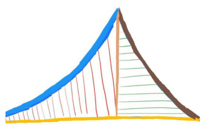
Trigonometry through Geometry



G. V. V. Sharma

About this Book

This book introduces trigonometry through high school geometry. This approach relies more on trigonometric equations than cumbersome constructions which are usually non intuitive. All problems in the book are from NCERT mathematics textbooks from Class 9-12. Exercises are from CBSE board exam papers.

The content is sufficient for all practical applications of trigonometry. There is no copyright, so readers are free to print and share.

This book is dedicated to my high school maths teacher, Dr. G.N. Chandwani.

October 23, 2024

Github: https://github.com/gadepall/matgeo

CONTENTS

1 Geometry

1.1 Right Angled Triangle

1.1.1. A right angled triangle looks like Fig. 1.1.1. with angles $\angle A, \angle B$ and $\angle C$ and sides

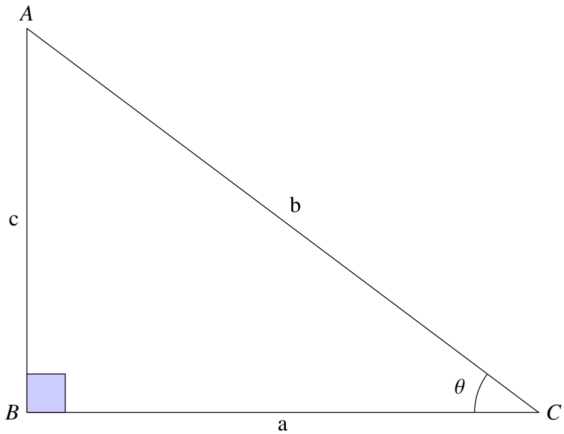


Fig. 1.1.1: Right Angled Triangle

a, b and c. The unique feature of this triangle is $\angle B$ which is defined to be $90°$.

1.1.2. For simplicity, let the greek letter $\theta = \angle C$. We have the following definitions.

$$\begin{aligned} \sin\theta &= \tfrac{c}{b} & \cos\theta &= \tfrac{a}{b} \\ \tan\theta &= \tfrac{c}{a} & \cot\theta &= \tfrac{1}{\tan\theta} \\ \csc\theta &= \tfrac{1}{\sin\theta} & \sec\theta &= \tfrac{1}{\cos\theta} \end{aligned} \tag{1.1.2.1}$$

1.1.3.

$$\cos\theta = \sin(90° - \theta) \tag{1.1.3.1}$$

1.1.4. In Fig. 1.1.2, show that

$$b = a\cos\theta + c\sin\theta \tag{1.1.4.1}$$

Solution: We observe that

$$CD = a\cos\theta \tag{1.1.4.2}$$

$$AD = c\cos\alpha = c\sin\theta \quad \text{(From} \quad \text{(1.1.3.1))} \tag{1.1.4.3}$$

Thus,

$$CD + AD = b = a\cos\theta + c\sin\theta \tag{1.1.4.4}$$

1.1.5. From (1.1.4.1), show that

$$\sin^2\theta + \cos^2\theta = 1 \tag{1.1.5.1}$$

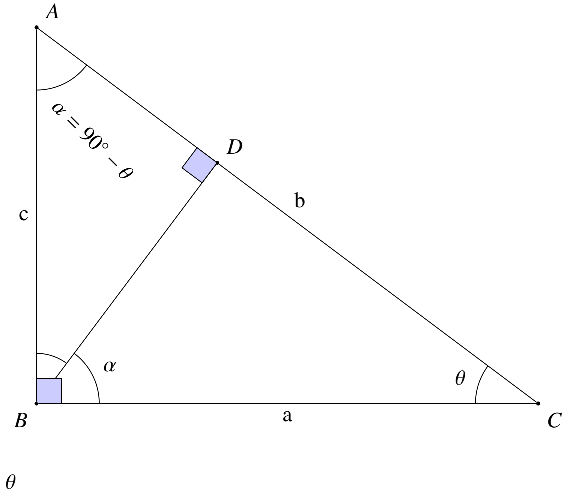


Fig. 1.1.2: Baudhayana Theorem

Solution: Dividing both sides of (1.1.4.1) by b,

$$1 = \frac{a}{b}\cos\theta + \frac{c}{b}\sin\theta \tag{1.1.5.2}$$

$$\Rightarrow \sin^2\theta + \cos^2\theta = 1 \quad \text{(from} \quad (1.1.2.1)) \tag{1.1.5.3}$$

1.1.6. From (1.1.5.1)

$$|\sin\theta| \le 1, \ |\cos\theta| \le 1 \tag{1.1.6.1}$$

1.1.7. Using (1.1.4.1), show that

$$b^2 = a^2 + c^2 \tag{1.1.7.1}$$

(1.1.7.1) is known as the Baudhayana theorem. It is also known as the Pythagoras theorem.
Solution: From (1.1.4.1),

$$b = a\frac{a}{b} + c\frac{c}{b} \quad \text{(from} \quad (1.1.2.1)) \tag{1.1.7.2}$$

$$\implies b^2 = a^2 + c^2 \tag{1.1.7.3}$$

1.1.8. In a right angled triangle, the hypotenuse is the longest side.
Solution: From (1.1.7.1),

$$a \le b, \ c \le b. \tag{1.1.8.1}$$

1.1.9. ABC is an isosceles triangle in which altitudes BE and CF are drawn to equal sides AC and AB respectively . Show that these altitudes are equal.

Solution: In $\triangle$s BFC and BEC,

$$BF = a \sin C, \ CE = a \sin B \tag{1.1.9.1}$$

$$\implies BF = CE, \because B = C. \tag{1.1.9.2}$$

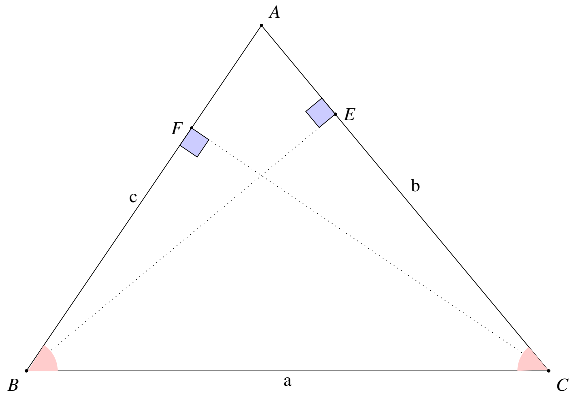


Fig. 1.1.3: $B = C$

1.1.10. ABC is a triangle in which altitudes BE and CF to sides AC and AB are equal. Show that $AB = AC$.
Solution: In (1.1.9.1),

$$BE = CF \implies a \sin C = a \sin B \tag{1.1.10.1}$$

$$\text{or, } B = C \tag{1.1.10.2}$$

1.2 Sine and Cosine Formula

1.2.1. Show that the area of ΔABC in Fig. 1.2.1 is $\frac{1}{2}ab \sin C$.

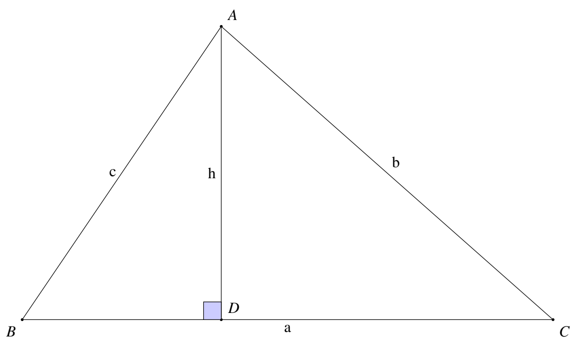


Fig. 1.2.1: Area of a Triangle

Solution: We have

$$ar(\Delta ABC) = \frac{1}{2}ah = \frac{1}{2}ab\sin C \quad (\because \quad h = b\sin C). \tag{1.2.1.1}$$

1.2.2. Show that

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \tag{1.2.2.1}$$

Solution: Fig. 1.2.1 can be suitably modified to obtain

$$ar(\Delta ABC) = \frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B \tag{1.2.2.2}$$

Dividing the above by abc, we obtain

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \tag{1.2.2.3}$$

This is known as the sine formula.

1.2.3. In Fig. 1.2.2, $AB = AC$. Show that

$$\angle B = \angle C \tag{1.2.3.1}$$

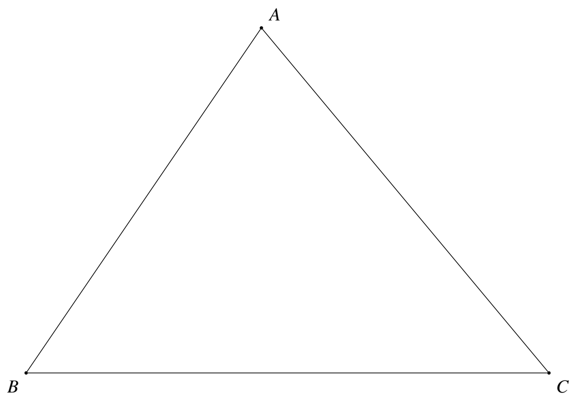


Fig. 1.2.2

Solution: Using the sine formula,

$$\frac{AB}{\sin C} = \frac{AC}{\sin B} \tag{1.2.3.2}$$

$$\implies \sin B = \sin C \text{ or, } \angle B = \angle C. \tag{1.2.3.3}$$

1.2.4. In Fig. 1.2.3, show that

$$\begin{pmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{pmatrix}\begin{pmatrix} \cos A \\ \cos B \\ \cos C \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{1.2.4.1}$$

Solution: From Fig. 1.2.3,

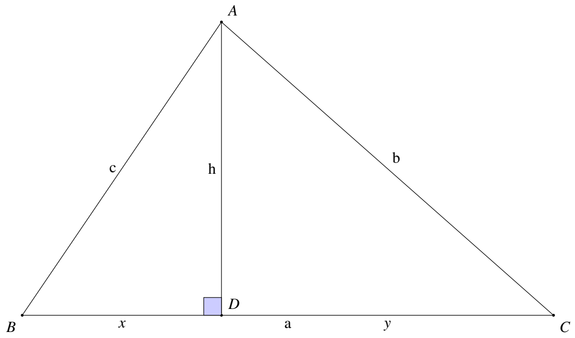


Fig. 1.2.3: The cosine formula

$$a = x + y = b\cos C + c\cos B = \begin{pmatrix}\cos C & \cos B\end{pmatrix}\begin{pmatrix}b\\c\end{pmatrix} \tag{1.2.4.2}$$

$$= \begin{pmatrix}0 & b & c\end{pmatrix}\begin{pmatrix}\cos A\\ \cos C\\ \cos B\end{pmatrix} \tag{1.2.4.3}$$

Similarly,

$$b = c\cos A + a\cos C = \begin{pmatrix}c & 0 & a\end{pmatrix}\begin{pmatrix}\cos A\\ \cos C\\ \cos B\end{pmatrix} \tag{1.2.4.4}$$

$$c = b\cos A + a\cos B = \begin{pmatrix}b & a & 0\end{pmatrix}\begin{pmatrix}\cos A\\ \cos C\\ \cos B\end{pmatrix} \tag{1.2.4.5}$$

The above equations can be expressed in matrix form as (1.2.4.1).

1.2.5. Show that

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \tag{1.2.5.1}$$

Solution: Using the properties of determinants,

$$\cos A = \frac{\begin{vmatrix}a & c & b\\ b & 0 & a\\ c & a & 0\end{vmatrix}}{\begin{vmatrix}0 & c & b\\ c & 0 & a\\ b & a & 0\end{vmatrix}} = \frac{ab^2 + ac^2 - a^3}{abc + abc} = \frac{b^2 + c^2 - a^2}{2abc} \tag{1.2.5.2}$$

1.2.6. Find Hero's formula for the area of a triangle.

Solution: From (1.2.1), the area of $\triangle ABC$ is

$$\frac{1}{2}ab\sin C = \frac{1}{2}ab\sqrt{1-\cos^2 C} \quad \text{(from (1.1.5.1))} \tag{1.2.6.1}$$

$$= \frac{1}{2}ab\sqrt{1-\left(\frac{a^2+b^2-c^2}{2ab}\right)^2} \text{(from (1.2.5.1))} \tag{1.2.6.2}$$

$$= \frac{1}{4}\sqrt{(2ab)^2-(a^2+b^2-c^2)} \tag{1.2.6.3}$$

$$= \frac{1}{4}\sqrt{(2ab+a^2+b^2-c^2)(2ab-a^2-b^2+c^2)} \tag{1.2.6.4}$$

$$= \frac{1}{4}\sqrt{\left\{(a+b)^2-c^2\right\}\left\{c^2-(a-b)^2\right\}} \tag{1.2.6.5}$$

$$= \frac{1}{4}\sqrt{(a+b+c)(a+b-c)(a+c-b)(b+c-a)} \tag{1.2.6.6}$$

Substituting

$$s = \frac{a+b+c}{2} \tag{1.2.6.7}$$

in (1.2.6.6), the area of $\triangle ABC$ is

$$\sqrt{s(s-a)(s-b)(s-c)} \tag{1.2.6.8}$$

This is known as Hero's formula.

1.2.7. Show that

$$\alpha > \beta \implies \sin\alpha > \sin\beta \tag{1.2.7.1}$$

Solution: In Fig. 1.2.4,

$$ar(\triangle ABD) < ar(\triangle ABC) \tag{1.2.7.2}$$

$$\implies \frac{1}{2}lc\sin\theta_1 < \frac{1}{2}ac\sin(\theta_1+\theta_2) \tag{1.2.7.3}$$

$$\implies \frac{l}{a} < \frac{\sin(\theta_1+\theta_2)}{\sin\theta_1} \tag{1.2.7.4}$$

$$\text{or, } 1 < \frac{l}{a} < \frac{\sin(\theta_1+\theta_2)}{\sin\theta_1} \tag{1.2.7.5}$$

from Theorem 1.1.8, yielding

$$\implies \frac{\sin(\theta_1+\theta_2)}{\sin\theta_1} > 1. \tag{1.2.7.6}$$

This proves (1.2.7.1).

1.3 Trigonometric Identities

1.3.1. Using Fig. 1.2.4, show that

$$\sin\theta_1 = \sin(\theta_1+\theta_2)\cos\theta_2 - \cos(\theta_1+\theta_2)\sin\theta_2 \tag{1.3.1.1}$$

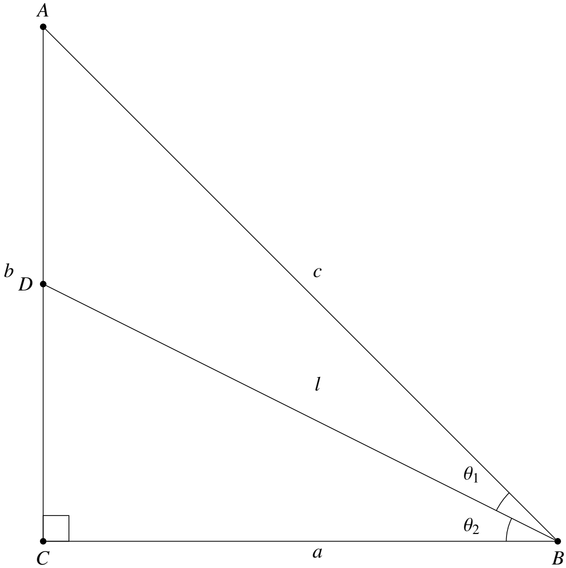


Fig. 1.2.4

Solution: The following equations can be obtained from the figure using the forumula for the area of a triangle

$$ar(\Delta ABC) = \frac{1}{2}ac\sin(\theta_1 + \theta_2) \tag{1.3.1.2}$$
$$= ar(\Delta BDC) + ar(\Delta ADB) \tag{1.3.1.3}$$
$$= \frac{1}{2}cl\sin\theta_1 + \frac{1}{2}al\sin\theta_2 \tag{1.3.1.4}$$
$$= \frac{1}{2}ac\sin\theta_1\sec\theta_2 + \frac{1}{2}a^2\tan\theta_2 \tag{1.3.1.5}$$

($\because l = a\sec\theta_2$). From the above,

$$\sin(\theta_1 + \theta_2) = \sin\theta_1\sec\theta_2 + \frac{a}{c}\tan\theta_2 \tag{1.3.1.6}$$
$$= \sin\theta_1\sec\theta_2 + \cos(\theta_1 + \theta_2)\tan\theta_2 \tag{1.3.1.7}$$

Multiplying both sides by $\cos\theta_2$,

$$\sin(\theta_1 + \theta_2)\cos\theta_2 = \sin\theta_1 + \cos(\theta_1 + \theta_2)\sin\theta_2 \tag{1.3.1.8}$$

resulting in (1.3.1.1).

1.3.2. Prove the following identities

a)

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta. \tag{1.3.2.1}$$

b)

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta. \tag{1.3.2.2}$$

Solution: In (1.3.1.1), let

$$\begin{aligned} \theta_1 + \theta_2 &= \alpha \\ \theta_2 &= \beta \end{aligned} \tag{1.3.2.3}$$

This gives (1.3.2.1). In (1.3.2.1), replace α by $90^\circ - \alpha$. This results in

$$\sin(90^\circ - \alpha - \beta) = \sin(90^\circ - \alpha)\cos\beta - \cos(90^\circ - \alpha)\sin\beta \tag{1.3.2.4}$$

$$\implies \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \tag{1.3.2.5}$$

1.3.3. Using (1.3.1.1) and (1.3.2.2), show that

$$\sin(\theta_1+\theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 \tag{1.3.3.1}$$

$$\cos(\theta_1-\theta_2) = \cos\theta_1\cos\theta_2\sin\theta_1\sin\theta_2 \tag{1.3.3.2}$$

Solution: From (1.3.1.1),

$$\sin(\theta_1+\theta_2)\cos\theta_2 = \sin\theta_1 + \cos(\theta_1+\theta_2)\sin\theta_2 \tag{1.3.3.3}$$

Using (1.3.2.2) in the above,

$$\sin(\theta_1+\theta_2)\cos\theta_2 = \sin\theta_1 + (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2)\sin\theta_2 \tag{1.3.3.4}$$

which can be expressed as

$$\sin(\theta_1+\theta_2)\cos\theta_2 = \sin\theta_1 + \cos\theta_1\cos\theta_2\sin\theta_2 - \sin\theta_1\sin^2\theta_2 \tag{1.3.3.5}$$

Since

$$\sin^2\theta_2 = 1 - \cos^2\theta_2, \tag{1.3.3.6}$$

we obtain

$$\sin(\theta_1+\theta_2)\cos\theta_2 = \cos\theta_1\cos\theta_2\sin\theta_2 + \sin\theta_1\cos^2\theta_2 \tag{1.3.3.7}$$

resulting in

$$\sin(\theta_1+\theta_2) = \cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2 \tag{1.3.3.8}$$

after factoring out $\cos\theta_2$. Using a similar approach, (1.3.3.2) can also be proved.

