

# Trigonometry through Geometry

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## ABOUT THIS BOOK

This book introduces trigonometry through high school geometry. This approach relies more on trigonometric equations than cumbersome constructions which are usually non intuitive. All problems in the book are from NCERT mathematics textbooks from Class 9-12. Exercises are from CBSE, JEE and Olympiad exam papers.

The content is sufficient for all practical applications of trigonometry. There is no copyright, so readers are free to print and share.

This book is dedicated to my Hindi teacher in school, Shri Mandavi.

February 7, 2025

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## 1 GEOMETRY

## 1.1 Right Angled Triangle

1.1.1. A right angled triangle looks like Fig. 1.1.1. with angles  $\angle A$ ,  $\angle B$  and  $\angle C$  and sides

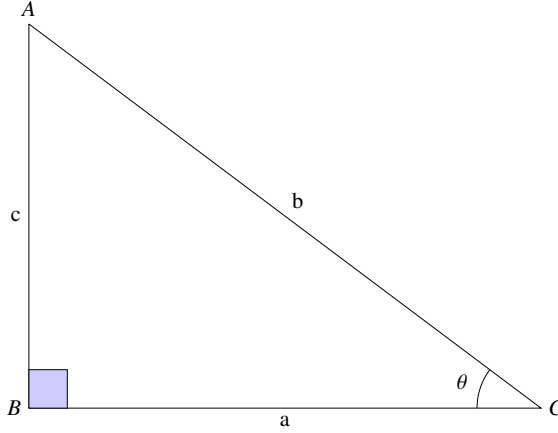


Fig. 1.1.1: Right Angled Triangle

$a, b$  and  $c$ . The unique feature of this triangle is  $\angle B$  which is defined to be  $90^\circ$ .

1.1.2. For simplicity, let the greek letter  $\theta = \angle C$ . We have the following definitions.

$$\begin{aligned} \sin \theta &= \frac{c}{b} & \cos \theta &= \frac{a}{b} \\ \tan \theta &= \frac{c}{a} & \cot \theta &= \frac{1}{\tan \theta} \\ \csc \theta &= \frac{1}{\sin \theta} & \sec \theta &= \frac{1}{\cos \theta} \end{aligned} \quad (1.1.2.1)$$

1.1.3.

$$\cos \theta = \sin (90^\circ - \theta) \quad (1.1.3.1)$$

1.1.4. In Fig. 1.1.2, show that

$$b = a \cos \theta + c \sin \theta \quad (1.1.4.1)$$

**Solution:** We observe that

$$CD = a \cos \theta \quad (1.1.4.2)$$

$$AD = c \cos \alpha = c \sin \theta \quad (\text{From } (1.1.3.1)) \quad (1.1.4.3)$$

Thus,

$$CD + AD = b = a \cos \theta + c \sin \theta \quad (1.1.4.4)$$

1.1.5. From (1.1.4.1), show that

$$\sin^2 \theta + \cos^2 \theta = 1 \quad (1.1.5.1)$$



Fig. 1.1.2: Baudhayana Theorem

**Solution:** Dividing both sides of (1.1.4.1) by  $b$ ,

$$1 = \frac{a}{b} \cos \theta + \frac{c}{b} \sin \theta \quad (1.1.5.2)$$

$$\Rightarrow \sin^2 \theta + \cos^2 \theta = 1 \quad (\text{from } (1.1.2.1)) \quad (1.1.5.3)$$

1.1.6. From (1.1.5.1)

$$|\sin \theta| \leq 1, \quad |\cos \theta| \leq 1 \quad (1.1.6.1)$$

1.1.7. Using (1.1.4.1), show that

$$b^2 = a^2 + c^2 \quad (1.1.7.1)$$

(1.1.7.1) is known as the Baudhayana theorem. It is also known as the Pythagoras theorem.

**Solution:** From (1.1.4.1),

$$b = a \frac{a}{b} + c \frac{c}{b} \quad (\text{from } (1.1.2.1)) \quad (1.1.7.2)$$

$$\Rightarrow b^2 = a^2 + c^2 \quad (1.1.7.3)$$

1.1.8. In a right angled triangle, the hypotenuse is the longest side.

**Solution:** From (1.1.7.1),

$$a \leq b, \quad c \leq b. \quad (1.1.8.1)$$

1.1.9.  $ABC$  is an isosceles triangle in which altitudes  $BE$  and  $CF$  are drawn to equal sides  $AC$  and  $AB$  respectively. Show that these altitudes are equal.

**Solution:** In  $\triangle BFC$  and  $BEC$ ,

$$BF = a \sin C, \quad CE = a \sin B \quad (1.1.9.1)$$

$$\implies BF = CE, \because B = C. \quad (1.1.9.2)$$

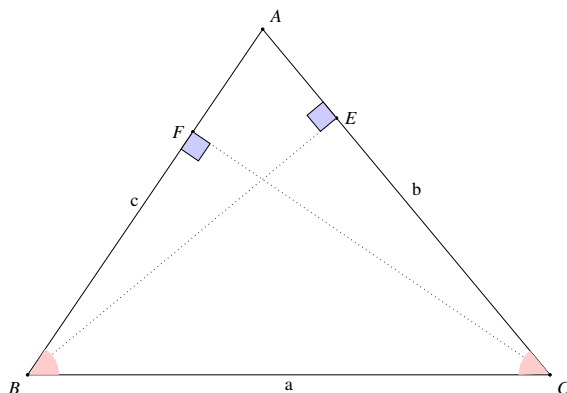


Fig. 1.1.3:  $B = C$

1.1.10.  $ABC$  is a triangle in which altitudes  $BE$  and  $CF$  to sides  $AC$  and  $AB$  are equal. Show that  $AB = AC$ .

**Solution:** In (1.1.9.1),

$$BE = CF \implies a \sin C = a \sin B \quad (1.1.10.1)$$

$$\text{or, } B = C \quad (1.1.10.2)$$

## 1.2 Sine and Cosine Formula

1.2.1. Show that the area of  $\triangle ABC$  in Fig. 1.2.1 is  $\frac{1}{2}ab \sin C$ .



Fig. 1.2.1: Area of a Triangle

**Solution:** We have

$$ar(\Delta ABC) = \frac{1}{2}ah = \frac{1}{2}ab \sin C \quad (\because h = b \sin C). \quad (1.2.1.1)$$

1.2.2. Show that

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \quad (1.2.2.1)$$

**Solution:** Fig. 1.2.1 can be suitably modified to obtain

$$ar(\Delta ABC) = \frac{1}{2}ab \sin C = \frac{1}{2}bc \sin A = \frac{1}{2}ca \sin B \quad (1.2.2.2)$$

Dividing the above by  $abc$ , we obtain

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \quad (1.2.2.3)$$

This is known as the sine formula.

1.2.3. In Fig. 1.2.2,  $AB = AC$ . Show that

$$\angle B = \angle C \quad (1.2.3.1)$$



Fig. 1.2.2

**Solution:** Using the sine formula,

$$\frac{AB}{\sin C} = \frac{AC}{\sin B} \quad (1.2.3.2)$$

$$\Rightarrow \sin B = \sin C \text{ or, } \angle B = \angle C. \quad (1.2.3.3)$$

1.2.4. In Fig. 1.2.3, show that

$$\begin{pmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{pmatrix} \begin{pmatrix} \cos A \\ \cos B \\ \cos C \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad (1.2.4.1)$$

**Solution:** From Fig. 1.2.3,



Fig. 1.2.3: The cosine formula

$$a = x + y = b \cos C + c \cos B = \begin{pmatrix} \cos C & \cos B \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix} \quad (1.2.4.2)$$

$$= \begin{pmatrix} 0 & b & c \end{pmatrix} \begin{pmatrix} \cos A \\ \cos C \\ \cos B \end{pmatrix} \quad (1.2.4.3)$$

Similarly,

$$b = c \cos A + a \cos C = \begin{pmatrix} c & 0 & a \end{pmatrix} \begin{pmatrix} \cos A \\ \cos C \\ \cos B \end{pmatrix} \quad (1.2.4.4)$$

$$c = b \cos A + a \cos B = \begin{pmatrix} b & a & 0 \end{pmatrix} \begin{pmatrix} \cos A \\ \cos C \\ \cos B \end{pmatrix} \quad (1.2.4.5)$$

The above equations can be expressed in matrix form as (1.2.4.1).

1.2.5. Show that

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \quad (1.2.5.1)$$

**Solution:** Using the properties of determinants,

$$\cos A = \frac{\begin{vmatrix} a & c & b \\ b & 0 & a \\ c & a & 0 \end{vmatrix}}{\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}} = \frac{ab^2 + ac^2 - a^3}{abc + abc} = \frac{b^2 + c^2 - a^2}{2abc} \quad (1.2.5.2)$$

1.2.6. Find Hero's formula for the area of a triangle.



**Solution:** From (1.2.1), the area of  $\triangle ABC$  is

$$\frac{1}{2}ab \sin C = \frac{1}{2}ab \sqrt{1 - \cos^2 C} \quad (\text{from (1.1.5.1)}) \quad (1.2.6.1)$$

$$= \frac{1}{2}ab \sqrt{1 - \left( \frac{a^2 + b^2 - c^2}{2ab} \right)^2} \quad (\text{from (1.2.5.1)}) \quad (1.2.6.2)$$

$$= \frac{1}{4} \sqrt{(2ab)^2 - (a^2 + b^2 - c^2)^2} \quad (1.2.6.3)$$

$$= \frac{1}{4} \sqrt{(2ab + a^2 + b^2 - c^2)(2ab - a^2 - b^2 + c^2)} \quad (1.2.6.4)$$

$$= \frac{1}{4} \sqrt{\{(a+b)^2 - c^2\} \{c^2 - (a-b)^2\}} \quad (1.2.6.5)$$

$$= \frac{1}{4} \sqrt{(a+b+c)(a+b-c)(a+c-b)(b+c-a)} \quad (1.2.6.6)$$

Substituting

$$s = \frac{a+b+c}{2} \quad (1.2.6.7)$$

in (1.2.6.6), the area of  $\triangle ABC$  is

$$\sqrt{s(s-a)(s-b)(s-c)} \quad (1.2.6.8)$$

This is known as Hero's formula.

1.2.7. Show that

$$\alpha > \beta \implies \sin \alpha > \sin \beta \quad (1.2.7.1)$$

**Solution:** In Fig. 1.2.4,

$$ar(\triangle ABD) < ar(\triangle ABC) \quad (1.2.7.2)$$

$$\implies \frac{1}{2}lc \sin \theta_1 < \frac{1}{2}ac \sin(\theta_1 + \theta_2) \quad (1.2.7.3)$$

$$\implies \frac{l}{a} < \frac{\sin(\theta_1 + \theta_2)}{\sin \theta_1} \quad (1.2.7.4)$$

$$\text{or, } 1 < \frac{l}{a} < \frac{\sin(\theta_1 + \theta_2)}{\sin \theta_1} \quad (1.2.7.5)$$

from Theorem 1.1.8, yielding

$$\implies \frac{\sin(\theta_1 + \theta_2)}{\sin \theta_1} > 1. \quad (1.2.7.6)$$

This proves (1.2.7.1).

### 1.3 Trigonometric Identities

1.3.1. Using Fig. 1.2.4, show that

$$\sin \theta_1 = \sin(\theta_1 + \theta_2) \cos \theta_2 - \cos(\theta_1 + \theta_2) \sin \theta_2 \quad (1.3.1.1)$$



Fig. 1.2.4

**Solution:** The following equations can be obtained from the figure using the formula for the area of a triangle

$$ar(\triangle ABC) = \frac{1}{2}ac \sin(\theta_1 + \theta_2) \quad (1.3.1.2)$$

$$= ar(\triangle BDC) + ar(\triangle ADB) \quad (1.3.1.3)$$

$$= \frac{1}{2}cl \sin \theta_1 + \frac{1}{2}al \sin \theta_2 \quad (1.3.1.4)$$

$$= \frac{1}{2}ac \sin \theta_1 \sec \theta_2 + \frac{1}{2}a^2 \tan \theta_2 \quad (1.3.1.5)$$

( $\because l = a \sec \theta_2$ ). From the above,

$$\sin(\theta_1 + \theta_2) = \sin \theta_1 \sec \theta_2 + \frac{a}{c} \tan \theta_2 \quad (1.3.1.6)$$

$$= \sin \theta_1 \sec \theta_2 + \cos(\theta_1 + \theta_2) \tan \theta_2 \quad (1.3.1.7)$$

Multiplying both sides by  $\cos \theta_2$ ,

$$\sin(\theta_1 + \theta_2) \cos \theta_2 = \sin \theta_1 + \cos(\theta_1 + \theta_2) \sin \theta_2 \quad (1.3.1.8)$$

resulting in (1.3.1.1).

1.3.2. Prove the following identities

a)

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta. \quad (1.3.2.1)$$

b)

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta. \quad (1.3.2.2)$$

**Solution:** In (1.3.1.1), let

$$\begin{aligned} \theta_1 + \theta_2 &= \alpha \\ \theta_2 &= \beta \end{aligned} \quad (1.3.2.3)$$

This gives (1.3.2.1). In (1.3.2.1), replace  $\alpha$  by  $90^\circ - \alpha$ . This results in

$$\sin(90^\circ - \alpha - \beta) = \sin(90^\circ - \alpha) \cos \beta - \cos(90^\circ - \alpha) \sin \beta \quad (1.3.2.4)$$

$$\implies \cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta \quad (1.3.2.5)$$

1.3.3. Using (1.3.1.1) and (1.3.2.2), show that

$$\sin(\theta_1 + \theta_2) = \sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2 \quad (1.3.3.1)$$

$$\cos(\theta_1 - \theta_2) = \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \quad (1.3.3.2)$$

**Solution:** From (1.3.1.1),

$$\sin(\theta_1 + \theta_2) \cos \theta_2 = \sin \theta_1 + \cos(\theta_1 + \theta_2) \sin \theta_2 \quad (1.3.3.3)$$

Using (1.3.2.2) in the above,

$$\sin(\theta_1 + \theta_2) \cos \theta_2 = \sin \theta_1 + (\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) \sin \theta_2 \quad (1.3.3.4)$$

which can be expressed as

$$\sin(\theta_1 + \theta_2) \cos \theta_2 = \sin \theta_1 + \cos \theta_1 \cos \theta_2 \sin \theta_2 - \sin \theta_1 \sin^2 \theta_2 \quad (1.3.3.5)$$

Since

$$\sin^2 \theta_2 = 1 - \cos^2 \theta_2, \quad (1.3.3.6)$$

we obtain

$$\sin(\theta_1 + \theta_2) \cos \theta_2 = \cos \theta_1 \cos \theta_2 \sin \theta_2 + \sin \theta_1 \cos^2 \theta_2 \quad (1.3.3.7)$$

resulting in

$$\sin(\theta_1 + \theta_2) = \cos \theta_1 \sin \theta_2 + \sin \theta_1 \cos \theta_2 \quad (1.3.3.8)$$

after factoring out  $\cos \theta_2$ . Using a similar approach, (1.3.3.2) can also be proved.

1.3.4. Show that

$$\sin \theta_1 + \sin \theta_2 = 2 \sin\left(\frac{\theta_1 + \theta_2}{2}\right) \cos\left(\frac{\theta_1 - \theta_2}{2}\right) \quad (1.3.4.1)$$

$$\cos \theta_1 + \cos \theta_2 = 2 \cos\left(\frac{\theta_1 + \theta_2}{2}\right) \cos\left(\frac{\theta_1 - \theta_2}{2}\right) \quad (1.3.4.2)$$

$$\sin \theta_1 - \sin \theta_2 = 2 \sin\left(\frac{\theta_1 - \theta_2}{2}\right) \cos\left(\frac{\theta_1 + \theta_2}{2}\right) \quad (1.3.4.3)$$

$$\cos \theta_1 - \cos \theta_2 = 2 \sin\left(\frac{\theta_1 + \theta_2}{2}\right) \cos\left(\frac{\theta_2 - \theta_1}{2}\right) \quad (1.3.4.4)$$

**Solution:** Let

$$\begin{aligned}\theta_1 &= \alpha + \beta \\ \theta_2 &= \alpha - \beta\end{aligned}\tag{1.3.4.5}$$

From (1.3.3.1),

$$\sin \theta_1 + \sin \theta_2 = \sin (\alpha + \beta) + \sin (\alpha - \beta) \tag{1.3.4.6}$$

$$= \sin \alpha \cos \beta + \cos \alpha \sin \beta + \sin \alpha \cos \beta - \cos \alpha \sin \beta \tag{1.3.4.7}$$

$$= 2 \sin \alpha \cos \beta \tag{1.3.4.8}$$

resulting in (1.3.4.1)

$$\therefore \alpha = \frac{\theta_1 + \theta_2}{2}, \beta = \frac{\theta_1 - \theta_2}{2} \tag{1.3.4.9}$$

from (1.3.4.5). Other identities may be proved similarly.

1.3.5. Show that

$$\sin 2\theta = 2 \sin \theta \cos \theta \tag{1.3.5.1}$$

$$\cos 2\theta = 1 - 2 \sin^2 \theta = 2 \cos^2 \theta - 1 \tag{1.3.5.2}$$

$$= \cos^2 \theta - \sin^2 \theta \tag{1.3.5.3}$$

#### 1.4 Incircle

1.4.1. In Fig. 1.4.1, the bisectors of  $\angle B$  and  $\angle C$  meet at **I**. Show that  $IA$  bisects  $\angle A$ .



Fig. 1.4.1: Incentre  $I$  of  $\triangle ABC$

**Solution:** Using sine formula in (1.2.2.3)

$$\frac{l_1}{\sin \frac{C}{2}} = \frac{l_3}{\sin (A - \theta)}, \quad \frac{l_3}{\sin \frac{B}{2}} = \frac{l_2}{\sin \frac{C}{2}}, \quad \frac{l_2}{\sin \theta} = \frac{l_1}{\sin \frac{B}{2}} \tag{1.4.1.1}$$

Multiplying the above equations,

$$\sin \theta = \sin (A - \theta) \quad (1.4.1.2)$$

$$\implies \theta = A - \theta \text{ or, } \theta = \frac{A}{2} \quad (1.4.1.3)$$

1.4.2. Fig. 1.4.2, is obtained from Fig. 1.4.1 with

$$ID \perp BC, IE \perp AC, IF \perp AB. \quad (1.4.2.1)$$

Show that

$$ID = IE = IF = r \quad (1.4.2.2)$$

**Solution:** In  $\triangle IDC$  and  $IEC$ ,



Fig. 1.4.2: Inradius  $r$  of  $\triangle ABC$

$$ID = IE = \frac{l_3}{\sin \frac{C}{2}} \quad (1.4.2.3)$$

Similarly, in  $\triangle IEA$  and  $IFA$ ,

$$IF = IE = \frac{l_1}{\sin \frac{A}{2}} \quad (1.4.2.4)$$

yielding (1.4.2.2)

1.4.3. In Fig. 1.4.2, show that

$$BD = BF, AE = AF, CD = CE \quad (1.4.3.1)$$

**Solution:** From Fig. 1.4.2, in  $\triangle IBD$  and  $IBF$ ,

$$x = BD = BF = r \cot \frac{B}{2} \quad (1.4.3.2)$$

Similarly, other results can be obtained.

1.4.4. The circle with centre **I** and radius  $r$  in Fig. 1.4.3 is known as the *incircle*.



Fig. 1.4.3: Incircle of  $\triangle ABC$

1.4.5. The lengths of tangents drawn from an external point to a circle are equal.

1.4.6. In an isosceles  $\triangle ABC$ , with  $AB = AC$ ,  $BE$  and  $CF$  are the bisectors of  $\angle B$  and  $\angle C$  respectively. Show that

$$BE = CF \quad (1.4.6.1)$$



Fig. 1.4.4

**Solution:** In  $\triangle s$   $BEC$  and  $BFC$ , using the sine formula,

$$\begin{aligned}\frac{BE}{\sin C} &= \frac{BC}{\sin\left(\frac{B}{2} + C\right)} \\ \frac{CF}{\sin B} &= \frac{BC}{\sin\left(\frac{C}{2} + B\right)}\end{aligned}\tag{1.4.6.2}$$

$\because B = C$ , from the above, we obtain (1.4.6.1).

1.4.7. Show that

$$\sin 5\theta = 5 \sin \theta - 20 \sin^3 \theta \cos^2 \theta + 16 \sin^5 \theta \tag{1.4.7.1}$$

$$\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta \tag{1.4.7.2}$$

1.4.8. In Fig. 1.4.4, if  $BE = CF$ , show that the triangle is isosceles.

**Solution:** From (1.4.6.2),

$$\sin C \sin\left(\frac{C}{2} + B\right) = \sin\left(\frac{B}{2} + C\right) \sin B \tag{1.4.8.1}$$

$$\implies 2 \sin C \sin\left(\frac{C}{2} + B\right) = 2 \sin B \sin\left(\frac{B}{2} + C\right) \tag{1.4.8.2}$$

$$\cos\left(B - \frac{C}{2}\right) - \cos\left(B + \frac{3C}{2}\right) = \cos\left(C - \frac{B}{2}\right) - \cos\left(C + \frac{3B}{2}\right) \tag{1.4.8.3}$$

using (1.3.4.4), which can be expressed as

$$\cos\left(C - \frac{B}{2}\right) - \cos\left(B - \frac{C}{2}\right) - \cos\left(C + \frac{3B}{2}\right) + \cos\left(B + \frac{3C}{2}\right) = 0 \tag{1.4.8.4}$$

which, using (1.3.4.4), yields

$$2 \sin\left(\frac{B+C}{2}\right) \sin\left[\frac{3(B-C)}{2}\right] + 2 \sin\left[5\frac{(B+C)}{2}\right] \sin\left[\frac{(B-C)}{2}\right] = 0 \tag{1.4.8.5}$$

Let

$$\theta = \frac{B-C}{2}, \quad \alpha = \frac{B+C}{2} \tag{1.4.8.6}$$

Substituting the above in (1.4.8.5),

$$\sin \alpha \sin 3\theta + \sin 5\alpha \sin \theta = 0 \tag{1.4.8.7}$$

Substituting from (1.4.7.2) in (1.4.8.7) and simplifying,

$$\sin \alpha \sin \theta (3 - 4 \sin^2 \theta + 5 - 20 \sin^2 \alpha \cos^2 \alpha + 16 \sin^4 \alpha) = 0 \tag{1.4.8.8}$$

One possible solution of the above equation is

$$3 - 4 \sin^2 \theta + 5 - 20 \sin^2 \alpha \cos^2 \alpha + 16 \sin^4 \alpha = 0 \tag{1.4.8.9}$$

$$4 - 4 \sin^2 \theta + 4 - 20 \sin^2 \alpha (1 - \sin^2 \alpha) + 16 \sin^4 \alpha = 0 \tag{1.4.8.10}$$

which, upon substituting from (1.1.5.1) results in

$$\cos^2 \theta + 1 - 5 \sin^2 \alpha + 36 \sin^4 \alpha = 0 \quad (1.4.8.11)$$

$$= \cos^2 \theta + (1 - 6 \sin^2 \alpha)^2 + 7 \sin^2 \alpha = 0 \quad (1.4.8.12)$$

For the above equation to have a solution,

$$\cos \theta = 0, \sin^2 \alpha = \frac{1}{6}, \sin \alpha = 0. \quad (1.4.8.13)$$

which is impossible. Another possible solution is

$$\sin \alpha = \sin \frac{B+C}{2} = 0 \quad (1.4.8.14)$$

$$\Rightarrow \cos \frac{A}{2} = 0, \text{ or, } A = \pi, \quad (1.4.8.15)$$

which is impossible. Hence, the only possible solution is

$$\sin \theta = \sin \frac{B-C}{2} = 0 \quad (1.4.8.16)$$

$$\Rightarrow \frac{B-C}{2} = 0, \text{ or, } B = C. \quad (1.4.8.17)$$

## 1.5 Circumcircle

1.5.1. In Fig. 1.5.1,



Fig. 1.5.1: Isosceles Triangle

$$OB = OC = R \quad (1.5.1.1)$$

Such a triangle is known as an isosceles triangle. Show that

$$\angle B = \angle C \quad (1.5.1.2)$$

**Solution:** Using (1.2.2.3),

$$\frac{\sin B}{R} = \frac{\sin C}{R} \quad (1.5.1.3)$$

$$\Rightarrow \sin B = \sin C \quad (1.5.1.4)$$

$$\text{or, } \angle B = \angle C. \quad (1.5.1.5)$$



1.5.2. In Fig. 1.5.1, show that

$$a = 2R \sin \frac{\theta}{2} \quad (1.5.2.1)$$

**Solution:** In  $\triangle OBC$ , using the cosine formula from (1.2.5.1),

$$\cos \theta = \frac{R^2 + R^2 - a^2}{2R^2} = 1 - \frac{a^2}{2R^2} \quad (1.5.2.2)$$

$$\Rightarrow \frac{a^2}{2R^2} = 2 \sin^2 \frac{\theta}{2} \quad (1.5.2.3)$$

yielding (1.5.2.1).

1.5.3. In Fig. 1.5.2,

$$OB = OC = R, BD = DC. \quad (1.5.3.1)$$

Show that  $OD \perp BC$ .



Fig. 1.5.2: Perpendicular bisector.

1.5.4. In Fig. 1.5.3,  $OD$  and  $OE$  are the perpendicular bisectors of sides  $BC$  and  $AC$  respectively. Show that  $OA = R$ .

1.5.5. In Fig. 1.5.3, show that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R. \quad (1.5.5.1)$$

**Solution:** From (1.5.10.1) and (1.5.2.1)

$$a = 2R \sin A \quad (1.5.5.2)$$

1.5.6. Fig. 1.5.4 shows the *circumcircle* of  $\triangle ABC$ .

1.5.7. Any point on the circle can be expressed as

$$\mathbf{x} = \mathbf{O} + R \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \quad 0 \in [0, 2\pi]. \quad (1.5.7.1)$$

where  $\mathbf{O}$  is the centre of the circle.

1.5.8. Let

$$R = 1, \mathbf{O} = \mathbf{0}, \mathbf{A} = \begin{pmatrix} \cos \theta_1 \\ \sin \theta_1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} \cos \theta_2 \\ \sin \theta_2 \end{pmatrix}, \quad (1.5.8.1)$$



Fig. 1.5.3: Perpendicular bisectors of  $\triangle ABC$  meet at **O**.



Fig. 1.5.4: Circumcircle of  $\triangle ABC$

Show that the distance

$$AB = \|\mathbf{A} - \mathbf{B}\| = 2 \sin\left(\frac{\theta_1 - \theta_2}{2}\right)$$

(1.5.8.2)

**Solution:** From (1.5.7.1).

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} \cos \theta_1 - \cos \theta_2 \\ \sin \theta_1 - \sin \theta_2 \end{pmatrix} \quad (1.5.8.3)$$

$$\implies \|\mathbf{A} - \mathbf{B}\|^2 = (\mathbf{A} - \mathbf{B})^\top (\mathbf{A} - \mathbf{B}) \quad (1.5.8.4)$$

$$= (\cos \theta_1 - \cos \theta_2)^2 + (\sin \theta_1 - \sin \theta_2)^2 \quad (1.5.8.5)$$

$$= 2 \{1 - \cos(\theta_1 - \theta_2)\} = 4 \sin^2 \left( \frac{\theta_1 - \theta_2}{2} \right) \quad (1.5.8.6)$$

yielding (1.5.8.2) from (1.3.5.3).

1.5.9. In Fig. 1.5.4, show that

$$\cos A = \frac{(\mathbf{A} - \mathbf{B})^\top (\mathbf{A} - \mathbf{B})}{\|\mathbf{A} - \mathbf{B}\| \|\mathbf{A} - \mathbf{C}\|}, \quad (1.5.9.1)$$

1.5.10. In Fig. 1.5.4, show that

$$\theta = 2A. \quad (1.5.10.1)$$

**Solution:** Let

$$\mathbf{C} = \begin{pmatrix} \cos \theta_3 \\ \sin \theta_3 \end{pmatrix} \quad (1.5.10.2)$$

Then, substituting from (1.5.8.2) in (1.2.5.1),

$$\cos A = \frac{4 \sin^2 \left( \frac{\theta_1 - \theta_2}{2} \right) + 4 \sin^2 \left( \frac{\theta_1 - \theta_3}{2} \right) - 4 \sin^2 \left( \frac{\theta_2 - \theta_3}{2} \right)}{8 \sin \left( \frac{\theta_1 - \theta_2}{2} \right) \sin \left( \frac{\theta_1 - \theta_3}{2} \right)} \quad (1.5.10.3)$$

$$= \frac{2 \sin^2 \left( \frac{\theta_1 - \theta_2}{2} \right) + \cos(\theta_2 - \theta_3) - \cos(\theta_1 - \theta_3)}{4 \sin \left( \frac{\theta_1 - \theta_2}{2} \right) \sin \left( \frac{\theta_1 - \theta_3}{2} \right)} \quad (1.5.10.4)$$

from (1.3.5.3).  $\therefore$  From (1.3.4.4),

$$\cos A = \frac{2 \sin^2 \left( \frac{\theta_1 - \theta_2}{2} \right) + 2 \sin \left( \frac{\theta_1 - \theta_2}{2} \right) \sin \left( \frac{\theta_1 + \theta_2}{2} - \theta_3 \right)}{4 \sin \left( \frac{\theta_1 - \theta_2}{2} \right) \sin \left( \frac{\theta_1 - \theta_3}{2} \right)} \quad (1.5.10.5)$$

$$= \frac{\sin \left( \frac{\theta_1 - \theta_2}{2} \right) + \sin \left( \frac{\theta_1 + \theta_2}{2} - \theta_3 \right)}{2 \sin \left( \frac{\theta_1 - \theta_3}{2} \right)} \quad (1.5.10.6)$$

From (1.3.4.1), the above equation can be expressed as

$$\cos A = \frac{2 \sin \left( \frac{\theta_1 - \theta_3}{2} \right) \cos \left( \frac{\theta_2 - \theta_3}{2} \right)}{2 \sin \left( \frac{\theta_1 - \theta_3}{2} \right)} = \cos \left( \frac{\theta_2 - \theta_3}{2} \right) \quad (1.5.10.7)$$

$$\implies 2A = \theta_2 - \theta_3 \quad (1.5.10.8)$$

Similarly,

$$\cos \theta = \frac{1 + 1 - 4 \sin^2 \left( \frac{\theta_2 - \theta_3}{2} \right)}{2} = \cos (\theta_2 - \theta_3) = \cos 2A \quad (1.5.10.9)$$

1.5.11. In Fig. 1.5.5, show that

$$\theta = \alpha \quad (1.5.11.1)$$

where  $CP$  is the tangent.

**Solution:** Let

$$\mathbf{O} = \mathbf{0}, \mathbf{A} = \begin{pmatrix} \cos \theta_1 \\ \sin \theta_1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} \cos \theta_2 \\ \sin \theta_2 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} \cos \theta_3 \\ \sin \theta_3 \end{pmatrix} \quad (1.5.11.2)$$

Without loss of generality, let

$$\theta_3 = \frac{\pi}{2} \quad (1.5.11.3)$$

Then,

$$\mathbf{C} - \mathbf{O} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \implies \mathbf{C} - \mathbf{P} \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (1.5.11.4)$$

$\therefore CO \perp CP$ . From (1.5.9.1), and (1.5.11.4),

$$\cos \theta = \frac{\begin{pmatrix} \cos \theta_3 - \cos \theta_1 & \sin \theta_3 - \sin \theta_1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}}{2 \sin \left( \frac{\theta_1 - \theta_3}{2} \right)} \quad (1.5.11.5)$$

$$= \sin \left( \frac{\theta_1 + \theta_3}{2} \right) = \cos \left( \frac{\pi}{2} - \frac{\theta_1 + \theta_3}{2} \right) = \cos \left( \frac{\pi}{4} - \frac{\theta_1}{2} \right) \quad (1.5.11.6)$$

upon substituting from (1.5.11.3). Similarly, from (1.5.10.7),

$$\cos \alpha = \cos \left( \frac{\theta_1 - \theta_3}{2} \right) = \cos \left( \frac{\pi}{4} - \frac{\theta_1}{2} \right) = \cos \theta \quad (1.5.11.7)$$

1.5.12. In Fig. 1.5.5, show that  $PA.PB = PC^2$ .

**Solution:** In  $\triangle s APC$  and  $BPC$ , using (1.5.11.1),

$$\frac{AP}{\sin \theta} = \frac{AC}{\sin P} \quad (1.5.12.1)$$

$$\frac{PC}{\sin \theta} = \frac{BC}{\sin P} \quad (1.5.12.2)$$

$$\implies \frac{PC}{AP} = \frac{BC}{AC} \left( = \frac{BP}{CP} \right) \quad (1.5.12.3)$$

which gives the desired result.  $\triangle s APC$  and  $BPC$  are said to be *similar*.

