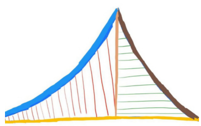
Trigonometry through Geometry



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About this Book

This book introduces trigonometry through high school geometry. This approach relies more on trigonometric equations than cumbersome constructions which are usually non intuitive. All problems in the book are from NCERT mathematics textbooks from Class 9-12. Exercises are from CBSE board exam papers.

The content is sufficient for all practical applications of trigonometry. There is no copyright, so readers are free to print and share.

This book is dedicated to my high school maths teacher, Dr. G.N. Chandwani.

September 30, 2024

Contents

1 Triangle

1.1 Formulae

1.1.1. A right angled triangle looks like Fig. 1.1.1.1. with angles $\angle A, \angle B$ and $\angle C$ and sides

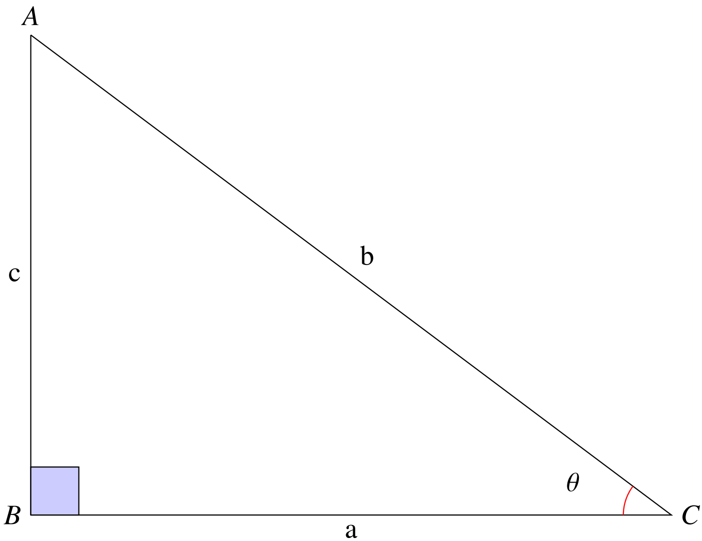

Fig. 1.1.1.1: Right Angled Triangle

a, b and c. The unique feature of this triangle is $\angle B$ which is defined to be 90°.

1.1.2. For simplicity, let the greek letter $\theta = \angle C$. We have the following definitions.

$$\begin{aligned} \sin\theta &= \frac{c}{b} & \cos\theta &= \frac{a}{b} \\ \tan\theta &= \frac{c}{a} & \cot\theta &= \frac{1}{\tan\theta} \\ \csc\theta &= \frac{1}{\sin\theta} & \sec\theta &= \frac{1}{\cos\theta} \end{aligned} \tag{1.1.2.1}$$

1.1.3.

$$\cos\theta = \sin(90^\circ - \theta) \tag{1.1.3.1}$$

1.2 Problems

1.2.1. $ABCD$ is a trapezium in which $AB \parallel DC$ and its diagonals intersect each other at the point O. Show that $\frac{AO}{BO} = \frac{CO}{DO}$

1.2.2. In an isosceles $\triangle ABC$, with $AB = AC$, the bisectors of $\angle B$ and $\angle C$ intersect each other at O. Join A to O. Show that :

a) $OB = OC$

b) AO bisects $\angle A$

1.2.3. ABC is an isosceles triangle in which altitudes BE and CF are drawn to equal sides AC and AB respectively . Show that these altitudes are equal.

1.2.4. ABC is a triangle in which altitudes BE and CF to sides AC and AB are equal. Show that

a) $\triangle ABE \cong \triangle ACF$
b) $AB = AC$, i.e., ABC is an isosceles triangle.

1.2.5. $ABCD$ is a trapezium in which $AB \parallel DC$ and its diagonals intersect each other at the point O. Show that $\frac{AO}{BO} = \frac{CO}{DO}$

1.2.6. D is a point on the side BC of a $\triangle ABC$ such that $\angle ADC = \angle BAC$. Show that $CA^2 = CB.CD$.

1.2.7. BL and CM are medians of a $\triangle ABC$ right angled at A. Prove that $4(BL^2 + CM^2) = 5BC^2$.

1.2.8. O is any point inside a rectangle $ABCD$. Prove that $OB^2 + OD^2 = OA^2 + OC^2$.

1.2.9. D is a point on side BC of $\triangle ABC$ such that $\frac{BD}{CD} = \frac{AB}{AC}$. Prove that AD is the bisector of $\angle BAC$.

1.2.10. $\mathbf{Q}$ is a point on the side $\mathbf{SR}$ of $\triangle\mathbf{PSR}$ such that $\mathbf{PQ} = \mathbf{PR}$. Prove that $\mathbf{PS} > \mathbf{PQ}$.

1.2.11. $\mathbf{S}$ is any point on side $\mathbf{QR}$ of a $\triangle\mathbf{PQR}$. Show that $\mathbf{PQ} + \mathbf{QR} + \mathbf{RP} > \mathbf{2PS}$.

1.2.12. $\mathbf{D}$ is any point on side $\mathbf{AC}$ of a $\triangle\mathbf{ABC}$ with $\mathbf{AB} = \mathbf{AC}$. Show that $\mathbf{CD} < \mathbf{BD}$.

1.2.13. AD is the bisector of $\angle BAC$. Prove that $AB > BD$.

1.2.14. Prove that sum of any two sides of a triangle is greater than twice the median with respect to the third side.

1.2.15. Prove that in a triangle, other than an equilateral triangle, angle opposite the longest side is greater than $\frac{2}{3}$ of a right angle.

1.2.16. P is a point in the interior of a parallelogram $ABCD$. Show that

a) $ar(APB) + ar(PCD) = \frac{1}{2}ar(ABCD)$
b) $ar(APD) + ar(PBC) = ar(APB) + ar(PCD)$

1.2.17. $PQRS$ and $ABRS$ are parallelograms and X is any point on side BR. show that

a) $ar(PQRS) = ar(ABRS)$
b) $ar(AXS) = \frac{1}{2}ar(PQRS)$

1.2.18. $ABCD$ is a quadrilateral and $BE \parallel AC$ and also BE meets DC produced at E. Show that area of $\triangle ADE$ is equal to the area of the quadrilateral $ABCD$.

1.2.19. E is any point on median AD of a $\triangle ABC$. Show that $ar(ABE) = ar(ACE)$.

1.2.20. In a $\triangle ABC, E$ is the mid-point of median AD. Show that $ar(BED) = \frac{1}{4}ar(ABC)$.

1.2.21. AB is a diameter of the circle, CD is a chord equal to the radius of the circle. AC and BD when extended intersect at a point E. Prove that $\angle AEB = 60°$.

1.2.22. If the non-parallel sides of a trapezium are equal, prove that it is cyclic.

1.2.23. Prove that the line of centres of two intersecting circles subtends equal angles at the two points of intersection.

2 Applications

2.0.1. A ladder is placed against a wall such that its foot is at a distance of 2.5 m from the wall and its top reaches a window 6 m above the ground. Find the length of the ladder.

2.0.2. A ladder 10 m long reaches a window 8 m above the ground. Find the distance of the foot of the ladder from base of the wall.

2.0.3. A guy wire attached to a vertical pole of height 18 m is 24 m long and has a stake attached to the other end. How far from the base of the pole should the stake be driven so that the wire will be taut?

2.0.4. An aeroplane leaves an airport and flies due north at a speed of 1000 km per hour. At the same time, another aeroplane leaves the same airport and flies due west at a speed of 1200 km per hour. How far apart will be the two planes after $1\frac{1}{2}$ hours?

2.0.5. Two poles of heights 6 m and 11 m stand on a plane ground. If the distance between the feet of the poles is 12 m, find the distance between their tops.

2.0.6. In $\triangle ABC, AB = 6\sqrt{3}cm, AC = 12cm$ and $BC = 6cm$. Find the angle B.

2.0.7. An aircraft is flying at a height of 3400 m above the ground. If the angle subtended at a ground observation point by the aircraft positions 10.0 s apart is 30°, what is the speed of the aircraft ?

2.0.8. A statue, 1.6 m tall, stands on the top of a pedestal. From a point on the ground, the angle of elevation of the top of the statue is 60° and from the same point the angle of elevation of the top of the pedestal is 45°. Find the height of the pedestal.

2.0.9. The angle of elevation of the top of a building from the foot of the tower is 30° and the angle of elevation of the top of the tower from the foot of the building is 60°. If the tower is 50 m high, find the height of the building.

2.0.10. Two poles of equal heights are standing opposite each other on either side of the road, which is 80 m wide. From a point between them on the road, the angles of elevation of the top of the poles are 60° and 30°, respectively. Find the height of the poles and the distances of the point from the poles.

2.0.11. A TV tower stands vertically on a bank of a canal. From a point on the other bank directly opposite the tower, the angle of elevation of the top of the tower is 60°. From another point 20 m away from this point on the line joing this point to the foot of the tower, the angle of elevation of the top of the tower is 30°. Find the height of the tower and the width of the canal.

2.0.12. From the top of a 7 m high building, the angle of elevation of the top of a cable tower is 60° and the angle of depression of its foot is 45°. Determine the height of the tower.

2.0.13. As observed from the top of a 75 m high lighthouse from the sea-level, the angles of depression of two ships are 30° and 45°. If one ship is exactly behind the other on the same side of the lighthouse, find the distance between the two ships.

2.0.14. A 1.2 m tall girl spots a balloon moving with the wind in a horizontal line at a height of 88.2 m from the ground. The angle of elevation of the balloon from the eyes of the girl at any instant is 60°. After some time, the angle of elevation reduces to 30°. Find the distance travelled by the balloon during the interval.

2.0.15. A straight highway leads to the foot of a tower. A man standing at the top of the tower observes a car at an angle of depression of 30°, which is approaching the foot of the tower with a uniform speed. Six seconds later, the angle of depression of the car is found to be 60°. Find the time taken by the car to reach the foot of the tower from this point.

2.0.16. The angles of elevation of the top of a tower from two points at a distance of 4 m and 9 m from the base of the tower and in the same straight line with it are complementary. Prove that the height of the tower is 6 m.

2.0.17. A girl of height 90 cm is walking away from the base of a lamp-post at a speed of 1.2 m/s. If the lamp is 3.6 m above the ground, find the length of her shadow after 4 seconds.

2.0.18. Nazima is fly fishing in a stream. The tip of her fishing rod is 1.8 m above the surface of the water and the fly at the end of the string rests on the water 3.6 m away and 2.4 m from a point directly under the tip of the rod. Assuming that her string (from the tip of her rod to the fly) is taut, how much string does she have out? If she pulls in the string at the rate of 5 cm per second, what will be the horizontal distance of the fly from her after 12 seconds?

2.0.19. A vertical pole of length 6 m casts a shadow 4 m long on the ground and at the same time a tower casts a shadow 28 m long. Find the height of the tower.

2.0.20. A circus artist is climbing a 20m long rope, which is tightly stretched and tied from the top of a vertical pole to the ground. Find the height of the pole, if the angle made by the rope with the ground level is $30°$.