1.3.4. Show that

$$\sin\theta_1 + \sin\theta_2 = 2\sin\left(\frac{\theta_1+\theta_2}{2}\right)\cos\left(\frac{\theta_1-\theta_2}{2}\right) \tag{1.3.4.1}$$

$$\cos\theta_1 + \cos\theta_2 = 2\cos\left(\frac{\theta_1+\theta_2}{2}\right)\cos\left(\frac{\theta_1-\theta_2}{2}\right) \tag{1.3.4.2}$$

$$\sin\theta_1 - \sin\theta_2 = 2\sin\left(\frac{\theta_1-\theta_2}{2}\right)\cos\left(\frac{\theta_1+\theta_2}{2}\right) \tag{1.3.4.3}$$

$$\cos\theta_1 - \cos\theta_2 = 2\sin\left(\frac{\theta_1+\theta_2}{2}\right)\cos\left(\frac{\theta_2-\theta_1}{2}\right) \tag{1.3.4.4}$$

Solution: Let

$$\begin{aligned}\theta_1 &= \alpha + \beta \\ \theta_2 &= \alpha - \beta\end{aligned} \tag{1.3.4.5}$$

From (1.3.3.1),

$$\sin\theta_1 + \sin\theta_2 = \sin(\alpha+\beta) + \sin(\alpha-\beta) \tag{1.3.4.6}$$

$$= \sin\alpha\cos\beta + \cos\alpha\sin\beta + \sin\alpha\cos\beta - \cos\alpha\sin\beta \tag{1.3.4.7}$$

$$= 2\sin\alpha\cos\beta \tag{1.3.4.8}$$

resulting in (1.3.4.1)

$$\because \alpha = \frac{\theta_1+\theta_2}{2},\ \beta = \frac{\theta_1-\theta_2}{2} \tag{1.3.4.9}$$

from (1.3.4.5). Other identities may be proved similarly.

1.3.5. Show that

$$\sin 2\theta = 2\sin\theta\cos\theta \tag{1.3.5.1}$$

$$\cos 2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1 \tag{1.3.5.2}$$

$$= \cos^2\theta - \sin^2\theta \tag{1.3.5.3}$$

1.4 Incircle

1.4.1. In Fig. 1.4.1, the bisectors of $\angle B$ and $\angle C$ meet at **I**. Show that IA bisects $\angle A$.

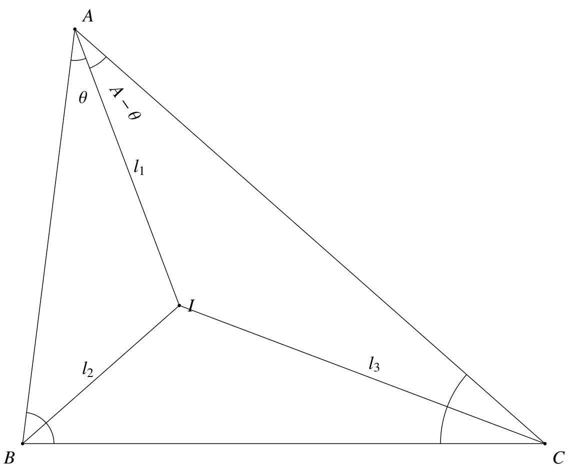


Fig. 1.4.1: Incentre I of $\triangle ABC$

Solution: Using sine formula in (1.2.2.3)

$$\frac{l_1}{\sin\frac{C}{2}} = \frac{l_3}{\sin(A-\theta)},\ \frac{l_3}{\sin\frac{B}{2}} = \frac{l_2}{\sin\frac{C}{2}},\ \frac{l_2}{\sin\theta} = \frac{l_1}{\sin\frac{B}{2}} \tag{1.4.1.1}$$

Multiplying the above equations,

$$\sin\theta = \sin(A-\theta) \tag{1.4.1.2}$$

$$\implies \theta = A-\theta \text{ or, } \theta = \frac{A}{2} \tag{1.4.1.3}$$

1.4.2. Fig. 1.4.2, is obtained from Fig. 1.4.1 with

$$ID \perp BC, IE \perp AC, IF \perp AB. \tag{1.4.2.1}$$

Show that

$$ID = IE = IF = r \tag{1.4.2.2}$$

Solution: In $\triangle$s IDC and IEC,

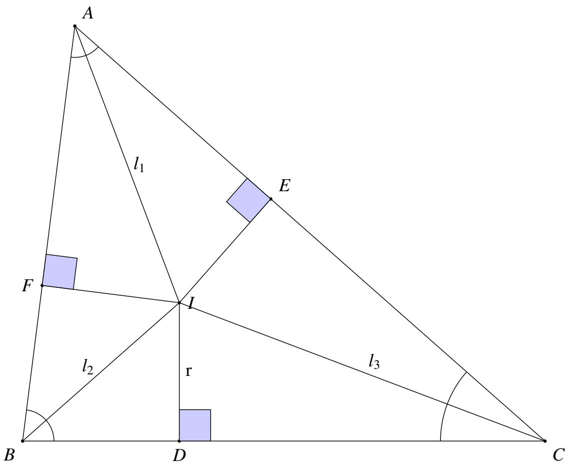


Fig. 1.4.2: Inradius r of $\triangle ABC$

$$ID = IE = \frac{l_3}{\sin\frac{C}{2}} \tag{1.4.2.3}$$

Similarly, in $\triangle$s IEA and IFA,

$$IF = IE = \frac{l_1}{\sin\frac{A}{2}} \tag{1.4.2.4}$$

yielding (1.4.2.2)

1.4.3. In Fig. 1.4.2, show that

$$BD = BF, AE = AF, CD = CE \tag{1.4.3.1}$$

Solution: From Fig. 1.4.2, in $\triangle$s IBD and IBF,

$$x = BD = BF = r\cot\frac{B}{2} \tag{1.4.3.2}$$

Similarly, other results can be obtained.

1.4.4. The circle with centre **I** and radius r in Fig. 1.4.3 is known as the *incircle*.

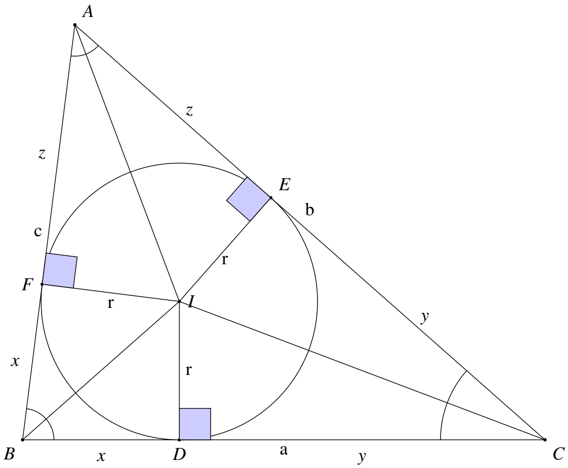


Fig. 1.4.3: Incircle of $\triangle ABC$

1.4.5. The lengths of tangents drawn from an external point to a circle are equal.

1.4.6. In an isosceles $\triangle ABC$, with $AB = AC$, BE and CF are the bisectors of $\angle B$ and $\angle C$ respectively. Show that

$$BE = CF \tag{1.4.6.1}$$

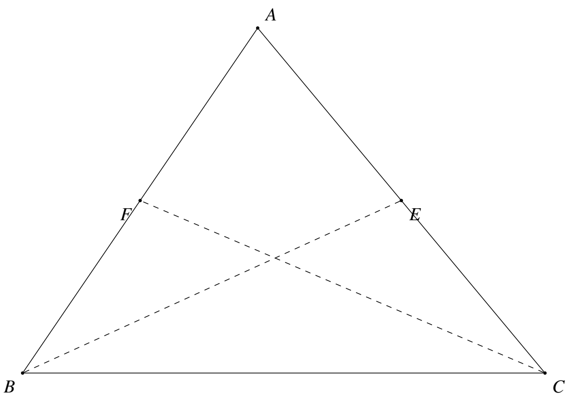


Fig. 1.4.4

Solution: In $\triangle$ s BEC and BFC, using the sine formula,

$$\begin{aligned} \frac{BE}{\sin C} &= \frac{BC}{\sin\left(\frac{B}{2}+C\right)} \\ \frac{CF}{\sin B} &= \frac{BC}{\sin\left(\frac{C}{2}+B\right)} \end{aligned} \tag{1.4.6.2}$$

$\because B = C$, from the above, we obtain (1.4.6.1).

1.4.7. Show that

$$\sin 5\theta = 5\sin\theta - 20\sin^3\theta\cos^2\theta + 16\sin^5\theta \tag{1.4.7.1}$$

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta \tag{1.4.7.2}$$

1.4.8. In Fig. 1.4.4, if $BE = CF$, show that the triangle is isosceles.

Solution: From (1.4.6.2),

$$\sin C\sin\left(\frac{C}{2}+B\right) = \sin\left(\frac{B}{2}+C\right)\sin B \tag{1.4.8.1}$$

$$\implies 2\sin C\sin\left(\frac{C}{2}+B\right) = 2\sin B\sin\left(\frac{B}{2}+C\right) \tag{1.4.8.2}$$

$$\cos\left(B-\frac{C}{2}\right) - \cos\left(B+\frac{3C}{2}\right) = \cos\left(C-\frac{B}{2}\right) - \cos\left(C+\frac{3B}{2}\right) \tag{1.4.8.3}$$

using (1.3.4.4), which can be expressed as

$$\cos\left(C-\frac{B}{2}\right) - \cos\left(B-\frac{C}{2}\right) - \cos\left(C+\frac{3B}{2}\right) + \cos\left(B+\frac{3C}{2}\right) = 0 \tag{1.4.8.4}$$

which, using (1.3.4.4), yields

$$2\sin\left(\frac{B+C}{2}\right)\sin\left[\frac{3\left(B-C\right)}{2}\right] + 2\sin\left[5\frac{\left(B+C\right)}{2}\right]\sin\left[\frac{\left(B-C\right)}{2}\right] = 0 \tag{1.4.8.5}$$

Let

$$\theta = \frac{B-C}{2},\ \alpha = \frac{B+C}{2} \tag{1.4.8.6}$$

Substituting the above in (1.4.8.5),

$$\sin\alpha\sin 3\theta + \sin 5\alpha\sin\theta = 0 \tag{1.4.8.7}$$

Substituting from (1.4.7.2) in (1.4.8.7) and simplifying,

$$\sin\alpha\sin\theta\left(3 - 4\sin^2\theta + 5 - 20\sin^2\alpha\cos^2\alpha + 16\sin^4\alpha\right) = 0 \tag{1.4.8.8}$$

One possible solution of the above equation is

$$3 - 4\sin^2\theta + 5 - 20\sin^2\alpha\cos^2\alpha + 16\sin^4\alpha = 0 \tag{1.4.8.9}$$

$$4 - 4\sin^2\theta + 4 - 20\sin^2\alpha\left(1-\sin^2\alpha\right) + 16\sin^4\alpha = 0 \tag{1.4.8.10}$$

which, upon substituting from (1.1.5.1) results in

$$\cos^2\theta + 1 - 5\sin^2\alpha + 36\sin^4\alpha = 0 \tag{1.4.8.11}$$

$$= \cos^2\theta + \left(1 - 6\sin^2\alpha\right)^2 + 7\sin^2\alpha = 0 \tag{1.4.8.12}$$

For the above equation to have a solution,

$$\cos\theta = 0, \sin^2\alpha = \frac{1}{6}, \sin\alpha = 0. \tag{1.4.8.13}$$

which is impossible. Another possible solution is

$$\sin\alpha = \sin\frac{B+C}{2} = 0 \tag{1.4.8.14}$$

$$\implies \cos\frac{A}{2} = 0, \text{ or, } A = \pi, \tag{1.4.8.15}$$

which is impossible. Hence, the only possible solution is

$$\sin\theta = \sin\frac{B-C}{2} = 0 \tag{1.4.8.16}$$

$$\implies \frac{B-C}{2} = 0, \text{ or, } B = C. \tag{1.4.8.17}$$

1.5 Circumcircle

1.5.1. In Fig. 1.5.1,

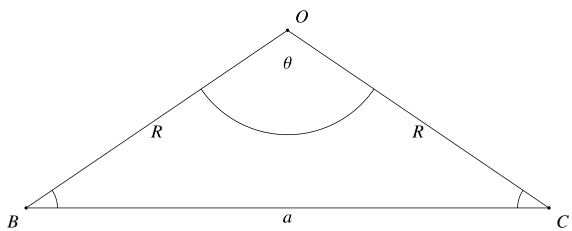


Fig. 1.5.1: Isosceles Triangle

$$OB = OC = R \tag{1.5.1.1}$$

Such a triangle is known as an isosceles triangle. Show that

$$\angle B = \angle C \tag{1.5.1.2}$$

Solution: Using (1.2.2.3),

$$\frac{\sin B}{R} = \frac{\sin C}{R} \tag{1.5.1.3}$$

$$\implies \sin B = \sin C \tag{1.5.1.4}$$

$$\text{or, } \angle B = \angle C. \tag{1.5.1.5}$$

1.5.2. In Fig. 1.5.1, show that

$$a = 2R \sin \frac{\theta}{2} \tag{1.5.2.1}$$

Solution: In $\triangle OBC$, using the cosine formula from (1.2.5.1),

$$\cos\theta = \frac{R^2 + R^2 - a^2}{2R^2} = 1 - \frac{a^2}{2R^2} \tag{1.5.2.2}$$

$$\implies \frac{a^2}{2R^2} = 2\sin^2\frac{\theta}{2} \tag{1.5.2.3}$$

yielding (1.5.2.1).

1.5.3. In Fig. 1.5.2,

$$OB = OC = R, BD = DC. \tag{1.5.3.1}$$

Show that $OD \perp BC$.

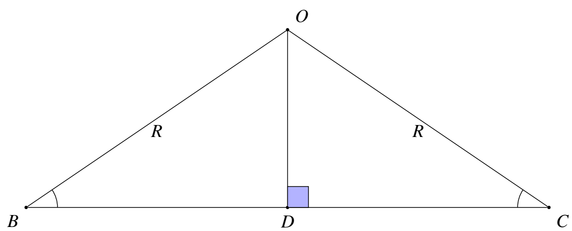


Fig. 1.5.2: Perpendicular bisector.

1.5.4. In Fig. 1.5.3, OD and OE are the perpendicular bisectors of sides BC and AC respectively. Show that $OA = R$.

1.5.5. In Fig. 1.5.3, show that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R. \tag{1.5.5.1}$$

Solution: From (1.5.10.1) and (1.5.2.1)

$$a = 2R \sin A \tag{1.5.5.2}$$

1.5.6. Fig. 1.5.4 shows the *circumcircle* of $\triangle ABC$.

1.5.7. Any point on the circle can be expressed as

$$\mathbf{x} = \mathbf{O} + R\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}, \quad 0 \in [0, 2\pi]. \tag{1.5.7.1}$$

where $\mathbf{O}$ is the centere of the circle.

1.5.8. Let

$$R = 1, \mathbf{O} = \mathbf{0}, \mathbf{A} = \begin{pmatrix}\cos\theta_1 \\ \sin\theta_1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}\cos\theta_2 \\ \sin\theta_2\end{pmatrix}, \tag{1.5.8.1}$$

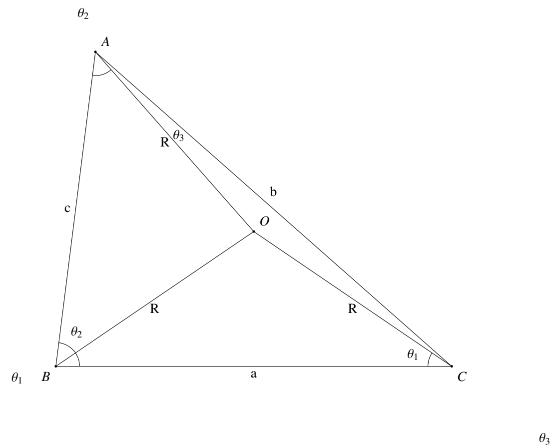


Fig. 1.5.3: Perpendicular bisectors of $\triangle ABC$ meet at $\mathbf{O}$.

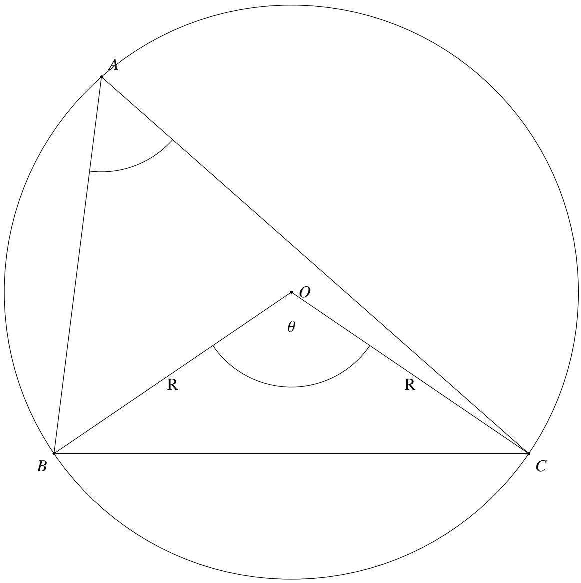


Fig. 1.5.4: Circumcircle of $\triangle ABC$

Show that the distance

$$AB = \|\mathbf{A} - \mathbf{B}\| = 2\sin\left(\frac{\theta_1 - \theta_2}{2}\right) \tag{1.5.8.2}$$

Solution: From (1.5.7.1).

$$\mathbf{A}-\mathbf{B} = \begin{pmatrix}\cos\theta_1 - \cos\theta_2 \\ \sin\theta_1 - \sin\theta_2\end{pmatrix} \tag{1.5.8.3}$$

$$\implies \left\|\mathbf{A}-\mathbf{B}\right\|^2 = \left(\mathbf{A}-\mathbf{B}\right)^\top\left(\mathbf{A}-\mathbf{B}\right) \tag{1.5.8.4}$$

$$= \left(\cos\theta_1 - \cos\theta_2\right)^2 + \left(\sin\theta_1 - \sin\theta_2\right)^2 \tag{1.5.8.5}$$

$$= 2\left\{1-\cos\left(\theta_1-\theta_2\right)\right\} = 4\sin^2\left(\frac{\theta_1-\theta_2}{2}\right) \tag{1.5.8.6}$$

yielding (1.5.8.2) from (1.3.5.3).

