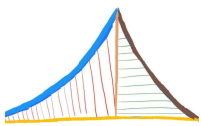
Geometry through Trigonometry



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About this Book

This book introduces trigonometry through high school geometry. This approach relies more on trigonometric equations than cumbersome constructions which are usually non intuitive. All problems in the book are from NCERT mathematics textbooks from Class 9-12. Exercises are from CBSE, JEE and Olympiad exam papers.

The content is sufficient for all practical applications of trigonometry. There is no copyright, so readers are free to print and share.

This book is dedicated to my Hindi teacher in school, Shri Mandavi.

March 28, 2025

Github: https://github.com/gadepall/matgeo

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1 Heights and Distances

1.1 Right Angled Triangle

1.1.1. A right angled triangle looks like Fig. 1.1.1. with angles $\angle A, \angle B$ and $\angle C$ and sides

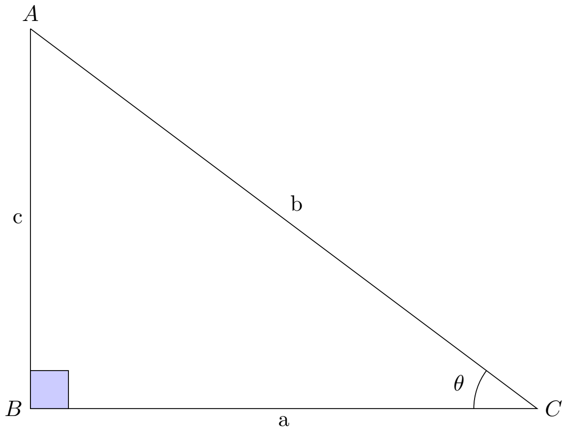


Fig. 1.1.1: Right Angled Triangle

a, b and c. The unique feature of this triangle is $\angle B$ which is defined to be $90°$.

1.1.2. For simplicity, let the greek letter $\theta = \angle C$. We have the following definitions.

$$\begin{aligned} \sin\theta &= \tfrac{c}{b} & \cos\theta &= \tfrac{a}{b} \\ \tan\theta &= \tfrac{c}{a} & \cot\theta &= \tfrac{1}{\tan\theta} \\ \csc\theta &= \tfrac{1}{\sin\theta} & \sec\theta &= \tfrac{1}{\cos\theta} \end{aligned} \tag{1.1.2.1}$$

1.1.3.

$$\cos\theta = \sin(90° - \theta) \tag{1.1.3.1}$$

1.1.4. In Fig. 1.1.2, show that

$$b = a\cos\theta + c\sin\theta \tag{1.1.4.1}$$

Solution: We observe that

$$CD = a\cos\theta \tag{1.1.4.2}$$

$$AD = c\cos\alpha = c\sin\theta \quad \text{(From} \quad (1.1.3.1)) \tag{1.1.4.3}$$

Thus,

$$CD + AD = b = a\cos\theta + c\sin\theta \tag{1.1.4.4}$$

1.1.5. From (1.1.4.1), show that

$$\sin^2\theta + \cos^2\theta = 1 \tag{1.1.5.1}$$

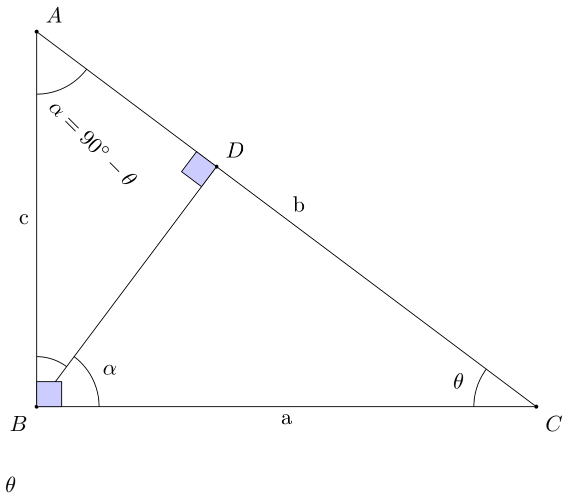


Fig. 1.1.2: Baudhayana Theorem

Solution: Dividing both sides of (1.1.4.1) by b,

$$1 = \frac{a}{b}\cos\theta + \frac{c}{b}\sin\theta \tag{1.1.5.2}$$

$$\Rightarrow \sin^2\theta + \cos^2\theta = 1 \quad (\text{from} \quad (1.1.2.1)) \tag{1.1.5.3}$$

1.1.6. From (1.1.5.1)

$$|\sin\theta| \le 1, \ |\cos\theta| \le 1 \tag{1.1.6.1}$$

1.1.7. Using (1.1.4.1), show that

$$b^2 = a^2 + c^2 \tag{1.1.7.1}$$

(1.1.7.1) is known as the Baudhayana theorem. It is also known as the Pythagoras theorem.
Solution: From (1.1.4.1),

$$b = a\frac{a}{b} + c\frac{c}{b} \quad (\text{from} \quad (1.1.2.1)) \tag{1.1.7.2}$$

$$\implies b^2 = a^2 + c^2 \tag{1.1.7.3}$$

1.1.8. In a right angled triangle, the hypotenuse is the longest side.
Solution: From (1.1.7.1),

$$a \le b, \ c \le b. \tag{1.1.8.1}$$

1.1.9. ABC is an isosceles triangle in which altitudes BE and CF are drawn to equal sides AC and AB respectively . Show that these altitudes are equal.

Solution: In $\triangle$s BFC and BEC,

$$BF = a\sin C,\ CE = a\sin B \tag{1.1.9.1}$$
$$\implies BF = CE, \because B = C. \tag{1.1.9.2}$$

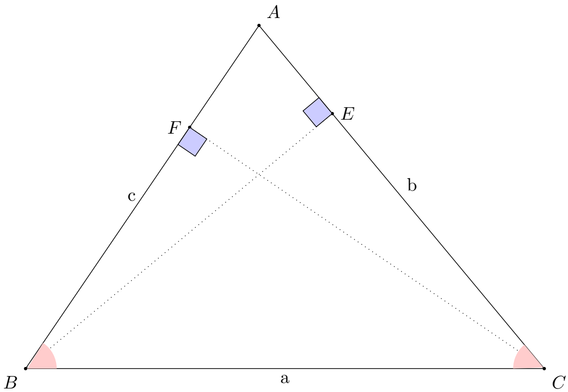


Fig. 1.1.3: $B = C$

1.1.10. ABC is a triangle in which altitudes BE and CF to sides AC and AB are equal. Show that $AB = AC$.
Solution: In (1.1.9.1),

$$BE = CF \implies a\sin C = a\sin B \tag{1.1.10.1}$$
$$\text{or, } B = C \tag{1.1.10.2}$$

1.1.11. A ladder is placed against a wall such that its foot is at a distance of $2.5m$ from the wall and its top reaches a window $6m$ above the ground. Find the length of the ladder.
1.1.12. A ladder $10m$ long reaches a window $8m$ above the ground. Find the distance of the foot of the ladder from base of the wall.
1.1.13. A guy wire attached to a vertical pole of height $18m$ is $24m$ long and has a stake attached to the other end. How far from the base of the pole should the stake be driven so that the wire will be taut?
1.1.14. An aeroplane leaves an airport and flies due north at a speed of $1000km$ per hour. At the same time, another aeroplane leaves the same airport and flies due west at a speed of $1200km$ per hour. How far apart will be the two planes after $1\frac{1}{2}$ hours?

1.2 Sine and Cosine Formula

1.2.1. Show that the area of ΔABC in Fig. 1.2.1 is $\frac{1}{2}ab\sin C$.
Solution: We have

$$ar(\Delta ABC) = \frac{1}{2}ah = \frac{1}{2}ab\sin C \quad (\because \quad h = b\sin C). \tag{1.2.1.1}$$

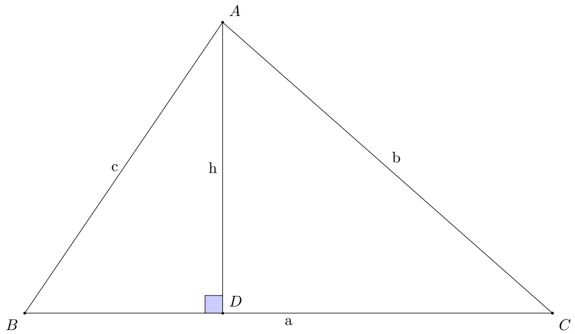


Fig. 1.2.1: Area of a Triangle

1.2.2. Show that

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \tag{1.2.2.1}$$

Solution: Fig. 1.2.1 can be suitably modified to obtain

$$ar\left(\Delta ABC\right) = \frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B \tag{1.2.2.2}$$

Dividing the above by *abc*, we obtain

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \tag{1.2.2.3}$$

This is known as the sine formula.

1.2.3. In Fig. 1.2.2, $AB = AC$. Show that

$$\angle B = \angle C \tag{1.2.3.1}$$

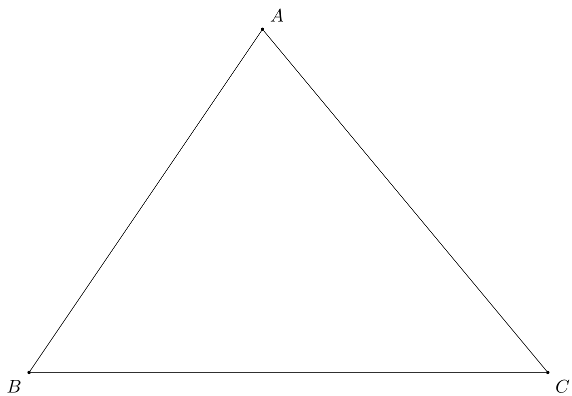


Fig. 1.2.2

Solution: Using the sine formula,

$$\frac{AB}{\sin C} = \frac{AC}{\sin B} \tag{1.2.3.2}$$

$$\implies \sin B = \sin C \text{ or, } \angle B = \angle C. \tag{1.2.3.3}$$

1.2.4. In Fig. 1.2.3, show that

$$\begin{pmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{pmatrix} \begin{pmatrix} \cos A \\ \cos B \\ \cos C \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{1.2.4.1}$$

Solution: From Fig. 1.2.3,

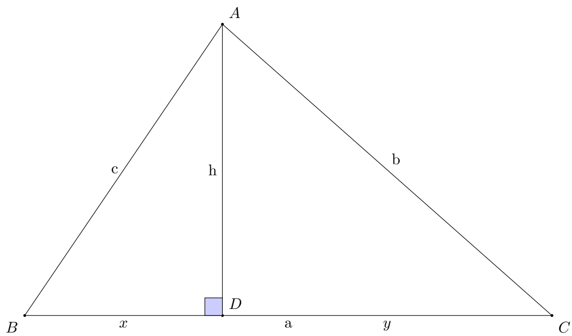


Fig. 1.2.3: The cosine formula

$$a = x + y = b\cos C + c\cos B = \begin{pmatrix} \cos C & \cos B \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix} \tag{1.2.4.2}$$

$$= \begin{pmatrix} 0 & b & c \end{pmatrix} \begin{pmatrix} \cos A \\ \cos C \\ \cos B \end{pmatrix} \tag{1.2.4.3}$$

Similarly,

$$b = c\cos A + a\cos C = \begin{pmatrix} c & 0 & a \end{pmatrix} \begin{pmatrix} \cos A \\ \cos C \\ \cos B \end{pmatrix} \tag{1.2.4.4}$$

$$c = b\cos A + a\cos B = \begin{pmatrix} b & a & 0 \end{pmatrix} \begin{pmatrix} \cos A \\ \cos C \\ \cos B \end{pmatrix} \tag{1.2.4.5}$$

The above equations can be expressed in matrix form as (1.2.4.1).

1.2.5. Show that

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \tag{1.2.5.1}$$

Solution: Using the properties of determinants,

$$\cos A = \frac{\begin{vmatrix} a & c & b \\ b & 0 & a \\ c & a & 0 \end{vmatrix}}{\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}} = \frac{ab^2 + ac^2 - a^3}{abc + abc} = \frac{b^2 + c^2 - a^2}{2abc} \tag{1.2.5.2}$$

1.2.6. Find Hero's formula for the area of a triangle.
Solution: From (1.2.1), the area of $\triangle ABC$ is

$$\frac{1}{2} ab \sin C = \frac{1}{2} ab \sqrt{1 - \cos^2 C} \quad \text{(from (1.1.5.1))} \tag{1.2.6.1}$$

$$= \frac{1}{2} ab \sqrt{1 - \left(\frac{a^2 + b^2 - c^2}{2ab}\right)^2} \text{ (from (1.2.5.1))} \tag{1.2.6.2}$$

$$= \frac{1}{4} \sqrt{(2ab)^2 - (a^2 + b^2 - c^2)} \tag{1.2.6.3}$$

$$= \frac{1}{4} \sqrt{(2ab + a^2 + b^2 - c^2)(2ab - a^2 - b^2 + c^2)} \tag{1.2.6.4}$$

$$= \frac{1}{4} \sqrt{\left\{(a + b)^2 - c^2\right\}\left\{c^2 - (a - b)^2\right\}} \tag{1.2.6.5}$$

$$= \frac{1}{4} \sqrt{(a + b + c)(a + b - c)(a + c - b)(b + c - a)} \tag{1.2.6.6}$$

Substituting

$$s = \frac{a + b + c}{2} \tag{1.2.6.7}$$

in (1.2.6.6), the area of $\triangle ABC$ is

$$\sqrt{s(s - a)(s - b)(s - c)} \tag{1.2.6.8}$$

This is known as Hero's formula.

1.2.7. Show that

$$\alpha > \beta \implies \sin\alpha > \sin\beta \tag{1.2.7.1}$$

Solution: In Fig. 1.2.4,

$$ar(\triangle ABD) < ar(\triangle ABC) \tag{1.2.7.2}$$

$$\implies \frac{1}{2} lc \sin\theta_1 < \frac{1}{2} ac \sin(\theta_1 + \theta_2) \tag{1.2.7.3}$$

$$\implies \frac{l}{a} < \frac{\sin(\theta_1 + \theta_2)}{\sin\theta_1} \tag{1.2.7.4}$$

$$\text{or, } 1 < \frac{l}{a} < \frac{\sin(\theta_1 + \theta_2)}{\sin\theta_1} \tag{1.2.7.5}$$

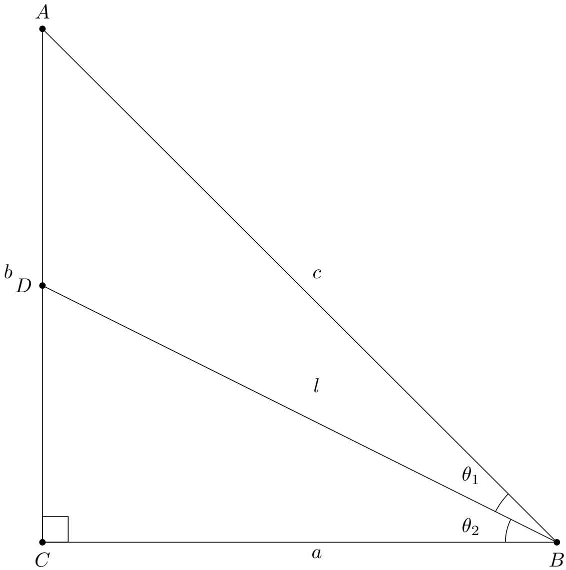


Fig. 1.2.4

from Theorem 1.1.8, yielding

$$\implies \frac{\sin(\theta_1 + \theta_2)}{\sin\theta_1} > 1. \tag{1.2.7.6}$$

This proves (1.2.7.1).

1.3 NCERT

1.3.1. An aircraft is flying at a height of 3400m above the ground. If the angle subtended at a ground observation point by the aircraft positions 10.0s apart is 30°, what is the speed of the aircraft?
Solution: See Fig. 1.3.1 and Table 1.3.1.

$$vt = h\tan\theta \implies v = \frac{h\tan\theta}{t} \tag{1.3.1.1}$$

$$\text{or, } v = \frac{3400\tan 30}{10} = 178.02\, m/s \tag{1.3.1.2}$$

Parameter	Description	Value
h	Height	$3400m$
v	Speed	?
t	Time	$10s$
θ	Angle	30

TABLE 1.3.1

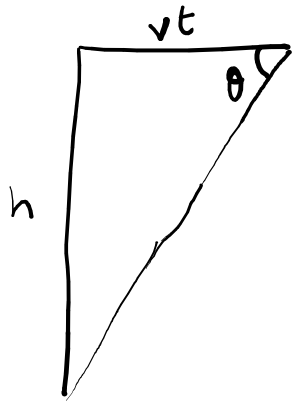


Fig. 1.3.1

1.3.2. A statue, $1.6m$ tall, stands on the top of a pedestal. From a point on the ground, the angle of elevation of the top of the statue is $60°$ and from the same point the angle of elevation of the top of the pedestal is $45°$. Find the height of the pedestal.
Solution: See Fig. 1.3.2 and Table 1.3.2.

$$d = h_1 \cot\theta_1 = h_2 \cot\theta_2 \tag{1.3.2.1}$$

$$\implies h_2 = \frac{h_1 \cot\theta_1}{\theta_2} = 0.92m \tag{1.3.2.2}$$

Parameter	Description	Value
h_1	Height of the statue	$1.6m$
h_2	Height of the pedestal	?
θ_1	Angle of elevation of the statue	$60°$
θ_2	Angle of elevation of the pedestal	$45°$
d	Distance of the observation point from the foot of the pedestal	?

TABLE 1.3.2

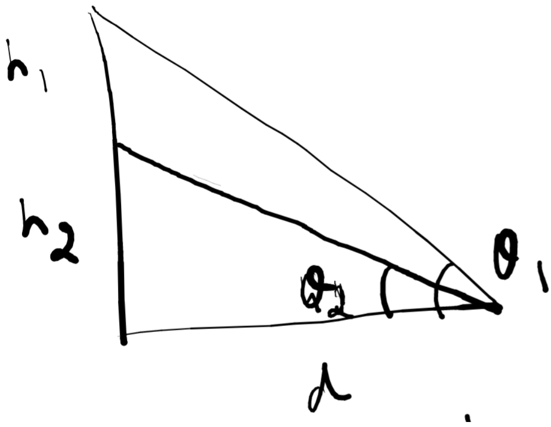


Fig. 1.3.2

1.3.3. Two poles of equal heights are standing opposite each other on either side of the road, which is $80m$ wide. From a point between them on the road, the angles of elevation of the top of the poles are $60°$ and $30°$, respectively. Find the height of the poles and the distances of the point from the poles.
Solution: See Fig. 1.3.3 and Table 1.3.3.

$$d = d_1 + d_2 = h\cot\theta_1 + h\cot\theta_2 \tag{1.3.3.1}$$

$$\implies h = \frac{d}{\cot\theta_1 + \cot\theta_2} = 50.72m \tag{1.3.3.2}$$

Parameter	Description	Value
h	Height of the poles	?
d_1	Distance from first pole	?
d_2	Distance from second pole	?
d	Distance between poles	$80m$
θ_1	Angle of elevation of first pole	$60°$
θ_2	Angle of elevation of second pole	$30°$

TABLE 1.3.3

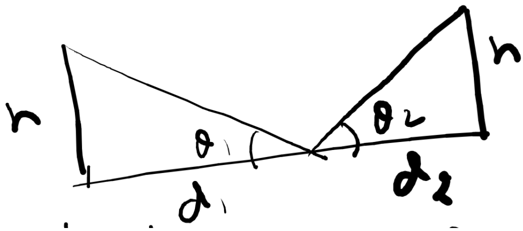


Fig. 1.3.3

1.3.4. A TV tower stands vertically on a bank of a canal. From a point on the other bank directly opposite the tower, the angle of elevation of the top of the tower is 60°. From another point $20m$ away from this point on the line joing this point to the foot of the tower, the angle of elevation of the top of the tower is 30°. Find the height of the tower and the width of the canal.
Solution: See Fig. 1.3.4 and Table 1.3.4.

$$d_1 + d_2 = h \cot\theta_2 \tag{1.3.4.1}$$

$$d_1 = h \cot\theta_1 \tag{1.3.4.2}$$

which can be expressed as the matrix equation

$$\begin{pmatrix} \cot\theta_2 & -1 \\ \cot\theta_1 & -1 \end{pmatrix} \begin{pmatrix} h \\ d_1 \end{pmatrix} = d_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.3.4.3}$$

yielding

$$\begin{pmatrix} h \\ d_1 \end{pmatrix} = \begin{pmatrix} 17.32 \\ 10 \end{pmatrix} \tag{1.3.4.4}$$

Parameter	Description	Value
h	Height of the tower	?
d_1	Width of the canal	?
d_2	Distance between the observation points	$20m$
θ_1	Angle of elevation from first observation point	$60°$
θ_2	Angle of elevation from second observation point	$30°$

TABLE 1.3.4

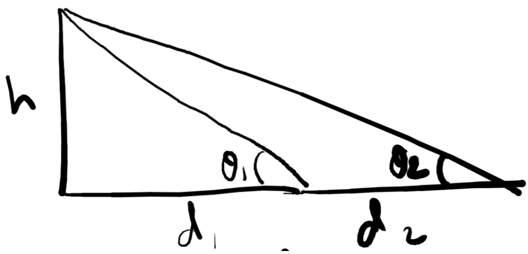


Fig. 1.3.4

1.3.5. From the top of a $7m$ high building, the angle of elevation of the top of a cable tower is $60°$ and the angle of depression of its foot is $45°$. Determine the height of the tower.

Solution: See Fig. 1.3.5 and Table 1.3.5. In $\triangle ABC$, using sine formula,

$$\frac{\sin ACB}{AB} = \frac{\sin BAC}{BC} \implies \frac{\cos\theta_1}{h_1 \csc\theta_2} = \frac{\sin(\theta_1+\theta_2)}{h_2} \tag{1.3.5.1}$$

yielding

$$h_2 = \frac{h_1 \csc\theta_2 \sin(\theta_1+\theta_2)}{\cos\theta_1} = 19.12m \tag{1.3.5.2}$$

Parameter	Description	Value
h	Height of the building	h_1
h_2	Height of the tower	?
d_2	Distance between the observation points	$20m$
θ_1	Angle of elevation from first observation point	$60°$
θ_2	Angle of elevation from second observation point	$30°$

TABLE 1.3.5

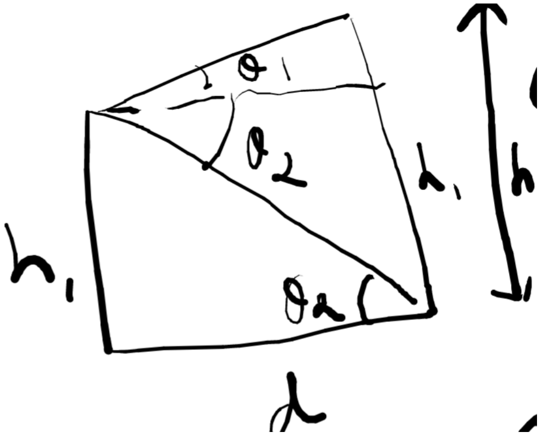


Fig. 1.3.5

1.3.6. As observed from the top of a 75m high lighthouse from the sea-level, the angles of depression of two ships are 30° and 45°. If one ship is exactly behind the other on the same side of the lighthouse, find the distance between the two ships.

1.3.7. A 1.2m tall girl spots a balloon moving with the wind in a horizontal line at a height of 88.2m from the ground. The angle of elevation of the balloon from the eyes of the girl at any instant is 60°. After some time, the angle of elevation reduces to 30°. Find the distance travelled by the balloon during the interval.

1.3.8. A straight highway leads to the foot of a tower. A man standing at the top of the tower observes a car at an angle of depression of 30°, which is approaching the foot of the tower with a uniform speed. Six seconds later, the angle of depression of the car is found to be 60°. Find the time taken by the car to reach the foot of the tower from this point.

1.3.9. The angles of elevation of the top of a tower from two points at a distance of 4m and 9m from the base of the tower and in the same straight line with it are complementary. Prove that the height of the tower is 6m.

1.3.10. A girl of height 90cm is walking away from the base of a lamp-post at a speed of 1.2 m/s. If the lamp is 3.6m above the ground, find the length of her shadow after 4 seconds.

1.3.11. Namrata is fly fishing in a stream. The tip of her fishing rod is 1.8m above the surface of the water and the fly at the end of the string rests on the water 3.6m away and 2.4m from a point directly under the tip of the rod. Assuming that her string (from the tip of her rod to the fly) is taut, how much string does she have out? If she pulls in the string at the rate of 5cm per second, what will be the horizontal distance of the fly from her after 12 seconds?

1.3.12. The angle of elevation of the top of a building from the foot of the tower is 30° and the angle of elevation of the top of the tower from the foot of the building is 60°. If the tower is 50m high, find the height of the building.

1.3.13. A vertical pole of length 6m casts a shadow 4m long on the ground and at the same time a tower casts a shadow 28m long. Find the height of the tower.

1.3.14. A circus artist is climbing a $20m$ long rope, which is tightly stretched and tied from the top of a vertical pole to the ground. Find the height of the pole, if the angle made by the rope with the ground level is $30°$.

1.3.15. A tree breaks due to storm and the broken part bends so that the top of the tree touches the ground making an angle of $30°$ with it. The distance between the foot of the tree to the point where the top touches the ground is $8m$. Find the height of the tree.

1.3.16. A contractor plans to install two slides for the children to play in a park. For the children below the age of 5 years, she prefers to have a slide whose top is at a height of $1.5m$, and is inclined at an angle of $30°$ to the ground, whereas for elder children she wants to have a steep slide at a height of $3m$, and inclined at an angle of $60°$ to the ground. What should be the length of the slide in each case?

1.3.17. The angle of elevation of the top of a tower from a point on the ground, which is $30m$ away from the foot of the tower, is $30°$. Find the height of the tower.

1.3.18. A kite is flying at a height of $60m$ above the ground. The string attached to the kite is temporarily tied to a point on the ground. The inclination of the string with the ground is $60°$. Find the length of the string, assuming that there is no slack in the string.

1.3.19. A $1.5m$ tall boy is standing at some distance from a $30m$ tall building. The angle of elevation from his eyes to the top of the building increases from $30°$ to $60°$ as he walks towards the building. Find the distance he walked towards the building.

1.3.20. From a point on the ground, the angles of elevation of the bottom and the top of a transmission tower fixed at the top of a $20m$ high building are $45°$ and $60°$ respectively. Find the height of the tower.

1.3.21. A girl walks $4km$ west, then she walks $3km$ in a direction $30°$ east of north and stops. Determine the girl's displacement from her initial point of departure.

1.3.22. The angles of depression of the top and the bottom of an $8m$ tall building from the top of a multi-storeyed building are $30°$ and $45°$ respectively. Find the height of the multi-storeyed building and the distance between the two buildings.

1.3.23. A tower stands vertically on the ground. From a point on the ground, which is $15m$ away from the foot of the tower, the angle of elevation of the top of the tower is found to be $60°$. Find the height of the tower.

1.3.24. An electrician has to repair an electric fault pole of height $5m$. She needs to reach a point $1.3m$ below the top of the pole to undertake the repair work. What should be the length of the ladder that she should use which, when inclined at an angle of $60°$ to the horizontal, would enable her to reach the required position? Also, how far from the foot of the pole should she place the foot of the ladder?

1.3.25. An observer $1.5m$ tall is $28.5m$ away from a chimney. The angle of elevation of the top of the chimney from her eyes is $45°$. What is the height of the chimney?

1.3.26. From a point P on the ground the angle of elevation of the top of a 10m tall building is $30°$. A flag is hoisted at the top of the building and the angle of elevation of the top of the flagstaff from P is $45°$. Find the length of the flagstaff and the distance of the building from the point P.

1.3.27. The shadow of a tower standing on a level ground is found to be $40m$ longer when the Sun's altitude is $30°$ than when it is $60°$. Find the height of the tower.

1.3.28. Two poles of heights $6m$ and $11m$ stand on a plane ground. If the distance between the feet of the poles is $12m$, find the distance between their tops.

1.4 CBSE

1.4.1. In Fig. 1.4.1, the angles of elevation of two kites from point C are found to be $30°$ and $60°$ respectively. Taking $AD = 50m$ and $BE = 60m$, find

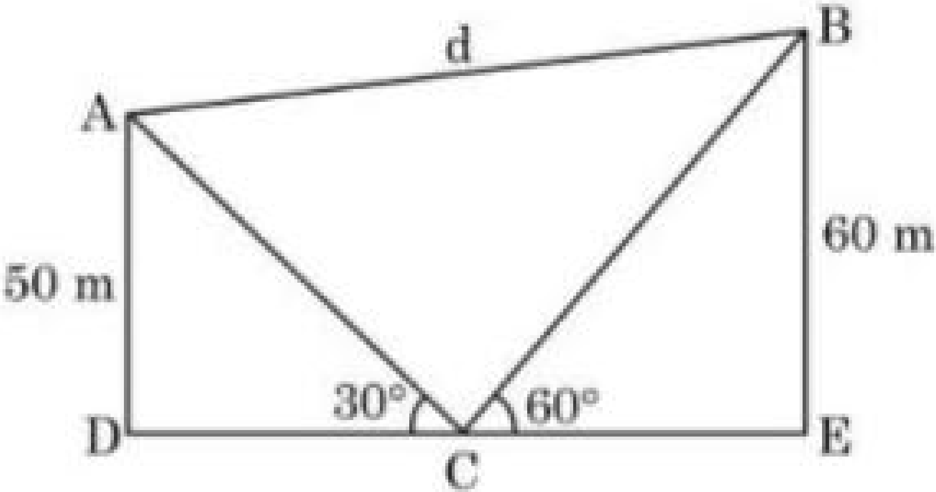


Fig. 1.4.1

a) The length of string used (take them straight) for kites A and B as shown in the figure.
b) The distance d between these two kites.

(10, 2022)

1.4.2. In Fig. 1.4.2, a tower stands vertically on the ground. From a point on the ground, which is $80m$ away from the foot of the tower, the angle of elevation of the tower is found to be $30°$. Find the height of the tower. (10, 2022)

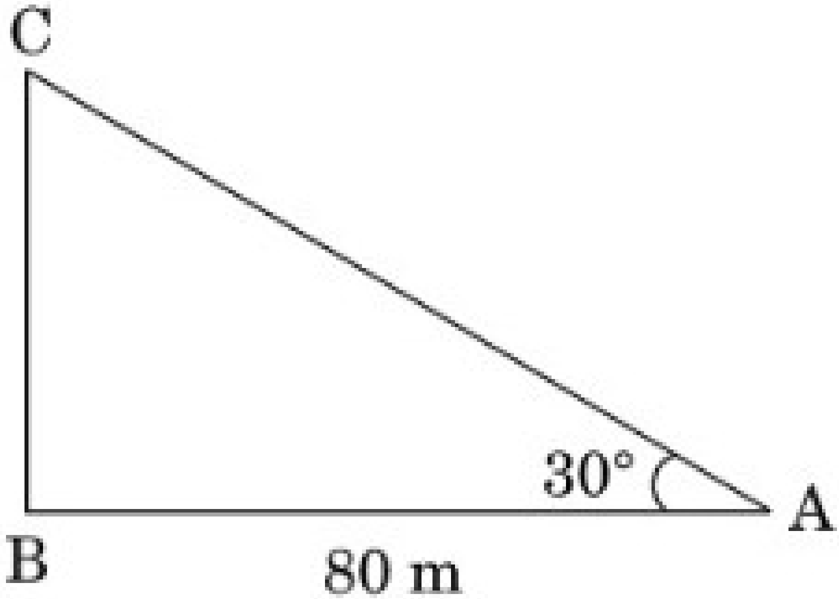


Fig. 1.4.2

1.4.3. The angles of depression of the top and bottom of a tower as seen from the top of a $60\sqrt{3}m$ high cliff are 45° and 60° respectively. Find the height of the tower. (Use $\sqrt{3} = 1.73$) (10, 2022)

1.4.4. The angle of elevation of the top of a building from the foot of the tower is 30° and the angle of elevation of the top of the tower from the foot of the building is 60°. If the tower is 50 meters high, then find the height of the building. (10, 2022)

1.4.5. From a point on a bridge across a river, the angles of depression of the banks on opposite sides of the river are 30° and 60° respectively. If the bridge is at a height of 3 meters from the banks, then find the width of the river. (10, 2022)

1.4.6. In Fig. 1.4.3, Gadisar Lake is located in the Jaisalmer district of Rajasthan. It was built by the King of Jaisalmer and rebuilt by Gadsi Singh in the 14th century. The lake has many Chhatris. One of them is shown below:

Fig. 1.4.3

Observe the picture. From a point A, h meters above the water level, the angle of elevation of the top of Chhatri (point B) is $45°$ and the angle of depression of its reflection in the water (point C) is $60°$. If the height of Chhatri above water level is (approximately) 10 meters, then (10, 2022)

a) Draw a well-labeled figure based on the above information.
b) Find the height h of the point A above water level. (Use $\sqrt{3} = 1.73$)

1.4.7. In Fig. 1.4.4, from a point on a bridge across a river, the angles of depression of the banks on opposite sides of the river are $30°$ and $45°$. If the bridge is at a height of 8 meters from the banks, then find the width of the river.

