{D}\| \tag{1.1.7}$$
$$= \frac{\sqrt{34}}{2} \tag{1.1.7}$$

8. Prove that the points $\begin{pmatrix}-1\\0\end{pmatrix}$, $\begin{pmatrix}0\\3\end{pmatrix}$, $\begin{pmatrix}3\\2\end{pmatrix}$ and $\begin{pmatrix}2\\-1\end{pmatrix}$ are the vertices of a square.
9. Prove that the points $\mathbf{A} = \begin{pmatrix}-1\\0\end{pmatrix}$, $\mathbf{B} = \begin{pmatrix}3\\1\end{pmatrix}$, $\mathbf{C} = \begin{pmatrix}2\\2\end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix}-2\\1\end{pmatrix}$ are the vertices of a parallelogram. Find $\mathbf{E}, \mathbf{F}, \mathbf{G}, \mathbf{H}$, the mid points of AB, BC, CD, AD respectively. Show that EG and FH bisect each other.
Solution:

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix}-4\\-1\end{pmatrix} \tag{1.1.9}$$
$$= -(\mathbf{C} - \mathbf{D}) \tag{1.1.9}$$
$$\mathbf{B} - \mathbf{C} = \begin{pmatrix}1\\-1\end{pmatrix}\mathbf{A} - \mathbf{D} \tag{1.1.9}$$
$$\implies AB \parallel CD, BC \parallel AD \tag{1.1.9}$$

Hence, $ABCD$ is a parallelogram. Also,

$$\mathbf{E} = \frac{\mathbf{A} + \mathbf{B}}{2} = \begin{pmatrix}1\\\frac{1}{2}\end{pmatrix} \tag{1.1.9}$$
$$\mathbf{F} = \frac{\mathbf{B} + \mathbf{C}}{2} = \begin{pmatrix}\frac{5}{2}\\\frac{3}{2}\end{pmatrix} \tag{1.1.9}$$
$$\mathbf{G} = \frac{\mathbf{C} + \mathbf{D}}{2} = \begin{pmatrix}0\\\frac{3}{2}\end{pmatrix} \tag{1.1.9}$$
$$\mathbf{H} = \frac{\mathbf{A} + \mathbf{D}}{2} = \begin{pmatrix}\frac{-3}{2}\\\frac{1}{2}\end{pmatrix} \tag{1.1.9}$$

and

$$\frac{\mathbf{E} + \mathbf{G}}{2} = \begin{pmatrix}\frac{1}{2}\\1\end{pmatrix} \tag{1.1.9}$$
$$= \frac{\mathbf{F} + \mathbf{H}}{2} \tag{1.1.9}$$

See Fig. 1.1.9.
10. Prove that the points $\begin{pmatrix}21\\-2\end{pmatrix}$, $\begin{pmatrix}15\\10\end{pmatrix}$, $\begin{pmatrix}-5\\0\end{pmatrix}$ and $\begin{pmatrix}1\\-12\end{pmatrix}$ are the vertices of a rectangle, and find the coordinates of its centre.

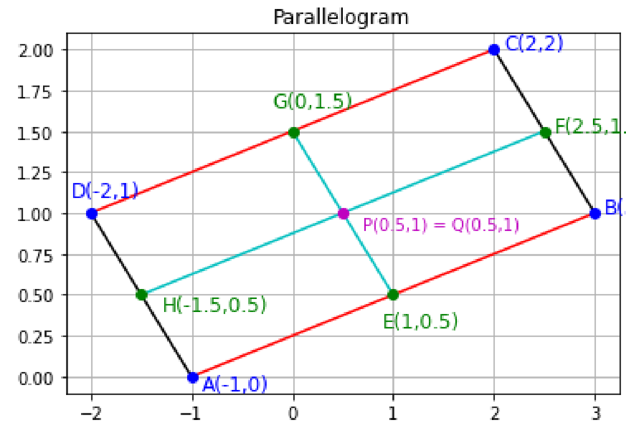

Fig. 1.1.9

Solution: Let

$$\mathbf{A} = \begin{pmatrix}21\\-2\end{pmatrix}, \mathbf{B} = \begin{pmatrix}15\\10\end{pmatrix}, \mathbf{C} = \begin{pmatrix}-5\\0\end{pmatrix}, \mathbf{D} = \begin{pmatrix}1\\-12\end{pmatrix} \tag{1.1.10}$$

Then,

$$\mathbf{A} - \mathbf{B}* = \begin{pmatrix}6\\-12\end{pmatrix} \tag{1.1.10}$$
$$\mathbf{B} - \mathbf{C} = \begin{pmatrix}20\\10\end{pmatrix} \tag{1.1.10}$$
$$\mathbf{C} - \mathbf{D} = \begin{pmatrix}-6\\12\end{pmatrix} \tag{1.1.10}$$
$$\mathbf{D} - \mathbf{A} = \begin{pmatrix}-20\\-10\end{pmatrix} \tag{1.1.10}$$

Since the directional vectors of **AB** and **CD** are in the same ratio, so **AB** and **CD** are parallel and also opposite to each other. Similarly, **BC** and **DA** are parallel and opposite. Hence ABCD is a parallelogram. Also,

$$(\mathbf{B} - \mathbf{A})^\top(\mathbf{C} - \mathbf{B}) = \begin{pmatrix}-6 & 12\end{pmatrix}\begin{pmatrix}-20\\-10\end{pmatrix} \tag{1.1.10}$$
$$= 0 \tag{1.1.10}$$

Therefore, one of the angle is right angle and ABCD is a rectangle. The center

$$\mathbf{O} = \frac{\mathbf{A} + \mathbf{C}}{2} \tag{1.1.10}$$
$$= \begin{pmatrix}8\\-1\end{pmatrix} \tag{1.1.10}$$

This is verified in Fig. 1.1.10.
11. Find the lengths of the medians of the triangle

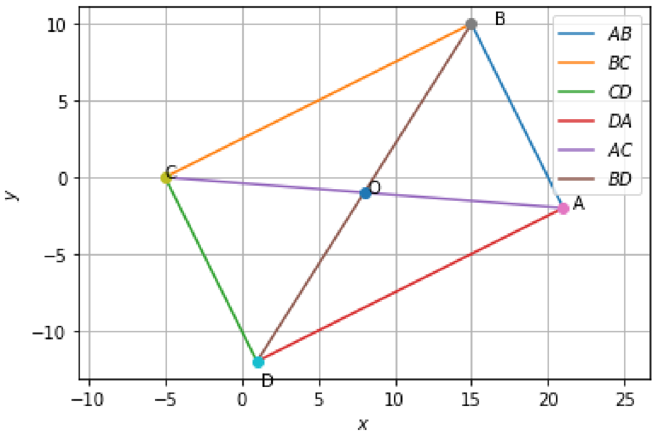

Fig. 1.1.10: plot

whose vertices are at the points $\begin{pmatrix}1\\2\end{pmatrix}$, $\begin{pmatrix}0\\3\end{pmatrix}$ and $\begin{pmatrix}-1\\-2\end{pmatrix}$.

12. Find the coordinates of the points that divide the line joining the points $\begin{pmatrix}-35\\-20\end{pmatrix}$ and $\begin{pmatrix}5\\-10\end{pmatrix}$ into four equal parts.
13. Find the coordinates of the points of trisection of the line joining the points $\begin{pmatrix}-5\\5\end{pmatrix}$ and $\begin{pmatrix}25\\10\end{pmatrix}$.
14. Prove that the middle point of the line joining the points $\begin{pmatrix}-5\\12\end{pmatrix}$ and $\begin{pmatrix}9\\-2\end{pmatrix}$ is a point of trisection of the line joining the points $\begin{pmatrix}-8\\-5\end{pmatrix}$ and $\begin{pmatrix}7\\10\end{pmatrix}$.

    Solution:
15. The points $\begin{pmatrix}8\\5\end{pmatrix}$, $\begin{pmatrix}-7\\-5\end{pmatrix}$ and $\begin{pmatrix}-5\\5\end{pmatrix}$ are three of the vertices of a parallelogram. Find the coordinates of the remaining vertex which is to be taken as opposite to $\begin{pmatrix}-7\\-5\end{pmatrix}$.
16. The point $\begin{pmatrix}2\\6\end{pmatrix}$ is the intesection of the diagonals of a parallelogram two of whose vertices are at the points $\begin{pmatrix}7\\16\end{pmatrix}$ and $\begin{pmatrix}10\\2\end{pmatrix}$. Find the coordinates of the remaining vertices.
17. Find the area of the triangle whose vertices are the points $\begin{pmatrix}2\\3\end{pmatrix}$, $\begin{pmatrix}-4\\7\end{pmatrix}$ and $\begin{pmatrix}5\\-2\end{pmatrix}$.
18. Find the coordinates of points which divide the join of $\begin{pmatrix}2\\3\end{pmatrix}$, $\begin{pmatrix}-4\\5\end{pmatrix}$ externally in the ratio 2 : 3, and also externally in the ratio 3 : 2.
19. Prove the centroid of $\triangle ABC$ is

$$\mathbf{O} = \frac{\mathbf{A}+\mathbf{B}+\mathbf{C}}{3} \tag{1.1.19}$$

1.2 Loci

1. A point moves so that its distance from the point $\begin{pmatrix}2\\1\end{pmatrix}$ is double its distance from the point $\begin{pmatrix}1\\2\end{pmatrix}$. Find the equation of its locus.
2. Find the equation of the perpendicular bisector of the line joining the points $\begin{pmatrix}3\\-4\end{pmatrix}$ and $\begin{pmatrix}-2\\3\end{pmatrix}$.
3. Find the equation of the circle of radius 5 with centre at $\begin{pmatrix}3\\-4\end{pmatrix}$.
4. A point moves so that its distance from the y-axis is equal to the distance from the point $\begin{pmatrix}2\\1\end{pmatrix}$. Find the equation of its locus.
5. A point moves so that the sum of the squares of its distance from the points $\begin{pmatrix}3\\4\end{pmatrix}$ and $\begin{pmatrix}4\\3\end{pmatrix}$ is constant. Find the equation of the locus.
6. A point moves so that its distance from the axis of x is twice its distance from the point $\begin{pmatrix}0\\1\end{pmatrix}$. Find the equation of the locus.
7. A point moves in such a way that with the points $\begin{pmatrix}2\\3\end{pmatrix}$ and $\begin{pmatrix}-3\\4\end{pmatrix}$ it forms a triangle of area 8.5. Show that its locus has an equation

$$\left\{\begin{pmatrix}1 & 5\end{pmatrix}\mathbf{x}\right\}\left\{\begin{pmatrix}1 & 5\end{pmatrix}\mathbf{x}-34\right\} = 0 \tag{1.2.7}$$

2 The Straight Line

2.1 Intercepts

1. Find the intercepts made on the axes by the straight lines whose equations are

    a) $\begin{pmatrix}2 & 3\end{pmatrix}\mathbf{x} = 2$
    b) $\begin{pmatrix}1 & -3\end{pmatrix}\mathbf{x} = -5$
    c) $\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = 0$
    d) $\begin{pmatrix}\frac{1}{a+b} & \frac{1}{a-b}\end{pmatrix}\mathbf{x} = \frac{1}{a^2-b^2}$
    e) $\begin{pmatrix}1 & -m\end{pmatrix}\mathbf{x} = -c$

2. Write down the equations of straight lines which make the following pairs of intercepts on the axes:

a) 3,-4
b) -5,6
c) $\frac{1}{a}, \frac{1}{b}$
d) $2a, -2a$

3. A straight line passes through a fixed point $\begin{pmatrix} h \\ k \end{pmatrix}$ and cuts the axes in $\mathbf{A}, \mathbf{B}$. Parallels to the axes through $\mathbf{A}$ and $\mathbf{B}$ intersect in $\mathbf{P}$. Find the equation of the locus of $\mathbf{P}$.

2.2 Line Equation

1. Find the equations of two straight lines at a distance 3 from the origin and making an angle of $120°$ with OX.
2. Find the equation of a straight line making an angle of $60°$ with OX and passing through the point $\begin{pmatrix} 2 \\ -2 \end{pmatrix}$. Transform the equation to the form

$$\begin{pmatrix} \cos\alpha & \sin\alpha \end{pmatrix} \mathbf{x} = p \qquad (2.2.2)$$

Solution: Let the straight line pass through the point $\mathbf{A} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$ and makes an angle of $60°$ with x-axis.
So slope of the line, $m = \tan 60° = \sqrt{3}$ and the direction vector $\begin{pmatrix} 1 \\ m \end{pmatrix} = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}$.

The vector form of the line passing through the point $\mathbf{A} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$ along the direction vector $\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}$ is given by:

$$\mathbf{X} = \begin{pmatrix} 2 \\ -2 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} \qquad (2.2.2)$$

The normal vector

$$\mathbf{n} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ m \end{pmatrix} = \begin{pmatrix} -\sqrt{3} \\ 1 \end{pmatrix} \qquad (2.2.2)$$

The equation of the line in terms of the normal vector is obtained as

$$\mathbf{n}^{\mathbf{T}} \left(x - A\right) = 0 \qquad (2.2.2)$$

$$\begin{pmatrix} -\sqrt{3} & 1 \end{pmatrix} x = \begin{pmatrix} -\sqrt{3} & 1 \end{pmatrix} A \qquad (2.2.2)$$

$$\begin{pmatrix} -\sqrt{3} & 1 \end{pmatrix} x = \begin{pmatrix} -\sqrt{3} & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \end{pmatrix} \qquad (2.2.2)$$

$$\begin{pmatrix} -\sqrt{3} & 1 \end{pmatrix} x = -2\sqrt{3} - 2 \qquad (2.2.2)$$

$$\begin{pmatrix} \sqrt{3}/2 & -1/2 \end{pmatrix} x = \sqrt{3} + 1 \qquad (2.2.2)$$

$$\begin{pmatrix} cos330° & sin330° \end{pmatrix} x = 2.732 \qquad (2.2.2)$$

See Fig. 2.2.2

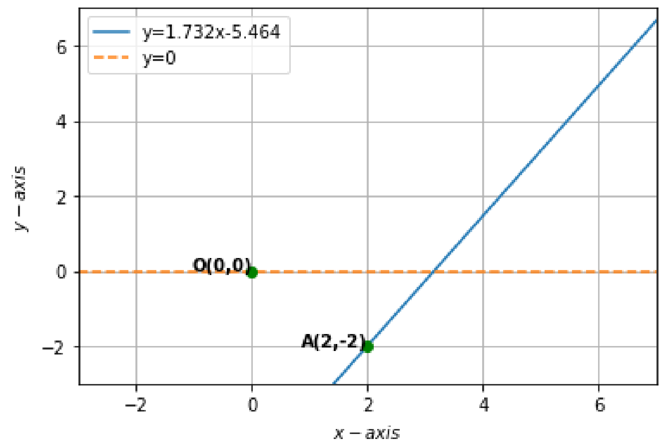

Fig. 2.2.2: This is the 2D diagram of the straight line passing through $\begin{pmatrix} 2 \\ -2 \end{pmatrix}$ and at an angle of $60°$ with the x axis

3. Find the equation of the straight line that passes through the points $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$. What is its inclination to OX?
4. Find the equation of the straight line through the point $\begin{pmatrix} 5 \\ 7 \end{pmatrix}$ that makes equal intercepts on the axes.
5. Find the equations of the sides of a triangle whose vertices are $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$, $\begin{pmatrix} -4 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$.
6. For the same triangle find the equations of the medians
7. Find the equation of a straight line passing through the point $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ parallel to the line $\begin{pmatrix} 4 & -1 \end{pmatrix} \mathbf{x} + 7 = 0$.
8. Find the intercepts on the axes made by a straight line which passes through the point

$\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ and makes an angle of 30° with OX.

9. Find the equation of the straight line through the points $\begin{pmatrix} 3 \\ -4 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and of the parallel line through $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$.
10. What is the distance from the origin of the line $\begin{pmatrix} 4 & -1 \end{pmatrix} \mathbf{x} = 7$? Write down the equation of a parallel line at double the distance.
11. Find the equation of the straight line through the point $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$ parallel to the line joining the origin to the point $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$.
12. Write down the equation of the straight line which makes intercepts 2 and -7 on the axes, and of the parallel line through the point $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$.
13. Find the equations of the straight line joining the points $\begin{pmatrix} 3 \\ 4 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ and of the parallel line through the origin.
14. ABC is a triangle and $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ are the points $\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 5 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -4 \\ 2 \end{pmatrix}$. Find the equation of the straight line through $\mathbf{A}$ parallel to BC.
15. Find the equation of a line parallel to $\begin{pmatrix} 2 & 5 \end{pmatrix} \mathbf{x} = 11$ passing through the middle point of the join of the points $\begin{pmatrix} -7 \\ 3 \end{pmatrix}, \begin{pmatrix} 5 \\ -11 \end{pmatrix}$.

Solution: General equation of straight line is given by:

$$\mathbf{n}^T \mathbf{x} = c \tag{2.2.15}$$

$\mathbf{n}$ will be same because both lines are parallel.

$$\mathbf{n} = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \tag{2.2.15}$$

Passing through mid point $\mathbf{M}$ of $\mathbf{A}, \mathbf{B}$:

$$\mathbf{M} = \frac{\mathbf{A} + \mathbf{B}}{2} \tag{2.2.15}$$

$$\mathbf{n}^T (\mathbf{x} - \mathbf{M}) = 0 \tag{2.2.15}$$

$$\mathbf{n}^T \mathbf{x} = \mathbf{n}^T \mathbf{M} \tag{2.2.15}$$

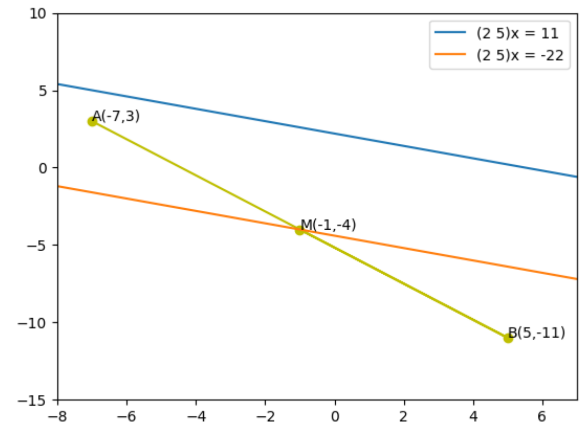

Fig. 2.2.15: Parallel Lines

So, the mid point $\mathbf{M}$ is :

$$\mathbf{M} = \frac{\begin{pmatrix} -7 \\ 3 \end{pmatrix} + \begin{pmatrix} 5 \\ -11 \end{pmatrix}}{2} = \begin{pmatrix} -1 \\ -4 \end{pmatrix} \tag{2.2.15}$$

$$\mathbf{n}^T \mathbf{x} = \begin{pmatrix} 2 & 5 \end{pmatrix} \begin{pmatrix} -1 \\ -4 \end{pmatrix} = -22 \tag{2.2.15}$$

So, the equation of line is :

$$\begin{pmatrix} 2 & 5 \end{pmatrix} \mathbf{x} = -22 \tag{2.2.15}$$

16. The base of a triangle passes through a fixed point $\begin{pmatrix} f \\ g \end{pmatrix}$ and the sides are bisected at right angles by the axes. Prove that the locus of the vertex is the line

$$\begin{pmatrix} g & f \end{pmatrix} \mathbf{x} = 0 \tag{2.2.16}$$

2.3 Point of Intersection

1. Find the vertices of the triangle whose sides are

$$\begin{pmatrix} 3 & 2 \end{pmatrix} \mathbf{x} + 6 = 0, \tag{2.3.1.1}$$

$$\begin{pmatrix} 2 & -5 \end{pmatrix} \mathbf{x} + 4 = 0, \tag{2.3.1.2}$$

$$\begin{pmatrix} 1 & -3 \end{pmatrix} \mathbf{x} - 6 = 0 \tag{2.3.1.3}$$

2. Prove that the lines

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} + 25 = 0, \tag{2.3.2.1}$$

$$\begin{pmatrix} 2 & 3 \end{pmatrix} \mathbf{x} + 7 = 0 \tag{2.3.2.2}$$

$$\begin{pmatrix} 3 & 5 \end{pmatrix} \mathbf{x} = 11 \tag{2.3.2.3}$$

are concurrent, and find the coordinates of their common point.

3. Find the equation of a line parallel to the line
$$\begin{pmatrix}2 & -1\end{pmatrix}\mathbf{x} = 3 \tag{2.3.3.1}$$
and passing through the intersection of the lines
$$\begin{pmatrix}3 & 1\end{pmatrix}\mathbf{x} = 7 \tag{2.3.3.2}$$
$$\begin{pmatrix}2 & -3\end{pmatrix}\mathbf{x} = 5 \tag{2.3.3.3}$$
4. Find the equation of the line joining the origin to the point of intersection of the lines
$$\begin{pmatrix}3 & -5\end{pmatrix}\mathbf{x} = 11 \tag{2.3.4.1}$$
$$\begin{pmatrix}2 & 7\end{pmatrix}\mathbf{x} + 4 = 0 \tag{2.3.4.2}$$
5. Find the acute angle between the lines
$$\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = -7 \tag{2.3.5.1}$$
$$\begin{pmatrix}2+\sqrt{3} & 1\end{pmatrix}\mathbf{x} = 11 \tag{2.3.5.2}$$
6. Find the angle between the lines
$$\begin{pmatrix}-2 & 1\end{pmatrix}\mathbf{x} = 5 \tag{2.3.6.1}$$
$$\begin{pmatrix}2 & 4\end{pmatrix}\mathbf{x} + 11 = 0 \tag{2.3.6.2}$$
7. Find the equation of a straight line through the point $\begin{pmatrix}2\\-4\end{pmatrix}$ at right angles to the line
$$\begin{pmatrix}5 & 7\end{pmatrix}\mathbf{x} + 12 = 0 \tag{2.3.7}$$
and find the point in which the lines intersect.
8. Find the equation of a straight line through the origin and at right angles to the line
$$\begin{pmatrix}a & b\end{pmatrix}\mathbf{x} + c = 0 \tag{2.3.8}$$
9. Find the equation of a straight line at right angles to the line
$$\begin{pmatrix}5 & -2\end{pmatrix}\mathbf{x} + 11 = 0 \tag{2.3.9.1}$$
and passing through the intersecton of the lines
$$\begin{pmatrix}1 & 2\end{pmatrix}\mathbf{x} + 1 = 0, \tag{2.3.9.2}$$
$$\begin{pmatrix}-1 & 1\end{pmatrix}\mathbf{x} = 7. \tag{2.3.9.3}$$
10. The origin is a corner of a square and two of its sides have equations
$$\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} = 0 \tag{2.3.10.1}$$
$$\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} = 3. \tag{2.3.10.2}$$
Find the equations of the other two sides.
11. Write down the equations of the perpendiculars from the origin to the lines
$$\begin{pmatrix}1 & 5\end{pmatrix}\mathbf{x} = 13, \tag{2.3.11.1}$$
$$\begin{pmatrix}5 & 1\end{pmatrix}\mathbf{x} = 13 \tag{2.3.11.2}$$
and find the equation of the line joining the feet of the perpendiculars.
12. Prove that the line
$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 11 \tag{2.3.12.1}$$
makes equal angles with the lines
$$\begin{pmatrix}1 & -\left(2-\sqrt{3}\right)\end{pmatrix}\mathbf{x} + 2 = 0, \tag{2.3.12.2}$$
$$\begin{pmatrix}\left(2-\sqrt{3}\right) & -1\end{pmatrix}\mathbf{x} + 5 = 0 \tag{2.3.12.3}$$
13. **A** is the point $\begin{pmatrix}-4\\0\end{pmatrix}$ and **B** is the point $\begin{pmatrix}3\\0\end{pmatrix}$. Find the locus of a point **P** such that the angles APO, OPB are equal, where **O** is the origin.

2.4 Perpendiculars and Bisectors

1. Find the distance of the point $\begin{pmatrix}4\\2\sqrt{3}\end{pmatrix}$ from the line
$$\begin{pmatrix}\cos 60^\circ & \sin 60^\circ\end{pmatrix}\mathbf{x} = 6 \tag{2.4.1}$$
2. Find the distance of the point $\begin{pmatrix}2\\3\end{pmatrix}$ from each of the straight lines
$$\begin{pmatrix}5 & 12\end{pmatrix}\mathbf{x} - 20 = 0 \tag{2.4.2.1}$$
$$\begin{pmatrix}4 & -3\end{pmatrix}\mathbf{x} + 11 = 0 \tag{2.4.2.2}$$
$$\begin{pmatrix}3 & 4\end{pmatrix}\mathbf{x} - 28 = 0. \tag{2.4.2.3}$$
3. Find the distance of the pont $\begin{pmatrix}3\\-1\end{pmatrix}$ from the line joining the points $\begin{pmatrix}2\\-3\end{pmatrix}, \begin{pmatrix}4\\1\end{pmatrix}$.
4. Are the points $\begin{pmatrix}-2\\3\end{pmatrix}, \begin{pmatrix}-2\\4\end{pmatrix}$ on the same or on opposite sides of the line
$$\begin{pmatrix}4 & 5\end{pmatrix}\mathbf{x} = 10? \tag{2.4.4}$$
5. Find the equations of the bisectors of the angles

between the lines

$$\begin{pmatrix}-4 & 2\end{pmatrix}\mathbf{x} = -9 \tag{2.4.5.1}$$

$$\begin{pmatrix}-1 & 2\end{pmatrix}\mathbf{x} = 4 \tag{2.4.5.2}$$

and state which equation refers to the angle which contains the origin.

6. Prove that the bisector of one of the angles between the lines

$$\begin{pmatrix}5 & 1\end{pmatrix}\mathbf{x} - 7 = 0 \tag{2.4.6.1}$$

$$\begin{pmatrix}1 & -5\end{pmatrix}\mathbf{x} + 7 = 0 \tag{2.4.6.2}$$

passes through the origin. What is the equation of the bisector of the other angle?

7. What is the condition that the point $\begin{pmatrix}x\\y\end{pmatrix}$ may be at unit distance from the line

$$\begin{pmatrix}3 & -4\end{pmatrix}\mathbf{x} + 10 = 0 \tag{2.4.7}$$

Write down the equations of two straight lines parallel to the given line and at unit distances from it, and state which of the two lies on the same side of the given line as the origin.

8. The sides AB, BC, CA of a triangle have equations

$$\begin{pmatrix}4 & -3\end{pmatrix}\mathbf{x} = 12 \tag{2.4.8.1}$$

$$\begin{pmatrix}3 & 4\end{pmatrix}\mathbf{x} = 24 \tag{2.4.8.2}$$

$$\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} = 2. \tag{2.4.8.3}$$

Find the coordinates of the centres of the inscribed circle and of the escribed circle opposite to the vertex **A**.

Solution:

If the three vertices are located at

$$\mathbf{A}, \mathbf{B}, \mathbf{C} \tag{2.4.8.4}$$

and the sides opposite these vertices have corresponding lengths

$$a, b \quad and \quad c \tag{2.4.8.5}$$

then:

The coordinates of the centre of the Inscribed circle

$$\mathbf{O} = \frac{1}{a+b+c}\left(a\mathbf{A} + b\mathbf{B} + c\mathbf{C}\right) \tag{2.4.8.6}$$

And the coordinates of the centre of the Escribed circle opposite to the vertex **A**

$$\mathbf{P} = \frac{1}{b+c-a}\left(b\mathbf{B} + c\mathbf{C} - a\mathbf{A}\right) \tag{2.4.8.7}$$

The vertices **A** is the intersection of the sides **AB** and **CA**.Thus **A** is obtained from

$$\begin{pmatrix}4 & -3\\0 & 1\end{pmatrix}x = \begin{pmatrix}12\\2\end{pmatrix} \tag{2.4.8.8}$$

The augmented matrix for the above equation is row reduced as follows

$$\begin{pmatrix}4 & -3 & 12\\0 & 1 & 2\end{pmatrix} \xleftrightarrow{R_1=R_1+3R_2} \begin{pmatrix}4 & 0 & 18\\0 & 1 & 2\end{pmatrix} \tag{2.4.8.9}$$

$$\xleftrightarrow{R_1=\frac{1}{4}R_1} \begin{pmatrix}1 & 0 & \frac{9}{2}\\0 & 1 & 2\end{pmatrix} \tag{2.4.8.10}$$

$$\implies \mathbf{A} = \begin{pmatrix}\frac{9}{2}\\2\end{pmatrix} \tag{2.4.8.11}$$

The vertices **B** is the intersection of the sides **AB** and **BC**.Thus **B** is obtained from

$$\begin{pmatrix}4 & -3\\3 & 4\end{pmatrix}x = \begin{pmatrix}12\\24\end{pmatrix} \tag{2.4.8.12}$$

The augmented matrix for the above equation is row reduced as follows

$$\begin{pmatrix}4 & -3 & 12\\3 & 4 & 24\end{pmatrix} \xleftrightarrow{R_2=4R_2-3R_1} \begin{pmatrix}4 & -3 & 12\\0 & 25 & 60\end{pmatrix} \tag{2.4.8.13}$$

$$\xleftrightarrow{R_2=\frac{1}{25}R_2} \begin{pmatrix}4 & -3 & 12\\0 & 1 & \frac{12}{5}\end{pmatrix} \xleftrightarrow{R_1=R_1+3R_2} \begin{pmatrix}4 & 0 & \frac{96}{5}\\0 & 1 & \frac{12}{5}\end{pmatrix} \tag{2.4.8.14}$$

$$\xleftrightarrow{R_1=\frac{1}{4}R_1} \begin{pmatrix}1 & 0 & \frac{24}{5}\\0 & 1 & \frac{12}{5}\end{pmatrix} \tag{2.4.8.15}$$

$$\implies \mathbf{B} = \begin{pmatrix}\frac{24}{5}\\\frac{12}{5}\end{pmatrix} \tag{2.4.8.16}$$

The vertices **C** is the intersection of the sides **BC** and**CA**.Thus **C** is obtained from

$$\begin{pmatrix}3 & 4\\0 & 1\end{pmatrix}x = \begin{pmatrix}24\\2\end{pmatrix} \tag{2.4.8.17}$$

The augmented matrix for the above equation is row reduced as follows

$$\begin{pmatrix}3 & 4 & 24\\0 & 1 & 2\end{pmatrix} \xleftrightarrow{R_1=R_1-4R_2} \begin{pmatrix}3 & 0 & 16\\0 & 1 & 2\end{pmatrix} \tag{2.4.8.18}$$

$$\xleftrightarrow{R_1=\frac{1}{3}R_1} \begin{pmatrix}1 & 0 & \frac{16}{3}\\0 & 1 & 2\end{pmatrix} \tag{2.4.8.19}$$

$$\implies \mathbf{C} = \begin{pmatrix}\frac{16}{3}\\2\end{pmatrix} \tag{2.4.8.20}$$

The length of the side **AB**

$$c = \|(\mathbf{B}-\mathbf{A})\| = \left\|\begin{pmatrix}\frac{24}{5}\\\frac{12}{5}\end{pmatrix} - \begin{pmatrix}\frac{9}{2}\\2\end{pmatrix}\right\| = 0.5 \tag{2.4.8.21}$$

The length of the side **BC**

$$a = \|(\mathbf{C}-\mathbf{B})\| = \left\|\begin{pmatrix}\frac{16}{3}\\2\end{pmatrix} - \begin{pmatrix}\frac{24}{5}\\\frac{12}{5}\end{pmatrix}\right\| = 0.666 \tag{2.4.8.22}$$

The length of the side **CA**

$$b = \|(\mathbf{A}-\mathbf{C})\| = \left\|\begin{pmatrix}\frac{9}{2}\\2\end{pmatrix} - \begin{pmatrix}\frac{16}{3}\\2\end{pmatrix}\right\| = 0.833 \tag{2.4.8.23}$$

Therefore the coordinates of the centre of the Inscribed circle

$$\mathbf{O} = \frac{1}{.666+.833+0.5}\left(0.666\begin{pmatrix}\frac{9}{2}\\2\end{pmatrix} + 0.833\begin{pmatrix}\frac{24}{5}\\\frac{12}{5}\end{pmatrix} + .5\begin{pmatrix}\frac{16}{3}\\2\end{pmatrix}\right) \tag{2.4.8.24}$$

$$= \begin{pmatrix}4.833\\2.166\end{pmatrix} \tag{2.4.8.25}$$

And the coordinates of the centre of the Escribed circle opposite to the vertex **A**

$$\mathbf{P} = \frac{1}{.833+0.5-.666}\left(0.833\begin{pmatrix}\frac{24}{5}\\\frac{12}{5}\end{pmatrix} + .5\begin{pmatrix}\frac{16}{3}\\2\end{pmatrix} - 0.666\begin{pmatrix}\frac{9}{2}\\2\end{pmatrix}\right) \tag{2.4.8.26}$$

$$= \begin{pmatrix}5.499\\2.499\end{pmatrix} \tag{2.4.8.27}$$

See Fig. 2.4.8

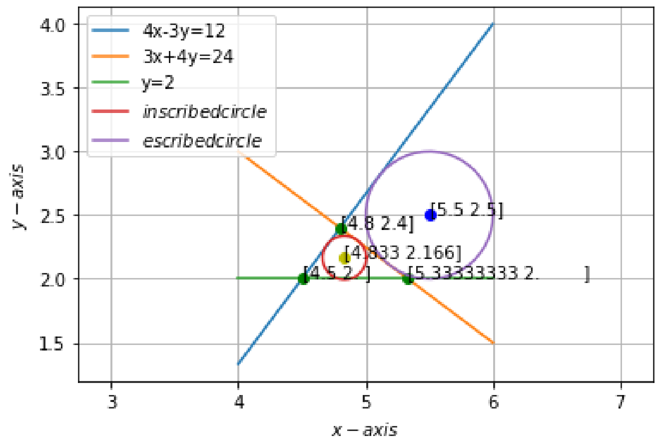

Fig. 2.4.8: This is the 2D diagram of the triangle , the inscribed circle and the escribed circle opposite to vertex **A**

9. Prove that the point $\begin{pmatrix}4\\4\end{pmatrix}$ lies outside the triangle whose sides are the lines

$$\begin{pmatrix}3 & 4\end{pmatrix}\mathbf{x} = 24 \tag{2.4.9.1}$$
$$\begin{pmatrix}5 & -3\end{pmatrix}\mathbf{x} = 15 \tag{2.4.9.2}$$
$$\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} = 0 \tag{2.4.9.3}$$

10. Find the equation of the line joining the origin to the point of intersection of the lines

$$\begin{pmatrix}1 & 7\end{pmatrix}\mathbf{x} - 11 = 0 \tag{2.4.10.1}$$
$$\begin{pmatrix}-2 & 1\end{pmatrix}\mathbf{x} = 3 \tag{2.4.10.2}$$

11. Find the equation of a line perpendicular to the line

$$\begin{pmatrix}3 & 5\end{pmatrix}\mathbf{x} + 11 = 0 \tag{2.4.11.1}$$

and passing through the intersection of the lines

$$\begin{pmatrix}5 & -6\end{pmatrix}\mathbf{x} = 1 \tag{2.4.11.2}$$
$$\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x} + 5 = 0 \tag{2.4.11.3}$$

12. Find the equation of a line through the intersection of the lines

$$\begin{pmatrix}2 & 5\end{pmatrix}\mathbf{x} = 1 \tag{2.4.12.1}$$
$$\begin{pmatrix}-4 & 1\end{pmatrix}\mathbf{x} = 9 \tag{2.4.12.2}$$

parallel to the line

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 1 \tag{2.4.12.3}$$

13. The vertices of a triangle are at the points

$$\mathbf{A}, \mathbf{B}, \mathbf{C} \tag{2.4.13}$$

Find the equations of the medians and prove that they meet in a point. What are the coordinates of their point of intersection?

14. For what multiples k, l, m is the equation
$$k\left\{\begin{pmatrix}2 & 3\end{pmatrix}\mathbf{x} - 13\right\} + l\left\{\begin{pmatrix}5 & -y\end{pmatrix}\mathbf{x} - 7\right\} + m\left\{\begin{pmatrix}1 & -4\end{pmatrix}\mathbf{x} + 10\right\} = 0 \quad (2.4.14)$$
an identity? In what point do the lines given by equating the three terms to zero concur?
15. Find the equations of the diagonals of the parallelogram
$$\begin{pmatrix}2 & -1\end{pmatrix}\mathbf{x} + 7 = 0 \quad (2.4.15)$$
$$\begin{pmatrix}2 & -1\end{pmatrix}\mathbf{x} - 5 = 0, \quad (2.4.15)$$
$$\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x} - 5 = 0 \quad (2.4.15)$$
$$\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x} + 4 = 0 \quad (2.4.15)$$
16. The vertices of a triangle are at the points
$$\begin{pmatrix}2\\3\end{pmatrix}, \begin{pmatrix}4\\-3\end{pmatrix}, \begin{pmatrix}-2\\1\end{pmatrix} \quad (2.4.16)$$
Find the equations of the perpendiculars to the sides through their middle points.
17. Work the same problem when the vertices of the triangle are at the points
$$\mathbf{A}, \mathbf{B}, \mathbf{C} \quad (2.4.17.1)$$
and show that the perpendiculars meet in a point.
18. The line
$$\begin{pmatrix}2 & -8\end{pmatrix}\mathbf{x} - 4 = 0 \quad (2.4.18.1)$$
is the perpendicular bisector of the line AB and $\mathbf{A}$ is the point $\begin{pmatrix}5\\6\end{pmatrix}$. What are the coordinates of $\mathbf{B}$?

2.5 Angle Between Lines

1. What lines are represented by the following equations:
   a) $\mathbf{x}^T\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\mathbf{x} = 0$
   b) $\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\mathbf{x} = 0$
   c) $\mathbf{x}^T\begin{pmatrix}6 & \frac{1}{2}\\\frac{1}{2} & -1\end{pmatrix}\mathbf{x} = 0$
   d) $\mathbf{x}^T\begin{pmatrix}-1 & -\tan\theta\\\tan\theta & 1\end{pmatrix}\mathbf{x} = 0$
   e) $x^3 + 3x^2y - 3xy^2 - y^3 = 0$
2. Find the angles between the pairs of straight lines represented by the following equations:
   a) $\mathbf{x}^T\begin{pmatrix}1 & -2\\2 & 1\end{pmatrix}\mathbf{x} = 0$
   b) $\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\mathbf{x} = 0$
   c) $\mathbf{x}^T\begin{pmatrix}1 & -\frac{5}{2}\\-\frac{5}{2} & 4\end{pmatrix}\mathbf{x} = 0$
   d) $\mathbf{x}^T\begin{pmatrix}1 & 2\\2 & 0\end{pmatrix}\mathbf{x} = 0$
   e) $\mathbf{x}^T\begin{pmatrix}1 & \frac{1}{2}\\\frac{1}{2} & -6\end{pmatrix}\mathbf{x} = 0$
3. Prove that the equations
$$\mathbf{x}^T\begin{pmatrix}1 & -2\\-2 & 1\end{pmatrix}\mathbf{x} = 0 \quad (2.5.3.1)$$
$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 3 \quad (2.5.3.2)$$
are the sides of an equilateral triangle.

2.6 Miscellaneous

1. Find the locus of a point which is equidistant from the points $\begin{pmatrix}6\\-3\end{pmatrix}, \begin{pmatrix}-4\\7\end{pmatrix}$.
2. Find the point on the line
$$\begin{pmatrix}2 & 5\end{pmatrix}\mathbf{x} + 7 = 0 \quad (2.6.2)$$
which is equidistant from the points $\begin{pmatrix}2\\-3\end{pmatrix}$, $\begin{pmatrix}-4\\1\end{pmatrix}$.
3. Find the coordinates of the circumcentre of the triangle whose corners are at the points $\begin{pmatrix}4\\3\end{pmatrix}$, $\begin{pmatrix}-1\\2\end{pmatrix}, \begin{pmatrix}2\\-2\end{pmatrix}$.
4. Find the equations of the lines through $\begin{pmatrix}3\\1\end{pmatrix}$ which are respectively parallel and perpendicular to the line joining the points $\begin{pmatrix}2\\4\end{pmatrix}, \begin{pmatrix}5\\-6\end{pmatrix}$.
5. Find the locus of a point at which the join of the points $\begin{pmatrix}2\\1\end{pmatrix}$ and $\begin{pmatrix}-3\\4\end{pmatrix}$ subtends a right angle.
6. Find the orthocentre of a triangle whose corners are at the points $\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}-3\\-4\end{pmatrix}, \begin{pmatrix}6\\2\end{pmatrix}$.
7. Prove that the line joining the points $\begin{pmatrix}2\\-1\end{pmatrix}, \begin{pmatrix}-3\\5\end{pmatrix}$ makes with the axes a triangle of area $\frac{49}{60}$.
8. $ABCD$ is a parallelogram and the coordinates of $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are $\begin{pmatrix}2\\4\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}$ and $\begin{pmatrix}4\\1\end{pmatrix}$ Find the coordinates of $\mathbf{D}$.