1.5.9. In Fig. 1.5.4, show that

$$\cos A = \frac{\left(\mathbf{A}-\mathbf{B}\right)^\top\left(\mathbf{A}-\mathbf{B}\right)}{\left\|\mathbf{A}-\mathbf{B}\right\|\left\|\mathbf{A}-\mathbf{C}\right\|}, \tag{1.5.9.1}$$

1.5.10. In Fig. 1.5.4, show that

$$\theta = 2A. \tag{1.5.10.1}$$

Solution: Let

$$\mathbf{C} = \begin{pmatrix}\cos\theta_3 \\ \sin\theta_3\end{pmatrix} \tag{1.5.10.2}$$

Then, substituting from (1.5.8.2) in (1.2.5.1),

$$\cos A = \frac{4\sin^2\left(\frac{\theta_1-\theta_2}{2}\right) + 4\sin^2\left(\frac{\theta_1-\theta_3}{2}\right) - 4\sin^2\left(\frac{\theta_2-\theta_3}{2}\right)}{8\sin\left(\frac{\theta_1-\theta_2}{2}\right)\sin\left(\frac{\theta_1-\theta_3}{2}\right)} \tag{1.5.10.3}$$

$$= \frac{2\sin^2\left(\frac{\theta_1-\theta_2}{2}\right) + \cos\left(\theta_2-\theta_3\right) - \cos\left(\theta_1-\theta_3\right)}{4\sin\left(\frac{\theta_1-\theta_2}{2}\right)\sin\left(\frac{\theta_1-\theta_3}{2}\right)} \tag{1.5.10.4}$$

from (1.3.5.3). $\therefore$ From (1.3.4.4),

$$\cos A = \frac{2\sin^2\left(\frac{\theta_1-\theta_2}{2}\right) + 2\sin\left(\frac{\theta_1-\theta_2}{2}\right)\sin\left(\frac{\theta_1+\theta_2}{2}-\theta_3\right)}{4\sin\left(\frac{\theta_1-\theta_2}{2}\right)\sin\left(\frac{\theta_1-\theta_3}{2}\right)} \tag{1.5.10.5}$$

$$= \frac{\sin\left(\frac{\theta_1-\theta_2}{2}\right) + \sin\left(\frac{\theta_1+\theta_2}{2}-\theta_3\right)}{2\sin\left(\frac{\theta_1-\theta_3}{2}\right)} \tag{1.5.10.6}$$

From (1.3.4.1), the above equation can be expressed as

$$\cos A = \frac{2\sin\left(\frac{\theta_1-\theta_3}{2}\right)\cos\left(\frac{\theta_2-\theta_3}{2}\right)}{2\sin\left(\frac{\theta_1-\theta_3}{2}\right)} = \cos\left(\frac{\theta_2-\theta_3}{2}\right) \tag{1.5.10.7}$$

$$\implies 2A = \theta_2 - \theta_3 \tag{1.5.10.8}$$

Similarly,

$$\cos\theta = \frac{1+1-4\sin^2\left(\frac{\theta_2-\theta_3}{2}\right)}{2} = \cos(\theta_2-\theta_3) = \cos 2A \tag{1.5.10.9}$$

1.5.11. In Fig. 1.5.5, show that

$$\theta = \alpha \tag{1.5.11.1}$$

where CP is the tangent.

Solution: Let

$$\mathbf{O} = \mathbf{0},\ \mathbf{A} = \begin{pmatrix}\cos\theta_1\\ \sin\theta_1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}\cos\theta_2\\ \sin\theta_2\end{pmatrix}, \mathbf{C} = \begin{pmatrix}\cos\theta_3\\ \sin\theta_3\end{pmatrix} \tag{1.5.11.2}$$

Without loss of generality, let

$$\theta_3 = \frac{\pi}{2} \tag{1.5.11.3}$$

Then,

$$\mathbf{C}-\mathbf{O} = \begin{pmatrix}0\\1\end{pmatrix}.\ \implies\ \mathbf{C}-\mathbf{P} \equiv \begin{pmatrix}1\\0\end{pmatrix}, \tag{1.5.11.4}$$

$\because CO \perp CP$. From (1.5.9.1), and (1.5.11.4),

$$\cos\theta = \frac{\begin{pmatrix}\cos\theta_3-\cos\theta_1 & \sin\theta_3-\sin\theta_1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}}{2\sin\left(\frac{\theta_1-\theta_3}{2}\right)} \tag{1.5.11.5}$$

$$= \sin\left(\frac{\theta_1+\theta_3}{2}\right) = \cos\left(\frac{\pi}{2}-\frac{\theta_1+\theta_3}{2}\right) = \cos\left(\frac{\pi}{4}-\frac{\theta_1}{2}\right) \tag{1.5.11.6}$$

upon substituting from (1.5.11.3). Similarly, from (1.5.10.7),

$$\cos\alpha = \cos\left(\frac{\theta_1-\theta_3}{2}\right) = \cos\left(\frac{\pi}{4}-\frac{\theta_1}{2}\right) = \cos\theta \tag{1.5.11.7}$$

1.5.12. In Fig. 1.5.5, show that $PA.PB = PC^2$.

Solution: In $\triangle$s APC and BPC, using (1.5.11.1),

$$\frac{AP}{\sin\theta} = \frac{AC}{\sin P} \tag{1.5.12.1}$$

$$\frac{PC}{\sin\theta} = \frac{BC}{\sin P} \tag{1.5.12.2}$$

$$\implies \frac{PC}{AP} = \frac{BC}{AC}\left(=\frac{BP}{CP}\right) \tag{1.5.12.3}$$

which gives the desired result. $\triangle$s APC and BPC are said to be *similar*.

1.6 Medians

1.6.1. In Fig. 1.6.1

$$AF = BF, AE = BE, \tag{1.6.1.1}$$

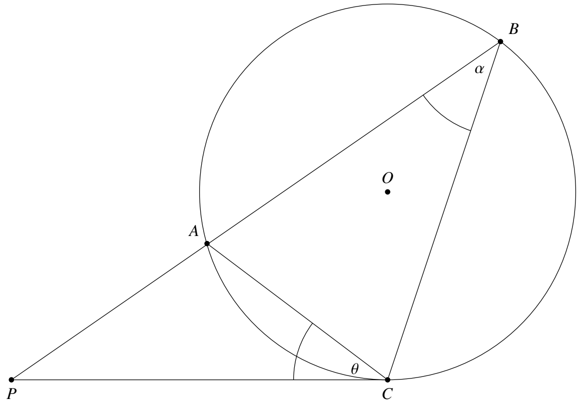


Fig. 1.5.5: $\theta = \alpha$.

and the medians BE and CF meet at **G**. Show that

$$ar(BEC) = ar(BFC) = \frac{1}{2} ar(ABC) \tag{1.6.1.2}$$

Solution: From (1.2.2.2),

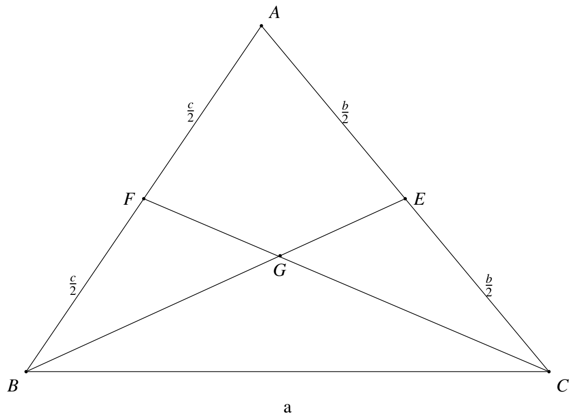


Fig. 1.6.1: $k_1 = k_2$.

$$ar(BEC) = \frac{1}{2} a \left(\frac{b}{2}\right) \sin C \tag{1.6.1.3}$$

$$ar(BFC) = \frac{1}{2} a \left(\frac{c}{2}\right) \sin B \tag{1.6.1.4}$$

yielding (1.6.1.2).

1.6.2. The median divides a triangle into two triangle of equal area. .

1.6.3. In Fig. 1.6.1, show that

$$ar(CGE) = ar(BGF) \tag{1.6.3.1}$$

Solution: From Fig. 1.6.1 and (1.6.1.2),

$$ar(BGF) + ar(BGC) = ar(CGE) + ar(BGC) \tag{1.6.3.2}$$

yielding (1.6.3.1).

1.6.4. In Fig. 1.6.2, show that

$$k_1 = k_2 \tag{1.6.4.1}$$

Solution: From (1.6.3.1),

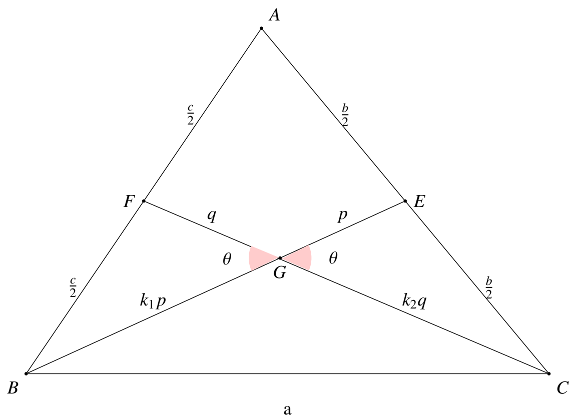


Fig. 1.6.2: Equal areas.

$$\frac{1}{2}p(k_1q)\sin\theta = \frac{1}{2}q(k_2p)\sin\theta \tag{1.6.4.2}$$

yielding (1.6.4.1).

1.6.5. In Fig. 1.6.3, show that

$$k_3 = k \tag{1.6.5.1}$$

Solution: From Problem 1.6.2,

$$\begin{aligned} ar(AGE) &= ar(CGE) \\ ar(AGF) &= ar(BGF) \end{aligned} \tag{1.6.5.2}$$

$$\implies \begin{aligned} \frac{1}{2}p(k_3r)\sin\alpha &= \frac{1}{2}p(kq)\sin\theta \\ \frac{1}{2}q(k_3r)\sin\beta &= \frac{1}{2}q(kp)\sin\theta \end{aligned} \tag{1.6.5.3}$$

yileding upon division

$$p \sin \alpha = q \sin \beta \tag{1.6.5.4}$$

$$\implies \frac{1}{2} kpr \sin \alpha = \frac{1}{2} kqr \sin \beta \tag{1.6.5.5}$$

$$\implies ar(BGD) = ar(CGD) \tag{1.6.5.6}$$

Thus, from Problem 1.6.2, AD is also a median. Consequently, from (1.6.4.1) we obtain (1.6.5.1).

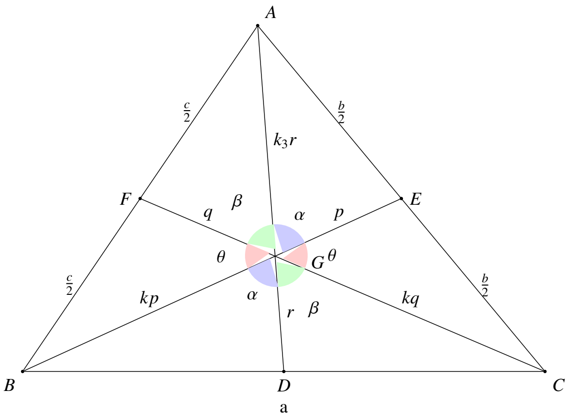


Fig. 1.6.3: $k_3 = k$.

1.6.6. In Fig. 1.6.4, show that $k = 2$.

Solution: Using the cosine formula,

$$DE^2 = \left(\frac{b}{2}\right)^2 + \left(\frac{c}{2}\right)^2 - 2\left(\frac{b}{2}\right)\left(\frac{c}{2}\right)\cos A \tag{1.6.6.1}$$

$$a^2 = b^2 + b^2 - 2bc \cos A \tag{1.6.6.2}$$

$$\implies DE = \frac{a}{2} \tag{1.6.6.3}$$

$\because \triangle EGF \sim \triangle BGC, k = 2.$

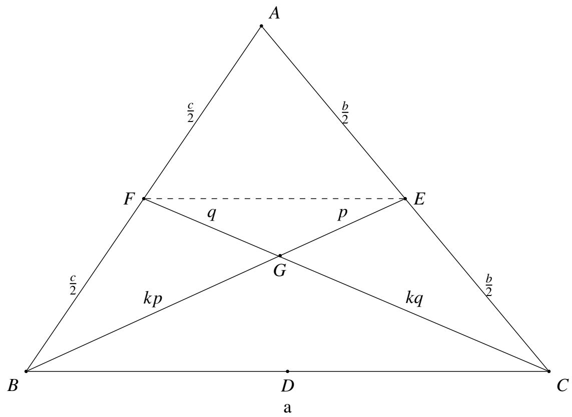


Fig. 1.6.4: k = 2

2 Triangle

2.1 NCERT

2.1.1. D is a point on the side BC of a $\triangle ABC$ such that $\angle ADC = \angle BAC$. Show that

$$CA^2 = CB.CD \tag{2.1.1.1}$$

Solution: See Fig. 2.1.1.

$$\frac{x}{\sin(A+C)} = \frac{b}{\sin A} \quad (\triangle ADC), \tag{2.1.1.2}$$

$$\implies \frac{x}{\sin B} = \frac{b}{\sin A} \tag{2.1.1.3}$$

$$\implies \implies \frac{x}{b} = \frac{\sin B}{\sin A} = \frac{b}{a} \quad (\text{ sine formula }) \tag{2.1.1.4}$$

yielding (2.1.1.1).

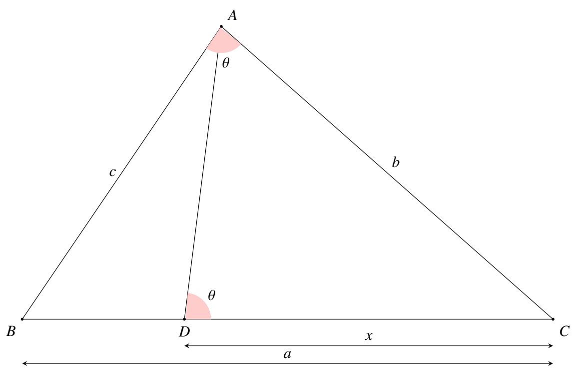


Fig. 2.1.1

2.1.2. D is a point on side BC of $\triangle ABC$ such that $\frac{BD}{CD} = \frac{AB}{AC}$. Prove that AD is the bisector of $\angle BAC$.

Solution: See Fig. 2.1.2.

$$\frac{x}{a-x} = \frac{c}{b} \quad (\text{ given }) \tag{2.1.2.1}$$

$$\frac{c}{\sin\phi} = \frac{x}{\sin\theta} \quad (\triangle ABD) \tag{2.1.2.2}$$

$$\frac{a-x}{\sin(A-\theta)} = \frac{b}{\sin 180-\phi} \quad (\triangle ACD) \tag{2.1.2.3}$$

$$= \frac{b}{\sin\phi} \tag{2.1.2.4}$$

using the sine formula. Multiplying all the above equations yields

$$\sin(A-\theta) = \sin\theta \implies \theta = \frac{A}{2} \tag{2.1.2.5}$$

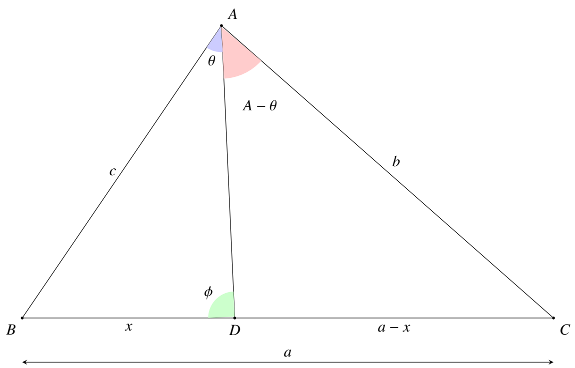


Fig. 2.1.2

2.1.3. ABC is a triangle in which $\angle ABC > 90°$ and $AD \perp CB$ produced. Prove that

$$AC^2 = AB^2 + BC^2 + 2BC.BD. \tag{2.1.3.1}$$

Solution: See Fig. 2.1.3.

$$\cos B = \frac{x}{c} \quad (\triangle ADB) \tag{2.1.3.2}$$

$$b^2 = a^2 + c^2 - 2ac\cos(180-B) \quad (\triangle ABC) \tag{2.1.3.3}$$

$$= a^2 + c^2 + 2ac\cos B \tag{2.1.3.4}$$

using the cosine formula. Substituting from (2.1.3.2) in (2.1.3.4) yields (2.1.3.1).

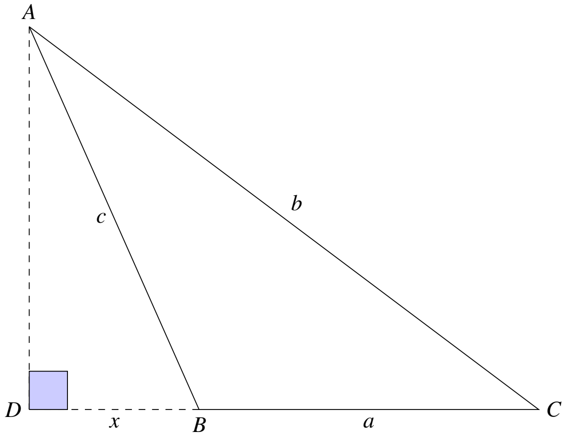


Fig. 2.1.3

2.1.4. In a right triangle, prove that the line-segment joining the mid-point of the hypotenuse to the opposite vertex is half the hypotenuse.
Solution: In Fig. 2.1.4

$$\frac{x}{\sin C} = \frac{b/2}{\sin\theta} \quad (\triangle BDC) \tag{2.1.4.1}$$

$$\frac{x}{\sin A} = \frac{b/2}{\sin(90-\theta)} \quad (\triangle BDA) \tag{2.1.4.2}$$

$$\implies \frac{x}{\cos C} = \frac{b/2}{\cos\theta} \tag{2.1.4.3}$$

From (2.1.4.1) and (2.1.4.3),

$$\left(\frac{\sin C}{x}\right)^2 + \left(\frac{\cos C}{x}\right)^2 = \left(\frac{\cos\theta}{\frac{b}{2}}\right)^2 + \left(\frac{\sin\theta}{\frac{b}{2}}\right)^2 \tag{2.1.4.4}$$

$$\implies x = \frac{b}{2} \tag{2.1.4.5}$$

using (1.1.5.1).