2.0.21. A tree breaks due to storm and the broken part bends so that the top of the tree touches the ground making an angle of $30°$ with it. The distance between the foot of the tree to the point where the top touches the ground is 8m. Find the height of the tree.

2.0.22. A contractor plans to install two slides for the children to play in a park. For the children below the age of 5 years, she prefers to have a slide whose top is at a height of 1.5m, and is inclined at an angle of $30°$ to the ground, whereas for elder children she wants to have a steep slide at a height of 3m, and inclined at an angle of $60°$ to the ground. What should be the length of the slide in each case?

2.0.23. The angle of elevation of the top of a tower from a point on the ground, which is 30m away from the foot of the tower, is $30°$. Find the height of the tower.

2.0.24. A kite is flying at a height of 60m above the ground. The string attached to the kite is temporarily tied to a point on the ground. The inclination of the string with the ground is $60°$. Find the length of the string, assuming that there is no slack in the string.

2.0.25. A 1.5m tall boy is standing at some distance from a 30m tall building. The angle of elevation from his eyes to the top of the building increases from $30°$ to $60°$ as he walks towards the building. Find the distance he walked towards the building.

2.0.26. From a point on the ground, the angles of elevation of the bottom and the top of a transmission tower fixed at the top of a 20 m high building are $45°$ and $60°$ respectively. Find the height of the tower.

2.0.27. A girl walks 4km west, then she walks 3km in a direction $30°$ east of north and stops. Determine the girl's displacement from her initial point of departure.

2.0.28. The angles of depression of the top and the bottom of an 8m tall building from the top of a multi-storeyed building are $30°$ and $45°$ respectively. Find the height of the multi-storeyed building and the distance between the two buildings.

2.0.29. A tower stands vertically on the ground. From a point on the ground, which is 15m away from the foot of the tower, the angle of elevation of the top of the tower is found to be $60°$. Find the height of the tower.

2.0.30. An electrician has to repair an electric fault pole of height 5m. She needs to reach a point 1.3m below the top of the pole to undertake the repair work. What should be the length of the ladder that she should use which, when inclined at an angle of 60° to the horizontal, would enable her to reach the required position? Also, how far from the foot of the pole should she place the foot of the ladder?

2.0.31. An observer 1.5m tall is 28.5m away from a chimney. The angle of elevation of the top of the chimney from her eyes is 45°. What is the height of the chimney?

2.0.32. From a point **P** on the ground the angle of elevation of the top of a 10m tall building is 30°. A flag is hoisted at the top of the building and the angle of elevation of the top of the flagstaff from **P** is 45°. Find the length of the flagstaff and the distance of the building from the point **P**.

2.0.33. The shadow of a tower standing on a level ground is found to be 40m longer when the Sun's altitude is 30° than when it is 60°. Find the height of the tower.

3 Inequalities

3.1. In a right angled triangle, the hypotenuse is the longest side.
Solution: From (C.2.1),

$$0 \le \sin\theta, \cos\theta \le 1 \tag{3.1.1}$$

Hence,

$$b\sin\theta \le b \implies c \le b \tag{3.1.2}$$

Similalry,

$$a \le b \tag{3.1.3}$$

3.2. D is a point on side BC of $\triangle ABC$ such that $AD = AC$. Show that $AB > AD$

3.3. Show that in a right angled triangle, the hypotenuse is the longest side.

3.4. Sides AB and AC of $\triangle ABC$ are extended to points P and Q respectively. Also, $\angle PBC < \angle QCB$. Show that $AC > AB$.

3.5. Line segments AD and BC intersect at O and form $\triangle OAB$ and $\triangle ODC$. $\angle B < \angle A$ and $\angle C < \angle D$. Show that $AD < BC$.

3.6. AB and CD are respectively the smallest and longest sides of a quadrilateral $ABCD$. Show that $\angle A > \angle C$ and $\angle B > \angle D$.

3.7. In $\triangle PQR, PR > PQ$ and PS bisects $\angle QPR$. Prove that $\angle PSR > \angle PSQ$.

4 Altitudes of a Triangle

4.1. In Fig. 4.2.1, $AD \perp BC$ and $BE \perp AC$ are defined to be the altitudes of $\triangle ABC$.

4.2. Let **H** be the intersection of the altitudes AD and BE as shown in Fig. 4.2.1. CH is extended to meet AB at **F**. Show that $CF \perp AB$.
Solution: From (B.6.2),

$$(\mathbf{B}-\mathbf{C})^\top(\mathbf{H}-\mathbf{A}) = 0 \tag{4.2.1}$$
$$(\mathbf{C}-\mathbf{A})^\top(\mathbf{H}-\mathbf{B}) = 0 \tag{4.2.2}$$

Adding both the above and simplifying,

$$(\mathbf{B}-\mathbf{A})^\top(\mathbf{H}-\mathbf{C}) = 0 \tag{4.2.3}$$

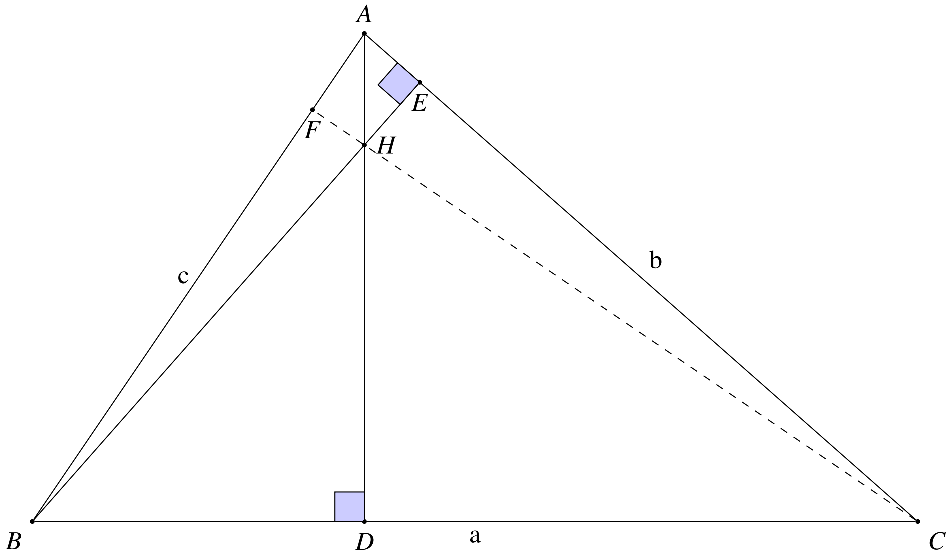

Fig. 4.2.1: Altitudes of a triangle meet at the orthocentre H

$\implies CH \perp AB$, or $CF \perp AB$.

4.3. Altitudes of a $\triangle$ meet at the *orthocentre* H.

5 Median

5.1. In Fig. 5.3.1

$$AF = BF, AE = BE, \tag{5.1.1}$$

and the medians BE and CF meet at **G**. Show that

$$ar(BEC) = ar(BFC) = \frac{1}{2}ar(ABC) \tag{5.1.2}$$

Solution: From (C.5.2),

$$ar(BEC) = \frac{1}{2}a\left(\frac{b}{2}\right)\sin C \tag{5.1.3}$$

$$ar(BFC) = \frac{1}{2}a\left(\frac{c}{2}\right)\sin B \tag{5.1.4}$$

yielding (5.1.2).

5.2. Show that

$$ar(CGE) = ar(BGF) \tag{5.2.1}$$

Solution: From Fig. 5.3.1 and (5.1.2),

$$ar(BGF) + ar(BGC) = ar(CGE) + ar(BGC) \tag{5.2.2}$$

yielding (5.2.1).

5.3. If **G** divides BE and CF in the ratios k_1 and k_2 respectively, show that

$$k_1 = k_2 \tag{5.3.1}$$

Solution: Let

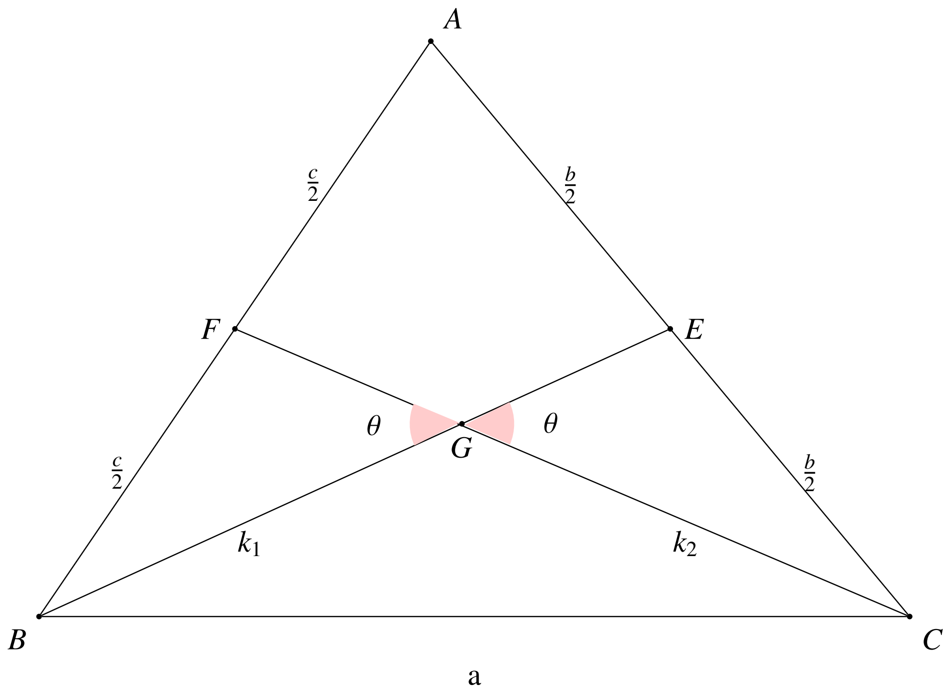

Fig. 5.3.1: $k_1 = k_2$.

$$GE = l_1, GF = l_2 \tag{5.3.2}$$

From (C.5.2) and (5.2.1),

$$\frac{1}{2}l_1 \left(k_2 l_2\right) \sin\theta = \frac{1}{2}l_2 \left(k_1 l_1\right) \sin\theta \tag{5.3.3}$$

yielding (5.3.1).

5.4. Show that

$$k_1 = k_2 = 2 \tag{5.4.1}$$

Solution: Let

$$k_1 = k_2 = k \tag{5.4.2}$$

Using (A.6.3),

$$\mathbf{G} = \frac{k\mathbf{E} + \mathbf{B}}{k+1} = \frac{k\mathbf{F} + \mathbf{C}}{k+1} \tag{5.4.3}$$

$$\implies k\left(\frac{\mathbf{A} + \mathbf{C}}{2}\right) + \mathbf{B} = k\left(\frac{\mathbf{A} + \mathbf{B}}{2}\right) + \mathbf{C} \tag{5.4.4}$$

$$\implies k(\mathbf{B} - \mathbf{C}) = 2(\mathbf{B} - \mathbf{C}) \tag{5.4.5}$$

resulting in (5.4.2).