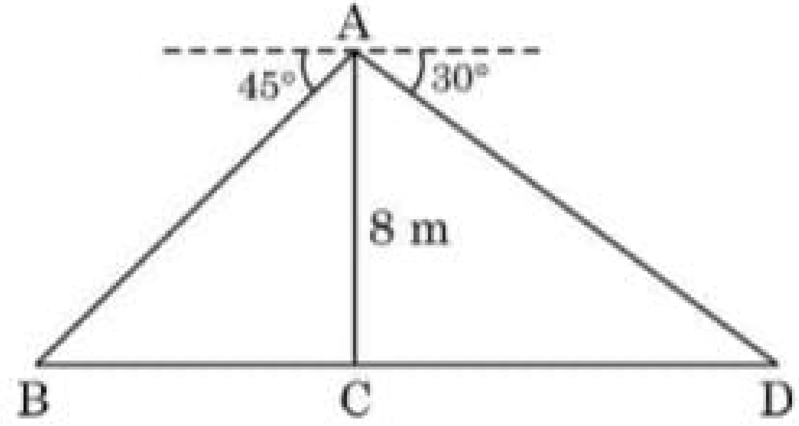


Fig. 1.4.4

(10, 2022)

1.4.8. Two boats are sailing in the sea 80 meters apart from each other towards a cliff AB. The angles of depression of the boats from the top of the cliff are 30° and 45° respectively, as shown in Fig. 1.4.5

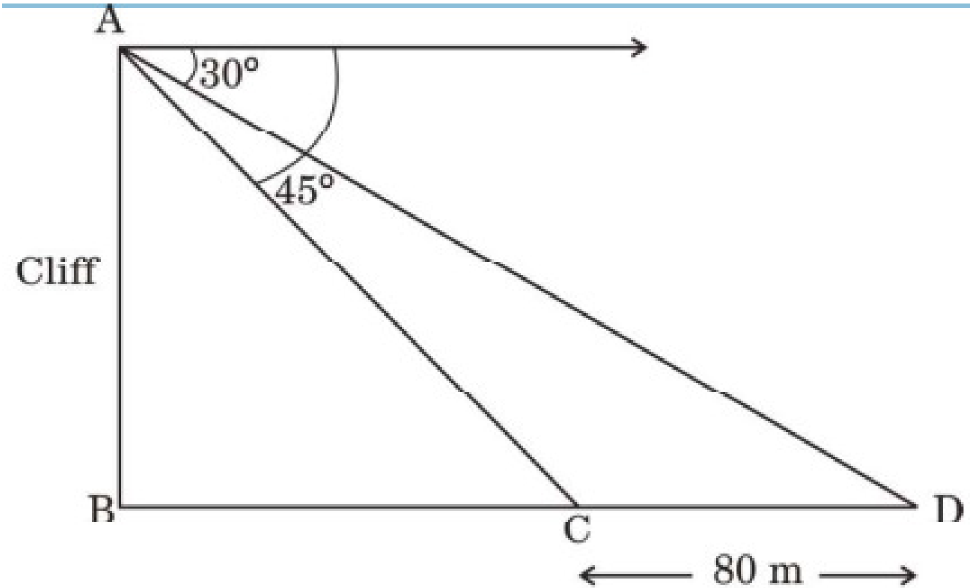


Fig. 1.4.5

Find the height of the cliff. (10, 2022)

1.4.9. The angle of elevation of the top Q of a vertical tower PQ from a point X on the ground is $60°$. From a point Y, 40 meters vertically above X, the angle of elevation of the top Q of tower PQ is $45°$. Find the height of the tower PQ and the distance PX. (Use $\sqrt{3} = 1.73$) (10, 2022)

1.4.10. An Aeroplane at an altitude of 200 meters observes the angles of depression of opposite points on the two banks of a river to be $45°$ and $60°$. Find the width of the river. (Use $\sqrt{3} = 1.732$) (10, 2022)

1.4.11. From the top of an 8 meter high building, the angle of elevation of the top of a cable tower is $60°$ and the angle of depression of its foot is $45°$. Determine the height of the tower. (Take $\sqrt{3} = 1.732$). (10, 2022)

1.4.12. As observed from the top of a lighthouse 60 meters high from the sea level, the angles of depression of two ships are $45°$ and $60°$. If one ship is exactly behind the other on the same side of the lighthouse, then find the distance between the two ships. (Use $\sqrt{3} = 1.732$) (10, 2022)

1.4.13. At a point on the level ground, the angle of elevation of the top of a vertical tower is found to be α, such that $\tan\alpha = \frac{5}{12}$. On walking 192 meters towards the tower, the angle of elevation β is such that $\tan\beta = \frac{3}{4}$. Find the height of the tower. (10, 2022)

1.4.14. A man on the top of a vertical tower observes a car moving at a uniform speed coming directly towards it. If it takes 18 minutes for the angle of depression to change from $30°$ to $60°$, how soon after this will the car reach the tower ? (10, 2021)

1.4.15. A girl on a ship standing on a wooden platform, which is $50m$ above water level, observes the angle of elevation of a top of a hill as $30°$ and the angle of depression of the base of the hill as $60°$. Calculate the distance of the hill from the platform and the height of the hill. (10, 2021)

1.4.16. The length of the shadow of a tower on the plane ground is $\sqrt{3}$ times the height of the tower. Find the angle of elevation of the sun. (10, 2023)

1.4.17. The angle of elevation of the top of a tower from a point on the ground which is $30m$ away from the foot of the tower, is $30°$. Find the height of the tower. (10, 2023)

1.4.18. As observed from the top of a $75m$ high lighthouse from the sea-level, the angles of depression of two ships are $30°$ and $60°$. If one ship is exactly behind the other on the same side of the lighthouse, find the distance between two ships. Use $\left(\sqrt{3} = 1.73\right)$ (10, 2023)

1.4.19. From a point on the ground,the angle of elevation of the bottom and top of a transmission tower fixed at the top of $30m$ high building are $30°$ and $60°$, respectively.Find the height of the transmission tower.Use $\left(\sqrt{3} = 1.73\right)$ (10, 2023)

1.4.20. A straight highway leads to the foot of a tower. A man standing on the top of the $75m$ high tower observes two cars at angles of depression of $30°$ and $60°$, which are approaching the foot of the tower. If one car is exactly behind the other on the same side of the tower, find the distance between the two cars. (10, 2023)

1.4.21. From the top of a $7m$ high building, the angle of elevation of the top of a cable tower is $60°$ and the angle of depression of its foot is $30°$. Determine the height of the tower. (take $\sqrt{3} = 1.73$) (10, 2023)

1.4.22. The angle of elevation of the top of a tower $24m$ high from the foot of another tower in the same plane is $60°$. The angle of elevation of the top of second tower from the foot of the first tower is 30°. Find the distance between two towers and the height of the other tower. Also, find the length of the wire attached to the tops of both the towers. (10, 2023)

1.4.23. A spherical balloon of radius r subtends an angle of 60° at the eye of an observer. If the angle of elevation of its centre is 45° from the same point, then prove that height of the centre of the balloon is $\sqrt{2}$ times its radius. (10, 2023)

1.4.24. A vertical pole is 100 metres high. Find the angle subtended by the pole at a point on the ground $100\sqrt{3}$ meters from the base of the pole. (10, 2021)

1.4.25. The angle of elevation of the top of a tower from a point is found to be 60°. At a point $40m$ above the first point, the angle of elevation of the top of the tower is 45°. Find the height of the tower. (10, 2021)

1.4.26. A statue 1.6m tall stands on the top of a pedestal. From a point on the ground, the angle of elevation of the top of statue is 60° and from the same point, the angle of elevation of the top of the pedestal is 45°.Find the height of the pedestal. (10, 2021)

1.4.27. Two poles, $6m$ and $11m$ high, stand vertically on the ground. If the distance between their feet is $12m$, find the distance between their tops. (10, 2021)

1.4.28. The angle of elevation of the top of a tower from a point on the ground,which is $30m$ away from the foot of the tower is 45°. What is the height of the tower ? (10, 2021)

1.4.29. Find the sun's altitude if the shadow of a $15m$ high tower is $15\sqrt{3}m$. (10, 2021)

1.4.30. From a point on the ground, $20m$ away from the foot of vertical tower, the angle of elevation of the top of the tower is 60°. Find the height of the tower. (10, 2021)

1.4.31. To explain how trignometry can be used measure the height of an inaccessible object, a teacher gave the following example to students : A TV tower stands vertically on the bank of a canal. From a point on the other bank direct opposite the tower, the angle of the elevation of the top of the tower is 60°. From another point $20m$ away from this point to the foot of the tower, the angle of elevation of the top of the tower is 30° (as shown in Fig. 1.4.6).

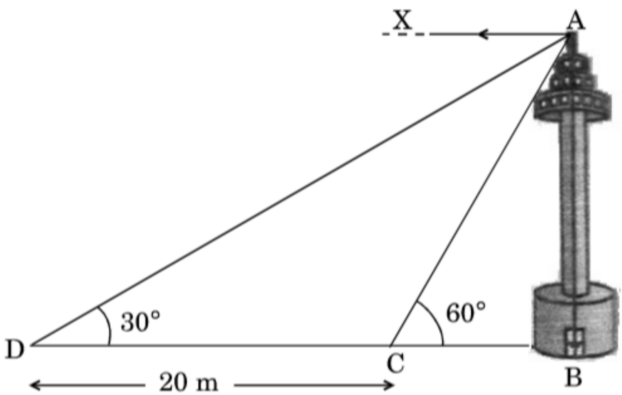


Fig. 1.4.6

Based on the above, answer the following questions

a) The width of the canal is

i) $10\sqrt{3}m$ ii) $20\sqrt{3}m$ iii) $10m$ iv) $20m$

b) Height of the tower is

i) $10\sqrt{3}m$ ii) $10m$ iii) $20\sqrt{3}m$ iv) $20m$

c) Distance of the foot of the tower from the point D is

i) $20m$ ii) $30m$ iii) $10m$ iv) $20\sqrt{3}m$

(10, 2021)

1.4.32. In Fig. 1.4.7, the angle of elevation of the top of a tower from a point C on the ground, which is $30m$ away from the foot of the tower, is $30°$. Find the height of the tower.

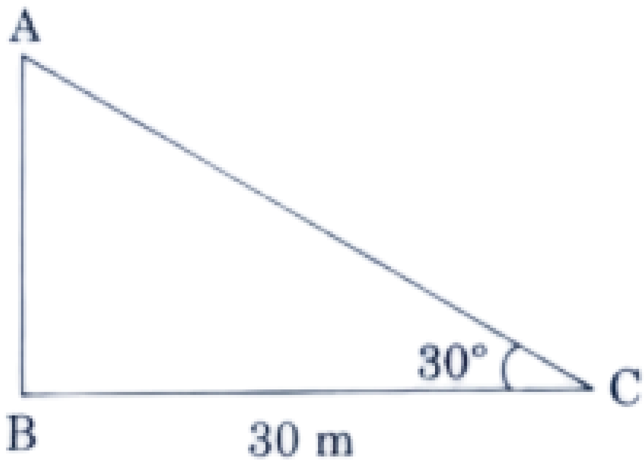


Fig. 1.4.7

(10, 2020)

1.4.33. A statue $1.6m$ tall, stands on the top of a pedestal. From a point on the ground, the angle of elevation of the top of the statue is $60°$ and from the same point the angle of elevation of the top of the pedestal is $45°$. Find the height of the pedestal.(Use $\sqrt{3} = 1.73$) (10, 2020)

1.4.34. A moving boat is observed from the top of a $150m$ high cliff moving away from the cliff. The angle of depression of the boat changes from $60°$ to $45°$ in 2 minutes. Find the speed of the boat in m/min. (10, 2019)

1.4.35. There are two poles, one each on either bank of a river just opposite to each other. One pole is $60m$ high. From the top of this pole, the angle of depression of the top and foot of the other pole are 30° and 60°respectively. Find the width of the river and height of the other pole. (10, 2019)

1.4.36. Two poles of equal heights are standing opposite to each other on either side of the road which is 80*m* wide. From a point *P* between them on the road, the angle of elevation of the top of a pole is 60° and the angle of depression from the top of the other pole of point *P* is 30°. Find the heights of the poles and the distance of the point *P* from the poles. (10, 2019)

1.4.37. Amit, standing on a horizontal plane, finds a bird flying at a distance of 200*m* from him at an elevation of 30°. Deepak standing on the roof of a 50*m* high building, finds the angle of elevation of the same bird to be 45°. Amit and Deepak are on opposite sides of the bird. Find the distance of the bird from Deepak. (10, 2019)

1.4.38. From a point *P* on the ground, the angle of elevation of the top of a tower is 30° and that of the top of the flag-staff fixed on the top of the tower is $\sqrt{5}$. If the length of the flag-staff is 5*m*, find the height of the tower. (Use $\sqrt{3} = 1.732$). (10, 2019)

1.4.39. The shadow of a tower standing on a level ground is found to be 40*m* longer when the Sun's altitude is 30° than when it was 60°. Find the height of the tower. Given ($\sqrt{3} = 1.732$) (10, 2019)

1.4.40. A man in a boat rowing away from a light house 100*m* high takes 2 minutes to change the angle of elevation of the top of the light house from 60° to 30°. Find the speed of the boat in metres per minute. [Use $\sqrt{3} = 1.732$] (10, 2019)

1.4.41. Two poles of equal heights are standing opposite each other on either side of the road, which is 80*m* wide. From a point between them on the road, the angles of elevation of the top of the poles are 60° to 30° respectively. Find the height of the poles and the distances of the point from the poles. (10, 2019)

1.4.42. As observed from the top of a 100*m* high light house from the sea level, the angles of depression of two ships are 30° and 45°. If one ship is exactly behind the other on the same side of the light house, find the distance between the two ships. Use ($\sqrt{3} = 1.732$) (10, 2018)

1.4.43. A statue, 1.46*m* tall, stands on a pedestal. From a point on the ground the angle of elevation of the top of the statue is 60° and from the same point angle of elevation of the top of the pedestal is 45°. Find the height of the pedestal. Use $\left(\sqrt{3} = 1.73\right)$ (10, 2018)

1.4.44. A ladder, leaning against a wall, makes an angle of 60° with the horizontal. If the foot of the ladder is 2.5*m* away from the wall, find the length of the ladder. (10, 2016)

1.4.45. A man standing on the deck of a ship, which is 10*m* above water level, observes the angle of elevation of the top of a hill as 60° and the angle of depression of the base of hill as 30°. Find the distance of the hill from the ship and the height of the hill. (10, 2016)

1.4.46. The angle of elevation of the top *Q* of a vertical tower *PQ* from a point *X* on the ground is 60°. From a point *Y*, 40*m* vertically above *X*, the angle of elevation of the top *Q* of tower is 45°. Find the height of the tower *PQ* and the distance *PX*. $\left(\text{Use } \sqrt{3} = 1.73\right)$ (10, 2016)

1.4.47. A boy standing on a horizontal plane finds a bird flying at a distance of 100*m* from him at an elevation of 30°. A girl standing on the roof of a 20*m* high building, finds the elevation of the same bird to be 45°. The boy and the girl are on the opposite sides of the bird. Find the distance of the bird from the girl. (Given $\sqrt{2} = 1.414$) (10, 2019)

1.4.48. The angle of elevation of an aeroplane from a point A on the ground is $60°$. After a flight of 30 seconds, the angle of elevation changes to $30°$. If the plane is flying at a constant height of $3600\sqrt{3}$ metres, find the speed of the aeroplane. (10, 2019)

1.4.49. If a tower $30m$ high, casts a shadow $10\sqrt{3}m$ long on a ground, then what is the angle of elevation of the sun ? (10, 2017)

1.4.50. A man observes a car from the top of a tower, which is moving towards the tower with a uniform speed. If the angle of depression of the car changes from $30°$ to $45°$ in 12 minutes, find the time taken by the car now to reach the tower. (10, 2017)

1.4.51. An aeroplane is flying at a height of $300m$ above the ground. Flying at this height, the angles of depression from the aeroplane of two points on both banks of a river in opposite directions are $45°$ and $60°$ respectively. Find the width of the river. Use $\left[\sqrt{3} = 1.732\right]$ (10, 2017)

1.4.52. On a straight line passing through the foot of a tower, two points C, D are at distances of $4m$ and $16m$ from the foot respectively. If the angles of elevation from C, D of the top tower are complementary, then find the height of the tower. (10, 2017)

1.4.53. From the top of a tower, $100m$ high, a man observes two cars on the opposite sides of the tower and in same straight line with its base, with angles of depression $30°$ and $45°$. Find the distance between the cars. Take $\left[\sqrt{3} = 1.732\right]$ (10, 2017)

1.4.54. At a point A, 20 metres above the level of water in a lake, the angle of elevation of a cloud is $30°$. The angle of depression of the reflection of the cloud in the lake, at A is $60°$. Find the distance of the cloud from A. (10, 2015)

1.4.55. In Figure 1.4.8, a tower AB is $20m$ high and BC, its shadow on the ground, is $20\sqrt{3}m$ long. Find the sun's altitude.

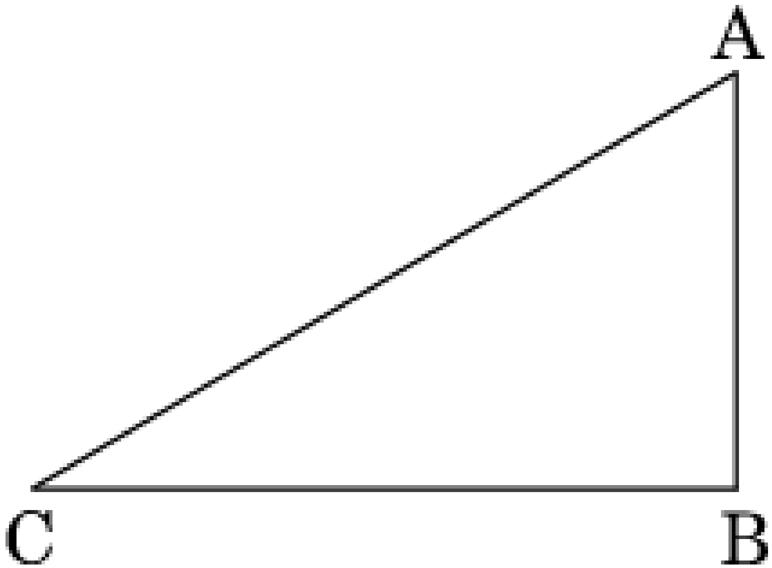


Fig. 1.4.8

(10, 2015)

1.4.56. The angle of elevation of an aeroplane from a point A on the ground is $60°$. After a flight of 15 seconds, the angle of elevation changes to $30°$. If the aeroplane is flying at a constant height of $1500\sqrt{3}$ m, find the speed of the plane in km/hr. (10, 2015)

1.4.57. A kite is flying at a height of $30m$ from the ground. The length of string from the kite to the ground is $60m$. Assuming that there is no slack in the string, the angle of elevation of the kite at the ground is ______. (10, 2012)

1.4.58. From a point on the ground, which is $15m$ away from the foot of a vertical tower, the angle of elevation of the top of the tower, is found to be $60°$. The height of the tower in (in metres) is ______. (10, 2012)

1.4.59. The length of shadow of a tower on the plane ground is $\sqrt{3}m$ times the height of the tower. The angle of elevation of sun is ______. (10, 2012)

1.4.60. The angles of depression of the top and bottom of a tower as seen from the top of a $60\sqrt{3}m$ high cliff are $45°$ and $60°$ respectively. Find the height of the tower. (10, 2012)

1.4.61. The angles of elevation and depression of the top and bottom of a light-house from the top of a $60m$ high building are $30°$ and $60°$ respectively. Find (10, 2012)

a) the difference between the heights of the light-house and the building.
b) the distance between light-house and building.

1.4.62. The angles of depression of two ships from the top of a light house and on the same side of it are found to be $45°$ and $30°$. if the ships are $200km$ apart, find the height of the light house. (10, 2012)

1.4.63. The angle of elevation of the top of a hill at the foot of a tower is $60°$ and the angle of depression from the top of the tower of the foot of the hill is $30°$. If the tower is $50m$ high, find the height of the hill. $(10, 2012)$

1.4.64. From the top of a tower $50m$ high, the angle of depression of the top of a pole is $45°$ and from the foot of the pole, the angle of elevation of the top of the tower is $60°$. find the height of the pole if the pole and tower stand on the same plane. $(10, 2012)$

1.4.65. The angle of depression from the top of a tower of a point A on the ground is $30°$. On moving a distance of $20m$ from the point A towards the foot of the tower to a point B the angle of elevation of the top of the tower from point B is $60°$. Find the height of the tower and its distance from point A. $(10, 2012)$

1.4.66. A tower stands vertically on the ground. From a point on the ground which is $25m$ away from the foot of the tower, the angle of elevation of the top of the tower is found to be $45°$. Then the height ($in\, meters$) of the tower is $(10, 2011)$

1.4.67. The angle of elevation of the top of a vertical tower from a point on the ground is $60°$. From another point $10m$ vertically above the first, its angle of elevation is $30°$. Find the height of the tower. $(10, 2011)$

1.4.68. From the top of a vertical tower, the angles of depression of two cars, in the same straight line with the base of the tower, at an instant are found to be $45°$ and $60°$. If the cars are $100m$ apart and are on the same side of the tower, find the height of the tower. [Use $\sqrt{3} = 1.73$] $(10, 2011)$

1.4.69. The angle of elevation of the top of a tower from a point on the ground, which is $30m$ away from the foot of the tower is $45°$. The height of the tower (in metres) is $(10, 2011)$

1.4.70. From the top of a tower $100m$ high, a man observes two cars on the opposite sides of the tower with angles of depression $30°$ and $45°$ respectively. Find the distance between the cars. [Use $\sqrt{3} = 1.73$]. $(10, 2011)$

1.4.71. Two poles of equal heights are standing opposite to each other on either side of the road, which is $100m$ wide. From a point between them on the road, the angles of elevation of the top of the poles are $60°$ and $30°$, respectively. Find the height of the poles. $(10, 2011)$

1.4.72. A man standing on the deck of a ship, which is $10m$ above the water level, observes the angle of elevation of the top of a hill as $60°$and the angle of depression of the base of the hill as $30°$. Calculate the distance of the hill from the ship and the height of the hill. $(10, 2006)$

1.4.73. From a window x meters high above the ground in a street, the angles of elevation and depression of the top and foot of the other house on the opposite side of the street are α and β respectively. Show that the height of the opposite house is $x(1 + \tan\alpha\cot\beta)$ meters. $(10, 2006)$

1.4.74. A pole $6m$ high is fixed on the top of a tower. The angle of elevation of the top of the pole observe d from a point P on the ground is $60°$ and the angle of depression of the point P from the top of the tower is $45°$. Find the height of the tower and the distance of point P from the foot of the tower $(10, 2024)$

1.4.75. The length of the shadow of a tower on the plane ground is $\sqrt{3}$ times the height of the tower. Find the angle of elevation of the sun. $(10, 2023)$

1.4.76. The angle of elevation of the top of a tower from a point on the ground which is $30m$ away from the foot of the tower, is $30°$. Find the height of the tower. $(10, 2023)$

1.4.77. As observed from the top of a $75m$ high lighthouse from the sea-level, the angles of depression of two ships are $30°$ and $60°$. If one ship is exactly behind the other on the same side of the lighthouse, find the distance between the two ships. Use $\left(\sqrt{3} = 1.73\right)$ (10, 2023)

1.4.78. From a point on the ground, the angle of elevation of the bottom and top of a transmission tower fixed at the top of $30m$ high building are $30°$ and $60°$, respectively. Find the height of the transmission tower. Use $\left(\sqrt{3} = 1.73\right)$. (10, 2023)

1.4.79. If a pole $6m$ high casts a shadow $2\sqrt{3}m$ long on the ground, then sun's elevation is

a) $60°$ b) $45°$ c) $30°$ d) $90°$

(10, 2023)

1.4.80. A straight highway leads to the foot of a tower. A man standing on the top of the $75m$ high tower observes two cars at angles of depression of $30°$ and $60°$, which are approaching the foot of the tower. If one car is exactky behind the other on the same side of the tower, find the distance between the two cars. Use $\left(\sqrt{3} = 1.73\right)$. (10, 2023)

1.4.81. From the top of a $7m$ building, the angle of elevation of the top a cable tower is $60°$and the angle of depression of its foot is $30°$. Determine the height of the tower. (10, 2023)

1.5 JEE

1.5.1 A person standing on the bank of a river observes that the angle of elevation of the top of a tree on the opposite bank of the river is $60°$ and when he retires 40 meters away from the tree, the angle of elevation becomes $30°$. The breadth of the river is (2004)

a) $60m$ b) $30m$ c) $40m$ d) $20m$

1.5.2 A tower stand at the centre of a circular park. A and B are two points on the boundary of the park such that AB $(= a)$ subtends an angle of $60°$ at the foot of the tower, and the angle of elevation of the top of the tower from A or B is $30°$. The height of the tower is (2007)

a) $\frac{a}{\sqrt{3}}$ b) $a\sqrt{3}$ c) $\frac{2a}{\sqrt{3}}$ d) $2a\sqrt{3}$

1.5.3 AB is a vertical pole with B at the ground level and A at the top. A man finds that the angle of elevation the the point A from a certain point C on the ground is $60°$. He moves away from the pole along the line BC to a point D such that CD $= 7m$. From D the angle of elevation of point A is $45°$. Then the height of the pole is (2008)

a) $\frac{7\sqrt{3}}{2\left(\sqrt{3}-1\right)}m$ b) $\frac{7\sqrt{3}}{2}\left(\sqrt{3}+1\right)m$ c) $\frac{7\sqrt{3}}{2}\left(\sqrt{3}-1\right)m$ d) $\frac{7\sqrt{3}}{2\left(\sqrt{3}+1\right)}m$

1.5.4 A bird is sitting on the top of a vertical pole $20m$ high and its elevation from a point O on the ground is $45°$. It flies off horizontally straight away from the point O. After one second, the elevation of the bird from O is reduced to 30°. Then the speed in (m/s) of the bird is (2014)

a) $20\sqrt{2}$ b) $20\left(\sqrt{3}-1\right)$ c) $40\left(\sqrt{2}-1\right)$ d) $40\left(\sqrt{3}-\sqrt{2}\right)$

1.5.5 If the angle of elevation of the top of a tower from three colinear points A, B and C on a line leading to foot of the tower, are $30°, 45°$ and $60°$ respectively, then the ratio, AB : BC, is: (2015)

a) $1:\sqrt{3}$ b) $2:3$ c) $\sqrt{3}:1$ d) $\sqrt{3}:\sqrt{2}$

1.5.6 Let a vertical tower AB have its end A on the level ground. Let C be the mid-point of AB and P be a point on the ground such that AP = 2AB. If $\angle BPC = \beta$, then $\tan\beta$ is equal to: (2017)

a) $\frac{4}{9}$ b) $\frac{6}{7}$ c) $\frac{1}{4}$ d) $\frac{2}{9}$

1.5.7 $\triangle PQR$ is a triangular park with PQ = PR = $200m$. A T.V. tower stands at the mid-point of QR. If the angles of the elevation of the top of the tower at P, Q and R are respectively $45°, 30°$ and $30°$, then the height of the tower in m is (2018)

a) 50 b) $100\sqrt{3}$ c) $50\sqrt{2}$ d) 100

1.5.8 From the top of a light-house 60 meter high with its base at the sea level the angle of depression of a boat is 15°. The distance of the boat from the foot of the light house. (1983)

a) $\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)60$ metres
b) $\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)60$ metres
c) $\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)^2 60$ metres
d) none of these

1.5.9 A pole stands vertically inside a triangular park $\triangle ABC$. If the angle of elevation of the top of the pole from each corner of the park is same, then in $\triangle ABC$ the foot of the pole is at the (2000)

a) centroid
b) circumcentre
c) incentre
d) orthocentre

1.5.10 A man from the top of a 100 metres high tower sees a car moving towards the tower at an angle of depression of 30°. After some time, the angle of depression becomes 60°. The distance (in metres) travelled by the car during this time is (2001)

a) $100\sqrt{3}$ b) $\frac{200\sqrt{3}}{3}$ c) $\frac{100\sqrt{3}}{3}$ d) $200\sqrt{3}$

1.5.11 A baloon is observed simultaneously from three points A, B and C on a straight road directly beneath it. The angular elevation at B is twice that at A and angular elevation at C is thrice that of A. If the distance between A and B is a and the distance between B and C is b, find height of baloon in terms of a and b. (1980)

1.5.12 PQ is a vertical tower. P is the foot and Q is the top of the tower. A, B, C are three points in the horizontal plane through P. The angles of elevation of Q from A, B, C are equal, and each is equal to θ. The sides of the triangle ABC are a, b, c; and the area of the triangle ABC is Δ. Show that the height of the tower is $\frac{abc\tan\theta}{4\Delta}$.

1.5.13 AB is a vertical pole. The end A is on the level ground. C is the middle point of AB. P is a point on the level ground. The portion CB subtends an angle β at P. If $AP = nAB$ then show that $\tan\beta = \frac{n}{2n^2+1}$. (1980)

1.5.14 A vertical pole stands at a point Q on a horizontal ground. A and B are points on the ground, d meters apart. The pole subtends angles α and β at A and B respectively. AB subtends an angle γ at Q. Find the height of the pole. (1982)

1.5.15 Four ships A, B, C and D are at sea in the following relative positions: B is on the straight line segment AC, B is due North of D and D is due west of C. The distance between B and D is $2km$. $\angle BDA = 40°$, $\angle BCD = 25°$. What is the distance between A and D? [Take $\sin 25° = 0.423$] (1983)

1.5.16 A ladder rests against a wall at an angle α to the horizontal. Its foot is pulled away from the wall through a distance a, so that it slides a distance b down the wall making an angle β with the horizontal. Show that $a = b\tan\frac{1}{2}(\alpha+\beta)$. (1985)

1.5.17 A sign-post in the form of an isosceles triangle ABC is mounted on a pole of height h fixed to the ground. The base BC of the triangle is parallel to the ground. A man standing on the ground at a distance d from the sign-post finds that the top vertex A of the triangle subtends an angle β and either of the other two vertices subtends the same angle α at his feet. Find the area of the triangle. (1988)

1.5.18 ABC is a triangular park with $AB = AC = 100$m. A television tower stands at the midpoint of BC. The angles of elevation of the top of the tower at A, B, C are 45°, 60°, 60°, respectively. Find the height of the tower. (1989)

1.5.19 A vertical tower PQ stands at a point P. Points A and B are located to the South and East of P respectively. M is the mid point of AB. PAM is an equilateral triangle; and N is the foot of the perpendicular from P on AB. Let $AN = 20$ metres and the angle of elevation of the top of the tower at N is $\tan^{-1} 2$. Determine the height of the tower and the angles of elevation of the top of the tower at A and B. (1990)

1.5.20 A man notices two objects in a straight line due west. After walking a distance c due north he observes that the objects subtend an angle α at his eye; and, after a further distance $2c$ due north, an angle β. Show that the distance between the objects is $\frac{8c}{3\cot\beta-\cot\alpha}$; the height of the man is being ignored. (1991)

2 Triangle

2.1 *Trigonometric Identities*

2.1.1. Using Fig. 1.2.4, show that

$$\sin\theta_1 = \sin(\theta_1+\theta_2)\cos\theta_2 - \cos(\theta_1+\theta_2)\sin\theta_2 \tag{2.1.1.1}$$

Solution: The following equations can be obtained from the figure using the forumula for the area of a triangle

$$ar(\Delta ABC) = \frac{1}{2}ac\sin(\theta_1 + \theta_2) \tag{2.1.1.2}$$
$$= ar(\Delta BDC) + ar(\Delta ADB) \tag{2.1.1.3}$$
$$= \frac{1}{2}cl\sin\theta_1 + \frac{1}{2}al\sin\theta_2 \tag{2.1.1.4}$$
$$= \frac{1}{2}ac\sin\theta_1\sec\theta_2 + \frac{1}{2}a^2\tan\theta_2 \tag{2.1.1.5}$$

($\because l = a\sec\theta_2$). From the above,

$$\sin(\theta_1 + \theta_2) = \sin\theta_1\sec\theta_2 + \frac{a}{c}\tan\theta_2 \tag{2.1.1.6}$$
$$= \sin\theta_1\sec\theta_2 + \cos(\theta_1 + \theta_2)\tan\theta_2 \tag{2.1.1.7}$$

Multiplying both sides by $\cos\theta_2$,

$$\sin(\theta_1 + \theta_2)\cos\theta_2 = \sin\theta_1 + \cos(\theta_1 + \theta_2)\sin\theta_2 \tag{2.1.1.8}$$

resulting in (2.1.1.1).

2.1.2. Prove the following identities

a)

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta. \tag{2.1.2.1}$$

b)

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta. \tag{2.1.2.2}$$

Solution: In (2.1.1.1), let

$$\begin{aligned} \theta_1 + \theta_2 &= \alpha \\ \theta_2 &= \beta \end{aligned} \tag{2.1.2.3}$$

This gives (2.1.2.1). In (2.1.2.1), replace α by $90° - \alpha$. This results in

$$\sin(90° - \alpha - \beta) = \sin(90° - \alpha)\cos\beta - \cos(90° - \alpha)\sin\beta \tag{2.1.2.4}$$
$$\implies \cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \tag{2.1.2.5}$$

2.1.3. Using (2.1.1.1) and (2.1.2.2), show that

$$\sin(\theta_1 + \theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 \tag{2.1.3.1}$$
$$\cos(\theta_1 - \theta_2) = \cos\theta_1\cos\theta_2\sin\theta_1\sin\theta_2 \tag{2.1.3.2}$$

Solution: From (2.1.1.1),

$$\sin(\theta_1 + \theta_2)\cos\theta_2 = \sin\theta_1 + \cos(\theta_1 + \theta_2)\sin\theta_2 \tag{2.1.3.3}$$

Using (2.1.2.2) in the above,

$$\sin(\theta_1 + \theta_2)\cos\theta_2 = \sin\theta_1 + (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2)\sin\theta_2 \tag{2.1.3.4}$$

which can be expressed as

$$\sin(\theta_1 + \theta_2)\cos\theta_2 = \sin\theta_1 + \cos\theta_1\cos\theta_2\sin\theta_2 - \sin\theta_1\sin^2\theta_2 \tag{2.1.3.5}$$

Since

$$\sin^2\theta_2 = 1 - \cos^2\theta_2, \tag{2.1.3.6}$$

we obtain

$$\sin(\theta_1 + \theta_2)\cos\theta_2 = \cos\theta_1\cos\theta_2\sin\theta_2 + \sin\theta_1\cos^2\theta_2 \tag{2.1.3.7}$$

resulting in

$$\sin(\theta_1 + \theta_2) = \cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2 \tag{2.1.3.8}$$

after factoring out $\cos\theta_2$. Using a similar approach, (2.1.3.2) can also be proved.

2.1.4. Show that

$$\sin\theta_1 + \sin\theta_2 = 2\sin\left(\frac{\theta_1 + \theta_2}{2}\right)\cos\left(\frac{\theta_1 - \theta_2}{2}\right) \tag{2.1.4.1}$$

$$\cos\theta_1 + \cos\theta_2 = 2\cos\left(\frac{\theta_1 + \theta_2}{2}\right)\cos\left(\frac{\theta_1 - \theta_2}{2}\right) \tag{2.1.4.2}$$

$$\sin\theta_1 - \sin\theta_2 = 2\sin\left(\frac{\theta_1 - \theta_2}{2}\right)\cos\left(\frac{\theta_1 + \theta_2}{2}\right) \tag{2.1.4.3}$$

$$\cos\theta_1 - \cos\theta_2 = 2\sin\left(\frac{\theta_1 + \theta_2}{2}\right)\cos\left(\frac{\theta_2 - \theta_1}{2}\right) \tag{2.1.4.4}$$

Solution: Let

$$\begin{aligned} \theta_1 &= \alpha + \beta \\ \theta_2 &= \alpha - \beta \end{aligned} \tag{2.1.4.5}$$

From (2.1.3.1),

$$\begin{aligned} \sin\theta_1 + \sin\theta_2 &= \sin(\alpha + \beta) + \sin(\alpha - \beta) && (2.1.4.6) \\ &= \sin\alpha\cos\beta + \cos\alpha\sin\beta + \sin\alpha\cos\beta - \cos\alpha\sin\beta && (2.1.4.7) \\ &= 2\sin\alpha\cos\beta && (2.1.4.8) \end{aligned}$$

resulting in (2.1.4.1)

$$\because \alpha = \frac{\theta_1 + \theta_2}{2}, \ \beta = \frac{\theta_1 - \theta_2}{2} \tag{2.1.4.9}$$

from (2.1.4.5). Other identities may be proved similarly.

2.1.5. Show that

$$\sin 2\theta = 2\sin\theta\cos\theta \tag{2.1.5.1}$$

$$\cos 2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1 \tag{2.1.5.2}$$

$$= \cos^2\theta - \sin^2\theta \tag{2.1.5.3}$$

2.2 Medians

2.2.1. In Fig. 2.2.1

$$AF = BF, AE = BE, \tag{2.2.1.1}$$

and the medians BE and CF meet at **G**. Show that

$$ar(BEC) = ar(BFC) = \frac{1}{2}ar(ABC) \tag{2.2.1.2}$$

Solution: From (1.2.2.2),

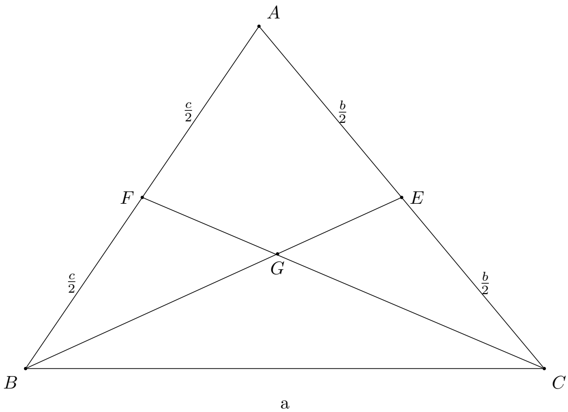


Fig. 2.2.1: $k_1 = k_2$.

$$ar(BEC) = \frac{1}{2}a\left(\frac{b}{2}\right)\sin C \tag{2.2.1.3}$$

$$ar(BFC) = \frac{1}{2}a\left(\frac{c}{2}\right)\sin B \tag{2.2.1.4}$$

yielding (2.2.1.2).