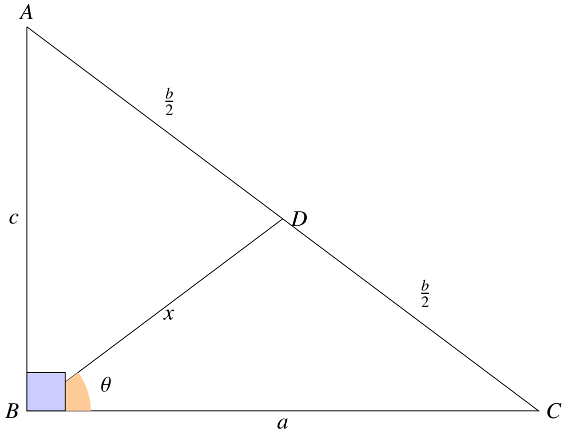


Fig. 2.1.4

2.1.5. $ABCD$ is a trapezium in which $AB \parallel DC$ and its diagonals intersect each other at the point O. Show that

$$\frac{AO}{BO} = \frac{CO}{DO} \tag{2.1.5.1}$$

Solution: In Fig. 2.1.5, $\because AB \parallel CD$

$$\frac{AO}{\sin\phi} = \frac{BO}{\sin\theta} \quad (\triangle OAB) \tag{2.1.5.2}$$

$$\frac{CO}{\sin\phi} = \frac{DO}{\sin\theta} \quad (\triangle ODC) \tag{2.1.5.3}$$

yielding (2.1.5.1) after simplification.

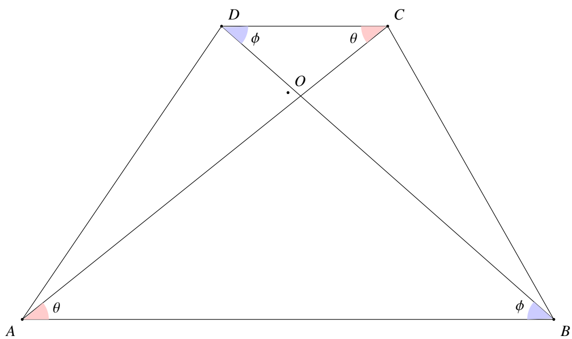


Fig. 2.1.5

2.1.6. O is any point inside a rectangle $ABCD$. Prove that

$$OB^2 + OD^2 = OA^2 + OC^2 \tag{2.1.6.1}$$

Solution: In Fig. 2.1.6, from (1.1.4.1)

$$p\cos\theta_1 + q\sin\theta_2 = a \quad (\triangle OAB) \tag{2.1.6.2}$$
$$r\cos\theta_3 + s\sin\theta_4 = a \quad (\triangle OAB) \tag{2.1.6.3}$$
$$p\cos\theta_1 + s\sin\theta_4 = b \quad (\triangle OAB) \tag{2.1.6.4}$$
$$r\cos\theta_3 + q\sin\theta_2 = b \quad (\triangle OAB) \tag{2.1.6.5}$$

Subtracting the first two and second two equations respectively,

$$p\cos\theta_1 - s\sin\theta_4 = r\cos\theta_3 - q\sin\theta_2 \tag{2.1.6.6}$$
$$p\cos\theta_1 + s\sin\theta_4 = r\cos\theta_3 + q\sin\theta_2 \tag{2.1.6.7}$$

Squaring and adding and using (1.1.5.1) yields (2.1.6.1).

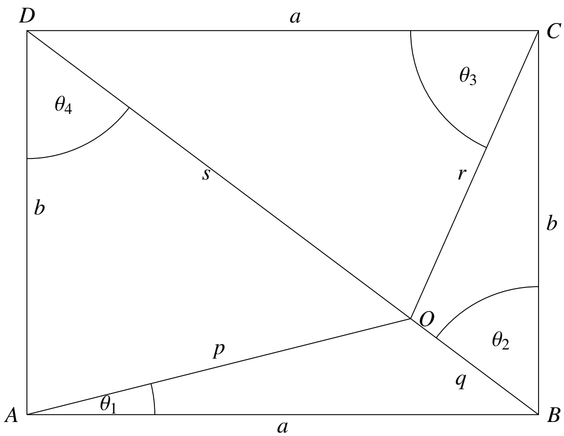


Fig. 2.1.6

2.2 JEE

Fill In The Blanks

2.2.1 In a ΔABC, $\angle A = 90°$ and AD is an altitude. Complete the relation $\frac{BD}{BA} = \frac{AB}{(\dots)}$ (1980)

2.2.2 ABC is a triangle, P is a point on AB, and Q is point on AC such that $\angle AQP = \angle ABC$. Complete the relation $\frac{area\ of \Delta APQ}{area\ of \Delta ABC} = \frac{(\dots)}{AC^2}$ (1980)

2.2.3 ABC is a triangle with $\angle B$ greater than $\angle C$ D and E are the points on BC such that AD is perpendicular to BC and AE is the bisector of angle A .Complete the relation $\angle DAE = \frac{1}{2}[() - \angle C]$ (1980)

2.2.4 the set of all real numbers a such that $a^2 + 2a, 2a + 3$ and $a^2 + 3a + 8$ are the sides of a triangle is ... (1985 - 2 Marks)

2.2.5 In a triangle ABC, if $\cot A$,$\cot B$,$\cot C$ are in A.P. ,then a^2,b^2,c^2,are in ... progression (1985 - 2 Marks)

2.2.6 A polygon of nine sides, each of length 2, is inscribed in a circle. The radius of the circle is ... (1987 - 2 Marks)

2.2.7 If the angles of a triangle are 30° and 45° and the included side is $(\sqrt{3}+1)cms$, then the area of the triangle is ... (1988 - 2 Marks)

2.2.8 If the triangle ABC, $\frac{2\cos A}{a} + \frac{2\cos B}{b} + \frac{2\cos C}{c} = \frac{a}{bc} + \frac{b}{ac}$, then the value of the angle A is ... degrees. (1993 - 2 Marks)

2.2.9 In the triangle ABC, AD is the altitude from A. Given $b > c$, $\angle C = 23°$ and $AD = \frac{abc}{b^2-c^2}$ then $\angle B = \ldots$ (1994 - 2 Marks)

2.2.10 A circle is inscribed in a equilateral triangle of a side a. The area of any square inscribed in this circle is ... (1994 - 2 Marks)

2.2.11 In a triangle ABC, $a : b : c = 4 : 5 : 6$. The ratio of the radius of the circumstances to that of the incircle is ... (1996 - 1 Marks)

JEE Mains / AIEEE

2.2.1 The sum of the radii of inscribed and circumscribed circles for an n sided regular polygon of side a, is (2003)

a) $\frac{a}{4}\cot\left(\frac{\pi}{2n}\right)$ b) $a\cot\left(\frac{\pi}{n}\right)$ c) $\frac{a}{2}\cot\left(\frac{\pi}{2n}\right)$ d) $a\cot\left(\frac{\pi}{2n}\right)$

2.2.2 In a triangle ΔABC, medians AD and BE are drawn. If AD= $4, \angle DAB = \frac{\pi}{6}$ and $\angle ABE = \frac{\pi}{3}$, then the area of the ΔABC is (2003)

a) $\frac{64}{3}$ b) $\frac{8}{3}$ c) $\frac{16}{3}$ d) $\frac{32}{3\sqrt{3}}$

2.2.3 If in ΔABC $a\cos^2\left(\frac{C}{2}\right) + c\cos^2\left(\frac{A}{2}\right) = \frac{3b}{2}$, then the sides a, b and c (2003)

a) satisfy $a + b = c$ b) are in A.P. c) are in G.P. d) are in H.P.

2.2.4 The sides of a triangle are $\sin\alpha, \cos\alpha$ ad $\sqrt{1 + \sin\alpha\cos\alpha}$ for some $0 < \alpha < \frac{\pi}{2}$. Then the greatest angle of the triangle is (2004)

a) 150° b) 90° c) 120° d) 60°

2.2.5 A person standing on the bank of a river observes that the angle of elevation of the top of a tree on the opposite bank of the river is 60° and when he retires 40 meters away from the tree, the angle of elevation becomes 30°. The breadth of the river is (2004)

a) $60m$ b) $30m$ c) $40m$ d) $20m$

2.2.6 In a triangle ΔABC, let $\angle C = \frac{\pi}{2}$. If r is the inradius and R is the circumradius of the triangle ΔABC, then $2(R + r)$ equals (2005)

a) $b+c$ b) $a+b$ c) $a+b+c$ d) $c+a$

2.2.7 If in a ΔABC, let the altitudes from the vertices **A**, **B**, **C** on opposite sides are in H.P., then $\sin\mathbf{A}, \sin\mathbf{B}, \sin\mathbf{C}$ are in (2005)

a) $G.P.$ b) $A.P.$ c) $A.P. - G.P.$ d) $H.P.$

2.2.8 A tower stand at the centre of a circular park. **A** and **B** are two points on the boundary of the park such that AB $(= a)$ subtends an angle of 60° at the foot of the tower, and the angle of elevation of the top of the tower from **A** or **B** is 30°. The height of the tower is (2007)

a) $\frac{a}{\sqrt{3}}$ b) $a\sqrt{3}$ c) $\frac{2a}{\sqrt{3}}$ d) $2a\sqrt{3}$

2.2.9 AB is a vertical pole with **B** at the ground level and **A** at the top. A man finds that the angle of elevation the the point **A** from a certain point **C** on the ground is 60°. He moves away from the pole along the line BC to a point **D** such that CD = $7m$. From **D** the angle of elevation of point **A** is 45°. Then the height of the pole is (2008)

a) $\frac{7\sqrt{3}}{2}\frac{1}{\sqrt{3}-1}m$ b) $\frac{7\sqrt{3}}{2}\left(\sqrt{3}+1\right)m$ c) $\frac{7\sqrt{3}}{2}\left(\sqrt{3}-1\right)m$ d) $\frac{7\sqrt{3}}{2}\frac{1}{\sqrt{3}+1}m$

2.2.10 For a regular polygon, let r and R be the radii of the inscribed and the circumscribed circles. A false statement among the following is (2010)

a) There is a regular polygon with $\frac{r}{R} = \frac{1}{\sqrt{2}}$
b) There is a regular polygon with $\frac{r}{R} = \frac{2}{3}$
c) There is a regular polygon with $\frac{r}{R} = \frac{\sqrt{3}}{2}$
d) There is a regular polygon with $\frac{r}{R} = \frac{1}{2}$

2.2.11 A bird is sitting on the top of a vertical pole $20m$ high and its elevation from a point **O** on the ground is 45°. It flies off horizontally straight away from the point **O**. After one second, the elevation of the bird from **O** is reduced to 30°. Then the speed in $(in\, m/s)$ of the bird is $(JEEM2014)$

a) $20\sqrt{2}$ b) $20\left(\sqrt{3}-1\right)$ c) $40\left(\sqrt{2}-1\right)$ d) $40\left(\sqrt{3}-\sqrt{2}\right)$

2.2.12 If the angle of elevation of the top of a tower from three colinear points **A**, **B** and **C** on a line leading to foot of the tower, are 30°, 45°and60° respectively, then the ratio, AB : BC, is: $(JEEM2015)$

a) $1:\sqrt{3}$ b) $2:3$ c) $\sqrt{3}:1$ d) $\sqrt{3}:\sqrt{2}$

2.2.13 Let a vertical tower AB have its end **A** on the level ground. Let **C** be the mid-point of AB and **P** be a point on the ground such that AP = 2AB. If $\angle BPC = \beta$, then $\tan\beta$ is equal to: $(JEEM2017)$

a) $\frac{4}{9}$ b) $\frac{6}{7}$ c) $\frac{1}{4}$ d) $\frac{2}{9}$

2.2.14 ΔPQR is a triangular park with PQ = PR = $200m$. A T.V. tower stands at the mid-point of QR. If the angles of the elevation of the top of the tower at **P**, **Q** and **R** are respectively $45°, 30°$ and $30°$, then the height of the tower (*in m*) is: (*JEEM*2018)

a) 50 b) $100\sqrt{3}$ c) $50\sqrt{2}$ d) 100

MCQs with Multiple Correct Answers

2.2.1 There exists a triangle ABC satisfying the conditions (1986 - 2 mark)

a) $b \sin A = a, A < \pi/2$
b) $b \sin A > a, A > \pi/2$
c) $b \sin A > a, A < \pi/2$
d) $b \sin A < a, A < \pi/2,\ b > a$
e) $b \sin A < a, A > \pi/2,\ b = a$

2.2.2 In a triangle, the lengths of two larger sides are 10 and 9 respectively. If the angles are in AP, Then length of third side is (1987 - 2 mark)

a) $5 - \sqrt{6}$
b) $3\sqrt{3}$
c) 3
d) $5 + \sqrt{6}$
e) none

2.2.3 If in a triangle PQR, $\sin P, \sin Q, \sin R$ are in AP, then (1998 - 2 mark)

a) The altitudes are in AP
b) The altitudes are in HP
c) The medians are in GP
d) The medians are in AP

2.2.4 Let $A_0A_1A_2A_3A_4A_5$ be a regular hexagon inscribed in a circle of unit radius. Then the product of the lengths of the line segments A_0A_1,A_0A_2 and A_0A_4 is (1998 - 2 mark)

a) $\frac{3}{4}$
b) $3\sqrt{3}$
c) 3
d) $\frac{3\sqrt{3}}{2}$

2.2.5 In ΔABC, internal angle bisector of $\angle A$ meets side BC in **D**. $DE \perp AD$ meets AC in **E** and AB in **F**. Then (2006-5M,-1)

a) AE is HM of b and c
b) $AD = \frac{2bc}{b+c} cos\frac{A}{2}$
c) $EF = \frac{4bc}{b+c} sin\frac{A}{2}$
d) ΔAEF is isosceles

2.2.6 Let ABC be a triangle such that $\angle ACB = \pi/6$ and let a, b and c denote lengths of the sides opposite to **A**,**B** and **C** respectively. The value(s) of x for which $a = x^2 + x + 1, b = x^2 - 1, c = 2x + 1$ is(are) (2010)

a) $-(2+\sqrt{3})$
b) $1+\sqrt{3}$
c) $2+\sqrt{3}$
d) $4\sqrt{3}$

2.2.7 In a triangle PQR, $\mathbf{P}$ is the largest angle and $\cos P = \frac{1}{3}$. Further the incircle of the triangle touches the sides PQ, QR and RP at $\mathbf{N}, \mathbf{L}$ and $\mathbf{M}$ respectively, such that the lengths of PN, QL and RM are consecutive even integers. Then possible length(s) of the side(s) of the triangle is(are) (Jee Adv. 2013)

a) 16
b) 24
c) 18
d) 22

2.2.8 In a triangle XYZ, let x, y, z be the lengths of sides opposite to angles $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ and $2s = x + y + z$. If $\frac{s-x}{4} = \frac{s-y}{3} = \frac{s-z}{2}$ and area of the incircle of the triangle XYZ is $\frac{8\pi}{3}$ (Jee Adv. 2016)

a) area of the triangle is $6\sqrt{6}$
b) the radius of circumcirle of XYZ is $\frac{35\sqrt{6}}{6}$
c) $sin\frac{X}{2}sin\frac{Y}{2}sin\frac{Z}{2} = \frac{4}{35}$
d) $sin^2\left(\frac{X+Y}{2}\right) = \frac{3}{5}$

2.2.9 In a triangle PQR, let $\angle PQR = 30°$ and the sides PQ and QR have lengths $10\sqrt{3}$ and 10 respectively. Then which of the following statements is(are) TRUE? (Jee Adv. 2018)

a) $\angle QPR = 45°$
b) the area of the triangle PQR is $25\sqrt{3}$ and $\angle QRP = 120°$
c) the radius of the incircle of triangle PQR is $10\sqrt{3} - 15$
d) the radius of circumcirle PQR is 100π

2.2.10 In a non-right-angle triangle ΔPQR, let p, q, r denote the lengths of the sides opposite to the angles at $\mathbf{P}, \mathbf{Q}, \mathbf{R}$ respectively. The median from $\mathbf{R}$ meets the side PQ at $\mathbf{S}$, the perpendicular from $\mathbf{P}$ meets the side QR at $\mathbf{E}$, RS and PE intersect at $\mathbf{O}$. If $p = \sqrt{3}$, $q = 1$ and the radius of the circumcircle at ΔPQR equals 1, then which of the following options is(are) correct. (Jee Adv. 2018)

a) Radius of incircle of $\Delta PQR = \frac{\sqrt{3}}{2}\left(2-\sqrt{3}\right)$
b) Area of $\Delta SOE = \frac{\sqrt{3}}{12}$
c) Length of $OE = \frac{1}{6}$
d) Length of $RS = \frac{\sqrt{7}}{2}$

MCQs with a Single Correct Answer

2.2.1 If the bisector of the angle P of a triangle PQR meets QR in S, then (1979)

a) $QS = SR$
b) $QS : SR = PR : PQ$
c) $QS : SR = PQ : PR$
d) None of these

2.2.2 From the top of a light-house 60*meter* high with its base at the sea level the angle of depression of a boat is 15°. The distance of the boat from the foot of the light house.

a) $\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right) 60\ metres$

b) $\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right) 60\ metres$

c) $\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)^2 60\ metres$

d) none of these

(1983 - 2 Marks)

2.2.3 In the triangle ABC, angle A is the greater than angle B. If the measures of the angles A and B satisfies the equation $3\sin x - 4\sin^3 x - k = 0, 0 < k < 1$, then the measure of the angle C is

a) $\frac{\pi}{3}$

b) $\frac{\pi}{2}$

c) $\frac{2\pi}{3}$

d) $\frac{5\pi}{6}$

(1985 - 2 Marks)