## 1.6 Medians

1.6.1. In Fig. 1.6.1

$$AF = BF, AE = BE, \quad (1.6.1.1)$$

Fig. 1.5.5:  $\theta = \alpha$ .

and the medians  $BE$  and  $CF$  meet at  $G$ . Show that

$$ar(BEC) = ar(BFC) = \frac{1}{2}ar(ABC) \quad (1.6.1.2)$$

**Solution:** From (1.2.2.2),

Fig. 1.6.1:  $k_1 = k_2$ .

$$ar(BEC) = \frac{1}{2}a\left(\frac{b}{2}\right)\sin C \quad (1.6.1.3)$$

$$ar(BFC) = \frac{1}{2}a\left(\frac{c}{2}\right)\sin B \quad (1.6.1.4)$$

yielding (1.6.1.2).

1.6.2. The median divides a triangle into two triangle of equal area. .

1.6.3. In Fig. 1.6.1, show that

$$ar(CGE) = ar(BGF) \quad (1.6.3.1)$$

**Solution:** From Fig. 1.6.1 and (1.6.1.2),

$$ar(BGF) + ar(BGC) = ar(CGE) + ar(BGC) \quad (1.6.3.2)$$

yielding (1.6.3.1).

1.6.4. In Fig. 1.6.2, show that

$$k_1 = k_2 \quad (1.6.4.1)$$

**Solution:** From (1.6.3.1),

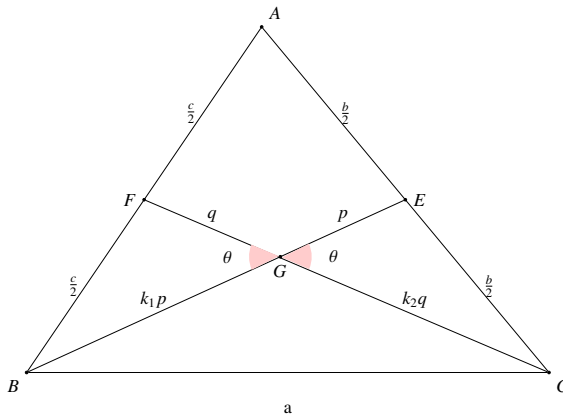


Fig. 1.6.2: Equal areas.

$$\frac{1}{2}p(k_1q)\sin\theta = \frac{1}{2}q(k_2p)\sin\theta \quad (1.6.4.2)$$

yielding (1.6.4.1).

1.6.5. In Fig. 1.6.3, show that

$$k_3 = k \quad (1.6.5.1)$$

**Solution:** From Problem 1.6.2,

$$ar(AGE) = ar(CGE) \quad (1.6.5.2)$$

$$ar(AGF) = ar(BGF)$$

$$\begin{aligned} \Rightarrow \frac{1}{2}p(k_3r)\sin\alpha &= \frac{1}{2}p(kq)\sin\theta \\ \frac{1}{2}q(k_3r)\sin\beta &= \frac{1}{2}q(kp)\sin\theta \end{aligned} \quad (1.6.5.3)$$

yileding upon division

$$p \sin \alpha = q \sin \beta \quad (1.6.5.4)$$

$$\Rightarrow \frac{1}{2} k p r \sin \alpha = \frac{1}{2} k q r \sin \beta \quad (1.6.5.5)$$

$$\Rightarrow ar(BGD) = ar(CGD) \quad (1.6.5.6)$$

Thus, from Problem 1.6.2,  $AD$  is also a median. Consequently, from (1.6.4.1) we obtain (1.6.5.1).



Fig. 1.6.3:  $k_3 = k$ .

1.6.6. In Fig. 1.6.4, show that  $k = 2$ .

**Solution:** Using the cosine formula,

$$DE^2 = \left(\frac{b}{2}\right)^2 + \left(\frac{c}{2}\right)^2 - 2\left(\frac{b}{2}\right)\left(\frac{c}{2}\right)\cos A \quad (1.6.6.1)$$

$$a^2 = b^2 + c^2 - 2bc \cos A \quad (1.6.6.2)$$

$$\Rightarrow DE = \frac{a}{2} \quad (1.6.6.3)$$

$\therefore \triangle EGF \sim \triangle BGC, k = 2$ .

Fig. 1.6.4:  $k = 2$ 

## 2 HEIGHTS AND DISTANCES

### 2.1 NCERT

- 2.1.1. A ladder is placed against a wall such that its foot is at a distance of  $2.5m$  from the wall and its top reaches a window  $6m$  above the ground. Find the length of the ladder.
- 2.1.2. A ladder  $10m$  long reaches a window  $8m$  above the ground. Find the distance of the foot of the ladder from base of the wall.
- 2.1.3. A guy wire attached to a vertical pole of height  $18m$  is  $24m$  long and has a stake attached to the other end. How far from the base of the pole should the stake be driven so that the wire will be taut?
- 2.1.4. An aeroplane leaves an airport and flies due north at a speed of  $1000km$  per hour. At the same time, another aeroplane leaves the same airport and flies due west at a speed of  $1200km$  per hour. How far apart will be the two planes after  $1\frac{1}{2}$  hours?
- 2.1.5. Two poles of heights  $6m$  and  $11m$  stand on a plane ground. If the distance between the feet of the poles is  $12m$ , find the distance between their tops.
- 2.1.6. An aircraft is flying at a height of  $3400m$  above the ground. If the angle subtended at a ground observation point by the aircraft positions  $10.0s$  apart is  $30^\circ$ , what is the speed of the aircraft?
- 2.1.7. A statue,  $1.6m$  tall, stands on the top of a pedestal. From a point on the ground, the angle of elevation of the top of the statue is  $60^\circ$  and from the same point the angle of elevation of the top of the pedestal is  $45^\circ$ . Find the height of the pedestal.
- 2.1.8. The angle of elevation of the top of a building from the foot of the tower is  $30^\circ$  and the angle of elevation of the top of the tower from the foot of the building is  $60^\circ$ . If the tower is  $50m$  high, find the height of the building.
- 2.1.9. Two poles of equal heights are standing opposite each other on either side of the road, which is  $80m$  wide. From a point between them on the road, the angles of elevation of the top of the poles are  $60^\circ$  and  $30^\circ$ , respectively. Find the height of the poles and the distances of the point from the poles.



- 2.1.10. A TV tower stands vertically on a bank of a canal. From a point on the other bank directly opposite the tower, the angle of elevation of the top of the tower is  $60^\circ$ . From another point  $20m$  away from this point on the line joining this point to the foot of the tower, the angle of elevation of the top of the tower is  $30^\circ$ . Find the height of the tower and the width of the canal.
- 2.1.11. From the top of a  $7m$  high building, the angle of elevation of the top of a cable tower is  $60^\circ$  and the angle of depression of its foot is  $45^\circ$ . Determine the height of the tower.
- 2.1.12. As observed from the top of a  $75m$  high lighthouse from the sea-level, the angles of depression of two ships are  $30^\circ$  and  $45^\circ$ . If one ship is exactly behind the other on the same side of the lighthouse, find the distance between the two ships.
- 2.1.13. A  $1.2m$  tall girl spots a balloon moving with the wind in a horizontal line at a height of  $88.2m$  from the ground. The angle of elevation of the balloon from the eyes of the girl at any instant is  $60^\circ$ . After some time, the angle of elevation reduces to  $30^\circ$ . Find the distance travelled by the balloon during the interval.
- 2.1.14. A straight highway leads to the foot of a tower. A man standing at the top of the tower observes a car at an angle of depression of  $30^\circ$ , which is approaching the foot of the tower with a uniform speed. Six seconds later, the angle of depression of the car is found to be  $60^\circ$ . Find the time taken by the car to reach the foot of the tower from this point.
- 2.1.15. The angles of elevation of the top of a tower from two points at a distance of  $4m$  and  $9m$  from the base of the tower and in the same straight line with it are complementary. Prove that the height of the tower is  $6m$ .
- 2.1.16. A girl of height  $90cm$  is walking away from the base of a lamp-post at a speed of  $1.2\text{ m/s}$ . If the lamp is  $3.6m$  above the ground, find the length of her shadow after 4 seconds.
- 2.1.17. Nazima is fly fishing in a stream. The tip of her fishing rod is  $1.8m$  above the surface of the water and the fly at the end of the string rests on the water  $3.6m$  away and  $2.4m$  from a point directly under the tip of the rod. Assuming that her string (from the tip of her rod to the fly) is taut, how much string does she have out? If she pulls in the string at the rate of  $5cm$  per second, what will be the horizontal distance of the fly from her after 12 seconds?
- 2.1.18. A vertical pole of length  $6m$  casts a shadow  $4m$  long on the ground and at the same time a tower casts a shadow  $28m$  long. Find the height of the tower.
- 2.1.19. A circus artist is climbing a  $20m$  long rope, which is tightly stretched and tied from the top of a vertical pole to the ground. Find the height of the pole, if the angle made by the rope with the ground level is  $30^\circ$ .
- 2.1.20. A tree breaks due to storm and the broken part bends so that the top of the tree touches the ground making an angle of  $30^\circ$  with it. The distance between the foot of the tree to the point where the top touches the ground is  $8m$ . Find the height of the tree.
- 2.1.21. A contractor plans to install two slides for the children to play in a park. For the children below the age of 5 years, she prefers to have a slide whose top is at a height of  $1.5m$ , and is inclined at an angle of  $30^\circ$  to the ground, whereas for elder children she wants to have a steep slide at a height of  $3m$ , and inclined at an angle of  $60^\circ$  to

the ground. What should be the length of the slide in each case?

- 2.1.22. The angle of elevation of the top of a tower from a point on the ground, which is  $30m$  away from the foot of the tower, is  $30^\circ$ . Find the height of the tower.
- 2.1.23. A kite is flying at a height of  $60m$  above the ground. The string attached to the kite is temporarily tied to a point on the ground. The inclination of the string with the ground is  $60^\circ$ . Find the length of the string, assuming that there is no slack in the string.
- 2.1.24. A  $1.5m$  tall boy is standing at some distance from a  $30m$  tall building. The angle of elevation from his eyes to the top of the building increases from  $30^\circ$  to  $60^\circ$  as he walks towards the building. Find the distance he walked towards the building.
- 2.1.25. From a point on the ground, the angles of elevation of the bottom and the top of a transmission tower fixed at the top of a  $20m$  high building are  $45^\circ$  and  $60^\circ$  respectively. Find the height of the tower.
- 2.1.26. A girl walks  $4km$  west, then she walks  $3km$  in a direction  $30^\circ$  east of north and stops. Determine the girl's displacement from her initial point of departure.
- 2.1.27. The angles of depression of the top and the bottom of an  $8m$  tall building from the top of a multi-storeyed building are  $30^\circ$  and  $45^\circ$  respectively. Find the height of the multi-storeyed building and the distance between the two buildings.
- 2.1.28. A tower stands vertically on the ground. From a point on the ground, which is  $15m$  away from the foot of the tower, the angle of elevation of the top of the tower is found to be  $60^\circ$ . Find the height of the tower.
- 2.1.29. An electrician has to repair an electric fault pole of height  $5m$ . She needs to reach a point  $1.3m$  below the top of the pole to undertake the repair work. What should be the length of the ladder that she should use which, when inclined at an angle of  $60^\circ$  to the horizontal, would enable her to reach the required position? Also, how far from the foot of the pole should she place the foot of the ladder?
- 2.1.30. An observer  $1.5m$  tall is  $28.5m$  away from a chimney. The angle of elevation of the top of the chimney from her eyes is  $45^\circ$ . What is the height of the chimney?
- 2.1.31. From a point  $P$  on the ground the angle of elevation of the top of a  $10m$  tall building is  $30^\circ$ . A flag is hoisted at the top of the building and the angle of elevation of the top of the flagstaff from  $P$  is  $45^\circ$ . Find the length of the flagstaff and the distance of the building from the point  $P$ .
- 2.1.32. The shadow of a tower standing on a level ground is found to be  $40m$  longer when the Sun's altitude is  $30^\circ$  than when it is  $60^\circ$ . Find the height of the tower.

## 2.2 CBSE

- 2.2.1. In Fig. 2.2.1, the angles of elevation of two kites from point  $C$  are found to be  $30^\circ$  and  $60^\circ$  respectively. Taking  $AD = 50m$  and  $BE = 60m$ , find



Fig. 2.2.1

- The length of string used (take them straight) for kites  $A$  and  $B$  as shown in the figure.
- The distance  $d$  between these two kites.

(10, 2022)

2.2.2. In Fig. 2.2.2, a tower stands vertically on the ground. From a point on the ground, which is  $80\text{ m}$  away from the foot of the tower, the angle of elevation of the tower is found to be  $30^\circ$ . Find the height of the tower.

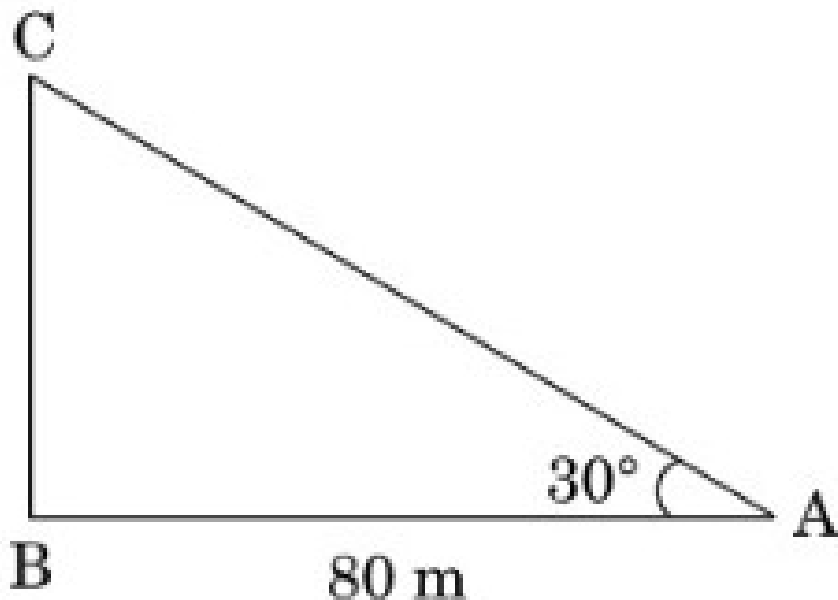


Fig. 2.2.2

(10, 2022)

2.2.3. The angles of depression of the top and bottom of a tower as seen from the top of a  $60\sqrt{3}m$  high cliff are  $45^\circ$  and  $60^\circ$  respectively. Find the height of the tower. (Use  $\sqrt{3} = 1.73$ ) (10, 2022)

2.2.4. The angle of elevation of the top of a building from the foot of the tower is  $30^\circ$  and the angle of elevation of the top of the tower from the foot of the building is  $60^\circ$ . If the tower is 50 meters high, then find the height of the building. (10, 2022)

2.2.5. From a point on a bridge across a river, the angles of depression of the banks on opposite sides of the river are  $30^\circ$  and  $60^\circ$  respectively. If the bridge is at a height of 3 meters from the banks, then find the width of the river. (10, 2022)

2.2.6. In Fig. 2.2.3, Gadisar Lake is located in the Jaisalmer district of Rajasthan. It was built by the King of Jaisalmer and rebuilt by Gadsingh in the 14th century. The lake has many Chhatris. One of them is shown below:



Fig. 2.2.3

Observe the picture. From a point  $A$   $h$  meters above the water level, the angle of elevation of the top of Chhatra (point  $B$ ) is  $45^\circ$  and the angle of depression of its reflection in the water (point  $C$ ) is  $60^\circ$ . If the height of Chhatra above water level is (approximately) 10 meters, then

- Draw a well-labeled figure based on the above information.
- Find the height ( $h$ ) of the point  $A$  above water level. (Use  $\sqrt{3} = 1.73$ )

(10, 2022)

2.2.7. In Fig. 2.2.4, from a point on a bridge across a river, the angles of depression of the banks on opposite sides of the river are  $30^\circ$  and  $45^\circ$ . If the bridge is at a height of 8 meters from the banks, then find the width of the river.



Fig. 2.2.4

(10, 2022)

2.2.8. Two boats are sailing in the sea 80 meters apart from each other towards a cliff  $AB$ . The angles of depression of the boats from the top of the cliff are  $30^\circ$  and  $45^\circ$  respectively, as shown in Fig. 2.2.5



Fig. 2.2.5

Find the height of the cliff. (10, 2022)

- 2.2.9. The angle of elevation of the top  $Q$  of a vertical tower  $PQ$  from a point  $X$  on the ground is  $60^\circ$ . From a point  $Y$ , 40 meters vertically above  $X$ , the angle of elevation of the top  $Q$  of tower  $PQ$  is  $45^\circ$ . Find the height of the tower  $PQ$  and the distance  $PX$ . (Use  $\sqrt{3} = 1.73$ ) (10, 2022)
- 2.2.10. An Aeroplane at an altitude of 200 meters observes the angles of depression of opposite points on the two banks of a river to be  $45^\circ$  and  $60^\circ$ . Find the width of the river. (Use  $\sqrt{3} = 1.732$ ) (10, 2022)
- 2.2.11. From the top of an 8 meter high building, the angle of elevation of the top of a cable tower is  $60^\circ$  and the angle of depression of its foot is  $45^\circ$ . Determine the height of the tower. (Take  $\sqrt{3} = 1.732$ ). (10, 2022)
- 2.2.12. As observed from the top of a lighthouse 60 meters high from the sea level, the angles of depression of two ships are  $45^\circ$  and  $60^\circ$ . If one ship is exactly behind the other on the same side of the lighthouse, then find the distance between the two ships. (Use  $\sqrt{3} = 1.732$ ) (10, 2022)
- 2.2.13. At a point on the level ground, the angle of elevation of the top of a vertical tower is found to be  $\alpha$ , such that  $\tan \alpha = \frac{5}{12}$ . On walking 192 meters towards the tower, the angle of elevation  $\beta$  is such that  $\tan \beta = \frac{3}{4}$ . Find the height of the tower. (10, 2022)
- 2.2.14. A man on the top of a vertical tower observes a car moving at a uniform speed coming directly towards it. If it takes 18 minutes for the angle of depression to change from  $30^\circ$  to  $60^\circ$ , how soon after this will the car reach the tower ? (10, 2021)
- 2.2.15. A girl on a ship standing on a wooden platform, which is 50m above water level, observes the angle of elevation of a top of a hill as  $30^\circ$  and the angle of depression of the base of the hill as  $60^\circ$ . Calculate the distance of the hill from the platform and the height of the hill. (10, 2021)
- 2.2.16. The length of the shadow of a tower on the plane ground is  $\sqrt{3}$  times the height of the tower. Find the angle of elevation of the sun. (10, 2023)
- 2.2.17. The angle of elevation of the top of a tower from a point on the ground which is 30m away from the foot of the tower, is  $30^\circ$ . Find the height of the tower. (10, 2023)
- 2.2.18. As observed from the top of a 75m high lighthouse from the sea-level, the angles of depression of two ships are  $30^\circ$  and  $60^\circ$ . If one ship is exactly behind the other on the same side of the lighthouse, find the distance between two ships. Use ( $\sqrt{3} = 1.73$ ) (10, 2023)
- 2.2.19. From a point on the ground, the angle of elevation of the bottom and top of a transmission tower fixed at the top of 30m high building are  $30^\circ$  and  $60^\circ$ , respectively. Find the height of the transmission tower. Use ( $\sqrt{3} = 1.73$ ) (10, 2023)
- 2.2.20. A straight highway leads to the foot of a tower. A man standing on the top of the 75m high tower observes two cars at angles of depression of  $30^\circ$  and  $60^\circ$ , which are approaching the foot of the tower. If one car is exactly behind the other on the same side of the tower, find the distance between the two cars. (10, 2023)
- 2.2.21. From the top of a 7m high building, the angle of elevation of the top of a cable tower is  $60^\circ$  and the angle of depression of its foot is  $30^\circ$ . Determine the height of the tower. (take  $\sqrt{3} = 1.73$ ) (10, 2023)
- 2.2.22. The angle of elevation of the top of a tower 24m high from the foot of another tower in the same plane is  $60^\circ$ . The angle of elevation of the top of second tower from the

foot of the first tower is  $30^\circ$ . Find the distance between two towers and the height of the other tower. Also, find the length of the wire attached to the tops of both the towers. (10, 2023)

- 2.2.23. A spherical balloon of radius  $r$  subtends an angle of  $60^\circ$  at the eye of an observer. If the angle of elevation of its centre is  $45^\circ$  from the same point, then prove that height of the centre of the balloon is  $\sqrt{2}$  times its radius. (10, 2023)
- 2.2.24. A vertical pole is 100 metres high. Find the angle subtended by the pole at a point on the ground  $100\sqrt{3}$  metres from the base of the pole. (10, 2021)
- 2.2.25. The angle of elevation of the top of a tower from a point is found to be  $60^\circ$ . At a point  $40m$  above the first point, the angle of elevation of the top of the tower is  $45^\circ$ . Find the height of the tower. (10, 2021)
- 2.2.26. A statue 1.6m tall stands on the top of a pedestal. From a point on the ground, the angle of elevation of the top of statue is  $60^\circ$  and from the same point, the angle of elevation of the top of the pedestal is  $45^\circ$ . Find the height of the pedestal. (10, 2021)
- 2.2.27. Two poles,  $6m$  and  $11m$  high, stand vertically on the ground. If the distance between their feet is  $12m$ , find the distance between their tops. (10, 2021)
- 2.2.28. The angle of elevation of the top of a tower from a point on the ground, which is  $30m$  away from the foot of the tower is  $45^\circ$ . What is the height of the tower? (10, 2021)
- 2.2.29. Find the sun's altitude if the shadow of a  $15m$  high tower is  $15\sqrt{3}m$ . (10, 2021)
- 2.2.30. From a point on the ground,  $20m$  away from the foot of vertical tower, the angle of elevation of the top of the tower is  $60^\circ$ . Find the height of the tower. (10, 2021)
- 2.2.31. To explain how trigonometry can be used to measure the height of an inaccessible object, a teacher gave the following example to students: A TV tower stands vertically on the bank of a canal. From a point on the other bank direct opposite the tower, the angle of elevation of the top of the tower is  $60^\circ$ . From another point  $20m$  away from this point to the foot of the tower, the angle of elevation of the top of the tower is  $30^\circ$  (as shown in Fig. 2.2.6).



Fig. 2.2.6

Based on the above, answer the following questions



a) The width of the canal is

- i)  $10\sqrt{3}m$       ii)  $20\sqrt{3}m$       iii)  $10m$       iv)  $20m$

b) Height of the tower is

- i)  $10\sqrt{3}m$       ii)  $10m$       iii)  $20\sqrt{3}m$       iv)  $20m$

c) Distance of the foot of the tower from the point  $D$  is

- i)  $20m$       ii)  $30m$       iii)  $10m$       iv)  $20\sqrt{3}m$

(10, 2021)

2.2.32. In Fig. 2.2.7, the angle of elevation of the top of a tower from a point  $C$  on the ground, which is  $30m$  away from the foot of the tower, is  $30^\circ$ . Find the height of the tower.

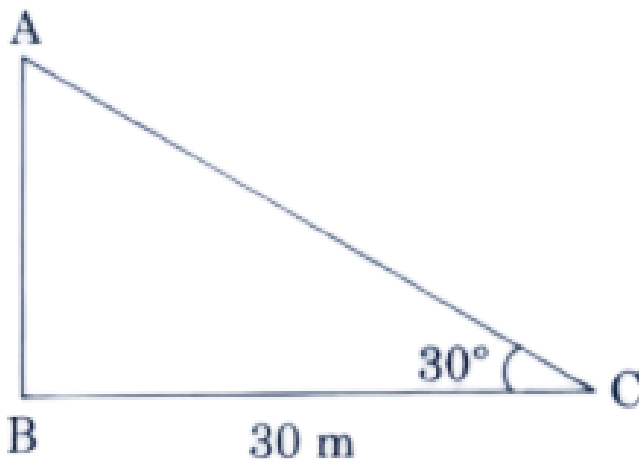


Fig. 2.2.7

(10, 2020)

2.2.33. A statue  $1.6m$  tall, stands on the top of a pedestal. From a point on the ground, the angle of elevation of the top of the statue is  $60^\circ$  and from the same point the angle of elevation of the top of the pedestal is  $45^\circ$ . Find the height of the pedestal. (Use  $\sqrt{3} = 1.73$ )

(10, 2020)

2.2.34. A moving boat is observed from the top of a  $150m$  high cliff moving away from the cliff. The angle of depression of the boat changes from  $60^\circ$  to  $45^\circ$  in 2 minutes. Find the speed of the boat in  $m/min$ .

(10, 2019)

2.2.35. There are two poles, one each on either bank of a river just opposite to each other. One pole is  $60m$  high. From the top of this pole, the angle of depression of the top

and foot of the other pole are  $30^\circ$  and  $60^\circ$  respectively. Find the width of the river and height of the other pole. (10, 2019)

- 2.2.36. Two poles of equal heights are standing opposite to each other on either side of the road which is  $80m$  wide. From a point  $P$  between them on the road, the angle of elevation of the top of a pole is  $60^\circ$  and the angle of depression from the top of the other pole of point  $P$  is  $30^\circ$ . Find the heights of the poles and the distance of the point  $P$  from the poles. (10, 2019)
- 2.2.37. Amit, standing on a horizontal plane, finds a bird flying at a distance of  $200m$  from him at an elevation of  $30^\circ$ . Deepak standing on the roof of a  $50m$  high building, finds the angle of elevation of the same bird to be  $45^\circ$ . Amit and Deepak are on opposite sides of the bird. Find the distance of the bird from Deepak. (10, 2019)
- 2.2.38. From a point  $P$  on the ground, the angle of elevation of the top of a tower is  $30^\circ$  and that of the top of the flag-staff fixed on the top of the tower is  $\sqrt{5}$ . If the length of the flag-staff is  $5m$ , find the height of the tower. (Use  $\sqrt{3} = 1.732$ ). (10, 2019)
- 2.2.39. The shadow of a tower standing on a level ground is found to be  $40m$  longer when the Sun's altitude is  $30^\circ$  than when it was  $60^\circ$ . Find the height of the tower. Given ( $\sqrt{3} = 1.732$ ) (10, 2019)
- 2.2.40. A man in a boat rowing away from a light house  $100m$  high takes 2 minutes to change the angle of elevation of the top of the light house from  $60^\circ$  to  $30^\circ$ . Find the speed of the boat in metres per minute. [Use  $\sqrt{3} = 1.732$ ] (10, 2019)
- 2.2.41. Two poles of equal heights are standing opposite each other on either side of the road, which is  $80m$  wide. From a point between them on the road, the angles of elevation of the top of the poles are  $60^\circ$  to  $30^\circ$  respectively. Find the height of the poles and the distances of the point from the poles. (10, 2019)
- 2.2.42. As observed from the top of a  $100m$  high light house from the sea level, the angles of depression of two ships are  $30^\circ$  and  $45^\circ$ . If one ship is exactly behind the other on the same side of the light house, find the distance between the two ships. Use ( $\sqrt{3} = 1.732$ ) (10, 2018)
- 2.2.43. A statue,  $1.46m$  tall, stands on a pedestal. From a point on the ground the angle of elevation of the top of the statue is  $60^\circ$  and from the same point angle of elevation of the top of the pedestal is  $45^\circ$ . Find the height of the pedestal. Use ( $\sqrt{3} = 1.73$ ) (10, 2018)
- 2.2.44. A ladder, leaning against a wall, makes an angle of  $60^\circ$  with the horizontal. If the foot of the ladder is  $2.5m$  away from the wall, find the length of the ladder. (10, 2016)
- 2.2.45. A man standing on the deck of a ship, which is  $10m$  above water level, observes the angle of elevation of the top of a hill as  $60^\circ$  and the angle of depression of the base of hill as  $30^\circ$ . Find the distance of the hill from the ship and the height of the hill. (10, 2016)
- 2.2.46. The angle of elevation of the top  $Q$  of a vertical tower  $PQ$  from a point  $X$  on the ground is  $60^\circ$ . From a point  $Y$ ,  $40m$  vertically above  $X$ , the angle of elevation of the top  $Q$  of tower is  $45^\circ$ . Find the height of the tower  $PQ$  and the distance  $PX$ . (Use  $\sqrt{3} = 1.73$ ) (10, 2016)
- 2.2.47. A boy standing on a horizontal plane finds a bird flying at a distance of  $100m$  from him at an elevation of  $30^\circ$ . A girl standing on the roof of a  $20m$  high building, finds the elevation of the same bird to be  $45^\circ$ . The boy and the girl are on the opposite

sides of the bird. Find the distance of the bird from the girl. (Given  $\sqrt{2} = 1.414$ ) (10, 2019)

- 2.2.48. The angle of elevation of an aeroplane from a point  $A$  on the ground is  $60^\circ$ . After a flight of 30 seconds, the angle of elevation changes to  $30^\circ$ . If the plane is flying at a constant height of  $3600\sqrt{3}$  metres, find the speed of the aeroplane. (10, 2019)
- 2.2.49. If a tower  $30m$  high, casts a shadow  $10\sqrt{3}m$  long on a ground, then what is the angle of elevation of the sun ? (10, 2017)
- 2.2.50. A man observes a car from the top of a tower, which is moving towards the tower with a uniform speed. If the angle of depression of the car changes from  $30^\circ$  to  $45^\circ$  in 12 minutes, find the time taken by the car now to reach the tower. (10, 2017)
- 2.2.51. An aeroplane is flying at a height of  $300m$  above the ground. Flying at this height, the angles of depression from the aeroplane of two points on both banks of a river in opposite directions are  $45^\circ$  and  $60^\circ$  respectively. Find the width of the river. Use  $[\sqrt{3} = 1.732]$  (10, 2017)
- 2.2.52. On a straight line passing through the foot of a tower, two points  $C$  and  $D$  are at distances of  $4m$  and  $16m$  from the foot respectively. If the angles of elevation from  $C$  and  $D$  of the top of the tower are complementary, then find the height of the tower. (10, 2017)
- 2.2.53. From the top of a tower,  $100m$  high, a man observes two cars on the opposite sides of the tower and in the same straight line with its base, with angles of depression  $30^\circ$  and  $45^\circ$ . Find the distance between the cars. Take  $[\sqrt{3} = 1.732]$  (10, 2017)
- 2.2.54. At a point  $A$ , 20 metres above the level of water in a lake, the angle of elevation of a cloud is  $30^\circ$ . The angle of depression of the reflection of the cloud in the lake, at  $A$  is  $60^\circ$ . Find the distance of the cloud from  $A$ . (10, 2015)
- 2.2.55. In Figure 2.2.8, a tower  $AB$  is  $20m$  high and  $BC$ , its shadow on the ground, is  $20\sqrt{3}m$  long. Find the sun's altitude.



Fig. 2.2.8

- (10, 2015)
- 2.2.56. The angle of elevation of an aeroplane from a point A on the ground is  $60^\circ$ . After a flight of 15 seconds, the angle of elevation changes to  $30^\circ$ . If the aeroplane is flying at a constant height of  $1500\sqrt{3}$  m, find the speed of the plane in  $km/hr$ . (10, 2015)
- 2.2.57. A kite is flying at a height of  $30m$  from the ground. The length of string from the kite to the ground is  $60m$ . Assuming that there is no slack in the string, the angle of elevation of the kite at the ground is \_\_\_\_\_. (10, 2012)
- 2.2.58. From a point on the ground, which is  $15m$  away from the foot of a vertical tower, the angle of elevation of the top of the tower, is found to be  $60^\circ$ . The height of the tower in (in metres) is \_\_\_\_\_. (10, 2012)
- 2.2.59. The length of shadow of a tower on the plane ground is  $\sqrt{3}m$  times the height of the tower. The angle of elevation of sun is \_\_\_\_\_. (10, 2012)
- 2.2.60. The angles of depression of the top and bottom of a tower as seen from the top of a  $60\sqrt{3}m$  high cliff are  $45^\circ$  and  $60^\circ$  respectively. Find the height of the tower. (10, 2012)
- 2.2.61. In a flight of  $2800km$ , an aircraft was slowed down due to bad weather. Its average speed is reduced by  $100km/h$  and time is increased by 30 minutes. Find the original duration of flight. (10, 2012)
- 2.2.62. The angles of elevation and depression of the top and bottom of a light-house from the top of a  $60m$  high building are  $30^\circ$  and  $60^\circ$  respectively. Find
- the difference between the heights of the light-house and the building.
  - the distance between light-house and building.