5.5. Substituting $k = 2$ in (5.4.4),

$$\mathbf{G} = \frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} \tag{5.5.1}$$

5.6. In Fig. 5.6.1, AG is extended to join BC at $\mathbf{D}$. Show that AD is also a median.

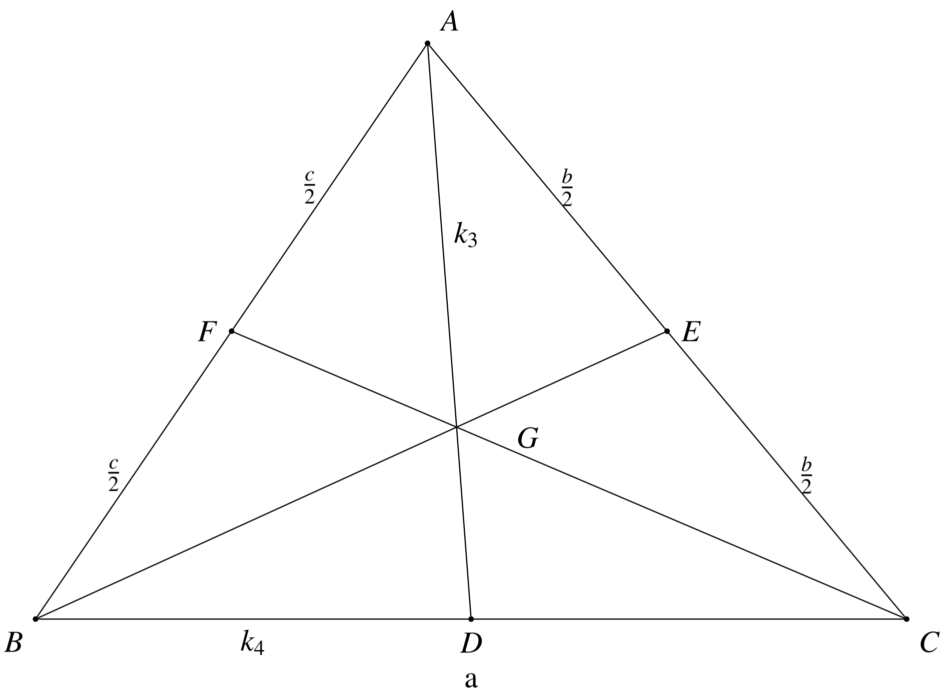

Fig. 5.6.1: $k_3 = 2, k_4 = 1$

Solution: Considering the ratios in Fig. 5.6.1,

$$\mathbf{G} = \frac{k_3\mathbf{D} + \mathbf{A}}{k_3 + 1} \tag{5.6.1}$$

$$\mathbf{D} = \frac{k_4\mathbf{C} + \mathbf{B}}{k_4 + 1} \tag{5.6.2}$$

Substituting from (5.5.1) in the above,

$$(k_3 + 1)\left(\frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3}\right) = k_3\left(\frac{k_4\mathbf{C} + \mathbf{B}}{k_4 + 1}\right) + \mathbf{A} \tag{5.6.3}$$

which can be expressed as

$$(k_3 + 1)(k_4 + 1)(\mathbf{A} + \mathbf{B} + \mathbf{C}) = \\ 3\{k_3(k_4\mathbf{C} + \mathbf{B}) + (k_4 + 1)\mathbf{A}\} \tag{5.6.4}$$

which can be expressed as

$$(k_3k_4 + k_3 - 2k_4 - 2)\mathbf{A} \\ -(-k_3k_4 - k_4 + 2k_3 - 1)\mathbf{B} \\ -(-k_3 - k_4 - 1 + 2k_3k_4)\mathbf{C} = \mathbf{0} \tag{5.6.5}$$

Comparing the above with (A.7.3),

$$p = -k_3k_4 - k_4 + 2k_3 - 1, q = -k_3 - k_4 - 1 + 2k_3k_4 \tag{5.6.6}$$

yielding

$$-k_3k_4 - k_4 + 2k_3 - 1 = 0 \tag{5.6.7}$$

$$-k_3 - k_4 - 1 + 2k_3k_4 = 0 \tag{5.6.8}$$

Subtracting (5.6.7) from (5.6.8),

$$3k_3(k_4 - 1) = 0 \tag{5.6.9}$$

$$\implies k_4 = 1 \tag{5.6.10}$$

which upon substituting in (5.6.7) yields

$$k_3 = 2 \tag{5.6.11}$$

6 Angle Bisectors

6.1. In Fig. 6.1.1, the bisectors of $\angle B$ and $\angle C$ meet at $\mathbf{I}$. Show that IA bisects $\angle A$.
Solution: Using sine formula in (C.5.3)

$$\frac{l_1}{\sin\frac{C}{2}} = \frac{l_3}{\sin(A - \theta)} \tag{6.1.1}$$

$$\frac{l_3}{\sin\frac{B}{2}} = \frac{l_2}{\sin\frac{C}{2}} \tag{6.1.2}$$

$$\frac{l_2}{\sin\theta} = \frac{l_1}{\sin\frac{B}{2}} \tag{6.1.3}$$

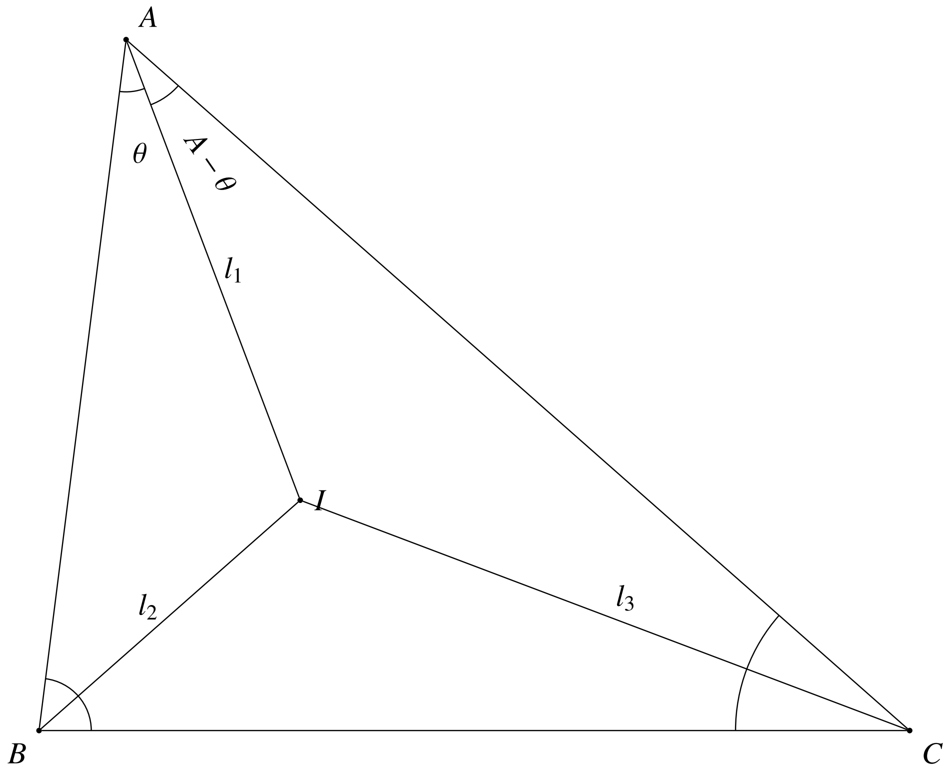

Fig. 6.1.1: Incentre I of $\triangle ABC$

Multiplying the above equations,

$$\sin\theta = \sin(A-\theta) \tag{6.1.4}$$

$$\implies \theta = A - \theta \tag{6.1.5}$$

$$\text{or, } \theta = \frac{A}{2} \tag{6.1.6}$$

6.2. In Fig. 6.2.1,

$$ID \perp BC,\ IE \perp AC,\ IF \perp AB. \tag{6.2.1}$$

Show that

$$ID = IE = IF = r \tag{6.2.2}$$

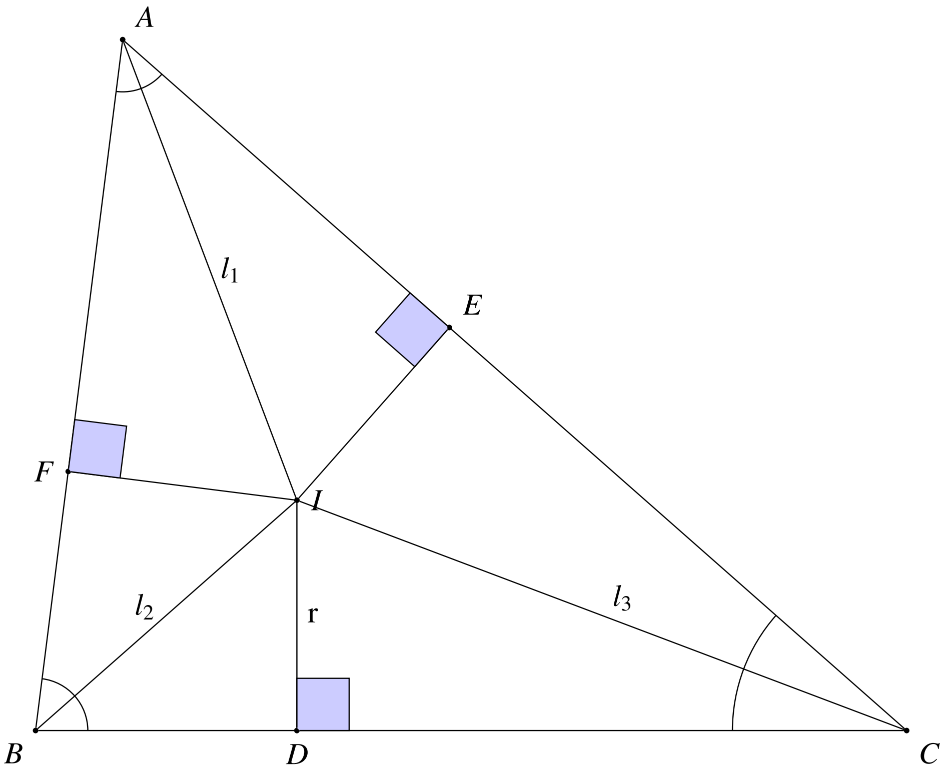

Fig. 6.2.1: Inradius r of $\triangle ABC$

Solution: In $\triangle$s IDC and IEC,

$$ID = IE = \frac{l_3}{\sin \frac{C}{2}} \tag{6.2.3}$$

Similarly, in $\triangle$s IEA and IFA,

$$IF = IE = \frac{l_1}{\sin \frac{A}{2}} \tag{6.2.4}$$

yielding (6.2.2)

6.3. In Fig. 6.2.1, show that

$$BD = BF, \; AE = AF, \; CD = CE \tag{6.3.1}$$

Solution: From Fig. 6.2.1, in $\triangle$s IBD and IBF,

$$x = BD = BF = r \cot \frac{B}{2} \tag{6.3.2}$$

Similarly, other results can be obtained.