9. Find the area of the triangle formed by the lines

$$\begin{pmatrix}3 & -2\end{pmatrix}\mathbf{x} = 5 \quad (2.6.9.1)$$

$$\begin{pmatrix}3 & 4\end{pmatrix}\mathbf{x} = 7 \quad (2.6.9.2)$$

$$\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} + 2 = 0 \quad (2.6.9.3)$$

10. Find the centre of the inscribed circle of the triangle whose sides are

$$\begin{pmatrix}3 & -4\end{pmatrix}\mathbf{x} = 0 \quad (2.6.10.1)$$

$$\begin{pmatrix}12 & -5\end{pmatrix}\mathbf{x} = 0, \quad (2.6.10.2)$$

$$\begin{pmatrix}4 & 3\end{pmatrix}\mathbf{x} = 8 \quad (2.6.10.3)$$

11. The ends of a diagonal of a square are on the coordinate axes at the points $\begin{pmatrix}2a\\0\end{pmatrix}$, $\begin{pmatrix}0\\a\end{pmatrix}$. Find the equations of the sides.
12. The sides of a triangle ABC are

$$AB = 3, BC = 5, CA = 4 \quad (2.6.12)$$

and $\mathbf{A}, \mathbf{B}$ are on the axes OX, OY respectively, while AC makes an angle θ with OX. Prove that the locus of $\mathbf{C}$, as θ varies, is given by the equation

$$\mathbf{x}^T\begin{pmatrix}16 & -12\\-12 & 25\end{pmatrix}\mathbf{x} = 256 \quad (2.6.12)$$

13. Prove that the locus of a point at which the join of the points $\begin{pmatrix}a\\0\end{pmatrix}$ and $\begin{pmatrix}-a\\0\end{pmatrix}$ subtends an angle of 45° is

$$\mathbf{x}^T\mathbf{x} - 2a\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} = a^2 \quad (2.6.13)$$

14. Prove that the line

$$\mathbf{n}^T\mathbf{x} + c = 0 \quad (2.6.14.1)$$

divides the line joining the points $\mathbf{x}_1, \mathbf{x}_2$ in the ratio

$$-\frac{\mathbf{n}^T\mathbf{x}_1 + c}{\mathbf{n}^T\mathbf{x}_2 + c} \quad (2.6.14.2)$$

15. Find the equation of the line joining the point $\mathbf{x}_1$, to the point of intersection of the lines

$$\mathbf{n}^T\mathbf{x} + c = 0 \quad (2.6.15.1)$$

$$\mathbf{n}_1^T\mathbf{x} + c_1 = 0 \quad (2.6.15.2)$$

16. Find the equations of the diagonals of the parallelogram whose sides are

$$\mathbf{n}^T\mathbf{x} + c = 0 \quad (2.6.16.1)$$

$$\mathbf{n}^T\mathbf{x} + d = 0 \quad (2.6.16.2)$$

$$\mathbf{n}_1^T\mathbf{x} + c_1 = 0 \quad (2.6.16.3)$$

$$\mathbf{n}_1^T\mathbf{x} + d_1 = 0 \quad (2.6.16.4)$$

17. Prove that for all values of k the line

$$\begin{pmatrix}2 + k & 1 - 2k\end{pmatrix}\mathbf{x} + 5 = 0 \quad (2.6.17)$$

passes through a fixed point, and find its coordinates.
18. Find the angle between the lines

$$\mathbf{x}^T\begin{pmatrix}1 & -\sec\theta\\-\sec\theta & 1\end{pmatrix}\mathbf{x} = 0 \quad (2.6.18)$$

19. Prove that the pairs of straight lines represented by

$$\mathbf{x}^T\begin{pmatrix}1 & \frac{1}{2}\\\frac{1}{2} & 0\end{pmatrix}\mathbf{x} = 0 \quad (2.6.19.1)$$

$$\mathbf{x}^T\begin{pmatrix}6 & -\frac{1}{2}\\-\frac{1}{2} & -1\end{pmatrix}\mathbf{x} = 0 \quad (2.6.19.2)$$

are such that the angles between one pair are equal to the angles between the other pair.
20. Find the angles between the lines

$$x^3 - 3x^2y - 3xy^2 + y^3 = 0 \quad (2.6.20)$$

21. Find the area of the triangle whose sides are given by

$$\mathbf{x}^T\begin{pmatrix}1 & -2\\-2 & 3\end{pmatrix}\mathbf{x} = 0 \quad (2.6.21.1)$$

$$\begin{pmatrix}3 & 4\end{pmatrix}\mathbf{x} = 7 \quad (2.6.21.2)$$

22. Show that the equation

$$\mathbf{x}^T\begin{pmatrix}6 & -\frac{1}{2}\\-\frac{1}{2} & -15\end{pmatrix}\mathbf{x} + \begin{pmatrix}-11 & 31\end{pmatrix}\mathbf{x} - 10 = 0 \quad (2.6.22)$$

represents two straight lines,and find the equations of the bisectors of the angles between them.

Solution:

construction Any quadratic equation in terms of x, y of the form $ax^2 + 2bxy + cy^2 + 2dx +$

$2ey + f = 0$,can be written as

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \quad (2.6.22)$$

$$\text{where,}\mathbf{V} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad (2.6.22)$$

$$\mathbf{u} = \begin{pmatrix} d \\ e \end{pmatrix} \quad (2.6.22)$$

The equation (2.6.22) represents two intersecting straight lines when

$$\begin{vmatrix} \mathbf{V} & u \\ u^T & f \end{vmatrix} = 0 \quad (2.6.22)$$

$$\left|\mathbf{V}\right| < 0 \quad (2.6.22)$$

explanation From equation (2.6.22) we get

$$\mathbf{V} = \begin{pmatrix} 6 & -\frac{1}{2} \\ -\frac{1}{2} & -15 \end{pmatrix} \quad (2.6.22)$$

$$\mathbf{u} = \begin{pmatrix} \frac{-11}{2} \\ \frac{31}{2} \end{pmatrix} \quad (2.6.22)$$

$$f = -10 \quad (2.6.22)$$

calculating the equation (2.6.22),we get

$$\begin{pmatrix} 6 & -\frac{1}{2} & \frac{-11}{2} \\ -\frac{1}{2} & -15 & \frac{31}{2} \\ \frac{-11}{2} & \frac{31}{2} & -10 \end{pmatrix} \xleftrightarrow{R_3=R_3+R_2+R_1} \begin{pmatrix} 6 & -\frac{1}{2} & \frac{-11}{2} \\ -\frac{1}{2} & -15 & \frac{31}{2} \\ 0 & 0 & 0 \end{pmatrix} \quad (2.6.22)$$

Therefore the determinant 0.And also the determinant of $\mathbf{V}$ is

$$\left|\mathbf{V}\right| = \begin{vmatrix} 6 & -\frac{1}{2} \\ -\frac{1}{2} & -15 \end{vmatrix} \quad (2.6.22)$$

$$= -90.25 \quad (2.6.22)$$

$$< 0 \quad (2.6.22)$$

Therefore the given equation represents the equation of two straight lines which intersect. point of intersection The point of intersection of the straight lines is given by

$$c = -\mathbf{V}^{-1}\mathbf{u} \quad (2.6.22)$$

The inverse of $\mathbf{V}$ can be found by using rref of augmented matrix of the matrices $\mathbf{V}$ and $\mathbf{I}$

$$\begin{pmatrix} 6 & -\frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & -15 & 0 & 1 \end{pmatrix} \xleftrightarrow{R_2=12R_2+R_1} \begin{pmatrix} 6 & -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{361}{2} & 1 & 12 \end{pmatrix} \quad (2.6.22)$$

$$\begin{pmatrix} 6 & -\frac{1}{2} & 1 & 0 \\ 0 & 1 & -\frac{2}{361} & -\frac{24}{361} \end{pmatrix} \xleftrightarrow{R_1=R_1+\frac{R_2}{2}} \begin{pmatrix} 6 & 0 & \frac{360}{361} & -\frac{12}{361} \\ 0 & 1 & -\frac{2}{361} & -\frac{24}{361} \end{pmatrix} \quad (2.6.22)$$

$$\mathbf{V}^{-1} = \begin{pmatrix} \frac{60}{361} & -\frac{2}{361} \\ -\frac{2}{361} & -\frac{24}{361} \end{pmatrix} \quad (2.6.22)$$

$$c = -\begin{pmatrix} \frac{60}{361} & -\frac{2}{361} \\ -\frac{2}{361} & -\frac{24}{361} \end{pmatrix}\begin{pmatrix} -\frac{11}{2} \\ \frac{31}{2} \end{pmatrix} \quad (2.6.22)$$

$$c = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (2.6.22)$$

Therefore the lines intersect at the point $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. eigenvectors The characteristic equation of the matrix $\mathbf{V}$ is

$$\left|\mathbf{V} - \lambda\mathbf{I}\right| = 0 \quad (2.6.22)$$

$$\begin{vmatrix} 6-\lambda & -\frac{1}{2} \\ -\frac{1}{2} & -15-\lambda \end{vmatrix} = 0 \quad (2.6.22)$$

$$\lambda^2 + 9\lambda - 90.25 = 0 \quad (2.6.22)$$

So the eigenvalues will be

$$\lambda_1 = \frac{-1}{2}\left(9 + \sqrt{442}\right) \quad (2.6.22)$$

$$\lambda_2 = \frac{-1}{2}\left(9 - \sqrt{442}\right) \quad (2.6.22)$$

The eigen vectors will be in the nullspace of $\mathbf{V} - \lambda_1\mathbf{I}$ and $\mathbf{V} - \lambda_2\mathbf{I}$.The eigen vector corresponding to eigen value λ_1 will be

$$\mathbf{V} - \lambda_1\mathbf{I} = \begin{pmatrix} 6 + \frac{1}{2}\left(9 + \sqrt{442}\right) & -\frac{1}{2} \\ -\frac{1}{2} & -15 + \frac{1}{2}\left(9 + \sqrt{442}\right) \end{pmatrix} \quad (2.6.22)$$

$$= \begin{pmatrix} \frac{1}{2}\left(21 + \sqrt{442}\right) & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2}\left(-21 + \sqrt{442}\right) \end{pmatrix} \quad (2.6.22)$$

$$\xleftrightarrow{R_2=\left(21+\sqrt{442}\right)R_2+R_1} \begin{pmatrix} \frac{1}{2}\left(21 + \sqrt{442}\right) & -\frac{1}{2} \\ 0 & 0 \end{pmatrix} \quad (2.6.22)$$

The above reduced matrix has one free variable.Let it be 1,then the eigen vector will be

$$p_1 = \begin{pmatrix} 1 \\ 21 + \sqrt{442} \end{pmatrix} \quad (2.6.22)$$

normalizing p_1,we get

$$p_1 = \frac{1}{\sqrt{884 + 42\sqrt{442}}} \begin{pmatrix} 1 \\ 21 + \sqrt{442} \end{pmatrix} \quad (2.6.22)$$

the eigen vector corresponding to eigen value λ_2 will be

$$\mathbf{V} - \lambda_1 \mathbf{I} = \begin{pmatrix} 6 + \frac{1}{2}\left(9 - \sqrt{442}\right) & -\frac{1}{2} \\ -\frac{1}{2} & -15 + \frac{1}{2}\left(9 - \sqrt{442}\right) \end{pmatrix} \quad (2.6.22)$$

$$= \begin{pmatrix} \frac{1}{2}\left(21 - \sqrt{442}\right) & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2}\left(-21 - \sqrt{442}\right) \end{pmatrix} \quad (2.6.22)$$

$$\xleftrightarrow{R_2 = (21 - \sqrt{442}) R_2 + R_1} \begin{pmatrix} \frac{1}{2}\left(21 - \sqrt{442}\right) & -\frac{1}{2} \\ 0 & 0 \end{pmatrix} \quad (2.6.22)$$

The above reduced matrix has one free variable.Let it be 1,then the eigen vector will be

$$p_2 = \begin{pmatrix} 1 \\ 21 - \sqrt{442} \end{pmatrix} \quad (2.6.22)$$

normalizing p_2,we get

$$p_2 = \frac{1}{\sqrt{884 - 42\sqrt{442}}} \begin{pmatrix} 1 \\ 21 - \sqrt{442} \end{pmatrix} \quad (2.6.22)$$

So the transformation matrix will be

$$\mathbf{P} = \begin{pmatrix} p_1 & p_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{884+42\sqrt{442}}} & \frac{1}{\sqrt{884-42\sqrt{442}}} \\ \frac{21+\sqrt{442}}{\sqrt{884+42\sqrt{442}}} & \frac{21-\sqrt{442}}{\sqrt{884-42\sqrt{442}}} \end{pmatrix} \quad (2.6.22)$$

affine transformation Doing the affine transformation on given quadratic equation, we get pair to intersecting straight lines passing through origin.
Let the affine transformation be $\mathbf{x} = \mathbf{P}\mathbf{y} + c$.The transformation will be

$$(\mathbf{P}\mathbf{y} + c)^T \mathbf{V} (\mathbf{P}\mathbf{y} + c) + 2\mathbf{u}^T (\mathbf{P}\mathbf{y} + c) + f = 0 \quad (2.6.22)$$

$$(2.6.22)$$

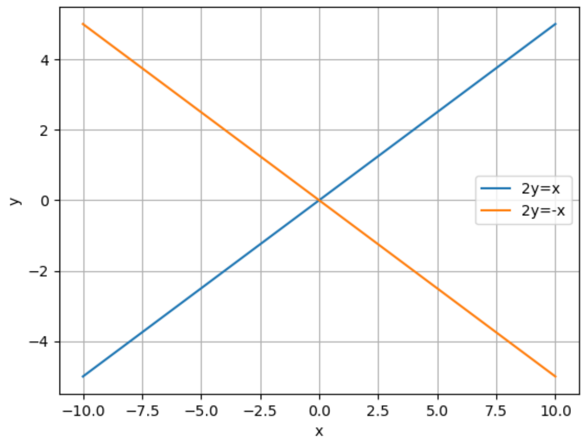

Fig. 2.6.22: straight lines after affine transformation passing through origin

$$\mathbf{y}^T \left(\mathbf{P}^T \mathbf{V} \mathbf{P}\right) \mathbf{y} + 2\left(c^T \mathbf{V} + \mathbf{u}^T\right) \mathbf{P}\mathbf{y} + c^T \mathbf{V} c + 2\mathbf{u}^T c + f = 0 \quad (2.6.22)$$

if the point c is taken as the point of intersection of the two lines.

$$c^T \mathbf{V} c + 2\mathbf{u}^T c + f = 0 \quad (2.6.22)$$

$$c^T \mathbf{V} + \mathbf{u}^T = 0 \quad (2.6.22)$$

So the affine transformation of the given lines will be

$$\mathbf{y}^T \left(\mathbf{P}^T \mathbf{V} \mathbf{P}\right) \mathbf{y} = 0 \quad (2.6.22)$$

$$\mathbf{y}^T \begin{pmatrix} -1.5 & 0 \\ 0 & 6 \end{pmatrix} \mathbf{y} = 0 \quad (2.6.22)$$

$$(x - 2y)(x + 2y) = 0 \quad (2.6.22)$$

Since the two lines are symmetric with respect to both $X - axis$ and $Y - axis$,the axes themselves are the bisectors of the transformed pair of lines.So the bisectors will be $x = 0$ and $y = 0$. Matrix notation will be of the form

$$\mathbf{y}^T \begin{pmatrix} 0 & 0.5 \\ 0.5 & 0 \end{pmatrix} \mathbf{y} = 0 \quad (2.6.22)$$

$$\mathbf{y}^T \mathbf{K} \mathbf{y} = 0 \quad (2.6.22)$$

$$\mathbf{K} = \begin{pmatrix} 0 & 0.5 \\ 0.5 & 0 \end{pmatrix} \quad (2.6.22)$$

bisectors Taking the inverse of the affine transformation of the equation $xy = 0$,will give the

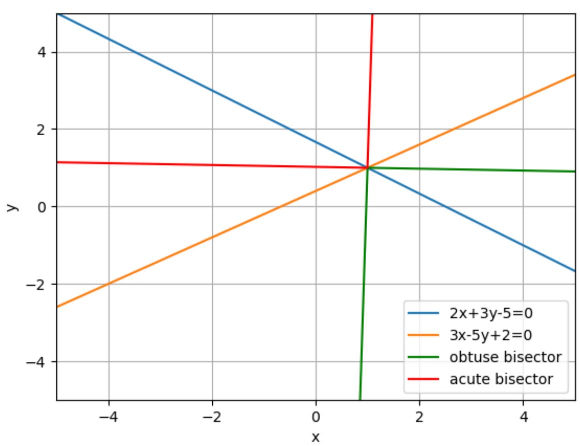

Fig. 2.6.22: Par of straight lines and their angular bisectors

angle bisectors.

$$\left(\mathbf{P}^{-1}\mathbf{x}-\mathbf{P}^{-1}c\right)^T\mathbf{K}\left(\mathbf{P}^{-1}\mathbf{x}-\mathbf{P}^{-1}c\right)=0 \tag{2.6.22}$$

$$\mathbf{x}^T\mathbf{P}\mathbf{K}\mathbf{P}^T\mathbf{x}-2c^T\mathbf{P}\mathbf{K}\mathbf{P}^T\mathbf{x}+c^T\mathbf{P}\mathbf{K}\mathbf{P}^Tc=0 \tag{2.6.22}$$

Substituting the values we get

$$\mathbf{x}^T\begin{pmatrix}\frac{1}{2\sqrt{442}} & \frac{21}{2\sqrt{442}}\\ \frac{21}{2\sqrt{442}} & -\frac{1}{2\sqrt{442}}\end{pmatrix}\mathbf{x} - \begin{pmatrix}\frac{22}{\sqrt{442}} & \frac{20}{\sqrt{442}}\end{pmatrix}\mathbf{x}+\frac{21}{\sqrt{442}}=0 \tag{2.6.22}$$

$$\frac{x^2}{2\sqrt{442}}+\frac{21}{\sqrt{442}}xy-\frac{y^2}{2\sqrt{442}} - \frac{22x}{\sqrt{442}}-\frac{20y}{\sqrt{442}}+\frac{21}{\sqrt{442}}=0 \tag{2.6.22}$$

$$x^2+42xy-y^2-44x-40y+42=0 \tag{2.6.22}$$

Therefore the equation of bisectors of the given line in quadratic form is

$$\mathbf{x}^T\begin{pmatrix}1 & 21\\ 21 & -1\end{pmatrix}\mathbf{x}-\begin{pmatrix}44 & 40\end{pmatrix}\mathbf{x}+42=0 \tag{2.6.22}$$

23. For what value of k does the equation

$$\mathbf{x}^T\begin{pmatrix}12 & \frac{7}{2}\\ \frac{7}{2} & k\end{pmatrix}\mathbf{x}+\begin{pmatrix}13 & -1\end{pmatrix}\mathbf{x}+3=0 \tag{2.6.23}$$

represent two straight lines? What is the angle between them?

24. For what value of k does the equation

$$\mathbf{x}^T\begin{pmatrix}6 & k/2\\ k/2 & -3\end{pmatrix}\mathbf{x}+\begin{pmatrix}4 & 5\end{pmatrix}\mathbf{x}-2=0 \tag{2.6.24}$$

represent a pair of straight lines?

Solution: (2.6.24) can also be written as

$$\mathbf{x}^T\begin{pmatrix}a & b\\ b & c\end{pmatrix}\mathbf{x}+\begin{pmatrix}d & e\end{pmatrix}\mathbf{x}+f=0 \tag{2.6.24}$$

$$\mathbf{V}=\begin{pmatrix}6 & k/2\\ k/2 & -3\end{pmatrix}$$
$$\mathbf{u}=\begin{pmatrix}2\\ 5/2\end{pmatrix} \tag{2.6.24}$$
$$f=-2$$

Block Matrix

$$=\begin{pmatrix}6 & k/2 & 2\\ k/2 & -3 & 5/2\\ 2 & 5/2 & -2\end{pmatrix} \tag{2.6.24}$$

Determinant of the Block Matrix

$$\Delta=\begin{vmatrix}6 & k/2 & 2\\ k/2 & -3 & 5/2\\ 2 & 5/2 & -2\end{vmatrix} \tag{2.6.24}$$

If the (2.6.24) represents a pair of straight lines then the Determinant is zero

$$\Delta=0$$
$$\implies \begin{vmatrix}6 & k/2 & 2\\ k/2 & -3 & 5/2\\ 2 & 5/2 & -2\end{vmatrix}=0$$
$$\implies 6\times(6-25/4)-k/2(-k-5)+2(5k/4+6)=0$$
$$\implies k^2+10k+21=0 \tag{2.6.24}$$

$$\implies \boxed{k=-3}$$
$$\implies \boxed{k=-7} \tag{2.6.24}$$

Substituting k=-3 in (2.6.24)

$$\mathbf{x}^T\begin{pmatrix}6 & -3/2\\ -3/2 & -3\end{pmatrix}\mathbf{x}+\begin{pmatrix}4 & 5\end{pmatrix}\mathbf{x}-2=0 \tag{2.6.24}$$

(2.6.24) can be represented as

$$\mathbf{V} = \begin{pmatrix} 6 & -3/2 \\ -3/2 & -3 \end{pmatrix}$$
$$\mathbf{u} = \begin{pmatrix} 2 \\ 5/2 \end{pmatrix} \tag{2.6.24}$$
$$f = -2$$

To find the separate equations of the straight lines we will use spectral decomposition. Characteristic equation of $\mathbf{V}$ is given by:

$$\left|V - \lambda\mathbf{I}\right| = \begin{vmatrix} 6-\lambda & -3/2 \\ -3/2 & -3-\lambda \end{vmatrix} = 0 \tag{2.6.24}$$
$$\implies \lambda^2 - 3\lambda - 81/4 = 0$$

The Eigen Values of $\mathbf{V}$ are:

$$\lambda_1 = \frac{3+3\sqrt{10}}{2}, \lambda_2 = \frac{3-3\sqrt{10}}{2} \tag{2.6.24}$$

Let $\mathbf{p}_1$ and $\mathbf{p}_2$ be the Eigen vector corresponding to λ_1 and λ_2 respectively
Eigen vector $\mathbf{p}$ is given as:

$$\mathbf{Vp} = \lambda\mathbf{p}$$
$$\implies (\mathbf{V} - \lambda\mathbf{I})\mathbf{p} = 0 \tag{2.6.24}$$

For $\lambda_1 = \frac{3+3\sqrt{10}}{2}$

$$(\mathbf{V} - \lambda_1\mathbf{I}) = \begin{pmatrix} \frac{9-3\sqrt{10}}{2} & -3/2 \\ -3/2 & \frac{-9-3\sqrt{10}}{2} \end{pmatrix} \tag{2.6.24}$$

To find $\mathbf{p}_1$ Use Augmented Matrix of $(\mathbf{V} - \lambda\mathbf{I})$

$$\begin{pmatrix} \frac{9-3\sqrt{10}}{2} & -3/2 & 0 \\ -3/2 & \frac{-9-3\sqrt{10}}{2} & 0 \end{pmatrix}$$
$$\xleftrightarrow{R_1 \to \frac{2}{9-3\sqrt{10}}R_1} \begin{pmatrix} 1 & 3+\sqrt{10} & 0 \\ -3/2 & \frac{-9-3\sqrt{10}}{2} & 0 \end{pmatrix}$$
$$\xleftrightarrow{R_1 \to 3/2R_1 + R_2} \begin{pmatrix} 1 & 3+\sqrt{10} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.6.24}$$

So we get,

$$x_1 + (3+\sqrt{10})x_2 = 0 \tag{2.6.24}$$

Therefore, Eigen Vector corresponding to λ_1

$$\mathbf{p}_1 = \frac{1}{\sqrt{20+6\sqrt{10}}}\begin{pmatrix} -(3+\sqrt{10}) \\ 1 \end{pmatrix} \tag{2.6.24}$$

Similarly for $\lambda_2 = \frac{3-3\sqrt{10}}{2}$

$$\mathbf{p}_2 = \frac{1}{\sqrt{20-6\sqrt{10}}}\begin{pmatrix} -(3-\sqrt{10}) \\ 1 \end{pmatrix} \tag{2.6.24}$$

We know that $\mathbf{V} = \mathbf{PDP}^T$ where $\mathbf{P}$ and $\mathbf{V}$ are given by:

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$
$$\implies \mathbf{D} = \begin{pmatrix} \frac{3+3\sqrt{10}}{2} & 0 \\ 0 & \frac{3-3\sqrt{10}}{2} \end{pmatrix} \tag{2.6.24}$$

Hence the rotation matrix P is

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix}$$
$$\implies \mathbf{P} = \begin{pmatrix} \frac{-(3+\sqrt{10})}{\sqrt{20+6\sqrt{10}}} & \frac{-(3-\sqrt{10})}{\sqrt{20-6\sqrt{10}}} \\ \frac{1}{\sqrt{20+6\sqrt{10}}} & \frac{1}{\sqrt{20-6\sqrt{10}}} \end{pmatrix} \tag{2.6.24}$$

We know that

$$\begin{pmatrix} \sqrt{|\lambda_1|} & \sqrt{|\lambda_2|} \end{pmatrix}\mathbf{P}^T(\mathbf{x} - \mathbf{c}) = 0 \tag{2.6.24}$$

where $\mathbf{c}$ is the point of intersection of the lines

$$\mathbf{Vc} = -\mathbf{u}$$
$$\begin{pmatrix} 6 & -3/2 \\ -3/2 & -3 \end{pmatrix}\mathbf{c} = -\begin{pmatrix} 2 \\ 5/2 \end{pmatrix} \tag{2.6.24}$$
$$\implies \mathbf{c} = \begin{pmatrix} -1/9 \\ 8/9 \end{pmatrix}$$

Substituting values in (2.6.24)

$$\begin{pmatrix} \sqrt{\frac{3+3\sqrt{10}}{2}} & \pm\sqrt{\frac{3-3\sqrt{10}}{2}} \end{pmatrix} \times$$
$$\begin{pmatrix} -\frac{3+\sqrt{10}}{\sqrt{20+6\sqrt{10}}} & \frac{1}{\sqrt{20+6\sqrt{10}}} \\ -\frac{3-\sqrt{10}}{\sqrt{20-6\sqrt{10}}} & -\frac{1}{\sqrt{20-6\sqrt{10}}} \end{pmatrix} \times \tag{2.6.24}$$
$$\begin{pmatrix} x + 1/9 \\ y - 8/9 \end{pmatrix} = 0$$

Simplifying (2.6.24) we get

$$3x - 3y + 3 = 0 \textit{ and } 2x + y - 2/3 = 0$$
$$\boxed{(3x - 3y + 3)(2x + y - 2/3) = 0} \tag{2.6.24}$$

Similarly substituting k=-7 in (2.6.24)

$$\mathbf{x}^T\begin{pmatrix} 6 & -7/2 \\ -7/2 & -3 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 4 & 5 \end{pmatrix}\mathbf{x} - 2 = 0 \tag{2.6.24}$$

Equation (2.6.24) can be represented as

$$\mathbf{V} = \begin{pmatrix} 6 & -7/2 \\ -7/2 & -3 \end{pmatrix}$$
$$\mathbf{u} = \begin{pmatrix} 2 \\ 5/2 \end{pmatrix} \tag{2.6.24}$$
$$f = -2$$

To find the separate equations of the straight lines we will use spectral decomposition.
Characteristic equation of $\mathbf{V}$ is given by:

$$\left|V - \lambda\mathbf{I}\right| = \begin{vmatrix} 6-\lambda & -7/2 \\ -7/2 & -3-\lambda \end{vmatrix} = 0 \tag{2.6.24}$$
$$\implies \lambda^2 - 3\lambda - 121/4 = 0$$

The Eigen Values of $\mathbf{V}$ are:

$$\lambda_1 = \frac{3+\sqrt{130}}{2}, \lambda_2 = \frac{3-3\sqrt{130}}{2} \tag{2.6.24}$$

Let $\mathbf{p}_1$ and $\mathbf{p}_2$ be the Eigen vector corresponding to λ_1 and λ_2 respectively
Eigen vector $\mathbf{p}$ is given as:

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p}$$
$$\implies (\mathbf{V} - \lambda\mathbf{I})\mathbf{p} = 0 \tag{2.6.24}$$

For $\lambda_1 = \frac{3+\sqrt{130}}{2}$

$$(\mathbf{V} - \lambda_1\mathbf{I}) = \begin{pmatrix} \frac{9-\sqrt{130}}{2} & -7/2 \\ -7/2 & \frac{-9-\sqrt{130}}{2} \end{pmatrix} \tag{2.6.24}$$

To find $\mathbf{p}_1$ Use Augmented Matrix of $(\mathbf{V} - \lambda\mathbf{I})$

$$\begin{pmatrix} \frac{9-\sqrt{130}}{2} & -7/2 & 0 \\ -7/2 & \frac{-9-\sqrt{130}}{2} & 0 \end{pmatrix}$$
$$\xleftrightarrow{R_1 \to \frac{2}{9-\sqrt{130}}R_1} \begin{pmatrix} 1 & \frac{9+\sqrt{130}}{7} & 0 \\ -7/2 & \frac{-9-\sqrt{130}}{2} & 0 \end{pmatrix} \tag{2.6.24}$$
$$\xleftrightarrow{R_1 \to 7/2R_1 + R_2} \begin{pmatrix} 1 & \frac{9+\sqrt{130}}{7} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So we get,

$$x_1 + (\frac{9+\sqrt{130}}{7})x_2 = 0 \tag{2.6.24}$$

Therefore, Eigen Vector corresponding to λ_1

$$\mathbf{p}_1 = \frac{7}{\sqrt{260+18\sqrt{130}}} \begin{pmatrix} -\frac{9+\sqrt{130}}{7}) \\ 1 \end{pmatrix} \tag{2.6.24}$$

Similarly for $\lambda_2 = \frac{3-\sqrt{130}}{2}$

$$\mathbf{p}_2 = \frac{7}{\sqrt{260-18\sqrt{130}}} \begin{pmatrix} -\frac{9-\sqrt{130}}{7} \\ 1 \end{pmatrix} \tag{2.6.24}$$

We know that $\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T$ where $\mathbf{P}$ and $\mathbf{V}$ are given by:

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \implies \mathbf{D} = \begin{pmatrix} \frac{3+\sqrt{130}}{2} & 0 \\ 0 & \frac{3-\sqrt{130}}{2} \end{pmatrix} \tag{2.6.24}$$

Hence the rotation matrix P is

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix} \tag{2.6.24}$$

$$\implies \mathbf{P} = \begin{pmatrix} -\frac{63+7\sqrt{130}}{\sqrt{260+18\sqrt{130}}} & -\frac{63-7\sqrt{130}}{\sqrt{260-18\sqrt{130}}} \\ \frac{7}{\sqrt{260+18\sqrt{10}}} & \frac{7}{\sqrt{260-18\sqrt{130}}} \end{pmatrix} \tag{2.6.24}$$

We know that

$$\begin{pmatrix} \sqrt{|\lambda_1|} & \sqrt{|\lambda_2|} \end{pmatrix} \mathbf{P}^T(\mathbf{x} - \mathbf{c}) = 0 \tag{2.6.24}$$

where $\mathbf{c}$ is the point of intersection of the lines

$$\mathbf{V}\mathbf{c} = -\mathbf{u}$$
$$\begin{pmatrix} 6 & -7/2 \\ -7/2 & -3 \end{pmatrix}\mathbf{c} = -\begin{pmatrix} 2 \\ 5/2 \end{pmatrix} \tag{2.6.24}$$
$$\implies \mathbf{c} = \begin{pmatrix} 1/11 \\ 8/11 \end{pmatrix}$$

Substituting values in (2.6.24)

$$\begin{pmatrix} \sqrt{\frac{3+\sqrt{130}}{2}} & \pm\sqrt{\frac{3-\sqrt{130}}{2}} \end{pmatrix} \times$$
$$\begin{pmatrix} -\frac{63+7\sqrt{130}}{\sqrt{260+18\sqrt{130}}} & \frac{7}{\sqrt{260+18\sqrt{130}}} \\ -\frac{63-7\sqrt{130}}{\sqrt{260-18\sqrt{130}}} & \frac{7}{\sqrt{260-18\sqrt{130}}} \end{pmatrix} \times \tag{2.6.24}$$
$$\begin{pmatrix} x - 1/11 \\ y - 8/11 \end{pmatrix} = 0$$

Simplifying (2.6.24) we get

$$2x - 3y + 2 = 0 \textit{ and } 3x + y - 1 = 0$$
$$\boxed{(2x - 3y + 2)(3x + y - 1) = 0} \tag{2.6.24}$$

3 Curves

3.1 Tangent

1. Find the equations of the tangents to the following curves at the points specified:

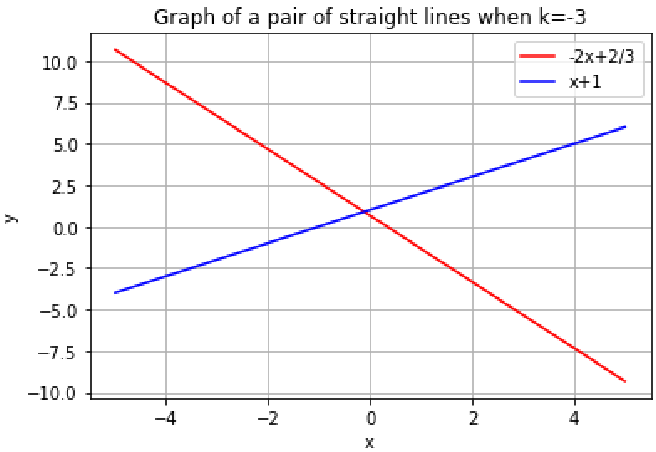

Fig. 2.6.24: Pair of straight lines when k=-3

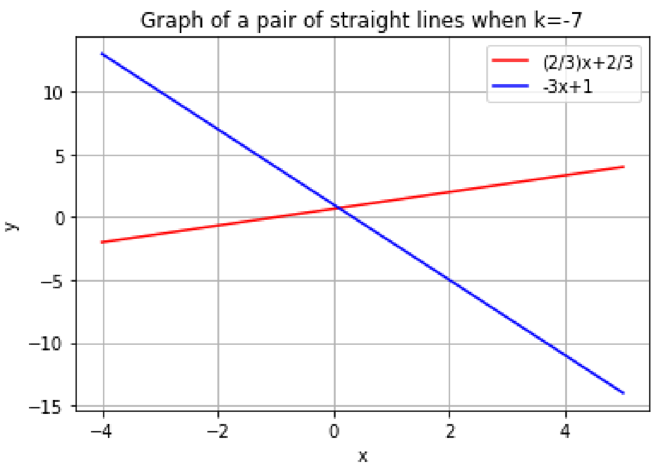

Fig. 2.6.24: Pair of straight lines when k=-7

a) $y = x\left(x^2 - 1\right), x = 2$
b) $y = x^2 + \frac{1}{x^2}, x = 1$
c) $y = x^3 + 2x, x = 0$
d) $y = \left(x + \frac{1}{x}\right)^3, x = 2$
e) $y = \left(x^2 - 1\right)^2, x = 1$
f) $y = x^3 - x + 1, x = 3$
g) $y = (x - a)^3, x = 2a$
h) $y = ax^2 + 2bx + c, (x_1, y_1)$
i) $y = \frac{x^3}{a^3} + \frac{a^3}{x^3}, x = a$
j) $y = \frac{x^2}{a} + \frac{a^2}{x}, x = a$

2. Find the tangents to the curve

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 0 \tag{3.1.2.1}$$

at the points where it is cut by the line

$$\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} + 4 = 0 \tag{3.1.2.2}$$

and find the point of intersection of the tangents.

3. Prove that the line

$$\begin{pmatrix} 3 & -4 \end{pmatrix} \mathbf{x} + 4 = 0 \tag{3.1.3.1}$$

touches the curve

$$\mathbf{x}^T \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix} \mathbf{x} + 1 = 0. \tag{3.1.3.2}$$

4. Find the points on the curve

$$3y = x^3 + 3x \tag{3.1.4.1}$$

at which the tangent is parallel to the line

$$\begin{pmatrix} 5 & -1 \end{pmatrix} \mathbf{x} = 0 \tag{3.1.4.2}$$

5. Find at what points on the curve

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 & -1 \end{pmatrix} \mathbf{x} + 9 = 0 \tag{3.1.5}$$

the tangents pass through the origin.

6. Show that there are three points on the curve

$$3y = 3x^4 + 8x^3 - 6x^2 \tag{3.1.6.1}$$

at which the tangents are parallel to the line

$$\begin{pmatrix} 8 & -1 \end{pmatrix} \mathbf{x} = 0 \tag{3.1.6.2}$$

7. Show that the line

$$\begin{pmatrix} 0 & 4 \end{pmatrix} \mathbf{x} = 17 \tag{3.1.7.1}$$

meets the curve

$$y = x^2 + \frac{1}{x^2} \tag{3.1.7.2}$$

in four points and that two of the points of intersection of the tangents at these four points are on the line

$$\begin{pmatrix} 0 & 4 \end{pmatrix} \mathbf{x} + 1 = 0, \tag{3.1.7.3}$$

and two are on the line

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 0. \tag{3.1.7.4}$$

3.2 More on Tangents

1. Find the equation of the tangents to the following curves at the points stated:

a) $\mathbf{x}^T \mathbf{x} = 25, \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

Solution: The given equation of the curve:

$$\mathbf{x}^\mathbf{T} \mathbf{x} = 25 \tag{3.2.1}$$

The general equation of a second degree can be expressed as:

$$\mathbf{x}^\mathbf{T} \mathbf{V} \mathbf{x} + 2\mathbf{u}^\mathbf{T} \mathbf{x} + f = 0 \tag{3.2.1}$$

Comparing (3.2.1) with (3.2.1):

$$\mathbf{V} = \mathbf{I}, \mathbf{u} = \begin{pmatrix}0\\0\end{pmatrix} and f = -25 \qquad (3.2.1)$$

For $\mathbf{V} = \mathbf{I}$, (3.2.1) represents a circle. $\mathbf{c}$ represent the center , r the radius and $\mathbf{q}$ the point of contact of the tangent to the circle .