(10, 2012)

- 2.2.63. The angles of depression of two ships from the top of a light house and on the same side of it are found to be  $45^\circ$  and  $30^\circ$ . if the ships are  $200\text{km}$  apart, find the height of the light house. (10, 2012)
- 2.2.64. The angle of elevation of the top of a hill at the foot of a tower is  $60^\circ$  and the angle of depression from the top of the tower of the foot of the hill is  $30^\circ$ . If the tower is  $50\text{m}$  high, find the height of the hill. (10, 2012)
- 2.2.65. From the top of a tower  $50\text{m}$  high, the angle of depression of the top of a pole is  $45^\circ$  and from the foot of the pole, the angle of elevation of the top of the tower is  $60^\circ$ . find the height of the pole if the pole and tower stand on the same plane. (10, 2012)
- 2.2.66. The angle of depression from the top of a tower of a point  $A$  on the ground is  $30^\circ$ . On moving a distance of  $20\text{m}$  from the point  $A$  towards the foot of the tower to a point  $B$  the angle of elevation of the top of the tower from point  $B$  is  $60^\circ$ . Find the height of the tower and its distance from point  $A$ . (10, 2012)
- 2.2.67. A tower stands vertically on the ground. From a point on the ground which is  $25\text{m}$  away from the foot of the tower, the angle of elevation of the top of the tower is found to be  $45^\circ$ . Then the height (*in meters*) of the tower is (10, 2011)
- 2.2.68. The angle of elevation of the top of a vertical tower from a point on the ground is  $60^\circ$ . From another point  $10\text{m}$  vertically above the first, its angle of elevation is  $30^\circ$ . Find the height of the tower. (10, 2011)
- 2.2.69. From the top of a vertical tower, the angles of depression of two cars, in the same straight line with the base of the tower, at an instant are found to be  $45^\circ$  and  $60^\circ$ . If the cars are  $100\text{m}$  apart and are on the same side of the tower, find the height of the tower. [Use  $\sqrt{3} = 1.73$ ] (10, 2011)
- 2.2.70. The angle of elevation of the top of a tower from a point on the ground, which is  $30\text{m}$  away from the foot of the tower is  $45^\circ$ . The height of the tower (in metres) is (10, 2011)
- 2.2.71. From the top of a tower  $100\text{m}$  high, a man observes two cars on the opposite sides of the tower with angles of depression  $30^\circ$  and  $45^\circ$  respectively. Find the distance between the cars. [Use  $\sqrt{3} = 1.73$ ]. (10, 2011)
- 2.2.72. Two poles of equal heights are standing opposite to each other on either side of the road, which is  $100\text{m}$  wide. From a point between them on the road, the angles of elevation of the top of the poles are  $60^\circ$  and  $30^\circ$ , respectively. Find the height of the poles. (10, 2011)
- 2.2.73. A man standing on the deck of a ship, which is  $10\text{m}$  above the water level, observes the angle of elevation of the top of a hill as  $60^\circ$  and the angle of depression of the base of the hill as  $30^\circ$ . Calculate the distance of the hill from the ship and the height of the hill. (10, 2006)
- 2.2.74. From a window  $x$  meters high above the ground in a street, the angles of elevation and depression of the top and foot of the other house on the opposite side of the street are  $\alpha$  and  $\beta$  respectively. Show that the height of the opposite house is  $x(1 + \tan \alpha \cot \beta)$  meters. (10, 2006)
- 2.2.75. A pole  $6\text{m}$  high is fixed on the top of a tower. The angle of elevation of the top of the pole observe d from a point  $P$  on the ground is  $60^\circ$  and the angle of depression of the point  $P$  from the top of the tower is  $45^\circ$ . Find the height of the tower and the distance of point  $P$  from the foot of the tower (10, 2024)

- 2.2.76. The length of the shadow of a tower on the plane ground is  $\sqrt{3}$  times the height of the tower. Find the angle of elevation of the sun. (10, 2023)
- 2.2.77. The angle of elevation of the top of a tower from a point on the ground which is  $30m$  away from the foot of the tower, is  $30^\circ$ . Find the height of the tower. (10, 2023)
- 2.2.78. As observed from the top of a  $75m$  high lighthouse from the sea-level, the angles of depression of two ships are  $30^\circ$  and  $60^\circ$ . If one ship is exactly behind the other on the same side of the lighthouse, find the distance between the two ships. Use  $(\sqrt{3} = 1.73)$  (10, 2023)
- 2.2.79. From a point on the ground, the angle of elevation of the bottom and top of a transmission tower fixed at the top of  $30m$  high building are  $30^\circ$  and  $60^\circ$ , respectively. Find the height of the transmission tower. Use  $(\sqrt{3} = 1.73)$ . (10, 2023)
- 2.2.80. If a pole  $6m$  high casts a shadow  $2 \times \sqrt{3}m$  long on the ground, then sun's elevation is

- a)  $60^\circ$                       b)  $45^\circ$                       c)  $30^\circ$                       d)  $90^\circ$

(10, 2023)

- 2.2.81. A straight highway leads to the foot of a tower. A man standing on the top of the  $75m$  high tower observes two cars at angles of depression of  $30^\circ$  and  $60^\circ$ , which are approaching the foot of the tower. If one car is exactly behind the other on the same side of the tower, find the distance between the two cars. Use  $(\sqrt{3} = 1.73)$ . (10, 2023)
- 2.2.82. From the top of a  $7m$  building, the angle of elevation of the top a cable tower is  $60^\circ$  and the angle of depression of its foot is  $30^\circ$ . Determine the height of the tower. (10, 2023)

## 2.3 JEE

### 3 TRIANGLE

#### 3.1 NCERT

- 3.1.1.  $D$  is a point on the side  $BC$  of a  $\triangle ABC$  such that  $\angle ADC = \angle BAC$ . Show that

$$CA^2 = CB \cdot CD \quad (3.1.1.1)$$

**Solution:** See Fig. 3.1.1.

$$\frac{x}{\sin(A+C)} = \frac{b}{\sin A} \quad (\triangle ADC), \quad (3.1.1.2)$$

$$\Rightarrow \frac{x}{\sin B} = \frac{b}{\sin A} \quad (3.1.1.3)$$

$$\Rightarrow \Rightarrow \frac{x}{b} = \frac{\sin B}{\sin A} = \frac{b}{a} \quad (\text{sine formula}) \quad (3.1.1.4)$$

yielding (3.1.1.1).

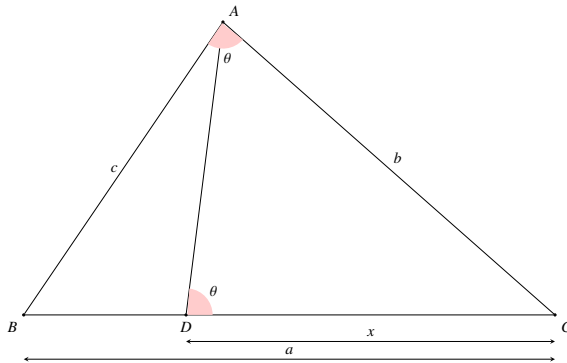


Fig. 3.1.1

3.1.2.  $D$  is a point on side  $BC$  of  $\triangle ABC$  such that  $\frac{BD}{CD} = \frac{AB}{AC}$ . Prove that  $AD$  is the bisector of  $\angle BAC$ .

**Solution:** See Fig. 3.1.2.

$$\frac{x}{a-x} = \frac{c}{b} \quad (\text{given}) \quad (3.1.2.1)$$

$$\frac{c}{\sin \phi} = \frac{x}{\sin \theta} \quad (\triangle ABD) \quad (3.1.2.2)$$

$$\frac{a-x}{\sin(A-\theta)} = \frac{b}{\sin 180-\phi} \quad (\triangle ACD) \quad (3.1.2.3)$$

$$= \frac{b}{\sin \phi} \quad (3.1.2.4)$$

using the sine formula. Multiplying all the above equations yields

$$\sin(A-\theta) = \sin \theta \implies \theta = \frac{A}{2} \quad (3.1.2.5)$$

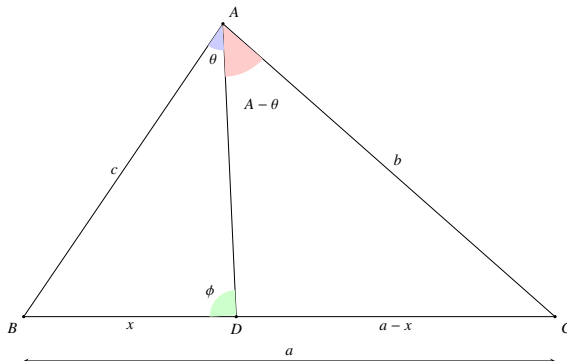


Fig. 3.1.2

3.1.3.  $ABC$  is a triangle in which  $\angle ABC > 90^\circ$  and  $AD \perp CB$  produced. Prove that

$$AC^2 = AB^2 + BC^2 + 2BC \cdot BD. \quad (3.1.3.1)$$

**Solution:** See Fig. 3.1.3.

$$\cos B = \frac{x}{c} \quad (\triangle ADB) \quad (3.1.3.2)$$

$$b^2 = a^2 + c^2 - 2ac \cos (180 - B) \quad (\triangle ABC) \quad (3.1.3.3)$$

$$= a^2 + c^2 + 2ac \cos B \quad (3.1.3.4)$$

using the cosine formula. Substituting from (3.1.3.2) in (3.1.3.4) yields (3.1.3.1).

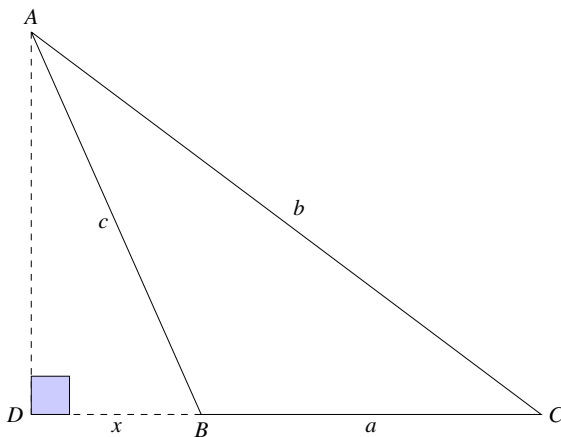


Fig. 3.1.3

3.1.4. In a right triangle, prove that the line-segment joining the mid-point of the hypotenuse to the opposite vertex is half the hypotenuse.

**Solution:** In Fig. 3.1.4

$$\frac{x}{\sin C} = \frac{b/2}{\sin \theta} \quad (\triangle BDC) \quad (3.1.4.1)$$

$$\frac{x}{\sin A} = \frac{b/2}{\sin (90 - \theta)} \quad (\triangle BDA) \quad (3.1.4.2)$$

$$\Rightarrow \frac{x}{\cos C} = \frac{b/2}{\cos \theta} \quad (3.1.4.3)$$

From (3.1.4.1) and (3.1.4.3),

$$\left( \frac{\sin C}{x} \right)^2 + \left( \frac{\cos C}{x} \right)^2 = \left( \frac{\cos \theta}{\frac{b}{2}} \right)^2 + \left( \frac{\sin \theta}{\frac{b}{2}} \right)^2 \quad (3.1.4.4)$$

$$\Rightarrow x = \frac{b}{2} \quad (3.1.4.5)$$

using (1.1.5.1).



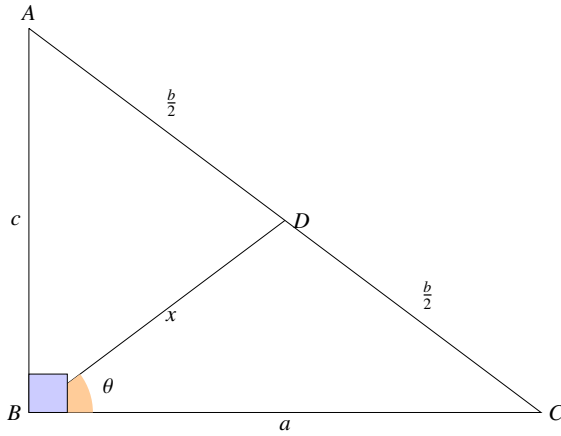


Fig. 3.1.4

3.1.5.  $ABCD$  is a trapezium in which  $AB \parallel DC$  and its diagonals intersect each other at the point  $O$ . Show that

$$\frac{AO}{BO} = \frac{CO}{DO} \quad (3.1.5.1)$$

**Solution:** In Fig. 3.1.5,  $\because AB \parallel CD$

$$\frac{AO}{\sin \phi} = \frac{BO}{\sin \theta} \quad (\triangle OAB) \quad (3.1.5.2)$$

$$\frac{CO}{\sin \phi} = \frac{DO}{\sin \theta} \quad (\triangle ODC) \quad (3.1.5.3)$$

yielding (3.1.5.1) after simplification.

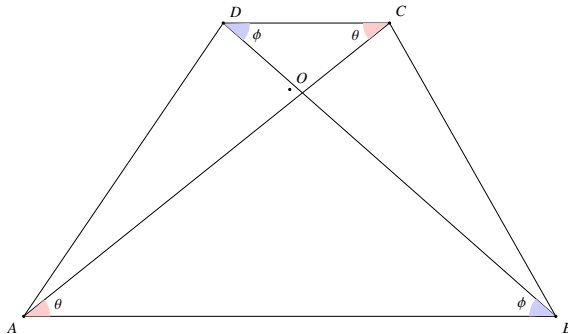


Fig. 3.1.5

3.1.6.  $O$  is any point inside a rectangle  $ABCD$ . Prove that

$$OB^2 + OD^2 = OA^2 + OC^2 \quad (3.1.6.1)$$

**Solution:** In Fig. 3.1.6, from (1.1.4.1)

$$p \cos \theta_1 + q \sin \theta_2 = a \quad (\triangle OAB) \quad (3.1.6.2)$$

$$r \cos \theta_3 + s \sin \theta_4 = a \quad (\triangle OCB) \quad (3.1.6.3)$$

$$p \cos \theta_1 + s \sin \theta_4 = b \quad (\triangle OAD) \quad (3.1.6.4)$$

$$r \cos \theta_3 + q \sin \theta_2 = b \quad (\triangle ODB) \quad (3.1.6.5)$$

Subtracting the first two and second two equations respectively,

$$p \cos \theta_1 - s \sin \theta_4 = r \cos \theta_3 - q \sin \theta_2 \quad (3.1.6.6)$$

$$p \cos \theta_1 + s \sin \theta_4 = r \cos \theta_3 + q \sin \theta_2 \quad (3.1.6.7)$$

Squaring and adding and using (1.1.5.1) yields (3.1.6.1).

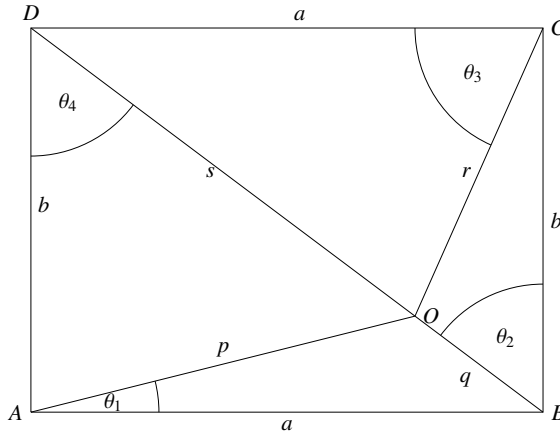


Fig. 3.1.6

3.1.7. In  $\triangle ABC$ ,  $AB = 6\sqrt{3}cm$ ,  $AC = 12cm$  and  $BC = 6cm$ . Find the angle  $B$ .

### 3.2 CBSE

- 1) In an equilateral  $\triangle ABC$ ,  $D$  is a point on side  $BC$  such that  $BD = \frac{1}{3}BC$ . Prove that  $9(AD)^2 = 7(AB)^2$ . (10, 2018)
- 2) Prove that the area of an equilateral triangle described on one side of the square is equal to half of the area of the equilateral triangle described on one of its diagonal. (10, 2018)
- 3) If the areas of two similar triangles are equal, prove that they are congruent. (10, 2018)
- 4) In Fig. 3.2.1,  $BN$  and  $CM$  are medians of a  $\triangle ABC$  right-angled at  $A$ . Prove that

$$4(BN^2 + CM^2) = 5BC^2$$

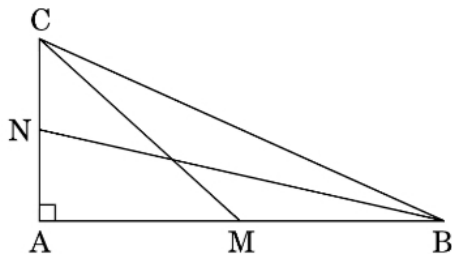


Fig. 3.2.1

(10, 2022)

- 5) If  $A$ ,  $B$  and  $C$  are interior angles of  $\triangle ABC$ , then show that

$$\cos\left(\frac{B+C}{2}\right) = \sin\left(\frac{A}{2}\right)$$

(10, 2020)

- 6) In  $\triangle ABC$ , right-angled at  $A$ , if  $AB = 7\text{cm}$  and  $AC = 24\text{cm}$ , then find  $\sin B$  and  $\tan C$ .

(10, 2021)

- 7) Two angles of a triangle are  $\cot^{-1} 2$  and  $\cot^{-1} 3$ . The third angle of the triangle is

(12, 2021)

- 8)  $A$ ,  $B$  and  $C$  are interior angles of a triangle  $ABC$ . Show that

a)  $\sin\left(\frac{B+C}{2}\right) = \cos\frac{A}{2}$

b) If  $\angle A = 90^\circ$ , then find the value of  $\tan\left(\frac{B+C}{2}\right)$ .

(10, 2019)

- 9) In  $\triangle ABC$ ,  $AB = 4\sqrt{3}\text{ cm}$ ,  $AC = 8\text{cm}$  and  $BC = 4\text{cm}$ . The angle  $B$  is

a)  $120^\circ$

b)  $90^\circ$

c)  $60^\circ$

d)  $45^\circ$

(10, 2021)

### 3.3 JEE

#### Fill In The Blanks

- 3.3.1 In a  $\triangle ABC$ ,  $\angle A = 90^\circ$  and  $AD$  is an altitude. Complete the relation

$$\frac{BD}{BA} = \frac{AB}{(\dots)}$$

(1980)

- 3.3.2  $ABC$  is a triangle,  $P$  is a point on  $AB$ , and  $Q$  is point on  $AC$  such that  $\angle AQP = \angle ABC$ .

Complete the relation  $\frac{\text{area of } \triangle APQ}{\text{area of } \triangle ABC} = \frac{(\dots)}{AC^2}$

(1980)

- 3.3.3  $ABC$  is a triangle with  $\angle B$  greater than  $\angle C$ .  $D$  and  $E$  are the points on  $BC$  such that  $AD$  is perpendicular to  $BC$  and  $AE$  is the bisector of angle  $A$ . Complete the relation

$$\angle DAE = \frac{1}{2}[\angle A - \angle C]$$

(1980)

- 3.3.4 the set of all real numbers  $a$  such that  $a^2 + 2a$ ,  $2a + 3$  and  $a^2 + 3a + 8$  are the sides of a triangle is ...

(1985 - 2 Marks)

- 3.3.5 In a triangle  $ABC$ , if  $\cot A, \cot B, \cot C$  are in A.P., then  $a^2, b^2, c^2$  are in ... progression

(1985 - 2 Marks)

- 3.3.6 A polygon of nine sides, each of length 2, is inscribed in a circle. The radius of the circle is ... (1987 - 2 Marks)
- 3.3.7 If the angles of a triangle are  $30^\circ$  and  $45^\circ$  and the included side is  $(\sqrt{3} + 1)\text{cms}$ , then the area of the triangle is ... (1988 - 2 Marks)
- 3.3.8 If the triangle  $ABC$ ,  $\frac{2\cos A}{a} + \frac{2\cos B}{b} + \frac{2\cos C}{c} = \frac{a}{bc} + \frac{b}{ac}$ , then the value of the angle  $A$  is ... degrees. (1993 - 2 Marks)
- 3.3.9 In the triangle  $ABC$ ,  $AD$  is the altitude from  $A$ . Given  $b > c$ ,  $\angle C = 23^\circ$  and  $AD = \frac{abc}{b^2 - c^2}$  then  $\angle B = \dots$  (1994 - 2 Marks)
- 3.3.10 A circle is inscribed in a equilateral triangle of a side  $a$ . The area of any square inscribed in this circle is ... (1994 - 2 Marks)
- 3.3.11 In a triangle  $ABC$ ,  $a : b : c = 4 : 5 : 6$ . The ratio of the radius of the circumferences to that of the incircle is ... (1996 - 1 Marks)

### JEE Mains / AIEEE

- 3.3.1 The sum of the radii of inscribed and circumscribed circles for an  $n$  sided regular polygon of side  $a$ , is (2003)
- a)  $\frac{a}{4} \cot\left(\frac{\pi}{2n}\right)$       b)  $a \cot\left(\frac{\pi}{n}\right)$       c)  $\frac{a}{2} \cot\left(\frac{\pi}{2n}\right)$       d)  $a \cot\left(\frac{\pi}{2n}\right)$
- 3.3.2 In a triangle  $\triangle ABC$ , medians  $AD$  and  $BE$  are drawn. If  $AD = 4$ ,  $\angle DAB = \frac{\pi}{6}$  and  $\angle ABE = \frac{\pi}{3}$ , then the area of the  $\triangle ABC$  is (2003)
- a)  $\frac{64}{3}$       b)  $\frac{8}{3}$       c)  $\frac{16}{3}$       d)  $\frac{32}{3\sqrt{3}}$
- 3.3.3 If in  $\triangle ABC$   $a \cos^2\left(\frac{C}{2}\right) + c \cos^2\left(\frac{A}{2}\right) = \frac{3b}{2}$ , then the sides  $a, b$  and  $c$  (2003)
- a) satisfy  $a + b = c$       b) are in A.P.      c) are in G.P.      d) are in H.P.
- 3.3.4 The sides of a triangle are  $\sin \alpha$ ,  $\cos \alpha$  and  $\sqrt{1 + \sin \alpha \cos \alpha}$  for some  $0 < \alpha < \frac{\pi}{2}$ . Then the greatest angle of the triangle is (2004)
- a)  $150^\circ$       b)  $90^\circ$       c)  $120^\circ$       d)  $60^\circ$
- 3.3.5 A person standing on the bank of a river observes that the angle of elevation of the top of a tree on the opposite bank of the river is  $60^\circ$  and when he retires 40 meters away from the tree, the angle of elevation becomes  $30^\circ$ . The breadth of the river is (2004)
- a)  $60\text{m}$       b)  $30\text{m}$       c)  $40\text{m}$       d)  $20\text{m}$
- 3.3.6 In a triangle  $\triangle ABC$ , let  $\angle C = \frac{\pi}{2}$ . If  $r$  is the inradius and  $R$  is the circumradius of the triangle  $\triangle ABC$ , then  $2(R + r)$  equals (2005)

- a)  $b + c$                       b)  $a + b$                       c)  $a + b + c$                       d)  $c + a$

3.3.7 If in a  $\triangle ABC$ , let the altitudes from the vertices **A**, **B**, **C** on opposite sides are in H.P., then  $\sin \mathbf{A}, \sin \mathbf{B}, \sin \mathbf{C}$  are in (2005)

- a)  $G.P.$                       b)  $A.P.$                       c)  $A.P. - G.P.$                       d)  $H.P.$

3.3.8 A tower stand at the centre of a circular park. **A** and **B** are two points on the boundary of the park such that  $AB (= a)$  subtends an angle of  $60^\circ$  at the foot of the tower, and the angle of elevation of the top of the tower from **A** or **B** is  $30^\circ$ . The height of the tower is (2007)

- a)  $\frac{a}{\sqrt{3}}$                       b)  $a\sqrt{3}$                       c)  $\frac{2a}{\sqrt{3}}$                       d)  $2a\sqrt{3}$

3.3.9 **AB** is a vertical pole with **B** at the ground level and **A** at the top. A man finds that the angle of elevation the the point **A** from a certain point **C** on the ground is  $60^\circ$ . He moves away from the pole along the line **BC** to a point **D** such that  $CD = 7m$ . From **D** the angle of elevation of point **A** is  $45^\circ$ . Then the height of the pole is (2008)

- a)  $\frac{7\sqrt{3}}{2} \frac{1}{\sqrt{3}-1}m$                       b)  $\frac{7\sqrt{3}}{2} (\sqrt{3} + 1)m$                       c)  $\frac{7\sqrt{3}}{2} (\sqrt{3} - 1)m$                       d)  $\frac{7\sqrt{3}}{2} \frac{1}{\sqrt{3}+1}m$

3.3.10 For a regular polygon, let  $r$  and  $R$  be the radii of the inscribed and the circumscribed circles. A false statement among the following is (2010)

- a) There is a regular polygon with  $\frac{r}{R} = \frac{1}{\sqrt{2}}$   
 b) There is a regular polygon with  $\frac{r}{R} = \frac{2}{3}$   
 c) There is a regular polygon with  $\frac{r}{R} = \frac{\sqrt{3}}{2}$   
 d) There is a regular polygon with  $\frac{r}{R} = \frac{1}{2}$

3.3.11 A bird is sitting on the top of a vertical pole  $20m$  high and its elevation from a point **O** on the ground is  $45^\circ$ . It flies off horizontally straight away from the point **O**. After one second, the elevation of the bird from **O** is reduced to  $30^\circ$ . Then the speed in (in  $m/s$ ) of the bird is (JEEM2014)

- a)  $20\sqrt{2}$                       b)  $20(\sqrt{3} - 1)$                       c)  $40(\sqrt{2} - 1)$                       d)  $40(\sqrt{3} - \sqrt{2})$

3.3.12 If the angle of elevation of the top of a tower from three colinear points **A**, **B** and **C** on a line leading to foot of the tower, are  $30^\circ, 45^\circ$  and  $60^\circ$  respectively, then the ratio,  $AB : BC$ , is: (JEEM2015)

- a)  $1 : \sqrt{3}$                       b)  $2 : 3$                       c)  $\sqrt{3} : 1$                       d)  $\sqrt{3} : \sqrt{2}$

3.3.13 Let a vertical tower **AB** have its end **A** on the level ground. Let **C** be the mid-point of **AB** and **P** be a point on the ground such that  $AP = 2AB$ . If  $\angle BPC = \beta$ , then  $\tan \beta$  is equal to: (JEEM2017)

- a)  $\frac{4}{9}$                       b)  $\frac{6}{7}$                       c)  $\frac{1}{4}$                       d)  $\frac{2}{9}$

3.3.14  $\Delta PQR$  is a triangular park with  $PQ = PR = 200m$ . A T.V. tower stands at the mid-point of  $QR$ . If the angles of the elevation of the top of the tower at **P**, **Q** and **R** are respectively  $45^\circ, 30^\circ$  and  $30^\circ$ , then the height of the tower (*in m*) is: (JEEM2018)

- a) 50                      b)  $100\sqrt{3}$                       c)  $50\sqrt{2}$                       d) 100

*MCQs with Multiple Correct Answers*

3.3.1 There exists a triangle  $ABC$  satisfying the conditions (1986 - 2 mark)

- a)  $b \sin A = a, A < \pi/2$   
 b)  $b \sin A > a, A > \pi/2$   
 c)  $b \sin A > a, A < \pi/2$   
 d)  $b \sin A < a, A < \pi/2, b > a$   
 e)  $b \sin A < a, A > \pi/2, b = a$

3.3.2 In a triangle, the lengths of two larger sides are 10 and 9 respectively. If the angles are in AP, Then length of third side is (1987 - 2 mark)

- a)  $5 - \sqrt{6}$                       d)  $5 + \sqrt{6}$   
 b)  $3\sqrt{3}$                       e) none  
 c) 3

3.3.3 If in a triangle  $PQR$ ,  $\sin P, \sin Q, \sin R$  are in AP, then (1998 - 2 mark)

- a) The altitudes are in AP  
 b) The altitudes are in HP  
 c) The medians are in GP  
 d) The medians are in AP

3.3.4 Let  $A_0A_1A_2A_3A_4A_5$  be a regular hexagon inscribed in a circle of unit radius. Then the product of the lengths of the line segments  $A_0A_1, A_0A_2$  and  $A_0A_4$  is (1998 - 2 mark)

- a)  $\frac{3}{4}$                       c) 3  
 b)  $3\sqrt{3}$                       d)  $\frac{3\sqrt{3}}{2}$

3.3.5 In  $\Delta ABC$ , internal angle bisector of  $\angle A$  meets side  $BC$  in **D**.  $DE \perp AD$  meets  $AC$  in **E** and  $AB$  in **F**. Then (2006-5M,-1)

- a)  $AE$  is HM of  $b$  and  $c$   
 b)  $AD = \frac{2bc}{b+c} \cos \frac{A}{2}$   
 c)  $EF = \frac{4bc}{b+c} \sin \frac{A}{2}$   
 d)  $\Delta AEF$  is isosceles

3.3.6 Let  $ABC$  be a triangle such that  $\angle ACB = \pi/6$  and let  $a, b$  and  $c$  denote lengths of the sides opposite to **A**, **B** and **C** respectively. The value(s) of  $x$  for which  $a = x^2 + x + 1, b = x^2 - 1, c = 2x + 1$  is(are) (2010)

- a)  $-(2 + \sqrt{3})$  c)  $2 + \sqrt{3}$   
 b)  $1 + \sqrt{3}$  d)  $4\sqrt{3}$

3.3.7 In a triangle  $PQR$ ,  $P$  is the largest angle and  $\cos P = \frac{1}{3}$ . Further the incircle of the triangle touches the sides  $PQ$ ,  $QR$  and  $RP$  at  $N$ ,  $L$  and  $M$  respectively, such that the lengths of  $PN$ ,  $QL$  and  $RM$  are consecutive even integers. Then possible length(s) of the side(s) of the triangle is(are) (Jee Adv. 2013)

- a) 16 c) 18  
 b) 24 d) 22

3.3.8 In a triangle  $XYZ$ , let  $x, y, z$  be the lengths of sides opposite to angles  $X, Y, Z$  and  $2s = x + y + z$ . If  $\frac{s-x}{4} = \frac{s-y}{3} = \frac{s-z}{2}$  and area of the incircle of the triangle  $XYZ$  is  $\frac{8\pi}{3}$  (Jee Adv. 2016)

- a) area of the triangle is  $6\sqrt{6}$   
 b) the radius of circumcircle of  $XYZ$  is  $\frac{35\sqrt{6}}{6}$   
 c)  $\sin \frac{X}{2} \sin \frac{Y}{2} \sin \frac{Z}{2} = \frac{4}{35}$   
 d)  $\sin^2 \left( \frac{X+Y}{2} \right) = \frac{3}{5}$

3.3.9 In a triangle  $PQR$ , let  $\angle PQR = 30^\circ$  and the sides  $PQ$  and  $QR$  have lengths  $10\sqrt{3}$  and 10 respectively. Then which of the following statements is(are) TRUE? (Jee Adv. 2018)

- a)  $\angle QPR = 45^\circ$   
 b) the area of the triangle  $PQR$  is  $25\sqrt{3}$  and  $\angle QRP = 120^\circ$   
 c) the radius of the incircle of triangle  $PQR$  is  $10\sqrt{3} - 15$   
 d) the radius of circumcircle  $PQR$  is  $100\pi$

3.3.10 In a non-right-angle triangle  $\Delta PQR$ , let  $p, q, r$  denote the lengths of the sides opposite to the angles at  $P, Q, R$  respectively. The median from  $R$  meets the side  $PQ$  at  $S$ , the perpendicular from  $P$  meets the side  $QR$  at  $E$ ,  $RS$  and  $PE$  intersect at  $O$ . If  $p = \sqrt{3}$ ,  $q = 1$  and the radius of the circumcircle at  $\Delta PQR$  equals 1, then which of the following options is(are) correct. (Jee Adv. 2018)

- a) Radius of incircle of  $\Delta PQR = \frac{\sqrt{3}}{2} (2 - \sqrt{3})$   
 b) Area of  $\Delta SOE = \frac{\sqrt{3}}{12}$   
 c) Length of  $OE = \frac{1}{6}$   
 d) Length of  $RS = \frac{\sqrt{7}}{2}$

### MCQs with a Single Correct Answer

3.3.1 If the bisector of the angle  $P$  of a triangle  $PQR$  meets  $QR$  in  $S$ , then

- a)  $QS = SR$   $PQ$   $PR$  (1979)  
 b)  $QS : SR = PR :$  c)  $QS : SR = PQ :$  d) None of these

3.3.2 From the top of a light-house 60meter high with its base at the sea level the angle of depression of a boat is  $15^\circ$ . The distance of the boat from the foot of the light house.