2.2.2. The median divides a triangle into two triangle of equal area. .

2.2.3. In Fig. 2.2.1, show that

$$ar(CGE) = ar(BGF) \tag{2.2.3.1}$$

Solution: From Fig. 2.2.1 and (2.2.1.2),

$$ar(BGF) + ar(BGC) = ar(CGE) + ar(BGC) \tag{2.2.3.2}$$

yielding (2.2.3.1).

2.2.4. In Fig. 2.2.2, show that

$$k_1 = k_2 \tag{2.2.4.1}$$

Solution: From (2.2.3.1),

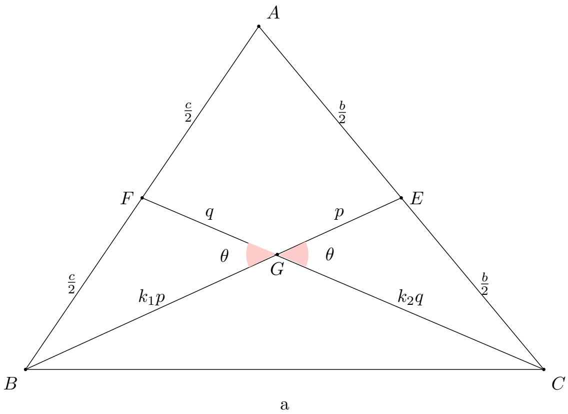


Fig. 2.2.2: Equal areas.

$$\frac{1}{2}p\,(k_1 q)\sin\theta = \frac{1}{2}q\,(k_2 p)\sin\theta \tag{2.2.4.2}$$

yielding (2.2.4.1).

2.2.5. In Fig. 2.2.3, show that

$$k_3 = k \tag{2.2.5.1}$$

Solution: From Problem 2.2.2,

$$\begin{aligned} ar\,(AGE) &= ar\,(CGE) \\ ar\,(AGF) &= ar\,(BGF) \end{aligned} \tag{2.2.5.2}$$

$$\implies \begin{aligned} \frac{1}{2}p\,(k_3 r)\sin\alpha &= \frac{1}{2}p\,(kq)\sin\theta \\ \frac{1}{2}q\,(k_3 r)\sin\beta &= \frac{1}{2}q\,(kp)\sin\theta \end{aligned} \tag{2.2.5.3}$$

yileding upon division

$$p\sin\alpha = q\sin\beta \tag{2.2.5.4}$$

$$\implies \frac{1}{2}kpr\sin\alpha = \frac{1}{2}kqr\sin\beta \tag{2.2.5.5}$$

$$\implies ar\,(BGD) = ar\,(CGD) \tag{2.2.5.6}$$

Thus, from Problem 2.2.2, AD is also a median. Consequently, from (2.2.4.1) we obtain (2.2.5.1).

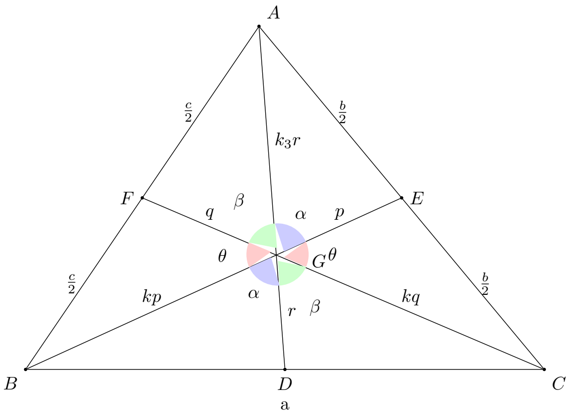


Fig. 2.2.3: $k_3 = k$.

2.2.6. In Fig. 2.2.4, show that $k = 2$.

Solution: Using the cosine formula,

$$DE^2 = \left(\frac{b}{2}\right)^2 + \left(\frac{c}{2}\right)^2 - 2\left(\frac{b}{2}\right)\left(\frac{c}{2}\right)\cos A \tag{2.2.6.1}$$

$$a^2 = b^2 + b^2 - 2bc\cos A \tag{2.2.6.2}$$

$$\implies DE = \frac{a}{2} \tag{2.2.6.3}$$

$\because \triangle EGF \sim \triangle BGC, k = 2.$

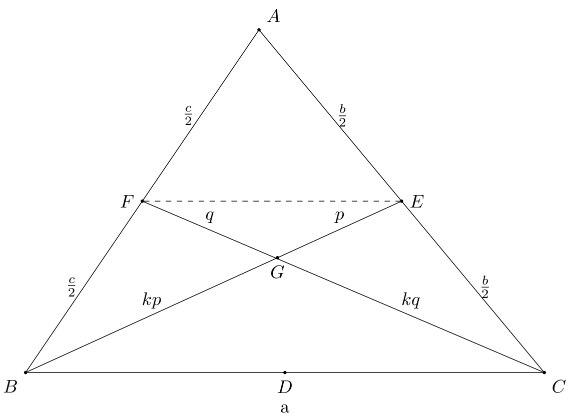


Fig. 2.2.4: k = 2

2.3 NCERT

2.3.1. D is a point on the side BC of a $\triangle ABC$ such that $\angle ADC = \angle BAC$. Show that

$$CA^2 = CB.CD \tag{2.3.1.1}$$

Solution: See Fig. 2.3.1.

$$\frac{x}{\sin(A+C)} = \frac{b}{\sin A} \quad (\triangle ADC), \tag{2.3.1.2}$$

$$\implies \frac{x}{\sin B} = \frac{b}{\sin A} \tag{2.3.1.3}$$

$$\implies \frac{x}{b} = \frac{\sin B}{\sin A} = \frac{b}{a} \quad (\text{ sine formula }) \tag{2.3.1.4}$$

yielding (2.3.1.1).

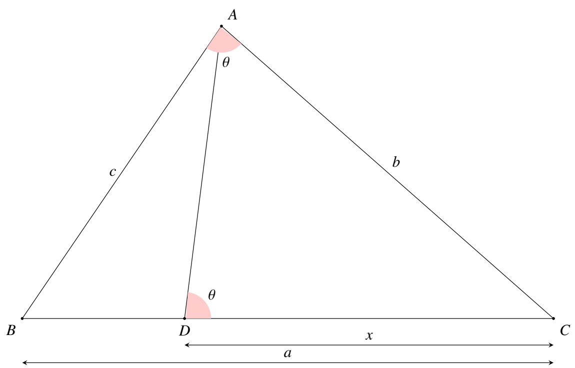


Fig. 2.3.1

2.3.2. D is a point on side BC of $\triangle ABC$ such that $\frac{BD}{CD} = \frac{AB}{AC}$. Prove that AD is the bisector of $\angle BAC$.
Solution: See Fig. 2.3.2.

$$\frac{x}{a-x} = \frac{c}{b} \quad (\text{ given }) \tag{2.3.2.1}$$

$$\frac{c}{\sin\phi} = \frac{x}{\sin\theta} \quad (\triangle ABD) \tag{2.3.2.2}$$

$$\frac{a-x}{\sin(A-\theta)} = \frac{b}{\sin 180-\phi} \quad (\triangle ACD) \tag{2.3.2.3}$$

$$= \frac{b}{\sin\phi} \tag{2.3.2.4}$$

using the sine formula. Multiplying all the above equations yields

$$\sin(A-\theta) = \sin\theta \implies \theta = \frac{A}{2} \tag{2.3.2.5}$$

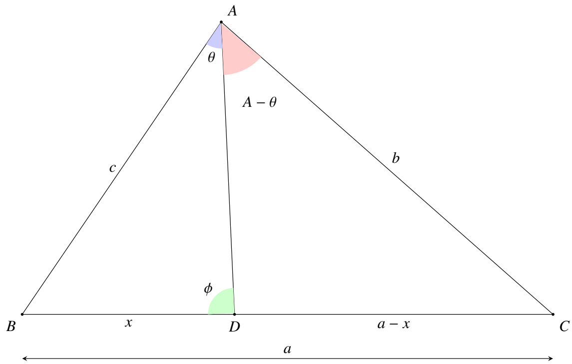


Fig. 2.3.2

2.3.3. ABC is a triangle in which $\angle ABC > 90°$ and $AD \perp CB$ produced. Prove that

$$AC^2 = AB^2 + BC^2 + 2BC.BD. \tag{2.3.3.1}$$

Solution: See Fig. 2.3.3.

$$\cos B = \frac{x}{c} \quad (\triangle ADB) \tag{2.3.3.2}$$

$$b^2 = a^2 + c^2 - 2ac\cos(180 - B) \quad (\triangle ABC) \tag{2.3.3.3}$$

$$= a^2 + c^2 + 2ac\cos B \tag{2.3.3.4}$$

using the cosine formula. Substituting from (2.3.3.2) in (2.3.3.4) yields (2.3.3.1).

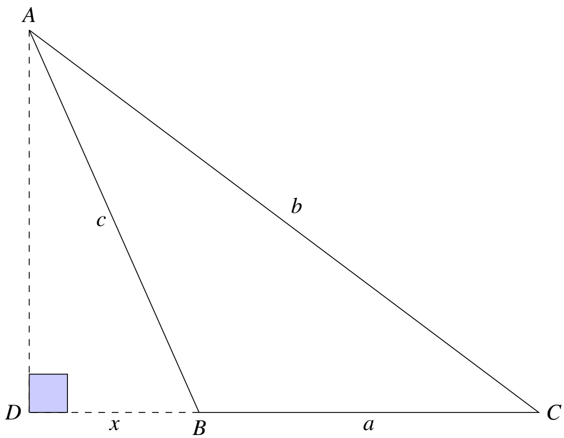


Fig. 2.3.3

2.3.4. In a right triangle, prove that the line-segment joining the mid-point of the hypotenuse to the opposite vertex is half the hypotenuse.

Solution: In Fig. 2.3.4

$$\frac{x}{\sin C} = \frac{b/2}{\sin\theta} \quad (\triangle BDC) \tag{2.3.4.1}$$

$$\frac{x}{\sin A} = \frac{b/2}{\sin(90-\theta)} \quad (\triangle BDA) \tag{2.3.4.2}$$

$$\implies \frac{x}{\cos C} = \frac{b/2}{\cos\theta} \tag{2.3.4.3}$$

From (2.3.4.1) and (2.3.4.3),

$$\left(\frac{\sin C}{x}\right)^2 + \left(\frac{\cos C}{x}\right)^2 = \left(\frac{\cos\theta}{\frac{b}{2}}\right)^2 + \left(\frac{\sin\theta}{\frac{b}{2}}\right)^2 \tag{2.3.4.4}$$

$$\implies x = \frac{b}{2} \tag{2.3.4.5}$$

using (1.1.5.1).

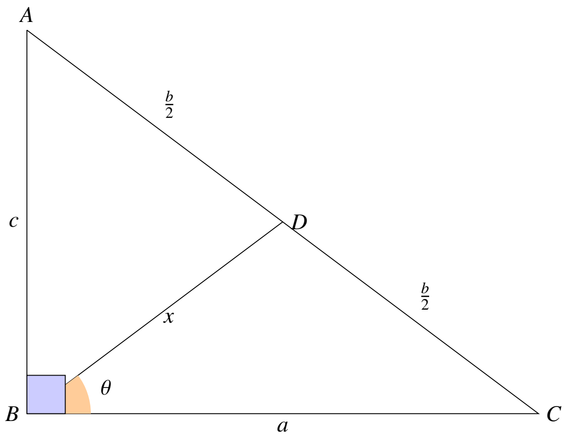


Fig. 2.3.4

2.3.5. $ABCD$ is a trapezium in which $AB \parallel DC$ and its diagonals intersect each other at the point O. Show that

$$\frac{AO}{BO} = \frac{CO}{DO} \tag{2.3.5.1}$$

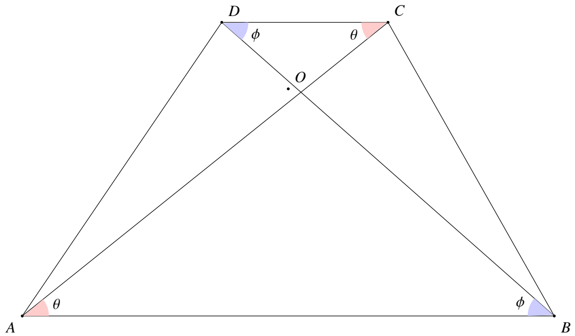


Fig. 2.3.5

Solution: In Fig. 2.3.5, $\because AB \parallel CD$

$$\frac{AO}{\sin\phi} = \frac{BO}{\sin\theta} \quad (\triangle OAB) \tag{2.3.5.2}$$

$$\frac{CO}{\sin\phi} = \frac{DO}{\sin\theta} \quad (\triangle ODC) \tag{2.3.5.3}$$

yielding (2.3.5.1) after simplification.

2.3.6. O is any point inside a rectangle $ABCD$. Prove that

$$OB^2 + OD^2 = OA^2 + OC^2 \tag{2.3.6.1}$$

Solution: In Fig. 2.3.6, from (1.1.4.1)

$$p\cos\theta_1 + q\sin\theta_2 = a \quad (\triangle OAB) \tag{2.3.6.2}$$

$$r\cos\theta_3 + s\sin\theta_4 = a \quad (\triangle OAB) \tag{2.3.6.3}$$

$$p\cos\theta_1 + s\sin\theta_4 = b \quad (\triangle OAB) \tag{2.3.6.4}$$

$$r\cos\theta_3 + q\sin\theta_2 = b \quad (\triangle OAB) \tag{2.3.6.5}$$

Subtracting the first two and second two equations respectively,

$$p\cos\theta_1 - s\sin\theta_4 = r\cos\theta_3 - q\sin\theta_2 \tag{2.3.6.6}$$

$$p\cos\theta_1 + s\sin\theta_4 = r\cos\theta_3 + q\sin\theta_2 \tag{2.3.6.7}$$

Squaring and adding and using (1.1.5.1) yields (2.3.6.1).

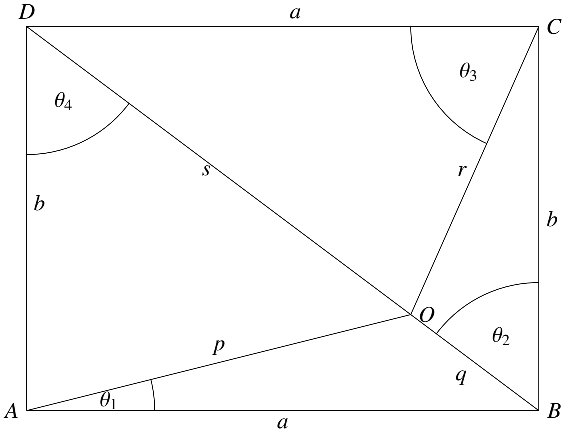


Fig. 2.3.6

2.3.7. In $\triangle ABC, AB = 6\sqrt{3}cm, AC = 12cm$ and $BC = 6cm$. Find the angle B.
Solution: Using (1.2.5.1),

$$\cos B = \frac{c^2 + a^2 - b^2}{2ca} = 0 \tag{2.3.7.1}$$

$$\implies B = 90^\circ \tag{2.3.7.2}$$

2.3.8. Prove that the line of centres of two intersecting circles subtends equal angles at the two points of intersection.
Solution: See Table 2.3.8 and Fig. 2.3.7. Using the cosine formula for θ_1 and θ_2

$$\cos\theta_1 = \cos\theta_1 = \frac{r_1^2 + r_2^2 - (r_1 + r_2)^2}{2r_1 r_2} \implies \theta_1 = \theta_2 \tag{2.3.8.1}$$

This shows that if corresponding sides of a triangle are equal, corresponding angles are also equal. This is known as *SSS* congruence.

Parameter	Description
r_1	Radius of first circle
r_2	Radius of second circle
θ_1	Angle subtended at first intersection
θ_2	Angle subtended at second intersection

TABLE 2.3.8

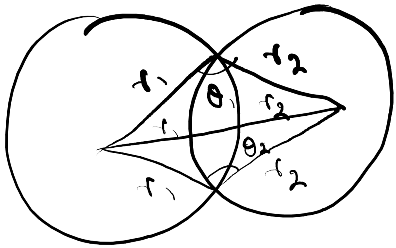


Fig. 2.3.7

2.4 CBSE

1) In an equilateral $\triangle ABC, D$ is a point on side BC such that $BD = \frac{1}{3}BC$. Prove that $9(AD)^2 = 7(AB)^2$. (10, 2018)
2) Prove that the area of an equilateral triangle described on one side of the square is equal to half of the area of the equilateral triangle described on one of its diagonal. (10, 2018)
3) If the areas of two similar triangles are equal, prove that they are congruent. (10, 2018)
4) In Fig. 2.4.1, BN and CM are medians of a $\triangle ABC$ right-angled at A. Prove that

$$4(BN^2 + CM^2) = 5BC^2$$

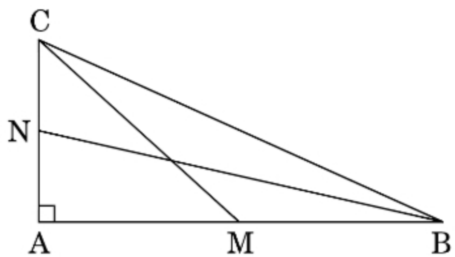


Fig. 2.4.1

(10, 2022)

5) If A, B and C are interior angles of $\triangle ABC$, then show that (10, 2020)

$$\cos\left(\frac{B+C}{2}\right) = \sin\left(\frac{A}{2}\right)$$

6) In $\triangle ABC$, right-angled at A, if $AB = 7cm$ and $AC = 24cm$, then find $\sin B$ and $\tan C$. (10, 2021)

7) Two angles of a triangle are $\cot^{-1} 2$ and $\cot^{-1} 3$. The third angle of the triangle is ______. (12, 2021)

8) A, B and C are interior angles of a triangle ABC. Show that (10, 2019)
   a) $\sin\left(\frac{B+C}{2}\right) = \cos\frac{A}{2}$
   b) If $\angle A = 90°$, then find the value of $\tan\left(\frac{B+C}{2}\right)$.

9) In $\triangle\, ABC$, $AB = 4\sqrt{3}$ cm, $AC = 8cm$ and $BC = 4cm$. The angle B is (10, 2021)

   a) $120°$ b) $90°$ c) $60°$ d) $45°$

2.5 JEE

2.5.1 In a $\triangle ABC$, $\angle A = 90°$ and AD is an altitude. Complete the relation

$$\frac{BD}{BA} = \frac{AB}{(\ldots)}.$$

(1980)

2.5.2 ABC is a triangle, P is a point on AB, and Q is point on AC such that $\angle AQP = \angle ABC$. Complete the relation

$$\frac{ar\left(\triangle APQ\right)}{ar\left(\triangle ABC\right)} = \frac{(\ldots)}{AC^2}$$

(1980)

2.5.3 ABC is a triangle with $\angle B$ greater than $\angle C$. D and E are the points on BC such that AD is perpendicular to BC and AE is the bisector of angle A. Complete the relation

$$\angle DAE = \frac{1}{2}[() - \angle C].$$

(1980)

2.5.4 The set of all real numbers a such that $a^2 + 2a, 2a + 3$ and $a^2 + 3a + 8$ are the sides of a triangle is ______. (1985)

2.5.5 In $\triangle ABC$, if $\cot A, \cot B, \cot C$ are in A.P., then a^2, b^2, c^2 are in ______ progression (1985)

2.5.6 If in the $\triangle ABC$,

$$\frac{2\cos A}{a} + \frac{2\cos B}{b} + \frac{2\cos C}{c} = \frac{a}{bc} + \frac{b}{ac},$$

then the value of the angle A is ______ degrees. (1993)

2.5.7 In the $\triangle ABC$, AD is the altitude from A. Given $b > c$, $\angle C = 23°$ and $AD = \frac{abc}{b^2-c^2}$ then $\angle B =$ ______. (1994)

2.5.8 In a $\triangle ABC$, medians AD and BE are drawn. If $AD = 4, \angle DAB = \frac{\pi}{6}$ and $\angle ABE = \frac{\pi}{3}$, then the area of the $\triangle ABC$ is (2003)

   a) $\frac{64}{3}$ b) $\frac{8}{3}$ c) $\frac{16}{3}$ d) $\frac{32}{3\sqrt{3}}$

2.5.9 If in $\triangle ABC, a\cos^2\left(\frac{C}{2}\right) + c\cos^2\left(\frac{A}{2}\right) = \frac{3b}{2}$, then the sides a, b and c (2003)

a) satisfy $a+b=c$ b) are in A.P. c) are in G.P. d) are in H.P.

2.5.10 The sides of a triangle are $\sin\alpha, \cos\alpha$ and $\sqrt{1+\sin\alpha\cos\alpha}$ for some $0<\alpha<\frac{\pi}{2}$. Then the greatest angle of the triangle is (2004)

a) $150°$ b) $90°$ c) $120°$ d) $60°$

2.5.11 In a $\triangle ABC$, let $\angle C=\frac{\pi}{2}$. If r is the inradius and R is the circumradius of the $\triangle ABC$, then $2(R+r)$ equals (2005)

a) $b+c$ b) $a+b$ c) $a+b+c$ d) $c+a$

2.5.12 If in a $\triangle ABC$, the altitudes from the vertices A, B, C on opposite sides are in H.P., then $\sin A, \sin B, \sin C$ are in (2005)

a) $G.P.$ b) $A.P.$ c) $A.P.-G.P.$ d) $H.P.$

2.5.13 There exists a $\triangle ABC$ satisfying the conditions (1986)

a) $b\sin A=a, A<\pi/2$
b) $b\sin A>a, A>\pi/2$
c) $b\sin A>a, A<\pi/2$
d) $b\sin A<a, A<\pi/2,\ b>a$
e) $b\sin A<a, A>\pi/2,\ b=a$

2.5.14 In a triangle, the lengths of two larger sides are 10 and 9 respectively. If the angles are in AP, Then length of third side is (1987)

a) $5-\sqrt{6}$
b) $3\sqrt{3}$
c) 3
d) $5+\sqrt{6}$
e) none

2.5.15 If in a $\triangle PQR$, $\sin P, \sin Q, \sin R$ are in AP, then (1998)

a) The altitudes are in AP
b) The altitudes are in HP
c) The medians are in GP
d) The medians are in AP

2.5.16 In $\triangle ABC$, internal angle bisector of $\angle A$ meets side BC in D. $DE\perp AD$ meets AC in E and AB in F. Then (2006)

a) AE is HM of b and c
b) $AD=\frac{2bc}{b+c}\cos\frac{A}{2}$
c) $EF=\frac{4bc}{b+c}\sin\frac{A}{2}$
d) $\triangle AEF$ is isosceles

2.5.17 Let ABC be a triangle such that $\angle ACB=\pi/6$ and let a, b and c denote lengths of the sides opposite to A,B and C respectively. The value(s) of x for which $a=x^2+x+1, b=x^2-1, c=2x+1$ is (are) (2010)

a) $-(2+\sqrt{3})$ b) $1+\sqrt{3}$ c) $2+\sqrt{3}$ d) $4\sqrt{3}$

2.5.18 If the bisector of the angle P of a $\triangle PQR$ meets QR in S, then (1979)

a) $QS = SR$
b) $QS : SR = PR : PQ$
c) $QS : SR = PQ : PR$
d) None of these

2.5.19 In the $\triangle ABC$, angle A is the greater than angle B. If the measures of the angles A and B satisfies the equation $3\sin x - 4\sin^3 x - k = 0, 0 < k < 1$, then the measure of the angle C is (1985)

a) $\frac{\pi}{3}$ b) $\frac{\pi}{2}$ c) $\frac{2\pi}{3}$ d) $\frac{5\pi}{6}$

2.5.20 If the lengths of the sides of a triangle are $3, 5, 7$ then the largest angle of the triangle is (1986)

a) $\frac{\pi}{2}$ b) $\frac{5\pi}{6}$ c) $\frac{2\pi}{3}$ d) $\frac{3\pi}{4}$

2.5.21 In a $\triangle ABC$, $\angle B = \frac{\pi}{3}$ and $\angle C = \frac{\pi}{4}$. Let D divide BC internally in the ratio $1: 3$ then $\frac{\sin \angle BAD}{\sin \angle CAD}$ is equal to (1995)

a) $\frac{1}{\sqrt{6}}$ b) $\frac{1}{3}$ c) $\frac{1}{\sqrt{3}}$ d) $\sqrt{\frac{2}{3}}$

2.5.22 In a $\triangle ABC, 2ac\sin\frac{1}{2}(A - B + C) =$ (2000)

a) $a^2 + b^2 - c^2$
b) $c^2 + a^2 - b^2$
c) $b^2 - c^2 - a^2$
d) $c^2 - a^2 - b^2$

2.5.23 In a $\triangle ABC$, let $\angle C = \frac{\pi}{2}$. If r is the inradius and R is the circumradius of the triangle, then $2(r + R)$ is equal to (2000)

a) $a + b$ b) $b + c$ c) $c + a$ d) $a + b + c$

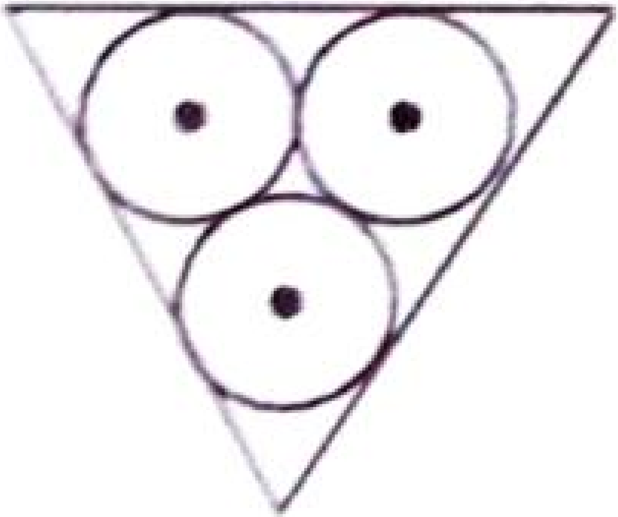
2.5.24 If the angles of a triangle are in the ratio $4: 1: 1$, then the ratio of the longest side to the perimeter is (2003)

a) $\sqrt{3}: 2+\sqrt{3}$ b) $1: 6$ c) $1: 2+\sqrt{3}$ d) $2: 3$

2.5.25 The sides of a triangle are in the ratio $1: \sqrt{3}: 2$, then the angles of the triangle are in the ratio (2004)

a) $1: 3: 5$
b) $2: 3: 4$
c) $3: 2: 1$
d) $1: 2: 3$

2.5.26 In an equilateral triangle, 3 coins of radii 1 unit each are kept so they touch each other and also the sides of the triangle. Area of the triangle is (2005)



a) $4+2\sqrt{3}$ b) $6+4\sqrt{3}$ c) $12+\frac{7\sqrt{3}}{4}$ d) $3+\frac{7\sqrt{3}}{4}$

2.5.27 In a $\triangle ABC$, a, b, c are the lengths of its sides and A, B, C are the angles of $\triangle ABC$. The correct relation is given by (2005)

a) $(b-c)\sin\left(\frac{B-C}{2}\right) = a\cos\left(\frac{A}{2}\right)$
b) $(b-c)\cos\left(\frac{A}{2}\right) = a\sin\left(\frac{B-C}{2}\right)$
c) $(b-c)\sin\left(\frac{B+C}{2}\right) = a\cos\left(\frac{A}{2}\right)$
d) $(b-c)\cos\left(\frac{A}{2}\right) = a\sin\left(\frac{B+C}{2}\right)$

2.5.28 If the angles A, B and C of a triangle are in an arithmetic progression and if a, b and c denote the lengths of the sides opposite to A, B and C respectively, then the value of the expression $\frac{a}{c}\sin 2C + \frac{c}{a}\sin 2A$ is (2010)

a) $\frac{1}{2}$ b) $\frac{\sqrt{3}}{2}$ c) 1 d) $\sqrt{3}$

2.5.29 Let PQR be a triangle of area $\triangle$ with $a = 2, b = \frac{7}{2}$ and $c = \frac{5}{2}$, where a, b and c are the lengths of the sides of the triangle opposite to the angles at P, Q and R respectively. Then $\frac{2\sin P - \sin 2P}{2\sin P + \sin 2P}$ equals (2012)

a) $\frac{3}{4\triangle}$ b) $\frac{45}{4\triangle}$ c) $\left(\frac{3}{4\triangle}\right)^2$ d) $\left(\frac{45}{4\triangle}\right)^2$

2.5.30 In a triangle the sum of two sides is x and the product of the same sides is y. If $x^2 - c^2 = y$, where c is the third side of the triangle, then the ratio of the inradius to the circum-radius of the triangle is (2014)

a) $\frac{3y}{2(x+c)}$
b) $\frac{3y}{2c(x+c)}$
c) $\frac{3y}{4x(x+c)}$
d) $\frac{3y}{4c(x+c)}$

2.5.31 A $\triangle ABC$ has sides $AB = AC = 5cm$ and $BC = 6cm$. $\triangle A'B'C'$ is the reflection of the $\triangle ABC$ in a line parallel to AB placed at a distance of 2 cm from AB, outside the $\triangle ABC$. $\triangle A''B''C''$ is the reflection of the $\triangle A'B'C'$ in a line parallel $B'C'$ placed at a distance of $2cm$ from $B'C'$ outside the $\triangle A'B'C'$. Find the distance between A and A'. (1978)

2.5.32 ABC is a triangle. D is the middle point of BC. If AD is perpendicular to AC, then prove that $\cos A \cos C = \frac{2(c^2-a^2}{3ac}$. (1980)

2.5.33 ABC is a triangle with $AB = AC$. D is any point on the side BC. E and F are points on the side AB and AC, respectively, such that DE is parallel to AC, and DF is parallel to AB. Prove that (1980)

$$DF + FA + AE + ED = AB + AC$$

2.5.34 Let the angles A, B, C of a $\triangle ABC$ be in A.P. and let $b : c = \sqrt{3} : \sqrt{2}$. Find the angle A. (1981)

2.5.35 The ex-radii r_1, r_2, r_3 of $\triangle ABC$ are in H.P. Show that its sides a, b, c are in A.P. (1983)

2.5.36 For a $\triangle ABC$ it is given that $\cos A + \cos B + \cos C = \frac{3}{2}$. Prove that the triangle is equilateral. (1984)

2.5.37 With usual notation, if in a $\triangle ABC$

$$\frac{b+c}{11} = \frac{c+a}{12} = \frac{a+b}{13}$$

then prove that

$$\frac{\cos A}{7} = \frac{\cos B}{19} = \frac{\cos C}{25}.$$

(1984)

2.5.38 In a $\triangle ABC$, the median to the side BC is of length $\frac{1}{\sqrt{11-6\sqrt{3}}}$ and it divides the angle A into angles $30°$ and $45°$. Find the length of the side BC. (1985)

2.5.39 If in a $\triangle ABC$, $\cos A \cos B + \sin A \sin B \sin C = 1$, show that $a : b : c = 1 : 1 : \sqrt{2}$. (1986)

2.5.40 The sides of a triangle are three consecutive natural numbers and its largest angle is twice the smallest one. Determine the sides of the triangle. (1991)

2.5.41 In a triangle of base a the ratio of the other two sides is $r\,(< 1)$. Show that the altitude of the triangle is less than or equal to $\frac{ar}{1-r^2}$. (1991)

2.5.42 If the angles of a triangle are $30°$ and $45°$ and the included side is $(\sqrt{3}+1)cm$, then the area of the triangle is ______. (1988)

2.5.43 The sides of a triangle in a given circle subtend angles α, β, γ. The minimum value of arithmetic mean of $\cos\left(\alpha + \frac{\pi}{2}\right)$, $\cos\left(\beta + \frac{\pi}{2}\right)$, $\cos\left(\gamma + \frac{\pi}{2}\right)$ is equal to ______. (1987)

2.5.44 $ABCD$ is a trapezium such that AB and CD are parallel and $BC \perp CD$. If $\angle ABD = \theta$, $BC=p$ and $CD=q$, then AB is equal to (2013)

a) $\frac{(p^2+q^2)\sin\theta}{p\cos\theta+q\sin\theta}$ b) $\frac{p^2+q^2\cos\theta}{p\cos\theta+q\sin\theta}$ c) $\frac{p^2+q^2}{p\cos^2\theta+q\sin^2\theta}$ d) $\frac{(p^2+q^2)\sin\theta}{(p\cos\theta+q\sin\theta)^2}$

2.5.45 In a $\triangle PQR$, $\angle R = \frac{\pi}{2}$. If $\tan\frac{P}{2}$ and $\tan\frac{Q}{2}$ are the roots of the equation $ax^2+bx+c = 0$ $(a \neq 0)$ then (1999)

a) $a+b=c$ b) $b+c=a$ c) $a+c=b$ d) $b=c$

2.5.46 Let O be the origin, and $\overrightarrow{OX}, \overrightarrow{OY}, \overrightarrow{OZ}$ be three unit vectors in the directions of the sides $\overrightarrow{QR}, \overrightarrow{RP}, \overrightarrow{PQ}$ respectively, of a triangle PQR. (2017)

a) $\left|\overrightarrow{OX} \times \overrightarrow{OY}\right| =$

i) $\sin(P+Q)$ ii) $\sin 2R$ iii) $\sin(P+R)$ iv) $\sin(Q+R)$

b) If the triangle PQR varies, then the minimum value of $\cos(P+Q)+\cos(Q+R)+\cos(R+P)$ is

i) $\frac{-5}{3}$ ii) $\frac{-3}{2}$ iii) $\frac{3}{2}$ iv) $\frac{5}{3}$

2.5.47 ABC is a triangle such that (1990)

$$\sin(2A+B) = \sin(C-A) = -\sin(B+2C) = \frac{1}{2}.$$

If A, B and C are in arithmetic progression, determine the values of A, B and C.