The center and radius is given by:

$$\mathbf{c} = -\mathbf{u} = \begin{pmatrix}0\\0\end{pmatrix} \qquad (3.2.1)$$

$$r = \sqrt{\mathbf{u^T u} - f} = \sqrt{0-(-25)} = 5 \qquad (3.2.1)$$

The given point of contact

$$\mathbf{q} = \begin{pmatrix}3\\4\end{pmatrix} \qquad (3.2.1)$$

The direction vector of the line joining the point $\mathbf{q}$ and the center $\mathbf{c}$ is:

$$\mathbf{n} = \mathbf{q} - \mathbf{c} \qquad (3.2.1)$$

$$\implies \mathbf{n} = \mathbf{q} + \mathbf{u} = \begin{pmatrix}3\\4\end{pmatrix} \qquad (3.2.1)$$

The vector $\mathbf{n}$ is normal to the tangent of the circle, drawn at $\mathbf{q}$

The equation of the tangent is

$$\mathbf{n^T}\left(\mathbf{x} - \mathbf{q}\right) = 0 \qquad (3.2.1)$$

$$\mathbf{n^T x} = c \qquad (3.2.1)$$

$$(3.2.1)$$

where

$$c = \mathbf{n^T q} \qquad (3.2.1)$$

$$\implies c = \begin{pmatrix}3 & 4\end{pmatrix}\begin{pmatrix}3\\4\end{pmatrix} = 25 \qquad (3.2.1)$$

Thus the equation of the tangent to the curve at $\mathbf{q}$ is

$$\mathbf{n^T x} = 25 \qquad (3.2.1)$$

$$\implies \begin{pmatrix}3 & 4\end{pmatrix}\mathbf{x} = 25 \qquad (3.2.1)$$

b) $\mathbf{x}^T\begin{pmatrix}4 & 0\\0 & 9\end{pmatrix}\mathbf{x} = 2, \begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\end{pmatrix}$

c) $\mathbf{x}^T\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\mathbf{x} - 4a\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} = 0, \begin{pmatrix}a\\2a\end{pmatrix}$

d) $\mathbf{x}^T\begin{pmatrix}b^2 & 0\\0 & a^2\end{pmatrix}\mathbf{x} = a^2b^2, \begin{pmatrix}a\cos\theta\\b\sin\theta\end{pmatrix}$

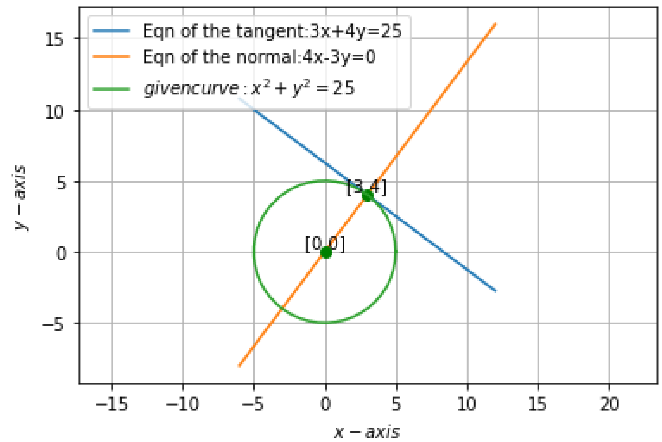

Fig. 3.2.1: This is the 2D diagram of the given curve $\mathbf{x^T x} = 25$ and the tangent to it at $\begin{pmatrix}3\\4\end{pmatrix}$

e) $\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\mathbf{x} = a^2, \begin{pmatrix}a\sec\theta\\b\tan\theta\end{pmatrix}$

f) $\mathbf{x}^T\begin{pmatrix}1 & -\frac{1}{2}\\-\frac{1}{2} & 1\end{pmatrix}\mathbf{x} = 4, \begin{pmatrix}2\\2\end{pmatrix}$

g) $\mathbf{x}^T\mathbf{x} + \begin{pmatrix}2 & 4\end{pmatrix}\mathbf{x} - 20 = 0, \begin{pmatrix}3\\1\end{pmatrix}$

h) $x^3 + y^3 - 3xy^2 + a^3 = 0, \begin{pmatrix}a\\a\end{pmatrix}$

2. Find the equation of the tangent at the point $\mathbf{p}$ on each of the following curves:

a) $\mathbf{x}^T\mathbf{x} = a^2$

b) $\mathbf{x}^T\begin{pmatrix}b^2 & 0\\0 & a^2\end{pmatrix}\mathbf{x} = a^2b^2$

c) $\mathbf{x}^T\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\mathbf{x} - 4a\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = 0$

d) $\mathbf{x}^T\begin{pmatrix}0 & \frac{1}{2}\\\frac{1}{2} & \end{pmatrix}\mathbf{x} - c^2 = 0$

e) $y^2\left(x^2 - a^2\right) = a^2\left(x^2 + a^2\right)$

f) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

3.3 Normal

1. Find the equations of the normals to the following curves at the given points

a) $\mathbf{x}^T\mathbf{x} - \begin{pmatrix}2 & 4\end{pmatrix}\mathbf{x} = 3, \begin{pmatrix}3\\4\end{pmatrix}$

b) $\mathbf{x}^T\begin{pmatrix}1 & 2\\2 & 1\end{pmatrix}\mathbf{x} = 13, \begin{pmatrix}1\\2\end{pmatrix}$

c) $\mathbf{x}^T\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\mathbf{x} - 4a\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = 0, \begin{pmatrix}a\\2a\end{pmatrix}$

d) $\mathbf{x}^T\begin{pmatrix}b^2 & 0\\0 & a^2\end{pmatrix}\mathbf{x} = a^2b^2, \begin{pmatrix}a\cos\theta\\b\sin\theta\end{pmatrix}$

e) $\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & -4 \end{pmatrix} \mathbf{x} = 4a^2, \begin{pmatrix} 2a\sec\theta \\ a\tan\theta \end{pmatrix}$

f) $x^3 - y^3 - 3xy^2 + a^2 = 0, \begin{pmatrix} a \\ -a \end{pmatrix}$

2. Find the equation of the normal at the point $\mathbf{p}$ on each of the following curves:

a) $\mathbf{x}^T\mathbf{x} = a^2$

b) $\mathbf{x}^T \begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix} \mathbf{x} = a^2b^2$

c) $\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - 4a \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 0$

d) $\mathbf{x}^T \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \mathbf{x} = c^2$

3. Prove that for the curve

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - 4a \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 0 \tag{3.3.3}$$

the subnormal is of constant length.

4. Prove that the portion of any tangent to the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ intercepted by the axes is of length a.
5. Prove that for the curve $ay^2 = x^3$ the subnormal varies as the square of the subtangent.
6. Prove that for the curve $y = ae^{\frac{x}{b}}$ the subtangent is of length b.
7. Prove that the area of the triangle formed by the axes and any tangent to the curve

$$\mathbf{x}^T \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \mathbf{x} = c^2 \tag{3.3.7}$$

is $2c^2$.

8. Prove that for the curve $x^m y^n = e^{m+n}$ the portion of a tangent intercepted by the axes is divided at the point of contact in the ratio $m : n$.
9. Prove that, if N is the foot of the ordinate and NT is the subtangent at a point on the curve

$$\mathbf{x}^T \begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix} \mathbf{x} = a^2b^2, \begin{pmatrix} a\cos\theta \\ b\sin\theta \end{pmatrix} \tag{3.3.9}$$

then $OT.ON = a^2$.

10. Prove that the perpendicular from the foot of the ordinate to the tangent to a curve is of length $\frac{y}{\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}}$. Show that for the curve $y = c\cosh\frac{x}{c}$, this perpendicular is of length c.
11. Find the equation of the tangent to the curve

$$2x^3 + 2y^3 - 9axy = 0 \tag{3.3.11.1}$$

at the point $\begin{pmatrix} 2a \\ a \end{pmatrix}$; and show that the tangent meets the curve again where

$$\begin{pmatrix} 4 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{3.3.11.2}$$

3.4 Affine Transformation: Exercises

1. What does the equation

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{x} - \begin{pmatrix} 4 & 6 \end{pmatrix} \mathbf{x} - 6 = 0 \tag{3.4.1}$$

become when the origin is moved to the point $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$?

Solution: Comparing the given equation with the form:

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{3.4.1}$$

We get:

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} -2 & -3 \end{pmatrix} \mathbf{x} - 6 = 0 \tag{3.4.1}$$

where,

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.4.1}$$

$$\mathbf{u} = \begin{pmatrix} -2 \\ -3 \end{pmatrix} \tag{3.4.1}$$

$$f = -6 \tag{3.4.1}$$

Here, $\left|\mathbf{V}\right| = -1$. Since $\left|\mathbf{V}\right| < 0$ the given equation represents a hyperbola with center:

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} \tag{3.4.1}$$

The characteristic equation of $\mathbf{V}$ is:

$$\left|V - \lambda\mathbf{I}\right| = 0 \tag{3.4.1}$$

$$\begin{vmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} = 0 \tag{3.4.1}$$

$$\implies \lambda^2 - 1 = 0 \tag{3.4.1}$$

$$\lambda_1 = 1, \lambda_2 = -1 \tag{3.4.1}$$

Finding the eigen vector matrix $\mathbf{P}$ such that $\mathbf{P}^T = \mathbf{P}^{-1}$. For $\lambda_1 = 1$

$$(\mathbf{V} - \lambda_1 \mathbf{I})\mathbf{p}_1 = 0 \tag{3.4.1}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} \mathbf{p}_1 = 0 \tag{3.4.1}$$

$$\implies \mathbf{p}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.4.1}$$

$$\tag{3.4.1}$$

For $\lambda_2 = -1$

$$(\mathbf{V} - \lambda_2 \mathbf{I})\mathbf{p}_2 = 0 \tag{3.4.1}$$

$$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{p}_2 = 0 \tag{3.4.1}$$

$$\implies \mathbf{p}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.4.1}$$

[Choosing Orthonormal eigen vectors]

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.4.1}$$

By affine transformation $\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c}$, Equation (3.4.1) can be written in the form:

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f \tag{3.4.1}$$

where,

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \tag{3.4.1}$$

Thus, we can write:

$$\lambda_1 y_1^2 - (-\lambda_2)\, y_1^2 = \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f \tag{3.4.1}$$

The equation (3.4.1) represents a modified hyperbola, The equation of the asymptotes for (3.4.1) is:

$$\begin{pmatrix} \sqrt{|\lambda_1|} & \pm\sqrt{|\lambda_2|} \end{pmatrix} \mathbf{y} = 0 \tag{3.4.1}$$

Putting the values of λ_1 and λ_2 in equation (3.4.1), we get the two asymptotes for (3.4.1):

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{y} = 0 \tag{3.4.1}$$

$$\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{y} = 0 \tag{3.4.1}$$

These are the asymptotes of our modified hyperbola. The asymptotes of our original hyberbola in equation (3.4.1) can be obtained using:

$$\begin{pmatrix} \sqrt{|\lambda_1|} & \pm\sqrt{|\lambda_2|} \end{pmatrix} \mathbf{P}^T (\mathbf{x} - \mathbf{c}) = 0 \tag{3.4.1}$$

Putting the values of λ_1, λ_2 and $\mathbf{P}$ in equation (3.4.1), we get the equations of the asymptotes of the original hyperbola with center at $\mathbf{c}$:

$$\begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \left(\mathbf{x} + \begin{pmatrix} 2 \\ -3 \end{pmatrix} \right) = 0 \tag{3.4.1}$$

$$\implies \boxed{\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 5} \tag{3.4.1}$$

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \left(\mathbf{x} + \begin{pmatrix} 2 \\ -3 \end{pmatrix} \right) = 0 \tag{3.4.1}$$

$$\implies \boxed{\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = -1} \tag{3.4.1}$$

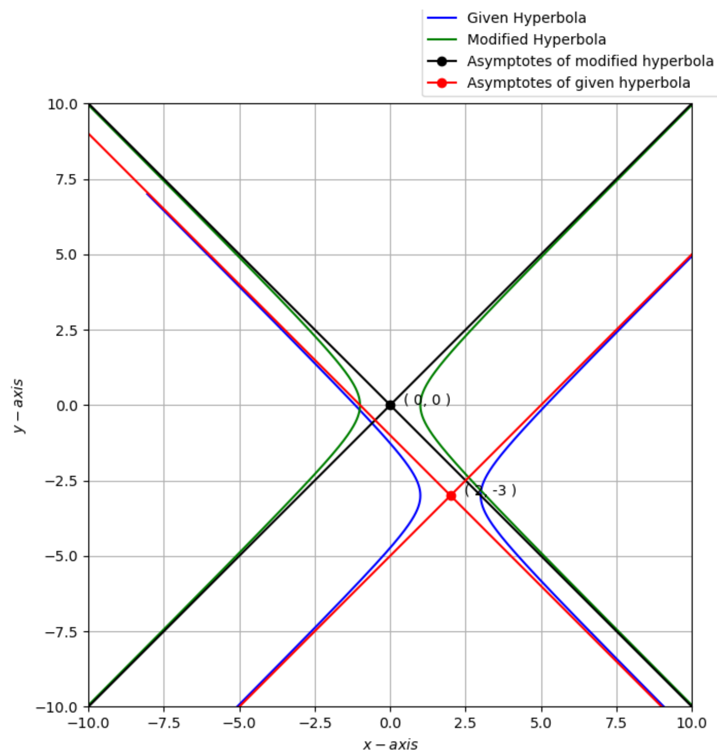

Fig. 3.4.1: Plot of the Asymtotes.

2. To what point must origin be shifted so that

$$\mathbf{x}^T \begin{pmatrix} 2 & \frac{-3}{2} \\ \frac{-3}{2} & 4 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 10 & -19 \end{pmatrix} \mathbf{x} + 23 = 0 \tag{3.4.2}$$

is transformed to

$$\mathbf{x}^T \begin{pmatrix} 2 & \frac{-3}{2} \\ \frac{-3}{2} & 4 \end{pmatrix} \mathbf{x} = 1 \tag{3.4.2}$$

Solution: Given,

$$\mathbf{x}^T\begin{pmatrix}2 & \frac{-3}{2}\\ \frac{-3}{2} & 4\end{pmatrix}\mathbf{x}+\begin{pmatrix}10 & -19\end{pmatrix}\mathbf{x}+23=0 \quad (3.4.2)$$

The general second degree equation can be expressed as follows,

$$\mathbf{x^T}\mathbf{V}\mathbf{x}+2\mathbf{u^T}\mathbf{x}+f=0 \quad (3.4.2)$$

From the given second degree equation we get,

$$\mathbf{V}=\begin{pmatrix}2 & \frac{-3}{2}\\ \frac{-3}{2} & 4\end{pmatrix} \quad (3.4.2)$$

$$\mathbf{u}=\begin{pmatrix}5\\ \frac{-19}{2}\end{pmatrix} \quad (3.4.2)$$

$$f=23 \quad (3.4.2)$$

Origin which is moved to the point is given by **c** The above equation (3.4.2) can be modified as

$$(\mathbf{x}+\mathbf{c})^T\mathbf{V}(\mathbf{x}+\mathbf{c})+2\mathbf{u}^T(\mathbf{x}+\mathbf{c})+23=0 \quad (3.4.2)$$

From equation (3.4.2) consider,

$$\implies (\mathbf{x}+\mathbf{c})^T\mathbf{V}(\mathbf{x}+\mathbf{c}) \quad (3.4.2)$$

$$\implies \mathbf{x}^T\mathbf{V}\mathbf{x}+\mathbf{c}^T\mathbf{V}\mathbf{x}+\mathbf{x}^T\mathbf{V}\mathbf{c}+\mathbf{c}^T\mathbf{V}\mathbf{c} \quad (3.4.2)$$

we know that

$$\mathbf{x}^T\mathbf{V}\mathbf{c}=\mathbf{c}^T\mathbf{V}\mathbf{x} \quad (3.4.2)$$

Substituting equation (3.4.2) in equation (3.4.2)

$$\implies \mathbf{x}^T\mathbf{V}\mathbf{x}+2\mathbf{c}^T\mathbf{V}\mathbf{x}+\mathbf{c}^T\mathbf{V}\mathbf{c} \quad (3.4.2)$$

Equation (3.4.2) is modified as

$$\implies \mathbf{x}^T\mathbf{V}\mathbf{x}+2\mathbf{c}^T\mathbf{V}\mathbf{x}+\mathbf{c}^T\mathbf{V}\mathbf{c}+2\mathbf{u}^T\mathbf{x}2\mathbf{u}^T\mathbf{c}+23=0 \quad (3.4.2)$$

Equating (3.4.2) and (3.4.2):

$$\implies \mathbf{x}^T\mathbf{V}\mathbf{x}+2\mathbf{c}^T\mathbf{V}\mathbf{x}+\mathbf{c}^T\mathbf{V}\mathbf{c}+2\mathbf{u}^T\mathbf{x}2\mathbf{u}^T\mathbf{c}+23=\mathbf{x}^T\mathbf{V}\mathbf{x}-1 \quad (3.4.2)$$

From above equation (3.4.2) we have,

$$2\mathbf{c}^T\mathbf{V}\mathbf{x}+2\mathbf{u}^T\mathbf{x}=0 \quad (3.4.2)$$

and

$$2\mathbf{u}^T\mathbf{x}+\mathbf{v}^T\mathbf{c}=-22 \quad (3.4.2)$$

From (3.4.2)

$$\mathbf{c}^T\mathbf{V}\mathbf{x}=-\mathbf{u}^T\mathbf{x} \quad (3.4.2)$$

$$\mathbf{c}^T\mathbf{V}=-\mathbf{u}^T \quad (3.4.2)$$

$$\mathbf{c}^T=-\mathbf{V}^{-1}\mathbf{u}^T \quad (3.4.2)$$

Adjoining **V** with identity matrix to compute inverse:

$$\begin{pmatrix}2 & \frac{-3}{2} & 1 & 0\\ \frac{-3}{2} & 4 & 0 & 1\end{pmatrix}\xleftrightarrow{R_1\leftarrow\frac{1}{2}R_1}\begin{pmatrix}1 & \frac{-3}{4} & \frac{1}{2} & 0\\ \frac{-3}{2} & 4 & 0 & 1\end{pmatrix} \quad (3.4.2)$$

$$\begin{pmatrix}1 & \frac{-3}{4} & \frac{1}{2} & 0\\ \frac{-3}{2} & 4 & 0 & 1\end{pmatrix}\xleftrightarrow{R_2\leftarrow R_2+\frac{3}{2}R_1}\begin{pmatrix}1 & \frac{-3}{4} & \frac{1}{2} & 0\\ 0 & \frac{23}{8} & \frac{3}{4} & 1\end{pmatrix} \quad (3.4.2)$$

$$\begin{pmatrix}1 & \frac{-3}{4} & \frac{1}{2} & 0\\ 0 & \frac{23}{8} & \frac{3}{4} & 1\end{pmatrix}\xleftrightarrow{R_2\leftarrow\frac{8}{23}R_2}\begin{pmatrix}1 & \frac{-3}{4} & \frac{1}{2} & 0\\ 0 & 1 & \frac{6}{23} & \frac{8}{23}\end{pmatrix} \quad (3.4.2)$$

$$\begin{pmatrix}1 & \frac{-3}{4} & \frac{1}{2} & 0\\ 0 & 1 & \frac{6}{23} & \frac{8}{23}\end{pmatrix}\xleftrightarrow{R_1\leftarrow R_1+\frac{3}{4}R_2}\begin{pmatrix}1 & 0 & \frac{16}{23} & \frac{6}{23}\\ 0 & 1 & \frac{6}{23} & \frac{8}{23}\end{pmatrix} \quad (3.4.2)$$

$$\mathbf{V}^{-1}=\begin{pmatrix}\frac{16}{23} & \frac{6}{23}\\ \frac{6}{23} & \frac{8}{23}\end{pmatrix} \quad (3.4.2)$$

From (3.4.2)

$$\mathbf{c}^T=\begin{pmatrix}\frac{-16}{23} & \frac{-6}{23}\\ \frac{-6}{23} & \frac{-8}{23}\end{pmatrix}\begin{pmatrix}5\\ \frac{-19}{2}\end{pmatrix} \quad (3.4.2)$$

From above we have :

$$\mathbf{c}^T=\begin{pmatrix}-1\\ 2\end{pmatrix} \quad (3.4.2)$$

Hence,

$$\mathbf{c}=\begin{pmatrix}-1 & 2\end{pmatrix} \quad (3.4.2)$$

From (3.4.2) and (3.4.2) when the origin is moved to the point $\mathbf{c}=\begin{pmatrix}-1 & 2\end{pmatrix}$ **V** doesn't change

$$\det(\mathbf{V})=5.75 \quad (3.4.2)$$

Since $\det(\mathbf{V})>0$ the given equation represents the ellipse. The characteristic equation of **V** is obtained by evaluating the determinant

$$\left| V - \lambda \mathbf{I} \right| = 0 \quad (3.4.2)$$

$$\begin{vmatrix} 2-\lambda & \frac{-3}{2} \\ \frac{-3}{2} & 4-\lambda \end{vmatrix} = 0 \quad (3.4.2)$$

$$\implies 4\lambda^2 - 24\lambda + 23 = 0 \quad (3.4.2)$$

The eigenvalues are the roots of equation 3.4.2 is given by

$$\lambda_1 = \frac{6+\sqrt{13}}{2} \quad (3.4.2)$$

$$\lambda_2 = \frac{6-\sqrt{13}}{2} \quad (3.4.2)$$

Hence from above:

$$\mathbf{D} = \begin{pmatrix} \frac{6+\sqrt{13}}{2} & 0 \\ 0 & \frac{6-\sqrt{13}}{2} \end{pmatrix} \quad (3.4.2)$$

The eigenvector $\mathbf{p}$ is defined as

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \quad (3.4.2)$$

$$\implies (\mathbf{V} - \lambda\mathbf{I})\mathbf{p} = 0 \quad (3.4.2)$$

For $\lambda_1 = \frac{6+\sqrt{13}}{2}$,

$$(\mathbf{V} - \lambda_1\mathbf{I}) = \begin{pmatrix} \frac{-\sqrt{13}-2}{2} & \frac{-3}{2} \\ \frac{-3}{2} & \frac{-\sqrt{13}+2}{2} \end{pmatrix} \quad (3.4.2)$$

$$\begin{pmatrix} \frac{-\sqrt{13}-2}{2} & \frac{-3}{2} \\ \frac{-3}{2} & \frac{-\sqrt{13}+2}{2} \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 \div \frac{-\sqrt{13}-2}{2}} \begin{pmatrix} 1 & \frac{-\sqrt{13}-2}{2} \\ \frac{-3}{2} & \frac{-\sqrt{13}-2}{2} \end{pmatrix} \quad (3.4.2)$$

$$\begin{pmatrix} 1 & \frac{-\sqrt{13}-2}{2} \\ \frac{-3}{2} & \frac{-\sqrt{13}-2}{2} \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 \frac{-3}{2} R_1} \begin{pmatrix} 1 & \frac{-\sqrt{13}-2}{2} \\ 0 & 0 \end{pmatrix} \quad (3.4.2)$$

$$\mathbf{p}_1 = \begin{pmatrix} \frac{-\sqrt{13}+2}{2} \\ 1 \end{pmatrix} \quad (3.4.2)$$

For $\lambda_2 = \frac{6-\sqrt{13}}{2}$,

$$(\mathbf{V} - \lambda_1\mathbf{I}) = \begin{pmatrix} \frac{\sqrt{13}-2}{2} & \frac{-3}{2} \\ \frac{-3}{2} & \frac{\sqrt{13}+2}{2} \end{pmatrix} \quad (3.4.2)$$

$$\begin{pmatrix} \frac{\sqrt{13}-2}{2} & \frac{-3}{2} \\ \frac{-3}{2} & \frac{\sqrt{13}+2}{2} \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 \div \frac{\sqrt{13}-2}{2}} \begin{pmatrix} 1 & \frac{-\sqrt{13}-2}{2} \\ \frac{-3}{2} & \frac{-\sqrt{13}-2}{2} \end{pmatrix} \quad (3.4.2)$$

$$\begin{pmatrix} 1 & \frac{-\sqrt{13}-2}{2} \\ \frac{-3}{2} & \frac{-\sqrt{13}-2}{2} \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 \frac{-3}{2} R_1} \begin{pmatrix} 1 & \frac{-\sqrt{13}-2}{2} \\ 0 & 0 \end{pmatrix} \quad (3.4.2)$$

$$\mathbf{p}_2 = \begin{pmatrix} \frac{\sqrt{13}+2}{2} \\ 1 \end{pmatrix} \quad (3.4.2)$$

Again,for ellipse

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1} \quad (3.4.2)$$

Where $\mathbf{D}$ is a diagonal matrix, we get,

$$\mathbf{P} = \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} \end{pmatrix} \quad (3.4.2)$$

$$\implies \mathbf{P} = \begin{pmatrix} \frac{-\sqrt{13}+2}{3} & \frac{\sqrt{13}+2}{3} \\ 1 & 1 \end{pmatrix} \quad (3.4.2)$$

$$\mathbf{D} = \begin{pmatrix} \frac{6+\sqrt{13}}{2} & 0 \\ 0 & \frac{6-\sqrt{13}}{2} \end{pmatrix} \quad (3.4.2)$$

Standard ellipse can be written in the form:

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f \quad (3.4.2)$$

Simplifying we get:

$$\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} = \begin{pmatrix} 5 & \frac{-19}{2} \end{pmatrix} \begin{pmatrix} \frac{16}{23} & \frac{6}{23} \\ \frac{6}{23} & \frac{8}{23} \end{pmatrix} \begin{pmatrix} 5 \\ \frac{-19}{2} \end{pmatrix} = 24 \quad (3.4.2)$$

substituting (3.4.2) in (3.4.2) we have :

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = 1 \quad (3.4.2)$$

To get $\mathbf{y}$,

$$\mathbf{y} = \mathbf{P}^T\mathbf{x} - \mathbf{P}^T\mathbf{c} \quad (3.4.2)$$

$$\mathbf{y} = \begin{pmatrix} \frac{-\sqrt{13}+2}{3} & 1 \\ \frac{\sqrt{13}+2}{3} & 1 \end{pmatrix}\mathbf{x} - \begin{pmatrix} \frac{-\sqrt{13}+2}{3} & 1 \\ \frac{\sqrt{13}+2}{3} & 1 \end{pmatrix} \quad (3.4.2)$$

Substituting equation (3.4.2), in equation (3.4.2)

$$\mathbf{y}^T \begin{pmatrix} \frac{6+\sqrt{13}}{2} & 0 \\ 0 & \frac{6-\sqrt{13}}{2} \end{pmatrix} \mathbf{y} = 1 \quad (3.4.2)$$

3. Show that the equation

$$\mathbf{x}^T\mathbf{x} = a^2 \quad (3.4.3)$$

remains unaltered by any rotation of the axes.

4. What does the equation

$$\mathbf{x}^T \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} \mathbf{x} = 2a^2 \quad (3.4.4)$$

become when the axes are turned through 30°?

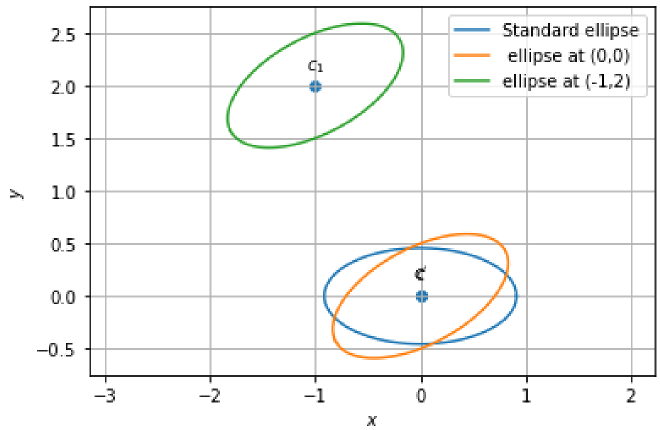

Fig. 3.4.2: Standard ellipse, Ellipse at (0,0) and ellipse at (-1,2)

Solution: The general second degree equation is expressed as follows,

$$\mathbf{x}^{\mathbf{T}}\mathbf{V}\mathbf{x} + 2\mathbf{u}^{\mathbf{T}}\mathbf{x} + f = 0 \tag{3.4.4}$$

Comparing (3.4.4) and (3.4.4), we get

$$\mathbf{V} = \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} \tag{3.4.4}$$

$$\mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.4.4}$$

$$f = -2a^2 \tag{3.4.4}$$

Now we find

$$\left|\mathbf{V}\right| = \begin{vmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{vmatrix} \tag{3.4.4}$$

$$\implies \left|\mathbf{V}\right| = -4 \tag{3.4.4}$$

$$\implies \left|\mathbf{V}\right| < 0 \tag{3.4.4}$$

Therefore the given equation (3.4.4) represents hyperbola.
Now from affine transformations,

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c} \tag{3.4.4}$$

We have to rotate the axes by $\theta = 30°$, Then using rotation matrix.

$$\mathbf{P} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{3.4.4}$$

$$\implies \mathbf{P} = \begin{pmatrix} \cos 30° & -\sin 30° \\ \sin 30° & \cos 30° \end{pmatrix} \tag{3.4.4}$$

$$\implies \mathbf{P} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \tag{3.4.4}$$

From eigenvalue decomposition,

$$\mathbf{P}^{\mathbf{T}}\mathbf{V}\mathbf{P} = \mathbf{D} \tag{3.4.4}$$

$$\implies \mathbf{D} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \tag{3.4.4}$$

$$\implies \mathbf{D} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{3.4.4}$$

$$\mathbf{P} = \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} \end{pmatrix} \tag{3.4.4}$$

$$\mathbf{P}^T = \mathbf{P}^{-1} \tag{3.4.4}$$

The equation (3.4.4) becomes as below due to affine transformation

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f \tag{3.4.4}$$

with

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \tag{3.4.4}$$

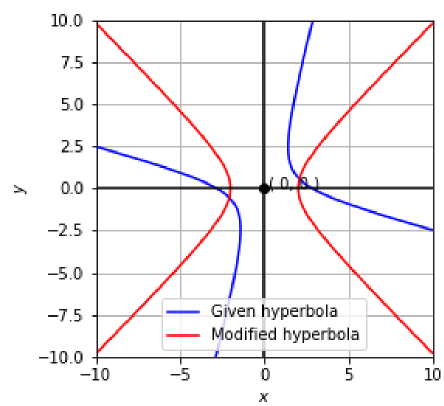

Fig. 3.4.4: For $a = 2$, Hyperbola rotated by 30°

Substitute (3.4.4),(3.4.4),(3.4.4),(3.4.4) in (3.4.4) and (3.4.4), we get

$$\mathbf{y}^T\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}\mathbf{y} = 2a^2 \tag{3.4.4}$$

with centre,

$$\mathbf{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (3.4.4)$$

Therefore the given equation (3.4.4) becomes (3.4.4) when the axes are turned through 30°. The plot is shown in Fig 3.4.4 for a=2.

5. What does the equation

$$\mathbf{x}^T \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \mathbf{x} - 4\sqrt{2}a \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 0 \quad (3.4.5)$$

become when the axes are turned through 45°?

Solution:

$$\mathbf{x}^T \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \mathbf{x} - 4\sqrt{2}a \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 0 \quad (3.4.5)$$

$$\mathbf{V} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \implies \left|\mathbf{V}\right| = 0, \mathbf{u} = \begin{pmatrix} -2\sqrt{2}a \\ -2\sqrt{2}a \end{pmatrix} \quad (3.4.5)$$

The characteristics equation of $\mathbf{V}$

$$\left|\lambda \mathbf{I} - \mathbf{V}\right| = \begin{vmatrix} \lambda - 1 & 1 \\ 1 & \lambda - 1 \end{vmatrix} = 0 \quad (3.4.5)$$

$$\implies \lambda^2 - 2\lambda = 0 \quad (3.4.5)$$

The eigen values are

$$\lambda_1 = 0, \lambda_2 = 2 \quad (3.4.5)$$

The (3.4.5) is equation of parabola as$\lambda_1 = 0 and \left|\mathbf{V}\right| = 0$. The eigen vector $\mathbf{p}$ is defined as

$$\mathbf{V}\mathbf{p} = \lambda \mathbf{p} \quad (3.4.5)$$

$$\implies (\lambda \mathbf{I} - \mathbf{V})\mathbf{p} = 0 \quad (3.4.5)$$

for $\lambda_1 = 0$

$$(\lambda_1 \mathbf{I} - \mathbf{V}) = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \xleftrightarrow{R_2 \to R_2 + R_1} \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \quad (3.4.5)$$

$$\mathbf{p_1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (3.4.5)$$

such that $\|\mathbf{p_1}\| = 1$ similarly the eigen vector for $\lambda_2 = 2$ can be find

$$\mathbf{p_2} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad (3.4.5)$$

$$\mathbf{P} = \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad (3.4.5)$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \quad (3.4.5)$$

The parabola parameters are given by

$$f = \frac{|\eta|}{|\lambda_2|}; \eta = 2\mathbf{p_1}^T \mathbf{u} \quad (3.4.5)$$

$$f = \frac{8a}{2} = 4a \quad (3.4.5)$$

$$\begin{pmatrix} 0 & -6\sqrt{2}a \\ 0 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -4a \\ -2\sqrt{2}a \\ 0 \end{pmatrix} \quad (3.4.5)$$

$$\mathbf{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} where \mathbf{c} is vertex \quad (3.4.5)$$

The axes are turned around 45°then

$$\mathbf{P} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \mathbf{x} \quad (3.4.5)$$

$$\mathbf{P} = \begin{pmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{pmatrix} \mathbf{x} \quad (3.4.5)$$

$$\mathbf{P} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \mathbf{x} \quad (3.4.5)$$

when $\mathbf{P}$ passes through the (3.4.5) we get

$$\mathbf{x}^T \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \mathbf{x} - 4\sqrt{2}a \begin{pmatrix} 1 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \mathbf{x} = 0 \quad (3.4.5)$$

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{x} - 4a \begin{pmatrix} 2 & 0 \end{pmatrix} \mathbf{x} = 0 \quad (3.4.5)$$

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \implies \left|\mathbf{V}\right| = 0 \quad (3.4.5)$$

Therefore it is parabola..

6. To what point must the origin be moved in order that the equation

$$\mathbf{x}^T \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 10 & -4 \end{pmatrix} \mathbf{x} = 0 \quad (3.4.6)$$

may become

$$\mathbf{x}^T \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix} \mathbf{x} = 1 \quad (3.4.6)$$

and through what angle must the axes be turned

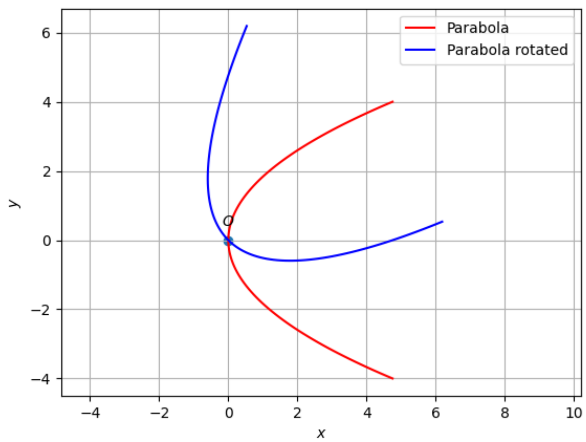

Fig. 3.4.5: parabola with a =1

in order to obtain

$$\mathbf{x}^T \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \mathbf{x} = 1 \tag{3.4.6}$$

Solution: The general second order equation can be expressed as follows,

$$\mathbf{x}^\mathbf{T}\mathbf{V}\mathbf{x} + 2\mathbf{u}^\mathbf{T}\mathbf{x} + f = 0 \tag{3.4.6}$$

Comparing (3.4.6) with (3.4.6),

$$\mathbf{V} = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix} \tag{3.4.6}$$

$$\mathbf{u} = \begin{pmatrix} 5 \\ -2 \end{pmatrix} \tag{3.4.6}$$

$$f = 0 \tag{3.4.6}$$

Let the point to which the origin is moved be **c**
The above equation (3.4.6) can be modified as

$$(\mathbf{x}+\mathbf{c})^T\mathbf{V}(\mathbf{x}+\mathbf{c}) + 2\mathbf{u}^T(\mathbf{x}+\mathbf{c}) = 0 \tag{3.4.6}$$

From equation (3.4.6) consider,

$$\implies (\mathbf{x}+\mathbf{c})^T\mathbf{V}(\mathbf{x}+\mathbf{c}) \tag{3.4.6}$$

$$\implies \mathbf{x}^T\mathbf{V}\mathbf{x} + \mathbf{c}^T\mathbf{V}\mathbf{x} + \mathbf{x}^T\mathbf{V}\mathbf{c} + \mathbf{c}^T\mathbf{V}\mathbf{c} \tag{3.4.6}$$

$$\implies \mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{c}^T\mathbf{V}\mathbf{x} + \mathbf{c}^T\mathbf{V}\mathbf{c} \tag{3.4.6}$$

Substituting (3.4.6) in equation (3.4.6)

$$\implies \mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{c}^T\mathbf{V}\mathbf{x} + \mathbf{c}^T\mathbf{V}\mathbf{c} + 2\mathbf{u}^T(\mathbf{x}+\mathbf{c}) = 0 \tag{3.4.6}$$

Comparing equations (3.4.6) and (3.4.6), we can write as,

$$\implies 2\mathbf{c}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} = 0 \tag{3.4.6}$$

$$\implies \mathbf{c}^T\mathbf{V} = -\mathbf{u}^T \tag{3.4.6}$$

$$\implies \mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} = -\begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{6} \end{pmatrix} \begin{pmatrix} 5 \\ -2 \end{pmatrix} \tag{3.4.6}$$

$$\implies \boxed{\mathbf{c} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}} \tag{3.4.6}$$

From (3.4.6), when the origin is moved to point **c**, the equation (3.4.6) becomes (3.4.6).

From equations (3.4.6) and (3.4.6), **V** doesn't change

$$\det(\mathbf{V}) = -6 \tag{3.4.6}$$

Since $\det(\mathbf{V}) < 0$ the given equation represents the hyperbola

From equation (3.4.6), the equation is of the form,

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + f = 0 \tag{3.4.6}$$

The matrix **V** can be decomposed as,

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T \quad \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{3.4.6}$$

where λ_1 and λ_2 are Eigen values of **V**, and **P** contains the Eigen vectors corresponding to the Eigen values λ_1 and λ_2. The affine transformation is given by,

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c} \tag{3.4.6}$$

where, **P** indicates the rotation of axes and **c** indicates the shift of origin.
Eigen values of **V** are,

$$\left|\lambda\mathbf{I} - \mathbf{v}\right| = 0 \tag{3.4.6}$$

$$\implies \begin{vmatrix} \lambda - 1 & -2 \\ -2 & \lambda + 2 \end{vmatrix} = 0 \tag{3.4.6}$$

$$\implies \lambda^2 + \lambda - 6 = 0 \tag{3.4.6}$$

$$\implies \lambda_1 = -3, \lambda_2 = 2 \tag{3.4.6}$$

$$\mathbf{D} = \begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix} \tag{3.4.6}$$

Eigen vector for λ_1=-3,

$$\lambda_1\mathbf{I}-\mathbf{v}=\begin{pmatrix}-4 & -2\\ -2 & -1\end{pmatrix}\xleftrightarrow{R_2\leftarrow R_2-\frac{R_1}{2}}\begin{pmatrix}-2 & -1\\ 0 & 0\end{pmatrix} \tag{3.4.6}$$

$$\implies \mathbf{P_1}=\begin{pmatrix}1\\ -2\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{5}}\\ \frac{-2}{\sqrt{5}}\end{pmatrix} \tag{3.4.6}$$

Eigen vector for λ_2=2,

$$\lambda_1\mathbf{I}-\mathbf{v}=\begin{pmatrix}1 & -2\\ -2 & 4\end{pmatrix}\xleftrightarrow{R_2\leftarrow R_2+2R_1}\begin{pmatrix}1 & -2\\ 0 & 0\end{pmatrix} \tag{3.4.6}$$

$$\implies \mathbf{P_2}=\begin{pmatrix}2\\ 1\end{pmatrix}=\begin{pmatrix}\frac{2}{\sqrt{5}}\\ \frac{1}{\sqrt{5}}\end{pmatrix} \tag{3.4.6}$$

$$\mathbf{P}=\begin{pmatrix}\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}}\\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\end{pmatrix} \tag{3.4.6}$$

Therefore $\mathbf{V}$ can be written as,

$$\mathbf{V}=\begin{pmatrix}\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}}\\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\end{pmatrix}\begin{pmatrix}-3 & 0\\ 0 & 2\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}}\\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\end{pmatrix} \tag{3.4.6}$$

Equation (3.4.6) can be written as,

$$\mathbf{x}^T\left[\begin{pmatrix}\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}}\\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\end{pmatrix}\begin{pmatrix}-3 & 0\\ 0 & 2\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}}\\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\end{pmatrix}\right]\mathbf{x}=1 \tag{3.4.6}$$

$$\left[\begin{pmatrix}\frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}}\\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\end{pmatrix}\mathbf{x}\right]^T\begin{pmatrix}-3 & 0\\ 0 & 2\end{pmatrix}\left[\begin{pmatrix}\frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}}\\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\end{pmatrix}\mathbf{x}\right]=1 \tag{3.4.6}$$

Consider the rotation transformation

$$\mathbf{x}=\mathbf{P}\mathbf{y} \tag{3.4.6}$$

$$\implies \mathbf{x}=\begin{pmatrix}\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}}\\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\end{pmatrix}\mathbf{y} \tag{3.4.6}$$

$$\mathbf{P}^{-1}\mathbf{x}=\mathbf{P}^{-1}\mathbf{P}\mathbf{y} \tag{3.4.6}$$

$$\implies \mathbf{y}=\mathbf{P}^{-1}\mathbf{x} \tag{3.4.6}$$

$$\text{But, } \mathbf{P}^{-1}=\mathbf{P}^T \tag{3.4.6}$$

$$\implies \mathbf{y}=\begin{pmatrix}\frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}}\\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\end{pmatrix}\mathbf{x} \tag{3.4.6}$$

Using (3.4.6) in (3.4.6), the equation can be rewritten as

$$\mathbf{y}^T\begin{pmatrix}-3 & 0\\ 0 & 2\end{pmatrix}\mathbf{y}=1 \tag{3.4.6}$$

Equation (3.4.6) is same as (3.4.6) with p=-3 and q=2.
From equation (3.4.6), the orthogonal matrix represents the rotation matrix in form of,

$$\mathbf{P}=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix} \tag{3.4.6}$$

Comparing (3.4.6) and (3.4.6),

$$\cos\theta=\frac{1}{\sqrt{5}} \tag{3.4.6}$$

$$\implies \boxed{\theta=63.43^\circ} \tag{3.4.6}$$

From equation (3.4.6),if the axes is turned by θ then the equation obtained would be (3.4.6).