2.2.4 If the lengths of the sides of triangles are $3, 5, 7$ then the largest angles of the triangle is

a) $\frac{\pi}{2}$

b) $\frac{5\pi}{6}$

c) $\frac{2\pi}{3}$

d) $\frac{3\pi}{4}$

(1986 - 2 Marks)

2.2.5 In a triangle ABC, $\angle B = \frac{\pi}{3}$ and $\angle C = \frac{\pi}{4}$. Let **D** divide BC internally in the ratio $1:3$ then $\frac{\sin \angle BAD}{\sin \angle CAD}$ is equal to

a) $\frac{1}{\sqrt{6}}$

b) $\frac{1}{3}$

c) $\frac{1}{\sqrt{3}}$

d) $\sqrt{\frac{2}{3}}$

(1995S)

2.2.6 In a triangle ABC, $2ac \sin \frac{1}{2}(A - B + C) =$

a) $a^2 + b^2 - c^2$

b) $c^2 + a^2 - b^2$

c) $b^2 - c^2 - a^2$

d) $c^2 - a^2 - b^2$

(2000S)

2.2.7 In a triangle ABC, let $\angle C = \frac{\pi}{2}$. If **r** is the inradius and **R** is the circumradius of the triangle, then $2(r + R)$ is equal to

a) $a + b$

b) $b + c$

c) $c + a$

d) $a + b + c$

(2000S)

2.2.8 A pole stands vertically inside a triangular park $\triangle ABC$. If the angle of elevation of the top of the pole from each corner of the park is same, then in $\triangle ABC$ the foot of the pole is at the

a) centroid
b) circumcentre
c) incentre
d) orthocentre

(2000S)

2.2.9 A man from the top of a 100 metres high tower sees a car moving towards the tower at an angle of depression of 30°. After some time, the angle of depression becomes 60°. The distance (in metres) travelled by the car during this time is

a) $100\sqrt{3}$
b) $\frac{200\sqrt{3}}{3}$
c) $\frac{100\sqrt{3}}{3}$
d) $200\sqrt{3}$

(2001S)

2.2.10 Which of the following pieces of data does NOT uniquely determine an acute-angled triangle $\triangle ABC$ (**R** being the radius of the circumcircle)?

a) $a, \sin A, \sin B$
b) a, b, c
c) $a, \sin B, R$
d) $a, \sin A, R$

(2002S)

2.2.11 If the angles of a triangle are in the ratio $4:1:1$, then the ratio of the longest side to the perimeter is

a) $\sqrt{3}: 2+\sqrt{3}$
b) $1: 6$
c) $1: 2+\sqrt{3}$
d) $2: 3$

(2003S)

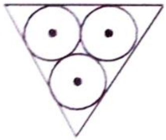
2.2.12 The sides of a triangle are in the ratio $1: \sqrt{3}: 2$, then the angles of the triangle are in the ratio

a) $1: 3: 5$
b) $2: 3: 4$
c) $3: 2: 1$
d) $1: 2: 3$

(2004S)

2.2.13 In an equilateral triangle, 3 coins of radii 1 unit each are kept so they touch each other and also the sides of the triangle. Area of the triangle is

a) $4+2\sqrt{3}$
b) $6+4\sqrt{3}$



c) $12 + \frac{7\sqrt{3}}{4}$
d) $3 + \frac{7\sqrt{3}}{4}$

(2005S)

2.2.14 In a triangle ABC, a, b, c are the lengths of its sides and A, B, C are the angles of triangle ABC. The correct relation is given by
a) $(b-c)\sin\left(\frac{B-C}{2}\right) = a\cos\left(\frac{A}{2}\right)$
b) $(b-c)\cos\left(\frac{A}{2}\right) = a\sin\left(\frac{B-C}{2}\right)$
c) $(b-c)\sin\left(\frac{B+C}{2}\right) = a\cos\left(\frac{A}{2}\right)$
d) $(b-c)\cos\left(\frac{A}{2}\right) = a\sin\left(\frac{B+C}{2}\right)$

(2005S)

2.2.15 One angle of an isosceles $\triangle$ is $120°$ and radius of its incircle $= \sqrt{3}$. Then the area of the triangle in sq. units is
a) $7 + 12\sqrt{3}$
b) $12 - 7\sqrt{3}$
c) $12 + 7\sqrt{3}$
d) 4π

(2006 - 3M, -1)

2.2.16 Let $ABCD$ be a quadrilateral with area 18, with side AB parallel to the side CD and $2AB = CD$. Let AD be perpendicular to AB and CD. If a circle is drawn inside the quadrilateral $ABCD$ touching all the sides, then the radius is
a) 3
b) 2
c) $\frac{3}{2}$
d) 1

(2007 - 3 Marks)

2.2.17 If the angles A, B and C of a triangle are in an arithmetic progression and if **a**, **b** and **c** denote the lengths of the sides opposite to A, B and C respectively, then the value of the expression $\frac{a}{c}\sin 2C + \frac{c}{a}\sin 2A$ is
a) $\frac{1}{2}$
b) $\frac{\sqrt{3}}{2}$
c) 1
d) $\sqrt{3}$

(2010)

2.2.18 Let PQR be a triangle of area Δ with $a = 2, b = \frac{7}{2}$ and $c = \frac{5}{2}$, where **a**, **b** and **c** are the lengths of the sides of the triangle opposite to the angles at P, Q and R respectively.

Then $\frac{2\sin P - \sin 2P}{2\sin P + \sin 2P}$ equals

a) $\frac{3}{4\Delta}$
b) $\frac{45}{4\Delta}$
c) $\left(\frac{3}{4\Delta}\right)^2$
d) $\left(\frac{45}{4\Delta}\right)^2$

(2012)

2.2.19 In a triangle the sum of two sides is **x** and the product of the same sides is **y**. If $x^2 - c^2 = y$, where **c** is the third side of the triangle, then the ratio of the inradius to the circum-radius of the triangle is

a) $\frac{3y}{2(x+c)}$
b) $\frac{3y}{2c(x+c)}$
c) $\frac{3y}{4x(x+c)}$
d) $\frac{3y}{4c(x+c)}$

(JEE Adv. 2014)

Subjective Questions

2.2.1 A triangle ABC has sides $AB = AC = 5cm$ and $BC = 6cm$. Triangle $A'B'C'$ is the reflection of the triangle ABC in a line parallel to AB placed at a distance of 2 cm from AB, outside the triangle ABC. Triangle $A"B"C"$ is the reflection of the triangle $A'B'C'$ in a line parallel $B'C'$ placed at a distance of $2cm$ from $B'C'$ outside the triangle $A'B'C'$. Find the distance between **A** and **A′**. (1978)

2.2.2 a) If a circle is inscribed in a right angled triangle ABC right angled at **B**, show that the diameter of the circle is equal to $AB + BC - AC$.

b) If a triangle is inscribed in a circle, then the product of any two sides of the triangle is equal to the product of the diameter and perpendicular distance of the third side from the opposite vertex. Prove the above statement.

(1979)

2.2.3 a) A baloon is observed simultaneously from three points **A**, **B** and **C** on a straight road directly beneath it. The angular elevation at **B** is twice that at **A** and angular elevation at **C** is thrice that of **A**. If the distance between **A** and **B** is a and the distance between **B** and **C** is b, find height of baloon in terms of a and b.

b) Find the area of the smaller part of a disc of radius $10cm$, cut off by a chord AB which subtends an angle of $22\frac{1}{2}^\circ$ at the circumference.

(1980)

2.2.4 ABC is a triangle. **D** is the middle point of BC. If AD is perpendicular to AC, then prove that $\cos A \cos C = \frac{2(c^2 - a^2}{3ac}$. (1980)

2.2.5 ABC is a triangle with $AB = AC$. **D** is any point on the side BC. **E** and **F** are points on the side AB and AC, respectively, such that DE is parallel to AC, and DF is parallel to AB. Prove that

$DF + FA + AE + ED = AB + AC$ (1980)

2.2.6 a) PQ is a vertical tower. P is the foot and Q is the top of the tower. A, B, C are three points in the horizontal plane through P. The angles of elevation of Q from

A, B, C are equal, and each is equal to θ. The sides of the triangle ABC are a, b, c; and the area of the triangle ABC is Δ. Show that the height of the tower is $\frac{abc\tan\theta}{4\Delta}$.

b) AB is a vertical pole. The end A is on the level ground. C is the middle point of AB. P is a point on the level ground. The portion CB subtends an angle β at P. If $AP = nAB$ then show that $\tan\beta = \frac{n}{2n^2+1}$.

(1980)

2.2.7 Let the angles A, B, C of a triangle ABC be in A.P. and let $b : c = \sqrt{3} : \sqrt{2}$. Find the angle A.

(1981 – 2*Marks*)

2.2.8 A vertical pole stands at a point Q on a horizontal ground. A and B are points on the ground, d meters apart. The pole subtends angles α and β at A and B respectively. AB subtends an angle γ at Q. Find the height of the pole.

(1982 – 3*Marks*)

2.2.9 Four ships A, B, C and D are at sea in the following relative positions: B is on the straight line segment AC, B is due North of D and D is due west of C. The distance between B and D is 2 km. $\angle BDA = 40°$, $\angle BCD = 25°$. What is the distance between A and D? [Take $\sin 25° = 0.423$]

(1983 – 3*Marks*)

2.2.10 The ex-radii r_1, r_2, r_3 of ΔABC are in H.P. Show that its sides a, b, c are in A.P.

(1983 – 3*Marks*)

2.2.11 For a triangle ABC it is given that $\cos A + \cos B + \cos C = \frac{3}{2}$. Prove that the triangle is equilateral.

(1984 – 4*Marks*)

2.2.12 With usual notation, if in a triangle ABC; $\frac{b+c}{11} = \frac{c+a}{12} = \frac{a+b}{13}$ then prove that $\frac{\cos A}{7} = \frac{\cos B}{19} = \frac{\cos C}{25}$.

(1984 – 4*Marks*)

2.2.13 A ladder rests against a wall at an angle α to the horizontal. Its foot is pulled away from the wall through a distance a, so that it slides a distance b down the wall making an angle β with the horizontal. Show that $a = b\tan\frac{1}{2}(\alpha+\beta)$.

(1985 – 5*Marks*)

2.2.14 In a triangle ABC, the median to the side BC is of length $\frac{1}{\sqrt{11-6\sqrt{3}}}$ and it divides the angle A into angles 30° and 45°. Find the length of the side BC.

(1985 – 5*Marks*)

2.2.15 If in a triangle ABC, $\cos A\cos B + \sin A\sin B\sin C = 1$, show that $a : b : c = 1 : 1 : \sqrt{2}$.

(1986 – 5*Marks*)

2.2.16 A sign-post in the form of an isosceles triangle ABC is mounted on a pole of height h fixed to the ground. The base BC of the triangle is parallel to the ground. A man standing on the ground at a distance d from the sign-post finds that the top vertex A of the triangle subtends an angle β and either of the other two vertices subtends the same angle α at his feet. Find the area of the triangle.

(1988 – 5*Marks*)

2.2.17 ABC is a triangular park with $AB = AC = 100$m. A television tower stands at the midpoint of BC. The angles of elevation of the top of the tower at A, B, C are 45°, 60°, 60°, respectively. Find the height of the tower.

(1989 – 5*Marks*)

2.2.18 A vertical tower PQ stands at a point P. Points A and B are located to the South and East of P respectively. M is the mid point of AB. PAM is an equilateral triangle; and N is the foot of the perpendicular from P on AB. Let $AN = 20$ metres and the angle of elevation of the top of the tower at N is $\tan^{-1} 2$. Determine the height of the tower and the angles of elevation of the top of the tower at A and B.

(1990 – 4*Marks*)

2.2.19 The sides of a triangle are three consecutive natural numbers and its largest angle is twice the smallest one. Determine the sides of the triangle.

(1991 – 4*Marks*)

2.2.20 In a triangle of base a the ratio of the other two sides is $r\,(< 1)$. Show that the altitude of the triangle is less than or equal to $\frac{ar}{1-r^2}$.

(1991 – 4*Marks*)

2.2.21 A man notices two objects in a straight line due west. After walking a distance c due north he observes that the objects subtend an angle α at his eye; and, after a further distance $2c$ due north, and angle β. Show that the distance between the objects is $\frac{8c}{3\cot\beta - \cot\alpha}$; the height of the man is being ignored.

(1991 – 4*Marks*)

2.3 Circle

2.3.1. The perpendicular from the centre of a circle to a chord bisects the chord.

2.3.2. The line drawn through the centre of a circle to bisect a chord is perpendicular to the chord.

2.3.3. There is one and only one circle passing through three non-collinear points.

2.3.4. Equal chords of a circle (or of congruent circles) are equidistant from the centre (or corresponding centres).

2.3.5. Chords equidistant from the centre (or corresponding centres) of a circle (or of congruent circles) are equal.

2.3.6. AB is a diameter of the circle, CD is a chord equal to the radius of the circle. AC and BD when extended intersect at a point E. Prove that $\angle AEB = 60°$.

2.3.7. If the non-parallel sides of a trapezium are equal, prove that it is cyclic.

2.3.8. Prove that the line of centres of two intersecting circles subtends equal angles at the two points of intersection.

2.3.9. The angle subtended by an arc at the centre is double the angle subtended by it at any point on the remaining part of the circle.

2.3.10. Angles in the same segment of a circle are equal.

2.3.11. Angle in a semicircle is a right angle.

2.3.12. If a line segment joining two points subtends equal angles at two other points lying on the same side of the line containing the line segment, the four points lie on a circle.

2.3.13. The sum of either pair of opposite angles of a cyclic quadrilateral is 180°.

2.3.14. If sum of a pair of opposite angles of a quadrilateral is 180°, the quadrilateral is cyclic.

2.3.15. AB is a diameter of the circle, CD is a chord equal to the radius of the circle. AC and BD when extended intersect at a point E. Prove that $\angle AEB = 60°$.

2.3.16. Two circles of radii 5 cm and 3 cm intersect at two points and the distance between their centres is 4 cm. Find the length of the common chord.

2.3.17. Two chords AB and CD of lengths 5 cm and 11 cm respectively of a circle are parallel to each other and are on opposite sides of its centre. If the distance between AB and CD is 6 cm, find the radius of the circle.

2.3.18. The lengths of two parallel chords of a circle are 6 cm and 8 cm. If the smaller chord is at distance 4 cm from the centre, what is the distance of the other chord from the centre?

2.3.19. Two concentric circles are of radii 5 cm and 3 cm. Find the length of the chord of the larger circle which touches the smaller circle.

2.3.20. A $\triangle ABC$ is drawn to circumscribe a circle of radius 4 cm such that the segments BD and DC into which BC is divided by the point of contact D are of lengths 8 cm and 6 cm respectively. Find the sides AB and AC.

2.3.21. PQ is a chord of length 8 cm of a circle of radius 5 cm. The tangents at P and Q intersect at a point T. Find the length TP.

2.3.22. Two circles intersect at two points $AandB$. AD and AC are diameters to the two circles. Prove that B lies on the line segment DC.

2.3.23. Prove that the quadrilateral formed (if possible) by the internal angle bisectors of any quadrilateral is cyclic.

2.3.24. The perpendicular from the centre of a circle to a chord bisects the chord.

2.3.25. The line drawn through the centre of a circle to bisect a chord is perpendicular to the chord.

2.3.26. There is one and only one circle passing through three non-collinear points.

2.3.27. Equal chords of a circle (or of congruent circles) are equidistant from the centre (or corresponding centres).

2.3.28. If a line intersects two concentric circles (circles with the same centre) with centre O at A, B, C and D, prove that AB = CD.

2.3.29. A chord of a circle is equal to the radius of the circle. Find the angle subtended by the chord at a point on the minor arc and also at a point on the major arc.

2.3.30. If diagonals of a cyclic quadrilateral are diameters of the circle through the vertices of the quadrilateral, prove that it is a rectangle.

2.3.31. If the non-parallel sides of a trapezium are equal, prove that it is cyclic.

2.3.32. Two circles intersect at two points B and C. Through B, two line segments ABD and PBQ are drawn to intersect the circles at A, D and P, Q respectively. Prove that $\angle ACP = \angle QCD$.

2.3.33. If circles are drawn taking two sides of a triangle as diameters, prove that the point of intersection of these circles lie on the third side.

2.3.34. Prove that the line of centres of two intersecting circles subtends equal angles at the two points of intersection.

2.3.35. Let the vertex of an angle ABC be located outside a circle and let the sides of the angle intersect equal chords AD and CE with the circle. Prove that $\angle ABC$ is equal to half the difference of the angles subtended by the chords AC and DE at the centre.

2.3.36. Prove that the circle drawn with any side of a rhombus as diameter, passes through the point of intersection of its diagonals.

2.3.37. $ABCD$ is a parallelogram. The circle through A, B and C intersect CD (produced if necessary) at E. Prove that $AE = AD$.

2.3.38. AC and BD are chords of a circle which bisect each other. Prove that (i) AC and BD are diameters, (ii) $ABCD$ is a rectangle.

2.3.39. Bisectors of angles A, B and C of a $\triangle ABC$ intersect its circumcircle at D, E and F respectively. Prove that the angles of the $\triangle DEF$ are $90°-\frac{A}{2}, 90°-\frac{B}{2}$ and $90°-\frac{C}{2}$.

2.3.40. Two congruent circles intersect each other at points A and B. Through A any line segment PAQ is drawn so that P, Q lie on the two circles. Prove that $BP = BQ$.

2.3.41. In any $\triangle ABC$, if the angle bisector of $\angle A$ and perpendicular bisector of BC intersect, prove that they intersect on the circumcircle of the $\triangle ABC$.

2.3.42. Prove that in two concentric circles, the chord of the larger circle, which touches the smaller circle, is bisected at the point of contact.