2.5.48 In any $\triangle ABC$, prove that (2000)

$$\cot\left(\frac{A}{2}\right)+\cot\left(\frac{B}{2}\right)+\cot\left(\frac{C}{2}\right) = \cot\left(\frac{A}{2}\right)\cot\left(\frac{B}{2}\right)\cot\left(\frac{C}{2}\right).$$

2.5.49 Let x, y and z be positive real numbers. Suppose x, y and z are the lengths of the sides of a triangle opposite to its angles X, Y and Z, respectively. If

$$\tan\left(\frac{X}{2}\right)+\tan\left(\frac{Z}{2}\right) = \frac{2y}{x+y+z}$$

then which of the following statements is/are TRUE? (2020)

a) $2Y = X+Z$
b) $Y = X+2$
c) $\tan\frac{X}{2} = \frac{x}{y+x}$
d) $x^2+z^2-y^2 = xz$

2.5.50 In a triangle ABC, let $AB = \sqrt{23}$, $BC = 3$, and $CA = 4$. Then the value of

$$\frac{\cot A+\cot C}{\cot B}$$

is ______. (2021)

2.5.51 Let $PQRS$ be a quadrilateral in a plane, where $QR = 1$, $\angle PQR = \angle QRS = 70°$, $\angle PQS = 15°$, and $\angle PRS = 40°$. If $\angle RPS = \theta°$, $PQ = \alpha$, and $PS = \beta$, then the interval(s) that contain(s) the value of $4\alpha\beta\sin\theta°$ is/are (2022)

a) $(0, \sqrt{2})$ b) $(1, 2)$ c) $(\sqrt{2}, 3)$ d) $(2\sqrt{2}, 3\sqrt{2})$

2.6 *Olympiad*

2.6.1 Let $ABCD$ be a convex quadrilateral with perpendicular diagonals. If $AB = 20$, $BC = 70$, and $CD = 90$, then what is the value of DA? (PRMO 2014)

2.6.2 In a triangle with integer side lengths, one side is three times as long as a second side, and the length of the third side is 17. What is the greatest possible perimeter of the triangle? (PRMO 2014)

2.6.3 In a triangle ABC, X and Y are points on the segments AB and AC, respectively, such that $AX : XB = 1 : 2$ and $AY : YC = 2 : 1$. If the area of triangle AXY is 10, then what is the area of triangle ABC? (PRMO 2014)

2.6.4 Let XOY be a triangle with $\angle XOY = 90°$. Let M and N be the midpoints of legs OX and OY, respectively. Suppose that $XN = 19$ and $YM = 22$. What is XY? (PRMO 2014)

2.6.5 In $\triangle ABC$, we have $AC = BC = 7$ and $AB = 2$. Suppose that D is a point on line AB such that B lies between A and D and $CD = 8$. What is the length of the segment BD? (PRMO 2012)

2.6.6 In rectangle $ABCD$, $AB = 5$ and $BC = 3$. Points F and G are on line segment CD so that $DF = 1$ and $GC = 2$. Lines AF and BG intersect at E. What is the area of $\triangle ABE$? (PRMO 2012)

2.6.7 A triangle with perimeter 7 has integer side lengths. What is the maximum possible area of such a triangle? (PRMO 2012)

2.6.8 $ABCD$ is a square and $AB = 1$. Equilateral triangles AYB and CXD are drawn such that X and Y are inside the square. What is the length of XY? (PRMO 2012)

2.6.9 A 2×3 rectangle and a 3×4 rectangle are contained within a square without overlapping at any interior point, and the sides of the square are parallel to the sides of the two given rectangles. What is the smallest possible area of the square? (PRMO 2015)

2.6.10 What is the greatest possible perimeter of a right-angled triangle with integer side lengths if one of the sides has length 12? (PRMO 2015)

2.6.11 In the acute-angled triangle ABC, let D be the foot of the altitude from A, and E be the midpoint of BC. Let F be the midpoint of AC. Suppose $\angle BAE = 40°$. If $\angle DAE = \angle DFE$, what is the magnitude of $\angle ADF$ in degrees? (PRMO 2015)

2.6.12 In an equilateral triangle of side length 6, pegs are placed at the vertices and also evenly along each side at a distance of 1 from each other. Four distinct pegs are chosen from the 15 interior pegs on the sides (that is, the chosen ones are not vertices of the triangle) and each peg is joined to the respective opposite vertex by a line segment. If N denotes the number of ways we can choose the pegs such that the drawn line segments divide the interior of the triangle into exactly nine regions, find the sum of the squares of the digits of N. (IOQM 2015)

2.6.13 In a triangle ABC, let E be the midpoint of AC and F be the midpoint of AB. The medians BE and CF intersect at G. Let Y and Z be the midpoints of BE and CF, respectively. If the area of triangle ABC is 480, find the area of triangle GYZ. (IOQM 2015)

2.6.14 Let X be the set of all even positive integers n such that the measure of the angle of some regular polygon is n degrees. Find the number of elements in X. (IOQM 2015)

2.6.15 Let ABC be a triangle in the xy-plane, where B is at the origin $(0,0)$. Let BC be produced to D such that $BC : CD = 1 : 1$, CA be produced to E such that $CA : AE = 1 : 2$, and AB be produced to F such that $AB : BF = 1 : 3$. Let $G(32,24)$ be the centroid of triangle ABC and K be the centroid of triangle DEF. Find the length GK. (IOQM 2015)

2.6.16 A trapezium in the plane is a quadrilateral in which a pair of opposite sides are parallel. A trapezium is said to be non-degenerate if it has positive area. Find the number of mutually non-congruent, non-degenerate trapeziums whose sides are four distinct integers from the set $\{5,6,7,8,9,10\}$. (IOQM 2015)

2.6.17 Consider the convex quadrilateral $ABCD$. The point P is the interior of $ABCD$. The following ratio equalities hold

$$\angle PAD : \angle PBA : \angle DPA = 1 : 2 : 3 = \angle CBP : \angle BAP : \angle BPC.$$

prove that the following three lines meet in a point: the internal bisectors of angles $\angle ADP$ and $\angle PCB$ and the perpendicular bisector of segment AB. (IMO 2020)

2.6.18 Three points X, Y, Z are on a straight line such that $XY = 10$ and $XZ = 3$. What is the product of all possible values of YZ? (PRMO 2013)

2.6.19 Let AD and BC be the parallel sides of a trapezium $ABCD$. Let P and Q be the midpoints of the diagonals AC and BD. If $AD = 16$ and $BC = 20$, what is the length of PQ? (PRMO 2013)

2.6.20 Let ABC be an equilateral triangle. Let P and S be points on AB and AC, respectively, and let Q and R be points on BC such that $PQRS$ is a rectangle. If $PQ = \sqrt{3}PS$ and the area of $PQRS$ is $\frac{28}{3}$, what is the length of PC? (PRMO 2013)

2.6.21 Let A_1, B_1, C_1, D_1 be the midpoints of the sides of a convex quadrilateral $ABCD$ and let A_2, B_2, C_2, D_2 be the midpoints of the sides of the quadrilateral $A_1B_1C_1D_1$. If $A_2B_2C_2D_2$ is a rectangle with sides 4 and 6, then what is the product of the lengths of the diagonals of $ABCD$? (PRMO 2013)

2.6.22 In a triangle ABC with $\angle BCA = 90°$, the perpendicular bisector of AB intersects segments AB and AC at X and Y, respectively. If the ratio of the area of quadrilateral $BXYC$ to the area of triangle ABC is 13:18 and $BC = 12$, then what is the length of AC? (PRMO 2013)

2.6.23 A convex hexagon has the property that for any pair of opposite sides the distance between their midpoints is $\frac{\sqrt{3}}{2}$ times the sum of their lengths Show that all the hexagon's angles are equal. (IMO 2003)

2.6.24 In a triangle ABC, let AP bisect $\angle BAC$, with P on BC, and let BQ bisect $\angle ABC$, with Q on CA. It is known that $\angle BAC = 60°$ and that $AB + BP = AQ + QB$. What are the possible angles of triangle ABC? (IMO 2001)

2.6.25 Let d be the sum of the lengths of all the diagonals of a plane convex polygon with n vertices ($n > 3$), and let p be its perimeter. Prove that

$$\ln -3 < \frac{2d}{p} < \left[\frac{n}{2}\right]\left[\frac{n+1}{2}\right] - 2,$$

Where $[x]$ denotes the greatest integer not exceeding x. (IMO 1984)

2.6.26 P is a point inside a given triangle ABC. D, E, F are the feet of the perpendiculars from P to the lines BC, CA, AB respectively. Find all P for which

$$\frac{BC}{PD} + \frac{CA}{PE} + \frac{AB}{PF}$$

is least. (IMO 1981)

2.6.27 The diagonals AC and CE of the regular hexagon $ABCDEF$ are divided by the inner points M and N, respectively, so that

$$\frac{AM}{AC} = \frac{CN}{CE} = r.$$

Determine r if B, M and N are collinear. (IMO 1982)

2.6.28 Let A, B be adjacent vertices of a regular n-gon ($n \leq 5$) in the plane having center at O. A triangle XYZ, which is congruent to and initially coincides with OAB, moves in the plane in such a way that Y and Z each trace out the whole boundary of the polygon, X remaining inside the polygon. Find the locus of X. (IMO 1986)

2.6.29 ABC is a triangle right-angled at A, and D is the foot of the altitude from A. The straight line joining the incenters of the triangles ABD, ACD intersects the sides AB, AC at the points K, L respectively. S and T denote the areas of the triangles ABC and AKL respectively. Show that $S \geq 2T$. (IMO 1988)

2.6.30 Let $ABCD$ be a convex quadrilateral such that the sides AB, AD, BC satisfy $AB = AD + BC$. There exists a point P inside the quadrilateral at a distance h from the line CD such that $AP = h + AD$ and $BP = h + BC$. Show that

$$\frac{1}{\sqrt{h}} \geq \frac{1}{\sqrt{AD}} + \frac{1}{\sqrt{BC}}$$

. (IMO 1989)

2.6.31 Prove that there exists a convex 1990-gon with the following two properties

a) All angles are equal.

b) The lengths of the 1990 sides are the numbers $1^2, 2^2, 3^2, \ldots 1990^2$ in some order. (IMO 1990)

2.6.32 Let ABC be a triangle and P an interior point of ABC. Show that at least one of the angles $\angle PAB, \angle PBC, \angle PCA$ is less than or equal to $30°$. (IMO 1991)

2.6.33 Equilateral triangles ABK, BCL, CDM, DAN are constructed inside the square $ABCD$. Prove that the midpoints of the four segments KL, LM, MN, NK and the midpoints of the eight segments $AKBK$, BL, CL, CM, DM, DN, AN are the twelve vertices of a regular dodecagon. (IMO 1977).

2.6.34 A triangle $A_1A_2A_3$ and a point P_0 are given in the plane. We define $A_s = A_s - 3$ for all $s \geq 4$. We construct a set of points P_1, P_2, P_3 ..., such that P_{k+1} is the image of P_k under a rotation with center A_{k+1} through angle $120°$ clockwise for ($k = 0, 1, 2, 3 \ldots$). Prove that if P_{1986}=P_0, then the triangle $A_1A_2A_3$ is equilateral. (IMO 1986)

2.6.35 Six points are chosen on the sides of an equilateral triangle ABC: A_1,A_2 on BC,B_1,B_2 on CA and C_1,C_2 on AB, such that they are the vertices of a convex hexagon A_1A_2 B_1B_2 C_1C_2 with equal side lengths. Prove that the line A_1B_2,B_1C_2 and C_1A_2 are concurrent. (IMO 2005)

2.6.36 Let P be a regular 2006-gon. A diagonal of P is called good if its endpoints divide the boundary of P into two parts, each composed of an odd number of sides of P. The sides of P are also called good. Suppose P has been dissected into triangles by 2003 diagonals, no two of which have a common point in the interior of P. Find the maximum number of isosceles triangles having two good sides that could appear in such a configuration. (IMO 2006)

2.6.37 Assign to each side b of a convex polygon P the maximum area of a triangle that has b as a side and is contained in P. Show that the sum of the areas assigned to the sides of P is at least twice the area of P. (IMO 2006)

2.6.38 Let $ABCDEF$ be a convex hexagon with $AB = BC = CD$ and $DE = EF = FA$, such that $\angle BCD = \angle EFA = \frac{\pi}{3}$. Suppose G and H are points in the interior of the hexagon such that $\angle AGB = \angle DHE = \frac{2\pi}{3}$. Prove that $AG + GB + GH + DH + HE \geq CF$. (IMO 1995)

2.6.39 Triangle BCF has a right angle at B. Let A be the point on line CF such that $FA = FB$ and F lies between A and C. Point D is chosen such that $DA = DC$ and AC is the bisector of $\angle DAB$. Point E is chosen such that $EA = ED$ and AD is the bisector of $\angle EAC$. Let M be the midpoint of CF. Let X be the point such that $AMXE$ is a parallelogram (where $AM \parallel EX$ and $AE \parallel MX$). Prove that lines BD, FX, and ME are concurrent. (IMO 2016)

2.6.40 A convex quadrilateral $ABCD$ satisfies

$$AB.CD = BC.DA.$$

Point X lies inside $ABCD$ so that

$$\angle XAB = \angle XCD \text{ and } \angle XBC = \angle XDA.$$

Prove that

$$\angle BXA + \angle DXC = 180^\circ.$$

(IMO 2018)

2.6.41 For three points P, Q, R in the plane, we define $m(PQR)$ as the minimum length of the three altitudes of $\triangle PQR$. (If the points are collinear, we set $m(PQR) = 0$.) Prove that for points A, B, C, X in the plane, (IMO 1993)

$$m(ABC) \leq m(ABX) + m(AXC) + m(XBC).$$

2.6.42 ABC is an isosceles triangle with $AB = AC$. Suppose that M is the midpoint of BC and O is the point on the line AM such that OB is perpendicular to AB. (IMO 1994)

a) Q is an arbitrary point on the segment BC different from B and C;
b) E lies on the line AB and F lies on the line AC such that E, Q, F are distinct and collinear.

Prove that OQ is perpendicular to EF if and only if $QE = QF$.

2.6.43 Four real constants a, b, A, B are given, and

$$f(\theta) = 1 - a\cos\theta - b\sin\theta - A\cos 2\theta - B\sin 2\theta$$

. Prove that if

$$f(\theta) \geq 0$$

for all real θ, then (IMO 1977)

$$a^2 + b^2 \leq 2 \text{ and } A^2 + B^2 \leq 1$$

3 Circle

3.1 Incircle

3.1.1. In Fig. 3.1.1, the bisectors of $\angle B$ and $\angle C$ meet at **I**. Show that IA bisects $\angle A$.

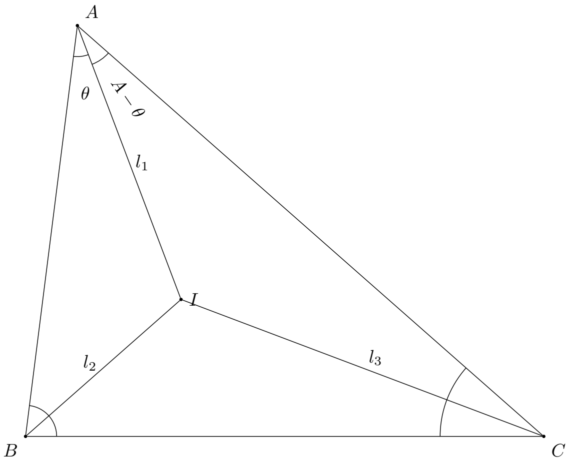


Fig. 3.1.1: Incentre I of $\triangle ABC$

Solution: Using sine formula in (1.2.2.3)

$$\frac{l_1}{\sin\frac{C}{2}} = \frac{l_3}{\sin(A-\theta)}, \quad \frac{l_3}{\sin\frac{B}{2}} = \frac{l_2}{\sin\frac{C}{2}}, \quad \frac{l_2}{\sin\theta} = \frac{l_1}{\sin\frac{B}{2}} \tag{3.1.1.1}$$

Multiplying the above equations,

$$\sin\theta = \sin(A-\theta) \tag{3.1.1.2}$$

$$\implies \theta = A - \theta \text{ or, } \theta = \frac{A}{2} \tag{3.1.1.3}$$

3.1.2. Fig. 3.1.2, is obtained from Fig. 3.1.1 with

$$ID \perp BC, IE \perp AC, IF \perp AB. \tag{3.1.2.1}$$

Show that

$$ID = IE = IF = r \tag{3.1.2.2}$$

Solution: In $\triangle$s IDC and IEC,

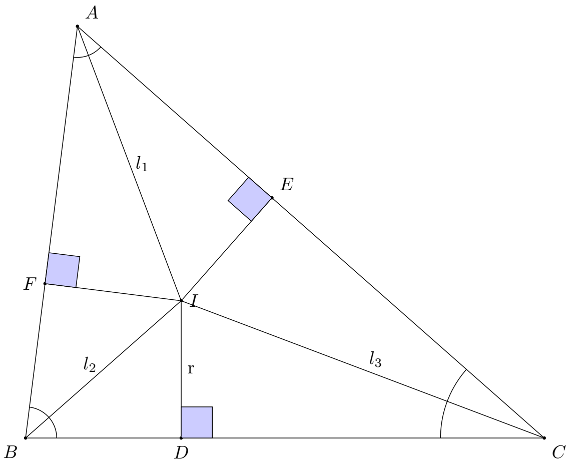


Fig. 3.1.2: Inradius r of $\triangle ABC$

$$ID = IE = \frac{l_3}{\sin \frac{C}{2}} \tag{3.1.2.3}$$

Similarly, in $\triangle$s IEA and IFA,

$$IF = IE = \frac{l_1}{\sin \frac{A}{2}} \tag{3.1.2.4}$$

yielding (3.1.2.2)

3.1.3. In Fig. 3.1.2, show that

$$BD = BF, AE = AF, CD = CE \tag{3.1.3.1}$$

Solution: From Fig. 3.1.2, in $\triangle$s IBD and IBF,

$$x = BD = BF = r \cot \frac{B}{2} \tag{3.1.3.2}$$

Similarly, other results can be obtained.

3.1.4. The circle with centre **I** and radius r in Fig. 3.1.3 is known as the *incircle*.

3.1.5. The lengths of tangents drawn from an external point to a circle are equal.

3.1.6. In an isosceles $\triangle ABC$, with $AB = AC$, BE and CF are the bisectors of $\angle B$ and $\angle C$ respectively. Show that

$$BE = CF \tag{3.1.6.1}$$

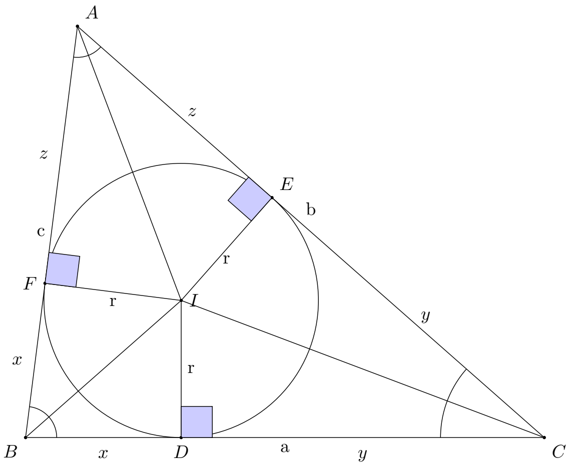


Fig. 3.1.3: Incircle of $\triangle ABC$

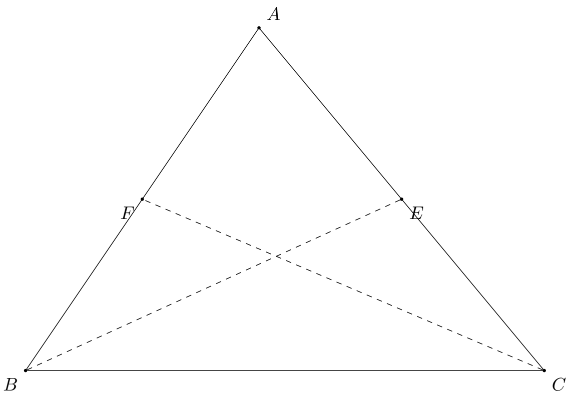


Fig. 3.1.4

Solution: In $\triangle$ s BEC and BFC, using the sine formula,

$$\begin{aligned} \frac{BE}{\sin C} &= \frac{BC}{\sin\left(\frac{B}{2}+C\right)} \\ \frac{CF}{\sin B} &= \frac{BC}{\sin\left(\frac{C}{2}+B\right)} \end{aligned} \tag{3.1.6.2}$$

$\because B = C$, from the above, we obtain (3.1.6.1).

3.1.7. Show that

$$\sin 5\theta = 5\sin\theta - 20\sin^3\theta\cos^2\theta + 16\sin^5\theta \tag{3.1.7.1}$$

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta \tag{3.1.7.2}$$

3.1.8. In Fig. 3.1.4, if $BE = CF$, show that the triangle is isosceles.
Solution: From (3.1.6.2),

$$\sin C \sin\left(\frac{C}{2} + B\right) = \sin\left(\frac{B}{2} + C\right)\sin B \tag{3.1.8.1}$$

$$\implies 2\sin C \sin\left(\frac{C}{2} + B\right) = 2\sin B \sin\left(\frac{B}{2} + C\right) \tag{3.1.8.2}$$

$$\cos\left(B - \frac{C}{2}\right) - \cos\left(B + \frac{3C}{2}\right) = \cos\left(C - \frac{B}{2}\right) - \cos\left(C + \frac{3B}{2}\right) \tag{3.1.8.3}$$

using (2.1.4.4), which can be expressed as

$$\cos\left(C - \frac{B}{2}\right) - \cos\left(B - \frac{C}{2}\right) - \cos\left(C + \frac{3B}{2}\right) + \cos\left(B + \frac{3C}{2}\right) = 0 \tag{3.1.8.4}$$

which, using (2.1.4.4), yields

$$2\sin\left(\frac{B+C}{2}\right)\sin\left[\frac{3(B-C)}{2}\right] + 2\sin\left[5\frac{(B+C)}{2}\right]\sin\left[\frac{(B-C)}{2}\right] = 0 \tag{3.1.8.5}$$

Let

$$\theta = \frac{B-C}{2}, \ \alpha = \frac{B+C}{2} \tag{3.1.8.6}$$

Substituting the above in (3.1.8.5),

$$\sin\alpha \sin 3\theta + \sin 5\alpha \sin\theta = 0 \tag{3.1.8.7}$$

Substituting from (3.1.7.2) in (3.1.8.7) and simplifying,

$$\sin\alpha \sin\theta\left(3 - 4\sin^2\theta + 5 - 20\sin^2\alpha\cos^2\alpha + 16\sin^4\alpha\right) = 0 \tag{3.1.8.8}$$

One possible solution of the above equation is

$$3 - 4\sin^2\theta + 5 - 20\sin^2\alpha\cos^2\alpha + 16\sin^4\alpha = 0 \tag{3.1.8.9}$$

$$4 - 4\sin^2\theta + 4 - 20\sin^2\alpha\left(1 - \sin^2\alpha\right) + 16\sin^4\alpha = 0 \tag{3.1.8.10}$$

which, upon substituting from (1.1.5.1) results in

$$\cos^2\theta + 1 - 5\sin^2\alpha + 36\sin^4\alpha = 0 \tag{3.1.8.11}$$

$$= \cos^2\theta + \left(1 - 6\sin^2\alpha\right)^2 + 7\sin^2\alpha = 0 \tag{3.1.8.12}$$

For the above equation to have a solution,

$$\cos\theta = 0, \sin^2\alpha = \frac{1}{6}, \sin\alpha = 0. \tag{3.1.8.13}$$

which is impossible. Another possible solution is

$$\sin\alpha = \sin\frac{B+C}{2} = 0 \tag{3.1.8.14}$$

$$\implies \cos\frac{A}{2} = 0, \ \text{or, } A = \pi, \tag{3.1.8.15}$$

which is impossible. Hence, the only possible solution is

$$\sin\theta = \sin\frac{B-C}{2} = 0 \tag{3.1.8.16}$$

$$\implies \frac{B-C}{2} = 0, \text{ or, } B = C. \tag{3.1.8.17}$$

3.2 Circumcircle

3.2.1. In Fig. 3.2.1,

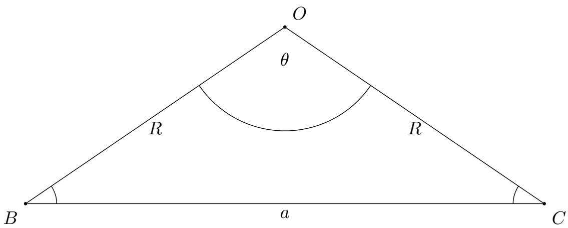


Fig. 3.2.1: Isosceles Triangle

$$OB = OC = R \tag{3.2.1.1}$$

Such a triangle is known as an isosceles triangle. Show that

$$\angle B = \angle C \tag{3.2.1.2}$$

Solution: Using (1.2.2.3),

$$\frac{\sin B}{R} = \frac{\sin C}{R} \tag{3.2.1.3}$$

$$\implies \sin B = \sin C \tag{3.2.1.4}$$

$$\text{or, } \angle B = \angle C. \tag{3.2.1.5}$$

3.2.2. In Fig. 3.2.1, show that

$$a = 2R\sin\frac{\theta}{2} \tag{3.2.2.1}$$

Solution: In $\triangle OBC$, using the cosine formula from (1.2.5.1),

$$\cos\theta = \frac{R^2 + R^2 - a^2}{2R^2} = 1 - \frac{a^2}{2R^2} \tag{3.2.2.2}$$

$$\implies \frac{a^2}{2R^2} = 2\sin^2\frac{\theta}{2} \tag{3.2.2.3}$$

yielding (3.2.2.1).

3.2.3. In Fig. 3.2.2,

$$OB = OC = R, BD = DC. \tag{3.2.3.1}$$

Show that $OD \perp BC$.

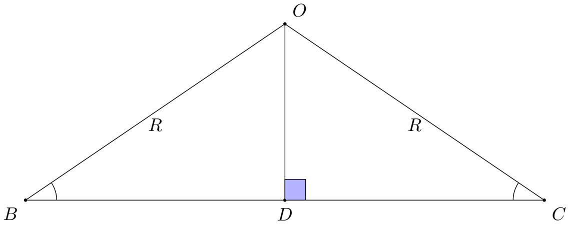


Fig. 3.2.2: Perpendicular bisector.

3.2.4. In Fig. 3.2.3, OD and OE are the perpendicular bisectors of sides BC and AC respectively. Show that $OA = R$.

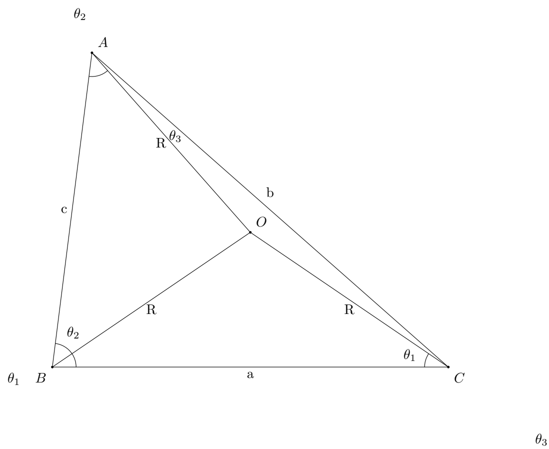


Fig. 3.2.3: Perpendicular bisectors of $\triangle ABC$ meet at **O**.

3.2.5. In Fig. 3.2.3, show that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R. \tag{3.2.5.1}$$

Solution: From (3.2.10.1) and (3.2.2.1)

$$a = 2R \sin A \tag{3.2.5.2}$$

3.2.6. Fig. 3.2.4 shows the *circumcircle* of $\triangle ABC$.

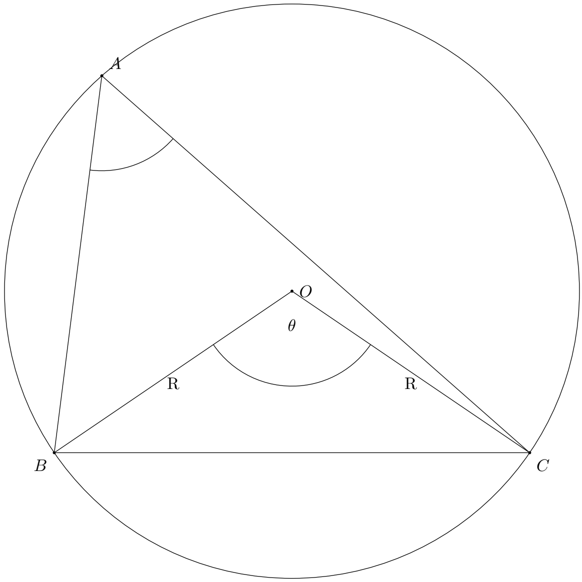


Fig. 3.2.4: Circumcircle of $\triangle ABC$

3.2.7. Any point on the circle can be expressed as

$$\mathbf{x} = \mathbf{O} + R\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}, \quad 0 \in [0, 2\pi]. \tag{3.2.7.1}$$

where **O** is the centere of the circle.

3.2.8. Let

$$R = 1,\ \mathbf{O} = \mathbf{0},\ \mathbf{A} = \begin{pmatrix}\cos\theta_1\\ \sin\theta_1\end{pmatrix},\ \mathbf{B} = \begin{pmatrix}\cos\theta_2\\ \sin\theta_2\end{pmatrix}, \tag{3.2.8.1}$$

Show that the distance

$$AB = \|\mathbf{A} - \mathbf{B}\| = 2\sin\left(\frac{\theta_1 - \theta_2}{2}\right) \tag{3.2.8.2}$$

Solution: From (3.2.7.1).

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix}\cos\theta_1 - \cos\theta_2\\ \sin\theta_1 - \sin\theta_2\end{pmatrix} \tag{3.2.8.3}$$

$$\implies \|\mathbf{A} - \mathbf{B}\|^2 = (\mathbf{A} - \mathbf{B})^\top (\mathbf{A} - \mathbf{B}) \tag{3.2.8.4}$$

$$= (\cos\theta_1 - \cos\theta_2)^2 + (\sin\theta_1 - \sin\theta_2)^2 \tag{3.2.8.5}$$

$$= 2\{1 - \cos(\theta_1 - \theta_2)\} = 4\sin^2\left(\frac{\theta_1 - \theta_2}{2}\right) \tag{3.2.8.6}$$

yielding (3.2.8.2) from (2.1.5.3).

3.2.9. In Fig. 3.2.4, show that

$$\cos A = \frac{(\mathbf{A}-\mathbf{B})^{\top}(\mathbf{A}-\mathbf{B})}{\|\mathbf{A}-\mathbf{B}\|\,\|\mathbf{A}-\mathbf{C}\|}, \tag{3.2.9.1}$$

3.2.10. In Fig. 3.2.4, show that

$$\theta = 2A. \tag{3.2.10.1}$$

The angle subtended by an arc at the centre is double the angle subtended by it at any point on the remaining part of the circle. **Solution:** Let

$$\mathbf{C} = \begin{pmatrix} \cos\theta_3 \\ \sin\theta_3 \end{pmatrix} \tag{3.2.10.2}$$

Then, substituting from (3.2.8.2) in (1.2.5.1),

$$\cos A = \frac{4\sin^2\left(\frac{\theta_1-\theta_2}{2}\right) + 4\sin^2\left(\frac{\theta_1-\theta_3}{2}\right) - 4\sin^2\left(\frac{\theta_2-\theta_3}{2}\right)}{8\sin\left(\frac{\theta_1-\theta_2}{2}\right)\sin\left(\frac{\theta_1-\theta_3}{2}\right)} \tag{3.2.10.3}$$

$$= \frac{2\sin^2\left(\frac{\theta_1-\theta_2}{2}\right) + \cos(\theta_2-\theta_3) - \cos(\theta_1-\theta_3)}{4\sin\left(\frac{\theta_1-\theta_2}{2}\right)\sin\left(\frac{\theta_1-\theta_3}{2}\right)} \tag{3.2.10.4}$$

from (2.1.5.3). $\therefore$ From (2.1.4.4),

$$\cos A = \frac{2\sin^2\left(\frac{\theta_1-\theta_2}{2}\right) + 2\sin\left(\frac{\theta_1-\theta_2}{2}\right)\sin\left(\frac{\theta_1+\theta_2}{2}-\theta_3\right)}{4\sin\left(\frac{\theta_1-\theta_2}{2}\right)\sin\left(\frac{\theta_1-\theta_3}{2}\right)} \tag{3.2.10.5}$$

$$= \frac{\sin\left(\frac{\theta_1-\theta_2}{2}\right) + \sin\left(\frac{\theta_1+\theta_2}{2}-\theta_3\right)}{2\sin\left(\frac{\theta_1-\theta_3}{2}\right)} \tag{3.2.10.6}$$

From (2.1.4.1), the above equation can be expressed as

$$\cos A = \frac{2\sin\left(\frac{\theta_1-\theta_3}{2}\right)\cos\left(\frac{\theta_2-\theta_3}{2}\right)}{2\sin\left(\frac{\theta_1-\theta_3}{2}\right)} = \cos\left(\frac{\theta_2-\theta_3}{2}\right) \tag{3.2.10.7}$$

$$\implies 2A = \theta_2 - \theta_3 \tag{3.2.10.8}$$

Similarly,

$$\cos\theta = \frac{1+1-4\sin^2\left(\frac{\theta_2-\theta_3}{2}\right)}{2} = \cos(\theta_2-\theta_3) = \cos 2A \tag{3.2.10.9}$$

3.2.11. Angles in the same segment of a circle are equal.

3.2.12. In Fig. 3.2.5, show that

$$\theta = \alpha \tag{3.2.12.1}$$

where CP is the tangent.

Solution: Let

$$\mathbf{O} = \mathbf{0}, \ \mathbf{A} = \begin{pmatrix}\cos\theta_1\\ \sin\theta_1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}\cos\theta_2\\ \sin\theta_2\end{pmatrix}, \mathbf{C} = \begin{pmatrix}\cos\theta_3\\ \sin\theta_3\end{pmatrix} \tag{3.2.12.2}$$

Without loss of generality, let

$$\theta_3 = \frac{\pi}{2} \tag{3.2.12.3}$$

Then,

$$\mathbf{C} - \mathbf{O} = \begin{pmatrix}0\\1\end{pmatrix}. \implies \mathbf{C} - \mathbf{P} \equiv \begin{pmatrix}1\\0\end{pmatrix}, \tag{3.2.12.4}$$

$\because CO \perp CP$. From (3.2.9.1), and (3.2.12.4),

$$\cos\theta = \frac{\begin{pmatrix}\cos\theta_3 - \cos\theta_1 & \sin\theta_3 - \sin\theta_1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}}{2\sin\left(\frac{\theta_1-\theta_3}{2}\right)} \tag{3.2.12.5}$$

$$= \sin\left(\frac{\theta_1+\theta_3}{2}\right) = \cos\left(\frac{\pi}{2} - \frac{\theta_1+\theta_3}{2}\right) = \cos\left(\frac{\pi}{4} - \frac{\theta_1}{2}\right) \tag{3.2.12.6}$$

upon substituting from (3.2.12.3). Similarly, from (3.2.10.7),

$$\cos\alpha = \cos\left(\frac{\theta_1-\theta_3}{2}\right) = \cos\left(\frac{\pi}{4} - \frac{\theta_1}{2}\right) = \cos\theta \tag{3.2.12.7}$$

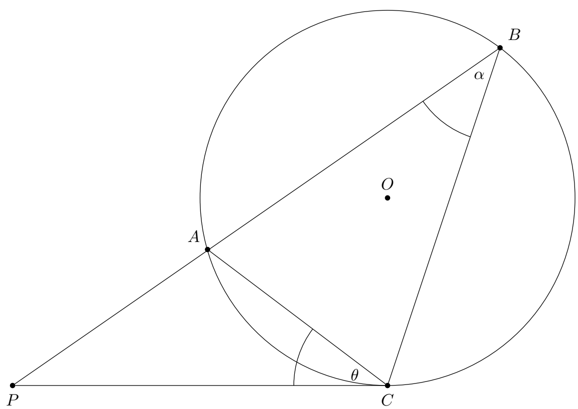


Fig. 3.2.5: $\theta = \alpha$.

3.2.13. In Fig. 3.2.5, show that $PA.PB = PC^2$.

Solution: In △s APC and BPC, using (3.2.12.1),

$$\frac{AP}{\sin\theta} = \frac{AC}{\sin P} \tag{3.2.13.1}$$

$$\frac{PC}{\sin\theta} = \frac{BC}{\sin P} \tag{3.2.13.2}$$

$$\implies \frac{PC}{AP} = \frac{BC}{AC}\left(= \frac{BP}{CP}\right) \tag{3.2.13.3}$$

which gives the desired result. △s APC and BPC are said to be *similar*.

3.2.14. The perpendicular from the centre of a circle to a chord bisects the chord.

3.2.15. The line drawn through the centre of a circle to bisect a chord is perpendicular to the chord.

3.3 NCERT

3.3.1 Equal chords of a circle are equidistant from the centre.
Solution: In Fig. 3.3.1,

$$l = 2r\sin\frac{\theta_1}{2} = 2r\sin\frac{\theta_2}{2} \tag{3.3.1.1}$$

$$\implies \theta_1 = \theta_2 = \theta \tag{3.3.1.2}$$

Thus, the distances

$$d_1 = r\cos\frac{\theta}{2} = d_2. \tag{3.3.1.3}$$

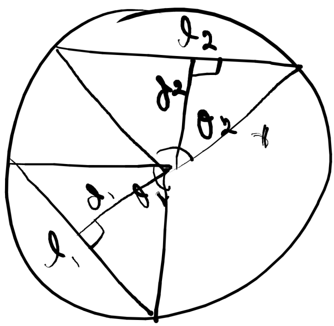


Fig. 3.3.1

3.3.2 Chords equidistant from the centre of a circle are equal.
Solution: In Fig. 3.3.2,

$$l_1 = l_2 = 2d\tan\theta \tag{3.3.2.1}$$

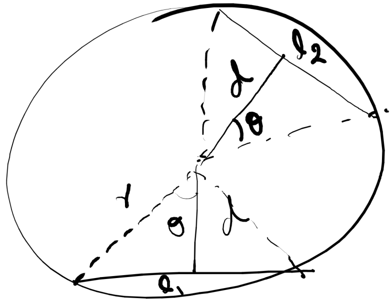


Fig. 3.3.2

3.3.3 Angle in a semicircle is a right angle.
Solution: In Fig. 3.3.3, considering a unit circle with

$$\mathbf{A} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \mathbf{B} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \mathbf{C} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.3.3.1}$$

$$(A - B)^\top (A - C)^\top = \begin{pmatrix} \cos\theta + 1 & \sin\theta \end{pmatrix} \begin{pmatrix} \cos\theta - 1 \\ \sin\theta \end{pmatrix} \tag{3.3.3.2}$$

$$= \cos^2\theta - 1 + \sin^2\theta = 0 \tag{3.3.3.3}$$

Thus, $AB \perp AC$.