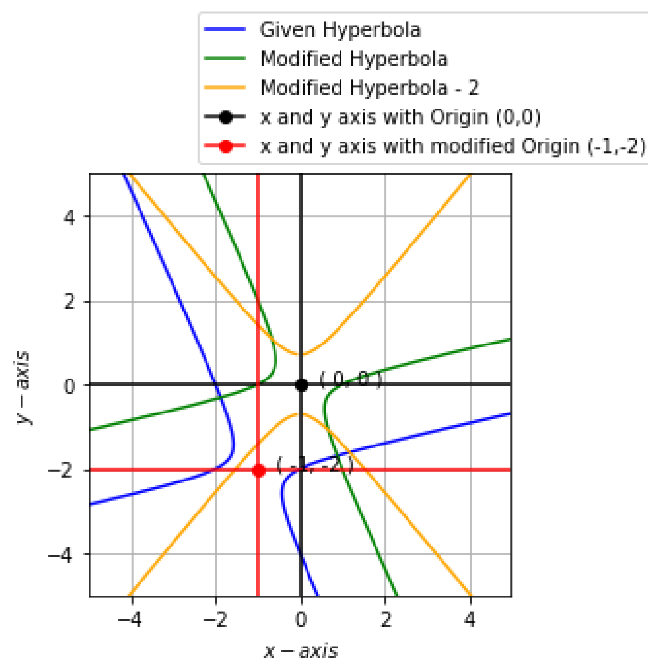

Fig. 3.4.6: Hyperbola plot when origin is shifted and rotated

7. Through what angle must the axes be turned to reduce the equation

$$\mathbf{x}^T\begin{pmatrix}1 & -1\\ -1 & -1\end{pmatrix}\mathbf{x}=1 \tag{3.4.7}$$

to the form

$$\mathbf{x}^T \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \mathbf{x} = c \tag{3.4.7}$$

where c is a constant.

8. Show that, by changing the origin, the equation

$$2\mathbf{x}^T\mathbf{x} + \begin{pmatrix} 7 & 5 \end{pmatrix}\mathbf{x} - 13 = 0 \tag{3.4.8}$$

can be transformed to

$$8\mathbf{x}^T\mathbf{x} = 89 \tag{3.4.8}$$

Solution: Eq (3.4.8) cab be written as

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix} \frac{7}{2} & \frac{5}{2} \end{pmatrix}\mathbf{x} - \frac{13}{2} = 0 \tag{3.4.8}$$

$$\implies \mathbf{x}^T\mathbf{x} + 2\begin{pmatrix} \frac{7}{4} & \frac{5}{4} \end{pmatrix}\mathbf{x} - \frac{13}{2} = 0 \tag{3.4.8}$$

The above eq (3.4.8) cab be compared with the circle equation gives as

$$\mathbf{x}^T\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{3.4.8}$$

then

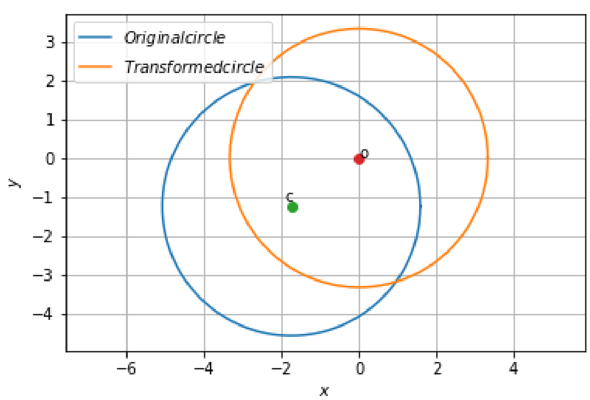

Fig. 3.4.8: Figure depicting transformation of circle

$$\mathbf{u} = \begin{pmatrix} \frac{7}{4} \\ \frac{5}{4} \end{pmatrix} \tag{3.4.8}$$

$$\implies centre, \mathbf{c} = \begin{pmatrix} \frac{-7}{4} \\ \frac{-5}{4} \end{pmatrix} \tag{3.4.8}$$

$$\|u\|^2 - r^2 = f \tag{3.4.8}$$

$$\implies r^2 = \|u\|^2 - f \tag{3.4.8}$$

$$\implies r^2 = \left(\frac{7}{4}\right)^2 + \left(\frac{5}{4}\right)^2 + \frac{13}{2} \tag{3.4.8}$$

$$\implies radius, r = \sqrt{\frac{89}{8}} \tag{3.4.8}$$

The eq (3.4.8) can be written by changing the origin as

$$(\mathbf{x} + \mathbf{c})^T(\mathbf{x} + \mathbf{c}) = \frac{89}{8} \tag{3.4.8}$$

$$\implies \mathbf{x}^T\mathbf{x} + \mathbf{x}^T\mathbf{c} + \mathbf{c}^T\mathbf{x} + \mathbf{c}^T\mathbf{c} = \frac{89}{8} \tag{3.4.8}$$

We know that

$$\mathbf{x}^T\mathbf{c} = \mathbf{c}^T\mathbf{x} \tag{3.4.8}$$

by substituting (3.4.8) in (3.4.8)

$$\mathbf{x}^T\mathbf{x} + 2\mathbf{c}^T\mathbf{x} + \mathbf{c}^T\mathbf{c} = \frac{89}{8} \tag{3.4.8}$$

substituting the orgin of (3.4.8) in above eq (3.4.8)

$$\mathbf{x}^T\mathbf{x} + 2\begin{pmatrix} \frac{7}{4} & \frac{5}{4} \end{pmatrix}\mathbf{x} + \begin{pmatrix} \frac{-7}{4} & \frac{-5}{4} \end{pmatrix}\begin{pmatrix} \frac{-7}{4} \\ \frac{-5}{4} \end{pmatrix} = \frac{89}{8} \tag{3.4.8}$$

$$\implies \mathbf{x}^T\mathbf{x} + \begin{pmatrix} \frac{7}{2} & \frac{5}{2} \end{pmatrix}\mathbf{x} + \frac{74}{16} - \frac{89}{8} = 0 \tag{3.4.8}$$

$$\implies \mathbf{x}^T\mathbf{x} + \begin{pmatrix} \frac{7}{2} & \frac{5}{2} \end{pmatrix}\mathbf{x} - \frac{13}{2} = 0 \tag{3.4.8}$$

$$\implies 2\mathbf{x}^T\mathbf{x} + \begin{pmatrix} 7 & 5 \end{pmatrix}\mathbf{x} - 13 = 0 \tag{3.4.8}$$

$\therefore$ It is proved that by changing the origin in (3.4.8) we obtained (3.4.8).

9. Show that, by rotating the axes, the equation

$$\mathbf{x}^T \begin{pmatrix} 3 & \frac{7}{2} \\ \frac{7}{2} & -3 \end{pmatrix} \mathbf{x} = 1 \tag{3.4.9}$$

can be reduced to

$$\sqrt{85}\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{x} = 2 \tag{3.4.9}$$

10. Show that, by rotating the axes, the equation

$$\mathbf{x}^T \begin{pmatrix} 41 & 12 \\ 12 & 34 \end{pmatrix} \mathbf{x} = 75 \tag{3.4.10}$$

can be reduced to

$$\mathbf{x}^T \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = 3 \tag{3.4.10}$$

Solution:

TThe given equation is of the form

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + f = 0 \tag{3.4.10}$$

The matrix $\mathbf{V}$ can be decomposed as

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T \tag{3.4.10}$$

$$\text{where } \mathbf{D} = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix} \tag{3.4.10}$$

λ_1 and λ_2 are Eigen values of $\mathbf{V}$, and $\mathbf{P}$ contains the Eigen vectors corresponding to the Eigen values λ_1 and λ_2

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c} \tag{3.4.10}$$

indicates the linear transformation where $\mathbf{P}$ indicates the rotation of axes and $\mathbf{c}$ gives the shift of origin.

$$\mathbf{V} = \begin{pmatrix}41 & 12\\ 12 & 34\end{pmatrix} \tag{3.4.10}$$

$$det(\mathbf{V}) = \begin{vmatrix}41 & 12\\ 12 & 34\end{vmatrix} > 0 \tag{3.4.10}$$

So,the given equation represents an ellipse
To find the Eigen values of $\mathbf{V}$

$$\left|\lambda\mathbf{I} - \mathbf{v}\right| = 0 \tag{3.4.10}$$

$$\implies \begin{vmatrix}\lambda - 41 & -12\\ -12 & \lambda - 34\end{vmatrix} = 0 \tag{3.4.10}$$

$$\implies \lambda^2 - 75\lambda + 1250 = 0 \tag{3.4.10}$$

$$\implies \lambda_1 = 50, \lambda_2 = 25 \tag{3.4.10}$$

$$\mathbf{D} = \begin{pmatrix}50 & 0\\ 0 & 25\end{pmatrix} \tag{3.4.10}$$

Finding Eigen vector $\mathbf{p}_1$,

$$\lambda_1\mathbf{I} - \mathbf{V} = \begin{pmatrix}9 & -12\\ -12 & 16\end{pmatrix} \xleftrightarrow[R_2 \leftarrow R_2/4]{R_1 \leftarrow R_1/3} \begin{pmatrix}3 & -4\\ -3 & 4\end{pmatrix} \tag{3.4.10}$$

$$\xleftrightarrow{R_2 \leftarrow R_1 + R_2} \begin{pmatrix}3 & -4\\ 0 & 0\end{pmatrix} \tag{3.4.10}$$

$$\implies \mathbf{p_1} = \frac{1}{\sqrt{4^2 + 3^2}}\begin{pmatrix}4\\ 3\end{pmatrix} = \begin{pmatrix}\frac{4}{5}\\ \frac{3}{5}\end{pmatrix} \tag{3.4.10}$$

Similarly,

$$\lambda_2\mathbf{I} - \mathbf{V} = \begin{pmatrix}-16 & -12\\ -12 & -9\end{pmatrix} \xleftrightarrow[R_2 \leftarrow R_2/-3]{R_1 \leftarrow R_1/-4} \begin{pmatrix}4 & 3\\ 4 & 3\end{pmatrix} \tag{3.4.10}$$

$$\xleftrightarrow{R_2 \leftarrow R_1 - R_2} \begin{pmatrix}4 & 3\\ 0 & 0\end{pmatrix} \tag{3.4.10}$$

$$\implies \mathbf{p_2} = \frac{1}{\sqrt{4^2 + 3^2}}\begin{pmatrix}-3\\ 4\end{pmatrix} = \begin{pmatrix}\frac{-3}{5}\\ \frac{4}{5}\end{pmatrix} \tag{3.4.10}$$

$$\text{Therefore, } \mathbf{P} = \begin{pmatrix}\mathbf{p_1} & \mathbf{p_2}\end{pmatrix} = \begin{pmatrix}\frac{4}{5} & \frac{-3}{5}\\ \frac{3}{5} & \frac{4}{5}\end{pmatrix} \tag{3.4.10}$$

From (3.4.10) $\mathbf{V}$ can be rewritten as

$$\mathbf{V} = \begin{pmatrix}\frac{4}{5} & \frac{-3}{5}\\ \frac{3}{5} & \frac{4}{5}\end{pmatrix}\begin{pmatrix}50 & 0\\ 0 & 25\end{pmatrix}\begin{pmatrix}\frac{4}{5} & \frac{3}{5}\\ \frac{-3}{5} & \frac{4}{5}\end{pmatrix} \tag{3.4.10}$$

(3.4.10) can be now rewritten as

$$25\left[\mathbf{x}^T\begin{pmatrix}\frac{4}{5} & \frac{-3}{5}\\ \frac{3}{5} & \frac{4}{5}\end{pmatrix}\begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\frac{4}{5} & \frac{3}{5}\\ \frac{-3}{5} & \frac{4}{5}\end{pmatrix}\mathbf{x}\right] = 75 \tag{3.4.10}$$

$$\left[\begin{pmatrix}\frac{4}{5} & \frac{3}{5}\\ \frac{-3}{5} & \frac{4}{5}\end{pmatrix}\mathbf{x}\right]^T\begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}\left[\begin{pmatrix}\frac{4}{5} & \frac{3}{5}\\ \frac{-3}{5} & \frac{4}{5}\end{pmatrix}\mathbf{x}\right] = 3 \tag{3.4.10}$$

Consider the rotation transformation

$$\mathbf{x} = \mathbf{P}\mathbf{y} \tag{3.4.10}$$

$$\implies \mathbf{x} = \begin{pmatrix}\frac{4}{5} & \frac{-3}{5}\\ \frac{3}{5} & \frac{4}{5}\end{pmatrix}\mathbf{y} \tag{3.4.10}$$

$$\mathbf{P}^{-1}\mathbf{x} = \mathbf{P}^{-1}\mathbf{P}\mathbf{y} \tag{3.4.10}$$

$$\implies \mathbf{y} = \mathbf{P}^{-1}\mathbf{x} \tag{3.4.10}$$

$$\text{But, } \mathbf{P}^{-1} = \mathbf{P}^T \tag{3.4.10}$$

$$\implies \mathbf{y} = \begin{pmatrix}\frac{4}{5} & \frac{3}{5}\\ \frac{-3}{5} & \frac{4}{5}\end{pmatrix}\mathbf{x} \tag{3.4.10}$$

Using (3.4.10) in (3.4.10), the ellipse equation can be rewritten as

$$\mathbf{y}^T\begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}\mathbf{y} = 3 \tag{3.4.10}$$

11. Show that, by a change of origin and the directions of the coordinate axes, the equation

$$\mathbf{x}^T\begin{pmatrix}5 & 1\\ 1 & 5\end{pmatrix}\mathbf{x} - \begin{pmatrix}14 & 22\end{pmatrix}\mathbf{x} + 27 = 0 \tag{3.4.11}$$

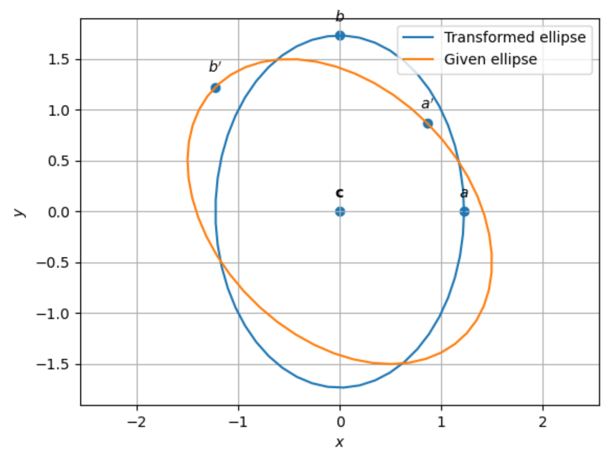

Fig. 3.4.10: plot showing the original and rotated ellipse

can be transformed to

$$\mathbf{x}^T\begin{pmatrix}3 & 0\\0 & 2\end{pmatrix}\mathbf{x} = 1 \tag{3.4.11}$$

or

$$\mathbf{x}^T\begin{pmatrix}2 & 0\\0 & 3\end{pmatrix}\mathbf{x} = 1 \tag{3.4.11}$$

Solution: The general second order equation can be expressed as follows,

$$\mathbf{x}^{\mathbf{T}}\mathbf{V}\mathbf{x} + 2\mathbf{u}^{\mathbf{T}}\mathbf{x} + f = 0 \tag{3.4.11}$$

Comparing (3.4.11) with (3.4.11),

$$\mathbf{V} = \begin{pmatrix}5 & 1\\1 & 5\end{pmatrix} \tag{3.4.11}$$

$$\mathbf{u} = \begin{pmatrix}-7\\-11\end{pmatrix} \tag{3.4.11}$$

$$f = 27 \tag{3.4.11}$$

Let $\mathbf{c}$ be the change in the origin. The equation (3.4.11) can be modified as

$$(\mathbf{x}+\mathbf{c})^{\mathbf{T}}\mathbf{V}(\mathbf{x}+\mathbf{c}) + 2\mathbf{u}^{\mathbf{T}}(\mathbf{x}+\mathbf{c}) + f = 0 \tag{3.4.11}$$

Considering (3.4.11)

$$\implies (\mathbf{x}+\mathbf{c})^{\mathbf{T}}\mathbf{V}(\mathbf{x}+\mathbf{c}) \tag{3.4.11}$$

$$\implies \mathbf{x}^{\mathbf{T}}\mathbf{V}\mathbf{x} + \mathbf{c}^{\mathbf{T}}\mathbf{V}\mathbf{x} + \mathbf{x}^{\mathbf{T}}\mathbf{V}\mathbf{c} + \mathbf{c}^{\mathbf{T}}\mathbf{V}\mathbf{c} \tag{3.4.11}$$

In the above equation

$$\mathbf{c}^{\mathbf{T}}\mathbf{V}\mathbf{x} = \mathbf{x}^{\mathbf{T}}\mathbf{V}\mathbf{c} \tag{3.4.11}$$

From (3.4.11) and (3.4.11) then (3.4.11) becomes

$$\mathbf{x}^{\mathbf{T}}\mathbf{V}\mathbf{x} + 2\mathbf{c}^{\mathbf{T}}\mathbf{V}\mathbf{x} + \mathbf{c}^{\mathbf{T}}\mathbf{V}\mathbf{c} + 2\mathbf{u}^{\mathbf{T}}\mathbf{x} + 2\mathbf{u}^{\mathbf{T}}\mathbf{c} + f = 0 \tag{3.4.11}$$

Comparing (3.4.11) and (3.4.11)

$$2\mathbf{c}^{\mathbf{T}}\mathbf{V}\mathbf{P}\mathbf{y} + 2\mathbf{u}^{\mathbf{T}}\mathbf{P}\mathbf{y} = 0 \tag{3.4.11}$$

$$\mathbf{c}^{\mathbf{T}}\mathbf{V}\mathbf{P}\mathbf{y} = -\mathbf{u}^{\mathbf{T}}\mathbf{P}\mathbf{y} \tag{3.4.11}$$

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \tag{3.4.11}$$

Substituting (3.4.11) and (3.4.11) in (3.4.11)

$$\mathbf{c} = \frac{-1}{24}\begin{pmatrix}5 & -1\\-1 & 5\end{pmatrix}\begin{pmatrix}-7\\-11\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix} \tag{3.4.11}$$

Hence (3.4.11) becomes

$$\mathbf{x}^{\mathbf{T}}\mathbf{V}\mathbf{x} + \mathbf{c}^{\mathbf{T}}\mathbf{V}\mathbf{c} + 2\mathbf{u}^{\mathbf{T}}\mathbf{c} + f = 0 \tag{3.4.11}$$

Substituting (3.4.11), (3.4.11) and (3.4.11) the above equation becomes

$$\mathbf{x}^{\mathbf{T}}\begin{pmatrix}5 & 1\\1 & 5\end{pmatrix}\mathbf{x} + \begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}5 & 1\\1 & 5\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix} + 2\begin{pmatrix}-7 & -11\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix} \tag{3.4.11}$$

$$+27 = 0 \tag{3.4.11}$$

$$\mathbf{x}^{\mathbf{T}}\begin{pmatrix}5 & 1\\1 & 5\end{pmatrix}\mathbf{x} + 29 - 58 + 27 = 0 \tag{3.4.11}$$

$$\mathbf{x}^{\mathbf{T}}\mathbf{V}\mathbf{x} - 2 = 0 \tag{3.4.11}$$

With change in the origin to point $\mathbf{c} = \begin{pmatrix}1\\2\end{pmatrix}$ but the $\mathbf{V}$ doesn't change.

$$\left|\mathbf{V}\right| = \begin{vmatrix}5 & 1\\1 & 5\end{vmatrix} = 24 \tag{3.4.11}$$

As $\left|\mathbf{V}\right| > 0$ it represents a ellipse.Hence $\mathbf{V}$ can be written as,

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^{\mathbf{T}} \tag{3.4.11}$$

The characteristic equation of $\mathbf{V}$ is given by

$$\left|\mathbf{V} - \mathbf{I}\lambda\right| = 0 \tag{3.4.11}$$

$$\begin{vmatrix}5-\lambda & 1\\1 & 5-\lambda\end{vmatrix} = 0 \tag{3.4.11}$$

$$\implies \lambda^2 - 10\lambda + 24 = 0 \tag{3.4.11}$$

Hence the eigen vales are,

$$\lambda_1 = 4 \tag{3.4.11}$$
$$\lambda_2 = 6 \tag{3.4.11}$$

Hence diagonal vector is given by,

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 6 \end{pmatrix} \tag{3.4.11}$$

The eigen vector $\mathbf{p}$ is given by

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \tag{3.4.11}$$
$$(\mathbf{V} - \lambda\mathbf{I})\mathbf{p} = 0 \tag{3.4.11}$$

For $\lambda_1 = 4$ the eigenvector is,

$$\mathbf{V} - \lambda_1\mathbf{I} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \tag{3.4.11}$$
$$\mathbf{p_1} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} \tag{3.4.11}$$

For $\lambda_1 = 6$ the eigenvector is,

$$\mathbf{V} - \lambda_1\mathbf{I} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 + R_1} \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \tag{3.4.11}$$
$$\mathbf{p_1} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \tag{3.4.11}$$

Hence,

$$\mathbf{P} = \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \tag{3.4.11}$$

Substituting (3.4.11) and (3.4.11) in (3.4.11)

$$\mathbf{V} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 6 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \tag{3.4.11}$$

Hence substituting (3.4.11) in (3.4.11)

$$\mathbf{x^T} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 6 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \mathbf{x} = 2 \tag{3.4.11}$$
$$\mathbf{y^T} \begin{pmatrix} 4 & 0 \\ 0 & 6 \end{pmatrix} \mathbf{y} = 2 \tag{3.4.11}$$
$$\mathbf{y^T} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \mathbf{y} = 1 \tag{3.4.11}$$

where $\mathbf{y}$ is given by Affine transformation

$$\mathbf{x} = \mathbf{P}\mathbf{y} \tag{3.4.11}$$
$$\mathbf{y} = \mathbf{P^T}\mathbf{x} \tag{3.4.11}$$

The rotation matrix $\mathbf{P}$ can be given by,

$$\mathbf{P} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{3.4.11}$$

Comparing (3.4.11) and (3.4.11)

$$\cos\theta = \frac{1}{\sqrt{2}} \tag{3.4.11}$$
$$\theta = \frac{\pi}{4} \tag{3.4.11}$$

But given the direction of coordinate axes changes so,

$$\theta = \pi + \frac{\pi}{4} \tag{3.4.11}$$

Subtituting (3.4.11) in (3.4.11) we get

$$\mathbf{P} = \begin{pmatrix} \cos\left(\pi + \frac{\pi}{4}\right) & \sin\left(\pi + \frac{\pi}{4}\right) \\ -\sin\left(\pi + \frac{\pi}{4}\right) & \cos\left(\pi + \frac{\pi}{4}\right) \end{pmatrix} \tag{3.4.11}$$
$$\mathbf{P} = \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} \tag{3.4.11}$$

From (3.4.11) we find the diagonal matrix

$$\mathbf{D} = \mathbf{P^T}\mathbf{V}\mathbf{P} = \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} \tag{3.4.11}$$
$$\mathbf{D} = \begin{pmatrix} 6 & 0 \\ 0 & 4 \end{pmatrix} \tag{3.4.11}$$

Hence using (3.4.11), (3.4.11) and (3.4.11) in (3.4.11) we get.

$$\mathbf{x^T} \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 6 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} \mathbf{x} = 2 \tag{3.4.11}$$

using (3.4.11) the above equation becomes,

$$\mathbf{y^T} \begin{pmatrix} 6 & 0 \\ 0 & 4 \end{pmatrix} \mathbf{y} = 2 \tag{3.4.11}$$
$$\mathbf{y^T} \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{y} = 1 \tag{3.4.11}$$

Hence from (3.4.11) and (3.4.11) proved that change of origin and the directions of the coordinate axes (3.4.11) can be tranformed to (3.4.11) or (3.4.11)

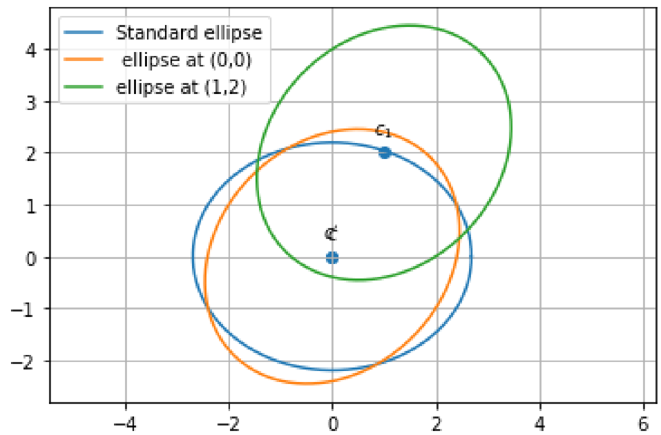

Fig. 3.4.11: Ellipse

4 Circle

4.1 Equation

1. Find the radius and the coordinates of the centre of each of the following circles:
   a) $3\mathbf{x}^T\mathbf{x} + \begin{pmatrix}-12 & 6\end{pmatrix}\mathbf{x} + 11 = 0$
   Solution:
   The general equation of circle can be expressed as

$$\mathbf{x}^T\mathbf{x} - 2\mathbf{c}^T\mathbf{x} + f = 0 \tag{4.1.1}$$

where $\mathbf{c}$ is the centre of the circle and radius of the circle is given as

$$r = \sqrt{\|\mathbf{c}\|^2 - f} \tag{4.1.1}$$

Given equation is

$$3\mathbf{x}^T\mathbf{x} + \begin{pmatrix}-12 & 6\end{pmatrix}\mathbf{x} + 11 = 0 \tag{4.1.1}$$

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}-4 & 2\end{pmatrix}\mathbf{x} + \frac{11}{3} = 0 \tag{4.1.1}$$

$$\mathbf{x}^T\mathbf{x} - 2\begin{pmatrix}2 & -1\end{pmatrix}\mathbf{x} + \frac{11}{3} = 0 \tag{4.1.1}$$

Compare Eq (4.1.1) and Eq (4.1.1)

$$\implies \mathbf{c}^T = \begin{pmatrix}2 & -1\end{pmatrix} \tag{4.1.1}$$

$$\implies \mathbf{c} = \begin{pmatrix}2\\-1\end{pmatrix} \tag{4.1.1}$$

$$f = \frac{11}{3} \tag{4.1.1}$$

From Eq (4.1.1),

$$\implies r = \sqrt{\left(2^2 + (-1)^2\right) - \frac{11}{3}} \tag{4.1.1}$$

$$= \sqrt{4 + 1 - \frac{11}{3}} \tag{4.1.1}$$

$$= \sqrt{\frac{4}{3}} \tag{4.1.1}$$

From Eq (4.1.1) and Eq (4.1.1)

$$\mathbf{c} = \begin{pmatrix}2\\-1\end{pmatrix} \tag{4.1.1}$$

$$r = \sqrt{\frac{4}{3}} \tag{4.1.1}$$

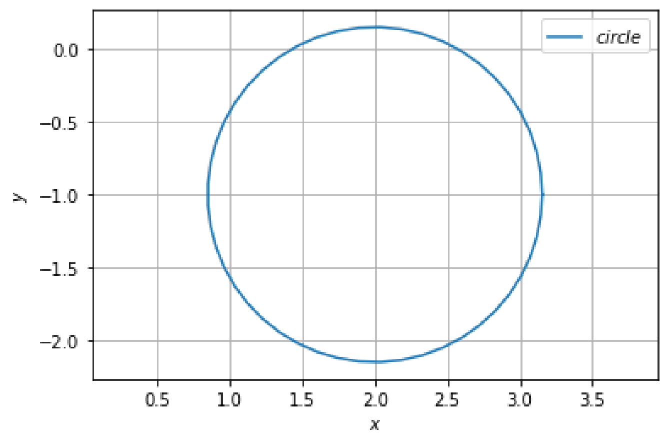

Fig. 4.1.1: Circle with radius 1.154 and center coordinates (2,-1)

   b) $\mathbf{x}^T\mathbf{x} = a^2 + b^2$
   c)

$$2\mathbf{x}^T\mathbf{x} + \begin{pmatrix}16 & -4\end{pmatrix}\mathbf{x} + 33 = 0 \tag{4.1.1}$$

Solution:
The general equation of a circle is given by

$$\mathbf{x}^T\mathbf{x} - 2\mathbf{c}^T\mathbf{x} + f = 0 \tag{4.1.1}$$

where $\mathbf{c}$ is the centre of the circle and r $= \sqrt{\|\mathbf{c}\|^2 - f}$ is the radius of the circle. Dividing (4.1.1) by 2 and rearranging terms (4.1.1) can be rewritten as

$$\mathbf{x}^T\mathbf{x} - 2\begin{pmatrix}-4\\1\end{pmatrix}^T\mathbf{x} + \frac{33}{2} = 0 \tag{4.1.1}$$

Comparing (4.1.1) and (4.1.1) we get

$$\mathbf{c} = \begin{pmatrix} -4 \\ 1 \end{pmatrix} \tag{4.1.1}$$

$$f = \frac{33}{2} \tag{4.1.1}$$

Then centre of the circle (4.1.1) is $\mathbf{c} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$ and radius

$$r = \sqrt{\|\mathbf{c}\|^2 - f} \tag{4.1.1}$$

$$= \sqrt{(-4)^2 + 1^2 - \frac{33}{2}} \tag{4.1.1}$$

$$= \sqrt{\frac{1}{2}} \tag{4.1.1}$$

$$= 0.7071 \tag{4.1.1}$$

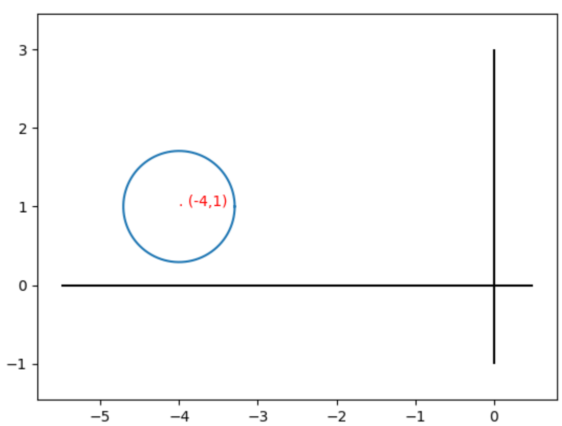

Fig. 4.1.1: Graph of $2x^2 + 2y^2 + 16x - 4y + 33 = 0$

d) $36\mathbf{x}^T\mathbf{x} - \begin{pmatrix} 36 & 24 \end{pmatrix}\mathbf{x} - 131 = 0$

2. Find the equation of the circle that passes through the points $\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Solution: The equation of circle can be expressed as

$$\mathbf{x}^T\mathbf{x} - 2\mathbf{c}^T\mathbf{x} + f = 0 \tag{4.1.2}$$

$\mathbf{c}$ is the centre and substituting the points in the equation of circle we get

$$2\begin{pmatrix} 1 & 2 \end{pmatrix}\mathbf{c} - f = 5 \tag{4.1.2}$$

$$2\begin{pmatrix} 2 & 1 \end{pmatrix}\mathbf{c} - f = 5 \tag{4.1.2}$$

$$2\begin{pmatrix} 0 & 0 \end{pmatrix}\mathbf{c} - f = 0 \tag{4.1.2}$$

can be expressed in matrix form

$$\begin{pmatrix} 2 & 4 & -1 \\ 4 & 2 & -1 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} \mathbf{c} \\ f \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \\ 0 \end{pmatrix} \tag{4.1.2}$$

Row reducing the augmented matrix

$$\begin{pmatrix} 2 & 4 & -1 & 5 \\ 4 & 2 & -1 & 5 \\ 0 & 0 & -1 & 0 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow 2R_1 - R_2} \begin{pmatrix} 2 & 4 & -1 & 5 \\ 0 & 6 & -1 & 5 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{4.1.2}$$

$$\xleftrightarrow[R_1 \leftarrow R_1 - R_3]{R_2 \leftarrow R_2 - R_3} \begin{pmatrix} 2 & 4 & 0 & 5 \\ 0 & 6 & 0 & 5 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{4.1.2}$$

$$\xleftrightarrow{R_1 \leftarrow 3R_1 - 2R_2} \begin{pmatrix} 6 & 0 & 0 & 5 \\ 0 & 6 & 0 & 5 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{4.1.2}$$

$$\mathbf{c} = \begin{pmatrix} \frac{5}{6} \\ \frac{5}{6} \end{pmatrix} \tag{4.1.2}$$

$$f = 0 \tag{4.1.2}$$

$$r = \sqrt{\|\mathbf{c}\|^2 - f} = \sqrt{\frac{50}{36}} \tag{4.1.2}$$

The required equation of circle is

$$\mathbf{x}^T\mathbf{x} - 2\begin{pmatrix} \frac{5}{6} & \frac{5}{6} \end{pmatrix}\mathbf{x} = 0 \tag{4.1.2}$$

See Fig. 4.1.2

3. Find the equation of the circle that passes through the points $\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ 1 \end{pmatrix}$.

Solution: The general equation of circle is represented as

$$\mathbf{x}^T\mathbf{x} - 2\mathbf{c}^T\mathbf{x} + f = 0 \tag{4.1.3}$$

where $\mathbf{c}$ is the center of the circle. Substituting the given points in the equation (4.1.3), we

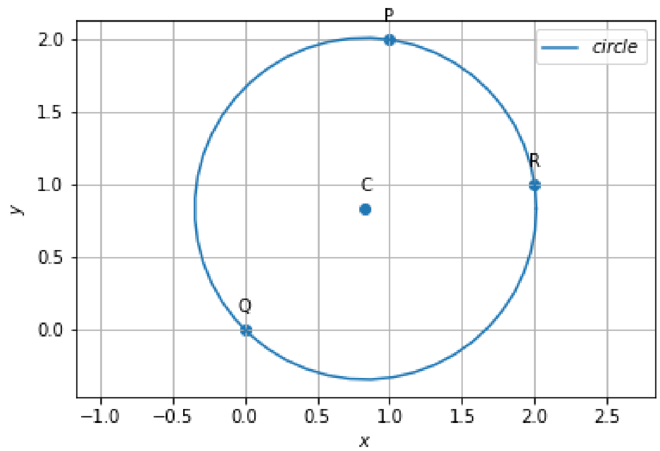

Fig. 4.1.2: Circle passing through point P,Q,R with centre C.

obtain

$$2\begin{pmatrix}2 & 3\end{pmatrix}\mathbf{c} - f = 13 \tag{4.1.3}$$

$$2\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{c} - f = 13 \tag{4.1.3}$$

$$2\begin{pmatrix}5 & 1\end{pmatrix}\mathbf{c} - f = 36 \tag{4.1.3}$$

can be expressed in matrix form as

$$\begin{pmatrix}4 & 6 & -1\\ 6 & 4 & -1\\ 10 & 2 & -1\end{pmatrix}\begin{pmatrix}\mathbf{c}\\ f\end{pmatrix} = \begin{pmatrix}13\\ 13\\ 26\end{pmatrix} \tag{4.1.3}$$

The augmented matrix for (4.1.3) can be row

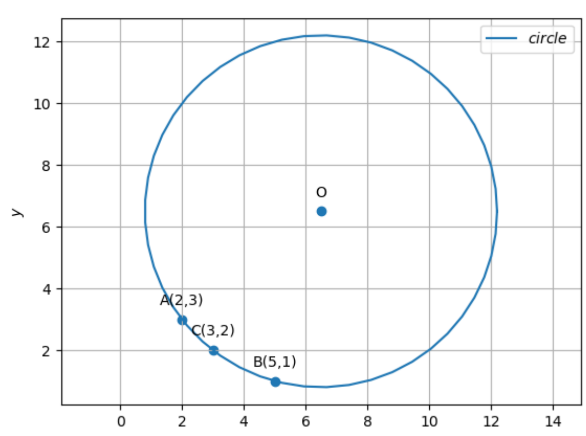

Fig. 4.1.3: Circle passing through the points A,B,C with center O

reduced as follows

$$\begin{pmatrix}4 & 6 & -1 & 13\\ 6 & 4 & -1 & 13\\ 10 & 2 & -1 & 26\end{pmatrix} \tag{4.1.3}$$

$$\xleftrightarrow[R_2 \leftarrow 4R_2 - 6R_1]{R_3 \leftarrow 4R_3 - 10R_1}\begin{pmatrix}4 & 6 & -1 & 13\\ 0 & -20 & 2 & -26\\ 0 & -52 & 6 & -26\end{pmatrix} \tag{4.1.3}$$

$$\xleftrightarrow{R_3 \leftarrow 5R_3 - 13R_2}\begin{pmatrix}4 & 6 & -1 & 13\\ 0 & -20 & 2 & -26\\ 0 & 0 & 4 & 208\end{pmatrix} \tag{4.1.3}$$

$$\xleftrightarrow[R_1 \leftarrow 4R_1 + R_3]{R_2 \leftarrow 2R_2 - R_3}\begin{pmatrix}16 & 24 & 0 & 260\\ 0 & -40 & 0 & -260\\ 0 & 0 & 4 & 208\end{pmatrix} \tag{4.1.3}$$

$$\xleftrightarrow{R_1 \leftarrow 5R_1 + 3R_2}\begin{pmatrix}80 & 0 & 0 & 520\\ 0 & -40 & 0 & -260\\ 0 & 0 & 4 & 208\end{pmatrix} \tag{4.1.3}$$

$$\xleftrightarrow[R_1 \leftarrow \frac{R_1}{40}]{R_2 \leftarrow \frac{R_2}{-20}, R_3 \leftarrow \frac{R_3}{4}}\begin{pmatrix}2 & 0 & 0 & 13\\ 0 & 2 & 0 & 13\\ 0 & 0 & 1 & 52\end{pmatrix} \tag{4.1.3}$$

From the matrix (4.1.3),

$$\mathbf{c} = \begin{pmatrix}\frac{13}{2}\\ \frac{13}{2}\end{pmatrix} \tag{4.1.3}$$

$$k = 52 \tag{4.1.3}$$

$$r = \sqrt{\|\mathbf{c}\|^2 - f} = 11 \tag{4.1.3}$$

Hence the circle equation can be written as,

$$\mathbf{x}^T\mathbf{x} - 2\begin{pmatrix}\frac{13}{2} & \frac{13}{2}\end{pmatrix}^T\mathbf{x} + 52 = 0 \tag{4.1.3}$$

4. Find the equation of the circle that passes through the points $\begin{pmatrix}2a\\ 0\end{pmatrix}, \begin{pmatrix}0\\ 2b\end{pmatrix}, \begin{pmatrix}a+b\\ a+b\end{pmatrix}$.
Solution: The equation of circle can be expressed as

$$\mathbf{x}^T\mathbf{x} - 2\mathbf{c}^T\mathbf{x} + f = 0 \tag{4.1.4}$$

$\mathbf{c}$ is the centre and substituting the points in the equation of circle we get

$$2\begin{pmatrix}2a & 0\end{pmatrix}\mathbf{c} - f = 4a^2 \tag{4.1.4}$$

$$2\begin{pmatrix}0 & 2b\end{pmatrix}\mathbf{c} - f = 4b^2 \tag{4.1.4}$$

$$2\begin{pmatrix}a+b & a+b\end{pmatrix}\mathbf{c} - f = 2(a+b)^2 \tag{4.1.4}$$

which can be expressed in matrix form

$$\begin{pmatrix}4a & 0 & -1\\ 0 & 4b & -1\\ 2(a+b) & 2(a+b) & -1\end{pmatrix}\begin{pmatrix}\mathbf{c}\\ f\end{pmatrix} = \begin{pmatrix}4a^2\\ 4b^2\\ 2(a+b)^2\end{pmatrix} \tag{4.1.4}$$

Row reducing the augmented matrix

$$\begin{pmatrix}4a & 0 & -1 & 4a^2\\ 0 & 4b & -1 & 4b^2\\ 2(a+b) & 2(a+b) & -1 & 2(a+b)^2\end{pmatrix} \tag{4.1.4}$$

$$\xleftrightarrow[R_3\leftarrow R_3-2(a+b)R_1]{R_1\leftarrow\frac{R_1}{4a}}\begin{pmatrix}1 & 0 & -\frac{1}{4a} & a\\ 0 & 4b & -1 & 4b^2\\ 0 & 2(a+b) & \frac{-a+b}{2a} & 2b(a+b)\end{pmatrix} \tag{4.1.4}$$

$$\xleftrightarrow[R_2\leftarrow\frac{R_2}{4b}]{R_3\leftarrow R_3-2(a+b)R_2}\begin{pmatrix}1 & 0 & -\frac{1}{4a} & a\\ 0 & 1 & -\frac{1}{4b} & b\\ 0 & 0 & \frac{a}{2b}+\frac{b}{2a} & 0\end{pmatrix} \tag{4.1.4}$$

$$\xleftrightarrow{R_3\leftarrow\frac{R_3}{\frac{a}{2b}+\frac{b}{2a}}}\begin{pmatrix}1 & 0 & -\frac{1}{4a} & a\\ 0 & 1 & -\frac{1}{4b} & b\\ 0 & 0 & 1 & 0\end{pmatrix} \tag{4.1.4}$$

$$\xleftrightarrow[R_1\leftarrow R_1-(-\frac{1}{4a})R_3]{R_2\leftarrow R_2-(-\frac{1}{4b})R_3}\begin{pmatrix}1 & 0 & 0 & a\\ 0 & 1 & 0 & b\\ 0 & 0 & 1 & 0\end{pmatrix} \tag{4.1.4}$$

$$\mathbf{c} = \begin{pmatrix}a\\ b\end{pmatrix} \tag{4.1.4}$$

$$f = 0 \tag{4.1.4}$$

$$r = \sqrt{\|\mathbf{c}\|^2 - f} = \sqrt{(a^2+b^2)} \tag{4.1.4}$$

The required equation of circle is

$$\mathbf{x}^T\mathbf{x} - 2\begin{pmatrix}a & b\end{pmatrix}\mathbf{x} = 0 \tag{4.1.4}$$