- a)  $\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right) 60 \text{ metres}$                       c)  $\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)^2 60 \text{ metres}$   
 b)  $\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right) 60 \text{ metres}$                       d) none of these

(1983 - 2 Marks)

3.3.3 In the triangle  $ABC$ , angle  $A$  is the greater than angle  $B$ . If the measures of the angles  $A$  and  $B$  satisfies the equation  $3 \sin x - 4 \sin^3 x - k = 0, 0 < k < 1$ , then the measure of the angle  $C$  is

- a)  $\frac{\pi}{3}$                                               c)  $\frac{2\pi}{3}$   
 b)  $\frac{\pi}{2}$                                               d)  $\frac{5\pi}{6}$

(1985 - 2 Marks)

3.3.4 If the lengths of the sides of triangles are 3, 5, 7 then the largest angles of the triangle is

- a)  $\frac{\pi}{2}$                                               c)  $\frac{2\pi}{3}$   
 b)  $\frac{5\pi}{6}$                                               d)  $\frac{3\pi}{4}$

(1986 - 2 Marks)

3.3.5 In a triangle  $ABC$ ,  $\angle B = \frac{\pi}{3}$  and  $\angle C = \frac{\pi}{4}$ . Let  $D$  divide  $BC$  internally in the ratio 1 : 3 then  $\frac{\sin \angle BAD}{\sin \angle CAD}$  is equal to

- a)  $\frac{1}{\sqrt{6}}$   
 b)  $\frac{1}{3}$   
 c)  $\frac{1}{\sqrt{3}}$   
 d)  $\sqrt{\frac{2}{3}}$

(1995S)

3.3.6 In a triangle  $ABC$ ,  $2ac \sin \frac{1}{2}(A - B + C) =$

- a)  $a^2 + b^2 - c^2$   
 b)  $c^2 + a^2 - b^2$   
 c)  $b^2 - c^2 - a^2$   
 d)  $c^2 - a^2 - b^2$

(2000S)

3.3.7 In a triangle  $ABC$ , let  $\angle C = \frac{\pi}{2}$ . If  $r$  is the inradius and  $R$  is the circumradius of the triangle, then  $2(r + R)$  is equal to

- a)  $a + b$   
 b)  $b + c$   
 c)  $c + a$   
 d)  $a + b + c$



(2000S)

3.3.8 A pole stands vertically inside a triangular park  $\triangle ABC$ . If the angle of elevation of the top of the pole from each corner of the park is same, then in  $\triangle ABC$  the foot of the pole is at the

- a) centroid
- b) circumcentre
- c) incentre
- d) orthocentre

(2000S)

3.3.9 A man from the top of a 100 metres high tower sees a car moving towards the tower at an angle of depression of  $30^\circ$ . After some time, the angle of depression becomes  $60^\circ$ . The distance (in metres) travelled by the car during this time is

- a)  $100\sqrt{3}$
- b)  $\frac{200\sqrt{3}}{3}$
- c)  $\frac{100\sqrt{3}}{3}$
- d)  $200\sqrt{3}$

(2001S)

3.3.10 Which of the following pieces of data does NOT uniquely determine an acute-angled triangle  $\triangle ABC$  ( $R$  being the radius of the circumcircle)?

- a)  $a, \sin A, \sin B$
- b)  $a, b, c$
- c)  $a, \sin B, R$
- d)  $a, \sin A, R$

(2002S)

3.3.11 If the angles of a triangle are in the ratio 4: 1: 1, then the ratio of the longest side to the perimeter is

- a)  $\sqrt{3}: 2 + \sqrt{3}$
- b) 1: 6
- c)  $1: 2 + \sqrt{3}$
- d) 2: 3

(2003S)

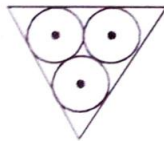
3.3.12 The sides of a triangle are in the ratio 1:  $\sqrt{3}$ : 2, then the angles of the triangle are in the ratio

- a) 1: 3: 5
- b) 2: 3: 4
- c) 3: 2: 1
- d) 1: 2: 3

(2004S)

3.3.13 In an equilateral triangle, 3 coins of radii 1 unit each are kept so they touch each other and also the sides of the triangle. Area of the triangle is

- a)  $4 + 2\sqrt{3}$
- b)  $6 + 4\sqrt{3}$



- c)  $12 + \frac{7\sqrt{3}}{4}$   
 d)  $3 + \frac{7\sqrt{3}}{4}$

(2005S)

3.3.14 In a triangle  $ABC$ ,  $a, b, c$  are the lengths of its sides and  $A, B, C$  are the angles of triangle  $ABC$ . The correct relation is given by

- a)  $(b - c) \sin\left(\frac{B-C}{2}\right) = a \cos\left(\frac{A}{2}\right)$   
 b)  $(b - c) \cos\left(\frac{A}{2}\right) = a \sin\left(\frac{B-C}{2}\right)$   
 c)  $(b - c) \sin\left(\frac{B+C}{2}\right) = a \cos\left(\frac{A}{2}\right)$   
 d)  $(b - c) \cos\left(\frac{A}{2}\right) = a \sin\left(\frac{B+C}{2}\right)$

(2005S)

3.3.15 One angle of an isosceles  $\triangle$  is  $120^\circ$  and radius of its incircle =  $\sqrt{3}$ . Then the area of the triangle in sq. units is

- a)  $7 + 12\sqrt{3}$   
 b)  $12 - 7\sqrt{3}$   
 c)  $12 + 7\sqrt{3}$   
 d)  $4\pi$

(2006 - 3M, -1)

3.3.16 Let  $ABCD$  be a quadrilateral with area 18, with side  $AB$  parallel to the side  $CD$  and  $2AB = CD$ . Let  $AD$  be perpendicular to  $AB$  and  $CD$ . If a circle is drawn inside the quadrilateral  $ABCD$  touching all the sides, then the radius is

- a) 3  
 b) 2  
 c)  $\frac{3}{2}$   
 d) 1

(2007 - 3 Marks)

3.3.17 If the angles  $A, B$  and  $C$  of a triangle are in an arithmetic progression and if  $\mathbf{a}, \mathbf{b}$  and  $\mathbf{c}$  denote the lengths of the sides opposite to  $A, B$  and  $C$  respectively, then the value of the expression  $\frac{a}{c} \sin 2C + \frac{c}{a} \sin 2A$  is

- a)  $\frac{1}{2}$   
 b)  $\frac{\sqrt{3}}{2}$   
 c) 1  
 d)  $\sqrt{3}$

(2010)

3.3.18 Let  $PQR$  be a triangle of area  $\Delta$  with  $a = 2, b = \frac{7}{2}$  and  $c = \frac{5}{2}$ , where  $\mathbf{a}, \mathbf{b}$  and  $\mathbf{c}$  are the lengths of the sides of the triangle opposite to the angles at  $P, Q$  and  $R$  respectively.

Then  $\frac{2 \sin P - \sin 2P}{2 \sin P + \sin 2P}$  equals

- a)  $\frac{3}{4\Delta}$
- b)  $\frac{45}{4\Delta}$
- c)  $\left(\frac{3}{4\Delta}\right)^2$
- d)  $\left(\frac{45}{4\Delta}\right)^2$

(2012)

3.3.19 In a triangle the sum of two sides is  $x$  and the product of the same sides is  $y$ . If  $x^2 - c^2 = y$ , where  $c$  is the third side of the triangle, then the ratio of the inradius to the circum-radius of the triangle is

- a)  $\frac{3y}{2(x+c)}$
- b)  $\frac{3y}{2c(x+c)}$
- c)  $\frac{3y}{4x(x+c)}$
- d)  $\frac{3y}{4c(x+c)}$

(JEE Adv. 2014)

### Subjective Questions

3.3.1 A triangle  $ABC$  has sides  $AB = AC = 5\text{cm}$  and  $BC = 6\text{cm}$ . Triangle  $A'B'C'$  is the reflection of the triangle  $ABC$  in a line parallel to  $AB$  placed at a distance of 2 cm from  $AB$ , outside the triangle  $ABC$ . Triangle  $A''B''C''$  is the reflection of the triangle  $A'B'C'$  in a line parallel to  $B'C'$  placed at a distance of 2 cm from  $B'C'$  outside the triangle  $A'B'C'$ . Find the distance between  $A$  and  $A''$ . (1978)

3.3.2 a) If a circle is inscribed in a right angled triangle  $ABC$  right angled at  $B$ , show that the diameter of the circle is equal to  $AB + BC - AC$ .

b) If a triangle is inscribed in a circle, then the product of any two sides of the triangle is equal to the product of the diameter and perpendicular distance of the third side from the opposite vertex. Prove the above statement.

(1979)

3.3.3 a) A balloon is observed simultaneously from three points  $A, B$  and  $C$  on a straight road directly beneath it. The angular elevation at  $B$  is twice that at  $A$  and angular elevation at  $C$  is thrice that of  $A$ . If the distance between  $A$  and  $B$  is  $a$  and the distance between  $B$  and  $C$  is  $b$ , find height of balloon in terms of  $a$  and  $b$ .

b) Find the area of the smaller part of a disc of radius 10 cm, cut off by a chord  $AB$  which subtends an angle of  $22\frac{1}{2}^\circ$  at the circumference.

(1980)

3.3.4  $ABC$  is a triangle.  $D$  is the middle point of  $BC$ . If  $AD$  is perpendicular to  $AC$ , then prove that  $\cos A \cos C = \frac{2(c^2 - a^2)}{3ac}$ . (1980)

3.3.5  $ABC$  is a triangle with  $AB = AC$ .  $D$  is any point on the side  $BC$ .  $E$  and  $F$  are points on the side  $AB$  and  $AC$ , respectively, such that  $DE$  is parallel to  $AC$ , and  $DF$  is parallel to  $AB$ . Prove that

$$DF + FA + AE + ED = AB + AC$$

(1980)

3.3.6 a)  $PQ$  is a vertical tower.  $P$  is the foot and  $Q$  is the top of the tower.  $A, B, C$  are three points in the horizontal plane through  $P$ . The angles of elevation of  $Q$  from

$A, B, C$  are equal, and each is equal to  $\theta$ . The sides of the triangle  $ABC$  are  $a, b, c$ ; and the area of the triangle  $ABC$  is  $\Delta$ . Show that the height of the tower is  $\frac{abc \tan \theta}{4\Delta}$ .

- b)  $AB$  is a vertical pole. The end  $A$  is on the level ground.  $C$  is the middle point of  $AB$ .  $P$  is a point on the level ground. The portion  $CB$  subtends an angle  $\beta$  at  $P$ . If  $AP = nAB$  then show that  $\tan \beta = \frac{n}{2n^2+1}$ .

(1980)

- 3.3.7 Let the angles  $A, B, C$  of a triangle  $ABC$  be in A.P. and let  $b : c = \sqrt{3} : \sqrt{2}$ . Find the angle  $A$ .

(1981 – 2Marks)

- 3.3.8 A vertical pole stands at a point  $Q$  on a horizontal ground.  $A$  and  $B$  are points on the ground,  $d$  meters apart. The pole subtends angles  $\alpha$  and  $\beta$  at  $A$  and  $B$  respectively.  $AB$  subtends an angle  $\gamma$  at  $Q$ . Find the height of the pole.

(1982 – 3Marks)

- 3.3.9 Four ships  $A, B, C$  and  $D$  are at sea in the following relative positions:  $B$  is on the straight line segment  $AC$ ,  $B$  is due North of  $D$  and  $D$  is due west of  $C$ . The distance between  $B$  and  $D$  is 2 km.  $\angle BDA = 40^\circ$ ,  $\angle BCD = 25^\circ$ . What is the distance between  $A$  and  $D$ ? [Take  $\sin 25^\circ = 0.423$ ]

(1983 – 3Marks)

- 3.3.10 The ex-radii  $r_1, r_2, r_3$  of  $\triangle ABC$  are in H.P. Show that its sides  $a, b, c$  are in A.P.

(1983 – 3Marks)

- 3.3.11 For a triangle  $ABC$  it is given that  $\cos A + \cos B + \cos C = \frac{3}{2}$ . Prove that the triangle is equilateral.

(1984 – 4Marks)

- 3.3.12 With usual notation, if in a triangle  $ABC$ ;  $\frac{b+c}{11} = \frac{c+a}{12} = \frac{a+b}{13}$  then prove that  $\frac{\cos A}{7} = \frac{\cos B}{19} = \frac{\cos C}{25}$ .

(1984 – 4Marks)

- 3.3.13 A ladder rests against a wall at an angle  $\alpha$  to the horizontal. Its foot is pulled away from the wall through a distance  $a$ , so that it slides a distance  $b$  down the wall making an angle  $\beta$  with the horizontal. Show that  $a = b \tan \frac{1}{2}(\alpha + \beta)$ .

(1985 – 5Marks)

- 3.3.14 In a triangle  $ABC$ , the median to the side  $BC$  is of length  $\frac{1}{\sqrt{11-6\sqrt{3}}}$  and it divides the angle  $A$  into angles  $30^\circ$  and  $45^\circ$ . Find the length of the side  $BC$ .

(1985 – 5Marks)

- 3.3.15 If in a triangle  $ABC$ ,  $\cos A \cos B + \sin A \sin B \sin C = 1$ , show that  $a : b : c = 1 : 1 : \sqrt{2}$ .

(1986 – 5Marks)

- 3.3.16 A sign-post in the form of an isosceles triangle  $ABC$  is mounted on a pole of height  $h$  fixed to the ground. The base  $BC$  of the triangle is parallel to the ground. A man standing on the ground at a distance  $d$  from the sign-post finds that the top vertex  $A$  of the triangle subtends an angle  $\beta$  and either of the other two vertices subtends the same angle  $\alpha$  at his feet. Find the area of the triangle.

(1988 – 5Marks)

- 3.3.17  $ABC$  is a triangular park with  $AB = AC = 100\text{m}$ . A television tower stands at the

midpoint of  $BC$ . The angles of elevation of the top of the tower at  $A, B, C$  are  $45^\circ, 60^\circ, 60^\circ$ , respectively. Find the height of the tower.

(1989 – 5Marks)

- 3.3.18 A vertical tower  $PQ$  stands at a point  $P$ . Points  $A$  and  $B$  are located to the South and East of  $P$  respectively.  $M$  is the mid point of  $AB$ .  $PAM$  is an equilateral triangle; and  $N$  is the foot of the perpendicular from  $P$  on  $AB$ . Let  $AN = 20$  metres and the angle of elevation of the top of the tower at  $N$  is  $\tan^{-1} 2$ . Determine the height of the tower and the angles of elevation of the top of the tower at  $A$  and  $B$ .

(1990 – 4Marks)

- 3.3.19 The sides of a triangle are three consecutive natural numbers and its largest angle is twice the smallest one. Determine the sides of the triangle.

(1991 – 4Marks)

- 3.3.20 In a triangle of base  $a$  the ratio of the other two sides is  $r (< 1)$ . Show that the altitude of the triangle is less than or equal to  $\frac{ar}{1-r^2}$ .

(1991 – 4Marks)

- 3.3.21 A man notices two objects in a straight line due west. After walking a distance  $c$  due north he observes that the objects subtend an angle  $\alpha$  at his eye; and, after a further distance  $2c$  due north, and angle  $\beta$ . Show that the distance between the objects is  $\frac{8c}{3 \cot \beta - \cot \alpha}$ ; the height of the man is being ignored.

(1991 – 4Marks)

### 3.4 Circle

- 3.4.1. The perpendicular from the centre of a circle to a chord bisects the chord.
- 3.4.2. The line drawn through the centre of a circle to bisect a chord is perpendicular to the chord.
- 3.4.3. There is one and only one circle passing through three non-collinear points.
- 3.4.4. Equal chords of a circle (or of congruent circles) are equidistant from the centre (or corresponding centres).
- 3.4.5. Chords equidistant from the centre (or corresponding centres) of a circle (or of congruent circles) are equal.
- 3.4.6.  $AB$  is a diameter of the circle,  $CD$  is a chord equal to the radius of the circle.  $AC$  and  $BD$  when extended intersect at a point  $E$ . Prove that  $\angle AEB = 60^\circ$ .
- 3.4.7. If the non-parallel sides of a trapezium are equal, prove that it is cyclic.
- 3.4.8. Prove that the line of centres of two intersecting circles subtends equal angles at the two points of intersection.
- 3.4.9. The angle subtended by an arc at the centre is double the angle subtended by it at any point on the remaining part of the circle.
- 3.4.10. Angles in the same segment of a circle are equal.
- 3.4.11. Angle in a semicircle is a right angle.
- 3.4.12. If a line segment joining two points subtends equal angles at two other points lying on the same side of the line containing the line segment, the four points lie on a circle.
- 3.4.13. The sum of either pair of opposite angles of a cyclic quadrilateral is  $180^\circ$ .
- 3.4.14. If sum of a pair of opposite angles of a quadrilateral is  $180^\circ$ , the quadrilateral is cyclic.

- 3.4.15.  $AB$  is a diameter of the circle,  $CD$  is a chord equal to the radius of the circle.  $AC$  and  $BD$  when extended intersect at a point  $E$ . Prove that  $\angle AEB = 60^\circ$ .
- 3.4.16. Two circles of radii 5 cm and 3 cm intersect at two points and the distance between their centres is 4 cm. Find the length of the common chord.
- 3.4.17. Two chords  $AB$  and  $CD$  of lengths 5 cm and 11 cm respectively of a circle are parallel to each other and are on opposite sides of its centre. If the distance between  $AB$  and  $CD$  is 6 cm, find the radius of the circle.
- 3.4.18. The lengths of two parallel chords of a circle are 6 cm and 8 cm. If the smaller chord is at distance 4 cm from the centre, what is the distance of the other chord from the centre?
- 3.4.19. Two concentric circles are of radii 5 cm and 3 cm. Find the length of the chord of the larger circle which touches the smaller circle.
- 3.4.20. A  $\triangle ABC$  is drawn to circumscribe a circle of radius 4 cm such that the segments  $BD$  and  $DC$  into which  $BC$  is divided by the point of contact  $D$  are of lengths 8 cm and 6 cm respectively. Find the sides  $AB$  and  $AC$ .
- 3.4.21.  $PQ$  is a chord of length 8 cm of a circle of radius 5 cm. The tangents at  $P$  and  $Q$  intersect at a point  $T$ . Find the length  $TP$ .
- 3.4.22. Two circles intersect at two points  $A$  and  $B$ .  $AD$  and  $AC$  are diameters to the two circles. Prove that  $B$  lies on the line segment  $DC$ .
- 3.4.23. Prove that the quadrilateral formed (if possible) by the internal angle bisectors of any quadrilateral is cyclic.
- 3.4.24. The perpendicular from the centre of a circle to a chord bisects the chord.
- 3.4.25. The line drawn through the centre of a circle to bisect a chord is perpendicular to the chord.
- 3.4.26. There is one and only one circle passing through three non-collinear points.
- 3.4.27. Equal chords of a circle (or of congruent circles) are equidistant from the centre (or corresponding centres).
- 3.4.28. If a line intersects two concentric circles (circles with the same centre) with centre  $O$  at  $A, B, C$  and  $D$ , prove that  $AB = CD$ .
- 3.4.29. A chord of a circle is equal to the radius of the circle. Find the angle subtended by the chord at a point on the minor arc and also at a point on the major arc.
- 3.4.30. If diagonals of a cyclic quadrilateral are diameters of the circle through the vertices of the quadrilateral, prove that it is a rectangle.
- 3.4.31. If the non-parallel sides of a trapezium are equal, prove that it is cyclic.
- 3.4.32. Two circles intersect at two points  $B$  and  $C$ . Through  $B$ , two line segments  $ABD$  and  $PBQ$  are drawn to intersect the circles at  $A, D$  and  $P, Q$  respectively. Prove that  $\angle ACP = \angle QCD$ .
- 3.4.33. If circles are drawn taking two sides of a triangle as diameters, prove that the point of intersection of these circles lie on the third side.
- 3.4.34. Prove that the line of centres of two intersecting circles subtends equal angles at the two points of intersection.
- 3.4.35. Let the vertex of an angle  $ABC$  be located outside a circle and let the sides of the angle intersect equal chords  $AD$  and  $CE$  with the circle. Prove that  $\angle ABC$  is equal to half the difference of the angles subtended by the chords  $AC$  and  $DE$  at the centre.
- 3.4.36. Prove that the circle drawn with any side of a rhombus as diameter, passes through

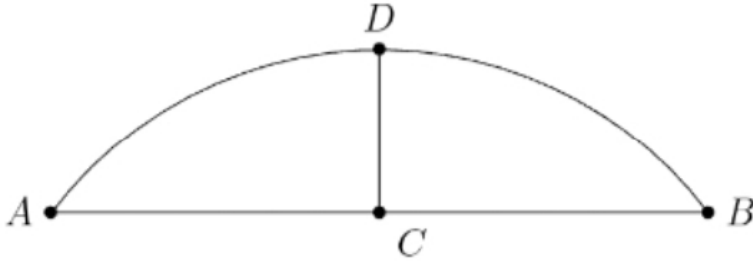
the point of intersection of its diagonals.

- 3.4.37.  $ABCD$  is a parallelogram. The circle through  $A, B$  and  $C$  intersect  $CD$  (produced if necessary) at  $E$ . Prove that  $AE = AD$ .
- 3.4.38.  $AC$  and  $BD$  are chords of a circle which bisect each other. Prove that (i)  $AC$  and  $BD$  are diameters, (ii)  $ABCD$  is a rectangle.
- 3.4.39. Bisectors of angles  $A, B$  and  $C$  of a  $\triangle ABC$  intersect its circumcircle at  $D, E$  and  $F$  respectively. Prove that the angles of the  $\triangle DEF$  are  $90^\circ - \frac{A}{2}$ ,  $90^\circ - \frac{B}{2}$  and  $90^\circ - \frac{C}{2}$ .
- 3.4.40. Two congruent circles intersect each other at points  $A$  and  $B$ . Through  $A$  any line segment  $PAQ$  is drawn so that  $P, Q$  lie on the two circles. Prove that  $BP = BQ$ .
- 3.4.41. In any  $\triangle ABC$ , if the angle bisector of  $\angle A$  and perpendicular bisector of  $BC$  intersect, prove that they intersect on the circumcircle of the  $\triangle ABC$ .
- 3.4.42. Prove that in two concentric circles, the chord of the larger circle, which touches the smaller circle, is bisected at the point of contact.

### 3.5 Olympiad

- $AB$  is tangent to the circles  $CAMN$  and  $NMBD$ .  $M$  lies between  $C$  and  $D$  on the line  $CD$ , and  $CD$  is parallel to  $AB$ . The chords  $NA$  and  $CM$  meet at  $P$ ; the chords  $NB$  and  $MD$  meet at  $Q$ . The rays  $CA$  and  $DB$  meet at  $E$ . Prove that  $PE = QE$ . (IMO 2000)
- Let  $ABCD$  be a convex quadrilateral with perpendicular diagonals. If  $AB = 20$ ,  $BC = 70$ , and  $CD = 90$ , then what is the value of  $DA$ ? (PRERMO 2014)
- In a triangle with integer side lengths, one side is three times as long as a second side, and the length of the third side is 17. What is the greatest possible perimeter of the triangle? (PRERMO 2014)
- In a triangle  $ABC$ ,  $X$  and  $Y$  are points on the segments  $AB$  and  $AC$ , respectively, such that  $AX : XB = 1 : 2$  and  $AY : YC = 2 : 1$ . If the area of triangle  $AXY$  is 10, then what is the area of triangle  $ABC$ ? (PRERMO 2014)
- Let  $XOY$  be a triangle with  $\angle XOY = 90^\circ$ . Let  $M$  and  $N$  be the midpoints of legs  $OX$  and  $OY$ , respectively. Suppose that  $XN = 19$  and  $YM = 22$ . What is  $XY$ ? (PRERMO 2014)
- $PS$  is a line segment of length 4 and  $O$  is the midpoint of  $PS$ . A semicircular arc is drawn with  $PS$  as diameter. Let  $X$  be the midpoint of this arc.  $Q$  and  $R$  are points on the arc  $PXS$  such that  $QR$  is parallel to  $PS$  and the semicircular arc drawn with  $QR$  as diameter is tangent to  $PS$ . What is the area of the region  $QXROQ$  bounded by the two semicircular arcs? (PRERMO 2012)
- $O$  and  $I$  are the circumcentre and incentre of  $\triangle ABC$  respectively. Suppose  $O$  lies in the interior of  $\triangle ABC$  and  $I$  lies on the circle passing through  $B, O$ , and  $C$ . What is the magnitude of  $\angle BAC$  in degrees? (PRERMO 2012)
- In  $\triangle ABC$ , we have  $AC = BC = 7$  and  $AB = 2$ . Suppose that  $D$  is a point on line  $AB$  such that  $B$  lies between  $A$  and  $D$  and  $CD = 8$ . What is the length of the segment  $BD$ ? (PRERMO 2012)
- In rectangle  $ABCD$ ,  $AB = 5$  and  $BC = 3$ . Points  $F$  and  $G$  are on line segment  $CD$  so that  $DF = 1$  and  $GC = 2$ . Lines  $AF$  and  $BG$  intersect at  $E$ . What is the area of  $\triangle ABE$ ? (PRERMO 2012)

- 10) A triangle with perimeter 7 has integer side lengths. What is the maximum possible area of such a triangle? (PRERMO 2012)
- 11)  $ABCD$  is a square and  $AB = 1$ . Equilateral triangles  $AYB$  and  $CXD$  are drawn such that  $X$  and  $Y$  are inside the square. What is the length of  $XY$ ? (PRERMO 2012)
- 12) The figure below shows a broken piece of a circular plate made of glass.



$C$  is the midpoint of  $AB$ , and  $D$  is the midpoint of arc  $AB$ . Given that  $AB = 24$  cm and  $CD = 6$  cm, what is the radius of the plate in centimeters? (The figure is not drawn to scale.) (PRERMO 2015)

- 13) A  $2 \times 3$  rectangle and a  $3 \times 4$  rectangle are contained within a square without overlapping at any interior point, and the sides of the square are parallel to the sides of the two given rectangles. What is the smallest possible area of the square? (PRERMO 2015)
- 14) What is the greatest possible perimeter of a right-angled triangle with integer side lengths if one of the sides has length 12? (PRERMO 2015)
- 15) In rectangle  $ABCD$ ,  $AB = 8$  and  $BC = 20$ . Let  $P$  be a point on  $AD$  such that  $\angle BPC = 90^\circ$ . If  $r_1, r_2, r_3$  are the radii of the incircles of triangles  $APB, BPC$ , and  $CPD$ , what is the value of  $r_1 + r_2 + r_3$ ? (PRERMO 2015)
- 16) In the acute-angled triangle  $ABC$ , let  $D$  be the foot of the altitude from  $A$ , and  $E$  be the midpoint of  $BC$ . Let  $F$  be the midpoint of  $AC$ . Suppose  $\angle BAE = 40^\circ$ . If  $\angle DAE = \angle DFE$ , what is the magnitude of  $\angle ADF$  in degrees? (PRERMO 2015)
- 17) The circle  $\omega$  touches the circle  $\Omega$  internally at  $P$ . The center  $O$  of  $\Omega$  is outside  $\omega$ . Let  $XY$  be a diameter of  $\Omega$  which is also tangent to  $\omega$ . Assume  $PY > PX$ . Let  $PY$  intersect  $\omega$  at  $Z$ . If  $YZ = 2PZ$ , what is the magnitude of  $\angle LPYX$  in degrees? (PRERMO 2015)
- 18) On each side of an equilateral triangle with side length  $n$  units, where  $n$  is an integer,  $1 \leq n \leq 100$ , consider  $n - 1$  points that divide the side into  $n$  equal segments. Through these points, draw lines parallel to the sides of the triangle, obtaining a net of equilateral triangles of side length one unit. On each of the vertices of these small triangles, place a coin head up. Two coins are said to be adjacent if the distance between them is 1 unit. A move consists of flipping over any three mutually adjacent coins. Find the number of values of  $n$  for which it is possible to turn all coins tail up after a finite number of moves. (IOQM 2015)



- 19) In an equilateral triangle of side length 6, pegs are placed at the vertices and also evenly along each side at a distance of 1 from each other. Four distinct pegs are chosen from the 15 interior pegs on the sides (that is, the chosen ones are not vertices of the triangle) and each peg is joined to the respective opposite vertex by a line segment. If  $N$  denotes the number of ways we can choose the pegs such that the drawn line segments divide the interior of the triangle into exactly nine regions, find the sum of the squares of the digits of  $N$ . (IOQM 2015)
- 20) In a triangle  $ABC$ , let  $E$  be the midpoint of  $AC$  and  $F$  be the midpoint of  $AB$ . The medians  $BE$  and  $CF$  intersect at  $G$ . Let  $Y$  and  $Z$  be the midpoints of  $BE$  and  $CF$ , respectively. If the area of triangle  $ABC$  is 480, find the area of triangle  $GYZ$ . (IOQM 2015)
- 21) The six sides of a convex hexagon  $A_1A_2A_3A_4A_5A_6$  are colored red. Each of the diagonals of the hexagon is colored either red or blue. If  $N$  is the number of colorings such that every triangle  $A_iA_jA_k$ , where  $1 \leq i < j < k \leq 6$ , has at least one red side, find the sum of the squares of the digits of  $N$ . (IOQM 2015)
- 22) Let  $X$  be the set of all even positive integers  $n$  such that the measure of the angle of some regular polygon is  $n$  degrees. Find the number of elements in  $X$ . (IOQM 2015)
- 23) Let  $ABCD$  be a unit square. Suppose  $M$  and  $N$  are points on  $BC$  and  $CD$ , respectively, such that the perimeter of triangle  $MCN$  is 2. Let  $O$  be the circumcenter of triangle  $MAN$ , and  $P$  be the circumcenter of triangle  $MON$ . If  $\left(\frac{OP}{OA}\right)^2 = \frac{m}{n}$  for some relatively prime positive integers  $m$  and  $n$ , find the value of  $m + n$ . (IOQM 2015)
- 24) Let  $ABC$  be a triangle in the  $xy$ -plane, where  $B$  is at the origin  $(0,0)$ . Let  $BC$  be produced to  $D$  such that  $BC : CD = 1 : 1$ ,  $CA$  be produced to  $E$  such that  $CA : AE = 1 : 2$ , and  $AB$  be produced to  $F$  such that  $AB : BF = 1 : 3$ . Let  $G(32, 24)$  be the centroid of triangle  $ABC$  and  $K$  be the centroid of triangle  $DEF$ . Find the length  $GK$ . (IOQM 2015)
- 25) In the coordinate plane, a point is called a lattice point if both of its coordinates are integers. Let  $A$  be the point  $(12, 84)$ . Find the number of right-angled triangles  $ABC$  in the coordinate plane where  $B$  and  $C$  are lattice points, having a right angle at the vertex  $A$  and whose incenter is at the origin  $(0, 0)$ . (IOQM 2015)
- 26) A trapezium in the plane is a quadrilateral in which a pair of opposite sides are parallel. A trapezium is said to be non-degenerate if it has positive area. Find the number of mutually non-congruent, non-degenerate trapeziums whose sides are four distinct integers from the set  $\{5, 6, 7, 8, 9, 10\}$ . (IOQM 2015)
- 27) In triangle  $ABC$ , point  $A_1$  lies on side  $BC$  and point  $B_1$  lies on side  $AC$ . Let  $P$  and  $Q$  be points on segments  $AA_1$  and  $BB_1$ , respectively, such that  $PQ \parallel AB$ . Let  $P_1$  be a point on line  $PB_1$  such that  $B_1$  lies strictly between  $P$  and  $P_1$ , and  $\angle PP_1C = \angle BAC$ . Similarly, let  $Q_1$  be a point on line  $QA_1$  such that  $A_1$  lies strictly between  $Q$  and  $Q_1$ , and  $\angle CQ_1Q = \angle CBA$ . Prove that points  $P, Q, P_1$ , and  $Q_1$  are concyclic. (IMO 2019)
- 28) Let  $I$  be the in center of acute triangle  $ABC$  with  $AB \neq AC$ . The incircle  $\omega$  of  $ABC$  is tangent to sides  $BC$ ,  $CA$ , and  $AB$  at points  $D$ ,  $E$ , and  $F$ , respectively. The line through  $D$  perpendicular to  $EF$  meets  $\omega$  again at  $R$ . Line  $AR$  meets  $\omega$  again at  $S$ . (IMO 2019)

again at  $P$ . The circumcircles of triangles  $PCE$  and  $PBF$  meet again at  $Q$ .