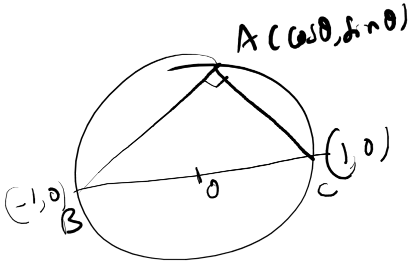


Fig. 3.3.3

3.3.4 Two circles of radii 5 cm and 3 cm intersect at two points and the distance between their centres is 4 cm. Find the length of the common chord.
Solution: In Fig. 3.3.4,

$$\cos\alpha = \frac{r_1^2 + d^2 - r_2^2}{2r_1 r_2} = 0.8 \tag{3.3.4.1}$$

$$\implies l = 2r_1 \sin\alpha = 6 \tag{3.3.4.2}$$

upon substituting numerical values.

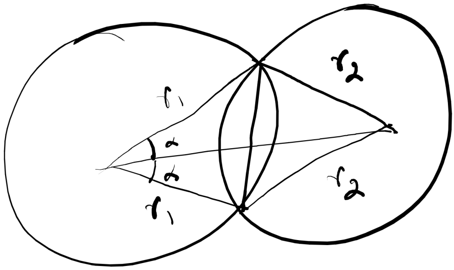


Fig. 3.3.4

3.3.5 Two chords AB and CD of lengths 5 cm and 11 cm respectively of a circle are parallel to each other and are on opposite sides of its centre. If the distance between AB and CD is 6 cm, find the radius of the circle.
Solution: In Fig. 3.3.5,

$$l_1 = 2r\sin\theta_1 \; l_2 = 2r\sin\theta_2 \tag{3.3.5.1}$$

$$d = r(\cos\theta_1 + \cos\theta_2) \tag{3.3.5.2}$$

yielding

$$2d = \sqrt{4r^2 - l_1^2} + \sqrt{4r^2 - l_2^2} \tag{3.3.5.3}$$

$$\implies \left(2d - \sqrt{4r^2 - l_1^2}\right)^2 = 4r^2 - l_2^2 \tag{3.3.5.4}$$

$$\text{or, } 4d^2 - l_1^2 + l_2^2 = 4d\sqrt{4r^2 - l_1^2} \tag{3.3.5.5}$$

$$\implies r = \frac{\sqrt{\left(\frac{4d^2 - l_1^2 + l_2^2}{4d}\right)^2 + l_1^2}}{2} = 5.59 \tag{3.3.5.6}$$

upon substituting numerical values.

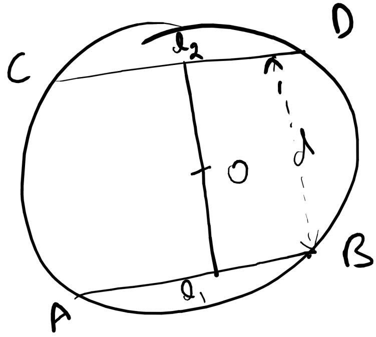


Fig. 3.3.5

3.3.6 The lengths of two parallel chords of a circle are 6 cm and 8 cm. If the smaller chord is at distance 4 cm from the centre, what is the distance of the other chord from the centre?

Solution: In Fig. 3.3.6,

$$l_1 = 2r\sin\theta_1\, l_2 = 2r\sin\theta_2 \tag{3.3.6.1}$$

$$d_1 = r\cos\theta_1\, d_2 = r\cos\theta_2 \tag{3.3.6.2}$$

yielding

$$\theta_1 = \tan^{-1}\left(\frac{l_1}{2d_1}\right) \tag{3.3.6.3}$$

$$\theta_2 = \sin^{-1}\left(\frac{l_2\cos\theta_1}{2d_1}\right) \tag{3.3.6.4}$$

$$\implies d_2 = \frac{l_2}{2}\cot\theta_2 = 3 \tag{3.3.6.5}$$

upon substituting numerical values.

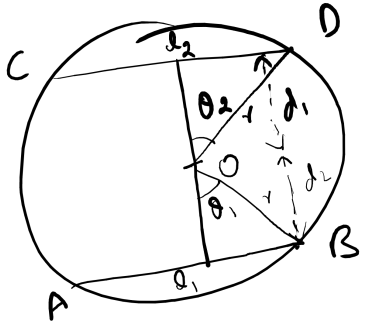


Fig. 3.3.6

3.3.7 Two concentric circles are of radii 5 cm and 3 cm. Find the length of the chord of the larger circle which touches the smaller circle.
Solution: In Fig. 3.3.7,

$$\theta = \cos^{-1}\left(\frac{r_2}{r_1}\right) \tag{3.3.7.1}$$

$$\implies l = 2r_1 \sin\theta = 8 \tag{3.3.7.2}$$

upon substituting numerical values.

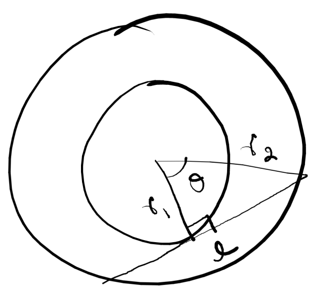


Fig. 3.3.7

3.3.8 A $\triangle ABC$ is drawn to circumscribe a circle of radius 4 cm such that the segments BD and DC into which BC is divided by the point of contact D are of lengths 8 cm and 6 cm respectively. Find the sides AB and AC.

Solution: In Fig. 3.3.8,

$$a = a_1 + a_2 \tag{3.3.8.1}$$
$$\frac{B}{2} = \tan^{-1}\frac{r}{a_1} \tag{3.3.8.2}$$
$$\implies B = 2\tan^{-1}\frac{r}{a_1} \tag{3.3.8.3}$$
$$\frac{C}{2} = \tan^{-1}\frac{r}{a_2} \tag{3.3.8.4}$$
$$\implies C = 2\tan^{-1}\frac{r}{a_2} \tag{3.3.8.5}$$
$$\text{and } A = \pi - B - C \tag{3.3.8.6}$$

Using sine formula,

$$b = a\frac{\sin B}{\sin A} = 13 \tag{3.3.8.7}$$
$$c = a\frac{\sin C}{\sin A} = 15 \tag{3.3.8.8}$$

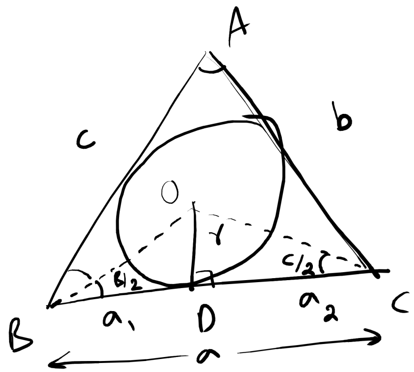


Fig. 3.3.8

3.3.9 PQ is a chord of length 8 cm of a circle of radius 5 cm. The tangents at P and Q intersect at a point T. Find the length TP.
Solution: In Fig. 3.3.9,

$$\frac{\theta}{2} = \sin^{-1}\frac{l}{2r} \tag{3.3.9.1}$$
$$\implies \theta = 2\sin^{-1}\frac{l}{2r} \tag{3.3.9.2}$$
$$\text{and } T = \pi - \theta \tag{3.3.9.3}$$

Also,

$$l = 2x \sin \frac{T}{2} = 2x \cos \frac{\theta}{2} \tag{3.3.9.4}$$

$$\implies x = \frac{l}{2} \sec \frac{\theta}{2} = 6.67 \tag{3.3.9.5}$$

upon substituting numerical values.

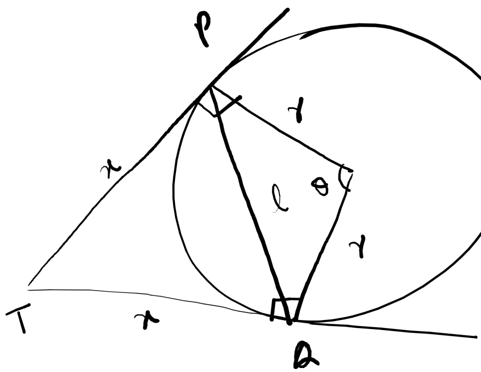


Fig. 3.3.9

3.3.10 If a circle is inscribed in a right angled triangle ABC right angled at **B**, show that the diameter of the circle is equal to $AB + BC - AC$.
Solution: In Fig. 3.3.10,

$$a = r\left(\cot \frac{B}{2} + \cot \frac{C}{2}\right) \tag{3.3.10.1}$$

$$b = r\left(\cot \frac{C}{2} + \cot \frac{A}{2}\right) \tag{3.3.10.2}$$

$$c = r\left(\cot \frac{A}{2} + \cot \frac{B}{2}\right) \tag{3.3.10.3}$$

$$\implies c + a - b = 2r \cot \frac{B}{2} = 2r \quad \because B = 90^\circ \tag{3.3.10.4}$$

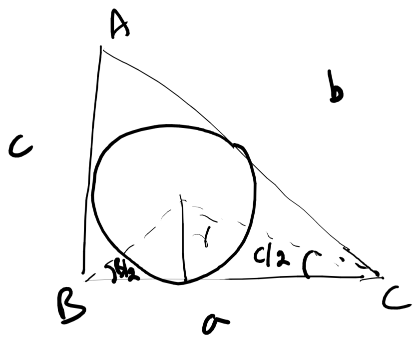


Fig. 3.3.10

3.4 JEE

3.4.1 A polygon of nine sides, each of length 2, is inscribed in a circle. The radius of the circle is ______. (1987)

3.4.2 A circle is inscribed in a equilateral triangle of a side a. The area of any square inscribed in this circle is ______. (1994)

3.4.3 In a triangle ABC, $a : b : c = 4 : 5 : 6$. The ratio of the radius of the circumstances to that of the incircle is ______. (1996)

3.4.4 The sum of the radii of inscribed and circumscribed circles for an n sided regular polygon of side a, is (2003)

a) $\frac{a}{4}\cot\left(\frac{\pi}{2n}\right)$ b) $a\cot\left(\frac{\pi}{n}\right)$ c) $\frac{a}{2}\cot\left(\frac{\pi}{2n}\right)$ d) $a\cot\left(\frac{\pi}{2n}\right)$

3.4.5 For a regular polygon, let r and R be the radii of the inscribed and the circumscribed circles. A false statement among the following is (2010)

a) There is a regular polygon with $\frac{r}{R} = \frac{1}{\sqrt{2}}$
b) There is a regular polygon with $\frac{r}{R} = \frac{2}{3}$
c) There is a regular polygon with $\frac{r}{R} = \frac{\sqrt{3}}{2}$
d) There is a regular polygon with $\frac{r}{R} = \frac{1}{2}$

3.4.6 Let $A_0A_1A_2A_3A_4A_5$ be a regular hexagon inscribed in a circle of unit radius. Then the product of the lengths of the line segments A_0A_1,A_0A_2 and A_0A_4 is (1998)

a) $\frac{3}{4}$ b) $3\sqrt{3}$ c) 3 d) $\frac{3\sqrt{3}}{2}$

3.4.7 In a triangle PQR, P is the largest angle and $\cos P = \frac{1}{3}$. Further the incircle of the triangle touches the sides PQ, QR and RP at N, L and M respectively, such that the lengths of PN, QL and RM are consecutive even integers. Then possible length(s) of the side(s) of the triangle is (are) (2013)

a) 16 b) 24 c) 18 d) 22

3.4.8 In a triangle XYZ, let x, y, z be the lengths of sides opposite to angles X, Y, Z and $2s = x + y + z$. If

$$\frac{s-x}{4} = \frac{s-y}{3} = \frac{s-z}{2}$$

and area of the incircle of the triangle XYZ is $\frac{8\pi}{3}$, (2016)

a) area of the triangle is $6\sqrt{6}$
b) the radius of circumcirle of XYZ is $\frac{35\sqrt{6}}{6}$
c) $\sin\frac{X}{2}\sin\frac{Y}{2}\sin\frac{Z}{2} = \frac{4}{35}$
d) $\sin^2\left(\frac{X+Y}{2}\right) = \frac{3}{5}$

3.4.9 In a triangle PQR, let $\angle PQR = 30°$ and the sides PQ and QR have lengths $10\sqrt{3}$ and 10 respectively. Then which of the following statements is (are) TRUE? (2018)

a) $\angle QPR = 45°$
b) the area of the triangle PQR is $25\sqrt{3}$ and $\angle QRP = 120°$
c) the radius of the incircle of triangle PQR is $10\sqrt{3} - 15$
d) the radius of circumcirle PQR is 100π

3.4.10 In a non-right-angle triangle $\triangle PQR$, let p, q, r denote the lengths of the sides opposite to the angles at P, Q, R respectively. The median from R meets the side PQ at S, the perpendicular from P meets the side QR at E, RS and PE intersect at O. If $p = \sqrt{3}$, $q = 1$ and the radius of the circumcircle at $\triangle PQR$ equals 1, then which of the following options is (are) correct. (2018)

a) Radius of incircle of $\triangle PQR = \frac{\sqrt{3}}{2}\left(2 - \sqrt{3}\right)$
b) Area of $\triangle SOE = \frac{\sqrt{3}}{12}$
c) Length of $OE = \frac{1}{6}$
d) Length of $RS = \frac{\sqrt{7}}{2}$

3.4.11 Which of the following pieces of data does NOT uniquely determine an acute-angled triangle $\triangle ABC$ (R being the radius of the circumcircle)? (2002)

a) $a, \sin A, \sin B$ b) a, b, c c) $a, \sin B, R$ d) $a, \sin A, R$

3.4.12 One angle of an isosceles $\triangle$ is $120°$ and radius of its incircle $= \sqrt{3}$. Then the area of the triangle in sq. units is (2006)

a) $7 + 12\sqrt{3}$ b) $12 - 7\sqrt{3}$ c) $12 + 7\sqrt{3}$ d) 4π

3.4.13 Let $ABCD$ be a quadrilateral with area 18, with side AB parallel to the side CD and $2AB = CD$. Let AD be perpendicular to AB and CD. If a circle is drawn inside the quadrilateral $ABCD$ touching all the sides, then the radius is (2007)

a) 3 b) 2 c) $\frac{3}{2}$ d) 1

3.4.14 If a triangle is inscribed in a circle, then the product of any two sides of the triangle is equal to the product of the diameter and perpendicular distance of the third side from the opposite vertex. Prove the above statement. (1979)

3.4.15 Find the area of the smaller part of a disc of radius $10cm$, cut off by a chord AB which subtends an angle of $22\frac{1}{2}^{\circ}$ at the circumference. (1980)

3.4.16 Let ABC be the triangle with $AB = 1, AC = 3$ and $\angle BAC = \frac{\pi}{2}$. If a circle of radius $r > 0$ touches the sides AB, AC and also touches internally the circumcircle of the triangle ABC, then the value of r is______. (2022)

3.4.17 Let G be a circle of radius $R > 0$. Let $G_1, G_2, \ldots, G_n$ be n circles of equal radius $r > 0$. Suppose each of the n circles $G_1, G_2, \ldots, G_n$ touches the circle G externally. Also, for $i = 1, 2, \ldots, n-1$, the circle G_i touches G_{i+1} externally, and G_n touches G_1 externally. Then, which of the following statements is/are TRUE? (2022)

a) If $n = 4$, then $(\sqrt{2} - 1)r < R$.
b) If $n = 5$, then $r < R$.
c) If $n = 8$, then $(\sqrt{2} - 1)r < R$.
d) If $n = 12$, then $\sqrt{2}(\sqrt{3} + 1)r > R$.

3.4.18 Consider an obtuse angled triangle ABC in which the difference between the largest and the smallest angle is $\frac{\pi}{2}$ and whose sides are in arithmetic progression. Suppose that the vertices of this triangle lie on a circle of radius 1. (2023)

a) Let a be the area of the triangle ABC. Then the value of $(64a)^2$ is ______.
b) The in radius of the triangle ABC is ______.

3.4.19 Let $A_1, A_2, A_3, \ldots, A_8$ be the vertices of a regular octagon that lie on a circle of radius 2. Let P be a point on the circle, and let PA_i denote the distance between the points P and A_i for $i = 1, 2, \ldots, 8$. If P varies over the circle, then the maximum value of the product $PA_1 \cdot PA_2 \cdot PA_3 \cdots PA_8$ is ______. (2023)

3.5 Olympiad

3.5.1 Let $ABCD$ be a unit square. Suppose M and N are points on BC and CD, respectively, such that the perimeter of triangle MCN is 2. Let O be the circumcenter of triangle MAN, and P be the circumcenter of triangle MON. If $\left(\frac{OP}{OA}\right)^2 = \frac{m}{n}$ for some relatively prime positive integers m and n, find the value of $m + n$. (IOQM 2015)

3.5.2 In triangle ABC, point A_1 lies on side BC and point B_1 lies on side AC. Let P and Q be points on segments AA_1 and BB_1, respectively, such that $PQ \parallel AB$. Let P_1 be a point on line PB_1 such that B_1 lies strictly between P and P_1, and $\angle PP_1C = \angle BAC$. Similarly, let Q_1 be a point on line QA_1 such that A_1 lies strictly between Q and Q_1, and $\angle CQ_1Q = \angle CBA$. Prove that points P, Q, P_1, and Q_1 are concyclic. (IMO 2019)

3.5.3 Let D be an interior point of the acute triangle ABC with $AB > AC$ so that $\angle DAB = \angle CAD$. The point E on the segment AC satisfies $\angle ADE = \angle BCD$, the point F on the segment AB satisfies $\angle FDA = \angle DBC$, and the point X on the line AC satisfies $CX = BX$. let O_1 and O_2 be the circumcentres of the triangles ADC and EXD, respectively. Prove that the lines BC, EF and O_1O_2 are concurrent. (IMO 2021)

3.5.4 $ABCD$ is cyclic. The feet of the perpendicular from D to the lines AB, BC, CA are P, Q, R respectively. Show that the angle bisectors of ABC and CDA meet on the line AC iff $RP = RQ$. (IMO 2003)

3.5.5 In the convex quadrilateral $ABCD$, the diagonals AC and BD are perpendicular and the opposite sides AB and DC are not parallel. Suppose that the point P, where the perpendicular bisectors of $AB$and DC meet, is inside $ABCD$. Prove that $ABCD$ is a cyclic quadrilateral if and only if the triangles ABP and CDP have equal areas. (IMO 1998)

3.5.6 Consider five points A, B, C, D and E such that $ABCD$ is a parallelogram and $BCED$ is a cyclic quadrilateral. Let l be a line passing through A. Suppose that l intersts the interior of the segment DC at F and intersects line BC at G. Suppose also that $EF = EG = EC$. Prove that l is the bisector of angle DAB. (IMO 2007)

3.5.7 Let P be a point inside triangle ABC such that

$$\angle APB - \angle ACB = \angle APC - \angle BC.$$

Let D, E be the incenters of triangles APB, APC, respectively. Show that AP, BD, CE meet at a point. (IMO 1996)

3.5.8 Let $ABCDEF$ be a convex hexagon such that A is parallel to DE, BC is parallel to EF, and CD is parallel to FA. Let R_A, R_C, R_E denote the circumradii of triangles FAB, BCD, DEF, respectively, and let P denote the perimeter of the hexagon. Prove that (IMO 1996)

$$R_A + R_C + R_E \geq \frac{p}{2}.$$

3.5.9 The angle at A is the smallest angle of triangle ABC. The point B and C divide the circumcircle of the triangle into two arcs. Let U be an interior point of the arc between B and C which does not contain A. The perpendicular bisectors of AB and AC meet the line AU at V and W, respctively. The lines BV and CW meet at T. Show that (IMO 1997)

$$AU = TB + TC.$$

3.5.10 Let $P = A_1A_2\ldots A_k$ be a convex polygon in the plane. The vertices $A_1, A_2, \ldots A_k$ have integral coordinates and lie on a circle. Let S be the area of P. An odd positive integer n is given such that the squares of the side lengths of P are integers divisible by n. Prove that $2S$ is an integer divisible by n. (IMO 2016)

3.5.11 Let I be the circumcircle of acute-angled triangle ABC. Points D and E lie on segments AB and AC respectively, such that $AD = AE$. The perpendicular bisectors of BD and CE intersect the minor arcs AB and AC of I at points F and G respectively. Prove that the lines DE and FG are parallel (or are the same line). (IMO 2018)

3.5.12 In the plane let C be a circle, L a line tangent to the circle C, and M a point on L. Find the locus of all points P with the following property: there exists two points Q, R on L such that M is the midpoint of QR and C is the inscribed circle of triangle PQR. (IMO 1992)

3.5.13 Let D be a point inside acute triangle ABC such that $\angle ADB = \angle ACB + \pi/2$ and $AC \cdot BD = AD \cdot BC$.

a) Calculate the ratio $(AB \cdot CD)/(AC \cdot B)$.
b) Prove that the tangents at C to the circumcircles of $\triangle ACD$ and $\triangle BCD$ are perpendicular. (IMO 1993)

3.5.14 In a triangle ABC, let I denote the incenter. Let the lines AI, BI, and CI intersect the incircle at P, Q, and R, respectively. If $\angle BAC = 40°$, what is the value of $\angle QPR$ in degrees? (PRMO 2014)

3.5.15 AB is tangent to the circles $CAMN$ and $NMBD$. M lies between C and D on the line CD, and CD is parallel to AB. The chords NA and CM meet at P; the chords NB and MD meet at Q. The rays CA and DB meet at E. Prove that $PE = QE$. (IMO 2000)

3.5.16 O and I are the circumcentre and incentre of $\triangle ABC$ respectively. Suppose O lies in the interior of $\triangle ABC$ and I lies on the circle passing through B, O, and C. What is the magnitude of $\angle BAC$ in degrees? (PRMO 2012)

3.5.17 In rectangle $ABCD$, $AB = 8$ and $BC = 20$. Let P be a point on AD such that $\angle BPC = 90°$. If r_1, r_2, r_3 are the radii of the incircles of triangles APB, BPC, and CPD, what is the value of $r_1 + r_2 + r_3$? (PRMO 2015)

3.5.18 The circle ω touches the circle Ω internally at P. The center O of Ω is outside ω. Let XY be a diameter of Ω which is also tangent to ω. Assume $PY > PX$. Let PY intersect ω at Z. If $YZ = 2PZ$, what is the magnitude of $\angle LPYX$ in degrees? (PRMO 2015)

3.5.19 Let I be the incentre of acute triangle ABC with $AB \neq AC$. The incircle ω of ABC is tangent to sides BC, CA, and AB at points D, E, and F, respectively. The line through D perpendicular to EF meets ω again at R. Line AR meets ω again at P. The circumcircles of triangles PCE and PBF meet again at Q. Prove that lines DI and PQ meet on the line through A that is perpendicular to AI. (IMO 2019)

3.5.20 Let r be a circle with centre I, and $ABCD$ a convex quadrilateral such that each of the segments AB, BC, CD and DA is a tangent to r. Let Ω be the circumcircle of the triangle AIC. The extension of BA beyond A meets Ω at X, and the extension of BC beyond C meets Ω at Z. The extensions of AD and CD beyond D meet Ω at Y and T, respectively. Prove that

$$AD + DT + TX + XA = CD + DY + YZ + ZC$$

(IMO 2021)

3.5.21 Let $ABCDE$ be a convex pentagon such that $BC = DE$. Assume that there is a point T inside $ABCDE$ with $TB = TD$, $TC = TE$ and $\angle ABT = \angle TEA$. Let line AB intersect lines CD and CT at points P and Q, respectively. Assume that the points P, B, A, Q occur on their line in that order. Let line AE intersect lines CD and DT at points R and S, respectively. Assume that the points R, E, A, S occur on their line in that order. Prove that the points P, S, Q, R lie on a circle. (IM0 2022)

3.5.22 Let ABC be an acute-angled triangle with $AB \leq AC$. Let Ω be the circumcircle of ABC. Let S be the midpoint of the arc CB of Ω containing A. The perpendicular from A to BC meets BS at D and meets Ω again at$E \neq A$. The line through D parallel to BC meets line BE at L. Denote the circumcircle of triangle BDL by ω. Let ω meet Ω again at $P \neq B$. Prove that the line tangent to ω at P meets line BS

on the internal angle bisector of $\angle BAC$. (IMO 2023)

3.5.23 Let ABC be an equilateral triangle. Let A_1, B_1, C_1 be interior points of ABC such that $BA_1 = A_1C, CB_1 = B_1A, AC_1 = C_1B$, and $\angle BAC + \angle CB_1A + \angle AC_1B = 480°$. Let BC_1 and CB_1 meet at A_2, let CA_1 and AC_1 meet at B_2, and let AB_1 and BA_1 meet at C_2. Prove that if triangle $A_1B_1C_1$ is scalene, then the three circumcircles of triangles AA_1A_2, BB_1B_2 and CC_1C_2 all pass through two common points. (Note: no 2 sides have equal length.) (IMO 2023)

3.5.24 Let ABC be a triangle with $AB \leq AC \leq BC$. Let the incentre and incircle of triangle ABC be I and ω, respectively. Let X be the point on line BC different from C such that the line through X parallel to AC is tangent to ω. Similarly, let Y be the point on line BC different from B such that the line through Y parallel to AB is tangent to ω. Let AI intersect the circumcircle of triangle ABC again at $P \neq A$. Let K and L be the midpoints of AC and AB, respectively. Prove that $\angle KIL + \angle YPX = 180°$. (IMO 2024)

3.5.25 In a triangle ABC, let H, I, and O be the orthocenter, incenter, and circumcenter, respectively. If the points B, H, I, and C lie on a circle, what is the magnitude of $\angle BOC$ in degrees? (PRMO 2013)

3.5.26 Let S be a circle with center O. A chord AB, not a diameter, divides S into two regions R_1 and R_2. Let S_1 be a circle with center in R_1 touching AB, the circle S internally. Let S_2 be a circle with center in R_2 touching AB at Y, the circle S internally, and passing through the center of S. The point X lies on the diameter passing through the center of S_2, and $\angle YXO = 30°$. If the radius of S_2 is 100, then what is the radius of S? (PRMO 2013)

3.5.27 BC is a diameter of a circle with center O. A is any point on the circle with $\angle AOC > 60°$. EF is the chord which is the perpendicular bisector of AO. D is the midpoint of the minor arc AB. The line through O parallel to AD meets AC at J. Show that J is the incenter of triangle CEF. (IMO 2002)

3.5.28 Let $ABCD$ be a convex quadrilateral such that the line CD is a tangent to the circle on AB as diameter. Prove that the line AB is a tangent to the circle on CD as diameter if and only if the lines BC and AD are parallel. (IMO 1984)

3.5.29 let A be one of the two distinct points of intersection of two unequal coplanar tangents to the circles C_1 and C_2 with centers O_1 and O_2, respectively. One of the common tangents to the circles touches C_1 at P_1 and C_2 at P_2, while the other touches C_1 at Q_1 and C_2 at Q_2. Let M_1 be the midpoint of P_1Q_1, M_2 be the midpoint of P_2Q_2. Prove that $\angle O_1AO_2 = \angle M_1AM_2$. (IMO 1983)

3.5.30 A circle has center on the side AB of the cyclic quadrilateral $ABCD$. The other three sides are tangent to the circle. Prove that $AD + BC = AB$. (IMO 1985)

3.5.31 A circle with center O passes through the vertices A and C of triangle ABC and intersects the segments AB and BC again at distinct points K and N respectively. The circumscribed circle of the triangle ABC and EBN intersect at exactly two distinct points B and M. Prove that angle OMB is a right angle. (IMO 1985)

3.5.32 Three congruent circles have a common point O and lie inside a given triangle. Each circle touches a pair of sides of the triangle. Prove that the incenter and the circumcenter of the triangle and the point O are collinear. (IMO 1981)

3.5.33 A non-isosceles triangle $A_1A_2A_3$ is given with sides a_1, a_2, a_3 (a_i is the side opposite

A_i). For all $i = 1, 2, 3, M_i$ is the midpoint of side a_i and T_i is the point where the incircle touches side a_i. Denote by S_i the reflection of T_i in the interior bisector of angle A_i. Prove that the lines M_1S_1, M_2S_2 and M_3S_3 are concurrent. (IMO 1982)

3.5.34 In an acute-angled triangle ABC the interior bisector of the angle A intersects BC at L and intersects the circumcircle of ABC again at N. From point L perpendiculars are drawn to AB and AC, the feet of these perpendiculars being K and M respectively. Prove that the quadrilateral $AKNM$ and the triangle ABC have equal areas. (IMO 1987)

3.5.35 Consider two coplanar circles of radii R and r ($R > r$) with the same center. Let P be a fixed point on the smaller circle and B a variable point on the larger circle. The line BP meets the larger circle again at C. The perpendicular l to BP at P meets the smaller circle again at A. (If l is tangent to the circle at P then $A = P$). (IMO 1988)

a) Find the set of values of $BC^2 + CA^2 + AB^2$

b) Find the locus of the midpoint of BC.

3.5.36 Let the excircle of triangle ABC opposite the vertex A be tangent to the side BC at the point A_1. Define the points B_1 on CA and C_1 on AB analogously, using the excircles opposite B and C respectively. Suppose that the circumcentre of triangle $A_1B_1C_1$, lies on the circumcircle of triangle ABC. Prove that triangle ABC is right-angled. (The excircle of triangle ABC opposite the vertex A is the circle that is tangent to the line segment BC, to the ray AB beyond B, and to the ray AC beyond C. The excircles opposite B and C are similarly defined. (IMO 2013)

3.5.37 Convex quadrilateral $ABCD$ has $\angle ABC = \angle CDA = 90°$. Point H is the foot of the perpendicular from A to BD. Points S and T lie on sides AB and AD respectively, such that H lies inside triangle SCT and $\angle CHS - \angle CSB = 90°$, $\angle THC - \angle DTC = 90°$. Prove that line BD is tangent to the circumcircle of triangle TSH. (IMO 2014)

3.5.38 Points P and Q lie on side BC of acute-angled triangle ABC so that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Points M and N lie on lines AP and AQ, respectively, such that P is the midpoint of AM, and Q is the midpoint of AN. Prove that lines BM and CN intersect on circumcircle of triangle ABC (IMO 2014)

3.5.39 Let ABC be an acute triangle with $AB > AC$. Let I be its circumcircle, H its orthocentre, and F the foot of the altitude from A. Let M be the midpoint of BC. Let Q be the point on T such that $\angle HQA = 90$, and let K be the point on T such that $\angle HKQ = 90°$. Assume that the points A, B, C, K and Q are all different, and lie on T in this order. Prove that the circumcircles of triangles KQH and FKM are tangent to each other. (IMO 2015)

3.5.40 Triangle ABC has circumcircle Ω and circumcentre O. A circle T with centre A intersects the segment BC at points D and E, such that B, D, E and C are all different and lie on line BC in this order. Let F and G be the points of intersection of T and Ω such that A, F, B, C and G lie on Ω in this order. Let K be the second point of intersection of the circumcircle of triangle BDF and the segment AB. Let L be the second point of intersection of the circumcircle of triangle CGE and the segment CA. Suppose that the lines FK and GL are different and intersect at the point X. Prove that X lies on the line AO. (IMO 2015)

3.5.41 In an acute-angled triangle ABC, the internal bisector of angle A meets the circumcircle of the triangle again at A_1. Points B_1 and C_1 are defined similarly. Let A_0 be the point of intersection of the line AA_1 with the external bisectors of angles B and C. Points B_0 and C_0 are defined similarly. Prove that

a) The area of the triangle $A_0\ B_0C_0$ is twice the area of the hexagon $AC_1BA_1CB_1$
b) The area of the triangle $A_0B_0C_0$ is at least four times the area of the triangle ABC.