5. A circle has its centre on the line $x = 2y$ and passes through the points $\begin{pmatrix}-1\\ 2\end{pmatrix}, \begin{pmatrix}3\\ -2\end{pmatrix}$. Find the coordinates of the centre and the equation of the circle.
Solution: The equation of circle can be expressed as

$$\mathbf{x}^T\mathbf{x} - 2\mathbf{c}^T\mathbf{x} + f = 0 \tag{4.1.5}$$

$\mathbf{c}$ is the centre and substituting the points in the

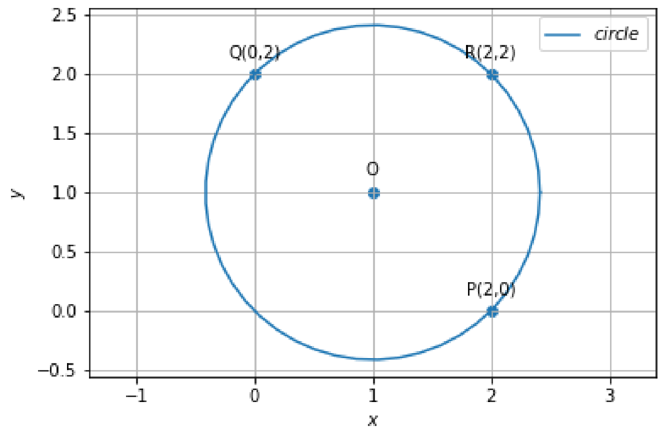

Fig. 4.1.4: Circle passing through point P and Q and R

equation of circle we get

$$2\begin{pmatrix}-1 & 2\end{pmatrix}\mathbf{c} - f = 5 \tag{4.1.5}$$

$$2\begin{pmatrix}3 & -2\end{pmatrix}\mathbf{c} - f = 13 \tag{4.1.5}$$

$$\begin{pmatrix}1 & -2\end{pmatrix}\mathbf{c} = 0 \tag{4.1.5}$$

can be expressed in matrix form

$$\begin{pmatrix}1 & -2 & 0\\ 6 & -4 & -1\\ -2 & 4 & -1\end{pmatrix}\begin{pmatrix}\mathbf{c}\\ f\end{pmatrix} = \begin{pmatrix}0\\ 13\\ 5\end{pmatrix} \tag{4.1.5}$$

Row reducing the augmented matrix

$$\begin{pmatrix}1 & -2 & 0 & 0\\ 6 & -4 & -1 & 13\\ -2 & 4 & -1 & 5\end{pmatrix}\xleftrightarrow[R_3\leftarrow R_3+2R_1]{R_2\leftarrow R_2-6R_1}\begin{pmatrix}1 & -2 & 0 & 0\\ 0 & 8 & -1 & 13\\ 0 & 0 & -1 & 5\end{pmatrix} \tag{4.1.5}$$

$$\xleftrightarrow[R_1\leftarrow R_1+R_2]{R_2\leftarrow R_2/4}\begin{pmatrix}1 & 0 & \frac{-1}{4} & \frac{13}{4}\\ 0 & 2 & \frac{-1}{4} & \frac{13}{4}\\ 0 & 0 & -1 & 5\end{pmatrix} \tag{4.1.5}$$

$$\xleftrightarrow[R_1\leftarrow R_1-R_3/4]{R_2\leftarrow R_2-R_3/4}\begin{pmatrix}1 & 0 & 0 & 2\\ 0 & 2 & 0 & 2\\ 0 & 0 & -1 & 5\end{pmatrix} \tag{4.1.5}$$

$$\mathbf{c} = \begin{pmatrix}2\\ 1\end{pmatrix} \tag{4.1.5}$$

$$f = -5 \tag{4.1.5}$$

$$r = \sqrt{\|\mathbf{c}\|^2 - f} = \sqrt{10} \tag{4.1.5}$$

The required equation of circle is

$$\mathbf{x}^T\mathbf{x} - 2\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} + 5 = 0 \tag{4.1.5}$$

See Fig. 4.1.5

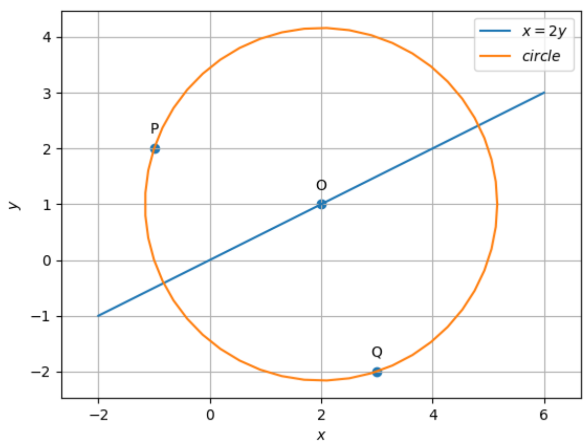

Fig. 4.1.5: Circle passing through point P and Q also centre lie on the line x=2y

6. Find the locus of the centre of a circle which touches the line

$$\begin{pmatrix}\cos\alpha & \sin\alpha\end{pmatrix}\mathbf{x} = p \tag{4.1.6.1}$$

and the circle

$$\|\mathbf{x} - \mathbf{c}\| = r \tag{4.1.6.2}$$

4.2 Tangent and Normal

1. Without drawing a figure, determine whether the points $\begin{pmatrix}-1\\2\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}3\\-4\end{pmatrix}$ lie outside, on the circumference, or inside the circle

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}-5 & 2\end{pmatrix}\mathbf{x} - 5 = 0 \tag{4.2.1.1}$$

Solution: Without drawing a figure, determine whether the points $\begin{pmatrix}-1\\2\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}3\\-4\end{pmatrix}$ lie outside, on the circumference, or inside the circle

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}-5 & 2\end{pmatrix}\mathbf{x} - 5 = 0 \tag{4.2.1.2}$$

The equation of circle with center $\mathbf{c}$ can be expressed as

$$\mathbf{x}^T\mathbf{x} - 2\mathbf{c}^T\mathbf{x} + f = 0 \tag{4.2.1.3}$$

Comparing (4.2.1.3) with (4.2.1.2)

$$\mathbf{c} = \begin{pmatrix}\frac{5}{2}\\-1\end{pmatrix}, f = -5 \tag{4.2.1.4}$$

$$r = \sqrt{\|\mathbf{c}\|^2 - f} = \sqrt{\frac{49}{4}} \tag{4.2.1.5}$$

a) Let a= $\begin{pmatrix}-1\\2\end{pmatrix}$

$$\|\mathbf{a} - \mathbf{c}\| = \sqrt{\frac{49}{4} + 9} = \sqrt{\frac{84}{4}} \implies \|\mathbf{a} - \mathbf{c}\| > r \tag{4.2.1.6}$$

Point a is outside the circle

b) Let b= $\begin{pmatrix}0\\0\end{pmatrix}$

$$\|\mathbf{b} - \mathbf{c}\| = \sqrt{\frac{25}{4} + 1} = \sqrt{\frac{29}{4}} \implies \|\mathbf{a} - \mathbf{c}\| < r \tag{4.2.1.7}$$

Point b is inside the circle.

c) Let d= $\begin{pmatrix}3\\-4\end{pmatrix}$

$$\|\mathbf{d} - \mathbf{c}\| = \sqrt{\frac{1}{4} + 9} = \sqrt{\frac{37}{4}} \implies \|\mathbf{d} - \mathbf{c}\| < r \tag{4.2.1.8}$$

Point d is inside the circle.

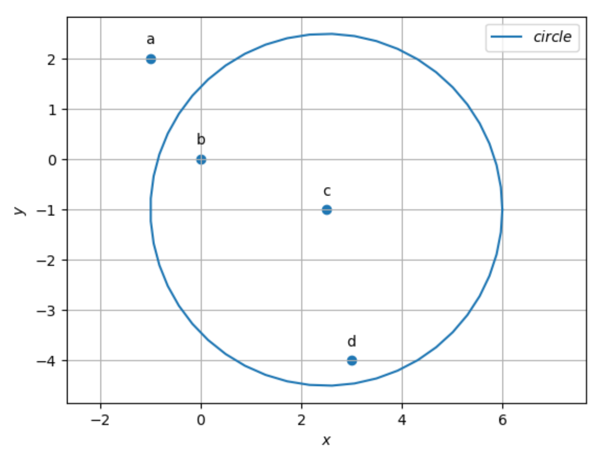

Fig. 4.2.1: Points a,b,d in the circle with center c

2. Find the points of intersection of the line

$$\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x} = 12 \tag{4.2.2.1}$$

and the circle

$$\|\mathbf{x}\|^2 = 13 \tag{4.2.2.2}$$

and for what values of c the line

$$\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x} = c \tag{4.2.2.3}$$

touches the circle.

Solution: If $\mathbf{P}$ be a point on the line and $\mathbf{n}$ is the normal vector, the equation of the line can be expressed as

$$\mathbf{n}^T(\mathbf{x}-\mathbf{P}) = 0 \tag{4.2.2.4}$$

$$\implies \mathbf{n}^T\mathbf{x} = c \tag{4.2.2.5}$$

where

$$c = \mathbf{n}^T\mathbf{P} \tag{4.2.2.6}$$

From (4.2.2.1) and (4.2.2.5),

$$\mathbf{n}^T = \begin{pmatrix}3 & 2\end{pmatrix} \tag{4.2.2.7}$$

We know,

$$\mathbf{m}^T\mathbf{n} = 0 \tag{4.2.2.8}$$

$$\implies \mathbf{m}^T\begin{pmatrix}3\\2\end{pmatrix} = 0 \tag{4.2.2.9}$$

$$\implies \mathbf{m}^T = \begin{pmatrix}-2 & 3\end{pmatrix} \tag{4.2.2.10}$$

Now,

$$\mathbf{n}^T\mathbf{P} = c \tag{4.2.2.11}$$

$$\implies \begin{pmatrix}3 & 2\end{pmatrix}\mathbf{P} = 12 \tag{4.2.2.12}$$

$\mathbf{P}$ can be

$$\begin{pmatrix}4\\0\end{pmatrix}, \begin{pmatrix}2\\3\end{pmatrix}, \begin{pmatrix}0\\6\end{pmatrix} \tag{4.2.2.13}$$

Let us take

$$\mathbf{P} = \begin{pmatrix}2\\3\end{pmatrix} = \mathbf{q} \tag{4.2.2.14}$$

The circle equation:

$$\mathbf{x}^T\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{4.2.2.15}$$

From (4.2.2.2),

$$\mathbf{u} = \begin{pmatrix}0\\0\end{pmatrix} \tag{4.2.2.16}$$

$$f = -13 \tag{4.2.2.17}$$

The points of intersection of the line

$$L : \mathbf{x} = \mathbf{q} + \mu\mathbf{m} \quad , \mu \in \mathbb{R} \tag{4.2.2.18}$$

with the conic section

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{4.2.2.19}$$

are given by

$$\mathbf{x}_i = \mathbf{q} + \mu_i\mathbf{m} \tag{4.2.2.20}$$

where,

$$\mu_i = \frac{1}{\mathbf{m}^T\mathbf{V}\mathbf{m}}\Big(-\mathbf{m}^T(\mathbf{V}\mathbf{q}+\mathbf{u}) \pm \sqrt{[\mathbf{m}^T(\mathbf{V}\mathbf{q}+\mathbf{u})]^2 - (\mathbf{q}^T\mathbf{V}\mathbf{q} + 2\mathbf{u}^T\mathbf{q} + f)(\mathbf{m}^T\mathbf{V}\mathbf{m})}\Big) \tag{4.2.2.21}$$

For circle,

$$\mathbf{V} = \mathbf{I} \tag{4.2.2.22}$$

$$\therefore \mu_i = \frac{1}{13}\left(-5 \pm \sqrt{25-(13-13)\,13}\right) \tag{4.2.2.23}$$

$$= \frac{1}{13}(-5 \pm 5) \tag{4.2.2.24}$$

$$= 0, -\frac{10}{13} \tag{4.2.2.25}$$

Using (4.2.2.20), the points of intersection are given by

$$\mathbf{x} = \begin{pmatrix}2\\3\end{pmatrix}, \begin{pmatrix}\frac{46}{13}\\\frac{9}{13}\end{pmatrix} \tag{4.2.2.26}$$

Points of contact are given by

$$\mathbf{q} = \mathbf{V}^{-1}(\kappa\mathbf{n} - \mathbf{u}) \tag{4.2.2.27}$$

$$\kappa = \pm\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\mathbf{n}^T\mathbf{V}^{-1}\mathbf{n}}} \tag{4.2.2.28}$$

Since for circle,

$$\mathbf{V} = \mathbf{I} \tag{4.2.2.29}$$
$$\therefore \mathbf{V}^{-1} = \mathbf{I} \quad \because \mathbf{I}^{-1} = \mathbf{I} \tag{4.2.2.30}$$
$$\therefore \kappa = \pm\sqrt{\frac{-f}{\mathbf{n}^T\mathbf{n}}} \quad \because \mathbf{u}^T\mathbf{u} = 0 \tag{4.2.2.31}$$
$$= \pm\sqrt{\frac{13}{\begin{pmatrix}3 & 2\end{pmatrix}\begin{pmatrix}3\\2\end{pmatrix}}} \tag{4.2.2.32}$$
$$= \pm\sqrt{\frac{13}{13}} \tag{4.2.2.33}$$
$$= \pm 1 \tag{4.2.2.34}$$
$$\therefore \mathbf{q} = \pm 1\begin{pmatrix}3\\2\end{pmatrix} \tag{4.2.2.35}$$
$$= \begin{pmatrix}3\\2\end{pmatrix}, \begin{pmatrix}-3\\-2\end{pmatrix} \tag{4.2.2.36}$$

From (4.2.2.3),

$$c = \begin{pmatrix}3 & 2\end{pmatrix}\begin{pmatrix}3\\2\end{pmatrix} = 13, \tag{4.2.2.37}$$
$$\begin{pmatrix}3 & 2\end{pmatrix}\begin{pmatrix}-3\\-2\end{pmatrix} = -13 \tag{4.2.2.38}$$

The line (4.2.2.3) touches the circle for $c = 13, -13$.

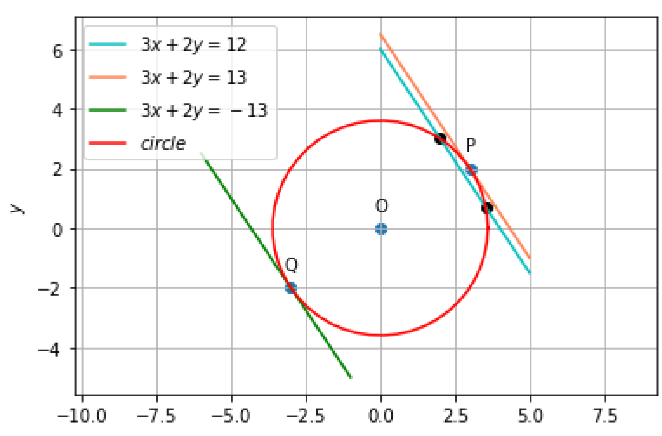

Fig. 4.2.2: Circle with tangent and intersection lines

3. Prove that the line

$$\begin{pmatrix}3 & 2\end{pmatrix}\mathbf{x} = 30 \tag{4.2.3.1}$$

touches the circle

$$\mathbf{x}^T\mathbf{x} - \begin{pmatrix}10 & 2\end{pmatrix}\mathbf{x} + 13 = 0 \tag{4.2.3.2}$$

and find the coordinates of the point of contact.

Solution:

The vector equation of a line can be expressed as

$$\mathbf{x} = \mathbf{q} + \mu\mathbf{m} \tag{4.2.3.3}$$

The general equation of a second degree can be expressed as :

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{4.2.3.4}$$

Comparing (4.2.3.2) with (4.2.3.4)

$$\mathbf{u} = \begin{pmatrix}-5\\-1\end{pmatrix}, f = 13 \tag{4.2.3.5}$$

If $\mathbf{n}$ is the normal vector of a line, equation of that line can be written as

$$\mathbf{n}^T\mathbf{x} = c \tag{4.2.3.6}$$

Comparing (4.2.3.1) with (4.2.3.6)

$$\mathbf{n} = \begin{pmatrix}3\\2\end{pmatrix} \tag{4.2.3.7}$$

The point of contact $\mathbf{q}$, of a line with a normal vector $\mathbf{n}$ to the conic in (4.2.3.4) is given by:

$$\mathbf{q} = \mathbf{V}^{-1}(\kappa\mathbf{n} - \mathbf{u}) \tag{4.2.3.8}$$
$$\kappa = \pm\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\mathbf{n}^T\mathbf{V}^{-1}\mathbf{n}}} \tag{4.2.3.9}$$

We know that, for a circle,

$$\mathbf{V} = \mathbf{I} \tag{4.2.3.10}$$

and from the properties of an Identity matrix,

$$\mathbf{I}^{-1} = \mathbf{I} \tag{4.2.3.11}$$
$$\mathbf{I}\mathbf{X} = \mathbf{X} \tag{4.2.3.12}$$

Solving for the point of contact using the above equations we get,

$$\kappa = \pm\sqrt{\frac{\begin{pmatrix}-5 & -1\end{pmatrix}\begin{pmatrix}-5\\-1\end{pmatrix} - 13}{\begin{pmatrix}3 & 2\end{pmatrix}\begin{pmatrix}3\\2\end{pmatrix}}} \tag{4.2.3.13}$$

$$= \pm\sqrt{\frac{26-13}{13}} \tag{4.2.3.14}$$

$$= \pm\sqrt{1} \tag{4.2.3.15}$$

$$q = \begin{pmatrix}3\\2\end{pmatrix} - \begin{pmatrix}-5\\-1\end{pmatrix} \tag{4.2.3.16}$$

$$= \begin{pmatrix}8\\3\end{pmatrix} \tag{4.2.3.17}$$

If the line in (4.2.3.3) touches (4.2.3.4) at exactly one point $\mathbf{q}$, then

$$\mathbf{m}^T(\mathbf{Vq} + \mathbf{u}) = 0 \tag{4.2.3.18}$$

It can seen that for the given line

$$\mathbf{m} = \begin{pmatrix}1\\-1.5\end{pmatrix} \tag{4.2.3.19}$$

Solving (4.2.3.18) for given line and circle, we get

$$= \begin{pmatrix}1 & -1.5\end{pmatrix}\left(\begin{pmatrix}8\\3\end{pmatrix} + \begin{pmatrix}-5\\-1\end{pmatrix}\right) \tag{4.2.3.20}$$

$$= \begin{pmatrix}1 & -1.5\end{pmatrix}\begin{pmatrix}3\\2\end{pmatrix} \tag{4.2.3.21}$$

$$= 0 \tag{4.2.3.22}$$

Hence, it is proved that the given line touches the given circle at only one point and so it is a tangent.

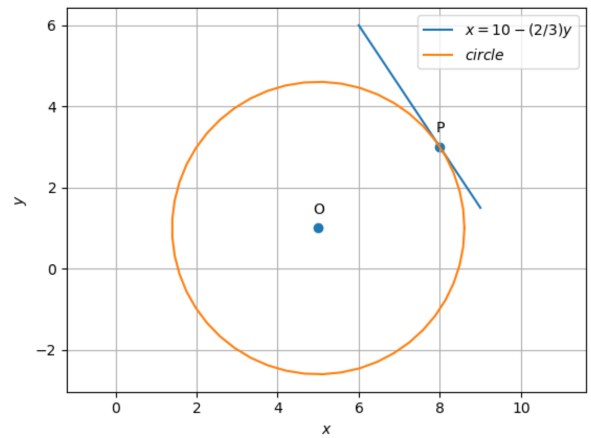

Fig. 4.2.3: Circle with center (5 1) and having the line (3 2)x =30 as tangent with (8 3) as point of contact.

4. For what values of m does the line

$$\begin{pmatrix}m & -1\end{pmatrix}\mathbf{x} = 0 \tag{4.2.4.1}$$

touch the circle

$$\mathbf{x}^T\mathbf{x} - \begin{pmatrix}6 & 2\end{pmatrix}\mathbf{x} + 8 = 0 \tag{4.2.4.2}$$

Solution: The general equation of a circle is

$$\implies \mathbf{x}^T\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{4.2.4.3}$$

$$\text{If } r \text{ is radius}, f = \mathbf{u}^T\mathbf{u} - r^2 \tag{4.2.4.4}$$

$$\text{center } \mathbf{c} = -\mathbf{u} \tag{4.2.4.5}$$

From equations 4.2.4.2 and 4.2.4.3,

$$\mathbf{u} = \begin{pmatrix}-3\\-1\end{pmatrix} \tag{4.2.4.6}$$

$$\implies \text{Center of the cirlce } \mathbf{c} = \begin{pmatrix}3\\1\end{pmatrix} \tag{4.2.4.7}$$

Also, radius can be determined as follows from 4.2.4.4

$$\implies 8 = \begin{pmatrix}-3 & -1\end{pmatrix}\begin{pmatrix}-3\\-1\end{pmatrix} - r^2 \tag{4.2.4.8}$$

$$\implies 8 = 10 - r^2 \tag{4.2.4.9}$$

$$\implies r = \sqrt{2} \tag{4.2.4.10}$$

Given equation of the line is

$$\begin{pmatrix}m & -1\end{pmatrix}\mathbf{x} = 0 \tag{4.2.4.11}$$

It can be expressed as:-

$$L: \quad \mathbf{x} = \mathbf{q} + \mu\mathbf{m} \quad \mu \in \mathbb{R} \tag{4.2.4.12}$$

(4.2.4.13)

The normal vector to the line is obtained as

$$\mathbf{n} = \mathbf{q} + \mathbf{u} \tag{4.2.4.14}$$

$$\implies \mathbf{q} = \mathbf{n} - \mathbf{u} \tag{4.2.4.15}$$

$$\mathbf{n} = \begin{pmatrix}m & -1\end{pmatrix}^T \text{ and } \mathbf{u} = \begin{pmatrix}-3 & -1\end{pmatrix}^T \tag{4.2.4.16}$$

$$\mathbf{q} = \begin{pmatrix}m+3\\0\end{pmatrix} \tag{4.2.4.17}$$

The point $\mathbf{q}$ also satisfies the equation of the

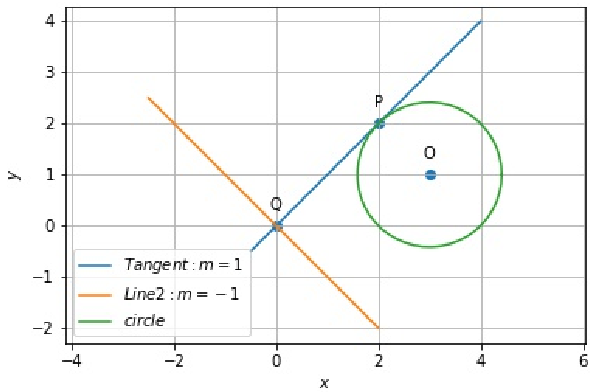

Fig. 4.2.4: Circle with tangent

circle at 4.2.4.2.

$$\mathbf{q}^T\mathbf{q} + 2\mathbf{u}^T\mathbf{q} + f = 0 \quad (4.2.4.18)$$
$$(4.2.4.19)$$

$$(\mathbf{n}-\mathbf{u})^T(\mathbf{n}-\mathbf{u}) + 2\mathbf{u}^T(\mathbf{n}-\mathbf{u}) + f = 0 \quad (4.2.4.20)$$

$$\|\mathbf{n}\|^2 - \mathbf{n}^T\mathbf{u} - \mathbf{u}^T\mathbf{n} + \|\mathbf{u}\|^2 + 2\mathbf{u}^T\mathbf{n} - 2\|\mathbf{u}\|^2 + f = 0 \quad (4.2.4.21)$$

$$\|\mathbf{n}\|^2 - \mathbf{n}^T\mathbf{u} + \mathbf{n}^T\mathbf{u} - \|\mathbf{u}\|^2 + f = 0 \quad (4.2.4.22)$$

$$\|\mathbf{n}\|^2 - \|\mathbf{u}\|^2 + f = 0 \quad (4.2.4.23)$$
$$(m^2+1) - 10 + 8 = 0 \quad (4.2.4.24)$$
$$m^2 - 1 = 0 \quad (4.2.4.25)$$
$$m = \pm 1 \quad (4.2.4.26)$$

For the line 4.2.4.1 to be a tangent to circle at equation 4.2.4.2, the values of m are ± 1

5. Prove that the circle

$$\mathbf{x}^T\mathbf{x} - 2a\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} + a^2 = 0 \quad (4.2.5)$$

touches the coordinate axes. **Solution:** Let r be the radius of the circle. Now for proving that the circle touches the co-ordinate axes we have to prove that it touches x axis and y axes at points such that:

$$\mathbf{Point_1} = \begin{pmatrix}r\\0\end{pmatrix} \quad (4.2.5)$$

$$\mathbf{Point_2} = \begin{pmatrix}0\\r\end{pmatrix} \quad (4.2.5)$$

The general equation of a circle is given by

$$\mathbf{x}^T\mathbf{x} - 2\mathbf{O}^T\mathbf{x} + \mathbf{f} = 0 \quad (4.2.5)$$

Where $\mathbf{O}$ is the centre and $\mathbf{r}$ is the radius of the circle.
Substituting (4.2.5) in (4.2.5), we rewrite (4.2.5)as:

$$\begin{pmatrix}r & 0\end{pmatrix}\begin{pmatrix}r\\0\end{pmatrix} - 2\begin{pmatrix}a & a\end{pmatrix}\begin{pmatrix}r\\0\end{pmatrix} + \mathbf{a}^2 = 0 \quad (4.2.5)$$

$$\implies \mathbf{r}^2 - 2(\mathbf{ar}) + \mathbf{a}^2 = 0 \quad (4.2.5)$$

$$\implies \mathbf{r}^2 - 2(\mathbf{ar}) + \mathbf{a}^2 = 0 \quad (4.2.5)$$

$$\implies (\mathbf{r}-\mathbf{a})^2 = 0 \quad (4.2.5)$$

$$\implies \mathbf{r} = \mathbf{a} \quad (4.2.5)$$

Similarly, substituting (4.2.5) in (4.2.5), we rewrite (4.2.5)as:

$$\begin{pmatrix}0 & r\end{pmatrix}\begin{pmatrix}0\\r\end{pmatrix} - 2\begin{pmatrix}a & a\end{pmatrix}\begin{pmatrix}0\\r\end{pmatrix} + \mathbf{a}^2 = 0 \quad (4.2.5)$$

$$\implies \mathbf{r}^2 - 2(\mathbf{ar}) + \mathbf{a}^2 = 0 \quad (4.2.5)$$

$$\implies \mathbf{r}^2 - 2(\mathbf{ar}) + \mathbf{a}^2 = 0 \quad (4.2.5)$$

$$\implies (\mathbf{r}-\mathbf{a})^2 = 0 \quad (4.2.5)$$

$$\implies \mathbf{r} = \mathbf{a} \quad (4.2.5)$$

Therefore, the circle touches x axis at $\mathbf{Point_1}$ i.e $\begin{pmatrix}a\\0\end{pmatrix}$ and y axis at $\mathbf{Point_2}$ i.e $\begin{pmatrix}0\\a\end{pmatrix}$
Hence,it is proved that the circle touches the co-ordinate axes

6. Show that two circles can be drawn to pass through the point $\begin{pmatrix}1\\2\end{pmatrix}$ and touch the coordinate axes, and find their equations. **Solution:** Let us consider we have a circle which passes through the point $\begin{pmatrix}1\\2\end{pmatrix}$ and touches x - axis at point $\begin{pmatrix}r\\0\end{pmatrix}$ and y - axis at $\begin{pmatrix}0\\r\end{pmatrix}$. Radius of the circle is $\mathbf{r}$ since it touches both axes. Hence we have 3

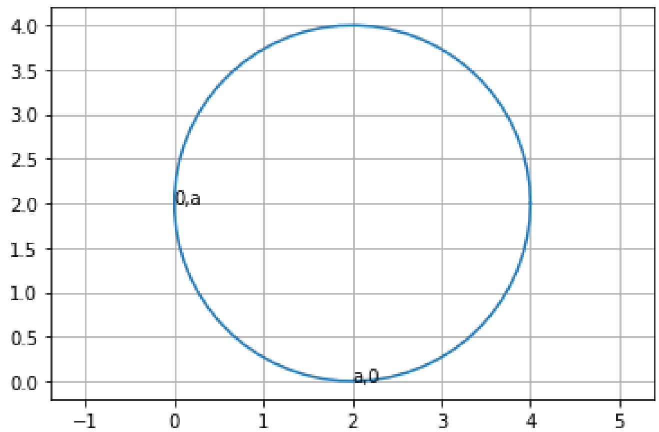

Fig. 4.2.5: Circle touching the co-ordinate axes a=2

points which are :

$$\mathbf{P_1} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{4.2.6}$$

$$\mathbf{p_2} = \begin{pmatrix} r \\ 0 \end{pmatrix} \tag{4.2.6}$$

$$\mathbf{P_3} = \begin{pmatrix} 0 \\ r \end{pmatrix} \tag{4.2.6}$$

The general equation of circle is :

$$\|\mathbf{x} - \mathbf{O}\| = r \tag{4.2.6}$$

Substituting the given coordinates:

$$\|\mathbf{P_2} - \mathbf{O}\|^2 = r^2 \tag{4.2.6}$$

$$\|\mathbf{P_3} - \mathbf{O}\|^2 = r^2 \tag{4.2.6}$$

$$\|\mathbf{P_1} - \mathbf{O}\|^2 = r^2 \tag{4.2.6}$$

From equation 4.2.6, 4.2.6 and 4.2.6 we have

$$\|\mathbf{P_2} - \mathbf{O}\|^2 - \|\mathbf{P_1} - \mathbf{O}\|^2 = 0 \tag{4.2.6}$$

And,

$$\|\mathbf{P_3} - \mathbf{O}\|^2 - \|\mathbf{P_1} - \mathbf{O}\|^2 = 0 \tag{4.2.6}$$

Simplifying 4.2.6 and 4.2.6,

$$\left(\mathbf{P_2} - \mathbf{O}\right)^T \left(\mathbf{P_2} - \mathbf{O}\right) - \left(\mathbf{P_1} - \mathbf{O}\right)^T \left(\mathbf{P_1} - \mathbf{O}\right) = \mathbf{0} \tag{4.2.6}$$

$$\implies \|\mathbf{P_2}\|^2 - 2\mathbf{P_2}^T\mathbf{O} - \|\mathbf{P_1}\|^2 + 2\mathbf{P_1}^T\mathbf{O} = \mathbf{0} \tag{4.2.6}$$

Substituting the value of $\|\mathbf{P_1}\|$ and $\|\mathbf{P_2}\|$ and other values then rearranging it, we get :

$$\begin{pmatrix} 2-2r & 4 \end{pmatrix} \begin{pmatrix} O \end{pmatrix} = 5 - r^2 \tag{4.2.6}$$

Similarly,

$$\left(\mathbf{P_3} - \mathbf{O}\right)^T \left(\mathbf{P_3} - \mathbf{O}\right) - \left(\mathbf{P_1} - \mathbf{O}\right)^T \left(\mathbf{P_1} - \mathbf{O}\right) = \mathbf{0} \tag{4.2.6}$$

$$\implies \|\mathbf{P_3}\|^2 - 2\mathbf{P_3}^T\mathbf{O} - \|\mathbf{P_1}\|^2 + 2\mathbf{P_1}^T\mathbf{O} = \mathbf{0} \tag{4.2.6}$$

Substituting the value of $\|\mathbf{P_1}\|$ and $\|\mathbf{P_3}\|$ and other values then rearranging it, we get :

$$\begin{pmatrix} 2 & 4-2r \end{pmatrix} \begin{pmatrix} O \end{pmatrix} = 5 - r^2 \tag{4.2.6}$$

Combining 4.2.6 and 4.2.6

$$\begin{pmatrix} 2-2r & 4 \\ 2 & 4-2r \end{pmatrix} \begin{pmatrix} O \end{pmatrix} = \begin{pmatrix} 5-r^2 \\ 5-r^2 \end{pmatrix} \tag{4.2.6}$$

Transforming the matrix into row-echelon form

$$\begin{pmatrix} 2-2r & 4 & 5-r^2 \\ 2 & 4-2r & 5-r^2 \end{pmatrix} \tag{4.2.6}$$

$$\begin{pmatrix} 2-2r & 4 & 5-r^2 \\ 2 & 4-2r & 5-r^2 \end{pmatrix} \xleftrightarrow{R1 \leftarrow \frac{R1}{2-2r}} \begin{pmatrix} 1 & \frac{-2}{r-1} & \frac{r^2-}{2(r-1)} \\ 2 & 4-2r & 5-r^2 \end{pmatrix} \xleftrightarrow{R2 \leftarrow R2 - 2R1} \begin{pmatrix} 1 & \frac{-2}{r-1} & \frac{r^2-5}{2(r-1)} \\ 0 & \frac{2r(r-3)}{r-1} & \frac{r\left(r^2-5\right)}{r-1} \end{pmatrix} \xleftrightarrow{R2 \leftarrow \left(\frac{1-r}{2r(r-3)}\right)R2} \begin{pmatrix} 1 & \frac{-2}{r-1} & \frac{r^2-5}{2(r-1)} \\ 0 & 1 & \frac{r^2-5}{2(r-3)} \end{pmatrix} \xleftrightarrow{R1 \leftarrow R1 - \left(\frac{-2}{r-1}\right)R2} \begin{pmatrix} 1 & 0 & \frac{r^2-5}{2(r-3)} \\ 0 & 1 & \frac{r^2-5}{2(r-3)} \end{pmatrix} \tag{4.2.6}$$

So,

$$\mathbf{O} = \begin{pmatrix} \frac{r^2-5}{2(r-3)} \\ \frac{r^2-5}{2(r-3)} \end{pmatrix} \quad (4.2.6)$$

Now subsituting the 4.2.6 in 4.2.6, we have

$$\|\mathbf{P_3} - \mathbf{O}\|^2 = r^2 \quad (4.2.6)$$

Subsituting the value of $\mathbf{O}$ in 4.2.6 and simplify,

$$\left(\mathbf{P_3} - \mathbf{O}\right)^T \left(\mathbf{P_3} - \mathbf{O}\right) = r^2 \quad (4.2.6)$$

$$\implies \|\mathbf{P_3}\|^2 - \mathbf{P_3}^T\mathbf{O} - \mathbf{P_3}\mathbf{O}^T + \|\mathbf{O}\|^2 = r^2 \quad (4.2.6)$$

Putting the values of $\mathbf{O}$ from 4.2.6 and $\|\mathbf{P_3}\|^2$

$$\implies -\mathbf{P_3}^T\mathbf{O} - \mathbf{P_3}\mathbf{O}^T = -\|\mathbf{O}\|^2 \quad (4.2.6)$$

$$\implies -\begin{pmatrix} 0 \\ r \end{pmatrix}^T \begin{pmatrix} \frac{r^2-5}{2(r-3)} \\ \frac{r^2-5}{2(r-3)} \end{pmatrix} - \begin{pmatrix} 0 \\ r \end{pmatrix} \begin{pmatrix} \frac{r^2-5}{2(r-3)} \\ \frac{r^2-5}{2(r-3)} \end{pmatrix}^T = -\|\mathbf{O}\|^2 \quad (4.2.6)$$

Subsituting the value of $\|\mathbf{O}\|^2$ and simplify it,

$$\implies 2r\left(\frac{r^2-5}{2(r-3)}\right) = 2\left(\frac{r^2-5}{2(r-3)}\right)^2 \quad (4.2.6)$$

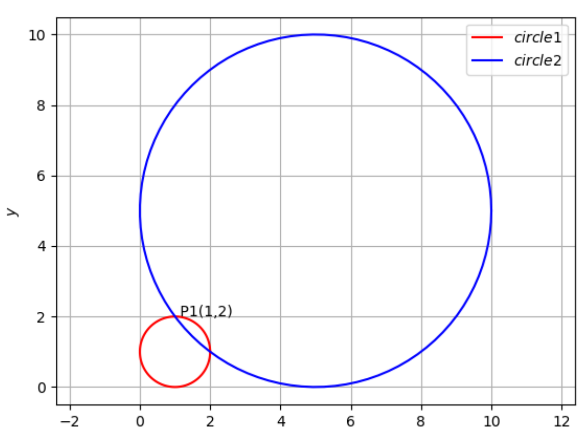

Fig. 4.2.6: Two circles passing through a common point

$$\implies r = \frac{r^2-5}{2(r-3)} \quad (4.2.6)$$

$$\implies r^2 - 6r + 5 = 0 \quad (4.2.6)$$

$$\implies (r-1)(r-5) = 0 \quad (4.2.6)$$

$$\implies r = 1, r = 5. \quad (4.2.6)$$

Hence,

$$\mathbf{O_1} = \begin{pmatrix} 5 \\ 5 \end{pmatrix} \text{and,} \mathbf{O_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (4.2.6)$$

Hence equation of circles are :

$$\left\|\mathbf{x} - \begin{pmatrix} 5 \\ 5 \end{pmatrix}\right\| = 5 \quad (4.2.6)$$

And,

$$\left\|\mathbf{x} - \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\| = 1 \quad (4.2.6)$$

7. Find the length of the tangent from the point $\begin{pmatrix} 7 \\ 4 \end{pmatrix}$ to the circle

$$\mathbf{x}^T\mathbf{x} - \begin{pmatrix} 4 & 6 \end{pmatrix}\mathbf{x} + 12 = 0 \quad (4.2.7)$$

Solution:
The general equation of a second degree can be expressed as :

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \quad (4.2.7)$$

Let the equation of the tangent be

$$\begin{pmatrix} -m & 1 \end{pmatrix}\mathbf{x} = c \quad (4.2.7)$$

We know that, for a circle,

$$\mathbf{V} = \mathbf{I} \quad (4.2.7)$$
$$\mathbf{c} = -\mathbf{u} \quad (4.2.7)$$

Comparing the equation (4.2.7) and (4.2.7) we get

$$\mathbf{u} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}, f = 12 \tag{4.2.7}$$

$$\mathbf{c} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \tag{4.2.7}$$

The normal vector to the line is obtained as

$$\mathbf{n} = \mathbf{q} + \mathbf{u} \tag{4.2.7}$$

$$\mathbf{q} = \mathbf{n} - \mathbf{u} \tag{4.2.7}$$

Comparing the equation (4.2.7)

$$\mathbf{n} = \begin{pmatrix} -m & 1 \end{pmatrix}^T \tag{4.2.7}$$

Given

$$\mathbf{u} = \begin{pmatrix} -2 \\ -3 \end{pmatrix} \tag{4.2.7}$$

$$\implies \mathbf{q} = \begin{pmatrix} -m+2 \\ 4 \end{pmatrix} \tag{4.2.7}$$

The point q also satisfies the equation of the circle at (4.2.7)

$$\mathbf{q}^T\mathbf{q} + 2\mathbf{u}^T\mathbf{q} + f = 0 \tag{4.2.7}$$

$$(\mathbf{n}-\mathbf{u})^T(\mathbf{n}-\mathbf{u}) + 2\mathbf{u}^T(\mathbf{n}-\mathbf{u}) + f = 0 \tag{4.2.7}$$

$$\|\mathbf{n}\|^2 - \mathbf{n}^T\mathbf{u} - \mathbf{u}^T\mathbf{n} + \|\mathbf{u}\|^2 + 2\mathbf{u}^T\mathbf{n} - 2\|\mathbf{u}\|^2 + f = 0 \tag{4.2.7}$$

$$\|\mathbf{n}\|^2 - \|\mathbf{u}\|^2 + f = 0 \tag{4.2.7}$$

$$m^2 + 1 - 13 + 12 = 0 \tag{4.2.7}$$

$$m^2 = 0 \tag{4.2.7}$$

$$m = 0 \tag{4.2.7}$$

Simplying (4.2.7) and (4.2.7) we get

$$\mathbf{q} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} \tag{4.2.7}$$

Let $\mathbf{p} = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$ The length of tangent is

$$\|\mathbf{p}-\mathbf{q}\| = \sqrt{(7-2)^2 + (4-4)^2} \tag{4.2.7}$$

$$= \sqrt{25} \tag{4.2.7}$$

$$= 5 \tag{4.2.7}$$

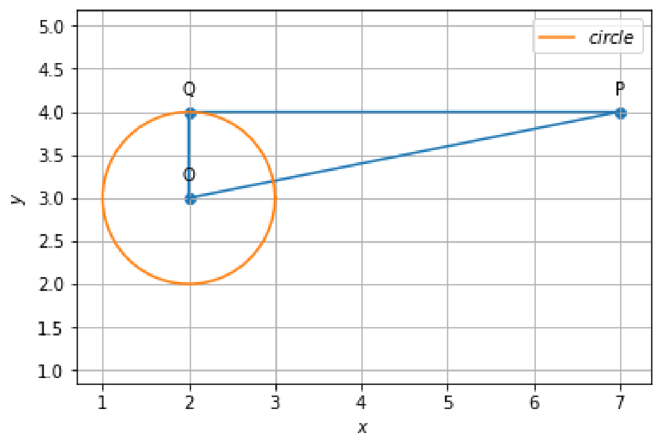

Fig. 4.2.7: Perpendicular Line

8. Find the equations of tangents to the circle

$$\mathbf{x}^T\mathbf{x} - \begin{pmatrix} 4 & 3 \end{pmatrix}\mathbf{x} + 5 = 0 \tag{4.2.8.1}$$

that are parallel to the line

$$\begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} = 0 \tag{4.2.8.2}$$

Solution: The vector equation of a line can be expressed as

$$\mathbf{x} = \mathbf{q} + \mu\mathbf{m} \tag{4.2.8.3}$$

The general equation of a second degree can be expressed as :

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{4.2.8.4}$$

Comparing (4.2.8.1) with (4.2.8.4)

$$\mathbf{u} = \begin{pmatrix} -2 \\ \frac{-3}{2} \end{pmatrix}, f = 5 \tag{4.2.8.5}$$

If $\mathbf{n}$ is the normal vector of a line, equation of that line can be written as

$$\mathbf{n}^T\mathbf{x} = c \tag{4.2.8.6}$$

Comparing (4.2.8.2) with (4.2.8.6)

$$\mathbf{n} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{4.2.8.7}$$

Since it is mentioned that the tangent is parallel to given line it will have same normal vector. The point of contact $\mathbf{q}$, of a line with a normal vector $\mathbf{n}$ to the conic in (4.2.8.4) is given by:

$$\mathbf{q} = \mathbf{V}^{-1}(\kappa\mathbf{n} - \mathbf{u}) \tag{4.2.8.8}$$

$$\kappa = \pm\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\mathbf{n}^T\mathbf{V}^{-1}\mathbf{n}}} \tag{4.2.8.9}$$

The point of contact $\mathbf{q}$, of a line with a normal vector $\mathbf{n}$ to the conic in (4.2.8.4) is given by:

$$\mathbf{q} = \mathbf{V}^{-1}(\kappa\mathbf{n} - \mathbf{u}) \tag{4.2.8.10}$$

$$\kappa = \pm\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\mathbf{n}^T\mathbf{V}^{-1}\mathbf{n}}} \tag{4.2.8.11}$$

We know that, for a circle,

$$\mathbf{V} = \mathbf{I} \tag{4.2.8.12}$$

and from the properties of an Identity matrix,

$$\mathbf{I}^{-1} = \mathbf{I} \tag{4.2.8.13}$$

$$\mathbf{IX} = \mathbf{X} \tag{4.2.8.14}$$

Solving for the point of contact using the above equations we get,

$$\kappa = \pm\sqrt{\frac{\begin{pmatrix}-2 & -1.5\end{pmatrix}\begin{pmatrix}-2\\-1.5\end{pmatrix} - 5}{\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}}} \tag{4.2.8.15}$$

$$= \pm\sqrt{\frac{6.25 - 5}{2}} \tag{4.2.8.16}$$

$$= \pm\sqrt{\frac{5}{8}} \tag{4.2.8.17}$$

$$\tag{4.2.8.18}$$

So there are two tangents to a circle which are parallel to given line which touch circle at two different points $\mathbf{q_1}$ and $\mathbf{q_2}$

$$\mathbf{q_1} = \begin{pmatrix}\sqrt{\frac{5}{8}}\\\sqrt{\frac{5}{8}}\end{pmatrix} + \begin{pmatrix}2\\\frac{3}{2}\end{pmatrix} \tag{4.2.8.19}$$

$$= \begin{pmatrix}\frac{279}{100}\\\frac{229}{100}\end{pmatrix} \tag{4.2.8.20}$$

$$\mathbf{q_2} = \begin{pmatrix}-\sqrt{\frac{5}{8}}\\\sqrt{\frac{5}{8}}\end{pmatrix} + \begin{pmatrix}2\\\frac{3}{2}\end{pmatrix} \tag{4.2.8.21}$$

$$= \begin{pmatrix}\frac{121}{100}\\\frac{71}{100}\end{pmatrix} \tag{4.2.8.22}$$

Since points $\mathbf{q_1}$ and $\mathbf{q_2}$ lie on tangent they satisfy the line equation of tangents, there are two different tangents with same normal vector

$$\mathbf{n}^T\mathbf{q_1} = c_1 \tag{4.2.8.23}$$

$$\mathbf{n}^T\mathbf{q_2} = c_2 \tag{4.2.8.24}$$

c_1 and c_2 are some constants

$$c_1 = \begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}\frac{279}{100}\\\frac{229}{100}\end{pmatrix} = \frac{127}{25} \tag{4.2.8.25}$$

$$c_2 = \begin{pmatrix}\frac{121}{100}\\\frac{71}{100}\end{pmatrix} = \frac{48}{25} \tag{4.2.8.26}$$

So line equations of tangents to the given circle which are parallel to line are

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = \frac{127}{25} \tag{4.2.8.27}$$

and

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = \frac{48}{25} \tag{4.2.8.28}$$

9. Find the equations of the tangents to the circle

$$\mathbf{x}^T\mathbf{x} - \begin{pmatrix}7 & 5\end{pmatrix}\mathbf{x} + 18 = 0 \tag{4.2.9}$$

at the points $\begin{pmatrix}4\\3\end{pmatrix}$ and $\begin{pmatrix}3\\2\end{pmatrix}$, showing that they are parallel.