Prove that lines  $DI$  and  $PQ$  meet on the line through  $A$  that is perpendicular to  $AI$ . (IMO 2019)

- 29) consider the convex quadrilateral  $ABCD$ . The point  $P$  is the interior of  $ABCD$ . The following ratio equalities hold:

$$\angle PAD : \angle PBA : \angle DPA = 1 : 2 : 3 = \angle CBP : \angle BAP : \angle BPC. \quad (29.1)$$

prove that the following three lines meet in a point: the internal bisectors of angles  $\angle ADP$  and  $\angle PCB$  and the perpendicular bisector of segment  $AB$  (IMO 2020)

- 30) Prove that there exists a positive constant  $c$  such that the following statement is true: Consider an integer  $n > 1$ , and a set  $S$  of  $n$  points in the plane such that the distance between any two different points in  $S$  is at least 1. It follows that there is a line  $l$  separating  $S$  such that the distance from any point of  $S$  to  $l$  is at least  $cn^{\frac{-1}{3}}$ . (A line  $l$  separates a set of points  $S$  if some segment joining two points in  $S$  crosses  $l$ .) Note. Weaker results with replaced by  $cn^\alpha$  may be awarded points depending on the value of the constant  $\alpha > 1/3$ . (IMO 2020)

- 31) Let  $D$  be an interior point of the acute triangle  $ABC$  with  $AB > AC$  so that  $\angle DAB = \angle CAD$ . The point  $E$  on the segment  $AC$  satisfies  $\angle ADE = \angle BCD$ , the point  $F$  on the segment  $AB$  satisfies  $\angle FDA = \angle DBC$ , and the point  $X$  on the line  $AC$  satisfies  $CX = BX$ . Let  $O_1$  and  $O_2$  be the circumcentres of the triangles  $ADC$  and  $EXD$ , respectively. Prove that the lines  $BC$ ,  $EF$ , and  $O_1O_2$  are concurrent (IMO 2021)

- 32) Let  $r$  be a circle with centre  $I$ , and  $ABCD$  a convex quadrilateral such that each of the segments  $AB$ ,  $BC$ ,  $CD$  and  $DA$  is a tangent to  $r$ . Let  $\Omega$  be the circumcircle of the triangle  $AIC$ . The extension of  $BA$  beyond  $A$  meets  $\Omega$  at  $X$ , and the extension of  $BC$  beyond  $C$  meets  $\Omega$  at  $Z$ . The extensions of  $AD$  and  $CD$  beyond  $D$  meet  $\Omega$  at  $Y$  and  $T$ , respectively. Prove that

$$AD + DT + TX + XA = CD + DY + YZ + ZC \quad (32.1)$$

(IMO 2021)

- 33) Let  $ABCDE$  be a convex pentagon such that  $BC = DE$ . Assume that there is a point  $T$  inside  $ABCDE$  with  $TB = TD$ ,  $TC = TE$  and  $\angle ABT = \angle TEA$ . Let line  $AB$  intersect lines  $CD$  and  $CT$  at points  $P$  and  $Q$ , respectively. Assume that the points  $P, B, A, Q$  occur on their line in that order. Let line  $AE$  intersect lines  $CD$  and  $DT$  at points  $R$  and  $S$ , respectively. Assume that the points  $R, E, A, S$  occur on their line in that order. Prove that the points  $P, S, Q, R$  lie on a circle. (IMO 2022)

- 34) Let  $ABC$  be an acute-angled triangle with  $AB \leq AC$ . Let  $\Omega$  be the circumcircle of  $ABC$ . Let  $S$  be the midpoint of the arc  $CB$  of  $\Omega$  containing  $A$ . The perpendicular from  $A$  to  $BC$  meets  $BS$  at  $D$  and meets  $\Omega$  again at  $E \neq A$ . The line through  $D$  parallel to  $BC$  meets line  $BE$  at  $L$ . Denote the circumcircle of triangle  $BDL$  by  $\omega$ . Let  $\omega$  meet  $\Omega$  again at  $P \neq B$ . Prove that the line tangent to  $\omega$  at  $P$  meets line  $BS$  on the internal angle bisector of  $\angle BAC$ . (IMO 2023)

- 35) Let  $ABC$  be an equilateral triangle. Let  $A_1, B_1, C_1$  be interior points of  $ABC$  such that  $BA_1 = A_1C$ ,  $CB_1 = B_1A$ ,  $AC_1 = C_1B$ , and  $\angle BAC + \angle CB_1A + \angle AC_1B = 480^\circ$ . Let  $BC_1$  and  $CB_1$  meet at  $A_2$ , let  $CA_1$  and  $AC_1$  meet at  $B_2$ , and let  $AB_1$  and

$BA_1$  meet at  $C_2$ . Prove that if triangle  $A_1B_1C_1$  is scalene, then the three circumcircles of triangles  $AA_1A_2$ ,  $BB_1B_2$  and  $CC_1C_2$  all pass through two common points. (Note: no 2 sides have equal length.) (IMO 2023)

- 36) Let  $ABC$  be a triangle with  $AB \leq AC \leq BC$ . Let the incentre and incircle of triangle  $ABC$  be  $I$  and  $\omega$ , respectively. Let  $X$  be the point on line  $BC$  different from  $C$  such that the line through  $X$  parallel to  $AC$  is tangent to  $\omega$ . Similarly, let  $Y$  be the point on line  $BC$  different from  $B$  such that the line through  $Y$  parallel to  $AB$  is tangent to  $\omega$ . Let  $AI$  intersect the circumcircle of triangle  $ABC$  again at  $P \neq A$ . Let  $K$  and  $L$  be the midpoints of  $AC$  and  $AB$ , respectively. Prove that  $\angle KIL + \angle YPX = 180^\circ$ . (IMO 2024)
- 37) Three points  $X, Y, Z$  are on a straight line such that  $XY = 10$  and  $XZ = 3$ . What is the product of all possible values of  $YZ$ ? (Prermo 2013)
- 38) Let  $AD$  and  $BC$  be the parallel sides of a trapezium  $ABCD$ . Let  $P$  and  $Q$  be the midpoints of the diagonals  $AC$  and  $BD$ . If  $AD = 16$  and  $BC = 20$ , what is the length of  $PQ$ ? (Prermo 2013)
- 39) In a triangle  $ABC$ , let  $H$ ,  $I$ , and  $O$  be the orthocenter, incenter, and circumcenter, respectively. If the points  $B$ ,  $H$ ,  $I$ , and  $C$  lie on a circle, what is the magnitude of  $\angle BOC$  in degrees? (Prermo 2013)
- 40) Let  $ABC$  be an equilateral triangle. Let  $P$  and  $S$  be points on  $AB$  and  $AC$ , respectively, and let  $Q$  and  $R$  be points on  $BC$  such that  $PQRS$  is a rectangle. If  $PQ = \sqrt{3} \times PS$  and the area of  $PQRS$  is  $\frac{28}{3}$ , what is the length of  $PC$ ? (Prermo 2013)
- 41) Let  $A_1, B_1, C_1, D_1$  be the midpoints of the sides of a convex quadrilateral  $ABCD$  and let  $A_2, B_2, C_2, D_2$  be the midpoints of the sides of the quadrilateral  $A_1B_1C_1D_1$ . If  $A_2B_2C_2D_2$  is a rectangle with sides 4 and 6, then what is the product of the lengths of the diagonals of  $ABCD$ ? (Prermo 2013)
- 42) Let  $S$  be a circle with center  $O$ . A chord  $AB$ , not a diameter, divides  $S$  into two regions  $R_1$  and  $R_2$ . Let  $S_1$  be a circle with center in  $R_1$  touching  $AB$ , the circle  $S$  internally. Let  $S_2$  be a circle with center in  $R_2$  touching  $AB$  at  $Y$ , the circle  $S$  internally, and passing through the center of  $S$ . The point  $X$  lies on the diameter passing through the center of  $S_2$ , and  $\angle YXO = 30^\circ$ . If the radius of  $S_2$  is 100, then what is the radius of  $S$ ? (Prermo 2013)
- 43) In a triangle  $ABC$  with  $\angle BCA = 90^\circ$ , the perpendicular bisector of  $AB$  intersects segments  $AB$  and  $AC$  at  $X$  and  $Y$ , respectively. If the ratio of the area of quadrilateral  $BXYC$  to the area of triangle  $ABC$  is 13:18 and  $BC = 12$ , then what is the length of  $AC$ ? (Prermo 2013)
- 44) A convex hexagon has the property that for any pair of opposite sides the distance between their midpoints is  $\frac{\sqrt{3}}{2}$  times the sum of their lengths. Show that all the hexagon's angles are equal. (IMO 2003)
- 45)  $ABCD$  is cyclic. The feet of the perpendiculars from  $D$  to the lines  $AB$ ,  $BC$ ,  $CA$  are  $P$ ,  $Q$ ,  $R$  respectively. Show that the angle bisectors of  $ABC$  and  $CDA$  meet on the line  $AC$  iff  $RP = RQ$ . (IMO 2003)
- 46) Let  $ABC$  be an acute-angled triangle with circumcentre  $O$ . Let  $P$  on  $BC$  be the foot of the altitude from  $A$ . Suppose that  $\angle BCS \leq \angle ABC + 30^\circ$ .

Prove that  $\angle CAB + \angle \leq \angle 90^\circ$ .

(IMO 2001)

- 47) In a triangle  $ABC$ , let  $AP$  bisect  $\angle BAC$ , with  $P$  on  $BC$ , and let  $BQ$  bisect  $\angle ABC$ , with  $Q$  on  $CA$ . It is known that  $\angle BAC = 60^\circ$  and that  $AB + BP = AQ + QB$ . What are the possible angles of triangle  $ABC$ ? (IMO 2001)
- 48)  $BC$  is a diameter of a circle center  $O$ .  $A$  is any point on the circle with  $\angle AOC > 60^\circ$ .  $EF$  is the chord which is the perpendicular bisector of  $AO$ .  $D$  is the midpoint of the minor arc  $AB$ . The line through  $O$  parallel to  $AD$  meets  $AC$  at  $J$ . Show that  $J$  is the incenter of triangle  $CEF$ . (IMO 2002)
- 49)  $n > 2$  circles of radius 1 are drawn in the plane so that no line meets more than two of the circles. Their centers are  $O_1, O_2, \dots, O_n$ . Show that  $\sum_{i < j} 1/O_i O_j \leq (n-1) \frac{\pi}{4}$ . (IMO 2002)
- 50) In the plane two different points  $O$  and  $A$  are given. For each point  $X$  of the plane, other than  $O$ , denote by  $a(X)$  the measure of the angle between  $OA$  and  $OX$  in radians counterclockwise from  $OA$  ( $0 \leq a(X) < 2\pi$ ). Let  $C(X)$  be the circle with center  $O$  and radius of length  $\frac{OX + a(X)}{OX}$ . Each point of the plane is colored by one of a finite number of colors. Prove that for each  $Y$  for which  $a(Y) > 0$  such that color appears on the circumference of the circle  $C(Y)$ . (IMO 1984)
- 51) Let  $ABCD$  be a convex quadrilateral such that the line  $CD$  is a tangent to the circle on  $AB$  as diameter. Prove that the line  $AB$  is a tangent to the circle on  $CD$  as diameter if and only if the lines  $BC$  and  $AD$  are parallel. (IMO 1984)
- 52) Let  $d$  be the sum of the lengths of all the diagonals of a plane convex polygon with  $n$  vertices ( $n > 3$ ), and let  $p$  be its perimeter. Prove that.

$$In - 3 < \frac{2d}{p} < \left(\frac{n}{2}\right) \left(\frac{n+1}{2}\right) - 2,$$

Where  $(x)$  denotes the greatest integer not exceeding  $x$

(IMO 1984)

- 53) Let  $A$  be one of the two distinct points of intersection of two unequal coplanar tangents to the circles  $C_1$  and  $C_2$  with centers  $O_1$  and  $O_2$ , respectively. One of the common tangents to the circles touches  $C_1$  at  $P_1$  and  $C_2$  at  $P_2$ , while the other touches  $C_1$  at  $Q_1$  and  $C_2$  at  $Q_2$ . Let  $M_1$  be the midpoint of  $P_1Q_1$ ,  $M_2$  be the midpoint of  $P_2Q_2$ . Prove that  $\angle O_1AO_2 = \angle M_1AM_2$ . (IMO 1983)
- 54) A circle has center on the side  $AB$  of the cyclic quadrilateral  $ABCD$ . The other three sides are tangent to the circle. Prove that  $AD + BC = AB$ . (IMO 1985)
- 55) A circle with center  $O$  passes through the vertices  $A$  and  $C$  of triangle  $ABC$  and intersects the segments  $AB$  and  $BC$  again at distinct points  $K$  and  $N$  respectively. The circumscribed circle of the triangle  $ABC$  and  $EBN$  intersect at exactly two distinct points  $B$  and  $M$ . Prove that angle  $OMB$  is a right angle. (IMO 1985)
- 56)  $P$  is a point inside a given triangle  $ABC$ .  $D, E, F$  are the feet of the perpendiculars from  $P$  to the lines  $BC, CA, AB$  respectively. Find all  $P$  for which  $\frac{BC}{PD} + \frac{CA}{PE} + \frac{AB}{PF}$  is least. (IMO 1981)
- 57) Three congruent circles have a common point  $O$  and lie inside a given triangle. Each circle touches a pair of sides of the triangle. Prove that the incenter and the circumcenter of the triangle and the point  $O$  are collinear. (IMO 1981)
- 58) A non-isosceles triangle  $A_1A_2A_3$  is given with sides  $a_1, a_2, a_3$  ( $a_i$  is the side opposite  $A_i$ ). For all  $i = 1, 2, 3$ ,  $M_i$  is the midpoint of side  $a_i$  and  $T_i$  is the point where the

incircle touches side  $a_i$ . Denote by  $S_i$  the reflection of  $T_i$  in the interior bisector of an angle  $A_i$ . Prove that the lines  $M_1S_1, M_2S_2$  and  $M_3S_3$  are concurrent. (IMO 1982)

- 59) The diagonals  $AC$  and  $CE$  of the regular hexagon  $ABCDEF$  are divided by the inner points  $M$  and  $N$ , respectively, so that

$$\frac{AM}{AC} = \frac{CN}{CE} = r.$$

Determine  $r$  if  $B, M$ , and  $N$  are collinear.

(IMO 1982)

- 60) Let  $S$  be a square with sides of length 100, and let  $L$  be a path within  $S$  which does not meet itself and which is composed of line segments  $A_0A_1, A_1A_2, \dots, A_{n-1}A_n$  with  $A_0 \neq A_n$ . Suppose that for every point  $P$  of the boundary of  $S$  there is a point of  $L$  at a distance from  $P$  not greater than  $\frac{1}{2}$ . Prove that there are two points  $X$  and  $Y$  in  $L$  such that the distance between  $X$  and  $Y$  is not greater than 1, and the length of that part of  $L$  which lies between  $X$  and  $Y$  is not smaller than 198. (IMO 1982)
- 61) A triangle  $A_1A_2A_3$  and a point  $P_0$  are given in the plane. We define  $A_s = A_s - 3$  for all  $s \geq 4$ . We construct a set of points  $P_1, P_2, P_3, \dots$ , such that  $P_{k+1}$  is the image of  $P_k$  under a rotation with center  $A_{k+1}$  through angle  $120^\circ$  clockwise (for  $k = 0, 1, 2, 3, \dots$ ). Prove that if  $P_{1986} = P_0$ , then the triangle  $A_1A_2A_3$  is equilateral. (IMO 1986)
- 62) Let  $A, B$  be adjacent vertices of a regular  $n$ -gon ( $n \leq 5$ ) in the plane having center at  $O$ . A triangle  $XYZ$ , which is congruent to and initially coincides with  $OAB$ , moves in the plane in such a way that  $Y$  and  $Z$  each trace out the whole boundary of the polygon,  $X$  remaining inside the polygon. Find the locus of  $X$ . (IMO 1986)
- 63) In an acute-angled triangle  $ABC$  the interior bisector of the angle  $A$  intersects  $BC$  at  $L$  and intersects the circumcircle of  $ABC$  again at  $N$ . From point  $L$  perpendiculars are drawn to  $AB$  and  $AC$ , the feet of these perpendiculars being  $K$  and  $M$  respectively. Prove that the quadrilateral  $AKNM$  and the triangle  $ABC$  have equal areas. (IMO 1987)
- 64) Prove that there is no function  $f$  from the set of non-negative integers into itself such that  $f(f(n)) = n + 1987$  for every  $n$ . (IMO 1987)
- 65) Consider two coplanar circles of radii  $R$  and  $r$  ( $R > r$ ) with the same center. Let  $P$  be a fixed point on the smaller circle and  $B$  a variable point on the larger circle. The line  $BP$  meets the larger circle again at  $C$ . The perpendicular  $l$  to  $BP$  at  $P$  meets the smaller circle again at  $A$ . (If  $l$  is tangent to the circle at  $P$  then  $A = P$ ) (i) Find the set of values of  $BC^2 + CA^2 + AB^2$  (ii) Find the locus of the midpoint of  $BC$ . (IMO 1988)
- 66)  $ABC$  is a triangle right-angled at  $A$ , and  $D$  is the foot of the altitude from  $A$ . The straight line joining the incenters of the triangles  $ABD, ACD$  intersects the sides  $AB, AC$  at the points  $K, L$  respectively.  $S$  and  $T$  denote the areas of the triangles  $ABC$  and  $AKL$  respectively. Show that  $S \geq 2T$ . (IMO 1988)
- 67) Problem 5. A configuration of 4027 points in the plane is called Colombian if it consists of 2013 red points and 2014 blue points, and no three of the points of the configuration are collinear. By drawing some lines, the plane is divided into several regions. An arrangement of lines is good for a Colombian configuration if the following two conditions are satisfied: \* no line passes through any point of the configuration; \* no region contains points of both colours

Find the least value of  $k$  such that for any Colombian configuration of 4027 points, there is a good arrangement of  $k$  lines (Imo 2013)

- 68) Problem 6. Let the excircle of triangle  $ABC$  opposite the vertex  $A$  be tangent to the side  $BC$  at the point  $A_1$ . Define the points  $B_1$ , on  $CA$  and  $C_1$ , on  $AB$  analogously, using the excircles opposite  $B$  and  $C$ , respectively. Suppose that the circumcentre of triangle  $A_1B_1C_1$ , lies on the circumcircle of triangle  $ABC$ . Prove that triangle  $ABC$  is right-angled. (Imo 2013)

The excircle of triangle  $ABC$  opposite the vertex  $A$  is the circle that is tangent to the line segment  $BC$ , to the ray  $AB$  beyond  $B$ , and to the ray  $AC$  beyond  $C$ . The excircles opposite  $B$  and  $C$  are similarly defined. (Imo 2013)

- 69) problem7 Let  $ABC$  be an acute-angled triangle with orthocentre  $H$ , and let  $W$  be a point on the side  $BC$ , lying strictly between  $B$  and  $C$ . The points  $M$  and  $N$  are the feet of the altitudes from  $B$  and  $C$ , respectively. Denote by  $w_1$  the circumcircle of  $BWN$ , and let  $X$  be the point on  $w_1$  such that  $WX$  is a diameter of  $w_1$ . Analogously, denote by  $w_2$  the circumcircle of  $CWM$ , and let  $Y$  be the point on  $w_2$  such that  $WY$  is a diameter of  $w_2$ . Prove that  $X$ ,  $Y$  and  $H$  are collinear. (Imo 2013)

- 70) Problem 8. Let  $Q_{>0}$  be the set of positive rational numbers. Let  $f : Q_{>0} \rightarrow R$  be a function satisfying the following three conditions:

- for all  $x, y \in Q_{>0}$ , we have  $f(x)f(y) \geq f(xy)$
  - for all  $x, y \in Q_{>0}$ , we have  $f(x+y) \geq f(x) + f(y)$
  - there exists a rational number  $a > 1$  such that  $f(a) = a$ .
- prove that  $F(x) = x$  for all  $x \in Q_{>0}$ .

(Imo 2013)

- 71) Problem 9. let  $n \geq 2$  be an integer. Consider an  $n \times n$  chessboard consisting of  $n^2$  unit squares. A configuration of  $n$  rooks on this board is peaceful if every row and every column contains exactly one rook. Find the greatest positive integer  $k$  such that, for each peaceful configuration of  $n$  rooks, there is a  $k \times k$  square which does not contain a rook on any of its  $k^2$  unit squares. (Imo 2014)

- 72) Problem 10. Convex quadrilateral  $ABCD$  has  $\angle ABC = \angle CDA = 90^\circ$ . Point  $H$  is the foot of the perpendicular from  $A$  to  $BD$ . Points  $S$  and  $T$  lie on sides  $AB$  and  $AD$ , respectively, such that  $H$  lies inside triangle  $SCT$  and  $\angle CHS - \angle CSB = 90^\circ$ ,  $\angle THC - \angle DTC = 90^\circ$ . Prove that line  $BD$  is tangent to the circumcircle of triangle  $TSH$ . (Imo 2014)

- 73) Problem 4. Points  $P$  and  $Q$  lie on side  $BC$  of acute-angled triangle  $ABC$  so that  $\angle PAB = \angle BCA$  and  $\angle CAQ = \angle ABC$ . Points  $M$  and  $N$  lie on lines  $AP$  and  $AQ$ , respectively, such that  $P$  is the midpoint of  $AM$ , and  $Q$  is the midpoint of  $AN$ . Prove that lines  $BM$  and  $CN$  intersect on circumcircle of triangle  $ABC$  (Imo 2014)

- 74) Problem 11. A set of lines in the plane is in general position if no two are parallel and no three pass through the same point. A set of lines in general position cuts the plane into regions, some of which have finite area; we call these its finite regions. Prove that for all sufficiently large  $n$ , in any set of  $n$  lines in general position it is possible to colour at least  $\sqrt{n}$  of the lines blue in such a way that none of its finite regions has a completely blue boundary.

Note: Results with  $\sqrt{n}$  replaced by  $c\sqrt{n}$  will be awarded points depending on the

value of the constant  $c$ .

(Imo 2014)

- 75) Problem 12. We say that a finite set  $S$  of points in the plane is balanced if, for any two different points  $A$  and  $B$  in  $S$ , there is a point  $C$  in  $S$  such that  $AC = BC$ . We say that  $S$  is centre-free if for any three different points  $A, B$  and  $C$  in  $S$ , there is no point  $P$  in  $S$  such that  $PA = PB = PC$

- Show that for all integers  $n \geq 3$ , there exists a balanced set consisting of  $n$  points.
- Determine all integers  $n \geq 3$  for which there exists a balanced centre-free set consisting of  $n$  points.

(Imo 2015)

- 76) Problem 13. Determine all triples  $(a, b, c)$  of positive integers such that each of the numbers  $ab - c, bc - a, ca - b$  is a power of 2

(A power of 2 is an integer of the form  $2^n$ , Where  $n$  is a non-negative integer). (Imo 2015)

- 77) Problem 14. Let  $ABC$  be an acute triangle with  $AB > AC$ . Let  $I$  be its circumcentre,  $H$  its orthocentre, and  $F$  the foot of the altitude from  $A$ . Let  $M$  be the midpoint of  $BC$ . Let  $Q$  be the point on  $AI$  such that  $\angle HQA = 90^\circ$ , and let  $K$  be the point on  $AI$  such that  $\angle HKQ = 90^\circ$ . Assume that the points  $A, B, C, K$  and  $Q$  are all different, and lie on  $AI$  in this order.

Prove that the circumcircles of triangles  $KQH$  and  $FKM$  are tangent to each other. (Imo2015)

- 78) Problem 15. Triangle  $ABC$  has circumcircle  $\Omega$  and circumcentre  $O$ . A circle  $T$  with centre  $A$  intersects the segment  $BC$  at points  $D$  and  $E$ , such that  $B, D, E$  and  $C$  are all different and lie on line  $BC$  in this order. Let  $F$  and  $G$  be the points of intersection of  $T$  and  $\Omega$ , such that  $A, F, B, C$  and  $G$  lie on  $\Omega$  in this order. Let  $K$  be the second point of intersection of the circumcircle of triangle  $BDF$  and the segment  $AB$ . Let  $L$  be the second point of intersection of the circumcircle of triangle  $CGE$  and the segment  $CA$ . Suppose that the lines  $FK$  and  $GL$  are different and intersect at the point  $X$ . Prove that  $X$  lies on the line  $AO$ . (Imo 2015)

- 79) Problem 16. Let  $R$  be the set of real numbers. Determine all functions  $f : R \rightarrow R$  satisfying the equation

$$f(x + f(x + y)) + f(xy) = x + f(x + y) + yf(x) \quad (79.1)$$

for all real numbers  $x$  and  $y$

(Imo2015)

- 80) problem 17 the sequence  $a_1, a_2, \dots$  of an integers satisfies the following conditions;

- $1 \leq a_j \leq 2015$  for all  $j \geq 1$ ;
- $k + a_k \neq l + a_l$  for all  $1 \leq k < l$ .

prove that there exist two positive integers  $b$  and  $N$  such that

$$\left| \sum_{j=m+1}^n (a_j - b) \right| \leq 1007^2$$

for all integers  $m$  and  $n$  satisfying  $n > m \geq N$

(Imo 2015)

- 81) Prove that the set  $\{1, 2, \dots, 1989\}$  can be expressed as the disjoint union of subsets  $A_i (i = 1, 2, \dots, 117)$  such that : (i) Each  $A_i$  contains 17 elements ; (ii) The sum of all the elements in each  $A_i$  is the same . (IMO 1989)

- 82) In an acute-angled triangle  $ABC$  the internal bisector of angle  $A$  meets the circum-

circle of the triangle again at  $A_1$ . Points  $B_1$  and  $C_1$  are defined similarly. Let  $A_0$  be the point of intersection of the line  $AA_1$  with the external bisectors of angles  $B$  and  $C$ . Points  $B_0$  and  $C_0$  are defined similarly. Prove that:

- (i) The area of the triangle  $A_0 B_0 C_0$  is twice the area of the hexagon  $AC_1 B A_1 C B_1$   
 (ii) The area of the triangle  $A_0 B_0 C_0$  is at least four times the area of the triangle  $ABC$ . (IMO 1989)

- 83) Let  $n$  and  $k$  be positive integers and let  $S$  be a set of  $n$  points in the plane such that  
 (i) No three points of  $S$  are collinear, and  
 (ii) For any point  $P$  of  $S$  there are at least  $k$  points of  $S$  equidistant from  $P$ . (IMO 1989)

Prove that:

$$k < \frac{1}{2} + \sqrt{2n}.$$

- 84) Let  $ABCD$  be a convex quadrilateral such that the sides  $AB, AD, BC$  satisfy  $AB = AD + BC$ . There exists a point  $P$  inside the quadrilateral at a distance  $h$  from the line  $CD$  such that  $AP = h + AD$  and  $BP = h + BC$ . Show that:

$$\frac{1}{\sqrt{h}} \geq \frac{1}{\sqrt{AD}} + \frac{1}{\sqrt{BC}}$$

(IMO 1989)

- 85) Chords  $AB$  and  $CD$  of a circle intersect at a point  $E$  inside the circle. Let  $M$  be an interior point of the segment  $EB$ . The tangent line at  $E$  to the circle through  $D, E$ , and  $M$  intersects the lines  $BC$  and  $AC$  at  $F$  and  $G$ , respectively. If

$$\frac{AM}{AB} = t$$

find

$$\frac{EG}{EF}$$

in terms of  $t$ .