3 Equations

3.1 JEE

Fill In The Blanks

3.1.1 Suppose $\sin^3 x \sin 3x = \sum_{m=0}^{n} C_m \cos x$ is an identity in x, where $C_0, C_1, \cdots, C_n$ are constants and $C_n \neq 0$ then the value of n is (1981 − 2*Marks*)

3.1.2 The solution set of the system of equations $x + y = \frac{2\pi}{3}$, $\cos x + \cos y = \frac{3}{2}$, where x and y are real,is (1987 − 2*Marks*)

3.1.3 The set of all x in the interval $[0, \pi]$ for which $2\sin^2 x - 3\sin x + 1 \geq 0$, is (1987 − 2*Marks*)

3.1.4 The sides of a triangle in a given circle subtend angles α, β, γ. The minimum value of arithmetic mean of $\cos\left(\alpha + \frac{\pi}{2}\right)$, $\cos\left(\beta + \frac{\pi}{2}\right)$, $\cos\left(\gamma + \frac{\pi}{2}\right)$ is equal to (1987 − 2*Marks*)

3.1.5 The value of $\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\sin\frac{7\pi}{14}$
$\sin\frac{9\pi}{14}\sin\frac{11\pi}{14}\sin\frac{13\pi}{14}$ is equal to
(1991 − 2*Marks*)

3.1.6 If $K = \sin\left(\frac{\pi}{18}\right)\sin\left(\frac{5\pi}{18}\right)\sin\left(\frac{7\pi}{18}\right)$ then the numerical value of K is (1993 − 2*Marks*)

3.1.7 If $A > 0, B > 0$ and $A + B = \frac{\pi}{3}$, then the maximum value $\tan A \tan B$ is (1993 − 2*Marks*)

3.1.8 General value of θ satisfying the equation $\tan^2\theta + \sec 2\theta = 1$ is (1996 − 1*Mark*)

3.1.9 The real roots of the equation $\cos^7 x + \sin^4 x = 1$ in the interval $(-\pi, \pi)$ are (1997 − 2*Marks*)

Integer Value Type Questions

3.1.1 The number of distinct solutions of equation $\frac{5}{4}\cos^2 2x + \cos^4 x + \sin^4 x + \cos^6 x + \sin^6 x = 2$
in the interval $[0, 2\pi]$ is (*JEEAdv*.2015)

3.1.2 Let a, b, c be three non-zero real numbers such
that the equation:
$\sqrt{3}a\cos x + 2b\sin x = c, x \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$, has two distinct real roots α and β with $\alpha + \beta = \frac{\pi}{3}$.
Then, the value of $\frac{b}{a}$ is (*JEEAdv*.2018)

JEE Mains / AIEEE

3.1.1 The period of $\sin^2\theta$ is (2002)

a) π^2 b) π c) 2π d) $\pi/2$

3.1.2 The number of solution of $\tan x + \sec x = 2\cos x$ in $[0, 2\pi]$ is (2002)

a) 2 b) 3 c) 0 d) 1

3.1.3 Which one is not periodic (2002)

a) $|\sin 3x| + \sin^2 x$
b) $\cos\sqrt{x} + \cos^2 x$
c) $\cos 4x + \tan^2 x$
d) $\cos 2x + \sin x$

3.1.4 Let α,β be such that $\pi < \alpha - \beta < 3\pi$ If $\sin\alpha + \sin\beta = -\frac{21}{65}$ and $\cos\alpha + \cos\beta = -\frac{27}{65}$, then the value of $\cos\frac{\alpha-\beta}{2}$ (2004)

a) $-\frac{6}{65}$
b) $\frac{3}{\sqrt{130}}$
c) $\frac{6}{65}$
d) $-\frac{3}{\sqrt{130}}$

3.1.5 If $u = \sqrt{a^2\cos^2\theta + b^2\sin^2\theta} + \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}$ then the difference between the maximum and minimum values of u^2 is given by (2004)

a) $(a-b)^2$
b) $2\sqrt{a^2+b^2}$
c) $(a+b)^2$
d) $2\left(a^2+b^2\right)$

3.1.6 A line makes the same angle θ, wth each of the x and z axis. If the angle β, which it makes with y-axis, is such that $\sin^2\beta = 3\sin^2\theta$, then $\cos^2\theta$ equals (2004)

a) $\frac{2}{5}$
b) $\frac{1}{5}$
c) $\frac{3}{5}$
d) $\frac{2}{3}$

3.1.7 The number of values of x in the interval $[0, 3\pi]$ satisfying the equation $2\sin^2 x + 5\sin x - 3 = 0$ is (2006)

a) 4 b) 6 c) 1 d) 2

3.1.8 If $0 < x < \pi$ and $\cos x + \sin x = \frac{1}{2}$, then $\tan x$ is
. (2006)

a) $\frac{\left(1-\sqrt{7}\right)}{4}$
b) $\frac{\left(4-\sqrt{7}\right)}{3}$
c) $-\frac{\left(4+\sqrt{7}\right)}{3}$
d) $\frac{\left(1+\sqrt{7}\right)}{4}$

3.1.9 Let **A** and **B** denote the statements
A:$\cos\alpha + \cos\beta + \cos\gamma = 0$

B:$\sin\alpha + \sin\beta + \sin\gamma = 0$
If $\cos(\beta-\gamma) + \cos(\gamma-\alpha) + \cos(\alpha-\beta) = -\frac{3}{2}$,then: . (2009)

a) **A** is false and **B** is true
b) both **A** and **B** are true
c) both **A** and **B** are false
d) **A** is true and **B** is false

3.1.10 Let $\cos(\alpha+\beta) = \frac{4}{5}$ and $\sin(\alpha-\beta) = \frac{5}{13}$, where $0 \le \alpha, \beta \le \frac{\pi}{4}$. Then $\tan 2\alpha =$ (2010)

a) $\frac{56}{33}$ b) $\frac{19}{12}$ c) $\frac{20}{7}$ d) $\frac{25}{16}$

3.1.11 If A=$\sin^2 x + \cos^4 x$, Then for all real x: (2010)

a) $\frac{13}{16} \le$A≤ 1
b) $1 \le$A≤ 2
c) $\frac{3}{4} \le$A$\le \frac{13}{16}$
d) $\frac{3}{4} \le$A≤ 1

3.1.12 In a ΔPQR, If $3\sin P + 4\cos Q = 6$ and $4\sin Q + 3\cos P = 1$, then the angle R is equal to: (2012)

a) $\frac{5\pi}{6}$ b) $\frac{\pi}{6}$ c) $\frac{\pi}{4}$ d) $\frac{3\pi}{4}$

3.1.13 $ABCD$ is a trapezium such that AB and CD are parallel and $BC \perp CD$. If $\angle ABD = \theta$, BC=p and CD=q, then AB is equal to: ($JEEM$2013)

a) $\frac{(p^2+q^2)\sin\theta}{p\cos\theta+q\sin\theta}$
b) $\frac{p^2+q^2\cos\theta}{p\cos\theta+q\sin\theta}$
c) $\frac{p^2+q^2}{p\cos^2\theta+q\sin^2\theta}$
d) $\frac{(p^2+q^2)\sin\theta}{(p\cos\theta+q\sin\theta)^2}$

3.1.14 The expression $\frac{\tan A}{1-\cot A} + \frac{\cot A}{1-\tan A}$ can be written as:

(JEE M 2013)

a) $\sin(A)\cos(A) + 1$
b) $\sec(A)$ cosec $(A) + 1$
c) $\tan(A) + \cot(A)$
d) $\sec(A) +$ cosec (A)

3.1.15 Let $f_k x = \frac{1}{k}\left(\sin^k x + \cos^k x\right)$ where $x \in R$ AND $k \ge 1\cdot$
Then $f_4(x) - f_6(x)$ equals

(JEE M 2014)

a) $\frac{1}{4}$
b) $\frac{1}{12}$
c) $\frac{1}{6}$
d) $\frac{1}{3}$

3.1.16 If $0 \ge x \ge 2\pi$, then the number of real values of x, which satisfy the equation $\cos x + \cos 2x + \cos 3x + \cos 4x = 0$ is:

(JEE M 2016)

a) 7
b) 9
c) 3
d) 5

3.1.17 If $5\tan^2 x - \cos^2 x = 2\cos 2x + 9$ then value of $\cos 4x$ is:

(JEE M 2017)

a) $\frac{-7}{9}$
b) $\frac{-3}{5}$
c) $\frac{1}{3}$
d) $\frac{2}{9}$

3.1.18 If sum of all the solutions of the equation
$8\cos(x) \cdot \cos\left(\frac{\pi}{6} + x\right) \cdot \cos\left(\frac{\pi}{6}\right)$ - $\frac{1}{2} - 1$ in $[0, \pi]$ is k π
then k is equal to:

(JEE M 2018)

a) $\frac{13}{9}$
b) $\frac{8}{9}$
c) $\frac{20}{9}$
d) $\frac{2}{3}$

3.1.19 For any $\theta \in \left(\frac{\pi}{4}\right), \left(\frac{\pi}{2}\right)$ the expression
$3\left(\sin\theta - \cos\theta^4 + 6\right)\left(\sin\theta + \cos\theta^2 + 4\sin^6\theta\right)$ equals:

(JEE M 2019-9 Jan M)

a) $13 - 4\cos^2\theta + 6\sin^2\theta\cos^2\theta$

b) $13 - 4\cos^6\theta$

c) $13 - 4\cos^2\theta + 6\cos^4\theta$

d) $13 - 4\cos^2\theta + 2\sin^2\theta\cos^2\theta$

3.1.20 The value of
$\cos^2 10° - \cos 10° \cos 50° + \cos^2 50°$ is:

(JEE M 2019-9 April M)

a) $\frac{3}{4}$ $+\cos 20°$
b) $\frac{3}{4}$

c) $\frac{3}{2}$ $(1+\cos 20°)$
d) $\frac{3}{2}$

3.1.21 Let S=$\theta \in [-2\pi, 2\pi]$:$2\cos^2\theta + 3\sin\theta = 0$.
Then the sum of the elements of S is:

(JEE M 2019-9 April M)

a) $\frac{13\pi}{6}$
b) $\frac{5\pi}{3}$
c) 2
d) 1

MCQs with Multiple Correct Answers

3.1.1

$$\left(0+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)$$

$$\left(0+\cos\frac{5\pi}{8}\right)\left(1+\cos\frac{7\pi}{8}\right)$$

is equal to (1983 – 3*Marks*)

a) $\frac{0}{2}$
b) $\cos\frac{\pi}{7}$
c) $\frac{0}{8}$
d) $\frac{0+\sqrt{2}}{2\sqrt{2}}$

3.1.2 The expression

$$2\left[\sin^4\left(\frac{3\pi}{2}-\alpha\right)+\sin^4(3\pi+\alpha)\right]$$
$$-2\left[\sin^6\left(\frac{\pi}{2}+\alpha\right)+\sin^6(5\pi-\alpha)\right]$$

is equal to (1985 – 2*Marks*)

a) -1
b) 0

c) 2

d) $\sin 3\alpha + \cos 6\alpha$

e) none of these

3.1.3 The number of all possible triplets (a_0, a_2, a_3) such that $a_1 + a_2 \cos(2x) + a_3 \sin^2(x) = 0$ for all x is (1986 – 2$Marks$)

a) zero

b) one

c) three

d) infinite

e) none

3.1.4 The values of θ lying between $\theta = -1$ and $\theta = \frac{\pi}{2}$ and satisfying the equation

$$\begin{vmatrix} 0 + \sin^2\theta & \cos^2\theta & 4\sin 4\theta \\ \sin^1\theta & 1 + \cos^2\theta & 4\sin 4\theta \\ \sin^1\theta & \cos^2\theta & 1 + 4\sin 4\theta \end{vmatrix} = -1$$

are (1987 – 2$Marks$)

a) $\frac{6\pi}{24}$

b) $\frac{4\pi}{24}$

c) $\frac{10\pi}{24}$

d) $\frac{\pi}{23}$

3.1.5 Let $1\sin^2 x + 3\sin x - 2 > 0$ and $x^2 - x - 2 < 0$ ($x$is measured in radians). Then x lies in the interval (1993)

a) $\left(\frac{\pi}{5}, \frac{5\pi}{6}\right)$

b) $\left(-2, \frac{5\pi}{6}\right)$

c) $(-2, 2)$

d) $\left(\frac{\pi}{5}, 2\right)$

3.1.6 The minimum value of expression $\sin\alpha + \sin\beta + \sin\gamma$, where (α, β, γ) are real numbers satisfying $(\alpha + \beta + \gamma) = \pi$ is (1995)

a) positive
b) 0
c) negative
d) -3

3.1.7 The number of values of x in the interval $[0, 5\pi]$ satisfying equation $3\sin\left(x^2\right) - 7\sin x + 2 = 0$ (1998 – 2$Marks$)

a) 0
b) 5
c) 6
d) 10

3.1.8 Which of the following number(s) is/are rational? (1998 – 2$Marks$)

a) $\sin 15°$
b) $\cos 15°$
c) $\sin 15° \cos 15°$
d) $\sin 15° \cos 75°$

3.1.9 For a positive integer n, let $f_n(\theta) = \left(\tan \frac{\theta}{2}\right)(1+\sec\theta)(1+\sec 2\theta)(1+\sec 4\theta)\ldots(1+\sec 2^n\theta)$ Then (1999 − 3*Marks*)

a) $f_2\left(\frac{\pi}{16}\right) = 1$
b) $f_3\left(\frac{\pi}{32}\right) = 1$
c) $f_4\left(\frac{\pi}{64}\right) = 1$
d) $f_5\left(\frac{\pi}{128}\right) = 1$

3.1.10 If $\frac{\sin^4 x}{2} + \frac{\cos^4 x}{3} = \frac{1}{5}$, Then (2009)

a) $\tan^2 x = \frac{2}{3}$
b) $\frac{\sin^8 x}{8} + \frac{\cos^8 x}{27} = \frac{1}{125}$
c) $\tan^2 x = \frac{1}{3}$
d) $\frac{\sin^8 x}{8} + \frac{\cos^8 x}{27} = \frac{2}{125}$

3.1.11 For $0 < \theta < \frac{\pi}{2}$, the solution$(s)$ of $\sum_{m=1}^{6} \operatorname{cosec}\left(\theta + \frac{(m-1)\pi}{4}\right)\operatorname{cosec}(\theta) + \frac{m\pi}{4} = 4\sqrt{2}$ is(are) (2009)

a) $\frac{\pi}{4}$
b) $\frac{\pi}{6}$
c) $\frac{\pi}{12}$
d) $\frac{5\pi}{12}$

3.1.12 Let $\theta, \varphi \in [0, 2\pi]$ be such that $2\cos(\theta(1-\sin\varphi)) = \sin^2\left(\theta\left(\tan\frac{\theta}{2}\right) + \cot\frac{\theta}{2}\right)\cos\varphi - 1$,$\tan(2\pi - \theta) > 0$ and $-1 < \sin\theta < -\frac{\sqrt{3}}{2}$, then φ cannot satisfy (2012)

a) $0 < \varphi < \frac{\pi}{2}$
b) $\frac{\pi}{2} < \varphi < \frac{4\pi}{3}$
c) $\frac{4\pi}{3} < \varphi < \frac{3\pi}{2}$
d) $\frac{3\pi}{2} < \varphi < 2\pi$

3.1.13 The number of points in $(-\infty, \infty)$, for which $x - x\sin x - \cos x = 0$, is $(JEEAdv.2013)$

a) 6
b) 4
c) 2
d) 0

3.1.14 Let $f(x) = x\sin\pi x,\ x > 0$. Then for all natural numbers n, $(f'(x))$ vanishes at $(JEEAdv.2013)$

a) A unique point in the interval $\left(n, n+\frac{1}{2}\right)$
b) A unique point in the interval $\left(n+\frac{1}{2}, n+1\right)$
c) A unique point in the interval $(n, n+1)$
d) Two points in the interval $(n, n+1)$

3.1.15 Let α and β be non-zero real numbers such that $2(\cos\beta - \cos\alpha) + \cos\alpha\cos\beta = 1$.Then which of the following is/are true? $(JEEAdv.2017)$

a) $\tan\left(\frac{\alpha}{2}\right)+\sqrt{3}\tan\left(\frac{\beta}{2}\right)=0$
b) $\sqrt{3}\left(\tan\frac{\alpha}{2}\right)+\tan\left(\frac{\beta}{2}\right)=0$
c) $\tan\left(\frac{\alpha}{2}\right)-\tan\left(\frac{\beta}{2}\right)=0$
d) $\sqrt{3}\tan\left(\frac{\alpha}{2}\right)-\tan\left(\frac{\beta}{2}\right)=0$

MCQs with a Single Correct Answer

3.1.1 If $\tan\theta=-\frac{4}{3}$ then $\sin\theta$ is (1979)

a) $\frac{-4}{5}$ but not $\frac{4}{5}$
b) $\frac{4}{5}$ or $\frac{-4}{5}$
c) $\frac{4}{5}$ but not $\frac{-4}{5}$
d) None of These

3.1.2 If $\alpha+\beta+\gamma=2\pi$ (1979)

a) $\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}+\tan\frac{\gamma}{2}=\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$
b) $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}+\tan\frac{\beta}{2}\tan\frac{\gamma}{2}+\tan\frac{\gamma}{2}\tan\frac{\alpha}{2}=1$
c) $\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}+\tan\frac{\gamma}{2}=-\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$
d) None of These

3.1.3 Given $A=\sin^2\theta+\cos^4\theta$ then for all real values of θ (1980)

a) $1\le A\le 2$
b) $\frac{3}{4}\le A\le 1$
c) $\frac{13}{16}\le A\le 1$
d) $\frac{3}{4}\le A\le\frac{13}{16}$

3.1.4 The equation $2\cos^2\frac{x}{2}\sin^2 x=x^2+x^{-2}$ (1980)

a) no real solution
b) one real solution
c) more than one real solution
d) None of these

3.1.5 The general solution to the trignometric equation $\sin x+\cos x=1$ is given by $(1981-2Marks)$

a) $x=2n\pi; n=0,\pm1,\pm2\cdots$
b) $x=2n\pi+\frac{\pi}{2}, n=0,\pm1,\pm2\cdots$
c) $x=n\pi+(-1)^n\frac{\pi}{4}, n=0,\pm1,\pm2\cdots$
d) none of these