6.4. The circle with centre **I** and radius r in Fig. 6.4.1 is known as the *incircle*.

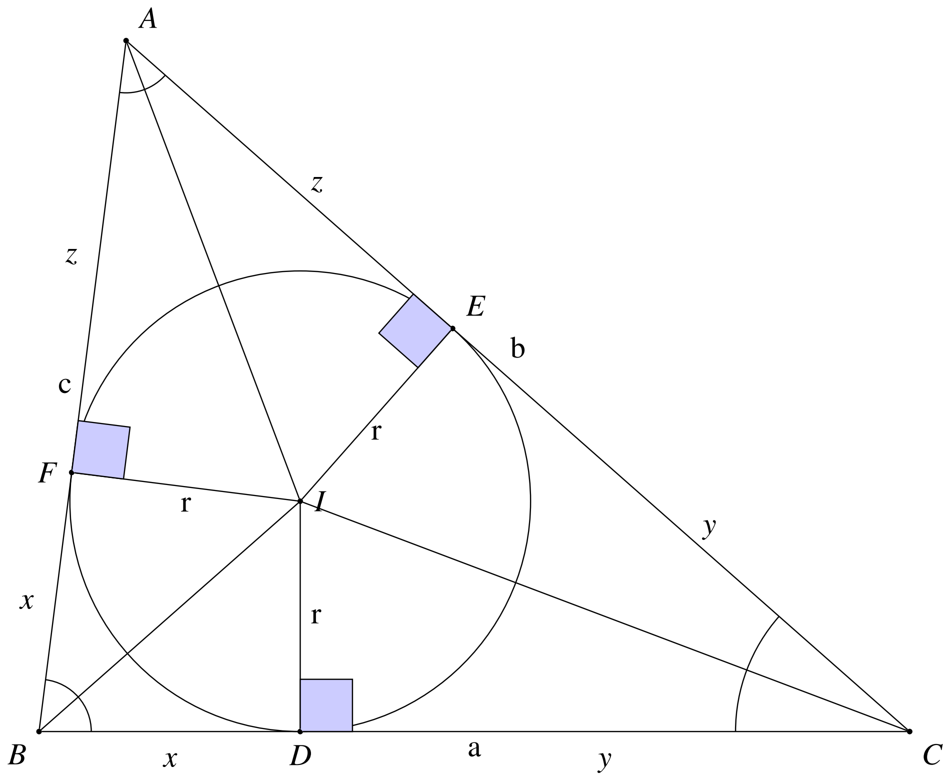

Fig. 6.4.1: Incircle of $\triangle ABC$

7 Perpendicular Bisectors

7.1. In Fig. 7.1.1,

$$OB = OC = R \tag{7.1.1}$$

Such a triangle is known as an isosceles triangle. Show that

$$\angle B = \angle C \tag{7.1.2}$$

Solution: Using (C.5.3),

$$\frac{\sin B}{R} = \frac{\sin C}{R} \tag{7.1.3}$$

$$\implies \sin B = \sin C \tag{7.1.4}$$

$$\text{or, } \angle B = \angle C. \tag{7.1.5}$$

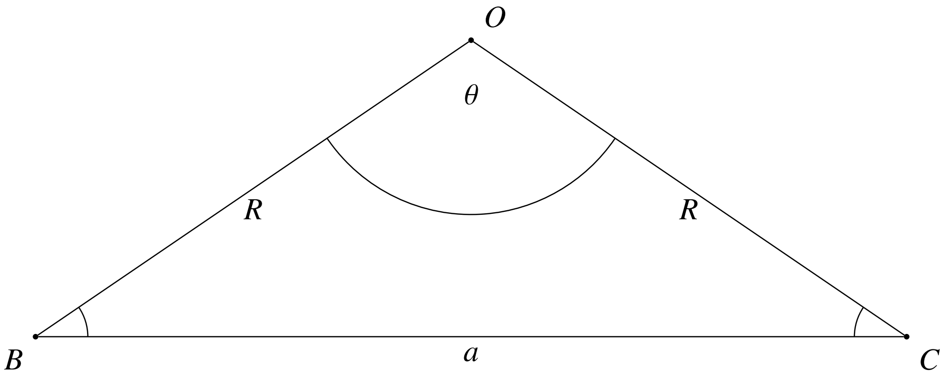

Fig. 7.1.1: Isosceles Triangle

7.2. In Fig. 7.1.1, show that

$$a = 2R\sin\frac{\theta}{2} \tag{7.2.1}$$

Solution: In $\triangle OBC$, using the cosine formula from (B.2.1),

$$\cos\theta = \frac{R^2 + R^2 - a^2}{2R^2} = 1 - \frac{a^2}{2R^2} \tag{7.2.2}$$

$$\implies \frac{a^2}{2R^2} = 2\sin^2\frac{\theta}{2} \tag{7.2.3}$$

yielding (7.2.1).

7.3. In Fig. 8.1.1, show that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R. \tag{7.3.1}$$

Solution: From (8.4.1) and (7.2.1)

$$a = 2R\sin A \tag{7.3.2}$$

7.4. In Fig. 7.4.1,

$$OB = OC = R, BD = DC. \tag{7.4.1}$$

Show that $OD \perp BC$.

Solution:

$$\|\mathbf{O} - \mathbf{C}\| = \|\mathbf{O} - \mathbf{B}\| = R \tag{7.4.2}$$

$$\implies \|\mathbf{O} - \mathbf{C}\|^2 = \|\mathbf{O} - \mathbf{B}\|^2 \tag{7.4.3}$$

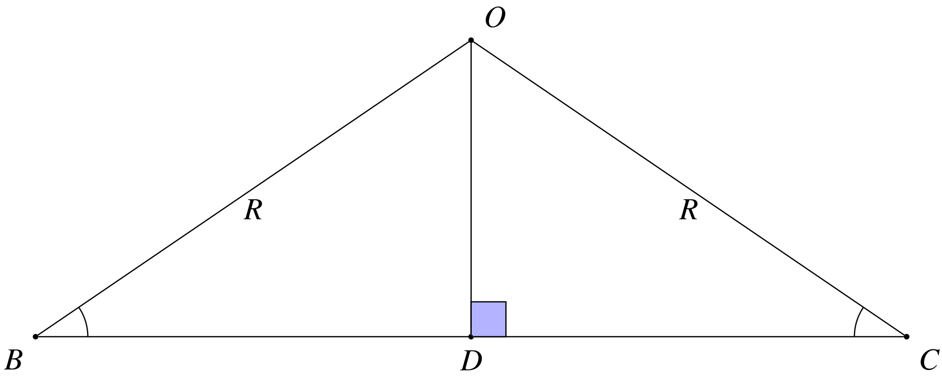

Fig. 7.4.1: Perpendicular bisector.

which can be expressed as

$$(\mathbf{O}-\mathbf{C})^{\top}(\mathbf{O}-\mathbf{C}) = (\mathbf{O}-\mathbf{B})^{\top}(\mathbf{O}-\mathbf{B}) \tag{7.4.4}$$

$$\|\mathbf{O}\|^2 - 2\mathbf{O}^{\top}\mathbf{C} + \|\mathbf{C}\|^2 = \|\mathbf{O}\|^2 - 2\mathbf{O}^{\top}\mathbf{B} + \|\mathbf{B}\|^2 \tag{7.4.5}$$

$$\implies (\mathbf{B}-\mathbf{C})^{\top}\mathbf{O} = \frac{\|\mathbf{B}\|^2 - \|\mathbf{C}\|^2}{2} \tag{7.4.6}$$

which can be simplified to obtain

$$(\mathbf{B}-\mathbf{C})^{\top}\left\{\mathbf{O} - \left(\frac{\mathbf{B}+\mathbf{C}}{2}\right)\right\} = 0 \tag{7.4.7}$$

$$\text{or,}\ \ (\mathbf{B}-\mathbf{C})^{\top}\{\mathbf{O}-\mathbf{D}\} = 0 \tag{7.4.8}$$

which proves the give result using (A.6.3) and (B.6.2).

7.5. In Fig. 7.5.1, OD and OE are the perpendicular bisectors of sides BC and AC respectively. Show that $OA = R$.
Solution: Tracing (7.4.8) backwards yields

$$OB = OC, OC = OA = R. \tag{7.5.1}$$

8 Circumcircle: Circle Equation

8.1. The equation of the circle in Fig. 8.1.1, is

$$\|\mathbf{x}-\mathbf{O}\| = R \tag{8.1.1}$$

This is known as the *circumcircle* of $\triangle ABC$.

8.2. Any point on the circle can be expressed as

$$\mathbf{x} = \mathbf{O} + R\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}, \quad 0 \in [0, 2\pi]. \tag{8.2.1}$$

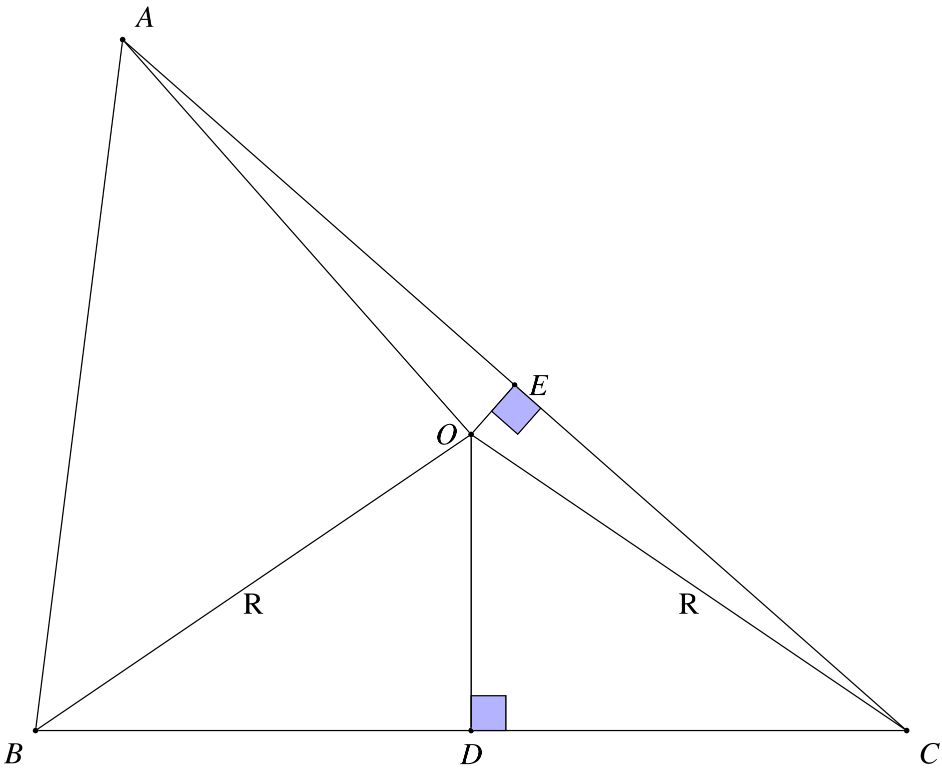

Fig. 7.5.1: Perpendicular bisectors of $\triangle ABC$ meet at $\mathbf{O}$.