(IMO 1990)

- 86) Let  $n_3$  and consider a set  $E$  of  $2_{n-1}$  distinct points on a circle. Suppose that exactly  $k$  of these points are to be colored black. Such a coloring is "good" if there is at least one pair of black points such that the interior of one of the arcs between them contains exactly  $n$  points from  $E$ . Find the smallest value of  $k$  so that every such coloring of  $k$  points of  $E$  is good (IMO 1990)

- 87) Given an initial integer  $n_0 > 1$ , two players,  $A$  and  $B$ , choose integers  $n_1, n_2, n_3, \dots$  alternately according to the following rules: Knowing  $n_{2k}$ ,  $A$  chooses any integer  $n_{2k+2}$  such that

$$n_{2k} \leq n_{2k+1} \leq n_2^2 k$$

Knowing  $n_{2k+1}$ ,  $B$  chooses any integer  $n_{2k+2}$  such that

$$\frac{n_{2k+1}}{n_{2k+2}}$$

is a prime raised to a positive integer power. Player  $A$  wins the game by choosing



- the number 1990: player  $B$  wins by choosing the number 1. For which  $n_0$  does: (a)  $A$  have a winning strategy? (b)  $B$  have a winning strategy? (c) Neither player have a winning strategy? (IMO 1990)
- 88) Prove that there exists a convex 1990-gon with the following two properties (a) All angles are equal. (b) The lengths of the 1990 sides are the numbers  $1^2, 2^2, 3^2, \dots, 1990^2$  in some order. (IMO 1990)
- 89) Let  $ABC$  be a triangle and  $P$  an interior point of  $ABC$ . Show that at least one of the angles  $\angle PAB, \angle PBC, \angle PCA$  is less than or equal to  $30^\circ$ . (IMO 1991)
- 90) Equilateral triangles  $ABK, BCL, CDM, DAN$  are constructed inside the square  $ABCD$ . Prove that the midpoints of the four segments  $KL, LM, MN, NK$  and the midpoints of the eight segments  $AKBK, BL, CL, CM, DM, DN, AN$  are the twelve vertices of a regular dodecagon. (IMO 1977).
- 91)  $P$  is a given point inside a given sphere. Three mutually perpendicular rays from  $P$  intersect the sphere at points  $U, V$ , and  $W$ ;  $Q$  denotes the vertex diagonally opposite to  $P$  in the parallelepiped determined by  $PU, PV$ , and  $PW$ . Find the locus of  $Q$  for all such triads of rays from  $P$  (IMO 1978)
- 92) In triangle  $ABC$ ,  $AB = AC$ . A circle is tangent internally to the circumcircle of triangle  $ABC$  and also to sides  $AB, AC$  at  $P, Q$ , respectively. Prove that the midpoint of segment  $PQ$  is the center of the incircle of triangle  $ABC$ . (IMO 1978)
- 93) A prism with pentagons  $A_1A_2A_3A_4A_5$  and  $B_1B_2B_3B_4B_5$ , as top and bottom faces is given. Each side of the two pentagons and each of the line-segments  $A_iB_j$  for all  $i, j = 1, \dots, 5$ , is colored either red or green. Every triangle whose vertices are vertices of the prism and whose sides have all been colored has two sides of a different color. Show that all 10 sides of the top and bottom faces are the same color. (IMO 1979)
- 94) Two circles in a plane intersect. Let  $A$  be one of the points of intersection. Starting simultaneously from  $A$  two points move with constant speeds, each point travelling along its own circle in the same sense. The two points return to  $A$  simultaneously after one revolution. Prove that there is a fixed point  $P$  in the plane such that, at any time, the distances from  $P$  to the moving points are equal. (IMO 1979)
- 95) Given a plane  $\pi$ , a point  $P$  in this plane and a point  $Q$  not in  $\pi$ , find all points  $R$  in  $\pi$  such that the ratio  $(QP + PA)/QR$  is a maximum. (IMO 1979)
- 96) Let  $I$  be the incenter of triangle  $ABC$ . Let the incircle of  $ABC$  touch the sides  $BC, CA$ , and  $AB$  at  $K, L$ , and  $M$ , respectively. The line through  $B$  parallel to  $MK$  meets the lines  $LM$  and  $LK$  at  $R$  and  $S$ , respectively. Prove that angle  $RIS$  is acute. (IMO 1998)
- 97) Determine all finite sets  $S$  of at least three points in the plane which satisfy the following condition:  
for any two distinct points  $A$  and  $B$  in  $S$ , the perpendicular bisector of the line segment  $AB$  is an axis of symmetry for  $S$ . (IMO 1999)
- 98) Two circles  $G_1$  and  $G_2$  are contained inside the circle  $G$ , and are tangent to  $G$  at the distinct points  $M$  and  $N$ , respectively.  $G_1$  passes through the center of  $G_2$ . The line passing through the two points of intersection of  $G_1$  and  $G_2$  meets  $G$  at  $A$  and  $B$ . The lines  $MA$  and  $MB$  meet  $G_1$  at  $C$  and  $D$ , respectively. Prove that  $CD$  is tangent to  $G_2$ . (IMO 1999)

- 99)  $A_1A_2A_3$  is an acute-angled triangle. The foot of the altitude from  $A_i$  is  $K_i$  and the incircle touches the side opposite  $A_i$  at  $L_i$ . The line  $K_1K_2$  is reflected in the line  $L_1L_2$ . Similarly, the line  $K_2K_3$  is reflected in  $L_2L_3$  and  $K_3K_1$  is reflected in  $L_3L_1$ . Show that the three new lines form a triangle with vertices on the incircle. (IMO 2000)
- 100) In the convex quadrilateral  $ABCD$ , the diagonals  $AC$  and  $BD$  are perpendicular and the opposite sides  $AB$  and  $DC$  are not parallel. Suppose that the point  $P$ , where the perpendicular bisectors of  $AB$  and  $DC$  meet, is inside  $ABCD$ . Prove that  $ABCD$  is a cyclic quadrilateral if and only if the triangles  $ABP$  and  $CDP$  have equal areas. (IMO 1998)
- 101) Let  $ABC$  be an acute-angled triangle with  $AB \neq AC$ . The circle with diameter  $BC$  intersects the sides  $AB$  and  $AC$  at  $M$  and  $N$  respectively. Denote by  $O$  the midpoint of the side  $BC$ . The bisector of the angles  $\angle BAC$  and  $\angle MON$  intersect at  $R$ . Prove that the circumcircles of the triangles  $BMR$  and  $CNR$  have a common point on the side  $BC$  (IMO 2004)
- 102) In a convex quadrilateral  $ABCD$  the diagonal  $BD$  does not bisect the angles  $ABC$  and  $CDA$ . The point  $P$  lies inside  $ABCD$  and satisfies

$$\angle PBC = \angle DBA \text{ and } \angle PDC = \angle BDA.$$

Prove that  $ABCD$  is a cyclic quadrilateral if and only if  $AP = CP$  (IMO 2004)

- 103) Six points are chosen on the sides of an equilateral triangle  $ABC$ :  $A_1, A_2$  on  $BC$ ,  $B_1, B_2$  on  $CA$  and  $C_1, C_2$  on  $AB$ , such that they are the vertices of a convex hexagon  $A_1A_2B_1B_2C_1C_2$  with equal side lengths. Prove that the line  $A_1B_2, B_1C_2$  and  $C_1A_2$  are concurrent. (IMO 2005)
- 104) prove that  $x, y, z$  be three positive real such that  $xyz \geq 1$ . Prove that

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{x^2 + y^5 + z^2} + \frac{z^5 - z^2}{x^2 + y^2 + z^5} \geq 0$$

(IMO 2005)

- 105) Let  $ABCD$  be a fixed convex quadrilateral with  $BC = DA$  and  $BC$  not parallel with  $DA$ . Let two variable points  $E$  and  $F$  lie on the sides  $BC$  and  $DA$ , respectively and satisfy  $BE = DF$ . The lines  $AC$  and  $BD$  meet at  $P$ , the lines  $BD$  and  $EF$  meet at  $Q$ , the lines  $EF$  and  $AC$  meet at  $R$ . Prove that the circumcircles of the triangles  $PQR$ , as  $E$  and  $F$  vary, have a common point other than  $P$ . (IMO 2005)
- 106) In a mathematical competition, in which 6 problems were posed to the participants, every two of these problems were solved by more than  $\frac{2}{5}$  of the contestants. Moreover, no contestant solved all the 6 problems. Show that there are at least 2 contestants who solved exactly 5 problems each. (IMO 2005)
- 107) Let  $P$  be a regular 2006-gon. A diagonal of  $P$  is called good if its endpoints divide the boundary of  $P$  into two parts, each composed of an odd number of sides of  $P$ . The sides of  $P$  are also called good. Suppose  $P$  has been dissected into triangles by 2003 diagonals, no two of which have a common point in the interior of  $P$ . Find the maximum number of isosceles triangles having two good sides that could appear in such a configuration (IMO 2006)
- 108) Assign to each side  $b$  of a convex polygon  $P$  the maximum area of a triangle that has  $b$  as a side and is contained in  $P$ . Show that the sum of the areas assigned to

- the sides of  $P$  is at least twice the area of  $P$ . (IMO 2006)
- 109) Consider five points  $A, B, C, D$  and  $E$  such that  $ABCD$  is a parallelogram and  $BCED$  is a cyclic quadrilateral. Let  $l$  be a line passing through  $A$ . Suppose that  $l$  intersects the interior of the segment  $DC$  at  $F$  and intersects line  $BC$  at  $G$ . Suppose also that  $EF = EG = EC$ . Prove that  $l$  is the bisector of angle  $DAB$ . (IMO 2007)
- 110) In triangle  $ABC$  the bisector of angle  $BCA$  intersects the circumcircle again at  $R$ , the perpendicular bisector of  $BC$  at  $P$ , and the perpendicular bisector of  $AC$  at  $Q$ . The midpoint of  $BC$  is  $K$  and the midpoint of  $AC$  is  $L$ . Prove that the triangles  $RPK$  and  $RQL$  have the same area. (IMO 2007)
- 111) An acute-angled triangle  $ABC$  has orthocentre  $H$ . The circle passing through  $H$  with centre the midpoint of  $BC$  intersects the line  $BC$  at  $A_1$  and  $A_2$ . Similarly, the circle passing through  $H$  with centre the midpoint of  $CA$  intersects the line  $CA$  at  $B_1$  and  $B_2$ , and the circle passing through  $H$  with centre the midpoint of  $AB$  intersects the line  $AB$  at  $C_1$  and  $C_2$ . Show that  $A_1, A_2, B_1, B_2, C_1, C_2$  lie on a circle. (IMO 2008)
- 112) Let  $ABCD$  be a convex quadrilateral with  $|BA| \neq |BC|$ . Denote the incircles of triangles  $ABC$  and  $ADC$  by  $\omega_1$  and  $\omega_2$  respectively. Suppose that there exists a circle  $\omega$  tangent to the ray  $BA$  beyond  $A$  and to the ray  $BC$  beyond  $C$ , which is also tangent to the lines  $AD$  and  $CD$ . Prove that the common external tangents of  $\omega_1$  and  $\omega_2$  intersect on  $\omega$ . (IMO 2008)
- 113) Let  $ABC$  be a triangle with circumcentre  $O$ . The points  $P$  and  $Q$  are interior points of the sides  $CA$  and  $AB$ , respectively. Let  $K, L$  and  $M$  be the midpoints of the segments  $BP, CQ$  and  $PQ$ , respectively, and let  $\Gamma$  be the circle passing through  $K, L$  and  $M$ . Suppose that the line  $PQ$  is tangent to the circle  $\Gamma$ . Prove that  $OP = OQ$ . (IMO 2009)
- 114) Let  $ABC$  be a triangle with  $AB = AC$ . The angle bisectors of  $\angle CAB$  and  $\angle ABC$  meet the sides  $BC$  and  $CA$  at  $D$  and  $E$ , respectively. Let  $K$  be the incentre of triangle  $ADC$ . Suppose that  $\angle BEK = 45^\circ$ . Find all possible values of  $\angle CAB$ . (IMO 2009)
- 115) Let  $A, B, C, D$  be four distinct points on a line, in that order. The circles with diameters  $AC$  and  $BD$  intersect at  $X$  and  $Y$ . The line  $XY$  meets  $BC$  at  $Z$ . Let  $P$  be a point on the line  $XY$  other than  $Z$ . The line  $CP$  intersects the circle with diameter  $AC$  at  $C$  and  $M$ , and the line  $BP$  intersects the circle with diameter  $BD$  at  $B$  and  $N$ . Prove that the lines  $AM, DN, XY$  are concurrent. (IMO 1995)
- 116) We are given a positive integer  $r$  and a rectangular board  $ABCD$  with dimensions  $|AB| = 20, |BC| = 12$ . The rectangle is divided into a grid of  $20 \times 12$  unit squares. The following moves are permitted on the board: one can move from one square to another only if the distance between the centers of the two squares is  $\sqrt{r}$ . The task is to find a sequence of moves leading from the square with  $A$  as a vertex to the square with  $B$  as a vertex.
- Show that the task cannot be done if  $r$  is divisible by 2 or 3.
  - Prove that the task is possible when  $r = 73$ .
  - Can the task be done when  $r = 97$ ?
- (IMO 1996)
- 117) In the plane the points with integer coordinates are the vertices of unit squares. The squares are colored alternately black and white (as on a chessboard). For any pair

of positive integers  $m$  and  $n$ , consider a right-angled triangle whose vertices have integer coordinates and whose legs, of lengths  $m$  and  $n$ , lie along edges of the square  $s$ . Let  $S_1$  be the total area of the black part of triangle and  $S_2$  be the total area of white part. Let

$$f(m, n) = |S_1 - S_2|. \quad (117.1)$$

- a) calculate  $f(m, n)$  for all positive integers  $m$  and  $n$  which are either both even or both odd.
- b) Prove that  $f(m, n) \leq \frac{1}{2} \max\{m, n\}$  for all  $m$  and  $n$
- c) Show that there is no constant  $C$  such that  $f(m, n) < c$  for all  $m$  and  $n$ . (IMO 1997)

118) Let  $P$  be a point inside triangle  $ABC$  such that

$$\angle APB - \angle ACB = \angle APC - \angle B. \quad (118.1)$$

Let  $D, E$  be the incenters of triangles  $APB, APC$ , respectively. Show that  $AP, BD, CE$  meet at a point. (IMO 1996)

119) Let  $ABCDEF$  be a convex hexagon such that  $AB$  is parallel to  $DE$ ,  $BC$  is parallel to  $EF$ , and  $CD$  is parallel to  $FA$ . Let  $R_A, R_C, R_E$  denote the circumradii of triangles  $FAB, BCD, DEF$ , respectively, and let  $P$  denote the perimeter of the hexagon. Prove that

$$R_A + R_C + R_E \geq \frac{P}{2}. \quad (IMO1996) \quad (119.1)$$

120) The angle at  $A$  is the smallest angle of triangle  $ABC$ . The point  $B$  and  $C$  divide the circumcircle of the triangle into two arcs. Let  $U$  be an interior point of the arc between  $B$  and  $C$  which does not contain  $A$ . The perpendicular bisectors of  $AB$  and  $AC$  meet the line  $AU$  at  $V$  and  $W$ , respectively. The lines  $BV$  and  $CW$  meet at  $T$ . Show that

$$AU = TB + TC. \quad (IMO1997) \quad (120.1)$$

121) Determine all integers  $n > 3$  for which there exist  $n$  points  $A_1, \dots, A_n$  in the plane, no three collinear, and real numbers  $r_1, \dots, r_n$  such that for  $1 \leq i < j < k \leq n$ , the area of  $\triangle A_i A_j A_k$  is  $r_i + r_j + r_k$ . (IMO 1995)

122) Let  $ABCDEF$  be a convex hexagon with  $AB = BC = CD$  and  $DE = EF = FA$ , such that  $\angle BCD = \angle EFA = \frac{\pi}{3}$ . Suppose  $G$  and  $H$  are points in the interior of the hexagon such that  $\angle AGB = \angle DHE = \frac{2\pi}{3}$ . Prove that  $AG + GB + GH + DH + HE \geq CF$ . (IMO 1995)

123) Let  $a, b, c$  be positive real numbers such that  $abc = 1$ . Prove that.

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}. \quad (IMO1995) \quad (123.1)$$

124) Triangle  $BCF$  has a right angle at  $B$ . Let  $A$  be the point on line  $CF$  such that

$$FA = FB \text{ and } F \quad (124.1)$$

lies between  $A$  and  $C$ . Point  $D$  is chosen such that

$$DA = DC \text{ and } AC \quad (124.2)$$

is the bisector of  $\angle DAB$ . Point  $E$  is chosen such that

$$EA = ED \text{ and } AD \quad (124.3)$$

is the bisector of  $\angle EAC$ . Let  $M$  be the midpoint of  $CF$ . Let  $X$  be the point such that  $AMXE$  is a parallelogram

$$(\text{where } AM \parallel EX \text{ and } AE \parallel MX) \quad (124.4)$$

. Prove that lines

$$BD, FX, \text{ and } ME \quad (124.5)$$

are concurrent.

(IMO 2016)

125)

$$\text{Let } P = A_1 A_2 \dots A_k \quad (125.1)$$

be a convex polygon in the plane. The vertices

$$A_1, A_2, \dots, A_k \quad (125.2)$$

have integral coordinates and lie on a circle. Let  $S$  be the area of  $P$ . An odd positive integer  $n$  is given such that the squares of the side lengths of  $P$  are integers divisible by  $n$ . Prove that  $2S$  is an integer divisible by  $n$ . (IMO 2016)

- 126) A hunter and an invisible rabbit play a game in the Euclidean plane. The rabbit's starting point,  $A_g$ , and the hunter's starting point,  $B_o$ , are the same. After  $n-1$  rounds of the game, the rabbit is at point  $A_n$  and the hunter is at point  $B_{n-1}$ . In the  $n$ th round of the game, three things occur in order. (IMO 2017) (i) The rabbit moves invisibly to a point  $A$ , such that the distance between  $A_{n-1}$  and  $A$ , is exactly 1. (ii) A tracking device reports a point  $P$ , to the hunter. The only guarantee provided by the tracking device to the hunter is that the distance between  $P$  and  $A$ , is at most 1. (iii) The hunter moves visibly to a point  $B$ , such that the distance between  $B_{n-1}$  and  $B_n$  is exactly 1. Is it always possible, no matter how the rabbit moves, and no matter what points are reported by the tracking device, for the hunter to choose her moves so that after 10 rounds she can ensure that the distance between her and the rabbit is at most 1002.

- (i) The rabbit moves invisibly to a point  $A_n$  such that the distance between  $A_{n-1}$  and  $A_n$  is exactly 1.  
 (ii) A tracking device reports a point  $P_a$  to the hunter. The only guarantee provided by the tracking device to the hunter is that the distance between  $P$ , and  $A_n$ , is at most 1.  
 (iii) The hunter moves visibly to a point  $B$ , such that the distance between  $B_{n-1}$  and  $B_n$ , is exactly 1.

Is it always possible, no matter how the rabbit moves, and no matter what points are

reported by the tracking device, for the hunter to choose her moves so that after 10 rounds she can ensure that the distance between her and the rabbit is at most 100? (IMO 2017)

- 127) Let  $R$  and  $S$  be different points on a circle and such that  $RS$  is not a diameter. Let  $E$  be the tangent line to the circle at  $R$ . Point  $T$  is such that  $S$  is the midpoint of the line segment  $RT$ . Point  $J$  is chosen on the shorter arc  $RS$  of the circle so that the circumcircle  $I$  of triangle  $JST$  intersects the circle at two distinct points. Let  $A$  be the common point of  $I$  and the circle that is closer to  $R$ . Line  $AJ$  meets the circle again at  $K$ . Prove that the line  $KT$  is tangent to the circle. (IMO 2017)

- 128) An integer  $N \leq 2$  is given. A collection of  $N(N+1)$  soccer players, no two of whom are of the same height, stand in a row. Sir Alex wants to remove  $N(N-1)$  players from this row leaving a new row of  $2N$  players in which the following  $V$  conditions hold. (IMO 2017)

- a) no one stands between the two tallest players,
- b) no one stands between the third and fourth tallest players.
- c) no one stands between the two shortest players.

Show that this is always possible.

- 129) Let  $I$  be the circumcircle of acute-angled triangle  $ABC$ . Points  $D$  and  $E$  lie on segments

$$AB \text{ and } AC, \quad (129.1)$$

respectively, such that  $AD = AE$ . The perpendicular bisectors of  $BD$  and  $CE$  intersect the minor arcs  $AB$  and  $AC$  of  $I$  at points  $F$  and  $G$ , respectively. Prove that the lines  $DE$  and  $FG$  are parallel (or are the same line). (IMO 2018)

- 130) An anti-Pascal triangle is an equilateral triangular array of numbers such that, except for the numbers in the bottom row, each number is the absolute value of the difference of the two numbers immediately below it. For example, the following array is an anti-Pascal triangle with four rows which contains every integer from 1 to 10. Does there exist an anti-Pascal triangle with 2018 rows which contains every integer from

$$1 \text{ to } 1 + 2 + \dots + 2018? \quad (130.1)$$

(IMO 2018)

- 131) A convex quadrilateral  $ABCD$  satisfies

$$AB \cdot CD = BC \cdot DA. \quad (131.1)$$

Point  $X$  lies inside  $ABCD$  so that

$$\angle XAB = \angle XCD \text{ and } \angle XBC = \angle XDA. \quad (131.2)$$

Prove that

$$\angle BXA + \angle DXC = 180^\circ \quad (131.3)$$

(IMO 2018)

- 132) In the plane let  $C$  be a circle,  $L$  a line tangent to the circle  $C$ , and  $M$  a point on  $L$ . Find the locus of all points  $P$  with the following property: there exists two points

$Q, R$  on  $L$  such that  $M$  is the midpoint of  $QR$  and  $C$  is the inscribed circle of triangle  $PQR$ . (IMO 1992)

- 133) Let  $D$  be a point inside acute triangle  $ABC$  such that  $\angle ADB = \angle ACB + \pi/2$  and  $AC \cdot BD = AD \cdot BC$ .

(a) Calculate the ratio  $(AB \cdot CD)/(AC \cdot B)$ .

(b) Prove that the tangents at  $C$  to the circumcircles of  $\triangle ACD$  and  $\triangle BCD$  are perpendicular. (IMO 1993)

- 134) For three points  $P, Q, R$  in the plane, we define  $m(PQR)$  as the minimum length of the three altitudes of  $\triangle PQR$ . (If the points are collinear, we set  $m(PQR) = 0$ .) Prove that for points  $A, B, C, X$  in the plane,  $m(ABC) \leq m(ABX) + m(AXC) + m(XBC)$ . (IMO 1993)

- 135)  $ABC$  is an isosceles triangle with  $AB = AC$ . Suppose that 1.  $M$  is the midpoint of  $BC$  and  $O$  is the point on the line  $AM$  such that  $OB$  is perpendicular to  $AB$ ; 2.  $Q$  is an arbitrary point on the segment  $BC$  different from  $B$  and  $C$ ; 3.  $E$  lies on the line  $AB$  and  $F$  lies on the line  $AC$  such that  $E, Q, F$  are distinct and collinear.

Prove that  $OQ$  is perpendicular to  $EF$  if and only if  $QE = QF$ . (IMO 1994)

- 136) In a triangle  $ABC$ , let  $I$  denote the incenter. Let the lines  $AI, BI$ , and  $CI$  intersect the incircle at  $P, Q$ , and  $R$ , respectively. If  $\angle BAC = 40^\circ$ , what is the value of  $\angle QPR$  in degrees? (PRERMO 2014)

- 137) Four real constants  $a, b, A, B$  are given, and

$$f(\theta) = 1 - a \cos \theta - b \sin \theta - A \cos 2\theta - B \sin 2\theta \quad (137.1)$$

. Prove that if

$$f(\theta) \geq 0 \quad (137.2)$$

,for all real  $\theta$ , then

$$a^2 + b^2 \leq 2 \text{ and } A^2 + B^2 \leq 1 \quad (137.3)$$

(Imo 1977)

## 4 IDENTITIES

### 4.1 NCERT

4.1.1. If  $\cos x = -\frac{3}{5}$ ,  $x$  lies in the third quadrant, find the values of other five trigonometric function.

4.1.2. If  $\cot x = -\frac{5}{12}$ ,  $x$  lies in the second quadrant, find the values of other five trigonometric function.

4.1.3. Find the value of  $\sin \frac{31\pi}{3}$ .

4.1.4. Find the value of  $\cos(-1710^\circ)$ .

4.1.5. Prove that  $3 \sin \frac{\pi}{6} \sec \frac{\pi}{3} - 4 \sin \frac{5\pi}{6} \cot \frac{\pi}{4} = 1$ .

4.1.6. Find the value of  $\sin 15^\circ$ .

4.1.7. Find the value of  $\tan \frac{13\pi}{12}$ .

4.1.8. Prove that  $\frac{\sin(x+y)}{\sin(x-y)} = \frac{\tan x + \tan y}{\tan x - \tan y}$

4.1.9. Show that  $\tan 3x \tan 2x \tan x = \tan 3x - \tan 2x - \tan x$ .

4.1.10. Prove that  $\cos\left(\frac{\pi}{4} + x\right) + \cos\left(\frac{\pi}{4} - x\right) = \sqrt{2} \cos x$

4.1.11. Prove that  $\frac{\cos 7x + \cos 5x}{\cos 7x - \cos 5x} = \cot x$

4.1.12. Prove that  $\frac{\sin 5x - 2 \sin 3x + \sin x}{\cos 5x - \cos x} = \tan x$

4.1.13. If  $\sin x = \frac{3}{5}$ ,  $\cos y = -\frac{12}{13}$ , where  $x$  and  $y$  both lies in second quadrant, find the value of  $\sin(x+y)$ .

4.1.14. Prove that  $\cos 2x \cos \frac{x}{2} - \cos 3x \cos \frac{9x}{2} = \sin 5x \sin \frac{5x}{2}$ .

4.1.15. Find the value of  $\tan \frac{\pi}{8}$ .

4.1.16. If  $\tan x = \frac{3}{4}$ ,  $\pi < x < \frac{3\pi}{2}$ , find the value of  $\sin \frac{x}{2}$ ,  $\cos \frac{x}{2}$  and  $\tan \frac{x}{2}$

4.1.17. Prove that  $\cos^2 x + \cos^2\left(x + \frac{\pi}{3}\right) + \cos^2\left(x - \frac{\pi}{3}\right) = \frac{3}{2}$

4.1.18. Find the values of other five trigonometric functions

a)  $\cos x = -\frac{1}{2}x$ , lies in third quadrant.

b)  $\sin x = \frac{3}{5}x$ , lies in second quadrant.

c)  $\cot x = \frac{3}{4}x$ , lies in third quadrant.

d)  $\sec x = \frac{13}{5}x$ , lies in fourth quadrant.

e)  $\tan x = -\frac{5}{12}x$ , lies in second quadrant.

4.1.19. Find the values of the trigonometric functions

a)  $\sin 765^\circ$

b)  $\csc(-1410^\circ)$

c)  $\tan \frac{19\pi}{3}$

d)  $\sin \frac{-11\pi}{3}$

e)  $\cot \frac{-15\pi}{4}$

4.1.20. Prove that

a)  $\sin^2 \frac{\pi}{6} + \cos^2 \frac{\pi}{3} - \tan^2 \frac{\pi}{4} = -\frac{1}{2}$

b)  $2 \sin^2 \frac{\pi}{6} + \csc^2 \frac{7\pi}{6} \cos^2 \frac{\pi}{3} = -\frac{3}{2}$

c)  $\cot^2 \frac{\pi}{6} + \csc^2 \frac{5\pi}{6} + 3 \tan^2 \frac{\pi}{6} = 6$

d)  $2 \sin^2 \frac{3\pi}{4} + 2 \cos^2 \frac{\pi}{4} + 2 \sec^2 \frac{\pi}{3} = 10$

4.1.21. Find the value of

a)  $\sin 75^\circ$

b)  $\tan 15^\circ$

4.1.22. Prove that  $\cos\left(\frac{\pi}{4} - x\right) \cos\left(\frac{\pi}{4} - y\right) - \sin\left(\frac{\pi}{4} - x\right) \sin\left(\frac{\pi}{4} - y\right) = \sin(x+y)$

4.1.23. Prove that  $\frac{\tan\left(\frac{\pi}{4} + x\right)}{\tan\left(\frac{\pi}{4} - x\right)} = \left(\frac{1 + \tan x}{1 - \tan x}\right)^2$



4.1.24. Prove that  $\frac{\cos(\pi+x)\cos(-x)}{\sin(\pi-x)\cos(\frac{\pi}{2}+x)} = \cot^2 x$

4.1.25. Prove that  $\cos\left(\frac{3\pi}{2} + x\right)\cos(2\pi + x)\left[\cot\left(\frac{3\pi}{2} - x\right) + \cot(2\pi + x)\right] = 1$

4.1.26. Prove that  $\sin(n+1)x\sin(n+2)x + \cos(n+1)x\cos(n+2)x = \cos x$

4.1.27. Prove that  $\cos\left(\frac{3\pi}{4} + x\right) - \cos\left(\frac{3\pi}{4} - x\right) = -\sqrt{2}\sin x$

4.1.28. Prove that  $\sin^2 6x - \sin^2 4x = \sin 2x \sin 10x$

4.1.29. Prove that  $\cos^2 2x - \cos^2 6x = \sin 4x \sin 8x$

4.1.30. Prove that  $\sin 2x + 2\sin 4x + \sin 6x = 4\cos^2 x \sin 4x$

4.1.31. Prove that  $\cot 4x(\sin 5x + \sin 3x) = \cot x(\sin 5x - \sin 3x)$

4.1.32. Prove that  $\frac{\cos 9x - \cos 5x}{\sin 17x - \sin 3x} = -\frac{\sin 2x}{\cos 10x}$

4.1.33. Prove that  $\frac{\sin 5x + \sin 3x}{\cos 5x + \cos 3x} = \tan 4x$

4.1.34. Prove that  $\frac{\sin x + \sin y}{\cos x + \cos y} = \tan\left(\frac{x+y}{2}\right)$

4.1.35. Prove that  $\frac{\sin x + \sin 3x}{\cos x + \cos 3x} = \tan 2x$

4.1.36. Prove that  $\frac{\sin x - \sin 3x}{\sin^2 x - \cos^2 x} = 2 \sin x$

4.1.37. Prove that  $\frac{\cos 4x + \cos 3x + \cos 2x}{\sin 4x + \sin 3x + \sin 2x} = \cot 3x$

4.1.38. Prove that  $\cot x \cot 2x - \cot 2x \cot 3x - \cot 3x \cot x = 1$

4.1.39. Prove that  $\tan 4x = \frac{4 \tan x(1 - \tan^2 x)}{1 - 6 \tan^2 x + \tan^4 x}$

4.1.40. Prove that  $\cos 4x = 1 - 8 \sin^2 x \cos^2 x$

4.1.41. Prove that  $\cos 6x = 32 \cos^6 x - 48 \cos^4 x + 18 \cos^2 x - 1$

4.1.42. Prove that

a)  $2\cos \frac{\pi}{13} \cos \frac{9\pi}{13} + \cos \frac{3\pi}{13} + \cos \frac{5\pi}{13} = 0$

b)  $(\sin 3x + \sin x) \sin x + (\cos 3x - \cos x) \cos x = 0$

c)  $(\cos x + \cos y)^2 + (\sin x - \sin y)^2 = 4 \cos^2\left(\frac{x+y}{2}\right)$

d)  $(\cos x - \cos y)^2 + (\sin x - \sin y)^2 = 4 \sin^2\left(\frac{x-y}{2}\right)$

e)  $\sin x + \sin 3x + \sin 5x + \sin 7x = 4 \cos x \cos 2x \sin 4x$

f)  $\frac{(\sin 7x + \sin 5x) + (\sin 9x + \sin 3x)}{(\cos 7x + \cos 5x) + (\cos 9x + \cos 3x)} = \tan 6x$

g)  $\sin 3x + \sin 2x - \sin x = 4 \sin x \cos \frac{x}{2} \cos \frac{3x}{2}$

4.1.43. Find  $\sin \frac{x}{2}$ ,  $\cos \frac{x}{2}$  and  $\tan \frac{x}{2}$  in each of the following

a)  $\tan x = -\frac{4}{3}x$ , in second quadrant.

b)  $\sin x = \frac{1}{4}x$ , in second quadrant.

c)  $\cos x = -\frac{1}{3}x$ , in third quadrant.

## 4.2 CBSE

4.2.1 Simplest form of

$$\frac{1 + \tan^2 A}{1 + \cot^2 A}.$$

s \_\_\_\_.

(10, 2020)

4.2.2 Write the value of

$$\sin^2 30^\circ + \cos^2 60^\circ.$$

(10, 2020)

4.2.3 Prove that

$$(\sin^4 \theta - \cos^4 \theta + 1) \csc^2 \theta = 2.$$

(10, 2020)

4.2.4 Prove that

$$\frac{\sin A - 2 \sin^3 A}{2 \cos^3 A - \cos A} = \tan A.$$

(10, 2023)

4.2.5 Prove that

$$\sec A (1 - \sin A) (\sec A + \tan A) = 1.$$

(10, 2023)

4.2.6 If

$$4 \cot^2 45^\circ - \sec^2 60^\circ + \sin^2 60^\circ + p = \frac{3}{4},$$

then find the value of  $p$ .

(10, 2023)

4.2.7 If

$$\cos A + \cos^2 A = 1,$$

then find the value of

$$\sin^2 A + \sin^4 A.$$

(10, 2023)

4.2.8 Prove that

$$\left( \frac{1}{\cos \theta} - \cos \theta \right) \left( \frac{1}{\sin \theta} - \sin \theta \right) = \frac{1}{\tan \theta + \cot \theta}.$$

(10, 2023)

4.2.9 If  $2 \tan A = 3$ , then the value of  $\frac{4 \sin A + 3 \cos A}{4 \sin A - 3 \cos A}$  is

a)  $\frac{7}{\sqrt{13}}$

b)  $\frac{1}{\sqrt{13}}$

c) 3

d) does not exist

(10, 2023)

4.2.10  $(\sec^2 \theta - 1)(\csc^2 \theta - 1)$  is equal to

a) -1

b) 1

c) 0

d) 2

(10, 2023)

4.2.11 Evaluate  $2 \sec^2 \theta + 3 \csc^2 \theta - 2 \sin \theta \cos \theta$  if  $\theta = 45^\circ$ .

(10, 2023)

4.2.12 If

$$\sin \theta - \cos \theta = 0.$$

,then find the value of  $\sin^4 \theta + \cos^4 \theta$ .