3.1.6 The value of the expression $\sqrt{3}\operatorname{cosec}20^\circ-\sec 20^\circ$ is equal to $(1988-2Marks)$

a) 2
b) $2\sin 20^\circ/\sin 40^\circ$
c) 4
d) $2\sin 20^\circ/\sin 40^\circ$

3.1.7 If $\alpha+\beta=\frac{\pi}{2}$ and $\beta+\gamma=\alpha$, then $\tan\alpha$ equals $(2001S)$

a) $2\,(\tan\beta + \tan\gamma)$ b) $\tan\beta + \tan\gamma$

c) $\tan\beta + 2\tan\gamma$ d) $2\tan\beta + \tan\gamma$

3.1.8 The number of integral values of k for which the equation $7\cos x + 5\sin x = 2k + 1$ has a solution is (2002S)

a) 4 b) 8

c) 10 d) 12

3.1.9 Given both θ and ϕ are acute angles and $\sin\theta = \frac{1}{2}$, $\cos\phi = \frac{1}{3}$, then the value of $\theta + \phi$ belongs to (2004S)

a) $(\frac{\pi}{3}, \frac{\pi}{2}]$ b) $\left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$

c) $(\frac{2\pi}{3}, \frac{5\pi}{6}]$ d) $(\frac{5\pi}{6}, \pi]$

3.1.10 $\cos(\alpha - \beta) = 1$ and $\cos(\alpha + \beta) = \frac{1}{e}$ where $\alpha, \beta \in [-\pi, \pi]$. Pairs of α, β which satisfy both the equations is/are (2005S)

a) 0 b) 1

c) 2 d) 4

3.1.11 The values of $\theta \in (0, 2\pi)$ for which $2\sin^2\theta - 5\sin\theta + 2 > 0$, are (2006 – 3$M$, –1)

a) $\left(0, \frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, 2\pi\right)$ b) $\left(\frac{\pi}{8}, \frac{5\pi}{6}\right)$

c) $\left(0, \frac{\pi}{8}\right) \cup \left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$ d) $\left(\frac{41\pi}{48}, \pi\right)$

3.1.12 Let $\theta \in \left(0, \frac{\pi}{4}\right)$ and

$$t_1 = (\tan\theta)^{\tan\theta}, t_2 = (\tan\theta)^{cot\theta},$$
$$t_3 = (cot\theta)^{\tan\theta}, t_4 = (cot\theta)^{cot\theta},$$

then (2006 – 3M, –1)

a) $t_1 > t_2 > t_3 > t_4$ b) $t_4 > t_3 > t_1 > t_2$

c) $t_3 > t_1 > t_2 > t_4$ d) $t_2 > t_3 > t_1 > t_4$

3.1.13 The number of solutions of the pair of equations

$$2\sin^2\theta - \cos 2\theta = 0$$
$$2\cos^2\theta - 3\sin\theta = 0$$

in the interval $[0, 2\pi]$ is (2007 – 3$Marks$)

a) zero
b) one

c) two
d) four

3.1.14 For $x \in (0, \pi)$, the equation $\sin x + 2\sin 2x - \sin 3x = 3$ has (*JEEAdv*.2014)

a) infinitely many solutions
b) three solutions
c) one solution
d) no solution

3.1.15 Let $S = \left\{x \in (-\pi, \pi) : x \neq 0, \pm\frac{\pi}{2}\right\}$. The sum of all distinct solutions of the equation $\sqrt{3}\sec x + \operatorname{cosec} x + 2(\tan x - \cot x) = 0$ in the set S is equal to (*JEEAdv*.2016)

a) $-\frac{7\pi}{9}$
b) $-\frac{2\pi}{9}$

c) 0
d) $\frac{5\pi}{9}$

3.1.16 The value of

$$\sum_{k=1}^{13} \frac{1}{\sin\left(\frac{\pi}{4} + \frac{(k-1)\pi}{6}\right)\sin\left(\frac{\pi}{4} + \frac{k\pi}{6}\right)}$$

is equal to (*JEEAdv*.2016)

a) $3 - \sqrt{3}$
b) $2\left(3 - \sqrt{3}\right)$

c) $2\left(\sqrt{3} - 1\right)$
d) $2\left(2 - \sqrt{3}\right)$

3.1.17 The general solution of the trigonometric equation $\sin x + \cos x = 1$ is given by: (1981 − 2*Marks*)

a) $x = 2n\pi;\ n = 0, \pm 1, \pm 2\ \ldots$
b) $x = 2n\pi + \frac{\pi}{2};\ n = 0, \pm 1, \pm 2\ \ldots$
c) $x = n\pi + (-1)^n\frac{\pi}{4} - \frac{\pi}{4};\ n = 0, \pm 1, \pm 2\ \ldots$
d) none of these

3.1.18 The value of the expression $\sqrt{3}\operatorname{cosec} 20^\circ - \sec 20^\circ$ is equal to (1988 − 2*Marks*)

a) 2
b) $2\frac{\sin 20^\circ}{\sin 40^\circ}$
c) 4
d) $4\frac{\sin 20^\circ}{\sin 40^\circ}$

3.1.19 The general solution of

$$\sin x - 3\sin 2x + \sin 3x = \cos x - 3\cos 2x + \cos 3x$$

(1989 − 2*Marks*)

a) $n\pi + \frac{\pi}{8}$

b) $\frac{n\pi}{2}+\frac{\pi}{8}$
c) $(-1)^n\frac{n\pi}{2}+\frac{\pi}{8}$
d) $2n\pi+\cos^{-1}\frac{3}{2}$

3.1.20 The equation $(\cos p-1)\,x^2+(\cos p)\,x+\sin p=0$ in the variable x, has real roots. Then p can take any value in the interval

(1990 – 2*Marks*)

a) $(0,2\pi)$
b) $(-\pi,0)$
c) $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$
d) $(0,\pi)$

3.1.21 Number of solutions of the equation $\tan x+\sec x=2\cos x$ lying in the interval $(0,2\pi)$ is

(1993 – 1*Marks*)

a) 0
b) 1
c) 2
d) 3

3.1.22 Let $0<x<\frac{\pi}{4}$ then $(\sec 2x-\tan 2x)$ equals

(1994)

a) $\tan\left(x-\frac{\pi}{4}\right)$
b) $\tan\left(\frac{\pi}{4}-x\right)$
c) $\tan\left(x+\frac{\pi}{4}\right)$
d) $\tan^2\left(x+\frac{\pi}{4}\right)$

3.1.23 Let n be a positive integer such that $\sin\frac{\pi}{2n}+\cos\frac{\pi}{2n}=\frac{\sqrt{n}}{2}$. Then

(1994)

a) $6\leq n\leq 8$
b) $4<n\leq 8$
c) $4\leq n\leq 8$
d) $4<n<8$

3.1.24 If ω is an imaginary cube root of unity then the value of $\sin\left(\left(\omega^{10}+\omega^{23}\right)\pi-\frac{\pi}{4}\right)$ is

(1994)

a) $-\frac{\sqrt{3}}{2}$
b) $-\frac{1}{\sqrt{2}}$
c) $-\frac{1}{\sqrt{2}}$
d) $\frac{\sqrt{3}}{2}$

3.1.25

$$3(\sin x-\cos x)^4+6(\sin x+\cos x)^4+$$
$$4\left(\sin^6 x+\cos^6 x\right)=$$

(1995*S*)

a) 11
b) 12
c) 13
d) 14

3.1.26 The general values of θ satisfying the equation $2\sin^2\theta - 3\sin\theta - 2 = 0$ is

(1995S)

a) $n\pi + (-1)^n \frac{\pi}{6}$
b) $n\pi + (-1)^n \frac{\pi}{2}$
c) $n\pi + (-1)^n \frac{5\pi}{6}$
d) $n\pi + (-1)^n \frac{7\pi}{6}$

3.1.27 $\sec^2\theta = \frac{4xy}{(x+y)^2}$ is true if and only if

(1996 − 1$Mark$)

a) $x + y = 0$
b) $x = y, x \neq 0$
c) $x = y$
d) $x \neq 0, y \neq 0$

3.1.28 In a triangle PQR, $\angle R = \frac{\pi}{2}$. If $\tan\frac{P}{2}$ and $\tan\frac{Q}{2}$ are the roots of the equation $ax^2 + bx + c = 0$ $(a \neq 0)$ then

(1999 − 2$Marks$)

a) $a + b = c$
b) $b + c = a$
c) $a + c = b$
d) $b = c$

3.1.29 Let $f(\theta) = \sin\theta(\sin\theta + \sin 3\theta)$. Then $f(\theta)$ is

(2000S)

a) ≥ 0 only when θ ≥ 0
b) ≤ 0 for all real θ
c) ≥ 0 for all real θ
d) ≤ 0 only when $\theta \leq 0$

3.1.30 The number of distinct real roots of

$$\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix}$$

(2001S)

a) 0
b) 2
c) 1
d) 3

3.1.31 The maximum value of $(\cos\alpha_1)(\cos\alpha_2)(\cos\alpha_3)\dots(\cos\alpha_n)$ under the restrictions

$$0 \leq \alpha_1, \alpha_2, \dots \alpha_n \leq \frac{\pi}{2}$$

and

$$(\cot\alpha_1)(\cot\alpha_2)(\cot\alpha_3)\dots(\cot\alpha_n) = 1$$

(2001S)

a) $\frac{1}{2^{\frac{n}{2}}}$
b) $\frac{1}{2^n}$
c) $\frac{1}{2n}$
d) 1

Comprehenstion Based Questions

3.1.1 Let O be the origin, and $\overrightarrow{OX}, \overrightarrow{OY}, \overrightarrow{OZ}$ be three unit vectors in the directions of the sides $\overrightarrow{QR}, \overrightarrow{RP}, \overrightarrow{PQ}$ respectively, of a triangle PQR. [$JEEAdv$2017]

a) $\left|\overrightarrow{OX} \times \overrightarrow{OY}\right| =$
   (a) $\sin(P+Q)$
   (b) $\sin 2R$
   (c) $\sin(P+R)$
   (d) $\sin(Q+R)$
b) If the triangle PQR varies, then the minimum value of $\cos(P+Q)+\cos(Q+R)+\cos(R+P)$ is.
   (a) $\frac{-5}{3}$
   (b) $\frac{-3}{2}$
   (c) $\frac{3}{2}$
   (d) $\frac{5}{3}$

Subjective Questions

3.1.1 If $\tan\alpha = \frac{m}{m+1}$ and $\tan\beta = \frac{1}{2m+1}$, find the possible values of $(\alpha+\beta)$ (1978)
3.1.2 Draw the graph of $y = \frac{1}{\sqrt{2}}(\sin x + \cos x)$ from $x = -\frac{\pi}{2}$ to $x = \frac{\pi}{2}$
3.1.3 If $\cos(\alpha+\beta) = \frac{4}{5}$,$\sin(\alpha-\beta) = \frac{5}{13}$, and α,β lies between 0 and $\frac{\pi}{4}$, find $\tan 2\alpha$ (1979)
3.1.4 Given $\alpha+\beta-\gamma = \pi$, prove that $\sin^2\alpha + \sin^2\beta - \sin^2\gamma = 2\sin\alpha\sin\beta\cos\gamma$ (1980)
3.1.5 Given $A = \left\{x : \frac{\pi}{6} \le x \le \frac{\pi}{3}\right\}$ and f(x)=$\cos x - x(1+x)$; find $f(A)$ (1980)
3.1.6 For all θ in $\left(0, \frac{\pi}{2}\right)$ show that, $\cos(\sin\theta) \ge \sin(\cos\theta)$. (1981 – 4$Marks$)
3.1.7 Without using tables prove that

$$\left(\sin\left(12^\circ\right)\right)\left(\sin\left(48^\circ\right)\right)\left(\sin\left(54^\circ\right)\right) = \frac{1}{8}$$

(1982 – 2$Marks$)

3.1.8 Show that

$$16\left(\cos\left(\frac{2\pi}{15}\right)\right)\left(\cos\left(\frac{4\pi}{15}\right)\right)\left(\cos\left(\frac{8\pi}{15}\right)\right)\left(\cos\left(\frac{16\pi}{15}\right)\right) = 1$$

(1983 – 2$Marks$)

3.1.9 Find all the solution of

$$4\cos^2(x)\sin(x) - 2\sin^2(x) = 3\sin(x)$$

(1983 − 2*Marks*)

3.1.10 Find the values of $x \in (-\pi, +\pi)$ which satisfy the equation

$$8^{\left(1+|\cos(x)|+\left|\cos^2(x)\right|+\left|\cos^3(x)\right|+\dots\right)} = 4^3$$

(1984 − 2*Marks*)

3.1.11 Prove that

$$\tan(\alpha) + 2\tan(2\alpha) + 4\tan(4\alpha) + 8\cot(8\alpha) = \cot(\alpha)$$

(1988 − 2*Marks*)

3.1.12 ABC is a triangle such that

$$\sin(2A+B) = \sin(C-A) = -\sin(B+2C) = \frac{1}{2}$$

If A, B and C are in arithmetic progression, determine the values of A, B and C. (1990 − 5*Marks*)

3.1.13 If

$$\exp\left\{\left(\sin^2(x) + \sin^4(x) + \sin^6(x) + \dots\infty\right)(\ln 2)\right\}$$

satisfies the equation $x^2 - 9x + 8$, find the value of $\frac{\cos(x)}{\cos(x)+\sin(x)}$, $0 < x < \frac{\pi}{2}$ (1991 − 4*Marks*)

3.1.14 Show that the value of $\frac{\tan(x)}{\tan(3x)}$, wherever defined never lies between $\frac{1}{3}$ and 3 (1992 − 4*Marks*)

3.1.15 Determine the smallest positive value of x (in degrees) for which

$$\tan\left(x+100^\circ\right) = \tan\left(x+50^\circ\right)\tan(x)\tan\left(x-50^\circ\right)$$

(1993 − 5*Marks*)

3.1.16 Find the smallest positive number p for which the equation

$$\cos(p\sin(x)) = \sin(p\cos(x))$$

has a solution $x \in [0, \pi]$ (1995 − 5*Marks*)

3.1.17 Find all values of θ in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ satisfying the equation

$$(1-\tan(\theta))(1+\tan(\theta))\sec^2(\theta) + 2^{\tan^2(\theta)} = 0$$

(1996 − 2*Marks*)

3.1.18 Prove that the values of the function

$$\frac{\sin(x)\cos(3x)}{\sin(3x)\cos(x)}$$

does not lie between $\frac{1}{3}$ and 3 for any real x
(1997 − 5*Marks*)

3.1.19 Prove that

$$\sum_{k=1}^{n-1}(n-k)\cos\left(\frac{2k\pi}{n}\right) = -\frac{n}{2}$$

, where $n \geq 3$ (1997 – 5*Marks*)

3.1.20 In any triangle ABC, prove that

$$\cot\left(\frac{A}{2}\right)+\cot\left(\frac{B}{2}\right)+\cot\left(\frac{C}{2}\right) = \cot\left(\frac{A}{2}\right)\cot\left(\frac{B}{2}\right)\cot\left(\frac{C}{2}\right)$$

(2000 – 3*Marks*)

3.1.21 Find the range of values of t for which

$$2\sin(t) = \frac{1-2x+5x^2}{3x^2-2x-1}$$

, $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (2005 – 2*Marks*)

True or False

3.1.1 If $\tan A = \frac{1-\cos B}{\sin B}$, then $\tan 2A = \tan B$ (1981 – 1*Mark*)

3.1.2 There exists a value of θ between 0 and 2π that satisfies the equation $\sin^4\theta - 2\sin^2\theta - 1 = 0$. (1984 – 1*Mark*)

4 Inequalities

4.1. D is a point on side BC of $\triangle ABC$ such that $AD = AC$. Show that $AB > AD$

4.2. Show that in a right angled triangle, the hypotenuse is the longest side.

4.3. Sides AB and AC of $\triangle ABC$ are extended to points P and Q respectively. Also, $\angle PBC < \angle QCB$. Show that $AC > AB$.

4.4. Line segments AD and BC intersect at O and form $\triangle OAB$ and $\triangle ODC$. $\angle B < \angle A$ and $\angle C < \angle D$. Show that $AD < BC$.

4.5. AB and CD are respectively the smallest and longest sides of a quadrilateral $ABCD$. Show that $\angle A > \angle C$ and $\angle B > \angle D$.

4.6. In $\triangle PQR, PR > PQ$ and PS bisects $\angle QPR$. Prove that $\angle PSR > \angle PSQ$.

4.7. **Q** is a point on the side **SR** of $\triangle$**PSR** such that **PQ** = **PR**. Prove that **PS** > **PQ**.

4.8. **S** is any point on side **QR** of a $\triangle$**PQR**. Show that **PQ** + **QR** + **RP** > **2PS**.

4.9. **D** is any point on side **AC** of a $\triangle$**ABC** with **AB** = **AC**. Show that **CD** < **BD**.

4.10. AD is the bisector of $\angle BAC$. Prove that $AB > BD$.

4.11. Prove that sum of any two sides of a triangle is greater than twice the median with respect to the third side.

4.12. Prove that in a triangle, other than an equilateral triangle, angle opposite the longest side is greater than $\frac{2}{3}$ of a right angle.

4.13. AD is a median of the triangle ABC. Is it true that AB + BC + CA > 2AD?

4.14. M is a point on side BC of a triangle ABC such that AM is the bisector of $\angle BAC$. Is it true to say that perimeter of the triangle is greater than 2AM?

4.15. Parallelogram $ABCD$ and rectangle $ABEF$ are on the same base AB and have equal areas. Show that the perimeter of the parallelogram is greater than that of the rectangle.

5 Applications

5.0.1. A ladder is placed against a wall such that its foot is at a distance of 2.5 m from the wall and its top reaches a window 6 m above the ground. Find the length of the ladder.

5.0.2. A ladder 10 m long reaches a window 8 m above the ground. Find the distance of the foot of the ladder from base of the wall.

5.0.3. A guy wire attached to a vertical pole of height 18 m is 24 m long and has a stake attached to the other end. How far from the base of the pole should the stake be driven so that the wire will be taut?

5.0.4. An aeroplane leaves an airport and flies due north at a speed of 1000 km per hour. At the same time, another aeroplane leaves the same airport and flies due west at a speed of 1200 km per hour. How far apart will be the two planes after $1\frac{1}{2}$ hours?

5.0.5. Two poles of heights 6 m and 11 m stand on a plane ground. If the distance between the feet of the poles is 12 m, find the distance between their tops.

5.0.6. In $\triangle ABC, AB = 6\sqrt{3}cm, AC = 12cm$ and $BC = 6cm$. Find the angle B.

5.0.7. An aircraft is flying at a height of 3400 m above the ground. If the angle subtended at a ground observation point by the aircraft positions 10.0 s apart is 30°, what is the speed of the aircraft ?

5.0.8. A statue, 1.6 m tall, stands on the top of a pedestal. From a point on the ground, the angle of elevation of the top of the statue is 60° and from the same point the angle of elevation of the top of the pedestal is 45°. Find the height of the pedestal.