(IMO 1989)

3.5.42 Chords AB and CD of a circle intersect at a point E inside the circle. Let M be an interior point of the segment EB. The tangent line at E to the circle through D, E and M intersects the lines BC and AC at F and G respectively. If $\frac{AM}{AB} = t$, find $\frac{EG}{EF}$ in terms of t. (IMO 1990)

3.5.43 In triangle ABC, $AB = AC$. A circle is tangent internally to the circumcircle of triangle ABC and also to sides AB, AC at P, Q, respectively. Prove that the midpoint of segment PQ is the center of the incircle of triangle ABC. (IMO 1978)

3.5.44 Two circles in a plane intersect. Let A be one of the points of intersection. Starting simultaneously from A two points move with constant speeds, each point travelling along its own circle in the same sense. The two points return to A simultaneously after one revolution. Prove that there is a fixed point P in the plane such that, at any time, the distances from P to the moving points are equal. (IMO 1979)

3.5.45 Let I be the incenter of triangle ABC. Let the incircle of ABC touch the sides BC,CA, and AB at K, L, and M, respectively. The line through B parallel to MK meets the lines LM and LK at R and S respectively. Prove that angle RIS is acute. (IMO 1998)

3.5.46 Two circles G_1 and G_2 are contained inside the circle G, and are tangent to G at the distinct points M and N, respectively. G_1 passes through the center of G_2. The line passing through the two points of intersection of G_1 and G_2 meets G at A and B. The lines MA and MB meet G_1 at C and D respectively. Prove that CD is tangent to G_2. (IMO 1999)

3.5.47 $A_1A_2A_3$ is an acute-angled triangle. The foot of the altitude from A_i is K_i and the incircle touches the side opposite A_i at L_i. The line K_1K_2 is reflected in the line L_1L_2. Similarly, the line K_2K_3 is reflected in L_2L_3 and K_3K_1 is reflected in L_3L_1. Show that the three new lines form a triangle with vertices on the incircle. (IMO 2000)

3.5.48 Let ABC be an acute-angled triangle with $AB \neq AC$. The circle with diameter BC intersects the sides AB and AC at M and N respectively. Denote by O the midpoint of the side BC. The bisectors of the angles BAC and MON intersect at R. Prove that the circumcircles of the triangles BMR and CNR have a common point on the side BC. (IMO 2004)

3.5.49 In a convex quadrilateral $ABCD$ the diagonal BD does not bisect the angles ABC and CDA. The point P lies inside $ABCD$ and satisfies

$$\angle PBC = \angle DBA \text{ and } \angle PDC = \angle BDA.$$

Prove that $ABCD$ is a cyclic quadrilateral if and only if $AP = CP$. (IMO 2004)

3.5.50 Let $ABCD$ be a fixed convex quadrilateral with $BC = DA$ and BC not parallel with DA. Let two variable points E and F lie on the sides BC and DA, respectively and satisfy $BEDF$. The lines AC and BD meet at P, the lines BD and EF meet at Q, the lines EF and AC meet at R. Prove that the circumcircles of the triangles PQR, as E and F vary, have a common point other than P. (IMO 2005)

3.5.51 In triangle ABC the bisector of angle BCA intersects the circumcircle again at R, the perpendicular bisector of BC at P, and the perpendicular bisector of AC at Q. The midpoint of BC is K and the midpoint of AC is L. Prove that the triangles RPK and RQL have the same area. (IMO 2007)

3.5.52 An acute-angled triangle ABC has orthocentre H. The circle passing through H with centre the midpoint of BC intersects the line BC at A_1 and A_2. Similarly, the circle passing through H with centre the midpoint of CA intersects the line CA at B_1 and B_2, and the circle passing through H with centre the midpoint of AB intersects the line AB at C_1 and C_2. Show that A_1, A_2, B_1, B_2, C_1, C_2 lie on a circle.(IMO 2008)

3.5.53 Let $ABCD$ be a convex quadrilateral with $|BA| \neq |BC|$. Denote the incircles of triangles ABC and ADC by ω_1 and ω_2 respectively. Suppose that there exists a circle ω tangent to the ray BA beyond A and to the ray BC beyond C, which is also tangent to the lines AD and CD. Prove that the common external tangents of ω_1 and ω_2 intersect on ω. (IMO 2008)

3.5.54 Let ABC be a triangle with circumcentre O. The points P and Q are interior points of the sides CA and AB, respectively. Let K, L and M be the midpoints of the segments BP, CQ and PQ, respectively, and let Γ be the circle passing through K, L and M. Suppose that the line PQ is tangent to the circle Γ. Prove that $OP = OQ$.
(IMO 2009)

3.5.55 Let ABC be a triangle with $AB = AC$. The angle bisectors of $\angle CAB$ and $\angle ABC$ meet the sides BC and CA at D and E, respectively. Let K be the incentre of triangle ADC. Suppose that $\angle BEK$=45°. Find all possible values of $\angle CAB$. (IMO 2009)

3.5.56 Let A,B,C,D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at X and Y. The line XY meets BC at Z. Let P be a point on the line XY other than Z. The line CP intersects the circle with diameter AC at C and M, and the line BP intersects the circle with diameter BD at B and N. Prove that the lines AM, DN, XY are concurrent. (IMO 1995)

3.5.57 PS is a line segment of length 4 and O is the midpoint of PS. A semicircular arc is drawn with PS as diameter. Let X be the midpoint of this arc. Q and R are points on the arc PXS such that QR is parallel to PS and the semicircular arc drawn with QR as diameter is tangent to PS. What is the area of the region $QXROQ$ bounded by the two semicircular arcs? (PRMO 2012)

3.5.58 The figure below shows a broken piece of a circular plate made of glass.
C is the midpoint of AB, and D is the midpoint of arc AB. Given that $AB = 24$ cm and $CD = 6$ cm, what is the radius of the plate in centimeters? (The figure is not drawn to scale.) (PRMO 2015)

3.5.59 In the coordinate plane, a point is called a lattice point if both of its coordinates are integers. Let A be the point $(12, 84)$. Find the number of right-angled triangles ABC in the coordinate plane where B and C are lattice points, having a right angle at the vertex A and whose incenter is at the origin $(0, 0)$. (IOQM 2015)

3.5.60 Let ABC be an acute-angled triangle with orthocentre H, and let W be a point on the side BC, lying strictly between B and C. The points M and N are the feet of the altitudes from B and C respectively. Denote by ω_1 the circumcircle of BWN, and let

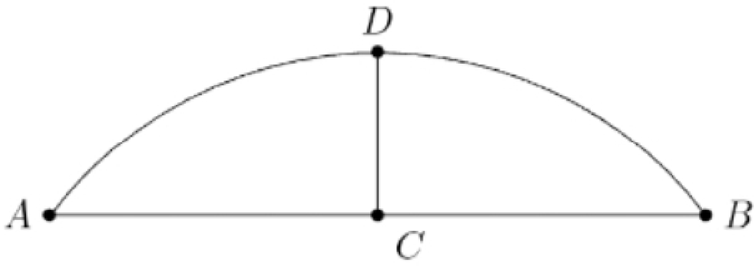


Fig. 3.5.1

X be the point on ω_1 such that WX is a diameter of ω_1. Analogously, denote by ω_2 the circumcircle of CWM and let Y be the point on ω_2 such that WY is a diameter of ω_2. Prove that X, Y and H are collinear. (IMO 2013)

3.5.61 Let ABC be an acute-angled triangle with circumcentre O. Let P on BC be the foot of the altitude from A. Suppose that $\angle BCA \geq \angle ABC + 30°$. Prove that $\angle CAB + \angle COP < \angle 90°$. (IMO 2001)

3.5.62 Let R and S be different points on a circle Ω such that RS is not a diameter. Let l be the tangent line to Ω at R. Point T is such that S is the midpoint of the line segment RT. Point J is chosen on the shorter arc RS of Ω so that the circumcircle Γ of triangle JST intersects l at two distinct points. Let A be the common point of Γ and l that is closer to R. Line AJ meets Ω again at K. Prove that the line KT is tangent to Γ. (IMO 2017)

4 Identities

4.1 NCERT

4.1.1 If $\cos x = -\frac{3}{5}$, x lies in the third quadrant, find the values of other five trigonometric function.

Solution: In Fig. 4.1.1,

$$a = -3, b = 5, c = -4 \tag{4.1.1.1}$$

$$\implies \cos x = -\frac{3}{5} \ \sin x = -\frac{4}{5} \ \tan x = \frac{-4}{-3} \tag{4.1.1.2}$$

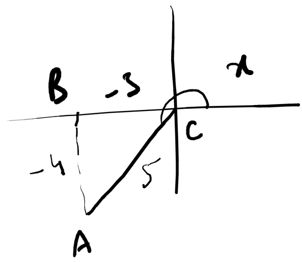


Fig. 4.1.1

4.1.2 If $\cot x = -\frac{5}{12}$, x lies in the second quadrant, find the values of other five trigonometric function.
Solution: In Fig. 4.1.2,

$$a = -5, b = 13, c = 12 \tag{4.1.2.1}$$

$$\implies \cos x = -\frac{5}{13} \; \sin x = \frac{12}{13} \; \tan x = -\frac{12}{5} \tag{4.1.2.2}$$

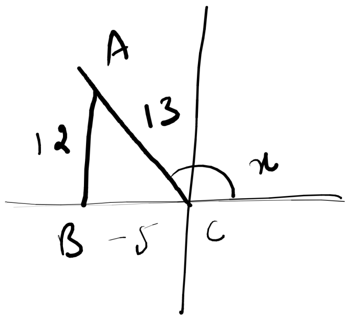


Fig. 4.1.2

4.1.3 Find the value of $\sin \frac{31\pi}{3}$.
Solution:

$$\sin \frac{31\pi}{3} = \sin\left(10\pi + \frac{\pi}{3}\right) \tag{4.1.3.1}$$

$$= \sin\left(\frac{\pi}{3}\right) = \frac{1}{2} \tag{4.1.3.2}$$

4.1.4 Find the value of $\cos(-1710°)$.

Solution:

$$\cos(-1710^\circ) = \cos(-5 \times 360^\circ + 90^\circ) \tag{4.1.4.1}$$
$$= \cos 90^\circ = 0 \tag{4.1.4.2}$$

4.1.5 Prove that $3\sin\frac{\pi}{6}\sec\frac{\pi}{3} - 4\sin\frac{5\pi}{6}\cot\frac{\pi}{4} = 1$.
Solution:

$$3\sin\frac{\pi}{6}\sec\frac{\pi}{3} - 4\sin\frac{5\pi}{6}\cot\frac{\pi}{4} = 3\frac{\sin\frac{\pi}{6}}{\cos\left(\frac{\pi}{2} - \frac{\pi}{6}\right)} - 4\sin\left(\pi - \frac{\pi}{6}\right) \tag{4.1.5.1}$$
$$= 3\frac{\sin\frac{\pi}{6}}{\sin\frac{\pi}{6}} - 4\sin\left(\frac{\pi}{6}\right) = 1 \tag{4.1.5.2}$$

upon substituting numerical values.

4.1.6 Find the value of $\sin 15^\circ$.
Solution:

$$1 - 2\sin^2 15 = \cos 30 = \frac{\sqrt{3}}{2} \tag{4.1.6.1}$$
$$\implies \sin 15 = \sqrt{\frac{1 - \frac{\sqrt{3}}{2}}{2}} = \frac{\sqrt{2 - \sqrt{3}}}{2} \tag{4.1.6.2}$$

4.1.7 Find the value of $\tan\frac{13\pi}{12}$.
Solution:

$$\tan\frac{13\pi}{12} = \tan\left(\pi + \frac{\pi}{12}\right) = \tan\frac{\pi}{12} \tag{4.1.7.1}$$

Since

$$2\cos^2 15 - 1 = \cos 30 = \frac{\sqrt{3}}{2}, \tag{4.1.7.2}$$
$$\cos 15 = \sqrt{\frac{1 + \frac{\sqrt{3}}{2}}{2}} = \frac{\sqrt{2 + \sqrt{3}}}{2} \tag{4.1.7.3}$$
$$\therefore \tan\frac{\pi}{12} = \frac{\sin 15}{\cos 15} = \frac{\sqrt{2 - \sqrt{3}}}{\sqrt{2 + \sqrt{3}}} \tag{4.1.7.4}$$

upon substituting from (4.1.6.2).

4.1.8 Prove that

$$\frac{\sin(x+y)}{\sin(x-y)} = \frac{\tan x + \tan y}{\tan x - \tan y}. \tag{4.1.8.1}$$

Solution:

$$\frac{\sin(x+y)}{\sin(x-y)} = \frac{\sin x\cos y + \sin y\cos x}{\sin x\cos y - \sin y\cos x} \tag{4.1.8.2}$$

Dividing the numerator and denominator by $\cos x \cos y$ yields (4.1.8.1).

4.1.9 Show that

$$\tan 3x \tan 2x \tan x = \tan 3x - \tan 2x - \tan x \tag{4.1.9.1}$$

Solution:

$$\tan 3x = \tan(2x + \tan x) = \frac{\tan 2x + \tan x}{1 - \tan x \tan 2x} \tag{4.1.9.2}$$

$$\implies \tan 3x(1 - \tan x \tan 2x) = \tan 2x + \tan x \tag{4.1.9.3}$$

yielding (4.1.9.1).

4.1.10 Prove that

$$\cos\left(\frac{\pi}{4} + x\right) + \cos\left(\frac{\pi}{4} - x\right) = \sqrt{2}\cos x. \tag{4.1.10.1}$$

Solution:

$$\cos\left(\frac{\pi}{4} + x\right) + \cos\left(\frac{\pi}{4} - x\right) = 2\cos\left(\frac{\pi}{4}\right)\cos x \tag{4.1.10.2}$$

yielding (4.1.10.1) after substituting numerical values.

4.1.11 Prove that

$$\frac{\cos 7x + \cos 5x}{\sin 7x - \sin 5x} = \cot x \tag{4.1.11.1}$$

Solution:

$$\frac{\cos 7x + \cos 5x}{\sin 7x - \sin 5x} = \frac{2\cos 6x \cos x}{2\cos 6x \cos x} \tag{4.1.11.2}$$

yielding (4.1.11.1).

4.1.12 Prove that

$$\frac{\sin 5x - 2\sin 3x + \sin x}{\cos 5x - \cos x} = \tan x. \tag{4.1.12.1}$$

Solution:

$$\frac{\sin 5x - 2\sin 3x + \sin x}{\cos 5x - \cos x} = -\frac{2\sin 3x \cos 2x - 2\sin 3x}{2\sin 3x \sin 2x} \tag{4.1.12.2}$$

$$= \frac{1 - \cos 2x}{\sin 2x} = \frac{2\sin^2 x}{2\sin x \cos x} \tag{4.1.12.3}$$

yielding (4.1.12.1).

4.1.13 If $\sin x = \frac{3}{5}, \cos y = -\frac{12}{13}$, where x and y both lies in second quadrant, find the value of $\sin(x + y)$.

Solution: From the given information,

$$\cos x = -\frac{4}{5}, \sin y = \frac{5}{13} \tag{4.1.13.1}$$

$$\implies \sin(x+y) = -\frac{3}{5} \times \frac{12}{13} - \frac{4}{5} \times \frac{5}{13} \tag{4.1.13.2}$$

$$= -\frac{56}{65} \tag{4.1.13.3}$$

4.1.14 Prove that

$$\cos 2x \cos \frac{x}{2} - \cos 3x \cos \frac{9x}{2} = \sin 5x \sin \frac{5x}{2}.. \tag{4.1.14.1}$$

Solution:

$$\cos 2x \cos \frac{x}{2} = \frac{1}{2}\left(\cos\left(2x + \frac{x}{2}\right) + \cos\left(2x - \frac{x}{2}\right)\right) \tag{4.1.14.2}$$

$$\cos 3x \cos \frac{9x}{2} = \frac{1}{2}\left(\cos\left(3x + \frac{9x}{2}\right) + \cos\left(3x - \frac{9x}{2}\right)\right) \tag{4.1.14.3}$$

Also,

$$\cos\left(2x + \frac{x}{2}\right) - \cos\left(3x + \frac{9x}{2}\right) = 2\sin 5x \sin \frac{5x}{2} \tag{4.1.14.4}$$

$$\cos\left(2x - \frac{x}{2}\right) - \cos\left(3x - \frac{9x}{2}\right) = 0 \tag{4.1.14.5}$$

yielding (4.1.14.1) after some algebra.

4.1.15 Find the value of $\tan \frac{\pi}{8}$.
Solution:

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} \tag{4.1.15.1}$$

For $\theta = \frac{\pi}{8}$, we obtain

$$\frac{2\tan\theta}{1-\tan^2\theta} = 1 \implies \tan^2\theta + 2\tan\theta - 1 = 0 \tag{4.1.15.2}$$

yielding

$$\tan\theta = \sqrt{2} - 1 \tag{4.1.15.3}$$

by taking the positive root of the quadratic.

4.1.16 If $\tan x = \frac{3}{4}, \pi < x < \frac{3\pi}{2}$, find the value of $\sin\frac{x}{2}, \cos\frac{x}{2}$ and $\tan\frac{x}{2}$.
Solution: $\frac{\pi}{2} < \frac{x}{2} < \frac{3\pi}{4}$, hence $\frac{x}{2}$ lies in the 2^{nd} quadrant. For $\theta = \frac{x}{2}$ in (4.1.15.1),

$$3\tan^2\theta + 8\tan\theta - 3 = 0 \tag{4.1.16.1}$$

$$\implies \tan\frac{x}{2} = \tan\theta = -3 \tag{4.1.16.2}$$

by taking the negative root. From Fig. 4.1.3,

$$\cos\frac{x}{2} = -\frac{1}{\sqrt{10}}, \sin\frac{x}{2} = \frac{3}{\sqrt{10}} \tag{4.1.16.3}$$

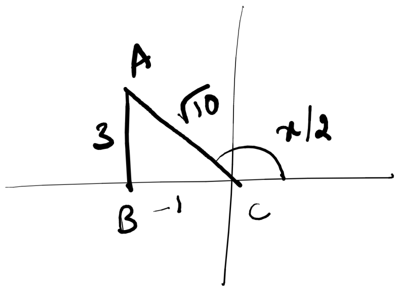


Fig. 4.1.3

4.1.17 Prove that $\cos^2 x + \cos^2\left(x + \frac{\pi}{3}\right) + \cos^2\left(x - \frac{\pi}{3}\right) = \frac{3}{2}$.
Solution: The LHS can be expressed as

$$\frac{1 + \cos 2x + 1 + \cos\left(2x + \frac{2\pi}{3}\right) + 1 + \cos\left(2x - \frac{2\pi}{3}\right)}{2}$$

$$= \frac{3}{2} + \frac{\cos 2x + 2\cos 2x \cos\left(\frac{2\pi}{3}\right)}{2} \tag{4.1.17.1}$$

yielding the RHS upon substituting numerical values.

4.1.18 Find the values of other five trigonometric functions

a) $\cos x = -\frac{1}{2}$, x lies in third quadrant.

b) $\sin x = \frac{3}{5}$,x lies in second quadrant.

c) $\cot x = \frac{3}{4}$,x lies in third quadrant.

d) $\sec x = \frac{13}{5}$, x lies in fourth quadrant.

e) $\tan x = -\frac{5}{12}$, x lies in second quadrant.

Solution:

a) See Fig. 4.1.4.

$$\sin x = -\frac{\sqrt{3}}{2}, \tan x = \sqrt{3} \tag{4.1.18.1}$$

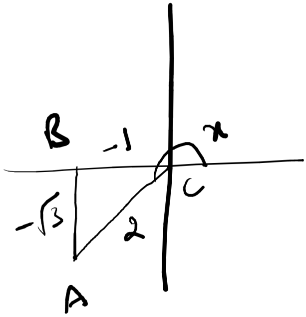


Fig. 4.1.4

b) See Fig. 4.1.5.

$$\cos x = -\frac{4}{5}, \tan x = -\frac{3}{4}. \qquad (4.1.18.2)$$

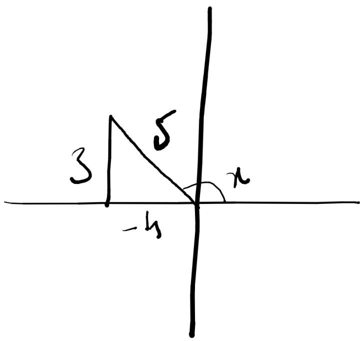


Fig. 4.1.5

c) See Fig. 4.1.6.

$$\cos x = -\frac{3}{5}, \sin x = -\frac{4}{5}, \tan x = \frac{4}{3}. \qquad (4.1.18.3)$$

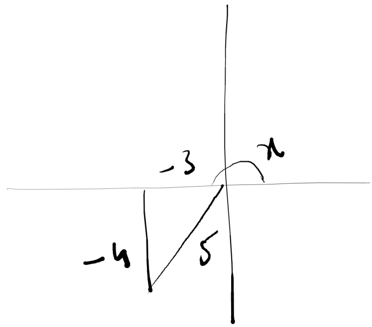


Fig. 4.1.6

d) See Fig. 4.1.7.

$$\cos x = \frac{5}{13}, \sin x = -\frac{12}{13}, \tan x = -\frac{12}{5}. \tag{4.1.18.4}$$

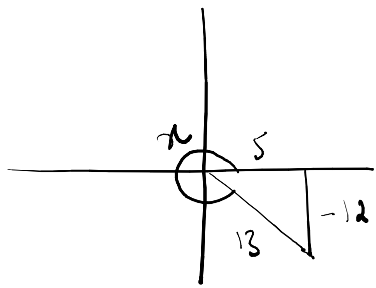


Fig. 4.1.7

e) See Fig. 4.1.8.

$$\cos x = -\frac{12}{13}, \sin x = \frac{5}{13} \tag{4.1.18.5}$$

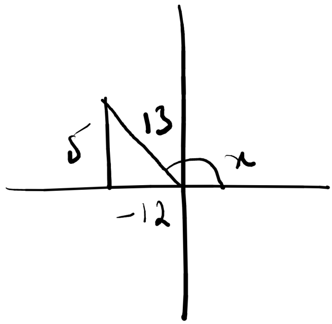


Fig. 4.1.8

4.1.19 Find the values of the trigonometric functions

a) $\sin 765°$

b) $\csc(-1410°)$

c) $\tan \frac{19\pi}{3}$

d) $\sin \frac{-11\pi}{3}$

e) $\cot \frac{-15\pi}{4}$

Solution:

a)

$$\sin 765° = \sin(2 \times 360° + 45°) \tag{4.1.19.1}$$
$$= \sin 45° = \frac{1}{\sqrt{2}} \tag{4.1.19.2}$$

b)

$$\csc(-1410°) = \csc(-4 \times 360° + 30°) \tag{4.1.19.3}$$
$$= \csc 30° = 2 \tag{4.1.19.4}$$

c)

$$\tan \frac{19\pi}{3} = \tan\left(6\pi + \frac{\pi}{3}\right) \tag{4.1.19.5}$$
$$= \tan \frac{\pi}{3} = \sqrt{3} \tag{4.1.19.6}$$

d)

$$\sin \frac{-11\pi}{3} = \sin\left(-4\pi + \frac{\pi}{3}\right) \tag{4.1.19.7}$$
$$= \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2} \tag{4.1.19.8}$$

e) $\cot \frac{-15\pi}{4}$

$$\cot \frac{-15\pi}{4} = \cot\left(-4\pi + \frac{\pi}{4}\right) \tag{4.1.19.9}$$
$$= \cot \frac{\pi}{4} = 1 \tag{4.1.19.10}$$

4.1.20 Prove that

a) $\sin^2 \frac{\pi}{6} + \cos^2 \frac{\pi}{3} - \tan^2 \frac{\pi}{4} = -\frac{1}{2}$

b) $2\sin^2 \frac{\pi}{6} + \csc^2 \frac{7\pi}{6} \cos^2 \frac{\pi}{3} = \frac{3}{2}$

c) $\cot^2 \frac{\pi}{6} + \csc \frac{5\pi}{6} + 3\tan^2 \frac{\pi}{6} = 6$

d) $2\sin^2 \frac{3\pi}{4} + 2\cos^2 \frac{\pi}{4} + 2\sec^2 \frac{\pi}{3} = 10$

Solution:

a) The LHS is

$$\sin^2 \frac{\pi}{6} + \sin^2 \frac{\pi}{6} - 1 = \sin^2 \frac{\pi}{6} - \cos^2 \frac{\pi}{6} \tag{4.1.20.1}$$
$$= -\cos \frac{\pi}{3} = -\frac{1}{2} \tag{4.1.20.2}$$

b) The LHS can be expressed as

$$2\sin^2 \frac{\pi}{6} + \frac{\cos^2 \frac{\pi}{3}}{\sin^2\left(\pi + \frac{\pi}{6}\right)} = 2\sin^2 \frac{\pi}{6} + \frac{\sin^2 \frac{\pi}{6}}{\sin^2 \frac{\pi}{6}} = \frac{1}{2} + 1 \tag{4.1.20.3}$$

c) The LHS equals

$$\tan^2 \frac{\pi}{6} + \cot^2 \frac{\pi}{6} + \csc\left(\pi - \frac{\pi}{6}\right) + 2\tan^2 \frac{\pi}{6} = \left(\tan \frac{\pi}{6} + \cot \frac{\pi}{6}\right)^2 - 2 + \csc \frac{\pi}{6} + 2\tan^2 \frac{\pi}{6} \tag{4.1.20.4}$$
$$= \sec^2 \frac{\pi}{6} \csc^2 \frac{\pi}{6} - 2 + 2 + 2\tan^2 \frac{\pi}{6} \tag{4.1.20.5}$$
$$= 4\sec^2 \frac{\pi}{6} + 2\tan^2 \frac{\pi}{6} \tag{4.1.20.6}$$
$$= 4 + 6\tan^2 \frac{\pi}{6} = 6 \tag{4.1.20.7}$$

d) The LHS can be expressed as

$$2\sin^2\left(\pi - \frac{\pi}{4}\right) + 2\cos^2 \frac{\pi}{4} + 2\sec^2 \frac{\pi}{3} = 2\left(\sin^2 \frac{\pi}{4} + \cos^2 \frac{\pi}{4}\right) + 8 \tag{4.1.20.8}$$

4.1.21 Find the value of

a) $\sin 75°$

b) $\tan 15°$

Solution:

a)

$$\sin 75° = \cos 15° \tag{4.1.21.1}$$

which is available in (4.1.7.3).

b) See (4.1.7.4).

4.1.22 Prove that $\cos\left(\frac{\pi}{4}-x\right)\cos\left(\frac{\pi}{4}-y\right)-\sin\left(\frac{\pi}{4}-x\right)\sin\left(\frac{\pi}{4}-y\right)=\sin(x+y)$.
Solution: The LHS can be expressed as

$$\cos\left(\frac{\pi}{4}-x+\frac{\pi}{4}-y\right)=\cos\left[\frac{\pi}{2}-(x+y)\right] \tag{4.1.22.1}$$

which is equal to the RHS.

4.1.23 Prove that

$$\frac{\tan\left(\frac{\pi}{4}+x\right)}{\tan\left(\frac{\pi}{4}-x\right)}=\left(\frac{1+\tan x}{1-\tan x}\right)^2.$$

Solution:

$$\tan\left(\frac{\pi}{4}+x\right)=\frac{\tan\frac{\pi}{4}+\tan x}{1-\tan\frac{\pi}{4}\tan x} \tag{4.1.23.1}$$

$$=\frac{1+\tan x}{1-\tan x} \tag{4.1.23.2}$$

$$\tan\left(\frac{\pi}{4}-x\right)=\frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\tan x} \tag{4.1.23.3}$$

$$=\frac{1-\tan x}{1+\tan x} \tag{4.1.23.4}$$

From the above, the desired result is obtained.

4.1.24 Prove that

$$\frac{\cos(\pi+x)\cos(-x)}{\sin(\pi-x)\cos\left(\frac{\pi}{2}+x\right)}=\cot^2 x.$$

Solution: The LHS can be expressed as

$$\frac{-\cos x\cos(-x)}{\sin x(-\sin x)} \tag{4.1.24.1}$$

yielding the RHS.

4.1.25 Prove that $\cos\left(\frac{3\pi}{2}+x\right)\cos(2\pi+x)\left[\cot\left(\frac{3\pi}{2}-x\right)+\cot(2\pi+x)\right]=1$.
Solution: The LHS can be expressed as

$$\sin x\cos x\left[\tan x+\cot x\right]=\sin x\cos x\sec x\csc x \tag{4.1.25.1}$$

yielding the RHS.

4.1.26 Prove that $\sin(n+1)x\sin(n+2)x+\cos(n+1)x\cos(n+2)x=\cos x$.
Solution: The LHS can be expressed as

$$\frac{1}{2}\left[\cos x-\cos\left(\frac{n+3}{2}x\right)+\cos x+\cos\left(\frac{n+3}{2}x\right)\right] \tag{4.1.26.1}$$

yielding the RHS.

4.1.27 Prove that $\cos\left(\frac{3\pi}{4}+x\right)-\cos\left(\frac{3\pi}{4}-x\right)=-\sqrt{2}\sin x$.

Solution: The LHS can be expressed as

$$-2\sin x \sin\frac{3\pi}{4} = -2\sin x \sin\frac{\pi}{4} \tag{4.1.27.1}$$

yielding the RHS.

4.1.28 Prove that $\sin^2 6x - \sin^2 4x = \sin 2x \sin 10x$.
Solution: The LHS can be expressed as

$$\frac{\cos 8x - \cos 12x}{2} \tag{4.1.28.1}$$

yielding the RHS.

4.1.29 Prove that $\cos^2 2x - \cos^2 6x = \sin 4x \sin 8x$.
Solution: The LHS can be expressed as

$$\frac{\cos 4x - \cos 12x}{2} \tag{4.1.29.1}$$

yielding the RHS.

4.1.30 Prove that $\sin 2x + 2\sin 4x + \sin 6x = 4\cos^2 x \sin 4x$.
Solution: The LHS can be expressed as

$$\sin 2x + \sin 6x + 2\sin 4x = 2\sin 4x \cos 2x + 2\sin 4x \tag{4.1.30.1}$$

$$= 2\sin 4x\,(1 + \cos 2x) \tag{4.1.30.2}$$

yielding the RHS.

4.1.31 Prove that $\cot 4x\,(\sin 5x + \sin 3x) = \cot x\,(\sin 5x - \sin 3x)$.
Solution:

$$LHS = 2\cot 4x \sin 4x \cos x \tag{4.1.31.1}$$

$$= 2\cos 4x \sin x \tag{4.1.31.2}$$

$$RHS = 2\cot x \cos 4x \sin x \tag{4.1.31.3}$$

4.1.32 Prove that

$$\frac{\cos 9x - \cos 5x}{\sin 17x - \sin 3x} = -\frac{\sin 2x}{\cos 10x}.$$

Solution:

$$LHS = -\frac{2\sin 7x \sin 2x}{2\cos 10x \sin 7x} = RHS \tag{4.1.32.1}$$

4.1.33 Prove that

$$\frac{\sin 5x + \sin 3x}{\cos 5x + \cos 3x} = \tan 4x.$$

Solution:

$$LHS = \frac{2\sin 4x \cos 2x}{2\cos 4x \cos 2x} = RHS \tag{4.1.33.1}$$

4.1.34 Prove that

$$\frac{\sin x + \sin y}{\cos x + \cos y} = \tan\left(\frac{x - y}{2}\right).$$

Solution:

$$LHS = \frac{2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)}{2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)} = RHS \tag{4.1.34.1}$$

4.1.35 Prove that

$$\frac{\sin x + \sin 3x}{\cos x + \cos 3x} = \tan 2x.$$

Solution:

$$LHS = \frac{2\sin 2x \cos x}{2\cos 2x \cos x} = RHS \tag{4.1.35.1}$$

4.1.36 Prove that

$$\frac{\sin x - \sin 3x}{\sin^2 x - \cos^2 x} = 2\sin x.$$

Solution:

$$LHS = \frac{-2\sin x \cos 2x}{-\cos 2x} = RHS \tag{4.1.36.1}$$

4.1.37 Prove that

$$\frac{\cos 4x + \cos 3x + \cos 2x}{\sin 4x + \sin 3x + \sin 2x} = \cot 3x.$$

Solution:

$$LHS = \frac{2\cos 3x \cos x + \cos 3x}{2\sin 3x \cos x + \sin 3x} \tag{4.1.37.1}$$

$$= \frac{\cos 3x\,(\cos x + 1)}{\sin 3x\,(\cos x + 1)} = RHS \tag{4.1.37.2}$$

4.1.38 Prove that

$$\cot x \cot 2x - \cot 2x \cot 3x - \cot 3x \cot x = 1. \tag{4.1.38.1}$$

Solution:

$$\cot x = \cot(3x - 2x) = \frac{\cot 3x \cot 2x + 1}{\cot 2x - \cot 3x} \tag{4.1.38.2}$$

$$\implies \cot x \cot 2x - \cot 3x \cot x = 1 + \cot 2x \cot 3x \tag{4.1.38.3}$$

yielding (4.1.38.1).

4.1.39 Prove that

$$\tan 4x = \frac{4\tan x\left(1 - \tan^2 x\right)}{1 - 6\tan^2 x + \tan^4 x}. \tag{4.1.39.1}$$

Solution:

$$\tan 4x = \frac{2\tan 2x}{1-\tan^2 2x} \tag{4.1.39.2}$$

$$\tan 2x = \frac{2\tan x}{1-\tan^2 x} \tag{4.1.39.3}$$

Substituting (4.1.39.2) in (4.1.39.2) yileds (4.1.39.1).

4.1.40 Prove that

$$\cos 4x = 1 - 8\sin^2 x\cos^2 x. \tag{4.1.40.1}$$

Solution:

$$\cos 4x = 1 - 2\sin^2 2x \tag{4.1.40.2}$$

$$= 1 - 2\left(2\sin x\cos x\right)^2 = RHS \tag{4.1.40.3}$$

4.1.41 Prove that

$$\cos 6x = 32\cos^6 x - 48\cos^4 x + 18\cos^2 x - 1. \tag{4.1.41.1}$$

Solution:

$$\cos 6x = 4\cos^3 2x - 3\cos 2x \tag{4.1.41.2}$$

$$= 4\left(2\cos^2 x - 1\right)^3 - 3\left(2\cos^2 x - 1\right) = RHS \tag{4.1.41.3}$$

after some algebra.

4.1.42 Prove that

a) $2\cos\frac{\pi}{13}\cos\frac{9\pi}{13} + \cos\frac{3\pi}{13} + \cos\frac{5\pi}{13} = 0$

b) $(\sin 3x + \sin x)\sin x + (\cos 3x - \cos x)\cos x = 0$

c) $(\cos x + \cos y)^2 + (\sin x - \sin y)^2 = 4\cos^2\left(\frac{x+y}{2}\right)$

d) $(\cos x - \cos y)^2 + (\sin x - \sin y)^2 = 4\sin^2\left(\frac{x-y}{2}\right)$

e) $\sin x + \sin 3x + \sin 5x + \sin 7x = 4\cos x\cos 2x\sin 4x$

f)

$$\frac{(\sin 7x + \sin 5x) + (\sin 9x + \sin 3x)}{(\cos 7x + \cos 5x) + (\cos 9x + \cos 3x)} = \tan 6x$$

g) $\sin 3x + \sin 2x - \sin x = 4\sin x\cos\frac{x}{2\cos\frac{3x}{2}}$

4.1.43 Find $\sin\frac{x}{2}, \cos\frac{x}{2}$ and $\tan\frac{x}{2}$ in each of the following

a) $\tan x = -\frac{4}{3}x$, in second quadrant.

b) $\sin x = \frac{1}{4}x$, in second quadrant.

c) $\cos x = -\frac{1}{3}x$, in third quadrant.