8.3. Let

$$R = 1,\ \mathbf{O} = \mathbf{0},\ \mathbf{A} = \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix},\ \mathbf{B} = \begin{pmatrix} \cos\theta_2 \\ \sin\theta_2 \end{pmatrix}, \tag{8.3.1}$$

Show that

$$\|\mathbf{A} - \mathbf{B}\| = 2\sin\left(\frac{\theta_1 - \theta_2}{2}\right) \tag{8.3.2}$$

Solution: From (8.2.1).

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} \cos\theta_1 - \cos\theta_2 \\ \sin\theta_1 - \sin\theta_2 \end{pmatrix} \tag{8.3.3}$$

$$\implies \|\mathbf{A} - \mathbf{B}\|^2 = (\cos\theta_1 - \cos\theta_2)^2 + (\sin\theta_1 - \sin\theta_2)^2 \tag{8.3.4}$$

$$= 2\left\{1 - \cos(\theta_1 - \theta_2)\right\} = 4\sin^2\left(\frac{\theta_1 - \theta_2}{2}\right) \tag{8.3.5}$$

yielding (8.3.2) from (C.12.3).

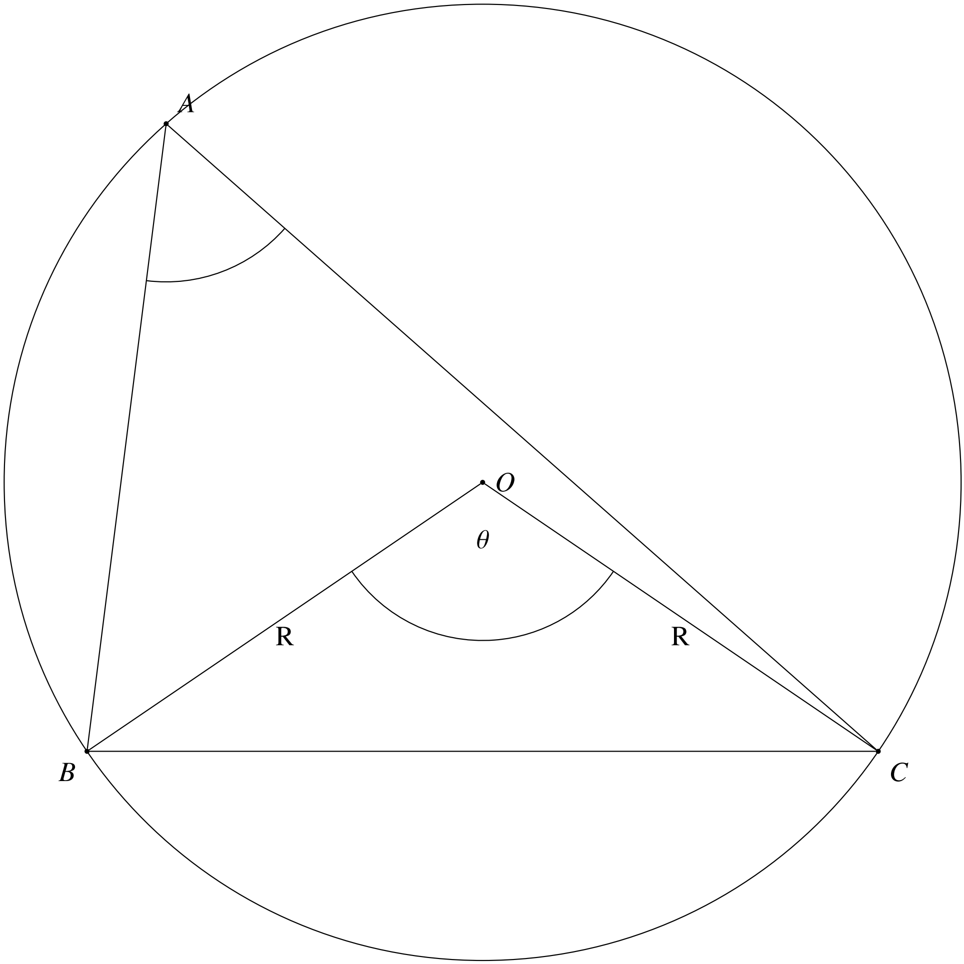

Fig. 8.1.1: Circumcircle of $\triangle ABC$

8.4. In Fig. 8.1.1, show that

$$\theta = 2A. \tag{8.4.1}$$

Solution: Let

$$\mathbf{C} = \begin{pmatrix} \cos\theta_3 \\ \sin\theta_3 \end{pmatrix} \tag{8.4.2}$$

Then, substituting from (8.3.2) in (B.5.2),

$$\cos A = \frac{4\sin^2\left(\frac{\theta_1-\theta_2}{2}\right) + 4\sin^2\left(\frac{\theta_1-\theta_3}{2}\right) - 4\sin^2\left(\frac{\theta_2-\theta_3}{2}\right)}{8\sin\left(\frac{\theta_1-\theta_2}{2}\right)\sin\left(\frac{\theta_1-\theta_3}{2}\right)} \tag{8.4.3}$$

$$= \frac{2\sin^2\left(\frac{\theta_1-\theta_2}{2}\right) + \cos(\theta_2-\theta_3) - \cos(\theta_1-\theta_3)}{4\sin\left(\frac{\theta_1-\theta_2}{2}\right)\sin\left(\frac{\theta_1-\theta_3}{2}\right)} \tag{8.4.4}$$

from (C.12.3). $\therefore$ from (C.11.4),

$$\cos A = \frac{2\sin^2\left(\frac{\theta_1-\theta_2}{2}\right) + 2\sin\left(\frac{\theta_1-\theta_2}{2}\right)\sin\left(\frac{\theta_1+\theta_2}{2} - \theta_3\right)}{4\sin\left(\frac{\theta_1-\theta_2}{2}\right)\sin\left(\frac{\theta_1-\theta_3}{2}\right)} \tag{8.4.5}$$

$$= \frac{\sin\left(\frac{\theta_1-\theta_2}{2}\right) + \sin\left(\frac{\theta_1+\theta_2}{2} - \theta_3\right)}{2\sin\left(\frac{\theta_1-\theta_3}{2}\right)} \tag{8.4.6}$$

From (C.11.1), the above equation can be expressed as

$$\cos A = \frac{2\sin\left(\frac{\theta_1-\theta_3}{2}\right)\cos\left(\frac{\theta_2-\theta_3}{2}\right)}{2\sin\left(\frac{\theta_1-\theta_3}{2}\right)} = \cos\left(\frac{\theta_2-\theta_3}{2}\right) \tag{8.4.7}$$

$$\implies 2A = \theta_2 - \theta_3 \tag{8.4.8}$$

Similarly,

$$\cos\theta = \frac{1 + 1 - 4\sin^2\left(\frac{\theta_2-\theta_3}{2}\right)}{2} = \cos(\theta_2-\theta_3) = \cos 2A \tag{8.4.9}$$

9 Tangent

9.1. In Fig. 9.1.1, OC is the radius and PC touches the circle at C. Show that

$$OC \perp PC. \tag{9.1.1}$$

Solution: The equation of PC can be expressed as

$$\mathbf{x} = \mathbf{C} + \mu\mathbf{m} \tag{9.1.2}$$

and the equation of the circle is

$$\|\mathbf{x} - \mathbf{O}\| = R \tag{9.1.3}$$

Substituting (9.1.2) in (9.1.3),

$$\|\mathbf{C} + \mu\mathbf{m} - \mathbf{O}\|^2 = R^2 \tag{9.1.4}$$

$$\implies \mu^2\|\mathbf{m}\|^2 + 2\mu\mathbf{m}^\top(\mathbf{C}-\mathbf{O}) + \|\mathbf{C}-\mathbf{O}\|^2 - R^2 = 0 \tag{9.1.5}$$

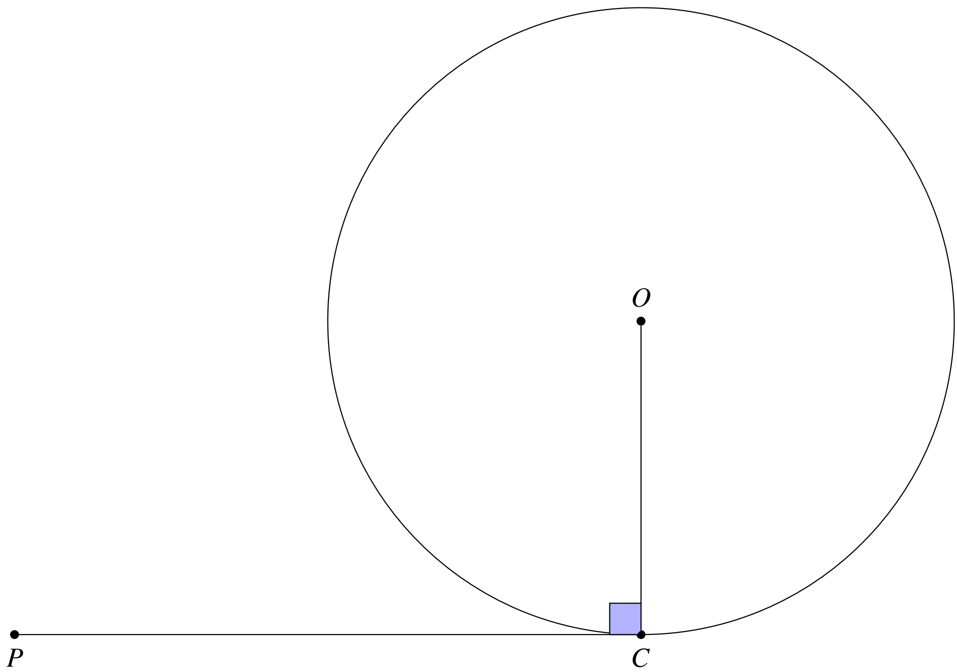

Fig. 9.1.1

The above equation has only one root. Hence the discriminant of the above quadratic should be zero. So,

$$\left\{\mathbf{m}^\top (\mathbf{C}-\mathbf{O})\right\}^2 - \|\mathbf{m}\|^2 \left\{\|\mathbf{C}-\mathbf{O}\|^2 - R^2\right\} = 0 \tag{9.1.6}$$

Since $\mathbf{C}$ is a point on the circle,

$$\|\mathbf{C}-\mathbf{O}\|^2 - R^2 = 0 \tag{9.1.7}$$

$$\implies \mathbf{m}^\top (\mathbf{C}-\mathbf{O}) = 0 \tag{9.1.8}$$

upon substituting in (9.1.6). Using the definition of the direction vector from (A.2.1)

$$\mathbf{m} = \mathbf{P}-\mathbf{C} \tag{9.1.9}$$

$$\implies (\mathbf{P}-\mathbf{C})^\top (\mathbf{C}-\mathbf{O}) = 0 \tag{9.1.10}$$

which is equivalent to (9.1.1).