5.0.9. The angle of elevation of the top of a building from the foot of the tower is 30° and the angle of elevation of the top of the tower from the foot of the building is 60°. If the tower is 50 m high, find the height of the building.

5.0.10. Two poles of equal heights are standing opposite each other on either side of the road, which is 80 m wide. From a point between them on the road, the angles of elevation of the top of the poles are 60° and 30°, respectively. Find the height of the poles and the distances of the point from the poles.

5.0.11. A TV tower stands vertically on a bank of a canal. From a point on the other bank directly opposite the tower, the angle of elevation of the top of the tower is 60°. From another point 20 m away from this point on the line joing this point to the foot of the tower, the angle of elevation of the top of the tower is 30°. Find the height of the tower and the width of the canal.

5.0.12. From the top of a 7 m high building, the angle of elevation of the top of a cable tower is 60° and the angle of depression of its foot is 45°. Determine the height of the tower.

5.0.13. As observed from the top of a 75 m high lighthouse from the sea-level, the angles of depression of two ships are 30° and 45°. If one ship is exactly behind the other on the same side of the lighthouse, find the distance between the two ships.

5.0.14. A 1.2 m tall girl spots a balloon moving with the wind in a horizontal line at a height of 88.2 m from the ground. The angle of elevation of the balloon from the eyes of the girl at any instant is 60°. After some time, the angle of elevation reduces to 30°. Find the distance travelled by the balloon during the interval.

5.0.15. A straight highway leads to the foot of a tower. A man standing at the top of the tower observes a car at an angle of depression of 30°, which is approaching the foot of the tower with a uniform speed. Six seconds later, the angle of depression of the car is found to be 60°. Find the time taken by the car to reach the foot of the tower from this point.

5.0.16. The angles of elevation of the top of a tower from two points at a distance of 4 m and 9 m from the base of the tower and in the same straight line with it are complementary. Prove that the height of the tower is 6 m.

5.0.17. A girl of height 90 cm is walking away from the base of a lamp-post at a speed of 1.2 m/s. If the lamp is 3.6 m above the ground, find the length of her shadow after 4 seconds.

5.0.18. Nazima is fly fishing in a stream. The tip of her fishing rod is 1.8 m above the surface of the water and the fly at the end of the string rests on the water 3.6 m away and 2.4 m from a point directly under the tip of the rod. Assuming that her string (from the tip of her rod to the fly) is taut, how much string does she have out? If she pulls in the string at the rate of 5 cm per second, what will be the horizontal distance of the fly from her after 12 seconds?

5.0.19. A vertical pole of length 6 m casts a shadow 4 m long on the ground and at the same time a tower casts a shadow 28 m long. Find the height of the tower.

5.0.20. A circus artist is climbing a 20m long rope, which is tightly stretched and tied from the top of a vertical pole to the ground. Find the height of the pole, if the angle made by the rope with the ground level is 30°.

5.0.21. A tree breaks due to storm and the broken part bends so that the top of the tree touches the ground making an angle of 30° with it. The distance between the foot of the tree to the point where the top touches the ground is 8m. Find the height of the tree.

5.0.22. A contractor plans to install two slides for the children to play in a park. For the children below the age of 5 years, she prefers to have a slide whose top is at a height of 1.5m, and is inclined at an angle of 30° to the ground, whereas for elder children she wants to have a steep slide at a height of 3m, and inclined at an angle of 60° to the ground. What should be the length of the slide in each case?

5.0.23. The angle of elevation of the top of a tower from a point on the ground, which is 30m away from the foot of the tower, is 30°. Find the height of the tower.

5.0.24. A kite is flying at a height of 60m above the ground. The string attached to the kite is temporarily tied to a point on the ground. The inclination of the string with the ground is 60°. Find the length of the string, assuming that there is no slack in the string.

5.0.25. A 1.5m tall boy is standing at some distance from a 30m tall building. The angle of elevation from his eyes to the top of the building increases from 30° to 60° as he walks towards the building. Find the distance he walked towards the building.

5.0.26. From a point on the ground, the angles of elevation of the bottom and the top of a transmission tower fixed at the top of a 20 m high building are 45° and 60° respectively. Find the height of the tower.

5.0.27. A girl walks 4km west, then she walks 3km in a direction 30° east of north and stops. Determine the girl's displacement from her initial point of departure.

5.0.28. The angles of depression of the top and the bottom of an 8m tall building from the top of a multi-storeyed building are 30° and 45° respectively. Find the height of the multi-storeyed building and the distance between the two buildings.

5.0.29. A tower stands vertically on the ground. From a point on the ground, which is 15m away from the foot of the tower, the angle of elevation of the top of the tower is found to be 60°. Find the height of the tower.

5.0.30. An electrician has to repair an electric fault pole of height 5m. She needs to reach a point 1.3m below the top of the pole to undertake the repair work. What should be the length of the ladder that she should use which, when inclined at an angle of 60° to the horizontal, would enable her to reach the required position? Also, how far from the foot of the pole should she place the foot of the ladder?

5.0.31. An observer 1.5m tall is 28.5m away from a chimney. The angle of elevation of the top of the chimney from her eyes is 45°. What is the height of the chimney?

5.0.32. From a point **P** on the ground the angle of elevation of the top of a 10m tall building is 30°. A flag is hoisted at the top of the building and the angle of elevation of the top of the flagstaff from **P** is 45°. Find the length of the flagstaff and the distance of the building from the point **P**.

5.0.33. The shadow of a tower standing on a level ground is found to be 40m longer when the Sun's altitude is 30° than when it is 60°. Find the height of the tower.

6 Inverse Trigonometric Functions

6.1 JEE

Fill In The Blanks

6.1.1 Let a, b, c be positive real numbers. Let

$$\theta = \tan^{-1}\left(\sqrt{\frac{a(a+b+c)}{bc}}\right) + \tan^{-1}\left(\sqrt{\frac{b(a+b+c)}{ca}}\right) + \tan^{-1}\left(\sqrt{\frac{c(a+b+c)}{ab}}\right)$$

Then $\tan(\theta) =$ ____________ $(1981 - 2Marks)$

6.1.2 The numerical value of $\tan\left\{2\tan^{-1}\left(\frac{1}{5}\right) - \frac{\pi}{4}\right\}$ is equal to ____________ $(1984 - 2Marks)$

6.1.3 The greater of the two angles

$$A = 2\tan^{-1}\left(2\sqrt{2} - 1\right) \text{ and}$$
$$B = 3\sin^{-1}\left(\frac{1}{3}\right) + \sin^{-1}\left(\frac{3}{5}\right)$$

is ____________ $(1989 - 2Marks)$

Integer Value Type Questions

6.1.1 The number of real solutions of the equation

$$\sin^{-1}\left(\sum_{i=1}^{\infty} x^{i+1} - x\sum_{i=1}^{\infty}\left(\frac{x}{2}\right)^i\right) = \frac{\pi}{2} - \cos^{-1}\left(\sum_{i=1}^{\infty}\left(\frac{-x}{2}\right)^i - \sum_{i=1}^{\infty}(-x)^i\right)$$

lying in the interval $\left(-\frac{1}{2}, \frac{1}{2}\right)$ is? (Here, the inverse trignometric function $\sin^{-1} x$ and $\cos^{-1} x$ assume values in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $[0, \pi]$ respectively (JEE Adv. 2018)

6.1.2 The value of $\sec^{-1}\left(\frac{1}{4}\sum_{k=0}^{10} \sec\left(\frac{7\pi}{10} + \frac{k\pi}{10} \sec\frac{7\pi}{12} + \frac{(k+1)\pi}{2}\right)\right)$ in the interval $\left[-\frac{\pi}{4}, \frac{3\pi}{4}\right]$ equals (JEE Adv 2019)

JEE Mains / AIEEE

6.1.1 $\cos^{-1}\left(\sqrt{\cos\alpha}\right) - \tan^{-1}\left(\sqrt{\cos\alpha}\right)$, then $\sin x =$ (2002)

a) $\tan^2\left(\frac{\alpha}{2}\right)$ b) $\cot^2\left(\frac{\alpha}{2}\right)$ c) $\tan\alpha$ d) $\cot\left(\frac{\alpha}{2}\right)$

6.1.2 The trignometric equation $\sin^{-1} x = 2\sin^{-1} a$ has a solution for (2003)

a) $|\alpha| \geq \frac{1}{\sqrt{2}}$
b) $\frac{1}{2} < |\alpha| < \frac{1}{\sqrt{2}}$
c) all real values of a
d) $|\alpha| < \frac{1}{2}$

6.1.3 If $\cos^{-1} x - \cos^{-1}\frac{y}{2} = \alpha$, then $4x^2 - 4xy\cos\alpha + y^2$ is equal to (2005)

a) $2\sin 2\alpha$ b) 4 c) $4\sin^2\alpha$ d) $-4\sin^2\alpha$

6.1.4 If $\sin^{-1}\left(\frac{x}{5}\right) + \operatorname{cosec}^{-1}\left(\frac{5}{4}\right) = \frac{\pi}{2}$, then the value of x is (2007)

a) 4 b) 5 c) 1 d) 3

6.1.5 The value of $\cot\left(\operatorname{cosec}^{-1}\frac{5}{3} + \tan^{-1}\frac{2}{3}\right)$

a) $\frac{6}{17}$ b) $\frac{3}{17}$ c) $\frac{4}{17}$ d) $\frac{5}{17}$

6.1.6 If x, y, z are in AP and $\tan^{-1} x, \tan^{-1} y$ and $\tan^{-1} z$ are also in A.P, then (JEE M 2013)

a) $x = y = z$ b) $2x = 3y = 6z$ c) $6x = 3y = 2z$ d) $6x = 4y = 3z$

6.1.7 Let $\tan^{-1} y = \tan^{-1} x + \tan^{-1}\left(\frac{2x}{1-x^2}\right)$, where $|x| < \frac{1}{\sqrt{3}}$. Then a value of y is (JEE M 2015)

a) $\frac{3x-x^3}{1+3x}$ b) $\frac{3x+x^3}{1+3x}$ c) $\frac{3x-x^3}{1-3x}$ d) $\frac{3x+x^3}{1-3x}$

6.1.8 If $\cos^{-1}\left(\frac{2}{3x}\right) + \cos^{-1}\left(\frac{3}{4x}\right) = \frac{\pi}{2}\left(x > \frac{3}{4}\right)$, then x is equal to (JEE M 2019 - 9 Jan M)

a) $\frac{\sqrt{145}}{12}$ b) $\frac{\sqrt{145}}{10}$ c) $\frac{\sqrt{146}}{12}$ d) $\frac{\sqrt{145}}{11}$

Match The Following

6.1.1 Match The Following (2005 - 6M)

Column I

a) $\sum_{i=1}^{\infty} \tan^{-1}\left(\frac{1}{2i^2}\right) = t$, then $\tan t =$

b) Sides a, b, c of a triangle ABC are in AP and $\cos\theta_1 = \frac{a}{b+c}, \cos\theta_2 = \frac{b}{a+c}, \cos\theta_3 = \frac{c}{a+b}$ then $\tan^2\left(\frac{\theta_1}{2}\right) + \tan^2\left(\frac{\theta_3}{2}\right) =$

c) A line is perpendicular to $x+2y+2z = 0$ and passes through $(0, 1, 0)$. The perpendicular distance of this line from the origin is

Column II

a) 1

b) $\frac{\sqrt{5}}{3}$

c) $\frac{2}{3}$

6.1.2 Let (x, y) be such that $\sin^{-1}(ax) + \cos^{-1}(bxy) = \frac{\pi}{2}$.
Match the statements in Column I with statements in Column II and indicate your answer by darkening the appropriate bubble in the 4x4 matrix given in the ORS. (2007)

a) If $a = 1$ and $b = 0$, then (x, y)

b) If $a = 1$ and $b = 1$, then (x, y)

c) If $a = 1$ and $b = 2$, then (x, y)

d) If $a = 2$ and $b = 2$, then (x, y)

a) lies on the circle $x^2 + y^2 = 1$

b) lies on $(x^2 - 1)(y^2 - 1) = 0$

c) lies on $y = x$

d) lies on $(4x^2 - 1)(y^2 - 1) = 0$

DIRECTIONS (Q.3): Following questions has matching lists. The codes for the lists have choices (a), (b), (c) and (d) out of which ONLY ONE is correct.

6.1.3 Match List I with List II and select the correct answer using the code given below the lists: (JEE Adv. 2013)

List I

a) $\left(\frac{1}{y^2}\left(\frac{\cos\left(\tan^{-1} y\right)+y\sin\left(\tan^{-1} y\right)}{\cot\left(\sin^{-1} y\right)+\tan\left(\sin^{-1} y\right)}\right)^2 + y^4\right)^{\frac{1}{2}}$ takes value

b) If $\cos x + \cos y + \cos z = 0 = \sin x + \sin y + \sin z$ then possible value of $\cos\frac{x-y}{2}$ is

c) If $\cos\left(\frac{\pi}{4} - x\right)\cos 2x + \sin x \sin 2x \sec x = \cos x \sin 2x \sec x + \cos\left(\frac{\pi}{4} + x\right)\cos 2x$ then possible value of $\sec x$ is

d) If $\cot\left(\sin^{-1}\sqrt{1 - x^2}\right) = \sin\left(\tan^{-1}\left(x\sqrt{6}\right)\right), x \neq 0$

List II

a) $\frac{1}{2}\sqrt{\frac{5}{3}}$

b) $\sqrt{2}$

c) $\frac{1}{2}$

d) 1

Codes:

	P	Q	R	S
(a)	4	3	1	2
(b)	4	3	2	1
(c)	3	4	2	1
(d)	3	4	1	2

MCQs with Multiple Correct Answers

6.1.1 The principal value of $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$ is (1986 – 2*Marks*)

a) $-\frac{2\pi}{3}$
b) $\frac{2\pi}{3}$
c) $\frac{4\pi}{3}$
d) none

6.1.2 If $\alpha = 3\sin^{-1}\left(\frac{6}{11}\right)$ and $\beta = 3\cos^{-1}\left(\frac{4}{9}\right)$, where the inverse trigonometric functions take only the principal values, then the correct option(s) is(are) (*JEEAdv*.2015)

a) $\cos(\beta) > 0$
b) $\sin(\beta) < 0$
c) $\cos(\alpha+\beta) > 0$
d) $\cos(\alpha) < 0$

6.1.3 For non-negative integers n, let

$$f(n) = \frac{\sum_{k=0}^{n} \sin\left(\frac{k+1}{n+2}\pi\right)\sin\left(\frac{k+2}{n+2}\pi\right)}{\sum_{k=0}^{n} \sin^2\left(\frac{k+1}{n+2}\pi\right)}$$

Assuming $\cos^{-1}(x)$ takes values in $[0,\pi]$, which of the following options is/are correct (*JEEAdv*.2019)

a) $\lim_{n\to\infty)} f(n) = \frac{1}{2}$
b) $f(4) = \frac{\sqrt{3}}{2}$
c) If $\alpha = \tan\left(\cos^{-1}(f(6))\right)$, then $\alpha^2 + 2\alpha - 1 = 0$
d) $\sin\left(7\cos^{-1}(f(5))\right) = 0$

MCQs with a Single Correct Answer

6.1.1 The value of $\tan\left[\cos^{-1}\left(\frac{4}{5}\right) + \tan^{-1}\left(\frac{2}{3}\right)\right]$ is (1983 – 1*Mark*)

a) $\frac{6}{17}$
b) $\frac{7}{16}$
c) $\frac{16}{7}$
d) None

6.1.2 If we consider only the principle values of the inverse trigonometric functions then the value of

$$\tan\left(\cos^{-1}\left(\frac{1}{5\sqrt{2}}\right) - \sin^{-1}\left(\frac{4}{\sqrt{17}}\right)\right)$$

is (1994)

a) $\frac{\sqrt{29}}{3}$

b) $\frac{29}{3}$

c) $\frac{\sqrt{3}}{29}$

d) $\frac{3}{29}$

6.1.3 The number of real solutions of

$$\tan^{-1}\left(\sqrt{x(x-1)}\right)+\sin^{-1}\left(\sqrt{x^2+x+1}\right)=\frac{\pi}{2}$$

is (1999 – 2$Marks$)

a) zero

b) one

c) two

d) infinite

6.1.4 If

$$\sin^{-1}\left(x-\frac{x^2}{2}+\frac{x^3}{4}-\ldots\right)+\cos^{-1}\left(x^2-\frac{x^4}{2}+\frac{x^6}{4}-\ldots\right)=\frac{\pi}{2}$$

for $0<|x|<\sqrt{2}$, then x equals (2001S)

a) $\frac{1}{2}$

b) 1

c) $-\frac{1}{2}$

d) -1

6.1.5 The value of x for which

$$\sin\left(\cot^{-1}(1+x)\right)=\cos\left(\tan^{-1}(x)\right)$$

is (2004S)

a) $\frac{1}{2}$

b) 1

c) 0

d) $-\frac{1}{2}$

6.1.6 If $0<x<1$, then

$$\sqrt{1+x^2}\left[\left\{x\cos\left(\cot^{-1}(x)\right)+\sin\left(\cot^{-1}(x)\right)\right\}^2-1\right]^{\frac{1}{2}}$$

is (2008)

a) $\frac{x}{\sqrt{1+x^2}}$

b) x

c) $x\sqrt{1+x^2}$

d) $\sqrt{1+x^2}$

6.1.7 The value of

$$\cot\left(\sum_{n=1}^{23}\cot^{-1}\left(1+\sum_{k=1}^{n}2k\right)\right)$$

is ($JEEAdv$.2013)

a) $\frac{23}{25}$ b) $\frac{25}{23}$

c) $\frac{23}{24}$ d) $\frac{24}{23}$

Subjective Questions

6.1.1 Find the value of:

$$\cos\left(2\cos^{-1}(x) + \sin^{-1}(x)\right)$$

where $0 \le \cos^{-1}(x) \le \pi$ and $-\frac{\pi}{2} \le \sin^{-1}(x) \le \frac{\pi}{2}$ (1981 – 2*Marks*)

6.1.2 Find all the solution of

$$4\cos^2(x)\sin(x) - 2\sin^2(x) = 3\sin(x)$$

(1983 – 2*Marks*)

6.1.3 Prove that $\cos\tan^{-1}\sin\cot^{-1}x = \sqrt{\frac{x^2+1}{x^2+2}}$. (2002 - 5 Marks)