4.2 CBSE

4.2.1 Simplest form of

$$\frac{1+\tan^2 A}{1+\cot^2 A}.$$

is ______. (10, 2020)

4.2.2 Write the value of

$$\sin^2 30^\circ + \cos^2 60^\circ.$$

(10, 2020)

4.2.3 Prove that

$$\left(\sin^4\theta - \cos^4\theta + 1\right)\csc^2\theta = 2.$$

(10, 2020)

4.2.4 Prove that

$$\frac{\sin A - 2\sin^3 A}{2\cos^3 A - \cos A} = \tan A.$$

(10, 2023)

4.2.5 Prove that

$$\sec A\,(1-\sin A)\,(\sec A + \tan A) = 1.$$

(10, 2023)

4.2.6 If

$$4\cot^2 45^\circ - \sec^2 60^\circ + \sin^2 60^\circ + p = \frac{3}{4},$$

then find the value of p. (10, 2023)

4.2.7 If

$$\cos A + \cos^2 A = 1,$$

then find the value of

$$\sin^2 A + \sin^4 A.$$

(10, 2023)

4.2.8 Prove that

$$\left(\frac{1}{\cos\theta} - \cos\theta\right)\left(\frac{1}{\sin\theta} - \sin\theta\right) = \frac{1}{\tan\theta + \cot\theta}.$$

(10, 2023)

4.2.9 If $2\tan A = 3$, then the value of

$$\frac{4\sin A + 3\cos A}{4\sin A - 3\cos A}$$

is

a) $\frac{7}{\sqrt{13}}$ b) $\frac{1}{\sqrt{13}}$ c) 3 d) does not exist

(10, 2023)

4.2.10 $\left(\sec^2\theta - 1\right)\left(\csc^2\theta - 1\right)$ is equal to

a) -1 b) 1 c) 0 d) 2

(10, 2023)

4.2.11 Evaluate $2\sec^2\theta + 3\csc^2\theta - 2\sin\theta\cos\theta$ if $\theta = 45^\circ$.

(10, 2023)

4.2.12 If

$$\sin\theta - \cos\theta = 0,$$

then find the value of $\sin^4\theta + \cos^4\theta$. (10, 2023)

4.2.13 If $\sin\theta = 0$, then the value of $\tan^2\theta + \cot^2\theta$ is

a) 2 b) 4 c) 1 d) $\frac{10}{9}$

(10, 2022)

4.2.14 $5\tan^2\theta - 5\sec^2\theta =$ ______. (10, 2022)

4.2.15 Show that

$$\cos(38^\circ)\cos(52^\circ) - \sin(38^\circ)\sin(52^\circ) = \cos(90^\circ).$$

(10, 2022)

4.2.16 Prove that

$$\frac{\sin\theta}{\cot\theta + \csc\theta} = 2 + \frac{\sin\theta}{\cot\theta - \csc\theta}.$$

(10, 2022)

4.2.17 Given $15\cot(A) = 8$, find the values of $\sin(A)$ and $\sec(A)$. (10, 2022)

4.2.18 Find $\tan^{-1}\frac{1}{\sqrt{3}} - \cot^{-1}\frac{-1}{\sqrt{3}}$. (10, 2022)

4.2.19 Simplify

$$\frac{\sin 30^\circ + \tan 45^\circ - \cos 60^\circ}{\sec 30^\circ + \cos 60^\circ + \cot 45^\circ}.$$

(10, 2021)

4.2.20 Prove that

$$\sec\theta(1 - \sin\theta)(\sec\theta + \tan\theta) = 1.$$

(10, 2021)

4.2.21 Prove that

$$\frac{1 + \sec A}{\sec A} = \frac{\sin^2 A}{1 - \cos A}.$$

(10, 2021)

4.2.22 If $\tan\theta = 4/3$, find the value

$$\frac{2\sin\theta - 3\cos\theta}{2\sin\theta + 3\cos\theta}.$$

(10, 2021)

4.2.23 If x= $a\cos\theta$ and y=$b\sin\theta$, then find the value of $b^2x^2 + a^2y^2$ (10, 2021)

4.2.24 Prove that

$$\frac{\tan\theta - \cot\theta}{\sin\theta\cos\theta} = \tan^2\theta - \cot^2\theta.$$

(10, 2021)

4.2.25 Prove that

$$(\sec\theta - \tan\theta)^2 = \frac{1+\sin\theta}{1-\sin\theta}.$$

(10, 2021)

4.2.26 If $3\sin A = 1$, then find the value of $\sec A$. (10, 2021)

4.2.27 Show that

$$\frac{1+\cot^2\theta}{1+\tan^2\theta} = \cot^2\theta.$$

(10, 2021)

4.2.28 Simplify

$$\csc^2 60^\circ \sin^2 30^\circ - \sec^2 60^\circ$$

(10, 2021)

4.2.29 If $\tan\theta + \cot\theta = \frac{4\sqrt{3}}{3}$, then find the value of $\tan^2\theta + \cot^2\theta$. (10, 2021)

4.2.30 Prove

$$\frac{1}{(\cot A)(\sec A) - \cot A} - \csc A = \csc A - \frac{1}{(\cot A)(\sec A) + \cot A}.$$

(10, 2021)

4.2.31 Prove

$$\sin^6 A + 3\sin^2 A\cos^2 A = 1 - \cos^6 A.$$

(10, 2021)

4.2.32 Prove that $2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{31}{17}$. (12, 2021)

4.2.33 $\sin\left[\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right]$ is equal to

a) $\frac{1}{2}$ b) $\frac{1}{3}$ c) -1 d) 1

(12, 2021)

4.2.34 $\sin\left(\tan^{-1}x\right)$,where $|x| \leq 1$,is equal to

a) $\frac{x}{\sqrt{1-x^2}}$ b) $\frac{1}{\sqrt{1-x^2}}$ c) $\frac{1}{\sqrt{1+x^2}}$ d) $\frac{x}{\sqrt{1+x^2}}$

(12, 2021)

4.2.35 Simplest form of

$$\tan^{-1}\left(\frac{\sqrt{1+\cos x}+\sqrt{1-\cos x}}{\sqrt{1+\cos x}-\sqrt{1-\cos x}}\right), \pi < x < \frac{3\pi}{2}$$

is

a) $\frac{\pi}{4}-\frac{x}{2}$ b) $\frac{3\pi}{2}-\frac{x}{2}$ c) $-\frac{x}{2}$ d) $\pi-\frac{x}{2}$

(12, 2021)

4.2.36 Prove that

$$\sin^{-1}\frac{4}{5}+\tan^{-1}\frac{5}{12}+\cos^{-1}\frac{63}{65}=\frac{\pi}{2}.$$

(12, 2019)

4.2.37 Find the value of $\sin\left(\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{2}{3}\right)$. (12, 2019)

4.2.38 Prove that

$$\cos^{-1}\left(\frac{12}{13}\right)+\sin^{-1}\left(\frac{3}{5}\right)=\sin^{-1}\left(\frac{56}{65}\right).$$

(12, 2019)

4.2.39 Evaluate $\frac{\tan 65^\circ}{\cot 25^\circ}$. (10, 2019)

4.2.40 Express $(\sin 67^\circ+\cos 75^\circ)$ in terms of trigonometric ratios of the angle between 0° and 45°. (10, 2019)

4.2.41 Prove that

$$(\sin\theta+1+\cos\theta)(\sin\theta-1+\cos\theta)\sec\theta\csc\theta=2.$$

(10, 2019)

4.2.42 Prove that

$$\sqrt{\frac{\sec\theta-1}{\sec\theta+1}}+\sqrt{\frac{\sec\theta+1}{\sec\theta-1}}=2\csc\theta.$$

(10, 2019)

4.2.43 If $\sec\theta+\tan\theta=m$, show that $\frac{m^2-1}{m^2+1}=\sin\theta$. (10, 2019)

4.2.44 Prove that

$$2\left(\sin^6\theta+\cos^6\theta\right)-3\left(\sin^4\theta+\cos^4\theta\right)+1=0.$$

(10, 2019)

4.2.45 Evaluate

$$\sin^2 60^\circ+2\tan 45^\circ-\cos^2 30^\circ.$$

(10, 2019)

4.2.46 Evaluate

$$\left(\frac{3\tan 41^\circ}{\cot 90^\circ}\right)^2-\left(\frac{\sin 3^\circ\sec 55^\circ}{\tan 10^\circ\tan 20^\circ\tan 60^\circ\tan 70^\circ\tan 80^\circ}\right)^2.$$

(10, 2019)

4.2.47 Prove that

$$\frac{\tan\theta}{1-\cot\theta}+\frac{\cot\theta}{1-\tan\theta}=1+\sec\theta\csc\theta.$$

(10, 2019)

4.2.48 Prove that

$$\frac{\sin\theta}{\cot\theta+\csc\theta}=2+\frac{\sin\theta}{\cot\theta-\csc\theta}.$$

(10, 2019)

4.2.49 Evaluate

$$\left(\frac{3\sin 43^\circ}{\cos 47^\circ}\right)^2-\frac{\cos 37^\circ\csc 53^\circ}{\tan 5^\circ\tan 25^\circ\tan 45^\circ\tan 65^\circ\tan 85^\circ}.$$

(10, 2019)

4.2.50 If $\sin A=\frac{3}{4}$, calculate $\sec A$. (10, 2019)

4.2.51 If $\tan\,\alpha=\frac{5}{12}$, find the value of $\sec\,\alpha$. (10, 2019)

4.2.52 If $1+\sin^2\theta=3\sin\theta\cos\theta$, then prove that $\tan\,\theta=1$ or $\tan\,\theta=\frac{1}{2}$. (10, 2019)

4.2.53 Prove that

$$\frac{\tan^3\theta}{1+\tan^2\theta}+\frac{\cot^3\theta}{1+\cot^2\theta}=\sec\theta\csc\theta-2\sin\theta\cos\theta.$$

(10, 2019)

4.2.54 Find the value of $\cos 48^\circ-\sin 42^\circ$. (10, 2019)

4.2.55 Prove that

$$\frac{\tan\theta}{1-\tan\theta}-\frac{\cot\theta}{1-\cot\theta}=\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}.$$

(10, 2019)

4.2.56 If $\cos\theta+\sin\theta=\sqrt{2}\cos\theta$, show that $\cos\theta-\sin\theta=\sqrt{2}\sin\theta$. (10, 2019)

4.2.57 Prove that

$$\frac{(1+\cot\theta+\tan\theta)\,(\sin\theta-\cos\theta)}{(\sec^3\theta-\csc^3\theta)}=\sin^2\theta\cos^2\theta.$$

(10, 2019)

4.2.58 Evaluate

$$\frac{\csc^2(90^\circ-\theta)-\tan^2\theta}{2\,(\cos^2 37^\circ+\cos^2 53^\circ)}-\frac{2\tan^2 30^\circ\sec^2 37^\circ\sin^2 53^\circ}{\csc^2 63^\circ-\tan^2 27^\circ}.$$

(10, 2019)

4.2.59 Prove that

$$(\sin\theta+\csc\theta)^2+(\cos\theta+\sec\theta)^2=7+\tan^2\theta+\cot^2\theta.$$

(10, 2019)

4.2.60 Prove that

$$(1 + \cot A - \csc A)(1 + \tan A + \sec A) = 2.$$

(10, 2019)

4.2.61 Prove that

$$\frac{\sin A - \cos A + 1}{\sin A + \cos A - 1} = \frac{1}{\sec A - \tan A}.$$

(10, 2019)

4.2.62 Find the value of

$$\left(\sin^2 33^\circ + \sin^2 57^\circ\right).$$

(10, 2019)

4.2.63 If $\sec\theta = x + \frac{1}{4x}$, where $x \neq 0$, find $(\sec\theta + \tan\theta)$. (10, 2019)

4.2.64 Prove that

$$\frac{\tan^2 A}{\tan^2 A - 1} + \frac{\csc^2 A}{\sec^2 A - \csc^2 A} = \frac{1}{1 - 2\cos^2 A}.$$

(10, 2019)

4.2.65 If 4 $\tan\theta = 3$, evaluate

$$\left(\frac{4\sin\theta - \cos\theta + 1}{4\sin\theta + \cos\theta - 1}\right).$$

(10, 2018)

4.2.66 What is the value of $\left(\cos^2 67^\circ - \sin^2 23^\circ\right)$? (10, 2018)

4.2.67 Prove that

$$\left(\frac{\sin A - 2sin^3 A}{2\cos^3 A - \cos A} = \tan A\right).$$

(10, 2018)

4.2.68 Find the value of

$$\tan^{-1}\sqrt{3} - \cot^{-1}\left(\sqrt{-3}\right).$$

(12, 2018)

4.2.69 Prove that

$$3\sin^{-1} x = \sin^{-1}\left(3x - 4x^3\right), x \in \left(\frac{-1}{2}, \frac{1}{2}\right).$$

(12, 2018)

4.2.70 Prove that

$$\cos^{-1}\left(\frac{12}{13}\right) + \sin^{-1}\left(\frac{3}{5}\right) = \sin^{-1}\left(\frac{56}{65}\right).$$

(12, 2018)

4.2.71 Prove that

$$\sin^{-1}\left(\frac{8}{17}\right)+\cos^{-1}\left(\frac{4}{5}\right)=\cot^{-1}\left(\frac{36}{77}\right).$$

(12, 2018)

4.2.72 Prove that

$$\sin^{-1}\frac{4}{5}+\tan^{-1}\frac{5}{12}+\cos^{-1}\frac{63}{65}=\frac{\pi}{2}.$$

(12, 2018)

4.2.73 Find the value of $\sin\left(\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{2}{3}\right)$. (12, 2018)

4.2.74 Prove that $2\sin^{-1}\left(\frac{3}{5}\right)-\tan^{-1}\left(\frac{17}{31}\right)=\frac{\pi}{4}$. (12, 2016)

4.2.75 Prove that

$$\tan^{-1}\left(\frac{6x-8x^3}{1-12x^2}\right)-\tan^{-1}\left(\frac{4x}{1-4x^2}\right)=\tan^{-1}2x; |2x|<\frac{1}{\sqrt{3}}.$$

(12, 2016)

4.2.76 Prove that

$$2\sin^{-1}\left(\frac{3}{5}\right)-\tan^{-1}\left(\frac{17}{31}\right)=\frac{\pi}{4}.$$

(12, 2016)

4.2.77 Prove that $2\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{7}\right)=\sin^{-1}\left(\frac{31}{25\sqrt{2}}\right)$. (12, 2015)

4.2.78 If $\sin\theta+\cos\theta=\sqrt{2}\cos(90^\circ-\theta)$, find the value of $\cot\theta$. (10, 2018)

4.2.79 Prove that

$$\frac{1}{\operatorname{cosec}\theta+\cot\theta}-\frac{1}{\sin\theta}=\frac{1}{\sin\theta}-\frac{1}{\operatorname{cosec}\theta-\cot\theta}.$$

(10, 2018)

4.2.80 If $\tan\theta+\sin\theta=m$, $\tan\theta-\sin\theta=n$, show that $m^2-n^2=4\sqrt{mn}$. (10, 2018)

4.2.81 Prove that

$$\left(\frac{\sin A}{1-\cos A}-\frac{1-\cos A}{\sin A}\right)\left(\frac{\cos A}{1-\sin A}-\frac{1-\sin A}{\cos A}\right)=4.$$

(10, 2018)

4.2.82 Prove that

$$\tan\left(\frac{6x-8x^3}{1-12x^2}\right)-\tan^{-1}\left(\frac{4x}{1-4x^2}\right)=\tan^{-1}2x, \quad |2x|<\frac{1}{\sqrt{3}}.$$

(12, 2016)

4.2.83 Write the principal value of $\sec^{-1}(-2)$. (12, 2010)

4.2.84 Prove the following

$$\cos\left[\tan^{-1}\left\{\sin\left(\cot^{-1}x\right)\right\}\right]=\sqrt{\frac{1+x^2}{2+x^2}}.$$

(12, 2010)

4.2.85 Prove the following

$$\tan^{-1} x + \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right).$$

(12, 2010)

4.2.86 Find the value of

$$\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) + \cot^{-1}\left(\frac{1}{\sqrt{3}}\right) + \tan^{-1}\left[\sin\left(-\frac{\pi}{2}\right)\right].$$

(10, 2024)

4.2.87 If $\sec\theta - \tan\theta = m$, then the value of $\sec\theta + \tan\theta$ is ______. (10, 2024)

4.2.88 If $\cos(\alpha+\beta) = 0$ then the value of $\cos\left(\frac{\alpha+\beta}{2}\right)$ is equal to ______. (10, 2024)

4.2.89 Simplify

$$\cos^{-1} x + \cos^{-1}\left[\frac{x}{2}\frac{\sqrt{3-3x^2}}{2}\right]; -\frac{1}{2} \le x \le 1.$$

(12, 2024)

4.2.90 Evaluate $2\sqrt{2}\cos 45^\circ \sin 10^\circ + 2\sqrt{3}\cos 30^\circ$. (10, 2024)

4.2.91 If $A = 60^\circ$ and $B = 30^\circ$, verify that $\sin(A+B) = \sin A\cos B + \cos A\sin B$. (10, 2024)

4.2.92 Prove that

$$\frac{\tan\theta}{1-\cot\theta} + \frac{\cot\theta}{1-\tan\theta} = 1 + \sec\theta\csc\theta.$$

(10, 2024)

4.2.93 If $a = \sin^{-1}\left(\frac{\sqrt{2}}{2}\right) + \cos^{-1}\left(\frac{-1}{2}\right)$ and $b = \tan^{-1}\left(\sqrt{3}\right) + \cot^{-1}\left(\frac{-1}{\sqrt{3}}\right)$, then find the value of $a+b$. (12, 2024)

4.2.94 Find the value k if

$$\sin^{-1}\left[k\tan\left(2\cos^{-1}\frac{\sqrt{3}}{2}\right)\right] = \frac{\pi}{3}.$$

(12, 2024)

4.2.95 If $4\cot^2 45^\circ - \sec^2 60^\circ + \sin^2 60^\circ + p = \frac{3}{4}$, then find the value of p. (10, 2023)

4.2.96 If $\cos A + \cos^2 A = 1$, then find the value of $\sin^2 A + \sin^4 A$. (10, 2023)

4.2.97 Prove that

$$\left(\frac{1}{\cos\theta} - \cos\theta\right)\left(\frac{1}{\sin\theta} - \sin\theta\right) = \frac{1}{\tan\theta + \cot\theta}.$$

(10, 2023)

4.2.98 $\left(\sec^2\theta - 1\right)\left(\operatorname{cosec}^2\theta - 1\right)$ is equal to

a) -1 b) 1 c) 0 d) 2

(10, 2023)

4.2.99 Evaluate $2\sec^2\theta + 3\csc^2\theta - 2\sin\theta\cos\theta$ if $\theta = 45^\circ$. (10, 2023)

4.2.100 If $\sin\theta - \cos\theta = 0$, then find the value of $\sin^4\theta + \cos^4\theta$. (10, 2023)

4.2.101 Prove that

$$\frac{\sin A - 2\sin^3 A}{2\cos^3 A - \cos A} = \tan A.$$

(10, 2023)

4.2.102 Prove that

$$\sec A\,(1 - \sin A)\,(\sec A + \tan A) = 1.$$

(10, 2023)

4.2.103 Write the principal value of $\sec^{-1}(-2)$. (12, 2010)

4.2.104 Prove the following

$$\cos\left[\tan^{-1}\left\{\sin\left(\cot^{-1} x\right)\right\}\right] = \sqrt{\frac{1+x^2}{2+x^2}}.$$

(12, 2010)

4.2.105 Prove the following

$$\tan^{-1} x + \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right).$$

(12, 2010)

4.3 JEE

4.3.1 Suppose

$$\sin^3 x \sin 3x = \sum_{m=0}^{n} C_m \cos x$$

is an identity in x, where $C_0, C_1, \cdots, C_n$ are constants and $C_n \neq 0$, then the value of n is ______. (1981)

4.3.2 The value of (1991)

$$\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\sin\frac{7\pi}{14}\sin\frac{9\pi}{14}\sin\frac{11\pi}{14}\sin\frac{13\pi}{14}$$

is equal to

4.3.3 If (1993)

$$K = \sin\left(\frac{\pi}{18}\right)\sin\left(\frac{5\pi}{18}\right)\sin\left(\frac{7\pi}{18}\right)$$

then the numerical value of K is

4.3.4 Let α, β be such that $\pi < \alpha - \beta < 3\pi$. If

$$\sin\alpha + \sin\beta = -\frac{21}{65}$$
$$\cos\alpha + \cos\beta = -\frac{27}{65},$$

then the value of $\cos\frac{\alpha-\beta}{2}$ is (2004)

a) $-\frac{6}{65}$ b) $\frac{3}{\sqrt{130}}$ c) $\frac{6}{65}$ d) $-\frac{3}{\sqrt{130}}$

4.3.5 The expression $\frac{\tan A}{1-\cot A}+\frac{\cot A}{1-\tan A}$ can be written as (2013)

a) $\sin(A)\cos(A)+1$
b) $\sec(A)$ cosec $(A)+1$
c) $\tan(A)+\cot(A)$
d) $\sec(A)+$ cosec (A)

4.3.6 Let

$$f_k(x)=\frac{1}{k}\left(\sin^k x+\cos^k x\right)$$

where $x\in R$ and $k\geq 1$. Then $f_4(x)-f_6(x)$ equals (2014)

a) $\frac{1}{4}$ b) $\frac{1}{12}$ c) $\frac{1}{6}$ d) $\frac{1}{3}$

4.3.7 For any $\theta\in\left(\frac{\pi}{4}\right),\left(\frac{\pi}{2}\right)$ the expression

$$3\left(\sin\theta-\cos\theta^4+6\right)\left(\sin\theta+\cos\theta^2+4\sin^6\theta\right)$$

equals (2019)

a) $13-4\cos^2\theta+6\sin^2\theta\cos^2\theta$
b) $13-4\cos^6\theta$
c) $13-4\cos^2\theta+6\cos^4\theta$
d) $13-4\cos^2\theta+2\sin^2\theta\cos^2\theta$

4.3.8 The value of

$$\cos^2 10^\circ-\cos 10^\circ\cos 50^\circ+\cos^2 50^\circ$$

is (2019)

a) $\frac{3}{4}+\cos 20^\circ$ b) $\frac{3}{4}$ c) $\frac{3}{2}\left(1+\cos 20^\circ\right)$ d) $\frac{3}{2}$

4.3.9

$$\left(0+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(0+\cos\frac{5\pi}{8}\right)\left(1+\cos\frac{7\pi}{8}\right)$$

is equal to ______. (1983)

4.3.10 The expression

$$2\left[\sin^4\left(\frac{3\pi}{2}-\alpha\right)+\sin^4(3\pi+\alpha)\right]-2\left[\sin^6\left(\frac{\pi}{2}+\alpha\right)+\sin^6(5\pi-\alpha)\right]$$

is equal to (1985)

a) -1
b) 0
c) 2
d) $\sin 3\alpha+\cos 6\alpha$
e) none of these

4.3.11 Let α and β be non-zero real numbers such that (2017)

$$2(\cos\beta-\cos\alpha)+\cos\alpha\cos\beta=1.$$

Then which of the following is/are true?

a) $\tan\left(\frac{\alpha}{2}\right) + \sqrt{3}\tan\left(\frac{\beta}{2}\right) = 0$

b) $\sqrt{3}\left(\tan\frac{\alpha}{2}\right) + \tan\left(\frac{\beta}{2}\right) = 0$

c) $\tan\left(\frac{\alpha}{2}\right) - \tan\left(\frac{\beta}{2}\right) = 0$

d) $\sqrt{3}\tan\left(\frac{\alpha}{2}\right) - \tan\left(\frac{\beta}{2}\right) = 0$

4.3.12 For a positive integer n, let (1999)

$$f_n(\theta) = \left(\tan\frac{\theta}{2}\right)(1+\sec\theta)(1+\sec 2\theta)(1+\sec 4\theta)\ldots(1+\sec 2^n\theta).$$

Then

a) $f_2\left(\frac{\pi}{16}\right) = 1$ b) $f_3\left(\frac{\pi}{32}\right) = 1$ c) $f_4\left(\frac{\pi}{64}\right) = 1$ d) $f_5\left(\frac{\pi}{128}\right) = 1$

4.3.13 If $\alpha+\beta+\gamma = 2\pi$, (1979)

a) $\tan\frac{\alpha}{2} + \tan\frac{\beta}{2} + \tan\frac{\gamma}{2} = \tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$

b) $\tan\frac{\alpha}{2}\tan\frac{\beta}{2} + \tan\frac{\beta}{2}\tan\frac{\gamma}{2} + \tan\frac{\gamma}{2}\tan\frac{\alpha}{2} = 1$

c) $\tan\frac{\alpha}{2} + \tan\frac{\beta}{2} + \tan\frac{\gamma}{2} = -\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$

d) None of These

4.3.14 The value of the expression $\sqrt{3}\,\mathrm{cosec}\,20^\circ - \sec 20^\circ$ is equal to ______. (1988)

4.3.15 Let $0 < x < \frac{\pi}{4}$. Then $(\sec 2x - \tan 2x)$ equals (1994)

a) $\tan\left(x - \frac{\pi}{4}\right)$ b) $\tan\left(\frac{\pi}{4} - x\right)$ c) $\tan\left(x + \frac{\pi}{4}\right)$ d) $\tan^2\left(x + \frac{\pi}{4}\right)$

4.3.16 If ω is an imaginary cube root of unity, then the value of (1994)

$$\sin\left(\left(\omega^{10} + \omega^{23}\right)\pi - \frac{\pi}{4}\right)$$

is

a) $-\frac{\sqrt{3}}{2}$ b) $-\frac{1}{\sqrt{2}}$ c) $-\frac{1}{\sqrt{2}}$ d) $\frac{\sqrt{3}}{2}$

4.3.17 The value of

$$\sum_{k=1}^{13} \frac{1}{\sin\left(\frac{\pi}{4} + \frac{(k-1)\pi}{6}\right)\sin\left(\frac{\pi}{4} + \frac{k\pi}{6}\right)}$$

is equal to (2016)

a) $3 - \sqrt{3}$ b) $2\left(3 - \sqrt{3}\right)$ c) $2\left(\sqrt{3} - 1\right)$ d) $2\left(2 - \sqrt{3}\right)$

4.3.18 Given $\alpha+\beta-\gamma = \pi$, prove that $\sin^2\alpha + \sin^2\beta - \sin^2\gamma = 2\sin\alpha\sin\beta\cos\gamma$. (1980)

4.3.19 Without using tables prove that (1982)

$$\sin\left(12^\circ\right)\sin\left(48^\circ\right)\sin\left(54^\circ\right) = \frac{1}{8}$$

4.3.20 Show that (1983)

$$16\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\cos\frac{16\pi}{15} = 1$$

4.3.21 Prove that (1988)

$$\tan(\alpha) + 2\tan(2\alpha) + 4\tan(4\alpha) + 8\cot(8\alpha) = \cot(\alpha)$$

4.3.22 Prove that (1997)

$$\sum_{k=1}^{n-1} (n-k)\cos\left(\frac{2k\pi}{n}\right) = -\frac{n}{2},$$

where $n \geq 3$.

4.3.23 (1995)

$$3(\sin x - \cos x)^4 + 6(\sin x + \cos x)^4 + 4\left(\sin^6 x + \cos^6 x\right) =$$

a) 11 b) 12 c) 13 d) 14

4.3.24 Let a, b, c be positive real numbers. Let

$$\theta = \tan^{-1}\left(\sqrt{\frac{a(a+b+c)}{bc}}\right) + \tan^{-1}\left(\sqrt{\frac{b(a+b+c)}{ca}}\right) + \tan^{-1}\left(\sqrt{\frac{c(a+b+c)}{ab}}\right)$$

Then $\tan(\theta) =$ ______. (1981)

4.3.25 The numerical value of $\tan\left\{2\tan^{-1}\left(\frac{1}{5}\right) - \frac{\pi}{4}\right\}$ is equal to ______. (1984)

4.3.26 The greater of the two angles

$$A = 2\tan^{-1}\left(2\sqrt{2} - 1\right) \text{ and}$$
$$B = 3\sin^{-1}\left(\frac{1}{3}\right) + \sin^{-1}\left(\frac{3}{5}\right)$$

is ______. (1989)

4.3.27 The value of

$$\sec^{-1}\left(\frac{1}{4}\sum_{k=0}^{10}\sec\left(\frac{7\pi}{10} + \frac{k\pi}{10}\sec\frac{7\pi}{12} + \frac{(k+1)\pi}{2}\right)\right)$$

in the interval $\left[-\frac{\pi}{4}, \frac{3\pi}{4}\right]$ equals (2019)

4.3.28 $x = \cos^{-1}\left(\sqrt{\cos\alpha}\right) - \tan^{-1}\left(\sqrt{\cos\alpha}\right)$, then $\sin x =$ (2002)

a) $\tan^2\left(\frac{\alpha}{2}\right)$ b) $\cot^2\left(\frac{\alpha}{2}\right)$ c) $\tan\alpha$ d) $\cot\left(\frac{\alpha}{2}\right)$

4.3.29 If $\cos^{-1} x - \cos^{-1}\frac{y}{2} = \alpha$, then $4x^2 - 4xy\cos\alpha + y^2$ is equal to (2005)

a) $2\sin 2\alpha$ b) 4 c) $4\sin^2\alpha$ d) $-4\sin^2\alpha$

4.3.30 The value of $\cot\left(\operatorname{cosec}^{-1}\frac{5}{3} + \tan^{-1}\frac{2}{3}\right)$ is

a) $\frac{6}{17}$ b) $\frac{3}{17}$ c) $\frac{4}{17}$ d) $\frac{5}{17}$

4.3.31 If x, y, z are in AP and $\tan^{-1} x, \tan^{-1} y$ and $\tan^{-1} z$ are also in A.P, then (2013)

a) $x = y = z$ b) $2x = 3y = 6z$ c) $6x = 3y = 2z$ d) $6x = 4y = 3z$

4.3.32 Let $\tan^{-1} y = \tan^{-1} x + \tan^{-1}\left(\frac{2x}{1-x^2}\right)$, where $|x| < \frac{1}{\sqrt{3}}$. Then a value of y is (2015)

a) $\frac{3x-x^3}{1+3x}$ b) $\frac{3x+x^3}{1+3x}$ c) $\frac{3x-x^3}{1-3x}$ d) $\frac{3x+x^3}{1-3x}$

4.3.33 Match The Following (2005)

a)
$$\sum_{i=1}^{\infty} \tan^{-1}\left(\frac{1}{2i^2}\right) = t,$$
then $\tan t =$

b) Sides a, b, c of a triangle ABC are in AP and
$$\cos\theta_1 = \frac{a}{b+c}, \cos\theta_2 = \frac{b}{a+c}, \cos\theta_3 = \frac{c}{a+b}$$
then
$$\tan^2\left(\frac{\theta_1}{2}\right) + \tan^2\left(\frac{\theta_3}{2}\right) =$$

c) A line is perpendicular to $x + 2y + 2z = 0$ and passes through $(0, 1, 0)$. The perpendicular distance of this line from the origin is

a) 1
b) $\frac{\sqrt{5}}{3}$
c) $\frac{2}{3}$

4.3.34 Let (x, y) be such that $\sin^{-1}(ax) + \cos^{-1}(bxy) = \frac{\pi}{2}$. Match the statements in Column I with statements in Column II. (2007)

a) If $a = 1$ and $b = 0$, then (x, y)
b) If $a = 1$ and $b = 1$, then (x, y)
c) If $a = 1$ and $b = 2$, then (x, y)
d) If $a = 2$ and $b = 2$, then (x, y)

a) lies on the circle $x^2 + y^2 = 1$
b) lies on $(x^2 - 1)(y^2 - 1) = 0$
c) lies on $y = x$
d) lies on $(4x^2 - 1)(y^2 - 1) = 0$

4.3.35 Match List I with List II. (2013)

a) $\left(\frac{1}{y^2}\left(\frac{\cos\left(\tan^{-1} y\right)+y\sin\left(\tan^{-1} y\right)}{\cot\left(\sin^{-1} y\right)+\tan\left(\sin^{-1} y\right)}\right)^2 + y^4\right)^{\frac{1}{2}}$

b) If $\cos x + \cos y + \cos z = 0 = \sin x + \sin y + \sin z$ then possible value of $\cos\frac{x-y}{2}$ is

c) If $\cos\left(\frac{\pi}{4} - x\right)\cos 2x + \sin x \sin 2x \sec x = \cos x \sin 2x \sec x + \cos\left(\frac{\pi}{4} + x\right)\cos 2x$ then possible value of $\sec x$ is

d) If $\cot\left(\sin^{-1}\sqrt{1-x^2}\right) = \sin\left(\tan^{-1}\left(x\sqrt{6}\right)\right)$, $x \neq 0$, then x is

a) $\frac{1}{2}\sqrt{\frac{5}{3}}$
b) $\sqrt{2}$
c) $\frac{1}{2}$
d) 1

4.3.36 The principal value of $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$ is (1986)

a) $-\frac{2\pi}{3}$ b) $\frac{2\pi}{3}$ c) $\frac{4\pi}{3}$ d) none

4.3.37 If $\alpha = 3\sin^{-1}\left(\frac{6}{11}\right)$ and $\beta = 3\cos^{-1}\left(\frac{4}{9}\right)$, where the inverse trigonometric functions take only the principal values, then the correct option(s) is(are) (2015)

a) $\cos(\beta) > 0$ b) $\sin(\beta) < 0$ c) $\cos(\alpha+\beta) > 0$ d) $\cos(\alpha) < 0$

4.3.38 For non-negative integers n, let

$$f(n) = \frac{\sum_{k=0}^{n} \sin\left(\frac{k+1}{n+2}\pi\right)\sin\left(\frac{k+2}{n+2}\pi\right)}{\sum_{k=0}^{n} \sin^2\left(\frac{k+1}{n+2}\pi\right)}$$

Assuming $\cos^{-1}(x)$ takes values in $[0,\pi]$, which of the following options is/are correct (2019)

a) $\lim_{n\to\infty)} f(n) = \frac{1}{2}$
b) $f(4) = \frac{\sqrt{3}}{2}$
c) If $\alpha = \tan\left(\cos^{-1}(f(6))\right)$, then $\alpha^2 + 2\alpha - 1 = 0$
d) $\sin\left(7\cos^{-1}(f(5))\right) = 0$

4.3.39 The value of $\tan\left[\cos^{-1}\left(\frac{4}{5}\right) + \tan^{-1}\left(\frac{2}{3}\right)\right]$ is (1983)

a) $\frac{6}{17}$ b) $\frac{7}{16}$ c) $\frac{16}{7}$ d) None

4.3.40 If we consider only the principal values of the inverse trigonometric functions, then the value of

$$\tan\left(\cos^{-1}\left(\frac{1}{5\sqrt{2}}\right) - \sin^{-1}\left(\frac{4}{\sqrt{17}}\right)\right)$$

is (1994)

a) $\frac{\sqrt{29}}{3}$ b) $\frac{29}{3}$ c) $\frac{\sqrt{3}}{29}$ d) $\frac{3}{29}$

4.3.41 If $0 < x < 1$, then

$$\sqrt{1+x^2}\left[\left\{x\cos\left(\cot^{-1}(x)\right) + \sin\left(\cot^{-1}(x)\right)\right\}^2 - 1\right]^{\frac{1}{2}}$$

is (2008)

a) $\frac{x}{\sqrt{1+x^2}}$ b) x c) $x\sqrt{1+x^2}$ d) $\sqrt{1+x^2}$

4.3.42 The value of

$$\cot\left(\sum_{n=1}^{23} \cot^{-1}\left(1 + \sum_{k=1}^{n} 2k\right)\right)$$

is (2013)

a) $\frac{23}{25}$ b) $\frac{25}{23}$ c) $\frac{23}{24}$ d) $\frac{24}{23}$

4.3.43 Find the value of:

$$\cos\left(2\cos^{-1}(x) + \sin^{-1}(x)\right)$$

where $0 \le \cos^{-1}(x) \le \pi$ and $-\frac{\pi}{2} \le \sin^{-1}(x) \le \frac{\pi}{2}$. (1981)

4.3.44 Prove that $\cos\tan^{-1}\sin\cot^{-1} x = \sqrt{\frac{x^2+1}{x^2+2}}$. (2002)

4.3.45 Let $f : [0,2] \to \mathbb{R}$ be the function defined by

$$f(x) = (3 - \sin(2\pi x))\sin(\pi x - \frac{\pi}{4}) - \sin(3\pi x + \frac{\pi}{4})$$

If $\alpha, \beta \in [0,2]$ are such that $\{x \in [0,2] : f(x) \ge 0\} = [\alpha, \beta]$, then the value of $\beta - \alpha$ is ______. (2020)

4.3.46 Considering only the principal values of the inverse trigonometric functions, the value of

$$\frac{3}{2}\cos^{-1}\sqrt{\frac{2}{2+\pi^2}} + \frac{1}{4}\sin^{-1}\frac{2\sqrt{2}\pi}{2+\pi^2} + \tan^{-1}\frac{\sqrt{2}}{\pi}$$

is ______. (2022)

4.3.47 Let α and β be real numbers such that

$$-\frac{\pi}{4} < \beta < 0 < \alpha < \frac{\pi}{4}.$$

If

$$\sin(\alpha + \beta) = \frac{1}{3} \text{ and } \cos(\alpha - \beta) = \frac{2}{3},$$

then the greatest integer less than or equal to

$$\left(\frac{\sin\alpha}{\cos\beta} + \frac{\cos\beta}{\sin\alpha} + \frac{\cos\alpha}{\sin\beta} + \frac{\sin\beta}{\cos\alpha}\right)^2$$

is ______. (2022)

4.3.48 Let $\frac{\pi}{2} < x < \pi$ be such that $\cot x = \frac{-5}{\sqrt{11}}$. Then

$$\left(\sin\frac{11x}{2}\right)(\sin 6x - \cos 6x) + \left(\cos\frac{11x}{2}\right)(\sin 6x + \cos 6x)$$

is equal to (2024)

a) $\frac{\sqrt{11}-1}{2\sqrt{3}}$ b) $\frac{\sqrt{11}+1}{2\sqrt{3}}$ c) $\frac{\sqrt{11}+1}{3\sqrt{2}}$ d) $\frac{\sqrt{11}-1}{3\sqrt{2}}$

5 Equations

5.1 NCERT

5.1.1 Find the principal solutions of the equation $\sin x = \frac{\sqrt{3}}{2}$.

5.1.2 Find the principal solutions of the equation $\tan x = -\frac{1}{\sqrt{3}}$.

5.1.3 Find the solution of $\sin x = -\frac{\sqrt{3}}{2}$.

5.1.4 Solve $\cos x = \frac{1}{2}$.
5.1.5 Solve $\tan 2x = -\cot\left(x + \frac{\pi}{3}\right)$.
5.1.6 Solve $\sin 2x - \sin 4x + \sin 6x = 0$.
5.1.7 Solve $2\cos^2 x + 3\sin x = 0$.
5.1.8 Find the general solution for each of the following equations

a) $\cos 4x = \cos 2x$.
b) $\cos 3x + \cos x - \cos 2x = 0$.
c) $\sin 2x + \cos x = 0$.
d) $\sec^2 2x = 1 - \tan 2x$.
e) $\sin x + \sin 3x + \sin 5x = 0$.