9.2. In Fig. 9.2.1 show that

$$\theta = \alpha \tag{9.2.1}$$

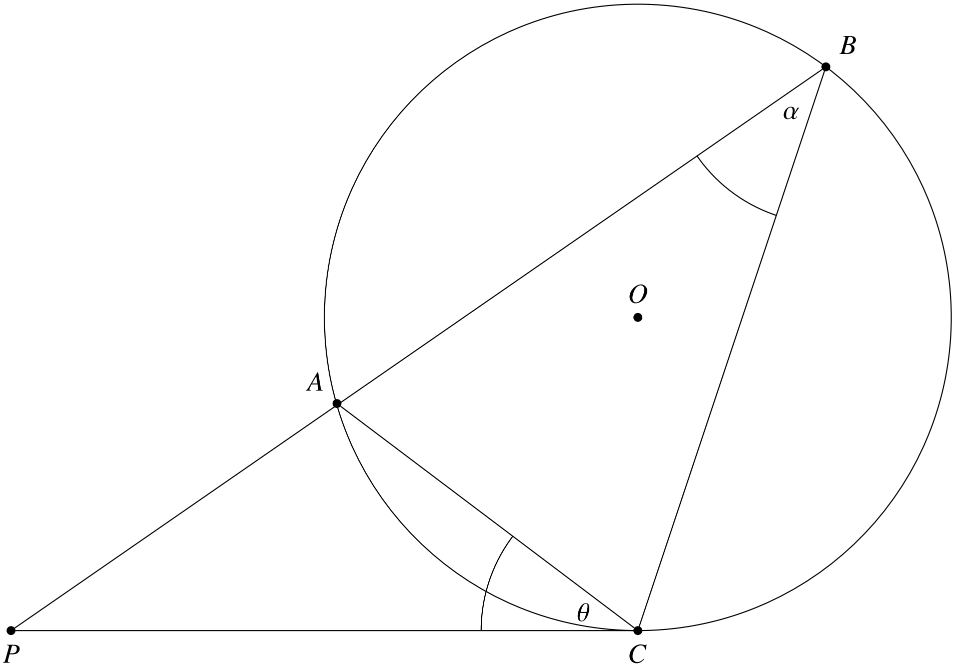

Fig. 9.2.1: $\theta = \alpha$.

Solution: Let Let

$$\mathbf{O} = \mathbf{0} \mathbf{A} = \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} \cos\theta_2 \\ \sin\theta_2 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} \cos\theta_3 \\ \sin\theta_3 \end{pmatrix} \tag{9.2.2}$$

Without loss of generality, let

$$\theta_3 = \frac{\pi}{2} \tag{9.2.3}$$

Then,

$$\mathbf{C} - \mathbf{O} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{9.2.4}$$

From from (9.1.10),

$$\mathbf{C} - \mathbf{P} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{9.2.5}$$

From (B.5.1) and (9.2.5),

$$\cos\theta = \frac{\begin{pmatrix}\cos\theta_3 - \cos\theta_1 & \sin\theta_3 - \sin\theta_1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}}{2\sin\left(\frac{\theta_1-\theta_3}{2}\right)} \tag{9.2.6}$$

$$= \sin\left(\frac{\theta_1+\theta_3}{2}\right) = \cos\left(\frac{\pi}{2} - \frac{\theta_1+\theta_3}{2}\right) = \cos\left(\frac{\pi}{4} - \frac{\theta_1}{2}\right) \tag{9.2.7}$$

upon substituting from (9.2.3). Similarly, from (8.4.7),

$$\cos\alpha = \cos\left(\frac{\theta_1-\theta_3}{2}\right) = \cos\left(\frac{\pi}{4} - \frac{\theta_1}{2}\right) = \cos\theta \tag{9.2.8}$$

9.3. In Fig. 9.2.1, show that $PA.PB = PC^2$.
Solution: In $\triangle$s APC and BPC, using (9.2.1),

$$\frac{AP}{\sin\theta} = \frac{AC}{\sin P} \tag{9.3.1}$$

$$\frac{PC}{\sin\theta} = \frac{BC}{\sin P} \tag{9.3.2}$$

$$\implies \frac{PC}{AP} = \frac{BC}{AC}\left(= \frac{BP}{CP}\right) \tag{9.3.3}$$

which gives the desired result. $\triangle$s APC and BPC are said to be *similar*.

Appendix A
Collinear Points

A.1. In Fig. A.1.1,

$$a = y\cot\theta + x \tag{A.1.1}$$

$$\implies \mathbf{D} = \begin{pmatrix}-x\\y\end{pmatrix} = \begin{pmatrix}-a + y\cot\theta\\y\end{pmatrix} \tag{A.1.2}$$

$$= \begin{pmatrix}-a\\0\end{pmatrix} + y\cot\theta\begin{pmatrix}1\\\tan\theta\end{pmatrix} \tag{A.1.3}$$

$$\text{or, } \mathbf{D} \equiv \mathbf{B} + \kappa\mathbf{m} \tag{A.1.4}$$

The above equation can be generalized for any point on the line AB as

$$\mathbf{x} = \mathbf{B} + \kappa\mathbf{m} \tag{A.1.5}$$

which is known as the *parametric* equation of a line. $\mathbf{m}$ is defined to be the *direction vector* of AB and

$$m = \tan\theta \tag{A.1.6}$$

is defined to be the *slope*.

A.2. The direction vector of the line AB is

$$\mathbf{A} - \mathbf{B} \equiv \mathbf{B} - \mathbf{A} \equiv \kappa\begin{pmatrix}1\\m\end{pmatrix}, \tag{A.2.1}$$

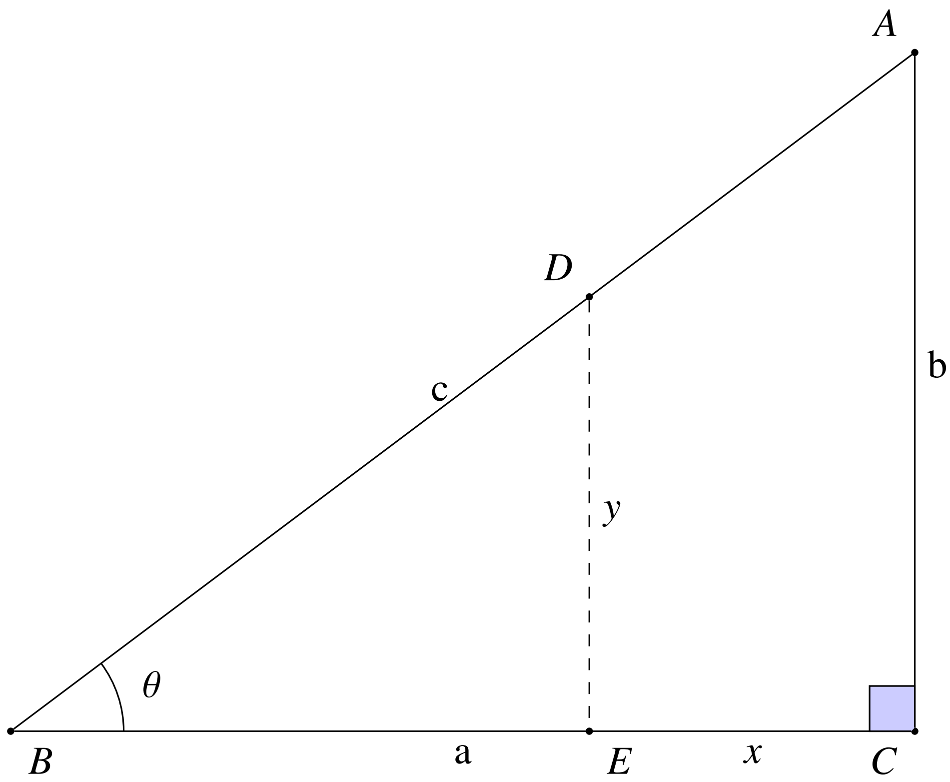

Fig. A.1.1: $k_1 = k_2 = 2$.

A.3. (A.1.1) can also be expressed as

$$a = y \cot\theta + x \tag{A.3.1}$$

$$\implies \begin{pmatrix} -\tan\theta & 1 \end{pmatrix} \begin{pmatrix} -x \\ y \end{pmatrix} = b \tag{A.3.2}$$

$$\text{or, } \mathbf{n}^\top \mathbf{x} = b \tag{A.3.3}$$

which is known as the *normal* equation of a line. Here,

$$\mathbf{n} = \begin{pmatrix} -m \\ 1 \end{pmatrix} \tag{A.3.4}$$

is defined to be the *normal vector* of the line. The vector product in (A.3.2) is known as the *inner product* or *dot product*

A.4. It is easy to verify that

$$\mathbf{n}^\top \mathbf{m} = 0 \tag{A.4.1}$$

and

A.5.

$$\mathbf{n} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{m} = \begin{pmatrix} \cos\left(\frac{\pi}{2}\right) & \sin\left(\frac{\pi}{2}\right) \\ \sin\left(\frac{\pi}{2}\right) & \cos\left(\frac{\pi}{2}\right) \end{pmatrix} \mathbf{m} \tag{A.5.1}$$

The matrix

$$\mathbf{R}_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{A.5.2}$$

is defined to be the *rotation matrix*. (A.5.1) implies that $\mathbf{n}$ can be obtained from $\mathbf{m}$ through a 90° clockwise rotation.

A.6. From (A.1.5), since $\mathbf{A}, \mathbf{D}$ and $\mathbf{C}$ are on the same line,

$$\begin{aligned} \mathbf{D} &= \mathbf{A} + q\mathbf{m} \\ \mathbf{B} &= \mathbf{D} + p\mathbf{m} \end{aligned} \tag{A.6.1}$$

$$\implies p(\mathbf{D} - \mathbf{A}) + q(\mathbf{D} - \mathbf{B}) = 0, \quad p, q \neq 0 \tag{A.6.2}$$

$$\implies \mathbf{D} = \frac{k\mathbf{A} + \mathbf{B}}{k+1}, \quad k = \frac{p}{q}. \tag{A.6.3}$$

which is known as *section formula*. $(\mathbf{D} - \mathbf{A}), (\mathbf{D} - \mathbf{B})$ are then said to be *linearly dependent*.