Solution:

The general equation of a second degree can be expressed as:

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{4.2.9}$$

Comparing 1.0.1 with 2.0.1

$$\mathbf{V} = \mathbf{I}, \mathbf{u} = \begin{pmatrix}\frac{-7}{2}\\\frac{-5}{2}\end{pmatrix}, f = 18 \tag{4.2.9}$$

The vector equation of a line can be expressed

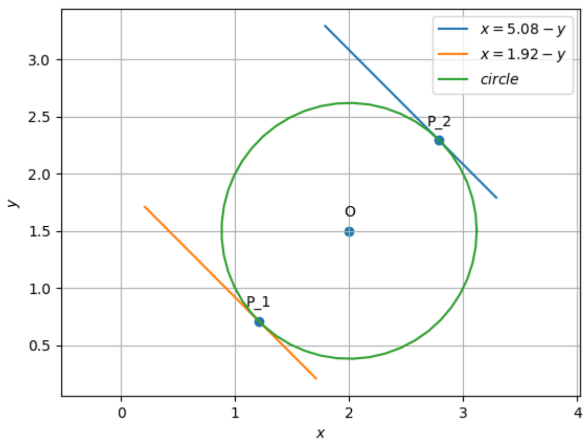

Fig. 4.2.8: Circle with center (2 1.5) and having the lines (1 1)x =5.08 and (1 1)x =1.92 as tangents with (2.79 2.29) and (1.21 0.71) as point of contact.

as

$$\mathbf{x} = \mathbf{q} + \mu \mathbf{m} \tag{4.2.9}$$

Tangent-1 Given point

$$\mathbf{q_1} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \tag{4.2.9}$$

The direction vector of the line joining the point $\mathbf{q_1}$ and the centre $\mathbf{c}$ expressed as:

$$\mathbf{n_1} = \mathbf{q_1} - \mathbf{c} \tag{4.2.9}$$

$$\implies \mathbf{n_1} = \mathbf{q_1} + \mathbf{u} \tag{4.2.9}$$

where,

$$\mathbf{c} = -\mathbf{u} \tag{4.2.9}$$

The vector $\mathbf{n_1}$ is normal to the tangent drawn at $\mathbf{q_1}$. From (2.0.2) and (2.1.1) we get,

$$\mathbf{n_1}^T = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix} \tag{4.2.9}$$

We know,

$$\mathbf{m}^T \mathbf{n_1} = 0 \tag{4.2.9}$$

$$\implies \mathbf{m}^T \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = 0 \tag{4.2.9}$$

$$\implies \mathbf{m}^T = \begin{pmatrix} \frac{-1}{2} & \frac{1}{2} \end{pmatrix} \tag{4.2.9}$$

If $\mathbf{q_1}$ be a point on the line and $\mathbf{n_1}$ is the normal vector then the equation of the line can be expressed From(2.0.3) is :

$$\mathbf{n_1}^T (\mathbf{x} - \mathbf{q_1}) = 0 \tag{4.2.9}$$

$$\implies \mathbf{n_1}^T \mathbf{x} = c \tag{4.2.9}$$

where

$$c = \mathbf{n_1}^T \mathbf{q_1} \tag{4.2.9}$$

Using the equations (2.1.1) and (2.1.5),

$$\implies c = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \frac{7}{2} \tag{4.2.9}$$

From (2.1.10), Line equation of Tangent-1 is:

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix} \mathbf{x} = \frac{7}{2} \tag{4.2.9}$$

$$\implies \boxed{\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 7} \tag{4.2.9}$$

Tangent-2 Now,

$$\mathbf{q_2} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \tag{4.2.9}$$

The direction vector of the line joining the point $\mathbf{q_2}$ and the centre $\mathbf{c}$ expressed as:

$$\mathbf{n_2} = \mathbf{q_2} - \mathbf{c} \tag{4.2.9}$$

$$\implies \mathbf{n_2} = \mathbf{q_2} + \mathbf{u} \tag{4.2.9}$$

where,

$$\mathbf{c} = -\mathbf{u} \tag{4.2.9}$$

The vector $\mathbf{n_2}$ is normal to the tangent drawn at $\mathbf{q_2}$. From (2.0.2) and (2.2.1),

$$\mathbf{n_2}^T = \begin{pmatrix} \frac{-1}{2} & \frac{-1}{2} \end{pmatrix} \tag{4.2.9}$$

We know,

$$\mathbf{m}^T \mathbf{n_2} = 0 \tag{4.2.9}$$

$$\implies \mathbf{m}^T \begin{pmatrix} \frac{-1}{2} \\ \frac{-1}{2} \end{pmatrix} = 0 \tag{4.2.9}$$

$$\implies \mathbf{m}^T = \begin{pmatrix} \frac{1}{2} & \frac{-1}{2} \end{pmatrix} \tag{4.2.9}$$

If $\mathbf{q_2}$ be a point on the line and $\mathbf{n_2}$ is the normal vector, the equation of the line can be expressed From (2.0.3) is:

$$\mathbf{n_2}^T(\mathbf{x}-\mathbf{q_2}) = 0 \tag{4.2.9}$$

$$\implies \mathbf{n_2}^T\mathbf{x} = c \tag{4.2.9}$$

where

$$c = \mathbf{n_2}^T\mathbf{q_2} \tag{4.2.9}$$

Using the equations 2.2.1 and 2.2.5,

$$\implies c = \begin{pmatrix}\frac{-1}{2} & \frac{-1}{2}\end{pmatrix}\begin{pmatrix}3\\2\end{pmatrix} = \frac{-5}{2} \tag{4.2.9}$$

From (2.2.10), Line equation of Tangent-2 is:

$$\begin{pmatrix}\frac{-1}{2} & \frac{-1}{2}\end{pmatrix}\mathbf{x} = \frac{-5}{2} \tag{4.2.9}$$

$$\implies \boxed{\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 5} \tag{4.2.9}$$

Result From the equations (2.1.14) and (2.2.14), normal vectors of Tangent-1 and Tangent-2 are equal.
Hence, the two tangents are parallel.

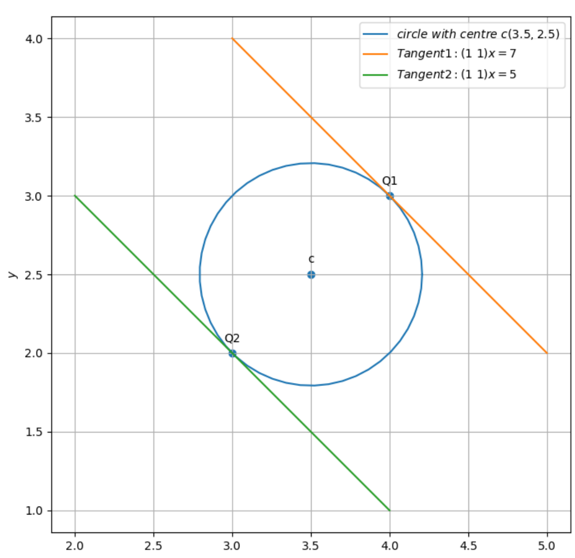

Fig. 4.2.9: Tangents to the circle at given points

10. Find the equations of the tangent and normal to the circle

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}-6 & 4\end{pmatrix}\mathbf{x} - 12 = 0 \tag{4.2.10}$$

at the point $\begin{pmatrix}6\\2\end{pmatrix}$.

11. Prove that the line

$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 1 \tag{4.2.11.1}$$

touches the circle

$$\mathbf{x^T}\mathbf{x} - \begin{pmatrix}8 & 6\end{pmatrix}\mathbf{x} + 7 = 0 \tag{4.2.11.2}$$

and find the equations of the parallel and perpendicular tangents.

Solution: The general equation of a second degree can be expressed as,

$$\mathbf{x^T}\mathbf{V}\mathbf{x} + 2\mathbf{u^T}\mathbf{x} + f = 0 \tag{4.2.11.3}$$

Comparing (4.2.11.2) and (4.2.11.3) we get,

$$\mathbf{u} = \begin{pmatrix}-4\\-3\end{pmatrix} \tag{4.2.11.4}$$

$$f = 7 \tag{4.2.11.5}$$

If $\mathbf{n}$ is the normal vector, $\mathbf{P}$ is a point on that line then equation of the line can be written as,

$$\mathbf{n^T}(\mathbf{x}-\mathbf{P}) = 0 \tag{4.2.11.6}$$

$$\implies \mathbf{n^T}\mathbf{x} = c \tag{4.2.11.7}$$

where c = $\mathbf{n^T}\mathbf{P}$. Comparing (4.2.11.1) and (4.2.11.7) we get,

$$\mathbf{n} = \begin{pmatrix}1\\1\end{pmatrix} and\ c = 1 \tag{4.2.11.8}$$

The point of contact q, of a line with a normal vector $\mathbf{n}$ to the conic in (4.2.11.3) is given by,

$$\mathbf{q} = \mathbf{V}^{-1}(\kappa\mathbf{n} - \mathbf{u}) \tag{4.2.11.9}$$

$$\kappa = \pm\sqrt{\frac{\mathbf{u^T}\mathbf{V}^{-1}\mathbf{u} - f}{\mathbf{n^T}\mathbf{V}^{-1}\mathbf{n}}} \tag{4.2.11.10}$$

For a circle,

$$\mathbf{V} = \mathbf{I} \tag{4.2.11.11}$$

where I is the Identity matrix.. Solving for κ using (4.2.11.10) we get,

$$\kappa = \pm 3 \tag{4.2.11.12}$$

$$i.e.\ \mathbf{q_1} = \begin{pmatrix}1\\0\end{pmatrix}\ for\ \kappa = -3 \tag{4.2.11.13}$$

and

$$\mathbf{q_2} = \begin{pmatrix} 7 \\ 6 \end{pmatrix} \ for\ \kappa = 3 \tag{4.2.11.14}$$

To prove that the line touches the circle at $\mathbf{q}$ need to check that

$$\mathbf{m}^{\mathbf{T}} \left(\mathbf{V}\mathbf{q} + \mathbf{u} \right) = 0 \tag{4.2.11.15}$$

We know that,

$$\mathbf{m}^{\mathbf{T}}\mathbf{n} = 0 \tag{4.2.11.16}$$

$$\implies m = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{4.2.11.17}$$

Using (4.2.11.13), (4.2.11.14) and (4.2.11.17), the expression in (4.2.11.15) holds true for both $\mathbf{q_1}$ and $\mathbf{q_2}$ which means that both those points lie on the circle i.e. there will be a tangent passing through each of them which can be found out using (4.2.11.7)

$$i.e.\ \mathbf{n}^{\mathbf{T}}\mathbf{q_1} = c_1 \tag{4.2.11.18}$$

$$\mathbf{n}^{\mathbf{T}}\mathbf{q_2} = c_2 \tag{4.2.11.19}$$

where,

$$c_1 = \mathbf{n}^{\mathbf{T}}\mathbf{q_1} = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \tag{4.2.11.20}$$

which was already obtained in (4.2.11.8) and

$$c_2 = \mathbf{n}^{\mathbf{T}}\mathbf{q_2} = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 7 \\ 6 \end{pmatrix} = 13 \tag{4.2.11.21}$$

Using (4.2.11.18) the given line in the question is obtained which is (4.2.11.1). Therefore, the tangent parallel to (4.2.11.1) is,

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 13 \tag{4.2.11.22}$$

And the line(s) perpendicular to (4.2.11.1) can be found out using (4.2.11.8) and here the normal vector for this line will be $\mathbf{m}$ which was calculated using (4.2.11.17) and its equation(s) will be,

$$\mathbf{m}^{\mathbf{T}}\mathbf{x} = c_3 \tag{4.2.11.23}$$

$$\mathbf{m}^{\mathbf{T}}\mathbf{x} = c_4 \tag{4.2.11.24}$$

where,

$$c_3 = \mathbf{m}^{\mathbf{T}}\mathbf{q_1} = \begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -1 \tag{4.2.11.25}$$

$$c_4 = \mathbf{m}^{\mathbf{T}}\mathbf{q_2} = \begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} 7 \\ 6 \end{pmatrix} = -1 \tag{4.2.11.26}$$

Therefore, the line perpendicular to (4.2.11.1) and also to (4.2.11.22) is,

$$\begin{pmatrix} -1 & 1 \end{pmatrix} \mathbf{x} = -1 \tag{4.2.11.27}$$

In Fig. 4.2.11. $\mathbf{C}$ is the center of the circle. $\mathbf{q_1}$ and $\mathbf{q_2}$ are points of contact with the circle.

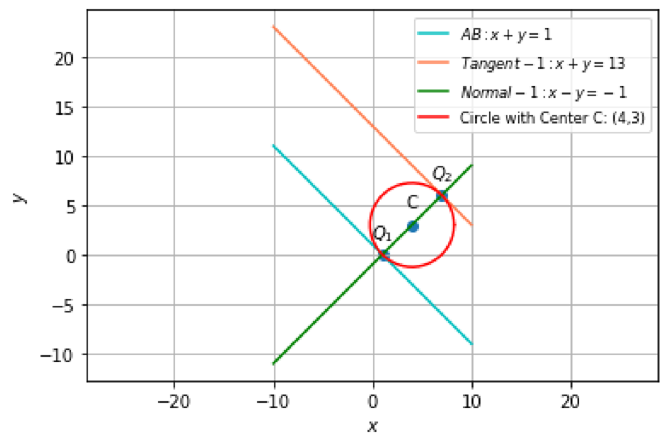

Fig. 4.2.11: Tangents and Normal on the Circle

Line 1 is (4.2.11.1), Line 2 is (4.2.11.22) and Line 3 is (4.2.11.27).

12. Find the equation of the tangent at the origin to the circle

$$\mathbf{x}^T\mathbf{x} + 2\begin{pmatrix} g & f \end{pmatrix}\mathbf{x} = 0 \tag{4.2.12}$$

13. Prove that the line

$$\begin{pmatrix} \cos\alpha & \sin\alpha \end{pmatrix} \mathbf{x} = p \tag{4.2.13.1}$$

touches the circle

$$\left\| \mathbf{x} - \begin{pmatrix} a \\ b \end{pmatrix} \right\| = r \tag{4.2.13.2}$$

if

$$r = \pm \left(p - a\cos\alpha - b\sin\alpha \right) \tag{4.2.13.3}$$

14. Find the points of contact of the tangents to the circle

$$\|\mathbf{x}\| = 5 \tag{4.2.14}$$

that pass through the point $\begin{pmatrix}7\\1\end{pmatrix}$ and write down the equations of the tangents.

15. Prove that the tangent to the circle

$$\|\mathbf{x}\|^2 = 5 \tag{4.2.15.1}$$

at the point $\begin{pmatrix}1\\-2\end{pmatrix}$ also touches the circle

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}-8 & 6\end{pmatrix}\mathbf{x} + 20 = 0 \tag{4.2.15.2}$$

and find the coordinates of the point of contact.

Solution:

From the given information,

$$\mathbf{c} = \mathbf{u} = \begin{pmatrix}0\\0\end{pmatrix}, \mathbf{q} = \begin{pmatrix}1\\-2\end{pmatrix} \tag{4.2.15.3}$$

$$\mathbf{f} = -5, r = \sqrt{5} \tag{4.2.15.4}$$

Given the point of contact $\mathbf{q}$, the equation of a tangent to the circle is

$$(\mathbf{V}\mathbf{q} + \mathbf{u})^T \mathbf{x} + \mathbf{u}^T\mathbf{q} + f = 0 \tag{4.2.15.5}$$

Using (4.2.15.5),the tangent at the point $P\begin{pmatrix}1\\-2\end{pmatrix}$ is

$$\implies \begin{pmatrix}1 & -2\end{pmatrix}\mathbf{x} = 5 \tag{4.2.15.6}$$

The equation of the tangent line is

$$\begin{pmatrix}1 & -2\end{pmatrix}\mathbf{x} = 5 \tag{4.2.15.7}$$

The parameters in (4.2.15.1) are

$$\mathbf{u} = \begin{pmatrix}-4\\3\end{pmatrix}, f = 20 \tag{4.2.15.8}$$

$$\mathbf{n} = \begin{pmatrix}1\\-2\end{pmatrix} \tag{4.2.15.9}$$

The point of contact $\mathbf{q}$, of a line with a normal vector $\mathbf{n}$ to the conic in (4.2.15.2) is given by:

$$\mathbf{q} = \mathbf{V}^{-1}(\kappa\mathbf{n} - \mathbf{u}) \tag{4.2.15.10}$$

$$\kappa = \pm\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\mathbf{n}^T\mathbf{V}^{-1}\mathbf{n}}} \tag{4.2.15.11}$$

Solving for the point of contact of (4.2.15.7) with (4.2.15.2) using the above equations,

$$\kappa = \pm\sqrt{\frac{\begin{pmatrix}-4 & 3\end{pmatrix}\begin{pmatrix}-4\\3\end{pmatrix} - 20}{\begin{pmatrix}1 & -2\end{pmatrix}\begin{pmatrix}1\\-2\end{pmatrix}}} \tag{4.2.15.12}$$

$$= \pm\sqrt{\frac{25-20}{5}} \tag{4.2.15.13}$$

$$= \pm 1 \tag{4.2.15.14}$$

$$\mathbf{q} = -\begin{pmatrix}1\\-2\end{pmatrix} - \begin{pmatrix}-4\\3\end{pmatrix} \tag{4.2.15.15}$$

$$= \begin{pmatrix}3\\-1\end{pmatrix} \tag{4.2.15.16}$$

Fig. 4.2.15 verifies this result.

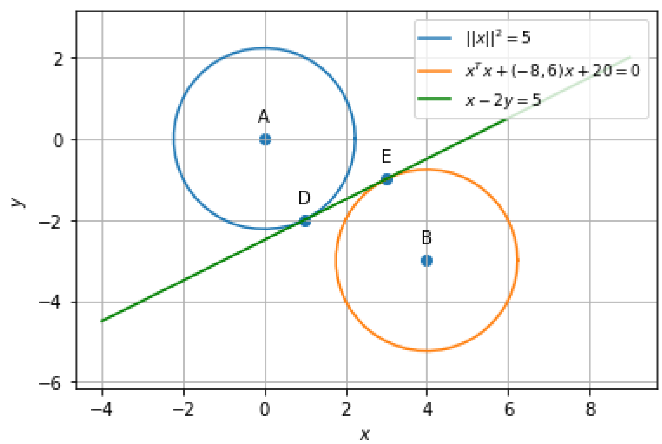

Fig. 4.2.15: Graphical illustration

16. Find the equations of the circles that touch the lines

$$\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} = 0 \tag{4.2.16.1}$$

$$\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} = 4 \tag{4.2.16.2}$$

$$\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} = 2 \tag{4.2.16.3}$$

17. Find the coordinates of the middle point of the chord

$$\begin{pmatrix}1 & 7\end{pmatrix}\mathbf{x} = 25 \tag{4.2.17.1}$$

of the circle

$$\|\mathbf{x}\| = 5 \tag{4.2.17.2}$$

18. Find the equation of the chord of the circle

$$\mathbf{x}^T\mathbf{x} - \begin{pmatrix}6 & 4\end{pmatrix}\mathbf{x} - 23 = 0 \tag{4.2.18}$$

which has the point $\begin{pmatrix}4\\1\end{pmatrix}$ as its middle point.

Solution:

The general equation of circle is

$$\mathbf{x}^T\mathbf{x} - 2\mathbf{c}^T\mathbf{x} + f = 0 \tag{4.2.18}$$

where $\mathbf{c}$ is the centre of the circle.

$$\mathbf{c} = \begin{pmatrix}3\\2\end{pmatrix} \tag{4.2.18}$$

$$\mathbf{O} = \mathbf{c} \tag{4.2.18}$$

$$\mathbf{M} = \begin{pmatrix}4\\1\end{pmatrix} \tag{4.2.18}$$

The line passing through the centre bisects any chord perpendicularly. The direction vector of **OM** is

$$\mathbf{OM} = \mathbf{M} - \mathbf{O} \tag{4.2.18}$$

$$= \begin{pmatrix}4\\1\end{pmatrix} - \begin{pmatrix}3\\2\end{pmatrix} \tag{4.2.18}$$

$$= \begin{pmatrix}1\\-1\end{pmatrix} \tag{4.2.18}$$

The normal vector $\mathbf{n}$ is

$$\mathbf{n} = \mathbf{OM} \tag{4.2.18}$$

The equation of line in terms of normal vector

$$\mathbf{n}^T(\mathbf{x} - \mathbf{M}) = 0 \tag{4.2.18}$$

$$\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = \begin{pmatrix}1 & -1\end{pmatrix}\mathbf{M} \tag{4.2.18}$$

$$\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = \begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}4\\1\end{pmatrix} \tag{4.2.18}$$

$$\implies \begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = 3 \tag{4.2.18}$$

See Fig. 4.2.18

19. Prove that the circle

$$\mathbf{x}^T\mathbf{x} - \begin{pmatrix}6 & 4\end{pmatrix}\mathbf{x} + 9 = 0 \tag{4.2.19}$$

subtends an angle $\tan^{-1}\frac{12}{5}$ at the origin.

20. Find the condition that the line

$$\begin{pmatrix}l & m\end{pmatrix}\mathbf{x} + n = 0 \tag{4.2.20.1}$$

should touch the circle

$$\left\|\mathbf{x} - \begin{pmatrix}a\\b\end{pmatrix}\right\| = r \tag{4.2.20.2}$$

21. Verify that the perpendicular bisector of the

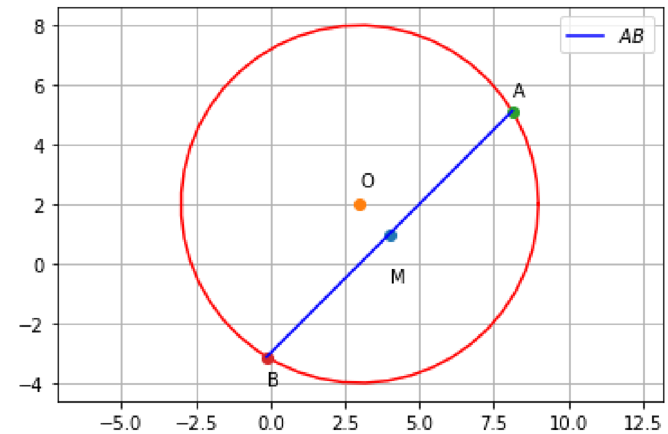

Fig. 4.2.18: Plot of the given points and circle

chord joining two points $\mathbf{x}_1, \mathbf{x}_2$ on the circle

$$\mathbf{x}^T\mathbf{x} + 2\begin{pmatrix}g & f\end{pmatrix}\mathbf{x} + c = 0 \tag{4.2.21}$$

passes through the centre.

Solution: Since $\mathbf{x_1}$ and $\mathbf{x_2}$ lie on the circle,

$$\mathbf{x_1}^\top\mathbf{x_1} + 2\begin{pmatrix}g & f\end{pmatrix}\mathbf{x_1} + c = 0 \tag{4.2.21}$$

$$\mathbf{x_2}^\top\mathbf{x_2} + 2\begin{pmatrix}g & f\end{pmatrix}\mathbf{x_2} + c = 0 \tag{4.2.21}$$

Subtracting the two, we get,

$$\mathbf{x_2}^\top\mathbf{x_2} - \mathbf{x_1}^\top\mathbf{x_1} = -2\begin{pmatrix}g & f\end{pmatrix}(\mathbf{x_2} - \mathbf{x_1}) \tag{4.2.21}$$

The midpoint of the chord $\mathbf{x_1x_2}$ is

$$\mathbf{M} = \frac{\mathbf{x_1} + \mathbf{x_2}}{2} \tag{4.2.21}$$

Since perpendicular bisector of $\mathbf{x_1x_2}$ is perpendicular to $\mathbf{x_1x_2}$ and passes through $\mathbf{M}$, any general point $\mathbf{P}$ on the perpendicular bisector can be given using the relation

$$(\mathbf{x_2} - \mathbf{x_1})^\top(\mathbf{P} - \mathbf{M}) = 0 \tag{4.2.21}$$

$$\implies (\mathbf{x_2} - \mathbf{x_1})^\top\mathbf{P} - (\mathbf{x_2}^\top - \mathbf{x_1}^\top)\left(\frac{\mathbf{x_1} + \mathbf{x_2}}{2}\right) = 0 \tag{4.2.21}$$

$$\implies (\mathbf{x_2} - \mathbf{x_1})^\top\mathbf{P} = \frac{\mathbf{x_2}^\top\mathbf{x_2} - \mathbf{x_1}^\top\mathbf{x_1} + \mathbf{x_2}^\top\mathbf{x_1} - \mathbf{x_1}^\top\mathbf{x_2}}{2} \tag{4.2.21}$$

Since $\mathbf{x_1}^\top\mathbf{x_2} = \mathbf{x_2}^\top\mathbf{x_1}$, the equation reduces to

$$2(\mathbf{x_2} - \mathbf{x_1})^\top\mathbf{P} = \mathbf{x_2}^\top\mathbf{x_2} - \mathbf{x_1}^\top\mathbf{x_1} \tag{4.2.21}$$

Using (4.2.21), we get

$$2\left(\mathbf{x_2}-\mathbf{x_1}\right)^{\top}\mathbf{P} = -2\begin{pmatrix} g & f \end{pmatrix}\left(\mathbf{x_2}-\mathbf{x_1}\right) \tag{4.2.21}$$

$$\implies \left(\mathbf{x_2}-\mathbf{x_1}\right)^{\top}\mathbf{P} = \left(\mathbf{x_2}-\mathbf{x_1}\right)^{\top}\begin{pmatrix} -g \\ -f \end{pmatrix} \tag{4.2.21}$$

Since the centre of the circle is $\mathbf{C} = \begin{pmatrix} -g \\ -f \end{pmatrix}$, we can clearly see that $\mathbf{P} = \mathbf{C}$ satisfies the equation of perpendicular bisector.
Hence, the perpendicular bisector of any chord of a circle passes through the centre of the circle.
For example, consider the circle

$$\mathbf{x}^{\top}\mathbf{x} + 2\begin{pmatrix} -4 & -5 \end{pmatrix}\mathbf{x} + 5 = 0 \tag{4.2.21}$$

and let $\mathbf{x_1} = \begin{pmatrix} 0 \\ 5-2\sqrt{5} \end{pmatrix}$ and $\mathbf{x_2} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$. We can clearly see in Fig. 4.2.21 that the perpendicular bisector the chord $\mathbf{x_1}\mathbf{x_2}$ passes through the centre of the circle.

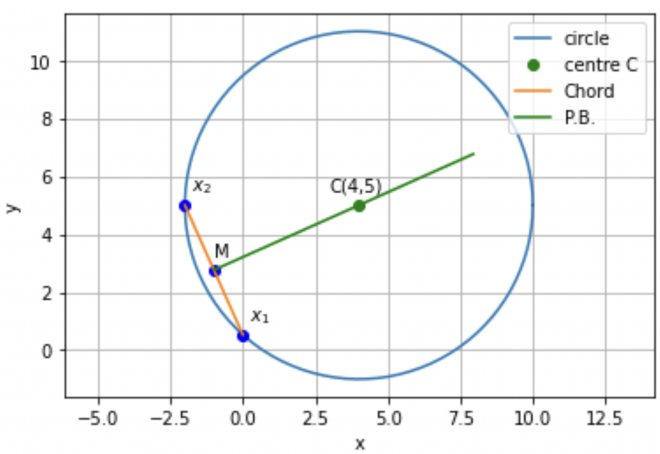

Fig. 4.2.21: Example figure

4.3 Pole and Polar

1. Write down the equations of the polars of the following points with regard to the circle

$$\|\mathbf{x}\|^2 = 6 \tag{4.3.1}$$

a) $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$
b) $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$

2. Find the poles of the following lines with regard to the circle

$$\|\mathbf{x}\| = 3 \tag{4.3.2}$$

a) $\begin{pmatrix} 3 & 4 \end{pmatrix}\mathbf{x} = 7$
b) $\begin{pmatrix} 5 & -1 \end{pmatrix}\mathbf{x} + 6 = 0$

and verify that the polar of their point of intersection is the line joining their poles.

3. Show that the points $\begin{pmatrix} 4 \\ -2 \end{pmatrix}, \begin{pmatrix} 3 \\ -6 \end{pmatrix}$ are conjugate with regard to the circle

$$\|\mathbf{x}\|^2 = 24 \tag{4.3.3}$$

4. Prove that the lines

$$\begin{pmatrix} 5 & 3 \end{pmatrix}\mathbf{x} = 40 \tag{4.3.4.1}$$

$$\begin{pmatrix} 7 & -5 \end{pmatrix}\mathbf{x} = 10 \tag{4.3.4.2}$$

are conjugate with regard to the circle

$$\|\mathbf{x}\|^2 = 20 \tag{4.3.4.3}$$

5. Find the polar of the point $\begin{pmatrix} 5 \\ 4 \end{pmatrix}$ with regard to the circle

$$\mathbf{x}^T\mathbf{x} - \begin{pmatrix} 4 & 3 \end{pmatrix}\mathbf{x} - 8 = 0 \tag{4.3.5}$$

6. Find the pole of

$$\begin{pmatrix} l & m \end{pmatrix}\mathbf{x} + n = 0 \tag{4.3.6}$$

with regard to

a) $\|\mathbf{x}\| = a$
b) $\mathbf{x}^T\mathbf{x} + 2\begin{pmatrix} g & f \end{pmatrix}\mathbf{x} + c = 0$

7. Prove that if two lines at right angles are conjugate with regard to a circle one of them must pass through the centre.
8. Prove that, if the chords of contact of pairs of tangents to a circle from $\mathbf{P}$ and $\mathbf{Q}$ intersect in in $\mathbf{R}$, then the line joining $\mathbf{R}$ to the centre is perpendicular to PQ.

4.4 Systems of Circles

1. Find the equation of a circle which passes through the points $\begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ and cuts orthogonally the circle

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix} -2 & 3 \end{pmatrix}\mathbf{x} - 5 = 0 \tag{4.4.1}$$

2. Find the equation of a circle which cuts orthogonally the three circles

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}4 & -5\end{pmatrix}\mathbf{x} + 6 = 0 \qquad (4.4.2.1)$$
$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}5 & -6\end{pmatrix}\mathbf{x} + 7 = 0 \qquad (4.4.2.2)$$
$$\mathbf{x}^T\mathbf{x} - \begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} - 1 = 0 \qquad (4.4.2.3)$$

3. Find the equation of a circle which cuts orthogonally the two circles

$$\mathbf{x}^T\mathbf{x} - \begin{pmatrix}2 & 2\end{pmatrix}\mathbf{x} + 1 = 0 \qquad (4.4.3.1)$$
$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}-3 & 6\end{pmatrix}\mathbf{x} - 2 = 0 \qquad (4.4.3.2)$$

and passes through the point $\begin{pmatrix}-3\\2\end{pmatrix}$.

4. Write down the equations of the radical axes of the following pairs of circles:

a)

$$\mathbf{x}^T\mathbf{x} - \begin{pmatrix}4 & -5\end{pmatrix}\mathbf{x} - 2 = 0 \qquad (4.4.4.1)$$
$$\mathbf{x}^T\mathbf{x} - \begin{pmatrix}5 & -6\end{pmatrix}\mathbf{x} = 0 \qquad (4.4.4.2)$$

b)

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}3 & -2\end{pmatrix}\mathbf{x} + 1 = 0 \qquad (4.4.4.3)$$
$$\mathbf{x}^T\mathbf{x} - \begin{pmatrix}3 & -5\end{pmatrix}\mathbf{x} + 2 = 0 \qquad (4.4.4.4)$$

c)

$$\mathbf{x}^T\mathbf{x} + 2g\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} + c = 0 \qquad (4.4.4.5)$$
$$\mathbf{x}^T\mathbf{x} + 2f\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} + c = 0 \qquad (4.4.4.6)$$

Solution:
Given, two circles with equations,

$$S = \mathbf{x}^T\mathbf{x} - \begin{pmatrix}4 & -5\end{pmatrix}\mathbf{x} - 2 = 0 \qquad (4.4.4.7)$$
$$S' = \mathbf{x}^T\mathbf{x} - \begin{pmatrix}5 & -6\end{pmatrix}\mathbf{x} = 0 \qquad (4.4.4.8)$$

We know, the radical axis for the pair of circles, $S = 0, S' = 0$ is given by $L = S - S' = 0$. Using (4.4.4.7), (4.4.4.8), the required equation is

$$\left(\mathbf{x}^T\mathbf{x} - \begin{pmatrix}4 & -5\end{pmatrix}\mathbf{x} - 2\right) - \left(\mathbf{x}^T\mathbf{x} - \begin{pmatrix}5 & -6\end{pmatrix}\mathbf{x} = 0\right) = 0 \qquad (4.4.4.9)$$
$$\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} - 2 = 0 \qquad (4.4.4.10)$$

$\therefore L = \begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} - 2 = 0$ is the equation of the required radical axis. See Fig. 4.4.4

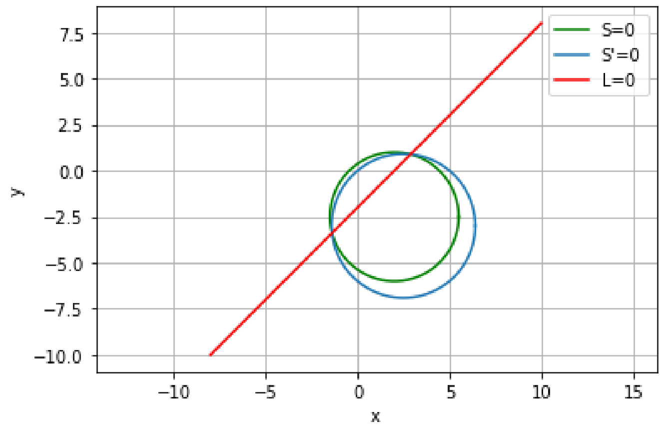

Fig. 4.4.4: Pair of Circles and their radical axis

5. Find the equation of a circle coaxal with

$$\mathbf{x}^T\mathbf{x} - \begin{pmatrix}2 & -3\end{pmatrix}\mathbf{x} - 1 = 0 \qquad (4.4.5.1)$$
$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}3 & -2\end{pmatrix}\mathbf{x} - 1 = 0 \qquad (4.4.5.2)$$

and passing through the point $\begin{pmatrix}2\\1\end{pmatrix}$.

6. Find the coordinates of the point from which the tanges to the three circles

$$\mathbf{x}^T\mathbf{x} - \begin{pmatrix}4 & 4\end{pmatrix}\mathbf{x} + 7 = 0 \qquad (4.4.6.1)$$
$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}4 & 0\end{pmatrix}\mathbf{x} + 3 = 0 \qquad (4.4.6.2)$$
$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}0 & 2\end{pmatrix}\mathbf{x} = 0 \qquad (4.4.6.3)$$

are of equal length.

7. Find the limiting points of the circles

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}0 & 2\end{pmatrix}\mathbf{x} - 4 = 0 \qquad (4.4.7.1)$$
$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}2 & 2\end{pmatrix}\mathbf{x} - 10 = 0 \qquad (4.4.7.2)$$

8. Prove that if a point moves so that the tangent from it to the circle

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}4 & -5\end{pmatrix}\mathbf{x} + 6 = 0 \qquad (4.4.8.1)$$

is double the length of the tangent to the circle

$$\|\mathbf{x}\| = 2, \qquad (4.4.8.2)$$

its locus is the circle

$$3\mathbf{x}^T\mathbf{x} - \begin{pmatrix}4 & -5\end{pmatrix}\mathbf{x} - 22 = 0 \qquad (4.4.8.3)$$

9. Prove that the locus of a point such that the lengths of the tangents from it to two given circles are in a constant ratio is a circle coaxal with the given circles.