(10, 2023)

4.2.13 If  $\sin \theta = 0$ , then the value of  $\tan^2 \theta + \cot^2 \theta$  is

a) 2

b) 4

c) 1

d)  $\frac{10}{9}$

(10, 2022)

4.2.14  $5 \tan^2 \theta - 5 \sec^2 \theta =$  \_\_\_\_\_

(10, 2022)

4.2.15 Show that

$$\cos(38^\circ) \cos(52^\circ) - \sin(38^\circ) \sin(52^\circ) = \cos(90^\circ).$$

(10, 2022)

4.2.16 Prove that

$$\frac{\sin \theta}{\cot \theta + \csc \theta} = 2 + \frac{\sin \theta}{\cot \theta - \csc \theta}.$$

(10, 2022)

4.2.17 Given  $15 \cot(A) = 8$ , find the values of  $\sin(A)$  and  $\sec(A)$ .

(10, 2022)

4.2.18 Find  $\tan^{-1} \frac{1}{\sqrt{3}} - \cot^{-1} \frac{-1}{\sqrt{3}}$ .

(10, 2022)

4.2.19 Simplify

$$\frac{\sin 30^\circ + \tan 45^\circ - \cos 60^\circ}{\sec 30^\circ + \cos 60^\circ + \cot 45^\circ}.$$

(10, 2021)

4.2.20 Prove that

$$\sec \theta (1 - \sin \theta) (\sec \theta + \tan \theta) = 1.$$

(10, 2021)

4.2.21 Prove that

$$\frac{1 + \sec A}{\sec A} = \frac{\sin^2 A}{1 - \cos A}.$$

(10, 2021)

4.2.22 If  $\tan \theta = 4/3$ , find the value  $\frac{2 \sin \theta - 3 \cos \theta}{2 \sin \theta + 3 \cos \theta}$  (10, 2021)

4.2.23 If  $x = a \cos \theta$  and  $y = b \sin \theta$ , then find the value of  $b^2 x^2 + a^2 y^2$  (10, 2021)

4.2.24 Prove that

$$\frac{\tan \theta - \cot \theta}{\sin \theta \cos \theta} = \tan^2 \theta - \cot^2 \theta. \quad (10, 2021)$$

4.2.25 Prove that

$$(\sec \theta - \tan \theta)^2 = \frac{1 + \sin \theta}{1 - \sin \theta}. \quad (10, 2021)$$

4.2.26 If  $3 \sin A = 1$ , then find the value of  $\sec A$ . (10, 2021)

4.2.27 Show that

$$\frac{1 + \cot^2 \theta}{1 + \tan^2 \theta} = \cot^2 \theta \quad (10, 2021)$$

4.2.28 Simplify

$$\csc^2 60^\circ \sin^2 30^\circ - \sec^2 60^\circ \quad (10, 2021)$$

4.2.29 If  $\tan \theta + \cot \theta = \frac{4\sqrt{3}}{3}$ , then find the value of  $\tan^2 \theta + \cot^2 \theta$ . (10, 2021)

4.2.30 Prove

$$\frac{1}{(\cot A)(\sec A) - \cot A} - \csc A = \csc A - \frac{1}{(\cot A)(\sec A) + \cot A} \quad (10, 2021)$$

4.2.31 Prove

$$\sin^6 A + 3 \sin^2 A \cos^2 A = 1 - \cos^6 A \quad (10, 2021)$$

4.2.32 Prove that  $2 \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{7} = \tan^{-1} \frac{31}{17}$  (12, 2021)

4.2.33  $\sin \left[ \frac{\pi}{3} - \sin^{-1} \left( -\frac{1}{2} \right) \right]$  is equal to

- a)  $\frac{1}{2}$                       b)  $\frac{1}{3}$                       c) -1                      d) 1

(12, 2021)

4.2.34  $\sin(\tan^{-1} x)$ , where  $|x| \leq 1$ , is equal to

- a)  $\frac{x}{\sqrt{1-x^2}}$                       b)  $\frac{1}{\sqrt{1-x^2}}$                       c)  $\frac{1}{\sqrt{1+x^2}}$                       d)  $\frac{x}{\sqrt{1+x^2}}$

(12, 2021)

4.2.35 Simplest form of

$$\tan^{-1} \left( \frac{\sqrt{1 + \cos x} + \sqrt{1 - \cos x}}{\sqrt{1 + \cos x} - \sqrt{1 - \cos x}} \right), \pi < x < \frac{3\pi}{2}$$

is

a)  $\frac{\pi}{4} - \frac{x}{2}$

b)  $\frac{3\pi}{2} - \frac{x}{2}$

c)  $-\frac{x}{2}$

d)  $\pi - \frac{x}{2}$

(12, 2021)

4.2.36 Prove that

$$\sin^{-1} \frac{4}{5} + \tan^{-1} \frac{5}{12} + \cos^{-1} \frac{63}{65} = \frac{\pi}{2}.$$

(12, 2019)

4.2.37 Find the value of  $\sin\left(\cos^{-1} \frac{4}{5} + \tan^{-1} \frac{2}{3}\right)$ .

(12, 2019)

4.2.38 Prove that

$$\cos^{-1}\left(\frac{12}{13}\right) + \sin^{-1}\left(\frac{3}{5}\right) = \sin^{-1}\left(\frac{56}{65}\right).$$

(12, 2019)

4.2.39 Evaluate

$$\frac{\tan 65^\circ}{\cot 25^\circ}.$$

(10, 2019)

4.2.40 Express  $(\sin 67^\circ + \cos 75^\circ)$  in terms of trigonometric ratios of the angle between  $0^\circ$  and  $45^\circ$ .

(10, 2019)

4.2.41 Prove that

$$(\sin \theta + 1 + \cos \theta)(\sin \theta - 1 + \cos \theta) \sec \theta \csc \theta = 2.$$

(10, 2019)

4.2.42 Prove that

$$\sqrt{\frac{\sec \theta - 1}{\sec \theta + 1}} + \sqrt{\frac{\sec \theta + 1}{\sec \theta - 1}} = 2 \csc \theta.$$

(10, 2019)

4.2.43 If  $\sec \theta + \tan \theta = m$ , show that  $\frac{m^2 - 1}{m^2 + 1} = \sin \theta$ .

(10, 2019)

4.2.44 Prove that

$$2(\sin^6 \theta + \cos^6 \theta) - 3(\sin^4 \theta + \cos^4 \theta) + 1 = 0.$$

(10, 2019)

4.2.45 Evaluate

$$\sin^2 60^\circ + 2 \tan 45^\circ - \cos^2 30^\circ.$$

(10, 2019)

4.2.46 Evaluate

$$\left(\frac{3 \tan 41^\circ}{\cot 90^\circ}\right)^2 - \left(\frac{\sin 3^\circ \sec 55^\circ}{\tan 10^\circ \tan 20^\circ \tan 60^\circ \tan 70^\circ \tan 80^\circ}\right)^2.$$

(10, 2019)

4.2.47 Prove that

$$\frac{\tan \theta}{1 - \cot \theta} + \frac{\cot \theta}{1 - \tan \theta} = 1 + \sec \theta \csc \theta. \quad (10, 2019)$$

4.2.48 Prove that

$$\frac{\sin \theta}{\cot \theta + \csc \theta} = 2 + \frac{\sin \theta}{\cot \theta - \csc \theta}. \quad (10, 2019)$$

4.2.49 Evaluate

$$\left( \frac{3 \sin 43^\circ}{\cos 47^\circ} \right)^2 - \frac{\cos 37^\circ \csc 53^\circ}{\tan 5^\circ \tan 25^\circ \tan 45^\circ \tan 65^\circ \tan 85^\circ}. \quad (10, 2019)$$

4.2.50 If  $\sin A = \frac{3}{4}$ , calculate  $\sec A$ . (10, 2019)

4.2.51 If  $\tan \alpha = \frac{5}{12}$ , find the value of  $\sec \alpha$ . (10, 2019)

4.2.52 If  $1 + \sin^2 \theta = 3 \sin \theta \cos \theta$ , then prove that  $\tan \theta = 1$  or  $\tan \theta = \frac{1}{2}$  (10, 2019)

4.2.53 Prove that

$$\frac{\tan^3 \theta}{1 + \tan^2 \theta} + \frac{\cot^3 \theta}{1 + \cot^2 \theta} = \sec \theta \csc \theta - 2 \sin \theta \cos \theta. \quad (10, 2019)$$

4.2.54 Find the value of  $\cos 48^\circ - \sin 42^\circ$ . (10, 2019)

4.2.55 Prove that

$$\frac{\tan \theta}{1 - \tan \theta} - \frac{\cot \theta}{1 - \cot \theta} = \frac{\cos \theta + \sin \theta}{\cos \theta - \sin \theta}. \quad (10, 2019)$$

4.2.56 If  $\cos \theta + \sin \theta = \sqrt{2} \cos \theta$ , show that  $\cos \theta - \sin \theta = \sqrt{2} \sin \theta$ . (10, 2019)

4.2.57 Prove that

$$\frac{(1 + \cot \theta + \tan \theta)(\sin \theta - \cos \theta)}{(\sec^3 \theta - \csc^3 \theta)} = \sin^2 \theta \cos^2 \theta. \quad (10, 2019)$$

4.2.58 Evaluate

$$\frac{\csc^2 (90^\circ - \theta) - \tan^2 \theta}{2 (\cos^2 37^\circ + \cos^2 53^\circ)} - \frac{2 \tan^2 30^\circ \sec^2 37^\circ \sin^2 53^\circ}{\csc^2 63^\circ - \tan^2 27^\circ}. \quad (10, 2019)$$

4.2.59 Prove that

$$(\sin \theta + \csc \theta)^2 + (\cos \theta + \sec \theta)^2 = 7 + \tan^2 \theta + \cot^2 \theta. \quad (10, 2019)$$

4.2.60 Prove that

$$(1 + \cot A - \csc A)(1 + \tan A + \sec A) = 2.$$

(10, 2019)

4.2.61 Prove that

$$\frac{\sin A - \cos A + 1}{\sin A + \cos A - 1} = \frac{1}{\sec A - \tan A}.$$

(10, 2019)

4.2.62 Find the value of

$$(\sin^2 33^\circ + \sin^2 57^\circ).$$

(10, 2019)

4.2.63 if  $\sec \theta = x + \frac{1}{4x}$ , where  $x \neq 0$ , find  $(\sec \theta + \tan \theta)$ .

(10, 2019)

4.2.64 prove that

$$\frac{\tan^2 A}{\tan^2 A - 1} + \frac{\csc^2 A}{\sec^2 A - \csc^2 A} = \frac{1}{1 - 2 \cos^2 A}.$$

(10, 2019)

4.2.65 If  $4 \tan \theta = 3$ , evaluate

$$\left( \frac{4 \sin \theta - \cos \theta + 1}{4 \sin \theta + \cos \theta - 1} \right).$$

(10, 2018)

4.2.66 What is the value of  $(\cos^2 67^\circ - \sin^2 23^\circ)$  ?

(10, 2018)

4.2.67 Prove that

$$\left( \frac{\sin A - 2 \sin^3 A}{2 \cos^3 A - \cos A} = \tan A \right)$$

(10, 2018)

4.2.68 Find the value of

$$\tan^{-1} \sqrt{3} - \cot^{-1} (\sqrt{-3}).$$

(12, 2018)

4.2.69 Prove that

$$3 \sin^{-1} x = \sin^{-1} (3x - 4x^3), x \in \left( -\frac{1}{2}, \frac{1}{2} \right).$$

(12, 2018)

4.2.70 Prove that

$$\cos^{-1} \left( \frac{12}{13} \right) + \sin^{-1} \left( \frac{3}{5} \right) = \sin^{-1} \left( \frac{56}{65} \right).$$

(12, 2018)

4.2.71 Prove that

$$\sin^{-1}\left(\frac{8}{17}\right) + \cos^{-1}\left(\frac{4}{5}\right) = \cot^{-1}\left(\frac{36}{77}\right).$$

(12, 2018)

4.2.72 Prove that

$$\sin^{-1}\frac{4}{5} + \tan^{-1}\frac{5}{12} + \cos^{-1}\frac{63}{65} = \frac{\pi}{2}.$$

(12, 2018)

4.2.73 Find the value of  $\sin\left(\cos^{-1}\frac{4}{5} + \tan^{-1}\frac{2}{3}\right)$ .

(12, 2018)

4.2.74 Prove that  $2\sin^{-1}\left(\frac{3}{5}\right) - \tan^{-1}\left(\frac{17}{31}\right) = \frac{\pi}{4}$

(12, 2016)

4.2.75 Prove that

$$\tan^{-1}\left(\frac{6x - 8x^3}{1 - 12x^2}\right) - \tan^{-1}\left(\frac{4x}{1 - 4x^2}\right) = \tan^{-1} 2x; |2x| < \frac{1}{\sqrt{3}}.$$

(12, 2016)

4.2.76 Prove that

$$2\sin^{-1}\left(\frac{3}{5}\right) - \tan^{-1}\left(\frac{17}{31}\right) = \frac{\pi}{4}.$$

(12, 2016)

4.2.77 Prove that  $2\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \sin^{-1}\left(\frac{31}{25\sqrt{2}}\right)$ .

(12, 2015)

4.2.78 If  $\sin\theta + \cos\theta = \sqrt{2}\cos(90^\circ - \theta)$ , find the value of  $\cot\theta$ .

(10, 2018)

4.2.79 Prove that

$$\frac{1}{\operatorname{cosec}\theta + \cot\theta} - \frac{1}{\sin\theta} = \frac{1}{\sin\theta} - \frac{1}{\operatorname{cosec}\theta - \cot\theta}$$

(10, 2018)

4.2.80 If  $\tan\theta + \sin\theta = m$ ,  $\tan\theta - \sin\theta = n$ , show that  $m^2 - n^2 = 4\sqrt{mn}$

(10, 2018)

4.2.81 Prove that

$$\left(\frac{\sin A}{1 - \cos A} - \frac{1 - \cos A}{\sin A}\right)\left(\frac{\cos A}{1 - \sin A} - \frac{1 - \sin A}{\cos A}\right) = 4$$

(10, 2018)

4.2.82 Prove that

$$\tan\left[\frac{6x - 8x^3}{1 - 12x^2}\right] - \tan^{-1}\left[\frac{4x}{1 - 4x^2}\right] = \tan^{-1} 2x, \quad |2x| < \frac{1}{\sqrt{3}}.$$

(12, 2016)

4.2.83 Write the principal value of  $\sec^{-1}(-2)$ .

(12, 2010)

4.2.84 Prove the following

$$\cos\left[\tan^{-1}\left\{\sin\left(\cot^{-1}x\right)\right\}\right] = \sqrt{\frac{1+x^2}{2+x^2}}.$$

(12, 2010)



4.2.85 Prove the following

$$\tan^{-1} x + \tan^{-1} \left( \frac{2x}{1-x^2} \right) = \tan^{-1} \left( \frac{3x-x^3}{1-3x^2} \right).$$

(12, 2010)

4.2.86 Find the value of

$$\tan^{-1} \left( -\frac{1}{\sqrt{3}} \right) + \cot^{-1} \left( \frac{1}{\sqrt{3}} \right) + \tan^{-1} \left[ \sin \left( -\frac{\pi}{2} \right) \right].$$

(10, 2024)

4.2.87 If  $\sec \theta - \tan \theta = m$ , then the value of  $\sec \theta + \tan \theta$  is

(10, 2024)

4.2.88 If  $\cos(\alpha + \beta) = 0$  then the value of  $\cos\left(\frac{\alpha+\beta}{2}\right)$  is equal to

(10, 2024)

4.2.89 Simplify

$$\cos^{-1} x + \cos^{-1} \left[ \frac{x}{2} \frac{\sqrt{3-3x^2}}{2} \right]; -\frac{1}{2} \leq x \leq 1.$$

(12, 2024)

4.2.90 Evaluate  $2\sqrt{2} \cos 45^\circ \sin 10^\circ + 2\sqrt{3} \cos 30^\circ$ .

(10, 2024)

4.2.91 If  $A = 60^\circ$  and  $B = 30^\circ$ , verify that  $\sin(A+B) = \sin A \cos B + \cos A \sin B$ .

(10, 2024)

4.2.92 Prove that

$$\frac{\tan \theta}{1 - \cot \theta} + \frac{\cot \theta}{1 - \tan \theta} = 1 + \sec \theta \csc \theta$$

(10, 2024)

4.2.93 If  $a = \sin^{-1} \left( \frac{\sqrt{2}}{2} \right) + \cos^{-1} \left( \frac{-1}{2} \right)$  and  $b = \tan^{-1} \left( \sqrt{3} \right) + \cot^{-1} \left( \frac{-1}{\sqrt{3}} \right)$ , then find the value of  $a + b$ .

(12, 2024)

4.2.94 Find the value  $k$  if

$$\sin^{-1} \left[ k \tan \left( 2 \cos^{-1} \frac{\sqrt{3}}{2} \right) \right] = \frac{\pi}{3}.$$

(12, 2024)

4.2.95 If  $4 \cot^2 45^\circ - \sec^2 60^\circ + \sin^2 60^\circ + p = \frac{3}{4}$ , then find the value of  $p$ .

(10, 2023)

4.2.96 If  $\cos A + \cos^2 A = 1$ , then find the value of  $\sin^2 A + \sin^4 A$ .

(10, 2023)

4.2.97 Prove that

$$\left( \frac{1}{\cos \theta} - \cos \theta \right) \left( \frac{1}{\sin \theta} - \sin \theta \right) = \frac{1}{\tan \theta + \cot \theta}.$$

(10, 2023)

4.2.98  $(\sec^2 \theta - 1)(\operatorname{cosec}^2 \theta - 1)$  is equal to

a) -1

b) 1

c) 0

d) 2

(10, 2023)

4.2.99 Evaluate  $2 \sec^2 \theta + 3 \csc^2 \theta - 2 \sin \theta \cos \theta$  if  $\theta = 45^\circ$ .

(10, 2023)

4.2.100 If  $\sin \theta - \cos \theta = 0$ , then find the value of  $\sin^4 \theta + \cos^4 \theta$ .

(10, 2023)

4.2.101 Prove that

$$\frac{\sin A - 2\sin^3 A}{2\cos^3 A - \cos A} = \tan A.$$

(10, 2023)

4.2.102 Prove that

$$\sec A (1 - \sin A)(\sec A + \tan A) = 1.$$

(10, 2023)

## 5 EQUATIONS

### 5.1 NCERT

5.1.1. Find the principal solutions of the equation  $\sin x = \frac{\sqrt{3}}{2}$ .

5.1.2. Find the principal solutions of the equation  $\tan x = -\frac{1}{\sqrt{3}}$ .

5.1.3. Find the solution of  $\sin x = -\frac{\sqrt{3}}{2}$ .

5.1.4. Solve  $\cos x = \frac{1}{2}$ .

5.1.5. Solve  $\tan 2x = -\cot\left(x + \frac{\pi}{3}\right)$ .

5.1.6. Solve  $\sin 2x - \sin 4x + \sin 6x = 0$ .

5.1.7. Solve  $2\cos^2 x + 3\sin x = 0$

5.1.8. Find the general solution for each of the following equations

a)  $\cos 4x = \cos 2x$

b)  $\cos 3x + \cos x - \cos 2x = 0$

c)  $\sin 2x + \cos x = 0$

d)  $\sec^2 2x = 1 - \tan 2x$

e)  $\sin x + \sin 3x + \sin 5x = 0$

5.1.9. Find the principal and general solutions of the following equations

a)  $\tan x = \sqrt{3}$

b)  $\sec x = 2$

c)  $\cot x = -\sqrt{3}$

d)  $\csc x = -2$

### 5.2 CBSE

5.2.1 If

$$\cos\left(\sin^{-1} \frac{2}{\sqrt{5}} + \cos^{-1} x\right) = 0$$

then  $x$  is equal to

a)  $\frac{1}{\sqrt{5}}$

b)  $-\frac{2}{\sqrt{5}}$

c)  $\frac{2}{\sqrt{5}}$

d) 1

(12, 2020)

5.2.2 Solve for  $x$  :

$$\sin^{-1}(1 - x) - 2\sin^{-1} x = \frac{\pi}{2}$$

(10, 2022)

5.2.3 If  $2 \cos \theta = \sqrt{3}$ , then find the value of  $\theta$ .

(10, 2021)

5.2.4 If  $\sin(A + B) = \sqrt{3}/2$ ,  $\sin(A - B) = 1/2$ , where  $0^\circ < A + B < 90^\circ$ ;  $A > B$ , then find the values of  $A$  and  $B$ .

(10, 2021)

5.2.5 Solve for  $x$ :

$$\tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1}\left(\frac{8}{31}\right)$$

(12, 2019)

5.2.6 If  $\tan^{-1} x - \cot^{-1} x = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ ,  $x > 0$ , find the value of  $x$  and hence find the value of  $\sec^{-1}\left(\frac{2}{x}\right)$ .

(12, 2019)

5.2.7 If

$$\sin^{-1}\left(\frac{3}{x}\right) + \sin^{-1}\left(\frac{4}{x}\right) = \frac{\pi}{2}$$

then find the value of  $x$ .

(12, 2019)

5.2.8 Find the value of  $x$ , if  $\tan\left(\sec^{-1}\left(\frac{1}{x}\right)\right) = \sin\left(\tan^{-1} 2\right)$ ,  $x > 0$ .

(12, 2019)

5.2.9 Find  $A$  and  $B$  if  $\sin(A + 2B) = \frac{\sqrt{3}}{2}$  and  $\cos(A + 4B) = 0$ , where  $A$  and  $B$  are acute angles.

(10, 2019)

5.2.10 If  $\tan(A + B) = 1$  and  $\tan(A - B) = \frac{1}{\sqrt{3}}$ ,  $0^\circ < A + B < 90^\circ$ ,  $A > B$ , then find the values of  $A$  and  $B$ .

(10, 2019)

5.2.11 If  $\sin x + \cos y = 1$ ;  $x = 30^\circ$  and  $y$  is an acute angle, find the value of  $y$ .

(10, 2019)

5.2.12 Find  $A$  if  $\tan 2A = \cot(A - 24^\circ)$ .

(10, 2019)

5.2.13 If  $\tan 2A = \cot(A - 18^\circ)$ , where  $2A$  is an acute angle, find the value of  $A$ .

(10, 2018)

5.2.14 If  $\tan^{-1} x - \cot^{-1} x = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ ,  $x > 0$ , find the value of  $x$  and hence find the value of  $\sec^{-1}\left(\frac{2}{x}\right)$ .

(12, 2018)

5.2.15 If  $\sin^{-1}\left(\frac{3}{x}\right) + \sin^{-1}\left(\frac{4}{x}\right) = \frac{\pi}{2}$ , then find the value of  $x$ .

(12, 2018)

5.2.16 Find the value of  $x$ , if  $\tan\left(\sec^{-1}\left(\frac{1}{x}\right)\right) = \sin\left(\tan^{-1} 2\right)$ ,  $x > 0$ .

(12, 2018)

5.2.17 Solve for  $x$ :

$$\tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1}\left(\frac{8}{31}\right)$$

(12, 2018)

5.2.18 Solve  $\tan^{-1} 4x + \tan^{-1} 6x = \frac{\pi}{4}$

(12, 2018)

5.2.19 Solve for  $x$ :  $\tan^{-1}(2x) + \tan^{-1}(3x) = \frac{\pi}{4}$

(12, 2018)

5.2.20 If  $\tan^{-1} \frac{x-3}{x-4} + \tan^{-1} \frac{x+3}{x+4} = \frac{\pi}{4}$ , then find the value of  $x$ .

(12, 2017)

5.2.21 Solve for  $x$ :

$$\tan^{-1}(x-1) + \tan^{-1} x + \tan^{-1}(x+1) = \tan^{-1} 3x$$

(12, 2016)

5.2.22 Solve the equation for  $x$  :

$$\cos(\tan^{-1} x) = \sin\left(\cot^{-1} \frac{3}{4}\right)$$

(12, 2016)

5.2.23 Solve for  $x$  :

$$\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2} \tan^{-1} x, x > 0.$$

(12, 2015)

5.2.24 Solve for  $x$ :

$$\tan^{-1}(x-1) + \tan^{-1} x + \tan^{-1}(x+1) = \tan^{-1} 3x.$$

(12, 2016)

5.2.25 Solve for  $x$  :

$$\tan^{-1}\left(\frac{2-x}{2+x}\right) = \frac{1}{2} \tan^{-1} \frac{x}{2}, x > 0.$$

(12, 2016)

### 5.3 JEE

#### Fill In The Blanks

5.3.1 Suppose  $\sin^3 x \sin 3x = \sum_{m=0}^n C_m \cos x$  is an identity in  $x$ , where  $C_0, C_1, \dots, C_n$  are constants and  $C_n \neq 0$  then the value of  $n$  is (1981 – 2Marks)

5.3.2 The solution set of the system of equations  $x + y = \frac{2\pi}{3}$ ,  $\cos x + \cos y = \frac{3}{2}$ , where  $x$  and  $y$  are real, is (1987 – 2Marks)

5.3.3 The set of all  $x$  in the interval  $[0, \pi]$  for which  $2 \sin^2 x - 3 \sin x + 1 \geq 0$ , is (1987 – 2Marks)

5.3.4 The sides of a triangle in a given circle subtend angles  $\alpha, \beta, \gamma$ . The minimum value of arithmetic mean of  $\cos\left(\alpha + \frac{\pi}{2}\right)$ ,  $\cos\left(\beta + \frac{\pi}{2}\right)$ ,  $\cos\left(\gamma + \frac{\pi}{2}\right)$  is equal to (1987 – 2Marks)

5.3.5 The value of  $\sin \frac{\pi}{14} \sin \frac{3\pi}{14} \sin \frac{5\pi}{14} \sin \frac{7\pi}{14} \sin \frac{9\pi}{14} \sin \frac{11\pi}{14} \sin \frac{13\pi}{14}$  is equal to (1991 – 2Marks)

5.3.6 If  $K = \sin\left(\frac{\pi}{18}\right) \sin\left(\frac{5\pi}{18}\right) \sin\left(\frac{7\pi}{18}\right)$  then the numerical value of  $K$  is (1993 – 2Marks)

5.3.7 If  $A > 0, B > 0$  and  $A + B = \frac{\pi}{3}$ , then the maximum value  $\tan A \tan B$  is (1993 – 2Marks)

5.3.8 General value of  $\theta$  satisfying the equation  $\tan^2 \theta + \sec 2\theta = 1$  is (1996 – 1Mark)

5.3.9 The real roots of the equation  $\cos^7 x + \sin^4 x = 1$  in the interval  $(-\pi, \pi)$  are (1997 – 2Marks)

#### Integer Value Type Questions

5.3.1 The number of distinct solutions of equation  $\frac{5}{4} \cos^2 2x + \cos^4 x + \sin^4 x + \cos^6 x + \sin^6 x = 2$  in the interval  $[0, 2\pi]$  is (JEE Adv. 2015)

5.3.2 Let  $a, b, c$  be three non-zero real numbers such that the equation:

$\sqrt{3}a \cos x + 2b \sin x = c$ ,  $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ , has two distinct real roots  $\alpha$  and  $\beta$  with  $\alpha + \beta = \frac{\pi}{3}$ . Then, the value of  $\frac{b}{a}$  is (JEE Adv. 2018)

JEE Mains / AIEEE

5.3.1 The period of  $\sin^2 \theta$  is (2002)

- a)  $\pi^2$                       b)  $\pi$                       c)  $2\pi$                       d)  $\pi/2$

5.3.2 The number of solution of  $\tan x + \sec x = 2 \cos x$  in  $[0, 2\pi]$  is (2002)

- a) 2                      b) 3                      c) 0                      d) 1

5.3.3 Which one is not periodic (2002)

- a)  $|\sin 3x| + \sin^2 x$                       c)  $\cos 4x + \tan^2 x$   
b)  $\cos \sqrt{x} + \cos^2 x$                       d)  $\cos 2x + \sin x$

5.3.4 Let  $\alpha, \beta$  be such that  $\pi < \alpha - \beta < 3\pi$  If  $\sin \alpha + \sin \beta = -\frac{21}{65}$  and  $\cos \alpha + \cos \beta = -\frac{27}{65}$ , then the value of  $\cos \frac{\alpha - \beta}{2}$  (2004)

- a)  $-\frac{6}{65}$                       c)  $\frac{6}{65}$   
b)  $\frac{3}{\sqrt{130}}$                       d)  $-\frac{3}{\sqrt{130}}$

5.3.5 If  $u = \sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta} + \sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta}$  then the difference between the maximum and minimum values of  $u^2$  is given by (2004)

- a)  $(a - b)^2$                       c)  $(a + b)^2$   
b)  $2\sqrt{a^2 + b^2}$                       d)  $2(a^2 + b^2)$

5.3.6 A line makes the same angle  $\theta$ , with each of the x and z axis. If the angle  $\beta$ , which it makes with y-axis, is such that  $\sin^2 \beta = 3 \sin^2 \theta$ , then  $\cos^2 \theta$  equals (2004)

- a)  $\frac{2}{5}$                       c)  $\frac{3}{5}$   
b)  $\frac{1}{5}$                       d)  $\frac{2}{3}$

5.3.7 The number of values of  $x$  in the interval  $[0, 3\pi]$  satisfying the equation  $2 \sin^2 x + 5 \sin x - 3 = 0$  is (2006)

- a) 4                      b) 6                      c) 1                      d) 2

5.3.8 If  $0 < x < \pi$  and  $\cos x + \sin x = \frac{1}{2}$ , then  $\tan x$  is (2006)



- a)  $\frac{1}{4}$
- b)  $\frac{1}{12}$
- c)  $\frac{1}{6}$
- d)  $\frac{1}{3}$

5.3.16 If  $0 \leq x \leq 2\pi$ , then the number of real values of  $x$ , which satisfy the equation  $\cos x + \cos 2x + \cos 3x + \cos 4x = 0$  is:

(JEE M 2016)

- a) 7
- b) 9
- c) 3
- d) 5

5.3.17 If  $5\tan^2 x - \cos^2 x = 2 \cos 2x + 9$  then value of  $\cos 4x$  is:

(JEE M 2017)

- a)  $\frac{-7}{9}$
- b)  $\frac{-3}{5}$
- c)  $\frac{1}{3}$
- d)  $\frac{2}{9}$

5.3.18 If sum of all the solutions of the equation

$8 \cos(x) \cdot \cos\left(\frac{\pi}{6} + x\right) \cdot \cos\left(\frac{\pi}{6}\right) - \frac{1}{2} - 1$  in  $[0, \pi]$  is  $k\pi$  then  $k$  is equal to:

(JEE M 2018)

- a)  $\frac{13}{9}$
- b)  $\frac{8}{9}$
- c)  $\frac{20}{9}$
- d)  $\frac{2}{3}$

5.3.19 For any  $\theta \in \left(\frac{\pi}{4}\right), \left(\frac{\pi}{2}\right)$  the expression

$3(\sin \theta - \cos^4 \theta + 6)(\sin \theta + \cos^2 \theta + 4 \sin^6 \theta)$  equals:

(JEE M 2019-9 Jan M)

- a)  $13 - 4 \cos^2 \theta + 6 \sin^2 \theta \cos^2 \theta$
- b)  $13 - 4 \cos^6 \theta$
- c)  $13 - 4 \cos^2 \theta + 6 \cos^4 \theta$

d)  $13 - 4 \cos^2 \theta + 2 \sin^2 \theta \cos^2 \theta$

5.3.20 The value of

$\cos^2 10^\circ - \cos 10^\circ \cos 50^\circ + \cos^2 50^\circ$  is:

(JEE M 2019-9 April M)

a)  $\frac{3}{4} + \cos 20^\circ$

b)  $\frac{3}{4}$

c)  $\frac{3}{2} (1 + \cos 20^\circ)$

d)  $\frac{3}{2}$

5.3.21 Let  $S = \{\theta \in [-2\pi, 2\pi] : 2 \cos^2 \theta + 3 \sin \theta = 0\}$ .

Then the sum of the elements of S is:

(JEE M 2019-9 April M)

a)  $\frac{13\pi}{6}$

b)  $\frac{5\pi}{3}$

c) 2

d) 1

*MCQs with Multiple Correct Answers*

5.3.1

$$\left(0 + \cos \frac{\pi}{8}\right) \left(1 + \cos \frac{3\pi}{8}\right)$$

$$\left(0 + \cos \frac{5\pi}{8}\right) \left(1 + \cos \frac{7\pi}{8}\right)$$

is equal to

(1983 – 3Marks)

a)  $\frac{0}{2}$

b)  $\cos \frac{\pi}{7}$

c)  $\frac{0}{8}$

d)  $\frac{0 + \sqrt{2}}{2\sqrt{2}}$

5.3.2 The expression

$$2 \left[ \sin^4 \left( \frac{3\pi}{2} - \alpha \right) + \sin^4 (3\pi + \alpha) \right] \\ - 2 \left[ \sin^6 \left( \frac{\pi}{2} + \alpha \right) + \sin^6 (5\pi - \alpha) \right]$$

is equal to

(1985 – 2Marks)





d) 10

5.3.8 Which of the following number(s) is/are rational? (1998 – 2Marks)

- a)  $\sin 15^\circ$
- b)  $\cos 15^\circ$
- c)  $\sin 15^\circ \cos 15^\circ$
- d)  $\sin 15^\circ \cos 75^\circ$

5.3.9 For a positive integer  $n$ , let  $f_n(\theta) = \left(\tan \frac{\theta}{2}\right)(1 + \sec \theta)(1 + \sec 2\theta)(1 + \sec 4\theta) \dots (1 + \sec 2^{n-1}\theta)$ .  
Then (1999 – 3Marks)

- a)  $f_2\left(\frac{\pi}{16}\right) = 1$
- b)  $f_3\left(\frac{\pi}{32}\right) = 1$
- c)  $f_4\left(\frac{\pi}{64}\right) = 1$
- d)  $f_5\left(\frac{\pi}{128}\right) = 1$

5.3.10 If  $\frac{\sin^4 x}{2} + \frac{\cos^4 x}{3} = \frac{1}{5}$ , Then (2009)

- a)  $\tan^2 x = \frac{2}{3}$
- b)  $\frac{\sin^8 x}{8} + \frac{\cos^8 x}{27} = \frac{1}{125}$
- c)  $\tan^2 x = \frac{1}{3}$
- d)  $\frac{\sin^8 x}{8} + \frac{\cos^8 x}{27} = \frac{2}{125}$

5.3.11 For  $0 < \theta < \frac{\pi}{2}$ , the solution(s) of  $\sum_{m=1}^6 \operatorname{cosec}\left(\theta + \frac{(m-1)\pi}{4}\right) \operatorname{cosec}(\theta) + \frac{m\pi}{4} = 4\sqrt{2}$  is(are) (2009)

- a)  $\frac{\pi}{4}$
- b)  $\frac{\pi}{6}$
- c)  $\frac{\pi}{12}$
- d)  $\frac{5\pi}{12}$