A.7. Consequently, points $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ form a triangle if

$$p(\mathbf{A} - \mathbf{B}) + q(\mathbf{C} - \mathbf{B}) \tag{A.7.1}$$

$$= (p+q)\mathbf{B} - p\mathbf{A} - q\mathbf{C} = 0 \tag{A.7.2}$$

$$\implies p = 0, q = 0 \tag{A.7.3}$$

Appendix B
Cosine Formula

B.1. In Fig. B.1.1, show that

$$\begin{pmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{pmatrix} \begin{pmatrix} \cos A \\ \cos B \\ \cos C \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{B.1.1}$$

Solution: From Fig. B.1.1,

$$a = x + y = b\cos C + c\cos B = \begin{pmatrix} \cos C & \cos B \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix} \tag{B.1.2}$$

$$= \begin{pmatrix} 0 & b & c \end{pmatrix} \begin{pmatrix} \cos A \\ \cos C \\ \cos B \end{pmatrix} \tag{B.1.3}$$

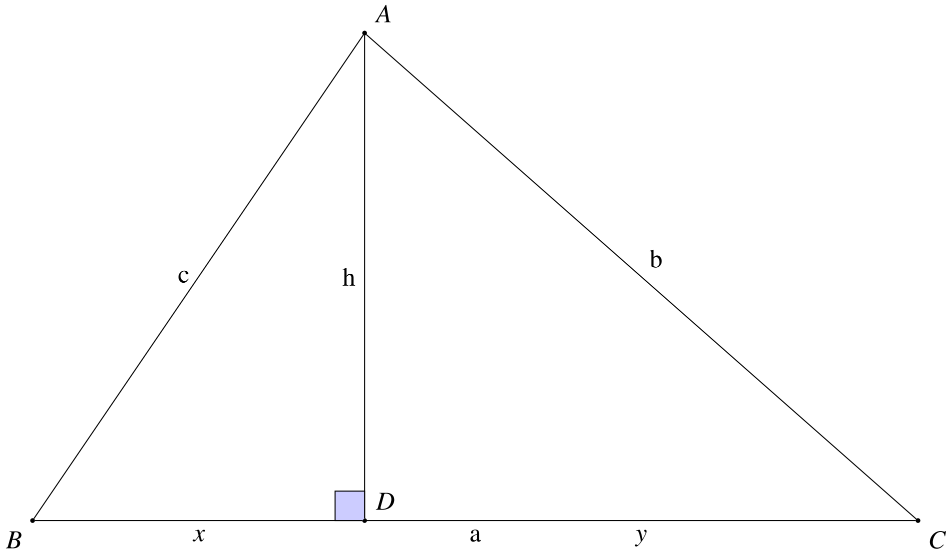

Fig. B.1.1: The cosine formula

Similarly,

$$b = c\cos A + a\cos C = \begin{pmatrix} c & 0 & a \end{pmatrix}\begin{pmatrix} \cos A \\ \cos C \\ \cos B \end{pmatrix} \tag{B.1.4}$$

$$c = b\cos A + a\cos B = \begin{pmatrix} b & a & 0 \end{pmatrix}\begin{pmatrix} \cos A \\ \cos C \\ \cos B \end{pmatrix} \tag{B.1.5}$$

The above equations can be expressed in matrix form as (B.1.1).

B.2. Show that

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \tag{B.2.1}$$

Solution: Using the properties of determinants,

$$\cos A = \frac{\begin{vmatrix} a & c & b \\ b & 0 & a \\ c & a & 0 \end{vmatrix}}{\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}} = \frac{ab^2 + ac^2 - a^3}{abc + abc} = \frac{b^2 + c^2 - a^2}{2abc} \tag{B.2.2}$$

B.3. The *norm* of $\mathbf{A}$ is defined as

$$\|\mathbf{A}\| = \sqrt{\mathbf{A}^\top \mathbf{A}} = \sqrt{a_1^2 + a_2^2} \tag{B.3.1}$$

B.4. In Fig. C.1.1 it is easy to verify that

$$\|\mathbf{A}-\mathbf{C}\|^2 = \begin{pmatrix}-a & c\end{pmatrix}\begin{pmatrix}-a \\ c\end{pmatrix} = a^2 + c^2 = b^2 \tag{B.4.1}$$

from (C.3.1). Thus, the distance betwen any two points **A** and **B** is given by

$$\|\mathbf{A}-\mathbf{B}\| \tag{B.4.2}$$

B.5. In Fig. B.1.1 show that

$$\cos A = \frac{(\mathbf{A}-\mathbf{B})^\top(\mathbf{A}-\mathbf{C})}{\|\mathbf{A}-\mathbf{B}\|\,\|\mathbf{A}-\mathbf{C}\|} \tag{B.5.1}$$

Solution: From (B.2.1), using (B.4.2),

$$\cos A = \frac{\|\mathbf{A}-\mathbf{B}\|^2 + \|\mathbf{A}-\mathbf{C}\|^2 - \|\mathbf{B}-\mathbf{C}\|^2}{2\,\|\mathbf{A}-\mathbf{B}\|\,\|\mathbf{A}-\mathbf{C}\|} \tag{B.5.2}$$

$$= \frac{\|\mathbf{A}\|^2 - \mathbf{A}^\top\mathbf{B} - \mathbf{A}^\top\mathbf{C} + \mathbf{B}^\top\mathbf{C}}{\|\mathbf{A}-\mathbf{B}\|\,\|\mathbf{A}-\mathbf{C}\|} \tag{B.5.3}$$

which can be expressed as (B.5.1).

B.6. For $A = 90°$,

$$\cos A = 0 \tag{B.6.1}$$

$$\implies (\mathbf{A}-\mathbf{B})^\top(\mathbf{A}-\mathbf{C}) = 0 \tag{B.6.2}$$

from (B.5.1).

Appendix C
Trigonometric Identities

C.1. In Fig. C.1.1, show that

$$b = a\cos\theta + c\sin\theta \tag{C.1.1}$$

Solution: We observe that

$$CD = a\cos\theta \tag{C.1.2}$$

$$AD = c\cos\alpha = c\sin\theta \quad \text{(From} \quad (1.1.3.1)) \tag{C.1.3}$$

Thus,

$$CD + AD = b = a\cos\theta + c\sin\theta \tag{C.1.4}$$

C.2. From (C.1.1), show that

$$\sin^2\theta + \cos^2\theta = 1 \tag{C.2.1}$$

Solution: Dividing both sides of (C.1.1) by b,

$$1 = \frac{a}{b}\cos\theta + \frac{c}{b}\sin\theta \tag{C.2.2}$$

$$\Rightarrow \sin^2\theta + \cos^2\theta = 1 \quad \text{(from} \quad (1.1.2.1)) \tag{C.2.3}$$

C.3. Using (C.1.1), show that

$$b^2 = a^2 + c^2 \tag{C.3.1}$$

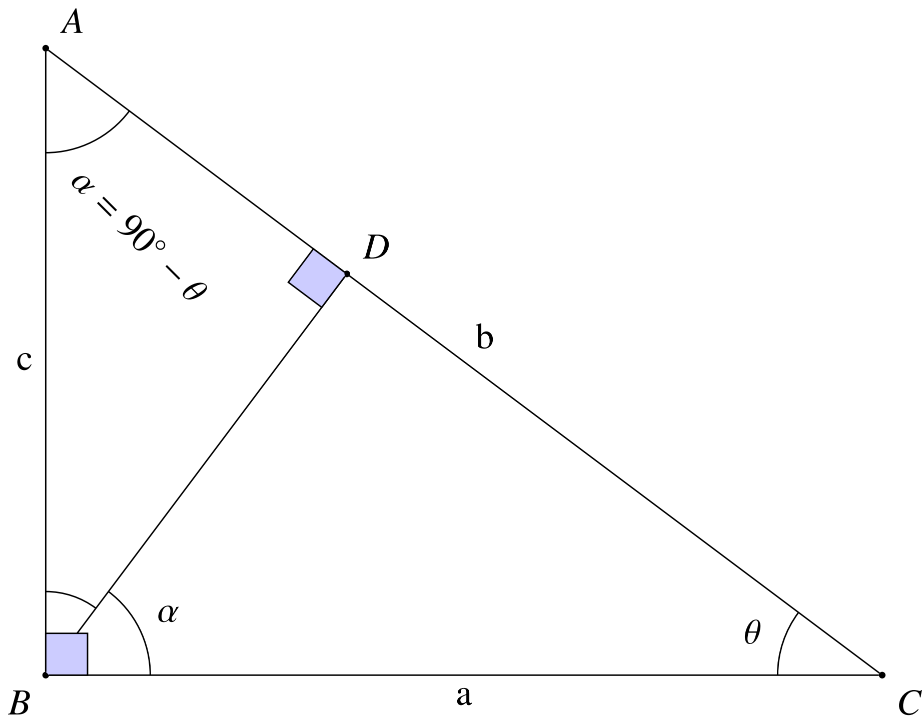

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Fig. C.1.1: Baudhayana Theorem

(C.3.1) is known as the Baudhayana theorem. It is also known as the Pythagoras theorem.

Solution: From (C.1.1),

$$b = a\frac{a}{b} + c\frac{c}{b} \quad \text{(from} \quad (1.1.2.1)) \tag{C.3.2}$$

$$\implies b^2 = a^2 + c^2 \tag{C.3.3}$$

C.4. Show that the area of ΔABC in Fig. C.4.1 is $\frac{1}{2}ab\sin C$.

Solution: We have

$$ar(\Delta ABC) = \frac{1}{2}ah = \frac{1}{2}ab\sin C \quad (\because \quad h = b\sin C). \tag{C.4.1}$$

C.5. Show that

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \tag{C.5.1}$$

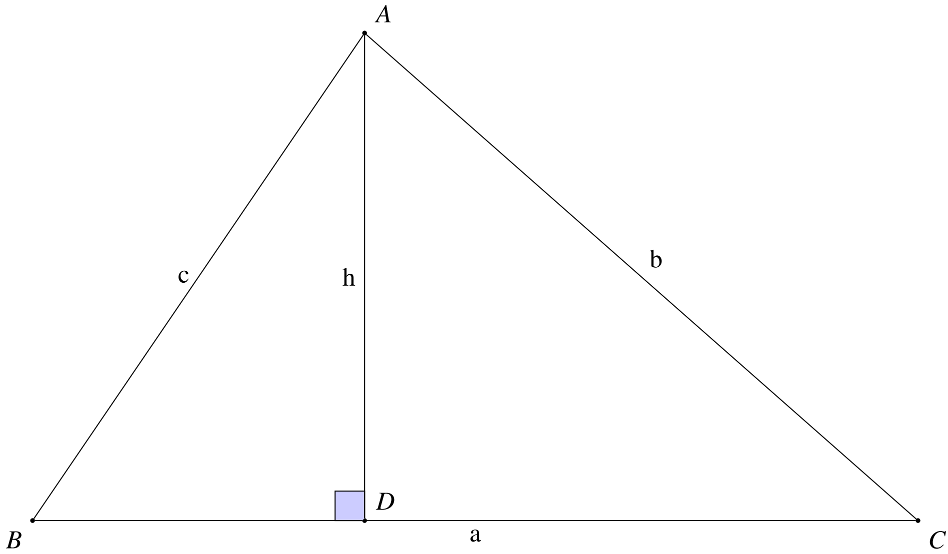

Fig. C.4.1: Area of a Triangle

Solution: Fig. C.4.1 can be suitably modified to obtain

$$ar\,(\Delta ABC) = \frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B \tag{C.5.2}$$

Dividing the above by abc, we obtain

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \tag{C.5.3}$$

This is known as the sine formula.