10. Find the equations of the two circles coaxal with

$$\mathbf{x}^T\mathbf{x} - \begin{pmatrix}8 & -10\end{pmatrix}\mathbf{x} + 2 = 0 \qquad (4.4.10.1)$$

$$\mathbf{x}^T\mathbf{x} - \begin{pmatrix}3 & -5\end{pmatrix}\mathbf{x} - 1 = 0 \qquad (4.4.10.2)$$

that touch the line

$$\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} - 3 = 0 \qquad (4.4.10.3)$$

11. Find the centre and radius of the circle which cuts orthogonally the three circles

$$\mathbf{x}^T\mathbf{x} - \begin{pmatrix}6 & 4\end{pmatrix}\mathbf{x} + 12 = 0 \qquad (4.4.11.1)$$

$$\mathbf{x}^T\mathbf{x} + 2\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} + 1 = 0 \qquad (4.4.11.2)$$

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}4 & -2\end{pmatrix}\mathbf{x} + 4 = 0 \qquad (4.4.11.3)$$

12. The line

$$\begin{pmatrix}1 & 3\end{pmatrix}\mathbf{x} + 2 = 0 \qquad (4.4.12.1)$$

is the radical axis of a family of coaxal circles of which one circle is

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}2 & 5\end{pmatrix}\mathbf{x} - 1 = 0. \qquad (4.4.12.2)$$

Find the equation of the member of the family that passes through the point $\begin{pmatrix}-3\\1\end{pmatrix}$.

4.5 Miscellaneous

1. Find the locus of a point which moves so that the sum of the squares of its distances from the sides of an equilateral triangle is constant.
2. Find the locus of a point which moves so that the sum of the squares of its distances from n fixed points is constant.
3. Find the locus of a point at which two given circles subtend equal angles.
4. A circle passes through the four points $\begin{pmatrix}a\\0\end{pmatrix}$, $\begin{pmatrix}b\\0\end{pmatrix}$, $\begin{pmatrix}0\\c\end{pmatrix}$, $\begin{pmatrix}0\\d\end{pmatrix}$. By what relation are a, b, c, d connected? Find the equation of the circle and show that the tangent at the point $\begin{pmatrix}a\\c+d\end{pmatrix}$ is

$$\begin{pmatrix}a-b & c+d\end{pmatrix}\mathbf{x} - a(a-b) - (c+d)^2 = 0 \qquad (4.5.4)$$

5. Write down the equations of the tangents to the circles

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}-2a & 0\end{pmatrix}\mathbf{x} - 5 = 0 \qquad (4.5.5.1)$$

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}0 & -2b\end{pmatrix}\mathbf{x} - 5 = 0 \qquad (4.5.5.2)$$

at their points of intersection and verify that they cut at right angles.

6. Find the equation of the tangent to the circle

$$\|\mathbf{x}\| = a \qquad (4.5.6.1)$$

at the point $\begin{pmatrix}a\cos\theta\\a\sin\theta\end{pmatrix}$ and show that the length of the tangent intercepted by the lines

$$\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\mathbf{x} = 0 \qquad (4.5.6.2)$$

is $\pm 2a\sec\theta$.

7. **A** and **B** are two fixed points $\begin{pmatrix}c\\0\end{pmatrix}$, $\begin{pmatrix}-c\\0\end{pmatrix}$, and **P** moves so that $PA = k.PB$. Find the locus of **P** and prove that it is cut orthogonally by any circle through **A** and **B**.
8. Show that the common chord of the circles

$$\mathbf{x}^T\mathbf{x} - \begin{pmatrix}6 & 4\end{pmatrix}\mathbf{x} + 9 = 0 \qquad (4.5.8.1)$$

$$\mathbf{x}^T\mathbf{x} - \begin{pmatrix}8 & 6\end{pmatrix}\mathbf{x} + 23 = 0 \qquad (4.5.8.2)$$

is a diameter of the latter circle and find the angle at which the circles cut.

9. Prove analytically that the tangents to a circle at the ends of a chord are equally inclined to the chord.
10. Prove that for different values of a the equation

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}-2a\operatorname{cosec}\alpha & 0\end{pmatrix}\mathbf{x} + a^2\cot^2\alpha = 0 \qquad (4.5.10.1)$$

represents a family of circles touching the lines

$$\begin{pmatrix}\pm\tan\alpha & 1\end{pmatrix}\mathbf{x} = 0 \qquad (4.5.10.2)$$

Prove also that the locus of the poles of the line

$$\begin{pmatrix}l & m\end{pmatrix}\mathbf{x} = 0 \qquad (4.5.10.3)$$

with regard to the circles is the line

$$\begin{pmatrix}m\sin^2\alpha & l\cos^2\alpha\end{pmatrix}\mathbf{x} = 0 \qquad (4.5.10.4)$$

11. Find the coordinates of the middle point of the chord

$$\begin{pmatrix}l & m\end{pmatrix}\mathbf{x} = 1 \qquad (4.5.11.1)$$

of the circle

$$\mathbf{x}^T\mathbf{x} + 2\begin{pmatrix} g & f \end{pmatrix}\mathbf{x} + c = 0 \qquad (4.5.11.2)$$

12. Prove that the points of intersection of the line

$$\begin{pmatrix} l & m \end{pmatrix}\mathbf{x} = 1 \qquad (4.5.12.1)$$

and the circle

$$\mathbf{x}^T\mathbf{x} + 2\begin{pmatrix} g & f \end{pmatrix}\mathbf{x} + c = 0 \qquad (4.5.12.2)$$

subtend a right angle at the origin if

$$c\left(l^2 + m^2\right) + 2gl + 2fm + 2 = 0 \qquad (4.5.12.3)$$

13. Prove that the equation of the circle having for diameter the portion of the line

$$\begin{pmatrix} \cos\alpha & \sin\alpha \end{pmatrix}\mathbf{x} = p \qquad (4.5.13.1)$$

intercepted by the circle

$$\|\mathbf{x}\| = a \qquad (4.5.13.2)$$

is

$$\mathbf{x}^T\mathbf{x} - 2p\begin{pmatrix} \cos\alpha & \sin\alpha \end{pmatrix}\mathbf{x} + 2p^2 - a^2 = 0 \qquad (4.5.13.3)$$

14. Prove that if a chord of the circle

$$\|\mathbf{x}\| = a \qquad (4.5.14.1)$$

subtends a right angle at a fixed point $\mathbf{x}_1$, the locus of the middle point of the chord is

$$2\mathbf{x}^T\mathbf{x} - 2\mathbf{x}_1^T\mathbf{x} + \|\mathbf{x}_1\|^2 - a^2 = 0 \qquad (4.5.14.2)$$

15. Prove that the equation of any tangent to the circle

$$\left\|\mathbf{x} - \begin{pmatrix} a \\ b \end{pmatrix}\right\| = r \qquad (4.5.15.1)$$

may be written in the form

$$\begin{pmatrix} \cos\theta & \sin\theta \end{pmatrix}\left(\mathbf{x} - \begin{pmatrix} a \\ b \end{pmatrix}\right) = r \qquad (4.5.15.2)$$

Deduce that the equation of the tangents from $\mathbf{x}_1$ to the circle is

$$r^2\|\mathbf{x} - \mathbf{x}_1\|^2 = \left[\left\{\mathbf{x} - \begin{pmatrix} a \\ b \end{pmatrix}\right\}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\left\{\mathbf{x}_1 - \begin{pmatrix} a \\ b \end{pmatrix}\right\}\right]^2 \qquad (4.5.15.3)$$

16. Prove that the distances of two points from the centre of a circle are proportional to the distance of each point from the polar of the other.
17. Prove that the tangents to the circles of a coaxal system drawn from a limiting point are bisected by the radical axis.
18. Show that a common tangent to the two circles is bisected by their radical axis and subtends a right angle at either limiting point.
19. Prove that if a point moves so that the difference of the squares of the tangents from it to two given circles is constant its locus is a straight line parallel to the radical axis of the circles.
20. Prove that the polars of a fixed point with regard to a family of coaxal circles all pass through another fixed point.
21. The circles

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix} -2a\sec\alpha & 0 \end{pmatrix}\mathbf{x} - a^2 = 0 \qquad (4.5.21.1)$$

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix} 0 & -2a\operatorname{cosec}\alpha \end{pmatrix}\mathbf{x} - a^2 = 0 \qquad (4.5.21.2)$$

where α is a given angle, both cut orthogonally every member of a coaxal family of circles. Find the radical axis and the limiting points of the family.
22. Prove that, if two points **P**, **Q** are conjugate with regard to a circle, the circle on PQ as diameter cuts the first circle orthogonally.
23. Prove that if **P**, **Q** are conjugate points with regard to a circle, the circles with **P**, **Q** as centres which cut the given circle orthogonally are orthogonal to one another.
24. Prove that, if PQ is a diameter of a circle, then **P**, **Q** are conjugate points with regard to any circle which cuts the given circle orthogonally.
25. Prove that if **P**, **Q** are conjugate points with regard to a circle, the square on PQ is equal to the sum of the squares on the tangents from **P**, **Q** to the circle.
26. The equation

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix} -2g & 0 \end{pmatrix}\mathbf{x} + 2g - 5 = 0 \qquad (4.5.26)$$

where g is a variable parameter, represents a family of coaxial circles. Show that the radius of the smallest circle of the family is 2.
27. Prove that, if perpendiculars are drawn from a fixed point **P** to the polars of **P** with regard to a family of coaxial circles, then the locus of the feet of these perpendiculars is a circle whose

centre lies on the radical axis of the family.

28. Prove that, if the points in which the line

$$\begin{pmatrix} l & m \end{pmatrix} \mathbf{x} + n = 0 \tag{4.5.28.1}$$

meets the circle,

$$\mathbf{x}^T\mathbf{x} + 2\begin{pmatrix} g & f \end{pmatrix} \mathbf{x} + c = 0 \tag{4.5.28.2}$$

and those in which the line

$$\begin{pmatrix} l_1 & m_1 \end{pmatrix} \mathbf{x} + n_1 = 0 \tag{4.5.28.3}$$

meets

$$\mathbf{x}^T\mathbf{x} + 2\begin{pmatrix} g_1 & f_1 \end{pmatrix} \mathbf{x} + c_1 = 0 \tag{4.5.28.4}$$

lie on a circle, then

$$2(g - g_1)(mn_1 - m_1 n) + 2(f - f_1) (nl_1 - n_1 l) + (c - c_1)(lm_1 - l_1 m) = 0 \tag{4.5.28.5}$$

29. Show that, if a diameter of a circle is the portion of the line

$$\begin{pmatrix} l & m \end{pmatrix} \mathbf{x} = 1 \tag{4.5.29.1}$$

intercepted by the lines

$$\mathbf{x}^T \begin{pmatrix} a & h \\ h & b \end{pmatrix} \mathbf{x} = 0 \tag{4.5.29.2}$$

then the equation of the circle is

$$\left(am^2 - 2hlm + bl^2\right)\mathbf{x}^T\mathbf{x} + 2\begin{pmatrix} (hm - bl) & (hl - am) \end{pmatrix} \mathbf{x} + a + b = 0 \tag{4.5.29.3}$$

30. Prove that, as k varies, the equation

$$\mathbf{x}^T\mathbf{x} + 2\begin{pmatrix} a & b \end{pmatrix} \mathbf{x} + c + 2k\left\{\begin{pmatrix} a & -b \end{pmatrix} \mathbf{x} + 1\right\} = 0 \tag{4.5.30.1}$$

represents a system of coaxial circles. Also prove that the orthogonal system is given by

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix} \frac{c+2}{2a} & \frac{c-2}{2b} \end{pmatrix} \mathbf{x} + h\left\{\begin{pmatrix} \frac{1}{2a} & \frac{1}{2b} \end{pmatrix} \mathbf{x} + 1\right\} = 0 \tag{4.5.30.2}$$

where h is a variable parameter.

5 Parabola

5.1 Tangent and Normal

1. Show that the line

$$\begin{pmatrix} 4 & -2 \end{pmatrix} \mathbf{x} + a = 0 \tag{5.1.1.1}$$

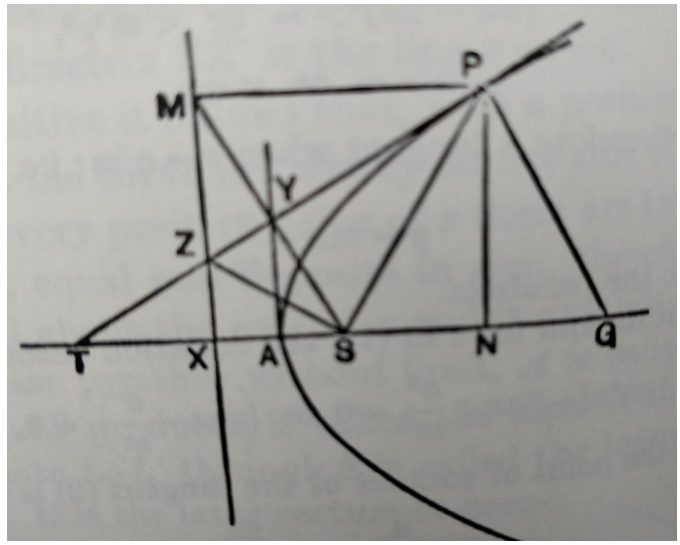

Fig. 5.1.0

touches the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - 4a\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 0 \tag{5.1.1.2}$$

and find the coordinates of the point of contact.

2. Find the point of intersection of the parabolas

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - 4a\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 0 \tag{5.1.2.1}$$

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} - 4a\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{5.1.2.2}$$

other than the origin, and prove that the tangents at this point are inclined at an angle $\tan^{-1} \frac{3}{4}$.

3. Find the points in which the line

$$\begin{pmatrix} 8 & -1 \end{pmatrix} \mathbf{x} = a \tag{5.1.3.1}$$

cuts the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - 4a\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 0 \tag{5.1.3.2}$$

and find the point where the tangents at these points intersect.

4. A line through the vertex $\mathbf{A}$ of the parabola

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - 4a\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 0 \tag{5.1.4}$$

makes an angle of 60° with the axis and cuts the curve again in $\mathbf{P}$. Find the equation of the tangent at $\mathbf{P}$, and show that the area of the triangle this tangent makes with the axes is $\frac{4a^2}{3\sqrt{3}}$.

5. Prove that in Fig. 5.1.0
$$SG = SP \tag{5.1.5}$$
6. Prove that $t_1t_2 = -1$ is the condition that the chord joining the points with parameters t_1, t_2 on a parabola shall pass through the focus.
7. Prove that the tangents drawn to a parabola from a point on the directrix are at right angles and that their chord of contact passes through the focus.
8. If in Fig. 5.1.0 PS cuts the curve again in $\mathbf{Q}$, prove that QA passes through $\mathbf{M}$.
9. Through the vertex $\mathbf{A}$ of a parabola chords AP, AP at right angles to one another are drawn. Prove that PA cuts the axis in a fixed point.
10. Find the coordinates of the other point in which the normal at $\begin{pmatrix} at^2 \\ 2at \end{pmatrix}$ meets the parabola
$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - 4a \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 0 \tag{5.1.10}$$
; and prove that two normal chords that cut at right angles divide one another in the ratio 1 : 3.
11. Three normals are drawn to a parabola from the point $\begin{pmatrix} h \\ k \end{pmatrix}$. Prove that the centroid of the triangle formed by their feet is the point $\begin{pmatrix} \frac{2}{3}(h-2a) \\ 0 \end{pmatrix}$.
12. Find the equation of the tangent to the parabola
$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - 4a \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 0 \tag{5.1.12.1}$$
that is parallel to the normal at $\mathbf{P} = \begin{pmatrix} at^2 \\ 2at \end{pmatrix}$; and prove that, if this tangent meets the axis in T and PN is the ordinate of P and A is the vertex, then
$$TA.AN = a^2 \tag{5.1.12.2}$$
13. Prove that the circle
$$\mathbf{x}^T\mathbf{x} + 2\begin{pmatrix} g & f \end{pmatrix} \mathbf{x} + c = 0 \tag{5.1.13.1}$$
cuts the parabola
$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - 4a \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 0 \tag{5.1.13.2}$$
in four points the sum of whose ordinates is zero; and conversely that if four ponits on a parabola be such that the sum of their ordinates is zero then the four points lie on a circle.
14. Prove that the orthocentre of a triangle whose sides all touch a parabola lies on the directrix.
15. A chord POQ of a parabola
$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - 4a \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 0 \tag{5.1.15.1}$$
cuts the axis in a fixed point $\mathbf{O}$. PN, QM are the ordinates of $\mathbf{P}$ and $\mathbf{Q}$, and $\mathbf{A}$ is the vertex. Prove that
$$NP.MQ + 4a.AO = 0 \tag{5.1.15.2}$$
16. From a point $\mathbf{P} = \begin{pmatrix} at_1^2 \\ 2at_1 \end{pmatrix}$ on the parabola
$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - 4a \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 0 \tag{5.1.16.1}$$
two chords PQ, PR are drawn normal to the curve at $\mathbf{Q}$ and $\mathbf{R}$. Prove that, if $\mathbf{Q}$, $\mathbf{R}$ are the points with parameters t_2, t_3 on the curve, then $t_2t_3 = 2$, and the equation of QR is
$$\begin{pmatrix} 2 & t_1 \end{pmatrix} \mathbf{x} + 4a = 0 \tag{5.1.16.2}$$
17. Prove that the normals to a parabola at the ends of a chord whose inclination to the axis is θ meet on the normal whose inclination is $\tan^{-1}(2\cot\theta)$.
18. Prove that, if two parabolas are on the same side of the same directrix and have their axes in the same line, then they intersect at a distance from the directrix equal to one-quarter of the sum of their latera recta.

5.2 *Miscellaneous*

In the following problems, the equation of a parabola is assumed to be the standard parabola and capital letters refer to Fig. 5.1.0 unless the contrary is stated.

1. Prove that as $\mathbf{P}$ moves along the curve GP^2 varies as SG.
2. Prove that, if PP_1 is a double ordinate and PX meets the curve in $\mathbf{Q}$, then P_1Q passes through $\mathbf{S}$.
3. Prove that, if PSP_1 is a focal chord and AP, AP_1 meet the latus rectum in $\mathbf{Q}$, $\mathbf{Q}_1$, the SQ, SQ_1 are equal to the ordinates of $\mathbf{P}_1$ and $\mathbf{P}$.

4. Prove that, if the tangents at **P**, **Q** intersect in **T**, then
$$ST^2 = SP.SQ. \tag{5.2.4}$$
5. Prove that, if the tangent at the end **Q** of a focal chord PSQ meets the latus rectum in **R**, then PGR is a right angle.
6. Tangents at **P**, **Q**, **R** on a parabola form a triangle UVW. Show that the centroids of the triangles PQR and UVW lie on the same diameter.
7. Prove that, if the difference of the ordinates of two points on a parabola is constant, then the locus of the point of intersection of the tangents at these points is an equal parabola.
8. Prove that, if two tangents intercept a fixed length on the tangent at the vertex, the locus of their intersections is an equal parabola.
9. The chord of contact of tangents from any point **Q** meets the tangent at the vertex in **R**. Prove that the tangent of the angle which AQ makes with the axis is $\frac{2a}{AR}$.
10. The parameters, t, t_1 of two points on a parabola are connected by the relation $t = k^2 t_1$, prove that the tangents at the points intersect on the curve
$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - \left(k + \frac{1}{k}\right)^2 a \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 0 \tag{5.2.10}$$
11. Show that the length of the normal chord at the point of parameter t is
$$\frac{4a}{t^2}\left(1 + t^2\right)^{\frac{3}{2}} \tag{5.2.11}$$
12. Prove that the locus of intersection of tangents at the ends of a normal chord is
$$(x + 2a)\, y^2 + 4a^2 = 0. \tag{5.2.12}$$
13. Prove that the locus of the point of intersection of perpendicular normals is the parabola
$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - a \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} + 3a^2 = 0 \tag{5.2.13}$$
14. Prove that if the tangents at two points on the parabola intersect in the point $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$, the corresponding normals intersect in the point
$$\begin{pmatrix} 2a - x_1 + \frac{y_1^2}{a} \\ -\frac{x_1 y_1}{a} \end{pmatrix}. \tag{5.2.14}$$
15. Show that, if the tangent at **P** meets the latus rectum in **K**, then SK is a mean proportional between the segments of the focal chord through **P**.
16. Show that, if the tangents from **Q** to the parabola form with the tangent at the vertex, a triangle of constant area c^2, then the locus of **Q** is the curve
$$x^2\left(y^2 - 4ax\right) = 4c^4. \tag{5.2.16}$$
17. Show that the normals at the ends of each of a series of parallel chords of a parabola intersect on a fixed straight line, itself a normal to the parabola.
18. **P**, **Q** are points on the parabola subtending a constant angle α at the vertex. Show that the locus of the intersection of the tangents at **P**, **Q** is the curve
$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - \begin{pmatrix} 4a + \frac{\tan^2 \alpha}{4} & 0 \end{pmatrix} \mathbf{x} = a^2 \tan^2 \alpha \tag{5.2.18}$$
19. Prove that the exterior angle between two tangents to a parabola is equal to the angle which either of them subtends at the focus.
20. Two perpendicular focal chords of a parabola meet the directrix in **T** and $\mathbf{T}_1$. Show that the tangents to the parabola which are parallel to these chords intersect in the middle point of TT_1.
21. Prove that, if the tangents at the points **Q**, **R** intersect at **P**, then
$$PQ^2 : PR^2 = SQ : SR \tag{5.2.21}$$
22. The tangents at any two points **P**, **Q** meet at **T** and the normals meet at **N**. Prove that the projection of TN on the axis is equal to the sum of the distances of **P** and **Q** from the directrix.
23. Prove that the circumscribing circle of the triangle formed by three tangents to a parabola passes through the focus.
24. PQ is a chord of a parabola normal at **P**; the circle on PQ as diameter cuts the parabola again in **R**. Prove that the projection of QR on the axis is twice the latus rectum.
25. Prove that the distance between a tangent and the parallel normal is $a \operatorname{cosec}\theta \sec^2\theta$, where θ is the angle which either makes with the axis.
26. Prove that, if the normals at **P** and **Q** intersect

on the curve, then PQ cuts the axis in a fixed point.

27. Prove that, if the normals at **P** and **Q** meet at the point **R** $\left(x_1, y_1\right)$ on the parabola, and the tangents at **P** and **Q** meet at **T**, then

$$TP.TQ = \frac{1}{2}\left(x_1 - 8a\right)\sqrt{y_1^2 + 4a^2}. \quad (5.2.27)$$

28. Show that, in the last problem, as **R** moves along the parabola, the middle point of PQ always lies on the parabola

$$\mathbf{x}^T\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\mathbf{x} - 2a\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = 4a^2 \quad (5.2.28)$$

29. Prove that the area of the triangle formed by the tangents at the points t_1, t_2, and their chord of contact is

$$\frac{1}{2}a^2\left(t_1 - t_2\right)^2 \quad (5.2.29)$$

30. Prove that the area of the triangle formed by three points t_1, t_2, t_3 on the parabola is

$$a^2\left(t_2 - t_3\right)\left(t_3 - t_1\right)\left(t_1 - t_2\right) \quad (5.2.30)$$

and that this is double the area of the triangle formed by the tangents at these points.

31. Prove that, if a line through any point $P\begin{pmatrix}x_1\\ y_1\end{pmatrix}$ making an angle θ with the axis meets the parabola at **Q** and **R**, then

$$PQ.PR = \left(y_1^2 - 4ax_1\right)\operatorname{cosec}^2\theta. \quad (5.2.31)$$

32. Two chords QR, Q_1R_1 of a parabola meet at **O**, and the diameters bisecting them meet the curve at **P** and P_1. Prove that

$$QO.OR : Q_1O.OR_1 = SP : SP_1 \quad (5.2.32)$$

33. Show that, if **P** is on the parabola, the length of the chord through **P** that makes an angle θ with the axis is

$$4a\sin\left(\alpha - \theta\right)\operatorname{cosec}^2\theta\operatorname{cosec}\alpha \quad (5.2.33)$$

where α is the inclination of the tangent at **P** to the axis.

34. Show that the locus of the middle point of a chord which passes through the fixed point $\begin{pmatrix}h\\ k\end{pmatrix}$ is the parabola

$$\mathbf{x}^T\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\mathbf{x} - \begin{pmatrix}2a & k\end{pmatrix}\mathbf{x} + 2ah = 0 \quad (5.2.34)$$

35. A tangent to the parabola

$$\mathbf{x}^T\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\mathbf{x} + 4b\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = 0 \quad (5.2.35)$$

meets the parabola

$$\mathbf{x}^T\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\mathbf{x} - 4a\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = 0 \quad (5.2.35)$$

at **P**, **Q**. Prove that the locus of the middle point of PQ is

$$\mathbf{x}^T\begin{pmatrix}0 & 0\\ 0 & 2a+b\end{pmatrix}\mathbf{x} - 4a^2\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = 0 \quad (5.2.35)$$

36. Prove that the polar of the focus of a parabola is the directrix.

37. Prove that, if a chord of the parabola subtends a right angle at the vertex, the locus of its pole is

$$\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} + 4a = 0 \quad (5.2.37)$$

38. Show that, if parabolas

$$\mathbf{x}^T\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\mathbf{x} - 4a\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = 0 \quad (5.2.38)$$

are drawn corresponding to different values of a, the feet of the perpendiculars from a fixed point on its polar lines all lie on a circle passing through the point.

39. Prove that, if from a point $\mathbf{Q} = \begin{pmatrix}x_1\\ y_1\end{pmatrix}$ a perpendicular be drawn to the polar of **Q** with regard to the parabola cutting it in **R** and the axis in G, then

$$SG = SR = x_1 + a \quad (5.2.39)$$

40. Prove that, if the diameter through a point **P** of a parabola meets any chord in **O** and the tangents at the ends of the chord in $\mathbf{T}, \mathbf{T}_1$, then

$$PO^2 = PT.PT_1 \quad (5.2.40)$$

41. QQ_1 is a chord of a parabola and TOR is a diameter which meets the tangent at **Q** in **T**, the curve in **O** and QQ_1 in **R**. Prove that

$$TO : OR = QR : RQ_1 \quad (5.2.41)$$

42. Prove that if the normals at three points **P**, **Q**, **R** on a parabola concur, then the points **P**, **Q**, **R** and the vertex of the parabola are concyclic.
43. Prove that in general, two members of the family of parabolas

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - 4a \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 4a^2 \tag{5.2.43}$$

, where a is the parameter specifying members of the family, pass through any assigned point of the plane, and that these two parabolas cut orthogonally at **P**.

6 The Ellipse

6.1 Properties

1. Find the length of the latus rectum, the eccentricity and the coordinates of the foci of the ellipses
   a) $\mathbf{x}^T \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix} \mathbf{x} = 36$
   b) $\mathbf{x}^T \begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix} \mathbf{x} = 36$
   c) $\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{x} = 8$
2. Find the equation of the ellipse whose foci are the points $\begin{pmatrix} 3 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \end{pmatrix}$ and eccentricity $\frac{1}{2}$. What are the equations of the directrices?
3. Find the equation of the ellipse of eccentricity $\frac{3}{4}$, which has its centre at the point $\begin{pmatrix} 4 \\ 0 \end{pmatrix}$ and touches the axis of y at the origin. What is the length of its latus rectum?
4. An ellipse has the axis of y for directrix and its centre at the point $\begin{pmatrix} 6 \\ 0 \end{pmatrix}$. Find its equation if its eccentricity is $\frac{3}{4}$.
5. Find the equation of the ellipse which has a focus at $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$, corresponding directrix the line

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 6 \tag{6.1.5.1}$$

   and eccentricity $\frac{1}{2}$. What are the lengths of its axes?
6. An ellipse has its centre at $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and a focus at $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and its eccentricity is $\frac{1}{2}$. Find its equation.
7. Are the points $\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ inside or outside the ellipse

$$\mathbf{x}^T \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix} \mathbf{x} = 20 \tag{6.1.7.1}$$

8. Prove that the line $y = x - 5$ touches the ellipse

$$\mathbf{x}^T \begin{pmatrix} 9 & 0 \\ 0 & 16 \end{pmatrix} \mathbf{x} = 144 \tag{6.1.8.1}$$

   and find the coordinates of the point of contact.
9. The line

$$\begin{pmatrix} 2 & 3 \end{pmatrix} \mathbf{x} = c \tag{6.1.9.1}$$

   touches the ellipse

$$\mathbf{x}^T \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix} \mathbf{x} = 1 \tag{6.1.9.2}$$

   Find the values of c.
10. Find the equation of the tangent to the ellipse

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \mathbf{x} = 8 \tag{6.1.10.1}$$

   at the point $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, and also the equations of the two tangents perpendicular to this.
11. Find the equations of the tangents to the ellipse

$$\mathbf{x}^T \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} \mathbf{x} = 2 \tag{6.1.11.1}$$

   which makes an angle of 60° with the major axis.
12. Find at what point the line

$$\begin{pmatrix} e & 1 \end{pmatrix} \mathbf{x} = a \tag{6.1.12.1}$$

   touches the ellipse

$$\mathbf{x}^T \begin{pmatrix} 1 - e^2 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = a^2 \left(1 - e^2\right) \tag{6.1.12.2}$$

13. Prove that the normal at an end of the latus rectum meets the major axis at a distance ae^3 from the centre.
14. Prove that the normal at an end L of the latus rectum meets the minor axis in g, and l is the projection of L on the minor axis, then

$$gl = a \tag{6.1.14.1}$$

15. Prove that the normal at $\begin{pmatrix} x \\ y \end{pmatrix}$ divides the major

axis into segments of lengths $a-e^2x$ and $a+e^2x$.

16. Prove that if the normal at $\mathbf{P}$ meets the major axis in $\mathbf{G}$, and the minor axis in $\mathbf{G}_1$, then
    a) $SG = eSP$;
    b) $PG : PG_1 = b^2 : a^2$

6.2 Pole and Polar

1. Find the polar of the point $\begin{pmatrix}5\\7\end{pmatrix}$ with regard to the ellipse
$$\mathbf{x}^T\begin{pmatrix}2 & 0\\0 & 3\end{pmatrix}\mathbf{x} = 6 \qquad (6.2.1.1)$$
2. Find the pole of the line
$$\begin{pmatrix}3 & 4\end{pmatrix}\mathbf{x} = 5 \qquad (6.2.2.1)$$
with regard to the ellipse
$$\mathbf{x}^T\begin{pmatrix}3 & 0\\0 & 4\end{pmatrix}\mathbf{x} = 5 \qquad (6.2.2.2)$$
3. Find the poles of the lines
$$\begin{pmatrix}2 & -1\end{pmatrix}\mathbf{x} = 1 \qquad (6.2.3.1)$$
$$\begin{pmatrix}1 & 3\end{pmatrix}\mathbf{x} = 4 \qquad (6.2.3.2)$$
with regard to the ellipse
$$\mathbf{x}^T\begin{pmatrix}1 & 0\\0 & 4\end{pmatrix}\mathbf{x} = 6, \qquad (6.2.3.3)$$
and verify that the line joining the poles is the polar of the point of intersection of the lines.
4. Prove that the points $\begin{pmatrix}4\\1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}$ are conjugate with regard to the ellipse
$$\mathbf{x}^T\begin{pmatrix}2 & 0\\0 & 3\end{pmatrix}\mathbf{x} = 5 \qquad (6.2.4.1)$$
5. Find the equatioin of a line through the point $\begin{pmatrix}3\\2\end{pmatrix}$ and conjugate to the line
$$\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 1 \qquad (6.2.5.1)$$
with regard to the ellipse
$$\mathbf{x}^T\begin{pmatrix}3 & 0\\0 & 4\end{pmatrix}\mathbf{x} = 2 \qquad (6.2.5.2)$$
6. Prove that, if the polar of a point is at right angles to the line joining the point to the centre of an ellipse, the point must lie on one of the axes.
7. Prove that, if $\begin{pmatrix}x_2\\y_2\end{pmatrix}$ is the pole of the normal at $\begin{pmatrix}x_1\\y_1\end{pmatrix}$, then
$$\frac{x_1x_2}{a^4} = -\frac{y_1y_2}{b^4} = \frac{1}{a^2-b^2} \qquad (6.2.7.1)$$
8. Prove that a directrix of an ellipse is the polar of the corresponding focus.

6.3 Eccentric Angles

The following problems refer to the ellipse whose equation is
$$\mathbf{x}^T\begin{pmatrix}b^2 & 0\\0 & a^2\end{pmatrix}\mathbf{x} = a^2b^2 \qquad (8)$$

1. Show that the eccentric angle of one end of a latus rectum is $\cos^{-1} e$.
2. If α is the eccentric angle of a point $\mathbf{P}$ on an ellipse, what are the coordinates of the corresponding point $\mathbf{Q}$ on the auxilary circle? Write down the equations of the tangents at $\mathbf{P}$ and $\mathbf{Q}$ and show that they intersect on the major axis.
3. Prove that, if CP, CD are conjugate radii of an ellipse and ω the angle between them, then $\sin^2\omega$ varies as $CP^{-2} + CD^{-2}$.
4. Prove that if a parallelogram is inscribed in an ellipse, its sides are parallel to the conjugate diameters.
5. Prove that if QQ_1 is a chord of an ellipse parallel to the tangent at $\mathbf{P}$ the eccentric angles of $\mathbf{Q}$ and $\mathbf{Q}_1$ differ from the eccentric angle at $\mathbf{P}$ by equal amounts.
6. Prove that the equation of the perpendicular bisector of the chord joining the points $\mathbf{Q}$, $\mathbf{Q}_1$ whose eccentric angles are $\alpha+\beta$, $\alpha-\beta$ is
$$\begin{pmatrix}a\sec\alpha & -b\ \operatorname{cosec}\ \alpha\end{pmatrix}\mathbf{x} = \begin{pmatrix}a^2-b^2\end{pmatrix}\cos\beta \qquad (6.3.6.1)$$
and deduce the equation of the normal at the point whose eccentric angle is α.
7. Prove that, if the same chord passes through a focus, then
$$\cos\beta = \pm e\cos\alpha \qquad (6.3.7.1)$$

8. Prove that the equation of a focal chord parallel to the tangent at the point whose eccentric angle is α is

$$\begin{pmatrix} \frac{\cos\alpha}{a} & \frac{\sin\alpha}{b} \end{pmatrix} \mathbf{x} = \pm e \cos\alpha \tag{6.3.8.1}$$

9. Prove that, if α is variable and β constant, the chord joining the points whose eccentric angles are $\alpha + \beta$ and $\alpha - \beta$ touches the ellipse

$$\mathbf{x}^T \begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix} \mathbf{x} = a^2 b^2 \cos^2\beta \tag{6.3.9.1}$$

and that the locus of the poles of the chord is

$$\mathbf{x}^T \begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix} \mathbf{x} = a^2 b^2 \sec^2\beta \tag{6.3.9.2}$$

10. Prove that the tangents at the points whose eccentric angles are α, $\alpha + \frac{\pi}{2}$ intersect on the ellipse

$$\mathbf{x}^T \begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix} \mathbf{x} = 2a^2 b^2 \tag{6.3.10.1}$$

11. Prove that, if **P**, **Q** are corresponding points on an ellipse and its auxiliary circle and the normals at **P**, **Q** intersect in R, then

$$CR = a + b \tag{6.3.11.1}$$

12. Prove that, if the line joining the ends of two equal conjugate radii of an ellipse passes through a focus the eccentricity is $\frac{1}{\sqrt{2}}$.
13. Prove that the chords that join the ends of conjugate radii all touch the ellipse

$$2\mathbf{x}^T \begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix} \mathbf{x} = a^2 b^2 \tag{6.3.13.1}$$

6.4 Miscellaneous

1. The following problems refer to the ellipse whose equation is

$$\mathbf{x}^T \begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix} \mathbf{x} = a^2 b^2 \tag{6.4.1.1}$$

and that C is its centre and S, S_1 its foci.
2. Prove that, if the tangent and normal at a point P on an ellipse meet the major axis in T, G, then the tangent from either end of the minor axis to the circle TPG is equal in length to half the major axis.
3. Show that if $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ are the coordinates of a point of intersection of the ellipses

$$\mathbf{x}^T \begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix} \mathbf{x} = a^2 b^2 \tag{6.4.3.1}$$

and

$$\mathbf{x}^T \begin{pmatrix} b_1^2 & 0 \\ 0 & a_1^2 \end{pmatrix} \mathbf{x} = a_1^2 b_1^2 \tag{6.4.3.2}$$

the equations of their common tangents are

$$\begin{pmatrix} \pm\frac{x_1}{aa_1} & \pm\frac{y_1}{bb_1} \end{pmatrix} \mathbf{x} = 1 \tag{6.4.3.3}$$

4. The normal at any point P of the ellipse meets the axis in G; a point Q is taken in the tangent so that $PQ = \lambda.PG$, where λ is constant; prove that the locus of Q is the ellipse

$$\mathbf{x}^T \begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix} \mathbf{x} = a^2 b^2 \frac{a^2 + \lambda b^2}{a^2} \tag{6.4.4.1}$$

5. Prove that the line

$$\begin{pmatrix} \frac{a}{k^2 - 1} & \frac{b}{2k} \end{pmatrix} \mathbf{x} + \frac{a^2 - b^2}{k^2 + 1} = 0 \tag{6.4.5.1}$$

is a normal to the ellipse

$$\mathbf{x}^T \begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix} \mathbf{x} = a^2 b^2 \tag{6.4.5.2}$$

for all values of k.
6. Prove that the foot of the focal perpendicular on the normal at any point of an ellipse is at a distance from the centre equal to the difference between the semi-major axis and the focal radius vector to the point at which the normal is drawn.
7. Prove that the locus of the poles of normal chords of the ellipse is the curve

$$\frac{a^6}{x^2} + \frac{b^6}{y^2} = \left(a^2 - b^2\right)^2. \tag{6.4.7.1}$$

8. P is a point $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ on the ellipse and PS, PS_1 meet the curve again in Q, R. Prove that the equation of QR is

$$\begin{pmatrix} \frac{x_1}{a^2} & \frac{y_1}{b^2}\frac{1+e^2}{1-e^2} \end{pmatrix} \mathbf{x} + 1 = 0, \tag{6.4.8.1}$$

where e is the eccentricity.
9. Show that two parallel tangents to an ellipse are met by any other tangent in points which lie on conjugate diameters.
10. Prove that if CP, CD be any two conjugate

semi-diameters of an ellipse and PF be drawn perpendicular to CD and produced both ways to E, E_1 so that $PE = PE_1 = CD$, then $CE.CE_1 = CS^2$, where S is a focus.