5.1.9 Find the principal and general solutions of the following equations

a) $\tan x = \sqrt{3}$.
b) $\sec x = 2$.
c) $\cot x = -\sqrt{3}$.
d) $\csc x = -2$.

5.2 CBSE

5.2.1 If

$$\cos\left(\sin^{-1}\frac{2}{\sqrt{5}} + \cos^{-1} x\right) = 0$$

then x is equal to

a) $\frac{1}{\sqrt{5}}$ b) $-\frac{2}{\sqrt{5}}$ c) $\frac{2}{\sqrt{5}}$ d) 1

(12, 2020)

5.2.2 Solve for x :

$$\sin^{-1}(1-x) - 2\sin^{-1} x = \frac{\pi}{2}$$

(10, 2022)

5.2.3 If $2\cos\theta = \sqrt{3}$, then find the value of θ. (10, 2021)

5.2.4 If $\sin(A+B) = \sqrt{3}/2, \sin(A-B) = 1/2$, where $0° < A + B < 90°; A > B$, then find the values of A and B. (10, 2021)

5.2.5 Solve for x :

$$\tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1}\left(\frac{8}{31}\right)$$

(12, 2019)

5.2.6 If $\tan^{-1} x - \cot^{-1} x = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right), x > 0$, find the value of x and hence find the value of $\sec^{-1}\left(\frac{2}{x}\right)$. (12, 2019)

5.2.7 If

$$\sin^{-1}\left(\frac{3}{x}\right)+\sin^{-1}\left(\frac{4}{x}\right)=\frac{\pi}{2}$$

then find the value of x. (12, 2019)

5.2.8 Find the value of x, if $\tan\left(\sec^{-1}\left(\frac{1}{x}\right)\right)=\sin\left(\tan^{-1}2\right), x>0$. (12, 2019)

5.2.9 Find A and B if $\sin(A+2B)=\frac{\sqrt{3}}{2}$ and $\cos(A+4B)=0$, where A and B are acute angles. (10, 2019)

5.2.10 If $\tan(A+B)=1$ and $\tan(A-B)=\frac{1}{\sqrt{3}}$, $0° < A+B < 90°$, $A>B$, then find the values of A and B. (10, 2019)

5.2.11 If $\sin x+\cos y=1$; $x=30°$ and y is an acute angle, find the value of y. (10, 2019)

5.2.12 Find A if $\tan 2A=\cot(A-24°)$. (10, 2019)

5.2.13 If $\tan 2A=\cot(A-18°)$, where $2A$ is an acute angle, find the value of A. (10, 2018)

5.2.14 If $\tan^{-1}x-\cot^{-1}x=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$, $x>0$, find the value of x and hence find the value of $\sec^{-1}\left(\frac{2}{x}\right)$. (12, 2018)

5.2.15 If $\sin^{-1}\left(\frac{3}{x}\right)+\sin^{-1}\left(\frac{4}{x}\right)=\frac{\pi}{2}$, then find the value of x. (12, 2018)

5.2.16 Find the value of x, if $\tan\left(\sec^{-1}\left(\frac{1}{x}\right)\right)=\sin\left(\tan^{-1}2\right), x>0$. (12, 2018)

5.2.17 Solve for x:

$$\tan^{-1}(x+1)+\tan^{-1}(x-1)=\tan^{-1}\left(\frac{8}{31}\right)$$

(12, 2018)

5.2.18 Solve $\tan^{-1}4x+\tan^{-1}6x=\frac{\pi}{4}$ (12, 2018)

5.2.19 Solve for x : $\tan^{-1}(2x)+\tan^{-1}(3x)=\frac{\pi}{4}$ (12, 2018)

5.2.20 If $\tan^{-1}\frac{x-3}{x-4}+\tan^{-1}\frac{x+3}{x+4}=\frac{\pi}{4}$, then find the value of x. (12, 2017)

5.2.21 Solve for x :

$$\tan^{-1}(x-1)+\tan^{-1}x+\tan^{-1}(x+1)=\tan^{-1}3x$$

(12, 2016)

5.2.22 Solve the equation for x :

$$\cos\left(\tan^{-1}x\right)=\sin\left(\cot^{-1}\frac{3}{4}\right)$$

(12, 2016)

5.2.23 Solve for x :

$$\tan^{-1}\left(\frac{1-x}{1+x}\right)=\frac{1}{2}\tan^{-1}x, x>0.$$

(12, 2015)

5.2.24 Solve for x:

$$\tan^{-1}(x-1)+\tan^{-1}x+\tan^{-1}(x+1)=\tan^{-1}3x.$$

(12, 2016)

5.2.25 Solve for x :

$$\tan^{-1}\left(\frac{2-x}{2+x}\right)=\frac{1}{2}\tan^{-1}\frac{x}{2}, x>0.$$

(12, 2016)

5.3 JEE

5.3.1 The solution set of the system of equations $x+y=\frac{2\pi}{3}$, $\cos x+\cos y=\frac{3}{2}$, where x and y are real, is ______. (1987)

5.3.2 The set of all x in the interval $[0,\pi]$ for which $2\sin^2 x-3\sin x+1\geq 0$, is ______. (1987)

5.3.3 General value of θ satisfying the equation $\tan^2\theta+\sec 2\theta=1$ is ______. (1996)

5.3.4 The real roots of the equation $\cos^7 x+\sin^4 x=1$ in the interval $(-\pi,\pi)$ are ______. (1997)

5.3.5 The number of distinct solutions of equation

$$\frac{5}{4}\cos^2 2x+\cos^4 x+\sin^4 x+\cos^6 x+\sin^6 x=2$$

in the interval $[0,2\pi]$ is ______. (2015)

5.3.6 Let a,b,c be three non-zero real numbers such that the equation

$$\sqrt{3}a\cos x+2b\sin x=c, x\in\left[\frac{-\pi}{2},\frac{\pi}{2}\right],$$

has two distinct real roots α and β with $\alpha+\beta=\frac{\pi}{3}$. Then, the value of $\frac{b}{a}$ is ______. (2018)

5.3.7 The period of $\sin^2\theta$ is ______. (2002)

a) π^2 b) π c) 2π d) $\pi/2$

5.3.8 The number of solutions of $\tan x+\sec x=2\cos x$ in $[0,2\pi]$ is ______. (2002)

a) 2 b) 3 c) 0 d) 1

5.3.9 Which one is not periodic? (2002)

a) $|\sin 3x|+\sin^2 x$ b) $\cos\sqrt{x}+\cos^2 x$ c) $\cos 4x+\tan^2 x$ d) $\cos 2x+\sin x$

5.3.10 A line makes the same angle θ, with each of the x and z axis. If the angle β, which it makes with Y axis, is such that $\sin^2\beta=3\sin^2\theta$, then $\cos^2\theta$ equals (2004)

a) $\frac{2}{5}$ b) $\frac{1}{5}$ c) $\frac{3}{5}$ d) $\frac{2}{3}$

5.3.11 The number of values of x in the interval $[0,3\pi]$ satisfying the equation

$$2\sin^2 x+5\sin x-3=0$$

is (2006)

a) 4 b) 6 c) 1 d) 2

5.3.12 If $0 < x < \pi$ and $\cos x + \sin x = \frac{1}{2}$, then $\tan x$ is ______. (2006)

a) $\frac{(1-\sqrt{7})}{4}$ b) $\frac{(4-\sqrt{7})}{3}$ c) $-\frac{(4+\sqrt{7})}{3}$ d) $\frac{(1+\sqrt{7})}{4}$

5.3.13 Let

$$A : \cos\alpha + \cos\beta + \cos\gamma = 0$$
$$B : \sin\alpha + \sin\beta + \sin\gamma = 0$$

If $\cos(\beta-\gamma) + \cos(\gamma-\alpha) + \cos(\alpha-\beta) = -\frac{3}{2}$, then (2009)

a) A is false and B is true
b) both A and B are true
c) both A and B are false
d) A is true and B is false

5.3.14 Let $\cos(\alpha+\beta) = \frac{4}{5}$ and $\sin(\alpha-\beta) = \frac{5}{13}$, where $0 \le \alpha, \beta \le \frac{\pi}{4}$. Then $\tan 2\alpha =$ (2010)

a) $\frac{56}{33}$ b) $\frac{19}{12}$ c) $\frac{20}{7}$ d) $\frac{25}{16}$

5.3.15 If $A = \sin^2 x + \cos^4 x$, then for all real x (2010)

a) $\frac{13}{16} \le A \le 1$ b) $1 \le A \le 2$ c) $\frac{3}{4} \le A \le \frac{13}{16}$ d) $\frac{3}{4} \le A \le 1$

5.3.16 In a $\triangle PQR$, if $3\sin P + 4\cos Q = 6$ and $4\sin Q + 3\cos P = 1$, then the angle R is equal to (2012)

a) $\frac{5\pi}{6}$ b) $\frac{\pi}{6}$ c) $\frac{\pi}{4}$ d) $\frac{3\pi}{4}$

5.3.17 If $0 \ge x \ge 2\pi$, then the number of real values of x, which satisfy the equation $\cos x + \cos 2x + \cos 3x + \cos 4x = 0$ is (2016)

a) 7 b) 9 c) 3 d) 5

5.3.18 If $5\tan^2 x - \cos^2 x = 2\cos 2x + 9$, then value of $\cos 4x$ is (2017)

a) $\frac{-7}{9}$ b) $\frac{-3}{5}$ c) $\frac{1}{3}$ d) $\frac{2}{9}$

5.3.19 If sum of all the solutions of the equation

$$8\cos(x)\cos\left(\frac{\pi}{6}+x\right)\cos\left(\frac{\pi}{6}\right)-\frac{1}{2} = 1$$

in $[0, \pi]$ is $k\pi$, then k is equal to (2018)

a) $\frac{13}{9}$ b) $\frac{8}{9}$ c) $\frac{20}{9}$ d) $\frac{2}{3}$

5.3.20 Let

$$S = \left\{\theta \in [-2\pi, 2\pi] : 2\cos^2\theta + 3\sin\theta = 0\right\}.$$

Then the sum of the elements of S is (2019)

a) $\frac{13\pi}{6}$ b) $\frac{5\pi}{3}$ c) 2 d) 1

5.3.21 The number of all possible triplets (a_0, a_2, a_3) such that

$$a_1 + a_2\cos(2x) + a_3\sin^2(x) = 0$$

for all x is (1986)

a) zero b) one c) three d) infinite e) none

5.3.22 The values of θ lying between $\theta = -1$ and $\theta = \frac{\pi}{2}$ and satisfying the equation (1987)

$$\begin{vmatrix} 1+\sin^2\theta & \cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & 1+\cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & \cos^2\theta & 1+4\sin 4\theta \end{vmatrix} = 0$$

are

a) $\frac{6\pi}{24}$ b) $\frac{4\pi}{24}$ c) $\frac{10\pi}{24}$ d) $\frac{\pi}{23}$

5.3.23 The number of values of x in the interval $[0, 5\pi]$ satisfying equation (1998)

$$3\sin\left(x^2\right) - 7\sin x + 2 = 0$$

a) 0 b) 5 c) 6 d) 10

5.3.24 Which of the following number(s) is (are) rational? (1998)

a) $\sin 15°$ b) $\cos 15°$ c) $\sin 15° \cos 15°$ d) $\sin 15° \cos 75°$

5.3.25 If

$$\frac{\sin^4 x}{2} + \frac{\cos^4 x}{3} = \frac{1}{5},$$

then (2009)

a) $\tan^2 x = \frac{2}{3}$ c) $\tan^2 x = \frac{1}{3}$
b) $\frac{\sin^8 x}{8} + \frac{\cos^8 x}{27} = \frac{1}{125}$ d) $\frac{\sin^8 x}{8} + \frac{\cos^8 x}{27} = \frac{2}{125}$

5.3.26 For $0 < \theta < \frac{\pi}{2}$, the solution(s) of

$$\sum_{m=1}^{6} \operatorname{cosec}\left(\theta + \frac{(m-1)\pi}{4}\right) \operatorname{cosec}(\theta) + \frac{m\pi}{4} = 4\sqrt{2}$$

is (are) (2009)

a) $\frac{\pi}{4}$ b) $\frac{\pi}{6}$ c) $\frac{\pi}{12}$ d) $\frac{5\pi}{12}$

5.3.27 Let $\theta, \varphi \in [0, 2\pi]$ be such that

$$2\cos(\theta(1-\sin\varphi)) = \sin^2\left(\theta\left(\tan\frac{\theta}{2}\right) + \cot\frac{\theta}{2}\right)\cos\varphi - 1,$$

$$\tan(2\pi-\theta) > 0 \text{ and } -1 < \sin\theta < -\frac{\sqrt{3}}{2},$$

then φ cannot satisfy (2012)

a) $0 < \varphi < \frac{\pi}{2}$ b) $\frac{\pi}{2} < \varphi < \frac{4\pi}{3}$ c) $\frac{4\pi}{3} < \varphi < \frac{3\pi}{2}$ d) $\frac{3\pi}{2} < \varphi < 2\pi$

5.3.28 The number of points in $(-\infty, \infty)$, for which $x - x\sin x - \cos x = 0$, is (2013)

a) 6 b) 4 c) 2 d) 0

5.3.29 Let $f(x) = x\sin\pi x, x > 0$. Then for all natural numbers n, $(f'(x))$ vanishes at (2013)

a) A unique point in the interval $\left(n, n+\frac{1}{2}\right)$
b) A unique point in the interval $\left(n+\frac{1}{2}, n+1\right)$
c) A unique point in the interval $(n, n+1)$
d) Two points in the interval $(n, n+1)$

5.3.30 If $\tan\theta = -\frac{4}{3}$ then $\sin\theta$ is (1979)

a) $\frac{-4}{5}$ but not $\frac{4}{5}$ b) $\frac{4}{5}$ or $\frac{-4}{5}$ c) $\frac{4}{5}$ but not $\frac{-4}{5}$ d) None of These

5.3.31 The equation $2\cos^2\frac{x}{2}\sin^2 x = x^2 + x^{-2}$ (1980)

a) no real solution
b) one real solution
c) more than one real solution
d) None of these

5.3.32 The general solution to the trignometric equation $\sin x + \cos x = 1$ is given by (1981)

a) $x = 2n\pi; n = 0, \pm1, \pm2\cdots$
b) $x = 2n\pi + \frac{\pi}{2}, n = 0, \pm1, \pm2\cdots$
c) $x = n\pi + (-1)^n\frac{\pi}{4}, n = 0, \pm1, \pm2\cdots$
d) none of these

5.3.33 The general solution of the trigonometric equation $\sin x + \cos x = 1$ is given by (1981)

a) $x = 2n\pi;\ n = 0, \pm1, \pm2\ \ldots$
b) $x = 2n\pi + \frac{\pi}{2};\ n = 0, \pm1, \pm2\ \ldots$
c) $x = n\pi + (-1)^n\frac{\pi}{4} - \frac{\pi}{4};\ n = 0, \pm1, \pm2\ \ldots$
d) none of these

5.3.34 The value of the expression $\sqrt{3}\ \mathrm{cosec}\, 20^\circ - \sec 20^\circ$ is equal to (1988)

a) 2 b) $2\frac{\sin 20^\circ}{\sin 40^\circ}$ c) 4 d) $4\frac{\sin 20^\circ}{\sin 40^\circ}$

5.3.35 The general solution of (1989)

$$\sin x - 3\sin 2x + \sin 3x = \cos x - 3\cos 2x + \cos 3x$$

a) $n\pi + \frac{\pi}{8}$ b) $\frac{n\pi}{2} + \frac{\pi}{8}$ c) $(-1)^n \frac{n\pi}{2} + \frac{\pi}{8}$ d) $2n\pi + \cos^{-1} \frac{3}{2}$

5.3.36 The equation $(\cos p - 1)x^2 + (\cos p)x + \sin p = 0$ in the variable x, has real roots. Then p can take any value in the interval (1990)

a) $(0, 2\pi)$ b) $(-\pi, 0)$ c) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ d) $(0, \pi)$

5.3.37 Number of solutions of the equation $\tan x + \sec x = 2\cos x$ lying in the interval $(0, 2\pi)$ is (1993)

a) 0 b) 1 c) 2 d) 3

5.3.38 Let n be a positive integer such that $\sin\frac{\pi}{2n} + \cos\frac{\pi}{2n} = \frac{\sqrt{n}}{2}$. Then (1994)

a) $6 \le n \le 8$ b) $4 < n \le 8$ c) $4 \le n \le 8$ d) $4 < n < 8$

5.3.39 The general values of θ satisfying the equation $2\sin^2\theta - 3\sin\theta - 2 = 0$ is (1995)

a) $n\pi + (-1)^n \frac{\pi}{6}$ b) $n\pi + (-1)^n \frac{\pi}{2}$ c) $n\pi + (-1)^n \frac{5\pi}{6}$ d) $n\pi + (-1)^n \frac{7\pi}{6}$

5.3.40 $\sec^2\theta = \frac{4xy}{(x+y)^2}$ is true if and only if (1996)

a) $x + y = 0$ b) $x = y, x \neq 0$ c) $x = y$ d) $x \neq 0, y \neq 0$

5.3.41 The number of distinct real roots of (2001)

$$\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix}$$

are

a) 0 b) 2 c) 1 d) 3

5.3.42 If $\alpha + \beta = \frac{\pi}{2}$ and $\beta + \gamma = \alpha$, then $\tan\alpha$ equals (2001)

a) $2(\tan\beta + \tan\gamma)$ b) $\tan\beta + \tan\gamma$ c) $\tan\beta + 2\tan\gamma$ d) $2\tan\beta + \tan\gamma$

5.3.43 The number of integral values of k for which the equation $7\cos x + 5\sin x = 2k + 1$ has a solution is (2002)

a) 4 b) 8 c) 10 d) 12

5.3.44 Given both θ and ϕ are acute angles and $\sin\theta = \frac{1}{2}$, $\cos\phi = \frac{1}{3}$, then the value of $\theta + \phi$ belongs to (2004)

a) $(\frac{\pi}{3}, \frac{\pi}{2}]$ b) $\left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$ c) $(\frac{2\pi}{3}, \frac{5\pi}{6}]$ d) $(\frac{5\pi}{6}, \pi]$

5.3.45 $\cos(\alpha - \beta) = 1$ and $\cos(\alpha + \beta) = \frac{1}{e}$ where $\alpha, \beta \in [-\pi, \pi]$. Pairs of α, β which satisfy both the equations is (are) (2005)

a) 0 b) 1 c) 2 d) 4

5.3.46 The number of solutions of the pair of equations

$$2\sin^2\theta - \cos 2\theta = 0$$
$$2\cos^2\theta - 3\sin\theta = 0$$

in the interval $[0, 2\pi]$ is (2007)

a) zero b) one c) two d) four

5.3.47 For $x \in (0, \pi)$, the equation $\sin x + 2\sin 2x - \sin 3x = 3$ has (2014)

a) infinitely many solutions
c) one solution
b) three solutions
d) no solution

5.3.48 Let

$$S = \left\{x \in (-\pi, \pi) : x \neq 0, \pm\frac{\pi}{2}\right\}.$$

The sum of all distinct solutions of the equation

$$\sqrt{3}\sec x + \operatorname{cosec} x + 2(\tan x - \cot x) = 0$$

in the set S is equal to (2016)

a) $-\frac{7\pi}{9}$ b) $-\frac{2\pi}{9}$ c) 0 d) $\frac{5\pi}{9}$

5.3.49 If $\tan\alpha = \frac{m}{m+1}$ and $\tan\beta = \frac{1}{2m+1}$, find the possible values of $(\alpha + \beta)$. (1978)
5.3.50 Draw the graph of $y = \frac{1}{\sqrt{2}}(\sin x + \cos x)$ from $x = -\frac{\pi}{2}$ to $x = \frac{\pi}{2}$.
5.3.51 If $\cos(\alpha + \beta) = \frac{4}{5}$, $\sin(\alpha - \beta) = \frac{5}{13}$, and α, β lies between 0 and $\frac{\pi}{4}$, find $\tan 2\alpha$. (1979)
5.3.52 Given $A = \left\{x : \frac{\pi}{6} \leq x \leq \frac{\pi}{3}\right\}$ and $f(x) = \cos x - x(1 + x)$, find $f(A)$ (1980)
5.3.53 Find all the solutions of (1983)

$$4\cos^2(x)\sin(x) - 2\sin^2(x) = 3\sin(x)$$

5.3.54 Find the values of $x \in (-\pi, +\pi)$ which satisfy the equation (1984)

$$8^{\left(1+|\cos(x)|+\left|\cos^2(x)\right|+\left|\cos^3(x)\right|+\dots\right)} = 4^3$$

5.3.55 If (1991)

$$\exp\left\{\left(\sin^2(x)+\sin^4(x)+\sin^6(x)+\ldots\infty\right)(\ln 2)\right\}$$

satisfies the equation $x^2-9x+8=0$, find the value of

$$\frac{\cos(x)}{\cos(x)+\sin(x)}, 0<x<\frac{\pi}{2}.$$

5.3.56 Determine the smallest positive value of x (in degrees) for which

$$\tan\left(x+100^{\circ}\right)=\tan\left(x+50^{\circ}\right)\tan(x)\tan\left(x-50^{\circ}\right).$$

(1993)

5.3.57 Find the smallest positive number p for which the equation (1995)

$$\cos(p\sin(x))=\sin(p\cos(x))$$

has a solution $x\in[0,\pi]$.

5.3.58 Find all values of θ in the interval $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ satisfying the equation (1996)

$$(1-\tan(\theta))(1+\tan(\theta))\sec^2(\theta)+2^{\tan^2(\theta)}=0$$

5.3.59 If $\tan A=\frac{1-\cos B}{\sin B}$, then $\tan 2A=\tan B$. (1981)

5.3.60 There exists a value of θ between 0 and 2π that satisfies the equation (1984)

$$\sin^4\theta-2\sin^2\theta-1=0.$$

5.3.61 The number of real solutions of the equation

$$\sin^{-1}\left(\sum_{i=1}^{\infty}x^{i+1}-x\sum_{i=1}^{\infty}\left(\frac{x}{2}\right)^i\right)=\frac{\pi}{2}-\cos^{-1}\left(\sum_{i=1}^{\infty}\left(\frac{-x}{2}\right)^i-\sum_{i=1}^{\infty}(-x)^i\right)$$

lying in the interval $\left(-\frac{1}{2},\frac{1}{2}\right)$ is? (Here, the inverse trignometric function $\sin^{-1}x$ and $\cos^{-1}x$ assume values in $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ and $[0,\pi]$ respectively (2018)

5.3.62 Find all the solutions of (1983)

$$4\cos^2(x)\sin(x)-2\sin^2(x)=3\sin(x).$$

5.3.63 The trignometric equation $\sin^{-1}x=2\sin^{-1}a$ has a solution for (2003)

a) $|\alpha|\geq\frac{1}{\sqrt{2}}$

b) $\frac{1}{2}<|\alpha|<\frac{1}{\sqrt{2}}$

c) all real values of a

d) $|\alpha|<\frac{1}{2}$

5.3.64 The number of real solutions of

$$\tan^{-1}\left(\sqrt{x(x-1)}\right)+\sin^{-1}\left(\sqrt{x^2+x+1}\right)=\frac{\pi}{2}$$

is (1999)

a) zero b) one c) two d) infinite

5.3.65 If $\sin^{-1}\left(\frac{x}{5}\right) + \operatorname{cosec}^{-1}\left(\frac{5}{4}\right) = \frac{\pi}{2}$, then the value of x is (2007)

a) 4 b) 5 c) 1 d) 3

5.3.66 If $\cos^{-1}\left(\frac{2}{3x}\right) + \cos^{-1}\left(\frac{3}{4x}\right) = \frac{\pi}{2}\left(x > \frac{3}{4}\right)$, then x is equal to (2019)

a) $\frac{\sqrt{145}}{12}$ b) $\frac{\sqrt{145}}{10}$ c) $\frac{\sqrt{146}}{12}$ d) $\frac{\sqrt{145}}{11}$

5.3.67 The value of x for which

$$\sin\left(\cot^{-1}(1+x)\right) = \cos\left(\tan^{-1}(x)\right)$$

is (2004)

a) $\frac{1}{2}$ b) 1 c) 0 d) $-\frac{1}{2}$

5.3.68 If

$$\sin^{-1}\left(x - \frac{x^2}{2} + \frac{x^3}{4} - \dots\right) + \cos^{-1}\left(x^2 - \frac{x^4}{2} + \frac{x^6}{4} - \dots\right) = \frac{\pi}{2}$$

for $0 < |x| < \sqrt{2}$, then x equals (2001)

a) $\frac{1}{2}$ b) 1 c) $-\frac{1}{2}$ d) -1

5.3.69 For any positive integer n, let $S_n : (0, \infty) \to \mathbb{R}$ be defined by

$$S_n(x) = \sum_{k=1}^{n} \cot^{-1}\left(\frac{1 + k(k+1)x^2}{x}\right),$$

where for any $x \in \mathbb{R}$, $\cot^{-1}(x) \in (0, \pi)$ and $\tan^{-1}(x) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then which of the following statements is (are) TRUE? (2021)

a) $S_{10}(x) = \frac{\pi}{2} - \tan^{-1}\left(\frac{1+11x^2}{10x}\right)$, for all $x > 0$
b) $\lim_{n\to\infty} \cot(S_n(x)) = x$, for all $x > 0$
c) The equation $S_3(x) = \frac{\pi}{4}$ has a root in $(0, \infty)$
d) $\tan(S_n(x)) \le \frac{1}{2}$, for all $n \ge 1$ and $x > 0$

5.3.70 Consider the following lists

(I) $x \in \left[-\frac{2\pi}{3}, \frac{2\pi}{3}\right] : \cos x + \sin x = 1$	(A) has two elements
(II) $x \in \left[-\frac{5\pi}{18}, \frac{5\pi}{18}\right] : \sqrt{3}\tan 3x = 1$	(B) has three elements
(III) $x \in \left[-\frac{6\pi}{5}, \frac{6\pi}{5}\right] : 2\cos(2x) = \sqrt{3}$	(C) has four elements
(IV) $x \in \left[-\frac{7\pi}{4}, \frac{7\pi}{4}\right] : \sin x - \cos x = 1$	(D) has five elements
	(E) has six elements

The correct option is (2022)

a) $(I) \to (A); (II) \to (D); (III) \to (A); (IV) \to (D)$
b) $(I) \to (A); (II) \to (A); (III) \to (E); (IV) \to (C)$
c) $(I) \to (B); (II) \to (A); (III) \to (E); (IV) \to (D)$
d) $(I) \to (B); (II) \to (D); (III) \to (A); (IV) \to (C)$

5.3.71 Let $\tan^{-1} x \in \left(\frac{\pi}{2}, \frac{\pi}{2}\right)$ for $x \in \mathbb{R}$. Then the number of real solutions of the equation

$$1 + \cos(2x) = 2\tan^{-1}(\tan x)$$

in the set $\left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right) \cup \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ is equal to ______. (2023)

5.3.72 For any $y \in \mathbb{R}$, let $\cot^{-1}(y) \in (0, \pi)$ and $\tan^{-1}(y) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then the sum of all the solutions of the equation

$$\tan^{-1}\left(\frac{6y}{9 - y^2}\right) + \cot^{-1}\left(\frac{9 - y^2}{6y}\right) = \frac{2\pi}{3}$$

for $0 < |y| < 3$ is equal to (2023)

a) $2\sqrt{3} - 3$ b) $3 - 2\sqrt{3}$ c) $4\sqrt{3} - 6$ d) $6 - 4\sqrt{3}$

6 Inequalities

6.1 NCERT

6.1. D is a point on side BC of $\triangle ABC$ such that $AD = AC$. Show that $AB > AD$

6.2. Show that in a right angled triangle, the hypotenuse is the longest side.

6.3. Sides AB and AC of $\triangle ABC$ are extended to points P and Q respectively. Also, $\angle PBC < \angle QCB$. Show that $AC > AB$.

6.4. Line segments AD and BC intersect at O and form $\triangle OAB$ and $\triangle ODC$. $\angle B < \angle A$ and $\angle C < \angle D$. Show that $AD < BC$.

6.5. AB and CD are respectively the smallest and longest sides of a quadrilateral $ABCD$. Show that $\angle A > \angle C$ and $\angle B > \angle D$.

6.6. In $\triangle PQR, PR > PQ$ and PS bisects $\angle QPR$. Prove that $\angle PSR > \angle PSQ$.

6.7. **Q** is a point on the side **SR** of $\triangle$**PSR** such that **PQ** = **PR**. Prove that **PS** > **PQ**.

6.8. **S** is any point on side **QR** of a $\triangle$**PQR**. Show that **PQ** + **QR** + **RP** > **2PS**.

6.9. **D** is any point on side **AC** of a $\triangle$**ABC** with **AB** = **AC**. Show that **CD** < **BD**.

6.10. AD is the bisector of $\angle BAC$. Prove that $AB > BD$.

6.11. Prove that sum of any two sides of a triangle is greater than twice the median with respect to the third side.

6.12. Prove that in a triangle, other than an equilateral triangle, angle opposite the longest side is greater than $\frac{2}{3}$ of a right angle.

6.13. AD is a median of the triangle ABC. Is it true that AB + BC + CA > 2AD?

6.14. M is a point on side BC of a triangle ABC such that AM is the bisector of $\angle BAC$. Is it true to say that perimeter of the triangle is greater than 2AM?

6.15. Parallelogram $ABCD$ and rectangle $ABEF$ are on the same base AB and have equal areas. Show that the perimeter of the parallelogram is greater than that of the rectangle.

6.2 JEE

6.2.1 Let $\sin^2 x + 3\sin x - 2 > 0$ and $x^2 - x - 2 < 0$ (x is measured in radians). Then x lies in the interval (1993)

a) $\left(\frac{\pi}{5}, \frac{5\pi}{6}\right)$ b) $\left(-2, \frac{5\pi}{6}\right)$ c) $(-2, 2)$ d) $\left(\frac{\pi}{5}, 2\right)$

6.2.2 The minimum value of expression $\sin\alpha + \sin\beta + \sin\gamma$, where (α, β, γ) are real numbers satisfying $(\alpha + \beta + \gamma) = \pi$ is (1995)

a) positive b) 0 c) negative d) -3

6.2.3 Given $A = \sin^2\theta + \cos^4\theta$ then for all real values of θ (1980)

a) $1 \le A \le 2$ b) $\frac{3}{4} \le A \le 1$ c) $\frac{13}{16} \le A \le 1$ d) $\frac{3}{4} \le A \le \frac{13}{16}$

6.2.4 Let $f(\theta) = \sin\theta(\sin\theta + \sin 3\theta)$. Then $f(\theta)$ is (2000)

a) ≥ 0 only when $\theta \ge 0$
b) ≤ 0 for all real θ
c) ≥ 0 for all real θ
d) ≤ 0 only when $\theta \le 0$

6.2.5 The maximum value of $(\cos\alpha_1)(\cos\alpha_2)(\cos\alpha_3)\dots(\cos\alpha_n)$ under the restrictions (2001)

$$0 \le \alpha_1, \alpha_2, \dots \alpha_n \le \frac{\pi}{2}$$

and

$$(\cot\alpha_1)(\cot\alpha_2)(\cot\alpha_3)\dots(\cot\alpha_n) = 1$$

a) $\frac{1}{2^{\frac{n}{2}}}$ b) $\frac{1}{2^n}$ c) $\frac{1}{2n}$ d) 1

6.2.6 The values of $\theta \in (0, 2\pi)$ for which $2\sin^2\theta - 5\sin\theta + 2 > 0$, are (2006)

a) $\left(0, \frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, 2\pi\right)$ b) $\left(\frac{\pi}{8}, \frac{5\pi}{6}\right)$ c) $\left(0, \frac{\pi}{8}\right) \cup \left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$ d) $\left(\frac{41\pi}{48}, \pi\right)$

6.2.7 Let $\theta \in \left(0, \frac{\pi}{4}\right)$ and

$$t_1 = (\tan\theta)^{\tan\theta}, t_2 = (\tan\theta)^{cot\theta},$$
$$t_3 = (cot\theta)^{\tan\theta}, t_4 = (cot\theta)^{cot\theta},$$

then (2006 − 3M, −1)

a) $t_1 > t_2 > t_3 > t_4$ b) $t_4 > t_3 > t_1 > t_2$ c) $t_3 > t_1 > t_2 > t_4$ d) $t_2 > t_3 > t_1 > t_4$

6.2.8 For all θ in $\left(0, \frac{\pi}{2}\right)$ show that, $\cos(\sin\theta) \ge \sin(\cos\theta)$. (1981)

6.2.9 Show that the value of $\frac{\tan(x)}{\tan(3x)}$, wherever defined never lies between $\frac{1}{3}$ and 3. (1992)

6.2.10 Prove that the values of the function

$$\frac{\sin(x)\cos(3x)}{\sin(3x)\cos(x)}$$

do not lie between $\frac{1}{3}$ and 3 for any real x. (1997)

6.2.11 Find the range of values of t for which

$$2\sin(t) = \frac{1-2x+5x^2}{3x^2-2x-1}, t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

(2005)

6.2.12 If $A > 0, B > 0$ and $A + B = \frac{\pi}{3}$, then the maximum value $\tan A \tan B$ is ______. (1993)

6.2.13 If

$$u = \sqrt{a^2\cos^2\theta + b^2\sin^2\theta} + \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}$$

then the difference between the maximum and minimum values of u^2 is given by (2004)

a) $(a-b)^2$ b) $2\sqrt{a^2+b^2}$ c) $(a+b)^2$ d) $2\left(a^2+b^2\right)$

6.2.14 Let $|\mathbf{M}|$ denote the determinant of a square matrix $\mathbf{M}$. Let g: $\left[0, \frac{\pi}{2}\right] \to \mathbb{R}$ be the function defined by

$$g(\theta) = \sqrt{f(\theta)-1} + \sqrt{f(\frac{\pi}{2}-\theta)-1}$$

where

$$f(\theta) = \frac{1}{2}\begin{vmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{vmatrix} + \begin{vmatrix} \sin\pi & \cos\left(\theta+\frac{\pi}{4}\right) & \tan\left(\theta-\frac{\pi}{4}\right) \\ \sin\left(\theta-\frac{\pi}{4}\right) & -\cos\frac{\pi}{2} & \log_e\left(\frac{4}{\pi}\right) \\ \cot\left(\theta+\frac{\pi}{4}\right) & \log_e\left(\frac{\pi}{4}\right) & \tan\pi \end{vmatrix}$$

Let $p(x)$ be a quadratic polynomial whose roots are the maximum and minimum values of the function $g(\theta)$ and $p(2) = 2-\sqrt{2}$. Then, which of the following is/are TRUE ? (2022)

a) $p\left(\frac{3+\sqrt{2}}{4}\right) < 0$ b) $p\left(\frac{1+3\sqrt{2}}{4}\right) > 0$ c) $p\left(\frac{5\sqrt{2}-1}{4}\right) > 0$ d) $p\left(\frac{5-\sqrt{2}}{4}\right) < 0$

6.2.15 Let

$$\alpha = \sum_{k=1}^{\infty} \sin^{2k}\left(\frac{\pi}{6}\right).$$

Let $g : [0,1] \to \mathbb{R}$ be the function defined by

$$g(x) = 2^{\alpha x} + 2^{\alpha(1-x)}.$$

Then, which of the following statements is/are TRUE? (2022)

a) The minimum value of $g(x)$ is $2^{7/6}$.
b) The maximum value of $g(x)$ is $1 + 2^{1/3}$.

c) The function $g(x)$ attains its maximum at more than one point.
d) The function $g(x)$ attains its minimum at more than one point.