C.6. Show that

$$\alpha > \beta \implies \sin\alpha > \sin\beta \tag{C.6.1}$$

Solution: In Fig. C.6.1,

$$ar\,(\triangle ABD) < ar\,(\triangle ABC) \tag{C.6.2}$$

$$\implies \frac{1}{2}lc\sin\theta_1 < \frac{1}{2}ac\sin(\theta_1 + \theta_2) \tag{C.6.3}$$

$$\implies \frac{l}{a} < \frac{\sin(\theta_1 + \theta_2)}{\sin\theta_1} \tag{C.6.4}$$

$$\text{or, } 1 < \frac{l}{a} < \frac{\sin(\theta_1 + \theta_2)}{\sin\theta_1} \tag{C.6.5}$$

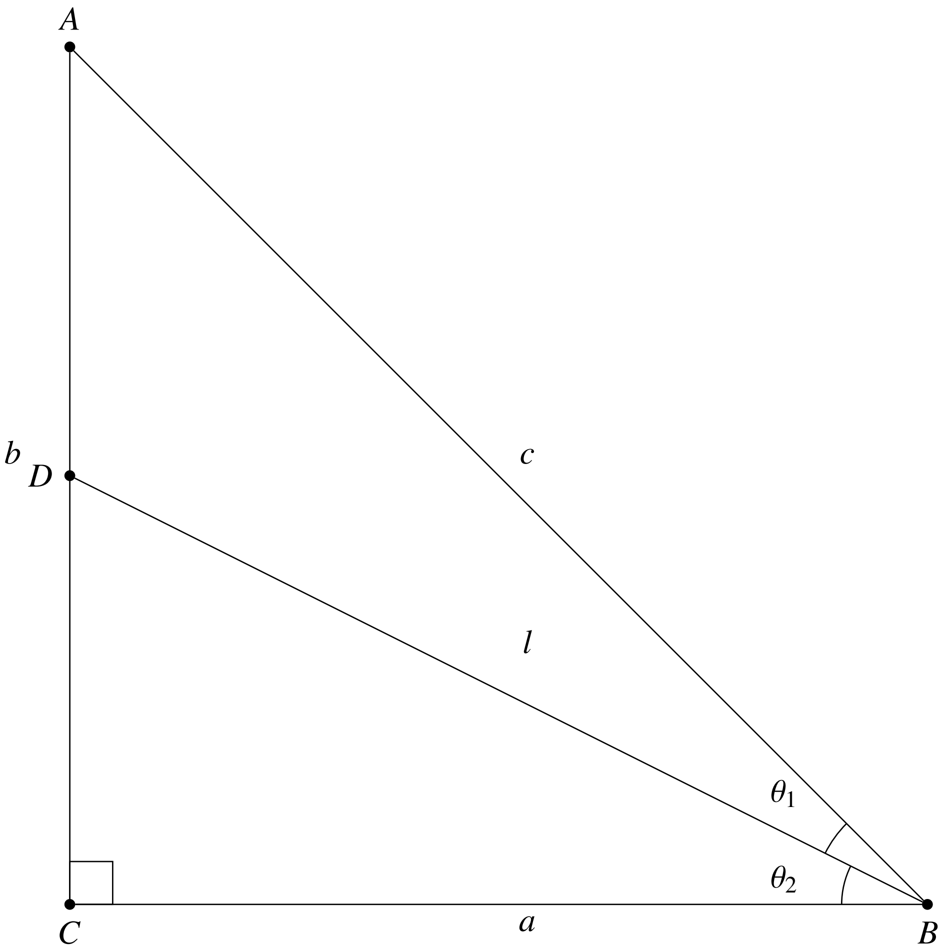

Fig. C.6.1

from Theorem 3.1, yielding

$$\implies \frac{\sin(\theta_1 + \theta_2)}{\sin\theta_1} > 1. \tag{C.6.6}$$

This proves (C.6.1).

C.7. Using Fig. C.6.1, show that

$$\sin\theta_1 = \sin(\theta_1 + \theta_2)\cos\theta_2 - \cos(\theta_1 + \theta_2)\sin\theta_2 \tag{C.7.1}$$

Solution: The following equations can be obtained from the figure using the forumula for the area of a triangle

$$ar(\Delta ABC) = \frac{1}{2}ac\sin(\theta_1 + \theta_2) \tag{C.7.2}$$

$$= ar(\Delta BDC) + ar(\Delta ADB) \tag{C.7.3}$$

$$= \frac{1}{2}cl\sin\theta_1 + \frac{1}{2}al\sin\theta_2 \tag{C.7.4}$$

$$= \frac{1}{2}ac\sin\theta_1\sec\theta_2 + \frac{1}{2}a^2\tan\theta_2 \tag{C.7.5}$$

($\because l = a\sec\theta_2$). From the above,

$$\sin(\theta_1 + \theta_2) = \sin\theta_1\sec\theta_2 + \frac{a}{c}\tan\theta_2 \tag{C.7.6}$$

$$= \sin\theta_1\sec\theta_2 + \cos(\theta_1 + \theta_2)\tan\theta_2 \tag{C.7.7}$$

Multiplying both sides by $\cos\theta_2$,

$$\sin(\theta_1 + \theta_2)\cos\theta_2 = \sin\theta_1 + \cos(\theta_1 + \theta_2)\sin\theta_2 \tag{C.7.8}$$

resulting in (C.7.1).

C.8. Find Hero's formula for the area of a triangle.
Solution: From (C.4), the area of $\triangle ABC$ is

$$\frac{1}{2}ab\sin C = \frac{1}{2}ab\sqrt{1 - \cos^2 C} \quad \text{(from (C.2.1))} \tag{C.8.1}$$

$$= \frac{1}{2}ab\sqrt{1 - \left(\frac{a^2 + b^2 - c^2}{2ab}\right)^2} \text{(from (B.2.1))} \tag{C.8.2}$$

$$= \frac{1}{4}\sqrt{(2ab)^2 - (a^2 + b^2 - c^2)} \tag{C.8.3}$$

$$= \frac{1}{4}\sqrt{(2ab + a^2 + b^2 - c^2)(2ab - a^2 - b^2 + c^2)} \tag{C.8.4}$$

$$= \frac{1}{4}\sqrt{\left\{(a+b)^2 - c^2\right\}\left\{c^2 - (a-b)^2\right\}} \tag{C.8.5}$$

$$= \frac{1}{4}\sqrt{(a+b+c)(a+b-c)(a+c-b)(b+c-a)} \tag{C.8.6}$$

Substituting

$$s = \frac{a+b+c}{2} \tag{C.8.7}$$

in (C.8.6), the area of $\triangle ABC$ is

$$\sqrt{s(s-a)(s-b)(s-c)} \tag{C.8.8}$$

This is known as Hero's formula.

C.9. Prove the following identities

a)

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta. \tag{C.9.1}$$

b)

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta. \tag{C.9.2}$$

Solution: In (C.7.1), let

$$\begin{aligned} \theta_1 + \theta_2 &= \alpha \\ \theta_2 &= \beta \end{aligned} \tag{C.9.3}$$

This gives (C.9.1). In (C.9.1), replace α by $90^\circ - \alpha$. This results in

$$\sin(90^\circ - \alpha - \beta) = \sin(90^\circ - \alpha)\cos\beta - \cos(90^\circ - \alpha)\sin\beta \tag{C.9.4}$$

$$\implies \cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \tag{C.9.5}$$

C.10. Using (C.7.1) and (C.9.2), show that

$$\sin(\theta_1 + \theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 \tag{C.10.1}$$

$$\cos(\theta_1 - \theta_2) = \cos\theta_1\cos\theta_2\sin\theta_1\sin\theta_2 \tag{C.10.2}$$

Solution: From (C.7.1),

$$\sin(\theta_1 + \theta_2)\cos\theta_2 = \sin\theta_1 + \cos(\theta_1 + \theta_2)\sin\theta_2 \tag{C.10.3}$$

Using (C.9.2) in the above,

$$\sin(\theta_1 + \theta_2)\cos\theta_2 = \sin\theta_1 + (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2)\sin\theta_2 \tag{C.10.4}$$

which can be expressed as

$$\sin(\theta_1 + \theta_2)\cos\theta_2 = \sin\theta_1 + \cos\theta_1\cos\theta_2\sin\theta_2 - \sin\theta_1\sin^2\theta_2 \tag{C.10.5}$$

Since

$$\sin^2\theta_2 = 1 - \cos^2\theta_2, \tag{C.10.6}$$

we obtain

$$\sin(\theta_1 + \theta_2)\cos\theta_2 = \cos\theta_1\cos\theta_2\sin\theta_2 + \sin\theta_1\cos^2\theta_2 \tag{C.10.7}$$

resulting in

$$\sin(\theta_1 + \theta_2) = \cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2 \tag{C.10.8}$$

after factoring out $\cos\theta_2$. Using a similar approach, (C.10.2) can also be proved.

C.11. Show that

$$\sin\theta_1 + \sin\theta_2 = 2\sin\left(\frac{\theta_1 + \theta_2}{2}\right)\cos\left(\frac{\theta_1 - \theta_2}{2}\right) \tag{C.11.1}$$

$$\cos\theta_1 + \cos\theta_2 = 2\cos\left(\frac{\theta_1 + \theta_2}{2}\right)\cos\left(\frac{\theta_1 - \theta_2}{2}\right) \tag{C.11.2}$$

$$\sin\theta_1 - \sin\theta_2 = 2\sin\left(\frac{\theta_1 - \theta_2}{2}\right)\cos\left(\frac{\theta_1 + \theta_2}{2}\right) \tag{C.11.3}$$

$$\cos\theta_1 - \cos\theta_2 = 2\sin\left(\frac{\theta_1 + \theta_2}{2}\right)\cos\left(\frac{\theta_2 - \theta_1}{2}\right) \tag{C.11.4}$$

Solution: Let

$$\begin{aligned}\theta_1 &= \alpha + \beta \\ \theta_2 &= \alpha - \beta\end{aligned} \tag{C.11.5}$$

From (C.10.1),

$$\sin\theta_1 + \sin\theta_2 = \sin(\alpha+\beta) + \sin(\alpha-\beta) \tag{C.11.6}$$

$$= \sin\alpha\cos\beta + \cos\alpha\sin\beta \tag{C.11.7}$$

$$+ \sin\alpha\cos\beta - \cos\alpha\sin\beta \tag{C.11.8}$$

$$= 2\sin\alpha\cos\beta \tag{C.11.9}$$

resulting in (C.11.1)

$$\because \alpha = \frac{\theta_1 + \theta_2}{2} \tag{C.11.10}$$

$$\beta = \frac{\theta_1 - \theta_2}{2} \tag{C.11.11}$$

from (C.11.5). Other identities may be proved similarly.

C.12. Show that

$$\sin 2\theta = 2\sin\theta\cos\theta \tag{C.12.1}$$

$$\cos 2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1 \tag{C.12.2}$$

$$= \cos^2\theta - \sin^2\theta \tag{C.12.3}$$