11. Tangents are drawn from points on the ellipse to the circle
$$\|\mathbf{x}\| = r \tag{6.4.11.1}$$
, show that the chords of contact touch the ellipse
$$\mathbf{x}^T \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix} \mathbf{x} = r^4 \tag{6.4.11.2}$$
12. Tangents TP, TQ_1 are drawn to an ellipse so that SP, S_1Q are parallel. Prove that CT is parallel to SP or S_1Q.
13. the normal to an ellipse at a point P passes through one end of the minor axis, and CD is the semidiameter conjugate to CP. The perpendicular from C to CD meets the auxiliary circle in E. Prove that DE is equal to half the distance betwen the directrices.
14. Prove that, if $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ be the middle point of a chord of the ellipse, the equation of the chord is
$$\begin{pmatrix} \frac{x_1}{a^2} & \frac{y_1}{b^2} \end{pmatrix} \mathbf{x} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} \tag{6.4.14.1}$$
15. If Q be the pole of the chord of the ellipse which is normal at a point P, and if CR drawn through the centre C perpendicular to CQ meet the normal at P in R, prove that the locus of R is
$$\mathbf{x}^T \begin{pmatrix} a^6 & 0 \\ 0 & b^6 \end{pmatrix} \mathbf{x} = a^4 b^4 \tag{6.4.15.1}$$
16. Prove that, if the point P lies on the ellipse
$$\mathbf{x}^T \begin{pmatrix} b_1^2 & 0 \\ 0 & a_1^2 \end{pmatrix} \mathbf{x} = a_1^2 b_1^2 \tag{6.4.16.1}$$
its polar with regard to the ellipse
$$\mathbf{x}^T \begin{pmatrix} b^2 & 0 \\ 0 & a^6 \end{pmatrix} \mathbf{x} = a^2 b^2 \tag{6.4.16.2}$$
touches the ellipse
$$\mathbf{x}^T \begin{pmatrix} a_1^2 b^4 & 0 \\ 0 & b_1^2 a^4 \end{pmatrix} \mathbf{x} = a^4 b^4 \tag{6.4.16.3}$$
17. Show that the polar with regard to the ellipse
$$\mathbf{x}^T \begin{pmatrix} b^2 & 0 \\ 0 & a^6 \end{pmatrix} \mathbf{x} = a^2 b^2 \tag{6.4.17.1}$$
of a point on the circle
$$\|\mathbf{x}\| = c \tag{6.4.17.2}$$
touches the ellipse
$$\mathbf{x}^T \begin{pmatrix} b^4 & 0 \\ 0 & a^4 \end{pmatrix} \mathbf{x} = \frac{a^4 b^4}{c^2} \tag{6.4.17.3}$$
18. Prove that, if the tangents to an ellipse at $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ meet at $\begin{pmatrix} x \\ y \end{pmatrix}$ and the normals at $\begin{pmatrix} \xi \\ \eta \end{pmatrix}$, then $a^2\xi = e^2 x x_1 x_2$ and $b^2\eta = -e^2 y y_1 y_2$, where e is the eccentricity.
19. Show that, if $\begin{pmatrix} x \\ y \end{pmatrix}$ is the middle point of a chord of the ellipse, and the tangents at the ends of the chord intersect in $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and the normals in $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$, then
$$\frac{a^2 x_2}{x_1} + \frac{b^2 y_2}{y_1} = \left(a^2 - b^2\right)\left(\frac{x x_1}{a^2} - \frac{y y_1}{b^2}\right). \tag{6.4.19.1}$$
20. Prove that, if the normals at two points P, Q on an ellipse intersect on the diameter that bisects PQ, then the two normals are at right angles.
21. Prove that if a chord of the ellipse subtends a right angle at the centre then it touches the circle
$$\|\mathbf{x}\| = \frac{ab}{\sqrt{a^2 + b^2}} \tag{6.4.21.1}$$
22. The locus of middle points of chords of the ellipse which subtend a right angle at its centre is
$$\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{a^2 + b^2}{a^2 b^2}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2 \tag{6.4.22.1}$$
23. Show that the tangents at the extremities of all chords of the ellipse which subtend a right angle at the centre, intersect on the ellipse
$$\mathbf{x}^T \begin{pmatrix} b^4 & 0 \\ 0 & a^4 \end{pmatrix} \mathbf{x} = a^2 b^2 \left(a^2 + b^2\right) \tag{6.4.23.1}$$

24. If P be any point on the ellipse whose axes are AA_1, BB_1 and if the parallel lines AP, BQ be drawn, Q being on the ellipse; Q will be one extremity of the diamter conjugate to that drawn from P.
25. From a any point T on one of the equiconjugate diamters of a conic whose centre is O, tangents TP, TQ are drawn to the conic. Show that O, P, Q, T are concyclic.
26. Prove that, if two conjugate radii of an ellipse cut the director circle in T, T_1 then TT_1 touches the ellipse.
27. If PSP_1, QSQ_1 be two focal chords and if PQ be parallel to the major axis, show that P_1Q_1 bisects the distance between S and the nearer directrix.
28. Tangents are drawn at the extremities of conjugate diameters of an ellipse, and meet in O. Prove that the perpendicular from O on the focal radius to a point of contact is half the minor axis.
29. Show that the area of the rectangle fromed by two parallel tangents and the corresponding normals to an ellipse is never greater than half the square on the line joining the foci.
30. Prove that the angle btween the normal and the central radius at a point on an ellipse is greatest when the point is the end of one of the equiconjugate diameters.
31. P, Q are two points on the ellipse, and PS, QS_1 intersect on the curve, prove that the locus of the pole of PQ is

$$\mathbf{x}^T \begin{pmatrix} \left(2a^2 - b^2\right)^2 & 0 \\ 0 & a^2b^2 \end{pmatrix} \mathbf{x} = a^2\left(2a^2 - b^2\right)^2 \tag{6.4.31.1}$$

32. Two conjugate diameters of an ellipse meet a fixed straight line

$$\begin{pmatrix} l & m \end{pmatrix} \mathbf{x} = 1 \tag{6.4.32.1}$$

in P, Q, and the straight lines through P, Q perpendicular to these diameters intersect in R; prove that the locus of R is the straight line

$$\begin{pmatrix} a^2l & b^2m \end{pmatrix} \mathbf{x} = a^2 + b^2 \tag{6.4.32.2}$$

33. Prove that if α, β are the eccentric angles of two points P, P_1 on an ellipse such that the focal distances SP, S_1P_1 are parallel, then

$$\tan\frac{\alpha}{2} : \tan\frac{\beta}{2} = 1 \pm e : 1 \mp e, \tag{6.4.33.1}$$

and PP_1 touches the ellipse

$$\mathbf{x}^T \begin{pmatrix} b^4 & 0 \\ 0 & a^4 \end{pmatrix} \mathbf{x} = a^2b^4 \tag{6.4.33.2}$$

34. Prove that the square of the sum of two conjugate radii is greatest when the radii are equal.
35. Prove that, if on the inward normal to an ellipse at P a length PQ be taken equal to the conjugate radius CD, the locus of Q is a circle of radius $a - b$.
36. Prove that, if P, Q are corresponding points on an ellipse and its auxiliary circle, and the normal at P to the ellipse meets the normal at Q to the circle in R, then the locus of R is a circle of radius $a + b$.
37. Prove that, if lines drawn from any poin on an ellipse to the ends of a diameter PCP_1 meet the conjugate diameter DCD_1 in M, M_1, then $CM.CM_1 = CD^2$.
38. Prove that, if an ellipse slides between two straight lines at right angles to one another, the locus of its centre is a circle.

6.5 Hyperbola

1. Show that the eccentricity of a rectangular hyperbola is $\sqrt{2}$
2. Prove that the locus of the centre of a circle which touches two given non-concentric circles of unequal radii is an ellipse or a hyperbola.
3. ABC is a triangle in which C is a right angle. On CA, CB points P, Q are taken such that $CP.CQ = CA.CB$. Prove that the locus of the centroid of the trianlge CPQ is a rectangular hyperbola.
4. Show that the tangents to a rectangular hyperbola at the extremities of its latera recta pass through the vertices of the conjugate hyperbola.
5. P is a point on a rectangular hyperbola whose centre is C and a line is drawn through C perpendicular to CP. Through Q, any point on the curve, lines are drawn parallel to the asymptotes meeting this line in L, M, show that LPM is a right angle.
6. Prove that the intercept on any tangent to a hyperbola, made by its asymptotes, subtends a constant angle at either focus.

7. Prove that the line $lx + my = n$ touches the rectangular hyperbola $xy = c^2$, if
$$n^2 = 4lmc^2$$
8. A tangent to a hyperbola of foci S, S_1 meets the asymptotes in L, L_1. Prove that the points S, L, S_1, L_1 are concyclic.
9. The tangents at two points P, P_1 on a rectangular hyperbola meet an asymptote in L, L_1 and P, P_1 meets it in K. Prove that
$$LK = KL_1$$
10. Prove that conjugate diameters of a rectangular hyperbola are equally inclined to the asymptotes.
11. Prove that the polars of a point with regard to two conjugate hyperbolas are parallel and equidistant from the centre.
12. Prove that in a rectangular hyperbola any chord PP_1 subtends at the ends of a diameter AA_1 angles which are either equal or suplementary.
13. Prove that if CP, CD are conjugate radii of a hyperbola the orthocentre of the triangle CPD lies on the line $ax = by$.
14. Prove that, if the tangent at P to a rectanglular hyperbola meets the asymptotes in L, L_1 and the normal at P meets the transverse axis in G, then LGL_1 is a right angle.
15. Prove that if an ellipse and a hyperbola have the same foci they cut one another at right angles.
16. The perpendiculars from a point P to the axes meet them in M, N, and the perpendicular bisector of MN passes through a fixed point C on one of the axes. Prove that the locus of P is a rectangular hyperbola with centre at C.
17. Show that the locus of the middle point of a chord of the rectangular hyperbola $xy = c^2$ of constant length $2a$ is
$$\left(xy - c^2\right)\left(x^2 + y^2\right) = a^2xy$$
18. Prove that, if the position of a point on a rectangular hyperbola is determined by the variable θ where $x = c\tan\theta$, $y = c\cot\theta$, the locus of the intersection of tangents at the points θ, $\theta + \alpha$ is
$$4\left(c^2 - xy\right) = (x + y)^2 \tan^2\alpha$$
α being a constant angle.
19. Tangents at right angles are drawn to a rectangular hyperbola and its conjugate. Show that they cut either asymptote in two points K, K_1 such that the rectangle $CK.CK_1$ is equal to twice the square on the semi-axis, where C is the centre.
20. Prove that the line joining the feet of the perpendiculars drawn to a pair of conjugate diameters of a rectangular hyperbola from any point P on the hyperbola is parallel to the normal at P.
21. The normal at P_1 on the hyperbola $xy - c^2 = 0$ meets the curve again at P_2, the normal at P_2 meets the curve again at P_3 and so on. Prove that if $y_1, y_2, \ldots, y_{n+1}$ are the ordinates of these points respectively,
$$y_1^2y_2 = y_2^2y_3 = \cdots = y_n^2y_{n+1} = c^4.$$
22. If PN be the ordinate and PG the normal of a point P of a hyperbola, whose centre is C, and the tangent at P intersect the asymptotes at L and L_1, show that half the sum of CL and CL_1 is the mean proportional between CN and CG.
23. At the point of intersection of the rectangular hyperbola $xy = k^2$, and of the parabola $y^2 = 4ax$, the tangents to the hyperbola and parabola make angles θ and ϕ respectively with the axis of x. Prove that
$$\tan\theta = -2\tan\phi.$$
24. Prove that, if A, B, C are three points on a rectangular hyperbola, the curve passes through the orthocentre of the triangle ABC.
25. Prove that, in a rectangular hyperbola, the product of the focal distances of a point is equal to the square of the distance of the point from the centre.
26. Prove that, if $(c\tan\theta, c\cot\theta)$ and $(c\tan\theta_1, c\cot\theta_1)$ are two points on the hyperbola $xy = c^2$, and $\theta + \theta_1$ is constant, then the chord joining the points passes through a fixed point on the conjugate axis of the hyperbola.
27. Show that the point whose coordinates are
$$\frac{a}{2}\left(t + \frac{1}{t}\right), \frac{b}{2}\left(t - \frac{1}{t}\right)$$
lies on the hyperbola $\frac{x^2}{a} - \frac{y^2}{b^2} = 1$. Prove that, if C is the centre of the hyperbola and S is either focus and if the tangent at the above point meets the asymptote $\frac{x}{a} = \frac{y}{b}$ at X and meets

the asymptote $\frac{x}{a} = -\frac{y}{b}$ at Y, then

$$t = \frac{CX}{CS} = \frac{CS}{CY}$$

28. From any point on the normal at a given point A on a rectangular hyperbola the other three normals to the curve are drawn. Sow that the centroid of the feet of these three normals lies on the diameter of the hyperbola parallel to the normal at A.
29. A circle is drawn passing through any point P on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and through the ends A, A_1 of the transverse axis. The ordinate NP is produced to meet the circle again in Q. Prove that, as the position of P on the hyperbola varies, the locus of Q is the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b_1^2} = 1$, where $b_1 = \frac{a^2}{b}$.

7 Polar Coordinates

1. Find the polar equations of the curves whose cartesian equations are:
   a) $x^2 + y^2 + 2gx + 2hy + c = 0$
   b) $x^2 + y^2 = ax$
   c) $x^2 - y^2 = c^2$
   d) $xy = \frac{c^2}{2}$
   e) $x^3 - 8xy^2 = a^3$.
2. Find the Cartesian equations corresponding to the following equations in polar coordinates:
   a) $r \sin\theta + a = 0$
   b) $r = a\sin\theta + b\cos\theta$
   c) $r^2 \cos 2\theta = a^2$
   d) $\frac{1}{r} = 1 + e\cos\theta$
   e) $r^3 \sin 3\theta = a^3$
   f) $r^2 = a^2 + b^2 \cos 2\theta$
3. Using the polar equation of a conic with focus as pole, show that the semi-latus rectum is a harmonic mean between the segments of a focal chord.
4. Draw the curve $r = a\cos 2\theta$, noting carefully when r changes sign and showing that the curve forms a double figure of eight.
5. Draw the curve
$$r = a(2 + \cos\theta)$$
6. Draw the curve $r = a(1 + 2\cos\theta)$, showing that it consists of two loops one within the other.
7. Deduce from the polar equations of the curves the pedal equations for the circle, parabola with focus as pole, ellipse with focus as pole and ellipse with centre as pole.
8. Find the pedal equation of:
   a) The rectangular hyperbola $xy = c^2$
   b) the lemniscate $r^2 = a^2 \cos 2\theta$
   c) the curve $r^n = a^n \cos n\theta$
   d) the spiral $r\theta = a$.
9. Prove that, for the curve $r\theta = a$, if a perpendicular through the pole be drawn to the radius vector at any point, the length intercepted on it by the tangent is constant.
10. Prove that, for the curve $r = a\cos(\theta - \alpha)$, $\phi = \frac{\pi}{2} + \theta - \alpha$.
11. Show that, for the curve $r = a(1 + \cos\theta)$, the locus of the foot of the perpendicular from the pole to the tangent is $r = 2a\cos^3\frac{\theta}{3}$.
12. Prove that, for the curve $r = a\sec^3\frac{\theta}{3}$, the locus of the foot of the perpendicular from the pole to the tangent is a parabola.
13. Trace the curve $r^2 = a^2\cos\theta$.
14. Prove that the pedal equation of the curve $r\cos n\theta = a$ is
$$\frac{1}{p^2} = \frac{1 - n^2}{r^2} + \frac{n^2}{a^2}$$

8 Easy Problems

8.1 The Straight Line

1. Find the distance between the pionts A, B whose coordinates are $(3, 7)$ and $(11, 13)$. Also find the coordinates of the point which divides AB in the ratio $3 : 1$.
2. Find the equations of the sides of a triangle whose corners are at the points $(3, 4)$, $(-3, 2)$, $(2, -4)$.
3. In the last problem, find the equations of the perpendiculars from the corners of the triangle to the opposite sides and verify that they meet in a point.
4. Find the distance between the points $A(2, 3)$ and $B(14, 8)$. Find also the equation of the line AB and the distance from it of the point $C(5, 6)$. What is the area of the triangle ABC?
5. The equations of the sides of a triangle are $x + y = 5$, $2x - y + 4 = 0$ and $x - 4y + 4 = 0$. Find the tangents of the angles of the triangle.
6. Find the equation of a line parallel to the line $5x + 4y = 9$ and making an intercept -5 on the x-axis.

7. Find the equations of two straight lines which pass through the intersection of the lines $3x + 4y = 7$, $5x - 2y = 3$ and are parallel and perpendicular respectively to the line $2x + 7y = 3$.
8. Find the value of k if the three lines $2x + 5y = 12$, $7x + 2y = 11$ and $kx - 3y = 10$ meet in a point.
9. Prove that the points (x_1, y_1), (x_2, y_2) and
$$\left(\frac{x_1 + kx_2}{1 + k}, \frac{y_1 + ky_2}{1 + k}\right)$$
are collinear.
10. Show that the points $(2, 3)$ and $(-1, 2)$ are on opposite sides of the line $2x - 3y = 7$. What are their distances from the line?
11. Two straight lines AB, CD bisect one another at right angles. The coordinates of A, B and C are $(1, 5)$, $(3, 1)$ and $\left(-1, 1\frac{1}{2}\right)$. Find the coordinates of D and the area $ABCD$.
12. Prove that the line $2x + 3y = 5$ divides the join of the points $(3, -5)$, $(2, 1)$ in the ratio $7 : 1$.
13. Find the equations of the two straight lines which pass through the point $(3, 2)$ and make angles of $45°$ with the line $4x - 5y = 6$.
14. Find the equation of the perpendicular bisector of the join of the points $(3, 5)$, $(-6, 7)$.
15. From a point $A(2, 5)$ a perpendicular AB is drawn to the line $3x - 4y = 8$. AB is then produced to C so that $BC = 3AB$. Find the coordinates of C.
16. The sides BC, CA, AB of a triangle have equations $3x + 4y = 9$, $4x - 3y + 6 = 0$, $5x - 12y + 12 = 0$. Find the equations of the bisectors of the interior angles of the triangle and the coordinates of their point of concurrence.
17. ABC and DBC are two triangles such that $DB = AB$ and $DC = AC$. The coordinates of A, B, C are $(3, 1)$, $(2, 2)$, $(-1, 3)$. Find the coordinates of D. Also find the equations of AD and BC and verify that these lines are at right angles.
18. Find the centre of the inscribed circle of the triangle formed by the lines $3x - 4y + 8 = 0$, $4x + 3y - 9 = 0$, $y = 6$.
19. Find the equations of two lines passing through the intersection of the lines $x + 3y = 5$ and $4x - y + 2 = 0$, and perpendicular and parallel respectively to the line $5x - 3y + 3 = 0$.
20. Find the equation of the line which passes through the intersection of the lines $2x + 3y = 4$ and $3x - y + 2 = 0$ and also through the intersection of the lines $x + y = 0$, $4x - y + 3 = 0$.
21. Prove that the points $A(4, 2)$ and $C(-3, 1)$ subtend a right angle at $B\left(-\frac{17}{13}, -\frac{20}{13}\right)$. Find the coordinates of D, the remaining corner of the rectangle $ABCD$, and verify that $DB = AC$.
22. The points $A(4, 2)$, $B(1, -2)$, $C(-3, 1)$ and D are the corners of a parallelogram. Find the coordinates of D and prove that the parallelogram is equal in area to the square on OD where O is the origin.
23. Show that the equation $6x^2 + 7xy - 3y^2 + x + 7y - 2 = 0$ represents two straight lines, and find the tangents of the angles between then.
24. Find the separate equations of the two straight lines represented by the equation $x^2 - 2xy \operatorname{cosec}\theta + y^2 = 0$. Also find the angles between the lines, and show that these angles are bisected by the lines $x^2 - y^2 = 0$.

8.2 *Curves and Loci*

1. Find the equations of the tangents to the curves $y = x^3 - x$ and $x^2 - 3x + 2$ at their point of intersection, and the angle at which the curves cut.
2. Prove that the line $3y = 9x - 16$ is a tangent to the curve $3y = x^3 - 3x$, and find the coordinates of the point of contact.
3. Prove that the tangents to the curve $y = x^3 - 6x^2 + 13x - 6$ at the points $(1, 2)$ and $(3, 6)$ are parallel, and that the tangent at the point $(2, 4)$ is inclined to them at an angle $\tan^{-1}\frac{3}{5}$.
4. Prove that the line $y = 5x$ is a tangent to the curve
$$y = x^3 - 2x^2 + x + 8$$
and find the coordinates of the point of contact. In what other point does the line $y = 5x$ meet the curve?
5. Find the equation of the tangent to the curve $y = 3x^2 - 2x - 1$ at the point $(2, 7)$. Also find the coordinates of the point on the curve the tangent at which is perpendicular to the tangent at $(2, 7)$.
6. Find the equation of the normal to the curve $y = 2x^2 - 3x - 5$ at the point $(2, -3)$, and find the length of the subnormal at this point.

7. Find the equation of the tangent to the curve
$$3y = x^3 - 6x^2 - 12x + 1$$
at the point $(-1, 2)$, and find the coordinates of another poin on the curve where the tangent is parallel to that at $(-1, 2)$.
8. Prove that at every point of the curve $y = x^3 - 3x^2 + 3x + 7$ the gradient is positive. Table the values of y and $\frac{dy}{dx}$ at the points $x = -1, 0, 1, 2, 3$ and make a rought drawing of the curve.
9. Find the equation of the tangent and normal to the curve $y = (x-1)^4$ at the point $(2, 1)$. Find the area of the triangle which the tangent and normal make with the x-axis.
10. Show that the curve $y = 1 - 2x + x^3$ cuts the x-axis in three real points. Find the gradient of the curve at the point $(1, 0)$; also find the points at which the gradient vanishes and make a rough drawing of the curve.
11. Find the gradients of the curve $y = (x-1)^2(x-2)$ at the points where it crosses the axes. Find also the points at which the gradient vanishes and make a rough drawing of the curve.
12. Find the points on the curve $y = x^3 - x^2 + 1$ at which the gradient is unity. Find the equations of the tangents and normals at these points.
13. Find the points in which the curve $y = 1 + 2x - 3x^2$ cuts the coordinate axes and find the gradients of the curve at these points. At what point does the gradient vanish? Make a rough drawing of the curve.
14. Show that the curve $y = \left(x^2 - 1\right)^2$ is symmetrical about the y-axis. Find its gradient and the points at which the gradient vanishes. In what points does the line $y = 9$ cut the curve? Make a rough drawing of the curve.
15. Show that the curve $y = x^4 - 4x^2$ is symmetrical about the y-axis. Find the gradients of the curve at its intersections with the x-axis. find also the points where the gradient vanishes and make a rough drawing of the curve.
16. Show that the curve $y = x^4 - 4x^2$ is symmetrical about the y-axis. Find the gradients of the curve at its intersections with the x-axis. Find also the points where the gradient vanishes and make a rough drawing of the curve.
17. Find the gradient of the curve $y = x^2 - 3x - 4$, and the point on the curve at which the gradient vanishes. Also find the gradient at the points where the curve crosses the x-axis and make a rough drawing of the curve. To what point on the x-axis must the origin be moved so that the curve may become symmetrical about the y-axis, and what will be the equation of the curve when the position of the origin is so changed?
18. Show that the curve whose equation is $27y = x^3 - 27x$ is symmetrical in opposite quadrants. Prove that the tangent to the curve at the origin is $x + y = 0$. Find the gradient at the other intersections with the x-axis. Also find at what points the tangent is parallel to the x-axis, and make a rough drawing of the curve.
19. Find the equation of the locus of a point which moves so that its distance from the point $(2, 1)$ is twice its distance from the line $3x + 4y = 2$.
20. Find the locus of a point which moves so that its distance from the origin is half the distance from the line $3x + 4y = 2$.
21. A point P moves so that $PA^2 + PB^2 = AB^2$, where A, B are the points $(2, 0)$, $(0, 3)$. Find the equation of the locus of P.
22. Find the locus of a point whose distance from the point $A(3, 0)$ is three times its distance from the point $B(-1, 0)$. Prove that the tangents to the locus at the points $(0, 0)$ and $(-3, 0)$ are parallel to the y-axis.
23. The coordiantes of a point are given in terms of a variable parameter t by the relations $x = \frac{t}{t+1}, y = \frac{t}{t-1}$. Find the equation of the locus of the point, and prove that the equation of the tangent at any point t on the locus is $(t+1)^2 x + (t-1)^2 y = 2t^2$.
24. The coordinates of a point are given by the relations $x = t + \frac{1}{t}$, $y = t - \frac{1}{t}$ and t is a variable parameter. Find the equation of the locus of the point for different values of t, and also the equations of the tangent at the point denoted by t.

8.3 The Circle

1. Prove that the circle of centre $(3, 4)$ and rdius 5 passes through the origin. Find the equations of the tangent at the origin and of the parallel tangent. IN what other points does the circle cut the axes?
2. find the equation of a circle having the points $(2, 1)$ and $(1, -3)$ for the ends of a diameter.

find the coordinates of the ends of the perpendicular diameter.

3. Find the equation of the tangent to the circle $x^2 + y^2 = 1$ at the point $\left(\frac{4}{5}, \frac{3}{5}\right)$. Find the two points in which this tangent cuts the axes and also find the equations of the other tangents to the circle that can be drawn from the last two points.
4. For what values of k does the line $y = kx$ touch the circle $x^2 + y^2 + 2x + 4y + 1 = 0$? find the coodinates of the points of contact.
5. Show that the points $(5, 8)$ and $(-1, 0)$ are the opposite ends of a diameter of the circle $x^2 + y^2 - 4x - 8y - 5 = 0$. Find the equations of the tangents at these points.
6. Find the equation of the circle through the points $(3, 0)$, $(4, 0)$ and $(1, 2)$. Find the equation of the tangent at the point $(3, 13)$.
7. Find the equation of the circle which passes through the points $(3, 0)$, $(0, 4)$, $(0, 6)$. In what other points does the circle cut the x-axis?
8. Find where the line $x + y = 5$ cuts the circle
$$3x^2 + 3y^2 - 11x - 11y + 16 = 0? \qquad (8.3.8.1)$$
What are the equations of the tangents parallel to the line?
9. Find the coordinaes of the points in which the line $x + 2y = 7$ cuts the circle $x^2 + y^2 - 13x - 13y + 52 = 0$. Find the equations of the tangents at these points and the coordinates of the point in which these tangents intersect.
10. Prove that if (x_1, y_1) is the middle point of a chord of a circle $x^2 + y^2 = a^2$, then the equation of the chord is $xx_1 + yy_1 = x_1^2 + y_1^2$.
11. Find the coordinates of the middle point of the chord of the circle $x^2 + y^2 = 25$ which lies along the line $3x - 4y = 7$. Also find the points of contact of the parallel tangents.
12. Find the coordinates of the middle point of the chord of the circle $x^2 + y^2 - 4x + 6y + 1 = 0$ which lies along the line $2x - 3y = 12$.
13. Prove that the line $2x + y - 7 = 0$ is the equation of a diameter of the circle $x^2 + y^2 - 6x - 2y + 5 = 0$, and find the equations of the tangents at the ends of this diameter.
14. Find the angle which the circle $x^2 + y^2 - 6x - 4y + 9 = 0$ subtends at the origin.
15. Find the equations of the circles which pass through the point $(-2, 1)$ and touch both the coordinate axes.
16. Prove that the locus of points at which the circle
$$x^2 + y^2 - 4x - 6y + 4 = 0 \qquad (8.3.16.1)$$
subtends a right angle is the circle $x^2 + y^2 - 4x - 6y - 5 = 0$.
17. Find the equations of the tangents to the circle
$$x^2 + y^2 - 4x + 2y - 8 = 0 \qquad (8.3.17.1)$$
which are parallel to the line $2x + 3y = c$.
18. Find the equation of the chord of the circle
$$x^2 + y^2 - 4x + 2y - 4 = 0 \qquad (8.3.18.1)$$
which has $(3, 1)$ as its middle point.
19. Show that the points $(0, 0)$, $(2, 1)$, $(3, 3)$ and $(1, 2)$ are the corners of a rhombus, and find the equation of its inscribed circle.
20. Find the equation of the circle which has the same centre as the circle $x^2 + y^2 + 6x - 10y + 18 = 0$ and passes through the point $(6, 7)$. Show that the origin is outside the given circle and inside the other.
21. Find the locus of the centre of a circle
$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad (8.3.21.1)$$
which is such that the tangets drawn to the circle from the origin are always at right angles.

8.4 The Parabola

1. Find the value of c if the parabola $y^2 = 4ax$ intercepts a length $4a$ on the line $y = x + c$.
2. What is the equation of a parabola which is symmetrical about the x-axis, touches the y-axis at the origin and has a latus rectum of length 8? What are the coordinates of the focus? What are the coordinates of a point on the curve at a distance 20 from the focus?
3. The parabola $y^2 = 4ax$ passes through the point $P(6, 6)$. If S is the focus find the coordinates of the other point in which PS meets the curve.
4. Find the value of a if the parabola $y^2 = 4ax$ touches the line $y = 3x + 4$. What are the coordinates of the point of contact?
5. Prove that the line $ty = x + at^2$ touches the parabola $y^2 = 4ax$, and find the coordinates of the point of contact. Also find the coordinates of the foot of the perpendicular to this tangent from the focus of the parabola.

6. Prove that, if P is a point on a parabola $y^2 = 4ax$ whose vertex is A, and PL at right angles to AP meets the x-axis in L, and PN is the ordinate of P, then $NL = 4a$.
7. PN is the ordinate at a point P on a parabola and the normal at P meets the axis in G. Prove that PG is equal to the ordinate that passes through the middle point of NG.
8. Find the locus of a point P which moves so that its distance from the point $(1, 0)$ is equal to its distance from the line $x + 1 = 0$. Also find the coordinates of the middle point of the chord of this locus which lies along the line $3y = 2x + 4$.
9. The normal at P to a parabola meets the axis in G and S is the focus. Prove that, if the triangle SPG is equilateral, then SP is equal to the latus rectum.
10. A is the vertex of a parabola $y^2 = 4ax$ and LL_1 is the latus rectum. Prove that the diameter of the circle LAL_1 is $5a$.
11. The chord joining the points (x_1, y_1), (x_2, y_2) on the parabola $y^2 = 4ax$ cuts the axis at C. Prove that, if A is the vertex, $x_1x_2 = AC^2$.
12. C is a fixed point $(0, c)$ on the axis Oy and Q is a variable point on the line through C parallel to Ox. A point P is taken on OQ so that the ordinate of P is equal to CQ. Prove that the locus of P is the parabola $y^2 = cx$.
13. the chord PQ of a parabola passes through the focus. If P is the point $\left(at^2, 2at\right)$, what are the coordinates of Q?
14. The chord PQ of a parabola is the normal at P. If P is the point $\left(at^2, 2at\right)$, what are the coordinates of Q?
15. Find the value of a if the parabola $y^2 = 4ax$ touches the line $3y = x + 8$. Find the equation of the normal at the point of contact, and the coordinates of the second point of intersection of the normal and the curve.
16. Find the equations of the tangents to the parabola $y^2 = 16x$ at the points $(36, 24)$, $\left(\frac{4}{9}, -\frac{8}{3}\right)$ and verify that they intersect on the directrix.
17. Prove that the line $x - 6y + 36 = 0$ touches the parabola $y^2 = 4x$ and find the coordinates of the point of contact. Find also the coordinates of the foot of the perpendicular drawn to this tangent from the focus of the parabola.
18. Prove that the lines $wy = 6x + 1$ and $8y = 32x + 3$ both touch the parabola $y^2 = 6x$, and find the equation of the line joining the points of contact.
19. Find the value of c if the parabola $y^2 = 8x$ intercepts a length 8 on the line $y = x + c$.
20. Points on a parabola are represented parametrically by the relations $x = at^2$, $y = 2at$, where t is a variable parameter. Normals are drawn at the points $t = 2$ and $t = 1$. Prove that they intersect on the curve.
21. Prove that, if the points (x_2, y_1), x_2, y_2 are the ends of a chord passing through the focus of a parabola $y^2 = 4ax$, then $x_1x_2 = a^2$ and $y_1y_2 = -4a^2$.
22. From the focus S of a parabola a line is drawn parallel to the tangent at $P\left(at^2, 2at\right)$ meeting the line $y = 2at$ in Q. Prove that the locus of Q is the parabola $y^2 = 2a(x - a)$.
23. Prove that the tangents and normal to a parabola at the points $\left(at^2, 2at\right)$, $\left(\frac{a}{t^2}, \frac{-2a}{t}\right)$ enclose a rectangle of area $a^2\left(\frac{t+1}{t^2}\right)$.

8.5 The Ellipse

1. An ellipse has its foci at the points $(2, 0)$, $(-2, 0)$ and passes through the point $(2, 3)$. Find its equation.
2. Prove that the line $x+2y = 8$ touches the ellipse $3x^2 + 4y^2 = 48$, and find the coordinates of the point of contact.
3. Find the equation of the ellipse which toches the line $2x + 3y = 9$ and has the points $(-2, 0)$, $(2, 0)$ as its foci. Find also the coordinates of the point of contact of the line and the ellipse.
4. Show that the distance of the point $(a\cos\theta, b\sin\theta)$ from the focus $(ae, 0)$ of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $a(1 - e\cos\theta)$.
5. Prove that if the normals at $P(6, 4)$ and $Q(-8, 3)$ on the ellipse $\frac{x^2}{100} + \frac{y^2}{25} = 1$ meet at G, then the diameter through G is perpendicular to PA.
6. Find the equation of an ellipse of eccentricity $\frac{1}{2}$ which has a focus at $(3, 0)$ and $x = 1$ for corresponding directrix.
7. Find the equations of the tangents to the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ which are parallel to the line $x = 2y$.
8. Find the equation of the ellipse of eccentricity $\frac{1}{2}$ which has its foci at the points $(-1, 0)$, $(1, 0)$. Find also the length of the latus rectum and

verify that the tangent at either end of the latus rectum cuts the major axis on the directrix.

9. Find the equation of an ellipse which has the point $(2, 3)$ as an end of a latus rectum and its axes along the coordinate axes. At what point does the line $x - 2y = 8$ touch the ellipse?
10. find the equation of an ellipse of eccentricity 0.8 which has its centre at the origin and the lines $x = \pm 25$ as directrices. Verify that the ellipse touches the line $9x + 20y = 300$.
11. The axes of an ellipse are the coordinate axes, its directrices pass through the points $\left(\pm, 5\frac{1}{3}, 0\right)$ and it touches the line $3x + 4y = 16$. Find its equation.
12. The major and minor axes of an ellipse lie along the lines $3x-4y+6 = 0$ and $4x+3y-17 = 0$ and the lengths of the semi-axes are 5 and 4. Find the eccentricity and the coordinates of the centre and foci.
13. Find the equation of an ellipse which has its axes along the coordinate axes and the line $3x-2y = 5$ as the normal at the point $(15, 20)$.
14. Prove that, if the tangent at an end of the minor axis of an ellipse cuts the latus rectum produced in D, and C is the centre, then a perpendicular to CD through D cuts the major axis on the directrix.
15. Prove that the tangent at the ends of the latera recta of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ form a quadrilateal of area $\frac{2a^2}{e}$, where e is the eccentricity.
16. Prove that, if a series of ellipses have the same major axis, the tangents at the ends of the latera recta pass through one or other of the two fixed points on the minor axis.
17. Find the equations of the tangents to the ellipse $4x^2 + 9y^2 = 180$ at the points $P(6, 2)$ and $P_1(-6, -2)$. Find also the equations of the tangents that are parallel to the line PP_1, and the coordinates of their points of contact.
18. Prove that the line $x+3y = 9$ touches the ellipse $\frac{x^2}{9}+\frac{y^2}{8} = 1$, and find the coordinates of the point of contact. Find the coordinates of the foci of the ellipse, and verify that the product of the distances of the foci from the above tangent is equal to the square on the minor axis.
19. An ellipse has its foci at the point $(\pm 3, 0)$ and passes through the point $P\left(2, 2\sqrt{6}\right)$. Prove that its eccentricity is $\frac{1}{2}$ and that the normal at P passes through the point $\left(\frac{1}{2}, 0\right)$.
20. An ellipse has a focus at the point $(3, 0)$, the y-axis is the corresponding directrix and the point $(6, 4)$ lies on the curve. Prove that the axes are in the ratio $6 : \sqrt{11}$.
21. Prove that, if the line $lx+my+n = 0$ is a normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then $\frac{a^2}{l^2} + \frac{b^2}{m^2} = \frac{\left(a^2-b^2\right)}{n^2}$.
22. Find the coordinates of the points on the ellipse $8x^2 + 25y^2 = 200$ at which the normals make angles of 60° with the major axis.
23. Find the values of c for which the line $5x-2y = c$ is normal to the ellipse $x^2 = 5y^2 = 9$.
24. Find the coodrinates of the four points on the ellipse

$$9x^2 + 16y^2 = 1 \tag{8.5.24.1}$$

the tangents at which are equally inclined to the coordinate axes; and prove that the normals at these points form a square of area $\frac{49}{1800}$.

25. Find the equation of the normal at the point $(2, 3)$ on the ellipse $3x^2 + 4y^2 = 48$ and the coordinates of the point in which the normal again cuts the curve. Show that the middle point of this normal chord is at a distance $\frac{\sqrt{73}}{19}$ from the centre of the ellipse.
26. Find the equation of an ellipse of eccentricity $\frac{1}{3}$ which touches the line $2x + 3y = 5$ and has its axes along the coordinate axes. find the coordinates of the point of contact of the ellipse with the given line.
27. Find the equations of the two ellipses which have their axes along the coordinate axes, pass through the point $(2, 1)$ and touch the line $6x+12y = 25$.
28. Prove that, if an ordinate NP to an ellipse is produced to meet the tangent at the end of the latus rectum in Q, then $QN = SP$, where S is the corresponding focus.
29. The tangent is drawn at the point $P(2, 1)$ on the ellipse $4x^2 + 9y^2 = 25$ whose centre is O, and the diameter DOD_1 of the ellipse is parallel to the tangent at P. Find the coordinates of D and D_1 and prove that the tangents at these points are parallel to the radius OP.
30. The tangent at P to an ellipse meets a directrix in T and S is the corresponding focus. Prove that PST is a right angle.
31. The foci of an ellipse are the points $(0, 0)$ and $(8, 6)$ and the eccentricity is $\frac{4}{5}$. Find the coordinates of the centre and teh equations and

lengths of the major and minor axes. Find also the equations of the directrices.

32. Show that, if the normal at a point ¶$brakx_1, y_1$ on an ellipse of focus S and eccentricity e meets the major axis in G and GL is the perpendicular to SP, then $GL = ey_1$ and $PL =$ the semi-latus rectum.
33. P denotes any point on an ellipse of which the major axis is AA_1. Prove that, if AP, A_1P cut the minor axis in M, M_1, then the tangent at P bisects MM_1.

8.6 The Hyperbola

1. Prove that, if a variable line moves with its ends on the coordinate axes so as to enclose with them a constant area, then the locus of the middle point of the line is a rectangular hyperbola.
2. Prove that, if a straight line moves with its ends on the coordinate axes so as to form with them a triangle of constant area c^2, then the line touches the rectangular hyperbola $2xy = c^2$.
3. Find the value of c^2 if the rectangular hyperbola $xy = c^2$ touches the line $3x + 5y = 80$.
4. Show that the equation of the tangent at (x_1, y_1) on the rectangular hyperbola $xy = c^2$ may be put in the form $\frac{x}{x_1} + \frac{y}{1} = 2$.
5. Prove that the line $3x + 4y = 24c$ touches the hyperbola $xy = 12c^2$. What is the area of the triangle whose sides are the asymptotes and this tangent?
6. Find the points in which the line $2x + y = 3c$ cuts the hyperbola $xy = c^2$, showing that one of the points of intersecton lies on the transverse axis of the hyperbola.
7. Prove that, if a and b are real numbers of opposite signs, the straight line $ax + by = 1$ cannot touch the rectangular hyperbola $xy = c^2$.
8. Prove that the straight line $x + t^2y = 2ct$ is a tangent to the rectangular hyperbola $xy = c^2$, and that no perpendicular line can touch the curve.
9. Prove that, if through any point on the curve $xy = c^2$ two perpendicular lines are drawn, their other intersections with the curve will lie on opposite brances.
10. The normal at $P(3, 4)$ on the rectangular hyperbola $xy = 12$ meets the curve again at Q. Prove that $PQ = \frac{125}{12}$.
11. Find the equations of the tangent and normal at the point $x = 4ct$, $y = \frac{3c}{t}$ on the rectangular hyperbola $xy = 12c^2$. Verify that the length of the tangent intercepted by the asymptotes is bisected at the point of contact.
12. Find the equation of the tangent at the point $x = 2ct$, $y = \frac{5c}{t}$ on the hyperbola $xy = 10c^2$, and show that the area of the triangle the tangent forms with the asymptotes is independent of t.
13. Find the equation of the chord joining the points $\left(ct_1, \frac{c}{t_1}\right)$ and $\left(ct_2 \frac{c}{t_2}\right)$ on the rectangular hyperbola $xy = c^2$. Find the equation of the tangent parallel to the chord, and the coordinates of its point of contact.
14. Show that the hyperbola $x^2 - y^2 = 20$ and $xy = 24$ cut at right angles at all the common points.
15. If the line $2x - ky = 8$ touches a rectangular hyperbola $xy = 9$, what is the value of k, and what are the coordinates of the point of contact?
16. Find the coordinates of the foci and the equations of the directrices of the hyperbola $xy = c^2$, referred to the asymptotes as axes.
17. A circle cuts the rectangular hyperbola $xy = 1$ in the points (x_r, y_r), $r = 1, 2, 3, 4$. Prove that $x_1x_2x_3x_4 = y_1y_2y_3y_4 = 1$.
18. Find the points of intersection of the line $8y - 2x = 15c$ and the rectangular hyperbola $xy = c^2$. Prove that the line is a normal to the curve at one of these intersections.
19. Prove that the normal at the point $P\left(ct, \frac{c}{t}\right)$ on the rectangular hyperbola $xy = c^2$ meets the curve again at Q so that
$$PQ = c\left(t^2 + \frac{1}{t^2}\right)^{\frac{3}{2}} \qquad (8.6.19.1)$$
20. Four points are taken on a rectangular hyperbola $xy = c^2$. Find the condition that the chord joining two of the points should be perpendicular to the chord joining the other two. Prove that, if the condition is satisfied for one pair of such chords, then it is true for all three pairs.
21. Prove that the locus of the middle points of chords of the rectangular hyperbola $xy = c^2$ which pass through the point $(2c, 2c)$ is an equal hyperbola and find the coordinates of its vertices.
22. The radius OP from the centre to a point P

on the rectangular hyperbola $xy = c^2$ makes an angle θ with the x-axis, and the normal at P cuts the axes of x and y in G and G_1. Prove that $\frac{PG}{PG_1} = \tan^2\theta$.