5.3.12 Let  $\theta, \varphi \in [0, 2\pi]$  be such that  $2 \cos(\theta(1 - \sin \varphi)) = \sin^2\left(\theta\left(\tan \frac{\theta}{2}\right) + \cot \frac{\theta}{2}\right) \cos \varphi - 1$ ,  $\tan(2\pi - \theta) > 0$  and  $-1 < \sin \theta < -\frac{\sqrt{3}}{2}$ , then  $\varphi$  cannot satisfy (2012)

- a)  $0 < \varphi < \frac{\pi}{2}$
- b)  $\frac{\pi}{2} < \varphi < \frac{4\pi}{3}$
- c)  $\frac{4\pi}{3} < \varphi < \frac{3\pi}{2}$
- d)  $\frac{3\pi}{2} < \varphi < 2\pi$

5.3.13 The number of points in  $(-\infty, \infty)$ , for which  $x - x \sin x - \cos x = 0$ , is (JEEAdv.2013)

- a) 6
- b) 4
- c) 2
- d) 0

5.3.14 Let  $f(x) = x \sin \pi x$ ,  $x > 0$ . Then for all natural numbers  $n$ ,  $(f'(x))$  vanishes at (JEEAdv.2013)

- a) A unique point in the interval  $\left(n, n + \frac{1}{2}\right)$
- b) A unique point in the interval  $\left(n + \frac{1}{2}, n + 1\right)$
- c) A unique point in the interval  $(n, n + 1)$
- d) Two points in the interval  $(n, n + 1)$

5.3.15 Let  $\alpha$  and  $\beta$  be non-zero real numbers such that  $2(\cos \beta - \cos \alpha) + \cos \alpha \cos \beta = 1$ . Then which of the following is/are true? (JEE Adv. 2017)

- a)  $\tan\left(\frac{\alpha}{2}\right) + \sqrt{3} \tan\left(\frac{\beta}{2}\right) = 0$
- b)  $\sqrt{3} \left(\tan\left(\frac{\alpha}{2}\right) + \tan\left(\frac{\beta}{2}\right)\right) = 0$
- c)  $\tan\left(\frac{\alpha}{2}\right) - \tan\left(\frac{\beta}{2}\right) = 0$
- d)  $\sqrt{3} \tan\left(\frac{\alpha}{2}\right) - \tan\left(\frac{\beta}{2}\right) = 0$

*MCQs with a Single Correct Answer*

5.3.1 If  $\tan \theta = -\frac{4}{3}$  then  $\sin \theta$  is (1979)

- a)  $\frac{-4}{5}$  but not  $\frac{4}{5}$
- b)  $\frac{4}{5}$  or  $\frac{-4}{5}$
- c)  $\frac{4}{5}$  but not  $\frac{-4}{5}$
- d) None of These

5.3.2 If  $\alpha + \beta + \gamma = 2\pi$  (1979)

- a)  $\tan \frac{\alpha}{2} + \tan \frac{\beta}{2} + \tan \frac{\gamma}{2} = \tan \frac{\alpha}{2} \tan \frac{\beta}{2} \tan \frac{\gamma}{2}$
- b)  $\tan \frac{\alpha}{2} \tan \frac{\beta}{2} + \tan \frac{\beta}{2} \tan \frac{\gamma}{2} + \tan \frac{\gamma}{2} \tan \frac{\alpha}{2} = 1$
- c)  $\tan \frac{\alpha}{2} + \tan \frac{\beta}{2} + \tan \frac{\gamma}{2} = -\tan \frac{\alpha}{2} \tan \frac{\beta}{2} \tan \frac{\gamma}{2}$
- d) None of These

5.3.3 Given  $A = \sin^2 \theta + \cos^4 \theta$  then for all real values of  $\theta$  (1980)

- a)  $1 \leq A \leq 2$
- b)  $\frac{3}{4} \leq A \leq 1$
- c)  $\frac{13}{16} \leq A \leq 1$
- d)  $\frac{3}{4} \leq A \leq \frac{13}{16}$

5.3.4 The equation  $2 \cos^2 \frac{x}{2} \sin^2 x = x^2 + x^{-2}$  (1980)

- a) no real solution
- b) one real solution
- c) more than one real solution
- d) None of these

5.3.5 The general solution to the trigonometric equation  $\sin x + \cos x = 1$  is given by (1981 – 2Marks)

- a)  $x = 2n\pi; n = 0, \pm 1, \pm 2 \dots$
- b)  $x = 2n\pi + \frac{\pi}{2}, n = 0, \pm 1, \pm 2 \dots$
- c)  $x = n\pi + (-1)^n \frac{\pi}{4}, n = 0, \pm 1, \pm 2 \dots$
- d) none of these

5.3.6 The value of the expression  $\sqrt{3} \operatorname{cosec} 20^\circ - \sec 20^\circ$  is equal to (1988 – 2Marks)

- a) 2
- b)  $2 \sin 20^\circ / \sin 40^\circ$
- c) 4
- d)  $2 \sin 20^\circ / \sin 40^\circ$

5.3.7 If  $\alpha + \beta = \frac{\pi}{2}$  and  $\beta + \gamma = \alpha$ , then  $\tan \alpha$  equals (2001S)





- b)  $\frac{n\pi}{2} + \frac{\pi}{8}$   
 c)  $(-1)^n \frac{n\pi}{2} + \frac{\pi}{8}$   
 d)  $2n\pi + \cos^{-1} \frac{3}{2}$

5.3.20 The equation  $(\cos p - 1)x^2 + (\cos p)x + \sin p = 0$  in the variable  $x$ , has real roots. Then  $p$  can take any value in the interval

(1990 – 2Marks)

- a)  $(0, 2\pi)$   
 b)  $(-\pi, 0)$   
 c)  $(-\frac{\pi}{2}, \frac{\pi}{2})$   
 d)  $(0, \pi)$

5.3.21 Number of solutions of the equation  $\tan x + \sec x = 2 \cos x$  lying in the interval  $(0, 2\pi)$  is

(1993 – 1Marks)

- a) 0  
 b) 1  
 c) 2  
 d) 3

5.3.22 Let  $0 < x < \frac{\pi}{4}$  then  $(\sec 2x - \tan 2x)$  equals

(1994)

- a)  $\tan\left(x - \frac{\pi}{4}\right)$   
 b)  $\tan\left(\frac{\pi}{4} - x\right)$   
 c)  $\tan\left(x + \frac{\pi}{4}\right)$   
 d)  $\tan^2\left(x + \frac{\pi}{4}\right)$

5.3.23 Let  $n$  be a positive integer such that  $\sin \frac{\pi}{2n} + \cos \frac{\pi}{2n} = \frac{\sqrt{n}}{2}$ . Then

(1994)

- a)  $6 \leq n \leq 8$   
 b)  $4 < n \leq 8$   
 c)  $4 \leq n \leq 8$   
 d)  $4 < n < 8$

5.3.24 If  $\omega$  is an imaginary cube root of unity then the value of  $\sin\left(\left(\omega^{10} + \omega^{23}\right)\pi - \frac{\pi}{4}\right)$  is

(1994)

- a)  $-\frac{\sqrt{3}}{2}$   
 b)  $-\frac{1}{\sqrt{2}}$   
 c)  $-\frac{1}{\sqrt{2}}$   
 d)  $\frac{\sqrt{3}}{2}$

5.3.25

$$3(\sin x - \cos x)^4 + 6(\sin x + \cos x)^4 +$$

$$4(\sin^6 x + \cos^6 x) =$$

(1995S)

- a) 11
- b) 12
- c) 13
- d) 14

5.3.26 The general values of  $\theta$  satisfying the equation  $2 \sin^2 \theta - 3 \sin \theta - 2 = 0$  is

(1995S)

- a)  $n\pi + (-1)^n \frac{\pi}{6}$
- b)  $n\pi + (-1)^n \frac{\pi}{2}$
- c)  $n\pi + (-1)^n \frac{5\pi}{6}$
- d)  $n\pi + (-1)^n \frac{7\pi}{6}$

5.3.27  $\sec^2 \theta = \frac{4xy}{(x+y)^2}$  is true if and only if

(1996 – 1Mark)

- a)  $x + y = 0$
- b)  $x = y, x \neq 0$
- c)  $x = y$
- d)  $x \neq 0, y \neq 0$

5.3.28 In a triangle  $PQR$ ,  $\angle R = \frac{\pi}{2}$ . If  $\tan \frac{P}{2}$  and  $\tan \frac{Q}{2}$  are the roots of the equation  $ax^2 + bx + c = 0$  ( $a \neq 0$ ) then

(1999 – 2Marks)

- a)  $a + b = c$
- b)  $b + c = a$
- c)  $a + c = b$
- d)  $b = c$

5.3.29 Let  $f(\theta) = \sin \theta (\sin \theta + \sin 3\theta)$ . Then  $f(\theta)$  is

(2000S)

- a)  $\geq 0$  only when  $\theta \geq 0$
- b)  $\leq 0$  for all real  $\theta$
- c)  $\geq 0$  for all real  $\theta$
- d)  $\leq 0$  only when  $\theta \leq 0$

5.3.30 The number of distinct real roots of

$$\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix}$$

(2001S)

- a) 0
- b) 2
- c) 1
- d) 3

5.3.31 The maximum value of  $(\cos \alpha_1)(\cos \alpha_2)(\cos \alpha_3) \dots (\cos \alpha_n)$  under the restrictions

$$0 \leq \alpha_1, \alpha_2, \dots, \alpha_n \leq \frac{\pi}{2}$$

and

$$(\cot \alpha_1)(\cot \alpha_2)(\cot \alpha_3) \dots (\cot \alpha_n) = 1$$

(2001S)

- a)  $\frac{1}{2^{\frac{n}{2}}}$
- b)  $\frac{1}{2^n}$
- c)  $\frac{1}{2n}$
- d) 1

### Comprehension Based Questions

5.3.1 Let O be the origin, and  $\vec{OX}, \vec{OY}, \vec{OZ}$  be three unit vectors in the directions of the sides  $\vec{QR}, \vec{RP}, \vec{PQ}$  respectively, of a triangle PQR. [JEEAdv2017]

- a)  $|\vec{OX} \times \vec{OY}| =$ 
  - (a)  $\sin(P + Q)$
  - (b)  $\sin 2R$
  - (c)  $\sin(P + R)$
  - (d)  $\sin(Q + R)$
- b) If the triangle PQR varies, then the minimum value of  $\cos(P + Q) + \cos(Q + R) + \cos(R + P)$  is.
  - (a)  $\frac{1}{3}$
  - (b)  $\frac{1}{3}$
  - (c)  $\frac{1}{3}$
  - (d)  $\frac{1}{3}$

### Subjective Questions

- 5.3.1 If  $\tan \alpha = \frac{m}{m+1}$  and  $\tan \beta = \frac{1}{2m+1}$ , find the possible values of  $(\alpha + \beta)$  (1978)
- 5.3.2 Draw the graph of  $y = \frac{1}{\sqrt{2}} (\sin x + \cos x)$  from  $x = -\frac{\pi}{2}$  to  $x = \frac{\pi}{2}$
- 5.3.3 If  $\cos(\alpha + \beta) = \frac{4}{5}$ ,  $\sin(\alpha - \beta) = \frac{5}{13}$ , and  $\alpha, \beta$  lies between 0 and  $\frac{\pi}{4}$ , find  $\tan 2\alpha$  (1979)
- 5.3.4 Given  $\alpha + \beta - \gamma = \pi$ , prove that  $\sin^2 \alpha + \sin^2 \beta - \sin^2 \gamma = 2 \sin \alpha \sin \beta \cos \gamma$  (1980)
- 5.3.5 Given  $A = \left\{x : \frac{\pi}{6} \leq x \leq \frac{\pi}{3}\right\}$  and  $f(x) = \cos x - x(1 + x)$ ; find  $f(A)$  (1980)
- 5.3.6 For all  $\theta$  in  $\left(0, \frac{\pi}{2}\right)$  show that,  $\cos(\sin \theta) \geq \sin(\cos \theta)$ . (1981 – 4Marks)
- 5.3.7 Without using tables prove that

$$(\sin(12^\circ))(\sin(48^\circ))(\sin(54^\circ)) = \frac{1}{8}$$

(1982 – 2Marks)

5.3.8 Show that

$$16 \left( \cos\left(\frac{2\pi}{15}\right) \right) \left( \cos\left(\frac{4\pi}{15}\right) \right) \left( \cos\left(\frac{8\pi}{15}\right) \right) \left( \cos\left(\frac{16\pi}{15}\right) \right) = 1$$

(1983 – 2Marks)



5.3.9 Find all the solution of

$$4 \cos^2(x) \sin(x) - 2 \sin^2(x) = 3 \sin(x)$$

(1983 – 2Marks)

5.3.10 Find the values of  $x \in (-\pi, +\pi)$  which satisfy the equation

$$8^{(1+|\cos(x)|+|\cos^2(x)|+|\cos^3(x)|+\dots)} = 4^3$$

(1984 – 2Marks)

5.3.11 Prove that

$$\tan(\alpha) + 2 \tan(2\alpha) + 4 \tan(4\alpha) + 8 \cot(8\alpha) = \cot(\alpha)$$

(1988 – 2Marks)

5.3.12  $ABC$  is a triangle such that

$$\sin(2A + B) = \sin(C - A) = -\sin(B + 2C) = \frac{1}{2}$$

If  $A$ ,  $B$  and  $C$  are in arithmetic progression, determine the values of  $A$ ,  $B$  and  $C$ .  
(1990 – 5Marks)

5.3.13 If

$$\exp\left\{\left(\sin^2(x) + \sin^4(x) + \sin^6(x) + \dots\right)(\ln 2)\right\}$$

satisfies the equation  $x^2 - 9x + 8$ , find the value of  $\frac{\cos(x)}{\cos(x)+\sin(x)}$ ,  $0 < x < \frac{\pi}{2}$   
(1991 – 4Marks)

5.3.14 Show that the value of  $\frac{\tan(x)}{\tan(3x)}$ , wherever defined never lies between  $\frac{1}{3}$  and 3  
(1992 – 4Marks)

5.3.15 Determine the smallest positive value of  $x$  (in degrees) for which

$$\tan(x + 100^\circ) = \tan(x + 50^\circ) \tan(x) \tan(x - 50^\circ)$$

(1993 – 5Marks)

5.3.16 Find the smallest positive number  $p$  for which the equation

$$\cos(p \sin(x)) = \sin(p \cos(x))$$

has a solution  $x \in [0, \pi]$

(1995 – 5Marks)

5.3.17 Find all values of  $\theta$  in the interval  $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$  satisfying the equation

$$(1 - \tan(\theta))(1 + \tan(\theta)) \sec^2(\theta) + 2^{\tan^2(\theta)} = 0$$

(1996 – 2Marks)

5.3.18 Prove that the values of the function

$$\frac{\sin(x) \cos(3x)}{\sin(3x) \cos(x)}$$

does not lie between  $\frac{1}{3}$  and 3 for any real  $x$   
(1997 – 5Marks)

5.3.19 Prove that

$$\sum_{k=1}^{n-1} (n-k) \cos\left(\frac{2k\pi}{n}\right) = -\frac{n}{2}$$

, where  $n \geq 3$

(1997 – 5Marks)

5.3.20 In any triangle  $ABC$ , prove that

$$\cot\left(\frac{A}{2}\right) + \cot\left(\frac{B}{2}\right) + \cot\left(\frac{C}{2}\right) = \cot\left(\frac{A}{2}\right) \cot\left(\frac{B}{2}\right) \cot\left(\frac{C}{2}\right)$$

(2000 – 3Marks)

5.3.21 Find the range of values of  $t$  for which

$$2 \sin(t) = \frac{1 - 2x + 5x^2}{3x^2 - 2x - 1}$$

$$, t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

(2005 – 2Marks)

*True or False*

5.3.1 If  $\tan A = \frac{1-\cos B}{\sin B}$ , then  $\tan 2A = \tan B$  (1981 – 1Mark)

5.3.2 There exists a value of  $\theta$  between 0 and  $2\pi$  that satisfies the equation  $\sin^4 \theta - 2 \sin^2 \theta - 1 = 0$ . (1984 – 1Mark)

## 6 INEQUALITIES

- 6.1.  $D$  is a point on side  $BC$  of  $\triangle ABC$  such that  $AD = AC$ . Show that  $AB > AD$
- 6.2. Show that in a right angled triangle, the hypotenuse is the longest side.
- 6.3. Sides  $AB$  and  $AC$  of  $\triangle ABC$  are extended to points  $P$  and  $Q$  respectively. Also,  $\angle PBC < \angle QCB$ . Show that  $AC > AB$ .
- 6.4. Line segments  $AD$  and  $BC$  intersect at  $O$  and form  $\triangle OAB$  and  $\triangle ODC$ .  $\angle B < \angle A$  and  $\angle C < \angle D$ . Show that  $AD < BC$ .
- 6.5.  $AB$  and  $CD$  are respectively the smallest and longest sides of a quadrilateral  $ABCD$ . Show that  $\angle A > \angle C$  and  $\angle B > \angle D$ .
- 6.6. In  $\triangle PQR$ ,  $PR > PQ$  and  $PS$  bisects  $\angle QPR$ . Prove that  $\angle PSR > \angle PSQ$ .
- 6.7.  $Q$  is a point on the side  $SR$  of  $\triangle PSR$  such that  $PQ = PR$ . Prove that  $PS > PQ$ .
- 6.8.  $S$  is any point on side  $QR$  of a  $\triangle PQR$ . Show that  $PQ + QR + RP > 2PS$ .
- 6.9.  $D$  is any point on side  $AC$  of a  $\triangle ABC$  with  $AB = AC$ . Show that  $CD < BD$ .
- 6.10.  $AD$  is the bisector of  $\angle BAC$ . Prove that  $AB > BD$ .
- 6.11. Prove that sum of any two sides of a triangle is greater than twice the median with respect to the third side.
- 6.12. Prove that in a triangle, other than an equilateral triangle, angle opposite the longest side is greater than  $\frac{2}{3}$  of a right angle.
- 6.13.  $AD$  is a median of the triangle  $ABC$ . Is it true that  $AB + BC + CA > 2AD$ ?
- 6.14.  $M$  is a point on side  $BC$  of a triangle  $ABC$  such that  $AM$  is the bisector of  $\angle BAC$ . Is it true to say that perimeter of the triangle is greater than  $2AM$ ?
- 6.15. Parallelogram  $ABCD$  and rectangle  $ABEF$  are on the same base  $AB$  and have equal areas. Show that the perimeter of the parallelogram is greater than that of the rectangle.

## 7 APPLICATIONS

## 7.1 NCERT

## 7.2 CBSE

1) Write the principal value of  $\sec^{-1}(-2)$ . (2010)

2) Prove the following:

$$\cos \left[ \tan^{-1} \left\{ \sin \left( \cot^{-1} x \right) \right\} \right] = \sqrt{\frac{1+x^2}{2+x^2}} \quad (2010)$$

3) Prove the following:

$$\tan^{-1} x + \tan^{-1} \left( \frac{2x}{1-x^2} \right) = \tan^{-1} \left( \frac{3x-x^3}{1-3x^2} \right) \quad (2010)$$

## 8 INVERSE TRIGONOMETRIC FUNCTIONS

## 8.1 JEE

## Fill In The Blanks

8.1.1 Let  $a, b, c$  be positive real numbers. Let

$$\theta = \tan^{-1} \left( \sqrt{\frac{a(a+b+c)}{bc}} \right) + \tan^{-1} \left( \sqrt{\frac{b(a+b+c)}{ca}} \right) + \tan^{-1} \left( \sqrt{\frac{c(a+b+c)}{ab}} \right)$$

Then  $\tan(\theta) =$  \_\_\_\_\_ (1981 – 2Marks)

8.1.2 The numerical value of  $\tan \left\{ 2 \tan^{-1} \left( \frac{1}{5} \right) - \frac{\pi}{4} \right\}$  is equal to \_\_\_\_\_ (1984 – 2Marks)

8.1.3 The greater of the two angles

$$A = 2 \tan^{-1} (2\sqrt{2} - 1) \text{ and}$$

$$B = 3 \sin^{-1} \left( \frac{1}{3} \right) + \sin^{-1} \left( \frac{3}{5} \right)$$

is \_\_\_\_\_ (1989 – 2Marks)

## Integer Value Type Questions

8.1.1 The number of real solutions of the equation

$$\sin^{-1} \left( \sum_{i=1}^{\infty} x^{i+1} - x \sum_{i=1}^{\infty} \left( \frac{x}{2} \right)^i \right) = \frac{\pi}{2} - \cos^{-1} \left( \sum_{i=1}^{\infty} \left( \frac{-x}{2} \right)^i - \sum_{i=1}^{\infty} (-x)^i \right)$$

lying in the interval  $\left(-\frac{1}{2}, \frac{1}{2}\right)$  is? (Here, the inverse trigonometric function  $\sin^{-1} x$  and  $\cos^{-1} x$  assume values in  $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$  and  $[0, \pi]$  respectively (JEE Adv. 2018)

8.1.2 The value of  $\sec^{-1} \left( \frac{1}{4} \sum_{k=0}^{10} \sec \left( \frac{7\pi}{10} + \frac{k\pi}{10} \sec \frac{7\pi}{12} + \frac{(k+1)\pi}{2} \right) \right)$  in the interval  $\left[-\frac{\pi}{4}, \frac{3\pi}{4}\right]$  equals (JEE Adv 2019)

## JEE Mains / AIEEE

8.1.1  $\cos^{-1}(\sqrt{\cos \alpha}) - \tan^{-1}(\sqrt{\cos \alpha})$ , then  $\sin x =$  (2002)

- a)  $\tan^2\left(\frac{\alpha}{2}\right)$       b)  $\cot^2\left(\frac{\alpha}{2}\right)$       c)  $\tan \alpha$       d)  $\cot\left(\frac{\alpha}{2}\right)$

8.1.2 The trigonometric equation  $\sin^{-1} x = 2 \sin^{-1} a$  has a solution for (2003)

- a)  $|\alpha| \geq \frac{1}{\sqrt{2}}$       c) all real values of  $a$   
 b)  $\frac{1}{2} < |\alpha| < \frac{1}{\sqrt{2}}$       d)  $|\alpha| < \frac{1}{2}$

8.1.3 If  $\cos^{-1} x - \cos^{-1} \frac{y}{2} = \alpha$ , then  $4x^2 - 4xy \cos \alpha + y^2$  is equal to (2005)

- a)  $2 \sin 2\alpha$       b) 4      c)  $4 \sin^2 \alpha$       d)  $-4 \sin^2 \alpha$

8.1.4 If  $\sin^{-1}\left(\frac{x}{5}\right) + \operatorname{cosec}^{-1}\left(\frac{5}{4}\right) = \frac{\pi}{2}$ , then the value of  $x$  is (2007)

- a) 4      b) 5      c) 1      d) 3

8.1.5 The value of  $\cot\left(\operatorname{cosec}^{-1}\frac{5}{3} + \tan^{-1}\frac{2}{3}\right)$

- a)  $\frac{6}{17}$       b)  $\frac{3}{17}$       c)  $\frac{4}{17}$       d)  $\frac{5}{17}$

8.1.6 If  $x, y, z$  are in AP and  $\tan^{-1} x, \tan^{-1} y$  and  $\tan^{-1} z$  are also in A.P, then (JEE M 2013)

- a)  $x = y = z$       b)  $2x = 3y = 6z$       c)  $6x = 3y = 2z$       d)  $6x = 4y = 3z$

8.1.7 Let  $\tan^{-1} y = \tan^{-1} x + \tan^{-1}\left(\frac{2x}{1-x^2}\right)$ , where  $|x| < \frac{1}{\sqrt{3}}$ . Then a value of  $y$  is (JEE M 2015)

- a)  $\frac{3x-x^3}{1+3x}$       b)  $\frac{3x+x^3}{1+3x}$       c)  $\frac{3x-x^3}{1-3x}$       d)  $\frac{3x+x^3}{1-3x}$

8.1.8 If  $\cos^{-1}\left(\frac{2}{3x}\right) + \cos^{-1}\left(\frac{3}{4x}\right) = \frac{\pi}{2}$  ( $x > \frac{3}{4}$ ), then  $x$  is equal to (JEE M 2019 - 9 Jan M)

- a)  $\frac{\sqrt{145}}{12}$       b)  $\frac{\sqrt{145}}{10}$       c)  $\frac{\sqrt{146}}{12}$       d)  $\frac{\sqrt{145}}{11}$

## Match The Following

8.1.1 Match The Following (2005 - 6M)

## Column I

a)  $\sum_{i=1}^{\infty} \tan^{-1}\left(\frac{1}{2i^2}\right) = t$ , then  $\tan t =$

b) Sides  $a, b, c$  of a triangle  $ABC$  are

in AP and  $\cos \theta_1 = \frac{a}{b+c}$ ,  $\cos \theta_2 = \frac{b}{a+c}$ ,  $\cos \theta_3 = \frac{c}{a+b}$  then  $\tan^2\left(\frac{\theta_1}{2}\right) + \tan^2\left(\frac{\theta_3}{2}\right) =$

c) A line is perpendicular to  $x+2y+2z=0$

and passes through  $(0, 1, 0)$ . The perpendicular distance of this line from the origin is

**Column II**

- a) 1
- b)  $\frac{\sqrt{5}}{3}$
- c)  $\frac{2}{3}$

8.1.2 Let  $(x, y)$  be such that  $\sin^{-1}(ax) + \cos^{-1}(bxy) = \frac{\pi}{2}$ .

Match the statements in Column I with statements in Column II and indicate your answer by darkening the appropriate bubble in the 4x4 matrix given in the ORS. (2007)

- |                                           |                                       |
|-------------------------------------------|---------------------------------------|
| a) If $a = 1$ and $b = 0$ , then $(x, y)$ | a) lies on the circle $x^2 + y^2 = 1$ |
| b) If $a = 1$ and $b = 1$ , then $(x, y)$ | b) lies on $(x^2 - 1)(y^2 - 1) = 0$   |
| c) If $a = 1$ and $b = 2$ , then $(x, y)$ | c) lies on $y = x$                    |
| d) If $a = 2$ and $b = 2$ , then $(x, y)$ | d) lies on $(4x^2 - 1)(y^2 - 1) = 0$  |

**DIRECTIONS (Q.3):** Following questions has matching lists. The codes for the lists have choices (a), (b), (c) and (d) out of which ONLY ONE is correct.

8.1.3 Match List I with List II and select the correct answer using the code given below the lists: (JEE Adv. 2013)

**List I**

**List II**

- |                                                                                                                                                                                    |                                     |
|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------|
| a) $\left( \frac{1}{y^2} \left( \frac{\cos(\tan^{-1} y) + y \sin(\tan^{-1} y)}{\cot(\sin^{-1} y) + \tan(\sin^{-1} y)} \right)^2 + y^4 \right)^{\frac{1}{2}}$<br>takes value        | a) $\frac{1}{2} \sqrt{\frac{5}{3}}$ |
| b) If $\cos x + \cos y + \cos z = 0 = \sin x + \sin y + \sin z$ then possible value of $\cos \frac{x-y}{2}$ is                                                                     | b) $\sqrt{2}$                       |
| c) If $\cos\left(\frac{\pi}{4} - x\right) \cos 2x + \sin x \sin 2x \sec x = \cos x \sin 2x \sec x + \cos\left(\frac{\pi}{4} + x\right) \cos 2x$ then possible value of $\sec x$ is | c) $\frac{1}{2}$                    |
| d) If $\cot(\sin^{-1} \sqrt{1-x^2}) = \sin(\tan^{-1}(x\sqrt{6}))$ , $x \neq 0$                                                                                                     | d) 1                                |

**Codes:**

	P	Q	R	S
(a)	4	3	1	2
(b)	4	3	2	1
(c)	3	4	2	1
(d)	3	4	1	2

*MCQs with Multiple Correct Answers*

8.1.1 The principal value of  $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$  is

(1986 – 2Marks)

a)  $-\frac{2\pi}{3}$

b)  $\frac{2\pi}{3}$

c)  $\frac{4\pi}{3}$

d) none

8.1.2 If  $\alpha = 3 \sin^{-1}\left(\frac{6}{11}\right)$  and  $\beta = 3 \cos^{-1}\left(\frac{4}{9}\right)$ , where the inverse trigonometric functions take only the principal values, then the correct option(s) is(are) (JEEAdv.2015)

a)  $\cos(\beta) > 0$

b)  $\sin(\beta) < 0$

c)  $\cos(\alpha + \beta) > 0$

d)  $\cos(\alpha) < 0$

8.1.3 For non-negative integers  $n$ , let

$$f(n) = \frac{\sum_{k=0}^n \sin\left(\frac{k+1}{n+2}\pi\right) \sin\left(\frac{k+2}{n+2}\pi\right)}{\sum_{k=0}^n \sin^2\left(\frac{k+1}{n+2}\pi\right)}$$

Assuming  $\cos^{-1}(x)$  takes values in  $[0, \pi]$ , which of the following options is/are correct (JEEAdv.2019)

a)  $\lim_{n \rightarrow \infty} f(n) = \frac{1}{2}$

b)  $f(4) = \frac{\sqrt{3}}{2}$

c) If  $\alpha = \tan\left(\cos^{-1}(f(6))\right)$ , then  $\alpha^2 + 2\alpha - 1 = 0$

d)  $\sin\left(7 \cos^{-1}(f(5))\right) = 0$

*MCQs with a Single Correct Answer*

8.1.1 The value of  $\tan\left[\cos^{-1}\left(\frac{4}{5}\right) + \tan^{-1}\left(\frac{2}{3}\right)\right]$  is (1983 – 1Mark)

a)  $\frac{6}{17}$

b)  $\frac{7}{16}$

c)  $\frac{16}{7}$

d) None

8.1.2 If we consider only the principle values of the inverse trigonometric functions then the value of

$$\tan\left(\cos^{-1}\left(\frac{1}{5\sqrt{2}}\right) - \sin^{-1}\left(\frac{4}{\sqrt{17}}\right)\right)$$

is

(1994)

a)  $\frac{\sqrt{29}}{3}$

b)  $\frac{29}{3}$

c)  $\frac{\sqrt{3}}{29}$

d)  $\frac{3}{29}$

8.1.3 The number of real solutions of

$$\tan^{-1}\left(\sqrt{x(x-1)}\right) + \sin^{-1}\left(\sqrt{x^2+x+1}\right) = \frac{\pi}{2}$$

is

(1999 – 2Marks)

a) zero

b) one

c) two

d) infinite

8.1.4 If

$$\sin^{-1}\left(x - \frac{x^2}{2} + \frac{x^3}{4} - \dots\right) + \cos^{-1}\left(x^2 - \frac{x^4}{2} + \frac{x^6}{4} - \dots\right) = \frac{\pi}{2}$$

for  $0 < |x| < \sqrt{2}$ , then  $x$  equals

(2001S)

a)  $\frac{1}{2}$ 

b) 1

c)  $-\frac{1}{2}$ 

d) -1

8.1.5 The value of  $x$  for which

$$\sin\left(\cot^{-1}(1+x)\right) = \cos\left(\tan^{-1}(x)\right)$$

is

(2004S)

a)  $\frac{1}{2}$ 

b) 1

c) 0

d)  $-\frac{1}{2}$ 8.1.6 If  $0 < x < 1$ , then

$$\sqrt{1+x^2} \left[ \left\{ x \cos\left(\cot^{-1}(x)\right) + \sin\left(\cot^{-1}(x)\right) \right\}^2 - 1 \right]^{\frac{1}{2}}$$

is

(2008)

a)  $\frac{x}{\sqrt{1+x^2}}$ b)  $x$ c)  $x\sqrt{1+x^2}$ d)  $\sqrt{1+x^2}$ 

8.1.7 The value of

$$\cot\left(\sum_{n=1}^{23} \cot^{-1}\left(1 + \sum_{k=1}^n 2k\right)\right)$$

is

(JEEAdv.2013)

a)  $\frac{23}{25}$ b)  $\frac{25}{23}$ c)  $\frac{23}{24}$ d)  $\frac{24}{23}$

*Subjective Questions*

8.1.1 Find the value of:

$$\cos \left( 2 \cos^{-1} (x) + \sin^{-1} (x) \right)$$

where  $0 \leq \cos^{-1} (x) \leq \pi$  and  $-\frac{\pi}{2} \leq \sin^{-1} (x) \leq \frac{\pi}{2}$

(1981 – 2Marks)

8.1.2 Find all the solution of

$$4 \cos^2 (x) \sin (x) - 2 \sin^2 (x) = 3 \sin (x)$$

(1983 – 2Marks)

8.1.3 Prove that  $\cos \tan^{-1} \sin \cot^{-1} x = \sqrt{\frac{x^2+1}{x^2+2}}$ .

(2002 - 5 Marks)